Chapter 8 – Revision of chapters 2–7

Solutions to 8A Short-answer questions

1 **a** Vertices $A(-2, 1), B(3, -4), C(5, 7)$

$$\text{Coordinates of M} = \left(\frac{-2+3}{2}, \frac{1+(-4)}{2}\right) = \left(\frac{1}{2}, -\frac{3}{2}\right)$$

$$\text{Coordinates of N} = \left(\frac{-2+5}{2}, \frac{1+7}{2}\right) = \left(\frac{3}{2}, 4\right)$$

b
$$\text{Gradient of MN} = \frac{4-\left(-\frac{3}{2}\right)}{\frac{3}{2}-\frac{1}{2}} = \frac{11}{2}$$

$$\text{Gradient of BC} = \frac{7-(-4)}{5-3} = \frac{11}{2}$$

$\therefore BC \parallel MN$

2 $P(x) = 8x^3 + 4x - 3$

a
$$P\left(-\frac{1}{2}\right) = 8\times\left(-\frac{1}{2}\right)^3 + 4\left(-\frac{1}{2}\right) - 3 = -1 - 2 - 3 = -6$$

b
$$P(2) = 8\times(2)^3 + 4(2) - 3 = 64 + 8 - 3 = 69$$

c
$$\begin{aligned} Q(x) &= P(x+1) \\ &= 8\times(x+1)^3 + 4(x+1) - 3 \\ &= 64 + 8 - 3 \\ &= 69 \end{aligned}$$

$$\begin{aligned} Q(-2) &= 8\times(-2+1)^3 + 4(-2+1) - 3 \\ &= -8 - 4 - 3 \\ &= -15 \end{aligned}$$

3 $g(x) = 3x^2 - 4$

a $g(2a) = 3(2a)^2 - 4 = 12a^2 - 4$

b
$$\begin{aligned} g(a-1) &= 3(a-1)^2 - 4 \\ &= 3(a^2 - 2a + 1) - 4 \\ &= 3a^2 - 6a - 1 \end{aligned}$$

c
$$\begin{aligned} &g(a+1) - g(a-1) \\ &= 3(a+1)^2 - 4 - (3(a-1)^2 - 4) \\ &= 3((a^2 + 2a + 1) - (a^2 - 2a + 1)) \\ &= 12a \end{aligned}$$

4 $f(x) = 4 - 5x$ and $g(x) = 7 + 2x$

a $f(2) + f(3) = -6 + (-11) = -17$
$f(2+3) = f(5) = -21$
$\therefore f(2) + f(3) \neq f(2+3)$

b
$$\begin{aligned} f(x) &= g(x) \\ 4 - 5x &= 7 + 2x \\ -3 &= 7x \\ x &= -\frac{3}{7} \end{aligned}$$

c
$$\begin{aligned} f(x) &\geq g(x) \\ 4-5x &\geq= 7+2x \\ -3 &\geq 7x \\ x &\leq -\frac{3}{7} \end{aligned}$$

d
$$\begin{aligned} f(2k) &= g(3k) \\ 4-5(2k) &= 7+2(3k) \\ 4-10k &= 7+6k \\ -3 &= 16k \\ k &= -\frac{3}{16} \end{aligned}$$

5
$$\begin{aligned} x+y &= 5\ldots(1) \\ (x+1)^2+(y+1)^2 &= 25\ldots(2) \end{aligned}$$

From equation (1) $y = 5 - x$
Substitute in equation (2)
$$\begin{aligned} (x+1)^2+(6-x)^2 &= 25 \\ x^2+2x+1+36-12x+x^2 &= 25 \\ 2x^2-10x+37 &= 25 \\ 2x^2-10x+12 &= 0 \\ x^2-5x+6 &= 0 \\ (x-3)(x-2) &= 0 \\ x = 3 \text{ or } x &= 2 \end{aligned}$$

From equation (1)
When $x = 3, y = 2$ and when $x = 2, y = 3$

6 $A(0,-5), B(-1,2), C(4,7), D(5,0)$
$$\begin{aligned} AB &= \sqrt{(7-2))^2+(4-(-1))^2} \\ &= \sqrt{25+25} \\ &= 5\sqrt{2} \end{aligned}$$
$$\begin{aligned} BC &= \sqrt{(2-(-5))^2+(-1-0)^2} \\ &= \sqrt{49+1} \\ &= 5\sqrt{2} \end{aligned}$$
$$\begin{aligned} CD &= \sqrt{(0-7)^2+(5-4)^2} \\ &= \sqrt{49+1} \\ &= 5\sqrt{2} \end{aligned}$$
$$\begin{aligned} DA &= \sqrt{(5-0))^2+(0-(-5))^2} \\ &= \sqrt{25+25} \\ &= 5\sqrt{2} \end{aligned}$$

This is sufficient to prove $ABCD$ is a rhombus.

7 a
$$\begin{aligned} y &= x^2+4x-9 \\ &= x^2+4x+4-4-9 \\ &= (x+2)^2-13 \end{aligned}$$

b
$$\begin{aligned} y &= x^2-3x-11 \\ &= x^2-3x+\frac{9}{4}-\frac{9}{4}-11 \\ &= \left(x-\frac{3}{2}\right)^2-\frac{53}{4} \end{aligned}$$

c
$$\begin{aligned} y &= 2x^2-3x+11 \\ &= 2\left[x^2-\frac{3}{2}x+\frac{11}{2}\right] \\ &= 2\left[x^2-\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}+\frac{11}{2}\right] \\ &= 2\left[\left(x-\frac{3}{4}\right)^2+\frac{79}{16}\right] \\ &= 2\left(x-\frac{3}{4}\right)^2+\frac{79}{8} \end{aligned}$$

8 a
$$\begin{aligned} y &= 4x+1\ldots(1) \\ y &= x^2+3x-9\ldots(2) \end{aligned}$$

Substitute in equation (2) from equation 1

$$4x + 1 = x^2 + 3x - 9$$

$$\therefore 0 = x^2 - x - 10$$

$$\therefore x^2 - x - 10 = 0$$

$$\therefore x^2 - x + \frac{1}{4} - \frac{1}{4} - 10 = 0$$

$$\therefore (x - \frac{1}{2})^2 = \frac{41}{4}$$

$$x = \frac{1}{2} \pm \frac{\sqrt{41}}{2}$$

$$x = \frac{1 \pm \sqrt{41}}{2}$$

From equation (1)

When $x = \dfrac{1 + \sqrt{41}}{2}$

$y = 2 + 2\sqrt{41} + 1 = 3 + 2\sqrt{41}$

When $x = \dfrac{1 - \sqrt{41}}{2}$

$y = 2 - 2\sqrt{41} + 1 = 3 - 2\sqrt{41}$

b $y = 2x + 2 \ldots (1)$

$y = x^2 - 2x + 6 \ldots (2)$

Substitute in equation (2) from equation 1

$$2x + 2 = x^2 - 2x + 6 \text{ From}$$

$$\therefore 0 = x^2 - 4x + 4$$

$$\therefore x^2 - 4x + 4 = 0$$

$$\therefore (x - 2)^2 = 0$$

$$x = 2$$

equation (1)

When $x = 2, y = 6$

c $y = -3x + 2 \ldots (1)$

$y = x^2 + 5x + 18 \ldots (2)$

$$-3x + 2 = x^2 + 5x + 18 \text{ From}$$

$$\therefore 0 = x^2 + 8x + 16$$

$$\therefore x^2 + 8x + 16 = 0$$

$$\therefore (x + 4)^2 = 0$$

$$\therefore x = -4$$

equation (1)

When $x = -4, y = 14$

9 **a** $x^2 + 3x - 5 > 0$

Consider

$$x^2 + 3x - 5 = 0$$

$$x^2 + 3x + \frac{9}{4} - \frac{9}{4} - 5 = 0$$

$$\left(x + \frac{3}{2}\right)^2 = \frac{29}{4}$$

$$x + \frac{3}{2} = \pm\frac{\sqrt{29}}{2}$$

$$x = \frac{-3 \pm \sqrt{29}}{2}$$

The coefficient of x^2 is positive.

