UNIT - IV

① show that $div(curl\ \bar{v}) = 0$?

(or)

P.T $div(curl\ \bar{f}) = 0$?

Pf:- ~~curl f~~

let $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$curl\ \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$$

$\therefore div(curl\ \bar{f}) = \nabla\cdot(\nabla\times\bar{f})$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \frac{\partial^2 f_3}{\partial x\partial y} - \frac{\partial^2 f_2}{\partial x\partial z} - \frac{\partial^2 f_3}{\partial y\partial x} + \frac{\partial^2 f_1}{\partial y\partial z} + \frac{\partial^2 f_2}{\partial z\partial x} - \frac{\partial^2 f_1}{\partial z\partial y}$$

$$\boxed{div(curl\ \bar{f}) = 0}$$

(2) Define divergence and solenoidal of vector?

Sol:-

Divergence of a vector:-

If $\bar{F}(x,y,z)$ is a continuously differentiable vector point function in a given region of space, then the divergence of $\bar{F}$ is defined by

$$\nabla\cdot\bar{F} = \operatorname{div}\bar{F} = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)(F_1\bar{i}+F_2\bar{j}+F_3\bar{k})$$

$$= \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$$

Solenoidal vector:-

A vector $\bar{F}$ is said to be solenoidal if $\operatorname{div}\bar{F}=0$ i.e $\nabla\cdot\bar{F}=0$.

(3) Find a unit normal vector to the surface $z=x^2+y^2$ at $(-1,-2,5)$

Sol:- Given $\varphi = x^2+y^2=z$

$$\varphi = x^2+y^2-z$$

$$\nabla\varphi = \bar{i}\frac{\partial\varphi}{\partial x}+\bar{j}\frac{\partial\varphi}{\partial y}+\bar{k}\frac{\partial\varphi}{\partial z}$$

$$= \bar{i}(2x)+\bar{j}(2y)+\bar{k}(-1)$$

$$(\nabla\varphi)_{(-1,-2,5)} = -2\bar{i}-4\bar{j}-\bar{k}$$

$$|\nabla\phi| = \sqrt{(-2)^2+(-4)^2+(-1)^2}$$

$$= \sqrt{4+16+1}$$

$$= \sqrt{21}$$

unit normal $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-2\bar{i}-4\bar{j}-\bar{k}}{\sqrt{21}}$

(4) Find the directional derivative of $f(x,y,z) = 2x^2+3y^2+z^2$ at $P(2,1,3)$ in the direction of $a = [1,0,-2]$ in which direction it is maximum and what is it's maximum value?

(or)

Find directional derivative of $f(x,y,z) = 2x^2+3y^2+z^2$ at the point $P(2,1,3)$ in the direction of the vector $\bar{a} = \bar{i}-2\bar{k}$ in which direction it is maximum and what is it's maximum value?

Sol:- Given $f(x,y,z) = 2x^2+3y^2+z^2$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$= i(4x)+j(6y)+k(2z)$$

$$(\nabla f)_{(2,1,3)} = 8\bar{i}+6\bar{j}+6\bar{k}$$

Given $\bar{a} = \bar{i} - 2\bar{k}$

$$|\bar{a}| = \sqrt{(1)^2+(-2)^2}$$

$$= \sqrt{1+4}$$

$$|\bar{a}| = \sqrt{5}$$

$$\bar{e} = \frac{\bar{a}}{|\bar{a}|} = \frac{\bar{i}-2\bar{k}}{\sqrt{5}}$$

$$D \cdot D = \bar{e} \cdot \nabla f$$

$$= \frac{1}{\sqrt{5}}(\bar{i}-2\bar{k})(8\bar{i}+6\bar{j}+6\bar{k})$$

$$= \frac{8-12}{\sqrt{5}}$$

$$= \frac{-4}{\sqrt{5}}$$

Greatest value or maximum value of ~~magnitude~~ D · D of $f = |\nabla f|$ at $(2,1,3)$

$$= \sqrt{64+36+36}$$

$$= \sqrt{136}$$

5) P.T $\operatorname{div}(\operatorname{curl} \bar{F}) = 0$

5 (A) P.T $\operatorname{div}(u \times v) = v \cdot \operatorname{curl} u - u \cdot \operatorname{curl} v$;

(or)

If $\bar{A}$ & $\bar{B}$ are vector point functions

Then $\nabla \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot (\nabla \times \bar{A}) - \bar{A} \cdot (\nabla \times \bar{B})$?

