Question 7 : HW2

Task : To prove the Laplacian operator is rotationally invariant.
Rotational invariance of an operator implies that we get the same result if we first apply rotation and then the Laplacian operator or if we first apply Laplacian and then rotate. Here we prove this by showing

$$f_{xx}+f_{yy} = f_{uu}+f_{vv} \qquad \left(f_{xx} = \frac{\partial^2 f}{\partial x^2}\right)$$

$u = x\cos\theta - y\sin\theta \quad ; \quad v = x\sin\theta + y\cos\theta$

From this : $\quad u\cos\theta + v\sin\theta = x(1) + 0$

$\therefore \; x = u\cos\theta + v\sin\theta$

$\quad\; y = v\cos\theta - u\sin\theta$

$$f_u = \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial f}{\partial x}\cdot\cos\theta + \frac{\partial f}{\partial y}\cdot(-\sin\theta)$$

$$f_v = \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial f}{\partial x}(\sin\theta) + \frac{\partial f}{\partial y}(\cos\theta)$$

$$f_{uu} = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\right)\cdot\cos\theta + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u}\right)\cdot(-\sin\theta)$$

$$= \cos\theta\left(\frac{\partial^2 f}{\partial x^2}\cdot\cos\theta - \frac{\partial^2 f}{\partial x\partial y}\sin\theta\right) + \left(\frac{\partial^2 f}{\partial y^2}(-\sin\theta) + \frac{\partial^2 f}{\partial y\partial x}\cos\theta\right)(-\sin\theta)$$

$$f_{vv} = \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}\right)\cdot\sin\theta + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial v}\right)\cdot\cos\theta$$

$$= \left(\frac{\partial^2 f}{\partial x^2}\sin\theta + \frac{\partial^2 f}{\partial x\partial y}\cos\theta\right)\sin\theta + \left(\frac{\partial^2 f}{\partial x\partial y}\sin\theta + \frac{\partial^2 f}{\partial y^2}\cos\theta\right)\cos\theta$$

$$f_{uu}+f_{vv} = \frac{\partial^2 f}{\partial x^2}\cdot\cos^2\theta - \cancel{\frac{\partial^2 f}{\partial x\partial y}\sin\theta\cos\theta} + \frac{\partial^2 f}{\partial y^2}\cdot\sin^2\theta + \cancel{\frac{\partial^2 f}{\partial y\partial x}(-\sin\theta\cos\theta)}$$

$$+ \frac{\partial^2 f}{\partial x^2}\sin^2\theta + \cancel{\frac{\partial^2 f}{\partial x\partial y}\sin\theta\cos\theta} + \cancel{\frac{\partial^2 f}{\partial x\partial y}\sin\theta\cos\theta} + \frac{\partial^2 f}{\partial y^2}\cos^2\theta$$

we get

$$f_{uu} + f_{vv} = \frac{\partial^2 f}{\partial x^2}(\cos^2\theta + \sin^2\theta) + \frac{\partial^2 f}{\partial y^2}(\cos^2\theta + \sin^2\theta)$$

$$= f_{xx} + f_{yy}$$