Section Two: Calculator-assumed

CALCULATOR ASSUMED

65% (96 Marks)

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(7 marks)**

The capacity, X mL, of glass bottles made in a factory can be modelled by a normal distribution with mean μ and standard deviation 1.5 mL.

(a) If $\mu = 364$, determine

(i) $P(X \geq 362)$. $= 0.9088$ ✓ correct probability. (1 mark)

(ii) $P(X < 362 \mid X < 363)$. (2 marks)

$$= \frac{P(X < 362)}{P(X < 363)}$$

$$= \frac{0.0912}{0.2525}$$

$$= 0.3612$$

✓ numerator
✓ probability.

(iii) the value of x, if $P(X \geq x) = \frac{3}{11}$. (1 mark)

$x = 364.9$ ✓ correct value

(b) Given that $P(X < k) = 0.119$,

(i) determine the value of μ in terms of k. (2 marks)

$$P\left(x < \left(\frac{K-\mu}{1.5}\right)\right) = 0.119$$

$$\frac{K-\mu}{1.5} = -1.18$$

$$\therefore \mu = K + 1.77$$

✓ correct z score $\frac{K-\mu}{1.5}$
✓ expression for μ.

(ii) determine μ if $k = 95$. (1 mark)

$\mu = 96.77\text{ mL}$

✓ correct value.

Question 10

(9 marks)

A fair die has one face numbered 1, three faces numbered 2 and two faces numbered 3.

(a) Determine the probability that the second even number occurs on the fourth throw of the dice.
(3 marks)

$${}^3C_2 \times \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) \times \frac{1}{2}$$

$$= \frac{3}{16}.$$

✓ $P(\text{even}) = \frac{1}{2}$

✓ calculates $P(1 \text{ even in } 3 \text{ throws}) = \frac{3}{8}$

✓ calculates Probability

(b) The die is thrown twice and X is the sum of the two scores.

(i) Complete the table below to show the probability distribution of X. (2 marks)

x	2	3	4	5	6
$P(X = x)$	$\frac{1}{36}$	$\frac{1}{6}$	$\frac{13}{36}$	$\frac{1}{3}$	$\frac{4}{36} = \frac{1}{9}$

	1	2	2	2	3	3
1	2					
2						
2						
2						
3						
3						

✓ $P(X=2)$

✓ $P(X=6)$

(ii) Determine $P(X = 5 \mid X \geq 5)$. (2 marks)

$$P(X=5 \mid X \geq 5) = \frac{P(X=5)}{P(X \geq 5)}$$

$$= \frac{\frac{1}{3}}{\frac{4}{9}}$$

$$= \frac{3}{4}$$

✓ $P(X \geq 5)$

✓ Probability.

(iii) Calculate $E(X)$. (2 marks)

$$E(X) = \frac{2}{36} + \frac{18}{36} + \frac{52}{36} + \frac{60}{36} + \frac{24}{36}$$

$$= \frac{156}{36}$$

$$= \frac{13}{3}$$

✓ sums $x \cdot P(x)$

✓ $E(X)$

Question 11 **(8 marks)**

From a random survey of 524 users of a free music streaming service, it was found that 386 would stop using it if they had to pay.

(a) Based on this survey, calculate the percentage of users who would stop using the service. (1 mark)

$\frac{386}{524}$ 73.7%.

✓ correct % calculated.

