Chapter 19 – Revision of chapters 17–18

Solutions to 19A Short-answer questions

1 $\dfrac{s(4)-s(2)}{4-2} = \dfrac{6(4)^2-6(2)^2}{2}$

$= \dfrac{96-24}{2}$

$= 36\,\text{cm}^2/\text{cm}$

2 **a** $x(0) = 0$ and $x(1) = 1$

Average velocity $= \dfrac{x(1)-x(0)}{1-0}$

$= 1$ cm/s

b $x(1) = 1$ and $x(4) = 124$

Average velocity $= \dfrac{x(4)-x(1)}{4-1}$

$= 41$ m/s

3 **a** **i** Average rate of change

$= \dfrac{0-8}{3-1} = -4$

ii Average rate of change

$= \dfrac{5-8}{2-1} = -3$

b Average rate of change

$= \dfrac{(9-(1+h)^2)-(9-1)}{1+h-1}$

$= \dfrac{9-(1+2h+h^2)-8}{h}$

$= \dfrac{-2h-h^2}{h}$

$= -2-h$

c -2

4 $\dfrac{f(x+h)-f(x)}{x+h-x}$

$= \dfrac{\frac{1}{2}(x+h)^2-(x+h)-(\frac{1}{2}x^2-x)}{h}$

$= \dfrac{xh+\frac{1}{2}h^2-h}{h}$

$= x+\frac{1}{2}h-1$

$\therefore f'(x) = x-1$

5 **a** Let $f(x) = 2x^3-x+1$

$\therefore f'(x) = 6x^2-1$

b Let $f(x) = (x-1)(x-2) = x^2+x-2$

$\therefore f'(x) = 2x+1$

c Let $f(x) = \dfrac{x^2+5x}{x} = x+5$

$\therefore f'(x) = 1$

6 **a** Let $y = 3x^4+x$

Then $\dfrac{dy}{dx} = 12x^3+1$

When $x = 1$, $\dfrac{dy}{dx} = 13$

Gradient $= 13$ at the point(1, 4)

b Let $y = 2x(1-x) = 2x-x^2$

Then $\dfrac{dy}{dx} = 2-2x$

When $x = -2$, $\dfrac{dy}{dx} = 10$

Gradient $= 10$ at the point(-2, -12)

7 **a** $f(x) = 0$

$x-2x^2 = 0$

$x(1-2x) = 0x = 0$ or $x = \dfrac{1}{2}$

b $f'(x) = 0$

$1 - 4x = 0$

$x = \frac{1}{4}$

c $f'(x) > 0$

$1 - 4x > 0$

$x < \frac{1}{4}$

d $f'(x) < 0$

$1 - 4x < 0$

$x > \frac{1}{4}$

e $f'(x) = 10$

$1 - 4x = 10$

$4x = 11\ x = \frac{11}{4}$

8 a $\frac{d}{dx}(2x^{-3} - x^{-1}) = -6x^{-4} + x^{-2}$

b $\frac{d}{dz}\left(\frac{3 - z}{z^3}\right) = \frac{d}{dz}(3z^{-3} - z^{-2}) =$
$-9z^{-4} + 2z^{-3} = \frac{2z - 9}{z^4}$

9 Let $y = x^2 - 5x$

$\frac{dy}{dx} = 2x - 5$

When $x = 1, \frac{dy}{dx} = -3$

When $x = 1, y = -4$

Therefore equation of tangent:

$y + 4 = -3(x - 1)$

$y = -3x - 1.$

Normal has gradient $\frac{1}{3}$

Equation of Normal $y = \frac{1}{3}x - \frac{13}{3}$

10 $x = \frac{1}{6}t^3 - \frac{1}{2}t^2$

$v = \frac{dx}{dt} = \frac{1}{2}t^2 - t$

$a = \frac{dv}{dt} = t - 1$

a $v = 0 \Rightarrow \frac{1}{2}t^2 - t = 0$

$\Rightarrow t(\frac{1}{2}t - 1) = 0$

$\Rightarrow t = 0$ and $t = 2$

b $t = 0, a = -1$ cm/s^2; $t = 2$,
$a = 1$ cm/s^2

c $a = 0 \Rightarrow t = 1 \Rightarrow v = -\frac{1}{2}$ cm/s

11 $y = 2(x^3 - 4x) = 2x^3 - 8x\ \frac{dy}{dx} = 6x^2 - 8$

$\frac{dy}{dx} = 0$

$\Rightarrow 3x^2 - 4 = 0$

$\Rightarrow x = \frac{2}{\sqrt{3}}$ or $x = -\frac{2}{\sqrt{3}}$

When $x = \frac{2}{\sqrt{3}}, y = \frac{32}{3\sqrt{3}}$

When $x = -\frac{2}{\sqrt{3}}, y = \frac{32}{3\sqrt{3}}$

Local minimum $\left(\frac{2}{\sqrt{3}}, -\frac{32}{3\sqrt{3}}\right)$

Local maximum $\left(-\frac{2}{\sqrt{3}}, \frac{32}{3\sqrt{3}}\right)$

Leading coefficient of the cubic is positive.

Solutions to 19B Multiple-choice questions

1 C 1st week: $t = 0$ to $t = 1$
2nd week: $t = 1$ to $t = 2$
3rd week: $t = 2$ to $t = 3$
4th week: $t = 3$ to $t = 4$
5th week: $t = 4$ to $t = 5$

$$\frac{P(5) - P(4)}{5 - 4} = \frac{10 \times 1.1^5 - 10 \times 1.1^4}{5 - 4}$$
$$= 1.4641$$

2 A Gradient $\approx \dfrac{0 - 60}{6 - 0} = -10$

3 B Av. speed $= \dfrac{3 - 0}{3 - 0} = 1$ m/s

4 A
$$\text{Av. rate} = \frac{f(2) - f(0)}{2 - 0} = \frac{13 - 1}{2} = 6$$

5 B
$$\text{Av. rate} = \frac{23.5 - 10}{12 - 7} = 2.7^\circ\text{C/h}$$

6 A $y = 5x^2 + 1 \therefore \dfrac{dy}{dx} = 10x$

7 D
$$f(5 + h) - f(5) = (5 + h)^2 - 5^2 = 10h + h^2$$

8 B Gradient = 0 at turning points
$x = -1, 1.5$

9 C $V = 3t^2 + 4t + 2, \therefore V' = 6t + 4$
$$\therefore V'(2) = 6(2) + 4 = 16 \text{ m}^3/\text{ min}$$

10 A $\dfrac{f(3 + h) - f(3)}{h} = 2h^2 + 2h$
$\therefore \lim\limits_{h \to 0} 2h^2 + 2h = 0$

11 C Curve increases for
$x \in (-\infty, -2) \cup (1, \infty)$

12 B
$$f(x) = x^3 - x^2 - 5$$
$$\therefore f'(x) = 3x^2 - 2x = x(3x - 2)$$
$$\therefore f' = 0, x = 0, \frac{2}{3}$$

13 A $f'(x) = \dfrac{0 - 3}{5 - 0} = -\dfrac{3}{5}$ for all x

14 C $y = 2x^3 - 3x^2, \therefore y' = 6x^2 - 6x$
$\therefore\ y'(1) = 6 - 6 = 0$

15 C $y = 7 + 2x - x^2, \therefore y' = 2(1 - x)$
Inverted parabola, so
y max. $= y(1) = 8$

16 A $s = 28t - 16t^2, \therefore v = 28 - 32t$
$$s_{\max} = s\frac{7}{8} = \frac{49}{2} - \frac{49}{4} = \frac{49}{4} \text{ m/s}$$

