Matrix :

The system of numbers, arranged in a rectangular array in rows and column and bounded by the brackets, is called a matrix.

e.g.-1 $A = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}_{3\times 3}$

* It has 3 rows and 3 column and in all $3\times 3 = 9$ elements.
* It is called as 3×3 matrix to be read as 3 by 3 matix.

e.g.-2 $A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 4 & 2 & 5 & 7 \\ 3 & 4 & 2 & 6 \end{bmatrix}_{3\times 4}$.

Types of matrices :

(i) Row Matrix : If a matrix has only one row and any number of columns, then it is called a Row matrix

e.g. $[1 \ 2 \ -1]$

(ii) <u>Column Matrix</u>: A matrix has only one column and any number of rows, is called a <u>Column matrix</u>

e.g. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(iii) <u>Null Matrix or Zero Matrix</u>:

Any matrix is called a <u>zero Matrix or Null matrix</u> if its all the elements are zeros.

e.g. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(iv) <u>Square Matrix</u>:

A matrix in which the numbers of rows and columns are equal, is called a <u>Square matrix</u>

e.g. $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix}$

(v) <u>Diagonal Matrix</u>: A square matrix is called a <u>diagonal matrix</u>, if all its non diagonal elements are zero.

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times 3}$

(vi) Scalar matrix

A diagonal matrix in which all the diagonal elements are equal to a scalar, say λ, is called a scalar matrix

e.g. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times 3}$ Here $\lambda = 3$.

(vii) Unit or Identity Matrix:

A square matrix is called unit matrix if all the diagonal elements are unity and non diagonal elements are zero.

e.g $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

(viii) Triangular Matrix:

(a) Upper triangular matrix:

A square matrix is called an upper triangular matrix if all of its elements below the leading diagonal are zero.

e.g $\begin{bmatrix} 1 & -2 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

(b) Lower triangular matrix:

A square matrix is called a lower triangular matrix if all of its elements above the leading diagonal are zero

e.g. $\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 6 & 7 \end{bmatrix}$

(ix) Transpose of a matrix:

If, we interchange the rows and the corresponding columns in a matrix A then the new matrix obtained is called the transpose of the matrix A. and denoted by A' or A^T

e.g. $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$, $A' = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

(X) Symmetric Matrix: A square matrix is ~~called~~ is said to be symmetric if

$\boxed{A' = A}$ i.e. $a_{ij} = a_{ji}, \forall i, j$

e.g. $\begin{bmatrix} f & a & b \\ a & g & c \\ b & c & h \end{bmatrix}$

(xi) **Skew Symmetric Matrix:**

A square matrix is said to be skew symmetric if $\boxed{A' = -A}$

i.e. $a_{ij} = -a_{ji}$, $\forall\, i, j$

e.g. $\begin{bmatrix} 0 & -a & b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$

(xii) **Orthogonal Matrix:** A square matrix A is called an orthogonal matrix if $\boxed{AA' = I}$ when I is an identity matrix.

(xiii) **Conjugate of a matrix:**

conjugate of a given matrix A is denoted by $\bar{A}$.

~~e.g.~~ if $A = \begin{bmatrix} 1 & 2+3i & 3+i \\ 2-3i & 2 & 1-2i \\ 3-i & 1+2i & 5 \end{bmatrix}$

then $\bar{A} = \begin{bmatrix} 1 & 2-3i & 3-i \\ 2+3i & 2 & 1+2i \\ 3+i & 1-2i & 5 \end{bmatrix}$

(xiv) Transpose of the conjugate of a matrix :

If A be a matrix then its transposed conjugate is denoted by A^{θ} or A^*.

Let $A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$

then $A^{\theta} = (\bar{A})' = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3+2i \end{bmatrix}$

(xv) Unitary Matrix :

A square matrix is said to be unitary if $\boxed{A^{\theta}A = I}$

e.g. $A = \begin{bmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) \\ \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \end{bmatrix}$.

(xvi) Hermitian Matrix :- A square matrix A is called Hermitian matrix if $\boxed{A = A^{\theta}}$

i.e. if $A = [a_{ij}]$ be Hermitian then

$$a_{ij} = \overline{a_{ji}} \quad , \quad \forall\, i, j \,.$$

e.g. $A = \begin{bmatrix} 1 & 2+3i & 3+i \\ 2-3i & 2 & 1-2i \\ 3-i & 1+2i & 5 \end{bmatrix}$

(xvii) Skew Hermitian Matrix :

A square matrix A is called a Skew Hermitian matrix if $\boxed{A = -A^{\theta}}$

i.e. if $A = [a_{ij}]$ be skew hermitian then
$$a_{ij} = -\bar{a}_{ji} \quad , \forall i, j$$

* All the diagonal elements of a skew hermitian matrix are either zeros or pure imaginary.

e.g $A = \begin{bmatrix} i & 2-3i & 4+5i \\ -2-3i & 0 & 2i \\ -4+5i & 2i & -3i \end{bmatrix}$.

(xviii) Singular Matrix :

A square matrix A is said to be singular if $|A| = 0$.

e.g. $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

otherwise non singular.

(xix) Idempotent Matrix :

A matrix is called Idempotent Matrix if $\boxed{A^2 = A}$

e.g. $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

(xx) Nilpotent Matrix :

A matrix is called a Nilpotent Matrix of index k if $A^k = 0$ where 0 is a null matrix & k is the least +'ve integer.

e.g. $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$

$A \cdot A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$

$\Rightarrow A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

$\Rightarrow$ A is nilpotent of index 2

(xxi) Involuntary Matrix :

A matrix A is called an Involuntary if $\boxed{A^2 = I}$ where $I \rightarrow$ unit matrix

I is always Involuntary since $I^2 = I$

(xxii) <u>Equal Matrices:</u>

Two matrices are said to be equal if they are of same order and the elements in the corresponding positions are equal.

e.g. $A = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$ & $B = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$

$\Rightarrow \boxed{A = B}$

<u>Addition of Matrices:</u>

If A and B be two matrices of the same order then their sum is denoted by A + B.

e.g. $A = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

$A + B = \begin{bmatrix} 2+1 & 5+2 \\ 1+3 & 4+6 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 4 & 10 \end{bmatrix}$

<u>Subtraction of Matrices:</u>

The difference between two matrices is obtained by subtracting the elements of second matrix from the corresponding element of first.

e.g. $A = \begin{bmatrix} 8 & 6 & 4 \\ 1 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 & 1 \\ 7 & 6 & 2 \end{bmatrix}$

$$A - B = \begin{bmatrix} 8-3 & 6-5 & 4-1 \\ 1-7 & 2-6 & 0-2 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 3 \\ -6 & -4 & -2 \end{bmatrix}$$

Multiplication:

The product of two matrices A and B is only possible if the number of columns in A is equal to the number of rows in B.

Ex. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$

Here AB is possible but BA is not defined.

Rank of a Matrix

Rank of a matrix A is r if there is at least one determinant of order 'r' which is not equal to zero and every determinant of order r+1 is zero.

The symbol for the rank of matrix A is $\rho[A]$.

Briefly we may say that the rank of a matrix is the order of any highest order non vanishing determinant of matrix.

Q.1 Find the rank of the matrix

$$A = \begin{bmatrix} 2 & 4 & 5 & 6 \\ 7 & 8 & 9 & 1 \end{bmatrix}$$

Soln: $\begin{vmatrix} 2 & 4 \\ 7 & 8 \end{vmatrix} = 16-28 = -12 \neq 0$

Hence rank of A is 2.

Q.2 Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 8 & 10 \\ 3 & 6 & 12 & 15 \end{bmatrix}$$

Ans. $\rho[A] = 1$.

Note: The rank of every nonzero matrix is ≥ 1 & only the rank of null matrix is 0

Elementary Transformations:

Any one of the following operations on a matrix is called an elementary transformation

① Interchange of any two rows (or columns)

$$R_i \leftrightarrow R_j \text{ or } (C_i \leftrightarrow C_j)$$

② The multiplication of the elements of any row (or column) by any non zero scalar.

$$R_i \to \lambda R_i \text{ or } (C_i \to \lambda C_i)$$

③ The addition to the elements of any row (or column), the same multiple of the corresponding elements of any other row or column.

$$R_i \to R_i + \lambda R_j \text{ or } C_i \to C_i + \lambda C_j$$

If matrix B is obtained from matrix A by these elementary transformations then B is called equivalent to A and symbolically written as $A \sim B$.

Echelon form (Triangular form) of a matrix:

A matrix A is said to be in Echelon form if

(1) all the zero rows or any zero (if it exists) follows then non zero rows.

(2) The number of zeros before the first non zero element in first, second, third row, ------ should be in the increasing order

The rank of matrix in echelon form is equal to the number of non-zero rows of the matrix

Q.1 find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix}$$

Solⁿ

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -7 \end{bmatrix} \begin{array}{l} R_3 \to R_3 - 3R_1 \\ R_2 \to R_2 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \quad R_3 \to R_3 + 5R_2$$

Here the no. of non zero rows = 3

so $\boxed{\rho[A] = 3}$

Q.2 find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix}$$

Solⁿ

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} R_3 \to R_3 - R_1 \\ R_2 \to R_2 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

which is in echelon form

so $\rho[A]$ = no. of non zero rows = 2

#

Q.3 $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$, find rank.

Solⁿ

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad \begin{matrix} R_4 \to R_4 - R_1 \\ R_3 \to R_3 - 3R_1 \\ R_2 \to R_2 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 + \frac{1}{2} R_3$$

which is in echelon form

so $\rho[A]$ = no. of nonzero rows = 3

Q-4 find the rank of the matrix

$$A = \begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Solⁿ:

$$A \sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 2 & 2 & 1 \\ 3 & -2 & 0 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 2 & 2 & 1 \\ 0 & 4 & 9 & -7 \\ 0 & 1 & 2 & 1 \end{bmatrix} R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 5 & -9 \\ 0 & 0 & 1 & 1/2 \end{bmatrix} \begin{matrix} R_4 \to R_4 - \frac{1}{2}R_2 \\ R_3 \to R_3 - 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 5 & -9 \\ 0 & 0 & 0 & 23/10 \end{bmatrix} R_4 \to R_4 - \frac{1}{5}R_3$$

which is in echelon form.

Q-5 find the rank of the following matrices

(a) $\begin{bmatrix} 1 & 2 & 3 & -4 \\ -2 & 3 & 7 & -1 \\ 1 & 9 & 16 & -13 \end{bmatrix}$. Ans : 2

(b) $\begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$. Ans: 3

The following form of the matrices are echelon form:

(a) $\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & g \end{bmatrix}$ (b) $\begin{bmatrix} a & b & c \\ 0 & 0 & f \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & b & c \\ 0 & 0 & f \\ 0 & 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Normal form (Canonical form):

A matrix of the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$ where I_r is unit matrix of order 'r'.

is called normal form

then $\boxed{\text{rank} = r}$

Q-1 Reduce the matrix A to normal form and find its rank.

Soln

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$, $C_4 \to C_4 + C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \\ 2 & 7 & 2 & 3 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 5 \\ 0 & 7 & 2 & 3 \end{bmatrix}$$

$C_2 \leftrightarrow C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 5 \\ 0 & 2 & 7 & 3 \end{bmatrix}$$

$C_3 \to C_3 + 2C_2, \; C_4 \to C_4 - 5C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 11 & -7 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 11 & -7 \end{bmatrix}$$

$C_3 \to \frac{1}{11} C_3, \; C_4 \to -\frac{1}{7} C_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$C_4 \to C_4 - C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [I_3 \; O]$$ which is normal form.

$\Rightarrow \boxed{\rho[A] = 3}$

Q.2 find the rank of the matrix

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$$

by reducing it to normal form.

Solⁿ: Given $A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & -7 & -5 & -2 \\ 0 & -14 & -10 & -4 \\ 0 & -21 & -16 & -6 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & -14 & -10 & -4 \\ 0 & -21 & -16 & -6 \end{bmatrix} \quad \begin{matrix} C_2 \to C_2 - 3C_1 \\ C_3 \to C_3 - 4C_1 \\ C_4 \to C_4 - 2C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 3R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} C_4 \to C_4 - \frac{2}{7}C_2 \\ C_3 \to C_3 - \frac{5}{7}C_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \to -R_3 \\ R_2 \to -\frac{1}{7}R_2 \end{matrix}$$

$$= \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \text{ which is normal form}$$

Hence $\boxed{\rho[A] = 3}$

Q-3 Using elementary transformations, reduce the following matrices into normal form and then find rank.

(a) $\begin{bmatrix} 3 & 2 & 5 & 7 & 12 \\ 1 & 1 & 2 & 3 & 5 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$ Ans. $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$, rank = 2

(b) $\begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$ Ans. $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$, rank = 3

Linear Equations

We can not solve every simultaneous equations of the type

$$\left.\begin{aligned} a_1x+b_1y+c_1z &= d_1 \\ a_2x+b_2y+c_2z &= d_2 \\ a_3x+b_3y+c_3z &= d_3 \end{aligned}\right\}$$

Or $$\left.\begin{aligned} a_1x+b_1y &= c_1 \\ a_2x+b_2y &= c_2 \end{aligned}\right\}$$

for example consider the simultaneous equations

$$2x+3y=2 \quad \text{—— (1)}$$
$$4x+6y=7 \quad \text{—— (2)}$$

Multiply equation (1) by 2 and then subtracting it from (2), we get $\boxed{0=3}$

Such equations are said to be inconsistent i.e. No, set of values of x and y are possible which can satisfy both these equations.

Now, let us consider another example

$$\left.\begin{aligned} x + y &= 4 \\ x - y &= 2 \end{aligned}\right\}$$

which gives $\boxed{x = 3, y = 1}$

so these equations are <u>not inconsistent</u>. Since there exist values of x and y which satisfy both these equations.

Hence the above set of equations are <u>"consistent"</u> and possess <u>unique solution</u>

Next consider the set of equations

$$\left.\begin{aligned} 2x + 3y &= 2 \\ 4x + 6y &= 4 \end{aligned}\right\}$$

These equations are also <u>'consistent'</u> Since there exist values of x & y which satisfy both these equations

Take $y = c$, we have $x = 1 - \frac{3}{2}c$

Since c is arbitrary so these equations have an <u>infinite number of solutions</u>.

Matrix representation of simultaneous equations:

consider a system of linear equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

The matrix representation of these system of equations is

$$\boxed{AX = B}$$

where A = matrix formed by the coefficients of the variables in the given equations

so $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

X = column vector formed by the variables only.

so $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

& B = Column matrix formed by number on R.H.S. of equations.

so $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Augmented matrix:

The matrix, which is obtained by adding an additional column to matrix A of the number $d_1, d_2, d_3 \ldots$ is called augmented matrix & ~~dllo~~ written as

$$[A, B] = \begin{bmatrix} a_1 & b_1 & c_1 & \cdots & d_1 \\ a_2 & b_2 & c_2 & \cdots & d_2 \\ a_3 & b_3 & c_3 & \cdots & d_3 \end{bmatrix}.$$

Note: The system of equations $AX = B$ is consistent i.e. possesses a solution if and only if the coefficient matrix A and the augmented matrix [A, B] are of same rank.

i.e. If $\rho[A] = \rho[A, B]$ then the given system of equations is consistent.

otherwise inconsistent.

i.e. If $\rho[A] \neq \rho[A, B]$ then given system of eqn's is inconsistent.

How to solve Given system of Non Homogeneous Equations:

consider,

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

(i) Write the matrix equation of given system of equations in the form $AX = B$.

(ii) Write augmented matrix $[A, B]$

(iii) Perform only E-row operations (transformations) ~~only~~ to reduce it to echelon form.

(iv) Determine ranks of A and [A,B].

<u>Case I:</u> If $\rho[A] \neq \rho[A,B]$ then given system of equations is <u>inconsistent</u>

<u>Case II:</u> If $\rho[A] = \rho[A,B]$ then given system of equations is <u>consistent</u>.

(i) when $\rho[A] = \rho[A,B] =$ no. of unknown then given system of equations will have unique solutions.

(ii) when $\rho[A] = \rho[A,B] <$ no. of unknowns then given system of equations will have infinite solutions.

[redacted]:

[redacted]

Q.1 Solve:

$$2x - y + 3z = 8$$
$$-x + 2y + z = 4$$
$$3x + y - 4z = 0$$

Solⁿ:

Matrix equation is,

$$AX = B$$

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

Augmented matrix is,

$$[A, B] = \begin{bmatrix} 2 & -1 & 3 & \cdots & 8 \\ -1 & 2 & 1 & \cdots & 4 \\ 3 & 1 & -4 & \cdots & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 2 & 1 & \cdots & 4 \\ 2 & -1 & 3 & \cdots & 8 \\ 3 & 1 & -4 & \cdots & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} -1 & 2 & 1 & \cdots & 4 \\ 0 & 3 & 5 & \cdots & 16 \\ 0 & 7 & -1 & \cdots & 12 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} -1 & 2 & 1 & \cdots & 4 \\ 0 & 3 & 5 & \cdots & 16 \\ 0 & 1 & -11 & \cdots & -20 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} -1 & 2 & 1 & \cdots & 4 \\ 0 & 1 & -11 & \cdots & -20 \\ 0 & 3 & 5 & \cdots & 16 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} -1 & 2 & 1 & \cdots & 4 \\ 0 & 1 & -11 & \cdots & -20 \\ 0 & 0 & 38 & \cdots & 76 \end{bmatrix} \quad R_3 \to R_3 - 3R_2$$

which is in echelon form

Here we see that,

$\rho[A] = 3 \quad \rho[A, B] = 3$

$\Rightarrow \rho[A] = \rho[A, B]$

so $\Rightarrow$ The give system of equations is <u>consistent</u>.

Since $\rho[A] = \rho[A, B]$ = no. of unknowns = 3

so given system of equations will have unique solution.

Now equivalent system of matrix equations is

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & -11 \\ 0 & 0 & 38 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 4 \\ -20 \\ 76 \end{bmatrix}$$

which gives

$$-x + 2y + z = 4 \quad \text{—— (i)}$$

$$y - 11z = -20 \quad \text{—— (ii)}$$

$$38z = 76 \Rightarrow \boxed{z = 2} \quad \text{—— (iii)}$$

(ii) & (iii) $\Rightarrow \boxed{y = 2}$

Putting the values of y & z in (i)

we have $\boxed{x = 2}$

Hence the unique solu^n is

$x = 2, \; y = 2, \; z = 2$ #

Q-2

Solve

$$x+2y+z=2$$
$$2x+4y+3z=3$$
$$3x+6y+5z=4$$

Soln

Matrix equation is

$$AX=B$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Augmented matrix

$$[A,B]=\begin{bmatrix} 1 & 2 & 1 & \cdots & 2 \\ 2 & 4 & 3 & \cdots & 3 \\ 3 & 6 & 5 & \cdots & 4 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & 2 & 1 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & -1 \\ 0 & 0 & 2 & \cdots & -2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2-2R_1 \\ R_3 \to R_3-3R_1 \end{matrix}$$

$$\sim\begin{bmatrix} 1 & 2 & 1 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad R_3 \to R_3-2R_2$$

which is in echelon form

Here $\rho[A]=\rho[A,B]$

then given system of equations is consistent

Since $\rho[A]=\rho[A,B]=2<$ no. of unknowns

so given system of equation will have infinite solutions.

Now equivalent matrix equation is

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

which gives

$$x + 2y + z = 2$$
$$z = -1$$

Let $y = k$ (arbitrary)

then $x = 3 - k$

Hence $\boxed{x = 3-k,\ y = k,\ z = -1}$

Since k is arbitrary, we have infinite solutions.

Q-3 Investigate for what values of λ and μ the system of simultaneous equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

has (i) no solution
(ii) a unique solution
(iii) an infinite solutions.

Solⁿ:

The matrix equation is

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$[A, B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right] R_3 \to R_3 - R_2$$

(i) when $\lambda = 3$, $\mu \neq 10$ then $\rho[A] = 2$ & $\rho[A, B] = 3 \Rightarrow \rho[A] \neq \rho[A, B]$
$\Rightarrow$ the system of equation will be inconsistent.

(ii) when $\lambda \neq 3$ & $\mu =$ any value (including 10)
then $\rho[A] = \rho[A, B] = 3$ = no. of unknowns
so $\Rightarrow$ the solution will be unique.

(iii) when $\lambda = 3$ and $\mu = 10$

then $\rho[A] = \rho[A, B] = 2 <$ no. of variables

Hence will have infinite solutions.

Q-~~3~~ 4 For what value of μ the equations

$$x + y + z = 1$$
$$x + 2y + 4z = \lambda$$
$$x + 4y + 10z = \lambda^2$$

has a solution and solve them completely in each case.

Solⁿ: Matrix equation is

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

$$[A, B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda - 1 \\ 0 & 3 & 9 & \lambda^2 - 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda - 1 \\ 0 & 0 & 0 & \lambda^2 - 3\lambda + 2 \end{array}\right] \quad R_3 \to R_3 - 3R_2$$

Thus $\rho[A]=2$

In order to make the system consistent, $\rho[A,B]$ must be 2 also.

which is possible only, when

$$\lambda^2-3\lambda+2=0$$

$$\Rightarrow \boxed{\lambda=2,1}$$

Solving for $\lambda=2$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

which gives

$$x+y+z=1$$
$$y+3z=1$$

Here one variable will have to be assigned arbitrary value.

Let $z=k$, then $y=1-2k$ & $x=k$

Similarly, for $\lambda=1$:

$x=1+2k,\ y=-3k,\ z=k$

#

Q.5 Test the consistency and hence solve the following set of equations

$$x_1 + 2x_2 + x_3 = 2$$
$$3x_1 + x_2 - 2x_3 = 1$$
$$4x_1 - 3x_2 - x_3 = 3$$
$$2x_1 + 4x_2 + 2x_3 = 4$$

Ans: $\left.\begin{array}{l} x_1 = 1 \\ x_2 = 0 \\ x_3 = 1 \end{array}\right\}$

Q.6 Solve:

$$4x_1 - x_2 = 12$$
$$-x_1 + 5x_2 - 2x_3 = 0$$
$$-2x_2 + 4x_3 = -8$$

Ans: $x_1 = \frac{44}{15}$

$x_2 = \frac{-4}{15}$

$x_3 = \frac{-32}{15}$

Q.7 Solve:

$$x_1 + 2x_2 - x_3 = 1$$
$$3x_1 - 2x_2 + 2x_3 = 2$$
$$7x_1 - 2x_2 + 3x_3 = 5$$

Ans: $x_3 = k$

$x_2 = \frac{5k+1}{8}$

$x_1 = \frac{3-k}{4}$

Solutions of Homogeneous system of equations

Consider $a_1 x + b_1 y + c_1 z = 0$

$a_2 x + b_2 y + c_2 z = 0$

$a_3 x + b_3 y + c_3 z = 0$

Write matrix equation

$$AX = 0.$$

These equations are always consistent, since ranks of coefficient matrix A and augmented matrix [A, B] are always same.

(i) When $\rho(A)$ = no. of variables (unknowns) then ~~general~~ solution will be unique but it is zero solution i.e. $x = 0, y = 0, z = 0$. It is called trivial solution

(ii) When $\rho(A) <$ no. of variables then, will be infinite solutions

Q.1
Solve:

$$x_1 - x_2 + x_3 = 0$$
$$x_1 + 2x_2 - x_3 = 0$$
$$2x_1 + x_2 + 3x_3 = 0$$

Solⁿ Coefficient matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 3 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 3 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

which is in echelon form

Here
$\rho[A] = 3$ = No. of variables
so given system of equations will have unique solutions

i.e. $\boxed{x_1 = 0,\ x_2 = 0,\ x_3 = 0}$

Q.2 Determine k such that the system of homogeneous equations

$$2x + y + 2z = 0$$
$$x + y + 3z = 0$$
$$4x + 3y + kz = 0$$

has (i) trivial solution
(ii) non-trivial solution.

Solu^n: Given
$$2x+y+2z=0$$
$$x+y+3z=0$$
$$4x+3y+kz=0$$

The matrix equation is
$$AX=0$$
$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & k \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A=\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & k \end{bmatrix}$$
$$\sim\begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & k \end{bmatrix} \quad R_1 \leftrightarrow R_2$$
$$\sim\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & -1 & k-12 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2-2R_1 \\ R_3 \to R_3-4R_1 \end{matrix}$$
$$\sim\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & k-8 \end{bmatrix} \quad R_3 \to R_3-R_2$$

(i) for trivial solu^n: $k \neq 8$ i.e. any value except 8.
(ii) for non trivial solu^n: $k=8$.

Q.3 Solve:

$$x - y - z + t = 0$$
$$x - y + 2z - t = 0$$
$$3x + y + t = 0$$

Solu^n: coefficient matrix

$$A = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 4 & 3 & -2 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 4 & 3 & -2 \\ 0 & 0 & 3 & -2 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

which is in echelon form

$\rho[A] = 3 <$ no. of variables 4.

Hence infinite solutions.

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 4 & 3 & -2 \\ 0 & 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - y - z + t = 0$$
$$4y + 3z - 2t = 0$$
$$3z - 2t = 0$$

Let $t = k$ then $z = \frac{2k}{3}$, $y = 0$ & $x = -\frac{k}{3}$

where k is arbitrary.

Q-4 find the value of k such that the system of equations

$$2x_1 + 3x_2 - 2x_3 = 0$$
$$3x_1 - x_2 + 3x_3 = 0$$
$$7x_1 + kx_2 - x_3 = 0$$

has non trivial solutions. find the solutions

Ans: $k = 5$

$x = -\frac{7}{11}a$

$y = \frac{12}{11}a$

$z = a$

where a is arbitrary

Q-5 find the value of k for which the following system of equations

$$4x + 2y - 5z = 0$$
$$x + ky + 2z = 0$$
$$2x + y - z = 0$$

has a non zero solution

Ans: $k = \frac{1}{2}$

Linear dependence and independence of Vectors:

Vectors $X_1, X_2, \cdots X_n$ are said to be 'dependent' if

(i) all the vectors (row or column matrices) are of same order.

(ii) n scalars $\lambda_1, \lambda_2, \cdots \lambda_n$ (not all zero) exist such that

$$\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n = 0$$

Otherwise they are linearly 'independent'.

Note: * If in a set of vectors, any vector of the set is the combination of the remaining vectors then the vectors are called 'dependent vectors'.

* If the rank of the matrix of the given vectors is equal to the number of vectors then the vectors are linearly independent.

* If the rank of the matrix of the given vectors is less than the number of vectors then the vectors are <u>linearly dependent</u>.

Q. 1 Examine the system of vectors

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = (1, -1, 1) = [1 \ -1 \ 1]^T$$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = (2, 1, 1) = [2 \ 1 \ 1]^T$$

$$X_3 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = (3, 0, 2) = [3 \ 0 \ 2]^T$$

for linear dependence. If dependent find the relation between them.

