Chapter 12 – Circular functions

Solutions to Exercise 12A

1 **a** $60^\circ = \frac{60\pi}{180} = \frac{\pi}{3}$

b $144^\circ = \frac{144\pi}{180} = \frac{4\pi}{5}$

c $240^\circ = \frac{240\pi}{180} = \frac{4\pi}{3}$

d $330^\circ = \frac{330\pi}{180} = \frac{11\pi}{6}$

e $420^\circ = \frac{420\pi}{180} = \frac{7\pi}{3}$

f $480^\circ = \frac{480\pi}{180} = \frac{8\pi}{3}$

2 **a** $\frac{2\pi}{3} = \frac{2\pi}{3}\frac{180^\circ}{\pi} = 120^\circ$

b $\frac{5\pi}{6} = \frac{5\pi}{63}\frac{180^\circ}{\pi} = 150^\circ$

c $\frac{7\pi}{6} = \frac{7\pi}{6}\frac{180^\circ}{\pi} = 210^\circ$

d $0.9\pi = 0.9\pi\frac{180^\circ}{\pi} = 162^\circ$

e $\frac{5\pi}{9} = \frac{5\pi}{9}\frac{180^\circ}{\pi} = 100^\circ$

f $\frac{9\pi}{5} = \frac{9\pi}{5}\frac{180^\circ}{\pi} = 324^\circ$

g $\frac{11\pi}{5} = \frac{11\pi}{5}\frac{180^\circ}{\pi} = 220^\circ$

h $1.8\pi = 1.8\pi\frac{180^\circ}{\pi} = 324^\circ$

3 From calculator:

a $0.6 = 34.38^\circ$

b $1.89 = 108.29^\circ$

c $2.9 = 166.16^\circ$

d $4.31 = 246.94^\circ$

e $3.72 = 213.14^\circ$

f $5.18 = 296.79^\circ$

g $4.73 = 271.01^\circ$

h $6.00 = 343.77^\circ$

4 From calculator:

a $38^\circ = 0.66$

b $73^\circ = 1.27$

c $107^\circ = 1.87$

d $161^\circ = 2.81$

e $84.1^\circ = 1.47$

f $228^\circ = 3.98$

g $136.4^\circ = 2.38$

h $329^\circ = 5.74$

5 **a** $-\dfrac{\pi}{3} = -\dfrac{\pi}{3}\dfrac{180°}{\pi} = -60°$

b $-4\pi = -4\pi\dfrac{180°}{\pi} = -720°$

c $-3\pi = -3\pi\dfrac{180°}{\pi} = -540°$

d $-\pi = -\pi\dfrac{180°}{\pi} = -180°$

e $\dfrac{5\pi}{3} = \dfrac{5\pi}{3}\dfrac{180°}{\pi} = 300°$

f $-\dfrac{11\pi}{6} = -\dfrac{11\pi}{6}\dfrac{180°}{\pi} = -330°$

g $\dfrac{23\pi}{6} = \dfrac{23\pi}{6}\dfrac{180°}{\pi} = 690°$

h $-\dfrac{23\pi}{6} = -\dfrac{23\pi}{6}\dfrac{180°}{\pi} = -690°$

6 **a** $-360° = -\dfrac{360\pi}{180} = -2\pi$

b $-540° = -\dfrac{540\pi}{180} = -3\pi$

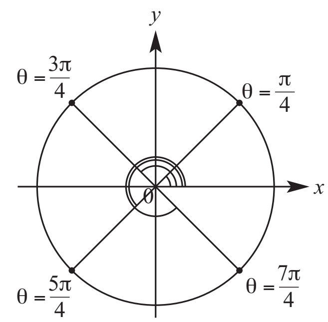
c $-240° = -\dfrac{240\pi}{180} = -\dfrac{4\pi}{3}$

d $-720° = -\dfrac{720\pi}{180} = -4\pi$

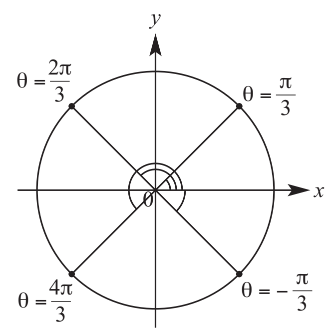
e $-330° = -\dfrac{330\pi}{180} = -\dfrac{11\pi}{6}$

f $-210° = -\dfrac{210\pi}{180} = -\dfrac{7\pi}{6}$

7 **a**



b



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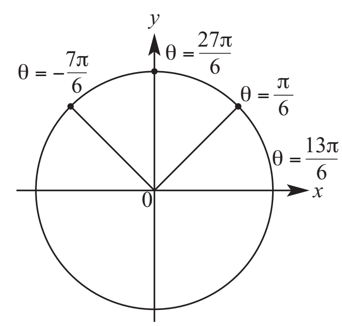


Solutions to Exercise 12B

1 **a** $t = 0; \sin t = 0; \cos t = 1$

b $t = \frac{3\pi}{2}; \sin t = -1; \cos t = 0$

c $t = -\frac{3\pi}{2}; \sin t = 1; \cos t = 0$

d $t = \frac{5\pi}{2}; \sin t = 1; \cos t = 0$

e $t = -3\pi; \sin t = 0; \cos t = -1$

f $t = \frac{9\pi}{2}; \sin t = 1; \cos t = 0$

g $t = \frac{7\pi}{2}; \sin t = -1; \cos t = 0$

h $t = 4\pi; \sin t = 0; \cos t = 1$

2 From calculator:

a $\sin 1.9 = 0.95$

b $\sin 2.3 = 0.75$

c $\sin 4.1 = -0.82$

d $\cos 0.3 = 0.96$

e $\cos 2.1 = -0.5$

f $\cos(-1.6) = -0.03$

g $\sin(-2.1) = -0.86$

h $\sin(-3.8) = 0.61$

3 **a** $\theta = 27\pi; \sin\theta = 0; \cos\theta = -1$

b $\theta = -\frac{5\pi}{2}; \sin\theta = -1; \cos\theta = 0$

c $\theta = \frac{27\pi}{2}; \sin\theta = -1; \cos\theta = 0$

d $\theta = -\frac{9\pi}{2}; \sin\theta = -1; \cos\theta = 0$

e $\theta = \frac{11\pi}{2}; \sin\theta = -1; \cos\theta = 0$

f $\theta = 57\pi; \sin\theta = 0; \cos\theta = -1$

g $\theta = 211\pi; \sin\theta = 0; \cos\theta = -1$

h $\theta = -53\pi; \sin\theta = 0; \cos\theta = -1$

Solutions to Exercise 12C

1 **a** $\tan\pi = \tan 0 = 0$

b $\tan(-\pi) = \tan 0 = 0$

c $\tan\left(\frac{7\pi}{2}\right) = \tan\frac{\pi}{2} = \text{undefined}$

d $\tan(-2\pi) = \tan 0 = 0$

e $\tan\left(\frac{5\pi}{2}\right) = \tan\frac{\pi}{2} = \text{undefined}$

f $\tan -\frac{\pi}{2} = \tan\frac{\pi}{2} = \text{undefined}$

2 From calculator:

a $\tan 1.6 = -34.23$

b $\tan(-1.2) = -2.57$

c $\tan 136° = -0.97$

d $\tan(-54°) = -1.38$

e $\tan 3.9 = 0.95$

f $\tan(-2.5) = 0.75$

g $\tan 239° = 1.66$

3 **a** $\tan 180° = \tan 0° = 0$

b $\tan 360° = \tan 0° = 0$

c $\tan 0° = 0$

d $\tan(-180°) = \tan 0° = 0$

e $\tan(-540°) = \tan 0° = 0$

f $\tan 720° = \tan 0° = 0$

Solutions to Exercise 12D

1 **a** $\theta = \cos^{-1}\left(\frac{3}{8}\right) = 67.98^\circ$ or $67^\circ 59'$

b $x = 5\ \cos 25^\circ = 4.5315$

c $x = 6\ \sin 25^\circ = 2.5357$

d $x = 10\ \cos 50^\circ = 6.4279$

e $\theta = \tan^{-1}\left(\frac{6}{5}\right) = 50.19^\circ$ or $50^\circ 12'$

f $x = 10\ \sin 20^\circ = 3.4202$

g $x = \dfrac{5}{\tan 65^\circ} = 2.3315$

h $x = 7\ \sin 70^\circ = 6.5778$

i $x = \dfrac{5}{\cos 40^\circ} = 6.5270$

Solutions to Exercise 12E

1 $\sin\theta = 0.42, \cos x = 0.7, \tan\alpha = 0.38$

a $\sin(\pi+\theta) = -\sin\theta = -0.42$

b $\cos(\pi - x) = -\cos x = -0.7$

c $\sin(2\pi - \theta) = -\sin\theta = -0.42$

d $\tan(\pi - \alpha) = -\tan\alpha = -0.38$

e $\sin(\pi - \theta) = \sin\theta = 0.42$

f $\tan(2\pi - \alpha) = -\tan\alpha = -0.38$

g $\cos(\pi + x) = -\cos x = -0.7$

h $\cos(2\pi - x) = \cos x = 0.7$

2 **a** $\cos x = -\cos\dfrac{\pi}{6}; \dfrac{\pi}{2} < x < \pi,$

$\therefore x = \pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}$

b $\cos x = -\cos\dfrac{\pi}{6}; \pi < x < \dfrac{3\pi}{2}$

$\therefore x = \pi + \dfrac{\pi}{6} = \dfrac{7\pi}{6}$

c $\cos x = \cos\dfrac{\pi}{6}; \dfrac{3\pi}{2} < x < 2\pi$

$\therefore x = 2\pi - \dfrac{\pi}{6} = \dfrac{11\pi}{6}$

3 $\sin\theta = \dfrac{\sqrt{3}}{2}, \cos\theta = \dfrac{1}{2}$ from diagram

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\sqrt{3}}{2} \div \dfrac{1}{2} = \sqrt{3}$

a $a = \cos(\pi - \theta) = -\cos\theta = -\dfrac{1}{2}$

b $b = \sin(\pi - \theta) = \sin\theta = \dfrac{\sqrt{3}}{2}$

c $c = \cos(-\theta) = \cos\theta = \dfrac{1}{2}$

d $d = \sin(-\theta) = -\sin\theta = -\dfrac{\sqrt{3}}{2}$

e $\tan(\pi - \theta) = -\tan\theta = -\sqrt{3}$

f $\tan(-\theta) = -\tan\theta = -\sqrt{3}$

4 $\sin\theta = \dfrac{\sqrt{3}}{2}, \cos\theta = -\dfrac{1}{2}$ from diagram

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\sqrt{3}}{2} \div -\dfrac{1}{2} = -\sqrt{3}$

a $d = \sin(\pi + \theta) = -\sin\theta = -\dfrac{\sqrt{3}}{2}$

b $c = \cos(\pi + \theta) = -\cos\theta = \dfrac{1}{2}$

c $\tan(\pi + \theta) = \tan\theta = -\sqrt{3}$

d $\sin(2\pi - \theta) = -\sin\theta = -\dfrac{\sqrt{3}}{2}$

e $\cos(2\pi - \theta) = \cos\theta = -\dfrac{1}{2}$

5 **a** $(a, b) = (\cos 40^\circ, \sin 40^\circ)$
$= (0.7660, 0.6428)$

b $(c, d) = (-\cos 40^\circ, \sin 40^\circ)$
$= (-0.7660, 0.6428)$

c **i** $\cos 140^\circ = -0.7660,$
$\sin 140^\circ = 0.6428$

ii $\cos 140^\circ = -\cos 40^\circ$

6 $\sin x^\circ = 0.7, \cos\theta = 0.6^\circ$ and

$\tan\alpha^\circ = 0.4$

a $\sin(180 + x)^\circ = -\sin x^\circ = -0.7$

b $\cos(180 + \theta)^\circ = -\cos\theta^\circ = -0.6$

c $\tan(360 - \alpha)^\circ = -\tan\alpha^\circ = -0.4$

d $\cos(180 - \theta)^\circ = -\cos\theta^\circ = -0.6$

e $\sin(360 - x)^\circ = -\sin x^\circ = -0.7$

f $\sin(-x)^\circ = -\sin x^\circ = -0.7$

g $\tan(360 + \alpha)^\circ = \tan\alpha^\circ = 0.4$

h $\cos(-\theta)^\circ = \cos\theta^\circ = 0.6$

7 **a** $\sin x = \sin 60^\circ$ and $90^\circ < x < 180^\circ$

$\therefore x = 180^\circ - 60^\circ = 120^\circ$

b

$\sin x = -\sin 60^\circ$ and $180^\circ < x < 270^\circ$

$\therefore x = 180^\circ + 60^\circ = 240^\circ$

c $\sin x = -\sin 60^\circ$ and $-90^\circ < x < 0^\circ$

$\therefore x = 0^\circ - 60^\circ = -60^\circ$

d

$\cos x^\circ = -\cos 60^\circ$ and $90^\circ < x < 180^\circ$

$\therefore x = 180^\circ - 60^\circ = 120^\circ$

e

$\cos x^\circ = -\cos 60^\circ$ and $180^\circ < x < 270^\circ$

$\therefore x = 180^\circ + 60^\circ = 240^\circ$

f $\cos x^\circ = \cos 60^\circ$ and $270^\circ < x < 360^\circ$

$\therefore x = 360^\circ - 60^\circ = 300^\circ$

Solutions to Exercise 12F

1

	x	$\sin x$	$\cos x$	$\tan x$
a	120°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
b	135°	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
c	210°	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
d	240°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
e	315°	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
f	390°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
g	420°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
h	−135°	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
i	−300	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
j	−60	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$

2 **a** $\sin\frac{2\pi}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

b $\cos\frac{3\pi}{4} = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

c $\tan\frac{5\pi}{6} = -\tan\frac{\pi}{6} = -\frac{\sqrt{3}}{3}$

d $\sin\frac{7\pi}{6} = -\sin\frac{\pi}{6} = -\frac{1}{2}$

e $\cos\frac{5\pi}{4} = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

f $\tan\frac{4\pi}{3} = \tan\frac{\pi}{3} = \sqrt{3}$

g $\sin\frac{5\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

h $\cos\frac{7\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

i $\tan\frac{11\pi}{6} = -\tan\frac{\pi}{6} = -\frac{\sqrt{3}}{3}$

3 **a**
$$\sin\left(-\frac{2\pi}{3}\right) = -\sin\left(\frac{2\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

b
$$\cos\frac{11\pi}{4} = \cos\frac{3\pi}{4} = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$$

c $\tan\frac{13\pi}{6} = \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$

d
$$\tan\frac{15\pi}{6} = \tan\frac{5\pi}{2} = \tan\frac{\pi}{2} = \text{undefined}$$

e
$$\cos\frac{14\pi}{4} = \cos\frac{7\pi}{2} = \cos\frac{3\pi}{2} = 0$$

f
$$\cos\left(-\frac{3\pi}{4}\right) = \cos\frac{3\pi}{4} = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$$

g
$$\sin\frac{11\pi}{4} = \sin\frac{3\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

h
$$\cos\left(-21\frac{\pi}{3}\right) = \cos(-7\pi) = \cos\pi = -1$$

Solutions to Exercise 12G

1 a $2\sin\theta$: per $= 2\pi$, ampl $= 2$

b $3\sin 2\theta$: per $= \dfrac{2\pi}{2} = \pi$, ampl $= 3$

c $\dfrac{1}{2}\cos 3\theta$: per $= \dfrac{2\pi}{3}$, ampl $= \dfrac{1}{2}$

d $3\sin\dfrac{\theta}{2}$: per $= \dfrac{2\pi}{\frac{1}{2}} = 4\pi$, ampl $= 3$

e $4\cos 3\theta$: per $= \dfrac{2\pi}{3}$, ampl $= 4$

f $-\dfrac{1}{2}\sin 4\theta$: per $= \dfrac{2\pi}{4} = \dfrac{\pi}{2}$, ampl $= \dfrac{1}{2}$

g $-2\cos\dfrac{\theta}{2}$: per $= \dfrac{2\pi}{\frac{1}{2}} = 4\pi$, ampl $= 2$

h $2\cos\pi t$: per $= \dfrac{2\pi}{\pi} = 2$, ampl $= 2$

i $-3\sin\left(\dfrac{\pi t}{2}\right)$: per $= \dfrac{2\pi}{\frac{\pi}{2}} = 4$, ampl $= 3$

