Question 1. [9 marks]

Differentiate the following with respect to x. Do not simplify unless specifically required.

a) $f(x) = 4x^2 + 4 + \frac{3}{x^2}$ [2]

$f'(x) = 8x - 6x^{-3}$

b) $y = \sqrt{(2x^2 + 5x)^3}$ [2]
(fully simplify)

$\frac{3}{2}(4x+5)(2x^2+5x)^{1/2}$

$\frac{(12x+15)\sqrt{2x^2+15x}}{2}$

c) $y = 2dx^4 + 3d^2$ [2]

$y' = 8dx^3$

d) $g(x) = \frac{6x-1}{3(4x+6)^2}$ [3]

$g'(x) = \frac{3(4x+6)^2(6) - (6x-1)[6(4x+6)]}{9(4x+6)^4}$

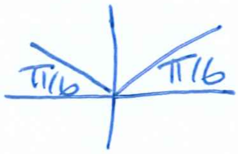
Question 2. [6 marks] $y = \cos(x + \pi)$

Find the points on the curve $y = \cancel{\sin^2 x}$ for $0 \le x \le 2\pi$ where the gradient of the curve is $\cancel{1}$ -1.

$y' = -2\sin(x + \pi)$

$-1 = -2\sin(x + \pi)$

$\frac{1}{2} = \sin(x + \pi)$



$x + \pi = \pi/6,\ \cancel{\frac{5\pi}{6}},\ \frac{13\pi}{6},\ \frac{17\pi}{6}$

$x = \frac{7\pi}{6},\ \frac{11\pi}{6}$

$x = \frac{7\pi}{6}$: $y = \cos\left(\frac{7\pi}{6} + \pi\right)$

$= \cos\left(\frac{13\pi}{6}\right)$

$= \cos\left(\frac{\pi}{6}\right)$

$= \frac{\sqrt{3}}{2}$

$x = 11\pi/6$:

$y = \cos\left(\frac{11\pi}{6} + \pi\right)$

$= \cos\left(\frac{17\pi}{6}\right)$

$= -\cos\left(\frac{\pi}{6}\right)$

$= -\frac{\sqrt{3}}{2}$

$\therefore$ pts $\left(\frac{7\pi}{6}, \frac{\sqrt{3}}{2}\right)$

& $\left(\frac{11\pi}{6}, -\frac{\sqrt{3}}{2}\right)$

Question 3. [9 marks]

a) Complete the following indefinite integral; [2]

$$\int (4x^3 + x^2 + 2)\, dx$$

$$x^4 + \frac{x^3}{3} + 2x + c.$$

b) Evaluate; $\int_0^4 (x+1)(2x-6)\, dx$ [3]

$$\int_0^4 2x^2 - 4x - 6\, dx$$

$$\left[\frac{2x^3}{3} - 2x^2 - 6x\right]_0^4$$

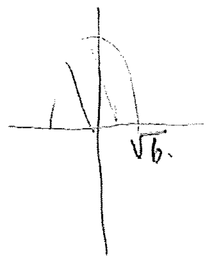
$$\left(\frac{128}{3} - 32 - 24.\right) - (0)$$

$$\frac{128}{3} - \frac{168}{3}$$

$$= \frac{-40}{3}.$$

c) Determine the **exact value** of the area bounded by the function $f(x) = -x^2 + 6$ and the x axis. [4]