Soln Let $\lambda_1, \lambda_2, \lambda_3$ be scalars such that

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 = 0 \quad \text{——— (1)}$$

$$\Rightarrow \lambda_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \lambda_1 + 2\lambda_2 + 3\lambda_3 \\ -\lambda_1 + \lambda_2 \\ \lambda_1 + \lambda_2 + 2\lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} \lambda_1 + 2\lambda_2 + 3\lambda_3 &= 0 \\ -\lambda_1 + \lambda_2 &= 0 \\ \lambda_1 + \lambda_2 + 2\lambda_3 &= 0 \end{aligned}\right\} \text{ which is the homogeneous system.}$$

Or $$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—— (2)}$$

Let $$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$R_3 \to R_3 - R_1$, $R_2 \to R_2 + R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & -1 & -1 \end{bmatrix}$$

$R_3 \to 3R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to \frac{1}{3} R_2$$

Here $\rho[A] = 2 <$ no. of vectors
so vectors X_1, X_2 & X_3 are <u>l.d.</u>
i.e. scalars λ_1, λ_2 & λ_3 not all zero.

Again, $$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \lambda_1 + 2\lambda_2 + 3\lambda_3 = 0 \quad \text{—— (i)}$$
$$\lambda_2 + \lambda_3 = 0 \quad \text{—— (ii)}$$

Let $\boxed{\lambda_3 = k}$ then (ii) $\Rightarrow \boxed{\lambda_2 = -k}$

(i) $\Rightarrow$ $\boxed{\lambda_1 = -k}$

Substituting the values of λ_1, λ_2 and λ_3 in (1) we get

$$-kX_1 - kX_2 + kX_3 = 0$$

$$\Rightarrow \boxed{X_1 + X_2 - X_3 = 0}$$

which is the relation between the given vectors.

Q.2 If $X_1 = (3, 1, -4)$ and $X_2 = (2, 2, -3)$ then show that vectors X_1 and X_2 are linearly independent.

Solⁿ: coefficient matrix

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \\ -4 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ -4 & -3 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 2 \\ 0 & -4 \\ 0 & 5 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 + 4R_1 \\ R_2 \to R_2 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 \\ 0 & -4 \\ 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + \frac{5}{4}R_2$$

Here

$\rho[A] = 2 =$ no. of vectors so X_1 and X_2 are linearly independent.

Q-3 Prove that the vectors $X_1 = (1,2,3)$, $X_2 = (3,-2,1)$ and $X_3 = (1,-6,-5)$ form a linearly dependent system. Also find the relation between them.

Ans: $2X_1 - X_2 + X_3 = 0$.

Q.4 Examine the following system of vectors for linear dependence. If dependent, find the relation between them.

(i) $X_1 = (1,2,3)$, $X_2 = (2,-2,6)$

Ans: linearly independent.

(ii) $X_1 = (1,1,1,3)$, $X_2 = (1,2,3,4)$, $X_3 = (2,3,4,7)$

Ans: dependent, $X_1 + X_2 - X_3 = 0$

Hint (ii)

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 = 0$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Characteristic equation

Let A be a square matrix and X be a column matrix then we can write matrix equation

$$AX = \lambda X$$, where λ is any scalar

or $AX - \lambda X = 0$

or $[A - \lambda I]X = 0$

Since it is system of homogeneous equation it will have non zero solution if coefficient matrix $[A-\lambda I]$ is singular.

i.e. $|A - \lambda I| = 0$

Let the roots of above equation be $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then for these values of λ, the matrix equation $AX = \lambda X$ has non-zero solution.

Definition :

* The matrix $[A - \lambda I]$ is called charateristic matrix of A.
* The determinant $|A - \lambda I|$ is called characteristic polynomial of A.

* The equation $|A-\lambda I|=0$ is called characteristic equation of A.
The roots of this equation are called characteristic roots or Eigen values or Latent roots.

* The column matrix X is called a vector:
Non zero vectors X which satisfy the matrix equation $AX=\lambda X$ for different values of scalar λ, are called 'characteristic vectors' of matrix A.

The set of characteristic vectors corresponding to a characteristic root is called 'characteristic space'

Q-1 find the characteristic roots of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Solⁿ: The characteristic equation of the matrix A is

$$|A-\lambda I| = 0 \quad \text{——} \ (1)$$

$$[A-\lambda I] = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}$$

Now,

$$(1) \Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)\left[(3-\lambda)(3-\lambda) - (-1)(-1)\right] - (-2)\left[(-2)(3-\lambda) - (-1)(2)\right] + 2\left[(-2)(-1) - (2)(3-\lambda)\right] = 0$$

or $$(6-\lambda)\left[9-6\lambda+\lambda^2-1\right] + 2\left[-6+2\lambda+2\right] + 2\left[2-6+2\lambda\right] = 0$$

$$\Rightarrow -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

$$\Rightarrow (\lambda-2)(\lambda-2)(\lambda-8) = 0$$

$\Rightarrow \lambda = 2, 2, 8$ are the <u>charateristic roots</u> or <u>eigen values</u>.

Q.2 find the eigen values of the following matrices

(i) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ Ans. 2, 3, 6

(ii) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ Ans. −2, 3, 6

Characteristic Vectors or Eigen vectors:

Q-1 find the eigen values and corresponding eigen vectors of the matrix

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

Solu^n: The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (-5-\lambda)(-2-\lambda) - (2)(2) = 0$

$\Rightarrow (5+\lambda)(2+\lambda) - 4 = 0$

$\Rightarrow \lambda^2 + 7\lambda + 6 = 0 \Rightarrow (\lambda+1)(\lambda+6) = 0$

$\Rightarrow \boxed{\lambda = -1, -6}$ which are eigen values of A.

Now consider, $[A - \lambda I]X = 0$ ——— (i)

for $\lambda = -1$: $\begin{bmatrix} -5+1 & 2 \\ 2 & -2+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 + 2R_1$ on coefficient matrix

$$\begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -4x_1 + 2x_2 = 0$

or $-2x_1 + x_2 = 0$

Let $\quad x_2 = k \Rightarrow x_1 = \frac{k}{2}$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k/2 \\ k \end{bmatrix} = \frac{k}{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

for $\lambda = -6$ from (i), we get

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, on coefficient matrix

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_1 + 2x_2 = 0$

Let $x_2 = k$ then $x_1 = -2k$

$$X_2 = \begin{bmatrix} -2k \\ k \end{bmatrix} = k\begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Hence, the eigen values are $-1, -6$.

and eigen vectors are

$$X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Successive Differentiation

Let $y = f(x)$ be a function of x.

$\frac{dy}{dx}, \frac{df}{dx}, Dy, f'(x), y_1, y' \rightarrow$ first differential coefficient of y with respect to x
or first derivative of y w.r. to x.

$\frac{d^2y}{dx^2}, \frac{d^2f}{dx^2}, D^2y, f''(x), y_2, y'' \rightarrow$ second differential coefficient of y with respect to x
or second derivative of y w.r. to x

In general,

$\frac{d^ny}{dx^n}, \frac{d^nf}{dx^n}, D^ny, f^{(n)}(x), y_n, y^{(n)} \rightarrow$ n^{th} differential coefficient of y with respect to x
or n^{th} derivative of y with respect to x.

The value of n^{th} derivative of $y = f(x)$ at a given point $x = a$ is denoted by

$\left(\frac{d^ny}{dx^n}\right)_{x=a}, \left(\frac{d^nf}{dx^n}\right)_{x=a}, (D^ny)_{x=a}, (y_n)_{x=a}, f^{(n)}(a)$

or $y_n(a)$.

Standard results:

(1). $D^n[(ax+b)^m] = m(m-1)(m-2)\cdots(m-n+1)\, a^n (ax+b)^{m-n}$

Proof: we know, by ordinary differentiation, that

$$D[(ax+b)^m] = m\, a\, (ax+b)^{m-1}$$

$$D^2[(ax+b)^m] = m(m-1)\, a^2 (ax+b)^{m-2}$$

$$D^3[(ax+b)^m] = m(m-1)(m-2)\, a^3 (ax+b)^{m-3}$$

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In general, - - - - - - - - & so on.

$$\boxed{D^n[(ax+b)^m] = m(m-1)(m-2)\cdots(m-n+1)\, a^n (ax+b)^{m-n}}$$

Case-I: Putting $a=1, b=0$, we get

$$\boxed{D^n x^m = m(m-1)(m-2)\cdots(m-n+1)\, x^{m-n}}$$

Case-II: If we put $m=-1$, we have

$$D^n[(ax+b)^{-1}] = (-1)(-1-1)(-1-2)\cdots(-\cancel{1}-n+\cancel{1})\, a^n (ax+b)^{-1-n}$$

$$= (-1)(-2)(-3)\cdots(-n)\, a^n (ax+b)^{-n-1}$$

$$\boxed{D^n(ax+b)^{-1} = (-1)^n n!\, a^n (ax+b)^{-n-1}}$$

(2). $D^n[e^{ax}] = a^n e^{ax}$

Proof: $D[e^{ax}] = a e^{ax}$, $D^2[e^{ax}] = a^2 e^{ax}$, $D^3[e^{ax}] = a^3 e^{ax}$,

---- & so on.

proceeding in this manner, we obtain

$$\boxed{D^n[e^{ax}] = a^n e^{ax}}$$

(3) $D^n[a^x] = (\log_e a)^n a^x$

Proof: $D[a^x] = (\log a)\, a^x$, $D^2(a^x) = (\log a)^2 a^x$, $D^3(a^x) = (\log a)^3 a^x$

proceeding in this way, we have

$$D^n(a^x) = \boxed{D^n[\cancel{e^{ax}}] = a^n \cancel{e^{ax}}} = (\log a)^n a^x$$

(4) $D^n[\log_e(ax+b)] = (-1)^{n-1}(n-1)!\, a^n (ax+b)^{-n}$

Proof: $D\{\log_e(ax+b)\} = a(ax+b)^{-1}$

$$D^{n-1}[D\{\log_e(ax+b)\}] = a[D^{n-1}(ax+b)^{-1}]$$

$$= a[(-1)^{n-1}(n-1)!\, a^{n-1}(ax+b)^{-n}]$$

or $$\boxed{D^n\{\log_e(ax+b)\} = (-1)^{n-1}(n-1)!\, a^n (ax+b)^{-n}}$$

(5) (i) $D^n[\sin(ax+b)] = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$

(ii) $D^n[\cos(ax+b)] = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$

Proof: (i) $D[\sin(ax+b)] = a\cos(ax+b)$

$$= a\sin\left(ax+b+\frac{\pi}{2}\right)$$

$$D^2[\sin(ax+b)] = a^2\cos\left(ax+b+\frac{\pi}{2}\right)$$

$$= a^2\sin\left(ax+b+\frac{2\pi}{2}\right)$$

$$D^3[\sin(ax+b)] = a^3\sin\left(ax+b+\frac{3\pi}{2}\right)$$

and so on.

Proceeding in this manner, we obtain

$$\boxed{D^n[\sin(ax+b)] = a^n\sin\left(ax+b+\frac{n\pi}{2}\right)}$$

(ii) The proof is similar to (i).

In particular,

if we take $a=1$ and $b=0$ in (i) & (ii) then we have $D^n[\sin x] = a^n \sin\left(x+\frac{n\pi}{2}\right)$

& $D^n[\cos x] = a^n\cos\left(x+\frac{n\pi}{2}\right)$ respectively

(6) (i) $D^n[e^{ax}\sin(bx+c)] = (a^2+b^2)^{\frac{n}{2}} e^{ax}\sin(bx+c+n\phi)$

(ii) $D^n[e^{ax}\cos(bx+c)] = (a^2+b^2)^{n/2} e^{ax}\cos(bx+c+n\phi)$

where $\phi = \tan^{-1}(b/a)$

Proof: Since,

$$D[e^{ax}\sin(bx+c)] = a e^{ax}\sin(bx+c) + b e^{ax}\cos(bx+c)$$

$$= e^{ax}[a\sin(bx+c) + b\cos(bx+c)] \quad \text{—— (A)}$$

Put $a = r\cos\phi$, $b = r\sin\phi$ in (A).

$\Rightarrow r^2 = a^2+b^2$, $\tan\phi = \frac{b}{a}$

$\Rightarrow r = (a^2+b^2)^{1/2}$, $\phi = \tan^{-1}(b/a)$

(A) $\Rightarrow D[e^{ax}\sin(bx+c)] = r e^{ax}[\sin(bx+c)\cos\phi + \cos(bx+c)\sin\phi]$

$$= r e^{ax}\sin(bx+c+\phi)$$

Again,

$$D^2[e^{ax}\sin(bx+c)] = r e^{ax}[a\sin(bx+c+\phi) + b\cos(bx+c+\phi)]$$

$$= r^2 e^{ax}[\sin(bx+c+\phi)\cos\phi + \cos(bx+c+\phi)\sin\phi]$$

$$= r^2 e^{ax}\sin(bx+c+2\phi)$$

Proceeding in this manner, we obtain

$$D^n[e^{ax}\sin(bx+c)] = r^n e^{ax}\sin(bx+c+n\phi)$$

or $\boxed{D^n[e^{ax}\sin(bx+c)] = (a^2+b^2)^{n/2} e^{ax}\sin(bx+c+n\phi)}$

where $\phi = \tan^{-1}(b/a)$.

(ii) The proof is similar to that of (i) above.

Q.1 find the 8th derivative of $(4x+5)^{12}$.

Solⁿ: Let $y = (4x+5)^{12}$

$$\frac{d^8y}{dx^8} = y_8 = 12(12-1)(12-2)\cdots\cdots(12-8+1)(4)^8(4x+5)^{12-8}$$
$$= 12.11.10.\cdots\cdots5.4^8(4x+5)^4.$$

Q.2 find the 7th derivative of $\log(3x+5)$

Solⁿ: $y = \log(3x+5)$

$$\Rightarrow y_7 = (-1)^{7-1}(7-1)!\,3^7(3x+5)^{-7}$$
$$= (-1)^6\,6!\,3^7(3x+5)^{-7}$$
$$= \frac{6!(3)^7}{(3x+5)^7}$$

Q.3 find the nth differential coefficient of $\log(ax^2+x^3)$.

Solⁿ: Let $y = \log(ax^2+x^3)$
$$= \log\{x^2(a+x)\}$$
$$= \log x^2 + \log(a+x)$$
$$= 2\log x + \log(a+x)$$

Differentiating n times, we get
$$D^ny = 2\left[D^n\log x\right] + D^n\left[\log(a+x)\right]$$
$$= 2(-1)^{n-1}(n-1)!\,x^{-n} + (-1)^{n-1}(n-1)!\,(x+a)^{-n}$$
$$= (-1)^{n-1}(n-1)!\left[2x^{-n} + (x+a)^{-n}\right].$$

Q.4. find y_n if $y = \sin^3x\cos^2x$

Solⁿ Since, $\cos 2x = 2\cos^2x - 1$
$\sin 3x = 3\sin x - 4\sin^3x$

So $y = \sin^3 x \cos^2 x$

$$= \frac{1}{2}(1+\cos 2x)\cdot\frac{1}{4}(3\sin x - \sin 3x)$$

$$= \frac{1}{8}\left[3\sin x + 3\sin x\cos 2x - \sin 3x - \sin 3x\cos 2x\right]$$

$$= \frac{1}{8}\left[3\sin x - \sin 3x + \frac{3}{2}\{2\sin x\cos 2x\} - \frac{1}{2}\{2\sin 3x\cos 2x\}\right]$$

$$= \frac{1}{8}\left[3\sin x - \sin 3x + \frac{3}{2}(\sin 3x - \sin x) - \frac{1}{2}(\sin 5x + \sin x)\right]$$

$\{$since $2\sin A\cos B = \sin(A+B) + \sin(A-B)\}$

$$\Rightarrow y = \frac{1}{8}\left[3\sin x + \frac{1}{2}\sin 3x - \frac{1}{2}\sin 5x\right]$$

Differentiating n-times,

$$D^n y = \frac{1}{8}\left[3(D^n \sin x) + \frac{1}{2}(D^n \sin 3x) - \frac{1}{2}(D^n \sin 5x)\right]$$

$$\Rightarrow y_n = \frac{1}{8}\left[3\sin\left(x+\frac{n\pi}{2}\right) + \frac{1}{2}\cdot 3^n \sin\left(3x+\frac{n\pi}{2}\right) - \frac{1}{2}\cdot 5^n \sin\left(5x+\frac{n\pi}{2}\right)\right].$$

Q-5 If $y = \sin rx + \cos rx$, prove that

$$y_n = r^n\left[1+(-1)^n \sin 2rx\right]^{1/2}.$$

Also find $y_0(\pi)$ where $r = \frac{1}{4}$.

Soln Given, $y = \sin rx + \cos rx$

$$D^n y = D^n(\sin rx + \cos rx)$$

$$\Rightarrow y_n = r^n\left[\sin\left(rx+\frac{n\pi}{2}\right) + \cos\left(rx+\frac{n\pi}{2}\right)\right]$$

$$= r^n \left[\left\{\sin\left(rx+\frac{n\pi}{2}\right)+\cos\left(rx+\frac{n\pi}{2}\right)\right\}^2\right]^{1/2}$$

$$= r^n \left[\sin^2\left(rx+\frac{n\pi}{2}\right)+\cos^2\left(rx+\frac{n\pi}{2}\right) + 2\sin\left(rx+\frac{n\pi}{2}\right)\cos\left(rx+\frac{n\pi}{2}\right)\right]^{1/2}$$

$$= r^n \left[1+\sin 2\left(rx+\frac{n\pi}{2}\right)\right]^{1/2}$$

$$= r^n \left[1+\sin(2rx+n\pi)\right]^{1/2}$$

$$\therefore y_n = r^n\left[1+(-1)^n \sin(2rx)\right]^{1/2}$$

$\{\text{Since } \sin(n\pi+\theta)=(-1)^n\sin\theta\}$

Put $x=\pi$, $n=8$ and $r=\frac{1}{4}$

$$y_8(\pi) = \left(\frac{1}{4}\right)^8\left[1+(-1)^8\sin\left(2\cdot\frac{1}{4}\cdot\pi\right)\right]^{1/2}$$

$$= \frac{1}{2^{16}}\left[1+1.\sin\frac{\pi}{2}\right]^{1/2}$$

$$= \frac{1}{2^{16}}\cdot 2^{1/2} = \frac{1}{2^{16-\frac{1}{2}}} = \frac{1}{2^{31/2}} = \left(\frac{1}{2}\right)^{31/2}$$

Q.6 find the n^{th} derivative of $e^x\sin^2 x$.

Solⁿ $y = e^x \sin^2 x$

$$= e^x\left[\frac{1}{2}(1-\cos 2x)\right]$$

$$= \frac{1}{2}\left[e^x - e^x\cos 2x\right]$$

[Since, $\cos 2x = 1-2\sin^2 x$
$\Rightarrow 2\sin^2 x = 1-\cos 2x$
$\sin^2 x = \frac{1}{2}(1-\cos 2x)$]

Now,

$$D^n y = \frac{1}{2}\left[D^n e^x - D^n(e^x\cos 2x)\right]$$

$$y_n = \frac{1}{2}\left[e^x - e^x(1^2+2^2)^{n/2}\cos(2x+n\phi)\right]$$

where $\phi = \tan^{-1}\left(\frac{2}{1}\right) = \tan^{-1} 2$

$$y_n = \frac{e^x}{2}\left[1 - 5^{n/2}\cos(2x + n\tan^{-1}2)\right]$$

Q.7 If $I = \frac{d^n}{dx^n}(x^n \log x)$, prove that

$$I_n = n I_{n-1} + (n-1)!$$

Also show that

$$I_n = n!\left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right].$$

Soln. Given, $I_n = \frac{d^n}{dx^n}[x^n \log x]$

$$= \frac{d^{n-1}}{dx^{n-1}}\left\{\frac{d}{dx}(x^n \log x)\right\}$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left\{n x^{n-1}\log x + x^n \cdot \frac{1}{x}\right\}$$

$$= n\left\{\frac{d^{n-1}}{dx^{n-1}}(x^{n-1}\log x)\right\} + \frac{d^{n-1}}{dx^{n-1}}(x^{n-1})$$

$$\Rightarrow \boxed{I_n = n I_{n-1} + (n-1)!} \quad \text{——— (1)}$$

[Since $D^n x^m = m(m-1)(m-2)\cdots(m-n+1)x^{m-n}$

Take $m = n$,

$$D^n x^n = n(n-1)(n-2)\cdots(\cancel{n}-\cancel{n}+1)x^{n-n}$$

$$= n(n-1)(n-2)\cdots 1 \cdot x^0$$

$$\Rightarrow \boxed{D^n x^n = n!}$$]

Now dividing both sides of (1) b $n!$ we get—

$$\frac{I_n}{n!} = \frac{(n-1)!}{n!} + \frac{n I_{n-1}}{n!}$$

or $$\frac{I_n}{n!} = \frac{1}{n} + \frac{I_{n-1}}{(n-1)!}$$

Putting $n = 2, 3, 4, \cdots, n-1, n$, we get

$$\frac{I_2}{2!} = \frac{1}{2} + I_1$$

$$\frac{I_3}{3!} = \frac{1}{3} + \frac{I_2}{2!}$$

$$\frac{I_4}{4!} = \frac{1}{4} + \frac{I_3}{3!}$$

$$\text{-- -- -- -- --}$$

$$\text{-- -- -- -- --}$$

$$\frac{I_{n-1}}{(n-1)!} = \frac{1}{n-1} + \frac{I_{n-2}}{(n-2)!}$$

$$\frac{I_n}{n!} = \frac{1}{n} + \frac{I_{n-1}}{(n-1)!}$$

Now adding all above equations, we get

$$\frac{I_n}{n!} = I_1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1} + \frac{1}{n}$$

$$\left\{ \begin{aligned} \text{since } I_1 &= \frac{d}{dx}(x \log x) \\ &= \log x + 1 \end{aligned} \right\}$$

$$\Rightarrow \quad I_n = n! \left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1} + \frac{1}{n} \right].$$

Q. 8 find the n^{th} derivative of following functions

(i) $\dfrac{x}{x^2 - a^2}$ (ii) $\dfrac{1}{x^2 + a^2}$

Soln: $y = \dfrac{x}{x^2 - a^2} = \dfrac{x}{(x+a)(x-a)} = \dfrac{1}{2}\left[\dfrac{1}{x+a} + \dfrac{1}{x-a}\right]$

$$D^n y = \frac{1}{2}\left[D^n (x+a)^{-1} + D^n (x-a)^{-1}\right]$$

$$\Rightarrow \quad y_n = \frac{(-1)^n n!}{2}\left[(x+a)^{-n-1} + (x-a)^{-n-1}\right].$$

(ii) $y = \frac{1}{x^2+a^2} = \frac{1}{(x+ia)(x-ia)}$

$= \frac{1}{2ia}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$

$D^n y = \frac{1}{2ia}\left[D^n(x-ia)^{-1} - D^n(x+ia)^{-1}\right]$

or $y_n = \frac{1}{2ia}(-1)^n n!\left[(x-ia)^{-n-1} - (x+ia)^{-n-1}\right]$

[To remove the complex number, we write

$x = r\cos\phi, \quad a = r\sin\phi$

then $x-ia = r(\cos\phi - i\sin\phi)$

$x+ia = r(\cos\phi + i\sin\phi)$

where $r^2 = a^2+x^2 \Rightarrow r = (a^2+x^2)^{1/2}$

$\tan\phi = \frac{a}{x} \Rightarrow \phi = \tan^{-1}(a/x)$.]