17 D $f'(x) > 0$ for $x < 1$, $f'(x) < 0$ for $x > 1$
$f'(1) = 0$; only **D** fits.

18 E $f'(-2) > 0, f'(-1) = 0, f'(0) < 0$
$f(x)$ has a local max. at $x = -1$.

19 C
$$y = \frac{x^2}{2}(x^2 + 2x - 4) = \frac{x^4}{2} + x^3 - 2x^2$$
$$\therefore \frac{dy}{dx} = 2x^3 + 3x^2 - 4x$$

20 B $\dfrac{d}{dx}(5 + 3x^2) = 6x$

21 E Negative slope for $x < -1, x > 1$

22 D $\text{Rise/run} = \dfrac{(1+h)^2 - 1}{1 + h - 1}$
$= 2 + h$

23 A $y = x^2(2x - 3) = 2x^3 - 3x^2$
$y' = 6x^2 - 6x < \therefore y'(1) = 0$

24 A $\text{Rise/run} = \dfrac{b^2 - a^2}{b - a} = b + a$

25 C $f(x) = 3x^3 + 6x^2 - x + 1$
$\therefore f'(x) = 9x^2 + 12x - 1$

26 D $y + 3x = 10, \therefore y = 10 - 3x$
$A = 4x(10 - 3x)$
$\therefore A' = 40 - 24x = 0$
$\therefore 5 - 3x = 0$

27 B $y = x^2 + 3, \therefore y' = 2x$
$\therefore y'(3) = 6$

28 B $y = x^3 + 5x^2 - 8x$
$\therefore y' = 3x^2 + 10x - 8$
$= (3x - 2)(x + 4)$

x	-5	-4	0	$\frac{2}{3}$	1
y'	$+$	0	$-$	0	$+$

$x = -4$ is a local maximum.
$x = \dfrac{2}{3}$ is a local minimum.

29 B $f'(1) = \lim\limits_{h \to 0} \dfrac{f(1+h) - f'(1)}{h}$

30 C $y = x^2 + 4x - 3$
$\therefore y' = 2(x + 2)$
$y\,\text{min} = y(-2) = -7$

31 D $y = x^2, \therefore y' = 2x$
$y'(2) = 4$
$\therefore$ gradient of normal $= -\dfrac{1}{4}$

32 E $y = \dfrac{2x + 5}{x} = 2 + 5x^{-1}$
$\therefore \dfrac{dy}{dx} = -5x^{-2} = -\dfrac{5}{x^2}$

33 A $y = x^2 - 3x - 4, \therefore y' = 2x - 3$
$y' < 0, \therefore x < \dfrac{3}{2}$

34 A $\lim\limits_{x \to 0} \dfrac{x^2 - x}{x} = x - 1 = -1$

35 C Graph is discontinuous at $x = 0, 2$ since in both cases the positive and negative limits are different.

36 C Graph is discontinuous at $x = -1, 1$ since in both cases the positive and negative limits are different.

37 D

38 D

39 D

Solutions to 19C Extended-response questions

1 a When the particle returns to ground level, $y = 0$

$$\therefore \quad x - 0.01x^2 = 0$$
$$\therefore \quad x(1 - 0.01x) = 0$$
$$\therefore \quad x = 0 \quad \text{or} \quad 1 - 0.01x = 0$$
$$0.01x = 1$$
$$x = 100$$

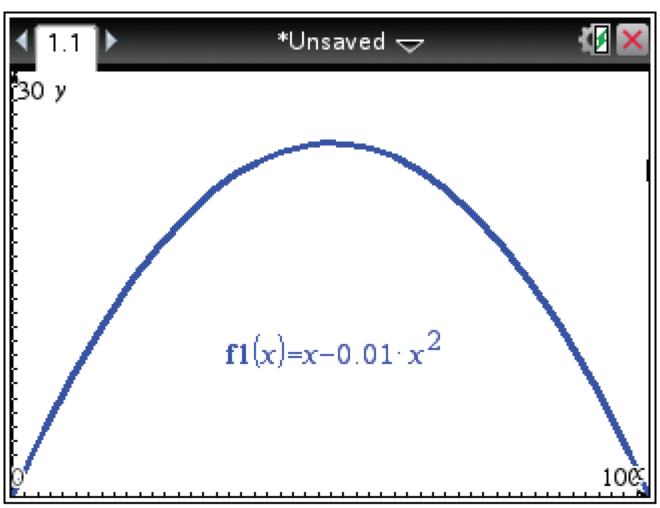
The particle travels 100 units horizontally before returning to ground level.

b
$$y = x - 0.01x^2$$
$$\therefore \quad \frac{dy}{dx} = 1 - 0.02x$$

c
$$\frac{dy}{dx} = 0$$
$$\therefore \quad 1 - 0.02x = 0$$
$$\therefore \quad 0.02x = 1$$
$$\therefore \quad x = 50$$

When $x = 50$,
$$\begin{aligned} y &= 50 - 0.01(50)^2 \\ &= 50 - 0.01 \times 2500 \\ &= 50 - 25 \\ &= 25 \end{aligned}$$

d



e **i** When $\frac{dy}{dx} = \frac{1}{2}$, $1 - 0.02x = \frac{1}{2}$

$\therefore \quad 0.02x = \frac{1}{2}$

$\therefore \quad x = 25$

When $x = 25$, $y = 25 - 0.01(25)^2$

$= 25 - 0.01 \times 625$

$= 25 - 6.25$

$= 18.75$

i.e. the coordinates of the point with gradient $\frac{1}{2}$ are (25, 18.75).

ii When $\frac{dy}{dx} = -\frac{1}{2}$, $1 - 0.02x = -\frac{1}{2}$

$\therefore \quad 0.02x = 1.5$

$\therefore \quad x = 75$

When $x = 75$, $y = 75 - 0.01(75)^2$

$= 75 - 0.01 \times 5625$

$= 75 - 56.25$

$= 18.75$

i.e. the coordinates of the point with gradient $-\frac{1}{2}$ are (75, 18.75).

