Chapter 6.

The exponential function.

Situation

The three Ripoff brothers, Muscles, Brains and Sneaky, run "The Concrete Boot Loan Company". They charge 100% per annum on all loans, with the interest compounded annually!
If you borrow $100 for one year then at the end of the year Muscles comes to collect what you owe. How much would that be?

> *"Hey,"* said Brains, *"I think we need to raise our interest rate. We're being too easy on the customers."*
> *"Yeah,"* agreed Sneaky, *"But we need to do it without them realising we've done it."*
> *"Ow you gonna do that Sneaky?"* asked Brains.
> *"Well, what if we still charge 100% per annum but compound it every six months. That way we charge 50% per six months which sounds the same but in fact we'll get more."*

How much would Muscles be sent to collect after one year if you have borrowed $100 for the year under this new scheme?

> *"But if that's gonna get us more why not charge 100% per annum compounded quarterly (i.e. four times per year),"* chipped in Muscles.
> Brains and Sneaky stared at Muscles in amazement. He was never one to put forward sensible suggestions but what he said seemed to have some merit.

How much would you owe at the end of the $100 one year loan under this new scheme?

> Sneaky had been thinking.
> *"Well if Muscles' idea is okay why not compound monthly, or weekly, or daily, or even by the hour, or the minute!! A customer borrowing $100 would still be thinking they'd be paying 100% per annum but if we compound weekly or daily or hourly they'd owe us a fortune at the end of the year!"*
> *"If only we could compound continuously,"* said Sneaky, *"we'd make heaps."*

How does it effect what you owe at the end of a $100 one year loan as this compounding period is reduced? Investigate and write a brief report of your findings.

The situation on the previous page involved the growth of a loan. Did you find that the number 2·71828, (approximately), played an important part?

You should have found that

- if \$$P$ is borrowed for one year at 100% per annum, compounded n times in the year, the amount owed at the end of the year is $P\left(1+\frac{1}{n}\right)^n$.

and that

- as $n \to \infty$ then $\left(1+\frac{1}{n}\right)^n \to 2{\cdot}71828$, correct to five decimal places.

Hence if we compound an infinite number of times in the year, i.e. **continuous** compounding, then the amount owed at the end of the year will be \$$P \times 2{\cdot}71828$.

If the loan continues for another year the amount owed would be \$$P \times (2{\cdot}71828)^2$.

A third year and the amount owed would be \$$P \times (2{\cdot}71828)^3$ etc.

Of course most financial institutions would not charge 100% interest per annum! However, this number, 2·71828, also arises when other rates are charged.

If the rate was 7% per annum it can be shown that with continuous compounding the amount owed at the end of 1 year would be \$$P \times (2{\cdot}71828)^{0{\cdot}07}$. After 2 years it would be \$$P \times (2{\cdot}71828^2)^{0{\cdot}07}$, after 3 years \$$P \times (2{\cdot}71828^3)^{0{\cdot}07}$ etc.

This number, 2·71828, came from consideration of $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]$.

We call this limiting value "e":

e is defined to be $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]$ and is approximately 2·71828.

- Many calculators have an e^x button. Use your calculator to confirm that
  $$e^1 \approx 2{\cdot}71828, \quad e^2 \approx 7{\cdot}38906, \quad e^{-0{\cdot}5} \approx 0{\cdot}60653.$$
- Investigate $\lim_{n\to\infty}\left[\left(1+\frac{a}{n}\right)^n\right]$ for $a \neq 1$.

The repetitive multiplication by some constant gives rise to expressions in which the variable appears as an index, power or exponent, e.g. 2^x, 3^{n-1} etc.

In this chapter we are particularly interested in **exponential expressions** having a **base** of e, for example e^x, e^{2x}, e^{2t}, $e^{-0{\cdot}4t}$, etc.

The constant e ($\approx 2{\cdot}71828$) allows us to describe mathematically many situations involving that amazing phenomenon – growth.

- ☞ Population, spread of disease, investments, demand for a resource such as oil are all examples in which the growth can be exponential.
- ☞ Radioactivity, the temperature of an object placed in cooler surroundings, the concentration of a drug in the bloodstream are all examples in which the decay can be exponential.

The number e can often be used to describe these situations mathematically.

Growth and decay.

Many growth and decay situations involve some variable, say A, growing, or decaying continuously, according to a rule of the form $A = A_0e^{kt}$

where A is the amount present at time t,

A_0 is the initial amount (i.e. the amount present at $t = 0$),

and k is some constant dependent on the situation.

Example 1

A certain culture of bacteria grows in such a way that t days after observation commences the number of bacteria present, N, is given by:

$$N \approx 2000e^{0{\cdot}75t}.$$

Determine the number of bacteria present

(a) when observation commenced,

(b) three days after observation commenced,

(c) ten days after observation commenced.

(a) If $t = 0$ $\quad N \approx 2000e^{0{\cdot}75\times 0}$
$= 2000$

When observation commenced there were approximately 2 000 bacteria.

(b) If $t = 3$ $\quad N \approx 2000e^{0{\cdot}75\times 3}$
≈ 18975

Three days after observation commenced there were approximately 19 000 bacteria present.

