Chapter 7 – Polynomials

Solutions to Exercise 7A

1 $P(x) = x^3 - 3x^2 - 2x + 1$

a $P(1) = 1 - 3 - 2 + 1 = -3$

b $P(-1) = (-1)^3 - 3(-1)^2 - 2(-1) + 1$
$= -1 - 3 + 2 + 1$
$= -1$

c $P(2) = (2)^3 - 3(2)^2 - 2(2) + 1$
$= 8 - 12 - 4 + 1$
$= -7$

d $P(-2) = (-2)^3 - 3(-2)^2 - 2(-2) + 1$
$= -8 - 12 + 4 + 1$
$= -15$

2 $P(x) = 8x^3 - 4x^2 - 2x + 1$

a $P(\frac{1}{2}) = 8(\frac{1}{2})^3 - 4(\frac{1}{2})^2 - 2(\frac{1}{2}) + 1$
$= 8 \times \frac{1}{8} - 4 \times \frac{1}{4} - 2 \times \frac{1}{2} + 1$
$= 0$

b
$P(-\frac{1}{2}) = 8(-\frac{1}{2})^3 - 4(-\frac{1}{2})^2 - 2(-\frac{1}{2}) + 1$
$= 8 \times (-\frac{1}{8}) - 4 \times \frac{1}{4} + 2 \times \frac{1}{2} + 1$
$= 0$

3 $P(x) = x^3 + 4x^2 - 2x + 6$

a $P(0) = (0)^3 + 4(0)^2 - 2(0) + 6$
$= 6$

b $P(1) = (1)^3 + 4(1)^2 - 2(1) + 6$
$= 9$

c $P(2) = (2)^3 + 4(2)^2 - 2(2) + 6$
$= 26$

d $P(-1) = (-1)^3 + 4(-1)^2 - 2(-1) + 6$
$= -1 + 4 + 2 + 6$
$= 11$

e $P(a) = (a)^3 + 4(a)^2 - 2(a) + 6$
$= a^3 + 4a^2 - 2a + 6$

f $P(2a) = (2a)^3 + 4(2a)^2 - 2(2a) + 6$
$= 8a^3 + 16a^2 - 4a + 6$

4 a $P(x) = x^3 + 5x^2 - ax - 20$ and
$P(2) = 0$
$\therefore 2^3 + 5 \times (2)^2 - 2a - 20 = 0$
$\therefore 8 - 2a = 0$
$\therefore a = 4$

b $P(x) = 2x^3 + ax^2 - 5x - 7$ and
$P(3) = 68$
$\therefore 2 \times 3^3 + a \times (3)^2 - 5 \times 3 - 7 = 68$
$\therefore 9a = 36$
$\therefore a = 4$

c $P(x) = x^4 + x^3 - 2x + c$ and $P(1) = 6$
$\therefore 1 + 1 - 2 + c = 6$
$\therefore c = 6$

d $P(x) = 3x^6 - 5x^3 + ax^2 + bx + 10$ and
$P(-1) = P(2) = 0$

$P(-1) = 0$ imples $a - b = -18 \ldots (1)$
$P(2) = 0$ implies $4a + 2b = -162$ and thus $2a + b = -81 \ldots (2)$
Add equations (1) and (2)
$3a = -99$
Hence $a = -33$
Substitute in (1) to find $b = -15$

e Let $P(x) = x^5 - 3x^4 + ax^3 + bx^2 + 24x - 36$
$P(3) = P(1) = 0$
$P(3)$
$= 3^5 - 3 \times 3^4 + 3^3a + 3^2b + 24 \times 3 - 36$
$= 243 - 243 + 27a + 9b + 72 - 36$
$= 9(3a + b + 4)$
$P(1)$
$= 1^5 - 3 \times 1^4 + 1^3a + 1^2b + 24 \times 1 - 36$
$= 1 - 3 + a + b + 24 - 36$
$= a + b - 14$
We have the simultaneous equations
$3a + b = -4 \ldots (1)$
$a + b = 14 \ldots (2)$
Subtract equation (1) from equation (2)
$2a = -18$
$\therefore a = -9$ and $b = 23$.

5 $f(x) = x^3 - 2x^2 + x, g(x) = 2 - 3x$ and $h(x) = x^2 + x$

a $f(x) + g(x) = x^3 - 2x^2 + x + 2 - 3x$
$= x^3 - 2x^2 - 2x + 2$

b $f(x) + h(x) = x^3 - 2x^2 + x + x^2 + x$
$= x^3 - x^2 + 2x$

c $f(x) - g(x) = x^3 - 2x^2 + x - (2 - 3x)$
$= x^3 - 2x^2 + 4x - 2$

d $3f(x) = 3(x^3 - 2x^2 + x)$
$= 3x^3 - 6x^2 + 3x$

e
$f(x)g(x) = (x^3 - 2x^2 + x)(2 - 3x$
$= 2(x^3 - 2x^2 + x) - 3x(x^3 - 2x^2 + x)$
$= -3x^4 + 8x^3 - 7x^2 + 2x$

f $g(x)h(x) = (2 - 3x)(x^2 + x)$
$= 2(x^2 + x) - 3x(x^2 + x)$
$= -3x^3 - x^2 + 2x$

g
$f(x) + g(x) + h(x) = x^3 - 2x^2 + x + 2 - 3x + x^2 + x$
$= x^3 - x^2 - x + 2$

h
$f(x)h(x) = (x^3 - 2x^2 + x)(x^2 + x)$
$= x^3(x^2 + x) - 2x^2(x^2 + x) + x(x^2 + x)$
$= x^5 - x^4 - x^3 + x^2$

6 a $(x - 2)(x^2 - 2x + 3)$
$= x(x^2 - 2x + 3) - 2(x^2 - 2x + 3)$
$= x^3 - 2x^2 + 3x - 2x^2 + 4x - 6$
$= x^3 - 4x^2 + 7x - 6$

b $(x - 4)(x^2 - 2x + 3)$
$= x(x^2 - 2x + 3) - 4(x^2 - 2x + 3)$
$= x^3 - 2x^2 + 3x - 4x^2 + 8x - 12$
$= x^3 - 6x^2 + 11x - 12$

c $(x - 1)(2x^2 - 3x - 4)$
$= x(2x^2 - 3x - 4) - 1(2x^2 - 3x - 4)$
$= 2x^3 - 3x^2 - 4x - 2x^2 + 3x + 4$
$= 2x^3 - 5x^2 - x + 4$

d $(x-2)(x^2+bx+c)$

$= x(x^2+bx+c) - 2(x^2+bx+c)$

$= x^3 + bx^2 + cx - 2x^2 - 2bx - 2c$

$= x^3 + (b-2)x^2 + (c-2b)x - 2c$

e $(2x+1)(x^2-4x-3)$

$= 2x(x^2-4x-3) + (x^2-4x-3)$

$= 2x^3 - 8x^2 - 6x + x^2 - 4x - 3$

$= 2x^3 - 7x^2 - 10x - 3$

7 a $(x+1)(x^2+bx+c)$

$= x(x^2+bx+c) + (x^2+bx+c)$

$= x^3 + bx^2 + cx + x^2 + bx + c$

$= x^3 + (b+1)x^2 + (c+b)x + c$

b Equating coefficients
$b+1 = -7$ (coefficients of x^2)
$\therefore b = -8$
Note that $c = 12$. Also as a check
note that: $c + b = 4$ (coefficients of x)
$\therefore c = 12$

c $x^3 - 7x^2 + 4x + 12$

$= (x+1)(x^2 - 8x + 12)$

$= (x+1)(x-6)(x-2)$

8 $x^2 + 6x - 2 = (x-b)^2 + c$

$= x^2 - 2bx + b^2 + c$
Equating coefficients
$-2b = 6$ and $b^2 + c = -2$.
$\therefore b = -3$ and $c = -11$.

9 a We know that
$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
$(a+b)^5 = (a+b)(a+b)^4$

$= a(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)$

$+ b(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)$

$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

b

$(a+b)^6 = (a+b)(a+b)^5$

$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

10 We know that
$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

a Let $a = x$ and $b = -y$
$(x-y)^4 = (x+(-y))^4$

$= x^4 + 4x^3(-y) + 6x^2(-y)^2 + 4x(-y)^3 + (-y)^4$

$= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

b Let $a = 2x$ and $b = y$
$(2x+y)^4$

$= (2x)^4 + 4(2x)^3y + 6(2x)^2y^2 + 4(2x)y^3 + y^4$

$= 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$

Solutions to Exercise 7B

1 a

$$\begin{array}{r|l} & x^2 + 2x + \dfrac{3}{x-1} \\ \hline x-1 & x^3 + x^2 - 2x + 3 \\ & \underline{x^3 - x^2} \\ & \quad 2x^2 - 2x \\ & \quad \underline{2x^2 - 2x} \\ & \qquad\qquad 0 \end{array}$$

b

$$\begin{array}{r|l} & 2x^2 - x - 3 + \dfrac{6}{x+1} \\ \hline x+1 & 2x^3 + x^2 - 4x + 3 \\ & \underline{2x^3 + 2x^2} \\ & \quad -x^2 - 4x \\ & \quad \underline{-x^2 - x} \\ & \qquad -3x + 3 \\ & \qquad \underline{-3x - 3} \\ & \qquad\qquad 6 \end{array}$$

c

$$\begin{array}{r|l} & 3x^2 - 10x + 22 - \dfrac{43}{x+2} \\ \hline x+2 & 3x^3 - 4x^2 + 2x + 1 \\ & \underline{3x^3 + 6x^2} \\ & \quad -10x^2 + 2x \\ & \quad \underline{-10x^2 - 20x} \\ & \qquad 22x + 1 \\ & \qquad \underline{22x + 44} \\ & \qquad\qquad -43 \end{array}$$

d

$$\begin{array}{r|l} & 2x^2 + 3x + 10 + \dfrac{28}{x-3} \\ \hline x-3 & 2x^3 - 3x^2 + x - 2 \\ & \underline{2x^3 - 6x^2} \\ & \quad 3x^2 + x \\ & \quad \underline{3x^2 - 9x} \\ & \qquad 10x - 2 \\ & \qquad \underline{10x - 30} \\ & \qquad\qquad 28 \end{array}$$

2 a

$$\begin{array}{r|l} & x^2 - x + 4 - \dfrac{8}{x+1} \\ \hline x+1 & x^3 + 0x^2 + 3x - 4 \\ & \underline{x^3 + x^2} \\ & \quad -x^2 + 3x \\ & \quad \underline{-x^2 - x} \\ & \qquad 4x - 4 \\ & \qquad \underline{4x + 4} \\ & \qquad\qquad -8 \end{array}$$

b

$$\begin{array}{r|l} & 2x^2 - 8x + 49 - \dfrac{181}{x+4} \\ \hline x+4 & 2x^3 + 0x^2 + 17x + 15 \\ & \underline{2x^3 + 8x^2} \\ & \quad -8x^2 + 17x \\ & \quad \underline{-8x^2 - 32x} \\ & \qquad 49x + 15 \\ & \qquad \underline{49x + 196} \\ & \qquad\qquad -181 \end{array}$$

c
$$\begin{array}{rl} & x^2+x-3+\dfrac{11}{x+3} \\ x+3 & \overline{)\,x^3+4x^2+0x+2} \\ & \underline{x^3+3x^2} \\ & \quad x^2+0x \\ & \quad \underline{x^2+3x} \\ & \qquad\quad -3x+2 \\ & \qquad\quad \underline{-3x-9} \\ & \qquad\qquad\quad 11 \end{array}$$

d
$$\begin{array}{rl} & x^2-x+4+\dfrac{8}{x-2} \\ x-2 & \overline{)\,x^3-3x^2+6x+0} \\ & \underline{x^3-2x^2} \\ & \quad -x^2+6x \\ & \quad \underline{-x^2+2x} \\ & \qquad\quad 4x+0 \\ & \qquad\quad \underline{4x-8} \\ & \qquad\qquad\quad 8 \end{array}$$

3 a
$$\begin{array}{rl} & x^2-2x+5 \\ x+1 & \overline{)\,x^3-x^2+3x+5} \\ & \underline{x^3+x^2} \\ & \quad -2x^2+3x \\ & \quad \underline{-2x^2-2x} \\ & \qquad\quad 5x+5 \\ & \qquad\quad \underline{5x+5} \\ & \qquad\qquad\quad 0 \end{array}$$

b
$$\begin{array}{rl} & 2x^2-2x-6 \\ x+4 & \overline{)\,2x^3+6x^2-14x-24} \\ & \underline{2x^3+8x^2} \\ & \quad -2x^2-14x \\ & \quad \underline{-2x^2-8x} \\ & \qquad\quad -6x-24 \\ & \qquad\quad \underline{-6x-24} \\ & \qquad\qquad\quad 0 \end{array}$$

c
$$\begin{array}{rl} & x^2-2x-6 \\ x-3 & \overline{)\,x^3-5x^2+0x+18} \\ & \underline{x^3-3x^2} \\ & \quad -2x^2+0x \\ & \quad \underline{-2x^2+6x} \\ & \qquad\quad -6x+18 \\ & \qquad\quad \underline{-6x+18} \\ & \qquad\qquad\quad 0 \end{array}$$

d
$$\begin{array}{rl} & 3x^2-x-6 \\ x-2 & \overline{)\,3x^3-7x^2-4x+12} \\ & \underline{-x^3-6x^2} \\ & \quad -x^2-4x \\ & \quad \underline{-x^2+2x} \\ & \qquad\quad -6x+12 \\ & \qquad\quad \underline{-6x+12} \\ & \qquad\qquad\quad 0 \end{array}$$

4 a
$$\begin{array}{rl} & x^2+0x-3 \\ x+2 & \overline{)\,x^3+2x^2-3x+1} \\ & \underline{x^3+2x^2} \\ & \quad 0x^2-3x \\ & \quad \underline{0x^2+0x} \\ & \qquad\quad -3x+1 \\ & \qquad\quad \underline{-3x-6} \\ & \qquad\qquad\quad 7 \end{array}$$

Quotient = x^2-3, Remainder = 7

b
$$\begin{array}{rl} & x^2+2x+15 \\ x-5 & \overline{)\,x^3-3x^2+5x-4} \\ & \underline{x^3-5x^2} \\ & \quad 2x^2+5x \\ & \quad \underline{2x^2-10x} \\ & \qquad\quad 15x-4 \\ & \qquad\quad \underline{15x-75} \\ & \qquad\qquad\quad 71 \end{array}$$

Quotient = $x^2+2x+15$,

Remainder = 71

c
$$\begin{array}{r}
2x^2 - 3x + 0 \\
x+1\overline{)\,2x^3 - x^2 - 3x - 7} \\
\underline{2x^3 + 2x^2} \\
-3x^2 - 3x \\
\underline{-3x^2 - 3x} \\
0x - 7 \\
\underline{0x + 0} \\
-7
\end{array}$$

Quotient = $2x^2 - 3x$,
Remainder = -7

d
$$\begin{array}{r}
5x^2 + 20x + 77 \\
x-4\overline{)\,5x^3 + 0x^2 - 3x + 7} \\
\underline{5x^3 - 20x^2} \\
20x^2 - 3x \\
\underline{20x^2 - 80x} \\
77x + 7 \\
\underline{77x - 308} \\
315
\end{array}$$

Quotient = $5x^2 + 20x + 77$,
Remainder = 315

5 a
$$\begin{array}{r}
\frac{1}{2}x^2 + \frac{7}{4}x - \frac{3}{8} + \frac{103}{8(2x+5)} \\
2x+5\overline{)\,x^3 + 6x^2 + 8x + 11} \\
\underline{2x^3 + \frac{5}{2}x^2} \\
\frac{7}{2}x^2 + 8x \\
\underline{\frac{7}{2}x^2 + \frac{35}{4}x} \\
-\frac{3}{4}x + 11 \\
\underline{-\frac{3}{8}x - \frac{15}{8}} \\
\frac{103}{8}
\end{array}$$

b
$$\begin{array}{r}
x^2 + 2x - 3 - \frac{2}{2x+1} \\
2x+1\overline{)\,2x^3 + 5x^2 - 4x - 5} \\
\underline{2x^3 + x^2} \\
4x^2 - 4x \\
\underline{4x^2 + 2x} \\
-6x - 5 \\
\underline{-6x - 3} \\
-2
\end{array}$$

c
$$\begin{array}{r}
x^2 + 2x - 15 \\
2x-1\overline{)\,2x^3 + 3x^2 - 32x + 15} \\
\underline{2x^3 - x^2} \\
4x^2 - 32x \\
\underline{4x^2 - 2x} \\
-30x + 15 \\
\underline{-30x + 15} \\
0
\end{array}$$

d
$$\begin{array}{r}
\frac{1}{3}x^2 - \frac{8}{9}x - \frac{8}{27} + \frac{19}{27(3x-1)} \\
3x-1\overline{)\,x^3 - 3x^2 + 0x + 1} \\
\underline{x^3 - \frac{1}{3}x^2} \\
-\frac{8}{3}x^2 + 0x \\
\underline{-\frac{8}{3}x^2 + \frac{8}{9}x} \\
-\frac{8}{9}x + 1 \\
\underline{-\frac{8}{9}x + \frac{8}{27}} \\
\frac{19}{27}
\end{array}$$