Sol :- $\nabla \cdot (\bar{A} \times \bar{B}) = \sum \bar{i} \cdot \frac{\partial}{\partial x} (\bar{A} \times \bar{B})$

$$= \sum \bar{i} \left(\bar{A} \times \frac{\partial \bar{B}}{\partial x} + \frac{\partial \bar{A}}{\partial x} \times \bar{B} \right)$$

$$= \sum \bar{i} \left(\bar{A} \times \frac{\partial \bar{B}}{\partial x} \right) + \sum \bar{i} \left(\frac{\partial \bar{A}}{\partial x} \times \bar{B} \right)$$

$$= -\left(\sum \bar{i} \times \frac{\partial \bar{B}}{\partial x} \right) \cdot \bar{A} + \left(\sum \bar{i} \times \frac{\partial \bar{A}}{\partial x} \right) \cdot \bar{B}$$

$$= -(\nabla \times \bar{B}) \cdot \bar{A} + (\nabla \times \bar{A}) \cdot \bar{B}$$

$$\therefore \boxed{\nabla \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot (\nabla \times \bar{A}) - \bar{A} \cdot (\nabla \times \bar{B})}$$

(6) (B) Find a potential $f = \text{grad}\, v$ for given $v = [4x^3, 3y^2, -6z]$?

(or)

P.T $\bar{F} = 4x^3\bar{i} + 3y^2\bar{j} - 6z\bar{k}$ is a scalar potential of $\bar{F}$?

Sol: Given ~~$\bar{F} = \nabla\phi$~~ $\bar{F} = 4x^3\bar{i} + 3y^2\bar{j} - 6z\bar{k}$

To find ϕ such that $\bar{F} = \nabla\phi$

$$\Rightarrow 4x^3\bar{i} + 3y^2\bar{j} - 6z\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

Equating the co-efficients of $\bar{i}$, $\bar{j}$ and $\bar{k}$ we get

$$\frac{\partial\phi}{\partial x} = 4x^3, \quad \frac{\partial\phi}{\partial y} = 3y^2 \quad \frac{\partial\phi}{\partial z} = -6z$$

Integrating the above equations partially with repect to x, y, z re

$$\int\frac{\partial\phi}{\partial x}dx = \int 4x^3 dx + f_1(y,z).$$

$$\phi = 4\left(\frac{x^4}{4}\right) + f_1(y,z)$$

$$\boxed{\phi = x^4 + f_1(y,z)}$$

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$$\int \frac{\partial \varphi}{\partial y}\, dy = \int 3y^2\, dy + f_2(x, z)$$

$$\Rightarrow \varphi = 3\left(\frac{y^3}{3}\right) + f_2(x, z)$$

$$\Rightarrow \boxed{\varphi = y^3 + f_2(x, z)}$$

$$\int \frac{\partial \varphi}{\partial z}\, dz = \int -6z\, dz + f_3(x, y)$$

$$\varphi = -\cancel{6}^{3}\left(\frac{z^2}{\cancel{2}}\right) + f_3(x, y)$$

$$\boxed{\varphi = -3z^2 + f_3(x, y)}$$

$$\therefore \varphi = x^4 + f_1(y, z)$$

$$\varphi = y^3 + f_2(x, z)$$

$$\varphi = -3z^2 + f_3(x, y)$$

$\therefore \varphi = x^4 + y^3 - 3z^2 + C$ where C is constant

$\therefore \varphi$ is scalar potential of $\vec{F}$