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Test 5

Continuous Random Variables

The Normal Disribution

Sample Proportions

Semester Two 2018
Year 12 Mathematics Methods
Calculator Assumed

Name:	Solutions

Teacher:

_______ Mr McClelland

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_______ Ms Cheng

_______ Mr Staffe

_______ Mr Strain

Date: Fri 17th Aug. **7:45am**

You may have a formula sheet for this section of the test.

Classpad Calculators

1 page of Notes

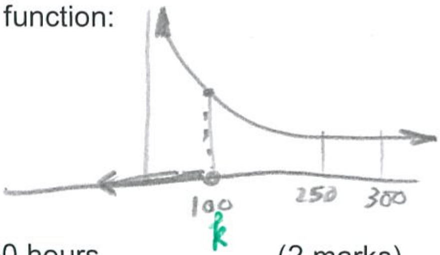
Total___________/46 **50 minutes**

/9

Question 1 **(5 marks)**

The life of an electronic component is given by the probability density function:

$$f(x) = \begin{cases} \dfrac{100}{x^2} & x > 100 \\ 0 & \text{otherwise} \end{cases}$$



Find:

(a) the probability that a component lasts for more than 250 hours. (2 marks)

$$1 - \int_{100}^{250} \frac{100}{x^2}\,dx = \underline{0.4}$$

(b) the median life of a component. (2 marks)

$$\int_{100}^{\infty} \frac{100}{x^2} = 0.5 \Rightarrow \left[-\frac{100}{x}\right]_k^{\infty} = 100\left[0-\left(-\frac{1}{k}\right)\right] \Rightarrow \frac{100}{k} = 0.5$$

$$\therefore \underline{k = 200}$$

(c) the lifetime for 95% of components. (1 mark)

$$\int_k^{\infty} \frac{100}{x^2}\,dx = 0.05;\ k = \underline{2000\text{ hrs}}$$

$P(100 < X \leq k) = 0.95$

$\therefore$ The "Lifetime is $\underline{100 < X \leq 2000}$

Question 2 **(4 marks)**

(a) Pr($Z < -0.376$), where Z is a standard normal random variable is: (1 mark)

$X \sim N(0,1) \Rightarrow \underline{0.3535}$

(b) If Z is a standard normal random variable, and Pr($Z > c$) = 0.75, then the value of c is? (1 mark)

$\underline{c = -0.6745}$

(c) If X is a normally distributed random variable with mean $\mu = 4$ and standard deviation, $\sigma = \sqrt{2}$, then the transformation that maps the curve of the density function of X, $f(x)$, to the curve of the standard normal distribution is: (2 marks)

$$Z = \frac{x-\mu}{\sigma} = \frac{x-4}{\sqrt{2}}$$

$$\therefore (x, y) \to \left(\frac{x-4}{\sqrt{2}}, \sqrt{2}y\right)$$

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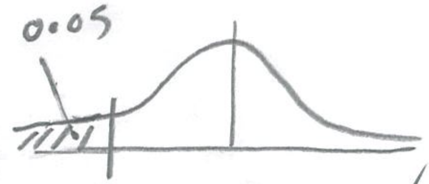
Question 3 **(2 marks)**

The weight of a population of teenage females is normally distributed with a mean of 55 kg and a standard deviation of 8 kg. If the lowest 5% of teenage females is classified as underweight, what is the cut-off weight for this group?

Solve [normal CDF $(-\infty, x, 8, 55) = 0.05$] ✓

0.05



$\therefore$ The cut-off weight is 41.84 kg ✓ accept 41kg

42kg is incorrect

Question 4 **(6 marks)**

A probability density function is given by

$$f(x) = Ax(6-x)^2 \qquad 0 < x < 6$$

2 marks 2 marks

Find the value of A and hence the mean and the standard deviation of this distribution.