(b) Calculate the approximate margin of error for a 90% confidence interval estimate of the proportion of users who would stop using the service. (3 marks)

$$E = 1.645\sqrt{\frac{0.7366(1-0.7366)}{524}}$$

$$= 0.0317$$

(Use $\frac{386}{524}$ in calculation)

✓ 1.645

✓ Show working in formula

✓ Margin of Error

(c) Determine a 90% confidence interval for the proportion of users who would stop using the service. (2 marks)

(0.705, 0.768) writes interval ✓✓

(d) If 50 identical surveys were carried out and a 90% confidence interval for the proportion was calculated from each survey, determine the probability that exactly 48 of the intervals will contain the true value of the proportion. (2 marks)

$X \sim B(50, 0.9)$

$P(X=48) = 0.0779$

✓ Identifies Binomial + gives parameters

✓ Calculates probability

Question 12 **(7 marks)**

The lifetime, T hours, of an electronic component is a continuous random variable with probability density function given by

$$f(t) = 0.005e^{-0.005t}, \qquad 0 \le t < \infty.$$

(a) Determine the probability that a randomly chosen component has a lifetime of less than 450 hours. (2 marks)

$$P(T < 450) = \int_0^{450} f(t)\,dt$$
$$= 0.8946$$

✓ writes integral
✓ evaluates.

(b) An engineer buys 12 of the components. If they operate independently of each other, determine the probability that at least 11 of them will not last 450 hours. (2 marks)

$$X \sim B(12, 0.8946)$$
$$P(X \geq 11) = 0.6342$$

✓ identifies Binomial and parameters

✓ probability

(c) A component has already been operating for exactly 440 hours. Determine the probability that it will fail within the next 36 hours. (3 marks)

$$P(T < 476 \mid T > 440)$$
$$= \frac{P(440 < T < 476)}{P(T > 440)}$$
$$= \frac{\int_{440}^{476} f(t)\,dt}{\int_{440}^{\infty} f(t)\,dt}$$
$$= \frac{0.01825}{0.110803} = 0.1647$$

✓ $P(T > 440)$
✓ $P(440 < T < 476)$
✓ Probability

Question 13

(8 marks)

160 black and 840 white spherical beads, identical except for their colour, are placed in a container and thoroughly mixed. $\frac{160}{1000} = 0{\cdot}16$

In experiment A, a bead is randomly selected, its colour noted and then replaced until a total of 20 beads have been selected.

(a) The random variable X is the number of black beads selected in experiment A. Determine $P(X > 5)$. (2 marks)

$X \sim B(20, 0{\cdot}16)$

$P(X \geq 6) = 0{\cdot}0870$

✓ Identifies Binomial RV Parameters

✓ Probability

(b) Experiment A is repeated 10 times. Determine the probability that at least one black bead is selected in each of these experiments. (2 marks)

$P(X \geq 1) = 0{\cdot}9694$

$(0{\cdot}9694)^{10} = 0{\cdot}7329$

✓ $P(X \geq 1)$

✓ Probability.

In experiment B, a bead is randomly selected, its colour noted and then replaced until a total of 65 beads have been selected.

Experiments A and B are repeated a large number of times, with the proportions of black beads in each experiment, $\hat{p}_A$ and $\hat{p}_B$ respectively, recorded.

(c) The distribution of which proportion, $\hat{p}_A$ or $\hat{p}_B$, is most likely to approximate normality? Explain your answer and state the mean and standard deviation of the normal distribution for the proportion you have chosen. (4 marks)

