SHENTON COLLEGE

2019 YEAR 12 MATHEMATICS: METHODS
Test 3 (Continuous Random Variables, Normal Distribution, Logarithms)

$\overline{78}$

NAME: SOLUTIONS

TEACHER: AI FRIDAY SMITH

Calculator-Free | Formula sheet provided | Working time: 25 minutes | Marks: 41

−1 units

−1 notation

QUESTION 1

[5 marks – 2, 3]

Evaluate the following logarithms.

a) $\frac{\log_5 8}{\log_5 32}$	b) $2\log_6 3 - \log_6 54 + 2$
$= \frac{\log_5 2^3}{\log_5 2^5}$ ✓ writes as powers of 2 $= \frac{3\log_5 2}{5\log_5 2}$ $= \frac{3}{5}$ ✓ applies log law and cancels c.f.	$= \log_6 9 - \log_6 54 + \log_6 36$ ✓ writes as base 6 $= \log_6\left(\frac{9(36)}{54}\right)$ ✓ applies log laws $= \log_6 6$ $= 1$ ✓ simplifies

QUESTION 2

[10 marks – 2, 3, 2, 3]

a) If $\log_a 3 = x$ and $\log_a 5 = y$, express the following in terms of x and y.

i) $\log_a(3\sqrt{5})$	ii) $\log_a\left(\frac{9}{5a}\right)$
$\log_a 3 + \log_a 5^{\frac{1}{2}}$ $= \log_a 3 + \frac{1}{2}\log_a 5$ ✓ applies log laws $= x + \frac{1}{2}y$ ✓ substitutes x, y	$\log_a 9 - \log_a 5 - \log_a a$ ✓ applies log laws $= 2\log_a 3 - \log_a 5 - 1$ ✓ $\log_a(a) = 1$ $= 2x - y - 1$ ✓ substitutes x, y

b) If $\log m = 7$ and $\log n = 4$, evaluate the following.

i) $\log(mn^3)$	ii) $\log\left(\frac{100\sqrt{m}}{n}\right)$
$\log m + 3\log n$ ✓ applies log laws $= 7 + 3(4)$ $= 19$ ✓ substitutes and evaluates	$= \log 100 + \log m^{\frac{1}{2}} - \log n$ ✓ applies log laws $= 2 + \frac{1}{2}\log m - \log n$ ✓ $\log 100 = 2$ $= 2 + \frac{1}{2}(7) - 4$ $= \frac{3}{2}$ ✓ substitutes and evaluates

QUESTION 3

[8 marks – 3, 2, 3]

a) Solve the following equation, stating your answer in terms of a **base ten logarithm**.

$$3^{7x-2} = 5^{x+1}$$

$\log 3^{7x-2} = \log 5^{x+1}$

$(7x-2)\log 3 = (x+1)\log 5$ ✓ applies $\log_{10}$ to both sides and brings powers down

$7x\log 3 - x\log 5 = \log 5 + 2\log 3$

$x(7\log 3 - \log 5) = \log 5 + 2\log 3$ ✓ expands and collects x's on one side

$x = \dfrac{\log 5 + 2\log 3}{7\log 3 - \log 5}$ ✓ correct solution

b) Solve the following equations, stating your answers in terms of **natural logarithms**.

i) $e^{x+1} = 19$

$\ln 19 = x + 1$ ✓ converts to ln form

$x = \ln 19 - 1$ ✓ correct solution

ii) $2e^{2x} - 3e^{x} = 2$

Let $y = e^{x}$

$2y^2 - 3y - 2 = 0$ ✓ substitutes $y = e^{x}$

$(2y+1)(y-2) = 0$

$y = -\frac{1}{2}$ (reject) , $\underline{y = 2}$ ✓ factorises and solves for y.

