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7

TERMINOLOGY

asymptote
base
common logarithm
decibel
exponent
exponential
index (indices)
logarithm
Naperian logarithm
natural logarithm
pH scale
power
Richter scale

THE LOGARITHMIC FUNCTION

LOGARITHMIC FUNCTIONS

Prior learning

LOGARITHMIC FUNCTIONS

- define logarithms as indices: $a^x = b$ is equivalent to $x = \log_a(b)$, i.e. $a^{\log_a(b)} = b$ (ACMMM151)
- establish and use the algebraic properties of logarithms (ACMMM152)
- recognise the inverse relationship between logarithms and exponentials: $y = a^x$ is equivalent to $x = \log_a(y)$ (ACMMM153)
- interpret and use logarithmic scales such as decibels in acoustics, Richter Scale for earthquake magnitude, octaves in music, pH in chemistry (ACMMM154)
- solve equations involving indices using logarithms (ACMMM155)
- recognise the qualitative features of the graph of $y = \log_a(x)$ $(a > 1)$ including asymptotes, and of its translations $y = \log_a(x) + b$ and $y = \log_a(x + c)$ (ACMMM156)
- solve simple equations involving logarithmic functions algebraically and graphically (ACMMM157)
- identify contexts suitable for modelling by logarithmic functions and use them to solve practical problems (ACMMM158)

CALCULUS OF LOGARITHMIC FUNCTIONS

- define the natural logarithm $\ln(x) = \log_e(x)$ (ACMMM159)
- recognise and use the inverse relationship of the functions $y = e^x$ and $y = \ln(x)$ (ACMMM160)
- establish and use the formula $\frac{d}{dx}\ln(x) = \frac{1}{x}$ (ACMMM161)
- establish and use the formula $\int \frac{1}{x}\,dx = \ln(x) + c$ for $x > 0$ (ACMMM162)
- use logarithmic functions and their derivatives to solve practical problems (ACMMM163)

7.01 INDICES AND LOGARITHMS

You already know that indices are useful for large numbers; $3^{16} = 43\,046\,721$. This means that when you multiply 3 by itself 16 times, you get 43 046 721.

What if you wanted to write it the other way around? This is done using a **logarithm**. You would write $\log_3(43\,046\,721)$, which means the power of 3 needed to get 42 046 721, so $\log_3(43\,046\,721) = 16$.

What is $\log_5(625)$?

This means the power of 5 needed to get 625.

Since $5^4 = 625$, $\log_5(625) = 4$.

The logarithm is actually just the index, not the whole power, so you should ask 'How many factors of 5 do I need to multiply?'

$5 \times 5 \times 5 \times 5 = 625$ so $5^4 = 625$ so $\log_5(625) = 4$.

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IMPORTANT

The **logarithm** of b to the **base** a, where $a, b > 0$ and $a \neq 1$ is the **exponent** x such that $a^x = b$.

In symbols: $a^x = b \Leftrightarrow \log_a(b) = x$, where $a, b > 0$ and $a \neq 1$.

x is also called the **index** of a^x and the expression is called the xth **power** of a.

The index statement $a^x = b$ means the same as the logarithm statement $\log_a(b) = x$. You say they are *equivalent*.

Example 1

Write the following in index form.

a $\log_2(64) = 6$

b $\log_7\left(\frac{1}{7}\right) = -1$

Solution

a Write the equivalence backwards.	$\log_a(b) = x \Leftrightarrow a^x = b$
Substitute $a = 2$, $b = 64$ and $x = 6$.	$\log_2(64) = 6 \Leftrightarrow 2^6 = 64$
Write the answer.	The index form of $\log_2(64) = 6$ is $2^6 = 64$
b Write the equivalence backwards.	$\log_a(b) = x \Leftrightarrow a^x = b$
Substitute $a = 7$, $b = \frac{1}{7}$ and $x = -1$.	$\log_7\left(\frac{1}{7}\right) = -1 \Leftrightarrow 7^{-1} = \frac{1}{7}$
Write the answer.	The index form of $\log_7\left(\frac{1}{7}\right) = -1$ is $7^{-1} = \frac{1}{7}$

Example 2

Write the following in logarithmic form.

a $5^2 = 25$

b $2^{-3} = \frac{1}{8}$

Solution

a Write the equivalence.	$a^x = b \Leftrightarrow \log_a(b) = x$
Substitute $a = 5$, $x = 2$ and $b = 25$.	$5^2 = 25 \Leftrightarrow \log_5(25) = 2$
Write the answer.	The logarithmic form of $5^2 = 25$ is $\log_5(25) = 2$
b Write the equivalence.	$a^x = b \Leftrightarrow \log_a(b) = x$
Substitute $a = 2$, $x = -3$ and $b = \frac{1}{8}$.	$2^{-3} = \frac{1}{8} \Leftrightarrow \log_2\left(\frac{1}{8}\right) = -3$
Write the answer.	The logarithmic form of $2^{-3} = \frac{1}{8}$ is $\log_2\left(\frac{1}{8}\right) = -3$

When you have to find the logarithm of a number to a particular base, you need to think 'What power of this will give the number?' or 'How many times do I need to multiply the base to get the number?'.

Example 3

Evaluate each of the following.

a $\log_4(64)$ **b** $\log_3(9)$ **c** $\log_{\frac{1}{2}}\left(\frac{1}{32}\right)$

d $\log_6\left(\frac{1}{216}\right)$ **e** $\log_{\frac{1}{5}}(25)$ **f** $\log_{96}(1)$

Solution

a Write the problem. $\log_4(64)$

Think $4^{???} = 64$, $4 \times 4 \times 4 = 64$. $4^3 = 64$, so $\log_4(64) = 3$

b Write the problem. $\log_3(9)$

Think $3^{???} = 9$, $3 \times 3 = 9$. $3^2 = 9$, so $\log_3(9) = 2$

c Write the problem. $\log_{\frac{1}{2}}\left(\frac{1}{32}\right)$

Think $\left(\frac{1}{2}\right)^{???} = \frac{1}{32}$, $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{32}$. $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$ so $\log_{\frac{1}{2}}\left(\frac{1}{32}\right) = 5$

d Write the problem. $\log_6\left(\frac{1}{216}\right)$

Think $6^{???} = \frac{1}{216}$. It must be negative.

Think $6^{???} = 216$, $6 \times 6 \times 6 = 216$. $6^{-3} = \frac{1}{216}$, so $\log_6\left(\frac{1}{216}\right) = -3$

e Write the problem. $\log_{\frac{1}{5}}(25)$

Think $\left(\frac{1}{5}\right)^{???} = 25$ It must be negative.

Think $5^{???} = 25$, $5 \times 5 = 25$. $\left(\frac{1}{5}\right)^{-2} = 25$, so $\log_{\frac{1}{5}}(25) = -2$

f Write the problem. $\log_{96}(1)$

Think $96^{???} = 1$. $96^0 = 1$, so $\log_{96}(1) = 0$

You can see from Example **3** that if $a > 1$ and $b < 1$ or $a < 1$ and $b > 1$, then $\log_a(b)$ is negative. If both a and b are less than 1, then $\log_a(b)$ is positive. Since the zeroth power of any number is 1, the log of 1 is 0, no matter what the base is.

EXERCISE 7.01 Indices and logarithms

Logarithms

Concepts and techniques

1 Example 1 Write the following in index form.

a $\log_5(25) = 2$ b $\log_4(16) = 2$ c $\log_5(125) = 3$ d $\log_2(16) = 4$
e $\log_3(3) = 1$ f $\log_7(49) = 2$ g $\log_2(128) = 7$ h $\log_5(1) = 0$

2 Write the following in index form.

a $\log_8(2) = \frac{1}{3}$ b $\log_4\left(\frac{1}{2}\right) = -\frac{1}{2}$ c $\log_4\left(\sqrt[4]{7}\right) = \frac{1}{4}$ d $\log_3\left(\frac{1}{\sqrt[3]{3}}\right) = -\frac{1}{3}$

e $\log_2\left(\frac{\sqrt{2}}{4}\right) = -\frac{3}{2}$ f $\log_a(b) = c$ g $\log_c\left(\sqrt{a}\right) = 3m$

3 Example 2 Write the following in logarithmic form.

a $7^2 = 49$ b $3^3 = 27$ c $2^4 = 16$
d $5^3 = 125$ e $11^0 = 1$ f $(2)^0 = 1$

4 Write the following in logarithmic form.

a $5^{-2} = \frac{1}{25}$ b $4^{-2} = \frac{1}{16}$ c $10^{-3} = \frac{1}{1000}$ d $\left(\frac{1}{3}\right)^4 = \frac{1}{81}$

e $\left(\frac{1}{4}\right)^3 = \frac{1}{64}$ f $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ g $6^{\frac{2}{3}} = \sqrt[3]{36}$ h $7^{\frac{3}{5}} = \sqrt[5]{343}$

i $a^k = m$ j $b^3 = d$

5 Example 3 Evaluate the following.

a $\log_2(64)$ b $\log_9(81)$ c $\log_3(81)$ d $\log_7(343)$
e $\log_6(216)$ f $\log_5(1)$ g $\log_3(3)$ h $\log_{10}(100\,000)$
i $\log_3(243)$ j $\log_4(1024)$

6 Evaluate the following.

a $\log_{\frac{1}{2}}\left(\frac{1}{16}\right)$ b $\log_5\left(\frac{1}{125}\right)$ c $\log_{\frac{1}{4}}(16)$ d $\log_{\frac{1}{4}}\left(\frac{1}{256}\right)$

e $\log_2\left(\frac{1}{128}\right)$ f $\log_{\frac{1}{2}}(512)$ g $\log_{\frac{1}{3}}\left(\frac{1}{81}\right)$ h $\log_7\left(\frac{1}{7}\right)$

i $\log_{\frac{1}{3}}(27)$ j $\log_{\frac{1}{10}}(0.001)$

Reasoning and communication

7 Evaluate each of the following.

a $\log_2\left(\sqrt{2}\right)$ b $\log_9\left(9\sqrt{9}\right)$ c $\log_4\left(\sqrt{64}\right)$ d $\log_7\left(\sqrt{343}\right)$ e $\log_6\left(\sqrt[3]{36}\right)$

8 Evaluate each of the following.

a $\log_4(8)$ b $\log_9(27)$ c $\log_8(16)$ d $\log_{64}(32)$ e $\log_{27}(243)$

9 If $x = \sqrt[m]{a^p}$ show that $\log_a\left(x^m\right) = p$

10 If $a = \frac{1}{\sqrt[y]{b^p}}$ show that $\log_b\left(a^{-y}\right) = p$

11 If $y = \sqrt[3a]{p^m}$ show that $\log_p\left(y^{3a}\right) = m$

7.02 PROPERTIES OF LOGARITHMS

In the last section, you found that $\log_5(1) = 0$, $\log_3(3) = 1$ and $\log_7(\frac{1}{7}) = -1$. In fact, it should be clear that will be true for any base, and this is easily proven from the definition of a logarithm.

