Green's Theorem

Green's theorem relates a line integral to the double integral taken over the region bounded by the closed curve.

Statement

If $M(x,y)$ and $N(x,y)$ are continuous functions with continuous, partial derivatives in a region R of the xy – plane bounded by a simple closed curve C, then

$\oint_{c} Mdx + Ndy = \iint_{R} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$,where C is the curve described in the positive direction.

Vector form of Green's theorem

$$\oint_{c} \vec{F} \cdot d\vec{r} = \iint_{R} (\nabla \times \vec{F}) \cdot \vec{k}\, dR$$

... vertices $(0,0), \left(\frac{\pi}{2},0\right)$ and $\left(\frac{\pi}{2},1\right)$ Ans: $\frac{\pi}{4}+\frac{2}{\pi}$

GAUSS DIVERGENCE THEOREM

This theorem enables us to convert a surface integral of a vector function on a closed surface into volume integral.

Statement of Gauss Divergence theorem

If V is the volume bounded by a closed surface S and if a vector function $\vec{F}$ is continuous and has continuous partial derivatives in V and on S, then

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

Where $\hat{n}$ is the unit outward normal to the surface S and $dV = dxdydz$

STOKE'S THEOREM

Statement of Stoke's theorem

If S is an open surface bounded by a simple closed curve C if $\vec{F}$ is continuous having continuous partial derivatives in S and C, then

$$\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\, \vec{F} \cdot \hat{n}\, ds$$

(or)

$$\int_c \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n}\, ds$$

$\hat{n}$ is the outward unit normal vector and C is traversed in the anti – clockwise direction.

(4) Calculate $\int_C F(r)\, dr$ where $F = [y^2 i, -x^2 j]$; $C: 4x^2$ from $(0,0)$ to $(1,4)$ in counter clock sense?

(or)

Calculate the work done by a particle under the influence of a force $y^2 i - x^2 j$ along the curve $y = 4x^2$ from $(0,0)$ to $(1,4)$?

Sol:- Given $\bar{F} = y^2 \bar{i} - x^2 \bar{j}$

$$d\bar{r} = dx\, \bar{i} + dy\, \bar{j}$$

$$\bar{F} \cdot d\bar{r} = y^2 dx - x^2 dy$$

Given c is $y = 4x^2$

$$\therefore dy = 8x\, dx$$

Along c, x varies from 0 to 1

$$\int_c \bar{F} \cdot d\bar{r} = \int_c (y^2 dx - x^2 dy)$$

$$= \int_0^1 (4x^2)^2 dx - x^2 (8x\, dx)$$

$$= \int_0^1 (16x^4 dx - 8x^3 dx)$$

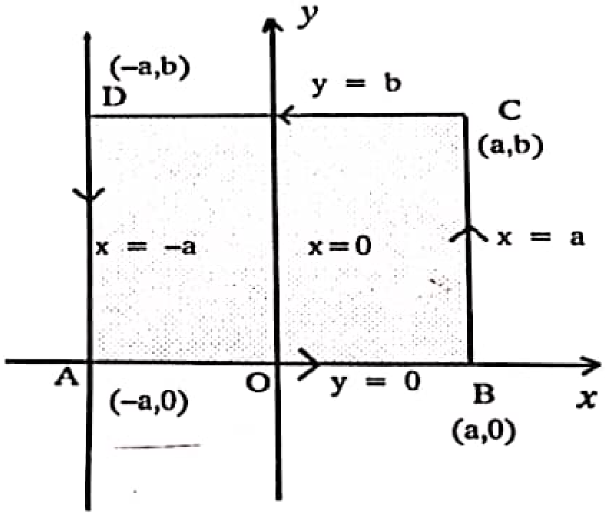
$$= \left[16\left(\frac{x^5}{5}\right) - \cancel{8}^{2}\left(\frac{x^4}{4}\right)\right]_0^1$$

$$= \left(\frac{16}{5} - 2\right)$$

$$= \frac{6}{5} = 1.2$$

Example: **Verify Stoke's theorem for $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\,\vec{j}$ taken around the rectangle bounded by the lines $x = \pm a, y = 0, y = b$.**

Solution:



By Stokes theorem, $\int_c \vec{F}\cdot d\vec{r} = \iint_S Curl\,\vec{F}\cdot\hat{n}\,dS$

Given $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\,\vec{j}$

$$Curl\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$$

$$= \vec{i}[0-0] - \vec{j}\,[0-0] + \vec{k}\,[-2y-2y]$$

$$= -4y\,\vec{k}$$

Since the region is in xoy plane we can take $\hat{n} = \vec{k}$ and $dS = dx\,dy$

Limits:

x varies from $-a$ to a.