$$2 \times \int_0^{\sqrt{6}} -x^2 + 6\, dx$$



$$2 \times \left[\frac{-x^3}{3} + 6x\right]_0^{\sqrt{6}}$$

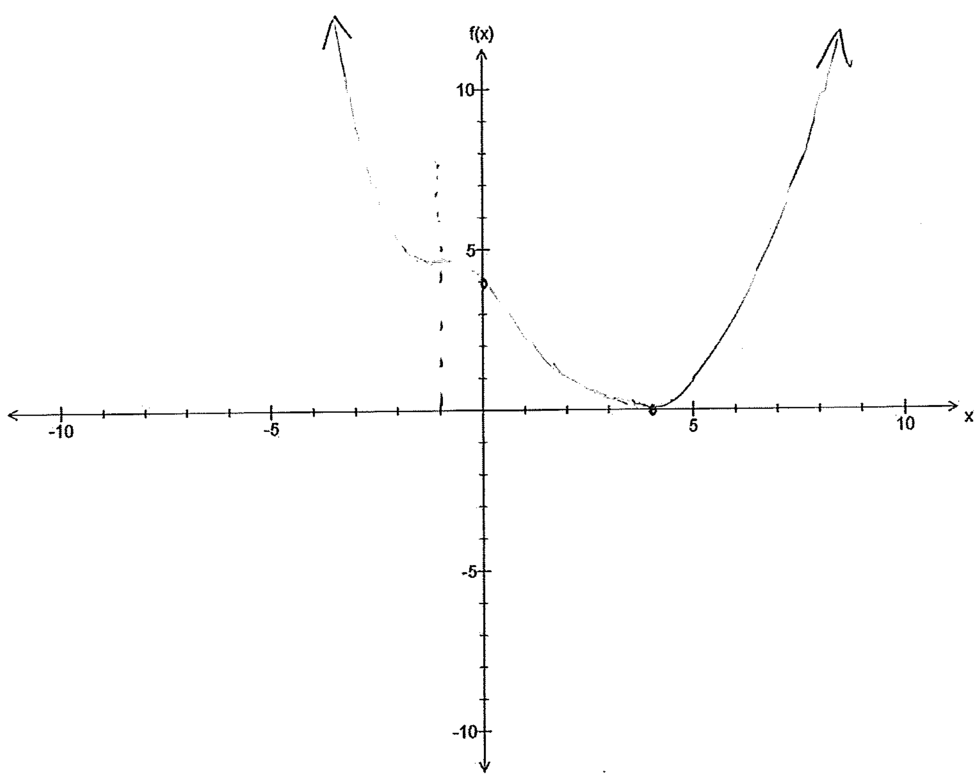
$$2 \times \left[\left(\frac{-6\sqrt{6}}{3} + 6\sqrt{6}\right) - 0\right]$$

$$8\sqrt{6} \text{ units}^2$$

Question 4. [5 marks]

Use the axis below to draw a sketch of a graph with the following characteristics.

- Both the x and y intercept are 4 and these are the only intercepts.
- $f'(x) = 0$ at $x = 4$
- $f'(-1) = f''(-1) = 0$
- Apart from $x = -1$ the graph has a negative gradient for $x < 4$
- The graph has a positive gradient when $x > 4$



Question 5. [8 marks]

(a) Simplify the following:

(i) $\frac{\log 16}{\log 2}$ $= \frac{\log 2^4}{\log 2} = \frac{4 \log 2}{\log 2}$

$= 4$

(ii) $\frac{2}{3}\log_2 8 + 6\log_2 \sqrt[3]{2} - \frac{1}{2}\log_2 \frac{1}{4}$

$= \frac{2}{3}\log_2 2^3 + 6\log_2 2^{1/3} - \frac{1}{2}\log_2 2^{-2}$

$= 3 \times \frac{2}{3} + 6 \times \frac{1}{3} - \frac{1}{2} \times -2$

$= 2 + 2 + 1$

$= 5$

(b) Solve the following equations:

(i) $6^{1-x} = 2^{3x+5}$

$(1-x)\log 6 = (3x+5)\log 2$

$\log 6 - x\log 6 = 3x\log 2 + 5\log 2$

$\log 6 - 5\log 2 = x(3\log 2 - \log 6)$

$x = \frac{\log(3/16)}{\log(4/3)}$

$\frac{6}{32} = \frac{3}{16}$

(ii) $6e^{1-2x} = 360$

$e^{1-2x} = 60$

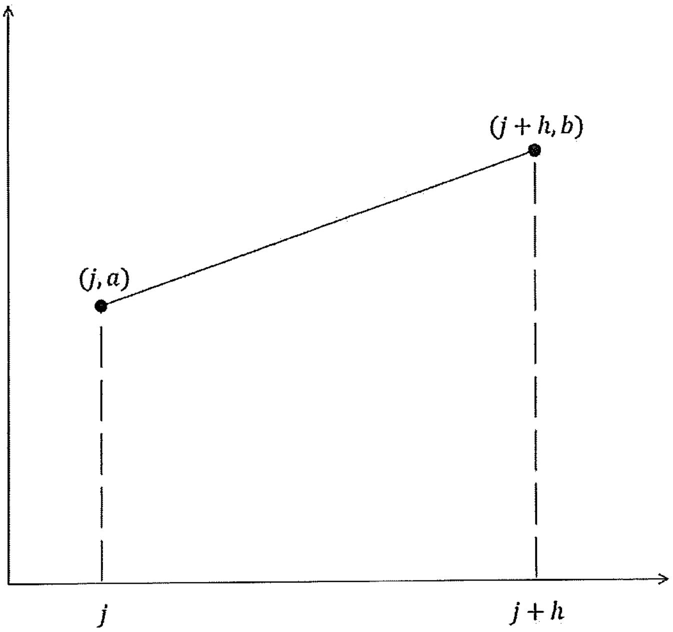
$1-2x = \ln 60$

$-2x = \ln 60 - 1$

$x = \frac{1 - \ln 60}{2}$

Question 6. [8 marks]

Consider the graph below of the function $f(x) = kx + n$ between the values of j and $j + h$.



a) Evaluate $\int_j^{j+h} f(x)\,dx$ (simplify your answer) [3]

$$\int_j^{j+h} kx + n \, dx.$$

$$= \left[\frac{kx^2}{2} + nx \right]_j^{j+h}$$

$$= \left[\frac{k(j+h)^2}{2} + n(j+h) \right] - \left[\frac{kj^2}{2} + nj \right]$$

$$= \cancel{\frac{kj^2}{2}} + kjh + \frac{kh^2}{2} + \cancel{nj} + hn - \cancel{\frac{kj^2}{2}} - \cancel{nj}$$

$$= kjh + \frac{kh^2}{2} + hn.$$

b) By determining the values of a and b in similar variables, show that the area of a trapezium is;

$$Area = \frac{1}{2}(a+b) \times perpendicular\ height$$ [5]

$f(j) = kj + n \Rightarrow a.$

$f(j+h) = k(j+h) + n \Rightarrow b.$

$$Area = \frac{1}{2}\left(kj+n + k(j+h)+n\right) \times h.$$
$$= \frac{1}{2}h\left(2kj + kh + 2n\right)$$
$$= kjh + \frac{kh^2}{2} + nh.$$
$$= \text{integral}.$$

Question 7. [5 marks]

The curve $y = px^3 + qx^2 - 4x$ has turning point at $x = -\frac{2}{3}$ and a point of inflection at $x = \frac{1}{6}$.
Determine the values of p and q.

$y' = 3px^2 + 2qx - 4.$

$0 = 3p\left(-\frac{2}{3}\right)^2 + 2q\left(-\frac{2}{3}\right) - 4.$

$0 = \frac{4}{3}p - \frac{4}{3}q - 4.$

$y'' = 6px + 2q$

$0 = 6p\left(\frac{1}{6}\right) + 2q$

$0 = p + 2q$

$-2q = p$

$\therefore \quad 0 = \frac{4}{3}(-2q) - \frac{4}{3}q - 4.$

$0 = -\frac{8q}{3} - \frac{4}{3}q - 4.$

$4 = \frac{-12q}{3}$

$\underline{\underline{-1. = q}}$

$\therefore \quad \underline{\underline{p = 2}}$

END OF SECTION ONE

Question 8. [7 marks]

A particle moves along a straight line such that its displacement, *y* metres at time *t* seconds is given by $y = 3\sin(2t) + 4$. Determine:

(a) An expression for the velocity of the particle at time *t*.

$V = 6\cos(2t)$

(b) The maximum velocity of the particle.

6 m/s.

(c) An expression for the acceleration of the particle at time *t*.

$a = -12\sin(2t)$

(d) The velocity of the particle when $t = \frac{\pi}{2}$.

