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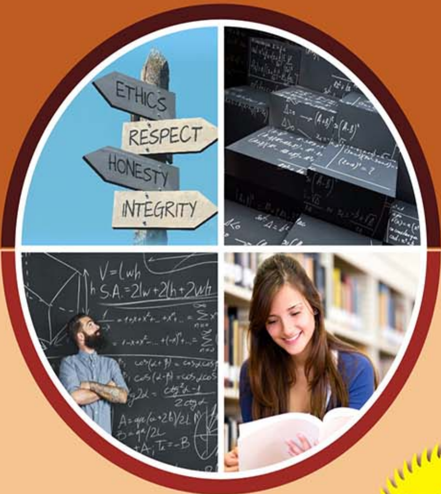

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MATHEMATICS - III

By
Kanika Dhama

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Mathematics –III
(Integral Transform & Discrete Maths)

(To be offered to CE and Allied Branches CE/EV)

Subject Code	KAS303/KAS403					
Category	Basic Science Course					
Subject Name	MATHEMATICS-III (Integral Transform & Discrete Maths)					
Scheme and Credits	**L-T-P**	**Theory Marks**	**Sessional**		**Total**	**Credit**
			Test	**Assig/Att.**		
	3—1—0	100	30	20	150	4
Pre- requisites (if any)	Knowledge of Mathematics I and II of B. Tech or equivalent					

Course Outcomes

The objective of this course is to familiarize the students with Laplace Transform, Fourier Transform, their application, logic group, sets, lattices, Boolean algebra and Karnaugh maps. It aims to present the students with standard concepts and tools at B.Tech first year to superior level that will provide them well towards undertaking a variety of problems in the concern discipline.

The students will learn:

- The idea of Laplace transform of functions and their application
- The idea of Fourier transform of functions and their applications
- The basic ideas of logic and Group and uses.
- The idea s of sets, relation, function and counting techniques.
- The idea of lattices, Boolean algebra, Tables and Karnaugh maps.

Laplace Transform **(8)**

Laplace transform, Existence theorem, Laplace transforms of derivatives and integrals, Initial and final value theorems, Unit step function, Dirac- delta function, Laplace transform of periodic function, Inverse Laplace transform, Convolution theorem, Application to solve simple linear and simultaneous differential equations.

MODULE II

Integral Transforms **(9)**

Fourier integral, Fourier Transform , Complex Fourier transform, Inverse Transforms, Convolution Theorems, Fourier sine and cosine transform, Applications of Fourier transform to simple one dimensional heat transfer equations, wave equations and Laplace equations, Z-Transform and its application to solve difference equations.

Module- III **(8)**

Formal Logic ,Group, Ring and Field: Introduction to First order logic, Proposition, Algebra of Proposition, Logical connectives, Tautologies, contradictions and contingency, Logical implication, Argument, Normal form, Rules of inferences, semi group, Monoid Group, Group, Cosets, Lagrange's theorem , Congruence relation , Cyclic and permutation groups, Properties of groups, Rings and Fields (definition, examples and standard results only)

Module- IV **(10)**

Set, Relation, function and Counting Techniques - Introduction of Sets, Relation and Function, Methods of Proof, Mathematical Induction, Strong Mathematical Induction, Discrete numeric function and Generating functions, recurrence relations and their solution , Pigeonhole principle.

Module- V **(10)**

Lattices and Boolean Algebra: Introduction, Partially ordered sets, Hasse Diagram, Maximal and Minimal element, Upper and Lower bounds, Isomorphic ordered sets, Lattices, Bounded Lattices and , Distributive Lattices.

Duality, Boolean Algebras as Lattices, Minimization of Boolean Expressions, prime Implicants, Logic Gates and Circuits, Truth Table, Boolean Functions, Karnaugh Maps.

Text Books

1. E. Kreyszig: Advanced Engineering Mathematics; John Wiley & Sons.
2. R.K. Jain & S.R.K. Iyenger: Adnanced Engineering Mathematics, Narosa Publishing House.
3. C.L.Liu: Elements of Discrete Mathematics; Tata McGraw- Hill Publishing Company Limited, New Delhi.
4. S. Lipschutz, M.L. Lipson and Varsha H. Patil: Discrete Mathematics; Tata McGraw-Hill Publishing Company Limited, New Delhi
5. B. Kolman , Robert C. Busby & S. C. Ross: Discrete Mathematical Structures' 5th Edition, Perason Education (Singapore), Delhi, India.

Reference Books

1. B.S. Grewal: Higher Engineering Mathematics; Khanna Publishers, New Delhi.
2. B.V. Ramana: Higher Engineering Mathematics; Tata McGraw- Hill Publishing Company Limited, New Delhi.
3. Peter V.O' Neil. Advanced Engineering Mathematics, Thomas (Cengage) Learning.
4. Kenneth H. Rosem: Discrete Mathematics its Application, with Combinatorics and Graph Theory; Tata McGraw- Hill Publishing Company Limited, New Delhi
5. K.D. Joshi: Foundation of Discrete Mathematics; New Age International (P) Limited, Publisher, New Delhi.

COURSE OUTCOMES

	Course Outcome (CO)	Bloom's Knowledge Level (KL)
	At the end of this course, the students will be able to:	
CO 1	Remember the concept of Laplace transform and apply in solving real life problems.	K_1 & K_3
CO 2	Understand the concept of Fourier and Z – transform to evaluate engineering problems	K_2 & K_4
CO 3	Remember the concept of Formal Logic ,Group and Rings to evaluate real life problems	K_1 & K_5
CO 4	Apply the concept of Set, Relation, function and Counting Techniques	K_3
CO 5	Apply the concept of Lattices and Boolean Algebra to create Logic Gates and Circuits, Truth Table, Boolean Functions, Karnaugh Maps	K_3 & K_6

K_1 – Remember, K_2 – Understand, K_3 – Apply, K_4 – Analyze, K_5 – Evaluate, K_6 – Create

Evaluation methodology to be followed:

The evaluation and assessment plan consists of the following components:

a. Class attendance and participation in class discussions etc.
b. Quiz.
c. Tutorials and assignments.
d. Sessional examination.
e. Final examination.

Award of Internal/External Marks:

Assessment procedure will be as follows:

1. These will be comprehensive examinations held on-campus (Sessionals).
2. Quiz.
   a. Quiz will be of type multiple choice, fill-in-the-blanks or match the columns.
   b. Quiz will be held periodically.
3. Tutorials and assignments
   a. The assignments/home-work may be of multiple choice type or comprehensive type at least one assignment from each Module/Unit.
   b. The grades and detailed solutions of assignments (of both types) will be accessible online after the submission deadline.
4. Final examinations.
   These will be comprehensive external examinations held on-campus or off campus (External examination) on dates fixed by the Dr. APJ Abdul Kalam Technical University, Lucknow.

1
UNIT

Laplace Transform

CONTENTS

PART-1

Laplace Transform, Existence Theorem.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.1. **Find the Laplace transforms of**

i. $\sin 2t \sin 3t$

ii. $\cos^2 2t$

iii. $\sin^3 2t$

Answer

i. Since $\sin 2t \sin 3t = \dfrac{1}{2}[\cos t - \cos 5t]$

$$\therefore \quad L(\sin 2\text{t} \sin 3\text{t}) = \frac{1}{2}[L(\cos t) - L(\cos 5t)]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+1^2} - \frac{s}{s^2+5^2}\right] = \frac{12s}{(s^2+1)(s^2+25)}$$

ii. Since $\cos^2 2t = \dfrac{1}{2}(1 + \cos 4t)$

$$\therefore \quad L(\cos^2 2t) = \frac{1}{2}[L(1) + L(\cos 4t)] = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+16}\right)$$

iii. Since $\sin 6t = 3 \sin 2t - 4 \sin^3 2t$

or $\sin^3 2t = \dfrac{3}{4}\sin 2t - \dfrac{1}{4}\sin 6t$

$$\therefore \quad L(\sin^3 2t) = \frac{3}{4}L(\sin 2t) - \frac{1}{4}L(\sin 6t)$$

$$= \frac{3}{4}\frac{2}{s^2+2^2} - \frac{1}{4}\frac{6}{s^2+6^2} = \frac{48}{(s^2+4)(s^2+36)}$$

Que 1.2. **Find the Laplace transform of**

i. $e^{-3t}(2\cos 5t - 3\sin 5t)$

ii. $e^{2t}\cos^2 t$

iii. $\sqrt{t}e^{3t}$

Answer

i. $L\{e^{-3t}(2\cos 5t - 3\sin 5t)\} = 2L(e^{-3t}\cos 5t) - 3L(e^{-3t}\sin 5t)$

$$= 2\frac{s+3}{(s+3)^2+5^2} - 3\frac{5}{(s+3)^2+5^2} = \frac{2s-9}{s^2+6s+34}$$

ii. Since $L(\cos^2 t) = \frac{1}{2}L(1+\cos 2t) = \frac{1}{2}\left\{\frac{1}{s}+\frac{s}{s^2+4}\right\}$

$\therefore$ By shifting property, we get

$$L(e^{2t}\cos^2 t) = \frac{1}{2}\left\{\frac{1}{s-2}+\frac{s-2}{(s-2)^2+4}\right\}$$

iii. Since $L\left(\sqrt{t}\right) = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{(1/2)\,\Gamma\pi}{s^{3/2}}$

$\therefore$ By shifting property, we obtain $L\left(e^{3t}\sqrt{t}\right) = \frac{\sqrt{\pi}}{2}\frac{1}{(s-3)^{3/2}}$

Que 1.3. **Find the Laplace transform of $f(t)$ defined as**

i. $f(t) = t/\tau$, **when** $0 < t < \tau$
$= 1$, **when** $t > \tau$.

ii. $f(t) = \begin{cases} 1, & 0 < t \le 1 \\ t, & 1 < t \le 2 \\ 0, & t > 2 \end{cases}$

Answer

i. $$Lf(t) = \int_0^{\tau} e^{-st}\frac{t}{\tau}dt + \int_{\tau}^{\infty} e^{-st}\,1\,dt = \frac{1}{\tau}\left[\left|t\frac{e^{-st}}{-s}\right|_0^{\tau} - \int_0^{\tau} 1\frac{e^{-st}}{-s}dt\right] + \left|\frac{e^{-st}}{-s}\right|_{\tau}^{\infty}$$

$$= \frac{1}{\tau}\left[\frac{\tau e^{-s\tau}-0}{-s} - \left|\frac{e^{-st}}{s^2}\right|_0^{\tau}\right] + \frac{0-e^{-s\tau}}{-s} = \frac{-e^{-s\tau}}{s} - \frac{e^{-s\tau}-1}{\tau s^2} + \frac{e^{-s\tau}}{s} = \frac{1-e^{-s\tau}}{\tau s^2}$$

ii. $$L\,|f(t)\,\} = \int_0^1 e^{-st}\,1\,dt + \int_1^2 e^{-st}\,t\,dt + \int_2^{\infty} e^{-st}\,(0)\,dt$$

$$= \left|\frac{e^{-st}}{-s}\right|_0^1 + \left|t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right|_1^2 = \frac{1-e^{-s}}{s} + \left\{\left(-\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2}\right) - \left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right)\right\}$$

$$= \frac{1}{s} - \frac{2e^{-2s}}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}$$

Que 1.4. **Write the linearity and condition for existence theorem of Laplace transformation.**

Answer

A. **Linearity of Laplace Transformation :** Let $f(t)$ and $g(t)$ be any two functions whose Laplace transforms exist. Then, for any two constants α and β, we have

$$L[\alpha f(t) + \beta g(t)] = \alpha L[f(t)] + \beta L[g(t)]$$

Proof : Using the definition, we have

$$L[\alpha f(t) + \beta g(t)] = \int_0^{\infty} e^{-st}[\alpha f(t) + \beta g(t)]dt$$

$$= \alpha \int_0^{\infty} e^{-st} f(t)dt + \beta \int_0^{\infty} e^{-st} g(t)dt$$

$$= \alpha L[f(t)] + \beta L[g(t)]$$

B. **Sufficient Conditions for Existence of Laplace Transform :** If $f(t)$ is a piecewise continuous function on the interval $[0, \infty)$ and is of exponential order α for $t \geq 0$, then $L[f(t)]$ exists for $s > \alpha$.

Proof : If $f(t)$ is piecewise continuous on $[0, T]$ for $T > 0$, then $e^{-st} f(t)$ is also piecewise continuous on $[0, T]$. Hence, $\int_0^{T} e^{-st} f(t)dt$ exists. Therefore, the existence of the Laplace transform of $f(t)$ depends on whether this integral converges or has a finite limit as $T \to \infty$. To prove the convergence, we use the following results from the theory of improper integrals.

If for all t $(t \geq t_0)$ the inequality $0 \leq f(t) \leq g(t)$ is satisfied and if $\int_{t_0}^{\infty} g(t)dt$ converges, then $\int_{t_0}^{\infty} f(t)dt$ also converges. If $\int_{t_0}^{\infty} |f(t)|\, dt$ converges, then the integral $\int_{t_0}^{\infty} f(t)dt$ also converges. In this case, the later integral is called an absolutely convergent integral. To use these results, we can determine a function $g(t)$ such that $\int_0^{\infty} g(t)dt$ converges and

$$|e^{-st} f(t)| \leq g(t),\ t \geq 0 \quad ...(1.4.1)$$

Now, choose some numbers a and M such that

$$|f(t)| \leq Me^{at}$$

Then, from equation (1.4.1) we obtain

$$|e^{-st} f(t)| \leq Me^{(a-s)t} = g(t)$$

Now, $\int_0^{\infty} g(t)dt = \int_0^{\infty} Me^{(a-s)t}\, dt$

converges if $s > a$. Therefore, if we choose $g(t) = Me^{(a-s)t}$, $s > a$, the integral $\int_0^\infty e^{-st} f(t)dt$ is absolutely convergent and hence convergent.

Hence, $L[f(t)]$ exists.

Alternatively, we have

$$|L(f)| = \left|\int_0^\infty e^{-st} f(t)dt\right| \le \int_0^\infty e^{-st} |f(t)| dt \le M\int_0^\infty e^{-st} e^{at} dt$$

$$= M\int_0^\infty e^{-(s-a)t} dt = \frac{M}{s-a}, s > a$$

proving the existence of the Laplace transform.

It is important to note that the above condition gives only the sufficient conditions for the existence of the Laplace transform. That is, a function may have Laplace transform even if it violates the existence conditions.

Que 1.5. **Find the Laplace transform of $t\, e^{-t} \sin 2t$.**

Answer

$$L\{t\} = \frac{1}{s^2}, s > 0$$

Using first translation or first shifting property *i.e.*, If $L\{F(t)\} = f(s)$ then $L\{e^{at} f(t)\} = f(s-a)$ we get

$$L\{te^{2it}\} = \frac{1}{(s-2i)^2} = \frac{(s+2i)^2}{(s-2i)^2(s+2i)^2}$$

$$= \frac{(s^2-4)+4is}{(s^2-4i^2)^2} = \frac{(s^2-4)+4is}{(s^2+4)^2}$$

Equating imaginary part on both sides, we get

$$L\{t \sin 2t\} = \frac{4s}{(s^2+4)^2}$$

Using first shifting property *i.e.*, $L\{F(t)\} = f(s)$ then $L\{e^{at} F(t)\} = f(s-a)$ we get,

$$L\{e^{-t} t \sin 2t\} = \frac{4(s+1)}{\{(s+1)^2+4\}^2} = \frac{4s+4}{(s^2+2s+5)^2}$$

Que 1.6. **Find the Laplace transform of the following :**

i. $\frac{1-\cos 2t}{t^2}$

ii. $\int_0^\infty t^2 e^{-t} \sin t\, dt$

Answer

i. $$L\{1-\cos 2t\} = L\{1\} - L\{\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds = \left[\ln s - \frac{1}{2}\ln(s^2+4)\right]_s^\infty$$

$$= \left[\ln s - \ln(s^2+4)^{1/2}\right]_s^\infty = \left[\ln \frac{s}{\sqrt{s^2+4}}\right]_s^\infty$$

$$= \left[\ln \frac{1}{\sqrt{1+\frac{4}{s^2}}}\right]_s^\infty = -\ln \frac{1}{\sqrt{1+\frac{4}{s^2}}}$$

$$= -\ln \frac{s}{\sqrt{s^2+4}}$$

$$= \ln \frac{\sqrt{s^2+4}}{s} = \ln \sqrt{1+\frac{4}{s^2}} = \frac{1}{2}\ln\left(1+\frac{4}{s^2}\right)$$

$$L\left\{\frac{1-\cos 2t}{t^2}\right\} = \int_s^\infty \frac{1}{2}\ln\left(1+\frac{4}{s^2}\right) ds$$

$$= \frac{1}{2}\left[\int_s^\infty \left\{\ln(s^2+4) - \ln s^2\right\} ds\right]$$

$$= \frac{1}{2}\left\{\int_s^\infty \left[\ln(s^2+4) - 2\ln s\right] ds\right\}$$

$$= \frac{1}{2}\left[\left\{\ln(s^2+4) - 2\ln s\right\} s - \int \left(\frac{2s}{s^2+4} - \frac{2}{s}\right) s\, ds\right]_s^\infty$$

$$= \left[\frac{s}{2}\ln\frac{s^2+4}{s^2}\right]_s^\infty - \frac{1}{2}\int_s^\infty \left(\frac{2s^2}{s^2+4} - 2\right) ds$$

$$= \left[\frac{s}{2}\ln\frac{s^2+4}{s^2}\right]_s^\infty - \frac{1}{2}\int_s^\infty \frac{2s^2 - 2s^2 - 8}{s^2+4}\, ds$$

$$= \left[\frac{s}{2}\ln\left(\frac{s^2+4}{s^2}\right)\right]_s^\infty + \frac{1}{2}\int_s^\infty \frac{8}{s^2+4}\, ds$$

$$= -\frac{s}{2}\ln\left(1+\frac{4}{s^2}\right) + \frac{4}{2}\left[\tan^{-1}\frac{s}{2}\right]_s^\infty$$

$$= -\frac{s}{2}\ln\left(1+\frac{4}{s^2}\right) + 2\left(\frac{\pi}{2} - \tan^{-1}\frac{s}{2}\right)$$

$$L\left\{\frac{1-\cos 2^{t}}{t^2}\right\} = 2\cot^{-1}\frac{s}{2}-\frac{s}{2}\ln\left(1+\frac{4}{s^2}\right)$$

ii. $$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\{t^2 \sin t\} = (-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right) = \frac{d}{ds}\left\{\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\}$$

$$= \frac{d}{ds}\left\{\frac{-2s}{(s^2+1)^2}\right\} = -\left\{\frac{(s^2+1)^2 2-2s\,2(s^2+1)2s}{(s^2+1)^4}\right\}$$

$$= -\left\{\frac{(s^2+1)\,(2s^2+2-8s^2)}{(s^2+1)^4}\right\} = \frac{6s^2-2}{(s^2+1)^3}$$

By definition of Laplace transform,

$$\int_0^\infty e^{-st}t^2\sin\ t = \frac{6s^2-2}{(s^2+1)^3}$$

Putting $s = 1$

$$\int_0^\infty e^{-t}t^2\sin\ t = \frac{6-2}{(1+1)^3} = \frac{4}{8} = \frac{1}{2}$$

Que 1.7. **Evaluate $\int_0^\infty \frac{e^{-3t}\ \sin t}{t}dt$**

Answer

We know that

$$L(\sin t) = \frac{1}{s^2+1}$$

Using division by t property,

$$L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{s^2+1}ds = [\tan^{-1} s]_s^\infty = \frac{\pi}{2}-\tan^{-1} s$$

$$L\left(\frac{\sin t}{t}\right) = \cot^{-1} s$$

$$\int_0^\infty e^{-st}\frac{\sin t}{t}dt = \cot^{-1} s$$

Putting $s = 3$

$$\int_0^\infty e^{-3t}\frac{\sin t}{t}dt = \cot^{-1} 3$$

Que 1.8. **Use Laplace transform to evaluate :**

$$\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}dt.$$

Answer

Since $$L\{e^{-at} - e^{-bt}\} = \frac{1}{s+a} - \frac{1}{s+b}$$

$$L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\} = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

$$= [\ln (s+a) - \ln (s-b)]_s^\infty$$

$$= \left[\ln \frac{(s+a)}{(s+b)}\right]_s^\infty = \left[\ln \frac{\left(1+\frac{a}{s}\right)}{\left(1+\frac{b}{s}\right)}\right]_s^\infty$$

$$= \ln 1 - \ln \left[\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right] = 0 - \ln \frac{s+a}{s+b} = \ln \frac{s+b}{s+a}$$

$$\int_0^\infty e^{-st}\left(\frac{e^{-at} - e^{-bt}}{t}\right) dt = \ln\left(\frac{s+b}{s+a}\right)$$

Put $s = 0$, $$\int_0^\infty \left(\frac{e^{-at} - e^{-bt}}{t}\right) dt = \ln \frac{b}{a}$$

PART-2

Laplace Transform of Derivatives and Integrals.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.9. **Find the Laplace transform of**

i. $\boldsymbol{(1 - e^t)/t}$ **ii.** $\boldsymbol{\dfrac{\cos at - \cos bt}{t}}$

Answer

i. Since $$L(1 - e^t) = L(1) - L(e^t) = \frac{1}{s} - \frac{1}{s-1}$$

$$L\left(\frac{1-e^t}{t}\right) = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds = [\log s - \log(s-1)]_0^\infty$$

$$= \left[\log\left(\frac{s}{s-1}\right)\right]_0^{\infty} = -\log\left[\frac{1}{1-1/s}\right] = \log\left(\frac{s-1}{s}\right)$$

ii. Since $L(\cos at - \cos bt) = \dfrac{s}{s^2+a^2} - \dfrac{s}{s^2+b^2}$

$$\therefore\ L\left(\frac{\cos at - \cos bt}{t}\right) = \int_s^{\infty}\left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^{\infty}$$

$$= \frac{1}{2}\underset{s\to\infty}{\text{Lt}}\ \log\frac{s^2+a^2}{s^2+b^2} - \frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}$$

$$= \frac{1}{2}\log\left(\frac{1+0}{1+0}\right) - \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) = \log\left(\frac{s^2+b^2}{s^2+a^2}\right)^{1/2}$$

Que 1.10. **Evaluate**

i. $L\left\{e^{-t}\int_0^t \frac{\sin t}{t}\, dt\right\}$ **ii.** $L\left\{t\int_0^t \frac{e^{-t}\sin t}{t}\, dt\right\}$

iii. $L\left\{\int_0^t\int_0^t\int_0^t (t\sin t)\, dt\, dt\, dt\right\}$

Answer

i. We know that $L(\sin t) = \dfrac{1}{s^2+1}$

$$L\left(\frac{\sin t}{t}\right) = \int_0^{\infty}\frac{1}{s^2+1}ds = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

$$\therefore\quad L\left\{\int_t^0 \frac{\sin t}{t}dt\right\} = \frac{1}{s}\cot^{-1}s$$

Thus by shifting property, $L\left\{e^{-t}\left(\int_t^0 \frac{\sin t}{t}dt\right)\right\} = \dfrac{1}{s+1}\cot^{-1}(s+1)$

ii. Since $L\left(\dfrac{\sin t}{t}\right) = \cot^{-1}s$

$$\therefore\quad L\left(e^{-t}\frac{\sin t}{t}\right) = \cot^{-1}(s+1)$$

and $L\left(\int_0^t e^{-t}\dfrac{\sin t}{t}dt\right) = \dfrac{1}{s}\cot^{-1}(s+1)$

Hence $L\left(t\int_0^t e^{-t}\,\dfrac{\sin t}{t}\,dt\right) = -\dfrac{d}{ds}\left\{\dfrac{\cot^{-1}(s+1)}{s}\right\}$

$$= -\frac{s\left[\dfrac{-1}{1+(s+1)^2}\right]-\cot^{-1}(s+1)}{s^2} = \frac{s+(s^2+2s+2)\cot^{-1}(s+1)}{s^2(s^2+2s+2)}$$

iii. Since $L(\sin t) = \dfrac{1}{s^2+1}$

$\therefore$ $L(t\sin t) = -\dfrac{d}{ds}\dfrac{1}{(s^2+1)} = \dfrac{2s}{(s^2+1)^2}$

Thus $L\left\{\int_0^t\int_0^t\int_0^t (t\sin t)\,dt\,dt\,dt\right\} = \dfrac{1}{s^3}L(t\sin t)$

$$= \frac{1}{s^3}\frac{2s}{(s^2+1)^2} = \frac{2}{s^2(s^2+1)^2}$$

Que 1.11. **Evaluate**

i. $\displaystyle\int_0^\infty te^{-2t}\sin t\,dt$ **ii.** $\displaystyle\int_0^\infty \frac{\sin mt}{dt}\,dt$

iii. $\displaystyle\int_0^\infty e^{-t}\left(\frac{\cos at-\cos bt}{t}\right)dt$ **iv.** $\displaystyle L\left\{\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt\right\}$.

Answer

i. $\displaystyle\int_0^\infty te^{-2t}\sin t\,dt = \int_0^\infty e^{-st}(t\sin t)\,dt$ where $s = 2$

$= L(t\sin t)$, by definition

$$= (-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2} = \frac{2\times 2}{(2^2+1)^2} = \frac{4}{25}$$

ii. Since $L(\sin mt) = m/(s^2+m^2) = f(s)$, say.

$\therefore$ $L\left(\dfrac{\sin mt}{t}\right) = \displaystyle\int_s^\infty f(s)\,ds = \int_0^\infty \frac{m\,ds}{s^2+m^2} = \left[\tan^{-1}\frac{s}{m}\right]_s^\infty$

$$\int_s^\infty e^{-st}\frac{\sin mt}{t}\,dt = \frac{\pi}{2}-\tan^{-1}\frac{s}{m}$$

Now $\underset{s\to 0}{\text{Lt}}\ \tan^{-1}(s/m) = 0$ if $m > 0$ or π if $m < 0$

Thus taking limits as $s \to 0$, we get

$$\int_0^\infty \frac{\sin mt}{t}\,dt = \frac{\pi}{2} \text{ if } m > 0 \quad\text{or}\quad -\pi/2 \text{ if } m < 0$$

iii. We know that

$$L(\cos at) = \frac{s}{s^2 + a^2} \text{ and } L(\cos bt) = \frac{s}{s^2 + b^2}$$

$$\therefore \quad L\,\frac{\cos at - \cos bt}{t} = \int_0^{\infty}\left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\left\{\log\left(\frac{s^2+a^2}{s^2+b^2}\right)_s^{\infty}\right\} = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

This implies that $\displaystyle\int_0^{\infty} e^{-st}\left(\frac{\cos at - \cos bt}{t}\right) dt = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$

Taking $s = 1$, we get $\displaystyle\int_0^{\infty}\left(e^{-t}\,\frac{\cos at - \cos bt}{t}\right) dt = \frac{1}{2}\log\left(\frac{1+b^2}{1+a^2}\right)$

iv. Since $\displaystyle L\left(\frac{\sin t}{t}\right) = \int_0^{\infty}\frac{ds}{s^2+1} = \tan^{-1} s = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s.$

$$\therefore \quad L\left\{e^t\left(\frac{\sin t}{t}\right)\right\} = \cot^{-1}(s-1), \text{ by shifting property.}$$

Thus $\displaystyle L\left[\int_0^t\left\{e^t\left(\frac{\sin t}{t}\right)\right\} dt\right] = \frac{1}{s}\cot^{-1}(s-1).$

PART-3

Initial and Final Value Theorems, Unit Step Function, Dirac-delta Function.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.12. **If $U(z) = \dfrac{2z^2 + 5z + 14}{(z-1)^4}$, evaluate u_2 and u_3.**

Answer

Writing $$U(z) = \frac{1}{z^2}\,\frac{2 + 5z^{-1} + 14z^{-2}}{(1 - z^{-1})^4}$$

By initial value theorem, $u_0 = \displaystyle\operatorname*{Lt}_{z \to \infty} U(z) = 0$

Similarly, $u_1 = \underset{z \to \infty}{\text{Lt}} \{z[U(z) - u_0]\} = 0$

Now $u_2 = \underset{z \to \infty}{\text{Lt}} \{z^2[U(z) - u_0 - u_1 z^{-1}]\} = 2 - 0 - 0 = 2$

and
$$u^3 = \underset{z \to \infty}{\text{Lt}}\ z^3[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}]$$
$$= \underset{z \to \infty}{\text{Lt}}\ z^3[U(z) - 0 - 0 - 2z^{-2}]$$
$$= \underset{z \to \infty}{\text{Lt}}\ z^3\left[\frac{2z^2 + 5z + 14}{(z-1)^4} - \frac{2}{z^2}\right]$$
$$= \underset{z \to \infty}{\text{Lt}}\ z^3\left[\frac{13z^2 + 2z^2 + 8z - 2}{z^2(z-1)^4}\right] = 13$$

Que 1.13. Using unit step function, find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ \sin 2t, & \pi \le t < 2\pi \\ \sin 3t, & t \ge 2\pi \end{cases}$$

Answer

$f(t) = \sin t[u(t-0) - u(t-\pi) + \sin 2t[u(t-\pi)] - u(t-2\pi)] + \sin 3t\ u(t-2\pi)$

$= \sin t + (\sin 2t - \sin t)\ u(t-\pi) + (\sin 3t - \sin 2t)\ u(t-2\pi)$

Since $L[f(t-a)\ u(t-a) = e^{-as}\ \bar{f}\ (s)$ and $L(\sin at) = \dfrac{a}{s^2 + a^2}$,

$$L[f(t)] = L(\sin t) + L[(\sin 2t - \sin t)\ .\ u(t-\pi)] + L[(\sin 3t - \sin 2t)\ .\ u(t-2\pi)]$$

$$= \frac{1}{s^2+1} + e^{-\pi s}\left(\frac{2}{s^2+4} - \frac{1}{s^2+1}\right) + e^{-2\pi s}\left(\frac{3}{s^2+9} - \frac{2}{s^2+4}\right)$$

Que 1.14.

i. Express the function (Fig. 1.14.1) in terms of unit steps function and find the Laplace transform.
ii. Obtain the Laplace transform of $e^{-t}[1 - u(t-2)]$.

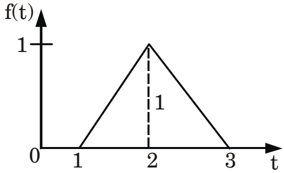

Fig. 1.14.1.

Answer

i. We have $f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \end{cases}$

$f(t) = (t-1)\{u(t-1) - u(t-2)\} + (3-t)\{u(t-2) - u(t-3)\}$
$= (t-1)\,u(t-1) - 2(t-2)\,u(t-2) + (t-3)\,u(t-3)$

Since $L\{(t-a)\,u(t-a)\} = e^{-as}\bar{f}(s)$

$$\therefore \quad L[f(t)] = e^{-s}\frac{1}{s^2} - 2e^{-2s}\frac{1}{s^2} + e^{-3s}\frac{1}{s^2} = \frac{e^{-s}(1-e^{-s})^2}{s^2}$$

$[\because \; f(t) = t]$

ii. $L\{e^{-t}[1 - u(t-2)]\} = L(e^{-t}) - L\{e^{-t}\,u(t-2)\}$

$$= \frac{1}{s+1} - e^{-2}\,L\{e^{-(t-2)}\,u(t-2)\}$$

Taking $f(t) = e^{-t}, \bar{f}(s) = \dfrac{1}{s+1}$ and using (λ) above,

$$L(e^{-(t-2)}u(t-2)) = e^{-2s}\frac{1}{s+1}$$

Hence $Le^{-t}[1 - u(t-2)] = \{1 - e^{-2(s+1)}\}/(s+1)$.

PART-4

Laplace Transform of Periodic Function.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.15. **Find the Laplace transform of the function**

$$f(t) = E \sin \omega t, \; 0 < t < \pi/\omega$$
$$= 0, \; \pi/\omega < t < 2\pi/\omega$$

Answer

Since $f(t)$ is a periodic function with period $2\pi/\omega$.

$$\therefore \quad L[f(t)] = \frac{1}{1-e^{-2\pi s/\omega}}\int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\int_0^{\pi/\omega} e^{-st} E\sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}\,0\,dt\right]$$

$$= \frac{E}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-st}(-s\sin\omega t - \omega\cos\omega t)}{s^2+\omega^2}\right]_0^{\pi/\omega}$$

$$= \frac{E(\omega e^{-\pi s/\omega} + \omega)}{(1 - e^{-2\pi s/\omega})(s^2 + \omega^2)} = \frac{E\omega}{(1 - e^{-\pi s/\omega})(s^2 + \omega^2)}$$

Que 1.16. **Draw the graph of the periodic function**

$$f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi \end{cases}$$

and find its Laplace transform.

Answer

Here the period of $f(t) = 2\pi$ and its graph is an in Fig. 1.16.1.

$$\therefore \quad L\, f(t) = \frac{1}{1 - e^{-2\pi s}} \left\{ \int_0^{\pi} e^{-st}\, t\, dt + \int_{\pi}^{2\pi} e^{-st} (\pi - t) dt \right\}$$

$$= \frac{1}{1 - e^{-2\pi s}} \left\{ \left| t\left(\frac{e^{-st}}{-s}\right) - 1\left(\frac{e^{-st}}{s^2}\right) \right|_0^{\pi} + \left| (\pi - t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right) \right|_{\pi}^{2\pi} \right\}$$

$$= \frac{1}{1 - e^{-2\pi s}} \left\{ \frac{-\pi e^{-\pi s}}{s} - \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} + \frac{\pi e^{-2\pi s}}{s} + \frac{e^{-2\pi s}}{s^2} - \frac{e^{-\pi s}}{s^2} \right\}$$

$$= \frac{1}{1 - e^{-2\pi s}} \left\{ \frac{\pi}{s}(e^{-2\pi s} - e^{-\pi s}) + \frac{1}{s^2}(1 + e^{-2\pi s} - 2e^{-\pi s}) \right\}$$

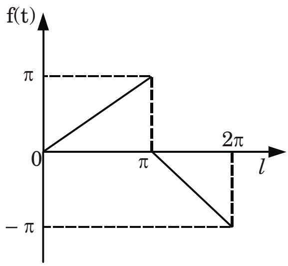

Fig. 1.16.1.

PART-5

Inverse Laplace Transform.

Questions-Answers
Long Answer Type and Medium Answer Type Questions

Que 1.17. **Find the inverse Laplace transforms of the following :**

i. $\dfrac{s^2}{(s-2)^3}$ **ii.** $\dfrac{s+2}{(s^2-4s+13)}$ **iii.** $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$

Answer

i. Since $s^2 = (s-2)^2 + 4(s-2) + 4$

$$\therefore \quad \frac{s^2}{(s-2)^3} = \frac{1}{s-2} + \frac{4}{(s-2)^2} + \frac{4}{(s-2)^3}$$

$$\therefore \quad L^{-1}\left\{\frac{s^2}{(s-2)^3}\right\} = L^{-1}\left\{\frac{1}{s-2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^3}\right\}$$

$$= e^{2t} + 4e^{2t}\,t + 2e^{2t}\,t^2 \qquad \text{[using shifting property]}$$

ii.

$$\frac{s+2}{s^2-4s+13} = \frac{s-2}{(s-2)^2+3^2} + \frac{4}{(s-2)^2+3^2}$$

$$\therefore \quad L^{-1}\left\{\frac{s+2}{s^2-4s+13}\right\} = L^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\} + \frac{4}{3}L^{-1}\left\{\frac{3}{(s-2)^2+3^2}\right\}$$

$$= e^{2t}\cos 3t + \frac{4}{3}e^{2t}\sin 3t$$

[using shifting property]

iii.

$$L^{-1}\frac{(s+2)^2}{(s^2+4s+8)^2} = L^{-1}\frac{(s+2)^2}{(s^2+4s+4+4)^2} = L^{-1}\frac{(s+2)^2}{[(s+2)^2+4]^2}$$

$$= e^{-2t}\,L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} = e^{-2t}\,L^{-1}\left\{\frac{s^2+4-4}{(s^2+4)^2}\right\}$$

$$= e^{-2t}\,L^{-1}\left\{\frac{1}{s^2+4} - \frac{4}{(s^2+4)^2}\right\}$$

$$= \frac{e^{-2t}\sin 2t}{2} - 4e^{-2t}\,L^{-1}\left\{\frac{1}{(s^2+4)^2}\right\}$$

$$= \frac{e^{-2t}\sin 2t}{2} - 4e^{-2t}\left\{\frac{1}{4}\left(\frac{\sin 2t}{4} - \frac{t\cos 2t}{2}\right)\right\}$$

$$= e^{-2t}\left\{\frac{\sin 2t}{2} - \frac{\sin 2t}{4} + \frac{t\cos 2t}{2}\right\}$$

$$= e^{-2t}\left\{\left(\frac{\sin 2t}{4} + \frac{t\cos 2t}{2}\right)\right\}$$

Que 1.18. **Find the inverse transform of**

i. $1/s\,(s^2 + a^2)$

ii. $1/s\,(s + a)^3$.

Answer

i. Since $L^{-1}\left(\dfrac{1}{s^2 + a^2}\right) = \dfrac{1}{a}\sin at$.

$$L^{-1}\left\{\frac{1}{s(s^2 + a^2)}\right\} = \int_0^t \frac{1}{a}\sin at\,dt = \frac{1}{a^2}[-\cos at]_0^t = (1 - \cos at)/a^2$$

ii. $$L^{-1}\left\{\frac{1}{s(s + a)^3}\right\} = L^{-1}\left\{\frac{1}{[(s + a) - a](s + a)^3}\right\} = e^{-at}\,L^{-1}\left\{\frac{1}{(s - a)s^3}\right\}$$

Now $$L^{-1}\left\{\frac{1}{s - a}\right\} = e^{at}$$

$$\therefore \quad L^{-1}\left\{\frac{1}{(s - a)s}\right\} = \int_0^t e^{at}\,dt = \frac{e^{at}}{a} - \frac{1}{a}$$

$$\therefore \quad L^{-1}\left\{\frac{1}{(s - a)s^2}\right\} = \frac{1}{a}\int_0^t (e^{at} - 1)dt = \frac{1}{a^2}(e^{at} - at - 1)$$

$$L^{-1}\left\{\frac{1}{(s - a)s^3}\right\} = \frac{1}{a^2}\int_0^t (e^{at} - at - 1)dt = \frac{1}{a^3}\left(e^{at} - \frac{a^2}{2}t^2 - at - 1\right)$$

Hence $L^{-1}\left\{\dfrac{1}{s(s + a)^3}\right\}$

$$= e^{-at}\,\frac{1}{a^3}\left(e^{at} - \frac{a^2t^2}{2} - at - 1\right) = \frac{1}{a^3}\left(1 - e^{-at} - ate^{-at} - \frac{a^2}{2}t^2e^{-at}\right)$$

PART-6

Convolution Theorem.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.19. **Find the function $f(t)$ whose Laplace transform is** $\log\left(1 + \dfrac{1}{s^2}\right)$.

Answer

We know that $$L\{t\,f(t)\} = (-1)\,\frac{d}{ds}\,F(s)$$

$$f(t) = -\frac{1}{t}\,L^{-1}\left[\frac{d}{ds}\,F(s)\right] = -\frac{1}{t}\,L^{-1}\left[\frac{d}{ds}\log\left(1+\frac{1}{s^2}\right)\right]$$

$$= -\frac{1}{t}\,L^{-1}\left[\frac{d}{ds}\{\log(s^2+1) - 2\log s\}\right]$$

$$= -\frac{1}{t}\,L^{-1}\left[\frac{2s}{s^2+1} - \frac{2}{s}\right]$$

$$= -\frac{1}{t}\left[L^{-1}\left(\frac{2s}{s^2+1}\right) - L^{-1}\left(\frac{2}{s}\right)\right] = -\frac{2}{t}\,(\cos t - 1)$$

$$f(t) = \frac{2\,(1-\cos t)}{t}$$

Que 1.20. **Using Convolution theorem, evaluate**

$$\boldsymbol{L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\}}$$

Answer

Let $$f(s) = \frac{s}{s^2+1},\, g(s) = \frac{1}{s^2+4}$$

$$L^{-1}\{f(s)\} = \cos t$$

$$L^{-1}\{g(t)\} = \frac{1}{2}\sin 2t$$

$$L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\} = \frac{1}{2}L^{-1}\left\{\left(\frac{s}{s^2+1}\right)\left(\frac{2}{s^2+4}\right)\right\}$$

Applying Convolution theorem,

$$\frac{1}{2}L^{-1}\left\{\left(\frac{s}{s^2+1}\right)\left(\frac{2}{s^2+4}\right)\right\} = \frac{1}{2}\int_0^t \sin 2x\,\cos(t-x)\,dx$$

$$= \frac{1}{2}\int_0^t 2\sin x\cos x(\cos t\cos x + \sin t\sin x)dx$$

$$= \int_0^t \cos t\,\sin x\,\cos^2 x\,dx + \int_0^t \sin t\,\cos x\,\sin^2 x\,dx$$

$$= \left[\cos t\left(\frac{-\cos^3 x}{3}\right) + \sin t\left(\frac{\sin^3 x}{3}\right)\right]_0^t$$

$$= \frac{-\cos^4 t}{3} + \frac{\sin^4 t}{3} + \frac{\cos t}{3}$$

$$= \frac{\cos t}{3} + \frac{1}{3}(\sin^4 t - \cos^4 t)$$

$$\Rightarrow \quad \frac{\cos t}{3} + \frac{1}{3}(\sin^2 t - \cos^2 t) = \frac{\cos t}{3} - \frac{\cos 2t}{3}$$

Que 1.21. **Apply Convolution theorem to evaluate**

i. $L^{-1}\dfrac{s}{(s^2+a^2)^2}$ **ii.** $L^{-1}\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

Answer

i. Since $f(t) = L^{-1}\left(\dfrac{s}{s^2+a^2}\right)$ = cos at and $g(t) = L^{-1}\left(\dfrac{s}{s^2+a^2}\right) = \dfrac{1}{a}$ sin at

$\therefore$ By Convolution theorem, we get

$$L^{-1}\left[\frac{s}{s^2+a^2}\,\frac{1}{s^2+a^2}\right] = \int_0^t \cos a\,u \frac{\sin a(t-u)}{a}\,du$$

$$\left[\begin{array}{l} \because f(u) = \cos au \\ g(t-u) = \dfrac{1}{a}\sin a(t-u) \end{array}\right.$$

$$= \frac{1}{2a}\int_0^t [\sin at - \sin(2au - at)]dt = \frac{1}{2a}\left|u\sin at + \frac{1}{2a}\cos(2au - at)\right|_0^t$$

$$= \frac{1}{2a}t\sin at$$

Hence $L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\} = \dfrac{1}{2a}t\sin at$

ii. Since $f(t) = L^{-1}\left(\dfrac{s}{s^2+a^2}\right)$ = cos at and $g(t) = L^{-1}\left(\dfrac{s}{s^2+b^2}\right)$ = cos bt,

$\therefore$ By Convolution theorem, we get

$$L^{-1}\left(\frac{s}{s^2+a^2}\,\frac{s}{s^2+b^2}\right) = \int_0^t \cos au \cos b(t-u)\, du$$

$[\because \quad f(u) = \cos au,\ g(t-u) = \cos b(t-u)]$

$$= \frac{1}{2}\int_0^t \{\cos[(a-b)u + bt] + \cos[(a+b)u - bt]\}\, du$$

$$= \frac{1}{2}\left|\frac{\sin[(a-b)u+bt]}{a-b} + \frac{\sin[(a+b)u-bt]}{a+b}\right|_0^t$$

$$= \frac{1}{2}\left\{\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b}\right\}$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2}$$

Que 1.22. Evaluate

i. $L^{-1}\dfrac{1}{(s^2+1)(s^2+9)}$ **ii.** $L^{-1}\dfrac{s}{(s^2+1)(s^2+4)(s^2+9)}$

Answer

i. Since $L^{-1}\left(\dfrac{1}{s^2+1}\right) = \sin t,\ L^{-1}\left(\dfrac{1}{s^2+9}\right) = \dfrac{\sin 3t}{3}$

$\therefore$ By Convolution theorem, we get

$$L^{-1}\left(\frac{1}{s^2+1}\,\frac{1}{s^2+9}\right) = \int_0^t \sin u\, \frac{\sin 3(t-u)}{3}\, du$$

$$= \frac{1}{6}\int_0^t [\cos(4u-3t) - \cos(3t-2u)]\, du = \frac{1}{6}\left|\frac{\sin(4u-3t)}{4} - \frac{\sin(3t-2u)}{-2}\right|_0^t$$

$$= \frac{1}{6}\left\{\frac{1}{4}(\sin t + \sin 3t) + \frac{1}{2}(\sin t - \sin 3t)\right\} = \frac{1}{8}\left[\sin t - \frac{1}{3}\sin 3t\right]$$

ii. Since $L^{-1}\left(\dfrac{s}{s^2+4}\right) = \cos 2t$

$$L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\} = \frac{1}{8}\left[\sin t - \frac{1}{3}\sin 3t\right]$$

$\therefore$ By Convolution theorem, we get

$$L^{-1}\frac{s}{(s^2+1)(s^2+4)(s^2+9)} = L^{-1}\left\{\frac{s}{(s^2+1)(s^2+9)}\,\frac{s}{s^2+4}\right\}$$

$$= \int_0^t \frac{1}{8}\left(\sin u - \frac{1}{3}\sin 3u\right)\cos 2\,(t-u)du$$

$$= \frac{1}{8}\int_0^t\left[\sin u\cos 2(t-u) - \frac{1}{3}\sin 3u\cos 2(t-u)du\right]$$

$$= \frac{1}{8}\int_0^t\left[\frac{1}{2}\{\sin(2t-u)-\sin(3u-2t)\} - \frac{1}{6}\{\sin(u+2t)-\sin(5u-2t)\}\right]du$$

$$= \frac{1}{16}\left[\left|\frac{-\cos(2t-u)}{-1} + \frac{\cos(3u-2t)}{3}\right|_0^t\right] - \frac{1}{48}\left[\left|-\cos(u+2t) + \frac{\cos(5u-2t)}{5}\right|_0^t\right]$$

$$= \frac{1}{12}\cos t - \frac{1}{10}\cos 2t + \frac{1}{60}\cos 3t$$

PART-7

Application to Solve Simple Linear and Simultaneous Differential Equations.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.23. **Use Convolution theorem to find the inverse Laplace transform of** $\mathbf{\dfrac{1}{s^3(s^2+1)}}$.

Answer

$$\frac{1}{s^3(s^2+1)} = \frac{1}{s^3}\cdot\frac{1}{s^2+1}$$

$$F_1(s) = \frac{1}{s^2+1},\ F_2(s) = \frac{1}{s^3}$$

$$L^{-1}\{F_1(s)\} = \sin t$$

$$L^{-1}\{F_2(s)\} = \frac{t^2}{2}$$

Using Convolution theorem

$$L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = L^{-1}\{F_1(s).\ F_2(s)\}$$

$$= \int_0^t \frac{(t-x)^2}{2} \sin x\, dx = \int_0^t \frac{(t^2+x^2-2xt)}{2} \sin x\, dx$$

$$= \frac{1}{2}\Big[(t^2+x^2-2xt)(-\cos x) - (2x-2t)(-\sin x) + 2\cos x\Big]_0^t$$

$$= \frac{1}{2}[0-0+2\cos t + t^2 + 0 - 2] = \frac{1}{2}[2\cos t + t^2 - 2] = \frac{t^2}{2} + \cos t - 1$$

Que 1.24. **Find the inverse Laplace transform of $\dfrac{1}{s^2(s+1)^2}$ using Convolution theorem.**

Answer

Inverse Laplace of $\left[\dfrac{1}{s^2(s+1)^2}\right]$ by Convolution theorem.

$$\frac{1}{s^2(s+1)^2} = \frac{1}{s^2}\cdot\frac{1}{(s+1)^2} = F(s)\cdot G(s)$$

$$F(s) = \frac{1}{s^2} \text{ and } G(s) = \frac{1}{(s+1)^2}$$

$$F(t) = L^{-1}[F(s)] = t$$

$$G(t) = \mathrm{L}^{-1}[F(s)] = t \cdot e^{-t} = e^{-t}\cdot t$$

$\therefore$ By Convolution theorem

$$L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \int_0^t x\cdot(t-x)\, e^{-(t-x)}\, dx$$

$$= \int_0^t [xte^{-(t-x)} - x^2e^{-(t-x)}]\, dx$$

$$= \int_0^t xte^{-(t-x)}dx - \int_0^t x^2\, e^{-(t-x)}dx$$

$$= \left[\frac{xte^{-(t-x)}}{1} - \int_0^t t \cdot 1 \cdot \frac{e^{-(t-x)}}{1} dx \right]_0^t - \left[\frac{x^2 e^{-(t-x)}}{1} - \int_0^2 \frac{2xe^{-(t-x)}}{1} dx \right]_0^t$$

$$= \left[xte^{-(t-x)} - \frac{te^{-(t-x)}}{1} \right]_0^t - \left[x^2 e^{-(t-x)} - \frac{2xe^{-(t-x)}}{1} + \frac{2e^{-(t-x)}}{1} \right]_0^t$$

$$= t^2 - t + te^{-t} - t^2 + 2t - 2 + 2e^{-t}$$

$$= t + te^{-t} + 2e^{-t} - 2$$

$$L^{-1} \left\{ \frac{1}{s^2 (s+1)^2} \right\} = t + (t + 2)e^{-t} - 2$$

Que 1.25. **Find the inverse Laplace transform of the function using Convolution theorem $\left[\frac{s}{(s^2 + a^2)^2} \right]$.**

Answer

$$\frac{s}{(s^2 + a^2)^2} = \frac{1}{(s^2 + a^2)} \cdot \frac{s}{(s^2 + a^2)}$$

$$F(s) = \frac{1}{(s^2 + a^2)} \text{ and } G(s) = \frac{s}{s^2 + a^2}$$

$$f(t) = L^{-1} [F(s)] = \frac{1}{a} \sin at$$

and $$g(t) = L^{-1} [G(s)] = \cos at$$

Now, $$f(\tau) = \frac{1}{a} \sin a\tau, \; g(t - \tau) = \cos a(t - \tau)$$

$\therefore$ By Convolution theorem, we have

$$L^{-1} \left[\frac{s}{(s^2 + a^2)^2} \right] = \int_0^t \frac{1}{a} \sin a\tau . \cos a(t - \tau) \, d\tau$$

$$= \frac{1}{2a} \int_0^t [\sin at + \sin (2a\tau - at)] \, d\tau$$

$$= \frac{1}{2a} \left[\tau \sin at - \frac{\cos (2a\tau - at)}{2a} \right]_0^t$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a}[t \sin at] = \frac{t}{2a}\sin at$$

Que 1.26. **State Convolution theorem of Laplace transform and using it find :**

$$L^{-1}\left\{\frac{1}{(s^2+4)(s+2)}\right\}$$

Answer

Statement : Laplace transform of the Convolution of two functions is equal to the product of their Laplace transforms.

$$L\left\{\int_0^t F(x)G(t-x)dx\right\} = \int_{t=0}^{\infty} e^{-st}\left[\int_{x=0}^{t} F(x)G(t-x)\,dx\right]dt$$

Given : $L^{-1}\left\{\dfrac{1}{(s^2+4)(s+2)}\right\}$

Let, $$f(s) = \frac{1}{(s^2+4)},\quad g(s) = \frac{1}{(s+2)}$$

$$L^{-1}(f(s)) = \frac{1}{2}\sin 2t,\ L^{-1}\{g(s)\} = e^{-2t}$$

$$\therefore L^{-1}\left\{\frac{1}{(s^2+4)(s+2)}\right\} = \frac{1}{2}L^{-1}\left[\frac{2}{(s^2+4)}\left(\frac{1}{(s+2)}\right)\right]$$

$$I = \frac{1}{2}\int_0^t \sin 2x\, e^{-2(t-x)}\,dx \qquad \text{...(1.26.1)}$$

(By using Convolution theorem)

$$I = \frac{1}{2}\left[\left|\frac{-1}{2}\sin 2x\ e^{-2(t-x)}\right|_0^t + \int_0^t \frac{2\cos\ 2x\, e^{-2(t-x)}\,dx}{2}\right]$$

$$I = \frac{1}{2}\left[\left|\frac{-1}{2}\sin\ 2x\, e^{-2(t-x)}\right|_0^t + \left|\cos\ 2x\left(\frac{-1}{2}\right)e^{-2(t-x)}\right|_0^t - \int_0^t \frac{2}{2}\sin\ 2x\, e^{-2(t-x)}dx\right]$$

$$I = \frac{1}{2}\left[\frac{-1}{2}\sin 2x\, e^{-2(t-x)} + \cos 2x\left(\frac{-1}{2}\right)e^{-2(t-x)}\right]_0^t - \frac{1}{2}\int_0^t \sin 2x\ e^{-2(t-x)}dx$$

...(1.26.2)

Adding equation (1.26.1) and equation (1.26.2),

$$2I = \left[\frac{-1}{4}\sin 2x\, e^{-2(t-x)} + \frac{1}{2}\left(\frac{-1}{2}\right)\cos 2x\, e^{-2(t-x)}\right]_0^t$$

$$2I = \frac{-1}{4}[\sin 2x e^{-2(t-x)} + \cos 2x\, e^{-2(t-x)}]_0^t$$

$$I = \frac{-1}{8}[\sin 2t - 0 + \cos 2t - e^{-2t}]$$

$$I = \frac{-1}{8}[\sin 2t + \cos 2t - e^{-2t}]$$

Que 1.27. **Solve by the method of transforms, the equation**

$$y''' + 2y' - y' - 2y = 0 \text{ given } y(0) = y'(0) = 0 \text{ and } y''(0) = 6$$

Answer

Taking the Laplace transform of both sides, we get

$$[s^3 \bar{y} - s^2 y(0) - sy'(0) - y''(0)] + 2[s^2 \bar{y} - sy(0) - y'(0)] - [s \bar{y} - y(0)] - 2\bar{y} = 0$$

Using the given conditions, it reduces to

$$(s^3 + 2s^2 - s - 2)\bar{y} = 6$$

$$\therefore \bar{y} = \frac{6}{(s-1)(s+1)(s+2)} = \frac{6}{(s-1)(6)} + \frac{6}{(-2)(s+1)} + \frac{6}{3(s+2)}$$

On inversion, we get $\quad y = L^{-1}\frac{1}{(s-1)} - 3L^{-1}\frac{1}{(s+2)} + 2L^{-1}\left(\frac{1}{s+2}\right)$

or $\quad y = e^t - 3e^{-t} + 2e^{-2t}$ which is the desired result

Que 1.28. **Use transform method to solve**

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t \text{ with } x = 2,\ \frac{dx}{dt} = -1 \text{ at } t = 0$$

Answer

Taking the Laplace transforms of both sides, we get

$$[s^2\bar{x} - sx(0) - x'(0)] - 2[s\bar{x} - x(0)] + \bar{x} = \frac{1}{s-1}$$

Using the given conditions, it reduces to

$$(s^2 - 2s + 1)\bar{x} = \frac{1}{s-1} + 2s - 5 = \frac{2s^2 - 7s + 6}{s-1}$$

$$\therefore \qquad \bar{x} = \frac{2s^2 - 7s + 6}{(s-1)^3} = \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}$$

on breaking into partial fractions

On inversion, we obtain

$$x = 2L^{-1}\left(\frac{1}{s-1}\right) - 3L^{-1}\frac{1}{(s-1)^2} + L^{-1}\frac{1}{(s-1)^3}$$

$$= 2e^t - \frac{3e^t\, t}{1!} + \frac{e^t\, t^2}{2!} = 2e^t - 3te^t + \frac{1}{2}t^2e^t$$

Que 1.29. **Solve $(D^2 + n^2)x = a \sin (nt + \alpha)$, $x = Dx = 0$ at $t = 0$**

Answer

Taking the laplace transforms of both sides, we get

$[s^2\,\bar{x} - sx(0)\, x'(0)] + n^2\,\bar{x} = aL\{\sin nt \cos \alpha + \cos nt \sin \alpha\}$

On using the given conditions,

$$(s^2 + n^2)\,\bar{x} = a \cos \alpha\, \frac{n}{s^2 + n^2} + a \sin \alpha\, \frac{s}{s^2 + n^2}$$

$$\bar{x} = an \cos \alpha\, \frac{1}{(s^2 + n^2)} + a \sin \alpha\, \frac{s}{(s^2 + n^2)^2}$$

On inversion, we obtain

$$x = an \cos \alpha\, \frac{1}{2n^3}(\sin nt - nt \cos nt)$$

$$+ \; a \sin \alpha\, \frac{t}{2n} \sin nt$$

$$= a\, \{\sin nt \cos \alpha - nt \cos (nt + \alpha)\}/2n^2.$$

Que 1.30. **Solve $y'' + 3y' + 2y = te^{-t}$, $y(0) = 1$, $y'(0) = 0$, using Laplace transform method.**

Answer

$$y'' + 3y' + 2y = te^{-t},\ y(0) = 1,\ y'(0) = 0$$

Taking Laplace transform of both sides,

Let $$L(y) = \bar{y}$$

$$L\{y''\} + 3L\{y'\} + 2L\{y\} = L\{te^{-t}\}$$

$$[s^2\,\bar{y} - sy\,(0) - y'\,(0)] + 3\,[s\,\bar{y} - y(0)] + 2\bar{y} = \frac{1}{(s+1)^2}$$

$$(s^2 + 3s + 2)\,\bar{y} = \frac{1}{(s+1)^2} + s + 3$$

$$\bar{y} = \frac{1}{(s+1)^3\,(s+2)} + \frac{(s+3)}{(s+1)(s+2)}$$

$$= \frac{1}{(s+1)^3(s+2)} + \frac{1}{s+1} + \frac{1}{(s+1)(s+2)}$$

$$\bar{y} = \frac{1}{(s+1)^3(s+2)} + \frac{1}{s+1} + \frac{1}{s+1} - \frac{1}{(s+2)}$$

$$= \frac{1}{(s+1)^3(s+2)} + \frac{2}{s+1} - \frac{1}{(s+2)}$$

Taking inverse Laplace transform of both sides,

$$y = L^{-1}\left[\frac{1}{(s+1)^3\,(s+2)}\right] + 2e^{-t} - e^{-2t} \qquad \text{...(1.30.1)}$$

Now $$L^{-1}\left[\frac{1}{(s+1)^3}\right] = e^{-t}\frac{t^2}{2!} \text{ and } L^{-1}\left[\frac{1}{(s+2)}\right] = e^{-2t}$$

Using Convolution theorem,

$$L^{-1}\left[\frac{1}{(s+1)^3\,(s+2)}\right] = \frac{1}{2}\int_0^t e^{-x}\,x^2 e^{-2(t-x)} dx$$

$$= \frac{1}{2}e^{-2t}\int_0^t x^2\,e^x\,dx = \frac{1}{2}e^{-2t}\,[x^2\,e^x - 2xe^x + 2e^x]_0^t$$

$$= \frac{1}{2} e^{-2t} [t^2 e^t - 2te^t + 2e^t - 2]$$

$$= \frac{1}{2} [t^2 e^{-t} - 2te^{-t} + 2e^{-t} - 2e^{-2t}]$$

From equation (1.30.1),

$$y = \frac{1}{2} (t^2 e^{-t} - 2te^{-t} + 2e^{-t} - 2e^{-2t}) + 2e^{-t} - e^{-2t}$$

$$y = \frac{1}{2} t^2 e^{-t} - te^{-t} + 3e^{-t} - 2e^{-2t}$$

Que 1.31. **Solve the following differential equation using Laplace transform :**

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = \sin x,\ y(0) = 1,\ y'(0) = 0 \text{ at } t = 0.$$

Answer

Given : $$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = \sin x$$

Taking Laplace on both sides $L\{y''\} + 4L\{y'\} + 8L\{y\} = L\{\sin x\}$

$$[s^2\overline{y} - sy(0) - y'(0)] + 4[s\overline{y} - y(0)] + 8\overline{y} = \frac{1}{s^2+1}$$

$$s^2\overline{y} - s + 4s\overline{y} - 4 + 8\overline{y} = \frac{1}{s^2+1}$$

$$\overline{y}(s^2 + 4s + 8) = \frac{1}{(s^2+1)} + s + 4$$

$$\overline{y} = \frac{1}{[(s^2+4s+4)+4](s^2+1)} + \frac{s+4}{(s^2+4s+4)+4}$$

$$\overline{y} = \frac{1}{((s+2)^2+2^2)(s^2+1)} + \frac{s+2}{((s+2)^2+2^2)} + \frac{2}{((s+2)^2+2^2)}$$

Taking Laplace inverse on both sides

$$L^{-1}\{\bar{y}\} = L^{-1}\left\{\frac{1}{((s+2)^2+2^2)(s^2+1)}\right\} + L^{-1}\left\{\frac{s+2}{((s+2)^2+2^2)}\right\}$$

$$+ L^{-1}\left\{\frac{1}{(s+2)^2+2^2}\right\}$$

$$y\ (t) = \frac{1}{2}e^{-2t}\sin 2t\ \sin t + e^{-2t}\cos 2t + e^{-2t}\sin 2t$$

$$y\ (t) = e^{-2t}\,\frac{1}{2}\sin 2t\ \sin t + e^{-2t}\cos 2t + e^{-2t}\sin 2t \qquad ...(1.31.1)$$

Using Convolution theorem for , $\frac{1}{2}e^{-2t}\sin 2t\sin t$

$$= \frac{1}{4}\int_0^t e^{-2x}\, 2\sin 2x\ \sin(t-x)\ dx$$

$$\{\because\ 2\sin A\sin B = \cos(A-B) - \cos(A+B)\}$$

$$= \frac{1}{4}\int_0^t e^{-2x}[\cos(3x-t) - \cos(x+t)]\ dx$$

$$= \frac{1}{4}\left[\int_0^t e^{-2x}\cos(3x-t)\ dx - \int_0^t e^{-2x}\cos(x+t)\ dx\right]$$

$$= \frac{1}{4}\left[\left|\frac{e^{-2x}}{4+9}(-2\cos(3x-t)+3\sin(3x-t))\right|_0^t - \left|\frac{e^{-2x}}{4+1}(-2\cos(x+t)+\sin(x+t))\right|_0^t\right]$$

$$= \frac{1}{4}\left[\frac{e^{-2t}}{13}\left\{-2\cos 2t + 3\sin 2t - \frac{1}{13}[-2\cos(-t)+3\sin(-t)]\right\}\right.$$

$$\left. - \frac{e^{-2t}}{5}\left\{-2\cos 2t + \sin 2t - \frac{e^0}{5}(-2\cos t + \sin t)\right\}\right]$$

$$= \frac{1}{4}\left[\frac{-2e^{-2t}}{13}\cos 2t + \frac{3}{13}e^{-2t}\sin 2t + \frac{2}{13}\cos t + \frac{3}{13}\sin t\right.$$

$$\left. + \frac{2}{5}e^{-2t}\cos 2t - \frac{1}{5}e^{-2t}\sin 2t - \frac{2}{5}\cos t + \frac{1}{5}\sin t\right]$$

$$= \frac{1}{4}\left[e^{-2t}\cos 2t\left[\frac{2}{5}-\frac{2}{13}\right]+e^{-2t}\sin 2t\left[\frac{3}{13}-\frac{1}{5}\right]+\cos t\left[\frac{2}{13}-\frac{2}{5}\right]\right.$$

$$\left.+\sin t\left[\frac{3}{13}+\frac{1}{5}\right]\right]$$

$$= \frac{1}{4}\left[\frac{16}{65}e^{-2t}\cos 2t+\frac{2}{65}e^{-2t}\sin 2t+\frac{28}{65}\sin t-\frac{16}{65}\cos t\right]$$

Now from equation (1.35.1),

$$y(t) = \frac{4}{65}e^{-2t}\cos 2t+\frac{1}{130}e^{-2t}\sin 2t+\frac{7}{65}\sin t-\frac{4}{65}\cos t$$

$$+ e^{-2t}\cos 2t + e^{-2t}\sin 2t$$

$$y(t) = \frac{69}{65}e^{-2t}\cos 2t+\frac{131}{130}e^{-2t}\sin 2t-\frac{4}{65}\cos t+\frac{7}{65}\sin t$$

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2
UNIT

Integral Transforms

CONTENTS

PART-1

Fourier Integral, Fourier Transform, Complex Fourier Transform.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.1. **Find the Fourier transform of the following function $f(x) = 1 - x^2$, if $|x| \leq 1$ and $f(x) = 0$, if $|x| > 1$.**

AKTU 2014-15 (III), Marks 10

Answer

$$F\{f(x)\} = F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)dx = \int_{-1}^{1} e^{isx}(1-x^2)dx$$

$$= \left[(1-x^2)\frac{e^{isx}}{is} - (-2x)\frac{e^{isx}}{-s^2} + (-2)\frac{e^{isx}}{-is^3}\right]_{-1}^{1}$$

$$= \frac{2e^{is}}{-s^2} + \frac{2e^{-is}}{-s^2} + \frac{2}{is^3}(e^{is} - e^{-is})$$

$$= -\frac{2}{s^2}(e^{is} - e^{-is}) + \frac{2}{is^3}(e^{is} - e^{-is}) = -\frac{4\cos s}{s^2} + \frac{4\sin s}{s^3}$$

$$F(s) = \frac{4}{s^3}(\sin s - s\cos s)$$

Now using inverse Fourier transform

$$F^{-1}\{F(s)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{s^3}(\sin s - s\cos s)\, e^{-isx}dx$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{(\sin s - s\cos s)}{s^3}(\cos sx - i\sin x)ds$$

$$f(x) = \frac{4}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\, ds$$

$$\text{Put } x = \frac{1}{2}$$

$$1 - \left(\frac{1}{2}\right)^2 = \frac{4}{\pi}\int_{0}^{\infty}\frac{\sin s - s\cos s}{s^3}\cos\left(\frac{s}{2}\right)ds$$

$$\int_0^{\infty}\frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}\, dx = -3\pi/16$$

Que 2.2. **Find the Fourier transform of** $F(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$

AKTU 2015-16 (III), Marks 10

AKTU 2017-18 (III), Marks 10

Answer

The Fourier transform of a function $f(x)$ is given by

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

Substituting the value of $f(x)$, we get

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1.e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is}\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{(is)} [e^{ias} - e^{-ias}]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2}{s} \cdot \frac{e^{ias} - e^{-ias}}{2i} = \frac{1}{\sqrt{2\pi}} \frac{2\sin sa}{s} = \sqrt{\frac{2}{\pi}} \frac{\sin sa}{s}$$

Que 2.3. **Find the Fourier transform of the following function defined for** $a > 0$ **by** $f(t) = e^{-at^2}$

AKTU 2016-17 (III), Marks 10

Answer

Given, $f(t) = e^{-at^2}$

Now first we need to find the Fourier transform of e^{-t^2}.