IMPORTANT

The logarithm of 1 with any base is zero: $\log_a(1) = 0$ for $a > 0$ and $a \neq 1$.

The logarithm of any number to its own base is 1: $\log_a(a) = 1$ for $a > 0$ and $a \neq 1$.

The logarithm of the reciprocal of any number to its own base is –1: $\log_a\left(\frac{1}{a}\right) = -1$ for $a > 0$ and $a \neq 1$.

Example 4

Evaluate the following.

a $\log_3(1)$ b $\log_7(7)$ c $\log_5(0)$

d $\log_2(-3)$ e $\log_1(5)$ f $\log_{10}(0.1)$

g $\log_{0.25}(4)$

Solution

a The log of 1 to any allowable base is 0. $\log_3(1) = 0$

b The log of any number to its own base is 1. $\log_7(7) = 1$

c A number has to be positive to have a log ($b > 0$). $\log_5(0)$ is undefined

d A number has to be positive to have a log ($b > 0$). $\log_2(-3)$ is undefined.

e The base of a log cannot be 1 ($a \neq 0$). $\log_1(5)$ is undefined.

f The logarithm of the reciprocal of any number to its own base is –1, and $0.1 = \frac{1}{10}$. $\log_{10}(0.1) = -1$

g The logarithm of the reciprocal of any number to its own base is –1, and $4 = \frac{1}{0.25}$. $\log_{0.25}(4) = -1$

Consider the values of $\log_3(9)$, $\log_3(81)$ and $\log_3(729)$. Since $3^2 = 9$, $3^4 = 81$ and $3^6 = 729$, $\log_3(9) = 2$, $\log_3(81) = 4$ and $\log_3(729) = 6$.

Notice that $729 = 9 \times 81$ and $6 = 2 + 4$ so $\log_3(9 \times 81) = \log_3(9) + \log_3(81)$.

This is a consequence of the index law $\mathbf{a}^m \times \mathbf{a}^n = \mathbf{a}^{m+n}$, so it will work in all cases.

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You can write the first index law in words as follows.

The product of two powers with the same base number is the same base number whose index is the sum of the other two indices.

Translating this to a log statement, you get the following.

IMPORTANT

First law of logarithms: logarithm of a product

The logarithm of a product is equal to the sum of the logarithms to the same base:

$$\log_a(xy) = \log_a(x) + \log_a(y) \text{ for } a, x, y > 0 \text{ and } a \neq 1.$$

This is proven as follows.

Let $m = \log_a(x)$ and $n = \log_a(y)$.

Then $a^m = x$ and $a^n = y$.

Now $xy = a^m \times a^n = a^{m+n}$.

In logarithmic form, this is $\log_a(xy) = m + n$.

But $m = \log_a(x)$ and $n = \log_a(y)$, so $\log_a(xy) = \log_a(x) + \log_a(y)$. QED

The index laws $\dfrac{a^m}{a^n} = a^{m-n}$ and $a^{nm} = (a^n)^m$ lead to the other log laws.

IMPORTANT

Second law of logarithms: logarithm of a quotient

The logarithm of a quotient is equal to the difference of the logarithms to the same base:

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y) \text{ for } a, x, y > 0 \text{ and } a \neq 1.$$

Third law of logarithms: logarithm of a power

The logarithm of a power is equal to the product of the power and the logarithm:

$$\log_a(x^p) = p\log_a(x) \text{ for } a, x > 0 \text{ and } a \neq 1.$$

You can use the definition, laws and other properties of logarithms to simplify expressions involving logarithms.

Example 5

Simplify the following without using a calculator.

a $\log_4(\sqrt{4})$

b $\log_6(18)+\log_6(12)-2$

c $\dfrac{\log_2(27)}{\log_2(81)}$

Solution

a Write $\sqrt{4}$ as a power. $\log_4(\sqrt{4})=\log_4\left(4^{\frac{1}{2}}\right)$

Use the definition. $=\frac{1}{2}$

b Write the expression. $\log_6(18)+\log_6(12)-2$

Use the product law in reverse. $=\log_6(18\times 12)-2$

Write as a product with 6s. $=\log_6(6\times 3\times 6\times 2)-2$

Write as a power. $=\log_6(6^3)-2$

Use the definition. $=3-2$

$=1$

c Write the expression. $\dfrac{\log_2(27)}{\log_2(81)}$

Write in terms of powers. $=\dfrac{\log_2(3^3)}{\log_2(3^4)}$

Use the power law. $=\dfrac{3\log_2(3)}{4\log_2(3)}$

Cancel $\log_2(3)$. $=\frac{3}{4}$

Example 6

Write the expressions below as single logarithms.

a $3\log_2(x)-4\log_2(x+3)+\log_2(y)$

b $2\log_5\left(\sqrt{x}\right)-\log_5(5x)$

Solution

a Write the expression. $3\log_2(x)-4\log_2(x+3)+\log_2(y)$

Use the power law in reverse. $=\log_2(x)^3-\log_2(x+3)^4+\log_2(y)$

Use the product and quotient laws in reverse. $=\log_2\left(\dfrac{x^3y}{(x+3)^4}\right)$

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b Write the expression. $2\log_5\left(\sqrt{x}\right) - \log_5(5x)$

Use the product and power laws. $= \log_5\left[\left(\sqrt{x}\right)^2\right] - [\log_5(5) + \log_5(x)]$

Simplify and use $\log_a(a) = 1$. $= \log_5(x) - 1 - \log_5(x)$

Simplify $= -1$

EXERCISE 7.02 Properties of logarithms

WS Logarithm laws

Concepts and techniques

1 Example 4 Evaluate the following.

a $\log_3(1)$ b $\log_2(1)$ c $2\log_5(1)$ d $\log_x(1), x > 0$
e $[\log_3(1)]^2$ f $\log_3(3)$ g $\log_2(2)$ h $2\log_5(5)$
i $\log_x(x), x > 0$ j $5\log_a(a), a > 0$

2 Evaluate the following.

a $\log_7(0)$ b $\log_2(0)$ c $2\log_5(0)$ d $\log_y(0), y > 0$
e $\log_7(-1)$ f $\log_6(-7)$ g $\log_4(-x), x > 0$

3 Example 5 Simplify the following without using a calculator.

a $\log_2(64)$ b $\log_4(64)$ c $\log_3\left(\sqrt{3}\right)$ d $\log_5\left(5\sqrt{5}\right)$
e $\log_7\left(\frac{1}{\sqrt{7}}\right)$ f $4\log_a\left(\sqrt{a}\right)$ g $2\log_a(a^3)$

4 Example 6 Write as single logarithms.

a $\log_4(10) + \log_4(2) - \log_4(5)$ b $\log_5(25) + \log_5(125) - \log_5(625)$
c $\log_{27}\left(\frac{1}{9}\right) + \log_8(4)$ d $\log_2(16) + \log_2(4) + \log_2(8)$
e $\log_4(40) - \log_4(10) - \log_4(4)$ f $\log_5(8) - \log_5(4) + 2$
g $\log_8(2) - \log_8\left(\frac{1}{4}\right)$ h $\log_6(125) - \log_6(32) - \log_6\left(\frac{2}{5}\right)$

5 Evaluate, without the use of a calculator.

a $\frac{\log_6(16)}{\log_6(2)}$ b $\frac{\log_2(81)}{\log_2(27)}$ c $\frac{\log_3(81)}{\log_3(\frac{1}{3})}$ d $\frac{\log_7(2)}{\log_7(0.25)}$

6 Write as single logarithms.

a $5\log_4(x) + \log_4\left(x^2\right) - \log_4\left(x^3\right)$ b $3\log_7(x) - 5\log_7(x) + 4\log_7(x)$
c $4\log_6(x) - \log_6\left(x^2\right) - \log_6\left(x^3\right)$ d $\log_2(x+2) + \log_2(x+2)^2$
e $\log_4[(x-1)^3] - \log_4[(x-1)^2]$ f $\log_3(x-3) + \log_3(x+3) - \log_3(x^2-9)$

7 Expand using the logarithmic laws.

a $\log\left(\frac{12a}{10}\right)$ b $\log_6\left[\left(\frac{a}{b}\right)^5\right]$ c $\log_3\left(\sqrt[5]{10x^3}\right)$ d $\log_4\left(\frac{\sqrt[3]{x^2a}}{y^2}\right)$

8 If $\log_6(3) = 0.613$, then correct to the nearest thousandth, find the approximate value of each of the following.

a $\log_6\left(\sqrt[4]{3}\right)$ b $\log_6(2)$ c $\log_6(108)$

Reasoning and communication

9 Given that $\log_p(7) + \log_p(k) = 0$, find k.

10 If $\log_a(x) = 4$ and $\log_a(y) = 5$, find the exact values of

a $\log_a(x^2y)$

b $\log_a(axy)$

c $\log_a\left(\frac{\sqrt{x}}{y}\right)$

11 Show that $\log_3\left(\sqrt[4]{\frac{x^2}{y^8z^6}}\right)$ can be expressed as $\frac{1}{2}\log_3(x) - 2\log_3(y) - \frac{3}{2}\log_3(z)$.

12 Use the definition of a logarithm to prove each of the following.

a $\log_a(1) = 0$ for $a > 0$ and $a \neq 1$

b $\log_a(a) = 1$ for $a > 0$ and $a \neq 1$.

13 Prove the quotient law of logarithms.

7.03 COMMON LOGARITHMS AND THE CHANGE OF BASE THEOREM

Logarithms were invented simultaneously by the Scotsman John Napier and Joost Burgi from Switzerland about the same time that decimal fractions were invented. Napier published his version in 1614 and then collaborated with the Englishman Henry Briggs to change them to base 10. The use of logarithms was taught in schools up until the 1970s for calculations, particularly for trigonometry.

IMPORTANT

A **common logarithm** (or **Briggsian logarithm**) is in base 10 and is normally written without the base, so $\log(16) = \log_{10}(16)$

Briggs published tables of common logarithms that were used for nearly 400 years for calculations until the invention of electronic calculators. Your calculator has two kinds of logarithms on it, common logarithms shown by log and **natural logarithms** shown by ln. If you want to find a logarithm to another base you need to use the **change of base** theorem.

IMPORTANT

Change of base theorem

The logarithm of a number to a given base is the quotient of the logarithms of the number and base to any other base: $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$ for $a, b, x > 0$ and $a, b \neq 0$.