$V = 6\cos(\pi)$

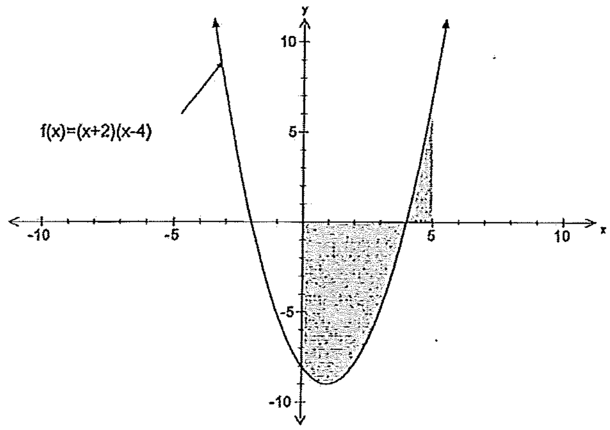
$= -6$ m/s.

Question 9. [4 marks]

a) Determine the area enclosed by the graphs of the two parabolas $f(x) = -x^2 + 5x + 1$ and $g(x) = 3x^2 - 15x + 17$ [2]

18 units^2.

b) Circle the integration statements that would give the <u>correct</u> answer to the area of the shaded region below. [2]



$\left|\int_0^5 f(x)dx\right|$

$-\int_0^4 f(x)\,dx + \int_4^5 f(x)\,dx$

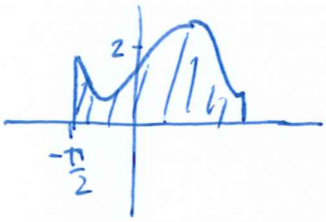
$\int_0^5 |f(x)|\,dx$

$\int_4^0 f(x)\,dx + \int_4^5 f(x)\,dx$

Question 10. [7 marks]

Using **calculus techniques**

(a) Find the exact area enclosed by the x-axis and the graph of $y = \sin(2x) + 2$ between $-\frac{\pi}{2} \le x \le \frac{3\pi}{4}$. [5]



$$\text{Area} = \int_{-\frac{\pi}{2}}^{3\pi/4} \sin(2x) + 2 \, dx$$

$$= -\frac{1}{2}\cos(2x) + 2x \Big]_{-\pi/2}^{3\pi/4}$$

$$= \left[-\frac{1}{2}\cos\left(\frac{3\pi}{2}\right) + \frac{6\pi}{4}\right] - \left(-\frac{1}{2}\cos(-\pi) - \pi\right]$$

$$= \frac{3\pi}{2} - \left(-\frac{2\pi+1}{2}\right)$$

$$= \frac{5\pi - 1}{2}$$

(b) Evaluate p if $\int_1^p \left(\frac{3}{2x-1}\right) dx = 2$ and $p > 1$.

$$\frac{3}{2} \ln|2x-1| \Big]_1^p = 2$$

$$\frac{3}{2}\left[\ln(2p-1) - \ln 0\right] = 2$$

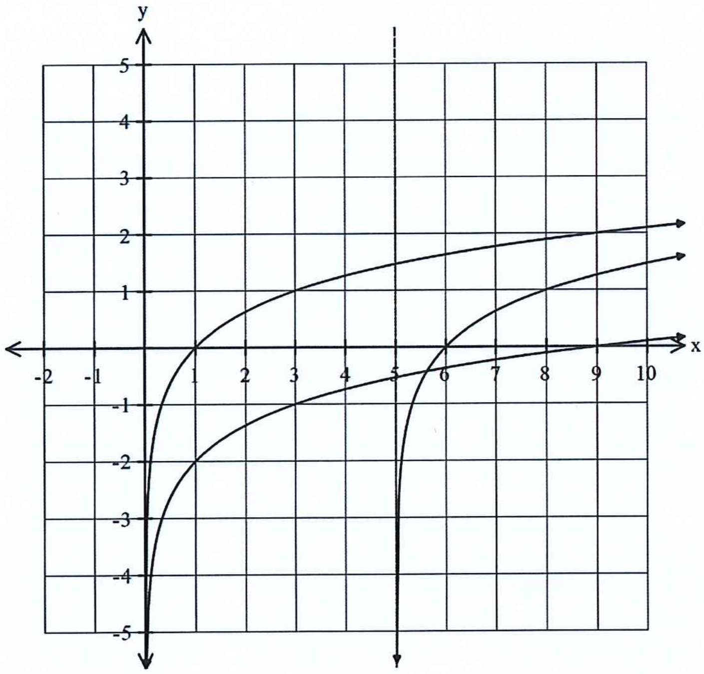
$$\ln(2p-1) = \frac{4}{3}$$

$$e^{4/3} = 2p - 1$$

$$p = \frac{e^{4/3} + 1}{2}$$

Question 11. [7 marks]

(a) On the axes below are the sketches of the functions $y = \log_a x$, $y = \log_a x + b$ and $y = \log_a (x - c)$. [3]



(i) Determine the value of *a*, *b* and *c*.