Now, $F(s) = F\{f(t)\} = \int_{-\infty}^{\infty} f(t).e^{-ist} dt$

$\therefore$ $$F(s) = \int_{-\infty}^{\infty} e^{-t^2}.e^{-ist}\, dt$$

or, $$= \int_{-\infty}^{\infty} e^{-(t^2+ist)}\, dt = \int_{-\infty}^{\infty} e^{-\left\{\left(t+\frac{is}{2}\right)^2 + \frac{s^2}{4}\right\}}\, dt$$

$$= e^{-\frac{s^2}{4}} \int_{-\infty}^{\infty} e^{-\left\{t+\frac{is}{2}\right\}^2}\, dt$$

$$F(s) = F\{f(x)\} = e^{-s^2/4} \int_{-\infty}^{\infty} e^{-x^2}\, dx \qquad \left(\begin{array}{l} \because (t + is/2) = x \\ \Rightarrow dt = dx \end{array}\right)$$

$$F(s) = e^{-s^2/4}\sqrt{\pi} \qquad ...(2.3.1) \qquad \left(\because \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}\right)$$

Now from the change of scale property of Fourier transform *i.e.*,

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

So for Fourier transform of $f(x) = e^{-ax^2}$, we get

$$F(e^{-ax^2}) = F\{f(\sqrt{a}x)\} = \frac{1}{\sqrt{a}}.F\left(\frac{s}{\sqrt{a}}\right)$$

$$F(e^{-ax^2}) = \frac{1}{\sqrt{a}}.\sqrt{\pi}\, e^{\frac{-\left(\frac{s}{\sqrt{a}}\right)^2}{4}} \qquad \text{(Using eq. (2.3.1))}$$

$$F(e^{-ax^2}) = \sqrt{\frac{\pi}{a}}.\, e^{\frac{-s^2}{4a}}$$

or, by changing the variable of function, we get

$$F(e^{-at^2}) = \sqrt{\frac{\pi}{a}}.\, e^{\frac{-s^2}{4a}}$$

Que 2.4. **Find the finite Fourier sine transform of**

$$f(x) = x(\pi - x) \text{ in } 0 < x < \pi$$

AKTU 2016-17 (III), Marks 10

Answer

Finite Fourier sine transform of $f(x)$ is

$$F_s(n) = \int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

$$= \int_0^\pi x(\pi - x).\sin nx dx$$

$$= x(\pi - x).\left[\frac{-\cos nx}{n}\right|_0^\pi - (\pi - 2x).\left(\frac{-\sin nx}{n^2}\right)\Bigg|_0^\pi + (-2).\frac{\cos nx}{n^3}\Bigg]_0^\pi$$

$$F_s(n) = 0 + 0 + \frac{2}{n^3}(1 - \cos n\pi) = \frac{2}{n^3}[1 - (-1)^n]$$

Que 2.5. **Find the Fourier transform of block function $f(t)$ of height 1 and duration defined by**

$$f(t) = \begin{Bmatrix} 1 & \text{for } |t| \le \dfrac{a}{2} \\ 0 & \text{otherwise} \end{Bmatrix}$$

AKTU 2015-16 (IV), Marks 05

Answer

Given :

$$f(t) = \begin{cases} 1 & \text{for } |t| \leq \dfrac{a}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$F\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-ist} dt = \int_{-a/2}^{a/2} e^{-ist} dt$$

($\because$ $f(t) = 0$ outside this limit)

$$= \frac{1}{-is} [e^{-ist}]_{-a/2}^{a/2} = \frac{1}{-is} \left[e^{-\frac{ias}{2}} - e^{+\frac{ias}{2}} \right]$$

On multiplying and dividing by $2i$,

$$= \frac{2i}{-is} \left[\frac{e^{-\frac{ias}{2}} - e^{\frac{ias}{2}}}{2i} \right] = \frac{2}{s} \sin \frac{as}{2}$$

Que 2.6. **Find the Fourier cosine transform of $\dfrac{1}{1+x^2}$ and hence find Fourier sine transform of $\dfrac{x}{1+x^2}$.**

AKTU 2014-15 (IV), 2016-17 (IV); Marks 10

AKTU 2017-18 (IV), Marks 10

Answer

$$f(x) = \frac{1}{1+x^2}$$

Fourier cosine transform of $f(x)$,

$$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx$$

$$I = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{1+x^2} \cos sx \, dx \qquad \text{...(2.6.1)}$$

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{-x}{1+x^2} \sin sx \, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{(1+x^2-1) \sin sx \, dx}{x(1+x^2)}$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin sx}{x} dx + \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin sx \, dx}{x(1+x^2)}$$

$$\frac{dI}{ds} = -\sqrt{\frac{2}{\pi}}\,\frac{\pi}{2} + \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{\sin sx\,dx}{(1+x^2)x} \quad ...(2.6.2)$$

$$\frac{d^2I}{ds^2} = 0 + \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{\cos sx\,dx}{1+x^2}$$

$$\frac{d^2I}{ds^2} = I \Rightarrow m = \pm 1$$

$$\text{C.F.} = C_1\,e^s + C_2\,e^{-s}$$

$$\text{P. I.} = 0$$

$$I = C_1 e^s + C_2 e^{-s} \quad ...(2.6.3)$$

$$\frac{dI}{ds} = C_1 e^s - C_2 e^{-s} \quad ...(2.6.4)$$

$$I = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{1}{1+x^2}\,dx = \sqrt{\frac{\pi}{2}}$$

From eq. (2.6.2),

$$\frac{dI}{ds} = -\sqrt{\frac{\pi}{2}}$$

Putting $s = 0$ in eq. (2.6.3) and eq. (2.6.4), we get

$$I = C_1 + C_2 \quad ...(2.6.5)$$

$$\frac{dI}{ds} = -\sqrt{\frac{\pi}{2}} = C_1 - C_2 \quad ...(2.6.6)$$

From eq. (2.6.5) and eq. (2.6.6), we get

$$C_1 = 0,\ C_2 = \sqrt{\frac{\pi}{2}}$$

Thus

$$I = \sqrt{\frac{\pi}{2}}\,e^{-s}$$

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{\cos sx\,dx}{1+x^2} = \sqrt{\frac{\pi}{2}}\,e^{-s}$$

On differentiating, we get

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{-x\sin sx}{1+x^2}\,dx = -\sqrt{\frac{\pi}{2}}\,e^{-s}$$

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{x\sin sx\,dx}{1+x^2} = \sqrt{\frac{\pi}{2}}\,e^{-s}$$

$$\int_0^{\infty} \frac{x\sin sx}{1+x^2}\,dx = \frac{\pi}{2}\,e^{-s}$$

Que 2.7. **Using the Fourier integral transformation , show that**

$$e^{-ax} = \frac{2a}{\pi}\int_0^{\infty} \frac{\cos sx}{s^2+a^2}\,ds,\ a > 0,\ x \geq 0$$

AKTU 2015-16 (III), Marks 05

Answer

Using Fourier cosine integral representation

$$F(x) = \frac{2}{\pi}\int_0^\infty \cos sx \int_0^\infty e^{-at}\cos st\, dt\, ds$$

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty \cos sx \left[\frac{e^{-at}}{a^2+s^2}(-a\cos st + s\sin st)\right]_0^\infty$$

$$= \frac{2}{\pi}\int_0^\infty \cos sx \left[\frac{-1}{a^2+s^2}(-a)\right] ds$$

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos sx}{a^2+s^2} ds$$

Que 2.8. **Evaluate :** $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$ **if** $a > |b|$

AKTU 2016-17 (IV), Marks 10

Answer

Consider the integration round a unit circle $C \equiv |z| = 1$

so that $z = e^{i\theta} \quad \therefore \quad d\theta = \frac{dz}{iz}$

Also, $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

Then the given integral reduces to

$$I = \oint_C \frac{1}{\left[a + \frac{b}{2i}\left(z - \frac{1}{z}\right)\right]}\left(\frac{dz}{iz}\right) = \oint_C \frac{2iz}{bz^2 + 2iaz - b}\left(\frac{dz}{iz}\right)$$

$$= \frac{2}{b}\oint_C \frac{dz}{z^2 + \frac{2ia}{b}z - 1}$$

Poles are given by

$$z^2 + \frac{2ia}{b}z - 1 = 0$$

$$z = \frac{-\frac{2ia}{b} \pm \sqrt{\frac{-4a^2}{b^2} + 4}}{2} = \frac{-ia}{b} \pm \frac{\sqrt{b^2 - a^2}}{b}$$

$$= \frac{-ia}{b} \pm \frac{i\sqrt{a^2 - b^2}}{b} = \alpha, \beta \text{ (simple poles)}$$

Where, $\alpha = \dfrac{-ia}{b} + \dfrac{i\sqrt{a^2-b^2}}{b}$ and $\beta = \dfrac{-ia}{b} - \dfrac{i\sqrt{a^2-b^2}}{b}$

Clearly, $|\beta| > 1$

But $\alpha\beta = -1$

$\therefore$ $|\alpha\beta| = 1$

$|\alpha||\beta| = 1$

$|\alpha| < 1$

Hence $z = \alpha$ is the only pole which lies inside circle $C \equiv |z| = 1$.

Residue of $f(z)$ at $(z = \alpha)$ is

$$R = \lim_{z\to\alpha}(z-\alpha) \times \frac{2}{b(z-\alpha)(z-\beta)} = \frac{2}{b(\alpha-\beta)}$$

$$= \frac{2}{b\left(\dfrac{2i\sqrt{a^2-b^2}}{b}\right)} = \frac{1}{i\sqrt{a^2-b^2}}$$

$\therefore$ By Cauchy's Residue theorem,

$$I = 2\pi i(R) = 2\pi i\left(\frac{1}{i\sqrt{a^2-b^2}}\right) = \frac{2\pi}{\sqrt{a^2-b^2}}$$

$$\therefore \quad \int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Que 2.9. **Using complex integration method, evaluate $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta}\, d\theta$.**

AKTU 2018-19 (III), Marks 05

Answer

Let $z = e^{i\theta}$, $dz = ie^{i\theta}\, d\theta$, $d\theta = -\dfrac{idz}{z}$, $\cos\theta = \dfrac{z + 1/z}{2}$

$$\cos 2\theta = \frac{e^{2i\theta} + e^{-2i\theta}}{2} = \frac{z^2 + 1/z^2}{2} \quad \text{Then}$$

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta}\, d\theta = -i\int_{C_1(0)} \frac{\frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)dz}{5 + 4\dfrac{\left(z + \frac{1}{z}\right)}{2}z}$$

$$= -\frac{i}{2}\int_{C_1(0)} \frac{z^4+1}{5z^2 + 2z^3 + 2z}\frac{dz}{z}$$

$$= -\frac{i}{2}\int_{C_1(0)} \frac{z^4+1}{z^2(2z^2+5z+2)}\, dz$$

$$= -\frac{i}{2} 2\pi i \sum_j \text{Res}\left(\frac{z^4+1}{z^2(2z^2+5z+2)}, z_j\right)$$

$$= \pi \sum_j \text{Res}\left(\frac{z^4+1}{z^2\,(2z^2+5z+2)}, z_j\right)$$

We have a pole of order 2 at 0 and possible more poles at the roots of $2z^2 + 5z + 2$.

$$\text{Res}\left(\frac{z^4+1}{z^2\,(2z^2+5z+2)}, 0\right) = \lim_{z\to 0} \frac{d}{dz} \frac{z^4+1}{(2z^2+5z+2)}$$

$$= \left.\frac{4z^3\,(2z^2+5z+2)-(z^4+1)\,(4z+5)}{(2z^2+5z+2)^2}\right|_{z=0} = -\frac{5}{4}$$

For non-zero poles, $2z^2 + 5z + 2 = 0$

$$z = -\frac{5 \pm 3}{4}$$

$$z = -\frac{1}{2}, -2$$

$$z_1 = -\frac{1}{2} \text{ is inside } C_1(0)$$

$$\text{Res}\left(\frac{z^4+1}{z^2\,(2z^2+5z+2)}, z_1\right) = \frac{z_1^4+1}{z_1^2} \frac{1}{\left.\frac{d}{dz}(2z^2+5z+2)\right|_{z_1}}$$

$$\frac{\left(\frac{1}{2}\right)^4 + 1}{\frac{1}{4}\left(4\left(-\frac{1}{2}\right)+5\right)} = \frac{17}{12}$$

Hence $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta = \pi\left(\frac{17}{12} - \frac{5}{4}\right) = \frac{\pi}{6}$

Que 2.10. **Using complex variable techniques evaluate the real integral $\displaystyle\int_0^{2\pi} \frac{\sin 2\theta}{5-4\cos\theta}\, d\theta$**

AKTU 2017-18 (III), Marks 10

Answer

$$\sin 2\theta = \frac{1}{2i}(e^{2i\theta} - e^{-2i\theta})$$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

The given integral, $I = \int_0^{2\pi} \frac{\sin 2\theta}{5 - 4\cos\theta}\, d\theta$ ...(2.10.1)

Putting $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$ in eq. (2.10.1), we get

$$I = \oint_C \frac{\frac{1}{2i}\left(z^2 - \frac{1}{z^2}\right)}{5 - 4 \times \frac{1}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz} = \frac{1}{2i} \oint_C \frac{z^4 - 1}{z^2\left(5 - 2\left(\frac{z^2+1}{z}\right)\right)} \frac{dz}{iz}$$

$$= \frac{1}{2i^2} \oint_C \frac{z^4 - 1}{z^2 \frac{(5z - 2z^2 - 2)}{z}} \frac{dz}{z}$$

$$= \frac{1}{2} \oint_C \frac{z^4 - 1}{z^2\,(2z^2 - 5z + 2)}\, dz = \frac{1}{2} \oint_C \frac{z^4 - 1}{z^2\,(2z - 1)\,(z - 2)}\, dz$$

$$= \frac{1}{2} \oint_C f(z) dz \text{ (where } C \text{ is the unit circle } |z| = 1)$$

Now $f(z)$ has a pole of order z at $z = 0$ and simple poles at $z = 1/2$ and $z = 2$ of these only $z = 0$ and $z = 1/2$ lie within the circle.

$$\therefore \text{Res } f\left(\frac{1}{2}\right) = \lim_{z \to 1/2} \left(z - \frac{1}{2}\right) \frac{(z^4 - 1)}{z^2\,(2z-1)\,(z-2)} = \lim_{z \to 1/2} \left[\frac{z^4 - 1}{2z^2\,(z-2)}\right]$$

$$= \frac{\frac{1}{16} - 1}{2 \times \frac{1}{4}\left(\frac{1}{2} - 2\right)} = \frac{\frac{-15}{16}}{\frac{1}{2} \times \left(\frac{-3}{2}\right)} = \frac{5}{4}$$

$$\text{Res } f(0) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\,[(z-0)^n\ f(z)]\right\}_{z=0}$$

$$= \frac{1}{(2-1)!} \int \frac{d^{2-1}}{dz^{2-1}} \left[(z-0)^2 \times \frac{z^4 - 1}{z^2(2z-1)\,(z-2)}\right]_{z=0}$$

$(\because n = 2)$

$$= \left[\frac{d}{dz} \times z^2 \frac{(z^4 - 1)}{z^2\,(2z-1)\,(z-2)}\right]_{z=0}$$

$$= \left[\frac{d}{dz} \frac{z^4 - 1}{(2z - 1)\,(z - 2)}\right]_{z=0}$$

$$= \left\{\frac{(2z-1)\,(z-2)\,(4z^3) - (z^4 - 1)\,[(2z-1) + (z-2)2]}{[(2z-1)\,(z-2)]^2}\right\}_{z=0}$$

$$= \frac{0 - (-1)\,(-1-4)}{[-1\,(-2)]^2} = \frac{-5}{4}$$

$$\text{Hence } I = \frac{1}{2}\,\{2\pi i\,[\text{Res } f\,(1/2) + \text{Res } f\,(0)]\} = 2i\left(\frac{5}{4} - \frac{5}{4}\right) = 0$$

Que 2.11. **Find the inverse Fourier sine transform of $\frac{1}{x}e^{-ax}$.**

AKTU 2015-16 (IV), Marks 10

Answer

$$F_s\left\{\frac{e^{-ax}}{x}\right\} = \int_0^\infty \frac{e^{-ax}}{x} \sin sx\, dx$$

Let,
$$I = \int_0^\infty \frac{e^{-ax}}{x} \sin sx\, dx$$

$$\frac{dI}{ds} = \int_0^\infty e^{-ax} \cos sx\, dx$$

$$= \left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx)\right]_0^\infty = \frac{a}{s^2+a^2}$$

$$I = \tan^{-1}\frac{s}{a} + A$$

at $s = 0, I = 0,$

Putting, $0 = \tan^{-1} 0 + A$

$\therefore$ $A = 0$

Thus, $I = \tan^{-1}\left(\frac{s}{a}\right)$

PART-2

Inverse Transforms, Convolution Theorem, Fourier Sine and Cosine Transform.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.12. **Find the inverse Z-transform of $Z(z) = \frac{z}{z-1}, |z| > 1$**

AKTU 2016-17 (IV), Marks 05

Answer

Given : $Z(z) = \frac{z}{z-1}, |z| > 1$

$$\frac{z}{z-1} = \frac{z}{z\left(1-\frac{1}{z}\right)} = \frac{z}{z(1-z^{-1})} = \frac{1}{(1-z^{-1})}$$

$$= (1 - z^{-1})^{-1}$$

$$= \left[1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z^2}\right) + \left(\frac{1}{z^3}\right) + ...\right]$$

$$= 1 + (1)z^{-1} + (1)^2 z^{-2} + (1)^3 z^{-3} + ... + (1)^k z^{-k}$$

$$= (1)^k z^{-k}$$

$$Z^{-1}\left[\frac{z}{z-1}\right] = (1)^k$$

Que 2.13. **State the Convolution theorem for Fourier transform. Prove that the Fourier transform of the convolution of the two functions equal to the product of their Fourier transforms.**

AKTU 2014-15 (III), Marks 10

Answer

Convolution Theorem for Fourier Transform : The convolution of two functions $F(x)$ and $G(x)$ over the interval $(-\infty, \infty)$ is defined as

$$F * G = \int_{-\infty}^{\infty} F(u)\, G(x-u)\, du$$

$$F\{F(x) * G(x)\} = F \int_{-\infty}^{\infty} F(u)\, G(x-u)\, du$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} F(u)\, G(x-u)\, du\right] e^{isx}\, dx$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} G(x-u)\, e^{isx}\, dx\right] F(u)\, du\,,$$

Put $x - u = t$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} G(t)\, e^{ist}\, dt\right] F(u)\, e^{isu}\, du$$

$$= \int_{-\infty}^{\infty} e^{isu}\, F(u).\, F\{G(t)\} = \int_{-\infty}^{\infty} e^{isx}\, F(x).\, F\{G(t)\}$$

$$F\,\{F(x) * G(x)\} = F\{F(x)\}\; F\{G(x)\}$$

Hence proved.

Que 2.14. **Using the convolution theorem, evaluate** $\boldsymbol{Z^{-1}\left\{\frac{z^2}{(z-1)(z-3)}\right\}}$

AKTU 2016-17 (IV), Marks 10

Answer

Let $$F(z) = \frac{z}{z-1} \text{ and } G(z) = \frac{z}{z-3}$$

We have

$$f_n = Z^{-1}\{F(z)\} = 1^n \quad \text{and} \quad g_n = Z^{-1}\{G(z)\} = 3^n$$

then by convolution,

$$Z^{-1}\left\{\frac{z^2}{(z-1)(z-3)}\right\} = Z^{-1}\{F(z)G(z)\} = \{f_n \times g_n\} = \{1^n \times 3^n\}$$

$$\sum_{k=0}^{n} 1^k 3^{n-k} = 3^n + 3^{n-1} + \cdots + 1 = \frac{1}{2}(3^{n+1} - 1)$$

PART-3

Applications of Fourier Transform to Simple One Dimensional Heat Transfer Equations, Wave Equations and Laplace Equations.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.15. **Solve one dimensional wave equation given by** $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$.

Answer

We know that

$$F\{f'(x)\} = \int_{-\infty}^{\infty} f'(x) e^{isx} dx = \left[e^{isx} f(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} is\, e^{isx} f(x) dx$$

Assuming that $f(x) \to 0$ as $x \to \pm\infty$

$$F\{f'(x)\} = -is\int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$F\{f'(x)\} = -is\, F\{f(x)\}$$

$$\therefore \quad F\left\{\frac{\partial u}{\partial x}\right\} = -is\, F\{u\} \qquad \left[f'(x) = \frac{\partial u}{\partial x}\right]$$

$$F\left\{\frac{\partial u}{\partial x}\right\} = -is\, u(s)$$

Similarly, $$F\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\right] = F\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{isx} dx = -s^2 u(s).$$

and
$$F\left[\frac{\partial^2 u}{\partial t^2}\right] = \frac{\partial^2}{\partial t^2}\{u(s)\} \quad ...(2.15.1)$$

Thus taking Fourier transform both sides of wave equation, we have
$$D^2\{u(s)\} = -s^2c^2\, u(s) \quad ...(2.15.2)$$

Eq. (2.15.2) is a second order ordinary differential equation for $u(s)$.

Solving eq. (2.15.2), we have
$$m^2 = -s^2c^2$$
$$m = \pm isc$$

Thus
$$u(s) = C_1\, e^{isct} + C_2\, e^{-isct}$$

General solution of eq. (2.15.2) is
$$u(s) = f(s)\, e^{isct} + g(s)\, e^{-isct} \quad ...(2.15.3)$$

$\because$ Corresponding to different values of s, eq. (2.15.2) has different values of C_1 and C_2.

To find $u(x)$, we now take inverse Fourier sine transform of eq. (2.15.3),
$$u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(s)e^{-isx}ds$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} [f(s)e^{isct} + g(s)e^{-isct}]e^{-isx}ds$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} [f(s)e^{-is(x-ct)} + g(s)e^{-is(x+ct)}]ds$$
$$u(x, t) = F(x - ct) + G(x + ct)$$

Que 2.16. **Use finite Fourier transformation to solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with the conditions**

i. $u(0, t) = 0$ **ii. $u(\pi, t) = 0$**

iii. $u(x, 0) = 2x$ where $0 < x < \pi$.

AKTU 2017-18 (III), Marks 10

Answer

Since $u(0, t)$ is given, take finite fourier sine transform.
$$\int_0^\pi \frac{\partial u}{\partial t}\sin\frac{p\pi x}{\pi}\,dx = \int_0^\pi \frac{\partial^2 u}{\partial x^2}\sin px\,dx$$
$$\frac{d}{dt}\bar{u}_s = F_s\frac{\partial^2 u}{\partial x^2} = -p^2\,\bar{u}_s + p[u(0, t) - (-1)^p\, u(\pi, t)]$$
$$= -p^2\,\bar{u}_s \qquad [\because u(0, t) = 0 \text{ and } u(\pi, t) = 0]$$
$$\frac{d\bar{u}_s}{\bar{u}_s} = -p^2 dt$$

On integrating,
$$\log\,\bar{u}_s = -p^2t + c$$

or
$$\bar{u}_s = Ae^{-p^2t} \quad ...(2.16.1)$$

Since $$u(x, 0) = 2x$$

$$A = \bar{u}_s(p, 0) = \int_0^{\pi} 2x \sin\left(\frac{p\pi x}{\pi}\right) dx$$

$$A = \int_0^{\pi} 2x \sin px \, dx = -\frac{2\pi}{n} \cos p\pi$$

Substituting the value of A in eq. (2.16.1), we have

$$\bar{u}_s = -\frac{2\pi}{n}(-1)^n e^{-p^2 t} \qquad [\because \cos p\pi = (-1)^n]$$

Now by inversion theorem,

$$u(x, t) = \frac{2}{\pi} \sum_{p=1}^{\infty} \frac{-2\pi}{n} (-1)^{p+1} e^{-p^2} \sin p\pi$$

Que 2.17. The temperature u in the semi-infinite rod $0 \leq x < \infty$ is determined by the differential equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ subject to conditions

i. $u = 0$ when $t = 0, x \geq 0$

ii. $\frac{\partial u}{\partial x} = -\mu$ (a constant) when $x = 0$ and $t > 0$

iii. $u(x, t)$ is bounded .
Determine the temperature $u(x, t)$

AKTU 2017-18 (IV), Marks 07

Answer

Given : $$\frac{\partial u}{\partial x} = k\frac{\partial^2 u}{\partial x^2}$$

Taking Fourier cosine transform on both sides of the given equation

$$F_c\left(\frac{\partial u}{\partial x}\right) = F_c\left(k\frac{\partial^2 u}{\partial x^2}\right)$$

$$\frac{\partial \bar{u}}{\partial t} = k\left(-s^2\bar{u} - \sqrt{\frac{2}{\pi}} \cdot \frac{\partial u}{\partial x}(0, t)\right)$$

$$= -ks^2\bar{u} + \sqrt{\frac{2}{\pi}} k\mu$$

$$\frac{d\bar{u}}{dt} + ks^2\bar{u} = \sqrt{\frac{2}{\pi}} k\mu$$

This is linear in $\bar{u}$. Therefore, solving

$$\bar{u}\ e^{ks^2t} = \int\sqrt{\frac{2}{\pi}}\,k\mu e^{ks^2t}\,dt = \sqrt{\frac{2}{\pi}}\,k\mu\,\frac{e^{ks^2t}}{ks^2} + c$$

$$\bar{u}\,(s, t) = \sqrt{\frac{2}{\pi}}\,\frac{\mu}{s^2} + ce^{-ks^2t} \qquad ...(2.17.1)$$

Since $u\,(x, 0) = 0 \quad$ for $x \geq 0.$

$$\bar{u}\,(s, 0) = 0$$

Using this in eq. (2.17.1), we get

$$\bar{u}\,(s, 0) = c + \sqrt{\frac{2}{\pi}}\,\frac{\mu}{s^2} = 0$$

$\therefore$

$$c = -\sqrt{\frac{2}{\pi}}\,\frac{\mu}{s^{2-}}$$

Substituting this in eq. (2.17.1)

$$\bar{u}\,(s, t) = \sqrt{\frac{2}{\pi}}\,\frac{\mu}{s^2}\,(1 - e^{-ks^2t})$$

By inversion theorem

$$u(x, t) = \frac{2}{\pi}\,.\,\mu\int_0^\infty \frac{1 - e^{-ks^2t}}{s^2}\,\cos sx\,ds.$$

PART-4

Z-transform and Its Applications to Solve Difference Equations.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.18. Using Z-transform, solve the difference equation $u_{n+2} - 4u_{n+1} + 3u_n = 5^n$ with $u_0 = u_1 = 1.$

AKTU 2015-16 (IV), Marks 05

Answer

Given :

$$u_{n+2} - 4u_{n+1} + 3_n = 5^n$$
$$u_0 = u_1 = 1$$

On taking Z-transform on both sides, we get

$$Z[u_{n+2} - 4u_{n+1} + 3u_n] = Z[5^n]$$

$$z^2\left[u(z) - u_0 - \frac{u_1}{z}\right] - 4z[u(z) - u_0] + 3\,u(z) = \frac{z}{z-5}$$

$$(z^2 - 4z + 3)\,u(z) + (-z^2 + 4z)\,u_0 - zu_1 = \frac{z}{z-5}$$

$$(z^2 - 4z + 3)\,u(z) + (-z^2 + 4z - z) = \frac{z}{z-5}$$

$$(z^2 - 4z + 3)\,u(z) = \frac{z}{z-5} + z^2 - 3z$$

$$\frac{u(z)}{z} = \frac{1}{(z^2-4z+3)} \times \left[\frac{1+z^2-5z-3z+15}{z-5}\right]$$

$$= \frac{1}{(z^2-4z+3)} \times \frac{(z^2-8z+16)}{(z-5)}$$

$$\therefore \qquad \frac{u(z)}{z} = \frac{z^2-8z+16}{(z-1)(z-3)(z-5)}$$

$$\frac{z^2-8z+16}{(z-1)(z-3)(z-5)} = \frac{A}{(z-1)} + \frac{B}{z-3} + \frac{C}{z-5}$$

$$A = \left.\frac{z^2-8z+16}{(z-3)(z-5)}\right|_{z=1} = \frac{1-8+16}{(1-3)(1-5)} = \frac{9}{-2\times-4} = \frac{9}{8}$$

$$B = \left.\frac{(z-4)^2}{(z-1)(z-5)}\right|_{z=3} = \frac{(3-4)^2}{(3-1)(3-5)} = \frac{1}{2\times-2} = \frac{-1}{4}$$

$$C = \left.\frac{(z-4)^2}{(z-1)(z-3)}\right|_{z=5} = \frac{(5-4)^2}{(5-1)(5-3)} = \frac{1}{4\times 2} = \frac{1}{8}$$

Then,
$$\frac{u(z)}{z} = \frac{9}{8}\frac{1}{(z-1)} - \frac{1}{4(z-3)} + \frac{1}{8}\frac{1}{(z-5)}$$

$$u(z) = \frac{9}{8}\frac{z}{(z-1)} - \frac{1}{4}\frac{z}{(z-3)} + \frac{1}{8}\frac{z}{(z-5)}$$

On taking inverse Z-transform,

$$u_n = \frac{9}{8}(1)^n - \frac{1}{4}(3)^n + \frac{1}{8}(5)^n$$

Que 2.19. **Using Z-transform, solve the following difference equation.**

$$u_{n+2} + 2u_{n+1} + u_n = n \text{ with } u_0 = u_1 = 0$$

AKTU 2016-17 (III), Marks 05

Answer

Taking Z-transform on both sides

$$Z(u_{n+2}) + 2Z(u_{n+1}) + Z(u_n) = Z(n)$$

$$z^2[U(z) - u_0 - u_1 z^{-1}] + 2z[U(z) - u_0] + U(z) = \frac{z}{(z-1)^2}$$

Putting $u_0 = u_1 = 0$ and solving

$$U(z) = \frac{z}{(z-1)^2(z+1)^2}$$

By partial fractions

$$\frac{1}{(z-1)^2(z+1)^2} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2} + \frac{C}{(z+1)} + \frac{D}{(z+1)^2}$$

We get $A = -\frac{1}{4}, B = C = D = \frac{1}{4}$ so

$$U(z) = \frac{1}{4}\left[-\frac{z}{z-1} + \frac{z}{(z-1)^2} + \frac{z}{z+1} + \frac{z}{(z+1)^2}\right]$$

Taking inverse Z-transform on either side

$$u_n = \frac{1}{4}[-1^n + n + (-1)^n - n(-1)^n]$$

$$u_n = \left(\frac{n-1}{4}\right)[1-(-1)^n].$$

Que 2.20. **Using Z-transform solve the following difference equation $Y_{n+2} - (2\cos\alpha)\, Y_{n+1} + Y_n = 7^n$ with the conditions that $Y_0 = 5, Y_1 = 1$.**

AKTU 2014-15 (III), Marks 10

Answer

Given : $Y_{n+2} - 2\cos\alpha\, Y_{n+1} + Y_n = 7^n$,
$Y_0 = 5, Y_1 = 1$

Taking Z-transform on both sides

$$z^2\,\bar{Y} - z^2 Y_0 - zY_1 - 2\cos\alpha\,(z\bar{Y} - zY_0) + \bar{Y} = \frac{z}{z-7}$$

$$(z^2 - 2\cos\alpha z + 1)\,\bar{Y} = \frac{z}{z-7} + 5z^2 + z + 10z\cos\alpha$$

$$\bar{Y} = \frac{z}{(z-7)(z^2 - 2\cos\alpha\, z + 1)} + \frac{5z^2 + z(1+10\cos\alpha)}{(z^2 - 2\cos\alpha\, z + 1)}$$

$$\bar{Y} = \frac{z}{(z-7)(z^2 - 2\cos\alpha\cdot z + 1)} + \frac{z(5z + 10\cos\alpha + 1)}{(z^2 - 2\cos\alpha\, z + 1)}$$

$$\text{or, } \bar{Y} = \frac{z}{(z-7)(z^2 - z(2\cos\alpha) + 1)} + \frac{z(5z + 10\cos\alpha + 1)}{(z^2 - 2z\cos\alpha\, z + 1)}$$

Poles are given by $z = 7$ and $z^2 - 2\cos\alpha\, z + 1 = 0$

$$z^2 - 2\left(\frac{e^{i\alpha} + e^{-i\alpha}}{2}\right)z + 1 = 0$$

$$\Rightarrow \quad z^2 - ze^{i\alpha} - e^{-i\alpha}z + 1 = 0$$

$$\Rightarrow \quad z(z - e^{i\alpha}) - e^{-i\alpha}(z - e^{i\alpha}) = 0$$

$$z = e^{i\alpha}, e^{-i\alpha}$$

Residue at $(z = 7)$ $= \left[\dfrac{z^{K-1}z}{(z^2 - 2z\cos\alpha + 1)}\right]_{z=7} = \dfrac{7^K}{50 - 14\cos\alpha}$

Residue at $(z = e^{i\alpha})$

$$= \left[\frac{z^{K-1}z}{(z-7)(z-e^{-i\alpha})}\right]_{z=e^{i\alpha}} + \left[\frac{z^{K-1}z\,(5z + 10\cos\alpha + 1)}{(z - e^{-i\alpha})}\right]_{z=e^{i\alpha}}$$

$$= \left[\frac{e^{i\alpha K}}{(e^{i\alpha} - 7)(e^{i\alpha} - e^{-i\alpha})}\right] + \left[\frac{e^{i\alpha K}\,(5e^{i\alpha} + 10\cos\alpha + 1)}{(e^{i\alpha} - e^{-i\alpha})}\right]$$

Residue at $(z = e^{-i\alpha})$

$$= \left[\frac{e^{-i\alpha K}}{(e^{-i\alpha} - 7)(e^{-i\alpha} - e^{i\alpha})}\right] + \left[\frac{e^{-i\alpha K}\,(5e^{-i\alpha} + 10\cos\alpha + 1)}{(e^{-i\alpha} - e^{-i\alpha})}\right]$$

Thus $\bar{Y}$ = Sum of all residues

$$\therefore \quad \bar{Y} = \frac{7^K}{(50 - 14\cos\alpha)} + \frac{\sin\alpha\,(1-K) + 7\sin\alpha K}{7\sin 2\alpha - 50\sin\alpha}$$

$$+ \frac{1}{\sin\alpha}\,[5\sin\alpha\,(1 + \mathrm{K}) + (10\cos\alpha + 1)\sin\alpha K]$$

Que 2.21. **Find the inverse Z-transform of**

$$\boldsymbol{Z(z) = \frac{z}{z-1},\ |z| > 1}$$

AKTU 2016-17 (III), Marks 05

Answer

Given : $Z(z) = \dfrac{z}{z-1}, \ |z| > 1$

$$\frac{z}{z-1} = \frac{z}{z\left(1 - \dfrac{1}{z}\right)} = \frac{z}{z(1 - z^{-1})} = \frac{1}{(1 - z^{-1})}$$

$$= (1 - z^{-1})^{-1}$$

$$= \left[1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z^2}\right) + \left(\frac{1}{z^3}\right) + \ldots\right]$$

$$= 1 + (1)z^{-1} + (1)^2 z^{-2} + (1)^3 z^{-3} + \ldots + (1)^k z^{-k}$$

$$= (1)^k z^{-k}$$

$$z^{-1}\left[\frac{z}{z-1}\right] = (1)^k$$

Que 2.22. **Find the inverse *Z*-transform of :**

$$f(z) = \frac{3z^2 - 18z + 26}{(z-2)(z-3)(z-4)}$$

AKTU 2014-15 (IV), Marks 10

Answer

The poles are given by, $z = 2, 3, 4$

$$\text{Residue at } (z=2) = \left[\frac{(z-2)\, z^{k-1}\,(3z^2 - 18z + 26)}{(z-2)(z-3)(z-4)}\right]_{z=2}$$

$$= \left[\frac{3z^{k+1} - 18z^k + 26z^{k-1}}{(z-3)(z-4)}\right]_{z=2} = 2^{k-1}$$

$$\text{Residue at } (z=3) = \left[\frac{(z-3)\, z^{k-1}\,(3z^2 - 18z + 26)}{(z-2)(z-3)(z-4)}\right]_{z=3} = 3^{k-1}$$

$$\text{Residue at } (z=4) = \left[\frac{(z-4)\, z^{k-1}\,(3z^2 - 18z + 26)}{(z-2)(z-3)(z-4)}\right]_{z=4} = 4^{k-1}$$

Hence, $f(k)$ = Sum of residues

$$y_k = 2^{k-1} + 3^{k-1} + 4^{k-1},\ k > 0$$

Que 2.23. **Using the *Z*-transform, solve the following difference equation :**

$$\mathbf{6y_{k+2} - y_{k+1} - y_k = 0 \text{ given that } y_{(0)} = 0,\ y_{(1)} = 1.}$$

AKTU 2017-18 (III), Marks 10

Answer

$$6y_{k+2} - y_{k+1} - y_k = 0$$

Taking the Z-transform of both sides, we get $Z[6y_{k+2} - y_{k+1} - y_k] = 0$

$$Z(6y_{k+2}) - Z(y_{k+1}) - Z(y_k) = 0$$

$$6[z^2\, Y(z) - z^2\, y(0) - zy(1)] - [zY(z) - zy\ (0)] - Y(z) = 0$$

On putting the values of $y\ (0)$ and $y\ (1)$, we get

$$6z^2\, Y(z) - 6\, z - z\, Y(z) - Y(z) = 0 \Rightarrow (6z^2 - z - 1)\, Y(z) = 6z$$

$$Y(z) = \frac{6z}{6z^2 - z - 1} = \frac{6z}{(3z+1)(2z-1)}$$

$$= \frac{z^{-1}}{\left(1 + \frac{z^{-1}}{3}\right)\left(1 - \frac{z^{-1}}{2}\right)} = \frac{\frac{6}{5}}{1 - \frac{z^{-1}}{2}} - \frac{\frac{6}{5}}{1 + \frac{z^{-1}}{3}}$$

$$y_k = Z^{-1}\left[\frac{\frac{6}{5}}{1 - \frac{z^{-1}}{2}}\right] - Z^{-1}\left[\frac{\frac{6}{5}}{1 + \frac{z^{-1}}{3}}\right]$$

$$= \frac{6}{5}\left(\frac{1}{2}\right)^k - \frac{6}{5}\left(-\frac{1}{3}\right)^k = \frac{6}{5}\left[\left(\frac{1}{2}\right)^k - \left(-\frac{1}{3}\right)^k\right]$$

Que 2.24. **Solve by *Z*-transform : $y_{k+2} - 4y_{k+1} + 3y_k = 5^k$**

AKTU 2016-17 (IV), Marks 10

Answer

$$u_{n+2} - 4u_{n+1} + 3_n = 5^n$$
$$u_0 = u_1 = 1$$

On taking Z transform on both sides we get,

$$Z[u_{n+2} - 4u_{n+1} + 3u_n] = Z[5^n]$$

$$z^2\left[u(z) - u_0 - \frac{u_1}{z}\right] - 4z[u(z) - u_0] + 3\,u(z) = \frac{z}{z-5}$$

$$(z^2 - 4z + 3)\,u(z) + (-z^2 + 4z)\,u_0 - zu_1 = \frac{z}{z-5}$$

$$(z^2 - 4z + 3)\,u(z) + (-z^2 + 4z - z) = \frac{z}{z-5}$$

$$(z^2 - 4z + 3)\,u(z) = \frac{z}{z-5} + z^2 - 3z$$

$$\frac{u(z)}{z} = \frac{1}{(z^2 - 4z + 3)} \times \left[\frac{1 + z^2 - 5z - 3z + 15}{z-5}\right]$$

$$= \frac{1}{(z^2 - 4z + 3)} \times \frac{(z^2 - 8z + 16)}{(z-5)}$$

$$\therefore \quad \frac{u(z)}{z} = \frac{z^2 - 8z + 16}{(z-1)(z-3)(z-5)}$$

$$\frac{z^2 - 8z + 16}{(z-1)(z-3)(z-5)} = \frac{A}{(z-1)} + \frac{B}{z-3} + \frac{C}{z-5}$$

$$A = \left.\frac{z^2 - 8z + 16}{(z-3)(z-5)}\right|_{z=1} = \frac{1 - 8 + 16}{(1-3)(1-5)} = \frac{9}{-2 \times -4} = \frac{9}{8}$$

$$B = \left.\frac{(z-4)^2}{(z-1)(z-5)}\right|_{z=3} = \frac{(3-4)^2}{(3-1)(3-5)} = \frac{1}{2 \times -2} = \frac{-1}{4}$$

$$C = \left.\frac{(z-4)^2}{(z-1)(z-3)}\right|_{z=5} = \frac{(5-4)^2}{(5-1)(5-3)} = \frac{1}{4 \times 2} = \frac{1}{8}$$

Then, $$\frac{u(z)}{z} = \frac{9}{8}\frac{1}{(z-1)} - \frac{1}{4(z-3)} + \frac{1}{8}\frac{1}{(z-5)}$$

$$u(z) = \frac{9}{8}\frac{z}{(z-1)} - \frac{1}{4}\frac{z}{(z-3)} + \frac{1}{8}\frac{z}{(z-5)}$$

On taking inverse Z-transform, $u_n = \frac{9}{8}(1)^n - \frac{1}{4}(3)^n + \frac{1}{8}(5)^n$

Que 2.25. **Using Z-transform, solve the following difference equation :**

$y_{k+2} + 4y_{k+1} + 3y_k = 3^k$, given that $y_0 = 0$ and $y_1 = 1$

AKTU 2017-18 (IV), Marks 10

AKTU 2015-16 (III), Marks 10

AKTU 2014-15 (IV), Marks 10

Answer

Given : $y_{k+2} + 4y_{k+1} + 3y_k = 3^k, y_0 = 0, y_1 = 1$

Taking Z-transform of both sides of given difference equation.

$$(z^2\,\bar{y} - z^2 y(0) - zy(1)) + 4\,[z\,\bar{y} - zy(0)] + 3\,\bar{y} = \frac{1}{1-3z^{-1}}$$

$$(z^2 + 4z + 3)\,\bar{y} = z + \frac{1}{1-3z^{-1}}$$

$$(z+3)(z+1)\;\bar{y} = z + \frac{1}{1-3z^{-1}}$$

$$\bar{y} = \frac{z}{(z+3)(z+1)} + \frac{1}{(1-3z^{-1})(z+3)(z+1)}$$

$$= \frac{z}{(z+3)(z+1)} + \frac{z}{(z-3)(z+3)(z+1)}$$

$$\bar{y} = z\left[\frac{z-3+1}{(z-3)(z+3)(z+1)}\right] = \frac{z(z-2)}{(z-3)(z+3)(z+1)}$$

Residue at pole $3, -3, -1$.

Residue at $(z = 3) = \left[\frac{z^{k-1}\,z(z-2)}{(z+3)(z+1)}\right]_{z=3} = \frac{3^k}{24}$

Residue at $(z = -3) = \left[\frac{z^{k-1}\,z(z-2)}{(z-3)(z+1)}\right]_{z=-3} = \frac{-5(-3)^k}{12}$

Residue at $(z = -1) = \left[\frac{z^{k-1}\,z(z-2)}{(z-3)(z+3)}\right]_{z=-1} = \frac{-3(-1)^k}{-8} = \frac{3}{8}(-1)^k$

Thus, $\bar{y}$ = Sum of all residues

$$\bar{y} = \frac{1}{24}(3^k) - \frac{5}{12}(-3)^k + \frac{3}{8}(-1)^k,\; k \geq 0$$

Que 2.26. **Using Cauchy Integral formula to evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$, where C is the circle $|z| = 3$.**

AKTU 2016-17 (IV), Marks 10

Answer

Poles are $z = -1$ of order 4 will lie in $|z| = 3$

Using Cauchy Integral formula, we get

$$\int \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3!} \left[\frac{d^3}{dz^3} (e^{2z}) \right]_{z=-1}$$

$$= \frac{2\pi i}{3!} (8e^{2z})_{z=-1} = \frac{16\pi i}{6} \times e^{-2} = \frac{8\pi i}{3e^2}$$

Que 2.27. **By residue method, find the inverse Z-transform of $\frac{z}{z^2 + 7z + 10}$.**

AKTU 2018-19 (III), Marks 10

Answer

$$F(z) = \frac{z}{z^2 + 7z + 10}$$

$$f(k) = \frac{1}{2\pi i} \int_C z^{k-1} F(z) dz = \text{sum of residues}$$

$$= \frac{1}{2\pi i} \int_C z^{k-1} \frac{z}{z^2 + 7z + 10} dz = \frac{1}{2\pi i} \int_C \frac{z^k}{(z+2)(z+5)} dz$$

Poles are $z = -2$, $z = -5$, these are simple poles.

$$\text{Residue (at } z = -2) = \lim_{z \to -2} (z+2) \frac{z^k}{(z+2)(z+5)} = \frac{(-2)^k}{3}$$

$$\text{Residue (at } z = -5) = \lim_{z \to -5} (z+5) \frac{z^k}{(z+2)(z+5)} = \frac{(-5)^k}{-3}$$

$$f(k) = \frac{(-2)^k}{3} + \frac{(-5)^k}{-3} = \frac{1}{3} \{(-2)^k - (-5)^k\}$$

Que 2.28. **Expand $\frac{1}{z^2 - 3z + z}$ in the region $1 < |z| < 2$.**

AKTU 2018-19 (III), Marks 10

Answer

$$f(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}\ldots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\ldots\right]$$

After rearranging, we get,

$$f(z) = \ldots\ldots - z^{-3} - z^{-2} - z^{-1} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 \ldots\ldots$$

Que 2.29. **Find the residue of $f(z) = \dfrac{z^3}{(z-1)^4 (z-2)(z-3)}$ at its pole and hence evaluate $\int_C f(z)dz$, where C is the circle $|z| = 5/2$.**

AKTU 2018-19 (III), Marks 10

Answer

The poles of $f(z)$ are given by $(z-1)^4 (z-2)(z-3) = 0$

$\therefore$ $z = 1$ is a pole of order 4, while $z = 2$ and $z = 3$ are simple poles.

$$\text{Res} f(1) = \frac{1}{3!}\frac{d^3}{dz^3}\left\{(z-1)^4 \frac{z^3}{(z-1)^4 (z-2)(z-3)}\right\}_{z=1}$$

$$= \frac{1}{6}\frac{d^3}{dz^3}\left\{\frac{z^3}{(z-2)(z-3)}\right\}_{z=1}$$

$$= \frac{1}{6}\frac{d^3}{dz^3}\left[z+5-\frac{8}{z-2}+\frac{27}{z-3}\right]$$

$$= \frac{1}{6}\left[-8\frac{(-1)^3 3!}{(z-2)} + \frac{27(-1)^3\, 3!}{(z-2)^4}\right]$$

$$= -\left[-8+\frac{27}{16}\right] = \frac{101}{16}$$

$$\text{Res} f(2) = \lim_{z\to 2}\left\{(z-2)\frac{z^3}{(z-1)^4 (z-2)(z-3)}\right\}$$

$$= \lim_{z\to 2}\left\{\frac{z^3}{(z-1)^4 (z-3)}\right\} = \frac{8}{(1)^4\,(-1)} = -8$$

$$\text{Res} f(3) = \lim_{z\to 3}\left\{(z-3)\frac{z^3}{(z-1)^4 (z-2)(z-3)}\right\}$$

$$= \frac{27}{(2)^4\, 1} = \frac{27}{16}$$

Now $$\oint_C f(z)\,dz = 2\pi i\ [\text{Res } f(1) + \text{Res } f(2)]$$

[$\because$ Pole $z = 3$ is outside C]

$$= 2\pi i\left(\frac{101}{16} - 8\right) = \frac{-27\pi i}{8}$$

Que 2.30. **Evaluate by Cauchy integral formula $\oint_C \frac{z^2 - 2z}{(z+1)^2(z^2+4)}\,dz$, where C is the circle $|z| = 3$.**

AKTU 2015-16 (III), Marks 10

Answer

Here, we have $\int_c \frac{z^2 - 2z}{(z+1)^2\,(z^2+4)}\,dz$

The poles are determined by putting the denominator equal to zero.

$$(z+1)^2\,(z^2+4) = 0$$

$$z = -1, -1 \text{ and } z = \pm 2\,i$$

The circle $|z| = 3$ with centre at origin and radius = 3 encloses a pole at $z = -1$ of second order and simple poles $z = \pm 2i$.

The given integral = $I_1 + I_2 + I_3$

$$I_1 = \int_{c_1} \frac{z^2 - 2z}{(z+1)^2\,(z^2+4)}\,dz = \int_{c_1} \frac{\dfrac{z^2 - 2z}{z^2+4}}{(z+1)^2}\,dz$$

$$= 2\pi i\left[\frac{d}{dz}\,\frac{z^2 - 2z}{z^2+4}\right]_{z=-1}$$

$$= 2\pi i\left[\frac{(z^2+4)\,(2z-2) - (z^2 - 2z)2z}{(z^2+4)^2}\right]_{z=-1}$$

$$= 2\pi i\left[\frac{(1+4)\,(-2-2) - (1+2)\,2(-1)}{(1+4)^2}\right]$$

$$= 2\pi i\left(-\frac{14}{25}\right) = \frac{-28\,\pi i}{25}$$

$$I_2 = \int_{c_2} \frac{\dfrac{z^2 - 2z}{(z+1)^2\,(z+2i)}}{(z-2i)}\,dz = 2\pi i\left[\frac{z^2 - 2z}{(z+1)^2\,(z+2i)}\right]_{z=2i}$$

$$= 2\pi i\left[\frac{-4 - 4i}{(2i+1)^2\,(2i+2i}\right] = 2\pi i\,\frac{(1+i)}{4+3i}$$

$$I_3 = \int_{c_3} \frac{\dfrac{z^2 - 2z}{(z+1)^2 (z-2i)}}{(z+2i)} dz = 2\pi i \left[\frac{z^2 - 2z}{(z+1)^2 (z-2i)}\right]_{z=-2i}$$

$$= 2\pi i \left[\frac{-4+4i}{(-2i+1)^2 (-2i-2i)}\right] = 2\,\pi i \frac{(i-1)}{(3i-4)}$$

$$\int_c \frac{z^2 - 2z}{(z+1)^2 (z^2+4)} dz = I_1 + I_2 + I_3$$

$$= \frac{-28\,\pi i}{25} + 2\pi i \left(\frac{1+i}{4+3i}\right) + 2\pi i \left(\frac{i-1)}{3i-4)}\right)$$

$$= 2\pi i \left[\frac{-14}{25} + \frac{1+i}{(4+3i)} + \frac{(i-1)}{(3i-4)}\right]$$

$$= 2\pi i \left[\frac{-14}{25} + \frac{(1+i)(3i-4) + (i-1)(4+3i)}{(-9-16)}\right]$$

$$= \frac{2\,\pi i}{-25} [14 + (3i - 4 - 3 - 4i) + (4i - 3 - 4 - 3i)]$$

$$= 0$$

Que 2.31. **Use Calculus of Residue to evaluate the following integral** $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} dx$

AKTU 2016-17 (III), Marks 05

Answer

We consider $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} dx = \int_C f(z) dz$

Where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to $+R$.