Therefore $x^2 + 3x - 5 > 0$ if and only if

$x \in \left(-\infty, \dfrac{-3 - \sqrt{29}}{2}\right) \cup \left(\dfrac{-3 + \sqrt{29}}{2}, \infty\right)$

b $2x^2 - 5x - 5 \geq 0$

Consider

$$2\left(x^2 - \frac{5}{2}x - \frac{5}{2}\right) = 0$$

$$x^2 - \frac{5}{2}x - \frac{5}{2} = 0$$

$$x^2 - \frac{5}{2}x + \frac{25}{16} - \frac{25}{16} - \frac{5}{2} = 0$$

$$\left(x - \frac{5}{4}\right)^2 = \frac{65}{16}$$

$$x - \frac{5}{4} = \pm\frac{\sqrt{65}}{4}$$

$$x = \frac{5 \pm \sqrt{65}}{4}$$

The coefficient of x^2 is positive.
Therefore $2x^2 - 5x - 5 \geq 0$ if and only if

$$x \in \left(-\infty, \frac{5 - \sqrt{65}}{4}\right] \cup \left[\frac{5 + \sqrt{65}}{4}, \infty\right)$$

10 **a** $\mathbb{R} \setminus \{\frac{5}{2}\}$

b $(-\infty, 5]$

c $\mathbb{R}$

11 Let $P(x) = 3x^3 + x^2 + px + 24$
$P(-4) = 0$ by the factor theorem.
Hence

$$3(-4)^3 + (-4)^2 + (-4)p + 24 = 0$$

$$-192 + 16 - 4p + 24 = 0$$

$$-4p = 152$$

$$\therefore p = -38$$

$\therefore P(x) = 3x^3 + x^2 - 38x + 24$
$3x^3 + x^2 - 38x + 24 = (x+4)(3x^2 + bx + 6)$
since $x + 4$ is a factor.
By equating coefficients of x^2
$1 = 12 + b, \therefore b = -11$

$$P(x) = (x + 4)(3x^2 - 11x + 6)$$

$$= (x + 4)(3x - 2)(x - 3)$$

12

$$5x^3 - 3x^2 + ax + 7 = (x + 2)Q_1(x) + R \ldots (1)$$

$$4x^3 + ax^2 + 7x - 4 = (x + 2)Q_2(x) + 2R \ldots (2)$$

Multiply (1) by 2 and subtract (1) from the result.
$6x^3 - (6 + a)x^2 + (2a - 7)x + 18 = (x + 2)(2Q_1 - Q_2)$
When $x = -2$
$6(-2)^3 - (6 + a)(-2)^2 + (2a - 7)(-2) + 18 = 0$
$\therefore -48 - 24 - 4a - 4a + 14 + 18 = 0$

$$\therefore -8a = 40$$

$$\therefore a = -5$$

Substitute in (1)
$5x^3 - 3x^2 - 5x + 7 = (x + 2)Q_1(x) + R$
Substitute $x = -2$
$R = 5(-2)^3 - 3(-2)^2 - 5(-2) + 7 = -35$

13 **a** $f : [1, 2] \to \mathbb{R}, f(x) = x^2$
Domain of $f = [1, 2]$
Range of $f = [1, 4]$
Let $y = x^2$
Interchange x and y.
$x = y^2$
Choose $y = \sqrt{x}$, (range of f)
$\therefore f^{-1} : [1, 4] \to \mathbb{R}, f^{-1}(x) = \sqrt{x}$

b $h : [-1, 2] \to \mathbb{R}, h(x) = 2 - x$
Domain of $h = [-1, 2]$
Range of $h = [0, 3]$
Let $y = 2 - x$
Interchange x and y.
$x = 2 - y$
$y = 2 - x$
$\therefore h^{-1} : [0, 3] \to \mathbb{R}, h^{-1}(x) = 2 - x$

c $g : \mathbb{R}^{-1} \to \mathbb{R}, g(x) = x^2 - 4$
Domain of $g = (-\infty, 0)$
Range of $g = (-4, \infty)$
Let $y = x^2 - 4$

Interchange x and y.
$x = y^2 - 4$
$y = -\sqrt{x+4}$ (range of g)
$\therefore g^{-1} : [0, 3] \to \mathbb{R}, g^{-1}(x) = -\sqrt{x+4}$

d $f : (-\infty, 2] \to \mathbb{R}, f(x) = \sqrt{2-x} + 3$
Domain of $f = (-\infty, 2]$
Range of $f = [3, \infty)$
Let $y = \sqrt{2-x} + 3$
Interchange x and y.
$x = \sqrt{2-y} + 3$
$y = -(x-3)^2 + 2$
$\therefore f^{-1} : [3, \infty) \to \mathbb{R}$,
$f^{-1}(x) = -(x-3)^2 + 2$

e $f : \to \mathbb{R}, f(x) = (x-2)^3 + 8$
Domain of $f = \mathbb{R}$
Range of $f = \mathbb{R}$
Let $y = (x-2)^3 + 8$
Interchange x and y.
$x = (y-2)^3 + 8$
$y = (x-8)^{\frac{1}{3}} + 2$
$\therefore f^{-1} : \mathbb{R} \to \mathbb{R}$,
$f^{-1}(x) = (x-8)^{\frac{1}{3}} + 2$

14 Let b be the cost of a Bob's burger.
Let f be the cost of a regular fries.

a $\therefore 3b + 2f = 18.20$

b If $b = 4.2$
$3 \times 4.20 + 2f = 18.20$
$\therefore 2f = 18.20 - 12.60$
$\therefore f = 2.80$
The cost of regular fries is \$2.80

15 $4x + ky = 7$ and $y = 3 - 4x$ The gradient of the line $4x + ky = 7$ is $-\frac{4}{k}$
The gradient of the line $y = 3 - 4x$ is -4

a If the lines are parallel, $-\frac{4}{k} = -4$
Hence $k = 1$

b If the lines are perpendicular
$-\frac{4}{k} \times -4 = -1$
$k = -16$

16 Line ℓ_1 has x-axis intercept $(5, 0)$ and y-axis intercept $(0, -2)$.

a Gradient of $\ell_1 = \frac{-2-0}{0-5} = \frac{2}{5}$

b Line ℓ_2 is perpendicular to line line ℓ_1
Hence gradient of ℓ_2 is $-\frac{5}{2}$
The line ℓ_2 has equation of the form
$y = -\frac{5}{2}x + c$
When $x = 1, y = 6 \therefore, 6 = -\frac{5}{2} + c$ and hence
$c = \frac{17}{2}$ and $y = -\frac{5}{2}x + \frac{17}{2}$
Rearranging as required
$5x + 2y - 17 = 0$

17 $\ell \propto \sqrt{n}$
$\ell = k\sqrt{n}$
$k = 2$
$\ell = 2\sqrt{4} = 2$

$14 = 2\sqrt{n}$
$n = 49$

18 **a** $ax^2 + 2x + a$

$= a(x^2 + \frac{2}{a}x + 1)$

$= a\left(x^2 + \frac{2}{a}x + \frac{1}{a^2} - \frac{1}{a^2} + 1\right)$

$= a\left(\left(x + \frac{1}{a}\right)^2 + \frac{a^2 - 1}{a^2}\right)$

$= a\left(x + \frac{1}{a}\right)^2 + \frac{a^2 - 1}{a}$

b $\left(-\frac{1}{a}, \frac{a^2 - 1}{a}\right)$

c Perfect square when $\Delta = 4 - 4a^2 = 0$.
That is when $a = \pm 1$

d There are two solutions when
$\Delta = 4 - 4a^2 > 0$.
That is when $-1 < a < 1$

19 **a** $y = 1 + \frac{1}{2 + x}$

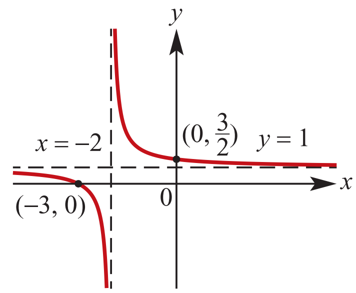
When $x = 0, y = \frac{3}{2}$ When

$y = 0, 1 + \frac{1}{2 + x} = 0$

That is, $\frac{1}{2 + x} = -1$ which implies
$x = -3$

The horizontal asymptote has equation $y = 1$

The vertical asymptote has equation $x = -2$



b $A\left(0, \frac{3}{2}\right), B(-3, 0)$

c $y = \frac{1}{2}x + \frac{3}{2}$

d The midpoint

$\left(\frac{0 + (-3)}{2}, \frac{\frac{3}{2} + 0}{2}\right) = \left(-\frac{3}{2}, \frac{3}{4}\right)$

e Gradient of line $AB = \dfrac{\frac{3}{2} - 0}{0 - (-3)} = \frac{1}{2}$

Gradient of a line perpendicular to AB is -2.

