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6

TERMINOLOGY

algebraic area
antiderivative
antidifferentiation
definite integral
difference function
differentiation
family of functions
indefinite integral
integration
linearity of integration
marginal rate of change
net signed area
physical area
primitive function
signed area
total change
velocity

INTEGRALS

APPLICATIONS OF INTEGRATION

Prior learning

ANTI-DIFFERENTIATION

- recognise anti-differentiation as the reverse of differentiation (ACMMM114)
- use the notation $\int f(x)dx$ for anti-derivatives or indefinite integrals (ACMMM115)
- establish and use the formula $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$ for $n \neq -1$ (ACMMM116)
- establish and use the formula $\int e^x\,dx = e^x + c$ (ACMMM117)
- establish and use the formulas $\int \sin(x)dx = -\cos(x) + c$ and $\int \cos(x)dx = \sin(x) + c$ (ACMMM118)
- recognise and use linearity of anti-differentiation (ACMMM119)
- determine indefinite integrals of the form $\int f(ax+b)dx$ (ACMMM120)
- identify families of curves with the same derivative function (ACMMM121)
- determine $f(x)$, given $f'(x)$ and an initial condition $f(a) = b$ (ACMMM122)
- determine displacement given velocity in linear motion problems (ACMMM123)

APPLICATIONS OF INTEGRATION

- calculate the area under a curve (ACMMM132)
- calculate total change by integrating instantaneous or marginal rate of change (ACMMM133)
- calculate the area between curves in simple cases (ACMMM134)
- determine positions given acceleration and initial values of position and velocity (ACMMM135) AC

6.01 INDEFINITE INTEGRALS

You know that the **definite integral** of a function $f(x)$ over the interval $[a, b]$, $\int_a^b f(x)dx$, is the area 'under' the graph of $f(x)$. If $f(x)$ is negative, the definite integral works out to be negative, so we say that the definite integral is the **signed area** between the curve, the x-axis, and the lines $x = a$ and $x = b$. The definite integral is given by the lmit of the sum of areas of rectangles as

$$\int_a^b f(x)dx = \lim_{\delta x \to 0} \sum f(x)\delta x = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\delta x_i.$$

You should recall the following about the **indefinite integral**.

IMPORTANT

An **indefinite integral** $F(x)$ of $f(x)$ is a function such that $F'(x) = f(x)$.

The indefinite integral is normally written as $F(x) + c$, where c is an arbitrary constant called the **constant of integration**, because indefinite integrals differ only by a constant.

An indefinite integral is also called an **antiderivative** or **primitive** and is written as

$$F(x) = \int f(x)dx$$

The definite integral can be calculated from an indefinite integral $F(x)$ using **The Fundamental Theorem of Calculus** as

$$\int_a^b f(x)dx = F(b) - F(a)$$

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You should already be familiar with the derivatives shown below.

IMPORTANT

Basic derivatives

$$\frac{d}{dx}x^n = nx^{n-1} \qquad \frac{d}{dx}kx^n = knx^{n-1} \qquad \frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\cos(x) = -\sin(x) \qquad \frac{d}{dx}\tan(x) = \frac{1}{\cos^2(x)}$$

$$\frac{d}{dx}kf(x) = kf'(x) \qquad \frac{d}{dx}f(ax+b) = af'(ax+b)$$

You should also be able to find the derivatives of combined functions using the **product rule**, **quotient rule** and **chain rule**. You can reverse these rules to make rules for many indefinite integrals.

INVESTIGATION Basic integrals

- Find the derivative of x^{n+1}
- Find the derivative of $\dfrac{x^{n+1}}{n+1}$
- Hence find the indefinite integral of x^n
- What is the derivative of $\sin(x)$?
- Hence find the indefinite integral of $\cos(x)$
- Find the derivative of $-\cos(x)$
- Hence find the indefinite integral of $\sin(x)$
- What is the derivative of e^x?
- Hence find the indefinite integral of e^x
- Find the derivative of $\dfrac{f(ax+b)}{a}$
- Hence find the indefinite integral of $f(ax+b)$

From the investigation, we have the following rules for basic indefinite integrals.

IMPORTANT

Indefinite integrals of basic functions

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c\ (n \neq -1) \qquad \int kx^n\,dx = \frac{kx^{n+1}}{n+1} + c\ (n \neq -1)$$

$$\int \sin(x)\,dx = -\cos(x) + c \qquad \int \sin(ax+b)\,dx = -\frac{\cos(ax+b)}{a} + c$$

$$\int \cos(x)\,dx = \sin(x) + c \qquad \int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a} + c$$

$$\int e^x\,dx = e^x + c \qquad \int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + c$$

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c\ (n \neq -1) \qquad \int kf'(x)\,dx = kf(x) + c$$

If you are given a value for the integral at a particular point, then you can find the constant of integration.

Example 1

Find the function $f(x)$ in each case if:

a $f'(x) = 4x^5$ and $f(1) = 5$

b $f'(x) = \frac{1}{x^3}$ and $f\left(\frac{1}{2}\right) = 3$

c $f'(x) = 5\sqrt{x}$ and $f(9) = 100$

Solution

a Write using integral notation. $\int 4x^5\ dx = 4\int x^5\ dx$

Apply the relevant rule. $f(x) = 4 \times \frac{x^6}{6} + c$

Simplify. $= \frac{2x^6}{3} + c$

Substitute in the values given. $5 = \frac{2 \times 1^6}{3} + c$

Find c. $c = 4\frac{1}{3}$

Write the function. $f(x) = \frac{2x^6}{3} + 4\frac{1}{3}$

b Write using integral notation. $\int \frac{1}{x^3}\ dx = \int x^{-3}\ dx$

Apply the relevant rule. $f(x) = \frac{x^{-2}}{-2} + c$

Write in the form given. $f(x) = -\frac{1}{2x^2} + c$

Substitute in the values given. $3 = -\frac{1}{2(\frac{1}{2})^2} + c$

Find c. $c = 5$

Write the function. $f(x) = 5 - \frac{1}{2x^2}$

c Write using integral notation. $\int 5\sqrt{x}\ dx = 5\int x^{\frac{1}{2}}\ dx$

Apply the relevant rule. $f(x) = 5 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$

Substitute in the values given. $100 = \frac{10 \times 9^{\frac{3}{2}}}{3} + c$

Find c. $c = 10$

Write in the form given. $f(x) = \frac{10\sqrt{x^3}}{3} + 10$ or $f(x) = \frac{10x\sqrt{x}}{3} + 10$

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Example 2

Find y in terms of x if:

a $\dfrac{dy}{dx} = e^{4x}$ **b** $\dfrac{dy}{dx} = 6\sin(3x)$ **c** $\dfrac{dy}{dx} = 8\cos\left(-\dfrac{x}{2}\right)$

Solution

a Write using integral notation.

$$y = \int e^{4x}\,dx$$

Apply the relevant rule.

$$= \frac{e^{4x}}{4} + c$$

b Write using integral notation.

$$y = \int 6\sin(3x)\,dx$$

$$= 6\int \sin(3x)\,dx$$

Apply the relevant rule.