$A \int_0^6 x(6-x)^2 = 1$ ✓

$A[108] = 1$

$\therefore \boxed{A = \frac{1}{108}}$ ✓ $= 0.009259$

$E(X) = \frac{1}{108}\int_0^6 x \times x(6-x)^2 dx$ ✓

$= 2.4$ ✓ $= \mu$

$Var(X) = \int_0^6 (x-\mu)^2 \times f(x)\,dx$

$= \int_0^6 (x-2.4)^2 \times \frac{1}{108}x(6-x)^2 dx$

$= 1.44$

$Var(X) = E(X^2) - \mu^2$

$= 7.2 - 2.4^2$

$= 1.44$ ✓

$\therefore \sigma_x = 1.2$

$E(X^2) = \int x^2 \times f(x)$

$= \frac{1}{108}\int_0^6 x^2 \times x(6-x)^2 dx$

$= 7.2$ ✓

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Question 5 **(10 marks)**

A taxi company determined that on an annual basis the distance travelled per taxi is normally distributed with a mean of 92 000 kilometres and a standard deviation of 23 500 kilometres.

(a) What is the probability, correct to four decimal places, that a taxi travels less than 75 000 kilometres per year?

$X \sim N(92000, 23500^2) \Rightarrow P(X < 75000) = \underline{0.2347}$ to 4dp ✓ ✓

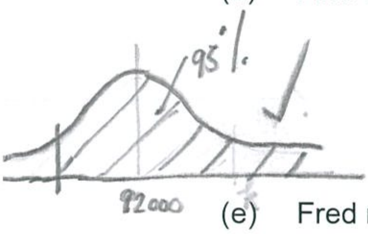
(b) What is the probability, correct to four decimal places, that a taxi travels more than 80 000 kilometres per year?

$P(X > 80000) = \underline{0.6952}$ to 4dp ✓ ✓ (Use fit)

(c) What is the probability, correct to four decimal places, that a taxi travels between 60 000 and 100 000 kilometres in the year?

$P(60000 \leq X \leq 100000) = \underline{0.5466}$ to 4dp ✓✓

(d) Find the minimum mileage that could be expected by 95% of taxis, to the nearest km.



$P(X > k) = 0.95$ ✓

$\therefore k = \underline{53346 \text{ km}}$ ✓

−1

(e) Fred runs a fleet of 10 taxis. What is the probability that at least four of the taxis travel more than 80 000 kilometres in a year?

$X \sim B(10, 0.6952)$

Bin CDF $(4, 10, 10, 0.6952)$ ✓

$= \underline{0.9884}$ ✓

Question 6 **(1 marks)**

A bag contains 4 black balls and three blue balls. If a random sample of four balls is taken from the bag, without replacement, the possible values of the sample proportion of blue balls in the sample are: **D** $\left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}$

We can have 0, 1, 2, or 3 Blue Balls

$\therefore D \cdot \left\{0, \frac{1}{4}, \frac{2}{4}, \text{ or } \frac{3}{4}\right\}$ ✓

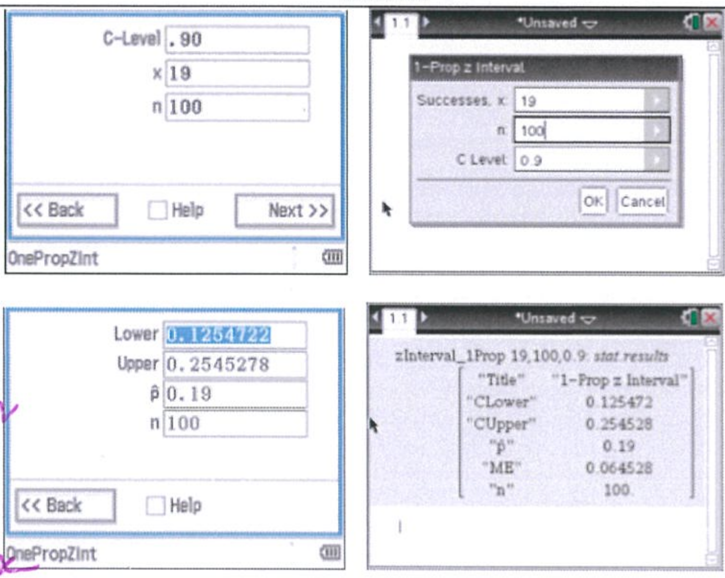
must have all 4 values

Question 7 **(~~8~~ 9 marks)**

A random sample of 100 people indicated that 19% had taken a plane flight in the last year.

