Exercise 1. A simple congruential random number generator

Goal: We implement a basic congruential random number generator (proposed by Lehmer in 1948).

Write a program that generates random numbers according to

$$x_i = (cx_{i-1}) \bmod p.$$

At first, consider $c = 3$ and $p = 31$.

Task 1: Check your generated random numbers for correlations using the square test, i. e. plot x_i vs x_{i+1}. Do you see a pattern?

Task 2: Repeat the same using the cube test, i. e. plot x_i vs x_{i+1} vs x_{i+2}.

Task 3: Repeat the same for a different random number generator. This can be achieved e.g. by choosing a different c and p.

Exercise 2. The χ^2-test

Goal: We get to know to another random number generator test.

Chapter 3.1.4 of the lecture notes: *The distribution [of a sequence of uniform random numbers] around the mean value should behave like a Gaussian distribution.*

Task 1: Understand why this is the case and verify this fact with a numerical experiment.

The χ^2-test is performed as follows:

1. Divide the range of random numbers into k bins i.e discrete intervals of the same size such that the probability of a random number to be in the interval i is given by $p_i = 1/k$.
2. Using each random number generator, generate at least one sequence of n numbers. For each sequence, measure the count N_i of random numbers in each interval i.
3. Compute the χ^2-value for one specific sequence of random numbers

$$\chi^2 = \sum_{i=1}^{k} \frac{(N_i - np_i)^2}{np_i}.$$

4. Use the table `chi_square_description.pdf` (from Donald E. Knuth, The Art of Computer Programming, Volume 2) to check the reliability of your random number generators.

Hint: A reasonable choice is $k = 10, n = 1000$.

Task 2: Test your random number generators from exercise 1 using the χ^2-test.

Exercise 3. Random numbers on the surface of a sphere

> *Goal: We learn how to distribute random numbers uniformly on the surface of a sphere.*

Task 1: Consider a sphere of radius R. Generate a homogeneous distribution of random points on the surface of the sphere using the transformation method.

Task 2: Visualize your results.

Hint: Be careful to consider the proper surface element dA and not only the angles φ and θ.

Exercise 4. Random numbers within an ellipse

> *Goal: We learn how to distribute points uniformly within an ellipse.*

Task 1: Use the rejection method to sample points uniformly within an ellipse with semi-major axis a and semi-minor axis b.

Task 2: Use the transformation method instead.

Hint: Consider the angle φ and the radius r separately. Find an inversion formula for φ first, and then one for r given a fix value of φ. This is possible because of the definition of conditional probabilities:

$$p(\varphi, r) = p(r|\varphi)p(\varphi).$$

Hint: The polar form of an ellipse describes the relationship between the radial and the angular polar coordinates for any point on the ellipse surface:

$$R(\varphi) = \frac{ab}{\sqrt{(b\cos\varphi)^2 + (a\sin\varphi)^2}}.$$

It might prove useful.

Hint:

$$\int \frac{1}{\cos^2\varphi + c^2\sin^2\varphi} d\varphi = \frac{1}{c}\arctan\left(c\tan(\varphi)\right) + \text{const.}$$

Hint: The area of an ellipse is $A = \pi ab$.

Exercise 5. The D^*-discrepancy

> *Goal: We understand Eq.* 3.10 *of the lecture notes and get a qualitative understanding of the quality of an RNG.*

Task: Implement the formula for $D_N^*(x_1, \ldots, x_N)$, Eq. 3.10 of the lecture notes, and compare the RNG from exercise 1 with a Halton sequence:

$$D_N^*(\mathbf{x}_1, ..., \mathbf{x}_N) = \sup_{\mathbf{v}} \left| \frac{1}{N} \sum_{i=1}^{N} \prod_{j=1}^{d} 1_{0 \le x_i^j \le v^j} - \prod_{j=1}^{d} v^j \right|.$$

Hint: You might want to install and use the Julia package "HaltonSequences". A Halton sequence can be generated e.g. with `h = Halton(29, start=1234, length=1024)`.