y varies from 0 to b.

$$\therefore \iint_S Curl\,\vec{F}\cdot\hat{n}\,dS = -4\int_0^b\int_{-a}^a y\,dx\,dy$$

$$= -4\int_0^b [xy]_{-a}^a\,dy$$

$$= -8a\left[\frac{y^2}{2}\right]_0^b = -4ab^2 \quad \dots(1)$$

$$\int_c \vec{F}\cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

Along AB: $y = 0, dy = 0, x$ varies from $-a$ to a

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$$

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_{-a}^{a} x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{2a^3}{3}$$

Along $BC, x = a, dx = 0,$ y varies from 0 to b

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_0^b (-2ay)\,dy$$

$$= -a[y^2]_0^b = -ab^2$$

Along $CD: y = b, dy = 0, x$ varies from a $to - a$

$$\int_{CD} \vec{F}\cdot d\vec{r} = \int_a^{-a} (x^2 + b^2)\,dx = \left[\frac{x^3}{3} + b^2x\right]_a^{-a}$$

$$= -\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 = -\frac{2a^3}{3} - 2ab^2$$

Along DC: $x = -a, dx = 0, y$ varies from b to 0

$$\int_{DC} \vec{F}\cdot d\vec{r} = \int_b^0 2ay\,dy$$

$$= a[y^2]_b^0 = -b^2a$$

$$\therefore \int_c \vec{F}\cdot d\vec{r} = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - b^2a$$

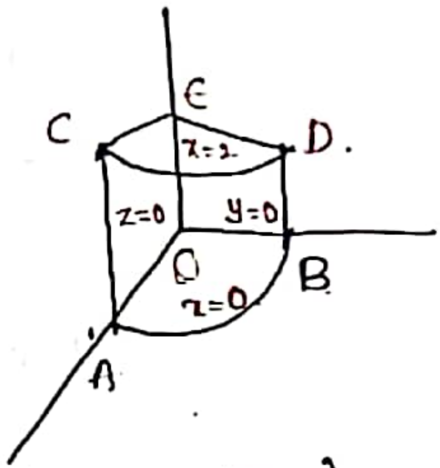
$$= -4ab^2 \qquad \ldots.(2)$$

From (1) and (2) $\int_c \vec{F}\cdot d\vec{r} = \iint_S Curl\,\vec{F}\cdot\vec{n}\,dS$

Hence Stoke's theorem is verified.

Verify Divergence theorem for $2x^2y\bar{i} - y^2\bar{j} + 4xz^2\bar{k}$ take over the region of first Octant of the cylinder $y^2+z^2=9$ and $x=2$.

$$\bar{F} = 2x^2y\bar{i} - y^2\bar{j} + 4xz^2\bar{k}$$



$$\int \text{div} f\, dv = \iint_S F.n\, ds$$

$$\text{div} f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\bar{F}$$

$$\text{div} f = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(2x^2y\bar{i} - y^2\bar{j} + 4xz^2\bar{k}\right)$$

$$= 4xy - 2y + 8xz$$

$$y^2+z^2=9$$

$$z^2 = 9-y^2$$

$$z = \sqrt{9-y^2}$$

$$z \to 0 \text{ to } \sqrt{9-y^2}.$$

$y=0$ then $y^2=9$

$$y=3$$

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$y \to 0$ to 3 and $x \to 0$ to 2.

$$\iiint \operatorname{div} f \, dv = \int_{x=0}^{2}\int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (4xy - 2y + 8xz)\, dx\, dy\, dz.$$

$$= \int_{x=0}^{2}\int_{y=0}^{3} (4xy-2y)(z)_0^{\sqrt{9-y^2}} \, dx\, dy + 8x \int_0^2\int_0^3 \left(\frac{z^2}{2}\right)_0^{\sqrt{9-y^2}} dx\, dy$$

$$= \int_{x=0}^{2}\int_0^3 (4x-2)\, y\sqrt{9-y^2}\, dx\, dy + \int_0^2\int_0^3 4x\left(\sqrt{9-y^2}\right)^2 dx\, dy$$

$$= \int_{x=0}^{2} (4x-2)\, dx \int_0^3 y\sqrt{9-y^2}\, dy + \int_0^2 4x\, dx \int_0^3 (9-y^2)\, dy$$

let $9-y^2 = t^2$

$-2y\,dy = 2t\,dt$

$y\,dy = -t\,dt$.

