DS-GA 3001.001

Probabilistic time series analysis

Lecture 5 Hidden Markov Models

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NYU, CNS & CDS

1. **Project proposals** due **next Wed (10/16, 6pm)**

2. **HW2** (LDS), due week **Oct 11;** Artie there for office hours (Thu)

3. Weird time for class next week: **TUE Oct.15!!!**

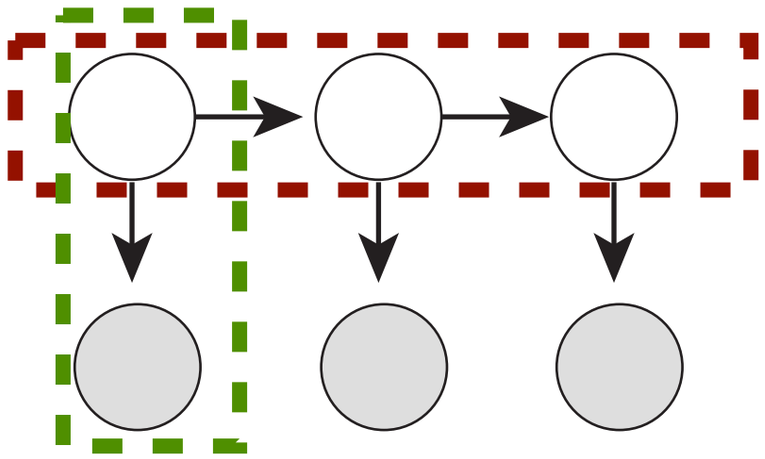
4. I'll need to change time for office for hours this and next week

feedback: https://nyu.qualtrics.com/jfe/form/SV_6JBVlDDk8xB5L8x

Latent state models

In the latent space the temporal dependencies are simple:

1st order Markov



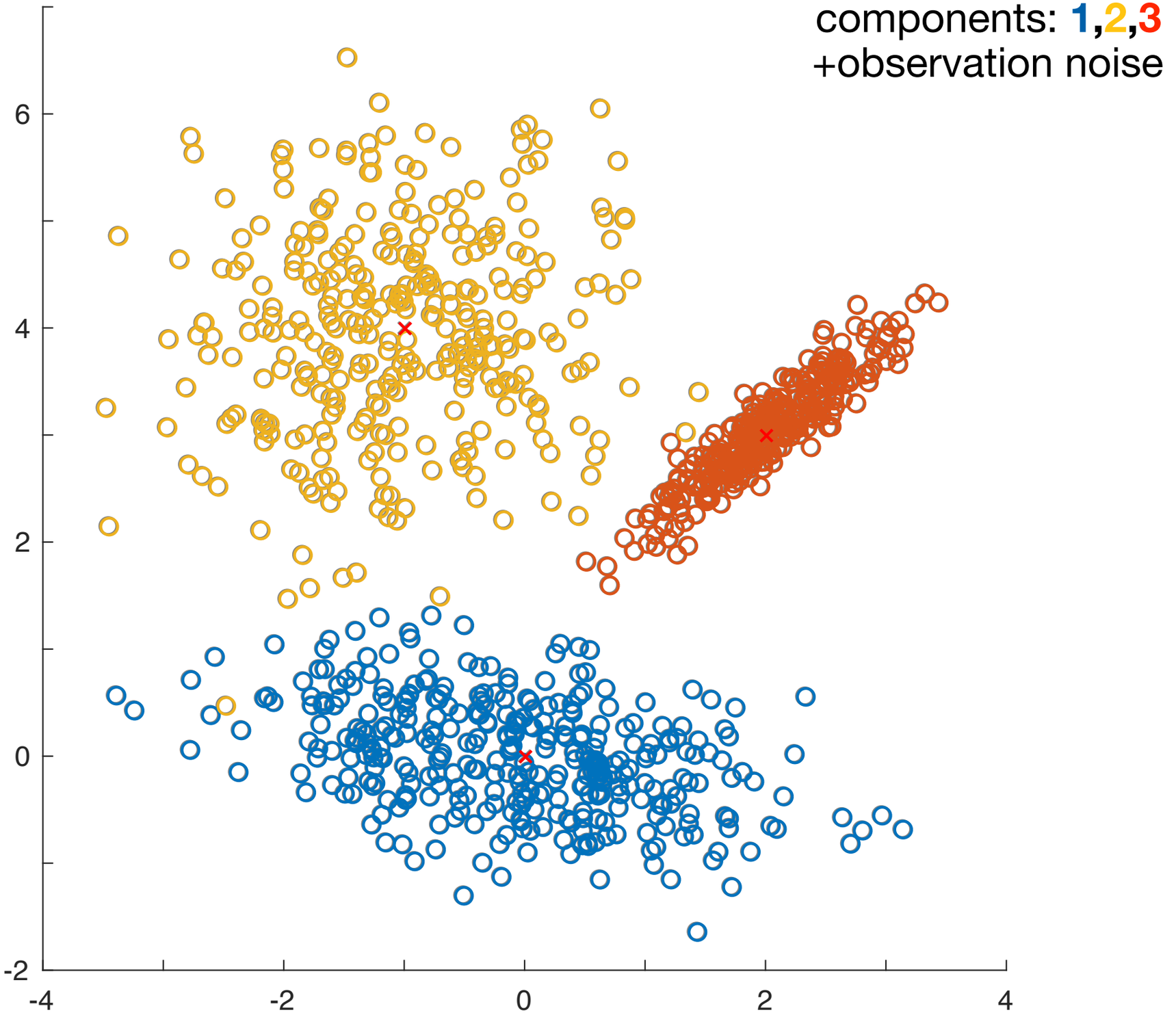
Goal: find a **latent** variable
that summarizes
the relevant **history**
while keeping math simple

continuous z, linear gaussian: **Kalman filtering**

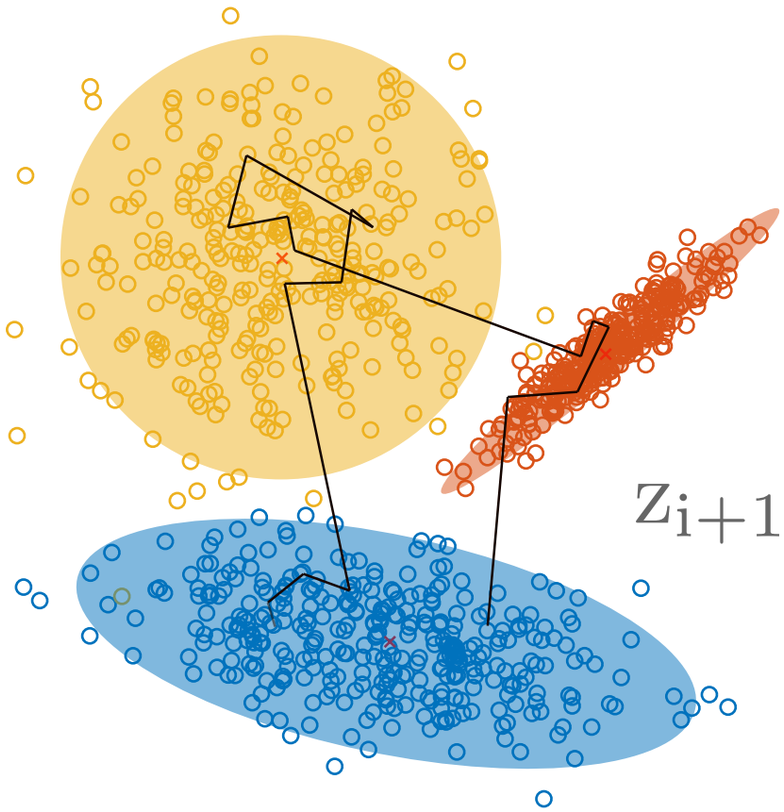
continuous z, (non)linear/(non)gaussian: **particle filtering**