The change of base theorem is proven from the definition.

Let $y = \log_a(x)$.

Then $x = a^y$.

By taking logarithms of both sides, we get $\log_b(x) = \log_b(a^y)$.

So from the power rule of logarithms, $\log_b(x) = y \log_b(a)$

But $y = \log_a(x)$, so $\log_b(x) = \log_a(x) \times \log_b(a)$.

Divide by $\log_b(a)$ to get $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$ QED

Example 7

CAS Find the value of $\log_5(27)$.

Solution

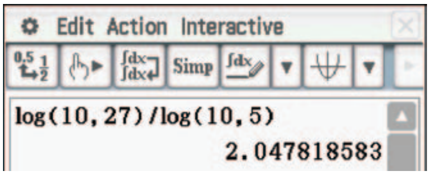
Use the change of base theorem. $\log_5(27) = \dfrac{\log(27)}{\log(5)}$

TI-Nspire CAS

Use a calculator page and set the calculation mode to approximate.
Press ctrl 10^x to get log. Leave the base blank and press enter.

ClassPad

Use the Main menu and the Math1 menu. It is easiest to tap $\log_{10}(\blacksquare)$, but note how the calculator writes base 10 logarithms. $\log_{10}(27)$ is written as log(10, 27).



Write the answer. $\log_5(27) \approx 2.047$

You can use the change of base theorem to determine the value of an exponent when given two bases.

For example, you can solve simple equations involving indices using logarithms.

Example 8

CAS Solve the equation $(0.993)^x = 0.5$ correct to 4 significant figures.

Solution

Write the problem.	$(0.993)^x = 0.5$
Change to logarithms.	$x = \log_{0.993}(0.5)$
Use the change of base theorem.	$= \dfrac{\log(0.5)}{\log(0.993)}$
Use your calculator.	$= 98.674\ldots$
Write the answer.	$x \approx 98.67$

EXERCISE 7.03 Common logarithms and the change of base theorem

Concepts and techniques

1 Example 7 Express in terms of common logarithms.

a $\log_4(9)$ b $\log_8(6)$ c $\log_2(20)$
d $\log_7(200)$ e $\log_9(0.2)$

2 Find the values of each of the following, correct to 4 significant features.

a $\log_3(6)$ b $\log_{12}(2)$ c $\log_5(15)$
d $\log_{25}(4)$ e $\log_8(1.3)$

3 Example 8 Solve the following equations, correct to 4 significant figures.

a $2^x = 100$ b $4^x = 9$ c $5^x = 70$
d $(0.75)^x = 0.01$ e $(1.045)^x = 2$

4 Solve for x in the following equations, correct to 3 decimal places.

a $3^x = 5$ b $7^x = 14.3$ c $3^x = 15$
d $5^x = 100$ e $6^x = 4$

5 Solve the following equations, correct to 4 significant figures.

a $3^{x+1} = 85.7$ b $9^{4x+1} = 64$ c $5^{2x+1} = 32$
d $3^{7x-2} = 13$ e $6^{5-3x} = 17$

Reasoning and communication

6 If $\log_3(a) = b$ and $\log_a(2) = c$, find $\log_a(48)$.

7 Prove that $\log_a(b) = \dfrac{1}{\log_b(a)}$

8 Given that $3^x = 4^y = 12^z$, show that $z = \dfrac{xy}{x+y}$

7.04 SOLVING EQUATIONS WITH LOGARITHMS

You may have to solve equations that are more involved than $2^x = 7$. You saw in the last section that you can do this by changing to logs and using the change of base theorem.

In most cases, you will need to use your general equation solving skills as well. You can rearrange some equations to find the value of a power such as 2^x. If you have to solve a quadratic as part of the process, you should check the solutions in the original equation.

Example 9

Solve $4^x = 2^{x+1} + 3$.

Solution

Write the equation.	$4^x = 2^{x+1} + 3$
Rewrite so that 2 is the only base.	$(2^2)^x = 2^{x+1} + 3$
Simplify.	$2^{2x} = 2^{x+1} + 3$
Write in terms of 2^x.	$(2^x)^2 = 2^x \times 2^1 + 3$
You may find it easier to substitute $a = 2^x$.	$a^2 = 2a + 3$
Write in standard form.	$a^2 - 2a - 3 = 0$
Factorise.	$(a-3)(a+1) = 0$
Solve for a.	$a = 3$ or $a = -1$
Substitute back 2^x.	$2^x = 3$ or $2^x = -1$
Eliminate the false solution.	But $2^x > 0$, so $2^x = 3$ only.
Write as logarithms.	$x = \log_2(3)$
Write the answer as a decimal.	≈ 1.584

In Example **9**, the same power was hidden in the equation by the fact that one base was a power of the other. Some equations involving several bases that are not powers of each other can still be solved by changing to only one power.

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Example 10

Solve $3^{x+1} = 5^{x-4}$ and then evaluate, correct to four significant figures.

Solution

Write the equation. $3^{x+1} = 5^{x-4}$

Write in terms of simple powers. $3^x \times 3^1 = \dfrac{5^x}{5^4}$

Isolate the variables. $\dfrac{5^x}{3^x} = 3 \times 5^4$

Express the variable as a single index. $\left(\dfrac{5}{3}\right)^x = 3 \times 5^4$

Write as logarithms. $x = \log_{\frac{5}{3}}(3 \times 5^4)$

Use the change of base theorem. $= \dfrac{\log(3 \times 5^4)}{\log(\frac{5}{3})}$

Change to simpler logs. $= \dfrac{\log(3) + 4\log(5)}{\log(5) - \log(3)}$

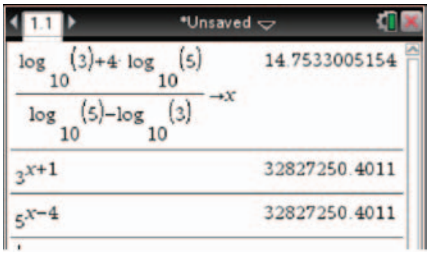
Evaluate. $= 14.753\ldots$

Write the answer to the required accuracy. $\dfrac{\log(3) + 4\log(5)}{\log(5) - \log(3)} \approx 14.75$

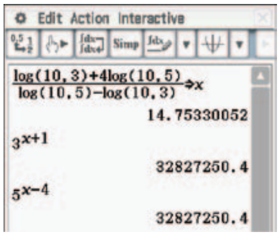
You could evaluate the expression $\dfrac{\log(3 \times 5^4)}{\log(\frac{5}{3})}$ without expressing it in the more elegant form $\dfrac{\log(3) + 4\log(5)}{\log(5) - \log(3)}$, but the latter is certainly preferred. You could also do this problem by taking common logs of both sides of the original equation or the equation $\dfrac{5^x}{3^x} = 3 \times 5^4$, and solving for x. Do you get the same answer?

You should use your CAS calculator to check the answers to problems like those in Examples **9** and **10** by substituting the answer into the original equation. Store the answer before you do the checking.

TI-Nspire CAS



ClassPad



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You can also have equations that involve a logarithm of the variable. In this case you need to find a single logarithm and then use the definition. You may have to solve an equation to find a single logarithm or solve an equation after converting it to powers.

Example 11

Solve the following, correct to 4 significant figures if necessary.

a $\log_3(x+4) - \log_3(x-2) = 2$

b $[\log(x)]^2 + \log(x) - 3 = 0$

Solution

a	Write the equation.	$\log_3(x+4) - \log_3(x-2) = 2$
	Write a single logarithm.	$\log_3\left(\dfrac{x+4}{x-2}\right) = 2$
	Change to a power.	$\dfrac{x+4}{x-2} = 3^2 = 9$
	Simplify.	$x + 4 = 9x - 18$
	Solve.	$x = 2\frac{3}{4}$
b	The equation is a quadratic in $\log(x)$.	$[\log(x)]^2 + \log(x) - 3 = 0$
	Substitute $y = \log(x)$ to make it easier.	$y^2 + y - 3 = 0$
	Write the quadratic formula.	$y = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
	Substitute $a = 1$, $b = 1$ and $c = -3$	$= \dfrac{-1 \pm \sqrt{1^2 - 4 \times 1 \times -3}}{2 \times 1}$
	Use your calculator.	$y = -2.302\ldots$ or $y = 1.302\ldots$
	Substitute $y = \log(x)$ back in.	$\log(x) = -2.302\ldots$ or $\log(x) = 1.302\ldots$
	Use the definition.	$x = 10^{-2.302\ldots}$ or $x = 10^{1.302\ldots}$
	Use your calculator again.	$x = 0.004\,979\ldots$ or $x = 20.080\ldots$
	Use your calculator to check the solutions.	Both solutions work.
	Write the answer.	$x \approx 0.0050$ or $x \approx 20.08$

Logarithms – Solving equations 1

Logarithms – Solving equations 2

EXERCISE 7.04 Solving equations with logarithms

Concepts and techniques

1 Example 9 Solve the following.

a $9^x + 3^x = 12$ b $5^{2x+1} + 5^x - 4 = 0$ c $1 + 6^{1-x} = 6^x$

d $2^{2x+1} + 20 = 3 \times 2^x$ e $11 \times 8^x - 30 = 8^{2x}$

2 Example 10 Find exact solutions to each of the following.

a $9^x = 5^{x+3}$ b $8^x = 49^{x-3}$ c $4^{x+5} = 350^{x-5}$

d $2^{3x} = 15^{x-1}$ e $7^{2x-1} = 17^{x+2}$

3 Solve the following, correct to 4 significant figures.

a $4^x = 7^{x-2}$ b $58^x = 4^{x+4}$ c $5^{x+2} = 46^{x-2}$

d $6^{2x} = 5^{x+3}$ e $28^{x+1} = 9^{2x-4}$

4 Example 11 Solve each of the following.

a $\log_3 (x-2) = 4$ b $\log (2x-10) = 2$

c $\log_2 (2x+12) - \log_2 (x) = \log_2 (4)$ d $\log_2 (2x+1) - \log_2 (x-1) = \log_2 (4x-4) + 2$

e $\log (3x+6) - \log (x+2) = \log (x-2)$ f $\log_3 (2x-4) - \log_3 (x-1) = \log_3 (x-2)$

5 Solve each of the following.

a $[\log (x)]^2 - 2 \log (x) - 3 = 0$ b $[\log_2(x)]^2 - 2 \log_2 (x) = 8$

c $[\log_2 (x)]^2 + \log_2 (x) - 2 = 0$ d $[\log_5 (x)]^2 = \log_5 (x) + 2$

e $[\log_3 (x)]^2 - \log_3 (x^4) + 3 = 0$ f $[\log_5 (x)]^2 - \log_5 (x^5) - 24 = 0$

6 Solve the following, correct to 3 significant figures.

a $3 [\log(x)]^2 + 5 \log (x) - 4 = 0$ b $[\log_2 (x)]^2 = 5 \log_2 (x) - 3$

c $4 [\log_3 (x)]^2 = 6 - \log_3 (x)$ d $2 \log (x) - 4 = 3 [\log (x)]^2$

e $[\log_5 (x)]^2 + 5 \log_5 (x) - 3 = 0$

Reasoning and communication

7 The compound interest formula is $A = P\left(1+\dfrac{i}{k}\right)^{kn}$, where A is the amount after n years, i is the interest rate as a decimal, P is the principal (starting amount) and k is the number of times that interest is calculated each year (the 'rest').
What interest rate would you need to turn \$1000 into \$5000 in 20 years if it was compounded half-yearly?

