Question 6 : HW 2

Task: Given 1D image $I(x) = cx + d$. We are required to

a) Derive expression for image $J \Rightarrow I$ filtered by 0-mean Gaussian.

Convolution of weights w with image f is given as

$$(w * f)(x, y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t) \cdot f(x-s, y-t)$$

while applying Gaussian blur, the gaussian function is the weight

$$G(x, y) = \frac{e^{-(x^2+y^2)/2\sigma^2}}{\sigma\sqrt{2\pi}}$$

$$G(x) = \frac{e^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$$

On convolution with the image $I(x) = cx + d$, we have

$$(G * I)(x) = \int_{-\infty}^{\infty} (c(x-t) + d) \cdot \frac{e^{-t^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \, dt$$

$$= \frac{(cx+d)}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2\sigma^2} \, dt + \int_{-\infty}^{\infty} \frac{(-t)\, e^{-t^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \, dt$$

$$= \frac{(cx+d)}{\cancel{\sigma\sqrt{2\pi}}} \cdot \cancel{\sqrt{2\pi}\,\sigma} + \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} -t\, e^{-t^2/2\sigma^2} \, dt$$

$$= (cx + d) + \left(\frac{-1}{\sigma\sqrt{2\pi}}\right) \cdot (0)$$

$\hookrightarrow$ since it is an odd function

$\therefore (G * I)(x) = cx + d$ (same image).

b) Bilateral filter.

$$BF[I]_p = \frac{1}{W_p} \sum_{q \in S} G_{\sigma_s}(\|p-q\|)\, G_{\sigma_r}(I_p - I_q)\, I_q$$

$$W_p = \sum_{q \in S} G_{\sigma_s} \|p-q\|\, G_{\sigma_r}(I_p - I_q)$$

σ_s: spatial extent of kernel

σ_r: range / min amplitude of an edge.

$$J(x) = \frac{1}{W} \cdot \sum_{i=-a}^{a} I(x+i) \cdot e^{-i^2/2\sigma_s^2} \cdot e^{-\frac{(I(x+i) - I(x))^2}{2\sigma_r^2}}$$

$$= \frac{1}{W} \sum_{i=-a}^{a} (cx + ci + d) \cdot e^{-i^2/2\sigma_s^2} \cdot e^{-\frac{(cx + ci + d - cx + d)^2}{2\sigma_r^2}}$$

$$\Rightarrow \frac{1}{W} \sum_{i=-a}^{a} ci \cdot e^{-i^2/2\sigma_s^2} \, e^{-(ci)^2/2\sigma_s^2} = 0$$

As this is an odd function of i

$$\therefore J(x) = \frac{1}{W} \sum_{i=-a}^{a} (cx + d) \cdot e^{-i^2/2\sigma_s^2} \cdot e^{-(ci)^2/2\sigma_r^2}$$

$$= \frac{(cx + d)}{W} \sum_{i=-a}^{a} e^{-i^2/2\sigma_s^2} \cdot e^{-(ci)^2/2\sigma_r^2}$$

we know, $W = \sum_{i=-a}^{a} e^{-i^2/2\sigma_s^2} \cdot e^{-(ci)^2/2\sigma_r^2}$

$$\therefore J(x) = cx + d = I(x) \,//$$

Hence we get the same image back.