Therefore using the general form $y - y_1 = m(x - x_1)$ we have

$y - \frac{3}{4} = -2\left(x + \frac{3}{2}\right)$

That is,

$y = -2x - \frac{9}{4}$

20 $\sqrt{2}$ cm

21 192 g

Solutions to 8B Multiple-choice questions

1 B $y = x^2 - ax$

$$= x^2 - ax + \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2$$

$$= \left(x - \frac{a}{2}\right)^2 - \frac{a^2}{4}$$

2 D $\Delta = 4a^2 - 4b = 0$

$a^2 = b$

$a = \sqrt{b}$ or $a = -\sqrt{b}$

But a and b are positive constants.

Therefore $a = \sqrt{b}$

3 C Gradients are the same when

$$\frac{2-m}{3} = \frac{-2}{m+2}$$

$$\frac{m-2}{3} = \frac{2}{m+2}$$

$m^2 - 4 = 6$

$m = \pm\sqrt{10}$

4 A $m = kn$

$9 = 4k$

$k = \frac{9}{4}$

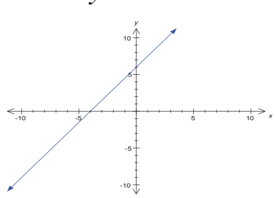
5 D $x = ky$

$8 = 2k$

$k = 4$

$x = 4 \times 7 = 28$

6 A $3x - 2y = -6$



7 D Only (1, 2) is on the line $y = 3x - 1$

8 D $x^3 - 8 = x^3 - 2^3$

$= (x-2)(x^2 + 2x + 4)$

9 C $2x^2 - 5x - 12 = (2x + a)(x - b)$

$a - 2b = -5;\ ab = 12$

$a = 3,\ b = 4 : f(x) = (2x + 3)(x - 4)$

10 C $P(x) = 4x^3 - 5x + 5$

$P(-\frac{3}{2}) = -1$

11 C $x^2 + y^2 + 6x - 2y + 6 = 0$

$\therefore\ x^2 + 6x + 9 + y^2 - 2y + 1 = 10 - 6$

$\therefore\ (x+3)^2 + (y-1)^2 = 2^2$

Radius = 2

12 A $2x + 4y - 6 = 0$

$\therefore\ 4y = -2x + 6$

$\therefore\ y = -\frac{1}{2}x + \frac{3}{2}$

Gradient $= -\frac{1}{2}$

13 E $2x + 4y = 3$

$\therefore\ 4y = -2x + 3$

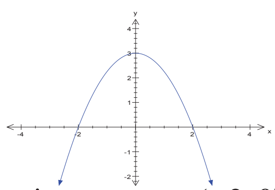
$\therefore\ y = -\frac{1}{2}x + \frac{3}{4}$

Line has gradient $= -\frac{1}{2}$, so perpendicular has gradient $m = 2$.
Using $(1, 2)$: $y - 2 = 2(x - 1)$
$\therefore\ y = 2x$

14 B $P(x) = x^3 + ax^2 - x - 6$
If $x - 3$ is a factor of $P(x)$ then $P(3) = 0$:
$P(3) = 27 + 9a - 3 - 6 = 0$
$\therefore 9a + 18 = 0,\ \therefore a = -2$

15 A
$$P(x) = x^3 + 8x^2 + 9x - 18$$
$$P(1) = 1 + 8 + 9 - 18 = 0$$
$$\therefore P(x) = (x - 1)(x^2 + 9x + 18)$$
$$= (x - 1)(x + 6)(x + 3)$$

16 E



x-intercepts at $(-2, 0)$ and $(2, 0)$, so
$y = a(x - 2)(x + 2)$
$\therefore\ y = a(x^2 - 4)$
Using y-intercept at $(0, 3)$, $a = -\frac{3}{4}$
$\therefore\quad y = -\frac{3}{4}(x - 2)(x + 2)$
OR $4y = -3(x - 2)(x + 2)$

17 B Perpendicular lines have gradients which multiply to -1
$\therefore\ -3m = -1,\ \therefore m = \frac{1}{3}$

18 D
$$f(x) = x^2 - 1$$
$$\therefore\ f(x - 1) = ((x - 1)^2 - 1)$$
$$= x^2 - 2x + 1 - 1$$
$$= x^2 - 2x$$

19 D $y = x^2 + kx + k + 8$ touches the x-axis. Therefore it is a perfect square and $\Delta = 0$:
$$\Delta = k^2 - 4(k + 8)$$
$$= k^2 - 4k - 32$$
$$= (k - 8)(k + 4)$$
$\Delta = 0$ when $k = -4$ or 8

20 E $P(x) = 3x^3 - 4x - k$
If $P(x)$ is divisible by $x - k$, then $P(k) = 0$: $P(k) = 3k^3 - 4k - k = 0$
$$= 3k^3 - 5k = 0$$
Remainder when $P(x)$ is divided by $x + k$:
$$P(-k) = -3k^3 + 4k - k$$
$$= -3k^3 + 3k$$
$$3k^3 - 5k = 0,\ \therefore\ -3k^3 + 5k = 0$$
$$\therefore\ P(-k) = 0 - 2k$$

21 B TP of $y = a(x - b)^2 + c$ is at (b, c)

22 D $y = 3 + 4x - x^2$
meets $y = k$ only once.
$\therefore\ -x^2 + 4x + (3 - k) = 0$ has $\Delta = 0$:
$$\Delta = 16 + 4(3 - k) = 0$$
$\therefore\ 3 - k = -4,\ \therefore\ k = 7$

23 E Midpoint of $(12,7)$ and $(-3, 5)$ is:
$\left(\frac{12 - 3}{2}, \frac{7 + 5}{2}\right) = \left(\frac{9}{2}, 6\right)$

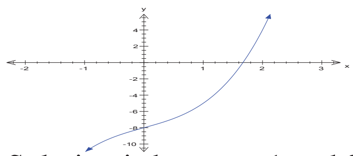
24 D X is at (a, b): $(7, -3) = \left(\frac{5+a}{2}, \frac{4+b}{2}\right)$

$5 + a = 14, \therefore a = 9$

$4 + b = -6, \therefore b = -10$

25 B $y = x^2 + 1$
dom $[-2, 1] \rightarrow$ range $[1, 5]$

26 D $x^3 + 2x - 8 = 0$
Use calculator:



Solution is between 1 and 2.

27 D $f(x) = x(x-2)$

$\therefore\ f(-3) = (-3)(-5) = 15$

28 A $x^2 + y^2 - 11x - 10y + 24 = 0$
Circle cuts the y-axis at M and N so $x = 0$

$$y^2 - 10y + 24 = 0$$
$$\therefore y^2 - 10y + 25 = 1$$
$$\therefore \quad y - 5 = \pm 1$$
$$\therefore \quad y = 4;\ 6$$

Distance between M and N is 2

29 B Distance between $(-4, -3)$ and $(-5, -10)$

$$= \sqrt{(-4-(-5))^2 + (-3-(-10))^2}$$
$$= \sqrt{1 + 49} = 5\sqrt{2}$$

30 D $y = x^2 + 4x - 3$ cuts the line $y = 4 - 2x$ at

$$x^2 + 4x - 3 = 4 - 2x$$
$$\therefore \quad x^2 + 6x - 7 = 0$$
$$\therefore (x+7)(x-1) = 0$$

$x = -7, y = 18$ and $x = 1, y = 2$
Distance between $(-7, 18)$ and $(1,2)$

$$= \sqrt{(-7-1)^2 + (18-2)^2}$$
$$= \sqrt{8^2 + 16^2} = \sqrt{320}$$

31 D $\{(x, y): y \leq 2x + 3\}$

A	$(1, 4)$: $4 < 5$	√
B	$(-1, 1)$: $1 = 1$	√
C	$\left(\frac{1}{2}, 3\frac{1}{2}\right)$: $3\frac{1}{2} < 4$	√
D	$(-\frac{1}{2}, 2\frac{1}{2})$: $2\frac{1}{2} > 2$	**X**
E	$(2, 5)$: $5 < 7$	√

32 B $y = k + 2x - x^2$
If the graph touches the x-axis then $\Delta = 0$:
$\Delta = 4 + 4k = 0, \therefore\ k = -1$

33 C Perpendicular lines have gradients which multiply to -1:

$$kx + y - 4 = 0, \therefore y = 4 - kx$$
$$x - 2y + 3 = 0, \therefore y = \frac{x+3}{2}$$
$$\therefore (-k)\left(\frac{1}{2}\right) = -1, \therefore k = 2$$