8 A sum of money doubles itself at compound interest in 15 years. In how many years will it become eight times as big?

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7.05 LOGARITHMIC GRAPHS

You can use the graphs of logarithmic functions to solve logarithm equations instead of doing them algebraically.

Example 12

Plot a graph to solve each of the following.

a $\log_3(x) = 1.6$

b $\log_{0.5}(x) = -1.5$

c CAS $\log_4(x) = 1.3$

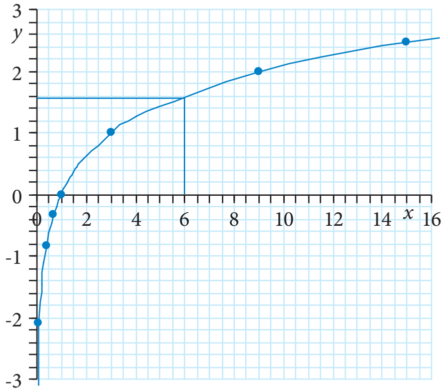
Solution

a Make a table of values.
Remember $x > 0$.

$y = \log_3(x)$

x	0.1	0.4	0.7	1	3	9	15
y	−2.1	−0.83	0.32	0	1	2	2.46

Plot the points and join with a smooth curve.



Use the graph.

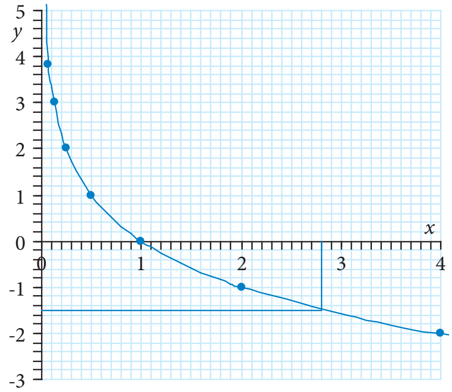
$\log_3(6) \approx 1.6$, so $x \approx 6$

b Make a table of values.
Remember $x > 0$

$y = \log_{0.5}(x)$

x	0.07	0.125	0.25	0.5	1	2	4
y	3.84	3	2	1	0	−1	−2

Plot the points and join with a smooth curve.

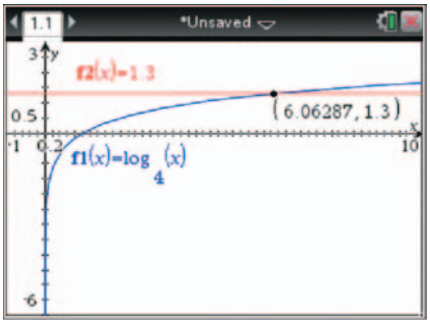


Use the graph.

$\log_3(2.8) \approx -1.5$, so $x \approx 2.8$

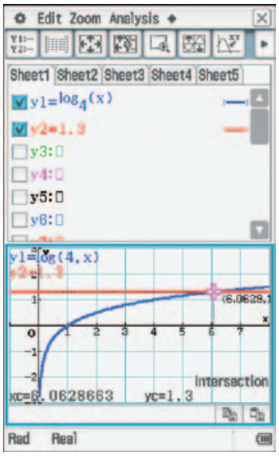
TI-Nspire CAS

Use a graph page.
Complete $f1(x) = \log_4(x)$ and change the Window Settings to appropriate values.
Use [menu], 3: Graph Entry/Edit and 1: Function to put in $f2(x) = 2.3$.
Then use [menu], 6: Analyse Graph and 4: Intersection to find the point needed.



ClassPad

Use the Graph&Table menu.
Enter the graphs in y1 and y2.
Tap [log■□] in the Math1 menu.
You may have to set the View Window appropriately.
Tap Analysis, G-Solve and Intersection.



Write the answer.

$x \approx 6.063$

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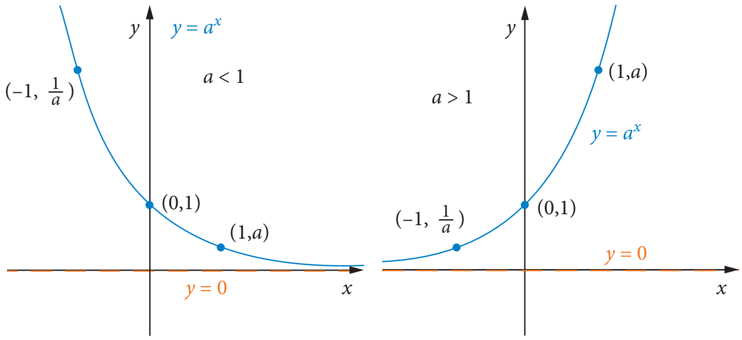
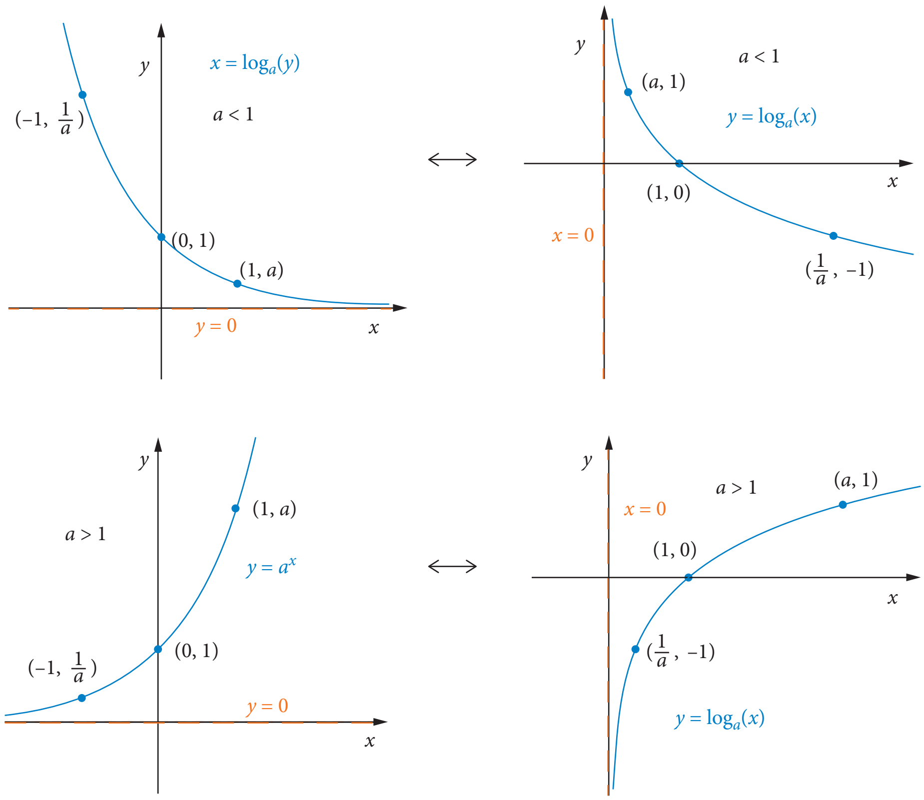
You already know how to sketch exponential functions from your work in Year 11. The domain of an exponential function such as $y = a^x$ has domain $\boldsymbol{R}$ and range $\boldsymbol{R}^+$, with y-intercept 1 and horizontal asymptote $y = 0$ (the x-axis). For $a > 1$ the graph is always increasing and for $a < 1$ it is always decreasing. The graph passes through the points $(1, a)$ and $(-1, -a)$. The graphs look like those below.



The definition of a logarithm means that $y = a^x$ and $x = \log_a(y)$ are equivalent, so the graphs above are also graphs of $x = \log_a(y)$.

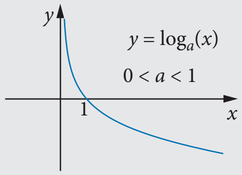
To change them to graphs of $y = \log_a(x)$, you only need to swap the x and y coordinates. This means that $\left(-1, \dfrac{1}{a}\right) \leftrightarrow \left(\dfrac{1}{a}, -1\right)$, $(0, 1) \leftrightarrow (1, 0)$ and $(1, a) \leftrightarrow (a, 1)$. The shape stays the same and the horizontal asymptote becomes a vertical asymptote.

The swapped over graphs are shown below.

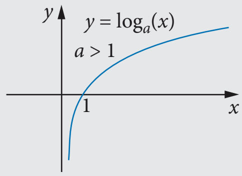


IMPORTANT

Graphs of logarithmic functions of the form $y = \log_a(x)$



Increasing logarithmic function



Decreasing logarithmic function

The graphs of exponential functions of the form $y = \log_a(x)$ have a zero of 1.

For increasing functions, as $x \to 0$, $y \to -\infty$.

For decreasing functions, as $x \to 0$, $y \to \infty$.

The y-axis is a vertical asymptote.

What shapes are logarithmic functions of the form $y = \log_a(x) + b$ or $y = \log_a(x + c)$?

INVESTIGATION Transforming logarithms

a Consider the function $f(x) = \log_2(x)$

- **i** State the domain and range.
- **ii** What is the zero for this function?
- **iii** What is the asymptote for this function?
- **iv** Find $f(2)$ and explain why this value is so important for this function.

b Now consider the function $f(x) = \log_2(x + 3)$.

Use your CAS calculator or computer software to draw the graph.

- **i** State the domain and range.
- **ii** What is the zero for this function?
- **iii** What is the asymptote for this function?
- **iv** Find $f(-1)$ and explain why this value is so important for this function.

c Now consider the function $f(x) = \log_2(x) + 4$.

Use your CAS calculator or computer software to draw the graph.

- **i** State the domain and range.
- **ii** What is the zero for this function?
- **iii** What is the asymptote for this function?
- **iv** Find $f(2)$ and explain why this value is so important for this function.

