# Chapter 18 – Applications of differentiation of polynomials

## Solutions to Exercise 18A

**1 a** $f(x) = x^2, \therefore f'(x) = 2x$

$$f'(2) = 4$$

Tangent at (2, 4) has equation:

$y - 4 = 4(x - 2)$

$\therefore \quad y = 4x - 4$

Normal at (2, 4) has equation:

$$y - 4 = -\frac{1}{4}(x - 2)$$

$$y = -\frac{1}{4}x + \frac{9}{2}$$

$\therefore \; 4x + y = 18$

**b** $f(x) = (2x - 1)^2 = 4x^2 - 4x + 1$

$\therefore f'(x) = 8x - 4$

$f'(2) = 12$

Tangent at (2, 9) has equation:

$y - 9 = 12(x - 2)$

$\therefore y = 12x - 15$

Normal at (2, 9) has equation:

$$y - 9 = -\frac{1}{12}(x - 2)$$

$$y = -\frac{1}{12}x + \frac{55}{6}$$

$\therefore \quad 12y + x = 110$

**c** $f(x) = 3x - x^2, \therefore f'(x) = 3 - 2x$

$$f'(2) = -1$$

Tangent at (2, 2) has equation:

$y - 2 = -(x - 2)$

$\therefore y = -x + 4$

Normal at (2,2) has equation:

$y - 2 = x - 2$

$\therefore \quad y = x$

**d** $f(x) = 9x - x^3, \therefore f'(x) = 9 - 3x^2$

$$f'(1) = 6$$

Tangent at (1, 8) has equation:

$y - 8 = 6(x - 1)$

$\therefore \quad y = 6x + 2$

Normal at (1, 8) has equation:

$$y - 8 = -\frac{1}{6}(x - 1)$$

$$y = -\frac{1}{6}x + \frac{49}{6}$$

$\therefore \quad 6y + x = 49$

**2** $y = 3x^3 - 4x^2 + 2x - 10$

$\therefore \dfrac{dy}{dx} = 9x^2 - 8x + 2$

Intersection with the $y$-axis is at $(0, -10)$

$\therefore$ gradient = 2

Tangent equation: $y + 10 = 2(x - 0)$

$$\therefore \quad y = 2x - 10$$

**3** $y = x^2, \therefore \dfrac{dx}{dy} = 2x$

Tangent at (1,1) has grad = 2 and equation:

$y - 1 = 2(x - 1)$

$\therefore \quad y = 2x - 1$

$y = \dfrac{x^3}{6}, \therefore \dfrac{dy}{dx} = \dfrac{x^2}{2}$

Tangent at $\left(2, \dfrac{4}{3}\right)$ has grad = 2 and equation:

$$y - \frac{4}{3} = 2(x - 2)$$

$$\therefore y = 2x - \frac{8}{3}$$

Tangents are parallel, since both have gradient = 2.
To find the perpendicular distance between them we need to measure the normal between, which has a gradient of $-\frac{1}{2}$.
From (1,1) the normal is:

$$y - 1 = -\frac{1}{2}(x - 1)$$
$$\therefore y = -\frac{x}{2} + \frac{3}{2}$$

This cuts the 2nd tangent where:

$$-\frac{x}{2} + \frac{3}{2} = 2x - \frac{8}{3}$$
$$\frac{5x}{2} = \frac{8}{3} + \frac{3}{2}$$
$$15x = 16 + 9, \therefore x = \frac{5}{3}$$

$\therefore$ Normal cuts 2nd tangent at $\left(\frac{5}{3}, \frac{2}{3}\right)$
Distance between (1,1) and $\left(\frac{5}{3}, \frac{2}{3}\right)$ is
$\sqrt{\left(\frac{5}{3} - 1\right)^2 + \left(\frac{2}{3} - 1\right)^2} = \frac{\sqrt{5}}{3}$

**4** $$y = x^3 - 6x^2 + 12x + 2$$
$\therefore \frac{dy}{dx} = 3x^2 - 12x + 12$
Tangents parallel to $y = 3x$ have gradient = 3

$$\therefore\ 3x^2 - 12x + 12 = 3$$
$$3x^2 - 12x + 9 = 0$$
$$3(x - 1)(x - 3) = 0, \therefore x = 1, 3$$

$y(1) = 9; y(3) = 11$
Tangents are:

$$y - 9 = 3(x - 1), \therefore y = 3x + 6$$
$$y - 11 = 3(x - 3), \therefore y = 3x + 2$$

**5** **a** $$y = (x - 2)(x - 3)(x - 4)$$
$$= x^3 - 9x^2 + 26x + 24$$
$\therefore \frac{dy}{dx} = 3x^2 - 18x + 26$
$\frac{dy}{dx} = 2$ at $P$, 4 at $R$ and $-1$ at $Q$.
Gradients at $P$ and $R$ are equal, so tangents are parallel.

**b** Normal at $Q(3, 0)$ has gradient = ±1:
$y = x - 3$ which cuts the $y$-axis at $(0, -3)$ .

**6** $y = x^2 + 3, \therefore \frac{dy}{dx} = 2x$
Gradient at $x = a$ is $2a$; $y(a) = a^2 + 3$
Tangent has equation:

$$y - (a^2 + 3) = 2a(x - a)$$
$$\therefore \quad y = 2ax - 2a^2 + a^2 + 3$$
$$= 2ax - a^2 + 3$$

Tangents pass through (2, 6)

$$\therefore \quad 6 = 2a(2) - a^2 + 3$$
$$a^2 - 4a + 3 = 0$$
$$(a - 1)(a - 3) = 0, \therefore a = 1, 3$$

If $a = 1$, the point is (1, 4)
If $a = 3$, the point is (3, 12)

**7** **a** $y = x^3 - 2x, \therefore \frac{dy}{dx} = 3x^2 - 2$
At (2, 4), gradient = 10
Equation of tangent:

$$y - 4 = 10(x - 2)$$
$$\therefore y = 10x - 16$$

**b** The tangent meets the curve again where

$$y = x^3 - 2x = 10x - 16$$

$$\therefore \quad x^3 - 12x + 16 = 0$$

$$(x-2)(x^2 + 2x - 8) = 0$$

$$(x-2)^2(x+4) = 0$$

$$\therefore \quad x = 2, -4$$

Tangent cuts the curve again at $x = -4$

$y(-4) = (-4)^3 - 2(-4) = -56$

Coordinates are $(-4, -56)$.

**8 a** $y = x^3 - 9x^2 + 20x - 8$

$$\therefore \frac{dy}{dx} = 3x^2 - 18x + 20$$

At $(1, 4)$, gradient $= 5$

Equation of tangent:

$$y - 4 = 5(x-1)$$

$$\therefore \quad y = 5x - 1$$

**b** $4x + y - 3 = 0$ has gradient $= -4$

$$\therefore \frac{dy}{dx} = 3x^2 - 18x + 20 = -4$$

$$3x^2 - 18x + 24 = 0$$

$$x^2 - 6x + 8 = 0$$

$$(x-2)(x-4) = 0$$

$$\therefore x = 2, 4$$

If $x = 2, y = 2^3 - 9(2)^2 + 20(2) - 8$
$= 4$

If $x = 4, y = 4^3 - 9(4)^2 + 20(4) - 8$
$= -8$

Coordinates are $(2, 4)$ and $(4, -8)$.

## Solutions to Exercise 18B

**1** **a** $y = 35 + 12x^2$

$\therefore y(2) = 83, y(1) = 47$

Av. rate of change

$= \dfrac{y(2) - y(1)}{2 - 1} = \dfrac{83 - 47}{1} = 36$

**b** $y(2 - h) = 35 + 12(2 - h)^2$

$= 35 + 12(4 - 4h + h^2)$

$= 83 - 48h + 12h^2$

Av. rate of change $= \dfrac{y(2) - y(2 - h)}{2 - (2 - h)}$

$= \dfrac{83 - \left(83 - 48h + 12h^2\right)}{h} = 48 - 12h$

**c** Rate of change at $x = 2$ is $y'(2)$:

$y'(x) = 24x, \therefore y'(2) = 48$

(Alternatively, let $h \to 0$ in **part b** answer)

**2** **a** $M = 200\,000 + 600\,t^2 - \dfrac{200}{3}t^3$

$\therefore \dfrac{dM}{dt} = 1200t - 200t^2 = 200t(6 - t)$

**b** At $t = 3$, $\dfrac{dM}{dt} = \$1800$/month

**c** $\dfrac{dM}{dt} = 0$ at $t = 0$ and $t = 6$

**3** **a** $R = 30P - 2P^2, \therefore \dfrac{dR}{dP} = 30 - 4P$

$\dfrac{dR}{dP}$ means the rate of change of profit per dollar increase in list price.

**b** $\dfrac{dR}{dP}$ is 10 at $P = 5$ and $-10$ at $P = 10$

**c** Revenue is rising for

$0 < P < 7.5\left(= \dfrac{30}{4}\right)$

**4** $P = 100(5 + t - 0.25t^2)$

$\therefore \dfrac{dP}{dt} = 100(1 - 0.5t)$

**a** At 1 year $\dfrac{dP}{dt} = 100(1 - 0.5) =$ 50 people/yr

**b** At 2 years $\dfrac{dP}{dt} = 100(1 - 1) =$ 0 people/yr

**c** At 3 years $\dfrac{dP}{dt} = 100(1 - 1.5) = -50$

i.e. decreasing by 50 people/yr

**5** **a** $V(t) = \dfrac{5}{8}\left(10t^2 - \dfrac{t^3}{3}\right), 0 \le t \le 20$

**i** $V(0) = 0$

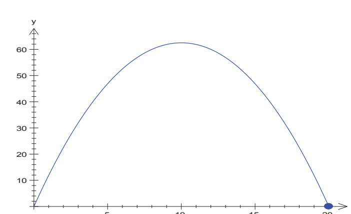
**ii** $V(t) = \dfrac{5}{8}\left(10(20)^2 - \dfrac{20^3}{3}\right)$

$= \dfrac{5}{8}\left(4000 - \dfrac{8000}{3}\right)$

$= \dfrac{2500}{3} = 833\dfrac{1}{3}$ mL

**b** $V'(t) = \dfrac{5}{8}(20t - t^2)$

**c**



**6** $A(t) = \frac{t}{2} + \frac{1}{10}t^2 \text{ km}^2$

$\therefore\ A'(t) = \frac{1}{2} + \frac{t}{5}\text{km}^2/\text{h}$

**a** $A(1) = \frac{1}{2} + \frac{1}{10} = 0.6 \text{ km}^2$

**b** $A'(1) = \frac{1}{2} + \frac{1}{5} = 0.7\text{km}^2/\text{hr}$

## Solutions to Exercise 18C

**1 a** $f(x) = x^2 - 6x + 3$

$\therefore f'(x) = 2x - 6$

$2x - 6 = 0, \therefore x = 3$

$f(3) = -6$

Coordinates of stationary pt are $(3, -6)$.

**b** $y = x^3 - 4x^2 - 3x + 20, x > 0$

$\therefore \ y'(x) = 3x^2 - 8x - 3$

$= (3x + 1)(x - 3)$

$y' = 0$ for $x = 3$ since $x = -\frac{1}{3} < 0$

$y(3) = 2$

Coordinates of stationary pt are $(3, 2)$.

**c** $z = x^4 - 32x + 50$

$\therefore \ z' = 4x^3 - 32$

$4x^3 - 32 = 0, \therefore x = 2$

$z(2) = 2$

Coordinates of stationary pt are $(2, 2)$.

**d** $q = 8t + 5t^2 - t^3, t > 0$

$\therefore \ q' = 8 + 10t - 3t^2$

$= (4 - t)(3t + 2)$

$q' = 0$ for $t = 4$ since $x = -\frac{2}{3} < 0$

$q(4) = 48$

Coordinates of stationary pt are $(4, 48)$.

**e** $y = 2x^2(x - 3)$

$= 2x^3 - 6x^2$

$\therefore \ y' = 6x^2 - 12x$

$= 6x(x - 2)$

$y' = 0$ for $x = 0, 2$

$y(0) = 0$; $y(2) = -8$

Stationary pts at $(0, 0)$ and $(2, -8)$ .

**f** $y = 3x^4 - 16x^3 + 24x^2 - 10$

$\therefore \ y = 12x^3 - 48x^2 + 48x$

$= 12x(x - 2)^2$

$y' = 0$ for $x = 0, 2$

$y(0) = -10$; $y(2) = 6$

Stationary pts at $(0, -10)$ and $(2, 6)$.

**2** $y = ax^2 + bx + c, \therefore \ y' = 2ax + b$

Using $(0, -1)$: $c = -1$

Using $(2, -9)$: $4a + 2b = -8$

$y'(2) = 0, \therefore \ 4a + b = 0$

$\therefore \ a = 2, b = -8, c = -1$

**3** $y = ax^2 + bx + c, \therefore y' = 2ax + b$

When $x = 0$, the slope of the curve is $45°$.

$y'(0) = 1, \qquad \therefore b = 1$

$y'(1) = 0, \qquad \therefore \ 2a + b = 0$

$a = -\frac{1}{2}$

$y(1) = 2, \qquad \therefore -\frac{1}{2} + 1 + c = 2$

$c = \frac{3}{2}$

$\therefore \ a = -\frac{1}{2}, \quad b = 1, c = \frac{3}{2}$

**4 a** $y = ax^2 + bx, \therefore y' = 2ax + b$

$y'(2) = 3, \therefore \ 4a + b = 3$

$y(2) = -2, \therefore \ 4a + 2b = -2$

$\therefore \ a = 2, b = -5$

**b** $y'(x) = 4x - 5 = 0, \therefore\ x = \dfrac{5}{4}$

$y\left(\dfrac{5}{4}\right) = 2\left(\dfrac{5}{4}\right)^2 - 5\left(\dfrac{5}{4}\right) = -\dfrac{25}{8}$

Coordinates of stationary pt are $\left(\dfrac{5}{4}, -\dfrac{25}{8}\right)$.

**5** $y = x^2 + ax + 3, \therefore\ y' = 2x + a$

$y' = 0$ when $x = 4$

$\therefore\ a = -8$

**6** $y = x^2 - ax + 4, \therefore\ y' = 2x - a$

$y' = 0$ when $x = 3$

$\therefore a = 6$

**7** **a** $y = x^2 - 5x - 6, \therefore\ y' = 2x - 5$

$y' = 0$ when $x = 2.5$: $y\left(\dfrac{5}{2}\right) = -12.25$

Stationary pt at $(2.5, -12.25)$.

**b**
$$\begin{aligned} y &= (3x - 2)(8x + 3) \\ &= 24x^2 - 7x - 6 \end{aligned}$$

$$y' = 48x - 7 = 0, \therefore\ x = \frac{7}{48}$$

$$\begin{aligned} y\left(\frac{7}{48}\right) &= \left(\frac{7}{16} - 2\right)\left(\frac{7}{6} + 3\right) \\ &= -\frac{625}{96} \end{aligned}$$

Stationary pt at $\left(\dfrac{7}{48}, -\dfrac{625}{96}\right)$.

**c** $y = 2x^3 - 9x^2 + 27$

$$\begin{aligned} \therefore\ y' &= 6x^2 - 18x \\ &= 6x(x - 3) \end{aligned}$$

$y' = 0$ at $x = 0, 3$: $y(0) = 27, y(3) = 0$

Stationary pts at $(0, 27)$ and $(3, 0)$.

**d** $y = x^3 - 3x^2 - 24x + 20$

$$\begin{aligned} \therefore\ y' &= 3x^2 - 6x - 24 \\ &= 3(x + 2)(x - 4) \end{aligned}$$

$y' = 0$ when $x = -2, 4$:

$y(-2) = -48, y(4) = -60$

Stationary pts at $(-2, 48)$ and $(4, -60)$.

**e**
$$\begin{aligned} y &= (x + 1)^2(x + 4) \\ &= x^3 + 6x^2 + 9x + 4 \end{aligned}$$

$$\begin{aligned} \therefore\ y' &= 3x^2 + 12x + 9 \\ &= 3(x + 1)(x + 3) \end{aligned}$$

$y' = 0$ when $x = -3, -1$:

$y(-3) = 4, y(-1) = 0$

Stationary pts at $(-3, 4)$ and $(-1, 0)$ .

**f**
$$\begin{aligned} y &= (x + 1)^2 + (x + 2)^2 \\ &= 2x^2 + 6x + 5 \end{aligned}$$

$\therefore\ y' = 4x + 6 = 0, x = -1.5$

$y(-1.5) = 0.5$

Stationary pt at $(-1.5, 0.5)$.

**8** $y = ax^2 + bx + 12, \therefore\ y' = 2ax + b$

$y' = 0$ at $x = 1$: $2a + b = 0$

Using (1, 13): $a + b = 1$

$\therefore a = -1, b = 2$

**9** $y = ax^3 + bx^2 + cx + d$

$\therefore\ y'(x) = 3ax^2 + 2bx + c$

$y' = -3$ at $x = 0$: $\quad c = -3$

$y' = 0$ at $x = 3$: $\quad 27a + 6b - 3 = 0$

$9a + 2b = 1 \ldots$ **(1)**

$y(0) = \frac{15}{2}: \quad d = \frac{15}{2}$

$y(3) = 6, \quad \therefore\ 27a + 9b - 9 + \frac{15}{2} = 6$

$9a + 3b = \frac{5}{2} \ldots \textbf{(2)}$

From **(1)** and **(2)**: $b = \frac{3}{2}, \therefore a = -\frac{2}{9}$

$a = -\frac{2}{9}, b = \frac{3}{2}, c = -3, d = \frac{15}{2}$

## Solutions to Exercise 18D

**1 a** $y = 9x^2 - x^3$

$\therefore\ y' = 18x - 3x^2 = 3x(6 - x)$

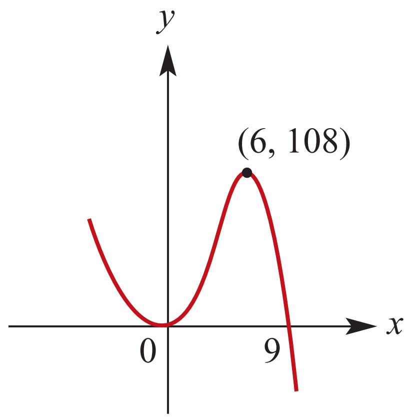
$y' = 0$ at $x = 0, 6$:

$y(0) = 0$; $y(6) = 108$

| $x$ | $-3$ | 0 | 3 | 6 | 9 |
|---|---|---|---|---|---|
| $y'$ | $-$ | 0 | $+$ | 0 | $-$ |

$(0, 0)$ is a local minimum.