(a) Determine a 90% confidence interval for the proportion of the population that had taken a plane flight in the last year. (3 marks)

Solution



0.125 : Lower

0.255 : Upper

Hence $0.125 \le p \le 0.255$

✓ identifies Z score

$Z = 1.645$

Alternative solution

$$\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \qquad n = 100, z = 1.645, \hat{p} = 0.19$$

$$0.19 - 1.645\sqrt{\frac{0.19(1-0.19)}{100}} \le p \le 0.19 + 1.645\sqrt{\frac{0.19(1-0.19)}{100}}$$

$0.125 \le p \le 0.255$

Specific behaviours
✓ correctly calculates lower value of confidence interval
✓ correctly calculated upper value of confidence interval

Assume the 19% sample proportion applies to the whole population.

* (b) A new sample of 200 people was taken and X= the number of people who had taken a plane flight in the last year was recorded. Give a range, using the 90% confidence internal, within which you would expect X to lie. (1 mark)

Solution
$200 \times 0.125 \le X \le 200 \times 0.254 \Rightarrow 25 \le X \le 51$
Specific behaviours
✓ correctly calculates upper and lower value of interval

* Accept :

$\hat{p} = 0.19$

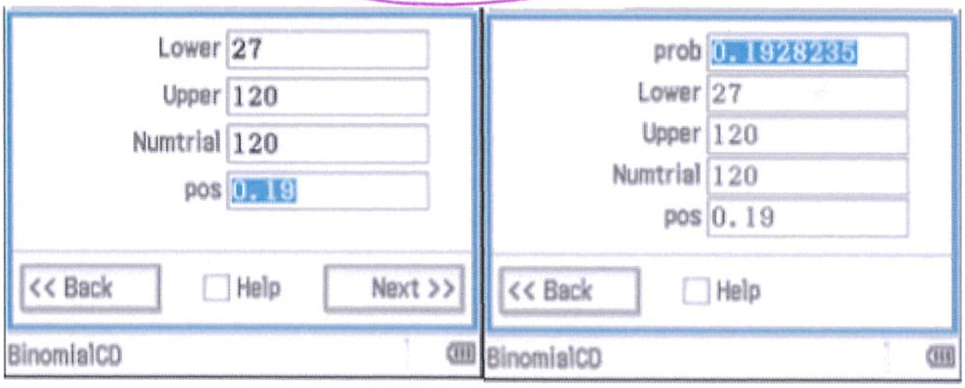
$\sigma = \sqrt{\frac{0.19 \times 0.81}{200}} = 0.02774$

$0.1444 \le p \le 0.2356$

$\therefore 29 \le p \le 47$

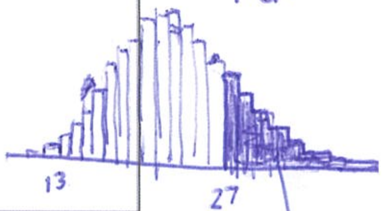
(c) Determine the probability that in a random sample of 120 people, the number who had taken a plane flight in the last year was greater than 26. (3 marks)

ie 27 → 120

Solution
The distribution is binomial with $p = 0.19$ and $n = 120$. $P(X > 26) = P(X \geq 27)$, since n is discrete

Hence the required probability is 0.1928 (to four decimal places)
Specific behaviours
✓ identifies the distribution as binomial – bin(120,0.19)   ✓ uses 27 as the lower bound in the binomial cumulative distribution   ✓ states the correct probability

If $n = 26$
prob = 0.2602
2 marks

Pdf



0.1928

(d) If seven surveys were taken and for each a 95% confidence interval for p was calculated, determine the probability that at least four of the intervals included the true value of p. (2 marks)