34 A $y = x^2 + k$ and $y = x$

$$\therefore \quad x^2 + k = x$$
$$\therefore x^2 - x + k = 0$$

For 1 solution $\Delta = 0$:
$\Delta = 1 - 4k = 0, \therefore k = \frac{1}{4}$

35 C Circle with centre at $(-4, 2)$:
$\therefore (x+4)^2 + (y-2)^2 = r^2$
Circle touches the y-axis so $r = 4$:

$$\therefore \quad x^2 + 8x + 16 + y^2 - 4y + 4 = 16$$
$$\therefore \quad x^2 + 8x + y^2 - 4y + 4 = 0$$

36 D $x \propto \frac{1}{y}$

$y = 5y$

$x \propto \frac{1}{5y}$

$x = \frac{x}{5}$

37 A $A = kb$ and $A = 14$ when $b = 2.4$

$14 = 2.4k$

$k = \frac{14}{2.4}$

$= \frac{140}{24} = \frac{35}{6}$

$A = \frac{35b}{6}$

When $A = 18$,

$18 = \frac{35b}{6}$

$b = \frac{18 \times 6}{35}$

≈ 3.086

38 A $2x - y + 3 = 0$ has gradient $= 2$.
If $ax + 3y - 1 = 0$ is parallel, its gradient $= 2$

$\therefore \quad 3y = 1 - ax$

$\therefore \quad y = \frac{1 - ax}{3}$

$\therefore \quad -\frac{a}{3} = 2, \therefore a = -6$

39 B $f(x) = \sqrt{4 - x^2}$ has max. dom. $[-2, 2]$

40 C $f(x) = 2x^2 + 3x + 4$

$= 2\left(x^2 + \frac{3}{2}x + 2\right)$

$= 2\left(x + \frac{3}{2}x + \frac{9}{16} + \frac{23}{16}\right)$

$= 2\left(x + \frac{3}{2}\right)^2 + \frac{23}{8}$

Range $= [\frac{23}{8}, \infty)$

41 D $P(x) = x^3 - kx^2 - 10kx + 25$

$P(2) = 8 - 4k - 20k + 25 = 9$

$\therefore 24k = 24, \therefore k = 1$

42 E $f(x) = x^2 - 7x + k$

$f(k) = k^2 - 7k + k = -9$

$\therefore k^2 - 6k + 9 = 0$

$\therefore \quad (k - 3)^2 = 0, \therefore k = 3$

$\therefore f(x) = x^2 - 7x + 3$

$\therefore f(-1) = 1 + 7 + 3 = 11$

43 E $2xy - x^2 - y^2$

$= -(x^2 - 2xy + y^2)$

$= -(x - y)^2$

44 C $x^2 - x - 12 \le 0$

$\therefore (x - 4)(x + 3) \le 0$

Upright parabola so $-3 \le x \le 4$

45 C $f(x) = \frac{1}{2}x(x - 1)$

$\therefore f(x) - f(x + 1)$

$= \frac{1}{2}x(x - 1) - \frac{1}{2}x(x + 1)$

$= \frac{x}{2}((x - 1) - (x + 1))$

$= \frac{x}{2}(-2) = -x$

46 C $2x^2 - 2 \le 0$

$\therefore x^2 \le 1, \therefore -1 \le x \le 1$

47 A $f(x) = -2\left(\left(x - \frac{1}{2}\right)^2 - 3\right)$

$= 6 - 2\left(x - \frac{1}{2}\right)^2$

Inverted parabola so max. value = 6

48 C $p = \dfrac{kx}{y^2}$

Set both x and $y = 1$ so that $p = k$.

When x and y are decreased,

$p = \dfrac{k \times 0.7}{0.8^2} = 1.09375\,k$

This has increased by approximately 9.4%.

49 E In the case of the tank, $P = krh$.

When $r = 5$ and $h = 4$, $P = 60$.

$60 = 5 \times 4 \times k$

$k = \dfrac{60}{20} = 3$

When $r = 4$ and $h = 6$,

$P = 3 \times 4 \times 6$

$= \$72$

Solutions to 8C Extended-response questions

1 a $$x^2 + y^2 + bx + cy + d = 0$$

At $(-4, 5)$, $16 + 25 - 4b + 5c + d = 0$

$\therefore \quad 4b - 5c - d = 41 \qquad (1)$

At $(-2, 7)$, $4 + 49 - 2b + 7c + d = 0$

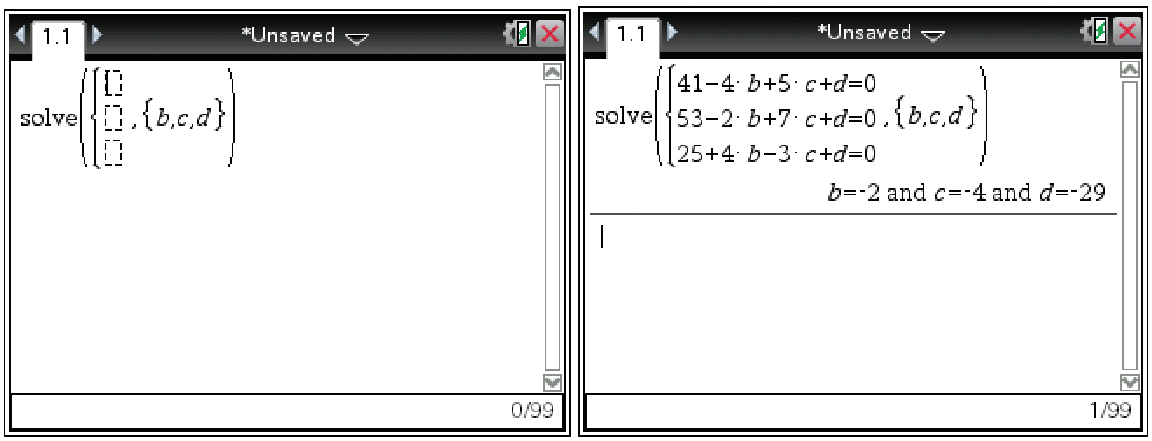
$\therefore \quad 2b - 7c - d = 53 \qquad (2)$

At $(4, -3)$, $16 + 9 + 4b - 3c + d = 0$

$\therefore \quad -4b + 3c - d = 25 \qquad (3)$

b Using a CAS calculator.

Change the number of equations to 3 and change the variables to b, c, d



The calculator yields $b = -2, c = -4$ and $d = -29$

Therefore the equation of the circle is $x^2 + y^2 - 2x - 4y - 29 = 0$

2 a $$x^2 + y^2 + bx + cy = 0 \qquad (1)$$

At $(4, 4)$, $16 + 16 + 4b + 4c = 0$

$\therefore \quad 32 + 4b + 4c = 0$

$\therefore \quad 4c = -4b - 32$

$\therefore \quad c = -b - 8 \qquad (2)$

b To find the x-axis intercept, let $y = 0$ in equation (1),

$\therefore \quad x^2 + bx = 0$

$\therefore \quad x(x + b) = 0$

$\therefore \quad x = 0$ or $x = -b$

c To find the y-axis intercept, let $x = 0$ in equation (1),

$\therefore \quad y^2 + cy = 0$

$\therefore \quad y(y + c) = 0$

$\therefore \quad y(y - b - 8) = 0 \qquad$ from (2)

$\therefore \quad y = 0$ or $y = b + 8$

d The circle touches the y-axis when there is one y-axis intercept, i.e. when $b + 8 = 0$, or $b = -8$.

3 a For $f(x) = \sqrt{a - x}$, the maximal domain is $x \le a$.

b At the point of intersection, $\sqrt{a - x} = x$

$\therefore$ &$a - x = x^2$

$\therefore \quad x^2 + x - a = 0$

Using the general quadratic formula, $x = \dfrac{-1 \pm \sqrt{1 + 4a}}{2}$.