$$= 6\left[\frac{-\cos(3x)}{3}\right] + c$$

Simplify.

$$= -2\cos(3x) + c$$

c Write using integral notation.

$$y = \int 8\cos\left(-\frac{x}{2}\right)dx$$

$$= 8\int \cos\left(-\frac{x}{2}\right)dx$$

Apply the relevant rule.

$$= 8\left[\frac{\sin\left(-\frac{x}{2}\right)}{-\frac{1}{2}}\right] + c$$

Simplify.

$$= -16\sin\left(-\frac{x}{2}\right) + c$$

$$= 16\sin\left(\frac{x}{2}\right) + c$$

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Example 3

Find the antiderivative of:

a $(2x+1)^4$ **b** $(5x-1)^{-3}$

Solution

a Apply the relevant rule. $\int (2x+1)^4\,dx = \dfrac{(2x+1)^5}{2\times 5} + c$

Simplify. $= \dfrac{1}{10}(2x+1)^5 + c$ or $\dfrac{(2x+1)^5}{10} + c$

b Write in index form. $\int \dfrac{1}{(5x-1)^3}\,dx = \int (5x-1)^{-3}\,dx$

Apply the relevant rule. $= \dfrac{(5x-1)^{-2}}{5\times(-2)} + c$

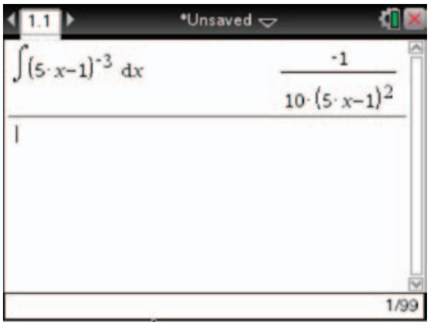
Simplify. $= -\dfrac{(5x-1)^{-2}}{10} + c$

Write in the original form. $= -\dfrac{1}{10(5x-1)^2} + c$

You can use a CAS calculator to find indefinite integrals.

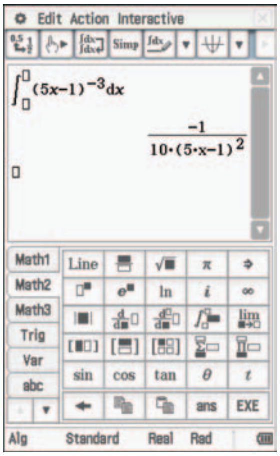
TI-Nspire CAS

In the calculation page, Press menu, 4:Calculus and 3:Integral.
To obtain the indefinite integral, leave out the bounds. Type in the expression and then x after the 'd'.



ClassPad

Use the Main menu and the Math1 soft keyboard. Set the calculator to Standard.
Tap [∫]. Fill in the blanks as shown on the right and dx. To obtain the indefinite integral, leave out the bounds.



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EXERCISE 6.01 Indefinite integrals

Concepts and techniques

WS Finding indefinite integrals 1

1 Example 1 The function $f(x)$ such that $f'(x) = 6x^2$ and $f(x) = 2$ when $x = -1$ is:

A $12x - 4$ **B** $2x^3 + 4$ **C** $2x^3 - 2$ **D** $3x^2 + 4$ **E** $2x^3 + 2$

2 Example 2 Given that $\dfrac{dy}{dx} = 4e^{-x}$ and $y = -9$ when $x = 0$, the function y is given by:

A $4e^{-x} + 5$ **B** $e^{-4x} + 5$ **C** $-e^{4x} - 4$ **D** $-4e^{-x} - 5$ **E** $-4e^{x} + 9$

3 The function $f(x)$ such that $f'(x) = -8\sin(4x)$ and $f\left(\dfrac{\pi}{4}\right) = -5$ is:

A $2\cos(4x) - 3$ **B** $16\cos(4x) - 9$ **C** $2\sin(4x) - 5$
D $-2\cos(4x) + 3$ **E** $-32\cos(4x) + 5$

4 Find the indefinite integral of each of the following, giving answers with positive indices.

a x **b** x^2 **c** x^6 **d** $2x^4$
e $5x^{-3}$ **f** $-3x^3$ **g** $-5x^{-4}$ **h** $\sqrt{x}$
i $5\sqrt{x}$ **j** $\dfrac{x^5}{7}$ **k** $\dfrac{x^3}{5}$ **l** $\dfrac{x^{-3}}{4}$
m $x^{\frac{1}{3}}$ **n** $3x^{\frac{2}{5}}$ **o** $x^{\frac{-3}{4}}$ **p** $\dfrac{4}{x^3}$
q $\dfrac{-5}{x^6}$ **r** $\dfrac{10}{\sqrt{x}}$ **s** $\dfrac{-6}{\sqrt[3]{x}}$ **t** $\dfrac{8}{x\sqrt{x}}$

5 Find the indefinite integrals of the following.

a e^{2x} **b** e^{4x} **c** e^{-x} **d** e^{5x}
e e^{-2x} **f** e^{4x+1} **g** $-3e^{5x}$ **h** e^{2t}
i $5e^{4x}$ **j** $-6e^{-2x}$ **k** $4e^{\frac{x}{2}}$ **l** $6e^{-\frac{x}{3}}$

6 Find the indefinite integrals of the following.

a $\cos(x)$ **b** $\sin(x)$ **c** $\sin(3x)$ **d** $-\sin(7x)$
e $\cos(x + 1)$ **f** $\sin(2x - 3)$ **g** $\cos(2x - 1)$ **h** $4\sin\left(\dfrac{x}{2}\right)$
i $-\sin(3 - x)$ **j** $3\cos\left(\dfrac{x}{4}\right)$ **k** $\sin(\pi - x)$ **l** $\cos(x + \pi)$
m $-2\sin\left(\dfrac{2x}{5}\right)$ **n** $4\cos\left(\dfrac{7x}{4}\right)$ **o** $2\cos\left(\dfrac{\pi x}{3}\right)$ **p** $-2\sin\left(\dfrac{-3x}{\pi}\right)$

7 Example 3 Find the antiderivative of each of the following.

a $(x + 1)^4$ **b** $(5x - 1)^9$ **c** $(3y - 2)^7$ **d** $(4 + 3x)^4$
e $(7x + 8)^{12}$ **f** $(1 - x)^6$ **g** $\sqrt{2x - 5}$ **h** $2(3x + 1)^{-4}$
i $3(x + 7)^{-2}$ **j** $\dfrac{1}{2(4x - 5)^3}$ **k** $\sqrt[3]{4x + 3}$ **l** $(2 - x)^{-\frac{1}{2}}$
m $\sqrt{(t + 3)^3}$ **n** $\sqrt{(5x + 2)^5}$ **o** $(4 - 5x)^{-4}$ **p** $-6(3 - 4x)^{-5}$

Reasoning and communication

8 For the curve $y = f(x)$, $f'(x) = -3x$. If $y = 2$ when $x = 2$, find the equation of the curve.

9 The gradient at a point (x, y) on a curve is given by $3e^{2x}$. Find the equation of the curve if it passes through the point (0, 5.5).