$2000e^{0.75\times 3}$	
	18975.47167
$2000e^{0.75\times 10}$	
	3616084.829

(c) If $t = 10$ $\quad N \approx 2000e^{0{\cdot}75\times 10}$
$\approx 3\,616\,085$

Ten days after observation commenced there were approximately 3 600 000 bacteria present.

Example 2

If \$1000 is invested at 12% per annum interest, compounded continuously, the investment will be worth \$$A$ after t years where

$$A = 1000\,e^{0 \cdot 12t}.$$

Find the value of t, correct to one decimal place, for which the value of the investment is \$8 000.

We are given $\qquad A = 1000\,e^{0 \cdot 12t}$

If A = 8 000 then $\qquad 8\,000 = 1000\,e^{0 \cdot 12t}$

Using the solve facility on a calculator

$$t = 17 \cdot 3 \text{ (correct to 1 d.p.)}$$

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solve(8 = e^(0.12×t), t)
                {t = 17.32867951}
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Thus the value of the investment is \$8 000 when $t = 17 \cdot 3$ (correct to 1 d.p.).

Note: When solving the equation of the previous question, some calculators, if set to give exact answers, may give the answer as 25 ln (2). This exact form uses the idea of a "logarithm", a concept we will meet in the next unit of *Mathematics Methods*. For now simply get your calculator to output the decimal answer.

Exercise 6A

1. If \$1000 is invested at 12% per annum interest, compounded continuously, the investment will be worth \$$A$ after t years where

$$A = 1000\,e^{0 \cdot 12t}.$$

Find the value of this investment after

(a) 5 years,

(b) 10 years,

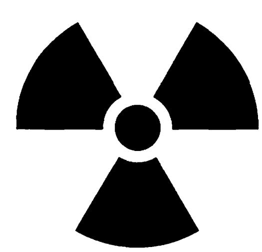
(c) 25 years.

2. If \$$P$ is invested at 8% per annum interest, compounded continuously, the investment grows to \$$Pe^{0 \cdot 08t}$ after t years.

If the investment is worth \$27 819.26 after ten years, find P.

3. A scientific experiment starts with 100 grammes of a particular radioactive element. The element decays such that the amount present t hours later is $100e^{-0 \cdot 03t}$.

How many grammes of the element will have decayed after ten hours?



4. Sales of a particular chocolate bar increase whilst an advertising campaign is in progress. At a time of t weeks after the campaign ceases, the sales have fallen to S bars per week, where $S \approx 2\,000\,000\,e^{-0\cdot15t}$.
   Determine the number of bars sold per week
   (a) when the campaign ceases,
   (b) 2 weeks after the campaign ceases,
   (c) 4 weeks after the campaign ceases,
   (d) 6 weeks after the campaign ceases.

5. A freefalling object falls such that its downward speed, t seconds after release is given by v m/sec where

$$v = 75(1 - e^{-0\cdot13t}) \text{ m/sec.}$$

   Find the downward speed of the body after (a) 5 seconds,
   (b) 20 seconds,
   (c) 40 seconds.

6. If $Y = 20 + \dfrac{40}{e^{0\cdot05x}}$ find x (correct to two decimal places if necessary) given that

   (a) $Y = 60$, (b) $Y = 30$, (c) $Y = 21$.

7. A disease is spreading through a particular community of people such that N, the number of people infected t days after the first reported case, is given by

$$N = \frac{3000}{1 + 2999e^{-0\cdot4t}}.$$

   After how many days should it be expected that 1000 people in this community are infected with the disease?

8. If a payment of $\$P$ is made every year into an account that attracts a fixed interest rate of $r\%$ per annum compounded continuously and the account is closed t years later the balance due will be:

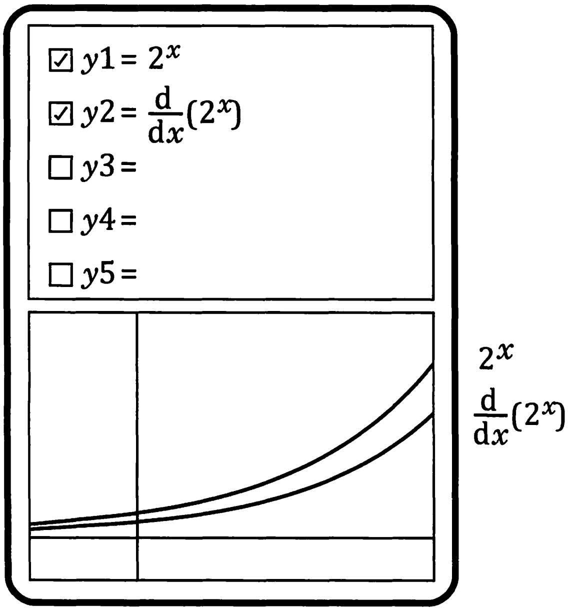
$$\frac{P(e^{0\cdot01rt} - 1)}{1 - e^{-0\cdot01r}}.$$