2 **a** $y = -0.0001(x^3 - 100x^2)$

$= -0.0001x^3 + 0.01x^2$

Highest point is reached where $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -0.0003x^2 + 0.02x$

When $\frac{dy}{dx} = 0$, $-0.0003x^2 + 0.02x = 0$

$\therefore \quad x(0.02 - 0.0003x) = 0$

$\therefore \quad x = 0$ or $0.02 - 0.0003x = 0$

$\therefore \quad 0.0003x = 0.02$

$\therefore \quad x = \frac{200}{3}$

$= 66\frac{2}{3}$

When $x = 0$, $\quad y = 0$

When $x = 66\frac{2}{3}$,

$$\begin{aligned} y &= -0.0001x^2(x - 100) \\ &= -0.0001\left(\frac{200}{3}\right)^2\left(\frac{200}{3} - 100\right) \\ &= -0.0001 \times \frac{40\,000}{9}\left(-\frac{100}{3}\right) \\ &= -\frac{4}{9} \times -\frac{100}{3} \\ &= \frac{400}{27} \\ &= 14\frac{22}{27} \end{aligned}$$

i.e. the coordinates of the highest point are $\left(66\frac{2}{3}, 14\frac{22}{27}\right)$.

b **i** At $x = 20$,

$$\begin{aligned} \frac{dy}{dx} &= x(0.02 - 0.0003x) \\ &= 20(0.02 - 0.0003 \times 20) \\ &= 20(0.02 - 0.006) \\ &= 20 \times 0.014 \\ &= 0.28 \end{aligned}$$

i.e. at $x = 20$, the gradient of the curve is 0.28.

ii At $x = 80$,

$$\begin{aligned} \frac{dy}{dx} &= x(0.02 - 0.0003x) \\ &= 80(0.02 - 0.0003 \times 80) \\ &= 80(0.02 - 0.024) \\ &= 80 \times -0.004 \\ &= -0.32 \end{aligned}$$

i.e. at $x = 80$, the gradient of the curve is -0.32.

iii At $x = 100$,

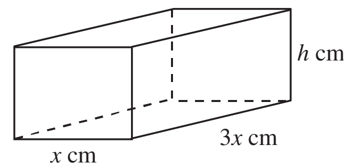
$$\begin{aligned} \frac{dy}{dx} &= x(0.02 - 0.0003x) \\ &= 100(0.02 - 0.0003 \times 100) \\ &= 100(0.02 - 0.03) \\ &= 100 \times -0.01 \\ &= -1 \end{aligned}$$

i.e. at $x = 100$, the gradient of the curve is -1.

c The rollercoaster begins with a gentle upwards slope until it reaches the turning point (its highest point). On its downward trip the rollercoaster has a steeper slope and by the end of the ride it has reached a very steep downward slope.

d It would be less dangerous if the steep slope at the end were smoothed out.

3 a Let h = height of the block.



$$\begin{aligned} \text{Now} \quad 4(3x + x + h) &= 20 \\ \therefore \quad 4(4x + h) &= 20 \\ \therefore \quad 4x + h &= 5 \\ \therefore \quad h &= 5 - 4x \end{aligned}$$

i.e. the height of the block is $(5 - 4x)$ cm.

b
$$\begin{aligned} V &= x \times 3x \times (5 - 4x) \\ &= 3x^2(5 - 4x) \\ &= 15x^2 - 12x^3 \text{ as required} \end{aligned}$$

c $x > 0$ and $V > 0$

$$\begin{aligned} \therefore \quad & 15x^2 - 12x^3 > 0 \\ \Longleftrightarrow \quad & 3x^2(5 - 4x) > 0 \\ \Longleftrightarrow \quad & 5 - 4x > 0 \text{ as } 3x^2 > 0 \text{ for all } x \\ \Longleftrightarrow \quad & 5 > 4x \\ \Longleftrightarrow \quad & \frac{5}{4} > x \end{aligned}$$

Domain is $\left\{x\colon 0 < x < \dfrac{5}{4}\right\}$

d $\dfrac{dV}{dx} = 30x - 36x^2$

e When $\dfrac{dV}{dx} = 0$, $\quad 30x - 36x^2 = 0$

$$\begin{aligned} \therefore \quad & 6x(5 - 6x) = 0 \\ \therefore \quad & 6x = 0 \quad \text{or} \quad 5 - 6x = 0 \\ \therefore \quad & x = 0 \quad \text{or} \quad x = \frac{5}{6} \\ \therefore \quad & x = \frac{5}{6} \text{ as } x > 0 \end{aligned}$$

When $x = \frac{5}{6}$, $\quad V = 3\left(\frac{5}{6}\right)^2\left(5 - 4 \times \frac{5}{6}\right)$

$$= 3 \times \frac{25}{36}\left(5 - \frac{10}{3}\right) = \frac{25}{12} \times \frac{5}{3}$$

$$= \frac{125}{36} = 3\frac{17}{36}$$

$$\frac{dV}{dx} = 6x(5 - 6x)$$

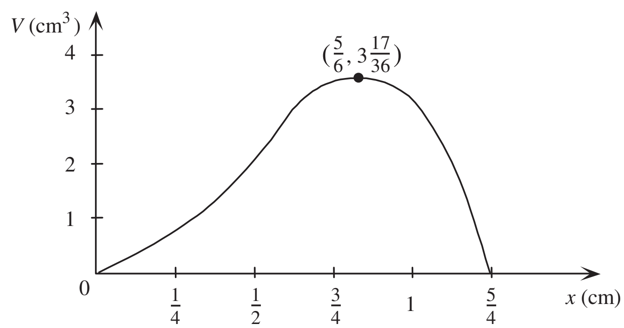
If $x < \frac{5}{6}$, e.g. $x = \frac{1}{6}$, $\frac{dV}{dx} > 0$.

If $x > \frac{5}{6}$, e.g. $x = 1$, $\frac{dV}{dx} < 0$.

$\therefore$ local maximum at $\left(\frac{5}{6}, \frac{125}{36}\right)$.

i.e. the maximum volume possible is $3\frac{17}{36}$ cm^3, for $x = \frac{5}{6}$.

f



4 a $\quad h = 30t - 5t^2$

$$\frac{dh}{dt} = 30 - 10t$$

b Maximum height is reached where $\frac{dh}{dt} = 0$

$\therefore \quad 30 - 10t = 0$

$\therefore \quad 10t = 30 \quad \therefore t = 3$

(a maximum, as it is a quadratic with negative coefficient of t^2)

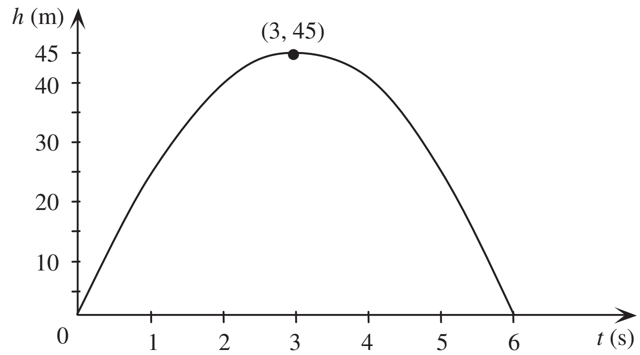
When $t = 3$, $\quad h = 30(3) - 5(3)^2$

$$= 90 - 5 \times 9$$

$$= 90 - 45 = 45$$

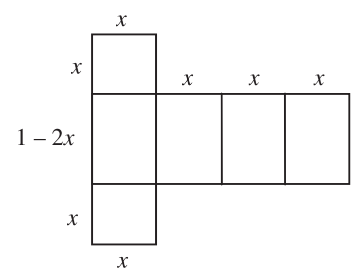
i.e. maximum height reached is 45 m after 3 seconds.