10 Differentiate e^{x^4} and use this result to find the indefinite integral of $2x^3e^{x^4}$.

11 Differentiate $(4x^2 + 1)^3$ and hence deduce the indefinite integral of $6(4x^2 + 1)^2$.

6.02 PROPERTIES OF INDEFINITE INTEGRALS

You have seen how the indefinite integrals of certain functions can be determined using rules.

The rule for the derivative of a linear combination of functions leads directly to the property of **linearity of integration**.

IMPORTANT

Linearity of integration

Linear combination: $\int [af(x)+bg(x)]dx = a\int f(x)dx + b\int g(x)dx$

Constant multiple: $\int kf(x)dx = k\int f(x)dx$

Sum of functions: $\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$

Difference of functions: $\int [f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$

Suppose $F(x)$ and $G(x)$ are primitives of $f(x)$ and $g(x)$ respectively.

Then $F'(x) = f(x)$ and $G'(x) = g(x)$.

Consider the derivative of $aF(x) + bG(x)$.

Then $\frac{d}{dx}[aF(x) + bG(x)] = aF'(x) + bG'(x)$ because a linear combination is preserved by differentiation.

Substituting $F'(x) = f(x)$ and $G'(x) = g(x)$ gives

$$\frac{d}{dx}[aF(x) + bG(x)] = af(x) + bg(x)$$

But this is just a statement that

$$\int [af(x)+bg(x)]dx = a\int f(x)dx + b\int g(x)dx$$

written in reverse form, so integration also preserves linear combinations.

Substitution of $a = k$ and $g(x)$ gives the constant multiple rule, $a = 1$ and $b = 1$ gives the addition rule and $a = 1$ and $b = -1$ gives the subtraction rule.

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Example 4

Find the indefinite integral of each of the following.

a $4x^7$ **b** $4x^3 - 7x^{-2} + 5$

Solution

a Apply the constant multiple rule. $\int 4x^7\,dx = 4\int x^7\,dx$

Apply the relevant rule of integration. $= 4 \times \dfrac{x^8}{8} + c$

Simplify. $= \dfrac{x^8}{2} + c$

b Apply the linear combination rule. $\int (4x^3 - 7x^{-2} + 5)dx = 4\int x^3\,dx - 7\int x^{-2}\,dx + \int 5dx$

Apply the relevant rule of integration. $= 4 \times \dfrac{x^4}{4} - 7 \times \dfrac{x^{-1}}{-1} + 5 \times x + c$

$= x^4 + \dfrac{7}{x} + 5x + c$

As you have seen before, when additional information is provided, we can determine the value of the constant, and hence the unique antiderivative.

Example 5

Given that $f'(x) = 3x^3 - 3x^2$ and $f(2) = 7$, find $f(x)$.

Solution

Find the indefinite integral. $\int (3x^3 - 3x^2)dx = \dfrac{3x^4}{4} - \dfrac{3x^3}{3} + c$

Write the expression for $f(x)$. $f(x) = \dfrac{3x^4}{4} - x^3 + c$

Find $f(2)$. $f(2) = \dfrac{3 \times 2^4}{4} - 2^3 + c$

$= 4 - c$

We know that $f(2) = 7$. $7 = 4 - c$

Solve. $c = -3$

State the result. $f(x) = \dfrac{3x^4}{4} - x^3 - 3$

The substitution of different values for the constant of integration leads to the formation of a **family of functions**.

Example 6

CAS **a** Find the indefinite integral $\int (4x-3)(x+1)dx$.

b Substitute values of $c=-1$, $c=2$ and $c=5$ to create $f1(x)$, $f2(x)$ and $f3(x)$.

c Draw the functions on the same graph and describe this family of functions.

Solution

a Expand the product.

$$\int (4x-3)(x+1)dx = \int (4x^2+4x-3x-3)dx$$

Collect like terms.

$$= \int (4x^2+x-3)dx$$

Apply the linear combination rule.

$$= \int 4x^2\,dx + \int x\,dx - \int 3\,dx$$

$$= \frac{4x^3}{3} + \frac{x^2}{2} - 3x + c$$

b Write $f1(x)$ by substituting $c=-1$.

$$f1(x) = \frac{4x^3}{3} + \frac{x^2}{2} - 3x - 1$$

Write $f2(x)$ by substituting $c=2$.

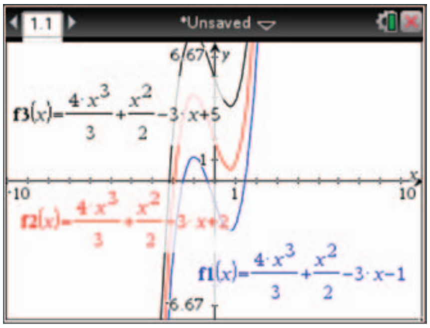
$$f2(x) = \frac{4x^3}{3} + \frac{x^2}{2} - 3x + 2$$

Write $f2(x)$ by substituting $c=5$.

$$f3(x) = \frac{4x^3}{3} + \frac{x^2}{2} - 3x + 5$$

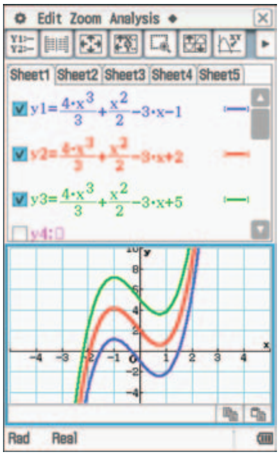
c **TI-Nspire CAS**

Start with a graph page. After entering $f1(x)$, use menu, 3: Graph Entry/Edit and 1: Function to enter $f2(x)$ and $f3(x)$.
Move the labels around to make it easier to distinguish between the graphs.



ClassPad

Use the Graph&Table menu.
You may need to reset Graph Format to default. First tap ???
Enter the functions in $y1$, $y2$ and $y3$.
Tap [graph icon] to see the graphs and tap [view window icon] to change the View Window if necessary.
Make sure that you tick the boxes to show the functions and set them to different lines.



Comment on the functions.

The functions are vertical translations of each other.

EXERCISE 6.02 Properties of indefinite integrals

Concepts and techniques

Finding indefinite integrals 2

1 Example 4 $\int (2x^2+x+5)dx$ is equal to:

A $\int (2x^2+x)dx+5$ B $2x^2+x+\int 5dx$ C $\int 2x^2\,dx+\int x\,dx+\int 5\,dx$