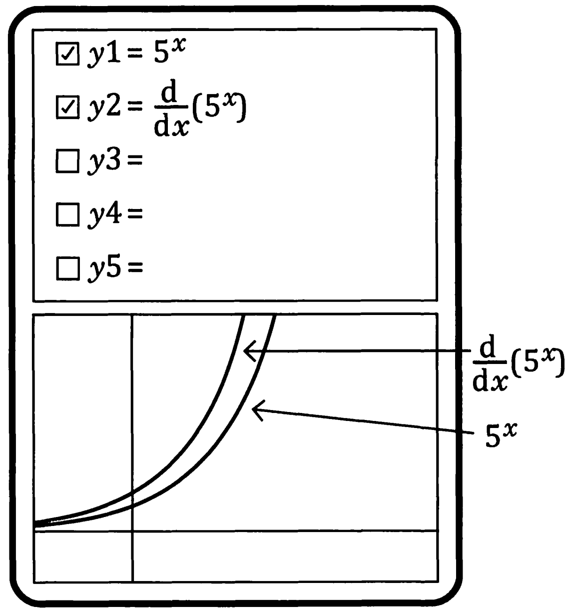
   (a) Find the balance due after 10 years if \$2000 is invested each year and the interest rate is a fixed 10% per annum compounded continuously.
   (b) Find the number of years the scheme must run if an investor wants to invest \$3000 per year and close the account when the balance reaches \$154 000, assuming a constant interest rate of 8% per year compounded continuously.

The derivative of e^x.

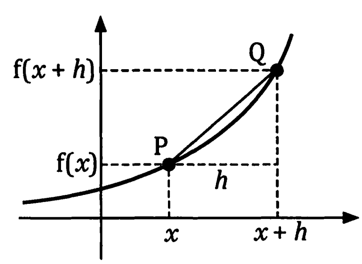
The display below left shows both $y = 2^x$ and $y = \frac{d}{dx}(2^x)$. Notice that the graph of the derivative also appears to be an exponential function but it lies "below" that of $y = 2^x$.

The display below right shows both $y = 5^x$ and $y = \frac{d}{dx}(5^x)$. Notice that the graph of the derivative again appears to be an exponential function but this time the graph of the derivative is "above" that of $y = 5^x$.





This suggests that for some value of a, between a = 2 and a = 5, the graphs of $y = a^x$ and $y = \frac{d}{dx}(a^x)$ coincide. I.e. for some value of a between a = 2 and a = 5, $\frac{d}{dx}(a^x) = a^x$.



However, as the preliminary work mentioned, if we want to differentiate a function for which we do not already have a rule, for example $f(x) = a^x$, we go back to the basic "limiting chord process".

i.e. Gradient at P, see diagram, $= \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

For $f(x) = a^x$,

$$\begin{aligned} \frac{d}{dx}(a^x) &= \lim_{h\to 0}\left[\frac{a^{x+h} - a^x}{h}\right] \\ &= \lim_{h\to 0}\left[\frac{a^x(a^h - 1)}{h}\right] \\ &= a^x \lim_{h\to 0}\left[\frac{a^h - 1}{h}\right] \qquad \leftarrow ① \end{aligned}$$

Examining values of $\dfrac{a^h - 1}{h}$, shown below correct to 7 decimal places, for various values of a, and as h gets smaller and smaller:

	$\dfrac{a^h - 1}{h}$		
h	a = 2	a = 2·5	a = 3
1	1·0000000	1·5000000	2·0000000
0·1	0·7177346	0·9595823	1·1612317
0·01	0·6955550	0·9205015	1·1046692
0·001	0·6933875	0·9167107	1·0992160
0·0001	0·6931712	0·9163327	1·0986726
0·00001	0·6931496	0·9162949	1·0986183
0·000001	0·6931474	0·9162912	1·0986129
0·0000001	0·6931472	0·9162908	1·0986123

Thus, once again, as the graphs on the previous page suggested, there exists a value of a between 2 and 5, and indeed now between 2·5 and 3, for which

$$\lim_{h\to 0}\left[\frac{a^h - 1}{h}\right] = 1$$

and hence for this value of a, from equation ① on the previous page,

$$\frac{d}{dx}(a^x) = a^x.$$

Given that the above discussion follows on from pages introducing "e", it probably didn't take you long to realise that the value of a for which the derivative of a^x is itself, is the number e.

Consider $\displaystyle\lim_{h\to 0}\left[\frac{e^h - 1}{h}\right]$:

If $h = 1$ $\qquad \dfrac{e^h - 1}{h} \approx 1{\cdot}71828$

If $h = 0{\cdot}1$ $\qquad \dfrac{e^h - 1}{h} \approx 1{\cdot}05171$

If $h = 0{\cdot}01$ $\qquad \dfrac{e^h - 1}{h} \approx 1{\cdot}00502$

If $h = 0{\cdot}001$ $\qquad \dfrac{e^h - 1}{h} \approx 1{\cdot}00050$

If $h = 0{\cdot}0001$ $\qquad \dfrac{e^h - 1}{h} \approx 1{\cdot}00005$

The above figures suggest that $\displaystyle\lim_{h\to 0}\left[\frac{e^h - 1}{h}\right] = 1$ and so, from ①, $\dfrac{d}{dx}(e^x) = e^x$.