Solution
$bin(7, 0.95) \Rightarrow P(4 \leq x \leq 7) = 0.9998$
Specific behaviours
✓ identifies the distribution as binomial – $bin(7, 0.95)$ ✓ calculates the probability correctly

binomial CDF $(4, 7, 7, 0.95) = 0.9998$

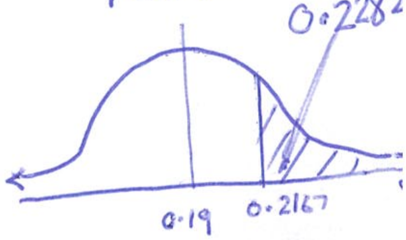
Accept: Normal Dist

$\sigma = \sqrt{\dfrac{0.19 \times 0.81}{120}} = 0.0358$

$\therefore P\left(X > \frac{26}{120}\right) \sim N(0.19, 0.0358^2)$

$= 0.2282$

Normal



* Classpad normCDf$\left(\frac{26}{120}, \infty, 0.0358, 0.19\right)$

0.2167

Question 8 **(10 marks)**

A random survey was conducted to estimate then proportion of mobile phone users who favoured standard smart phones over the new *phablet* style smart phones. It was found that 283 out of 412 people surveyed preferred the new *phablet* style smart phones.

(a) Determine the sample proportion $\hat{p}$ of those in the survey who preferred a phablet style smart phone. (1 mark)

Solution
$\hat{p} = \dfrac{283}{412} = 0.6869$
Specific behaviours
✓ calculates $\hat{p}$ correctly

(b) Use the survey results to estimate the standard deviation of $\hat{p}$, for the sample proportions. (2 marks)

Solution
Standard deviation $= \sqrt{\dfrac{\frac{283}{412}(1-\frac{283}{412})}{412}} = 0.0228$
Specific behaviours
✓ substitutes correctly into standard deviation formula ✓ calculates standard deviation correctly

(c) A follow – up survey is to be conducted to confirm the results of the initial survey. Working with a confidence interval of 95%, estimate the sample size necessary to ensure margin of error of at most 4%. (3 marks)

0.6869

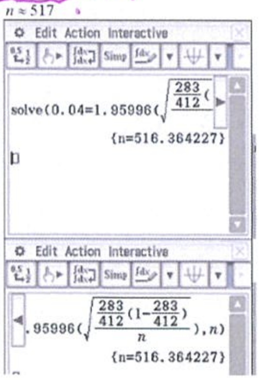
0.3131

$$0.04 = 1.96\sqrt{\frac{\frac{283}{412}(1-\frac{283}{412})}{n}}$$

$n = 516.366$

$n \approx 517$

$n = 517$



Specific behaviours
✓ writes an equation to evaluate n from the margin of error ✓ solves correctly for n ✓ rounds n up to the nearest whole number

The 90% confidence interval of the sample proportion $\hat{p}$, from the initial survey is $0.649 \le \hat{p} \le 0.725$.

(d) Use the 90% confidence interval of the initial sample to compare the following samples:

(i) A random sample of 365 people at a shopping centre found that 258 had a preference for the phablet style smart phone. (2 marks)

Solution
$\hat{p} = \frac{258}{365} = 0.71$ ✓ and $0.668 \le \hat{p} \le 0.746$ ✓ The confidence interval for this second survey overlaps, significantly, the 90% confidence interval of the initial survey so this indicates we are sampling from the same population.
Specific behaviours
✓ calculates 90% confidence interval for $\hat{p}$ correctly ✓ states the similarity of results

8 d (ii)

$\hat{p} = \frac{52}{78}$ and $0.5789 \le \hat{p} \le 0.7545$ ✓

$= \underline{0.667}$

Again the $\hat{p}$ falls within the C.I. and is similar to initial survey results so sampling from the <u>same</u> population.

(No need to talk about Bias : • Maths Teachers • Inside Retirement Village)

<u>Ch 5</u> ? oops

Any reasonable Comment ✓.