WESLEY COLLEGE

By daring & by doing

YEAR 12 MATHEMATICS METHODS

Differentiation Techniques & the Exponential Function

Test 1

Name: Solutions

Marks: /45

Time: 55 minutes

Calculator Free (25 marks)

1. [7 marks]

Differentiate the following functions and simplify:

a) $y=(1+3x^3)^5$ [2]

$y' = 5(1+3x^3)^4 \cdot 9x^2$

$= 45x^2(1+3x^3)^4$

b) $y=\sqrt{\pi}\, e^{x^2+1}$ [2]

$y' = \sqrt{\pi}\ 2x\ e^{x^2+1}$

$= 2\sqrt{\pi}\, x\, e^{x^2+1}$

c) $y=(1-x^2)e^{4x}$ [3]

$y' = 4e^{4x}(1-x^2) + e^{4x}(-2x)$

$= 2e^{4x}(2-2x^2-x)$

or $-2e^{4x}(2x^2+x-2)$

$\overline{7}$

2. [5 marks]

a) Consider $f(x)=\dfrac{(x-2)^2}{e^{x-2}}$, clearly show that $f'(x)=\dfrac{-x^2+6x-8}{e^{x-2}}$ [3]

$$f'(x) = \frac{2(x-2)e^{x-2} - e^{x-2}(x-2)^2}{(e^{x-2})^2}$$

$$= \frac{\cancel{e^{x-2}}^{1}(x-2)(2-x+2)}{(e^{x-2})^{\cancel{2}\,1}}$$

$$= \frac{(x-2)(4-x)}{e^{x-2}} = \frac{-x^2+6x-8}{e^{x-2}}$$

b) Determine the x-ordinates of the point(s) where the gradient of the curve is zero. [2]

$$y'=0 \Rightarrow -x^2+6x-8=0$$

$$(x-2)(4-x)=0$$

$$x=2,4$$

3. [3 marks]

Determine the equation of the tangent to the curve $y=3x^2+e^{2x}+3$ at the point $(1,\ 6+e^2)$.

$$y' = 6x + 2e^{2x}$$

$$y'(1) = 6 + 2e^2$$

$\therefore$ eqn of tangent:

$$y=(6+2e^2)x+c$$

$(1, 6+e^2)$

ie $6+e^2 = (6+2e^2).1 + c$

$$c=-e^2$$

$\therefore$ eqn is $y=(6+2e^2)x-e^2$

8

4. [3 marks]

The curve $y = a\sqrt{x} + 3x$ has a gradient of 4 when $x = 1$.

Calculate the value of 'a'.

$$y' = \frac{1}{2} \cdot a \cdot x^{-\frac{1}{2}} + 3$$

$$= \frac{a}{2\sqrt{x}} + 3$$

$$y'(1) = 4 \Rightarrow \frac{a}{2\sqrt{1}} + 3 = 4$$

$$\frac{a}{2} = 1$$

$$a = 2.$$

5. [4 marks]

If $z = 6 - x^2$ and $y = \sqrt{z}$ determine:

a) $\frac{dz}{dx} = -2x$ [1]

b) $\frac{dy}{dz} = \frac{1}{2\sqrt{z}}$ [1]

c) $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$ [2]

$$= \frac{1}{\cancel{2}\sqrt{z}} \times -\cancel{2}x$$

$$= \frac{-x}{\sqrt{z}}$$

$$= \frac{-x}{\sqrt{6-x^2}}$$

$\frac{}{7}$

6. [3 marks]

Given $y = x + \sqrt{x^2 - 4}$ show that $\dfrac{d^2y}{dx^2} = \dfrac{-4}{\left(\sqrt{x^2-4}\right)^3}$

$$y' = 1 + \frac{1}{\cancel{2}}(x^2-4)^{-\frac{1}{2}} \cdot \cancel{2}x$$

$$= 1 + \frac{x}{\sqrt{x^2-4}}$$

$$y'' = \frac{1\sqrt{x^2-4} - x\,\frac{1}{\cancel{2}}(x^2-4)^{-\frac{1}{2}} \cdot \cancel{2}x}{x^2-4}$$

$$= \frac{\sqrt{x^2-4} - \dfrac{x^2}{\sqrt{x^2-4}}}{x^2-4}$$

$$= \frac{\cancel{x^2} - 4 - \cancel{x^2}}{\sqrt{x^2-4}\,(x^2-4)}$$

$$= \frac{-4}{(x^2-4)^{3/2}}$$

$$= \frac{-4}{\left(\sqrt{x^2-4}\right)^3}$$

Calculator Section (10 marks)

7. [6 marks]

The temperature, $T\,^\circ C$, of a bronze casting t seconds after being removed from a kiln was modelled by $T = T_0 e^{-0.0034t}$ for $0 \le t \le 800$.

a) How long, to the nearest second, did it take for the initial temperature of the casting to halve? [2]

ie $T = 0.5\,T_0$

ie $e^{-0.0034t} = 0.5$

$\therefore\ t = 203.866$

$\sim 204\ s$

b) Determine the initial temperature of the casting, given that it had cooled to $787\,^\circ C$ after one minute. [2]

<u>NB</u> one minute = 60s

$787 = T_0 e^{-0.0034 \times 60}$

$\therefore\ T_0 = 965.0966$

$\sim 965\,^\circ C$

c) Can the above rate of change model be used to calculate how long it takes the temperature of the casting to fall below $40\,^\circ C$? Explain your answer. [2]

$T < 40$ for $0 \leq t \leq 800$?

$965.0966\ e^{-0.0034t} < 40$

$\therefore\ t > 936.279$

outside domain

$\therefore$ model can't be used.

6

7. 8 [4 marks]

The rate of decay of a radio-active material is proportional to the amount present

i.e. $\frac{dM}{dt} = -kM$ where M is the amount of radio-active material in grams and t is in years.

Given that it takes 100 years for ten grams of the materials to decay to eight grams, determine:

a) the mass present after 50 years, if ten grams were originally present

$\frac{dm}{dt} = -km \Rightarrow m = M_0 e^{-kt}$

$8 = 10 e^{-100k}$

$k = 2.2314 \times 10^{-3}$

$m(50) = 10 e^{2.2314 \times 10^{-3}(50)}$

$= 8.944 \text{ g}$

b) the material's half-life.

half-life $\Rightarrow e^{-kt} = 0.5$

$\therefore t = 310.628$

≈ 310.63 years.

4