Thus if $y = e^x$ then $\frac{dy}{dx} = e^x$. The exponential function, e^x, differentiates to itself!

$$\boxed{\text{If } y = e^x \text{ then } \frac{dy}{dx} = e^x.}$$

Example 3

Differentiate (a) $x^3 + e^x$ (b) $5e^x$ (c) $e^{x^2 - 5x + 1}$

(a) If
$$y = x^3 + e^x$$
$$\frac{dy}{dx} = 3x^2 + e^x$$

(b) If
$$y = 5e^x$$
$$\frac{dy}{dx} = 5e^x$$

(c) If
$$y = e^{x^2 - 5x + 1}$$

Let $u = x^2 - 5x + 1$ then $y = e^u$.

$$\frac{du}{dx} = 2x - 5 \quad \text{and} \quad \frac{dy}{du} = e^u.$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{(Chain rule)}$$
$$= e^u(2x - 5)$$
$$= (2x - 5)\,e^{x^2 - 5x + 1}$$

$$\frac{d}{dx}(x^3 + e^x)$$
$$e^x + 3 \cdot x^2$$
$$\frac{d}{dx}(5e^x)$$
$$5 \cdot e^x$$
$$\frac{d}{dx}(e^{x^2 - 5x + 1})$$
$$(2 \cdot x - 5)e^{x^2 - 5 \cdot x + 1}$$

The general statement of example 3 part (c) is:

$$\boxed{\text{If } y = e^{f(x)} \text{ then, by the chain rule, } \frac{dy}{dx} = f'(x)\,e^{f(x)}.}$$

Example 4

Differentiate (a) e^{5x-2} (b) e^{x^2+x} (c) $x^2 e^x$.

(a) If
$$y = e^{5x-2}$$
$$\frac{dy}{dx} = 5e^{5x-2}$$

(b) If
$$y = e^{x^2+x}$$
$$\frac{dy}{dx} = (2x+1)e^{x^2+x}$$

(c) If
$$y = x^2 e^x$$
$$\frac{dy}{dx} = e^x(2x) + x^2(e^x)$$
$$= x e^x(2+x)$$

$\frac{d}{dx}(e^{5x-2})$

$5 \cdot e^{5 \cdot x - 2}$

$\frac{d}{dx}(e^{x^2+x})$

$(2 \cdot x + 1) \cdot e^{x^2+x}$

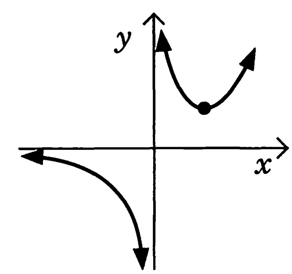
$\frac{d}{dx}(x^2 e^x)$

$x^2 \cdot e^x + 2 \cdot x \cdot e^x$

Example 5

The sketch on the right shows part of the curve $y = \frac{e^x}{x}$.

Use calculus to prove that the local minimum shown is the only stationary point on the curve and to determine its exact location.



If $y = \frac{e^x}{x}$ then, using the quotient rule,
$$\frac{dy}{dx} = \frac{xe^x - e^x}{x^2}$$
$$= \frac{e^x(x-1)}{x^2}$$

Thus $\frac{dy}{dx} = 0$ for $x = 1$ (and *only* for $x = 1$).

Now if $x = 1$ then $y = e$.

The minimum point shown in the diagram is the only stationary point on the curve and it has coordinates $(1, e)$.

Exercise 6B

Differentiate each of the following with respect to x.

1. e^x
2. $7e^x$
3. $3e^x$
4. $6e^x$
5. $9e^x$
6. $-8e^x$
7. e^{5x}
8. e^{7x}
9. e^{-2x}
10. $5e^{3x}$
11. $4e^{0 \cdot 5x}$
12. $-2e^{-0 \cdot 5x}$
13. $6e^x + 2x^3 + 3x^2$
14. $2e^x + \sqrt{x}$
15. $e^{5x} + e^{2x}$
16. $2e^{4x}$
17. $2e^{3x} + 3e^{2x}$
18. $5e^{3x} + x^4$
19. e^{3x-1}
20. e^{x^2+3}
21. e^{5x-1}
22. e^{3x^2+2x-1}
23. e^{x^3}
24. xe^{2x}
25. x^3e^x
26. $e^x\sqrt{x}$
27. $\dfrac{e^x}{2x}$
28. $e^x(1+2x)^3$
29. $e^x(1-2x)^5$
30. $\dfrac{1}{e^{3x}}$

31. Find the exact gradient of $y = e^{2x} + x^2$ at the point $(1, e^2 + 1)$.

32. Find the exact gradient of $y = xe^x$ at the point $(1, e)$.

33. Find the equation of the tangent to $y = 5e^{2x}$ at the point $(0, 5)$.

34. If \$100 is invested at 8% per annum, compounded continuously the account grows to $\$100e^{0 \cdot 08t}$ after t years. What is the instantaneous rate of growth, in dollars per year correct to two decimal places, when
(a) $t = 1$, (b) $t = 10$, (c) $t = 20$, (d) $t = 40$?