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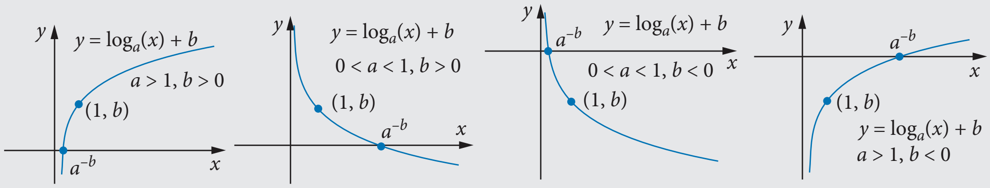
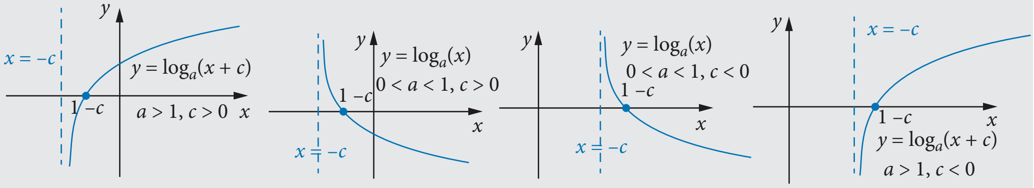
From your investigation you should be able to see the following.

IMPORTANT

The graph of $y = \log_a(x) + b$ is a vertical translation of $y = \log_a(x)$ by b. For b positive, the translation is upwards, and for b negative it is down. The asymptote is still the vertical axis, but the zero becomes $(a^{-b}, 0)$.



The graph of $y = \log_a(x + c)$ is a horizontal translation of $y = \log_a(x)$ by c. For c positive, the translation is to the left, and for b negative it is to the right. The zero becomes $(1 - c, 0)$ and the vertical asymptote becomes $x = -c$.



Example 13

Sketch the graph of $f(x) = \log_2(x) - 3$, labelling important features.

Solution

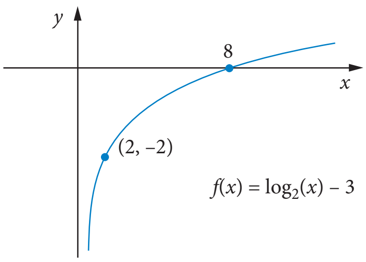
State the translation. $\log_2(x)$ is translated 3 units down.

State the zero. The zero is $(2^3, 0) = (8, 0)$.

State the asymptote. The asymptote is at $x = 0$.

State another point. The point $(2, -2)$ is on the graph.

Sketch the graph.



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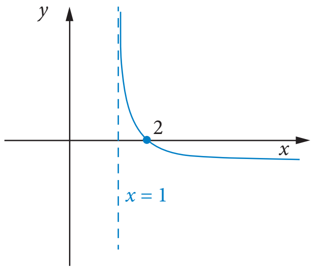
Example 14

Sketch the graph of $f(x) = \log_{\frac{1}{4}}(x-1)$, labelling important features.

Solution

State the translation.	$\log_{\frac{1}{4}}(x)$ is translated 1 unit to the right.
State the zero.	The zero is (2, 0).
State the asymptote.	The asymptote is at $x = 1$.
Sketch the graph.	

EXERCISE 7.05 Logarithmic graphs

Concepts and techniques

Plotting log functions

1 Example 12 Plot a graph to solve each of the following.

a $\log_2(x) = 2.3$ b $\log_{\frac{1}{6}}(y) = 1.4$ c $\log_5(z) = -0.8$ d $\log_{0.6}(k) = -1.7$

2 CAS Use a graph to solve each of the following.

a $\log_4(x) = 3.1$ b $\log_{0.3}(y) = 2.5$ c $\log_7(z) = -1.6$ d $\log_{0.2}(k) = -0.4$

3 Example 13 Sketch the graphs of the following, labelling important features.

a $f(x) = \log_2(x) - 2$ b $f(x) = \log_{0.5}(x) - 2$

c $f(x) = \log_4(x) + 1$ d $f(x) = \log_{0.6}(x) + 3$

e $f(x) = \log_3(x) - 2$ f $f(x) = \log_{0.2}(x) + 2$

4 Example 14 Sketch the graphs of the following, labelling important features.

a $f(x) = \log_{\frac{1}{6}}(x-2)$ b $f(x) = \log_4(x+3)$

c $f(x) = \log_{0.8}(x-3)$ d $f(x) = \log_3(x+2)$

e $f(x) = \log_{\frac{1}{6}}(x-1)$ f $f(x) = \log_2(x+1)$

5 Find the equations of the new functions produced by the following.

a Translation of $\log_7(x)$ by 3 units up.

b Translation of $\log_{0.5}(x)$ by 2 units down.

c Translation of $\log_{\frac{1}{6}}(x)$ by 1 unit up.

d Translation of $\log_3(x)$ by 4 units down.

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6 Find the equations of the new functions produced by the following.
   a Translation of $\log_5(x)$ by 4 units to the left.
   b Translation of $\log_{0.3}(x)$ by 2 units to the right.
   c Translation of $\log_4(x)$ by 3 units to the left.
   d Translation of $\log_{0.8}(x)$ by 5 units to the right.

Reasoning and communication

7 Find the equations of the new functions produced by the following.
   a Translation of $y = \log_2(x)$ by 4 units to the left and 3 units up.
   b Translation of $y = \log_{0.1}(x)$ by 2 units to the right and 1 unit up.
   c Translation of $y = \log_4(x)$ by 3 units to the left and 4 units down.
   d Translation of $y = \log_{0.6}(x)$ by 1 unit to the right and 2 units down.

8 Sketch the graphs of the following, labelling important features.
   a $y = \log_2(x+3)+1$
   b $y = \log_{0.5}(x-2)-2$
   c $y = \log_3(x+1)-2$
   d $y = \log_4(x-2)+1$

7.06 APPLICATIONS OF LOGARITHMS

You can use logarithmic models in a number of physical applications, such as determining the magnitude of earthquakes, intensity of sound, and acidity of a solution. A logarithmic model initially changes rapidly but the growth slows as it continues.

The intensity of earthquakes were originally measured by a scale of physical experiences such as 'like vibrations from heavy traffic' or 'chimney fall'. The Richter scale is a logarithmic scale based on measurements by a seismograph to measure the magnitude of an earthquake. The intensity refers to the local effect of an earthquake, but the magnitude measures the energy.

One of the Richter formulas used to measure the magnitude of an earthquake is $M(x) = \log\left(\frac{x}{x_0}\right)$, where x_0 is defined as the magnitude of an earthquake (x_0) with a seismographic reading of 0.001 millimetre at a distance of 100 kilometres from the epicenter. Modern seismographs are calibrated to this scale

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Example 15

The San Francisco Earthquake of 1906 would have given a seismographic reading of 7.943 metres 100 kilometres from the centre. What was its magnitude?

Solution

Write the formula. $M(x) = \log\left(\frac{x}{x_0}\right)$

Substitute in $x = 7943$, $x_0 = 0.001$ $= \log\left(\frac{7943}{0.001}\right)$

Evaluate. $= 6.9$

Write the answer. The magnitude was 6.9.

Example 16

The pH scale measures the acidity of a solution using the concentration of hydrogen ions. In pure water, the concentration is 10^{-7} moles/litre and the pH is defined as $-\log([H^+])$, where $[H^+]$ is the concentration in moles/litre.

a What is the pH of pure water?

b In 0.1 M hydrochloric acid solution, all the molecules are disassociated so the concentration of hydrogen ions is 0.1 moles/litre. What is the pH of this solution?

Solution

a Write the formula. $\text{pH} = -\log([H^+])$

Substitute $[H^+] = 10^{-7}$. $= -\log(10^{-7})$

Use the definition of $\log(x)$. $= 7$

Write the answer. Pure water has a pH of 7.

b Find the concentration of H^+ ions. $[H^+] = 10^{-7} + 0.1\ (\approx 0.1)$

Substitute $[H^+] = 0.1$. $= -\log(0.1)$

Use the definition of $\log(x)$. $= 1$

Write the answer. 0.1 molar hydrochloric acid has a pH of 1.

EXERCISE 7.06 Applications of logarithms

Reasoning and communication

1 Examples 15, 16 The wind speed s (in km/h) near the center of a tornado is related to the distance d (in thousands of km) the tornado has travelled over a warm ocean surface. The tornado travels according to the equation $s = 930\log(d) + 65$. A tornado whose wind speed is about 280 kilometres per hour struck the coast. How far had this tornado travelled over warm ocean water?

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2 The pH of a solution is given by pH = –log ([H^+]), where [H^+] is the molar concentration of hydrogen ions in moles/litre. The equation for the dissociation of water molecules is [H^+] [OH^-] = 10^{-14}.

a What is the pH of a sodium hydroxide solution with [OH^-] = 10^{-1}?

b When ingested, dishwasher detergent dissolves in the moisture in the mouth, oesophagus and stomach to give a pH of 15. What is the concentration of OH^- ions?

3 Pure sulfuric acid is carried in steel tankers because it needs water to produce hydrogen ions. A tanker carrying sulfuric acid spills some on a wet road and the pH of the resulting solution was measured to be –2 as it burned through the road. What was the concentration of hydrogen ions?

4 Biologists use the logarithmic model $n = k\log(A)$ to estimate the number of species (n) that live in a region of area (A km^2). In the model, k represents a constant. If 2800 species live in a rain forest of 500 square kilometres, then how many species will be left when half of this rainforest is destroyed by logging?

5 The intensity of a sound wave is interpreted by our ears as its loudness. The weakest sound wave that a human ear can hear has an intensity of 1×10^{-12} watts/m^2 and is called the threshold of human hearing, I_0. To compare relative sound intensities, S, we use a scale called decibels, dB, which is calculated with the formula $S = 10\log\left(\frac{I}{I_0}\right)$. In this formula, the threshold of human hearing has a decibel reading of 0 dB. What is the intensity in watt/m^2 of a sound wave that has a sound level reading of 125 dB, the loudness of an average fire alarm?

6 A telescope is limited in its usefulness by the brightness of the star that it is aimed at and by the diameter of its lens. One measure of a star's brightness is its magnitude – the dimmer the star, the larger its magnitude. A formula for the limiting magnitude L of a telescope, that is, the magnitude of the dimmest star that it can be used to view, is given by $L = 9 + 5.1\log(d)$, where d is the diameter (in cm) of the lens.

a What is the limiting magnitude of a telescope 6 cm in diameter?

b What diameter is required to view a star of magnitude 10?