$\hat{p}_B$, because sample size is larger

Mean $= 0{\cdot}16$

sd $= \sqrt{\frac{(0{\cdot}16)(0{\cdot}84)}{65}}$

$= 0{\cdot}045$

✓ select $\hat{p}_B$

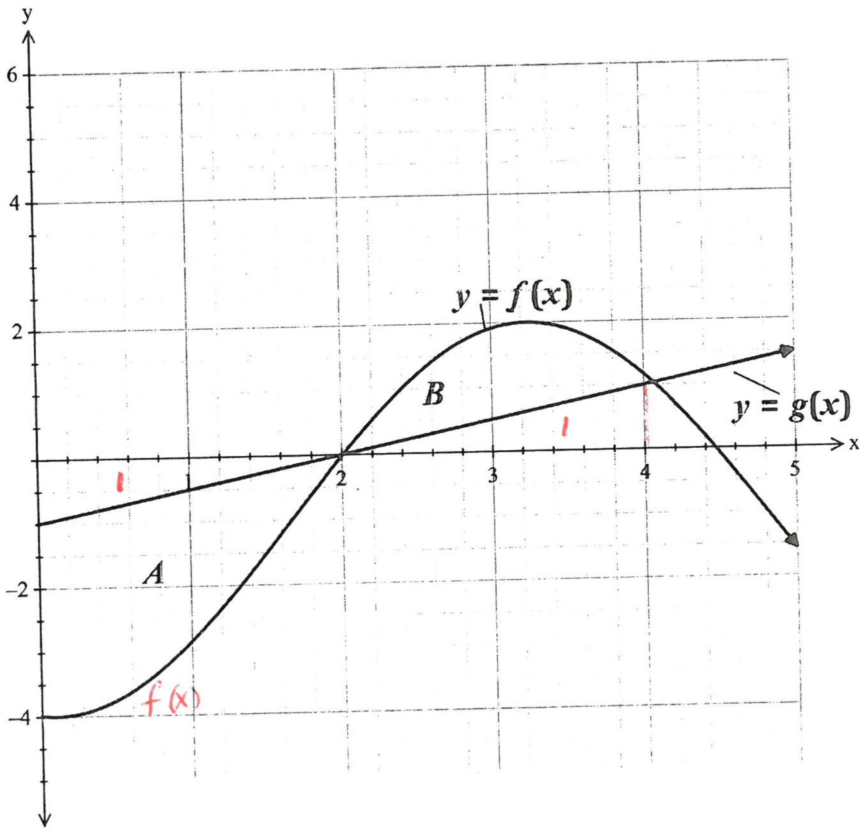
✓ explains why

✓ mean

✓ st devn

Question 14 **(6 marks)**

Region A, on the graph below, is defined as the area enclosed between the $y-$ axis, $g(x)$ and $f(x)$ while **Region B**, is defined as the area enclosed between $g(x)$ and $f(x)$ and $2 \le x \le 4$.



The following information is known:

$$\int_0^2 f(x)dx = -5.1$$

$$\int_0^4 f(x)dx = -2.18$$

(i) Determine the area of Region A. (2 marks)

✓ Area Δ
✓ Area Region A

RegionA $= \left(-\int_0^2 f(x)\,dx\right) - 1$
$= 5.1 - 1$
$= 4.1$

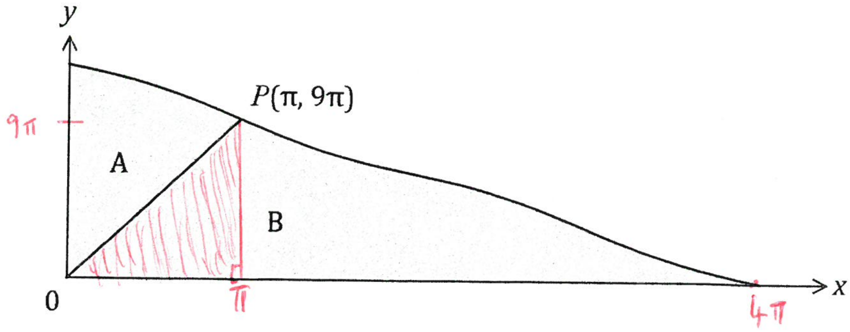
(ii) Show that the area of Region B = 1.92 (4 marks)

✓ Defines region B using integrals
✓ Rearranges to use info given + shows use
✓ shows resut

Region B $= \int_2^4 f(x)\,dx - \int_2^4 g(x)\,dx$
$= \int_0^4 f(x)\,dx - \int_0^2 f(x)\,dx - \int_2^4 g(x)\,dx$
$= -2.18 - (-5.1) - 1$
$= 1.92$

Question 15 **(7 marks)**

The curve $y = 12\pi - 3x + \sin x$ is shown below passing through $P(\pi, 9\pi)$.



