Q1)

a) i) Let $g(x_k) = x_k - f(x_k)/d$. For convergence, we need to find its derivative, so $g'(x_k) = 1 - f'(x_k)/d$. We'll use fixed point iteration method here where $x_{k+1} = g(x_k)$ as per equation. Let x^* be the fixed point.

Now for convergence, the condition for the derivative is

$$|g'(x^*)| = |1 - f'(x^*)/d| < 1$$

=> $$-1 < 1 - f'(x^*)/d < 1$$

=> $$0 < f'(x^*)/d < 2$$

ii) Convergence rate will be linear with a constant $c = |1 - f'(x^*)/d|$

iii) For quadratic convergence, the condition is that $g'(x^*) = 0$ or $1 - f'(x^*)/d = 0$. Therefore, $d = f'(x^*)$

b) For part (i), the convergence rate ≈ 2, so this equation has quadratic convergence.
For part (ii), the convergence rate ≈ 1, so this equation has linear convergence.
For part (iii), the convergence rate ≈ 2, so this equation has quadratic convergence.

Q2)

c) When relative residual is small, relative error will only be small when the matrix is well conditioned. These three are bound by a simple relation:

=> $$(\text{relative residual}) \leq (\text{condition number})*(\text{relative error})$$

In some cases however, we'll find divergence from this relation. This is due to dissimilar range in some cases.

Q3.

(a) Given → $F_n(t) = \cos(n\cos^{-1}(t))$

Chebyshev 3 term recurrence →

$$F_0(t) = 1 \quad -①$$

$$F_1(t) = t \quad -②$$

$$F_{n+1}(t) = 2t\,F_n(t) - F_{n-1}(t) \quad -③$$

→ Now, ∀ ① $n=0$

$$F_0(t) = \cos(0) = 1$$

So ① is satisfied

→ Now ∀ ②

$$F_1(t) = \cos(\cos^{-1}(t))$$

$$= t \qquad \forall\, t \in [-1,1]$$

∴, ② is satisfied.

→ Now ∀ ③

$$F_{n+1}(t) = \cos((n+1)(\cos^{-1}(t)))$$

To solve this, we first consider let

$$y = \cos^{-1}(t)$$

Now

$$\cos((n+1)y) = \cos(ny + y)$$

$$= \cos(ny)\cos(y) - \sin(ny)\sin(y)$$

Also

$$\cos((n-1)y) = \cos(ny - y)$$

$$= \cos(ny)\cos(y) + \sin(ny)\sin(y)$$

Adding both

$$\cos((n+1)y) + \cos((n-1)y) = 2\cos(ny)\cos(y)$$

Substituting in the form of $F_n(t)$

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$\Rightarrow \quad F_{n+1}(t) + F_{n-1}(t) = 2t\, F_n(t)$

$$\Rightarrow \quad \boxed{F_{n+1}(t) = 2t\, F_n(t) - F_{n-1}(t)}$$

$\therefore$ (3) is satisfied.

Hence, $F_n(t)$ satisfies the Chebyshev 3-term reccurrence.

(b) from (a) $\quad f_n(t) = \cos(ny)$

Expanding this

$$= \cos^n y - {}^nC_2 \cos^{n-2} y\,(1-\cos^2 y) + {}^nC_4 \cos^{n-4} y\,(1-\cos^2 y)^2 - \ldots$$

