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DIGITAL SIGNAL PROCESSING

By
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KEC-503	DIGITAL SIGNAL PROCESSING	3L:1T:0P	4 Credits

Unit	Topics	Lectures
I	**Introduction to Digital Signal Processing**: Basic elements of digital signal processing, advantages and disadvantages of digital signal processing, Technology used for DSP. **Realization of Digital Systems**: Introduction- basic building blocks to represent a digital system, recursive and non-recursive systems, basic structures of a digital system: Canonic and Non-Canonic structures. **IIR Filter Realization**: Direct form, cascade realization, parallel form realization, Ladder structures- continued fraction expansion of H (z), example of continued fraction, realization of a ladder structure, design examples. **FIR Filter Realization**: Direct, Cascade, FIR Linear Phase Realization and design examples.	8
II	**Infinite Impulse Response Digital (IIR) Filter Design:** Introduction to Filters, Impulse Invariant Transformation, Bi-Linear Transformation, All- Pole Analog Filters: Butterworth and Chebyshev, Design of Digital Butterworth and Chebyshev Filters, Frequency Transformations.	6
III	**Finite Impulse Response Filter (FIR) Design**: Windowing and the Rectangular Window, Gibb's phenomenon, Other Commonly Used Windows (Hamming, Hanning, Bartlett, Blackmann, Kaiser), Examples of Filter Designs Using Windows. **Finite Word length effects in digital filters**: Coefficient quantization error, Quantization noise – truncation and rounding, Limit cycle oscillations-dead band effects.	4 2
IV	**Discrete Fourier Transform:** Concept and relations for DFT/IDFT, Twiddle factors and their properties, computational burden on direct DFT, DFT/IDFT as linear transformations, DFT/IDFT matrices, computation of DFT/IDFT by matrix method, multiplication of DFTs, circular convolution, computation of circular convolution by graphical, DFT/IDFT and matrix methods, linear filtering using DFT, aliasing error, filtering of long data sequences – Overlap-Save and Overlap-Add methods with examples. **Fast Fourier Transform:** Radix-2 algorithm, decimation-in-time, decimation-in-frequency algorithms, signal flow graphs, Butterflies, computations in one place, bit reversal, examples for DIT & DIF FFT Butterfly computations with examples.	6 6
V	**Multirate Digital Signal Processing (MDSP)**: Introduction, Decimation, Interpolation, Sampling rate conversion: Single and Multistage, applications of MDSP- Subband Coding of Speech signals, Quadrature mirror filters, Advantages of MDSP. **Adaptive Filter:** Introduction & Example of adaptive Filter, The window LMS Algorithm, Recursive Least Square Algorithm. The Forward-Backward Lattice and Gradient Adaptive Lattic Method. **Digital Signal Processors:** Introduction, Architecture, Features, Addressing Formats, Functional modes. Introduction to Commercial Digital Signal Processors	4 4 2

Text Books:

1. John G Prokias, Dimitris G Manolakis, Digital Signal Processing. Pearson , 4th Edition, 2007
2. S. Salivahanan, Digital Signal Processing, McGraw Hill, 4th Edition 2017.
3. Johny R. Johnson, Digital Signal Processing, PHI Learning Pvt Ltd., 2009.

Reference Books:

1. Oppenheim & Schafer, Digital Signal Processing. Pearson Education 2015
2. S.K. Mitra, 'Digital Signal Processing–A Computer Based Approach, McGraw Hill, 4th Edition.

Course Outcomes: At the end of this course students will demonstrate the ability to:

1. Design and describe different types of realizations of digital systems (IIR and FIR) and their utilities.
2. Select design parameters of analog IIR digital filters (Butterworth and Chebyshev filters) and implement various methods such as impulse invariant transformation and bilinear transformation of conversion of analog to digital filters.
3. Design FIR filter using various types of window functions.
4. Define the principle of discrete Fourier transform & its various properties and concept of circular and linear convolution. Also, students will be able to define and implement FFT i.e. a fast computation method of DFT.
5. Define the concept of decimation and interpolation. Also, they will be able to implement it in various practical applications.

1 UNIT

Introduction to Digital Signal Processing

CONTENTS

PART-1

Basic Elements of Digital Signal Processing, Advantages and Disadvantages of Digital Signal Processing, Technology Used for DSP.

CONCEPT OUTLINE

- There are two types of signal processing.

i. Analog signal processing.

ii. Digital signal processing (DSP).

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.1. Explain basic elements of digital signal processing with the help of block diagram.

Answer

Fig. 1.1.1 shows the basic elements of digital signal processing system. Input is given through analog digital converter and output is obtained through digital to analog converter.

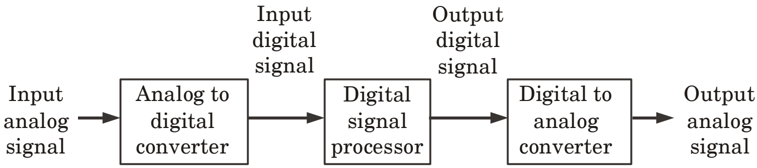

Fig. 1.1.1. Basic elements of digital signal processing.

Basic elements of digital signal processing :

i. Analog to digital converter :

i. The A/D converter converts analog input to digital input. This signal is processed by a DSP system.

2. The A/D converter determines sampling rate and quantization error in digitizing operation.

ii. Digital signal processor :

1. It performs amplification, attenuation, filtering, spectral analysis, feature extraction etc operations on digital data.
2. The digital signal processor consists of ALU, shifter, serial ports, interrupts, address generators etc. for its functioning.
3. The DSP processor has special architectural features due to which DSP operations are implemented fast on it compared to general purpose microprocessors.

ii. Digital to analog converter :

1. Some of the processed signals are required back in their analog form. For example sound, image, video signals and required in analog form. Hence the DSP processor output is given to digital to analog converter.
2. The D/A converter convert digital output of DSP processor to its analog equivalent. Such analog output is processed signal.

Que 1.2. **Write advantages, disadvantages and applications of DSP.**

Answer

A. Advantage of DSP :

1. Digital signal processing systems are flexible.
2. Accuracy of digital signal processing systems is much higher that analog systems.
3. The digital signals can be easily stored on the storage media such as magnetic tapes, disks etc.
4. The digital signal processing systems are easily upgradable since they are software controlled.
5. The digital signal processing system are small size and more reliable.

B. Disadvantages of DSP :

1. When the analog signals have wide bandwidth, then high speed A/D converter are required. Such high speeds of A/D conversion are difficult to achieve for same signals. For such applications, analog system must be used.
2. The digital signal processing systems are expensive for small applications. Hence the selection is done on the basis of cost complexity and performance.

C. Applications of DSP :

1. **DSP for voice and speech :** Speech recognition, voice mail, speech vocoding etc.
2. **DSP for consumer applications :** Digital audio / video / television / music systems, music synthesizer, toys etc.

3. **DSP for military/defence :** Radar processing, Sonar processing, Navigation, missile guidance, RF modems, secure communications.
4. **DSP for industrial applications :** Robotics, CNC security access and power line monitors etc.

Que 1.3. What is the technology used for DSP ?

Answer

i. Dedicated processor based DSP :

1. In such systems the DSP processors are used. The DSP processor of analog devices, Texas instruments and Motorola are commonly used. These DSP processors are designed especially for array operations and multiply-accumulate operations.
2. The DSP processors based systems are stand alone, portable, low cost and suitable for real time applications.

ii. General purpose processor based DSP :

1. Such systems use general purpose micro processors or computers. The software is developed to perform DSP operations on computers.

   For example '*C*' programs can be developed for digital filtering, z-transform, fourier transform, FFT etc. which run on computer.
2. Thus utility of computers can be increased. Such systems are flexible and easily upgradable.
3. The technology of computers such as networking, storage, display, printing etc. can be shared. But such systems are computationally inefficient.
4. If only DSP operations are to be performed then it is better to use dedicated processor based systems.

PART-2

Realization of Digital Systems : Introduction-Basic Building Blocks to Represent a Digital System, Recursive and Non-recursive Systems, Basic Structures of a Digital System : Canonic and Non-canonic Structures.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.4. **What do you understand by realization of digital filters ? What are the basic building blocks ? Discuss the importance of realization of digital system.**

Answer

A. Realization of digital filters :

1. For designing of digital filters, the transfer function $H(z)$ or the impulse response $h(n)$ must be specified. After knowing the $H(z)$ or $h(n)$, the difference equations can be directly obtained from $H(z)$ or $h(n)$.
2. Using these equations the digital filter structure can be implemented or synthesized in hardware or software.
3. Each difference equation or computational algorithm may be implemented by using a digital computer or special purpose digital hardware or special programmable integrated circuit.

B. Basic building blocks :

i. Addition : When two or more than two inputs are applied to a system then the output $y(n)$ is given as

$$y(n) = \sum_{i=1}^{n} x_i(n)$$

The structure is given as

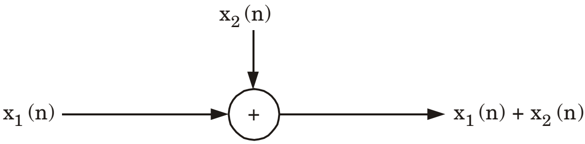

Fig. 1.4.1.

ii. Multiplier : It is given as

Fig. 1.4.2.

or it can also be given as

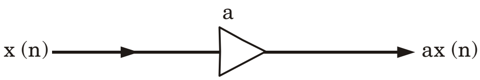

Fig. 1.4.3.

iii. Delay : The delay by 'k' unit is given as

$$x(n) \longrightarrow \boxed{z^{-k}} \longrightarrow x(n-k)$$

Fig. 1.4.4.

The unit delay : If $k = 1$, then it is unit delay and is given by

$$x(n) \longrightarrow \boxed{z^{-1}} \longrightarrow x(n-1)$$

Fig. 1.4.5.

C. Importance of realization of digital system :

1. System can be realized in terms of block diagram and signal flow graph.
2. System transfer function can be determined easily from difference equation corresponding block diagram or signal flow graph representation.
3. Complexity of digital system is reduced.

Que 1.5. What do you mean by recursive and non-recursive system ?

Answer

A. Recursive system : When the output $y(n)$ of the system depends upon present and past inputs as well as past output then it is called as recursive system.

Mathematically,

$$y(n) = F[y(n-1), y(n-2), \ldots\ldots., y(n-N), x(n), x(n-1), \ldots\ldots., x(n-M)]$$

B. Non-recursive system : When the output $y(n)$ of the system depends upon present and past inputs, then it is called as non-recursive system.

Mathematically,

$$y(n) = F[x(n), x(n-1), \ldots\ldots., x(n-M)]$$

Que 1.6. Define canonic and non-canonic structure.

Answer

A. Canonic structure : If the number of delays in the basic realisation block diagram is equal to the order of the difference equation or the order of the transfer function of a digital filter, then the realisation structure is known as canonic structure.

B. Non-canonic structure : If the number of delay in the structure is not same as order, then it is called non-canonic realisation or structure.

PART-3

IIR Filter Realization : Direct Form, Cascade Realization, Parallel Form Realization.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.7. **What are the different basic structures for digital linear systems ? Also obtain the transfer function of IIR and FIR systems.**

Answer

A. Basic structures for digital linear system :

The basic structures for digital linear systems are classified as :

- IIR System
  - Direct form (I and II)
  - Cascade form
  - Parallel structure
  - Transposed structure
  - All pass structure
- FIR System
  - Direct form
  - Cascade form
  - Linear phase filters
  - Frequency sampling

Fig. 1.7.1.

B. IIR systems :

1. The causal systems are characterized by the constant coefficient difference equation as given below :

   Let $x(n)$ is the input and $y(n)$ is the corresponding output.

$$y(n) = -\sum_{k=1}^{N} A_k\, y(n-k) + \sum_{k=0}^{M} B_k x(n-k) \qquad ...(1.7.1)$$

   where A_k and B_k are constant with $A_0 \neq 0$ and $M \leq N$.

2. Taking z-transform on both side of eq. (1.7.1), then we get,

$$Y(z) = -\sum_{k=1}^{N} A_k z^{-k} Y(z) + \sum_{k=0}^{M} B_k z^{-k} X(z)$$

3. Hence the transfer function of IIR system is given by,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} B_k z^{-k}}{1 + \sum_{k=1}^{N} A_k z^{-k}} \qquad ...(1.7.2)$$

4. It is clear from eq. (1.7.1) and (1.7.2) that the realization of IIR systems involves a recursive computational algorithm.

C. FIR system :

1. An FIR system does not have feedback. Hence the past output term $y(n-k)$ will be absent. Hence output of FIR system is given as

$$y(n) = \sum_{k=0}^{M} B_k x(n-k) \quad \text{...(1.7.3)}$$

2. If there are M coefficients then eq. (1.7.3) becomes,

$$y(n) = \sum_{k=0}^{M-1} B_k x(n-k) \quad \text{...(1.7.4)}$$

3. Taking z-transform on both side of eq. (1.7.4)

$$Y(z) = \sum_{k=0}^{M-1} B_k\, z^{-k} X(z)$$

$$\frac{Y(z)}{X(z)} = \sum_{k=0}^{M-1} B_k\, z^{-k}$$

$$H(z) = \sum_{k=0}^{M-1} B_k\, z^{-k}$$

4. Taking inverse z-transform, $h(n) = \begin{cases} B_n & \text{for } 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}$

Que 1.8. Explain the direct form-I structure for IIR system.

Answer

1. The digital system structures determined directly from equation

$$y(n) = -\sum_{k=1}^{N} A_k y(n-k) + \sum_{k=0}^{M} B_k x(n-k)$$

or equation
$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} B_K\, z^{-k}}{1 + \sum_{k=1}^{N} A_K\, z^{-k}}$$

are called the direct form-I structure. In the case of direct form-I, the system function is divided in two parts, and these two parts are connected in cascade *i.e.*,

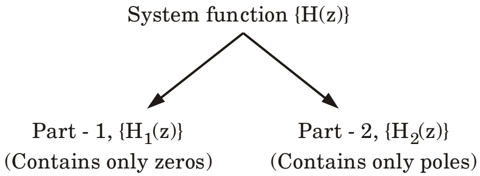

Fig. 1.8.1.

Thus $$H(z) = H_1(z) \, . \, H_2(z)$$

2. The block diagram is shown in Fig. 1.8.2.

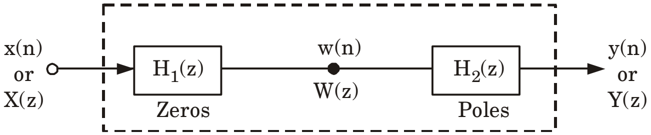

Fig. 1.8.2. Possible IIR system direct form-I realization

3. An intermediate sequence $w(n)$ is introduced which represents the output of the first part and input of the second part.
4. The equation for direct form-I are given as

$$H_1(z) = \frac{W(z)}{X(z)}, \; H_2(z) = \frac{Y(z)}{W(z)}$$

Let $$w(n) = \sum_{k=0}^{M} B_k x(n-k) \qquad \text{...(1.8.1)}$$

and $$y(n) = -\sum_{k=1}^{N} A_k y(n-k) + w(n) \qquad \text{...(1.8.2)}$$

5. Taking z-transform of eq. (1.8.1) and (1.8.2), we get

$$W(z) = X(z)\sum_{k=0}^{M} B_k z^{-k} \qquad \text{...(1.8.3)}$$

and $$Y(z) = -Y(z)\sum_{k=1}^{N} A_k z^{-k} + W(z) \qquad \text{...(1.8.4)}$$

or $$Y(z) = \frac{W(z)}{1 + \sum_{k=1}^{N} A_k z^{-k}} \qquad \text{...(1.8.5)}$$

6. Eq. (1.8.3) can be represented as shown in Fig. 1.8.3.

i.e., $$H_1(z) = \frac{W(z)}{X(z)}$$

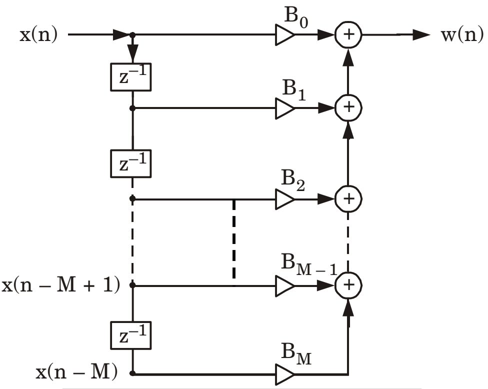

Fig. 1.8.3. Block diagram of $H_1(z)$.

6. Eq. (1.8.5) can be represented in as shown Fig. 1.8.4.

i.e., $$H_2(z) = \frac{Y(z)}{W(z)}$$

7. After joining, the $w(n)$ links of Fig.1.8.3 and Fig. 1.8.4, we will get the complete direct form-I structure.

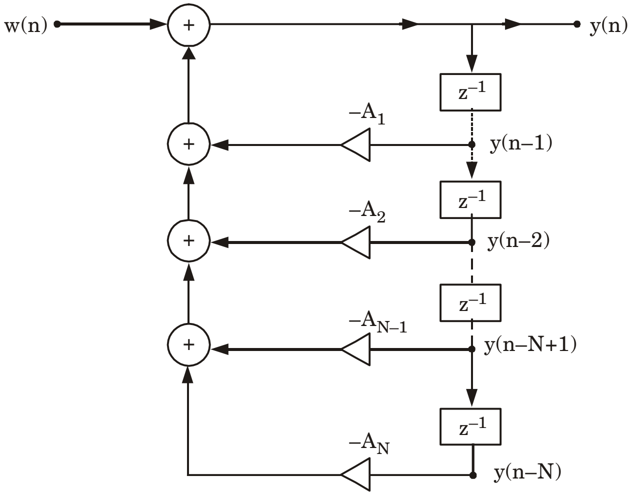

Fig 1.8.4. Block diagram of $H_2(z)$.

Que 1.9. **Draw the block diagram for the following system with input $x(n)$ and output $y(n)$**

$$w(n) = x(n) + \frac{1}{2}\, x(n-1) \text{ and}$$

$$y(n) + \frac{1}{4}\, y(n-1) = w(n)$$

Answer

1. Given, $$w(n) = x(n) + \frac{1}{2}\, x(n-1)$$

   where, $x(n)$ input sequence

2. Given, $y(n) + \dfrac{1}{4} y(n-1) = w(n)$

   $\Rightarrow$ $$y(n) = w(n) - \frac{1}{4} y(n-1)$$

3. So, the block diagram of the system is shown in Fig. 1.9.1.

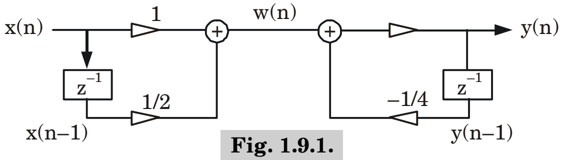

Fig. 1.9.1.

Que 1.10. **Obtain the parallel form realization**

$$H(z) = \frac{(1 + 1/2z^{-1})}{(1 - z^{-1} + 1/4z^{-2})\,(1 - z^{-1} + 1/2z^{-2})}$$

AKTU 2018-19, Marks 07

Answer

1. $$H(z) = \frac{1 + 1/2z^{-1}}{(1 - z^{-1} + 1/4z^{-2})\,(1 - z^{-1} + 1/4z^{-2})}$$

$$= \frac{1 + 1/2z^{-1}}{(1 - 1/2z^{-1})^2\,(1 - z^{-1} + 1/2z^{-2})}$$

$$= \frac{A}{(1 - 1/2z^{-1})} + \frac{B}{(1 - 1/2z^{-1})^2} + \frac{Cz^{-1} + D}{(1 - z^{-1} + 1/2z^{-2})}$$

2. On solving we get,

$$A = 2,\ B = 2,\ C = 2, \text{ and } D = -3.$$

3. So, $$H(z) = \frac{2}{(1 - 1/2z^{-1})} + \frac{2}{(1 - 1/2z^{-1})^2} + \frac{2z^{-1} - 3}{(1 - z^{-1} + 1/2z^{-2})}$$

$$= \frac{2}{(1 - 1/2z^{-1})} + \frac{2}{(1 - z^{-1} + 1/4z^{-2})} + \frac{2z^{-1} - 3}{(1 - z^{-1} + 1/2z^{-2})}$$

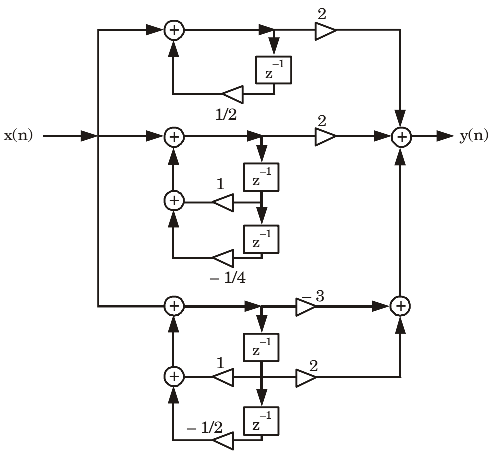

Fig. 1.10.1.

Que 1.11. **Obtain the parallel form realization for the system function given below :**

$$H(z) = \frac{(1 + 0.25\, z^{-1})}{(1 + 0.5\, z^{-1})\,(1 + 0.5\, z^{-1} + 0.25\, z^{-2})}$$

AKTU 2019-20, Marks 07

Answer

1. Given,

$$H(z) = \frac{(1 + 0.25\, z^{-1})}{(1 + 0.5\, z^{-1})\,(1 + 0.5\, z^{-1} + 0.25\, z^{-2})}$$

$$\frac{1 + 0.25z^{-1}}{(1 + 0.5z^{-1})\,(1 + 0.5z^{-1} + 0.25z^{-2})} = \frac{A}{(1 + 0.5z^{-1})} + \frac{Bz^{-1} + C}{(1 + 0.5z^{-1} + 0.25z^{-2})}$$

2. On solving, we get, $A = \frac{1}{2}$, $B = -\frac{1}{4}$ and $C = \frac{1}{2}$.

3. So, $H(z) = \dfrac{\frac{1}{2}}{(1 + 0.5z^{-1})} + \dfrac{-\frac{1}{4}z^{-1} + \frac{1}{2}}{(1 + 0.5z^{-1} + 0.25z^{-2})}$

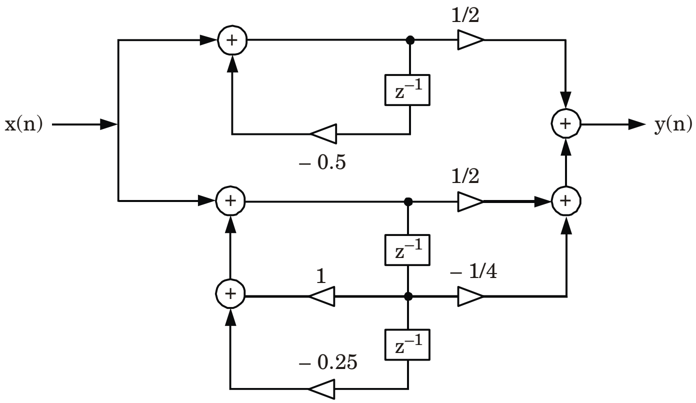

Fig. 1.11.1.

Que 1.12. **Obtain the parallel form realization for the transfer function $H(z)$ given below :**

$$H(z) = \frac{2 + z^{-1} + 1/4\, z^{-2}}{\left(1 + \frac{1}{2} z^{-1}\right)\left(1 + z^{-1} + \frac{1}{2} z^{-2}\right)}$$

AKTU 2016-17, Marks 10

Answer

1. Given, $H(z) = \dfrac{2 + z^{-1} + \frac{1}{4} z^{-2}}{\left(1 + \frac{1}{2} z^{-1}\right)\left(1 + z^{-1} + \frac{1}{2} z^{-2}\right)}$

2. $H(z)$ can be written as

$$H(z) = \frac{8z^3 + 4z^2 + z}{(2z + 1)\,(2z^2 + 2z + 1)}$$

3. $$\frac{H(z)}{z} = \frac{8z^2 + 4z + 1}{(2z + 1)\,(2z^2 + 2z + 1)}$$

$$= \frac{A}{2z + 1} + \frac{Bz + C}{2z^2 + 2z + 1} \quad \text{...(1.12.1)}$$

4. On solving eq. (1.12.1), we get,

$$A = 2, B = 2, C = -1$$

5. Now eq. (1.12.1) can be written as

$$\frac{H(z)}{z} = \frac{2}{2z + 1} + \frac{(2z - 1)}{2z^2 + 2z + 1}$$

$$H(z) = \frac{2z}{2z+1} + \frac{z(2z-1)}{2z^2+2z+1}$$

So,
$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{1-(1/2)z^{-1}}{1+z^{-1}+\frac{1}{2}z^{-2}}$$

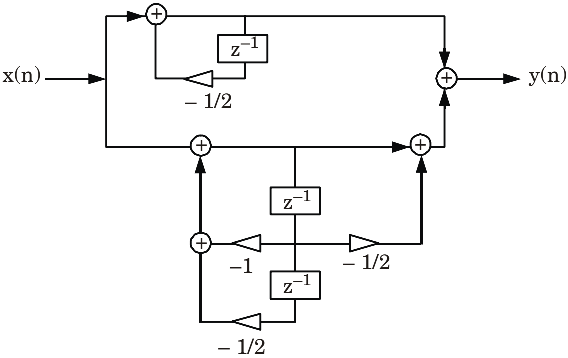

Fig. 1.12.1.

Que 1.13. **Obtain the Cascade form realization :**

***y*(*n*) = *y*(*n* – 1) – 1/2 [*y*(*n* – 2)] + 1/4 [*y*(*n* – 2)] + *x*(*n*) – *x*(*n* – 1) + *x* (*n* – 2)**

AKTU 2018-19, Marks 07

Answer

1. Given,
$$y(n) = y(n-1) - 1/2\,[y(n-2)] + 1/4\,[y(n-2)] + x(n) - x(n-1) + x\,(n-2)$$
$$y(n) = y(n-1) - 1/4\,[y(n-2)] + x(n) - x(n-1) + x\,(n-2)$$

2. Taking z-transform on both sides
$$Y(z) = z^{-1}\,Y(z) - 1/4\,[z^{-2}\,Y(z)] + X(z) - z^{-1}\,X(z) + z^{-2}\,X(z)$$
$$Y(z) - z^{-1}\,Y(z) + 1/4\,[z^{-2}\,Y(z)] = X(z)\,[1 - z^{-1} + z^{-2}]$$
$$\frac{Y(z)}{X(z)} = \frac{1-z^{-1}+z^{-2}}{1-z^{-1}+1/4z^{-2}}$$
$$\frac{Y(z)}{X(z)} = \frac{(1-z^{-1}+z^{-2})}{(1-z^{-1}/2)\,(1-z^{-1}/2)}$$
$$= \frac{1}{(1-z^{-1}/2)} \cdot \frac{(1-z^{-1}+z^{-2})}{(1-z^{-1}/2)}$$

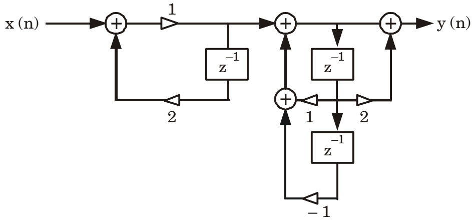

Fig. 1.13.1.

Que 1.14. **Obtain the Direct form I and II form realization**

$$H(z) = \frac{(1+z^{-1})(1+2z^{-1})}{(1+1/\ 2z^{-1})(1-1/4\,z^{-1})(1+1/8z^{-1})}$$

AKTU 2018-19, Marks 07

Answer

1. Given,

$$H(z) = \frac{(1-z^{-1})(1+2z^{-1})}{(1+1/2\,z^{-1})(1-1/4\,z^{-1})\left(1+1/8\,z^{-1}\right)}$$

$$H(z) = \frac{(1+3z^{-1}+2z^{-2})}{(1+1/8\,z^{-1}-3/32\,z^{-2}-1/64\,z^{-3})}$$

i. Direct form I :

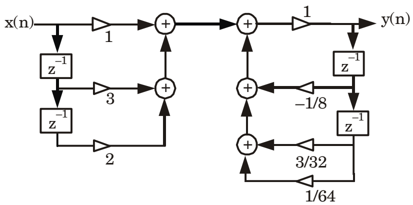

Fig. 1.14.1.

ii. Direct form II :

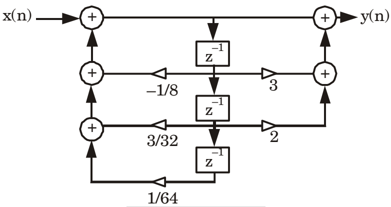

Fig. 1.14.2.

Que 1.15. **Obtain direct form I, direct form II and parallel form structures for the following filter.**

$$y(n) = \frac{3}{4}y(n-1) + \frac{3}{32}y(n-2) + \frac{1}{64}y(n-3) + x(n) + 3x(n-1) + 2x(n-2)$$

AKTU 2015-16, Marks 10

Answer

1. Given,

$$y(n) = \frac{3}{4}y(n-1)+\frac{3}{32}y(n-2)+\frac{1}{64}y(n-3) + x(n) + 3x(n-1) + 2x(n-2) \quad ...(1.15.1)$$

2. Taking z-transform of both side of eq. (1.15.1)

$$Y(z) = \frac{3}{4}z^{-1}Y(z)+\frac{3}{32}z^{-2}Y(z)+\frac{1}{64}z^{-3}Y(z) + X(z) + 3z^{-1}X(z) + 2z^{-2}X(z)$$

$$Y(z)\left(1-\frac{3}{4}z^{-1}-\frac{3}{32}z^{-2}-\frac{1}{64}z^{-3}\right) = X(z)(1+3z^{-1}+2z^{-2})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1+3z^{-1}+2z^{-2})}{\left(1-\frac{3}{4}z^{-1}-\frac{3}{32}z^{-2}-\frac{1}{64}z^{-3}\right)}$$

i. Direct form I :

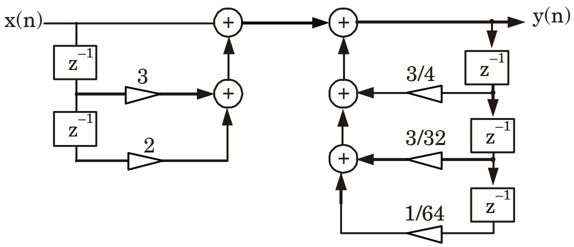

Fig. 1.15.1.

ii. Direct form II :

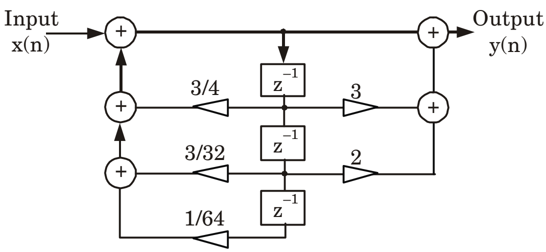

Fig. 1.15.2.

iii. Parallel realization :

$$H(z) = \frac{(1+3z^{-1}+2z^{-2})}{\left(1-\frac{3}{4}z^{-1}-\frac{3}{32}z^{-2}-\frac{1}{64}z^{-3}\right)} = \frac{z(z^2+3z+2)}{\left(z^3-\frac{3}{4}z^2-\frac{3}{32}z-\frac{1}{64}\right)}$$

$$\frac{H(z)}{z} = \frac{(z+1)(z+2)}{64z^3-48z^2-6z-1} = \frac{(z+1)(z+2)}{(z-0.87)(z+0.06)^2}$$

Now, $$\frac{(z+1)(z+2)}{(z-0.87)(z^2+0.06)^2} = \frac{A}{(z-0.87)} + \frac{Bz+C}{(z+0.06)^2}$$

On solving, we get

$A = 6.2, B = -5.2, C = -2.27$

So, $$\frac{H(z)}{z} = \frac{6.2}{(z-0.87)} + \frac{(-5.2z-2.27)}{(z+0.06)^2}$$

So, $$H(z) = \frac{6.2\,z}{(z-0.87)} + \frac{(-5.2z-2.27)z}{(z+0.06)^2}$$

$$H(z) = \frac{6.2}{(1-0.87z^{-1})} + \frac{(-5.2-2.27z^{-1})}{(1+0.12z^{-1}+0.0036z^{-2})}$$

$$H(z) = H_1(z) + H_2(z)$$

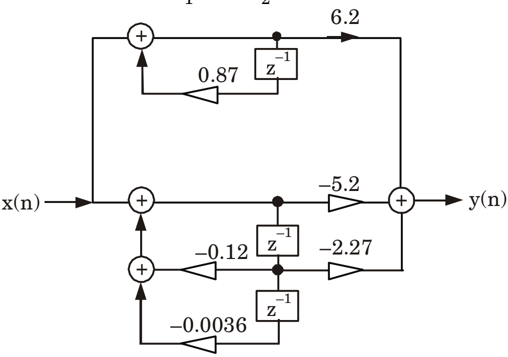

Fig. 1.15.3. Parallel realization of H(z).

Que 1.16. **Obtain the direct form I, direct form II, cascade and parallel form realization for the following system :**

$$\boldsymbol{y(n) = -\,0.1\,y(n-1) + 0.2\,y(n-2) + 3\,x(n) + 3.6\,x(n-1) + 0.6\,x(n-2)}$$

AKTU 2017-18, Marks 10

Answer

1. Given, $y(n) = 0.1y(n-1) + 0.2y(n-2) + 3x(n) + 3.6x(n-1) + 0.6x(n-2)$

2. Taking z-transform on both sides,

$$Y(z) = -\,0.1z^{-1}\,Y(z) + 0.2z^{-2}\,Y(z) + 3X(z) + 3.6z^{-1}X(z) + 0.6z^{-2}X(z)$$

$$\frac{Y(z)}{X(z)} = \left(\frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}\right)$$

A. Direct form I :

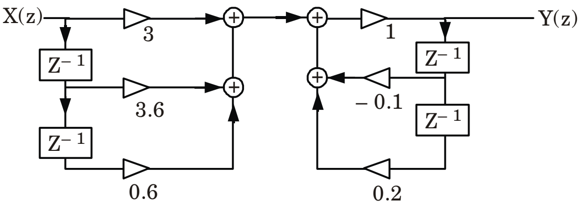

Fig. 1.16.1.

B. Direct form II :

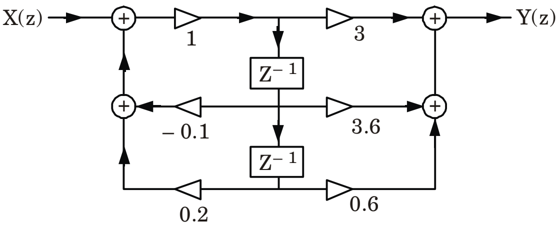

Fig. 1.16.2.

C. Cascade form :

1. We have,
$$H(z) = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}} \quad ...(1.16.1)$$

2. After rearranging the eq. (1.16.1)
$$H(z) = \frac{3z^2 + 3.6z + 0.6}{z^2 + 0.1z - 0.2}$$
$$H(z) = \frac{(3z + 0.6)(z + 1)}{(z - 0.4)(z + 0.5)}$$
$$H(z) = \frac{(3 + 0.6z^{-1})(1 + z^{-1})}{(1 - 0.4z^{-1})(1 + 0.5z^{-1})}$$
$$H(z) = H_1(z) \,.\, H_2(z)$$

where
$$H_1(z) = \frac{3 + 0.6z^{-1}}{1 - 0.4z^{-1}}, \; H_2 = \frac{1 + z^{-1}}{1 + 0.5z^{-1}}$$

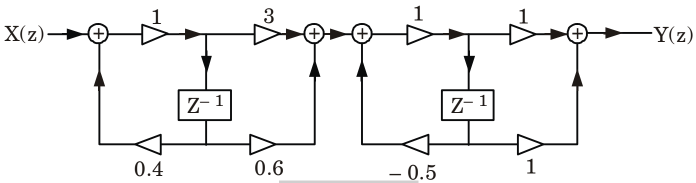

Fig. 1.16.3.

D. Parallel form :

1. We have,

$$H(z) = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

$$H(z) = \frac{3z^2 + 3.6z + 0.6}{z^2 + 0.1z - 0.2}$$

$$H(z) = 3 + \frac{3.3z + 1.2}{(z - 0.4)(z + 0.5)}$$

2. Taking partial fraction

$$H(z) = 3 + \frac{A}{(z - 0.4)} + \frac{B}{(z + 0.5)}$$

3. By solving and we get,

$$A = 2.8, \quad B = 0.5$$

$$H(z) = 3 + \frac{2.8}{z - 0.4} + \frac{0.5}{z + 0.5}$$

$$H(z) = 3 + \frac{2.8}{z - 0.4z^{-1}} + \frac{0.5}{z + 0.5z^{-1}}$$

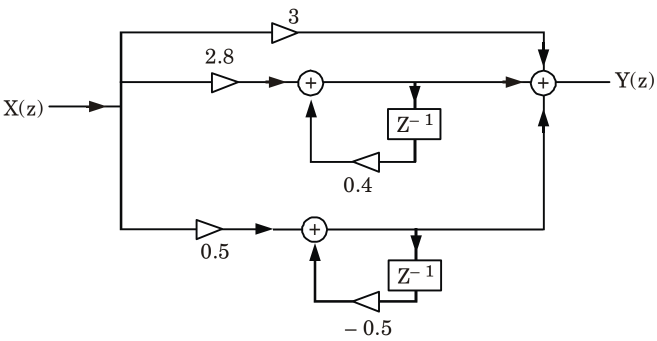

Fig. 1.16.4.

Que 1.17. **Consider the causal linear shift invariant filter with the system function**

$$H(z) = \frac{1 + 0.875z^{-1}}{(1 + 0.2z^{-1} + 0.9z^{-2})(1 - 0.7z^{-1})}$$

Obtain following realization :

a. Direct form II

b. A cascade of first order and second order system realized in transposed direct form II

c. A parallel connection of first order and second order system realized in direct form II

AKTU 2015-16, Marks 10

Answer

1. Given, $H(z) = \dfrac{(1 + 0.875z^{-1})}{(1 + 0.2z^{-1} + 0.9z^{-2})(1 - 0.7z^{-1})}$

a. Direct form II :

$$H(z) = \frac{(1 + 0.875z^{-1})}{(1 + 0.2z^{-1} + 0.9z^{-2} - 0.7z^{-1} - 0.14z^{-2} - 0.63z^{-3})}$$

$$H(z\ ) = \frac{(1 + 0.875z^{-1})}{(1 - 0.5z^{-1} + 0.76z^{-2} - 0.63z^{-3})}$$

For direct form II :

$$H(z) = H_1(z).\ H_2(z)$$

$$H_1(z) = \frac{1}{(1 - 0.5z^{-1} + 0.76z^{-2} - 0.63z^{-3})}$$

and $H_2(z) = (1 + 0.875z^{-1})$

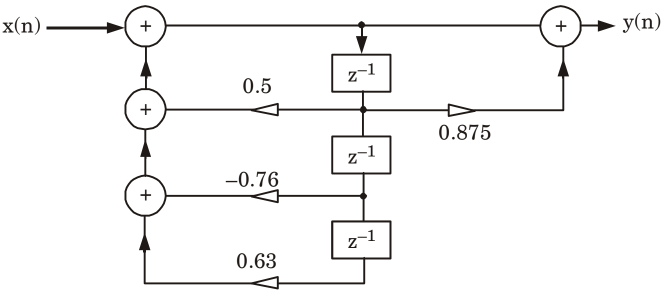

Fig. 1.17.1. Direct form II realization of H(z).

b. Cascade Form :

1. Given, $$H(z) = \frac{(1+0.875z^{-1})}{(1+0.2z^{-1}+0.9z^{-2})(1-0.7z^{-1})}$$

2. Let $$H_1(z) = \frac{1}{(1+0.2z^{-1}+0.9z^{-2})}$$

and $$H_2(z) = \frac{(1+0.875z^{-1})}{(1-0.7z^{-1})}$$

$$H(z) = H_1(z)\, H_2(z)$$

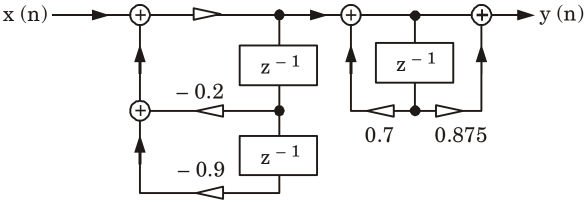

Fig. 1.17.2. Cascade form.

Cascade realization in transpose direct form II :

The transpose direct form II is obtained by making the following changes :

i. Reverse all signal flow graph direction.

ii. Interchange input and output.

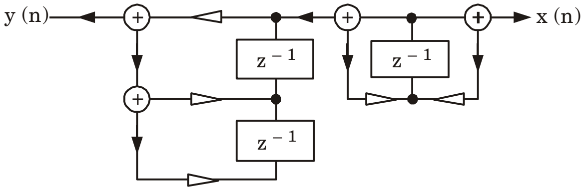

Fig. 1.17.3. Cascade realization in transpose direct form II.

c. Parallel realization :

$$H(z) = \frac{(1+0.875z^{-1})}{(1+0.2z^{-1}+0.9z^{-2})(1-0.7z^{-1})}$$

Now,

$$\frac{(1+0.875z^{-1})}{(1+0.2z^{-1}+0.9z^{-2})(1-0.7z^{-1})} = \frac{A}{(1-0.7\,z^{-1})} + \frac{Bz^{-1}+C}{(1+0.2\,z^{-1}+0.9\,z^{-2})}$$

On solving, we get, $A = 0.72$, $B = 0.93$, $C = 0.28$

So, $$H(z) = \frac{0.72}{(1-0.7\,z^{-1})} + \frac{(0.93\,z^{-1}+0.28)}{(1+0.2\,z^{-1}+0.9z^{-2})}$$

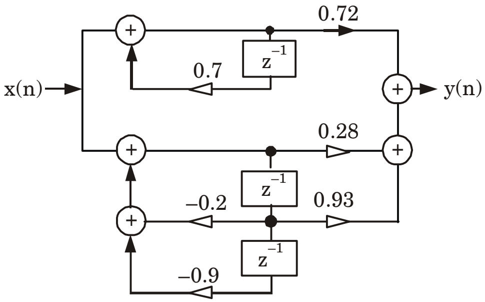

Fig. 1.17.4. Parallel realization.

PART-4

Ladder Structures : Continued Fraction Expansion of H(z), Example of Continued Fraction, Realization of a Ladder Structure, Design Example.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.18. **Discuss the realization of a ladder structure and its advantages.**

Answer

A. Realization of ladder structure :

1. In order to develop a digital ladder structure, the system function $H(z)$ can be written as

$$H(z) = \frac{a_N z^{-N} + a_{N-1} z^{-N+1} + ... + a_1 z^{-1} + a_0}{b_N z^{-N} + b_{N-1} z^{-N+1} + ... + b_1 z^{-1} + b_0} \quad ...(1.18.1)$$

2. The first ladder structure can be obtained from the continued fraction expansion

$$H(z) = \alpha_0 + \cfrac{1}{\beta_1 z^{-1} + \cfrac{1}{\alpha_1 + \cfrac{1}{\beta_2 z^{-1} + \ddots + \cfrac{1}{\alpha_N}}}} \quad ...(1.18.2)$$

3. To determine the coefficients in eq. (1.18.2), Routh array is developed.

$$\begin{array}{lllllll}
z^{-N} & a_N & a_{N-1} & a_{N-2} & --- & a_1 & a_0 \\
z^{-N} & b_N & b_{N-1} & b_{N-2} & --- & b_1 & b_0 \\
z^{-N+1} & c_{N-1} & c_{N-2} & c_{N-3} & --- & c_0 & \\
z^{-N+1} & d_{N-1} & d_{N-2} & d_{N-3} & --- & d_0 & \\
z^{-N+2} & e_{N-2} & e_{N-3} & e_{N-4} & & & \\
z^{-N+2} & f_{N-3} & f_{N-4} & & & & \\
- & - & & & & & \\
- & - & & & & & \\
- & - & & & & & \\
1 & g_0 & & & & & \\
1 & h_0 & & & & &
\end{array}$$

4. The first row of the array contains the numerator coefficients and second row contains denominator coefficients. The continued fraction coefficients are given by

$$\alpha_0 = a_N/b_N,\ \beta_1 = b_N/c_{N-1},\ \alpha_1 = c_{N-1}/d_{N-1}$$

5. Eq. (1.18.2), can be written as

$$H(z) = \left[\alpha_0 + \cfrac{1}{\beta_1 z^{-1} + \cfrac{1}{T_1(z)}}\right] \quad ...(1.18.3)$$

where,
$$T_1(z) = \alpha_1 + \cfrac{1}{\beta_1 z^{-1} + \ddots} \quad ...(1.18.4)$$

6. The transfer function $H(z)$ can be written as

$$H(z) = \frac{Y(z)}{X(z)}$$

$$Y(z) = \left[\alpha_0 + \cfrac{1}{\beta_1 z^{-1} + \cfrac{1}{T_1(z)}}\right] X(z)$$

$$Y(z) = \alpha_0 X(z) + \left[\cfrac{1}{\beta_1 z^{-1} + \cfrac{1}{T_1(z)}}\right] X(z) \quad ...(1.18.5)$$

7. Let
$$H_1(z) = \cfrac{1}{\beta_1 z^{-1} + \cfrac{1}{T_1(z)}} \quad ...(1.18.6)$$

8. Eq. (1.18.5) becomes

$$Y(z) = \alpha_0 X(z) + H_1(z) X(z)$$
$$Y_1(z) = H_1(z) X(z)$$
$$Y_1(z) = \cfrac{1}{\beta_1 z^{-1} + \cfrac{1}{T_1(z)}} X(z) \quad ...(1.18.7)$$

$$\left[\beta_1 z^{-1} + \frac{1}{T_1(z)}\right] Y_1(z) = X(z)$$

9. For realization purpose

$$Y_1(z) = T_1(z)\,[X(z) - \beta_1 z^{-1}\, Y_1(z)]$$

$$= T_1(z)\, X_1(z) \qquad ...(1.18.8)$$

where, $$X_1(z) = X(z) - \beta_1 z^{-1}\, Y_1(z) \qquad ...(1.18.9)$$

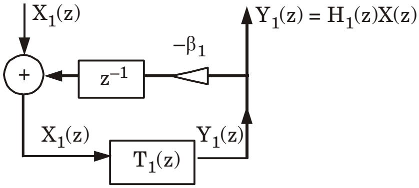

Fig. 1.18.1. A realization of $H_1(z)$.

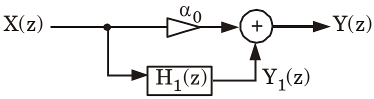

Fig. 1.18.2.

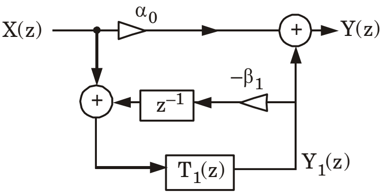

Fig. 1.18.3. Structure for $H(z) = \alpha_0 + 1/[\beta_1 z^{-1} + 1/T_1(z)]$.

10. The expression for $T_1(z)$ is given as

$$T_1(z) = \frac{Y_1(z)}{X_1(z)} = \alpha_1 + \cfrac{1}{\beta_2 z^{-1} + \cfrac{1}{T_2(z)}}$$

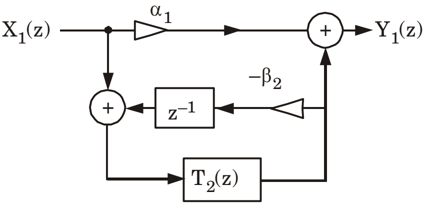

Fig. 1.18.4. Structure for $T_1(z)$.

11. With continued fractional realization of $H(z)$, the resultant structure is shown in Fig. 1.18.5.

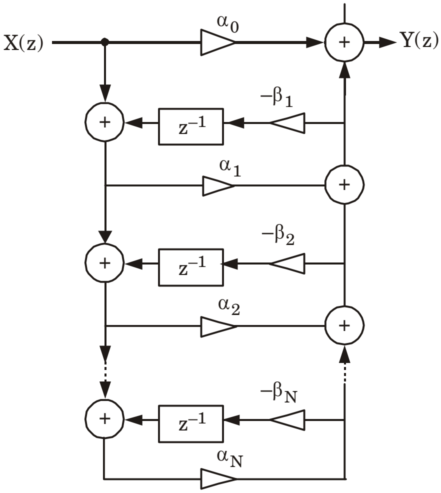

Fig. 1.18.5. Continued fraction realization of $H(z)$.

B. Advantages :

1. The ladder form of realization provides desirable coefficient sensitivity properties.
2. Digital filters can also be realized using structure that is similar to those obtained for analog filters.
3. Ladder structures required minimum amount of memory.

Que 1.19. Draw the Ladder structure for the system with system function

$$H(z) = \frac{5z^{-3} + 2z^{-2} + 3z^{-1} + 1}{z^{-3} + z^{-2} + z^{-1} + 1}$$

Answer

1. Given, $$H(z) = \frac{5z^{-3} + 2z^{-2} + 3z^{-1} + 1}{z^{-3} + z^{-2} + z^{-1} + 1}$$
2. For the given system, Routh array is

z^{-3}	5	2	3	1
z^{-3}	1	1	1	1
z^{-2}	– 3	– 2	– 4	0
z^{-2}	– 1/3	– 1/3	1	
z^{-1}	– 5	5	0	
z^{-1}	0			
1				
1				

3. Since there is a zero in the first column of Routh array, the filter coefficients will tend to infinity which is not possible. Hence the ladder structure is not possible for the given transfer function.

Que 1.20. **Obtain the ladder structure for the system function $H(z)$ given below :**

$$H(z) = \frac{2 + 8z^{-1} + 6z^{-2}}{1 + 8z^{-1} + 12z^{-2}}$$

AKTU 2016-17, Marks 7.5

AKTU 2019-20, Marks 07

Answer

1. For the given system, obtain the Routh array

z^{-2}	6	8	2
z^{-2}	12	8	1
z^{-1}	4	3/2	
z^{-1}	7/2	1	
1	5/14	0	
1	1		

2. The ladder structure parameters are

$$\alpha_0 = \frac{6}{12} = \frac{1}{2},\ \beta_1 = \frac{12}{4} = 3,\ \alpha_1 = \frac{4}{7/2} = \frac{8}{7},\ \beta_2 = \frac{7/2}{5/14} = \frac{49}{5},$$

$$\alpha_2 = \frac{5/14}{1} = \frac{5}{14}$$

$$H(z) = \frac{1}{2} + \cfrac{1}{3z^{-1} + \cfrac{1}{\cfrac{8}{7} + \cfrac{1}{(49/5)z^{-1} + \cfrac{1}{5/14}}}}$$

3. The ladder structure is shown in Fig. 1.20.1.

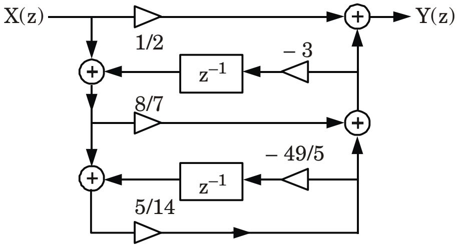

Fig. 1.20.1.

Que 1.21. **A system function is given as under :**

$$H(z) = \frac{(1 + 8z^{-1} + 6z^{-2})}{(1 + 8z^{-1} + 12z^{-2})}$$

realize the system function using ladder structure.

AKTU 2017-18, Marks 05

Answer

1. For the given system, obtain the Routh array

z^{-2}	6	8	1
z^{-2}	12	8	1
z^{-1}	4	1/2	
z^{-1}	13/2	1	
1	– 3/26	0	
1	1		

2. The ladder structure parameters are,

$$\alpha_0 = \frac{6}{12} = \frac{1}{2},\ \beta_1 = \frac{12}{4} = 3,\ \alpha_1 = \frac{4}{13/2} = 0.62\ \beta_2 = \frac{13/2}{-3/26} = -56.34$$

$$\alpha_2 = \frac{-3/26}{1} = -0.12$$

$$H(z) = \frac{1}{2} + \cfrac{1}{3z^{-1} + \cfrac{1}{\cfrac{8}{13} + \cfrac{1}{\cfrac{169}{3}z^{-1} + \cfrac{1}{\cfrac{-3}{26}}}}}$$

3. Ladder structure is shown in Fig. 1.21.1.

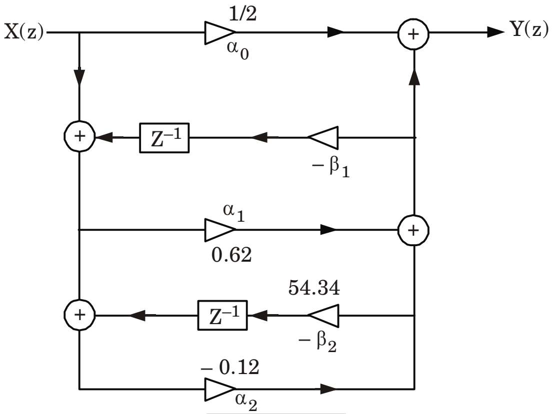

Fig. 1.21.1.

PART-5

FIR Filter Realization : Direct, Cascade, FIR Linear Phase Realization and Design Examples.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.22. **Explain the direct form realization of FIR systems.**

Answer

1. The system response of FIR system is given by

$$y(n) = \sum_{k=0}^{M-1} h(k)\, x(n-k) \qquad \text{...(1.22.1)}$$

where, $y(n)$ = Output sequence

$x(n)$ = Input sequence

2. Eq. (1.22.1) can be achieved by the equation of IIR system. The equation of IIR system is given as

$$y(n) = \sum_{k=1}^{N} A_k y(n-k) + \sum_{k=0}^{M} B_k x(n-k) \qquad \text{...(1.22.2)}$$

3. Comparing eq. (1.22.1) and (1.22.2), we get

$B_k = h(k)$

and $A_k = 0$

where, $k = 0, 1, 2, \ldots\ldots, M-1$

4. The direct form realization for eq. (1.22.1) is given in Fig. 1.22.1.

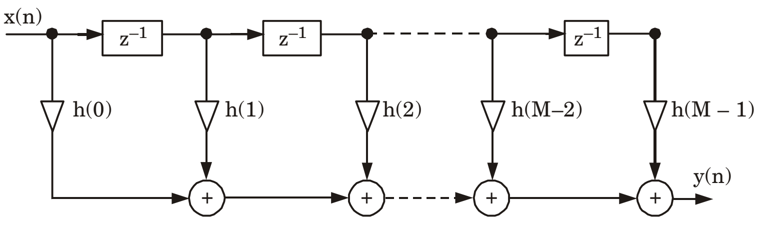

Fig. 1.22.1. Direct form realization structure of an FIR system.

i. The direct form realization structure for FIR system is the special case of the direct form realization structure for IIR system.

ii. The direct form FIR realization is similar to direct form structure realization of IIR systems except $A_k = 0$.

Que 1.23. **Give two different realization of the system described by system function :**

$$H(z) = \frac{1}{2} + \frac{1}{4}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{2}z^{-3}$$

and compare them.

Answer

1. Given, $$H(z) = \frac{1}{2} + \frac{1}{4}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{2}z^{-3}$$

i. Direct form realization :

The given function is all zero function, so it represents a FIR filter. Here,

$$b_0 = \frac{1}{2}, b_1 = \frac{1}{4}, b_2 = \frac{1}{4}, b_3 = \frac{1}{2}$$

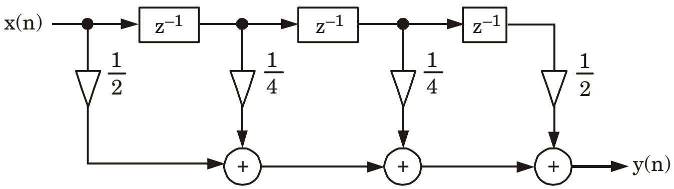

Fig. 1.23.1. Direct form realization of $H(z)$.

ii. Cascade form realization :

$$H(z) = \frac{(2 + z^{-1} + z^{-2} + 2z^{-3})z^3}{4\,z^3} = \frac{2z^3 + z^2 + z + 2}{4\,z^3}$$

$$= \frac{(z+1)(2z^2 - z + 2)}{4\,z^3} = \frac{(z+1)}{z} \cdot \frac{(2z^2 - z + 2)}{4z^2}$$

$$= (1 + z^{-1})\left(\frac{1}{2} - \frac{1}{4}z^{-1} + \frac{1}{2}z^{-2}\right) = H_1(z) \,.\, H_2(z).$$

Here, $$H_1(z) = 1 + z^{-1}$$

and $$H_2(z) = \frac{1}{2} - \frac{1}{4}z^{-1} + \frac{1}{2}z^{-2}$$

The cascade form of realization is shown in Fig. 1.23.2

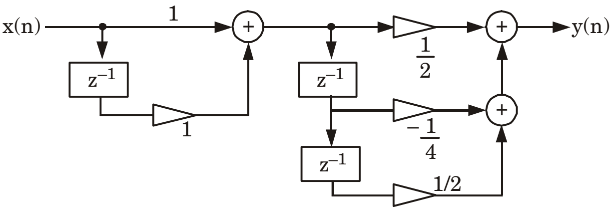

Fig. 1.23.2. Cascade realization of $H(z)$.

iii. Comparison :

S. No.	Direct form realization	Cascade form realization
1.	It can be obtained directly from the difference equation or system function.	It is obtained by first decomposing the function $H(z)$ into the product of several transfer function.
2.	Simple to construct.	Difficult to design.
3.	No additional computation is required.	Additional computation is required.
4.	Computational accuracy is low.	Computational accuracy is good.

Que 1.24. **Obtain a linear phase and cascade realization of the system**

$$H(z) = (1 + 0.5\, z^{-1} + z^{-2})\,(1 + 0.5\, z^{-1} + z^{-2})$$

AKTU 2019-20, Marks 07

Answer

1. Given, $H(z) = (1 + 0.5z^{-1} + z^{-2})\,(1 + 0.5z^{-1} + z^{-2})$
2. Assume $H(z) = H_1(z).H_2(z)$

   where $H_1(z) = (1 + 0.5z^{-1} + z^{-2})$

   $H_2(z) = (1 + 0.5z^{-1} + z^{-2})$
3. The cascade realization of the system is shown in Fig. 1.24.1.

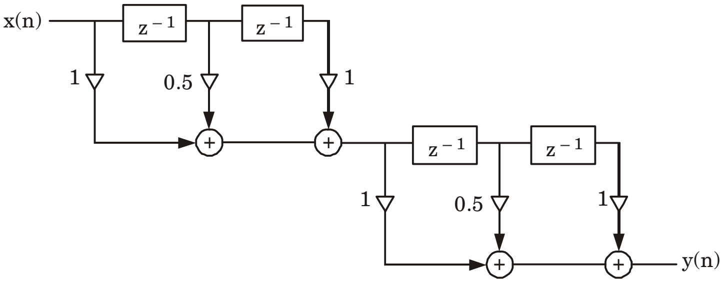

Fig. 1.24.1.

Que 1.25. **Compare FIR and IIR filter.**

Answer

S. No.	FIR filter	IIR filter
1.	These filters can be easily designed to have perfectly linear phase.	These filters do not have linear phase.
2.	FIR filters can be realized recursively and non-recursively.	IIR filters are easily realized recursively.
3.	Greater flexibility to control the shape of their magnitude response.	Less flexibility, usually limited to specific king of filters.
4.	Errors due to round off noise are less severe in FIR filters, mainly because feedback is not used.	The round off noise in IIR filters is more.

Que 1.26. **Explain linear phase FIR filter with its mathematical expression.**

Answer

1. For linear phase FIR filter, signal falling entirely in the passband will be reproduced with the delay equal to the slope of the phase curve.
2. One of the most important features of FIR digital filters is that they can be designed to have exactly linear phase.
3. The unit-sample response for a causal FIR digital filter with linear phase has the following property.

$$h(n) = h(N-1-n) \quad \text{...(1.26.1)}$$

4. However, since the coefficient are equal in pairs, $h(n) = h(N-1-n)$, it is possible to use this symmetry to reduce the number of multipliers required in the realization.
5. System function $H(z)$ is given by

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} \quad \text{...(1.26.2)}$$

i.e., if the impulse response or unit sample response $h(n)$ is N samples in duration, then $H(z)$ is polynomial in z^{-1} of degree $N-1$.

6. To see that the condition $h(n) = h(N-1-n)$ implies linear phase, we can write eq. (1.26.2) as

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n}$$

$$= \underbrace{\sum_{n=0}^{\left(\frac{N}{2}-1\right)} h(n)z^{-n}}_{\text{For Ist half point of } h(n)} + \underbrace{\sum_{n=\frac{N}{2}}^{N-1} h(n)z^{-n}}_{\text{For IInd half point for } h(n)} \quad ...(1.26.3)$$

7. Putting, $n = N - 1 - n$

Lower limit, $n = \dfrac{N}{2}$ then $n = N - 1 - \dfrac{N}{2} = \dfrac{N}{2} - 1$

and upper limit, $n = N - 1$ then, $n = N - 1 - (N - 1)$
$= N - 1 - N + 1 = 0$

8. Replacing these limits in IInd part of eq. (1.26.3), we get

$$H(z) = \sum_{n=0}^{\left(\frac{N}{2}-1\right)} h(n)z^{-n} + \sum_{\frac{N}{2}-1}^{n=0} h(N-1-n)z^{-(N-1-n)}$$

or
$$H(z) = \sum_{n=0}^{\left(\frac{N}{2}-1\right)} h(n)z^{-n} + \sum_{n=0}^{\left(\frac{N}{2}-1\right)} h(N-1-n)z^{-(N-1-n)} \quad ...(1.26.4)$$

i. If N is even and using the condition $h(n) = h(N - 1 - n)$ in eq. (1.26.3), we get

$$H(z) = \sum_{n=0}^{\left(\frac{N}{2}-1\right)} h(n)\,[z^{-n} + z^{-(N-1-n)}] \quad ...(1.26.5)$$

ii. If N is odd then $H(z)$ can be expressed as

$$H(z) = h\left(\frac{N-1}{2}\right) z^{-[(N-1)/2]} + \sum_{n=0}^{[(N-1)/2]-1} h(n)[z^{-n} + z^{-(N-1-n)}] \quad ...(1.26.6)$$

iii Frequency response $H(e^{j\omega})$ for N even.

Putting $z = e^{j\omega}$ in eq. (1.26.5), we get $H(j\omega)$ for N even as

$$H(e^{j\omega}) = \sum_{n=0}^{\left(\frac{N}{2}-1\right)} h(n)\,[e^{-j\omega n} + e^{-j\omega(N-1-n)}]$$

or
$$H(e^{j\omega}) = e^{-j\omega[(N-1)/2]}\left\{\sum_{n=0}^{\left(\frac{N}{2}-1\right)} 2h(n)\cos\left[\omega\left(n - \frac{N-1}{2}\right)\right]\right\} \quad ...(1.26.7)$$

iv. Frequency response $H(e^{j\omega})$ for N odd. Putting $z = e^{j\omega}$ in eq. (1.26.6), we get $H(e^{j\omega})$ for N odd as

$$H(e^{j\omega}) = h\left(\frac{N-1}{2}\right)e^{-j\omega[(N-1)/2]} + \sum_{n=0}^{[(N-1)/2]-1} h(n)\,[e^{-j\omega n + e^{-j\omega(N-1-n)}}]$$

or $$H(e^{j\omega}) = e^{-j\omega[(N-1)/2]}$$

$$\left\{h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{[(N-3)/2]} 2h(n)\cos\left[\omega\left(n - \frac{N-1}{2}\right)\right]\right\} \quad ...(1.26.8)$$

9. In both cases of frequency responses given by eq. (1.26.7) and eq. (1.26.8), the sum in bracket are real. This real sum implies a linear phase shift corresponding to delay of (N – 1/2) samples. Here, we note that for N even, (N – 1/2) is not an integer.

10. Direct-form network complementation of FIR filter structure of system functions given by eq. (1.26.5) and eq. (1.26.6) are shown in Fig. (1.26.1) and Fig. (1.26.2), respectively.

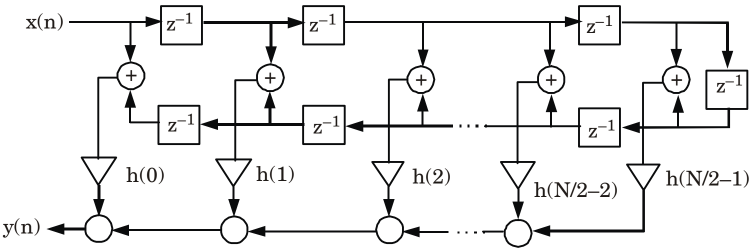

Fig. 1.26.1. Direct-form realization of a linear phase FIR digital system of filter for N even.

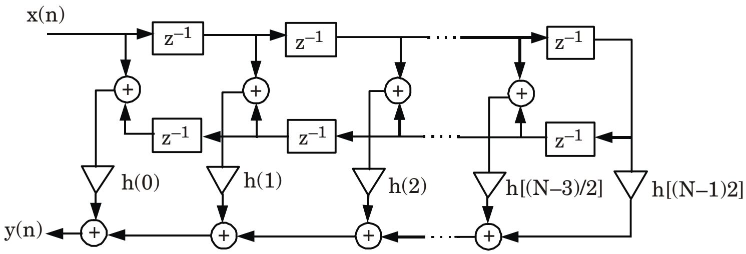

Fig. 1.26.2. Direct-form realization of a linear-phase FIR digital system of filter for N odd.

Que 1.27. **An FIR filter has following symmetry in the impulse response :**

$$\boldsymbol{h(n) = h(M-1-n) \text{ for } M \text{ odd.}}$$

Derive its frequency response and show that it has linear phase.

AKTU 2015-16, Marks 7.5

AKTU 2019-20, Marks 07

Answer

1. The discrete-time fourier transform (DTFT) of the impulse response $h(n)$ is given by,

$$H(e^{j\omega}) = \sum_{n=0}^{M-1} h(n)\, e^{-j\omega nT} = |H(e^{j\omega})|\, e^{j\phi(\omega)} \quad \text{...(1.27.1)}$$

2. If the length M of filters is odd, then eq. (1.27.1) can be written as

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-3}{2}} h(n)\, e^{-j\omega nT} + h\left(\frac{M-1}{2}\right) e^{-j\omega\left(\frac{M-1}{2}\right)T} \quad \text{...(1.27.2)}$$

3. We have $h(n) = h(M-1-n)$ for $0 < n < M-1$ ...(1.27.3)
Eq. (1.27.2) can be written as by using eq. (1.27.3).

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-3}{2}} h(n)\, [e^{-j\omega nT} + e^{-j\omega(M-1-n)T}]$$

$$+h\left(\frac{M-1}{2}\right) e^{-j\omega\left(\frac{M-1}{2}\right)T} \quad \text{...(1.27.4)}$$

4. Factorising $e^{-j\omega(M-1)T/2}$ in eq. (1.27.4),

$$H(e^{j\omega T}) = e^{-j\omega\left(\frac{M-1}{2}\right)T} \left\{ \sum_{n=0}^{\frac{M-3}{2}} h(n) \left[e^{j\omega\left(\frac{M-1}{2}-n\right)T} + e^{-j\omega\left(\frac{M-1}{2}-n\right)T} \right] \right.$$

$$\left. +h\left(\frac{M-1}{2}\right) \right\} \quad \text{...(1.27.5)}$$

5. Put $k = \left(\frac{M-1}{2}\right) - n$ in eq. (1.27.5)

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{M-1}{2}\right)T} \left\{ \sum_{k=1}^{\frac{M-1}{2}} h\left(\frac{M-1}{2} - k\right) [e^{j\omega kT} + e^{-j\omega kT}] \right.$$

$$\left. +h\left(\frac{M-1}{2}\right) \right\} \quad \text{...(1.27.6)}$$

or

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{M-1}{2}\right)T} \left\{ \sum_{k=0}^{\frac{M-1}{2}} b(k) \cos \omega kT \right\}$$

where, $b(k) = 2h\left(\frac{M-1}{2} - k\right)$; for $1 \le k \le \left(\frac{M-1}{2}\right)$

It gives $b(0) = h\left(\frac{M-1}{2}\right)$

6. The $H(e^{j\omega})$ can be written as

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{M-1}{2}\right)T}\{M(\omega)\} \quad ...(1.27.7)$$

Here magnitude response

$$M(\omega) = \sum_{k=0}^{(M-1)/2} b(k)\cos(\omega k T) \quad ...(1.27.8)$$

And phase response function

$$\phi(\omega) = -\omega(M-1)/2 \quad ...(1.27.9)$$

7. The eq. (1.27.9) represents the $(M-1)/2$ units delay in sampling time. Hence, FIR filter will have constant phase and group delays and thus the phase of the filter will be linear.

VERY IMPORTANT QUESTIONS

Following questions are very important. These questions may be asked in your SESSIONALS as well as UNIVERSITY EXAMINATION.

Q. 1. Obtain the parallel form realization

$$H(z) = \frac{(1 + 1/2z^{-1})}{(1 - z^{-1} + 1/4z^{-2})(1 - z^{-1} + 1/2z^{-2})}$$

Ans. Refer Q. 1.10.

Q. 2. Obtain the parallel form realization for the system function given below :

$$H(z) = \frac{(1 + 0.25\, z^{-1})}{(1 + 0.5\, z^{-1})(1 + 0.5\, z^{-1} + 0.25\, z^{-2})}$$

Ans. Refer Q. 1.11.

Q. 3. Obtain the parallel form realization for the transfer function $H(z)$ given below :

$$H(z) = \frac{2 + z^{-1} + 1/4\, z^{-2}}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 + z^{-1} + \frac{1}{2}z^{-2}\right)}$$

Ans. Refer Q. 1.12.

Q. 4. Obtain the Cascade form realization :

$$y(n) = y(n-1) - 1/2\,[y(n-2)] + 1/4\,[y(n-2)] + x(n) - x(n-1) + x(n-2)$$

Ans. Refer Q. 1.13.

Q. 5. Obtain direct form I, direct form II and parallel form structures for the following filter.

$$y(n) = \frac{3}{4}y(n-1) + \frac{3}{32}y(n-2) + \frac{1}{64}y(n-3) + x(n) + 3x(n-1) + 2x(n-2)$$

Ans. Refer Q. 1.15.

Q. 6. Obtain the direct form I, direct form II, cascade and parallel form realization for the following system :

$$y(n) = -0.1\,y(n-1) + 0.2\,y(n-2) + 3\,x(n) + 3.6\,x(n-1) + 0.6\,x(n-2)$$

Ans. Refer Q. 1.16.

Q. 7. Discuss the realization of a ladder structure and its advantages.

Ans. Refer Q. 1.18.

Q. 8. Obtain the ladder structure for the system function $H(z)$ given below :

$$H(z) = \frac{2 + 8z^{-1} + 6z^{-2}}{1 + 8z^{-1} + 12z^{-2}}$$

Ans. Refer Q. 1.20.

Q. 9. Explain the direct form realization of FIR systems.

Ans. Refer Q. 1.22.

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2 UNIT

IIR Filter Design

CONTENTS

PART-1

Introduction to Filters, Impulse Invariant Transformation, Bilinear Transformation.

CONCEPT OUTLINE :

- The transformation formula for IIR filter design by approximation of derivatives, $s = \frac{1 - z^{-1}}{T}$
- For bilinear transformation, $s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$
- For impulse invariant transformation, $\frac{1}{s - p_i} = \frac{1}{1 - e^{p_i T} z^{-1}}$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.1. **What is filter ? Explain its types.**

Answer

A. Filter : A filter is one which rejects unwanted frequencies from the input signal and allows the desired frequencies. The range of frequencies of signal that are passed through the filter is called passband (P.B.) and those frequencies that are blocked is called stop band (S.B.).

B. Types :

i. Low-pass filter : It passes the frequency from zero to some designated frequency, called as cut-off frequency. After this frequency, it will not allow any signal to pass through it.

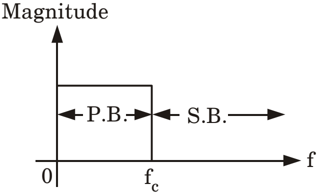


Fig. 2.1.1.

ii. High-pass filter : It passes the frequency above some designated frequency, called as cut-off frequency. If input signal frequency is less than the cut-off frequency, then this signal is not allowed to pass through it.

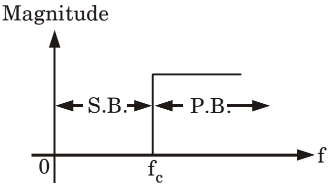


Fig. 2.1.2.

iii. **Band-pass filter :** It allows the frequencies between two designated frequencies *i.e.*, f_{c1} and f_{c2}.

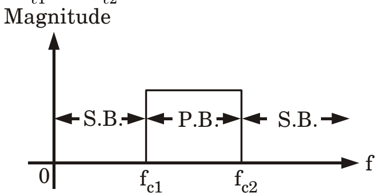


Fig. 2.1.3.

iv. **Band-reject filter :** It attenuates all frequencies between two designated cut-off frequencies. At the same time it passes all other frequencies.

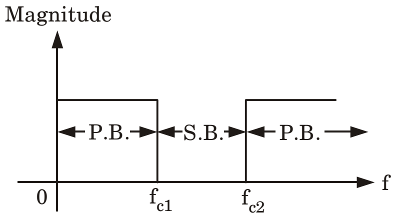


Fig. 2.1.4.

v. **All-pass filter :** It passes all the frequencies. By using this filter the phase of input signal can be modified.

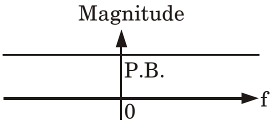


Fig. 2.1.5.

Que 2.2. **How a digital filter is designed from analog filter ?**

OR

Prove that physically realizable IIR filter cannot have linear phase.

Answer

1. The system function describing an analog filter can be written as

$$H_a(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \quad ...(2.2.1)$$

where a_k and $b_k \rightarrow$ Filter coefficients

2. The impulse response of this analog filter $h(t)$ is given by Laplace transform of $H_a(s)$ as follows,

$$H_a(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt \quad ...(2.2.2)$$

3. The rational system function $H(s)$ of eq. (2.2.1) can also be described by the linear constant coefficient differential equation, as follows :

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \quad ...(2.2.3)$$

where, $x(t)$ = Input signal

$y(t)$ = Output signal

4. **Digital filter design from analog filter :** The digital filter can be obtained by transforming the analog filter by following three methods :

i. IIR filter design by approximation of derivatives.

ii. IIR filter design by impulse invariant method.

iii. IIR filter design by the bilinear transformation method.

5. Properties of conversion techniques : The conversion techniques should have following properties :

a. Property-1 : The mapping of s-plane should be done on to the unit circle in the z-plane. This mapping of s-plane to z-plane gives direct relationship between two domains.

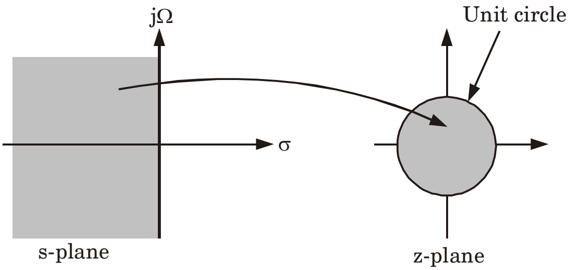


Fig. 2.2.1. Mapping from s-plane to z-plane.

b. Property-2 :

1. For converting a stable analog filter into a stable digital filter by just mapping left-half plane of the s-plane into the inside of the unit circle in the z-plane.

2. Condition for IIR filter to be linear : For a filter having linear phase should have following condition.

   i.e., $$h(n) = h(M-1-n) \quad ...(2.2.4)$$

i. A physically realizable and stable IIR filter cannot have a linear phase.

ii. Since the condition $h(n) = h(M-1-n)$ provides a mirror image pole outside the unit circle. But in this, every pole inside the unit circle. This results in an unstable filter.

Que 2.3. Describe the method to design IIR filters by approximation of derivatives.

Answer

1. This method is used to convert analog filter transfer function into a digital filter transfer function. The backward difference formula for the derivative $dy(t)/dt$ at time $t = nT$ is given as

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT) - y(nT-T)}{T} = \frac{y(n) - y(n-1)}{T} \quad ...(2.3.1)$$

   where T = Sampling interval

   and $y(n) = y(nT)$

2. The system function of an analog differentiator with an output dy/dt is given as

$$H(s) = s \quad ...(2.3.2)$$

3. The digital output for this system can be calculated by taking z-transform of $[y(n) - y(n-1)]/T$ we get

$$H(z) = \frac{(1-z^{-1})}{T} \quad ...(2.3.3)$$

4. Since both filter have same transfer function except their operator.

5. Thus on comparing eq. (2.3.2) with eq. (2.3.3) we will get the condition for conversion of analog filter into digital filter such as

$$s = \frac{(1-z^{-1})}{T} \quad ...(2.3.4)$$

6. The second derivative of $y(t)$ *i.e.*, $d^2y(t)/dt^2$ is replaced by the second backward difference such as

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=nT} = \frac{d}{dt}\left[\frac{dy(t)}{dt}\right]_{t=nT}$$

$$= \frac{\dfrac{[y(nT) - y(nT-T)]}{T} - \dfrac{[y(nT-T) - y(nT-2T)]}{T}}{T}$$

$$= \frac{y(n) - y(n-1) - y(n-1) + y(n-2)}{T^2}$$

$$= \frac{y(n) - 2y(n-1) + y(n-2)}{T^2} \quad ...(2.3.5)$$

7. Taking z-transform (*i.e.*, in frequency domain) of eq. (2.3.5) we get

$$= Y(z)\frac{[1 - 2z^{-1} + z^{-2}]}{T^2} = Y(z)\frac{[1 - z^{-1}]^2}{T^2} \quad ...(2.3.6)$$

or

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t = nT} \xrightarrow{LT} s^2Y(s) \quad ...(2.3.7)$$

8. On comparison of eq. (2.3.6) and, eq. (2.3.7) we get

$$s^2 = \frac{[1 - z^{-1}]^2}{T^2} = \left(\frac{1 - z^{-1}}{T}\right)^2 \quad ...(2.3.8)$$

9. The i^{th} derivative of $y(t)$ give the equivalent frequency domain conversion relationship.

$$s^i = \left(\frac{1 - z^{-1}}{T}\right)^i \quad ...(2.3.9)$$

10. Using this method the system function of digital filter can be obtained from the system function of analog filter.

$$H(z) = H_a(s)\Big|_{s = (1 - z^{-1})/T} \quad ...(2.3.10)$$

where $H_a(s) \rightarrow$ System function of analog filter.

11. Mapping of s-plane to z-plane :
The eq. (2.3.4) can be written as

$$z = \frac{1}{1 - sT} \quad ...(2.3.11)$$

12. Substituting $s = j\Omega$ in eq. (2.3.11), we get

$$z = \frac{1}{1 - j\Omega T} = \frac{1 \times (1 + j\Omega T)}{(1 - j\Omega T)(1 + j\Omega T)}$$

$$= \frac{1 + j\Omega T}{(1 + \Omega^2 T^2)}$$

$$= \frac{1}{1 + \Omega^2 T^2} + j\frac{\Omega T}{1 + \Omega^2 T^2} \quad ...(2.3.12)$$

13. If varies from $-\infty$ to ∞ the corresponding locus of points in the z-plane is a circle with radius 1 / 2 and with centre at z = 1 / 2, as shown in Fig. 2.3.1.

14. From Fig. 2.3.1, it is clear that the left-half plane of s-domain into the corresponding points inside the circle of radius 0.5 and centre of $z = 0.5$. The right side of s-plane is mapped outside the unit circle.

i. This mapping is used only for transforming stable analog filter into stable digital filter.

ii. This method is used only for transforming analog low-pass filters and bandpass filters which are having smaller resonant frequencies.

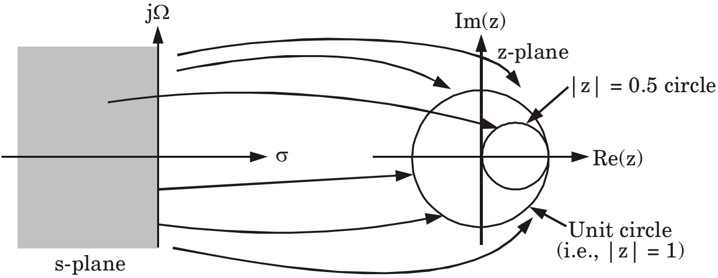


Fig. 2.3.1. Mapping of eq. (2.3.12) into the z-plane.

15. The transformation formula for mapping is given as

$$s = \frac{1}{T}\sum_{k=1}^{N} a_k\,(z^k - z^{-k}) \quad ...(2.3.13)$$

Que 2.4. Discuss the impulse invariant method of IIR filter designing and write its properties.

Answer

A. Impulse invariance technique :

1. The desired impulse response of the digital filter is obtained by uniformly sampling the impulse response of the equivalent analog filter.
   i.e., $$h(n) = h_a(nT) \quad ...(2.4.1)$$
   where T = Sampling interval
2. The system function of analog filter having distinct pole is given as

$$H_a(s) = \sum_{i=1}^{M}\frac{A_i}{s - p_i} \quad ...(2.4.2)$$

3. Taking inverse Laplace transform of eq. (2.4.2) we get impulse response

$$h_a(t) = \sum_{i=1}^{M} A_i\, e^{p_i t} u_a(t) \quad ...(2.4.3)$$

   where $u_a(t)$ = Unit step function in continuous time
4. By uniformly sampling $h_a(t)$ we get the impulse response $h(n)$ of the equivalent digital filter is given as

$$h(n) = h_a(nT) = \sum_{i=1}^{M} A_i\, e^{p_i nT} u_a(nT) \quad ...(2.4.4)$$

5. By taking the z-transform of eq. (2.4.4) we get response of digital system.

$$H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n} = \sum_{n=0}^{\infty}\left[\sum_{i=1}^{M} A_i\, e^{p_i nT} u_a(nT)\right] z^{-n} \quad ...(2.4.5)$$

6. Interchanging the order of summation.

$$H(z) = \sum_{i=1}^{M} \left[\sum_{n=0}^{\infty} A_i \, e^{p_i nT} u_a(nT) \right] z^{-n}$$

or $$H(z) = \sum_{i=1}^{M} \frac{A_i}{1 - e^{p_i T} z^{-1}} \quad ...(2.4.6)$$

7. On comparing eq. (2.4.2) with eq. (2.4.6) we get

$$\frac{1}{s - p_i} = \frac{1}{1 - e^{p_i T} z^{-1}} \quad ...(2.4.7)$$

Mapping of s-plane to z-plane

8. It is clear from eq. (2.4.7) that the analog pole at $s = p_i$ is mapped into a digital pole at $z = e^{piT}$, thus the digital and analog poles are related as

$$z = e^{sT} \quad ...(2.4.8)$$

By substituting $s = \sigma + j\Omega$

And replacing complex variable z in polar form as

$$z = re^{j\omega}$$

9. Therefore eq. (2.4.8) will become

$$re^{j\omega} = e^{(\sigma + j\Omega)T} = e^{\sigma T} \times e^{j\Omega T}$$

Therefore we have

$$r = e^{\sigma T} \quad ...(2.4.9)$$

$$\omega = \Omega T$$

10. The eq. (2.4.9) gives three cases,

Case I : when $\sigma < 0$

then $0 < r < 1$

Case II : when $\sigma > 0$

then $r > 1$

Case III : when $\sigma = 0$

then $r = 1$

11. It means that the left-half of s-plane is mapped inside the unit circle in the z-plane and right half of s-plane is mapped outside the unit circle.

i. Mapping of $j\Omega$ axis is not one to one. The mapping $\omega = \Omega T$ implies that the interval $-\pi/T \le \Omega \le \pi/T$ maps into the corresponding value of $-\pi \le \omega \le \pi$. The frequency interval $\pi/T \le \Omega \le 3\pi/T$ again mapped into the interval $-\pi \le \omega \le \pi$ in the z-plane.

ii. The mapping from the analog frequency Ω to the frequency variable ω in the digital domain is many to-one.

iii. Due to many to one mapping aliasing effect occurs in impulse response of filter the mapping is given in Fig. 2.4.1.