D $\int 2x^2dx+x+5$ E $2x^2+\int (x+5)dx$

2 Example 5 The function $f(x)$ such that $f'(x)=4x+3$ and $f(1)=7$ is:

A $2x^3+3x^2+2x$ B $\frac{4x}{2}+3x^2+1$ C $2x^2+3x+2$

D $4x^2+3x-2$ E $2x^2-3x-2$

3 Example 6 $\int x^2(2x-7)dx$ is equal to:

A $\int (2x^3-7x^2)dx$ B $\int x^2\,dx\int (2x-7)dx$ C $(2x-7)\int x^2\,dx$

D $x^2\int (2x-7)dx$ E $\int (x^2+2x-7)dx$

4 Given that $\frac{dy}{dx}=\frac{ax^2}{3}-x$ and $y=2$ when $x=0$, an expression for y is:

A $\frac{ax^3}{3}-ax+2$ B $ax^3-\frac{x}{2}+2$ C $\frac{ax^3}{3}-\frac{x}{2}+2$

D $\frac{ax^3}{9}-\frac{x}{2}+2$ E $ax^3-ax+2a$

5 Find each indefinite integral.

a $\int (m+1)dm$ b $\int (t^2-7)dt$ c $\int (h^2+5)dh$

d $\int (y-3)dy$ e $\int (2x+4)dx$ f $\int (b^2+b)db$

g $\int (a^3-a-1)da$ h $\int (x^2+2x+5)dx$ i $\int (4x^3-3x^2+8x-1)dx$

j $\int (6x^5+x^4+2x^3)dx$ k $\int (x^7-3x^6-9)dx$ l $\int (2x^3+x^2-x-2)dx$

m $\int (x^5+x^3+4)dx$ n $\int (4x^2-5x-8)dx$ o $\int (3x^4-2x^3+x)dx$

p $\int (6x^3+5x^2-4)dx$ q $\int (3x^{-4}+x^{-3}+2x^{-2})dx$ r $\int (7x^{\frac{3}{2}}-4x+6x^{-\frac{1}{3}})dx$

6 Find each indefinite integral

a $\int \frac{x^6-3x^5+2x^4}{x^3}dx$ b $\int (1-2x)^2\,dx$ c $\int (x-2)(x+5)dx$

d $\int \frac{4x^3-x^5-3x^2+7}{x^5}\,dx$ e $\int (y^2-y^{-7}+5)dy$ f $\int (t^2-4)(t-1)dt$

g $\int \sqrt{x}\left(1+\frac{1}{\sqrt{x}}\right)dx$ h $\int \frac{(x+5)(x-2)}{x^4}dx$ i $\int \frac{2x^2-4x+3}{\sqrt{x}}dx$

7 Example 6 For each of the following, find y in terms of x.

a $\frac{dy}{dx}=2x-5$ and $y=8$ when $x=-1$.

b $\frac{dy}{dx}=3x^{\frac{1}{2}}-4x$ and $y=-6$ when $x=4$.

c $\frac{dy}{dx}=3x^2-x+2$ and $y=0$ when $x=2$.

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8 Find the equation of the curve $f(x)$ given that:

a $f'(x) = 6x - 1$ and the curve passes through the point (0, 5)

b $f'(x) = 7 - 4x$ and the curve passes through the point (–1, 1)

c $f'(x) = 3x^{-2} + 2$ and the curve passes through the point (1, 5)

d $f'(x) = \frac{2}{\sqrt{x}} + 3x$ and $f(1) = 3$

e $f'(x) = x^{\frac{1}{3}} + 6x^2 - 10$ and $f(1) = -7$

9 CAS a Find the indefinite integral $\int -6x\,dx$.

b Substitute in values of $c = 0$, $c = 1$ and $c = 2$ to create $f1(x)$, $f2(x)$ and $f3(x)$.

c Draw the functions on the same graph and describe this family of functions.

10 CAS a Find the indefinite integral $\int (3x^2 + 2x)dx$.

b Substitute in values of $c = -2$, $c = 1$ and $c = 3$ to create $f1(x)$, $f2(x)$ and $f3(x)$.

c Draw the functions on the same graph and describe this family of functions.

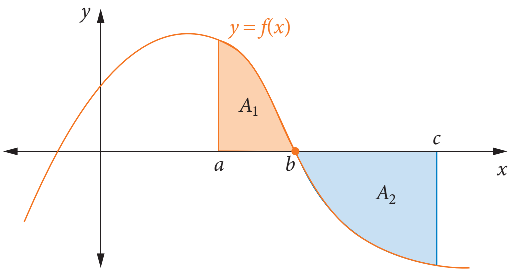
Reasoning and communication

11 The curve of the function $f(x)$ has a stationary point at (4, –2) and a gradient of $\frac{3x}{2} + k$, where k is a constant. Calculate $f(2)$.

12 The curve of the function $f(x)$ has a stationary point at $(\frac{1}{4}, 8)$ and a gradient of $\frac{16x - \sqrt{x}}{x^3}$. Find $f(x)$.

6.03 AREAS UNDER CURVES

An area must be positive. However, the definite integral is also positive for areas above the x-axis and negative for areas below the x-axis. We say that the definite integral calculates the **signed area**, so when we want the actual area, we need to change the sign for areas below the x-axis or for areas calculated from right to left. This is illustrated below for areas above and below the x-axis.



The definite integrals for the regions above are as follows.

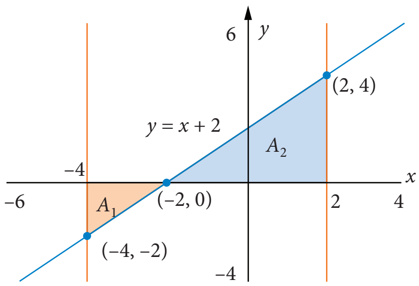
$\int_a^b f(x)dx = A_1$, $\int_b^c f(x)dx = -A_2$ and $\int_a^c f(x)dx = A_1 - A_2$

Note also that $\int_b^a f(x)dx = -A_1$, and $\int_c^b f(x)dx = A_2$.

You should think of the integral from a to c is as total signed area or **net signed area** between the curve and the x-axis.

Consider the function $f(x) = x + 2$ and the area between the graph of $f(x)$ for $-4 \le x < 2$.

The graph of $y = x + 2$ is shown here.



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Using the area of a triangle, you get:

$A_1 = \frac{1}{2} \times 2 \times 2 = 2$ square units and

$A_2 = \frac{1}{2} \times 4 \times 4 = 8$ square units.

The net signed area $= A_2 - A_1 = 8 - 2 = 6$ square units

Using integrals with the same function, you get:

$$\begin{aligned}\int_{-4}^{2} (x+2)\,dx &= \left[\frac{x^2}{2} + 2x\right]_{-4}^{2} \\ &= \left(\frac{2^2}{2} + 2\times 2\right) - \left(\frac{(-4)^2}{2} + 2\times(-4)\right) \\ &= 6 - 0 \\ &= 6 \text{ square units}\end{aligned}$$

Also,

$$\int_{-4}^{-2} (x+2)\,dx = \left[\frac{x^2}{2} + 2x\right]_{-4}^{-2} = -2$$

showing that the signed area below the axis (calculated from left to right as usual) works out to be negative.

Recall the following properties of definite integrals from Chapter 4.

IMPORTANT

Properties of definite integrals

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

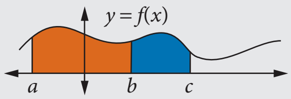
$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$\int_a^b [cf(x) + dg(x)]\,dx = c\int_a^b f(x)dx + d\int_a^b g(x)dx$$



Example 7

Find the definite integrals of:

a $f(x) = 6x^2 - 10x + 3$ between 4 and 7

b $f(x) = \dfrac{16}{(2x+3)^3}$ between 1 and 2

Solution

a Write in symbols.

$$\int_4^7 (6x^2 - 10x + 3)\,dx$$

Find the indefinite integral and insert the limits.

$$= \left[2x^3 - 5x^2 + 3x\right]_4^7$$

Find the values at each limit.

$$= (2 \times 7^3 - 5 \times 7^2 + 3 \times 7) - (2 \times 4^3 - 5 \times 4^2 + 3 \times 4)$$

Simplify.

$$= 462 - 60$$

$$= 402 \text{ square units}$$

Write the answer.