35. Damage to a poorly maintained grain store causes the marketable weight of grain in the store to fall from its initial amount of A_0 tonnes to an amount A_t, t weeks later, according to the rule:

$$A_t = 100\, e^{-0 \cdot 1t} \text{ tonnes}$$

(a) Determine A_0.
(b) What is the marketable weight of grain in the store when $t = 5$? (To the nearest tonne.)

At what rate is A falling, in tonnes per week, when (c) $t = 2$,
(d) $t = 5$,
(e) $t = 8$?

More on growth and decay.

Notice that if $y = Ae^{kt}$ then $\dfrac{dy}{dt} = kAe^{kt}$

i.e. $\dfrac{dy}{dt} = ky$. ← ①

Thus, from equation ①, in functions of the form $y = Ae^{kt}$ the rate of change of y with respect to t is proportional to y itself. This sentence is repeated below. Read it again carefully to take in what it means:

In functions of the form $y = Ae^{kt}$ the rate of change of y with respect to t is proportional to y itself.

What this sentence is telling us explains why functions of the form $y = Ae^{kt}$ describe growth or decay situations. A population will tend to reproduce itself at a rate proportional to its size and will continue this constant proportion unless some special factors are introduced that may stimulate or inhibit growth. If country A has a larger population than country B then we would expect the number of babies born in country A in one year to be more than the number of babies born in country B in that year, all other factors being equal.

It is the fact that functions of the form $y = Ae^{kt}$ are such that $\dfrac{dy}{dt} = ky$ that makes them suitable functions for describing many growth and decay situations.

Any growth or decay situation in which the rate of change of the population is proportional to the population itself, i.e. $\dfrac{dP}{dt} = kP$, can be modelled by an equation of the form $P = P_0 e^{kt}$ where P_0 is the population at time $t = 0$.

Thus :

If $\dfrac{dP}{dt} = kP$ then $P = P_0e^{kt}$ where P_0 is the value of P when $t = 0$.

Or, in terms of x and y:

If $\dfrac{dy}{dx} = ky$ then $y = y_0e^{kx}$ where y_0 is the value of y when $x = 0$.

Example 6

Demographers monitored a particular country's population growth over a 30 year period from 1985, when the population was 2 000 000. They found that the population was continuously growing with the instantaneous rate of increase in the population per year, $\frac{dP}{dt}$, always close to $\frac{P}{20}$.

(a) Estimate the population of this country at the end of the 30 year period.

(b) If this pattern of growth continues estimate the population in the years 2025, 2040 and 2065.

(a) Let the population t years after 1985 be P.

We are told that $$\frac{dP}{dt} \approx 0{\cdot}05\,P$$

[Remember that if $\frac{dP}{dt} = kP$ then $P = P_0\,e^{kt}$]

Hence $$P = P_0 e^{0{\cdot}05t}$$

Taking $t = 0$ at 1985 then $P_0 = 2\,000\,000$, the $t = 0$ population.

Thus $$P = 2\,000\,000\,e^{0{\cdot}05t}$$

When $t = 30$ $$P = 2\,000\,000\,e^{0{\cdot}05(30)} \approx 8\,960\,000$$

The population of this country at the end of the 30 year period was approximately nine million.

(b) By 2025, $t = 40$ and so $$P = 2\,000\,000\,e^{0{\cdot}05(40)} \approx 15\,000\,000$$

By 2040, $t = 55$ and so $$P = 2\,000\,000\,e^{0{\cdot}05(55)} \approx 31\,000\,000$$

By 2065, $t = 80$ and so $$P = 2\,000\,000\,e^{0{\cdot}05(80)} \approx 109\,000\,000$$

Assuming the pattern of growth continues the population estimates for 2025, 2040 and 2065 would be 15 million, 31 million and 109 million respectively.

If the situation involves a quantity **decaying** rather than growing then the rate of change of the quantity with respect to time will be negative, rather than positive. (See the next example.)

Example 7
A particular radioactive isotope decays continuously at a rate of 9% per year. One kilogram of this isotope is produced in a particular industrial process. How much remains undecayed after 20 years?

If A kg remains undecayed after t years then $\quad \dfrac{dA}{dt} = -0{\cdot}09\,A$

This is of the form $\dfrac{dA}{dt} = kA$ and so $\quad A = A_0 e^{-0{\cdot}09t}$

When $t = 0$, $A = 1$. Thus $\quad A = 1e^{-0{\cdot}09t}$

When $t = 20$ $\quad A = e^{-0{\cdot}09 \times 20}$

$\approx 0{\cdot}165$

Approximately 165 grams remain undecayed after 20 years.

Example 8
A savings account is opened with a deposit of \$400 and attracts interest at a rate of 8% per annum compounded continuously.

(a) If the interest rate is maintained for five years what will be the balance of the account at the end of this time?

(b) How many years (correct to one decimal place) will it take for the balance in the account to be treble the initial deposit?