The integral has simple poles at

$$z = \pm ai, z = \pm bi$$

of which $z = ai, bi$ only lie inside C.

$$\text{The residue (at } z = ai) = \lim_{z \to ai} (z - ai) \frac{\cos z\, dz}{(z^2 + a^2)(z^2 + b^2)}$$

$$= \lim_{z \to ai} (z - ai) \frac{\cos z\, dz}{(z - ai)(z + ai)(z^2 + b^2)}$$

$$= \lim_{z \to ai} \frac{\cos z\, dz}{(z + ai)(z^2 + b^2)}$$

$$= \left[\frac{\cos ai}{(ai + ai)((ai)^2 + b^2)}\right] = \frac{\cos ai}{2ai(b^2 - a^2)}$$

The residue (at $z = bi$) $= \lim_{z \to bi} (z - bi) \dfrac{\cos z \, dz}{(z^2 + a^2)(z - bi)(z + bi)}$

$$= \lim_{z \to bi} \frac{\cos z \, dz}{(z^2 + a^2)(z + bi)}$$

$$= \left[\frac{\cos bi}{((bi)^2 + a^2)(bi + bi)} \right] = \frac{\cos bi}{(a^2 - b^2)2bi}$$

Sum of Residues $(R) = \dfrac{\cos ai}{2ai(b^2 - a^2)} + \dfrac{\cos bi}{(a^2 - b^2)2bi}$

$$= \frac{1}{2i} \left[\frac{\cos ai}{a(b^2 - a^2)} + \frac{\cos bi}{b(a^2 - b^2)} \right]$$

$$= \frac{1}{2i} \left[-\frac{\cos ai}{a(a^2 - b^2)} + \frac{\cos bi}{b(a^2 - b^2)} \right]$$

$$= \frac{1}{2i} \left[\frac{\cos bi}{b(a^2 - b^2)} - \frac{\cos ai}{a(a^2 - b^2)} \right]$$

$$= \frac{1}{2i(a^2 - b^2)} \left[\frac{\cos bi}{b} - \frac{\cos ai}{a} \right]$$

$\therefore$ Using Cauchy's Residue theorem,

$$\int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2 + a^2)(x^2 + b^2)} = 2\pi i . \frac{1}{2i(a^2 - b^2)} \left[\frac{\cos bi}{b} - \frac{\cos ai}{a} \right]$$

$$= \text{Re} \left[\frac{\pi}{(a^2 - b^2)} \left(\frac{\cos bi}{b} - \frac{\cos ai}{a} \right) \right]$$

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3
UNIT

Formal Logic, Group, Ring and Field

CONTENTS

PART-1

Introduction to First Order Logic, Proposition, Algebra of Proposition.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.1. Write a short note on first order logic.

Answer

1. First order logic is the extension of propositional logic by generalizing and quantifying the propositions over given universe of discourse.
2. In first order logic every individual has the property p (say).
3. It is also called first order predicate calculus.
4. Predicate calculus is generalization of propositional calculus. Predicate calculus allows us to manipulate statements about all or something.
5. Universe of Discourse (UD) : It is the set of all possible values that can be substituted in place of predicate variable.

Que 3.2. Define the term proposition. Also, explain compound proposition with example.

Answer

Proposition : Proposition is a statement which is either true or false but not both. It is a declarative statement. It is usually denoted by lower case letters p, q, r, s, t etc. They are called Boolean variable or logic variable.
For example :

1. Dr. A.P.J. Abdul Kalam was Prime Minister of India.
2. Roses are red.
3. Delhi is in India.

(1) proposition is false whereas (2) and (3) are true.

Compound proposition : A compound proposition is formed by composition of two or more propositions called components or sub-propositions.

For example :

1. Risabh is intelligent and he studies hard.
2. Sky is blue and clouds are white.

Here first statement contain two propositions “Risabh is intelligent” and “he studies hard” whereas second statement contain propositions “sky is blue” and “clouds are white”. As both statements are formed using two propositions. So they are compound propositions.

Que 3.3. **Write short note on algebra of propositions.**

Answer

Proposition satisfies various laws which are useful in simplifying complex expressions. These laws are listed as :

1. Idempotent laws :
   a. $p \vee p \equiv p$
   b. $p \wedge p \equiv p$
2. Associative laws :
   a. $(p \vee q) \vee r \equiv p \vee (q \vee r)$
   b. $(p \wedge \mathrm{q}) \wedge r \equiv p \wedge (q \wedge r)$
3. Commutative laws :
   a. $p \vee q \equiv q \vee p$
   b. $p \wedge q \equiv q \wedge p$
4. Distributive laws :
   a. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
   b. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
5. Identity laws :
   a. $p \vee F \equiv p$
   b. $p \vee T \equiv T$
   c. $p \wedge F \equiv F$
   d. $p \wedge T \equiv p$
6. Complement laws :
   a. $p \vee \sim P \equiv T$
   b. $p \wedge \sim p \equiv F$
   c. $\sim T \equiv F$
   d. $\sim F \equiv T$
7. Involution law :
   a. $\sim(\sim p) \equiv p$
8. De Morgan's laws :
   a. $\sim (p \vee q) \equiv \sim p \wedge \sim q$
   b. $\sim (p \wedge q) \equiv \sim p \vee \sim q$
9. Absorption laws :
   a. $p \vee (p \wedge q) \equiv p$
   b $p \wedge (p \vee q) \equiv p$

These laws can easily be verified using truth table.

PART-2

Logical Connectives, Tautologies, Contradictions and Contingency, Logical Implication, Argument Normal forms, Rules of Inferences.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.4. **Discuss connectives in detail with truth tables.**

Answer

1. The words or phrases used to form compound proposition are called connectives.
2. There are five basic connectives.
   i. **Negation :** If P is a proposition then negation of P is a proposition which is true when p is false and false when p is true. It is denoted by $\sim p$ or $\neg$ or p' or $\bar{p}$.

   Truth table :

p	$\sim p$
T	F
F	T

   ii. **Conjunction :** If p and q are two propositions then conjunction of p and q is a proposition which is true when both p and q are true otherwise false. It is denoted by $p \wedge q$.

   Truth table :

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

   iii. **Disjunction :** If p and q be two propositions, then disjunction of p and q is a proposition which is true when either one of p or q or both are true and is false when both p and q are false and it is denoted by $p \vee q$.

   Truth table :

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

iv. **Implication :** If p and q are two proposition then implication of p and q is true if both p and q are true or if p is false. It is false if p is true and q is false. It is denoted by $p \to q$.

Truth table :

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

v. **Biconditional :** If p and q are two proposition then biconditional of p and q is true either both p and q are true or both p and q are false, else it is false. It is denoted by $p \leftrightarrow q$.

Truth table :

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Que 3.5. **Explain tautologies, contradictions, satisfiability and contingency.**

Answer

1. **Tautology :** Tautology is defined as a compound proposition that is always true for all possible truth values of its propositional variables and it contains T in last column of its truth table.
   Propositions like,
   i. The doctor is either male or female.
   ii. Either it is raining or not.
   are always true and are tautologies.
2. **Contradiction :** Contradiction is defined as a compound proposition that is always false for all possible truth values of its propositional variables and it contains F in last column of its truth table.
   Propositions like,
   i. x is even and x is odd number.
   ii. Tom is good boy and Tom is bad boy.
   are always false and are contradiction.
3. **Satisfiability :** A compound statement formula $A(P_1, P_2, \ldots P_n)$ is said to be satisfiable, if it has the truth value T for at least one combination of truth value of $P_1, P_2, \ldots. P_n$.

4. **Contingency :** A proposition which is neither tautology nor contradiction is called contingency.
   Here the last column of truth table contains both T and F.

Que 3.6. **What is a tautology, contradiction and contingency ? Show that $(p \vee q) \vee (\neg p \vee r) \rightarrow (q \vee r)$ is a tautology, contradiction or contingency.**

AKTU 2018-19, Marks 07

Answer

Tautology, contradiction and contingency : Refer Q. 3.5, Page 3–5B, Unit-3.

Proof : $((p \vee q) \vee (\sim p \vee r)) \rightarrow (q \vee r)$

p	q	r	$\sim P$	$(p \vee q) = A$	$(\sim p \vee r) = B$	$(A \vee B) = C$	$(q \vee r) = D$	$C \rightarrow D$
F	F	F	T	F	T	T	F	F
F	F	T	T	F	T	T	T	T
F	T	F	T	T	T	T	T	T
F	T	T	T	T	T	T	T	T
T	F	F	F	T	F	T	F	F
T	F	T	F	T	T	T	T	T
T	T	F	F	T	F	T	T	T
T	T	T	F	T	T	T	T	T

So, $((p \vee q) \vee (\sim p \vee r)) \rightarrow (q \vee r)$ is contingency.

Que 3.7. **Explain the following terms with suitable example :**

i. Conjunction
ii. Disjunction
iii. Conditional
iv. Converse
v. Contrapositive

AKTU 2014-15, Marks 10

OR

Define inverse.

Answer

i. **Conjunction :** Refer Q. 3.4, Page 3–4B, Unit-3.
   Example :
   p **:** Ram is healthy.
   q **:** He has blue eyes.
   $p \wedge q$ **:** Ram is healthy and he has blue eyes.

ii. **Disjunction :** Refer Q. 3.4, Page 3–4B, Unit-3.
   Example :
   p **:** Ram will go to Delhi.

q : Ram will go to Calcutta.
$p \vee q$: Ram will go to Delhi or Calcutta.

iii. Conditional : Refer Q. 3.4, Page 3–4B, Unit-3.

Example :
p : Ram works hard.
q : He will get good marks.
$p \rightarrow q$: If Ram works hard then he will get good marks.

For converse and contrapositive :

Let p : It rains.
q : The crops will grow.

iv. Converse : If $p \Rightarrow q$ is an implication then its converse is given by $q \Rightarrow p$ states that S : If the crops grow, then there has been rain.

v. Contrapositive : If $p \Rightarrow q$ is an implication then its contrapositive is given by $\sim q \Rightarrow \sim p$ states that,
t : If the crops do not grow then there has been no rain.

Inverse :

If $p \Rightarrow q$ is implication the inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$.
Consider the statement
p : It rains.
q : The crops will grow.
The implication $p \Rightarrow q$ states that,
r : If it rains then the crops will grow.
The inverse of the implication $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$ states that,
u : If it does not rain then the crops will not grow.

Que 3.8. What do you mean by valid argument ? Are the following arguments valid ? If valid, construct a formal proof; if not, explain why. For students to do well in discrete structure course, it is necessary that they study hard. Students who do well in courses do not skip classes. Student who study hard do well in courses. Therefore students who do well in discrete structure course do not skip class.

Answer

Valid arguments :

1. An argument $P_1, P_2, ..., P_n \vdash Q$ is said to be valid if Q is true whenever all the premises $P_1, P_2, ..., P_n$ are true.
2. For example : Consider the argument : $p \rightarrow q, q \vdash p$.

C	P	P
p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

where P denotes the premise and C denotes the conclusion.

3. From the truth table we can see in first and third rows both the premises q and $p \rightarrow q$ are true, but the conclusion p is false in third row. Therefore, this is not a valid argument.
4. First and third rows are called critical rows.
5. This method to determine whether the conclusion logically follows from the given premises by constructing the relevant truth table is called truth table technique.
6. Also, we can say the argument $P_1, P_2, ..., P_n \vdash Q$ is valid if and only if the proposition $P_1 \wedge P_2 \wedge ... \wedge P_n$ is true or we can say if $P_1 \wedge P_2 \wedge ... \wedge P_n \rightarrow Q$ is a tautology.

   For example : Consider the argument $p \rightarrow q, p \vdash q$.

   Then from the truth table :

p	q	$p \rightarrow q$	$p \wedge p \rightarrow q$	$p \wedge (p \rightarrow q) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

$p \wedge (p \rightarrow q) \rightarrow q$ is a tautology since the last column contains T only.

$\therefore \quad p \rightarrow q, p \vdash q$ is a valid argument.

Numerical :

Let the propositional variables be :

$p \rightarrow$ Do well in the course.

$q \rightarrow$ They study hard.

$r \rightarrow$ Do not skip classes.

1. For students to do well in discrete structure course, it is necessary that they study hard : $p \rightarrow q$
2. Students who do well in the courses do not skip classes : $p \rightarrow r$
3. Students who study hard do well in courses : $q \rightarrow p$
4. Therefore, students who do well in discrete structure course do not skip classes : $p \rightarrow r$

   Therefore, we have,

Given : $\underset{\text{I}}{p \rightarrow q}$ $\quad \underset{\text{II}}{p \rightarrow r}$ $\quad \underset{\text{III}}{q \rightarrow p}$ $\quad$ **Conclusion :** $\underset{\text{IV}}{p \rightarrow r}$

Proof : Taking III and II together we get

$q \rightarrow p, p \rightarrow r$ gives $\underset{\text{V}}{q \rightarrow r}$ (Using hypothetical syllogism)

Now taking I and V

$p \rightarrow q$ and $q \rightarrow r$ we get $p \rightarrow r$ (Using hypothetical syllogism)

Hence, $p \rightarrow r$ is conclusion, so it is valid.

Yes, the statement is valid.

Que 3.9. Discuss theory of inference in propositional logic.

Answer

Rules of inference are the laws of logic which are used to reach the given conclusion without using truth table. Any conclusion which can be derived using these laws is called valid conclusion and hence the given argument is valid argument.

1. **Modus ponens (Law of detachment) :** By this rule if an implication $p \to q$ is true and the premise p is true then we can always conclude that q is also true.
   The argument is of the form :

$$\begin{array}{l} p \to q \\ \underline{p \qquad\quad} \\ \therefore \quad q \end{array}$$

2. **Modus tollens (Law of contraposition) :** By this rule if an implication $p \to q$ is true and conclusion q is false then the premise p must be false.
   The argument is of the form :

$$\begin{array}{r} p \to q \\ \underline{\sim q} \\ \therefore \sim p \end{array}$$

3. **Hypothetical syllogism :** By this rule whenever the two implications $p \to q$ and $q \to r$ are true then the implication $p \to r$ is also true.
   The argument is of the form :

$$\begin{array}{r} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$$

4. **Disjunctive syllogism :** By this rule if the premises $p \vee q$ and $\sim q$ are true then p is true.
   The argument is of the form :

$$\begin{array}{r} p \vee q \\ \underline{\sim q} \\ \therefore \quad p \end{array}$$

5. **Addition :** By this rule if p is true then $p \vee q$ is true regardless the truth value of q.
   The argument is of the form :

$$\begin{array}{c} \underline{\quad p \quad} \\ \therefore p \vee q \end{array}$$

6. **Simplification :** By this rule if $p \wedge q$ is true then p is true.
   The argument is of form :

$$\frac{p \wedge q}{\therefore \ p} \text{ or } \frac{p \wedge q}{\therefore \ q}$$

7. **Conjunction :** By this rule if p and q are true then $p \wedge q$ is true.
The argument is of the form :

$$\begin{array}{c} p \\ q \\ \hline \therefore\ p \wedge q \end{array}$$

8. **Constructive dilemma :** By this rule if $(p \to q) \wedge (r \to s)$ and $p \vee r$ are true then $q \vee s$ is true.
The argument is of form :

$$\begin{array}{r} (p \to q) \wedge (r \to s) \\ p \vee r \\ \hline \therefore\ q \vee s \end{array}$$

9. **Destructive dilemma :** By this rule if $(p \to q) \wedge (r \to s)$ and $\sim q \wedge s$ are true.
The argument is of the form :

$$\begin{array}{r} (p \to q) \wedge (r \to s) \\ \sim q \wedge s \\ \hline \therefore\ \sim p \wedge r \end{array}$$

10. **Absorption :** By this rule if $p \to q$ is true then $p \to (p \wedge q)$ is true.
The argument is of the form :

$$\begin{array}{c} p \to q \\ \hline \therefore\ p \to (p \wedge q) \end{array}$$

Que 3.10. **Show that : $(r \to \sim q, r \vee s, s \to \sim q, p \to q) \leftrightarrow \sim p$ are inconsistent.** **AKTU 2017-18, Marks 07**

Answer

Following the indirect method, we introduce p as an additional premise and show that this additional premise leads to a contradiction.

{1}	(1) $p \to q$	Rule P
{2}	(2) p	Rule P (assumed)
{1, 2}	(3) q	Rule T, (1), (2) and modus ponens
{4}	(4) $s \to \bar{q}$	Rule P
{1, 2, 4}	(5) $\bar{s}$	Rule T, (3), (4) and modus tollens
{6}	(6) $r \vee s$	Rule P
{1, 2, 4, 6}	(7) r	Rule T, (5), (6) disjunctive syllogism
{8}	(8) $r \to \bar{q}$	Rule P
{8}	(9) $\bar{r} \vee \bar{q}$	Rule T, (8) and EQ_{16} $(p \to q \equiv \bar{p} \vee q)$
{8}	(10) $\overline{r \wedge q}$	Rule T, (8) and De Morgan's law
{1, 2, 4, 6}	(11) $r \wedge q$	Rule T, (7), (3) and conjunction
{1, 2, 4, 6, 8}	(12) $r \wedge q \wedge \overline{r \wedge q}$	Rule T, (10), (11) and conjunction.

Since we know that set of formula is inconsistent if their conjunction implies contradiction.
Hence it leads to a contradiction. So, it is inconsistent.

Que 3.11.

i. Show that $((p \vee q) \wedge \sim (\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \vee r)$ is a tautology without using truth table.

ii. Rewrite the following arguments using quantifiers, variables and predicate symbols :

a. All birds can fly.

b. Some men are genius.

c. Some numbers are not rational.

d. There is a student who likes mathematics but not geography.

AKTU 2014-15, Marks 10

OR

Show that $((p \vee q) \wedge \sim (\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \vee r)$ is a tautology without using truth table. **AKTU 2018-19, Marks 07**

Answer

i. We have

$((p \vee q) \wedge \sim (\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \vee r)$

$\equiv ((p \vee q) \wedge \sim (\sim p \wedge \sim (q \wedge r))) \vee (\sim (p \vee q) \vee \sim (p \vee r))$

(Using De Morgan's Law)

$\equiv [(p \vee q)] \wedge (p \vee (q \wedge r)) \vee \sim ((p \vee q) \wedge (p \vee r))$

$\equiv [(\mathrm{p} \vee q) \wedge (p \vee q) \wedge (p \wedge r)] \vee \sim ((p \vee q) \wedge (p \vee r))$

(Using distributive law)

$\equiv [((p \vee q) \wedge (p \vee q)] \wedge (p \vee r) \vee \sim ((p \vee q) \wedge (p \vee r))$

$\equiv ((p \vee q) \wedge (p \vee r)) \vee \sim ((p \vee q) \wedge (p \vee r))$

$\equiv x \vee \sim x$ where $x = (p \vee q) \wedge (p \wedge r)$

$\equiv T$

ii. a. $\forall x\ [B(x) \Rightarrow F(x)]$

b. $\exists x\ [M(x) \wedge G(x)]$

c. $\sim [\ \exists(x)\ (N(x) \wedge \mathrm{R}(x)]$

d. $\exists x\ [S(x) \wedge M(x) \wedge \sim G(x)]$

Que 3.12. **Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip." and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."**

AKTU 2018-19, Marks 07

Answer

i. The compound proposition will be : $(p \wedge q \wedge r) \Leftrightarrow s$

ii. Let p be the proposition "It is sunny this afternoon", q be the proposition "It is colder than yesterday", r be the proposition "We will go swimming", s be the proposition, "We will take a canoe trip", and t be the proposition "We will be home by sunset".

Then the hypothesis becomes $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$. The conclusion is simply t.

We construct an argument to show that our hypothesis lead to the conclusion as follows :

S. No.	Step	Reason
1.	$\neg p \wedge q$	Hypothesis
2.	$\neg p$	Simplification using step 1
3.	$r \rightarrow p$	Hypothesis
4.	$\neg r$	Modus tollens using steps 2 and 3
5.	$\neg r \rightarrow s$	Hypothesis
6.	s	Modus ponens using steps 4 and 5
7.	$s \rightarrow t$	Hypothesis
8.	t	Modus ponens using steps 6 and 7

Que 3.13. **"If the labour market is perfect then the wages of all persons in a particular employment will be equal. But it is always the case that wages for such persons are not equal therefore the labour market is not perfect". Test the validity of this argument using truth table.** **AKTU 2014-15, Marks 10**

Answer

Let p_1 : The labour market is perfect.

p_2 : Wages of all persons in a particular employment will be equal.

$\sim p_2$: Wages for such persons are not equal.

$\sim p_1$: The labour market is not perfect.

The premises are $p_1 \Rightarrow p_2$, $\sim p_2$ and the conclusion is $\sim p_1$. The argument $p_1 \Rightarrow p_2, \sim p_2 \Rightarrow \sim p_1$ is valid if $((p_1 \Rightarrow p_2) \wedge \sim p_2) \Rightarrow \sim p_1$ is a tautology.

Its truth table is,

p_1	p_2	$\sim p_1$	$\sim p_2$	$p_1 \Rightarrow p_2$	$(p_1 \Rightarrow p_2) \wedge \sim p_2$	$(p_1 \Rightarrow p_2 \wedge \sim p_2) \Rightarrow \sim p_1$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Since $((p_1 \Rightarrow p_2) \wedge \sim p_2) \Rightarrow \sim p_1$ is a tautology. Hence, this is valid argument.

Que 3.14. **Write the symbolic form and negate the following statements :**

a. Everyone who is healthy can do all kinds of work.

b. Some people are not admired by everyone.

c. Everyone should help his neighbours, or his neighbours will not help him.

AKTU 2016-17, Marks 10

Answer

a. **Symbolic form :**

Let $P(x)$: x is healthy and $Q(x)$: x do all work

$\forall x(P(x) \rightarrow Q(x))$

Negation : $\neg(\forall x (P(x) \rightarrow Q(x))$

b. **Symbolic form :**

Let $P(x)$: x is a person

$A(x, y)$: x admires y

The given statement can be written as "There is a person who is not admired by some person" and it is $(\exists x)(\exists y)[P(x) \wedge P(y) \wedge \neg A(x, y)]$

Negation : $(\exists x)(\exists y)[P(x) \wedge P(y) \wedge A(x, y)]$

c. **Symbolic form :**

Let $N(x, y)$: x and y are neighbours

$H(x, y)$: x should help y

$P(x, y)$: x will help y

The statement can be written as "For every person x and every person y, if x and y are neighbours, then either x should help y or y will not help x" and it is $(\forall x)(\forall y)[N(x, y) \rightarrow (H(x, y) \;\; \neg P(y, x))]$

Negation : $(\forall x)(\forall y)[N(x, y) \rightarrow \neg (H(x, y) \; P(y, x))]$

Que 3.15. **Translate the following sentences in quantified expressions of predicate logic.**

i. All students need financial aid.

ii. Some cows are not white.

iii. Suresh will get if division if and only if he gets first div.

iv. If water is hot, then Shyam will swim in pool.

v. All integers are either even or odd integer.

AKTU 2017-18, Marks 07

Answer

i. $\forall x\,[S(x) \Rightarrow F(x)]$

ii. $\sim[\exists(x)\,(C(x) \wedge W(x))]$

iii. Sentence is incorrect so cannot be translated into quantified expression.

iv. $W(x)$: x is water

$H(x)$: x is hot

$S(x)$: x is Shyam
$P(x)$: x will swim in pool
$\forall x\ [((W(x) \wedge H(x)) \Rightarrow (S(x) \wedge P(x))]$

v. $E(x)$: x is even
$O(x)$: x is odd
$\forall x\ (E(x) \vee O(x))$

PART-3

Semigroup, Monoid Group, Group, Coset.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.16. Write a short note on semigroup and monoid group.

Answer

Semigroup : An algebraic structure $(S, *)$ is called a semigroup if the following conditions are satisfied :

a. The binary operation * is a closed operation *i.e.*, $a * b \in S$ for all $a, b \in S$ (closure law).
b. The binary operation * is an associative operation *i.e.*, $a * (b * c) = (a * b) * c$ for all $a, b, c \in S$ (associative law).

Monoid : An algebraic structure $(S, *)$ is called a monoid if the following conditions are satisfied :

a. The binary operation * is a closed operation (closure law).
b. The binary operation * is an associative operation (associative law).
c. There exists an identity element, *i.e.*, for some $e \in S$, $e * a = a * e = a$ for all $a \in S$.

Thus a monoid is a semigroup $(S, *)$ that has an identity element.

Que 3.17. Write short notes on :

i. Group
ii. Abelian group
iii. Finite and infinite group
iv. Order of group
v. Groupoid

Answer

i. **Group :** Let $(G, *)$ be an algebraic structure where * is binary operation then $(G, *)$ is called a group if following properties are satisfied :

1. $a * b \in G\ \forall\ a, b \in G$ [closure property]
2. $a * (b * c) = (a * b) * c \quad \forall\, a, b, c \in G$ [associative property]
3. There exist an element $e \in G$ such that for any $a \in G$
$a * e = e * a = e$ [existence of identity]

4. For every $a \in G$, $\exists$ element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$

For example : $(Z, +)$, $(R, +)$, and $(Q, +)$ are all groups.

ii. **Abelian group :** A group $(G, *)$ is called abelian group or commutative group if binary operation $*$ is commutative *i.e.*, $a * b = b * a \ \forall \ a, b \in G$

For example : $(Z, +)$ is an abelian group.

iii. **Finite group :** A group $\{G, *\}$ is called a finite group if number of elements in G are finite.

For example : $G = \{0, 1, 2, 3, 4, 5\}$ under $\otimes_6$ is a finite group.

Infinite group : A group $\{G, *\}$ is called infinite group if number of element in G are infinite.

For example : $(Z, +)$ is infinite group.

iv. **Order of group :** Order of group G is the number of elements in group G. It is denoted by $o(G)$ or $|G|$. A group of order 1 has only the identity element.

v. **Groupoid :** Let $(S, *)$ be an algebraic structure in which S is a non-empty set and $*$ is a binary operation on S. Thus, S is closed with the operation $*$. Such a structure consisting of a non-empty set S and a binary operation defined in S is called a groupoid.

Que 3.18. **Prove that $(Z_6, (+_6))$ is an abelian group of order 6, where $Z_6 = \{0, 1, 2, 3, 4, 5\}$.**

AKTU 2014-15, Marks 05

Answer

The composition table is :

$+_6$	**0**	**1**	**2**	**3**	**4**	**5**
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Since $2 +_6 1 = 3$

$4 +_6 5 = 3$

From the table we get the following observations :

Closure : Since all the entries in the table belong to the given set Z_6. Therefore, Z_6 is closed with respect to addition modulo 6.

Associativity : The composition '$+_6$' is associative. If a, b, c are any three elements of Z_6,

$$a +_6 (b +_6 c) = a +_6 (b + c) \quad [\because b +_6 c = b + c \text{ (mod 6)}]$$

= least non-negative remainder when $a + (b + c)$ is divided by 6.

= least non-negative remainder when $(a + b) + c$ is divided by 6.

= $(a + b) +_6 c = (a +_6 b) +_6 c$.

Identity : We have $0 \in Z_6$. If a is any element of Z_6, then from the composition table we see that

$$0 +_6 a = a = a +_6 0$$

Therefore, 0 is the identity element.

Inverse : From the table we see that the inverse of 0, 1, 2, 3, 4, 5 are 0, 5, 4, 3, 2, 1 respectively. For example $4 +_6 2 = 0 = 2 +_6 4$ implies 4 is the inverse of 2.

Commutative : The composition is commutative as the elements are symmetrically arranged about the main diagonal. The number of elements in the set Z_6 is 6.

$\therefore$ $(Z_6, +_6)$ is a finite abelian group of order 6.

Que 3.19. **Let $G = \{1, -1, i, -i\}$ with the binary operation multiplication be an algebraic structure, where $i = \sqrt{-1}$. Determine whether G is an abelian or not.** **AKTU 2018-19, Marks 07**

Answer

The composition table of G is

$*$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	-1	1

1. **Closure property :** Since all the entries of the composition table are the elements of the given set, the set G is closed under multiplication.
2. **Associativity :** The elements of G are complex numbers, and we know that multiplication of complex numbers is associative.
3. **Identity :** Here, 1 is the identity element.
4. **Inverse :** From the composition table, we see that the inverse elements of $1, -1, i, -i$ are $1, -1, -i, i$ respectively.
5. **Commutativity :** The corresponding rows and columns of the table are identical. Therefore the binary operation is commutative. Hence, $(G, *)$ is an abelian group.

Que 3.20. **Let G be a group and let $a, b \in G$ be any elements. Then**

i. $(a^{-1})^{-1} = a$ ii. $(a * b)^{-1} = b^{-1} * a^{-1}$. **AKTU 2014-15, Marks 05**

Answer

i. Let e be the identity element for $*$ in G.
Then we have $a * a^{-1} = e$, where $a^{-1} \in G$.

Also $(a^{-1})^{-1} * a^{-1} = e$

Therefore, $(a^{-1})^{-1} * a^{-1} = a * a^{-1}$.

Thus, by right cancellation law, we have $(a^{-1})^{-1} = a$.

ii. Let a and $b \in G$ and G is a group for $*$, then $a * b \in G$ (closure)

Therefore, $(a * b)^{-1} * (a * b) = e$. ....(3.20.1)

Let a^{-1} and b^{-1} be the inverses of a and b respectively, then a^{-1}, $b^{-1} \in G$.

Therefore, $(b^{-1} * a^{-1}) * (a * b) = b^{-1} * (a^{-1} * a) * b$ (associativity)

$= b^{-1} * e * b = b^{-1} * b = e$ ...(3.20.2)

From (3.20.1) and (3.20.2) we have,

$$(a * b)^{-1} * (a * b) = (b^{-1} * a^{-1}) * (a * b)$$

$$(a * b)^{-1} = b^{-1} * a^{-1} \quad \text{(by right cancellation law)}$$

Que 3.21. **Let G be the set of all non-zero real number and let $a * b = ab/2$. Show that $(G, *)$ be an abelian group.**

AKTU 2015-16, Marks 10

Answer

i. Closure property : Let $a, b \in G$.

$$a * b = \frac{ab}{2} \in G \text{ as } ab \neq 0$$

$\Rightarrow$ $*$ is closure in G.

ii. Associativity : Let $a, b, c \in G$

Consider $$a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{a(bc)}{4} = \frac{abc}{4}$$

$$(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{(ab)c}{4} = \frac{abc}{4}$$

$\Rightarrow$ $*$ is associative in G.

iii. Existence of the identity : Let $a \in G$ and $\exists\, e$ such that

$$a * e = \frac{ae}{2} = a$$

$\Rightarrow$ $ae = 2a$

$\Rightarrow$ $e = 2$

$\therefore$ 2 is the identity element in G.

iv. Existence of the inverse : Let $a \in G$ and $b \in G$ such that $a * b = e = 2$

$\Rightarrow$ $\frac{ab}{2} = 2$

$\Rightarrow$ $ab = 4$

$\Rightarrow$ $b = \frac{4}{a}$

$\therefore$ The inverse of a is $\frac{4}{a}, \forall\, a \in G$.

v. Commutative : Let $a, b \in G$

$$a * b = \frac{ab}{2}$$

and $$b * a = \frac{ba}{2} = \frac{ab}{2}$$

$\Rightarrow$ * is commutative.

Thus, $(G, *)$ is an abelian group.

Que 3.22. **Prove that the intersection of two subgroups of a group is also subgroup.**

AKTU 2014-15, Marks 05

Answer

Let H_1 and H_2 be any two subgroups of G. Since at least the identity element e is common to both H_1 and H_2.

$\therefore$ $$H_1 \cap H_2 \neq \phi$$

In order to prove that $H_1 \cap H_2$ is a subgroup, it is sufficient to prove that

$a \in H_1 \cap H_2, b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$

Now $a \in H_1 \cap H_2 \Rightarrow a \in H_1$ and $a \in H_2$

$b \in H_1 \cap H_2 \Rightarrow b \in H_1$ and $b \in H_2$

But H_1, H_2 are subgroups. Therefore,

$a \in H_1, b \in H_1 \Rightarrow ab^{-1} \in H_1$

$a \in H_2, b \in H_2 \Rightarrow ab^{-1} \in H_2$

Finally, $ab^{-1} \in H_1, ab^{-1} \in H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$

Thus, we have shown that

$a \in H_1 \cap H_2, b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$.

Hence, $H_1 \cap H_2$ is a subgroup of G.

Que 3.23. **Prove that inverse of each element in a group is unique.**

AKTU 2015-16, Marks 10

Answer

Let (if possible) b and c be two inverses of element $a \in G$.

Then by definition of group :

$$b * a = a * b = e$$

and $$a * c = c * a = e$$

where e is the identity element of G

Now
$$\begin{aligned} b &= e * b = (c * a) * b \\ &= c * (a * b) \\ &= c * c \\ &= c \\ b &= c \end{aligned}$$

Therefore, inverse of an element is unique in $(G, *)$.

Que 3.24. **Define cosets. Write and prove properties of cosets.**

Answer

Let H be a subgroup of group G and let $a \in G$ then the set $Ha = \{ha : h \in H\}$ is called right coset generated by H and a.
Also the set $aH = \{ah : h \in H\}$ is called left coset generated by a and H.
Properties of cosets : Let H be a subgroup of G and let a and b belong to G. Then

1. $a \in a\,H$
   Proof : $a = a\,e \in a\,H$
   Since e is identity element of G.
2. $aH = H$ iff $a \in H$.
   Proof : Let $aH = H$.
   Then $a = ae \in aH = H$ (e is identity in G and so is in H)
   $\Rightarrow a \in H$
3. $aH = bH$ or $aH \cap bH = \phi$
   Proof : Let $aH = bH$ or $aH \cap bH = \phi$
   and to prove that $aH = bH$.
   Let $aH \cap bH$
   Then there exists $h_1, h_2 \in H$ such that
   $x = ah_1$ and $x = bh_2$
   $a = xh_1^{-1} = bh_2\,h_1^{-1}$
   Since H is a subgroup, we have $h_2\,h_1^{-1} \in H$
   let $h_2\,h_1^{-1} = h \in H$
   Now, $aH = bh_2h_1^{-1}\,H = (bh)\,H = b\,(hH) = bh$ ($\because$ $hH = H$ by property 2)
   $\therefore \quad aH = bH$ if $a\,H \cap bH \neq \phi$
   Thus, either $a\,H \cap bH = \phi$ or $aH = bH$.
4. $aH = bH$ iff $a^{-1}\,b \in H$.
   Proof : Let
   $$aH = bH.$$
   $$a^{-1}\,aH = a^{-1}\,bH$$
   $$eH = a^{-1}\,bH \qquad (e \text{ is identity in } G)$$
   $$H = (a^{-1}\,b)\,H$$
   Therefore by property (2); $a^{-1}\,b \in H$.
   Conversely, now if $a^{-1}\,b \in H$.
   Then consider $bH = e\,(bH) = (a\,a^{-1})\,(bH) = a\,(1^{-1}\,b)\,H = aH$
   Thus $\quad aH = bH$ iff $a^{-1}\,b \in H$.
5. aH is a subgroup of G iff $a \in H$.
   Proof : Let aH is a subgroup of G then it contains the identity e of G.
   Thus, $aH \cap eH \neq \phi$
   then by property (3) ; $aH = eH = H$
   $aH = H \Rightarrow a \in H$
   Conversely, if $a \in H$ then by property (2); $aH = H$.

PART-4

Lagrange's Theorem, Congruence Relation, Cyclic and Permutation Groups, Properties of Groups.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.25. **State and explain Lagrange's theorem.**

Answer

Lagrange's theorem :

If G is a finite group and H is a subgroup of G then o(H) divides o(G). Moreover, the number of distinct left (right) cosets of H in G is o(G)/o(H).

Proof : Let H be subgroup of order m of a finite group G of order n.

Let $H \{h_1, h_2, ..., h_m\}$

Let $a \in G$. Then aH is a left coset of H in G and $aH = \{ah_1, ah_2, ..., ah_m\}$ has m distinct elements as $ah_i = ah_j \Rightarrow h_i = h_j$ by cancellation law in G.

Thus, every left coset of H in G has m distinct elements.

Since G is a finite group, the number of distinct left cosets will also be finite. Let it be k. Then the union of these k-left cosets of H in G is equal to G.

i.e., if $a_1H, a_2H, ..., a_kH$ are right cosets of H in G then

$$G = a_1H \cup a_2H \cup ... \cup a_kH.$$

$\therefore$ $\quad o(G) = o(a_1H) + o(a_2H) + ... + o(a_kH)$

(Since two distinct left cosets are mutually disjoint.)

$$\Rightarrow \qquad n = m + m + ... + m \ (k \text{ times})$$

$$\Rightarrow \qquad n = mk \quad \Rightarrow \quad k = \frac{n}{m}$$

$$\therefore \qquad k = \frac{o(G)}{o(H)}.$$

Thus order of each subgroup of a finite group G is a divisor of the order of the group.

Cor 1 : If H has m different cosets in G then by Lagrange's theorem :

$$o(G) = m\ o(H)$$

$$\Rightarrow \qquad m = \frac{o(G)}{o(H)}$$

$$\therefore \qquad [G : H] = \frac{o(G)}{o(H)}$$

Cor 2 : If $|G| = n$ and $a \in G$ then $a^n = e$

Let $|a| = m \quad \Rightarrow \quad a^m = e$

Now, the subset H of G consisting of all integral powers of a is a subgroup of G and the order of H is m.

Then by Lagrange's theorem, m is divisor of n.

Let $\qquad n = mk$, then

$$a^n = a^{mk} = (a^m)^k = e^k = e$$

Que 3.26. **State and prove Lagrange's theorem for group. Is the converse true ?**

AKTU 2016-17, Marks 10

Answer

Lagrange's theorem :

Statement : The order of each subgroup of a finite group is a divisor of the order of the group.

Proof : Let G be a group of finite order n. Let H be a subgroup of G and let $o(H) = m$. Suppose $h_1, h_2 ...., h_m$ are the m members of H.

Let $a \in G$, then Ha is the right coset of H in G and we have

$$Ha = \{h_1 a, h_2 a, ...h_m a\}$$

Ha has m distinct members, since $= h_i a = h_j a \Rightarrow h_i = h_j$

Therefore, each right coset of H in G has m distinct members. Any two distinct right cosets of H in G are disjoint *i.e.*, they have no element in common. Since G is a finite group, the number of distinct right cosets of H in G will be finite, say, equal to k. The union of these k distinct right cosets of H in G is equal to G.

Thus, if $Ha_1, Ha_2, ..., Ha_k$ are the k distinct right cosets of H in G. Then $G = Ha_1 \cup Ha_2 \cup Ha_3 \cup .... \cup Ha_k$

$\Rightarrow$ the number of elements in G = the number of elements in Ha_1 + ... + the number of elements in Ha_2 +......+ the number of elements in Ha_k

$$\Rightarrow \quad o(G) = km$$

$$\Rightarrow \quad n = km$$

$$\Rightarrow \quad k = \frac{n}{m}$$

$\Rightarrow$ m is a divisor of n.

$\Rightarrow$ $o(H)$ is a divisor of $o(G)$.

Proof of converse : If G be a finite group of order n and $n \in G$, then

$$a^n = e$$

Let $o(a) = m$ which implies $a^m = e$.

Now, the subset H of G consisting of all the integral power of a is a subgroup of G and the order of H is m.

Then, by the Lagrange's theorem, m is divisor of n.

Let $n = mk$, then $\quad a^n = a^{mk} = (a^m)^k = e^k = e$

$\therefore$ Yes, the converse is true.

Que 3.27. **Write and prove the Lagrange's theorem. If a group $G = \{....,-3, 2, -1, 0, 1, 2, 3,.....\}$ having the addition as binary operation. If H is a subgroup of group G where $x^2 \in H$ such that $x \in G$. What is H and its left coset w.r.t 1 ?**

AKTU 2014-15, Marks 05

Answer

Lagrange's theorem : Refer Q. 3.25, Page 3–20B, Unit-3.

Numerical :

$$H = \{x^2 : x \in G\} = \{0, 1, 4, 9, 16, 25 ....\}$$

Left coset of H will be $1 + H = \{1, 2, 5, 10, 17, 26, ...\}$

Que 3.28. Write a short note on congruence relation with its properties.

Answer

1. If a and b are integers and m is a positive integer then a is said to be congruent to b modulo m, if m divides $a - b$, *i.e.*, $m \mid (a - b)$.
2. Symbolically, this is expressed as

$$a \equiv b \pmod{m} \qquad ...(3.28.1)$$

3. Expression 3.28.1 is called congruence relation, m is called modulus of the congruence, and b is called a residue of $a \pmod{m}$.
4. For all integers a, b and c :
   i. Reflexive : $a \equiv a \pmod{m}$.
   ii. Symmetric : If $a \equiv b \pmod{m}$ then $b \equiv a \pmod{m}$.
   iii. Transitive : If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$.

Que 3.29. Define cyclic group with suitable example.

Answer

A group G is called a cyclic group if $\exists$ at least one element a in G such that every element $x \in G$ is of the form a^n, where n is some integer. The element $a \in G$ is called the generator of G.

For example :

Show that the multiplicative group $G = \{1, -1, i, -i\}$ is cyclic. Also find its generators.

We have, $\quad i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1.$

and $\quad (-i)^1 = -i, (-i)^2 = -1, (-i)^3 = i, (-i)^4 = 1$

Thus, every element in G be expressed as i^n or $(-i)^n$

$\therefore$ G is cyclic group and its generators are i and $-i$.

Que 3.30.

a. Prove that every cyclic group is an abelian group.

b. Obtain all distinct left cosets of {(0), (3)} in the group (Z_6, $+_6$) and find their union.

c. Find the left cosets of {[0], [3]} in the group (Z_6, $+_6$).

AKTU 2016-17, Marks 10

Answer

a. Let G be a cyclic group and let a be a generator of G so that

$$G = \langle a \rangle = \{a^n : n \in Z\}$$

If g_1 and g_2 are any two elements of G, there exist integers r and s such that $g_1 = a^r$ and $g_2 = a^s$. Then

$$g_1 g_2 = a^r a^s = a^{r+s} = a^{s+r} = a^s \cdot a^r = g_2 g_1$$

So, G is abelian.

b. $\therefore$ $[0] + H = [3] + H$, $[1] + [4] + H$ and $[2] + H = [5] + H$ are the three distinct left cosets of H in $(Z_6, +_6)$.

We would have the following left cosets :

$$g_1 H = \{g_1 h, h \in H\}$$
$$g_2 H = \{g_2 h, h \in H\}$$
$$g_n H = \{g_n h, h \in H\}$$

The union of all these sets will include all the g' s, since for each set

$$g_k = \{g_k h, h \in H\}$$

we have $g_e \in g_k = \{g_k h, h \in H\}$

where e is the identity.

Then if we make the union of all these sets we will have at least all the elements of g. The other elements are merely g_h for some h. But since $g_h \in G$ they would be repeated elements in the union. So, the union of all left cosets of H in G is G, *i.e.*,

$$Z_6 = \{[0], [1], [2], [3], [4], [5]\}$$

c. Let $Z_6 = \{[0], [1], [2], [3], [4], [5]\}$ be a group.

$H = \{[0], [3]\}$ be a subgroup of $(Z_6, +_6)$.

The left cosets of H are,

$$[0] + H = \{[0], [3]\}$$
$$[1] + H = \{[1], [4]\}$$
$$[2] + H = \{[2], [5]\}$$
$$[3] + H = \{[3], [0]\}$$
$$[4] + H = \{[4], [1]\}$$
$$[5] + H = \{[5], [2]\}$$

Que 3.31. Prove that every group of prime order is cyclic.

AKTU 2018-19, Marks 07

Answer

1. Let G be a group whose order is a prime p.
2. Since $P > 1$, there is an element $a \in G$ such that $a \neq e$.
3. The group $<a>$ generated by 'a' is a subgroup of G.
4. By Lagrange's theorem, the order of 'a' divides $|G|$.
5. But the only divisors of $|G| = p$ are 1 and p. Since $a \neq e$ we have $|<a>| > 1$, so $|<a>| = p$.
6. Hence, $<a> = G$ and G is cyclic.

Que 3.32. Show that every group of order 3 is cyclic.

AKTU 2014-15, Marks 05

Answer

1. Suppose G is a finite group whose order is a prime number p, then to prove that G is a cyclic group.

2. An integer p is said to be a prime number if $p \neq 0$, $p \neq \pm 1$, and if the only divisors of p are ± 1, $\pm p$.
3. Some G is a group of prime order, therefore G must contain at least 2 element. Note that 2 is the least positive prime integer.
4. Therefore, there must exist an element $a \in G$ such that $a \neq$ the identity element e.
5. Since a is not the identity element, therefore $o(a)$ is definitely ≥ 2. Let $o(a) = m$. If H is the cyclic subgroup of G generated by a then $o\ (H = o\ (a) = m)$.
6. By Lagrange's theorem m must be a divisor of p. But p is prime and $m \geq 2$. Hence, $m = p$.
7. $\therefore H = G$. Since H is cyclic therefore G is cyclic and a is a generator of G.

Que 3.33. **Show that $G = [(1, 2, 4, 5, 7, 8), X_9]$ is cyclic. How many generators are there ? What are they ?** **AKTU 2015-16, Marks 7.5**

Answer

Composition table for X_9 is

X_9	1	2	4	5	7	8
1	2	3	4	5	7	8
2	2	4	8	1	5	7
4	4	8	7	2	1	5
5	5	1	2	7	8	4
7	7	5	1	8	4	2
8	8	7	5	4	2	1

1 is identity element of group G

$$2^1 = 2 \equiv 2 \bmod 9$$
$$2^2 = 4 \equiv 4 \bmod 9$$
$$2^3 = 8 \equiv 8 \bmod 9$$
$$2^4 = 16 \equiv 7 \bmod 9$$
$$2^5 = 32 \equiv 5 \bmod 9$$
$$2^6 = 64 \equiv 1 \bmod 9$$

Therefore, 2 is generator of G. Hence G is cyclic.
Similarly, 5 is also generator of G.
Hence there are two generators 2 and 5.

Que 3.34. **Let H be a subgroup of a finite group G. Prove that order of H is a divisor of order of G.** **AKTU 2018-19, Marks 10**

Answer

1. Let H be any sub-group of order m of a finite group G of order n. Let us consider the left coset decomposition of G relative to H.
2. We will show that each coset aH consists of m different elements.

Let $H = \{h_1, h_2, \ldots.., h_m\}$

3. Then $ah_1, ah_2, \ldots., ah_m$, are the members of aH, all distinct.
For, we have $ah_i = ah_j \Rightarrow h_i = h_j$
by cancellation law in G.

4. Since G is a finite group, the number of distinct left cosets will also be finite, say k. Hence the total number of elements of all cosets is k_m which is equal to the total number of elements of G.
Hence $n = mk$
This show that m, the order of H, is a divisor of n, the order of the group G.
We also find that the index k is also a divisor of the order of the group.

Que 3.35. **If the permutation of the elements of {1, 2, 3, 4, 5} are given by a = (1 2 3) (4 5), b = (1) (2) (3) (4 5), c = (1 5 2 4) (3). Find the value of x, if $ax = b$. And also prove that the set Z_4 = (0, 1, 2, 3) is a commutative ring with respect to the binary modulo operation $+_4$ and $*_4$.**

AKTU 2015-16, Marks 10

Answer

$ax = b \quad \Rightarrow \quad (123)\,(45)\,x = (1)\,(2)\,(3)\,(4, 5)$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix} x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix} x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 1 & 5 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 4 & 5 \end{pmatrix}$$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\times_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

We find from these tables :

i. All the entries in both the tables belong to Z_4. Hence, Z_4 is closed with respect to both operations.

ii. **Commutative law :** The entries of 1st, 2nd, 3rd, 4th rows are identical with the corresponding elements of the 1st, 2nd, 3rd, 4th columns respectively in both the tables. Hence, Z_4 is commutative with respect to both operations.

iii. **Associative law :** The associative law for addition and multiplication
$a +_4(b +_4c) = (a +_4b) +_4c$ for all $a, b, c \in Z_4$
$a \times_4(b \times_4c) = (a \times_4b) \times_4c$, for all $a, b, c \in Z_4$
can easily be verified.

iv. **Existence of identity :** 0 is the additive identity and 1 is multiplicative identity for Z_4.

v. **Existence of inverse :** The additive inverses of 0, 1, 2, 3 are 0, 3, 2, 1 respectively. Multiplicative inverse of non-zero element 1, 2, 3 are 1, 2, 3 respectively.

vi. **Distributive law :** Multiplication is distributive over addition *i.e.*,

$$a \times_4(b +_4c) = a \times_4b + a \times_4c$$
$$(b +_4c) \times_4a = b \times_4a + c \times_4a$$

For, $a \times_4(b +_4c) = a \times_4(b + c)$ for b $+_4$ $c = b + c$ (mod 4)
= least positive remainder when $a \times (b + c)$ is divided by 4
= least positive remainder when $ab + ac$ is divided by 4
$= ab +_4ac = a \times_4b +_4a \times_4c$

For, $a \times_4b = a \times b$ (mod 4)

Since $(Z_4, +_4)$ is an abelian group, $(Z_4, \times_4)$ is a semigroup and the operation is distributive over addition. The $(Z_4, +_4, \times_4)$ is a ring. Now $(Z_4, \times_4)$ is commutative with respect to $\times_4$. Therefore, it is a commutative ring.

Que 3.36. Write the properties of group. Show that the set (1, 2, 3, 4, 5) is not group under addition and multiplication modulo 6.

AKTU 2017-18, Marks 07

Answer

Properties of a group : Refer Q. 3.17, Page 3–14B, Unit-3.

Numerical :

Addition modulo 6 ($+_6$) : Composition table of S = {1, 2, 3, 4, 5} under operation $+_6$ is given as :

$+_6$	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

Since, $1 +_6 5 = 0$ but $0 \notin S$ *i.e.*, S is not closed under addition modulo 6.
So, S is not a group.

Multiplication modulo 6 ($*_6$) :

Composition table of $S = \{1, 2, 3, 4, 5\}$ under operation $*_6$ is given as

$*_6$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

Since, $2 *_6 3 = 0$ but $0 \notin S$ *i.e.*, S is not closed under multiplication modulo 6.
So, S is not a group.

PART-5

Rings and Fields (Definition, Examples and Standard Results Only).

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.37. **Write a short note on rings and fields.**

Answer

A ring $(R, +, .)$ is a set R together with two binary operation + (Addition) and . (Multiplication) defined on R such that the following axioms are satisfied :

(R_1) $(a + b) + c = a + (b + c)$ for all $a, b, c \in R$.

(R_2) $a + b = b + a$ for all $a, b \in R$.

(R_3) there exists an element 0 in R such that $a + 0 = a$ for all $a \in R$.

(R_4) for all $a \in R$, there exists an element $-a \in R$ such that

$$a + (-a) = 0.$$

(R_5) $(a . b) . c = a . (b . c)$ for all $a, b, c \in R$.

(R_6) $a . (b + c) = (a . b) + (a . c)$ for all $a, b, c \in R$. (Left distributive law)

(R_7) $(b + c) . a = (b . a) + (c . a)$ for all $a, b, c \in R$. (Right distributive law)

Que 3.38. **If R is a ring such that $a^2 = a \ \forall \ a \in R$ prove that**

i. $a + a = 0 \ \forall \ a \in R$ *i.e.*, each element of R is its own additive inverse

ii. $a + b = q \ \Rightarrow \ a = b$.

iii. R is a commutative ring.

Answer

i. $a \in R \Rightarrow a = a \in R.$

Now $(a + a)^2 = (a + a)$

$\Rightarrow (a + a)(a + a) = a + a$ [by assumption $a^2 = a$]

$\Rightarrow (a + a)a + (a + a)a = a + a$ [by left distributive law]

$\Rightarrow (a^2 + a^2) + (a^2 + a^2) = a + a$ [by right distributive law]

$\Rightarrow (a + a) + (a + a) = a + a$ $(\because \ a^2 = a)$

$\Rightarrow (a + a) + (a + a) = (a + a) + 0$ $(\because \ a + 0 = a)$

$\Rightarrow (a + a) = 0$

[by left cancellation law for addition in R]

ii. $a + b = 0 \Rightarrow a + b = a + a$ [using (i)]

$\Rightarrow b = a$ [by left cancellation law]

$\Rightarrow a = b$

iii. We have

$(a + b)^2 = (a + b)$

$\Rightarrow (a + b)(a + b) = (a + b)$

$\Rightarrow (a + b)a + (a + b)b = a + b$ [by left distributive law]

$\Rightarrow (a^2 + ba) + (ab + b^2) = a + b$ [by right distributive law]

$\Rightarrow (a + ba) + (ab + b) = a + b$ $(\because \ a^2 = a, b^2 = b)$

$\Rightarrow (a + b) + (ba + ab) = (a + b) + 0$

[by commutativity and associativity of addition]

$\Rightarrow ba + ab = 0$ [by left cancellation law]

$\Rightarrow ab = ba.$ [by ii]

$\therefore$ R is commutative ring.

Que 3.39. Consider a ring $(R, +, \bullet)$ defined by $a \bullet a = a$, determine whether the ring is commutative or not.

AKTU 2014-15, Marks 05

Answer

Let $a, b \in R$ $(a + b)^2 = (a + b)$

$\Rightarrow (a + b)(a + b) = (a + b)$

$(a + b)a + (a + b)b = (a + b)$

$(a^2 + ba) + (ab + b^2) = (a + b)$

$(a + ba) + (ab + b) = (a + b)$ $(\because \ a^2 = a$ and $b^2 = b)$

$(a + b) + (ba + ab) = (a + b) + 0$

$\Rightarrow ba + ab = 0$

$a + b = 0 \Rightarrow a + b = a + a$ [being every element of its own additive inverse]

$\Rightarrow b = a$

$\Rightarrow ab = ba$

$\therefore$ R is commutative ring.

Que 3.40. What is meant by ring ? Give examples of both commutative and non-commutative rings.

AKTU 2018-19, Marks 07

Answer

Ring : Refer Q. 3.37, Page 3–27B, Unit-3.

Example of commutative ring : Refer Q. 3.39, Page 3–28B, Unit-3.

Example of non-commutative ring : Consider the set R of 2 × 2 matrix with real element. For $A, B, C \in$ R

$$A * (B + C) = (A * B) + (A * C)$$

also, $$(A + B) * C = (A * C) + (B * C)$$

$\therefore$ $*$ is distributive over +.

$\therefore$ $(R, +, *)$ is a ring.

We know that $AB \neq BA$, Hence $(R, +, *)$ is non-commutative ring.

Que 3.41. Every field is an integral domain. Explain.

Answer

Since a field F is a commutative ring with unity. Therefore, to show F is an integral domain we need to show that a field, F has no zero divisors.

Let $a, b \in F$ with $a \neq 0$ such that $ab = 0$.

Since $a \neq 0$, a^{-1} exists.

Now we have $ab = 0$

$\Rightarrow$ $$a^{-1}(ab) = a^{-1}\,0$$

$\Rightarrow$ $$(a^{-1}a)\,b = 0$$

$\Rightarrow$ $$1b = 0 \qquad [\because a^{-1}a = 1]$$

$\Rightarrow$ $$b = 0 \qquad [\because 1b = b]$$

Similarly, let $ab = 0$ and $b \neq 0$

Since $b \neq 0$, b –1 exists

Again we have $ab = 0$

$\Rightarrow$ $$(ab)\,b^{-1} = 0\,b^{-1}$$

$\Rightarrow$ $$a(bb^{-1}) = 0$$

$\Rightarrow$ $$a1 = 0$$

$\Rightarrow$ $$a = 0$$

Thus in a field $ab = 0 \Rightarrow a = 0 \Rightarrow b = 0$.

Therefore, a field has no zero divisors.

Hence, every field is an integral domain.

☺☺☺

4
UNIT

Sets, Relation, Function and Counting Techniques

CONTENTS

PART-1

Introduction of Sets.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.1. **Define set. Explain all types of sets.**

Answer

1. A set is a collection of well defined objects, called elements or members of the set.
2. These elements may be anything like numbers, letters of alphabets, points etc.
3. Sets and their elements are denoted by capital letters and lower case letters respectively.
4. If an object x is an element of set A, we write it as $x \in A$ and read it as 'x belongs to A' otherwise $x \notin A$ (x does not belong to A).

Types of sets :

1. **Finite set :** A set with finite or countable number of elements is called finite set.
2. **Infinite set :** A set with infinite number of elements is called infinite set.
3. **Null set :** A set which contains no element at all is called a null set. It is denoted by ϕ or { }. It is also called empty or void set.
4. **Singleton set :** A set which has only one element is called singleton set.
5. **Subset :** Let A and B be two sets, if every elements of A also belongs to B *i.e.*, if every element of set A is also an element of set B, then A is called subset of B and it is denoted by $A \subseteq B$.
   Symbolically, $A \subseteq B$ if $x \in A \Rightarrow x \in B$.
6. **Superset :** If A is subset of a set B, then B is called superset of A.
7. **Proper subset :** Any subset A is said to be proper subset of another set B, if there is at least one element of B which does not belong to A, *i.e.*, if $A \subseteq B$ but $A \neq B$. It is denoted by $A \subset B$.
8. **Universal set :** In many applications of sets, all the sets under consideration are considered as subsets of one particular set. This set is called universal set and is denoted by U.
9. **Equal set :** Two set A and B are said to be equal if every element of A belong to set B and every element of B belong to set A. It is denoted as $A = B$.
   Symbolically, $A = B$ if $x \in A$ and $x \in B$.
10. **Disjoint set :** Let A and B be two sets, if there is no common element between A and B, then they are said to be disjoint.

Que 4.2. Give different types of operations on sets.