2 a $g(x) = 3\sin x$: dilation of 3 from x-axis, amplitude $= 3$, period $= 2\pi$

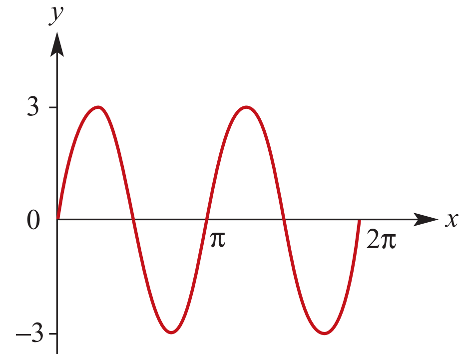
b $g(x) = \sin(5x)$: dilation of $\dfrac{1}{5}$ from y-axis, amplitude $= 1$, period $= \dfrac{2\pi}{5}$

c $g(x) = \sin\left(\dfrac{x}{3}\right)$: dilation of 3 from y-axis, amplitude $= 1$, period $= 6\pi$

d $g(x) = 2\sin 5x$: dilation of 2 from x-axis, dilation of $\dfrac{1}{5}$ from y-axis, amplitude $= 2$, period $= \dfrac{2\pi}{5}$

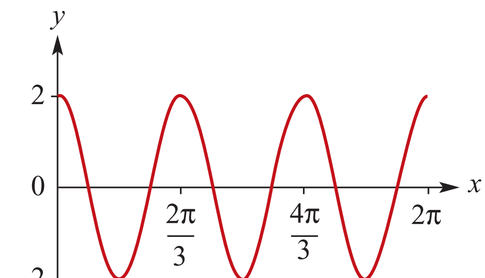
3 a $y = 3\sin 2x$:

per $= \pi$, ampl $= 3$, x-intercepts $0, \dfrac{\pi}{2}, \pi$



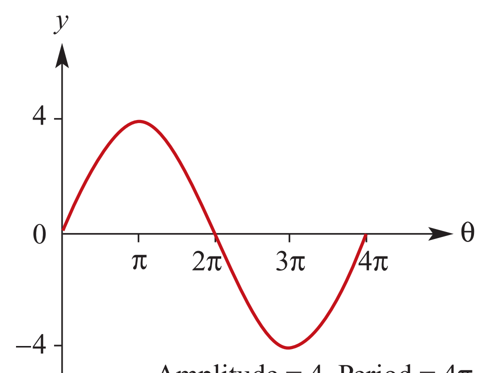
Amplitude = 3, Period = π

b $y = 2\cos 3\theta$:
per $= \dfrac{2\pi}{3}$, ampl $= 2$,
θ intercepts $0, \dfrac{\pi}{6}, \dfrac{\pi}{2}$



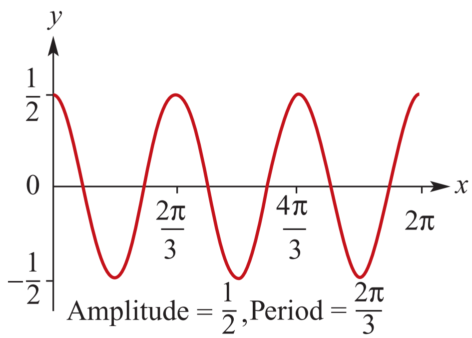
Amplitude = 2, Period = $\dfrac{2\pi}{3}$

c $y = 4\cos\dfrac{\theta}{2}$:
per $= 4\pi$, ampl $= 4$,
θ intercepts $0, \pi, 3\pi$

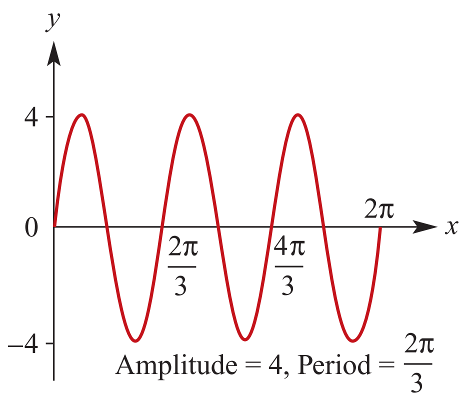


Amplitude = 4, Period = 4π

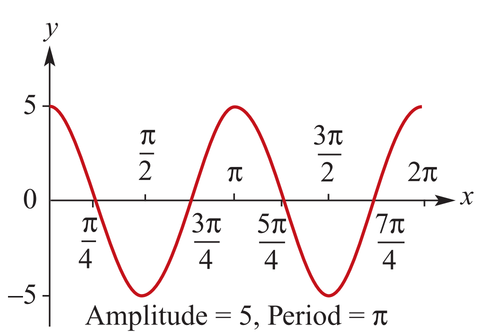
d $y = \frac{1}{2}\cos 3x$:
per $= \frac{2\pi}{3}$, ampl $= \frac{1}{2}$,
x intercepts $\frac{\pi}{6}, \frac{\pi}{2}$



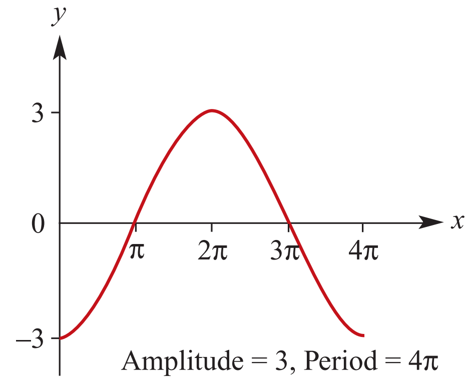
e $y = 4\sin 3x$:
per $= \frac{2\pi}{3}$, ampl $= 4$,
x intercepts $0, \frac{2\pi}{3}, \frac{4\pi}{3}$



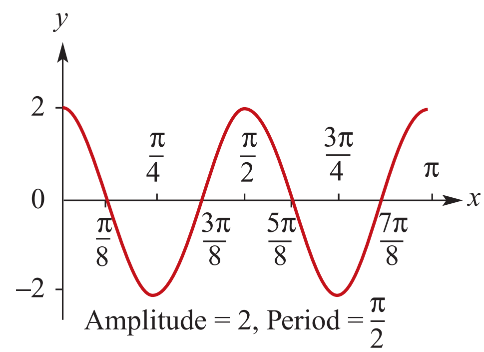
f $y = 5\cos 2x$:
per $= \pi$, ampl $= 5$,
x intercepts $0, \frac{\pi}{4}, \frac{3\pi}{4}$



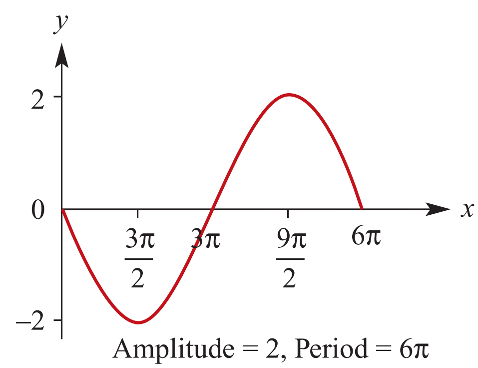
g $y = -3\cos\frac{\theta}{2}$:
per $= 4\pi$, ampl $= 3$,
θ intercepts $0, \pi, 3\pi$



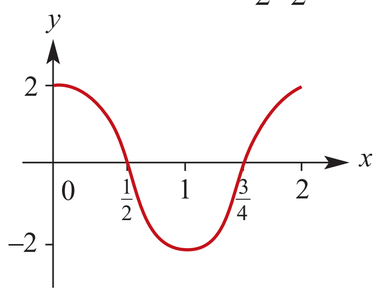
h $y = 2\cos 4\theta$:
per $= \frac{\pi}{2}$, ampl $= 2$,
θ intercepts $0, \frac{\pi}{8}, \frac{3\pi}{8}$



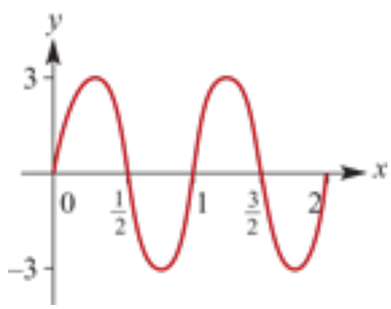
i $y = -2\sin\frac{\theta}{3}$:
per $= 6\pi$, ampl $= 2$,
θ intercepts $0, 3\pi, 6\pi$



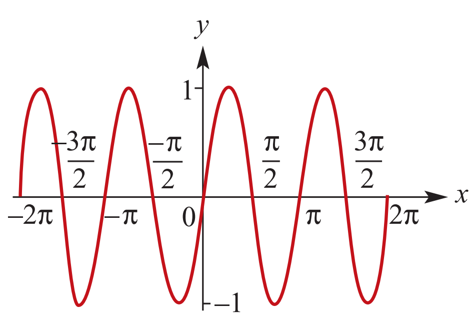
4 a $f\colon [0, 2] \to R, f(t) = 2\cos \pi t$
per $= \dfrac{2\pi}{\pi} = 2$, ampl $= 2$,
range $= [-2, 2]$,
endpoints $(0, 2)$ and
$(2, 2)$; x-intercepts $\dfrac{1}{2}, \dfrac{3}{2}$



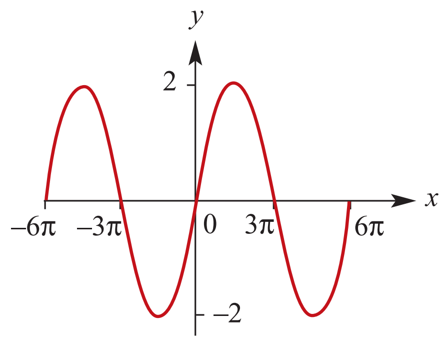
b $f\colon [0, 2] \to R, f(t) = 3\sin(2\pi t)$
per $= \dfrac{2\pi}{2\pi} = 1$, ampl $= 3$,
range $= [-3, 3]$,
endpoints $(0, 0)$ and $(2, 0)$;
x-intercepts $\dfrac{1}{2}, \dfrac{3}{2}, \dfrac{5}{2}$



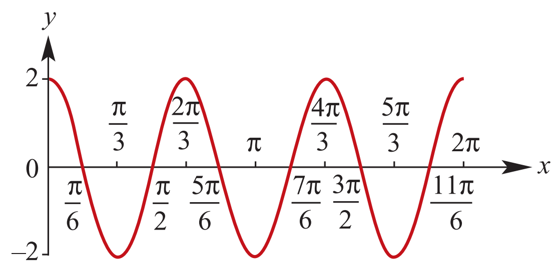
5 a $f(x) = \sin 2x$ for $x \in [-2\pi, 2\pi]$:
endpoints: $(-2\pi, 0), (2\pi, 0)$
x-intercepts: $-\dfrac{3\pi}{2}, -\pi, -\dfrac{\pi}{2}, 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$
ampl $= 1$, range $[-1, 1]$



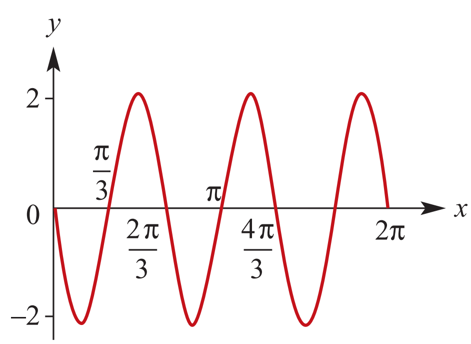
b $f(x) = 2\sin\dfrac{x}{3}$ for $x \in [-6\pi, 6\pi]$:
endpoints: $(-6\pi, 0), (6\pi, 0)$
x-intercepts: $-3\pi, 0, 3\pi$
ampl $= 2$, range $[-2, 2]$



c $f(x) = 2\cos 3x$ for $x \in [0, 2\pi]$:
endpoints: $(0, 1), (2\pi, 1)$
x-intercepts: $\dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{3\pi}{2}, \dfrac{11\pi}{6}$
ampl $= 2$, range $[-2, 2]$



d $f(x) = -2\sin 3x$ for $x \in [0, 2\pi]$:
endpoints: $(0, 0), (2\pi, 0)$
x-intercepts: $\dfrac{\pi}{3}, \dfrac{2\pi}{3}, \pi, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$
ampl $= 2$, range $[-2, 2]$

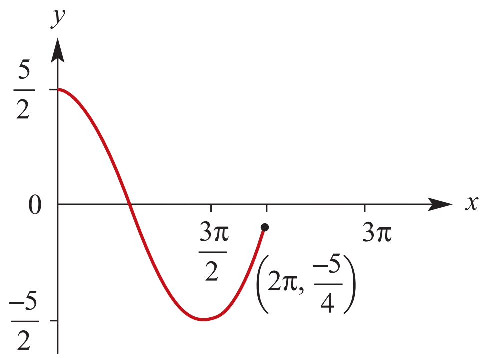


6 $f\colon [0, 2\pi] \to R, f(x) = \frac{5}{2}\cos\left(\frac{2x}{3}\right)$:

endpoints: $f(0) = \frac{5}{2}$ and $f(2\pi) = -\frac{5}{4}$

per $= \frac{2\pi}{\frac{2}{3}} = 3\pi$ so we only have $\frac{2}{3}$ period

ampl $= \frac{5}{2}$, range $= \left[-\frac{5}{2}, \frac{5}{2}\right]$



7 **a** $g(x) = -\sin 5x$: dilation of $\frac{1}{5}$ from y-axis, reflection in x-axis, amplitude $= 1$, period $= \frac{2\pi}{5}$