The value of the integral of $f(x) = 6x^2 - 10x + 3$ between 4 and 7 is 402.

b Write in symbols.

$$\int_1^2 \frac{16}{(2x+3)^3}\,dx$$

Rewrite so that the expression is not a fraction.

$$= \int_1^2 16(2x+3)^{-3}\,dx$$

Find the integral using the relevant rule.

$$= \left[\frac{16(2x+3)^{-2}}{2 \times -2}\right]_1^2$$

Express with a positive power and simplify.

$$= \left[\frac{-4}{(2x+3)^2}\right]_1^2$$

Substitute in the values of the limits.

$$= \left[\frac{-4}{7^2}\right] - \left[\frac{-4}{5^2}\right]$$

Evaluate.

$$= \frac{-4}{49} + \frac{4}{25}$$

$$= \frac{96}{1225}$$

Write the answer.

The integral of $f(x) = \dfrac{16}{(2x+3)^3}$ between 1 and 2 is $\dfrac{96}{1225}$ square units.

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Example 8

Calculate $\int_{0.1}^{3} \frac{dx}{x^2}$.

Solution

Write the function in index form.

$$\int_{0.1}^{3} \frac{dx}{x^2} = \int_{0.1}^{3} x^{-2}\,dx$$

Integrate and insert the limits.

$$= \left[-x^{-1}\right]_{0.1}^{3}$$

Find the values at each limit.

$$= (-1 \times 3^{-1}) - (-1 \times 0.1^{-1})$$

Evaluate.

$$= \left(-1\times\frac{1}{3}\right) - \left(-1\times\frac{1}{0.1}\right)$$

$$= -\frac{1}{3} + 10$$

$$= 9\frac{2}{3}$$

Write the answer.

$$\int_{0.1}^{3} \frac{dx}{x^2} = 9\,\frac{2}{3} \text{ square units}$$

Example 9

Calculate the following.

a $\int_{0}^{\frac{\pi}{2}} \sin(2x)\,dx$ **b** $\int_{-1.5}^{1.5} 6e^{3x}\,dx$

Solution

a Integrate and insert the limits.

$$\int_{0}^{\frac{\pi}{2}} \sin(2x)\,dx = \left[-\frac{1}{2}\cos(2x)\right]_{0}^{\frac{\pi}{2}}$$

Find the values at each limit.

$$= -\frac{1}{2}\times\cos(\pi) - \left[-\frac{1}{2}\times\cos(0)\right]$$

Evaluate.

$$= \left[-\frac{1}{2}\times(-1)\right] - \left(-\frac{1}{2}\times 1\right)$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1$$

Write the answer.

$$\int_{0}^{\frac{\pi}{2}} \sin(2x)\,dx = 1$$

b Integrate and insert limits.

$$\int_{-1.5}^{1.5} 6e^{3x}\,dx = \left[2e^{3x}\right]_{-1.5}^{1.5}$$

Find the values at each limit.

$$= 2e^{4.5} - 2e^{-4.5}$$

Evaluate.

$$= 180.0342\ldots - 0.0222\ldots$$

$$\approx 180.01$$

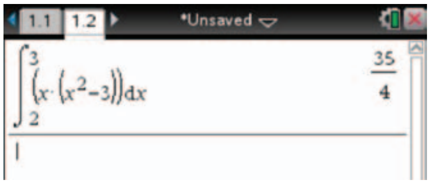
Write the answer.

$$\int_{-1.5}^{1.5} 6e^{3x}\,dx \approx 180.01 \text{ square units}$$

You can use your CAS calculator to calculate definite integrals. Instead of leaving the bounds blank like you did for indefinite integrals, put them in, as shown for $\int_2^3 x(x^2-3)dx$ below.

TI-Nspire CAS

Use a calculator page.
Use menu, 4:Calculus and 3:Integral.
Complete the integral, including the bounds and dx.



ClassPad

Use the Main menu and the Math1 soft keyboard. Set the calculator to Standard. Tap . Fill in the function including the bounds and dx.

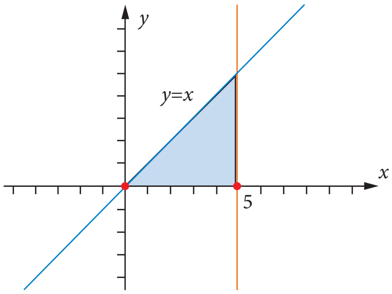
On either calculator, if you put in an integral that cannot be calculated exactly, such as $\int_0^3 e^{-x^2}\,dx$, then even if you have the calculator set to give an exact answer, the answer will be given as a decimal. However, $\int_0^3 \frac{1}{\sqrt{x}}dx$ will be given as either $2\sqrt{3}$ or 3.464…, depending whether you set the calculator for an exact or approximate answer.

EXERCISE 6.03 Areas under curves

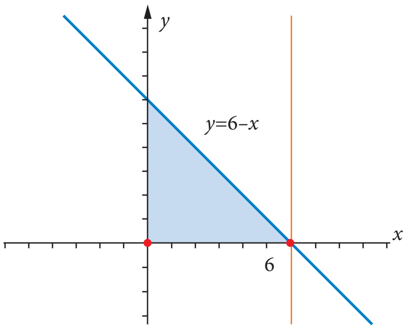
WS
Finding definite integrals

Concepts and techniques

1 Find the area of the triangle on the right:
  a geometrically
  b using integration



2 Find the area of the triangle on the right:
  a geometrically
  b using integration



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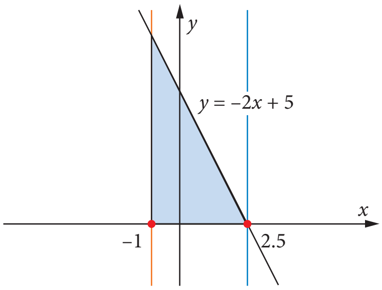
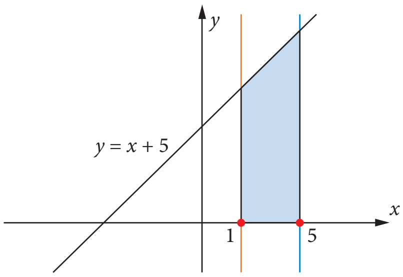
3 Express the following shaded areas as definite integrals.