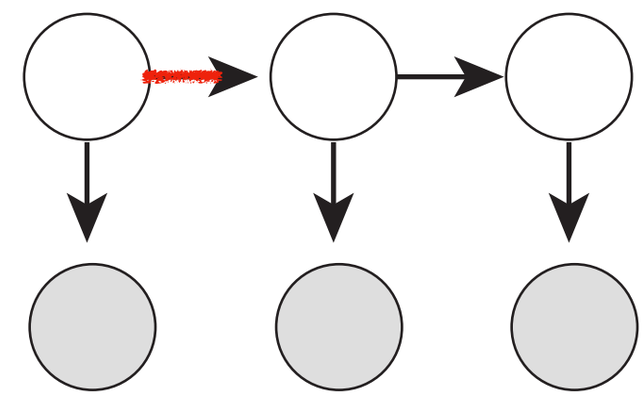
discrete z: **hidden Markov models** (HMMs)

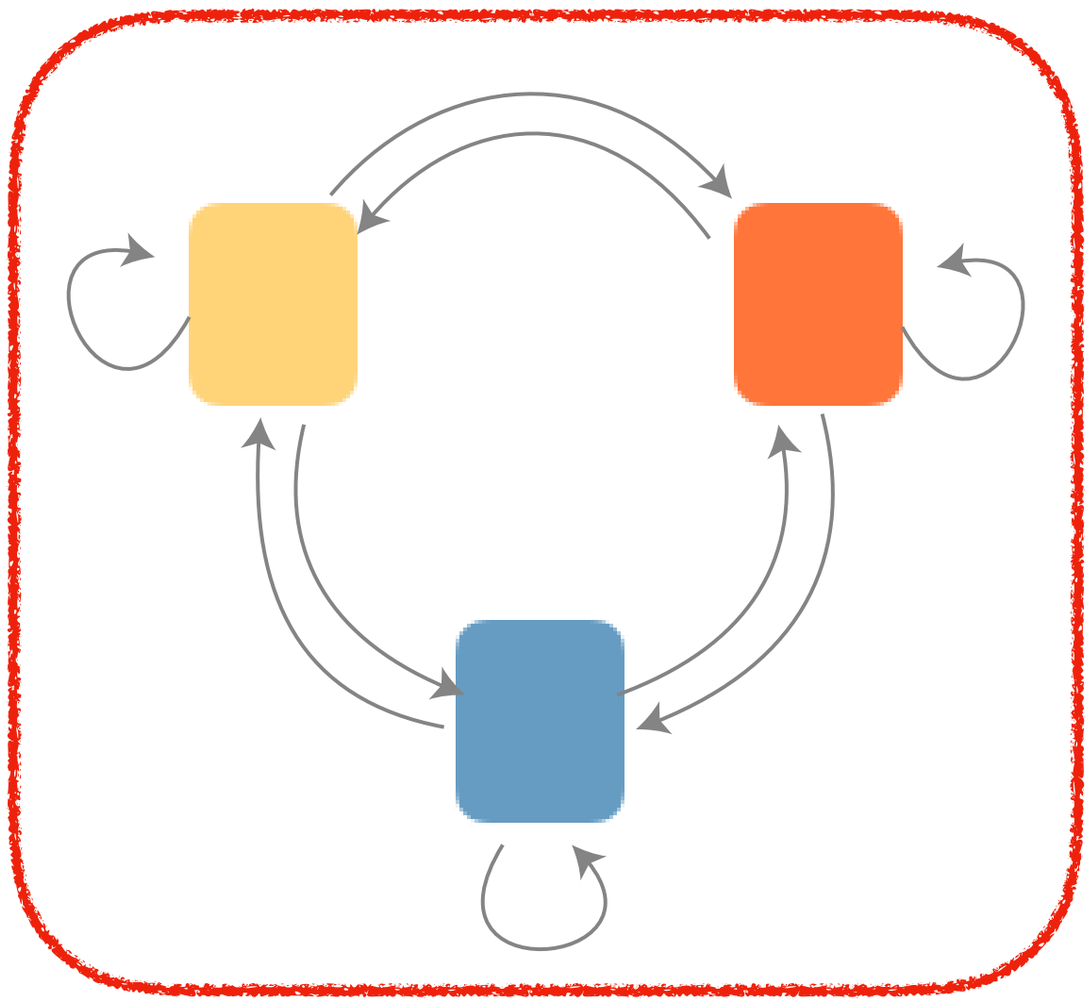
Reminder: mixture models



Hidden Markov models: mixture models, with **temporal dependencies** for components