$a = 3$ $b = -2$ $c = 5$

(b) The formula $\text{pH} = \log[\text{H}^+]$ calculates the pH level where H^+ is the hydrogen ion concentration in moles/L.

(i) Calculate the hydrogen ion concentration if the pH is 6.89. [2]

$6.89 = \log H^+$

$H^+ = 7762471$

(ii) Calculate the pH if the hydrogen concentration in 1.25×10^{-8} . [2]

-7.9

Question 12. [4 marks]

Use your knowledge of antidifferentiation to determine $f(x)$ given that $f(3) = 72$, $f'(-2) = -20$ and $f''(x) = -12x$

$f'(x) = -6x^2 + c$

$f'(-2) = -20 = -6(-2)^2 + c$

$c = 4$

$f(x) = -2x^3 + 4x + d$

$f(3) = 72 = -2(3)^3 + 4(3) + d$

$d = 114$

$\therefore f(x) = -2x^3 + 4x + 114$

Question 13. [3 marks]

Evaluate $$\int_2^5 \frac{d}{dx}\left[\frac{x^2}{1-x^2}\right] dx$$

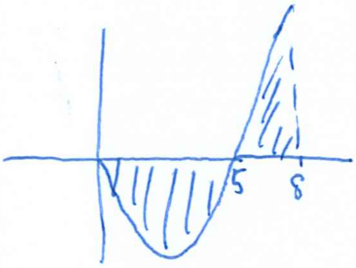
$= \left[\frac{25}{1-25}\right] - \left[\frac{4}{1-4}\right]$

$= \frac{25}{-24} - \frac{4}{-3}$

$= \frac{7}{24}$

Question 14. [12 marks]

Consider the functions $f(x)=\dfrac{\sqrt{x}}{2}(x^2-5x)$. Using calculus techniques, determine the area bound by the function and the x-axis for $0 \le x \le 8$.



$$\text{Area} = -\int_0^5 \frac{\sqrt{x}}{2}(x^2-5x)\,dx + \int_5^8 \frac{\sqrt{x}}{2}(x^2-5x)\,dx$$

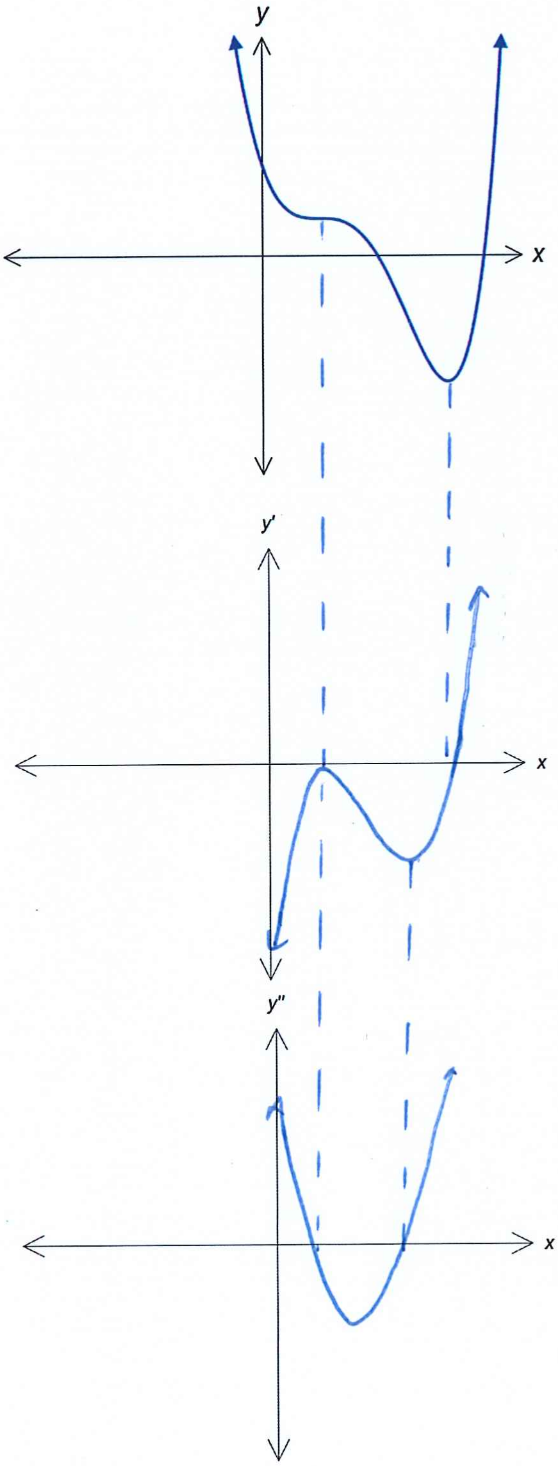
$$= -\left[\frac{x^{7/2}}{7} - x^{5/2}\right]_0^5 + \left[\frac{x^{7/2}}{7} - x^{5/2}\right]_5^8$$