(a) The principal grows continuously at 8% p.a. $\therefore \quad \dfrac{dP}{dt} = 0{\cdot}08P.$

This is of the form $\dfrac{dP}{dt} = kP$ and so $\quad P = P_0 e^{0{\cdot}08t}$

When $t = 0$, $P = 400$. Thus $\quad P = 400e^{0{\cdot}08t}$

When $t = 5$ $\quad P = 400e^{0{\cdot}08 \times 5}$

$\approx \$596{\cdot}73$

After five years the account balance will be \$596·73.

(b) If $P = 1200$ then $\quad 1200 = 400e^{0{\cdot}08t}$

i.e. $\quad 3 = e^{0{\cdot}08t}$

Solving with a calculator gives $\quad t = 13{\cdot}7$ (1 d.p.)

The initial deposit will treble after approximately 13·7 years.

Exercise 6C

For this exercise use the fact that if $\frac{dP}{dt} = kP$ then $P = P_0 e^{kt}$.

1. If $\frac{dA}{dt} = 2{\cdot}5A$ and $A = 50$ when $t = 0$, find A when (a) $t = 1$, (b) $t = 3$.

2. If $\frac{dP}{dt} = 0{\cdot}01P$ and $P = 2000$ when $t = 0$, find P when (a) $t = 10$, (b) $t = 50$.

3. If $\frac{dQ}{dt} = \frac{3Q}{100}$ and $Q = 150$ when $t = 0$, find Q when (a) $t = 2$, (b) $t = 25$.

4. If $\frac{dA}{dt} = -0{\cdot}1A$ and $A = 20\,000$ when $t = 0$, find A when (a) $t = 10$, (b) $t = 20$.

5. If $\frac{dX}{dt} = \frac{X}{2}$ and $X = 6$ million when $t = 5$, find X when (a) $t = 10$, (b) $t = 20$.

6. If $\frac{dP}{dt} = 0{\cdot}025P$ and $P = 2000$ when $t = 10$, find P when (a) $t = 11$, (b) $t = 20$.

7. A particular country has a population of 250 million. Records indicate that the population growth rate is 3% per year, i.e. $\frac{dP}{dt} = 0{\cdot}03P$.
   Estimate the population of the country after a further (a) 10 yrs, (b) 50 yrs.

8. Repeat question 7 but now for a growth rate of $2{\cdot}5\%$.

9. A particular radioactive isotope decays continuously at a rate of 12% per year. Three kilograms of this isotope are produced in a particular industrial process. How much remains undecayed after 20 years?

10. A 30 year old person makes a "one off" payment of \$5000 to a savings plan with the intention of leaving it untouched for 25 years. If the investment attracts a fixed guaranteed interest rate of 11% per annum, compounded continuously, find the value of the investment at the end of the 25 years.

11. How much does a person need to deposit in an account attracting a constant interest rate of 12% per annum, compounded continuously, for it to have grown to \$20 000 after 20 years?

12. Let us suppose that the cost of goods is rising continuously at 5% per annum. The rate of change in the cost of an article costing \$P would then be such that
$$\frac{dP}{dt} = 0{\cdot}05\text{P}.$$
    Under these conditions what would be the cost in 100 years of a chocolate bar now costing 80 cents?

13. Repeat question 12 if costs rise continuously at 8% per year rather than 5%.

14. The instantaneous rate of decline of a particular species of frog is 5% per month. If the current population is 10 000 what will it be in (a) 5 months, (b) 10 months?

15. The population of a particular country varies with time. The rate of change of the population from 2000 onwards was found to be proportional to the population itself, i.e. $\frac{dP}{dt} = kP$. P is the population of the country t years after 2000 and k maintains a roughly constant value.
    (a) If the instantaneous growth rate was always roughly 2% per year find k.
    (b) If the population in 2000 was 20 million, and assuming the growth rate remains constant, when will the population reach 50 million?

16. N, the number of people in the world known to be suffering from a particular disease, was thought to be increasing such that $\frac{dN}{dt} = 0{\cdot}05N$, with t the time in years.

    If the number of known sufferers was 1500 000 in 2000, and nothing is done to alter the rate of increase, estimate the number of known sufferers in each of the years (a) 2025, (b) 2050.

17. A colony of bacteria grows such that $\frac{dP}{dt} = 1{\cdot}2P$ where P is the number of organisms present t hours after observation commenced.
    When observation commenced $P \approx 1000$.
    After approximately how many hours will the population be (a) 1 million, (b) 2 million?
    (c) What is the "doubling" time for this population?
    (d) What is the "quadrupling" time for this population?

18. The instantaneous rate of decline in the number of rabbits on a particular property is 25% per month. If there were 2000 rabbits on the property two months ago how many will there be in two months time?

19. A company finds that S, the number of sales per week of a particular product, falls such that $\frac{dS}{dt} = -0{\cdot}24S$, where t is the number of weeks since the end of the company's promotion and advertising campaign. They decide to repeat the campaign when weekly sales fall to $0{\cdot}45S_0$, where S_0 is the weekly sales at the end of the first campaign. How many weeks after the end of the first campaign should they expect to have to launch the repeat campaign?

Integrating exponential functions.

We now know that if $y = e^x$ then $\frac{dy}{dx} = e^x$.

