Test 2 : Thursday 17th March
Applications of Differentiation

This assessment contributes 4% towards the final year mark.

40 minutes are allocated for this test.

No notes of ANY nature are permitted.

CAS and scientific calculators are permitted for this section.

Full marks may not be awarded to correct answers unless sufficient justification is given.

Name : Solutions

Score : (out of 40)

Calculator Assumed

Do NOT turn over this page until you are instructed to do so.

Q1 **(15 marks)**

Consider the function $f(x) = e^x \sin(x)$ on the domain $-\pi \leq x \leq \pi$.

(a) ***Use a calculus method*** to answer the following questions:

i. Determine the gradient of $f(x)$ at the points $x = -\pi/4$ and $x = \frac{3\pi}{4}$. (3 marks)

$$f'(x) = e^x \sin x + e^x \cos x$$ ✓ Differentiates

$$f'\left(-\frac{\pi}{4}\right) = 0 \qquad f\left(\frac{3\pi}{4}\right) = 0$$ ✓✓ Evaluates each correctly

ii. Determine the concavity of $f(x)$ at the points $x = -\pi/4$ and $x = \frac{3\pi}{4}$. (3 marks)

$$f''(x) = 2e^x \cos x$$ ✓ uses f''

✓ evaluates both

$$f''\left(\frac{-\pi}{4}\right) = \sqrt{2}\, e^{-\pi/4} > 0 \qquad f''\left(\frac{3\pi}{4}\right) = -\sqrt{2}\, e^{3\pi/4} < 0$$

$\therefore$ concave up $\qquad$ $\therefore$ concave down

✓ concludes correctly

iii. Interpret your results from part i and ii above. (2 marks)

Local Min at $x = -\frac{\pi}{4}$ ✓ concludes correctly.

Local Max at $x = \frac{3\pi}{4}$ ✓

Q1 continued

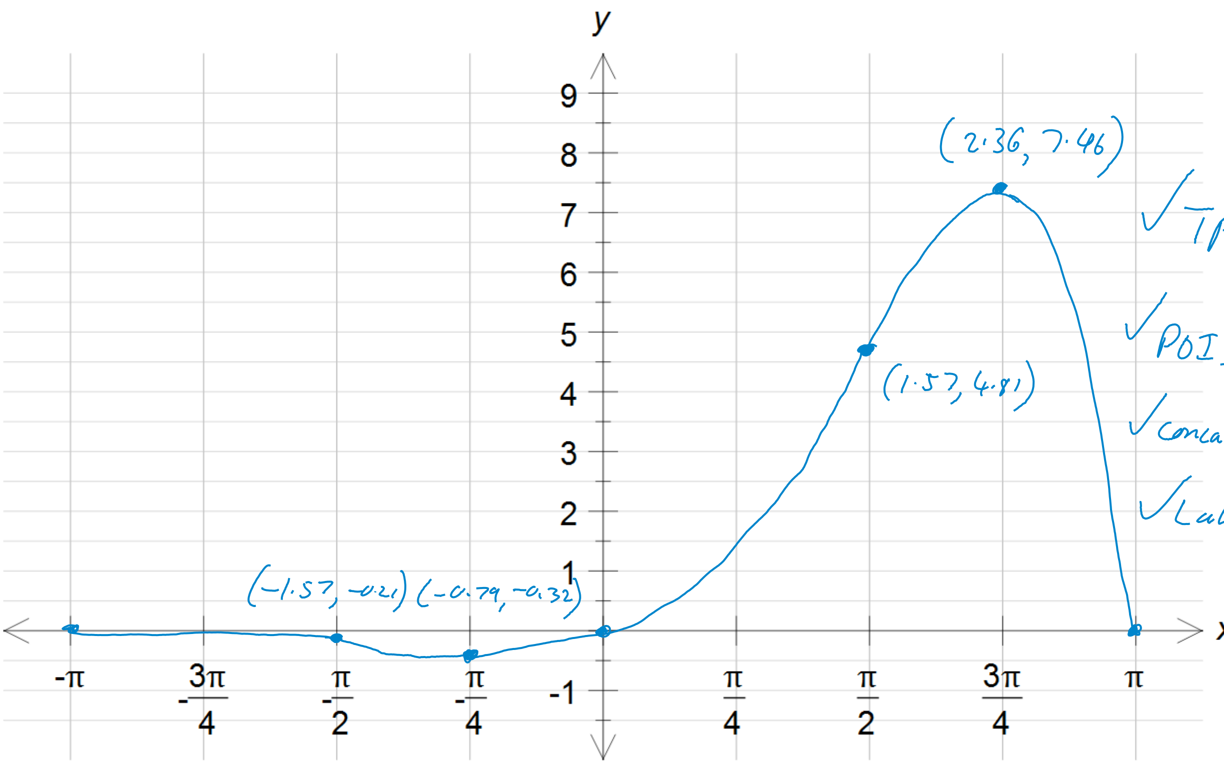
(b) The graph of $f(x)$ has two points of inflection on the domain $-\pi \le x \le \pi$.
Use calculus to determine the x-coordinates of each point of inflection. (3 marks)

$f''(x) = 0$ ✓ Recognises $f'' = 0$

$0 = 2e^{x} \cos x$

$x = -\frac{\pi}{2}, \frac{\pi}{2}$ ✓✓ solves for x

(c) Use the results above with the help of your CAS calculator to sketch the graph of $y = f(x)$ on the axes below.
Label the coordinates of the key features found above correct to 2 decimal places. (4 marks)



✓ TPs

✓ POIs

✓ concavity

✓ Labels

Q2. (4 marks)

Given that $y = x^{\frac{1}{3}}$, use $x = 1000$ and the increments formula $\delta y \approx \frac{dy}{dx} \times \delta x$ to determine an approximate value for $\sqrt[3]{1008}$. State your final answer accurate to 4 decimal places.