$$= -\left[-\frac{50\sqrt{5}}{7} - 0\right] + \left[\frac{128\sqrt{2}}{7} - -\frac{50\sqrt{5}}{7}\right]$$

$$= \frac{100\sqrt{5} + 128\sqrt{2}}{7}$$

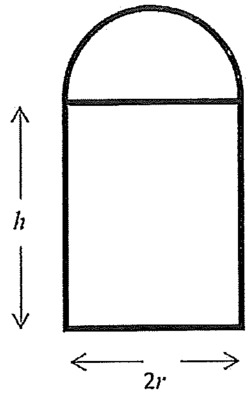
$$\approx 57.8 \text{ units}^2$$

(b) Sketch the first and second derivative of the following.



Question 15. [14 marks]

The diagram shows an arched church wooden window frame, to be made from 10m of timber.



a) Find an expression for h in terms of r. [2]

$$2h = 10 - 4r - \pi r$$

$$h = 5 - 2r - \frac{\pi r}{2}.$$

b) Show that the area of the window is $A = 10r - r^2\left(4 + \frac{\pi}{2}\right)$ [3]

$$A = \frac{1}{2}\pi r^2 + 2r\left(5 - 2r - \frac{\pi r}{2}\right)$$

$$= \frac{1}{2}\pi r^2 + 10r - 4r^2 - \pi r^2$$

$$= 10r - r^2\left(4 + \frac{\pi}{2}\right)$$

Hence, or otherwise,

c) show that the exact value of r that maximises the area is $r = \frac{10}{8+\pi}$ [4]

$$A' = 10 - 2\left(4 + \frac{\pi}{2}\right)r$$

$$0 = 10 - 2\left(4 + \frac{\pi}{2}\right)r$$

$$10 = (8 + \pi)r$$

$$r = \frac{10}{8+\pi}$$

d) Suppose the radius (r) is increased by 10cm. Find the approximate change, using calculus methods, in the height of the window if the 10m of timber restriction still applies. [3]

$$h = 5 - 2r - \frac{\pi r}{2} \qquad \delta r = 0.1$$

$$\frac{\delta h}{\delta r} = -2 - \frac{\pi}{2}$$

$$\delta h = \left(-2 - \frac{\pi}{2}\right) \times 0.1$$

$$= \frac{-4 - \pi}{20}$$

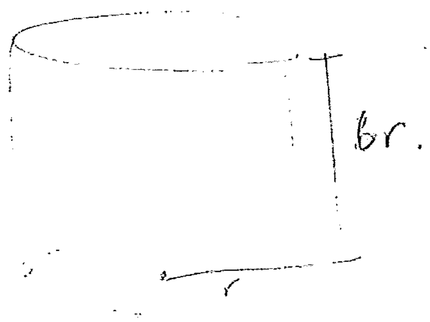
e) Interpret your answer in part (d). [1]

the height would decrease by more than the radius increase.