6 a Using equating coefficients.

$$x^3 + 2x^2 + 5x + 1 = (x - 1)(x^2 + 3x + 8) + 9.$$

$$\therefore \frac{x^3 + 2x^2 + 5x + 1}{x - 1} = x^2 + 3x + 8 + \frac{9}{x - 1}$$

$\therefore a = 9.$

b Using equating coefficients.

$$2x^3 - 2x^2 + 5x + 3 = (2x - 1)(x^2 - \frac{x}{2} + \frac{9}{4}) + \frac{21}{4}.$$

$$\therefore \frac{2x^3 - 2x^2 + 5x + 3}{2x - 1} = x^2 - \frac{x}{2} + \frac{9}{4} + \frac{21}{4(2x - 1)}$$

$\therefore a = \dfrac{21}{4}.$

7 a
$$\begin{array}{r}
2x-6 \\
x^2+0x-2\,\overline{\big)\,2x^3-6x^2-4x+12} \\
\underline{2x^3+0x^2-4x} \\
-6x^2+0x+12 \\
\underline{-6x^2+0x+12} \\
0
\end{array}$$

b
$$\begin{array}{r}
x-6 \\
x^2+0x+1\,\overline{\big)\,x^3-6x^2+x-8} \\
\underline{x^3+0x^2+x} \\
-6x^2+0x-8 \\
\underline{-6x^2+0x-6} \\
-2
\end{array}$$

c
$$\begin{array}{r}
2x-6 \\
x^2+0x-2\,\overline{\big)\,2x^3-6x^2-4x+54} \\
\underline{2x^3+0x^2-4x} \\
-6x^2+0x+54 \\
\underline{-6x^2+0x+12} \\
42
\end{array}$$

d
$$\begin{array}{r}
x^2-4x+2 \\
x^2+2x-1\,\overline{\big)\,x^4-2x^3-7x^2+7x+5} \\
\underline{x^4+2x^3-x^2} \\
-4x^3-6x^2+7x \\
\underline{-4x^3-8x^2+4x} \\
2x^2+3x+5 \\
\underline{2x^2+4x-2} \\
-x+7
\end{array}$$

e
$$\begin{array}{r}
x^2-3x+7 \\
x^2+2x-1\,\overline{\big)\,x^4-x^3+0x^2+7x+2} \\
\underline{x^4+2x^3-x^2} \\
-3x^3+x^2+7x \\
\underline{-3x^3-6x^2+3x} \\
7x^2+4x+2 \\
\underline{7x^2+14x-7} \\
-10x+9
\end{array}$$

f
$$\begin{array}{r}
x^2+x-\tfrac{3}{2} \\
2x^2-x+4\,\overline{\big)\,2x^4+x^3+0x^2+13x+10} \\
\underline{2x^4-x^3+4x^2} \\
2x^3-4x^2+13x \\
\underline{2x^3-x^2+4x} \\
-3x^2+9x+10 \\
\underline{-3x^2+\tfrac{3}{2}x-6} \\
\dfrac{15}{2}x+16
\end{array}$$

Solutions to Exercise 7C

Use the Remainder Theorem.

1 a $P(x) = x^3 - x^2 - 3x + 1$
Divide by $x - 1$: remainder $= P(1)$
$= 1^3 - 1^2 - 3(1) + 1 = -2$

b $P(x) = x^3 - 3x^2 + 4x - 1$
Divide by $x + 2$: remainder $= P(-2)$
$= (-2)^3 - 3(-2)^2 + 4(-2) - 1 = -29$

c $P(x) = 2x^3 - 2x^2 + 3x + 1$
Divide by $x - 2$: remainder $= P(2)$
$= 2(2)^3 - 2(2)^2 + 3(2) + 1 = 15$

d $P(x) = x^3 - 2x + 3$
Divide by $x + 1$: remainder $= P(-1)$
$= (-1)^3 - 2(-1) + 3 = 4$

e $P(x) = x^3 + 2x - 5$
Divide by $x - 2$: remainder $= P(2)$
$= (2)^3 + 2(2) - 5 = 7$

f $P(x) = 2x^3 + 3x^2 + 3x - 2$
Divide by $x + 2$: remainder $= P(-2)$
$= 2(-2)^3 + 3(-2)^2 + 3(-2) - 2 = -12$

g $P(x) = 6 - 5x + 9x^2 + 10x^3$
Divide by $2x + 3$: remainder $= P(-\frac{3}{2})$
$$= 6 - 5\left(-\frac{3}{2}\right) + 9\left(-\frac{3}{2}\right)^2$$
$$+ 10\left(-\frac{3}{2}\right)^3 = 0$$

h $P(x) = 10x^3 - 3x^2 + 4x - 1$
Divide by $2x + 1$: remainder
$$= P\left(-\frac{1}{2}\right)$$
$$= 10\left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2$$
$$+4\left(-\frac{1}{2}\right) - 1 = -5$$

i $P(x) = 108\, x^3 - 27x^2 - 1$
Divide by $3x + 1$: remainder $= P(-\frac{1}{3})$
$$= 108\left(-\frac{1}{3}\right)^3 - 27\left(-\frac{1}{3}\right)^2 - 1 = -8$$

2 a $P(x) = x^3 + ax^2 + 3x - 5$
Remainder -3 when divided by $x - 2$
$\therefore\ P(2) = 8 + 4a + 6 - 5 = -3$
$\therefore\ 4a = -12$
$\therefore\ a = -3$

b $P(x) = x^3 + x^2 - 2ax + a^2$
Remainder 8 when divided by $x - 2$
$\therefore\ P(2) = 8 + 4 - 4a + a = 8$
$\therefore\ a - 4a = -4$
$\therefore\ (a - 2)^2 = 0$
$\therefore\ a = 2$

c $P(x) = x^3 - 3x^2 + ax + 5$
Remainder 17 when divided by $x - 3$
$\therefore\ P(3) = 27 - 27 + 3a + 5 = 17$
$\therefore\ 3a = 12$
$\therefore\ a = 4$

d $P(x) = x^3 + x^2 + ax + 8$
Remainder 0 when divided by $x - 1$
$\therefore\ P(1) = 1 + 1 + a + 8 = 0$
$\therefore\ a = -10$

Use the Factor Theorem.

3 a $P(x) = x^3 - x^2 + x - 1$
$\therefore\ P(1) = 1 - 1 + 1 - 1 = 0$
Therefore $P(x)$ is exactly divisible by $x - 1$

b $P(x) = x^3 + 3x^2 - x - 3$
$\therefore\ P(1) = 1 + 3 - 1 - 3 = 0$

Therefore $P(x)$ is exactly divisible by $x-1$

c $P(x) = 2x^3 - 3x^2 - 11x + 6$

$\therefore\ P(-2) = -16 - 12 + 22 + 6 = 0$

Therefore $P(x)$ is exactly divisible by $x+2$

d $P(x) = 2x^3 - 13x^2 + 27x - 18$

$\therefore\ P\left(\frac{3}{2}\right) = \frac{27}{4} - \frac{117}{4} + \frac{81}{2} - 18 = 0$

Therefore $P(x)$ is exactly divisible by $2x-3$

4 a $P(x) = x^3 - 4x^2 + x + m$

$P(3) = 27 - 36 + 3 + m = 0$

$\therefore\ m = 6$

b $P(x) = 2x^3 - 3x^2 - (m+1)x - 30$

$P(5) = 250 - 75 - 5(m+1) - 30 = 0$

$\therefore\ 5(m+1) = 145$

$\therefore\ m + 1 = 29, \therefore\ m = 28$

c $P(x) = x^3 - (m+1)x^2 - x + 30$

$P(-3) = -27 - 9(m+1) + 3 + 30 = 0$

$\therefore 9(m+1) = 6$

$\therefore\ m + 1 = \frac{2}{3}, \therefore\ m = -\frac{1}{3}$

5 a $2x^3 + x^2 - 2x - 1$

$= x^2(2x+1) - (2x+1)$

$= (2x+1)(x^2-1)$

$= (2x+1)(x+1)(x-1)$

b $x^3 + 3x^2 + 3x + 1$

$= (x+1)^3$

c $P(x) = 6x^3 - 13x^2 + 13x - 6$

$P(1) = 6 - 13 + 13 - 6 = 0$

$(x-1)$ is a factor.

Long division or calculator:

$P(x) = (x-1)(6x^2 - 7x + 6)$

No more factors since $\Delta < 0$ for the quadratic term.

d $P(x) = x^3 - 21x + 20$

$P(1) = 1 - 21 + 20 = 0$

$(x-1)$ is a factor.

Long division or calculator:

$P(x) = (x-1)(x^2 + x - 20)$

$\therefore\ P(x) = (x-1)(x-4)(x+5)$

e $P(x) = 2x^3 + 3x^2 - 1$

$P(-1) = -2 + 3 - 1 = 0$

$(x+1)$ is a factor.

Long division or calculator:

$P(x) = (x+1)(2x^2 + x - 1)$

$\therefore\ P(x) = (x+1)(x+1)(2x-1)$

$= (2x-1)(x+1)^2$

f $P(x) = x^3 - x^2 - x + 1$

$= x^2(x-1) - (x-1)$

$= (x-1)(x^2-1)$

$= (x+1)(x-1)^2$

g $P(x) = 4x^3 + 3x - 38$

$P(2) = 32 + 6 - 38 = 0$

$(x-2)$ is a factor.

Long division or calculator:

$P(x) = (x-2)(4x^2 + 8x + 19)$

No more factors since $\Delta < 0$ for the quadratic term.

h $P(x) = 4x^3 + 4x^2 - 11x - 6$

$P(-2) = -32 + 16 + 22 - 6 = 0$

$(x+2)$ is a factor.

Long division or calculator:

$P(x) = (x+2)(4x^2 - 4x - 3)$

$= (x+2)(2x+1)(2x-3)$

6 Let $P(x) = (1+x)^4$. Then

$P(-2) = (-2)^4 = 1$
The remainder is 1.

7 **a** $P(x) = 2x^3 - 7x^2 + 16x - 15$
Note that $P(x) = 0$ has no integer solutions. Check this by using the factor theorems.
The factor of 2 to be considered is2
The factors of 15 to be considered are $\pm 3, \pm 5, \pm 15, \pm 1$.
The values to check using the factor theorem are $\pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{15}{2}, \pm\frac{1}{2}$
Then using the factor theorem.
$P(\frac{3}{2}) = 0$. No need to try another. We can factorise since we know $2x - 3$ is a factor.
Using the equating coefficients method we find:
$P(x) = (2x - 3)(x^2 - 2x + 5)$

b $P(x) = 2x^3 - 7x^2 + 8x + 5$
Note that $P(x) = 0$ has no integer solutions. Check this by using the factor theorems.
The factor of 2 to be considered is2
The factors of 5 to be considered are $\pm 5, \pm 1$.
The values to check using the factor theorem are $\pm\frac{5}{2}, \pm\frac{1}{2}$
Then using the factor theorem.
$P(\frac{5}{2}) \neq 0$ but $P\left(-\frac{1}{2}\right) = 0$. No need to try another. We can factorise since we know $2x + 1$ is a factor.
Using the equating coefficients method we find:
$P(x) = (2x + 1)(x^2 - 2x + 5)$

c $P(x) = 2x^3 - 3x^2 - 12x - 5$
Note that $P(x) = 0$ has no integer solutions. Check this by using the factor theorems.
The factor of 2 to be considered is2
The factors of -5 to be considered are $\pm 5, \pm 1$.
The values to check using the factor theorem are $\pm\frac{5}{2}, \pm\frac{1}{2}$
Then using the factor theorem.
$P(\frac{5}{2}) \neq 0$ but $P\left(-\frac{1}{2}\right) = 0$. No need to try another. We can factorise since we know $2x + 1$ is a factor.
Using the equating coefficients method we find:
$P(x) = (2x + 1)(x^2 - 2x - 5)$

d $P(x) = 2x^3 - x^2 - 8x - 3$
Note that $P(x) = 0$ has no integer solutions. Check this by using the factor theorems.
The factor of 2 to be considered is2
The factors of -3 to be considered are $\pm 3, \pm 1$.
The values to check using the factor theorem are $\pm\frac{3}{2}, \pm\frac{1}{2}$
Then using the factor theorem.
$P(\frac{1}{2}) \neq 0$ but $P\left(-\frac{3}{2}\right) = 0$. No need to try another. We can factorise since we know $2x + 3$ is a factor.
Using the equating coefficients method we find:
$P(x) = (2x + 3)(x^2 - 2x - 1)$

8 Sum/difference of two cubes:
$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

a $x^3 - 1 = (x - 1)(x^2 + x + 1)$

b $x^3 + 64 = (x + 4)(x^2 - 4x + 16)$

c $27x^3 - 1 = (3x - 1)(9x^2 + 3x + 1)$

d $64x^3 - 125 = (4x - 5)(16x^2 + 20x + 25)$

e $1 - 125x^3 = (1 - 5x)(1 + 5x + 25x^2)$

f $8 + 27x^3 = (2 + 3x)(4 - 6x + 9x^2)$

g $64m^3 - 27n^3 = (4m - 3n)(16m^2 + 12mn + 9n^2)$

h $27b^3 + 8a^3 = (3b + 2a)(9b^2 - 6ab + 4a^2)$

9 **a** $P(x) = x^3 + x^2 - x + 2$
$P(-2) = -8 + 4 + 2 + 2 = 0$
$\therefore\ P(x) = (x + 2)(x^2 - x + 1)$
No more factors since $\Delta < 0$, for the quadratic term.

b $P(x) = 3x^3 - 7x^2 + 4$
$P(1) = 3 - 7 + 4 = 0$
$\therefore\ P(x) = (x - 1)(3x^2 - 4x - 4)$
$= (x - 1)(3x + 2)(x - 2)$

c $P(x) = x^3 - 4x^2 + x + 6$
$P(-1) = -1 - 4 - 1 + 6 = 0$
$\therefore P(x) = (x + 1)(x - 5x + 6)$
$= (x + 1)(x - 2)(x - 3)$

d $P(x) = 6x^3 + 17x^2 - 4x - 3$
$P(-3) = -162 + 153 + 12 - 3 = 0$
$\therefore\ P(x) = (x + 3)(6x^2 - x - 1)$
$= (x + 3)(3x + 1)(2x - 1)$

10 $P(x) = x^3 + ax^2 - x + b$
$P(x)$ is divisible by $x - 1$ and $x + 3$:
$P(1) = 1 + a - 1 + b = 0$
$\therefore a + b = 0$
$P(-3) = -27 + 9a + 3 + b = 0$
$\therefore\ 9a + b = 24$
$a = 3, b = -3$
So $P(x) = x^3 + 3x^2 - x - 3$
$= x^2(x + 3) - (x + 3)$
$= (x + 3)(x^2 - 1)$
$= (x + 3)(x - 1)(x + 1)$

11 **a** $P(x) = x^n - a^n$
$P(a) = a^n - a^n = 0$
By the Factor Theorem, $(x - a)$ is a factor of $P(x)$

b $Q(x) = x^n + a^n$

i If $(x + a)$ is a factor of $Q(x)$, then $Q(-a) = (-a)^n + a^n$,
which is zero if n is an odd number.

ii If $(x + a)$ is a factor of $P(x)$, then $P(-a) = (-a)^n - a^n$,
which is zero if n is an even number.