A straight line joins the origin to P, dividing the shaded area into two regions, A and B.

(a) Show that when $x = 4\pi$, $y = 0$. (1 mark)

$y = 12\pi - 3(4\pi) + \sin(4\pi)$ ✓ substitutes to show

$= 0$

(b) Determine the exact value of $\int_0^{\pi} (12\pi - 3x + \sin x)\, dx$. (2 marks)

$= \frac{21\pi^2}{2} + 2$ (105.63)

✓ evaluates
✓ exactly.

(c) Determine the ratio of the area of region A to the area of region B in the form $1{:}k$. (4 marks)

$A = \int_0^{\pi} (12\pi - 3x + \sin x)\, dx - \left(\frac{1}{2}\pi \cdot 9\pi\right)$

$= \left(\frac{21\pi^2}{2} + 2\right) - \frac{9\pi^2}{2}$ ✓ Area A

$= 6\pi^2 + 2$ ✓ A

$B = \int_{\pi}^{4\pi} (12\pi - 3x + \sin x)\, dx + \frac{9\pi^2}{2}$ ✓ B

$= \left(\frac{27\pi^2}{2} - 2\right) + \frac{9\pi^2}{2}$ ✓ 1:K ratio

$= 18\pi^2 - 2$

Ratio A : B $\frac{6\pi^2+2}{18\pi^2-2}$ $1 : 2.869$

Question 16 **(8 marks)**

A researcher wants to estimate the proportion of Western Australian school-aged students who participate in organised sport during school holidays. The researcher plans to collect sample data by visiting schools and asking students.

(a) Discuss two different sources of bias that may occur when the researcher collects their sample data and suggest a procedure to avoid bias. (4 marks)

- All students should have equal chance of selection, otherwise could favour age, gender etc. ie under-coverage
- Some students may choose not to answer

✓ discuss source of bias (1)
✓ discuss source of bias (2)

To Avoid Bias – Simple Random sampling Give each student a number & select at random
– Systematic sampling - number every student and select every k^{th} student.

✓ suggests suitable sampling technique to avoid bias
✓ explains procedure suggested

(b) Determine, to the nearest 10, the sample size the researcher should use to ensure that the margin of error of a 90% confidence interval is no more than 6%. (3 marks)

$$0.06 = 1.645\sqrt{\frac{(0.5)(0.5)}{n}}$$

187.92

$$n = 188$$

Sample size 190.

✓ assumes $\hat{p} = 0.5$

✓ shows equation

✓ solves for n to nearest 10

(c) Comment on how your answer to (b) would change if the researcher had a reliable estimate that the population proportion was close to 20%. (1 mark)

Sample size would decrease

✓ states decrease

Question 17 **(8 marks)**

The mass, X g, of wasted metal when a cast is made is a random variable with probability density function given by

$$f(x) = \begin{cases} \dfrac{x}{2a^2} & 0 \le x \le 2a, \\ 0 & \text{elsewhere,} \end{cases}$$

where a is a positive constant.

(a) Determine $\mathrm{E}(X)$ in terms of a. (2 marks)

$E(X) = \int_0^{2a} x \cdot \left(\frac{x}{2a^2}\right) dx$ ✓ correct integral

$= \frac{4a}{3}$ ✓ evaluates integral In terms of a

(b) The total mass of wasted metal from a random sample of 25 casts was 500 g. Estimate the value of a. (2 marks)

Mean $= \frac{500}{25} = 20$ ✓ calculates Sample mean

$\therefore \frac{4a}{3} = 20$

ie $a = 15$. ✓ determines a

(c) If $a = 9$, determine

(i) $P(X \ge 12)$. $\int_{12}^{18} \frac{x}{162} dx = \frac{5}{9}$ ✓ probability. (1 mark)

(ii) $\mathrm{Var}(X)$. $E(X) = \frac{4(9)}{3} = 12$ ✓ E(X) (3 marks)

$Var(X) = \int_0^{18} \frac{x}{162} \cdot (x-12)^2 \, dx$ ✓ correct ∫

$= 18$ ✓ evaluates Var(X)

Question 18 **(7 marks)**

A polynomial function $f(x)$ is such that $\int_2^6 4f(x)\,dx = 12$.