Since we know $y = \cos^{-1}(t)$

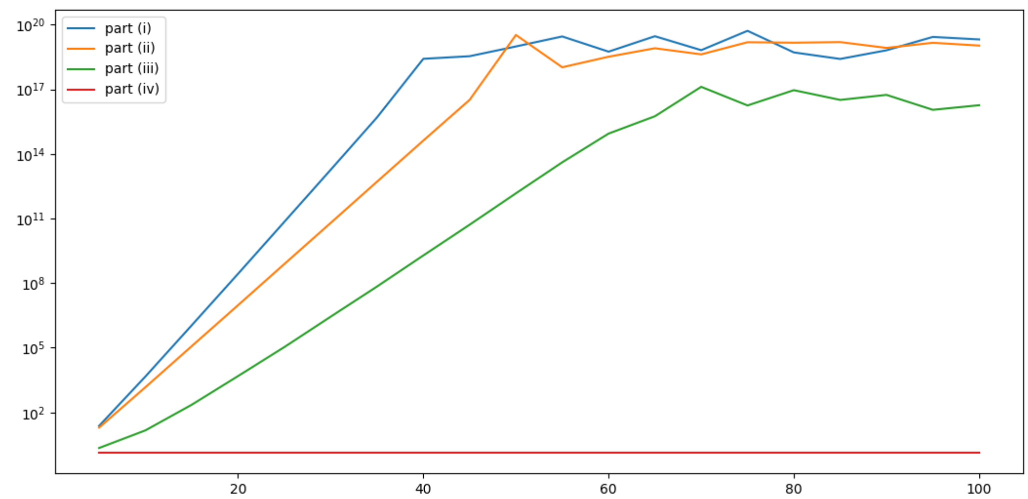
$\Rightarrow \quad t = \cos(y)$

Substituting.

$$\Rightarrow \quad f_n(t) = t^n - {}^nC_2\, t^{n-2}(1-t^2) + {}^nC_4\, t^{n-4}(1-t^2)^2 - \ldots$$

The f^n on RHS is clearly a polynomial of t

$\therefore$ Chebyshev f^n $f_n(t)$ is a polynomial.



3d) As we can see from the plot, there is negligible change in the condition number (remains same) as n increases in part (iv) (Chebyshev nodes with Chebyshev polynomial). Hence this combination performs best.

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Q4.

(a) $$f[t_1, t_2, \ldots, t_k] := \frac{f[t_2, t_3, \ldots, t_k] - f[t_1, \ldots, t_{k-1}]}{t_k - t_1}$$

$$f[t_j] = f(t_j)$$

Proof by induction → We are going to prove this using strong induction

Consider the polynomial

$$p(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1})$$

When $k = 0$

we see $p(x_0) = a_0$

And p interpolates f on x_0

$\therefore \quad p(x_0) = f(x_0)$

So, $\quad a_0 = f(x_0) = f[x_0]$

IH → Assuming $a_j = f[x_0, x_1, \ldots, x_j]$ $\forall\, j = 0, \ldots, k-1$ all true

IS → We need to show $\forall$ a_k

Now, $a_0, \ldots, a_{k-1}$ are known by divided differences

Now we know $f(x_k) = p_k(x_k)$

So, $\quad f(x_k) = a_0 + a_1(x_k - x_0) + \cdots$

$\Rightarrow \quad \dfrac{f(x_k) - a_0}{x_k - x_0} = a_1 + a_2(x_k - x_1) + \cdots$

Since $a_0 = f(x_k)$ $\Rightarrow$ LHS $= f[x_0, x_k]$

Keep on repeating this k-1 times, we'll get

$$f[x_0, \dots, x_{k-2}, x_k] = a_{k-1} + a_k (x_k - x_{k-1})$$

So,

$$\frac{f[x_0, \dots, x_{k-2}, x_k] - a_{k-1}}{x_k - x_{k-1}} = a_k$$

∴, this approach gives the coefficient of the j^{th} basis function using Newton interpolation polynomial

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(b) Given $\rightarrow$ 3 points $(-1,1)$ $(0,0)$ $(1,1)$

(i) We'll determine the interpolating polynomial using monomial basis

Now, linear system using monomial basis

$$Ax = y$$

$$\begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Using data points given above

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Solving this, we get

$$x = [0 \quad 0 \quad 1]^T$$

$\therefore$, the interpolating polynomial is :

$$\boxed{p(t) = t^2}$$

(ii) We'll determine the interpolating polynomial using lagrange basis, which states:

$$p(t) = y_1 \frac{(t-t_2)(t-t_3)}{(t_1-t_2)(t_1-t_3)} + y_2 \frac{(t-t_1)(t-t_3)}{(t_2-t_1)(t_2-t_3)} + y_3 \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)}$$