Answer

Different types of operations on sets are :

1. **Union :** Let A and B be two sets, then the union of sets A and B is a set that contain those elements that are either in A or B or in both. It is denoted by $A \cup B$ and is read as 'A union B'.
   Symbolically, $A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$
   For example : $A = \{1, 2, 3, 4\}$
   $B = \{3, 4, 5, 6\}$
   $A \cup B = \{1, 2, 3, 4, 5, 6\}$
2. **Intersection :** Let A and B be two sets, then intersection of A and B is a set that contain those elements which are common to both A and B. It is denoted by $A \cap B$ and is read as 'A intersection B'.
   Symbolically, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
   For example : $A = \{1, 2, 3, 4\}$
   $B = \{2, 4, 6, 7\}$
   then $A \cap B = \{2, 4\}$
3. **Complement :** Let U be the universal set and A be any subset of U, then complement of A is a set containing elements of U which do not belong to A. It is denoted by A^c or A' or $\overline{A}$.
   Symbolically, $A^c = \{x \mid x \in U \text{ and } x \notin A\}$
   For example : $U = \{1, 2, 3, 4, 5, 6\}$
   and $A = \{2, 3, 5\}$
   then $A^c = \{1, 4, 6\}$
4. **Difference of sets :** Let A and B be two sets. Then difference of A and B is a set of all those elements which belong to A but do not belong to B and is denoted by $A - B$.
   Symbolically, $A - B = \{x \mid x \in A \text{ and } x \notin B\}$
   For example : Let $A = \{2, 3, 4, 5, 6, 7\}$
   and $B = \{4, 5, 7\}$
   then $A - B = \{2, 3, 6\}$
5. **Symmetric difference of set :** Let A and B be two sets. The symmetric difference of A and B is a set containing all the elements that belong to A or B but not both. It is denoted by $A \oplus B$ or $A \Delta B$.
   Also $A \oplus B = (A \cup B) - (A \cap B)$
   For example : Let $A = \{2, 3, 4, 6\}$
   $B = \{1, 2, 5, 6\}$
   then $A \oplus B = \{1, 3, 4, 5\}$

Que 4.3. List down laws of algebra of sets.

Answer

Let A, B, C be any three sets and U be the universal set, then following are the laws of algebra of sets :

1. **Idempotent laws :**
   a. $A \cup A = A$ b. $A \cap A = A$
2. **Commutative laws :**
   a. $A \cup B = B \cup A$ b. $A \cap B = B \cap A$
3. **Associative laws :**
   a. $A \cup (B \cup C) = (A \cup B) \cup C$
   b. $A \cap (B \cap C) = (A \cap B) \cap C$
4. **Distributive laws :**
   a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. **Identity laws :**
   a. $A \cup \phi = A$ b. $A \cap U = A$
   c. $A \cup U = U$ d. $A \cap \phi = \phi$
6. **Involution law :**
   a. $(A^c)^c = A$
7. **Complement laws :**
   a. $A \cup A^c = U$ b. $A \cap A^c = \phi$
   c. $U^c = \phi$ d. $\phi^c = U$
8. **De Morgan's laws :**
   a. $(A \cup B)^c = A^c \cap B^c$ b. $(A \cap B)^c = A^c \cup B^c$
9. **Absorption laws :**
   a. $A \cup (A \cap B) = A$ b. $A \cap (A \cup B) = A$

Que 4.4. Define cardinality, countable and uncountable sets.

Answer

Cardinality : Cardinality of a set is defined as the total number of elements in a finite set.

Countable set : A set S is called countable if there exist a one-to-one function $f : S \to N$, *i.e.*, f is function from set S to set of natural numbers N. If f is onto also thus making f bijective, then S is called countably infinite.

Uncountable set : A set S is uncountable if and only if any of the following conditions holds true :

1. There is no one-to-one function from S to set of natural number N.
2. The set S has cardinality strictly greater than cardinality of natural numbers.

Que 4.5. Prove for any two sets A and B that, $(A \cup B)' = A' \cap B'$.

AKTU 2014-15, Marks 05

Answer

Let $x \in (A \cup B)'$

$\Rightarrow$ $x \notin A \cup B$

$\Rightarrow$ $x \notin A$ and $x \notin B$

$\Rightarrow$ $x \in A'$ and $x \in B'$

$\Rightarrow$ $x \in A' \cap B'$

$\Rightarrow \quad (A \cup B)' \subseteq A' \cap B'$ ...(4.5.1)

Now, let $\quad x \in A' \cap B'$

$\Rightarrow \quad x \in A'$ and $x \in B'$

$\Rightarrow \quad x \notin A$ and $x \notin B$

$\Rightarrow \quad x \notin (A \cup B)$

$x \in (A \cup B)'$

$(A' \cap B') \subseteq (A \cup B)'$ ...(4.5.2)

From eq. (4.5.1) and (4.5.2), $(A \cup B)' = A' \cap B'$

Que 4.6. A total of 1232 student have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken least one of Spanish, French and Russian, how many students have taken a course in all three languages ?

AKTU 2018-19, Marks 07

Answer

Let S be the set of students who have taken a course in Spanish, F be the set of students who have taken a course in French, and R be the set of students who have taken a course in Russian. Then, we have

$|S| = 1232$, $|F| = 879$, $|R| = 114$, $|S \cap F| = 103$, $|S \cap R| = 23$, $|S \cap R| = 14$, and $|S \cup F \cup R| = 23$.

Using the equation

$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |S \cap R| + |S \cup F \cup R|$,

$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$,

$|S \cap F \cap R| = 7$.

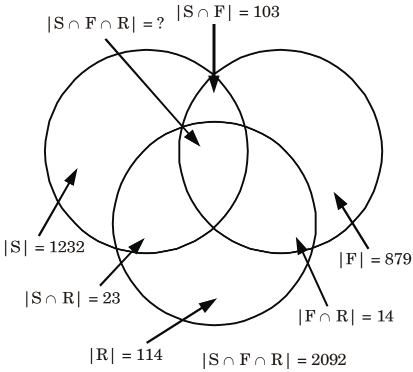


Fig. 4.6.1.

PART-2

Relation.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.7. **Define relation. Discuss its types.**

Answer

Let A and B be two non-empty sets, then R is relation from A to B if R is subset of $A \times B$ and is set of ordered pair (a, b) where $a \in A$ and $b \in B$. It is denoted by aRb and read as "a is related to b by R".

Symbolically, $R = \{(a, b) : a \in A, b \in B, a\,R\,b\}$

If $(a, b) \notin R$ then $a \not R\, b$ and read as "a is not related to b by R".

For example :

Let $A = \{1, 2, 3, 4\}$, $B = \{1, 2\}$ and aRb iff $a \times b$ = even number

Then $R = \{(1, 2), (2, 1), (2, 2), (3, 2), (4, 1), (4, 2)\}$

Types of relation :

1. **Universal relation :** A relation R is called universal relation on A if $R = A \times A$. In case where R is defined from A to B, then R is universal relation if $R = A \times B$.

   For example :

   If $A = \{1, 2, 3\}$, then

   $R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

   is universal relation over A.

2. **Identity relation :** A relation R is called identity relation on A if $R = \{(a, a) \mid a \in A\}$. It is denoted by I_A or Δ_A or Δ. It is also called diagonal relation.

   For example :

   If $A = \{1, 2, 3\}$, then $I_A = \{(1, 1), (2, 2), (3, 3)\}$

   is identity relation on A.

3. **Void relation :** A relation R is called a void relation on A if $R = \phi$. It is also called null relation.

   For example :

   If $A = \{1, 2, 3\}$ and R is defined as $R = \{(a + b) \mid a + b > 5\}$, $a, b \in A$ then $R = \phi$.

4. **Inverse relation :** A relation R defined from B to A is called inverse relation of R defined from A to B if

$$R^{-1} = \{(b, a) : b \in B \text{ and } a \in A \text{ and } (a, b) \in R\}.$$

For example : Consider relation

$$R = \{(1, 1), (1, 2), (1, 3), (3, 2)\}$$

then $$R^{-1} = \{(1, 1), (2, 1), (3, 1), (2, 3)\}$$

5. **Complement of a relation :** Let relation R is defined from A to B, then complement R is set of ordered pairs $\{(a, b) : (a, b) \notin R\}$. It is also called complementary relation.

   For example :

   Let $A = \{1, 2, 3\}$ $B = \{4, 5\}$

   Then $A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$

   Let R be defined as

   $$R = \{(1, 4), (3, 4), (3, 5)\}$$

   Then $$R^c = \bar{R} = \{(1, 5), (2, 4), (2, 5)\}$$

Que 4.8. **Give properties of relation.**

Answer

Properties of relation are :

1. **Reflexive relation :** A binary relation R on set A is said to be reflexive if every element of set A is related to itself.

   i.e., $\forall\, a \in A, (a, a) \in R$ or aRa.

   For example : Let $R = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\}$ be a relation defined on set $A = \{1, 2, 3\}$. As $(1, 1) \in R$, $(2, 2) \in R$ and $(3, 3) \in R$. Therefore, R is reflexive relation.
2. **Irreflexive relation :** A binary relation R defined on set A is said to irreflexive if there is no element in A which is related to itself *i.e.*, $\forall\, a \in A$ such that $(a, a) \notin R$.

   For example : Let $R = \{(1, 2), (2, 1), (3, 1)\}$ be a relation defined on set $A = \{1, 2, 3\}$. As $(1, 1) \notin R$, $(2, 2) \notin R$ and $(3, 3) \notin R$. Therefore, R is irreflexive relation.
3. **Non-reflexive relation :** A relation R defined on set A is said to be non-reflexive if it is neither reflexive nor irreflexive *i.e.*, some elements are related to itself but there exist at least one element not related to itself.
4. **Symmetric relation :** A binary relation on a set A is said to be symmetric if $(a, b) \in R \Rightarrow (b, a) \in R$.
5. **Asymmetric relation :** A binary relation on a set A is said to be asymmetric if $(a, b) \in R \Rightarrow (b, a) \notin R$.
6. **Antisymmetric relation :** A binary relation R defined on a set A is said to antisymmetric relation if $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ *i.e.*, aRb and $bRa \Rightarrow a = b$ for $a, b \in R$.
7. **Transitive relation :** A binary relation R on a set A is transitive whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

   i.e., aRb and $bRc \Rightarrow aRc$.

Que 4.9. **Write short notes on :**

a. **Equivalence relation** **b.** **Composition of relation**

Answer

a. Equivalence relation :

1. A relation R on a set A is said to be equivalence relation if it is reflexive, symmetric and transitive.
2. The two elements a and b related by an equivalence relation are called equivalent.
3. So, a relation R is called equivalence relation on set A if it satisfies following three properties :
   - i. $(a, a) \in R \ \forall \ a \in A$ (Reflexive)
   - ii. $(a, b) \in R \Rightarrow (b, a) \in R$ (Symmetric)
   - iii. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ (Transitive)

b. Composition of relation :

1. Let R be a relation from a set A to B and S be a relation from set B to C then composition of R and S is a relation consisting of ordered pair (a, c) where $a \in A$ and $c \in C$ provided that there exist $b \in B$ such that $(a, b) \in R \subseteq A \times B$ and $(b, c) \in S \subseteq B \times C$. It is denoted by $R \circ S$.
2. Symbolically, $R \circ S = \{(a, c) \mid \exists \ b \in B$ such that $(a, b) \in R$ and $(b, c) \in S\}$

Que 4.10. **Show that $R = \{(a, b) \mid a \equiv b \pmod{m}\}$ is an equivalence relation on Z. Show that if $x_1 \equiv y_1$ and $x_2 \equiv y_2$ then $(x_1 + x_2) \equiv (y_1 + y_2)$.**

AKTU 2014-15, Marks 05

Answer

$R = \{(a, b) \mid a \equiv b \pmod{m}\}$

For an equivalence relation it has to be :

Reflexive : For reflexive $\forall \ a \in Z$ we have $(a, a) \in R$ *i.e.*,

$a \equiv a \pmod{m}$

$\Rightarrow \quad a - a$ is divisible by m *i.e.*, 0 is divisible by m

Therefore aRa, $\forall \ a \in Z$, it is reflexive.

Symmetric : Let $(a, b) \in Z$ and we have

$(a, b) \in R$ *i.e.*, $a \equiv b \pmod{m}$

$\Rightarrow \quad a - b$ is divisible by m

$\Rightarrow \quad a - b = km$, k is an integer

$\Rightarrow \quad (b - a) = (-k)\, m$

$\Rightarrow \quad (b - a) = p\, m$, p is also an integer

$\Rightarrow \quad b - a$ is also divisible by m

$\Rightarrow \quad b \equiv a \pmod{m} \Rightarrow (b, a) \in R$

It is symmetric.

Transitive : Let $(a, b) \in R$ and $(b, c) \in R$ then

$(a, b) \in R \Rightarrow a - b$ is divisible by m

$\Rightarrow \quad a - b = tm$, t is an integer ...(4.10.1)

$(b, c) \in R \Rightarrow b - c$ is divisible by m

$\Rightarrow \quad b - c = sm$, s is an integer ...(4.10.2)

From eq. (4.10.1) and (4.10.2)

$$a - b + b - c = (t + s)\,m$$

$$a - c = lm, \ l \text{ is also an integer}$$

$a - c$ is divisible by m

$$a \equiv c \pmod{m}, \text{ yes it is transitive.}$$

Hence, R is an equivalence relation.

To show : $(x_1 + x_2) \equiv (y_1 + y_2)$:

It is given $x_1 \equiv y_1$ and $x_2 \equiv y_2$

i.e., $x_1 - y_1$ divisible by m

$x_2 - y_2$ divisible by m

Adding above equation :

$(x_1 - y_1) + (x_2 - y_2)$ is divisible by m

$\Rightarrow \quad (x_1 + x_2) - (y_1 + y_2)$ is divisible by m

i.e., $(x_1 + x_2) \equiv (y_1 + y_2)$

Que 4.11. **Let R be binary relation on the set of all strings of 0's and 1's such that $R = \{(a, b) \mid a$ and b are strings that have the same number of 0's$\}$. Is R is an equivalence relation and a partial ordering relation ?**

AKTU 2014-15, Marks 05

Answer

For equivalence relation :

Reflexive : $a\,R\,a \Rightarrow (a, a) \in R \ \forall a \in R,$ where a is a string of 0's and 1's.
Always a is related to a because both a has same number of 0's.
It is reflexive.

Symmetric : Let $(a, b) \in R$, then a and b both have same number of 0's which indicates that again both b and a will also have same number of zeros. Hence $(b, a) \in R$. It is symmetric.

Transitive : Let $(a, b) \in R$, $(b, c) \in R$

$(a, b) \in R \Rightarrow a$ and b have same number of zeros.

$(b, c) \in R \Rightarrow b$ and c have same number of zeros.

Therefore a and c also have same number of zeros, hence $(a, c) \in R$.
It is transitive.

$\therefore \quad R$ is an equivalence relation.

For partial order, it has to be reflexive, antisymmetric and transitive. Since, symmetricity and antisymmetricity cannot hold together. Therefore, it is not partial order relation.

Que 4.12. **Let A $\{1, 2, 3, \ldots\ldots\ldots, 13\}$. Consider the equivalence relation on $A \times A$ defined by $(a, b)\ R\ (c, d)$ if $a + d = b + c$. Find equivalence classes of $(5, 8)$.**

AKTU 2014-15, Marks 05

Answer

$$A = \{1, 2, 3, \ldots., 13\}$$

$$
\begin{aligned}
[(5, 8)] &= [(a, b) : (a, b)\ R\ (5, 8), (a, b) \in A \times A] \\
&= [(a, b) : a + 8 = b + 5] \\
&= [(a, b) : a + 3 = b] \\
[5, 8] &= \{(1, 4), (2, 5), (3, 6), (4, 7) \\
&\quad (5, 8), (6, 9), (7, 10), (8, 11) \\
&\quad (9, 12), (10, 13)\}
\end{aligned}
$$

Que 4.13. **The following relation on $A = \{1, 2, 3, 4\}$. Determine whether the following :**

a. $R = \{(1, 3), (3, 1), (1, 1), (1, 2), (3, 3), (4, 4)\}$

b. $R = A \times A$

Is an equivalence relation or not ? **AKTU 2015-16, Marks 10**

Answer

a. $R = \{(1, 3), (3, 1), (1, 1), (1, 2), (3, 3), (4, 4)\}$

Reflexive : $(a, a) \in R\ \forall\ a \in A$

$\because$ $(1, 1) \in R$, $(2, 2) \notin R$

$\therefore$ R is not reflexive.

Symmetric : Let $(a, b) \in R$ then $(b, a) \in R$.

$\because$ $(1, 3) \in R$ so $(3, 1) \in R$

$\because$ $(1, 2) \in R$ but $(2, 1) \notin R$

$\therefore$ R is not symmetric.

Transitive : Let $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

$\because$ $(1, 3) \in R$ and $(3, 1) \in R$ so $(1, 1) \in R$

$\because$ $(2, 1) \in R$ and $(1, 3) \in R$ but $(2, 3) \notin R$

$\therefore$ R is not transitive.

Since, R is not reflexive, not symmetric, and not transitive. So, R is not an equivalence relation.

b. $R = A \times A$

Since, $A \times A$ contains all possible elements of set A. So, R is reflexive, symmetric and transitive. Hence, R is an equivalence relation.

Que 4.14. **Let n be a positive integer and S a set of strings. Suppose that R_n is the relation on S such that sR_nt if and only if $s = t$, or both s and t have at least n characters and first n characters of s and t are the same. That is, a string of fewer than n characters is related only to itself; a string s with at least n characters is related to a string t if and only if t has at least n characters and t beings with the n characters at the start of s.** **AKTU 2018-19, Marks 07**

Answer

We have to show that the relation R_n is reflexive, symmetric, and transitive.

1. **Reflexive :** The relation R_n is reflexive because $s = s$, so that sR_ns whenever s is a string in S.

2. **Symmetric :** If $sR_n\ t$, then either $s = t$ or s and t are both at least n characters long that begin with the same n characters. This means that $tR_n\ s$. We conclude that R_n is symmetric.
3. **Transitive :** Now suppose that $sR_n\ t$ and $tR_n\ u$. Then either $s = t$ or s and t are at least n characters long and s and t begin with the same n characters, and either $t = u$ or t and u are at least n characters long and t and u begin with the same n characters. From this, we can deduce that either $s = u$ or both s and u are n characters long and s and u begin with the same n characters, *i.e.*, $sR_n\ u$. Consequently, R_n is transitive.

Que 4.15. **Let $X = \{1, 2, 3, ..., 7\}$ and $R = \{(x, y) \mid (x - y)$ is divisible by 3$\}$. Is R equivalence relation ? Draw the digraph of R.**

AKTU 2017-18, Marks 07

Answer

Given : $X = \{1, 2, 3, 4, 5, 6, 7\}$, and
$R = \{(x, y) : (x - y) \text{ is divisible by } 3\}$

Then R is an equivalence relation if

i. **Reflexive :** $\forall\ x \in X \Rightarrow (x - x)$ is divisible by 3

So, $(x, x) \in X\ \forall\ x \in X$ or, R is reflexive.

ii. **Symmetric :** Let $x, y \in X$ and $(x, y) \in R$
$\Rightarrow (x - y)$ is divisible by 3 $\Rightarrow (x - y) = 3n_1$, ($n_1$ being an integer)
$\Rightarrow (y - x) = -3n_2 = 3n_2$, ($n_2$ is also an integer)
So, $y - x$ is divisible by 3 or R is symmetric.

iii. **Transitive :** Let $x, y, z \in X$ and $(x, y) \in R$, $(y, z) \in R$
Then $x - y = 3n_1$, $y - z = 3n_2$, (n_1, n_2 being integers)
$\Rightarrow x - z = 3(n_1 + n_2)$, ($n_1 + n_2 = n_3$ be any integer)
So, $(x - z)$ is also divisible by 3 or $(x, z) \in R$
So, R is transitive.
Hence, R is an equivalence relation.

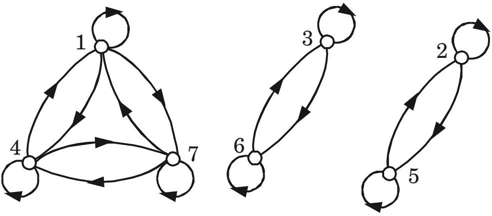


Fig. 4.15.1. Diagraph of R.

PART-3

Function.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.16. **Define function. Explain different types of function.**

Answer

Function :
1. Let X and Y be any two non-empty sets. A function from X to Y is a rule that assigns to each element $x \in X$ a unique element $y \in Y$.
2. If f is a function from X to Y we write $f : X \to Y$.
3. Functions are denoted by f, g, h, i etc.
4. It is also called mapping or transformation or correspondence.

Domain and co-domain of a function : Let f be a function from X to Y. Then set X is called domain of function f and Y is called co-domain of function f.

Range of function : The range of f is set of all images of elements of X.

i.e., Range $(f) = \{y : y \in Y \text{ and } y = f(x) \ \forall \ x \in X\}$. Also Range $(f) \subseteq Y$

Different types of function are :

1. **One-to-one function (Injective function or injection) :** Let $f : X \to Y$ then f is called one-to-one function if for distinct elements of X there are distinct image in Y *i.e.*, f is one-to-one iff

$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2 \ \forall \ x_1, x_2, \in X$$

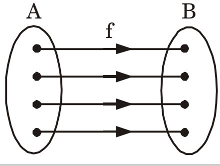


Fig. 4.16.1. One-to-one.

2. **Onto function (Surjection or surjective function) :** Let $f : X \to Y$ then f is called onto function iff for every element $y \in Y$ there is an element $x \in X$ with $f(x) = y$ or f is onto if Range $(f) = Y$.

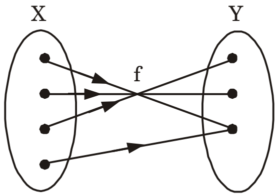


Fig. 4.16.2. Onto.

3. **One-to-one onto function (Bijective function or bijection) :** A function which is both one-to-one and onto is called one-to-one onto function or bijective function.

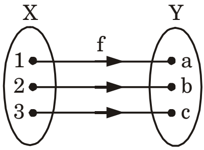


Fig. 4.16.3. One-to-one onto.

4. **Many one function :** A function which is not one-to-one is called many one function *i.e.*, two or more elements in domain have same image in co-domain *i.e.*,
   If $f : X \rightarrow Y$ then $f(x_1) = f(x_2) \Rightarrow x_1 \neq x_2$.

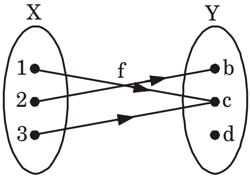


Fig. 4.16.4. Many one.

5. **Identity function :** Let $f : X \rightarrow X$ then f is called identity function if $f(a) = a \ \forall \ a \in X$ *i.e.*, every element of X is image of itself. It is denoted by I.
6. **Inverse function (Invertible function) :** Let f be a bijective function from X to Y. The inverse function of f is the function that assigns an element $y \in Y$, a unique element $x \in X$ such that $f(x) = y$ and inverse of f denoted by f^{-1}. Therefore if $f(x) = y$ implies $f^{-1}(y) = x$.

Que 4.17. **Determine whether each of these functions is a bijective from R to R.**

a. $f(x) = x^2 + 1$ **b.** $f(x) = x^3$

c. $f(x) = (x^2 + 1)/(x^2 + 2)$

AKTU 2015-16, Marks 15

Answer

a. $f(x) = x^2 + 1$

Let $x_1, x_2 \in R$ such that

$$f(x_1) = f(x_2)$$
$$x_1^2 + 1 = x_2^2 + 1$$
$$x_1^2 = x_2^2$$
$$x_1 = \pm x_2$$

Therefore, if $x_2 = 1$ then $x_1 = \pm 1$

So, f is not one-to-one.

Hence, f is not bijective.

b. Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$$x_1^3 = x_2^3$$
$$x_1 = x_2$$

$\therefore$ f is one-to-one.

Let $y \in R$ such that

$$y = x^3$$
$$x = (y)^{1/3}$$

For $\forall\ y \in R\ \exists$ a unique $x \in R$ such that $y = f(x)$

$\therefore$ f is onto.

Hence, f is bijective.

c. Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$$\Rightarrow \quad \frac{x_1^2 + 1}{x_1^2 + 2} = \frac{x_2^2 + 1}{x_2^2 + 2}$$

If $x_1 = 1, x_2 = -1$ then $f(x_1) = f(x_2)$

but $x_1 \neq x_2$

$\therefore f$ is not one-to-one.

Hence, f is not bijective.

Que 4.18. **If $f: A \rightarrow B$, $g: B \rightarrow C$ are invertible functions, then show that $g \circ f: A \rightarrow C$ is invertible and $(g\ o\ f)^{-1} = f^{-1} \circ g^{-1}$.**

AKTU 2014-15, Marks 05

Answer

If $f: A \rightarrow B$ and $g : B \rightarrow C$ be one-to-one onto functions, then gof is also one-onto and $(gof)^{-1} = f^{-1}og^{-1}$

Proof. Since f is one-to-one, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for $x_1, x_2 \in R$

Again since g is one-to-one, $g(y_1) = g(y_2) \Rightarrow y_1 = y_2$ for $y_1, y_2 \in R$

Now g o f is one-to-one, since $(gof)(x_1) = (gof)(x_2) \Rightarrow g\ [f(x_1)] = g\ [f(x_2)]$

$\Rightarrow \quad f(x_1) = f(x_2)$ [g is one-to-one]

$\Rightarrow \quad x_1 = x_2$ [f is one-to-one]

Since g is onto, for $z \in C$, there exists $y \in B$ such that $g(y) = z$. Also f being onto there exists $x \in A$ such that $f(x) = y$. Hence $z = g(y)$

$$= g\ [f(x)] = (gof)(x)$$

This shows that every element $z \in C$ has pre-image under gof. So, gof is onto.

Thus, gof is one-to-one onto function and hence $(gof)^{-1}$ exists.

By the definition of the composite functions, $gof : A \rightarrow C$. So, $(gof)^{-1} : C \rightarrow A$.

Also $g^{-1} : C \rightarrow B$ and $f^{-1} : B \rightarrow A$.

Then by the definition of composite functions, $f^{-1}og^{-1} : C \rightarrow A$.

Therefore, the domain of $(gof)^{-1}$ = the domain of $f^{-1}og^{-1}$.

Now
$$\begin{aligned}(gof)^{-1}(z) = x &\Leftrightarrow (gof)(x) = z\\ &\Leftrightarrow g(f(x)) = z\\ &\Leftrightarrow g(y) = z \text{ where } y = f(x)\\ &\Leftrightarrow y = g^{-1}(z)\\ &\Leftrightarrow f^{-1}(y) = f^{-1}(g^{-1}(z)) = (f^{-1}og^{-1})(z)\\ &\Leftrightarrow x = (f^{-1}og^{-1})(z) \qquad [f^{-1}(y) = x]\end{aligned}$$

Thus, $(gof)^{-1}(z) = (f^{-1}og^{-1})(z)$.

So, $(gof)^{-1} = f^{-1}og^{-1}$.

PART-4

Methods of Proof, Mathematical Induction, Strong Mathematical Induction.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.19. What are the different methods of proof ?

Answer

Different methods of proof are :

1. **Direct proof :** In this method, we will assume that hypothesis p is true from the implication $p \to q$. We can proof that q is true by using rules of interference or some other theorem. This will show $p \to q$ is true. The combination of p is true and q is false will never occur.
2. **Indirect methods :** These are of five types as follows :
   a. Proof by contrapositive.
   b. Proof by contradiction.
   c. Proof by exhaustive cases.
   d. Proof by cases.
   e. Proof by counter example.

Que 4.20. Explain proof by contradiction with example.

Answer

In this method, we assume that q is false *i.e.*, $\sim q$ is true. Then by using rules of inference and other theorems we will show the given statement is true as well as false *i.e.*, we will reach at a contradiction. Therefore, q must be true.

For example : Prove that $\sqrt{3}$ is irrational.

Let $\sqrt{3}$ is a rational number. Then $\exists$ positive prime integer p and q such that

$$\sqrt{3} = \frac{p}{q}$$

$$\Rightarrow \quad 3 = \frac{p^2}{q^2} \quad \Rightarrow \quad p^2 = 3q^2 \qquad \text{...(4.20.1)}$$

$$\Rightarrow \quad 3/p^2 \text{ (3 divides } p^2)$$

$$\Rightarrow \quad 3/p \qquad (\because \quad p \text{ is an integer})$$

$$\therefore \quad p = 3x \text{ for some } x \in Z$$

$$\Rightarrow \quad p^2 = 9x^2 \qquad \text{...(4.20.2)}$$

From eq. (4.20.1) and (4.20.2) we get,

$$9x^2 = 3q^2$$

$$\Rightarrow \quad q^2 = 3x^2$$

$$\Rightarrow \quad 3/q^2 \text{ (3 divides } q^2)$$

$$\Rightarrow \quad 3/q$$

$\therefore$ 3 divides p and q which is contradiction to our assumption that p and q are prime. Therefore, $\sqrt{3}$ is irrational.

Que 4.21. **Write short note on the following with example :**

i. Proof by contrapositive.
ii. Proof by exhaustive cases.
iii. Proof by cases.

Answer

i. **Proof by contrapositive :** In this we can prove $p \rightarrow q$ is true by showing $\sim q \rightarrow \sim p$ is true. It is also called proof by contraposition.

Example : Using method of contraposition if n is integer and $3n + 2$ is even then n is even.

Let p : $3n + 2$ is even

q : n is even

Let $\sim q$ is true *i.e.*, n is odd

$$\Rightarrow \quad n = 2k + 1, \text{ where } k \in Z$$

Now, $$3n + 2 = 3(2k + 1) + 2 = 2(3k + 2) + 1$$

$$= 2m + 1 \text{ where } m = 3k + 2 \in Z$$

$\therefore$ $(3n + 2)$ is odd $\Rightarrow$ $\sim p$ is true

Hence $\sim q \rightarrow \sim p$ is true.

$\therefore$ By method of contraposition $p \rightarrow q$ is true.

ii. **Proof by exhaustive cases :** Some proofs proceed by exhausting all the possibilities. We will examine a relatively small number of examples to prove the theorem. Such proofs are called exhaustive proofs.

Example : Prove that $n^2 + 1 \geq 2^n$ where n is a positive integer and $1 \leq n \leq 4$.

We will verify the given inequality $n^2 + 1 \geq 2^n$ for $n = 1, 2, 3, 4$.

For $n = 1$; $1 + 1 = 2$ which is true

For $n = 2$; $4 + 1 > 2^2$ which is true

For $n = 3$; $9 + 1 > 2^3$ which is true

For $n = 4$; $16 + 1 > 2^4$ which is true

In each of these four cases $n^2 + 1 \geq 2^n$ holds true. Therefore, by method of exhaustive cases $n^2 + 1 \geq 2^n$, where n is the positive integer and $1 \leq n \leq 4$ is true.

iii. **Proof by cases :** In this method, we will cover up all the possible cases that we come across while proving the theorem.

Example : Prove that if x and y are real numbers, then max (x, y) min $(x, y) = x + y$.

Case I : If $x \leq y$, then max (x, y) + min $(x, y) = y + x = x + y$.

Case II : if $x \geq y$, then max (x, y) + min $(x, y) = x + y$.

Que 4.22. **Describe mathematical induction.**

Answer

1. Mathematical induction is a technique of proving a proposition over the positive integers.
2. It is the most basic method of proof used for proving statements having a general pattern.
3. A formal statement of principle of mathematical induction can be stated as follows :

Let $S(n)$ be statement that involve positive integer $n = 1, 2, ...$ then

Step I : Verify $S(1)$ is true. (Inductive base)

Step II : Assume that $S(k)$ is true for some arbitrary k. (Inductive hypothesis)

Step III : Verify $S(k + 1)$ is true using basis of inductive hypothesis. (Inductive step)

Que 4.23. **Explain principle of strong mathematical induction.**

Answer

1. According to principle of strong mathematical induction, we will use stronger hypothesis *i.e.*, instead of assuming only $S(k)$ is true, we will assume that $S(1), S(2), ..., S(k)$ are true.
2. A formal statement of principle of mathematical induction can be stated as follows :

Let $S(n)$ be statement involving positive integer $n = 1, 2, ...$ then

Step I : Verify $S(1)$ is true. (Inductive base)

Step II : Assume that $S(1), S(2), ..., S(k)$ is true for some arbitrary k. (Strong inductive hypothesis)

Step III : Verify show that $S(k + 1)$ is true using strong inductive hypothesis. (Inductive step)

Que 4.24. **Prove by induction :** $\boldsymbol{\dfrac{1}{1.2} + \dfrac{1}{2.3} + ... + \dfrac{1}{n\,(n+1)} = \dfrac{n}{(n+1)}}$.

AKTU 2016-17, Marks 10

Answer

Let the given statement be denoted by $S(n)$.

Step I : Inductive base : For $n = 1$

$$\frac{1}{1.2} = \frac{1}{1+1} = \frac{1}{2}$$

Hence, $S(1)$ is true.

Step II : Inductive hypothesis : Assume that $S(k)$ is true *i.e.*,

$$\frac{1}{1.2} + \frac{1}{2.3} + ... + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

Step III : Inductive step : We wish to show that the statement is true for $n = k + 1$ *i.e.*,

$$\frac{1}{1.2}+\frac{1}{2.3}+.....+\frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

$$\text{Now, } \frac{1}{1.2}+\frac{1}{2.3}+.....+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

Thus, $S(k + 1)$ is true whenever $S(k)$ is true. By principle of mathematical induction, $S(n)$ is true for all positive integer n.

Que 4.25. **Prove by the principle of mathematical induction, that the sum of finite number of terms of a geometric progression, $a + ar + ar^2 + ... ar^{n-1} = a(r^n - 1)/(r - 1)$ if $r \neq 1$.**

AKTU 2014-15, Marks 05

Answer

Basis : True for $n = 1$ *i.e.*,

$$\text{L.H.S} = a$$

$$\text{R.H.S} = \frac{a\,(r-1)}{r-1} = a$$

Therefore, L.H.S. = R.H.S.

Induction : Let it be true for $n = k$ *i.e.*,

$$a + ar + ar^2 + ... + ar^{k-1} = \frac{a\,(r^k-1)}{r-1} \quad ...(4.25.1)$$

Now we will show that it is true for $n = k + 1$ using eq. (4.25.1)

i.e., $a + ar + ar^2 +.... + ar^{k-1} + ar^k$

Using eq. (4.25.1), we get

$$\frac{a\,(r^k-1)}{r-1} + ar^k$$

$$= \frac{ar^k - a + ar^{k+1} - ar^k}{r-1} = \frac{a\,(r^{k+1}-1)}{r-1}$$

which is R.H.S. for $n = k + 1$, hence it is true for $n = k + 1$.

By mathematical induction, it is true for all n.

Que 4.26. **Prove that if n is a positive integer, then 133 divides $11^{n+1} + 12^{2n-1}$.**

AKTU 2018-19, Marks 07

Answer

We prove this by induction on n.

Base case : For $n = 1$, $11^{n+1} + 12^{2n-1} = 11^2 + 12^1 = 133$ which is divisible by 133.

Inductive step : Assume that the hypothesis holds for $n = k$, *i.e.*, $11^{k+1} + 12^{2k-1} = 133A$ for some integer A. Then for $n = k + 1$,

$$\begin{aligned} 11^{n+1} + 12^{2n-1} &= 11^{k+1+1} + 12^{2(k+1)-1} \\ &= 11^{k+2} + 12^{2k+1} \\ &= 11 * 11^{k+1} + 144 * 12^{2k-1} \\ &= 11 * 11^{k+1} + 11 * 12^{2k-1} + 133 * 12^{2k-1} \\ &= 11[11^{k+1} + 12^{2k-1}] + 133 * 12^{2k-1} \\ &= 11 * 133A + 133 * 12^{2k-1} \\ &= 133[11A + 12^{2k-1}] \end{aligned}$$

Thus if the hypothesis holds for $n = k$ it also holds for $n = k + 1$. Therefore, the statement given in the equation is true.

Que 4.27. **Prove by mathematical induction $n^4 - 4n^2$ is divisible by 3 for all $n >= 2$.**

AKTU 2017-18, Marks 07

Answer

Base case : If $n = 0$, then $n^4 - 4n^2 = 0$, which is divisible by 3.

Inductive hypothesis : For some $n \geq 0$, $n^4 - 4n^2$ is divisible by 3.

Inductive step : Assume the inductive hypothesis is true for n. We need to show that $(n + 1)^4 - 4(n + 1)^2$ is divisible by 3. By the inductive hypothesis, we know that $n^4 - 4n^2$ is divisible by 3.

Hence $(n + 1)^4 - 4(n + 1)^2$ is divisible by 3 if

$(n + 1)^4 - 4(n + 1)^2 - (n^4 - 4n^2)$ is divisible by 3.

Now $(n + 1)^4 - 4(n + 1)^2 - (n^4 - 4n^2)$

$= n^4 + 4n^3 + 6n^2 + 4n + 1 - 4n^2 - 8n - 4 - n^4 + 4n^2$

$= 4n^3 + 6n^2 - 4n - 3$,

which is divisible by 3 if $4n^3 - 4n$ is. Since $4n^3 - 4n = 4n(n + 1)(n - 1)$, we see that $4n^3 - 4n$ is always divisible by 3.

Going backwards, we conclude that $(n + 1)^4 - 4(n + 1)^2$ is divisible by 3, and that the inductive hypothesis holds for $n + 1$.

By the Principle of Mathematical Induction, $n^4 - 4n^2$ is divisible by 3, for all $n \in N$.

Que 4.28. **Prove that $n^3 + 2n$ is divisible by 3 using principle of mathematical induction, where n is natural number.**

AKTU 2015-16, Marks 10

Answer

Let $S(n) : n^3 + 2n$ is divisible by 3.

Step I : Inductive base : For $n = 1$

$(1)^3 + 2.1 = 3$ which is divisible by 3

Thus, $S(1)$ is true.

Step II : Inductive hypothesis : Let $S(k)$ is true *i.e.*, $k^3 + 2k$ is divisible by 3 holds true.

or $k^3 + 2k = 3s$ for $s \in N$

Step III : Inductive step : We have to show that $S(k + 1)$ is true

i.e., $(k + 1)^3 + 2(k + 1)$ is divisible by 3

Consider $(k + 1)^3 + 2(k + 1)$

$$= k^3 + 1 + 3k^2 + 3k + 2k + 2$$
$$= (k^3 + 2k) + 3(k^2 + k + 1)$$
$$= 3s + 3l \text{ where } l = k^2 + k + 1 \in N$$
$$= 3(s + l)$$

Therefore, $S(k + 1)$ is true

Hence by principle of mathematical induction $S(n)$ is true for all $n \in N$.

Que 4.29. **Prove by mathematical induction**

$$\mathbf{3 + 33 + 333 + \dots\dots\dots\dots + 3333 = (10^{n+1} - 9n - 10)/27}$$

AKTU 2017-18, Marks 07

Answer

$3 + 33 + 333 + \dots\dots\dots\dots + 3333 = (10^{n+1} - 9n - 10)/27$

Let given statement be denoted by $S(n)$

1. **Inductive base :** For $n = 1$

$$3 = \frac{(10^2 - 9(1) - 10)}{27}, \ 3 = \frac{100 - 19}{27} = \frac{81}{27} = 3$$

$3 = 3$. Hence $S(1)$ is tree.

2. **Inductive hypothesis :** Assume that $S(k)$ is true *i.e.*,

   $3 + 33 + 333 + \dots\dots\dots\dots + 3333 = (10^{k+1} - 9k - 10)/27$

3. **Inductive steps :** We have to show that $S(k + 1)$ is also true *i.e.*,

   $3 + 33 + 333 + \dots\dots\dots\dots \ (10^{k+2} - 9^{(k+1)} - 10)/27$

   Now, $3 + 33 + \dots\dots\dots\dots + 33 \dots\dots\dots\dots 3$

   $= 3 + 33 + 333 + \dots\dots\dots\dots + 3 \dots\dots\dots\dots 3$

   $= (10^{k+1} - 9k - 10)/27 + 3(10^{k+1} - 1)/9$

   $= (10^{k+1} + 9k - 10 + 9.10^{k+1} - 9)/27$

   $= (10^{k+1} + 9.10^{k+1} - 9k - 8 - 10)/27 = (10^{k+2} - 9(k + 1) - 10)/27$

   Thus $S(k + 1)$ is true whenever $S(k)$ is true. By the principle of mathematical induction $S(n)$ true for all positive integer n.

PART-5

Discrete Numeric Function and Generating Functions, Recurrence Relation and their Solution.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.30. **What do you mean generating function ? Solve the recurrence relation :**
$a_n = 2\,a_{n-1} - a_{n-2}$, $n \geq 2$ given $a_0 = 3$, $a_1 = -2$
using generating function.

Answer

Generating function : The generating function for the sequence $a_0, a_1, \ldots a_k, \ldots$ of real numbers is infinite series given by

$$G(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_kx^k + \ldots\ldots = \sum_{k=0}^{\infty} a_k x^k$$

x is considered just a symbol called indeterminate and is replaced by numbers belonging to same domain.
The given recurrence relation is,

$$a_n = 2a_{n-1} - a_{n-2},\ n \geq 2 \qquad \ldots(4.30.1)$$

Multiply by x^n and take summation from $n = 2$ to ∞, we get

$$\sum_{n=2}^{\infty} a_n x^n = 2\sum_{n=2}^{\infty} a_{n-1}\, x^n - \sum_{n=2}^{\infty} a_{n-2}\ x^n \qquad \ldots(4.30.2)$$

We know, $$G(x) = \sum_{n=0}^{\infty} a_n\, x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \qquad \ldots(4.30.3)$$

From eq. (4.30.2), we have

$$(a_2x^2 + a_3\,x^3 + \ldots) = 2\,(a_1x^2 + a_2\,x^3 + \ldots) - (a_0\,x^2 + a_1x^3 + \ldots.)$$
$$(a_2x^2 + a_3\,x^3 + \ldots) = 2x\,(a_1x + a_2\,x^2 + \ldots) - x^2\,(a_0 + a_1\,x + \ldots.)$$

Using eq. (4.30.3), we get

$$G(x) - a_0 - a_1x = 2x\,(G(x) - a_0) - x^2\,G(x)$$
$$G(x) - 3 + 2x = 2x\,(G(x) - 3) - x^2\,G(x)$$
$$G(x)[1 - 2x + x^2] = 3 - 8x$$
$$G(x) = \frac{3 - 8x}{x^2 - 2x + 1} = \frac{3 - 8x}{(x - 1)^2} = \frac{3 - 8x}{(1 - x)^2} = \frac{3}{(1 - x)^2} - \frac{8x}{(1 - x)^2}$$

$\therefore$ $$a_n = 3(n + 1) - 8n = 3 - 5n$$

Que 4.31. **Solve the recurrence relation $y_{n+2} - 5y_{n+1} + 6y_n = 5^n$ subject to the condition $y_0 = 0$, $y_1 = 2$.** **AKTU 2016-17, Marks 10**

Answer

Let $G(t) = \sum_{n=0}^{\infty} a_n t^n$ be generating function of sequence $\{a_n\}$.

Multiplying given equation by t^n and summing from $n = 0$ to ∞, we have

$$\sum_{n=0}^{\infty} a_{n+2}\, t^n - 5 \sum_{n=0}^{\infty} a_{n+1}\, t^n + 6 \sum_{n=0}^{\infty} a_n\, t^n = \sum_{n=0}^{\infty} 5^n\, t^n$$

$$\frac{G(t) - a_0 - a_1 t}{t^2} - 5\left[\frac{G(t) - a_0}{t}\right] + 6\,G(t) = \frac{1}{1-5t}$$

Put $a_0 = 0$ and $a_1 = 2$

$$G(t) - 2t - 5t\,G(t) + 6t^2\,G(t) = \frac{t^2}{1-5t}$$

$$G(t) - 5t\,G(t) + 6t^2\,G(t) = \frac{t^2}{1-5t} + 2t$$

$$G(t)\,(1 - 5t + 6t^2) = \frac{t^2}{1-5t} + 2t$$

$$(6t^2 - 5t + 1)\,G(t) = \frac{t^2}{1-5t} + 2t$$

$$G(t) = \frac{t^2}{(1-5t)(3t-1)(2t-1)} + \frac{2t}{(3t-1)(2t-1)}$$

$$= \frac{t^2}{(1-5t)(1-3t)(1-2t)} + \frac{2t}{(1-3t)(1-2t)}$$

Let $$\frac{t^2}{(1-5t)(1-3t)(1-2t)} = \frac{A}{(1-5t)} + \frac{B}{(1-3t)} + \frac{C}{(1-2t)}$$

$$A = (1-5t)\left.\frac{t^2}{(1-5t)(1-3t)(1-2t)}\right|_{t=1/5}$$

$$= \left.\frac{t^2}{(1-3t)(1-2t)}\right|_{t=1/5}$$

$$= \frac{1/25}{(1-3/5)(1-2/5)} = \frac{1}{6}$$

$$B = (1-3t)\left.\frac{t^2}{(1-5t)(1-3t)(1-2t)}\right|_{t=1/3}$$

$$= \left.\frac{t^2}{(1-5t)(1-2t)}\right|_{t=1/3} = \frac{1/9}{\left(\frac{3-5}{3}\right)\left(\frac{3-2}{3}\right)}$$

$$= -\frac{1}{2}$$

$$C = (1-2t)\left.\frac{t^2}{(1-5t)(1-3t)(1-2t)}\right|_{t=1/2}$$

$$= \left.\frac{t^2}{(1-5t)(1-3t)}\right|_{t=1/2} = \frac{1/4}{\frac{(2-5)}{2} \times \frac{(2-3)}{2}}$$

$$= \frac{1}{3}$$

Again, $$\frac{2t}{(1-3t)(1-2t)} = \frac{D}{(1-3t)} + \frac{E}{(1-2t)}$$

$$D = (1-3t)\left.\frac{2t}{(1-3t)(1-2t)}\right|_{t=1/3}$$

$$= \left.\frac{2t}{(1-2t)}\right|_{t=1/3} = \frac{2/3}{\frac{(3-2)}{3}} = 2$$

$$E = (1-2t)\left.\frac{2t}{(1-3t)(1-2t)}\right|_{t=1/2}$$

$$= \left.\frac{2t}{(1-3t)}\right|_{t=1/2} = \frac{2/2}{\frac{2-3}{2}} = -2$$

$$G(t) = \frac{1/6}{(1-5t)} - \frac{1/2}{(1-3t)} + \frac{1/3}{(1-2t)} + \frac{2}{(1-3t)} - \frac{2}{(1-2t)}$$

$$= \frac{1/6}{1-5t} + \frac{3/2}{(1-3t)} - \frac{5/3}{1-2t}$$

$$\sum_{n=0}^{\infty} a_n t^n = \frac{1}{6}\sum_{n=0}^{\infty}(5t)^n + \frac{3}{2}\sum_{n=0}^{\infty}(3t)^n - \frac{5}{3}\sum_{n=0}^{\infty}(2t)^n$$

$\therefore$ $$a_n = \frac{1}{6}(5)^n + \frac{3}{2}(3)^n - \frac{5}{3}(2)^n$$

Que 4.32. **Solve the recurrence relation by the method of generating function :**
$a_r - 7a_{r-1} + 10a_{r-2} = 0, r \geq 2$. Given $a_0 = 3$ and $a_1 = 3$.

AKTU 2014-15, Marks 10

OR

Solve the recurrence relation using generating function :

$a_n - 7a_{n-1} + 10a_{n-2} = 0$ with $a_0 = 3, a_1 = 3$.

AKTU 2015-16, Marks 10

Answer

$a_r - 7a_{r-1} + 10\, a_{r-2} = 0, r \geq 2$

Multiply by x^r and take sum from 2 to ∞.

$$\sum_{r=2}^{\infty} a_r\, x^r - 7 \sum_{r=2}^{\infty} a_{r-1}\, x^r + 10 \sum_{r=2}^{\infty} a_{r-2}\, x^r = 0$$

$$(a_2\, x^2 + a_3\, x^3 + a_4\, x^4 + ...) - 7\,(a_1\, x^2 + a_2\, x^3 + ...)$$
$$+ 10\,(a_0\, x^2 + a_1\, x^3 + ...) = 0$$

We know that $G(x) = \sum_{r=0}^{\infty} a_r\, x^r = a_0 + a_1\, x + ...$

$$G(x) - a_0 - a_1\, x - 7\, x\,(G(x) - a_0) + 10\, x^2\, G(x) = 0$$
$$G(x)\,[1 - 7\,x + 10\,x^2] = a_0 + a_1\, x - 7\, a_0\, x$$
$$= 3 + 3x - 21\, x = 3 - 18\, x$$
$$G(x) = \frac{3 - 18x}{10\,x^2 - 7x + 1} = \frac{3 - 18x}{10\,x^2 - 5x - 2x + 1}$$
$$= \frac{3 - 18x}{5x\,(2\,x - 1) - 1\,(2\,x - 1)} = \frac{3 - 18x}{(5x - 1)(2\,x - 1)}$$

Now, $\dfrac{3 - 18x}{(5x - 1)\,(2x - 1)} = \dfrac{A}{5\,x - 1} + \dfrac{B}{2\,x - 1}$

$$3 - 18\,x = A\,(2\,x - 1) + B\,(5\,x - 1)$$

put $x = \dfrac{1}{2}$

$$3 - 9 = B\left(\frac{5}{2} - 1\right) \Rightarrow -6 = \frac{3}{2}\,B \Rightarrow B = -4$$

put $x = \dfrac{1}{5}$

$$3 - \frac{18}{5} = A\left(\frac{2}{5} - 1\right) \Rightarrow -\frac{3}{5} = -\frac{3}{5}A = 1 \Rightarrow \text{A} = 1$$

$\therefore$ $G(x) = \dfrac{1}{5\,x - 1} - \dfrac{4}{2x - 1} = \dfrac{4}{1 - 2x} - \dfrac{1}{1 - 5x}$

$\therefore$ $a^r = 4.2^r - 5^r$

Que 4.33. **Solve the recurrence relation $a_{r+2} - 5a_{r+1} + 6\, a_r = (r + 1)^2$.**

AKTU 2014-15, Marks 10

Answer

$a_{r+2} - 5\, a_{r+1} + 6\, a_r = (r + 1)^2 = r^2 + 2r + 1$ ...(4.33.1)

Now the characteristic equation is : $x^2 - 5\, x + 6 = 0$

$$(x - 3)\,(x - 2) = 0 \Rightarrow x = 3,\, 2$$

The homogeneous solution is : $a_r^{(h)} = C_1 2^r + C_2\, 3^r$

Let the particular solution be : $a_r^{(p)} = A_0 + A_1 r + A_2 r^2$

From eq. (4.33.1)

$$A_0 + A_1\,(r + 2) + A_2\,(r + 2)^2 - 5\,\{A_0 + A_1\,(r + 1)\} + A_2\,(r + 1)^2\}$$
$$+ 6A_0 + 6A_1 r + 6A_2\, r^2$$

$$= r^2 + 2r + 1$$

$$(A_0 + 2A_1 + 4A_2 - 5A_0 - 5A_1 - 5A_2 + 6A_0) + r(A_1 + 4A_2 - 5A_1 - 10A_2 + 6A_1) + r^2(A_2 - 5A_2 + 6A_2) = r^2 + 2r + 1$$

Comparing both sides, we get

$$2A_0 - 3A_1 - A_2 = 1 \quad ...(4.33.2)$$

$$2A_1 - 6A_2 = 2 \quad ...(4.33.3)$$

$$2A_2 = 1 \Rightarrow A_2 = 1/2$$

From eq. (4.33.3), $2A_1 - 3 = 2, \quad A_1 = \dfrac{5}{2}$

From eq. (4.33.2)

$$2A_0 - \frac{15}{2} - \frac{1}{2} = 1$$

$$2A_0 - 8 = 1 \Rightarrow A_0 = \frac{9}{2}$$

$$\therefore \quad a_r^{(p)} = \frac{9}{2} + \frac{5}{2}r + \frac{r^2}{2}$$

The final solution is, $\quad a_r = a_r^{(h)} + a_r^{(p)} = C_1\, 2^r + C_2\, 3^r + \dfrac{9}{2} + \dfrac{5}{2}r + \dfrac{r^2}{2}$

Que 4.34. **Solve $a_r - 6a_{r-1} + 8a_{r-2} = r.4^r$, given $a_0 = 8$, and $a_1 = 1$.**

AKTU 2017-18, Marks 07

Answer

$a_r - 6a_{r-1} + 8a_{r-2} = r4^r$

The characteristic equation is, $x^2 - 6x + 8 = 0, x^2 - 2x - 4x + 8 = 0$

$$(x - 2)(x - 4) = 0, x = 2, 4$$

The solution of the associated non-homogeneous recurrence relation is,

$$a_r^{(h)} = B_1(2)^r + B_2(4)^r \quad ...(4.34.1)$$

Let particular solution of given equation is, $a_r^{(p)} = r^2(A_0 + A_1 r)4^r$

Substituting in the given equation, we get

$$\Rightarrow \quad r^2(A_0 + A_1 r)4^r - 6(r-1)^2 (A_0 + A_1(r-1))4^{r-1} + 8(r-2)^2 (A_0 + A_1(r-2)4^{r-2} = r4^r$$

$$\Rightarrow r^2 A_0 + A_1 r^3 - \frac{6}{4}\,[(A_0 r^2 - 2A_0 r + A_0) + (A_1 r^3 - A_1 - 3A_1 r^2 + 3A_1 r)^2]$$

$$+ \frac{8}{4^2}\,[(A_0 r^2 - 4rA_0 + 4A_0) + (A_1 r^3 - 8A_1 - 6A_1 r^{\backslash 2} + 12A_1 r)] = r$$

$$\Rightarrow \quad rA_0 + A_1 r^3 - \frac{3}{2}A_0 r^2 + 3A_0 r - \frac{3}{2}A_0 - \frac{3}{2}A_1 r^3 + \frac{3}{2}A_1$$

$$+ \frac{9}{2}A_1 r^2 - \frac{9}{2}A_1 r + \frac{1}{2}A_0 r^2 - 2A_0 r + 2A_0\, \frac{1}{2}A_1 r^3 - 4A_1 - 3A_1 r^2 - 6A_1 r = r$$

$$\Rightarrow \quad 2A_0 r - A_0 r^2 - \frac{1}{2}A_0 - \frac{5}{2}A_1 + \frac{3}{2}A_1 r^2 + \frac{3}{2}A_1 r = r$$

Comparing both sides, we get

$$2A_0 + \frac{3}{2} A_1 = 1 \quad ...(4.34.2)$$

$$A_0 + 5A_1 = 0 \quad ...(4.34.3)$$

Solving equation (4.34.2) and (4.34.3), we get $A_1 = \frac{-2}{17}$ $A_0 = \frac{-10}{17}$

To find the value of B_1 and B_2 put $r = 0$ and $r = 1$ in equation (4.34.1)

$r = 0$ $a_0 = B_1 + B_2$ $B_1 + B_2 = 8$ ...(4.34.4)

$r = 1$ $a_1 = 2B_1 + 4B_2$ $2B_1 + 4B_2 = 1$ ...(4.34.5)

Solving equations (4.34.4) and (4.34.5), we get $B_1 = \frac{31}{2}$ $B_2 = \frac{-15}{2}$

Complete solution is, $a_r = a_r^{(h)} + a_r^{(p)}$

$$a_r = \frac{31}{2} 2^r - \frac{15}{2} 4^r + r^2 \left[\left(\frac{-10}{17}\right) + \left(\frac{-2}{17}\right) r\right] 4^r$$

Que 4.35. **Solve the recurrence relation : $a_r + 4a_{r-2} + 4a_{r-2} = r^2$.**

AKTU 2017-18, Marks 07

Answer

$a_r + 4a_{r-1} + 4a_{r-2} = r^2$

The characteristic equation is : $x^2 + 4x + 4 = 0$,

$$(x + 2)^2 = 0, \quad x = -2, -2$$

The homogeneous solution is, $a^{(h)} = (A_0 + A_1 r)(-2)^r$

The particular solution be, $a^{(p)} = (A_0 + A_1 r)\, r^2$

Put a_r, a_{r-1} and a_{r-2} from $a^{(p)}$ in the given equation, we get

$$r^2 A_0 + A_1 r^3 + 4A_0(r-1)^2 + 4A_1(r-1)^3 + 4A_0(r-2)^2 + 4A_1(r-2)^3 = r^2$$

$$A_0(r^2 + 4r^2 - 8r + 4 + 4r^2 - 16r + 16) +$$

$$A_1(r^3 + 4r^3 - 4 - 12r^2 + 12r + 4r^3 - 32 - 24r^2 + 48r) = r^2$$

$$A_0(9r^2 - 24r + 20) + A_1(9r^3 - 48r^2 + 60r - 36) = r^2$$

Comparing the coefficient of same power of r, we get

$$9A_0 - 48A_1 = 1 \quad ...(4.35.1)$$

$$20A_0 - 36A_1 = 0 \quad ...(4.35.2)$$

Solving equation (4.35.1) and (4.35.2) $A_0 = \frac{-3}{53}$ $A_1 = \frac{-5}{159}$

The complete solution is,

$$a_r = a_r^{(p)} + a_r^{(h)} = (A_0 + A_1 r)(-2)^r + \left[\left(\frac{-3}{53}\right) + \left(\frac{-5}{159}\right) r\right] r^2$$

Que 4.36. **Suppose that a valid codeword is an n-digit number in decimal notation containing an even number of 0's. Let a_n denote the number of valid codeword's of length n satisfying the recurrence**

relation $a_n = 8a_{n-1} + 10^{n-1}$ and the initial condition $a_1 = 9$. Use generating functions to find an explicit formula for a_n.

AKTU 2018-19, Marks 07

Answer

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $a_0, a_1, a_2 ....$

We sum both sides of the last equations starting with $n = 1$. To find that

$$G(x) - 1 = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (8a_{n-1}x^n + 10^{n-1} x^n)$$

$$= 8\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1} x^n$$

$$= 8x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} 10^{n-1} x^{n-1}$$

$$= 8x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 10^n x^n$$

$$= 8xG(x) + x/(1 - 10x)$$

Therefore, we have $G(x) - 1 = 8xG(x) + x/(1 - 10x)$

Expanding the right hand side of the equation into partial fractions gives

$$G(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right)$$

This is equivalent to $G(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right)$

$$= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n) x^n$$

$$a_n = \frac{1}{2}(8^n + 10^n)$$

PART-6

Pigeonhole Principle.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.37. **Write short notes on Pigeonhole principle.**

Answer

Pigeonhole principle : The pigeonhole principle is sometime useful in counting methods.