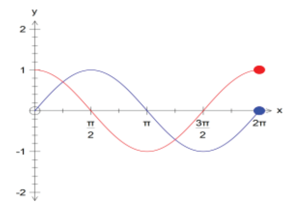
b $g(x) = \sin(-x)$: reflection in y-axis, amplitude $= 1$, period $= 2\pi$

c $g(x) = 2\sin(\frac{x}{3})$: dilation of 3 from y-axis, dilation of 2 from x-axis, amplitude $= 2$, period $= 6\pi$

d $g(x) = -4\sin\left(\frac{x}{2}\right)$: dilation of 2 from y-axis, dilation of 4 from x-axis, reflection in x-axis, amplitude $= 4$, period $= 4\pi$

e $g(x) = 2\sin\left(-\frac{x}{3}\right)$: dilation of 3 from y-axis, dilation of 2 from x-axis, reflection in y-axis, amplitude $= 2$, period $= 6\pi$

8 **a** $f\colon [0, 2\pi] \to R, f(x) = \sin x$;
per $= 2\pi$, ampl $= 1$, range $= [-1, 1]$,
endpoints $(0, 0)$ and $(2\pi, 0)$,
other x-intercept at π



$g\colon [0, 2\pi] \to R, g(x) = \cos x$:
per $= 2\pi$, ampl $= 1$, range $= [-1, 1]$,
endpoints at $(0, 1)$ and $(2\pi, 1)$,
x-intercepts at $\frac{\pi}{2}, \frac{3\pi}{2}$

b $\sin x = \cos x$ when $x = \frac{\pi}{4}$ and $\frac{5\pi}{4}$

Solutions to Exercise 12H

1 **a**

$$\cos x = \frac{1}{2}$$
$$x = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 2\pi + \frac{\pi}{3}, 4\pi - \frac{\pi}{3}, \ldots$$
$$= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$$

b

$$\sin x = \frac{1}{\sqrt{2}}$$
$$x = \frac{\pi}{4}, \pi - \frac{\pi}{4}, 2\pi + \frac{\pi}{4}, 3\pi - \frac{\pi}{4}, \ldots$$
$$= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$$

c

$$\sin x = \frac{\sqrt{3}}{2}$$
$$x = \frac{\pi}{3}, \pi - \frac{\pi}{3}, 2\pi + \frac{\pi}{3}, 3\pi - \frac{\pi}{3}, \ldots$$
$$= \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$$

2 Solve over $[0, 2\pi]$:

a $\sin x = 0.8, \therefore\ x = 0.93, \pi - 0.93$
$= 0.93, 2.21$

b $\cos x = -0.4, \therefore\ x = \pi \pm 1.16$
$= 1.98, 4.30$

c $\sin x = -0.35,$
$\therefore\ x = \pi + 0.36, 2\pi - 0.36$
$= 3.50, 5.93$

d $\sin x = 0.4, \therefore\ x = 0.41, \pi - 0.41$
$= 0.41, 2.73$

e $\cos x = -0.7, \therefore\ x = \pi \pm 0.80$
$= 2.35, 3.94$

f $\cos x = -0.2, \therefore\ x = \pi \pm 1.39$
$= 1.77, 4.51$

3 Solve over $[0, 360°]$:

a $\cos\theta° = -\frac{\sqrt{3}}{2}, \therefore\ \theta = 180 \pm 30$
$= 150, 210$

b $\sin\theta° = \frac{1}{2}, \therefore\ \theta = 30, 180 - 30$
$= 30, 150$

c $\cos\theta° = -\frac{1}{2}, \therefore\ \theta = 180 \pm 60$
$= 120, 240$

d $2\cos\theta° + 1 = 0, \therefore \cos\theta° = -\frac{1}{2}$
$\therefore\ \theta = 120, 240$

e $2\sin\theta° = \sqrt{3}, \therefore\ \sin\theta° = \frac{\sqrt{3}}{2}$
$\therefore\ \theta = 60, 180 - 60$
$= 60, 120$

f $\sqrt{2}\sin\theta° - 1 = 0, \therefore\ \sin\theta° = \frac{1}{\sqrt{2}}$
$\theta = 45, 180 - 45$
$= 45, 135$

4 **a**

$$2\cos x = \sqrt{3}$$
$$\therefore\quad \cos x = \frac{\sqrt{3}}{2}$$
$$x = \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$$
$$= \frac{\pi}{6}, \frac{11\pi}{6}$$

b $\sqrt{2}\sin x + 1 = 0$

$$\therefore \quad \sin x = -\frac{1}{\sqrt{2}}$$

$$x = \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$$

$$= \frac{5\pi}{4}. \frac{7\pi}{4}$$

c $\sqrt{2}\cos x - 1 = 0$

$$\therefore \quad \cos x = \frac{1}{\sqrt{2}}$$

$$x = \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$$

$$= \frac{\pi}{4}, \frac{7\pi}{4}$$

5 Solve over $[-\pi, \pi]$:

a $\cos x = -\frac{1}{\sqrt{2}}, \therefore \ x = \pi - \frac{\pi}{4}, -\pi + \frac{\pi}{4}$

$$= -\frac{3\pi}{4}, \frac{3\pi}{4}$$

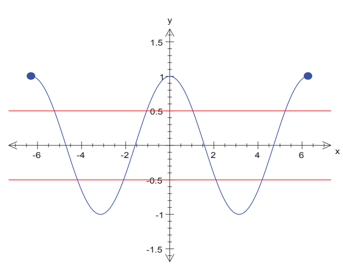
b $\sin x = \frac{\sqrt{3}}{2}, \therefore x = \frac{\pi}{3}, \pi - \frac{\pi}{3}$

$$= \frac{\pi}{3}, \frac{2\pi}{3}$$

c $\cos x = -\frac{1}{2}, \therefore x = \pi - \frac{\pi}{3}, -\pi + \frac{\pi}{3}$

$$= -\frac{2\pi}{3}, \frac{2\pi}{3}$$

6 **a** $f : [-2\pi, 2\pi] \to R, f(x) = \cos x$



b Line marked at $y = \frac{1}{2}$, x-values are at:

$$x = \pm\frac{\pi}{3}, \pm\left(2\pi - \frac{\pi}{3}\right) = \pm\frac{\pi}{3}, \pm\frac{5\pi}{3}$$

c Line marked at $y = -\frac{1}{2}$, x-values are at:

$$x = \pm\left(\pi \pm \frac{\pi}{3}\right) = \pm\frac{2\pi}{3}, \pm\frac{4\pi}{3}$$

7 Solve over $[0, 2\pi]$:

a

$$\sin(2\theta) = -\frac{1}{2}$$

$$\therefore \quad 2\theta = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}, 3\pi + \frac{\pi}{6}, 4\pi - \frac{\pi}{6}$$

$$\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$$

b

$$\cos(2\theta) = \frac{\sqrt{3}}{2}$$

$$\therefore \quad 2\theta = \frac{\pi}{6}, 2\pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 4\pi - \frac{\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$$

c

$$\sin(2\theta) = \frac{1}{2}$$

$$\therefore \quad 2\theta = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

d

$$\sin(3\theta) = -\frac{1}{\sqrt{2}}$$

$$\therefore 3\theta = \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}, 3\pi + \frac{\pi}{4}, 4\pi - \frac{\pi}{4} \ldots$$

$$\theta = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{15\pi}{12}, \frac{21\pi}{12}, \frac{23\pi}{12}$$

e

$$\cos(2\theta) = -\frac{\sqrt{3}}{2}$$

$$\therefore \quad 2\theta = \pi \pm \frac{\pi}{6}, 3\pi \pm \frac{\pi}{6}$$

$$\theta = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$$

f

$$\sin(2\theta) = -\frac{1}{\sqrt{2}}$$

$$\therefore \quad 2\theta = \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}, 3\pi + \frac{\pi}{4}, 4\pi - \frac{\pi}{4}$$

$$\theta = \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$$

8 Solve over $[0, 2\pi]$:

a

$$\sin(2\theta) = -0.8$$

$$\therefore \quad 2\theta = \pi + 0.927, 2\pi - 0.927,$$

$$3\pi + 0.927, 4\pi - 0.927$$

$$\theta = 2.034, 2.678, 5.176, 5.820$$

b

$$\sin(2\theta) = -0.6$$

$$\therefore \quad 2\theta = \pi + 0.644, 2\pi - 0.644,$$

$$3\pi + 0.644, 4\pi - 0.644$$

$$\theta = 1.892, 2.820, 5.034, 5.961$$

c

$$\cos(2\theta) = 0.4$$

$$\therefore \quad 2\theta = 1.159, 2\pi \pm 1.159, 4\pi - 1.159$$

$$\theta = 0.580, 2.562, 3.721, 5.704$$

d $\cos(3\theta) = 0.6$

$$\therefore \quad 3\theta = 0.927, 2\pi \pm 0.927;$$

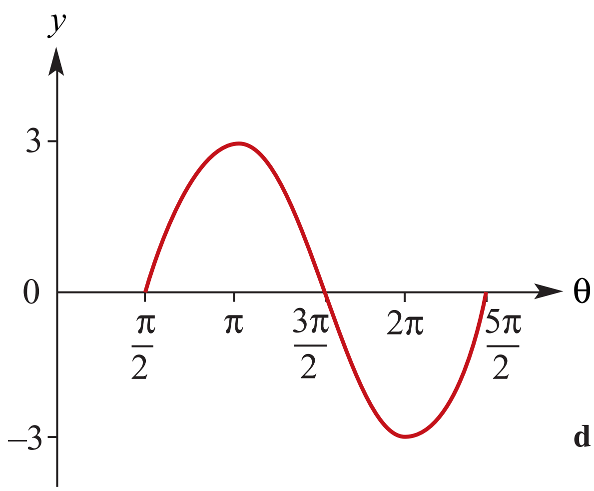
$$4\pi \pm 0.927, 6\pi - 0.927$$

$$\theta = 0.309, 1.785, 2.403,$$

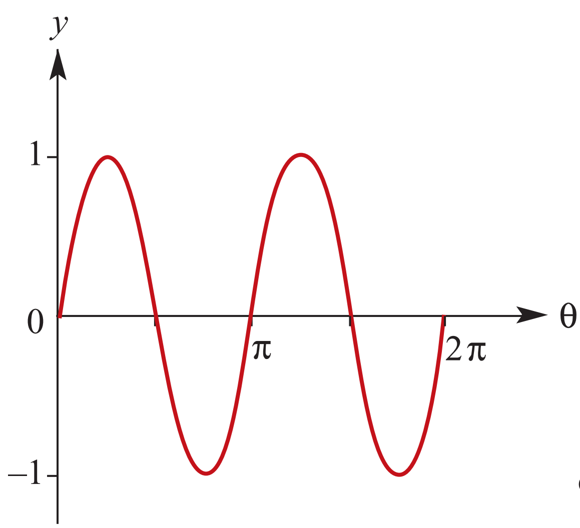
$$3.880, 4.498, 5.974$$

Solutions to Exercise 12I

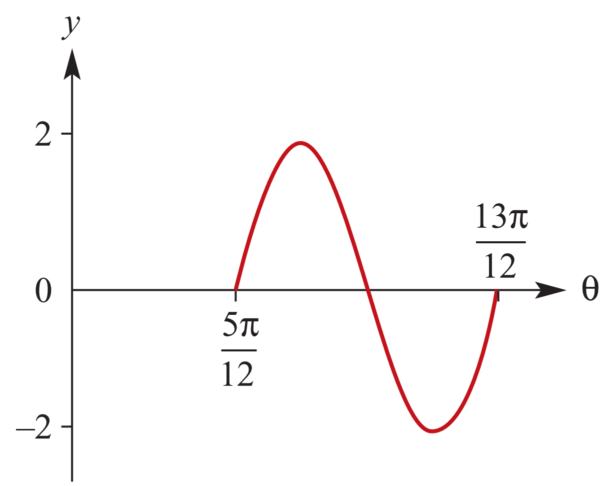
1 **a** $y = 3\sin\left(\theta - \frac{\pi}{2}\right)$:
per $= 2\pi$, ampl $= 3$, range $= [-3, 3]$



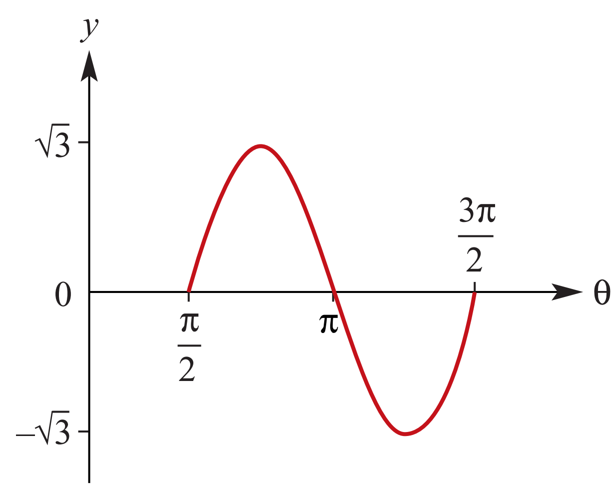
b $y = \sin 2(\theta + \pi)$:
per $= \pi$, ampl $= 1$, range $= [-1, 1]$



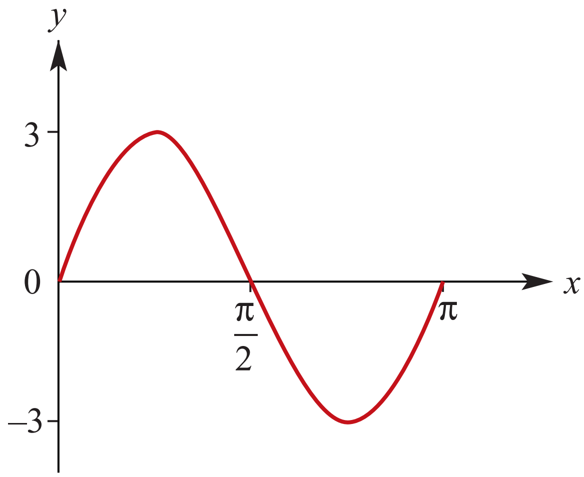
c $y = 2\sin 3\left(\theta + \frac{\pi}{4}\right)$:
per $= \frac{2\pi}{3}$, ampl $= 2$, range $= [-2, 2]$



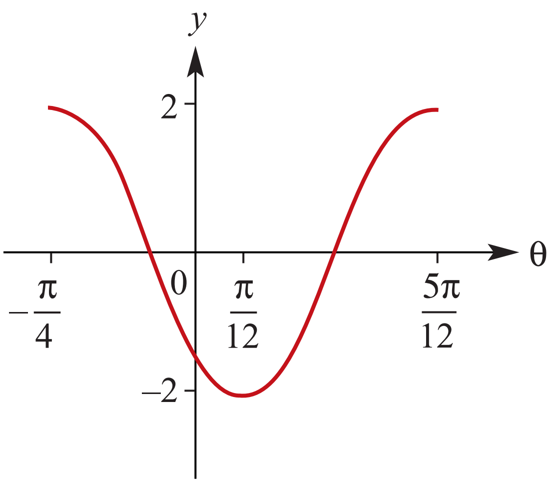
d $y = \sqrt{3}\sin 2\left(\theta - \frac{\pi}{2}\right)$:
per $= \pi$, ampl $= \sqrt{3}$,
range $= [-\sqrt{3}, \sqrt{3}]$