Thus

$$\boxed{\int e^x \, dx = e^x + c}$$

Also, if $y = e^{f(x)}$ we let $u = f(x)$ and so $y = e^u$

Then, by the chain rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= e^u \times f'(x) \\ &= f'(x)\, e^{f(x)} \end{aligned}$$

Thus

$$\boxed{\int f'(x)\, e^{f(x)} \, dx = e^{f(x)} + c}$$

Example 9

Find (a) $\int e^{6x} \, dx$ (b) $\int 10x e^{x^2} \, dx$ (c) $\int_0^1 8e^{2x} \, dx$.

(a) <u>Method one.</u> (Making an intelligent guess then adjusting.)

Try $y = e^{6x}$

Then $\frac{dy}{dx} = 6e^{6x}$

Thus our initial trial needs to be divided by 6.

$$\therefore \quad \int e^{6x} \, dx = \frac{e^{6x}}{6} + c.$$

<u>Method two.</u> (Rearranging to set up $\int f'(x)\, e^{f(x)} \, dx$.)

$$\begin{aligned} \int e^{6x} \, dx &= \int \frac{1}{6} \times 6e^{6x} \, dx \\ &= \frac{1}{6} \times \int 6e^{6x} \, dx \\ &= \frac{1}{6} \times e^{6x} + c \\ &= \frac{e^{6x}}{6} + c. \end{aligned}$$

$\int e^{6x} dx$

$\frac{e^{6 \cdot x}}{6}$

(b) Method one. (Making an intelligent guess then adjusting.)

Try $\quad y = e^{x^2}$

Then $\quad \dfrac{dy}{dx} = 2xe^{x^2}$

Thus our initial trial needs to be multiplied by 5.

$$\therefore \int 10xe^{x^2}\,dx = 5e^{x^2} + c.$$

Method two. (Rearranging.)

$$\begin{aligned}\int 10xe^{x^2}\,dx &= \int 5 \times 2xe^{x^2}\,dx \\ &= 5 \times \int 2xe^{x^2}\,dx \\ &= 5e^{x^2} + c\end{aligned}$$

(c) $$\begin{aligned}\int_0^1 8e^{2x}\,dx &= \left[4e^{2x}\right]_0^1 \\ &= 4e^2 - 4 \\ &= 4(e^2 - 1).\end{aligned}$$

$\int 10xe^{x^2}\,dx$	
	$5\cdot e^{x^2}$
$\int_0^1 8e^{2x}\,dx$	
	$4\cdot(e^2-1)$

Exercise 6D Attempt each question without the assistance of your calculator - then use your calculator to check your answer if you wish.

Find the following indefinite integrals.

1. $\int 6e^{3x}\,dx$
2. $\int 6e^{2x}\,dx$
3. $\int e^{5x}\,dx$
4. $\int 3e^{9x}\,dx$
5. $\int 5e^{3x}\,dx$
6. $\int \dfrac{5}{e^x}\,dx$
7. $\int 4\sqrt{e^x}\,dx$
8. $\int \dfrac{1}{e^{2x}}\,dx$
9. $\int (4e^{2x} + 2x)\,dx$
10. $\int (e^{3x} + e^{2x})\,dx$
11. $\int 3e^{-2x}\,dx$
12. $\int \left(\dfrac{4}{e^{2x}} + \dfrac{e^{2x}}{4}\right)dx$

13. $\int 2xe^{x^2}\,dx$

14. $\int 6e^{2x+1}\,dx$

15. $\int (8xe^{x^2+5})\,dx$

Evaluate the following definite integrals, giving exact answers.

16. $\int_0^2 5e^x\,dx$

17. $\int_0^1 e^{5x}\,dx$

18. $\int_1^2 (e^x + 4e^{2x})\,dx$

19. $\int_0^2 2(x + e^{2x})\,dx$

20. $\int_{-1}^0 \frac{1}{e^x}\,dx$

21. $\int_0^2 6(\sqrt{e^x} + x^2)\,dx$

22. If $\frac{dA}{dt} = 5e^{2t}$, and $A = 3$ when $t = 0$, find

    (a) A in terms of t,

    (b) the exact value of A, when $t = 0{\cdot}5$.

23. If $f'(x) = 6(x^2 - 2e^{3x})$, and $f(0) = 1$, find (a) $f(x)$,
    (b) $f(2)$ as an exact value.

24. (a) Find the area between $y = e^x$ and the x-axis from $x = 0$ to $x = 3$ giving your answer correct to one decimal place.

    (b) Find the area between $y = e^x - e$ and the x-axis from $x = 0$ to $x = 3$ giving your answer as an exact value.

Miscellaneous Exercise Six.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

1. Clearly showing your use of the product rule, find the equation of the tangent to
$$y = (2x - 1)(3x + 2)$$
at the point $(1, 5)$.

2. Without the assistance of your calculator, find $\frac{dy}{dx}$ for each of the following.

    (a) $y = (x + 2)^5$

    (b) $y = (2x + 1)^5$

    (c) $y = \dfrac{x - 5}{x + 5}$

    (d) $y = \dfrac{5x - 1}{x + 5}$

    (e) $y = 4x^3 - e^x + 5$

    (f) $y = e^{5x} + 5x$

3. The tangent to the curve $y = ax^3$ at the point $(5, b)$ has a gradient of 30. Find the values of the constants a and b.

