1) Weak form derivation

$\nabla^2 u(\underline{r}) = f(\underline{r})$, boundary values include Dirichlet on Γ_D and Neumann on Γ_N .

Multiply from left by the test function $w(\underline{r})$ and integrate over domain

$$\int_\Omega w(\underline{r})\, \nabla^2 u(\underline{r})\, d\underline{r} = \int_\Omega w(\underline{r})\, f(\underline{r})\, d\underline{r} \Big| \text{ Greens identity}$$

$$-\int_\Omega \nabla w(\underline{r}) \cdot \nabla u(\underline{r})\, d\underline{r} + \int_{\partial\Omega} w(\underline{r})\, \nabla u(\underline{r}) \cdot d\underline{S} = \int_\Omega w(\underline{r})\, f(\underline{r})\, d\underline{r}$$

Neumann boundary conditions: $\nabla u(\underline{r}) \cdot \underline{n} = N$ (on Γ_N)

$$-\int_\Omega \nabla w \cdot \nabla u\, d\underline{r} + \int_{\partial\Gamma_N} wN\, dS + \int_{\partial\Gamma_D} w \nabla u\, d\underline{S} = \int_\Omega wf\, d\underline{r}$$

Function spaces: $w \in V = \{ w \in H^1(\Omega) : w|_{\Gamma_D} = 0 \}$

$u \in U = \{ u \in H^1(\Omega) ; u|_{\Gamma_D} = D \}$

So the final weak form is

$$-\int_\Omega \nabla w \cdot \nabla u\, d\underline{r} + \int_{\partial\Gamma_N} wN\, dS = \int_\Omega wf\, d\underline{r}$$

b) $u'' + 2u' + u = f$, Dirichlet boundary: $u(\pm L) = 0$

$w(\pm L) = 0$

$$\Rightarrow \int_\Omega w(u'' + 2u' + u)\,dx = \int_\Omega wf\,dx$$

$$\underbrace{\Big/_{-L}^{L} u'w}_{=0} - \int_\Omega w'u'\,dx + \int_\Omega w2u'\,dx + \int_\Omega wu\,dx = \int_\Omega wf\,dx$$

$$\int_{-L}^{L} (-w'u' + 2wu' + wu)\,dx = \int_{-L}^{L} wf\,dx$$

c) $-\frac{1}{2}\frac{\partial^2}{\partial x^2}\phi(x) + V(x)\phi(x) = E\phi(x)$, $x \in [0, L]$

$\phi(0) = \phi(L) = 0$

$$\int_0^L \left(-\frac{1}{2}w\phi'' + wV\phi\right)dx = \int_0^L wE\phi\,dx$$

$$\underbrace{\Big/_0^L -\frac{1}{2}\cdot w\phi'}_{=0} + \int_0^L \left(\frac{1}{2}w'\phi' + wV\phi\right)dx = \int_0^L wE\phi\,dx$$

$$\int_0^L \frac{1}{2}w'\phi'\,dx + \int_0^L wV\phi\,dx = \int_0^L wE\phi\,dx$$

d) $\nabla^2 u(\underline{r}) = f(\underline{r}) + v(\underline{r}) \quad (1)$ Neumann: $\nabla u \cdot \underline{n} = N_u$

$\nabla^2 v(\underline{r}) = g(\underline{r}) + u(\underline{r}) \quad (2)$ $\qquad \nabla v \cdot \underline{n} = N_v$

(1): $\int_\Omega w_1(\underline{r}) \nabla^2 u(\underline{r}) d\underline{r} = \int_\Omega w_1(\underline{r}) f(\underline{r}) d\underline{r} + \int_\Omega w_1(\underline{r}) v(r) d\underline{r}$

$-\int_\Omega \nabla w_1 \nabla u \, d\underline{r} + \int_{\partial\Omega} w_1 \nabla u \, d\underline{S} = \int_\Omega w_1 f \, d\underline{r} + \int_\Omega w_1 v \, d\underline{r}$ | Neumann boundary

$-\int_\Omega \nabla w_1 \nabla u \, d\underline{r} + \int_{\partial\Gamma_N} w_1 N_u \, dS + \int_{\partial\Gamma_D} w_1 \nabla u \, d\underline{S} = \int_\Omega w_1 f \, d\underline{r} + \int_\Omega w_1 v \, d\underline{r}$

(2): (Similarly)

$-\int_\Omega \nabla w_2 \nabla v \, d\underline{r} + \int_{\partial\Gamma_N} w_2 N_v \, dS + \int_{\partial\Gamma_D} w_2 \nabla v \, d\underline{S} = \int_\Omega w_2 g \, d\underline{r} + \int_\Omega w_2 u \, d\underline{r}$

$$\Rightarrow \begin{cases} -\int_\Omega \nabla w_1 \nabla u \, d\underline{r} + \int_{\partial\Gamma_N} w_1 N \, dS + \int_{\partial\Gamma_D} w_1 \nabla u \, d\underline{S} = \int_\Omega w_1 (f+v) \, d\underline{r} \\ -\int_\Omega \nabla w_2 \nabla v \, d\underline{r} + \int_{\partial\Gamma_N} w_2 N \, dS + \int_{\partial\Gamma_D} w_2 \nabla v \, d\underline{S} = \int_\Omega w_2 (g+u) \, d\underline{r} \end{cases}$$