12 **a** $P(x) = (x - 1)(x - 2)Q(x) + ax + b$

$P(1) = a + b = 2$

$P(2) = 2a + b = 3$
$a = b = 1$

b **i** If $P(x)$ is a cubic with x^3 coefficient $= 1$:
$P(x) = (x - 1)(x - 2)(x + c) + x + 1$
Since -1 is a solution to $P(1) = 0$:

$$P(-1) = (-2)(-3)(-1 + c) - 1 + 1 = 0$$

$$\therefore\ c = 1$$

$$\begin{aligned}P(x) &= (x-1)(x-2)(x+1) + x + 1\\ &= (x+1)((x-1)(x-2)+1)\\ &= (x+1)(x^2 - 3x + 2 + 1)\\ &= (x+1)(x^2 - 3x + 3)\\ &= x^3 - 2x^2 + 3\end{aligned}$$

ii $P(x) = (x + 1)(x^2 - 3x + 3)$ includes a quadratic where $\Delta < 0$, so $x = -1$ is the only real solution to $P(x) = 0$

Solutions to Exercise 7D

1 **a** $(x-1)(x+2)(x-4)=0$

$x=1,-2,4$

b $(x-4)(x-4)(x-6)=0$

$x=4,6$

c $(2x-1)(x-3)(3x+2)=0$

$x=\frac{1}{2},3,-\frac{2}{3}$

d $x(x+3)(2x-5)=0$

$x=0$ or $x=-3$ or $x=\frac{5}{2}$

2 **a** $x^3-2x^2-8x=0$

$\therefore\ x(x^2-2x-8)=0$

$\therefore\ x(x+2)(x-4)=0$

$x=-2,0,4$

b $x^3+2x^2-11x=0$

$\therefore x(x^2+2x-11)=0$

$\therefore x(x+1-2\sqrt{3})(x+1+2\sqrt{3})=0$

$x=0,-1\pm 2\sqrt{3}$

c $x^3-3x^2-40x=0$

$\therefore x(x^2-3x-40)=0$

$\therefore x(x-8)(x+5)=0$

$x=-5,0,8$

d $x^3+2x^2-16x=0$

$\therefore x(x^2+2x-16)=0$

$\therefore x(x+1-\sqrt{17})(x+1+\sqrt{17})=0$

$x=0,-1\pm\sqrt{17}$

3 **a** $x^3-x^2+x-1=0$

$\therefore\ x^2(x-1)+(x-1)=0$

$\therefore\ (x-1)(x^2+1)=0$

$x=1$; no other real solutions since $\Delta<0$ for the quadratic term.

b $x^3+x^2+x+1=0$

$\therefore\ x^2(x+1)+(x+1)=0$

$\therefore\ (x+1)(x^2+1)=0$

$x=-1$; no other real solutions since $\Delta<0$ for the quadratic term.

c $x^3-5x^2-10x+50=0$

$\therefore\ x^2(x-5)-10(x-5)=0$

$\therefore\ (x-5)(x^2-10)=0$

$\therefore\ (x-5)(x-\sqrt{10})(x+\sqrt{10})=0$

$x=5,\pm\sqrt{10}$

d $x^3-ax^2-16x+16a=0$

$\therefore\ x^2(x-a)-16(x-a)=0$

$\therefore\ x^2(x-a)(x-16)=0$

$\therefore\ (x-a)(x-4)(x+4)=0$

$x=a,\pm4$

4 **a** $x^3-19x+30=0$

$P(2)=8-38+30=0$

$\therefore\ P(x)=(x-2)(x^2+2x-15)=0$

$=(x-2)(x-3)(x+5)=0$

$x=-5,2,3$

b $P(x)=3x^3-4x^2-13x-6=0$

$P(-1)=-3-4+13-6=0$

$\therefore\ P(x)=(x+1)(3x^2-7x-6)=0$

$=(x+1)(3x+2)(x-3)=0$

$x=-1,\ -\frac{2}{3},3$

c $x^3-x^2-2x+2=0$

$\therefore\ x^2(x-1)-2(x-1)=0$

$\therefore\ (x-1)(x^2-2)=0$

$\therefore\ (x-1)(x-\sqrt{2})(x+\sqrt{2})=0$

$x=1,\pm\sqrt{2}$

d $P(x)=5x^3+12x^2-36x-16=0$

$P(2) = 40 + 48 - 72 - 16 = 0$
$\therefore\ P(x) = (x-2)(5x \ + 22x + 8) = 0$
$= (x-2)(5x+2)(x+4) = 0$
$x = -4, -\frac{2}{5}, 2$

e $P(x) = 6x^3 - 5x^2 - 2x + 1 = 0$
$P(1) = 6 - 5 - 2 + 1 = 0$
$\therefore\ P(x) = (x-1)(6x^2 + x - 1) = 0$
$= (x-1)(3x-1)(2x+1) = 0$
$x = -\frac{1}{2}, \frac{1}{3}, 1$

f $P(x) = 2x^3 - 3x^2 - 29x - 30 = 0$
$P(-2) = -16 - 12 + 58 - 30 = 0$
$\therefore\ P(x) = (x+2)(2x^2 - 7x - 15) = 0$
$= (x+2)(2x+3)(x-5) = 0$
$x = -2, -\frac{3}{2}, 5$

5 a $P(x) = x^3 + x^2 - 24x + 36 = 0$
$P(2) = 8 + 4 - 48 + 36 = 0$
$\therefore\ P(x) = (x-2)(x + 3x - 18) = 0$
$= (x-2)(x-3)(x+6) = 0$
$x = -6, 2, 3$

b $P(x) = 6x^3 + 13x^2 - 4 = 0$
$P(-2) = -48 + 52 - 4 = 0$
$\therefore\ P(x) = (x+2)(6x + x - 2) = 0$
$= (x+2)(2x-1)(3x+2) = 0$
$x = -2, -\frac{2}{3}, \frac{1}{2}$

c $P(x) = x^3 - x^2 - 2x - 12 = 0$
$P(3) = 27 - 9 - 6 - 12 = 0$
$\therefore\ P(x) = (x-3)(x^2 + 2x + 4) = 0$
$x = 3$; no other real solutions since
$\Delta < 0$ for the quadratic term.

d $P(x) = 2x^3 + 3x^2 + 7x + 6 = 0$
$P(-1) = -2 + 3 - 7 + 6 = 0$
$\therefore\ P(x) = (x+1)(2x^2 + x + 6) = 0$
$x = -1$; no other real solutions since
$\Delta < 0$ for the quadratic term.

e $P(x) = x^3 - x^2 - 5x - 3 = 0$
$P(3) = 27 - 9 - 15 - 3 = 0$
$\therefore\ P(x) = (x-3)(x^2 + 2x + 1) = 0$
$= (x-3)(x+1)^2 = 0$
$x = -1, 3$

f $P(x) = x^3 + x^2 - 11x - 3 = 0$
$P(3) = 27 + 9 - 33 - 3 = 0$
$\therefore\ P(x) = (x-3)(x^2 + 4x + 1) = 0$
$= (x-3)(x+2-\sqrt{3})(x+2+\sqrt{3})$
$= 0$
$x = 3, -2 \pm \sqrt{3}$

6 a $2x^3 = 16x$
$\therefore\ 2x^3 - 16x = 0$
$\therefore\ 2x(x^2 - 8) = 0$
$\therefore\ 2x(x - 2\sqrt{2})(x + 2\sqrt{2}) = 0$
$x = 0, \pm 2\sqrt{2}$

b $2(x-1)^3 = 32$
$\therefore\ (x-1)^3 = 16$
$x - 1 = 2\sqrt[3]{2}$
$x = 1 + 2\sqrt[3]{2}$

c $x^3 + 8 = 0$
$\therefore\ (x+2)(x^2 - 2x + 4)$
$x = -2$; no other real solutions since
$\Delta < 0$ for the quadratic term

d $2x^3 + 250 = 0$
$\therefore\ 2(x^3 + 125) = 0$
$\therefore\ 2(x+5)(x^2 - 5x + 25) = 0$
$x = -5$; no other real solutions since
$\Delta < 0$ for the quadratic term.

e $1000 = \dfrac{1}{x^3}$

$\therefore \quad 1000x^3 = 1$

$\therefore \quad (10x)^3 = 1$

$\therefore \quad 10x = 1, \therefore\ x = \dfrac{1}{10}$

7 **a** $2x^3 - 22x^2 - 250x + 2574$
$= 2(x-9)(x^2 - 2x - 143)$
$= 2(x-9)(x-13)(x+11)$

b $2x^3 + 27x^2 + 52x - 33$
$= (x+3)(2x^2 + 15x - 11)$
$= (x+3)(2x^2 + 21x - 11)$
$= (x+3)(2x-1)(x+11)$

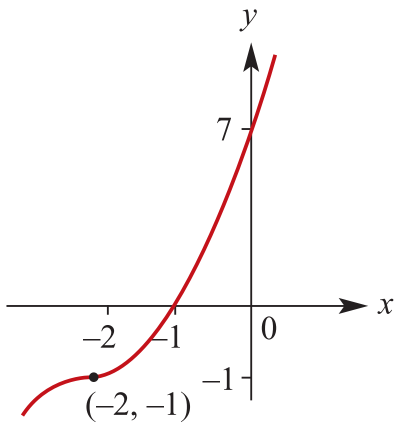
c $2x^3 - 9x^2 - 242x + 1089$
$= (x-11)(2x^2 + \ 13\ x - 99)$
$= (x-11)(2x-9)(x+11)$

d $2x^3 + 51x^2 + 304x - 165$
$= (x+11)(2x^2 + 29x - 15)$
$= (x+11)(2x-1)(x+15)$

Solutions to Exercise 7E

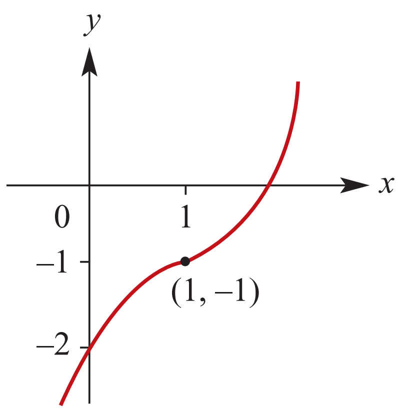
1 **a** $y = (x + 2)^3 - 1$

Stationary point of inflection at $(-2, -1)$



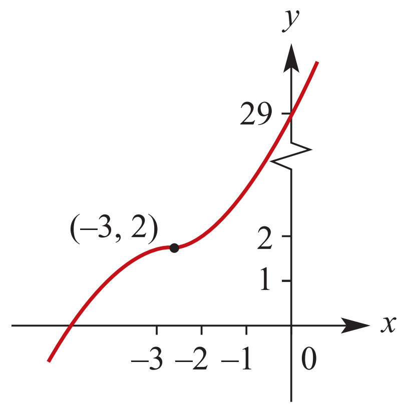
b $y = (x - 1)^3 - 1$

Stationary point of inflection at $(1, -1)$



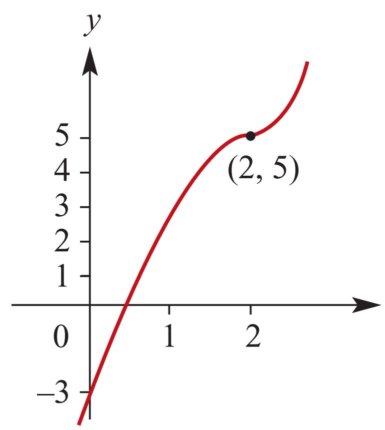
c $y = (x + 3)^3 + 2$

Stationary point of inflection at $(-3, 2)$



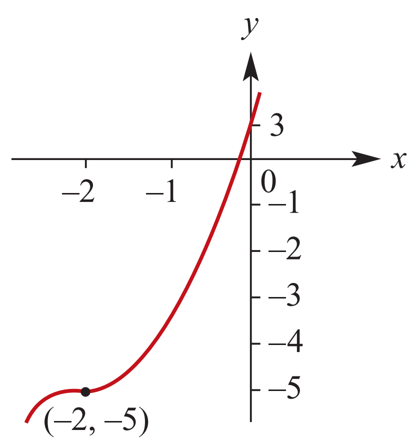
d $y = (x - 2)^3 + 5$

Stationary point of inflection at $(2, 5)$



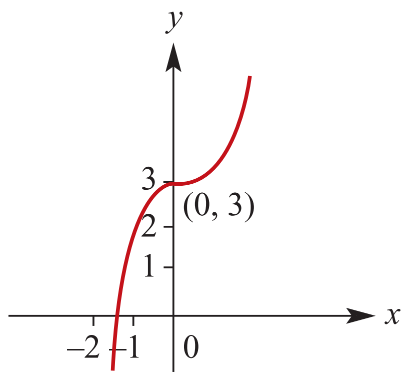
e $y = (x + 2)^3 - 5$

Stationary point of inflection at $(-2, -5)$



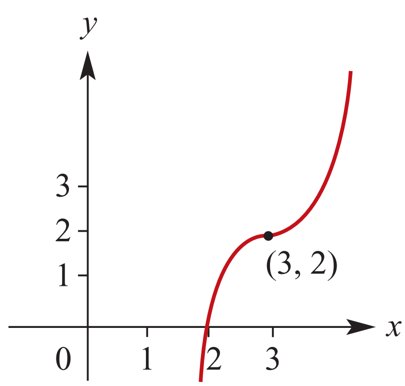
2 **a** $y = 2x^3 + 3$

Stationary point of inflection at $(0, 3)$



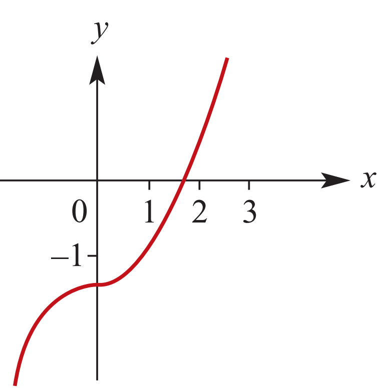
b $y = 2(x - 3)^3 + 2$

Stationary point of inflection at $(3, 2)$



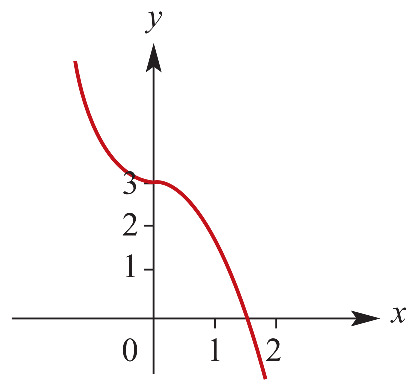
c $3y = x^3 - 5$

Stationary point of inflection at $(0, -\frac{5}{3})$



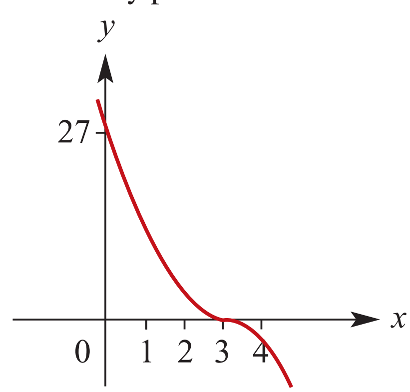
d $y = 3 - x^3$

Stationary point of inflection at $(0, 3)$



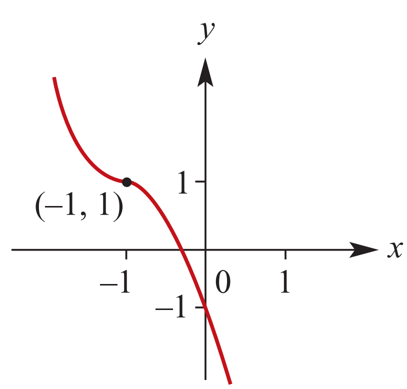
e $y = (3 - x)^3$

Stationary point of inflection at $(3, 0)$



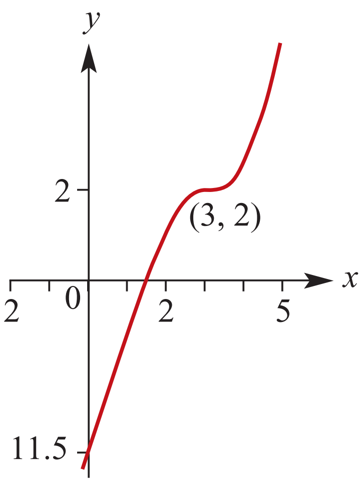
f $y = -2(x + 1)^3 + 1$

Stationary point of inflection at $(-1, 1)$



g $y = \frac{1}{2}(x-3)^3 + 2$

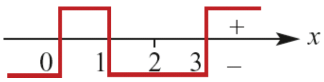
Stationary point of inflection at (3,2)