$y_n = \frac{1}{2ia}(-1)^n n!\, r^{-n-1}\left[\{\cos\phi - i\sin\phi\}^{-(n+1)} - \{\cos\phi + i\sin\phi\}^{-(n+1)}\right]$

$= \frac{1}{2ia}(-1)^n n!\, r^{-n-1}\left[\{\cancel{\cos(n+1)\phi} + i\sin(\ \ )\phi\} - \{\cancel{\cos(n+1)\phi} - i\sin(n+1)\phi\}\right]$

$= \frac{1}{\cancel{2i}a}(-1)^n n!\, r^{-n-1}\left[\cancel{2i}\sin(n+1)\phi\right]$

$= (-1)^n n!\, a^{-1} r^{-n-1}\sin(n+1)\phi$

$= (-1)^n n!\, a^{-1}\cdot a^{-n-1}\sin^{n+1}\phi\,\sin(n+1)\phi$

$\boxed{y_n = (-1)^n n!\, a^{-n-2}\sin^{n+1}\phi\,\sin(n+1)\phi}$

where $\phi = \tan^{-1}(a/x)$

[Since $r = \sqrt{a^2+x^2}$

$= \sqrt{a^2+a^2\cot^2\phi}$

$= a\sqrt{1+\cot^2\phi}$

$= a\,\mathrm{cosec}\,\phi$

$r^{-n-1} = a^{-n-1}(\mathrm{cosec}\,\phi)^{-(n+1)}$

$= a^{-n-1}\sin^{n+1}\phi$

Leibnitz's Theorem:

If u and v are two functions of x such that their derivatives of n^{th} order exist then

$$D^n(uv) = u(D^n v) + {}^nC_1 (Du)(D^{n-1}v) + {}^nC_2 (D^2u)(D^{n-2}v) + \cdots + {}^nC_r (D^r u)(D^{n-r}v) + \cdots + {}^nC_n (D^n u)v.$$

where $${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$\Rightarrow \quad {}^nC_1 = \frac{n!}{1!\,(n-1)!} = n$$

$${}^nC_2 = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)}{2!}$$

$${}^nC_3 = \frac{n!}{3!\,(n-3)!} = \frac{n(n-1)(n-2)}{3!}$$

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$${}^nC_n = \frac{n!}{n!} = 1$$

so, also

$$D^n(uv) = u(D^n v) + n(Du)(D^{n-1}v) + \frac{n(n-1)}{2!}(D^2u)(D^{n-2}v) + \cdots + (D^n u)v.$$

Q.-1 Find the n^{th} derivative of the following functions:

(i) $x\cos 2x$ (ii) $x^2 e^x$ (iii) $x^3 \log x$

Solⁿ (i) $y = x\cos 2x$

Differentiating n times using Leibnitz's theorem,

$$D^n y = D^n (x \cos 2x)$$

or $$y_n = x(D^n \cos 2x) + n(Dx)(D^{n-1}\cos 2x)$$

$$= x\, 2^n \cos\left(2x + \frac{n\pi}{2}\right) + n \cdot 1\; 2^{n-1} \cos\left(2x + \frac{(n-1)\pi}{2}\right)$$

$$= 2^n x \cos\left(2x + \frac{n\pi}{2}\right) + 2^{n-1} n \cos\left(2x + \frac{n\pi}{2} - \frac{\pi}{2}\right)$$

$$= 2^n x \cos\left(2x + \frac{n\pi}{2}\right) + 2^{n-1} n \sin\left(2x + \frac{n\pi}{2}\right)$$

$$= 2^{n-1}\left[x \cos\left(2x + \frac{n\pi}{2}\right) + \frac{n}{2} \sin\left(2x + \frac{n\pi}{2}\right)\right]$$

(ii) $$y = x^2 e^x$$

$$D^n y = D^n (x^2 e^x)$$

$$\Rightarrow y_n = x^2 (D^n e^x) + n(Dx^2)(D^{n-1} e^x) + \frac{n(n-1)}{2!}(D^2 x^2)(D^{n-2} e^x)$$

$$= x^2 e^x + n(2x)\, e^x + \frac{n(n-1)}{\cancel{2}} \cdot \cancel{2} \cdot e^x$$

$$= e^x (x^2 + 2nx + n(n-1))$$

(iii) $$y = x^3 \log x$$

$$y_n = D^n (x^3 \log x)$$

$$= x^3 (D^n \log x) + n(Dx^3)(D^{n-1} \log x)$$
$$+ \frac{n(n-1)}{2!}(D^2 x^3)(D^{n-2} \log x) + \frac{n(n-1)(n-2)}{3!}(D^3 x^3)$$
$$(D^{n-3} \log x)$$

$$= x^3 \cdot (-1)^{n-1}(n-1)!\, x^{-n} + n \cdot (3x^2)(-1)^{n-2}(n-2)!\, x^{-n+1}$$
$$+ \frac{n(n-1)}{\cancel{2}}(3 \cdot \cancel{2} \cdot x)(-1)^{n-3}(n-3)!\, x^{-n+2}$$
$$+ \frac{n(n-1)(n-2)}{\cancel{6}}(\cancel{3} \cdot \cancel{2} \cdot 1)(-1)^{n-4}(n-4)!\, x^{-n+3}$$

$$= (-1)^n (n-4)!\, x^{-n+3}\left[-(n-1)(n-2)(n-3) + 3n(n-2)(n-3) - 3n(n-1)(n-3) + n(n-1)(n-2)\right]$$

$$= (-1)^n (n-4)!\, x^{-n+3}\,[\,6\,]$$

$$= (-1)^n\, 6\,(n-4)!\, x^{-n+3}.$$

Q.2 If $y = x^n \log x$, prove that $y_{n+1} = \frac{n!}{x}$

Solu^n: $y = x^n \log x$ ——— (1)

Differentiating with respect to x, we get

$$y_1 = n x^{n-1} \log x + x^n \cdot \frac{1}{x}$$

$$\Rightarrow x y_1 = n(x^n \log x) + x^n$$

$$x y_1 = n y + x^n \quad \text{——— (2)}$$

Differentiating (2) n times with respect to x, by Leibnitz's theorem, we get

$$D^n [x y_1] = n(D^n y) + (D^n x^n)$$

$$x(D^n y_1) + n(Dx)(D^{n-1} y_1) = n(D^n y) + n!$$

$$x\, y_{n+1} + \cancel{n.1.\, y_n} = \cancel{n y_n} + n!$$

$$\Rightarrow \boxed{y_{n+1} = \frac{n!}{x}}$$

Q-3 Differentiate n-times the following equations

(i) $x^2 y_2 + x y_1 + y = 0$

(ii) $(1+x^2) y_2 + x y_1 = 0$

Solⁿ: (i) $x^2 y_2 + x y_1 + y = 0$

Differentiating n times with respect to x by Leibnitz theorem, we get

$$D^n[x^2 y_2] + D^n[x y_1] + D^n[y] = 0$$

$$\left\{ \text{Since, } D^n[uv] = u(D^n v) + n(Du)(D^{n-1} v) + \frac{n(n-1)}{2!}(D^2 u)(D^{n-2} v) + \cdots \right\}$$

$$\left[x^2 (D^n y_2) + n(Dx^2)(D^{n-1} y_2) + \frac{n(n-1)}{2!}(D^2 x^2)(D^{n-2} y_2)\right]$$
$$+ \left[x(D^n y_1) + n(Dx)(D^{n-1} y_1)\right] + D^n[y] = 0$$

$$\left\{ D^n y_2 = y_{n+2},\ D^{n-1} y_2 = y_{n+1},\ D^{n-2} y_2 = y_n,\ D^n y_1 = y_{n+1},\ D^{n-1} y_1 = y_n,\ D^n y = y_n \right\}$$

$$\Rightarrow \left[x^2 y_{n+2} + n(2x) y_{n+1} + \frac{n(n-1)}{\cancel{2}}(\cancel{2}) y_n\right]$$
$$+ \left[x y_{n+1} + n(1) y_n\right] + y_n = 0$$

$$\Rightarrow x^2 y_{n+1} + (2nx + x) y_{n+1} + (n^2 - \cancel{n} + \cancel{n} + 1) y_n = 0$$

$$\Rightarrow \boxed{x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0}$$

(ii) $(1+x^2) y_2 + x y_1 = 0$

Differentiating n times using Leibnitz theorem,

$$D^n[(1+x^2) y_2] + D^n[x y_1] = 0$$

$$\left[(1+x^2)(D^n y_2) + n\{D(1+x^2)\}\{D^{n-1} y_2\} + \frac{n(n-1)}{2!}\{D^2(1+x^2)\}(D^{n-2} y_2)\right]$$
$$+ \left[x(D^n y_1) + n(Dx)(D^{n-1} y_1)\right] = 0$$

$$\left[(1+x^2) y_{n+2} + n(2x) y_{n+1} + \frac{n(n-1)}{\cancel{2}}(\cancel{2}) y_n\right] + \left[x y_{n+1} + n(1) y_n\right] = 0$$

$$\Rightarrow (1+x^2) y_{n+2} + [2nx + x] y_{n+1} + [n^2 - \cancel{n} + \cancel{n}] y_n = 0$$

$$\Rightarrow \boxed{(1+x^2) y_{n+2} + (2n+1) x y_{n+1} + n^2 y_n = 0}$$

Q:-4 If $y = \sin^{-1} x$, show that

$$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - n^2 y_n = 0.$$

Solⁿ: Given $y = \sin^{-1} x$. —— (1)

Differentiating with respect to x, we have

$$y_1 = \frac{1}{\sqrt{1-x^2}}$$

or $$y_1^2 = \frac{1}{1-x^2}$$

$$\Rightarrow (1-x^2) y_1^2 = 1$$

or $$(1-x^2) y_1^2 - 1 = 0 \quad \text{—— (2)}$$

Differentiating (2) with respect to x again,

$$(1-x^2)(2y_1y_2) + (-2x) y_1^2 - 0 = 0$$

$$\Rightarrow (1-x^2) y_2 - x y_1 = 0 \quad \text{—— (3)}$$

$$\left[\frac{d}{dx}(y_1^2) = \frac{d}{dx}\left(\frac{dy}{dx}\right)^2 = 2\left(\frac{dy}{dx}\right) \cdot \frac{d^2y}{dx^2} = 2y_1y_2\right]$$

Now differentiating n-times using Leibnitz's theorem, we get

$$\left[(1-x^2)(D^n y_2) + n\{D(1-x^2)\}\{D^{n-1} y_2\} + \frac{n(n-1)}{2!}\{D^2(1-x^2)\}\{D^{n-2} y_2\}\right] - \left[x(D^n y_1) + n(Dx)(D^{n-1} y_1)\right] = 0$$

$$\Rightarrow \left[(1-x^2) y_{n+2} + n(-2x) y_{n+1} + \frac{n(n-1)}{\cancel{2}}(-\cancel{2}) y_n\right] - \left[x y_{n+1} + n(1) y_n\right] = 0$$

$$(1-x^2)y_{n+2} + [-2nx - x]y_{n+1} + [-n^2 + \cancel{n} - \cancel{n}]y_n = 0$$

$$\Rightarrow \boxed{(1-x^2)y_{n+1} - (2n+1)x\,y_{n+1} - n^2 y_n = 0}$$

Q-5 If $y = a\cos(\log x) + b\sin(\log x)$ then show that $x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1)y_n = 0$

Sol^n Given $y = a\cos(\log x) + b\sin(\log x)$ —— (1)

Diff,
$$y_1 = -a\sin(\log x)\cdot\frac{1}{x} + b\cos(\log x)\cdot\frac{1}{x}$$

or $xy_1 = -a\sin(\log x) + b\cos(\log x)$ —— (2)

Again diff,
$$xy_2 + y_1 = -a\cos(\log x)\frac{1}{x} - b\sin(\log x)\frac{1}{x}$$

or $x^2y_2 + xy_1 = -[a\cos(\log x) + b\sin(\log x)]$

or $x^2y_2 + xy_1 = -y$ (from (1))

$\Rightarrow x^2y_2 + xy_1 + y = 0$

from Q-3 (i), we have

$$x^2y_{n+2} + (2n+1)x\,y_{n+1} + (n^2+1)y_n = 0.$$

Q.6 If $y^{1/m} + y^{-1/m} = 2x$, prove that

$$(x^2-1)y_{n+2} + (2n+1)x\,y_{n+1} + (n^2-m^2)y_n = 0$$

Sol^n Given $y^{1/m} + y^{-1/m} = 2x$ —— (1)

multiplying both side by $y^{1/m}$, we get

$$(y^{1/m})^2 + 1 = 2x\,y^{1/m}$$

$$\Rightarrow (y^{1/m})^2 - 2x(y^{1/m}) + 1 = 0 \quad \text{—— (2)}$$

Let $y^{1/m} = t$

then $t^2 - 2xt + 1 = 0$

$$\Rightarrow t = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$$

$$\Rightarrow y^{1/m} = x + \sqrt{x^2-1} \quad \text{(taking '+' sign)}$$

or $y = [x+\sqrt{x^2-1}]^m$ ——— (3)

Diff,
$$y_1 = m\,(x+\sqrt{x^2-1})^{m-1}\left(1+\frac{1}{\cancel{2}\sqrt{x^2-1}}\cdot\cancel{2}x\right)$$

$$= m\,(x+\sqrt{x^2-1})^{m-1}\left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right)$$

$$\sqrt{x^2-1}\,y_1 = m\,(x+\sqrt{x^2-1})^m$$

or $\sqrt{x^2-1}\,y_1 = my$

or $(x^2-1)y_1^2 = m^2y^2$

or $(x^2-1)y_1^2 - m^2y^2 = 0$

Again diff,
$$(x^2-1)(2y_1y_2) + (2x)y_1^2 - m^2(2yy_1) = 0 \qquad \left[\frac{d(y^2)}{dx} = 2y\frac{dy}{dx} = 2yy_1\right]$$

or $(x^2-1)y_2 + xy_1 - m^2y = 0$

Differentiating n-times with respect to x by Leibnitz theorem, we get

$$\left[(x^2-1)y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{\cancel{2}}(\cancel{2})y_n\right] + [xy_{n+1} + n\cdot1\cdot y_n] - m^2y_n = 0$$

$$(x^2-1)y_{n+2} + [2nx+x]y_{n+1} + [n^2-\cancel{n}+\cancel{n}-m^2]y_n = 0$$

or $\boxed{(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0}$

The value of n^{th} derivative of $y = f(x)$ at $x = 0$.

Q-1. If $y = x^2 \sin x$ then find $y_n(0)$.

Solu^n Given $y = x^2 \sin x$

Differentiating n times using Leibnitz theorem

$$D^n y = D^n [x^2 \sin x]$$

$$\Rightarrow y_n = x^2 (D^n \sin x) + n (Dx^2)(D^{n-1} \sin x) + \frac{n(n-1)}{2!} (D^2 x^2)(D^{n-2} \sin x)$$

$$\Rightarrow y_n = x^2 \sin\left(x + \frac{n\pi}{2}\right) + n(2x) \sin\left(x + \frac{(n-1)\pi}{2}\right) + \frac{n(n-1)}{\cancel{2}} (\cancel{2}) \sin\left(x + \frac{(n-2)\pi}{2}\right)$$

Putting $x = 0$

$$y_n(0) = 0 + 0 + n(n-1) \sin\left(\frac{(n-2)\pi}{2}\right)$$

$$= n(n-1) \sin\left(\frac{n\pi}{2} - \pi\right)$$

or $$\boxed{y_n(0) = -n(n-1) \sin \frac{n\pi}{2}}$$

Q.2 If $y = e^{m \sin^{-1} x}$ then find $(y_n)_0$.

Solu^n Given $y = e^{m \sin^{-1} x}$ ——— (1)

Diff, $y_1 = e^{m \sin^{-1} x} \cdot \frac{m}{\sqrt{1-x^2}}$

$$\sqrt{1-x^2}\, y_1 = my \Rightarrow (1-x^2) y_1^2 = m^2 y^2$$

or $(1-x^2) y_1^2 - m^2 y^2 = 0$ ——— (2)

Again diff.

$$(1-x^2)(2y_1y_2) - 2xy_1^2 - m^2(2yy_1) = 0$$

$$(1-x^2)y_2 - xy_1 - m^2y = 0 \quad \text{——— (3)}$$

Now differentiating n times using Leibnitz's theorem

$$D^n[(1-x^2)y_2] - D^n[xy_1] - m^2[D^ny]$$

$$\left[(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2!}(-2)y_n\right]$$

$$-[xy_{n+1} + n.1\,y_n] - m^2y_n = 0$$

$$(1-x^2)y_{n+2} + [-2nx - x]y_{n+1} + (-n^2 + \cancel{n} - \cancel{n} - m^2)y_n = 0$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0. \quad \text{— (4)}$$

Putting $x=0$ in (1), (2), (3) & (4), we get

$$(y)_0 = 1, \quad (y_1)_0 = m,$$

$$(3) \Rightarrow (1-0)(y_2)_0 - 0 - m^2(y)_0 = 0$$

$$\Rightarrow (y_2)_0 = m^2.1 = m^2$$

$$(4) \Rightarrow (1-0)(y_{n+2})_0 - 0 - (n^2+m^2)(y_n)_0 = 0$$

$$\boxed{(y_{n+2})_0 = (n^2+m^2)(y_n)_0} \quad \text{——— (5)}$$

Case I: When n is even:

then putting $n = n-2, n-4, \ldots 6, 4, 2$ in (5)

$$(y_n)_0 = \{(n-2)^2 + m^2\}(y_{n-2})_0$$

$$(y_{n-2})_0 = \{(n-4)^2 + m^2\}(y_{n-4})_0$$

$$(y_{n-4})_0 = \{(n-6)^2 + m^2\}(y_{n-6})_0$$

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$(y_8)_0 = (6^2+m^2)(y_6)_0$

$(y_6)_0 = (4^2+m^2)(y_4)_0$

$(y_4)_0 = (2^2+m^2)(y_2)_0$

$\Rightarrow (y_n)_0 = \{(n-2)^2+m^2\}\{(n-4)^2+m^2\} \cdots$
$\cdots (6^2+m^2)(4^2+m^2)(2^2+m^2)(y_2)_0$

or $\boxed{(y_n)_0 = \{(n-2)^2+m^2\} \cdots (4^2+m^2)(2^2+m^2)\, m^2}$

Case II when n is odd:

Then putting $n = n-2, n-4, \cdots 5, 3, 1$ in (5)

$(y_n)_0 = \{(n-2)^2+m^2\}(y_{n-2})_0$

$(y_{n-2})_0 = \{(n-4)^2+m^2\}(y_{n-4})_0$

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$(y_7)_0 = (5^2+m^2)(y_5)_0$

$(y_5)_0 = (3^2+m^2)(y_3)_0$

$(y_3)_0 = (1^2+m^2)(y_1)_0$

$\Rightarrow (y_n)_0 = \{(n-2)^2+m^2\}\{(n-4)^2+m^2\} \cdots$
$\cdots (5^2+m^2)(3^2+m^2)(1^2+m^2)(y_1)_0$

or $\boxed{(y_n)_0 = \{(n-2)^2+m^2\} \cdots\cdots (3^2+m^2)(1^2+m^2) \cdot m}$

Q.3 If $y = \cos(m\sin^{-1}x)$ then prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0.$$

Also find $y_n(0)$.

Solⁿ Given $y = \cos(m\sin^{-1}x)$ —— ①

$$y_1 = -\sin(m\sin^{-1}x)\cdot\frac{m}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,y_1 = -m\sin(m\sin^{-1}x)$$

Squaring both sides, we have

$$(1-x^2)y_1^2 = m^2\sin^2(m\sin^{-1}x)$$
$$= m^2[1-\cos^2(m\sin^{-1}x)]$$
$$= m^2[1-y^2)$$

$$\Rightarrow (1-x^2)y_1^2 = m^2 - m^2y^2 \quad \text{—— ②}$$

Again differentiating with respect to x, we get

$$(1-x^2)(2y_1y_2) - 2xy_1^2 = 0 - m^2(2yy_1)$$

$$\Rightarrow (1-x^2)y_2 - xy_1 + m^2y = 0 \quad \text{—— ③}$$

Now differentiating n times using Leibnitz's theorem,

$$D^n[(1-x^2)y_2] - D^n[xy_1] + m^2[D^ny] = 0$$

$$\left[(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2!}(-\cancel{2})y_n\right]$$
$$-[xy_{n+1} + n\cdot 1\cdot y_n] + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} + [-2nx - x]y_{n+1} + [-n^2 + \cancel{n} - \cancel{n} + m^2]y_n = 0$$

$$\Rightarrow \boxed{(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0} \quad \text{—— ④}$$

Putting $x=0$ in ①, ②, ③ & ④, we get

$\boxed{(y)_0 = 1}$, $\boxed{(y_1)_0 = 0}$, $(y_2)_0 - 0 + m^2 (y)_0 = 0$

$\Rightarrow \boxed{(y_2)_0 = -m^2}$

④ $\Rightarrow$ $(y_{n+2})_0 - 0 + (m^2 - n^2)(y_n)_0 = 0$

$\Rightarrow \boxed{(y_{n+2})_0 = (n^2 - m^2)(y_n)_0}$ —— ⑤

Case I: when n is even:

Putting $n = n-2, n-4, \text{-----} 4, 2$ in ⑤, we get

$(y_n)_0 = \{(n-2)^2 - m^2\} (y_{n-2})_0$

$(y_{n-2})_0 = \{(n-4)^2 - m^2\} (y_{n-4})_0$

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$(y_6)_0 = (4^2 - m^2)(y_4)_0$

$(y_4)_0 = (2^2 - m^2)(y_2)_0$

$$\boxed{(y_n)_0 = \{(n-2)^2 - m^2\}\{(n-4)^2 - m^2\} \text{---} (4^2 - m^2)(2^2 - m^2)(-m^2)}$$

Case-II: when n is odd:

Putting $n = n-2, n-4, \text{---} 5, 3, 1$ in ⑤, we get

$(y_n)_0 = \{(n-2)^2 - m^2\} (y_{n-2})_0$

$(y_{n-2})_0 = \{(n-4)^2 - m^2\} (y_{n-4})_0$

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$(y_5)_0 = (3^2 - m^2)(y_3)_0$

$(y_3)_0 = (1^2 - m^2)(y_1)_0$

$$\boxed{(y_n)_0 = \{(n-2)^2 - m^2\} \text{----} (3^2 - m^2)(1^2 - m^2)(y_1)_0 = 0}$$

Q-4 If $y = (\sin^{-1}x)^2$ then find $(y_n)_0$

Solu^n: Given $y = (\sin^{-1}x)^2$ —— ①

Diff, $y_1 = 2\sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\, y_1 = 2\sin^{-1}x$

Squaring, $(1-x^2)\, y_1^2 = 4(\sin^{-1}x)^2$

Again diff,

$(1-x^2)(2y_1y_2) - 2x\, y_1^2 = 4y_1$

$\Rightarrow (1-x^2)\, y_2 - xy_1 - 2 = 0$ —— ③

Differentiating n times w.r.to x, by Leibnitz's theorem,

$D^n[(1-x^2)y_2] - D^n[xy_1] - D^n[2] = 0$

$\left[(1-x^2)\, y_{n+2} + n(-2x)\, y_{n+1} + \frac{n(n-1)}{2}(-2)\, y_n\right]$

$- [x\, y_{n+1} + n(1)\, y_n] - 0 = 0$

$\Rightarrow (1-x^2)\, y_{n+2} + [-2nx - x]\, y_{n+1} + [-n^2 + \cancel{n} - \cancel{n}]\, y_n = 0$

$\Rightarrow (1-x^2)\, y_{n+2} - (2n+1)x\, y_{n+1} - n^2 y_n = 0$ —— ④

Putting $x=0$ in ①, ②, ③ & ④ we get

$y)_0 = 0$, $(y_1)_0 = 0$, $(y_2)_0 = 2$

④ $\Rightarrow$ $\boxed{(y_{n+2})_0 = n^2 (y_n)_0}$ —— ⑤

Case I: when n is even:

then putting $n = n-2, n-4, \ldots 4, 2$ in ⑤, we get

$$(y_n)_0 = (n-2)^2 (y_{n-2})_0$$
$$(y_{n-2})_0 = (n-4)^2 (y_{n-4})_0$$
$$(y_{n-4})_0 = (n-6)^2 (y_{n-6})_0$$

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$$(y_8)_0 = 6^2 (y_6)_0$$
$$(y_6)_0 = 4^2 (y_4)_0$$
$$(y_4)_0 = 2^2 (y_2)_0$$

$$(y_n)_0 = (n-2)^2 . (n-4)^2 \text{-----} 4^2 \, 2^2 (y_2)_0$$

$$\Rightarrow \boxed{(y_n)_0 = (n-2)^2 . (n-4)^2 \text{-----} 4^2 . 2^2 . 2}$$

Case-II. when n is odd.

then $\boxed{(y_n)_0 = 0}$ Since $(y_1)_0 = 0$

Partial differentiation

If $z = f(x,y)$ be function of two independent variables x and y then
the derivative of z with respect to x (keeping y as constant) is called partial derivative of z w.r.t. 'x' and denoted by

$$\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, f_x(x,y).$$

i.e. $\left(\frac{\partial z}{\partial x}\right)_{y \to \text{constant}} \rightarrow$ Partial derivative of z w.r. to x.

Similarly

$\left(\frac{\partial z}{\partial y}\right)_{x \to \text{constant}} \longrightarrow$ Partial derivative of z w.r. to y.

it is also denoted by $\frac{\partial f}{\partial y}$, $f_y(x,y)$.

Second order partial derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}$$

similarly, third order partial derivatives:

$$\frac{\partial^3 z}{\partial x^3}, \frac{\partial^3 z}{\partial y^3}, \frac{\partial^3 z}{\partial y \partial x^2}, \frac{\partial^3 z}{\partial x \partial y^2}$$

& so on.

Q.1 If $z = x^2 + xy^2$ then find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}$ & $\frac{\partial^2 z}{\partial x \partial y}$.

Soln. Given $z = x^2 + xy^2$ —— (1)

Differentiating (1) partially w.r.to x, we get

$$\frac{\partial z}{\partial x} = 2x + y^2 \quad \text{—— (2)}$$

Differentiating (1) w.r.to y partially, we get

$$\frac{\partial z}{\partial y} = 0 + x(2y) = 2xy \quad \text{—— (3)}$$

Now,
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\right]$$
$$= \frac{\partial}{\partial x}\left[2x + y^2\right] = 2(1) + 0 = 2$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = 2 \quad \text{—— (4)}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial y}\right] = \frac{\partial}{\partial y}\left[2xy\right] = 2x$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = 2x \quad \text{—— (5)}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right] = \frac{\partial}{\partial x}\left[2xy\right] = 2y$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = 2y \quad \text{—— (6)}$$

Q.2 If $u = e^{xyz}$ then find the value of $\frac{\partial^3 u}{\partial x \partial y \partial z}$.

Sol^n Given $u = e^{xyz}$ —— ①

$$\frac{\partial u}{\partial z} = e^{xyz}[x.y.1] = x.y\,e^{xyz}$$

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial z}\right]$$

$$= \frac{\partial}{\partial y}\left[x.y.e^{xyz}\right]$$

$$= x\frac{\partial}{\partial y}\left[y e^{xyz}\right]$$

$$= x\left[1.e^{xyz} + \{e^{xyz}.xz\}y\right]$$

$$= e^{xyz}\left[x + x^2yz\right]$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left[\frac{\partial^2 u}{\partial y \partial z}\right]$$

$$= \frac{\partial}{\partial x}\left[e^{xyz}(x + x^2yz)\right]$$

$$= e^{xyz}(1 + 2xyz) + \{e^{xyz}(yz)\}(x + x^2yz)$$

$$= e^{xyz}\left[1 + 2xyz + xyz + x^2y^2z^2\right]$$

$$= e^{xyz}\left[1 + 3xyz + x^2y^2z^2\right].$$

Q.3 Prove that $y = f(x+at) + g(x-at)$ satisfies $\frac{\partial^2 y}{\partial t^2} = a^2\left(\frac{\partial^2 y}{\partial x^2}\right)$.