$(6, 108)$ is a local maximum.




**b** $y = x^3 - 3x^2 - 9x$

$\therefore\ y' = 3x^2 - 6x - 9 = 3(x + 1)(x - 3)$

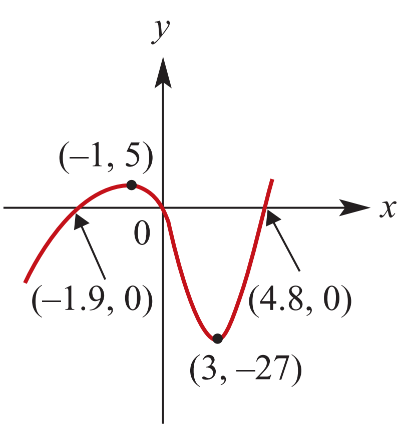
$y' = 0$ at $x = -1, 3$:

$y(-1) = 5$; $y(3) = -27$

| $x$ | $-2$ | $-1$ | 0 | 3 | 4 |
|---|---|---|---|---|---|
| $y'$ | $+$ | 0 | $-$ | 0 | $+$ |

$(-1, 5)$ is a local maximum.

$(3, -27)$ is a local minimum.




**c** $y = x^4 - 4x^3$

$\therefore\ y' = 4x^3 - 12x^2 = 4x^2(x - 3)$

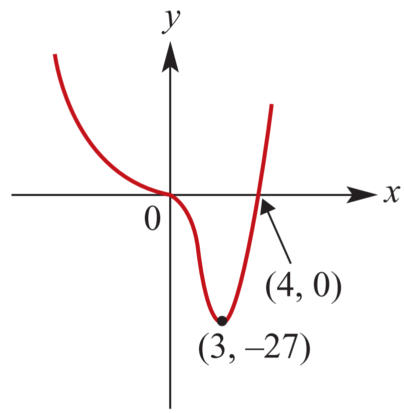
$y' = 0$ at $x = 0, 3$:

$y(0) = 0$; $y(3) = -27$

| $x$ | $-1$ | 0 | 1 | 3 | 4 |
|---|---|---|---|---|---|
| $y'$ | $-$ | 0 | $-$ | 0 | $+$ |

$(0, 0)$ is a stationary pt of inflexion.

$(3, -27)$ is a local minimum.




**2 a** $y = x^2(x - 4) = x^3 - 4x^2$

$\therefore\ y' = 3x^2 - 8x = x(3x - 8)$

$y' = 0$ at $x = 0, \frac{8}{3}$:

$y(0) = 0$; $y\left(\frac{8}{3}\right) = -\frac{256}{27}$

| $x$ | $-1$ | 0 | 1 | $\frac{8}{3}$ | 3 |
|---|---|---|---|---|---|
| $y'$ | $+$ | 0 | $-$ | 0 | $+$ |

$(0, 0)$ is a local maximum.

$\left(\frac{8}{3}, -\frac{256}{27}\right)$ is a local minimum.

**b** $y = x^2(3 - x) = 3x^2 - x^3$

$\therefore\ y' = 6x - 3x = 3x(2 - x)$

$y' = 0$ at $x = 0, 2$

$y(0) = 0$; $y(2) = 4$

| $x$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $y'$ | $-$ | 0 | $+$ | 0 | $-$ |

$(0, 0)$ is a local minimum

(2, 4) is a local maximum

**c** $y = x^4$

$\therefore\ y' = 4x^3$

$y' = 0$ at $x = 0$; $y(0) = 0$

$y'(-1) = -4$; $y'(1) = 4$

(0, 0) is a local minimum.

**d** $y = x^5(x-4) = x^6 - 4x^5$

$\therefore\ y' = 6x^5 - 20x^4 = 2x^4(3x - 10)$

$y' = 0$ at $x = 0, \dfrac{10}{3}$

$y(0) = 0; y\left(\dfrac{10}{3}\right) = \left(\dfrac{10}{3}\right)^5\left(-\dfrac{2}{3}\right)$

$= -\dfrac{200\,000}{729}$

| $x$ | $-1$ | 0 | 1 | $\frac{10}{3}$ | 4 |
|---|---|---|---|---|---|
| $y'$ | $-$ | 0 | $-$ | 0 | $+$ |

(0, 0) is a stationary pt of inflexion.

$\left(\dfrac{10}{3}, -\dfrac{200\,000}{729}\right)$ is a local minimum.

**e** $y = x^3 - 5x^2 + 3x + 2$

$\therefore\ y' = 3x^2 - 10x + 3$

$= (3x-1)(x-3) = 0,$

$\therefore x = \dfrac{1}{3}, 3$

$y\left(\dfrac{1}{3}\right) = \dfrac{67}{27}; y(3) = -7$

| $x$ | 0 | $\frac{1}{3}$ | 1 | 3 | 4 |
|---|---|---|---|---|---|
| $y'$ | $+$ | 0 | $-$ | 0 | $+$ |

$\left(\dfrac{1}{3}, \dfrac{67}{27}\right)$ is a local maximum.

$(3, -7)$ is a local minimum.

**f** $y = x(x-8)(x-3)$

$= x^3 - 11x^2 + 24x$

$\therefore\ y' = 3x^2 - 22x + 24$

$= (3x-4)(x-6)$

$y' = 0$ at $x = \dfrac{4}{3}, 6$

$y\left(\dfrac{4}{3}\right) = \dfrac{4}{3}\left(-\dfrac{20}{3}\right)\left(-\dfrac{5}{3}\right) = \dfrac{400}{27}$

$y(6) = -36$

| $x$ | 0 | $\frac{4}{3}$ | 2 | 6 | 9 |
|---|---|---|---|---|---|
| $y'$ | $+$ | 0 | $-$ | 0 | $+$ |

$\left(\dfrac{4}{3}, \dfrac{400}{27}\right)$ is a local maximum.

$(6, -36)$ is a local minimum.

**3** **a** $y = 2 + 3x - x^3 = (x+1)^2(2-x)$

Axis intercepts at (0, 2), (−1, 0) and (2, 0)

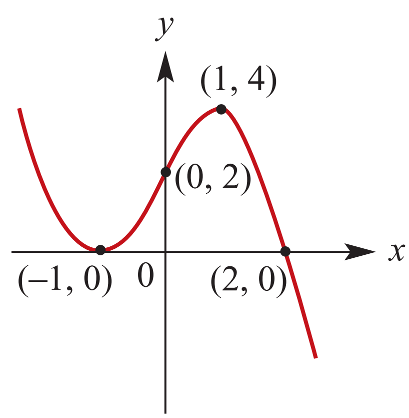
$y' = 3 - 3x^2 = 0, \therefore x = \pm 1$

$y(-1) = 0; y(1) = 4$

| $x$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y'$ | $-$ | 0 | $+$ | 0 | $-$ |

(−1, 0) is a local minimum.

(1, 4) is a local maximum.




**b** $y = 2x^2(x-3) = 2x^3 - 6x^2$

Axis intercepts at (0, 0) and (3, 0)

$y' = 6x^2 - 12x = 6x(x-2)$

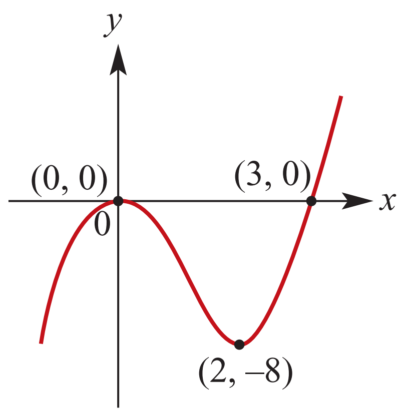
$y' = 0$ when $x = 0, 2$:

$y(0) = 0$; $y(2) = -8$

| $x$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $y'$ | $+$ | 0 | $-$ | 0 | $+$ |

(0, 0) is a local maximum.

(2, −8) is a local minimum.




**c**

$y = x^3 - 3x^2 - 9x + 11$

$= (x - 1)(x^2 - 2x - 11)$

$= (x - 1)(x - 1 - 2\sqrt{3})(x - 1 + 2\sqrt{3})$

Axis intercepts at $(0, 11)$, $(1, 0)$, $(1 - 2\sqrt{3}, 0)$ and $(1 + 2\sqrt{3}, 0)$.

$y' = 3x - 6x - 9$

$= 3(x + 1)(x - 3)$

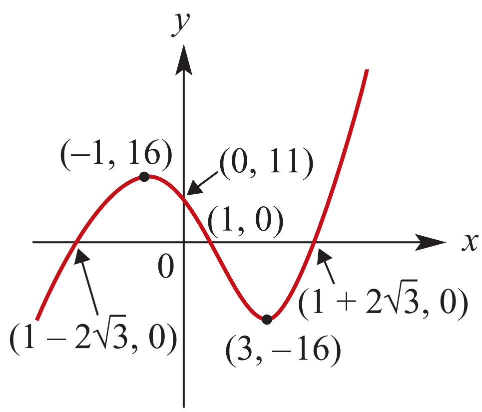
$y' = 0$ when $x = -1, 3$:

$y(-1) = 16; y(3) = -16$

| $x$ | $-2$ | $-1$ | $0$ | $3$ | $4$ |
|---|---|---|---|---|---|
| $y'$ | $+$ | $0$ | $-$ | $0$ | $+$ |

$(-1, 16)$ is a local maximum.

$(3, -16)$ is a local minimum.




**4** Graphs with a stationary point at $(-2, 10)$

**a** $y = 2x^3 + 3x^2 - 12x - 10$

$\therefore\ y' = 6x^2 + 6x - 12$

$= 6(x + 2)(x - 1)$

| $x$ | $-3$ | $-2$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $y'$ | $+$ | $0$ | $-$ | $0$ | $+$ |

$(-2, 10)$ is a local maximum.

**b** $y = 3x^4 + 16x^3 + 24x^2 - 6$

$\therefore\ y = 12x^3 + 48x^2 + 48$

$= 12(x + 2)^2$

$y' > 0; x \neq -2$

$(-2, 10)$ is a stationary pt of inflexion.

**5** $y = x^3 - 6x^2 + 9x + 10$

**a** $y' = 3x^2 - 12x + 9$

$= 3(x - 1)(x - 3)$

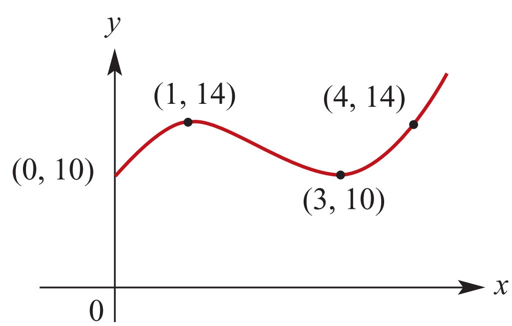
| $x$ | $0$ | $1$ | $2$ | $3$ | $4$ |
|---|---|---|---|---|---|
| $y'$ | $+$ | $0$ | $-$ | $0$ | $+$ |

$\{x: \dfrac{dy}{dx} > 0\} = \{x: x < 1\} \cup \{x: x > 3\}$

**b** $y(1) = 14, y(3) = 10$

$(1, 14)$ is a local maximum.

$(3, 10)$ is a local minimum.




**6** $f(x) = 1 + 12x - x^3$

$f'(x) = 12 - 3x^2$

$= 3(2 - x)(2 + x)$

| $x$ | $-3$ | $-2$ | $0$ | $2$ | $3$ |
|---|---|---|---|---|---|
| $f'$ | $-$ | $0$ | $+$ | $0$ | $-$ |

$\{x: f'(x) > 0\} = \{x: -2 < x < 2\}$

**7** $f(x) = 3 + 6x - 2x^3$

**a** $f'(x) = 6 - 6x^2$

$= 6(1 - x)(1 + x)$

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f'$ | $-$ | $0$ | $+$ | $0$ | $-$ |

$\{x: f'(x) > 0\} = \{x: -1 < x < 1\}$

**b** $(-\infty, -1) \cup (1, \infty)$

**8** **a** $f(x) = x(x + 3)(x - 5)$

$= x^3 - 2x^2 - 15x$

$\therefore\ f'(x) = 3x^2 - 4x - 15$

$= (3x + 5)(x - 3)$

$f'(x) = 0$ for $x = -\dfrac{5}{3}, 3$

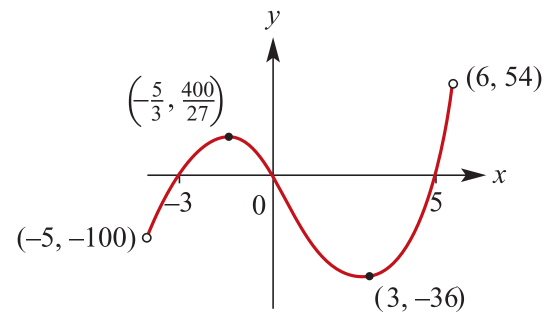
**b** Axis intercepts at $(0, -15)$, $(-3, 0)$, $(0, 0)$ and $(5, 0)$

$f'(-\frac{5}{3}) = (-\frac{5}{3})(\frac{4}{3})(-\frac{20}{3})$ $f(3) = -36$

$= \dfrac{400}{27}$

| $x$ | $-2$ | $-\frac{5}{3}$ | $0$ | $3$ | $4$ |
|---|---|---|---|---|---|
| $f'$ | $+$ | $0$ | $-$ | $0$ | $+$ |

$(-\frac{5}{3}, \frac{400}{27})$ is a local maximum.
$(3, -36)$ is a local minimum.




**9**

$y = x^3 - 6x^2 + 9x - 4$

$= (x - 1)^2(x - 4)$

Axis intercepts at $(0, -4)$, $(1, 0)$ and $(4, 0)$

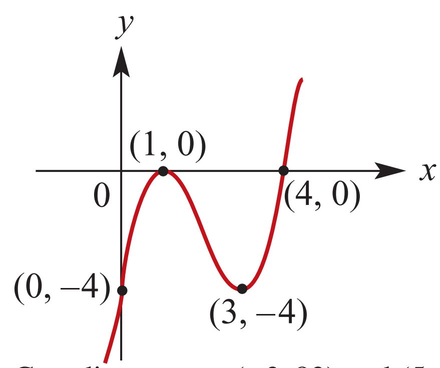
$y' = 3x^2 - 12x + 9$

$= 3(x - 1)(x - 3)$

$y(1) = 0; y(3) = -4$

| $x$ | $0$ | $1$ | $2$ | $3$ | $4$ |
|---|---|---|---|---|---|
| $y'$ | $+$ | $0$ | $-$ | $0$ | $+$ |

$(1, 0)$ is a local maximum.
$(3, -4)$ is a local minimum.




Coordinates are: $(-3, 83)$ and $(5, -173)$.

**10** $y = x^3 - 3x^2 - 45x + 2$

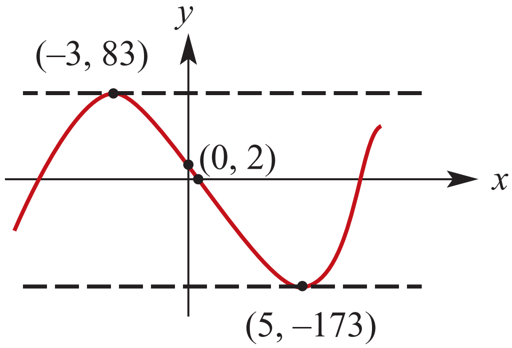
$\therefore\ y' = 3x^2 - 6x - 45$

$= 3(x + 3)(x - 5)$

If tangent is parallel to the $x$-axis then $y' = 0$

$\therefore x = -3, 5$

$y(-3) = 83; y(5) = -173$




**11** $f(x) = x^3 - 3x^2$

$\therefore\ f'(x) = 3x^2 - 6x = 3x(x-2)$

$f'(x) = 0$ for $x = 0, 2$

$f(0) = 0; f(2) = -4$

| $x$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|
| $f'$ | $+$ | $0$ | $-$ | $0$ | $+$ |

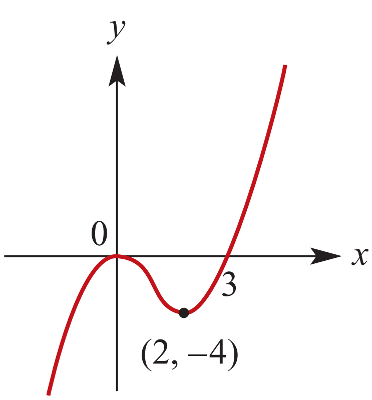
$(0, 0)$ is a local maximum.
$(2, -4)$ is a local minimum.