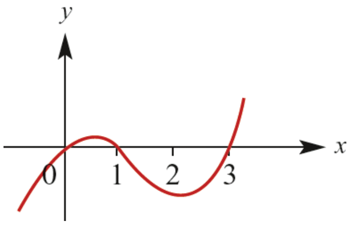


Solutions to Exercise 7F

1 **a** $y = x(x-1)(x-3)$

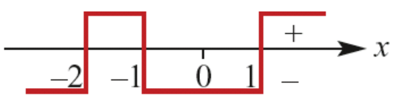
Axis intercepts: $(0,0)$, $(1,0)$ and $(3,0)$

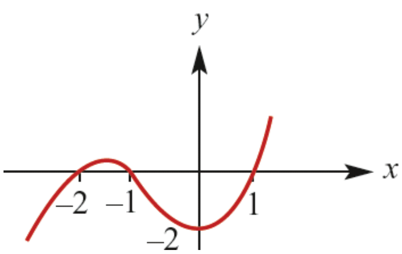




b $y = (x-1)(x+1)(x+2)$

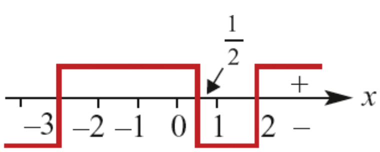
Axis intercepts: $(-2,0)$, $(-1,0)$, $(1,0)$ and $(0,6)$

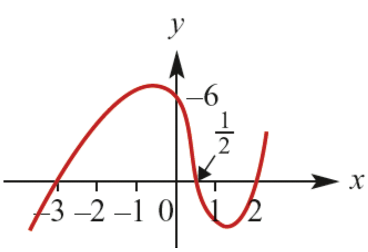




c $y = (2x-1)(x-2)(x+3)$

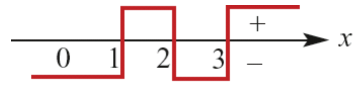
Axis intercepts: $(-3,0)$, $(\frac{1}{2},0)$, $(2,0)$ and $(0,6)$

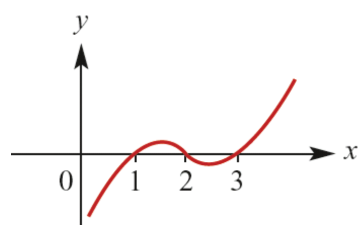




d $y = (x-1)(x-2)(x-3)$

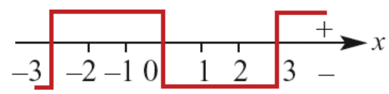
Axis intercepts: $(1,0)$, $(2,0)$, $(3,0)$ and $(0,6)$

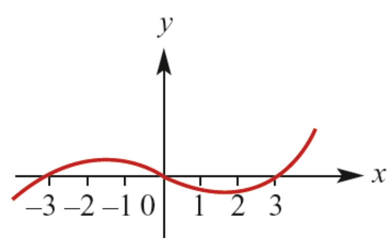




2 **a** $y = x^3 - 9x = x(x-3)(x+3)$

Axis intercepts: $(0,0)$, $(-3,0)$ and $(3,0)$

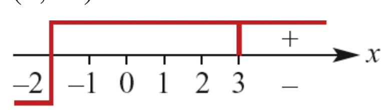


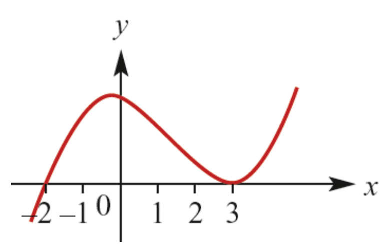


b $y = x^3 - 4x^2 - 3x + 18$

$\therefore y = (x-3)^2(x+2)$

Axis intercepts: $(-2,0)$, $(3,0)$ and $(0,18)$





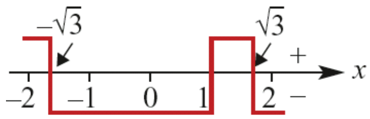
c $y = -x^3 + x^2 + 3x - 3$

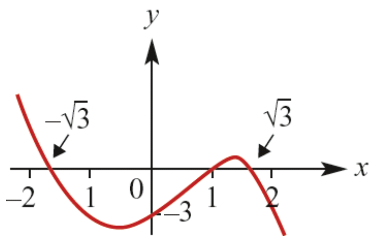
$\therefore\ y = (1-x)(x^2-3)$

$= (1-x)(x-\sqrt{3})(x+\sqrt{3})$

Axis intercepts:

$(1, 0), (-\sqrt{3}, 0), (\sqrt{3}, 0)$ and $(0, -3)$

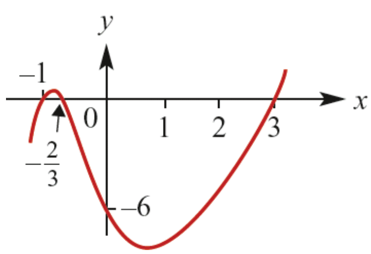




d $y = 3x^3 - 4x^2 - 13x - 6$

$\therefore y = (3x + 2)(x + 1)(x - 3)$

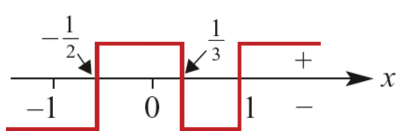
Axis intercepts: $(-1, 0), (-\frac{2}{3}, 0), (3, 0)$ and $(0,1)$

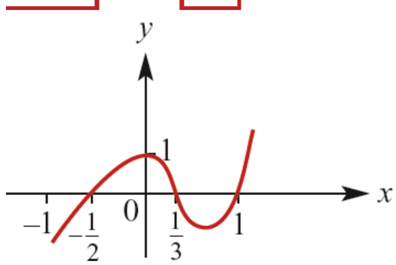


e $y = 6x^3 - 5x^2 - 2x + 1$

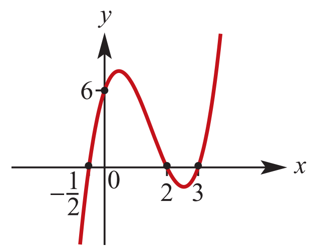
$y = (x - 1)(3x - 1)(2x + 1)$

Axis intercepts: $(-\frac{1}{2}, 0), (\frac{1}{3}, 0), (1, 0)$ and $(0, 1)$

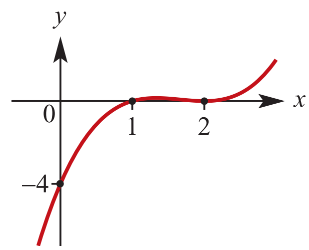




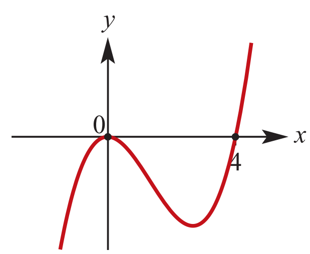
f



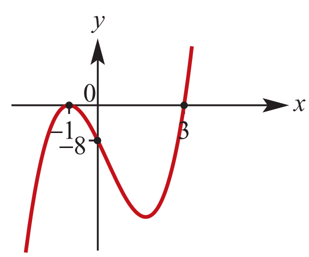
3 a



b

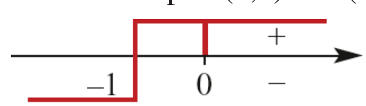


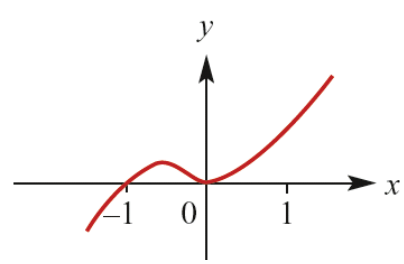
c



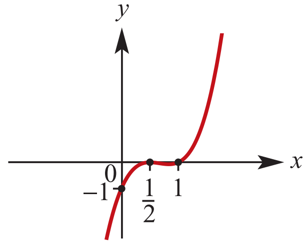
d $y = x^3 + x^2 = x^2(x + 1)$

Axis intercepts: $(0,0)$ and $(-1, 0)$





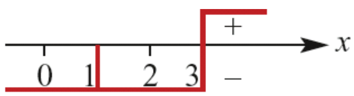
e

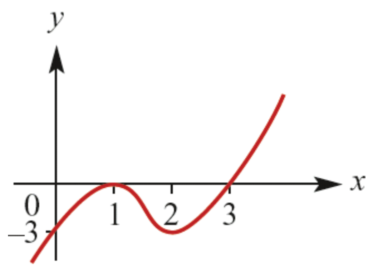


f $y = x^3 - 5x^2 + 7x - 3$

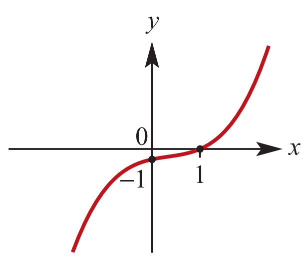
$\therefore\ y = (x-1)^2(x-3)$

Axis intercepts: $(1, 0), (3, 0)$ and $(0, -3)$

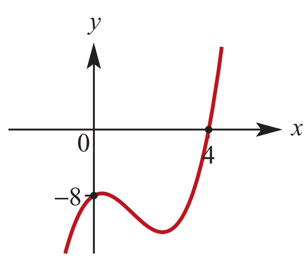




4 a



b

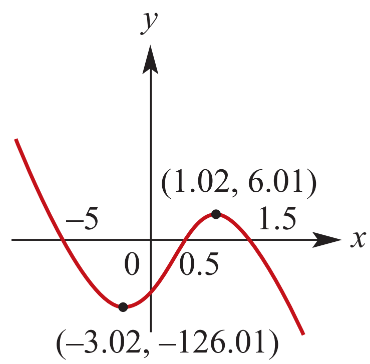


5 a $y = -4x^3 - 12x^2 + 37x - 15$

Intercepts: $(-5, 0)$, $(\frac{1}{2}, 0)$, $(\frac{2}{3}, 0)$ and $(0, -15)$

Max. : $(1.02, 6.01)$

Min. : $(-3.02, -126.01)$

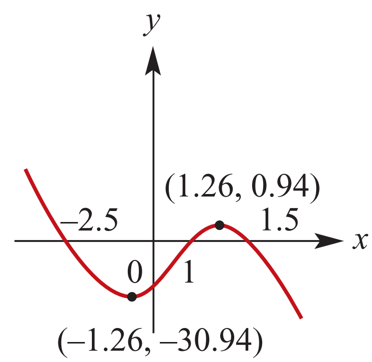


b $y = -4x^3 + 19x - 15$

Intercepts: $(-\frac{5}{2}, 0), (1, 0), (\frac{3}{2}, 0)$ and $(0, -15)$

Max. : $(1.26, 0.94)$

Min. : $(-1.26, -30.94)$

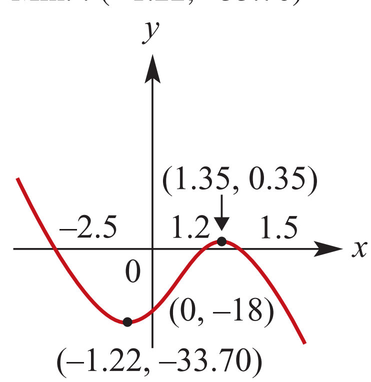


c $y = -4x^3 + 0.8x^2 + 19.8x - 18$

Intercepts: $(-2.5, 0)$, $(1.2, 0)$, $(1.5, 0)$ and $(0, -18)$

Max. : $(1.35, 0.35)$

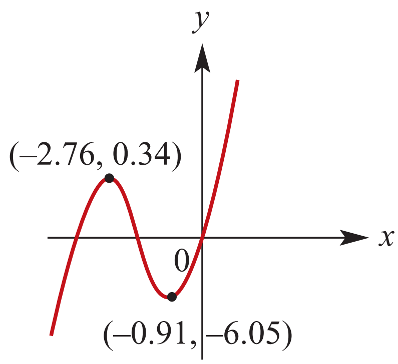
Min. : $(-1.22, -33.70)$



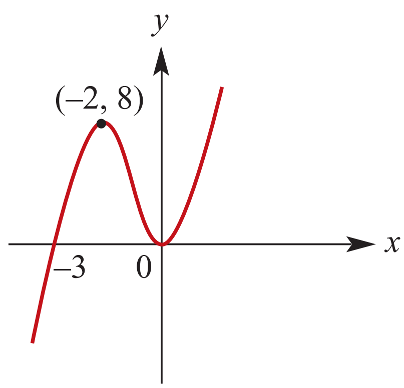
d $y = 2x^3 + 11x^2 + 15x$

Intercepts: $(-3, 0), (-\frac{5}{2}, 0)$, and $(0, 0)$

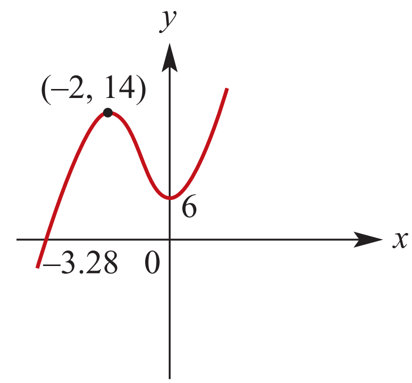
Max : $(-2.76, 0.34)$
Min : $(-0.91, -6.05)$



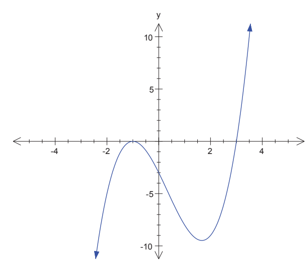
e $y = 2x^3 + 6x^2$
Intercepts: $(-3, 0)$ and $(0, 0)$
Max : $(-2, 8)$
Min : $(0, 0)$



f $y = 2x^3 + 6x^2 + 6$
Intercepts: $(-3.28, 0)$ and $(0, 6)$
Max : $(-2, 14)$
Min : $(0, 6)$



6
$$\begin{aligned} f(x) &= x^3 - x^2 - 5x - 3 \\ &= (x-3)(x^2 + 2x + 1) \\ &= (x-3)(x+1)^2 \end{aligned}$$
$f(x)$ cuts the axis at $x = 3$ and touches the axis at the repeated root $x = -1$.



Solutions to Exercise 7G

1 a $y = a(x-3)^3 + 1$

When $x = 4, y = 12$ $12 = a(4-3)^2 + 1$

$\therefore a = 11$

b $y = a(x-2)(x+3)(x-1)$

When $x = 3, y = 24$

$24 = a(3-2)(3+3)(3-1)$

$\therefore a = 2$

c $y = ax^3 + bx$

When $x = 1, y = 16$

When $x = 2, y = 40$

$$16 = a + b \ldots (1)$$

$$40 = 8a + 2b \ldots (2)$$

Multiply (1) by 2 and subtract from (2)

$$8 = 6a$$

$$\therefore a = \frac{4}{3}$$

$$\therefore b = \frac{44}{3}$$

2 a Equation is of the form

$y = -a(x+2)^3$.

$x = 0, y = -1$:

$-1 = -8a$, so $a = \dfrac{1}{8}$

So $y = -\dfrac{1}{8}(x+2)^3$

b Equation is of the form

$y = -a(x-3)^3 + 2$

$x = 5, y = 0 : 0 = -8a + 2$, so $a = \dfrac{1}{4}$

$\therefore\ y = 2 - \dfrac{1}{4}(x-3)^3$

3 The graph has a repeated root at $(2, 0)$ and cuts $(0, 0)$, $\therefore\ y = ax(x-2)^2$

Using $(3, 6)$: $\quad 3a(3-2) = 6$

$\therefore \quad a = 2$

$\therefore\ y = 2x(x-2)^2$

4 Repeated root at $x = -4$, cuts at $(0, 0)$

$\therefore\ y = ax(x+4)^2$

Using $(-3, 6)$: $\quad -3a(-3+4)^2 = 6$

$\therefore \quad -3a = 6$

$\therefore \quad a = -2$

$\therefore\ y = -2x(x+4)^2$

5 $y = a(x-1)(x-3)(x+1)$

When $x = 0, y = -6$

$\therefore -6 = a(-1)(-3)(1)$

$\therefore a = -2$

$y = -2(x-1)(x-3)(x+1)$

6 $f(x) = (x^2 + a)(x-3)$

$f(6) = 216$

$\therefore 216 = (36 + a)(3)$

$\therefore 72 = 36 + a$

$\therefore a = 36$

7 a $y = a(x-h)^3 + k$

Stationary point of inflection at $(3, 2)$, so $h = 3$.