Since the range off (x) is $[0, \infty)$, the point of intersection of the graphs of $y = f(x)$ and $y = x$ is $\left(\dfrac{-1 + \sqrt{1 + 4a}}{2}, \dfrac{-1 + \sqrt{1 + 4a}}{2}\right)$.

c When $\left(\dfrac{-1 + \sqrt{1 + 4a}}{2}, \dfrac{-1 + \sqrt{1 + 4a}}{2}\right) = (1, 1)$,

$$\frac{-1 + \sqrt{1 + 4a}}{2} = 1$$

$\therefore \quad -1 + \sqrt{1 + 4a} = 2$

$\therefore \quad \sqrt{1 + 4a} = 3$

$\therefore \quad 1 + 4a = 9$

$\therefore \quad 4a = 8$

$\therefore \quad a = 2$

d When $\left(\dfrac{-1 + \sqrt{1 + 4a}}{2}, \dfrac{-1 + \sqrt{1 + 4a}}{2}\right) = (2, 2)$,

$$\frac{-1 + \sqrt{1 + 4a}}{2} = 2$$

$\therefore \quad -1 + \sqrt{1 + 4a} = 4$

$\therefore \quad \sqrt{1 + 4a} = 5$

$\therefore \quad 1 + 4a = 25$

$\therefore \quad 4a = 24$

$\therefore \quad a = 6$

e When $\left(\dfrac{-1+\sqrt{1+4a}}{2}, \dfrac{-1+\sqrt{1+4a}}{2}\right) = (c, c)$,

$$\frac{-1+\sqrt{1+4a}}{2} = c$$

$$\therefore \quad -1+\sqrt{1+4a} = 2c$$

$$\therefore \quad \sqrt{1+4a} = 2c+1$$

$$\therefore \quad 1+4a = (2c+1)^2$$

$$\therefore \quad 1+4a = 4c^2+4c+1$$

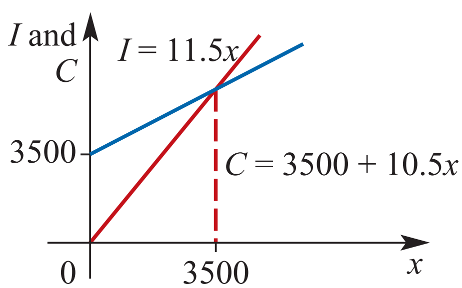
$$\therefore \quad 4a = 4c^2+4c$$

$$\therefore \quad a = c^2+c$$

4 a $C = 3500 + 10.5x$

b $I = 11.5x$

c



d $I = C$

$\therefore \quad 11.5x = 3500 + 10.5x$

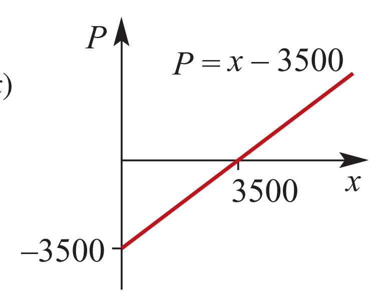
$\therefore \quad x = 3500$

e $P = I - C$

$= 11.5x - (3500 + 10.5x)$

$= x - 3500$

P = profit



f $P = 2000$

$\therefore x - 3500 = 2000 \qquad \therefore \quad x = 5500$

5500 plates must be sold for a profit of \$2000 to be made.

5 a $V = 45\,000 + 40m$

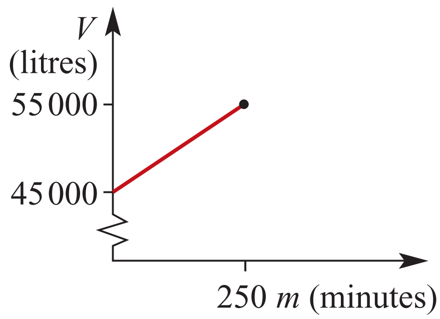
b $45\,000 + 40m = 55\,000$

$\therefore \quad m = \dfrac{10\,000}{40} = 250$

$250\,\text{min} = 4\text{h } 10\,\text{min}$

The pool will reach its maximum capacity after 4 hours 10 minutes.

c



6 a When $t = 10$, $V = 20 \times 10 = 200$ litres.

b For uniform rate, the gradient of the graph is given by the rate.

Hence, $a = 20$

When $t = 10$, $V = 200$ and $b = 15$

Thus $V = bt + c$ gives

$200 = 15 \times 10 + c, \therefore c = 50$

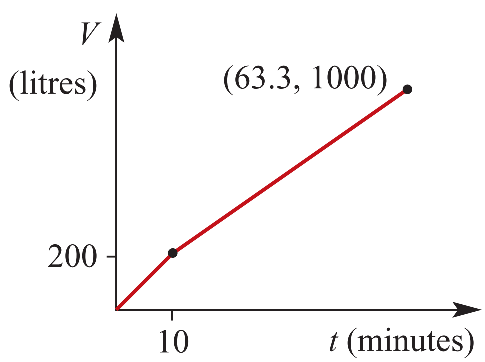
and $V = \begin{cases} 20t & 0 \le t \le 10 \\ 15t + 50 & 10 < t \le \dfrac{190}{3} \end{cases}$

Note: $d = \dfrac{190}{3}$ as $15t + 50 = 1000$

$\Rightarrow \quad 15t = 950$

$\Rightarrow \quad t = \dfrac{190}{3}$

c



7 a For rectangle, length = $3x$ cm, width = $2x$ cm, area = $6x^2$cm^2

b Side length of square $= \frac{1}{4}(42 - 10x)$

$= \frac{1}{2}(21 - 5x)$ cm

Area of square $= \left(\frac{1}{2}(21 - 5x)\right)^2$

$= (10.5 - 2.5x)^2$ cm^2

c $0 \le 10x \le 42$

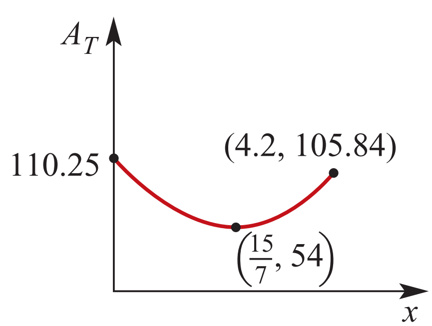
$\therefore \quad 0 \le x \le 4.2$

d
$$
\begin{aligned}
A_T &= 6x^2 + (10.5 - 2.5x)^2 \\
&= 6x^2 + \frac{25}{4}x^2 - \frac{105}{2}x + \frac{441}{4} \\
&= \frac{49}{4}x^2 - \frac{105}{2}x + \frac{441}{4} \\
&= \frac{49}{4}\left(x^2 - \frac{4}{49} \times \frac{105}{2}x + \frac{4}{49} \times \frac{441}{4}\right) \\
&= \frac{49}{4}\left(x^2 - \frac{30}{7}x + \left(\frac{15}{7}\right)^2 - \frac{225}{49} + \frac{441}{49}\right)
\end{aligned}
$$

$$\therefore A_T = \frac{49}{4}\left(x - \frac{15}{7}\right)^2 + \frac{49}{4} \times \frac{216}{49}$$

$$\therefore A_T = \left(\frac{49}{4}\left(x - \frac{15}{7}\right)^2 + 54\right)\text{cm}^2, \text{ or : } A = (12.25x^2 - 52.5x + 110.25)\text{ cm}^2$$

e



f Maximum total area = 110.25 cm^2 (area of rectangle equals zero)

g $$\frac{49}{4}x^2 - \frac{105}{2}x + \frac{441}{4} = 63$$

$$\therefore \quad \frac{49}{4}x^2 - \frac{105}{2}x + \frac{441}{4} - \frac{252}{4} = 0$$

$$\therefore \quad \frac{49}{4}x^2 - \frac{105}{2}x + \frac{189}{4} = 0$$

$$\therefore \quad 49x^2 - 210x + 189 = 0$$

$$\therefore \quad 7(7x^2 - 30x + 27) = 0$$

$$\therefore \quad 7(7x - 9)(x - 3) = 0$$

$$\therefore \quad x = \frac{9}{7} \text{ or } x = 3$$

When $x = \frac{9}{7}$, the rectangle has dimensions $3x = \frac{27}{7} \approx 3.9$ and $2x = \frac{18}{7} \approx 2.6$,
i.e. 3.9 cm × 2.6 cm, and the square has dimensions $\frac{1}{2}\left(21 - 5 \times \frac{9}{7}\right) = \frac{51}{7} \approx 7.3$,
i.e. 7.3 cm ×7.3 cm.
When $x = 3$, the rectangle has dimensions $3x = 9$ and $2x = 6$,
i.e. 9 cm × 6 cm, and the square has dimensions $\frac{1}{2}(21 - 5 \times 3) = 3$,
i.e. 3 cm × 3 cm.

8 $y = -\frac{1}{10}(x + 10)(x - 20),\ x \geq 0$

a When $x = 0$, $y = -\frac{1}{10}(10)(-20)$
$= 20$m, the height at the point of projection.

b When $y = 0$, $x = 20$ m, the horizontal distance travelled, ($x \neq -10$ as $x \geq 0$).

c
$$\begin{aligned} y &= -\frac{1}{10}(x^2 - 10x - 200) \\ &= -\frac{1}{10}(x^2 - 10x + 25 - 225) \\ &= -\frac{1}{10}(x - 5)^2 + 22.5 \end{aligned}$$

When $x = 5, y = 22.5$ m, the maximum height reached by the stone.