c



5 a Let A = surface area of the net

$$A = 4x(1 - 2x) + 2x^2$$
$$= 4x - 8x^2 + 2x^2$$
$$= 4x - 6x^2$$



b

$$V = x \times x \times (1 - 2x)$$
$$= x^2(1 - 2x)$$
$$= x^2 - 2x^3$$

c $x > 0$ and $V > 0$

$\therefore \quad x^2 - 2x^3 > 0$

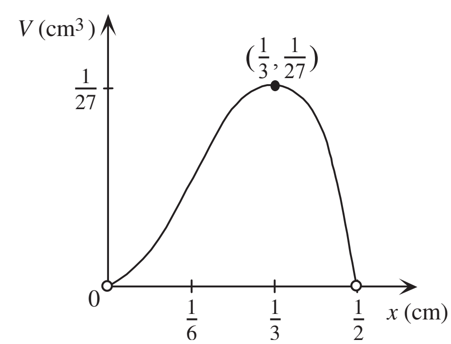
$\Longleftrightarrow \quad x^2(1 - 2x) > 0$

$\Longleftrightarrow \quad 1 - 2x > 0$

(as $x^2 > 0$ for all x)

$\therefore \quad x < \frac{1}{2}$

Domain $\left\{x\colon 0 < x < \frac{1}{2}\right\}$



When $x = 0$, $V = 0$

When $x = \frac{1}{2}$, $V = 0$

$$\frac{dV}{dx} = 2x - 6x^2$$

When $\frac{dV}{dx} = 0$, $\quad 2x - 6x^2 = 0$

$\therefore \quad 2x(1 - 3x) = 0$

$\therefore \quad x = 0$ or $x = \frac{1}{3}$

When $x = \frac{1}{3}$, $\quad V = \left(\frac{1}{3}\right)^2 - 2\left(\frac{1}{3}\right)^3 = \frac{1}{9} - \frac{2}{27} = \frac{1}{27}$

If $x < \frac{1}{3}$, e.g. $x = \frac{1}{6}, \frac{dV}{dx} > 0$.
If $x > \frac{1}{3}$, e.g. $x = \frac{1}{2}, \frac{dV}{dx} < 0$.
$\therefore$ a local maximum at $\left(\frac{1}{3}, \frac{1}{27}\right)$

d A box with dimensions $\frac{1}{3}$ cm $\times \frac{1}{3}$ cm $\times \frac{1}{3}$ cm will give a maximum volume of $\frac{1}{27}$ cm^3.

6 a i Using Pythagoras' theorem:

$$x^2 + r^2 = 1^2$$
$$\therefore \quad r^2 = 1 - x^2$$
$$\therefore \quad r = \sqrt{1 - x^2}$$

ii $h = 1 + x$

b $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(1 - x^2)(1 + x)$

$= \frac{\pi}{3}(1 + x - x^2 - x^3)$ as required

c $x > 0$ and $V > 0$
For $V > 0, \frac{\pi}{3}(1 - x^2)(1 + x) > 0$

$\iff \quad 1 - x^2 > 0$ as $1 + x > 0$ for all $x > 0$

$\iff \quad -1 < x < 1$

$\therefore \quad V > 0$ for $-1 < x < 1$
To satisfy $x > 0$ and $V > 0$, domain is $\{ x\colon 0 < x < 1 \}$.

d i $\frac{dV}{dx} = \frac{\pi}{3}(1 - 2x - 3x^2)$

ii When $\frac{dV}{dx} = 0, \quad \frac{\pi}{3}(1 - 2x - 3x^2) = 0$

$\therefore \quad \frac{-\pi}{3}(3x^2 + 2x - 1) = 0$

$\therefore \quad \frac{-\pi}{3}(3x - 1)(x + 1) = 0$

$\therefore \quad 3x - 1 = 0 \quad \text{or} \quad x + 1 = 0$

$\therefore \quad 3x = 1 \qquad\qquad x = -1$

$\therefore \quad x = \frac{1}{3}$

$\therefore \quad x = \frac{1}{3}$, as $x > 0$

i.e. $\left\{x\colon \frac{dV}{dx} = 0\right\} = \left\{x\colon x = \frac{1}{3}\right\}$

iii When $x = \frac{1}{3}$,

$$\begin{aligned} V &= \frac{\pi}{3}\left(1 - \left(\frac{1}{3}\right)^2\right)\left(1 + \frac{1}{3}\right) \\ &= \frac{\pi}{3}\left(1 - \frac{1}{9}\right)\left(\frac{4}{3}\right) \\ &= \frac{\pi}{3} \times \frac{8}{9} \times \frac{4}{3} \\ &= \frac{32\pi}{81} \\ &\approx 1.24 \end{aligned}$$

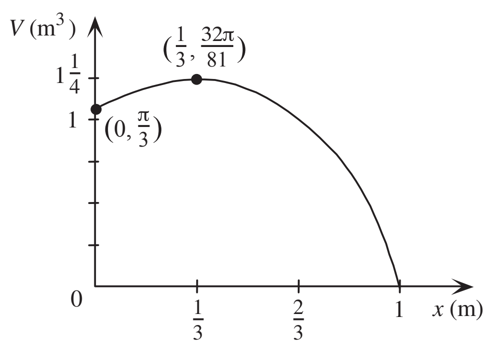
If $x < \frac{1}{3}$, e.g. $x = \frac{1}{6}$, $\frac{dV}{dx} > 0$.

If $x < \frac{1}{3}$, e.g. $x = \frac{1}{2}$, $\frac{dV}{dx} < 0$.

$\therefore$ local maximum at $\left(\frac{1}{3}, \frac{32\pi}{81}\right)$.

i.e. the maximum volume of the cone is $\frac{32\pi}{81}$ m^3 or approximately 1.24 m^3.

e



7 **a** When $t = 0$, $\quad P(0) = 1000 \times 2^{\frac{0}{20}}$

$\qquad = 1000$

On 1 January 1993, there were 1000 insects in the colony.

b When $t = 9$,
$$\begin{aligned} P(9) &= 1000 \times 2^{\frac{9}{20}} \\ &= 1000 \times 2^{0.45} \\ &\approx 1366 \end{aligned}$$
On 10 January, there were approximately 1366 insects in the colony.