7 The Richter scale can be given by $R = 0.67\log(0.37E) + 1.46$, where E is the energy (in kilowatt-hours) released by the earthquake.

a An earthquake releases 15 500 000 000 kilowatt-hours of energy. What is the earthquake's magnitude?

b How many kilowatt-hours of energy would an earthquake have to release in order to be a 8.5 on the Richter scale?

8 Desalination is the process of producing fresh water from salt water using the formula $y = a + b\log_2(t)$, where y is the amount of fresh water produced (in litres) in time, t (hours). How much fresh water can be produced after 10 hours from a desalination process, given that after 1 hour, 18.27 litres of fresh water can be produced and after 2 hours, 25.41 litres of fresh water can be produced?

9 In the modern scale of musical notes the note names repeat every octave, and each note is double the frequency of the note of the same name in the octave below. The A note below middle C has a standard frequency of 440 Hertz. There are actually 12 different notes, including sharps and flats, in an octave. This is called the chromatic scale and the ratio of the frequency of one note to the previous note in the chromatic scale is a constant.

a What is the ratio of a musical note to the previous note in the chromatic scale?

b What is the frequency of middle C?

7.07 THE NATURAL LOGARITHM AND ITS DERIVATIVE

The exponential function is given by e^x, where $e = 2.728\ 281\ldots$ is the number such that $\frac{d}{dx}(e^x) = e^x$. This number is also the most important base for logarithms.

IMPORTANT

The **natural logarithm** is given by $\ln(x) = \log_e(x)$.

The natural logarithm is the inverse of the exponential function.

Remember that inverse functions must be 1-to-1 and they are such that if $y = f(x)$, then $x = f^{-1}(y)$, where f^{-1} is the inverse of f.

Since $e > 1$, $y = e^x$ is equivalent to $x = \ln(y)$ by the definition of logarithms, so the functions must be inverses of each other. Calculators normally have a ln button for calculating the natural logarithm of a number.

The exponential function and the natural logarithm are extremely important in calculus.

IMPORTANT

The derivative of $\ln(x)$ is given by $\frac{d}{dx}\ln(x) = \frac{1}{x}$.

This can be proven as follows.

Let $y = \ln(x)$.

Then $x = e^y$.

Differentiate both sides to get $\frac{d}{dx}(x) = \frac{d}{dx}(e^y)$.

But $\frac{d}{dx}(x) = 1$, so $1 = \frac{d}{dx}(e^y)$.

Using the chain rule, $\frac{d}{dx}(e^y) = \frac{d}{dy}(e^y) \times \frac{dy}{dx}$.

But $\frac{d}{dy}(e^y) = e^y$, so $1 = e^y \times \frac{dy}{dx}$.

But $x = e^y$, so $1 = x \times \frac{dy}{dx}$ and this gives $\frac{dy}{dx} = \frac{1}{x}$. QED

All the normal derivative rules apply to the derivative of $\ln(x)$.

Therefore, if $f(x) = \ln(kx)$, then $f'(x) = \frac{1}{x}$, $x > 0$

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Example 17

Differentiate the following.

a $f(x) = \ln(5x)$ **b** $\log_7(x)$

Solution

a	Write as a function of a function.	Write $f(x) = \ln(u)$, where $u(x) = 5x$
	Find the derivatives.	$f'(u) = \dfrac{1}{u}$ and $u'(x) = 5$
	Use the chain rule.	$f'(x) = f'(u) \times u'(x)$
	Substitute the derivatives.	$= \dfrac{1}{u} \times 5$
	Substitute $u = 5x$.	$= \dfrac{1}{5x} \times 5$
	Cancel the 5s.	$= \dfrac{1}{x}$
b	Write the problem.	$\dfrac{d}{dx}\log_7(x)$
	Use the change of base theorem.	$= \dfrac{d}{dx}\left[\dfrac{\log_e(x)}{\log_e(7)}\right]$
	Rewrite to emphasise that $\dfrac{1}{\log_e(7)}$ is a constant.	$= \dfrac{d}{dx}\left[\dfrac{1}{\log_e(7)} \times \log_e(x)\right]$
	Use the rule $\dfrac{d}{dx}[kf(x)] = k\dfrac{d}{dx}[f(x)]$	$= \dfrac{1}{\log_e(7)} \cdot \dfrac{d}{dx}[\log_e(x)]$
	Use the derivative of $\ln(x)$.	$= \dfrac{1}{\ln(7)} \times \dfrac{1}{x}$
	Simplify.	$= \dfrac{1}{x\ln(7)}$

Form part **a**, you should be able to see that it doesn't matter what the constant is in the derivative of $\ln(kx)$, the derivative will still be the same, even though it seems a bit odd at first. That is also true for part **b**.

IMPORTANT

The derivative of $\ln(kx)$ is given by $\dfrac{d}{dx}\ln(kx) = \dfrac{1}{x}$.

The derivative of $\log_a(x)$ is given by $\dfrac{d}{dx}\log_a(x) = \dfrac{1}{x\ln(a)}$ for $a > 0$.

Example 18

Differentiate $y = \ln(2x - 5)^2$

Solution

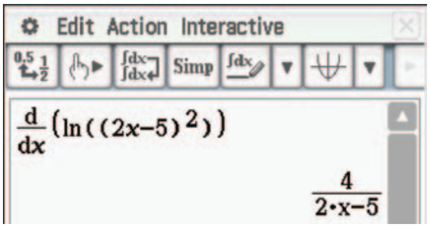
Write the problem.	$\dfrac{d}{dx}\ln(2x-5)^2$
Use the chain rule.	$= \dfrac{d}{du}\ln(u) \times \dfrac{d}{dx}(2x-5)^2$ where $u = (2x-5)^2$
Find the first derivative and use the chain rule again.	$= \dfrac{1}{u} \times \dfrac{d}{dx}(v^2) \times \dfrac{d}{dx}(2x-5)$ where $v = 2x - 5$
Find the last two derivatives.	$= \dfrac{1}{u} \times 2v \times 2$
Substitute u and v.	$= \dfrac{1}{(2x-5)^2} \times 2(2x-5) \times 2$
Simplify.	$= \dfrac{4}{2x-5}$

TI-Nspire CAS

Use a calculator page.
Use menu, 4: Calculus and 1: Derivative.

ClassPad

Use the Main menu and Math2. You may have to clear all variables (Edit menu) first.



You are expected to become competent in finding the derivatives of a wide variety of functions. However, you should get into the habit of checking complex derivatives with your CAS calculator if it is available.

It is worth noting the following rule, which is easily proven using the chain rule.

IMPORTANT

The derivative of $\ln[f(x)]$ is given by $\dfrac{d}{dx}\ln[f(x)] = \dfrac{f'(x)}{f(x)}$

Example 19

Find the derivative of $f(x) = x^2 \ln(x)$.

Solution

Write as a product.	Let $f(x) = u(x)v(x)$ where $u(x) = x^2$ and $v(x) = \ln(x)$
Write the product rule.	$f'(x) = u'(x)v(x) + u(x)v'(x)$
Substitute.	$= 2x\ln(x) + x^2 \times \dfrac{1}{x}$
Simplify.	$= x + 2x\ln(x)$

EXERCISE 7.07 The natural logarithm and its derivative

WS Differentiating exponential and logarithmic functions

Concepts and techniques

1 Example 17 Differentiate each of the following.

a $y = \ln(x)$ b $y = \ln(10x)$ c $y = 3\ln(2x)$

d $y = \ln(0.3x)$ e $y = 6\ln(9x)$ f $y = \ln\left(\dfrac{x}{2}\right)$

g $y = 4\ln\left(\dfrac{x}{3}\right)$ h $y = 2\ln\left(-\dfrac{2x}{3}\right)$

2 Find the derivatives of the following.

a $\log_4(x)$ b $\log(x)$ c $\log_9(x)$ d $\log_2(x)$ e $\log_{0.2}(x)$

3 Example 18 Find the derivatives of the following.

a $y = \ln(3x-1)$ b $y = \ln(2x+7)$ c $y = 2\ln(4x-3)$ d $y = 5\ln(6x+7)$

e $y = \ln(2x+1)$ f $y = 3\ln(5x-1)$ g $y = 6\ln(3-4x)$ h $y = 12\ln(5-8x)$

4 Find the derivatives of the following.

a $y = \ln(3x^5)$ b $y = \ln(4x^3)$ c $y = \ln(2x^2+1)$

d $y = 2\ln(5-8x^2)$ e $y = \ln(x^3-2x^2+3x-4)$ f $y = 3\ln(2x^4-7x^5+x)$

5 Differentiate the following.

a $y = \ln\left(\sqrt{3x+1}\right)$ b $y = \ln\left(\sqrt{5-7x}\right)$ c $y = \ln\left(\sqrt[3]{4x+9}\right)$

d $y = \ln\left(\sqrt[5]{8-x}\right)$ e $y = \ln(3x-7)^4$ f $y = \ln(5x-2)^3$

g $y = \ln\left(\dfrac{1}{x+2}\right)$ h $y = \ln\left(\dfrac{2}{5-3x}\right)$

i $y = \ln\left(\dfrac{3}{6x+1}\right)^{-2}$ j $y = \ln\left(\dfrac{7}{4-x}\right)^{-5}$

6 Find the derivatives of the following.

a $y = \ln (x^2 + 2)^2$

b $y = \ln (3 - x^2)^2$

c $y = \ln (x^3 - 2x + 3)^3$

d $y = \ln (2x^3 - 3x^2 + 4x - 1)^3$

7 Example 19 Find the derivatives of the following.

a $(x^2 - 2x + 1) \ln (x)$

b $(x^3 + 3x^2 + 5) \ln (x^3 + 3x^2 + 5)$

c $x \ln (x)$

d $e^x \ln (x)$

e $\ln (x) \sin (x)$

f $\ln (x) \cos (x) + \dfrac{\sin(x)}{x}$

Reasoning and communication

8 Given $f(x) = 6 \ln (3 - 4x)$, find:

a $f'(x)$

b $f'(2)$

c x such that $f'(x) = 2$

9 Given $f(x) = 6 \ln \left(\sqrt{x^2 - 1}\right)$, find:

a $f'(x)$

b $f'(2)$

c x such that $f'(x) = 6$

10 Given $f(x) = 4x^2 + 3 \ln (x^2 + 2x)$, find:

a $f'(x)$

b $f'(2)$

c x such that $f'(x) = 2$

11 If $g(x) = \ln\left[f(x)\right], f(1) = 3$ and $f'(1) = 6$ find the derivative of $g(x)$ when $x = 1$.

12 Prove that $\dfrac{d}{dx} \ln[f(x)] = \dfrac{f'(x)}{f(x)}$.

7.08 THE INTEGRAL OF $\dfrac{1}{x}$

The integral of $\dfrac{1}{x}$ is just the reverse of the derivative of $\ln (x)$.