If n pigeons are assigned to m pigeonholes then at least one pigeonhole contains two or more pigeons ($m < n$).

Proof :

1. Let m pigeonholes be numbered with the numbers 1 through m.
2. Beginning with the pigeon 1, each pigeon is assigned in order to the pigeonholes with the same number.
3. Since $m < n$ *i.e.*, the number of pigeonhole is less than the number of pigeons, n-m pigeons are left without having assigned a pigeonhole.
4. Thus, at least one pigeonhole will be assigned to a more than one pigeon.
5. We note that the pigeonhole principle tells us nothing about how to locate the pigeonhole that contains two or more pigeons.
6. It only asserts the existence of a pigeonhole containing two or more pigeons.
7. To apply the principle one has to decide which objects will play the role of pigeon and which objects will play the role of pigeonholes.

Que 4.38. **How many different rooms are needed to assign 500 classes, if there are 45 different time periods during in the university time table that are available ?**

Answer

Using pigeonhole principle :

Here $n = 500, m = 45 = \left[\frac{n-1}{m}\right] + 1 = \left[\frac{500-1}{45}\right] + 1$

At least 12 rooms are needed.

☺☺☺

5
UNIT

Lattices and Boolean Algebra

CONTENTS

PART-1

Introduction, Partially Ordered Set, Hasse Diagram.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.1. Define poset. What is totally or linearly ordered set ?

Answer

Let R be a relation on a set A satisfying following properties :

i. For any $a \in A$ $(a, a) \in R$ *i.e.*, aRa (Reflexive property)

ii. For $a, b \in A$ if aRb and bRa then $a = b$ (Antisymmetric property)

iii. For $a, b, c \in A$ if aRb and bRc then aRc (Transitive property)

Then R is called partial order relation or simply order relation or R is said to define a partial ordering of A. A set A together with relation R (partial order relation) is called partially ordered set or poset denoted by (A, R).

A partially ordered relation is denoted by $\preccurlyeq$.

$a \preccurlyeq b$ is read as "a precedes b".

Also $a \prec b$ is read as "a strictly precedes b".

For example :

The relation '/' of divisibility is not an ordering relation on set Z of integers. Since it is not antisymmetric as $7/-7 = -7/7$ but $7 \neq -7$.

Totally or linearly ordered set : An ordered set A is said to be linearly or totally ordered if every pair of element in A are comparable. A totally ordered set is also called a chain.

For example : The poset $(N, \leq)$ is totally ordered set since every two natural numbers are comparable.

Que 5.2. Define Hasse diagram. Also, explain how Hasse diagram is constructed ?

Answer

1. Let A be a poset and $a, b \in A$. Then a is immediate predecessor of b or b is immediate successor of a if $a < b$, but no element of A lies between a and b denoted by $a << b$. We can also say that b is cover of a.
2. Hasse diagram of a poset A is a directed graph whose vertices are elements of A and there is a directed edge from a to b whenever $a << b$.
3. In Hasse diagram, we will place b higher than a and draw a line between them to indicate succession instead of drawing an arrow.

Constructing a Hasse diagram : We can represent a partial ordering on a finite set using the following procedure :

1. Start with a directed graph of the relation.
2. Remove the loops at all the vertices *i.e.*,

Fig. 5.2.1.

3. If aRb, then b appear above the element a and the element a is connected to element b by an edge with arrows pointing upwards. Remove all the arrows.

Fig. 5.2.2.

4. Remove all edges whose existence is implied by the transitive property aRb and $bRc \Rightarrow aRc$.

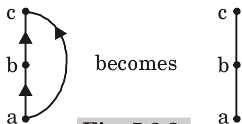


Fig. 5.2.3.

Que 5.3. **Draw the Hasse diagram of [P (a, b, c), $\subseteq$] (Note : '$\subseteq$' stands for subset). Find greatest element, least element, minimal element and maximal element.** **AKTU 2015-16, Marks 10**

Answer

Hasse diagram of [$P(a, b, c), \subseteq$] or [$P(a, b, c), \leq$] is shown in Fig. 5.3.1.

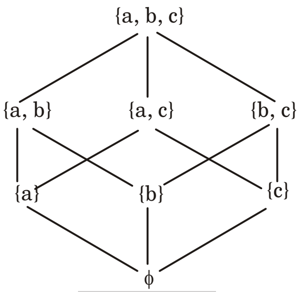


Fig. 5.3.1.

Greatest element is $\{a, b, c\}$ and it is maximal element too.
The least element is ϕ and is minimal element too.

Que 5.4. **Draw the Hasse diagram of $(A, \leq)$, where $A = \{3, 4, 12, 24, 48, 72\}$ and relation $\leq$ be such that $a \leq b$ if a divides b.**

AKTU 2017-18, Marks 07

Answer

Hasse diagram of $(A, \leq)$ where $A = \{3, 4, 12, 24, 48, 72\}$

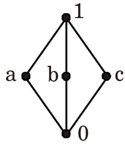


Fig. 5.4.1.

Que 5.5. **Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S. Draw the Hasse diagram for inclusion on the set $P(S)$, where $S = \{a, b, c, d\}$. Also determine whether $(P(S), \subseteq)$ is a lattice.**

AKTU 2018-19, Marks 07

Answer

The inclusion relation $(\subseteq)$ is partial ordering on the power set of a set S if it satisfies the conditions :

Reflexivity : $A \subseteq A$ whenever A is a subset of S.

Antisymmetry : If A and B are positive integers with $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Transitivity : If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Hasse diagram :

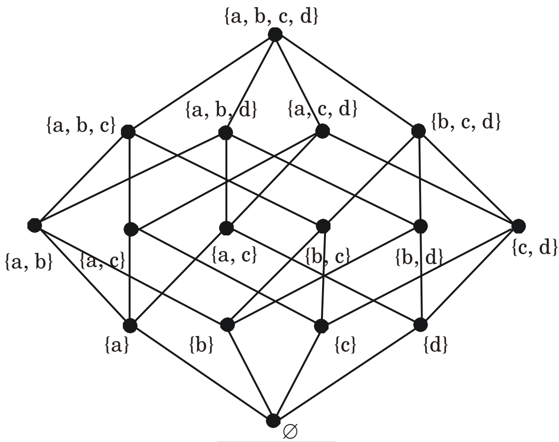


Fig. 5.5.1.

$(P(S), \subseteq)$ is not a lattice because $(\{a, b\},\{b, d\})$ has no *lub* and *glb*.

PART-2

Maximal and Minimal Element, Upper and Lower Bounds.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.6. **Write short note on :**

a. Minimal and maximal element
b. Least (first) and greatest (last) element
c. Upper and lower bound

Answer

a. Minimal and maximal element :

i. Let A be a poset. An element is called minimal element if no other element of A strictly precedes a. In Hasse diagram, a is minimal element if no edge enters a from below.

ii. An element $b \in A$ is called maximal element if no other element of A strictly succeeds b.

b. Least and greatest element :

i. Let A be a poset. An element $a \in A$ is called least (first) element if a precedes every other element in A. *i.e.*, $\forall\, x \in A, a \preccurlyeq x$

ii. An element $b \in A$ is called greatest (last) element if b succeeds every other element of A. *i.e.*, $\forall\, y \in A, y \preccurlyeq b$.

c. Upper and lower bound :

i. Let A_1 be a subset of poset A. An element $M \in A$ is called an upper bound of A_1 if M succeeds every element of A_1. *i.e.*, $\forall\, x \in A$, if $x \preccurlyeq M$.

ii. An element $m \in A$ is called a lower bound of A_1 if m precedes every element for A_1. *i.e.*, $\forall\, y \in A$ if $m \preccurlyeq y$.

Que 5.7. **The directed graph G for a relation R on set A = {1, 2, 3, 4} is shown below :**

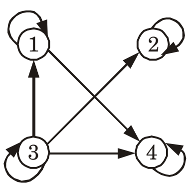


Fig. 5.7.1.

i. Verify that (A, R) is a poset and find its Hasse diagram.
ii. Is this a lattice ?
iii. How many more edges are needed in the Fig. 5.7.1 to extend (A, R) to a total order ?
iv. What are the maximal and minimal elements ?

AKTU 2014-15, Marks 10

Answer

i. The relation R corresponding to the given directed graph is,
$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (3, 1), (3, 4), (1, 4), (3, 2)\}$
R is a partial order relation if it is reflexive, antisymmetric and transitive.
Reflexive : Since $aRa, \forall a \in A$. Hence, it is reflexive.
Antisymmetric : Since aRb and bRa then we get $a = b$ otherwise aRb or bRa. Hence, it is antisymmetric.
Transitive: For every aRb and bRc we get aRc. Hence, it is transitive.
Therefore, we can say that (A, R) is poset. Its Hasse diagram is :

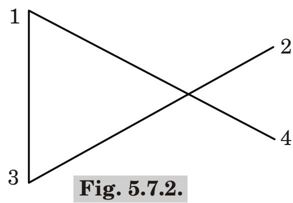


Fig. 5.7.2.

ii. Since there is no *lub* of 1 and 2 and same for 2 and 4. The given poset is not a lattice.

$\vee$	1	2	3	4
1	1	–	1	1
2	–	2	2	–
3	1	2	3	1
4	1	–	1	4

iii. Only one edge (4, 2) is included to make it total order.
iv. Maximal are {1, 2} and minimal are {3, 4}.

PART-3

Isomorphic Ordered Sets, Lattices, Bounded Lattices and Distribution Lattices.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.8. **Define lattice. Explain types of lattice.**

Answer

Lattice :

A lattice is a poset $(L, \leq)$ in which every subset $\{a, b\}$ consisting of 2 elements has least upper bound (lub) and greatest lower bound (glb). Least upper bound of $\{a, b\}$ is denoted by $a \vee b$ and is known as join of a and b. Greatest lower bound of $\{a, b\}$ is denoted by $a \wedge b$ and is known as meet of a and b.

Lattice is generally denoted by $(L, \wedge, \vee)$.

Types of lattice :

1. **Bounded lattice :** A lattice L is said to be bounded if it has a greatest element 1 and a least element 0.
2. **Complemented lattice :** Let L be a bounded lattice with greatest element 1 and least element 0. Let $a \in L$ then an element $a' \in L$ is complement of a if,
$$a \vee a' = 1 \text{ and } a \wedge a' = 0$$
A lattice L is called complemented if it is bounded and if every element in L has a complement.
3. **Distributive lattice :** A lattice L is said to be distributive if for any element a, b and c of L following properties are satisfied :
   i. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
   ii. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

   otherwise L is non-distributive lattice.
4. **Complete lattice :** A lattice L is called complete if each of its non-empty subsets has a least upper bound and greatest lower bound.
5. **Modular lattice :** A lattice $(L, \leq)$ is called modular lattice if,
$a \vee (b \wedge c) = (a \vee b) \wedge c$ whenever $a \leq c$ for all $a, b, c \in L$.

Que 5.9. **If the lattice is represented by the Hasse diagram given below :**

i. Find all the complements of 'e'.

ii. Prove that the given lattice is bounded complemented lattice.

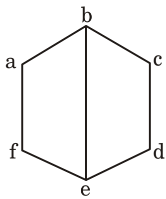


Fig. 5.9.1.

AKTU 2014-15, Marks 10

Answer

i. Complements of e are c and d which are as follows :

$c \vee e = b \quad , \quad c \wedge e = f$

$d \vee e = b \quad , \quad d \wedge e = f$

ii. A lattice is bounded if it has greatest and least elements. Here b is greatest and f is least element.

Que 5.10. Define a lattice. For any a, b, c, d in a lattice $(A, \leq)$ if $a \leq b$ and $c \leq d$ then show that $a \vee c \leq b \vee d$ and $a \wedge c \leq b \wedge d$.

AKTU 2018-19, Marks 07

Answer

Lattice : Refer Q. 5.8, Page 5–7B, Unit-5.

Numerical :

As $a \leq b$ and $c \leq d$, $a \leq b \leq b \vee d$ and $c \leq d \leq b \vee d$.

By transtivity of $\leq$, $a \leq b \vee d$ and $c \leq b \vee d$.

So, $b \vee d$ is an upper bound of a and c.

So, $a \vee c \leq b \vee d$.

As $a \wedge c \leq a$ and $a \wedge c \leq c$, $a \wedge c \leq a \leq b$ and $a \wedge c \leq c \leq d$.

Hence, $a \wedge c$ is a lower bound of b and d. So $a \wedge c \leq b \wedge d$.

So, $a \wedge c \leq b \wedge d$.

Que 5.11. Let L be a bounded distributed lattice, prove if a complement exists, it is unique. Is D_{12} a complemented lattice ? Draw the Hasse diagram of [P (a, b, c), $\underline{<}$], (Note : '$\underline{<}$' stands for subset). Find greatest element, least element, minimal element and maximal element.

AKTU 2015-16, Marks 15

Answer

Let a_1 and a_2 be two complements of an element $a \in L$.

Then by definition of complement

$$\left.\begin{aligned} a \vee a_1 &= I \\ a \wedge a_1 &= 0 \end{aligned}\right\} \quad \text{...(5.11.1)}$$

$$\left.\begin{aligned} a \vee a_2 &= I \\ a \wedge a_2 &= 0 \end{aligned}\right\} \quad \text{...(5.11.2)}$$

Consider

$$\begin{aligned} a_1 &= a_1 \vee 0 \\ &= a_1 \vee (a \wedge a_2) && \text{[from (5.11.2)]} \\ &= (a_1 \vee a) \wedge (a_1 \vee a_2) && \text{[Distributive property]} \\ &= (a \vee a_1) \wedge (a_1 \vee a_2) && \text{[Commutative property]} \\ &= I \wedge (a_1 \vee a_2) && \text{[from (5.11.1)]} \end{aligned}$$

$$= a_1 \vee a_2 \quad ...(5.11.3)$$

Now Consider

$$a_2 = a_2 \vee 0$$
$$= a_2 \vee (a \wedge a_1) \quad \text{[from (5.11.2)]}$$
$$= (a_2 \vee a) \wedge (a_2 \vee a_1) \quad \text{[Distributive property]}$$
$$= (a \vee a_2) \wedge (a_1 \vee a_2) \quad \text{[Commutative property]}$$
$$= I \wedge (a_1 \vee a_2) \quad \text{[from (5.11.1)]}$$
$$= a_1 \vee a_2 \quad ...(5.11.4)$$

Hence, from (5.11.3) and (5.11.4),

$$a_1 = a_2$$

So, for bounded distributive lattice complement is unique.

Hasse diagram : Refer Q. 5.3, Page 5–3B, Unit-5.

Que 5.12.

a. Prove that every finite subset of a lattice has an LUB and a GLB.

b. Give an example of a lattice which is a modular but not a distributive.

AKTU 2016-17, Marks 10

Answer

a.

1. The theorem is true if the subset has 1 element, the element being its own *glb* and *lub*.
2. It is also true if the subset has 2 elements.
3. Suppose the theorem holds for all subsets containing 1, 2, ..., k elements, so that a subset $a_1, a_2, ..., a_k$ of L has a *glb* and a *lub*.
4. If L contains more than k elements, consider the subset $\{a_1, a_2, ..., a_{k+1}\}$ of L.
5. Let $w = lub\ (a_1, a_2, ..., a_k)$.
6. Let $t = lub\ (w, a_{k+1})$.
7. If s is any upper bound of $a_1, a_2, ..., a_{k+1}$, then s is $\geq$ each of $a_1, a_2, ..., a_k$ and therefore $s \geq w$.
8. Also, $s \geq a_{k+1}$ and therefore s is an upper bound of w and a_{k+1}. Hence $s \geq t$.
9. That is, since $t \geq$ each a_1, t is the *lub* of $a_1, a_2, ..., a_{k+1}$.
10. The theorem follows for the *lub* by finite induction.
11. If L is finite and contains m elements, the induction process stops when $k + 1 = m$.

b.

1. The diamond is modular, but not distributive.
2. The distributive lattices are closed under sublattices and every sublattice of a distributive lattice is itself a distributive lattice.

3. If the diamond can be embedded in a lattice, then that lattice has a non-distributive sublattice, hence it is not distributive.

Que 5.13. Explain modular lattice, distributive lattice and bounded lattice with example and diagram.

AKTU 2017-18, Marks 10

Answer

Modular distributive and bounded lattice : Refer Q. 5.8, Page 5–7B, Unit-5.

Example : Let consider a Hasse diagram :

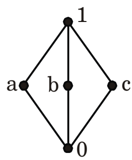


Fig. 5.13.1.

Modular lattice :

$0 \leq a$ *i.e.*, taking $b = 0$

$b \vee (a \wedge c) = 0 \vee 0 = 0$, $a \wedge (b \vee c) = a \wedge c = 0$

Distributive lattice : For a set S, the lattice $P(S)$ is distributive, since union and intersection each satisfy the distributive property.

Bounded lattice : Since, the given lattice has 1 as greatest and 0 as least element so it is bounded lattice.

Que 5.14. For any positive integer D36, then find whether (D36, '|') is lattice or not ?

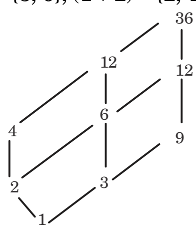
AKTU 2017-18, Marks 07

Answer

D36 = Divisor of 36 = {1, 2, 3, 4, 6, 9, 12, 18, 36}

Hasse diagram :

$(1 \vee 3) = \{3, 6\}$, $(1 \vee 2) = \{2, 4\}$, $(2 \vee 6) = \{6, 18\}$, $(9 \vee 4) = \{\phi\}$



Since, $9 \vee 4 = \{\phi\}$

So, D36 is not a lattice.

Que 5.15. **In a lattice if $a \le b \le c$, then show that**

a. $a \vee b = b \wedge c$

b. $(a \vee b) \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = b$

AKTU 2016-17, Marks 10

Answer

a. **Given :** $a \le b \le c$

Now $a \vee b$ = least upper bound of a, b

= least {all upper bounds of a, b} [using $a \le b \le c$]

$= b$ ...(5.15.1)

and $b \wedge c$ = greatest lower bound of b, c

= maximum {all lower bounds of b, c} [using $a \le b \le c$]

$= b$ ...(5.15.2)

Eq. (5.15.1) and (5.15.2) gives, $a \vee b = b \wedge c$

b. $(a \vee b) \vee (b \wedge c) \Rightarrow (a \vee b) \wedge (a \vee c) = b$

Consider, $(a \vee b) \vee (b \wedge c)$

$= b \vee b$ [using $a \le b \le c$ and definition of $\vee$ and $\wedge$]

$= b$ ...(5.15.3)

and $(a \vee b) \wedge (a \vee c) = b \wedge c$

$= b$ ...(5.15.4)

From eq. (5.15.3) and (5.15.4), $(a \vee b) \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = b$.

PART-4

Duality, Boolean Algebras as Lattices, Minimization of Boolean Expression.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.16. **What is Boolean algebra ? Write the axioms of Boolean algebra. Also, describe the theorems of it.**

Answer

1. A Boolean algebra is generally denoted by $(B, +, ., 0, 1)$ where $(B, +, .)$ is a lattice with binary operations '+' and '.' called the join and meet respectively and $(')$ is unary operation in B.
2. The elements 0 and 1 are zero (least) and unit (greatest) elements of lattice $(B, +, .)$.

3. B is called a Boolean algebra if the following axioms are satisfied for all a, b, c in B.

Axioms of Boolean algebra :

If $a, b, c \in B$, then

1. Commutative laws :
   a. $a + b = b + a$ b. $a.b = b.a$
2. Distributive laws :
   a. $a + (b.c) = (a + b).(a + c)$ b. $a.(b + c) = (a.b) = (a.c)$
3. Identity laws :
   a. $a + 0 = a$ b. $a.1 = a$
4. Complement laws :
   a. $a + a' = 1$ b. $a.a' = 0$

Basic theorems :

Let $a, b, c \in B$, then

1. Idempotent laws :
   a. $a + a = a$ b. $a.a = a$
2. Boundedness (Dominance) laws :
   a. $a + 1 = 1$ b. $a.0 = 0$
3. Absorption laws :
   a. $a + (a.b) = a$ b. $a.(a + b) = a$
4. Associative laws :
   a. $(a + b) + c = a + (b + c)$ b. $(a.b).c = a.(b.c)$
5. Uniqueness of complement :
   $a + x = 1$ and $a.x = 0$, then $x = a$
6. Involution law : $(a')' = a$
7. De-Morgan's laws :
   a. $(a + b)' = a'.b'$ b. $(a.b)' = a' + b'$

Que 5.17. **Prove the following theorems :**

a. Absorption law : Prove that $\forall\ a, b, \in B$
i. $a.(a + b) = a$ **ii. $a + a.b = a$**

b. Idempotent law : Prove that $\forall\ a \in B$, $a + a = a$ and $a.a = a$.

c. De Morgan's law : Prove that $\forall\ a, b, \in B$
i. $(a + b)' = a'.b'$ **ii. $(a.\ b)' = a' + b'$**

d. Prove that $0' = 1$ and $1' = 0$.

Answer

a. Absorption law :

i. To prove : $a.(a + b) = a$

Let
$a.(a + b) = (a + 0).(a + b)$ by Identity law
$= a + 0.b$ by Distributive law
$= a + b.0$ by Commutative law
$= a + 0$ by Dominance law
$= a$ by Identity law

ii. To prove : $a + a.b = a$

Let $a + a.b = a.1 + a.b$ by Identity law

$= a.(1 + b)$ by Distributive law
$= a.(b + 1)$ by Commutative law
$= a.1$ by Dominance law
$= a$ by Identity law

b. Idempotent law :

To prove : $a + a = a$ and $a.a = a$

Let $a = a + 0$ by Identity law
$= a + a.a'$ by Complement law
$= (a + a).(a + a')$ by Distributive law
$= (a + a).1$ by Complement law
$= a + a$ by Identity law

Now let $a = a.1$ by Identity law
$= a.(a + a')$ by Complement law
$= a.a + a.a'$ by Distributive law
$= a.a + 0$ by Complement law
$= a.a$ by Identity law

c. De Morgan's law :

i. To prove : $(a + b)' = a'.b'$

To prove the theorem we will show that

$$(a + b) + a'.b' = 1$$

Consider $(a + b) + a'.b' = \{(a + b) + a'\}.\{(a + b) + b'\}$ by Distributive law
$= \{(b + a) + a'\}.\{(a + b) + b'\}$
by Commutative law
$= \{b + (a + a')\}.\{a + (b + b')\}$
by Associative law
$= (b + 1).(a + 1)$ by Complement law
$= 1.1$ by Dominance law
$= 1$ ...(5.17.1)

Also consider $(a + b).a'b' = a'b'.(a + b)$ by Commutative law
$= a'b'.a + a'b'.b$ by Distributive law
$= a.(a'b') + a'.(b'b)$ by Commutative law
$= (a.a').b' + a'.(b.b')$ by Associative law
$= 0.b' + a'.0$ by Complement law
$= b'.0 + a'.0$ by Commutative law
$= 0 + 0$ by Dominance law
$= 0$ ...(5.17.2)

From eq. (5.17.1) and (5.17.2), we get,
$a'b'$ is complement of $(a + b)$ *i.e.* $(a + b)' = a'b'$.

ii. To prove : $(a.b)' = a' + b'$

Follows from principle of duality, that is, interchange operations + and • and interchange the elements 0 and 1.

d. To prove : $0' = 1$ and $1' = 0$.

$0' = (a a')'$ by Complement law
$= a' + (a')'$ by De Morgan's law
$= a' + a$ by Involution law
$= a + a'$ by Commutative law
$= 1$ by Complement law

Now $(0')' = 1'$

$\Rightarrow$ $0 = 1'$

$\Rightarrow$ $1' = 0.$

Que 5.18. **Consider the Boolean function.**

$$f(x_1, x_2, x_3, x_4) = x_1 + (x_2.\,(x'_1 + x_4) + x_3.\,(x'_2 + x'_4))$$

a. i. Simplify f algebraically

ii. Draw the logic circuit of the f and the reduction of the f.

b. Write the expressions $E_1 = (x + x * y) + (x/y)$ and $E_2 = x + ((x * y + y)/y)$, into

i. Prefix notation

ii. Postfix notation

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Answer

a. i. $f(x_1, x_2, x_3, x_4) = x_1 + (x_2.(x'_1 + x_4) + x_3.(x'_2 + x'_4)$

$$= x_1 + x_2.x'_1 + x_2.x_4 + x_3.x'_2 + x_3.x'_4$$
$$= x_1 + x_2 + x_2.x_4 + x_3.x'_2 + x_3.x'_4$$
$$= x_1 + x_2.(1 + x_4) + x_3.x'_2 + x_3.x'_4$$
$$= x_1 + x_2 + x_3.x'_2 + x_3.x'_4$$
$$= x_1 + x_2 + x_3 + x_3.x'_4$$
$$= x_1 + x_2 + x_3.(1 + x'_4)$$
$$= x_1 + x_2 + x_3$$

ii. Logic circuit :

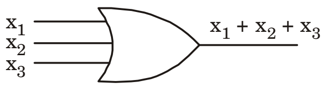

Fig. 5.18.1.

Reduction of f :

$$f(x_1, x_2, x_3, x_4) = x_1 + (x_2.(x'_1 + x_4) + x_3.(x'_2 + x'_4)$$
$$= x_1 + (x_2.x'_1 + x_2.x_4) + (x_3.x'_2 + x_3.x_4)$$
$$= x_1 + x_2.x'_1 + x_2.x_4 + x_3.x'_2 + x_3.x_4$$

b. $E_1 = (x + x * y) + (x/y)$

Binary tree is :

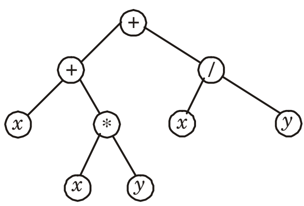

Fig. 5.18.2.

Prefix : $++ x * x\,y\,/\,xy$

Postfix : $xx\,y * + xy\,/ +$

$E_2 = x + ((x * y + y)/y)$

Binary tree is :

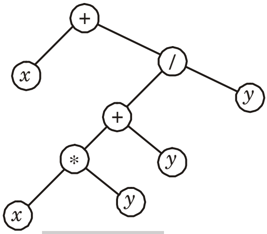

Fig. 5.18.3.

Prefix : $+ x / + * x y y y$

Postfix : $x x y * y + y / +$

Que 5.19. **Find the Boolean algebra expression for the following system.**

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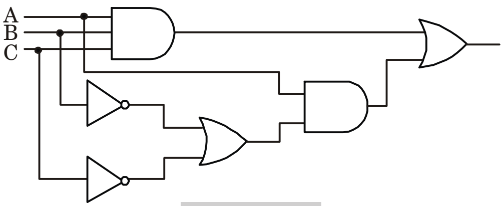


Fig. 5.19.1.

Answer

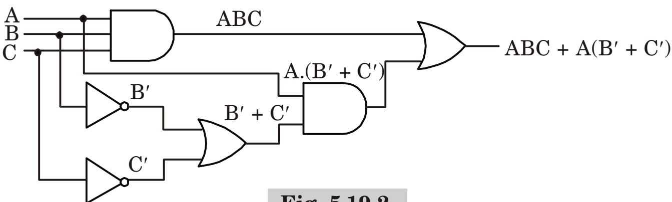


Fig. 5.19.2.

PART-5

Prime Implicants, Logic Gates and Circuits, Truth Table, Boolean Functions, Karnaugh Map.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.20. **Define a Boolean function of degree *n*. Simplify the following Boolean expression using Karnaugh maps $xyz + xy'z + x'y'z + x'yz + x'y'z'$**

Answer

Boolean function of degree *n* :

1. Let $B = \{0, 1\}$. Then $B^n = \{(x_1, x_2 \,...., x_n) \,|\, x_i\, B \text{ for } 1 \le i \le n\}$ is the set of all possible n-tuples of 0*s* and 1*s*.
2. The variable x is called a Boolean variable if it assumes values only from B, that is, if its only possible values are 0 and 1.
3. A function from B^n to B is called a Boolean function of degree n.

Numerical : The Karnaugh map for the given function is :

$xyz + xy'z + x'y'z + x'yz + x'y'z'$

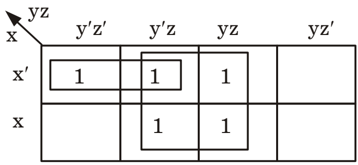


Fig. 5.20.1.

Then the simplified expression is : $z + x'y'$.

Que 5.21. **Explain the term prime implicant with example.**

Answer

A fundamental produce P is called prime implicant of Boolean expression E if $P + E = E$ but no other fundamental product contained in P has this property.

For example : Let $E = xy' + xyz' + x'yz'$

then $xz' + E = E$

as

$$xz' + E = xz' + xy' + xyz' + x'yz'$$
$$= xy' + xyz' + x'yz' + xz'$$
$$= xy' + xz' + x'yz' \qquad (\because \; xz' \text{ contained in } xyz)$$
$$= xy' + xz'\,(y + y') + x'yz'$$
$$= xy' + xyz' + xz'y' + x'yz'$$
$$= xy' + xyz' + x'yz' \qquad (\because \; xy' + xz'y' = xy')$$

But $x + E \ne E$ and $z' + E \ne E$

$\therefore$ xz' is prime implicant of E.

Que 5.22. **Write short notes on following :**

a. K-map **b. Logic gates**

Answer

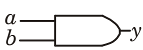
a. K-map (Karnaugh map) :

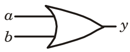
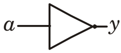
1. Karnaugh map is a pictorial method for finding prime implicants and minimal forms of a Boolean expression involving variables.
2. If the Boolean expression contains n variables then the K-map will have 2^n squares each of which represents a minterm.
3. **Step for minimization of Boolean expression using K-maps :**
   a. K-maps are constructed by placing (✓) or (1) in squares corresponding to minterms present in the given expression.
   b. All the 1's that cannot be combined with any other 1's are identified and looped.
   c. All the 1's that combine in a loop of two but do not make a loop of four are looped.
   d. All those 1's that combine in a loop of four but do not make a loop of eight are looped.
   e. The process stops when all 1's have been covered.
4. The Boolean expression which is to be simplified will contain two groups of minterms–one group of minterms which are to be necessarily included in the function and other group of minterms which may or may not be included in the function. The second group of minterms are called don't care terms.
5. In K-map, don't care terms are represented by d.

b. Logic gates :

1. A logic gates is an electronic circuit that operates on one or more input signals to produce an output signal. Gates are digital circuits and are called as logic circuits.
2. There are three basic logic gates which are given as :

Table 5.22.1.

S. No.	Name	Graphic symbol	Truth Table	Boolean formulas
1.	AND		a b y 0 0 0 0 1 0 1 0 0 1 1 1	$y = a.b$

2.	OR		a b y 0 0 0 0 1 1 1 0 1 1 1 1	$y = a + b$
3.	NOT		a y 0 1 1 0	$y = a'$

Que 5.23. **Simplify the following Boolean functions using K-map :**

$F(x, y, z) = \Sigma(0, 2, 3, 7)$ **AKTU 2017-18, Marks 07**

Answer

x \ yz	00	01	11	10
0	1		1	1
1			1	

$$F = \bar{x}\,\bar{z} + yz$$

Que 5.24. **Simplify the following Boolean expressions using K-map :**

a. $Y = ((AB)' + A' + AB)'$

b. $A'\,B'\,C'\,D' + A'\,B'\,C'\,D + A'\,B'\,CD + A'\,B'\,CD' = A'\,B'$

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Answer

a. $Y = ((AB)' + A' + AB)'$

$$= ((AB)')'\,(A' + (AB))' = (AB)\,((A')'\,(AB)')$$
$$= AB\,(A\,(A' + B')) = AB\,(AA' + AB')$$
$$= AB(0 + AB') = AB\,AB' = ABB' = 0$$

Here, we find that the expression is not in minterm. Hence, no need to use K-map for further simplification.

b. $A'\,B'\,C'\,D' + A'\,B'\,C'\,D + A'\,B'\,CD + A'\,B'\,CD' = A'\,B'$

$$= A'\,B'C'D' + A'B'C'D + A'B'CD + A'B'CD'$$

CD \ AB				
A′B′C′D′	1			
A′B′C′D	1			
A′B′CD	1			
A′B′CD′	1			

on simplification by K-map, we get $A'B'$ corresponding to all the four one's.

Que 5.25. **Find the Sum-Of-Products and Product-Of-sum expansion of the Boolean function $F(x, y, z) = (x + y)\, z'$.**

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Answer

$$F(x, y, z) = (x + y)z'$$

x	y	z	$x + y$	z'	$(x + y)z'$
1	1	1	1	0	0
1	1	0	1	1	1
1	0	1	1	0	0
1	0	0	1	1	1
0	1	1	1	0	0
0	1	0	1	1	1
0	0	1	0	0	0
0	0	0	0	1	0

Sum-Of-Product :

$$F(x, y, z) = xyz' + xy'z' + x'yz'$$

Product-Of-Sum :

$$F(x, y, z) = (x + y + z)(x + y' + z)(x' + y + z)(x' + y' + z)(x' + y' + z')$$

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1 UNIT

Laplace Transform (2 Marks Questions)

1.1. Define Laplace transform.

Ans. If the kernel $k(s, t)$ is defined as

$$k(s, t) = \begin{cases} e^{-st} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Then the integral transform

$$f(s) = L\{F(t)\} = \int_{-\infty}^{\infty} k(s, t)\, F(t)\, dt \text{ becomes}$$

$$f(s) = L\{F(t)\} = \int_{0}^{\infty} F(t)\, e^{-st}\, dt$$

This is known as Laplace transform of the function $F(t)$ and is denoted as $f(s)$.

1.2. Find Laplace transform of 1.

Ans. If $\quad F(t) = 1$

$$L\{F(t)\} = \int_0^{\infty} e^{-st}\, F(t)\, dt = \int_0^{\infty} e^{-st}\, dt$$

$$f(s) = \left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \frac{1}{s}, s > 0$$

1.3. If Laplace transform of $L\{f(t)\} = \dfrac{e^{-1/s}}{s}$, then find $L\{e^{-t} f(3t)\}$.

Ans.

$$L\{f(t)\} = \frac{e^{-1/s}}{s} = f(s) \qquad \{\because\ a = 3\}$$

$$L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

$$L\{f(3t)\} = \frac{1}{3} \frac{e^{-3/s}}{\left(\frac{s}{3}\right)} = \frac{e^{-3/s}}{s}$$

$$L\{e^{-t} f(3t)\} = \frac{e^{-3/(s+1)}}{(s+1)} \qquad [\because L(e^{at} f(t)) = F(s-a)]$$

1.4. Find the Laplace transform of $f(t) = t^4 e^{2t}$.

Ans. We know that, $L\{t^4\} = \dfrac{4!}{s^5}$ $\left[\because L\{t^n\} = \dfrac{n!}{s^{n+1}}\right]$

Using first shifting property,

$$L\{t^4 e^{2t}\} = \frac{4!}{(s-2)^5}$$

1.5. Find the Laplace transform of $\dfrac{\sin at}{t}$. Does the Laplace transform of $\dfrac{\cos at}{t}$ exist ?

Ans. Since $\lim\limits_{t \to 0} \dfrac{\sin at}{t} = \lim\limits_{t \to 0} a \cos at$ [By L Hospital rule]

$= a$, therefore transform of $\dfrac{\sin at}{t}$ exists.

$$L[\sin at] = \frac{a}{s^2 + a^2}$$

$$\therefore \qquad L\left\{\frac{\sin at}{t}\right\} = \int_s^{\infty} \frac{a}{s^2 + a^2} ds$$

$$= \left[\tan^{-1}\left(\frac{s}{a}\right)\right]_s^{\infty} = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \cot^{-1}\left(\frac{s}{a}\right) = \tan^{-1}\left(\frac{a}{s}\right)$$

Since the function $\dfrac{\cos at}{t}$ is discontinuous at $t = 0$. Therefore, its Laplace transform does not exist.

1.6. Find the Laplace transform of $f(t) = t \sin \sqrt{7}t$.

Ans.
$$L\{\sin \sqrt{7}t\} = \frac{\sqrt{7}}{s^2 + 7}$$

$$L\{t \sin \sqrt{7}t\} = (-1)^1 \frac{d}{ds}\left(\frac{\sqrt{7}}{s^2 + 1}\right)$$

$$= -\left[\frac{0 - 2s \times \sqrt{7}}{(s^2+1)^2}\right]$$

$$= \frac{2\sqrt{7}s}{(s^2+1)^2}$$

1.7. Find the Laplace transform of $\left[\int_0^t\int_0^t \sin u \, du \, du\right]$

AKTU 2014-15 (II), Marks 02

Ans.

$$L\left\{\int_0^t \sin u \, du\right\} = \frac{1}{s(s^2+1)}$$

$$L\left\{\int_0^t\int_0^t \sin u \, du \, du\right\} = \frac{1}{s^2(s^2+1)}$$

1.8. Find the Laplace transform of $t^3 \delta (t - 4)$.

AKTU 2015-16 (II), Marks 02

Ans.

$$L\{t^3 \delta (t-4)\} = \int_0^\infty e^{-st} t^3 \delta (t-4) dt = 4^3 e^{-as}$$

1.9. Find the Laplace transform of unit step function $u(t - a)$.

Ans. The unit step function $u(t-a)$ is defined as

$$u(t-a) = \begin{cases} 0 \ , & t < a \\ 1 \ , & t \geq a \end{cases}$$

The Laplace transform of unit step function is given as

$$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a) dt$$

$$= \int_0^a e^{-st} u(t-a) dt + \int_a^\infty e^{-st} 1 \, dt$$

$$= 0 + \int_a^\infty e^{-st} dt = \left[\frac{-e^{-st}}{s}\right]_a^\infty$$

$$= \frac{e^{-as}}{s}, \left[\lim_{t\to\infty} \frac{e^{-st}}{-s} = 0\right]$$

1.10. Find the Laplace transform of $F(t) = \dfrac{e^{-t} \sin t}{t}$.

Ans.

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\,ds = \left[\tan^{-1} s\right]_s^\infty = \tan^{-1}\infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$$L\left\{\frac{e^{-t}\sin t}{t}\right\} = \cot^{-1}(s+1) \qquad \text{[Using first shifting property]}$$

1.11. Evaluate the Laplace transform of Integral of a function $L\left\{\int_0^t f(t)\,dt\right\}$.

AKTU 2016-17 (II), Marks 02

Ans. Let

$$G(t) = \int_0^t f(t)\,dt, \text{ then}$$

$$G'(t) = f(t) \text{ and } G(0) = 0$$

Taking Laplace transform, we get

$$L\{G'(t)\} = sL\{G(t)\} - G(0) = sL\{G(t)\}$$

$$\therefore \qquad L\{G(t)\} = \frac{1}{s}L\{G'(t)\} = \frac{1}{s}L\{f(t)\} = \frac{1}{s}f(s)$$

i.e., $$L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}f(s)$$

1.12. Find the inverse Laplace transform of $\log\left(\frac{s+1}{s-1}\right)$.

Ans. The inverse Laplace transform of $\log\left(\frac{s+1}{s-1}\right)$ is

$$L^{-1}\left(\log\left(\frac{s+1}{s-1}\right)\right) = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\log\left(\frac{s+1}{s-1}\right)\right]$$

$$= \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\log(s+1) - \frac{d}{ds}\log(s-1)\right]$$

$$= \frac{-1}{t}L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$= \frac{-1}{t}[e^{-t} - e^{t}] = \frac{2}{t}\left(\frac{e^{t} - e^{-t}}{2}\right) = \frac{2}{t}\sinh t$$

1.13. Find the inverse Laplace transform of $\frac{1}{s(s+1)^3}$.

Ans. We know that,

$$L^{-1}\left\{\frac{1}{(s+1)^3}\right\} = e^{-t}\, L^{-1}\left\{\frac{1}{s^3}\right\} = e^{-t}\,\frac{t^2}{2!}$$

$$L^{-1}\left\{\frac{1}{s(s+1)^3}\right\} = \int_0^t e^{-t}\,\frac{t^2}{2!}dt = \frac{1}{2}\int_0^t e^{-t}t^2 dt$$

$$= \frac{1}{2}\left[t^2(-e^{-t}) - 2t(e^{-t}) + 2(-e^{-t})\right]_0^t$$

$$= \frac{1}{2}\left[(-t^2 - 2t - 2)e^{-t} + 2\right]$$

$$L^{-1}\left\{\frac{1}{s(s+1)^3}\right\} = 1 - e^{-t}\left(1 + t + \frac{t^2}{2}\right)$$

1.14. Find the function whose Laplace transform is

$$\boldsymbol{f(s) = \frac{8}{(s^2 - s - 2)}}$$

Ans.

$$F(t) = L^{-1}f(s)$$

$$= L^{-1}\left\{\frac{8}{s^2 - s - 2}\right\}$$

$$= L^{-1}\left\{\frac{8}{\left(s^2 + \frac{1}{4} - s\right) - 2 - \frac{1}{4}}\right\}$$

$$= L^{-1}\left\{\frac{8}{\left(s - \frac{1}{2}\right)^2 - \frac{9}{4}}\right\}$$

$$= L^{-1}\left\{\frac{8}{\left(s - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\right\}$$

$$= L^{-1}\left\{\frac{8 \times \frac{3}{2}}{\frac{3}{2}\left[\left(s - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right]}\right\}$$

$$F(t) = \frac{16}{3}\, e^{\frac{1}{2}t}\, \sinh \frac{3}{2}t$$

1.15. Find the inverse Laplace transform of $\boldsymbol{\frac{e^{-\pi s}}{s^2 + 1}}$

AKTU 2014-15 (II), Marks 02

Ans.

$$L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

$$L^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\} = \begin{cases}\sin(t-\pi) & ; \quad t>\pi \\ 0 & ; \quad t<\pi\end{cases} = \sin(t-\pi)\, u(t-\pi)$$

1.16. Find inverse Laplace transform of the function

$$f(s) = \frac{s}{2s^2+8}$$

AKTU 2015-16 (II), Marks 02

Ans.

$$f(s) = L^{-1}\left\{\frac{s}{2s^2+8}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{s}{s^2+2^2}\right\} = \frac{1}{2}\cos 2t$$

1.17. Find the inverse Laplace transform of : $\dfrac{e^{-2\pi s}}{s(s^2+1)}$.

Ans. Since we have, $L^{-1}\left\{\dfrac{1}{s^2+1}\right\} = \sin t$

$$L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin u\, du = [-\cos u]_0^t = -(\cos t - 1) = 1 - \cos t$$

$$\therefore L^{-1}\left\{\frac{e^{-2\pi s}}{s(s^2+1)}\right\} = \begin{cases}1-\cos(t-2\pi), & t>2\pi \\ \quad 0 \quad , & t<2\pi\end{cases}$$

1.18. Express the following function in terms of unit step function and find its Laplace transform.

$$f(t) = \begin{cases}E & a<t<b \\ 0 & t \geq b\end{cases}$$

Ans.

$$f(t) = \begin{cases}E & a<t<b \\ 0 & t\geq b\end{cases}$$

$$f(t) = E\begin{cases}1 & a<t<b \\ 0 & t\geq b\end{cases}$$

$$E = [u(t-a) - u(t-b)]$$

$$L\{f(t)\} = E\left[\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\right]$$

1.19. Express the following function in terms of unit step function :

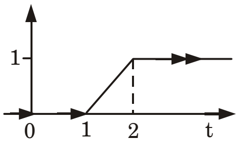

Fig. 1.19.1.

Ans. The function shown in the Fig. 1.19.1 is expressed in algebraic form

$$f(t) = \begin{cases} 0 & , \quad 0 < t < 1 \\ t - 1 & , \quad 1 < t < 2 \\ 1 & , \quad 2 < t \end{cases}$$

$$\begin{aligned} f(t) &= (t-1)\,[u\,(t-1) - u\,(t-2)] + u\,(t-2) \\ &= (t-1)\,u\,(t-1) - u\,(t-2)\,\{t-1-1\} \\ &= (t-1)\,u\,(t-1) - (t-2)\,u\,(t-2) \end{aligned}$$

1.20. Evaluate $\int_0^\infty \sin 2t\, \delta\left(t - \frac{\pi}{4}\right) dt$.

Ans. We know that $\int_0^\infty f(t)\,\delta\,(t-a)\,dt = f(a)$

$$\int_0^\infty \sin 2t\, \delta\left(t - \frac{\pi}{4}\right) dt = \sin\left(2.\frac{\pi}{4}\right) = 1$$

1.21. Evaluate $L\left[\frac{1}{t}\delta\,(t-a)\right]$.

Ans.
$$L[\delta\,(t-a)] = e^{-as}$$

$$L\left[\frac{1}{t}\delta\,(t-a)\right] = \int_s^\infty L\,[\delta\,(t-a)]\,ds = \int_s^\infty e^{-as}\,ds$$

$$= \left[\frac{e^{-as}}{-a}\right]_s^\infty = \frac{1}{a}e^{-as}$$

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2 UNIT

Integral Transforms (2 Marks Questions)

2.1. If $F(x)$ is the complex Fourier transform of $f(x)$, then find the Fourier transform of $f(ax)$.

Ans. Using change of scale property, the Fourier transform of $f(ax)$ is given as

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right), a \neq 0$$

2.2. State the shifting property of Fourier transform of $f(x)$.

Ans. If $F\{f(x)\} = F(s)$

then $F\{(x-a)\} = e^{ias} F(s)$.

2.3. Prove that Modulation theorem

$$F\{f(x) \cos ax\} = \frac{1}{2}[f(s+a) + f(s-a)]$$

AKTU 2016-17 (IV), Marks 02

Ans. Taking L.H.S,

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x) \left[\frac{e^{iax} + e^{-iax}}{2}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \times \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{isx} e^{iax} f(x) dx + \int_{-\infty}^{\infty} e^{isx} e^{-iax} f(x) dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s+a)x} f(x) dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s-a)x} f(x) dx\right]$$

$$= \frac{1}{2}[f(s+a) + f(s-a)]$$

$=$ R.H.S.

2.4. Define the *Z*-transform. **AKTU 2017-18 (III), Marks 02**

Ans. If the function y_n is defined for discrete value ($n = 0, 1, 2, ...$) and $f_n = 0$ for $n < 0$, then the Z-transform of f_n is defined as

$$Z\{f_n\} = F(z) = \sum_{n=0}^{\infty} f_n z^{-n}$$

whenever the infinite series converges and the inverse Z-transform is given as $Z^{-1}\{F(z)\} = f_n$.

2.5. Find Z-transformation of $f(k) = \begin{cases} 1, & k = 0 \\ 1, & k \neq 0 \end{cases}$

AKTU 2015-16 (III), Marks 02

Ans.

$$Z[f(k)] = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$
$$= [\ldots 0 + 0 + 1 + 0 + 0 \ldots] = 1$$

2.6. Find the *Z*-transform of $U_n = \{a^n\}$.

AKTU 2015-16 (IV), Marks 02

Ans. Given,

$$U_n = a^n$$
$$Z\{U_n\} = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{z}{z-a}$$

2.7. Find the *Z*-transform of the sequence $\{a_n\}$.

AKTU 2016-17 (III), Marks 02

Ans. The Z-transform of the sequence $\{a_n\}$ is given as

$$Z[\{a_n\}] = F(z) = \sum_{n=-\infty}^{\infty} f(n) z^{-n} = \sum_{n=-\infty}^{\infty} \frac{f(n)}{z^n} = \sum_{n=-\infty}^{\infty} \frac{a_n}{z^n}$$

Where, z = Complex number,
Z = Operator of Z-transform, and
$F(z)$ = Z-transform of $\{f(n)\}$ or a_n.

2.8. State the Convolution theorem for inverse *Z*-transform.

AKTU 2016-17 (III), Marks 02

Ans. **Convolution Theorem :**

If $Z^{-1}[U(z)] = u_n$ and $Z^{-1}[V(z)] = v_n$

Then, $Z^{-1}[U(z).V(z)] = u_n \times v_n$ = convolution of u_n and v_n

$$= \sum_{m=0}^{n} u_m v_{n-m}$$

2.9. Find inverse Z-transformation of $\dfrac{8z - z^3}{(4 - z)^3}$.

AKTU 2015-16 (III), Marks 02

Ans. Since,
$$f(z) = \frac{8z - z^3}{(4 - z)^3}$$

Poles are :
$$(4 - z)^3 = 0$$
$$z = 4, 4, 4$$

Residue at $(z = 4)$

$$= \left[\frac{1}{(3-1)!}\frac{d^{3-1}}{dz^{3-1}}(4 - z)^3 \, z^{k-1} \, \frac{8z - z^3}{(4 - z)^3}\right]_{z = 4}$$

$$= \frac{1}{2}\left[\frac{d^2}{dz^2}\{z^{k-1}\,(8z - z^3)\}\right]_{z = 4} = \frac{1}{2}\left[\frac{d^2}{dz^2}\{8z^k - z^{k+2}\}\right]_{z = 4}$$

$$= \frac{1}{2}[8\,k(k-1).z^{k-2} - (k+2)\,(k+1)z^k]_{z=4}$$

$$= \frac{1}{2}\,[8\,k\,(k-1)\,4^{k-2} - (k+2)\,(k+1)\,4^k]$$

Hence
$$f(k) = z^{-1}\,f(z) = \frac{1}{2}[8k\,(k-1)\,4^{k-2} - (k+2)\,(k+1)\,4^k]$$

2.10. If $u(x, y) = x^2 - y^2$, prove that the u satisfies Laplace equations.

AKTU 2015-16 (III), Marks 02

Ans. **Given :** $u(x, y) = x^2 - y^2$

On differentiating partially with respect to x,
$$\frac{\partial u}{\partial x} = 2x$$

Again differentiate with respect to x,
$$\frac{\partial^2 u}{\partial x^2} = 2 \quad ...(2.10.1)$$

Similarly on double differentiation of u (partially) with respect to y,
$$\frac{\partial^2 u}{\partial y^2} = -2 \quad ...(2.10.2)$$

Laplace equation $= \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

On adding eq. (2.10.1) and eq. (2.10.2),
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

So, $u(x, y) = x^2 - y^2$ is satisfying Laplace equation.

2.11. Solve the Z-transform : $y_{k+2} + y_{k+1} - 2y_k = 0, y_0 = 4\ y_1 = 0$

AKTU 2016-17 (IV), Marks 02

Ans. **Given :** $y_{k+2} + y_{k+1} - 2y_k = 0$

Taking Z-transform on both sides, we get

$$Z[y_{k+2} + y_{k+1} - 2y_k] = Z[0]$$

$$Z(y_{k+2}) + Z(y_{k+1}) - 2Z(y_k) = 0$$

$$[z^2 y(z) - z^2 y_0 - zy_1] + [zyz - zy_0] - 2y(z) = 0$$

$$z^2 y(z) - 4z^2 - 0 + zy(z) - 4z - 2y(z) = 0 \quad (\because\ y_0 = 4 \text{ and } y_1 = 0)$$

$$y(z)(z^2 + z - 2) = 4z^2 + 4z$$

$$y(z) = \frac{4(z^2+z)}{z^2+z-2} = \frac{4(z^2+z)}{z^2+2z-z-2}$$

$$y(z) = \frac{4(z^2+z)}{(z+2)(z-1)}$$

Taking inverse Z-transform, we get

$$y_k = Z^{-1}\left[\frac{4(z^2+z)}{(z+2)(z-1)}\right]$$

Now by Residue method solving the above inverse Z-transform.

The poles are determined by

$$(z+2)(z-1) = 0 \quad \Rightarrow \quad z = -2, 1$$

There are two poles. Let us consider the contour $|z| > 1$.

$$\text{Residue at } (z = 1) = \left[\frac{(z-1)z^{k-1}\,4(z^2+z)}{(z+2)(z-1)}\right]_{z=1} = \frac{(1)^{k-1}\,4(1^2+1)}{(1+2)} = \frac{4\times 2}{3}$$

$$= \frac{8}{3}$$

$$\text{Residue at } (z = -2) = \left[\frac{(z+2)z^{k-1}\,4(z^2+z)}{(z+2)(z-1)}\right]_{z=-2} = 4\left[\frac{z^{k+1}+z^k}{z-1}\right]_{z=-2}$$

$$= 4\left[\frac{z^k(z+1)}{z-1}\right]_{z=-2} = 4\left[\frac{(-2)^k(-2+1)}{-2-1}\right]$$

$$= \frac{4}{3}(-2)^k$$

Hence $\quad y_k$ = sum of the residues

$$y_k = \frac{8}{3} + \frac{4}{3}(-2)^k = \frac{4}{3}(2 + (-2)^k)$$

2.12. Find the Fourier coefficient for the function $f(x) = x^2; 0<x<2\pi$.

AKTU 2016-17 (II), Marks 02

Ans. The Fourier coefficients for the given function are as follows :

i. $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$a_0 = \frac{8\pi^2}{3}$$

ii. $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2 \sin nx}{n}\right)_0^{2\pi} - \int_0^{2\pi} 2x\,\frac{\sin nx\,dx}{n}\right] \quad (\because \quad \sin n\pi = 0)$$

$$= -\frac{2}{\pi n}\int_0^{2\pi} x \sin nx\,dx$$

$$= -\frac{2}{\pi n}\left[\left\{x\left(-\frac{\cos nx}{n}\right)\right\}_0^{2\pi} - \int_0^{2\pi} 1\left(-\frac{\cos nx}{n}\right)dx\right]$$

$$= -\frac{2}{\pi n}\left[-\frac{2\pi}{n} - 0\right] = \frac{4}{n^2}$$

iii. $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\left[x^2\left(-\frac{\cos nx}{n}\right)\right]_0^{2\pi} - \int_0^{2\pi} 2x\,\frac{\cos nx}{n}\,dx\right\}$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2}{n} - 0\right] = -\frac{4\pi}{n}$$

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3 UNIT

Formal Logic, Group, Ring and Field (2 Marks Questions)

3.1. What is compound proposition ?

Ans. A proposition obtained from the combinations of two or more propositions by means of logical operators or connectives of two or more propositions or by negating a single proposition is referred to as composite or compound proposition.

3.2. Show the implications without constructing the truth table $(P \to Q) \to Q \Rightarrow P \vee Q$. **AKTU 2016-17, Marks 02**

Ans. $(P \to Q) \to Q \Rightarrow P \vee Q$

Take L.H.S

$$\begin{aligned}(P \to Q) \to Q &= (\sim P \vee Q) \to Q \\ &= (\sim(\sim P \vee Q)) \vee Q \\ &= (P \vee \sim Q) \vee Q \\ &= (P \vee Q) \vee (\sim Q \vee Q) \\ &= (P \vee Q) \wedge T \\ &= P \vee Q\end{aligned}$$

It is equivalent.

3.3. Prove that $(P \vee Q) \to (P \wedge Q)$ is logically equivalent to $P \leftrightarrow Q$. **AKTU 2015-16, Marks 02**

Ans. $(P \vee Q) \to (P \wedge Q) = P \leftrightarrow Q$

P	Q	$P \vee Q$	$P \wedge Q$	$(P \vee Q) \leftrightarrow (P \wedge Q)$	$P \leftrightarrow Q$
T	T	T	T	T	T
T	F	T	F	F	F
F	T	T	F	F	F
F	F	F	F	T	T

3.4. The converse of a statement is : If a steel rod is stretched, then it has been heated. Write the inverse of the statement. **AKTU 2015-16, Marks 02**

Ans. The statement corresponding to the given converse is "If a steel rod is stretched, then it has been heated". Now the inverse of this statement is "If a steel rod is not stretched then it has not been heated".

3.5. Give truth table for converse, contrapositive and inverse.

Ans. The truth table of the four propositions are as follows :

p	q	Conditional $p \Rightarrow q$	Converse $q \Rightarrow p$	Inverse $\sim p \Rightarrow \sim q$	Contrapositive $\sim q \Rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

3.6. Show that contrapositive are logically equivalent; that is $\sim q \Rightarrow \sim p \equiv p \Rightarrow q$

Ans. The truth table of $\sim q \Rightarrow \sim p$ and $p \Rightarrow q$ are shown in the below table and the logical equivalence is established by the last two columns of the table, which are identical.

p	q	$\sim p$	$\sim q$	$\sim q \Rightarrow \sim p$	$p \Rightarrow q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

3.7. Find the contrapositive of "If he has courage, then he will win". **AKTU 2017-18, Marks 02**

Ans. If he will not win then he does not have courage.

3.8. Verify that the proposition $p \wedge (q \wedge \sim p)$ is a contradiction.

Ans.

p	q	$\sim p$	$q \wedge \sim p$	$p \wedge (q \wedge \sim p)$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	F
F	F	T	F	F

3.9. What are the contrapositive, converse, and the inverse of the conditional statement : "The home team wins whenever it is raining" ? **AKTU 2018-19, Marks 02**

Ans. **Given :** The home team wins whenever it is raining.
q(conclusion) : The home team wins.
p(hypothesis) : It is raining.
Contrapositive : $\sim q \to \sim p$ is "if the home team does not win then it is not raining".
Converse : $q \to p$ is "if the home team wins then it is raining".
Inverse : $\sim p \to \sim q$ is "if it is not raining then the home team does not win".

3.10. Show that $[((p \vee q) \to r) \wedge (\sim p)] \to (q \wedge r)$ is tautology or contradiction. **AKTU 2015-16, Marks 02**

Ans.

p	q	r	$p \vee q$	$\sim p$	$q \wedge r$	$((p \vee q) \to r)$	$((p \vee q) \to r) \wedge (\sim p)$	$[((p \vee q) \to r) \wedge (\sim p)] \to (q \wedge r)$
T	T	T	T	F	T	T	F	T
T	T	F	T	F	F	F	F	T
T	F	T	T	F	F	T	F	T
T	F	F	T	F	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F	T
F	F	T	F	T	F	T	T	F
F	F	F	F	T	F	T	T	F

Question is incorrect. Since the result of the question is contingency.

3.11. If a and b are any two elements of group G then prove $(a * b)^{-1} = (b^{-1} * a^{-1})$ **AKTU 2015-16, Marks 02**

Ans. Consider $(a * b) * (b^{-1} * a^{-1})$

$$= a * (b * b^{-1}) * a^{-1}$$
$$= a * e * a^{-1}$$
$$= a * a^{-1} = e$$

Also $(b^{-1} * a^{-1}) * (a * b) = b^{-1} * (a^{-1} * a) * b$

$$= b^{-1} * e * b$$
$$= b^{-1} * b = e$$

Therefore $(a * b)^{-1} = b^{-1} * a^{-1}$ for any $a, b \in G$

3.12. Define ring and give an example of a ring with zero divisors.

AKTU 2016-17, Marks 02

OR

Define ring and field. **AKTU 2018-19, Marks 02**

OR

Define rings and write its properties.