**a** **i** $\{x: f'(x) < 0\} = \{x: 0 < x < 2\}$

**ii** $\{x: f'(x) > 0\} = \{x: x < 0\} \cup \{x: x > 2\}$

**iii** $\{x: f'(x) = 0\} = \{0, 2\}$

**b**




**12** $y = x^3 - 9x^2 + 27x - 19$

$= (x-1)(x^2 - 8x + 19)$

Axis intercepts: $(0, -19)$ and $(1, 0)$

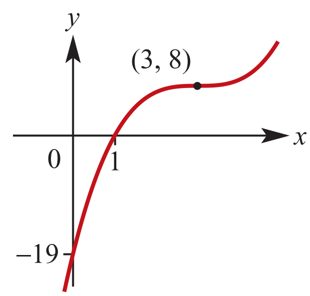
$y' = 3x^2 - 18x + 27$

$= 3(x-3)^2$

$y' = 0$ when $x = 3$; $y(3) = 8$

$y' > 0$ for all $x \neq 0$

Stationary pt of inflexion at $(3, 8)$




**13** $y = x^4 - 8x^2 + 7$

$= (x^2 - 1)(x^2 - 7)$

$= (x-1)(x+1)(x-\sqrt{7})(x+\sqrt{7})$

Axis intercepts: $(0, 7)$, $(-\sqrt{7}, 0)$, $(-1, 0)$, $(1, 0)$ and $(\sqrt{7}, 0)$

$y' = 4x^3 - 16x$

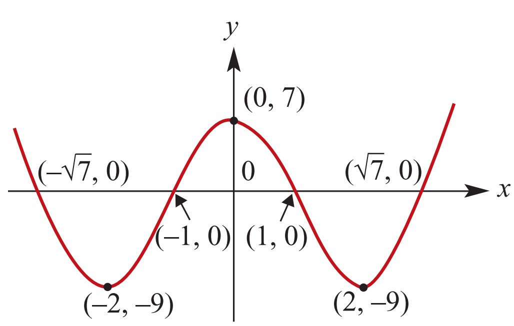
$= 4x(x-2)(x+2)$

$y' = 0$ when $x = -2, 0, 2$

$y(-2) = -9, y(0) = 7, y(2) = -9$

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $y'$ | $-$ | $0$ | $+$ | $0$ | $-$ | $0$ | $+$ |

$(-2, -9)$ is a local minimum.
$(0, 7)$ is a local maximum.
$(2, -9)$ is a local minimum.



## Solutions to Exercise 18E

**1** Let $x$ cm be the width and $y$ cm be the the length.
Then $2x + 2y = 200$ which implies that $y = 100 - x$
We note that $0 \le x \le 100$

$$\begin{aligned}\text{Area} &= xy\\ &= x(100 - x)\\ &= 100x - x^2\end{aligned}$$

Turning point of parabola with negative coefficient of $x^2$. Therefore a maximum.

$$\frac{dA}{dx} = 100 - 2x$$

$$\frac{dA}{dx} = 0 \text{ implies that}$$

$$100 - 2x = 0$$

$$\therefore x = 50$$

Maximum area of $50 \times 50 = 2500$ cm$^2$ when $x = 50$

**2** Let $P = x(10 - x) = 10x - x^2$

Then $\dfrac{dP}{dx} = 10 - 2x$

$$\frac{dP}{dx} = 0 \text{ implies that}$$

$$10 - 2x = 0$$

$$\therefore x = 5$$

Turning point of parabola with negative coefficient of $x^2$. Therefore a maximum.
Maximum value of $P = 25$

**3** Let $M = x^2 + y^2$and it is given that $x + y = 2$
$\therefore y = 2 - x$ and $M = x^2 + (2 - x)^2 = 2x^2 - 4x + 4$

Then $\dfrac{dM}{dx} = 4x - 4$ Turn-

$$\frac{dM}{dx} = 0 \text{ implies that}$$

$$4 - 4x = 0$$

$$\therefore x = 1$$

ing point of parabola with positive coefficient of $x^2$. Therefore a minimum.
Therefore minimum value of $M = 1 + 1 = 2$

**4** **a** Let $x$ cm be the length of the sides of the squares which are being removed.
The base of the box is a square with side lengths $6 - 2x$ cm and the height of the box is $x$ cm.
Therefore the volume $V$cm$^3$ is given by

$$\begin{aligned}V &= (6 - 2x)^2 x\\ &= (36 - 24x + 4x^2)x\\ &= 36x - 24x^2 + 4x^3\end{aligned}$$

Note that $0 \le x \le 3$

**b**
$$\frac{dV}{dx} = 12x^2 - 48x + 36$$

$$\frac{dV}{dx} = 0 \text{ implies that}$$

$$12x^2 - 48x + 36 = 0$$

$$\therefore x^2 - 4x + 3 = 0$$

$$\therefore (x - 1)(x - 3) = 0$$

$$\therefore x = 1 \text{ or } x = 3$$

The maximum value occurs when $x = 1$
We note that $V(3) = 0$
Maximum value $= V(1) = 16$ .
The maximum value of the volume of the box is 16 cm$^3$

**5** $y(x) = \dfrac{x^2}{400}(20 - x),\ 0 \le x \le 20$

**a** **i** $y(5) = \dfrac{5^2}{400}(20 - 5)$

$= \dfrac{15}{16} = 0.9375$ m

**ii** $y(10) = \dfrac{10^2}{400}(20 - 10)$

$= \dfrac{5}{2} = 2.5$ m

**iii** $y(15) = \dfrac{15^2}{400}(20 - 15)$

$= 2.8125$ m

**b** Use a CAS calculator to find the gradient function:

$$y'(x) = \frac{x(20 - x)}{200} - \frac{x^2}{400}$$
$$= \frac{x(40 - 3x)}{400}$$

$y' = 0$ when $x = 0, \dfrac{40}{3}$

$(0, 0)$ is the local minimum.

$$y(\tfrac{40}{3}) = \frac{40^2}{3600}(20 - \frac{40}{3})$$
$$= (\frac{4}{9})(\frac{20}{3}) = \frac{80}{27}$$

Local maximum at $\left(\dfrac{40}{3}, \dfrac{80}{27}\right)$.

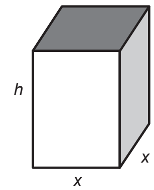
**c** **i** $y' = \dfrac{x(40 - 3x)}{400} = \dfrac{1}{8}$

$\therefore 40x - 3x^2 - 50 = 0$
$x = 1.396, 11.397$

**ii** $y' = \dfrac{x(40 - 3x)}{400} = -\dfrac{1}{8}$

$\therefore 40x - 3x^2 + 50 = 0$
$x = 14.484$ (since $x > 0$)

**6** TSA = 150 cm$^2$




**a** Area of top & base
$= 2x^2$
Area of 4 sides
$= 4xh$
$\therefore\ 2x^2 + 4xh = 150$ cm$^2$

$$2xh = 150 - 4x^2$$
$$h = \frac{75 - x^2}{2x}$$

**b** $V(x) = x^2h = x^2\left(\dfrac{75 - x^2}{2x}\right)$

$$= \frac{x}{2}(75 - x^2)$$
$$= \frac{1}{2}(75x - x^3)$$

**c** $V'(x) = \dfrac{1}{2}(75 - 3x^2)$
$v'(x) = 0, \therefore x^2 = 25$

$x = 5$ cm

$\therefore\ V(5) = 125$ cm$^3$
$V'(4) > 0;\ V'(6) < 0$
$\therefore$ stationary pt must be a maximum.

**d** Since $5 > 4$ and $V$ is still increasing at $x = 4$,
$V$ max . is $V(4) = \dfrac{4}{2}(75 - 4^2)$

$= 118$ cm$^3$

**7** $V = \pi r^2 h$ and $r + h = 12$
$h = 12 - r$
$\therefore V = \pi r^2(12 - r) = \pi(12r^2 - r^3)$
Note $0 \le r \le 12$

$$\frac{dV}{dr} = \pi(24r - 3r^2)$$
$$\frac{dV}{dr} = 0 \text{ implies that}$$

$24r - 3r^2 = 0$

$\therefore 3r(8 - r) = 0$

$\therefore r = 0$ or $r = 8$

Maximum occurs when $r = 8$

Maximum volume$= 8^2(12 - 8)\pi = 256\pi$

**8** The lengths of the sides of the base of the tray are $50 - 2x$ cm and $40 - 2x$ cm. The height of the tray is $x$ cm.

Therefore the volume $V\text{cm}^3$ of the tray is given by

$V = (50 - 2x)(40 - 2x)x =$

$4(x^3 - 45x^2 + 500x)$

We note: $20 \leq x \leq 25$

$$\frac{dV}{dx} = 4(3x^2 - 90x + 500)$$

$\frac{dV}{dx} = 0$ implies that

$$3x^2 - 90x + 500 = 0$$

$$\therefore x = \frac{5(9 - \sqrt{21})}{3} \text{ or } x = \frac{5(9 + \sqrt{21})}{3}$$

Maximum occurs when $x = \dfrac{5(9 - \sqrt{21})}{3}$

**9** $f(x) = 2 - 8x^2, -2 \leq x \leq 2$

$\therefore f'(x) = -16x = 0, \therefore x = 0$

For $x < 0, f'(x) > 0$; for $x > 0, f(x) < 0$

Local and absolute maximum for $f(0) = 2$ Absolute minimum at $f(\pm 2) = -30$.

**10** $f(x) = x^3 + 2x + 3, -2 \leq x \leq 1$

$\therefore f'(x) = 3x^2 + 2 > 0, x \in R$

Function is constantly increasing, so absolute maximum is $f(1) = 6$.

Absolute minimum $\neq f(-2) = -9$.

**11** $f(x) = 2x^3 - 6x^2, 0 \leq x \leq 4$

$\therefore f'(x) = 6x^2 - 12x = 6x(x - 2)$

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f'$ | 0 | $-$ | 0 | $+$ |

$x = 0$ is a local maximum $f(0) = 0$, but $f(4) = 32$, so the absolute maximum is 32. $x = 2$ is an absolute minimum of $f(2) = -8$.

**12** $f(x) = 2x^4 - 8x^2, -2 \leq x \leq 5$

$\therefore f'(x) = 8x^3 - 16x$

$= 8x(x - \sqrt{2})(x + \sqrt{2})$

| $x$ | $-2$ | $-\sqrt{2}$ | $-1$ | 0 | 1 | $\sqrt{2}$ | 2 |
|---|---|---|---|---|---|---|---|
| $f'$ | $-$ | 0 | $+$ | 0 | $-$ | 0 | $+$ |

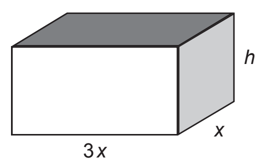
$x = 0$ is a local maximum $f(0) = 0$, but $f(5) = 1050$, so the absolute maximum is 1050.

At the other boundary condition,

$f(-2) = 0 < 1050$.

$f(\pm\sqrt{2}) = -8$ are local and absolute minima.

**13**




Total edges: $4h + 4x + 12x = 20$ cm.

$$\therefore h = \frac{20 - 16x}{4} = 5 - 4x$$

**a** $v = x(3x)h = 3x^2(5 - 4x)$

$= 15x^2 - 12x^3$

**b** $\dfrac{dV}{dx} = 30x - 36x^2 = 6x(5 - 6x)$

**c** Sign diagram for $x \in [0, 1.25]$:

| $x$ | 0 | $\frac{1}{2}$ | $\frac{5}{6}$ | 1 |
|---|---|---|---|---|
| $V'$ | 0 | $+$ | 0 | $-$ |

Local maximum $= V(\frac{5}{6}) = \frac{125}{36}$ cm$^3$

**d** If $x \in [0, 0.8]$, then $0.8 < \frac{5}{6}$ and V is

still increasing
$\therefore V \max = V(0.8) = \frac{432}{125}$ cm$^3$

**e** If $x \in [0, 1]$, $V \max = V(\frac{5}{6}) = \frac{125}{36}$ cm$^3$

**14** $x + y = 20, \therefore y = 20 - x$

**a** If $x \in [2, 5], y \in [15, 18]$

**b** $z = xy = x(20 - x) = 20x - x^2$
$\frac{dz}{dx} = 20 - 2x = 0$
$\therefore\ x = 10$ for a stationary point.
However, with $x$ restricted to [2, 5], $\frac{dz}{dx} > 0$ So the minimum value of $z$ is $z(2) = 36$ and maximum value is $z(5) = 75$.

**15** $2x + y = 50, \therefore y = 50 - 2x$

$\therefore z = x^2y = x^2(50 - 2x) = 50x^2 - 2x^3$

$\frac{dz}{dx} = 100x - 6x^2 = 2x(50 - 3x)$
Inverted cubic, so $z$ has a local minimum at $(0, 0)$ and a local maximum at $\left(\frac{50}{3}, \frac{125\,000}{27}\right)$

**a** $x \in [0, 25]$, So max. $z = \frac{125\,000}{27}$

**b** $x \in [0, 10]$, so max. $z = z(10) = 3000$

**c** $x \in [5, 20]$, so max. $z = \frac{125\,000}{27}$

**16** **a** 1st piece has length $x$ metres, so 2nd piece has length $(10 - x)$ metres, each folded into 4 to make a square:

$$\begin{aligned}\text{Total area } A &= (\frac{x}{4})^2 + (\frac{10 - x}{4})^2 \\ &= \frac{x^2}{16} + \frac{100 - 20x + x^2}{16} \\ &= \frac{1}{8}(x^2 - 10x + 50) \text{ m}^2\end{aligned}$$

**b** $\frac{dA}{dx} = \frac{1}{8}(2x - 10)$
$= \frac{x - 5}{4}$

**c** Upright parabola, so turning point is a minimum.
$\frac{dA}{dx} = \frac{x - 5}{4} = 0$
$\therefore\ x = 5$

**d** For $x \in [4,\ 7]$, check end points.
$A(4) = \frac{26}{8}$ and $A(7) = \frac{29}{8}$
Now check stationary point:
$A(5) = \frac{25}{8}$
Hence, $A_{\min} = \frac{25}{8}$

## Solutions to Exercise 18F

**1 a** The particle is at $O$ when it crosses the horizontal axis, i.e. $t = 2$, $t = 3$ and $t = 8$ seconds.

**b** Velocity is positive when the gradient is positive. This occurs when $0 < t < 2.3$ and $t > 6$ seconds (approximately).

**c** The velocity is equal to zero at the stationary points. This occurs at $t = 2.3$ and $t = 6$ seconds (approximately).

**2** $x = t^2 - 12t + 11, t \geq 0$

**a** Velocity $= v = \dfrac{dx}{dt} = 2t - 12$
When $t = 0, v = -12$
The particle is moving to the left at 12 cm/s

**b** $v = 0$ implies $2t - 12 = 0$. That is $t = 6$
When $t = 6, x = 36 - 72 + 11 = -25$.
The particle is 25 cm to the left of $O$.

**c** Average velocity for the first three seconds =

$$\frac{x(3) - x(0)}{3 - 0} = -\frac{27}{3} = -9 \text{ cm/s}.$$

**d** The particle moves to the left for the first three seconds and doesn't change direction. The speed is 9 cm/s

**3 a** The particle is stationary at the stationary point. This occurs at $t = 2$ seconds.

**b** The particle is moving to the right when the velocity is positive. This occurs when $0 < t < 2$.

**c** The furthest the particle travels to the right occurs at the maximum point on this graph. The distance is 8 meters.

**d** The particle returns to $O$ at the second $t$-intercept. This is at $t = 4$ seconds.

**e** The graph has equation $x = at(t - 4)$.
To find $a$, substitute $(2, 8)$:
$8 = a \times 2 \times (2 - 4)$

$a = -2$
The equation is $x = -2t(t - 4) = -2t^2 + 8t$
$p = -2$, $q = 8$ and $r = 0$

**f** The velocity of the particle is given by the gradient at $t = 3$:

$$\begin{aligned}\frac{dx}{dt} &= -4t + 8\\ &= -4 \times 3 + 8 \text{ when } t = 3\\ &= -4\end{aligned}$$

The velocity at $t = 3$ is $-4$ ms$^{-1}$.

**4** $x = \dfrac{1}{3}t^3 - 12t + 6, t \geq 0$

**a** Therefore $\dfrac{dx}{dt} = t^2 - 12$ When $t = 3, v = \dfrac{dx}{dt} = -3$

**b**

$$\begin{aligned}\frac{dx}{dt} &= 0\\ \Rightarrow t^2 &= 12\end{aligned}$$

$\Rightarrow t = \pm 2\sqrt{3}$

But $t \geq 0$. Therefore $t = 2\sqrt{3}$
The velocity is zero at time $t = 2\sqrt{3}$ seconds

**5** $x = 4t^3 - 6t^2 + 5$

Velocity : $v = \dfrac{dx}{dt} = 12t^2 - 12t$

Acceleration : $a = \dfrac{dv}{dt} = 24t - 12$

**a** When $t = 0, x = 5, v = 0, a = -12$
The particle is initially at rest at $x = 5$ and starts moving to the left.

**b** It is instantaneously at rest when $12t(t-1) = 0$. That is, when $t = 0$ and $t = 1$
When $t > 1$ it is moving to the right.

**6** $s = t^4 + t^2$

$v = \dfrac{ds}{dt} = 4t^3 + 2t$

$a = \dfrac{dv}{dt} = 12t^2 + 2$

**a** When $t = 0$, acceleration is $2\text{m/s}^2$

**b** When $t = 2$, acceleration is $50\text{m/s}^2$

**7** **a** $s = 10 + 15t - 4.9t^2$

$\therefore v = \dfrac{ds}{dt} = 15 - 9.8t$ m/s

**b** $a = \dfrac{dv}{dt} = -9.8 \text{ m/s}^2$

**8** $x(t) = t^2 - 7t + 10, t \geq 0$

**a** $v(t) = 2t - 7 = 0$

$\therefore t = 3.5$ s

**b** $a(t) = 2 \text{ m/s}^2$ at all times

**c** 14.5 m

**d** $v(t) = 2t - 7 = -2$

$\therefore \quad t = 2.5$ s

$x(2.5) = 2.5^2 - 7(2.5) + 10$ cm

$= 1.25$ m to the left of $O$.

**9** **a** $s = t^3 - 3t^2 + 2t$

$= t(t-1)(t-2)$

$\therefore s = 0$ at $t = 0, 1$ and $2$

**b** $v(t) = 3t^2 - 6t + 2; a(t) = 6t - 6$
$t = 0: v = 2$ m/s and $a = -6 \text{ m/s}^2$
$t = 1: v = -1$ m/s and $a = 0 \text{ m/s}^2$
$t = 2: v = 2$ m/s and $a = 6 \text{ m/s}^2$

**c** Av. $v$ in 1st second
$= s(1) - s(0) = 0$ m/s

**10** **a** $x = t^2 - 7t + 12$
$\therefore x(0) = 12$ cm to the right of $O$.

**b** $x(5) = 5^2 - 7(5) + 12$

$= 2$ cm to the right of $O$.

**c**

$v(t) = 2t - 7$

$\therefore v(0) = -7$

$= 7$ cm/s moving to the left of $O$.