IMPORTANT

$\displaystyle\int \frac{1}{x}\, dx = \ln (x) + c$ for $x > 0$.

$\displaystyle\int \frac{1}{x}\, dx$ is sometimes written as $\displaystyle\int \frac{dx}{x}$.

Since you already know how to find the integral of x^n for $n \neq -1$, knowing the integral of $\dfrac{1}{x} = x^{-1}$ means that you can integrate x^n for any value of n.

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Example 20

Find $\int \frac{3}{5x} dx$

Solution

Write the problem. $\int \frac{3}{5x} dx$

Use the rule for $\int kf(x)dx$. $= \frac{3}{5} \int \frac{1}{x} dx$

Integrate. $= \frac{3}{5} \ln(x) + c$ for $x > 0$

If you have to integrate any function involving a denominator, then you should test the derivative of the natural logarithm of the denominator.

Example 21

Find the integral of $\frac{2x-3}{x^2-3x+5}$

Solution

Examine the denominator and numerator. $\frac{d}{dx}(x^2 - 3x + 5) = 2x - 3$

Test the log of the denominator. $\frac{d}{dx}\ln(x^2 - 3x + 5)$

Write as a function of a function with $u = (x^2 - 3x + 5)$ $= \frac{d}{du}\ln(u) \times \frac{d}{dx}(x^2 - 3x + 5)$

Find the derivatives. $= \frac{1}{u} \times (2x - 3)$

Substitute u back in. $= \frac{1}{x^2 - 3x + 5} \times (2x - 3)$

Simplify. $= \frac{2x-3}{x^2-3x+5}$

Use the meaning of an indefinite integral. $\int \frac{2x-3}{x^2-3x+5} dx = \ln(x^2 - 3x + 5) + c$

Example **21** uses the method of trying to find the function whose derivative is the function you want to integrate. You can use this method for many functions that have a denominator that is itself a function. You should also remember that the integral of x^{-1} exists only for $x > 0$. In the case of Example **21**, the integral exists for all values of x because $x^2 - 3x + 5 > 0$ for $x \in \boldsymbol{R}$. You should always state any restrictions on the domain.

Example 22

Find the integral of $\dfrac{x^2-2x+5}{x-3}$

Solution

Divide to simplify the problem.

$$\begin{array}{r} x+1 \\ x-3\,\overline{)\,x^2-2x-5} \\ \underline{x^2-3x} \\ x+5 \\ \underline{x-3} \\ 8 \end{array}$$

Write the simplification.

$$\frac{x^2-2x+5}{x-3}=x+1+\frac{8}{x-3}$$

Test the derivative of the denominator.

$$\frac{d}{dx}\ln(x-3)=\frac{1}{x-3}\times 1$$
$$=\frac{1}{x-3}$$

Now do the integral.

$$\int\frac{x^2-2x+5}{x-3}\,dx$$

Use the simplification.

$$=\int\left(x+1+\frac{8}{x-3}\right)dx$$

Write as a sum and take out the 8.

$$=\int(x+1)\,dx+8\int\frac{1}{x-3}\,dx$$

Complete the integral.

$$=\frac{1}{2}x^2+x+8\ln(x-3)+c$$

Work out the restriction of the domain.

For $x-3>0$, $x>3$

State the final answer.

$$\int\frac{x^2-2x+5}{x-3}\,dx$$
$$=\frac{1}{2}x^2+x+8\ln(x-3)+c \text{ for } x>3$$

EXERCISE 7.08 The integral of $\frac{1}{x}$

WS Integration of reciprocals

Concepts and techniques

1 Example 20 Find the integrals of the following for $x > 0$.

a $\frac{2}{x}$ b $\frac{7}{x}$ c $\frac{6}{5x}$

d $\frac{4}{7x}$ e $-\frac{8}{11x}$ f $-\frac{9}{4x}$

2 Example 21 Find the integrals of the following with an appropriate restriction of the domain.

a $\frac{1}{x+4}$ b $\frac{1}{x-2}$ c $\frac{1}{3x+1}$ d $\frac{1}{5x-9}$

e $\frac{11}{7x-9}$ f $\frac{13}{4x-1}$ g $\frac{6}{5-2x}$ h $\frac{7}{3-x}$

Reasoning and communication

3 Example 22 Find the following, stating any restrictions of the domain.

a $\int \frac{x^3+x^2}{x^3}dx$ b $\int \frac{4x^4-3x^2+x}{x^3}dx$

c $\int \frac{5x+2x^3-1}{x^2}dx$ d $\int \frac{4x^2+8x^5-2x}{2x^3}dx$

e $\int \frac{3x^{10}-2x^4+15x^2}{x^3}dx$

4 Find the equation of the curve $f(x)$ given that $f'(x)=\frac{1}{x-2}$ and the curve passes through (3, 6).

5 Find the equation of the curve $f(x)$ given that $f'(x)=\frac{7}{5-3x}$ and $f(2)=7$.

6 Find $\frac{d}{dx}[\ln(x^2+2)]$. Hence find $\int \frac{4x}{x^2+2}dx$.

7 Find $\frac{d}{dx}[\ln(x^2-5)]$. Hence find $\int \frac{x}{x^2-5}dx$.

8 Show that $\frac{4}{x^2-4}=\frac{1}{x-2}-\frac{1}{x+2}$ and hence find $\int \frac{4}{x^2-4}dx$.

7.09 APPLICATIONS OF NATURAL LOGARITHMS

Logarithmic growth models apply to phenomena where the rate of increase decreases over time. For example, training to improve a skill generally follows a logarithmic model because you make a lot of progress to start with, but as you improve, it gets harder to go further. If you start from zero on day 0 of a program, the function must be of the type $a\ln(t+1)$. If you start with a level of, c, then it becomes $a\ln(t+1)+c$. More complex models may be modelled as $a\ln[b(t+1)]+c$.

Example 23

Paula is learning to speak Spanish before going to South America for 12 months. After 3 days of a crash course she has a vocabulary of 150 words and can use them to communicate in Spanish. 2 days later she has learned another 120. She needs a very basic vocabulary of 600 words before the trip, which will be in 5 months time.

Shutterstock.com/Leanne Vorrias

a Make a simple logarithmic model for her vocabulary after t days.
b How long will it take her to learn the very basic vocabulary?
c Find an expression for her rate of learning.
d How long would it take before her rate of learning dropped below 1 word per day?

Solution

a	Write the basic model to start at 3 days.	Let $w(3) = a \ln(t - 2) + 150$
	Substitute $w(5) = 150 + 120 = 270$	$270 = a \ln(5 - 2) + 150$
	Solve for a.	$a = \dfrac{120}{\ln(3)} = 109.2\ldots$
b	Write the equation of the model.	$w(t) = 109.2\ldots \ln(t - 2) + 150,\ t \geq 3$
	Substitute $w(t) = 600$.	$600 = 109.2\ldots \ln(t - 2) + 150$
	Solve to find t.	$t = 63.54\ldots$
	Write the answer.	At this rate, it would take her a bit over 2 months to acquire enough Spanish.
c	Find the derivative.	$w'(t) = \dfrac{109.2\ldots}{t - 2},\ t \geq 3$
d	Substitute $w'(t) = 1$.	$1 = \dfrac{109.2\ldots}{t - 2}$
	Solve for t.	$t = 111.2\ldots$
	Write the answer.	She would be learning less than 1 word a day after 111 days (nearly 16 weeks).

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EXERCISE 7.09 Applications of natural logarithms

Reasoning and communication

1 Example 23 A psychologist uses the function $L(t) = 100 \ln (kt)$ to measure the amount learnt, $L(t)$ at time t minutes where k is a constant. The psychologist determines that a student learnt 129 words after 20 minutes.

a Determine the exact value of k

b How many words will the student have learnt after 10 minutes?

c How many words will the student have learnt after 60 minutes?

d How long does it take for the student to learn 180 words?

e At what rate is the student learning after 45 minutes?

2 A small colony of black peppered moths live on a small isolated island. In summer the population begins to increase. If t is the number of days after 12 midnight on 1 January, the equation that best models the number of moths in the colony at any given time is

$$N(t) = 500 \ln (21t + 3),\ t \in [0, 40]$$

a What is the population of the species on 1 January?

b What is the population of moths after 30 days?

c On which day is the population first greater than 2000?

A related species, the white peppered moth, shares the same habitat with the black peppered moth. It reproduces in a similar pattern to the black peppered moth, with its population modelled by

$$P(t) = P \ln (Qt + 3),\ t \in [0, 40]$$

d The initial population of white peppered moths is 769 and the population when $t = 15$ is 2750. Find the value of P, correct to the nearest whole number and Q, correct to 3 decimal places.

e Sketch the graphs of $P(t)$ and $N(t)$ on the same set of axes, labelling all special features.

f Using your graph, find an approximate time when the populations of the black and white peppered moths will be the same.

g What are the population growth rates at this time?

3 A swimmer is training for a 100 m freestyle race and wants to get under 50 seconds. With ordinary training his best result is 1 minute. After 12 days of intensive training he is taking 55 seconds to swim 100 metres.

a Construct a model using the function $T(t) = 60 - a \log (t - b)$ for his time.

b How long will it take to get under 50 seconds?

c What is his rate of reduction of his 100 m time at this point?

d How long would it take him to be an Olympic champion contender (under 46 s), assuming his body could stand the training regime?

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4 Weight loss on strict diets generally follows a logarithmic pattern such as $W(t) = W_0 - a \ln (t + 1)$, where W_0 is the mass at time $t = 0$. After 30 days on a very strict diet, a man who began the diet with a weight of 185 kg claims that although he lost a massive amount of weight in the first week, he is now losing only 0.2 kg per day. How long would it take him to get down to 100 kg?

5 David can currently make about 5 porcelain figures in a day. He starts to improve his productivity by focussing more clearly on the task and after 2 weeks has increased his productivity by 2 figures per day.

Thinkstock/Dario Lo Presti

a Construct a simple productivity model for his productivity t weeks after starting.
b How long will it take him to get his productivity up to 10 per day?
c What will be his rate of improvement after 4 weeks?
d What will be his rate of improvement after 10 weeks?