4. A particle is performing rectilinear motion with x metres and v m/s the displacement and velocity of the particle, with respect to an origin O, at time t seconds. The motion is such that $v = e^{0.1t}$ and when $t = 0$, $x = 12$.
Find (a) the initial acceleration of the particle,
(b) the acceleration of the particle when $t = 20$,
(c) the displacement of the particle when $t = 10$, to the nearest centimetre.

5. Use first principles, i.e. $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, to show that if $y = x^2 + 3x$ then

$$\frac{dy}{dx} = 2x + 3.$$

6. The curve $y = x^3 + 3x^2 - 10x$ cuts the x-axis in three places. Find the coordinates of each of these points and determine the gradient of the curve at each one.
Find the total area of any regions enclosed by the curve and the x-axis.

7. The tangent to the curve $y = ax^2 + 5$ at the point $(-1, b)$ is perpendicular to the line $2y = -x + 8$. Given that if a line with gradient m_1 is perpendicular to a line with gradient m_2 then $m_1 m_2 = -1$, find the values of the constants a and b.

8. Evaluate the following definite integrals without the assistance of your calculator.

(a) $\displaystyle\int_2^{10} x \, dx$ (b) $\displaystyle\int_1^2 \frac{1}{x^2} \, dx$ (c) $\displaystyle\int_0^1 e^x \, dx$

(d) $\displaystyle\int_0^1 6e^{2x} \, dx$ (e) $\displaystyle\int_{-1}^2 (3x^2 + 4x) \, dx$ (f) $\displaystyle\int_2^3 \frac{4x}{(x^2 - 3)^2} \, dx$

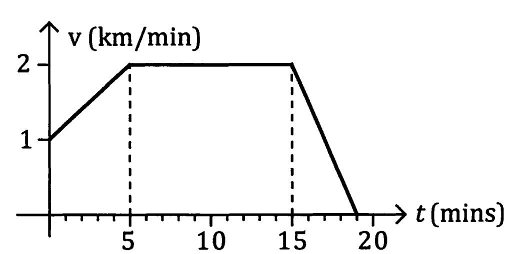
9. Differentiate each of the following with respect to x, without the assistance of your calculator, and then use your calculator to check your answers.

(a) $2x^3 + 4\sqrt{x}$ (b) $x^3 + e^x$ (c) $\dfrac{2x - 1}{x + 3}$

(d) $x^4 e^x$ (e) $(2x^3 + 4\sqrt{x})^5$ (f) $\displaystyle\int_5^x \frac{e^{5t}}{t} \, dt$

10. [Hint: What do we get if we integrate velocity with respect to time?
So what will the area "under" a velocity time graph give?]

The velocity-time graph for the motion of a train along a straight track is shown on the right.



(a) When $t = 0$ the train is at position A and when $t = 5$ the train is at B. Find the distance from A to B.
(b) When $t = 15$ the train is at a position C. Find the distance from A to C.
(c) When $t = 19$ the train is at a station D. Find the distance from A to D.

11. A falling object does not keep accelerating indefinitely but, due to air resistance, reaches a terminal speed. Suppose that the speed of such an object, t seconds after the fall commences is v m/s where

$$v = \frac{200}{3}\left(1 - e^{-0\cdot15t}\right).$$

Find the speed of the object after five seconds.
What is the terminal speed?

12. For δx, a small change in x, then δy, the associated small change in y, can be determined using $\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$.
Use the above statement to determine the approximate change in the exterior surface area of a closed cylindrical tin when the base radius changes from 10 cm to 10·2 cm with the height remaining unchanged on 20 cm.

13. An initial "one off" investment of \$500 grows continuously in such a way that

$$\frac{dP}{dt} = 0\cdot08P$$

where \$$P$ is in the account t years after the investment was opened.
How much is the investment worth after (a) 5 yrs, (b) 15 yrs?

14. A metal bar, temperature 120°C, is placed in an environment with temperature 25°C. The temperature of the bar, t minutes later, is approximately T°C where

$$T = 25 + 95e^{-0\cdot3t}.$$

Find the rate at which the temperature of the bar is falling, in °C/minute correct to one decimal place, after (a) 1 min, (b) 3 mins, (c) 15 mins.

15. Without the assistance of a graphic calculator produce a sketch of each of the following, clearly indicating on your sketch:
- the coordinates of any points where the graph cuts the axes,
- the exact coordinates of all turning points,
- the behaviour of the curve as $x \to \pm\infty$.

(a) $y = x^2e^x$ (b) $y = \frac{e^x}{x^2}$ (c) $y = \frac{1}{1 + e^x}$

16. A new rowing oar is being investigated. The shape of the blade is as shown shaded on the right The y-axis forms the left hand boundary, the x-axis is a line of symmetry and 1 cm = 1 unit on each axis. Find the shaded area, giving your answer correct to the nearest square cm.