**d** $v = 0$ when $t = 3.5$ s

$x(3.5) = 3.5^2 - 7(3.5) + 12$

$= -0.25$

$= 0.25$ cm to the left.

**e** Av.$v = \dfrac{x(5) - x(0)}{5}$

$= \dfrac{2 - 12}{5}$

$= -2$ cm/s

**f** Total distance traveled

$= x(0) - x(3.5) + x(5) - x(3.5)$

$= 12.25 + 2.25 = 14.5$ cm.

Total time = 5 seconds ∴ av. speed

$= \dfrac{14.5}{5} = 2.9$ cm/s

**11** $s = t^4 + 3t^2$

$v = \dfrac{ds}{dt} = 4t^3 + 6t$

$a = \dfrac{dv}{dt} = 12t^2 + 6$

**a** When $t = 1$, acceleration is $18 \text{m/s}^2$
When $t = 2$, acceleration is $54 \text{ m/s}^2$
When $t = 3$, acceleration is $114 \text{ m/s}^2$

**b** Average acceleration

$= \dfrac{v(3) - v(1)}{3 - 1} = \dfrac{116}{2} = 58 \text{ m/s}^2$

**12** $x(t) = t^3 - 11t^2 + 24t - 3, t \geq 0$

**a** $v(t) = 3t^2 - 22t + 24, t \geq 0$

$x(0) = -3$ cm; $v(0) = 24$ cm/s
Particle is 3 cm to the left of $O$
moving to the right at 24 cm/s.

**b** See **a**: $v(t) = 3t^2 - 22t + 24, t \geq 0$

**c** $v(t) = 3t^2 - 22t + 24 = 0$

$= (3t - 4)(t - 6) = 0,$

$\therefore \quad t = \dfrac{4}{3}, 6$ s

**d** $x\left(\dfrac{4}{3}\right) = \left(\dfrac{4}{3}\right)^3 - 11\left(\dfrac{4}{3}\right)^2 + 24\left(\dfrac{4}{3}\right) - 3$

$= \dfrac{64}{27} - \dfrac{176}{9} + 32 - 3$

$= \dfrac{319}{27}$ cm right of $O$

$x(6) = (6)^3 - 11(6)^2 + 24(6) - 3$

$= 216 - 396 + 144 - 3$

$= 39$ cm left of $O$

**e** Velocity negative for $t \in \left(\dfrac{4}{3}, 6\right)$,

i.e. for $\dfrac{14}{3}$ s $= 4\dfrac{2}{3}$ s.

**f** $a(t) = 6t - 22 \text{ cm/s}^2$

**g** $a(t) = 6t - 22 = 0, \therefore \; t = \dfrac{11}{3}$s

$x\left(\dfrac{11}{3}\right) = \left(\dfrac{11}{3}\right)^3 - 11\left(\dfrac{11}{3}\right)^2 + 24\left(\dfrac{11}{3}\right) - 3$

$= \dfrac{1331}{27} - \dfrac{1331}{9} + \dfrac{264}{3} - 3$

$= -\dfrac{313}{27}$

$= \dfrac{313}{27}$ cm to the left of $O$

$v\left(\dfrac{11}{3}\right) = 3\left(\dfrac{11}{3}\right)^2 - 22\left(\dfrac{11}{3}\right) + 24$

$= \dfrac{121}{3} - \dfrac{242}{3} + 24$

$= -\dfrac{49}{3}$

$= \dfrac{49}{3}$ cm/s moving to the left.

**13** $x(t) = 2t^3 - 5t^2 + 4t - 5, t \geq 0$

$\therefore\ v(t) = 6t^2 - l0t + 4, t \geq 0$

$\therefore\ a(t) = 12t - 10, t \geq 0$

**a** $v(t) = 6t^2 - 10t + 4 = 0$

$2(3t - 2)(t - 1) = 0$

$\therefore t = \dfrac{2}{3}, 1$

$$a\left(\frac{2}{3}\right) = 12\left(\frac{2}{3}\right) - 10$$

$$= -2 \text{ cm/s}^2$$

$$a(1) = 12 - 10 = 2 \text{ cm/s}^2$$

**b** $a(t) = 12t - 10 = 0, \quad \therefore t = \dfrac{5}{6}$ s

$$v\left(\frac{5}{6}\right) = 6\left(\frac{5}{6}\right)^2 - 10\left(\frac{5}{6}\right) + 4$$

$$= \frac{25}{6} - \frac{50}{6} + 4 = -\frac{1}{6}$$

Particle is moving to the left at $\dfrac{1}{6}$ cm/s

**14** $x(t) = t^3 - 13t^2 + 46t - 48, t \geq 0$

$\therefore\ v(t) = 3t^2 - 26t + 46, t \geq 0$

$\therefore\ a(t) = 6t - 26, t \geq 0$

The particle passes through $O$ where $x = 0$:

$x(t) = (t - 2)(t - 3)(t - 8) = 0$

$\therefore \quad t = 2, 3, 8$ s

At $t = 2$: $v = 6$ cm/s, $a = -14$ cm/s$^2$
At $t = 3$: $v = -5$ cm/s, $a = -8$ cm/s$^2$
At $t = 8$: $v = 30$ cm/s, $a = 22$ cm/s$^2$

**15** P1: $x(t) = t + 2, \therefore v(t) = 1$
P2: $x(t) = t^2 - 2t - 2, \therefore v(t) = 2t - 2$

**a** Particles at same position when

$t + 2 = t^2 - 2t - 2$

$t^2 - 3t - 4 = 0$

$(t - 4)(t + 1) = 0$

$\therefore \quad t = -1, 4$

(No restricted domain: both correct)

**b** Velocities equal when $2t - 2 = 1$ so $t = 1.5$ s

## Solutions to Exercise 18G

**1** $v = 4t - 6; t \geq 0$

**a** $x = \displaystyle\int 4t - 6dt = 2t^2 - 6t;\ v(0) = 0$

**b** $x(3) = 18 - 18 = 0$ cm
Body is at $O$.

**c** $v = 0,\ \therefore t = 1.5$ s
$x(1.5) = \dfrac{9}{2} - 9 = -4.5$ cm
$x(3) = 0$
Body goes 4.5 cm in each direction = 9 cm total.

**d** Av. $v$ over $[0, 3] = 0$ since $x(3) = x(0)$

**e** Av. speed over $[0, 3] = \dfrac{9}{3} = 3$ cm/s

**2** $v = 3t^2 - 8t + 5; t \geq 0$
$x(0) = 4$ m + direction to the right of $O$.

**a** $x = t^3 - 4t^2 + 5t + 4$

$a = 6t - 8$

**b** $v(t) = (3t - 5)(t - 1) = 0, \therefore t = 1,\ \dfrac{5}{3}$
$x(1) = 1 - 4 + 5 + 4 = 6$ m
$x\left(\dfrac{5}{3}\right) = \dfrac{125}{27} - \dfrac{100}{9} + \dfrac{25}{3} + 4$
$= \dfrac{158}{27}$ m

**c** $a(1) = -2$ m/s$^2$; $a\left(\dfrac{5}{3}\right) = 2$ m/s$^2$

**3** $a = 2t - 3; t \geq 0$
$\therefore\ v = t^2 - 3t + 3;\ v(0) = 3$
$\therefore\ x = \dfrac{1}{3}t^3 - \dfrac{3}{2}t^2 + 3t + 2; x(0) = 2$

$x(10) = \dfrac{1000}{3} - 150 + 30 + 2$
$= \dfrac{646}{3}$ m

**4** $a = -10$ m/s$^2$

**a** $v(0) = 25,\ \therefore v(t) = 25 - 10t$ m/s

**b** $h(0) = 0,\ \therefore h(t) = 25t - 5t^2$ m

**c** $h$ is max. when $v = 0$ at $t = 2.5$ s

**d** $h\left(\dfrac{5}{2}\right) = \dfrac{125}{4} = 31.25$ m

**e** $h = 0,\ \therefore 5t(5 - t) = 0$

$t = 0, 5$
Body returns to its start after 5 seconds.

**5** $a = \dfrac{t - 5}{9},\ \therefore v = \dfrac{t^2}{18} - \dfrac{5t}{9} + c$ m/s

$v(0) = -8,\ \therefore c = -8$

$\therefore\ v(t) = \dfrac{t^2}{18} - \dfrac{5t}{9} - 8$ m/s

$\therefore\ x(t) = \dfrac{t^3}{54} - \dfrac{5t^2}{18} - 8t + 300$ m
since $x(0) = 300$ m
$v = 0$ when $\dfrac{t^2}{18} - \dfrac{5t}{9} = 8$
$\therefore\ t^2 - 10t - 144 = 0$

$(t - 18)(t + 10) = 0$

$t = 18;\ t > 0$
$x(18) = 108 - 90 - 144 + 300$

$= 174$ m
$\dfrac{174}{6} = 29,\ \therefore$ lift stops at the 29th floor.

## Solutions to Exercise 18H

**1** $f(x) = (x-2)^2(x-b), b > 2$

**a** Use CAS calculator to find gradient function.
$f'(x) = (x-2)(3x-2(b+l))$

**b** For stationary points $f'(x) = 0$
$\therefore x = 2, c$ where $c = \frac{2}{3}(b+1)$

$$f(2) = 0; f(c) = (c-2)^2(c-b)$$
$$= -\frac{4}{27}(b-2)^3$$

$(2,0)$ and $(\frac{2}{3}(b+1), -\frac{4}{27}(b-2)^3)$

**c** $f(x)$ is an upright cubic and the lst stationary pt is always a maximum.
Since $\frac{2}{3}(b+1) > 0$ by definition, this is the 2nd stationary pt and is thus a minimum.
A sign diagram confirms this:

| $x$ | 0 | 2 | | $c$ | |
|---|---|---|---|---|---|
| $f'$ | + | 0 | − | 0 | + |

$\therefore (2,0)$ is always a local maximum.

**d** $\frac{2}{3}(b+1) = 4$
$\therefore \quad b+1 = 6$
$b = 5$

**2** **a** $y = x^4 - 12x^3$
$$\frac{dy}{dx} = 4x^3 - 36x^2 = 4x^2(x-9)$$
$$\frac{dy}{dx} = 0$$
$\Rightarrow 4x^2(x-9) = 0$
$\Rightarrow x = 0$ or $x = 9$
$\frac{dy}{dx} > 0$ for $x > 9$ and $\frac{dy}{dx} < 0$ for $x < 9$. There is a stationary point of inflexion at $(0,0)$ and a local minimum at $(9, -2187)$.

**b** $(a, b)$and $(9+a, -2187+b)$.

**3** $f(x) = x - ax^2, a > 0$
$\therefore f'(x) = 1 - 2ax$

**a** **i** $f$ is an increasing function if $1 - 2ax > 0$
$$\therefore 2ax < 1, \therefore x < \frac{1}{2a}$$
(since $a > 0$)

**ii** $f$ is a decreasing function if $1 - 2ax < 0$
$$\therefore 2ax > 1, \therefore x > \frac{1}{2a}$$
(since $a > 0$)

**b** Tangent at $(\frac{1}{a}, 0)$ has gradient $= -1$
$\therefore \; y - 0 = -1\left(x - \frac{1}{a}\right)$
$y = \frac{1}{a} - x$

**c** Normal at $(\frac{1}{a}, 0)$ has gradient $= 1$:
$\therefore \; y = x - \frac{1}{a}$

**d** Local maximum occurs at $x = \frac{1}{2a}$
$f\left(\frac{1}{2a}\right) = \frac{1}{2a} - \frac{a}{4a^2} = \frac{1}{4a}$
$\therefore$ Range of $f = (-\infty, \frac{1}{4a}]$

**4** **a** Using a CAS:

$f'(x) = (x-a)(x-a+2(x-1))$

$= (x-a)(3x-a-2)$

$f'(x) = 0, \therefore x = a, \dfrac{a+2}{3}$

$f(a) = 0;$

$$f\left(\frac{a+2}{3}\right) = \left(\frac{2}{3}\right)^2 (a-1)^2 \left(\frac{a-1}{3}\right)$$

$$= \frac{4}{27}(a-1)^3$$

Turning pts at $(a, 0)$ and $\left(\dfrac{a+2}{3}, \dfrac{4}{27}(a-1)^3\right)$

**b** $(a, 0)$ is a local minimum. $\left(\dfrac{a+2}{3}, \dfrac{4}{27}(a-1)^3\right)$ is a local maximum.

**c** **i** Tangent at $x = 1$ has gradient $(a-1)^2$:
$y(1) = 0, \therefore y = (a-1)^2(x-1)$

**ii** Tangent at $x = a$ has gradient 0:
$y(0) = 0, \therefore y = 0$

**iii** Tangent at $x = \dfrac{a+1}{2}$ has gradient:

$$= \left(\frac{a+1}{2} - a\right)\left(\frac{3}{2}(a+1) - a - 2\right)$$

$$= \frac{1-a}{2}\left(\frac{a-1}{2}\right)$$

$$= -\frac{1}{4}(a-1)^2$$

$$y\left(\frac{a+1}{2}\right) = \left(\frac{a+1}{2} - a\right)^2\left(\frac{a+1}{2} -\right)$$

$$= \left(\frac{1-a}{2}\right)^2\left(\frac{a-1}{2}\right)$$

$$= \frac{1}{8}(a-1)^3$$

Tangent equation:

$$y - \frac{1}{8}(a-1)^3$$

$$= -\frac{1}{4}(a-1)^2\left(x - \frac{a+1}{2}\right)$$

$$\therefore y = -\frac{1}{4}(a-1)^2\left(x - \frac{a+1}{2}\right) + \frac{1}{8}(a-1)^3$$

$$= -\frac{1}{4}(a-1)^2\left(x - \frac{a+1}{2} - \frac{a-1}{2}\right)$$

$$= -\frac{1}{4}(a-1)^2(x-a)$$

**5** $y = (x-2)^2$
$y = mx + c$ is a tangent to the curve at point $P$.

**a** **i** $y'(x) = 2(x-2)$
$\therefore y'(a) = 2(a-2)$
where $0 \le a < 2$

**ii** $m = 2(a-2)$

**b** $P = (a, (a-2)^2)$

**c** $y - (a-2)^2 = 2(a-2)(x-a)$

$$\therefore y = 2(a-2)x - 2a(a-2) + (a-2)^2$$

$$= 2(a-2)x + (a-2)(a-2-2a)$$

$$= 2(a-2)x + (a-2)(-a-2)$$

$$= 2(a-2)x + 4 - a^2$$

**d** $x$-axis intercept of the tangent is where $y = 0$
$2(a-2)x + 4 - a^2 = 0$
$\therefore x = \dfrac{a^2-4}{2(a-2)} = \dfrac{a+2}{2}$
(since $a \neq 2$)

**6** **a** $f(x) = x^3 \rightarrow y = f(x+h)$

$$f(1+h) = 27, \therefore\ (1+h)^3 = 27$$
$$1 + h = 3$$
$$h = 2$$

**b** $f(x) = x^3 \rightarrow y = f(ax)$

$f(ax)$ passes through (1, 27)

$\therefore ax = 3$

$\therefore\ a = 3$ since $x = 1$

**c** $y = ax^3 - bx^2 = x^2(ax - b)$

$\therefore\ y' = 3ax^2 - 2bx = x(3ax - 2b)$

Using (1, 8): a − b = 8 . . . **(1)**

$y'(1) = 0, \therefore\ 3a - 2b = 0 \ldots$ **(2)**

From **(1)**: $3a - 3b = 24$

$\therefore a = -16, b = -24$

**7** $y = x^4 + 4x^2$

Translation + $a$ in $x$ direction, and + $b$ in $y$ direction:

$y = (x-a)^4 + 4(x-a)^2 + b$

**a** $y' = 4x^3 + 8x = 4x(x^2 + 2)$

Turning pt at$(0, 0)$ only, since $x^2 + 2 > 0; x \in R$

**b** Turning point of image = $(a, b)$

**8** **a**

$$f(x) = (x-1)^2(x-b)^2, b > 1$$
$$\therefore\ f'(x) = 2(x-1)(x-b)^2 + 2(x-b)^2(x-1)$$
$$= 2(x-1)(x-b)(2x-b-1)$$

Use a CAS calculator to determine the gradient function.

**b** $f'(x) = 0$ when $x = 1, b, \dfrac{b+1}{2}$

$f(1) = f(b) = 0$

$$f'\left(\frac{b+1}{2}\right) = \left(\frac{b+1}{2} - 1\right)^2\left(\frac{b+1}{2} - b\right)^2$$
$$= \left(\frac{b-1}{2}\right)^2\left(\frac{1-b}{2}\right)^2$$
$$= \frac{1}{16}(b-1)^4$$

Turning pts: $(1, 0)$, $(b, 0)$ and $\left(\dfrac{b+1}{2}, \dfrac{1}{16}(b-1)^4\right)$

**c** Turning pt at (2, 1) must mean $\dfrac{b+1}{2} = 2$

$\therefore\ b + 1 = 4, \therefore\ b = 3$

## Solutions to Review: Short-answer questions

**1** $y = 4x - x^2$

**a** $\frac{dy}{dx} = 4 - 2x$

**b** Gradient at $Q(1, 3) = 4 - 2 = 2$

**c** Tangent at $Q : y - 3 = 2(x - 1)$
$\therefore\ y = 2x + 1$

**2** $y = x^3 - 4x^2$

**a** $\frac{dy}{dx} = 3x^2 - 8x$

**b** Gradient at $(2, -8) = 3(2)^2 - 8(2)$
$= -4$

**c** Tangent at $(2, -8) = y + 8$
$= -4(x - 2)$
$y = -4x$

**d** Tangent meets curve when
$y = x^3 - 4x^2$
$= -4x$
$\therefore x(x - 2)^2 = 0$
$x = 0, 2$

Tangent cuts curve again at $(0, 0)$.