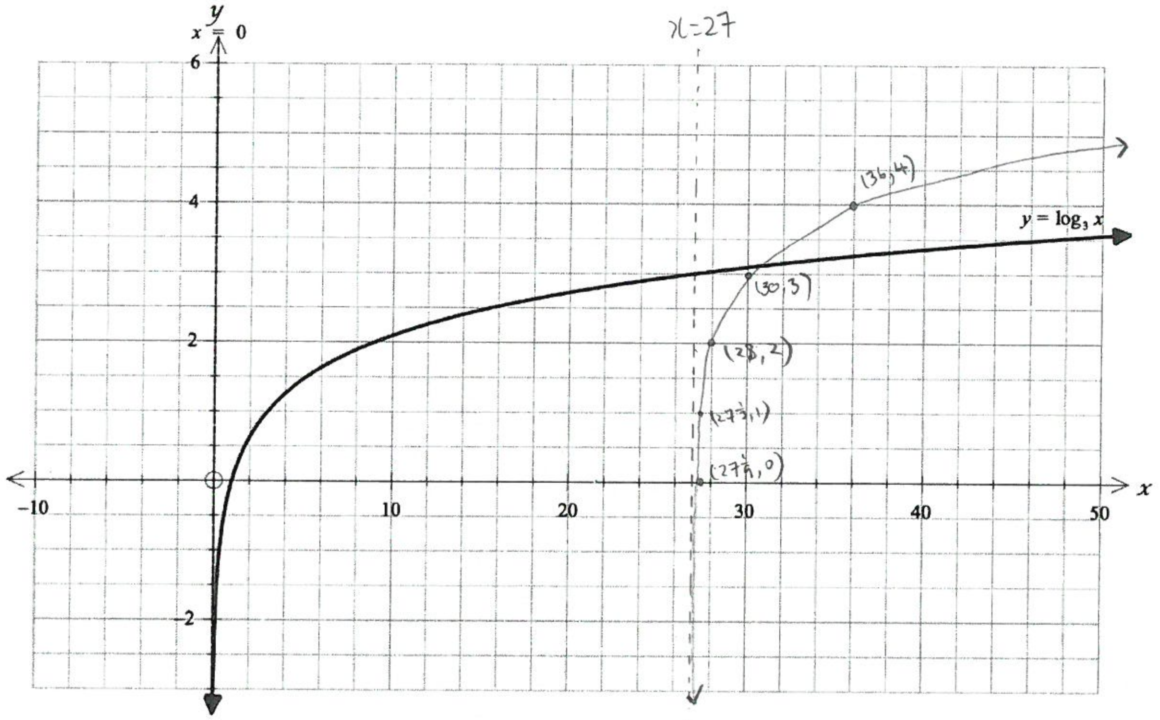
$e^{x} = 2$

$\therefore x = \ln 2$ ✓ solves for x.

QUESTION 4

[9 marks – 1, 2, 2, 2, 2]

The graph of $y = \log_3 x$ is shown below.



a) Use the graph above to solve for the approximate solution to $\log_3 x = 2.5$.

$x \approx 16$ ✓

b) Use the graph above to approximate the solutions to $\log_3(x - 8) = 3.25$.

$x - 8 \approx 36$ ✓

$x \approx 44$ ✓

c)

i) If $y = \log_3 x$ is translated 27 units to the right and 2 units up, state its new equation.

$y = \log_3(x - 27) + 2$ ✓ ✓

ii) State the equation of the asymptote and the coordinates of the x-intercept of the new function.

Asymptote at $x = 27$ ✓ correct asymptote

x-int when $y = 0$:

$-2 = \log_3(x-27)$

$x - 27 = 3^{-2} = \frac{1}{9}$

$x = 27 + \frac{1}{9}$

$= 27\frac{1}{9}$

$\therefore$ x-int at $(27\frac{1}{9}, 0)$ ✓ correct x-int.

iii) Add the sketch of the translated function onto the axes above, labelling its key features. Also label the coordinates of two other points.

✓ labels asymptote / x-int.

✓ two points labelled and accurate, smooth curve.

QUESTION 5

[6 marks – 2, 1, 1, 1, 1]

A uniform continuous random variable X is defined over the interval $5 \leq x \leq 15$.

a) State its probability density function.

$10k = 1 \quad \Rightarrow \quad k = \frac{1}{10}$ ✓ calculates $\frac{1}{10}$

$$f(x) = \begin{cases} \frac{1}{10} & 5 \leq x \leq 15 \\ 0 & \text{elsewhere} \end{cases}$$

✓ writes as piecewise function

b) State the mean of X.