$\delta x = 8$

$\frac{dy}{dx} = \frac{1}{3x^{3/2}}$ ✓ Differentiates

$x = 1000$: $\frac{dy}{dx} = \frac{1}{300}$ ✓ Evaluates $\frac{dy}{dx}$ at $x = 1000$

$$\delta y \approx \frac{dy}{dx} \times \delta x$$

$$\approx \frac{1}{300} \times 8$$

$$\approx 0.02\dot{6}$$

✓ Evaluates δy

$y = \sqrt[3]{1000} = 10$

$y + \delta y \approx 10 + 0.02\dot{6}$

$\approx \underline{\underline{10.0267}}$ (4dp)

✓ States approximation to 4dp

Q3. **(5 marks)**

Some drinking cordial with an initial sugar concentration of 7 g/100mL is placed into a large jug of water and the rate of change of the concentration as measured at the point of entry is given by:

$$\frac{dC}{dt} = -0.63C \text{ g/100mL per minute}$$

Where $C = C(t)$ is the concentration of the sugar t minutes after being placed into the jug.

(a) Find the equation for C as a function t. (2 marks)

$$C(t) = 7e^{-0.63t}$$ ✓$_k$

✓$_{C_0}$

(b) Determine the initial rate of change of the sugar concentration, correct to 2 decimal places. (2 marks)

$$C'(0) = -0.63 \times C(0)$$
$$= -0.63 \times 7$$
$$= \underline{\underline{-4.41}} \text{ g/100mL/min}$$

✓-4.41 ✓units

(c) Determine how long it takes for the concentration of sugar to fall below 0.1 g/100mL, in minutes to 2 decimal places. (1 mark)

$$7e^{-0.63t} = 0.1$$

Solve: $t = \underline{\underline{6.74 \text{ min}}}$ ✓

Q4. (8 marks)
The height (h) of the water above the ground sprayed from the fountain at Elizabeth Quay rises and falls according to the equation:

$$h = A\sin(2t) - B\cos(2t) + k$$

Where t is the time in seconds after the fountain has been turned on and h is the height in metres above ground level. A, B and k are <u>positive</u> constants.

(a) Determine the initial height of the fountain in terms of A, B and k. (1 mark)

$h(0) = -B + k$

✓

(b) Describe the rate of change of the height of the fountain at $\frac{\pi}{4}$ seconds. (3 marks)

$h'(t) = 2A\cos(2t) + 2B\sin(2t)$ ✓ h'

$h'\left(\frac{\pi}{4}\right) = 2B > 0$ $(B > 0)$ ✓ Evaluates

$\therefore$ Height is increasing ✓ Concludes

(c) The fountain first reaches its minimum height when $t = \frac{\pi}{8}$ seconds.
Use the second derivative to show that $B > A$. (4 marks)

$h''(t) = -4A\sin(2t) + B\cos(2t)$ ✓ $h''(t)$

$h''\left(\frac{\pi}{8}\right) = -2\sqrt{2}A + 2\sqrt{2}B$ ✓ $h''(\pi/8)$

Minimum $\Rightarrow$ concave up

✓ links to concavity or $h'' > 0$

$\Rightarrow -2\sqrt{2}A + 2\sqrt{2}B > 0$

$2\sqrt{2}B > 2\sqrt{2}A$

$\therefore B > A$ ✓ Justifies Inequality

Q5. (8 marks)
A local church has 90 regular attendees at its Sunday service. The average donation per person each Sunday is $7.

It is estimated that as the church becomes more crowded, the generosity of each donation will decrease by $0.10 for every 5 extra attendees.

(a) Write an equation for the total amount of money, T, donated on Sundays in terms of the number of attendees, x. (2 marks)

$$\text{Donation} = 7 - 0.1\left(\frac{x-90}{5}\right)$$ ✓ donation amount

$$T(x) = x\left(7 - 0.1\left(\frac{x-90}{5}\right)\right)$$ ✓ $x \times$ donation amount

(b) Use calculus to determine the ideal amount of attendees required to maximise the donations given. (5 marks)

$$T'(x) = \frac{220-x}{25} = 0$$ ✓ $T'(x)$ ✓ $T'(x)=0$

$$x = 220$$ ✓ Finds stationary point

$$T''(x) = -\frac{1}{25} < 0$$ ✓ Proves maximum

$\therefore$ Concave down

$\therefore$ Local Maximum

$\therefore$ 220 people gives Maximum ✓ concludes.

(c) What is the largest possible amount that could be donated? (1 mark)

$$T(220) = \$968$$ ✓

END OF QUESTIONS