**3** $y = x^3 - 12x + 2$

**a** $\frac{dy}{dx} = 3x^2 - 12$
$= 3(x - 2)(x + 2)$
$\frac{dy}{dx} = 0, \therefore\ x = \pm 2$
$y(-2) = 18, y(2) = -14$

**b** Upright cubic.
$(-2, 18)$ is a local maximum and $(2, -14)$ is a local minimum.

**c** Upright cubic.
$(-2, 18)$ is a local maximum and $(2, -14)$ is a local minimum.

**4** **a** $\frac{dy}{dx} = 3x^2$
Stationary pt of inflexion at $x = 0$:

| $x$ | $-1$ | 0 | 1 |
|---|---|---|---|
| $\frac{dy}{dx}$ | + | 0 | + |

**b** $\frac{dy}{dx} = -3x^3$
Local maximum at $x = 0$:

| $x$ | $-1$ | 0 | 1 |
|---|---|---|---|
| $\frac{dy}{dx}$ | + | 0 | − |

**c** $f'(x) = (x - 2)(x - 3)$

| $x$ | 0 | 2 | 2.5 | 3 | 4 |
|---|---|---|---|---|---|
| $f'$ | + | 0 | − | 0 | + |

Local maximum at $x = 2$, minimum at $x = 3$

**d** $f'(x) = (x - 2)(x + 2)$

| $x$ | $-3$ | $-2$ | 0 | 2 | 3 |
|---|---|---|---|---|---|
| $f'$ | + | 0 | − | 0 | + |

Local maximum at $x = -2$, minimum at $x = 2$

**e** $f'(x) = (2 - x)(x + 2)$

| $x$ | $-3$ | $-2$ | 0 | 2 | 3 |
|---|---|---|---|---|---|
| $f'$ | − | 0 | + | 0 | − |

Local minimum at $x = -2$, maximum at $x = 2$

**f** $f'(x) = -(x - 1)(x - 3)$

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $f'$ | − | 0 | + | 0 | − |

Local minimum at $x = 1$, maximum at $x = 3$

**g** $\dfrac{dy}{dx} = -x^2 + x + 12 = (4 - x)(x + 3)$

| $x$ | $-4$ | $-3$ | $0$ | $4$ | $5$ |
|---|---|---|---|---|---|
| $\dfrac{dy}{dx}$ | $-$ | $0$ | $+$ | $0$ | $-$ |

Local minimum at $x = -3$, maximum at $x = 4$

**h** $\dfrac{dy}{dx} = 15 - 2x - x^2 = (3 - x)(x + 5)$

| $x$ | $-6$ | $-5$ | $0$ | $3$ | $4$ |
|---|---|---|---|---|---|
| $\dfrac{dy}{dx}$ | $-$ | $0$ | $+$ | $0$ | $-$ |

Local minimum at $x = -5$, maximum at $x = 3$

**5 a** $y = 4x - 3x^3, \therefore\ y' = 4 - 9x^2$

$y' = 0, \therefore x = \pm\dfrac{2}{3}$

$y\left(-\dfrac{2}{3}\right) = -\dfrac{16}{9}, y\left(\dfrac{2}{3}\right) = \dfrac{16}{9}$

Inverted cubic:

$\left(-\dfrac{2}{3}, -\dfrac{16}{9}\right)$ is a local minimum,

$\left(\dfrac{2}{3}, \dfrac{16}{9}\right)$ is a local maximum.

**b** $y = 2x^3 - 3x^2 - 12x - 7$

$\therefore\ y' = 6x^2 - 6x - 12$

$= 6(x - 2)(x + 1)$

$y' = 0, \therefore x = -1, 2$

$y(-1) = 0, y(2) = -27$

Upright cubic:

$(-1, 0)$ is a local maximum,

$(2, -27)$ is a local minimum.

**c** $y = x(2x - 3)(x - 4)$

$= 2x^3 - 11x^2 + 12x$

$\therefore y' = 6x^2 - 22x + 12$

$= 2(3x - 2)(x - 3)$

$y' = 0, \therefore x = \dfrac{2}{3}, 3$

$y(3) = -9,$

$y\left(\dfrac{2}{3}\right) = \dfrac{2}{3}\left(-\dfrac{5}{3}\right)\left(-\dfrac{10}{3}\right)$

$= \dfrac{100}{27}$

Upright cubic:

$\left(\dfrac{2}{3}, \dfrac{100}{27}\right)$ is a local maximum,

$(3, -9)$ is a local minimum.

**6 a** $y = 3x^2 - x^3$

$= x^2(3 - x)$

Axis intercepts at $(0, 0)$ and $(3, 0)$.

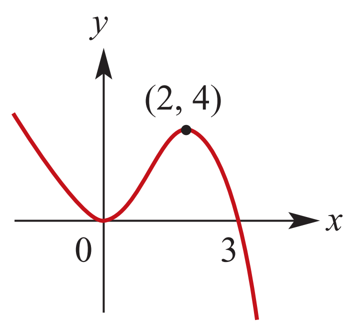
$y' = 6x - 3x^2$

$= 3x(2 - x)$

Stationary pts at $(0, 0)$ and $(2, 4)$.

Inverted cubic:

local min. at $(0, 0)$, max. at $(2, 4)$.




**b** $y = x^3 - 6x^2$

$= x^2(x - 6)$

Axis intercepts at $(0, 0)$ and $(6, 0)$.

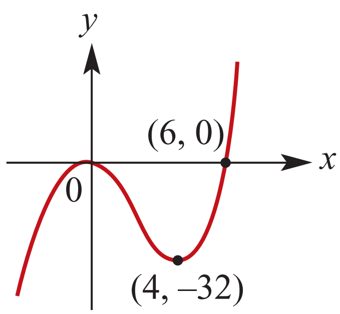
$y' = 3x^2 - 12x$

$= 3x(x - 4)$

Stationary pts at $(0, 0)$ and $(4, -32)$.

Upright cubic:

local max. at $(0, 0)$, min. at $(4, -32)$.




**c** $y = (x+1)^2(2-x)$

$\quad = 2 + 3x - x^3$

Axis intercepts at $(0, 2)$, $(-1, 0)$ and $(2, 0)$.

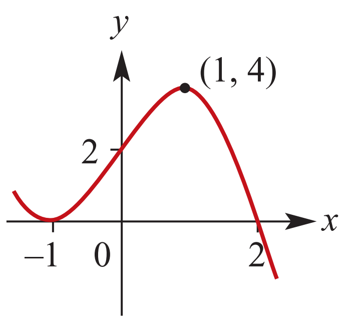
$y' = 3 - 3x^2$

$\quad = 3(1-x)(1+x)$

Stationary pts at $(-1, 0)$ and $(1, 4)$.

Inverted cubic:

local min. at $(-1, 0)$, max. at $(1, 4)$.




**d** $y = 4x^3 - 3x$

$\quad = x(2x - \sqrt{3})(2x + \sqrt{3})$

Axis intercepts at $(0, 0)$, $(-\sqrt{3}/2, 0)$ and $(\sqrt{3}/2, 0)$.

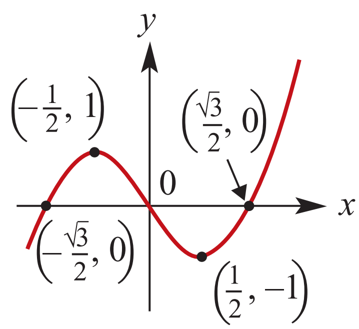
$y' = 12x^2 - 3$

$\quad = 3(2x-1)(2x+1)$

Stationary pts at $(-\frac{1}{2}, 1)$ and $(\frac{1}{2}, -1)$.

Upright cubic:

local max. $(-\frac{1}{2}, 1)$ min. $(\frac{1}{2}, -1)$.




**e** $y = x^3 - 12x^2$

$\quad = x^2(x - 12)$

Axis intercepts at $(0, 0)$ and $(12, 0)$.

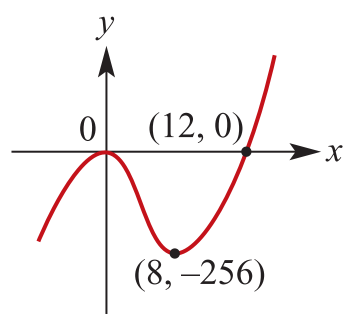
$y' = 3x^2 - 24x$

$\quad = 3x(x - 8)$

Stationary pts at $(0, 0)$ and $(8, -256)$.

Inverted cubic:

local max . $(0, 0)$, min . $(8, -256)$.




**7** **a** C

**b** A

**c** B

**8** $h = 20t - 5t^2$

$$v = \frac{dh}{dt} = 20 - 10t$$

**a**
$$\frac{dh}{dt} = 0$$
$$\Rightarrow 20 - 10t = 0$$
$$\Rightarrow t = 2$$

Stone reaches the maximum height when $t = 2$

$h(2) = 4020 = 20$

The maximum height is 20 m.

**b** $20t - 5t^2 = -60$

$\Leftrightarrow 5t^2 - 20t - 60 = 0$

$\Leftrightarrow t^2 - 4t - 12 = 0$

$\Leftrightarrow (t-6)(t+2) = 0$

$\Leftrightarrow t = 6$ or $t = -2$

$t \geq 0, \therefore t = 6$

It takes 6 seconds to hit the beach.

**c** When $t = 6, v = 20 - 60$

The speed is 40 m/s

**9** $x + y = 12 \Rightarrow y = 12 - x$

Let $M = x^2 + y^2$

Then $M = x^2 + 144 - 24x + x^2$

$= 2x^2 - 24x + 144$

Minimum value when $\dfrac{dM}{dx} = 0$

$\dfrac{dM}{dx} = 4x - 24$

$\therefore$ minimum value when $x = 6$

Therefore minimum value is 72

**10 a** $\dfrac{dv}{dt} = a = 4 - t$

$\therefore v = 4t - \dfrac{1}{2}t^2 + c$

When $t = 0, v = 0$

$\therefore v = 4t - \dfrac{1}{2}t^2$

$v = \dfrac{dx}{dt} = 4t - \dfrac{1}{2}t^2$

$\therefore x = 2t^2 - \dfrac{1}{6}t^3 + c$

When $t = 0, v = 0$

$\therefore x = 2t^2 - \dfrac{1}{6}t^3$

When $t = 3, v = \dfrac{15}{2}$ m/s

**b** Comes to rest when $v = 0$

$4t - \dfrac{1}{2}t^2 = 0$

$t(4 - \dfrac{1}{2}t) = 0$

$t = 0$ or $t = 8$

When $t = 8$

$$x = 2 \times 8^2 - \frac{1}{6} \times 8^3$$

$$x = \frac{128}{3} \text{ m}$$

**c** When $t = 12, x = 0$

**d** Moves to the right for the first 8 seconds of motion. $x(8) = \dfrac{128}{3}$

It then returns to the origin in the next 4 seconds.

Hence the total distance travelled $= \dfrac{256}{3}$ m

The average speed $= \dfrac{256}{36} = \dfrac{64}{9}$ m/s

## Solutions to Review: Multiple-choice questions

**1 D** $y = x^3 + 2x, \therefore\ y' = 3x^2 + 2$
Tangent at (1, 3) has gradient
$y'(1) = 5$
$y - 3 = 5(x - 1)$
$\therefore y = 5x - 2$

**2 E** Normal at (1, 3) has gradient $= -\frac{1}{5}$
$$y - 3 = -\frac{1}{5}(x - 1)$$
$$\therefore\ y = -\frac{1}{5}x + \frac{16}{5}$$

**3 E** $y = 2x - 3x^3, \therefore\ y' = 2 - 9x^2$
Tangent at (0,0) has gradient
$y'(0) = 2$
$\therefore y = 2x$

**4 A** $f(x) = 4x - x^2$
Av. rate of change over [0, 1]
$$= \frac{f(1) - f(0)}{1} = 3$$

**5 C**
$$S(t) = 4t^3 + 3t - 7$$
$$\therefore\ S'(t) = 12t^2 + 3$$
$$\therefore\ S(0) = 3\,\text{m/s}$$

**6 D**
$$y = x^3 - 12x$$
$$\therefore\ y' = 3x^2 - 12$$
$$= 3(x - 2)(x + 2)$$
$$y' = 0 \text{ for } x = \pm 2$$

**7 D**
$$y = 2x^3 - 6x$$
$$\therefore y' = 6x^2 - 6 = 6$$
So $6x^2 = 12$
$$\therefore x = \pm\sqrt{2}$$

**8 A**
$$f(x) = 2x^3 - 5x^2 + x$$
$$\therefore f'(x) = 6x^2 - 10x + 1$$
$$\therefore f'(2) = 5$$

**9 A** $y = \frac{1}{2}x^4 + 2x^2 - 5$
Av. rate of change over [−2, 2]
$$= \frac{y(-2) - y(-2)}{2 - (-2)} = 0$$

**10 C**
$$y = x^2 - 8x + 1$$
$$\therefore y' = 2x - 8$$
Minimum value is $y(4) = -15$

**11 A** The particle has a velocity of zero at the stationary points.

**12 A** The particle has a negative velocity when the gradient is negative.

# Solutions to Review: Extended-response questions

**1** **a** $s = 2 + 10t - 4t^2$

$$v = \frac{ds}{dt} = 10 - 8t, \text{ where } v \text{ is velocity}$$

When $t = 3$, $v = 10 - 8(3)$

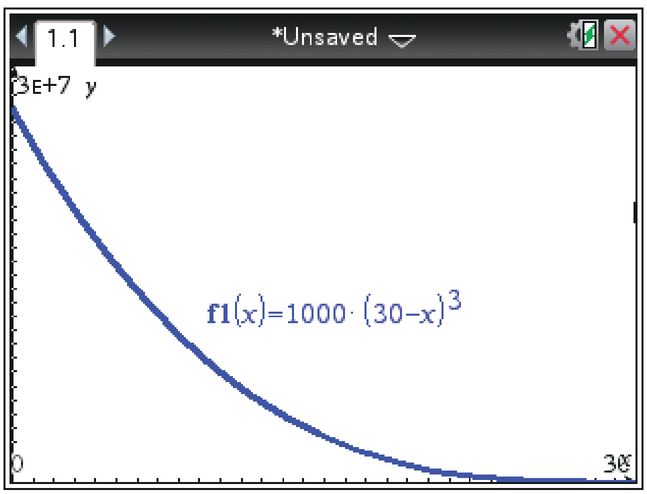
$= -14$

After 3 seconds, the velocity of the stone is $-14$ m/s (i.e. the stone is falling).

**b** $a = \dfrac{dv}{dt} = -8$

The acceleration due to gravity is $-8$ m/s$^2$.

**2** **a**




**b** **i**

$$2\,000\,000 = 1000(30 - t)^3$$
$$2000 = (30 - t)^3$$
$$(2000)^{\frac{1}{3}} = 30 - t$$
$$\therefore \quad t = 30 - (2000)^{\frac{1}{3}}$$
$$\approx 17.4 \text{ min}$$

**ii**

$$20\,000\,000 = 1000(30 - t)^3$$
$$20\,000 = (30 - t)^3$$
$$(20\,000)^{\frac{1}{3}} = 30 - t$$
$$\therefore \quad t = 30 - (20000)^{\frac{1}{3}}$$
$$\approx 2.9 \text{ min}$$

**c** $V = 1000(30 - t)^3$

$= 1000(30 - t)(900 - 60t + t^2)$

$= 1000(27\,000 - 1800t + 30t^2 - 900t + 60t^2 - t^3)$

$= 1000(27\,000 - 2700t + 90t^2 - t^3)$

$= 27\,000\,000 - 2700\,000t + 90\,000t^2 - 1000t^3$

$\frac{dV}{dt} = -2700\,000 + 180\,000t - 3000t^2$

$= -3000(900 - 60t + t^2)$

$= -3000(30 - t)^2, \qquad t \geq 0$

At any time $t$, the dam is being emptied at the rate of $3000(30 - t)^2$ litres/min.

**d** When $V = 0$, $1000(30 - t)^3 = 0$

$\therefore\ 30 - t = 0$

$\therefore\ \quad t = 30$

It takes 30 minutes to empty the dam.

**e** When $\frac{dV}{dt} = -8000 \qquad -3000(30 - t)^2 = -8000$

$\therefore \quad (30 - t)^2 = \frac{8}{3}$

$\therefore \quad 30 - t = \pm\frac{\sqrt{8}}{\sqrt{3}}$

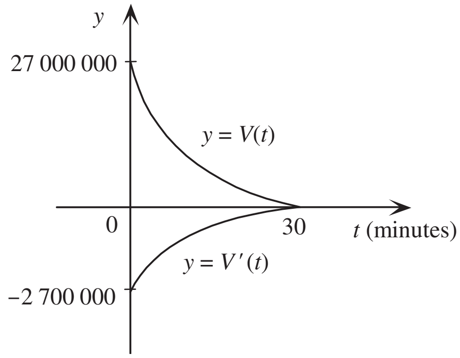
$\therefore \quad t = 30 \pm \frac{2\sqrt{2}}{\sqrt{3}}$

$\therefore \quad t = 30 - \frac{2\sqrt{2}}{\sqrt{3}}$, as $t \leq 30$

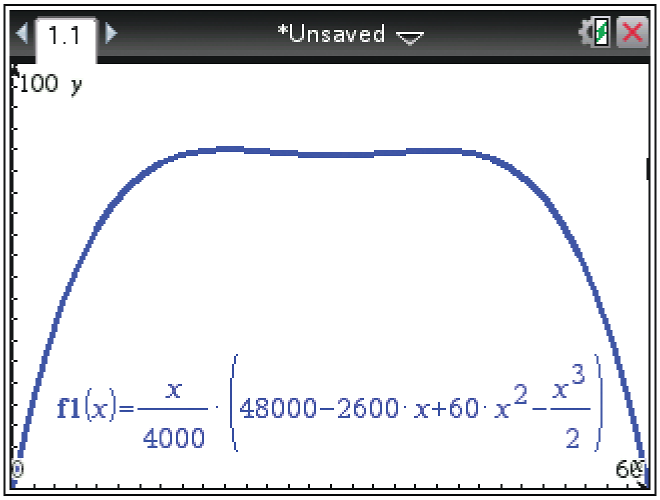
$\therefore \quad t \approx 28.37$

Water is flowing out of the dam at 8000 litres per minute when $t$ is approximately 28.37 minutes.