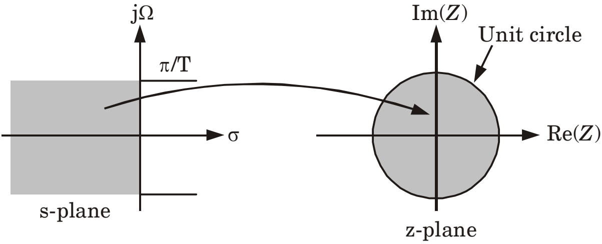


Fig. 2.4.1. The mapping from s to z plane *i.e.*, $z = e^{sT}$.

B. Properties :

i. $$\frac{1}{(s+s_i)^m} \to \frac{(-1)^{m-1}}{(m-1)!}\frac{d^{m-1}}{ds^{m-1}}\left[\frac{1}{1-e^{-sT}z^{-1}}\right]; s \to s_i$$

ii. $$\frac{s+a}{(s+a)^2+b^2} \to \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

iii. $$\frac{b}{(s+a)^2+b^2} \to \frac{e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

Que 2.5. **Determine $H(z)$ using the impulse invariant technique for the analog system function**

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)}$$

AKTU 2016-17, Marks 10

Answer

Given : $H(s) = \dfrac{1}{(s+0.5)(s^2+0.5s+2)}$

To Find : $H(z)$.

1. Using partial fractions, $H(s)$ can be written as,

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)} = \frac{A}{s+0.5} + \frac{Bs+C}{s^2+0.5s+2}$$

Therefore, $A(s^2 + 0.5s + 2) + (Bs + C)(s + 0.5) = 1$ ...(2.5.1)

2. Comparing the coefficients of s^2, s and the constants term on either side of the eq. (2.5.1), we get,

$$A + B = 0 \quad ...(2.5.2)$$
$$0.5A + 0.5B + C = 0 \quad ...(2.5.3)$$
$$2A + 0.5C = 1 \quad ...(2.5.4)$$

3. Solving the simultaneous eq. (2.5.2), eq. (2.5.3) and eq. (2.5.4), we get $A = 0.5$, $B = 0.5$ and $C = 0$

4. The system response can be written as,

$$H(s) = \frac{0.5}{s+0.5} - \frac{0.5s}{s^2+0.5s+2} = \frac{0.5}{s+0.5} - 0.5\left\{\frac{s}{(s+0.25)^2+(1.3919)^2}\right\}$$

$$= \frac{0.5}{s+0.5} - 0.5\left\{\frac{s+0.25}{(s+0.25)^2+(1.3919)^2} - \frac{0.25}{(s+0.25)^2+(1.3919)^2}\right\}$$

$$= \frac{0.5}{s+0.5} - 0.5\left\{\frac{s+0.25}{(s+0.25)^2+(1.3919)^2}\right\} + 0.0898\left\{\frac{1.3919}{(s+0.25)^2+(1.3919)^2}\right\}$$

5. As we know that,

$$\frac{s+a}{(s+a)^2+b^2} \rightarrow \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-at}(\cos bT)z^{-1}+e^{-2aT}z^{-2}} \quad ...(2.5.5)$$

$$\frac{b}{(s+a)^2+b^2} \rightarrow \frac{e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}} \quad ...(2.5.6)$$

6. Using eq. (2.5.5) and (2.5.6), we get,

$$H(z) = \frac{0.5}{1-e^{-0.5T}z^{-1}} - 0.5\left[\frac{1-e^{-0.25T}(\cos 1.3919T)z^{-1}}{1-2e^{-0.25T}(\cos 1.3919T)z^{-1}+e^{-0.5T}z^{-2}}\right] + 0.0898\left[\frac{e^{-0.25T}(\sin 1.3919T)z^{-1}}{1-2e^{-0.25T}(\cos 1.3919T)z^{-1}+e^{-0.5T}z^{-2}}\right]$$

Let $T = 1$ s,

$$H(z) = \frac{0.5}{1-0.6065\,z^{-1}} - 0.5\left(\frac{1-0.1385\,z^{-1}}{1-0.277\,z^{-1}+0.606\,z^{-2}}\right) + 0.0898\left[\frac{0.7663\,z^{-1}}{1-02771\,z^{-1}+0.606\,z^{-2}}\right]$$

Que 2.6. **The system function of analog filter is given by**

$$\boldsymbol{H(s) = \frac{(s+0.1)}{(s+0.1)^2+16}}$$

Obtain the system function of digital filter by using impulse invariant technique. Assume $T = 1$ sec.

AKTU 2019-20, Marks 07

Answer

1. Given, $$H(s) = \frac{(s+0.1)}{(s+0.1)^2+16} = \frac{(s+0.1)}{(s+0.1)^2+4^2} \quad ...(2.6.1)$$

2. We know that,

$$\frac{s+a}{(s+a)^2+b^2} \rightarrow \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}} \quad ...(2.6.2)$$

3. Comparing eq. (2.6.1) and eq. (2.6.2) then we get,

$$a = 0.1,\ b = 4$$

4. $$H(z) = \frac{1 - e^{-0.1T}(\cos 4T)z^{-1}}{1 - 2e^{-0.1T}(\cos 4T)z^{-1} + e^{-2\times 0.1T}z^{-2}}$$

5. At $T = 1$ sec,

$$H(z) = \frac{1 - e^{-0.1}(\cos 4)z^{-1}}{1 - 2e^{-0.1}(\cos 4)z^{-1} + e^{-0.2}z^{-2}}$$

$$H(z) = \frac{1 - 0.89z^{-1}}{1 - 1.78z^{-1} + 0.818z^{-2}}$$

Que 2.7. Discuss the bilinear transformation method of converting analog IIR filter into digital IIR filter.

Answer

1. Bilinear transformation is a one to one mapping from the s-domain to the z-domain.
2. Bilinear transformation is a conformal mapping that transforms the $j\Omega$-axis into the unit circle in the z-plane only once. Thus the aliasing effect is avoided.
3. The transformation of a stable analog filter result in a stable digital filter as all the poles in the left half of the s-plane are mapped onto points inside the unit circle of the z-domain.
4. It is obtained by using the trapezoidal formula of numerical integration.

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right) \quad ...(2.7.1)$$

5. For mapping of s-plane to z-plane, substitute $z = re^{j\omega}$ and $s = \sigma + j\Omega$ in eq. (2.7.1) we get

$$\sigma + j\Omega = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = \frac{2}{T}\left(\frac{re^{j\omega} - 1}{re^{j\omega} + 1}\right) = \frac{2}{T}\left[\frac{r(\cos\omega + j\sin\omega) - 1}{r(\cos\omega + j\sin\omega) + 1}\right]$$

$$= \frac{2}{T}\left[\frac{(r\cos\omega - 1 + j\,r\sin\omega)}{(r\cos\omega + 1 + j\,r\sin\omega)}\right]$$

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{r\cos\omega - 1 + jr\sin\omega}{r\cos\omega + 1 + jr\sin\omega}\right]\left[\frac{r\cos\omega + 1 - jr\sin\omega}{r\cos\omega + 1 - jr\sin\omega}\right]$$

$$= \frac{2}{T}\left[\frac{(r^2\cos^2\omega - 1 + r^2\sin^2\omega + j2\,r\sin\omega)}{(r\cos\omega + 1)^2 + r^2\sin^2\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r^2\cos^2\omega - 1 + r^2\sin^2\omega + j2r\sin\omega}{1 + r^2\cos^2\omega + 2r\cos\omega + r^2\sin^2\omega}\right]$$

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega} + j\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right] \quad ...(2.7.2)$$

6. On comparison real and imaginary part we get,

$$\sigma = \frac{2}{T}\left(\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega}\right) \quad ...(2.7.3)$$

and $$\Omega = \frac{2}{T}\left(\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right) \quad ...(2.7.4)$$

7. On the basis of value of r three cases can be discussed as below :
   Case I : If $r < 1$, then $\sigma < 0$
   Case II : If $r > 1$, then $\sigma > 0$
   Case III : If $r = 1$, then $\sigma = 0$
   From all three cases it is clear that the left half of the s-plane is mapped on to the points inside the unit circle in the z-plane.

Que 2.8. **Discuss the bilinear transformation method of converting analog IIR filter into digital IIR filter. What is frequency warping ?**

AKTU 2015-16, Marks 7.5

OR

What is frequency warping effect ? How this problem is overcome in bilinear transform method of IIR filter design ? Also write down the advantages and disadvantages of bilinear transformation.

AKTU 2019-20, Marks 07

Answer

A. Bilinear transformation method : Refer Q. 2.7, Page 2–11C, Unit-2.

B. Frequency warping :

1. We know, $$\Omega = \frac{2}{T_s} \times \frac{2r\sin\omega}{r^2+2r\cos\omega+1} \quad ...(2.8.1)$$

2. For the unit circle, $r = 1$. Thus putting $r = 1$ in the eq. (2.8.1) we get,

$$\Omega = \frac{2}{T_s} \times \frac{2\sin\omega}{1+2\cos\omega+1}$$

$$\therefore \quad \Omega = \frac{2}{T_s} \times \frac{2\sin\omega}{2+2\cos\omega}$$

$$\Omega = \frac{2}{T}\left(\frac{\sin\omega}{1+\cos\omega}\right)$$

$$= \frac{2}{T}\left[\frac{2\sin(\omega/2)\cos(\omega/2)}{\cos^2(\omega/2)+\sin^2(\omega/2)+\cos^2(\omega/2)-\sin^2(\omega/2)}\right]$$

or $$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

or $$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$$

3. In this method the entire range in Ω is mapped only once into the range $-\pi \le \omega \le \pi$. This mapping is non-linear. The lower frequencies in analog domain are expanded in digital domain while the higher frequencies in analog domain are compressed in digital domain.

4. Non-linearity in mapping is due to arc tangent function and this is known as frequency warping.

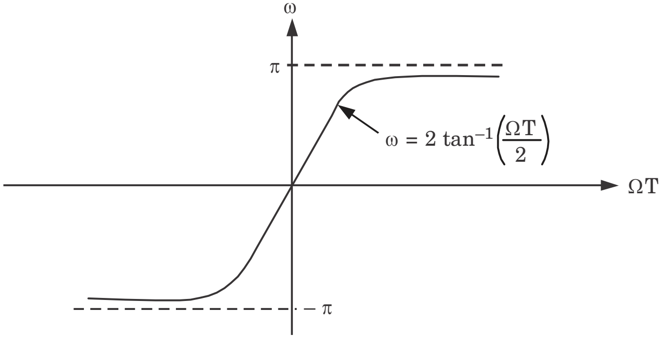


Fig. 2.8.1. Frequency warping.

C. Advantages of bilinear transformation :

1. The mapping is one to one.
2. There is no aliasing effect.
3. Stable analog filter is transformed into the stable digital filter.
4. There is no restriction one type of filter that can be transformed.
5. There is one to one transformation from the *s*-domain to the *Z*- domain.

Disadvantages of bilinear transformation :

1. In this method, the mapping is non-linear because of this frequency warping effect takes place.

Que 2.9. **Convert the analog filter with system function**

$$H(s) = \frac{s + 0.1}{(s + 0.1)^2 + 9}$$

into digital filter with a resonant frequency of $\omega_r = \pi/4$ of using bilinear transformation. **AKTU 2016-17, Marks 10**

Answer

1. Given, $$H(s) = \frac{s + 0.1}{(s + 0.1)^2 + 9} \quad ...(2.9.1)$$
2. From eq. (2.9.1), $\Omega_c = 3$
3. The sampling period T can be determine by,

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_r}{2}$$

$$T = \frac{2}{\Omega_c}\tan\frac{\omega_r}{2} = \frac{2}{3}\tan\left(\frac{\pi}{8}\right) = 0.276 \text{ s}$$

4. Using bilinear transformation,

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$H(z) = \frac{\frac{2}{T}\frac{(z-1)}{(z+1)}+0.1}{\left[\frac{2}{T}\frac{(z-1)}{(z+1)}+0.1\right]^2+9}$$

$$= \frac{\frac{2}{T}(z-1)(z+1)+0.1(z+1)^2}{\left[\left(\frac{2}{T}\right)(z-1)+0.1(z+1)\right]^2+9(z+1)^2} \quad ...(2.9.2)$$

5. Substituting the value of T in eq. (2.9.2) then we get

$$H(z) = \frac{1+0.27z^{-1}-0.973z^{-2}}{8.572-11.84z^{-1}+8.177z^{-2}}$$

Que 2.10. **The system function of the analog filter is given as :**

$$H(s) = \frac{s+0.1}{(s+0.1)^2+16}$$

obtain the system function of digital filter using bilinear transformation which is resonant at $\omega_r = \pi/2$.

AKTU 2017-18, Marks 10

Answer

The procedure is same as Q. 2.9, Page 2–13C, unit-2.

$$\left[\textbf{Ans.} \quad H(z) = \frac{4.1z^2+0.2z-3.9}{32.81z^2+0.02z+31.21}\right]$$

Que 2.11. **Use bilinear transformation to convert low pass filter, $H(s) = 1/s^2 + \sqrt{2}\,s + 1$ into a high pass filter with pass band edge at 100 Hz and f_s = 1 kHz.**

AKTU 2017-18, Marks 10

Answer

Given : f_P = 100 Hz, f_s = 1 kHz, $H(s) = \dfrac{1}{s^2+\sqrt{2}s+1}$

To Convert : Low pass into high pass filter

Step I : We have

$$\omega_P = 2\pi f_P = 200\,\pi = \omega_c$$

$$\Omega_c^* = \frac{2}{T_s}\tan\left(\frac{\omega_P T_s}{2}\right)$$

$$T_s = \frac{1}{f_s} = \frac{1}{1 \times 10^3} = 10^{-3}$$

$$\therefore \qquad \Omega_c^* = \frac{2}{1/10^3} \tan\left(\frac{200\pi}{2 \times 10^3}\right)$$

$$\therefore \qquad \Omega_c^* = 2 \times 10^3 \tan(0.1\pi)$$

$$\Omega_c^* = 2 \times 10^3 \times 0.325$$

$$\Omega_c^* = 650$$

Step II : Equation of $H^*(s)$ is obtained by replacing s by $\dfrac{\Omega_c^*}{s}$ in equation of $H(s)$.

$$\therefore \qquad H^*(s) = \frac{1}{\left(\frac{\Omega_c^*}{s}\right)^2 + \sqrt{2}\left(\frac{\Omega_c^*}{s}\right) + 1}$$

$$H^*(s) = \frac{1}{\left(\frac{650}{s}\right)^2 + \sqrt{2}\left(\frac{650}{s}\right) + 1}$$

$$H^*(s) = \frac{1}{0.422 \times 10^6 \, s^{-2} + 0.92 \times 10^3 \, s^{-1} + 1}$$

Step III :

$H(z)$ is obtained by replacing s by $\dfrac{2}{T_s}\left[\dfrac{z-1}{z+1}\right]$

$$H^*(z) = \frac{1}{0.422 \times 10^6 \left[\frac{2}{T_S}\left(\frac{z-1}{z+1}\right)\right]^{-2} + 0.92 \times 10^3 \left[\frac{2}{T_S}\left(\frac{z-1}{z+1}\right)\right]^{-1} + 1}$$

$$H^*(z) = \frac{1}{0.422 \times 10^6 \times \frac{10^{-6}}{2}\left(\frac{z+1}{z-1}\right)^2 + 0.92 \times 10^3 \times \frac{10^{-3}}{2}\left(\frac{z+1}{z-1}\right) + 1}$$

$$H^*(z) = \frac{1}{0.21\left(\frac{z+1}{z-1}\right)^2 + 0.46\left(\frac{z+1}{z-1}\right) + 1}$$

Que 2.12. Using bilinear transformation, design a Butterworth filter which satisfies the following condition

$$0.8 \le |H(e^{j\omega})| \le 1, \quad 0 \le \omega \le 0.2\pi$$
$$|H(e^{j\omega})| \le 0.2, \quad 0.6\pi \le \omega \le \pi$$

AKTU 2018-19, Marks 07

Answer

Given : $0.8 \leq |H(e^{j\omega})| \leq 1 \qquad 0 \leq \omega \leq 0.2\,\pi$

$|H(e^{j\omega})| \leq 0.2 \qquad 0.6\,\pi \leq \omega \leq \pi$

$A_p = 0.8$, $A_s = 0.2$, $\delta_p = 0.2\pi$, $\delta_s = 0.6\pi$

To Design : Butterworth low pass filter.

Step 1 : For bilinear transformation, we have

$$\Omega_s = \frac{2}{T_s}\tan\frac{\delta_s}{2}$$

$$\Omega_p = \frac{2}{T_s}\tan\frac{\delta_p}{2}$$

Assume, $T_s = 1$ s

$$\Omega_p = 2\tan\frac{\delta_p}{2} = 2\tan\frac{0.2\pi}{2} = 0.65 \text{ rad/sec}$$

$$\Omega_s = 2\tan\frac{\delta_s}{2} = 2\tan\frac{0.6\pi}{2} = 2.75 \text{ rad/sec}$$

Thus, specifications of analog filter are :

$$A_p = 0.8,\ \Omega_p = 0.65$$
$$A_s = 0.2,\ \Omega_s = 2.75$$

Step 2 : Order of filter, N, is given by,

$$N \geq \frac{\log\left[\dfrac{\left(\dfrac{1}{A_s^2}-1\right)}{\left(\dfrac{1}{A_p^2}-1\right)}\right]}{2\log\left(\dfrac{\Omega_s}{\Omega_p}\right)} \geq \frac{\log\left[\dfrac{\left(\dfrac{1}{(0.2)^2}-1\right)}{\left(\dfrac{1}{(0.8)^2}-1\right)}\right]}{2\log\left(\dfrac{2.75}{0.65}\right)}$$

$$\geq \frac{\log(24/0.5625)}{2\log(4.231)} = \frac{1.63}{1.253} = 1.3$$

Thus, we can take $N = 1$ (approx).

Step 3 : Cut-off frequency, can be determined by,

$$\Omega_c = \frac{\Omega_p}{\left(\dfrac{1}{A_p^2}-1\right)^{1/2N}} = \frac{0.65}{\left(\dfrac{1}{(0.85)^2}-1\right)^{1/2}}$$

$$= \frac{0.650}{0.75} = 0.867$$

Step 4 : The poles of $H(s)$ are given by,

$$p_K = \pm\ \Omega_c\, e^{j(N+2K+1)\pi/2N},\ K = 0, 1, 2........N-1$$

$$p_K = \pm\ 0.867\, e^{j(1+2K+1)\pi/2}$$

$$p_K = \pm\ 0.867\, e^{j(2+2K)\pi/2}$$

$$p_K = \pm\, 0.867 \left[\cos\left(\frac{(2+2K)\pi}{2}\right) + j\sin\left(\frac{(2+2K)\pi}{2}\right)\right]$$

Range of K is 0 to $N - 1$, thus $K = 0$

For $K = 0$, $p_0 = \pm\, 0.867\ [\cos \pi + j \sin \pi]$

$= \pm\, 0.867\ (-1) = -\, 0.867,\ 0.867$

Step 5 : Calculation of $H(s)$: For the stability of filter, select the poles on the L.H.S of s-plane.

So we will select the pole $s = -\, 0.867$.

The system function of second order Butterworth low pass filter is given as

$$H(s) = \frac{\Omega_c^N}{(s - s_1)(s - s_1{}^*)} = \frac{0.867}{(s + 0.867)} \qquad ...(2.12.1)$$

Step 6 : The transfer function for digital filter is obtained by putting,

$$s = \frac{2}{T_s}\left(\frac{z-1}{z+1}\right) = 2\left(\frac{z-1}{z+1}\right)$$

Put value of s in eq. (2.12.1), we get

$$H(z) = \frac{0.867}{\left[2\left(\frac{z-1}{z+1}\right) + 0.867\right]} = \frac{0.867}{\left[\frac{2z - 2 + (z+1)0.867}{z+1}\right]}$$

$$= \frac{0.867\,(z+1)}{2.867z - 1.133}$$

PART-2

All Pole Analog Filters : Butterworth and Chebyshev, Design of Digital Butterworth and Chebyshev Filters, Frequency Transformation.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.13. What are all pole filters ? Give their classification.

Answer

1. The transfer function of all pole filters is given as

$$H_a(s) = \frac{A}{P(s)} \qquad ...(2.13.1)$$

where $P(s)$ = Polynomial of degree N

A = Constant

2. The transfer function of all pole filter only consists poles. These filters are most commonly realized in cascaded form, so that the transfer function of filter can be written in factored form.
3. Two cases can be discussed according to filter order.

i. Case I : When N = even (*i.e.*, N = 2, 4, 6, ...)

$$H_a(s) = \prod_{k=1}^{N/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2} \quad \text{...(2.13.2)}$$

ii. Case II : When N = odd (*i.e.*, N = 3, 5, 7.....)

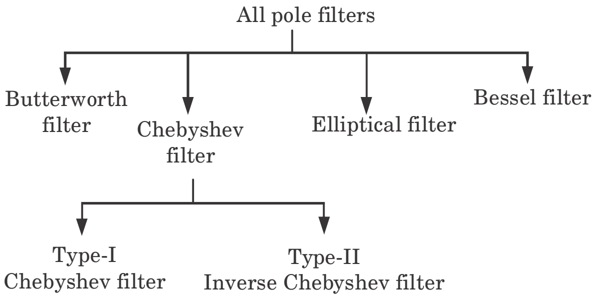
$$H_a(s) = \frac{B_o \Omega_c}{s + c_o \Omega_c} \prod_{k=1}^{(N-1)/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2} \quad \text{...(2.13.3)}$$

where Ω_c = Cut-off frequency

In normalized case

Ω_c = 1 radian/s

4. Classification of all pole filters :



5. The filter specifications are given in terms of the magnitude response *i.e.*, $|H_a(j\Omega)|$.

Que 2.14. Explain the designing of low pass Butterworth filter.

Answer

1. It is characterized by the magnitude squared frequency response.

$$|H(\Omega)|^2 = \frac{A}{1 + (\Omega/\Omega_c)^{2N}} \quad \text{...(2.14.1)}$$

$$|H(j\Omega)|^2 = \frac{A}{1 + \epsilon^2 (\Omega/\Omega_p)^{2N}} \quad \text{...(2.14.2)}$$

where

$\Omega_p = \epsilon^{1/N}\, \Omega_c$

N = Order of filter

Ω_c = 3 dB cut-off frequency

Ω_P = Pass-band edge frequency

A = Constant

$$\frac{1}{1+\epsilon^2} = \text{Band edge value of } |H(\Omega)|^2$$

2. Since $H(s)\,H(-s)$ evaluated at $s = j\Omega$ is simply equal to $|H(j\Omega)|^2$ if follows that

$$H(s)\,H(-s) = \frac{1}{1+\left(-\dfrac{\delta^2}{\Omega_c^2}\right)^N} \qquad \{\text{taking } A = 1\}$$

3. The pole of $H(s)$ and $H(-s)$ occur on a circle of radius Ω_c at equally spaced points.
4. The magnitude response of this filter is given in Fig. 2.14.1

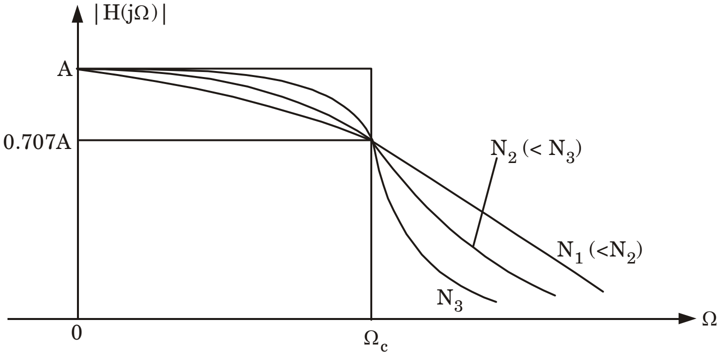


Fig. 2.14.1 Magnitude response of a Butterworth LPF.

5. **Designing of Butterworth filter :** The design parameters of this filter are obtained by considering the LPF with the desired specifications as given below.

$$\delta_1 \le |H(e^{j\omega})| \le 1 \qquad \text{for } 0 \le \omega \le \omega_1 \qquad \ldots(2.14.3)$$

$$|H(e^{j\omega})| \le \delta_2 \qquad \text{for } \omega_2 \le \omega \le \pi \qquad \ldots(2.14.4)$$

6. The corresponding analog magnitude response is to be obtained in the design process. Using eq. (2.14.1) with $A = 1$ in eq. (2.14.3) and eq. (2.14.4).

We get
$$\delta^2{}_1 \le \frac{1}{1+\left(\dfrac{\Omega_1}{\Omega_c}\right)^{2N}} \le 1 \qquad \ldots(2.14.5)$$

$$\frac{1}{1+\left(\dfrac{\Omega_2}{\Omega_c}\right)^{2N}} \le \delta^2{}_2 \qquad \ldots(2.14.6)$$

7. Eq. (2.14.5) and (2.14.6) can be written as

$$\left(\frac{\Omega_1}{\Omega_c}\right)^{2N} \le \frac{1}{\delta_1^2} - 1 \qquad \ldots(2.14.7)$$

$$\left(\frac{\Omega_2}{\Omega_c}\right)^{2N} \geq \frac{1}{\delta_2^2} - 1 \quad \text{...(2.14.8)}$$

8. Dividing both eq. (2.14.8) and (2.14.7), we get

$$\left(\frac{\Omega_2}{\Omega_1}\right)^{2N} = \frac{\left(\frac{1}{\delta_2^2} - 1\right)}{\left(\frac{1}{\delta_1^2} - 1\right)} \quad \text{...(2.14.9)}$$

9. From eq. (2.14.9) the order of filter is given as

$$N = \frac{1}{2} \frac{\log\left\{\left[\frac{\left(\frac{1}{\delta_2^2} - 1\right)}{\left(\frac{1}{\delta_1^2} - 1\right)}\right]\right\}}{\log\left(\frac{\Omega_2}{\Omega_1}\right)} \quad \text{...(2.14.10)}$$

10. The value of N is chosen to be next nearest integer value of N. The eq. (2.14.7) will give the cut-off frequency as

$$\Omega_c = \frac{\Omega_1}{\left(\frac{1}{\delta_1^2} - 1\right)^{1/2N}}$$

11. For the bilinear transformations we obtain

$$\frac{\Omega_c T}{2} = \frac{\tan(\omega_1/2)}{[(1/\delta_1)^2 - 1]^{1/2N}} \qquad \left[\because \Omega_1 = \frac{2}{T}\tan\left(\frac{\omega_1}{2}\right)\right]$$

12. For the impulse invariant transformation

$$\Omega_c = \frac{\omega_1/T}{[(1/\delta_1)^2 - 1]^{1/2N}}$$

13. Transfer function of Butterworth filter : This filter is usually written in factored form. There are two cases to be discussed :

i. **Case-I :** When N = even *i.e.*, N = 2, 4, 6, ……

$$H_a(s) = \prod_{k=1}^{N/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2}$$

ii. **Case-II :** When N = odd *i.e.*, N = 3, 5, 7, 9, …..

$$H_a(s) = \frac{B_0 \Omega_c}{s + c_0 \Omega_c} \prod_{k=1}^{(N-1)/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2}$$

14. The coefficients b_k and c_k are given as

$$b_k = 2 \sin[(2k - 1)\pi/2N] \text{ and } c_k = 1$$

15. The parameter B_k can be obtained from

$$A = \prod_{k=1}^{N/2} B_k, \text{ for } N = \text{even}$$

And $$A = \prod_{k=1}^{(N-1)/2} B_k \text{ , for } N = \text{odd}$$

After getting $H_a(s)$, we can transform it by any method to get digital filter.

Que 2.15. Design a digital Butterworth filter that satisfied the following constraints, using impulse invariant transformation.

$$\mathbf{0.9 \le |H(e^{j\omega})| \le 1 \; ; 0 \le \omega \le \pi/2}$$
$$\mathbf{|H(e^{j\omega})| \le 0.2 \; ; 3\pi/4 \le \omega \le \pi}$$

AKTU 2017-18, Marks 10

Answer

Given : $0.9 \le |H(e^{j\omega})| \le 1 \; ; \quad 0 \le \omega \le \pi/2$

$|H(e^{j\omega})| \le 0.2 \; ; \quad 3\pi/4 \le \omega \le \pi$

$\delta_1 = 0.9$, $\delta_2 = 0.2$, $\omega_1 = \pi/2$ and $\omega_2 = 3\pi/4$

1. Assume $T = 1$ sec.

$$\Omega_1 = \frac{\omega_1}{2} = \pi/4$$

$$\Omega_2 = \frac{\omega_2}{2} = 3\,\pi/8$$

$$\Omega_2/\Omega_1 = \frac{\pi/4}{3\pi/8} = 0.67$$

2. Order of the filter

$$N \ge \frac{1}{2} \frac{\log \{[(1/\delta_2)^2 - 1]/[(1/\delta_1)^2 - 1]\}}{\log(\Omega_2/\Omega_1)}$$

$$N \ge \frac{1}{2} \frac{\log \{24/0.2346\}}{\log(2.41)} = \frac{1}{2}\left(\frac{2}{0.382}\right)$$

$$N \ge 2.62$$

$$N \simeq 3$$

3. Determination of – 3 dB cut-off frequency,

$$\Omega_c = \frac{\Omega_1}{[1/\delta_1^2 - 1]^{1/2N}} = \frac{\pi/4}{[1/0.9^2 - 1]^{1/6}} = 1$$

4. Now determination of $H_a(s)$,

$$H_a(s) = \frac{B_0\,\Omega_c}{s + c_0\,\Omega_c} \prod_{k=1}^{(N-1)/2} \frac{B_k\,\Omega_c^2}{s^2 + b_k\,\Omega_c\,s + c_k\,\Omega_c^2}$$

$$= \frac{B_0\,\Omega_c^2}{s + c_0\,\Omega_c}\left(\frac{B_1\,\Omega_c^2}{s^2 + b_1\,\Omega_c\,s + c_1\,\Omega_c^2}\right) \qquad [\because \text{For } k = 1]$$

$$b_1 = 2\sin\frac{\pi}{6} = 1,\ c_0 = 1,\ c_1 = 1$$

$$B_0 B_1 = 1, \text{ therefore, } B_0 = B_1 = 1$$

$$\therefore \qquad H(s) = \left(\frac{1}{s+1}\right)\left(\frac{1}{s^2+s+1}\right)$$

5. Now doing partial fraction expansion,

$$H(s) = \frac{A}{(s+1)} + \frac{Bs+C}{(s^2+s+1)}$$

$$1 = A\,(s^2+s+1) + (Bs+C)\,(s+1)$$

Put $s = -1$

$$1 = A\,(1)$$

$$A = 1$$

Compare coefficients of polynomial,

$$0 = A + B \qquad \text{[coefficients of } s^2\text{]}$$

$$B = -1$$

$$0 = A + B + C \qquad \text{[coefficients of } s\text{]}$$

$$C = -(A+B) = 0$$

$$H(s) = \frac{1}{(s+1)} - \frac{s}{(s^2+s+1)}$$

$$= \frac{1}{(s+1)} - \frac{s}{s^2+s+1+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}$$

$$H(s) = \frac{1}{(s+1)} - \left[\frac{s+1/2-1/2}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

6. Apply impulse invariance transformation,

$$H_D(z) = \left[\frac{1}{1-e^{-T}z^{-1}}\right] - \left[\frac{1-e^{-0.5T}\cos\frac{\sqrt{3}}{2}Tz^{-1}}{1-2e^{0.5T}\cos\frac{\sqrt{3}}{2}Tz^{-1}+e^{-T}z^{-2}}\right] + \frac{\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)}\left[\frac{e^{-0.5T}\sin\frac{\sqrt{3}}{2}Tz^{-1}}{1-2e^{-0.5T}\cos\frac{\sqrt{3}}{2}Tz^{-1}+e^{-T}z^{-2}}\right]$$

We have assumed $T = 1$ sec.

$$H_D(z) = \left[\frac{1}{1-0.3678\,z^{-1}}\right] - \left[\frac{1-0.1261\,z^{-1}}{1-0.7858\,z^{-1}+0.3678\,z^{-2}}\right]$$

Que 2.16. Determine $H(z)$ for a Butterworth filter satisfying the following constraints

$$\sqrt{0.5} \le |H(e^{j\omega})| \le 1, \quad 0 \le \omega \le \pi/2$$
$$|H(e^{j\omega})| \le 0.2, \quad 3\pi/4 \le \omega \le \pi$$

with T = 1 sec. Apply impulse invariant transformation.

AKTU 2016-17, Marks 10

Answer

Given : $\sqrt{0.5} \le |H(e^{j\omega})| \le 1, \quad 0 \le \omega \le \pi/2$

$|H(e^{j\omega})| \le 0.2, \quad 3\pi/4 \le \omega \le \pi$

$\delta_1 = \sqrt{0.5} = 0.707$, $\delta_2 = 0.2$, $\omega_1 = \pi/2$ and $\omega_2 = 3\pi/4$.

To Find : $H(z)$.

Step I : Determination of analog filter's edge frequencies.

$$\Omega_1 = \frac{\omega_1}{T} = \frac{\pi}{2} \text{ and } \Omega_2 = \frac{\omega_2}{T} = \frac{3\pi}{4}$$

Therefore, $\Omega_2/\Omega_1 = 1.5$

Step II : Determination of the order of the filter.

$$N \ge \frac{1}{2}\frac{\log\{((1/\delta_2^2)-1)/((1/\delta_1^2)-1)\}}{\log(\Omega_2/\Omega_1)}$$

$$N \simeq 4$$

Step III : Determination of – 3 dB cut-off frequency :

$$\Omega_c = \frac{\Omega_1}{[(1/\delta_1^2-1)]^{1/2N}} = \frac{\pi/2}{[(1/0.707^2)-1]^{1/8}} = \frac{\pi}{2}$$

Step IV :

1. Determination of $H_a(s)$:

$$H(s) = \prod_{k=1}^{N/2} \frac{B_k\Omega_c^2}{s^2 + b_k\Omega_c s + c_k\Omega_c^2}$$

$$= \left(\frac{B_1\Omega_c^2}{s^2 + b_1\Omega_c s + c_1\Omega_c^2}\right)\left(\frac{B_2\Omega_c^2}{s^2 + b_2\Omega_c s + c_2\Omega_c^2}\right)$$

$$b_1 = 2\sin\frac{\pi}{8} = 0.76536, \; c_1 = 1$$

$$b_2 = 2\sin\frac{3\pi}{8} = 1.84776, \; c_2 = 1$$

$\because \quad B_1 B_2 = 1$, so $B_1 = B_2 = 1$.

2. Therefore, $$H(s) = \left(\frac{2.467}{s^2 + 1.2022s + 2.467}\right)\left(\frac{2.467}{s^2 + 2.9025s + 2.467}\right) \quad ...(2.14.1)$$

3. Using partial fractions,

$$H(s) = \left(\frac{As + B}{s^2 + 1.2022s + 2.467}\right) + \left(\frac{Cs + D}{s^2 + 2.9025s + 2.467}\right) \quad ...(2.14.2)$$

4. Comparing eq. (2.14.1) and (2.14.2), we get

$$6.086 = (s^2 + 2.9025s + 2.467)(As + B) + (s^2 + 1.2022s + 2.467)(Cs + D)$$

5. Comparing the coefficients of s^3, s^2, s and the constants, we get

$A + C = 0$

$2.9025\, A + B + 1.2022\, C + D = 0$ ...(2.14.3)

$2.467\, A + 2.9025\, B + 2.467\, C + 1.2022\, D = 0$ ...(2.14.4)

$B + D = 2.467$ ...(2.14.5)

6. Solving eq. (2.14.3), eq. (2.14.4) and eq. (2.14.5) then we get,

$A = -1.4509$, $B = -1.7443$, $C = 1.4509$ and $D = 4.2113$

$$H(s) = -\left(\frac{1.4509s + 1.7443}{s^2 + 1.2022s + 2.467}\right) + \left(\frac{1.4509s + 4.2113}{s^2 + 2.9025s + 2.467}\right)$$

7. Let $H(s) = H_1(s) + H_2(s)$,

where $$H_1(s) = -\left(\frac{1.4509s + 1.7443}{s^2 + 1.2022s + 2.467}\right)$$

and $$H_2(s) = \left(\frac{1.4509s + 4.2113}{s^2 + 2.9025s + 2.467}\right)$$

8. Rearranging $H_1(s)$ into the standard from,

$$H_1(s) = -\left(\frac{1.4509s + 1.7443}{s^2 + 1.2022s + 2.467}\right)$$

$$= -1.4509\left(\frac{s + 1.2022}{(s + 0.601)^2 + 1.451^2}\right)$$

$$= (-1.4509)\left[\frac{s + 0.601}{(s + 0.601)^2 + 1.451^2} + \frac{0.601}{(s + 0.601)^2 + 1.451^2}\right]$$

$$= (-1.4509)\left(\frac{s + 0.601}{(s + 0.601)^2 + 1.451^2}\right) - (0.601)\left(\frac{1.451}{(s + 0.601)^2 + 1.451^2}\right)$$

9. Similarly, $H_2(s)$ can be written as

$$H_2(s) = (1.4509)\left(\frac{s + 1.45}{(s + 1.45)^2 + 0.604^2}\right) + (3.4903)\left(\frac{0.604}{(s + 1.45)^2 + 0.604^2}\right)$$

Step V : Determination of $H(z)$

$$H_1(z) = (-1.4509)\frac{1 - e^{-0.601T}(\cos 1.451T)z^{-1}}{1 - 2e^{-0.601T}(\cos 1.451T)z^{-1} + e^{-1.202T}z^{-2}}$$

$$- (0.601)\frac{e^{-0.601T}(\sin 1.451T)z^{-1}}{1-2e^{-0.601T}(\cos 1.451T)z^{-1}+e^{-1.202T}z^{-2}}$$

and

$$H_2(z) = (1.4509)\frac{1-e^{-1.45T}(\cos 0.604T)z^{-1}}{1-2e^{-1.45T}(\cos 0.604T)z^{-1}+e^{-2.9T}z^{-2}}$$

$$+ (3.4903)\frac{e^{-1.45T}(\sin 0.604T)z^{-1}}{1-2e^{-1.45T}(\cos 0.604T)z^{-1}e^{-2.9T}z^{-2}}$$

Here $H(z) = H_1(z) + H_2(z)$.

$$H(z) = \frac{-1.4509-0.2321z^{-1}}{1-0.1310z^{-1}+0.3006z^{-2}} + \frac{1.4509+0.1848z^{-1}}{1-0.3862z^{-1}+0.055z^{-2}}$$

Que 2.17. **Find the order and cut-off frequency of a digital filter with the following specification**

$$0.89 \le |H(e^{j\omega})| \le 1,\ 0 \le \omega \le 0.4\pi$$
$$|H(e^{j\omega})| \le 0.18,\quad 0.6\pi \le \omega \le \pi$$

use the impulse invariance method. **AKTU 2018-19, Marks 07**

Answer

The procedure is same as Q. 2.16, Page 2–23C, Unit-2.
(**Ans.** Order = 1, Ω_c = 0.75)

Que 2.18. **Design a Butterworth low pass analog filter for the following specification :**

i. Pass band gain required : 0.9

ii. Frequency up to which pass band gain must remain more or less steady : 100 rad/sec.

iii. Gain in attenuation band : 0.4

iv. Frequency from which the attenuation must start : 200 rad/sec. **AKTU 2019-20, Marks 07**

Answer

The design steps are given below :

Step 1 : Determination of n and ω_c :

1. We have,
$$H(\omega) = \left[\frac{1}{(1+(\omega/\omega_c)^{2n}}\right]^{1/2} \quad ...(2.18.1)$$

2. Substituting the value of $H(\omega) = 0.9$ and $\omega = 100$ rad/s in eq. (2.18.1) then we get,
$$0.9 = \left[\frac{1}{1+(100/\omega_c)^{2n}}\right]^{1/2} \quad ...(2.18.2)$$

3. Similarly at $\omega = 200$ rad/s and $H(\omega) = 0.4$, we have

$$0.4 = \left[\frac{1}{1+(200/\omega_c)^{2n}}\right]^{1/2} \qquad ...(2.18.3)$$

4. Squaring in both sides of eq. (2.18.2), we get

$$0.81 = \frac{1}{1+(100/\omega_c)^{2n}} \qquad ...(2.18.4)$$

5. Inverting eq. (2.18.4), we get

$$1+\left(\frac{100}{\omega_c}\right)^{2n} = +\frac{1}{0.81} \qquad ...(2.18.5)$$

6. Eq. (2.18.5) may be simplified to get :

$$\left(\frac{100}{\omega_c}\right)^{2n} = 1.234 - 1 = 0.234 \qquad ...(2.18.6)$$

7. Similarly eq. (2.18.3) may be simplified as :

$$\left(\frac{200}{\omega_c}\right)^{2n} = \frac{1}{0.16} - 1 = 5.25 \qquad ...(2.18.7)$$

8. Dividing eq. (2.18.7) with eq. (2.18.6), we get

$$\frac{5.25}{0.234} = \left(\frac{200}{100}\right)^{2n} = 2^{2n} \qquad ...(2.18.8)$$

9. Taking logarithm of both sides of eq. (2.18.8), we have

$$2n \log 2 = \log 22.436 \qquad ...(2.18.9)$$

10. From eq. (2.18.9), we find that

$$n = 2.244 \qquad ...(2.18.10)$$

11. However, it is quite customary to specify the order of a given filter in integers or whole numbers.

$$n = 3$$

12. Substituting the value of n in eq. (2.18.7)

$$\left(\frac{200}{\omega_c}\right)^{2\times 3} = 5.25$$

$$\left(\frac{200}{\omega_c}\right) = (5.25)^{1/6}$$

$$\omega_c = \left(\frac{200}{1.318}\right) = 151.7$$

Step 2 : Determination of the poles of $H(s)$:

1. Since $n = 3$, which is odd and the first pole is at 0°.

Angle between poles $\theta = \frac{360°}{2n} = \frac{360°}{6} = 60°$

2. Hence, the second pole is at $\theta = 60°$. And we get the other poles at 120°, 180°, 240° and 300° respectively.

Step 3 : Determination of valid poles of $H(s)$:

1. The poles must be lie in between 90° and 270°. So the valid poles are 120°, 180° and 240°.
2. The location of the poles are,

Pole $B_1 \rightarrow (\cos 180° + j \sin 180°) \times \omega_c = -1 \times 151.7 = -151.7$

Pole $B_2 \rightarrow (\cos 120° + j \sin 120°) \times \omega_c = \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)(151.7)$

$= -75.85 + j\,131.376$

Pole $B_3 \rightarrow (\cos 240° + j \sin 240°) \times \omega_c = \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)(151.7)$

$= -75.85 - j\,131.376$

Step 4 : Finding the expression for $H(s)$:

1. We have,
$$H(s) = \frac{\omega_c^{\,2}}{(s + a + jb)(s + a - jb)}\frac{\omega_c}{(s + \omega_c)} \quad ...(2.18.11)$$
2. Substituting the values of poles in eq. (2.18.11) and we get,
$$H(s) = \frac{(151.7)^2}{(s + 75.85 + j131.376)(s + 75.85 - j131.376)} \times \frac{(151.7)}{(s + 151.7)}$$
$$H(s) = \frac{(151.7)^3}{(s + 75.85 + j131.376)(s + 75.85 - j131.37)(s + 1)} \quad ...(2.18.12)$$
3. Eq. (2.18.12) can be simplified to :
$$H(s) = \frac{3.491 \times 10^6}{(s^2 + 151.7s + 0.02301 \times 10^6)(s + 151.7)} \quad ...(2.18.13)$$

Que 2.19. **Discuss the type-I Chebyshev filter.**

Answer

1. The magnitude response is given as
$$|H(\Omega)|^2 = \frac{A}{[1 + \in^2 C_N^2(\Omega/\Omega_p)]^{0.5}} \quad ...(2.19.1)$$
where, $C_N(x)$ = N^{th} order Chebyshev polynomial

$\in$ = Parameter of the filter related to the ripple in the passband.
2. Chebyshev polynomial : It is given as
$$C_N(x) = \begin{cases} \cos(N \cos^{-1} x) & \text{for } |x| \le 1 \\ \cosh(N \cosh^{-1} x) & \text{for } |x| \ge 1 \end{cases} \quad ...(2.19.2)$$
3. The reoccurrence relationship $C_N(x) = 2\,xC_{N-1}(x) - C_{N-2}(x)$ ...(2.19.3)
4. The phase response of the Chebyshev filter is more non-linear than the Butterworth filter for a given filter length N.

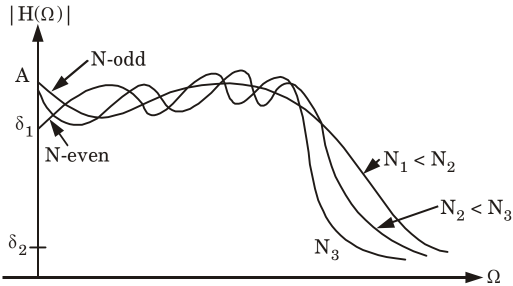


Fig. 2.19.1. Magnitude response of LPF Chebyshev filter.

Designing of Chebyshev filter :

1. $$\delta_1 \leq |H(e^{j\omega})| \leq 1 \quad \text{for } 0 \leq \omega \leq \omega_1 \quad \text{...(2.19.4)}$$
$$|H(e^{j\omega})| \leq \delta_2 \quad \text{for } \omega_2 \leq \omega \leq \pi \quad \text{...(2.19.5)}$$

as we know $$\delta^2{}_1 \leq \frac{1}{1+\in^2 C_N^2(\Omega_1/\Omega_c)} \leq 1 \quad \text{...(2.19.6)}$$

$$\frac{1}{1+\in^2 C_N^2(\Omega_2/\Omega_c)} \leq \delta^2{}_2 \quad \text{...(2.19.7)}$$

2. Assuming $\Omega_c = \Omega_1$
we will have $C_N(\Omega_1/\Omega_c) = C_N(1) = 1$
3. Therefore eq. (2.19.6) and eq. (2.19.7) will be
$$\delta^2{}_1 \leq \frac{1}{1+\in^2} \quad \text{...(2.19.8)}$$
4. Assuming equality in the eq. (2.19.8) and the expression for $\in$ is
$$\in = \left(\frac{1}{\delta_1^2} - 1\right)^{0.5}$$
5. Assuming $\Omega_c = \Omega_1$, the order of the analog filter is given as
$$C_N(\Omega_2/\Omega_1) \geq \frac{1}{\in}\left(\frac{1}{\delta_2^2} - 1\right)^{0.5}$$
6. Since $\Omega_2 > \Omega_1$, $$\cosh\left[N \cosh^{-1}\left(\frac{\Omega_2}{\Omega_1}\right)\right] \geq \frac{1}{\in}\left(\frac{1}{\delta_2^2} - 1\right)^{0.5}$$

or $$N \geq \frac{\cosh^{-1}\left[\frac{1}{\in}\left(\frac{1}{\delta_2^2} - 1\right)^{0.5}\right]}{\cosh^{-1}\left(\frac{\Omega_2}{\Omega_1}\right)}$$

7. The coefficient b_k and c_k are given as
$$b_k = 2y_N \sin[(2k-1)\,\pi/2N]$$
$$c_k = y_N^2 + \cos^2\frac{(2k-1)\pi}{2N}$$
$$c_0 = y_N$$
8. The parameter y_N is given by

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\epsilon^2}+1\right)^{0.5}+\frac{1}{\epsilon}\right]^{\frac{1}{N}}-\left[\left(\frac{1}{\epsilon^2}+1\right)^{0.5}+\frac{1}{\epsilon}\right]^{-\frac{1}{N}}\right\}$$

9. The parameter B_k can be obtained from

$$\frac{A}{(1+\epsilon^2)^{0.5}} = \prod_{k=1}^{N/2}\frac{B_k}{C_k} \quad \text{for } N\text{-even}$$

or
$$A = \prod_{k=0}^{\frac{(N-1)}{2}}\frac{B_k}{C_k} \quad \text{for } N\text{-odd}$$

Que 2.20. **Discuss Chebyshev type-II filter.**

Answer

1. The response is given as : $H(j\Omega) = \dfrac{\epsilon C_N(\Omega_z/\Omega)}{[1+\epsilon^2 C_N^2(\Omega_z/\Omega)]^{0.5}}$

where, ϵ = Constant
Ω_c = 3 dB cut-off frequency
and $C_N(x)$ = Chebyshev polynomial

2. The parameter of Chebyshev filter are obtained by considering the low-pass filter with the desired specifications as given below :

$$0.707 \le |H(j\Omega)| \le 1 \quad 0 \le \Omega \le \Omega_c$$
$$|H(j\Omega)| \le \delta_z \quad \Omega \ge \Omega_z$$

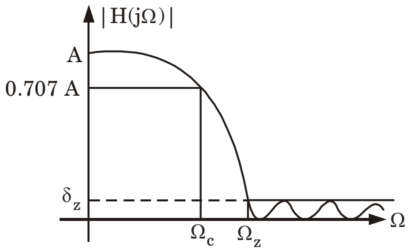


Fig. 2.20.1.

Que 2.21. **Compare the characteristics of Butterworth and Chebyshev filter. Determine the parameters of a Chebyshev filter for which $A_1 = \frac{1}{\sqrt{2}}$, $A_2 = 0.1$, $\Omega_1 = 2$ rad/s and $\Omega_2 = 4$ rad/s.**

Answer

A. Comparison :

S. No.	Characteristics	Butterworth filter	Chebyshev filter
1.	Magnitude frequency squared response $\lvert H_a j(\Omega)\rvert^2$	$\dfrac{1}{1+\left(\dfrac{\Omega}{\Omega_c}\right)^{2N}}$	$\dfrac{1}{1+\in^2 C_N^2(\Omega)}$ for $\Omega_c = 1$
2.	Transition band	Transition band is broader than Chebyshev for given order.	Transition band is narrower than Butterworth for given order.
3.	Frequency response.	Monotonically decreasing.	Ripples in passband and monotonic in stopband.
4.	Phase response	Good linear phase response.	Relatively non-linear phase response than Butterworth.
5.	Order of specification for a given set.	Higher.	Lower.
6.	Poles of $H_a(s)$ (location).	Poles of $H_a(s)$ lie on the circle of radius Ω_c in the s-plane.	Poles of $H_a(s)$ lie on the ellipse in the s-plane.

B. Numerical :

Given : $A_1 = \dfrac{1}{\sqrt{2}}$, $A_2 = 0.1$, $\Omega_1 = 2$ rad/s, $\Omega_2 = 4$ rad/s

To Find : Ω_c, $\in$, N.

1. Cut-off frequency, $\Omega_c = \Omega_1 = 2$ rad/s

   The required value of $\in$ is

$$\in = \left[\frac{1}{A_1^2} - 1\right]^{0.5} = \left[\frac{1}{(1/\sqrt{2})^2} - 1\right]^{0.5}$$

$$\in = \sqrt{1} = 1$$

2. Order of filter,

$$N \geq \frac{\cosh^{-1}\left\{\dfrac{1}{\in}\left[\dfrac{1}{A_2^2} - 1\right]^{0.5}\right\}}{\cosh^{-1}\left(\dfrac{\Omega_2}{\Omega_1}\right)}$$

$$N \geq \frac{\cosh^{-1}\left\{1\left(\frac{1}{(0.1)^2}-1\right)^{0.5}\right\}}{\cosh^{-1}(4/2)} = \frac{\cosh^{-1}(9.95)}{\cosh^{-1}(2)} = \frac{2.988}{1.317} = 2.269.$$

So, order of filter, $N = 3$ (approx.)

Que 2.22. **Design a digital Chebyshev filter to satisfy the constraints**

$$0.707 \leq |H(e^{j\omega})| \leq 1,\ 0 \leq \omega \leq 0.2\pi$$
$$|H(e^{j\omega})| \leq 0.1,\ 0.5\pi \leq \omega \leq \pi$$

Using bilinear transformation with $T = 1$ s.

AKTU 2016-17, Marks 7.5

Answer

Given : $0.707 \leq |H(e^{j\omega})| \leq 1,\ 0 \leq \omega \leq 0.2\pi$

$|H(e^{j\omega})| \leq 0.1,\ 0.5\pi \leq \omega \leq \pi$

$\delta_1 = 0.77$, $\delta_2 = 0.1$, $\omega_1 = 0.2\pi$, $\omega_2 = 0.5\pi$, $T = 1$ s.

To Design : Digital Chebyshev filter.

1. Determination of the analog filter's edge frequencies.

$$\Omega_c = \Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2} = 2\tan 0.1\pi = 0.6498$$

$$\Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = 2\tan 0.25\pi = 2$$

Therefore, $\Omega_2/\Omega_1 = 3.0779$

2. Determination of the order of this filter.

$$\varepsilon = \left[\frac{1}{\delta_1^2}-1\right]^{0.5} = \left[\frac{1}{0.77^2}-1\right]^{0.5} = 0.828$$

Thus $$N \geq \frac{\cosh^{-1}\left\{\frac{1}{\in}\left[\frac{1}{\delta_2^2}-1\right]^{0.5}\right\}}{\cosh^{-1}(\Omega_2/\Omega_1)} = \frac{\cosh^{-1}\{1.207[100-1]^{0.5}\}}{\cosh^{-1}(2/0.6498)}$$

$$= \frac{\cosh^{-1}\{1.207\times 9.949\}}{\cosh^{-1}(3.078)} = \frac{\cosh^{-1}(12)}{\cosh^{-1}(3.078)} = \frac{3.176}{1.789} = 1.775$$

$N = 2$

3. Determination of $H(s)$

$$H(s) = \prod_{k=1}^{N/2}\frac{B_k\Omega_c^2}{s^2+b_k\Omega_c s+c_k\Omega_c^2} = \frac{B_1\Omega_c^2}{s^2+b_1\Omega_c s+c_1\Omega_c^2} \quad [\because \text{For k = 1}]$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\in^2}+1\right)^{0.5}+\frac{1}{\in}\right]^{1/N} - \left[\left(\frac{1}{\in^2}+1\right)^{0.5}+\frac{1}{\in}\right]^{-1/N}\right\}$$

$$y_2 = \frac{1}{2}\{[2.775]^{1/2}-[2.775]^{-1/2}\} = \frac{1}{2}[1.666-0.600] = 0.533$$

$$b_1 = 2y_2 \sin [(2k-1)\pi/2N] = 0.754$$

$$c_1 = y_2^2 + \cos^2 \frac{(2k-1)\pi}{2N} = 0.784$$

4. For N even $\quad \prod_{k=1}^{N/2} \frac{B_k}{c_k} = \frac{A}{(1+\varepsilon^2)^{0.5}} = 0.770$

i.e., $\quad \frac{B_1}{c_1} = 0.770$

Thus $\quad B_1 = (0.770)\, c_1 = 0.604$

5. The system function is $H(s) = \dfrac{0.604(0.6498)^2}{s^2 + (0.754)(0.6498)s + (0.784)(0.6498)^2}$

On simplifying we get

$$H(s) = \frac{0.255}{s^2 + 0.49s + 0.331}$$

6. Determination of $H(z)$ using bilinear transformation

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{(z-1)}{(z+1)}}$$

For $T = 1$ s

$$H(z) = \frac{0.255}{\left\{2\frac{(z-1)}{(z+1)}\right\}^2 + 0.49\left\{2\frac{(z-1)}{(z+1)}\right\} + 0.331}$$

$$= \frac{0.255\,(z+1)^2}{4(z-1)^2 + 0.98(z-1)(z+1) + 0.331(z+1)^2}$$

$$= \frac{0.255\,(z+1)^2}{4z^2 - 8z + 4 + 0.98z^2 - 0.98 + 0.331z^2 + 0.662z + 0.331}$$

$$H(z) = \frac{0.255(z+1)^2}{5.311z^2 - 7.338z + 3.51}$$

7. By dividing z^2 in numerator and denominator, we get

$$H(z) = \frac{0.255(1+z^{-1})^2}{5.311(1 - 1.381z^{-1} + 0.6309z^{-2})}$$

$$H(z) = \frac{0.048(1+z^{-1})^2}{1 - 1.3816z^{-1} + 0.6309z^{-2}}$$

Que 2.23. Why is frequency transformation needed ? What are the different types of frequency transformations ?

OR

What is the difference between Butterworth and Chebyshev ? Explain the frequency transformation is done.

AKTU 2018-19, Marks 07

Answer

A. Difference between Butterworth and Chebyshev : Refer Q. 2.21, Page 2–29C, Unit-2.

B. Need of frequency transformation :

1. Frequency transformation technique is used to design high-pass, band-pass and band-reject filters.
2. It becomes very easy to design a filter in terms of low-pass characteristics and to transform it to get other types of characteristics such as high-pass, band-pass, band-stop.

C. Types of frequency transformation :

a. Analog frequency transformation : The frequency transformation formulae used to convert a prototype low-pass filter into a low-pass (with a different cut-off frequency), high-pass, band-pass or band-stop is given below :

i. Low-pass with cut-off frequency Ω_c to low-pass with a new cut-off frequency Ω_c^*.

$$s \rightarrow \frac{\Omega_c}{\Omega_c^*} s \qquad ...(2.23.1)$$

Thus, if the system response of the prototype filter is $H_P(s)$, the system response of the new low-pass filter will be

$$H(s) = H_P\left(\frac{\Omega_c}{\Omega_c^*} s\right) \qquad ...(2.23.2)$$

ii. Low-pass with cut-off frequency Ω_c to high-pass with cut-off frequency Ω_c^* :

$$s \rightarrow \frac{\Omega_c \Omega_c^*}{s} \qquad ...(2.23.3)$$

The system function of the high-pass filter is then,

$$H(s) = H_P\left(\frac{\Omega_c\ \Omega_c^*}{s}\right) \qquad ...(2.23.4)$$

iii. Low-pass with cut-off frequency Ω_c to band-pass with lower cut-off frequency Ω_1 and higher cut-off frequency Ω_2.

$$s \rightarrow \Omega_c \frac{s^2 + \Omega_1\ \Omega_2}{s(\Omega_2 - \Omega_1)} \qquad ...(2.23.5)$$

The system function of the band-pass filter is then

$$H(s) = H_P\left(\Omega_c \frac{s^2 + \Omega_1\ \Omega_2}{s(\Omega_2 - \Omega_1)}\right) \qquad ...(2.23.6)$$

iv. Low-pass with cut-off frequency Ω_c to band-stop with lower cut-off frequency Ω_1 and higher cut-off frequency Ω_2 :

$$s \rightarrow \Omega_c \frac{s\,(\Omega_2 - \Omega_1)}{s^2 + \Omega_1\ \Omega_2}$$

The system function of the band-stop filter is then,

$$H(s) = H_P\left(\Omega_c \frac{s\,(\Omega_2 - \Omega_1)}{s^2 + \Omega_1 \Omega_2}\right)$$

b. Digital frequency transformation :

i. The frequency transformation is done in the digital domain by replacing the variable z^{-1} by a function of z^{-1}, *i.e.*, $f(z^{-1})$.

ii. This mapping must take into account the stability criterion. All the poles lying within the unit circle must map onto itself and the unit circle must also map onto itself.

iii. For the unit circle to map onto itself, the implication is that for $r = 1$,

$$e^{-j\omega} = f(e^{-j\omega}) = |f(e^{-j\omega})|\, e^{j\arg[f(e^{-j\omega})]}$$

iv. Hence, we must have $|f(e^{-j\omega})| = 1$ for all frequencies. So, the mapping is that of an all pass filter and of the form,

$$f(z^{-1}) = \pm \prod_{k=1}^{n} \frac{z^{-1} - a_k}{1 - a_k z^{-1}} \qquad \text{...(2.23.7)}$$

v. To get a stable filter from the stable prototype filter, we must have $|a_k| < 1$. The transformation formulae can be obtained from eq. (2.23.7) for converting the prototype low-pass digital filter into a digital low-pass, high-pass, band-pass or band-stop filter.

Que 2.24. Explain frequency transformation with LPF to HPF conversion formula.

Answer

1. The frequency transformation formulae used to convert a prototype lowpass filter into a lowpass (with a different cut-off frequency), high pass, band pass or band stop are given below.
2. To convert low-pass with cut-off frequency Ω_c to high-pass with cut-off frequency Ω_c^*. $s \to \dfrac{\Omega_c \, \Omega_c^*}{s}$

   The system function of the high-pass filter is then, $H(s) = H_p\left(\dfrac{\Omega_c \, \Omega_c^*}{s}\right)$
3. In the digital domain the frequency transformation is done replacing the variable z^{-1} by a function of z^{-1}, *i.e.*, $f(z^{-1})$. This mapping must take into account the stability criterion.

   In low-pass $\qquad z^{-1} \to \dfrac{z^{-1} - a}{1 - az^{-1}}$

   In high-pass $\qquad z^{-1} \to \dfrac{z^{-1} + a}{1 - az^{-1}}$

Que 2.25. Transform the prototype LPF with system function

$$\boldsymbol{H_{LP}(s) = \frac{\Omega_p}{s + \Omega_p}} \text{ into a}$$

i. HPF with cut-off frequency Ω_p.

ii. BPF with upper and lower cut-off frequencies Ω_u and Ω_l respectively.

AKTU 2015-16, Marks 10

Answer

1. From the analog frequency transformation of LPF to HPF is given as

$$s \to \frac{\Omega_c\, \Omega_c^*}{s}$$

2. Thus, the system function of HPF is given as

$$H_{HPF}(s) = H_p(s)\Big|_{s=\frac{\Omega_c\Omega_c^*}{s}} = \left[\frac{\Omega_c}{s+\Omega_c}\right]\Bigg|_{s=\frac{\Omega_c\Omega_c^*}{s}}$$

$$= \frac{\Omega_c}{\frac{\Omega_c\Omega_c^*}{s}+\Omega_c} = \frac{\Omega_c\, s}{\Omega_c\, s+\Omega_c\,\Omega_c^*} = \frac{\Omega_c\, s}{\Omega_c\,(s+\Omega_c^*)} = \frac{s}{s+\Omega_c^*}$$

i. HPF with cut-off frequency Ω_p.

1. For LPF to HPF, we use $s \to \dfrac{\Omega_c\, \Omega_{HP}}{s}$

$$H_{HP} = \frac{\Omega_p}{\frac{\Omega_c.\Omega_{HP}}{s}+\Omega_p} = \frac{\Omega_p.s}{\Omega_c\Omega_{HP}+\Omega_p.s} = \frac{s}{s+\frac{\Omega_c\,\Omega_{HP}}{\Omega_p}}$$

ii. BPF with upper and lower cut-off frequencies Ω_u and Ω_l respectively.

1. Band-pass filter with upper and lower cut-off frequency Ω_u and Ω_l.

$$s \to \Omega_c.\frac{s^2+\Omega_l\Omega_u}{s(\Omega_u-\Omega_l)}$$

So,
$$H_{BP} = \frac{\Omega_p}{\Omega_c\left(\frac{s^2+\Omega_l\Omega_u}{s(\Omega_u-\Omega_l)}\right)+s} = \frac{(\Omega_p/\Omega_c)s(\Omega_u-\Omega_l)}{s^2+\Omega_l\Omega_u+\Omega_P.s(\Omega_u-\Omega_l)}$$

$$H_{BP} = \frac{s(\Omega_p/\Omega_c)(\Omega_u-\Omega_l)}{s^2+s.\Omega_p(\Omega_u-\Omega_l)+\Omega_l\Omega_u}$$

VERY IMPORTANT QUESTIONS

Following questions are very important. These questions may be asked in your SESSIONALS as well as UNIVERSITY EXAMINATION.

Q. 1. Determine $H(z)$ using the impulse invariant technique for the analog system function

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)}$$

Ans. Refer Q. 2.5.

Q. 2. The system function of analog filter is given by

$$H(s) = \frac{(s+0.1)}{(s+0.1)^2+16}$$

Obtain the system function of digital filter by using impulse invariant technique. Assume T = 1 sec.

Ans. Refer Q. 2.6.

Q. 3. Discuss the bilinear transformation method of converting analog IIR filter into digital IIR filter. What is frequency warping ?

Ans. Refer Q. 2.8.

Q. 4. Convert the analog filter with system function

$$H(s) = \frac{s+0.1}{(s+0.1)^2+9}$$

into digital filter with a resonant frequency of $\omega_r = \pi/4$ of using bilinear transformation.

Ans. Refer Q. 2.9.

Q. 5. Use bilinear transformation to convert low pass filter, $H(s) = 1/s^2 + \sqrt{2}\,s + 1$ into a high pass filter with pass band edge at 100 Hz and F_s = 1 kHz.

Ans. Refer Q. 2.11.

Q. 6. Design a digital Butterworth filter that satisfied the following constraints, using impulse invariant transformation.

$$0.9 \le |H(e^{j\omega})| \le 1 \quad ; \qquad 0 \le \omega \le \pi/2$$
$$|H(e^{j\omega})| \le 0.2 \; ; \qquad 3\pi/4 \le \omega \le \pi$$

Ans. Refer Q. 2.15.

Q. 7. Determine $H(z)$ for a Butterworth filter satisfying the following constraints

$$\sqrt{0.5} \le |H(e^{j\omega})| \le 1, \; 0 \le \omega \le \pi/2$$
$$|H(e^{j\omega})| \le 0.2, \; 3\pi/4 \le \omega \le \pi$$

with T = 1 sec. Apply impulse invariant transformation.

Ans. Refer Q. 2.16.

Q. 8. What is the difference between Butterworth and Chebyshev ? Explain the frequency transformation is done.

Ans. Refer Q. 2.23.

Q. 9. Transform the prototype LPF with system function

$$H_{LP}(s) = \frac{\Omega_p}{s+\Omega_p} \text{ into a}$$

i. HPF with cut-off frequency Ω_p.

ii. BPF with upper and lower cut-off frequencies Ω_u and Ω_l respectively.

Ans. Refer Q. 2.25.

☺☺☺

3 UNIT

FIR Filter Design

CONTENTS

PART-1

Windowing and the Rectangular Window, Gibb's Phenomenon, Other Commonly Used Windows (Hamming, Hanning, Bartlett, Blackmann, Kaiser), Examples of Filter Design Using Windows.

CONCEPT OUTLINE

- Rectangular window function, $w_R(n) = \begin{cases} 1 & , \text{for } |n| \leq \dfrac{M-1}{2} \\ 0 & , \text{otherwise} \end{cases}$
- Hamming window function,

$$w_H(n) = \begin{cases} 0.54 - 0.46 \cos \dfrac{2\pi n}{M-1} & , 0 \leq n < M-1 \\ 0 & , \text{otherwise} \end{cases}$$

- Kaiser window function,

$$w_k(n) = \begin{cases} \dfrac{I_0(\beta)}{I_0(\alpha)} & , \text{for } |n| \leq \dfrac{M-1}{2} \\ 0 & , \text{otherwise} \end{cases}$$

where α is an independent variable determined by Kaiser.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.1. **What is reason that FIR Filters are always stable ? Also write the properties of FIR filter.**

Answer

A. Reason :

1. The impulse response of FIR filter is $h(n) = a^n\, u(n)$
2. It has all non-zero for $n \geq 0$ *i.e.*, it has infinite number of non-zero terms. The difference equation for FIR filter is :

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

where M = Length of the filter

3. The filter response is given as,

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k) \quad ...(3.1.1)$$

Eq. (3.1.1) only has zeros in right half and poles. Thus FIR filters are always stable.

B. Properties of FIR filters :

i. They have linear phase.

ii. They are always stable.

iii. Their design methods are generally linear.

iv. They can be realized efficiently in hardware.

v. The filter start-up transients have finite duration.

Que 3.2. What are the different window functions used for windowing ? Explain the effects of using different window functions for designing FIR filter on the filter response.

AKTU 2017-18, Marks 10

Answer

A. Window Techniques :

1. Windowing techniques, to design, the FIR filter is the easiest method. In this method the impulse response of an IIR filter is truncated by some function.
2. Suppose

$$h_d(n) = \text{Impulse response of IIR filter}$$

and
$$h(n) = \text{Impulse response of FIR filter.}$$

3. Using windowing technique the $h(n)$ is given as,

$$h(n) = \begin{cases} h_d(n) & ; \quad N_1 \le n \le N_2 \\ 0 & ; \quad \text{otherwise} \end{cases} \quad ...(3.2.1)$$

4. In general $h(n) = h_d(n)\ w(n)$ ...(3.2.2)

where, $w(n)$ = Window function.

5. Using Fourier transform the eq. (3.2.2) can be given as

$$H(e^{j\omega}) = H_d(e^{j\omega}) * w(e^{j\omega}) \quad ...(3.2.3)$$

or

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})\, w(e^{j(\omega-\theta)})\, d\theta \quad ...(3.2.4)$$

B. Characteristics of window function :

i. It should be an even function about $n = 0$.

ii. It should have zero in the range $0 \le n \le N-1$.

iii. As $\omega \to \pi$, the energy of sidelobes of the frequency response is low.

iv. The narrow width of the main lobe means it has high total energy.

C. Types of window function :

i. Rectangular window function :

$$w_R(n) = \begin{cases} 1 & ; \text{for } |n| \leq \left(\dfrac{M-1}{2}\right) \\ 0 & ; \text{otherwise} \end{cases} \qquad \text{...(3.2.5)}$$

The frequency response of rectangular window is given as

$$w_R(e^{j\omega T}) = \sum_{n=0}^{(M-1)} e^{-j\omega nT} = e^{-j\omega\frac{(M-1)}{2}T} \frac{\sin\left(\dfrac{\omega MT}{2}\right)}{\sin\left(\dfrac{\omega T}{2}\right)} \qquad \text{...(3.2.6)}$$

From eq. (3.2.6) the linear phase response of

a. Causal filter $\theta(\omega) = \omega(M-1)\,T/2$

b. Non-causal filter $\theta(\omega) = 0$

Effect : In the magnitude spectrum of rectangular window, there is one main lobe and many sidelobes. As M increase, the main lobe becomes narrower. The area under the sidelobes remains same irrespective of change in value of M.

ii. Hamming window function :

a. Causal Hamming window :

$$w_{\text{Hamm}}(n) = \begin{cases} 0.54 - 0.46\cos\dfrac{2\pi n}{M-1} & ; \quad 0 \leq n < M-1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

b. Non-causal Hamming window :

$$w_{\text{Hamm}}(n) = \begin{cases} 0.54 + 0.46\cos\dfrac{2\pi n}{M-1} & ; \quad 0 \leq n < M-1/2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Effect : In magnitude response of FIR filter using hamming window, the sidelobes get reduced but slightly increased main lobe.

iii. Hanning window function :

a. Causal Hanning window :

$$w_{\text{Hann}}(n) = \begin{cases} 0.5 - 0.5\cos\dfrac{2\pi n}{M-1} & ; \quad 0 \leq n < M-1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

b. Non-causal Hanning window :

$$w_{\text{Hann}}(n) = \begin{cases} 0.5 + 0.5\cos\dfrac{2\pi n}{M-1} & ; 0 \leq n < M-1/2 \\ 0 & ; \text{otherwise} \end{cases}$$

Effect : It has narrow main lobe and first few sidelobes are significant, after then sidelobes reduce rapidly.

iv. Blackman window function :

a. Causal blackman window : It is given as

$$w_{\text{black}}(n) = \begin{cases} 0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1} & ; \quad 0 \le n < M-1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

b. Non-causal blackman window : It is given as :

$$w_{\text{black}}(n) = \begin{cases} 0.42 + 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1} & ; 0 \le n < M-1/2 \\ 0 & ; \text{otherwise} \end{cases}$$

Effect : In magnitude response, the width of main lobe is increased and it has very small sidelobes.

v. Bartlett window function : For non-causal Bartlett window the function is given as,

$$w_{\text{bart}}(n) = \begin{cases} 1+n\,; & -\dfrac{M-1}{2} < n < 1 \\ 1-n\,; & 1 < n < \dfrac{M-1}{2} \end{cases}$$

Effect : The sidelobe is smaller than that of rectangular window and the main lobe become twice that of the rectangular window.

vi. Kaiser window function :

$$w_k(n) = \begin{cases} \dfrac{I_0(\beta)}{I_0(\alpha)} & ; \quad \text{for } n \le \dfrac{M-1}{2} \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Effect : It has reduced sidelobes and transition band is also narrow. Here, note that width of main lobe is inversely proportional to the length of the filter.

Que 3.3. **Explain rectangular window with mathematical expression.**

Answer

1. Rectangular window is shown in Fig. 3.3.1. It is denoted by $w_R(n)$. Its magnitude is 1 for the range, $n = 0$ to $M - 1$. Let $h_d(n)$ be the impulse response having infinite duration.
2. If $h_d(n)$ is multiplied by $w_R(n)$ then a finite impulse response is obtained. Since the shape of the window function is rectangular, it is called as rectangular window.

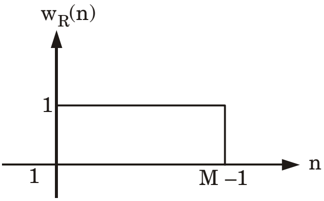


Fig. 3.3.1 Rectangular window

3. The rectangular window is defined as

$$w_R(n) = \begin{cases} 1 & ; \text{for } n = 0,1,2,\ldots, M-1 \\ 0 & ; \text{ otherwise} \end{cases} \quad \ldots(3.3.1)$$

4. Let $h_d(n)$ be infinite duration impulse response. We know that the finite duration impulse response $h(n)$ is obtained by multiplying $h_d(n)$ by $w_R(n)$.

$$\therefore \quad h(n) = h_d(n) \,.\, w_R(n) \quad \ldots(3.3.2)$$

5. By Fourier transform we can write,

$$W_R(\omega) = \sum_{n=0}^{M-1} w_R(n) e^{-j\omega n} \quad \ldots(3.3.3)$$

6. But the value of $w_R(n)$ is 1 for range $n = 0$ to $M - 1$.

$$\therefore \quad W_R(\omega) = \sum_{n=0}^{M-1} 1.e^{-j\omega n} \quad \ldots(3.3.4)$$

7. We can represent the window sequence as,

$$w_R(n) = u(n) - u(n - M)$$

Here $u(n)$ is unit step having duration $n = 0$ to $n \to (\infty)$ and $u(n - M)$ is delay unit step.

8. Thus eq. (3.3.3) becomes,

$$W_R(\omega) = \sum_{n=0}^{\infty} [u(n) - u(n-M)] e^{-j\omega n}$$

$$W_R(\omega) = \sum_{n=0}^{\infty} u(n) e^{-j\omega n} - \sum_{n=0}^{\infty} u(n-M) e^{-j\omega n} \quad \ldots(3.3.5)$$

9. Consider the first term at RHS. It represents Fourier transform of unit step.

$$\text{F.T. of } u(n) = \sum_{n=0}^{\infty} 1 \cdot e^{-j\omega n} = \sum_{n=0}^{\infty} (e^{-j\omega})^n = \frac{1}{1 - e^{-j\omega}}$$

10. Now consider the second term. It represents the Fourier transform of delayed unit step.

Fourier transform of

$$u(n - M) \longleftrightarrow e^{-j\omega M} F[u(n)]$$

$$= e^{-j\omega M} \frac{1}{1 - e^{-j\omega}} = \frac{e^{-j\omega M}}{1 - e^{-j\omega}}$$

11. Thus eq. (3.3.5) becomes,

$$W_R(\omega) = \frac{1}{1 - e^{-j\omega}} - \frac{e^{-j\omega M}}{1 - e^{-j\omega}}$$

$$\therefore \quad W_R(\omega) = \frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}} \quad \ldots(3.3.6)$$

12. Rearrange eq. (3.3.6) as follows,

$$W_R(\omega) = \frac{e^{\frac{-j\omega M}{2}} e^{\frac{+j\omega M}{2}} - e^{\frac{-j\omega M}{2}} e^{\frac{-j\omega M}{2}}}{e^{\frac{-j\omega}{2}} e^{\frac{+j\omega}{2}} - e^{\frac{-j\omega}{2}} e^{\frac{-j\omega}{2}}}$$

$$W_R(\omega) = \frac{e^{\frac{-j\omega M}{2}}\left(e^{\frac{+j\omega M}{2}} - e^{\frac{-j\omega M}{2}}\right)}{e^{\frac{-j\omega}{2}}\left(e^{\frac{+j\omega}{2}} - e^{\frac{-j\omega}{2}}\right)} \quad ...(3.3.7)$$

13. We know that, $\dfrac{e^{j\theta} - e^{-j\theta}}{j} = 2 \sin \theta$

$$W_R(\omega) = \frac{e^{\frac{-j\omega M}{2}}\left[2 \sin\left(\frac{\omega M}{2}\right)\right]}{e^{\frac{-j\omega}{2}}\left[2 \sin\left(\frac{\omega}{2}\right)\right]} = e^{\frac{-j\omega M}{2}} e^{\frac{j\omega}{2}} \frac{\sin\left(\frac{\omega M}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

$$\therefore \quad W_R(\omega) = \frac{\sin\left(\frac{\omega M}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\omega\left(\frac{M-1}{2}\right)} \quad ...(3.3.8)$$

14. Now $W_R(\omega)$ can be expressed in terms of magnitude and angle as,

$$W_R(\omega) = |W_R(\omega)| \angle W_R(\omega) \quad ...(3.3.9)$$

15. By comparing eq. (3.3.8) and eq. (3.3.9), we have

$$|W_R(\omega)| = \left|\frac{\sin\left(\frac{\omega M}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}\right|$$

Que 3.4. **Using a rectangular window technique design a low pass filter with passband gain of unity, cut-off frequency of 1000 Hz and working at a sampling frequency of 5 kHz. The length of the impulse response should be 7.** **AKTU 2018-19, Marks 07**

Answer

Given : $f_c = 1000$ Hz, $f_s = 5$ kHz, $M = 7$
To Design : FIR digital filter.

1. The desired response of the ideal low-pass filter is given by

$$H_d(e^{j\omega}) = \begin{cases} 1, & 0 \le f \le 1000 \text{ Hz} \\ 0, & f > 1000 \text{ Hz} \end{cases}$$

2. The above response can be equivalently specified in terms of the normalised ω_c.

$$\omega_c = 2\pi f_c / f_s = 2\pi(1000)/(5000) = 1.26 \text{ rad/sec.}$$

Hence, the desired response is

$$H_d(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le 1.26 \\ 0, & 1.26 < |\omega| \le \pi \end{cases}$$

3. The filter coefficients are given by

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-1.26}^{1.26} e^{j\omega n}d\omega$$

$$h_d(n) = \frac{\sin 1.26\, n}{\pi n},\ n \neq 0 \text{ and } h_d(0) = \frac{1.26}{\pi} = 0.40$$

4 Using the rectangular window function and for $M = 7$,

$$h(n) = h_d(n)\ w(n), \qquad n = 0, 1, 2, 3, 4, 5, 6$$

Therefore, $h(0) = 0.40$, $h(4) = -0.0753$

$h(1) = 0.303$, $h(5) = 0.0069$

$h(2) = 0.093$ $h(6) = 0.0067$

$h(3) = -0.063$

Que 3.5. **Explain the following phenomenon :**

i. Gibbs oscillations,

ii. Frequency warping **AKTU 2016-17, Marks 7.5**

OR

Explain the Gibbs phenomenon. Find the response of rectangular window and explain it. **AKTU 2019-20, Marks 07**

Answer

A. Gibbs oscillations :

1. The impulse response of FIR filter in terms of rectangular window is given by, $h(n) = h_d(n)\ w_R(n)$ ...(3.5.1)
2. The frequency response of the filter is obtained by taking Fourier transform of eq. (3.5.1)

$$\therefore \qquad H(\omega) = \text{FT}\{h_d(n)\ w_R(n)\} = H_d(\omega) * W_R(\omega)$$

This shows that the frequency response of FIR filter is equal to the convolution of desired frequency response, $H_d(\omega)$ and the Fourier transform of window function.

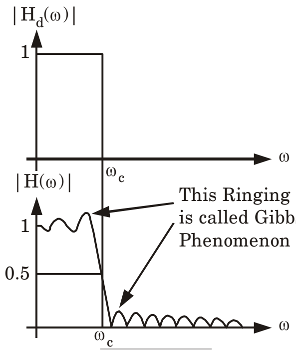


Fig. 3.5.1.

3. The sidelobes are present in the frequency response. Because of these sidelobes, the ringing is observed in the frequency response of FIR filter. This ringing is predominantly present near the band edge of the filter.
4. This oscillatory behaviour (*i.e.*, ringing effect) near the band edge of the filter is called Gibbs phenomenon.

B. Frequency warping : Refer Q. 2.8, Page 2–12C, Unit-2.

C. Response of rectangular window : Refer Q. 3.3., Page 3–5C, Unit-3.

Que 3.6. Design a low-pass filter with the following desired frequency response

$$H_d(e^{j\omega}) = \begin{cases} e^{-j2\omega} & , \quad \dfrac{-\pi}{4} \le \omega \le \dfrac{\pi}{4} \\ 0 & , \quad \dfrac{\pi}{4} < |\omega| < \pi \end{cases}$$

and using window function

$$w(n) = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

AKTU 2016-17, Marks 10

Answer

Given : $w(n) = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$, $H_d(e^{j\omega}) = \begin{cases} e^{-j2\omega}; & -\dfrac{\pi}{4} \le \omega \le \dfrac{\pi}{4} \\ 0\ ; & \dfrac{\pi}{4} < \omega \le \pi \end{cases}$

To Design : Low-pass filter.

1. We know, $$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} e^{-j2\omega}.e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} e^{j\omega(n-2)}d\omega = \frac{1}{\pi(n-2)}\left[\frac{e^{j(n-2)\frac{\pi}{4}} - e^{-j(n-2)\frac{\pi}{4}}}{2j}\right]$$

$$= \frac{1}{\pi(n-2)}\sin\frac{\pi}{4}(n-2), \quad n \ne 2 \qquad ...(3.6.1)$$

2. For $n = 2$, the filter coefficient can be obtained by applying L-Hospital's rule to eq. (3.6.1)

Thus, $h_d(2) = 1/4$

3. The other filter coefficients are given by

$$h_d(0) = \frac{1}{2\pi} = h_d(4) \text{ and } h_d(1) = \frac{1}{\sqrt{2}\pi} = h_d(3)$$

4. The filter coefficients of the filter would be then

$$h(n) = h_d(n)\ w(n)$$

Therefore, $h(0) = \dfrac{1}{2\pi} = h(4)$, $h(1) = \dfrac{1}{\sqrt{2}\pi} = h(3)$ and $h(2) = \dfrac{1}{4}$

5. The frequency response $H(e^{j\omega})$ is given by

$$H(e^{j\omega}) = \sum_{n=0}^{4} h(n)e^{-j\omega n}$$

$$= h(0) + h(1)e^{-j\omega} + h(2)e^{-j2\omega} + h(3)e^{-j3\omega} + h(4)e^{-j4\omega}$$

$$= e^{-j2\omega}[h(0)e^{j2\omega} + h(1)e^{j\omega} + h(2) + h(3)e^{-j\omega} + h(4)e^{-j2\omega}]$$

$$= e^{-j2\omega}\{h(2) + h(0)[e^{j2\omega} + e^{-j2\omega}] + h(1)[e^{j\omega} + e^{-j\omega}]\}$$

$$= e^{-j2\omega}\left\{\frac{1}{4} + \frac{1}{2\pi}[e^{j2\omega} + e^{-j2\omega}] + \frac{1}{\sqrt{2}\pi}[e^{j\omega} + e^{-j\omega}]\right\}$$

6. The frequency response of the designed low pass filter is then,

$$H(e^{j\omega}) = e^{-j2\omega}\left\{\frac{1}{4} + \frac{\sqrt{2}}{\pi}\cos\omega + \frac{1}{\pi}\cos 2\omega\right\}$$

Que 3.7. **A filter is to be designed with the following desired frequency response :**

$$H_d(e^{j\omega}) = \begin{cases} 0 & \dfrac{-\pi}{4} \le \omega \le \dfrac{\pi}{4} \\ e^{-j2\omega} & \dfrac{\pi}{4} \le |\omega| \le \pi \end{cases}$$

AKTU 2015-16, Marks 10

Answer

Given : $H_d(e^{j\omega}) = \begin{cases} 0 & \dfrac{-\pi}{4} \le \omega \le \dfrac{\pi}{4} \\ e^{-j2\omega} & \dfrac{\pi}{4} \le |\omega| \le \pi \end{cases}$

To Design : Filter.