9 **a** If height = x cm, width = $(x + 2)$ cm, length = $2(x + 2)$ cm

$$\begin{aligned} A &= 2x(x + 2) + 2x \times 2(x + 2) + 2(x + 2) \times 2(x + 2) \\ &= 2x^2 + 4x + 4x^2 + 8x + 4x^2 + 16x + 16 \\ &= 10x^2 + 28x + 16 \end{aligned}$$

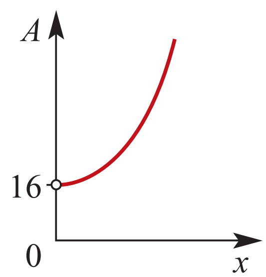
b **i** When $x = 1$,
$$\begin{aligned} A &= 10(1)^2 + 28(1) + 16 \\ &= 10 + 28 + 16 \\ &= 54\,\text{cm}^2 \end{aligned}$$

ii When $x = 2$,
$$\begin{aligned} A &= 10(2)^2 + 28(2) + 16 \\ &= 40 + 56 + 16 \\ &= 112\,\text{cm}^2 \end{aligned}$$

c
$$\begin{aligned} 10x^2 + 28x + 16 &= 190 \\ \therefore \quad 10x^2 + 28x - 174 &= 0 \\ \therefore \quad 2(5x^2 + 14x - 87) &= 0 \\ \therefore \quad (5x + 29)(x - 3) &= 0 \\ \therefore \quad x &= \frac{-29}{5} \text{ or } 3 \end{aligned}$$

But $x > 0$, $\therefore x = 3\text{cm}$

d $A = 10x^2 + 28x + 16$



e
$$\begin{aligned} V &= 2(x + 2)x(x + 2) \\ &= 2x(x + 2)^2 \\ &= 2x(x^2 + 4x + 4) \\ &= 2x^3 + 8x^2 + 8x \end{aligned}$$

f
$$2x^3 + 8x^2 + 8x = 150$$
$$\therefore 2x^3 + 8x^2 + 8x - 150 = 0$$

$$\begin{aligned}
P(0) &= -150 && \neq 0\\
P(1) &= 2(1)^3 + 8(1)^2 + 8(1) - 150 &&\\
&= -132 && \neq 0\\
P(2) &= 2(2)^3 + 8(2)^2 + 8(2) - 150 &&\\
&= 16 + 32 + 16 - 150 &&\\
&= -86 && \neq 0\\
P(3) &= 2(3)^3 + 8(3)^2 + 8(3) - 150 &&\\
&= 54 + 72 + 24 - 150 && = 0
\end{aligned}$$

$\therefore$ $(x - 3)$ is a factor of $2x^3 + 8x^2 + 8x - 150$

When $V = 150$, $x = 3$

$$\begin{array}{r|l}
\multicolumn{2}{r}{2x^2 + 14x + 50} \\ \cline{2-2}
x - 3 & 2x^3 + 8x^2 + 8x - 150 \\
& 2x^3 - 6x^2 \\ \cline{2-2}
& 14x^2 + 8x - 150 \\
& 14x^2 - 42x \\ \cline{2-2}
& 50x - 150 \\
& 50x - 150 \\ \cline{2-2}
& 0
\end{array}$$

$$\begin{aligned}
\therefore \quad 2x^3 + 8x^2 + 8x - 150 &= (x - 3)(2x^2 + 14x + 50)\\
\text{But} \quad 2x^2 + 14x + 50 &\neq 0\\
\text{as} \quad \Delta &= 196 - 400\\
&= -204 < 0\\
\therefore \quad x &= 3
\end{aligned}$$

g The answer can be found using a CAS calculator.
Input $\mathbf{Y_1 = 2X\char`^3 + 8X\char`^2 + 8X}$ and $\mathbf{Y_2 = 1000}$.
The point of intersection is (6.6627798, 1000). Therefore the volume of the block is 1000 cm^3 when $x = 6.66$, correct to 2 decimal places.

10 a i $A = 10y + (y - x)x$

$= 10y + yx - x^2$

ii $P = 2y + 20 + 2x$

$= 2(y + 10 + x)$

b i If $P = 100$

$$100 = 2(y + 10 + x)$$

$$\therefore \quad 50 = y + 10 + x$$

$$\therefore \quad y = 40 - x$$

$$\therefore \quad A = (10 + x)(40 - x) - x^2$$

$$= 400 + 30x - x^2 - x^2$$

$$= 400 + 30x - 2x^2$$

ii

$$A = -2(x^2 - 15x - 200)$$

$$= -2\left(x^2 - 15x + \frac{225}{4} - 200 - \frac{225}{4}\right)$$

$$\therefore \quad A = -2\left(\left(x - \frac{15}{2}\right)^2 - \frac{1025}{4}\right)$$

$$= -2(x - \frac{15}{2})^2 + \frac{1025}{2}$$

$$\therefore \text{ maximum possible area } = \frac{1025}{2}\text{ m}^2$$

$$= 512.5\text{ m}^2$$

iii $A > 0$ and $y > 0$ and $x \geq 0$ and $y - x \geq 0$

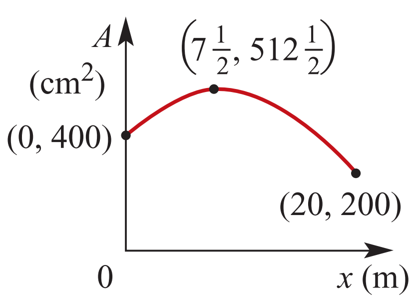
Considering the last inequality, $y \geq x$

$\therefore \quad 40 - x \geq x$

$\therefore \quad x \leq 20$

As $x \geq 0$, the largest possible domain is $0 \leq x \leq 20$.

iv



11 a Let: $A_T(\mathrm{m}^2)$ be the total area of the window.

$$\begin{aligned} A_T &= (2x + y)(3x + 2y) \\ &= 6x^2 + 7xy + 2y^2 \end{aligned}$$

b Let $A_W(\mathrm{m}^2)$ be the total area of the dividing wood.

$$\begin{aligned} A_W &= xy + xy + xy + xy + xy + xy + xy + y^2 + y^2 \\ &= 7xy + 2y^2 \end{aligned}$$

c i Area of glass, $A_G = 1.5$

$$\begin{aligned} \therefore \quad 6x^2 &= 1.5 \\ \therefore \quad x^2 &= \frac{3}{2} \times \frac{1}{6} = \frac{1}{4} \\ \therefore \quad x &= \frac{1}{2} \text{ or } 0.5 \text{ (as } x \geq 0) \end{aligned}$$

ii Area of wood, $A_w = 1$

$$\begin{aligned} \therefore \quad 7xy + 2y^2 &= 1 \\ \text{As } x = \frac{1}{2}, 7 \times \frac{1}{2} \times y + 2y^2 - 1 &= 0 \\ \therefore \quad 2y^2 + \frac{7}{2}y - 1 &= 0 \\ \therefore \quad 4y^2 + 7y - 2 &= 0 \\ \therefore \quad (4y - 1)(y + 2) &= 0 \\ \therefore \quad y &= \frac{1}{4} \text{ or } y = -2 \end{aligned}$$

But $y > 0$, $\quad \therefore y = \frac{1}{4} = 0.25$

12 a

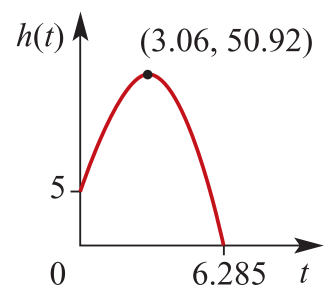
$$\begin{aligned} h(3) &= -4.9(3)^2 + 30(3) + 5 \\ &= -4.9(9) + 90 + 5 \\ &= -44.1 + 95 = 50.9 \end{aligned}$$

The drop will be at a height of 50.9 m after 3 seconds.

b
$$\begin{aligned}
-4.9t^2 + 30t + 5 &= 5 \\
\therefore \quad -4.9t^2 + 30t &= 0 \\
\therefore \quad t(30 - 4.9t) &= 0 \\
\therefore \quad t = 0 \quad \text{or} \quad 30 - 4.9t &= 0 \\
\therefore \quad 4.9t &= 30 \\
\therefore \quad t &\approx 6.12
\end{aligned}$$

The drop will be back at the spout height after approximately 6.12 seconds.

c Turning point at
$$\begin{aligned}
x &= \frac{-b}{2a} \\
&= \frac{-30}{2(-4.9)} = \frac{300}{98} \\
&= \frac{150}{49} \\
&\approx 3.06
\end{aligned}$$



$$\begin{aligned}
h\left(\frac{150}{49}\right) &= -4.9\left(\frac{150}{49}\right)^2 + 30\left(\frac{150}{49}\right) + 5 \\
&= \frac{2495}{49} \approx 50.92
\end{aligned}$$

d When $h(t) = 0$,
$$\begin{aligned}
t &= \frac{-30 \pm \sqrt{(30)^2 - 4(-4.9)(5)}}{2(-4.9)} \\
&= \frac{-30 \pm \sqrt{900 + 98}}{-9.8} \\
&\approx \frac{-30 \pm 31.59}{-9.8} \\
&\approx \frac{-61.59}{-9.8} \text{ or } \frac{1.5.9}{-9.8} \approx 6.285 \text{ or } -0.162
\end{aligned}$$

But as $t \geq 0$ $\quad t = 6.285$

It will take a drop of water 6.285 seconds to hit the ground.