Question 16. [7 marks]

Consider a cylinder with a height that is three times its diameter.

a) Draw a diagram of the cylinder showing all measurements in terms of the radius (r). [1]



b) Given that the volume of a cylinder is given by, $V_{Cylinder} = Area\ of\ Base \times Height$, determine and expression for the volume of this cylinder in terms of radius (r). [2]

$$V = \pi r^2 \times 6r$$
$$= 6\pi r^3$$

c) Determine the percentage change in height when the volume of the cylinder increases by 4%. [4]

$$\frac{\delta V}{V} = 0.04.$$

$$V = 6\pi r^3$$

$$\frac{\delta V}{\delta r} = 18\pi r^2$$

$$\frac{\delta r}{\delta V} = \frac{1}{18\pi r^2}$$

$$\delta r = \frac{\delta V}{18\pi r^2}$$

$$\frac{\delta r}{r} = \frac{\delta V}{18\pi r^3}$$

$$= \frac{\delta V}{3V}$$

$$= 1/3 \cdot 0.04 = 4/300$$

$\therefore$ 4/3 % increase.

Question 17 [7 marks]

(a) If $y = \dfrac{4}{h^2+1}$ and $h = x^5 + x$, use the chain rule to determine $\dfrac{dy}{dx}$. [4]

$$\frac{dy}{dx} = \frac{dy}{dh} \times \frac{dh}{dx}$$

$$= \frac{-8h}{(h^2+1)^2} \times (5x^4+1)$$

$$= \frac{-8(x^5+x)(5x^4+1)}{((x^5+x)+1)^2}$$

$$\frac{6x^2-4x}{e^{1-x}}$$

(b) For $\dfrac{dy}{dx} = \dfrac{6x^2-4x}{x^3-2x^2} - \dfrac{1}{e^{1-x}}$, determine the change in y when x changes from x=2 to x = 5.

[3]

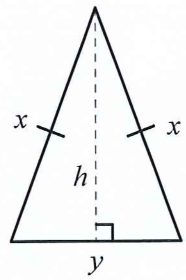
$$\int_2^5 \frac{6x^2-4x}{e^{1-x}}\,dx$$

$$\approx 4673.7.$$

Question 18. [8 marks]

An isosceles triangle has a perimeter of 80cm. If the two equal sides are labeled *x*, the third side *y*, and the perpendicular height *h*:

a. If it is known that $y = 80 - 2x$, show that $h = \sqrt{80x - 1600}$ [3]



$$h = \sqrt{x^2 - \left(\tfrac{1}{2}y\right)^2}$$
$$= \sqrt{x^2 - \tfrac{1}{4}(80-2x)^2}$$
$$= \sqrt{x^2 - \tfrac{1}{4}(6400 - 320x + 4x^2)}$$
$$= \sqrt{x^2 - 1600 + 80x - x^2}$$
$$= \sqrt{80x - 1600}$$

b. Using Calculus, determine the values of *x* and *y* if the area of the triangle is maximized. [5]