Solⁿ: Given $y = f(x+at) + g(x-at)$ —— ①

Differentiating ① partially w.r.to x, we get,

$$\frac{\partial y}{\partial x} = f'(x+at)\{1+0\} + g'(x-at)\{1-0\}$$
$$= f'(x+at) + g'(x-at)$$

Again differentiating w.r.to x, partially,

$$\frac{\partial^2 z}{\partial x^2} = f''(x+at) + g''(x-at) \quad \text{—— ②}$$

Differentiating ① w.r.to 't' partially, we get

$$\frac{\partial y}{\partial t} = f'(x+at)\{0+a.1\} + g'(x-at)\{0-a.1\}$$
$$= af'(x+at) - ag'(x-at)$$

Again differentiating with respect to 't',

$$\frac{\partial^2 y}{\partial t^2} = a^2 f''(x+at) + a^2 g''(x-at)$$
$$= a^2[f''(x+at) + g''(x-at)]$$

$$\Rightarrow \boxed{\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}}$$

Q.4 If $v = (x^2+y^2+z^2)^{\frac{m}{2}}$, then find the value of m $(m \neq 0)$ which will make

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0.$$

Solⁿ: Given $v = (x^2+y^2+z^2)^{m/2}$ —— (1)

$$\frac{\partial v}{\partial x} = \frac{m}{2}(x^2+y^2+z^2)^{\frac{m}{2}-1}(2x+0+0)$$

$$= mx(x^2+y^2+z^2)^{\frac{m}{2}-1}$$

$$\frac{\partial^2 v}{\partial x^2} = m\frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{\frac{m}{2}-1}\right]$$

$$= m\left[\left(\frac{m}{2}-1\right)\cdot x(x^2+y^2+z^2)^{\frac{m}{2}-2}(2x+0+0) + 1.(x^2+y^2+z^2)^{\frac{m}{2}-1}\right]$$

$$= m\frac{(m-2)}{\cancel{2}}\cdot\cancel{2}x^2(x^2+y^2+z^2)^{\frac{m}{2}-2} + m(x^2+y^2+z^2)^{\frac{m}{2}-1}$$

$$= m(x^2+y^2+z^2)^{\frac{m}{2}-2}\left[(m-2)x^2+(x^2+y^2+z^2)\right] \quad \text{—— (2)}$$

Similarly,

$$\frac{\partial^2 v}{\partial y^2} = m(x^2+y^2+z^2)^{\frac{m}{2}-2}\left[(m-2)y^2+(x^2+y^2+z^2)\right] \quad \text{—— (3)}$$

$$\frac{\partial^2 v}{\partial z^2} = m(x^2+y^2+z^2)^{\frac{m}{2}-2}\left[(m-2)z^2+(x^2+y^2+z^2)\right] \quad \text{—— (4)}$$

Now (2) + (3) + (4), we get

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2} = m(x^2+y^2+z^2)^{\frac{m}{2}-2}\left[(m-2)(x^2+y^2+z^2)+3(x^2+y^2+z^2)\right]$$

$$= m(x^2+y^2+z^2)^{\frac{m}{2}-2}(x^2+y^2+z^2)(m-2+3)$$

$$= m(x^2+y^2+z^2)^{\frac{m}{2}-1}(m+1).$$

Since $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

$$\Rightarrow m(m+1)\,(x^2+y^2+z^2)^{\frac{m}{2}-1} = 0$$

$$\Rightarrow m(m+1) = 0$$

$$\Rightarrow m = 0, -1$$

So $\Rightarrow \boxed{m = -1}$

Q.5 If $x^x y^y z^z = c$, show that at $x = y = z$

$$\frac{\partial^2 z}{\partial x \partial y} = -\,[x \log(ex)]^{-1}$$

Soln: Given $x^x y^y z^z = c$ ——— (1)

Taking log on both sides, we get

$$x \log x + y \log y + z \log z = \log c \quad \text{———} \quad (2)$$

Differentiating (2) w.r. to x partially, we have

$$\left(x \cdot \frac{1}{x} + \log x\right) + 0 + \left(z \cdot \frac{1}{z} + \log z\right)\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow (1 + \log x) + (1 + \log z)\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{1 + \log x}{1 + \log z} \quad \text{———} \quad (3)$$

Similarly differentiating (2) w.r. to y partially, we get

$$\frac{\partial z}{\partial x} = -\frac{1 + \log y}{1 + \log z} \quad \text{———} \quad (4)$$

Now differentiating (4) w.r.to x partially, we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[-\frac{1+\log y}{1+\log z}\right]$$

$$= -(1+\log y)\left\{-\frac{1}{(1+\log z)^2}\cdot\frac{1}{z}\cdot\frac{\partial z}{\partial x}\right\}$$

$$= \frac{1+\log y}{z(1+\log z)^2}\left[-\frac{1+\log x}{1+\log z}\right]$$

$$= -\frac{(1+\log x)(1+\log y)}{z(1+\log z)^3}$$

At $x=y=z$, we have

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{(1+\log x)(1+\log x)}{x(1+\log x)^3}$$

$$= -\frac{1}{x(1+\log x)}$$

$$= -\frac{1}{x(\log e+\log x)}$$

$$= -\frac{1}{x\log(ex)}$$

$$\Rightarrow \boxed{\frac{\partial^2 z}{\partial x \partial y} = -[x\log(ex)]^{-1}}$$

Q-6 If $z = \dfrac{x^2+y^2}{x+y}$ show that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

$\underline{\underline{\text{Solu}^n}}$: Given $z = \dfrac{x^2+y^2}{x+y}$ —— ①

$$\frac{\partial z}{\partial x} = \frac{(x+y)(2x)-(x^2+y^2)(1+0)}{(x+y)^2} = \frac{x^2+2yx-y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x+y)(2y)-(x^2+y^2)\cdot 1}{(x+y)^2} = \frac{-x^2+2xy+y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{x^2+2yx-y^2}{(x+y)^2} - \frac{-x^2+2xy+y^2}{(x+y)^2}$$

$$= \frac{2x^2-2y^2}{(x+y)^2} = \frac{2(x-y)(x+y)}{(x+y)^2}$$

$$= \frac{2(x-y)}{(x+y)}$$

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \frac{4(x-y)^2}{(x+y)^2} \text{ —— ②}$$

$$4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right) = 4\left(1-\left\{\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\right\}\right)$$

$$= 4\left(1-\left\{\frac{\cancel{x^2}+2xy-\cancel{y^2}-\cancel{x^2}+2xy+\cancel{y^2}}{(x+y)^2}\right\}\right)$$

$$= 4\left(1-\frac{4xy}{(x+y)^2}\right)$$

$$= 4\left(\frac{(x+y)^2-4xy}{(x+y)^2}\right)$$

$$= 4\,\frac{(x-y)^2}{(x+y)^2} \text{ —— ③}$$

from ② & ③

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)$$

Q.-7 If $z = y f(x^2-y^2)$ show that

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = \frac{xz}{y}$$

Solⁿ. Given $z = y f(x^2-y^2)$ —— ①

$$\frac{\partial z}{\partial x} = y f'(x^2-y^2)\,(2x-0) = 2xy\, f'(x^2+y^2)$$

$$y\frac{\partial z}{\partial x} = 2xy^2 f'(x^2+y^2) \text{ —— ②}$$

$$\frac{\partial z}{\partial y} = f(x^2-y^2).1 + y f'(x^2-y^2)(0-2y)$$

$$= f(x^2-y^2) - 2y^2 f'(x^2-y^2)$$

$$x\frac{\partial z}{\partial y} = x f(x^2-y^2) - 2xy^2 f'(x^2-y^2) \text{ —— ③}$$

Now ② + ③,

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x f(x^2-y^2)$$

$$= \frac{xz}{y}$$

Q.8 If $u = \log(x^3+y^3+z^3-3xyz)$

Show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$$

Solⁿ: $u = \log(x^3+y^3+z^3-3xyz)$ —— ①

$$\frac{\partial u}{\partial x} = \frac{3x^2+0+0+3yz}{x^3+y^3+z^3-3xyz} = \frac{3x^2+3yz}{x^3+y^3+z^3-3xyz} \text{ —— ②}$$

Similarly $\frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3+y^3+z^3-3xyz}$ —— (3)

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3+y^3+z^3-3xyz} \quad \text{—— (4)}$$

Adding (2), (3) & (4)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3\cancel{(x^2+y^2+z^2-xy-yz-zx)}}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-zx)}}$$

$$= \frac{3}{x+y+z}$$

$$\Rightarrow \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u = \frac{3}{x+y+z} \quad \text{—— (5)}$$

Now

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u\right]$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= -3\frac{1}{(x+y+z)^2}\{1+0+0\} - 3\frac{1}{(x+y+z)^2}\{0+1+0\} - 3\frac{1}{(x+y+z)^2}\{0+0+1\}$$

$$= -\frac{3}{(x+y+z)^2}[1+1+1] = -\frac{9}{(x+y+z)^2}$$

\#

Homogeneous function:

A function $f(x,y)$ is said to be homogeneous function in which the power of each term is the same.

consider the expression

$$f(x,y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_ny^n$$

in which every terms is of degree n. Such an expression is known as a homogeneous expression of degree 'n'.

The above expression can also be written as

$$f(x,y) = x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \cdots + a_n\left(\frac{y}{x}\right)^n\right]$$

$$\Rightarrow \boxed{f(x,y) = x^n\,\phi\left(\frac{y}{x}\right).}$$

so, An function $f(x,y)$ is said to be homogeneous function of degree 'n' in x and y if it can be written in the form $x^n\,\phi\left(\frac{y}{x}\right)$.

for Example:

(1) $f(x,y) = ax^2 + 2hxy + by^2$

$$= x^2\left[a + 2h\left(\frac{y}{x}\right) + b\left(\frac{y}{x}\right)^2\right]$$

$$= x^2\,\phi(y/x)$$

$\Rightarrow f(x,y)$ is homogeneous expression of degree 2.

(2) $f(x,y) = \dfrac{\sqrt{x}+\sqrt{y}}{x^2+y^2}$

$$= \frac{\sqrt{x}\left[1+(y/x)^{1/2}\right]}{x^2\left[1+\left(\frac{y}{x}\right)^2\right]}$$

$$= x^{-3/2}\,\phi(y/x)$$

$\Rightarrow f(x,y)$ is a homogeneous function of order $-\frac{3}{2}$.

(3). $f(x,y) = \tan^{-1}\left(\dfrac{x^2+y^2}{x+y}\right)$

$$= \tan^{-1}\left[\frac{x^2\{1+(y/x)^2\}}{x\{1+(y/x)\}}\right]$$

$$= \tan^{-1}\left[x\,\phi(y/x)\right]$$

$\Rightarrow f(x,y)$ is not a homogeneous function.

* A function $f(x_1, x_2, x_3)$ is said to be homogeneous function of degree k in x_1, x_2, x_3 if it is possible to write the given function in the form

$$f(x_1, x_2, x_3) = x_1^k \phi\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right)$$

$$\text{or } x_2^k \phi\left(\frac{x_1}{x_2}, \frac{x_3}{x_2}\right)$$

$$\text{or } x_3^k \phi\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right)$$

for Example:

$$f(x,y,z) = ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$

$$= x^2\left[a+b\left(\frac{y}{x}\right)^2+c\left(\frac{z}{x}\right)^2+2f\left(\frac{y}{x}\right)\left(\frac{z}{x}\right) + 2g\left(\frac{z}{x}\right)+2h\left(\frac{y}{x}\right)\right]$$

$$= x^2 \phi\left(\frac{y}{x}, \frac{z}{x}\right)$$

$\Rightarrow f(x,y,z)$ is a homogeneous function of degree 2.

Euler's theorem on homogeneous function:

If z is a homogeneous function of x, y of order 'n' then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

Proof: Given z be a homogeneous function of x, y of order n

so z can be written as

$$z = x^n \phi(y/x) \quad \text{——— (1)}$$

Differentiating (1) w.r.t. x partially, we get

$$\frac{\partial z}{\partial x} = n x^{n-1}\phi(y/x) + x^n \cdot \phi'(y/x)\left(-\frac{y}{x^2}\right)$$

$$= n x^{n-1}\phi(y/x) - x^{n-2} y\, \phi'(y/x)$$

Multiplying both sides by x,

$$x\frac{\partial z}{\partial x} = n x^n \phi(y/x) - x^{n-1} y\, \phi'(y/x) \quad \text{——— (2)}$$

Again, Differentiating (1) w.r.t. y partially, we get

$$\frac{\partial z}{\partial y} = x^n \phi'(y/x) \cdot \frac{1}{x} = x^{n-1} \phi'(y/x)$$

Multiplying both sides by 'y',

$$y \frac{\partial z}{\partial y} = x^{n-1} y \phi'(y/x) \quad \text{——— (3)}$$

Now (2) + (3),

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = n x^n \phi(y/x)$$

$$\Rightarrow \boxed{x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = n z}$$

Similarly, if u be a homogeneous function of x, y, z of degree 'n' then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = n u.$$

Deduction I :

If z is a homogeneous function of x, y of degree n and $z = f(u)$, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

Proof: Given z is a homogeneous function of degree 'n' then by Euler's theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \quad \text{——— (1)}$$

Also. $z = f(u)$

$$\frac{\partial z}{\partial x} = f'(u)\frac{\partial u}{\partial x}, \quad \frac{\partial z}{\partial y} = f'(u)\frac{\partial u}{\partial y}$$

Putting these values of $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ in (1) we get

$$xf'(u)\frac{\partial u}{\partial x} + yf'(u)\frac{\partial u}{\partial y} = nf(u)$$

$$\Rightarrow \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}}$$

Deduction II :

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

where $g(u) = n \frac{f(u)}{f'(u)}$

Proof: By deduction formula (I);

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)} \quad \text{--- (1)}$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = g(u) \quad \text{--- (2)}$$

Differentiating (2) partially w.r. to x, we get

$$\left(x \frac{\partial^2 u}{\partial x^2} + 1 \cdot \frac{\partial u}{\partial x}\right) + y \frac{\partial^2 u}{\partial x \partial y} = g'(u) \frac{\partial u}{\partial x}$$

$$\Rightarrow x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = [g'(u) - 1] \frac{\partial u}{\partial x} \quad \text{--- (3)}$$

Again differentiating (2) partially w.r. to y, we get

$$y \frac{\partial^2 u}{\partial y^2} + x \frac{\partial^2 u}{\partial y \partial x} = [g'(u) - 1] \frac{\partial u}{\partial y} \quad \text{--- (4)}$$

Now (3)x + (4)y,

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = [g'(u) - 1]\left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right]$$

$$= [g'(u) - 1]\, g(u)$$

Q.-1 Verify Euler's theorem for the function

$$u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}(y/x)$$

Soln. Given $u = x^0 \sin^{-1}\left(\frac{1}{y/x}\right) + x^0 \tan^{-1}(y/x)$.

$$= x^0 \phi(y/x). \qquad \text{--- (1)}$$

Here, u is a homogeneous function of degree 0 so by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \qquad \text{--- (2)}$$

Verification:

Differentiating (1) w.r.to x partially,

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{1}{y}\right) + \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)$$

$$\frac{\partial u}{\partial x} = \frac{\cancel{y}}{\sqrt{y^2-x^2}} \cdot \frac{1}{\cancel{y}} + \frac{\cancel{x^2}}{x^2+y^2}\left(-\frac{y}{\cancel{x^2}}\right)$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \quad \text{—— (3)}$$

Differentiating (1) w.r. to y, we get

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(-\frac{x}{y^2}\right) + \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x}$$

$$\Rightarrow y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \quad \text{—— (4)}$$

Now (3) + (4),

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0} \quad \text{—— (5)}$$

from (2) & (5), the theorem is verified.

Q.2 If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$$

Solⁿ Given $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ —— (1)

or $\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

Let $z = \frac{x+y}{\sqrt{x}+\sqrt{y}} \Rightarrow z = \cos u = f(u).$

$$= \frac{x}{\sqrt{x}}\left[\frac{1+\frac{y}{x}}{1+\sqrt{y/x}}\right]$$

$$= x^{1/2}\,\phi(y/x) \quad \text{—— (2)}$$

from (1) & (2), we ~~see~~ see that u is not a homogeneous function but z is a homogeneous function of degree $\frac{1}{2}$. Now by Euler's theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}z \quad \text{—— (3)}$$

Since $z = \cos u$

$$\Rightarrow \frac{\partial z}{\partial x} = -\sin u \frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = -\sin u \frac{\partial u}{\partial y}$$

Putting in (3), we get

$$x\left\{-\sin u \frac{\partial u}{\partial x}\right\} + y\left\{-\sin u \frac{\partial u}{\partial y}\right\} = \frac{1}{2}\cos u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\left(\frac{\cos u}{-\sin u}\right)$$

$$= -\frac{1}{2}\cot u$$

Q.3 If $u = \log\left(\frac{x^2+y^2}{x+y}\right)$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$$

Solution Given $u = \log\left(\frac{x^2+y^2}{x+y}\right)$ — (1)

u is not a homogeneous function.

so (1) $\Rightarrow$

$$e^u = \frac{x^2+y^2}{x+y}$$

Let $z = \dfrac{x^2+y^2}{x+y} \Rightarrow z = e^u = f(u)$

$$= x\left[\frac{1+y^2/x^2}{1+y/x}\right] = x^1\phi(y/x)$$

Here we see that
z is a homogeneous function of degree 1.

Now by Euler's deduction formula I,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

$$= 1 \cdot \frac{e^u}{e^u} = 1.$$

$$\Rightarrow \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1}$$

Q. 4 Show that if $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

$= 2\cos 3u \sin u$

Solu^n Given $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ —— (1)

Here u is not a homogeneous function

Now $\tan u = \frac{x^3+y^3}{x-y}$

Let, $z = \frac{x^3+y^3}{x-y} \Rightarrow z = \tan u = f(u)$

$= x^2 \phi(y/x)$

$\Rightarrow z$ is a homogeneous function of degree 2

By deduction formula (I),

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}$$

$$=2\,\frac{\tan u}{\sec^2 u}$$

$$=\frac{2\sin u\cos^2 u}{\cos u}$$

$$=2\sin u\cos u=\sin 2u$$

$$\Rightarrow \boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u}$$

(ii) By deduction formule(II),

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1]$$

since $g(u)=\sin 2u$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$$

$$=\sin 2u\,[2\cos 2u-1]$$

$$=2\sin 2u\cos 2u-\sin 2u$$

$$=\sin 4u-\sin 2u$$

$$=2\cos 3u\sin u$$

Q.5 If z be a homogeneous function of degree 'n', show that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

Solu^n From Euler's theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \quad \text{——— (1)}$$

Diff. (1) w.r.to x partially,

$$x\frac{\partial^2 z}{\partial x^2} + 1.\frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial x \partial y} = n\frac{\partial z}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x \partial y} = (n-1)\frac{\partial z}{\partial x} \quad \text{——— (2)}$$

Similarly,

Diff. (1) w.r.to y partially, we get

$$y\frac{\partial^2 z}{\partial y^2} + x\frac{\partial^2 z}{\partial x \partial y} = (n-1)\frac{\partial z}{\partial y} \quad \text{——— (3)}$$

Now

(2) $\times x$ + (3) $\times y$, we have

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

$$= (n-1)nz \quad \{\text{from (1)}\}$$

$$= n(n-1)z$$

Q-6 If $u(x,y,z) = \log(\tan x + \tan y + \tan z)$
prove that

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$$

Soln. Given $u(x,y,z) = \log(\tan x + \tan y + \tan z)$ —— (1)

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z}\{\sec^2 x + 0 + 0\}$$

$$= \frac{\sec^2 x}{\tan x + \tan y + \tan z} \quad \text{—— (2)}$$

Similarly.

$$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y + \tan z} \quad \text{—— (3)}$$

$$\frac{\partial u}{\partial z} = \frac{\sec^2 z}{\tan x + \tan y + \tan z} \quad \text{—— (4)}$$

Now $[(\sin 2x) \times (2)] + [(\sin 2y) \times (3)] + [(\sin 2z) \times (4)]$,
we have

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$$

$$= \frac{\sin 2x \sec^2 x + \sin 2y \sec^2 y + \sec^2 z . \sin 2z}{\tan x + \tan y + \tan z}.$$

$$= \frac{2\sin x\cos x \cdot \sec^2 x + 2\sin y\cos y\,\sec^2 y + 2\sin z\cos z\,\sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{2(\tan x + \tan y + \tan z)}{(\tan x + \tan y + \tan z)} = 2$$

Q-7 If $x = r\cos\theta$, $y = r\sin\theta$, find

(i) $\left(\frac{\partial x}{\partial r}\right)_\theta$ (ii) $\left(\frac{\partial y}{\partial \theta}\right)_r$ (iii) $\left(\frac{\partial r}{\partial x}\right)_y$ (iv) $\left(\frac{\partial \theta}{\partial y}\right)_x$

Solⁿ Given $x = r\cos\theta$, $y = r\sin\theta$

(i) $\left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta$

(ii) $\left(\frac{\partial y}{\partial \theta}\right)_r = r\cos\theta$

(iii) $x^2 + y^2 = r^2$

$\Rightarrow r = \sqrt{x^2+y^2} = (x^2+y^2)^{1/2}$

$$\left(\frac{\partial r}{\partial x}\right)_y = \frac{1}{\cancel{2}(x^2+y^2)^{1/2}}(\cancel{2}x) = \frac{x}{\sqrt{x^2+y^2}}$$

(iv) $\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}(y/x)$

$$\left(\frac{\partial \theta}{\partial y}\right)_x = \frac{1}{1+\left(\frac{y^2}{x^2}\right)} \cdot \frac{1}{x} = \frac{x^2}{x^2+y^2} \cdot \frac{1}{x}$$

$$= \frac{x}{x^2+y^2}$$

Taylor's and Maclaurin's theorem:

Taylor's theorem:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$$

If we take

$$a+h = x \Rightarrow h = x-a$$

then $f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$

Maclaurin's theorem:

If $f(x)$ be a continuous function and its derivatives upto n^{th} and higher order exist then

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots$$

Q. Expand $\sin x$ in powers of x

Solⁿ

$f(x) = \sin x$

$f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$

$f^{IV}(x) = \sin x$, $f^{(V)}(x) = \cos x$

$\Rightarrow f(0) = \sin 0 = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$

$f^{IV}(0) = 0$, $f^{(V)}(0) = 1$.

By Maclaurin's expansion,

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{\underline{4}} f^{(IV)}(0)$$
$$+ \frac{x^5}{5!} f^{(V)}(0) + \cdots$$

$$\sin x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{\underline{4}}(0)$$
$$+ \frac{x^5}{5!}(1) + \cdots$$

$$\Rightarrow \boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}$$

Q-2 Expand $\log x$ in powers of $(x-1)$ by Taylor's theorem.

Solⁿ. $f(x) = \log x$, $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$

$$f^{(IV)}(x) = -\frac{6}{x^4}$$

$\Rightarrow f(1) = 0$

$\Rightarrow f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$, $f^{(IV)}(1) = -6$

Now By Taylor's theorem,

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a)$$
$$+ \cdots$$

$$\Rightarrow f(x) = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \cdots$$

$$\Rightarrow \log x = 0 + (x-1).1 + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \cdots$$

$$= (x-1) - \frac{(x-1)^2}{2!} + 2\frac{(x-1)^3}{3!} - \cdots$$

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$$

<u>Note</u>: Expand $f(x)$ in powers of $(x-a)$

or Expand $f(x)$ in the nbd of $x = a$

or Expand $f(x)$ about the point 'a'

or Expand $f(x)$ near the point 'a'

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

Q-3 Expand $\sin\left(\frac{\pi}{4}+\theta\right)$ in powers of θ.

Solⁿ Here
$$f(\theta)=\sin\theta \Rightarrow f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$
$$f'(\theta)=\cos\theta \Rightarrow f'\left(\frac{\pi}{4}\right)=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$
$$f''(\theta)=-\sin\theta \Rightarrow f''(\pi/4)=-\frac{1}{\sqrt{2}}$$
$$f'''(\theta)=-\cos\theta \Rightarrow f'''(\pi/4)=-1/\sqrt{2}$$
$$f^{(IV)}(\theta)=\sin\theta \Rightarrow f^{(IV)}(\pi/4)=1/\sqrt{2}$$
$$f^{(V)}(\theta)=\cos\theta \Rightarrow f^{(V)}(\pi/4)=1/\sqrt{2}$$

Thus from Taylor's theorem,
$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\cdots$$
Take $a=\frac{\pi}{4}$, $h=\theta$
$$f\left(\frac{\pi}{4}+\theta\right)=f(\pi/4)+\theta f'(\pi/4)+\frac{\theta^2}{2!}f''(\pi/4)+\frac{\theta^3}{3!}f'''(\pi/4)$$
$$+\frac{\theta^4}{4!}f^{(IV)}(\pi/4)+\cdots$$

$$\sin\left(\frac{\pi}{4}+\theta\right)=\frac{1}{\sqrt{2}}+\theta\frac{1}{\sqrt{2}}+\frac{\theta^2}{2!}\left(-\frac{1}{2!}\right)+\frac{\theta^3}{3!}\left(-\frac{1}{\sqrt{2}}\right)$$
$$+\frac{\theta^4}{4!}\left(\frac{1}{\sqrt{2}}\right)+\cdots$$
$$=\frac{1}{\sqrt{2}}\left[1+\theta-\frac{\theta^2}{2!}-\frac{\theta^3}{3!}+\frac{\theta^4}{4!}-\cdots\right]$$

Taylor's Series of two variables :

If $f(x,y)$ and all its partial derivatives upto the nth order are finite and continuous for all points where

$$a \le x \le a+h \,, \quad b \le y \le b+k$$

Then

$$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{(a,b)} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)} + \frac{1}{3!}\left(h^3\frac{\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2\frac{\partial^3 f}{\partial x \partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}\right)_{(a,b)} + \cdots$$

or

$$f(a+h, b+k) = f(a,b) + \left(h f_x^{(a,b)} + k f_y^{(a,b)}\right) + \frac{1}{2!}\left(h^2 f_{xx}^{(a,b)} + 2hk f_{xy}^{(a,b)} + k^2 f_{yy}^{(a,b)}\right) + \frac{1}{3!}\left(h^3 f_{xxx}^{(a,b)} + 3h^2k f_{xxy}^{(a,b)} + 3hk^2 f_{xyy}^{(a,b)} + k^3 f_{yyy}^{(a,b)}\right) + \cdots$$

or

$$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f^{(a,b)} + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f^{(a,b)} \cdots\cdots$$

Note: (i) On Putting $a=0, b=0, h=x, k=y$.

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \cdots$$

which is the Maclaurin's expansion of $f(x,y)$.

(ii) Expand $f(x,y)$ in powers of $(x-a)$ and $(y-b)$

or Expand $f(x,y)$ in the nbd of the point (a,b)

or Expand $f(x,y)$ near the point $x=a, y=b$

or Expand $f(x,y)$ about the point (a,b).

then

$$\boxed{\begin{aligned} f(x,y) &= f[a+(x-a),\ b+(y-b)] \\ &= f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)] \\ &\quad + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) \\ &\quad + (y-b)^2 f_{yy}(a,b)] + \cdots \end{aligned}}$$

Q-1 Expand $e^x \sin y$ in powers of x and y. as far as terms of third degree.

Soln.

$f(x,y) = e^x \sin y \Rightarrow f(0,0) = 0$

$f_x(x,y) = e^x \sin y \Rightarrow f_x(0,0) = 0$

$f_y(x,y) = e^x \cos y \Rightarrow f_y(0,0) = 1$

$f_{xx}(x,y) = e^x \sin y \Rightarrow f_{xx}(0,0) = 0$

$f_{xy}(x,y) = e^x \cos y \Rightarrow f_{xy}(0,0) = 1$

$f_{yy}(x,y) = -e^x \sin y \Rightarrow f_{yy}(0,0) = 0$

$f_{xxx}(x,y) = e^x \sin y \Rightarrow f_{xxx}(0,0) = 0$

$f_{xxy}(x,y) = e^x \cos y \Rightarrow f_{xxy}(0,0) = 1$

$f_{xyy}(x,y) = -e^x \sin y \Rightarrow f_{xyy}(0,0) = 0$

$f_{yyy}(x,y) = -e^x \cos y \Rightarrow f_{yyy}(0,0) = -1$

By Taylor's Series,

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \frac{1}{3!}[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)] + \cdots\cdots$$

$$e^x \sin y = 0 + x(0) + y(1) + \frac{1}{2!}\left[x^2(0) + 2xy(1) + y^2(0)\right] + \frac{1}{3!}\left[x^3(0) + 3x^2y(1) + 3xy^2(0) + y^3(-1)\right] + \cdots$$

$$\boxed{e^x \sin y = y + xy + \frac{1}{2}x^2y - \frac{1}{6}y^3 + \cdots\cdots}$$

Q.2 Expand $\sin(xy)$ in powers of $(x-1)$ and $(y-\frac{\pi}{2})$ as far as the terms of second degree.