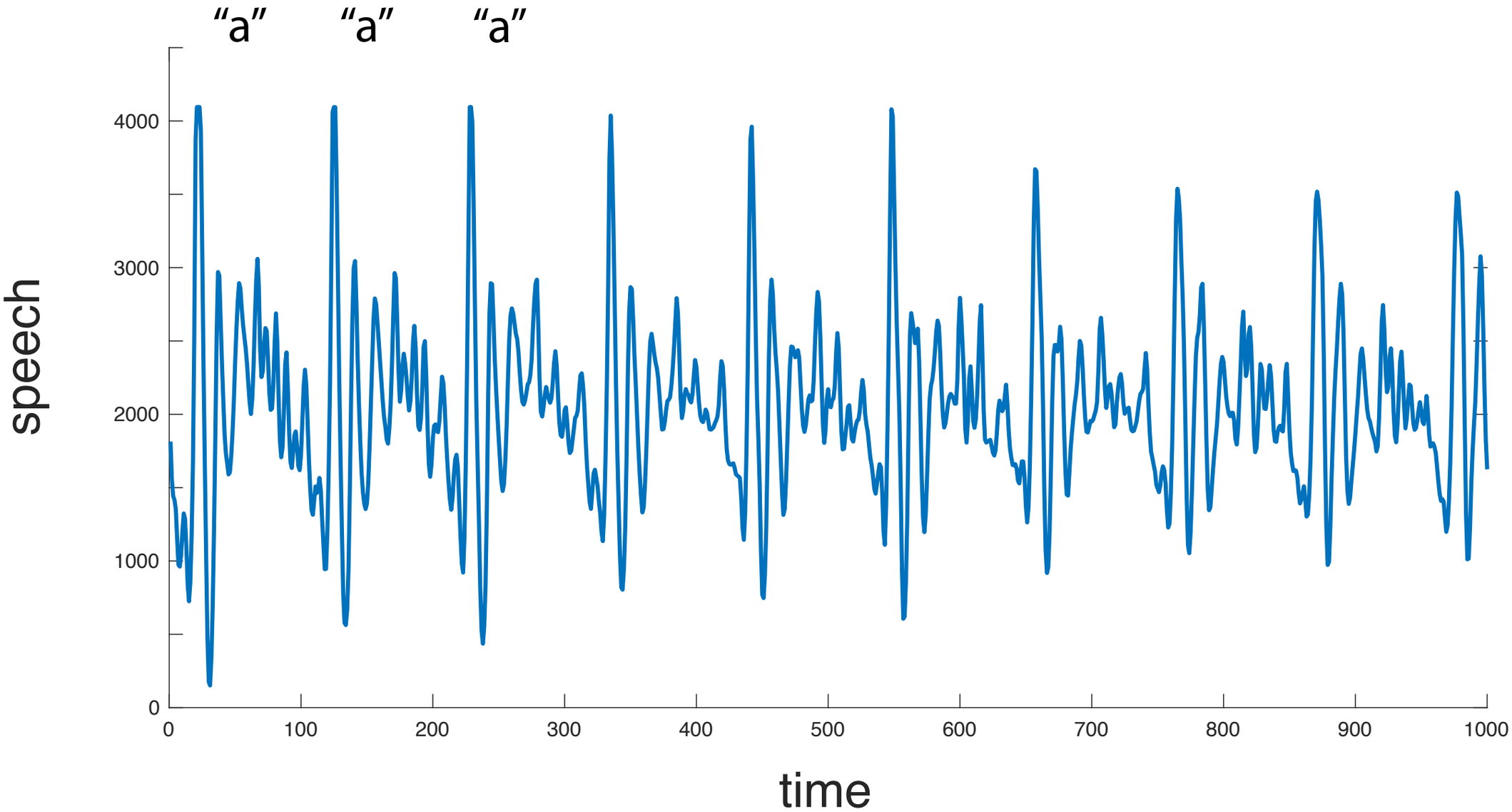




Markov(1913) used frequency counts to compute the probability that the next letter is a vowel?

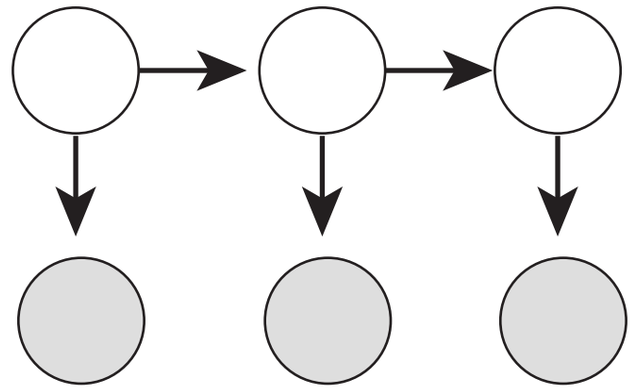
Applications:

speech recognition
natural language modeling
online handwritten recognition
structure of proteins, DNA



Voice recognition

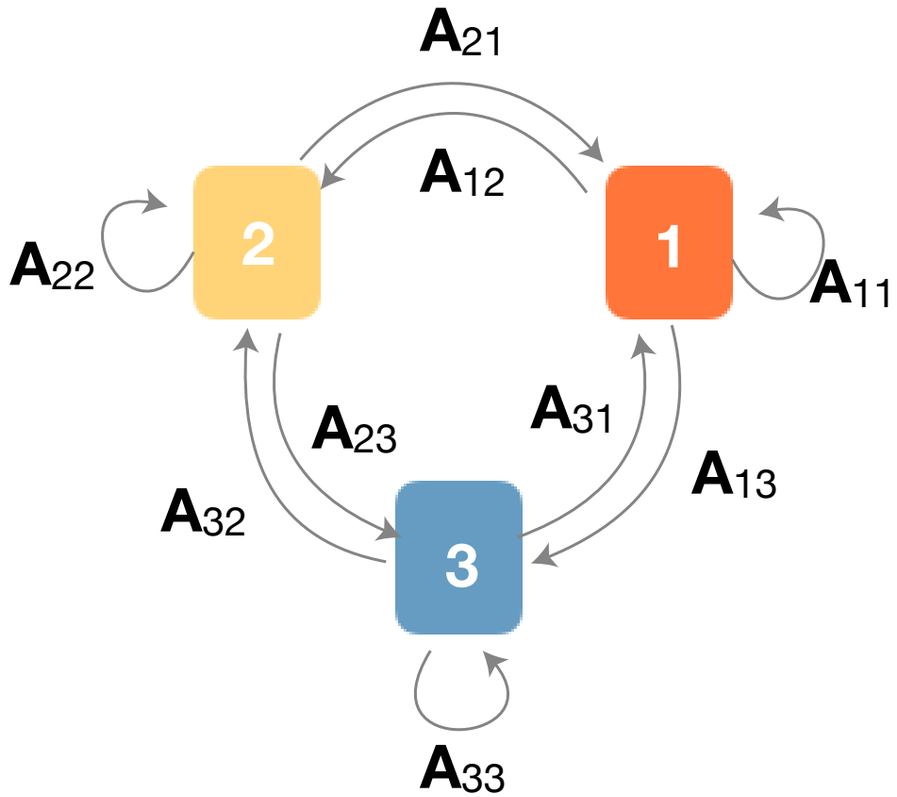
Hidden Markov models: details



K district latents, $z_i = \{1,2,\ldots,K\}$

Transition probabilities: A

$$A_{jk} = \mathrm{P}(z_{i+1} = k | z_i = j)$$



Emissions model for observations $\mathbf{x}_i$

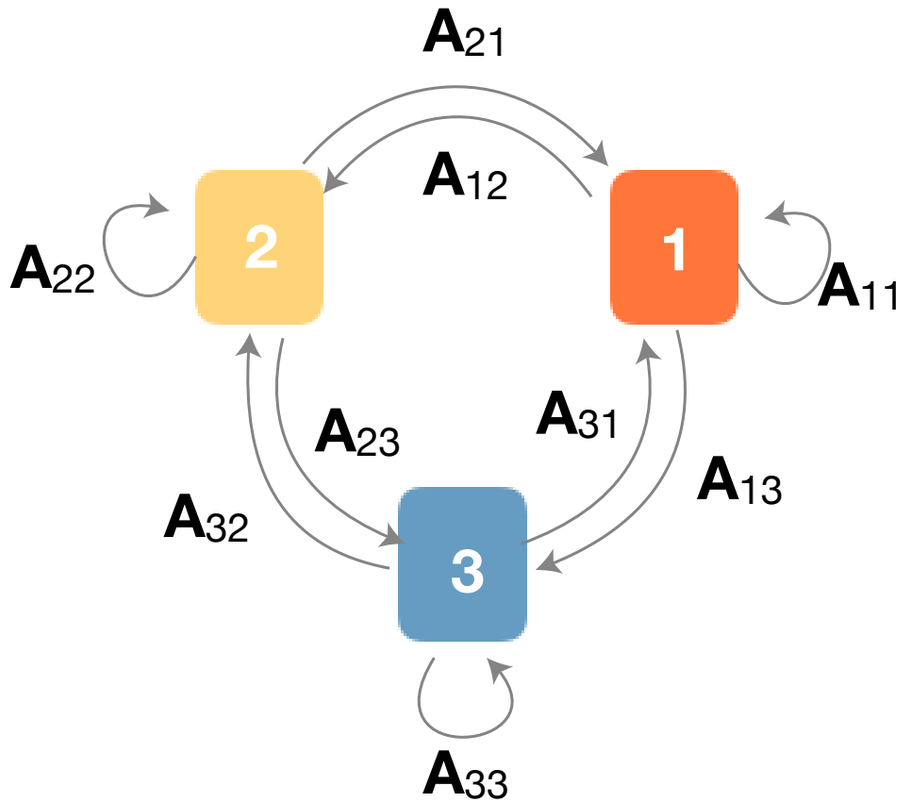
$$\mathrm{P}(\mathbf{x}_i | \mathbf{z}_i)$$