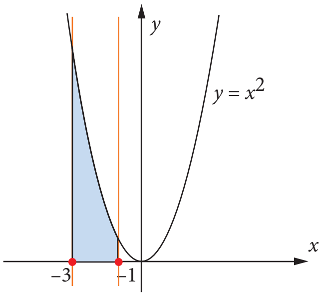
a



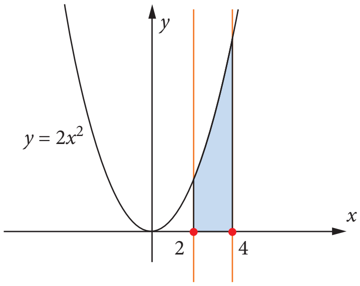
b



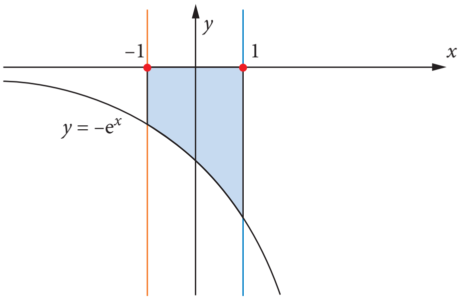
c



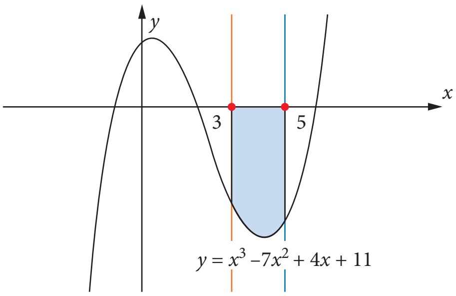
d



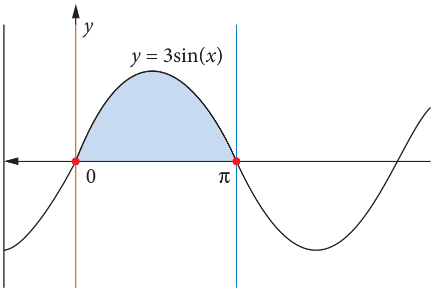
e



f



g



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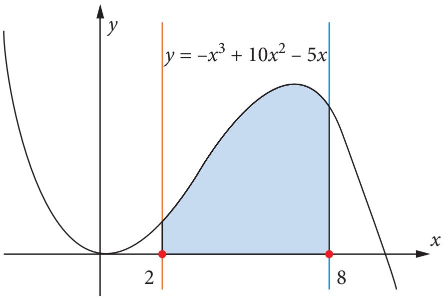

4 Example 7 Evaluate the following definite integrals.

a $\int_{1}^{10}(9x+7)\,dx$ b $\int_{0}^{6}8\,dx$ c $\int_{2}^{10}5x^3\,dx$

d $\int_{-3}^{3}x^6\,dx$ e $\int_{0}^{8}6x^3\,dx$ f $\int_{-5}^{0}(2x^2-x)\,dx$

g $\int_{-12}^{12}(20-m)\,dm$ h $\int_{1}^{2}(4t-7)\,dt$ i $\int_{-3}^{4}(2-x)^2\,dx$

j $\int_{-1}^{4}(3x^2-2x)\,dx$ k $\int_{1}^{3}(4x^2+6x-3)\,dx$ l $\int_{0}^{1}(x^3-3x^2+4x)\,dx$

5 Evaluate the following definite integrals.

a $\int_1^3 \frac{1}{(3x+1)^3}dx$ b $\int_0^1 \frac{1}{(2x-3)^2}dx$ c $\int_0^2 \frac{1}{(2x-5)^3}dx$

d $\int_0^1 \frac{3}{(2x+1)^4}dx$ e $\int_{-1}^0 \frac{2}{(3x+4)^4}dx$ f $\int_2^4 \frac{1}{\sqrt{2x+4}}dx$

6 Example 8 Evaluate the following definite integrals.

a $\int_1^3 \frac{1}{x^2}dx$ b $\int_1^2 \frac{1}{x^3}dx$ c $\int_5^{10} 4x^{-2}\,dx$

d $\int_{2.4}^{5.8} \frac{3}{x^4}dx$ e $\int_2^6 2x^{-3}\,dx$ f $\int_1^3 \frac{3x^2+2x}{x^4}dx$

7 Example 9 Evaluate in exact form.

a $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{3x}\,dx$ b $\int_3^8 5e^n\,dn$ c $\int_0^1 e^{5x}\,dx$

d $\int_0^2 -e^{-x}\,dx$ e $\int_1^4 2e^{3x+4}\,dx$ f $\int_2^3 (3x^2-e^{2x})dx$

g $\int_0^2 (e^{2x}+1)dx$ h $\int_1^2 (e^x-x)dx$ i $\int_0^3 (e^{2x}-e^{-x})dx$

8 Evaluate correct to 2 decimal places.

a $\int_1^4 e^{3V}\,dV$ b $\int_1^3 e^{-x}\,dx$ c $\int_0^2 2e^{3y}\,dy$

d $\int_5^6 (e^{x+5}+2x-3)dx$ e $\int_0^1 (e^{3t+4}-t)dt$ f $\int_1^2 (e^{4x}+e^{2x})dx$

9 Evaluate, giving exact answers where appropriate.

a $\int_0^{\pi} \sin(x)dx$ b $\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \cos(2x)dx$ c $\int_{\frac{\pi}{2}}^{\pi} \sin\left(\frac{x}{2}\right)dx$

d $\int_0^{\frac{\pi}{2}} \cos(3x)dx$ e $\int_0^{\frac{1}{2}} \sin(\pi x)dx$ f $\int_0^{\frac{\pi}{8}} \sec^2(2x)dx$

g $\int_0^{\frac{\pi}{12}} 3\cos(2x)dx$ h $\int_0^{\frac{\pi}{10}} -\sin(5x)dx$

10 Evaluate the following.

a $\int_2^4 (5t^2+4t+5)dt$ b $\int_0^3 (v^5-4v^3+2v)dv$ c $\int_{-3}^3 (6u^5+5u^4+4)du$

d $\int_{-1}^1 \frac{72}{(4y+5)^7}dy$ e $\int_1^8 \sqrt[4]{x}\,dx$ f $\int_4^9 \frac{dt}{t^2\sqrt{t}}$

g $\int_0^2 4e^{2t-3}\,dt$ h $\int_4^6 \frac{35}{(5h-9)^2}dh$ i $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 3\sin\left(6x+\frac{\pi}{3}\right)dx$

j $\int_4^6 (e^x-x^3)dx$ k $\int_4^6 \sqrt{4x+1}\,dx$ l $\int_2^4 16(5-4v)^3\,dv$

m $\int_0^{\frac{\pi}{3}} 6\sin\left(3x-\frac{\pi}{4}\right)dx$ n $\int_{-\pi}^{\pi} [\sin(x)-\cos(x)]dx$ o $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{3}{4}\cos\left(\frac{1}{2}x+\frac{\pi}{2}\right)dx$

Reasoning and communication

11 Why can't you calculate the following?

a $\int_0^3 \frac{1}{x^2}dx$ b $\int_0^5 \frac{1}{(x-5)^2}dx$ c $\int_{-1}^3 \frac{1}{(x+1)^3}dx$

12 Explain why the value that you get for the integral $\int_0^4 \frac{1}{(x-2)^2}dx$ is not valid.

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13 What is wrong with $\int_{-2}^{2} \frac{1}{x} dx$?