1. Impulse response of the filter is given as :

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\pi/4} e^{-j\omega 2}.e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\pi/4}^{\pi} e^{-j\omega 2}.e^{j\omega n}d\omega$$

$$h_d(n) = \frac{1}{\pi(n-2)}\{\sin \pi(n-2) - \sin (n-2)\pi/4\},\ n \ne 2$$

2. Let us assume the window function, $w(n) = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$

So
$$h_d(0) = \frac{1}{2\pi}, \; h_d(4) = \frac{1}{2\pi}$$
$$h_d(2) = \frac{3}{4}, \; h_d(1) = h_d(3) = \frac{1}{\sqrt{2}\pi}$$
$$h(n) = h_d(n) \times \omega(n)$$

3.
$$h(0) = h(4) = \frac{1}{2\pi} \text{ and } h(2) = \frac{3}{4}$$
$$h(1) = h(3) = \frac{1}{\sqrt{2}\pi}$$

4. The frequency response $H(e^{j\omega})$ is given by
$$H(e^{j\omega}) = \sum_{n=0}^{4} h(n)\, e^{-j\omega n}$$
$$= h(0) + h(1)\, e^{-j\omega} + h(2)\, e^{-j2\omega} + h(3)\, e^{-j3\omega} + h(4)\, e^{-j4\omega}$$
$$\therefore \quad H(e^{j\omega}) = e^{-j2\omega}\left[0.75 - \frac{\sqrt{2}}{\pi}\cos\omega - \frac{1}{11}\cos 2\omega\right]$$

Que 3.8. **The desired response of a low pass filter is $H_d(e^{j\omega}) = e^{-3j\omega}$, $-3\pi/4 \le \omega \le 3\pi/4$ determine $H(e^{j\omega})$ for $M = 7$ using a hamming window.**

AKTU 2018-19, Marks 07

Answer

Given : $H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega} & ; \; \dfrac{-3\pi}{4} \le \omega \le 3\pi/4 \\ 0 & ; \; \dfrac{3\pi}{4} < |\omega| \le \pi \end{cases}$

To Find : $H(e^{j\omega})$ for $M = 7$

1. The filter coefficients are given by
$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-3\pi/4}^{3\pi/4} e^{-j3\omega}e^{j\omega n}d\omega$$
$$h_d(n) = \frac{\sin 3\pi(n-3)/4}{\pi(n-3)}, \; n \ne 3 \text{ and } h_d(3) = \frac{3}{4}$$

The filter coefficients are,
$$h_d(0) = 0.0750, \; h_d(1) = -0.1592, \; h_d(2) = 0.2251,$$
$$h_d(3) = 0.75, \; h_d(4) = 0.2251, \; h_d(5) = -0.1592,$$
$$h_d(6) = 0.0750$$

2. The Hamming window function is,

$$w(n) = \begin{cases} 0.54 - 0.46 \cos \dfrac{2\pi n}{M-1} & ; \quad 0 \le n \le M-1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Therefore, with $M = 7$,

$$w(0) = 0.08, w(1) = 0.31, w(2) = 0.77, w(3) = 1,$$
$$w(4) = 0.77, w(5) = 0.31, w(6) = 0.08.$$

3. The filter coefficients of the resultant filter are then,

$$h\{n) = h_d(n)\, w(n), n = 0, 1, 2, 3, 4, 5, 6.$$

Therefore,
$$h(0) = 0.006$$
$$h(1) = -0.0494$$
$$h(2) = 0.1733$$
$$h(3) = 0.75$$
$$h(4) = 0.1733$$
$$h(5) = -0.0494$$
and
$$h(6) = 0.006$$

4. The frequency response is given by

$$H(e^{j\omega}) = \sum_{n=0}^{6} h(n)\, e^{-j\omega n}$$

$$= e^{-j3\omega}[h(3) + 2h(0)\cos 3\omega + 2h(1) \cos 2\omega + 2h(2) \cos \omega]$$
$$= e^{-j3\omega}[0.75 + 0.012 \cos 3\omega - 0.0988 \cos 2\omega + 0.3466 \cos \omega]$$

Que 3.9. **Design a linear phase FIR (high pass) filter of order seven with cut-off frequency π/4radian/sec using Hanning window.**

AKTU 2017-18, Marks 10

Answer

1. Equation of $h_d(n)$ is

$$h_d(n) = \begin{cases} \dfrac{\sin(n-3)}{\pi(n-3)} & \text{for } n \neq 3 \\ \dfrac{1}{\pi} & \text{for } n = 3 \end{cases} \quad ...(3.9.1)$$

2. Now, Hanning window is defined as

$$W_h(n) = \frac{1}{2}\left[1 - \cos\frac{2\pi\, n}{M-1}\right] \quad \text{[Given, } M = 7]$$

$$\therefore \quad W_h(n) = \frac{1}{2}\left[1 - \cos\frac{2\pi\, n}{6}\right] \quad ...(3.9.2)$$

3. Now the coefficients of FIR filter that means $h(n)$ is obtained by using the equation,

$$h(n) = h_d(n)\,.\,W_h(n) \quad ...(3.9.3)$$

4. Using eq. (3.9.3) we can obtain the values of it as shown in Table. 3.9.1.

Magnitude and phase of $h_d(n)$:

$$H_d(\omega) = \begin{cases} 1\, e^{-j\left(\frac{M-1}{2}\right)} & \text{for } |\omega| >|\omega_c| \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \quad |H_d(\omega)| = \begin{cases} 1 & \text{for } |\omega| >|\omega_c| \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and } \angle H_d(\omega) = -\omega\left(\frac{M-1}{2}\right) = -\omega\left(\frac{7-1}{2}\right) = -3\omega$$

Table 3.9.1.

Value of n	$h_d(n) = \begin{cases} \frac{\sin(n-3)}{\pi(n-3)} & \text{for } n \neq 3 \\ \frac{1}{\pi} & \text{for } n = 3 \end{cases}$	$W_h(n) = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{6}\right)\right]$	$h(n) = h_d(n).W_h(n)$
0	0.015	0	0
1	0.145	0.25	0.03625
2	0.268	0.75	0.201
3	0.318	1	0.318
4	0.268	0.75	0.201
5	0.145	0.25	0.03625
6	0.129	0	0

Que 3.10. **Design an FIR filter to meet the following specification :**

Pass band edge = 2 kHz
Stop band edge = 5 kHz
Stop band attenuation = 42 dB
Sampling frequency = 20 kHz

Use Hanning window. **AKTU 2017-18, Marks 10**

Answer

Given : δ_1 = 2 kHz, δ_2 = 5 kHz, A_s = 42 dB, f_s = 20 kHz
To Design : FIR Filter

1. $$\omega_1 = 2\pi \frac{\delta_1}{f_s} = 2\pi \times \frac{2 \text{ kHz}}{20 \text{ kHz}} = 0.2\pi \text{ rad/sample}$$

$$\omega_2 = 2\pi \frac{\delta_2}{f_s} = 2\pi \times \frac{5 \text{ kHz}}{20 \text{ kHz}} = 0.5\pi \text{ rad/sample}$$

$$\omega_c = \frac{1}{2}[\omega_1 + \omega_2] = \frac{1}{2}[0.2 + 0.5] = 0.35\pi \text{ rad/sample}$$

2. In Hanning window, it provides – 44 dB to stopband attenuation and we have 42 dB of stopband attenuation. So the window type is Hanning window.
3. In Hanning window the width of the main lobe is

$$k\left(\frac{2\pi}{M}\right) = \frac{8\pi}{M}$$

$$k = 4$$

4. The order of the filter is given by equation as

$$N = k\left(\frac{2\pi}{\omega_2 - \omega_1}\right)$$

$$N = 4\left[\frac{2\pi}{(0.5 - 0.2)\pi}\right] = \frac{8}{0.3} = 26.67$$

$$N \approx 27$$

$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{M-1}{2}\right)} & \text{for } \omega_c \le \omega \le \omega_c \\ 0 & \text{elsewhere} \end{cases} \quad ...(3.10.1)$$

$$h_d(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$$

$$h_d(\omega) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\left(\frac{M-1}{2}\right)}\, e^{j\omega n}\, d\omega \quad ...(3.10.2)$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega\left(n - \frac{M-1}{2}\right)}}{j\left(n - \frac{M-1}{2}\right)}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega_c}\left(n - \frac{M-1}{2}\right) - e^{-j\omega_c\left(n - \frac{M-1}{2}\right)}}{j\left(n - \frac{M-1}{2}\right)}\right]$$

$$= \frac{\sin\left[\omega_c\left(n - \frac{M-1}{2}\right)\right]}{\pi\left(n - \frac{M-1}{2}\right)} \text{ for } n \ne \frac{M-1}{2}$$

5. For $n = \dfrac{M-1}{2}$, eq. (3.10.1) becomes,

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{1}{2\pi}[\omega]_{-\omega_c}^{\omega_c} = \frac{\omega_c}{\pi}$$

$$h_d(n) = \begin{cases} \dfrac{\sin\left[\omega_c\left(n - \dfrac{M-1}{2}\right)\right]}{\pi\left(n - \dfrac{M-1}{2}\right)} & \text{for } n \neq \dfrac{M-1}{2} \\[2ex] \dfrac{\omega_c}{\pi} & \text{for } n = \dfrac{M-1}{2} \end{cases}$$

6. For Hanning window

$$W(n) = \frac{1}{2}\left[1 - \cos\frac{2\pi n}{M-1}\right]$$

$h(n)$ is given as

$$h(n) = h_d(n)\ W(n)$$

7. Putting value of $h_d(n)$ and $W(n)$

$$h(n) = \begin{cases} \dfrac{\sin\left[\omega_c\left(n - \dfrac{M-1}{2}\right)\right]}{\pi\left(n - \dfrac{M-1}{2}\right)} \cdot \dfrac{1}{2}\left[1 - \dfrac{\cos 2\pi n}{M-1}\right] & \text{for } n \neq \dfrac{M-1}{2} \\[2ex] \dfrac{\omega_c}{\pi} \cdot \dfrac{1}{2}\left[1 - \dfrac{\cos 2\pi n}{M-1}\right] & \text{for } n = \dfrac{M-1}{2} \end{cases}$$

Que 3.11. **Write a short note on the following :**

i. Bartlett window.
ii. Blackman window.

Answer

i. Bartlett window :

1. It is also called as triangular window. It is expressed mathematically as :

$$w_T(n) = \begin{cases} 1 - \dfrac{2\left|n - \dfrac{M-1}{2}\right|}{M-0} & \text{for } n = 0, 1, ... M-1 \\ 0 & \end{cases} \quad ...(3.11.1)$$

2. Fig. 3.11.1(a), shows the sketch of this window. Fig. 3.11.1(b), shows the magnitude response of this window.

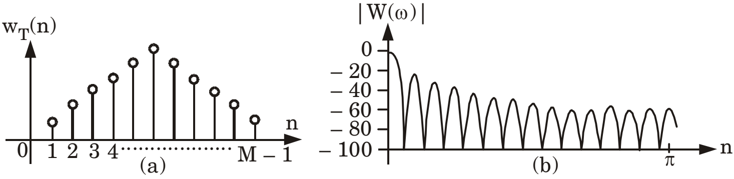


Fig. 3.11.1. Bartlett window (a) Time domain graph (b) Magnitude response.

ii. Blackman window :

1. It is expressed mathematically as :

$$w_B(n) = \begin{cases} 0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.8\cos\dfrac{4\pi n}{M-1} & \text{for } n = 0, 1, \ldots.\ M-1 \\ 0 & \text{Otherwise} \end{cases}$$

2. Fig. 3.11.2(a), shows the graph of Blackman window. Fig. 3.11.2(b) shows its magnitude response. In this, the width of the main lobe is increased. However, it has very small sidelobes.

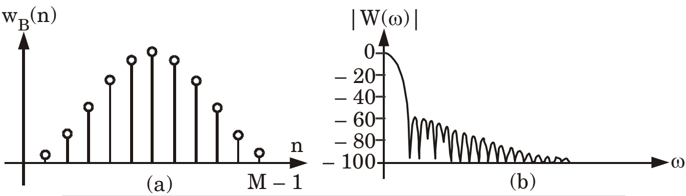


Fig. 3.11.2. Blackman window (a) Time domain graph (b) Magnitude response.

Que 3.12. **Explain the Kaiser window.**

Answer

1. All the window functions have variable width of main lobe. It is inversely proportional to the length of the filter.

$$i.e., \text{ width of main lobe } (\theta) \propto \frac{1}{\text{length of filter } (N)}$$

If $N \rightarrow$ increased

$\theta \rightarrow$ will be narrowed

It means at higher value of N the transition band is reduced considerably.

2. The attenuation in the sidelobe should be independent of the length and type of the window. Therefore to achieve desired stop band attenuation, the selection of proper window function is required.

3. A desired window function should have finite duration in the time domain and its fourier transform has maximum energy in the main lobe or a given peak sidelobe amplitude.
4. Kaiser has developed a simple approximation to these functions in terms of 0^{th} order modified bessel functions of the first kind.
5. Here the sidelobe energy level is controlled with respect to the main lobe peak by just varying a parameter 'α'. By adjusting the length of the filter, the width of the main lobe can be varied.
6. The Kaiser window is defined as,

$$w_k(n) = \begin{cases} \dfrac{I_0(\beta)}{I_0(\alpha)} & \text{for } |n| \leq \dfrac{M-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

where α is an independent variable determined by Kaiser.
7. The parameter β is expressed by

$$\beta = \alpha \left[1 - \left(\frac{2n}{M-1} \right)^2 \right]^{0.5}$$

8. The modified bessel function of the first kind is given as,

$$I_o(x) = 1 + \sum_{k=1}^{\infty} \left[\frac{1}{k!} \left(\frac{x}{2} \right)^k \right]^2$$

$$= 1 + \frac{0.25x^2}{(1!)^2} + \frac{(0.25x^2)^2}{(2!)^2} + \frac{(0.25x^2)^3}{(3!)^2} + \ldots$$

Que 3.13. What is a Kaiser window ? In what way it is superior to another window function ? Explain the procedure for designing a FIR filter using Kaiser window.

Answer

A. Kaiser window : Refer Q. 3.12, Page 3–16C, Unit-3.

B. Reason :
1. The first and last samples of Kaiser window are not zero.
2. The shape of the window depends upon values of '*M*' and 'β'.
3. Fig. 3.13.1(b) shows the magnitude response of Kaiser window.

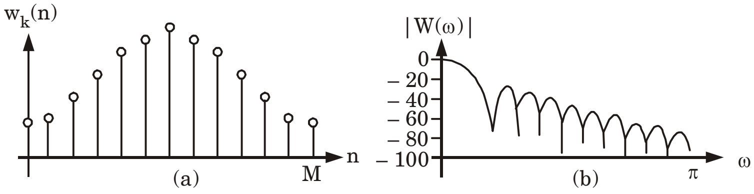


Fig. 3.13.1. Kaiser window
(a) Time domain graph (b) Magnitude response.

4. The shapes of main lobe and sidelobes can be adjusted by selection of 'M' and 'β'. Therefore, Kaiser window is most commonly used window in digital FIR filtering.

C. Design of FIR filter using Kaiser window function :

i. Design specifications :

1. Deciding of filter type (*i.e.*, low-pass filter, high-pass filter, band-pass and band-reject filter).
2. Frequencies of passband and stopband in hertz.
3. Minimum stop band attenuation (A_s') and pass band ripple (A_p') in positive decibels.
4. Sampling frequency (f_s) in Hertz.
5. Order of filter (M) taken as odd.

ii. Design procedure :

1. The actual pass band ripple (peak-to-peak) is given as,

$$A_p = 20 \log_{10} \frac{1+\delta_p}{1-\delta_p} \text{ dB} \quad ...(3.13.1)$$

2. The minimum stop band attenuation is given as,

$$A_s = -20 \log_{10} (\delta_s) \text{ dB} \quad ...(3.13.2)$$

3. The transition bandwidth is given as,

$$\Delta f = f_s - f_p \quad ...(3.13.3)$$

4. Suppose the specified pass band ripple and minimum stop band attenuation are given by A_p' and A_s' respectively, they are related with A_p and A_s are as follows :

$$A_p \le A_p' \quad ...(3.13.4)$$

$$A_s \le A_s' \quad ...(3.13.5)$$

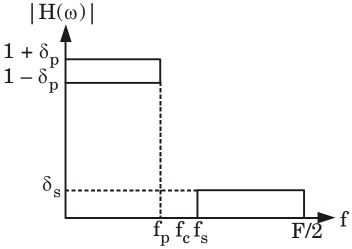


Fig. 3.13.2 Frequency response of LPF.

Step 1 : From eq. (3.13.1) and (3.13.2) we get

$$\delta_s = 10^{-0.05A'_s}$$

and

$$\delta_p = \frac{10^{0.05A_p'} - 1}{10^{0.05A_p'} + 1} \quad ...(3.13.6)$$

Determine δ according to eq. (3.13.1), eq. (3.13.2), eq. (3.13.4) and eq. (3.13.5) the actual parameter can be determined from :

$$\delta = \min (\delta_p, \delta_s) \quad ...(3.13.7)$$

Step 2 : Calculate A_s from eq. (3.13.2).

Step 3 : From Kaiser's design equation determine the parameter 'α' as follows

$$\alpha = \begin{cases} 0 & ; \text{ for } A_s \leq 21 \\ 0.5842(A_s - 21)^{0.4} + 0.07886(A_s - 21) & ; \text{ for } 21 < A_s \leq 50 \\ 0.1102(A_s - 8.7) & ; \text{ for } A_s > 50 \end{cases} \quad ...(3.13.8)$$

Step 4 : From Kaiser design equation calculate parameter D as follows

$$D = \begin{cases} 0.9222 & ; \text{ for } A_s \leq 21 \\ \dfrac{A_s - 7.95}{14.36} & ; \text{ for } A_s > 21 \end{cases} \quad ...(3.13.9)$$

Step 5 : Calculate the order of filter for the lowest odd value of M,

$$M \geq \frac{FD}{\Delta F} + 1 \quad ...(3.13.10)$$

Step 6 : Now using window technique the modified impulse response is computed as,

$$h(n) = w_k(n)\, h_d(n), \text{ for } |n| \leq \frac{M-1}{2} \quad ...(3.13.11)$$

Step 7 : Compute the transfer function of filter as follows,

$$H(z) = z^{-(M-1)/2}\left[h(0) + 2\sum_{n=1}^{(M-1)/2} h(n)\,(z^n + z^{-n})\right] \quad ...(3.13.12)$$

where, $h(0) = w_k(0)\,.\,h_d(0)$

$$h(n) = w_k(n)\,.\,h_d(n) \quad ...(3.13.13)$$

The magnitude response can be obtained from eq. (3.13.4) as given below,

$$M(\omega) = h(0) + 2\sum_{n=1}^{(M-1)/2} h(n)\cos(2\pi f n T) \quad ...(3.13.14)$$

Que 3.14. Design a low pass discrete time filter with following specification :

$0.99 \leq |H(e^{j\omega})| \leq 1.01$ $|\omega| \leq 0.4\pi$

$|H(e^{j\omega})| \leq 0.01$ $0.6\pi \leq |\omega| \leq 0.4\pi$

Use Kaiser Window for design.

AKTU 2019-20, Marks 07

Answer

Given : $0.99 \leq |H(e^{j\omega})| \leq 1.01 \,;\, 0 \leq |\omega| \leq 0.4\pi$

$|H(e^{j\omega})| \leq 0.01 \,;\, 0.6\pi \leq |\omega| \leq 0.4\pi$

To Design : FIR linear phase filter.

Step 1 : On comparing with the given specification as given below

$$1 - \delta_p \leq |H(e^{j\omega})| \leq 1 + \delta_p \,;\, \text{for } 0 \leq \omega \leq \omega_p$$

$$|H(e^{j\omega})| \leq \delta_s \,;\, \text{for } \omega_s \leq \omega \leq \pi$$

On comparison, we get

$$\delta_p = 0.01 \text{ and } \delta_s = 0.01$$

$$\omega_p = 0.4\pi \text{ and } \omega_s = 0.6\pi$$

Therefore the transition width '$\Delta\omega$' can be calculated as

$$\Delta\omega = \omega_s - \omega_p = 0.6\,\pi - 0.4\,\pi = 0.2\,\pi$$

The minimum value of ripple is given as

$$\delta = \min(\delta_p, \delta_s) = 0.01$$

The minimum stop band attenuation can be calculated as

$$A_s = -20\log_{10}\delta = -20\log_{10}(0.01) = 40$$

Step 2 : Calculation of cut-off frequency (ω_c)

It is calculated as

$$\omega_c = \frac{\omega_p + \omega_s}{2} = \frac{0.6\,\pi + 0.4\,\pi}{2} = 0.5\,\pi$$

Step 3 : Calculation of α and M.

Here $A_s = 40$, which lies in the range of 21 to 50.

Hence α can be obtained as

$$\begin{aligned}\alpha &= 0.5842\,(A_s - 21)^{0.4} + 0.07886\,(A_s - 21)\\ &= 0.5842\,(40-21)^{0.4} + 0.07886\,(40-21)\\ &= 3.395\end{aligned}$$

and M is given as

$$M = \frac{A_s - 8}{2.285(0.02\pi)} = \frac{40-8}{2.285(0.02\pi)} = 222.88$$

i.e., $M = 223$

Taking value of M to next integer value

Step 4 : Obtain equation of Kaiser window by taking

$$\beta = \frac{M}{2} = \frac{223}{2} = 111.5$$

Therefore window function will be

$$w_{\text{kaiser}}(n) = \begin{cases} \dfrac{I_o\left[\alpha\left\{1-\left(\dfrac{n-\beta}{\beta}\right)^2\right\}^{1/2}\right]}{I_o(\alpha)} & ;\text{ for } 0 \le n \le 223 \\ 0 & ;\text{ otherwise} \end{cases}$$

$$w_{\text{kaiser}}(n) = \begin{cases} \dfrac{I_o\left[3.395\left\{1-\left(\dfrac{n-111.5}{111.5}\right)^2\right\}^{1/2}\right]}{I_o(3.395)} & ;\text{ for } 0 \le n \le 223 \\ 0 & ;\text{ otherwise} \end{cases}$$

Step 5 : To obtain $h_d(n)$

Since this is a LPF. The ideal desired frequency response is given by equation,

$$H_d(\omega) = \begin{cases} e^{-j\omega(M-1)/2} & ; \quad \text{for } -\omega_c \le \omega \le \omega_c \\ 0 & ; \quad \text{otherwise} \end{cases} \qquad \text{...(3.14.1)}$$

$h_d(n)$ can be calculated by just taking inverse fourier transform of eq. (3.14.1).

$$h_d(n) = \begin{cases} \dfrac{\sin\left[\omega_c\left(n - \dfrac{M-1}{2}\right)\right]}{\pi\left(n - \dfrac{M-1}{2}\right)} & ; \quad \text{for } n \ne \dfrac{M-1}{2} \\ \dfrac{\omega_c}{\pi} & ; \quad \text{for } n = \dfrac{M-1}{2} \end{cases} \qquad \text{...(3.14.2)}$$

Since in the beginning the equation is derived by taking filter order $(M+1)$. The eq. (3.14.2) is derived for filter length M. Hence by replacing $M + 1$ in place of M in eq. (3.14.2) we get

$$h_d(n) = \begin{cases} \dfrac{\sin\left[\omega_c\left(n - \dfrac{M}{2}\right)\right]}{\pi\left(n - \dfrac{M}{2}\right)} & ; \text{for } n \ne \dfrac{M}{2} \\ \dfrac{\omega_c}{\pi} & ; \text{for } n = \dfrac{M}{2} \end{cases} \qquad \text{...(3.14.3)}$$

Putting here,

$M = 223$, hence $M/2 = 111.5$ *i.e.*, not an integer, so $h_d(n)$ cannot be calculated to this value. If filter length is taken as even *i.e.*, $M + 1 = 224$, the $h_d(n)$ will be given as

$$h_d(n) = \frac{\sin\left[\omega_c\left(n - \dfrac{M}{2}\right)\right]}{\pi\left(n - \dfrac{M}{2}\right)}; \text{ for } 0 \le n \le M \qquad \text{...(3.14.4)}$$

Putting for $\omega_c = 0.5\,\pi$ and $M = 223$, the $h_d(n)$ will become

$$h_d(n) = \frac{\sin\left[0.5\,\pi\left(n - \dfrac{223}{2}\right)\right]}{\pi\left(n - \dfrac{223}{2}\right)}$$

$$h_d(n) = \frac{\sin[0.5\,\pi\,(n - 111.5)]}{\pi(n - 111.5)}; \text{ for } 0 \le n \le 223 \qquad \text{...(3.14.5)}$$

Eq. (3.14.5) also holds good for linear phase requirement and gives desired unit sample response of LPF.

Step 6 : To obtain $h(n)$

The unit sample response of FIR filter is obtained by

$$h(n) = h_d(n)\, w_{\text{kaiser}}(n)$$

$$= \begin{cases} \dfrac{\sin\left[0.5\,\pi(n-111.5)\right]}{\pi(n-111.5)} \dfrac{I_o\left[3.395\left\{1-\left(\dfrac{n-111.5}{111.5}\right)^2\right\}^{1/2}\right]}{I_o(3.395)} & ; \text{for } 0 \le n \le 223 \\ 0 & ; \text{otherwise} \end{cases}$$

PART-2

Finite Word Length Effects in Digital Filters : Coefficient Quantization Error, Quantization Noise-Truncation and Rounding, Limit Cycle Oscillations-Deadband Effect.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.15. Discuss the Finite Word length effects in digital filters.

AKTU 2018-19, Marks 07

Answer

1. The fundamental operations in digital filters are multiplication and addition. When these operations are performed in a digital system, the input data as well as the product and sum (output data) have to be represented in finite word length, which depend on the size (length) of the resistor used to store the data.
2. In digital computation the input and output data (sum and product) are quantized by rounding or truncation to convert them into finite word size. This creates error (in noise) in the output or creates oscillations (limit cycle) in the output.
3. These effects due to finite precision representative of number in digital system are called as finite word length effects.
4. List some of the finite word length effects in digital filters :

i. Errors due to quantization of the input data.

ii. Errors due to quantization of the filter coefficients.

iii. Errors due to rounding the product in multiplications.

iv. Limit cycle due to product quantization and overflow in addition.

Que 3.16. Discuss the effect of coefficient quantization.

Answer

A. Effects of coefficient quantization in IIR system :

1. When the parameters of a rational system function or corresponding difference equation are quantized, the poles and zeros of the system function move to new position in the z-plane. Also, the frequency response is perturbed from its original value.
2. If the system implementation structure is highly sensitive to perturbations of the coefficients, the resulting system may no longer meet the original design specification or system might even become unstable.
3. For example, system function representation corresponding to both direct forms is the ratio of polynomials

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

4. Coefficient $\{a_k\}$ and $\{b_k\}$ are the ideal infinite-precision coefficients in both direct-form implementation structures. If we quantize these coefficients, we obtain the system function

$$\hat{H}(z) = \frac{\sum_{k=0}^{M} \hat{b}_k z^{-k}}{1 - \sum_{k=1}^{N} \hat{a}_k z^{-k}}$$

where $\hat{a}_k = a_k + \Delta a_k$ and $\hat{b}_k = b_k + \Delta b_k$ are the quantized coefficients that differ from the original coefficients by the quantization errors Δa_k and Δb_k.

5. Thus the roots of the denominator and numerator polynomials (the poles and zeros of $H(z)$) are affected by the error in the coefficients. Thus, each pole and zero will be affected by quantization errors in the denominator and numerator polynomials, respectively.

B. Effects of coefficient quantization in FIR systems :

1. System function for a direct-form FIR system

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n}$$

2. Suppose that the coefficients $\{h[n]\}$ are quantized resulting in a new set of coefficients $\{h[n] = h[n] + \Delta h[n]\}$. The system function for the quantized system is

$$\hat{H}(z) = \sum_{n=0}^{M} \hat{h}[n] z^{-n} = H(z) + \Delta H(z)$$

$$\Delta H(z) = \sum_{n=0}^{M} \Delta h[n] z^{-n}$$

3. Thus, the system function of the quantized system is linearly related to the quantization errors in the impulse-response coefficients.
4. Quantized system can be represented as in Fig. 3.16.1, which shows the unquantized system in parallel with an error system whose impulse response is the sequence of quantization error samples $\{\Delta h[n]\}$ and whose system function is the corresponding z-transform, $\Delta H(z)$.

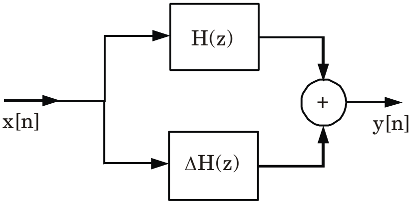


Fig. 3.16.1. Representation of coefficient quantization in FIR systems.

Que 3.17. What is the effect of rounding and truncation in fixed-point representation ?

Answer

1. The effect of rounding and truncation is to introduce an error whose value depends on the number of bits in the original number relative to the number of bits after quantization.
2. The characteristics of the errors introduced through either truncation or rounding depend on the particular form of number representation.

A. Effect of truncation :

1. Let us consider a fixed-point representation in which a number x is quantized from b_u bits to b bits. Thus the number

$$x = \overbrace{0.1011\ldots01}^{b_u}$$

consisting of b_u bits prior to quantization is represented as

$$x = \overbrace{0.101\ldots1}^{b}$$

after quantization, where $b < b_u$.

2. If x represents the sample of an analog signal, then b_u may be taken as infinite. In any case if the quantizer truncates the value of x, the truncation error is defined as

$$E_t = Q_t(x) - x$$

3. First, we consider the range of values of the error of sign-magnitude and two's complement representation. In both of these representations, the positive numbers have identical representations.
4. For positive numbers, truncation results in a number that is smaller than the unquantized number. Consequently, the truncation error resulting from a reduction of the number of significant bits from b_u to b is

$$-(2^{-b} - 2^{-b_u}) \leq E_t \leq 0$$

where the largest number error arises from discarding $b_u - b$ bits, all of which are ones.

5. In the case of negative fixed-point number based on the sign-magnitude representation, the truncation error is positive, since truncation basically reduces the magnitude of the numbers. Consequently, for negative number, we have

$$0 \leq E_t \leq (2^{-b} - 2^{-b_u})$$

6. In the two's complement representation, the negative of a number is obtained by subtracting the corresponding positive number from 2. As a consequence, the effect of truncation on a negative number is to increase the magnitude of the negative number. Consequently, $x > Q_t(x)$ and hence

$$-(2^{-b} - 2^{-b_u}) \leq E_t \leq 0$$

7. Hence the truncation error for the sign-magnitude representation is symmetric about zero and falls in the range

$$-(2^{-b} - 2^{-b_u}) \leq E_t \leq (2^{-b} - 2^{-b_u})$$

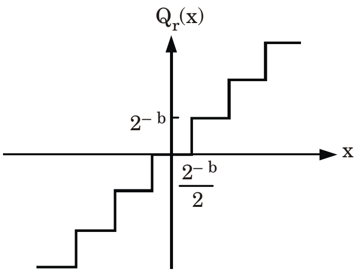
B. Effect of rounding :

1. A number x, represented by b_u bits before quantization and b bits after quantization, incurs a quantization error

$$E_r = Q_r(x) - x$$

2. Rounding involves only the magnitude of the number and, consequently, the round-off error is independent of the type of fixed-point representation.
3. The maximum error that can be introduced through rounding is $(2^{-b} - 2^{-b_u})/2$ and this can be either positive or negative, depending on the value of x.
4. Therefore, the round-off error is symmetric about zero and falls in the range

$$-\frac{1}{2}(2^{-b} - 2^{-b_u}) \leq E_r \leq -\frac{1}{2}(2^{-b} - 2^{-b_u})$$



$E_r = Q_r(x) - x$

$-\frac{1}{2} \cdot 2^{-b} \leq E_r \leq -\frac{1}{2}\, 2^{-b}$

(a)

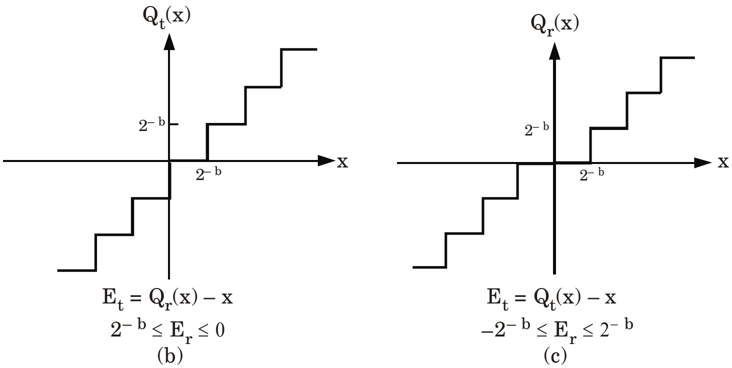


Fig. 3.17.1. Quantization errors in rounding and truncation :(a) rounding (b) truncation in two's complement ; (c) truncation is sign-magnitude.

Que 3.18. **Write a note on round-off noise in Digital filter.**

Answer

1. The output of a multiplier has more bits than its inputs. To store the output it has to be (re)quantized. An error called quantization noise or round off noise is added at that point.
2. It is often assumed that the quantization noise at each multiplier output is white (independent from sample to sample).
3. It is also assumed that it is independent between multipliers, so that the noise variances add.
4. The assumption of whiteness is actually a very poor model if the signal is narrowband, but it is reasonable for large amplitude wideband signals. The assumption of independence can also be a poor model.
5. The quantization noise from the multipliers of an FIR filter therefore adds white noise directly to the output signal.
6. In IIR filters, the white quantization noise from the feedback multipliers filter is fed to the input of the filter, so the resulting noise spectrum at the filter output is coloured.
7. Its spectrum is proportional to the square of the filter's frequency response magnitude.
8. Round-off noise level is affected by data wordlengths, filter response, filter structure and (to an extent) by section ordering in cascade structures.

Que 3.19. **Write short note on limit cycle oscillations.**

Answer

1. There are two types of limit cycles, zero input limit cycle and overflow limit cycle. Zero input limit cycles are usually of lower amplitudes in comparison with overflow limit cycles.

2. Let us consider a system with the difference equation

$$y(n) = 0.8\, y(n-1) + x(n) \quad (3.19.1)$$

with zero input, *i.e.*, $x(n) = 0$ and initial condition $y(-1) = 10$.

3. A comparison between the exact values of $y(n)$ as given by eq. 3.19.1, using unquantized arithmetic and the rounded value of $y(n)$ as obtained arithmetic.

4. It can be observed that for zero input the unquantized output $y(n)$ decays exponentially to zero with increasing.

5. However, the rounded-off (quantized) output $y(n)$ get sucks at a value of two and never decays further.

6. Thus, the output is finite even when no input is applied. This is referred to as zero input limit cycle effect.

7. It can also be seen that for any value of the input condition $|y(-1)| \leq 2$, the output $y(n) = y(-1)$, $n \geq 0$, when the input is zero. Thus, the deadband in this case is the interval $[-2, 2]$.

8. The effects of limit cycles in first-order and second-order systems were studied by Jackson using an "effective value" model. It was realised that limit cycles can occur only if the result of rounding leads to poles on the unit circle.

9. Consider the first-order difference equation $y(n) = x(n) - [ay(n-1)]^*$ where $[.]^*$ denotes rounding to the nearest integer with $x(n) = 0$, $n \geq 0$. The deadband in which limit cycles can exist is the range $[-l, l]$, where l is the largest integer satisfying,

$$l \leq \frac{0.5}{1-|\alpha|}$$

10. If a is negative, the limit cycle will have constant magnitude and sign. If a is positive, the limit cycle will have constant magnitude by alternating sign.

11. Consider the second-order system with the difference equation

$$Y(n) = x(n) — [b_1 y(n — 1)]^* - [b_2 y(n-2)]^*$$

12. The deadband for this system is the region $[-l, l]$ where l is the largest integer satisfying

$$l \leq \frac{0.5}{1-|b_2|} \; (0 < b_2 < 1)$$

VERY IMPORTANT QUESTIONS

Following questions are very important. These questions may be asked in your SESSIONALS as well as UNIVERSITY EXAMINATION.

Q. 1. What are the different window functions used for windowing ? Explain the effects of using different window functions for designing FIR filter on the filter response.

Ans. Refer Q. 3.2.

Q. 2. Using a rectangular window technique design a low pass filter with passband gain of unity, cut-off frequency of 1000 Hz and working at a sampling frequency of 5 kHz. The length of the impulse response should be 7.

Ans. Refer Q. 3.4.

Q. 3. Explain the following phenomenon :
i. Gibbs oscillations,
ii. Frequency warping

Ans. Refer Q. 3.5.

Q. 4. A filter is to be designed with the following desired frequency response :

$$H_d(e^{j\omega}) = \begin{cases} 0 & \dfrac{-\pi}{4} \leq \omega \leq \dfrac{\pi}{4} \\ e^{-j2\omega} & \dfrac{\pi}{4} \leq |\omega| \leq \pi \end{cases}$$

Ans. Refer Q. 3.7.

Q. 5. Design a linear phase FIR (high pass) filter of order seven with cut-off frequency $\dfrac{\pi}{4}$ radian/sec using Hanning window.

Ans. Refer Q. 3.9.

Q. 6. Design an FIR filter to meet the following specification :
Pass band edge = 2 kHz
Stop band edge = 5 kHz
Stop band attenuation = 42 dB
Sampling frequency = 20 kHz
Use Hanning window.

Ans. Refer Q. 3.10.

Q. 7. Design a low pass discrete time filter with following specification :

$$0.99 \leq |H(e^{j\omega})| \leq 1.01 \qquad |\omega| \leq 0.4\,\pi$$
$$|H(e^{j\omega})| \leq 0.01 \qquad 0.6\,\pi \leq |\omega| \leq 0.4\,\pi$$

Use Kaiser Window for design.

Ans. Refer Q. 3.14.

Q. 8. Discuss the Finite Word length effects in digital filters.

Ans. Refer Q. 3.15.

☺☺☺

4 UNIT

DFT & FFT

CONTENTS

PART-1

DFT : Definitions, Properties of the DFT, Circular Convolution, Linear Convolution Using Circular Convolution.

CONCEPT OUTLINE :

- **Circular convolution :** If $x_1(n) \xleftrightarrow{\text{DFT}} X_1(k)$ and $x_2(n) \xleftrightarrow{\text{DFT}} X_2(k)$ both sequences having same length 'N', then, circular convolution, will be given by $x_3(n)$,

$$x_3(n) = x_1(n) \; \textcircled{N} \; x_2(n)$$

Let $\quad x_3(n) \xleftrightarrow{\text{DFT}} X_3(k)$

Then, $\quad X_3(k) = X_1(k)\, X_2(k)$

For circular convolution all the sequences should be periodic.

- **Linear convolution :** Let $x(n)$ be the finite duration sequence of length N_1 which is given as input to FIR system with impulse response $h(n)$ of length N_2, then output is given by,

$$y(n) = x(n) * h(n)$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.1. **Write a short note on DFT.**

Answer

1. The discrete fourier transform computes the values of the z-transform for evenly spaced points around the unit circle for a given sequence.
2. If the sequence to be represented is of finite duration *i.e.*, has only a finite number of non-zero values, the transformation used is DFT.
3. Let $x(n)$ be a finite duration sequence. The N-point DFT of the sequence $x(n)$ is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}, \qquad k = 0, 1, \ldots\ldots N-1$$

and the corresponding IDFT is,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi nk/N}, \; n = 0, 1, \ldots\ldots N-1$$

Que 4.2. **What is the relation between DTFT and DFT. Explain the properties of DFT with examples.** **AKTU 2019-20, Marks 07**

Answer

A. Relation between DTFT and DFT :

Since DFT is defined for finite length causal sequences, let us consider a $x(n)$ which is causal and is of length N.

DTFT : $$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(nT)e^{-j\omega nT} \quad ...(4.2.1)$$

DFT : $$X(k) = \sum_{n=0}^{N-1} x(nT)e^{-j\frac{2k}{N}nk} \; ; k = 0,1,2, ...,(N-1) \quad ...(4.2.2)$$

From the eq. (4.2.1) and (4.2.2), it is clear that

$$X(k) = X(e^{j\omega T})\Big|_{\omega \to \frac{2\pi}{NT}\cdot k = \left(\frac{\omega_1}{N}\right)k} \; ; k = 0, 1, 2, ..., (N-1)$$

B. Properties of DFT :

1. **Periodicity :** Let $x(n) \xleftrightarrow{\text{DFT}} X(k)$

   Then if $x(n + N) = x(n)$ for all n

   Then $X(k + N) = X(k)$ for all k

2. **Linearity :**

   If $x_1(n) \xleftrightarrow[N]{\text{DFT}} X_1(k)$

   and $x_2(n) \xleftrightarrow[N]{\text{DFT}} X_2(k)$

   then this property gives

   $$a_1x_1(n) + a_2x_2(n) \xleftrightarrow[N]{\text{DFT}} a_1X_1(k) + a_2X_2(k) \quad ...(4.2.3)$$

3. **Circular symmetries of a sequence :** If periodic sequence $x_p(n)$ is given as,

   $$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) \quad ...(4.2.4)$$

   The DFT is given as

   $$x(n) \xleftrightarrow{\text{DFT}} X(k)$$

   $$x_p(n) \xleftrightarrow{\text{DFT}} X_p(k)$$

   Thus $x(n)$ and $x_p(n)$ are related by eq. (4.2.4). The relation between $x(n)$ and $x_p(n)$ is given as,

   $$x(n) = \begin{cases} x_p(n), & \text{for } 0 \le n \le (N-1) \\ 0 \quad , & \text{otherwise} \end{cases}$$

   Suppose $x_p(n)$ is shifted by 'k' units to the right. Then new sequence $x'_p(n)$ be given as,

   $$x'_p(n) = x_p(n - k)$$

$$= \sum_{l=-\infty}^{\infty} x(n-k-lN)$$

The $x'_p(n)$ will give $x(n)$ and it is given as,

$$x'(n) = \begin{cases} x'_p(n), & \text{for } 0 \le n \le (N-1) \\ 0 \quad , & \text{otherwise} \end{cases}$$

or simply $$x'(n) = \sum_{l=-\infty}^{\infty} x(n-k-lN)$$

The $x'(n)$ is related to $x(n)$ by a circular shift and it is given as,

$$x'(n) = x(n-k, \text{ modulo } N)$$

The short hand notation for $(n-k$, modulo $N)$ is $((n-k))_N$

or $$x'(n) = x\,((n-k))_N$$

4. **Time reversal of a sequence :**

If $$x(n) \xleftrightarrow{\text{DFT}} X(k)$$

Then this property states that

$$x((-n))_N = x(N-n) \xleftrightarrow{\text{DFT}} X((-k))_N$$

i.e., $$x((-k))_N \xleftrightarrow{\text{DFT}} X(N-k)$$

5. **Complex conjugate property :**

If $$x(n) \xleftrightarrow{\text{DFT}} X(k)$$

Then this property states that

$$x^*(n) \leftrightarrow X^*(N-k) = X^*((-k))_N$$

6. **Circular convolution :**

If $$x_1(n) \xleftrightarrow{\text{DFT}} X_1(k)$$

and $$x_2(n) \xleftrightarrow{\text{DFT}} X_2(k)$$

This property states that the circular convolution is given as,

$x_1(-n)$ Ⓝ $x_2(n) = X_1(k)X_2(k)$ where Ⓝ denote circular convolution

7. **Shifting property :**

If $$x(n) \xleftrightarrow{\text{DFT}} X(k)$$

Two shifting properties are given here,

a. **Circular shifting in time :**

This property states that

$$x((n-l))_N \xleftrightarrow{\text{DFT}} X(k)\, e^{-j2\pi k\ell/N}$$

b. **Circular shifting in frequency :**

The property states that

$$x(n)\, e^{j2\pi\ell n/N} \xleftrightarrow{\text{DFT}} X((k-l))_N$$

8. **Circular correlation :**

If $$x_1(n) \xleftrightarrow{\text{DFT}} X_1(k)$$

and $$x_2(n) \xleftrightarrow{\text{DFT}} X_2(k)$$

Then this property states that

$$r_{x_1x_2}(l) \xleftrightarrow{\text{DFT}} \frac{1}{N} R_{x_1x_2}(k) = X_1(k)\, X_2^*(k)$$

where $r_{x_1x_2}(l)$ is the circular cross correlation and it is given as

$$r_{x_1x_2}(l) = \sum_{n=0}^{N-1} x_1(n)x_2^*((n-l))_N$$

9. Multiplication of two sequences :

If $\quad x_1(n) \xleftrightarrow{\text{DFT}} X_1(k)$

and $\quad x_2(n) \xleftrightarrow{\text{DFT}} X_2(k)$

then, $\quad x_1(n) \cdot x_2(n) \xleftrightarrow{\text{DFT}} \dfrac{1}{N} X_1(k) \; N \; X_2(k)$

10. Parseval's theorem : If $x_1(n)$ and $x_2(n)$ are two complex valued sequences and their DFT are given as

$$x_1(n) \xleftrightarrow{\text{DFT}} X_1(k)$$

$$x_2(n) \xleftrightarrow{\text{DFT}} X_2(k)$$

Then $$\sum_{n=0}^{N-1} x_1(n)x_2^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\cdot X_2^*(k) \qquad ...(4.2.5)$$

When $\quad x_1(n) = x_2(n) = x(n)$

Then eq. (4.2.5) will become

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2 \qquad ...(4.2.6)$$

Eq. (4.2.5) and (4.2.6) are Parseval's theorem. Eq. (4.2.6) gives energy of finite duration sequence in terms of its frequency components.

Que 4.3. **Calculate the DFT of $x(n) = \cos an$.**

AKTU 2016-17, Marks 10

Answer

Given : $x(n) = \cos an$

To Find : DFT.

1. $$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N}, \; k = 0, 1, ..., N-1$$

$$= \sum_{n=0}^{N-1} (\cos an)\, e^{-j2\pi nk/N} = \sum_{n=0}^{N-1}\left(\frac{e^{jan} + e^{-jan}}{2}\right) e^{-j2\pi nk/N}$$

$$= \frac{1}{2}\left[\sum_{n=0}^{N-1}\left(e^{j(a-2\pi k/N)n}\right) + \sum_{n=0}^{N-1}\left(e^{-j(a+2\pi k/N)n}\right)\right]$$

2. Putting $n = 0, 1, ..., N-1$

$$= \frac{1}{2}\left[\frac{1-(e^{j(a-2\pi k/N)})^N}{1-e^{j(a-2\pi k/N)}} + \frac{1-(e^{-j(a+2\pi k/N)})^N}{1-e^{-j(a+2\pi k/N)}}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{jaN}}{1-e^{j(a-2\pi k/N)}} + \frac{1-e^{-jaN}}{1-e^{-j(a+2\pi k/N)}}\right]$$

Que 4.4. **Compute the DFT coefficients of a finite duration sequence (0, 1, 2, 3, 0, 0, 0, 0).**

Answer

Given : $x(n) = \{0, 1, 2, 3, 0, 0, 0, 0\}$
To Find : DFT.

1. $$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}}$$
2. $$X(k) = \sum_{n=0}^{7} x(n)\, e^{\frac{-j2\pi nk}{8}}$$
3. For $k = 0$,
$$X(0) = \sum_{n=0}^{7} x(n) = 0 + 1 + 2 + 3 + 0 + 0 + 0 + 0 = 6$$
4. For $k = 1$,
$$X(1) = \sum_{n=0}^{7} x(n)\, e^{\frac{-j\pi n}{4}}$$
$$= x(1)e^{-j\pi/4} + x(2)\, e^{-j\pi/2} + x(3)\, e^{-j3\pi/4}$$
$$= 1\left(\cos\frac{\pi}{4} - j\sin\frac{\pi}{4}\right) + 2(-j) + 3\left(\cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}\right)$$
$$= 0.707 - j\,0.707 - 2j + 3\,(-\,0.707 - j\,0.707)$$
$$= -\,1.414 - j\,4.828$$
5. For $k = 2$,
$$X(2) = \sum_{n=0}^{7} x(n)\, e^{\frac{-j\pi n}{2}} = x(1)\, e^{-j\pi/2} + x(2)\, e^{-j\pi} + x(3)\, e^{-j3\pi/2}$$
$$= 1(-j) + 2\,(-1) + 3\,(+j) = -\,2 + 2j$$
6. For $k = 3$,
$$X(3) = \sum_{n=0}^{7} x(n)\, e^{-\frac{j\,3\pi n}{4}}$$
$$= x(1)\, e^{-j3\pi/4} + x\,(2)\, e^{-j3\pi/2} + x(3)\; e^{\frac{-j9\pi}{4}}$$
$$= 1(-\,0.707 - j\,0.707) + 2\,(+j) + 3\,(0.707 - j\,0.707)$$
$$= 1.414 - j\,0.828$$
7. For $k = 4$,
$$X(4) = \sum_{n=0}^{7} x(n)\, e^{-j\pi n}$$
$$= x(1)\, e^{-j\pi} + x(2)\, e^{-j2\pi} + x(3)\, e^{-j3\pi}$$
$$= 1\,(-\,1) + 2\,(1) + 3\,(-\,1) = -\,2$$
8. For $k = 5$,
$$X(5) = \sum_{n=0}^{7} x(n)\, e^{\frac{-\,j5\pi n}{4}}$$

$$= x(1)\ e^{\frac{-j5\pi}{4}} + x(2)\, e^{\frac{-j5\pi}{2}} + x(3)\, e^{\frac{-j15\pi}{4}}$$
$$= 1\,(-0.707 + j\,0.707) + 2\,(-j) + 3\,(0.707 + j\,0.707)$$
$$= 1.414 + j\,0.828$$

9. For $k = 6$,

$$X(6) = \sum_{n=0}^{7} x(n)\, e^{\frac{-j\,3\pi n}{2}}$$
$$= x(1)\ e^{\frac{-j\,3\pi}{2}} + x(2)\, e^{-j3\pi} + x(3)\, e^{-\frac{j9\pi}{2}}$$
$$X(6) = 1\,(+j) + 2\,(-1) + 3\,(-j) = -2 - j2$$

10. For $k = 7$,

$$X(7) = \sum_{n=0}^{7} x(n)\, e^{\frac{-j7\pi n}{4}}$$
$$= x(1)\, e^{\frac{-j7\pi}{4}} + x(2)\, e^{\frac{-j7\pi}{2}} + x(3)\, e^{\frac{-j21\pi}{4}}$$
$$= 1\,(0.707 + j0.707) + 2\,(+j) + 3\,(-0.707 + j\,0.707)$$
$$X(7) = -1.414 + \mathrm{j}\ 4.828$$

So, $X(k) = \{6, -1.414 - j4.828, -2 + j2, 1.414 - j0.828, -2, 1.414 + j\,0.828, -2 - j2, -1.414 + j\,4.828\}$

Que 4.5. **The first five point of the 8-point DFT of a real valued sequence are : {0.25, 0.125 – j0.3018, 0, 0.125 – j0.0518, 0}. Determine the remaining three points.**

AKTU 2017-18, Marks 10

Answer

Given : $x(n)_8 = \{0.25, 0.125 - j0.3018, 0, 0.125 - j0.0518, 0\}$
To Find : Remaining three points.

1. Given, $X[0] = 0.25$, $X[1] = 0.125 - j0.3018$, $X[2] = 0$,
$X[3] = 0.125 - j0.0158$, $X[4] = 0$
2. By property of DFT,
$$X[k] = {}^*X[N-k]$$
Or
$$X[N-k] = {}^*X[k]$$
3. Here, $N = 8$,
$$X[k] = {}^*X[8-k]$$
$$X[5] = {}^*X[8-5] = {}^*X[3] = 0.125 + j0.0158$$
$$X[6] = {}^*X[8-6] = {}^*X[2] = 0$$
$$X[7] = {}^*X[8-7] = {}^*X[1] = 0.125 + j0.3018$$

Que 4.6. **Find the 10-point DFT of the following sequence :**

i. $x(n) = \delta(n) + \delta(n - 5)$

ii. $x(n) = u(n) - u(n - 6)$

AKTU 2015-16, Marks 10

Answer

i. Given : $x(n) = \delta(n) + \delta(n - 5)$
To Find : 10 point DFT.

1. For 10 point DFT, $N = 10$

So, $$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad ; k = 0,1,2......, N-1$$

$$X(k) = \sum_{n=0}^{9} \{\delta(n) + \delta(n-5)\} e^{-j2\pi kn/N} \; ; k = 0 \text{ to } 9$$

2. For $k = 0$, $X(0) = e^0 + 0 + 0 + 0 + 0 + e^0 + 0 + 0 + 0 + 0$
$X(0) = 1 + 1 = 2$
3. For $k = 1$, $X(1) = e^0 + 0 + 0 + 0 + 0 + e^{-j\pi} + 0 + 0 + 0 + 0$
$= 1 + (-1) = 0$
4. For $k = 2$, $X(2) = e^0 + 0 + 0 + 0 + 0 + e^{-j2\pi} + 0 + 0 + 0 + 0$
$= 1 + 1 = 2$
5. For $k = 3$, $X(3) = e^0 + 0 + 0 + 0 + 0 + e^{-j3\pi} + 0 + 0 + 0 + 0$
$= 1 + (-1) = 0$
6. For $k = 4$, $X(4) = e^0 + 0 + 0 + 0 + 0 + e^{-j4\pi} + 0 + 0 + 0 + 0$
$= 1 + 1 = 2$
7. For $k = 5$, $X(5) = e^0 + e^{-j5\pi} = 1 + (-1) = 0$
8. For $k = 6$, $X(6) = e^0 + e^{-j6\pi} = 1 + 1 = 2$
9. For $k = 7$, $X(7) = e^0 + e^{-j7\pi} = 1 + (-1) = 0$
10. For $k = 8$, $X(8) = e^0 + e^{-j8\pi} = 1 + 1 = 2$
11. For $k = 9$, $X(9) = e^0 + e^{-j9\pi} = 1 + (-1) = 0$
12. So, 10 point DFT of signal $x(n)$ is
$X(k) = \{2, 0, 2, 0, 2, 0, 2, 0, 2, 0\}$

ii. Given : $x(n) = u(n) - u(n - 6)$
To Find : 10 point DFT.

1. So, $$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad ; \; k = 0,1,2,3.....N-1$$

$$X(k) = \sum_{n=0}^{9} \{u(n) - u(n-6)\} e^{-j2\pi kn/N} \; ; \; k = 0,1,2,........9$$

2. For $k = 0$, $X(0) = 1 + 1 + 1 + 1 + 1 + 1 + 0 + 0 + 0 + 0$
$X(0) = 6$

3. For $k = 1$, $X(1) = 1.e^0 + 1.e^{-j\frac{2\pi}{10}} + 1.e^{-j\frac{4\pi}{10}} + 1.e^{-j\frac{6\pi}{10}} + 1.e^{-j\frac{8\pi}{10}} + 1.e^{-j\pi} + 0 + 0$
$X(1) = -j\, 3.0834$

4. For $k = 2$, $X(2) = e^0 + e^{-j\frac{4\pi}{10}} + e^{-j\frac{8\pi}{10}} + e^{-j\frac{12\pi}{10}} + e^{-j\frac{16\pi}{10}} + e^{-j2\pi} + 0 + 0 = (1 - j0)$
5. For $k = 3$, $X(3) = e^0 + e^{-j\frac{6\pi}{10}} + e^{-j\frac{12\pi}{10}} + e^{-j\frac{18\pi}{10}} + e^{-j\frac{24\pi}{10}} + e^{-j3\pi} + 0 + 0 = -j0.7266$
6. For $k = 4$, $X(4) = e^0 + e^{-j\frac{8\pi}{10}} + e^{-j\frac{16\pi}{10}} + e^{-j\frac{24\pi}{10}} + e^{-j\frac{32\pi}{10}} + e^{-j4\pi} + 0 + 0 = 1$
7. For $k = 5$, $X(5) = e^0 + e^{-j\pi} + e^{-j2\pi} + e^{-j3\pi} + e^{-j4\pi} + e^{-j5\pi} + 0 + 0 = 0$
8. For $k = 6$, $X(6) = e^0 + e^{-j\frac{12\pi}{10}} + e^{-j\frac{24\pi}{10}} + e^{-j\frac{36\pi}{10}} + e^{-j\frac{48\pi}{10}} + e^{-j\frac{60\pi}{10}} + 0 + 0 = 1$
9. For $k = 7$, $X(7) = e^0 + e^{-j\frac{14\pi}{10}} + e^{-j\frac{28\pi}{10}} + e^{-j\frac{42\pi}{10}} + e^{-j\frac{56\pi}{10}} + e^{-j\frac{70\pi}{10}} + 0 + 0 = j0.7245$
10. For $k = 8$, $X(8) = e^0 + e^{-j\frac{16\pi}{10}} + e^{-j\frac{32\pi}{10}} + e^{-j\frac{48\pi}{10}} + e^{-j\frac{64\pi}{10}} + e^{-j8\pi} + 0 + 0 = 1$
11. For $k = 9$, $X(9) = e^0 + e^{-j\frac{18\pi}{10}} + e^{-j\frac{36\pi}{10}} + e^{-j\frac{54\pi}{10}} + e^{-j\frac{72\pi}{10}} + e^{-j9\pi} + 0 + 0 = j3.077$
12. So, 10-point DFT of $x(n)$ is

$$X(k) = \{6, -j3.0834, 0, -j0.7266, 1, 0, 1, j\,0.7245, 1, j3.0775\}$$

Que 4.7. **Determine the 4-point discrete time sequence from its DFT $X(k) = \{4, 1 - j, -2, 1 + j\}$.**

AKTU 2016-17, Marks 7.5

Answer

Given : $X(k) = \{4, 1 - j, -2, 1 + j\}$

To Find : IDFT $(x(n))$

1. IDFT is defined as

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j2\pi nk/N}, \; 0 \le n \le N - 1$$

Here, $N = 4$

$$x(n) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, e^{j\pi nk/2}, \; 0 \le n \le 3$$

2. When $n = 0$, $$x(0) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, e^0$$

$$= \frac{1}{4}\,[4 + (1 - j) + (-2) + (1 + j)] = 1$$

3. When $n = 1$, $$x(1) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, e^{j\pi k/2}$$

$$= \frac{1}{4}\,[4 + (1 - j)e^{j\pi/2} + (-2)e^{j\pi} + (1 + j)e^{j3\pi/2}]$$

$$= \frac{1}{4}\,[4 + (1 - j)j + 2 + (1 + j)(-j)] = \frac{1}{4} \times 8 = 2$$

4. When $n = 2$,
$$x(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{j\pi k}$$
$$= \frac{1}{4}[4 + (1-j)\, e^{j\pi} + (-2)\, e^{j2\pi} + (1+j)\, e^{j3\pi}]$$
$$= \frac{1}{4}[4 + (1-j)(-1) + (-2) + (1+j)(-1)] = 0$$

5. When $n = 3$,
$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{j3\pi k/2}$$
$$= \frac{1}{4}[4 + (1-j)\, e^{j3\pi/2} + (-2)\, e^{j3\pi} + (1+j)\, e^{j9\pi/2}]$$
$$= \frac{1}{4}[4 + (1-j)(-j) + 2 + (1+j)(j)] = 1$$

6. Therefore, the IDFT of the given DFT produces the following four point discrete time sequence.
$$x(n) = \{1, 2, 0, 1\}$$

Que 4.8. **Draw a transformation matrix of size 5 × 5 and explain the properties of twiddle factor.**

Answer

A. Transformation matrix of size 5 × 5 :

$$= \begin{bmatrix} W_5^0 & W_5^0 & W_5^0 & W_5^0 & W_5^0 \\ W_5^0 & W_5^1 & W_5^2 & W_5^3 & W_5^4 \\ W_5^0 & W_5^2 & W_5^4 & W_5^6 & W_5^8 \\ W_5^0 & W_5^3 & W_5^6 & W_5^9 & W_5^{12} \\ W_5^0 & W_5^4 & W_5^8 & W_5^{12} & W_5^{16} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0.31 + j0.95 & -0.81 + j0.58 & -0.81 - j0.58 & 31 - j0.95 \\ 1 & -0.81 + j0.58 & 0.31 - j0.59 & 0.31 + j0.59 & -0.81 - j0.58 \\ 1 & -0.81 - j0.58 & 0.31 + j0.59 & 0.31 - j0.59 & -0.81 + j0.58 \\ 1 & 0.31 - j0.59 & -0.81 - j0.58 & -0.81 + j0.58 & 0.31 + j0.59 \end{bmatrix}$$

B. Properties of twiddle factor :

1. The property of periodicity for W_N (twiddle factor) is given by
$$W_N^{k+N/2} = -W_N^k$$
from this property W_N is repeated with inverse sign after a half period $N/2$.

2. The property of symmetry for W_N is given by $W_N^{k+N} = W_N^k$
from this property W_N is repeated after a period N.

Que 4.9.

i. **Compute 4-point DFT of the following sequence using linear transformation matrix $x(n) = (1, 1, -2, -2)$.**

ii. **Find IDFT $x(n)$ from $X(k)$ calculated in part (i).**

AKTU 2015-16, Marks 7.5

Answer

Given : $x(n) = (1, 1, -2, -2)$

To Find : DFT and IDFT.

i. **DFT :**

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

Here, $N = 4$

$$[X(k)] = [W_4^{kn}][x(n)]$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1+1-2-2 \\ 1-j+2-2j \\ 1-1-2+2 \\ 1+j+2+2j \end{bmatrix} = \begin{bmatrix} -2 \\ 3-3j \\ 0 \\ 3+3j \end{bmatrix}$$

So, $X(k) = \{-2, 3-3j, 0, 3+3j\}$

ii. **IDFT :**

$$\text{IDFT} = x(n) = \frac{1}{N}[W_N^*].X(k)$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} -2 \\ 3-3j \\ 0 \\ 3+3j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -2+3-3j+0+3+3j \\ -2+3j+3+0-3j+3 \\ -2-3+3j+0-3-3j \\ -2-3j-3+0+3j-3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 4 \\ -8 \\ -8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -2 \end{bmatrix}$$

$$x(n) = (1, 1, -2, -2)$$

Que 4.10. **Derive the relation between DFT and *z*-transform of a discrete time sequence $x(n)$.**

AKTU 2016-17, Marks 7.5

Answer

1. The z-transform of $x(n)$ is given as $X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$
2. If $X(z)$ is sampled at the N equally spaced points on the unit circle, these points will be

$$z = e^{j2\pi k/N},\ k = 0, 1, 2, ..., N-1, \text{ then}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi nk/N} \quad ...(4.10.1)$$

$$X(k) = X(z)\big|_{z=e^{j\omega\pi k/N}},\ k = 0, 1, ..., N-1$$

3. The R.H.S. of eq. (4.10.1) is DFT $X(k)$, thus the relationship between DFT and z-transform is

$$X(k) = X(z)\big|_{z=e^{j2\pi k/N}}$$

Que 4.11. **State and prove circular convolution property of DFT.**

AKTU 2017-18, Marks 05

Answer

A. Statement : The multiplication of two DFTs equivalent to the circular convolution of their sequences in time domain.

Mathematical equation :

If $x_1(n) \xleftrightarrow[N]{\text{DFT}} X_1(k)$ and $x_2(n) \xleftrightarrow[N]{\text{DFT}} X_2(k)$ then,

$$x_1(n) \; N \; x_2(n) \xleftrightarrow[N]{\text{DFT}} X_1(k)\cdot X_2(k) \quad ...(4.11.1)$$

Here N indicates circular convolution.

B. Proof :

1. By definition two DFTs, $X_1(k)$ and $X_2(k)$ are given as,

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)e^{-j2\pi kn/N},\ k = 0, 1, ... N-1 \quad ...(4.11.2)$$

and $$X_2(k) = \sum_{l=0}^{N-1} x_2(l)e^{-j2\pi kl/N},\ k = 0, 1, ... N-1 \quad ...(4.11.3)$$

2. Let $X_3(k)$ be equal to multiplication of $X_1(k)$ and $X_2(k)$,
i.e., $X_3(k) = X_1(k)\cdot X_2(k)$ ...(4.11.4)
3. Let $x_3(m)$ be the sequence whose DFT is $X_3(k)$. Then $x_3(m)$ can be obtained from $X_3(k)$ by taking IDFT. *i.e.*,

$$x_3(m) = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)e^{j2\pi km/N} = \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\cdot X_2(k)e^{j2\pi km/N} \quad ...(4.11.5)$$

4. Substitute for $X_1(k)$ and $X_2(k)$ in eq. (4.11.5)

$$x_3(m) = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1(n)\, e^{-j2\pi kn/N}\right]\left[\sum_{l=0}^{N-1} x_2(l)\, e^{-j2\pi kl/N}\right] e^{j2\pi km/N}$$

5. Rearranging the summations and terms,

$$x_3(m) = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left\{\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N}\right\} \quad \text{...(4.11.6)}$$

Now, $\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N} = \begin{cases} N & \text{When } (m-n-l) \text{ is multiple of } N \\ 0 & \text{otherwise} \end{cases}$

...(4.11.7)

6. Putting this value in eq. (4.11.6) we get,

$$x_3(m) = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l)\cdot N \quad \text{When } (m-n-l) \text{ is multiple on } N$$

$$= \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \quad \text{When } (m-n-l) \text{ is multiple on } N$$

...(4.11.8)

7. Let $(m-n-l) = pN$

Here p is some integer.

Since integer multiple can be positive or negative both, we can write condition for convenience as,

$$m - n - l = -pN \quad i.e., \; l = m - n + pN \quad \text{...(4.11.9)}$$

8. Putting for l from eq. (4.11.9) in eq. (4.11.8) we get,

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2(m-n+pN) \quad \text{...(4.11.10)}$$

9. In the eq. (4.11.10), $x_2(m-n+pN)$ represents it is periodic sequence with period N. This periodic sequence is delayed by 'n' samples. $x_2(m-n+pN)$ represents sequence $x_2(m)$ shifted circularly by 'n' samples.

$$\therefore \quad x_2(m-n+pN) = x_2(m-n, \text{ modulo } N) = x_2((m-n))_N \quad \text{...(4.11.11)}$$

10. Putting this sequence in eq. (4.11.10) we get,

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N,\; m = 0, 1, \ldots N-1 \quad \text{...(4.11.12)}$$

11. Circular convolution of $x_1(n)$ and $x_2(n)$ is denoted by $x_1(n)$ N $x_2(n)$ and it is given by eq. (4.11.12) as,

$$x_3(m) = x_1(n) \; N \; x_2(n) = \sum_{n=0}^{N-1} x_1(n) x_2((m-n))_N,$$

where, $m = 0, 1, \ldots N-1$

Thus circular convolution property is proved.

Que 4.12. **Compute the circular convolution of two discrete time sequences $x_1(n) = \{1, 2, 1, 2\}$ and $x_2(n) = \{3, 2, 1, 4\}$.**

AKTU 2016-17, Marks 7.5

Answer

Given : $x_1(n) = \{1, 2, 1, 2\}$, $x_2(n) = \{3, 2, 1, 4\}$

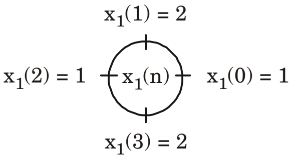
To Find : Circular convolution.

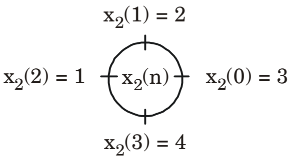
1. Circular convolution, $x_3(n) = x_1(n)$ ⓝ $x_2(n)$

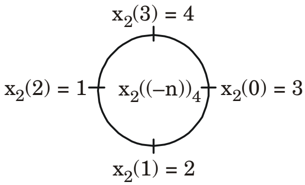
$$x_3(m) = \sum_{n=0}^{3} x_1(n)\, x_2\, ((m-n))_4$$

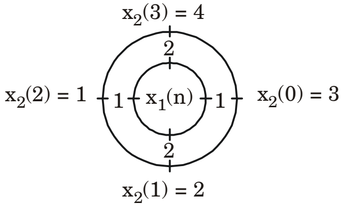
where, $m = 0, 1, 2, 3$

2. For $m = 0$, $$x_3(0) = \sum_{n=0}^{3} x_1(n)\, x_2\, ((-n))_4$$



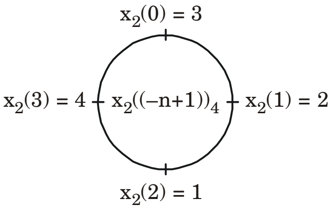


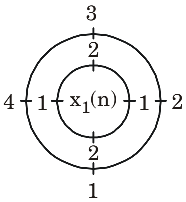




$\therefore$ $$x_3(0) = 1 \times 3 + 2 \times 4 + 1 \times 1 + 2 \times 2$$
$$= 3 + 8 + 1 + 4 = 16$$

3. For $m = 1$, $$x_3(1) = \sum_{n=0}^{3} x_1(n)\, x_2\, ((1-n))_4$$

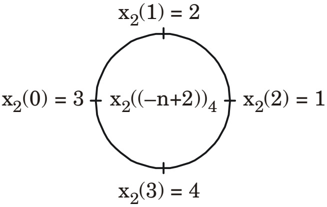


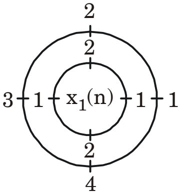


$\therefore$ $$x_3(1) = 1 \times 2 + 2 \times 3 + 1 \times 4 + 2 \times 1$$

$$= 2 + 6 + 4 + 2 = 14$$

4. For $m = 2$, $$x_3(2) = \sum_{n=0}^{3} x_1(n)\, x_2\, ((2 - n))_4$$

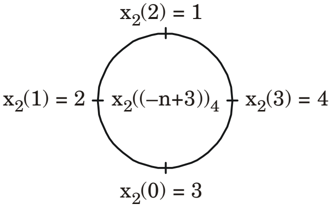


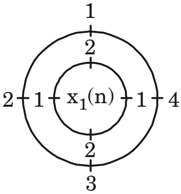


$\therefore$ $$x_3(2) = 1 \times 1 + 2 \times 2 + 3 \times 1 + 4 \times 2$$
$$= 1 + 4 + 3 + 8 = 16$$

5. For $m = 3$,

$$x_3(3) = \sum_{n=0}^{3} x_1(n)\, x_2\, ((3 - n))_4$$





$\therefore$ $$x_3(3) = 1 \times 4 + 2 \times 1 + 1 \times 2 + 2 \times 3$$
$$= 4 + 2 + 2 + 6 = 14$$

6. Therefore, $x_3(n) = \{16, 14, 16, 14\}$

Que 4.13. Find circular convolution of the following sequences using concentric circle method.

$$\boldsymbol{x_1(n) = (1, 2, 2, 1)}$$
$$\boldsymbol{x_2(n) = (1, 2, 3, 4)}$$

AKTU 2015-16, Marks 10

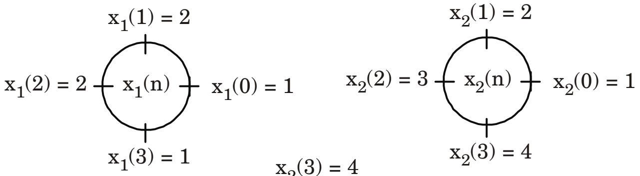
Answer

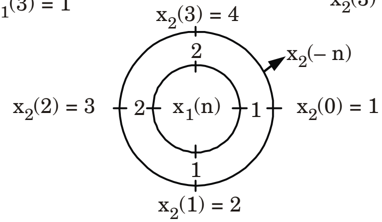
Given : $x_1(n) = (1, 2, 2, 1)$, $x_2(n) = (1, 2, 3, 4)$

To Find : Circular Convolution.

1. $$x_3(n) = x_1(n) \otimes x_2(n)$$
$$x_3(m) = \sum_{n=0}^{3} x_1(n).x_2(m - n)_4 \; ; m = 0, 1, 2, 3$$

2. For $m = 0$,

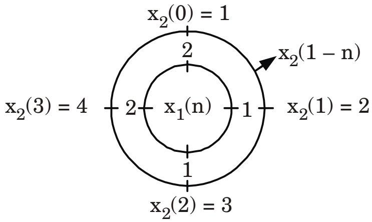




$$x_3(0) = \sum_{n=0}^{3} x_1(n).x_2(-n)_4$$

$$x_3(0) = 1 \times 1 + 4 \times 2 + 3 \times 2 + 1 \times 2 = 17$$

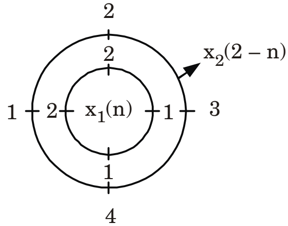
3. For $m = 1$,



$$x_3(1) = \sum_{n=0}^{3} x_1(n)\, x_2(1-n)$$

$$x_3(1) = 1 \times 2 + 2 \times 4 + 1 \times 3 + 1 \times 2 = 15$$

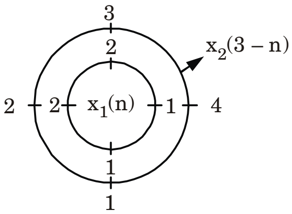
4. For $m = 2$,



$$x_3(2) = \sum_{n=0}^{3} x_1(n)\, x_2(2-n)$$

$$x_3(2) = 1 \times 3 + 2 \times 2 + 2 \times 1 + 1 \times 4 = 13$$

5. For $m = 1$,



$$x_3(3) = \sum_{n=0}^{3} x_1(n)\, x_2(3-n)$$

$$x_3(3) = 1 \times 1 + 1 \times 4 + 2 \times 3 + 2 \times 2 = 15$$

So, $$x_3(m) = \{17, 15, 13, 15\}$$

Que 4.14. **Calculate the circular convolution of $s_1(n) = \{1, 2, 1, 2\}$ and $s_2(n) = \{1, 2, 3, 4\}$ using Stockham method.**

Answer

Given : $s_1(n) = \{1, 2, 1, 2\}$, $s_2(n) = \{1, 2, 3, 4\}$

To Find : Circular convolution.

1. DFT of $s_1(n)$ is given by

$$S_1(k) = \sum_{n=0}^{3} s_1(n) W_4^{nk} = \sum_{n=0}^{3} s_1(n) e^{-j2\pi kn/4}$$

For $n = 0, 1, 2, 3$

$$= s_1(0) + s_1(1)e^{-j2\pi k.1/4} + s_1(2)e^{-j2\pi k.2/4} + s_1(3)e^{-j2\pi k.3/4}$$
$$= 1 + 2e^{-j\pi k/2} + e^{-j\pi k} + 2e^{-j3\pi k/2}$$

For $k = 0$,

$$S_1(0) = 1 + 2 + 1 + 2 = 6$$

For $k = 1$,

$$S_1(1) = 1 + 2e^{-j\pi/2} + e^{-j\pi} + 2e^{-j3\pi/2}$$
$$= 1 + 2[0 - j(1)] + [-1 - j0] + 2[0 - j(-1)]$$
$$= 1 - 2j - 1 + 2j = 0$$

For $k = 2$,

$$S_1(2) = 1 + 2e^{-j\pi} + e^{-j\pi 2} + 2e^{-j3\pi}$$
$$= 1 + (-2) + 1 + (-2) = -2$$

For $k = 3$,

$$S_1(3) = 1 + 2e^{-j3\pi/2} + e^{-j\pi 3} + 2e^{-j9\pi/2}$$
$$= 1 + 2j - 1 - 2j = 0$$
$$S_1(k) = \{6, 0, -2, 0\}$$

2. Now, DFT of $S_2(n)$ is given by

$$S_2(k) = \sum_{n=0}^{3} s_2(n) e^{-j2\pi kn/4}$$

For $n = 0, 1, 2, 3$

$$S_2(k) = s_2(0) + s_2(1)e^{-j2\pi k.1/4} + s_2(2)e^{-j2\pi k.2/4} + s_2(3)e^{-j2\pi k.3/4}$$
$$= 1 + 2e^{-j\pi k/2} + 3e^{-j\pi k} + 4e^{-j3\pi k/2}$$

For $k = 0$,

$$S_2(0) = 1 + 2 + 3 + 4 = 10$$

For $k = 1$,

$$S_2(1) = 1 + 2e^{-j\pi/2} + 3e^{-j\pi} + 4e^{-j3\pi/2} = -2 + j2$$

For $k = 2$,

$$S_2(2) = 1 + 2e^{-j\pi} + 3e^{-j2\pi} + 4e^{-j3\pi} = -2$$

For $k = 3$,

$$S_2(3) = 1 + 2e^{-j3\pi/2} + 3e^{-j3\pi} + 4e^{-j3\pi.3/2} = -2 - j2$$

So, $S_2(k) = \{10, -2 + j2, -2, -2 - j2\}$

3. We know that

$$S_3(k) = S_1(k) \,.\, S_2(k)$$
$$= \{6, 0, -2, 0\}\{10, -2 + j2, -2, -2 - j2\}$$
$$= \{60, 0, 4, 0\}$$

4. Taking inverse DFT of $S_3(k)$ for computing $s_3(n)$,

$$s_3(n) = \text{IDFT}\,[S_3(k)]$$
$$= \frac{1}{4}\sum_{k=0}^{3} S_3(k)e^{j2\pi kn/4} \qquad \text{for } n = 0, 1, 2, 3$$

$$= \frac{1}{4}\{S_3(0) + S_3(1)e^{j2\pi 1.n/4} + S_3(2)e^{(j2\pi 2.n)/4} + S_3(3)e^{(j3\pi n/2)}\}$$

$$= \frac{1}{4}[60 + 4e^{j\pi n}] = 15 + e^{j\pi n}$$

i.e., $s_3(n) = 15 + e^{j\pi n}$

for, $n = 0$, $s_3(0) = 15 + e^{j\pi 0} = 15 + 1 = 16$

$n = 1$, $s_3(1) = 15 + e^{j\pi} = 14$

$n = 2$, $s_3(2) = 15 + e^{j2\pi} = 16$

$n = 3$, $s_3(3) = 15 + e^{j3\pi} = 14$

$\therefore$ $s_3(n) = \{16, 14, 16, 14\}$

Que 4.15. Use the 4 point DFT and IDFT to determine circular convolution of the following sequence :

$$\boldsymbol{x(n) = \{1, 2, 3, 1\}}$$

$$\boldsymbol{h(n) = \{4, 3, 2, 2\}}$$

AKTU 2017-18, Marks 10

Answer

The procedure is same as Q. 4.14, Page 4–17C, Unit-4.

[**Ans.** $x(n)$ N $h(n) = \{17, 19, 22, 20\}$]

Que 4.16. Prove that multiplication of the DFTs of two sequences is equivalent to the circular convolution of the two sequences in the time domain.

AKTU 2015-16, Marks 7.5

AKTU 2019-20, Marks 07

Answer

1. Consider the two sequences $x(n)$ and $y(n)$ which are of finite duration. Let $X(k)$ and $Y(k)$ be the N-point DFTs of the two sequences respectively and they are given by

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}, \; k = 0, 1, ..., N-1.$$

$$Y(k) = \sum_{n=0}^{N-1} y(n)e^{-j2\pi nk/N}, \; k = 0, 1, ..., N-1.$$

2. Let $x_3(m)$ be another sequence of length N and its N-point DFT be $X_3(k)$ which is a product of $X(k)$ and $Y(k)$,

i.e., $\quad X_3(k) = X(k)\,Y(k), k = 0, 1,..., N-1.$

The sequence $x_3(m)$ can be obtained by taking the inverse DFT of $X_3(k)$,

i.e., $\quad x_3(m) = \text{IDFT}[X_3(k)]$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)e^{j2\pi mk/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y(k)e^{j2\pi mk/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}\right]\left[\sum_{l=0}^{N-1} y(l)e^{-j2\pi lk/N}\right]e^{j2\pi mk/N}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\sum_{l=0}^{N-1} y(l)\left[\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N}\right] \quad ...(4.16.1)$$

4. Consider the term within the brackets in eq. (4.16.1), it has the form

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & \text{for } a = 1 \\ \dfrac{1-a^N}{1-a}, & \text{for } a \neq 1 \end{cases}$$

where, $a = e^{j2\pi(m-n-l)/N}$.

5. When $(m-n-l)$ is a multiple of N, then $a = 1$, otherwise $a^N = 1$ for any value of $a \neq 0$.

Therefore,

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & l = m-n+pN, N = (m-n)(\text{mod } N), p : \text{integer}) \\ 0, & \text{otherwise} \end{cases}$$

6. Now, $x_3(m)$ becomes

$$x_3(m) = \sum_{n=0}^{N-1} x(n)y(m-n, (\text{mod } N)), N = 0, 1, ..., N-1$$

where, $y((m-n) \text{ mod } N)$ is the reflected and circularly shifted version of $y(m)$ and n represents the number of indices that the sequence $x(n)$ is shifted to the right.

Que 4.17. **Find the linear convolution using circular convolution of the following sequence :**

$$x(n) = \{1, 2, 1\}, h(n) = \{1, 2\}.$$

AKTU 2018-19, Marks 07

Answer

Given : $x(n) = \{1, 2, 1,\}$, $h(n) = \{1, 2\}$

To Find : Linear convolution.