Solu^n

$f(x,y) = \sin(xy) \Rightarrow f(1,\frac{\pi}{2}) = 1$

$f_x(x,y) = y\cos(xy) \Rightarrow f_x(1,\frac{\pi}{2}) = 0$

$f_y(x,y) = x\cos(xy) \Rightarrow f_y(1,\frac{\pi}{2}) = 0$

$f_{xx}(x,y) = -y^2\sin(xy) \Rightarrow f_{xx}(1,\frac{\pi}{2}) = -\frac{\pi^2}{4}$

$f_{xy}(x,y) = \cos(xy) - xy\sin(xy) \Rightarrow f_{xy}(1,\frac{\pi}{2}) = -\frac{\pi}{2}$

$f_{yy}(x,y) = -x^2\sin(xy) \Rightarrow f_{yy}(x,y) = -1$

By Taylor's theorem,

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots\cdots$$

Take, $a = 1$, $b = \frac{\pi}{2}$, then

$$f(x,y) = f(1,\tfrac{\pi}{2}) + \left[(x-1) f_x(1,\tfrac{\pi}{2}) + (y-\tfrac{\pi}{2}) f_y(1,\tfrac{\pi}{2})\right]$$
$$+ \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,\tfrac{\pi}{2}) + 2(x-1)(y-\tfrac{\pi}{2}) f_{xy}(1,\tfrac{\pi}{2})\right.$$
$$\left. + (y-\tfrac{\pi}{2})^2 f_{yy}(1,\tfrac{\pi}{2})\right] + \cdots\cdots$$

$$\sin(xy) = 1 + \left[(x-1)(0) + (y-\tfrac{\pi}{2})(0)\right] + \frac{1}{2!}\left[(x-1)^2\left(-\frac{\pi^2}{4}\right)\right.$$
$$\left. + 2(x-1)(y-\tfrac{\pi}{2})\left(-\frac{\pi}{2}\right) + (y-\tfrac{\pi}{2})^2(-1)\right] + \cdots\cdots$$
$$= 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)(y-\tfrac{\pi}{2}) - \frac{1}{2}(y-\tfrac{\pi}{2})^2$$
$$+ \cdots\cdots\cdots$$

Q.3 If $f(x,y) = \tan^{-1}(x, y)$, compute an approximate value of $f(0.9, -1.2)$.

Soln. $f(0.9, -1.2) = f(1-0.1, -1-0.2)$

$$= f(1,-1) + \left[(-0.1) f_x(1,-1) + (-0.2) f_y(1,-1)\right] + \frac{1}{2!}\left[\right.$$
$$(-0.1)^2 f_{xx}(1,-1) + 2(-0.1)(-0.2) f_{xy}(1,-1) + (-0.2)^2 f_{yy}(1,-1)$$
$$+ \cdots\cdots \quad \text{———} (1)$$

Given $f(x,y) = \tan^{-1}(xy) \Rightarrow f(1,-1) = -\frac{\pi}{4}$

$f_x(x,y) = \frac{y}{1+x^2y^2} \Rightarrow f_x(1,-1) = -\frac{1}{2}$

$f_y(x,y) = \frac{x}{1+x^2y^2} \Rightarrow f_y(1,-1) = \frac{1}{2}$

$f_{xx}(x,y) = -\frac{2xy^3}{(1+x^2y^2)^2} \Rightarrow f_{xx}(1,-1) = \frac{1}{2}$

$f_{xy}(x,y) = \frac{(1+x^2y^2)\cdot 1 - y(0+2x^2y)}{(1+x^2y^2)^2}$

or $f_{xy}(x,y) = \frac{1-x^2y^2}{(1+x^2y^2)^2} \Rightarrow f_{xy}(1,-1) = 0$

$f_{yy}(x,y) = -\frac{2x^3y}{(1+x^2y^2)^2} \Rightarrow f_{yy}(1,-1) = \frac{1}{2}$

Putting these values in (1), we get—

$$f(0.9,-1.2) = -\frac{\pi}{4} + \left\{(-0.1)\left(-\frac{1}{2}\right) + (-0.2)\left(\frac{1}{2}\right)\right\}$$
$$+ \frac{1}{2!}\left[(-0.1)^2\left(\frac{1}{2}\right) + 2(-0.1)(-0.2)(0) + (-0.2)^2\left(\frac{1}{2}\right)\right] + \cdots$$
$$= -\frac{22}{28} + 0.05 - 0.1 + \frac{1}{2}(0.005 + 0.02)$$
$$= -0.786 + 0.05 - 0.1 + 0.0125$$
$$= -0.8235.$$

Jacobians

If u and v are functions of the two independent variables x and y then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the Jacobian of u, v with respect to x, y and is denoted as

$$\frac{\partial(u,v)}{\partial(x,y)} \text{ or } J(u,v) \text{ or } J\left(\frac{u,v}{x,y}\right)$$

Similarly, the Jacobian of u, v, w with respect to x, y, z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Q-1 If $x = r\cos\theta$, $y = r\sin\theta$ then find $\frac{\partial(x,y)}{\partial(r,\theta)}$.

Soln Given $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (r\cos^2\theta) - (-r\sin^2\theta) = r(\cos^2\theta + \sin^2\theta)$$

$$= r(1) = r$$

Q-2 If $y_1 = \dfrac{x_2 x_3}{x_1}$, $y_2 = \dfrac{x_3 x_1}{x_2}$, $y_3 = \dfrac{x_1 x_2}{x_3}$ then show that the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 is 4.

Solⁿ: Given $y_1 = \dfrac{x_2 x_3}{x_1}$, $y_2 = \dfrac{x_3 x_1}{x_2}$, $y_3 = \dfrac{x_1 x_2}{x_3}$

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} & \dfrac{\partial y_1}{\partial x_3} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} & \dfrac{\partial y_2}{\partial x_3} \\ \dfrac{\partial y_3}{\partial x_1} & \dfrac{\partial y_3}{\partial x_2} & \dfrac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -\dfrac{x_2 x_3}{x_1^2} & \dfrac{x_3}{x_1} & \dfrac{x_2}{x_1} \\ \dfrac{x_3}{x_2} & -\dfrac{x_3 x_1}{x_2^2} & \dfrac{x_1}{x_2} \\ \dfrac{x_2}{x_3} & \dfrac{x_1}{x_3} & -\dfrac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2 x_3 & x_3 x_1 & x_1 x_2 \\ x_2 x_3 & -x_3 x_1 & x_1 x_2 \\ x_2 x_3 & x_3 x_1 & -x_1 x_2 \end{vmatrix}$$

$$= \frac{\cancel{x_1^2}\cancel{x_2^2}\cancel{x_3^2}}{\cancel{x_1^2}\cancel{x_2^2}\cancel{x_3^2}} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -1[1-1] - 1[-1-1] + 1[1+1]$$

$$= 0 + 2 + 2$$

$$= 4 \ \#.$$

* If $u_1 = f_1(x_1)$

$u_2 = f_2(x_1, x_2)$

$u_3 = f_3(x_1, x_2, x_3)$

then
$$\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & 0 & 0 \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & 0 \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

$$= \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \frac{\partial u_3}{\partial x_3}.$$

Q-3 If $u = \cos x$, $v = \sin x \cos y$, $w = \sin x \sin y \cos z$ then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

Solⁿ Given $u = \cos x$, $v = \sin x \cos y$, $w = \sin x \sin y \cos z$

$\frac{\partial u}{\partial x} = -\sin x$, $\frac{\partial u}{\partial y} = 0$, $\frac{\partial u}{\partial z} = 0$, $\frac{\partial v}{\partial x} = \cos x \cos y$

$\frac{\partial v}{\partial y} = -\sin x \sin y$, $\frac{\partial w}{\partial x} = \cos x \sin y \cos z$

$\frac{\partial w}{\partial y} = \sin x \cos y \cos z$, $\frac{\partial w}{\partial z} = -\sin x \sin y \sin z$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & 0 & 0 \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & 0 \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \frac{\partial w}{\partial z}$$

$$= (-\sin x)(-\sin x \sin y)(-\sin x \sin y \sin z)$$

$$= -\sin^3 x \sin^2 y \sin z$$

Properties:

(I) If u and v are the functions of x and y then $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1.$

i.e. $JJ' = 1.$

Q-4 If $x = u(1-v)$, $y = uv$ prove that $JJ' = 1$ where $J = \frac{\partial(x,y)}{\partial(u,v)}$ & $J' = \frac{\partial(u,v)}{\partial(x,y)}$

Soln. Given $x = u(1-v)$ —— (1)

$y = uv$ —— (2)

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u(1-v) - (v)(-u)$$

$$= u - \cancel{uv} + \cancel{vu}$$

$$= u$$

$$\Rightarrow J = \frac{\partial(x,y)}{\partial(u,v)} = u \quad \text{——— (3)}$$

Now from (1) & (2),

$$x = u - uv$$

$$\Rightarrow x = u - y \Rightarrow \underline{u = x+y} \quad \text{——— (4)}$$

Putting this value of u in (2),

$$y = uv \Rightarrow \underline{v = \frac{y}{x+y}} \quad \text{——— (5)}$$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 1, \quad \frac{\partial v}{\partial x} = \frac{-y}{(x+y)^2}, \quad \frac{\partial v}{\partial y} = \frac{(x+y).1 - y(1)}{(x+y)^2}$$

$$= \frac{x}{(x+y)^2}.$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2} = \frac{x+y}{(x+y)^2} = \frac{1}{x+y} = \frac{1}{u}$$

$$\Rightarrow J' = \frac{1}{u} \quad \text{——— (6)}$$

from (3) & (6), we get

$$JJ' = \cancel{u} \cdot \frac{1}{\cancel{u}} = 1$$

Q5 If $u = xyz$, $v = x^2+y^2+z^2$, $w = x+y+z$

find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

Solⁿ: Given $u = xyz$, $v = x^2+y^2+z^2$, $w = x+y+z$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} yz & zx & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$= \begin{vmatrix} z(y-x) & x(z-y) & xy \\ 2(x-y) & 2(y-z) & 2z \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2(x-y)(y-z) \begin{vmatrix} -z & -x & xy \\ 1 & 1 & z \\ 0 & 0 & 1 \end{vmatrix}$$

$C_1 \to C_1 - C_2$

$$= 2(x-y)(y-z) \begin{vmatrix} x-z & -x & xy \\ 0 & 1 & z \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2(x-y)(y-z)(x-z)\begin{vmatrix} 1 & -x & xy \\ 0 & 1 & z \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2(x-y)(y-z)(x-z)\left[1(1-0)+x(0-0)+xy(0-0)\right]$$

$$= 2(x-y)(y-z)(x-z)\left[1+0+0\right]$$

$$= -2(x-y)(y-z)(z-x)$$

Since $JJ' = 1$.

i.e. $$\frac{\partial(u,v,w)}{\partial(x,y,z)} \cdot \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1$$

$$\Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{J} = -\frac{1}{2(x-y)(y-z)(z-x)}$$

(II) Chain rule:

If u, v are the functions of r, s and r, s are functions of ~~___~~ x, y

i.e. $u \equiv u(r,s)$ & $r \equiv r(x,y)$
$v \equiv v(r,s)$ $\quad s \equiv s(x,y)$

then $$\boxed{\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}}$$

Q-6 find the value of the Jacobian $\frac{\partial(u,v)}{\partial(r,\theta)}$

where $u = x^2 - y^2$, $v = 2xy$ and

$x = r\cos\theta$, $y = r\sin\theta$.

Solⁿ

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)} \quad \text{—— (1)}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$$

$$= 4(x^2+y^2).$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(1) = r$$

Now (1) $\Rightarrow$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = 4(x^2+y^2) \cdot r \qquad \left\{ \text{since } x^2+y^2 = r^2 \right\}$$

$$= 4r^2 \cdot r = 4r^3.$$

(III)

著 The functions u, v, w of three independent variables x, y, z, are not independent i.e. there exist a relation between u, v, w $[f(u,v,w)=0]$ iff

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=0$$

<u>Q.-1</u> If $u=xy+yz+zx$, $v=x^2+y^2+z^2$ and $w=x+y+z$, determine whether there is a functional relationship between u, v, w and if so, find it.

<u>Solⁿ</u>: Given $u=xy+yz+zx$

$$V=x^2+y^2+z^2$$

$$w=x+y+z$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}=\begin{vmatrix}y+z & z+x & x+y\\ 2x & 2y & 2z\\ 1 & 1 & 1\end{vmatrix}$$

$$=2\begin{vmatrix}x+y+z & x+y+z & x+y+z\\ x & y & z\\ 1 & 1 & 1\end{vmatrix} \quad R_1\to R_1+R_2$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2(x+y+z)[0]$$

$$= 0$$

Hence, there exist a relation between u, v, w.

Now.
$$w^2 = (x+y+z)^2$$
$$= x^2+y^2+z^2+2xy+2yz+2zx$$
$$= (x^2+y^2+z^2)+2(xy+yz+zx)$$

$$\boxed{w^2 = v+2u}$$

Q-2 Verify whether the following functions are functionally dependent and if so find the relation between them

$$u = \frac{x+y}{1-xy}, \quad v = \tan^{-1}x + \tan^{-1}y$$

Soln: $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \quad \text{——— (1)}$$

$$\frac{\partial u}{\partial x} = \frac{(1-xy)(1+0) - (x+y)(0-y)}{(1-xy)^2} = \frac{1-\cancel{xy}+\cancel{xy}+y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2}, \quad \frac{\partial v}{\partial x} = \frac{1}{1+x^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

Now, $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{\cancel{1+y^2}}{(1-xy)^2} \times \frac{1}{\cancel{1+y^2}} - \frac{1}{\cancel{1+x^2}} \times \frac{\cancel{1+x^2}}{(1-xy)^2}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

Hence u, v are functionally related.

Since we know that

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \quad v = \tan^{-1}u$$

or $\boxed{u = \tan v}$

Jacobian of implicit function:

The variables x, y, u, v are connected by implicit functions

$$f_1(x, y, u, v) = 0 \quad \text{--- (1)}$$

$$f_2(x, y, u, v) = 0 \quad \text{--- (2)}$$

where u, v are functions of x, y.

then $$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\left|\frac{\partial(f_1,f_2)}{\partial(x,y)}\right|}{\left|\frac{\partial(f_1,f_2)}{\partial(u,v)}\right|}$$

Similarly, if u, v, w are the functions of x, y, z such that

$$f_1(x, y, z, u, v, w) = 0$$

$$f_2(x, y, z, u, v, w) = 0$$

$$f_3(x, y, z, u, v, w) = 0$$

then $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\left|\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}\right|}{\left|\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}\right|}$$

Q.-1 If $x^2+y^2+u^2-v^2=0$ and $uv+xy=0$ prove that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{x^2-y^2}{u^2+v^2}$$

Solu^n^ Let $f_1 = x^2+y^2+u^2-v^2$

$f_2 = uv+xy$

$$\frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix} = 2(x^2-y^2)$$

$$\frac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = 2(u^2+v^2)$$

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\left|\frac{\partial(f_1,f_2)}{\partial(x,y)}\right|}{\left|\frac{\partial(f_1,f_2)}{\partial(u,v)}\right|} = \frac{\cancel{2}(x^2-y^2)}{\cancel{2}(u^2+v^2)} = \frac{x^2-y^2}{u^2+v^2}$$

Q.2 If $x+y+z=u$, $y+z=uv$, $z=u.v.w$

show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$.

Solu^n^: $f_1 = x+y+z-u$

$f_2 = y+z-uv$

$f_3 = z-uvw$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ -v & -u & 0 \\ -vw & -uw & -uv \end{vmatrix}$$

$$= -1\left[(-u)(-uv) - 0\right] = -u^2v$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\left|\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}\right|}{\left|\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}\right|} = -\frac{-1}{u^2v} = \frac{1}{u^2v}$$

Since $JJ' = 1$

$$\Rightarrow \frac{\partial(u,v,w)}{\partial(x,y,z)} \cdot \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1$$

$$\Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\left(\frac{1}{u^2v}\right)} = u^2v \quad \#$$

Q.3 Given $x = u+v+w,\ y = u^2+v^2+w^2,\ z = u^3+v^3+w^3$

find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$.

Solⁿ:
$$f_1 \equiv u+v+w-x = 0$$
$$f_2 \equiv u^2+v^2+w^2-y = 0$$
$$f_3 \equiv u^3+v^3+w^3-z = 0$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & 1 & 1 \\ 0 & 2v & 2w \\ 0 & 3v^2 & 3w^2 \end{vmatrix}$$

$$= -1\left[(2v)(3w^2) - (3v^2)(2w)\right]$$
$$= -6vw(w-v) = 6vw(v-w)$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2u & 2v & 2w \\ 3u^2 & 3v^2 & 3w^2 \end{vmatrix}$$

$$\left\{\begin{matrix} C_1 \to C_1 - C_2 \\ C_2 \to C_2 - C_3 \end{matrix}\right\}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ 2(u-v) & 2(v-w) & 2w \\ 3(u+v)(u-v) & 3(v+w)(v-w) & 3w^2 \end{vmatrix}$$

$$= 6(u-v)(v-w)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & w \\ u+v & v+w & w^2 \end{vmatrix}$$

$$= 6\,(u-v)(v-w)\left[(v+w)-(u+v)\right]$$

$$= 6\,(u-v)(v-w)\left[\cancel{v}+w-u-\cancel{v}\right]$$

$$= 6\,(u-v)(v-w)(w-u)$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\left|\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}\right|}{\left|\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}\right|}$$

$$= (-1)\frac{\left[\cancel{6}vw(v-w)\right]}{\cancel{6}(u-v)(v-w)(w-u)}$$

$$= -\frac{vw}{(u-v)(w-u)}$$

Extrema of functions of Several Variables:

Maximum Value:

A function $f(x,y)$ is said to have a maximum value at $x=a$, $y=b$ if there exists a small neighbourhood of (a,b) such that

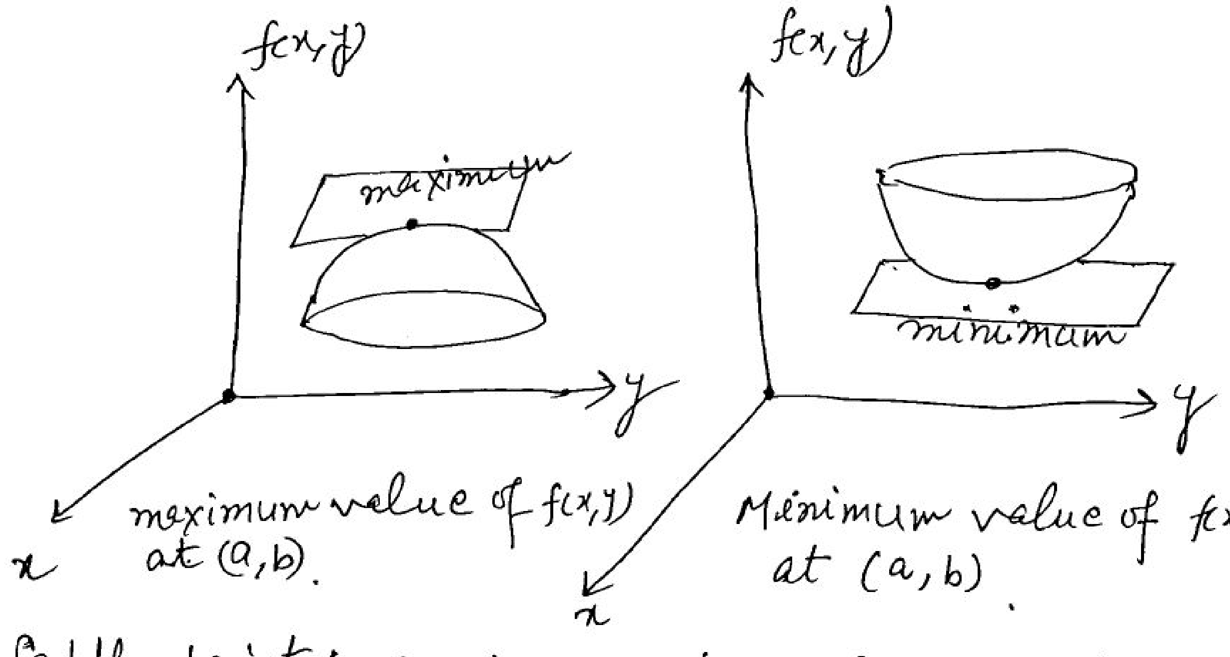
$$f(a,b) > f(a+h, b+k)$$

Minimum Value:

A function $f(x,y)$ is said to have a minimum value at $x=a$, $y=b$ if there exists a small neighbourhood of (a,b) such that

$$f(a,b) < f(a+h, b+k).$$

The maximum and minimum values of a function are also called extreme or extremum values of the function.



maximum value of $f(x,y)$ at (a,b).

Minimum value of $f(x,y)$ at (a,b).

Saddle point: It is a point where a function is neither maximum nor minimum.

Working rule to find extreme values :

Let $z = f(x,y)$.

1. Differentiate $f(x,y)$ and find
$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}.$$

2. Solve $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$. Let (a,b) be the values of (x,y). The point (a,b) is called stationary point.

3. Evaluate $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ at (a,b).

4. (I) If $rt - s^2 > 0$,
   (i) $r < 0$ then $f(x,y)$ has a maximum value
   (ii) $r > 0$ then $f(x,y)$ has a minimum value

   (II) If $rt - s^2 < 0$
   then $f(x,y)$ has no extremum value at the point (a,b).

   (III) If $rt - s^2 = 0$ then the case is doubtful.
   ~~and needs further investigation.~~

Q.1 Discuss the maximum and minimum of
$$x^2 + y^2 + 6x + 12.$$

Soln Given $f(x,y) = x^2 + y^2 + 6x + 12$

$$\frac{\partial f}{\partial x} = 2x + 6, \quad \frac{\partial f}{\partial y} = 2y.$$

$$r = \frac{\partial^2 f}{\partial x^2} = 2, \quad s = \frac{\partial^2 f}{\partial x \partial y} = 0, \quad t = \frac{\partial^2 f}{\partial y^2} = 2.$$

for maxima or minima,

$$\frac{\partial f}{\partial x}=0,\ \frac{\partial f}{\partial y}=0$$

$\Rightarrow\ 2x+6=0,\ 2y=0$

$\Rightarrow\ x=-3,\ y=0$

$\Rightarrow\ (x,y)=(-3,0)$

At $(-3,0)$: $\quad rt-s^2 = 2\times 2-(0)^2 = 4>0$

$\quad\quad \& \ r=2>0$

Hence, $f(x,y)$ is minimum at $(-3,0)$

Minimum value $=(-3)^2+(0)^2+6(-3)+12$
$= 9+0-18+12 = 3$ #

Q-2 Show that the minimum value of $f(x,y)=xy+a^3\left(\frac{1}{x}+\frac{1}{y}\right)$ is $3a^2$.

Solu^n: Given $f(x,y)=xy+a^3\left(\frac{1}{x}+\frac{1}{y}\right)$

$$\frac{\partial f}{\partial x}=y-\frac{a^3}{x^2},\quad \frac{\partial f}{\partial y}=x-\frac{a^3}{y^2}$$

$$r=\frac{\partial^2 f}{\partial x^2}=\frac{2a^3}{x^3},\quad s=\frac{\partial^2 f}{\partial x\partial y}=1,\ t=\frac{\partial^2 f}{\partial y^2}=\frac{2a^3}{y^3}$$

for maxima and minima,

$$\frac{\partial f}{\partial x}=0,\ \frac{\partial f}{\partial y}=0$$

$$y-\frac{a^3}{x^2}=0,\quad x-\frac{a^3}{y^2}=0$$

$\Rightarrow\ x^2y=a^3$ ——(i)

$\quad xy^2=a^3$ ——(ii)

Putting the value of x from () in (i), we have

$$\left(\frac{a^3}{y^2}\right)^2 y = a^3 \Rightarrow \frac{a^6}{y^4}\cdot y = a^3$$

$$\Rightarrow a^3 = y^3 \Rightarrow \boxed{y = a}$$

Putting this value of y in (i), we get

$$x^2 a = a^3 \Rightarrow x^2 = a^2 \Rightarrow \boxed{x = \pm a}$$

$$\Rightarrow \boxed{(x,y) = (a,a),\ (-a,a)}$$

At (a,a):

$$r = \frac{2a^3}{x^3} = \frac{2a^3}{a^3} = 2$$

$$s = 1,\quad t = \frac{2a^3}{y^3} = \frac{2a^3}{a^3} = 2$$

Now $rt - s^2 = (2)(2) - (1)^2 = 4 - 1 = 3 > 0$

& $r = 2 > 0$

$\Rightarrow f(x,y)$ is minimum at (a,a)

Minimum value $= a^2 + a^3\left(\frac{1}{a} + \frac{1}{a}\right)$

$$= a^2 + a^3 \cdot \frac{2}{a} = a^2 + 2a^2 = 3a^2$$

At $(-a,a)$:

$$r = -2,\quad s = 1,\quad t = 2$$

$$rt - s^2 = (-2)(2) - (1)^2 = -4 - 1 = -5 < 0$$

$\Rightarrow f(x,y)$ has neither maxima nor minima at $(-a, a)$

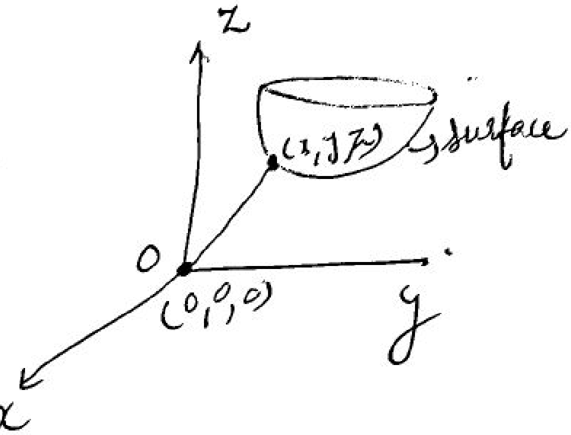
Q.3 find the points on the surface $z^2 = xy+1$ nearest to the origin.

Soln. Let point on the surface $z^2 = xy+1$ — (1) is (x,y,z).

Its distance from origin

$$r = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$$

$$\Rightarrow r^2 = x^2+y^2+z^2$$



Let $u = x^2+y^2+z^2$ — (2)

Now we find the value of (x,y,z) for which u is minimum.

Putting the value of z^2 from (1) in (2).

we get

$$u = x^2+y^2+xy+1 \quad \text{——— (3)}$$

for maxima or minima:

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0$$

$$\Rightarrow 2x+y=0, \quad 2y+x=0$$

$$\Rightarrow \boxed{x=0, y=0}$$

Putting $x=0, y=0$ in (1), we get

$$z^2 = 0+1$$

$$\Rightarrow \boxed{z = \pm 1}$$

Now

$$r = \frac{\partial^2 u}{\partial x^2} = 2, \quad s = \frac{\partial^2 u}{\partial x \partial y} = 1$$

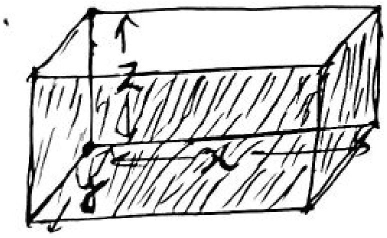
$$t = \frac{\partial^2 u}{\partial y^2} = 2, \quad rt - s^2 = 2\times 2 - (1)^2 = 3 > 0 \; \& \; r > 0$$

so u is minimum at $(0,0,\pm 1)$.