AKTU 2017-18, Marks 02

Ans. **Ring :** A non-empty set R is a ring if it is equipped with two binary operations called addition and multiplication and denoted by '+' and '.' respectively *i.e.*, for all $a, b \in R$ we have $a + b \in R$ and $a. b \in R$ and it satisfies the following properties :

i. Addition is associative *i.e.*,

$(a + b) + c = a + (b + c) \; \forall \; a,b,c \in R$

ii. Addition is commutative *i.e.*,

$a + b = b + a \; \forall \; a,b \in R$

iii. There exists an element $0 \in R$ such that

$0 + a = a = a + 0, \; \forall \; a \in R$

iv. To each element a in R there exists an element $-a$ in R such that $a + (-a) = 0$

v. Multiplication is associative *i.e.*,

$a.(b.c) = (a.b).c, \; \forall \; a \; b,c \in R$

vi. Multiplication is distributive with respect to addition *i.e.*, for all $a, b, c \in R$,

Example of ring with zero divisors : R = {a set of 2 × 2 matrices}.

Field : A ring R with at least two elements is called a field if it has following properties :

i. R is commutative

ii. R has unity

iii. R is such that each non-zero element possesses multiplicative inverse.

For example : The rings of real numbers and complex numbers are also fields.

3.13. Define group with suitable example.

Ans. An algebraic structure $(G, *)$ is a group if the binary operation * satisfies the following properties :

a. Closure property : $a * b \in G \;\; \forall \;\; a, b \in G$

b. Associativity : $(a * b) * c = a * (b * c) \;\; \forall \;\; a, b, c, \in G$

c. Existence of identity : There exists an element $e \in G$ such that $e * a = a = a * e \;\; \forall \;\; a \in G$. The element e is called identity.

d. Existence of inverse : Each element of G possess inverse. For $a \in G$ there exists an element $b \in G$ such that $a * b = e = b * a$. The element b is called the inverse of a and we write $b = a^{-1}$

e. Commutativity : $a * b = b * a \;\; \forall \;\; a, b \in G$. If the algebraic structure also satisfies the commutative property then it is called an abelian group or commutative group.

3.14. Define Lagrange's theorem. What is the use of the theorem ?

Ans. **Lagrange's theorem :**

Statement : The order of each subgroup of a finite group is a divisor of the order of the group.

Use of theorem :

i. It can be used to find the subgroup of any order for the symmetric group.

ii. It tells that the number of subgroups of the cyclic group of order p, when p is prime then there is only one subgroup and that is $\{\phi\}$.

3.15. Show that $\neg (p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

AKTU 2018-19, Marks 02

Ans. **To prove :** $(p + q)' = p'.q'$

To prove the theorem we will show that

$$(p + q) + p'.q' = 1$$

Consider

$$\begin{aligned}
(p + q) + p'.q' &= \{(p + q) + p'\}.\{(p + q) + q'\} && \text{by Distributive law} \\
&= \{(q + p) + p'\}.\{(p + q) + q'\} && \text{by Commutative law} \\
&= \{q + (p + p')\}.\{p + (q + q')\} && \text{by Associative law} \\
&= (q + 1).(p + 1) && \text{by Complement law} \\
&= 1.1 && \text{by Dominance law} \\
&= 1 && \ldots(3.15.1)
\end{aligned}$$

Also consider

$$\begin{aligned}
(p + q).p'q' &= p'q'.(p + q) && \text{by Commutative law} \\
&= p'q'.p + p'q'.q && \text{by Distributive law} \\
&= p.(p'q') + p'.(q'q) && \text{by Commutative law} \\
&= (p.p').q' + p'.(q.q') && \text{by Associative law} \\
&= 0.q' + p'.0 && \text{by Complement law} \\
&= q'.0 + p'.0 && \text{by Commutative law} \\
&= 0 + 0 && \text{by Dominance law} \\
&= 0 && \ldots(3.15.2)
\end{aligned}$$

From (3.15.1) and (3.15.2), we get,

$p'q'$ is complement of $(p + q)$ *i.e.*, $(p + q)' = p'q'$.

☺☺☺

4 UNIT

Sets, Relations, Functions and Counting Techniques (2 Marks Questions)

4.1. What do you understand by partition of a set ?

Ans. A partition of a set A is a collection of non-empty subsets A_1, A_2, ..., A_n called blocks, such that each element of A is in exactly one of the blocks. *i.e.*,

i. A is the union of all subsets $A_1 \cup A_2 \cup ..... \cup A_n = A$.

ii. The subsets are pairwise disjoint, $A_i \cap A_j = \varnothing$ for $i \neq j$.

4.2. Define transitive closure with suitable example.

Ans. The relation obtained by adding the least number of ordered pairs to ensure transitivity is called the transitive closure of the relation. The transitive closure of R is denoted by R^+.

4.3. Let R be a relation on the set of natural numbers N, as $R = \{(x, y) : x, y \in N, 3x + y = 19\}$. Find the domain and range of R. Verify whether R is reflexive. **AKTU 2016-17, Marks 02**

Ans. By definition of relation,

$R = \{(1, 16), (2, 13), (3, 10), (4, 7), (5, 4), (6, 1)\}$

$\therefore$ Domain = {1, 2, 3, 4, 5, 6}

$\therefore$ Range = {16, 13, 10, 7, 4, 1}

R is not reflexive since $(1, 1) \notin R$.

4.4. Show that the relation R on the set Z of integers given by $R = \{(a, b) : 3$ divides $a - b\}$, is an equivalence relation.

AKTU 2016-17, Marks 02

Ans. **Reflexive :** $a - a = 0$ is divisible by 3

$\therefore$ (a, a) $\in R \;\forall\; a \in Z$

$\therefore$ R is reflexive.

Symmetric : Let $(a, b) \in R \Rightarrow a - b$ is divisible by 3

$\Rightarrow$ $-(a - b)$ is divisible by 3

$\Rightarrow$ $b - a$ is divisible by 3

$\Rightarrow$ $(b, a) \in R$

$\therefore$ R is symmetric.

Transitive : Let $(a, b) \in R$ and $(b, c) \in R$

$a - b$ is divisible by 3 and $b - c$ is divisible by 3
Then $a - b + b - c$ is divisible by 3
$a - c$ is divisible by 3
$\therefore$ $(a, c) \in R$
$\therefore$ R is transitive.
Hence, R is equivalence relation.

4.5. How many symmetric and reflexive binary relations are possible on a set S with cardinality n ?

Ans. There are $2^{n(n+1)/2}$ symmetric binary relations and $2^{n(n-1)}$ reflexive binary relations are possible on a set S with cardinality.

4.6. Show that if set A has 3 elements, then we can have 2^6 symmetric relations on A. **AKTU 2015-16, Marks 02**

Ans. Number of elements in set = 3
Number of symmetric relations if number of elements is $n = 2^{n(n+1)/2}$
Here, $n = 3$
$\therefore$ Number of symmetric relations

$$= 2^{3(3+1)/2}$$
$$= 2^{3(4)/2}$$
$$= 2^6$$

Hence proved.

4.7. If $f : A \to B$ is one-to-one onto mapping, then prove that $f^{-1} : B \to A$ will be one-to-one onto mapping.

AKTU 2015-16, Marks 02

Ans. Proof : Here $f : A \to B$ is one-to-one and onto.
$a_1, a_2 \in A$ and $b_1, b_2 \in B$ so that

$$b_1 = f(a_1),\ b_2 = f(a_2) \text{ and } a_1 = f^{-1}(b_1),\ a_2 = f^{-1}(b_2)$$

As f is one-to-one

$$f(a_1) = f(a_2) \Leftrightarrow a_1 = a_2$$
$$b_1 = b_2 \Leftrightarrow f^{-1}(b_1) = f^{-1}(b_2)$$

i.e., $f^{-1}(b_1) = f^{-1}(b_2) \Rightarrow b_1 = b_2$
$\therefore$ f^{-1} is one-to-one function.
As f is onto.
Every element of B is associated with a unique element of A *i.e.*, for any $a \in A$ is pre-image of some $b \in B$ where $b = f(a) \Rightarrow a = f^{-1}(b)$ *i.e.*, for $b \in B$, there exists f^{-1} image $a \in A$.
Hence, f^{-1} is onto.

4.8. Define multiset and power set. Determine the power set A = {1, 2}. **AKTU 2015-16, Marks 02**

Ans. Multiset : Multisets are sets where an element can occur as a member more than once.

For example : $A = \{p, p, p, q, q, q, r, r, r, r\}$
$B = \{p, p, q, q, q, r\}$
are multisets.

Power set : A power set is a set of all subsets of the set.
The power set of A = {1, 2} is {ϕ}, {1}, {2}.

4.9. Define union and intersection of multiset and find for $A = [1, 1, 4, 2, 2, 3]$, $B = [1, 2, 2, 6, 3, 3]$

AKTU 2017-18, Marks 02

Ans. **Union :** Let A and B be two multisets. Then, $A \cup B$, is the multiset where the multiplicity of an element in the maximum of its multiplicities in A and B.

Intersection : The intersection of A and B, $A \cap B$, is the multiset where the multiplicity of an element is the minimum of its multiplicities in A and B.

Numerical :

$$A = \{1, 1, 4, 2, 2, 3\}, B = \{1, 2, 2, 6, 3, 3\}$$

Union : $A \cup B = \{1, 2, 3, 4, 6\}$

Intersection : $A \cap B = \{1, 2, 2, 3\}$

4.10. Let A = (2, 4, 5, 7, 8) = B, aRb if and only if $a + b$ <= 12. Find relation matrix.

AKTU 2017-18, Marks 02

Ans. $R = \{(2, 4), (2, 5), (2, 7), (2, 8)\ (4, 2), (4, 5), (4, 7), (4, 8)\ (5, 2), (5, 4), (5, 7), (7, 2), (7, 4), (7, 5), (8, 2), (8, 4), (2, 2), (4, 4), (5, 5)\}$

$$m_{ij} = \begin{array}{c} \\ 2 \\ 4 \\ 5 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccccc} 2 & 4 & 5 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

4.11. Find the power set of each of these sets, where a and b are distinct elements.

i. $\{a\}$

ii. $\{a, b\}$

iii. $\{\phi, \{\phi\}\}$

iv. $\{a, \{a\}\}$

AKTU 2018-19, Marks 02

Ans.

i. Power set of $\{a\} = \{\{\phi\}, \{a\}\}$

ii. Power set of $\{a, b\} = \{\{\phi\}, \{a\}, \{b\}, \{a, b\}\}$

iii. Power set of $\{\phi, \{\phi\}\} = \{\phi\}$

iv. Power set of $\{a, \{a\}\} = \{\{\phi\}, \{a\}, \{\{a\}\}, \{a, \{a\}\}\}$

4.12. Define injective, surjective and bijective function.

AKTU 2018-19, Marks 02

Ans.

1. **One-to-one function (Injective function or injection) :** Let $f : X \rightarrow Y$ then f is called one-to-one function if for distinct elements of X there are distinct image in Y *i.e.*, f is one-to-one iff

$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2 \;\; \forall \;\; x_1, x_2, \in X$$

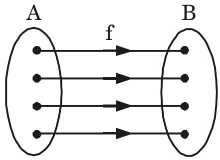

Fig. 4.12.1. One-to-one.

2. **Onto function (Surjection or surjective function) :** Let $f : X \rightarrow Y$ then f is called onto function iff for every element $y \in Y$ there is an element $x \in X$ with $f(x) = y$ or f is onto if Range $(f) = Y$.

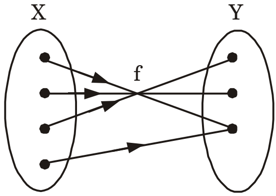

Fig. 4.12.2. Onto.

3. **One-to-one onto function (Bijective function or bijection) :** A function which is both one-to-one and onto is called one-to-one onto function or bijective function.

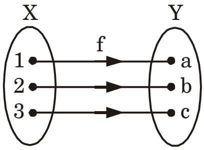

Fig. 4.12.3. One-to-one onto.

4.13. Find the recurrence relation from $y_n = A2^n + B(-3)^n$.

AKTU 2016-17, Marks 02

Ans. Given : $y_n = A2^n + B\,(-3)^n$

Therefore, $y_{n+1} = A(2)^{n+1} + B\,(-3)^{n+1}$

$= 2A\,(2)^n - 3B\,(-3)^n$

and $y_{n+2} = A(2)^{n+2} + B\,(-3)^{n+2}$

$= 4A\,(2)^n + 9B\,(-3)^n$

Eliminating A and B from these equations, we get

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & -3 \\ y_{n+2} & 4 & 9 \end{vmatrix} = 0$$

$= y_{n+2} - y_{n+1} - 6y_n = 0$ which is the required recurrence relation.

4.14. State and prove pigeonhole principle.

AKTU 2015-16, Marks 02

Ans. **Pigeonhole principle :** If n pigeons are assigned to m pigeonholes then at least one pigeon hole contains two or more pigeons ($m < n$).

Proof :

1. Let m pigeonholes be numbered with the numbers 1 through m.
2. Beginning with the pigeon 1, each pigeon is assigned in order to the pigeonholes with the same number.
3. Since $m < n$ *i.e.*, the number of pigeonhole is less than the number of pigeons, $n - m$ pigeons are left without having assigned a pigeon hole.
4. Thus, at least one pigeonhole will be assigned to more than one pigeon.

4.15. How many 4-digit numbers can be formed by using the digits 2, 4, 6, 8 when repetition of digits is allowed ?

AKTU 2015-16, Marks 02

Ans. When repetition is allowed :

The thousands place can be filled by 4 ways.

The hundreds place can be filled by 4 ways.

The tens place can be filled by 4 ways.

The units place can be filled by 4 ways.

$\therefore$ Total number of 4-digit number = $4 \times 4 \times 4 \times 4 = 256$

4.16. How many bit strings of length eight either start with a '1' bit or end with the two bit '00' ?

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Ans.

1. Number of bit strings of length eight that start with a 1 bit : $2^7 = 128$.
2. Number of bit strings of length eight that end with bits 00 : $2^6 = 64$.
3. Number of bit strings of length eight $2^5 = 32$ that start with a 1 bit and end with bits 00 : $2^5 = 32$

Hence, the number is $128 + 64 - 32 = 160$.

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5 UNIT

Lattices and Boolean Algebra (2 Marks Questions)

5.1. Define maximal and minimal element.

Ans. **Maximal element :** An element 'a' in the poset is called a maximal element of P if $a \le x$ for no 'x' in P, that is, if no element of P strictly succeeds 'a'.

Minimal element : An element 'b' in P is called a minimal element of P if $x \le b$ for no 'x' in P.

5.2. Determine :

i. All maximal and minimal elements

ii. Greatest and least element

iii. Upper and lower bounds of 'a' and 'b', 'c' and 'd'.

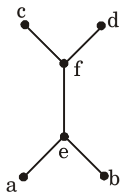

Fig. 5.2.1.

Ans.

i. Maximal elements = c, d, Minimal element = a, b

ii. Greatest and least elements do not exist.

iii. Upper bound for a,b are e, f, c, d.
Upper bound for c, d are does not exist.
Lower bound for a,b are does not exist.
Lower bound for c,d are f, e, a, b.

5.3. Consider $A = \{x \in R : 1 < x < 2\}$ with $\le$ as the partial order. Find

i. All the upper and lower bounds of A.

ii. Greatest lower bound and least upper bound of A.

Ans. i. Every real number ≥ 2 is an upper bound of A and every real number ≤ 1 is a lower bound of A.

ii. 1 is a greatest lower bound and 2 is the least upper bound of A.

5.4. Explain lattice homomorphism and lattice isomorphism.

Ans. **Lattice homomorphism :** Let $(L, *, \oplus)$ and $(S, \wedge, \vee)$ be two lattices. A mapping $g : L \to S$ is called a lattice homomorphism from the lattice $(L, *, \oplus)$ to $(S, \wedge, \vee)$ if for any $a, b \in L$

$$g(a * b) = g(a) \wedge g(b) \text{ and } (g \oplus b) = g(a) \vee g(b).$$

Lattice isomorphism : If a homomorphism $g : L \to S$ of two lattices $(L, *, \oplus)$ and $(S, \wedge, \vee)$ is bijective *i.e.*, one-to-one onto, then g is called an isomorphism.

5.5. Show that the relation $\geq$ is a partial ordering on the set of integers, Z.

Ans. Since :

1. $a \geq a$ for every a, $\geq$ is reflexive.
2. $a \geq b$ and $b \geq a$ imply $a = b$, $\geq$ is antisymmetric.
3. $a \geq b$ and $b \geq c$ imply $a \geq c$, $\geq$ is transitive.

It follows that $\geq$ is a partial ordering on the set of integers and $(Z, \geq)$ is a poset.

5.6. Let $(A, \leq)$ be a distributive lattice. Show that if $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ for some a then $x = y$.

Ans. We have

$$\begin{aligned} x &= x \vee (x \wedge a) = x \vee (y \wedge a) && (\because \text{ Given condition}) \\ &= (x \vee y) \wedge (x \vee a) \\ &= (x \vee y) \wedge (y \vee a) && (\because \text{ Distributive property}) \\ &= y \vee (x \wedge a) \\ x &= x \vee (y \wedge a) \\ x &= y \end{aligned}$$

5.7. Show that the "greater than or equal" relation (>=) is a partial ordering on the set of integers.

AKTU 2016-17, Marks 02

Ans. **Reflexive :**

$a \geq a \ \forall \ a \in Z$ (set of integer)

$(a, a) \in A$

$\therefore$ R is reflexive.

Antisymmetric : Let $(a, b) \in R$ and $(b, a) \in R$

$\Rightarrow$ $a \geq b$ and $b \geq a$

$\Rightarrow$ $a = b$

$\therefore$ R is antisymmetric.

Transitive : Let $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $a \geq$ b and b $\geq c$

$\Rightarrow$ $a \geq c \Rightarrow (a, c) \in R$

$\therefore$ R is transitive.

Hence, R is partial order relation.

5.8. Distinguish between bounded lattice and complemented lattice.

AKTU 2016-17, Marks 02

Ans. **Bounded lattice :** A lattice which has both elements 0 and 1 is called a bounded lattice.

Complemented lattice : A lattice L is called complemented lattice if it is bounded and if every element in L has complement.

5.9. Write the following in DNF $(x + y)(x' + y')$.

AKTU 2015-16, Marks 02

Ans. **Given :** $(x + y)(x' + y')$

The complete CNF in two variables (x, y)

$$= (x + y)(x' + y')(x + y')(x' + y')$$

Hence, $f'(x, y) = (x' + y)(x + y')$

$\therefore$ $[f'(x, y)]' = [(x' + y)(x + y')]'$

$$= xy' + x'y$$

which is the required DNF.

5.10. Draw the Hasse diagram of D_{30}.

AKTU 2018-19, Marks 02

Ans.

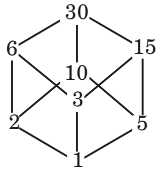

Fig. 5.10.1.

5.11. What is principle of duality ?

Ans. The principle of duality theorem says that a Boolean relation can derive another Boolean relation by,

1. Changing each OR sign to an AND sign
2. Changing each AND sign to an OR sign and
3. Complementing any 0 or 1 appearing in the expression.

For example : Dual of relation $A + \overline{A} = 1$ is $A . \overline{A} = 0$

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B. Tech.
(SEM. III) ODD SEMESTER THEORY EXAMINATION, 2014-15
DISCRETE STRUCTURE AND GRAPH THEORY

Time : 3 Hours **Max. Marks : 100**

1. Attempt any **four** parts : **(5 × 4 = 20)**

a. **Show that $R = \{(a, b) \mid a \equiv b \pmod{m}\}$ is an equivalence relation on Z. Show that if $x_1 \equiv y_1$ and $x_2 \equiv y_2$ then $(x_1 + x_2) \equiv (y_1 + y_2)$.**

b. **Prove for any two sets A and B that, $(A \cup B)' = A' \cap B'$.**

c. **Let R be binary relation on the set of all strings of 0's and 1's such that $R = \{(a, b) \mid a$ and b are strings that have the same number of 0's$\}$. Is R is an equivalence relation and a partial ordering relation ?**

d. **If f : $A \to B$, $g : B \to C$ are invertible functions, then show that gof : $A \to C$ is invertible and $(\text{go}f)^{-1} = f^{-1}\,\text{o}g^{-1}$.**

e. **Prove by the principle of mathematical induction, that the sum of finite number of terms of a geometric progression,**
$a + ar + ar^2 + \ldots ar^{n-1} = a(r^n - 1)/(r - 1)$ if $r \neq 1$.

f. **Let A {1, 2, 3,..........,13}. Consider the equivalence relation on $A \times A$ defined by $(a, b)\ R\ (c, d)$ if $a + d = b + c$. Find equivalence classes of (5,8).**

2. Attempt any **four** parts : **(5 × 4 = 20)**

a. **Prove that $(Z_6, (+_6))$ is an abelian group of order 6, where $Z_6 = \{0, 1, 2, 3, 4, 5\}$.**

b. **Let G be a group and let $a, b \in G$ be any elements. Then**
i. **$(a^{-1})^{-1} = a$**
ii. **$(a * b)^{-1} = b^{-1} * a^{-1}$.**

c. **Prove that the intersection of two subgroups of a group is also subgroup.**

d. Write and prove the Lagrange's theorem. If a group $G = \{...., -3, 2, -1, 0, 1, 2, 3, ...\}$ having the addition as binary operation. If H is a subgroup of group G where $x^2 \in H$ such that $x \in G$. What is H and its left coset w.r.t 1 ?

e. Consider a ring $(R, +, \bullet)$ defined by $a \cdot a = a$, determine whether the ring is commutative or not.

f. Show that every group of order 3 is cyclic.

3. Attempt any **two** parts : **$(10 \times 2 = 20)$**

a. The directed graph G for a relation R on set $A = \{1, 2, 3, 4\}$ is shown below :

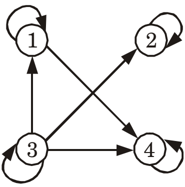


Fig. 1.

i. Verify that (A, R) is a poset and find its Hasse diagram.

ii. Is this a lattice ?

iii. How many more edges are needed in the Fig. 1 to extend (A, R) to a total order ?

iv. What are the maximal and minimal elements ?

b. If the lattice is represented by the Hasse diagram given below :

i. Find all the complements of 'e'.

ii. Prove that the given lattice is bounded complemented lattice.

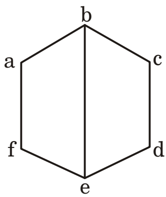


Fig. 2.

c. Consider the Boolean function

a. $f(x_1, x_2, x_3, x_4) = x_1 + (x_2 . (x'_1 + x_4) + x_3 . (x'_2 + x'_4))$

i. Simplify f algebraically.

ii. Draw the logic circuit of the f and the reduction of the f.

b. Write the expressions $E1 = (x + xy) + (x/y)$ and $E2 = x + ((xy + y)/y)$, into

i. Prefix notation ii. Postfix notation

4. Attempt any **two** parts : **(10 × 2 = 20)**

a. i. Show that $((p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \vee r)$ is a tautology without using truth table.

ii. Rewrite the following arguments using quantifiers, variables and predicate symbols :

a. All birds can fly.

b. Some men are genius.

c. Some numbers are not rational.

d. There is a student who likes mathematics but not geography.

b. "If the labour market is perfect then the wages of all persons in a particular employment will be equal. But it is always the case that wages for such persons are not equal therefore the labour market is not perfect". Test the validity of this argument using truth table.

c. Explain the following terms with suitable example :

i. Conjunction

ii. Disjunction

iii. Conditional

iv. Converse

v. Contrapositive

5. Attempt any **two** parts : **(10 × 2 = 20)**

a. Solve the recurrence relation by the method of generating function

$a_r - 7a_{r-1} + 10a_{r-2} = 0, r \geq 2$

Given $a_0 = 3$ and $a_1 = 3$.

b. Solve the recurrence relation

$a_{r+2} - 5a_{r+1} + 6a_r = (r + 1)^2$

c. Explain the following terms with example :

i. Homomorphism and Isomorphism graph

ii. Euler graph and Hamiltonian graph

iii. Planar and Complete bipartite graph

☺☺☺

SOLUTION OF PAPER (2014-15)

1. Attempt any **four** parts : **(5 × 4 = 20)**

a. Show that $R = \{(a, b) | a \equiv b \pmod m\}$ is an equivalence relation on Z. Show that if $x_1 \equiv y_1$ and $x_2 \equiv y_2$ then $(x_1 + x_2) \equiv (y_1 + y_2)$.

Ans. $R = \{(a, b) \mid a \equiv b \pmod m\}$

For an equivalence relation it has to be reflexive, symmetric and transitive.

Reflexive : For reflexive $\forall\, a \in Z$ we have $(a, a) \in R$ *i.e.*,

$a \equiv a \pmod m$

$\Rightarrow$ $a - a$ is divisible by m *i.e.*, 0 is divisible by m

Therefore aRa, $\forall\, a \in Z$, it is reflexive.

Symmetric : Let $(a, b) \in Z$ and we have

$(a, b) \in R$ *i.e.*, $a \equiv b \pmod m$

$\Rightarrow$ $a - b$ is divisible by m

$\Rightarrow$ $a - b = km$, k is an integer

$\Rightarrow$ $(b - a) = (-k)\, m$

$\Rightarrow$ $(b - a) = p\, m$, p is also an integer

$\Rightarrow$ $b - a$ is also divisible by m

$\Rightarrow$ $b \equiv a \pmod m \Rightarrow (b, a) \in R$

It is symmetric.

Transitive : Let $(a, b) \in R$ and $(b, c) \in R$ then

$(a, b) \in R \Rightarrow a - b$ is divisible by m

$\Rightarrow$ $a - b = t\, m$, t is an integer ...(1)

$(b, c) \in R \Rightarrow b - c$ is divisible by m

$\Rightarrow$ $b - c = s\, m$, s is an integer ...(2)

From eq. (1) and (2)

$$a - b + b - c = (t + s)\, m$$

$$a - c = lm, \; l \text{ is also an integer}$$

$a - c$ is divisible by m

$a \equiv c \pmod m$, yes it is transitive.

R is an equivalence relation.

To show : $(x_1 + x_2) \equiv (y_1 + y_2)$:

It is given $x_1 \equiv y_1$ and $x_2 \equiv y_2$

i.e., $x_1 - y_1$ divisible by m

$x_2 - y_2$ divisible by m

Adding above equation :

$(x_1 - y_1) + (x_2 - y_2)$ is divisible by m

$\Rightarrow$ $(x_1 + x_2) - (y_1 + y_2)$ is divisible by m

i.e., $(x_1 + x_2) \equiv (y_1 + y_2)$

b. Prove for any two sets A and B that, $(A \cup B)' = A' \cap B'$.

Ans.

Let $x \in (A \cup B)'$

$\Rightarrow$ $x \notin A \cup B$

$\Rightarrow \quad x \notin A$ and $x \notin B$

$\Rightarrow \quad x \in A'$ and $x \in B'$

$\Rightarrow \quad x \in A' \cap B'$

$\Rightarrow \quad (A \cup B)' \subseteq A' \cap B'$ ...(1)

Now, let $\quad x \in A' \cap B'$

$\Rightarrow \quad x \in A'$ and $x \in B'$

$\Rightarrow \quad x \notin A$ and $x \notin B$

$\Rightarrow \quad x \notin (A \cup B)$

$x \in (A \cup B)'$

$(A' \cap B') \subseteq (A \cup B)'$ ...(2)

From eq. (1) and (2), $(A \cup B)' = A' \cap B'$

c. Let R be binary relation on the set of all strings of 0's and 1's such that $R = \{(a, b) \mid a$ and b are strings that have the same number of 0's$\}$. Is R is an equivalence relation and a partial ordering relation ?

Ans. **For equivalence relation :**

Reflexive : $a\,R\,a \Rightarrow (a, a) \in R\ \forall a \in R$

where a is a string of 0's and 1's.

Always a is related to a because both a has same number of 0's.

It is reflexive.

Symmetric : Let $(a, b) \in R$

then a and b both have same number of 0's which indicates that again both b and a will also have same number of zeros. Hence $(b, a) \in R$. It is symmetric.

Transitive : Let $(a, b) \in R$, $(b, c) \in R$

$(a, b) \in R \Rightarrow a$ and b have same number of zeros.

$(b, c) \in R \Rightarrow b$ and c have same number of zeros.

Therefore a and c also have same number of zeros, hence $(a, c) \in R$.

It is transitive.

$\therefore \quad R$ is an equivalence relation.

For partial order, it has to be reflexive, antisymmetric and transitive. Since, symmetricity and antisymmetricity cannot hold together. Therefore, it is not partial order relation.

d. If $f : A \to B$, $g : B \to C$ are invertible functions, then show that $g \circ f : A \to C$ is invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Ans. If $f: A \to B$ and $g : B \to C$ be one-to-one onto functions, then $g \circ f$ is also one-onto and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Proof. Since f is one-to-one, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for $x_1, x_2 \in R$

Again since g is one-to-one, $g(y_1) = g(y_2) \Rightarrow y_1 = y_2$ for $y_1, y_2 \in R$

Now $g \circ f$ is one-to-one, since $(g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow g[f(x_1)] = g[f(x_2)]$

$\Rightarrow \quad f(x_1) = f(x_2)$ [g is one-to-one]

$\Rightarrow \quad x_1 = x_2$ [f is one-to-one]

Since g is onto, for $z \in C$, there exists $y \in B$ such that $g(y) = z$. Also f being onto there exists $x \in A$ such that $f(x) = y$. Hence $z = g(y)$

$= g\ [f(x)] = (g \circ f)(x)$

This shows that every element $z \in C$ has pre-image under $g \circ f$. So, $g \circ f$ is onto.

Thus, $g \circ f$ is one-to-one onto function and hence $(g \circ f)^{-1}$ exists.

By the definition of the composite functions, $g \circ f : A \to C$. So, $(g \circ f)^{-1} : C \to A$.

Also $g^{-1} : C \to B$ and $f^{-1} : B \to A$.

Then by the definition of composite functions, $f^{-1} \circ g^{-1} : C \to A$.

Therefore, the domain of $(g \circ f)^{-1}$ = the domain of $f^{-1} \circ g^{-1}$.

$$\begin{aligned}
\text{Now}(g \circ f)^{-1}(z) = x &\Leftrightarrow (g \circ f)(x) = z \\
&\Leftrightarrow g(f(x)) = z \\
&\Leftrightarrow g(y) = z \text{ where } y = f(x) \\
&\Leftrightarrow y = g^{-1}(z) \\
&\Leftrightarrow f^{-1}(y) = f^{-1}(g^{-1}(z)) = (f^{-1} \circ g^{-1})(z) \\
&\Leftrightarrow x = (f^{-1} \circ g^{-1})(z) \qquad [f^{-1}(y) = x]
\end{aligned}$$

Thus, $(g \circ f)^{-1}(z) = (f^{-1} \circ g^{-1})(z)$.

So, $\quad (g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

e. Prove by the principle of mathematical induction, that the sum of finite number of terms of a geometric progression,

$a + ar + ar^2 + \dots ar^{n-1} = a(r^n - 1)/(r - 1)$ if $r \neq 1$.

Ans. **Basis :** True for $n = 1$ *i.e.*,

$$\text{L.H.S} = a$$

$$\text{R.H.S} = \frac{a(r-1)}{r-1} = a$$

Therefore, L.H.S. = R.H.S.

Induction : Let it be true for $n = k$ *i.e.*,

$$a + ar + ar^2 + \dots + ar^{k-1} = \frac{a(r^k - 1)}{r - 1} \qquad \dots(1)$$

Now we will show that it is true for $n = k + 1$ using eq. (1)

i.e., $a + ar + ar^2 + \dots + ar^{k-1} + ar^k$

Using eq. (1), we get

$$\frac{a(r^k - 1)}{r - 1} + ar^k$$

$$= \frac{ar^k - a + ar^{k+1} - ar^k}{r - 1} = \frac{a(r^{k+1} - 1)}{r - 1}$$

which is R.H.S. for $n = k + 1$, hence it is true for $n = k + 1$.

By mathematical induction, it is true for all n.

f. Let A {1, 2, 3,..........,13}. Consider the equivalence relation on $A \times A$ defined by $(a, b)\ R\ (c, d)$ if $a + d = b + c$. Find equivalence classes of (5,8).

Ans. $A = \{1, 2, 3, \dots, 13\}$

$$\begin{aligned}[(5, 8)] &= [(a, b) : (a, b)\, R\, (5, 8), (a, b) \in A \times A] \\ &= [(a, b) : a + 8 = b + 5] \\ &= [(a, b) : a + 3 = b] \\ [5, 8] &= \{(1, 4), (2, 5), (3, 6), (4, 7) \\ &\quad (5, 8), (6, 9), (7, 10), (8, 11) \\ &\quad (9, 12), (10, 13)\}\end{aligned}$$

2. Attempt any **four** parts : **(5 × 4 = 20)**

a. Prove that (Z_6, ($+_6$)) is an abelian group of order 6, where Z_6 = {0, 1, 2, 3, 4, 5}.

Ans. The composition table is :

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Since $2 +_6 1 = 3$

$4 +_6 5 = 3$

From the table we get the following observations :

Closure : Since all the entries in the table belong to the given set Z_6. Therefore, Z_6 is closed with respect to addition modulo 6.

Associativity : The composition '$+_6$' is associative. If a, b, c are any three elements of Z_6,

$a +_6 (b +_6 c) = a +_6 (b + c)$ $\quad [\because\ b +_6 c = b + c \pmod 6]$

= least non-negative remainder when $a + (b + c)$ is divided by 6.

= least non-negative remainder when $(a + b) + c$ is divided by 6.

$= (a + b) +_6 c = (a +_6 b) +_6 c$.

Identity : We have $0 \in Z_6$. If a is any element of Z_6, then from the composition table we see that

$0 +_6 a = a = a +_6 0$

Therefore, 0 is the identity element.

Inverse : From the table we see that the inverse of 0, 1, 2, 3, 4, 5 are 0, 5, 4, 3, 2, 1 respectively. For example $4 +_6 2 = 0 = 2 +_6 4$ implies 4 is the inverse of 2.

Commutative : The composition is commutative as the elements are symmetrically arranged about the main diagonal. The number of elements in the set Z_6 is 6.

$\therefore$ (Z_6, $+_6$) is a finite abelian group of order 6.

b. Let G be a group and let $a, b \in G$ be any elements. Then

i. $(a^{-1})^{-1} = a$

ii. $(a * b)^{-1} = b^{-1} * a^{-1}$.

Ans.

i. Let e be the identity element for * in G.

Then we have $a * a^{-1} = e$, where $a^{-1} \in G$.
Also $(a^{-1})^{-1} * a^{-1} = e$
Therefore, $(a^{-1})^{-1} * a^{-1} = a * a^{-1}$.
Thus, by right cancellation law, we have $(a^{-1})^{-1} = a$.

ii. Let a and $b \in G$ and G is a group for $*$, then $a * b \in G$ (closure)
Therefore, $(a * b)^{-1} * (a * b) = e$. ....(1)
Let a^{-1} and b^{-1} be the inverses of a and b respectively, then a^{-1}, $b^{-1} \in G$.
Therefore, $(b^{-1} * a^{-1}) * (a * b) = b^{-1} * (a^{-1} * a) * b$ (associativity)
$= b^{-1} * e * b = b^{-1} * b = e$ ...(2)
From eq. (1) and (2) we have,
$(a * b)^{-1} * (a * b) = (b^{-1} * a^{-1}) * (a * b)$
$(a * b)^{-1} = b^{-1} * a^{-1}$ (by right cancellation law)

c. Prove that the intersection of two subgroups of a group is also subgroup.

Ans. Let H_1 and H_2 be any two subgroups of G. Since at least the identity element e is common to both H_1 and H_2.
$\therefore \quad H_1 \cap H_2 \neq \phi$
In order to prove that $H_1 \cap H_2$ is a subgroup, it is sufficient to prove that
$a \in H_1 \cap H_2,\ b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$
Now $a \in H_1 \cap H_2 \Rightarrow a \in H_1$ and $a \in H_2$
$b \in H_1 \cap H_2 \Rightarrow b \in H_1$ and $b \in H_2$
But H_1, H_2 are subgroups. Therefore,
$a \in H_1,\ b \in H_1 \Rightarrow ab^{-1} \in H_1$
$a \in H_2,\ b \in H_2 \Rightarrow ab^{-1} \in H_2$
Finally, $ab^{-1} \in H_1,\ ab^{-1} \in H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$
Thus, we have shown that
$a \in H_1 \cap H_2,\ b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$.
Hence, $H_1 \cap H_2$ is a subgroup of G.

d. Write and prove the Lagrange's theorem. If a group $G = \{...., -3, 2, -1, 0, 1, 2, 3,.....\}$ having the addition as binary operation. If H is a subgroup of group G where $x^2 \in H$ such that $x \in G$. What is H and its left coset w.r.t 1 ?

Ans. **Lagrange's theorem :**
If G is a finite group and H is a subgroup of G then o(H) divides o(G). Moreover, the number of distinct left (right) cosets of H in G is o(G)/o(H).
Proof : Let H be subgroup of order m of a finite group G of order n.
Let $H \{h_1, h_2, ..., h_m\}$
Let $a \in G$. Then aH is a left coset of H in G and $aH = \{ah_1, ah_2, ..., ah_m\}$ has m distinct elements as $ah_i = ah_j \Rightarrow h_i = h_j$ by cancellation law in G.
Thus, every left coset of H in G has m distinct elements.

Since G is a finite group, the number of distinct left cosets will also be finite. Let it be k. Then the union of these k-left cosets of H in G is equal to G.

i.e., if $a_1H, a_2H, ..., a_kH$ are right cosets of H in G then

$$G = a_1H \cup a_2H \cup ... \cup a_kH.$$

$$\therefore \quad o(G) = o(a_1H) + o(a_2H) + ... + o(a_kH)$$

(Since two distinct left cosets are mutually disjoint.)

$$\Rightarrow \quad n = m + m + ... + m \ (k \text{ times})$$

$$\Rightarrow \quad n = mk \quad \Rightarrow \quad k = \frac{n}{m}$$

$$\therefore \quad k = \frac{o(G)}{o(H)}.$$

Thus order of each subgroup of a finite group G is a divisor of the order of the group.

Numerical :

$$H = \{x^2 : x \in G\} = \{0, 1, 4, 9, 16, 25 ....\}$$

Left coset of H will be $1 + H = \{1, 2, 5, 10, 17, 26, ...\}$

e. Consider a ring $(R, +, \bullet)$ defined by $a \cdot a = a$, determine whether the ring is commutative or not.

Ans. Let $a, b \in R$ $(a + b)^2 = (a + b)$

$$\Rightarrow \quad (a + b)(a + b) = (a + b)$$

$$(a + b)a + (a + b)b = (a + b)$$

$$(a^2 + ba) + (ab + b^2) = (a + b)$$

$$(a + ba) + (ab + b) = (a + b) \qquad (\because a^2 = a \text{ and } b^2 = b)$$

$$(a + b) + (ba + ab) = (a + b) + 0$$

$$\Rightarrow \quad ba + ab = 0$$

$a + b = 0 \Rightarrow a + b = a + a$ [being every element of its own additive inverse]

$$\Rightarrow \quad b = a$$

$$\Rightarrow \quad ab = ba$$

$\therefore R$ is commutative ring.

f. Show that every group of order 3 is cyclic.

Ans.

1. Suppose G is a finite group whose order is a prime number p, then to prove that G is a cyclic group.
2. An integer p is said to be a prime number if $p \neq 0$, $p \neq \pm 1$, and if the only divisors of p are $\pm 1, \pm p$.
3. Some G is a group of prime order, therefore G must contain at least 2 element. Note that 2 is the least positive prime integer.
4. Therefore, there must exist an element $a \in G$ such that $a \neq$ the identity element e.

5. Since a is not the identity element, therefore $o(a)$ is definitely ≥ 2. Let $o(a) = m$. If H is the cyclic subgroup of G generated by a then $o\ (H = o\ (a) = m)$.
6. By Lagrange's theorem m must be a divisor of p. But p is prime and $m \geq 2$. Hence, $m = p$.
7. $\therefore H = G$. Since H is cyclic therefore G is cyclic and a is a generator of G.

3. Attempt any **two** parts : **(10 × 2 = 20)**

a. The directed graph G for a relation R on set $A = \{1, 2, 3, 4\}$ is shown below :

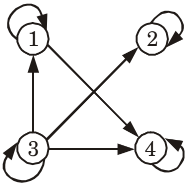


Fig. 1.

i. Verify that (A, R) is a poset and find its Hasse diagram.
ii. Is this a lattice ?
iii. How many more edges are needed in the Fig. 1 to extend (A, R) to a total order ?
iv. What are the maximal and minimal elements ?

Ans.

i. The relation R corresponding to the given directed graph is,
$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (3, 1), (3, 4), (1, 4), (3, 2)\}$
R is a partial order relation if it is reflexive, antisymmetric and transitive.

Reflexive : Since $aRa, \forall\, a \in A$. Hence, it is reflexive.

Antisymmetric : Since aRb and bRa then we get $a = b$ otherwise aRb or bRa.
Hence, it is antisymmetric.
Transitive: For every aRb and bRc we get aRc. Hence, it is transitive.
Therefore, we can say that (A, R) is poset. Its Hasse diagram is :

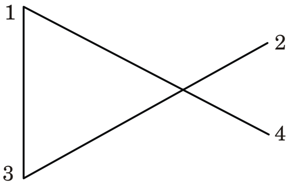


Fig. 2.

ii. Since there is no *lub* of 1 and 2 and same for 2 and 4. The given poset is not a lattice.

$\vee$	1	2	3	4
1	1	–	1	1
2	–	2	2	–
3	1	2	3	1
4	1	–	1	4

iii. Only one edge (4, 2) is included to make it total order.

iv. Maximals are {1, 2} and minimals are {3, 4}.

b. If the lattice is represented by the Hasse diagram given below :

i. Find all the complements of 'e'.

ii. Prove that the given lattice is bounded complemented lattice.

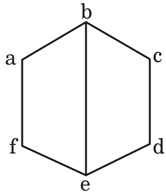


Fig. 3.

Ans.

i. In a given lattice, greatest element is b and least element is e. An element x in lattice is called a complement of element y if

$y \vee x = b$ and $y \wedge x = e$

For element e,

$e \vee b = b$, $e \wedge b = e$

So, complement of e is b

ii. **Proof :**

For bounded complemented lattice, every element in lattice has a complement and lattice is bounded. Since, given lattice have greatest and least element. So, the given lattice is bounded.

Now the complement of all elements is given below :

Complement of a = $\{c, d\}$

Complement of b = $\{e\}$

Complement of c = $\{a, f\}$

Complement of d = $\{a, f\}$

Complement of e = $\{b\}$

Complement of f = $\{c, d\}$

Since, complement of every element exists and lattice is bounded. So, the given lattice is bounded complemented lattice.

Hence proved.

c. Consider the Boolean function

a. $f(x_1, x_2, x_3, x_4) = x_1 + (x_2 . (x'_1 + x_4) + x_3 . (x'_2 + x'_4))$

i. Simplify f algebraically.

ii. Draw the logic circuit of the f and the reduction of the f.

b. Write the expressions $E1 = (x + xy) + (x/y)$ and $E2 = x + ((xy + y)/y)$, into

i. Prefix notation ii. Postfix notation

Ans.

a. i.
$$\begin{aligned} f(x_1, x_2, x_3, x_4) &= x_1 + (x_2.(x_1' + x_4) + x_3.(x_2' + x_4') \\ &= x_1 + x_2.x_1' + x_2.x_4 + x_3.x_2' + x_3.x_4' \\ &= x_1 + x_2 + x_2.x_4 + x_3.x_2' + x_3.x_4' \\ &= x_1 + x_2.(1 + x_4) + x_3.x_2' + x_3.x_4' \\ &= x_1 + x_2 + x_3.x_2' + x_3.x_4' \\ &= x_1 + x_2 + x_3 + x_3.x_4' \\ &= x_1 + x_2 + x_3.(1 + x_4') \\ &= x_1 + x_2 + x_3 \end{aligned}$$

ii. Logic circuit :

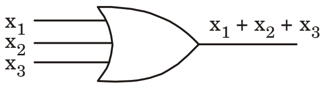


Fig. 4.

Reduction of f :
$$\begin{aligned} f(x_1, x_2, x_3, x_4) &= x_1 + (x_2.(x_1' + x_4) + x_3.(x_2' + x_4') \\ &= x_1 + (x_2.x_1' + x_2.x_4) + (x_3.x_2' + x_3.x_4) \\ &= x_1 + x_2.x_1' + x_2.x_4 + x_3.x_2' + x_3.x_4 \end{aligned}$$

b. $E_1 = (x + x * y) + (x/y)$

Binary tree is :

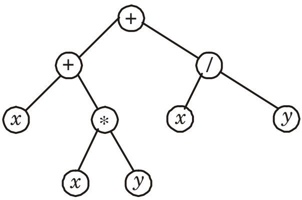


Fig. 5.

Prefix : $+ + x * x\ y\ /\ xy$

Postfix : $xx\ y * + xy / +$

$E_2 = x + ((x * y + y)/y)$

Binary tree is :

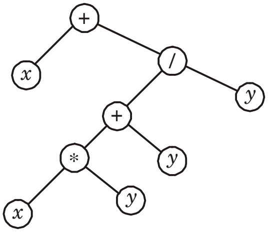


Fig. 6.

Prefix : $+ x / + * x y y y$
Postfix : $x x y * y + y / +$

4. Attempt any **two** parts : **(10 × 2 = 20)**

a. i. Show that $((p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \vee r)$ is a tautology without using truth table.

ii. Rewrite the following arguments using quantifiers, variables and predicate symbols :

a. All birds can fly.

b. Some men are genius.

c. Some numbers are not rational.

d. There is a student who likes mathematics but not geography.

Ans.

i. We have

$((p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \vee r)$

$\equiv ((p \vee q) \wedge \sim(\sim p \wedge \sim(q \wedge r))) \vee (\sim(p \vee q) \vee \sim(p \vee r))$

(Using De Morgan's Law)

$\equiv [(p \vee q)] \wedge (p \vee (q \wedge r)) \vee \sim((p \vee q) \wedge (p \vee r))$

$\equiv [(\mathrm{p} \vee q) \wedge (p \vee q) \wedge (p \wedge r)] \vee \sim((p \vee q) \wedge (p \vee r))$

(Using Distributive Law)

$\equiv [((p \vee q) \wedge (p \vee q)] \wedge (p \vee r) \vee \sim((p \vee q) \wedge (p \vee r))$

$\equiv ((p \vee q) \wedge (p \vee r)) \vee \sim((p \vee q) \wedge (p \vee r))$

$\equiv x \vee \sim x$ where $x = (p \vee q) \wedge (p \wedge r)$

$\equiv T$

ii.

a. $\forall x\ [B(x) \Rightarrow F(x)]$

b. $\exists x\ [M(x) \wedge G(x)]$

c. $\sim [\ \exists(x)\ (N(x) \wedge \mathrm{R}(x)]$

d. $\exists x\ [S(x) \wedge M(x) \wedge \sim G(x)]$

b. "If the labour market is perfect then the wages of all persons in a particular employment will be equal. But it is always the case that wages for such persons are not equal therefore the labour market is not perfect". Test the validity of this argument using truth table.

Ans. Let p_1 : The labour market is perfect.

p_2 : Wages of all persons in a particular employment will be equal.

$\sim p_2$: Wages for such persons are not equal.

$\sim p_1$: The labour market is not perfect.

The premises are $p_1 \Rightarrow p_2$, $\sim p_2$ and the conclusion is $\sim p_1$. The argument $p_1 \Rightarrow p_2, \sim p_2 \Rightarrow \sim p_1$ is valid if $((p_1 \Rightarrow p_2) \wedge \sim p_2) \Rightarrow \sim p_1$ is a tautology. Its truth table is,

p_1	p_2	$\sim p_1$	$\sim p_2$	$p_1 \Rightarrow p_2$	$(p_1 \Rightarrow p_2) \wedge \sim p_2$	$(p_1 \Rightarrow p_2 \wedge \sim p_2) \Rightarrow \sim p_1$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Since $((p_1 \Rightarrow p_2) \wedge \sim p_2) \Rightarrow \sim p_1$ is a tautology. Hence, this is valid argument.

c. Explain the following terms with suitable example :
i. Conjunction
ii. Disjunction
iii. Conditional
iv. Converse
v. Contrapositive

Ans.

i. Conjunction : If p and q are two statements, then conjunction of p and q is the compound statement denoted by $p \wedge q$ and read as "p and q". Its truth table is,

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example :

***p* :** Ram is healthy.

***q* :** He has blue eyes.

$p \wedge q$: Ram is healthy and he has blue eyes.

ii. Disjunction : If p and q are two statements, the disjunction of p and q is the compound statement denoted by $p \vee q$ and it is read as "p or q". Its truth table is,

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example :
***p* :** Ram will go to Delhi.
***q* :** Ram will go to Calcutta.
$p \vee q$: Ram will go to Delhi or Calcutta.

iii. **Conditional :** If p and q are propositions. The compound proposition if p then q denoted by $p \Rightarrow q$ or $p \to q$ and is called conditional proposition or implication. It is read as "If p then q" and its truth table is,

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example :
***p* :** Ram works hard.
***q* :** He will get good marks.
$p \to q$: If Ram works hard then he will get good marks.

For converse and contrapositive :
Let p : It rains.
q : The crops will grow.

iv. **Converse :** If $p \Rightarrow q$ is an implication then its converse is given by $q \Rightarrow p$ states that S : If the crops grow, then there has been rain.

v. **Contrapositive :** If $p \Rightarrow q$ is an implication then its contrapositive is given by $\sim q \Rightarrow \sim p$ states that,
t : If the crops do not grow then there has been no rain.

Inverse :
If $p \Rightarrow q$ is implication the inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$.
Consider the statement
p : It rains.
q : The crops will grow
The implication $p \Rightarrow q$ states that,
r : If it rains then the crops will grow.
The inverse of the implication $p \Rightarrow q$, namely $\sim p \Rightarrow \sim q$ states that.
u : If it does not rain then the crops will not grow.

5. Attempt any **two** parts : **(10 × 2 = 20)**

a. Solve the recurrence relation by the method of generating function

$a_r - 7\,a_{r-1} + 10a_{r-2} = 0,\ r \geq 2$

Given $a_0 = 3$ and $a_1 = 3$.

Ans. $a_r - 7a_{r-1} + 10\,a_{r-2} = 0,\ r \geq 2$

Multiply by x^r and take sum from 2 to ∞.

$$\sum_{r=2}^{\infty} a_r\, x^r - 7\sum_{r=2}^{\infty} a_{r-1}\, x^r + 10\sum_{r=2}^{\infty} a_{r-2}\, x^r = 0$$

$(a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + ...) - 7\,(a_1\,x^2 + a_2\,x^3 + ...)$

$+ 10\,(a_0\,x^2 + a_1\,x^3 + ...) = 0$

We know that

$$G(x) = \sum_{r=0}^{\infty} a_r\, x^r = a_0 + a_1\, x + ...$$

$G(x) - a_0 - a_1\,x - 7\,x\,(G(x) - a_0) + 10\,x^2\,G(x) = 0$

$G(x)\,[1 - 7\,x + 10\,x^2] = a_0 + a_1\,x - 7\,a_0\,x$

$= 3 + 3x - 21\,x = 3 - 18\,x$

$$G(x) = \frac{3-18x}{10\,x^2 - 7x + 1} = \frac{3-18x}{10\,x^2 - 5x - 2x + 1}$$

$$= \frac{3-18x}{5x\,(2\,x-1) - 1\,(2\,x-1)} = \frac{3-18x}{(5x-1)(2\,x-1)}$$

Now,

$$\frac{3-18x}{(5x-1)\,(2x-1)} = \frac{A}{5\,x-1} + \frac{B}{2\,x-1}$$

$$3 - 18\,x = A\,(2\,x-1) + B\,(5\,x-1)$$

put $x = \dfrac{1}{2}$

$$3 - 9 = B\left(\frac{5}{2} - 1\right) \Rightarrow -6 = \frac{3}{2}\,B \Rightarrow B = -4$$

put $x = \dfrac{1}{5}$

$$3 - \frac{18}{5} = A\left(\frac{2}{5} - 1\right) \Rightarrow -\frac{3}{5} = -\frac{3}{5}A = 1 \Rightarrow \text{A} = 1$$

$$\therefore \quad G(x) = \frac{1}{5\,x-1} - \frac{4}{2x-1} = \frac{4}{1-2x} - \frac{1}{1-5x}$$

$$\therefore \quad a^r = 4.2^r - 5^r$$

b. Solve the recurrence relation

$a_{r+2} - 5a_{r+1} + 6\,a_r = (r+1)^2$

Ans. $a_{r+2} - 5\,a_{r+1} + 6\,a_r = (r+1)^2 = r^2 + 2r + 1$ ...(1)

Now the characteristic equation is :

$$x^2 - 5x + 6 = 0$$

$$(x-3)(x-2) = 0 \Rightarrow x = 3, 2$$

The homogeneous solution is :

$$a_r^{(h)} = C_1 2^r + C_2 3^r$$

Let the particular solution be :

$$a_r^{(p)} = A_0 + A_1 r + A_2 r^2$$

From eq. (1)

$$A_0 + A_1(r+2) + A_2(r+2)^2 - 5\{A_0 + A_1(r+1)\} + A_2(r+1)^2\} + 6A_0 + 6A_1 r + 6A_2 r^2$$

$$= r^2 + 2r + 1$$

$$(A_0 + 2A_1 + 4A_2 - 5A_0 - 5A_1 - 5A_2 + 6A_0) + r(A_1 + 4A_2 - 5A_1 - 10A_2 + 6A_1)$$

$$+ r^2(A_2 - 5A_2 + 6A_2) = r^2 + 2r + 1$$

Comparing both sides, we get,

$$2A_0 - 3A_1 - A_2 = 1 \qquad ...(2)$$

$$2A_1 - 6A_2 = 2 \qquad ...(3)$$

$$2A_2 = 1 \Rightarrow A_2 = 1/2$$

From eq. (3), $2A_1 - 3 = 2$

$$A_1 = \frac{5}{2}$$

From eq. (2)

$$2A_0 - \frac{15}{2} - \frac{1}{2} = 1$$

$$2A_0 - 8 = 1 \Rightarrow A_0 = \frac{9}{2}$$

$$\therefore \qquad a_r^{(p)} = \frac{9}{2} + \frac{5}{2}r + \frac{r^2}{2}$$

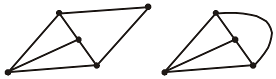
The final solution is,

$$a_r = a_r^{(h)} + a_r^{(p)} = C_1 2^r + C_2 3^r + \frac{9}{2} + \frac{5}{2}r + \frac{r^2}{2}$$

c. Explain the following terms with example :
i. Homomorphism and Isomorphism graph
ii. Euler graph and Hamiltonian graph
iii. Planar and Complete bipartite graph

Ans.

i. **Homomorphism of graph :** Two graphs are said to be homomorphic if one graph can be obtained from the other by the creation of edges in series (*i.e.*, by insertion of vertices of degree two) or by the merger of edges in series.



(*a*) (*b*) (*c*)

Fig. 7.

Isomorphism of graph : Two graphs are isomorphic to each other if :

i. Both have same number of vertices and edges.
ii. Degree sequence of both graphs are same (degree sequence is the sequence of degrees of the vertices of a graph arranged in non-increasing order).

Example :

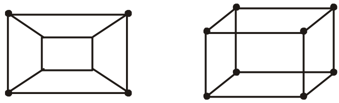


Fig. 8.

ii. **Eulerian path :** A path of graph *G* which includes each edge of *G* exactly once is called Eulerian path.

Eulerian circuit : A circuit of graph *G* which include each edge of *G* exactly once.

Eulerain graph : A graph containing an Eulerian circuit is called Eulerian graph.

For example : Graphs given below are Eulerian graphs.

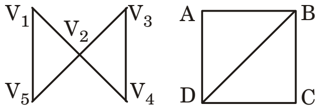
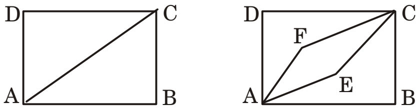


Fig. 9.

Hamiltonian graph : A Hamiltonian circuit in a graph *G* is a closed path that visit every vertex in *G* exactly once except the end vertices. A graph *G* is called Hamiltonian graph if it contains a Hamiltonian circuit.

For example : Consider graphs given below :



(*a*) (*b*)

Fig. 10.

Graph given is Fig. 10(*a*) is a Hamiltonian graph since it contains a Hamiltonian circuit $A-B-C-D-A$ while graph in Fig 10(*b*) is not a Hamiltonian graph.

Hamiltonian path : The path obtained by removing any one edge from a Hamiltonian circuit is called Hamiltonian path. Hamiltonian path is subgraph of Hamiltonian circuit. But converse is not true.

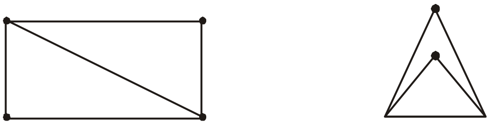
The length of Hamiltonian path in a connected graph of n vertices is $n - 1$ if it exists.

iii. Planar graph :

A graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect except only at the common vertex.

i. A graph is said a planar graph, if it cannot be drawn on a plane without a crossover between its edges crossing.

ii. The graphs shown in Fig. 11(a) and (b) are planar graphs.



(*a*) (*b*)

Fig. 11. Some planar graph.

Complete bipartite graph : The complete bipartite graph on m and n vertices, denoted $K_{m,\, n}$ is the graph, whose vertex set is partitioned into sets V_1 with m vertices and V_2 with n vertices in which there is an edge between each pair of vertices ν_1 and ν_2 where ν_1 is in V_1 and ν_2 is in V_2. The complete bipartite graphs $K_{2,3}$, $K_{2,4}$, $K_{3,3}$, $K_{3,5}$, and $K_{2,6}$

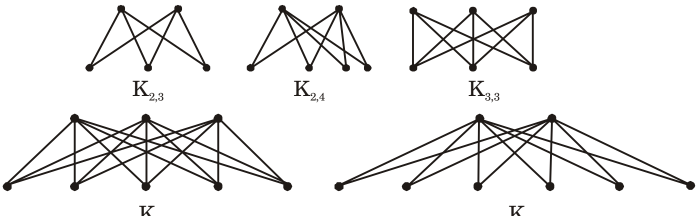


Fig. 12. Some complete bipartite graphs.

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B.Tech.