14 Differentiate xe^{x^2} and hence find $\int_0^1 \left(2x^2 e^{x^2} + e^{x^2}\right) dx$.

15 During a flood, the flow rate of a small creek under a road bridge increased from its normal level of 5 m^3/h to a peak after 5 hours, according to the equation $f = 5 + 30t^2$, where t is the time after the increase began. Find the amount of water that flowed under the bridge in the 5 hours before it reached its peak.

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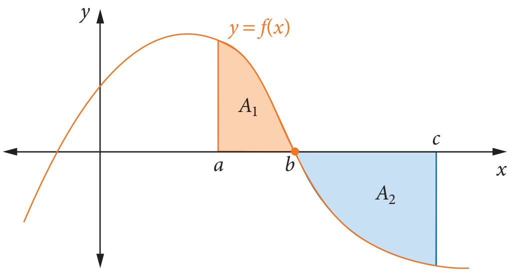
6.04 PHYSICAL AREAS

In the previous section you saw that the **signed** or **algebraic area** between a curve and the x-axis is given by the definite integral. It is the difference between the physical areas above and below the x-axis.

The **physical area** between a curve and the x-axis is always positive. Areas above and below the x-axis are calculated separately when using integrals to find a physical area.



In the diagram, the physical areas A_1 and A_2 are both positive, but the integral for A_2 is negative.

In terms of the integrals,

$$A_1 = \int_a^b f(x)dx > 0 \text{ and } A_2 = \int_b^c f(x)dx < 0$$

$$\begin{aligned} \text{So, physical area} &= A_1 - A_2 \\ &= \int_a^b f(x)dx - \int_b^c f(x)dx \end{aligned}$$

When finding physical areas by integration, it is necessary to find the zeros to check where the graph crosses the x-axis and hence where the areas change sign.

Example 10

Find the area enclosed by $y = x^2 - 3x - 4$, $x = 2$, $x = 5$ and $y = 0$.

Solution

$x = 2$ and $x = 5$ are just vertical lines, and $y = 0$ is the x-axis, so the question is actually asking for the physical area between $y = x^2 - 3x - 4$ and the x-axis from $x = 2$ to $x = 5$.

Since the question has several parts, set out your solution with headings.

Find the zeros.
Let $y = 0$.

Zeros

$x^2 - 3x - 4 = 0$

Factorise.

$(x + 1)(x - 4) = 0$

State the zeros.

The zeros are at $x = -1$ and $x = 4$

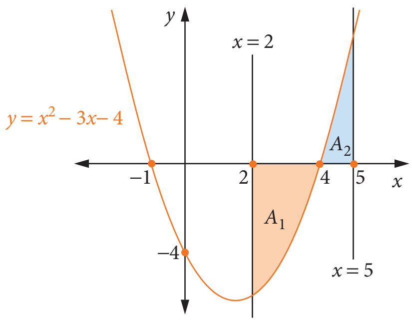
Let $x = 0$

***y*-intercept**

$y = -4$

The function is a quadratic with a positive coefficient of x^2, so it has a minimum.
Draw in the lines $x = 2$ and $x = 5$.
Shade the required areas.
Shade the areas needed as A_1 and A_2.

Sketch of graph



Write the integrals to find the area.

$$\text{Area} = -\int_2^4 (x^2 - 3x - 4)dx + \int_4^5 (x^2 - 3x - 4)dx$$

Integrate.

$$= -\left[\frac{x^3}{3} - \frac{3x^2}{2} - 4x\right]_2^4 + \left[\frac{x^3}{3} - \frac{3x^2}{2} - 4x\right]_4^5$$

Evaluate at the limits.

$$= -[-18\tfrac{2}{3} - (-11\tfrac{1}{3})] + [-15\tfrac{5}{6} - (-18\tfrac{2}{3})]$$

Finish the calculation.

$$= -(-7\tfrac{1}{3}) + 2\tfrac{5}{6} = 10\tfrac{1}{6}$$

Write the answer.

The area enclosed by $y = x^2 - 3x - 4$, $x = 2$, $x = 5$ and $y = 0$ is $10\frac{1}{6}$ square units.

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Example 11

Find the area enclosed by $f(x) = 24 - 2x - 2x^2$ and the x-axis.

Solution

The area will be between the zeros.
Let $f(x) = 0$.

Zeros

$-2(x^2 + x - 12) = 0$

Factorise and rearrange.

$-2(x - 3)(x + 4) = 0$

Write the zeros.

The zeros are at $x = -4$ and $x = 3$

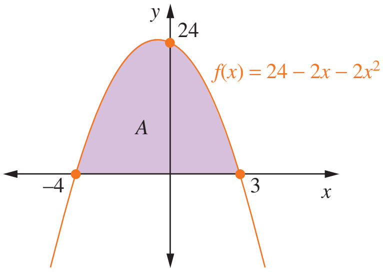
Let $x = 0$

***y*-intercept**

$y = 24$

The function is a quadratic with a negative coefficient of x^2, so it has a maximum.
Shade the required area as A.

Sketch of graph



Write the integral to find the area.

$$\text{Area } (A) = \int_{-4}^{3} (24 - 2x - 2x^2)\,dx$$

Integrate.

$$= \left[24x - x^2 - \frac{2x^3}{3} \right]_{-4}^{3}$$

Evaluate.

$$= 45 - \left(-69\tfrac{1}{3}\right)$$

$$= 114\tfrac{1}{3}$$

Write the answer.

The area enclosed by $f(x) = 24 - 2x - 2x^2$ and the x-axis is $114\frac{1}{3}$ square units.

INVESTIGATION Integrals

1 Find the following integrals.

$$\int_{-1}^{1} 1\,dx,\ \int_{-1}^{1} x\,dx,\ \int_{-1}^{1} x^2\,dx,\ \int_{-1}^{1} x^3\,dx,\ \int_{-1}^{1} x^4\,dx \text{ and } \int_{-1}^{1} x^5\,dx$$

2 Predict the result of $\int_{-1}^{1} x^n\,dx$ for n even and for n odd.

3 Use graphs to relate your integration results to the shapes of the functions.

4 Investigate further by changing the limits of integration to −5 and 5.

5 Write a report on your findings, including the integrations, graphs and predictions.

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You can use your CAS calculator to calculate the area under a curve. This is especially useful when the zeros of the function are not easily found or not exact.

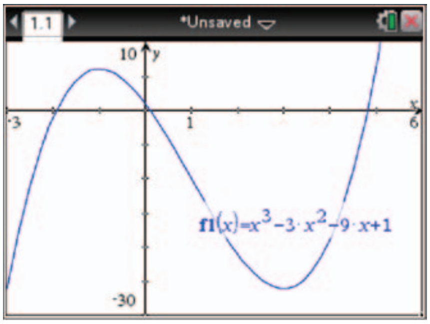
Example 12

CAS Find the physical area that is cut off the curve $f(x) = x^3 - 3x^2 - 9x + 1$ by the x-axis.