$$A = \tfrac{1}{2} y h$$
$$= \tfrac{1}{2}(80 - 2x)\sqrt{80x - 1600}$$

$$\frac{dA}{dx} = \frac{-(6\sqrt{5}x - 160\sqrt{5})}{\sqrt{x-20}} = 0$$

$$x = \frac{80}{3}$$

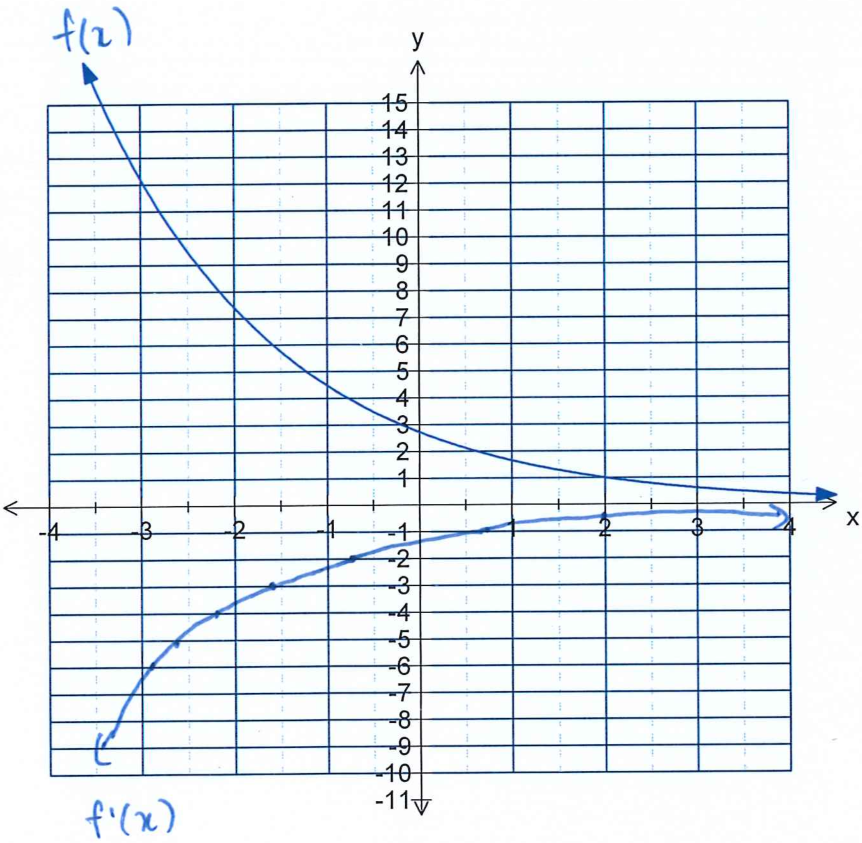
$x = \frac{80}{3}$ $\quad \frac{d^2A}{dx^2} = -3\sqrt{3}$ $\quad \therefore$ max.

$$\therefore x = \frac{80}{3} \qquad y = \frac{80}{3}$$

Question19. [6 marks]

(a) The following shows the graph of the function $f(x)=e^{-0.5(x-2)}$. On the same set of axes draw a sketch of its derivative, $f'(x)$

$f'(x) = -0.5e^{-0.5(x-2)}$



(b) Given that $y=e^{3x}$, prove that $\frac{d^2y}{dx^2}-\frac{dy}{dx}-6y=0$

$\frac{dy}{dx} = 3e^{3x}$ $\frac{d^2y}{dx^2} = 9e^{3x}$

$9e^{3x} - 3e^{3x} - 6e^{3x} = 0$

Question 20. [8 marks]

The Mass M (in grams) of a substance decaying after t years can be represented by $\frac{dM}{dt} = -kM$ where k is a positive constant. There is 250 grams of the substance initially and after 2 years the mass of the substance has decayed to 190 grams.

a. If $M(t) = Ae^{-kt}$ for some constant A, show that $\frac{dM}{dt} = -kM$. [2]

$$\frac{dM}{dt} = -kAe^{-kt}$$
$$= -kM$$

b. Determine the value of A and the value of k to 4 decimal places. [2]

$$190 = 250e^{-kt}$$
$$k = 0.1372$$

c. How long will it take for the mass of the substance to reduce to 80 grams? [2]

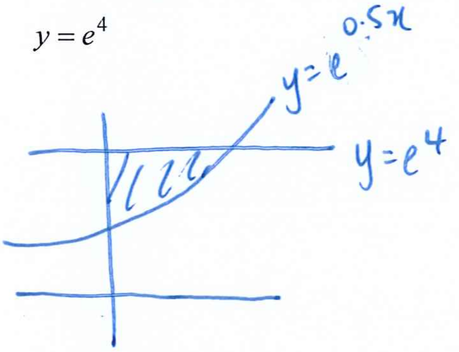
$$80 = 250e^{0.1372t}$$
$$t = 8.304 \text{ years}$$

d. Determine the amount of time for the mass to reduce by half. [2]

$$\frac{1}{2} = e^{-0.1372t}$$
$$t = 5.0514 \text{ years.}$$

Question 21. [6 marks]

Find the **exact** area of the region trapped between the curve $y = e^{0.5x}$, the y-axis and the line $y = e^4$



$$e^4 = e^{0.5x}$$

$$x = 8$$

$$\text{Area} = 8e^4 - \int_0^8 e^{0.5x}\,dx$$

$$= 8e^4 - [2e^4 - 2]$$

$$= 6e^4 + 2 \text{ units}^2$$

END OF SECTION TWO