1. Length of $x(n) = L = 3$

   Length of $h(n) = M = 2$

3. Therefore $N = L + M - 1$

   $= 3 + 2 - 1 = 5 - 1 = 4$

   Thus $N = 4$

3. Let us make length of $x(n)$ and $h(n)$ equal to 4 by adding zeros at end.

$$x(n) = \{1, 2, 1, 0\}$$
$$h(n) = \{1, 2, 0, 0\}$$

4. As we know $X(k) = W_N x_N$ ...(4.17.1)

   Here $W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$ and $x_N = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$

5. Substituting the values of W_4 and x_N in eq. (4.17.1), we get

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2j \\ 0 \\ j \end{bmatrix}$$

   Therefore $X(k) = \{4, -2j, 0, j\}$

6. To find $H(k)$, $H(k) = W_N h_N$

$$H(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1-2j \\ -1 \\ 1+2j \end{bmatrix}$$

$$Y(k) = X(k)\, H(k)$$
$$= \{4, -2j, 0, j\} \{3, 1 - 2j, -1, 1 + 2j\}$$
$$X(k) = \{12, -2j - 4, 0, j - 2\}$$

Que 4.18. **Determine the circular convolution of the following sequences and compare the result with linear convolution :**

$$x(n) = (1, 2, 3, 4)$$

$$h(n) = (1, 2, 1)$$ **AKTU 2017-18, Marks 10**

Answer

i. **Circular convolution :** The procedure is same as Q. 4.12, Page 4–14C, Unit-4.

ii. **Linear convolution :**

We know that

$$y(n) = x(n) * h(n)$$
$$h(n) = \{1, 2, 1, 0\}$$

Using matrix representation method,

$y(n) =$

$h(n)$ \ $x(n)$	1	2	3	4
1	1	2	3	4
2	2	4	6	8
1	1	2	3	4
0	0	0	0	0

Linear convolution of two sequences

i.e., $y(n) = x(n) * h(n)$

$$y(n) = \{1, 4, 8, 12, 11, 4, 0\}$$

Que 4.19. **If the 10-point DFT of $x(n) = \delta(n) - \delta(n-1)$ and $h(n) = u(n) - u(n-10)$ are $X(k)$ and $H(k)$ respectively, find the sequence $W(n)$ that corresponds to the 10-point inverse DFT of the product $H(k)\,X(k)$.** **AKTU 2015-16, Marks 7.5**

Answer

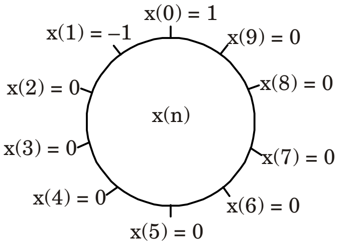
Given : $x(n) = \delta(n) - \delta(n-1)$, $h(n) = u(n) - u(n-10)$
To Find : $w(n)$

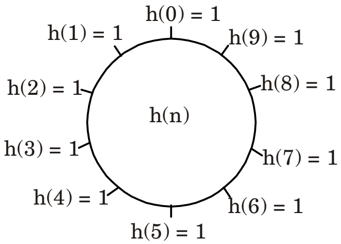
1. Since, product of $H(k)$ and $X(k)$ correspond to the circular convolution of $x(n)$ and $h(n)$.

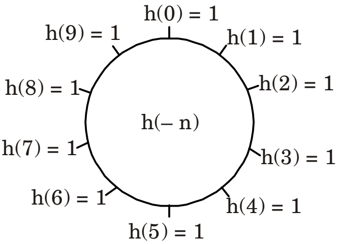
$$x(n) = \delta(n) - \delta(n-1)$$

$$x(n) = \{1, -1, 0, 0, 0, 0, 0, 0, 0, 0\}$$

and $$h(n) = u(n) - u(n - 10) = \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$$







2. Using circular convolution :

$$w(n) = x(n) * h(n)$$

so,
$$w(0) = 1 - 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$
$$w(1) = 1 - 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$
$$w(2) = 1 - 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$
$$w(3) = 1 - 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$
$$w(4) = 1 - 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$
$$w(5) = 1 - 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$
$$w(6) = 1 - 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$
$$w(7) = 1 - 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$
$$w(8) = 1 - 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$
$$w(9) = 1 - 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$

Hence, $$w(n) = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0\} = 0$$

PART-2

FFT : Definition, Decimation in Time (DIT) Algorithm, Decimation in Frequency (DIF) Algorithm.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.20. **What is FFT ? Explain it.**

Answer

1. FFT is an algorithm used to compute the DFT. It uses the two properties of twiddle factor (W_N) as follows :

i. Symmetry property

$$W_N^{k+N/2} = -W_N^k$$

ii. Periodicity property

$$W_N^{k+N} = W_N^k$$

2. The basic principle of FFT algorithm is to decompose DFT in small DFTs successively. This is illustrated in Fig. 4.20.1.

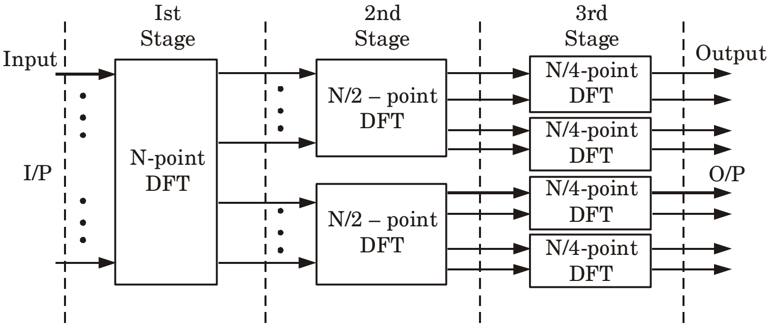


Fig. 4.20.1. FFT algorithm.

Stage 1 : In stage-1, we are having an N-point DFT.

Stage 2 : Here N-point DFT is divided into two equal $N/2$-point DFTs. The output of first stage is input of 2nd stage.

Stage 3 : Here the $N/2$-point DFTs of 2nd stage are also break into two small $N/4$-point DFTs.

3. The bigger DFTs are break into small DFTs upto the smallest DFT which is equal to the radix of FFT.

4. **Radix-*r* FFT :** The N-point DFT is decomposed into successively smaller size DFTs. If N is factored as

$$N = r_1 r_2 r_3 \,\ldots\ldots\, r_l$$

where, l = Number of stages.

$$r_1 = r_2 = r_3 = \,\ldots\, r_l = r$$

Then $N = r^l$

5. Thus the smallest DFT in FFT algorithm will be of size 'r'. This number 'r' is called the radix of the FFT algorithm. Most widely used radix is radix-2 FFT algorithm.

Que 4.21. **Write a short note on the following :**

i. Butterfly computation

ii. Inplace computation

iii. Bit reversal

AKTU 2017-18, Marks 10

Answer

i. Butterfly computation :

1. This is the fundamental or basic computation in FFT algorithms. Fig. 4.21.1 shows this operation.

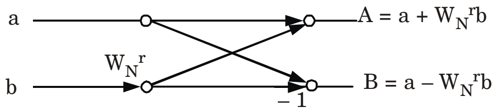


Fig. 4.21.1. Butterfly computation in FFT.

2. In the Fig. 4.21.1 observe that two values a and b are available as input. From these two values A and B are computed at output as shown in Fig 4.21.1. This operation is called butterfly operation.

ii. Inplace computation :

1. Observe the butterfly computation of Fig. 4.21.1 used in FFT. From values a and b new values A and B are computed. Once A and B are computed, there is no need to store a and b. Thus same memory locations can be used to store A, B where a, b was stored.
2. Since A, B or a, b are complex number, they are need two memory locations each. Thus for computation of one butterfly, four memory locations are required, *i.e.*, two for a or A and two for b or B.
   Memory locations for one butterfly = $2 \times 2 = 4$
3. In such computation, A is stored in place of a and B is stored in place of b. This is called is inplace computation.

iii. Bit reversal :

1. Consider the signal flow graph of 8-point DIT FFT algorithm. Observe that the sequence of input data is shuffled as $x(0)$, $x(4)$, $x(2)$, $x(6)$, $x(1)$, $x(5)$, $x(3)$, $x(7)$. And the DFT sequence $X(k)$ at the output is in proper order, *i.e.*, $x(0)$, $x(1)$, ... $x(7)$. The shuffling of the input sequence has well defined format.
2. In the Table 4.21.1 observe that the data point $x(1) = x\ (001)$ is to be placed at $m = 4$ *i.e.*, $(100)^{th}$ position in the decimated array. The last column of Table 4.21.1 shows the order in which the data is required.
3. Thus the input data should be stored in the bit reversed order then the DFT will be obtained in natural sequence.

Table 4.21.1

Memory address of $x(n)$ in decimal	Memory address of $x(n)$ in binary			Memory address in bit reversed order			New memory address of $x(n)$ according to reversed order of bits
n	n_2	n_1	n_0	n_0	n_1	n_2	m
0	0	0	0	0	0	0	0
1	0	0	1	1	0	0	4
2	0	1	0	0	1	0	2
3	0	1	1	1	1	0	6
4	1	0	0	0	0	1	1
5	1	0	1	1	0	1	5
6	1	1	0	0	1	1	3
7	1	1	1	1	1	1	7

Que 4.22. **Explain the decimation in time DIT-FFT algorithm.**

Answer

1. Assuming radix-2 FFT algorithm in DIT-FFT algorithm, the N-point DFT is divided into two $N/2$-point DFT in 2^{nd} stage, then into $N/4$-point DFT in 3^{rd} stage and so on until the 2-point DFT is not achieved.
   For radix-2 FFT
   $$N = 2.2.2 \ldots\ldots 2_l = 2^l.$$
2. Let $x(n)$ is the sequence of N values. The given sequence $x(n)$ is decimated (broken) into two $N/2$-point DFTs consisting of the even numbered values of $x(n)$ and the odd numbered values of $x(n)$.
3. The N-point DFT of sequence $x(n)$ is given as
   $$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}\ ,\ 0 \le k \le N-1 \qquad \ldots(4.22.1)$$
4. In eq. (4.22.1) breaking $x(n)$ into even and odd numbered values, we have
   $$X(k) = \sum_{n \text{ even}}^{N-1} x(n)\, W_N^{kn} + \sum_{n \text{ odd}}^{N-1} x(n)\, W_N^{kn} \qquad \ldots(4.22.2)$$
   Substituting $n = 2r$; for n even
   $n = 2r + 1$; for n odd
   we get
   $$X(k) = \sum_{r=0}^{N/2-1} x(2r)\, W_N^{2rk} + \sum_{r=0}^{(N/2-1)} x(2r+1)\, W_N^{(2r+1)k}$$

$$= \sum_{r=0}^{(N/2-1)} x(2r)\,(W_N^2)^{kr} + W_N^k \sum_{r=0}^{(N/2-1)} x(2r+1)\,(W_N^2)^{kr} \qquad ...(4.22.3)$$

5. Since $W_N^2 = [e^{-j2\pi/N}]^2 = (e^{-j(2\pi/(N/2))}) = W_{N/2}$

   Thus eq. (4.22.3) can be written as

$$X(k) = \sum_{r=0}^{(N/2-1)} x(2r)\,W_{N/2}^{kr} + W_N^k \sum_{r=0}^{(N/2-1)} x(2r+1)\,W_{N/2}^{kr} \qquad ...(4.22.4)$$

6. In eq. (4.22.4) the first part is the $N/2$-point DFT of $x(2r)$ say $A(k)$, and $x(2r + 1)$ is $B(k)$.

   Then eq. (4.22.4) can be written as

$$X(k) = A(k) + W_N^k\ B(k) \qquad ...(4.22.5)$$

   where $k = 0, 1, ... N/2 - 1$

7. The sum of $A(k)$ and $B(k)$ are computed for $0 \le k \le N/2 - 1$ because $A(k)$ and $B(k)$ are considered periodic with the period $N/2$. For periodicity the eq. (4.22.5) can be written as

$$X(k) = \begin{cases} A(k) + W_N^k B(k) & ; 0 \le k \le \dfrac{N}{2} - 1 \\ A\left(k + \dfrac{N}{2}\right) + W_N^{(k+N/2)}\ B\left(k + \dfrac{N}{2}\right) & ; \dfrac{N}{2} \le k \le N - 1 \end{cases} \qquad ...(4.22.6)$$

8. Using the symmetry property of $W_N^{k+N/2} = -W_N^k$, then eq. (4.22.6) will become

$$X(k) = \begin{cases} A(k) + W_N^k B(k) & ; 0 \le k \le \dfrac{N}{2} - 1 \\ A\left(k + \dfrac{N}{2}\right) - W_N^k\ B\left(k + \dfrac{N}{2}\right) & ; \dfrac{N}{2} \le k \le N - 1 \end{cases}$$

Que 4.23. **Develop a DIT FFT algorithm for $N = 8$ using a 4-points DFT and a 2-point DFT. Compare the number of multiplications with the algorithm using only 2-point DFTs.**

Answer

8-point DIT-FFT,

Here $N = 8$ *i.e.*, $N = 2 \times 2 \times 2 = 2^l = 2^3$

$l = 3$

Thus number of stages are three here,

Stage 1 :

1. We have two $N/2$ DFTs *i.e.*, two 4-points DFTs.

   For $N = 8$, $k = 0, 1, 2, 3, ..., 7$ and $n = 0, 1, 2, 3, ..., 7$

2. $X(k) = A(k) + W_N^k B(k)$ can be written as

   $X(0) = A(0) + W_8^0 B(0)$, $X(1) = A(1) + W_8^1 B(1)$

   $X(2) = A(2) + W_8^2 B(2)$, $X(3) = A(3) + W_8^3 B(3)$

$$X(4) = A(0) - W_8^0 B(0), \quad X(5) = A(1) - W_8^1 B(1)$$
$$X(6) = A(2) - W_8^2 B(2), \quad X(7) = A(3) - W_8^3 B(3)$$

3. Then $x(n)$ can be given as for $N = 8$

$$x(n) = [x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)]$$

4. The odd numbered and even numbered values of $x(n)$ are given as,

$$x(2r + 1) = [x(1), x(3), x(5), x(7)]$$

and $$x(2r) = [x(0), x(2), x(4), x(6)]$$

5. Thus the above equations can be plotted in signal flow graph as shown in Fig. 4.23.1.

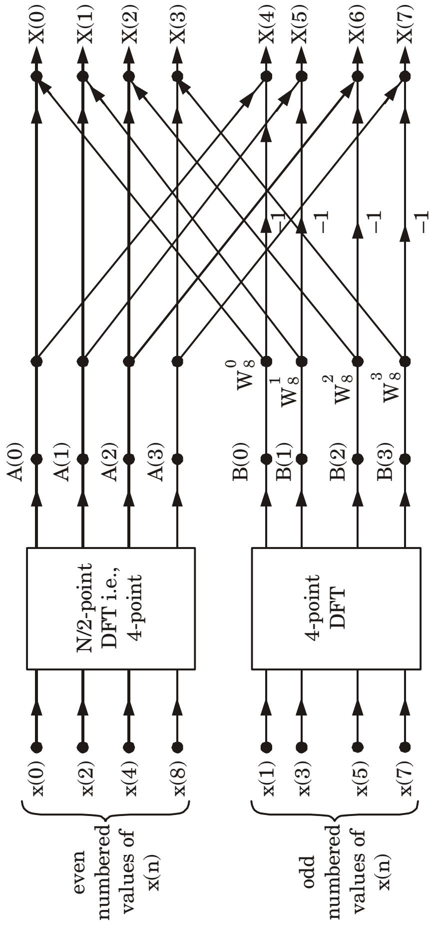


Fig. 4.23.1. Stage 1 signal flow graph for N = 8.

Stage 2 :

1. In stage-2, the 4-point DFTs are again broken into two 2-point DFTs *i.e.*, $A(k)$ and $B(k)$ sequence are again broken into even and odd numbered values. Suppose they are given as,

$$A(k) = C(k) + W_{N/2}^{k} D(k)$$

$$B(k) = E(k) + W_{N/2}^{k} F(k) \qquad \text{...(4.23.1)}$$

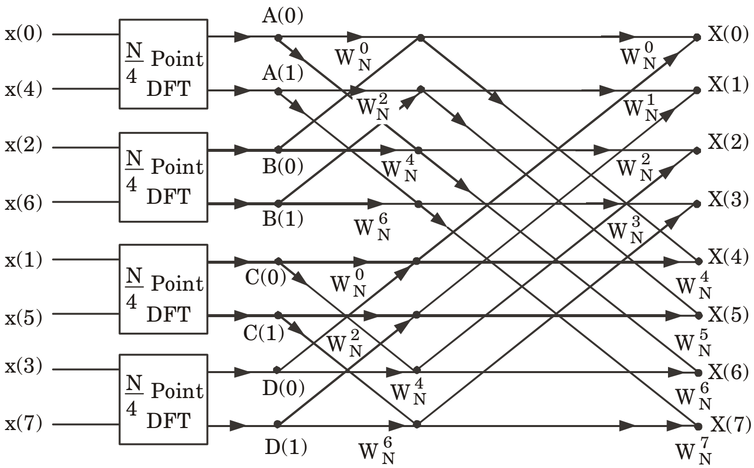


Fig. 4.23.2. Flow graph of the second stage Decimation-in-time FFT algorithm for N = 8.

2. For periodicity eq. (4.23.1) can also be written as

$$A(k) = \begin{cases} C(k) + W_N^{2k} D(k) & ;\, 0 \le k \le \dfrac{N}{4} - 1 \\ C\left(k + \dfrac{N}{4}\right) - W_N^{2k} D\left(k + \dfrac{N}{4}\right) & ;\, \dfrac{N}{4} \le k \le \dfrac{N}{2} - 1 \end{cases} \qquad \text{...(4.23.2)}$$

$$B(k) = \begin{cases} E(k) + W_N^{2k} F(k) & ;\, 0 \le k \le \dfrac{N}{4} - 1 \\ E\left(k + \dfrac{N}{4}\right) - W_N^{2k} F\left(k + \dfrac{N}{4}\right) & ;\, \dfrac{N}{4} \le k \le \dfrac{N}{2} - 1 \end{cases} \qquad \text{...(4.23.3)}$$

For $N/2 = 4$, $k = 0, 1, 2, 3$, and $n = 0, 1, 2, 3$.

3. Eq. (4.23.2) and eq. (4.23.3) can be written as

$$A(0) = C(0) + W_8^0 D(0), \quad B(0) = E(0) + W_8^0 F(0)$$
$$A(1) = C(1) + W_8^2 D(1), \quad B(1) = E(1) + W_8^2 F(0)$$
$$A(2) = C(0) - W_8^0 D(2), \quad B(2) = E(0) - W_8^0 F(0)$$
$$A(3) = C(1) - W_8^2 D(3), \quad B(3) = E(1) - W_8^2 F(1)$$

We have, $x(2r) = [x(0), x(2), x(4), x(6)]$

4. Breaking this into even and odd numbered values

$$x(2p) = [x(2), x(6)]$$

and $$x(2p+1) = [x(0), x(4)]$$

5. Similarly for $x\ (2r + 1) = [x(1), x(3), x(5), x(7)]$

We have $$x(2q) = [x(3), x(7)]$$

and $$x(2q + 1) = [x(1), x(5)]$$

Stage 3 :

1. In 3rd stage $N/4 = 2$, *i.e.*, 2-point DFTs are left. The $C(k)$, $D(k)$, $E(k)$ and $F(k)$ are broken into 2-point DFTs as

$$C(k) = \begin{cases} G(k) + W_{N/4}^{k}\ H(k) \\ G\left(k + \dfrac{N}{8}\right) - W_{N/4}^{k}\ H\left(k + \dfrac{N}{8}\right) \end{cases} \quad ...(4.23.4)$$

$$C(k) = \begin{cases} G(k) + W_{N}^{4k}\ H(k) & ; 0 \le k \le \dfrac{N}{8} - 1 \\ G\left(k + \dfrac{N}{8}\right) - W_{N}^{4k}\ H\left(k + \dfrac{N}{8}\right) & ; \dfrac{N}{8} \le k \le \dfrac{N}{4} - 1 \end{cases} \quad ...(4.23.5)$$

Here $N/4 = 2$, *i.e.*, $k = 0, 1$, and $n = 0, 1$.

Therefore

$$C(0) = G(0) + W_8^0\ H(0) = G(0) + H(0)$$

$$C(1) = G(0) - W_8^0\ H(0) = G(0) - H(0)$$

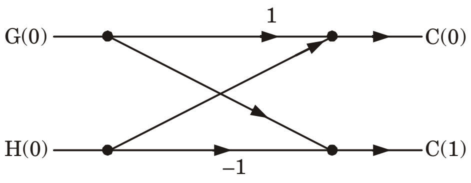


Fig. 4.23.3. 2-point DFT SFG (Butterfly structure).

2. The structure shown in Fig. 4.23.2 is known as butterfly structure. For drawing a signal flow graph (SFG) for 2-point DFT, this structure is used.

3. Thus the final signal flow graph (SFG) for 8-point DFT using DIT-FFT is given in Fig. 4.23.4.

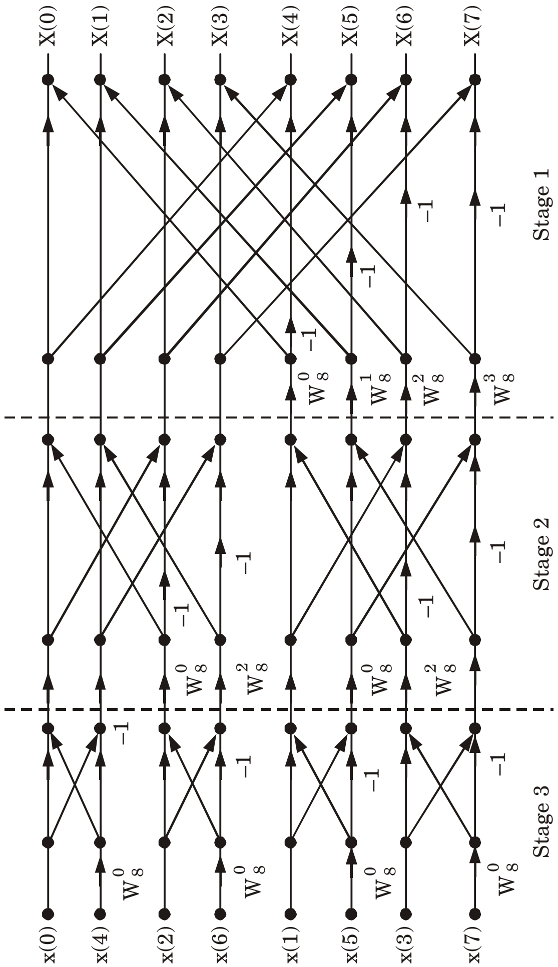


Fig. 4.23.4. SFG for an 8-points DIT-FFT.

4. The number of complex multiplications and additions required was $N + 2(N/2)^2$ for the original N-point transform when decomposed into two $(N/2)$-point transformation.
5. When the $(N/2)$-point transforms are decomposed into $N/4$-point transforms, then the number of complex multiplications and additions required will be $2N + 4\,(N/4)^2$.

Que 4.24. **Given $x(n) = 2^n$ and $N = 8$, find $X(K)$ using DIT-FFT algorithm. Also calculate the computational reduction factor.**

AKTU 2016-17, Marks 10

OR

Use Radix-2 DIT algorithm for efficient computation of 8 point DFT of $x(n) = 2^n$.

AKTU 2015-16, Marks 10

OR

Find the 8-pint DFT of $x(n) = 2^n$ by using DIT FFT algorithm.

AKTU 2019-20, Marks 07

Answer

Given : $x(n) = 2^n$ and $N = 8$

To Find : $X(k)$ and reduction factor.

1. $$x(0) = 1,\ x(1) = 2,\ x(2) = 4,\ x(3) = 8$$
$$x(4) = 16,\ x(5) = 32,\ x(6) = 64,\ x(7) = 128$$
$$x(n) = \{1, 2, 4, 8, 16, 32, 64, 128\}$$
2. We know,
$$W_8^0 = 1$$
$$W_8^1 = 0.707 - j0.707$$
$$W_8^2 = -j$$
$$W_8^3 = -0.707 - j0.707$$
3. Using DIT FFT algorithm, we can find $X(k)$ from the given sequence $x(n)$ as shown in Fig. 4.24.1.
4. Computation reduction factor

$$= \frac{\text{Number of complex multiplication requried for direct (DFT)}}{\text{Number of complex multiplications required for FFT algorithm}}$$

$$= \frac{N^2}{\frac{N}{2}\log_2(N)} = \frac{8^2}{\frac{8}{2}\log_2 8} = \frac{64}{12} = 5.33$$

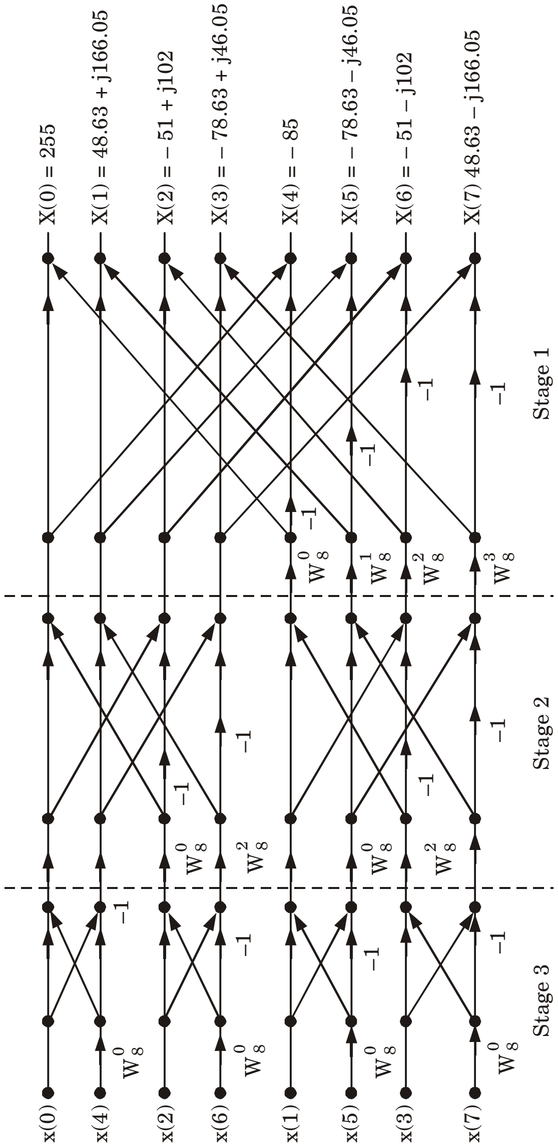

X(0) = 255
X(1) = 48.63 + j166.05
X(2) = – 51 + j102
X(3) = – 78.63 + j46.05
X(4) = – 85
X(5) = – 78.63 – j46.05
X(6) = – 51 – j102
X(7) 48.63 – j166.05
x(0)
x(4)
x(2)
x(6)
x(1)
x(5)
x(3)
x(7)
Stage 1
Stage 2
Stage 3

Fig. 4.24.1.

Que 4.25. **Find the 8 point DFT of the sequence $x(n) = \{1, 1, 1, 1, 1, 0, 0, 0\}$ using DIT FFT.**

AKTU 2018-19, Marks 07

Answer

The procedure is same as Q. 4.24, Page 4–32C, Unit-4.

(**Ans.** $X(K) = \{5, -j(1+\sqrt{2}), 1, j(1-\sqrt{2}), 1, -j(1-\sqrt{2}), 1, j(1+\sqrt{2})\}$)

Que 4.26. **Explain the decimation in frequency DIF-FFT algorithm.**

Answer

1. In DIF-FFT algorithm the output sequence $X(k)$ is divided into smaller and smaller subsequences.
2. In DIF-FFT the input sequence $x(n)$ is divided into two equal sequences of $N/2$ length. The first sequence $x_1(n)$ consists first $N/2$ values of sequence $x(n)$ while the second sequence $x_2(n)$ consists last $N/2$ values of sequence.
3. Therefore, the N-point DFT can be written as

$$X(k) = \sum_{n=0}^{(N/2-1)} x(n)\, W_N^{kn} + \sum_{n=N/2}^{N-1} x(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{(N/2-1)} x(n)\, W_N^{kn} + \sum_{n=0}^{(N/2-1)} x(n+N/2)\, W_N^{(n+N/2)k}$$

$$X(k) = \sum_{n=0}^{(N/2-1)} x(n)\, W_N^{kn} + W_N^{(N/2)k}\left[\sum_{n=0}^{N/2-1} x(n+N/2)\, W_N^{kn}\right]$$

Since $W_N^{(N/2)k} = e^{-j2\pi Nk/2.N} = e^{-j\pi k} = (-1)^k$

$$X(k) = \sum_{n=0}^{(N/2-1)} x(n)\, W_N^{kn} + (-1)^k \sum_{n=0}^{N/2-1} x(n+N/2)\, W_N^{kn}$$

$$X(k) = \sum_{n=0}^{(N/2-1)} [x(n) + (-1)^k x(n+N/2)]\, W_N^{kn}$$

Case I : When 'k' is even.

If $X(2p)$ is even numbered value of $X(k)$, then we have,

$$X(2p) = \sum_{n=0}^{N/2-1} [x(n) + (-1)^{2p} x(n+N/2)]\, W_N^{2pn}$$

$$X(2p) = \sum_{n=0}^{N/2-1} [x(n) + x(n+N/2)]\, W_{N/2}^{pn}$$

$$X(2p) = \sum_{n=0}^{N/2-1} h(n)\, W_{N/2}^{pn}$$

where $h(n) = [x(n) + x(n+N/2)]$

Case II : When 'k' is odd.

If $X(2p + 1)$ is odd numbered values sequence of $X(k)$.
then we have

$$X(2p+1) = \sum_{n=0}^{N/2-1} [x(n) + (-1)^{2p+1} x(n+N/2)]\, W_N^{(2p+1)n}$$

$$= \sum_{n=0}^{N/2-1} [x(n) - x(n+N/2)]\, W_N^{n} \cdot W_{N/2}^{pn}$$

$$X(2p+1) = \sum_{n=0}^{N/2-1} W_N^{n} f(n)\, W_{N/2}^{pn}$$

where $f(n) = [x(n) - x(n + N/2)]$

Que 4.27. **Derive and draw the flow graph for DIF-FFT algorithm for N = 8.**

AKTU 2016-17, 2017-18; Marks 10

Answer

1. Here $N = 8$, thus, n and $k = 0, 1, 2, \ldots, 7$

$$X(k) = [X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)]$$

Stage I :

i. $$X(2p) = [X(0), X(2), X(4), X(6)]$$
$$X(2p+1) = [X(1), X(3), X(5), X(7)]$$
$$h(n) = [x(n) + x(n + N/2)]$$
and $$f(n) = [x(n) - x(n + N/2)]$$

we get

$h(0) = x(0) + x(4)$ $f(0) = x(0) - x(4)$

$h(1) = x(1) + x(5)$ $f(1) = x(1) - x(5)$

$h(2) = x(2) + x(6)$ $f(2) = x(2) - x(6)$

$h(3) = x(3) + x(7)$ $f(3) = x(3) - x(7)$

ii. The first stage of DIF FFT algorithm can be drawn in signal flow graph (SFG) form as shown in Fig. 4.27.1.

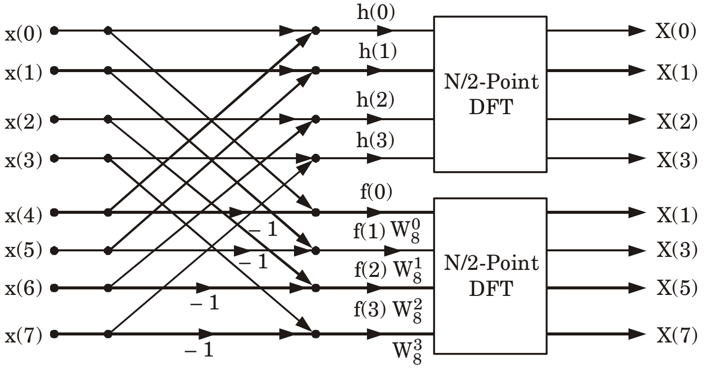


Fig. 4.27.1. SFG of first stage of DIF-FFT for N = 8.

Stage II :

i. Now $X(2p)$ and $X(2p + 1)$ are divided into even and odd numbered values sequence.

ii. Taking $X(2p)$ first we have

$$X(2p) = \sum_{n=0}^{\frac{N}{4}-1} h(n)W_N^{2pn} + \sum_{n=N/4}^{\frac{N}{2}-1} h(n)W_N^{2pn}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} h(n)W_N^{2pn} + \sum_{n=0}^{\frac{N}{4}-1} h\left(n+\frac{N}{4}\right)W_N^{2p\left(n+\frac{N}{4}\right)}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} h(n)W_{N/2}^{pn} + W_N^{\left(2p\frac{N}{4}\right)} \sum_{n=0}^{\frac{N}{4}-1} h\left(n+\frac{N}{4}\right)W_{N/2}^{pn} \quad ...(4.27.1)$$

Since $W_N^{N/2} = -1$

iii. Thus eq. (4.27.1) can be written as

$$X(2p) = \sum_{n=0}^{\frac{N}{4}-1} h(n)W_N^{2pn} + (-1)^p \sum_{n=0}^{\frac{N}{4}-1} h\left(n+\frac{N}{4}\right)W_N^{2pn} \quad ...(4.27.2)$$

2. Again for even and odd values of 'p' we have two cases :

Case I : When p = even, $p = 2r$ $(-1)^{2r} = 2r$

Then eq. (4.27.2) will become as

$$X(4r) = \sum_{n=0}^{\frac{N}{4}-1} \left[h(n) + h\left(n+\frac{N}{4}\right)\right] W_N^{4rn}$$

$$X(4r) = \sum_{n=0}^{\frac{N}{4}-1} g(n)W_N^{4rn} \quad ...(4.27.3)$$

where, $$g(n) = h(n) + h\left(n+\frac{N}{4}\right) \quad ...(4.27.4)$$

Case II :

a. When p = odd *i.e.*, $p = (2r + 1)$ then $(-1)^{2r+1} = -1$

$$X(4r+2) = \sum_{n=0}^{\frac{N}{4}-1} \left[h(n) - h\left(n+\frac{N}{4}\right)\right] W_N^{(2r+1)2n}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[h(n) - h\left(n+\frac{N}{4}\right)\right] W_N^{2n} \cdot W_N^{4rn}$$

$$X(4r+2) = \sum_{n=0}^{\frac{N}{4}-1} [A(n)W_N^{4rn}]\, W_N^{2n}$$

where, $$A(n) = h(n) - h\left(n+\frac{N}{4}\right) \quad ...(4.27.5)$$

b. Therefore for 2nd stage we get

$$g(0) = h(0) + h(2)$$
$$g(1) = h(1) + h(3)$$
$$A(0) = h(0) - h(2)$$
$$A(1) = h(1) - h(3)$$

Similarly we get

$$X(2p + 1) = \sum_{n=0}^{\frac{N}{2}-1} [f(n)W_N^{2pn}]\, W_N^n$$

c. Dividing $X(2p + 1)$ into even and odd parts.

$$X(2p + 1) = W_N^n \left[\sum_{n=0}^{\frac{N}{4}-1} f(n)W_N^{2pn} + \sum_{n=N/4}^{\frac{N}{2}-1} f(n)\, W_N^{2pn} \right]$$

$$= \left[\sum_{n=0}^{\frac{N}{4}-1} f(n)W_N^{2pn} \cdot W_N^n + \sum_{n=0}^{\frac{N}{4}-1} f\left(n+\frac{N}{4}\right) W_N^{\left(n+\frac{N}{4}\right)} \cdot W_N^{2p\left(n+\frac{N}{4}\right)} \right]$$

$$= \left[\sum_{n=0}^{\frac{N}{4}-1} f(n)W_N^n \cdot W_N^{2pn} + \sum_{n=0}^{\frac{N}{4}-1} f\left(n+\frac{N}{4}\right) W_N^n \cdot W_N^{2pn} \cdot W_N^{(2p+1)N/4} \right]$$

$$X(2p + 1) = \sum_{n=0}^{\frac{N}{4}-1} \left[f(n) + W_N^{(2p+1)N/4}\, f\left(n+\frac{N}{4}\right) \right] W_N^{(2p+1)n} \quad \text{...(4.27.6)}$$

3. Again this can also be discussed for even and odd values of 'p'. The results will be same as given in eq. (4.27.3) and (4.27.6).

Case I : $2p + 1 = 2l$ (even)

$$X(2l) = \sum_{n=0}^{\frac{N}{4}-1} \left[f(n) + W_N^{2lN/4}\, f\left(n+\frac{N}{4}\right) \right] W_N^{2ln}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[f(n) + (-1)^{2l} f\left(n+\frac{N}{4}\right) \right] W_N^{2ln}$$

$$X(2l) = \sum_{n=0}^{\frac{N}{4}-1} B(n)\, W_N^{2ln} \quad \text{...(4.27.7)}$$

where $$B(n) = f(n) + f\left(n+\frac{N}{4}\right) \quad \text{...(4.27.8)}$$

Case II :

a. Taking $(2p + 1) = (2l + 1)$ odd

we get

$$X(2l + 1) = \sum_{n=0}^{\frac{N}{4}-1} \left[f(n) + W_N^{(2l+1)N/4} \cdot f\left(n+\frac{N}{4}\right) \right] W_N^{(2l+1)n}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[f(n) - f\left(n+\frac{N}{4}\right) \right] W_N^{2n} \cdot W_N^{2ln}$$

$$X(2l+1) = \sum_{n=0}^{\frac{N}{4}-1} [C(n)W_N^{2ln}] \cdot W_N^{2n} \qquad ...(4.27.9)$$

where, $$C(n) = f(n) - f\left(n + \frac{N}{4}\right)$$

b. Therefore for this stage we get

$$B(0) = f(0) + f(2)$$
$$C(0) = f(0) - f(2)$$
$$B(1) = f(1) + f(3)$$
$$C(1) = f(1) - f(3)$$

c. The SFG for 8-point DFT using DIF FFT algorithm is shown in Fig. 4.27.3.

d. The decimation process is continued until the size of the last DFT is not equal to radix of the FFT. Number of decimation stages (l) is given as

$$l = \log_2 N$$
or $$N = 2^l$$

e. The $X(2l)$ and $X(2l+1)$ are again decomposed in even and odd parts.

f. Since for the 3rd stage, we have $N/4 = 8/4 = 2$ point DFT. This DFT can be drawn by using the butterfly structure as given in DIT-FFT.

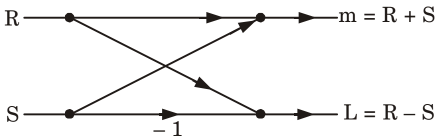


Fig. 4.27.2.

4. The total SFG for all three stages DIF-FFT is shown in Fig. 4.27.3.

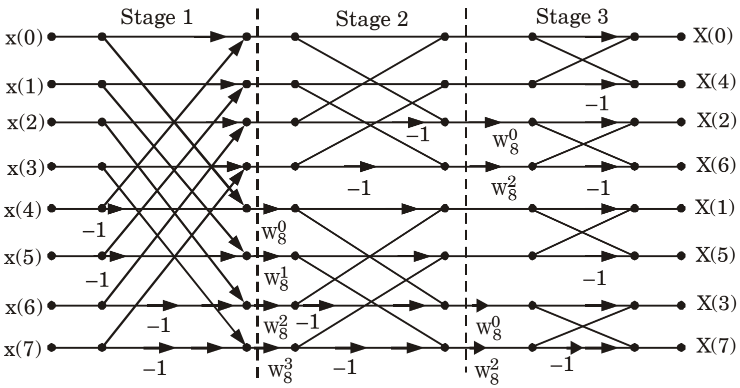


Fig. 4.27.3 SFG for 8-point DFT using DIF-FFT.

Que 4.28. **Determine the 8-point DFT of the following sequence using DIF FFT algorithm :**

$$x(n) = \{1, 2, 3, 4\}$$

AKTU 2017-18, Marks 10

Answer

1. The total flow graph is shown in Fig. 4.28.1.

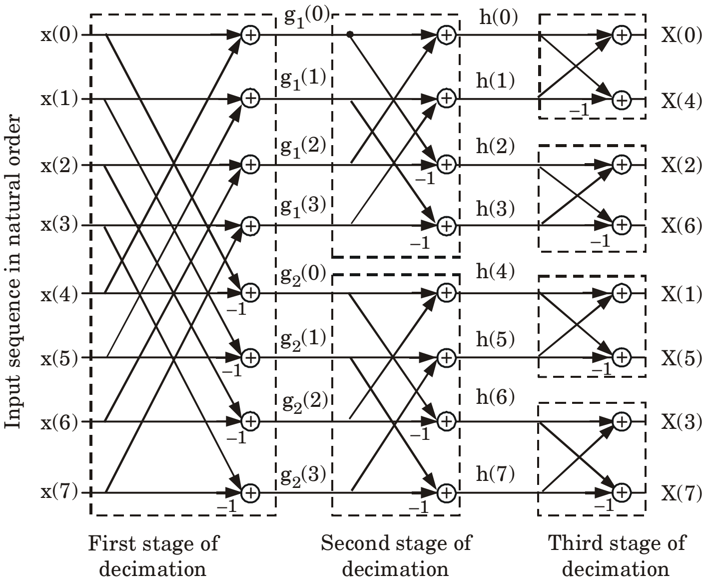


Fig. 4.28.1.

Here $g(n)$ is output of first stage and $h(n)$ is output of second stage.

2. The values of twiddle factor are as follows :

$$W_8^0 = 1,\ W_8^1 = e^{-j\pi/4} = 0.707 - j\,0.707$$

$$W_8^2 = e^{-j\pi/2} = -j \text{ and } W_8^3 = -0.707 - j\,0.707$$

Output of stage 1 :

$$g(0) = X(0) + X(4) = 1 + 0 = 1$$
$$g(1) = X(1) + X(5) = 2 + 0 = 2$$
$$g(2) = X(2) + X(6) = 3 + 0 = 3$$
$$g(3) = X(1) + X(7) = 4 + 0 = 4$$

$$g(4) = [X(0) - X(4)]\ W_8^0 = [1 - 0]\ 1 = 1$$

$$g(5) = [X(1) - X(5)]\ W_8^1 = [2 - 0]\ [0.707 - j\,0.707]$$
$$= 1.414 - j\,1.414$$

$$g(6) = [X(2) - X(6)]\ W_8^2 = [3 - 0]\ (-j) = -3j$$

$$g(7) = [X(3) - X(7)]\ W_8^3 = [4 - 0]\ (-0.707 - j\ 0.707)]$$
$$= -2.828 - j\ 2.828$$

Output of stage 2 :

$$h(0) = g(0) + g(2) = 1 + 3 = 4$$
$$h(1) = g(1) + g(3) = 2 + 4 = 6$$
$$h(2) = [g(0) - g(2)]\ W_8^0 = [1 - 3]\ 1 = -2$$
$$h(3) = [X(2) - X(6)]\ W_8^2 = [2 - 4]\ (-j) = 2j$$
$$h(4) = g(4) + g(6) = 1 - 3j$$
$$h(5) = g(5) + g(7) = 1.414 - j\ 1.414 - 2.828 - j\ 2.828$$
$$= -1.414 - j\ 4.242$$
$$h(6) = [g(4) - g(6)]\ W_8^0 = [1 + 3j] = 1 + 3j$$
$$h(7) = [g(5) - g(7)]\ W_8^2$$
$$= [1.414 - j\ 1.414 + 2.828 + j\ 2.828]$$
$$= (4.242 + j\ 1.414)\ (-j)$$
$$= 1.414 - j\ 4.242$$

Final Output :

$$X(0) = h(0) + h(1) = 4 + 6 = 10$$
$$X(1) = h(4) + h(5) = 1 - 3j - 1.414 - j\ 4.242$$
$$= -0.414 - j\ 7.242$$
$$X(2) = h(2) + h(3) = -2 + 2j$$
$$X(3) = [h(6) + h(7)]\ W_8^0 = [1 + 3j + 1.414 - j\ 4.242]\ (1)$$
$$= 2.414 - j\ 1.242$$
$$X(4) = [h(0) - h(1)]\ W_8^0 = [4 - 6]\ (1) = -2$$
$$X(5) = [h(4) - h(5)]\ W_8^0 = [1 - 3j + 1.414 + j\ 4.242]\ (1)$$
$$= [2.414 + j\ 1.242]$$
$$X(6) = [h(2) + h(3)]\ W_8^0 = [-2 - 2\,j]\ (1) = -2 - 2j$$
$$X(7) = [h(6) - h(7)]\ W_8^0 = [1 + 3j - 1.414 + j\ 4.242]\ (1)$$
$$= -0.414 + j\ 7.242$$

Que 4.29. **Compute the DFT of following 8 point sequence using 4 point radix-2 DIF algorithm.**

$$\boldsymbol{x(n) = \{2, 2, 2, 2, 1, 1, 1, 1\}}$$

AKTU 2015-16, Marks 10

Answer

Given : $x(n) = \{2, 2, 2, 2, 1, 1, 1, 1\}$, $N = 8$

To Find : DFT.

1. $$W_8^0 = e^0 = 1$$
$$W_8^1 = e^{-j\frac{\pi}{4}} = 0.707 - j0.707$$
$$W_8^2 = e^{-j\frac{\pi}{2}} = -j$$
$$W_8^3 = -0.707 - j0.707$$
2. The output of various stages can be easily calculated from 8-point butterfly diagram.
3. **Stage-1 outputs :**
$$g(0) = x(0) + x(4) = 2 + 1 = 3$$
$$g(1) = x(1) + x(5) = 2 + 1 = 3$$
$$g(2) = x(2) + x(6) = 2 + 1 = 3$$
$$g(3) = x(3) + x(7) = 2 + 1 = 3$$
$$g(4) = [x(0) - x(4)]W_8^1 = 1$$
$$g(5) = [x(1) - x(5)]W_8^1 = 0.707 - j0.707$$
$$g(6) = [x(2) - x(6)]W_8^2 = -j$$
$$g(7) = [x(3) - x(7)]W_8^3 = -0.707 - j0.707$$
4. **Stage-2 outputs :**
$$h(0) = g(0) + g(2) = 6$$
$$h(1) = g(1) + g(3) = 6$$
$$h(2) = [g(0) - g(2)]W_8^0 = 0$$
$$h(3) = [g(1) - g(3)]W_8^2 = 0$$
$$h(4) = g(4) + g(6) = 1 - j$$
$$h(5) = g(5) + g(7) = -j1.44$$
$$h(6) = [g(4) + g(6)]W_8^0 = 1 + j$$
$$h(7) = [g(5) - g(7)]W_8^2 = -j1.44$$
5. **Stage-3 outputs or Final outputs :**
$$X(0) = h(0) + h(1) = 12$$
$$X(1) = h(4) + h(5) = 1 - j\,2.414$$
$$X(2) = h(2) + h(3) = 0$$
$$X(3) = [h(6) + h(7)]W_8^0 = 1 - j\,0.414$$
$$X(4) = [h(0) - h(1)]W_8^0 = 0$$
$$X(5) = [h(4) - h(5)]W_8^0 = 1 + j\,0.414$$
$$X(6) = [h(2) - h(3)]W_8^0 = 0$$
$$X(7) = [h(6) - h(7)]W_8^0 = 1 + j\,2.414$$
6. Thus, $$X(k) = \{12, 1 - j\,2.414, 0, 1 - j\,0.414, 0, 1 + j\,0.414, 0, 1 + j\,2.414\}$$

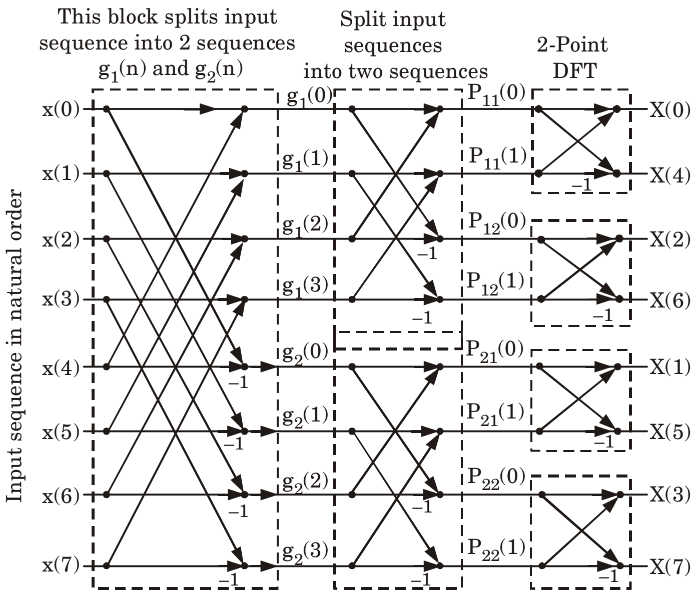


Fig. 4.29.1. Signal flow graph for 8-point DIF FFT.

Que 4.30. **Find the 8 point DFT of the sequence $x(n)$ = {1, 2, 3, 4, 4, 3, 2, 1} using DIF FFT.**

AKTU 2018-19, Marks 07

Answer

The procedure is same as Q. 4.29, Page 4–39C, Unit-4.

(**Ans.** $X(K) = \{20, -3-2\sqrt{2}-j(1+\sqrt{2}), 0, -3+2\sqrt{2}+j(1-\sqrt{2}), 0, -3+2\sqrt{2}-j(1-\sqrt{2}), 0, -3-2\sqrt{2}+j(1+\sqrt{2})\}$)

Que 4.31. **Find the 4-point circular convolution of $x(n)$ and $h(n)$ given by $x(n)$ = {1, 1, 1, 1) and $h(n)$ = { 1, 0, 1, 0} using radix-2 DIF-FFT algorithm.**

AKTU 2019-20, Marks 07

Answer

1. Let
$$y(n) = x(n) \otimes h(n) = \text{IFFT}\{X(k)\,H(k)\}$$
2. Using DIF FFT algorithm, we can find $X(k)$ from the given sequence $x(n)$ as shown in Fig. 4.31.1.

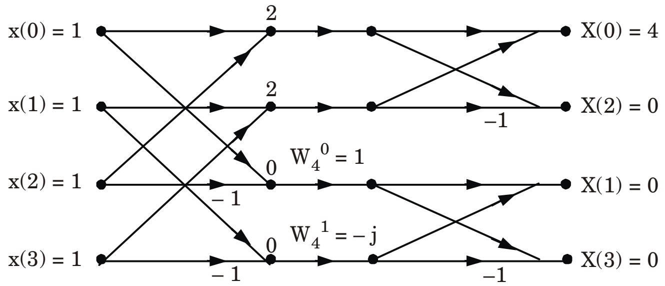


Fig. 4.31.1.

Therefore, $X(k) = (4, 0, 0, 0)$

3. Using DIF FFT algorithm we can find $H(k)$ from the given sequence $h(n)$ as shown in Fig. 4.31.1.

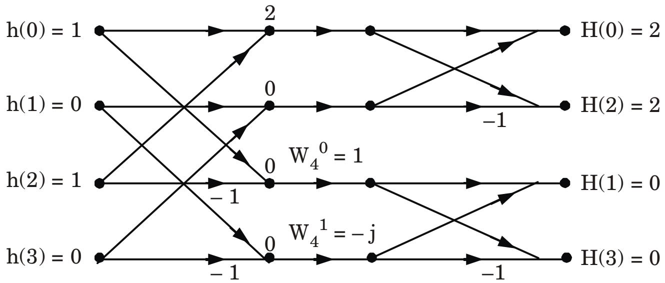


Fig. 4.31.2.

$H(k) = (2, 0, 2, 0)$

4. And $Y(k) = X(k)\, H(k)$

$= (4, 0, 0, 0)\,(2, 0, 2, 0) = (8, 0, 0, 0)$

5. Using IFFT algorithm, we can find $y(n)$ from the $Y(k)$ as shown in Fig. 4.31.3.

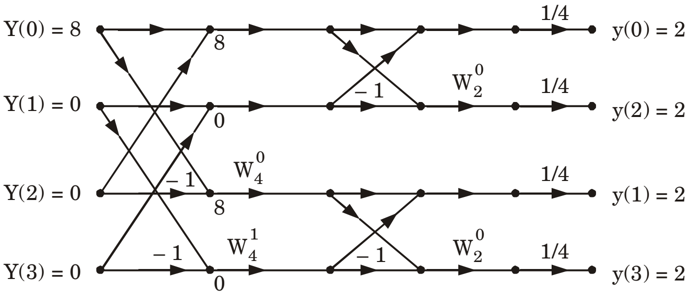


Fig. 4.31.3.

Thus, $y(n) = (2, 2, 2, 2)$

Que 4.32.

i. Compute 4-point DFT of the following sequence using DIF algorithm

$$x(n) = \cos\frac{n\pi}{2}$$

ii. Show that the same algorithm can be used to compute IDFT of $X(k)$ calculated in part (i).

AKTU 2015-16, Marks 10

Answer

Given : $N = 4$ and $x(n) = \cos\dfrac{n\pi}{2}$

To Find : DFT and IDFT.

i. DFT :

1. The sequence $x(n)$ can be obtained by putting $n = 0, 1, 2, 3$

$$x(0) = \cos 0 = 1 \qquad x(2) = \cos\frac{2\pi}{2} = -1$$

$$x(1) = \cos\frac{\pi}{2} = 0 \qquad x(3) = \cos\frac{3\pi}{2} = 0$$

$x(n) = \{1, 0, -1, 0\}$

$$W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k}$$

$$W_4^0 = 1 \text{ and } W_4^1 = e^{-j\pi/2} = -j$$

2. Using DIF FFT algorithm, we can find $X(k)$ from the given sequence $x(n)$ as shown in Fig. 4.32.1.

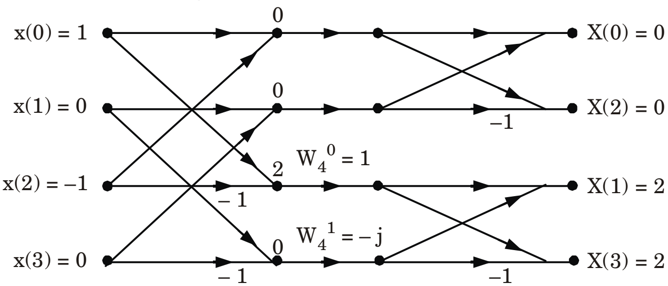


Fig. 4.32.1.

3. $$X(k) = \sum_{k=0}^{N-1} x(n) W_N^{kn}, \; n = 0, 1, \ldots\ldots\ldots, N-1$$

$\therefore$ $$X(k) = \{0, 2, 0, 2\}$$

ii. IDFT :

1. IDFT differs from DFT by

a. Multiplication by $\frac{1}{N}$

b. Negative sign of imaginary point of (W_N)

$$W_4^0 = 1, \quad W_4^{-1} = j$$

2. We know that

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}\ , n = 0,1,.........., N-1 \quad ...(4.32.1)$$

Here, $N = 4$

Therefore, $x(n) = \{1, 0, -1, 0\}$

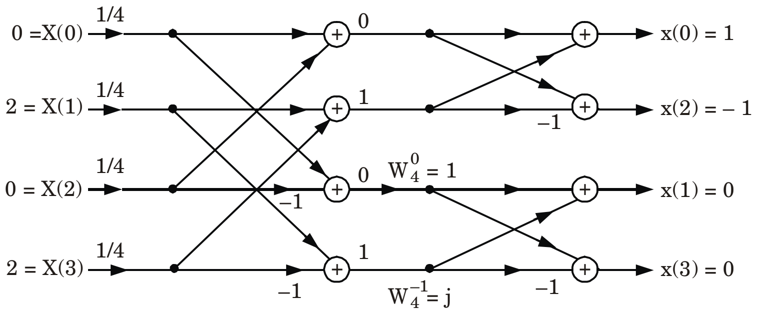


Fig. 4.32.2. SFG of IDFT of X(k).

Que 4.33. **Find the inverse DFT of the sequence :**
$X(k) = \{6, -2 + j2, -2, -2 - j2\}$, using DFT–FFT algorithm.

AKTU 2017-18, Marks 10

Answer

Given : $X(k) = \{6, -2 + j2, -2, -2 - j2\}$
To Find : IDFT.

1. The flow graph for calculation of 4 point IFFT is shown in Fig. 4.33.1.
$X(0) = 6, X(1) = -2 + 2j, X(2) = -2, X(3) = -2 - 2j$

Now,
$$S_0 = \frac{1}{2}X(0) + \frac{1}{4}X(2) = \frac{1}{4}(6) + \frac{1}{4}(-2) = 1$$
$$S_1 = \frac{1}{4}X(0) - \frac{1}{4}X(2) = \frac{1}{4}(6) - \frac{1}{4}(-2) = 2$$
$$S_2 = \frac{1}{4}X(1) + \frac{1}{4}X(3) = \frac{1}{4}(-2+2j) + \frac{1}{4}(-2-2j) = -1$$

$$S_3 = \left[\frac{1}{4}X(1) + \frac{1}{4}X(3)\right](-j)$$

$$= \left[\frac{1}{4}(-2+2j) - \frac{1}{4}(-2-2j)\right](-j) = -1$$

2. The final output is,

$$x(0) = S_0 + S_2 = 1 - 1 = 0$$
$$x(1) = S_1 + S_3 = 2 - 1 = 1$$
$$x(2) = S_0 - S_2 = 1 + 1 = 2$$
$$x(3) = S_1 - S_3 = 2 + 1 = 3$$
$$\therefore \quad x(n) = \{0, 1, 2, 3\}$$

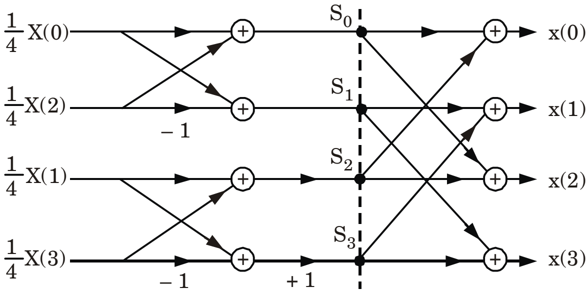


Fig. 4.33.1.

VERY IMPORTANT QUESTIONS

Following questions are very important. These questions may be asked in your SESSIONALS as well as UNIVERSITY EXAMINATION.

Q. 1. What is the relation between DTFT and DFT. Explain the properties of DFT with examples.

Ans. Refer Q. 4.2.

Q. 2. Calculate the DFT of $x(n) = \cos an$.

Ans. Refer Q. 4.3.

Q. 3. The first five point of the 8-point DFT of a real valued sequence are : {0.25, 0.125 – j0.3018, 0, 0.125 – j0.0518, 0}. Determine the remaining three points.

Ans. Refer Q. 4.5.

Q. 4.

i. Compute 4-point DFT of the following sequence using linear transformation matrix $x(n) = (1, 1, -2, -2)$.

ii. Find IDFT $x(n)$ from $X(k)$ calculated in part (i).

Ans. Refer Q. 4.9.

Q. 5. Find circular convolution of the following sequences using concentric circle method.

$$x_1(n) = (1, 2, 2, 1)$$
$$x_2(n) = (1, 2, 3, 4)$$

Ans. Refer Q. 4.13.

Q. 6. Prove that multiplication of the DFTs of two sequences is equivalent to the circular convolution of the two sequences in the time domain.

Ans. Refer Q. 4.16.

Q. 7. If the 10-point DFT of $x(n) = \delta(n) - \delta(n-1)$ and $h(n) = u(n) - u(n-10)$ are $X(k)$ and $H(k)$ respectively, find the sequence $W(n)$ that corresponds to the 10-point inverse DFT of the product $H(k)\,X(k)$.

Ans. Refer Q. 4.19.

Q. 8. Given $x(n) = 2^n$ and $N = 8$, find $X(K)$ using DIT-FFT algorithm. Also calculate the computational reduction factor.

Ans. Refer Q. 4.24.

Q. 9. Derive and draw the flow graph for DIF-FFT algorithm for $N = 8$.

Ans. Refer Q. 4.27.

Q. 10. Find the 4-point circular convolution of $x(n)$ and $h(n)$ given by $x(n) = \{1, 1, 1, 1)$ and $h(n) = \{1, 0, 1, 0\}$ using radix-2 DIF-FFT algorithm.

Ans. Refer Q. 4.31.

☺☺☺

5
UNIT

Multirate Digital Signal Processing (MDSP)

CONTENTS

PART-1

Multirate Digital Signal Processing (MDSP) : Introduction, Decimation, Interpolation.

CONCEPT OUTLINE

- The process of converting a signal from a given rate to different rate is called sampling rate conversion.
- The systems that employ multiple sampling rates in the processing of digital signals are called multirate digital signal processing.
- The process of reducing the sampling rate by a factor D (downsampling by D) is called decimation.
- The process of increasing the sampling rate by an integer factor I (upsampling by I) is called interpolation.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.1. Explain multirate digital signal processing and write its application.

OR

What is multirate digital signal processing ? Discuss about application areas of it.

AKTU 2018-19, Marks 07

Answer

A. Multirate digital signal processing :

1. The process of converting a signal from a given rate to a different rate is called sampling rate conversion. The systems that employ multiple sampling rates in the processing of digital signals are called multirate digital signal processing systems.
2. Different sampling rates can be obtained using an upsampler and downsampler. The basic operations in multirate processing to achieve this are decimation and interpolation.
3. Decimation is used for reducing the sampling rate and interpolation is for increasing the sampling rate.

B. Applications :

i. Sub-band coding (speech, image)

ii. Voice privacy using analog phone lines

iii. Signal compression by subsampling

iv. A/D, D/A converters.

C. Application area of multirate signal processing :

There are various areas in which multirate signal processing is used :

i. Communication systems

ii. Speech and audio processing systems

iii. Antenna systems, and

iv. Radar systems

Que 5.2. Write a short note on decimation.

Answer

1. The process of reducing the sampling rate by a factor D (downsampling by D) is called decimation.
2. The decimation process is illustrated in Fig. 5.2.1. The input sequence $x(n)$ is passed through a low pass filter, characterized by the impulse response $h(n)$ and a frequency response $H_D(\omega)$, which ideally satisfies the condition

$$H_D(\omega) = \begin{cases} 1, & |\omega| \le \pi / D \\ 0, & \text{otherwise} \end{cases} \quad \text{...(5.2.1)}$$

3. Thus the filter eliminates the spectrum of $X(\omega)$ in the range $\pi/D < \omega < \pi$. Of course, the implication is that only the frequency components of $x(n)$ in the range $|\omega| \le \pi/D$ are of interest in the further processing of the signal.
4. The output of the filter is a sequence $v(n)$ given as

$$v(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k) \quad \text{...(5.2.2)}$$

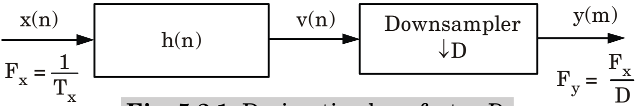


Fig. 5.2.1. Decimation by a factor D.

which is then downsampled by the factor D to produce $y(m)$. Thus

$$y(m) = v(mD)$$

$$= \sum_{k=0}^{\infty} h(k)\, x(mD - k) \quad \text{...(5.2.3)}$$

5. Although the filtering operation on $x(n)$ is linear and time invariant, the downsampling operation in combination with the filtering result in a time-variant system.

   For example : $x(n)$ is a input signal as shown in Fig. 5.2.2(a) and the decimation factor 3 then the output signal (decimated signal) $y(n)$ as shown in Fig. 5.2.2(b).

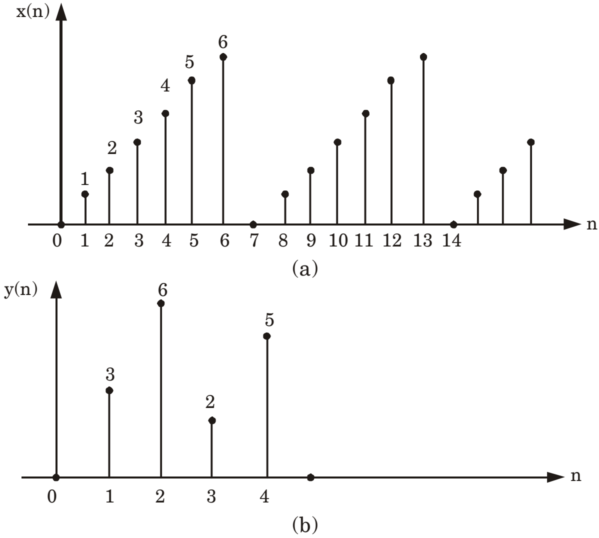


Fig. 5.2.2. (a) Input signal (b) Output signal (decimated signal).

Que 5.3. **Discuss polyphase implementation of decimation filters.**

Answer

1. One of the important applications of the polyphase decomposition is in the implementation of filters whose output is then downsampled as indicated in Fig. 5.3.1.
2. In the most straightforward implementation of Fig. 5.3.1, the filter computes an output sample at each value of n, but then only one of every D output points is retained.

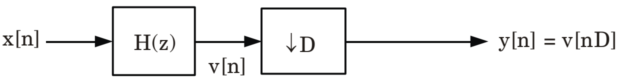


Fig. 5.3.1. Decimation system.

3. To obtain a more efficient implementation, we can use polyphase decomposition of the filter by expressing $h[n]$ in polyphase form with polyphase components

i.e., $e_k[n] = h[nD + k]$ ...(5.3.1)

$$H(z) = \sum_{k=0}^{M-1} E_k(z^M)z^{-k} \quad ...(5.3.2)$$

with this decomposition and the fact that downsampling commutes with addition.

4. Fig. 5.3.1 can be redrawn as shown in Fig. 5.3.2. To illustrate the advantage of Fig. 5.3.3 compared with Fig. 5.3.2, suppose that the input $x[n]$ is clocked at a rate of 1 sample per unit time and that $H(z)$ is an N-point FIR filter.
5. In the straightforward implementation of Fig. 5.3.2, we require N multiplications and $(N-1)$ additions per unit time. In the system of Fig. 5.3.3, each of the filters $E_k(z)$ is of length N/D, and their inputs are clocked at a rate of 1 per D units of time.

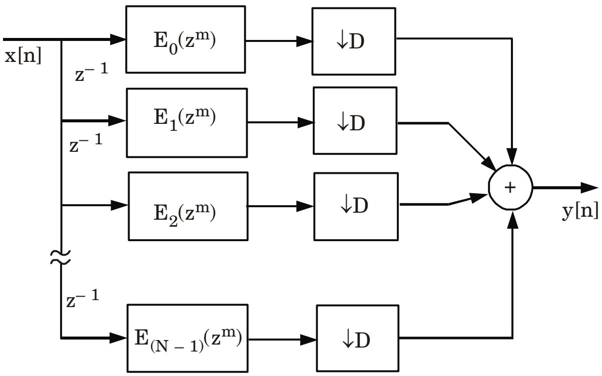


Fig. 5.3.2. Implementation of decimation filter using polyphase decomposition.

6. Consequently, each filter requires $\frac{1}{D}\left(\frac{N}{D}\right)$ multiplications per unit time and $\frac{1}{D}\left(\frac{N}{D}-1\right)$ additions per unit time, and the entire system then requires (N/M) multiplications and $\left(\frac{N}{D}-1\right)+(D-1)$ additions per unit time. Thus we can achieve a significant savings for some values of D and N.

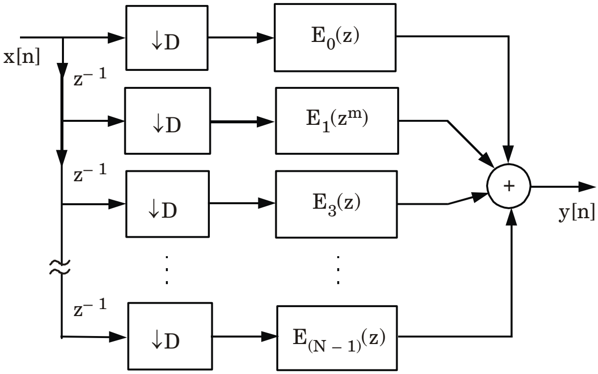


Fig. 5.3.3. Implementation of decimation filter after applying the downsampling identity to the polyphase decomposition.

Que 5.4. **A one stage decimator is characterized by the following :**

Decimation factor = 3

Anti-aliasing filter coefficients

$$h(0) = -0.06 = h(4)$$

$$h(1) = 0.30 = h(3)$$

$$h(2) = 0.62$$

Given the data, $x(n)$ with successive values [6, – 2, – 3, 8, 6, 4, – 2], calculate and list the filtered output $w(n)$ and the output of the decimator $y(m)$.

Answer

Given : $x(n) = [6, -2, -3, 8, 6, 4, -2]$, $D = 3$; $h(0) = -0.06 = h(4)$, $h(1) = 0.30 = h(3)$, $h(2) = 0.62$
To Find : Filtered output $w(n)$, output of decimator $y(m)$

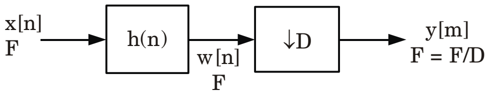


Fig. 5.4.1. Decimation process for the factor D = 3.

1. Filtered output, $w(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$

2. Decimator output, $y(m) = \sum_{k=-\infty}^{\infty} h(Dm-n)\, x(n)$

$$D = 3;\ x[n] = [6, -2, -3, 8, 6, 4, -2]$$

Filtered output :

$$w(0) = h(0)x(0) = (-0.06)(6) = -0.36$$
$$w(1) = h(0)x(1) + h(1)x(0) = (-0.06) + (-2)(0.3)(6) = 1.92$$
$$w(2) = h(0)x(2) + h(1)x(1) + h(2)x(0)$$
$$= (-0.06)(-3) + (0.3)(-2) + (0.62)(6) = 3.3$$
$$w(3) = h(0)x(3) + h(1)x(2) + h(2)x(1) + h(3)x(0)$$
$$= (-0.06)(8) + (0.3)(-3) + (0.06)(-2) + (0.3)(6) = -0.82$$
$$w(4) = h(0)x(4) + h(1)x(3) + h(2)x(2) + h(3)x(1) + h(4)x(0)$$
$$= (-0.06)(6) + (0.3)(8) + (0.62)(-3) + (0.3)(-2) + (-0.06)(-6) = -0.78$$
$$w(5) = h(0)x(5) + h(1)x(3) + h(2)x(2) + h(3)x(2) + h(3)x(2) + h(4)x(1)$$
$$= (-0.06)(4) + (0.3)(6) + (0.62)(8) + (0.3)(-3) + (-0.06)(-2) = 5.74$$
$$w(6) = h(0)x(6) + h(1)x(5) + h(2)x(4) + h(3)x(3) + h(4)x(2)$$
$$= -(0.06)(-2) + (0.3)(4) + (0.62)(6) + (0.3)(8) + (-0.06)(-3) = 7.62$$
$$w(7) = h(1)x(6) + h(2)x(5) + h(3)x(4) + h(4)x(3)$$
$$= (0.3)(-2) + (0.62)(4) + (0.3)(6) + (-0.06)(8) = 3.2$$
$$w(8) = h(2)x(6) + h(3)x(5) + h(4)x(4)$$
$$= (0.62)(-2) + (0.3)(4) + (-0.06)(6) = -0.4$$
$$w(9) = h(3)x(6) + h(4)x(5) = (0.3)(-2) + (-0.06)(4) = -0.84$$
$$w(10) = h(4)x(6) = (-0.06)(-2) = 0.12$$

Decimator output :

$$y(0) = h(0)x(0) = -0.36$$
$$y(1) = h(3)x(0) + h(2)x(1) + h(1)x(2) + h(0)x(3)$$
$$= (0.3)(6) + (0.62)(-2) + (0.3)(-3) + (-0.06)(8) = -0.82$$
$$y(2) = h(4)x(2) + h(3)x(3) + h(2)x(4) + h(1)x(5) + h(0)x(6)$$
$$= (-0.06)(-3) + (0.3)(8) + (0.62)(6) + (0.3)(4) + (-0.62)(-2) = 7.62$$
$$y(3) = h(4)x(5) + h(3)x(6) = (-0.06)(4) + (0.3)(-2) = -0.84$$

Filtered output,

$$w(n) = [-0.36, 1.92, 3.3, -0.82, -0.78, 5.74, 7.62, 3.2, -3.2, -0.4, -0.84, 0.12]$$

Decimator output, $y(m) = [-0.36, -0.82, 7.62, -0.84]$

Que 5.5. **Explain the phenomenon decimation and interpolation by suitable example.**

AKTU 2019-20, Marks 07

Answer

A. Decimation : Refer Q. 5.2, Page 5–3C, Unit-5.

B. Interpolation :

1. The process of increasing the sampling rate by an integer factor I (upsampling by I) is called interpolation.
2. An increase in the sampling rate by an integer factor of I can be accomplished by interpolating $I - 1$ new samples between successive values of the signal.

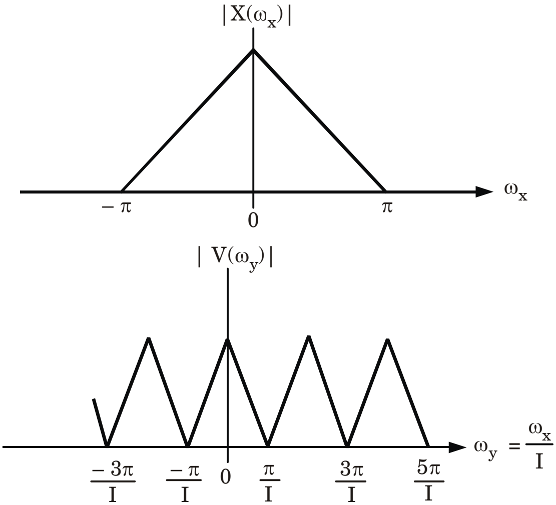


Fig. 5.5.1. Spectra of $x(n)$ and $v(n)$ where $V(\omega_y) = X(\omega_y I)$.

3. Let $v(m)$ denote a sequence with a rate $F_y = I\,F_x$, which is obtained from $x(n)$ by adding $I - 1$ zeros between successive values of $x(n)$.

Thus
$$V(m) = \begin{cases} x(m/I), & m = 0, \pm I, \pm 2I, \ldots \\ 0, & \text{otherwise} \end{cases} \quad \ldots(5.5.1)$$

and its sampling rate is identical to the rate of $y(m)$. This sequence has a z-transform

$$V(m) = \sum_{m=-\infty}^{\infty} v(m) z^{-m} = \sum_{m=-\infty}^{\infty} x(m) z^{-mI} \quad \ldots(5.5.2)$$
$$= X(z^{I})$$

4. The corresponding spectrum of $v(m)$ is obtained by evaluating eq. (5.5.2) on the unit circle.

Thus
$$V(\omega_y) = X(\omega_y I) \quad \ldots(5.5.3)$$

where ω_y denotes the frequency variable relative to the new sampling rate F_y (*i.e.*, $\omega_y = 2\pi F / F_y$).

5. Now the relationship between sampling rates is $F_y = I F_x$ and hence, the frequency variables ω_x and ω_y are related according to the formula

$$\omega_y = \frac{\omega_x}{I} \qquad \text{...(5.5.4)}$$

6. The spectra $X(\omega_x)$ and $V(\omega_y)$ are illustrated in Fig. 5.5.1. We observed that the sampling rate increase, obtained by the addition of $I - 1$ zero samples between successive values of $x(n)$, results in a signal whose spectrum $V(\omega_y)$ is an I-fold periodic repetition of the input signal spectrum $X(\omega_x)$.

Que 5.6. **Explain the process of polyphase implementation of interpolation filters.**

Answer

1. The efficient implementation of an interpolator, which is realized by first inserting $I - 1$ zeros between successive samples of $x(n)$ and then filtering the resulting sequence as shown in Fig. 5.6.1.
2. The major problem with this structure is that the filter computations are performed at the high sampling rate IF_x.
3. The desired simplification is achieved by first replacing the filter in Fig. 5.6.1 with the transpose polyphase structure in Fig 5.6.2, as illustrated in Fig. 5.6.2(*a*). Then, we use the second noble identity (see Fig. 5.6.1) to obtain the structure in Fig. 5.6.2(*b*).

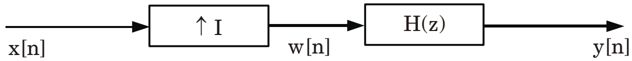
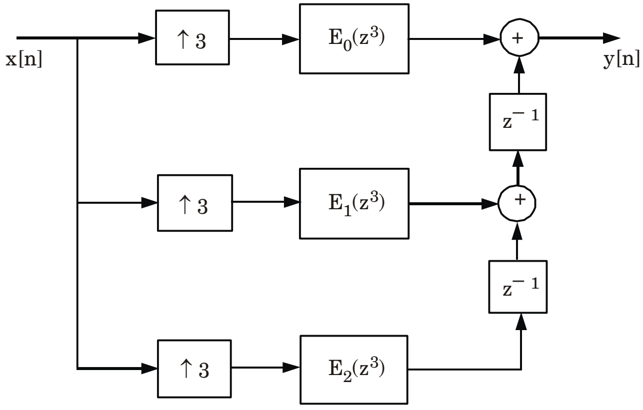
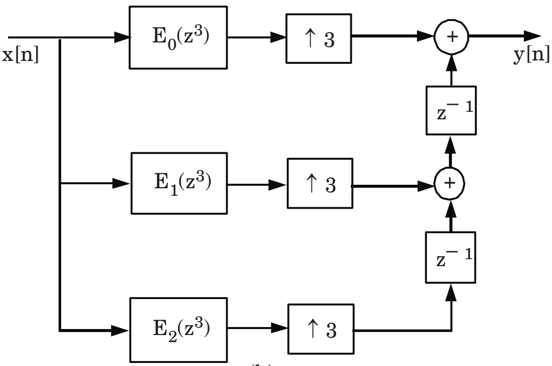


Fig. 5.6.1. Interpolation system.



(a)



(b)

Fig. 5.6.2. Implementation of an interpolation system using a polyphase structure before (a) and after (b) the use of the second noble indentity.

4. Thus, all the filtering multiplications are performed at the low rate F_x. So the structure for an interpolator can be obtained by transposing the structure of decimator, and vice versa.
5. For every input sample, the polyphase filters produce I output samples $y_0(n), y_1(n),...., y_{I-1}(n)$. Because the output $y_i(n)$ of the i^{th} filter is followed by $(I - 1)$ zeros and it is delayed by i^{th} samples, the polyphase filters contribute non-zero samples at different time slots.

Que 5.7. **Obtain the two-fold expanded signal $y(n)$ of the input signal $x(n)$**

$$x(n) = \begin{cases} n, & n > 0 \\ 0, & \text{otherwise} \end{cases}$$

Answer

1. The output signal $y(n)$ is given by

$$y(n) = \begin{cases} x(n / I) & , \quad n = \text{multiples of } I \\ 0 & , \quad \text{otherwise} \end{cases}$$

where $M = 2$.

$$x(n) = 0, 1, 2, 3, 4, 5, ...$$

$$y(n) = 0, 0, 1, 0, 2, 0, 3, 0, 4, 5, 0, ...$$

2. In general, to obtain the expanded signal $y(n)$ by a factor I, $(I - 1)$ zeros are inserted between the samples of the original signal $x(n)$.

3. The z-transform of the expanded signal is

$$Y(z) = X(z^I),\ I = 2.$$

4. The input and output signals are shown in Fig. 5.7.1.

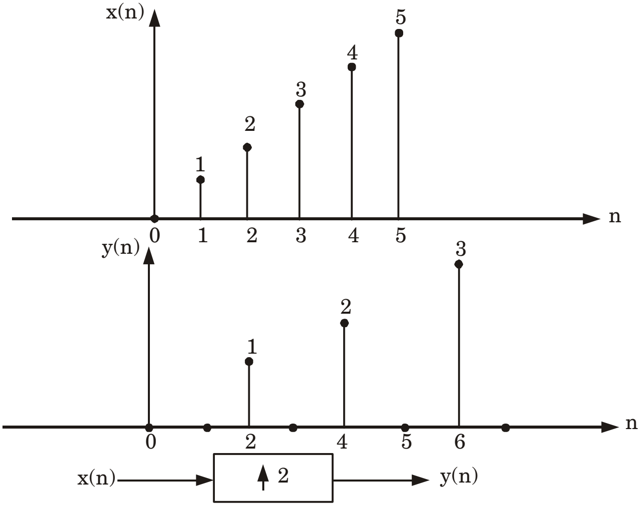


Fig. 5.7.1. Interpolation process with a factor 2.

Que 5.8. **Discuss modulation free method for decimation and interpolation.**

Answer

A. Modulation free method for decimation :

1. Decimation of the sampled bandpass signal whose spectrum is shown in Fig. 5.8.1. Note that the signal spectrum is confined to the frequency range,

$$\frac{m\pi}{D} \leq \omega \leq \frac{(m+1)\pi}{D} \quad ...(5.8.1)$$

where m is a positive integer.

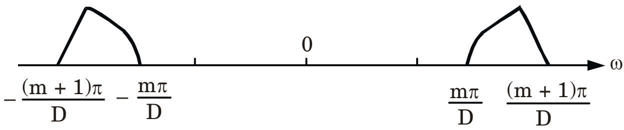


Fig. 5.8.1. Spectrum of a bandpass signal.

2. A bandpass filter would normally be used to eliminate signal frequency components outside the desired frequency range. Then direct decimation of the bandpass signal by the factor D results in the spectrum shown in Fig. 5.8.2(a), for m odd, and Fig. 5.8.2(b) for m even.

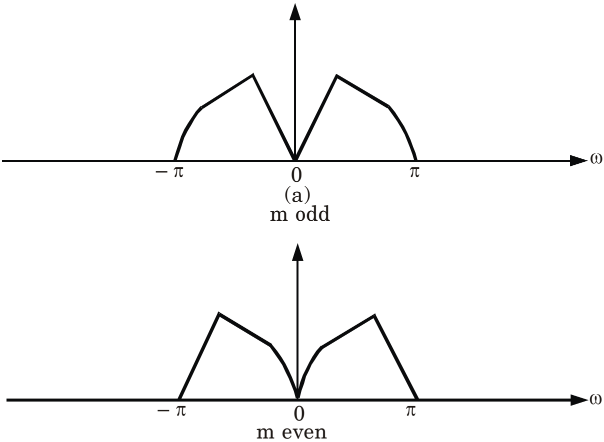


Fig. 5.8.2. Spectrum of decimated bandpass signal.

3. In the case where m is odd, there is an inversion of the spectrum of the signal. This inversion can be undone by multiplying each sample of the decimation signal by $(-1)_n$, n =0, 1, ...

B. Modulation free method for interpolation :

1. Modulation free interpolation of a bandpass signal by an integer factor I can be accomplished in a similar manner. The process of upsampling by inserting zeros between samples of $x(n)$ produces I image in the band $0 \leq \omega \leq \pi$. The desired image can be selected by bandpass filtering.
2. Note that the process of interpolation also provides us with the opportunity to achieve frequency translation of the spectrum.

PART-2

Sampling Rate Conversion : Single and Multistage, Applications of MSDP-Subband Coding of Speech Signals, Quadrature Mirror Filters, Advantages of MSDP.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.9. **Discuss about interpolation and sampling rate conversion in detail.**

AKTU 2018-19, Marks 07

Answer

A. Interpolation : Refer Q. 5.5, Page 5–7C, Unit-5.

B. Sampling rate conversion :

1. The process of sampling rate conversion in the digital domain can be viewed as a linear filtering operation, as illustrated in Fig. 5.9.1(a).
2. The input signal $x(n)$ is characterized by the sampling rate $F_x = 1/T_x$ and the output signal $y(m)$ is characterized by the sampling rate $F_y = 1/T_y$, where T_x and T_y are the corresponding sampling intervals.
3. In the main part of our treatment, the ratio F_y / F_x is constrained to be rational,

$$\frac{F_y}{F_x} = \frac{I}{D}$$

where D and I are relatively prime integers.

4. We shall show that the linear filter is characterized by a time-variant impulse response, denoted as $h(n, m)$. Hence the input $x(n)$ and the output $y(m)$ are related by the convolution summation for time-variant systems.

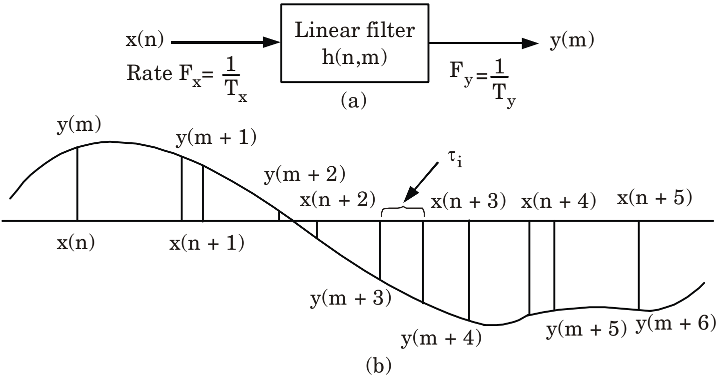


Fig. 5.9.1. Sampling rate conversion viewed as a linear filtering process.