(a) Evaluate $\int_6^2 f(x)\,dx$ (2 marks)

$= -\frac{1}{4}\int_2^6 f(x)\,dx$

$= -\frac{12}{4}$

$= -3$

✓ reverse limits and −

✓ evaluates.

(b) Determine the value of $\int_2^3 (f(x)+3x^2)\,dx + \int_3^6 (1+f(x))\,dx$. (5 marks)

$\int_2^3 f(x)\,dx + \int_3^6 f(x)\,dx + \int_2^3 3x^2\,dx + \int_3^6 1\,dx$

$= \int_2^6 f(x)\,dx + \int_2^3 3x^2\,dx + \int_3^6 1\,dx$

$= 3 + 19 + 3$

$= 25$

✓ Shows use of linearity to split integrals

✓ $\int_2^6 f(x)\,dx$

✓ $\int_2^3 3x^2\,dx$

✓ $\int_3^6 1\,dx$

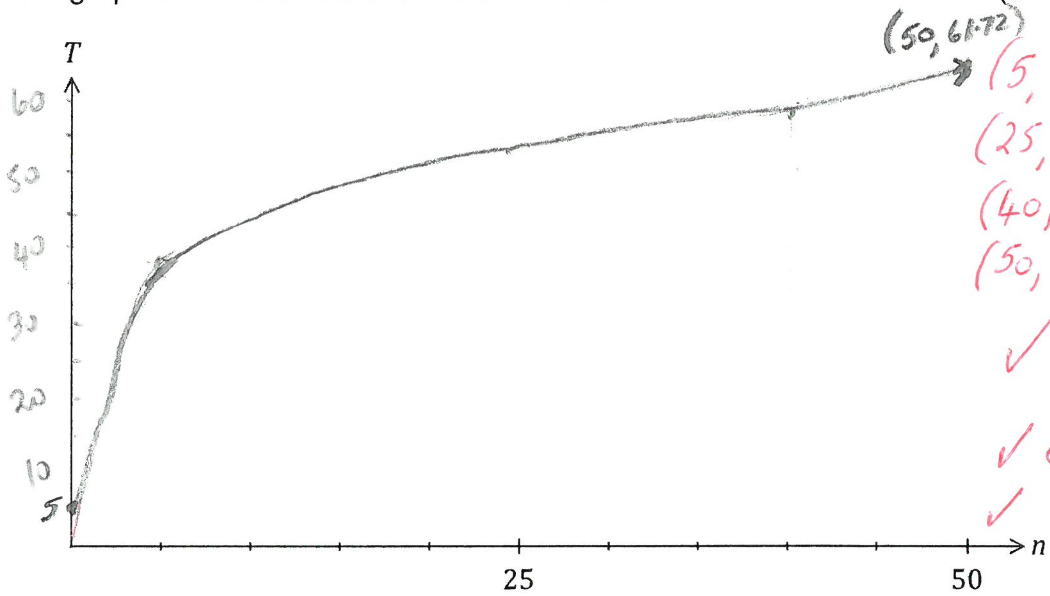
✓ Correct sum

Question 19 **(8 marks)**

Hick's law, shown below, models the average time, T seconds, for a person to make a selection when presented with n equally probable choices.

$$T = a + b\log_2(n+1), \text{ where } a \text{ and } b \text{ are positive constants.}$$

(a) Draw the graph of T vs n on the axes below when $a = 5$ and $b = 10$. (3 marks)



(5, 30.85)
(25, 52.004)
(40, 58.5)
(50, 61.72)
✓ add scale to y-axis
✓ endpoints
✓ log-shaped curve.