Using $(3, 2)$: $k = 2$

Using $(0, -25)$:

$$a(-3)^3 + 2 = -25$$

$$\therefore \quad 27a = -27$$

$$\therefore \quad a = 1$$

$\therefore\ y = (x-3)^3 + 2$

b $y = ax^3 + bx^2$

$\therefore\ y = x^2(ax + b)$

Using $(1, 5)$: $a + b = 5$

Using $(-3, -1)$: $9(-3a + b) = -1$

$$\therefore \quad 3a - b = \frac{1}{9}$$

$$a + b = 5$$

$$4a = \frac{46}{9}$$

$\therefore \ a = \frac{23}{18}; b = \frac{67}{18}$

$y = \frac{1}{18}(23x^3 + 67x^2)$

c $y = ax^3$

Using$(1, 5)$: $a(1)^3 = 5, \therefore a = 5$

$y = 5x^3$

8 **a** Graph has axis intercepts at (0,0) and $(\pm 2, 0)$:

$y = ax(x - 2)(x + 2)$

Using $(1, 1)$:

$a(1 - 2)(1 + 2) = 1$

$\therefore \quad -3a = 1$

$\therefore \quad a = -\frac{1}{3}$

$y = -\frac{1}{3}x(x - 2)(x + 2)$

OR $y = -\frac{1}{3}x^3 + \frac{4}{3}x$

b $y = ax^3 + bx^2 + cx$

Using $(2, 3)$: $8a + 4b + 2c = 3$

Using $(-2, -3)$: $-8a + 4b - 2c = -3$

$$8b = 0$$

$\therefore \ b = 0$

$\therefore \ y = ax^3 + cx$

and $8a + 2c = 3 \therefore 4a + c = 1.5$

Using $(1, 0.75)$: $a + c = 0.75$

$$3a = 0.75$$

$\therefore \ a = 0.25$

$\therefore \ c = 0.5$

$\therefore \ y = \frac{1}{4}x^3 + \frac{1}{2}x = \frac{1}{4}x(x^2 + 2)$

9 $y = ax^3 + bx^2 + cx + d$

Use CAS calculator **Solve** function.

a $(0, 270)(1, 312)(2, 230)(3, 0)$

$y = -4x^3 - 50x^2 + 96x + 270$

b $(-2, -406)(0, 26)(1, 50)(2, -22)$

$y = 4x^3 - 60x^2 + 80x + 26$

c $(-2, -32)(2, 8)(3, 23)(8, 428)$

$y = x^3 - 2x^2 + 6x - 4$

d $(1, -1)(2, 10)(3, 45)(4, 116)$

$y = 2x^3 - 3x$

e $(-3, -74)(-2, -23)(-1, -2)(1, -2)$

$y = 2x^3 - 3x^2 - 2x + 1$

f $(-3, -47)(-2, -15)(1, -3)(2, -7)$

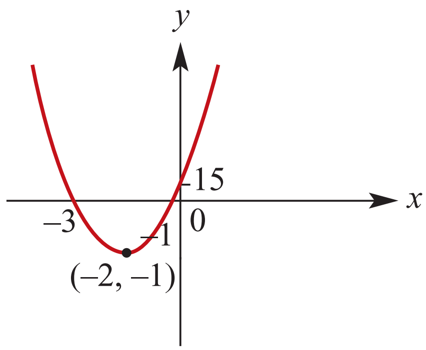
$y = x^3 - 3x^2 - 2x + 1$

g $(-4, 25)(-3, 7)(-2, 1)(1, -5)$

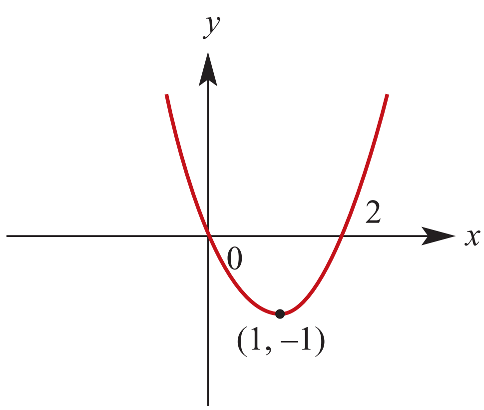
$y = -x^3 - 3x^2 - 2x + 1$

Solutions to Exercise 7H

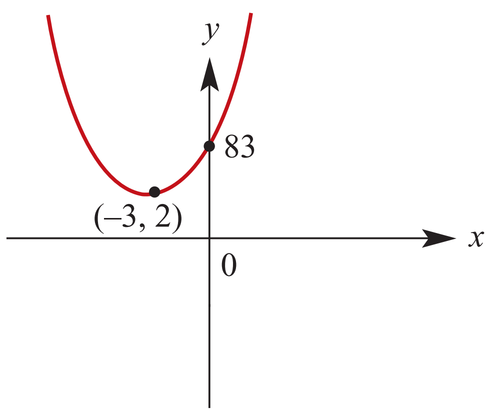
1 a $y = (x + 2)^4 - 1$; vertex at $(-2, -1)$



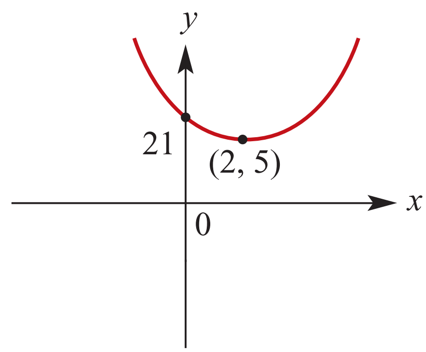
b $y = (x - 1)^4 - 1$; vertex at $(1, -1)$



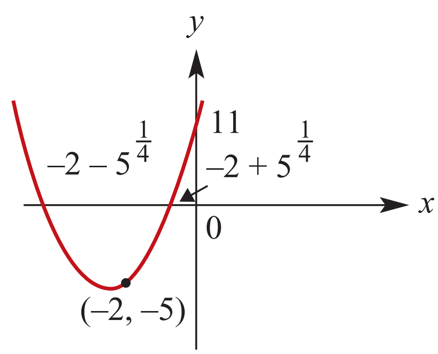
c $y = (x + 3)^4 + 2$; vertex at $(-3, 2)$



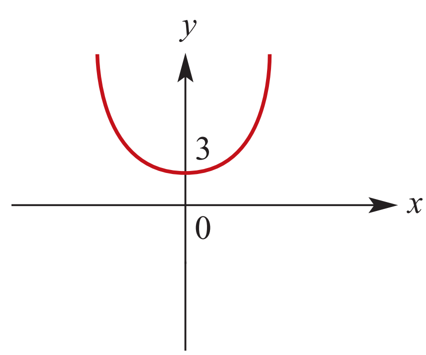
d $y = (x - 2)^4 + 5$; vertex at $(2, 5)$



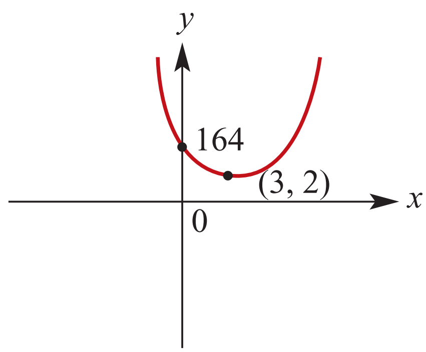
e $y = (x + 2)^4 - 5$; vertex at $(-2, -5)$



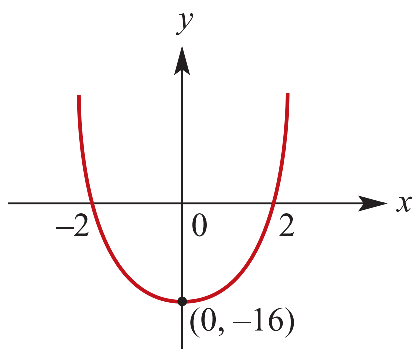
2 a $y = 2x^4 + 3$; vertex at $(0, 3)$



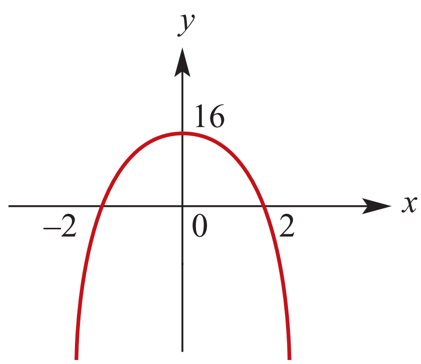
b $y = 2(x - 3)^4 + 2$; vertex at $(3, 2)$



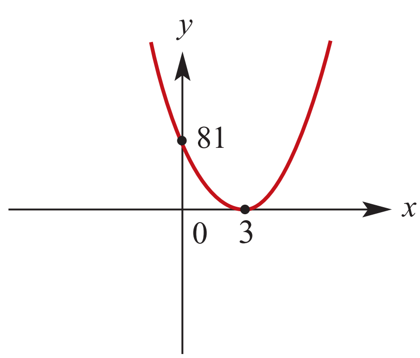
c $y = x^4 - 16$; vertex at $(0, -16)$



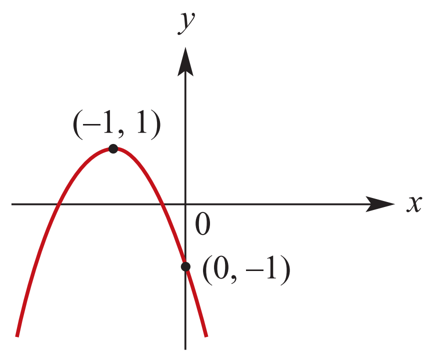
d $y = 16 - x^4$; vertex at $(0, 16)$



e $y = (3 - x)^4$; vertex at $(3,0)$



f $y = -2(x + 1)^4 + 1$; vertex at $(-1, 1)$



3 a $x^4 - 27x = 0$

$\therefore \quad x(x^3 - 27) = 0$

$\therefore \quad x(x - 3)(x^2 + 3x + 9) = 0$

$x = 0, 3$; quadratic has no real solutions

b $(x^2 - x - 2)(x^2 - 2x - 15) = 0$

$\therefore \quad (x - 2)(x + 1)(x - 5)(x + 3) = 0$

$x = -3, -1, 2, 5$

c $x^4 + 8x = 0$

$\therefore \quad (x^3 + 8) = 0$

$\therefore \quad x(x + 2)(x^2 - 2x + 4) = 0$

$x = 0, -2$; quadratic has no real solutions

d $x^4 - 6x^3 = 0$

$\therefore x^3(x - 6) = 0$

$x = 0, 6$

e $x^4 - 9x^2 = 0$

$\therefore \quad x^2(x^2 - 9) = 0$

$\therefore \quad x^2(x - 3)(x + 3) = 0$

$x = 0, \pm 3$

f $81 - x^4 = 0$

$\therefore \quad x^4 - 81 = 0$

$\therefore \quad (x^2 - 9)(x^2 + 9) = 0$

$\therefore \quad (x - 3)(x + 3)(x^2 + 9) = 0$

$x = \pm 3$; quadratic has no real solutions

g $x^4 - 16x^2 = 0$

$\therefore \quad x^2(x^2 - 16) = 0$

$\therefore \quad x^2(x - 4)(x + 4) = 0$

$x = 0, \pm 4$

h $x^4 - 7x^3 + 12x^2 = 0$

$\therefore \quad x^2(x^2 - 7x + 12) = 0$

$\therefore \quad x^2(x - 3)(x - 4) = 0$

$x = 0, 3, 4$

i $x^4 - 9x^3 + 20x^2 = 0$

$\therefore \quad x^2(x^2 - 9x + 20) = 0$

$\therefore \quad x^2(x - 4)(x - 5) = 0$

$x = 0, 4, 5$

j $(x^2 - 4)(x^2 - 9) = 0$

$\therefore \ (x - 2)(x + 2)(x - 3)(x + 3) = 0$

$x = \pm 2, \pm 3$

k $(x - 4)(x^2 + 2x + 8) = 0$

$x = 4$; quadratic has no real solutions

l $(x + 4)(x^2 + 2x - 8) = 0$

$\therefore \quad (x + 4)(x - 2)(x + 4) = 0$

$\therefore \quad (x + 4)^2(x - 2) = 0$

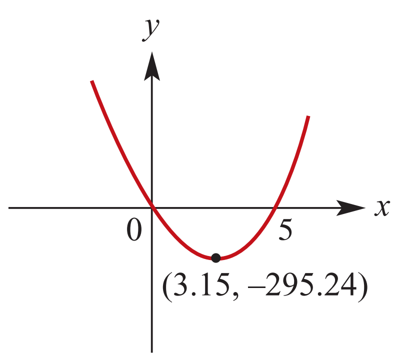
$x = -4, 2$

4 a $y = x^4 - 125x$

$\therefore \ y = x(x^3 - 125)$

x-intercepts: (0, 0) and (5, 0)

TP: (3.15, −295.24)



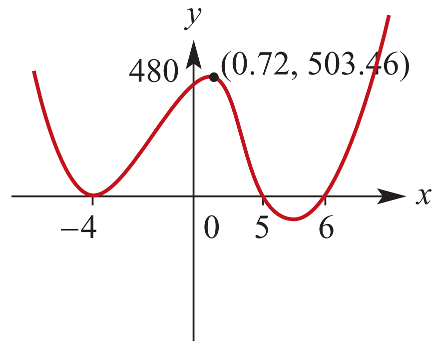
b $y = (x^2 - x - 20)(x^2 - 2x - 24)$

$= (x - 5)(x + 4)(x + 4)(x - 6)$

$= (x - 5)(x + 4)^2(x - 6)$

x-intercepts: (−4, 0), (5, 0) and (6, 0)

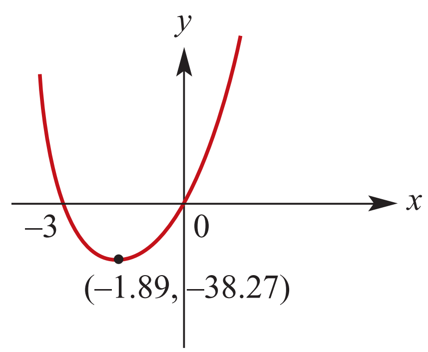
TPs: (−4, 0), (0.72, 503.5) and (5.53, −22.62)



c $y = x^4 + 27x$

x intercepts: (0,0) and (−3, 0)

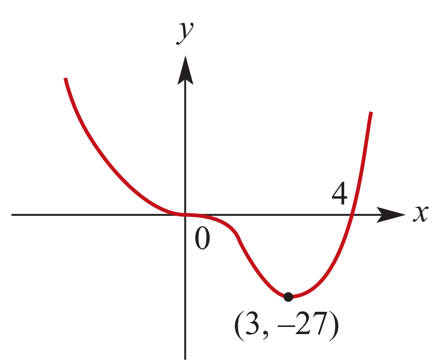
TP: (−1.89, −38.27)



d $y = x^4 - 4x^3$

x-intercepts: (0, 0) and (4, 0)

TP: (3, −27)