c **i** When $P(t) = 4000$,
$$\begin{aligned} 1000 \times 2^{\frac{t}{20}} &= 4000 \\ \therefore \quad 2^{\frac{t}{20}} &= 4 \\ \therefore \quad 2^{\frac{t}{20}} &= 2^2 \\ \therefore \quad \frac{t}{20} &= 2 \\ \therefore \quad t &= 40 \end{aligned}$$

ii When $P(t) = 6000$,
$$\begin{aligned} 1000 \times 2^{\frac{t}{20}} &= 6000 \\ \therefore \quad 2^{\frac{t}{20}} &= 6 \\ \therefore \quad \log_{10} 2^{\frac{t}{20}} &= \log_{10} 6 \\ \therefore \quad \frac{t}{20} &= \frac{\log_{10} 6}{\log_{10} 2} \\ \therefore \quad t &= \frac{20 \log_{10} 6}{\log_{10} 2} \\ &\approx 51.70 \end{aligned}$$

d
$$\begin{aligned} P(20) &= 1000 \times 2^{\frac{20}{20}} \\ &= 1000 \times 2 \\ &= 2000 \end{aligned}$$

$$\begin{aligned} P(15) &= 1000 \times 2^{\frac{15}{20}} \\ &\approx 1000 \times 1.681\,792\,831 \\ &\approx 1681.792\,831 \end{aligned}$$
Average rate of change of P with respect to time, for the interval of time $[15, 20]$
$$\begin{aligned} &= \frac{P(20) - P(15)}{20 - 15} \\ &\approx \frac{2000 - 1681.792\,831}{5} \\ &\approx \frac{318.207\,169\,5}{5} \approx 63.64 \end{aligned}$$

e **i** Average rate of change $= \dfrac{P(15+h) - P(15)}{15+h-15}$

$$= \frac{1000 \times 2^{\frac{15+h}{20}} - 1000 \times 2^{\frac{15}{20}}}{h}$$

$$= \frac{1000 \times 2^{\frac{3}{4}} \times 2^{\frac{h}{20}} - 1000 \times 2^{\frac{3}{4}}}{h}$$

$$= \frac{1000 \times 2^{\frac{3}{4}}\left(2^{\frac{h}{20}} - 1\right)}{h}, h \neq 0$$

ii Consider h decreasing and approaching zero:

Let $h = 0.0001$

Average rate of change $\approx \dfrac{1681.792\,831(2^{0.000\,005} - 1)}{0.0001}$

$\approx 58.286\,566\,86$

Let $h = 0.00001$

Average rate of change $\approx \dfrac{1681.792\,831(2^{0.000\,000\,5} - 1)}{0.000\,01}$

$\approx 58.285\,894\,14$

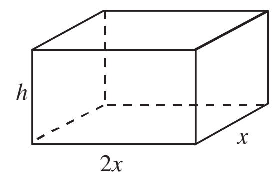
Let $h = 0.000\,001$

Average rate of change $\approx \dfrac{1681.792\,831(2^{0.000\,000\,05} - 1)}{0.000\,001}$

$\approx 58.286\,566\,86$

Hence as $h \to 0$, the instantaneous rate of change is approaching 58.287 insects per day.

8 a Let A (m^2) be the total surface area of the block.



Now $\quad A = 300$

and $\quad A = 2(2xh + 2x^2 + xh)$

$\quad = 2(2x^2 + 3xh)$

$\therefore \quad 2(2x^2 + 3xh) = 300$

$\therefore \quad 2x^2 + 3xh = 150$

$\therefore \quad 3xh = 150 - 2x^2$

$\therefore \quad h = \dfrac{150 - 2x^2}{3x}$

b $V = h \times 2x \times x$

$= \dfrac{150 - 2x^2}{3x} \times 2x^2$

$= \dfrac{2}{3}x(150 - 2x^2)$

c $\quad V = 100x - \dfrac{4}{3}x^3$

$\therefore \quad \dfrac{dV}{dx} = 100 - 4x^2$

d When $V = 0$, $\quad \dfrac{2}{3}x(150 - 2x^2) = 0$

$\therefore \quad \dfrac{2}{3}x = 0 \quad$ or $\quad 150 - 2x^2 = 0$

$\therefore \quad x = 0 \quad$ or $\quad 2x^2 = 150$

$\therefore \quad x^2 = 75$

$\therefore \quad x = \pm 5\sqrt{3}$

When $x = 1$, $\quad V = \dfrac{2}{3} \times 1(150 - 2(1)^2)$

$= \dfrac{2}{3}(148) = \dfrac{296}{3} > 0$

$\therefore \quad V > 0$ for $0 < x < 5\sqrt{3}$

Note also, for $x > 0$

$$\frac{2}{3}x(150 - 2x^2) > 0$$

$$\iff \quad 150 - 2x^2 > 0$$

$$\iff \quad 75 > x^2$$

$$\iff \quad 5\sqrt{3} > x$$

e Maximum value of V occurs when $\frac{dV}{dx} = 0$

$\therefore \quad 100 - 4x^2 = 0$

$\therefore \quad 4x^2 = 100$

$\therefore \quad x^2 = 25$

$\therefore \quad x = \pm\sqrt{25}$

$x = 5$ as $x > 0$

When $x = 5$, $V = \frac{2}{3} \times 5(150 - 2(5)^2)$

$= \frac{10}{3}(150 - 50)$

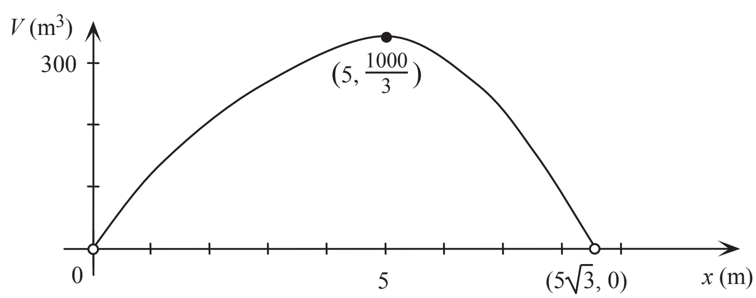
$= \frac{1000}{3} = 333\frac{1}{3}$

When $x < 5$, e.g. $x = 4$, $\frac{dV}{dx}$ and when $x > 5$, e.g. $x = 6$, $\frac{dV}{dx} < 0$

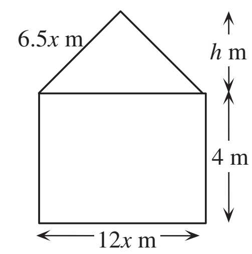
$\therefore$ a local maximum at $\left(5, \frac{1000}{3}\right)$.

i.e. when $x = 5$ m, the block has its maximum volume of $\frac{1000}{3}$ m^3 or $333\frac{1}{3}$ m^3.

f



9 **a**



$$12x + y + y + 6.5x + 6.5x = 70$$

$\therefore$ $$25x + 2y = 70$$

If $x = 2$, $$25(2) + 2y = 70$$

$\therefore$ $$50 + 2y = 70$$

$\therefore$ $$y = 10$$

b $$25x + 2y = 70$$

$$2y = 70 - 25x$$

$\therefore$ $$y = \frac{70 - 25x}{2} \text{ as required}$$

c **i** Using Pythagoras' theorem:

$$h^2 + (6x)^2 = (6.5x)^2$$

$\therefore$ $$h^2 + 36x^2 = 42.25x^2$$

$\therefore$ $$h^2 = 6.25x^2$$

$\therefore$ $$h = \sqrt{6.25x^2}$$

$$= 2.5x \text{ as } x > 0$$

ii Let A (m) be the area of the front face of the building.

$$A = \text{area of rectangle} + \text{area of triangle}$$

$$= 12x \times y + \frac{1}{2} \times 12x \times 2.5x$$

$$= 12xy + 15x^2$$

$$= 15x^2 + 12xy \text{ as required}$$

d Let V (cm^3) be the volume of the building.

$$V = A \times 40$$

$$= 40(15x^2 + 12xy)$$

$$= 40\left(15x^2 + 12x\left(\frac{70 - 25x}{2}\right)\right)$$

$$= 40(15x^2 + 6x(70 - 25x))$$

$$= 40(15x^2 + 420x - 150x^2)$$

$$= 600x(28 - 9x)$$

e **i** $$V = 600(28x - 9x^2)$$

Volume is a maximum when $\frac{dV}{dx} = 0$

$$\therefore \quad \frac{dV}{dx} = 600(28 - 18x)$$

$$\therefore \quad 600(28 - 18x) = 0$$

$$\therefore \quad 28 - 18x = 0$$

$$\therefore \quad 18x = 28$$

$$\therefore \quad x = \frac{28}{18} = \frac{14}{9}$$

When $x = \frac{14}{9}$,

$$\begin{aligned} y &= \frac{70 - 25\left(\frac{14}{9}\right)}{2} \\ &= \frac{70 - \frac{350}{9}}{2} \\ &= \frac{630 - 350}{18} \\ &= \frac{280}{18} = \frac{140}{9} \end{aligned}$$

When $x < \frac{14}{9}$, e.g. $x = 1$, $\frac{dV}{dx} > 0$.