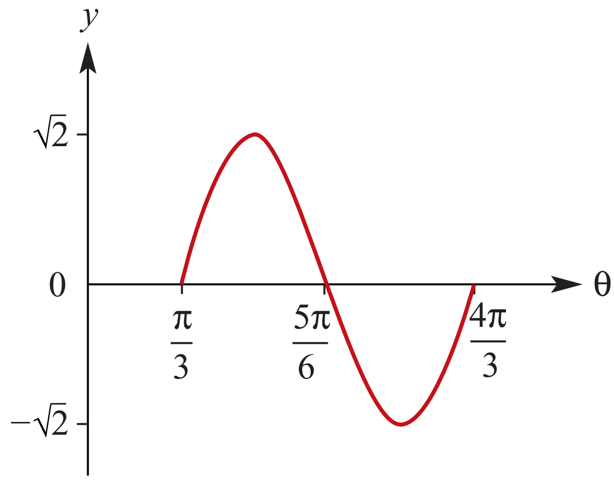
e $y = 3\sin(2x)$:
per $= \pi$, ampl $= 3$, range $= [-3, 3]$



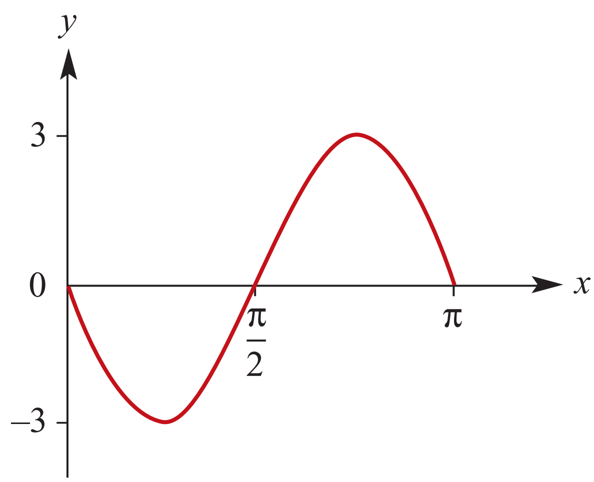
f $y = 2\cos 3\left(\theta + \frac{\pi}{4}\right)$:
per $= \frac{2\pi}{3}$, ampl $= 2$, range $= [-2, 2]$



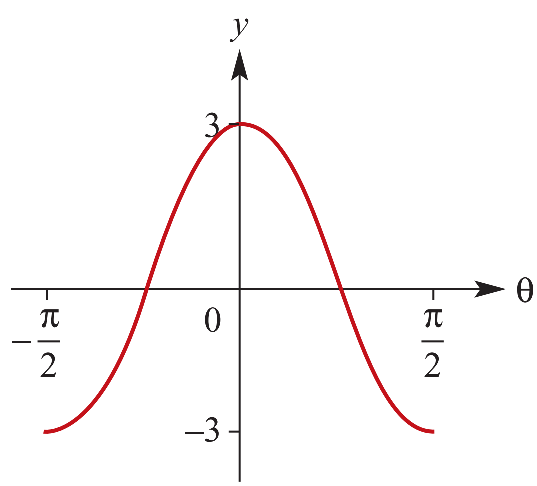
g $y = \sqrt{2}\sin 2\left(\theta - \frac{\pi}{3}\right)$:
per $= \pi$, ampl $= \sqrt{2}$,
range $= [-\sqrt{2}, \sqrt{2}]$



h $y = -3\sin(2x)$:
per $= \pi$, ampl $= 3$, range $= [-3, 3]$



i $y = -3\cos 2\left(\theta + \frac{\pi}{2}\right)$:
per $= \pi$, ampl $= 3$, range $= [-3, 3]$



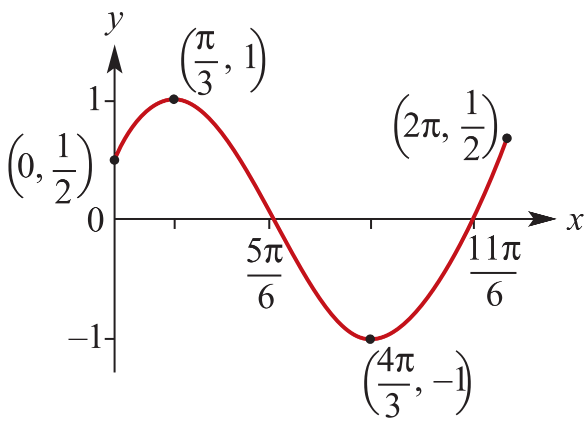
2 $f: [0, 2\pi] \to R, f(x) = \cos\left(x - \frac{\pi}{3}\right)$

a $f(0) = \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$

$f(2\pi) = \cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$

b per $= 2\pi$, ampl $= 1$, range $[-1, 1]$

x-intercepts at $\frac{5\pi}{6}, \frac{11\pi}{6}$



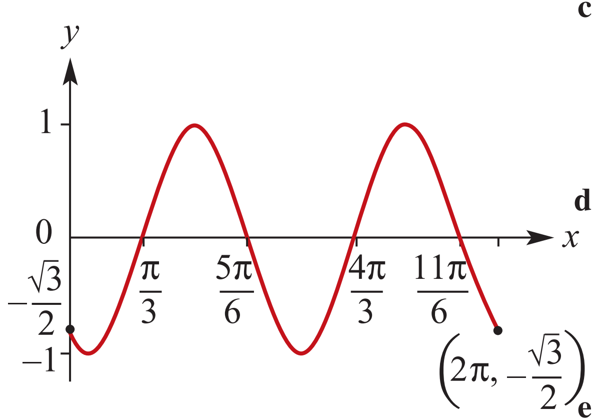
3 $f: [0, 2\pi] \to R, f(x) = \sin 2\left(x - \frac{\pi}{3}\right)$

a $f(0) = \sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$f(2\pi) = \sin\left(\frac{10\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

b per $= \pi$, ampl $= 1$, range $[-1, 1]$

x-intercepts at $\frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$



4 $f: [-\pi, \pi] \to R, f(x) = \sin\ 3\left(x + \frac{\pi}{4}\right)$:

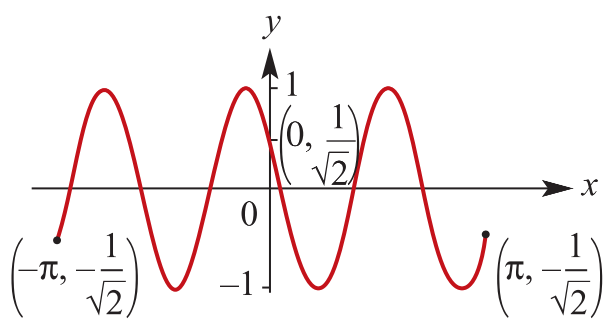
a $f(-\pi) = \sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f(\pi) = \sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

b per $= \frac{2\pi}{3}$, ampl $= 1$, range $[-1, 1]$,

x-intercepts at

$-\frac{11\pi}{12}, -\frac{7\pi}{12}, -\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$



5 $y = \sin x$

a Dilation of 2 from y-axis: $y = \sin\left(\frac{x}{2}\right)$;
dilation of 3 from x-axis:
$y = 3\sin\left(\frac{x}{2}\right)$

b Dilation of $\frac{1}{2}$ from y-axis: $y = \sin 2x$;
dilation of 3 from x-axis: $y = 3\sin 2x$

c Dilation of 3 from y-axis: $y = \sin\left(\frac{x}{3}\right)$;
dilation of 2 from x-axis:
$y = 2\sin\left(\frac{x}{3}\right)$

d Dilation of $\frac{1}{2}$ from y-axis: $y = \sin 2x$;
translation of $+\frac{\pi}{3}$ (x-axis):
$y = \sin 2\left(x - \frac{\pi}{3}\right)$

e Dilation of 2 from the y-axis:

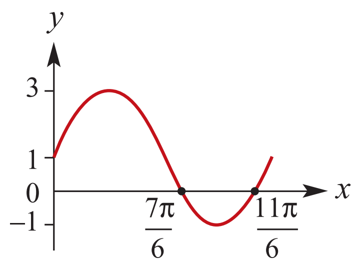
$y = \sin\left(\frac{x}{2}\right)$; $\quad y = \sin\frac{1}{2}\left(x + \frac{\pi}{3}\right)$

translation of $-\frac{\pi}{3}$ (x-axis):

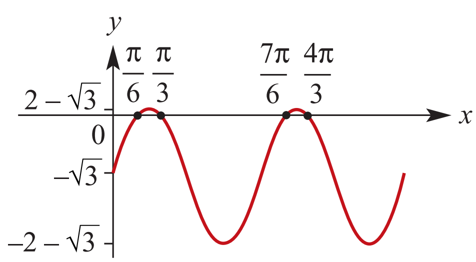
Solutions to Exercise 12J

1 Sketch over $[0, 2\pi]$:

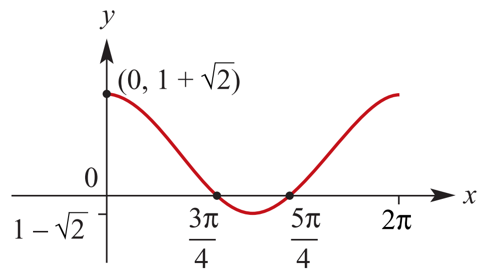
a $y = 2\sin x + 1$;
per $= 2\pi$, ampl $= 2$, range $= [-1, 3]$,
endpoints at $(0, 1)$ and $(2\pi, 0)$
$y = 0$ when $\sin x = -\dfrac{1}{2}$,
i.e. when $x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$



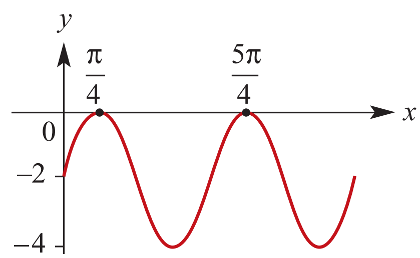
b $y = 2\sin 2x - \sqrt{3}$;
per $= \pi$, ampl $= 2$,
range $= [-2 - \sqrt{3}, 2 - \sqrt{3}]$,
endpoints at $(0, -\sqrt{3})$ and $(2\pi, -\sqrt{3})$
$y = 0$ when $\sin 2x = \dfrac{\sqrt{3}}{2}$,
i.e. when $x = \dfrac{\pi}{6}, \dfrac{\pi}{3}, \dfrac{7\pi}{6}, \dfrac{4\pi}{3}$



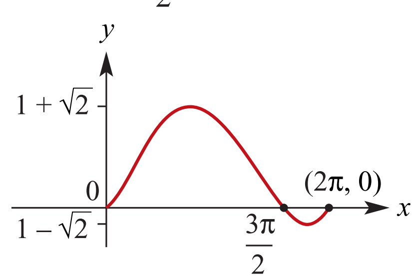
c $y = \sqrt{2}\cos x + 1$;
per $= 2\pi$, ampl $= \sqrt{2}$,
range $= [-\sqrt{2} + 1, \sqrt{2} + 1]$,
endpoints at $(0, \sqrt{2} + 1)$ and
$(2\pi, \sqrt{2} + 1)$
$y = 0$ when $\cos x = -\dfrac{1}{\sqrt{2}}$
i.e. when $x = \dfrac{3\pi}{4}, \dfrac{5\pi}{4}$



d $y = 2\sin 2x - 2$;
per $= \pi$, ampl $= 2$, range $= [-4, 0]$,
endpoints at $(0, -2)$ and $(2\pi, -2)$
$y = 0$ when $\sin 2x = 1$,
i.e. when $x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$



e $y = \sqrt{2}\sin\left(x - \dfrac{\pi}{4}\right) + 1$
per $= 2\pi$, ampl $= \sqrt{2}$,
range $= [1 - \sqrt{2}, 1 + \sqrt{2}]$,
endpoints at $(0, 0)$ and $(2\pi, 0)$
$y = 0$ when $\sin\left(x - \dfrac{\pi}{4}\right) = -\dfrac{1}{\sqrt{2}}$
i.e. when $x - \dfrac{\pi}{4} = -\dfrac{\pi}{4}, \pi + \dfrac{\pi}{4}, 2\pi - \dfrac{\pi}{4}$
i.e. $x = 0, \dfrac{3\pi}{2}, 2\pi$



2 Sketch over $[-2\pi, 2\pi]$:

a $y = 2\sin 3x - 2;$

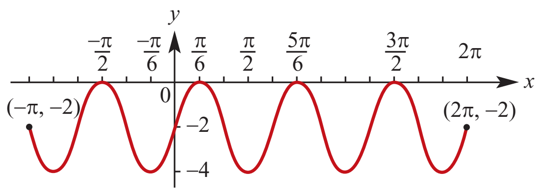
per $= \dfrac{2\pi}{3}$, ampl $= 2$, range $= [-4, 0]$,

endpoints at $(-2\pi, -2)$ and $(2\pi, -2)$

$y = 0$ when $\sin 3x = 1$

i.e. when $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}\dfrac{3\pi}{2},$

$-\dfrac{\pi}{2}, -\dfrac{7\pi}{6}, -\dfrac{11\pi}{6}$



b $y = 2\cos 3\left(x - \dfrac{\pi}{4}\right);$

per $= \dfrac{2\pi}{3}$, ampl $= 2$, range $= [-2, 2]$,

endpoints at $(-2\pi, -\sqrt{2})$ and

$(2\pi, -\sqrt{2})$

$y = 0$ when $\cos 3\left(x - \dfrac{\pi}{4}\right) = 0$

$$\therefore 3\left(x - \frac{\pi}{4}\right) = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \cdots \pm \frac{11\pi}{2}$$

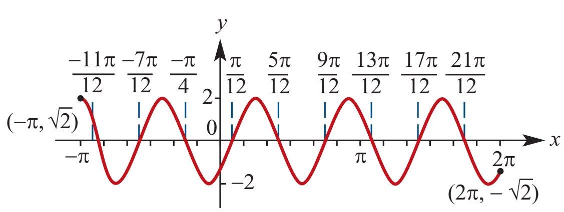
$$x - \frac{\pi}{4} = \pm\frac{\pi}{6}, \pm\frac{\pi}{2}, \pm\frac{5\pi}{6} \cdots \pm \frac{11\pi}{6}$$

$$x = -\frac{23\pi}{12}, -\frac{19\pi}{12}, -\frac{5\pi}{4}, -\frac{11\pi}{12}, -\frac{7\pi}{12},$$

$$-\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}$$

The $\dfrac{11\pi}{6}$ solution will drop out, since adding $\dfrac{\pi}{4}$ to it will take the resulting number over 2π.

It must be replaced by the solution $\dfrac{\pi}{4} - \dfrac{13\pi}{6}$.