#.

Q.4 A rectangular box, open at the top, is to have a volume of 32 c.c. find the dimensions of the box requiring least material for its construction.

Solⁿ:



Let x, y and z be the length, breadth and height of the box respectively.

Let S is its surface area and V is volume

then given $V = 32$ c.c.

$$\Rightarrow xyz = 32 \Rightarrow y = \frac{32}{xz} \quad \text{——— (1)}$$

Since $S = 2xz + 2yz + xy$

Let $f = 2(x+y)z + xy$ ——— (2)

Putting the value of y from (1) in (2), we get

$$f = 2\left(x + \frac{32}{xz}\right)z + x\left(\frac{32}{xz}\right)$$

so $f = 2xz + \frac{64}{x} + \frac{32}{z}$ ——— (3)

$$\frac{\partial f}{\partial x} = 2z - \frac{64}{x^2}, \quad \frac{\partial f}{\partial z} = 2x - \frac{32}{z^2}$$

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{128}{x^3}, \quad s = \frac{\partial^2 f}{\partial x \partial z} = 2, \quad t = \frac{\partial^2 f}{\partial z^2} = \frac{64}{z^3}$$

for maxima and minima;

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial z} = 0$$

$$\Rightarrow 2z - \frac{64}{x^2} = 0 \Rightarrow z x^2 = 32 \quad \text{—— (4)}$$

$$\& \quad 2x - \frac{32}{z^2} = 0 \Rightarrow x z^2 = 16 \quad \text{—— (5)}$$

Solving (4) and (5), we get

$\boxed{x = 4,\ z = 2}$

Putting $x = 4$ & $z = 2$ in $y = \frac{32}{xz}$ we have

$\boxed{y = 4}$

Now

$$r = \frac{128}{x^3} = \frac{128}{64} = 2$$

$$s = 2$$

$$t = \frac{64}{z^3} = \frac{64}{8} = 8$$

$$rt - s^2 = (2 \times 8) - (2)^2 = 12 > 0$$

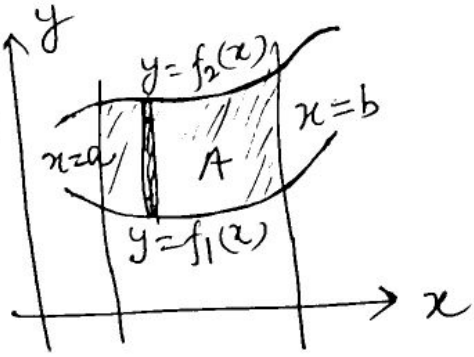
$$\& \quad r = 2 > 0$$

So S is minimum for $x = 4, y = 4, z = 2$.

Double integral in cartesian co-ordinates

Double integral over region A is evaluated by two successive integrations

Let us consider a function $f(x,y)$ of two variables x and y defined in the finite region A of x-y plane.



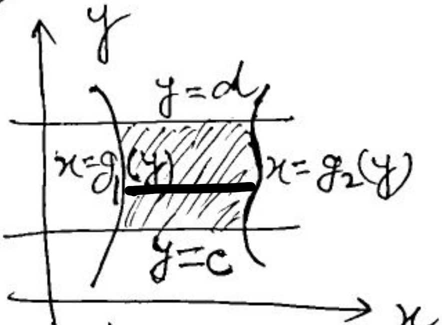
$$* \iint_A f(x,y)\,dx\,dy = \int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dx\,dy$$

$$= \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy \right] dx$$ keeping x-constant

Or.

$$* \iint f(x,y)\,dx\,dy = \int_{y=c}^{d} \int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$$

$$= \int_{y=c}^{d} \left[\int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dx \right] dy$$ Keeping y-constant



$$* \int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\,dx\,dy = \int_{x=a}^{b} \left[\int_{y=c}^{d} f(x,y)\,dy \right] dx$$

Q.1. Evaluate $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx$

Soln

$$\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx = \int_0^1 \left[\int_0^{x^2} e^{y/x}\, dy \right] dx$$

$$= \int_0^1 \left[\frac{e^{y/x}}{1/x} \right]_0^{x^2} dx = \int_0^1 x\,(e^x - 1)\, dx$$

$$= \int_0^1 x e^x\, dx - \int_0^1 x\, dx$$

$$= \left(x e^x - e^x\right)_0^1 - \left(\frac{x^2}{2}\right)_0^1$$

$$= \left[(1.e^1 - e^1) - (0 - e^0)\right] - \frac{1}{2}(1-0)$$

$$= [0+1] - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}$$

#.

Q.2 Evaluate $\int_0^1 \int_0^y xy\, e^{-x^2}\, dx\, dy$

Soln:

$$\int_0^1 \int_0^y xy\, e^{-x^2}\, dx\, dy = \int_0^1 y \left[\int_0^y x e^{-x^2}\, dx \right] dy$$

$$= \int_0^1 y \left[-\frac{e^{-x^2}}{2} \right]_0^y dy = \int_0^1 -y \left[\frac{e^{-y^2}}{2} - \frac{e^0}{2} \right] dy$$

$$= \frac{1}{2} \int_0^1 \left(y - y e^{-y^2} \right) dy$$

$$= \frac{1}{2} \left[\left(\frac{y^2}{2} \right)_0^1 - \left(\frac{-e^{-y^2}}{2} \right)_0^1 \right]$$

$$= \frac{1}{2} \times \frac{1}{2} \left[\{(1)^2 - 0\} + (e^{-1} - e^0) \right]$$

$$= \frac{1}{4} \left[\cancel{1} + e^{-1} - \cancel{1} \right]$$

$$= \frac{e^{-1}}{4}$$

#.

Q.3 Evaluate $\int_0^1\int_0^1 (x+y)\,dx\,dy$

Solⁿ $\int_0^1\int_0^1 (x+y)\,dx\,dy = \int_0^1 \left[\int_0^1 (x+y)\,dy\right] dx$

$$= \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^1 dx$$

$$= \int_0^1 \left[\left(x+\frac{1}{2}\right) - 0\right] dx = \int_0^1 \left(x+\frac{1}{2}\right) dx$$

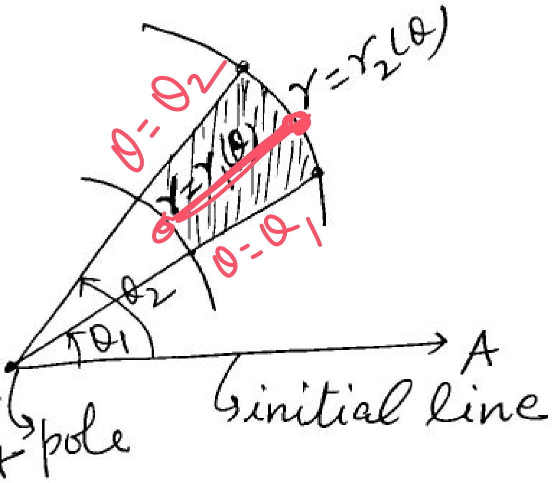
$$= \left(\frac{x^2}{2} + \frac{1}{2}x\right)_0^1 = \left(\frac{1}{2}+\frac{1}{2}\right) - (0+0)$$

$$= 1 - 0 = 1 \; \#.$$

Double integrals in polar co-ordinates:

$$\int_{\theta=\theta_1}^{\theta_2}\int_{r=r_1(\theta)}^{r_2(\theta)} f(r,\theta)\,dr\,d\theta$$

$$= \int_{\theta=\theta_1}^{\theta_2}\left[\int_{r=r_1(\theta)}^{r_2(\theta)} f(r,\theta)\,dr\right] d\theta$$ keeping θ-constant



Q.1 Evaluate $\int_0^\pi\int_0^{a(1-\cos\theta)} r^2 \sin\theta\,dr\,d\theta$

Solⁿ: $\int_0^\pi\int_0^{a(1-\cos\theta)} r^2 \sin\theta\,dr\,d\theta = \int_0^\pi \sin\theta\left[\int_0^{a(1-\cos\theta)} r^2\,dr\right] d\theta$

$$= \int_0^\pi \sin\theta \left[\frac{r^3}{3}\right]_0^{a(1-\cos\theta)} d\theta = \frac{1}{3}\int_0^\pi a^3(1-\cos\theta)^3 \sin\theta\,d\theta$$

Putting $1-\cos\theta = t \Rightarrow \sin\theta\,d\theta = dt$

$$= \frac{a^3}{3}\int_0^2 t^3\,dt = \frac{a^3}{3}\left(\frac{t^4}{4}\right)_0^2 = \frac{a^3}{12}(2^4-0) = \frac{4}{3}a^3 \; \#$$

Evaluate:

Q.2 $\int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$

Solu^n: $\int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$

$= \int_0^{\pi/2}\left[\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\right]d\theta$

Put $a^2-r^2=t \Rightarrow -2r\,dr = dt$

$\Rightarrow r\,dr = -\frac{1}{2}dt.$

$r=0 \Rightarrow \boxed{t=a^2}$

$r=a\cos\theta \Rightarrow t = a^2-a^2\cos^2\theta$

$= a^2(1-\cos^2\theta)$

$\boxed{r = a^2\sin^2\theta}$

$= \int_0^{\pi/2}\left[\int_{a^2}^{a^2\sin^2\theta}\sqrt{t}\left(-\frac{1}{2}dt\right)\right]d\theta$

$= -\frac{1}{2}\int_0^{\pi/2}\left(\frac{t^{3/2}}{3/2}\right)_{a^2}^{a^2\sin^2\theta}d\theta$

$= -\frac{1}{\cancel{2}}\times\frac{\cancel{2}}{3}\int_0^{\pi/2}\left[(a^2\sin^2\theta)^{3/2}-(a^2)^{3/2}\right]d\theta$

$= -\frac{1}{3}a^3\left[\int_0^{\pi/2}\sin^3\theta\,d\theta - \int_0^{\pi/2}1.\,d\theta\right]$

$\Big[$ Since $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\Rightarrow 4\sin^3\theta = 3\sin\theta - \sin 3\theta$

$\Rightarrow \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

$$= -\frac{a^3}{3}\left[\int_0^{\pi/2} -\left(\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta\right) d\theta - (\theta)_0^{\pi/2}\right]$$

$$= -\frac{a^3}{12}\left[3\int_0^{\pi/2}\sin\theta\, d\theta - \int_0^{\pi/2}\sin 3\theta\, d\theta - 4\left(\frac{\pi}{2}-0\right)\right]$$

$$= -\frac{a^3}{12}\left[3\{-\cos\theta\}_0^{\pi/2} - \left\{-\frac{\cos 3\theta}{3}\right\}_0^{\pi/2} - \frac{4\pi}{2}\right]$$

$$= -\frac{a^3}{12}\left[3\left\{-\cos\frac{\pi}{2} + \cos 0\right\} - \frac{1}{3}\left\{-\cos\frac{3\pi}{2} + \cos 0\right\} - \frac{4\pi}{2}\right]$$

$$= -\frac{a^3}{12}\left[3\{-0+1\} - \frac{1}{3}\{0+1\} - \frac{4\pi}{2}\right]$$

$$= -\frac{a^3}{12}\left[3 - \frac{1}{3} - \frac{4\pi}{2}\right]$$

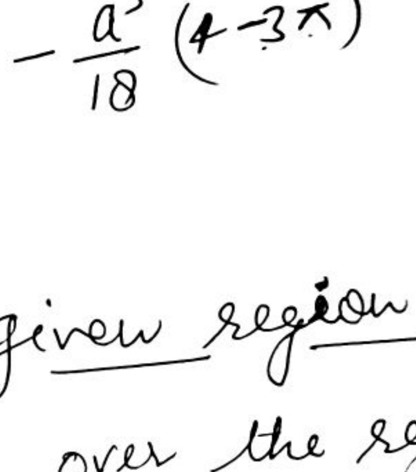
$$= -\frac{a^3}{12}\left[\frac{8}{3} - \frac{4\pi}{2}\right] = -\frac{a^3}{12}\times 4\left[\frac{2}{3} - \frac{\pi}{2}\right]$$

$$= -\frac{a^3}{3}\left[\frac{4-3\pi}{6}\right] = -\frac{a^3}{18}(4-3\pi)$$

$$= \frac{a^3}{18}(3\pi-4)$$

The integral over a given region:

Q. 1 Evaluate $\iint xy\,dx\,dy$ over the region in the positive quadrant for which $x+y \le 1$.



Soln: $\iint_A xy\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x} xy\,dx\,dy$

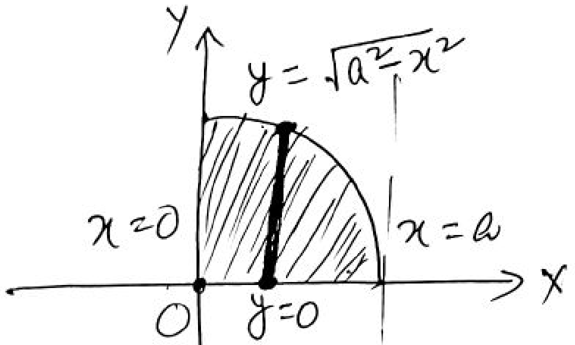
$$= \int_0^1 x\left[\int_0^{1-x} y\,dy\right]dx$$

$$= \int_0^1 x\left[\frac{y^2}{2}\right]_0^{1-x} dx = \frac{1}{2}\int_0^1 x(1-x)^2\,dx$$

$$= \frac{1}{2}\int_0^1 (x-2x^2+x^3)\,dx = \frac{1}{2}\left[\frac{x^2}{2} - 2\frac{x^3}{3} + \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{2} \times \frac{1}{12} = \frac{1}{24}$$

Q.2 Evaluate $\iint_R xy\,dx\,dy$, where R is the quadrant of circle $x^2+y^2=a^2$ where $x \geq 0, y \geq 0$.



Solⁿ:

$$\iint_R xy\,dx\,dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy\,dx\,dy = \int_0^a x\left[\int_0^{\sqrt{a^2-x^2}} y\,dy\right]dx$$

$$= \int_0^a x\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx = \frac{1}{2}\int_0^a x(a^2-x^2)\,dx$$

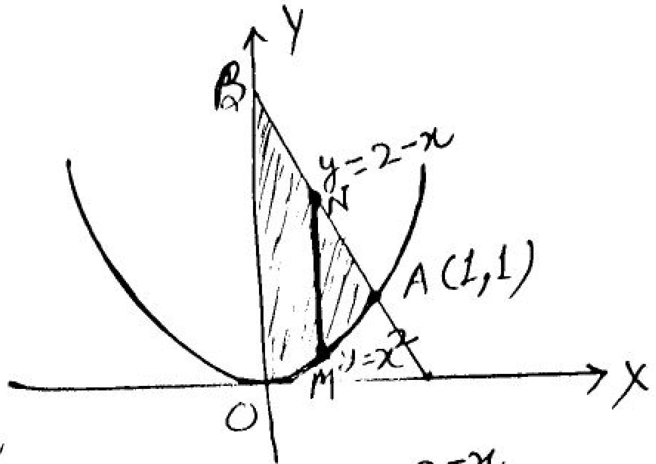
$$= \frac{1}{2}\int_0^a (a^2x - x^3)\,dx$$

$$= \frac{1}{2}\left(a^2\frac{x^2}{2} - \frac{x^4}{4}\right)_0^a = \frac{1}{2}\left[a^2 \cdot \frac{a^2}{2} - \frac{a^4}{4}\right]$$

$$= \frac{1}{2}\left[\frac{a^4}{4}\right] = \frac{a^4}{8}$$

Q.3 Evaluate $\iint y\,dx\,dy$ over the area bounded by $x=0, y=x^2, x+y=2$ in the first quadrant.

Solⁿ $\iint y\,dx\,dy = \int_0^1\int_{x^2}^{2-x} y\,dx\,dy$



$$\int_0^1\int_{x^2}^{2-x} y\,dx\,dy = \int_0^1\left[\int_{x^2}^{2-x} y\,dy\right]dx$$

$$= \int_0^1 \left(\frac{y^2}{2}\right)_{x^2}^{2-x} dx = \int_0^1\left[\frac{(2-x)^2}{2} - \frac{x^4}{2}\right]dx$$

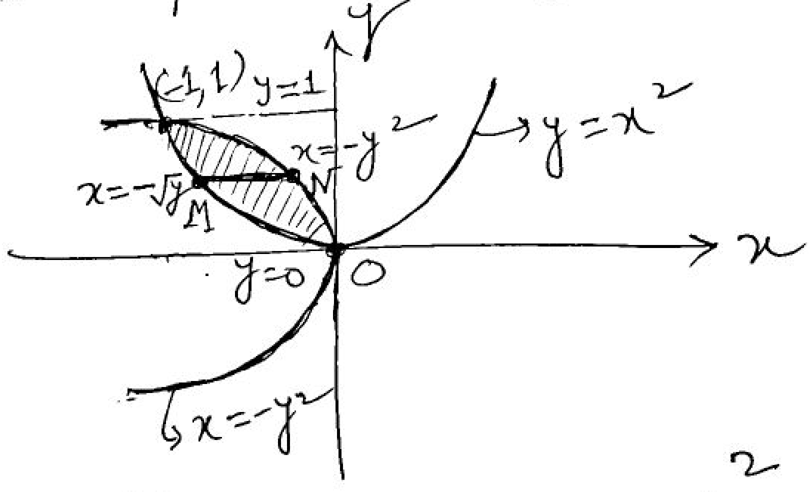
$$= \frac{1}{2}\int_0^1 (4-4x+x^2-x^4)\,dx$$

$$= \frac{1}{2}\left(4x - 4\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^5}{5}\right)_0^1$$

$$= \frac{1}{2}\left(\frac{32}{15}\right) = \frac{16}{15} \quad \#$$

Q-4 Evaluate $\iint xy\,dx\,dy$ over the area bounded by the parabolas $y=x^2$ and $x=-y^2$

Sol^n



$$\iint xy\,dx\,dy$$

$$= \int_0^1\int_{-\sqrt{y}}^{-y^2} xy\,dy\,dx$$

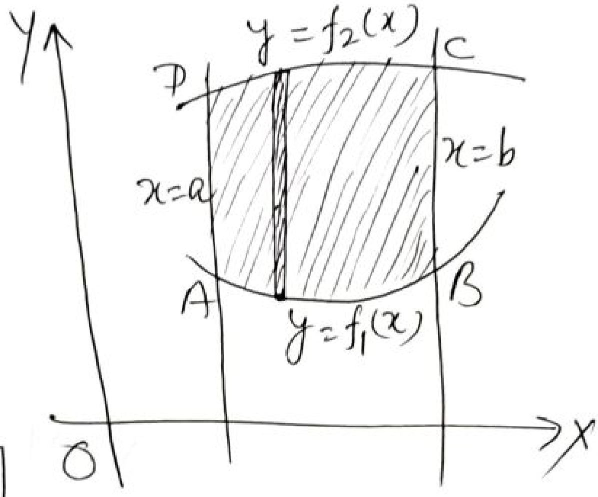
$$= \int_0^1 y\left[\int_{-\sqrt{y}}^{-y^2} x\,dx\right]dy = \int_0^1 y\left(\frac{x^2}{2}\right)_{-\sqrt{y}}^{-y^2} dx$$

$$= \frac{1}{2}\int_0^1 y\,[y^4 - y]\,dy = \frac{1}{2}\int_0^1 (y^5 - y^2)\,dy$$

$$= \frac{1}{2}\left[\frac{y^6}{6} - \frac{y^3}{3}\right]_0^1 = \frac{1}{2}\left(\frac{1}{6}\right) = -\frac{1}{12} \; \#$$

Area in Cartesian co-ordinates :

Area enclosed by the two curves $y = f_1(x)$ and $y = f_2(x)$ and $x = a$ and $x = b$ is ABCD.

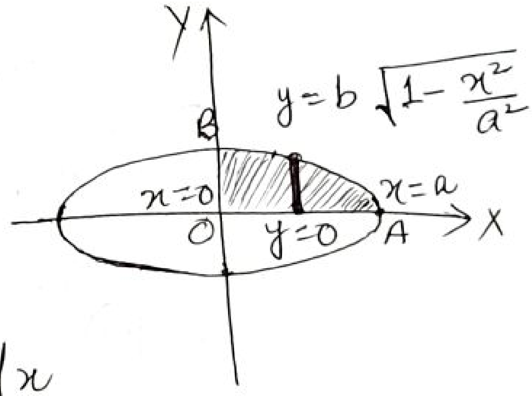


The area ABCD,

$$= \int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} dx\,dy$$

Q.1 find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Soln :



$$\text{Area OAB} = \int_{x=0}^{a} \int_{y=0}^{b\sqrt{1-x^2/a^2}} dx\,dy$$

$$= \int_0^a \left[\int_0^{b\sqrt{1-x^2/a^2}} 1 . \, dy \right] dx$$

$$= \int_0^a [y]_0^{b\sqrt{1-x^2/a^2}} dx$$

$$= \int_0^a b\sqrt{1-x^2/a^2}\, dx = \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\, dx$$

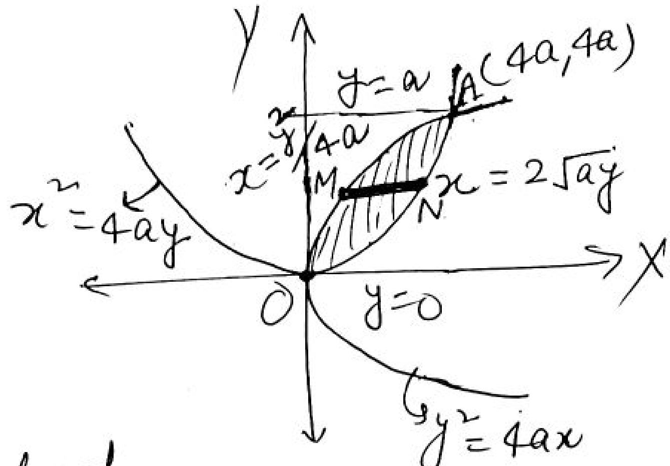
$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{b}{a}\left[\left(0+\frac{a^2}{2}\sin^{-1}1\right) - (0+0)\right] = \frac{b}{a} \times \frac{a^2}{2} \times \frac{\pi}{2} = \frac{\pi ab}{4}$$

Now required volume $= 4(\text{Area OAB}) = \pi ab$ #

Q.2 find the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Solⁿ



Required area ONAMO

$$= \int_{y=0}^{4a} \int_{x=y^2/4a}^{2\sqrt{ay}} dx\,dy$$

$$= \int_0^{4a} [x]_{y^2/4a}^{2\sqrt{ay}} dy = \int_0^{4a} \left(2\sqrt{ay} - y^2/4a\right) dy$$

$$= \left(2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{1}{4a}\frac{y^3}{3}\right)_0^{4a}$$

$$= \left(\frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{1}{12a}(4a)^3\right) - 0$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2 \quad \#$$

Q.3 find by double integration the area enclosed by the curve $y = x^2$ and $x + y = 2$.

Solⁿ Given parabola, $y = x^2$ ——— (1)

Straight line, $x + y = 2$ ——— (2)

Solving (1) & (2)

$x + x^2 = 2$

$\Rightarrow x^2 + x - 2 = 0$

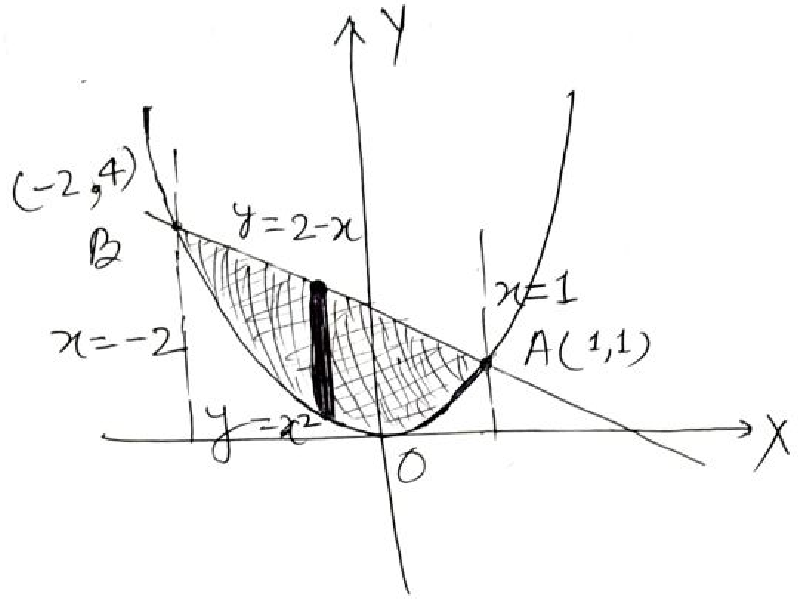
$\Rightarrow x = 1, -2$

$x = 1 \Rightarrow y = 1$

$x = -2 \Rightarrow y = 4$

$\Rightarrow A \to (1,1)$

$B \to (-2,4)$



The required area $= \iint dx\,dy$

$$= \int_{x=-2}^{1} \int_{y=x^2}^{2-x} dx\,dy$$

$$= \int_{-2}^{1} \left[\int_{x^2}^{2-x} 1\, dy \right] dx = \int_{-2}^{1} (y)_{x^2}^{2-x} dx$$

$$= \int_{-2}^{1} (2 - x - x^2)\, dx$$

$$= \left(2x - \frac{x^2}{2} - \frac{x^3}{3}\right)_{-2}^{1}$$

$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - \frac{4}{2} + \frac{8}{3}\right)$$

$$= 2 - \frac{1}{2} - \frac{1}{3} + 4 + 2 - \frac{8}{3}$$

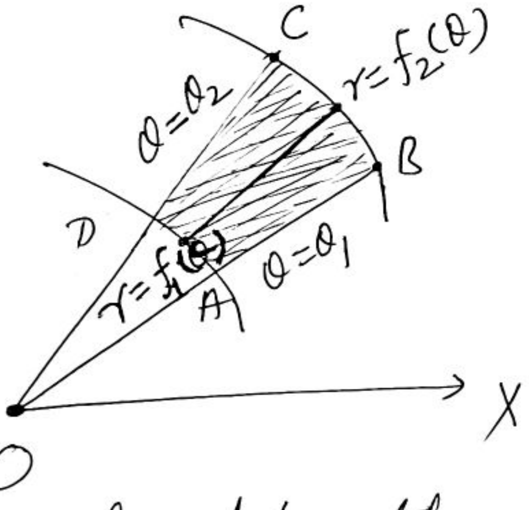
$$= \frac{9}{2}$$

#.