When $x > \frac{14}{9}$, e.g. $x = 2$, $\frac{dV}{dx} < 0$.

$\therefore$ a local maximum at $\left(\frac{14}{9}, \frac{140}{9}\right)$.

i.e. the volume is a maximum when $x = \frac{14}{9}$ and $y = \frac{140}{9}$.

ii When $x = \frac{14}{9}$,

$$\begin{aligned} V &= 40\left(420\left(\frac{14}{9}\right) - 135\left(\frac{14}{9}\right)^2\right) \\ &= 13\,066\frac{2}{3}\ \text{m}^3 \end{aligned}$$

i.e. the maximum volume of the building is $13\,066\frac{2}{3}\text{m}^3$.

10 **a** $y = kx^2(a - x)$

At $(200, 0)$ $\quad 0 = k \times 200^2(a - 200)$

$\therefore$ either $k = 0$ or $a = 200$

At $(170, 8.67)$ $\quad 8.67 = k \times 170^2(a - 170) \quad (1)$

$\therefore \quad k \neq 0 \quad \therefore \quad a = 200 \quad (2)$

Substitute (2) into (1) $\quad 8.67 = k \times 170^2(200 - 170)$

$$\therefore \quad 8.67 = 28\,900k \times 30$$

$$\therefore \quad k = \frac{8.67}{28\,900 \times 30}$$

$$= 0.000\,01$$

$$\therefore \quad y = 0.000\,01x^2(200 - x)$$

b **i**

$$y = 0.000\,01x^2(200 - x)$$

$$\therefore \quad = 0.002x^2 - 0.000\,01x^3$$

At the local maximum, $\dfrac{dy}{dx} = 0$

and $\dfrac{dy}{dx} = 0.004x - 0.000\,03x^2$

$$\therefore \quad 0.004x - 0.000\,03x^2 = 0$$

$$\therefore \quad 0.001x(4 - 0.03x) = 0$$

$$\therefore \quad x = 0 \quad \text{or} \quad 4 - 0.03x = 0$$

$$\therefore \quad 0.03x = 4$$

$$\therefore \quad x = \frac{400}{3}$$

If $x < \dfrac{400}{3}$, e.g. $x = 100$, $\dfrac{dy}{dx} > 0$.

If $x > \dfrac{400}{3}$, e.g. $x = 150$, $\dfrac{dy}{dx} < 0$.

Therefore a local maximum when $x = \dfrac{400}{3}$.

ii When $x = \dfrac{400}{3}$, $\quad y = 0.000\,01\left(\dfrac{400}{3}\right)^2\left(200 - \dfrac{400}{3}\right)$

$$= \frac{16}{90} \times \frac{200}{3}$$

$$= \frac{320}{27}$$

c **i** When $x = 105$,
$$\begin{aligned} y &= 0.000\,01(105)^2(200 - 105) \\ &= \frac{1}{100\,000} \times 11\,025 \times 95 \\ &= \frac{104\,737\,5}{100\,000} \\ &= \frac{8379}{800} \\ &= 10\frac{379}{800} \quad (= 10.473\,75) \end{aligned}$$

ii When $x = 105$,
$$\begin{aligned} \frac{dy}{dx} &= 0.001(105)(4 - 0.03 \times 105) \\ &= \frac{105}{1000}(4 - 3.15) \\ &= \frac{105}{1000} \times \frac{85}{100} \\ &= \frac{8925}{100\,000} = \frac{357}{4000} \end{aligned}$$

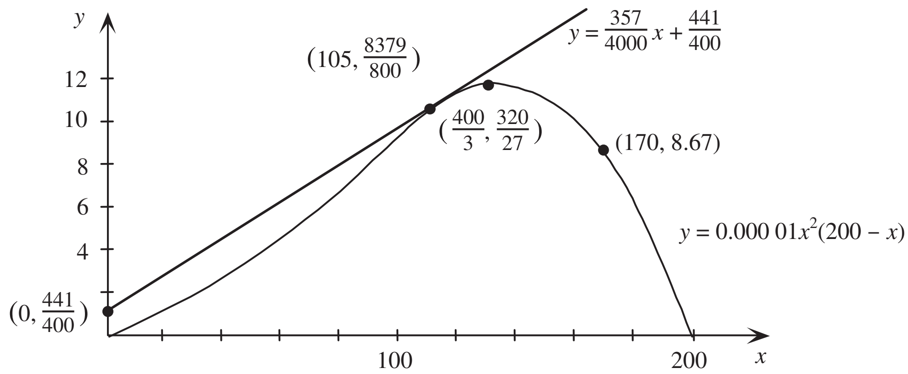
d **i**
$$\begin{aligned} y - y_1 &= m(x - x_1) \\ \therefore \quad y &= \frac{357}{4000}(x - 105) + \frac{8379}{800} \\ \therefore \quad y &= \frac{357}{4000}x - \frac{37485}{4000} + \frac{41895}{4000} \\ \therefore \quad y &= \frac{357}{4000}x + \frac{441}{400} \end{aligned}$$

i.e. the equation of the tangent at $x = 105$ is $y = \frac{357}{4000}x + \frac{441}{400}$.

ii The y-axis intercept of the tangent is $\frac{441}{400}$.

e Average rate of change
$$\begin{aligned} &= \frac{\frac{8379}{800} - 0}{105 - 0} \\ &= \frac{8379}{800 \times 105} \\ &= 0.099\,75 \end{aligned}$$

f $y = 0.000\,01x^2(200 - x)$



11 **a** In the centre of the city $\quad r = 0$

and $\quad P = 10 + 40(0) - 20(0)^2$

$\quad = 10$

i.e. the population density is 10 000 people per square kilometre.