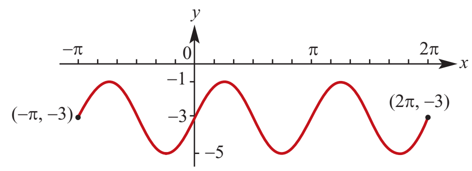


c $y = 2\sin 2x - 3;$

per $= \pi$, ampl $= 2$, range $= [-5, -1]$,

endpoints at $(-2\pi, -3)$ and $(2\pi, -3)$

No x-intercepts since $y < 0$ for all real x



d $y = 2\cos 2x + 1;$

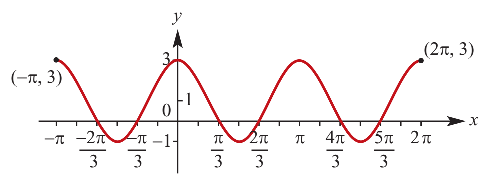
per $= \pi$, ampl $= 2$, range $= [-1, 3]$,

endpoints at $(-2\pi, 3)$ and $(2\pi, 3)$

$y = 0$ when $\cos 2x = -\dfrac{1}{2}$

i.e. when $2x = \pm\left(\pi \pm \dfrac{\pi}{3}, \pm 3\pi \pm \dfrac{\pi}{3}\right)$

$\therefore\ x = \pm\dfrac{\pi}{3}, \pm\dfrac{2\pi}{3}, \pm\dfrac{4\pi}{3}, \pm\dfrac{5\pi}{3}$



e $y = 2\cos 2\left(x - \dfrac{\pi}{3}\right) - 1;$

per $= \pi$, ampl $= 2$, range $= [-3, 1]$,

endpoints at $(-2\pi, -2)$ and $(2\pi, -2)$

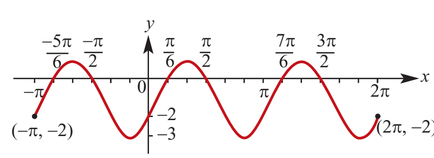
$y = 0$ when $\cos 2(x - \dfrac{\pi}{3}) = \dfrac{1}{2}$

$$\therefore 2\left(x - \frac{\pi}{3}\right) = \pm\frac{\pi}{3}, \pm\left(2\pi \pm \frac{\pi}{3}\right), \pm\left(4\pi \pm \frac{\pi}{3}\right)$$

$$x - \frac{\pi}{3} = \pm\frac{\pi}{6}, \pm\left(\pi \pm \frac{\pi}{6}\right), \pm\left(2\pi \pm \frac{\pi}{6}\right)$$

$$x = -\frac{11\pi}{6}, -\frac{3\pi}{2}, -\frac{5\pi}{6}, -\frac{\pi}{2},$$

$$\frac{\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}$$



f $y = 2\sin 2\left(x + \dfrac{\pi}{6}\right) + 1;$

per $= \pi$, ampl $= 2$, range $= [-1, 3]$,

endpoints at $(-2\pi, 1 + \sqrt{3})$ and

$(2\pi, 1 + \sqrt{3})$

$y = 0$ when $\sin 2(x + \frac{\pi}{6}) = -\frac{1}{2}$

Positive solutions:

$$2\left(x + \frac{\pi}{6}\right) = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}, 3\pi + \frac{\pi}{6}, 4\pi - \frac{\pi}{6}$$

$$x + \frac{\pi}{6} = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$$

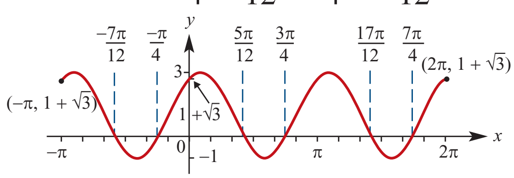
$$x = \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{17\pi}{12}, \frac{7\pi}{4}$$

Negative solutions:

$$2\left(x + \frac{\pi}{6}\right) = -\frac{\pi}{6}, -\pi + \frac{\pi}{6}, -2\pi - \frac{\pi}{6}, -3\pi + \frac{\pi}{6}$$

$$x + \frac{\pi}{6} = -\frac{\pi}{12}, -\frac{5\pi}{12}, -\frac{13\pi}{12}, -\frac{17\pi}{12}$$

$$x = -\frac{\pi}{4}, -\frac{7\pi}{12}, -\frac{5\pi}{4}, -\frac{19\pi}{12}$$



b $y = -2\sin 2\left(x + \frac{\pi}{6}\right) + 1$;

per $= \pi$, ampl $= 2$, range $= [-1, 3]$, endpoints at $(-\pi, 1 - \sqrt{3})$ and $(\pi, 1 - \sqrt{3})$

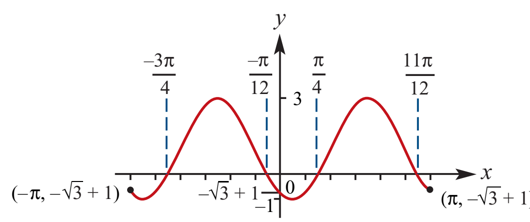
$y = 0$ when $\sin 2(x + \frac{\pi}{3}) = \frac{1}{2}$

$$\therefore 2\left(x + \frac{\pi}{6}\right) = -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$$

$$x + \frac{\pi}{6} = -\frac{7\pi}{12}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}$$

$$x = -\frac{3\pi}{4}, -\frac{\pi}{12}, \frac{\pi}{4}, \frac{11\pi}{12}$$

As with **Q.2b**, $-\frac{11\pi}{6}$ drops out, replaced by $\frac{13\pi}{6}$.



3 Sketch over $[-\pi, \pi]$:

a $y = 2\sin 2\left(x + \frac{\pi}{3}\right) + 1$;

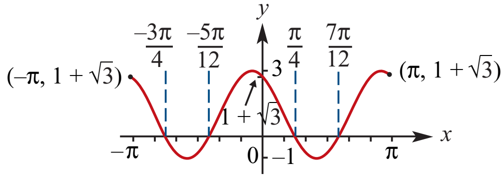
per $= \pi$, ampl $= 2$, range $= [-1, 3]$, endpoints at $(-\pi, 1 + \sqrt{3})$ and $(\pi, 1 + \sqrt{3})$

$y = 0$ when $\sin 2(x + \frac{\pi}{3}) = -\frac{1}{2}$

$$\therefore 2\left(x + \frac{\pi}{3}\right) = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$x + \frac{\pi}{3} = -\frac{5\pi}{12}, -\frac{\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$$

$$x = -\frac{3\pi}{4}, -\frac{5\pi}{12}, \frac{\pi}{4}, \frac{7\pi}{12}$$



c $y = 2\cos 2\left(x + \frac{\pi}{4}\right) + \sqrt{3}$;

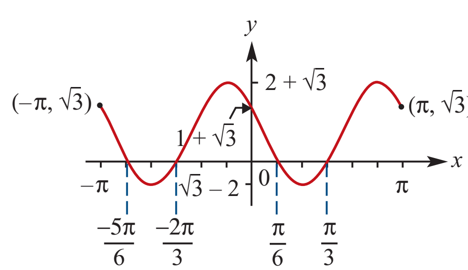
per $= \pi$, ampl $= 2$, range $= [-1, 3]$, endpoints at $(-\pi, \sqrt{3})$ and $(\pi, \sqrt{3})$

$y = 0$ when $\cos 2\left(x + \frac{\pi}{4}\right) = -\frac{\sqrt{3}}{2}$

$$\therefore 2\left(x + \frac{\pi}{4}\right) = -\frac{7\pi}{6}, -\frac{5\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$$

$$x + \frac{\pi}{4} = -\frac{7\pi}{12}, -\frac{5\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}$$

$$x = -\frac{5\pi}{6}, -\frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{3}$$



Solutions to Exercise 12K

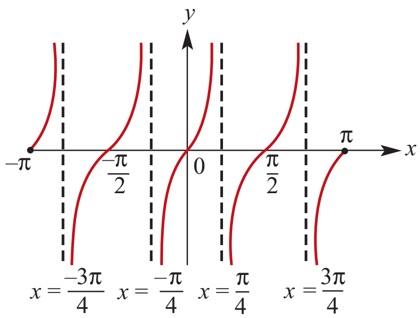
1 **a** $y = \tan(4x)$, per $= \dfrac{\pi}{4}$

b $y = \tan(\dfrac{2x}{3})$, per $= \dfrac{\pi}{\frac{2}{3}} = \dfrac{3\pi}{2}$

c $y = -3\tan(2x)$, per $= \dfrac{\pi}{2}$

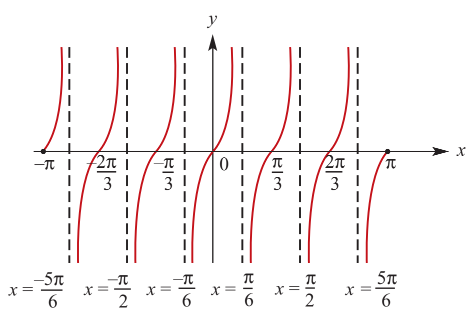
2 **a** $y = \tan(2x)$:
x-intercepts at $0, \pm\dfrac{\pi}{2}, \pm\pi$

Vertical asymptotes at $x = \pm\dfrac{\pi}{4}, \pm\dfrac{3\pi}{4}$

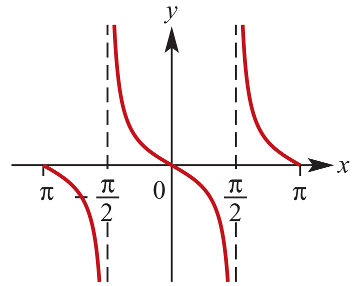


b $y = 2\tan(3x)$:

x-intercepts at $0, \pm\dfrac{\pi}{3}, \pm\dfrac{2\pi}{3}, \pm\pi$
Vertical asymptotes at
$x = \pm\dfrac{\pi}{6}, \pm\dfrac{\pi}{2}, \pm\dfrac{5\pi}{6}$



c $y = -2\tan(3x)$:
x-intercepts at $0, \pm\dfrac{\pi}{3}, \pm\dfrac{2\pi}{3}, \pm\pi$
Vertical asymptotes at
$x = \pm\dfrac{\pi}{6}, \pm\dfrac{\pi}{2}, \pm\dfrac{5\pi}{6}$



3 Solve over $[-\pi, \pi]$:

a $2\tan 2x = 2, \therefore \tan 2x = 1$
$\therefore 2x = -\dfrac{7\pi}{4}, -\dfrac{3\pi}{4}, \dfrac{\pi}{4}, \dfrac{5\pi}{4}$
$x = -\dfrac{7\pi}{8}, -\dfrac{3\pi}{8}, \dfrac{\pi}{8}, \dfrac{5\pi}{8}$

b $3\tan 3x = \sqrt{3}, \therefore \tan 3x = \dfrac{\sqrt{3}}{3}$
$\therefore 3x = -\dfrac{17\pi}{6}, -\dfrac{11\pi}{6}, -\dfrac{5\pi}{6}, \dfrac{\pi}{6}, \dfrac{7\pi}{6}, \dfrac{13\pi}{6}$
$x = -\dfrac{17\pi}{18}, -\dfrac{11\pi}{18}, -\dfrac{5\pi}{18}, \dfrac{\pi}{18}, \dfrac{7\pi}{18}, \dfrac{13\pi}{18}$

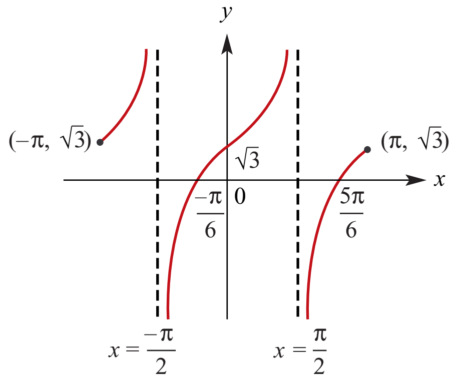
c $2\tan 2x = 2\sqrt{3} \tan 2x = \sqrt{3}$
$\therefore 2x = -\dfrac{5\pi}{3}; -\dfrac{2\pi}{3}; \dfrac{\pi}{3}; \dfrac{4\pi}{3}$
$\therefore x = -\dfrac{5\pi}{6}; -\dfrac{\pi}{3}; \dfrac{\pi}{6}; \dfrac{2\pi}{3}$

d $3\tan 3x = -\sqrt{3}, \therefore \tan 3x = -\dfrac{\sqrt{3}}{3}$
$\therefore 3x = -\dfrac{13\pi}{6}, -\dfrac{7\pi}{6}, -\dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{11\pi}{6}, \dfrac{17\pi}{6}$
$x = -\dfrac{13\pi}{18}, -\dfrac{7\pi}{18}, -\dfrac{\pi}{18}, \dfrac{5\pi}{18}, \dfrac{11\pi}{18}, \dfrac{17\pi}{18}$

4 Sketch over $[-\pi, \pi]$:

a $y = 3\tan x + \sqrt{3}$
x-intercepts where $\tan x = -\dfrac{1}{\sqrt{3}}$
$\therefore\ x = -\dfrac{\pi}{6}, \dfrac{5\pi}{6}$

Vertical asymptotes at $x = \pm\dfrac{\pi}{2}$

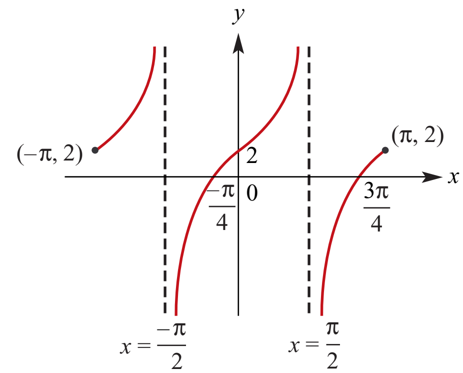


b $y = 2\tan x + 2$

x-intercepts where $\tan x = -1$

$\therefore\ x = -\dfrac{\pi}{4}, \dfrac{3\pi}{4}$

Vertical asymptotes at $x = \pm\dfrac{\pi}{2}$

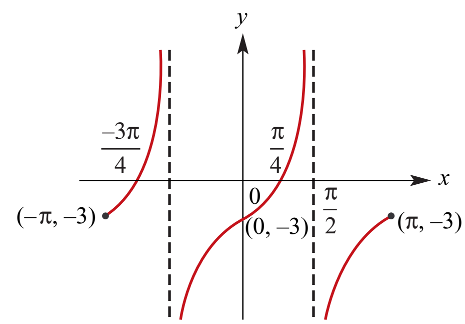


c $y = 3\tan x - 3$

x-intercepts where $\tan x = 1$

$\therefore\ x = \dfrac{\pi}{4}, -\dfrac{3\pi}{4}$

Vertical asymptotes at $x = \pm\dfrac{\pi}{2}$



Solutions to Exercise 12L

1 $\sin x = 0.3, \cos\alpha = 0.6$ and $\tan\theta = 0.7$

a $\cos(-\alpha) = \cos\alpha = 0.6$

b $\sin(\frac{\pi}{2} + \alpha) = \cos\alpha = 0.6$

c $\tan(-\theta) = -\tan\theta = -0.7$

d $\cos(\frac{\pi}{2} - x) = \sin x = 0.3$

e $\sin(-x) = -\sin x = -0.3$

f $\tan(\frac{\pi}{2} - \theta) = \dfrac{1}{0.7} = \dfrac{10}{7}$

g $\cos(\frac{\pi}{2} + x) = -\sin x = -0.3$

h $\sin(\frac{\pi}{2} - \alpha) = \cos\alpha = 0.6$

i $\sin(\frac{3\pi}{2} + \alpha) = -\cos\alpha = -0.6$

j $\cos(\dfrac{3\pi}{2} - x) = -\sin x = -0.3$

2 $0 < \theta < \dfrac{\pi}{2}$

a $\cos\theta = \sin\dfrac{\pi}{6}$

$\therefore \theta = (\dfrac{\pi}{2} - \dfrac{\pi}{6}) = \dfrac{\pi}{3}$

b $\sin\theta = \cos\dfrac{\pi}{6}$

$\therefore \theta = (\dfrac{\pi}{2} - \dfrac{\pi}{6}) = \dfrac{\pi}{3}$

c $\cos\theta = \sin\dfrac{\pi}{12}$

$\therefore \theta = (\dfrac{\pi}{2} - \dfrac{\pi}{12}) = \dfrac{5\pi}{12}$

d $\sin\theta = \cos\dfrac{3\pi}{7}$

$\therefore \theta = (\dfrac{\pi}{2} - \dfrac{3\pi}{7}) = \dfrac{\pi}{14}$

3 $\cos x = \dfrac{3}{5}, \dfrac{3\pi}{2} < x < 2\pi$:

$\sin x = \pm\sqrt{1 - \left(\dfrac{3}{5}\right)^2} = \pm\dfrac{4}{5}$

4th quadrant: $\sin x = -\dfrac{4}{5}$

$\tan x = -\dfrac{4}{5} \div \dfrac{3}{5} = -\dfrac{4}{3}$

4 $\sin x = \dfrac{5}{13}, \dfrac{\pi}{2} < x < \pi$:

$\cos x = \pm\sqrt{1 - \left(\dfrac{5}{13}\right)^2} = \pm\dfrac{12}{13}$

2nd quadrant: $\cos x = -\dfrac{12}{13}$

$\tan x = \dfrac{5}{13} \div -\dfrac{12}{13} = -\dfrac{5}{12}$

5 $\cos x = \dfrac{1}{5}, \dfrac{3\pi}{2} < x < 2\pi$:

$\sin x = \pm\sqrt{1 - \left(\dfrac{1}{5}\right)^2} = \pm\dfrac{\sqrt{24}}{5} = \pm\dfrac{2}{5}\sqrt{6}$

4th quadrant: $\sin x = -\dfrac{2}{5}\sqrt{6}$

$\tan x = -\dfrac{2}{5}\sqrt{6} \div \dfrac{1}{5} = -2\sqrt{6}$

Solutions to Exercise 12M

1 Different angles may be used.

a
$$\begin{aligned}\cos 15^\circ &= \cos(45^\circ - 30^\circ)\\ &= \cos 45^\circ \cos 30^\circ\\ &\quad + \sin 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

b
$$\begin{aligned}\cos 105^\circ &= \cos(45+60)^\circ\\ &= \cos 45^\circ \cos 60^\circ\\ &\quad - \sin 45^\circ \sin 60^\circ\\ &= \frac{1}{\sqrt{2}} \times \frac{1}{2} - \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\\ &= \frac{1-\sqrt{3}}{2\sqrt{2}}\\ &= \frac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

2 Different angles may be used.

a
$$\begin{aligned}\sin 165^\circ &= \sin(180-15)^\circ\\ &= \sin 15^\circ\end{aligned}$$
$$\begin{aligned}\sin 15^\circ &= \sin(45-30)^\circ\\ &= \sin 45^\circ \cos 30^\circ\\ &\quad - \cos 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

b
$$\begin{aligned}\tan 75^\circ &= \tan(45+30)^\circ\\ &= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}\\ &= \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} \times \frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}\\ &= \frac{3+2\sqrt{3}+1}{3-1} = 2+\sqrt{3}\end{aligned}$$

3 Different angles may be used.

a
$$\begin{aligned}\cos\frac{5\pi}{12} &= \cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)\\ &= \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

b $\sin\dfrac{11\pi}{12} = \sin\left(\pi - \dfrac{\pi}{12}\right)$

$= \sin\dfrac{\pi}{12}$

$\sin\dfrac{\pi}{12} = \sin\left(\dfrac{\pi}{3} - \dfrac{\pi}{4}\right)$

$= \sin\dfrac{\pi}{3}\cos\dfrac{\pi}{4} - \cos\dfrac{\pi}{3}\sin\dfrac{\pi}{4}$

$= \dfrac{\sqrt{3}}{2} \times \dfrac{1}{\sqrt{2}} - \dfrac{1}{2} \times \dfrac{1}{\sqrt{2}}$

$= \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$

$= \dfrac{\sqrt{6} - \sqrt{2}}{4}$

c $\tan\left(-\dfrac{\pi}{12}\right) = \tan\left(\dfrac{\pi}{4} - \dfrac{\pi}{3}\right)$

$= \dfrac{\tan\frac{\pi}{4} - \tan\frac{\pi}{3}}{1 + \tan\frac{\pi}{4}\tan\frac{\pi}{3}}$

$= \dfrac{1 - \sqrt{3}}{1 + \sqrt{3}} \times \dfrac{1 - \sqrt{3}}{1 - \sqrt{3}}$

$= \dfrac{1 - 2\sqrt{3} + 3}{1 - 3}$

$= -2 + \sqrt{3}$

4 $\cos^2 u = 1 - \sin^2 u$

$= 1 - \dfrac{144}{169} = \dfrac{25}{169}$

$\cos u = \pm\dfrac{5}{13}$

$\cos^2 v = 1 - \sin^2 v$

$= 1 - \dfrac{9}{25} = \dfrac{16}{25}$

$\cos v = \pm\dfrac{4}{5}$

$\sin(u + v) = \sin u\cos v + \cos u\sin v$

$= \pm\dfrac{3}{5} \times \dfrac{5}{13} \pm \dfrac{4}{5} \times \dfrac{12}{13}$

$= \dfrac{\pm 15 \pm 48}{65}$

There are four possible answers:

$\dfrac{63}{65}, \dfrac{33}{65}, -\dfrac{33}{65}, -\dfrac{63}{65}$

5 **a** $\sin\left(\theta + \dfrac{\pi}{6}\right) = \sin\theta\cos\dfrac{\pi}{6} + \cos\theta\sin\dfrac{\pi}{6}$

$= \dfrac{\sqrt{3}}{2}\sin\theta + \dfrac{1}{2}\cos\theta$

b $\cos\left(\pi - \dfrac{\pi}{4}\right)$

$= \cos\phi\cos\dfrac{\pi}{4} + \sin\phi\sin\dfrac{\pi}{4}$

$= \dfrac{1}{\sqrt{2}}\cos\phi + \dfrac{1}{\sqrt{2}}\sin\phi$

$= \dfrac{1}{\sqrt{2}}(\cos\phi + \sin\phi)$

c $\tan\left(\theta + \dfrac{\pi}{3}\right) = \dfrac{\tan\theta + \tan\frac{\pi}{3}}{1 - \tan\theta\tan\frac{\pi}{3}}$

$= \dfrac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta}$

d $\sin\left(\theta - \dfrac{\pi}{4}\right) = \sin\theta\cos\dfrac{\pi}{4} - \cos\theta\sin\dfrac{\pi}{4}$

$= \dfrac{1}{\sqrt{2}}\sin\theta - \dfrac{1}{\sqrt{2}}\cos\theta$

$= \dfrac{1}{\sqrt{2}}(\sin\theta - \cos\theta)$

6 **a** $\sin(v + (u - v)) = \sin u$

b $\cos((u + v) - v) = \cos u$

7 $\cos^2\theta = 1 - \sin^2\theta$

$$= 1 - \frac{9}{25} = \frac{16}{25}$$

$\cos\theta = -\frac{4}{5}$

(Since $\cos\theta < 0$)

$\sin^2\phi = 1 - \cos^2\phi$

$$= 1 - \frac{25}{169} = \frac{144}{169}$$

$\sin\phi = \frac{12}{13}$

(Since $\sin\theta > 0$)

a $\cos 2\phi = \cos^2\phi - \sin^2\phi$

$$= \frac{25}{169} - \frac{144}{169}$$

$$= -\frac{119}{169}$$

b $\sin 2\theta = 2\sin\theta\cos\theta$

$$= 2 \times -\frac{3}{5} \times -\frac{4}{5}$$

$$= \frac{24}{25}$$

c $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$= \frac{-3}{-4} = \frac{3}{4}$$

$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

$$= \frac{\frac{3}{2}}{1 - \frac{9}{16}}$$

$$= \frac{3}{2} \times \frac{16}{7}$$

$$= \frac{24}{7}$$

d $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

$$= -\frac{3}{5} \times -\frac{5}{13} + -\frac{4}{5} \times \frac{12}{13}$$

$$= \frac{14 - 48}{65}$$

$$= -\frac{33}{65}$$

e $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

$$= -\frac{4}{5} \times -\frac{5}{13} + -\frac{3}{5} \times \frac{12}{13}$$

$$= \frac{20 - 36}{65}$$

$$= -\frac{16}{65}$$

8 **a** $\tan(u + v)$

$$= \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

$$= \left(\frac{4}{3} + \frac{5}{12}\right) \div \left(1 - \frac{4}{3} \times \frac{5}{12}\right)$$

$$= \frac{21}{12} \div \frac{4}{9}$$

$$= \frac{21}{12} \times \frac{9}{4}$$

$$= \frac{63}{16}$$

b $\tan 2u = \frac{2\tan u}{1 - \tan^2 u}$

$$= \frac{\frac{8}{3}}{1 - \frac{16}{9}}$$

$$= \frac{8}{3} \times \frac{9}{-7}$$

$$= -\frac{24}{7}$$

c
$$\begin{aligned}\sec^2 u &= 1 + \tan^2 u\\ &= 1 + \frac{16}{9} = \frac{25}{9}\\ \cos^2 u &= \frac{9}{25}\\ \cos u &= \frac{3}{5} \text{ (since } u \text{ is acute)}\\ \sec^2 v &= 1 + \tan^2 v\\ &= 1 + \frac{25}{144} = \frac{169}{144}\\ \cos^2 v &= \frac{144}{169}\\ \cos v &= \frac{12}{13} \text{ (since } v \text{ is acute)}\\ \cos(u-v) &= \cos u \cos v + \sin u \sin v\\ &= \frac{3}{5}\times\frac{12}{13} + \frac{4}{5}\times\frac{5}{13}\\ &= \frac{56}{65}\end{aligned}$$

d
$$\begin{aligned}\sin 2u &= 2\sin u \cos u\\ &= 2\times\frac{4}{5}\times\frac{3}{5}\\ &= \frac{24}{25}\end{aligned}$$

9
$$\begin{aligned}\cos\alpha &= -\frac{4}{5}\\ \cos^2\beta &= 1 - \sin^2\beta\\ &= 1 - \frac{576}{625} = \frac{29}{625}\\ \cos\beta &= -\frac{7}{25}\\ \cos^2\alpha &= 1 - \sin^2\alpha\\ &= 1 - \frac{9}{25} = \frac{16}{25}\end{aligned}$$

a
$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha\\ &= \frac{16}{25} - \frac{9}{25}\\ &= \frac{7}{25}\end{aligned}$$

b
$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta\\ &= \frac{3}{5}\times-\frac{7}{25} - -\frac{4}{5}\times\frac{24}{25}\\ &= \frac{75}{125} = \frac{3}{5}\end{aligned}$$

c
$$\begin{aligned}\tan\alpha &= \frac{\sin\alpha}{\cos\alpha}\\ &= -\frac{3}{4}\\ \tan\beta &= \frac{\sin\beta}{\cos\beta}\\ &= -\frac{24}{7}\\ \tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\\ &= \frac{-\frac{3}{4} + -\frac{24}{7}}{1 - -\frac{3}{4}\times\frac{24}{7}}\\ &= -\frac{117}{28}\times-\frac{7}{11}\\ &= \frac{117}{44}\end{aligned}$$

d
$$\begin{aligned}\sin 2\beta &= 2\sin\beta\cos\beta\\ &= 2\times\frac{7}{25}\times-\frac{24}{25}\\ &= -\frac{336}{625}\end{aligned}$$

10 a
$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\times-\frac{\sqrt{3}}{2}\times\frac{1}{2}\\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

b
$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= \frac{1}{4} - \frac{3}{4}\\ &= -\frac{1}{2}\end{aligned}$$

11 **a** $(\sin\theta - \cos\theta)^2$

$= \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta$

$= 1 - \sin 2\theta$

b

$\cos^4(\theta) - \sin^4(\theta)$

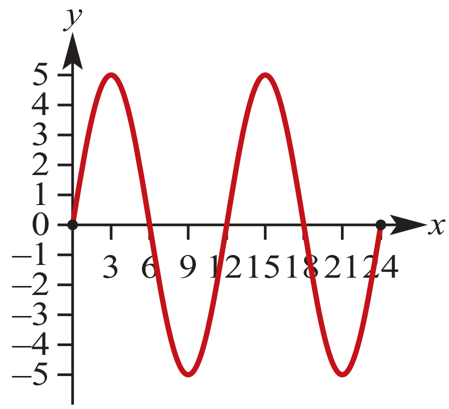
$= (\cos^2(\theta) - \sin^2(\theta))(\cos^2(\theta) + \sin^2(\theta))$

$= \cos 2\theta \times 1$

$= \cos 2\theta$

Solutions to Exercise 12N

1 a



b Maximum values occur when $\sin\left(\frac{\pi}{6}t\right) = 1$

That is when $t = 3$ and $t = 15$

c $h(3) = h(15) = 5$. The maximum height is 5 m above mean sea level

d $h(2) = 5\sin\left(\frac{\pi}{3}\right) = \frac{5\sqrt{3}}{2}$ m above mean sea level

e $h(14) = 5\sin\left(\frac{7\pi}{3}\right) = \frac{5\sqrt{3}}{2}$ m above mean sea level

f
$$5\sin\left(\frac{\pi}{6}t\right) = 2.5$$
$$\sin\left(\frac{\pi}{6}t\right) = \frac{1}{2}$$
$$\frac{\pi}{6}t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$
$$t = 1, 5, 13, 17$$

Tide is higher than 2.5 m for $t \in [1, 5] \cup [13, 17]$

2 a $x = 3 + 2\sin 3t$

When $\sin 3t = 1$, $x = 3 + 2 = 5$, the greatest distance from O.

b When $\sin 3t = -1$, $x = 3 - 2 = 1$, the least distance from O.

c When $x = 5$, $3 + 2\sin 3t = 5$

$\therefore\ \sin 3t = 1$

$\therefore\ 3t = \frac{\pi}{2}$ or $\frac{5\pi}{2}$ or $\frac{9\pi}{2}$ or ...