5. The sampling rate conversion process can also be understood from the point of view of digital resampling of the same analog signal. Let $x(t)$ be the analog signal that is sampled at the first rate F_x to generate $x(n)$.
6. The goal of rate conversion is to obtain another sequence $y(m)$ directly from $x(n)$, which is equal to the sampled values of $x(t)$ at a second rate F_y.

7. In Fig. 5.9.1(b), $y(m)$ is a time-shifted version of $x(n)$. Such a time shift can be realized by using a linear filter that has a flat magnitude response and a linear phase response (*i.e.*, it has frequency response of $e^{-j\tau\omega}$, where τ_i is the time delay generate by the filter).
8. If the two sampling rates are not equal, the required amount of time shifting will vary from sample to sample, as shown in Fig. 5.9.1(b).
9. Thus, the rate converter can be implemented using a set of linear filters that have the same flat magnitude response but generate different time delays.
10. Before considering the general case of sampling rate conversion, we shall consider two special cases. One is the case of sampling rate reduction by an integer factor D and the second is the case of a sampling rate increase by an integer factor I.

Que 5.10. Discuss the method for sampling rate conversion by a rational factor of *I/D*.

Answer

1. The general case of sampling rate conversion by the rational factor I/D is accomplished by cascading an interpolation with a decimator, as illustrated in Fig. 5.10.1.
2. We emphasize that the importance of performing the interpolation first and the decimation second, is to preserve the desired spectral characteristics of $x(n)$.
3. Furthermore, with the cascade configuration illustrated in Fig. 5.10.1, the two filter with impulse response $\{h_u(I)\}$ and $\{h_d(I)\}$ are operated at the same rate, namely IF_x and hence can be combined into a signal low pass filter with impulse response $h(I)$ as illustrated in Fig. 5.10.2.
4. The frequency response $H(\omega_v)$ of the combined filter must incorporate the filtering operations for both interpolation and decimation, and hence it should ideally posses the frequency response characteristic.

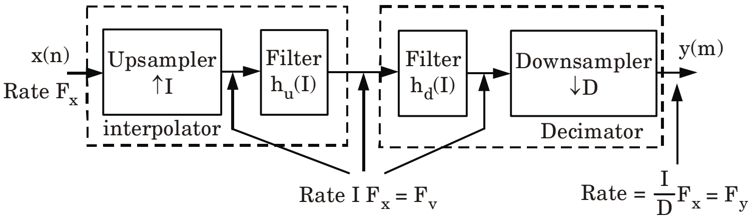


Fig. 5.10.1. Mehtod for sampling rate conversion by a factor *I/D*.

$$H(\omega_v) = \begin{cases} I, & 0 \le |\omega_v| \le \min(\pi / D, \pi / I) \\ 0, & \text{otherwise} \end{cases} \quad ...(5.10.1)$$

where $\omega_v = 2\pi F/F_v = 2\pi F/I F_x = \omega_x / I.$

5. In the time domain, the output of the upsampler is the sequence

$$v(l) = \begin{cases} x(l/I), & l = 0, \pm I, \pm 2I, ... \\ 0, & \text{otherwise} \end{cases} \quad ...(5.10.2)$$

and the output of the linear time-invariant filter is

$$w(l) = \sum_{k=-\infty}^{\infty} h(l-k)\, v(k)$$

$$= \sum_{k=-\infty}^{\infty} h(l-kI)\, x(k) \quad ...(5.10.3)$$

6. Finally, the output of the sampling rate converter is the sequence $\{y(m)\}$, which is obtained by downsampling the sequence $\{w(l)\}$ by a factor of D. Thus

$$y(m) = w(mD)$$

$$= \sum_{k=-\infty}^{\infty} h(mD-k)\, x(k) \quad ...(5.10.4)$$

7. It is illuminating to express eq. (5.10.4) in a different form by making a change in variable.

Let,

$$k = \left[\frac{mD}{I}\right] - n \quad ...(5.10.5)$$

With this change in variable, eq. (5.10.4) becomes

$$y(m) = \sum_{n=-\infty}^{\infty} h\left(mD - \left[\frac{mD}{I}\right]I + nI\right) x\left(\left[\frac{mD}{I}\right] - n\right) \quad ...(5.10.6)$$

we note that

$$mD - \left[\frac{mD}{I}\right]I = mD \quad \text{modulo } I$$

$$= (mD)_I$$

8. Consequently, eq. (5.10.6) can be expressed as

$$y(m) = \sum_{n=-\infty}^{\infty} h(nI + (mD)_I) \times \left(\left[\frac{mD}{I}\right] - n\right) \quad ...(5.10.7)$$

9. It is apparent from this form that the output $y(m)$ is obtained by passing the input sequence $x(n)$ through a time-variant filter with impulse response

$$g(n, m) = h(nI + (mD)_I) \qquad -\infty < m, n < \infty$$

where $h(k)$ is the impulse response of the time-invariant low pass filter operating at the sampling rate IF_x.

10. We further observe, that for any integer K,

$$g(n, m + KI) = h(nI + (mD + kDI)_I)$$

$$= h(nI + (mD)_I) \quad \text{...(5.10.8)}$$
$$= g(n, m)$$

11. Hence $g(n, m)$ is periodic in the variable m with period I.
12. The frequency-domain relationship can be obtained by combining the results of the interpolation and decimation processes. Thus the spectrum at the output of the linear filter with impulse response $h(l)$ is

$$V(\omega_v) = H(\omega_v)\, X(\omega_v I)$$
$$= \begin{cases} IX(\omega_v I), & 0 \le |\omega_v| \le \min(\pi/D, \pi/I) \\ 0, & \text{otherwise} \end{cases} \quad \text{...(5.10.9)}$$

13. The spectrum of the output sequence $y(m)$, obtained by decimating the sequence $v(n)$ by a factor of D, is

$$Y(\omega_y) = \frac{1}{D}\sum_{k=0}^{D-1} V\left(\frac{\omega_y - 2\pi k}{D}\right) \quad \text{...(5.10.10)}$$

where $\omega_y = D\omega_v$.

14. Since the linear filter prevents aliasing as implied by (5.10.10), the spectrum of the output sequence given by (5.10.11) reduces to

$$Y(\omega_y) = \begin{cases} \dfrac{1}{D} X\left(\dfrac{\omega_y}{D}\right), & 0 \le |\omega_y| \le \min\left(\pi, \dfrac{\pi D}{I}\right) \\ 0, & \text{otherwise} \end{cases} \quad \text{...(5.10.11)}$$

Que 5.11. What is the process of sampling rate conversion by an arbitrary factor (first-order approximation) ?

Answer

1. Let us denote the arbitrary conversion rate by r and suppose that the input to the rate converter is the sequence $\{x(n)\}$. We need to generate a sequence of output samples separated in time by T_x / r, where T_x is the sample interval for $\{x(n)\}$.
2. By constructing a polyphase filter with a large number of subfilters as just described. We can approximate such a sequence with a non-uniformly spaced sequence. Without loss of generally, we can express $1/r$ as

$$\frac{1}{r} = \frac{k}{I} + \beta$$

where k and I are positive integers and β is a number in the range.

$$0 < \beta < \frac{1}{I}$$

3. Consequently, $1/r$ is bounded from above and below as

$$\frac{k}{I} < \frac{1}{r} < \frac{k+1}{I}$$

4. I correspond to the interpolation factor, which will be determined to satisfy the specification on the amount of tolerable distortion introduced by rate conversion. I is also equal to the number of polyphase filters.
5. In general, to perform rate conversion by a factor r, we employ a polyphase filter to perform interpolation and therefore to increase the frequency of the original sequence of a factor of I. The time spacing between the samples of the interpolated sequence is equal to T_x/I.
6. If the ideal sampling time of the m^{th} sample, $y(m)$, of the desired output sequence is between the sampling times of two samples of the interpolated sequence, we select the sample closer to $y(m)$ as its approximation.
7. Let us assume that the m^{th} selected sample is generated by the i^{th} subfilter using the input samples $x(n)$, $x(n-1)$, ... , $x(n-K+1)$ in the delay line.
8. The normalized sampling time error (*i.e.*, the time difference between the selected sampling time and the desired sampling time normalized by T_x) is denoted by t_m.

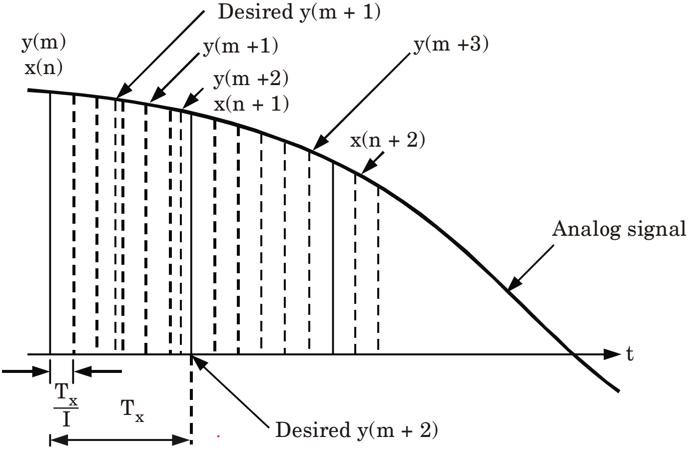


Fig. 5.11.1. Sample rate conversion by use of first-order approximation.

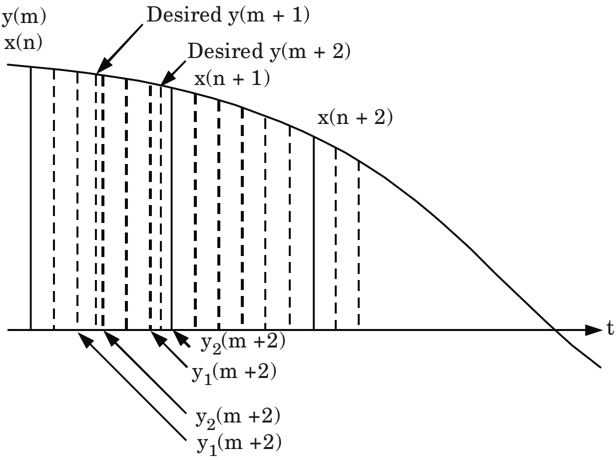
9. The sign of t'', is positive if the desired sampling time leads the selected sampling time, and negative otherwise. It is easy to show that $|t_m| < 0.5/I$. The normalized time advance from the m^{th} output $y(m)$ to the $(m+1)^{\text{st}}$ output $y(m+1)$ is equal to $(1/r) + t_m$.

10. To compute the next output, we first determine a number closest to $i_m/I + 1/r + t_m + k_m/I$ that is of the form $l_{m+1} + i_{m+1}/I$, where both l_{m+1} and i_{m+1} are integers and $i_{m+1} < I$.
11. Then, the $(m + 1)^{th}$ output $y(m + 1)$ is computed using the $(i_{m+I})^{th}$ subfilter after shifting the signal in the delay line by l_{m+1} input samples. The normalized timing error for the $(m + 1)^{th}$ sample is $t_{m+1} = (i_m /I + 1/r + t_m) - (l_{m+1} + i_{m+1} /I)$.
12. By increasing the number of subfilters used, we can arbitrarily increase the conversion accuracy. However, we also require more memory to store the large number of filter coefficients.

Que 5.12. **Describe linear interpolation method.**

Answer

1. Linear interpolation method is also called second-order approximation. The disadvantage of the first-order approximation method is the large number of subfilters needed to achieve a specified distortion requirement.
2. In linear interpolation, we achieve the same performance with reduced number of subfilters. If we denote these two samples by $y_1(m)$ and $y_2(m)$ and use linear interpolation, we can compute the approximation to the desired output as
$$y(m) = (1 - \alpha_m)\, y_1(m) + \alpha_m y_2(m) \quad \text{...(5.12.1)}$$
where $\alpha_m = I\, t_m$. Note that $0 \le \alpha_m \le 1$.
3. The implementation of linear interpolation is similar to that for the first order approximation. Normally, both $y_1(m)$ and $y_2(m)$ are computed using the i^{th} and $(i - 1)^{th}$ subfilters, respectively, with the same set of input data samples in the delay line.
4. The only exception is in the boundary case, where $i = I - 1$, in this case we use the $(I - 1)^{th}$ subfilter to compute $y_1(m)$, but the second sample $y_2(m)$ is computed using the zeroth subfilter after new input data are shifted into the delay line.
5. To analyze the error introduced by the second-order approximation, we first write the frequency responses of the desired filter and the two subfilters used to compute $y_1(m)$ and $y_2(m)$, as $e^{j\tau\omega}$, $e^{j\tau(\omega - t_m)}$, and $e^{j\omega(t - t_m + 1/r)}$, respectively. Because linear interpolation is used for a linear operation.
6. We can also use linear interpolation to compute the frequency response of the filter that generates $y(m)$ as
$$(1 - I\, t_m)e^{j\omega(\tau - t_m)} + I\, t_m e^{j\omega(\tau - t_m + 1/I)} = e^{j\omega\tau}[(1 - \alpha_m)e^{-j\omega t_m} + a_m e^{j\omega(-t_m + 1/I)}] \quad \text{...(5.12.2)}$$
$$= e^{j\omega\tau}(1 - \alpha_m)(\cos \omega t_m - j \sin \omega t_m) + e^{j\omega\tau}\, \alpha_m[\cos \omega(1/I - t_m) + j \sin \omega(1/I - t_m)]$$
7. By ignoring high-order errors, we can write the difference between the desired frequency responses and the one given by eq. (5.12.2) as



$$y(m+1) = (1-\alpha_m)\, y_1(m) + \alpha_m y_2(m)$$

$$\alpha_m = I\, t_m$$

Fig. 5.12.1. Sample rate conversion by use of linear interpolation.

$$e^{j\omega\tau} - (1-\alpha_m)\, e^{j\omega(\tau - t_m)} - \alpha_m e^{j\omega(t - t_m + 1/I)}$$

$$= e^{j\omega\tau}\{[1-(1-\alpha_m)\cos\omega t_m - a_m \cos\omega(1/I - t_m)]$$

$$+\ j[(1-\alpha_m)\sin\omega t_m - \alpha_m \sin\omega(1/I - t_m)]\} \quad ...(5.12.3)$$

$$\approx e^{\tau\omega t}\left[\omega^2(1-\alpha_m)\frac{\alpha_m}{I^2}\right]$$

8. Using $(1-\alpha_m)\alpha_m \leq \dfrac{1}{4}$, we obtain an upper bound for the total error power as

$$p_e = \frac{1}{2\pi}\int_{-\omega_x}^{\omega_x} |X(\omega)[e^{j\omega\tau} - (1-\alpha_m)e^{j\omega(\tau - t_m)} - \alpha_m e^{j\omega(\tau - t_m + 1/I)}]|^2\, d\omega$$

$$p_e \approx \frac{1}{2\pi}\int_{-\omega_x}^{\omega_x} \left| X(\omega)e^{j\omega\tau}\left[\omega^2(1-\alpha_m)\frac{\alpha_m}{I^2}\right]\right|^2 d\omega \quad ...(5.12.4)$$

$$p_e \leq \frac{1}{2\pi}\int_{-\omega_x}^{\omega_x} A^2\left(\frac{0.25}{I^2}\right)^2 \omega^4 d\omega = \frac{A^2\omega_x^5}{80\pi\, I^4}$$

9. This result indicates that the error magnitude is inversely proportional to I^2. Hence we call the approximation using linear interpolation a second-order approximation.
10. Using eq. (5.12.4) and eq. (5.12.1), the ratio of signal-to-distortion due to a sampling time error for the second-order approximation, denoted by SD_1R2, is bounded from below as

$$SD_1R2 = \frac{P_s}{P_e} \geq \frac{80 I^4}{\omega_x^4} \quad ...(5.12.5)$$

11. Therefore, the signal-to-distortion ratio is proportional to the fourth power of the number of subfilters.

Que 5.13. **Suppose that the input signal has a flat spectrum between – 0.8π and 0.8π. Determine the number of subfilters to achieve a signal to distortion ratio of 50 dB when linear interpolation is applied.**

Answer

Given : SD_1R = 50 dB, $\omega_x = 0.8\pi$

To Find : Number of subfilter.

1. Given, $SD_1R = 50$ dB
2. We know. $SD_1R = 80\, I^4/\omega_x^4$

$$10^5 = 80\, I^4/\omega_x^4$$

3. Thus $I \approx \omega_x \sqrt[4]{\dfrac{10^5}{80}} \approx 15$ subfilters.

Que 5.14. **Write a note on multistage implementation of sampling rate conversion.**

Answer

1. In practical applications of sampling rate conversion we often encounter decimation factor and interpolation factors that are much larger than unity. In this section, we consider methods for performing sampling rate conversion for either $D >> 1$ and/or $I >> 1$ in multiple stages.
2. First, let us consider interpolation by a factor $I >> 1$ and let us assume that I can be factored into a product of positive integers as

$$I = \sum_{i=i}^{L} D_i \qquad ...(5.14.1)$$

can be implemented as a cascade of J stages of filtering and decimation as illustrated in Fig. 5.14.1. Thus the sampling rate at the output of the i^{th} stage is

$$F_t = \frac{F_i - 1}{D_i} \quad [i = 1, 2, \dots, J] \qquad ...(5.14.2)$$

where the input rate for the sequence $\{x(n)\}$ is $F_0 = F_x$.

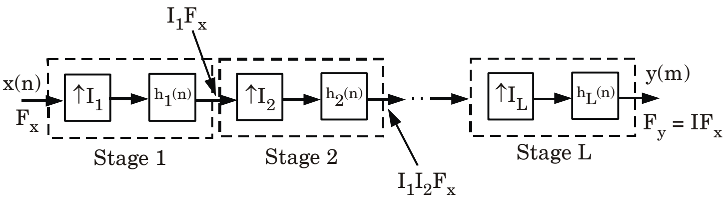


Fig. 5.14.1. Multistage implementation of interpolation by a factor *I*.

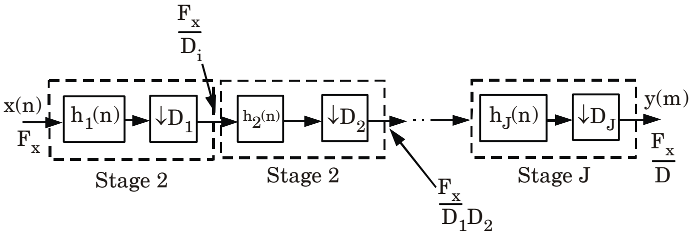


Fig. 5.14.2. Multistage implementation of decimation by a factor *D*.

3. To ensure that no aliasing occurs in the overall decimation process, we can design each filter stage to avoid aliasing within the frequency band of interest. To elaborate, let us define as the desired passband and the transition band in the overall decimator as

$$\text{Passband} = 0 \le F \le F_{pc}$$

$$\text{Transition band : } F_{pc} \le F \le F_{sc} \quad ...(5.14.3)$$

where $F_{sc} \le F_x/2D$.

4. Then, aliasing in the band $0 \le F \le F_{sc}$ is avoided by selecting the frequency band of each filter stage as follows :

$$\text{Passband : } 0 \le F \le F_{pc}$$

$$\text{Transition band : } F_{pc} \le F \le F_i - F_{sc}$$

$$\text{Stopband : } F_1 \le F_{sc} \le F \le F_0 / 2 \quad ...(5.14.4)$$

5. After decimation by *D*1, there is aliasing from the signal components that fall in the filter transition band, but the aliasing occurs at frequencies above F_{sc}. Thus there is no aliasing in the frequency band $0 \le F \le F_{sc}$.
6. By designing the filter in the subsequent stages to satisfy the specification given in eq. (5.14.1), we ensure that no aliasing occurs in the primary frequency band $0 \le F \le F_{sc}$.

Que 5.15. **Explain subband coding of speech signals.**

Answer

1. Subband coding is a method, where the speech signal is subdivided into several frequency bands and each band is digitally encoded separately.

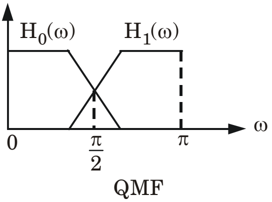


Fig. 5.15.1. Filter characteristics for subband coding.

2. Filter design is particularly important in achieving good performance in subband coding. Aliasing resulting from decimation of the subband signals must be negligible.
3. A practical solution to the aliasing problem is to use Quadrature Mirror Filters (QMF), which have the frequency response characteristics shown in Fig. 5.15.1.
4. The synthesis method for the subband encoded speech signal is basically the reverse of the encoding process.
5. The signals in adjacent low-pass and high-pass frequency bands are interpolated, filtered, and combined as shown in Fig. 5.15.1. A pair of (QMF) is used in the signal synthesis for each octave of the signal.
6. Subband coding is also an effective method to achieve data compression in image signal processing by combining subband coding with vector quantization for each subband signal.

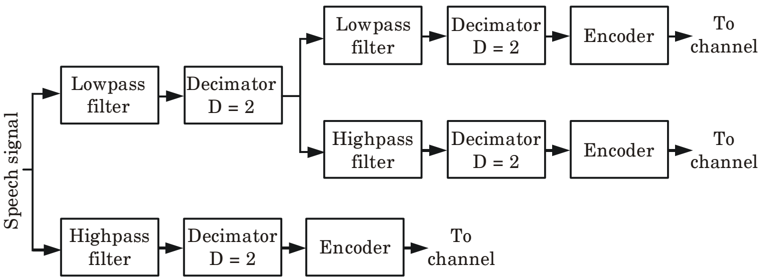


Fig. 5.15.2. Block diagram of subband speech coder.

7. In general, subband coding of signals is an effective method for achieving bandwidth compression in a digital representation of the signal, when the signal energy is concentrated in a particular region of the frequency band.

Que 5.16. **Discuss about quadrature mirror filters in detail.**

AKTU 2018-19, Marks 07

OR

Write a short note on :

i. Subband coding of speech signal

ii. Quardrature mirror filter. **AKTU 2019-20, Marks 07**

Answer

i. **Subband coding :** Refer Q. 5.15, Page 5–21C, Unit-5.

ii. **Quardrature mirror filter :**

1. The subband signals are down sampled before processing. The signals are up sampled after processing.
2. The structure used for this is known as Quadrature Mirror Filter (QMF) bank.
3. If the decimation and interpolation factors are equal, then the characteristics of $x(n)$ will be available in $y(n)$ if the filters are properly selected.
4. If this properly is satisfied, then the filter bank can be called as a critically sampled filter bank.
5. To list a few applications where QMF filters are used.

i. Efficient coding of the signal $x(n)$

ii. Analog voice privacy for secure telephone communication.

6. The two-channel QMF filter bank is shown in Fig. 5.16.1. The analysis filter $H_0(z)$ is a low-pass filter and $H_1(z)$ is a high-pass filter.
7. The cut-off frequency is taken as $\pi/2$ for these filters.
8. The subband signals $\{v_k(n)\}$ are down sampled. After down sampling, these signals are processed (encoded).
9. In the receiving side the signals are decoded, up-sampled and then passed through the synthesis filters $G_0(z)$ and $G_1(z)$ to get the output $y(n)$.
10. The encoding and decoding processes are not shown in Fig. 5.16.1.
11. For perfect reconstruction, the QMF filter banks should be properly selected.

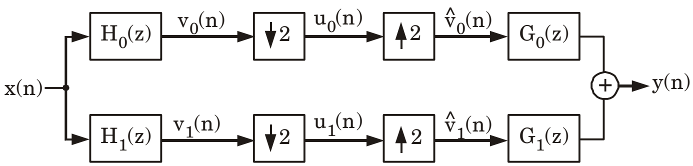


Fig. 5.16.1. Two-channel quadrature mirror filter bank.

Que 5.17. **What do you mean by multirate digital signal processing and write its advantages ?**

Answer

A. Multirate digital signal processing : Refer Q. 5.1, Page 5–2C, Unit-5.

B. Advantages :

i. Computational requirements is less
ii. Storage for filter coefficients is less
iii. Finite arithmetic effects are less
iv. Filter order required in multirate application is low, and
v. Sensitivity to filter coefficient lengths is less.

V ERY IMPORTANT QUESTIONS

Following questions are very important. These questions may be asked in your SESSIONALS as well as UNIVERSITY EXAMINATION.

Q. 1. What is multirate digital signal processing ? Discuss about application areas of it.

Ans. Refer Q. 5.1.

Q. 2. Explain the phenomenon decimation and interpolation by suitable example.

Ans. Refer Q. 5.5.

Q. 3. Discuss about interpolation and sampling rate conversion in detail.

Ans. Refer Q. 5.9.

Q. 4. What is the process of sampling rate conversion by an arbitrary factor (first-order approximation) ?

Ans. Refer Q. 5.11.

Q. 5. Suppose that the input signal has a flat spectrum between – 0.8π and 0.8π. Determine the number of subfilters to achieve a signal to distortion ratio of 50 dB when linear interpolation is applied.

Ans. Refer Q. 5.13.

Q. 6. Discuss about quadrature mirror filters in detail.

Ans. Refer Q. 5.16.

☺☺☺

1 UNIT

Introduction to Digital Signal Processing (2 Marks Questions)

1.1. Define digital signal processing.

AKTU 2016-17, Marks 02

Ans. Digital signal processing is a method of extracting information from digital signal with the help of digital processor. Output signals are also digital in nature.

1.2. Draw the block diagram of digital signal processing.

AKTU 2016-17, Marks 02

Ans.

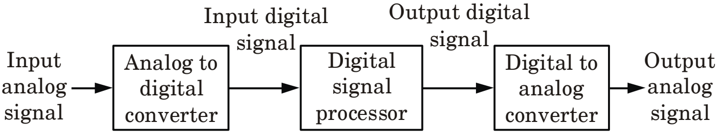


Fig. 1.2.1. Block diagram of digital signal processing.

1.3. Explain the basic elements required for realization of digital system.

AKTU 2016-17, Marks 02

Ans.

1. Analog to digital converter.
2. Digital signal processor.
3. Digital to analog converter.

1.4. What are the advantages for representing the digital system in block diagram form ?

AKTU 2019-20, Marks 02

Ans.

i. Just by inspection, the computation algorithm can be easily written.
ii. The hardware requirements can be easily determined.
iii. A variety of equivalent block diagram representations can be easily developed from the transfer function.

iv. The relationship between the output and the input can be determined.

1.5. List any five applications of DSP.

Ans.

i. Telecommunication.
ii. Military.
iii. Consumer electronics.
iv. Image processing.
v. Speech processing.

1.6. Enumerate the advantages of DSP over ASP.

AKTU 2016-17, Marks 02

Ans.

1. DSP is more flexible than ASP.
2. Accuracy of DSP is greater than ASP.
3. DSP has better storage capability over ASP.
4. DSP has mathematical processing function that is not present in ASP.

1.7. Define the canonic and non-canonic structures.

Ans.

A. Canonic structure : If the number of delays in the basic realisation block diagram is equal to the order of the difference equation or the order of the transfer function of a digital filter, then the realisation structure is known as canonic structure.

B. Non-canonic structure : If the number of delay in the structure is not same as order, then it is called non-canonic realisation or structure.

1.8. How an IIR filter is different than FIR filter ?

AKTU 2015-16, Marks 02

OR

Differentiate between IIR and FIR filters.

AKTU 2016-17, Marks 02

OR

Compare FIR and IIR filter. **AKTU 2018-19, Marks 02**

OR

What is difference between IIR and FIR filter ?

AKTU 2017-18, Marks 02

Ans.

S. No.	FIR filter	IIR filter
1.	They have finite impulse response.	They have infinite impulse response.
2.	Always stable.	Sometimes unstable.
3.	Have exact linear phase response.	Non-linear phase response.
4.	It requires more memory, higher computational complexity and involves more parameters.	It requires less memory, lower computational complexity and involves fewer parameters.

1.9. What is the main disadvantage of direct form realization ?

AKTU 2018-19, Marks 02

Ans.

i. They are lack hardware flexibility.
ii. Due to finite precision arithmetic, the sensitivity of the coefficients to quantization effect increases with the order of filter.

1.10. How can we obtain parallel realization of IIR system ? Also where it is used ?

Ans. By using the partial fraction expansion, the transfer function of an IIR system can be realized in a parallel form.
The parallel realization is useful for high speed filtering applications.

1.11. What are the advantages of ladder structures ?

Ans.

i. The ladder form of realization provides desirable coefficient sensitivity properties.
ii. They require minimum amount of memory.

1.12. Why parallel realization is used for high speed filtering applications ?

Ans. Because the filter operation is performed in parallel *i.e.*, the processing is performed simultaneously.

1.13. What is the fundamental time period of the signal $x(t) = \sin 15\pi t$.

AKTU 2016-17, Marks 02

Ans.

Given : $x(t) = \sin 15\pi t$
To Find : T.

1. $$\omega = 15\pi$$

2. $$T = \frac{2\pi}{\omega} = \frac{2\pi}{15\pi} = 0.133 \text{ sec}$$

1.14. For the given system function,

$$H(z) = (1 + z^{-1})\left(1 + \frac{3}{4}z^{-1} + \frac{3}{4}z^{-2} + z^{-3}\right)$$

Obtain cascade realization with minimum number of multipliers. **AKTU 2015-16, Marks 02**

Ans.

Given : $H(z) = (1 + z^{-1})\left(1 + \frac{3}{4}z^{-1} + \frac{3}{4}z^{-2} + z^{-3}\right)$

To Design : Cascade form realization.

For cascade realization

$$H(z) = H_1(z) \,.\, H_2(z)$$

so, $$H_1(z) = (1 + z^{-1})$$

and $$H_2(z) = \left(1 + \frac{3}{4}z^{-1} + \frac{3}{4}z^{-2} + z^{-3}\right)$$

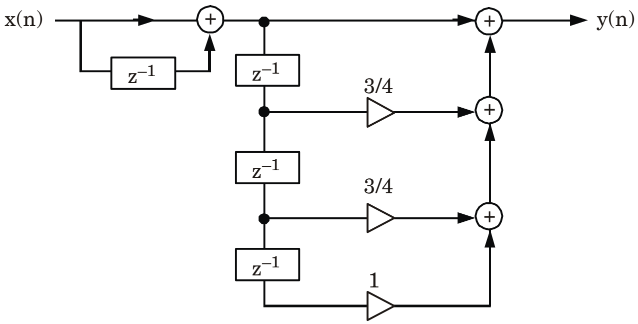


Fig. 1.14.1. Cascade form realization of H(z).

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2 UNIT

IIR Digital Filter Design (2 Marks Questions)

2.1. Define filter.

Ans. A filter is one which rejects unwanted frequencies from the input signal and allows the desired frequencies.

2.2. Name the methods or techniques used in designing of IIR filters.

Ans.

i. Approximation of derivatives.
ii. Impulse invariant method.
iii. Bilinear transformation.

2.3. What are the main disadvantages of designing IIR filters using windowing technique ? **AKTU 2015-16, Marks 02**

Ans.

i. Impulse response is infinite.
ii. It is hard to optimize than FIR.
iii. It is non-stable.

2.4. What do you mean by impulse invariant method of IIR filter design ? Give its transformation formula also.

Ans.

1. In this technique, the desired impulse response of the digital filter is obtained by uniformly sampling the impulse response of the equivalent analog filter.

   i.e., $h(n) = h_a(nT)$

   where T is the sampling interval.
2. Its transformation formula is given by,

$$\frac{1}{(s+s_i)^m} \rightarrow \frac{(-1)^{m-1}}{(m-1)!}\frac{d^{m-1}}{ds^{m-1}}\left[\frac{1}{1-e^{-sT}z^{-1}}\right]; s \rightarrow s_i$$

2.5. What are the limitations of impulse invariant and approximation of derivatives methods of design ?

Ans. These techniques are not suitable for the design of high-pass and band-reject filters.

2.6. How the limitation of impulse invariant and approximation of derivatives method is overcome ?

Ans. Limitation of these methods can be overcome in the mapping technique called as bilinear transformation.

2.7. What is bilinear transformation ?

Ans.

1. Bilinear transformation is a one to one mapping from the s-domain to the z-domain *i.e.*, the bilinear transformation is a conformal mapping that transforms the $j\Omega$-axis into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components.
2. The bilinear transformation is obtained by using the trapezoidal formula for numerical integration.

2.8. Write down the advantages and disadvantages of bilinear transformation.

AKTU 2019-20, Marks 02

Ans. **Advantages of bilinear transformation method :**

i. The mapping is one to one.
ii. There is no aliasing effect.
iii. Stable analog filter is transformed into the stable digital filter.
iv. There is no restriction one type of filter that can be transformed.

Disadvantages of bilinear transformation method :

1. In this method, the mapping is non-linear because of this frequency warping effect takes place.

2.9. What is the warping effect ?

AKTU 2018-19, Marks 02

Ans. The mapping of frequency from Ω to ω is approximately linear for small value of Ω and ω. For the higher frequencies, however the relation between Ω and ω becomes highly non-linear. This introduces the distortion in the frequency scale of digital filter relative to analog filter. This effect is known as warping effect.

2.10. What is the equation for order of Butterworth filter ?

AKTU 2017-18, Marks 02

Ans. The Butterworth low-pass filter has a magnitude response given by,

$$|H(j\Omega)| = \frac{A}{[1+(\Omega/\Omega_c)^{2N}]^{0.5}}$$

where A is the filter gain and Ω_c is the 3 dB cut-off frequency and N is the order of the filter.

2.11. What do you mean by Chebyshev filters ?

Ans.

1. The Chebyshev low-pass filters has a magnitude response given by,

$$|H(j\Omega)| = \frac{A}{[1+\epsilon^2 C_N^2(\Omega/\Omega_c)]^{0.5}}$$

where A is the filter gain, ϵ is constant and Ω_c is 3 dB cut-off frequency.

2. The Chebyshev polynomial of the I^{th} kind of N^{th} order, $C_N(x)$ is given by

$$C_N(x) = \begin{cases} \cos(N\cos^{-1}x) & \text{for } |x| \le 1 \\ \cosh(N\cosh^{-1}x) & \text{for } |x| \ge 1 \end{cases}$$

2.12. What are the properties of Chebyshev polynomial ?

Ans.

i. $C_N(x) \le 1$ for all $|x| \le 1$
ii. $C_N(1) = 1$ for all N
iii. The roots of the polynomials $C_N(x)$ occur in the interval $-1 \le x \le 1$.

2.13. A digital filter is to be designed from analog filter whose system response is

$$H(s) = \frac{\Omega_c}{s+\Omega_c}$$

Use bilinear transformation and obtain $H(z)$ for $\Omega_c = \dfrac{0.828}{T}$.

Ans.

Given : $H(s) = \dfrac{\Omega_c}{s+\Omega_c}$

To Design : Digital filter

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}} = \frac{\frac{0.828}{T}}{\frac{2}{T}\frac{(z-1)}{(z+1)}+\frac{0.828}{T}} = \frac{0.828\,(z+1)}{2(z-1)+0.828\,(z+1)}$$

2.14. Write the frequency transformation rule for the conversion of LP to HP filter. **AKTU 2017-18, Marks 02**

Ans. To convert low-pass with cut-off frequency Ω_c to high-pass with cut-off frequency Ω_c,

$$s \to \frac{\Omega_c\,\Omega_c^*}{s}$$

The system function of the high-pass filter is then,

$$H(s) = H_p\left(\frac{\Omega_c\,\Omega_c^*}{s}\right)$$

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3
UNIT

FIR Filter Design (2 Marks Questions)

3.1. What is window and why it is necessary ?

AKTU 2018-19, Marks 02

Ans. Window : A window function is a mathematical function that is zero valued outside of some chosen interval, normally symmetric around the middle of the interval.

Necessity : Enhance the ability of an FFT to extract spectral data from signals.

3.2. What are advantages and disadvantages of window methods ?

AKTU 2019-20, Marks 02

Ans. Advantages of window design method :

i. Simple method for design.
ii. Various window functions can be used depending upon the application.

Disadvantages of window design method :

i. Lack of precise control of the critical frequencies ω_p and ω_s.
ii. ω_p and ω_s depend upon type of window and filter length M.

3.3. What are the desirable characteristics of the window function ?

Ans.

i. The fourier transform of the window function $W(e^{j\omega})$ should have a small width of main lobe containing as much of the total energy as possible.
ii. The fourier transform of the window function $W(e^{j\omega})$ should have sidelobes that decrease in energy rapidly as ω tends to π.

3.4. What are the types of window functions ?

Ans.

i. Rectangular window function.
ii. Hanning window function.
iii. Hamming window function.
iv. Blackman window function.
v. Bartlett window function.
vi. Kaiser window function.

3.5. Write Gibbs phenomena. **AKTU 2017-18, Marks 02**

Ans. The abrupt truncation of the fourier series results in oscillations in the passband and stopband. These oscillations are due to slow convergence of the fourier series, particularly near the points of discontinuity. This effect is known as the Gibbs phenomenon.

3.6. How can we reduce the Gibbs oscillations ?

Ans. The undesirable oscillations (Gibbs oscillations) can be reduced by multiplying the desired impulse response coefficients by an appropriate window function.

3.7. What is Blackman window function ?

Ans.

1. Causal Blackman window function

$$w_B(n) = \begin{cases} 0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1} & ;\ 0 \le n \le M-1 \\ 0 & ;\ \text{otherwise} \end{cases}$$

2. Non-causal Blackman window function

$$w_B(n) = \begin{cases} 0.42 + 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1} & ;\ \text{for } |n| < \dfrac{M-1}{2} \\ 0 & ;\ \text{otherwise} \end{cases}$$

3.8. Write the expression for Blackman and Bartlett window.

AKTU 2019-20, Marks 02

Ans. **Expression for Blackman window :** Refer Q. 3.7, Page SQ–9C, Unit-3, 2 Marks Questions.

Expression for Bartlett window :

$$w_{\text{Bart}}(n) = \begin{cases} 1 + n & ;\ -\dfrac{M-1}{2} < n < 1 \\ 1 - n & ;\ 1 < n < \dfrac{M-1}{2} \end{cases}$$

3.9. Write the expression for Hamming window.

AKTU 2017-18, Marks 02

Ans.

a. Causal Hamming window :

$$w_{\text{Hamm}}(n) = \begin{cases} 0.54 - 0.46\cos\dfrac{2\pi n}{M-1} & ;\ 0 \le n < M-1 \\ 0 & ;\ \text{otherwise} \end{cases}$$

b. Non-causal Hamming window :

$$w_{\text{Hamm}}(n) = \begin{cases} 0.54 + 0.46 \cos \dfrac{2\pi n}{M-1} & ; \quad 0 \le n < (M-1)/2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

3.10. What are the advantages of frequency sampling method of FIR filter design ?

Ans.

i. Simple method of design.
ii. Simple realization in frequency domain.
iii. Better control over critical frequencies ω_p and ω_s.

3.11. What is the criteria for selecting a window ?

Ans. The particular window is selected depending upon the minimum stop band attenuation.

3.12. Suppose the axis of symmetry of impulse response *h*(*n*) lies half way between two samples. Identify the kind of applications in which this type of impulse response can be used.

Ans. Such type of impulse response may be used to design Hilbert transformers and differentiators.

3.13. Why transition bands are provided ?

Ans. Transition bands are provided since ideal filter (zero transition band) is not physically realizable.

3.14. What are the types of limit cycles ?

Ans. There are two types of limit cycles :
i. Zero input limit cycle.
ii. Overflow limit cycle.

3.15. What are the advantages of Kaiser window ?

AKTU 2018-19, Marks 02

Ans. Keeping the window length constant, we can adjust the shape factor, to design for the pass band and stop band ripples.

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4 UNIT

DFT & FFT (2 Marks Questions)

4.1. What is discrete time fourier transform and how it is related to discrete fourier transform ?

AKTU 2015-16, Marks 02

Ans.

A. DTFT :

1. The discrete-time fourier transform (DTFT) or, simply the fourier transform of a discrete time sequence $x(n)$ is represented by the complex exponential sequence $[e^{-j\omega n}]$ where ω is the real frequency variable.
2. This transform is useful to map the time-domain sequence into a continuous function of a frequency variable.

B. Relation between DTFT and DFT :

1. The DTFT $X(e^{j\omega})$ of a sequence $x(n)$ is defined by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \quad ...(4.1.1)$$

2. DFT of $x(n)$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \quad ...(4.1.2)$$

3. Comparing eq. (4.1.1) and eq. (4.1.2), we can say that DFT is obtained from DTFT by substituting $\omega = \dfrac{2\pi k}{N}$.
4. Hence, $X(k) = X(\omega)\Big|_{\omega = \frac{2\pi k}{N}}$

4.2. Define time reversal of sequence in DFT.

AKTU 2017-18, Marks 02

Ans. If $x(n) \xleftrightarrow{\text{DFT}} X(k)$

then, $x(-n, (\text{mod } N)) = x(N-n) \xleftrightarrow[\text{N}]{\text{DFT}} X(-k, (\text{mod } N)) = X(N-k)$.

Hence, when the N-point sequence in time is reversed, it is equivalent to reversing the DFT values.

4.3. Compare DTFT and DFT.

Ans.

S. No.	DTFT	DFT
1.	$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$	$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$ $k = 0, 1, \ldots\ldots N-1$
2.	DTFT is double sided.	DFT is single sided.
3.	The sequence $x(n)$ need not to be periodic.	The sequence $x(n)$ is assumed to be periodic.

4.4. Define linear convolution and its physical significance.

AKTU 2016-17, 2019-20; Marks 02

Ans.

A. Linear convolution :

Let us consider two finite duration sequences $x(n)$ and $h(n)$, the linear convolution of $x(n)$ and $h(n)$ is given as

$$y(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

B. Physical significance : Linear convolution can be used to find the response of a filter.

4.5. What is the difference between circular convolution and linear convolution ?

AKTU 2017-18, Marks 02

Ans.

S. No.	Linear convolution	Circular convolution
1.	If the length of one signal is N_1 and the length of other signal is N_2 then the length of output signal $y(n)$ is $N_1 + N_2 - 1$.	If the length of one signal is N_1 and the length of other signal is N_2 then the length of output signal is maximum of both signal (either N_1 or N_2).
2.	Multiplication of two sequences in time domain is called as linear convolution.	Multiplication of two sequences in frequency domain is called as circular convolution.

4.6. What is zero padding and what are its uses ?

AKTU 2015-16, Marks 02

Ans.

A. Zero padding : Let the sequence $x(n)$ has a length L. If we want to find the N-point DFT ($N > L$) of the sequence $x(n)$, we have to add ($N - L$) zeros to the sequence $x(n)$. This is known as zero padding.

B. Uses :

i. We can get "better display" of the frequency spectrum.

ii. With zero padding, the DFT can be used in linear filtering.

4.7. What is spectral leakage ? Give remedy to this problem.

AKTU 2015-16, Marks 02

Ans.

1. The spectrum of a product is the convolution between $S(f)$ and another function, which inevitably creates the new frequency components. But the term 'leakage' usually refers to the effect of windowing, which is the product of $s(t)$ with a different kind of the window function.
2. A smoothing window applied to the data before it is transformed into the frequency domain minimizes spectral leakage.

4.8. Establish the relation between *z*-transform and DFT.

AKTU 2015-16, Marks 02

Ans. The z-transform for a sequence $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \quad \text{...(4.8.1)}$$

The DFT of $x(n)$ is given by

$$X(k) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi nk/N} \quad \text{...(4.8.2)}$$

Comparing eq. (4.8.1) and eq. (4.8.2) we can say that DFT is obtained from z-transform by substituting $z = e^{j2\pi k/N}$

Hence, $$X(k) = X(z)\big|_{z = e^{j2\pi k/N}}$$

4.9. What is twiddle factor in DFT ? **AKTU 2017-18, Marks 02**

Ans. $W_N = e^{-j\frac{2\pi}{N}k}$ is called twiddle factor.

4.10. Draw a transformation matrix of size 4 × 4 and explain the properties of twiddle factor. **AKTU 2016-17, Marks 02**

Ans.

A. Transformation matrix :

For $N = 4$, W_4^{nk} is 4 × 4 matrix and is given as

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

B. Properties of twiddle factor :

i. Symmetry property

$$W_N^{k+N/2} = -W_N^k$$

ii. Periodicity property

$$W_N^{k+N} = -W_N^k$$

4.11. Write the expression for computation efficiency of an FFT.

AKTU 2016-17, Marks 02

Ans.

$$\eta_{comp} = \frac{\text{Number of complex multiplication required in FFT algorithm}}{\text{Number of complex multiplications required in direct DFT}}$$

$$= \frac{\frac{N}{2}\log_2 N}{N \times N} = \frac{\log_2 N}{2N}$$

4.12. Compare DIT and DIF FFT algorithm.

Ans.

S. No.	DIT FFT	DIF FFT
1.	The time domain sequence is decimated.	The DFT $X(k)$ is decimated.
2.	Input sequence is to be given in bit reversed order.	The DFT at the output is in bit reversed order.
3.	Suitable for calculating inverse DFT.	Suitable for calculating DFT.

4.13. Explain bit-reversal and in-place computation.

AKTU 2015-16, Marks 02

Ans.

A. Bit-reversal : Bit-reversal is helpful in shifting the data in decimation-in-time (DIT) and decimation-in-frequency (DIF) Fast Fourier Transform (FFT) algorithms. In bit-reversal, order of bits are reversed.

B. Inplace computation : Inplace algorithm is an algorithm which transforms input using no auxiliary data structure. However a small amount of extra storage space is allowed for auxiliary variables. The input is usually overwritten by the output as the algorithm executes.

4.14. Compute $X(0)$ if $X(k)$ is 4-point DFT of the following sequence

$$\boldsymbol{x(n) = \{1, 0, -1, 0\}}$$

AKTU 2015-16, Marks 02

Ans. Given, $x(n) = \{1, 0, -1, 0\}$

$$X(k) = \sum_{n=0}^{N-1} x(n).e^{-j\frac{2\pi kn}{N}}$$

$$k = 0, 1, 2,\ldots\ldots\ldots\ldots\ldots\ldots, N-1$$

Here, $N = 4$

$$X_4 = [W_4].x(n)$$

$$X_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+-1+0 \\ 1+0+1+0 \\ 1+0-1+0 \\ 1+0+1+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

$$X(k) = \{0, 2, 0, 2\}$$

$\therefore$ $X(0) = 0$

4.15. What is the DFT of $\delta(n)$?

AKTU 2017-18, Marks 02

Ans.

$$\text{DFT } \delta(n) = \sum_{n=0}^{N-1} \delta(n)\, e^{-j2\pi kn/N}$$

We know $\delta(0) = 1$

$$\text{DFT } \delta(n) = \delta(0)\,.\,1 = 1$$

4.16. Calculate the DFT of the sequence $s(n) = \{1, 2, 1, 3\}$.

AKTU 2016-17, Marks 02

Ans.

Given : $n = \{1, 2, 1, 3\}$
To Find : DFT

1. DFT of a finite duration sequence $s(n)$ is defined as

$$S(k) = \sum_{n=0}^{N-1} s(n)\, e^{-j2\pi nk/N}, \quad k = 0, 1, \ldots\ldots, N-1$$

Here, $N = 4$.

2. For $k = 0$,

$$S(0) = \sum_{n=0}^{3} s(n)\, e^{-j2\pi n(0)/4} = \sum_{n=0}^{3} s(n) = 1 + 2 + 1 + 3 = 7$$

3. For $k = 1$,

$$S(1) = \sum_{n=0}^{3} s(n)\, e^{-j2\pi n(1)/4}$$

$$= 1 + 2e^{-j\pi/2} + 1e^{-j\pi} + 3e^{-j3\pi/2}$$

$$= 1 + 2\,(-j) + 1(-1) + 3(j) = j$$

4. For $k = 2$,

$$S(2) = \sum_{n=0}^{3} s(n)\, e^{-j2\pi n(2)/4}$$

$$= 1 + 2e^{-j\pi} + 1e^{-j2\pi} + 3e^{-j3\pi}$$

$$= 1 + 2(-1) + 1(1) + 3(-1) = -3$$

5. For $k = 3$,

$$S(3) = \sum_{n=0}^{3} s(n)\, e^{-j2\pi n(3)/4}$$

$$= 1 + 2e^{-j3\pi/2} + 1e^{-j3\pi} + 3e^{-j9\pi/2}$$

$$= 1 + 2(j) + 1(-1) + 3(-j) = -j$$

Hence, $S(k) = \{7, j, -3, -j\}$

4.17. Calculate number of multiplications needed in calculation of DFT and FFT of 32 point sequence and also calculate speed improvement factor. **AKTU 2015-16, Marks 02**

Ans.

Given : 32 point sequence
To Find : i. Number of multiplication needed in DFT and FFT
ii. Speed improvement factor

1. Number of multiplications needed in a 32 point DFT

$$= N^2$$
$$= (32)^2$$
$$= 32 \times 32 = 1024$$

2. Number of multiplications needed in a 32 point FFT

$$= \frac{N}{2}\log_2 N$$

$$= \frac{32}{2}\log_2 32$$

$$= 16 \log_2 2^5 = 16 \times 5 = 80$$

3. Speed improvement factor is given as

$$= \frac{N^2}{\frac{N}{2}\log_2(N)} = \frac{1024}{80} = 12.8$$

4.18. If $x(n)$ = {6, 5, 4, 3} what will be $x((2 - n))_4$.

AKTU 2017-18, Marks 02

Ans. Given, $x(n) = \{6, 5, 4, 3\}$

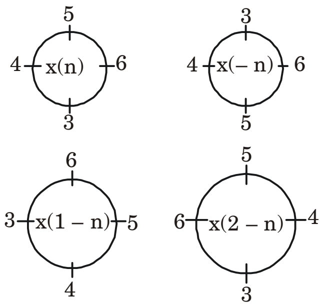

Fig. 4.18.1.

$$x((2 - n))_4 = \{4, 5, 6, 3\}$$

4.19. If $x(n)$ = (1, 5, 2, 3) what will be $x((3 - n))_4$.

AKTU 2019-20, Marks 02

Ans. Given, $x(n) = \{1, 5, 2, 3\}$

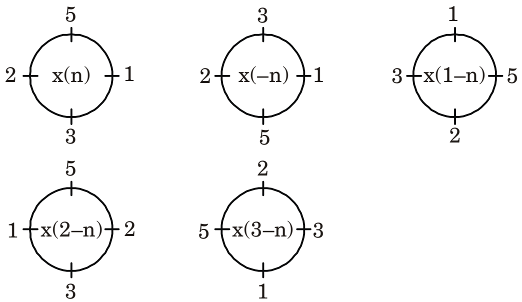

Fig. 4.19.1.

$$x((3 - n))_4 = \{3, 2, 5, 1\}$$

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5 UNIT

Multirate Digital Signal Processing (2 Marks Questions)

5.1. What is multirate digital signal processing ?

Ans. The process of converting a signal from a given rate to a different rate is called sampling rate conversion. The systems that employ multiple sampling rates in the processing of digital signals are called multirate digital signal processing systems.

5.2. Write uses of multirate digital signal processing.

Ans.

i. It is used in design of phase shifter.
ii. The digital systems operating at different sampling rates can be interfaced by multirate signal processing system.

5.3. Define decimation **AKTU 2018-19, Marks 02**

Ans. The process of reducing the sampling rate by a factor D (downsampling by D) is called decimation.

5.4. Write a short note on interpolation.

Ans. The process of increasing the sampling rate by an integer factor I (upsampling by I) is called interpolation.

5.5. What is downsampling and upsampling ?

AKTU 2018-19, Marks 02

Ans. **Downsampling :** It is the process of reducing the sampling rate of a signal. This is usually done to reduce the data rate or the size of the data.

Upsampling : It is the process of inserting zero-valued samples between original samples to increase the sampling rate.

5.6. Write the advantages of multirate digital signal processing.

Ans.

1. Reduction in Computations and Memory requirements.
2. Multirate DSP is more efficient, distortion less and flexible type of signal processing.

5.7. What is use of Decimation ?

Ans.
1. Decrease the ADC data rate to reasonable levels for data capture.
2. Maintain high output sampling rate for more flexible frequency planning.

5.8. What is sampling rate conversion ?

Ans. The process of conversion a signal from a given rate to a different rate is called sampling rate conversion.

5.9. Draw the block diagram of sampling rate conversion.

Ans.

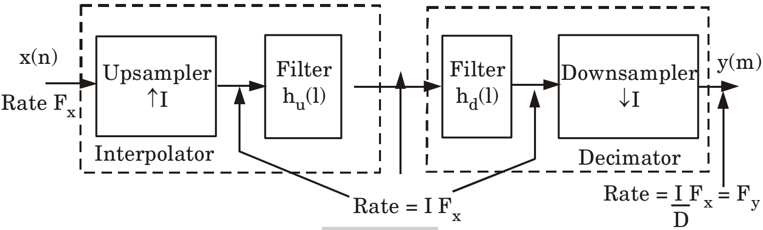

Fig. 5.9.1.

5.10. Write a short note on subband coding of speech signals.

Ans. It is a method, where the speech signal is subdivided into several frequency bands and each band is digitally encoded separately.

5.11. What do you mean by quadrature mirror filter bank ?

Ans. The subband signals are downsmapled before processing. The signals are upsampled after processing. The structure used for this is known as Quadrature Mirror Filter (QMF) bank.

5.12. What is the uses of quadrature mirror filters ?

Ans.
1. Quadrature mirror filter is a filter most commonly used to implement a filter bank that splits an input signal into two bands.
2. The resulting high-pass and low-pass signals are often reduced by a factor of 2, giving a critically sampled two channel representation of the original signal.

5.13. If $x(n)$ = {4, 3, 5, 7, 4, 6} and upsampling factor = 3, then what will be the value of upsampler output.

AKTU 2019-20, Marks 02

Ans. Given, $x(n) = \{4, 3, 5, 7, 4, 6\}$

And, $M = 3$

Output $y(n) = x(n/3)$

$y(n) = \{4, 0, 0, 3, 0, 0, 5, 0, 0, 7, 0, 0, 4, 0, 0, 6\}$

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B. Tech.
(SEM. VI) EVEN SEMESTER THEORY EXAMINATION, 2015-16
DIGITAL SIGNAL PROCESSING

Time : 3 Hours **Max. Marks : 100**

Note : Attempt all of the sections. Assume missing data suitable, if any.

SECTION-A

1. Attempt **all** parts. All parts carry equal marks. Write answer of each part in short. **(2 × 10 = 20)**

a. What is discrete time fourier transform and how it is related to discrete fourier transform ?

b. Establish the relation between z-transform and DFT.

c. What is zero padding ? What are its uses ?

d. Calculate number of multiplications needed in calculation of DFT and FFT of 32 point sequence and also calculate speed improvement factor.

e. Explain bit-reversal and inplace computation.

f. How an IIR filter is different than FIR filter ?

g. Compute $X(0)$ if $X(k)$ is 4-point DFT of the following sequence

$$x(n) = \{1, 0, -1, 0\}$$

h. For the given system function,

$$H(z) = (1 + z^{-1})\left(1 + \frac{3}{4}z^{-1} + \frac{3}{4}z^{-2} + z^{-3}\right)$$

Obtain cascade realization with minimum number of multipliers.

i. What is spectral leakage ? Give remedy to this problem.

j. What are the main disadvantages of designing IIR filters using windowing technique ?

SECTION-B

2. Attempt any **five** questions from this section. **(10 × 5 = 50)**

a. Find the 10-point DFT of the following sequence :

i. $x(n) = \delta(n) + \delta(n-5)$

ii. $x(n) = u(n) - u(n-6)$

b. Find circular convolution of the following sequences using concentric circle method.

$$x_1(n) = (1, 2, 2, 1)$$
$$x_2(n) = (1, 2, 3, 4)$$

c. i. Compute 4-point DFT of the following sequence using DIF algorithm $x(n) = \cos \dfrac{n\pi}{2}$

ii Show that the same algorithm can be used to compute IDFT of $X(k)$ calculated in part (i).

d. Compute the DFT of following 8 point sequence using 4 point radix-2 DIF algorithm.

$$x(n) = \{2, 2, 2, 2, 1, 1, 1, 1\}$$

e. Obtain direct form I, direct form II and parallel form structures for the following filter.

$$y(n) = \frac{3}{4}y(n-1) + \frac{3}{32}y(n-2) + \frac{1}{64}y(n-3) + x(n) + 3x(n-1) + 2x(n-2)$$

f. Consider the causal linear shift invariant filter with the system function

$$H(z) = \frac{1 + 0.875z^{-1}}{(1 + 0.2z^{-1} + 0.9z^{-2})(1 - 0.7z^{-1})}$$

Obtain following realization :

a. Direct form II

b. A cascade of first order and second order system realized in transposed direct form II.

c. A parallel connection of first order and second order system realized in direct form II.

g. A filter is to be designed with the following desired frequency response :

$$H_d(e^{j\omega}) = \begin{cases} 0 & \dfrac{-\pi}{4} \le \omega \le \dfrac{\pi}{4} \\ e^{-j2\omega} & \dfrac{\pi}{4} \le |\omega| \le \pi \end{cases}$$

h. Transform the prototype LPF with system function

$$H_{LP}(s) = \frac{\Omega_p}{s + \Omega_p} \text{ into a}$$

i. HPF with cut-off frequency Ω_p.

ii. BPF with upper and lower cut-off frequencies Ω_1 and Ω_2 respectively.

SECTION-C

Attempt any **two** questions from this section. **(15 × 2 = 30)**

3. a. Prove that multiplication of the DFTs of two sequences is equivalent to the circular convolution of the two sequences in the time domain.

b. If the 10-point DFT of $x(n) = \delta(n) - \delta(n-1)$ and $h(n) = u(n) - u(n-10)$ are $X(k)$ and $H(k)$ respectively, find the sequence $W(n)$ that corresponds to the 10-point inverse DFT of the product $H(k)\,X(k)$.

4. a.

i. Compute 4 point DFT of the following sequence using linear transformation matrix $x(n) = (1, 1, -2, -2)$

ii. Find IDFT $x(n)$ from $X(k)$ calculated in part(i).

b. Use Radix-2 DIT algorithm for efficient computation of 8 point DFT of $x(n) = 2^n$.

5. a. An FIR filter has following symmetry in the impulse response :

$$h(n) = h(M - 1 - n) \text{ for } M \text{ odd.}$$

Derive its frequency response and show that it has linear phase.

b. Discuss the bilinear transformation method of converting analog IIR filter into digital IIR filter. What is frequency warping ?

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SOLUTION OF PAPER (2015-16)

Note : Attempt all of the sections. Assume missing data suitable, if any.

SECTION-A

1. Attempt **all** parts. All parts carry equal marks. Write answer of each part in short. **(2 × 10 = 20)**

a. What is discrete time fourier transform and how it is related to discrete fourier transform ?

Ans.

A. DTFT :

1. The discrete-time fourier transform (DTFT) or, simply the fourier transform of a discrete time sequence $x(n)$ is represented by the complex exponential sequence $[e^{-j\omega n}]$ where ω is the real frequency variable.
2. This transform is useful to map the time-domain sequence into a continuous function of a frequency variable.

B. Relation between DTFT and DFT :

1. The DTFT $X(e^{j\omega})$ of a sequence $x(n)$ is defined by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \quad ...(1)$$

2. DFT of $x(n)$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \quad ...(2)$$

3. Comparing eq. (1) and eq. (2), we can say that DFT is obtained from DTFT by substituting $\omega = \frac{2\pi k}{N}$.
4. Hence, $X(k) = X(\omega)\big|_{\omega = \frac{2\pi k}{N}}$

b. Establish the relation between z-transform and DFT.

Ans. The z-transform for a sequence $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \quad ...(1)$$

The DFT of $x(n)$ is given by

$$X(k) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi nk/N} \quad ...(2)$$

Comparing eq. (1) and eq. (2) we can say that DFT is obtained from z-transform by substituting $z = e^{j2\pi k/N}$

Hence, $X(k) = X(z)\big|_{z = e^{j2\pi k/N}}$

c. What is zero padding ? What are its uses ?

Ans.

A. Zero padding : Let the sequence $x(n)$ has a length L. If we want to find the N-point DFT ($N > L$) of the sequence $x(n)$, we have to add $(N - L)$ zeros to the sequence $x(n)$. This is known as zero padding.

B. Uses :

i. We can get "better display" of the frequency spectrum.

ii. With zero padding, the DFT can be used in linear filtering.

d. Calculate number of multiplications needed in calculation of DFT and FFT of 32 point sequence and also calculate speed improvement factor.

Ans.

Given : 32 point sequence

To Find : i. Number of multiplication needed in DFT and FFT

ii. Speed improvement factor

1. Number of multiplications needed in a 32 point DFT

$$= N^2 = (32)^2 = 32 \times 32 = 1024$$

2. Number of multiplications needed in a 32 point FFT

$$= \frac{N}{2}\log_2 N = \frac{32}{2}\log_2 32 = 16 \log_2 2^5 = 16 \times 5 = 80$$

3. Speed improvement factor is given as

$$= \frac{N^2}{\frac{N}{2}\log_2(N)} = \frac{1024}{80} = 12.8$$

e. Explain bit-reversal and inplace computation.

Ans.

A. Bit-reversal : Bit-reversal is helpful in shifting the data in decimation-in-time (DIT) and decimation-in-frequency (DIF) Fast Fourier Transform (FFT) algorithms. In bit-reversal, order of bits are reversed.

B. Inplace computation : Inplace algorithm is an algorithm which transforms input using no auxiliary data structure. However a small amount of extra storage space is allowed for auxiliary variables. The input is usually overwritten by the output as the algorithm executes.

f. How an IIR filter is different than FIR filter ?

Ans.

S. No.	FIR filter	IIR filter
1.	They have finite impulse response.	They have infinite impulse response.
2.	Always stable.	Sometimes unstable.
3.	Have exact linear phase response.	Non-linear phase response.
4.	It requires more memory, higher computational complexity and involves more parameters.	It requires less memory, lower computational complexity and involves fewer parameters.

g. Compute $X(0)$ if $X(k)$ is 4-point DFT of the following sequence

$$x(n) = \{1, 0, -1, 0\}$$

Ans. Given, $x(n) = \{1, 0, -1, 0\}$

$$X(k) = \sum_{n=0}^{N-1} x(n).e^{-j\frac{2\pi kn}{N}}$$

$$k = 0, 1, 2, \ldots\ldots\ldots\ldots\ldots\ldots, N-1$$

Here, $N = 4$

$$X_4 = [W_4].x(n)$$

$$X_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+-1+0 \\ 1+0+1+0 \\ 1+0-1+0 \\ 1+0+1+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

$$X(k) = \{0, 2, 0, 2\}$$

$\therefore$ $X(0) = 0$

h. For the given system function,

$$H(z) = (1+z^{-1})\left(1+\frac{3}{4}z^{-1}+\frac{3}{4}z^{-2}+z^{-3}\right)$$

Obtain cascade realization with minimum number of multipliers.

Ans.

Given : $H(z) = (1+z^{-1})\left(1+\frac{3}{4}z^{-1}+\frac{3}{4}z^{-2}+z^{-3}\right)$

To Design : Cascade form realization.

For cascade realization

$$H(z) = H_1(z) \, . \, H_2(z)$$

so, $H_1(z) = (1 + z^{-1})$

and $H_2(z) = \left(1 + \frac{3}{4}z^{-1} + \frac{3}{4}z^{-2} + z^{-3}\right)$

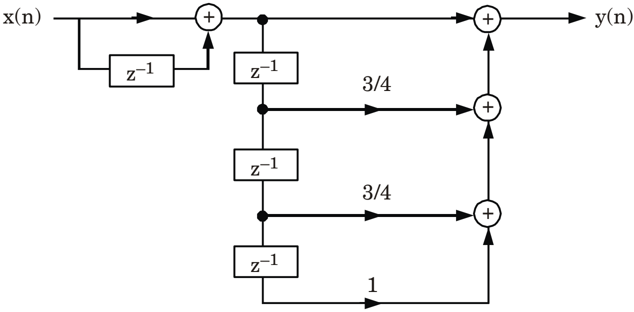


Fig. 1. Cascade form realization of H(z).

i. What is spectral leakage ? Give remedy to this problem.

Ans.

1. Spectral leakage is a term used to describe the phenomenon of energy from one DFT bin leaking into another DFT bin (or more generally into many other DFT bins).
2. Leakage happens because of the lack of orthogonality between some frequency components in the signal and the set of basis vectors in the DFT.
3. A smoothing window applied to the data before it is transformed into the frequency domain minimizes spectral leakage.

j. What are the main disadvantages of designing IIR filters using windowing technique ?

Ans.

i. Impulse response is infinite.

ii. It is hard to optimize than FIR.

iii. It is non-stable.

SECTION-B

2. Attempt any **five** questions from this section. **(10 × 5 = 50)**

a. Find the 10-point DFT of the following sequence :

i. $\boldsymbol{x(n) = \delta(n) + \delta(n-5)}$

ii. $\boldsymbol{x(n) = u(n) - u(n-6)}$

Ans.

i. Given : $x(n) = \delta(n) + \delta(n-5)$

To Find : 10 point DFT.

1. For 10 point DFT, $N = 10$

So, $$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \quad ; k = 0,1,2......, N-1$$

$$X(k) = \sum_{n=0}^{9} \{\delta(n) + \delta(n-5)\}e^{-j2\pi kn/N} \; ; k = 0 \text{ to } 9$$

2. For $k = 0$, $X(0) = e^0 + 0 + 0 + 0 + 0 + e^0 + 0 + 0 + 0 + 0$
$X(0) = 1 + 1 = 2$
3. For $k = 1$, $X(1) = e^0 + 0 + 0 + 0 + 0 + e^{-j\pi} + 0 + 0 + 0 + 0$
$= 1 + (-1) = 0$
4. For $k = 2$, $X(2) = e^0 + 0 + 0 + 0 + 0 + e^{-j2\pi} + 0 + 0 + 0 + 0$
$= 1 + 1 = 2$
5. For $k = 3$, $X(3) = e^0 + 0 + 0 + 0 + 0 + e^{-j3\pi} + 0 + 0 + 0 + 0$
$= 1 + (-1) = 0$
6. For $k = 4$, $X(4) = e^0 + 0 + 0 + 0 + 0 + e^{-j4\pi} + 0 + 0 + 0 + 0$
$= 1 + 1 = 2$
7. For $k = 5$, $X(5) = e^0 + e^{-j5\pi} = 1 + (-1) = 0$
8. For $k = 6$, $X(6) = e^0 + e^{-j6\pi} = 1 + 1 = 2$
9. For $k = 7$, $X(7) = e^0 + e^{-j7\pi} = 1 + (-1) = 0$
10. For $k = 8$, $X(8) = e^0 + e^{-j8\pi} = 1 + 1 = 2$
11. For $k = 9$, $X(9) = e^0 + e^{-j9\pi} = 1 + (-1) = 0$
12. So, 10 point DFT of signal $x(n)$ is
$X(k) = \{2, 0, 2, 0, 2, 0, 2, 0, 2, 0\}$

ii. Given : $x(n) = u(n) - u(n-6)$
To Find : 10 point DFT.

1. So, $$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \quad ; \; k = 0,1,2,3.....N-1$$

$$X(k) = \sum_{n=0}^{9} \{u(n) - u(n-6)\}e^{-j2\pi kn/N} \; ; \; k = 0,1,2,........9$$

2. For $k = 0$, $X(0) = 1 + 1 + 1 + 1 + 1 + 1 + 0 + 0 + 0 + 0$
$X(0) = 6$
3. For $k = 1$, $X(1) =$
$$1.e^0 + 1.e^{-j\frac{2\pi}{10}} + 1.e^{-j\frac{4\pi}{10}} + 1.e^{-j\frac{6\pi}{10}} + 1.e^{-j\frac{8\pi}{10}} + 1.e^{-j\pi} + 0 + 0$$
$X(1) = -j\,3.0834$
4. For $k = 2$, $X(2) = e^0 + e^{-j\frac{4\pi}{10}} + e^{-j\frac{8\pi}{10}} + e^{-j\frac{12\pi}{10}} + e^{-j\frac{16\pi}{10}} + e^{-j2\pi} + 0 + 0$
$= (1 - j0)$
5. For $k = 3$, $X(3) = e^0 + e^{-j\frac{6\pi}{10}} + e^{-j\frac{12\pi}{10}} + e^{-j\frac{18\pi}{10}} + e^{-j\frac{24\pi}{10}} + e^{-j3\pi} + 0 + 0$
$= -j0.7266$
6. For $k = 4$, $X(4) = e^0 + e^{-j\frac{8\pi}{10}} + e^{-j\frac{16\pi}{10}} + e^{-j\frac{24\pi}{10}} + e^{-j\frac{32\pi}{10}} + e^{-j4\pi} + 0 + 0 = 1$
7. For $k = 5$, $X(5) = e^0 + e^{-j\pi} + e^{-j2\pi} + e^{-j3\pi} + e^{-j4\pi} + e^{-j5\pi} + 0 + 0 = 0$

8. For $k = 6$, $X(6) =$

$$e^0 + e^{-j\frac{12\pi}{10}} + e^{-j\frac{24\pi}{10}} + e^{-j\frac{36\pi}{10}} + e^{-j\frac{48\pi}{10}} + e^{-j\frac{60\pi}{10}} + 0 + 0 = 1$$

9. For $k = 7$, $X(7) = e^0 + e^{-j\frac{14\pi}{10}} + e^{-j\frac{28\pi}{10}} + e^{-j\frac{42\pi}{10}} + e^{-j\frac{56\pi}{10}} + e^{-j\frac{70\pi}{10}} + 0 + 0$

$= j0.7245$

10. For $k = 8$, $X(8) = e^0 + e^{-j\frac{16\pi}{10}} + e^{-j\frac{32\pi}{10}} + e^{-j\frac{48\pi}{10}} + e^{-j\frac{64\pi}{10}} + e^{-j8\pi} + 0 + 0 = 1$

11. For $k = 9$, $X(9) = e^0 + e^{-j\frac{18\pi}{10}} + e^{-j\frac{36\pi}{10}} + e^{-j\frac{54\pi}{10}} + e^{-j\frac{72\pi}{10}} + e^{-j9\pi} + 0 + 0$

$= j3.077$

12. So, 10-point DFT of $x(n)$ is

$$X(k) = \{6, -j3.0834, 0, -j0.7266, 1, 0, 1, j\,0.7245, 1, j3.0775\}$$

b. Find circular convolution of the following sequences using concentric circle method.

$$\boldsymbol{x_1(n) = (1, 2, 2, 1)}$$
$$\boldsymbol{x_2(n) = (1, 2, 3, 4)}$$

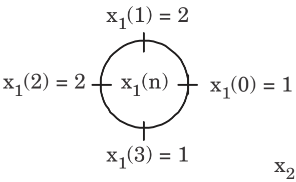
Ans.

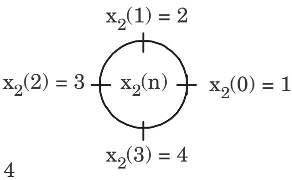
Given : $x_1(n) = (1, 2, 2, 1)$, $x_2(n) = (1, 2, 3, 4)$
To Find : Circular Convolution.

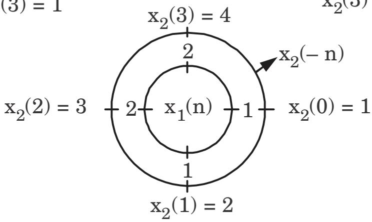
1. $$x_3(n) = x_1(n) \otimes x_2(n)$$

$$x_3(m) = \sum_{n=0}^{3} x_1(n).x_2(m-n)_4 \; ; m = 0, 1, 2, 3$$

2. For $m = 0$,



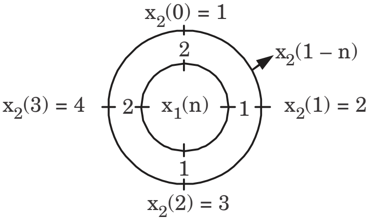




$$x_3(0) = \sum_{n=0}^{3} x_1(n).x_2(-n)_4$$

$$x_3(0) = 1 \times 1 + 4 \times 2 + 3 \times 2 + 1 \times 2 = 17$$

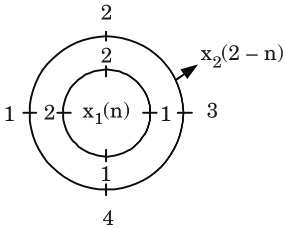
3. For $m = 1$,



$$x_3(1) = \sum_{n=0}^{3} x_1(n) x_2(1-n)$$

$$x_3(1) = 1 \times 2 + 2 \times 4 + 1 \times 3 + 1 \times 2 = 15$$

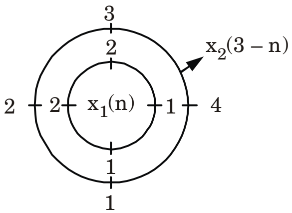
4. For $m = 2$,



$$x_3(2) = \sum_{n=0}^{3} x_1(n)\, x_2(2-n)$$

$$x_3(2) = 1 \times 3 + 2 \times 2 + 2 \times 1 + 1 \times 4 = 13$$

5. For $m = 1$,



$$x_3(3) = \sum_{n=0}^{3} x_1(n)\, x_2(3-n)$$

$$x_3(3) = 1 \times 1 + 1 \times 4 + 2 \times 3 + 2 \times 2 = 15$$

So, $\quad x_3(m) = \{17, 15, 13, 15\}$

c. i. Compute 4-point DFT of the following sequence using DIF algorithm $x(n) = \cos \dfrac{n\pi}{2}$

ii Show that the same algorithm can be used to compute IDFT of $X(k)$ calculated in part (i).

Ans.

Given : $N = 4$ and $x(n) = \cos \dfrac{n\pi}{2}$

To Find : DFT and IDFT.

i. DFT :

1. The sequence $x(n)$ can be obtained by putting $n = 0, 1, 2, 3$

$$x(0) = \cos 0 = 1 \qquad x(2) = \cos \frac{2\pi}{2} = -1$$

$$x(1) = \cos \frac{\pi}{2} = 0 \qquad x(3) = \cos \frac{3\pi}{2} = 0$$

$$x(n) = \{1, 0, -1, 0\}$$

$$W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k}$$

$$W_4^0 = 1 \text{ and } W_4^1 = e^{-j\pi/2} = -j$$

2. Using DIF FFT algorithm, we can find $X(k)$ from the given sequence $x(n)$ as shown in Fig. 2.

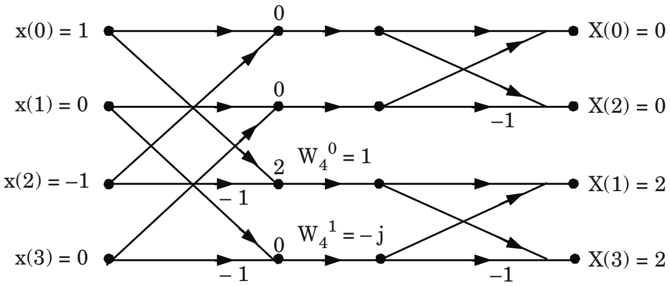


Fig. 2.

3. $$X(k) = \sum_{k=0}^{N-1} x(n) W_N^{kn}, \; n = 0, 1, \ldots\ldots\ldots, N-1$$

$\therefore$ $$X(k) = \{0, 2, 0, 2\}$$

ii. IDFT :

1. IDFT differs from DFT by

a. Multiplication by $\dfrac{1}{N}$

b. Negative sign of imaginary point of (W_N)

$$W_4^0 = 1, \qquad W_4^{-1} = j$$

2. We know that

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \; n = 0, 1, \ldots\ldots\ldots, N-1 \qquad \ldots(1)$$

Here, $N = 4$

Therefore, $x(n) = \{1, 0, -1, 0\}$

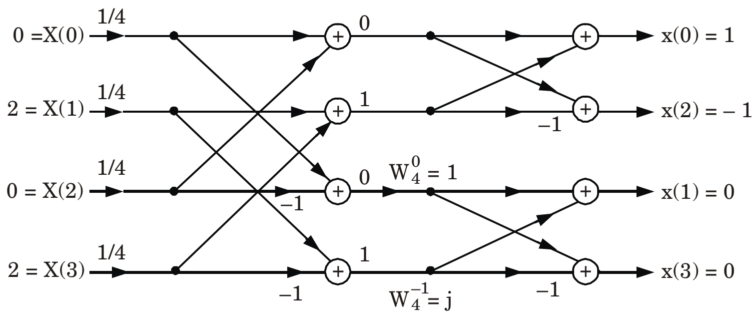


Fig. 3. SFG of IDFT of X(k).

d. Compute the DFT of following 8 point sequence using 4 point radix-2 DIF algorithm.

$$x(n) = \{2, 2, 2, 2, 1, 1, 1, 1\}$$

Ans.

Given : $x(n) = \{2, 2, 2, 2, 1, 1, 1, 1\}$, $N = 8$

To Find : DFT.

1. $$W_8^0 = e^0 = 1$$
$$W_8^1 = e^{-j\frac{\pi}{4}} = 0.707 - j0.707$$
$$W_8^2 = e^{-j\frac{\pi}{2}} = -j$$
$$W_8^3 = -0.707 - j0.707$$
2. The output of various stages can be easily calculated from 8-point butterfly diagram.
3. **Stage-1 outputs :**
$$g(0) = x(0) + x(4) = 2 + 1 = 3$$
$$g(1) = x(1) + x(5) = 2 + 1 = 3$$
$$g(2) = x(2) + x(6) = 2 + 1 = 3$$
$$g(3) = x(3) + x(7) = 2 + 1 = 3$$
$$g(4) = [x(0) - x(4)]W_8^1 = 1$$
$$g(5) = [x(1) - x(5)]W_8^1 = 0.707 - j0.707$$
$$g(6) = [x(2) - x(6)]W_8^2 = -j$$
$$g(7) = [x(3) - x(7)]W_8^3 = -0.707 - j0.707$$
4. **Stage-2 outputs :**
$$h(0) = g(0) + g(2) = 6$$
$$h(1) = g(1) + g(3) = 6$$
$$h(2) = [g(0) - g(2)]W_8^0 = 0$$
$$h(3) = [g(1) - g(3)]W_8^2 = 0$$
$$h(4) = g(4) + g(6) = 1 - j$$
$$h(5) = g(5) + g(7) = -j1.44$$

$$h(6) = [g(4) + g(6)]W_8^0 = 1 + j$$

$$h(7) = [g(5) - g(7)]W_8^2 = -j1.44$$

5. Stage-3 outputs or Final outputs :

$$X(0) = h(0) + h(1) = 12$$

$$X(1) = h(4) + h(5) = 1 - j\,2.414$$

$$X(2) = h(2) + h(3) = 0$$

$$X(3) = [h(6) + h(7)]W_8^0 = 1 - j\,0.414$$

$$X(4) = [h(0) - h(1)]W_8^0 = 0$$

$$X(5) = [h(4) - h(5)]W_8^0 = 1 + j\,0.414$$

$$X(6) = [h(2) - h(3)]W_8^0 = 0$$

$$X(7) = [h(6) - h(7)]W_8^0 = 1 + j\,2.414$$

6. Thus, $X(k) = \{12, 1 - j\,2.414, 0, 1 - j\,0.414, 0, 1 + j\,0.414, 0, 1 + j\,2.414\}$

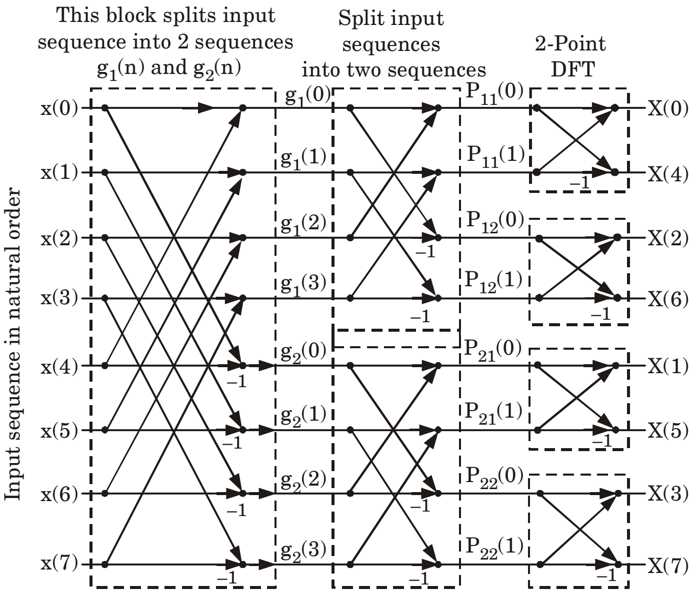


Fig. 4. Signal flow graph for 8-point DIF FFT.

e. Obtain direct form I, direct form II and parallel form structures for the following filter.

$$\boldsymbol{y(n) = \frac{3}{4}y(n-1) + \frac{3}{32}y(n-2) + \frac{1}{64}y(n-3) + x(n) + 3x(n-1) + 2x(n-2)}$$

Ans.

1. Given,
$$y(n) = \frac{3}{4}y(n-1) + \frac{3}{32}y(n-2) + \frac{1}{64}y(n-3) + x(n) + 3x(n-1) + 2x(n-2) \quad ...(1)$$

2. Taking z-transform of both side of eq. (1)
$$Y(z) = \frac{3}{4}z^{-1}Y(z) + \frac{3}{32}z^{-2}Y(z) + \frac{1}{64}z^{-3}Y(z) + X(z) + 3z^{-1}X(z) + 2z^{-2}X(z)$$
$$Y(z)\left(1 - \frac{3}{4}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}\right) = X(z)(1 + 3z^{-1} + 2z^{-2})$$
$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1 + 3z^{-1} + 2z^{-2})}{\left(1 - \frac{3}{4}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}\right)}$$

i. Direct form I :

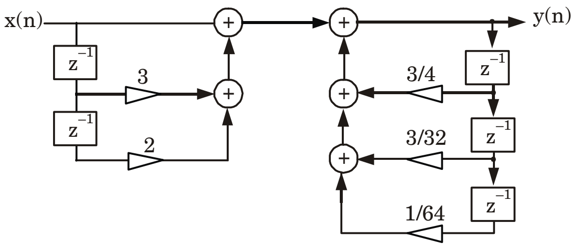


Fig. 5.

ii. Direct form II :

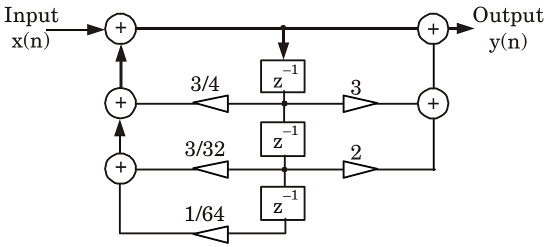


Fig. 6.

iii. Parallel realization :

$$H(z) = \frac{(1 + 3z^{-1} + 2z^{-2})}{\left(1 - \frac{3}{4}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}\right)} = \frac{z(z^2 + 3z + 2)}{\left(z^3 - \frac{3}{4}z^2 - \frac{3}{32}z - \frac{1}{64}\right)}$$
$$\frac{H(z)}{z} = \frac{(z+1)(z+2)}{64z^3 - 48z^2 - 6z - 1} = \frac{(z+1)(z+2)}{(z-0.87)(z+0.06)^2}$$

$$\text{Now, } \frac{(z+1)(z+2)}{(z-0.87)(z^2+0.06)^2} = \frac{A}{(z-0.87)} + \frac{Bz+C}{(z+0.06)^2}$$

On solving, we get

$A = 6.2$, $B = -5.2$, $C = -2.27$

$$\text{So, } \quad \frac{H(z)}{z} = \frac{6.2}{(z-0.87)} + \frac{(-5.2z-2.27)}{(z+0.06)^2}$$

$$\text{So, } \quad H(z) = \frac{6.2\,z}{(z-0.87)} + \frac{(-5.2z-2.27)z}{(z+0.06)^2}$$

$$H(z) = \frac{6.2}{(1-0.87z^{-1})} + \frac{(-5.2-2.27z^{-1})}{(1+0.12z^{-1}+0.0036z^{-2})}$$

$$H(z) = H_1(z) + H_2(z)$$

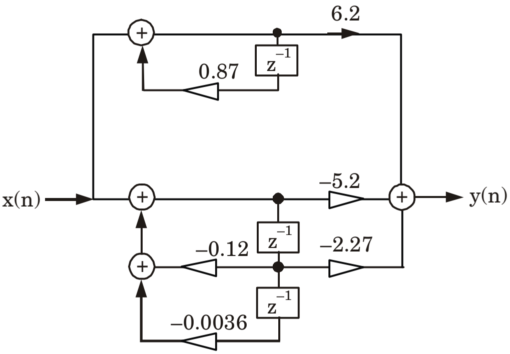


Fig. 7. Parallel realization of H(z).

f. Consider the causal linear shift invariant filter with the system function

$$H(z) = \frac{1+0.875z^{-1}}{(1+0.2z^{-1}+0.9z^{-2})(1-0.7z^{-1})}$$

Obtain following realization :

a. Direct form II

b. A cascade of first order and second order system realized in transposed direct form II.

c. A parallel connection of first order and second order system realized in direct form II.