Initial state, $\mathbf{z}_0$ $\quad \mathrm{P}(\mathbf{z}_1)$

$$0 \leq \mathbf{A}_{j,k} \leq 1$$
$$\sum_k \mathbf{A}_{j,k} = 1$$

Alternative **1-in-K** (one hot) representation*:

$\mathbf{z}_i$ - K-dimensional binary vector

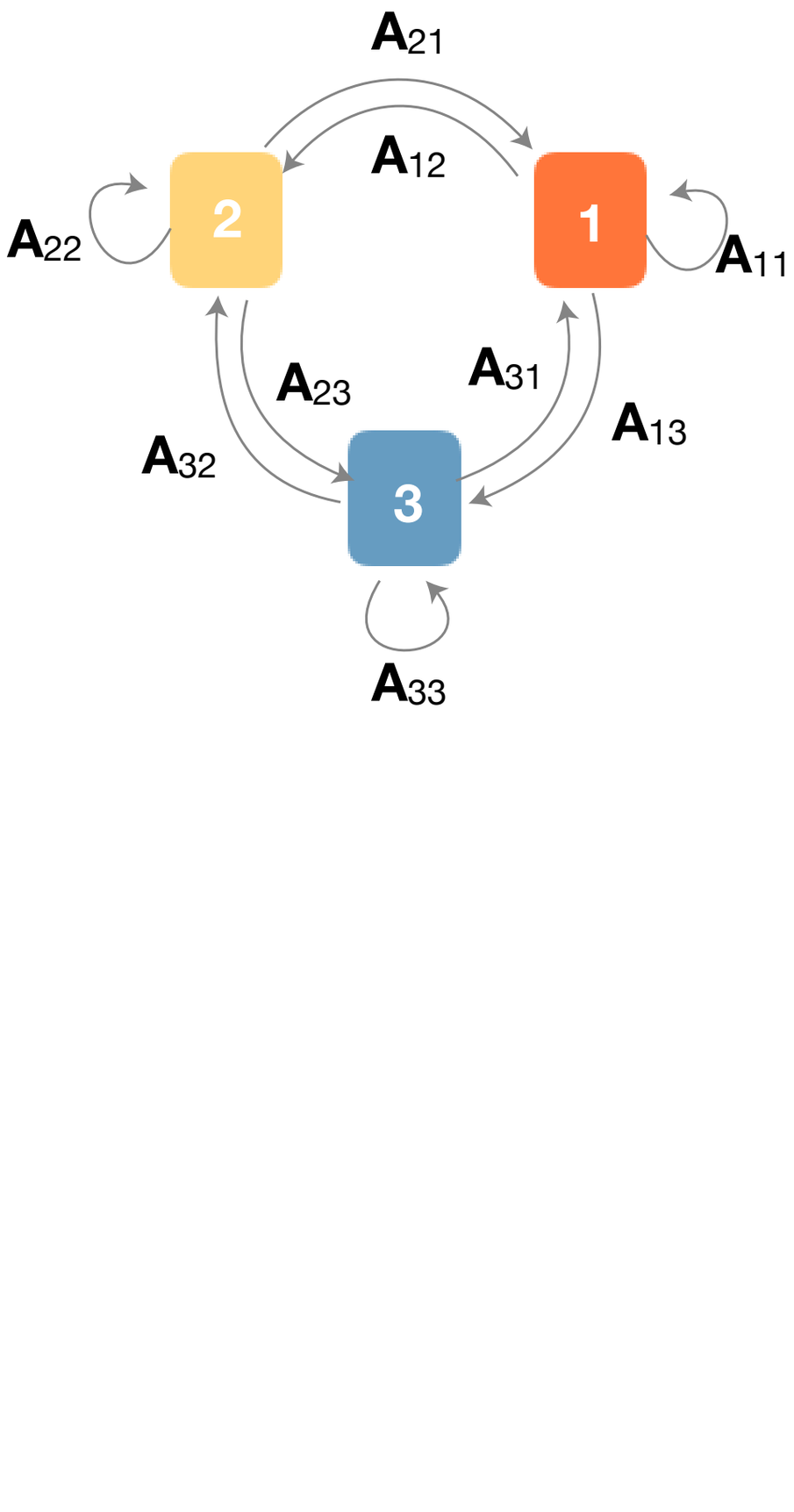


$$A_{jk} = \mathrm{P}(\mathbf{z}_{i+1,k} = 1 | \mathbf{z}_{i,j} = 1)$$

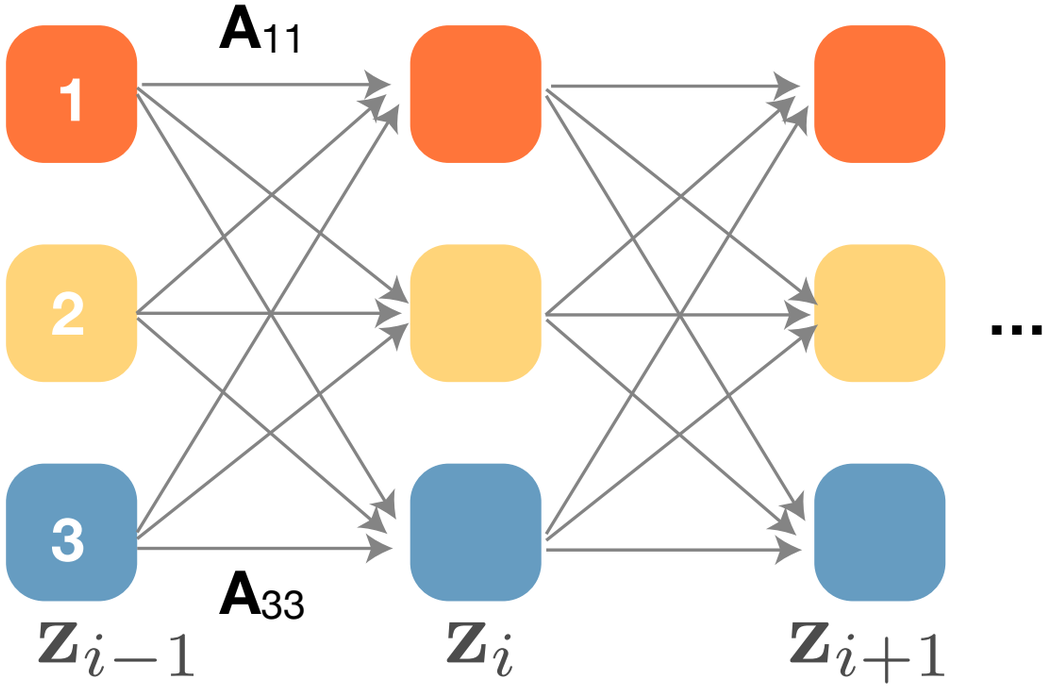
$$\mathrm{P}(\mathbf{z}_{i+1} | \mathbf{z}_{i\cdot}) =$$

Initial state, $\mathbf{z}_0$ $\qquad \mathrm{P}(\mathbf{z}_1) \;\; =$

*Note: This seems unnecessarily complicated at this point, but will turn out convenient when we want to represent beliefs about the state of the latent variables