$E(X) = 10$ ✓

c) The variance of X is $\frac{280}{3}$. Write the definite integral that can be used to obtain this value.

$$\int_5^{15} \frac{1}{10}(x-10)^2 \, dx$$ ✓

d) The continuous random variable of Y is such that $Y = 3X + 2$

i) State the mean of Y

$E(Y) = 3E(X) + 2$

$= 3(10) + 2 = 32$ ✓

ii) State the variance of Y

$Var(Y) = 3^2 \, Var(X)$

$= 9\left(\frac{280}{3}\right)$

$= 3(280) = 840$ ✓

QUESTION 6

[3 marks – 1, 2]

Use the 68%, 95%, 99.7% rule to calculate the following probabilities for $X \sim N(0,1)$.

a) $P(X \geq 3)$	b) $P(-2 < X < 1)$
$\frac{0.003}{2} = 0.0015$ ✓	$0.68 + \left(\frac{0.95 - 0.68}{2}\right)$ ✓ $= 0.68 + 0.135$ $= 0.815$ ✓

End of Calculator Free Section

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Calculator-Assumed | Formula sheet provided | Working time: 25 minutes | Marks: 37 marks

QUESTION 7

[4 marks –2, 2]

Calculate the exact value of a in each of the following probability density functions of continuous random variables.

a) $p(x) = \begin{cases} ax^2 & 1 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$

$\int_1^3 ax^2 \, dx = 1$ ✓

$a = \frac{3}{26}$ ✓

b) $p(x) = \begin{cases} 3e^{-2x} & 0 \le x \le a \\ 0 & x < 0 \end{cases}$

$\int_0^a 3e^{-2x} \, dx = 1$ ✓

$a = \frac{\ln 3}{2}$ ✓

QUESTION 8

[3 marks – 1, 2]

A continuous random variable X, as the probability density function given by

$$p(x) = \begin{cases} \frac{1}{2}\cos x & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 0 & \text{elsewhere} \end{cases}$$

Calculate the following probabilities correct to **four decimal places**. −1 if not 4dp.

a) $P(X > \frac{\pi}{3})$	b) $P(X < \frac{\pi}{4} \mid X > -\frac{\pi}{6})$
$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}\cos x \, dx$ $= 0.0670$ (4dp) ✓	$= \dfrac{P(-\frac{\pi}{6} < X < \frac{\pi}{4})}{P(-\frac{\pi}{6} < X < \frac{\pi}{2})}$ ✓ $= 0.8047$ (4dp) ✓

QUESTION 9

[10 marks – 2, 2, 2, 2, 2]

A continuous random variable X has a probability density function given by

$$p(x) = \begin{cases} \frac{1}{4}(2x+1) & 1 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

a) Calculate the mean of X.

$$E(X) = \int_1^2 \frac{1}{4}x(2x+1)\,dx \quad ✓$$

$$= 1.54 \text{ (2dp)} \quad ✓$$

b) Calculate the standard deviation of X.

$$Var(X) = \int_1^2 (x-1.54)^2\left(\frac{1}{4}(2x+1)\right)dx$$ ✓ correct variance

$$= 0.08160$$

$$SD(X) = \sqrt{Var(X)} = 0.29 \text{ (2dp)}$$ ✓ correct s.d.

c) Calculate the median of X.

$$\int_1^a \frac{1}{4}(2x+1)\,dx = 0.5$$ ✓ recognises median ⇒ 0.5 area.

$$a = 1.56 \text{ (2dp)}$$ ✓ solves for median

d) State the cumulative distribution function, $P(x)$.

$$\int_1^x \frac{1}{4}(2t+1)\,dt$$

$$= \frac{x^4}{4} + \frac{x}{4} - \frac{1}{2}$$ ✓ correct equation

$$P(x) = \begin{cases} 0 & x < 1 \\ \frac{x^2}{4} + \frac{x}{4} - \frac{1}{2} & 1 \le x \le 2 \\ 1 & x > 2 \end{cases}$$ ✓ correct piecewise form

e) Show how you would use the cumulative distribution function to calculate $P(1.2 < X < 1.7)$.

$$P(1.2 < X < 1.7) = P(1.7) - P(1.2) \quad ✓$$

$$= \frac{1.7^2}{4} + \frac{1.7}{4} - \frac{1}{2} - \left(\frac{1.2^2}{4} + \frac{1.2}{4} - \frac{1}{2}\right)$$

$$= 0.4875 \quad ✓$$

QUESTION 10

[3 marks – 1, 2]

The heights of 50 Year 12 students are displayed in the table below.