Substituting the points

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$$\Rightarrow p(t) = 1\cdot\frac{(t-0)(t-1)}{(-1-0)(-1-1)} + 0\cdot\frac{(t+1)(t-1)}{(0+1)(0-1)} + 1\cdot\frac{(t+1)(t-0)}{(1+1)(1-0)}$$

$$= \frac{t(t-1)}{2} + \frac{t(t+1)}{2}$$

$$= \frac{t^2 - t^2 + t^2 + t}{2}$$

$$\boxed{p(t) = t^2}$$

(iii) We'll determine the interpolating polynomial using Newton basis, which has the form $\rightarrow$

$$p_{n-1}(t) = x_1 + x_2(t-t_1) + x_3(t-t_1)(t-t_2) + \dots + x_n(t-t_1)\dots(t-t_{n-1})$$

In matrix form

$$Ax = y$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & t_2-t_1 & 0 \\ 1 & t_3-t_1 & (t_3-t_1)(t_3-t_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Solving this, we get

$$x = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T$$

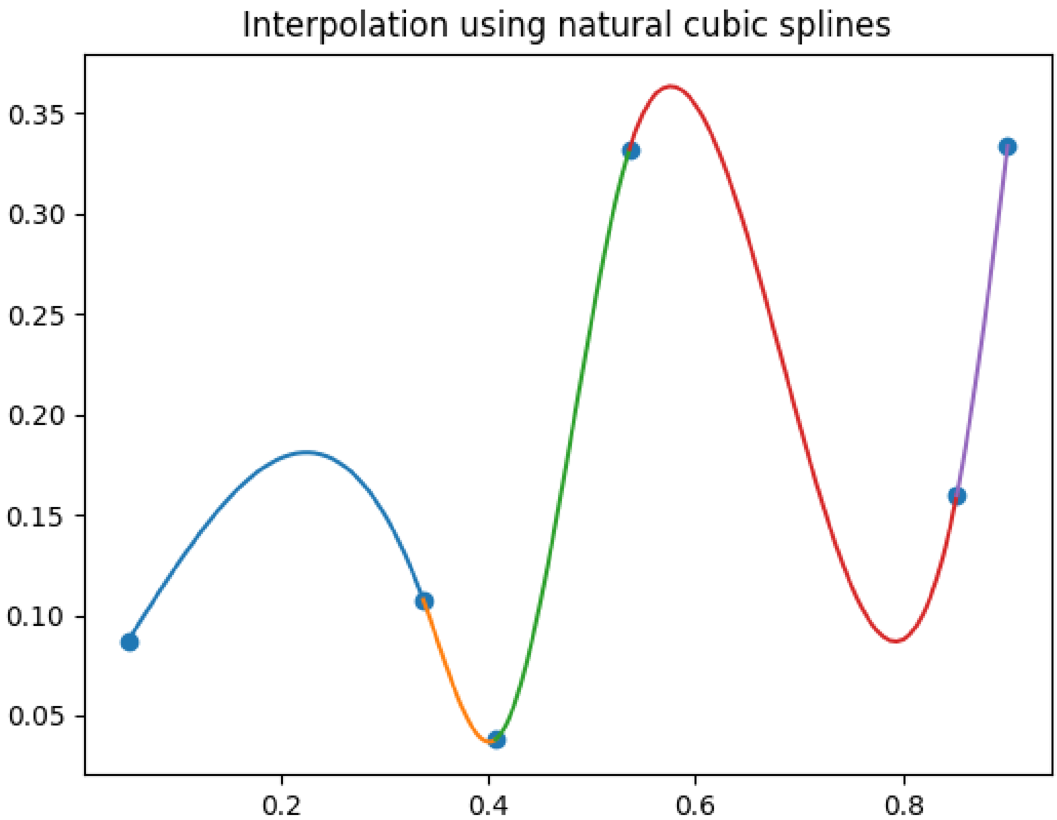
Now, the interpolating polynomial is

$$p(t) = 1 - 1\cdot(t+1) + 1\cdot(t+1)(t)$$

$$\Rightarrow \boxed{p(t) = t^2}$$

(iv) As done in (i), (ii), (iii) all 3 basis give the same polynomial.

Q4)



c)

6 random points have been plotted and interpolation has been performed