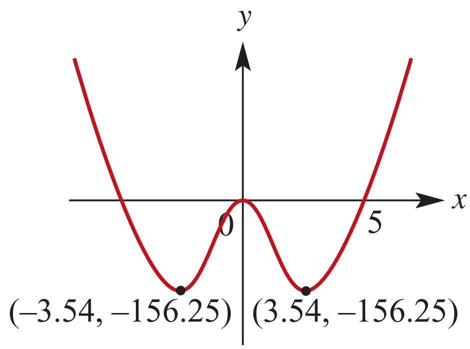
e $y = x^4 - 25x^2$

$= x^2(x^2 - 25)$

$= x^2(x - 5)(x + 5)$

x-intercepts: $(0, 0)(-5, 0)$ and $(5, 0)$

TPs: $(0, 0)$, $(-3.54, -156.25)$ and $(3.54, -156.25)$



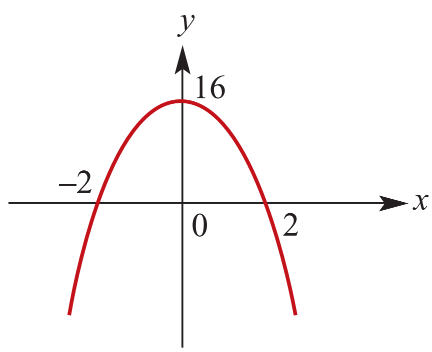
f $y = 16 - x^4$

$= (4 - x^2)(4 + x^2)$

$= (2 - x)(2 + x)(4 + x^2)$

x-intercepts: $(-2, 0)$ and $(2, 0)$

TP: $(0, 16)$



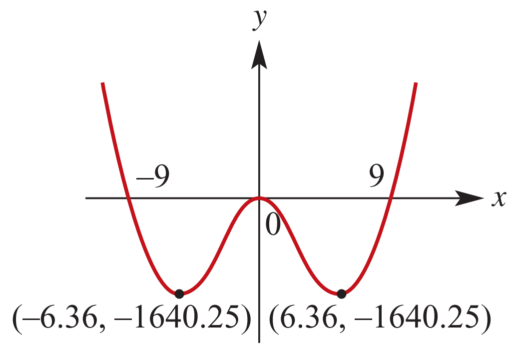
g $y = x^4 - 81x^2$

$= x^2(x^2 - 81)$

$= x^2(x - 9)(x + 9)$

x-intercepts: $(0, 0)$, $(-9, 0)$ and $(9, 0)$

TPs: $(0,0)$, $(-6.36, -1640.25)$ and $(6.36, -1640.25)$



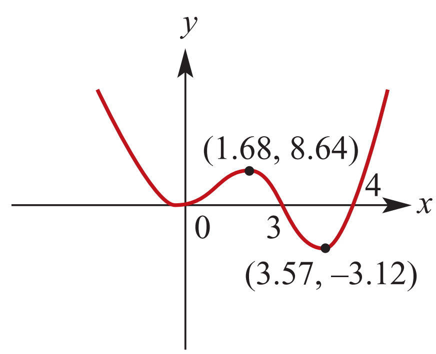
h $y = x^4 - 7x^3 + 12x^2$

$= x^2(x^2 - 7x + 12)$

$= x^2(x - 3)(x - 4)$

x-intercepts: $(0, 0)$, $(3, 0)$ and $(4, 0)$

TPs: $(0, 0)$, $(1.68, 8.64)$ and $(3.57, -3.12)$



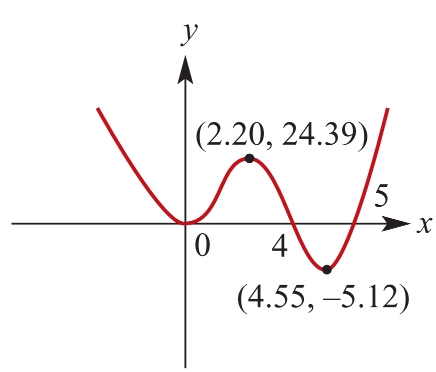
i $y = x^4 - 9x^3 + 20x^2$

$= x^2(x^2 - 9x + 20)$

$= x^2(x - 4)(x - 5)$

x-intercepts: $(0, 0)$, $(4, 0)$ and $(5, 0)$

TPs: $(0, 0)$, $(2.20, 24.39)$ and $(4.55, -5.12)$

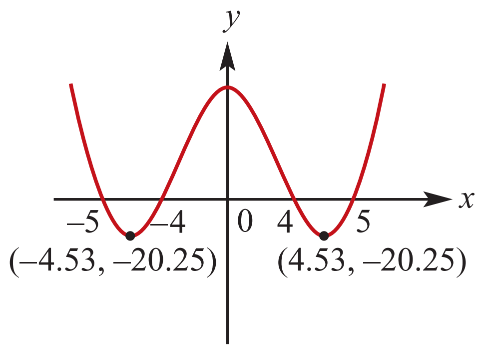


j $y = (x^2 - 16)(x^2 - 25)$

$= (x - 4)(x + 4)(x - 5)(x + 5)$

x-intercepts: $(-5, 0)$, $(-4, 0)$, $(4, 0)$ and $(5, 0)$

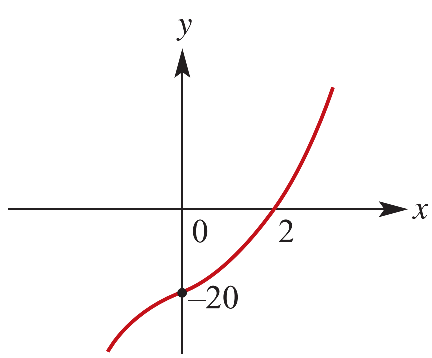
TPs: $(0, 400)$, $(-4.53, -20.25)$ and $(4.53, -20.25)$



k $y = (x - 2)(x^2 + 2x + 10)$ x-intercept: $(2, 0)$

Quadratic has no real solutions.

No Turning points, as shown by reference to a CAS graph.

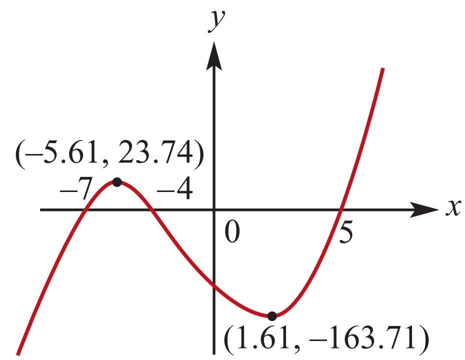


l $y = (x + 4)(x^2 + 2x - 35)$

$= (x + 4)(x + 7)(x - 5)$

x-intercepts: $(-7, 0)$, $(-4, 0)$ and $(5, 0)$

TPs: $(-5.61, 23.74)$ and $(1.61, -163.71)$



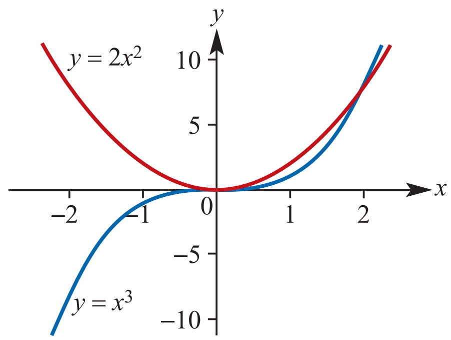
5 a
$$\begin{aligned} f(x) &= 5x^2 - 3x^2 \\ \therefore f(-x) &= 5(-x)^2 - 3(-x)^2 \\ &= 5x^2 - 3x^2 \\ &= f(x) \end{aligned}$$
$\therefore f(x)$ is even.

b
$$\begin{aligned} f(x) &= 7x^{11} - x^3 + 2x \\ \therefore f(-x) &= 7(-x)^{11} - (-x)^3 + 2(-x) \\ &= -7x^{11} + x^3 - 2x \\ &= -f(x) \end{aligned}$$
$\therefore f(x)$ is odd.

c
$$\begin{aligned} f(x) &= x^4 - 3x^2 + 2 \\ \therefore f(-x) &= (-x)^4 - 3(-x)^2 + 2 \\ &= x^4 - 3x^2 + 2 \\ &= f(x) \end{aligned}$$
$\therefore f(x)$ is even.

d
$$\begin{aligned} f(x) &= x^5 - 4x^3 \\ \therefore f(-x) &= (-x)^5 - 4(-x)^3 \\ &= -x^5 + 4x^3 \\ &= -f(x) \end{aligned}$$
$\therefore f(x)$ is odd.

6 a



b

$$f(x) = g(x)$$
$$x^3 = 2x^2$$
$$x^3 - 2x^2 = 0$$
$$x^2(x-2) = 0$$
$$x = 0 \text{ or } x = 2$$

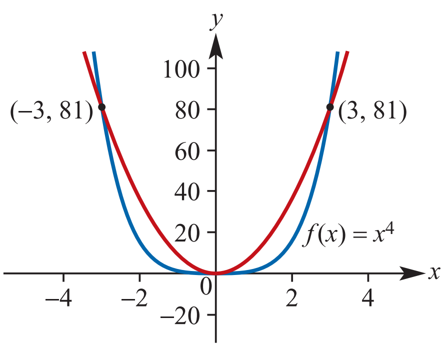
c $f(x) \le g(x)$

$\Leftrightarrow x^3 \le 2x^2$

$\Leftrightarrow x^2(x-2) \le 0$

$\Leftrightarrow x \in (-\infty, 2]$

7 a



b

$$f(x) = g(x)$$
$$x^4 = 9x^2$$
$$x^4 - 9x^2 = 0$$
$$x^2(x^2 - 9) = 0$$
$$x^2(x-3)(x+3) = 0$$
$$x = -3 \text{ or } x = 3 \text{ or } x = 0$$

c $f(x) \le g(x)$

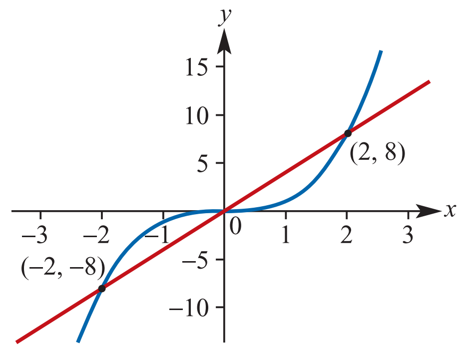
$\Leftrightarrow x^4 \le 9x^2$

$\Leftrightarrow x^2(x^2 - 9) \le 0$

$\Leftrightarrow x^2(x-3)(x+3) \le 0$

$\Leftrightarrow x \in [-3, 3]$

8 a



b

$$f(x) = g(x)$$
$$x^3 = 4x$$
$$x^3 - 4x = 0$$
$$x(x^2 - 4) = 0$$
$$x(x-2)(x+2) = 0$$
$$x = -2 \text{ or } x = 2 \text{ or } x = 0$$

c $f(x) \le g(x)$

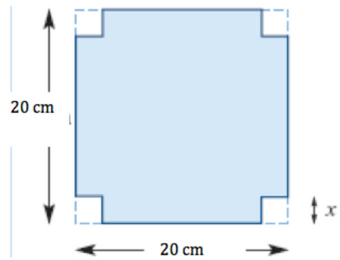
$\Leftrightarrow x^3 \le 4x$

$\Leftrightarrow x(x^2 - 4) \le 0$

$\Leftrightarrow x \in (-\infty, -2] \cup [0, 2]$

Solutions to Exercise 7I

1



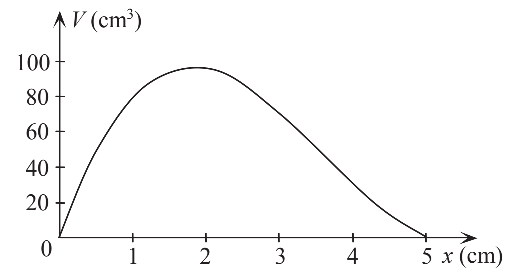
a $20 - 2x$

b $V = x(20 - 2x)^2$

c When $x = 5$,
$V = 5(20 - 2 \times 5)^2 = 500 \text{ cm}^3$

d
$$\begin{aligned} x(20 - 2x)^2 &= 500 \\ 4x(100 - 20x + x^2) &= 500 \\ 100x - 20x^2 + x^3 &= 125 \end{aligned}$$
$x^3 - 20x^2 + 100x - 125 = 0$
We know that $x - 5$ is a factor
Hence
$(x - 5)(x^2 - 15x + 25) = 0$
$$(x - 5)\left(x^2 - 15x + \left(\frac{15}{2}\right)^2 - \left(\frac{15}{2}\right)^2 + 25\right) = 0$$
$$(x - 5)\left(x - \frac{15}{2}\right)^2 - \frac{125}{4} = 0$$
$$(x - 5)\left(x - \frac{15}{2} - \frac{5\sqrt{5}}{2}\right)\left(x - \frac{15}{2} + \frac{5\sqrt{5}}{2}\right) = 0$$
The required other value is
$$x = \frac{15 - 5\sqrt{5}}{2}$$

c



d When $x = 1$,
$$\begin{aligned} V &= (12 - 2(1))(10 - 2(1))(1) \\ &= 10 \times 8 \\ &= 80 \end{aligned}$$

e On the CAS calculator, sketch the graphs of $\mathbf{Y_1 = 4X(6 - X)(5 - X)}$ and $\mathbf{Y_2 = 50}$.
The points of intersection are (0.50634849, 50) and (3.5608171, 50).
Therefore $V = 50$ when $x = 0.51$ or $x = 3.56$, correct to 2 decimal places.

f With $\boldsymbol{f1 = 4x \times (6 - x)(5 - x)}$ to yield (1.810745, 96.770576).
Therefore the maximum volume is 96.77 cm^3 and occurs when $x = 1.81$, correct to 2 decimal places.
Alternatively, use the CAS calculator to give the maximum when
$x = \dfrac{11 - \sqrt{31}}{3} \approx 1.81$; then
maximum volume is 96.77 cm^3.

2 a $l = 12 - 2x \qquad w = 10 - 2x$

b
$$\begin{aligned} V &= \text{length} \times \text{width} \times \text{height} \\ &= (12 - 2x)(10 - 2x)x \\ &= 4x(6 - x)(5 - x) \end{aligned}$$

3 a Surface area $x^2 + 4xh$

b $x^2 + 4xh = 75$
$$\therefore h = \frac{75 - x^2}{4x}$$

c $V = x^2h = \dfrac{x(75 - x^2)}{4}$

d **i** When $x = 2, V = \dfrac{71}{2}$

ii When $x = 5, V = \dfrac{125}{2}$

iii When $x = 8, V = 22$

e It is given that $x = 4$ is a solution of the equation:

$\dfrac{x(75 - x^2)}{4} = 59$

Rearranging we have:

$$\begin{aligned} x(75 - x^2) &= 236 \\ x^3 - 75x + 236 &= 0 \\ (x - 4)(x^2 + 4x - 59) &= 0 \\ (x - 4)(x^2 + 4x + 4 - 63) &= 0 \\ (x - 4)((x + 2)^2 - 63) &= 0 \\ (x - 4)(x + 2 - 3\sqrt{7})(x + 2 + 3\sqrt{7}) &= 0 \end{aligned}$$

The required solution is $x = 3\sqrt{7} - 2$

4 The base is a right-angled triangle $(5x, 12x, 13x)$

a The sum of all the lengths of the prisms edges is180 cm

$\therefore 2(5x + 12x + 13x) + 3h = 180$

$\therefore 60x + 3h = 180$

$\therefore h = \dfrac{180 - 60x}{3} = 60 - 20x$

b The area of the base is $30x^2$.

$\therefore V = 30x^2(60 - 20x) = 600x^2(3 - x)$

c When $x = 3, V = 0$

d

$$\begin{aligned} 600x^2(3 - x) &= 1200 \\ x^2(3 - x) &= 2 \\ 3x^2 - x^3 &= 2 \\ x^3 - 3x^2 + 2 &= 0 \\ (x - 1)(x^2 - 2x - 2) &= 0 \\ (x - 1)(x^2 - 2x + 1 - 3) &= 0 \\ (x - 1)((x - 1)^2 - 3) &= 0 \\ (x - 1)((x - 1 - \sqrt{3})(x - 1 + \sqrt{3}) &= 0 \end{aligned}$$

Required solutions $x = 1 + \sqrt{3}$ and $x = 1$

5 **a** Using Pythagoras' theorem,

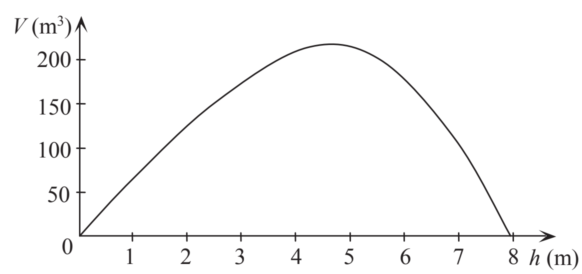
$x^2 + h^2 = 8^2$

$x = \sqrt{64 - h^2}$

b $V = \dfrac{1}{3}\pi x^2 h$

$= \dfrac{1}{3}\pi(64 - h^2)h$

c



d Domain $= \{h : 0 < h < 8\}$

e When $h = 4$, $V = \dfrac{1}{3}\pi(64 - 4^2)(4)$

$= 64\pi$

f On the CAS calculator, sketch the graphs of $\boldsymbol{f1 = 1/3\pi(64 - x^2) \times x}$ and $\boldsymbol{f2 = 150}$. The points of intersection are (2.4750081, 150) and (6.4700086, 0.150).
Therefore $V = 150$ when $h = 2.48$ or $h = 6.47$, correct to 2 decimal places.

g With $\boldsymbol{f1 = 1/3\pi(64 - x\text{^}2) \times x}$, to yield (4.6187997, 206.37006).
Therefore the maximum volume is 206.37 m^3 and occurs when $h = 4.62$, correct to 2 decimal places.
Alternatively, use
fMax(1/3π(64 − x^2) × x, x, 0, 8)
to give the maximum when
$h = \frac{8\sqrt{3}}{3} \approx 4.62$; then maximum volume is 206.37m.