b

$$P > 0$$

$$\therefore \quad 10 + 40r - 20r^2 > 0$$

$$\therefore \quad -10(2r^2 - 4r - 1) > 0$$

When $P = 0$, $\quad 2r^2 - 4r - 1 = 0$

$$\therefore \quad r = \frac{4 \pm \sqrt{4^2 - 4(2)(-1)}}{2 \times 2}$$

$$= \frac{4 \pm \sqrt{16 + 8}}{4}$$

$$= \frac{4 \pm 2\sqrt{6}}{4}$$

$$= \frac{2 \pm \sqrt{6}}{2}$$

and, as $r \geq 0$ $\quad r = \dfrac{2 + \sqrt{6}}{2}$

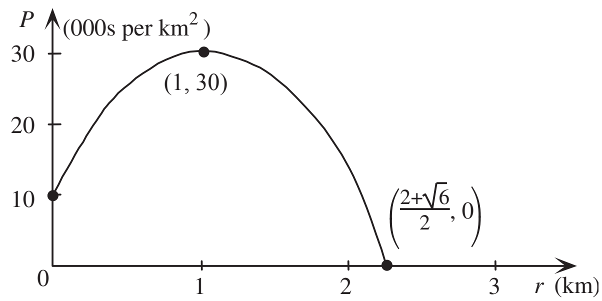
When $r = 1$, $\quad P = 10 + 40(1) - 20(1)^2$

$$= 10 + 40 - 20$$

$$= 30 > 0$$

$$\therefore \quad P > 0 \text{ for } 0 \leq r \leq \frac{2 + \sqrt{6}}{2}$$

c



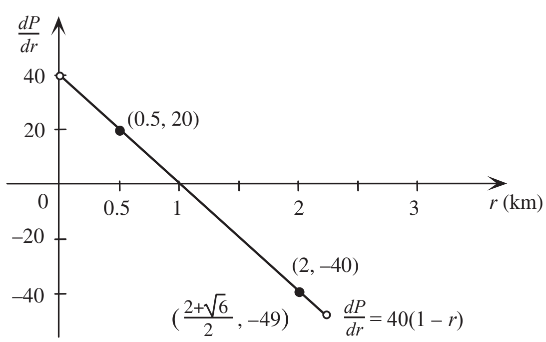
d **i** $\dfrac{dP}{dr} = 40 - 40r$

ii When $r = 0.5$,
$$\begin{aligned}\frac{dP}{dr} &= 40 - 40(0.5)\\ &= 40 - 20\\ &= 20\end{aligned}$$

When $r = 1$,
$$\begin{aligned}\frac{dP}{dr} &= 40 - 40(1)\\ &= 40 - 40\\ &= 0\end{aligned}$$

When $r = 2$,
$$\begin{aligned}\frac{dP}{dr} &= 40 - 40(2)\\ &= 40 - 80\\ &= -40\end{aligned}$$

iii



e The population density is greatest at a 1 km radius from the city centre.

12 **a**
$$\begin{aligned}y &= x(a - x)\\ &= ax - x^2\end{aligned}$$

b $0 < x < a$

c Maximum value of y is found where $\frac{dy}{dx} = 0$.

$$\frac{dy}{dx} = a - 2x$$

$\therefore \quad a - 2x = 0$

$\therefore \quad 2x = a$

$\therefore \quad x = \frac{a}{2}$

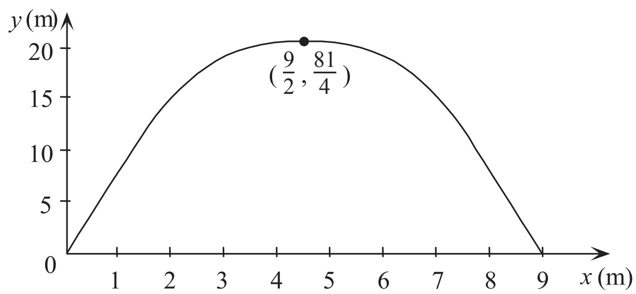
When $x = \frac{a}{2}$, $\quad y = \frac{a}{2}(a - \frac{a}{2})$

$$= \frac{a}{2} \times \frac{a}{2} = \frac{1}{4}a^2$$

So the maximum value of y is $\frac{1}{4}a^2$ when $x = \frac{a}{2}$.

d $y = \frac{1}{4}a^2$ is a maximum because the coefficient of the x^2 term is negative.

e **i** When $a = 9$, $y = x(9 - x)$



ii $0 < y \leq \frac{81}{4}$

13 **a** $V(t) = 0.6\left(20t^2 - \frac{2t^3}{3}\right)$

i When $t = 0$, $\quad V(0) = 0.6\left(20(0)^2 - \frac{2(0)^3}{3}\right)$

$$= 0.6(0 - 0)$$

$$= 0$$

ii When $t = 20$, $\quad V(20) = 0.6\left(20(20)^2 - \dfrac{2(20)^3}{3}\right)$

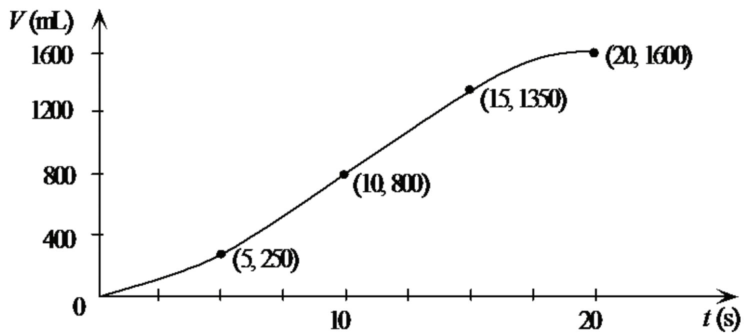
$$= 0.6\left(8000 - \frac{16000}{3}\right)$$

$$= 0.6 \times \frac{8000}{3}$$

$$= 1600$$

b $V'(t) = 0.6(40t - 2t^2) = 1.2t(20 - t)$

c



When $V'(t) = 0$, $\quad 1.2t(20 - t) = 0$

$\therefore \quad t = 0 \quad$ or $\quad 20 - t = 0$

$$t = 20$$

When $t = 10$, $\quad V = 0.6\left(20 \times 10^2 - \dfrac{2 \times 10^3}{3}\right)$

$$= 0.6\left(2000 - \frac{2000}{3}\right)$$

$$= 800$$

When $t = 5$, $\quad V = 0.6\left(20 \times 5^2 - \dfrac{2 \times 5^3}{3}\right)$

$$= 0.6\left(500 - \frac{250}{3}\right)$$

$$= 250$$

When $t = 15$, $\quad V = 0.6\left(20 \times 15^2 - \dfrac{2 \times 15^3}{3}\right)$

$$= 0.6\left(4500 - \frac{6750}{3}\right)$$

$$= 1350$$

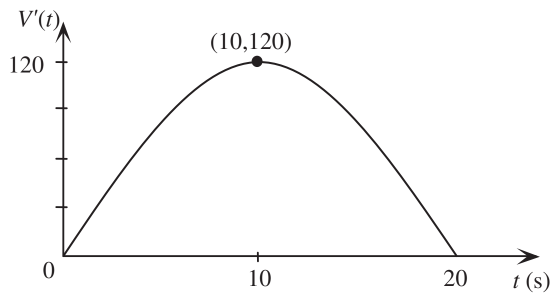
d

$$V'(t) = 1.2t(20 - t),\ t \in [0, 20]$$
$$= 24t - 1.2t^2$$

When $t = 0$, $V'(0) = 0$

When $t = 20$,
$$V'(20) = 24 \times 20 - 1.2(20)^2$$
$$= 480 - 480$$
$$= 0$$

When $t = 10$,
$$V'(10) = 24 \times 10 - 1.2(10)^2$$
$$= 240 - 120$$
$$= 120$$



14 a $y = ax^3 + bx^2$

At $(1, -1)$, $-1 = a(1)^3 + b(1)^2$

$\therefore$ $a + b + 1 = 0$ (1)

b

$$\frac{dy}{dx} = 3ax^2 + 2bx$$

At $(1, -1)$, $\frac{dy}{dx} = 0$

$\therefore$ $3a(1)^2 + 2b(1) = 0$

$\therefore$ $3a + 2b = 0$ (2)