Area in polar co-ordinates:

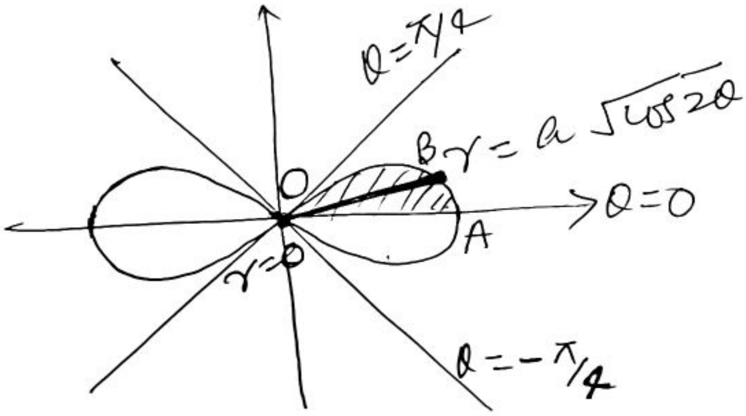
Area ABCD

$$= \int_{\theta=\theta_1}^{\theta_2} \int_{r=f_1(\theta)}^{f_2(\theta)} r \, d\theta \, dr.$$



Q-1 find the total area enclosed by the lemniscate of Bernoulli $r^2 = a^2 \cos 2\theta$.

Solⁿ:



Required area

$= 4$ (area OABO)

$$= 4 \int_{\theta=0}^{\pi/4} \int_{r=0}^{a\sqrt{\cos 2\theta}} r \, d\theta \, dr$$

$$= 4 \int_0^{\pi/4} \left[\int_0^{a\sqrt{\cos 2\theta}} r \, dr \right] d\theta$$

$$= 4 \int_0^{\pi/4} \left(\frac{r^2}{2} \right)_0^{a\sqrt{\cos 2\theta}} d\theta$$

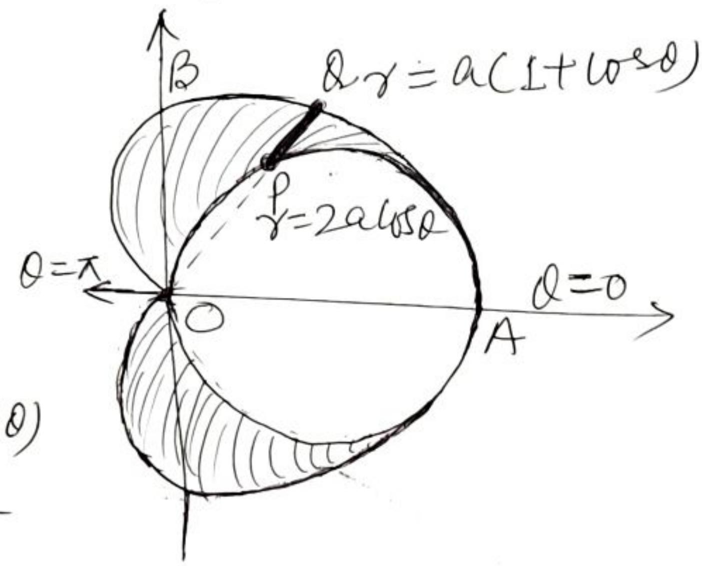
$$= \frac{4}{2} \int_0^{\pi/4} (a^2 \cos 2\theta - 0) \, d\theta$$

$$= 2a^2 \left(\frac{\sin 2\theta}{2} \right)_0^{\pi/4} = a^2 \sin \frac{\pi}{2} = a^2$$

Q.2 Find the area inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = 2a\cos\theta$.

Solⁿ: Given Cardioid, $r = a(1+\cos\theta)$ —— (1)

Circle, $r = 2a\cos\theta$ —— (2)



Required area

$$= 2\int_0^{\pi}\int_{2a\cos\theta}^{a(1+\cos\theta)} r\,d\theta\,dr$$

$$= 2\int_0^{\pi}\left(\frac{r^2}{2}\right)_{2a\cos\theta}^{a(1+\cos\theta)} d\theta$$

$$= \frac{\cancel{2}}{\cancel{2}}\int_0^{\pi}\left[a^2(1+\cos\theta)^2 - 4a^2\cos^2\theta\right]d\theta$$

$$= a^2\int_0^{\pi}\left[1+\cos^2\theta+2\cos\theta - 4\cos^2\theta\right]d\theta$$

$$= a^2\int_0^{\pi}\left(1-3\cos^2\theta+2\cos\theta\right)d\theta$$

$$= a^2\int_0^{\pi}\left[1-\frac{3}{2}(1+\cos 2\theta)+2\cos\theta\right]d\theta$$

$$= a^2\int_0^{\pi}\left(-\frac{1}{2}-\frac{3}{2}\cos 2\theta+2\cos\theta\right)d\theta$$

$$= a^2\left[-\frac{1}{2}\theta - \frac{3}{2}\frac{\sin 2\theta}{2} + 2\sin\theta\right]_0^{\pi}$$

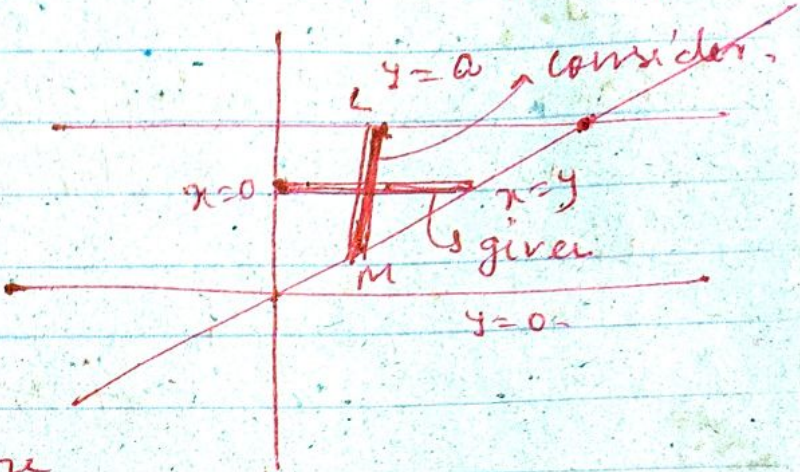
$$= a^2\left[\left(-\frac{\pi}{2} - \frac{3}{4}\sin 2\pi + 2\sin\pi\right) - 0\right] = -\frac{\pi a^2}{2} = \frac{\pi a^2}{2}$$

(neglecting $-$ve sign)

Change the order of integration

1. (a) $I = \int_0^a \int_0^y f(x,y)\, dx\, dy$

The region of integration is bounded by lines $y=0$, $y=a$ & $x=0$, $x=y$



consider a strip LM parallel to y-axis. Since the extremities of this strip lie on $y=x$ & $y=a$ so the limit of y are x to a. Also x varies from 0 to a.

Thus the integral after changing the order of integration takes the form

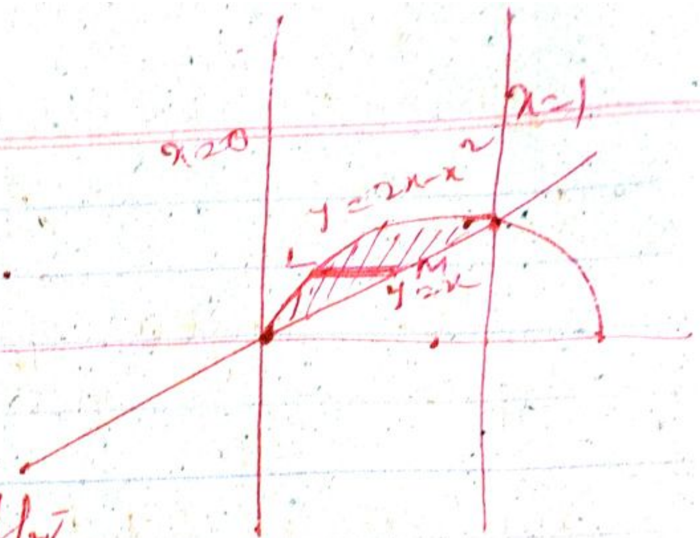
$$\int_0^a \int_x^a f(x,y)\, dy\, dx$$

(b) $I = \int_0^1 \int_x^{x(2-x)} f(x,y)\, dx\, dy$

The region of integration is bounded by $x=0$, $x=1$ & $y=x$, $y=2x-x^2$

consider a strip LM parallel to x-axis. whose extremities lie on $y=2x-x^2$ & $y=x$.

so the limit of x are $1-\sqrt{1-y}$ to y.



& y varies from 0 to 1

Thus the integral after changing the order of integration takes the form

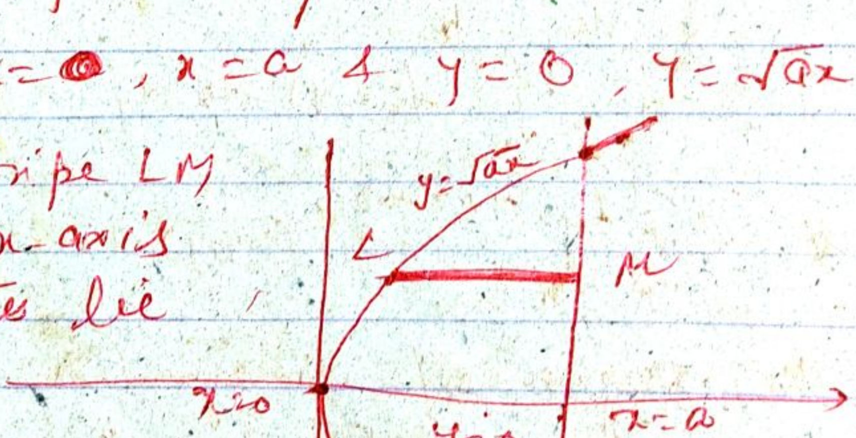
$$\int_0^1 \int_{1-\sqrt{1-y}}^{y} f(x,y)\, dy\, dx$$

(c) $I = \int_0^a \int_0^{\sqrt{ax}} (x^2+y^2)\, dx\, dy$

The region of integration is bounded by

$x=0$, $x=a$ & $y=0$, $y=\sqrt{ax}$ i.e. $y^2=ax$

Consider a stripe LM parallel to x-axis whose extremities lie on $y^2=ax$ & $x=a$



So the limit of x are $\frac{y^2}{a}$ to a

& y varies from 0 to a.

Thus the integral after changing the order of integration takes the form

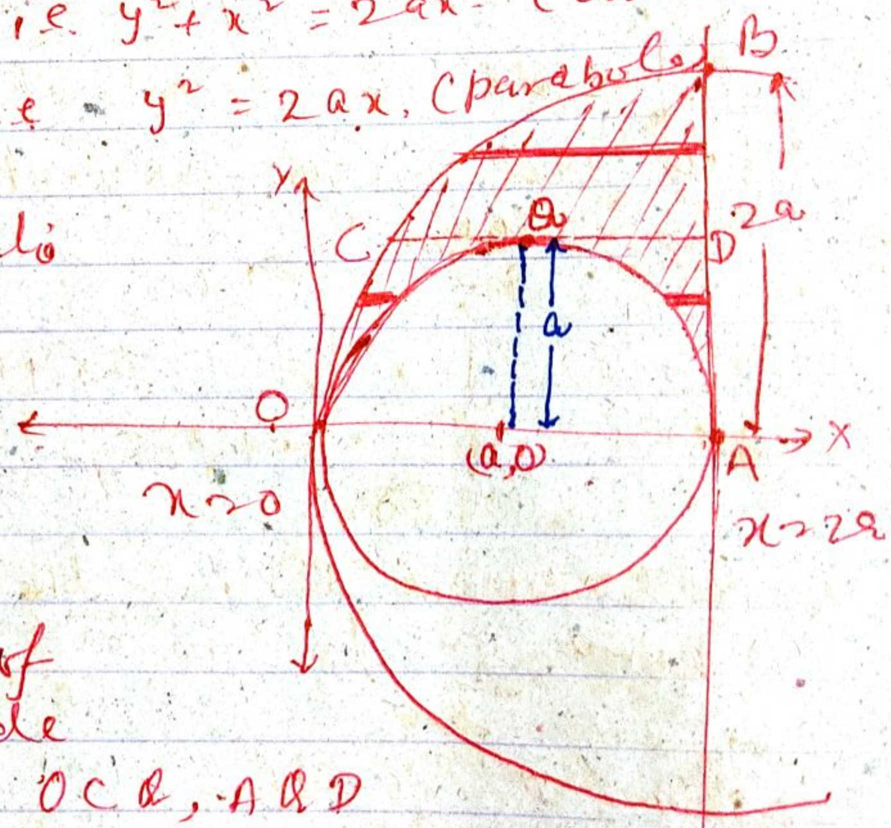
$$\int_0^a \int_{y^2/a}^{a} (x^2+y^2)\, dy\, dx.$$

(d) $I = \int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\, dx\, dy$

The region of integration is bounded by $x=0$, $x=2a$

& $y=\sqrt{2ax-x^2}$ i.e. $y^2+x^2=2ax$ (circle)

$y=\sqrt{2ax}$ i.e. $y^2=2ax$ (parabola)



The strips parallel to x axis change their character at Q so through Q draw a line CD parallel to x-axis. then the region of integration divide into three parts OCQ, AQD & CBD.

In region OCQ, the limits of x are from $\frac{y^2}{2a}$ to $a - \sqrt{a^2-y^2}$ & limits of y are from 0 to a.

Thus In region OCQ the integral after changing the order of integration takes the form

$$\int_0^a \int_{\frac{y^2}{2a}}^{a-\sqrt{a^2-y^2}} f(x,y)\, dy\, dx$$

In region AQD,

the limit of x are from $a+\sqrt{a^2-y^2}$ to 2a & limits of y are from 0 to a.

Thus in region AQD the integral takes the form

$$\int_0^a \int_{a+\sqrt{a^2-y^2}}^{2a} f(x,y)\, dy\, dx$$

In region CBP, the limit of x from $\frac{y^2}{2a}$ to 2a & limits of y are from a to 2a.

Thus in this region the integral becomes

$$\int_a^{2a} \int_{y^2/2a}^{2a} f(x,y)\, dy\, dx$$

Hence the integral after changing the order of integration,

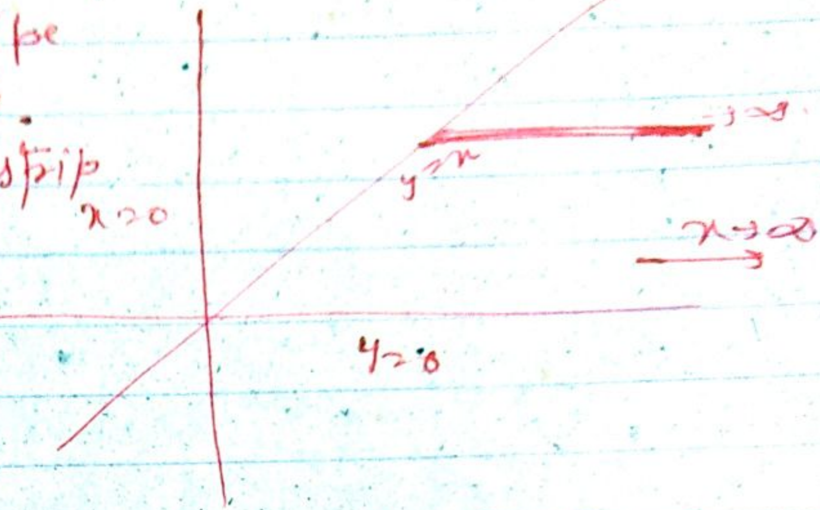
$$I = \int_0^a \int_{y^2/2a}^{a-\sqrt{a^2-y^2}} f(x,y)\,dy\,dx$$

$$+ \int_0^a \int_{a+\sqrt{a^2-y^2}}^{2a} f(x,y)\,dy\,dx$$

$$+ \int_a^{2a} \int_{y^2/2a}^{2a} f(x,y)\,dy\,dx$$

(2) $I = \int_0^\infty \int_0^x x e^{-x^2/y} dx dy$

The region of the integration is bounded by $y=0$, $y=x$ & $x=0$ $x=\infty$



Let us consider a stripe parallel to x-axis. one side of (extremity) strip is on $y=x$ & other side tends to ∞. So limit of x from y to ∞.

& ~~extremity~~ since y varies from 0 to ∞ so limit of y are from 0 to ∞.

Hence the integral after changing the order of integration becomes

$$I = \int_0^\infty \int_y^\infty x e^{-x^2/y} dy\, dx$$

Let $\frac{x^2}{y} = t$

$\Rightarrow \frac{2x dx}{y} = dt$

$\Rightarrow x dx = \frac{y}{2} dt$

$$= \int_0^\infty \left\{ \int_y^\infty x e^{-x^2/y} dx \right\} dy$$

$$= \int_0^\infty \frac{y}{2} \left\{ \int_y^\infty e^{-t} dt \right\} dy$$

$$= \int_0^\infty \frac{y}{2} \left\{ -e^{-t} \right\}_y^\infty dy$$

$$= \int_0^\infty \frac{y}{2} \left\{ -\left(e^{-\infty} - e^{-y} \right) \right\} dy$$

$$= \frac{1}{2} \int_0^\infty y e^{-y} \, dy$$

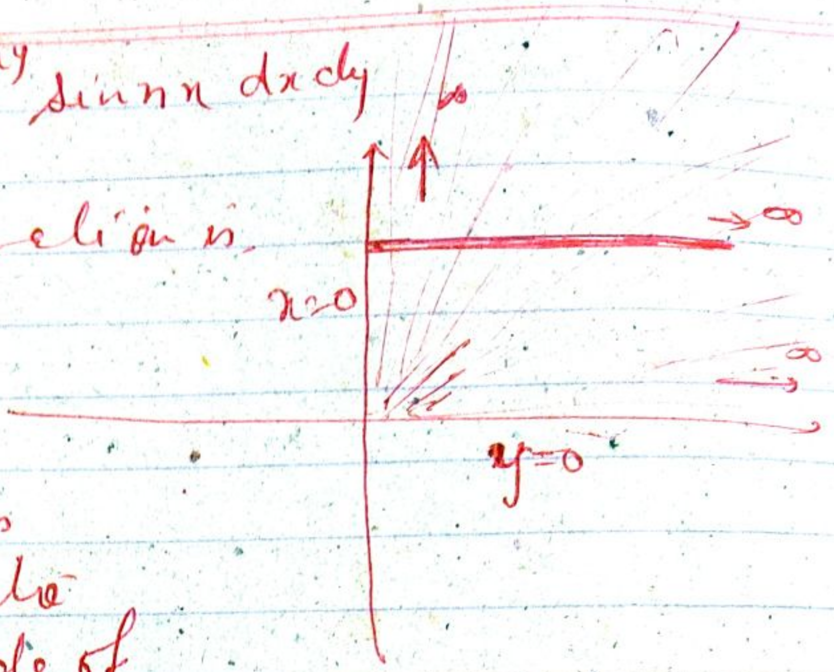
$$= \frac{1}{2} \left\{ y \cdot \left(-e^{-y} \right) - e^{-y} \right\}_0^\infty$$

$$= \frac{1}{2} \left\{ 0 + 1 \right\}$$

$$= \frac{1}{2} \quad \#$$

(7)

$$I = \int_0^\infty \int_0^\infty e^{-xy} \sin nx \, dx\, dy$$



The region of integration is

$x=0$ to $x \to \infty$

& $y=0$, $y \to \infty$.

Let us consider a strip parallel to x-axis. One side of strip is on $x=0$ & other side tends to ∞.

So limit of x are from 0 to ∞.

Also range of y is from 0 to ∞

So limit of y are from 0 to ∞

Hence the integral after changing the order of integration becomes

$$I = \int_{y=0}^{y\to\infty} \int_{x=0}^{x\to\infty} e^{-xy} \sin nx \, dy\, dx.$$

Now

$$I = \int_0^\infty \sin nx \left\{ \int_0^\infty e^{-xy}\, dy \right\} dx$$

$$= \int_0^\infty \sin nx \left(-\frac{1}{x}\right) \left(e^{-xy}\right)_0^\infty$$

$$= \int_0^\infty \frac{\sin nx}{x} \{-0+1\}\, dx$$

$$= \int_0^\infty \frac{\sin nx}{x}\,dx \quad \text{——} (1)$$

Also

$$\int_0^\infty \int_0^\infty e^{-xy} \sin nx \, dx \, dy$$

$$= \int_0^\infty \left[\int_0^\infty e^{-xy} \sin nx \, dx \right] dy$$

$$\left\{ \begin{array}{l} \text{Using formula} \\ \int e^{-ax} \sin bx \, dx = -\frac{e^{-ax}}{a^2+b^2} [a \sin bx + b \cos bx] \end{array} \right\}$$

$$= \int_0^\infty \left[-\frac{e^{-xy}}{n^2+y^2} (y \sin nx + n \cos nx) \right]_0^\infty dy$$

$$= \int_0^\infty \frac{n}{n^2+y^2} dy = \left[\tan^{-1}\left(\frac{y}{n}\right) \right]_0^\infty$$

$$= \tan^{-1}\infty - \tan^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2} \quad \text{——} (2)$$

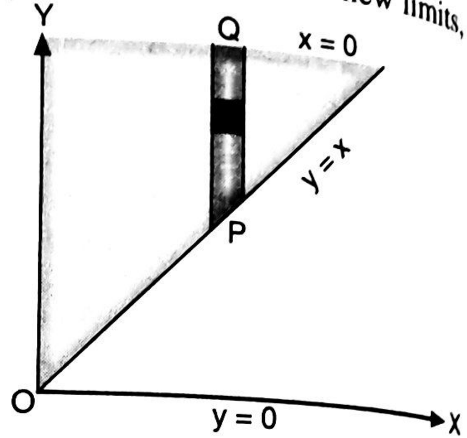
From (1) & (2)

$$\boxed{\int_0^\infty \frac{\sin nx}{x}\,dx = \frac{\pi}{2}}$$

14.1 CHANGE OF ORDER OF INTEGRATION

On changing the order of integration, the limits of integration change. To find the new limits, we draw the rough sketch of the region of integration.

Some of the problems connected with double integrals, which seem to be complicated, can be made easy to handle by a change in the order of integration.

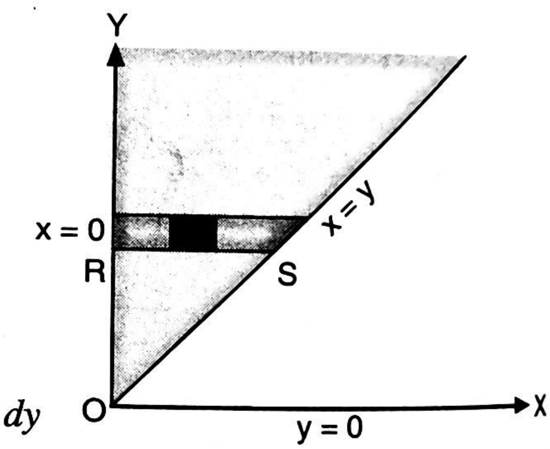


Example 1. *Evaluate :* $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy$.

Solution. We have, $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy$

Here the elementary strip PQ extends from $y = x$ to $y = \infty$ and this vertical strip slides from $x = 0$ to $x = \infty$. The shaded portion of the figure is, therefore, the region of integration.

On changing the order of integration, we first integrate w.r.t. x along a horizontal strip RS which extends from $x = 0$ to $x = y$. To cover the given region, we then integrate w.r.t. 'y' from $y = 0$ to $y = \infty$.



Thus
$$\int_0^\infty dx \int_x^\infty \frac{e^{-y}}{y}\, dy = \int_0^\infty \frac{e^{-y}}{y}\, dy \int_0^y dx$$

$$= \int_0^\infty \frac{e^{-y}}{y}\, dy\, [x]_0^y$$

$$= \int_0^\infty y\frac{e^{-y}}{y}\, dy = \int_0^\infty e^{-y}\, dy$$

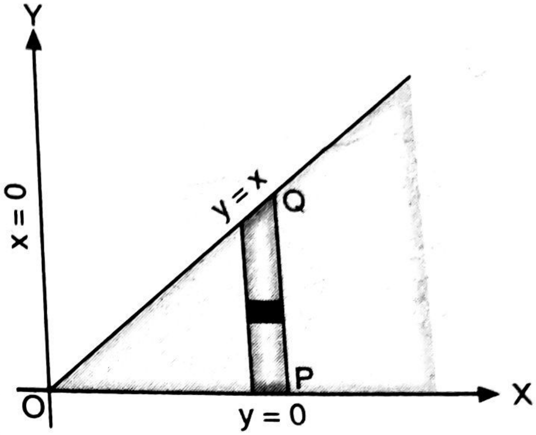
$$= \left[\frac{e^{-y}}{-1}\right]_0^\infty = -\left[\frac{1}{e^y}\right]_0^\infty = -\left[\frac{1}{\infty} - 1\right] = 1$$ **Ans.**

Example 2. *Evaluate the integral* $\int_0^\infty \int_0^x x \exp\left(-\frac{x^2}{y}\right) dx\, dy$ *by changing the order of integration.*

(U.P. I Semester, Dec., 2005)

Solution. Limits are given

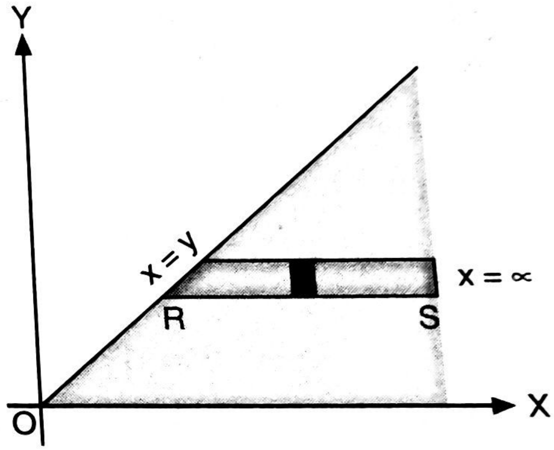
$$y = 0 \text{ and } y = x$$
$$x = 0 \text{ and } x = \infty$$



Here, the elementary strip PQ extends from $y = 0$ to $y = x$ and this vertical strip slides from $x = 0$ to $x = \infty$.

The region of integration is shown by shaded portion in the figure bounded by $y = 0$, $y = x$, $x = 0$ and $x = \infty$.

On changing the order of integration, we first integrate with respect to x along a horizontal strip RS which extends from $x = y$ to $x = \infty$ and this horizontal strip slides from $y = 0$ to $y = \infty$ to cover the given region of integration.



New limits:

$$x = y \quad \text{and} \quad x = \infty$$
$$y = 0 \quad \text{and} \quad y = \infty$$

On changing the order we first integrate with respect to x and then y.