CHAPTER SUMMARY
LOGARITHMIC FUNCTIONS

- The **logarithm** of b to the **base** a, where $a, b > 0$ and $a \neq 1$ is the **exponent** x such that $a^x = b$. In symbols: $a^x = b \Leftrightarrow \log_a(b) = x$, where $a, b > 0$ and $a \neq 1$. x is also called the **index** of a^x and the expression is called the xth **power** of a.
- The logarithm of 1 with any base is zero: $\log_a(1) = 0$ for $a > 0$ and $a \neq 1$.
- The logarithm of any number to its own base is 1: $\log_a(a) = 1$ for $a > 0$ and $a \neq 1$.
- The logarithm of the reciprocal of any number to its own base is -1:

  $$\log_a\left(\frac{1}{a}\right) = -1 \text{ for } a > 0 \text{ and } a \neq 1.$$
- **First law of logarithms: logarithm of a product**

  The logarithm of a product is the sum of the logarithms to the same base:

  $\log_a(xy) = \log_a(x) + \log_a(y)$ for $a, x, y > 0$ and $a \neq 1$.
- **Second law of logarithms: logarithm of a quotient**

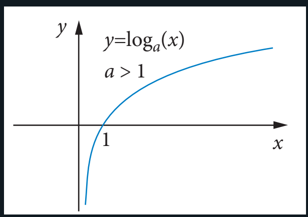
  The logarithm of a quotient is the difference of the logarithms to the same base:

  $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$ for $a, x, y > 0$ and $a \neq 1$.
- **Third law of logarithms: logarithm of a power**

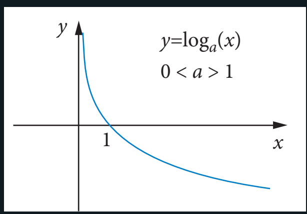
  The logarithm of a power is the the product of the power and the logarithm:

  $\log_a(x^p) = p\log_a(x)$ for $a, x > 0$ and $a \neq 1$.
- A **common logarithm** (or **Briggsian logarithm**) is in base 10 and is normally written without the base, so $\log(16) = \log_{10}(16)$
- **Change of base theorem**

  The logarithm of a number to a given base is the quotient of the logarithms of the number and base to any other base: $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$ for $a, b, x > 0$ and $a, b \neq 0$.
- **Graphs of logarithmic functions of the form $y = \log_a(x)$**



Increasing logarithmic function



Decreasing logarithmic function

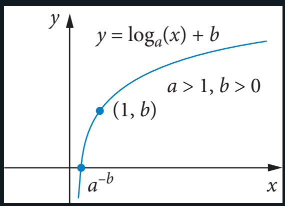
The graphs of exponential functions of the form $y = \log_a(x)$ have a zero of 1.

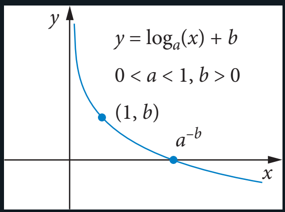
For increasing functions, as $x \to 0$, $y \to -\infty$.

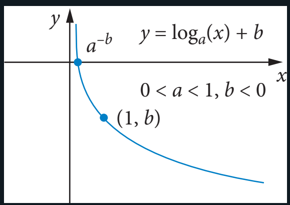
For decreasing functions, as $x \to 0$, $y \to \infty$.

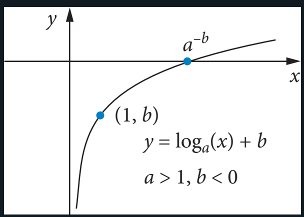
The y-axis is a vertical asymptote.

- The graph of $y = \log_a(x) + b$ is a vertical translation of $y = \log_a(x)$ by b. For b positive, the translation is upwards, and for b negative, it is down. The asymptote is still the vertical axis, but the zero becomes $(a^{-b}, 0)$.



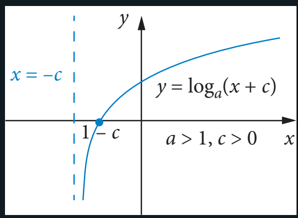


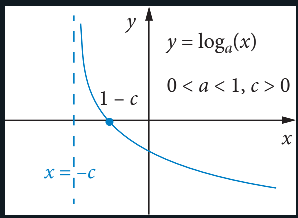


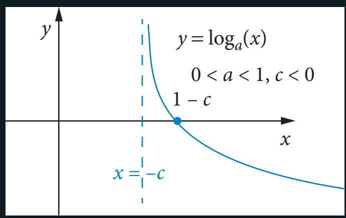


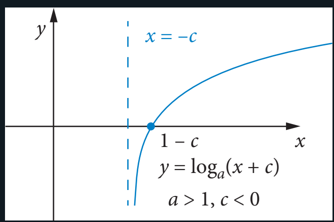
- The **natural logarithm** is given by $\ln(x) = \log_e(x)$. The natural logarithm is the inverse of the exponential function.
- The derivative of $\ln(x)$ is given by $\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$.
- The derivative of $\ln(kx)$ is given by $\dfrac{d}{dx}\ln(kx) = \dfrac{1}{x}$.
- The derivative of $\log_a(x)$ is given by $\dfrac{d}{dx}\log_a(x) = \dfrac{1}{x\ln(a)}$ for $a > 0$.

- The graph of $y = \log_a(x + c)$ is a horizontal translation of $y = \log_a(x)$ by c. For c positive the translation is to the left, and for b negative it is to the right. The zero becomes $(1 - c, 0)$ and the vertical asymptote becomes $x = -c$.









- The derivative of $\ln[f(x)]$ is given by $\dfrac{d}{dx}\ln[f(x)] = \dfrac{f'(x)}{f(x)}$
- $\int \frac{1}{x}dx = \ln(x) + c$ for $x > 0$. $\int \frac{1}{x}dx$ is sometimes written as $\int \frac{dx}{x}$.
- A simple logarithmic model is of the form $a\ln(t + 1) + c$, where c is the initial value and a is a constant determined by the situation.

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CHAPTER REVIEW
LOGARITHMIC FUNCTIONS

7

Multiple choice

1 Example 3 What is the value of $\log_2(8)$?

A $-\frac{1}{4}$ B $\frac{1}{4}$ C 3 D 4 E 6

2 Example 4 What is the value of $\log_3(-2)$?

A undefined B $\frac{1}{9}$ C $-\frac{1}{9}$ D $\frac{1}{8}$ E -9

3 Example 4 State the domain for $y = \log_2(x - 5)$

A R B $x < -5$ C $x > 2$ D $x > 5$ E $(2, 5)$

4 Example 5 $\log_4(2) + \log_4(8) + \log_4\left(\frac{1}{4}\right) =$

A 0 B 1 C 2 D 3 E 4

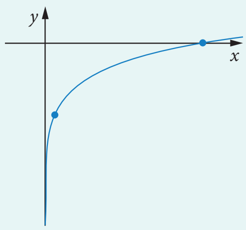
5 Example 6 Write $5\log(x) + 6\log(x + 6)$ as a single logarithm.

A $\log[x(x + 6)]$ B $30\log[x(x + 6)]$ C $\log[x^5(x + 6)^6]$

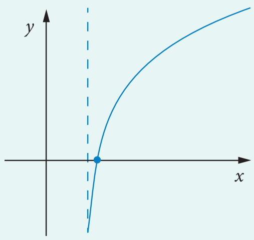
D $11\log(2x + 6)$ E None of these

6 Examples 13, 14 The graph of $y = \log_2(x) + 4$ is most like:

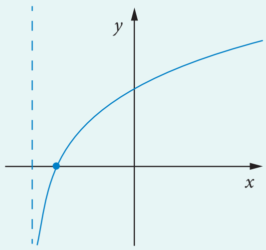
A



B



C



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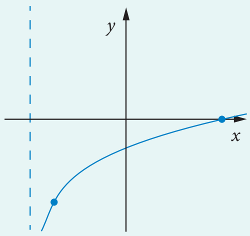

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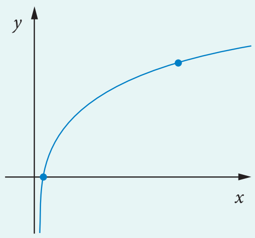


7 Example 17 What is the derivative of $\ln(2x-3)$?

A $\dfrac{2}{2x-3}$ B $\dfrac{-3}{2x-3}$ C $\dfrac{1}{x}$ D $\dfrac{1}{2x-3}$ E $\dfrac{1}{2x}$

8 Example 20 What is the integral of $\dfrac{1}{5x}$?

A $\ln(x)$ B $\log_5(x)$ C $\frac{1}{5}\ln(x)$ D $\ln(5x)$ E $\ln(\frac{1}{5}x)$

Short answer

9 Example 1 Write $\log_3(81) = 4$ in index form.

10 Example 2 Write $5^{-2} = 0.04$ in logarithmic form.

11 Example 7 CAS What is the value of $\log_6(15)$?

12 Example 8 CAS Solve $4^x = 23$ correct to 3 decimal places.

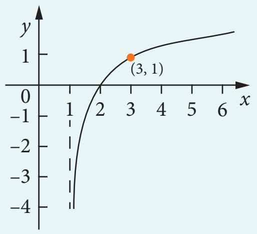
13 Example 9 Solve $3^{2x+1} + 3^x + 4 = 3^{x+2}$.

14 Example 10 Solve $4^{3x+2} = 6^{2x-1}$ and then evaluate correct to 4 significant figures.

15 Example 11 Solve $2\log_3(x) + \log_3(2x-1) - \log_3(x) = 1$

16 Example 12 Plot a graph to solve $\log_3(x) = -0.7$

17 Examples 13, 14 The function f has domain $(1, \infty)$ and its graph is as shown below. Given that $y = \log_a(x+b)$, find the equation of this function.



18 Examples 18, 19 Differentiate:

a $\log_6(x)$

b $\log_e[(3x^2+8)]$

c $(3x^4 - x^3 + 5)\ln(7x+1)$

d $\dfrac{\ln(x)}{\ln(3x-5)}$

19 Example 21 Find the integral of $\dfrac{3x^2-4}{x^3-4x+1}$.

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Application

20 Sketch the graph of $f(x) = \log_3(x + 2) + 2$, labelling important features.

21 The *Shade number* of welding glass is given by $-\frac{7}{3}\log\left(\frac{T}{I}\right) + 1$, where T measures the intensity of light transmitted through the glass when light of intensity I is incident on the glass. What percentage of light is transmitted through welding glass rated at shade factor 10?

22 Find the integral of $y = \dfrac{x^3 - 3x^2 + 2x - 4}{x + 1}$.

23 A patient is given a dose of a drug. At a later stage, the patient is given a *second* dose of the drug, and the amount x units of a metaboloid in the patient's bloodstream t minutes from administering this second dose is modelled by $x = \log_e(2t + e^2)$, $t \geq 0$.

a How much of the metaboloid is present in the patient's bloodstream:

i at the instant the second dose is given?

ii 5 hours later (to the nearest tenth of a unit)?

b Find the rate of change of the amount of the metaboloid in the patient's bloodstream after 3 hours, correct to 3 decimal places.

Practice quiz