$y=0,\ 9=t^2 \Rightarrow t=3,\ y=3 \quad 9-9=t^2$

$t=0$

$$= \left[4\frac{x^2}{2} - 2x\right]_0^2 \int_3^0 \sqrt{t^2}(-t)\, dt + \left(\frac{4x^2}{2}\right)_0^2 \left[9y - \frac{y^3}{3}\right]_0^3$$

$$= \left[4\left(\frac{4}{2}\right) - 2(2)\right]\left[-\frac{t^3}{3}\right]_3^0 + \left(2(4)\right)\left[9(3) - \frac{\cancel{27}^{9}}{\cancel{3}}\right]$$

$$= (8-4)\left(\frac{\cancel{27}^{9}}{\cancel{3}}\right) + 8(18)$$

$$= 36 + 144$$

$$\int \operatorname{div} f\, dv = 180.$$

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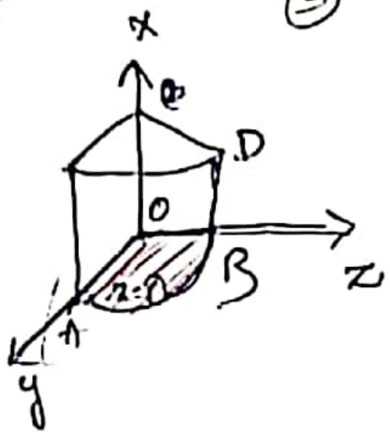
$\int_S F \cdot N \, ds$



$F = 2x^2 y \bar{i} - y^2 \bar{j} + 4xz^2 \bar{k}$

Surface 1 :

face OAB $n = -\bar{i}$

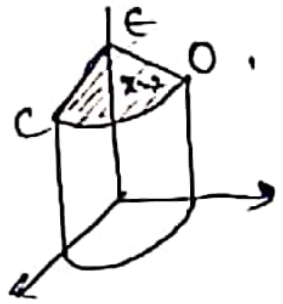
$Fn = (2x^2 y \bar{i} - y^2 \bar{j} + 4xz^2 k)(-\bar{i})$

$= 2x^2 y.$

$x = 0.$

$$\boxed{\int_{OAB} \bar{F} \cdot n = 0 \rightarrow I}$$

Surface II :



face CDE : $x = 2$

$n = \bar{i}$

$\bar{F}\bar{n} = (2x^2 y i - y^2 \bar{j} + 4xz^2 k)(i)$

$\bar{F} \cdot \bar{n} = 2x^2 y$

$x = 2$

$\bar{F} \cdot \bar{n} = 2(2)^2 y = 8y.$

yz plane.

$\int_S F N \, ds = \iint F \cdot n \frac{dy\,dz}{|n \cdot i|} = \iint 8y \, dy \, dz.$

$y^2 + z^2 = 9 \Rightarrow z^2 = 9 - y^2.$

$z = \sqrt{9 - y^2}.$

$y = 0$ to 3

$$= \int_{y=0}^{3} \int_{z=0}^{\sqrt{9-y^2}} 8y \, dy \, dz$$

$$= \int_{y=0}^{3} 8y (z)_0^{\sqrt{9-y^2}} \, dy$$

$$= \int_{y=0}^{3} 8y \left(\sqrt{9-y^2}\right) dy$$

$9-y^2 = t^2 \Rightarrow$

$-\cancel{2}y \, dy = \cancel{2}t \, dt$

$y \, dy = t \, dt$.

$y=0$ then $9-y^2 = t^2$

$9 = t^2$

$t = 3$

If $y=3$ then $9-9 = t^2$

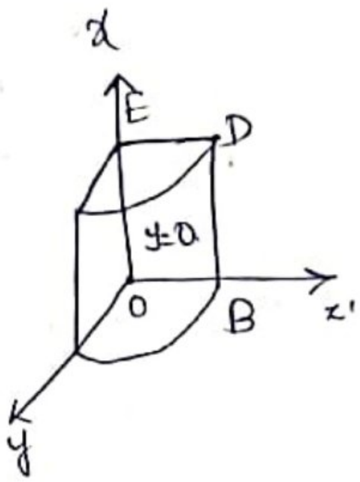
$t = 0$

$$= \int_{t=3}^{t=0} 8\sqrt{t^2}(-t \, dt)$$

$$= -8\left[\frac{t^3}{3}\right]_3^0$$

$$= -8\left(-\frac{\cancel{27}^{\,9}}{\cancel{3}}\right)$$

$$\boxed{\int_{CDE} F \cdot n \, ds = 72} \rightarrow 2.$$



Surface III : OBDE, $y=0$, $n=-j$

$$\int_S F \cdot n \, ds$$

$F = (2x^2y\,i - y^2 j + 4xz^2 k)(-j)$

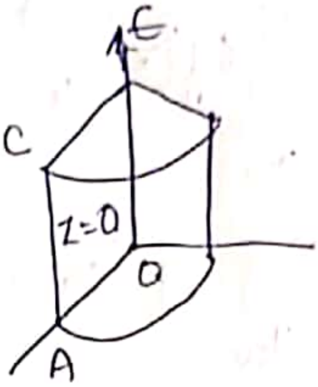
$F = y^2$

$y=0$.