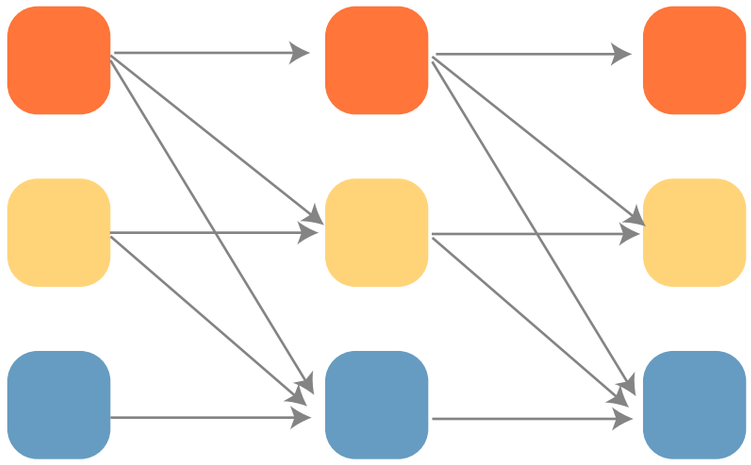


unfold over time:

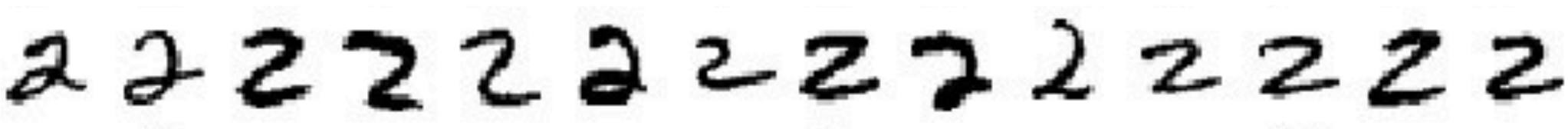


trellis representation

natural ordering of components:
left-to-right HMMs

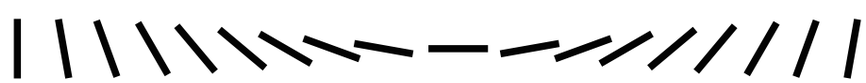


Example: real time digit recognition



probabilistic transition between components: rescaling, variation in length

z: lines fixes length, different orientations



1. Inference

the full posterior is $\mathrm{P}(\mathbf{z}_{1:t}|\mathbf{x}_{1:t})$

$\mathbf{z}_{1:t}$ has K^t possible configurations, so computationally intractable

2. Parameter learning

This is just more **EM**

What we can do is to compute exact **marginals**: **alpha-beta algorithm** (Baum-Welch)
(as we did in LDS with Kalman smoother)

We may also want to determine the **most likely sequence (MAP)**
Viterbi algorithm

1. Inference: marginals

We represent **posterior marginals** and joints using

$$\begin{aligned} \gamma(\mathbf{z}_i) &= \mathrm{P}\left(\mathbf{z}_i|\mathbf{x}_*, \theta^{\mathrm{old}}\right) \\ \xi(\mathbf{z}_i, \mathbf{z}_{i+1}) &= \mathrm{P}\left(\mathbf{z}_i, \mathbf{z}_{i+1}|\mathbf{x}_*, \theta^{\mathrm{old}}\right) \end{aligned}$$

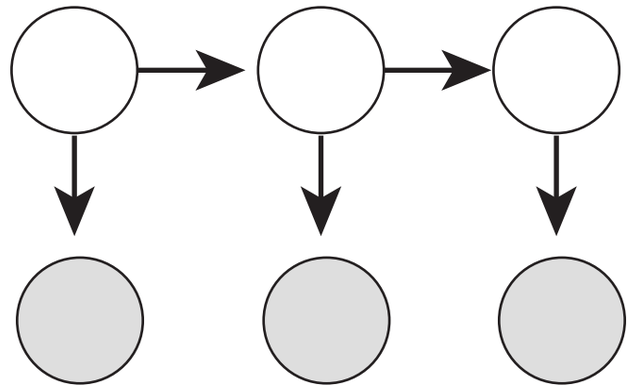
dimensionality?

With the corresponding **expectations**:

$$\begin{aligned} \gamma(z_{ik}) &= \mathbb{E}[z_{i,k}|\mathbf{x}_*, \theta^{\mathrm{old}}] = \sum_{\mathbf{z}_i} \gamma(\mathbf{z}_i) z_{i,k} \\ \xi(z_{i,j}, z_{i+1,k}) &= \mathbb{E}[z_{i,j} z_{i+1,k}|\mathbf{x}_*, \theta^{\mathrm{old}}] = \sum_{\mathbf{z}_i, \mathbf{z}_{i+1}} \xi(\mathbf{z}_i, \mathbf{z}_{i+1}) z_{i,j} z_{i+1,k} \end{aligned}$$

We seek an efficient way of computing these by using recursions
(dynamic programming, as for LDS)

Reminder: useful conditional independencies



$$
\begin{aligned}
\mathrm{P}\left(\mathbf{x}_{1:t}|\mathbf{z}_i\right) &= \mathrm{P}\left(\mathbf{x}_{1:i}|\mathbf{z}_i\right)\mathrm{P}\left(\mathbf{x}_{i+1:t}|\mathbf{z}_i\right) \\
\mathrm{P}\left(\mathbf{x}_{1:i}|\mathbf{z}_{i+1},\mathbf{x}_{i+1}\right) &= \mathrm{P}\left(\mathbf{x}_{1:i}|\mathbf{z}_{i+1}\right) \\
\mathrm{P}\left(\mathbf{x}_{i+2:t}|\mathbf{z}_{i+1},\mathbf{x}_{i+1}\right) &= \mathrm{P}\left(\mathbf{x}_{i+2:t}|\mathbf{z}_{i+1}\right) \\
\mathrm{P}\left(\mathbf{x}_{1:i}|\mathbf{z}_i,\mathbf{z}_{i+1}\right) &= \mathrm{P}\left(\mathbf{x}_{1:i}|\mathbf{z}_i\right) \\
\mathrm{P}\left(\mathbf{x}_{i+1:t}|\mathbf{z}_i,\mathbf{z}_{i+1}\right) &= \mathrm{P}\left(\mathbf{x}_{i+1:t}|\mathbf{z}_{i+1}\right) \\
\mathrm{P}\left(\mathbf{x}_{1:t}|\mathbf{z}_i,\mathbf{z}_{i+1}\right) &= \mathrm{P}\left(\mathbf{x}_{1:i}|\mathbf{z}_i\right)\mathrm{P}\left(\mathbf{x}_{i+1}|\mathbf{z}_{i+1}\right)\mathrm{P}\left(\mathbf{x}_{i+2:t}|\mathbf{z}_{i+1}\right)
\end{aligned}
$$

First, let's look at **marginals**

$$\mathrm{P}\left(\mathbf{z}_i|\mathbf{x}_{1:t},\theta^{\mathrm{old}}\right) = \frac{\mathrm{P}\left(\mathbf{x}_{1:t}|\mathbf{z}_i\right)\mathrm{P}\left(\mathbf{z}_i\right)}{\mathrm{P}\left(\mathbf{x}_{1:t}\right)} = \frac{\mathrm{P}\left(\mathbf{x}_{1:i},\mathbf{z}_i\right)\mathrm{P}\left(\mathbf{x}_{i+1:t}|\mathbf{z}_i\right)}{\mathrm{P}\left(\mathbf{x}_{1:t}\right)} = \frac{\alpha(\mathbf{z}_i)\beta(\mathbf{z}_i)}{\mathrm{P}\left(\mathbf{x}_{1:t}\right)}$$

where we defined the 'messages'

$$\begin{aligned}\alpha(\mathbf{z}_i) &= \mathrm{P}\left(\mathbf{x}_{1:i},\mathbf{z}_i\right)\\ \beta(\mathbf{z}_i) &= \mathrm{P}\left(\mathbf{x}_{i+1:t}|\mathbf{z}_i\right)\end{aligned}$$