Ans.

1. Given, $\quad H(z) = \dfrac{(1+0.875z^{-1})}{(1+0.2z^{-1}+0.9z^{-2})(1-0.7z^{-1})}$

a. Direct form II :

$$H(z) = \frac{(1+0.875z^{-1})}{(1+0.2z^{-1}+0.9z^{-2}-0.7z^{-1}-0.14z^{-2}-0.63z^{-3})}$$

$$H(z) = \frac{(1+0.875z^{-1})}{(1-0.5z^{-1}+0.76z^{-2}-0.63z^{-3})}$$

For direct form II :

$$H(z) = H_1(z).\, H_2(z)$$

$$H_1(z) = \frac{1}{(1-0.5z^{-1}+0.76z^{-2}-0.63z^{-3})}$$

and $\quad H_2(z) = (1 + 0.875z^{-1})$

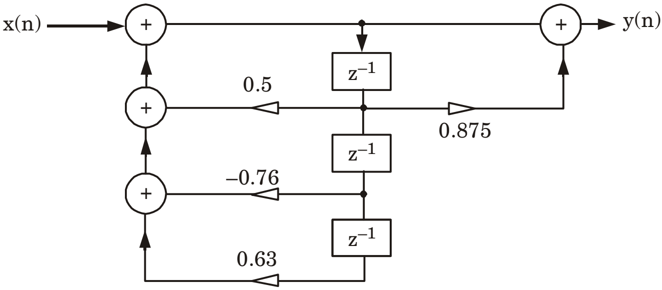


Fig. 8. Direct form II realization of H(z).

b. Cascade Form :

1. Given, $\quad H(z) = \dfrac{(1+0.875z^{-1})}{(1+0.2z^{-1}+0.9z^{-2})(1-0.7z^{-1})}$

2. Let $\quad H_1(z) = \dfrac{1}{(1+0.2z^{-1}+0.9z^{-2})}$

and $\quad H_2(z) = \dfrac{(1+0.875z^{-1})}{(1-0.7z^{-1})}$

$$H(z) = H_1(z)\, H_2(z)$$

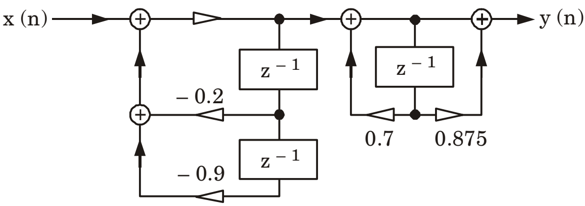


Fig. 9. Cascade form.

Cascade realization in transpose direct form II :

The transpose direct form II is obtained by making the following changes :

i. Reverse all signal flow graph direction.

ii. Interchange input and output.

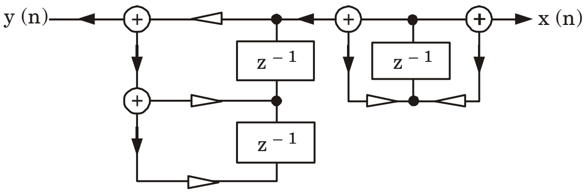


Fig. 10. Cascade realization in transpose direct form II.

c. Parallel realization :

$$H(z) = \frac{(1+0.875z^{-1})}{(1+0.2z^{-1}+0.9z^{-2})(1-0.7z^{-1})}$$

Now,

$$\frac{(1+0.875z^{-1})}{(1+0.2z^{-1}+0.9z^{-2})(1-0.7z^{-1})} = \frac{A}{(1-0.7\,z^{-1})} + \frac{Bz^{-1}+C}{(1+0.2\,z^{-1}+0.9\,z^{-2})}$$

On solving, we get, $A = 0.72,\ B = 0.93,\ C = 0.28$

So, $$H(z) = \frac{0.72}{(1-0.7\,z^{-1})} + \frac{(0.93\,z^{-1}+0.28)}{(1+0.2\,z^{-1}+0.9z^{-2})}$$

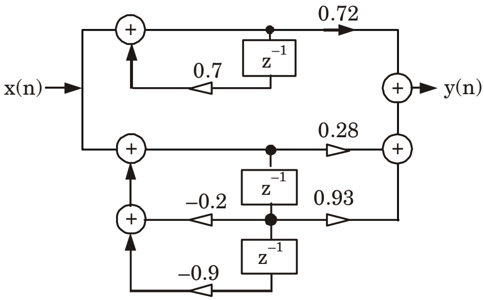


Fig. 11. Parallel realization.

g. A filter is to be designed with the following desired frequency response :

$$H_d(e^{j\omega}) = \begin{cases} 0 & \frac{-\pi}{4} \le \omega \le \frac{\pi}{4} \\ e^{-j2\omega} & \frac{\pi}{4} \le |\omega| \le \pi \end{cases}$$

Ans.

Given : $H_d(e^{j\omega}) = \begin{cases} 0 & \frac{-\pi}{4} \le \omega \le \frac{\pi}{4} \\ e^{-j2\omega} & \frac{\pi}{4} \le |\omega| \le \pi \end{cases}$

To Design : Filter.

1. Impulse response of the filter is given as :

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\pi/4} e^{-j\omega 2}.e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\pi/4}^{\pi} e^{-j\omega 2}.e^{j\omega n}d\omega$$

$$h_d(n) = \frac{1}{\pi(n-2)}\{\sin \pi(n-2) - \sin (n-2)\pi/4\}, n \neq 2$$

2. Let us assume the window function, $w(n) = \begin{cases} 1, & 0 \leq n \leq 4 \\ 0, & \text{otherwise} \end{cases}$

So $h_d(0) = \frac{1}{2\pi}$, $h_d(4) = \frac{1}{2\pi}$

$$h_d(2) = \frac{3}{4}, h_d(1) = h_d(3) = \frac{1}{\sqrt{2}\pi}$$

$$h(n) = h_d(n) \times \omega(n)$$

3. $$h(0) = h(4) = \frac{1}{2\pi} \text{ and } h(2) = \frac{3}{4}$$

$$h(1) = h(3) = \frac{1}{\sqrt{2}\pi}$$

4. The frequency response $H(e^{j\omega})$ is given by

$$H(e^{j\omega}) = \sum_{n=0}^{4} h(n)\, e^{-j\omega n}$$

$$= h(0) + h(1)\, e^{-j\omega} + h(2)\, e^{-j2\omega} + h(3)\, e^{-j3\omega} + h(4)\, e^{-j4\omega}$$

$$\therefore \quad H(e^{j\omega}) = e^{-j2\omega}\left[0.75 - \frac{\sqrt{2}}{\pi}\cos\omega - \frac{1}{11}\cos 2\,\omega\right]$$

h. Transform the prototype LPF with system function

$$H_{LP}(s) = \frac{\Omega_p}{s+\Omega_p} \text{ into } a$$

i. HPF with cut-off frequency Ω_p.

ii. BPF with upper and lower cut-off frequencies Ω_1 and Ω_2 respectively.

Ans.

1. From the analog frequency transformation of LPF to HPF is given as

$$s \to \frac{\Omega_c\,\Omega_c^*}{s}$$

2. Thus, the system function of HPF is given as

$$H_{HPF}(s) = H_p(s)\Big|_{s=\frac{\Omega_c\Omega_c^*}{s}} = \left[\frac{\Omega_c}{s+\Omega_c}\right]\Bigg|_{s=\frac{\Omega_c\Omega_c^*}{s}}$$

$$= \frac{\Omega_c}{\frac{\Omega_c\Omega_c^*}{s} + \Omega_c} = \frac{\Omega_c\, s}{\Omega_c\, s + \Omega_c\, \Omega_c^*} = \frac{\Omega_c\, s}{\Omega_c\,(s + \Omega_c^*)}$$

$$= \frac{s}{s + \Omega_c^*}$$

i. HPF with cut-off frequency Ω_p.

1. For LPF to HPF, we use $s \to \frac{\Omega_c\,\Omega_{HP}}{s}$

$$H_{HP} = \frac{\Omega_p}{\frac{\Omega_c.\Omega_{HP}}{s} + \Omega_p} = \frac{\Omega_p.s}{\Omega_c\Omega_{HP} + \Omega_p.s} = \frac{s}{s + \frac{\Omega_c\,\Omega_{HP}}{\Omega_p}}$$

ii. BPF with upper and lower cut-off frequencies Ω_u and Ω_l respectively.

1. Band-pass filter with upper and lower cut-off frequency Ω_u and Ω_l.

$$s \to \Omega_c.\frac{s^2 + \Omega_l\Omega_u}{s(\Omega_u - \Omega_l)}$$

So, $$H_{BP} = \frac{\Omega_p}{\Omega_c\left(\frac{s^2 + \Omega_l\Omega_u}{s(\Omega_u - \Omega_l)}\right) + s} = \frac{(\Omega_p / \Omega_c)s(\Omega_u - \Omega_l)}{s^2 + \Omega_l\Omega_u + \Omega_P.s(\Omega_u - \Omega_l)}$$

$$H_{BP} = \frac{s(\Omega_p / \Omega_c)(\Omega_u - \Omega_l)}{s^2 + s.\Omega_p(\Omega_u - \Omega_l) + \Omega_l\Omega_u}$$

SECTION-C

Attempt any **two** questions from this section. **(15 × 2 = 30)**

3. a. Prove that multiplication of the DFTs of two sequences is equivalent to the circular convolution of the two sequences in the time domain.

Ans.

1. Consider the two sequences $x(n)$ and $y(n)$ which are of finite duration. Let $X(k)$ and $Y(k)$ be the N-point DFTs of the two sequences respectively and they are given by

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}\ , k = 0, 1, ..., N-1.$$

$$Y(k) = \sum_{n=0}^{N-1} y(n)e^{-j2\pi nk/N}\ , k = 0, 1, ..., N-1.$$

2. Let $x_3(m)$ be another sequence of length N and its N-point DFT be $X_3(k)$ which is a product of $X(k)$ and $Y(k)$,
i.e., $X_3(k) = X(k)\, Y(k), k = 0, 1,..., N-1.$
The sequence $x_3(m)$ can be obtained by taking the inverse DFT of $X_3(k)$,

i.e., $x_3(m) = \text{IDFT}[X_3(k)]$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)e^{j2\pi mk/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y(k)e^{j2\pi mk/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}\right]\left[\sum_{l=0}^{N-1} y(l)e^{-j2\pi lk/N}\right]e^{j2\pi mk/N}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\sum_{l=0}^{N-1} y(l)\left[\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N}\right] \quad ...(1)$$

4. Consider the term within the brackets in eq. (1), it has the form

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & \text{for } a = 1 \\ \dfrac{1-a^N}{1-a}, & \text{for } a \neq 1 \end{cases}$$

where, $a = e^{j2\pi(m-n-l)/N}$.

5. When $(m-n-l)$ is a multiple of N, then $a = 1$, otherwise $a^N = 1$ for any value of $a \neq 0$.

Therefore,

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & l = m-n+pN, N = (m-n)(\text{mod } N), p : \text{integer}) \\ 0, & \text{otherwise} \end{cases}$$

6. Now, $x_3(m)$ becomes

$$x_3(m) = \sum_{n=0}^{N-1} x(n)y(m-n, (\text{mod } N)), N = 0, 1, ..., N-1$$

where, $y((m-n) \text{ mod } N)$ is the reflected and circularly shifted version of $y(m)$ and n represents the number of indices that the sequence $x(n)$ is shifted to the right.

b. If the 10-point DFT of $x(n) = \delta(n) - \delta(n-1)$ and $h(n) = u(n) - u(n-10)$ are $X(k)$ and $H(k)$ respectively, find the sequence $W(n)$ that corresponds to the 10-point inverse DFT of the product $H(k)X(k)$.

Ans.

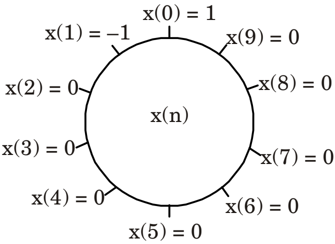
Given : $x(n) = \delta(n) - \delta(n-1)$, $h(n) = u(n) - u(n-10)$
To Find : $w(n)$

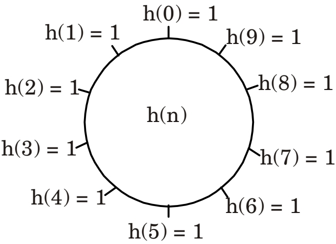
1. Since, product of $H(k)$ and $X(k)$ correspond to the circular convolution of $x(n)$ and $h(n)$.

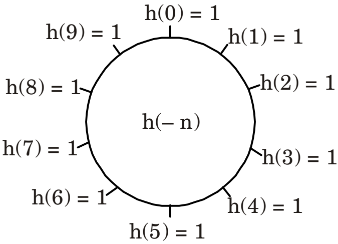
$$x(n) = \delta(n) - \delta(n-1)$$

$$x(n) = \{1, -1, 0, 0, 0, 0, 0, 0, 0, 0\}$$

and $$h(n) = u(n) - u(n-10) = \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$$







2. Using circular convolution :

$$w(n) = x(n) * h(n)$$

so,

$$w(0) = 1 - 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$
$$w(1) = 1 - 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$
$$w(2) = 1 - 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$
$$w(3) = 1 - 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$
$$w(4) = 1 - 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$
$$w(5) = 1 - 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$
$$w(6) = 1 - 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$
$$w(7) = 1 - 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$
$$w(8) = 1 - 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$
$$w(9) = 1 - 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$

Hence, $w(n) = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0\} = 0$

4. a.

i. Compute 4 point DFT of the following sequence using linear transformation matrix $x(n) = (1, 1, -2, -2)$

ii. Find IDFT $x(n)$ from $X(k)$ calculated in part(i).

Ans.

Given : $x(n) = (1, 1, -2, -2)$
To Find : DFT and IDFT.

i. DFT :

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

Here, $N = 4$

$$[X(k)] = [W_4^{kn}][x(n)]$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1+1-2-2 \\ 1-j+2-2j \\ 1-1-2+2 \\ 1+j+2+2j \end{bmatrix} = \begin{bmatrix} -2 \\ 3-3j \\ 0 \\ 3+3j \end{bmatrix}$$

So, $X(k) = \{-2, 3-3j, 0, 3+3j\}$

ii. IDFT :

$$\text{IDFT} = x(n) = \frac{1}{N}[W_N^*].X(k)$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} -2 \\ 3-3j \\ 0 \\ 3+3j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -2+3-3j+0+3+3j \\ -2+3j+3+0-3j+3 \\ -2-3+3j+0-3-3j \\ -2-3j-3+0+3j-3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 4 \\ -8 \\ -8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -2 \end{bmatrix}$$

$x(n) = (1, 1, -2, -2)$

b. Use Radix-2 DIT algorithm for efficient computation of 8 point DFT of $x(n) = 2^n$.

Ans.

Given : $x(n) = 2^n$ and $N = 8$

To Find : $X(k)$ and reduction factor.

1. $x(0) = 1, x(1) = 2, x(2) = 4, x(3) = 8$
   $x(4) = 16, x(5) = 32, x(6) = 64, x(7) = 128$
   $x(n) = \{1, 2, 4, 8, 16, 32, 64, 128\}$
2. We know, $W_8^0 = 1$
   $W_8^1 = 0.707 - j0.707$
   $W_8^2 = -j$
   $W_8^3 = -0.707 - j0.707$
3. Using DIT FFT algorithm, we can find $X(k)$ from the given sequence $x(n)$ as shown in Fig. 12.
4. Computation reduction factor

$$= \frac{\text{Number of complex multiplication requried for direct (DFT)}}{\text{Number of complex multiplications required for FFT algorithm}}$$

$$= \frac{N^2}{\frac{N}{2}\log_2(N)} = \frac{8^2}{\frac{8}{2}\log_2 8} = \frac{64}{12} = 5.33$$

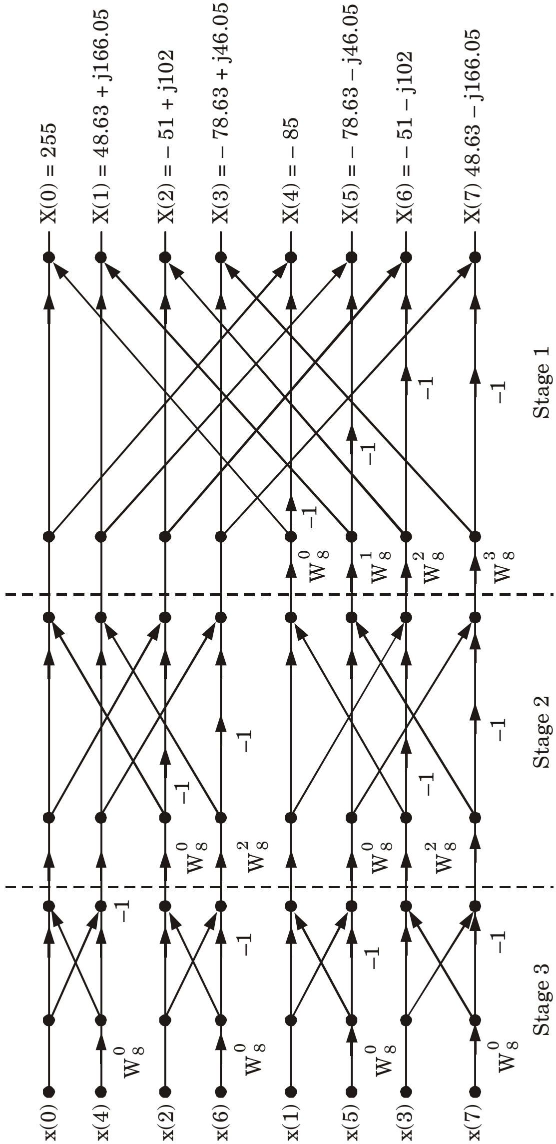


Fig. 12.

5. a. An FIR filter has following symmetry in the impulse response :

$$\boldsymbol{h(n) = h(M - 1 - n)} \textbf{ for } \boldsymbol{M} \textbf{ odd.}$$

Derive its frequency response and show that it has linear phase.

Ans.

1. The discrete-time fourier transform (DTFT) of the impulse response $h(n)$ is given by,

$$H(e^{j\omega}) = \sum_{n=0}^{M-1} h(n)\, e^{-j\omega nT} = |H(e^{j\omega})|\, e^{j\phi(\omega)} \quad \text{...(1)}$$

2. If the length M of filters is odd, then eq. (1) can be written as

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-3}{2}} h(n)\, e^{-j\omega nT} + h\left(\frac{M-1}{2}\right) e^{-j\omega\left(\frac{M-1}{2}\right)T} \quad \text{...(2)}$$

3. We have $h(n) = h(M-1-n)$ for $0 < n < M-1$ ...(3)

Eq. (2) can be written as by using eq. (3).

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-3}{2}} h(n)\, [e^{-j\omega nT} + e^{-j\omega(M-1-n)T}] + h\left(\frac{M-1}{2}\right) e^{-j\omega\left(\frac{M-1}{2}\right)T} \quad \text{...(4)}$$

4. Factorising $e^{-j\omega(M-1)T/2}$ in eq. (4),

$$H(e^{j\omega T}) = e^{-j\omega\left(\frac{M-1}{2}\right)T} \left\{ \sum_{n=0}^{\frac{M-3}{2}} h(n) \left[e^{j\omega\left(\frac{M-1}{2}-n\right)T} + e^{-j\omega\left(\frac{M-1}{2}-n\right)T} \right] + h\left(\frac{M-1}{2}\right) \right\} \quad \text{...(5)}$$

5. Put $k = \left(\frac{M-1}{2}\right) - n$ in eq. (5)

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{M-1}{2}\right)T} \left\{ \sum_{k=1}^{\frac{M-1}{2}} h\left(\frac{M-1}{2} - k\right) [e^{j\omega kT} + e^{-j\omega kT}] + h\left(\frac{M-1}{2}\right) \right\} \quad \text{...(6)}$$

or
$$H(e^{j\omega}) = e^{-j\omega\left(\frac{M-1}{2}\right)T} \left\{ \sum_{k=0}^{\frac{M-1}{2}} b(k) \cos \omega kT \right\}$$

where,
$$b(k) = 2h\left(\frac{M-1}{2} - k\right) ; \text{ for } 1 \le k \le \left(\frac{M-1}{2}\right)$$

It gives $$b(0) = h\left(\frac{M-1}{2}\right)$$

6. The $H(e^{j\omega})$ can be written as

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{M-1}{2}\right)T}\{M(\omega)\} \quad ...(7)$$

Here magnitude response

$$M(\omega) = \sum_{k=0}^{(M-1)/2} b(k)\cos(\omega kT) \quad ...(8)$$

And phase response function

$$\phi(\omega) = -\omega(M-1)/2 \quad ...(9)$$

7. The eq. (9) represents the $(M-1)/2$ units delay in sampling time. Hence, FIR filter will have constant phase and group delays and thus the phase of the filter will be linear.

b. Discuss the bilinear transformation method of converting analog IIR filter into digital IIR filter. What is frequency warping ?

Ans.

A. Bilinear transformation method :

1. Bilinear transformation is a one to one mapping from the s-domain to the z-domain.
2. Bilinear transformation is a conformal mapping that transforms the $j\Omega$-axis into the unit circle in the z-plane only once. Thus the aliasing effect is avoided.
3. The transformation of a stable analog filter result in a stable digital filter as all the poles in the left half of the s-plane are mapped onto points inside the unit circle of the z-domain.
4. It is obtained by using the trapezoidal formula of numerical integration.

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right) \quad ...(1)$$

5. For mapping of s-plane to z-plane, substitute $z = re^{j\omega}$ and $s = \sigma + j\Omega$ in eq. (1) we get

$$\sigma + j\Omega = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = \frac{2}{T}\left(\frac{re^{j\omega}-1}{re^{j\omega}+1}\right) = \frac{2}{T}\left[\frac{r(\cos\omega + j\sin\omega)-1}{r(\cos\omega + j\sin\omega)+1}\right]$$

$$= \frac{2}{T}\left[\frac{(r\cos\omega - 1 + j\,r\sin\omega)}{(r\cos\omega + 1 + j\,r\sin\omega)}\right]$$

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{r\cos\omega - 1 + jr\sin\omega}{r\cos\omega + 1 + jr\sin\omega}\right]\left[\frac{r\cos\omega + 1 - jr\sin\omega}{r\cos\omega + 1 - jr\sin\omega}\right]$$

$$= \frac{2}{T}\left[\frac{(r^2\cos^2\omega - 1 + r^2\sin^2\omega + j2\,r\sin\omega)}{(r\cos\omega + 1)^2 + r^2\sin^2\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r^2\cos^2\omega - 1 + r^2\sin^2\omega + j2r\sin\omega}{1 + r^2\cos^2\omega + 2r\cos\omega + r^2\sin^2\omega}\right]$$

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega} + j\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right] \quad ...(2)$$

6. On comparison real and imaginary part we get,

$$\sigma = \frac{2}{T}\left(\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega}\right) \quad ...(3)$$

and $$\Omega = \frac{2}{T}\left(\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right) \quad ...(4)$$

7. On the basis of value of r three cases can be discussed as below :
   Case I : If $r < 1$, then $\sigma < 0$
   Case II : If $r > 1$, then $\sigma > 0$
   Case III : If $r = 1$, then $\sigma = 0$
   From all three cases it is clear that the left half of the s-plane is mapped on to the points inside the unit circle in the z-plane.

B. Frequency warping :

1. We know, $$\Omega = \frac{2}{T_s} \times \frac{2r\sin\omega}{r^2 + 2r\cos\omega + 1} \quad ...(1)$$
2. For the unit circle, $r = 1$. Thus putting $r = 1$ in the eq. (1) we get,

$$\Omega = \frac{2}{T_s} \times \frac{2\sin\omega}{1 + 2\cos\omega + 1}$$

$\therefore$ $$\Omega = \frac{2}{T_s} \times \frac{2\sin\omega}{2 + 2\cos\omega}$$

$$\Omega = \frac{2}{T}\left(\frac{\sin\omega}{1 + \cos\omega}\right)$$

$$= \frac{2}{T}\left[\frac{2\sin(\omega/2)\cos(\omega/2)}{\cos^2(\omega/2) + \sin^2(\omega/2) + \cos^2(\omega/2) - \sin^2(\omega/2)}\right]$$

or $$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

or $$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$$

3. In this method the entire range in Ω is mapped only once into the range $-\pi \le \omega \le \pi$. This mapping is non-linear. The lower frequencies in analog domain are expanded in digital domain while the higher frequencies in analog domain are compressed in digital domain.
4. Non-linearity in mapping is due to arc tangent function and this is known as frequency warping.

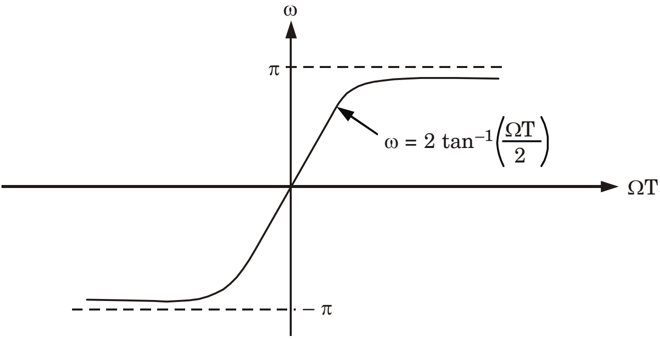


Fig. 13. Frequency warping.

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B. Tech.
(SEM. VI) EVEN SEMESTER THEORY EXAMINATION, 2016-17
DIGITAL SIGNAL PROCESSING

Time : 3 Hours **Max. Marks : 100**

Note : Be precise in your answer. In case of numerical problem assume data wherever not provided.

SECTION-A

1. Attempt the following questions : **(2 × 10 = 20)**

a. Define digital signal processing.

b. Draw the block diagram of digital signal processing.

c. Explain the basic elements required for realization of digital system.

d. Define linear convolution and its physical significance.

e. What is the fundamental time period of the signal $x(t) = \sin 15\pi t$?

f. Draw a transformation matrix of size 4 × 4 and explain the properties of twiddle factor.

g. Differentiate between IIR and FIR filters.

h. Enumerate the advantages of DSP over ASP.

i. Write the expression for computation efficiency of an FFT.

j. Calculate the DFT of the sequence $s(n) = \{1, 2, 1, 3\}$.

SECTION-B

2. Attempt any **five** of the following questions : **(10 × 5 = 50)**

a. Obtain the parallel form realization for the transfer function $H(z)$ given below :

$$H(z) = \frac{2 + z^{-1} + 1/4\, z^{-2}}{\left(1 + \frac{1}{2} z^{-1}\right)\left(1 + z^{-1} + \frac{1}{2} z^{-2}\right)}$$

b. Calculate the DFT of $x(n) = \cos an$

c. Derive and draw the flow graph for DIF FFT algorithm for $N = 8$.

d. Determine $H(z)$ using the impulse invariant technique for the analog system function

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)}$$

e. Determine $H(z)$ for a Butterworth filter satisfying the following constraints

$$\sqrt{0.5} \le |H(e^{j\omega})| \le 1 \quad ; 0 \le \omega \le \frac{\pi}{2}$$

$$|H(e^{j\omega})| \le 0.2 \, ; \frac{3\pi}{4} \le \omega \le \pi$$

with $T = 1$ sec. Apply impulse invariant transformation.

f. Given $x(n) = 2^n$ and $N = 8$, find $X(k)$ using DIT FFT algorithm. Also calculate the computational reduction factor.

g. Design a low-pass filter with the following desired frequency response

$$H_d(e^{jw}) = \begin{cases} e^{-j2\omega}, & \frac{-\pi}{4} \le \omega \le \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |\omega| < \pi \end{cases}$$ **and using window function**

$$w(n) = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

h. Convert the analog filter with system function $H(s) = \dfrac{s+0.1}{(s+0.1)^2+9}$ into digital filter with a resonant frequency of $\omega_r = \pi/4$ using bilinear transformation.

SECTION-C

Attempt any **two** of the following questions : **(15 × 2 = 30)**

3. i. Obtain the ladder structure for the system function $H(z)$ given below.

$$H(z) = \frac{2+8z^{-1}+6z^{-2}}{1+8z^{-1}+12z^{-2}}$$

ii. Compute the circular convolution of two discrete time sequences $x_1(n) = \{1, 2, 1, 2\}$ and $x_2(n) = \{3, 2, 1, 4\}$.

4. a. Determine the 4-point discrete time sequence from its DFT $X(k) = \{4, 1 - j, - 2, 1 + j\}$

b. Explain the following phenomenon :

i. Gibbs oscillations,

ii. Frequency warping

5. a. Derive the relation between DFT and z-transform of a discrete time sequence $x(n)$.

b. Design a digital Chebyshev filter to satisfy the constraints

$$0.707 \le |H(e^{j\omega})| \le 1, \; 0 \le \omega \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.1, \; 0.5\pi \le \omega \le \pi$$

Using bilinear transformation with $T = 1$ s.

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SOLUTION OF PAPER (2016-17)

Note : Be precise in your answer. In case of numerical problem assume data wherever not provided.

SECTION-A

1. Attempt the following questions : **(2 × 10 = 20)**

a. Define digital signal processing.

Ans. Digital signal processing is a method of extracting information from digital signal with the help of digital processor. Output signals are also digital in nature.

b. Draw the block diagram of digital signal processing.

Ans.

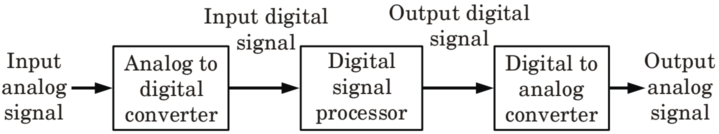


Fig. 1. Block diagram of digital signal processing.

c. Explain the basic elements required for realization of digital system.

Ans.

1. Analog to digital converter.
2. Digital signal processor.
3. Digital to analog converter.

d. Define linear convolution and its physical significance.

Ans.

A. Linear convolution :

Let us consider two finite duration sequences $x(n)$ and $h(n)$, the linear convolution of $x(n)$ and $h(n)$ is given as

$$y(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

B. Physical significance : Linear convolution can be used to find the response of a filter.

e. What is the fundamental time period of the signal $x(t) = \sin 15\,\pi t$?

Ans.

Given : $x(t) = \sin 15\pi t$
To Find : T.

1. $$\omega = 15\pi$$
2. $$T = \frac{2\pi}{\omega} = \frac{2\pi}{15\pi} = 0.133 \text{ sec}$$

f. Draw a transformation matrix of size 4 × 4 and explain the properties of twiddle factor.

Ans.

A. Transformation matrix :

For $N = 4$, W_4^{nk} is 4 × 4 matrix and is given as

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

B. Properties of twiddle factor :

i. Symmetry property

$$W_N^{k+N/2} = -W_N^k$$

ii. Periodicity property

$$W_N^{k+N} = -W_N^k$$

g. Differentiate between IIR and FIR filters.

Ans.

S. No.	FIR filter	IIR filter
1.	They have finite impulse response.	They have infinite impulse response.
2.	Always stable.	Sometimes unstable.
3.	Have exact linear phase response.	Non-linear phase response.
4.	It requires more memory, higher computational complexity and involves more parameters.	It requires less memory, lower computational complexity and involves fewer parameters.

h. Enumerate the advantages of DSP over ASP.

Ans.

1. DSP is more flexible than ASP.
2. Accuracy of DSP is greater than ASP.
3. DSP has better storage capability over ASP.
4. DSP has mathematical processing function that is not present in ASP.

i. Write the expression for computation efficiency of an FFT.

Ans.

$$\eta_{comp} = \frac{\text{Number of complex multiplication required in FFT algorithm}}{\text{Number of complex multiplications required in direct DFT}}$$

$$= \frac{\frac{N}{2}\log_2 N}{N \times N} = \frac{\log_2 N}{2N}$$

j. Calculate the DFT of the sequence $s(n) = \{1, 2, 1, 3\}$.

Ans.

Given : $n = \{1, 2, 1, 3\}$
To Find : DFT

1. DFT of a finite duration sequence $s(n)$ is defined as

$$S(k) = \sum_{n=0}^{N-1} s(n)\, e^{-j2\pi nk/N}, \quad k = 0, 1, \ldots\ldots, N-1$$

Here, $N = 4$.

2. For $k = 0$,

$$S(0) = \sum_{n=0}^{3} s(n)\, e^{-j2\pi n(0)/4} = \sum_{n=0}^{3} s(n) = 1 + 2 + 1 + 3 = 7$$

3. For $k = 1$,

$$S(1) = \sum_{n=0}^{3} s(n)\, e^{-j2\pi n(1)/4}$$

$$= 1 + 2e^{-j\pi/2} + 1e^{-j\pi} + 3e^{-j3\pi/2}$$

$$= 1 + 2(-j) + 1(-1) + 3(j) = j$$

4. For $k = 2$,

$$S(2) = \sum_{n=0}^{3} s(n)\, e^{-j2\pi n(2)/4}$$

$$= 1 + 2e^{-j\pi} + 1e^{-j2\pi} + 3e^{-j3\pi}$$

$$= 1 + 2(-1) + 1(1) + 3(-1) = -3$$

5. For $k = 3$,

$$S(3) = \sum_{n=0}^{3} s(n)\, e^{-j2\pi n(3)/4}$$

$$= 1 + 2e^{-j3\pi/2} + 1e^{-j3\pi} + 3e^{-j9\pi/2}$$

$$= 1 + 2(j) + 1(-1) + 3(-j) = -j$$

Hence, $S(k) = \{7, j, -3, -j\}$

SECTION-B

2. Attempt any **five** of the following questions : **(10 × 5 = 50)**

a. Obtain the parallel form realization for the transfer function $H(z)$ given below :

$$H(z) = \frac{2 + z^{-1} + 1/4\, z^{-2}}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 + z^{-1} + \frac{1}{2}z^{-2}\right)}$$

Ans.

1. Given, $H(z) = \dfrac{2 + z^{-1} + \frac{1}{4}z^{-2}}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 + z^{-1} + \frac{1}{2}z^{-2}\right)}$

2. $H(z)$ can be written as

$$H(z) = \frac{8z^3 + 4z^2 + z}{(2z+1)(2z^2+2z+1)}$$

3. $$\frac{H(z)}{z} = \frac{8z^2 + 4z + 1}{(2z+1)(2z^2+2z+1)}$$

$$= \frac{A}{2z+1} + \frac{Bz + C}{2z^2 + 2z + 1} \qquad ...(1)$$

4. On solving eq. (1), we get,

$$A = 2, B = 2, C = -1$$

5. Now eq. (1) can be written as

$$\frac{H(z)}{z} = \frac{2}{2z+1} + \frac{(2z-1)}{2z^2+2z+1}$$

$$H(z) = \frac{2z}{2z+1} + \frac{z(2z-1)}{2z^2+2z+1}$$

So, $$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{1 - (1/2)z^{-1}}{1 + z^{-1} + \frac{1}{2}z^{-2}}$$

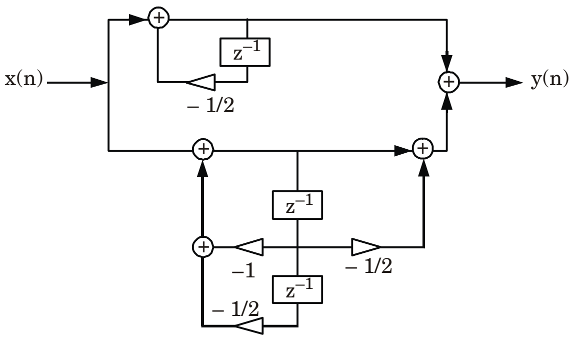


Fig. 2.

b. Calculate the DFT of $x(n)$ = cos an

Ans.

Given : $x(n) = \cos an$
To Find : DFT.

1.
$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N}, \quad k = 0, 1, ..., N-1$$

$$= \sum_{n=0}^{N-1} (\cos an)\, e^{-j2\pi nk/N} = \sum_{n=0}^{N-1} \left(\frac{e^{jan} + e^{-jan}}{2}\right) e^{-j2\pi nk/N}$$

$$= \frac{1}{2}\left[\sum_{n=0}^{N-1} (e^{j(a-2\pi k/N)n}) + \sum_{n=0}^{N-1} (e^{-j(a+2\pi k/N)n})\right]$$

2. Putting $n = 0, 1, ..., N-1$

$$= \frac{1}{2}\left[\frac{1-(e^{j(a-2\pi k/N)})^N}{1-e^{j(a-2\pi k/N)}} + \frac{1-(e^{-j(a+2\pi k/N)})^N}{1-e^{-j(a+2\pi k/N)}}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{jaN}}{1-e^{j(a-2\pi k/N)}} + \frac{1-e^{-jaN}}{1-e^{-j(a+2\pi k/N)}}\right]$$

c. Derive and draw the flow graph for DIF FFT algorithm for $N = 8$.

Ans.

1. Here $N = 8$, thus, n and $k = 0, 1, 2, ..., 7$

$$X(k) = [X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)]$$

Stage I :

i.
$$X(2p) = [X(0), X(2), X(4), X(6)]$$
$$X(2p+1) = [X(1), X(3), X(5), X(7)]$$
$$h(n) = [x(n) + x(n + N/2)]$$

and
$$f(n) = [x(n) - x(n + N/2)]$$

we get

$$h(0) = x(0) + x(4) \quad f(0) = x(0) - x(4)$$
$$h(1) = x(1) + x(5) \quad f(1) = x(1) - x(5)$$
$$h(2) = x(2) + x(6) \quad f(2) = x(2) - x(6)$$
$$h(3) = x(3) + x(7) \quad f(3) = x(3) - x(7)$$

ii. The first stage of DIF FFT algorithm can be drawn in signal flow graph (SFG) form as shown in Fig. 3.

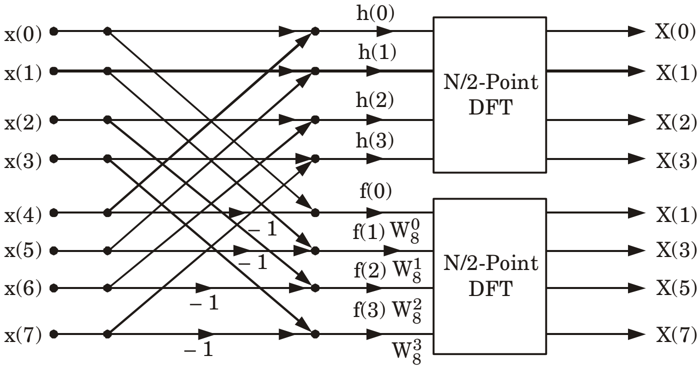


Fig. 3. SFG of first stage of DIF-FFT for N = 8.

Stage II :

i. Now $X(2p)$ and $X(2p + 1)$ are divided into even and odd numbered values sequence.

ii. Taking $X(2p)$ first we have

$$X(2p) = \sum_{n=0}^{\frac{N}{4}-1} h(n)W_N^{2pn} + \sum_{n=N/4}^{\frac{N}{2}-1} h(n)W_N^{2pn}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} h(n)W_N^{2pn} + \sum_{n=0}^{\frac{N}{4}-1} h\left(n+\frac{N}{4}\right)W_N^{2p\left(n+\frac{N}{4}\right)}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} h(n)W_{N/2}^{pn} + W_N^{\left(2p\frac{N}{4}\right)} \sum_{n=0}^{\frac{N}{4}-1} h\left(n+\frac{N}{4}\right)W_{N/2}^{pn} \quad ...(1)$$

Since $W_N^{N/2} = -1$

iii. Thus eq. (1) can be written as

$$X(2p) = \sum_{n=0}^{\frac{N}{4}-1} h(n)W_N^{2pn} + (-1)^p \sum_{n=0}^{\frac{N}{4}-1} h\left(n+\frac{N}{4}\right)W_N^{2pn} \quad ...(2)$$

2. Again for even and odd values of 'p' we have two cases :

Case I : When p = even, $p = 2r$ $(-1)^{2r} = 2r$

Then eq. (2) will become as

$$X(4r) = \sum_{n=0}^{\frac{N}{4}-1} \left[h(n) + h\left(n+\frac{N}{4}\right)\right] W_N^{4rn}$$

$$X(4r) = \sum_{n=0}^{\frac{N}{4}-1} g(n)W_N^{4rn} \quad ...(3)$$

where, $$g(n) = h(n) + h\left(n+\frac{N}{4}\right) \quad ...(4)$$

Case II :

a. When p = odd *i.e.*, $p = (2r + 1)$ then $(-1)^{2r+1} = -1$

$$X(4r+2) = \sum_{n=0}^{\frac{N}{4}-1}\left[h(n) - h\left(n+\frac{N}{4}\right)\right] W_N^{(2r+1)2n}$$

$$= \sum_{n=0}^{\frac{N}{4}-1}\left[h(n) - h\left(n+\frac{N}{4}\right)\right] W_N^{2n} \cdot W_N^{4rn}$$

$$X(4r+2) = \sum_{n=0}^{\frac{N}{4}-1} [A(n) W_N^{4rn}]\, W_N^{2n}$$

where, $A(n) = h(n) - h\left(n+\frac{N}{4}\right)$ ...(5)

b. Therefore for 2nd stage we get

$$g(0) = h(0) + h(2)$$
$$g(1) = h(1) + h(3)$$
$$A(0) = h(0) - h(2)$$
$$A(1) = h(1) - h(3)$$

Similarly we get

$$X(2p+1) = \sum_{n=0}^{\frac{N}{2}-1} [f(n) W_N^{2pn}]\, W_N^{n}$$

c. Dividing $X(2p + 1)$ into even and odd parts.

$$X(2p+1) = W_N^n \left[\sum_{n=0}^{\frac{N}{4}-1} f(n) W_N^{2pn} + \sum_{n=N/4}^{\frac{N}{2}-1} f(n)\, W_N^{2pn}\right]$$

$$= \left[\sum_{n=0}^{\frac{N}{4}-1} f(n) W_N^{2pn} \cdot W_N^{n} + \sum_{n=0}^{\frac{N}{4}-1} f\left(n+\frac{N}{4}\right) W_N^{\left(n+\frac{N}{4}\right)} \cdot W_N^{2p\left(n+\frac{N}{4}\right)}\right]$$

$$= \left[\sum_{n=0}^{\frac{N}{4}-1} f(n) W_N^{n} \cdot W_N^{2pn} + \sum_{n=0}^{\frac{N}{4}-1} f\left(n+\frac{N}{4}\right) W_N^{n} \cdot W_N^{2pn} \cdot W_N^{(2p+1)N/4}\right]$$

$$X(2p+1) = \sum_{n=0}^{\frac{N}{4}-1}\left[f(n) + W_N^{(2p+1)N/4} f\left(n+\frac{N}{4}\right)\right] W_N^{(2p+1)n} \quad ...(6)$$

3. Again this can also be discussed for even and odd values of 'p'. The results will be same as given in eq. (3) and (6).

Case I : $2p + 1 = 2l$ (even)

$$X(2l) = \sum_{n=0}^{\frac{N}{4}-1}\left[f(n) + W_N^{2lN/4} f\left(n+\frac{N}{4}\right)\right] W_N^{2ln}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[f(n) + (-1)^{2l} f\left(n + \frac{N}{4}\right) \right] W_N^{2ln}$$

$$X(2l) = \sum_{n=0}^{\frac{N}{4}-1} B(n)\, W_N^{2ln} \quad ...(7)$$

where $$B(n) = f(n) + f\left(n + \frac{N}{4}\right) \quad ...(8)$$

Case II :

a. Taking $(2p + 1) = (2l + 1)$ odd

we get

$$X(2l + 1) = \sum_{n=0}^{\frac{N}{4}-1} \left[f(n) + W_N^{(2l+1)N/4} \cdot f\left(n + \frac{N}{4}\right) \right] W_N^{(2l+1)n}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[f(n) - f\left(n + \frac{N}{4}\right) \right] W_N^{2n} \cdot W_N^{2ln}$$

$$X(2l + 1) = \sum_{n=0}^{\frac{N}{4}-1} [C(n) W_N^{2ln}] \cdot W_N^{2n} \quad ...(9)$$

where, $$C(n) = f(n) - f\left(n + \frac{N}{4}\right)$$

b. Therefore for this stage we get

$$B(0) = f(0) + f(2)$$

$$C(0) = f(0) - f(2)$$

$$B(1) = f(1) + f(3)$$

$$C(1) = f(1) - f(3)$$

c. The SFG for 8-point DFT using DIF FFT algorithm is shown in Fig. 4.

d. The decimation process is continued until the size of the last DFT is not equal to radix of the FFT. Number of decimation stages (l) is given as

$$l = \log_2 N$$

or $$N = 2^l$$

e. The $X(2l)$ and $X(2l + 1)$ are again decomposed in even and odd parts.

f. Since for the 3rd stage, we have $N/4 = 8/4 = 2$ point DFT. This DFT can be drawn by using the butterfly structure as given in DIT-FFT.

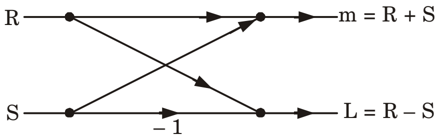


Fig. 4.

4. The total SFG for all three stages DIF-FFT is shown in Fig. 5.

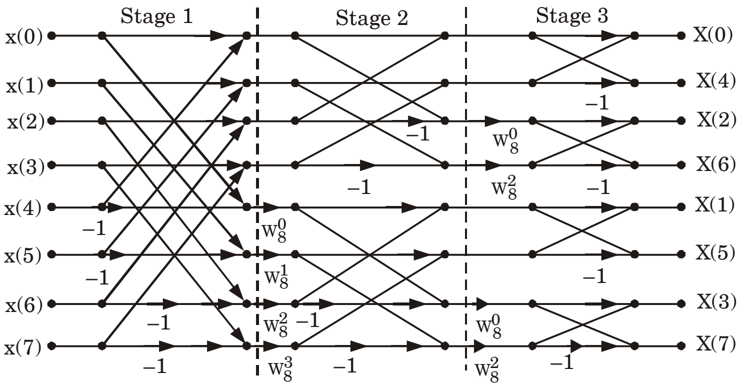


Fig. 5. SFG for 8-point DFT using DIF-FFT.

d. Determine *H*(*z*) using the impulse invariant technique for the analog system function

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)}$$

Ans.

Given : $H(s) = \dfrac{1}{(s+0.5)(s^2+0.5s+2)}$

To Find : $H(z)$.

1. Using partial fractions, $H(s)$ can be written as,

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)} = \frac{A}{s+0.5} + \frac{Bs+C}{s^2+0.5s+2}$$

Therefore, $A(s^2 + 0.5s + 2) + (Bs + C)(s + 0.5) = 1$ ...(1)

2. Comparing the coefficients of s^2, s and the constants term on either side of the eq. (1), we get,

$$A + B = 0 \quad ...(2)$$
$$0.5A + 0.5B + C = 0 \quad ...(3)$$
$$2A + 0.5C = 1 \quad ...(4)$$

3. Solving the simultaneous eq. (2), eq. (3) and eq. (4), we get $A = 0.5$, $B = 0.5$ and $C = 0$

4. The system response can be written as,

$$H(s) = \frac{0.5}{s+0.5} - \frac{0.5s}{s^2+0.5s+2} = \frac{0.5}{s+0.5} - 0.5\left\{\frac{s}{(s+0.25)^2+(1.3919)^2}\right\}$$

$$= \frac{0.5}{s+0.5} - 0.5\left\{\frac{s+0.25}{(s+0.25)^2+(1.3919)^2} - \frac{0.25}{(s+0.25)^2+(1.3919)^2}\right\}$$

$$= \frac{0.5}{s+0.5} - 0.5\left\{\frac{s+0.25}{(s+0.25)^2+(1.3919)^2}\right\}$$

$$+ 0.0898\left\{\frac{1.3919}{(s+0.25)^2+(1.3919)^2}\right\}$$

5. As we know that,

$$\frac{s+a}{(s+a)^2+b^2} \to \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-at}(\cos bT)z^{-1}+e^{-2aT}z^{-2}} \quad ...(5)$$

$$\frac{b}{(s+a)^2+b^2} \to \frac{e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}} \quad ...(6)$$

6. Using eq. (5) and (6), we get,

$$H(z) = \frac{0.5}{1-e^{-0.5T}z^{-1}} - 0.5\left[\frac{1-e^{-0.25T}(\cos 1.3919T)z^{-1}}{1-2e^{-0.25T}(\cos 1.3919T)z^{-1}+e^{-0.5T}z^{-2}}\right]$$

$$+ 0.0898\left[\frac{e^{-0.25T}(\sin 1.3919T)z^{-1}}{1-2e^{-0.25T}(\cos 1.3919T)z^{-1}+e^{-0.5T}z^{-2}}\right]$$

Let $T = 1$ s,

$$H(z) = \frac{0.5}{1-0.6065\,z^{-1}} - 0.5\left(\frac{1-0.1385\,z^{-1}}{1-0.277\,z^{-1}+0.606\,z^{-2}}\right)$$

$$+ 0.0898\left[\frac{0.7663\,z^{-1}}{1-02771\,z^{-1}+0.606\,z^{-2}}\right]$$

e. Determine $H(z)$ for a Butterworth filter satisfying the following constraints

$$\sqrt{0.5} \le |H(e^{j\omega})| \le 1 \quad ; 0 \le \omega \le \frac{\pi}{2}$$

$$|H(e^{j\omega})| \le 0.2 \; ; \frac{3\pi}{4} \le \omega \le \pi$$

with T = 1 sec. Apply impulse invariant transformation.

Ans.

Given : $\sqrt{0.5} \le |H(e^{j\omega})| \le 1,\ 0 \le \omega \le \pi/2$

$|H(e^{j\omega})| \le 0.2,\ 3\pi/4 \le \omega \le \pi$

$\delta_1 = \sqrt{0.5} = 0.707$, $\delta_2 = 0.2$, $\omega_1 = \pi/2$ and $\omega_2 = 3\pi/4$.

To Find : $H(z)$.

Step I : Determination of analog filter's edge frequencies.

$$\Omega_1 = \frac{\omega_1}{T} = \frac{\pi}{2} \text{ and } \Omega_2 = \frac{\omega_2}{T} = \frac{3\pi}{4}$$

Therefore, $\Omega_2/\Omega_1 = 1.5$

Step II : Determination of the order of the filter.

$$N \geq \frac{1}{2} \frac{\log\left\{((1/\delta_2^2)-1)/((1/\delta_1^2)-1)\right\}}{\log(\Omega_2/\Omega_1)}$$

$$N \simeq 4$$

Step III : Determination of – 3 dB cut-off frequency :

$$\Omega_c = \frac{\Omega_1}{[(1/\delta_1^2 - 1)]^{1/2N}} = \frac{\pi/2}{[(1/0.707^2)-1]^{1/8}} = \frac{\pi}{2}$$

Step IV :

1. Determination of $H_a(s)$:

$$H(s) = \prod_{k=1}^{N/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2}$$

$$= \left(\frac{B_1 \Omega_c^2}{s^2 + b_1 \Omega_c s + c_1 \Omega_c^2}\right)\left(\frac{B_2 \Omega_c^2}{s^2 + b_2 \Omega_c s + c_2 \Omega_c^2}\right)$$

$$b_1 = 2 \sin \frac{\pi}{8} = 0.76536, c_1 = 1$$

$$b_2 = 2 \sin \frac{3\pi}{8} = 1.84776, c_2 = 1$$

$\because$ $B_1 B_2 = 1$, so $B_1 = B_2 = 1$.

2. Therefore, $$H(s) = \left(\frac{2.467}{s^2 + 1.2022s + 2.467}\right)\left(\frac{2.467}{s^2 + 2.9025s + 2.467}\right) \quad ...(1)$$

3. Using partial fractions,

$$H(s) = \left(\frac{As + B}{s^2 + 1.2022s + 2.467}\right) + \left(\frac{Cs + D}{s^2 + 2.9025s + 2.467}\right) \quad ...(2)$$

4. Comparing eq. (1) and (2), we get

$$6.086 = (s^2 + 2.9025s + 2.467)(As + B) + (s^2 + 1.2022s + 2.467)(Cs + D)$$

5. Comparing the coefficients of s^3, s^2, s and the constants, we get

$$A + C = 0$$

$$2.9025\,A + B + 1.2022\,C + D = 0 \quad ...(3)$$

$$2.467\,A + 2.9025\,B + 2.467\,C + 1.2022\,D = 0 \quad ...(4)$$

$$B + D = 2.467 \quad ...(5)$$

6. Solving eq. (3), eq. (4) and eq. (5) then we get,

$$A = -1.4509, B = -1.7443, C = 1.4509 \text{ and } D = 4.2113$$

$$H(s) = -\left(\frac{1.4509s + 1.7443}{s^2 + 1.2022s + 2.467}\right) + \left(\frac{1.4509s + 4.2113}{s^2 + 2.9025s + 2.467}\right)$$

7. Let $$H(s) = H_1(s) + H_2(s),$$

where $$H_1(s) = -\left(\frac{1.4509s + 1.7443}{s^2 + 1.2022s + 2.467}\right)$$

and $$H_2(s) = \left(\frac{1.4509s + 4.2113}{s^2 + 2.9025s + 2.467}\right)$$

8. Rearranging $H_1(s)$ into the standard from,

$$H_1(s) = -\left(\frac{1.4509s + 1.7443}{s^2 + 1.2022s + 2.467}\right)$$

$$= -1.4509\left(\frac{s + 1.2022}{(s + 0.601)^2 + 1.451^2}\right)$$

$$= (-1.4509)\left[\frac{s + 0.601}{(s+0.601)^2 + 1.451^2} + \frac{0.601}{(s+0.601)^2 + 1.451^2}\right]$$

$$= (-1.4509)\left(\frac{s + 0.601}{(s + 0.601)^2 + 1.451^2}\right) - (0.601)\left(\frac{1.451}{(s + 0.601)^2 + 1.451^2}\right)$$

9. Similarly, $H_2(s)$ can be written as

$$H_2(s) = (1.4509)\left(\frac{s + 1.45}{(s + 1.45)^2 + 0.604^2}\right) + (3.4903)\left(\frac{0.604}{(s + 1.45)^2 + 0.604^2}\right)$$

Step V : Determination of $H(z)$

$$H_1(z) = (-1.4509)\frac{1 - e^{-0.601T}(\cos 1.451T)z^{-1}}{1 - 2e^{-0.601T}(\cos 1.451T)z^{-1} + e^{-1.202T}z^{-2}} - (0.601)\frac{e^{-0.601T}(\sin 1.451T)z^{-1}}{1 - 2e^{-0.601T}(\cos 1.451T)z^{-1} + e^{-1.202T}z^{-2}}$$

and $$H_2(z) = (1.4509)\frac{1 - e^{-1.45T}(\cos 0.604T)z^{-1}}{1 - 2e^{-1.45T}(\cos 0.604T)z^{-1} + e^{-2.9T}z^{-2}} + (3.4903)\frac{e^{-1.45T}(\sin 0.604T)z^{-1}}{1 - 2e^{-1.45T}(\cos 0.604T)z^{-1}e^{-2.9T}z^{-2}}$$

Here $H(z) = H_1(z) + H_2(z)$.

$$H(z) = \frac{-1.4509 - 0.2321z^{-1}}{1 - 0.1310z^{-1} + 0.3006z^{-2}} + \frac{1.4509 + 0.1848z^{-1}}{1 - 0.3862z^{-1} + 0.055z^{-2}}$$

f. Given $x(n) = 2^n$ and $N = 8$, find $X(k)$ using DIT FFT algorithm. Also calculate the computational reduction factor.

Ans.

Given : $x(n) = 2^n$ and $N = 8$

To Find : $X(k)$ and reduction factor.

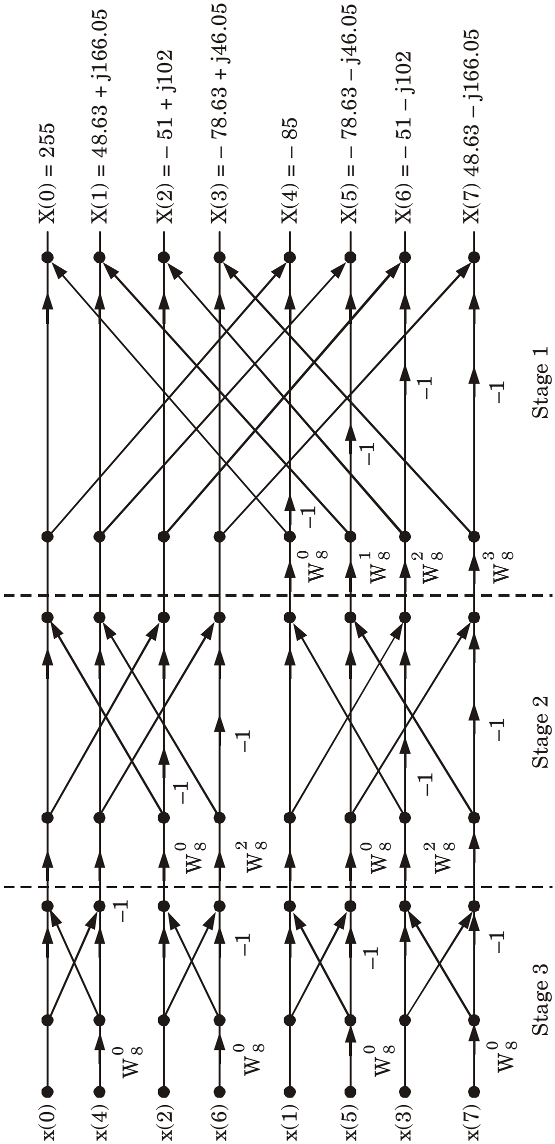


Fig. 6.

1. $$x(0) = 1, x(1) = 2, x(2) = 4, x(3) = 8$$
$$x(4) = 16, x(5) = 32, x(6) = 64, x(7) = 128$$
$$x(n) = \{1, 2, 4, 8, 16, 32, 64, 128\}$$

2. We know, $W_8^0 = 1$

$$W_8^1 = 0.707 - j0.707$$
$$W_8^2 = -j$$
$$W_8^3 = -0.707 - j0.707$$

3. Using DIT FFT algorithm, we can find $X(k)$ from the given sequence $x(n)$ as shown in Fig. 6.
4. Computation reduction factor

$$= \frac{\text{Number of complex multiplication requried for direct (DFT)}}{\text{Number of complex multiplications required for FFT algorithm}}$$

$$= \frac{N^2}{\frac{N}{2}\log_2(N)} = \frac{8^2}{\frac{8}{2}\log_2 8} = \frac{64}{12} = 5.33$$

g. Design a low-pass filter with the following desired frequency response

$$H_d(e^{jw}) = \begin{cases} e^{-j2\omega}, & \frac{-\pi}{4} \le \omega \le \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |\omega| < \pi \end{cases}$$ **and using window function**

$$w(n) = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

Ans.

Given : $w(n) = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$, $H_d(e^{j\omega}) = \begin{cases} e^{-j2\omega}; & -\frac{\pi}{4} \le \omega \le \frac{\pi}{4} \\ 0\ ; & \frac{\pi}{4} < \omega \le \pi \end{cases}$

To Design : Low-pass filter.

1. We know, $h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{-j2\omega}.e^{j\omega n}d\omega$

$$= \frac{1}{2\pi}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} e^{j\omega(n-2)}d\omega = \frac{1}{\pi(n-2)}\left[\frac{e^{j(n-2)\frac{\pi}{4}} - e^{-j(n-2)\frac{\pi}{4}}}{2j}\right]$$

$$= \frac{1}{\pi(n-2)}\sin\frac{\pi}{4}(n-2), \quad n \ne 2 \qquad \text{...(1)}$$

2. For $n = 2$, the filter coefficient can be obtained by applying L-Hospital's rule to eq. (1)

   Thus, $h_d(2) = 1/4$
3. The other filter coefficients are given by

$$h_d(0) = \frac{1}{2\pi} = h_d(4) \text{ and } h_d(1) = \frac{1}{\sqrt{2}\pi} = h_d(3)$$

4. The filter coefficients of the filter would be then

$$h(n) = h_d(n)\ w(n)$$

Therefore, $h(0) = \frac{1}{2\pi} = h(4),\ h(1) = \frac{1}{\sqrt{2}\pi} = h(3)$ and $h(2) = \frac{1}{4}$

5. The frequency response $H(e^{j\omega})$ is given by

$$H(e^{j\omega}) = \sum_{n=0}^{4} h(n)e^{-j\omega n}$$

$$= h(0) + h(1)e^{-j\omega} + h(2)e^{-j2\omega} + h(3)e^{-j3\omega} + h(4)e^{-j4\omega}$$

$$= e^{-j2\omega}[h(0)e^{j2\omega} + h(1)e^{j\omega} + h(2) + h(3)e^{-j\omega} + h(4)e^{-j2\omega}]$$

$$= e^{-j2\omega}\{h(2) + h(0)[e^{j2\omega} + e^{-j2\omega}] + h(1)[e^{j\omega} + e^{-j\omega}]\}$$

$$= e^{-j2\omega}\left\{\frac{1}{4} + \frac{1}{2\pi}[e^{j2\omega} + e^{-j2\omega}] + \frac{1}{\sqrt{2}\pi}[e^{j\omega} + e^{-j\omega}]\right\}$$

6. The frequency response of the designed low pass filter is then,

$$H(e^{j\omega}) = e^{-j2\omega}\left\{\frac{1}{4} + \frac{\sqrt{2}}{\pi}\cos\omega + \frac{1}{\pi}\cos 2\omega\right\}$$

h. Convert the analog filter with system function $H(s) = \frac{s+0.1}{(s+0.1)^2+9}$ into digital filter with a resonant frequency of $\omega_r = \pi/4$ using bilinear transformation.

Ans.

1. Given, $$H(s) = \frac{s+0.1}{(s+0.1)^2+9} \quad \text{...(1)}$$

2. From eq. (1), $\Omega_c = 3$
3. The sampling period T can be determine by,

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_r}{2}$$

$$T = \frac{2}{\Omega_c}\tan\frac{\omega_r}{2} = \frac{2}{3}\tan\left(\frac{\pi}{8}\right) = 0.276 \text{ s}$$

4. Using bilinear transformation,

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{(z-1)}{(z+1)}} = \frac{\frac{2}{T}\frac{(z-1)}{(z+1)} + 0.1}{\left[\frac{2}{T}\frac{(z-1)}{(z+1)} + 0.1\right]^2 + 9}$$

$$= \frac{\frac{2}{T}(z-1)(z+1) + 0.1(z+1)^2}{\left[\left(\frac{2}{T}\right)(z-1) + 0.1(z+1)\right]^2 + 9(z+1)^2} \quad \text{...(2)}$$

5. Substituting the value of T in eq. (2) then we get

$$H(z) = \frac{1 + 0.27z^{-1} - 0.973z^{-2}}{8.572 - 11.84z^{-1} + 8.177z^{-2}}$$

SECTION-C

Attempt any **two** of the following questions : **(15 × 2 = 30)**

3. i. Obtain the ladder structure for the system function $H(z)$ given below.

$$\boldsymbol{H(z) = \frac{2 + 8z^{-1} + 6z^{-2}}{1 + 8z^{-1} + 12z^{-2}}}$$

Ans.

1. For the given system, obtain the Routh array

z^{-2}	6	8	2
z^{-2}	12	8	1
z^{-1}	4	3/2	
z^{-1}	7/2	1	
1	5/14	0	
1	1		

2. The ladder structure parameters are

$$\alpha_0 = \frac{6}{12} = \frac{1}{2},\ \beta_1 = \frac{12}{4} = 3,\ \alpha_1 = \frac{4}{7/2} = \frac{8}{7},\ \beta_2 = \frac{7/2}{5/14} = \frac{49}{5},$$

$$\alpha_2 = \frac{5/14}{1} = \frac{5}{14}$$

$$H(z) = \frac{1}{2} + \cfrac{1}{3z^{-1} + \cfrac{1}{\frac{8}{7} + \cfrac{1}{(49/5)z^{-1} + \cfrac{1}{5/14}}}}$$

3. The ladder structure is shown in Fig. 7.

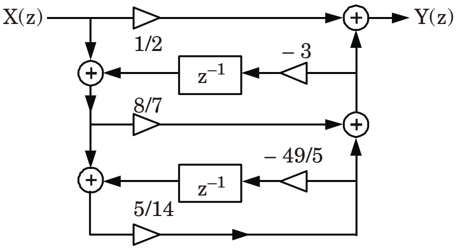


Fig. 7.

ii. Compute the circular convolution of two discrete time sequences $x_1(n) = \{1, 2, 1, 2\}$ and $x_2(n) = \{3, 2, 1, 4\}$.

Ans.

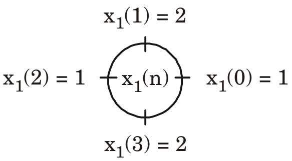
Given : $x_1(n) = \{1, 2, 1, 2\}$, $x_2(n) = \{3, 2, 1, 4\}$
To Find : Circular convolution.

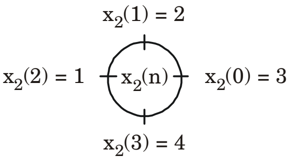
1. Circular convolution, $x_3(n) = x_1(n) \; (N) \; x_2(n)$

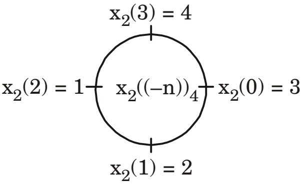
$$x_3(m) = \sum_{n=0}^{3} x_1(n)\, x_2\, ((m-n))_4$$

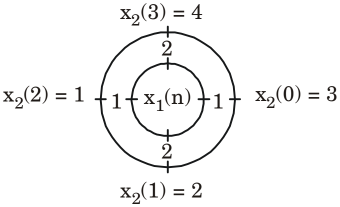
where, $m = 0, 1, 2, 3$

2. For $m = 0$, $x_3(0) = \sum_{n=0}^{3} x_1(n)\, x_2\, ((-n))_4$



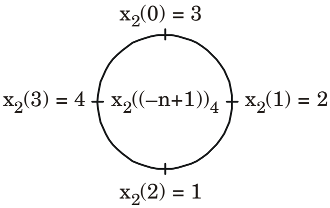


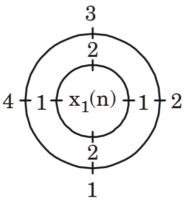




$\therefore$ $x_3(0) = 1 \times 3 + 2 \times 4 + 1 \times 1 + 2 \times 2$
$= 3 + 8 + 1 + 4 = 16$

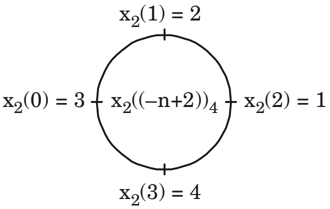
3. For $m = 1$, $x_3(1) = \sum_{n=0}^{3} x_1(n)\, x_2\, ((1-n))_4$

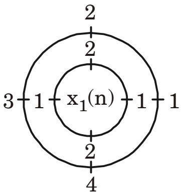




$\therefore$ $x_3(1) = 1 \times 2 + 2 \times 3 + 1 \times 4 + 2 \times 1$
$= 2 + 6 + 4 + 2 = 14$

4. For $m = 2$, $x_3(2) = \sum_{n=0}^{3} x_1(n)\, x_2\, ((2-n))_4$



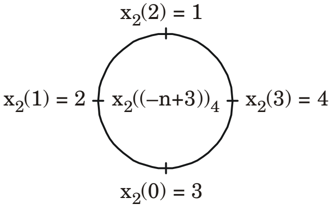


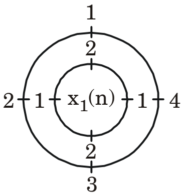
$\therefore \quad x_3(2) = 1 \times 1 + 2 \times 2 + 3 \times 1 + 4 \times 2$

$= 1 + 4 + 3 + 8 = 16$

5. For $m = 3$,

$$x_3(3) = \sum_{n=0}^{3} x_1(n)\, x_2\, ((3-n))_4$$





$\therefore \quad x_3(3) = 1 \times 4 + 2 \times 1 + 1 \times 2 + 2 \times 3$

$= 4 + 2 + 2 + 6 = 14$

6. Therefore, $x_3(n) = \{16, 14, 16, 14\}$

4. a. Determine the 4-point discrete time sequence from its DFT $X(k) = \{4, 1 - j, -2, 1 + j\}$

Ans.

Given : $X(k) = \{4, 1 - j, -2, 1 + j\}$

To Find : IDFT $(x(n))$

1. IDFT is defined as

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j2\pi nk/N}, \; 0 \le n \le N-1$$

Here, $N = 4$

$$x(n) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, e^{j\pi nk/2}, \;\; 0 \le n \le 3$$

2. When $n = 0$, $x(0) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, e^{0}$

$$= \frac{1}{4}\, [4 + (1 - j) + (-2) + (1 + j)] = 1$$

3. When $n = 1$, $x(1) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, e^{j\pi k/2}$

$$= \frac{1}{4} [4 + (1-j)e^{j\pi/2} + (-2)e^{j\pi} + (1+j)e^{j3\pi/2}]$$

$$= \frac{1}{4} [4 + (1-j)j + 2 + (1+j)(-j)] = \frac{1}{4} \times 8 = 2$$

4. When $n = 2$, $x(2) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{j\pi k}$

$$= \frac{1}{4} [4 + (1-j) e^{j\pi} + (-2) e^{j2\pi} + (1+j) e^{j3\pi}]$$

$$= \frac{1}{4} [4 + (1-j)(-1) + (-2) + (1+j)(-1)] = 0$$

5. When $n = 3$, $x(3) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{j3\pi k/2}$

$$= \frac{1}{4} [4 + (1-j) e^{j3\pi/2} + (-2) e^{j3\pi} + (1+j) e^{j9\pi/2}]$$

$$= \frac{1}{4} [4 + (1-j)(-j) + 2 + (1+j)(j)] = 1$$

6. Therefore, the IDFT of the given DFT produces the following four point discrete time sequence.

$$x(n) = \{1, 2, 0, 1\}$$

b. Explain the following phenomenon :
i. Gibbs oscillations,
ii. Frequency warping

Ans.

i. Gibbs oscillations :

1. The impulse response of FIR filter in terms of rectangular window is given by, $h(n) = h_d(n)\ w_R(n)$ ...(1)
2. The frequency response of the filter is obtained by taking Fourier transform of eq. (1)

$$\therefore \qquad H(\omega) = \text{FT}\{h_d(n)\ w_R(n)\} = H_d(\omega) * W_R(\omega)$$

This shows that the frequency response of FIR filter is equal to the convolution of desired frequency response, $H_d(\omega)$ and the Fourier transform of window function.
3. The sidelobes are present in the frequency response. Because of these sidelobes, the ringing is observed in the frequency response of FIR filter. This ringing is predominantly present near the band edge of the filter.
4. This oscillatory behaviour (*i.e.*, ringing effect) near the band edge of the filter is called Gibbs phenomenon.

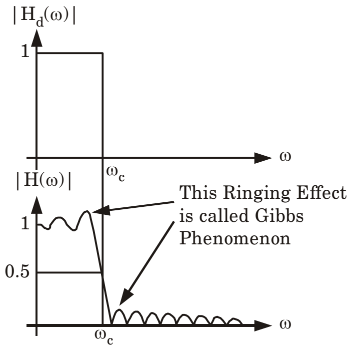


Fig. 8.

ii. Frequency warping :

1. We know, $\Omega = \dfrac{2}{T_s} \times \dfrac{2r \sin \omega}{r^2 + 2r \cos \omega + 1}$ ...(1)

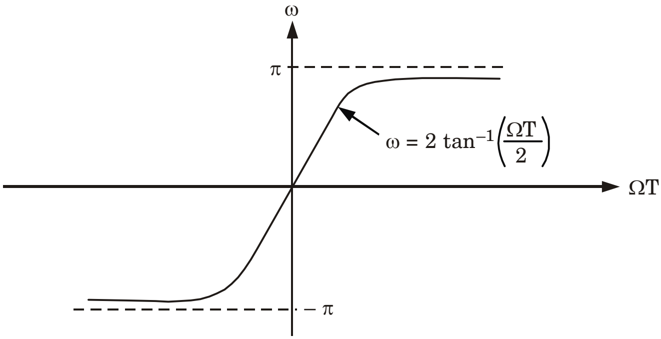


Fig. 9. Frequency warping.

2. For the unit circle, $r = 1$. Thus putting $r = 1$ in the eq. (1) we get,

$$\Omega = \frac{2}{T_s} \times \frac{2 \sin \omega}{1 + 2 \cos \omega + 1}$$

$$\therefore \quad \Omega = \frac{2}{T_s} \times \frac{2 \sin \omega}{2 + 2 \cos \omega}$$

$$\Omega = \frac{2}{T}\left(\frac{\sin \omega}{1 + \cos \omega}\right)$$

$$= \frac{2}{T}\left[\frac{2 \sin (\omega/2) \cos (\omega/2)}{\cos^2(\omega/2) + \sin^2(\omega/2) + \cos^2(\omega/2) - \sin^2(\omega/2)}\right]$$

or $$\Omega = \frac{2}{T} \tan\left(\frac{\omega}{2}\right)$$

or $$\omega = 2 \tan^{-1}\left(\frac{\Omega T}{2}\right)$$

3. In this method the entire range in Ω is mapped only once into the range $-\pi \leq \omega \leq \pi$. This mapping is non-linear. The lower frequencies in analog domain are expanded in digital domain while the higher frequencies in analog domain are compressed in digital domain.
4. Non-linearity in mapping is due to arc tangent function and this is known as frequency warping.

5. a. Derive the relation between DFT and *z*-transform of a discrete time sequence $x(n)$.

Ans.

1. The z-transform of $x(n)$ is given as $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$
2. If $X(z)$ is sampled at the N equally spaced points on the unit circle, these points will be

$$z = e^{j2\pi k/N},\ k = 0, 1, 2, \ldots, N-1, \text{ then}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi nk/N} \quad \ldots(1)$$

$$X(k) = X(z)\Big|_{z = e^{j\omega\pi k/N}},\ k = 0, 1, \ldots, N-1$$

3. The R.H.S. of eq. (1) is DFT $X(k)$, thus the relationship between DFT and z-transform is

$$X(k) = X(z)\Big|_{z = e^{j2\pi k/N}}$$

b. Design a digital Chebyshev filter to satisfy the constraints $0.707 \leq |H(e^{j\omega})| \leq 1,\ 0 \leq \omega \leq 0.2\pi$
$|H(e^{j\omega})| \leq 0.1,\ 0.5\pi \leq \omega \leq \pi$
Using bilinear transformation with $T = 1$ s.

Ans.

Given : $0.707 \leq |H(e^{j\omega})| \leq 1,\ 0 \leq \omega \leq 0.2\pi$
$|H(e^{j\omega})| \leq 0.1,\ 0.5\pi \leq \omega \leq \pi$
$\delta_1 = 0.77,\ \delta_2 = 0.1,\ \omega_1 = 0.2\pi,\ \omega_2 = 0.5\pi,\ T = 1$ s.

To Design : Digital Chebyshev filter.

1. Determination of the analog filter's edge frequencies.

$$\Omega_c = \Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2} = 2\tan 0.1\pi = 0.6498$$

$$\Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = 2\tan 0.25\pi = 2$$

Therefore, $\Omega_2/\Omega_1 = 3.0779$

2. Determination of the order of this filter.

$$\varepsilon = \left[\frac{1}{\delta_1^2} - 1\right]^{0.5} = \left[\frac{1}{0.77^2} - 1\right]^{0.5} = 0.828$$

Thus $$N \geq \frac{\cosh^{-1}\left\{\frac{1}{\epsilon}\left[\frac{1}{\delta_2^2}-1\right]^{0.5}\right\}}{\cosh^{-1}(\Omega_2/\Omega_1)}$$

$$= \frac{\cosh^{-1}\{1.207[100-1]^{0.5}\}}{\cosh^{-1}(2/0.6498)}$$

$$= \frac{\cosh^{-1}\{1.207\times 9.949\}}{\cosh^{-1}(3.078)}$$

$$= \frac{\cosh^{-1}(12)}{\cosh^{-1}(3.078)} = \frac{3.176}{1.789} = 1.775$$

$$N = 2$$

3. Determination of $H(s)$

$$H(s) = \prod_{k=1}^{N/2} \frac{B_k\Omega_c^2}{s^2 + b_k\Omega_c s + c_k\Omega_c^2} = \frac{B_1\Omega_c^2}{s^2 + b_1\Omega_c s + c_1\Omega_c^2} \qquad [\because \text{For k = 1}]$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\epsilon^2}+1\right)^{0.5} + \frac{1}{\epsilon}\right]^{1/N} - \left[\left(\frac{1}{\epsilon^2}+1\right)^{0.5} + \frac{1}{\epsilon}\right]^{-1/N}\right\}$$

$$y_2 = \frac{1}{2}\{[2.775]^{1/2} - [2.775]^{-1/2}\} = \frac{1}{2}[1.666 - 0.600] = 0.533$$

$$b_1 = 2y_2 \sin\,[(2k-1)\,\pi/2N] = 0.754$$

$$c_1 = y_2^2 + \cos^2\frac{(2k-1)\pi}{2N} = 0.784$$

4. For N even $\displaystyle\prod_{k=1}^{N/2}\frac{B_k}{c_k} = \frac{A}{(1+\varepsilon^2)^{0.5}} = 0.770$

i.e., $$\frac{B_1}{c_1} = 0.770$$

Thus $$B_1 = (0.770)\, c_1 = 0.604$$

5. The system function is

$$H(s) = \frac{0.604(0.6498)^2}{s^2 + (0.754)(0.6498)s + (0.784)(0.6498)^2}$$

On simplifying we get

$$H(s) = \frac{0.255}{s^2 + 0.49s + 0.331}$$

6. Determination of $H(z)$ using bilinear transformation

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}}$$

For $T = 1$ s

$$H(z) = \frac{0.255}{\left\{2\frac{(z-1)}{(z+1)}\right\}^2 + 0.49\left\{2\frac{(z-1)}{(z+1)}\right\} + 0.331}$$

$$= \frac{0.255\,(z+1)^2}{4(z-1)^2 + 0.98(z-1)(z+1) + 0.331(z+1)^2}$$

$$= \frac{0.255\,(z+1)^2}{4z^2 - 8z + 4 + 0.98z^2 - 0.98 + 0.331z^2 + 0.662z + 0.331}$$

$$H(z) = \frac{0.255(z+1)^2}{5.311z^2 - 7.338z + 3.51}$$

7. By dividing z^2 in numerator and denominator, we get

$$H(z) = \frac{0.255(1+z^{-1})^2}{5.311(1-1.381z^{-1} + 0.6309z^{-2})}$$

$$H(z) = \frac{0.048(1+z^{-1})^2}{1-1.3816z^{-1} + 0.6309z^{-2}}$$

☺☺☺

B. Tech.
(SEM. VI) EVEN SEMESTER THEORY EXAMINATION, 2017-18
DIGITAL SIGNAL PROCESSING

Time : 3 Hours **Max. Marks : 100**

Note : Attempt all of the sections. Assume missing data suitable, if any.

SECTION-A

1. Attempt the following questions : **(2 × 10 = 20)**

a. **If $x(n) = \{6, 5, 4, 3\}$ what will be $x((2 - n))_4$.**

b. **What is the DFT of $\delta(n)$?**

c. **What is the equation for order of Butterworth filter ?**

d. **What is difference between IIR and FIR filter ?**

e. **Write Gibbs phenomena.**

f. **Define time reversal of sequence in DFT.**

g. **What is twiddle factor in DFT ?**

h. **Write the frequency transformation rule for the conversion of LP to HP filter.**

i. **What is the difference between circular convolution and linear convolution ?**

j. **Write the expression for Hamming window.**

SECTION-B

2. Attempt any **three** of the following questions : **(10× 3 = 30)**

a. **Use the 4 point DFT and IDFT to determine circular convolution of the following sequence :**

$$x(n) = \{1, 2, 3, 1\}$$
$$h(n) = \{4, 3, 2, 2\}$$

b. **Determine the 8-point DFT of the following sequence using DIF FFT algorithm :**

$$x(n) = \{1, 2, 3, 4\}$$

c. Write a short note on the following :

i. Butterfly computation

ii. Inplace computation

iii. Bit reversal

d. Use bilinear transformation to convert low pass filter, $H(s) = 1/s^2 + \sqrt{2}\, s + 1$ into a high pass filter with pass band edge at 100 Hz and F_s = 1 kHz.

e. Design a digital Butterworth filter that satisfied the following constraints, using impulse invariant transformation.

$$0.9 \le |H(e^{j\omega})| \le 1;\ 0 \le \omega \le \pi/2$$

$$|H(e^{j\omega})| \le 0.2;\ 3\pi/4 \le \omega \le \pi$$

SECTION-C

3. Attempt any **one** of the following questions : **(10 × 1 = 10)**

a. i. A system function is given as under :

$$H(z) = \frac{(1 + 8z^{-1} + 6z^{-2})}{(1 + 8z^{-1} + 12z^{-2})}$$

realize the system function using ladder structure.

ii. State and prove the circular convolution theorem.

b. Design a linear phase FIR (high pass) filter of order seven with cut-off frequency π/4 radian/sec using Hanning window.

4. Attempt any **one** of the following questions : **(10 × 1 = 10)**

a. Determine the circular convolution of the following sequences and compare the result with linear convolution :

$$x(n) = (1, 2, 3, 4)$$

$$h(n) = (1, 2, 1)$$

b. The first five point of the 8-point DFT of a real valued sequence are : {0.25, 0.125 – j0.3018, 0, 0.125 – j0.0518, 0}. Determine the remaining three points.

5. Attempt any **one** of the following questions : **(10 × 1 = 10)**

a. The system function of the analog filter is given as :

$$H(s) = \frac{s + 0.1}{(s + 0.1)^2 + 16}$$

obtain the system function of digital filter using bilinear transformation which is resonant at $\omega_r = \pi/2$.

b. Design an FIR filter to meet the following specification :
Pass band edge = 2 kHz
Stop band edge = 5 kHz
Stop band attenuation = 42 dB
Sampling frequency = 20 kHz
Use Hanning window.