6 a $x + x + h = 160$

$2x + h = 160$

$h = 160 - 2x$

b $V = x \times x \times h$

$= x^2(160 - 2x)$

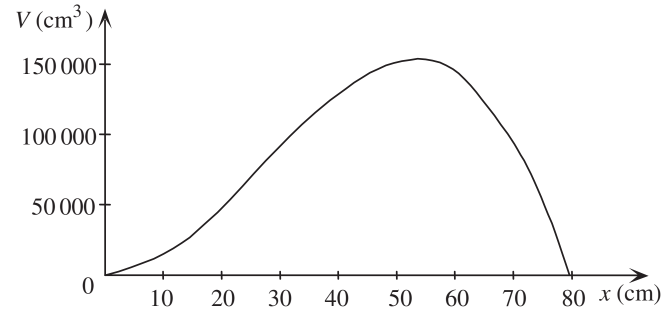
When $V = 0$, $x^2 = 0$ or $160 - 2x = 0$

$\therefore$ $x = 0$ or $160 = 2x$

$\therefore$ $x = 80$

c $\therefore$ Domain $V = \{x : 0 < x < 80\}$

d



e On the CAS calculator, sketch the graphs of $\boldsymbol{f1 = x\text{^}2(160 - 2x)}$ and $\boldsymbol{f2 = 50\,000}$. The points of intersection are (20.497586, 50000) and (75.629199, 50 000).
Therefore $V = 50\,000$ when $x = 20.50$ or $x = 75.63$, correct to 2 decimal places.

f With $\boldsymbol{f1 = x\text{^}2(160 - 2x)}$, to yield (53.333336, 151703.7).
Therefore the maximum volume is 151 704 cm^3 (to the nearest cm^3).
Alternatively, use
fMax(x^2(160 − 2x), x, 0, 80)
to give the maximum when
$h = \frac{160}{3} \approx 53\frac{1}{3}$; then maximum volume is 151703.7 cm^3.

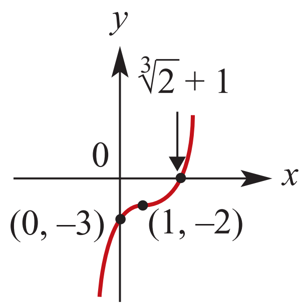
Solutions to Review: Short-answer questions

1 a $y = (x - 1)^3 - 2$

Stationary point of inflection at $(1, -2)$

x-intercept at $(1 + \sqrt[3]{2}, 0)$

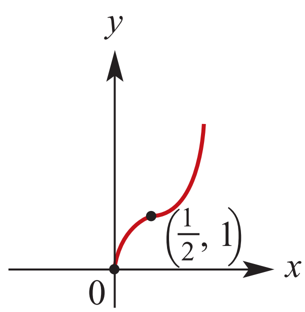
y-intercept at $(0, -3)$



b $y = (2x - 1)^3 + 1$

Stationary point of inflection at $\left(\frac{1}{2}, 1\right)$

Axis intercept at $(0, 0)$

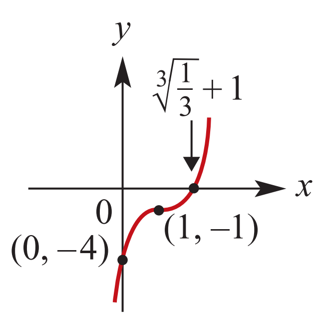


c $y = 3(x - 1)^3 - 1$

Stationary point of inflection at $(1, -1)$

x-intercept at $\left(1 + \sqrt[3]{\frac{1}{3}}, 0\right)$

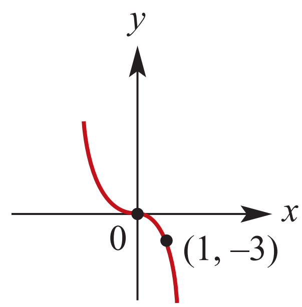
y-intercept at $(0, -4)$



d $y = -3x^3$

Stationary point of inflection at $(0, 0)$

Axis intercept at $(0, 0)$

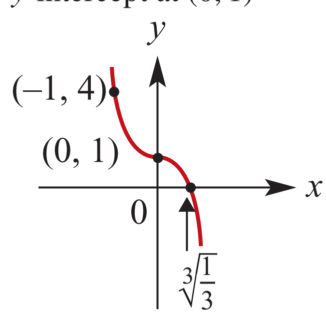


e $y = -3x^3 + 1$

Stationary point of inflection at $(0, 1)$

x-intercept at $\left(\sqrt[3]{\frac{1}{3}}, 0\right)$

y-intercept at $(0, 1)$

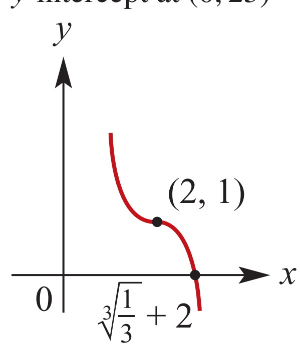


f $y = -3(x - 2)^3 + 1$

Stationary point of inflection at $(2,1)$

x-intercept at $\left(2 + \sqrt[3]{\frac{1}{3}}, 0\right)$

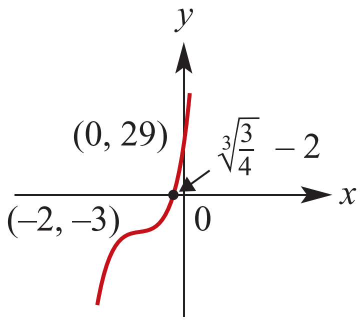
y-intercept at $(0, 25)$



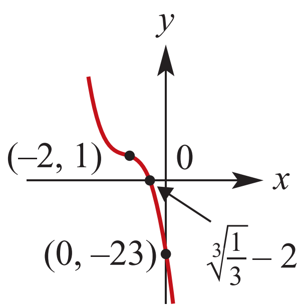
g $y = 4(x + 2)^3 - 3$

Stationary point of inflection at $(-2, -3)$

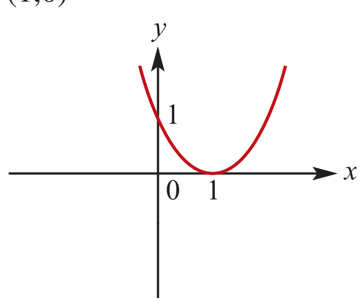
x-intercept at $(-2 + \sqrt[3]{\frac{3}{4}}, 0)$
y-intercept at $(0, 29)$



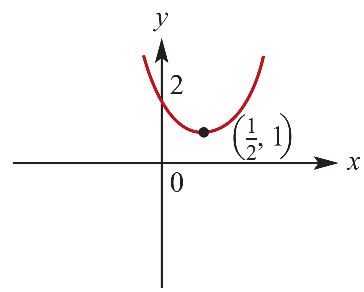
h $y = 1 - 3(x + 2)^3$
Stationary point of inflection at $(-2, 1)$
x-intercept at $(-2 + \sqrt[3]{\frac{1}{3}}, 0)$
y-intercept at $(0, -23)$



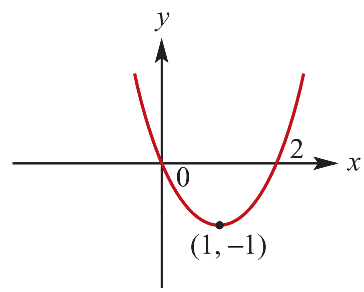
2 **a** $y = (x - 1)^4$
Turning point at $(1, 0)$
y-intercept at $(0,1)$, x-intercept at $(1,0)$



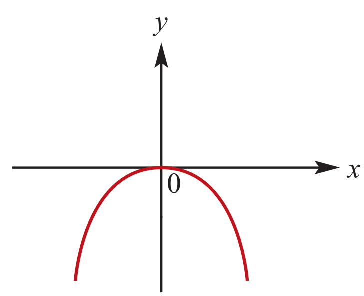
b $y = (2x - 1)^4 + 1$
Turning point at $(\frac{1}{2}, 1)$
y-intercept at $(0,2)$, no x-intercept



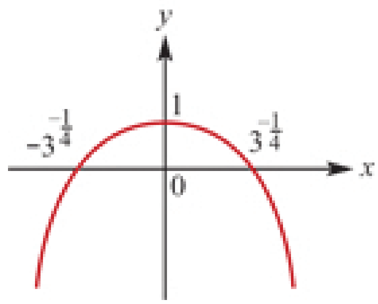
c $y = (x - 1)^4 - 1$
Turning point at $(1, -1)$
y-intercept at $(0, 0)$,
x-intercept at $(0, 0)$ and $(2, 0)$



d $y = -2x^4$
Turning point and axis intercept at $(0, 0)$



e $y = -3x^4 + 1$
Turning point at $(0, 1)$
x-intercepts at $(\pm\sqrt[4]{\frac{1}{3}}, 0)$
y-intercept at $(0, 1)$

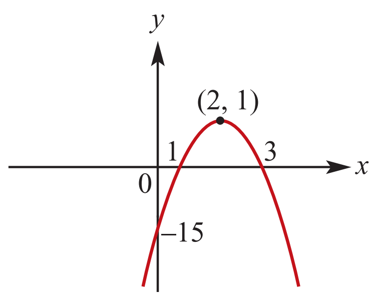


f $y = -(x-2)^4 + 1$

Turning point at $(2, 1)$

y-intercept at $(0, -15)$

x-intercept $(1, 0)$ and $(3, 0)$

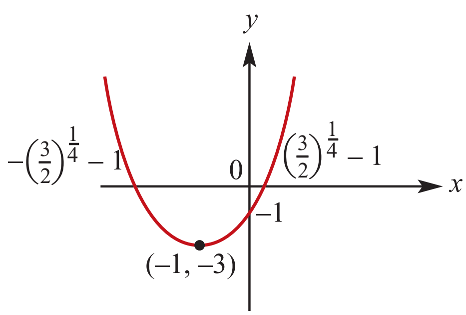


g $y = 2(x+1)^4 - 3$

Turning point at $(-1, -3)$

x-intercepts at $\left(-1 \pm \sqrt[4]{\frac{3}{2}}, 0\right)$

y-intercept at $(0, 1)$

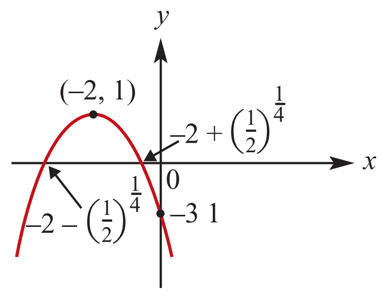


h $y = 1 - 2(x+2)^4$

Turning point at $(-2, 1)$

x-intercepts at $(-2 \pm \sqrt[4]{\frac{1}{2}}, 0)$

y-intercept at $(0, -31)$



3 a
$$2x^3 + 3x^2 = 11x + 6$$
$$2x^3 + 3x^2 - 11x - 6 = 0$$
$$(2x+1)(x^2 + x - 6) = 0$$
$$(2x+1)(x+3)(x-2) = 0$$
$$x = -\frac{1}{2} \text{ or } x = -3 \text{ or } x = 2$$

b
$$x^2(5 - 2x) = 4$$
$$5x^2 - 2x^3 - 4 = 0$$
$$2x^3 - 5x^2 + 4 = 0$$
$$(x-2)(2x^2 - x - 2) = 0$$
$$x = 2 \text{ or } 2x^2 - x - 2 = 0$$
$$x = 2 \text{ or } x = \frac{1 \pm \sqrt{17}}{4}$$

c
$$x^3 - 7x^2 + 4x + 12 = 0$$
$$(x-6)(x^2 - x - 2) = 0$$
$$(x-6)(x-2)(x+1) = 0$$
$$x = 6 \text{ or } x = 2 \text{ or } x = -1$$

4 a $P(x) = 6x^3 + 5x^2 - 17x - 6$

$P(-2) = 6(-8) + 5(4) - 17(-2) - 6 = 0$

So $x + 2$ is a factor of $P(x)$.

$P\left(\frac{3}{2}\right) = 6\left(\frac{27}{8}\right) + 5\left(\frac{9}{4}\right) - 17\left(\frac{3}{2}\right) - 6 = 0$

So $2x - 3$ is a factor of $P(x)$.

$\therefore\ P(x) = (x+2)(2x-3)(ax+b)$

$= (ax+b)(2x^2 + x - 6)$

Matching coefficients with $P(x)$:

$2a = 6, \therefore\ a = 3$
$-6b = -6, \therefore\ b = 1$
So the other factor is $3x + 1$.

b $P(x) = 2x^3 - 3x^2 - 11x + 6 = 0$
$P(-2) = 0$, so $(x + 2)$ is a factor.
$P(3) = 0$, so $(x - 3)$ is a factor.
$P(x) = (ax + b)(x + 2)(x - 3)$
$= (ax + b)(x^2 - x - 6)$
Matching coefficients with $P(x)$:
$a = 2$
$-6b = 6, \therefore\ b = -1$
$\therefore\ P(x) = (2x - 1)(x + 2)(x - 3)$
$x = -2, \frac{1}{2}, 3$

c $x^3 + x^2 - 11x - 3 = 8$
$\therefore\ P(x) = x^3 + x^2 - 11x - 11 = 0$
$P(-1) = 0$, so $(x + 1)$ is a factor.
$\therefore\ P(x) = x^2(x + 1) - 11(x + 1) = 0$
$= (x + 1)(x^2 - 11) = 0$
$x = -1, \pm\sqrt{11}$

d **i** $P(x) = 3x^3 + 2x^2 - 19x + 6$
$P\left(\frac{1}{3}\right) = \frac{3}{27} + \frac{2}{9} - \frac{19}{3} + 6 = 0$
so $(3x - 1)$ is a factor.

ii $P(2) = 24 + 8 - 38 + 6 = 0$
so $(x - 2)$ is a factor.
$P(x) = (ax + b)(x - 2)(3x - 1)$
$= (ax + b)(3x^2 - 7x + 2)$
Matching coefficients:
$a = 1,\ b = 3$
$\therefore P(x) = (x + 3)(x - 2)(3x - 1)$

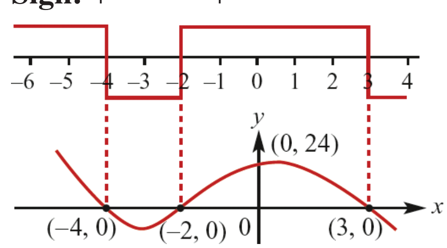
5 **a** $f(x) = x^3 - kx^2 + 2kx - k - 1$
$\therefore\ f(1) = 1 - k + 2k - k - 1 = 0$
By the Factor Theorem, $f(x)$ is divisible by $x - 1$.

b
$$\begin{array}{r|l} & x^2 + (1-k)x + (k+1) \\ \hline x - 1 & x^3 - kx^2 + 2kx - k - 1 \\ & x^3 - x^2 \\ \hline & (1-k)x^2 + 2kx \\ & (1-k)x^2 - (1-k)x \\ \hline & (k+1)x - (k+1) \\ & (k+1)x - (k+1) \\ \hline & 0 \end{array}$$
$f(x) = (x - 1)(x^2 + (1 - k)x + k + 1)$

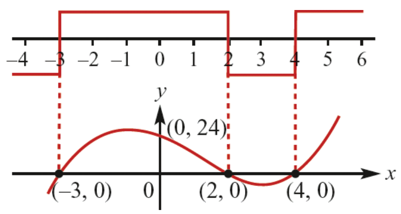
6 $P(x) = x^3 + ax^2 - 10x + b$
$P(x)$ is divisible by $Q(x) = x^2 + x - 12$
$Q(x) = (x - 3)(x + 4)$, so
$P(3) = P(-4) = 0$
$P(3) = 27 + 9a - 30 + b = 0$
$\therefore\ 9a + b = 3$
$P(-4) = -64 + 16a + 40 + b = 0$
$\therefore\ 16a + b = 24$
$\therefore\ 7a = 21$
$\therefore\ a = 3; b = -24$

7 Arrange in left-to-right order first.

a $y = (x + 4)(x + 2)(3 - x)$
Inverted cubic.
Axis intercepts: $(-4, 0), (-2, 0), (3, 0)$ and $(0,24)$
Sign: + − + −



b $y = (x + 3)(x - 2)(x - 4)$
Upright cubic.
Axis intercepts: $(-3, 0), (2, 0), (4, 0)$ and $(0, 24)$
Sign: + − + −



c $y = 6x^3 + 13x^2 - 4$

Upright cubic.