$\therefore\ t = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ or $\frac{9\pi}{6}$ or ...

$\therefore\ t = 0.524$ or 2.618 or 4.712 seconds for $t \in [0, 5]$

d When $x = 3, 3 + 2\sin 3t = 3$

$\therefore\ \sin 3t = 0$

$\therefore\ 3t = 0$ or π or 2π or ...

$\therefore\ t = 0$ or $\frac{\pi}{3}$ or $\frac{2\pi}{3}$ or ...

$\therefore\ t = 0$ or 1.047 or 2.094 seconds for $t \in [0, 3]$

e Particle oscillates about $x = 3$, from $x = 1$ to $x = 5$.

3 $x = 5 + 2\sin(2\pi t)$. Note that the particle oscillates between $x = 3$ and $x = 7$

a Greatest distance from O when $\sin(2\pi t) = 1$. Therefore greatest distance from O is 7 m

b Least distance from O when $\sin(2\pi t) = -1$. Therefore least distance from O is 3 m

c
$$\begin{aligned} 5 + 2\sin(2\pi t) &= 7 \\ 2\sin(2\pi t) &= 2 \\ \sin(2\pi t) &= 1 \\ 2\pi t &= \frac{\pi}{2}, \frac{5\pi}{2}, \ldots \\ t &= \frac{1}{4}, \frac{5}{4}, \frac{9}{4}, \frac{13}{4}, \frac{17}{4} \end{aligned}$$

d
$$\begin{aligned} 5 + 2\sin(2\pi t) &= 6 \\ 2\sin(2\pi t) &= 1 \\ \sin(2\pi t) &= \frac{1}{2} \\ 2\pi t &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}, \ldots \\ t &= \frac{1}{12}, \frac{5}{12}, \frac{13}{12}, \frac{17}{12}, \frac{25}{12}, \frac{29}{12} \end{aligned}$$

e Particle oscillates between $x = 3$ and $x = 7$

4 $h(t) = 10\sin\left(\frac{\pi t}{3}\right) + 10$

a **i** $h(0) = 10\sin\left(0\right) + 10 = 10$

ii $h(1) = 10\sin\left(\frac{\pi}{3}\right) + 10 = 10 + 5\sqrt{3}$

iii $h(2) = 10\sin\left(\frac{2\pi}{3}\right) + 10 = 10 + 5\sqrt{3}$

iv $h(4) = 10\sin\left(\frac{4\pi}{3}\right) + 10 = 10 - 5\sqrt{3}$

v $h(5) = 10\sin\left(\frac{5\pi}{3}\right) + 10 = 10 - 5\sqrt{3}$

b Period $= 2\pi \div \frac{\pi}{3} = 6$seconds

c Greatest height = 20 m

d
$$\begin{aligned} 10\sin\left(\frac{\pi t}{3}\right) + 10 &= 15 \\ 10\sin\left(\frac{\pi t}{3}\right) &= 5 \\ \sin\left(\frac{\pi t}{3}\right) &= \frac{1}{2} \\ \frac{\pi t}{3} &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \dots \\ t &= \frac{1}{2}, \frac{5}{2}, \frac{13}{2}, \frac{17}{2} \end{aligned}$$

e
$$\begin{aligned} 10\sin\left(\frac{\pi t}{3}\right) + 10 &= 5 \\ 10\sin\left(\frac{\pi t}{3}\right) &= -5 \\ \sin\left(\frac{\pi t}{3}\right) &= -\frac{1}{2} \\ \frac{\pi t}{3} &= \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6} \dots \\ t &= \frac{7}{2}, \frac{11}{2}, \frac{19}{2}, \frac{23}{2} \end{aligned}$$

5 $T = 17 - 8\cos\left(\frac{\pi t}{12}\right)$

a $T(0) = 17 - 8\cos\left(0\right) = 9$
The temperature was 9° C at midnight

b Maximum temperature 25°

Minimum temperature 9°

c $17 - 8\cos\left(\frac{\pi t}{12}\right) = 20$

$-8\cos\left(\frac{\pi t}{12}\right) = 3$

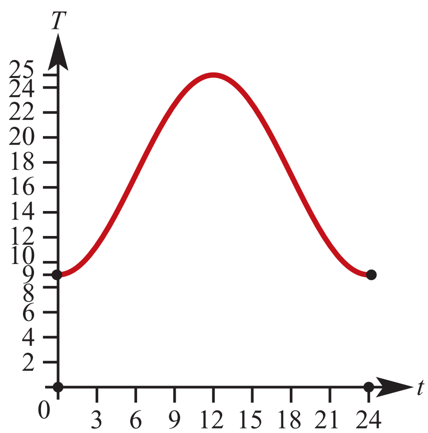
$\cos\left(\frac{\pi t}{12}\right) = -\frac{3}{8}$

$\frac{\pi t}{12} = \pi - \cos^{-1}\frac{3}{8}, \pi + \cos^{-1}\frac{3}{8}, \ldots$

$t = 7.468\ldots, 16.53\ldots$

That is between 7 : 28 and 16 : 32

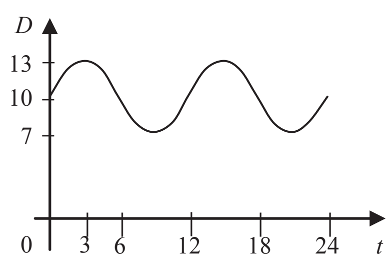
d



6 **a** $D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right), 0 \leq t \leq 24$

period $= \frac{2\pi}{\frac{\pi}{6}} = 12$; amplitude $= 3$;

translation in the positive direction of the $D(t)$-axis $= 10$



b When $D(t) = 8.5$, $10 + 3\sin\left(\frac{\pi t}{6}\right) = 8.5$

$\therefore\ 3\sin\left(\frac{\pi t}{6}\right) = -1.5$

$\therefore\ \sin\left(\frac{\pi t}{6}\right) = -\frac{1}{2}$

$\therefore \dfrac{\pi t}{6} = \dfrac{7\pi}{6}$ or $\dfrac{11\pi}{6}$ or $\dfrac{19\pi}{6}$ or $\dfrac{23\pi}{6}$ or ...

$\therefore t = 7$ or 11 or 19 or 23 or ...

From the graph, $D(t) \geq 8.5$ implies

$0 \leq t \leq 7$, or $11 \leq t \leq 19$, or $23 \leq t \leq 24$, for $0 \leq t \leq 24$

$\therefore \{t : D(t) \geq 8.5\} = \{t : 0 \leq t \leq 7\} \cup \{t : 11 \leq t \leq 19\} \cup \{t : 23 \leq t \leq 24\}$

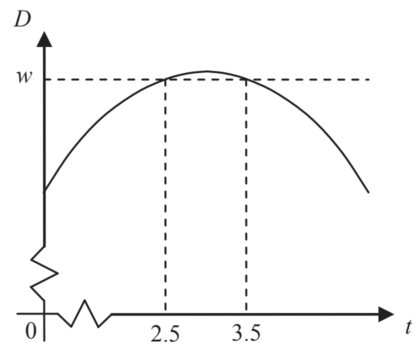
c The maximum depth is 13 m.

From the graph, the required period of time is $[2.5, 3.5]$.

The largest value of w occurs for $t = 2.5$.

$w = 10 + 3\sin\left(\dfrac{2.5\pi}{6}\right) \approx 12.9$

The largest value of w is 12.9, correct to 1 decimal place.



7 a period $= 2 \times 6$, and also period $= \dfrac{360}{r}$

$\therefore \dfrac{360}{r} = 12 \qquad \therefore r = 30$

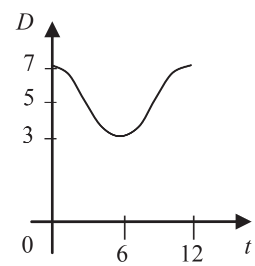
translation parallel to D-axis $= \dfrac{7+3}{2} = 5$

$\therefore p = 5$

amplitude $= \dfrac{7-3}{2} = 2 \qquad \therefore q = 2$

b When $t = 0, D = 7$

When $t = 6, D = 3$



c $D = 5 + 2\cos(30t)^\circ$

When $D = 4$, $5 + 2\cos(30t)^\circ = 4$

$\therefore 2\cos(30t)^\circ = -1 \qquad \therefore \cos(30t)^\circ = -\dfrac{1}{2}$

$\therefore (30t)^\circ = 120^\circ$ or 240°

$\therefore t = 4$ or 8 (from graph, only two values required)

Low tide is at $t = 6$, hence it will be 2 hours before the ship can enter the harbour.

Solutions to Review: Short-answer questions

1 **a** $330° = 330\left(\frac{\pi}{180}\right) = \frac{11\pi}{6}$

b $810° = 810\left(\frac{\pi}{180}\right) = \frac{9\pi}{2}$

c $1080° = 1080\left(\frac{\pi}{180}\right) = 6\pi$

d $1035° = 1035\left(\frac{\pi}{180}\right) = \frac{23\pi}{4}$

e $135° = 135\left(\frac{\pi}{180}\right) = \frac{3\pi}{4}$

f $405° = 405\left(\frac{\pi}{180}\right) = \frac{9\pi}{4}$

g $390° = 390\left(\frac{\pi}{180}\right) = \frac{13\pi}{6}$

h $420° = 420\left(\frac{\pi}{180}\right) = \frac{7\pi}{3}$

i $80° = 80\left(\frac{\pi}{180}\right) = \frac{4\pi}{9}$

2 **a** $\frac{5\pi}{6} = \frac{5\pi}{6}\left(\frac{180°}{\pi}\right) = 150°$

b $\frac{7\pi}{4} = \frac{7\pi}{4}\left(\frac{180°}{\pi}\right) = 315°$

c $\frac{11\pi}{4} = \frac{11\pi}{4}\left(\frac{180°}{\pi}\right) = 495°$

d $\frac{3\pi}{12} = \frac{3\pi}{12}\left(\frac{180°}{\pi}\right) = 45°$

e $\frac{15\pi}{2} = \frac{15\pi}{2}\left(\frac{180°}{\pi}\right) = 1350°$

f $-\frac{3\pi}{4} = -\frac{3\pi}{4}\left(\frac{180°}{\pi}\right) = -135°$

g $-\frac{\pi}{4} = -\frac{\pi}{4}\left(\frac{180°}{\pi}\right) = -45°$

h $-\frac{11\pi}{4} = -\frac{11\pi}{4}\left(\frac{180°}{\pi}\right) = -495°$

i $-\frac{23\pi}{4} = -\frac{23\pi}{4}\left(\frac{180°}{\pi}\right) = -1035°$

3 **a** $\sin\frac{11\pi}{4} = \sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$

b $\cos\left(-\frac{7\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

c $\sin\frac{11\pi}{6} = -\sin\frac{\pi}{6} = -\frac{1}{2}$

d $\cos\left(-\frac{7\pi}{6}\right) = \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

e $\cos\left(\frac{13\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

f $\sin\frac{23\pi}{6} = -\sin\frac{\pi}{6} = -\frac{1}{2}$

g $\cos\left(-\frac{23\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$

h $\sin\left(-\frac{17\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

4 **a** $2\sin\left(\frac{\theta}{2}\right)$

Ampl $= 2$, per $= \frac{2\pi}{\frac{1}{2}} = 4\pi$

b $-3\sin 4\theta$

Ampl $= 3$, per $= \frac{2\pi}{4} = \frac{\pi}{2}$

c $\frac{1}{2}\sin 3\theta$

Ampl $= \frac{1}{2}$, per $= \frac{2\pi}{3}$

d $-3\cos 2x$

Ampl $= 3$, per $= \dfrac{2\pi}{2} = \pi$

e $-4\sin\left(\dfrac{x}{3}\right)$

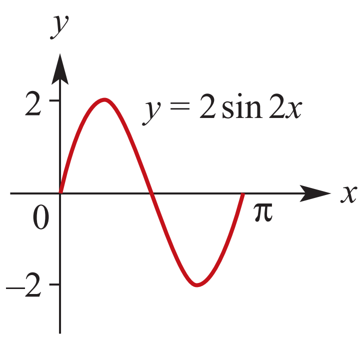
Ampl $= 4$, per $= \dfrac{2\pi}{\frac{1}{3}} = 6\pi$

f $\dfrac{2}{3}\sin\left(\dfrac{2x}{3}\right)$

Ampl $= \dfrac{2}{3}$, per $= \dfrac{2\pi}{\frac{2}{3}} = 3\pi$

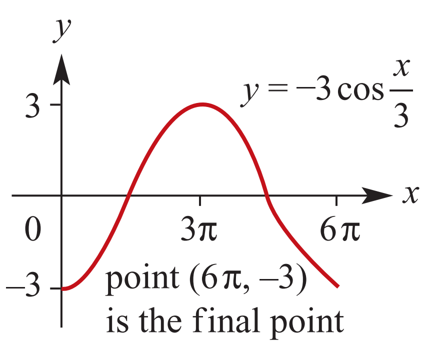
5 a $y = 2\sin 2x$

Per $= \pi$, ampl $= 2$, x-intercepts at $0, \dfrac{\pi}{2}, \pi$



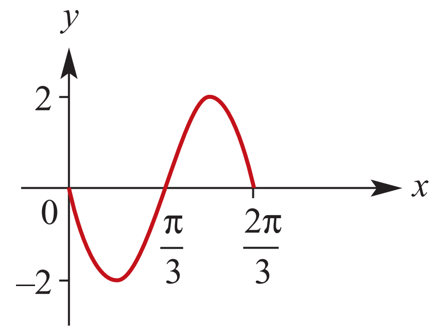
b $y = -3\cos\left(\dfrac{x}{3}\right)$

Per $= 6\pi$, ampl $= 3$, x-intercepts at $\dfrac{3\pi}{2}, \dfrac{9\pi}{2}$



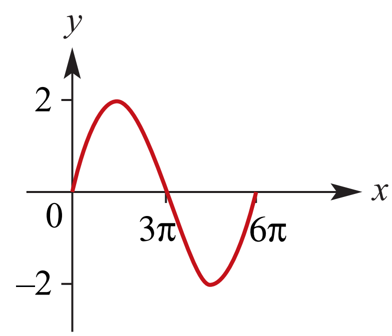
c $y = -2\sin 3x$

Per $= \dfrac{2\pi}{3}$, ampl $= 2$, x-intercepts at $0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}$



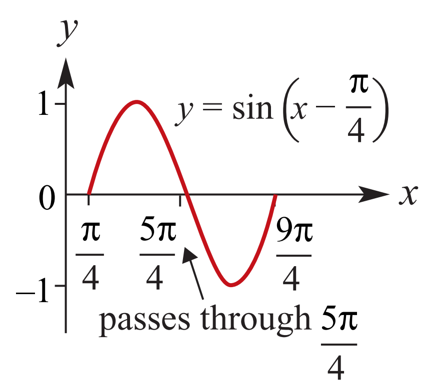
d $y = 2\sin\left(\dfrac{x}{3}\right)$

Per $= 6\pi$, ampl $= 2$, x-intercepts at $0, 3\pi, 6\pi$



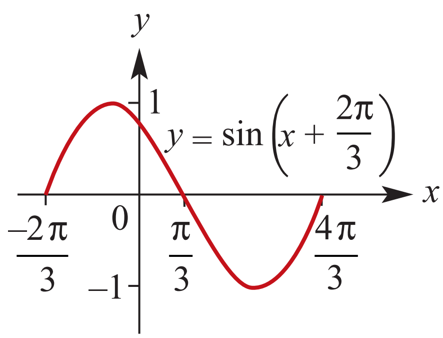
e $y = \sin\left(x - \dfrac{\pi}{4}\right)$