6. Attempt any **one** of the following questions : **(10 × 1 = 10)**

a. Obtain the direct form I, direct form II, cascade and parallel form realization for the following system :
$$y(n) = -0.1\, y(n-1) + 0.2\, y(n-2) + 3\, x(n) + 3.6\, x(n-1) + 0.6\, x(n-2)$$

b. Find the inverse DFT of the sequence :
$X(k) = \{6, -2 + j2, -2, -2 - j2\}$, **using DFT-FFT algorithm.**

7. Attempt any **one** of the following questions : **(10 × 1 = 10)**

a. What is the different window functions used for windowing ? Explain the effect of using different window functions for designing FIR filter on the filter response.

b. Derive and draw the flow graph for DIF FFT algorithm for $N = 8$.

☺☺☺

SOLUTION OF PAPER (2017-18)

Note : Attempt all of the sections. Assume missing data suitable, if any.

SECTION-A

1. Attempt the following questions : **(2 × 10 = 20)**

a. If $x(n)$ = {6, 5, 4, 3} what will be $x((2-n))_4$.

Ans. Given, $x(n) = \{6, 5, 4, 3\}$

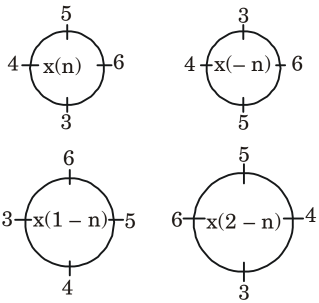


Fig. 1.

$$x((2-n))_4 = \{4, 5, 6, 3\}$$

b. What is the DFT of δ(*n*)?

Ans.
$$\text{DFT } \delta(n) = \sum_{n=0}^{N-1} \delta(n)\, e^{-j2\pi kn/N}$$

We know $\delta(0) = 1$

$$\text{DFT } \delta(n) = \delta(0) \,.\, 1 = 1$$

c. What is the equation for order of Butterworth filter ?

Ans. The Butterworth low-pass filter has a magnitude response given by,

$$|H(j\Omega)| = \frac{A}{[1 + (\Omega / \Omega_c)^{2N}]^{0.5}}$$

where A is the filter gain and Ω_c is the 3 dB cut-off frequency and N is the order of the filter.

d. What is difference between IIR and FIR filter ?

Ans.

S. No.	FIR filter	IIR filter
1.	They have finite impulse response.	They have infinite impulse response.
2.	Always stable.	Sometimes unstable.
3.	Have exact linear phase response.	Non-linear phase response.
4.	It requires more memory, higher computational complexity and involves more parameters.	It requires less memory, lower computational complexity and involves fewer parameters.

e. Write Gibbs phenomena.

Ans. The abrupt truncation of the fourier series results in oscillations in the passband and stopband. These oscillations are due to slow convergence of the fourier series, particularly near the points of discontinuity. This effect is known as the Gibbs phenomenon.

f. Define time reversal of sequence in DFT.

Ans. If $x(n) \xleftrightarrow{\text{DFT}} X(k)$

then, $x(-n, (\text{mod } N)) = x(N-n) \xleftrightarrow[\text{N}]{\text{DFT}} X(-k, (\text{mod } N)) = X(N-k)$.

Hence, when the N-point sequence in time is reversed, it is equivalent to reversing the DFT values.

g. What is twiddle factor in DFT ?

Ans. $W_N = e^{-j\frac{2\pi}{N}k}$ is called twiddle factor.

h. Write the frequency transformation rule for the conversion of LP to HP filter.

Ans. To convert low-pass with cut-off frequency Ω_c to high-pass with cut-off frequency Ω_c,

$$s \rightarrow \frac{\Omega_c \, \Omega_c^*}{s}$$

The system function of the high-pass filter is then,

$$H(s) = H_p\left(\frac{\Omega_c \, \Omega_c^*}{s}\right)$$

i. What is the difference between circular convolution and linear convolution ?

Ans.

S. No.	Linear convolution	Circular convolution
1.	If the length of one signal is N_1 and the length of other signal is N_2 then the length of output signal $y(n)$ is $N_1 + N_2 - 1$.	If the length of one signal is N_1 and the length of other signal is N_2 then the length of output signal is maximum of both signal (either N_1 or N_2).
2.	Multiplication of two sequences in time domain is called as linear convolution.	Multiplication of two sequences in frequency domain is called as circular convolution.

j. Write the expression for Hamming window.

Ans.

a. Causal Hamming window :

$$w_{\text{Hamm}}(n) = \begin{cases} 0.54 - 0.46 \cos \dfrac{2\pi n}{M-1} & ; \quad 0 \le n < M-1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

b. Non-causal Hamming window :

$$w_{\text{Hamm}}(n) = \begin{cases} 0.54 + 0.46 \cos \dfrac{2\pi n}{M-1} & ; \quad 0 \le n < (M-1)/2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

SECTION-B

2. Attempt any **three** of the following questions : **(10× 3 = 30)**

a. Use the 4 point DFT and IDFT to determine circular convolution of the following sequence :

$$\boldsymbol{x(n) = \{1, 2, 3, 1\}}$$
$$\boldsymbol{h(n) = \{4, 3, 2, 2\}}$$

Ans.

Given : $x(n) = \{1, 2, 3, 1\}$, $h(n) = \{4, 3, 2, 2\}$

To Find : Circular convolution.

1. DFT of $x(n)$ is given by,

$$X(k) = \sum_{n=0}^{3} x(n) W_4^{nk} = \sum_{n=0}^{3} x(n) e^{-j2\pi kn/4}$$

For $n = 0, 1, 2, 3$

$$= x(0) + x(1)e^{-j2\pi k.1/4} + x(2)e^{-j2\pi k.2/4} + x(3)e^{-j2\pi k.3/4}$$

For $k = 0$

$$X(0) = 1 + 2 + 3 + 1 = 7$$

For $k = 1$

$$X(1) = 1 + 2e^{-j\pi/2} + 3e^{-j\pi} + 1e^{-j3\pi/2}$$
$$= 1 + 2[0 - j(1)] + 3[-1 - j0] + 1[0 - j(-1)]$$

$= 1 - 2j - 3 + j = -2 - j$

For $k = 2$

$$X(2) = 1 + 2e^{-j\pi} + 3e^{-j\pi 2} + 1e^{-j3\pi}$$
$$= 1 - 2 + 3 + (-1) = 1$$

For $k = 3$

$$X(3) = 1 + 2e^{-j3\pi/2} + 3e^{-j\pi 3} + 1e^{-j9\pi/2}$$
$$= 1 + 2j - 3 - j = -2 + j$$
$$X(k) = \{7, -2 - j, 1, -2 + j\}$$

2. Now, DFT of $h(n)$ is given by

$$H(k) = \sum_{n=0}^{3} h(n)e^{-j2\pi kn/4}$$

For $n = 0, 1, 2, 3$

$$H(k) = h(0) + h(1)e^{-j2\pi k.1/4} + h(2)e^{-j2\pi k.2/4} + h(3)e^{-j2\pi k.3/4}$$
$$= 4 + 3e^{-j\pi k/2} + 2e^{-j\pi k} + 2e^{-j3\pi k/2}$$

For $k = 0$

$$X(0) = 4 + 3 + 2 + 2 = 11$$

For $k = 1$

$$H(1) = 4 + 3e^{-j\pi/2} + 2e^{-j\pi} + 2e^{-j3\pi/2}$$
$$= 4 - 3j - 2 + 2j = 2 - j$$

For $k = 2$

$$H(2) = 4 + 3e^{-j\pi} + 2e^{-j\pi 2} + 2e^{-j3\pi}$$
$$= 4 - 3 + 2 - 2 = 1$$

For $k = 3$

$$H(3) = 4 + 3e^{-j3\pi/2} + 2e^{-j\pi 3} + 2e^{-j9\pi/2}$$
$$= 4 + 3j - 2 - 2j = 2 + j$$

Thus, $H(k) = \{11, 2 - j, 1, 2 + j\}$

3. We know that

$$Y(k) = X(k) \,.\, H(k)$$
$$= \{7, -2 - j, 1, -2 + j\} \{11, 2 - j, 1, 2 + j\}$$
$$Y = \{77, -5, 1, 5\}$$

4. Taking inverse DFT of $Y(k)$ for computing $y(n)$,

$$y(n) = \text{IDFT } [Y(k)]$$
$$= \frac{1}{4}\sum_{k=0}^{3} Y(k)e^{j2\pi kn/4} \qquad \text{for } n = 0, 1, 2, 3$$
$$= \frac{1}{4}\{Y(0) + Y(1)e^{j2\pi 1.n/4} + Y(2)e^{j2\pi 2.n/4} + Y(3)e^{j2\pi 3.n/4}\}$$
$$y(n) = \frac{1}{4}[77 - 5e^{jn\pi/2} + e^{jn\pi} + 5e^{j3n\pi/2}]$$

For $n = 0$,

$$y(0) = \frac{1}{4}\,[77 - 5 + 1 + 5] = 19.5$$

For $n = 1$,

$$y(1) = \frac{1}{4}\,[77 - 5e^{j\pi/2} + e^{j\pi} + 5e^{j3\pi/2}]$$
$$= \frac{1}{4}\,[77 - 5j - 1 - 5j] = 19 - \frac{5}{2}\,j$$

For $n = 2$,

$$y(2) = \frac{1}{4}\,[77 - 5e^{j\pi} + e^{j\pi 2} + 5e^{j3\pi}]$$

$$= \frac{1}{4}\,[77 + 5 + 1 - 5] = 19.5$$

For $n = 3$,

$$y(3) = \frac{1}{4}\,[77 - 5e^{j3\pi/2} + e^{j3\pi} + 5e^{j9\pi/2}]$$

$$= \frac{1}{4}\,[77 + 5j - 1 + 5j] = 19 + \frac{5}{2}\,j$$

$$\therefore \quad y(n) = \left\{19.5,\, 19 - \frac{5}{2}\,j,\, 19.5,\, 19 + \frac{5}{2}\,j\right\}$$

b. Determine the 8-point DFT of the following sequence using DIF FFT algorithm :

$x(n) = \{1, 2, 3, 4\}$

Ans.

1. The total flow graph is shown in Fig. 2.

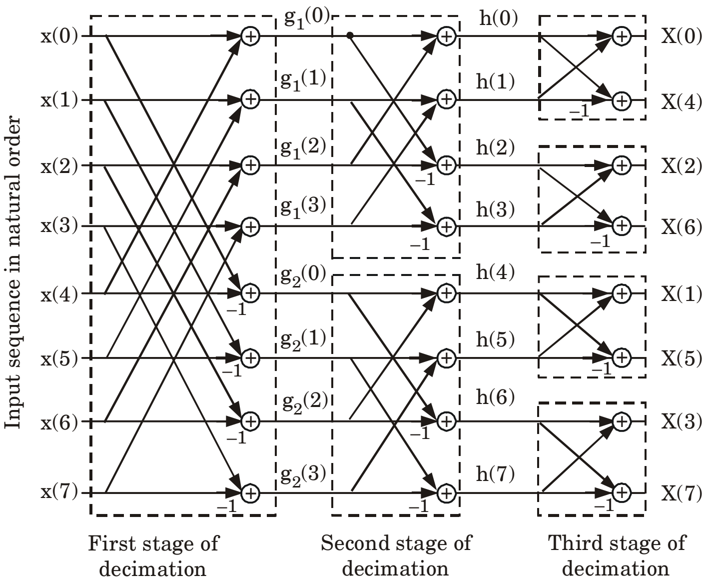


Fig. 2.

Here $g(n)$ is output of first stage and $h(n)$ is output of second stage.

2. The values of twiddle factor are as follows :

$$W_8^0 = 1,\ W_8^1 = e^{-j\pi/4} = 0.707 - j\,0.707$$

$$W_8^2 = e^{-j\pi/2} = -j \text{ and } W_8^3 = -0.707 - j\,0.707$$

Output of stage 1 :

$$g(0) = X(0) + X(4) = 1 + 0 = 1$$
$$g(1) = X(1) + X(5) = 2 + 0 = 2$$
$$g(2) = X(2) + X(6) = 3 + 0 = 3$$
$$g(3) = X(1) + X(7) = 4 + 0 = 4$$
$$g(4) = [X(0) - X(4)]\ W_8^0 = [1 - 0]\ 1 = 1$$
$$g(5) = [X(1) - X(5)]\ W_8^1 = [2 - 0]\ [0.707 - j\,0.707]$$
$$= 1.414 - j\,1.414$$
$$g(6) = [X(2) - X(6)]\ W_8^2 = [3 - 0]\ (-j) = -3j$$
$$g(7) = [X(3) - X(7)]\ W_8^3 = [4 - 0]\ (-0.707 - j\,0.707)]$$
$$= -2.828 - j\,2.828$$

Output of stage 2 :

$$h(0) = g(0) + g(2) = 1 + 3 = 4$$
$$h(1) = g(1) + g(3) = 2 + 4 = 6$$
$$h(2) = [g(0) - g(2)]\ W_8^0 = [1 - 3]\ 1 = -2$$
$$h(3) = [X(2) - X(6)]\ W_8^2 = [2 - 4]\ (-j) = 2j$$
$$h(4) = g(4) + g(6) = 1 - 3j$$
$$h(5) = g(5) + g(7) = 1.414 - j\,1.414 - 2.828 - j\,2.828$$
$$= -1.414 - j\,4.242$$
$$h(6) = [g(4) - g(6)]\ W_8^0 = [1 + 3j] = 1 + 3j$$
$$h(7) = [g(5) - g(7)]\ W_8^2$$
$$= [1.414 - j\,1.414 + 2.828 + j\,2.828]$$
$$= (4.242 + j\,1.414)\ (-j)$$
$$= 1.414 - j\,4.242$$

Final Output :

$$X(0) = h(0) + h(1) = 4 + 6 = 10$$
$$X(1) = h(4) + h(5) = 1 - 3j - 1.414 - j\,4.242$$
$$= -0.414 - j\,7.242$$
$$X(2) = h(2) + h(3) = -2 + 2j$$
$$X(3) = [h(6) + h(7)]\ W_8^0 = [1 + 3j + 1.414 - j\,4.242]\ (1)$$
$$= 2.414 - j\,1.242$$
$$X(4) = [h(0) - h(1)]\ W_8^0 = [4 - 6]\ (1) = -2$$
$$X(5) = [h(4) - h(5)]\ W_8^0 = [1 - 3j + 1.414 + j\,4.242]\ (1)$$

$$= [2.414 + j\,1.242]$$

$$X(6) = [h(2) + h(3)]\,W_8^0 = [-2 - 2j]\,(1) = -2 - 2j$$

$$X(7) = [h(6) - h(7)]\,W_8^0 = [1 + 3j - 1.414 + j\,4.242]\,(1)$$

$$= -0.414 + j\,7.242$$

c. Write a short note on the following :
i. Butterfly computation
ii. Inplace computation
iii. Bit reversal

Ans.

i. Butterfly computation :

1. This is the fundamental or basic computation in FFT algorithms. Fig. 3 shows this operation.
2. In the Fig. 3 observe that two values *a* and *b* are available as input. From these two values *A* and *B* are computed at output as shown in Fig 3. This operation is called butterfly operation.

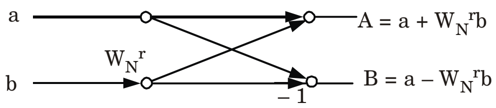


Fig. 3. Butterfly computation in FFT.

ii. Inplace computation :

1. Observe the butterfly computation of Fig. 3 used in FFT. From values *a* and *b* new values *A* and *B* are computed. Once *A* and *B* are computed, there is no need to store *a* and *b*. Thus same memory locations can be used to store *A*, *B* where *a*, *b* was stored.
2. Since *A*, *B* or *a*, *b* are complex number, they are need two memory locations each. Thus for computation of one butterfly, four memory locations are required, *i.e.*, two for *a* or *A* and two for *b* or *B*.
   Memory locations for one butterfly = 2 × 2 = 4
3. In such computation, *A* is stored in place of *a* and *B* is stored in place of *b*. This is called is inplace computation.

iii. Bit reversal :

1. Consider the signal flow graph of 8-point DIT FFT algorithm. Observe that the sequence of input data is shuffled as $x(0)$, $x(4)$, $x(2)$, $x(6)$, $x(1)$, $x(5)$, $x(3)$, $x(7)$. And the DFT sequence $X(k)$ at the output is in proper order, *i.e.*, $x(0)$, $x(1)$, ... $x(7)$. The shuffling of the input sequence has well defined format.
2. In the Table 1 observe that the data point $x(1) = x\,(001)$ is to be placed at $m = 4$ *i.e.*, $(100)^{th}$ position in the decimated array. The last column of Table 1 shows the order in which the data is required.
3. Thus the input data should be stored in the bit reversed order then the DFT will be obtained in natural sequence.

Table 1.

Memory address of $x(n)$ in decimal	Memory address of $x(n)$ in binary			Memory address in bit reversed order			New memory address of $x(n)$ according to reversed order of bits
n	n_2	n_1	n_0	n_0	n_1	n_2	m
0	0	0	0	0	0	0	0
1	0	0	1	1	0	0	4
2	0	1	0	0	1	0	2
3	0	1	1	1	1	0	6
4	1	0	0	0	0	1	1
5	1	0	1	1	0	1	5
6	1	1	0	0	1	1	3
7	1	1	1	1	1	1	7

d. Use bilinear transformation to convert low pass filter, $H(s) = 1/s^2 + \sqrt{2}\, s + 1$ into a high pass filter with pass band edge at 100 Hz and F_s = 1 kHz.

Ans.

Given : f_P = 100 Hz, f_s = 1 kHz, $H(s) = \dfrac{1}{s^2 + \sqrt{2}s + 1}$

To Convert : Low pass into high pass filter

$$\omega_P = 2\pi f_P = 200\,\pi = \omega_c$$

Step I : We have

$$\Omega_c^* = \frac{2}{T_s} \tan\left(\frac{\omega_P T_s}{2}\right)$$

$$T_s = \frac{1}{f_s} = \frac{1}{1 \times 10^3} = 10^{-3}$$

$$\therefore \qquad \Omega_c^* = \frac{2}{1/10^3} \tan\left(\frac{200\pi}{2 \times 10^3}\right)$$

$$\therefore \qquad \Omega_c^* = 2 \times 10^3 \tan(0.1\pi)$$

$$\Omega_c^* = 2 \times 10^3 \times 0.325$$

$$\Omega_c^* = 650$$

Step II : Equation of $H^*(s)$ is obtained by replacing s by $\frac{\Omega_c^*}{s}$ in equation of $H(s)$.

$$\therefore \quad H^*(s) = \frac{1}{\left(\frac{\Omega_c^*}{s}\right)^2 + \sqrt{2}\left(\frac{\Omega_c^*}{s}\right) + 1}$$

$$H^*(s) = \frac{1}{\left(\frac{650}{s}\right)^2 + \sqrt{2}\left(\frac{650}{s}\right) + 1}$$

$$H^*(s) = \frac{1}{0.422 \times 10^6\, s^{-2} + 0.92 \times 10^3\, s^{-1} + 1}$$

Step III :

$H(z)$ is obtained by replacing s by $\frac{2}{T_s}\left[\frac{z-1}{z+1}\right]$

$$H^*(z) = \frac{1}{0.422 \times 10^6 \left[\frac{2}{T_S}\left(\frac{z-1}{z+1}\right)\right]^{-2} + 0.92 \times 10^3 \left[\frac{2}{T_S}\left(\frac{z-1}{z+1}\right)\right]^{-1} + 1}$$

$$H^*(z) = \frac{1}{0.422 \times 10^6 \times \frac{10^{-6}}{2}\left(\frac{z+1}{z-1}\right)^2 + 0.92 \times 10^3 \times \frac{10^{-3}}{2}\left(\frac{z+1}{z-1}\right) + 1}$$

$$H^*(z) = \frac{1}{0.21\left(\frac{z+1}{z-1}\right)^2 + 0.46\left(\frac{z+1}{z-1}\right) + 1}$$

e. Design a digital Butterworth filter that satisfied the following constraints, using impulse invariant transformation.

$$\mathbf{0.9 \le |H(e^{j\omega})| \le 1;\ 0 \le \omega \le \pi/2}$$
$$\mathbf{|H(e^{j\omega})| \le 0.2;\ 3\pi/4 \le \omega \le \pi}$$

Ans.

Given : $0.9 \le |H(e^{j\omega})| \le 1\ ;\ 0 \le \omega \le \pi/2$

$|H(e^{j\omega})| \le 0.2\ ;\ 3\pi/4 \le \omega \le \pi$

$\delta_1 = 0.9$, $\delta_2 = 0.2$, $\omega_1 = \pi/2$ and $\omega_2 = 3\pi/4$

1. Assume $T = 1$ sec.

$$\Omega_1 = \frac{\omega_1}{2} = \pi/4$$

$$\Omega_2 = \frac{\omega_2}{2} = 3\,\pi/8$$

$$\Omega_2/\Omega_1 = \frac{\pi/4}{3\pi/8} = 0.67$$

2. Order of the filter

$$N \geq \frac{1}{2}\frac{\log\{[(1/\delta_2)^2 - 1]/[(1/\delta_1)^2 - 1]\}}{\log(\Omega_2/\Omega_1)}$$

$$N \geq \frac{1}{2}\frac{\log\{24/0.2346\}}{\log(2.41)} = \frac{1}{2}\left(\frac{2}{0.382}\right)$$

$$N \geq 2.62$$

$$N \simeq 3$$

3. Determination of – 3 dB cut-off frequency,

$$\Omega_c = \frac{\Omega_1}{[1/\delta_1^2 - 1]^{1/2N}} = \frac{\pi/4}{[1/0.9^2 - 1]^{1/6}} = 1$$

4. Now determination of $H_a(s)$,

$$H_a(s) = \frac{B_0\,\Omega_c}{s + c_0\,\Omega_c}\prod_{k=1}^{(N-1)/2}\frac{B_k\,\Omega_c^2}{s^2 + b_k\,\Omega_c\,s + c_k\,\Omega_c^2}$$

$$= \frac{B_0\,\Omega_c^2}{s + c_0\,\Omega_c}\left(\frac{B_1\,\Omega_c^2}{s^2 + b_1\,\Omega_c\,s + c_1\,\Omega_c^2}\right) \quad [\because \text{For } k = 1]$$

$$b_1 = 2\sin\pi/6 = 1,\ c_0 = 1,\ c_1 = 1$$

$B_0B_1 = 1$, therefore, $B_0 = B_1 = 1$

$$\therefore \quad H(s) = \left(\frac{1}{s+1}\right)\left(\frac{1}{s^2 + s + 1}\right)$$

5. Now doing partial fraction expansion,

$$H(s) = \frac{A}{(s+1)} + \frac{Bs + C}{(s^2 + s + 1)}$$

$$1 = A\,(s^2 + s + 1) + (Bs + C)\,(s + 1)$$

Put $s = -1$

$$1 = A\,(1)$$

$$A = 1$$

Compare coefficients of polynomial,

$$0 = A + B \quad [\text{coefficients of } s^2]$$

$$B = -1$$

$$0 = A + B + C \quad [\text{coefficients of } s]$$

$$C = -(A + B) = 0$$

$$H(s) = \frac{1}{(s+1)} - \frac{s}{(s^2 + s + 1)}$$

$$= \frac{1}{(s+1)} - \frac{s}{s^2 + s + 1 + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}$$

$$H(s) = \frac{1}{(s+1)} - \left[\frac{s+1/2-1/2}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

6. Apply impulse invariance transformation,

$$H_D(z) = \left[\frac{1}{1-e^{-T}z^{-1}}\right] - \left[\frac{1-e^{-0.5T}\cos\frac{\sqrt{3}}{2}Tz^{-1}}{1-2e^{0.5T}\cos\frac{\sqrt{3}}{2}Tz^{-1}+e^{-T}z^{-2}}\right]$$

$$+\frac{\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)}\left[\frac{e^{-0.5T}\sin\frac{\sqrt{3}}{2}Tz^{-1}}{1-2e^{-0.5T}\cos\frac{\sqrt{3}}{2}Tz^{-1}+e^{-T}z^{-2}}\right]$$

We have assumed $T = 1$ sec.

$$H_D(z) = \left[\frac{1}{1-0.3678\,z^{-1}}\right]-\left[\frac{1-0.1261\,z^{-1}}{1-0.7858\,z^{-1}+0.3678\,z^{-2}}\right]$$

SECTION-C

3. Attempt any **one** of the following questions : **(10 × 1 = 10)**

a. i. A system function is given as under :

$$H(z) = \frac{(1+8z^{-1}+6z^{-2})}{(1+8z^{-1}+12z^{-2})}$$

realize the system function using ladder structure.

Ans.

1. For the given system, obtain the Routh array

z^{-2}	6	8	1
z^{-2}	12	8	1
z^{-1}	4	1/2	
z^{-1}	13/2	1	
1	– 3/26	0	
1	1		

2. The ladder structure parameters are,

$$\alpha_0 = \frac{6}{12} = \frac{1}{2},\ \beta_1 = \frac{12}{4} = 3,\ \alpha_1 = \frac{4}{13/2} = 0.62\ \beta_2 = \frac{13/2}{-3/26} = -56.34$$

$$\alpha_2 = \frac{-3/26}{1} = -0.12$$

$$H(z) = \frac{1}{2} + \cfrac{1}{3z^{-1} + \cfrac{1}{\cfrac{8}{13} + \cfrac{1}{\cfrac{169}{3}z^{-1} + \cfrac{1}{\cfrac{-3}{26}}}}}$$

3. Ladder structure is shown in Fig. 4.

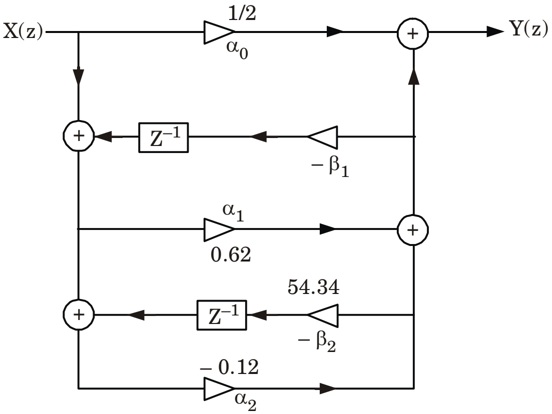


Fig. 4.

ii. State and prove the circular convolution theorem.

Ans.

A. Statement : The multiplication of two DFTs equivalent to the circular convolution of their sequences in time domain.

Mathematical equation :

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$ and $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$ then,

$$x_1(n) \; (N) \; x_2(n) \xleftrightarrow[N]{DFT} X_1(k) \cdot X_2(k) \quad ...(1)$$

Here (N) indicates circular convolution.

B. Proof :

1. By definition two DFTs, $X_1(k)$ and $X_2(k)$ are given as,

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi kn/N}, \; k = 0, 1, \ldots N-1 \quad ...(2)$$

and
$$X_2(k) = \sum_{l=0}^{N-1} x_2(l) e^{-j2\pi kl/N}, \; k = 0, 1, \ldots N-1 \quad ...(3)$$

2. Let $X_3(k)$ be equal to multiplication of $X_1(k)$ and $X_2(k)$,

i.e.,
$$X_3(k) = X_1(k) \cdot X_2(k) \quad ...(4)$$

3. Let $x_3(m)$ be the sequence whose DFT is $X_3(k)$. Then $x_3(m)$ can be obtained from $X_3(k)$ by taking IDFT. *i.e.*,

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_3(k) e^{j2\pi km/N} = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) \cdot X_2(k) e^{j2\pi km/N} \quad ...(5)$$

4. Substitute for $X_1(k)$ and $X_2(k)$ in eq. (5)

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} x_1(n) \, e^{-j2\pi kn/N} \right] \left[\sum_{l=0}^{N-1} x_2(l) \, e^{-j2\pi kl/N} \right] e^{j2\pi km/N}$$

5. Rearranging the summations and terms,

$$x_3(m) = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left\{\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N}\right\} \quad ...(6)$$

Now, $$\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N} = \begin{cases} N & \text{When } (m-n-l) \text{ is multiple of } N \\ 0 & \text{otherwise} \end{cases} \quad ...(7)$$

6. Putting this value in eq. (6) we get,

$$x_3(m) = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \cdot N \text{ When } (m-n-l) \text{ is multiple on } N$$

$$= \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \qquad \text{When } (m-n-l) \text{ is multiple on } N \quad ...(8)$$

7. Let $(m-n-l) = pN$

Here p is some integer.

Since integer multiple can be positive or negative both, we can write condition for convenience as,

$$m - n - l = -pN \quad i.e., \; l = m - n + pN \quad ...(9)$$

8. Putting for l from eq. (9) in eq. (8) we get,

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2(m-n+pN) \quad ...(10)$$

9. In the eq. (10), $x_2(m - n + pN)$ represents it is periodic sequence with period N. This periodic sequence is delayed by 'n' samples. $x_2(m-n+pN)$ represents sequence $x_2(m)$ shifted circularly by 'n' samples.

$$\therefore \quad x_2(m-n+pN) = x_2(m-n, \text{ modulo } N) = x_2((m-n))_N \quad ...(11)$$

10. Putting this sequence in eq. (10) we get,

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N,\; m = 0, 1, \ldots N-1 \quad ...(12)$$

11. Circular convolution of $x_1(n)$ and $x_2(n)$ is denoted by $x_1(n)$ N $x_2(n)$ and it is given by eq. (12) as,

$$x_3(m) = x_1(n) \; N \; x_2(n) = \sum_{n=0}^{N-1} x_1(n) x_2((m-n))_N,$$

where, $m = 0, 1, \ldots N-1$

Thus circular convolution property is proved.

b. Design a linear phase FIR (high pass) filter of order seven with cut-off frequency $\pi/4$ radian/sec using Hanning window.

Ans.

1. Equation of $h_d(n)$ is

$$h_d(n) = \begin{cases} \dfrac{\sin(n-3)}{\pi(n-3)} & \text{for } n \neq 3 \\ \dfrac{1}{\pi} & \text{for } n = 3 \end{cases} \quad ...(1)$$

2. Now, Hanning window is defined as

$$W_h(n) = \frac{1}{2}\left[1 - \cos\frac{2\pi n}{M-1}\right] \quad \text{[Given, } M = 7\text{]}$$

$$\therefore \quad W_h(n) = \frac{1}{2}\left[1 - \cos\frac{2\pi n}{6}\right] \quad ...(2)$$

3. Now the coefficients of FIR filter that means $h(n)$ is obtained by using the equation,

$$h(n) = h_d(n) \,.\, W_h(n) \quad ...(3)$$

4. Using eq. (3) we can obtain the values of it as shown in Table 2.

Magnitude and phase of $h_d(n)$:

$$H_d(\omega) = \begin{cases} 1\, e^{-j\left(\frac{M-1}{2}\right)} & \text{for } |\omega| >|\omega_c| \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \quad |H_d(\omega)| = \begin{cases} 1 & \text{for } |\omega| >|\omega_c| \\ 0 & \text{otherwise} \end{cases}$$

and $$\angle H_d(\omega) = -\omega\left(\frac{M-1}{2}\right) = -\omega\left(\frac{7-1}{2}\right) = -3\omega$$

Table 2.

Value of n	$h_d(n) = \begin{cases} \dfrac{\sin(n-3)}{\pi(n-3)} & \text{for } n \neq 3 \\ \dfrac{1}{\pi} & \text{for } n = 3 \end{cases}$	$W_h(n) = \dfrac{1}{2}\left[1 - \cos\left(\dfrac{2\pi n}{6}\right)\right]$	$h(n) = h_d(n).\, W_h(n)$
0	0.015	0	0
1	0.145	0.25	0.03625
2	0.268	0.75	0.201
3	0.318	1	0.318
4	0.268	0.75	0.201
5	0.145	0.25	0.03625
6	0.129	0	0

4. Attempt any **one** of the following questions : **(10 × 1 = 10)**

a. Determine the circular convolution of the following sequences and compare the result with linear convolution :

$$\boldsymbol{x(n) = (1, 2, 3, 4)}$$
$$\boldsymbol{h(n) = (1, 2, 1)}$$

Ans.

i. Circular convolution :

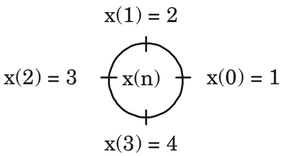
Given : $x(n) = \{1, 2, 3, 4\}$, $x(n) = \{1, 2, 1, 0\}$
To Find : Circular convolution.

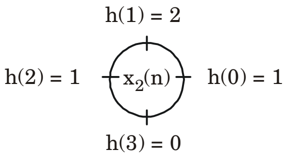
1. Circular convolution, $y(m) = x(n)$ N $h(n)$

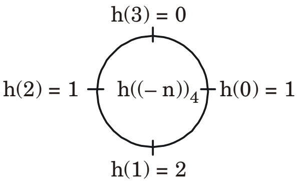
$$y(m) = \sum_{n=0}^{3} x(n)\, h((m-n))_4$$

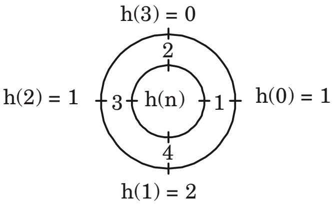
where, $m = 0, 1, 2, 3$

2. For $m = 0$, $y(0) = \sum_{n=0}^{3} x(n)\, h((-n))_4$



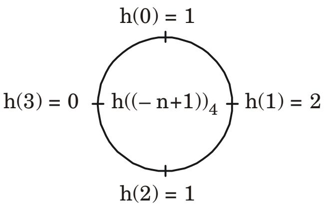


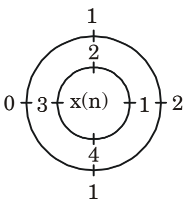




$$y(0) = 1 \times 1 + 2 \times 0 + 3 \times 1 + 2 \times 4$$
$$= 1 + 0 + 3 + 8 = 12$$

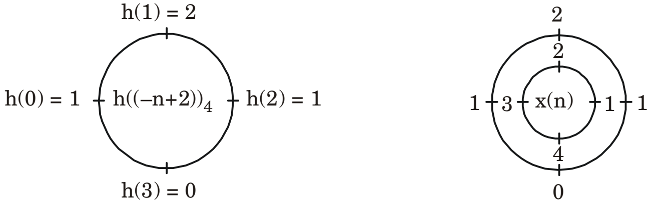
3. For $m = 1$, $y(1) = \sum_{n=0}^{3} x(n)\, h((1-n))_4$





$$y(1) = 2 \times 1 + 1 \times 2 + 0 \times 3 + 4 \times 1$$
$$= 2 + 2 + 0 + 4 = 8$$

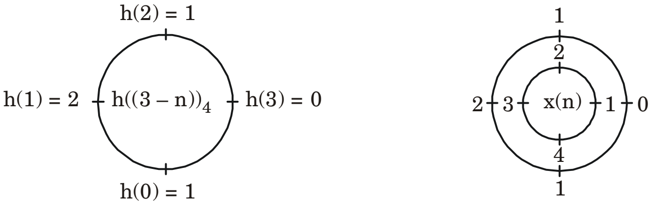
4. For $m = 2$, $\quad y(2) = \sum_{n=0}^{3} x(n)\, h((2-n))_4$



$$y(2) = 1 \times 1 + 2 \times 2 + 1 \times 3 + 0 \times 4$$
$$= 1 + 4 + 3 + 0 = 8$$

5. For $m = 3$,

$$y(3) = \sum_{n=0}^{3} x(n)\, h((3-n))_4$$



$$y(3) = 0 \times 1 + 2 \times 1 + 2 \times 3 + 1 \times 4$$
$$= 0 + 2 + 6 + 4 = 12$$

6. Therefore, $y(n) = \{12, 8, 8, 12\}$

ii. Linear convolution :

We know that

$$y(n) = x(n) * h(n)$$
$$h(n) = \{1, 2, 1, 0\}$$

Using matrix representation method,

$y(n) =$

h(n) \ x(n)	1	2	3	4
1	1	2	3	4
2	2	4	6	8
1	1	2	3	4
0	0	0	0	0

Linear convolution of two sequences

i.e., $\quad y(n) = x(n) * h(n)$

$$y(n) = \{1, 4, 8, 12, 11, 4, 0\}$$

b. The first five point of the 8-point DFT of a real valued sequence are : {0.25, 0.125 – *j*0.3018, 0, 0.125 – *j*0.0518, 0}. Determine the remaining three points.

Ans.

Given : $x(n)_8 = \{0.25, 0.125 - j0.3018, 0, 0.125 - j0.0518, 0\}$
To Find : Remaining three points.

1. Given, $X[0] = 0.25, X[1] = 0.125 - j0.3018, X[2] = 0,$

$$X[3] = 0.125 - j0.0158, X[4] = 0$$

2. By property of DFT,

$$X[k] = {}^*X[N-k]$$

or $X[N-k] = {}^*X[k]$

3. Here, $N = 8,$

$$X[k] = {}^*X[8-k]$$

$$X[5] = {}^*X[8-5] = {}^*X[3] = 0.125 + j0.0158$$

$$X[6] = {}^*X[8-6] = {}^*X[2] = 0$$

$$X[7] = {}^*X[8-7] = {}^*X[1] = 0.125 + j0.3018$$

5. Attempt any **one** of the following questions : **(10 × 1 = 10)**

a. The system function of the analog filter is given as :

$$\boldsymbol{H(s) = \frac{s+0.1}{(s+0.1)^2+16}}$$

obtain the system function of digital filter using bilinear transformation which is resonant at $\omega_r = \pi/2$.

Ans.

1. Given, $H(s) = \dfrac{s+0.1}{(s+0.1)^2+16}$ ...(1)

2. From eq. (1), $\Omega_c = 4$

3. The sampling period T can be determine by,

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_r}{2}$$

$$T = \frac{2}{\Omega_c}\tan\frac{\omega_r}{2} = \frac{2}{4}\tan\left(\frac{\pi}{8}\right) = 0.207 \text{ s}$$

4. Using bilinear transformation,

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$H(z) = \frac{\frac{2}{T}\frac{(z-1)}{(z+1)}+0.1}{\left[\frac{2}{T}\frac{(z-1)}{(z+1)}+0.1\right]^2+16}$$

$$= \frac{\frac{2}{T}(z-1)(z+1)+0.1(z+1)^2}{\left[\left(\frac{2}{T}\right)(z-1)+0.1(z+1)\right]^2+16(z+1)^2} \quad ...(2)$$

5. Substituting the value of T in eq. (2) then we get

$$H(z) = \frac{4.1z^2+0.2z-3.9}{32.81z^2+0.02z+31.21}$$

b. Design an FIR filter to meet the following specification :
Pass band edge = 2 kHz
Stop band edge = 5 kHz
Stop band attenuation = 42 dB
Sampling frequency = 20 kHz
Use Hanning window.

Ans.

Given : δ_1 = 2 kHz, δ_2 = 5 kHz, A_s = 42 dB, f_s = 20 kHz

To Design : FIR Filter

1.
$$\omega_1 = 2\pi\frac{\delta_1}{f_s} = 2\pi\times\frac{2\text{ kHz}}{20\text{ kHz}} = 0.2\pi \text{ rad/sample}$$

$$\omega_2 = 2\pi\frac{\delta_2}{f_s} = 2\pi\times\frac{5\text{ kHz}}{20\text{ kHz}} = 0.5\pi \text{ rad/sample}$$

$$\omega_c = \frac{1}{2}[\omega_1+\omega_2] = \frac{1}{2}[0.2+0.5] = 0.35\pi \text{ rad/sample}$$

2. In Hanning window, it provides – 44 dB to stopband attenuation and we have 42 dB of stopband attenuation. So the window type is Hanning window.
3. In Hanning window the width of the main lobe is

$$k\left(\frac{2\pi}{M}\right) = \frac{8\pi}{M}$$

$$k = 4$$

4. The order of the filter is given by equation as

$$N = k\left(\frac{2\pi}{\omega_2-\omega_1}\right)$$

$$N = 4\left[\frac{2\pi}{(0.5-0.2)\pi}\right] = \frac{8}{0.3} = 26.67$$

$$N \approx 27$$

$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{M-1}{2}\right)} & \text{for } \omega_c \le \omega \le \omega_c \\ 0 & \text{elsewhere} \end{cases} \quad ...(1)$$

$$h_d(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$$

$$h_d(\omega) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\left(\frac{M-1}{2}\right)}\, e^{j\omega n}\, d\omega \quad ...(2)$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega\left(n-\frac{M-1}{2}\right)}}{j\left(n-\frac{M-1}{2}\right)}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega_c}\left(n-\frac{M-1}{2}\right) - e^{-j\omega_c\left(n-\frac{M-1}{2}\right)}}{j\left(n-\frac{M-1}{2}\right)}\right]$$

$$= \frac{\sin\left[\omega_c\left(n-\frac{M-1}{2}\right)\right]}{\pi\left(n-\frac{M-1}{2}\right)} \text{ for } n \neq \frac{M-1}{2}$$

5. For $n = \frac{M-1}{2}$, eq. (1) becomes,

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{1}{2\pi}[\omega]_{-\omega_c}^{\omega_c} = \frac{\omega_c}{\pi}$$

$$h_d(n) = \begin{cases} \dfrac{\sin\left[\omega_c\left(n-\frac{M-1}{2}\right)\right]}{\pi\left(n-\frac{M-1}{2}\right)} & \text{for } n \neq \dfrac{M-1}{2} \\ \dfrac{\omega_c}{\pi} & \text{for } n = \dfrac{M-1}{2} \end{cases}$$

6. For Hanning window

$$W(n) = \frac{1}{2}\left[1 - \cos\frac{2\pi n}{M-1}\right]$$

$h(n)$ is given as

$$h(n) = h_d(n)\; W(n)$$

7. Putting value of $h_d(n)$and $W(n)$

$$h(n) = \begin{cases} \dfrac{\sin\left[\omega_c\left(n - \dfrac{M-1}{2}\right)\right]}{\pi\left(n - \dfrac{M-1}{2}\right)} \cdot \dfrac{1}{2}\left[1 - \dfrac{\cos 2\pi n}{M-1}\right] & \text{for } n \neq \dfrac{M-1}{2} \\ \dfrac{\omega_c}{\pi} \cdot \dfrac{1}{2}\left[1 - \dfrac{\cos 2\pi n}{M-1}\right] & \text{for } n = \dfrac{M-1}{2} \end{cases}$$

6. Attempt any **one** of the following questions : **(10 × 1 = 10)**

a. Obtain the direct form I, direct form II, cascade and parallel form realization for the following system :

$y(n) = -0.1\,y(n-1) + 0.2\,y(n-2) + 3\,x(n) + 3.6\,x(n-1) + 0.6\,x(n-2)$

Ans.

1. Given, $y(n) = 0.1y(n-1) + 0.2y(n-2) + 3x(n) + 3.6x(n-1) + 0.6x(n-2)$
2. Taking z-transform on both sides,

$$Y(z) = -0.1z^{-1}Y(z) + 0.2z^{-2}Y(z) + 3X(z) + 3.6z^{-1}X(z) + 0.6z^{-2}X(z)$$

$$\frac{Y(z)}{X(z)} = \left(\frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}\right)$$

A. Direct form I :

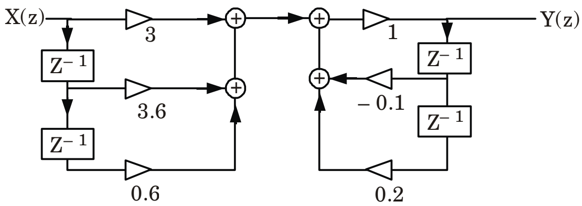


Fig. 5.

B. Direct form II :

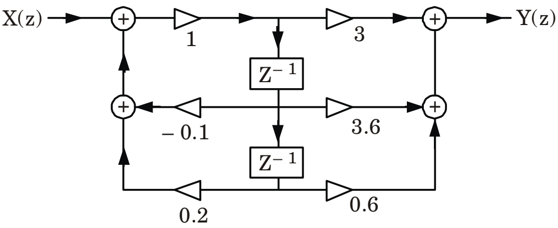


Fig. 6.

C. Cascade form :

1. We have, $$H(z) = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}} \quad ...(1)$$

2. After rearranging the eq. (1)

$$H(z) = \frac{3z^2 + 3.6z + 0.6}{z^2 + 0.1z - 0.2}$$

$$H(z) = \frac{(3z + 0.6)(z + 1)}{(z - 0.4)(z + 0.5)}$$

$$H(z) = \frac{(3 + 0.6z^{-1})(1 + z^{-1})}{(1 - 0.4z^{-1})(1 + 0.5z^{-1})}$$

$$H(z) = H_1(z) \,.\, H_2(z)$$

where $$H_1(z) = \frac{3 + 0.6z^{-1}}{1 - 0.4z^{-1}}, \; H_2 = \frac{1 + z^{-1}}{1 + 0.5z^{-1}}$$

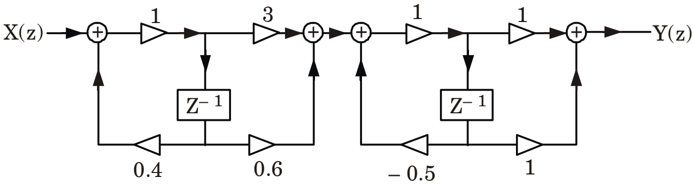


Fig. 7.

D. Parallel form :

1. We have, $$H(z) = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

$$H(z) = \frac{3z^2 + 3.6z + 0.6}{z^2 + 0.1z - 0.2}$$

$$H(z) = 3 + \frac{3.3z + 1.2}{(z - 0.4)(z + 0.5)}$$

2. Taking partial fraction

$$H(z) = 3 + \frac{A}{(z - 0.4)} + \frac{B}{(z + 0.5)}$$

3. By solving and we get,

$$A = 2.8, \quad B = 0.5$$

$$H(z) = 3 + \frac{2.8}{z - 0.4} + \frac{0.5}{z + 0.5}$$

$$H(z) = 3 + \frac{2.8}{z - 0.4z^{-1}} + \frac{0.5}{z + 0.5z^{-1}}$$

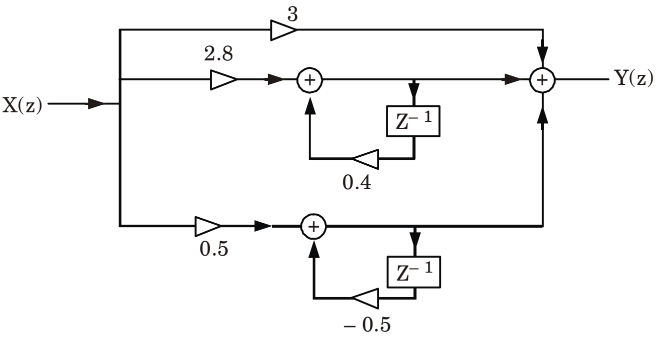


Fig. 8.

b. Find the inverse DFT of the sequence :
***X*(*k*) = {6, – 2 + *j*2, – 2, –2 – *j*2}, using DFT-FFT algorithm.**

Ans.

Given : $X(k) = \{6, -2 + j2, -2, -2 - j2\}$
To Find : IDFT.

1. The flow graph for calculation of 4 point IFFT is shown in Fig. 9.

$$X(0) = 6, X(1) = -2 + 2j, X(2) = -2, X(3) = -2 - 2j$$

Now,
$$S_0 = \frac{1}{2} X(0) + \frac{1}{4} X(2) = \frac{1}{4}(6) + \frac{1}{4}(-2) = 1$$

$$S_1 = \frac{1}{4} X(0) - \frac{1}{4} X(2) = \frac{1}{4}(6) - \frac{1}{4}(-2) = 2$$

$$S_2 = \frac{1}{4} X(1) + \frac{1}{4} X(3) = \frac{1}{4}(-2 + 2j) + \frac{1}{4}(-2 - 2j) = -1$$

$$S_3 = \left[\frac{1}{4} X(1) + \frac{1}{4} X(3)\right](-j)$$

$$= \left[\frac{1}{4}(-2 + 2j) - \frac{1}{4}(-2 - 2j)\right](-j) = -1$$

2. The final output is,

$$x(0) = S_0 + S_2 = 1 - 1 = 0$$
$$x(1) = S_1 + S_3 = 2 - 1 = 1$$
$$x(2) = S_0 - S_2 = 1 + 1 = 2$$
$$x(3) = S_1 - S_3 = 2 + 1 = 3$$
$$\therefore \quad x(n) = \{0, 1, 2, 3\}$$

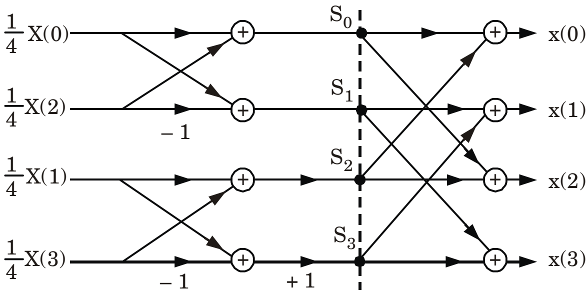


Fig. 9.

7. Attempt any **one** of the following questions : **(10 × 1 = 10)**

a. What is the different window functions used for windowing ? Explain the effect of using different window functions for designing FIR filter on the filter response.

Ans.

A. Window Techniques :

1. Windowing techniques, to design, the FIR filter is the easiest method. In this method the impulse response of an IIR filter is truncated by some function.
2. Suppose

$$h_d(n) = \text{Impulse response of IIR filter}$$

and $$h(n) = \text{Impulse response of FIR filter.}$$

3. Using windowing technique the $h(n)$ is given as,

$$h(n) = \begin{cases} h_d(n) & ; \quad N_1 \le n \le N_2 \\ 0 & ; \quad \text{otherwise} \end{cases} \qquad ...(1)$$

4. In general $h(n) = h_d(n)\ w(n)$ ...(2)

where, $w(n)$ = Window function.

5. Using Fourier transform the eq. (2) can be given as

$$H(e^{j\omega}) = H_d(e^{j\omega}) \ast w(e^{j\omega}) \qquad ...(3)$$

or $$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})\, w(e^{j(\omega-\theta)})\, d\theta \qquad ...(4)$$

B. Characteristics of window function :

i. It should be an even function about $n = 0$.

ii. It should have zero in the range $0 \le n \le N - 1$.

iii. As $\omega \to \pi$, the energy of sidelobes of the frequency response is low.

iv. The narrow width of the main lobe means it has high total energy.

C. Types of window function :

i. Rectangular window function :

$$w_R(n) = \begin{cases} 1 & ; \text{for } |n| \le \left(\dfrac{M-1}{2}\right) \\ 0 & ; \text{otherwise} \end{cases} \quad ...(5)$$

The frequency response of rectangular window is given as

$$w_R(e^{j\omega T}) = \sum_{n=0}^{(M-1)} e^{-j\omega nT} = e^{-j\omega\frac{(M-1)}{2}T} \frac{\sin\left(\dfrac{\omega MT}{2}\right)}{\sin\left(\dfrac{\omega T}{2}\right)} \quad ...(6)$$

From eq. (6) the linear phase response of

a. Causal filter $\theta(\omega) = \omega(M-1)\,T/2$

b. Non-causal filter $\theta(\omega) = 0$

Effect : In the magnitude spectrum of rectangular window, there is one main lobe and many sidelobes. As M increase, the main lobe becomes narrower. The area under the sidelobes remains same irrespective of change in value of M.

ii. Hamming window function :

a. Causal Hamming window :

$$w_{\text{Hamm}}(n) = \begin{cases} 0.54 - 0.46 \cos \dfrac{2\pi n}{M-1} & ; \quad 0 \le n < M-1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

b. Non-causal Hamming window :

$$w_{\text{Hamm}}(n) = \begin{cases} 0.54 + 0.46 \cos \dfrac{2\pi n}{M-1} & ; \quad 0 \le n < M-1/2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Effect : In magnitude response of FIR filter using hamming window, the sidelobes get reduced but slightly increased main lobe.

iii. Hanning window function :

a. Causal Hanning window :

$$w_{\text{Hann}}(n) = \begin{cases} 0.5 - 0.5 \cos \dfrac{2\pi n}{M-1} & ; \quad 0 \le n < M-1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

b. Non-causal Hanning window :

$$w_{\text{Hann}}(n) = \begin{cases} 0.5 + 0.5 \cos \dfrac{2\pi n}{M-1} & ; 0 \le n < M-1/2 \\ 0 & ; \text{otherwise} \end{cases}$$

Effect : It has narrow main lobe and first few sidelobes are significant, after then sidelobes reduce rapidly.

iv. Blackman window function :

a. Causal blackman window : It is given as

$$w_{\text{black}}(n) = \begin{cases} 0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1} & ; \quad 0 \le n < M-1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

b. Non-causal blackman window : It is given as :

$$w_{\text{black}}(n) = \begin{cases} 0.42 + 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1} & ; 0 \le n < M-1/2 \\ 0 & ; \text{otherwise} \end{cases}$$

Effect : In magnitude response, the width of main lobe is increased and it has very small sidelobes.

v. Bartlett window function : For non-causal Bartlett window the function is given as,

$$w_{\text{bart}}(n) = \begin{cases} 1+n; & -\dfrac{M-1}{2} < n < 1 \\ 1-n; & 1 < n < \dfrac{M-1}{2} \end{cases}$$

Effect : The sidelobe is smaller than that of rectangular window and the main lobe become twice that of the rectangular window.

vi. Kaiser window function :

$$w_k(n) = \begin{cases} \dfrac{I_0(\beta)}{I_0(\alpha)} & ; \quad \text{for } n \le \dfrac{M-1}{2} \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Effect : It has reduced sidelobes and transition band is also narrow. Here, note that width of main lobe is inversely proportional to the length of the filter.

b. Derive and draw the flow graph for DIF FFT algorithm for $N = 8$.

Ans.

1. Here $N = 8$, thus, n and $k = 0, 1, 2, \ldots\ldots, 7$

$$X(k) = [X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)]$$

Stage I :

i.
$$X(2p) = [X(0), X(2), X(4), X(6)]$$
$$X(2p+1) = [X(1), X(3), X(5), X(7)]$$
$$h(n) = [x(n) + x(n + N/2)]$$

and
$$f(n) = [x(n) - x(n + N/2)]$$

we get

$$h(0) = x(0) + x(4) \quad f(0) = x(0) - x(4)$$
$$h(1) = x(1) + x(5) \quad f(1) = x(1) - x(5)$$
$$h(2) = x(2) + x(6) \quad f(2) = x(2) - x(6)$$
$$h(3) = x(3) + x(7) \quad f(3) = x(3) - x(7)$$

ii. The first stage of DIF FFT algorithm can be drawn in signal flow graph (SFG) form as shown in Fig. 10.

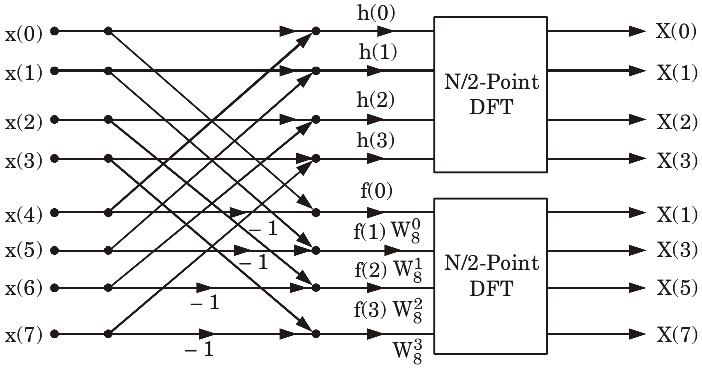


Fig. 10. SFG of first stage of DIF-FFT for N = 8.

Stage II :

i. Now $X(2p)$ and $X(2p + 1)$ are divided into even and odd numbered values sequence.

ii. Taking $X(2p)$ first we have

$$X(2p) = \sum_{n=0}^{\frac{N}{4}-1} h(n)W_N^{2pn} + \sum_{n=N/4}^{\frac{N}{2}-1} h(n)W_N^{2pn}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} h(n)W_N^{2pn} + \sum_{n=0}^{\frac{N}{4}-1} h\left(n+\frac{N}{4}\right)W_N^{2p\left(n+\frac{N}{4}\right)}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} h(n)W_{N/2}^{pn} + W_N^{\left(2p\frac{N}{4}\right)} \sum_{n=0}^{\frac{N}{4}-1} h\left(n+\frac{N}{4}\right)W_{N/2}^{pn} \quad ...(1)$$

Since $W_N^{N/2} = -1$

iii. Thus eq. (1) can be written as

$$X(2p) = \sum_{n=0}^{\frac{N}{4}-1} h(n)W_N^{2pn} + (-1)^p \sum_{n=0}^{\frac{N}{4}-1} h\left(n+\frac{N}{4}\right)W_N^{2pn} \quad ...(2)$$

2. Again for even and odd values of 'p' we have two cases :

Case I : When p = even, $p = 2r$ $(-1)^{2r} = 2r$

Then eq. (2) will become as

$$X(4r) = \sum_{n=0}^{\frac{N}{4}-1} \left[h(n)+h\left(n+\frac{N}{4}\right)\right] W_N^{4rn}$$

$$X(4r) = \sum_{n=0}^{\frac{N}{4}-1} g(n)W_N^{4rn} \quad ...(3)$$

where, $$g(n) = h(n)+h\left(n+\frac{N}{4}\right) \quad ...(4)$$

Case II :

a. When p = odd *i.e.*, $p = (2r + 1)$ then $(-1)^{2r+1} = -1$

$$X(4r+2) = \sum_{n=0}^{\frac{N}{4}-1}\left[h(n) - h\left(n + \frac{N}{4}\right)\right] W_N^{(2r+1)2n}$$

$$= \sum_{n=0}^{\frac{N}{4}-1}\left[h(n) - h\left(n + \frac{N}{4}\right)\right] W_N^{2n} \cdot W_N^{4rn}$$

$$X(4r+2) = \sum_{n=0}^{\frac{N}{4}-1} [A(n) W_N^{4rn}]\, W_N^{2n}$$

where, $$A(n) = h(n) - h\left(n + \frac{N}{4}\right) \quad ...(5)$$

b. Therefore for 2nd stage we get

$$g(0) = h(0) + h(2)$$

$$g(1) = h(1) + h(3)$$

$$A(0) = h(0) - h(2)$$

$$A(1) = h(1) - h(3)$$

Similarly we get

$$X(2p+1) = \sum_{n=0}^{\frac{N}{2}-1} [f(n) W_N^{2pn}]\, W_N^{n}$$

c. Dividing $X(2p + 1)$ into even and odd parts.

$$X(2p+1) = W_N^n \left[\sum_{n=0}^{\frac{N}{4}-1} f(n) W_N^{2pn} + \sum_{n=N/4}^{\frac{N}{2}-1} f(n)\, W_N^{2pn}\right]$$

$$= \left[\sum_{n=0}^{\frac{N}{4}-1} f(n) W_N^{2pn} \cdot W_N^{n} + \sum_{n=0}^{\frac{N}{4}-1} f\left(n + \frac{N}{4}\right) W_N^{\left(n+\frac{N}{4}\right)} \cdot W_N^{2p\left(n+\frac{N}{4}\right)}\right]$$

$$= \left[\sum_{n=0}^{\frac{N}{4}-1} f(n) W_N^{n} \cdot W_N^{2pn} + \sum_{n=0}^{\frac{N}{4}-1} f\left(n + \frac{N}{4}\right) W_N^{n} \cdot W_N^{2pn} \cdot W_N^{(2p+1)N/4}\right]$$

$$X(2p+1) = \sum_{n=0}^{\frac{N}{4}-1}\left[f(n) + W_N^{(2p+1)N/4} f\left(n + \frac{N}{4}\right)\right] W_N^{(2p+1)n} \quad ...(6)$$

3. Again this can also be discussed for even and odd values of 'p'. The results will be same as given in eq. (3) and (6).

Case I : $2p + 1 = 2l$ (even)

$$X(2l) = \sum_{n=0}^{\frac{N}{4}-1}\left[f(n) + W_N^{2lN/4} f\left(n + \frac{N}{4}\right)\right] W_N^{2ln}$$

$$= \sum_{n=0}^{\frac{N}{4}-1}\left[f(n)+(-1)^{2l} f\left(n+\frac{N}{4}\right)\right] W_N^{2ln}$$

$$X(2l) = \sum_{n=0}^{\frac{N}{4}-1} B(n)\, W_N^{2ln} \quad ...(7)$$

where $$B(n) = f(n)+f\left(n+\frac{N}{4}\right) \quad ...(8)$$

Case II :

a. Taking $(2p + 1) = (2l + 1)$ odd
we get

$$X(2l+1) = \sum_{n=0}^{\frac{N}{4}-1}\left[f(n)+W_N^{(2l+1)N/4}\cdot f\left(n+\frac{N}{4}\right)\right] W_N^{(2l+1)n}$$

$$= \sum_{n=0}^{\frac{N}{4}-1}\left[f(n)-f\left(n+\frac{N}{4}\right)\right] W_N^{2n}\cdot W_N^{2ln}$$

$$X(2l+1) = \sum_{n=0}^{\frac{N}{4}-1} [C(n)W_N^{2ln}]\cdot W_N^{2n} \quad ...(9)$$

where, $$C(n) = f(n)-f\left(n+\frac{N}{4}\right)$$

b. Therefore for this stage we get

$$B(0) = f(0) + f(2)$$
$$C(0) = f(0) - f(2)$$
$$B(1) = f(1) + f(3)$$
$$C(1) = f(1) - f(3)$$

c. The SFG for 8-point DFT using DIF FFT algorithm is shown in Fig. 11.

d. The decimation process is continued until the size of the last DFT is not equal to radix of the FFT. Number of decimation stages (l) is given as

$$l = \log_2 N$$
or $$N = 2^l$$

e. The $X(2l)$ and $X(2l + 1)$ are again decomposed in even and odd parts.

f. Since for the 3rd stage, we have $N/4 = 8/4 = 2$ point DFT. This DFT can be drawn by using the butterfly structure as given in DIT-FFT.

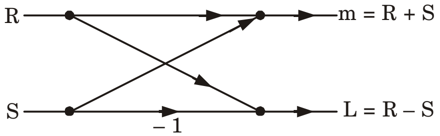


Fig. 11.

4. The total SFG for all three stages DIF-FFT is shown in Fig. 12.

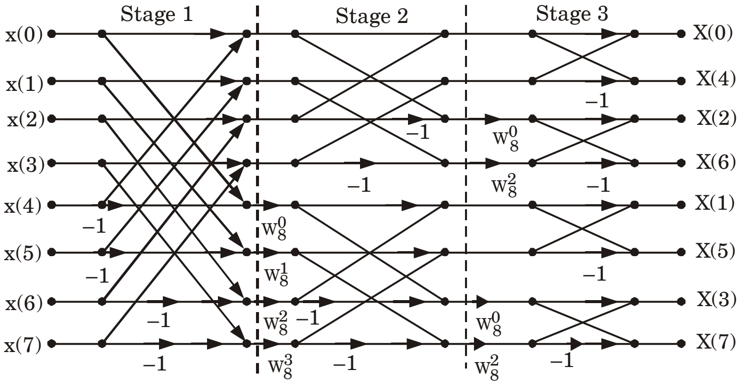


Fig. 12. SFG for 8-point DFT using DIF-FFT.

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B. Tech.

(SEM. V) ODD SEMESTER THEORY EXAMINATION, 2018-19

DIGITAL SIGNAL PROCESSING

Time : 3 Hours **Max. Marks : 70**

Note : Attempt **all** section. If required any missing data; then choose suitably.

SECTION-A

1. Attempt **all** questions in brief. **(2 × 7 = 14)**

a. **What is the main disadvantage of direct form realization ?**

b. **What is the warping effect ?**

c. **Compare FIR and IIR filter.**

d. **What are the advantages of Kaiser window ?**

e. **What is window and why it is necessary ?**

f. **What is downsampling and upsampling ?**

g. **Define decimation.**

SECTION-B

2. Attempt any **three** of the following : **(7 × 3 = 21)**

a. **Obtain the cascade form realization :**

$$y(n) = y(n-1) - 1/2\,[y(n-2)] + 1/4\,[y(n-2)] + x(n) - x(n-1) + x(n-2)$$

b. **Find the order and cut-off frequency of a digital filter with the following specification**

$$0.89 \le |H(e^{j\omega})| \le 1, \quad 0 \le \omega \le 0.4\,\pi$$

$$|H(e^{j\omega})| \le 0.18, \quad 0.6\,\pi \le \omega \le \pi$$

use the impulse invariance method.

c. **The desired response of a low pass filter is** $H_d(e^{j\omega}) = e^{-3j\omega}$, $-3\pi/4 \le \omega \le 3\pi/4$ **determine** $H(e^{j\omega})$ **for** $M = 7$ **using a hamming window.**

d. Find the 8 point DFT of the sequence $x(n) = \{1, 1, 1, 1, 1, 0, 0, 0\}$ using DIT FFT.

e. Discuss about quadrature mirror filters in detail.

SECTION-C

3. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Obtain the parallel form realization

$$H(z) = \frac{(1 + 1/2z^{-1})}{(1 - z^{-1} + 1/4z^{-2})(1 - z^{-1} + 1/2z^{-2})}$$

b. Obtain the direct form I and II form realization

$$H(z) = \frac{(1 + z^{-1})(1 + 2z^{-1})}{(1 + 1/\ 2z^{-1})(1 - 1/4\, z^{-1})(1 + 1/8z^{-1})}$$

4. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Using bilinear transformation, design a Butterworth filter which satisfies the following condition

$$0.8 \le |H(e^{j\omega})| \le 1, \quad 0 \le \omega \le 0.2\,\pi$$
$$|H(e^{j\omega})| \le 0.2, \quad 0.6\,\pi \le \omega \le \pi$$

b. What is the difference between Butterworth and Chebyshev ? Explain the frequency transformation is done.

5. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Using a rectangular window technique design a low pass filter with passband gain of unity, cut-off frequency of 1000 Hz and working at a sampling frequency of 5 kHz. The length of the impulse response should be 7.

b. Discuss the finite word length effects in digital filters.

6. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Find the linear convolution using circular convolution of the following sequence $x(n) = \{1, 2, 1\}$, $h(n) = \{1, 2\}$.

b. Find the 8 point DFT of the sequence $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$ using DIF FFT.

7. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. What is multirate digital signal processing ? Discuss about application areas of it.

b. Discuss about interpolation and sampling rate conversion in detail.

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SOLUTION OF PAPER (2018-19)

Note : Attempt **all** section. If required any missing data; then choose suitably.

SECTION-A

1. Attempt **all** questions in brief. **(2 × 7 = 14)**

a. What is the main disadvantage of direct form realization ?

Ans.

i. They are lack hardware flexibility.

ii. Due to finite precision arithmetic, the sensitivity of the coefficients to quantization effect increases with the order of filter.

b. What is the warping effect ?

Ans. The mapping of frequency from Ω to ω is approximately linear for small value of Ω and ω. For the higher frequencies, however the relation between Ω and ω becomes highly non-linear. This introduces the distortion in the frequency scale of digital filter relative to analog filter. This effect is known as warping effect.

c. Compare FIR and IIR filter.

Ans.

S. No.	FIR filter	IIR filter
1.	They have finite impulse response.	They have infinite impulse response.
2.	Always stable.	Sometimes unstable.
3.	Have exact linear phase response.	Non-linear phase response.
4.	It requires more memory, higher computational complexity and involves more parameters.	It requires less memory, lower computational complexity and involves fewer parameters.

d. What are the advantages of Kaiser window ?

Ans. Keeping the window length constant, we can adjust the shape factor, to design for the pass band and stop band ripples.

e. What is window and why it is necessary ?

Ans. Window : A window function is a mathematical function that is zero valued outside of some chosen interval, normally symmetric around the middle of the interval.

Necessity : Enhance the ability of an FFT to extract spectral data from signals.

f. What is downsampling and upsampling ?

Ans. **Downsampling :** It is the process of reducing the sampling rate of a signal. This is usually done to reduce the data rate or the size of the data.

Upsampling : It is the process of inserting zero-valued samples between original samples to increase the sampling rate.

g. Define decimation.

Ans. The process of reducing the sampling rate by a factor D (downsampling by D) is called decimation.

SECTION-B

2. Attempt any **three** of the following : **(7 × 3 = 21)**

a. Obtain the cascade form realization :
$$\boldsymbol{y(n) = y(n-1) - 1/2\,[y(n-2)] + 1/4\,[y(n-2)] + x(n) - x(n-1) + x(n-2)}$$

Ans.

1. Given,
$$y(n) = y(n-1) - 1/2\,[y(n-2)] + 1/4\,[y(n-2)] + x(n) - x(n-1) + x\,(n-2)$$
$$y(n) = y(n-1) - 1/4\,[y(n-2)] + x(n) - x(n-1) + x(n-2)$$

2. Taking z-transform on both sides
$$Y(z) = z^{-1}\,Y(z) - 1/4\,[z^{-2}\,Y(z)] + X(z) - z^{-1}\,X\,(z) + z^{-2}\,X(z)$$
$$Y(z) - z^{-1}\,Y(z) + 1/4\,[z^{-2}\,Y(z)] = X(z)\,[1 - z^{-1} + z^{-2}]$$
$$\frac{Y(z)}{X(z)} = \frac{1 - z^{-1} + z^{-2}}{1 - z^{-1} + 1/4z^{-2}}$$
$$\frac{Y(z)}{X(z)} = \frac{(1 - z^{-1} + z^{-2})}{(1 - z^{-1}/2)\,(1 - z^{-1}/2)}$$
$$= \frac{1}{(1 - z^{-1}/2)} \cdot \frac{(1 - z^{-1} + z^{-2})}{(1 - z^{-1}/2)}$$

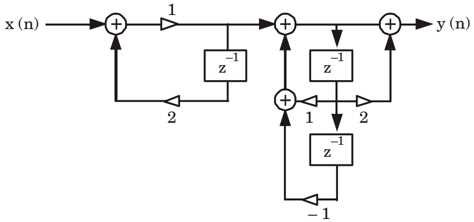

Fig. 1.

b. Find the order and cut-off frequency of a digital filter with the following specification

$$0.89 \le |H(e^{j\omega})| \le 1,\ 0 \le \omega \le 0.4\pi$$
$$|H(e^{j\omega})| \le 0.18,\quad 0.6\pi \le \omega \le \pi$$

use the impulse invariance method.

Ans.

Given : $0.89 \le |H(e^{j\omega})| \le 1,\ 0 \le \omega \le 0.4\pi$

$|H(e^{j\omega})| \le 0.18,\ 0.6\pi \le \omega \le \pi$

$\delta_1 = 0.89$, $\delta_2 = 0.18$, $\omega_1 = 0.4\pi$ and $\omega_2 = 0.6\pi$.

To Find : Order, cut-off frequency.

Step I : Determination of analog filter's edge frequencies.

$$\Omega_1 = \frac{\omega_1}{T} = 0.4\pi \text{ and } \Omega_2 = \frac{\omega_2}{T} = 0.6\pi$$

Therefore, $\Omega_2/\Omega_1 = 1.5$

Step II : Determination of the order of the filter.

$$N \ge \frac{1}{2}\frac{\log\left\{((1/\delta_2^2)-1)/((1/\delta_1^2)-1)\right\}}{\log(\Omega_2/\Omega_1)}$$

$$N \simeq 2$$

Step III : Determination of – 3 dB cut-off frequency :

$$\Omega_c = \frac{\Omega_1}{[(1/\delta_1^2-1)]^{1/2N}} = \frac{0.6\pi}{[(1/0.89^2)-1]^{1/8}} = 2.63$$

c. The desired response of a low pass filter is $H_d(e^{j\omega}) = e^{-3j\omega}$, $-3\pi/4 \le \omega \le 3\pi/4$ determine $H(e^{j\omega})$ for $M = 7$ using a hamming window.

Ans.

Given : $H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega} & ; \dfrac{-3\pi}{4} \le \omega \le 3\pi/4 \\ 0 & ; \text{Otherwise} \end{cases}$

To Find : $H(e^{j\omega})$ for $M = 7$

1. The filter coefficients are given by

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-3\pi/4}^{3\pi/4} e^{-j3\omega}e^{j\omega n}d\omega$$

$$h_d(n) = \frac{\sin 3\pi(n-3)/4}{\pi(n-3)},\ n \ne 3 \text{ and } h_d(3) = \frac{3}{4}$$

The filter coefficients are,

$h_d(0) = 0.0750$, $h_d(1) = -0.1592$, $h_d(2) = 0.2251$,

$h_d(3) = 0.75$, $h_d(4) = 0.2251$, $h_d(5) = -0.1592$,

$h_d(6) = 0.0750$

2. The Hamming window function is,

$$w(n) = \begin{cases} 0.54 - 0.46 \cos \dfrac{2\pi n}{M-1} & ; \quad 0 \le n \le M-1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Therefore, with $M = 7$,

$w(0) = 0.08$, $w(1) = 0.31$, $w(2) = 0.77$, $w(3) = 1$,

$w(4) = 0.77$, $w(5) = 0.31$, $w(6) = 0.08$.

3. The filter coefficients of the resultant filter are then,

$h\{n\} = h_d(n)\, w(n)$, $n = 0, 1, 2, 3, 4, 5, 6$.

Therefore, $h(0) = 0.006$

$h(1) = -0.0494$

$h(2) = 0.1733$

$h(3) = 0.75$

$h(4) = 0.1733$

$h(5) = -0.0494$

and $h(6) = 0.006$

4. The frequency response is given by

$$H(e^{j\omega}) = \sum_{n=0}^{6} h(n)\, e^{-j\omega n}$$

$= e^{-j3\omega}[h(3) + 2h(0)\cos 3\omega + 2h(1) \cos 2\omega + 2h(2) \cos \omega]$

$= e^{-j3\omega}[0.75 + 0.012 \cos 3\omega - 0.0988 \cos 2\omega + 0.3466 \cos \omega]$

d. Find the 8 point DFT of the sequence $x(n) = \{1, 1, 1, 1, 1, 0, 0, 0\}$ using DIT FFT.

Ans.

1. Given, $x(n) = \{1, 1, 1, 1, 1, 0, 0, 0\}$

2. Stage-3 outputs :

$$v_{11}(0) = x(0) + W_8^0\ x(4) = 1 + 1(1) = 2$$

$$v_{11}(1) = x(0) - W_8^0\ x(4) = 1 - 1(1) = 0$$

$$v_{12}(0) = x(2) + W_8^0\ x(6) = 1 + 1(0) = 1$$

$$v_{12}(1) = x(2) - W_8^0\ x(6) = 1 - 1(0) = 1$$

$$v_{21}(0) = x(1) + W_8^0\ x(5) = 1 + 1(0) = 1$$

$$v_{21}(1) = x(1) - W_8^0\ x(5) = 1 - 1(0) = 1$$

$$v_{22}(0) = x(3) + W_8^0\ x(7) = 1 + 1(0) = 1$$

$$v_{22}(1) = x(3) - W_8^0\ x(7) = 1 - 1(0) = 1$$

3. Stage-2 outputs :

$$F_1(0) = v_{11}(0) + W_8^0 \; v_{12}(0)$$
$$= 2 + 1(1) = 3$$

$$F_1(1) = v_{11}(1) + W_8^2 \; v_{12}(1)$$
$$= 0 - j(1) = -j$$

$$F_1(2) = v_{11}(0) - W_8^0 \; v_{12}(0)$$
$$= 2 - 1(1) = 1$$

$$F_1(3) = v_{11}(1) - W_8^2 \; v_{12}(1)$$
$$= 0 - (-j)(1) = j$$

$$F_2(0) = v_{21}(0) + W_8^0 \; v_{22}(0)$$
$$= 1 + 1(1) = 2$$

$$F_2(1) = v_{21}(1) + W_8^2 \; v_{22}(1)$$
$$= 1 + (-j)(1) = 1 - j$$

$$F_2(2) = v_{21}(0) - W_8^0 \; v_{22}(0)$$
$$= 1 - 1(1) = 0$$

$$F_2(3) = v_{21}(1) - W_8^2 \; v_{22}(1)$$
$$= 1 - (-j)(1) = 1 + j$$

4. Stage-1 or final outputs :

$$X(0) = F_1(0) + W_8^0 \; F_2(0)$$
$$= 3 + 1(2) = 5$$

$$X(1) = F_1(1) + W_8^1 \; F_2(1)$$
$$= -j + (0.707 - j0.707)\,(1 - j)$$
$$= -j2.414$$

$$X(2) = F_1(2) + W_8^2 \; F_2(2)$$
$$= 1 + (-j)(0) = 1$$

$$X(3) = F_1(3) + W_8^3 \; F_2(3)$$
$$= j + (-0.707 - j0.707)\,(1 + j)$$
$$= -j0.414$$

$$X(4) = F_1(0) - W_8^0 \; F_2(0)$$
$$= 3 - 2 = 1$$

$$X(5) = F_1(1) - W_8^1 \; F_2(1)$$
$$= -j - (0.707 - j0.707) = (1 - j)$$
$$= -j + j1.414 = j0.414$$

$$X(6) = F_1(2) - W_8^2 \; F_2(2)$$
$$= 1 - (-j)\,(0) = 1$$

$$X(7) = F_1(3) - W_8^3 \; F_2(3)$$
$$= j - (-0.707 - j0.707)\,(1 + j)$$
$$= j + j1.414 = j2.414$$

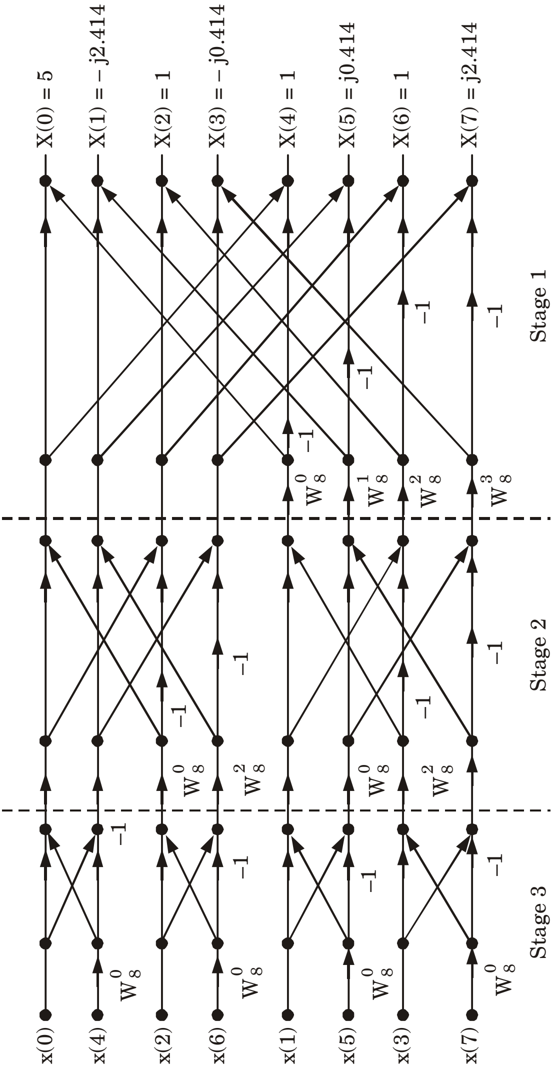

Fig. 2.

e. Discuss about quadrature mirror filters in detail.

Ans.

1. The subband signals are down sampled before processing. The signals are up sampled after processing.

2. The structure used for this is known as Quadrature Mirror Filter (QMF) bank.
3. If the decimation and interpolation factors are equal, then the characteristics of $x(n)$ will be available in $y(n)$ if the filters are properly selected.
4. If this properly is satisfied, then the filter bank can be called as a critically sampled filter bank.
5. To list a few applications where QMF filters are used.
i. Efficient coding of the signal $x(n)$
ii. Analog voice privacy for secure telephone communication.
6. The two-channel QMF filter bank is shown in Fig. 3. The analysis filter $H_0(z)$ is a low-pass filter and $H_1(z)$ is a high-pass filter.
7. The cut-off frequency is taken as $\pi/2$ for these filters.
8. The subband signals $\{v_k(n)\}$ are down sampled. After down sampling, these signals are processed (encoded).
9. In the receiving side the signals are decoded, up-sampled and then passed through the synthesis filters $G_0(z)$ and $G_1(z)$ to get the output $y(n)$.
10. The encoding and decoding processes are not shown in Fig. 3.
11. For perfect reconstruction, the QMF filter banks should be properly selected.

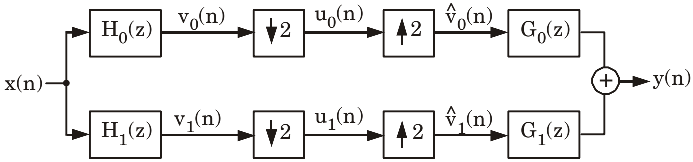

Fig. 3. Two-channel quadrature mirror filter bank.

SECTION-C

3. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Obtain the parallel form realization

$$\boldsymbol{H(z) = \frac{(1 + 1/2z^{-1})}{(1 - z^{-1} + 1/4z^{-2})(1 - z^{-1} + 1/2z^{-2})}}$$

Ans.

1. $$H(z) = \frac{1 + 1/2z^{-1}}{(1 - z^{-1} + 1/4z^{-2})(1 - z^{-1} + 1/4z^{-2})}$$

$$= \frac{1 + 1/2z^{-1}}{(1 - 1/2z^{-1})^2 (1 - z^{-1} + 1/2z^{-2})}$$

$$= \frac{A}{(1 - 1/2z^{-1})} + \frac{B}{(1 - 1/2z^{-1})^2} + \frac{Cz^{-1} + D}{(1 - z^{-1} + 1/2z^{-2})}$$

2. On solving we get,

$$A = 2, B = 2, C = 2, \text{ and } D = -3.$$

3. So, $$H(z) = \frac{2}{(1-1/2z^{-1})} + \frac{2}{(1-1/2z^{-1})^2} + \frac{2z^{-1}-3}{(1-z^{-1}+1/2z^{-2})}$$

$$= \frac{2}{(1-1/2z^{-1})} + \frac{2}{(1-z^{-1}+1/4z^{-2})} + \frac{2z^{-1}-3}{(1-z^{-1}+1/2z^{-2})}$$

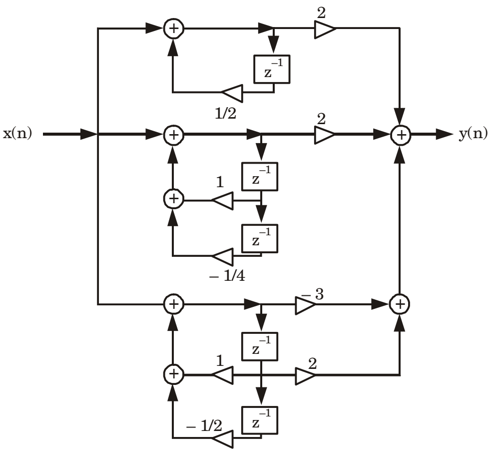

Fig. 4.

b. Obtain the direct form I and II form realization

$$\boldsymbol{H(z) = \frac{(1+z^{-1})(1+2z^{-1})}{(1+1/\ 2z^{-1})(1-1/4\,z^{-1})(1+1/8z^{-1})}}$$

Ans.