(b) When a pizzeria had 12 choices of pizza, the average time for patrons to make a choice was 40 seconds. After increasing the number of choices by 15, the average time to make their choice increased by 20%.

Modelling the relationship with Hick's law, predict the average time to make a choice if patrons were offered a choice of 16 pizzas. (5 marks)

$40 = a + b\log_2 13$ ✓ Eqn 1

$48 = a + b\log_2 28$ ✓ Eqn 2

$T = 13.256 + 7.227\log_2(16+1)$ ✓ solves for a

$T = 42.797$ ✓ for b

≈ 43 seconds ✓ states time to nearest second.

Question 20 **(7 marks)**

The acceleration of a particle is given by $a = 3\sin(2t)$ where distance is measure in metres and time, t, in seconds. Initially, the velocity of the particle is 4 m/s and its distance from the equilibrium position is 2 metres.

(a) Determine an equation for the velocity of the particle. (3 marks)

$$\frac{dx}{dt} = \int 3\sin(2t)\,dt$$

$$= -\frac{3\cos(2t)}{2} + c$$

✓ integration

$t = 0$
$v = 4$

$$4 = \frac{-3\cos 0}{2} + c$$

✓ evaluates c

$$c = 4 + \frac{3}{2}$$

$$= \frac{11}{2}$$

✓ equation for v

$$\frac{dx}{dt} = \frac{-3\cos(2t)}{2} + \frac{11}{2}$$

(b) Determine the displacement of the particle when $t = 2$ (accurate to 2 decimal places). (4 marks)

$$x = \int\left(\frac{-3\cos(2t)}{2} + \frac{11}{2}\right)dt$$

$$= -\frac{3\sin(2t)}{4} + \frac{11t}{2} + c$$

✓ correct integration

$t = 0$
$x = \pm 2.$

$$2 = -\frac{3\sin 0}{4} + 0 + c$$

✓ c values

$$c = 2$$

$$-2 = \frac{-3\sin 0}{4} + 0 + c$$

✓ x correct

✓ both c values

$$c = -2$$

$t = 2$

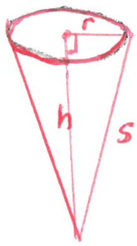
$$x = -\frac{3\sin(2t)}{4} + \frac{11t}{2} + 2 = 13.57\text{ m}$$

$$x = -\frac{3\sin(2t)}{4} + \frac{11t}{2} - 2 = 9.57\text{ m}$$

Question 21 **(6 marks)**

A popcorn container of capacity 660 mL is made from paper and has the shape of an open inverted cone of radius r and height h.

Determine the least area of paper required to make the container.



$A = \pi r s$

$s = \sqrt{h^2 + r^2}$

$\therefore A = \pi . r . \sqrt{h^2 + r^2}$ ✓ expresses A in terms of r and h

$V = \frac{1}{3}\pi r^2 h$

$660 = \frac{1}{3}\pi r^2 h$ ✓ expresses h in terms of r

$\therefore h = \dfrac{1980}{\pi r^2}$

$A = \pi r \sqrt{\left(\dfrac{3(660)}{\pi r^2}\right)^2 + r^2}$ ✓ expresses A in terms of r

$A = \pi r \sqrt{\left(\dfrac{1980}{\pi r^2}\right)^2 + r^2}$

$A = \pi r \sqrt{\dfrac{3920400}{\pi^2 r^4} + r^2}$

$\dfrac{dA}{dr} = \dfrac{2x^6\pi^2 - 3920400}{r^2\sqrt{r^6\pi^2 + 3920400}}$ ✓ differentiates A

$\dfrac{dA}{dr} = 0$ when $r = 7.638$ ✓ positive r for when $\dfrac{dA}{dr} = 0$

$A = 317.5 \text{ cm}^2$

✓ Min Area

End of Questions

Additional working space

Question number: _________