**f**



**3 a**




**b** Sketch the graph of $f2 = 50$ and

**TI:** Press **Menu → 6:Analyze Graph → 4:Intersection**

**CP:** Tap **Analysis → G-Solve → Intersect**

After 5.71 days; quantity drops below this after 54.29 days.

**c**

$$W = \frac{x}{4000}\left(48000 - 2600x + 60x^2 - \frac{x^3}{2}\right)$$

$$= 12x - \frac{13}{20}x^2 + \frac{3}{200}x^3 - \frac{1}{8000}x^4$$

$$\frac{dW}{dx} = 12 - \frac{26}{20}x + \frac{9}{200}x^2 - \frac{4}{8000}x^3$$

$$= 12 - \frac{13}{10}x + \frac{9}{200}x^2 - \frac{1}{2000}x^3$$

When $x = 20$, $$\frac{dW}{dx} = 12 - \frac{13}{10}(20) + \frac{9}{200}(20)^2 - \frac{1}{2000}(20)^3$$

$$= 12 - 26 + 18 - 4$$

$$= 0$$

When $x = 40$, $$\frac{dW}{dx} = 12 - \frac{13}{10}(40) + \frac{9}{200}(40)^2 - \frac{1}{2000}(40)^3$$

$$= 12 - 52 + 72 - 32$$

$$= 0$$

When $x = 60$, $$\frac{dW}{dx} = 12 - \frac{13}{10}(60) + \frac{9}{200}(60)^2 - \frac{1}{2000}(60)^3$$

$$= 12 - 78 + 162 - 108$$

$$= -12$$

The rate of increase of $W$, when $x = 20, 40$ and $60$ is $0, 0$ and $-12$ tonnes per day respectively.

**d** When $x = 30$, $W = 12(30) - \frac{13}{20}(30)^2 + \frac{3}{200}(30)^3 - \frac{1}{8000}(30)^4$

$= 360 - 585 + 405 - 101.25 = 78.75$

**4 a** When $t = 0$, $y = 15 + \frac{1}{80}(0)^2(30 - 0) = 15$

When $t = 0$, the temperature is 15°C.

**b**
$$\begin{aligned} y &= 15 + \frac{1}{80}t^2(30 - t) \\ &= 15 + \frac{3}{8}t^2 - \frac{1}{80}t^3 \\ \frac{dy}{dt} &= \frac{3}{4}t - \frac{3}{80}t^2 \end{aligned}$$

When $t = 0$, $\frac{dy}{dt} = \frac{3}{4}(0) - \frac{3}{80}(0)^2 = 0$

When $t = 5$, $\frac{dy}{dt} = \frac{3}{4}(5) - \frac{3}{80}(5)^2$

$= \frac{15}{4} - \frac{75}{80} = \frac{45}{16}$

When $t = 10$, $\frac{dy}{dt} = \frac{3}{4}(10) - \frac{3}{80}(10)^2$

$= \frac{30}{4} - \frac{300}{80} = \frac{15}{4}$

When $t = 15$, $\frac{dy}{dt} = \frac{3}{4}(15) - \frac{3}{80}(15)^2$

$= \frac{45}{4} - \frac{675}{80} = \frac{45}{16}$

When $t = 20$, $\frac{dy}{dt} = \frac{3}{4}(20) - \frac{3}{80}(20)^2$

$= \frac{60}{4} - \frac{1200}{80} = 0$

The rate of increase of $y$ with respect to $t$ when $t = 0, 5, 10, 15$ and $20$ is 0, $\frac{45}{16}, \frac{15}{4}, \frac{45}{16}$ and 0°C per minute respectively.

**c**

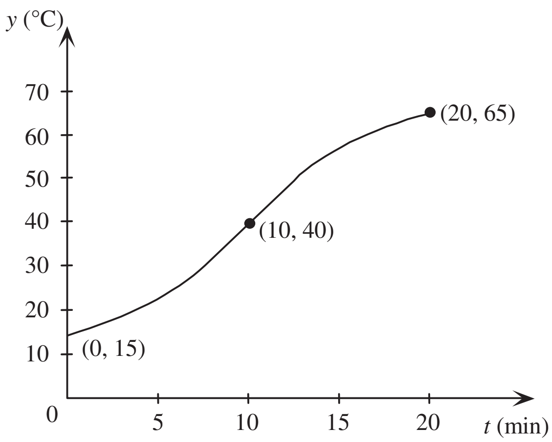
| $t$ | 0 | 5 | 10 | 15 | 20 |
|---|---|---|---|---|---|
| $y$ | 15 | 22.8125 | 40 | 57.1875 | 65 |

When $t = 5$, $y = 15 + \frac{1}{80}(5)^2(30 - 5)$

$= 22.8125$

When $t = 20$, $y = 15 + \frac{1}{80}(20)^2(30 - 20)$

$= 65$




**5 a**

$$\begin{aligned} S &= 4000 + (t-16)^3 \\ &= 4000 + (t-16)(t^2 - 32t + 256) \\ &= 4000 + t^3 - 32t^2 + 256t - 16t^2 + 512t - 4096 \\ &= t^3 - 48t^2 + 768t - 96 \\ \frac{dS}{dt} &= 3t^2 - 96t + 768 \end{aligned}$$

When $t = 0$, $\frac{dS}{dt} = 3(0)^2 - 96(0) + 768$

$$= 768$$

Sweetness was increasing by 768 units/day when $t = 0$.

**b** When $t = 4$, $\frac{dS}{dt} = 3(4)^2 - 96(4) + 768$

$$\begin{aligned} &= 48 - 384 + 768 \\ &= 432 \end{aligned}$$

When $t = 8$, $\frac{dS}{dt} = 3(8)^2 - 96(8) + 768$

$$\begin{aligned} &= 192 - 768 + 768 \\ &= 192 \end{aligned}$$

When $t = 12$, $\frac{dS}{dt} = 3(12)^2 - 96(12) + 768$

$$\begin{aligned} &= 432 - 1152 + 768 \\ &= 48 \end{aligned}$$

When $t = 16$, $\frac{dS}{dt} = 3(16)^2 - 96(16) + 768$

$$\begin{aligned} &= 768 - 1536 + 768 \\ &= 0 \end{aligned}$$

**c** The rate of increase of sweetness is zero after 16 days.

**d** When $t = 0$, $S = 4000 + (0 - 16)^3$

$= -96$

When $t = 4$, $S = 4000 + (4 - 16)^3$

$= 2272$

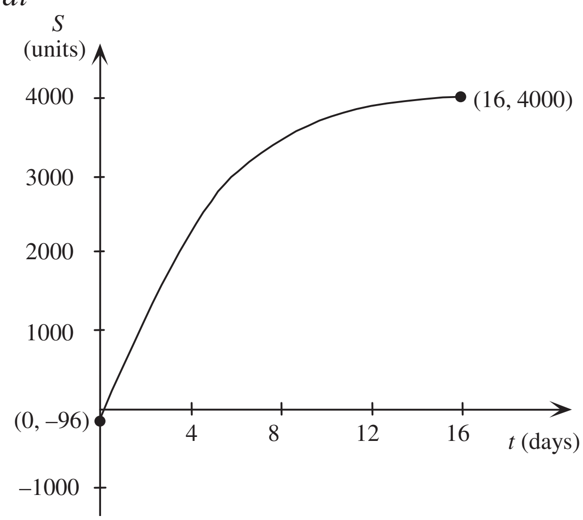
When $t = 8$, $S = 4000 + (8 - 16)^3$

$= 3488$

When $t = 16$, $S = 4000 + (16 - 16)^3$

$= 4000$

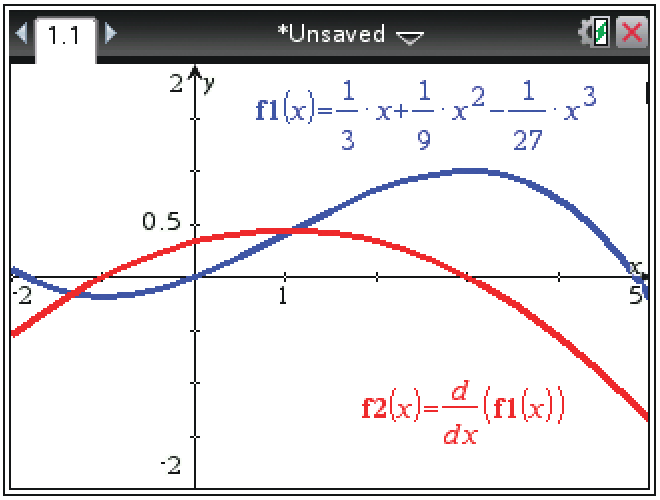
Note that $\frac{dS}{dt} = 3t^2 - 96t + 768 = 3(t^2 - 32t + 256) = 3(t - 16)^2$ which implies $\frac{dS}{dt} > 0$ for $0 \leq t < 16$.




**6 a** $\frac{ds}{dt} = \frac{1}{3} + \frac{2}{9}t - \frac{1}{9}t^2$

$= -\frac{1}{9}(t^2 - 2t - 3)$

$= -\frac{1}{9}(t - 3)(t + 1)$




**b** When the train stops at stations,

$$\frac{ds}{dt} = 0$$

$$\therefore \quad -\frac{1}{9}(t-3)(t+1) = 0$$

$$\therefore \quad t = 3 \text{ or } t = -1$$

When $t = -1$, the time is 1 minute before noon, i.e. 11.59 am is the time of departure from the first station.

When $t = 3$, the time is 3 minutes past noon, i.e. 12.03 pm is the time of arrival at the second station.

**c** When $t = -1$,

$$s = \frac{1}{3}(-1) + \frac{1}{9}(-1)^2 - \frac{1}{27}(-1)^3$$

$$= -\frac{1}{3} + \frac{1}{9} + \frac{1}{27} = -\frac{5}{27}$$

The first station is $\frac{5}{27}$ km before the signal box.

When $t = 3$,

$$s = \frac{1}{3}(3) + \frac{1}{9}(3)^2 - \frac{1}{27}(3)^3$$

$$= 1 + 1 - 1 = 1$$

The second station is 1 km after the signal box.

**d** $$\text{Average velocity} = \frac{s_2 - s_1}{t_2 - t_1}$$

where $s_2 = 1, s_1 = \frac{5}{27}, t_2 = 3, t_1 = -1$

$$\text{average velocity} = \frac{1 - \frac{-5}{27}}{3 - (-1)}$$

$$= \frac{\frac{32}{27}}{4} = \frac{8}{27}$$

$$\frac{8}{27}\text{ km/min} = \left(\frac{8}{27} \times 60\right)\text{ kW/h} = \frac{160}{9}\text{km/h}$$

$$\therefore \quad \text{average velocity} = 17\frac{7}{9}\text{km/h}$$

The average velocity between the stations is $17\frac{7}{9}$km/h.

**e** $$v = \frac{ds}{dt} = -\frac{1}{9}(t-3)(t+1)$$

When the train passes the signal box, $t = 0$

i.e. $$v = -\frac{1}{9}(0-3)(0+1) = \frac{1}{3}$$

$$\text{velocity} = \frac{1}{3}\text{km/min} = \left(\frac{1}{3} \times 60\right)\text{km/h} = 20\text{km/h}$$

The train passes the signal box at 20 km/h.

**7 a** $$V(t) \geq 0$$

$$\therefore \quad 1000 + (2-t)^3 \geq 0$$

$$\therefore \quad (2-t)^3 \geq -1000$$

$$\therefore \quad 2 - t \geq -10$$

$$\therefore \quad 2 \geq t - 10$$

$$\therefore \quad t \leq 12$$

Now $t \geq 0$ so the possible values of $t$ are $0 \leq t \leq 12$.

**b** Rate of change in volume over time $= \dfrac{dV}{dt}$

Now $V = 1000 + (2 - t)^3$

$= 1000 + (2 - t)(4 - 4t + t^2)$

$= 1000 + 8 - 8t + 2t^2 - 4t + 4t^2 - t^3$

$= 1008 - 12t + 6t^2 - t^3$

$\therefore \quad \frac{dV}{dt} = -12 + 12t - 3t^2$

$= -3(t^2 - 4t + 4)$

$= -3(t - 2)^2$

**i** When $t = 5$, $\frac{dV}{dt} = -3(5 - 2)^2 = -27$
The rate of draining is 27 L/h when $t = 5$.

**ii** When $t = 10$, $\frac{dV}{dt} = -3(10 - 2)^2 = -192$
The rate of draining is 192 L/h when $t = 10$.

**8 a** When $x = 0$, $y = \frac{1}{5}(4(0)^3 - 8(0)^2 + 192(0) + 144)$

$= \frac{1}{5} \times 144$

$= \frac{144}{5}$

$= 28.8$

The start of the track is 28.8 m above sea level.

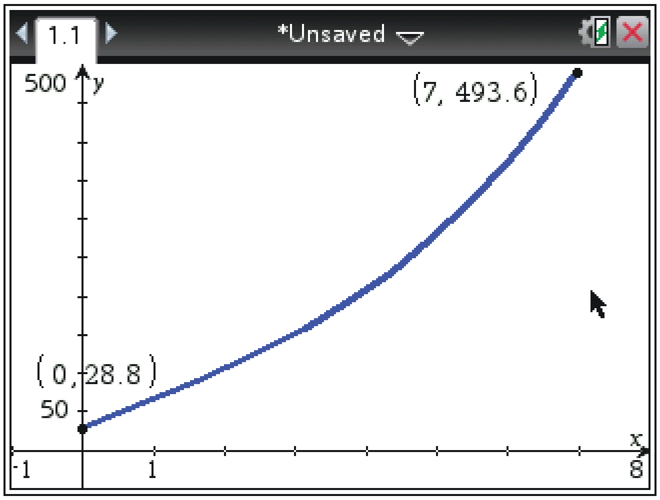
**b** When $x = 6$, $y = \frac{1}{5}(4(6)^3 - 8(6)^2 + 192(6) + 144)$

$= \frac{1}{5}(864 - 288 + 1152 + 144)$

$= \frac{1870}{5}$

$= 374.4$

**c**




**d** The graph gets very steep for $x > 7$, which would not be practical.

**e**

$$y = \frac{4}{5}x^3 - \frac{8}{5}x^2 + \frac{192}{5}x + \frac{144}{5}$$

$$\text{Gradient} = \frac{dy}{dx} = \frac{12}{5}x^2 \frac{16}{5}x + \frac{192}{5}$$

**i** When $x = 0$,

$$\begin{aligned}\frac{dy}{dx} &= \frac{12}{5}(0)^2 - \frac{16}{5}(0) + \frac{192}{5}\\ &= 38.4\end{aligned}$$

$$\begin{aligned}38.4\ \text{m/km} &= \left(38.4\frac{1}{1000}\right)\ \text{m/m}\\ &= 0.0384\ \text{m/m}\end{aligned}$$

The gradient of the graph is 0.0384 for $x = 0$.

**ii** When $x = 3$,

$$\begin{aligned}\frac{dy}{dx} &= \frac{12}{5}(3)^2 - \frac{16}{5}(3) + \frac{192}{5}\\ &= \frac{108}{5} - \frac{48}{5} + \frac{192}{5}\\ &= 50.4\end{aligned}$$

$$\begin{aligned}50.4\ \text{m/km} &= \left(50.4 \times \frac{1}{1000}\right)\ \text{m/m}\\ &= 0.0504\ \text{m/m}\end{aligned}$$

The gradient of the graph is 0.0504 for $x = 3$.

**iii** When $x = 7$,

$$\begin{aligned}\frac{dy}{dx} &= \frac{12}{5}(7)^2 - \frac{16}{5}(7) + \frac{192}{5}\\ &= \frac{588}{5} - \frac{112}{5} + \frac{192}{5}\\ &= \frac{668}{5}\\ &= 133.6\end{aligned}$$

$$\begin{aligned}133.6\text{m/km} &= \left(133.6 \times \frac{1}{1000}\right)\text{ m/m}\\ &= 0.1336\text{ m/m}\end{aligned}$$

The gradient of the graph is 0.1336 for $x = 7$.

**9 a** $y = x^3$

Point of inflexion at $(0, 0)$

When $x = -1$, $\quad y = -1 \quad (-1, -1)$

When $x = 1$, $\quad y = 1 \quad (1, 1)$

$$\begin{aligned}y &= 2 + x - x^2\\ &= -(x^2 - x - 2)\\ &= -(x - 2)(x + 1)\end{aligned}$$

When $x = 0$,

$$\begin{aligned}y &= -(0 - 2)(0 + 1)\\ &= 2\end{aligned}$$

$\therefore$ $y$-axis intercept is 2.

When $y = 0$, $\quad -(x - 2)(x + 1) = 0$

$\therefore \quad x - 2 = 0 \quad$ or $\quad x + 1 = 0$

$\therefore \quad x = 2 \quad$ or $\quad x = -1$

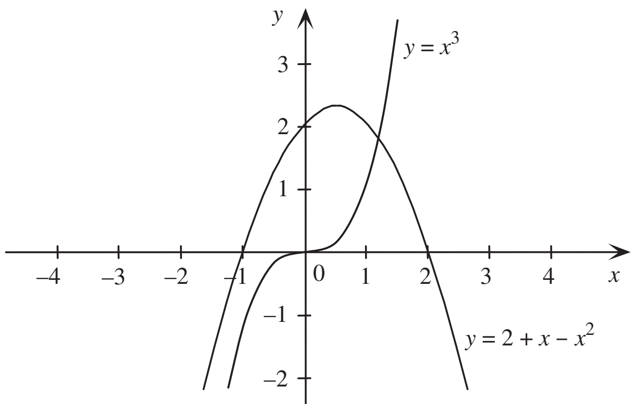
$\therefore$ $x$-axis intercepts are $-1$ and 2.

By symmetry, turning point is at $x = \dfrac{2 + -1}{2} = \dfrac{1}{2}$.

When $x = \dfrac{1}{2}$,

$$\begin{aligned}y &= -\left(\frac{1}{2} - 2\right)\left(\frac{1}{2} + 1\right)\\ &= -\left(\frac{-3}{2}\right)\left(\frac{3}{2}\right)\\ &= \frac{9}{4}\end{aligned}$$

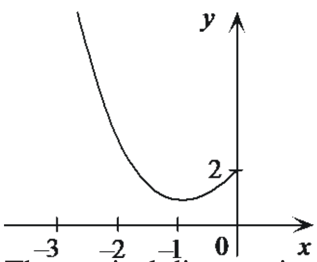
$\therefore$ turning point is $\left(\dfrac{1}{2}, \dfrac{9}{4}\right)$.