(SEM. III) ODD SEMESTER THEORY EXAMINATION, 2015-16

DISCRETE STRUCTURE AND GRAPH THEORY

Time : 3 Hours **Max. Marks : 100**

Note : Attempt **all** parts. All parts carry equal marks. Write answer of each part in short. **(2 × 10 = 20)**

SECTION – A

1. a. Define multiset and power set. Determine the power set A = {1, 2}.

b. Show that $[((p \vee q) \rightarrow r) \wedge (\sim p)] \rightarrow (q \wedge r)$ is tautology or contradiction.

c. State and prove pigeonhole principle.

d. Show that if set A has 3 elements, then we can have 2^6 symmetric relation on A.

e. Prove that $(P \vee Q) \rightarrow (P \wedge Q)$ is logically equivalent to $P \leftrightarrow Q$.

f. How many 4 digit numbers can be formed by using the digits 2, 4, 6, 8 when repetition of digits is allowed ?

g. The converse of a statement is : If a steel rod is stretched, then it has been heated. Write the inverse of the statement.

h. If a and b are any two elements of group G then prove $(a * b)^{-1} = (b^{-1} * a^{-1})$.

i. If $f : A \rightarrow B$ is one-one onto mapping, then prove that $f^{-1} : B \rightarrow A$ will be one-one onto mapping.

j. Write the following in DNF $(x + y)(x' + y')$.

SECTION – B

2. Prove that $n^3 + 2n$ is divisible by 3 using principle of mathematical induction, where n is natural number.

3. Solve the recurrence relation using generating function : $a_n - 7a_{n-1} + 10a_{n-2} = 0$ with $a_0 = 3$, $a_1 = 3$.

4. Express the following statements using quantifiers and logical connectives.

a. Mathematics book that is published in India has a blue cover.

b. All animals are mortal. All human being are animal. Therefore, all human being are mortal.

c. There exists a mathematics book with a cover that is not blue.

d. He eats crackers only if he drinks milk.

e. There are mathematics books that are published outside India.

f. Not all books have bibliographies.

5. Draw the Hasse diagram of $[P(a, b, c), \subseteq]$ (Note : '$\subseteq$' stands for subset). Find greatest element, least element, minimal element and maximal element.

6. Simplify the following boolean expressions using k-map :

a. $Y = ((AB)' + A' + AB)'$

b. $A'B'C'D' + A'B'C'D + A'B'CD + A'B'CD' = A'B'$

7. Let G be the set of all non-zero real number and let $a*b = ab/2$. Show that $(G*)$ be an abelian group.

8. The following relation on A = {1, 2, 3, 4}. Determine whether the following :

a. R = {(1, 3), (3, 1), (1, 1), (1, 2), (3, 3), (4, 4)}.

b. R = AXA

Is an equivalence relation or not.

9. If the permutation of the elements of {1, 2, 3, 4, 5} are given by $a = (1\ 2\ 3)\ (4\ 5)$, $b = (1)\ (2)\ (3)\ (4\ 5)$, $c = (1\ 5\ 2\ 4)\ (3)$. Find the value of x, if $ax = b$. And also prove that the set $Z_4 = (0, 1, 2, 3)$ is a commutative ring with respect to the binary modulo operation $+_4$ and $*_4$.

SECTION – C

10. **Let L be a bounded distributed lattice, prove if a complement exists, it is unique. Is D_{12} a complemented lattice ? Draw the Hasse diagram of [P (a, b, c), $\leq$], (Note : '$\leq$' stands for subset). Find greatest element, least element, minimal element and maximal element.**

11. **Determine whether each of these functions is a bijective from *R* to *R*.**

a. $f(x) = x^2 + 1$

b. $f(x) = x^3$

c. $f(x) = (x^2 + 1)/(x^2 + 2)$

12. a. **Prove that inverse of each element in a group is unique.**

b. **Show that G = [(1, 2, 4, 5, 7, 8), $\times_9$] is cyclic. How many generators are there ? What are they ?**

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SOLUTION OF PAPER (2015-16)

Note : Attempt **all** parts. All parts carry equal marks. Write answer of each part in short. **(2 × 10 = 20)**

SECTION – A

1. a. Define multiset and power set. Determine the power set A = {1, 2}.

Ans. **Multiset :** Multisets are sets where an element can occur as a member more than once.

For example : $A = \{p, p, p, q, q, q, r, r, r, r\}$
$B = \{p, p, q, q, q, r\}$

are multisets.

Power set : A power set is a set of all subsets of the set.

The power set of $A = \{1, 2\}$ is $\{\{\phi\}, \{1\}, \{2\}\}$.

b. Show that $[((p \vee q) \rightarrow r) \wedge (\sim p)] \rightarrow (q \wedge r)$ is tautology or contradiction.

Ans.

p	q	r	$p \vee q$	$\sim p$	$q \wedge r$	$((p \vee q) \rightarrow r)$	$((p \vee q) \rightarrow r) \wedge (\sim p)$	$[((p \vee q) \rightarrow r) \wedge (\sim p)] \rightarrow (q \wedge r)$
T	T	T	T	F	T	T	F	T
T	T	F	T	F	F	F	F	T
T	F	T	T	F	F	T	F	T
T	F	F	T	F	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F	T
F	F	T	F	T	F	T	T	F
F	F	F	F	T	F	T	T	F

Question is incorrect. Since the result of the question is contingency.

c. State and prove pigeonhole principle.

Ans. **Pigeonhole principle :** If n pigeons are assigned to m pigeonholes then at least one pigeon hole contains two or more pigeons ($m < n$).

Proof :

1. Let m pigeonholes be numbered with the numbers 1 through m.
2. Beginning with the pigeon 1, each pigeon is assigned in order to the pigeonholes with the same number.

3. Since $m < n$ *i.e.*, the number of pigeonhole is less than the number of pigeons, $n - m$ pigeons are left without having assigned a pigeon hole.
4. Thus, at least one pigeonhole will be assigned to a more than one pigeon.

d. Show that if set A has 3 elements, then we can have 2^6 symmetric relation on A.

Ans. Number of elements in set = 3

Number of symmetric relations if number of elements is $n = 2^{n(n+1)/2}$

Here, $n = 3$

$\therefore$ Number of symmetric relations

$$= 2^{3(3+1)/2}$$
$$= 2^{3(4)/2}$$
$$= 2^6$$

Hence proved.

e. Prove that $(P \vee Q) \rightarrow (P \wedge Q)$ is logically equivalent to $P \leftrightarrow Q$.

Ans. $(P \vee Q) \rightarrow (P \wedge Q) = P \leftrightarrow Q$

P	Q	$P \vee Q$	$P \wedge Q$	$(P \vee Q) \leftrightarrow (P \wedge Q)$	$P \leftrightarrow Q$
T	T	T	T	T	T
T	F	T	F	F	F
F	T	T	F	F	F
F	F	F	F	T	T

f. How many 4 digit numbers can be formed by using the digits 2, 4, 6, 8 when repetition of digits is allowed ?

Ans. When repetition is allowed :

The thousands place can be filled by 4 ways.

The hundreds place can be filled by 4 ways.

The tens place can be filled by 4 ways.

The units place can be filled by 4 ways.

$\therefore$ Total number of 4 digit number = $4 \times 4 \times 4 \times 4 = 256$

g. The converse of a statement is : If a steel rod is stretched, then it has been heated. Write the inverse of the statement.

Ans. The statement corresponding to the given converse is "If a steel rod is stretched, then it has been heated". Now the inverse of this statement is "If a steel rod is not stretched then it has not been heated".

h. If a and b are any two elements of group G then prove $(a * b)^{-1} = (b^{-1} * a^{-1})$.

Ans. Consider $(a * b) * (b^{-1} * a^{-1})$

$$= a * (b * b^{-1}) * a^{-1}$$
$$= a * e * a^{-1}$$
$$= a * a^{-1} = e$$

Also $(b^{-1} * a^{-1}) * (a * b) = b^{-1} * (a^{-1} * a) * b$

$$= b^{-1} * e * b$$
$$= b^{-1} * b = e$$

Therefore $(a * b)^{-1} = b^{-1} * a^{-1}$ for any $a, b \in G$

i. If $f : A \to B$ is one-one onto mapping, then prove that $f^{-1} : B \to A$ will be one-one onto mapping.

Ans. Proof : Here $f : A \to B$ is one-to-one and onto.

$a_1, a_2 \in A$ and $b_1, b_2 \in B$ so that

$$b_1 = f(a_1),\ b_2 = f(a_2) \text{ and } a_1 = f^{-1}(b_1),\ a_2 = f^{-1}(b_2)$$

As f is one-to-one

$$f(a_1) = f(a_2) \Leftrightarrow a_1 = a_2$$
$$b_1 = b_2 \Leftrightarrow f^{-1}(b_1) = f^{-1}(b_2)$$

i.e., $\quad f^{-1}(b_1) = f^{-1}(b_2) \Rightarrow b_1 = b_2$

$\therefore \quad f^{-1}$ is one-to-one function.

As f is onto.

Every element of B is associated with a unique element of A *i.e.*, for any $a \in A$ is pre-image of some $b \in B$ where $b = f(a) \Rightarrow a = f^{-1}(b)$ *i.e.*, for $b \in B$, there exists f^{-1} image $a \in A$.

Hence, f^{-1} is onto.

j. Write the following in DNF $(x + y)(x' + y')$.

Ans. Given : $(x + y)(x' + y')$

The complete CNF in two variables (x, y)

$$= (x + y)(x' + y')(x + y')(x' + y')$$

Hence, $\quad f'(x, y) = (x' + y)(x + y')$

$\therefore \quad [f'(x, y)]' = [(x' + y)(x + y')]'$

$$= xy' + x'y$$

which is the required DNF.

SECTION – B

2. Prove that $n^3 + 2n$ is divisible by 3 using principle of mathematical induction, where n is natural number.

Ans. Let $S(n) : n^3 + 2n$ is divisible by 3.

Step I : Inductive base : For $n = 1$

$(1)^3 + 2.1 = 3$ which is divisible by 3

Thus, $S(1)$ is true.

Step II : Inductive hypothesis : Let $S(k)$ is true *i.e.*, $k^3 + 2k$ is divisible by 3 holds true.

or $k^3 + 2k = 3s$ for $s \in N$

Step III : Inductive step : We have to show that $S(k+1)$ is true *i.e.*, $(k+1)^3 + 2(k+1)$ is divisible by 3

Consider $(k+1)^3 + 2(k+1)$

$$= k^3 + 1 + 3k^2 + 3k + 2k + 2$$
$$= (k^3 + 2k) + 3(k^2 + k + 1)$$
$$= 3s + 3l \text{ where } l = k^2 + k + 1 \in N$$
$$= 3(s + l)$$

Therefore, $S(k+1)$ is true

Hence by principle of mathematical induction $S(n)$ is true for all $n \in N$.

3. Solve the recurrence relation using generating function : $a_n - 7a_{n-1} + 10a_{n-2} = 0$ with $a_0 = 3$, $a_1 = 3$.

Ans. $a_n - 7a_{n-1} + 10\, a_{n-2} = 0$,

Let in assume $n \geq 2$

Multiply by x^n and take sum from 2 to ∞.

$$\sum_{n=2}^{\infty} a_n\, x^n - 7 \sum_{n=2}^{\infty} a_{n-1}\, x^n + 10 \sum_{n=2}^{\infty} a_{n-2}\, x^n = 0$$

$(a_2 x^2 + a_3 x^3 + a_4 x^4 + ...) - 7\,(a_1 x^2 + a_2 x^3 + ...)$
$+ 10\,(a_0 x^2 + a_1 x^3 + ...) = 0$

We know that

$$G(x) = \sum_{n=0}^{\infty} a_n\, x^n = a_0 + a_1\, x + ...$$

$G(x) - a_0 - a_1 x - 7x\,(G(x) - a_0) + 10\,x^2\, G(x) = 0$

$G(x)\,[1 - 7x + 10x^2] = a_0 + a_1 x - 7\,a_0\, x$

$$= 3 + 3x - 21\,x = 3 - 18\,x$$

$$G(x) = \frac{3 - 18x}{10\,x^2 - 7x + 1} = \frac{3 - 18x}{10\,x^2 - 5x - 2x + 1}$$

$$= \frac{3 - 18x}{5x\,(2\,x - 1) - 1\,(2\,x - 1)} = \frac{3 - 18x}{(5x - 1)(2\,x - 1)}$$

Now,

$$\frac{3 - 18x}{(5x - 1)\,(2x - 1)} = \frac{A}{5\,x - 1} + \frac{B}{2\,x - 1}$$

$$3 - 18\,x = A\,(2\,x - 1) + B\,(5\,x - 1)$$

put $\quad x = \frac{1}{2}$

$$3 - 9 = B\left(\frac{5}{2} - 1\right) \Rightarrow -6 = \frac{3}{2}\,B \Rightarrow B = -4$$

put $\quad x = \frac{1}{5}$

$$3 - \frac{18}{5} = A\left(\frac{2}{5} - 1\right) \Rightarrow -\frac{3}{5} = -\frac{3}{5}A = 1 \Rightarrow \text{A} = 1$$

$$\therefore \qquad G(x) = \frac{1}{5x-1} - \frac{4}{2x-1} = \frac{4}{1-2x} - \frac{1}{1-5x}$$

$$\therefore \qquad a^n = 4.2^n - 5^n$$

4. Express the following statements using quantifiers and logical connectives.

a. Mathematics book that is published in India has a blue cover.

b. All animals are mortal. All human being are animal. Therefore, all human being are mortal.

c. There exists a mathematics book with a cover that is not blue.

d. He eats crackers only if he drinks milk.

e. There are mathematics books that are published outside India.

f. Not all books have bibliographies.

Ans.

a. $P(x)$: x is a mathematic book published in India
$Q(x)$: x is a mathematic book of blue cover
$\forall x\, P(x) \to Q(x)$.

b. $P(x)$: x is an animal
$Q(x)$: x is mortal
$\forall x \;\; P(x) \to Q(x)$
$R(x)$: x is a human being
$\therefore \quad \forall x \;\; R(x) \to P(x)$.

c. $P(x)$: x is a mathematics book
$Q(x)$: x is not a blue color
$\exists x,\, P(x) \wedge Q(x)$.

d. $P(x)$: x drinks milk
$Q(x)$: x eats crackers
for x, if $P(x)$ then $Q(x)$.
or x, $P(x) \Rightarrow Q(x)$.

e. $P(x)$: x is a mathematics book
$Q(x)$: x is published outside India
$\exists x\, P(x) \wedge Q(x)$.

f. $P(x)$: x is a book having bibliography $\sim \;\forall x,\; P(x)$.

5. Draw the Hasse diagram of [P (a, b, c), $\subseteq$] (Note : '$\subseteq$' stands for subset). Find greatest element, least element, minimal element and maximal element.

Ans. Let a_1 and a_2 be two complements of an element $a \in L$. Then by definition of complement

$$\left.\begin{aligned} a \vee a_1 &= I \\ a \wedge a_1 &= 0 \end{aligned}\right\} \qquad \ldots(1)$$

$$\left.\begin{aligned} a \vee a_2 &= I \\ a \wedge a_2 &= 0 \end{aligned}\right\} \quad ...(2)$$

Consider $a_1 = a_1 \vee 0$

$= a_1 \vee (a \wedge a_2)$ [from (2)]

$= (a_1 \vee a) \wedge (a_1 \vee a_2)$ [Distributive property]

$= (a \vee a_1) \wedge (a_1 \vee a_2)$ [Commutative property]

$= I \wedge (a_1 \vee a_2)$ [from (1)]

$= a_1 \vee a_2$ ...(3)

Now Consider $a_2 = a_2 \vee 0$

$= a_2 \vee (a \wedge a_1)$ [from (2)]

$= (a_2 \vee a) \wedge (a_2 \vee a_1)$ [Distributive property]

$= (a \vee a_2) \wedge (a_1 \vee a_2)$ [Commutative property]

$= I \wedge (a_1 \vee a_2)$ [from (1)]

$= a_1 \vee a_2$ ...(4)

Hence, from (3) and (4),

$$a_1 = a_2$$

So, for bounded distributive lattice complement is unique.

Hasse diagram of $[P(a, b, c), \subseteq]$ is shown in Fig. 1.

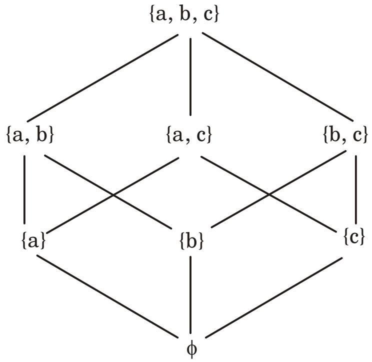


Fig. 1.

Greatest element is $\{a, b, c\}$ and maximal element is $\{a, b, c\}$.
The least element is ϕ and minimal element is ϕ.

6. Simplify the following boolean expressions using k-map :

a. $Y = ((AB)' + A' + AB)'$

b. $A'B'C'D' + A'B'C'D + A'B'CD + A'B'CD' = A'B'$

Ans.

a. $Y = ((AB)' + A' + AB)'$

$= ((AB)')' (A' + (AB))'$

$= (AB) ((A')' (AB)')$

$$= AB\,(A\,(A' + B'))$$
$$= AB\,(AA' + AB')$$
$$= AB(0 + AB') = AB\,AB'$$
$$= ABB'$$
$$= 0$$

Here, we find that the expression is not in minterm. For getting minterm, we simplify and find that its value is already zero. Hence, no need to use *K*-map for further simplification.

b. $\boldsymbol{A'\,B'\,C'\,D' + A'\,B'\,C'\,D + A'\,B'\,CD + A'\,B'\,CD' = A'\,B'}$

$$= A'\,B'C'D' + A'B'C'D + A'B'CD + A'B'CD'$$

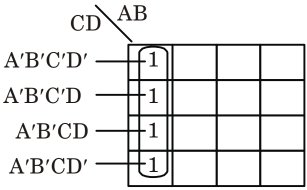


Fig. 2.

On simplification by *K*-map, we get $A'B'$ corresponding to all the four one's.

7. Let *G* be the set of all non-zero real number and let *a*b = ab*/2. Show that (*G) be an abelian group.**

Ans.

i. Closure property : Let $a, b \in G$.

$$a * b = \frac{ab}{2} \in G \text{ as ab} \neq 0$$

$\Rightarrow$ * is closure in *G*.

ii. Associativity : Let $a, b, c \in$ G

Consider $a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{a(bc)}{4} = \frac{abc}{4}$

$$(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{(ab)c}{4} = \frac{abc}{4}$$

$\Rightarrow$ * is associative in *G*.

iii. Existence of the identity : Let $a \in G$ and $\exists\, e$ such that

$$a * e = \frac{ae}{2} = a$$

$\Rightarrow$ $ae = 2a$

$\Rightarrow$ $e = 2$

$\therefore$ 2 is the identity element in *G*.

iv. Existence of the inverse : Let $a \in G$ and $b \in G$ such that $a * b = e = 2$

$\Rightarrow$ $\dfrac{ab}{2} = 2$

$$\Rightarrow \quad ab = 4$$

$$\Rightarrow \quad b = \frac{4}{a}$$

$\therefore$ The inverse of a is $\frac{4}{a}, \forall\, a \in G$.

v. **Commutative :** Let $a, b \in G$

$$a * b = \frac{ab}{2}$$

and $$b * a = \frac{ba}{2} = \frac{ab}{2}$$

$\Rightarrow$ * is commutative.

Thus, $(G, *)$ is an abelian group.

8. The following relation on A = {1, 2, 3, 4}. Determine whether the following :

a. R = {(1, 3), (3, 1), (1, 1), (1, 2), (3, 3), (4, 4)}.

b. R = A × A

Is an equivalence relation or not.

Ans.

a. $R = \{(1, 3), (3, 1), (1, 1), (1, 2), (3, 3), (4, 4)\}$

Reflexive : $(a, a) \in R \ \forall\, a \in A$

$\because$ $(1, 1) \in R, (2, 2) \notin R$

$\therefore$ R is not reflexive.

Symmetric : Let $(a, b) \in R$ then $(b, a) \in R$.

$\because$ $(1, 3) \in R$ so $(3, 1) \in R$

$\because$ $(1, 2) \in R$ but $(2, 1) \notin R$

$\therefore$ R is not symmetric.

Transitive : Let $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

$\because$ $(1, 3) \in R$ and $(3, 1) \in R$ so $(1, 1) \in R$

$\because$ $(2, 1) \in R$ and $(1, 3) \in R$ but $(2, 3) \notin R$

$\therefore$ R is not transitive.

Since, R is not reflexive, not symmetric, and not transitive so R is not an equivalence relation.

b. $R = A \times A$

Since, $A \times A$ contains all possible elements of set A. So, R is reflexive, symmetric and transitive. Hence R is an equivalence relation.

9. If the permutation of the elements of {1, 2, 3, 4, 5} are given by a = (1 2 3) (4 5), b = (1) (2) (3) (4 5), c = (1 5 2 4) (3). Find the value of x, if $ax = b$. And also prove that the set Z_4 = (0, 1, 2, 3) is a commutative ring with respect to the binary modulo operation $+_4$ and $*_4$.

Ans. $ax = b \quad \Rightarrow \quad (123)\,(45)\,x = (1)\,(2)\,(3)\,(4, 5)$

$$\Rightarrow \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix} x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix} x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 1 & 5 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 4 & 5 \end{pmatrix}$$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\times_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

We find from these tables :

i. All the entries in both the tables belong to Z_4. Hence, Z_4 is closed with respect to both operations.

ii. **Commutative law :** The entries of 1st, 2nd, 3rd, 4th rows are identical with the corresponding elements of the 1st, 2nd, 3rd, 4th columns respectively in both the tables. Hence, Z_4 is commutative with respect to both operations.

iii. **Associative law :** The associative law for addition and multiplication $a +_4 (b +_4 c) = (a +_4 b) +_4 c$ for all $a, b, c \in Z_4$
$a \times_4 (b \times_4 c) = (a \times_4 b) \times_4 c$, for all $a, b, c \in Z_4$
can easily be verified.

iv. **Existence of identity :** 0 is the additive identity and 1 is multiplicative identity for Z_4.

v. **Existence of inverse :** The additive inverses of 0, 1, 2, 3 are 0, 3, 2, 1 respectively.

Multiplicative inverse of non-zero element 1, 2, 3 are 1, 2, 3 respectively.

vi. **Distributive law :** Multiplication is distributive over addition *i.e.*,

$$a \times_4 (b +_4 c) = a \times_4 b + a \times_4 c$$
$$(b +_4 c) \times_4 a = b \times_4 a + c \times_4 a$$

For, $a \times_4 (b +_4 c) = a \times_4 (b + c)$ for b $+_4$ $c = b + c \pmod 4$

= least positive remainder when $a \times (b + c)$ is divided by 4

= least positive remainder when $ab + ac$ is divided by 4

$= ab +_4 ac$

$= a \times_4 b +_4 a \times_4 c$

For $a \times_4 b = a \times b \pmod 4)$

Since $(Z_4, +_4)$ is an abelian group, $(Z_4, \times_4)$ is a semigroup and the operation is distributive over addition. The $(Z_4, +_4, \times_4)$ is a ring. Now $(Z_4, \times_4)$ is commutative with respect to $\times_4$. Therefore, it is a commutative ring.

SECTION – C

10. Let L be a bounded distributed lattice, prove if a complement exists, it is unique. Is D_{12} a complemented lattice ? Draw the Hasse diagram of [P (a, b, c), $\leq$], (Note : '$\leq$' stands for subset). Find greatest element, least element, minimal element and maximal element.

Ans. Let a_1 and a_2 be two complements of an element $a \in L$.
Then by definition of complement

$$\left.\begin{aligned} a \vee a_1 &= I \\ a \wedge a_1 &= 0 \end{aligned}\right\} \quad ...(1)$$

$$\left.\begin{aligned} a \vee a_2 &= I \\ a \wedge a_2 &= 0 \end{aligned}\right\} \quad ...(2)$$

Consider

$$\begin{aligned} a_1 &= a_1 \vee 0 \\ &= a_1 \vee (a \wedge a_2) && \text{[from (2)]} \\ &= (a_1 \vee a) \wedge (a_1 \vee a_2) && \text{[Distributive property]} \\ &= (a \vee a_1) \wedge (a_1 \vee a_2) && \text{[Commutative property]} \\ &= I \wedge (a_1 \vee a_2) && \text{[from (1)]} \\ &= a_1 \vee a_2 && ...(3) \end{aligned}$$

Now Consider

$$\begin{aligned} a_2 &= a_2 \vee 0 \\ &= a_2 \vee (a \wedge a_1) && \text{[from (2)]} \\ &= (a_2 \vee a) \wedge (a_2 \vee a_1) && \text{[Distributive property]} \\ &= (a \vee a_2) \wedge (a_1 \vee a_2) && \text{[Commutative property]} \\ &= I \wedge (a_1 \vee a_2) && \text{[from (1)]} \\ &= a_1 \vee a_2 && ...(4) \end{aligned}$$

Hence, from (3) and (4),

$$a_1 = a_2$$

So, for bounded distributive lattice complement is unique.

Hasse diagram of $[P(a, b, c), \leq]$ is shown in Fig. 3.

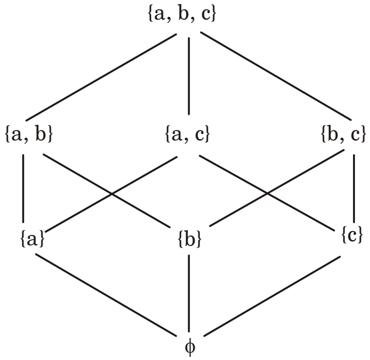


Fig. 3.

Greatest element is $\{a, b, c\}$ and maximal element is $\{a, b, c\}$.
The least element is ϕ and minimal element is ϕ.

11. Determine whether each of these functions is a bijective from *R* to *R*.

a. $f(x) = x^2 + 1$

b. $f(x) = x^3$

c. $f(x) = (x^2 + 1)/(x^2 + 2)$

Ans.

a. $f(x) = x^2 + 1$

Let $x_1, x_2 \in R$ such that

$$f(x_1) = f(x_2)$$
$$x_1^2 + 1 = x_2^2 + 1$$
$$x_1^2 = x_2^2$$
$$x_1 = \pm x_2$$

Therefore, if $x_2 = 1$ then $x_1 = \pm 1$

So, f is not one-to-one.

Hence, f is not bijective.

b. Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$$x_1^3 = x_2^3$$
$$x_1 = x_2$$

$\therefore$ f is one-to-one.

Let $y \in R$ such that

$$y = x^3$$
$$x = (y)^{1/3}$$

For $\forall\ y \in R\ \exists$ a unique $x \in R$ such that $y = f(x)$

$\therefore$ f is onto.

Hence, f is bijective.

c. Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$$\Rightarrow \qquad \frac{x_1^2 + 1}{x_1^2 + 2} = \frac{x_2^2 + 1}{x_2^2 + 2}$$

If $x_1 = 1, x_2 = -1$ then $f(x_1) = f(x_2)$
but $x_1 \neq x_2$
$\therefore$ f is not one-to-one.
Hence, f is not bijective.

12. a. Prove that inverse of each element in a group is unique.
b. Show that $G = [(1, 2, 4, 5, 7, 8), \times_9]$ is cyclic. How many generators are there ? What are they ?

Ans.
a. Let (if possible) b and c be two inverses of element $a \in G$.
Then by definition of group :

$$b * a = a * b = e$$

and

$$a * c = c * a = e$$

where e is the identity element of G

Now

$$\begin{aligned} b &= e * b = (c * a) * b \\ &= c * (a * b) \\ &= c * c \\ &= c \\ b &= c \end{aligned}$$

Therefore, inverse of an element is unique in $(G, *)$.

b. Composition table for X_9 is

X_9	1	2	4	5	7	8
1	2	3	4	5	7	8
2	2	4	8	1	5	7
4	4	8	7	2	1	5
5	5	1	2	7	8	4
7	7	5	1	8	4	2
8	8	7	5	4	2	1

1 is identity element of group G

$$2^1 = 2 \equiv 2 \bmod 9$$
$$2^2 = 4 \equiv 4 \bmod 9$$
$$2^3 = 8 \equiv 8 \bmod 9$$
$$2^4 = 16 \equiv 7 \bmod 9$$
$$2^5 = 32 \equiv 5 \bmod 9$$
$$2^6 = 64 \equiv 1 \bmod 9$$

Therefore, 2 is generator of G. Hence G is cyclic.
Similarly, 5 is also generator of G.
Hence there are two generators 2 and 5.

B.Tech.

(SEM. III) ODD SEMESTER

THEORY EXAMINATION, 2016-17

DISCRETE STRUCTURES AND GRAPH THEORY

Time : 3 Hours **Max. Marks : 100**

Section-A

Note : Attempt **all** parts. All parts carry equal marks. Write answer of each part in short. **(2 × 10 = 20)**

1. a. **Let R be a relation on the set of natural numbers N, as $R = \{(x, y) : x, y \in N, 3x + y = 19\}$. Find the domain and range of R. Verify whether R is reflexive.**

b. **Show that the relation R on the set Z of integers given by $R = \{(a, b) : 3 \text{ divides } a - b\}$, is an equivalence relation.**

c. **Show the implications without constructing the truth table $(P \to Q) \to Q \Rightarrow P \vee Q$.**

d. **Show that the "greater than or equal" relation (>=) is a partial ordering on the set of integers.**

e. **Distinguish between bounded lattice and complemented lattice.**

f. **Find the recurrence relation from $y_n = A2^n + B(-3)^n$.**

g. **Define ring and give an example of a ring with zero divisors.**

h. **State the applications of binary search tree.**

i. **Define multigraph. Explain with example in brief.**

j. **Let G be a graph with ten vertices. If four vertices has degree four and six vertices has degree five, then find the number of edges of G.**

Section-B

Note : Attempt any **five** questions from this section. **(10 × 5 = 50)**

2. Write the symbolic form and negate the following statements :

- **Everyone who is healthy can do all kinds of work.**
- **Some people are not admired by everyone.**
- **Everyone should help his neighbours, or his neighbours will not help him.**

3. In a lattice if $a \leq b \leq c$, then show that

a. $a \vee b = b \wedge c$

b. $(a \vee b) \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = b$

4. State and prove Lagrange's theorem for group. Is the converse true ?

5. Prove that a simple graph with n vertices and k components can have at most $\dfrac{(n-k)(n-k+1)}{2}$ edges.

6. Prove by induction : $\dfrac{1}{1.2} + \dfrac{1}{2.3} + \ldots + \dfrac{1}{n\,(n+1)} = \dfrac{n}{(n+1)}$.

7. Solve the recurrence relation $y_{n+2} - 5y_{n+1} + 6y_n = 5^n$ subject to the condition $y_0 = 0, y_1 = 2$.

8. a. Prove that every finite subset of a lattice has an LUB and a GLB.

b. Give an example of a lattice which is a modular but not a distributive.

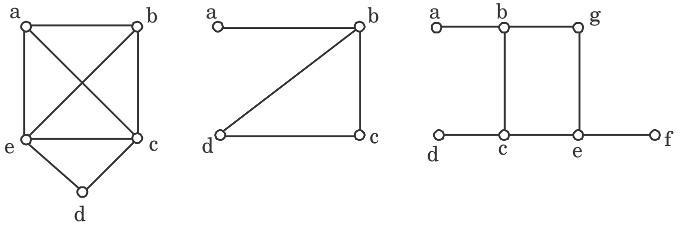
9. Explain in detail about the binary tree traversal with an example.

Section-C

Note : Attempt any **two** questions from this section. **(15 × 2 = 30)**

10. a. Prove that a connected graph G is Euler graph if and only if every vertex of G is of even degree.

b. Which of the following simple graph have a Hamiltonian circuit or, if not a Hamiltonian path ?



G1 G2 G3

Fig. 1.

11. a. Find the Boolean algebra expression for the following system.

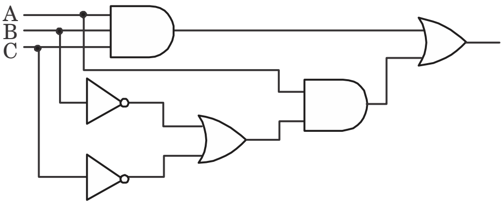


Fig. 2.

b. Suppose that a cookie shop has four different kinds of cookies. How many different way can six cookies be chosen ?

12. a. Prove that every cyclic group is an abelian group.

b. Obtain all distinct left cosets of {(0), (3)} in the group (Z_6, $+_6$) and find their union.

c. Find the left cosets of {[0], [3]} in the group (Z_6, $+_6$).

☺☺☺

SOLUTION OF PAPER (2016-17)

Section-A

Note : Attempt **all** parts. All parts carry equal marks. Write answer of each part in short. **(2 × 10 = 20)**

1. a. Let R be a relation on the set of natural numbers N, as $R = \{(x, y) : x, y \in N, 3x + y = 19\}$. Find the domain and range of R. Verify whether R is reflexive.

Ans. By definition of relation,

$R = \{(1, 16), (2, 13), (3, 10), (4, 7), (5, 4), (6, 1)\}$

$\therefore$ Domain = {1, 2, 3, 4, 5, 6}

$\therefore$ Range = {16, 13, 10, 7, 4, 1}

R is not reflexive since $(1, 1) \notin R$.

b. Show that the relation R on the set Z of integers given by $R = \{(a, b) : 3$ divides $a - b\}$, is an equivalence relation.

Ans. **Reflexive :** $a - a = 0$ is divisible by 3

$\therefore$ $(a, a) \in R \ \forall\ a \in Z$

$\therefore$ R is reflexive.

Symmetric : Let $(a, b) \in R \Rightarrow a - b$ is divisible by 3

$\Rightarrow$ $-(a - b)$ is divisible by 3

$\Rightarrow$ $b - a$ is divisible by 3

$\Rightarrow$ $(b, a) \in R$

$\therefore$ R is symmetric.

Transitive : Let $(a, b) \in R$ and $(b, c) \in R$

$a - b$ is divisible by 3 and $b - c$ is divisible by 3

Then $a - b + b - c$ is divisible by 3

$a - c$ is divisible by 3

$\therefore$ $(a, c) \in R$

$\therefore$ R is transitive.

Hence, R is equivalence relation.

c. Show the implications without constructing the truth table $(P \to Q) \to Q \Rightarrow P \vee Q$.

Ans. $(P \to Q) \to Q \Rightarrow P \vee Q$

Take L.H.S

$$\begin{aligned}(P \to Q) \to Q &= (\sim P \vee Q) \to Q \\ &= (\sim(\sim P \vee Q)) \vee Q \\ &= (P \vee \sim Q) \vee Q \\ &= (P \vee Q) \vee (\sim Q \vee Q) \\ &= (P \vee Q) \wedge T = P \vee Q\end{aligned}$$

It is equivalent.

d. Show that the "greater than or equal" relation (>=) is a partial ordering on the set of integers.

Ans. **Reflexive :**

$a \geq a \ \forall \ a \in Z$ (set of integer)

$(a, a) \in A$

$\therefore$ R is reflexive.

Antisymmetric : Let $(a, b) \in R$ and $(b, a) \in R$

$\Rightarrow$ $a \geq b$ and $b \geq a$

$\Rightarrow$ $a = b$

$\therefore$ R is antisymmetric.

Transitive : Let $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $a \geq$ b and b $\geq c$

$\Rightarrow$ $a \geq c \Rightarrow (a, c) \in R$

$\therefore$ R is transitive.

Hence, R is partial order relation.

e. Distinguish between bounded lattice and complemented lattice.

Ans. **Bounded lattice :** A lattice which has both elements 0 and 1 is called a bounded lattice.

Complemented lattice : A lattice L is called complemented lattice if it is bounded and if every element in L has complement.

f. Find the recurrence relation from $y_n = A2^n + B(-3)^n$.

Ans. Given : $y_n = A2^n + B(-3)^n$

Therefore, $y_{n+1} = A(2)^{n+1} + B(-3)^{n+1}$

$= 2A(2)^n - 3B(-3)^n$

and $y_{n+2} = A(2)^{n+2} + B(-3)^{n+2}$

$= 4A(2)^n + 9B(-3)^n$

Eliminating A and B from these equations, we get

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & -3 \\ y_{n+2} & 4 & 9 \end{vmatrix} = 0$$

$= y_{n+2} - y_{n+1} - 6y_n = 0$ which is the required recurrence relation.

g. Define ring and give an example of a ring with zero divisors.

Ans. **Ring :** A non-empty set R is a ring if it is equipped with two binary operations called addition and multiplication and denoted by '+' and '.' respectively *i.e.*, for all $a, b \in R$ we have $a + b \in R$ and $a.b \in R$ and it satisfies the following properties :

i. Addition is associative *i.e.*,

$(a + b) + c = a + (b + c) \ \forall \ a, b, c \in R$

ii. Addition is commutative *i.e.*,

$a + b = b + a \ \forall \ a, b \in R$

iii. There exists an element $0 \in R$ such that

$0 + a = a = a + 0, \ \forall a \in R$

iv. To each element a in R there exists an element $-a$ in R such that $a + (-a) = 0$

v. Multiplication is associative *i.e.*,

$a.(b.c) = (a.b).c, \forall a\ b, c \in R$

vi. Multiplication is distributive with respect to addition *i.e.*, for all $a, b, c \in R$,

Example of ring with zero divisors : R = {a set of 2 × 2 matrices}.

Field : A ring R with at least two elements is called a field if it has following properties :

i. R is commutative

ii. R has unity

iii. R is such that each non-zero element possesses multiplicative inverse.

For example : The rings of real numbers and complex numbers are also fields.

h. State the applications of binary search tree.

Ans. One of the most common applications is to efficiently store data in sorted form in order to access and search stored elements quickly. For example, std :: map or std :: set in C++ Standard Library. Binary tree as data structure is useful for various implementations of expression parsers and expression solvers.

i. Define multigraph. Explain with example in brief.

Ans. A multigraphs $G(V, E)$ consists of a set of vertices V and a set of edges E such that edge set E may contain multiple edges and self loops.

Example :

a. Undirected multigraph :

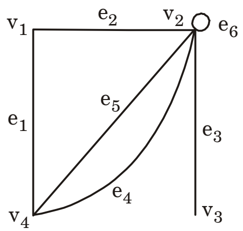


Fig. 1.

b. Directed multigraph :

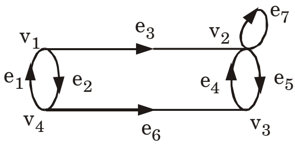


Fig. 2.

j. Let G be a graph with ten vertices. If four vertices has degree four and six vertices has degree five, then find the number of edges of G.

Ans. We know that

$$\sum_i \deg(v_i) = 2e$$

$$4 + 4 + 4 + 4 + 5 + 5 + 5 + 5 + 5 + 5 = 2e$$

$$16 + 30 = 2e$$

$$2e = 46$$

$$e = 23$$

Section-B

Note : Attempt any **five** questions from this section. **(10 × 5 = 50)**

2. Write the symbolic form and negate the following statements :

- **Everyone who is healthy can do all kinds of work.**
- **Some people are not admired by everyone.**
- **Everyone should help his neighbours, or his neighbours will not help him.**

Ans.

a. Symbolic form :

Let $P(x)$: x is healthy and $Q(x)$: x do all work

$\forall x(P(x) \to Q(x))$

Negation : $\neg (\forall x (P(x) \to Q(x))$

b. Symbolic form :

Let $P(x)$: x is a person

$A(x, y)$: x admires y

The given statement can be written as "There is a person who is not admired by some person" and it is $(\exists x)\ (\exists\ y)[P(x) \wedge P(y) \wedge \neg A(x, y)]$

Negation : $(\exists\ x)\ (\exists\ y)\ [P\ (x) \wedge P(y) \wedge A(x, y)]$

c. Symbolic form :

Let $N(x, y)$: x and y are neighbours

$H(x, y)$: x should help y

$P(x, y)$: x will help y

The statement can be written as "For every person x and every person y, if x and y are neighbours, then either x should help y or y

will not help x" and it is $(\forall x)(\forall y)[N(x, y) \rightarrow (H(x, y) \neg P(y, x))]$

Negation : $(\forall x)(\forall y)[N(x, y) \rightarrow \neg (H(x, y) P(y, x))]$

3. In a lattice if a ≤ b ≤ c, then show that

a. $\boldsymbol{a \vee b = b \wedge c}$

b. $\boldsymbol{(a \vee b) \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = b}$

Ans.

a. **Given :** $a \leq b \leq c$

Now $a \vee b$ = least upper bound of a, b
= least {all upper bounds of a, b}
= least $\{b, c, ...\}$ [using $a \leq b \leq c$]
= b ...(1)

and $b \wedge c$ = greatest lower bound of b, c
= maximum {all lower bounds of b, c}
= maximum $\{b, a, ...\}$ [using $a \leq b \leq c$]
= b ...(2)

Eq. (1) and (2) gives, $a \vee b = b \wedge c$

b. $(a \vee b) \vee (b \wedge c) \Rightarrow (a \vee b) \wedge (a \vee c) = b$

Consider, $(a \vee b) \vee (b \wedge c)$
= $b \vee b$ [using $a \leq b \leq c$ and definition of $\vee$ and $\wedge$]
= b ...(3)

and$(a \vee b) \wedge (a \vee c)$ = $b \wedge c$
= b ...(4)

From eq. (3) and (4), $(a \vee b) \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = b$.

4. State and prove Lagrange's theorem for group. Is the converse true ?

Ans. **Lagrange's theorem :**

Statement : The order of each subgroup of a finite group is a divisor of the order of the group.

Proof : Let G be a group of finite order n. Let H be a subgroup of G and let $O(H) = m$. Suppose $h_1, h_2 ....\ ..., h_m$ are the m members of H. Let $a \in G$, then Ha is the right coset of H in G and we have

$$Ha = \{h_1 a, h_2 a, ... h_m a\}$$

Ha has m distinct members, since $= h_i a = h_j a \Rightarrow h_i = h_j$

Therefore, each right coset of H in G has m distinct members. Any two distinct right cosets of H in G are disjoint *i.e.*, they have no element in common. Since G is a finite group, the number of distinct right cosets of H in G will be finite, say, equal to k. The union of these k distinct right cosets of H in G is equal to G.

Thus, if $Ha_1, Ha_2, ..., Ha_k$ are the k distinct right cosets of H in G.

Then $G = Ha_1 \cup Ha_2 \cup Ha_3 \cup .... \cup Ha_k$

$\Rightarrow$ the number of elements in G = the number of elements in Ha_1 + ... + the number of elements in Ha_2 +......+ the number of elements in Ha_k

$\Rightarrow \quad O(G) = km$

$\Rightarrow \quad n = km$

$\Rightarrow \quad k = \dfrac{n}{m}$

$\Rightarrow$ m is a divisor of n.

$\Rightarrow$ $O(H)$ is a divisor of $O(G)$.

Proof of converse : If G be a finite group of order n and $n \in G$, then

$$a^n = e$$

Let o $(a) = m$ which implies $a^m = e$.

Now, the subset H of G consisting of all the integral power of a is a subgroup of G and the order of H is m.

Then, by the Lagrange's theorem, m is divisor of n.

Let $n = mk$, then

$$a^n = a^{mk} = (a^m)^k = e^k = e$$

$\therefore$ Yes, the converse is true.

5. Prove that a simple graph with n vertices and k components can have at most $\dfrac{(n-k)(n-k+1)}{2}$ edges.

Ans. Let the number of vertices in each of the k-components of a graph G be $n_1, n_2, \ldots., n_k$, then we get

$n_1 + n_2 + \ldots + n_k = n$ where $n_i \geq 1$ $(i = 1, 2, \ldots, k)$

Now, $\displaystyle\sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k$

$$\therefore \quad \left(\sum_{i=1}^{k}(n_i - 1)\right)^2 = n^2 + k^2 - 2nk$$

or $\displaystyle\sum_{i=1}^{k}(n_i - 1)^2 + 2\sum_{i=1}^{k}\sum_{\substack{j=1 \\ i \neq j}}(n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk$

or $\displaystyle\sum_{i=1}^{k}(n_i - 1)^2 + 2(\text{non-negative terms}) = n^2 + k^2 - 2nk$

$[\because n_i - 1 \geq 0, n_j - 1 \geq 0]$

or $\displaystyle\sum_{i=1}^{k}(n_i - 1)^2 \leq n^2 + k^2 - 2nk$

or $\displaystyle\sum_{i=1}^{k} n_i^2 + \sum_{i=1}^{k} 1 - 2\sum_{i=1}^{k} n_i \leq n^2 + k^2 - 2nk$

or $\displaystyle\sum_{i=1}^{k} n_i^2 + k - 2n \leq n^2 + k^2 - 2nk$

or $\displaystyle\sum_{i=1}^{k} n_i^2 - n \leq n^2 + k^2 - 2nk - k + n$

$$= n(n - k + 1) - k\,(n - k + 1)$$
$$= (n - k)(n - k + 1) \quad ...(1)$$

We know that the maximum number of edges in the i^{th} component of

$$G = {}^{n_i}C_2 = \frac{n_i(n_i - 1)}{2}$$

Therefore, the maximum number of edges in G is :

$$\frac{1}{2}\sum n_i(n_i - 1) = \frac{1}{2}\left(\sum n_i^2 - \sum n_i\right) = \frac{1}{2}\left(\sum n_i^2 - n\right)$$
$$\leq \frac{1}{2}(n - k)(n - k + 1) \text{ by using eq. (1)}$$

6. Prove by induction : $\frac{1}{1.2} + \frac{1}{2.3} + ... + \frac{1}{n\,(n+1)} = \frac{n}{(n+1)}$.

Ans. Let the given statement be denoted by $S(n)$.

1. **Inductive base :** For $n = 1$

$$\frac{1}{1.2} = \frac{1}{1+1} = \frac{1}{2}$$

Hence S (1) is true.

2. **Inductive hypothesis :** Assume that S (k) is true *i.e.*,

$$\frac{1}{1.2} + \frac{1}{2.3} + ... + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

3. **Inductive step :** We wish to show that the statement is true for $n = k + 1$ *i.e.*,

$$\frac{1}{1.2} + \frac{1}{2.3} + ... + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

Now, $\frac{1}{1.2} + \frac{1}{2.3} + ... + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k^2 + 2k + 1}{(k+1)(k+2)}$$
$$= \frac{k+1}{k+2}$$

Thus, $S(k + 1)$ is true whenever $S(k)$ is true. By principle of mathematical induction, $S(n)$ is true for all positive integer n.

7. Solve the recurrence relation $y_{n+2} - 5y_{n+1} + 6y_n = 5^n$ subject to the condition $y_0 = 0$, $y_1 = 2$.

Ans.

Let $G(t) = \sum_{n=0}^{\infty} a_n t^n$ be generating function of sequence $\{a_n\}$.

Multiplying given equation by t^n and summing from $n = 0$ to ∞, we have

$$\sum_{n=0}^{\infty} a_{n+2}\, t^n - 5\sum_{n=0}^{\infty} a_{n+1}\, t^n + 6\sum_{n=0}^{\infty} a_n\, t^n = \sum_{n=0}^{\infty} 5^n\, t^n$$

$$\frac{G(t) - a_0 - a_1 t}{t^2} - 5\left[\frac{G(t) - a_0}{t}\right] + 6\,G(t) = \frac{1}{1-5t}$$

Put $a_0 = 0$ and $a_1 = 2$

$$G(t) - 2t - 5t\,G(t) + 6t^2\,G(t) = \frac{t^2}{1-5t}$$

$$G(t) - 5t\,G(t) + 6t^2\,G(t) = \frac{t^2}{1-5t} + 2t$$

$$G(t)\,(1 - 5t + 6t^2) = \frac{t^2}{1-5t} + 2t$$

$$(6t^2 - 5t + 1)\,G(t) = \frac{t^2}{1-5t} + 2t$$

$$G(t) = \frac{t^2}{(1-5t)(3t-1)(2t-1)} + \frac{2t}{(3t-1)(2t-1)}$$

$$= \frac{t^2}{(1-5t)(1-3t)(1-2t)} + \frac{2t}{(1-3t)(1-2t)}$$

Let

$$\frac{t^2}{(1-5t)(1-3t)(1-2t)} = \frac{A}{(1-5t)} + \frac{B}{(1-3t)} + \frac{C}{(1-2t)}$$

$$A = (1-5t)\left.\frac{t^2}{(1-5t)(1-3t)(1-2t)}\right|_{t=1/5}$$

$$= \left.\frac{t^2}{(1-3t)(1-2t)}\right|_{t=1/5}$$

$$= \frac{1/25}{(1-3/5)(1-2/5)} = \frac{1}{6}$$

$$B = (1-3t)\left.\frac{t^2}{(1-5t)(1-3t)(1-2t)}\right|_{t=1/3}$$

$$= \left.\frac{t^2}{(1-5t)(1-2t)}\right|_{t=1/3} = \frac{1/9}{\left(\frac{3-5}{3}\right)\left(\frac{3-2}{3}\right)}$$

$$= -\frac{1}{2}$$

$$C = (1-2t)\frac{t^2}{(1-5t)(1-3t)(1-2t)}\Bigg|_{t=1/2}$$

$$= \frac{t^2}{(1-5t)(1-3t)}\Bigg|_{t=1/2} = \frac{1/4}{\frac{(2-5)}{2}\times\frac{(2-3)}{2}}$$

$$= \frac{1}{3}$$

Again,

$$\frac{2t}{(1-3t)(1-2t)} = \frac{D}{(1-3t)} + \frac{E}{(1-2t)}$$

$$D = (1-3t)\frac{2t}{(1-3t)(1-2t)}\Bigg|_{t=1/3}$$

$$= \frac{2t}{(1-2t)}\Bigg|_{t=1/3} = \frac{2/3}{\frac{(3-2)}{3}} = 2$$

$$E = (1-2t)\frac{2t}{(1-3t)(1-2t)}\Bigg|_{t=1/2}$$

$$= \frac{2t}{(1-3t)}\Bigg|_{t=1/2} = \frac{2/2}{\frac{2-3}{2}} = -2$$

$$G(t) = \frac{1/6}{(1-5t)} - \frac{1/2}{(1-3t)} + \frac{1/3}{(1-2t)} + \frac{2}{(1-3t)} - \frac{2}{(1-2t)}$$

$$= \frac{1/6}{1-5t} + \frac{3/2}{(1-3t)} - \frac{5/3}{1-2t}$$

$$\sum_{n=0}^{\infty} a_n t^n = \frac{1}{6}\sum_{n=0}^{\infty}(5t)^n + \frac{3}{2}\sum_{n=0}^{\infty}(3t)^n - \frac{5}{3}\sum_{n=0}^{\infty}(2t)^n$$

$$\therefore \quad a_n = \frac{1}{6}(5)^n + \frac{3}{2}(3)^n - \frac{5}{3}(2)^n$$

8. a. Prove that every finite subset of a lattice has an LUB and a GLB.

b. Give an example of a lattice which is a modular but not a distributive.

Ans.

a.

1. The theorem is true if the subset has 1 element, the element being its own *glb* and *lub*.
2. It is also true if the subset has 2 elements.

3. Suppose the theorem holds for all subsets containing 1, 2, ..., k elements, so that a subset $a_1, a_2, ..., a_k$ of L has a *glb* and a *lub*.
4. If L contains more than k elements, consider the subset $\{a_1, a_2, ..., a_{k+1}\}$ of L.
5. Let $w = lub\ (a_1, a_2, ..., a_k)$.
6. Let $t = lub\ (w, a_{k+1})$.
7. If s is any upper bound of $a_1, a_2, ..., a_{k+1}$, then s is $\geq$ each of $a_1, a_2, ..., a_k$ and therefore $s \geq w$.
8. Also, $s \geq a_{k+1}$ and therefore s is an upper bound of w and a_{k+1}.
9. Hence $s \geq t$.
10. That is, since $t \geq$ each a_1, t is the *lub* of $a_1, a_2, ..., a_{k+1}$.
11. The theorem follows for the *lub* by finite induction.
12. If L is finite and contains m elements, the induction process stops when $k + 1 = m$.

b.

1. The diamond is modular, but not distributive.
2. Obviously the pentagon cannot be embedded in it.
3. The diamond is not distributive :

$$y \vee (x \wedge z) = y\ (y \vee x) \wedge (y \vee z) = 1$$

4. The distributive lattices are closed under sublattices and every sublattice of a distributive lattice is itself a distributive lattice.
5. If the diamond can be embedded in a lattice, then that lattice has a non-distributive sublattice, hence it is not distributive.

9. Explain in detail about the binary tree traversal with an example.

Ans. Tree traversal : A traversal of tree is a process in which each vertex is visited exactly once in a certain manner. For a binary tree we have three types of traversal :

1. Preorder traversal : Each vertex is visited in the following order :

a. Visit the root (N).
b. Visit the left child (or subtree) of root (L).
c. Visit the right child (or subtree) of root (R).

2. Postorder traversal :

a. Visit the left child (subtree) of root.
b. Visit the right child (subtree) of root.
c. Visit the root.

3. Inorder traversal :

a. Visit the left child (subtree) of root.
b. Visit the root.
c. Visit the right child (subtree) of root.

A binary tree with 12 vertices :

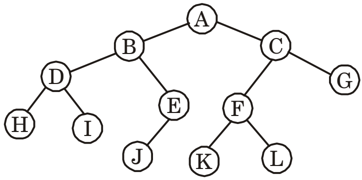


Fig. 3.

Preorder (NLR) : *A B D H I E J C F K L G*

Inorder (LNR) : *H D I B J E A K F L C G*

Postorder (LRN) : *H I D J E B K L F G C A*

Section-C

Note : Attempt any **two** questions from this section. **(15 × 2 = 30)**

10. a. Prove that a connected graph *G* is Euler graph if and only if every vertex of *G* is of even degree.

b. Which of the following simple graph have a Hamiltonian circuit or, if not a Hamiltonian path ?

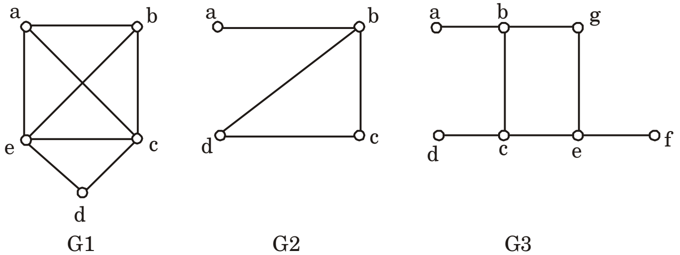


Fig. 4.

Ans.

a.

1. First of all we shall prove that if a non-empty connected graph is Eulerian then it has no vertices of odd degree.
2. Let G be Eulerian.
3. Then G has an Eulerian trail which begins and ends at u.
4. If we travel along the trail then each time we visit a vertex. We use two edges, one in and one out.
5. This is also true for the start vertex because we also end there.
6. Since an Eulerian trail uses every edge once, the degree of each vertex must be a multiple of two and hence there are no vertices of odd degree.
7. Now we shall prove that if a non-empty connected graph has no vertices of odd degree then it is Eulerian.
8. Let every vertex of G have even degree.
9. We will now use a proof by mathematical induction on $|E(G)|$, the number of edges of G.

Basis of induction :

Let $|E(G)| = 0$, then G is the graph K_1, and G is Eulerian.

Inductive step :

1. Let $P(n)$ be the statement that all connected graphs on n edges of even degree are Eulerian.
2. Assume $P(n)$ is true for all $n < |E(G)|$.
3. Since each vertex has degree at least two, G contains a cycle C.
4. Delete the edges of the cycle C from G.
5. The resulting graph, G' say, may not be connected.
6. However, each of its components will be connected, and will have fewer than $|E(G)|$ edges.
7. Also, all vertices of each component will be of even degree, because the removal of the cycle either leaves the degree of a vertex unchanged, or reduces it by two.
8. By the induction assumption, each component of G' is therefore Eulerian.
9. To show that G has an Eulerian trail, we start the trail at a vertex, u say, of the cycle C and traverse the cycle until we meet a vertex, c_1 say, of one of the components of G'.
10. We then traverse that component's Eulerian trail, finally returning to the cycle C at the same vertex, c_1.
11. We then continue along the cycle C, traversing each component of G' as it meets the cycle.
12. Eventually, this process traverses all the edges of G and arrives back at u, thus producing an Eulerian trail for G.
13. Thus, G is Eulerian by the principle of mathematical induction.

b. **G1 :** The graph G1 shown in Fig. 4 contains Hamiltonian circuit, *i.e.*, $a - b - c - d - e - a$ and also a Hamiltonian path, *i.e.*, $abcde$.

G2 : The graph G2 shown in Fig. 4 does not contain Hamiltonian circuit since every cycle containing every vertex must contain the edge e twice. But the graph does have a Hamiltonian path $a - b - c - d$.

G3 : The graph G3 shown in Fig. 4 neither have Hamiltonian circuit nor have Hamiltonian path because any traversal does not cover all the vertices.

11. a. Find the Boolean algebra expression for the following system.

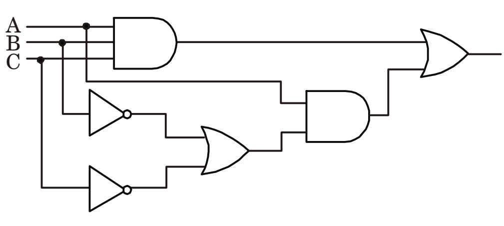


Fig. 5.

Ans.

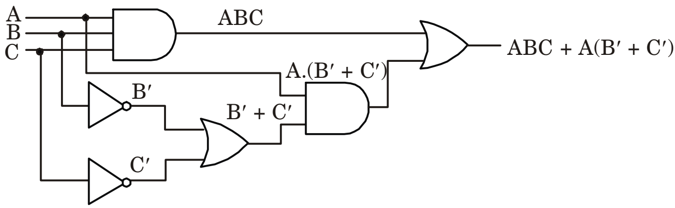


Fig. 6.

b. Suppose that a cookie shop has four different kinds of cookies. How many different way can six cookies be chosen ?

Ans. As the order in which each cookie is chosen does not matter and each kind of cookies can be chosen as many as 6 times, the number of ways these cookies can be chosen is the number of 6-combination with repetition allowed from a set with 4 distinct elements.

The number of ways to choose six cookies in the bakery shop is the number of 6 combinations of a set with four elements.

$$C(4 + 6 - 1, 6) = C(9, 6)$$

Since $C(9, 6) = C(9, 3) = (9 \cdot 8 \cdot 7) / (1 \cdot 2 \cdot 3) = 84$

Therefore, there are 84 different ways to choose the six cookies.