$y(-2) = -48 + 52 - 4 = 0$

So $(x + 2)$ is a factor.

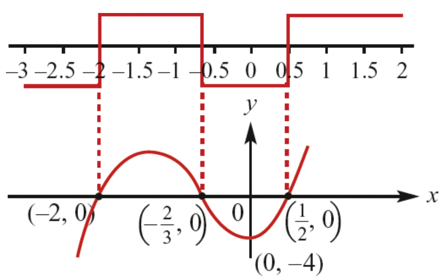
$\therefore\ y = (x + 2)(6x^2 + x - 2)$

$= (x + 2)(3x + 2)(2x - 1)$

Axis intercepts:

$(-2, 0), (-\frac{2}{3}, 0), (\frac{1}{2}, 0)$ and $(0, -4)$

Sign: + − + −



d $y = x^3 + x^2 - 24x + 36$

Upright cubic.

$y(2) = 8 + 4 - 48 + 36 = 0$

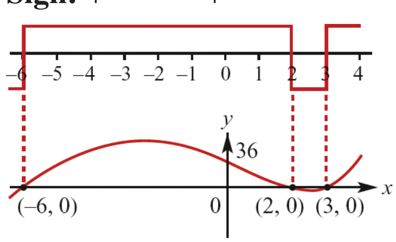
So $(x - 2)$ is a factor.

$\therefore\ y = (x - 2)(x^2 + 3x - 18)$

$= (x + 6)(x - 2)(x - 3)$

Axis intercepts: $(-6, 0), (2, 0), (3, 0)$ and $(0, 36)$

Sign: + − + −



8 a $P(x) = x^3 + 4x^2 - 5x + 1$.

Remainder after division by

$(x + 6) = P(-6) = -41$

b $P(x) = 2x^3 - 3x^2 + 2x + 4$.

Remainder after division by

$(x - 2) = P(2) = 12$

c $P(x) = 3x^3 + 2x + 4$.

Remainder after division by

$(3x - 1) = P(\frac{1}{3}) = \frac{43}{9}$

9 $y = a(x + 2)(x - 1)(x - 5)$ accounts for the x intercepts.

At $x = 0, y = a(2)(-1)(-5) = -4$

$\therefore\ a = -\frac{2}{5}$

$y = -\frac{2}{5}(x + 2)(x - 1)(x - 5)$

10 Cubic passes through the origin and touches the x-axis at $(-4, 0)$

$\therefore\ y = ax(x + 4)^2$

Using $(5, 10)$:

$5a(5 + 4)^2 = 10, \therefore\ a = \frac{2}{81}$

$\therefore\ y = \frac{2}{81}x(x + 4)^2$

11 a $f(x) = 2x^3 + ax^2 - bx + 3$

$f(1) = 2 + a - b + 3 = 0$

$\therefore\ b - a = 5 \ldots \mathbf{(1)}$

$f(2) = 16 + 4a - 2b + 3 = 15$

$\therefore\ 4a - 2b = -4$

$\therefore\ b - 2a = 2 \ldots \mathbf{(2)}$

(1) − (2) gives $a = 3,\ b = 8$

b $f(x) = 2x^3 + 3x^2 - 8x + 3$

$= (x - 1)(2x^2 + 5x - 3)$

$= (x - 1)(2x - 1)(x + 3)$

12 $f(x) = x^3$

a Dilation by a factor of 2 from x-axis:
$y = 2x^3$
Translation 1 unit in positive x and 3 units in positive y:
$y = 2(x - 1)^3 + 3$

b Reflection in x-axis:
$y = -x^3$
Translation 1 unit in negative x and 2 units in positive y:
$y = -(x + 1)^3 + 2$

c Dilation by a factor of $\frac{1}{2}$ from y-axis:
$y = (2x)^3$
Translation $\frac{1}{2}$ unit in negative x and 2 units in negative y:
$y = (2(x + \frac{1}{2}))^3 - 2$
$= (2x + 1)^3 - 2$

Solutions to Review: Multiple-choice questions

1 B $P(x) = x^3 + 3x^2 + x - 3$

$\therefore\ P(-2) = (-2)^3 + 3(-2)^2 + (-2) - 3$

$= -1$

2 D $P(x) = (x-a)^2(x-b)(x-c)$,

$a > b > c$

Graph of $y = P(x)$ is an upright quartic with a repeated root at $x = a$, so $P(x) < 0$ only between c and b.

3 A $y = x^3$

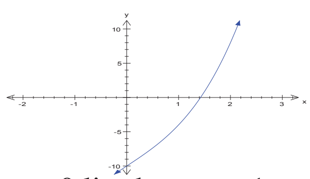
Dilation $\times 2$ from y-axis: $y = \left(\dfrac{x}{2}\right)^3$

reflection in the y-axis: $y = \left(-\dfrac{x}{2}\right)^3$

translation of 4 units in negative direction of y-axis:

$y = \left(-\dfrac{x}{2}\right)^3 - 4 = -\dfrac{x^3}{8} - 4$

4 D $y = x^3 + 5x - 10$



$y = 0$ lies between 1 and 2

5 A $P(x) = x^4 + ax^2 - 4$

$P(x) = 0$ if $x = -a \pm \sqrt{\dfrac{a^2}{4} + 4}$

If $P(x) = 0$ when $x = \pm\sqrt{2}$, then $a = 0$

6 C $P(x) = x^3 + ax^2 + bx - 9$

$P(x) = 0$ has zeros at $x = 1$ and $x = -3$.

$\therefore\ P(1) = 1 + a + b - 9 = 0$

$\therefore\ a + b = 8 \ldots \mathbf{(1)}$

$P(-3) = -27 + 9a - 3b - 9 = 0$

$\therefore\ 9a - 3b = 36$

$\therefore\ 3a - b = 12 \ldots \mathbf{(2)}$

(1) + **(2)** gives:

$4a = 20$

$\therefore\ a = 5; b = 3$

7 B $P(x) = ax^3 + 2x^2 + 5$ is divisible by $x + 1$

$\therefore\ P(-1) = -a + 2 + 5 = 0$

$\therefore a = 7$

8 B $P(x) = x^3 + 2x^2 - 5x + d$

$\dfrac{P(x)}{x-2}$ has a remainder of 10

$\therefore\ P(2) = 10$

$P(2) = 8 + 8 - 10 + d = 10$

$\therefore\ d = 4$

9 D The diagram shows an inverted cubic with a repeated root at $x = b$ and a single root at $x = a$.

$\therefore\ y = -(x-a)(x-b)^2$

10 B The graph of $y = -f(x)$ is a reflection in the x-axis. The graph of $y = 1 - f(x)$ is then a translation up by 1 unit. Only the graph in **B** satisfies these two features.

Solutions to Review: Extended-response questions

1 **a** $V = \pi r^2 h$

$r + h = 6$

$\therefore V = \pi r^2(6 - r)$

b $0 \le r \le 6$

c $V(3) = 27\pi$

d
$$\begin{aligned} \pi r^2(6 - r) &= 27\pi \\ 6r^2 - r^3 - 27 &= 0 \\ r^3 - 6r^2 + 27 &= 0 \\ (r - 3)(r^2 - 3r - 9) &= 0 \\ \Leftrightarrow r = 3 \text{ or } r &= \frac{3 \pm 3\sqrt{5}}{2} \end{aligned}$$

In the context of the question

$r = 3$ or $r = \dfrac{3 + 3\sqrt{5}}{2}$

e Maximum ≈ 100.53

2 **a** At $t = 900$, all the energy is used up.

The point with coordinates $(900, 0)$ is the vertex of the parabola.

Equation of the parabola is $v = a(t - 900)^2 + 0$

$$= a(t - 900)^2$$

When $t = 0$, $v = 25$

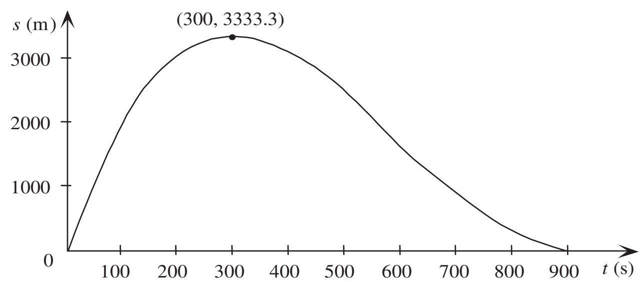
$$\begin{aligned} \therefore \quad 25 &= a(0 - 900)^2 \\ \therefore \quad a &= \frac{25}{810\,000} \\ &= \frac{1}{32\,400} \\ \therefore \quad v &= \frac{1}{32\,400}(t - 900)^2 \end{aligned}$$

b $s = vt$ and $\quad v = \dfrac{1}{32\,400}(t - 900)^2$

$$\therefore \quad s = \frac{t}{32\,400}(t - 900)^2$$

(Remember, t is the time in which all energy is used up; v is constant for a given t.)

c



d The maximum distance the t axis can travel is $3\frac{1}{3}$ km, so a proposal to place power sources at 3.5 km intervals is not feasible.

e If the power sources are at 2 km intervals, $v_{\max}$ and v min are given for values of t at which $s = 2000$. From the graph, when $s = 2000$, $t1 \approx 105$ and $t2 \approx 560$.

When $t1 = 105$, $v_{\max} \approx \dfrac{2000}{105}$

≈ 19

When $t2 = 560$, $v_{\min} = \dfrac{2000}{560}$

≈ 3.6

Hence, the maximum and minimum speeds recommended for drivers are approximately 19 m/s and 3.6 m/s respectively.

3 a The 'flat spot' is the point of inflexion $\therefore\ (h, k) = (5, 10)$
Hence $R - 10 = a(x - 5)^3$

b At $(0, 0)$, $\quad 0 - 10 = a(0 - 5)^3$

$$\therefore \quad -10 = -125a$$

$$\therefore \quad a = \frac{10}{125} = \frac{2}{25}$$

$$\therefore \quad R - 10 = \frac{2}{25}(x - 5)^3$$

c If $(h, k) = (7, 12)$, then $\quad R - 12 = a(x - 7)^3$

At $(0, 0)$, $\quad 0 - 12 = a(0 - 7)^3$

$$\therefore \quad -12 = -343a \quad \therefore a = \frac{12}{343}$$

$$\therefore \quad R - 12 = \frac{12}{343}(x - 7)^3$$

4 a Area of net $= \text{length} \times \text{width}$

$$= (l + w + l + w) \times \left(\frac{w}{2} + h + \frac{w}{2}\right)$$
$$= 2(l + w)(w + h)$$
$$= 2(35 + 20)(20 + 23)$$
$$= 2 \times 55 \times 43$$
$$= 4730$$

The area of the net is $4730\ \text{cm}^2$.

b Let V = volume of the box $\therefore V = h \times l \times w$ (1)

Now $2(l + w)(w + h) = 4730$ and $h = l$ (2)

$\therefore\ 2(l + w)(l + w) = 4730$

$\therefore\ (l + w)^2 = 2365$

$\therefore\ l + w = \sqrt{2365}$ (as $l > 0, w > 0$)

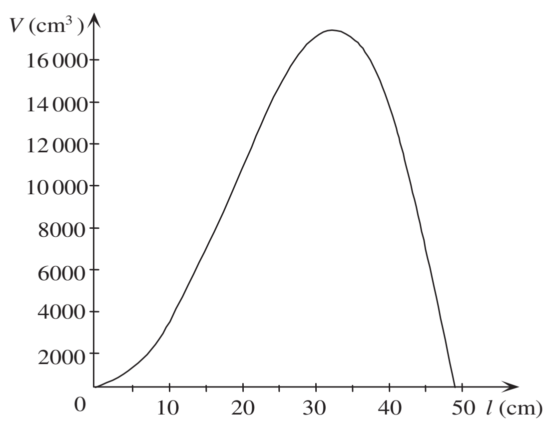
$\therefore\ w = \sqrt{2365} - l$ (3)

Substitute (2) and (3) in (1)

$V = l \times l \times (\sqrt{2365} - l)$

$\therefore\ V = l^2(\sqrt{2365} - l)$

c



d i On the CAS calculator, sketch $\boldsymbol{f1 = x\hat{}2(\sqrt{(2365)} - x)}$ and $\boldsymbol{f2 = 14\,000}$. Points of intersection are (23.694127, 14 000) and (39.787591, 14 000). Therefore the volume is $14\,000\ \text{cm}^3$ when $l = 23.69$ or $l = 39.79$.

ii Repeat **d i** using $\boldsymbol{f2 = 10\,000}$. The points of intersection are (18.096981, 10 000) and (43.296841, 10 000). Therefore the volume is 1 litre when $l = 43.3$ or $l = 18.1$, correct to 1 decimal place..

e With $f1 = x$^2($\sqrt{}$(2365) − x),
TI: Press **Menu → 6:Analyze Graph → 3:Maximum**
to yield (32.420846, 17038.955). The maximum volume is 17039 cm^3 (to the nearest cm^3) and occurs when $l \approx 32.42$.

5 a **TI:** Press **Menu → 1: Actions → 1: Define** then type $f(x) = a \times x^3 + b \times x^2 + c \times x + d$ followed by ENTER.
Now type the following then press ENTER
solve ($f(0) = 15.8$ **and** $f(10) = 14.5$ **and** $f(15) = 15.6$ **and** $f(20) = 15, \{a, b, c, d\}$)
solve($\{f(0) = 15.8, f(10) = 14.5, f(15) = 15.6, f(20) = 15\}, \{a, b, c, d\}$)The

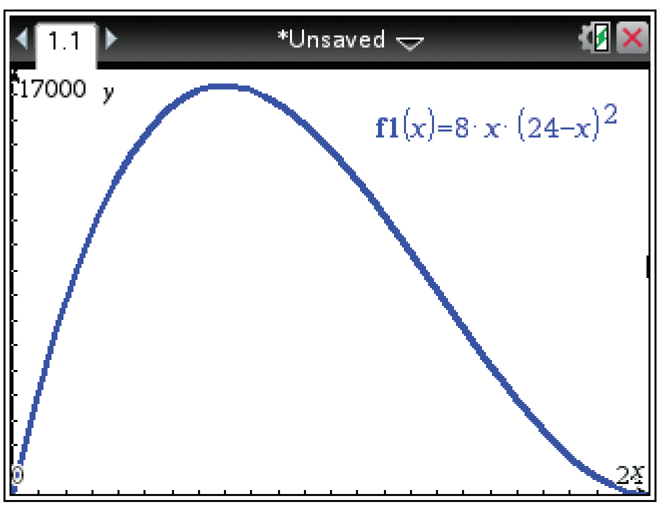
screen gives $a = -0.00287$, $b = 0.095$, $c = -0.793$ and $d = 15.80$.

b **i** With $f1 - 0.00287x$^3 + $0.095x$^2 − $0.793x + 15.8$
TI: Press **Menu → 6:Analyze Graph → 2:Minimum**
to get (5.59, 13.83) as the coordinates of the point closest to the ground.

ii **TI:** In a Calculator page type **f1(0)** followed by ENTER
CP: Tap **Analysis → G-Solve → y-Cal** and input 0 for the x-value to get (0, 15.8) as the point furthest from the ground.

6 a The length of the box (in cm) $= 96 - 4x = 4(24 - x)$.
The width of the box (in cm) $= 48 - 2x = 2(24 - x)$.
The height of the box (in cm) $= x$.
Therefore $V = 4(24 - x) \times 2(24 - x) \times x = 8x(24 - x)^2$

b



i The domain of V is $\{x : 0 < x < 24\}$.

ii With $f1 = 8x \times (24 - x)$^2,
TI: Press **Menu → 6:Analyze Graph → 3:Maximum**

CP: Tap **Analysis → G-Solve → Max** to yield (8.000002, 16 384).
The maximum volume is 16384 cm^3 (to the nearest cm^3) and occurs when $x \approx 8.00$.

c The volume of the box, when $x = 10$, is $V = 8 \times 10(24 - 10)^2 = 15\,680$ cm^3

d The volume is a maximum when $x = 5$. When $x = 5$, $V = 14\,440$.

e The volume is a minimum when $x = 15$. When $x = 15$, $V = 9720$.