1. Given, $$H(z) = \frac{(1-z^{-1})(1+2z^{-1})}{(1+1/2\,z^{-1})(1-1/4\,z^{-1})\left(1+1/8\,z^{-1}\right)}$$

$$H(z) = \frac{(1+3z^{-1}+2z^{-2})}{(1+1/8\,z^{-1}-3/32\,z^{-2}-1/64\,z^{-3})}$$

i. Direct form I :

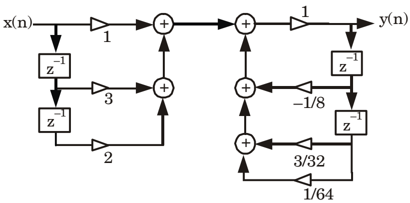

Fig. 5.

ii. Direct form II :

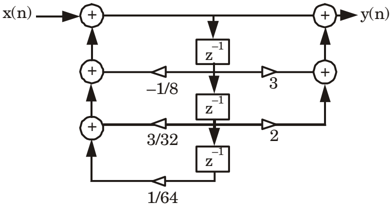

Fig. 6.

4. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Using bilinear transformation, design a Butterworth filter which satisfies the following condition

$$0.8 \le |H(e^{j\omega})| \le 1, \quad 0 \le \omega \le 0.2\pi$$
$$|H(e^{j\omega})| \le 0.2, \quad 0.6\pi \le \omega \le \pi$$

Ans.

Given : $0.8 \le |H(e^{j\omega})| \le 1, \quad 0 \le \omega \le 0.2\pi$

$|H(e^{j\omega})| \le 0.2, \quad 0.6\pi \le \omega \le \pi$

$A_p = 0.8,\ A_s = 0.2,\ \delta_p = 0.2\pi,\ \delta_s = 0.6\pi$

To Design : Butterworth low pass filter.

Step 1 : For bilinear transformation, we have

$$\Omega_s = \frac{2}{T_s} \tan \frac{\delta_s}{2}$$

$$\Omega_p = \frac{2}{T_s} \tan \frac{\delta_p}{2}$$

Assume, $T_s = 1$ s

$$\Omega_p = 2 \tan \frac{\delta_p}{2} = 2 \tan \frac{0.2\pi}{2} = 0.65 \text{ rad/sec}$$

$$\Omega_s = 2 \tan \frac{\delta_s}{2} = 2 \tan \frac{0.6\pi}{2} = 2.75 \text{ rad/sec}$$

Thus, specifications of analog filter are :

$$A_p = 0.8,\ \Omega_p = 0.65$$
$$A_s = 0.2,\ \Omega_s = 2.75$$

Step 2 : Order of filter, N, is given by,

$$N \ge \frac{\log\left[\dfrac{\left(\dfrac{1}{A_s^2} - 1\right)}{\left(\dfrac{1}{A_p^2} - 1\right)}\right]}{2 \log\left(\dfrac{\Omega_s}{\Omega_p}\right)} \ge \frac{\log\left[\dfrac{\left(\dfrac{1}{(0.2)^2} - 1\right)}{\left(\dfrac{1}{(0.8)^2} - 1\right)}\right]}{2 \log\left(\dfrac{2.75}{0.65}\right)}$$

$$\geq \frac{\log(24/0.5625)}{2\log(4.231)} = \frac{1.63}{1.253} = 1.3$$

Thus, we can take $N = 1$ (approx).

Step 3 : Cut-off frequency, can be determined by,

$$\Omega_c = \frac{\Omega_p}{\left(\dfrac{1}{A_p^2} - 1\right)^{1/2N}} = \frac{0.65}{\left(\dfrac{1}{(0.85)^2} - 1\right)^{1/2}}$$

$$= \frac{0.650}{0.75} = 0.867$$

Step 4 : The poles of $H(s)$ are given by,

$$p_K = \pm\ \Omega_c\, e^{j(N+2K+1)\pi/2N},\ K = 0,\ 1,\ 2........N-1$$

$$p_K = \pm\ 0.867\, e^{j(1+2K+1)\pi/2}$$

$$p_K = \pm\ 0.867\, e^{j(2+2K)\pi/2}$$

$$p_K = \pm\, 0.867\left[\cos\left(\frac{(2+2K)\pi}{2}\right) + j\sin\left(\frac{(2+2K)\pi}{2}\right)\right]$$

Range of K is 0 to $N-1$, thus $K = 0$

For $K = 0$, $\quad p_0 = \pm\, 0.867\,[\cos\pi + j\sin\pi]$

$$= \pm\, 0.867\,(-1) = -\,0.867,\ 0.867$$

Step 5 : Calculation of $H(s)$: For the stability of filter, select the poles on the L.H.S of s-plane.

So we will select the pole $s = -\,0.867$.

The system function of second order Butterworth low pass filter is given as

$$H(s) = \frac{\Omega_c^N}{(s - s_1)(s - s_1{}^*)} = \frac{0.867}{(s + 0.867)} \qquad ...(1)$$

Step 6 : The transfer function for digital filter is obtained by putting,

$$s = \frac{2}{T_s}\left(\frac{z-1}{z+1}\right) = 2\left(\frac{z-1}{z+1}\right)$$

Put value of s in eq. (2.12.1), we get

$$H(z) = \frac{0.867}{\left[2\left(\dfrac{z-1}{z+1}\right) + 0.867\right]} = \frac{0.867}{\left[\dfrac{2z - 2 + (z+1)0.867}{z+1}\right]}$$

$$= \frac{0.867\,(z+1)}{2.867z - 1.133}$$

b. What is the difference between Butterworth and Chebyshev ? Explain the frequency transformation is done.

Ans.

A. Difference between Butterworth and Chebyshev :

S. No.	Characteristics	Butterworth filter	Chebyshev filter
1.	Magnitude frequency squared response $\lvert H_a j(\Omega)\rvert^2$	$\dfrac{1}{1+\left(\dfrac{\Omega}{\Omega_c}\right)^{2N}}$	$\dfrac{1}{1+\in^2 C_N^2(\Omega)}$ for $\Omega_c = 1$
2.	Transition band	Transition band is broader than Chebyshev for given order.	Transition band is narrower than Butterworth for given order.
3.	Frequency response.	Monotonically decreasing.	Ripples in passband and monotonic in stopband.
4.	Phase response	Good linear phase response.	Relatively non-linear phase response than Butterworth.
5.	Order of specification for a given set.	Higher.	Lower.
6.	Poles of $H_a(s)$ (location).	Poles of $H_a(s)$ lie on the circle of radius Ω_c in the s-plane.	Poles of $H_a(s)$ lie on the ellipse in the s-plane.

B. Frequency transformation : There are two types of frequency transformations :

a. Analog frequency transformation : The frequency transformation formulae used to convert a prototype low-pass filter into a low-pass (with a different cut-off frequency), high-pass, band-pass or band-stop is given below :

i. Low-pass with cut-off frequency Ω_c to low-pass with a new cut-off frequency Ω_c^*.

$$s \to \frac{\Omega_c}{\Omega_c^*} s \qquad ...(1)$$

Thus, if the system response of the prototype filter is $H_P(s)$, the system response of the new low-pass filter will be

$$H(s) = H_P\left(\frac{\Omega_c}{\Omega_c^*} s\right) \qquad ...(2)$$

ii. Low-pass with cut-off frequency Ω_c to high-pass with cut-off frequency Ω_c^* :

$$s \to \frac{\Omega_c \Omega_c^*}{s} \quad ...(3)$$

The system function of the high-pass filter is then,

$$H(s) = H_P\left(\frac{\Omega_c\ \Omega_c^*}{s}\right) \quad ...(4)$$

iii. Low-pass with cut-off frequency Ω_c to band-pass with lower cut-off frequency Ω_1 and higher cut-off frequency Ω_2.

$$s \to \Omega_c\ \frac{s^2 + \Omega_1\ \Omega_2}{s(\Omega_2 - \Omega_1)} \quad ...(5)$$

The system function of the band-pass filter is then

$$H(s) = H_P\left(\Omega_c\ \frac{s^2 + \Omega_1\ \Omega_2}{s(\Omega_2 - \Omega_1)}\right) \quad ...6)$$

iv. Low-pass with cut-off frequency Ω_c to band-stop with lower cut-off frequency Ω_1 and higher cut-off frequency Ω_2 :

$$s \to \Omega_c\ \frac{s\ (\Omega_2 - \Omega_1)}{s^2 + \Omega_1\ \Omega_2}$$

The system function of the band-stop filter is then,

$$H(s) = H_P\left(\Omega_c\ \frac{s\ (\Omega_2 - \Omega_1)}{s^2 + \Omega_1\Omega_2}\right)$$

b. Digital frequency transformation :

i. The frequency transformation is done in the digital domain by replacing the variable z^{-1} by a function of z^{-1}, *i.e.*, $f(z^{-1})$.

ii. This mapping must take into account the stability criterion. All the poles lying within the unit circle must map onto itself and the unit circle must also map onto itself.

iii. For the unit circle to map onto itself, the implication is that for $r = 1$,

$$e^{-j\omega} = f(e^{-j\omega}) = |f(e^{-j\omega})|\, e^{j\arg[f(e^{-j\omega})]}$$

iv. Hence, we must have $|f(e^{-j\omega})| = 1$ for all frequencies. So, the mapping is that of an all pass filter and of the form,

$$f(z^{-1}) = \pm \prod_{k=1}^{n} \frac{z^{-1} - a_k}{1 - a_k\ z^{-1}} \quad ...(7)$$

v. To get a stable filter from the stable prototype filter, we must have $|a_k| < 1$. The transformation formulae can be obtained from eq. (7) for converting the prototype low-pass digital filter into a digital low-pass, high-pass, band-pass or band-stop filter.

5. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Using a rectangular window technique design a low pass filter with passband gain of unity, cut-off frequency of 1000 Hz and working at a sampling frequency of 5 kHz. The length of the impulse response should be 7.

Ans.

Given : f_c = 1000 Hz, f_s = 5 kHz, M = 7
To Design : FIR digital filter.

1. The desired response of the ideal low-pass filter is given by
$$H_d(e^{j\omega}) = \begin{cases} 1, & 0 \le f \le 1000 \text{ Hz} \\ 0, & f > 1000 \text{ Hz} \end{cases}$$
2. The above response can be equivalently specified in terms of the normalised ω_c.
$$\omega_c = 2\pi f_c / f_s = 2\pi\,(1000)/(5000) = 1.26 \text{ rad/sec.}$$
Hence, the desired response is
$$H_d(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le 1.26 \\ 0, & 1.26 < |\omega| \le \pi \end{cases}$$
3. The filter coefficients are given by
$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-1.26}^{1.26} e^{j\omega n}d\omega$$
$$h_d(n) = \frac{\sin 1.26\, n}{\pi n},\ n \ne 0 \text{ and } h_d(0) = \frac{1.26}{\pi} = 0.40$$
4 Using the rectangular window function and for M = 7,
$$h(n) = h_d(n)\ w(n), \qquad n = 0, 1, 2, 3, 4, 5, 6$$
Therefore,
$h(0) = 0.40$, $h(4) = -0.0753$
$h(1) = 0.303$, $h(5) = 0.0069$
$h(2) = 0.093$ $h(6) = 0.0067$
$h(3) = -0.063$

b. Discuss the finite word length effects in digital filters.

Ans.
1. The fundamental operations in digital filters are multiplication and addition. When these operations are performed in a digital system, the input data as well as the product and sum (output data) have to be represented in finite word length, which depend on the size (length) of the resistor used to store the data.
2. In digital computation the input and output data (sum and product) are quantized by rounding or truncation to convert them into finite word size. This creates error (in noise) in the output or creates oscillations (limit cycle) in the output.
3. These effects due to finite precision representative of number in digital system are called as finite word length effects.
4. List some of the finite word length effects in digital filters :
   i. Errors due to quantization of the input data.
   ii. Errors due to quantization of the filter coefficients.
   iii. Errors due to rounding the product in multiplications.
   iv. Limit cycle due to product quantization and overflow in addition.

6. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Find the linear convolution using circular convolution of the following sequence $x(n) = \{1, 2, 1\}$, $h(n) = \{1, 2\}$.

Ans.

Given : $x(n) = \{1, 2, 1,\}$, $h(n) = \{1, 2\}$

To Find : Linear convolution.

1. Length of $x(n) = L = 3$

   Length of $h(n) = M = 2$

3. Therefore $N = L + M - 1$

   $= 3 + 2 - 1 = 5 - 1 = 4$

   Thus $N = 4$

3. Let us make length of $x(n)$ and $h(n)$ equal to 4 by adding zeros at end.

$$x(n) = \{1, 2, 1, 0\}$$

$$h(n) = \{1, 2, 0, 0\}$$

4. As we know $X(k) = W_N\, x_N$ ...(1)

Here
$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \text{ and } x_N = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

5. Substituting the values of W_4 and x_N in eq. (1), we get

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2j \\ 0 \\ j \end{bmatrix}$$

Therefore $X(k) = \{4, -2j, 0, j\}$

6. To find $H(k)$, $H(k) = W_N\, h_N$

$$H(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1-2j \\ -1 \\ 1+2j \end{bmatrix}$$

$$Y(k) = X(k)\, H(k)$$
$$= \{4, -2j, 0, j\}\,\{3, 1-2j, -1, 1+2j\}$$
$$X(k) = \{12, -2j-4, 0, j-2\}$$

b. Find the 8 point DFT of the sequence $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$ using DIF FFT.

Ans.

Given : $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$

To Find : DFT.

1. $$W_8^0 = e^0 = 1$$
$$W_8^1 = e^{-j\pi/8} = 0.707 - j0.707$$
$$W_8^2 = e^{-j\pi/2} = e^{-j\pi/2} = -j$$
$$W_8^3 = -0.707 - j0.707$$
2. The output of various stages can be easily calculated from 8-point butterfly diagram.
3. **Stage-1 outputs :**
$$g(0) = x(0) + x(4) = 1 + 4 = 5$$
$$g(1) = x(1) + x(5) = 2 + 3 = 5$$
$$g(2) = x(2) + x(6) = 3 + 2 = 5$$
$$g(3) = x(3) + x(7) = 4 + 1 = 5$$
$$g(4) = [x(0) - x(4)]\ W_8^0 = -3$$
$$g(5) = [x(1) - x(5)]\ W_8^1 = -0.707 + j0.707$$
$$g(6) = [x(2) - x(6)]\ W_8^2 = -j$$
$$g(7) = [x(3) - x(7)]\ W_8^3 = -2.121 - j2.121$$
4. **Stage-2 outputs :**
$$h(0) = g(0) + g(2) = 5 + 5 = 10$$
$$h(1) = g(1) + g(3) = 5 + 5 = 10$$
$$h(2) = [g(0) - g(2)]\ W_8^0 = 5 - 5 = 0$$
$$h(3) = [g(1) - g(3)]\ W_8^2 = 0$$
$$h(4) = g(4) + g(6) = -3 - j$$
$$h(5) = g(5) + g(7) = -2.828 - j1.414$$
$$h(6) = [g(4) - g(6)]\ W_8^0 = -3 + j$$
$$h(7) = [g(5) - g(7)]\ W_8^2 = +2.828 - j1.414$$
5. **Stage-3 outputs or final outputs :**
$$X(0) = h(0) + h(1) = 10 + 10 = 20$$
$$X(1) = h(4) + h(5) = -3 - j - 2.828 - j1.414$$
$$= -5.828 - j2.414$$
$$X(2) = h(2) + h(3) = 0$$
$$X(3) = [h(6) + h(7)]\ W_8^0 = -3 + j + 2.828 - j1.414$$
$$= -0.172 + j0.414$$

$$X(4) = [h(0) - h(1)]\, W_8^0 = 0$$

$$X(5) = [h(4) - h(5)]\, W_8^0 = -3 - j + 2.828 + j1.414$$
$$= -0.172 + j0.414$$

$$X(6) = [h(2) - h(3)]\, W_8^0 = 0$$

$$X(7) = [h(6) - h(7)]\, W_8^0 = -3 + j - 2.828 + j1.414$$
$$= -5.828 + j2.414$$

6. Thus, $X(k) = \{20, -5.828 - j2.414, 0, -0.172 + j0.414, 0, -0.172 + j0.414, 0, -5.828 + j2.414\}$

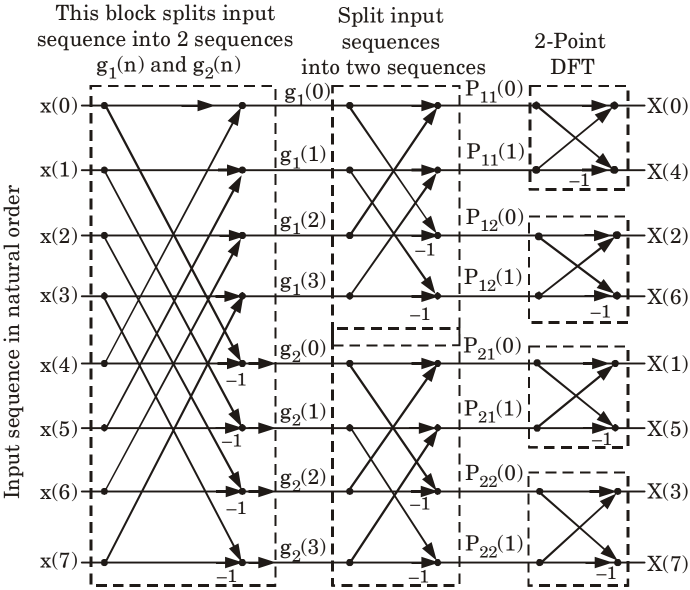

Fig. 7.

7. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. What is multirate digital signal processing ? Discuss about application areas of it.

Ans.

A. Multirate digital signal processing :

1. The process of converting a signal from a given rate to a different rate is called sampling rate conversion. The systems that employ multiple sampling rates in the processing of digital signals are called multirate digital signal processing systems.
2. Different sampling rates can be obtained using an upsampler and downsampler. The basic operations in multirate processing to achieve this are decimation and interpolation.

3. Decimation is used for reducing the sampling rate and interpolation is for increasing the sampling rate.

B. Application area of multirate signal processing :

There are various areas in which multirate signal processing is used :

i. Communication systems
ii. Speech and audio processing systems
iii. Antenna systems, and
iv. Radar systems

b. Discuss about interpolation and sampling rate conversion in detail.

Ans.

A. Interpolation :

1. The process of increasing the sampling rate by an integer factor I (upsampling by I) is called interpolation.
2. An increase in the sampling rate by an integer factor of I can be accomplished by interpolating $I - 1$ new samples between successive values of the signal.

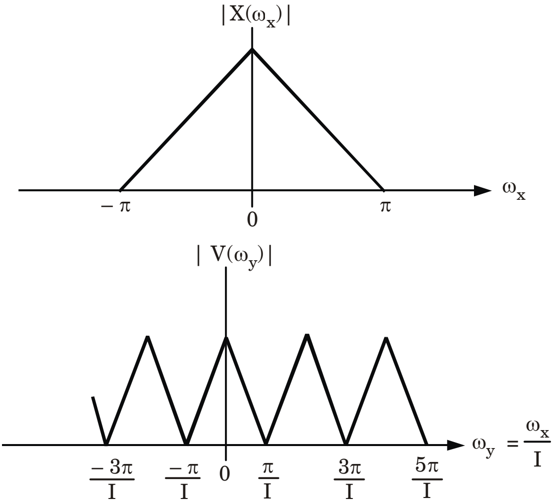

Fig. 8. Spectra of $x(n)$ and $v(n)$ where $V(\omega_y) = X(\omega_y I)$.

3. Let $v(m)$ denote a sequence with a rate $F_y = I F_x$, which is obtained from $x(n)$ by adding $I - 1$ zeros between successive values of $x(n)$.

Thus $$V(m) = \begin{cases} x(m/I), & m = 0, \pm I, \pm 2I, \ldots \\ 0, & \text{otherwise} \end{cases} \quad \ldots(1)$$

and its sampling rate is identical to the rate of $y(m)$. This sequence has a z-transform

$$V(m) = \sum_{m=-\infty}^{\infty} v(m)z^{-m} = \sum_{m=-\infty}^{\infty} x(m)z^{-mI} \quad ...(2)$$

$$= X(z^{I})$$

4. The corresponding spectrum of $v(m)$ is obtained by evaluating eq. (2) on the unit circle.

   Thus $\quad V(\omega_y) = X(\omega_y I) \quad ...(3)$

   where ω_y denotes the frequency variable relative to the new sampling rate F_y (*i.e.*, $\omega_y = 2\pi F / F_y$).
5. Now the relationship between sampling rates is $F_y = I\, F_x$ and hence, the frequency variables ω_x and ω_y are related according to the formula

$$\omega_y = \frac{\omega_x}{I} \quad ...(4)$$

6. The spectra $X(\omega_x)$ and $V(\omega_y)$ are illustrated in Fig. 1. We observed that the sampling rate increase, obtained by the addition of $I - 1$ zero samples between successive values of $x(n)$, results in a signal whose spectrum $V(\omega_y)$ is an I-fold periodic repetition of the input signal spectrum $X(\omega_x)$.

B. Sampling rate conversion :

1. The process of sampling rate conversion in the digital domain can be viewed as a linear filtering operation, as illustrated in Fig. 1(a).
2. The input signal $x(n)$ is characterized by the sampling rate $F_x = 1/T_x$ and the output signal $y(m)$ is characterized by the sampling rate $F_y = 1/T_y$, where T_x and T_y are the corresponding sampling intervals.
3. In the main part of our treatment, the ratio F_y / F_x is constrained to be rational,

$$\frac{F_y}{F_x} = \frac{I}{D}$$

   where D and I are relatively prime integers.
4. We shall show that the linear filter is characterized by a time-variant impulse response, denoted as $h(n, m)$. Hence the input $x(n)$ and the output $y(m)$ are related by the convolution summation for time-variant systems.
5. The sampling rate conversion process can also be understood from the point of view of digital resampling of the same analog signal. Let $x(t)$ be the analog signal that is sampled at the first rate F_x to generate $x(n)$.
6. The goal of rate conversion is to obtain another sequence $y(m)$ directly from $x(n)$, which is equal to the sampled values of $x(t)$ at a second rate F_y.

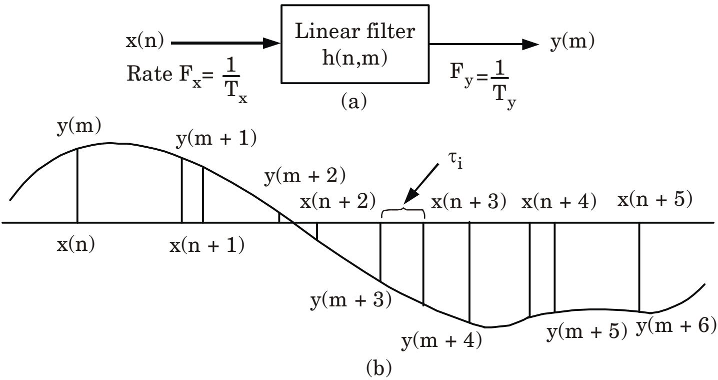

Fig. 9. Sampling rate conversion viewed as a linear filtering process.

7. In Fig. 9(b), $y(m)$ is a time-shifted version of $x(n)$. Such a time shift can be realized by using a linear filter that has a flat magnitude response and a linear phase response (*i.e.*, it has frequency response of $e^{-j\omega\tau}$, where τ_i is the time delay generate by the filter).
8. If the two sampling rates are not equal, the required amount of time shifting will vary from sample to sample, as shown in Fig. 9(b).
9. Thus, the rate converter can be implemented using a set of linear filters that have the same flat magnitude response but generate different time delays.
10. Before considering the general case of sampling rate conversion, we shall consider two special cases. One is the case of sampling rate reduction by an integer factor D and the second is the case of a sampling rate increase by an integer factor I.

☺☺☺

B. Tech.

(SEM. V) ODD SEMESTER THEORY EXAMINATION, 2019-20

DIGITAL SIGNAL PROCESSING

Time : 3 Hours **Max. Marks : 70**

Note : Attempt **all** Section. Assume any missing data.

SECTION-A

1. Attempt **all** questions in brief. **(2 × 7 = 14)**

a. Define linear convolution and its physical significance.

b. What are advantages and disadvantages of window methods ?

c. What are the advantages for representing the digital system in block diagram form ?

d. Write the expression for Blackman and Bartlett window.

e. If $x(n) = \{4, 3, 5, 7, 4, 6\}$ and upsampling factor = 3, then what will be the value of upsampler output.

f. If $x(n) = (1, 5, 2, 3)$ what will be $x((3 - n))_4$.

g. Write down the advantages and disadvantages of bilinear transformation.

SECTION-B

2. Attempt any **three** of the following : **(7 × 3 = 21)**

a. Obtain the parallel form realization for the system function given below :

$$H(z) = \frac{(1 + 0.25\, z^{-1})}{(1 + 0.5\, z^{-1})\,(1 + 0.5\, z^{-1} + 0.25\, z^{-2})}$$

b. What is the relation between DTFT and DFT. Explain the properties of DFT with examples.

c. Explain the Gibbs phenomenon. Find the response of rectangular window and explain it.

d. Find the 4-point circular convolution of $x(n)$ and $h(n)$ given by $x(n) = \{1, 1, 1, 1)$ and $h(n) = \{ 1, 0, 1, 0\}$ using radix-2 DIF-FFT algorithm.

e. The system function of analog filter is given by

$$H(s) = \frac{(s + 0.1)}{(s + 0.1)^2 + 16}$$

Obtain the system function of digital filter by using impulse invariant technique. Assume $T = 1$ sec.

SECTION-C

3. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Obtain the ladder structure of a given transfer function :

$$H(z) = \frac{2 + 8z^{-1} + 6z^{-2}}{1 + 8z^{-1} + 12z^{-2}}$$

b. Obtain a linear phase and cascade realization of the system $H(z) = (1 + 0.5\, z^{-1} + z^{-2})\ (1 + 0.5\, z^{-1} + z^{-2})$

4. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Design a Butterworth low pass analog filter for the following specification :

i. Pass band gain required : 0.9

ii. Frequency up to which pass band gain must remain more or less steady : 100 rad/sec.

iii. Gain in attenuation band : 0.4

iv. Frequency from which the attenuation must start : 200 rad/sec.

b. What is frequency warping effect ? How this problem is overcome in bilinear transform method of IIR filter design ? Also write down the advantages and disadvantages of bilinear transformation.

5. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. A FIR filter has following symmetry in impulse response ; $h(n) = h(M - 1 - n)$ for M odd. Derive its frequency response and show that it has linear phase.

b. Design a low pass discrete time filter with following specification :

$$0.99 \le |H(e^{j\omega})| \le 1.01 \qquad |\omega| \le 0.4\,\pi$$

$$|H(e^{j\omega})| \le 0.01 \qquad 0.6\,\pi \le |\omega| \le 0.4\,\pi$$

Use Kaiser window for design.

6. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Find the 8-pint DFT of $x(n) = 2^n$ by using DIT FFT algorithm.

b. Prove that multiplication of DFTs of two sequences is equivalent to the circular convolution of the two sequences in time domain.

7. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Write a short note on :

i. Subband coding of speech signal

ii. Quardrature mirror filter.

b. Explain the phenomenon decimation and interpolation by suitable example.

☺☺☺

SOLUTION OF PAPER (2019-20)

Note : Attempt **all** Section. Assume any missing data.

SECTION-A

1. Attempt **all** questions in brief. **(2 × 7 = 14)**

a. Define linear convolution and its physical significance.

Ans.

A. Linear convolution :

Let us consider two finite duration sequences $x(n)$ and $h(n)$, the linear convolution of $x(n)$ and $h(n)$ is given as

$$y(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

B. Physical significance : Linear convolution can be used to find the response of a filter.

b. What are advantages and disadvantages of window methods ?

Ans. Advantages of window design method :

i. Simple method for design.

ii. Various window functions can be used depending upon the application.

Disadvantages of window design method :

i. Lack of precise control of the critical frequencies ω_p and ω_s.

ii. ω_p and ω_s depend upon type of window and filter length M.

c. What are the advantages for representing the digital system in block diagram form ?

Ans.

i. Just by inspection, the computation algorithm can be easily written.

ii. The hardware requirements can be easily determined.

iii. A variety of equivalent block diagram representations can be easily developed from the transfer function.

iv. The relationship between the output and the input can be determined.

d. Write the expression for Blackman and Bartlett window.

Ans. Expression for Blackman window :

1. Causal Blackman window function

$$w_B(n) = \begin{cases} 0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1} & ; 0 \le n \le M-1 \\ 0 & ; \text{ otherwise} \end{cases}$$

2. Non-causal Blackman window function

$$w_B(n) = \begin{cases} 0.42 + 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1} & ; \text{for } |n| < \dfrac{M-1}{2} \\ 0 & ; \text{otherwise} \end{cases}$$

Expression for Bartlett window :

$$w_{\text{Bart}}(n) = \begin{cases} 1+n & ; \quad -\dfrac{M-1}{2} < n < 1 \\ 1-n & ; \quad 1 < n < \dfrac{M-1}{2} \end{cases}$$

e. If $x(n) = \{4, 3, 5, 7, 4, 6\}$ and upsampling factor = 3, then what will be the value of upsampler output.

Ans. Given, $x(n) = \{4, 3, 5, 7, 4, 6\}$

And, $M = 3$

Output $y(n) = x(n/3)$

$y(n) = \{4, 0, 0, 3, 0, 0, 5, 0, 0, 7, 0, 0, 4, 0, 0, 6\}$

f. If $x(n) = (1, 5, 2, 3)$ what will be $x((3-n))_4$.

Ans. Given, $x(n) = \{1, 5, 2, 3\}$

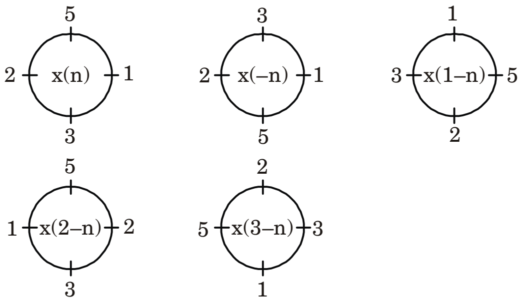

Fig. 1.

$x((3-n))_4 = \{3, 2, 5, 1\}$

g. Write down the advantages and disadvantages of bilinear transformation.

Ans. Advantages of bilinear transformation method :

i. The mapping is one to one.
ii. There is no aliasing effect.
iii. Stable analog filter is transformed into the stable digital filter.
iv. There is no restriction one type of filter that can be transformed.

Disadvantages of bilinear transformation method :

1. In this method, the mapping is non-linear because of this frequency warping effect takes place.

SECTION-B

2. Attempt any **three** of the following : **(7 × 3 = 21)**

a. Obtain the parallel form realization for the system function given below :

$$H(z) = \frac{(1 + 0.25\,z^{-1})}{(1 + 0.5\,z^{-1})\,(1 + 0.5\,z^{-1} + 0.25\,z^{-2})}$$

Ans.

1. Given, $$H(z) = \frac{(1 + 0.25\,z^{-1})}{(1 + 0.5\,z^{-1})\,(1 + 0.5\,z^{-1} + 0.25\,z^{-2})}$$

$$\frac{1 + 0.25z^{-1}}{(1 + 0.5z^{-1})(1 + 0.5z^{-1} + 0.25z^{-2})} = \frac{A}{(1 + 0.5z^{-1})} + \frac{Bz^{-1} + C}{(1 + 0.5z^{-1} + 0.25z^{-2})}$$

2. On solving, we get, $A = \frac{1}{2}$, $B = -\frac{1}{4}$ and $C = \frac{1}{2}$.

3. So, $$H(z) = \frac{\frac{1}{2}}{(1 + 0.5z^{-1})} + \frac{-\frac{1}{4}z^{-1} + \frac{1}{2}}{(1 + 0.5z^{-1} + 0.25z^{-2})}$$

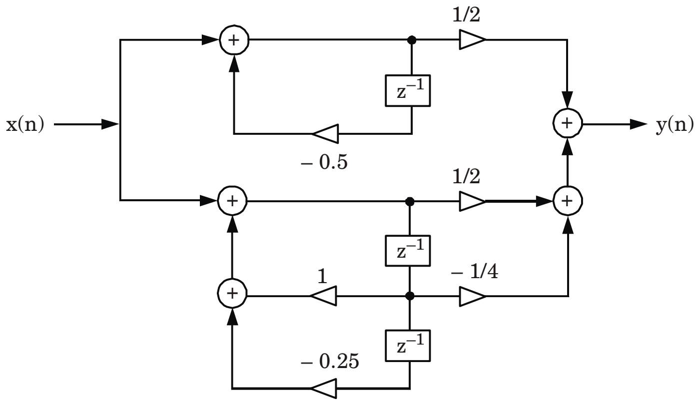

Fig. 2.

b. What is the relation between DTFT and DFT. Explain the properties of DFT with examples.

Ans.

A. Relation between DTFT and DFT :

Since DFT is defined for finite length causal sequences, let us consider a $x(n)$ which is causal and is of length N.

DTFT : $$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(nT)e^{-j\omega nT} \quad ...(1)$$

DFT : $$X(k) = \sum_{n=0}^{N-1} x(nT)e^{-j\frac{2k}{N}nk} \; ; k = 0,1,2, ...,(N-1) \quad ...(2)$$

From the eq. (1) and (2), it is clear that

$$X(k) = X(e^{j\omega T})\Big|_{\omega \to \frac{2\pi}{NT}.k = \left(\frac{\omega_1}{N}\right)k} \; ; k = 0, 1, 2, ..., (N-1)$$

B. Properties of DFT :

1. Periodicity : Let $x(n) \xleftrightarrow{\text{DFT}} X(k)$

Then if $x(n + N) = x(n)$ for all n

Then $X(k + N) = X(k)$ for all k

2. Linearity :

If $$x_1(n) \xleftrightarrow[N]{\text{DFT}} X_1(k)$$

and $$x_2(n) \xleftrightarrow[N]{\text{DFT}} X_2(k)$$

then this property gives

$$a_1x_1(n) + a_2x_2(n) \xleftrightarrow[\text{N}]{\text{DFT}} a_1X_1(k) + a_2X_2(k) \quad ...(3)$$

3. Circular symmetries of a sequence : If periodic sequence $x_p(n)$ is given as,

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) \quad ...(4)$$

The DFT is given as

$$x(n) \xleftrightarrow{\text{DFT}} X(k)$$

$$x_p(n) \xleftrightarrow{\text{DFT}} X_p(k)$$

Thus $x(n)$ and $x_p(n)$ are related by eq. (4.2.4). The relation between $x(n)$ and $x_p(n)$ is given as,

$$x(n) = \begin{cases} x_p(n), & \text{for } 0 \le n \le (N-1) \\ 0 \quad , & \text{otherwise} \end{cases}$$

Suppose $x_p(n)$ is shifted by 'k' units to the right. Then new sequence $x'_p(n)$ be given as,

$$x'_p(n) = x_p(n - k)$$

$$= \sum_{l=-\infty}^{\infty} x(n - k - lN)$$

The $x'_p(n)$ will give $x(n)$ and it is given as,

$$x'(n) = \begin{cases} x'_p(n), & \text{for } 0 \le n \le (N-1) \\ 0 \quad , & \text{otherwise} \end{cases}$$

or simply $$x'(n) = \sum_{l=-\infty}^{\infty} x(n - k - lN)$$

The $x'(n)$ is related to $x(n)$ by a circular shift and it is given as,

$$x'(n) = x(n - k, \text{ modulo } N)$$

The short hand notation for $(n - k$, modulo $N)$ is $((n - k))_N$

or $$x'(n) = x\,((n - k))_N$$

4. **Time reversal of a sequence :**

If $$x(n) \xleftrightarrow{\text{DFT}} X(k)$$

Then this property states that

$$x((-n))_N = x(N - n) \xleftrightarrow{\text{DFT}} X((-k))_N$$

i.e., $$x((-k))_N \xleftrightarrow{\text{DFT}} X(N - k)$$

5. **Complex conjugate property :**

If $$x(n) \xleftrightarrow{\text{DFT}} X(k)$$

Then this property states that

$$x^*(n) \leftrightarrow X^*(N - k) = X^*((-k))_N$$

6. **Circular convolution :**

If $$x_1(n) \xleftrightarrow{\text{DFT}} X_1(k)$$

and $$x_2(n) \xleftrightarrow{\text{DFT}} X_2(k)$$

This property states that the circular convolution is given as,

$x_1(-n)$ N $x_2(n) = X_1(k)X_2(k)$ where N denote circular convolution

7. **Shifting property :**

If $$x(n) \xleftrightarrow{\text{DFT}} X(k)$$

Two shifting properties are given here,

a. **Circular shifting in time :**

This property states that

$$x((n - l))_N \xleftrightarrow{\text{DFT}} X(k)\, e^{-j2\pi k\ell/N}$$

b. **Circular shifting in frequency :**

The property states that

$$x(n)\, e^{j2\pi\ell n/N} \xleftrightarrow{\text{DFT}} X((k - l))_N$$

8. **Circular correlation :**

If $$x_1(n) \xleftrightarrow{\text{DFT}} X_1(k)$$

and $$x_2(n) \xleftrightarrow{\text{DFT}} X_2(k)$$

Then this property states that

$$r_{x_1x_2}(l) \xleftrightarrow{\text{DFT}} \frac{1}{N} R_{x_1x_2}(k) = X_1(k)\, X_2^*(k)$$

where $r_{x_1x_2}(l)$ is the circular cross correlation and it is given as

$$r_{x_1x_2}(l) = \sum_{n=0}^{N-1} x_1(n) x_2^*((n - l))_N$$

9. **Multiplication of two sequences :**

If $$x_1(n) \xleftrightarrow{\text{DFT}} X_1(k)$$

and $$x_2(n) \xleftrightarrow{\text{DFT}} X_2(k)$$

then, $$x_1(n) \,.\, x_2(n) \xleftrightarrow{\text{DFT}} \frac{1}{N} X_1(k)\; N\; X_2(k)$$

10. **Parseval's theorem :** If $x_1(n)$ and $x_2(n)$ are two complex valued sequences and their DFT are given as

$$x_1(n) \xleftrightarrow{\text{DFT}} X_1(k)$$
$$x_2(n) \xleftrightarrow{\text{DFT}} X_2(k)$$

Then $$\sum_{n=0}^{N-1} x_1(n) x_2^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) \cdot X_2^*(k) \quad ...(5)$$

When $x_1(n) = x_2(n) = x(n)$

Then eq. (5) will become

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2 \quad ...(6)$$

Eq. (5) and (6) are Parseval's theorem. Eq. (6) gives energy of finite duration sequence in terms of its frequency components.

c. Explain the Gibbs phenomenon. Find the response of rectangular window and explain it.

Ans.

A. Gibbs phenomenon :

1. The impulse response of FIR filter in terms of rectangular window is given by, $h(n) = h_d(n)\ w_R(n)$ ...(1)
2. The frequency response of the filter is obtained by taking Fourier transform of eq. (1)

$\therefore$ $$H(\omega) = \text{FT}\{h_d(n)\ w_R(n)\} = H_d(\omega) * W_R(\omega)$$

This shows that the frequency response of FIR filter is equal to the convolution of desired frequency response, $H_d(\omega)$ and the Fourier transform of window function.

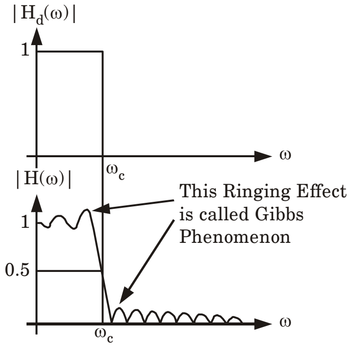

Fig. 3.

3. The sidelobes are present in the frequency response. Because of these sidelobes, the ringing is observed in the frequency response of FIR filter. This ringing is predominantly present near the band edge of the filter.

4. This oscillatory behaviour (*i.e.*, ringing effect) near the band edge of the filter is called Gibbs phenomenon.

B. Response of rectangular window :

1. Rectangular window is shown in Fig. 4. It is denoted by $w_R(n)$. Its magnitude is 1 for the range, $n = 0$ to $M - 1$. Let $h_d(n)$ be the impulse response having infinite duration.
2. If $h_d(n)$ is multiplied by $w_R(n)$ then a finite impulse response is obtained. Since the shape of the window function is rectangular, it is called as rectangular window.

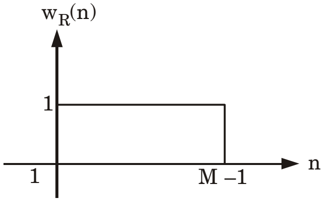

Fig. 4. Rectangular window.

3. The rectangular window is defined as

$$w_R(n) = \begin{cases} 1 & ; \text{for } n = 0, 1, 2, \ldots, M-1 \\ 0 & ; \text{ otherwise} \end{cases} \quad \ldots(1)$$

4. Let $h_d(n)$ be infinite duration impulse response. We know that the finite duration impulse response $h(n)$ is obtained by multiplying $h_d(n)$ by $w_R(n)$.

$\therefore$ $$h(n) = h_d(n) \cdot w_R(n) \quad \ldots(2)$$

5. By Fourier transform we can write,

$$W_R(\omega) = \sum_{n=0}^{M-1} w_R(n) e^{-j\omega n} \quad \ldots(3)$$

6. But the value of $w_R(n)$ is 1 for range $n = 0$ to $M - 1$.

$\therefore$ $$W_R(\omega) = \sum_{n=0}^{M-1} 1.e^{-j\omega n} \quad \ldots(4)$$

7. We can represent the window sequence as,

$$w_R(n) = u(n) - u(n - M)$$

Here $u(n)$ is unit step having duration $n = 0$ to $n \to (\infty)$ and $u(n - M)$ is delay unit step.

8. Thus eq. (3) becomes,

$$W_R(\omega) = \sum_{n=0}^{\infty} [u(n) - u(n-M)] e^{-j\omega n}$$

$$W_R(\omega) = \sum_{n=0}^{\infty} u(n) e^{-j\omega n} - \sum_{n=0}^{\infty} u(n-M) e^{-j\omega n} \quad \ldots(5)$$

9. Consider the first term at RHS. It represents Fourier transform of unit step.

$$\text{F.T. of } u(n) = \sum_{n=0}^{\infty} 1 \cdot e^{-j\omega n} = \sum_{n=0}^{\infty} (e^{-j\omega})^n = \frac{1}{1-e^{-j\omega}}$$

10. Now consider the second term. It represents the Fourier transform of delayed unit step.
Fourier transform of

$$u(n-M) \longleftrightarrow e^{-j\omega M} F[u(n)]$$

$$= e^{-j\omega M} \frac{1}{1-e^{-j\omega}} = \frac{e^{-j\omega M}}{1-e^{-j\omega}}$$

11. Thus eq. (5) becomes,

$$W_R(\omega) = \frac{1}{1-e^{-j\omega}} - \frac{e^{-j\omega M}}{1-e^{-j\omega}}$$

$$\therefore \quad W_R(\omega) = \frac{1-e^{-j\omega M}}{1-e^{-j\omega}} \quad \ldots(6)$$

12. Rearrange eq. (6) as follows,

$$W_R(\omega) = \frac{e^{\frac{-j\omega M}{2}} e^{\frac{+j\omega M}{2}} - e^{\frac{-j\omega M}{2}} e^{\frac{-j\omega M}{2}}}{e^{\frac{-j\omega}{2}} e^{\frac{+j\omega}{2}} - e^{\frac{-j\omega}{2}} e^{\frac{-j\omega}{2}}}$$

$$W_R(\omega) = \frac{e^{\frac{-j\omega M}{2}}\left(e^{\frac{+j\omega M}{2}} - e^{\frac{-j\omega M}{2}}\right)}{e^{\frac{-j\omega}{2}}\left(e^{\frac{+j\omega}{2}} - e^{\frac{-j\omega}{2}}\right)} \quad \ldots(7)$$

13. We know that, $\dfrac{e^{j\theta} - e^{-j\theta}}{j} = 2 \sin \theta$

$$W_R(\omega) = \frac{e^{\frac{-j\omega M}{2}}\left[2 \sin\left(\frac{\omega M}{2}\right)\right]}{e^{\frac{-j\omega}{2}}\left[2 \sin\left(\frac{\omega}{2}\right)\right]} = e^{\frac{-j\omega M}{2}} e^{\frac{j\omega}{2}} \frac{\sin\left(\frac{\omega M}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

$$\therefore \quad W_R(\omega) = \frac{\sin\left(\frac{\omega M}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\omega\left(\frac{M-1}{2}\right)} \quad \ldots(8)$$

14. Now $W_R(\omega)$ can be expressed in terms of magnitude and angle as,
$W_R(\omega) = |W_R(\omega)| \angle W_R(\omega)$ ...(9)

15. By comparing eq. (8) and eq. (9), we have

$$|W_R(\omega)| = \left|\frac{\sin\left(\frac{\omega M}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}\right|$$

d. Find the 4-point circular convolution of $x(n)$ and $h(n)$ given by $x(n) = \{1, 1, 1, 1)$ and $h(n) = \{1, 0, 1, 0\}$ using radix-2 DIF-FFT algorithm.

Ans.

1. Let $y(n) = x(n) \otimes h(n)$

$$= \text{IFFT}\{X(k)\, H(k)\}$$

2. Using DIF FFT algorithm, we can find $X(k)$ from the given sequence $x(n)$ as shown in Fig. 5.

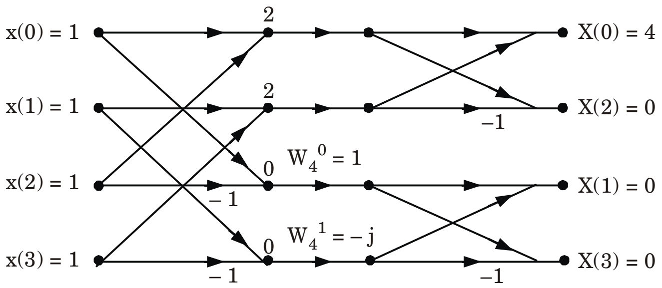

Fig. 5.

Therefore, $X(k) = (4, 0, 0, 0)$

3. Using DIF FFT algorithm we can find $H(k)$ from the given sequence $h(n)$ as shown in Fig. 6.

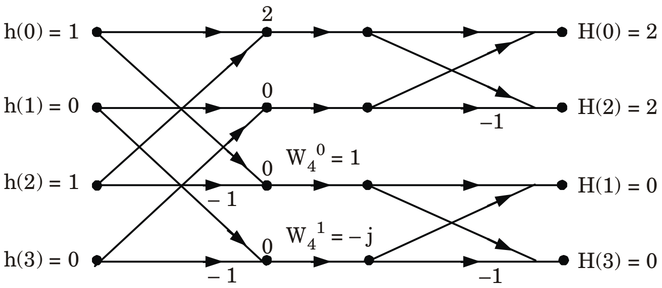

Fig. 6.

$$H(k) = (2, 0, 2, 0)$$

4. And $Y(k) = X(k)\, H(k)$

$$= (4, 0, 0, 0)\,(2, 0, 2, 0) = (8, 0, 0, 0)$$

5. Using IFFT algorithm, we can find $y(n)$ from the $Y(k)$ as shown in Fig. 7.

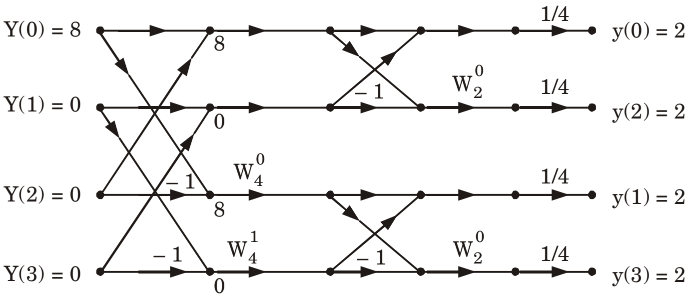

Fig. 7.

Thus, $y(n) = (2, 2, 2, 2)$

e. The system function of analog filter is given by

$$H(s) = \frac{(s + 0.1)}{(s + 0.1)^2 + 16}$$

Obtain the system function of digital filter by using impulse invariant technique. Assume $T = 1$ sec.

Ans.

1. Given, $$H(s) = \frac{(s + 0.1)}{(s + 0.1)^2 + 16} = \frac{(s + 0.1)}{(s + 0.1)^2 + 4^2} \quad ...(1)$$

2. We know that,

$$\frac{s + a}{(s + a)^2 + b^2} \rightarrow \frac{1 - e^{-aT}(\cos bT)z^{-1}}{1 - 2e^{-aT}(\cos bT)z^{-1} + e^{-2aT}z^{-2}} \quad ...(2)$$

3. Comparing eq. (1) and eq. (2) then we get,

$$a = 0.1, b = 4$$

4. $$H(z) = \frac{1 - e^{-0.1T}(\cos 4T)z^{-1}}{1 - 2e^{-0.1T}(\cos 4T)z^{-1} + e^{-2 \times 0.1T}z^{-2}}$$

5. At $T = 1$ sec,

$$H(z) = \frac{1 - e^{-0.1}(\cos 4)z^{-1}}{1 - 2e^{-0.1}(\cos 4)z^{-1} + e^{-0.2}z^{-2}}$$

$$H(z) = \frac{1 - 0.89z^{-1}}{1 - 1.78z^{-1} + 0.818z^{-2}}$$

SECTION-C

3. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Obtain the ladder structure of a given transfer function :

$$H(z) = \frac{2 + 8z^{-1} + 6z^{-2}}{1 + 8z^{-1} + 12z^{-2}}$$

Ans.

1. For the given system, obtain the Routh array

z^{-2}	6	8	2
z^{-2}	12	8	1
z^{-1}	4	3/2	
z^{-1}	7/2	1	
1	5/14	0	
1	1		

2. The ladder structure parameters are

$$\alpha_0 = \frac{6}{12} = \frac{1}{2}, \beta_1 = \frac{12}{4} = 3, \alpha_1 = \frac{4}{7/2} = \frac{8}{7}, \beta_2 = \frac{7/2}{5/14} = \frac{49}{5},$$

$$\alpha_2 = \frac{5/14}{1} = \frac{5}{14}$$

$$H(z) = \frac{1}{2} + \cfrac{1}{3z^{-1} + \cfrac{1}{\frac{8}{7} + \cfrac{1}{(49/5)z^{-1} + \cfrac{1}{5/14}}}}$$

3. The ladder structure is shown in Fig. 8.

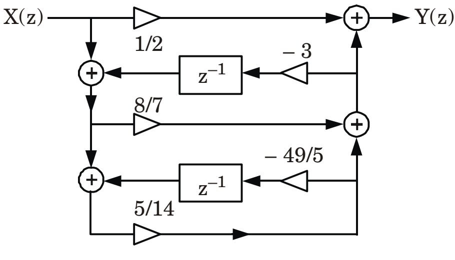

Fig. 8.

b. Obtain a linear phase and cascade realization of the system $H(z) = (1 + 0.5\, z^{-1} + z^{-2})\,(1 + 0.5\, z^{-1} + z^{-2})$

Ans.

1. Given, $H(z) = (1 + 0.5z^{-1} + z^{-2})\,(1 + 0.5z^{-1} + z^{-2})$
2. Assume $H(z) = H_1(z).H_2(z)$

   where $H_1(z) = (1 + 0.5z^{-1} + z^{-2})$

   $H_2(z) = (1 + 0.5z^{-1} + z^{-2})$
3. The cascade realization of the system is shown in Fig. 9.

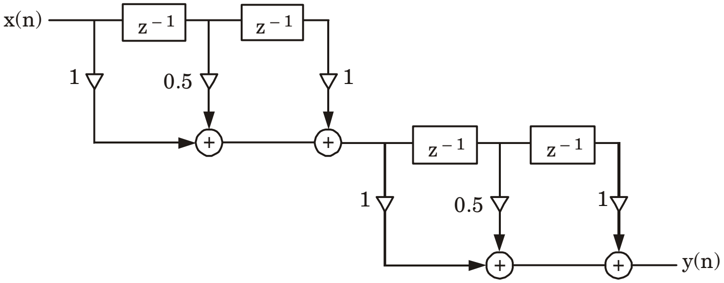

Fig. 9.

4. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Design a Butterworth low pass analog filter for the following specification :

i. Pass band gain required : 0.9

ii. Frequency up to which pass band gain must remain more or less steady : 100 rad/sec.

iii. Gain in attenuation band : 0.4

iv. Frequency from which the attenuation must start : 200 rad/sec.

Ans. The design steps are given below :

Step 1 : Determination of n and ω_c :

1. We have, $$H(\omega) = \left[\frac{1}{(1 + (\omega / \omega_c)^{2n}}\right]^{1/2} \quad ...(1)$$

2. Substituting the value of $H(\omega) = 0.9$ and $\omega = 100$ rad/s in eq. (1) then we get,

$$0.9 = \left[\frac{1}{1 + (100 / \omega_c)^{2n}}\right]^{1/2} \quad ...(2)$$

3. Similarly at $\omega = 200$ rad/s and $H(\omega) = 0.4$, we have

$$0.4 = \left[\frac{1}{1 + (200 / \omega_c)^{2n}}\right]^{1/2} \quad ...(3)$$

4. Squaring in both sides of eq. (2), we get

$$0.81 = \frac{1}{1 + (100 / \omega_c)^{2n}} \quad ...(4)$$

5. Inverting eq. (4), we get

$$1 + \left(\frac{100}{\omega_c}\right)^{2n} = + \frac{1}{0.81} \quad ...(5)$$

6. Eq. (5) may be simplified to get :

$$\left(\frac{100}{\omega_c}\right)^{2n} = 1.234 - 1 = 0.234 \quad ...(6)$$

7. Similarly eq. (3) may be simplified as :

$$\left(\frac{200}{\omega_c}\right)^{2n} = \frac{1}{0.16} - 1 = 5.25 \qquad ...(7)$$

8. Dividing eq. (7) with eq. (6), we get

$$\frac{5.25}{0.234} = \left(\frac{200}{100}\right)^{2n} = 2^{2n} \qquad ...(8)$$

9. Taking logarithm of both sides of eq. (8), we have

$$2n \log 2 = \log 22.436 \qquad ...(9)$$

10. From eq. (9), we find that

$$n = 2.244 \qquad ...(10)$$

11. However, it is quite customary to specify the order of a given filter in integers or whole numbers.

$$n = 3$$

12. Substituting the value of n in eq. (7)

$$\left(\frac{200}{\omega_c}\right)^{2 \times 3} = 5.25$$

$$\left(\frac{200}{\omega_c}\right) = (5.25)^{1/6}$$

$$\omega_c = \left(\frac{200}{1.318}\right) = 151.7$$

Step 2 : Determination of the poles of $H(s)$:

1. Since $n = 3$, which is odd and the first pole is at 0°.

$$\text{Angle between poles } \theta = \frac{360°}{2n} = \frac{360°}{6} = 60°$$

2. Hence, the second pole is at $\theta = 60°$. And we get the other poles at 120°, 180°, 240° and 300° respectively.

Step 3 : Determination of valid poles of $H(s)$:

1. The poles must be lie in between 90° and 270°. So the valid poles are 120°, 180° and 240°.
2. The location of the poles are,

Pole $B_1 \rightarrow (\cos 180° + j \sin 180°) \times \omega_c = -1 \times 151.7 = -151.7$

Pole $B_2 \rightarrow (\cos 120° + j \sin 120°) \times \omega_c = \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)(151.7)$

$$= -75.85 + j\,131.376$$

Pole $B_3 \rightarrow (\cos 240° + j \sin 240°) \times \omega_c = \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)(151.7)$

$$= -75.85 - j\,131.376$$

Step 4 : Finding the expression for $H(s)$:

1. We have,

$$H(s) = \frac{\omega_c^2}{(s + a + jb)(s + a - jb)} \frac{\omega_c}{(s + \omega_c)} \qquad ...(11)$$

2. Substituting the values of poles in eq. (11) and we get,

$$H(s) = \frac{(151.7)^2}{(s + 75.85 + j131.376)\,(s + 75.85 - j131.376)} \times \frac{(151.7)}{(s + 151.7)}$$

$$H(s) = \frac{(151.7)^3}{(s + 75.85 + j131.376)(s + 75.85 - j131.37)(s + 1)} \quad ...(12)$$

3. Eq. (12) can be simplified to :

$$H(s) = \frac{3.491 \times 10^6}{(s^2 + 151.7s + 0.02301 \times 10^6)\,(s + 151.7)} \quad ...(13)$$

b. What is frequency warping effect ? How this problem is overcome in bilinear transform method of IIR filter design ? Also write down the advantages and disadvantages of bilinear transformation.

Ans.

A. Frequency warping :

1. We know, $$\Omega = \frac{2}{T_s} \times \frac{2r \sin \omega}{r^2 + 2r \cos \omega + 1} \quad ...(1)$$

2. For the unit circle, $r = 1$. Thus putting $r = 1$ in the eq. (1) we get,

$$\Omega = \frac{2}{T_s} \times \frac{2 \sin \omega}{1 + 2 \cos \omega + 1}$$

$$\therefore \quad \Omega = \frac{2}{T_s} \times \frac{2 \sin \omega}{2 + 2 \cos \omega}$$

$$\Omega = \frac{2}{T}\left(\frac{\sin \omega}{1 + \cos \omega}\right)$$

$$= \frac{2}{T}\left[\frac{2 \sin (\omega/2) \cos (\omega/2)}{\cos^2(\omega/2) + \sin^2(\omega/2) + \cos^2(\omega/2) - \sin^2(\omega/2)}\right]$$

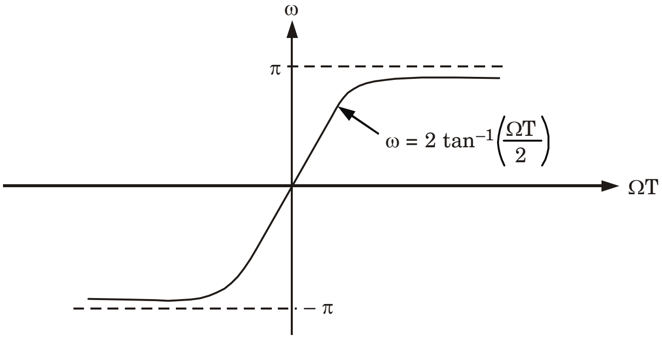

Fig. 10. Frequency warping.

or $$\Omega = \frac{2}{T} \tan\left(\frac{\omega}{2}\right)$$

or $$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$$

3. In this method the entire range in Ω is mapped only once into the range $-\pi \le \omega \le \pi$. This mapping is non-linear. The lower frequencies in analog domain are expanded in digital domain while the higher frequencies in analog domain are compressed in digital domain.
4. Non-linearity in mapping is due to arc tangent function and this is known as frequency warping.

B. Overcome frequency warping effect :

1. Bilinear transformation is a one to one mapping from the s-domain to the z-domain.
2. Bilinear transformation is a conformal mapping that transforms the $j\Omega$-axis into the unit circle in the z-plane only once. Thus the aliasing effect is avoided.
3. The transformation of a stable analog filter result in a stable digital filter as all the poles in the left half of the s-plane are mapped onto points inside the unit circle of the z-domain.

C. Advantages of bilinear transformation :

1. The mapping is one to one.
2. There is no aliasing effect.
3. Stable analog filter is transformed into the stable digital filter.
4. There is no restriction one type of filter that can be transformed.
5. There is one to one transformation from the s-domain to the Z- domain.

D. Disadvantages of bilinear transformation :

1. In this method, the mapping is non-linear because of this frequency warping effect takes place.

5. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. A FIR filter has following symmetry in impulse response ; $h(n) = h(M-1-n)$ for M odd. Derive its frequency response and show that it has linear phase.

Ans.

1. The discrete-time fourier transform (DTFT) of the impulse response $h(n)$ is given by,

$$H(e^{j\omega}) = \sum_{n=0}^{M-1} h(n)\, e^{-j\omega nT} = |H(e^{j\omega})|\, e^{j\phi(\omega)} \quad ...(1)$$

2. If the length M of filters is odd, then eq. (1) can be written as

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-3}{2}} h(n)\, e^{-j\omega nT} + h\left(\frac{M-1}{2}\right) e^{-j\omega\left(\frac{M-1}{2}\right)T} \quad ...(2)$$

3. We have $h(n) = h(M-1-n)$ for $0 < n < M-1$ ...(3)

Eq. (2) can be written as by using eq. (3).

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-3}{2}} h(n)\,[e^{-j\omega nT} + e^{-j\omega(M-1-n)T}]$$

$$+h\left(\frac{M-1}{2}\right) e^{-j\omega\left(\frac{M-1}{2}\right)T} \quad \ldots(4)$$

4. Factorising $e^{-j\omega(M-1)T/2}$ in eq. (4),

$$H(e^{j\omega T}) = e^{-j\omega\left(\frac{M-1}{2}\right)T} \left\{ \sum_{n=0}^{\frac{M-3}{2}} h(n) \left[e^{j\omega\left(\frac{M-1}{2}-n\right)T} + e^{-j\omega\left(\frac{M-1}{2}-n\right)T} \right] \right.$$

$$\left. +h\left(\frac{M-1}{2}\right) \right\} \quad \ldots(5)$$

5. Put $k = \left(\frac{M-1}{2}\right) - n$ in eq. (5)

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{M-1}{2}\right)T} \left\{ \sum_{k=1}^{\frac{M-1}{2}} h\left(\frac{M-1}{2} - k\right) [e^{j\omega kT} + e^{-j\omega kT}] \right.$$

$$\left. +h\left(\frac{M-1}{2}\right) \right\} \quad \ldots(6)$$

or $$H(e^{j\omega}) = e^{-j\omega\left(\frac{M-1}{2}\right)T} \left\{ \sum_{k=0}^{\frac{M-1}{2}} b(k) \cos \omega kT \right\}$$

where, $$b(k) = 2h\left(\frac{M-1}{2} - k\right); \text{ for } 1 \le k \le \left(\frac{M-1}{2}\right)$$

It gives $$b(0) = h\left(\frac{M-1}{2}\right)$$

6. The $H(e^{j\omega})$ can be written as

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{M-1}{2}\right)T} \{M(\omega)\} \quad \ldots(7)$$

Here magnitude response

$$M(\omega) = \sum_{k=0}^{(M-1)/2} b(k) \cos(\omega kT) \quad \ldots(8)$$

And phase response function

$$\phi(\omega) = -\omega(M-1)/2 \quad \ldots(9)$$

7. The eq. (9) represents the $(M-1)/2$ units delay in sampling time. Hence, FIR filter will have constant phase and group delays and thus the phase of the filter will be linear.

b. Design a low pass discrete time filter with following specification :

$$\mathbf{0.99 \le |H(e^{j\omega})| \le 1.01 \; ; \; 0 \le |\omega| \le 0.4\pi}$$
$$\mathbf{|H(e^{j\omega})| \le 0.01 \; ; \; 0.6\pi \le |\omega| \le 0.4\pi}$$

Use Kaiser window for design.

Ans.

Given : $0.99 \le |H(e^{j\omega})| \le 1.01 \; ; \; 0 \le |\omega| \le 0.4\pi$

$|H(e^{j\omega})| \le 0.01 \; ; \; 0.6\pi \le |\omega| \le 0.4\pi$

To Design : FIR linear phase filter.

Step 1 : On comparing with the given specification as given below

$$1 - \delta_p \le |H(e^{j\omega})| \le 1 + \delta_p \; ; \text{ for } 0 \le \omega \le \omega_p$$
$$|H(e^{j\omega})| \le \delta_s \; ; \text{ for } \omega_s \le \omega \le \pi$$

On comparison, we get

$$\delta_p = 0.01 \text{ and } \delta_s = 0.01$$
$$\omega_p = 0.4\pi \text{ and } \omega_s = 0.6\pi$$

Therefore the transition width '$\Delta\omega$' can be calculated as

$$\Delta\omega = \omega_s - \omega_p = 0.6\pi - 0.4\pi = 0.2\pi$$

The minimum value of ripple is given as

$$\delta = \min(\delta_p, \delta_s) = 0.01$$

The minimum stop band attenuation can be calculated as

$$A_s = -20 \log_{10} \delta = -20 \log_{10}(0.01) = 40$$

Step 2 : Calculation of cut-off frequency (ω_c)

It is calculated as

$$\omega_c = \frac{\omega_p + \omega_s}{2} = \frac{0.6\pi + 0.4\pi}{2} = 0.5\pi$$

Step 3 : Calculation of α and M.

Here $A_s = 40$, which lies in the range of 21 to 50.

Hence α can be obtained as

$$\alpha = 0.5842\,(A_s - 21)^{0.4} + 0.07886\,(A_s - 21)$$
$$= 0.5842\,(40 - 21)^{0.4} + 0.07886\,(40 - 21)$$
$$= 3.395$$

and M is given as

$$M = \frac{A_s - 8}{2.285(0.02\pi)} = \frac{40 - 8}{2.285(0.02\pi)} = 222.88$$

i.e., $M = 223$

Taking value of M to next integer value

Step 4 : Obtain equation of Kaiser window by taking

$$\beta = \frac{M}{2} = \frac{223}{2} = 111.5$$

Therefore window function will be

$$w_{\text{kaiser}}(n) = \begin{cases} \dfrac{I_o\left[\alpha\left\{1 - \left(\dfrac{n-\beta}{\beta}\right)^2\right\}^{1/2}\right]}{I_o(\alpha)} & ; \text{for } 0 \le n \le 223 \\ 0 & ; \text{otherwise} \end{cases}$$

$$w_{\text{kaiser}}(n) = \begin{cases} \dfrac{I_o\left[3.395\left\{1-\left(\dfrac{n-111.5}{111.5}\right)^2\right\}^{1/2}\right]}{I_o(3.395)} & ; \text{for } 0 \le n \le 223 \\ 0 & ; \text{otherwise} \end{cases}$$

Step 5 : To obtain $h_d(n)$

Since this is a LPF. The ideal desired frequency response is given by equation,

$$H_d(\omega) = \begin{cases} e^{-j\omega(M-1)/2} & ; \quad \text{for } -\omega_c \le \omega \le \omega_c \\ 0 & ; \quad \text{otherwise} \end{cases} \quad ...(1)$$

$h_d(n)$ can be calculated by just taking inverse fourier transform of eq. (1).

$$h_d(n) = \begin{cases} \dfrac{\sin\left[\omega_c\left(n-\dfrac{M-1}{2}\right)\right]}{\pi\left(n-\dfrac{M-1}{2}\right)} & ; \quad \text{for } n \ne \dfrac{M-1}{2} \\ \dfrac{\omega_c}{\pi} & ; \quad \text{for } n = \dfrac{M-1}{2} \end{cases} \quad ...(2)$$

Since in the beginning the equation is derived by taking filter order $(M + 1)$. The eq. (2) is derived for filter length M. Hence by replacing $M + 1$ in place of M in eq. (2) we get

$$h_d(n) = \begin{cases} \dfrac{\sin\left[\omega_c\left(n-\dfrac{M}{2}\right)\right]}{\pi\left(n-\dfrac{M}{2}\right)} & ; \text{for } n \ne \dfrac{M}{2} \\ \dfrac{\omega_c}{\pi} & ; \text{for } n = \dfrac{M}{2} \end{cases} \quad ...(3)$$

Putting here,

$M = 223$, hence $M/2 = 111.5$ *i.e.*, not an integer, so $h_d(n)$ cannot be calculated to this value. If filter length is taken as even *i.e.*, $M + 1 = 224$, the $h_d(n)$ will be given as

$$h_d(n) = \frac{\sin\left[\omega_c\left(n-\dfrac{M}{2}\right)\right]}{\pi\left(n-\dfrac{M}{2}\right)} ; \text{for } 0 \le n \le M \quad ...(4)$$

Putting for $\omega_c = 0.5\,\pi$ and $M = 223$, the $h_d(n)$ will become

$$h_d(n) = \frac{\sin\left[0.5\,\pi\left(n-\dfrac{223}{2}\right)\right]}{\pi\left(n-\dfrac{223}{2}\right)}$$

$$h_d(n) = \frac{\sin[0.5\pi(n-111.5)]}{\pi(n-111.5)} \text{ ; for } 0 \le n \le 223 \qquad \text{...(5)}$$

Eq. (5) also holds good for linear phase requirement and gives desired unit sample response of LPF.

Step 6 : To obtain $h(n)$

The unit sample response of FIR filter is obtained by

$$h(n) = h_d(n)\; w_{\text{kaiser}}(n)$$

$$= \begin{cases} \dfrac{\sin\left[0.5\,\pi\left(n-111.5\right)\right]}{\pi\left(n-111.5\right)} \dfrac{I_o\left[3.395\left\{1-\left(\dfrac{n-111.5}{111.5}\right)^2\right\}^{1/2}\right]}{I_o(3.395)} & ;\text{ for } 0 \le n \le 223 \\ 0 & ;\text{ otherwise} \end{cases}$$

6. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Find the 8-pint DFT of $x(n) = 2^n$ by using DIT FFT algorithm.

Ans.

Given : $x(n) = 2^n$ and $N = 8$

To Find : $X(k)$ and reduction factor.

1. $$x(0) = 1,\ x(1) = 2,\ x(2) = 4,\ x(3) = 8$$
$$x(4) = 16,\ x(5) = 32,\ x(6) = 64,\ x(7) = 128$$
$$x(n) = \{1, 2, 4, 8, 16, 32, 64, 128\}$$

2. We know, $W_8^0 = 1$
$$W_8^1 = 0.707 - j0.707$$
$$W_8^2 = -j$$
$$W_8^3 = -0.707 - j0.707$$

3. Using DIT FFT algorithm, we can find $X(k)$ from the given sequence $x(n)$ as shown in Fig. 10.
4. Computation reduction factor

$$= \frac{\text{Number of complex multiplication requried for direct (DFT)}}{\text{Number of complex multiplications required for FFT algorithm}}$$

$$= \frac{N^2}{\dfrac{N}{2}\log_2(N)} = \frac{8^2}{\dfrac{8}{2}\log_2 8} = \frac{64}{12} = 5.33$$

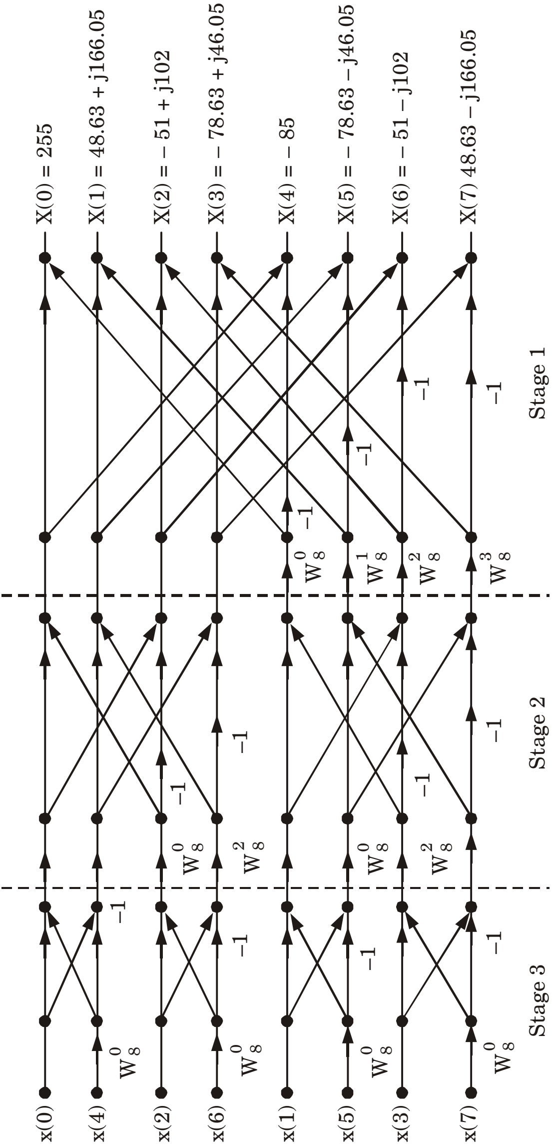
X(0) = 255
X(1) = 48.63 + j166.05
X(2) = – 51 + j102
X(3) = – 78.63 + j46.05
X(4) = – 85
X(5) = – 78.63 – j46.05
X(6) = – 51 – j102
X(7) 48.63 – j166.05
Stage 1
Stage 2
Stage 3
x(0)
x(4)
x(2)
x(6)
x(1)
x(5)
x(3)
x(7)

Fig. 11.

b. Prove that multiplication of DFTs of two sequences is equivalent to the circular convolution of the two sequences in time domain.

Ans.

1. Consider the two sequences $x(n)$ and $y(n)$ which are of finite duration. Let $X(k)$ and $Y(k)$ be the N-point DFTs of the two sequences respectively and they are given by

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}, \; k = 0, 1, ..., N-1.$$

$$Y(k) = \sum_{n=0}^{N-1} y(n)e^{-j2\pi nk/N}, \; k = 0, 1, ..., N-1.$$

2. Let $x_3(m)$ be another sequence of length N and its N-point DFT be $X_3(k)$ which is a product of $X(k)$ and $Y(k)$,

i.e., $\quad X_3(k) = X(k)\, Y(k), k = 0, 1,..., N-1.$

The sequence $x_3(m)$ can be obtained by taking the inverse DFT of $X_3(k)$,

i.e., $\quad x_3(m) = \text{IDFT}[X_3(k)]$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)e^{j2\pi mk/N} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y(k)e^{j2\pi mk/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}\right]\left[\sum_{l=0}^{N-1} y(l)e^{-j2\pi lk/N}\right]e^{j2\pi mk/N}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\sum_{l=0}^{N-1} y(l)\left[\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N}\right] \quad ...(1)$$

4. Consider the term within the brackets in eq. (1), it has the form

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & \text{for } a = 1 \\ \dfrac{1-a^N}{1-a}, & \text{for } a \neq 1 \end{cases}$$

where, $a = e^{j2\pi(m-n-l)/N}$.

5. When $(m - n - l)$ is a multiple of N, then $a = 1$, otherwise $a^N = 1$ for any value of $a \neq 0$.

Therefore,

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & l = m-n+pN, N = (m-n)(\text{mod } N), p : \text{integer}) \\ 0, & \text{otherwise} \end{cases}$$

6. Now, $x_3(m)$ becomes

$$x_3(m) = \sum_{n=0}^{N-1} x(n)y(m-n, (\text{mod } N)), N = 0, 1, ..., N-1$$

where, $y((m - n) \bmod N)$ is the reflected and circularly shifted version of $y(m)$ and n represents the number of indices that the sequence $x(n)$ is shifted to the right.

7. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Write a short note on :

i. Subband coding of speech signal

ii. Quardrature mirror filter.

Ans.

i. Subband coding :

1. Subband coding is a method, where the speech signal is subdivided into several frequency bands and each band is digitally encoded separately.

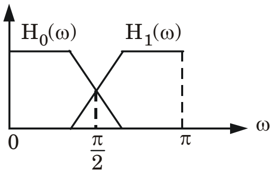

Fig. 12. Filter characteristics for subband coding.

2. Filter design is particularly important in achieving good performance in subband coding. Aliasing resulting from decimation of the subband signals must be negligible.
3. A practical solution to the aliasing problem is to use Quadrature Mirror Filters (QMF), which have the frequency response characteristics shown in Fig. 12.
4. The synthesis method for the subband encoded speech signal is basically the reverse of the encoding process.
5. The signals in adjacent low-pass and high-pass frequency bands are interpolated, filtered, and combined as shown in Fig. 13. A pair of (QMF) is used in the signal synthesis for each octave of the signal.
6. Subband coding is also an effective method to achieve data compression in image signal processing by combining subband coding with vector quantization for each subband signal.

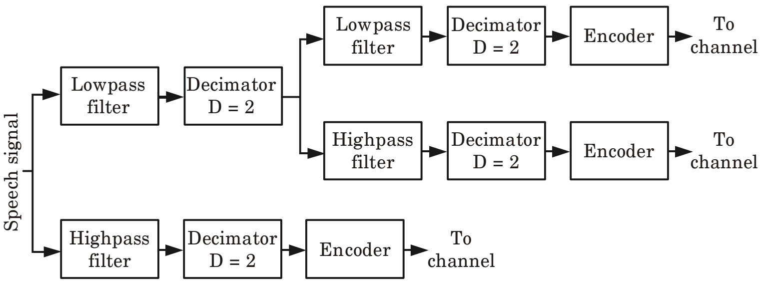

Fig. 13. Block diagram of subband speech coder.

7. In general, subband coding of signals is an effective method for achieving bandwidth compression in a digital representation of

the signal, when the signal energy is concentrated in a particular region of the frequency band.

ii. Quardrature mirror filter :

1. The subband signals are down sampled before processing. The signals are up sampled after processing.
2. The structure used for this is known as Quadrature Mirror Filter (QMF) bank.
3. If the decimation and interpolation factors are equal, then the characteristics of $x(n)$ will be available in $y(n)$ if the filters are properly selected.
4. If this properly is satisfied, then the filter bank can be called as a critically sampled filter bank.
5. To list a few applications where QMF filters are used.

   i. Efficient coding of the signal $x(n)$

   ii. Analog voice privacy for secure telephone communication.
6. The two-channel QMF filter bank is shown in Fig. 14. The analysis filter $H_0(z)$ is a low-pass filter and $H_1(z)$ is a high-pass filter.
7. The cut-off frequency is taken as $\pi/2$ for these filters.
8. The subband signals $\{v_k(n)\}$ are down sampled. After down sampling, these signals are processed (encoded).
9. In the receiving side the signals are decoded, up-sampled and then passed through the synthesis filters $G_0(z)$ and $G_1(z)$ to get the output $y(n)$.
10. The encoding and decoding processes are not shown in Fig. 14.
11. For perfect reconstruction, the QMF filter banks should be properly selected.

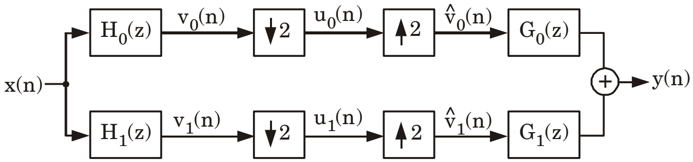

Fig. 14. Two-channel quadrature mirror filter bank.

b. Explain the phenomenon decimation and interpolation by suitable example.

Ans.

A. Decimation :

1. The process of reducing the sampling rate by a factor D (downsampling by D) is called decimation.
2. The decimation process is illustrated in Fig. 15. The input sequence $x(n)$ is passed through a low pass filter, characterized by the impulse response $h(n)$ and a frequency response $H_D(\omega)$, which ideally satisfies the condition

$$H_D(\omega) = \begin{cases} 1, & |\omega| \leq \pi / D \\ 0, & \text{otherwise} \end{cases} \quad \text{...(1)}$$

3. Thus the filter eliminates the spectrum of $X(\omega)$ in the range $\pi/D < \omega < \pi$. Of course, the implication is that only the frequency components of $x(n)$ in the range $|\omega| \le \pi/D$ are of interest in the further processing of the signal.
4. The output of the filter is a sequence $v(n)$ given as

$$v(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k) \qquad ...(2)$$

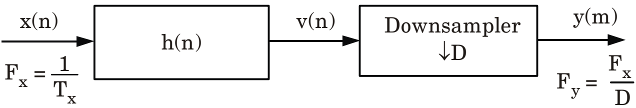

Fig. 15. Decimation by a factor D.

which is then downsampled by the factor D to produce $y(m)$. Thus

$$y(m) = v(mD)$$

$$= \sum_{k=0}^{\infty} h(k)\, x(mD-k) \qquad ...(3)$$

5. Although the filtering operation on $x(n)$ is linear and time invariant, the downsampling operation in combination with the filtering result in a time-variant system.
For example : $x(n)$ is a input signal as shown in Fig. 16(a) and the decimation factor 3 then the output signal (decimated signal) $y(n)$ as shown in Fig. 16(b).

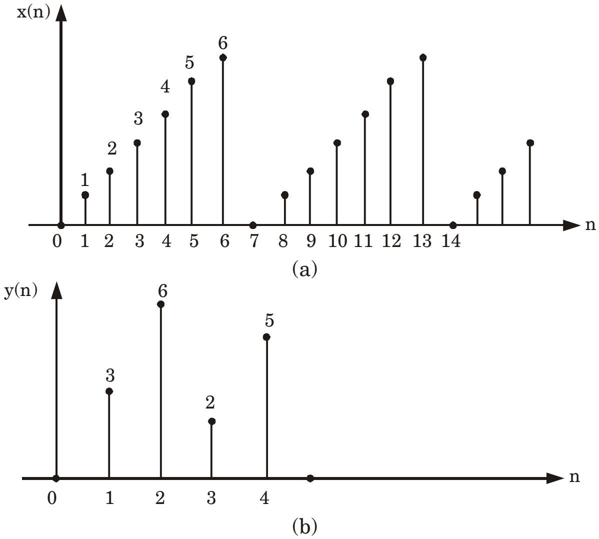

Fig. 16. (a) Input signal (b) Output signal (decimated signal).

B. Interpolation :

1. The process of increasing the sampling rate by an integer factor I (upsampling by I) is called interpolation.
2. An increase in the sampling rate by an integer factor of I can be accomplished by interpolating $I - 1$ new samples between successive values of the signal.

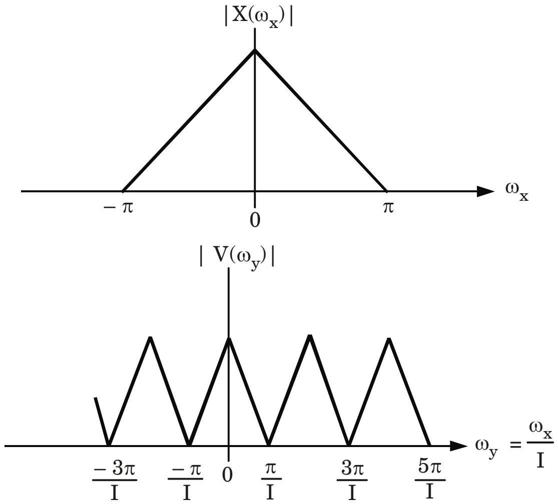

Fig. 17. Spectra of $x(n)$ and $v(n)$ where $V(\omega_y) = X(\omega_y I)$.

3. Let $v(m)$ denote a sequence with a rate $F_y = I F_x$, which is obtained from $x(n)$ by adding $I - 1$ zeros between successive values of $x(n)$.

Thus $$V(m) = \begin{cases} x(m/I), & m = 0, \pm I, \pm 2I, \ldots \\ 0, & \text{otherwise} \end{cases} \quad ...(1)$$

and its sampling rate is identical to the rate of $y(m)$. This sequence has a z-transform

$$V(m) = \sum_{m=-\infty}^{\infty} v(m) z^{-m} = \sum_{m=-\infty}^{\infty} x(m) z^{-mI} \quad ...(2)$$
$$= X(z^{I})$$

4. The corresponding spectrum of $v(m)$ is obtained by evaluating eq. (2) on the unit circle.

Thus $$V(\omega_y) = X(\omega_y I) \quad ...(3)$$
where ω_y denotes the frequency variable relative to the new sampling rate F_y (*i.e.*, $\omega_y = 2\pi F / F_y$).

5. Now the relationship between sampling rates is $F_y = I F_x$ and hence, the frequency variables ω_x and ω_y are related according to the formula

$$\omega_y = \frac{\omega_x}{I} \qquad \ldots(4)$$

6. The spectra $X(\omega_x)$ and $V(\omega_y)$ are illustrated in Fig. 17. We observed that the sampling rate increase, obtained by the addition of $I - 1$ zero samples between successive values of $x(n)$, results in a signal whose spectrum $V(\omega_y)$ is an I-fold periodic repetition of the input signal spectrum $X(\omega_x)$.

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