And the normalizing constant is computed by marginalizing $\mathbf{z}_i$:

$$\mathrm{P}\left(\mathbf{x}_{1:t}\right) = \sum_{\mathbf{z}_k} \alpha(\mathbf{z}_k)\beta(\mathbf{z}_k)$$

Recursion equations:

$$\alpha(\mathbf{z}_i) = \mathrm{P}\left(\mathbf{x}_{1:i}|\mathbf{z}_i\right)\mathrm{P}\left(\mathbf{z}_i\right) = \mathrm{P}\left(\mathbf{x}_i|\mathbf{z}_i\right)\sum_{\mathbf{z}_{i-1}}\alpha(\mathbf{z}_{i-1})\mathrm{P}\left(\mathbf{z}_i|\mathbf{z}_{i-1}\right)$$

$$\beta(\mathbf{z}_i) = \mathrm{P}\left(\mathbf{x}_{i+1:t}|\mathbf{z}_i\right) = \sum_{\mathbf{z}_{i+1}}\beta(\mathbf{z}_{i+1})\mathrm{P}\left(\mathbf{x}_{i+1}|\mathbf{z}_{i+1}\right)\mathrm{P}\left(\mathbf{z}_{i+1}|\mathbf{z}_i\right)$$

With initial conditions:

$$\alpha(\mathbf{z}_1) = \mathrm{P}\left(\mathbf{x}_1|\mathbf{z}_1\right)\mathrm{P}\left(\mathbf{z}_1\right) = \prod_k(\pi_k\mathrm{P}\left(\mathbf{x}_1|\phi_k\right))^{z_{1k}}$$

$$\beta(\mathbf{z}_t) = 1$$

These quantities are enough to also compute the joint:

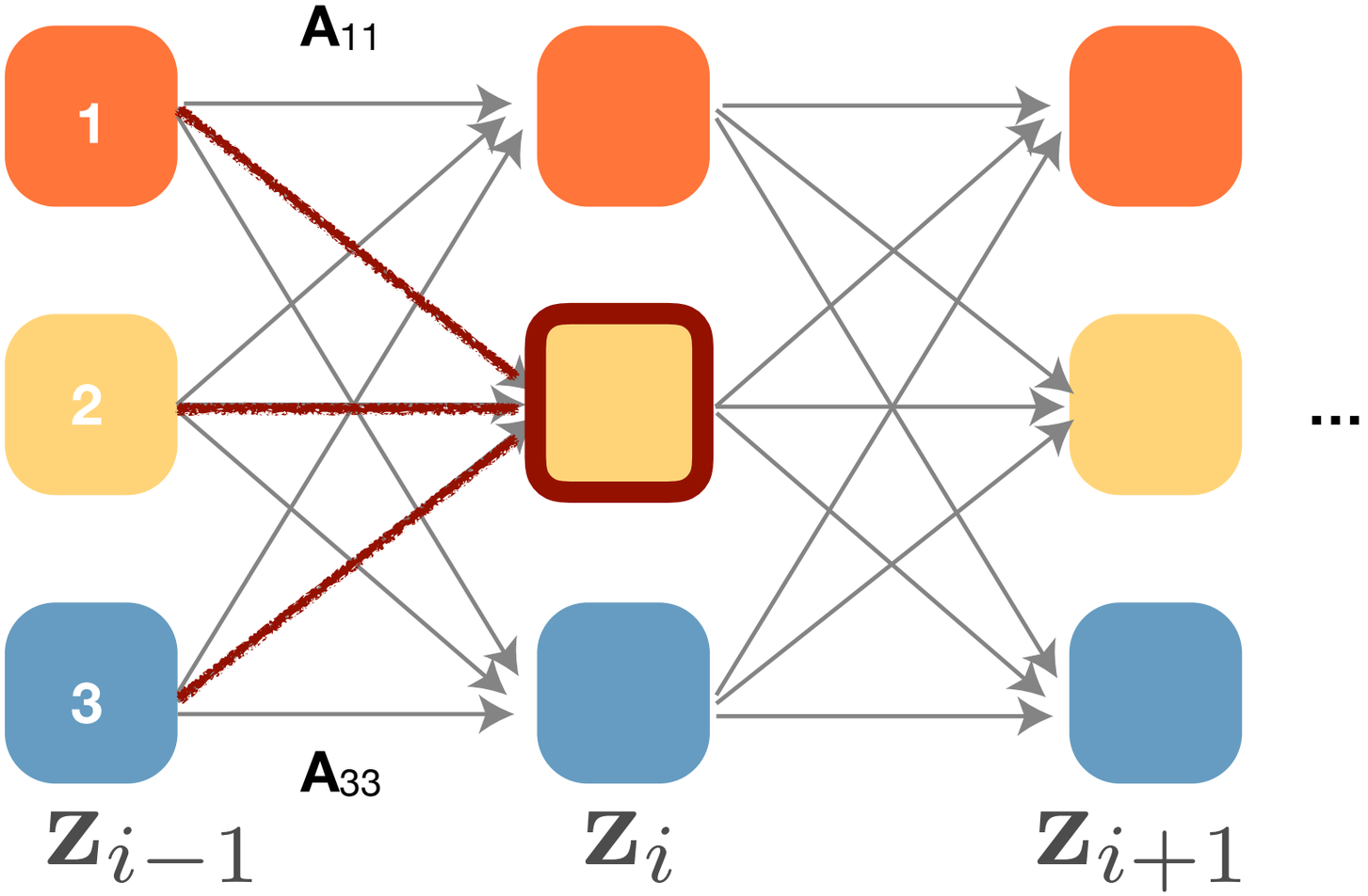
$$\mathrm{P}\left(\mathbf{z}_i, \mathbf{z}_{i+1}|\mathbf{x}_{1:t}\right) = \frac{\alpha(\mathbf{z}_i)\mathrm{P}\left(\mathbf{z}_{i+1}|\mathbf{z}_i\right)\mathrm{P}\left(\mathbf{x}_{i+1}|\mathbf{z}_{i+1}\right)\beta(\mathbf{z}_{i+1})}{\mathrm{P}\left(\mathbf{x}_{1:t}\right)}$$