Per $= 2\pi$, ampl $= 1$, x-intercepts at $\dfrac{\pi}{4}, \dfrac{5\pi}{4}$



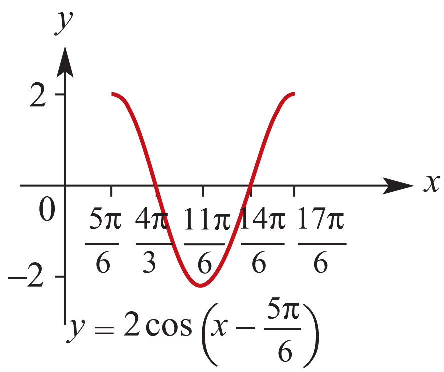
f $y = \sin\left(x + \dfrac{2\pi}{3}\right)$

Per $= 2\pi$, ampl $= 1$, x-intercepts at $\dfrac{\pi}{3}, \dfrac{4\pi}{3}$



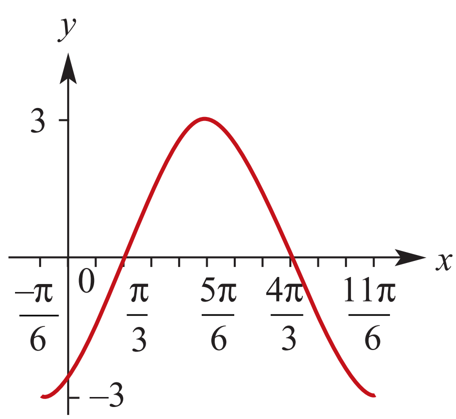
g $y = 2\cos\left(x - \frac{5\pi}{6}\right)$

Per $= 2\pi$, ampl $= 2$, x-intercepts at $\frac{\pi}{3}, \frac{4\pi}{3}$



h $y = -3\cos\left(x + \frac{\pi}{6}\right)$

Per $= 2\pi$, ampl $= 3$, x-intercepts at $\frac{\pi}{3}, \frac{4\pi}{3}$



6 a $\sin\theta = -\frac{\sqrt{3}}{2}, \therefore\ \theta = -\frac{2\pi}{3}, -\frac{\pi}{3}$

(No solutions over $[0, \pi]$)

b

$$\sin(2\theta) = -\frac{\sqrt{3}}{2}$$

$$\therefore\quad 2\theta = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

$$\theta = -\frac{\pi}{3}, -\frac{\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}$$

c

$$\sin\left(\theta - \frac{\pi}{3}\right) = -\frac{1}{2}$$

$$\therefore\quad \theta - \frac{\pi}{3} = -\frac{\pi}{6}, \frac{7\pi}{6}$$

$$\theta = \frac{\pi}{6}, \frac{3\pi}{2}$$

d

$$\sin\left(\theta + \frac{\pi}{3}\right) = -1$$

$$\therefore\quad \theta + \frac{\pi}{3} = \frac{3\pi}{2}$$

$$\theta = \frac{7\pi}{6}$$

(Only 1 solution for −1 and 1)

e

$$\sin\left(\frac{\pi}{3} - \theta\right) = -\frac{1}{2}$$

$$\therefore\quad \frac{\pi}{3} - \theta = -\frac{\pi}{6}, -\frac{5\pi}{6}$$

$$-\theta = -\frac{\pi}{2}, -\frac{7\pi}{6}$$

$$\theta = \frac{\pi}{2}, \frac{7\pi}{6}$$

7 $\cos^2 A = 1 - \sin^2 A$

$$= 1 - \frac{25}{169} = \frac{144}{169}$$

$\cos A = \frac{12}{13}$ (Since A is acute)

$\cos^2 B = 1 - \sin^2 B$

$$= 1 - \frac{64}{289} = \frac{225}{289}$$

$\cos B = \frac{15}{17}$ (Since B is acute)

a $\cos(A + B)$

$= \cos A \cos B - \sin A \sin B$

$= \dfrac{12}{13} \times \dfrac{15}{17} - \dfrac{5}{13} \times \dfrac{8}{17}$

$= \dfrac{140}{221}$

b $\sin(A - B)$

$= \sin A \cos B - \cos A \sin B$

$= - \times \dfrac{5}{13} \times \dfrac{15}{17} - \dfrac{12}{13} \times \dfrac{8}{17}$

$= -\dfrac{21}{221}$

c
$$\begin{aligned}
\tan A &= \frac{\sin A}{\cos A} = \frac{5}{12} \\
\tan B &= \frac{\sin B}{\cos B} = \frac{8}{15} \\
\tan(A + B) &= \frac{\tan a + \tan B}{1 - \tan A \tan B} \\
&= \left(\frac{5}{12} + \frac{8}{15}\right) \\
&\quad \div \left(1 - \frac{5}{12} \times \frac{8}{15}\right) \\
&= \frac{57}{60} \div \frac{7}{9} \\
&= \frac{19}{20} \times \frac{9}{7} \\
&= \frac{171}{140}
\end{aligned}$$

8
$$\begin{aligned}
\tan(\theta + A) &= 4 \\
\frac{\tan\theta + \tan A}{1 - \tan\theta \tan A} &= 4 \\
\frac{\tan\theta + 2}{1 - 2\tan\theta} &= 4 \\
\tan\theta + 2 &= 4(1 - 2\tan\theta) \\
&= 4 - 8\tan\theta \\
9\tan\theta &= 2 \\
\tan\theta &= \frac{2}{9}
\end{aligned}$$

Solutions to Review: Multiple-choice questions

1 C $\sin^{-1}\left(\frac{3}{5}\right) \approx 37°$

2 D $3 - 10\cos 2x$ has range $[3 - 10, 3 + 10]$.
So the minimum value is $3 - 10$

3 E $4\sin\left(2x - \frac{\pi}{2}\right)$ has range $[-4, 4]$.

4 C $3\sin\left(\frac{x}{2} - \pi\right) + 4$ has per $= \frac{2\pi}{\frac{1}{2}} = 4\pi$

5 E $y = \sin x$:
Dilation of $\frac{1}{2}$ from y-axis:
$y = \sin 2x$
Translated $+\frac{\pi}{4}$ units in x-axis:
$y = \sin 2\left(x - \frac{\pi}{4}\right)$

6 D $f(x) = a\sin(bx) + c$: per $= \frac{2\pi}{b}$

7 E $y = \tan ax$ has vertical asymptotes at $y = \pm\frac{\pi}{2a}$
If $\frac{\pi}{2a} = \frac{\pi}{6}$, then a could be 3

8 E $3\sin x + 1 = b$
If $b > 0$ the only value of b possible is 4, since the only positive value of y for $\sin x$ with one solution over a period is 1.
$3\sin x + 1 = 4, \therefore \sin x = 1$

9 C $b = a\sin x, x \in [-2\pi, 2\pi], a > b > 0$
2 periods, each with 2 solutions = 4

10 B $D(t) = 8 + 2\sin\left(\frac{\pi t}{6}\right), 0 \le t \le 24$
Find primary solution for $D = 9$:
$$8 + 2\sin\frac{\pi t}{6} = 9$$
$$\sin\frac{\pi t}{6} = \frac{1}{2}$$
$$\frac{\pi t}{6} = \frac{\pi}{6}$$
$\therefore\ t = 1$

11 A
$$\begin{aligned}\sin 2A &= 2\sin A\cos A\\ m &= 2\sin A \times n\\ \sin A &= \frac{m}{2n}\\ \tan A &= \frac{\sin A}{\cos A}\\ &= \frac{m}{2n} \times \frac{1}{n}\\ &= \frac{m}{2n^2}\end{aligned}$$

Solutions to Review: Extended-response questions

1 a i When $t = 5.7$, $\quad d = 12 + 12\cos\frac{1}{6}\pi\left(5.7 + \frac{1}{3}\right)$

$= 0.001\,83 = 1.83 \times 10^{-3}$ hours

ii When $t = 2.7$, $\quad d = 12 + 12\cos\frac{1}{6}\pi\left(2.7 + \frac{1}{3}\right)$

$= 11.79$ hours

b When $d = 5$, $\quad 12 + 12\cos\frac{1}{6}\pi\left(t + \frac{1}{3}\right) = 5$

$\therefore \quad 12\cos\frac{1}{6}\pi\left(t + \frac{1}{3}\right) = -7$

$\therefore \quad \cos\frac{1}{6}\pi\left(t + \frac{1}{3}\right) = -\frac{7}{12}$

$\therefore \quad \frac{1}{6}\pi\left(t + \frac{1}{3}\right) = 2.193\,622\,912, 4.089\,562\,395$

(first two positive values required)

$\therefore \quad t = \frac{2.193\,622\,912 \times 6}{\pi} - \frac{1}{3}, \frac{4.089\,562\,395 \times 6}{\pi} - \frac{1}{3}$

$\therefore \quad t = 3.856, 7.477$

When $t = 3.856$, the date is 26 April. When $t = 7.477$, the date is 14 August.

2 a When $t = 4$, $\quad A = 21 - 3\cos\left(\frac{4\pi}{12}\right) = 19.5$

The temperature inside the house is 19.5°C at 8 am.

b

$$D = A - B = 21 - 3\cos\left(\frac{\pi t}{12}\right) - \left(22 - 5\cos\left(\frac{\pi t}{12}\right)\right)$$

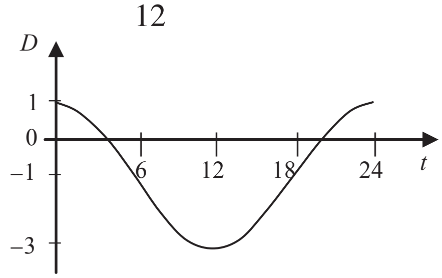
$$= 21 - 3\cos\left(\frac{\pi t}{12}\right) - 22 + 5\cos\left(\frac{\pi t}{12}\right)$$

$$\therefore \quad D = 2\cos\left(\frac{\pi t}{12}\right) - 1, 0 \le t \le 24$$

c amplitude = 2,

translation in positive direction of D-axis $= -1$,

period $= \dfrac{2\pi}{\frac{\pi}{12}} = 24$



d When $A < B$, $D < 0$

When $D = 0$, $2\cos\left(\frac{\pi t}{12}\right) - 1 = 0$

$\therefore \quad \cos\left(\frac{\pi t}{12}\right) = \frac{1}{2}$

$\therefore \quad \frac{\pi t}{12} = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ or $\ldots$

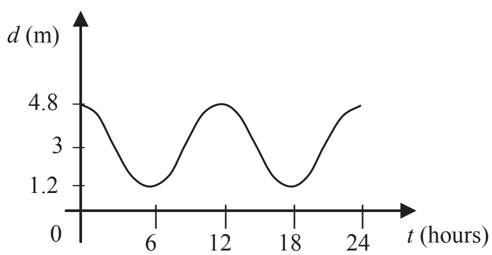
$\therefore \quad t = 4$ or 20 for $t \in [0, 24]$

When $D < 0, 4 < t < 20$

$\therefore \{t\colon A < B\} = \{t\colon 4 < t < 20\}$

3 **a** $d = 3 + 1.8\cos\left(\frac{\pi t}{6}\right)$

amplitude = 1.8; period = $2\pi \div \frac{\pi}{6} = 12$



b High tides occur when $t = 0, t = 12$ and $t = 24$, i.e. at 3 am, 3 pm and 3 am.

c Low tides occur when $t = 6$ and $t = 18$, i.e. at 9 am and 9 pm.

d The ferry operates from $t = 5$ to $t = 17$.

Consider $3 + 1.8\cos\left(\frac{\pi t}{6}\right) = 2$

$$\therefore \quad \cos\left(\frac{\pi t}{6}\right) = \frac{-1}{18}$$

$$= \frac{-5}{9}$$

$$\therefore \quad \frac{\pi t}{6} = \pi - \cos^{-1}\left(\frac{5}{9}\right), \pi + \cos^{-1}\left(\frac{5}{9}\right), 3\pi - \cos^{-1}\left(\frac{5}{9}\right), 3\pi + \cos^{-1}\left(\frac{5}{9}\right)$$

$$t = 6 - \frac{6}{\pi}\cos^{-1}\left(\frac{5}{9}\right), 6 + \frac{6}{\pi}\cos^{-1}\left(\frac{5}{9}\right), 18 - \frac{6}{\pi}\cos^{-1}\left(\frac{5}{9}\right) \text{ or } 18 + \frac{6}{\pi}\cos^{-1}\left(\frac{5}{9}\right)$$

$$\approx 4.125 \text{ or } 7.875 \text{ or } 16.125 \text{ or } 19.875$$

$\therefore$ earliest time, $7.875 - \frac{50}{60} = 7.04$

$\therefore$ ferry can leave Main Beach at 10.03 am.

e **i** Ferry must be in and out harbour by $t = 16.125$. It must leave 55 minutes earlier, i.e. at $t = 15.208\ldots$ It can leave Main Beach no later than 6.12 pm.

ii Starts at 10.03 am and last trip leaves at 6.12 pm. Five trips are possible.

4 $D = p - 2\cos(rt)$

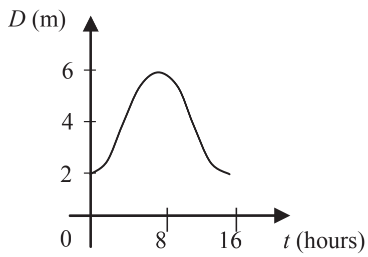
a Low tide depth is 2 m. High tide is 8 hours later, and the depth is 6 m.

period is 16 $\quad \therefore \quad \dfrac{2\pi}{r} = 16$

$\therefore \quad r = \dfrac{\pi}{8}$

The centre is 4, as the upper value is 6 and the lower value is 2, $\therefore\ p = 4$.
The amplitude is 2.

b



c The first low tide is at 4 am. The second low tide will be at 8 pm.

d The depth is equal to 4 metres when $t = 4$ and $t = 12$, i.e. at 8 am and 4 pm.

e **i** $7.5 - 6 = 1.5$ metres

ii At 2 pm, $t = 10$, and the depth is $5.414\ldots$ metres.

$7.5 - 5.414\ldots = 2.085\ldots$

The length of pole exposed = 2.086 m.

f When $d = 3.5, t = 3.356\ldots \therefore$ by symmetry,

total time $= 6.713\ldots = 6$ hours 42 minutes 47 seconds

$\therefore \quad$ time covered $= 16 - 6.713 = 9.287$

$= 9$ hours 17 minutes