13 **a** height = 7 cm, breadth = x cm, length = $(x + 5)$ cm

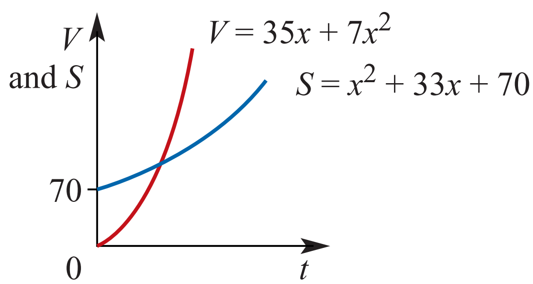
b
$$\begin{aligned}
V &= 7x(x + 5) \\
&= 7x^2 + 35x
\end{aligned}$$

c $S = 7x + 7x + 7(x+5) + 7(x+5) + x(x+5)$

$= 14x + 14x + 70 + x^2 + 5x$

$= x^2 + 33x + 70$

d



e Let $S = V$, $\therefore x^2 + 33x + 70 = 7x^2 + 35x$

$\therefore \quad 6x^2 + 2x - 70 = 0$

$$\therefore \quad x = \frac{-2 \pm \sqrt{4 - 4(6)(-70)}}{12}$$

$$= \frac{-2 \pm \sqrt{1684}}{12}$$

$$= \frac{-2 \pm 41.0366}{12}$$

$$= -3.59, 3.25$$

But $x \geq 0$

$\therefore V = S$ when $x = 3.25$, correct to 2 decimal places.

f Let $S = 500$, $\therefore x^2 + 33x + 70 = 500$

$\therefore \quad x^2 + 33x - 430 = 0$

$\therefore \quad (x - 10)(x + 43) = 0$

$\therefore \quad x = 10$ or $x = -43$

But $x \geq 0$, $\quad \therefore x = 10$

14 a Midpoint of $AC = \left(\frac{1+7}{2}, \frac{3+7}{2}\right) = (4, 5)$

Gradient of $AC = \frac{7-3}{7-1} = \frac{4}{6} = \frac{2}{3}$

Gradient of a line perpendicular to $AC = \frac{-1}{\frac{2}{3}} = \frac{-3}{2}$

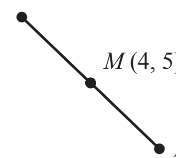
Equation of perpendicular bisector of AC

$$y - 5 = \frac{-3}{2}(x - 4) = \frac{-3x}{2} + 6$$

$\therefore\ 2y + 3x = 22$

b **i** When $x = 0, y = 11$

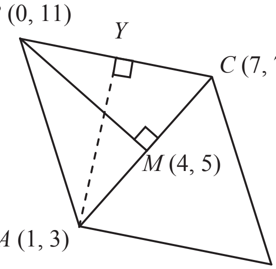
$\therefore$ B has coordinates $(0, 11)$



ii $x = 2 \times 4 - 0 = 8$

$y = 2 \times 5 - 11 = -1$

D has coordinates $(8, -1)$



c Area = length $AC \times$ length BM

$$= \sqrt{6^2 + 4^2} \times \sqrt{4^2 + 6^2}$$

$$= \sqrt{52} \times \sqrt{52}$$

$$= 52 \text{ units}^2$$

d Area $\Delta ABC = \frac{1}{2}$ area of the rhombus

$$= 26 \text{ units}^2$$

Area $\Delta ABC = \frac{1}{2}BC \times AY$

So $\quad AY = \dfrac{2 \times 2b}{BC}$

$$= \frac{52}{\sqrt{7^2 + 4^2}}$$

$$= \frac{52}{\sqrt{65}}$$

$$\approx 6.45$$

The length of AY (the perpendicular distance of A from BC) ≈ 6.45 units.

15 **a** Let x hours be the time for the first journey, $\therefore V = \dfrac{300}{x}$.

The time for the second journey = $(x - 2)$ hours

$$\begin{aligned}
\therefore \quad V + 5 &= \frac{300}{x - 2} \\
\therefore \quad \frac{300}{x} + 5 &= \frac{300}{x - 2} \\
\therefore \quad \left(\frac{300}{x} + 5\right)(x - 2) &= 300 \\
\therefore \quad 300 + 5x - \frac{600}{x} - 10 &= 300 \\
\therefore \quad 5x^2 - 600 - 10x &= 0 \\
\therefore \quad 5x^2 - 10x - 600 &= 0 \\
\therefore \quad 5(x^2 - 2x - 120) &= 0 \\
\therefore \quad (x - 12)(x + 10) &= 0 \\
\therefore \quad x &= 12 \text{ or } x = -10 \\
\text{But } x \geq 0, \quad \therefore x &= 12 \\
\therefore \quad V &= \frac{300}{12} = 25
\end{aligned}$$

The speed of the train travelling at the slower speed is 25 km/h.

b Let t_A minutes be the time it takes tap A to fill the tank, and t_B minutes be the time it takes tap B to fill the tank. $t_B = t_A + 15$
When the taps are running together, it takes $33\frac{1}{3}$ minutes to fill the tank.
Let R_A units/min be the rate of flow of tap A, and R_B units/min be the rate of flow of tap B.

$$\begin{aligned}
\text{Volume to be filled} &= \frac{100}{3}R_A + \frac{100}{3}R_B \\
R_A &= \frac{\text{volume to be filled}}{t_A} \\
R_B &= \frac{\text{volume to be filled}}{t_B}
\end{aligned}$$

Let V be the volume to be filled.

$$V = \frac{100}{3} \times \frac{V}{t_A} + \frac{100}{3} \times \frac{V}{t_B}$$

$$\therefore \quad \frac{3}{100} = \frac{1}{t_A} + \frac{1}{t_A + 15}$$

$$\therefore \quad 3(t_A + 15)t_A = 100(t_A + 15) + 100t_A$$

$$\therefore \quad 3t_A^2 + 45t_A = 200t_A + 1500$$

i.e. $3t_A^2 - 155t_A - 1500 = 0$

$$t_A = \frac{155 \pm \sqrt{42025}}{6}$$

$$\therefore \quad = \frac{155 \pm 205}{6} = 60 \text{ or } \frac{-25}{3}$$

Tap A takes 60 minutes to fill the tank by itself.
Tap B takes 75 minutes to fill the tank by itself.

c Let x cm be the length of a side of a square tile.
Let A be the floor area of the hall.

Then $\quad A = 200x^2$

and $\quad A = 128(x + 1)^2$

$$\therefore \quad 200x^2 = 128x^2 + 256x + 128$$

$$\therefore \quad 72x^2 - 256x - 128 = 0$$

$$\therefore \quad 8(9x^2 - 32x - 16) = 0$$

$$\therefore \quad 8(9x + 4)(x - 4) = 0$$

$$\therefore \quad x = \frac{-4}{9} \text{ or } x = 4$$

But $x \geq 0$, $\quad \therefore x = 4$

The smaller tiles are $(4 \times 4)\,\text{cm}^2$ and the larger tiles are $(5 \times 5)\,\text{cm}^2$.