Proofs: in handout hmm.pdf

$$
\begin{aligned}
\alpha(\mathbf{z}_i) &= \mathrm{P}\left(\mathbf{x}_{1:i}, \mathbf{z}_i\right) = \mathrm{P}\left(\mathbf{x}_{1:i} \middle| \mathbf{z}_i\right) \mathrm{P}\left(\mathbf{z}_i\right) \\
&= \mathrm{P}\left(\mathbf{x}_{1:i-1} \middle| \mathbf{z}_i\right) \mathrm{P}\left(\mathbf{x}_i \middle| \mathbf{z}_i\right) \mathrm{P}\left(\mathbf{z}_i\right) \\
&= \mathrm{P}\left(\mathbf{x}_i \middle| \mathbf{z}_i\right) \sum_{\mathbf{z}_{i-1}} \mathrm{P}\left(\mathbf{x}_{1:i-1}, \mathbf{z}_{i-1}, \mathbf{z}_i\right) \\
&= \mathrm{P}\left(\mathbf{x}_i \middle| \mathbf{z}_i\right) \sum_{\mathbf{z}_{i-1}} \mathrm{P}\left(\mathbf{x}_{1:i-1} \middle| \mathbf{z}_{i-1}, \mathbf{z}_i\right) \mathrm{P}\left(\mathbf{z}_i, \mathbf{z}_{i-1}\right) \\
&= \mathrm{P}\left(\mathbf{x}_i \middle| \mathbf{z}_i\right) \sum_{\mathbf{z}_{i-1}} \mathrm{P}\left(\mathbf{x}_{1:i-1} \middle| \mathbf{z}_{i-1}\right) \mathrm{P}\left(\mathbf{z}_i \middle| \mathbf{z}_{i-1}\right) \mathrm{P}\left(\mathbf{z}_{i-1}\right) \\
&= \mathrm{P}\left(\mathbf{x}_i \middle| \mathbf{z}_i\right) \sum_{\mathbf{z}_{i-1}} \mathrm{P}\left(\mathbf{x}_{1:i-1}, \mathbf{z}_{i-1}\right) \mathrm{P}\left(\mathbf{z}_i \middle| \mathbf{z}_{i-1}\right) \\
&= \mathrm{P}\left(\mathbf{x}_i \middle| \mathbf{z}_i\right) \sum_{\mathbf{z}_{i-1}} \alpha(\mathbf{z}_{i-1}) \mathrm{P}\left(\mathbf{z}_i \middle| \mathbf{z}_{i-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
\beta(\mathbf{z}_i) &= \mathrm{P}\left(\mathbf{x}_{i+1:t} \middle| \mathbf{z}_i\right) \\
&= \sum_{\mathbf{z}_{i+1}} \mathrm{P}\left(\mathbf{x}_{i+1:t}, \mathbf{z}_i \middle| \mathbf{z}_i\right) \\
&= \sum_{\mathbf{z}_{i+1}} \mathrm{P}\left(\mathbf{x}_{i+1:t} \middle| \mathbf{z}_{i+1}, \mathbf{z}_i\right) \mathrm{P}\left(\mathbf{z}_{i+1} \middle| \mathbf{z}_i\right) \\
&= \sum_{\mathbf{z}_{i+1}} \mathrm{P}\left(\mathbf{x}_{i+1} \middle| \mathbf{z}_{i+1}\right) \mathrm{P}\left(\mathbf{x}_{i+2:t} \middle| \mathbf{z}_{i+1}\right) \mathrm{P}\left(\mathbf{z}_{i+1} \middle| \mathbf{z}_i\right) \\
&= \sum_{\mathbf{z}_{i+1}} \beta(\mathbf{z}_{i+1}) \mathrm{P}\left(\mathbf{x}_{i+1} \middle| \mathbf{z}_{i+1}\right) \mathrm{P}\left(\mathbf{z}_{i+1} \middle| \mathbf{z}_i\right)
\end{aligned}
$$

Interpretation: sum-product



$$\alpha(\mathbf{z}_i) \quad = \quad \mathrm{P}\left(\mathbf{x}_i|\mathbf{z}_i\right) \sum_{\mathbf{z}_{i-1}} \alpha(\mathbf{z}_{i-1}) \mathrm{P}\left(\mathbf{z}_i|\mathbf{z}_{i-1}\right)$$

consider **all possible ways** to reach state $\mathbf{z}_{i,k}$ and **sum** them up,
then combine with local evidence

Implementational caveat (a rather important one)

$$\alpha(\mathbf{z}_i) \quad = \quad \mathrm{P}\left(\mathbf{x}_{1:i}, \mathbf{z}_i\right)$$

joint distribution over increasingly many things,
the probability of any particular configuration is vanishingly small,
easy to underflow machine precision

Solution: rescale messages to keep things in sensible range

$$\hat{\alpha}(\mathbf{z}_i) \quad = \quad \mathrm{P}\left(\mathbf{z}_i | \mathbf{x}_{1:i}\right) = \frac{\alpha(\mathbf{z}_i)}{\mathrm{P}\left(\mathbf{x}_{1:i}\right)}$$

$$\hat{\beta}(\mathbf{z}_i) \quad = \quad \frac{\mathrm{P}\left(\mathbf{x}_{i+1:t} | \mathbf{z}_i\right)}{\mathrm{P}\left(\mathbf{x}_{\mathbf{i+1:t}} | \mathbf{x}_{1:i}\right)} = \frac{\beta(\mathbf{z}_i)}{\prod_{j=i+1:t} c_j}$$

where we've introduced an intermediate variable

$$c_i \quad = \quad \mathrm{P}\left(\mathbf{x}_i | \mathbf{x}_{1:i-1}\right)$$

$$\mathrm{P}\left(\mathbf{x}_{1:i}\right) \quad = \quad \prod_{j=1:i} c_j$$

The updated recursion equation for $\hat{\alpha}$, and $\hat{\beta}$ become:

$$
\begin{aligned}
c_i\hat{\alpha}(\mathbf{z}_i) &= \mathrm{P}\left(\mathbf{x}_i|\mathbf{z}_i\right)\sum_{\mathbf{z}_{i-1}}\hat{\alpha}(\mathbf{z}_{i-1})\mathrm{P}\left(\mathbf{z}_i|\mathbf{z}_{i-1}\right) \\
c_{i+1}\hat{\beta}(\mathbf{z}_i) &= \sum_{\mathbf{z}_{i+1}}\hat{\beta}(\mathbf{z}_{i+1})\mathrm{P}\left(\mathbf{x}_{i+1}|\mathbf{z}_{i+1}\right)\mathrm{P}\left(\mathbf{z}_{i+1}|\mathbf{z}_i\right)
\end{aligned}
$$

check at home!

Finally the new expressions for the posterior marginals are:

$$
\begin{aligned}
\gamma(\mathbf{z}_i) &= \hat{\alpha}(\mathbf{z}_i)\hat{\beta}(\mathbf{z}_i) \\
\xi(z_{i,j}, z_{i+1,k}) &= {c_{i+1}}^{-1}\hat{\alpha}(\mathbf{z}_i)\mathrm{P}\left(\mathbf{z}_{i+1}|\mathbf{z}_i\right)\mathrm{P}\left(\mathbf{x}_{i+1}|\mathbf{z}_{i+1}\right)\hat{\beta}(\mathbf{z}_{i+1})
\end{aligned}
$$

Formal link to LDS:

Kalman filtering: $\hat{\alpha}(\mathbf{z}_i)$

Kalman smoothing: $\gamma(\mathbf{z}_i) = \hat{\alpha}(\mathbf{z}_i)\hat{\beta}(\mathbf{z}_i)$

1. Inference: MAP

$$\hat{\mathbf{z}}_{1:t} = \text{argmax}_{\mathbf{z}_{1:t}} \{\text{P}\left(\mathbf{z}_{1:t} | \mathbf{x}_{1:t}\right)\}$$

The Viterbi algorithm
(this is also an instance of dynamic programming)

General idea: somebody told you the probability of the most likely configuration is up to point i, and its end point,
can you use that to figure out the most likely configuration up to $\mathbf{z}_{i+1}$

Yet another instance of forward message passing:

why not conditional?
why log?