Thus,

$$\int_0^\infty \int_0^x x \cdot e^{-\frac{x^2}{y}}\, dy\, dx = \int_0^\infty dy \int_y^\infty x\, e^{-\frac{x^2}{y}}\, dx = \int_0^\infty dy \int_y^\infty -\frac{y}{2}\left(-\frac{2x}{y}\, e^{-\frac{x^2}{y}}\right) dx$$

$$= \int_0^\infty dy \left[-\frac{y}{2} e^{-\frac{x^2}{y}}\right]_y^\infty = \int_0^\infty dy \left[0 + \frac{y}{2} e^{-\frac{y^2}{y}}\right] = \int_0^\infty \frac{y}{2} e^{-y}\, dy$$

$$= \left[\frac{y}{2}(-e^{-y}) - \left(\frac{1}{2}\right)(e^{-y})\right]_0^\infty \qquad \text{(Integrating by parts)}$$

$$= \left[(0-0) + \left(0 + \frac{1}{2}\right)\right] = \frac{1}{2} \qquad \textbf{Ans.}$$

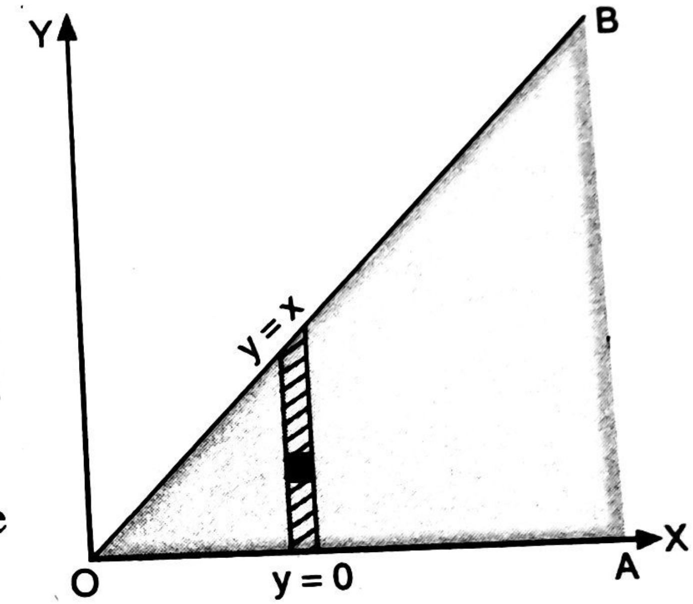
Example 5. *Change the order of the integration*

$$\int_0^{\infty}\int_0^{x} e^{-xy}\; y\,dy\,dx$$

(*B.P.U.T.; I Semester 2008*)

Solution. Here, we have

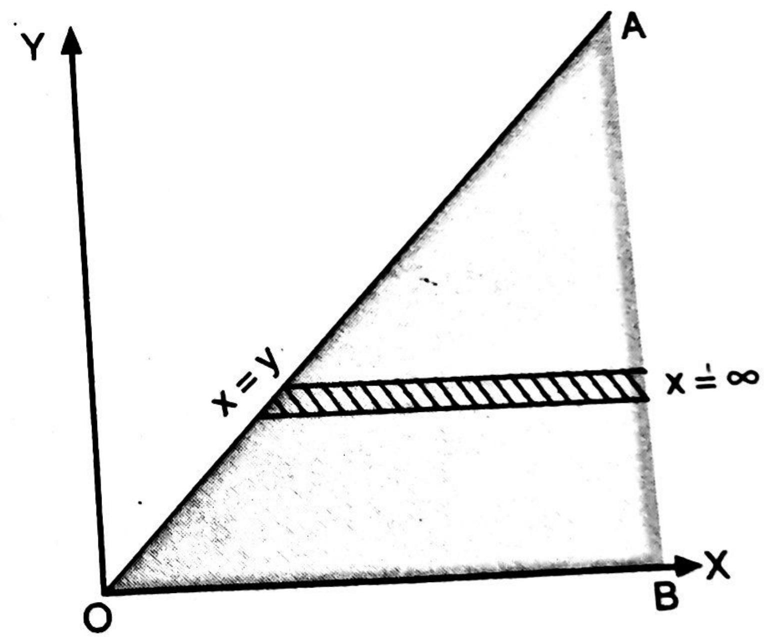
$$\int_0^{\infty}\int_0^{x} e^{-xy}\; y\,dy\,dx$$



Here the region OAB of integration is bounded by $y=0$ (x-axis), $y=x$ (a straight line), $x=0$, *i.e.*, y axis. A strip is drawn parallel to y-axis, y vares 0 to x and x varies 0 to ∞.

On changing the order of integration, first we integrate w.r.t. x and then w.r.t. y.

A strip is drawn parallel to x-axis. On this strip x varies from y to ∞ and y varies from 0 to ∞.



$$\text{Hence}\quad \int_0^{\infty}\int_0^{x} e^{-xy}\; y\,dy\,dx = \int_0^{\infty} y\,dy \int_y^{\infty} e^{-xy}dx$$

$$= \int_0^{\infty} y\,dy\left(\frac{e^{-xy}}{-y}\right)_y^{\infty}$$

$$= \int_0^{\infty} \frac{y\,dy}{-y}\,[0-e^{y^2}]$$

$$= \int_0^{\infty} e^{-y^2}\,dy = \frac{1}{2}\sqrt{\pi}$$

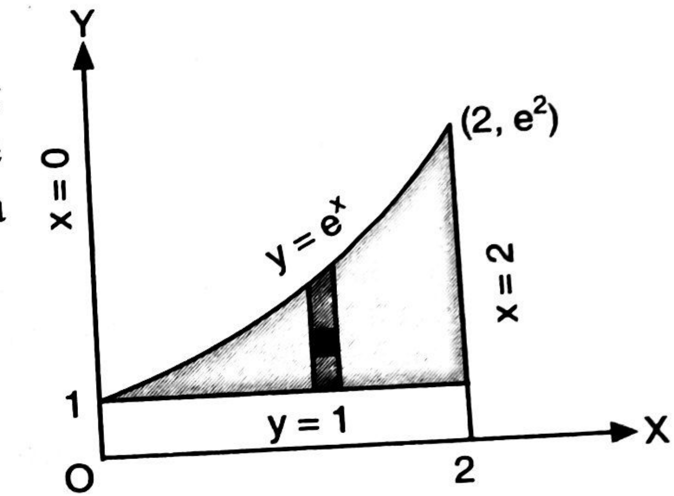
Ans.

Example 10. *Evaluate* $\int_0^2 \int_1^{e^x} dx\, dy$ *by changing the order of integration.*

(U.P., I Semester, 2003)

Solution. The given limits show that the region of integration is bounded by curves $x = 0$, $x = 2$, $y = 1$ and $y = e^x$.

The area of integration is shaded in the diagram.



On chaning the order of integration we integrate first w.r.t x and then y. A horizontal strip is drawn parallel to x axis. At the ends of the strip $x = \log y$ and $x = 2$. To cover the whole area of integration the strip slides from $y = 1$ and $y = e^x$.

$$\therefore \text{Given integral} = \int_1^{e^2} dy \int_{\log y}^{2} dx$$

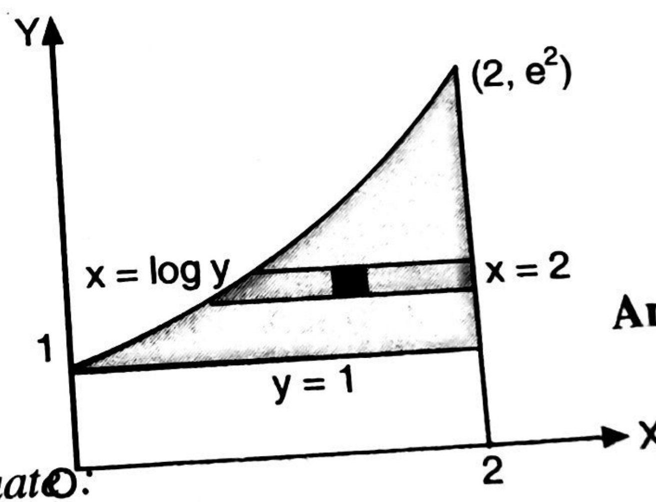
$$= \int_1^{e^2} [x]_{\log y}^{2}\, dy = \int_1^{e^2} (2 - \log y)\, dy$$

$$= (2y - y \log y + y)_1^{e^2}$$

$$= (3y - y \log y)_1^{e^2} \qquad [\log e^2 = 2]$$

$$= (3e^2 - 2e^2) - 3 = e^2 - 3.$$

Ans.



A

f integration, evaluate:

$$= \int_0 \int_0 f(x, y)\, dy\, dx + \int_2 \int_0 f(x, y)\, dy\, dx.$$

Ans.

Example 20. *Change the order of integration in*

$$I = \int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy$$

and hence evaluate the same. (A.M.I.E.T.E. June 2009, U.P., I Sem., Dec., 2004)

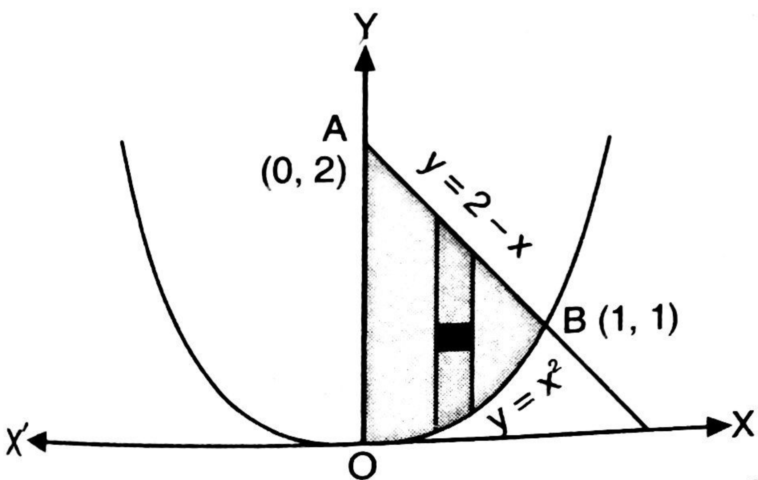
Solution. We have $$\mathrm{I} = \int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy$$

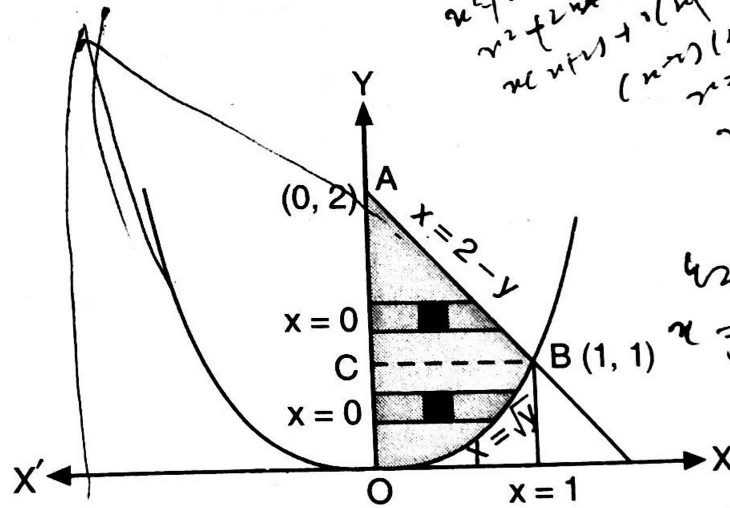
The region of integration is shown by shaded portion in the figure bounded by parabola $y = x^2$, $y = 2 - x$ (st line), $x = 0$ (y – axis)

The point of intersection of the parabola $y = x^2$ and the line $y = 2 - x$ is B (1, 1).

In the figure below (left) we draw a strip parallel to y-axis and the strip y, varies from x^2 to $2 - x$ and x varies from 0 to 1.

$$\int_0^1 x\, dx \int_{x^2}^{2-x} y\, dy$$





On changing the order of integration we have taken a strip parallel to x–axis in the area OBC and second strip in the area CBA. The limits of x in the area OBC are 0 and $\sqrt{y}$ and the limits of x in the area CBA are 0 and $2 - y$.

So, the given integral is

$$= \int_0^1 y\,dy \int_0^{\sqrt{y}} x\,dx + \int_1^2 y\,dy \int_0^{2-y} x\,dx = \int_0^1 y\,dy \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} + \int_1^2 y\,dy \left[\frac{x^2}{2}\right]_0^{2-y}$$

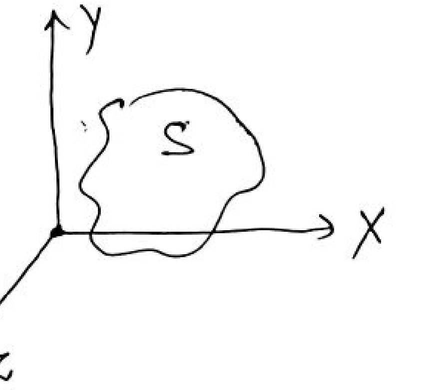
$$= \frac{1}{2}\int_0^1 y^2\,dy + \frac{1}{2}\int_1^2 y(2-y)^2\,dy = \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_1^2 (4y-4y^2+y^3)$$

$$= \frac{1}{6} + \frac{1}{2}\left[2y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right]_1^2 = \frac{1}{6} + \frac{1}{2}\left[8 - \frac{32}{3} + 4 - 2 + \frac{4}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{96-128+48-24+16-3}{12}\right] = \frac{1}{6} + \frac{5}{24} = \frac{9}{24} = \frac{3}{8}$$ **Ans.**

Triple integration:

Let a function $f(x,y,z)$ be continuous at every point of a finite region S of three dimensional space.



$$\int_x\int_y\int_z f(x,y,z)\,dx\,dy\,dz$$

$$= \int_{x=a}^{b}\int_{y=\phi_1(x)}^{\phi_2(x)}\int_{z=\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_{x=a}^{b}\left[\int_{y=\phi_1(x)}^{\phi_2(x)}\left\{\int_{z=\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)\,dz\right\}dy\right]dx$$

Q-1 Evaluate $\int_0^1\int_1^2\int_2^3 (x+y+z)\,dx\,dy\,dz$

Solⁿ: $\int_0^1\int_1^2\int_2^3 (x+y+z)\,dx\,dy\,dz$

$$= \int_0^1\int_1^2\left[\int_2^3 (x+y+z)\,dz\right]dx\,dy$$

$$= \int_0^1\int_1^2\left[\frac{(x+y+z)^2}{2}\right]_2^3 dx\,dy$$

$$= \frac{1}{2}\int_0^1\int_1^2\left[(x+y+3)^2-(x+y+2)^2\right]dx\,dy$$

$$= \frac{1}{2}\int_0^1\int_1^2\{(x+y+3)+(x+y+2)\}\{\cancel{x}+\cancel{y}+3-\cancel{x}-\cancel{y}-2\}\,dx\,dy$$

$$= \frac{1}{2}\int_0^1\int_1^2 (2x+2y+5)\cdot 1\,dx\,dy$$

$$= \frac{1}{2}\int_0^1 \left[\int_1^2 (2x+2y+5)\,dy\right] dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{(2x+2y+5)^2}{4}\right]_1^2 dx$$

$$= \frac{1}{8}\int_0^1 \left[(2x+2\cdot 2+5)^2 - (2x+2\cdot 1+5)^2\right] dx$$

$$= \frac{1}{8}\int_0^1 \left[(2x+9)^2 - (2x+7)^2\right] dx$$

$$= \frac{1}{8}\int_0^1 \left[(2x+9)+(2x+7)\right]\left[(2x+9)-(2x+7)\right] dx$$

$$= \frac{1}{8}\int_0^1 [4x+16][2]\, dx$$

$$= \frac{\cancel{4}\times\cancel{2}}{\cancel{8}}\int_0^1 (x+4)\,dx$$

$$= \left[\frac{(x+4)^2}{2}\right]_0^1 = \frac{1}{2}\left[(1+4)^2 - (0+4)^2\right]$$

$$= \frac{1}{2}(25-16) = \frac{9}{2} \quad \#.$$

Q.2 Evaluate

$$\iiint_R (x-2y+z)\,dx\,dy\,dz$$

where $R: 0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le x+y$

Solu^n: $\iiint_R (x-2y+z)\,dx\,dy\,dz$

$$= \int_0^1 \int_0^{x^2} \int_0^{x+y} (x-2y+z)\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^{x^2} \left[\int_0^{x+y} (x-2y+z)\,dz\right] dx\,dy$$

$$= \int_0^1 \int_0^{x^2} \left[\frac{(x-2y+z)^2}{2}\right]_0^{x+y} dx\,dy$$

$$= \frac{1}{2}\int_0^1 \int_0^{x^2} [(x-2y+x+y)^2 - (x-2y+0)^2]\, dx\, dy$$

$$= \frac{1}{2}\int_0^1 \int_0^{x^2} [(2x-y)^2 - (x-2y)^2]\, dx\, dy$$

$$= \frac{1}{2}\int_0^1 \int_0^{x^2} [(2x-y+x-2y)(2x-y-x+2y)]\, dx\, dy$$

$$= \frac{1}{2}\int_0^1 \int_0^{x^2} (3x-3y)(x+y)\, dx\, dy$$

$$= \frac{3}{2}\int_0^1 \int_0^{x^2} (x^2-y^2)\, dx\, dy$$

$$= \frac{3}{2}\int_0^1 \left[\int_0^{x^2} (x^2-y^2)\, dy \right] dx$$

$$= \frac{3}{2}\int_0^1 \left[x^2 y - \frac{y^3}{3} \right]_0^{x^2} dx$$

$$= \frac{3}{2}\int_0^1 \left(x^2 \cdot x^2 - \frac{x^6}{3} \right) dx$$

$$= \frac{3}{2}\int_0^1 \left(x^4 - \frac{x^6}{3} \right) dx$$

$$= \frac{3}{2}\left(\frac{x^5}{5} - \frac{x^7}{21} \right)_0^1$$

$$= \frac{3}{2}\left(\frac{1}{5} - \frac{1}{21} \right)$$

$$= \frac{8}{35}$$

Ans.

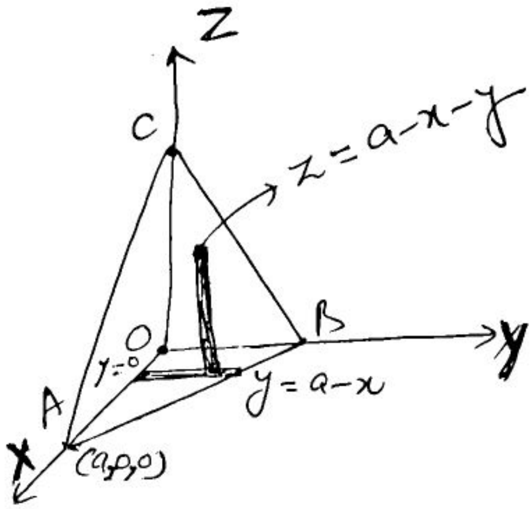
Volume of the solids :

$$\boxed{V = \iiint dx\, dy\, dz}$$

Q.1 find the volume of the tetrahedran bounded by the planes $x=0, y=0, z=0$ and $x+y+z=a$.

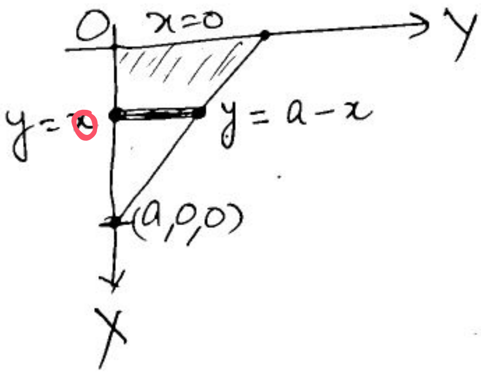
Solu^n

$$V = \int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y} dx\, dy\, dz$$



$$= \int_{x=0}^{a}\int_{y=0}^{a-x}\left[\int_{z=0}^{a-x-y} 1.dz\right] dx\,dy$$

$$= \int_0^a\int_0^{a-x}\left[Z\right]_0^{a-x-y} dx\,dy$$



$$= \int_0^a\int_0^{a-x}(a-x-y)\,dx\,dy$$

$$= \int_0^a\left[\int_0^{a-x}(a-x-y)\,dy\right]dx$$

$$= \int_0^a\left[\frac{(a-x-y)^2}{2}\right]_0^{a-x} dx$$

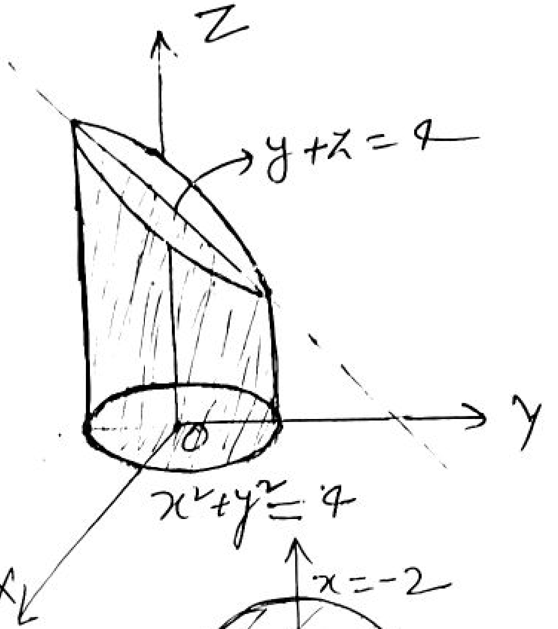
$$= \frac{1}{2}\int_0^a\left[(\cancel{a}-\cancel{x}-\cancel{a}+\cancel{x})^2-(a-x-0)^2\right]dx$$

$$= \frac{1}{2}\int_0^a\left[0-(a-x)^2\right]dx = \frac{1}{2}\left[-\frac{(a-x)^3}{3}\right]_0^a = -\frac{1}{6}\left[0-a^3\right]$$

$$= \frac{a^3}{6} \ \#$$

Q-2 Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$

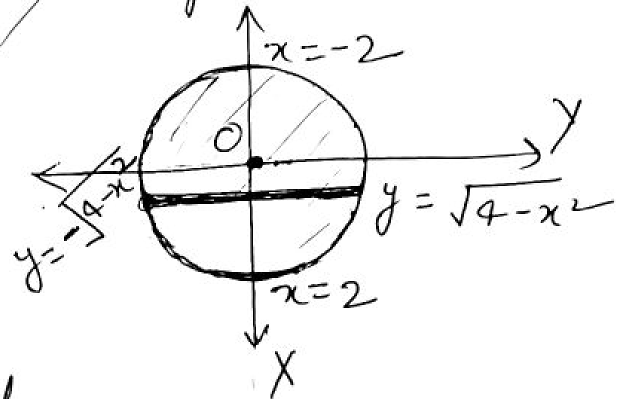
Solu^n. Given



$$x^2+y^2=4 \Rightarrow y=\pm\sqrt{4-x^2}$$

$$y+z=4 \Rightarrow z=4-y$$

$$z=0$$



$$V=\iiint dx\,dy\,dz$$

$$=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-y} dx\,dy\,dz$$

$$=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[\int_0^{4-y} 1\,dz\right]dx\,dy$$

$$=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[z\right]_0^{4-y} dx\,dy$$

$$=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4-y)\,dx\,dy$$

$$=\int_{-2}^{2}\left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4-y)\,dy\right]dx$$

$$= -\frac{1}{2}\int_{-2}^{2} \left[(4-y)^2\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= -\frac{1}{2}\int_{-2}^{2} \left[\left(4-\sqrt{4-x^2}\right)^2 - \left(4+\sqrt{4-x^2}\right)^2\right] dx$$

$$= -\frac{1}{2}\int_{-2}^{2} \left[\left(4-\sqrt{4-x^2}\right)+\left(4+\sqrt{4-x^2}\right)\right]\left[\left(4-\sqrt{4-x^2}\right)-\left(4+\sqrt{4-x^2}\right)\right] dx$$

$$= -\frac{1}{2}\int_{-2}^{2} [8]\cdot\left[-2\sqrt{4-x^2}\right] dx$$

$$= \frac{8}{\cancel{2}}\times\cancel{2}\int_{-2}^{2} \sqrt{4-x^2}\, dx$$

$$= 8\int_{-2}^{2} \sqrt{(2)^2-(x)^2}\, dx$$

$$= 8\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}$$

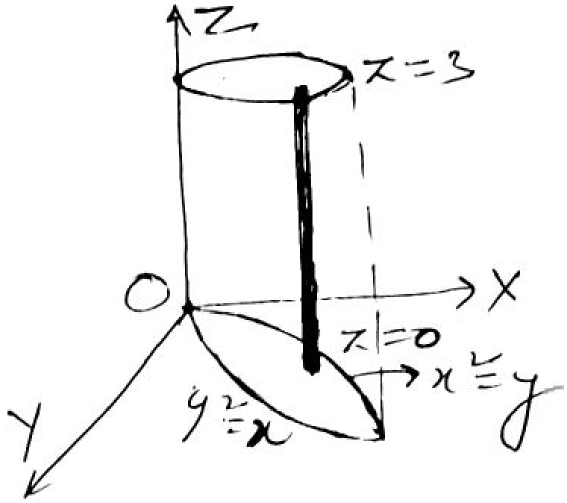
$$= 8\left[\left\{0+2\sin^{-1}(1)\right\}-\left\{0+2\sin^{-1}(-1)\right\}\right]$$

$$= 8\left[\left(\cancel{2}\frac{\pi}{\cancel{2}}\right)-\left(-\frac{\cancel{2}\pi}{\cancel{2}}\right)\right]$$

$$= 8\left[\pi+\pi\right)$$

$$= 8\times 2\pi = 16\pi$$

Q.3 Find the volume of the region bounded by the surface $y = x^2$, $x = y^2$ and the planes $z = 0$, $z = 3$.

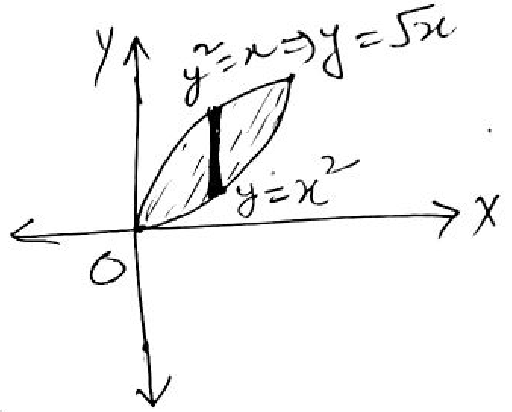


Solun:

Given

$y = x^2$, $x = y^2$, $z = 0$, $z = 3$

Volume



$$V = \iiint dx\,dy\,dz$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^3 dx\,dy\,dz$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} [z]_0^3 \,dx\,dy$$

$$= 3\int_0^1 \left[\int_{x^2}^{\sqrt{x}} 1\,dy\right] dx = 3\int_0^1 [y]_{x^2}^{\sqrt{x}} dx$$

$$= 3\int_0^1 (\sqrt{x} - x^2)\,dx$$

$$= 3\left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right]_0^1$$

$$= 3\left[\frac{2}{3} - \frac{1}{3}\right]$$

$$= \cancel{3}\left[\frac{1}{\cancel{3}}\right]$$

$$= 1$$