**b** For $x \leq 0, 2 + x - x^2 \geq x^3$.

The vertical distance between the two curves is given by $y = 2 + x - x^2 - x^3, x \leq 0$.

| | | | | |
|---|---|---|---|---|
| $x$ | −3 | −2 | −1 | 0 |
| $y$ | 17 | 4 | 1 | 2 |




The vertical distance is a minimum when $y$ is a minimum.

This occurs where $\dfrac{dy}{dx} = 0$.

Now $\quad \dfrac{dy}{dx} = 1 - 2x - 3x^2$

Consider $\quad 0 = 1 - 2x - 3x^2$

$\therefore \quad 0 = (-3x + 1)(x + 1)$

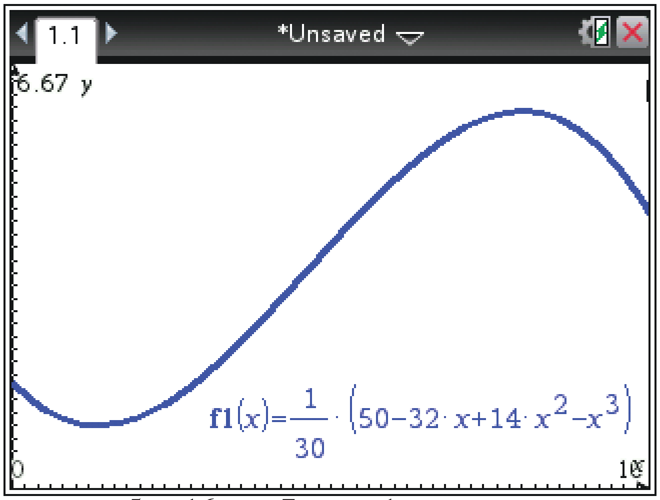
$\therefore \quad x + 1 = 0 \quad$ or $\quad -3x + 1 = 0$

$\therefore \quad x = -1 \quad$ or $\quad x = \dfrac{1}{3}$

But $x \leq 0$, so $x = -1$ and $y = 2 + (-1) - (-1)^2 - (-1)^3 = 1$.

Hence the minimum distance between the two curves is 1 unit when $x = -1$.

**10**




$M(x) = \frac{5}{3} - \frac{16}{15}x + \frac{7}{15}x^2 - \frac{1}{30}x^3 \le x \le 10$

$M'(x) = -\frac{16}{15} + \frac{14}{15}x - \frac{1}{10}x^2$

Stationary points occur where $M'(x) = 0$

$\therefore \quad -\frac{16}{15} + \frac{14}{15}x - \frac{1}{10}x^2 = 0$

$\therefore \quad -\frac{1}{10}x^2 + \frac{14}{15}x - \frac{16}{15} = 0$

$\therefore \quad -\frac{1}{30}(3x^2 - 28x + 32) = 0$

$\therefore \quad -\frac{1}{30}(3x - 4)(x - 8) = 0$

$\therefore \quad 3x - 4 = 0$ or $x - 8 = 0$

$x = \frac{4}{3}$ or $x = 8$

| $x$ | 0 | $\frac{4}{3}$ | 2 | 8 | 10 |
|---|---|---|---|---|---|
| Sign of $M'(x)$ | – | 0 | + | 0 | – |
| Shape | \ | — | / | — | \ |

Hence the minimum number of mosquitoes is produced when rainfall is $\frac{4}{3}$ mm and the maximum number is produced when rainfall is 8 mm.

**11 a** $x + y = 5$

$\therefore \quad y = 5 - x$

**b** $P = xy$

$\therefore \quad P = x(5 - x)$

**c** $P = 5x - x^2$

$$\frac{dP}{dx} = 5 - 2x$$

Stationary points occur where $\frac{dP}{dx} = 0$

$\therefore \quad 5 - 2x = 0$

$\therefore \quad x = 2.5$

As coefficient of $x^2$ is negative, there is a local maximum at $x = 2.5$.

When $x = 2.5$, $y = 5 - 2.5$

$= 2.5$

and $P = xy$

$= 2.5 \times 2.5$

$= 6.25$, the maximum value of $P$.

**12 a** $2x + y = 10$

$\therefore \quad y = 10 - 2x$

**b** $A = x^2 y$

$\therefore \quad A = x^2(10 - 2x)$

**c** $A = 10x^2 - 2x^3$

$$\frac{dA}{dx} = 20x - 6x^2$$

Stationary points occur where $\frac{dA}{dx} = 0$

$\therefore \quad 20x - 6x^2 = 0$

$\therefore \quad 2x(10 - 3x) = 0$

$\therefore \quad x = 0 \quad \text{or} \quad 10 - 3x = 0$

$x = \frac{10}{3}$

| $x$ | 0 | 1 | $\frac{10}{3}$ | 4 |
|---|---|---|---|---|
| Sign of $\frac{dA}{dx}$ | 0 | + | 0 | − |
| Shape | — | / | — | \ |

The maximum value of $A$ occurs at $x = \frac{10}{3}$.

When $x = \dfrac{10}{3}$, $\quad y = 10 - 2x = 10 - 2\left(\dfrac{10}{3}\right) = \dfrac{10}{3}$

$$\begin{aligned} A &= x^2y \\ &= \left(\frac{10}{3}\right)^2 \times \frac{10}{3} = \frac{1000}{27} \end{aligned}$$

Maximum value of $A$ of $\dfrac{1000}{27}$ occurs where $x = y = \dfrac{10}{3}$.

**13** $xy = 10 \quad \therefore \quad y = \dfrac{10}{x}$

$$\begin{aligned} T &= 3x^2y - x^3 \\ &= 3x^2 \times \frac{10}{x} - x^3 \\ &= 30x - x^3 \\ \frac{dT}{dx} &= 30 - 3x^2 \end{aligned}$$

Stationary points occur where $\dfrac{dT}{dx} = 0$

$$\begin{aligned} 30 - 3x^2 &= 0 \\ 30 &= 3x^2 \\ x^2 &= 10 \\ x &= \pm\sqrt{10} \end{aligned}$$

$\therefore \quad x = \sqrt{10} \quad$ as $0 < x < \sqrt{30}$

| $x$ | 0 | $\sqrt{10}$ | 4 |
|---|---|---|---|
| Sign of $\dfrac{dT}{dx}$ | + | 0 | − |
| Shape | / | — | \ |

Hence the maximum value of $T$ occurs when $x = \sqrt{10}$.

When $x = \sqrt{10}$, $\quad T = 30\sqrt{10} - (\sqrt{10})^3$

$$\begin{aligned} &= \sqrt{10}(30 - 10) \\ &= 20\sqrt{10} \\ &\approx 63.25 \end{aligned}$$

**14** **a** $x + y = 8 \quad \therefore \quad y = 8 - x$

**b**

$$\begin{aligned} s &= x^2 + y^2 \\ &= x^2 + (8 - x)^2 \end{aligned}$$

**c**

$$\begin{aligned} s &= x^2 + (8 - x)^2 \\ &= x^2 + 64 - 16x + x^2 \\ &= 2x^2 - 16x + 64 \\ \frac{ds}{dx} &= 4x - 16 \end{aligned}$$

Stationary points occur where $\frac{ds}{dx} = 0$

$\therefore$ $4x - 16 = 0$

$$\begin{aligned} 4x &= 16 \\ x &= 4 \end{aligned}$$

| $x$ | 0 | 4 | 5 |
|---|---|---|---|
| Sign of $\frac{ds}{dx}$ | – | 0 | + |
| Shape | \ | — | / |

**or** $x = 4$ is a local minimum because coefficient of $x^2$ is positive.
Hence the least value of the sum of the squares occurs at $x = 4$.
When $x = 4$,

$$\begin{aligned} s &= x^2 + (8 - x)^2 \\ &= 4^2 + (8 - 4)^2 \\ &= 16 + 4^2 \\ &= 16 + 16 \\ &= 32 \end{aligned}$$

**15** Let $x$ and $y$ be the two numbers.

$x + y = 4$ $\quad\therefore y = 4 - x$

$s = x^3 + y^2$ $\quad\therefore s = x^3 + (4 - x)^2$

$$\begin{aligned} &= x^3 + 16 - 8x + x^2 \\ &= x^3 + x^2 - 8x + 16 \\ \frac{ds}{dx} &= 3x^2 + 2x - 8 \end{aligned}$$

When $\frac{ds}{dx} = 0$, $\quad 3x^2 + 2x - 8 = 0$

$\therefore$ $\quad (3x - 4)(x + 2) = 0$

$\therefore 3x - 4 = 0$ $\quad$ or $x + 2 = 0$

$x = \frac{4}{3}$ $\quad$ or $x = -2$

| $x$ | $-3$ | $-2$ | $0$ | $\frac{4}{3}$ | $2$ |
|---|---|---|---|---|---|
| Sign of $\frac{ds}{dx}$ | + | 0 | – | 0 | + |
| Shape | / | — | \ | — | / |

Note: Positive numbers are considered, so $x = -2$ need not be considered.

$s$ will be as small as possible when $\quad x = \frac{4}{3}$

and
$$\begin{aligned} y &= 4 - x \\ &= 4 - \frac{4}{3} = \frac{8}{3} \end{aligned}$$

Hence the two numbers are $\frac{4}{3}$ and $\frac{8}{3}$.

**16** Let $x$ be the length, $y$ the width and $A$ the area of the rectangle.

$$\begin{aligned} \therefore \quad 2(x + y) &= 100 \\ \therefore \quad x + y &= 50 \\ \therefore \quad y &= 50 - x \\ A &= xy \\ &= x(50 - x) = 50x - x^2 \\ \frac{dA}{dx} &= 50 - 2x \end{aligned}$$

When $\frac{dA}{dx} = 0, \quad 50 - 2x = 0 \quad \therefore x = 25$

A local maximum at $x = 25$, as the coefficient of $x^2$ is negative.

When $x = 25, \quad y = 50 - x = 25$

The area is a maximum (625 $m^2$) when the rectangle is a square of side length 25 m.

**17** Let $y$ be the second number and $P$ the product of the two numbers.

$$\begin{aligned} x + y &= 24 \\ \therefore \quad y &= 24 - x \\ P &= xy \\ &= x(24 - x) \\ &= 24x - x^2 \\ \frac{dP}{dx} &= 24 - 2x \end{aligned}$$

When $\frac{dP}{dx} = 0, \quad 24 - 2x = 0$

$\therefore \quad x = 12$

There is a local maximum at $x = 12$, as the coefficient of $x^2$ is negative.
Hence, the product of the two numbers is a maximum when $x = 12$.

**18** Let $C$ = overhead costs (\$/h)

$$\therefore \quad C = 400 - 16n + \frac{1}{4}n^2$$

$$\frac{dC}{dn} = -16 + \frac{1}{2}n$$

When $\frac{dC}{dn} = 0$, $\quad -16 + \frac{1}{2}n = 0$

$$\therefore \quad n = 32$$

There is a local minimum at $n = 32$, as the coefficient of $n^2$ is positive.
Hence, 32 items should be produced per hour to keep costs to a minimum.

**19**

$$x + y = 100$$

$$\therefore \quad y = 100 - x$$

$$\begin{aligned} P &= xy \\ &= x(100 - x) \\ &= 100x - x^2 \end{aligned}$$

$$\frac{dP}{dx} = 100 - 2x$$

When $\frac{dP}{dx} = 0$, $\quad 100 - 2x = 0$

$$\therefore \quad x = 50$$

There is a local maximum at $x = 50$, as the coefficient of $x^2$ is negative.

When $x = 50$, $\quad \begin{aligned} y &= 100 - x \\ &= 100 - 50 \\ &= 50 \end{aligned}$

Hence $\quad x = y$

When $x = 50$, $\quad \begin{aligned} P &= xy \\ &= 50 \times 50 \\ &= 2500, \text{ the maximum value of } P. \end{aligned}$

**20** Let $x$ be the length of river (in km) to be used as a side of the enclosure and let $y$ be the side length of the rectangle (in km) perpendicular to the river.

$$\therefore \quad x + 2y = 4$$

$$\therefore \quad y = \frac{1}{2}(4 - x)$$

Let $A = xy$, the area of the land enclosed.

$$\therefore \quad A = x \times \frac{1}{2}(4 - x)$$

$$= 2x - \frac{1}{2}x^2$$

$$\frac{dA}{dx} = 2 - x$$

When $\frac{dA}{dx} = 0$, $\quad 2 - x = 0$

$$x = 2$$

There is a local maximum at $x = 2$, as the coefficient of $x^2$ is negative.

When $x = 2$, $\quad y = \frac{1}{2}(4 - x)$

$$= \frac{1}{2}(4 - 2) = 1$$

Hence the maximum area of land of 2 km$^2$ will be enclosed if the farmer uses a 2 km stretch of river and a width of 1 km for his land.

**21** $\quad p^3q = 9$ and $p, q > 0$

$$\therefore \quad q = \frac{9}{p^3}$$

$$= 9p^{-3}$$

$$z = 16p + 3q$$

$$= 16p + 3(9p^{-3})$$

$$= 16p + 27p^{-3}$$

We know that $\frac{d}{dx}(x^n) = yx^{n-1}$ when n = 1, 2, 3

Now suppose this also true for $n = -1, -2, -3, \ldots$

So $\frac{d}{dx}(p^{-3}) = -3x^{-4}$

$$\text{Hence } \frac{dz}{dp} = 16 - 81p^{-4}$$

When $\frac{dz}{dp} = 0$, $\quad 16 - 81p^{-4} = 0$

$$16 = 81p^{-4}$$

$$p^{-4} = \frac{16}{81}$$

$$p^4 = \frac{81}{16}$$

$$p = \frac{\sqrt[4]{81}}{\sqrt[4]{16}} = \frac{3}{2}$$

$$= \pm\frac{3}{2}$$

| $p$ | 1 | $\frac{3}{2}$ | 2 |
|---|---|---|---|
| Sign of $\frac{dz}{dp}$ | − | 0 | + |
| Shape | \ | — | / |

Hence $z$ is a minimum when $p = \frac{3}{2}$

and
$$\begin{aligned} q &= \frac{9}{p^3} \\ &= \frac{9}{(\frac{3}{2})^3} \\ &= \frac{9 \times 8}{27} \\ &= \frac{8}{3} \end{aligned}$$

So $p = \frac{3}{2}$ and $q = \frac{8}{3}$.

**22** **a** $2(x + y) = 120$

$x + y = 60$

$y = 60 - x$

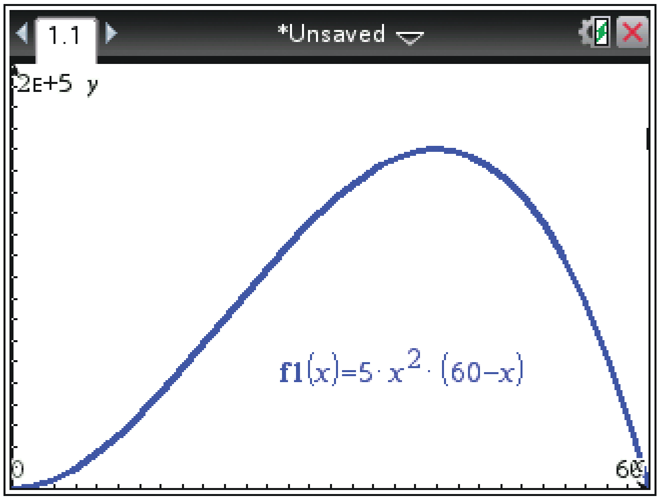
**b** $S = 5x^2y$

$= 5x^2(60 - x)$

**c** $S > 0$, $\therefore$ $5x^2(60 - x) > 0$

$\therefore$ $0 < x < 60$

**d**




**e** $S = 5x^2(60 - x)$

$= 300x^2 - 5x^3$

$\dfrac{dS}{dx} = 600x - 15x^2$

$= 0$

if $x = 0$ or $x = 40$

From the graph, the maximum occurs at $x = 40$

$\therefore \quad y = 60 - x$

$= 60 - 40$

$= 20$

Hence $x = 40$ and $y = 20$ give the strongest beam.

**f** For $x \leq 19$, the maximum strength of the beam occurs when $x = 19$.

$\therefore \quad S = 5 \times 19^2(60 - 19)$

$= 74\,005$

The maximum strength of the beam, if the cross-sectional depth of the beam must be less than 19 cm, is 74 005.

**23**

$$s'(x) = -3x^2 + 6x + 360$$
$$= -3(x^2 - 2x - 120)$$
$$= -3(x + 10)(x - 12)$$

When $s'(x) = 0, -3(x + 10)(x - 12) = 0$

$$\therefore x + 10 = 0 \text{ or } x - 12 = 0$$
$$\therefore x = -10 \text{ or } x = 12$$

But $6 \leq x \leq 20$, so $x = 12$.

| $x$ | 10 | 12 | 14 |
|---|---|---|---|
| Sign of $s'(x)$ | + | 0 | – |
| Shape | / | — | \ |

Hence the maximum number of salmon swimming upstream occurs when the water temperature is 12°C.

**24** **a** Let $x$ (cm) be the breadth of the box, $2x$ (cm) be the length of the box, and $h$ (cm) be the height of the box.

$$4(x + 2x) + 4h = 360$$

$$\therefore \quad 4h = 360 - 4(3x)$$

$$\therefore \quad h = 90 - 3x$$

$$\begin{aligned} V &= x \times 2x \times h \\ &= 2x^2(90 - 3x) \\ &= 180x^2 - 6x^3 \text{ as required} \end{aligned}$$

**b** $$V = 6x^2(30 - x)$$

$$\therefore \quad \text{Domain } V = \{x: 0 < x < 30\}$$

**c**

$$\begin{aligned} V &= 180x^2 - 6x^3 \\ \frac{dV}{dx} &= 360x - 18x^2 \\ &= 18x(20 - x) \end{aligned}$$

When $\frac{dV}{dx} = 0 \qquad 18x(20 - x) = 0$

$$\therefore \quad 18x = 0 \quad \text{or} \quad 20 - x = 0$$

$$\therefore \quad x = 0 \quad \text{or} \quad x = 20$$

| $x$ | −10 | 0 | 10 | 20 | 30 |
|---|---|---|---|---|---|
| Sign of $\frac{dV}{dx}$ | – | 0 | + | 0 | – |
| Shape | \ | — | / | — | \ |

Hence $V$ is a minimum when $x = 0$ and a maximum when $x = 20$.