16 a $\quad 4(x + 2x + h) = 400$

$$\therefore \quad 3x + h = 100$$

$$\therefore \quad h = 100 - 3x$$

b $V = x \times 2x \times h$

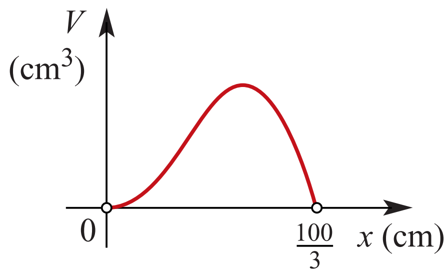
$\quad = 2x^2(100 - 3x)$

c When $V = 0$, $2x^2(100 - 3x) = 0$

$\therefore \quad x = 0$ or $x = \dfrac{100}{3}$

Now $V > 0$, $\quad \therefore 0 < x < \dfrac{100}{3}$

d



e **i** On a CAS calculator, set ***f*1=2x^2(100-3x)** and ***f*2=30 000**. The points of intersection are (18.142, 30 000) and (25.852, 30 000), correct to 3 decimal places. Thus volume is 30 000 cm^3 when $x = 18.142$ or $x = 25.852$, correct to 3 decimal places.

ii Repeat **e i**, using ***f*2 = 20 000**. Volume is 20 000 cm^3 when $x = 12.715$ or $x = 29.504$, correct to 3 decimal places.

f $V_{max} = 32\,921.811$ cm^3 when $x = 22.222$

g **i**
$$\begin{aligned} S &= 2(x \times 2x + x \times h + 2x \times h) \\ &= 2(2x^2 + x(100 - 3x) + 2x(100 - 3x)) \\ &= 2(2x^2 + 100x - 3x^2 + 200x - 6x^2) \\ &= 2(300x - 7x^2) \\ &= 600x - 14x^2 \end{aligned}$$

ii On a CAS calculator, sketch ***f*1 = 600*x* − 14*x*^2**.
$S_{max} = \dfrac{45000}{7}$ cm^2 when $x = \dfrac{150}{7}$

h Sketch ***f*1=600*x* − 14*x*^2** and ***f*2=2*x*^2(100 − 3*x*)** on a CAS calculator. The points of intersection are (3.068, 1708.802) and (32.599, 4681.642). Therefore $S = V$ when $x \approx 3.068$ or $x \approx 32.599$.

17 **a** Use a CAS calculator This yieldsgives $a = \dfrac{19}{250\,000} = 7.6 \times 10^{-5}$, $b = -\dfrac{69}{2500} = 0.0276$, $c = \dfrac{233}{100} = 2.33$ and $d = 0$.
Therefore the function which passes through the given points is
$y = (7.6 \times 10^{-5})x^3 - 0.0276x^2 + 2.33x$.

b $y = (7.6 \times 10^{-5})x^3 - 0.0276x^2 + 2.33x + 5$

c On a CAS calculator, sketch $\boldsymbol{f1=19/250\,000x^\wedge 3 - 69/2500x^\wedge 2+2.33x}$. The largest deviation from the x-axis is 57.31 metres, perpendicular to the x-axis and correct to 2 decimal places.

18 **a** The equation of BC is $y = \frac{3}{4}x - 4$

b The equation of AD is $y - 6 = -\frac{4}{3}(x - 5) = -\frac{4}{3}x + \frac{20}{3}$

$\therefore \quad y = -\frac{4}{3}x + \frac{38}{3}$ or $3y + 4x = 38$

c D is on the lines $y = \frac{3}{4}x - 4$ and $3y + 4x = 38$

Substituting $y = \frac{3}{4}x - 4$ into $3y + 4x = 38$ gives

$$3\left(\frac{3}{4}x - 4\right) + 4x = 38$$

$$\therefore \quad \frac{9}{4}x - 12 + 4x = 38$$

$$\therefore \quad \frac{25}{4}x = 50$$

$$\therefore \quad 25x = 200$$

$$\therefore \quad x = 8$$

$$\therefore \quad y = \frac{3}{4}(8) - 4$$

$$= 6 - 4$$

$$= 2$$

The coordinates of D are (8, 2).

d Length of $AD = \sqrt{(8 - 5)^2 + (6 - 2)^2}$

$= \sqrt{25}$

$= 5$units

e Area of $\Delta ABC = 2 \times$ area of ΔABD

$$= 2 \times \frac{1}{2} \times BD \times AD$$

$$= \sqrt{(8-0)^2 + (2-(-4))^2} \times 5$$

$$= 5\sqrt{64 + 36}$$

$$= 5\sqrt{100}$$

$= 50$ square units

19 a $P \propto mh$

$\therefore \quad P = kmh$ for a constant $k \in R \setminus \{0\} P = 5kh$

When $m = 5,\ P = 5kh$

$$\therefore \quad k = \frac{P}{5h}$$

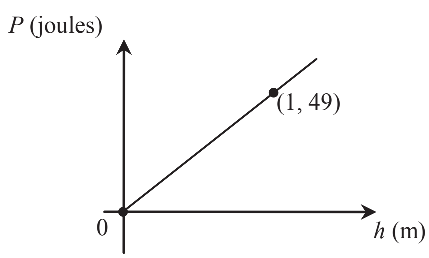
i When $P = 980,\ h = 20,$

$$k = \frac{980}{5 \times 20}$$

$$= 9.8$$

$$\therefore \quad P = 5 \times 9.8h$$

$$= 49h$$

ii 

iii When $h = 23.2,$

$P = 49 \times 23.2$

$= 1136.8$

b **i** $P = kmh$

$$\therefore \quad k = \frac{P}{mh}$$

When $P = 980,\ h = 20,\ m = 5,$

$$k = \frac{980}{5 \times 20}$$

$$\therefore \quad = 9.8$$

$$P = 9.8mh$$

ii Let $P_1 = 9.8mh$,

$$\therefore \quad P_2 = 9.8m \times (2h)$$

$$= 19.6mh$$

$$= 2P_1$$

Percentage change in potential energy $= \dfrac{P_2 - P_1}{P_1} \times 100$

$$= \frac{2P_1 - P_1}{P_1} \times 100$$

$$= 100$$

The potential energy has increased by 100%.

iii Let $P_1 = 9.8mh$

$$\therefore \quad P_2 = 9.8 \times 2m \times \frac{1}{4}h$$

$$= 4.9mh$$

$$= \frac{1}{2}P_1$$

Percentage change in potential energy $= \dfrac{P_2 - P_1}{P_1} \times 100$

$$= \frac{\frac{1}{2}P_1 - P_1}{P_1} \times 100$$

$$= -50$$

The potential energy has decreased by 50%.

c **i** When $h = 10$,

$$V = \sqrt{19.6 \times 10}$$

$$= 14$$

ii When $h = 90$,

$$V = \sqrt{19.6 \times 90}$$

$$= 42$$

d Let $V_1 = \sqrt{19.6h_1}$

$$\begin{aligned}\therefore V_2 &= 2V_1\\ &= 2\sqrt{19.6h_1}\\ &= \sqrt{19.6 \times 4h_1}\\ &= \sqrt{19.6h_2} \text{ where } h_2 = 4h_1\end{aligned}$$

The height must be increased by a factor of 4.

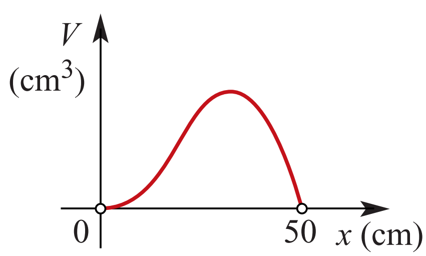
20 a i

$$\begin{aligned}2y + 6x + 4x &= 500\\ \therefore \quad y + 5x &= 250\\ \therefore \quad y &= 5(50 - x)\end{aligned}$$

ii

$$\begin{aligned}V &= x \times x \times y\\ &= x^2 \times 5(50 - x)\\ &= 5x^2(50 - x)\end{aligned}$$

b



c Domain = $(0, 50)$

d Sketch ***f*1=5*x*^2(50 – *x*)** and ***f*2=25 000** on a CAS calculator. The points of intersection are (11.378 052, 25 000) and (47.812 838, 25 000).
Therefore $V = 25000$ for $x = 11.38$ and $x = 47.81$, correct to 2 decimal places.

e Use a CAS calculator to yield the coordinates (33.333 331, 92 592.593). Therefore the maximum volume is 92 592.59 cm^3 when $x = 33.33$, correct to 2 decimal places.
When $x = 33.333\ldots$, $y = 5(50 - 33.333\ldots) \approx 83.33$.

21 a i $A \propto a^3$

$\therefore \quad A = ka^3 \qquad$ for some $k \in R \setminus \{0\}$

$$\therefore \quad k = \frac{A}{a^3}$$

$$= \frac{4}{3} \div 2^3 \qquad \text{when } A = \frac{4}{3},\ a = 2$$

$$= \frac{4}{3} \times \frac{1}{8}$$

$$= \frac{1}{6}$$

$$\therefore \quad A = \frac{a^3}{6}$$

ii When $a = 3,\ A = \dfrac{3^3}{6}$

$= 4.5$

iii $a = \sqrt[3]{6A}$

When $A = 4500,\ a = \sqrt[3]{6 \times 4500}$

$= 30$