$(2) - 2 \times (1)$
$$\begin{array}{r} 3a + 2b = 0 \\ -\ 2a + 2b + 2 = 0 \\ \hline a - 2 = 0 \end{array}$$

$\therefore$ $a = 2$

Substitute $a = 2$ into (1) $2 + b + 1 = 0$

$\therefore$ $b = -3$

$\therefore$ $y = 2x^3 - 3x^2$

c x-axis intercept when $y = 0$

$$\therefore \quad 2x^3 - 3x^2 = 0$$

$$\therefore \quad x^2(2x - 3) = 0$$

$$\therefore \quad x = 0 \quad \text{or} \quad x = \frac{3}{2}$$

$$\frac{dy}{dx} = 6x^2 - 6x$$

$$= 6x(x - 1)$$

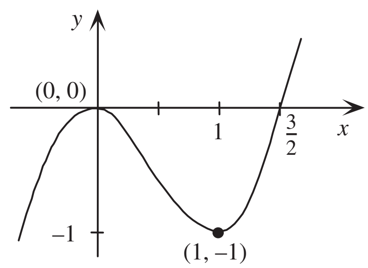
Stationary points where $\dfrac{dy}{dx} = 0$

$$\therefore \quad 6x(x - 1) = 0$$

$$\therefore \quad 6x = 0 \quad \text{or} \quad x - 1 = 0$$

$$\therefore \quad x = 0 \quad \text{or} \quad x = 1$$

At $x = 0,\ y = 0$
At $x = 1,\ y = -1$



	$x < 0$	0	$0 < x < 1$	1	$x > 1$
Sign of $\dfrac{dy}{dx}$	+	0	−	0	+
Shape	/	—	\	—	/

$\therefore$ there is a local minimum at $(1, -1)$ and a local maximum at $(0, 0)$.

15 **a** **i** $AD + AB + CB = 80$

$$\therefore \quad x + AB + x = 80$$

$$\therefore \quad AB = 80 - 2x$$

ii $\sin 60^\circ = \dfrac{h}{x}$

$$\therefore \quad h = x \sin 60^\circ$$

$$h = \frac{\sqrt{3}x}{2}$$

b Let area of trapezoid = A

$$\begin{aligned}
\therefore \quad A &= \text{area of rectangle} + 2(\text{area of triangle}) \\
&= \frac{\sqrt{3}}{2}x(80-2x) + 2\left(\frac{1}{2} \times \frac{\sqrt{3}}{2}x \times x \sin 30^\circ\right) \\
&= \frac{80\sqrt{3}}{2}x - \sqrt{3}x^2 + \frac{\sqrt{3}}{2}x \times \frac{x}{2} \\
&= \frac{80\sqrt{3}}{2}x - \sqrt{3}x^2 + \frac{\sqrt{3}}{4}x^2 \\
&= \frac{80\sqrt{3}}{2}x - \frac{3\sqrt{3}}{4}x^2 \\
&= \frac{\sqrt{3}}{4}x(160-3x)
\end{aligned}$$

(Formula for the area of a trapezium may also be used.)

c

$$\begin{aligned}
A &= \frac{\sqrt{3}}{4}x(160-3x) \\
&= 40\sqrt{3}x - \frac{3\sqrt{3}}{4}x^2 \\
\frac{dA}{dx} &= 40\sqrt{3} - \frac{3\sqrt{3}}{4}x
\end{aligned}$$

When $\frac{dA}{dx} = 0$, $\quad 40\sqrt{3} - \frac{3\sqrt{3}}{2}x = 0$

$$\therefore \quad \frac{3\sqrt{3}}{2}x = 40\sqrt{3}$$

$$\therefore \quad x = \frac{40\sqrt{3} \times 2}{3\sqrt{3}} = \frac{80}{3}$$

The area is a maximum for $x = \frac{80}{3}$, as $A = \frac{\sqrt{3}}{4}x(160-3x)$ is a quadratic function with negative coefficient of x^2.

16 a Total amount of cardboard = $x^2 + 4xy + x^2 + 8x$

$$\therefore \quad 2x^2 + 4xy + 8x = 1400$$

$$\therefore \quad y = \frac{1400 - 2x^2 - 8x}{4x}$$

b

$$\begin{aligned}
V &= x^2y \\
&= x^2\left(\frac{1400 - 2x^2 - 8x}{4x}\right) \\
&= \frac{-x^3}{2} - 2x^2 + 350x
\end{aligned}$$

c $$V = \frac{-x^3}{2} - 2x^2 + 350x$$
$$\frac{dV}{dx} = \frac{-3}{2}x^2 - 4x + 350$$

d $\frac{dV}{dx} = 0$ implies
$$\frac{-3}{2}x^2 - 4x + 350 = 0$$
$$\therefore \quad 3x^2 + 8x - 700 = 0$$
$$\therefore \quad x = \frac{-8 \pm \sqrt{64 + 8400}}{6} = \frac{-8 \pm 92}{6}$$
$$\therefore \quad x = 14, \text{ as } x \text{ is positive.}$$

e,f When $x = 14$, $\quad V = 3136$

Maximum volume is 3136 cm^3.

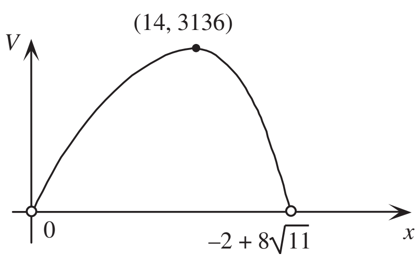
From part **b**, $$V = x^2\left(\frac{1400 - 2x^2 - 8x}{4x}\right)$$
$$= \frac{x}{4}(1400 - 2x^2 - 8x)$$

defined if $x > 0$ and $V > 0$

i.e. $$-2x^2 - 8x + 1400 > 0$$
$$x^2 + 4x - 700 > 0$$

Consider $$x = \frac{-4 \pm \sqrt{16 + 2800}}{2} = -2 \pm \sqrt{704} = -2 \pm 8\sqrt{11}$$

V is defined for $0 < x < -2 + 8\sqrt{11}$.



g On a CAS calculator, with **$f1 = x/4(1400 - 2x$^2 $- 8x$)** and **$f2$=1000**.

From the CAS calculator, when $V = 1000$,
$$x = 22.827\ldots \quad \text{or} \quad x = 2.943\ldots$$
$$\therefore \quad y = 1.919\ldots \quad \text{or} \quad y = 115.452\ldots$$