12. a. Prove that every cyclic group is an abelian group.

b. Obtain all distinct left cosets of {(0), (3)} in the group (Z_6, $+_6$) and find their union.

c. Find the left cosets of {[0], [3]} in the group (Z_6, $+_6$).

Ans.

a. Let G be a cyclic group and let a be a generator of G so that

$$G = \langle a \rangle = \{a^n : n \in Z\}$$

If g_1 and g_2 are any two elements of G, there exist integers r and s such that $g_1 = a^r$ and $g_2 = a^s$. Then

$$g_1 g_2 = a^r a^s = a^{r+s} = a^{s+r} = a^s \,.\, a^r = g_2 g_1$$

So, G is abelian.

b. $\therefore$ $[0] + H = [3] + H$, $[1] + [4] + H$ and $[2] + H = [5] + H$ are the three distinct left cosets of H in $(Z_6, +_6)$.

We would have the following left cosets :

$$g_1 H = \{g_1 h, h \in H\}$$
$$g_2 H = \{g_2 h, h \in H\}$$
$$g_n H = \{g_n h, h \in H\}$$

The union of all these sets will include all the g' s, since for each set

$$g_k = \{g_k h, h \in H\}$$

we have $g_e \in g_k = \{g_k h, h \in H\}$

where e is the identity.

Then if we make the union of all these sets we will have at least all the elements of g. The other elements are merely g_h for some h.

But since $g_h \in G$ they would be repeated elements in the union. So, the union of all left cosets of H in G is G, *i.e.*,

$$Z_6 = \{[0], [1], [2], [3], [4], [5]\}$$

c. Let $Z_6 = \{[0], [1], [2], [3], [4], [5]\}$ be a group.

$H = \{[0], [3]\}$ be a subgroup of $(Z_6, +_6)$.

The left cosets of H are,

$$[0] + H = \{[0], [3]\}$$
$$[1] + H = \{[1], [4]\}$$
$$[2] + H = \{[2], [5]\}$$
$$[3] + H = \{[3], [0]\}$$
$$[4] + H = \{[4], [1]\}$$
$$[5] + H = \{[5], [2]\}$$

☺☺☺

B.Tech.

(SEM. III) ODD SEMESTER THEORY EXAMINATION, 2017-18 DISCRETE STRUCTURES & THEORY OF LOGIC

Time : 3 Hours **Max. Marks : 70**

Note : **1.** Attempt **all** Sections. If require any missing data; then choose suitably.

2. Any special paper specific instructions.

SECTION – A

1. Attempt **all** questions in brief. **(2 × 7 = 14)**

a. Define Eulerian path, circuit and graph.

b. Let A = (2, 4, 5, 7, 8) = B, aRb if and only if $a + b$ <= 12. Find relation matrix.

c. Explain edge colouring and k-edge colouring.

d. Define chromatic number and isomorphic graph.

e. Define union and intersection of multiset and find for A = [1, 1, 4, 2, 2, 3], B = [1, 2, 2, 6, 3, 3]

f. Find the contrapositive of "If he has courage, then he will win".

g. Define rings and write its properties.

SECTION-B

2. Attempt any **three** of the following : **(7 × 3 = 21)**

a. Prove by mathematical induction
$3 + 33 + 333 + \ldots\ldots\ldots\ldots 3333 = (10^{n+1} - 9n - 10)/27$

b. Define the following with one example :

i. Bipartite graph **ii. Complete graph**

iii. How many edges in K_7 and $K_{3,6}$ **iv. Planar graph**

c. For any positive integer D36, then find whether (D36, '|') is lattice or not ?

d. **Let $X = \{1, 2, 3 ...... 7\}$ and $R = \{(x, y) \mid (x - y)$ is divisible by 3\}. Is R equivalence relation. Draw the digraph of R.**

e. **Simplify the following Boolean function using K-map :**
$$F(x, y, z) = \Sigma(0, 2, 3, 7)$$

SECTION-C

3. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. **Solve $a_r - 6a_{r-1} + 8a_{r-2} = r.4^r$, given $a_0 = 8$, and $a_1 = 1$.**

b. **Show that : $(r \to \sim q,\ r \vee S,\ S \to \sim q,\ p \to q) \leftrightarrow \sim p$ are inconsistent.**

4. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. **Write the properties of group. Show that the set (1, 2, 3, 4, 5) is not group under addition and multiplication modulo 6.**

b. **Prove by mathematical induction**
$n^4 - 4n^2$ is divisible by 3 for all $n >= 2$.

5. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. **Explain modular lattice, distribute lattice and bounded lattice with example and diagram.**

b. **Draw the Hasse diagram of $(A, \leq)$, where $A = \{3, 4, 12, 24, 48, 72\}$ and relation $\leq$ be such that $a \leq b$ if a divides b.**

6. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. **Given the inorder and postorder traversal of a tree T :**
Inorder : *HFEABIGDC* Postorder : *BEHFACDGI*
Determine the tree T and it's Preorder.

b. **Translate the following sentences in quantified expressions of predicate logic.**

i. **All students need financial aid.** **ii.** **Some cows are not white.**

iii. **Suresh will get if division if and only if he gets first div.**

iv. **If water is hot, then Shyam will swim in pool.**

v. **All integers are either even or odd integer.**

7. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. **Define and explain any two the following :**

1. **BFS and DFS in trees**
2. **Euler graph**
3. **Adjacency matrix of a graph**

b. **Solve the recurrence relation : $a_r + 4a_{r-2} + 4a_{r-2} = r^2$.**

☺☺☺

SOLUTION OF PAPER (2017-18)

Note : **1.** Attempt **all** Sections. If require any missing data; then choose suitably.
2. Any special paper specific instructions.

SECTION – A

1. Attempt **all** questions in brief. **(2 × 7 = 14)**

a. Define Eulerian path, circuit and graph.

Ans. **Eulerian path :** A path of graph G which includes each edge of G exactly once is called Eulerian path.

Eulerian circuit : A circuit of graph G which include each edge of G exactly once.

Eulerain graph : A graph containing an Eulerian circuit is called Eulerian graph.

b. Let A = (2, 4, 5, 7, 8) = B, aRb if and only if $a + b$ <= 12. Find relation matrix.

Ans. R = {(2, 4), (2, 5), (2, 7), (2, 8) (4, 2), (4, 5), (4, 7), (4, 8) (5, 2), (5, 4), (5, 7), (7, 2), (7, 4), (7, 5), (8, 2), (8, 4), (2, 2), (4, 4), (5, 5)}

$$\begin{array}{cc} & \begin{array}{ccccc} 2 & 4 & 5 & 7 & 8 \end{array} \\ m_{ij} = \begin{array}{c} 2 \\ 4 \\ 5 \\ 7 \\ 8 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

c. Explain edge colouring and k-edge colouring.

Ans. **Edge coloring :** An edge coloring of a graph G may also be thought of as equivalent to a vertex coloring of the line graph $L(G)$, the graph that has a vertex for every edge of G and an edge for every pair of adjacent edges in G.

***k*-edge coloring :** A proper edge coloring with k different colors is called a (proper) k-edge coloring.

d. Define chromatic number and isomorphic graph.

Ans. **Chromatic number :** The minimum number of colours required for the proper colouring of a graph so that no two adjacent vertices have the same colour, is called chromatic number of a graph.

Isomorphic graph : If two graphs are isomorphic to each other then :

i. Both have same number of vertices and edges.

ii. Degree sequence of both graphs are same (degree sequence is the sequence of degrees of the vertices of a graph arranged in non-increasing order).

e. Define union and intersection of multiset and find for $A = [1, 1, 4, 2, 2, 3], B = [1, 2, 2, 6, 3, 3]$

Ans. **Union :** Let A and B be two multisets. Then, $A \cup B$, is the multiset where the multiplicity of an element in the maximum of its multiplicities in A and B.

Intersection : The intersection of A and B, $A \cap B$, is the multiset where the multiplicity of an element is the minimum of its multiplicities in A and B.

Numerical :

$$A = \{1, 1, 4, 2, 2, 3\}, B = \{1, 2, 2, 6, 3, 3\}$$

Union : $A \cup B = \{1, 2, 3, 4, 6\}$

Intersection : $A \cap B = \{1, 2, 2, 3\}$

f. Find the contrapositive of "If he has courage, then he will win".

Ans. If he will not win then he does not have courage.

g. Define rings and write its properties.

Ans. **Ring :** A non-empty set R is a ring if it is equipped with two binary operations called addition and multiplication and denoted by '+' and '.' respectively *i.e.*, for all $a, b \in R$ we have $a + b \in R$ and $a.b \in R$ and it satisfies the following properties :

i. Addition is associative *i.e.*,

$(a + b) + c = a + (b + c) \ \forall\ a,b,c \in R$

ii. Addition is commutative *i.e.*,

$a + b = b + a \ \forall\ a,b \in R$

iii. There exists an element $0 \in R$ such that

$0 + a = a = a + 0, \ \forall\ a \in R$

iv. To each element a in R there exists an element $-a$ in R such that $a + (-a) = 0$

v. Multiplication is associative *i.e.*,

$a.(b.c) = (a.b).c, \forall\ a\ b,c \in R$

vi. Multiplication is distributive with respect to addition *i.e.*, for all $a, b, c \in R$,

SECTION-B

2. Attempt any **three** of the following : **(7 × 3 = 21)**

a. Prove by mathematical induction
$3 + 33 + 333 + \ldots\ldots\ldots\ldots 3333 = (10^{n+1} - 9n - 10)/27$

Ans. $3 + 33 + 333 + \text{...........} + 3333 \ldots = (10^{n+1} - 9n - 10)/27$
Let given statement be denoted by $S(n)$

1. **Inductive base :** For $n = 1$

$$3 = \frac{(10^2 - 9(1) - 10)}{27}, \; 3 = \frac{100 - 19}{27} = \frac{81}{27} = 3$$

$3 = 3$. Hence $S(1)$ is tree.

2. **Inductive hypothesis :** Assume that $S(k)$ is true *i.e.*,
$3 + 33 + 333 + \text{...........} + 3333 = (10^{k+1} - 9k - 10)/27$

3. **Inductive steps :** We have to show that $S(k + 1)$ is also true *i.e.*,
$3 + 33 + 333 + \text{...........} \; (10^{k+2} - 9^{(k+1)} - 10)/27$
Now, $3 + 33 + \text{...........} + 33 \text{...........} 3$
$= 3 + 33 + 333 + \text{...........} + 3 \text{...........} 3$
$= (10^{k+1} - 9k - 10)/27 + 3(10^{k+1} - 1)/9$
$= (10^{k+1} + 9k - 10 + 9.10^{k+1} - 9)/27$
$= (10^{k+1} + 9.10^{k+1} - 9k - 8 - 10)/27 = (10^{k+2} - 9(k + 1) - 10)/27$
Thus $S(k + 1)$ is true whenever $S(k)$ is true. By the principle of mathematical induction $S(n)$ true for all positive integer n.

b. Define the following with one example :
i. Bipartite graph **ii. Complete graph**
iii. How many edges in K_7 and $K_{3,6}$ **iv. Planar graph**

Ans.

i. Bipartite graph : A graph $G = (V, E)$ is bipartite if the vertex set V can be partitioned into two subsets (disjoint) V_1 and V_2 such that every edge in E connects a vertex in V_1 and a vertex V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). (V_1, V_2) is called a bipartition of G.

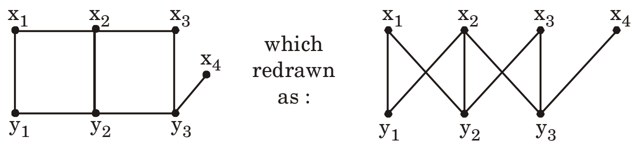


Fig. 1. Some bipartite graphs.

ii. Complete graph : A simple graph, in which there is exactly one edge between each pair of distinct vertices is called a complete graph. The complete graph of n vertices is denoted by K_n. The graphs K_1 to K_5 are shown below in Fig. 2.

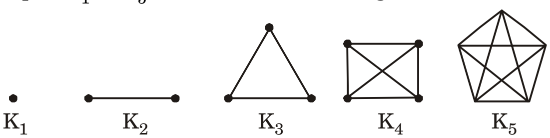


Fig. 2.

K_n has exactly $\frac{n(n-1)}{2} = {}^nC_2$ edges

iii. Number of edge in K_7 : Since, K_n is complete graph with n vertices.

Number of edge in $K_7 = \dfrac{7(7-1)}{2} = \dfrac{7 \times 6}{2} = 21$

Number of edge in $K_{3,6}$:

Since, $K_{n,m}$ is a complete bipartite graph with $n \in V_1$ and $m \in V_2$

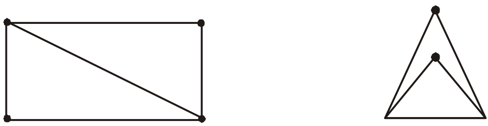
Number of edge in $K_{3,6} = 3 \times 6 = 18$

iv. Planar graph :

A graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect except only at the common vertex.

i. A graph is said a planar graph, if it cannot be drawn on a plane without a crossover between its edges crossing.

ii. The graphs shown in Fig. 3(a) and (b) are planar graphs.



(a) (b)

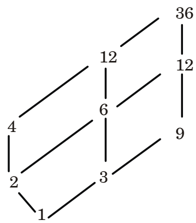
Fig. 3. Some planar graph.

c. For any positive integer D36, then find whether (D36, '|') is lattice or not ?

Ans. D36 = Divisor of 36 = {1, 2, 3, 4, 6, 9, 12, 18, 36}

Hasse diagram :

$(1 \vee 3) = \{3, 6\}, (1 \vee 2) = \{2, 4\}, (2 \vee 6) = \{6, 18\}, (9 \vee 4) = \{\phi\}$



Since, $9 \vee 4 = \{\phi\}$

So, D36 is not a lattice.

d. Let $X = \{1, 2, 3 \ldots\ldots 7\}$ and $R = \{(x, y) \mid (x - y)$ is divisible by 3$\}$. Is R equivalence relation. Draw the digraph of R.

Ans. Given that $X = \{1, 2, 3, 4, 5, 6, 7\}$

and $R = \{(x, y) : (x - y)$ is divisible by 3 $\}$

Then R is an equivalence relation if

i. Reflexive : $\forall\ x \in X \Rightarrow (x - x)$ is divisible by 3

So, $(x, x) \in X\ \forall\ x \in X$ or, R is reflexive.

ii. Symmetric : Let $x, y \in X$ and $(x, y) \in R$

$\Rightarrow (x - y)$ is divisible by 3 $\Rightarrow (x - y) = 3n_1$, ($n_1$ being an integer)

$\Rightarrow (y - x) = - 3n_2 = 3n_2$, n_2 is also an integer
So, $y - x$ is divisible by 3 or R is symmetric.

iii. Transitive : Let $x, y, z \in X$ and $(x, y) \in R$, $(y, z) \in R$
Then $x - y = 3n_1, y - z = 3n_2$, $\quad n_1, n_2$ being integers
$\Rightarrow x - z = 3(n_1 + n_2)$, $\quad n_1 + n_2 = n_3$ be any integer
So, $(x - z)$ is also divisible by 3 or $(x, z) \in R$
So, R is transitive.
Hence, R is an equivalence relation.

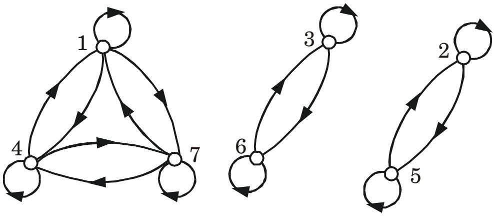
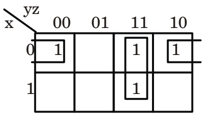


Fig. 4. Diagraph of R.

e. Simplify the following Boolean function using K-map :
$$\boldsymbol{F(x, y, z) = \Sigma(0, 2, 3, 7)}$$

Ans.



$$F = \overline{x}\,\overline{z} + yz$$

SECTION-C

3. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Solve $a_r - 6a_{r-1} + 8a_{r-2} = r.4^r$, given $a_0 = 8$, and $a_1 = 1$.

Ans. $a_r - 6a_{r-1} + 8a_{r-2} = r4^r$
The characteristic equation is, $x^2 - 6x + 8 = 0$, $x^2 - 2x - 4x + 8 = 0$
$(x - 2)(x - 4) = 0$, $x = 2, 4$
The solution of the associated non-homogeneous recurrence relation is,
$$a_r^{(h)} = B_1(2)^r + B_2(4)^r \quad ...(1)$$
Let particular solution of given equation is, $a_r^{(p)} = r^2(A_0 + A_1r)4^r$
Substituting in the given equation, we get
$$\Rightarrow \quad r^2(A_0 + A_1r)4^r - 6(r-1)^2(A_0 + A_1(r-1))4^{r-1} + 8(r-2)^2(A_0 + A_1(r-2)4^{r-2} = r4^r$$
$$\Rightarrow r^2A_0 + A_1r^3 - \frac{6}{4}\left[(A_0r^2 - 2A_0r + A_0) + (A_1r^3 - A_1 - 3A_1r^2 + 3A_1r)^2\right]$$
$$+ \frac{8}{4^2}\left[(A_0r^2 - 4rA_0 + 4A_0) + (A_1r^3 - 8A_1 - 6A_1r^2 + 12A_1r)\right] = r$$

$$\Rightarrow \quad rA_0 + A_1r^3 - \frac{3}{2}A_0r^2 + 3A_0r - \frac{3}{2}A_0 - \frac{3}{2}A_1r^3 + \frac{3}{2}A_1 + \frac{9}{2}A_1r^2 - \frac{9}{2}A_1r + \frac{1}{2}A_0r^2 - 2A_0r + 2A_0$$
$$\frac{1}{2}A_1r^3 - 4A_1 - 3A_1r^2 - 6A_1r = r$$

$$\Rightarrow \quad 2A_0r - A_0r^2 - \frac{1}{2}A_0 - \frac{5}{2}A_1 + \frac{3}{2}A_1r^2 + \frac{3}{2}A_1r = r$$

Comparing both sides, we get

$$2A_0 + \frac{3}{2}A_1 = 1 \quad \text{...(2)}$$
$$A_0 + 5A_1 = 0 \quad \text{...(3)}$$

Solving equation (2) and (3), we get $A_1 = \frac{-2}{17} \quad A_0 = \frac{-10}{17}$

To find the value of B_1 and B_2 put $r = 0$ and $r = 1$ in equation (1)

$r = 0 \quad a_0 = B_1 + B_2 \quad B_1 + B_2 = 8$ ...(4)

$r = 1 \quad a_1 = 2B_1 + 4B_2 \quad 2B_1 + 4B_2 = 1$ ...(5)

Solving equations (4) and (5), we get $B_1 = \frac{31}{2} \quad B_2 = \frac{-15}{2}$

Complete solution is, $a_r = a_r^{(h)} + a_r^{(p)}$

$$a_r = \frac{31}{2}2^r - \frac{15}{2}4^r + r^2\left[\left(\frac{-10}{17}\right) + \left(\frac{-2}{17}\right)r\right]4^r$$

b. Show that : $(r \to \sim q,\ r \vee S,\ S \to \sim q,\ p \to q) \leftrightarrow \sim p$ are inconsistent.

Ans. Following the indirect method, we introduce p as an additional premise and show that this additional premise leads to a contradiction.

{1}	(1) $p \to q$	Rule P
{2}	(2) p	Rule P (assumed)
{1, 2}	(3) q	Rule T, (1), (2) and modus ponens
{4}	(4) $s \to \overline{q}$	Rule P
{1, 2, 4}	(5) $\overline{s}$	Rule T, (3), (4) and modus tollens
{6}	(6) $r \vee s$	Rule P
{1, 2, 4, 6}	(7) r	Rule T, (5), (6) disjunctive syllogism
{8}	(8) $r \to \overline{q}$	Rule P
{8}	(9) $\overline{r} \vee \overline{q}$	Rule T, (8) and EQ_{16} $(p \to q \equiv \overline{p} \vee q)$
{8}	(10) $\overline{r \wedge q}$	Rule T, (8) and De Morgan's law
{1, 2, 4, 6}	(11) $r \wedge q$	Rule T, (7), (3) and conjunction
{1, 2, 4, 6, 8}	(12) $r \wedge q \wedge \overline{r \wedge q}$	Rule T, (10), (11) and conjunction.

Since, we know that set of formula is inconsistent if their conjunction implies contradiction. Hence it leads to a contradiction. So, it is inconsistent.

4. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Write the properties of group. Show that the set (1, 2, 3, 4, 5) is not group under addition and multiplication modulo 6.

Ans. Properties of group :

Following are the properties of group :

1. $a * b \in G \ \forall \ a, b \in G$ [closure property]
2. $a * (b * c) = (a * b) * c \quad \forall \ a, b, c \in G$ [associative property]
3. There exist an element $e \in G$ such that for any $a \in G$
   $a * e = e * a = e$ [existence of identity]
4. For every $a \in G$, $\exists$ element $a^{-1} \in G$
   such that $a * a^{-1} = a^{-1} * a = e$

For example : $(Z, +)$, $(R, +)$, and $(Q, +)$ are all groups.

Numerical :

Addition modulo 6 ($+_6$) : Composition table of $S = \{1, 2, 3, 4, 5\}$ under operation $+_6$ is given as :

$+_6$	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

Since, $1 +_6 5 = 0$ but $0 \notin S$ *i.e.*, S is not closed under addition modulo 6.
So, S is not a group.

Multiplication modulo 6 ($*_6$) :

Composition table of $S = \{1, 2, 3, 4, 5\}$ under operation $*_6$ is given as

$*_6$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

Since, $2 *_6 3 = 0$ but $0 \notin S$ *i.e.*, S is not closed under multiplication modulo 6.
So, S is not a group.

b. Prove by mathematical induction
$n^4 - 4n^2$ is divisible by 3 for all $n >= 2$.

Ans. Base case : If $n = 0$, then $n^4 - 4n^2 = 0$, which is divisible by 3.
Inductive hypothesis : For some $n \geq 0$, $n^4 - 4n^2$ is divisible by 3.
Inductive step : Assume the inductive hypothesis is true for n.

We need to show that $(n + 1)^4 - 4(n + 1)^2$ is divisible by 3. By the inductive hypothesis, we know that $n^4 - 4n^2$ is divisible by 3.
Hence $(n + 1)^4 - 4(n + 1)^2$ is divisible by 3 if
$(n + 1)^4 - 4(n + 1)^2 - (n^4 - 4n^2)$ is divisible by 3.
Now $(n + 1)^4 - 4(n + 1)^2 - (n^4 - 4n^2)$

$$= n^4 + 4n^3 + 6n^2 + 4n + 1 - 4n^2 - 8n - 4 - n^4 + 4n^2$$
$$= 4n^3 + 6n^2 - 4n - 3,$$

which is divisible by 3 if $4n^3 - 4n$ is. Since $4n^3 - 4n = 4n(n + 1)(n - 1)$, we see that $4n^3 - 4n$ is always divisible by 3.
Going backwards, we conclude that $(n + 1)^4 - 4(n + 1)^2$ is divisible by 3, and that the inductive hypothesis holds for $n + 1$.
By the Principle of Mathematical Induction, $n^4 - 4n^2$ is divisible by 3, for all $n \in N$.

5. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Explain modular lattice, distribute lattice and bounded lattice with example and diagram.

Ans. **Modular distributive and bounded lattice :**
Types of lattice :

1. **Modular lattice :** A lattice $(L, \leq)$ is called modular lattice if, $a \vee (b \wedge c) = (a \vee b) \wedge c$ whenever $a \leq c$ for all $a, b, c \in L$.
2. **Distributive lattice :** A lattice L is said to be distributive if for any element a, b and c of L following properties are satisfied :
   i. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
   ii. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

   otherwise L is non-distributive lattice.
3. **Bounded lattice :** A lattice L is said to be bounded if it has a greatest element 1 and a least element 0. In such lattice we have
   $a \vee 1 = 1, a \wedge 1 = a$
   $a \vee 0 = a, a \wedge 0 = 0$
   $\forall\ a \in L$and $0 \leq a \leq$ I

Example :

Let consider a Hasse diagram :

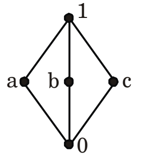


Fig. 5.

Modular lattice :

$0 \leq a$ *i.e.*, taking $b = 0$

$b \vee (a \wedge c) = 0 \vee 0 = 0, a \wedge (b \vee c) = a \wedge c = 0$

Distributive lattice :

For a set S, the lattice $P(S)$ is distributive, since union and intersection each satisfy the distributive property.

Bounded lattice : Since, the given lattice has 1 as greatest and 0 as least element so it is bounded lattice.

b. Draw the Hasse diagram of $(A, \leq)$, where $A = \{3, 4, 12, 24, 48, 72\}$ and relation $\leq$ be such that $a \leq b$ if a divides b.

Ans. Hasse diagram of $(A, \leq)$ where $A = \{3, 4, 12, 24, 48, 72\}$

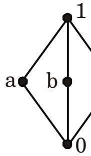


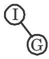
Fig. 6.

6. Attempt any **one** part of the following : **(7 × 1 = 7)**

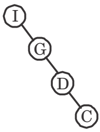
a. Given the inorder and postorder traversal of a tree *T* :
Inorder : *HFEABIGDC* Postorder : *BEHFACDGI*
Determine the tree *T* and it's Preorder.

Ans. The root of tree is *I*.

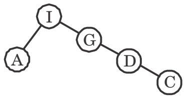
Now elements on right of *I* are *D*, *G*, *C* and *G* comes last of all in postorder traversal.



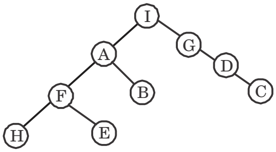
Now *D* and *C* are on right of *G* and *D* comes last of *G* and *I* postorder traversal so



Now element left of *I* are *H F E A B* in inorder traversal and *A* comes last of all in postorder traversal. Therefore tree will be



Now *H F E* are on left of *A* in inorder traversal and *B* comes last of all and continuing in same manner. We will get final binary tree as



Preorder traversal of above binary tree is *CAFHEBDGI*

b. Translate the following sentences in quantified expressions of predicate logic.

i. All students need financial aid. ii. Some cows are not white.

iii. Suresh will get if division if and only if he gets first div.

iv. If water is hot, then Shyam will swim in pool.

v. All integers are either even or odd integer.

Ans.

i. $\forall x\ [S(x) \Rightarrow F(x)]$

ii. $\sim [\exists(x)\ (C(x) \wedge W(x))]$

iii. Sentence is incorrect so cannot be translated into quantified expression.

iv. $W(x)$: x is water

$H(x)$: x is hot

$S(x)$: x is Shyam

$P(x)$: x will swim in pool

$\forall x\ [((W(x) \wedge H(x)) \Rightarrow (S(x) \wedge P(x))]$

v. $E(x)$: x is even

$O(x)$: x is odd

$\forall x\ (E(x) \vee O(x))$

7. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Define and explain any two the following :

1. BFS and DFS in trees

2. Euler graph

3. Adjacency matrix of a graph

Ans.

1. **Breadth First Search (BFS) :** Breadth First Search (BFS) is an algorithm for traversing or searching tree or graph data structures. It starts at the tree root and explores the neighbour nodes first, before moving to the next level neighbours.

Algorithmic steps :

Step 1 : Push the root node in the queue.

Step 2 : Loop until the queue is empty.

Step 3 : Remove the node from the queue.

Step 4 : If the removed node has unvisited child nodes, mark them as visited and insert the unvisited children in the queue.

Depth First Search (DFS) :

Depth First Search (DFS) is an algorithm for traversing or searching tree or graph data structures. One starts at the root (selecting some arbitrary node as the root in the case of a graph) and explores as far as possible along each branch before backtracking.

Algorithmic steps :

Step 1 : Push the root node in the stack.

Step 2 : Loop until stack is empty.

Step 3 : Pick the node of the stack.

Step 4 : If the node has unvisited child nodes, get the unvisited child node, mark it as traversed and push it on stack.

Step 5 : If the node does not have any unvisited child nodes, pop the node from the stack.

2. **Eulerian graph :** A graph containing an Eulerian circuit is called Eulerian graph.

For example : Graphs given below are Eulerian graphs.

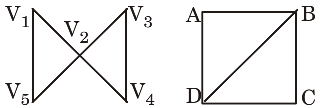


Fig. 7.

Eulerian path : A path of graph G which includes each edge of G exactly once is called Eulerian path.

Eulerian circuit : A circuit of graph G which include each edge of G exactly once.

The existence of Eulerian paths or Eulerian circuits in a graph is related to the degree of vertices.

a. **Adjacency matrix :**

i. **Representation of undirected graph :**

The adjacency matrix of a graph G with n vertices and no parallel edges is a $n \times n$ matrix $A = [a_{ij}]$ whose elements are given by

$$a_{ij} = 1, \text{ if there is an edge between } i^{\text{th}} \text{ and } j^{\text{th}} \text{ vertices}$$
$$= 0, \text{ if there is no edge between them}$$

ii. **Representation of directed graph :**

The adjacency matrix of a digraph D, with n vertices is the matrix

$$A = [a_{ij}]_{n \times n} \text{ in which}$$
$$a_{ij} = 1 \text{ if arc } (v_i, v_j) \text{ is in } D$$
$$= 0 \text{ otherwise}$$

For example :

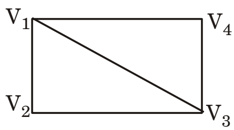


Fig. 8.

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

b. Solve the recurrence relation : $a_r + 4a_{r-2} + 4a_{r-2} = r^2$.

Ans. $a_r + 4a_{r-1} + 4a_{r-2} = r^2$

The characteristic equation is,

$$x^2 + 4x + 4 = 0$$

$$(x + 2)^2 = 0$$

$$x = -2, -2$$

The homogeneous solution is, $a^{(h)} = (A_0 + A_1 r)(-2)^r$

The particular solution be, $a^{(p)} = (A_0 + A_1 r)\, r^2$

Put a_r, a_{r-1} and a_{r-2} from $a^{(p)}$ in the given equation, we get

$$r^2A_0 + A_1r^3 + 4A_0(r-1)^2 + 4A_1(r-1)^3 + 4A_0(r-2)^2 + 4A_1(r-2)^3 = r^2$$

$$A_0(r^2 + 4r^2 - 8r + 4 + 4r^2 - 16r + 16) + A_1(r^3 + 4r^3 - 4 - 12r^2 + 12r + 4r^3 - 32 - 24r^2 + 48r) = r^2$$

$$A_0(9r^2 - 24r + 20) + A_1(9r^3 - 48r^2 + 60r - 36) = r^2$$

Comparing the coefficient of same power of r, we get

$$9A_0 - 48A_1 = 1 \quad ...(1)$$

$$20A_0 - 36A_1 = 0 \quad ...(2)$$

Solving equation (1) and (2) $A_0 = \dfrac{-3}{53}$ $\quad A_1 = \dfrac{-5}{159}$

The complete solution is,

$$a_r = a_r^{(p)} + a_r^{(h)} = (A_0 + A_1 r)(-2)^r + \left[\left(\frac{-3}{53}\right) + \left(\frac{-5}{159}\right) r\right] r^2$$

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B.Tech.

(SEM. III) ODD SEMESTER THEORY EXAMINATION, 2018-19 DISCRETE STRUCTURES AND THEORY OF LOGIC

Time : 3 Hours **Max. Marks : 70**

Note : **1.** Attempt **all** Sections. If require any missing data; then choose suitably.

2. Any special paper specific instructions.

SECTION – A

1. Attempt **all** questions in brief. **(2 × 7 = 14)**

a. Find the power set of each of these sets, where a and b are distinct elements.

i. $\{a\}$ **ii.** $\{a, b\}$

iii. $\{\phi, \{\phi\}\}$ **iv.** $\{a, \{a\}\}$

b. Define ring and field.

c. Draw the Hasse diagram of D_{30}.

d. What are the contrapositive, converse, and the inverse of the conditional statement : "The home team wins whenever it is raining" ?

e. How many bit strings of length eight either start with a '1' bit or end with the two bit '00' ?

f. Define injective, surjective and bijective function.

g. Show that $\neg (p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

SECTION-B

2. Attempt any **three** of the following : **(7 × 3 = 21)**

a. A total of 1232 student have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken least one of Spanish,

French and Russian, how many students have taken a course in all three languages ?

b. i. Let H be a subgroup of a finite group G. Prove that order of H is a divisor of order of G.

ii. Prove that every group of prime order is cyclic.

c. Define a lattice. For any a, b, c, d in a lattice $(A, \leq)$ if $a \leq b$ and $c \leq d$ then show that $a \vee c \leq b \vee d$ and $a \wedge c \leq b \wedge d$.

d. Show that $((p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \vee r)$ is a tautology without using truth table.

e. Define a binary tree. A binary tree has 11 nodes. It's inorder and preorder traversals node sequences are :
Preorder : $A\ B\ D\ H\ I\ E\ J\ L\ C\ F\ G$
Inorder : $H\ D\ I\ B\ J\ E\ K\ A\ F\ C\ G$
Draw the tree.

3. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Prove that if n is a positive integer, then 133 divides $11^{n+1} + 12^{2n-1}$.

b. Let n be a positive integer and S a set of strings. Suppose that R_n is the relation on S such that sR_nt if and only if $s = t$, or both s and t have at least n characters and first n characters of s and t are the same. That is, a string of fewer than n characters is related only to itself; a string s with at least n characters is related to a string t if and only if t has at least n characters and t beings with the n characters at the start of s.

4. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Let $G = \{1, -1, i, -i\}$ with the binary operation multiplication be an algebraic structure, where $i = \sqrt{-1}$. Determine whether G is an abelian or not.

b. What is meant by ring ? Give examples of both commutative and non-commutative rings.

5. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S. Draw the Hasse diagram for inclusion on the set $P(S)$, where $S = \{a, b, c, d\}$. Also determine whether $(P(S), \subseteq)$ is a lattice.

b. Find the Sum-Of-Products and Product-Of-sum expansion of the Boolean function $F(x, y, z) = (x + y)\, z'$.

6. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. What is a tautology, contradiction and contingency ? Show that $(p \vee q) \vee (\neg p \vee r) \rightarrow (q \vee r)$ is a tautology, contradiction or contingency.

b. Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip." and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

7. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. What are different ways to represent a graph. Define Euler circuit and Euler graph. Give necessary and sufficient conditions for Euler circuits and paths.

b. Suppose that a valid codeword is an n-digit number in decimal notation containing an even number of 0's. Let a_n denote the number of valid codewords of length n satisfying the recurrence relation $a_n = 8a_{n-1} + 10^{n-1}$ and the initial condition $a_1 = 9$. Use generating functions to find an explicit formula for a_n.

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SOLUTION OF PAPER (2018-19)

Note : **1.** Attempt **all** Sections. If require any missing data; then choose suitably.
2. Any special paper specific instructions.

SECTION – A

1. Attempt **all** questions in brief. **(2 × 7 = 14)**

a. Find the power set of each of these sets, where *a* and *b* are distinct elements.

i. $\{a\}$ **ii.** $\{a, b\}$
iii. $\{\phi, \{\phi\}\}$ **iv.** $\{a, \{a\}\}$

Ans.
i. Power set of $\{a\} = \{\{\phi\}, \{a\}\}$
ii. Power set of $\{a, b\} = \{\{\phi\}, \{a\}, \{b\}, \{a, b\}\}$
iii. Power set of $\{\phi, \{\phi\}\} = \{\phi\}$
iv. Power set of $\{a, \{a\}\} = \{\{\phi\}, \{a\}, \{\{a\}\}, \{a, \{a\}\}\}$

b. Define ring and field.

Ans. **Ring :** A non-empty set R is a ring if it is equipped with two binary operations called addition and multiplication and denoted by '+' and '.' respectively *i.e.*, for all $a, b \in R$ we have $a + b \in R$ and $a.b \in R$ and it satisfies the following properties :

i. Addition is associative *i.e.*,
$$(a + b) + c = a + (b + c) \ \forall\, a, b, c \in R$$
ii. Addition is commutative *i.e.*,
$$a + b = b + a \ \forall\, a, b \in R$$
iii. There exists an element $0 \in R$ such that
$$0 + a = a = a + 0, \ \forall\, a \in R$$
iv. To each element a in R there exists an element $-a$ in R such that $a + (-a) = 0$
v. Multiplication is associative *i.e.*,
$$a.(b.c) = (a.b).c, \ \forall\, a\ b, c \in R$$
vi. Multiplication is distributive with respect to addition *i.e.*, for all $a, b, c \in R$,

Field : A ring R with at least two elements is called a field if it has following properties :
i. R is commutative
ii. R has unity
iii. R is such that each non-zero element possesses multiplicative inverse.

c. Draw the Hasse diagram of D_{30}.

Ans.

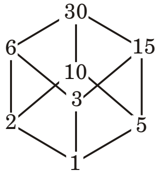


Fig. 1.

d. What are the contrapositive, converse, and the inverse of the conditional statement : "The home team wins whenever it is raining" ?

Ans. **Given :** The home team wins whenever it is raining.

***q*(conclusion) :** The home team wins.

***p*(hypothesis) :** It is raining.

Contrapositive : $\sim q \to \sim p$ is "if the home team does not win then it is not raining".

Converse : $q \to p$ is "if the home team wins then it is raining".

Inverse : $\sim p \to \sim q$ is "if it is not raining then the home team does not win".

e. How many bit strings of length eight either start with a '1' bit or end with the two bit '00' ?

Ans.

1. Number of bit strings of length eight that start with a 1 bit : $2^7 = 128$.
2. Number of bit strings of length eight that end with bits 00 : $2^6 = 64$.
3. Number of bit strings of length eight $2^5 = 32$ that start with a 1 bit and end with bits 00 : $2^5 = 32$

   Hence, the number is $128 + 64 - 32 = 160$.

f. Define injective, surjective and bijective function.

Ans.

1. **One-to-one function (Injective function or injection) :** Let $f : X \to Y$ then f is called one-to-one function if for distinct elements of X there are distinct image in Y *i.e.*, f is one-to-one iff

$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2 \ \forall \ x_1, x_2, \in X$$

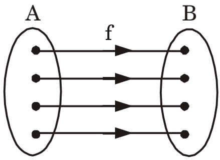


Fig. 2. One-to-one.

2. **Onto function (Surjection or surjective function) :** Let $f : X \to Y$ then f is called onto function iff for every element $y \in Y$ there is an element $x \in X$ with $f(x) = y$ or f is onto if Range $(f) = Y$.

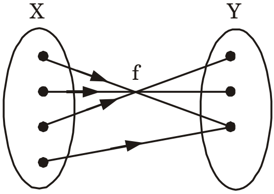


Fig. 3. Onto.

3. **One-to-one onto function (Bijective function or bijection) :** A function which is both one-to-one and onto is called one-to-one onto function or bijective function.

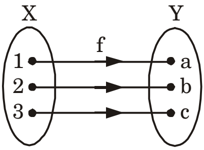


Fig. 4. One-to-one onto.

g. Show that $\neg (p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Ans. To prove : $(p + q)' = p'.q'$

To prove the theorem we will show that

$$(p + q) + p'.q' = 1$$

Consider $(p + q) + p'.q' = \{(p + q) + p'\}.\{(p + q) + q'\}$ by Distributive law

$= \{(q + p) + p'\}.\{(p + q) + q'\}$ by Commutative law

$= \{q + (p + p')\}.\{p + (q + q')\}$ by Associative law

$= (q + 1).(p + 1)$ by Complement law

$= 1.1$ by Dominance law

$= 1$ ...(1)

Also consider

$(p + q).p'q' = p'q'.(p + q)$ by Commutative law

$= p'q'.p + p'q'.q$ by Distributive law

$= p.(p'q') + p'.(q'q)$ by Commutative law

$= (p.p').q' + p'.(q.q')$ by Associative law

$= 0.q' + p'.0$ by Complement law

$= q'.0 + p'.0$ by Commutative law

$= 0 + 0$ by Dominance law

$= 0$ ...(2)

From (1) and (2), we get,

$p'q'$ is complement of $(p + q)$ *i.e.*, $(p + q)' = p'q'$.

SECTION-B

2. Attempt any **three** of the following : **(7 × 3 = 21)**

a. A total of 1232 student have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken least one of Spanish, French and Russian, how many students have taken a course in all three languages ?

Ans. Let S be the set of students who have taken a course in Spanish, F be the set of students who have taken a course in French, and R be the set of students who have taken a course in Russian. Then, we have

$|S| = 1232$, $|F| = 879$, $|R| = 114$, $|S \cap F| = 103$, $|S \cap R| = 23$, $|S \cap R| = 14$, and $|S \cup F \cup R| = 23$.

Using the equation

$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |S \cap R| + |S \cup F \cup R|$,

$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$,

$|S \cap F \cap R| = 7$.

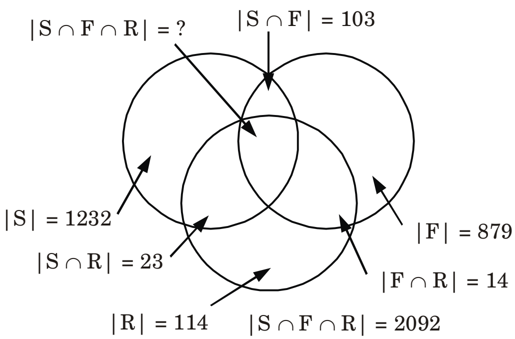


Fig. 5.

b. i. Let H be a subgroup of a finite group G. Prove that order of H is a divisor of order of G.

Ans.

1. Let H be any sub-group of order m of a finite group G of order n. Let us consider the left coset decomposition of G relative to H.
2. We will show that each coset aH consists of m different elements. Let $H = \{h_1, h_2, ..., h_m\}$
3. Then $ah_1, ah_2, ..., ah_m$, are the members of aH, all distinct.

For, we have

$$ah_i = ah_j \Rightarrow h_i = h_j$$

by cancellation law in G.

4. Since G is a finite group, the number of distinct left cosets will also be finite, say k. Hence the total number of elements of all cosets is k_m which is equal to the total number of elements of G.
Hence

$$n = mk$$

This show that m, the order of H, is a divisor of n, the order of the group G.
We also find that the index k is also a divisor of the order of the group.

ii. Prove that every group of prime order is cyclic.

Ans.

1. Let G be a group whose order is a prime p.
2. Since $P > 1$, there is an element $a \in G$ such that $a \neq e$.
3. The group $\langle a \rangle$ generated by 'a' is a subgroup of G.
4. By Lagrange's theorem, the order of 'a' divides $|G|$.
5. But the only divisors of $|G| = p$ are 1 and p. Since $a \neq e$ we have $|\langle a \rangle| > 1$, so $|\langle a \rangle| = p$.
6. Hence, $\langle a \rangle = G$ and G is cyclic.

c. Define a lattice. For any a, b, c, d in a lattice $(A, \leq)$ if $a \leq b$ and $c \leq d$ then show that $a \vee c \leq b \vee d$ and $a \wedge c \leq b \wedge d$.

Ans. **Lattice :** A lattice is a poset $(L, \leq)$ in which every subset $\{a, b\}$ consisting of 2 elements has least upper bound (lub) and greatest lower bound (glb). Least upper bound of $\{a, b\}$ is denoted by $a \vee b$ and is known as join of a and b. Greatest lower bound of $\{a, b\}$ is denoted by $a \wedge b$ and is known as meet of a and b.
Lattice is generally denoted by $(L, \wedge, \vee)$.

Numerical :

As $a \leq b$ and $c \leq d$, $a \leq b \leq b \vee d$ and $c \leq d \leq b \vee d$.
By transtivity of $\leq$, $a \leq b \vee d$ and $c \leq b \vee d$.
So $b \vee d$ is an upper bound of a and c.
So $a \vee c \leq b \vee d$.
As $a \wedge c \leq a$ and $a \wedge c \leq c$, $a \wedge c \leq a \leq b$ and $a \wedge c \leq c \leq d$.
Hence $a \wedge c$ is a lower bound of b and d. So $a \wedge c \leq b \wedge d$.
So $a \wedge c \leq b \wedge d$.

d. Show that $((p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \vee r)$ is a tautology without using truth table.

Ans.

i. We have

$((p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \vee r)$

$\equiv ((p \vee q) \wedge \sim(\sim p \wedge \sim(q \wedge r))) \vee (\sim(p \vee q) \vee \sim(p \vee r))$

(Using De Morgan's Law)

$\equiv [(p \vee q)] \wedge (p \vee (q \wedge r)) \vee \sim ((p \vee q) \wedge (p \vee r))$

$\equiv [(\text{p} \vee q) \wedge (p \vee q) \wedge (p \wedge r)] \vee \sim ((p \vee q) \wedge (p \vee r))$

(Using Distributive Law)

$\equiv [((p \vee q) \wedge (p \vee q)] \wedge (p \vee r) \vee \sim ((p \vee q) \wedge (p \vee r))$

$\equiv ((p \vee q) \wedge (p \vee r)) \vee \sim ((p \vee q) \wedge (p \vee r))$

$\equiv x \vee \sim x$ where $x = (p \vee q) \wedge (p \wedge r)$

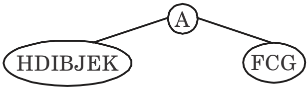
$\equiv T$

e. Define a binary tree. A binary tree has 11 nodes. It's inorder and preorder traversals node sequences are :
Preorder : ***A B D H I E J L C F G***
Inorder : ***H D I B J E K A F C G***
Draw the tree.

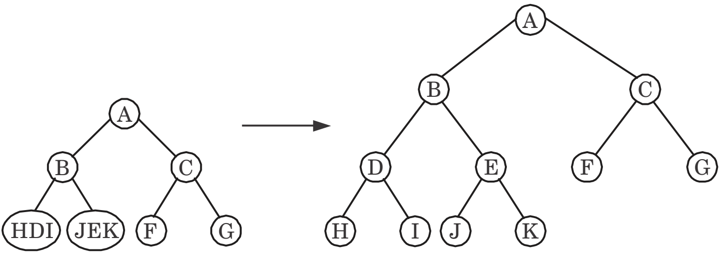
Ans. **Binary tree :** Binary tree is the tree in which the degree of every node is less than or equal to 2. A tree consisting of no nodes is also a binary tree.

Numerical :

Step 1 : In preorder sequence, leftmost element is the root of the tree. By searching A in 'Inorder Sequence' we can find out all the elements on the left and right sides of 'A'.



Step 2 : We recursively follow the above steps and we get



3. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Prove that if n is a positive integer, then 133 divides $11^{n+1} + 12^{2n-1}$.

Ans. We prove this by induction on n.

Base case : For $n = 1$, $11^{n+1} + 12^{2n-1} = 11^2 + 12^1 = 133$ which is divisible by 133.

Inductive step : Assume that the hypothesis holds for $n = k$, *i.e.*, $11^{k+1} + 12^{2k-1} = 133A$ for some integer A. Then for $n = k + 1$,

$11^{n+1} + 12^{2n-1} = 11^{k+1+1} + 12^{2(k+1)-1}$

$= \quad 11^{k+2} + 12^{2k+1}$

$=11 * 11^{k+1} + 144 * 12^{2k-1}$
$=11 * 11^{k+1} + 11 * 12^{2k-1} + 133 * 12^{2k-1}$
$=11[11^{k+1} + 12^{2k-1}] + 133 * 12^{2k-1}$
$=11 * 133A + 133 * 12^{2k-1}$
$= 133[11A + 12^{2k-1}]$

Thus if the hypothesis holds for $n = k$ it also holds for $n = k + 1$. Therefore, the statement given in the equation is true.

b. Let n be a positive integer and S a set of strings. Suppose that R_n is the relation on S such that sR_nt if and only if $s = t$, or both s and t have at least n characters and first n characters of s and t are the same. That is, a string of fewer than n characters is related only to itself; a string s with at least n characters is related to a string t if and only if t has at least n characters and t beings with the n characters at the start of s.

Ans. We have to show that the relation R_n is reflexive, symmetric, and transitive.

1. **Reflexive :** The relation R_n is reflexive because $s = s$, so that sR_ns whenever s is a string in S.
2. **Symmetric :** If $sR_n t$, then either $s = t$ or s and t are both at least n characters long that begin with the same n characters. This means that $tR_n s$. We conclude that R_n is symmetric.
3. **Transitive :** Now suppose that $sR_n t$ and $tR_n u$. Then either $s = t$ or s and t are at least n characters long and s and t begin with the same n characters, and either $t = u$ or t and u are at least n characters long and t and u begin with the same n characters. From this, we can deduce that either $s = u$ or both s and u are n characters long and s and u begin with the same n characters, *i.e.*, $sR_n u$. Consequently, R_n is transitive.

4. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Let $G = \{1, -1, i, -i\}$ with the binary operation multiplication be an algebraic structure, where $i = \sqrt{-1}$. Determine whether G is an abelian or not.

Ans. The composition table of G is

*	1	– 1	i	– i
1	1	– 1	i	– i
– 1	– 1	1	– i	i
i	i	– i	– 1	1
– i	– i	i	– 1	1

1. **Closure property :** Since all the entries of the composition table are the elements of the given set, the set G is closed under multiplication.
2. **Associativity :** The elements of G are complex numbers, and we know that multiplication of complex numbers is associative.
3. **Identity :** Here, 1 is the identity element.
4. **Inverse :** From the composition table, we see that the inverse elements of $1, -1, i, -i$ are $1, -1, -i, i$ respectively.
5. **Commutativity :** The corresponding rows and columns of the table are identical. Therefore the binary operation is commutative.

Hence, $(G, *)$ is an abelian group.

b. What is meant by ring ? Give examples of both commutative and non-commutative rings.

Ans. Ring : A ring is an algebraic system $(R, +, \bullet)$ where R is a non empty set and + and • are two binary operations (which can be different from addition and multiplication) and if the following conditions are satisfied :

1. $(R, +)$ is an abelian group.
2. $(R, \bullet)$ is semigroup *i.e.*, $(a \bullet b) \bullet c = a \bullet (b \bullet c)\ \forall\ a, b, c \in R$.
3. The operation • is distributive over +.

i.e., for any $a, b, c \in R$

$a \bullet (b + c) = (a \bullet b) + (a \bullet c)$ or $(b + c) \bullet a = (b \bullet a) + (c \bullet a)$

Example of commutative ring :

Let $a, b \in R\ (a + b)^2 = (a + b)$

$\Rightarrow (a + b)\,(a + b) = (a + b)$

$(a + b)a + (a + b)b = (a + b)$

$(a^2 + ba) + (ab + b^2) = (a + b)$

$(a + ba) + (ab + b) = (a + b)$ $\quad (\because\ a^2 = a$ and $b^2 = b)$

$(a + b) + (ba + ab) = (a + b) + 0$

$\Rightarrow \quad ba + ab = 0$

$a + b = 0 \Rightarrow a + b = a + a$ [being every element of its own additive inverse]

$\Rightarrow \quad b = a$

$\Rightarrow \quad ab = ba$

$\therefore$ R is commutative ring.

Example of non-commutative ring : Consider the set R of 2×2 matrix with real element. For $A, B, C \in$ R

$A * (B + C) = (A * B) + (A * C)$

also, $(A + B) * C = (A * C) + (B * C)$

$\therefore$ * is distributive over +.

$\therefore$ $(R, +, *)$ is a ring.

We know that $AB \neq BA$, Hence $(R, +, *)$ is non-commutative ring.

5. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. Show that the inclusion relation ⊆ is a partial ordering on the power set of a set S. Draw the Hasse diagram for

inclusion on the set $P(S)$, where $S = \{a, b, c, d\}$. Also determine whether $(P(S), \subseteq)$ is a lattice.

Ans. Show that the inclusion relation ($\subseteq$) is a partial ordering on the power set of a set S.

Reflexivity : $A \subseteq A$ whenever A is a subset of S.

Antisymmetry : If A and B are positive integers with $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Transitivity : If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Hasse diagram :

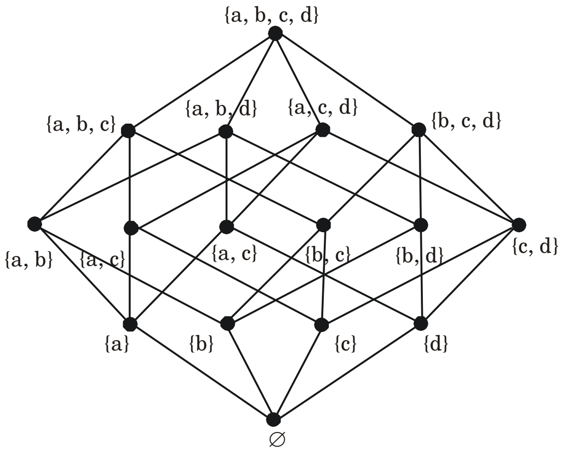


Fig. 6.

$(P(S), \subseteq)$ is not a lattice because $(\{a, b\},\{b, d\})$ has no *lub* and *glb*.

b. Find the Sum-Of-Products and Product-Of-sum expansion of the Boolean function $F(x, y, z) = (x + y)\, z'$.

Ans. $F(x, y, z) = (x + y)z'$

x	y	z	$x + y$	z'	$(x + y)z'$
1	1	1	1	0	0
1	1	0	1	1	1
1	0	1	1	0	0
1	0	0	1	1	1
0	1	1	1	0	0
0	1	0	1	1	1
0	0	1	0	0	0
0	0	0	0	1	0

Sum-Of-Product :

$$F(x, y, z) = xyz' + xy'z' + x'yz'$$

Product-Of-Sum :

$$F(x, y, z) = (x + y + z)(x + y' + z)(x' + y + z)(x' + y' + z)(x' + y' + z')$$

6. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. What is a tautology, contradiction and contingency ? Show that $(p \vee q) \vee (\neg p \vee r) \rightarrow (q \vee r)$ is a tautology, contradiction or contingency.

Ans. Tautology, contradiction and contingency :

1. Tautology : Tautology is defined as a compound proposition that is always true for all possible truth values of its propositional variables and it contains T in last column of its truth table.

Propositions like,

i. The doctor is either male or female.

ii. Either it is raining or not.

are always true and are tautologies.

2. Contradiction : Contradiction is defined as a compound proposition that is always false for all possible truth values of its propositional variables and it contains F in last column of its truth table.

Propositions like,

i. x is even and x is odd number.

ii. Tom is good boy and Tom is bad boy.

are always false and are contradiction.

3. Contingency : A proposition which is neither tautology nor contradiction is called contingency.

Here the last column of truth table contains both T and F.

Proof : $((p \vee q) \vee (\sim p \vee r)) \rightarrow (q \vee r)$

p	q	r	$\sim P$	$(p \vee q) = A$	$(\sim p \vee r) = B$	$(A \vee B) = C$	$(q \vee r) = D$	$C \rightarrow D$
F	F	F	T	F	T	T	F	F
F	F	T	T	F	T	T	T	T
F	T	F	T	T	T	T	T	T
F	T	T	T	T	T	T	T	T
T	F	F	F	T	F	T	F	F
T	F	T	F	T	T	T	T	T
T	T	F	F	T	F	T	T	T
T	T	T	F	T	T	T	T	T

So, $((p \vee q) \vee (\sim p \vee r)) \rightarrow (q \vee r)$ is contingency.

b. Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip." and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

Ans.

i. The compound proposition will be : $(p \wedge q \wedge r) \Leftrightarrow s$

ii. Let p be the proposition "It is sunny this afternoon", q be the proposition "It is colder than yesterday", r be the proposition "We will go swimming", s be the proposition "We will take a canoe trip", and t be the proposition "We will be home by sunset".
Then the hypothesis becomes $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$. The conclusion is simply t.
We construct an argument to show that our hypothesis lead to the conclusion as follows :

S. No.	Step	Reason
1.	$\neg p \wedge q$	Hypothesis
2.	$\neg p$	Simplification using step 1
3.	$r \rightarrow p$	Hypothesis
4.	$\neg r$	Modus tollens using steps 2 and 3
5.	$\neg r \rightarrow s$	Hypothesis
6.	s	Modus ponens using steps 4 and 5
7.	$s \rightarrow t$	Hypothesis
8.	t	Modus ponens using steps 6 and 7

7. Attempt any **one** part of the following : **(7 × 1 = 7)**

a. What are different ways to represent a graph. Define Euler circuit and Euler graph. Give necessary and sufficient conditions for Euler circuits and paths.

Ans. **Representation of graph :**
Graph can be represented in following two ways :

1. **Matrix representation :**
Matrices are commonly used to represent graphs for computer processing. Advantages of representing the graph in matrix lies in the fact that many results of matrix algebra can be readily applied to study the structural properties of graph from an algebraic point of view.

a. **Adjacency matrix :**

i. Representation of undirected graph

ii. Representation of directed graph

b. Incidence matrix :

i. Representation of undirected graph

ii. Representation of directed graph

2. Linked representation : In this representation, a list of vertices adjacent to each vertex is maintained. This representation is also called adjacency structure representation. In case of a directed graph, a care has to be taken, according to the direction of an edge, while placing a vertex in the adjacent structure representation of another vertex.

Euler circuit and Euler graph :

Eulerian circuit : A circuit of graph G which include each edge of G exactly once.

Eulerain graph : A graph containing an Eulerian circuit is called Eulerian graph.

For example : Graphs given below are Eulerian graphs.

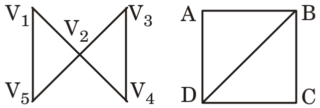


Fig. 7.

Necessary and sufficient condition for Euler circuits and paths :

1. A graph has an Euler circuit if and only if the degree of every vertex is even.
2. A graph has an Euler path if and only if there are at most two vertices with odd degree.

b. Suppose that a valid codeword is an n-digit number in decimal notation containing an even number of 0's. Let a_n denote the number of valid codewords of length n satisfying the recurrence relation $a_n = 8a_{n-1} + 10^{n-1}$ and the initial condition $a_1 = 9$. Use generating functions to find an explicit formula for a_n.

Ans. Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence a_0, a_1, a_2 .... we sum both sides of the last equations starting with $n = 1$. To find that

$$G(x) - 1 = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (8a_{n-1}x^n + 10^{n-1} x^n)$$

$$= 8\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1} x^n$$

$$= 8x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + x \sum_{n=1}^{\infty} 10^{n-1} x^{n-1}$$

$$= 8x \sum_{n=0}^{\infty} a_n x^n + x \sum_{n=0}^{\infty} 10^n x^n$$

$$= 8xG(x) + x/(1 - 10x)$$

Therefore, we have

$$G(x) - 1 = 8xG(x) + x/(1 - 10x)$$

Expanding the right hand side of the equation into partial fractions gives

$$G(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right)$$

This is equivalent to

$$G(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n) x^n$$

$$a_n = \frac{1}{2}(8^n + 10^n)$$

☺☺☺

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