Height (cm) x	Frequency
$140 \leq x < 150$	2
$150 \leq x < 160$	10
$160 \leq x < 170$	19
$170 \leq x < 180$	15
$180 \leq x < 190$	3
$190 \leq x < 200$	1

Use the data in the table to calculate the following probabilities.

a) $P(160 < X < 180)$

$\frac{34}{50} = 0.68$

b) $P(X < 150 | X < 170)$

$\frac{P(140 < X < 150)}{P(140 < X < 170)} = \frac{2}{31} = 0.0645$ (4dp)

QUESTION 11

[5 marks – 2, 1, 2]

Each note on a piano keyboard is one semi-tone apart. The ratio of frequencies between each semitone is 5.946%.

This means that if one note has a frequency of f_1 and another higher note has a frequency of f_2, then

$$1.05946^x = \frac{f_2}{f_1}$$

where x the number of semitones between the two notes.

a) Apply logarithms of base ten to both sides of the above equation and hence obtain a rule for x in terms of f_1 and f_2.

$x \log 1.05946 = \log\left(\frac{f_1}{f_2}\right)$

$x = \frac{\log\left(\frac{f_2}{f_1}\right)}{\log 1.05946}$

Middle C has a frequency of 261.63 Hz.

b) The next C on the keyboard, which is an octave higher, has a frequency of 523.25 Hz. Show the use of your formula from part a) to verify that there are 12 semitones in an octave.

$\frac{\log\left(\frac{523.25}{261.63}\right)}{\log(1.05946)} \approx 12$

c) An interval between two notes is called a "perfect fifth" if they are 7 semi-tones apart. Calculate the frequency of the note that is a perfect fifth higher than middle C.

$7 = \frac{\log\left(\frac{f_2}{261.63}\right)}{\log(1.05946)} \Rightarrow f_2 = 391.99$ Hz (2dp)

QUESTION 12

[4 marks – 2, 2]

a) If $X \sim N(\mu, 4)$ and it is known that $P(X < 28.5) = 0.225$, calculate the value of μ.

In $Z \sim N(0,1)$: $P(X < z) = 0.225$

$z = -0.7554$ (4dp) ✓ calculates z-score

$z = \frac{x-\mu}{\sigma}$

$-0.7554 = \frac{28.5-\mu}{2} \Rightarrow \mu = 30.0$ (1dp) ✓ correct μ.

b) Calculate the 85th percentile for the same random variable X from part a).

$X \sim N(30.0, 4)$

$P(X \leq x) = 0.85$ ✓ $\Rightarrow x = 32.1$ ✓

QUESTION 13

[8 marks – 1, 2, 2, 3]

Loaves of bread made in a particular bakery are found to follow a normal distribution X with mean 250g and standard deviation 30g. $X \sim N(250, 30^2)$

a) Calculate the probability that a randomly selected loaf of bread is greater than 215g.

$P(X > 215) = 0.8783$ (4dp) ✓

b) If there are 120 loaves baked on a particular day, how many would you expect to have a weight between 215g and 275g?

$P(215 < X < 275) = 0.6760$ (4dp) ✓ correct probability

$120 \times 0.6760 \approx 81$ loaves ✓ correct no. of loaves

c) 3% of loaves are rejected for being underweight and 4% of loaves are rejected for being overweight. What is the range of weights of a loaf of bread such that it should be accepted?

$P(X < x) = 0.03 \Rightarrow x = 193.58$

$P(X > y) = 0.04 \Rightarrow y = 302.52$ ✓ lower bound

✓ upper bound

$\therefore$ From 193.58g to 302.52g

d) Calculate the probability that out of 50 loaves of bread, at least 45 of them will have a weight greater than 215 g.

$Y \sim Bin(50, 0.8783)$ ✓ recognises binomial distribution

$P(Y \geq 45) = 0.4211$ (4dp) ✓ correct probability

✓ interprets "at least 45" correctly

End of Calculator Assumed Section