$$\boxed{\int_{OBDE} \bar{F}\cdot n = 0} \rightarrow \text{III}$$

(3)

Surface IV OECA



$z=0, \quad n=-\bar{k}$

$\bar{F}\cdot n = (2x^2y\bar{i} - y^2\bar{j} + 4xz^2\bar{k})(-\bar{k})$

$\bar{F}\cdot n = -4xz^2$

$$\boxed{\int_{OECA} \bar{F}\cdot n\,ds = 0} \rightarrow \text{IV}$$

Surface V.

$\bar{F} = 2x^2y\bar{i} - y^2\bar{j} + 4xz^2\bar{k}$

$\phi = y^2 + z - 9$

$\nabla\phi = 2y\bar{j} + 2z\bar{k}$

$$n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2y\bar{j} + 2z\bar{k}}{\sqrt{4y^2+4z^2}} = \frac{2(y\bar{j}+z\bar{k})}{2\sqrt{y^2+z^2}}$$

$$= \frac{y\bar{j}+z\bar{k}}{\sqrt{9}}$$

$$\bar{n} = \frac{y\bar{j}+z\bar{k}}{3}$$

$$\bar{F}\cdot\bar{n} = (2x^2y\bar{i} - y^2\bar{j} + 4xz^2\bar{k})\left(\frac{y\bar{j}+z\bar{k}}{3}\right)$$

$$= \frac{-y^3 + 4xz^3}{3}$$

xy plane :- $\int_S \bar{F}\cdot n\, ds = \iint \bar{F}\cdot n \dfrac{dx\,dy}{|n\cdot k|}$

$\bar{n}\cdot k = \dfrac{z}{3}$

$= \iint \dfrac{-y^3+4xz^3}{\cancel{3}} \cdot \dfrac{dx\,dy}{z/\cancel{3}}$

$y^2+z^2=9$

$z=0$

$y^2=9$

$y=3$

$y \to 0$ to 3

$x \to 0$ to 2

$= \int_{x=0}^{2}\int_{y=0}^{3} \dfrac{-y^3+4xz^3}{z}\, dx\,dy$

$y^2+z^2=9 \Rightarrow z^2 = 9-y^2$

$z = \sqrt{9-y^2}$

$= \int_{x=0}^{2}\int_{y=0}^{3} \dfrac{-y^3+4x(\sqrt{9-y^2})^3}{\sqrt{9-y^2}}\, dx\,dy$

$= \int_{x=0}^{2}\int_{y=0}^{3} \dfrac{-y^3}{\sqrt{9-y^2}}\, dx\,dy + \int_{x=0}^{2}\int_{y=0}^{3} \dfrac{4x(9-y^2)\cancel{\sqrt{9-y^2}}}{\cancel{\sqrt{9-y^2}}}\, dx\,dy$

$= \int_{x=0}^{2}\int_{y=0}^{3} \dfrac{-y^3\,dy\,dx}{\sqrt{9-y^2}} + \int_0^2 4x\,dx \cdot \int_0^3 (9-y^2)\,dy$

$= \int_{x=0}^{2} dx \int_{y=0}^{3} \dfrac{-y^2(y\,dy)}{\sqrt{9-y^2}} + \int_0^2 4x\,dx \int_0^3 9-y^2\,dy$

$9-y^2 = t^2$

$-\cancel{2}y\,dy = \cancel{2}t\,dt$

$y\,dy = -t\,dt$

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$y=0$ then $9=t^2$

$t=3$

$y=3$ then $9-9=t^2$

$t=0$

$$= (x)_0^2 \int_3^0 \frac{(9-t^2)(t\,dt)}{\sqrt{t^2}} + \left(\frac{4z^2}{2}\right)_0^2 \left[9y-\frac{y^3}{3}\right]_0^3$$

$$= 2\left[9t-\frac{t^3}{3}\right]_3^0 + 8(27-9)$$

$$= 2\left[-27+\frac{27}{3}\right] + 8(18)$$

$$= 2(-18)+144$$

$$= -36+144$$

$$\boxed{\int F\cdot n\,ds = 108} \rightarrow 5.$$

$$\int F\cdot n\,ds = \text{surface } 1 + \text{Surface } 2 + \text{Surface } 3 + \text{Surface } 4 + \text{Surface } 5$$

$$= 0+72+0+0+108$$

$$= 180 \rightarrow$$

$\therefore$ Gauss Divergence theorem is verified.