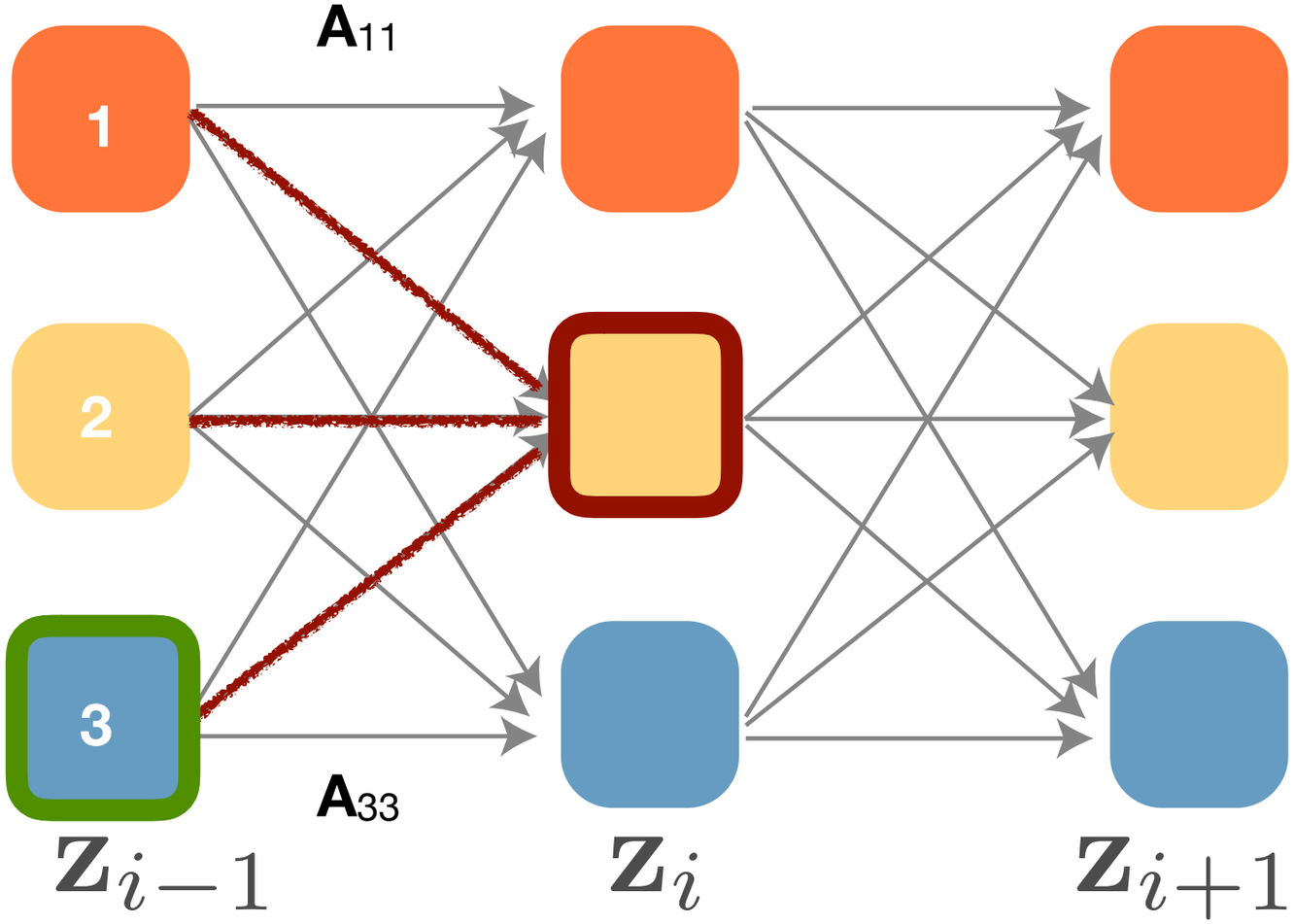
With recursion:

$$w_{i+1} = \log \text{P}\left(\mathbf{x}_{i+1} | \mathbf{z}_{i+1}\right) + \max_{\mathbf{z}_i} \{\log \text{P}\left(\mathbf{z}_{i+1} | \mathbf{z}_i\right) + w_i(\mathbf{z}_i)\}$$

And initial condition: $w_1 = \log \text{P}\left(\mathbf{z}_1\right) + \log \text{P}\left(\mathbf{x}_1 | \mathbf{z}_1\right)$

nterpretation: max-product

$$k_i^{\max} = \text{argmax}_{\mathbf{z}_i}\{\log \mathrm{P}\left(\mathbf{z}_{i+1}|\mathbf{z}_i\right) + w_i(\mathbf{z}_i)\}$$



...

+need to remember
how we got there

$$w_{i+1} = \log \mathrm{P}\left(\mathbf{x}_{i+1}|\mathbf{z}_{i+1}\right) + \max_{\mathbf{z}_i}\{\log \mathrm{P}\left(\mathbf{z}_{i+1}|\mathbf{z}_i\right) + w_i(\mathbf{z}_i)\}$$

consider **all possible ways** to reach state $\mathbf{z}_{i,k}$ and **pick the best**,
then combine with local evidencet

2. Parameter learning: EM

Goal: find parameters that maximize $\mathrm{P}\left(\mathbf{x}|\theta\right) = \int_Z \mathrm{P}\left(\mathbf{x}, \mathbf{z}|\theta\right) \mathbf{dz}$

Remember the **general recipe**:

E-step: $q(\mathbf{z}) = \mathrm{P}\left(\mathbf{z}|\mathbf{x}, \theta^{\mathrm{old}}\right)$ fixed parameters θ^{old}

This we do using alpha-beta

M-step: $\mathcal{Q}(\theta, \theta^{\mathrm{old}}) = \int_Z \mathrm{P}\left(\mathbf{z}|\mathbf{x}, \theta^{\mathrm{old}}\right) \log \mathrm{P}\left(\mathbf{x}, \mathbf{z}|\theta\right)$

fixed q, optimize parameters

Separate terms for observation model, latent state transitions and initial state, optimize one at a time

Learning: EM

Log likelihood: $\mathcal{L}(\theta) = \log \mathrm{P}(\mathbf{x}_*|\theta)$

$$\mathrm{P}(\mathbf{x}_*, \mathbf{z}_*|\theta) = \mathrm{P}_\theta(\mathbf{z}_0) \prod_i \mathrm{P}_\theta(\mathbf{z}_{i+1}|\mathbf{z}_i) \prod_i \mathrm{P}_\theta(\mathbf{x}_i|\mathbf{z}_i)$$

In general:

$$\begin{aligned} \pi_k^{\mathrm{new}} &= \frac{\gamma(z_{1,k})}{\sum_j \gamma(z_{1,j})} \\ A_{jk}^{\mathrm{new}} &= \frac{\sum_i \xi(z_{i,j}, z_{i+1,k})}{\sum_{i,l} \xi(z_{i,j}, z_{i+1,l})} \end{aligned}$$

For a simple *gaussian* observation model

$$\begin{aligned} \mu_k^{\mathrm{new}} &= \frac{1}{\sum_i \gamma(z_{i,k})} \sum_i \gamma(z_{i,k}) \mathbf{x}_i \\ \Sigma_k^{\mathrm{new}} &= \frac{1}{\sum_i \gamma(z_{i,k})} \sum_i \gamma(z_{i,k}) \mathbf{x}_i \mathbf{x}_i^{\mathrm{t}} - \mu_k \mu_k^{\mathrm{t}} \end{aligned}$$

\

What happens next:

HMMs lab: real data! (supervised setting, so model fit easier, no EM)
but you get to implement Viterbi inference

next lecture: putting what we've learned in a bigger picture,
variations, what can and can't we do with these models,
where do we go when they are not enough