When $x = 0$, $\quad V = 180(0)^2 - 6(0)^3$

$$= 0$$

$\therefore$ $y$-axis intercept is 0.

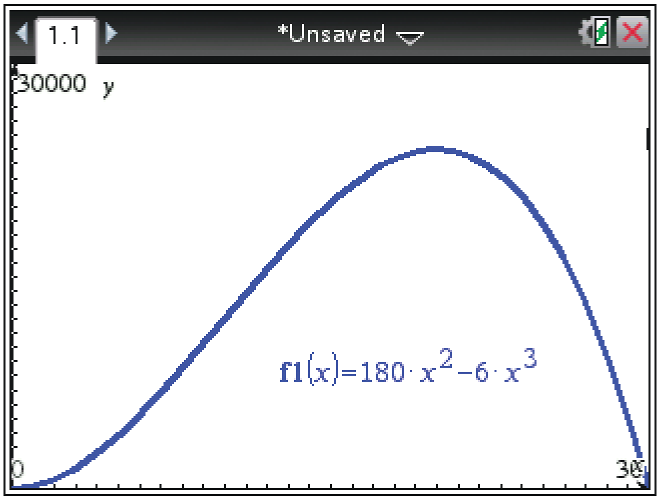
When $V = 0$, $180x^2 - 6x^3 = 0$

$\therefore \quad 6x^2(30 - x) = 0$

$\therefore \quad 6x^2 = 0$ or $30 - x = 0$

$x = 0$ or $x = 30$

$\therefore$ $x$-axis intercepts are 0 and 30.




**d** From part **c**, $V$ is a maximum when $x = 20$.

$$\therefore \quad h = 90 - 3x$$
$$= 90 - 60$$
$$= 30$$

Hence the dimensions giving the greatest volume are 20 cm × 30 cm × 40 cm.

**e** On a CAS calculator, with ***f*1 = 180*x*^2 − 6*x*^3** and ***f*1 = 20 000**,
The $x$-coordinates of the points of intersection are 14.817 02 and 24.402 119. Hence the values of $x$ for which $V = 20\,000$ are 14.82 and 24.40, correct to 2 decimal places.

**25 a**

$$A = 8x \times y + 2\left(\frac{1}{2} \times 4x \times \sqrt{(5x)^2 - (4x)^2}\right)$$
$$= 8xy + 4x \times \sqrt{25x^2 - 16x^2}$$
$$= 8xy + 4x \times \sqrt{9x^2}$$
$$= 8xy + 4x \times 3x \text{ (only positive square root appropriate)}$$
$$= 8xy + 12x^2$$

Also

$$8x + y + y + 5x + 5x = 90$$

$$\therefore \quad 18x + 2y = 90$$

$$2y = 90 - 18x$$

$$y = 45 - 9x$$

$$\therefore \quad A = 8x(45 - 9x) + 12x^2$$

$$= 360x - 72x^2 + 12x^2$$

$$\therefore \quad A = 360x - 60x^2 \text{ as required}$$

**b** $A = 360x - 60x^2$, $\quad \therefore \dfrac{dA}{dx} = 360 - 120x$

When $\dfrac{dA}{dx} = 0$, $\quad 360 - 120x = 0$

$$x = 3$$

Area is a maximum at $x = 3$, as the coefficient of $x^2$ is negative.

When $x = 3$, $\quad y = 45 - 9x$

$$= 45 - 27$$

$$= 18$$

**26 a** Let $r$ (cm) be the radius of the circle and $x$ (cm) be the side length of the square.

$$2\pi r + 4x = 100$$

$$2\pi r = 100 - 4x$$

$$r = \frac{50 - 2x}{\pi}$$

As $r > 0$, $\quad 50 - 2x > 0$

i.e. $\quad x < 25$

Let $A$ be the sum of the areas of the circle and the square.

$$\therefore \quad A = \pi r^2 + x^2$$

$$= \pi\left(\frac{50 - 2x}{\pi}\right)^2 + x^2$$

$$= \frac{1}{\pi}(2500 - 200x + 4x^2) + x^2$$

$$= \frac{2500}{\pi} - \frac{200}{\pi}x + \frac{4}{\pi}x^2 + x^2$$

i.e. $$A = \frac{4+\pi}{\pi}x^2 - \frac{200}{\pi}x + \frac{2500}{\pi}, x \in [0, 25]$$

$$\frac{dA}{dx} = \frac{2(4+\pi)}{\pi}x - \frac{200}{\pi}$$

When $\frac{dA}{dx} = 0$,

$$\frac{2(4+\pi)}{\pi}x - \frac{200}{\pi} = 0$$

$$2(4+\pi)x = 200$$

$$x = \frac{100}{4+\pi}$$

The area is a minimum when $x = \frac{100}{4+\pi}$, as the coefficient of $x^2$ is positive.

When $x = \frac{100}{4+\pi}$, $$4x = \frac{400}{4+\pi} \approx 56$$

and

$$\begin{aligned} 2\pi r &= 2\pi\left(\frac{50-2x}{\pi}\right) \\ &= 100 - 4x \\ &= 100 - 4\left(\frac{100}{4+\pi}\right) \\ &= 100 - \frac{400}{4+\pi} \\ &\approx 44 \end{aligned}$$

Hence the wire should be cut into a 56 cm strip to form the square, and 44 cm to form the circle.

**b** When $x = 0$,

$$\begin{aligned} A &= \frac{4+\pi}{\pi}(0)^2 - \frac{200}{\pi}(0) + \frac{2500}{\pi} \\ &= \frac{2500}{\pi} \\ &\approx 796 \end{aligned}$$

When $x = 25$, $$A < \frac{2500}{\pi}$$

Hence the maximum area occurs when $x = 0$ and all the wire is used to form the circle.

**27**

$$\begin{aligned} 2(x + 2x + x) + 2x + 4y &= 36 \\ 2(4x) + 2x + 4y &= 36 \\ 10x + 4y &= 36 \\ 4y &= 36 - 10x \\ y &= 9 - \frac{5}{2}x \end{aligned}$$

Let $A$ (m$^2$) be the area of the court.

$$A = 4xy$$
$$= 4x(9 - \tfrac{5}{2}x)$$
$$= 36x - 10x^2$$
$$\frac{dA}{dx} = 36 - 20x$$

When $\frac{dA}{dx} = 0$, $\quad 36 - 20x = 0$

$$20x = 36$$
$$x = \frac{9}{5}$$

Area is a maximum when $x = \frac{9}{5}$, as the coefficient of $x^2$ is negative.

When $x = \frac{9}{5}$, $\quad$ length $= 4x$

$$= 4 \times \frac{9}{5}$$
$$= 7.2$$

and $\quad$ width $= y$

$$= 9 - \frac{5}{2}x$$
$$= 9 - \frac{5}{2} \times \frac{9}{5}$$
$$= 4.5$$

The length is 7.2 m and the width is 4.5 m.

**28** **a** $A = xy$

**b** $x + 2y = 16$

$$2y = 16 - x$$
$$y = 8 - \frac{x}{2}$$
$$A = \left(8 - \frac{x}{2}\right)x$$

**c** When $A = 0$, $\quad x(8 - \frac{1}{2}x) = 0$

$x = 0 \quad$ or $\quad x = 16$

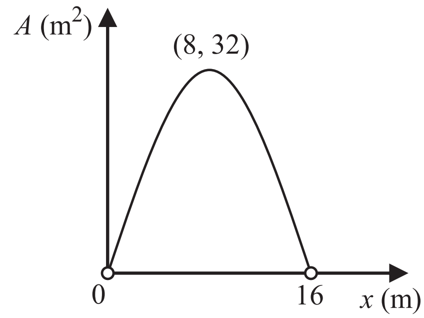
$\therefore \quad$ Domain $= \{x\colon 0 < x < 16\}$

**d** Turning point is at $x = \dfrac{0 + 16}{2} = 8$

When $x = 8$, $A = x(8 - \frac{1}{2}x)$

$$= 8(8 - \tfrac{1}{2}(8))$$

$$= 32$$




**e** Calculus could be used, but as the graph is a parabola with turning point (8, 32), the maximum is 32. Therefore, the largest area of ground which could be covered is 32 m$^2$.

**29**
$$2a + h + h + 2a + 2a = 8000$$
$\therefore$ $$6a + 2h = 8000$$
$\therefore$ $$2h = 8000 - 6a$$
$\therefore$ $$h = 4000 - 3a$$

Let $A$ be the area of the shape and $v$ be the vertical height of the triangle.

$$\begin{aligned} v &= \sqrt{(2a)^2 - a^2} \\ &= \sqrt{4a^2 - a^2} \\ &= \sqrt{3a^2} \\ &= \sqrt{3}a \end{aligned}$$

$$\begin{aligned} A &= 2ah + \frac{1}{2}(2a)v \\ &= 2a(4000 - 3a) + a \times \sqrt{3}a \\ &= 8000a - 6a^2 + \sqrt{3}a^2 \\ &= (\sqrt{3} - 6)a^2 + 8000a \end{aligned}$$

$$\frac{dA}{da} = 2(\sqrt{3} - 6)a + 8000$$

When $\dfrac{dA}{da} = 0$, $\quad 2(\sqrt{3} - 6)a + 8000 = 0$

$\therefore$
$$\begin{aligned} a &= \frac{8000}{2(6 - \sqrt{3})} \\ &= \frac{4000}{6 - \sqrt{3}} \\ &\approx 937 \end{aligned}$$

The area is a maximum when $a = 937$, as the coefficient of $a^2$ is negative.

When $a = \dfrac{4000}{6-\sqrt{3}}$,

$$\begin{aligned} h &= 4000 - 3a \\ &= 4000 - \frac{3\times 4000}{6-\sqrt{3}} \\ &\approx 4000 - 2812 = 1188 \end{aligned}$$

The maximum amount of light passes through when $a = 931$ and $h = 1188$.

**30 a**

$$\pi x + y = 10$$

$$\therefore \quad y = 10 - \pi x$$

**b** When $x = 0$,

$$\begin{aligned} y &= 10 - \pi(0) \\ &= 10 \end{aligned}$$

When $y = 0$,

$$10 - \pi x = 0$$

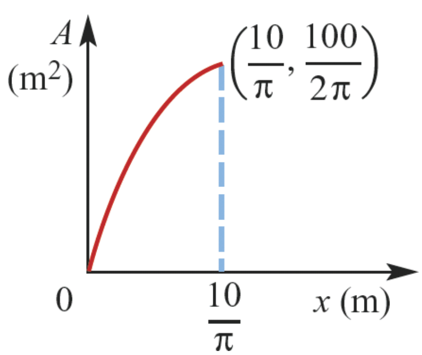
$$\therefore \quad x = \frac{10}{\pi}$$

Hence all possible values of $x$ are $0 \le x \le \dfrac{10}{\pi}$.

**c**

$$\begin{aligned} A &= xy + \frac{\pi}{2}x^2 \\ &= x(10-\pi x) + \frac{\pi}{2}x^2 \\ &= 10x - \pi x^2 + \frac{\pi}{2}x^2 \\ &= \frac{x}{2}(20 - \pi x) \end{aligned}$$

**d**



**e** $$A = 10x - \frac{\pi}{2}x^2$$

$\therefore$ $$\frac{dA}{dx} = 10 - \pi x$$

When $\frac{dA}{dx} = 0$, $10 - \pi x = 0$

$\therefore$ $$x = \frac{10}{\pi}$$

The value of $x$ which maximises $A$ is $\frac{10}{\pi}$.

**f** When $x = \frac{10}{\pi}$, $y = 10 - \pi \times \frac{10}{\pi} = 0$

The capacity of the drain is a maximum when the cross-section is a semi-circle.

**31 a** $$\text{Surface area } = 2\pi xh + 2\pi x^2$$

$\therefore$ $$1000 = 2\pi xh + 2\pi x^2$$

$\therefore$ $$500 = \pi xh + \pi x^2$$

$\therefore$ $$h = \frac{500 - \pi x^2}{\pi x} = \frac{500}{\pi x} - x$$

**b** $$\begin{aligned} V &= \pi x^2 h \\ &= \pi x^2\left(\frac{500 - \pi x^2}{\pi x}\right) \\ &= x(500 - \pi x^2) = 500x - \pi x^3 \end{aligned}$$

**c** $$\frac{dV}{dx} = 500 - 3\pi x^2$$

**d** $$\frac{dV}{dx} = 0$$

implies $500 - 3\pi x^2 = 0$

$\therefore$ $$x = \frac{\sqrt{500}}{\sqrt{3\pi}} \text{ as } x > 0$$

$\therefore$ $$x = \frac{10\sqrt{5}}{\sqrt{3\pi}} \approx 7.28$$

**e** $h > 0$ and $x > 0$

$$\therefore \quad \frac{500 - \pi x^2}{\pi x} > 0$$

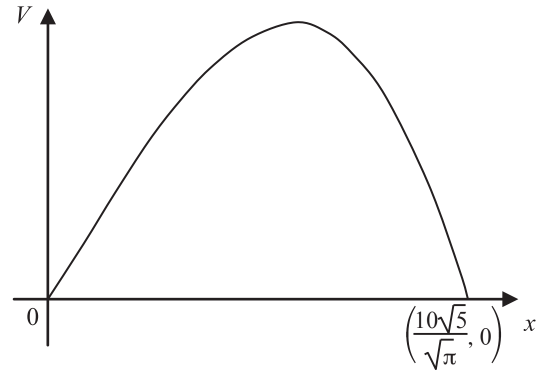
$$\therefore \quad 500 > \pi x^2$$

$$\therefore \quad \frac{500}{\pi} > x^2$$

$$\therefore \quad x < \frac{\sqrt{500}}{\sqrt{\pi}} \text{ for } x > 0$$

$$\therefore \quad x < \frac{10\sqrt{5}}{\sqrt{\pi}}$$

$$\therefore \quad \text{domain } = \left(0, \frac{10\sqrt{5}}{\sqrt{\pi}}\right)$$




**f** When $x = \dfrac{10\sqrt{5}}{\sqrt{3\pi}}$ (from part **d**)

$$\begin{aligned} V &= 5000 \times \frac{\sqrt{5}}{\sqrt{3\pi}} - \pi \times \left(\frac{5}{3\pi}\right)^{\frac{3}{2}} \times 1000 \\ &= \frac{\sqrt{5}}{\sqrt{3\pi}}\left(5000 - \frac{\pi \times 5000}{3\pi}\right) \\ &= \frac{\sqrt{5}}{\sqrt{3\pi}} \times \frac{10\,000}{3} \\ &= \frac{10\,000\sqrt{5}}{3\sqrt{3\pi}} \text{ cm}^3 \end{aligned}$$

The maximum volume is 2427.89 $\text{cm}^3$, correct to 2 decimal places.

**g** On a CAS calculator, with $\boldsymbol{f1 = x(500 - \pi x\hat{}2)}$ and $\boldsymbol{f2 = 1000}$, to find $x = 2.05$ and $x = 11.46$

Corresponding values of $h$ are $h = 75.41$ and $h = 2.42$

**32** **a** Let $x$ (cm) be the radius, $h$ (cm) be the height, $S\,(\text{cm}^2)$ be the surface area of the can.

$$\pi x^2 h = 500$$

$$\therefore \quad h = \frac{500}{\pi x^2}$$

$$\begin{aligned} S &= 2\pi x h + 2\pi x^2 \\ &= 2\pi x\left(\frac{500}{\pi x^2}\right) + 2\pi x^2 \\ &= \frac{1000}{x} + 2\pi x^2 \\ &= 1000x^{-1} + 2\pi x^2 \end{aligned}$$

We know that $\frac{d}{dx}(x^n) = yx^{n-1}$ when n = 1, 2, 3

Now suppose this also true for $n = -1, -2, -3, \ldots$

So $\frac{d}{dx}(x^{-1}) = -x^{-2}$

$$\begin{aligned} \text{Hence } \frac{dS}{dx} &= -1000x^{-2} + 4\pi x \\ &= \frac{-1000}{x^2} + 4\pi x \end{aligned}$$

When $\frac{dS}{dx} = 0$, $\quad \frac{-1000}{x^2} + 4\pi x = 0$

$$\therefore \quad 4\pi x = \frac{1000}{x^2}$$

$$\therefore \quad x^3 = \frac{1000}{4\pi}$$

$$\therefore \quad x = \frac{10}{(4\pi)^{\frac{1}{3}}}$$

$$\approx 4.3$$

| $x$ | 4 | 4.3 | 5 |
|---|---|---|---|
| Sign of $\frac{dS}{dx}$ | − | 0 | + |
| Shape | \ | — | / |

The surface area is a minimum when the radius is 4.3 cm,

and
$$\begin{aligned} h &= \frac{500}{\pi x^2} \\ &= \frac{500}{\pi\left(\dfrac{10}{(4\pi)^{\frac{1}{3}}}\right)^2} \\ &= \frac{500}{\pi\left(\dfrac{100}{(4\pi)^{\frac{2}{3}}}\right)} \\ &= \frac{500 \times (4\pi)^{\frac{2}{3}}}{100\pi} \end{aligned}$$

$$\therefore \quad \begin{aligned} h &= \frac{5(4\pi)^{\frac{2}{3}}}{\pi} \\ &\approx 8.6 \end{aligned}$$

The minimum surface area occurs when the radius is approximately 4.3 cm and the height is approximately 8.6 cm.

**b** If the radius of the can must be no greater than 5 cm, the minimum surface area occurs when the radius is approximately 4.3 cm and the height is approximately 8.6 cm.