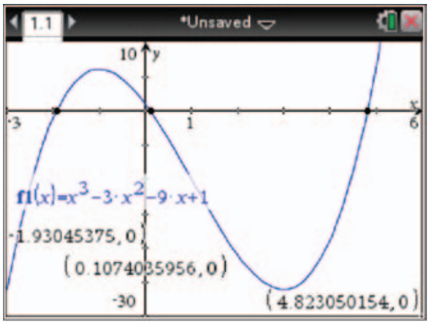
Solution

TI-Nspire CAS

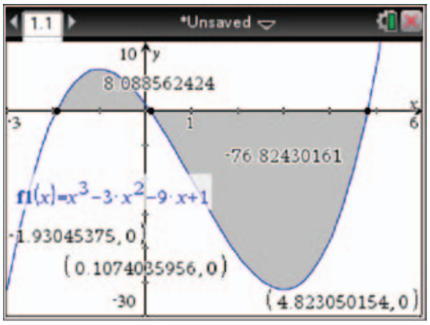
Use a Graph page.
Draw the graph and change the window settings to suitable values.
Use menu, 9: Settings to change the accuracy to, say, 10 digits.



Use menu, 6:Analyse Graph and 1:Zero to find the zeros in succession. The calculator will ask you to position the point to lower and upper bounds within which to find each zero. Grab each zero label and move them out of the way.

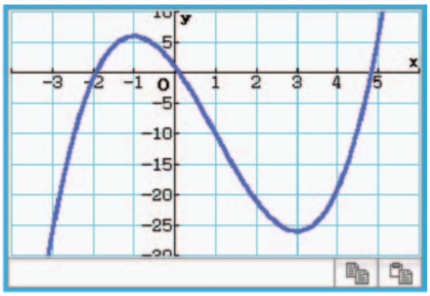


Press menu, 6:Analyse Graph and 7:Integral to find the first area. When asked for the lower bound, position the pointer at the first zero so that it says 'Intersection point' to get the exact point.
Complete with the upper bound and repeat for the second area.



ClassPad

Use the Graph&Table menu. Draw the graph and change the View Window settings to suitable values.



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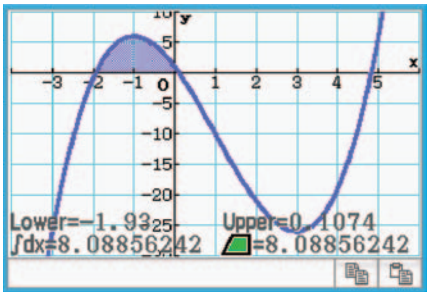
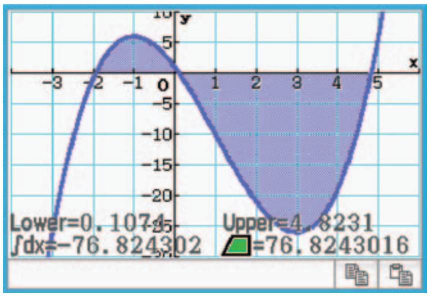
To find the first area, tap Analysis, G-solve, Integral, and ∫dx root. The smallest zero will be found to start the integral. Tap EXE to confirm and use the right arrow to move to the next zero for the upper bound.



Repeat for the second area, but when asked for the lower bound, use the right arrow to go to the second root before confirming.



Add the physical areas.

Physical area ≈ 8.088 562… + 76.824 301…

Write the answer.

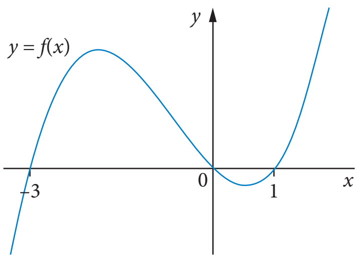
The physical area cut off the curve $f(x) = x^3 - 3x^2 - 9x + 1$ by the x-axis is about 84.91 square units.

EXERCISE 6.04 Physical areas

Calculating physical areas

Concepts and techniques

Use the diagram on the right to answer Questions **1** and **2**.



1 Example 10 The area between the graph of $f(x)$, the x-axis and the lines $x = -2$ and $x = -1$ is equal to:

A $\int_1^2 f(x)dx$

B $\int_{-1}^{-2} f(x)dx$

C $\int_2^1 f(x)dx$

D $\int_{-2}^{-1} f(x)dx$

E $\int_{-3}^1 f(x)dx$

2 The area between the graph of $f(x)$, the x-axis and the lines $x = -3$ and $x = 1$ is equal to:

A $\int_0^{-3} f(x)dx + \int_0^1 f(x)dx$

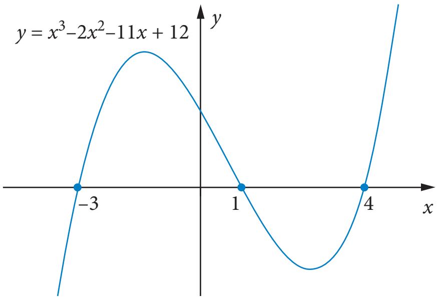
B $\int_{-3}^0 f(x)dx - \int_0^1 f(x)dx$

C $\int_{-3}^1 f(x)dx$

D $\int_{-3}^0 f(x)dx + \int_0^1 f(x)dx$

E $\int_{-3}^0 f(x)dx + \int_0^1 f(x)dx$

Use the diagram on the right to answer Questions **3** to **5**.



3 The area between the graph of $y = x^3 - 2x^2 - 11x + 12$ and the x-axis from $x = -3$ to $x = 1$ is equal to:

A $15\frac{2}{3}$ B $21\frac{1}{3}$ C $23\frac{1}{3}$

D $51\frac{1}{3}$ E $53\frac{1}{3}$

4 The area between the graph of $y = x^3 - 2x^2 - 11x + 12$ and the x-axis from $x = 1$ to $x = 4$ is equal to:

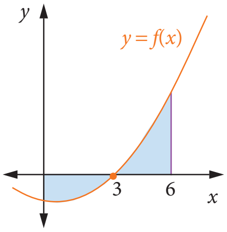
A $15\frac{1}{4}$ B $20\frac{1}{2}$ C $21\frac{1}{4}$ D $24\frac{3}{4}$ E $25\frac{1}{2}$

5 The area between the graph of $y = x^3 - 2x^2 - 11x + 12$ and the x-axis from $x = -3$ to $x = 4$ is equal to:

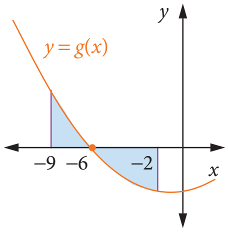
A $-48\frac{5}{12}$ B $-31\frac{1}{2}$ C $28\frac{7}{12}$ D $75\frac{3}{4}$ E $78\frac{1}{12}$

6 For each of the following, state how the shaded physical area can be calculated using integrals.

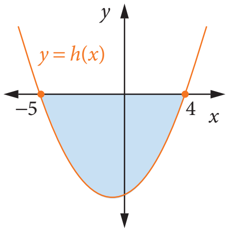
a



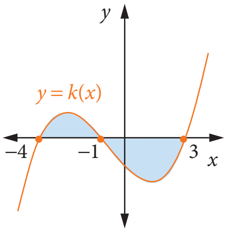
b



c



d



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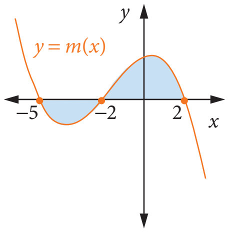

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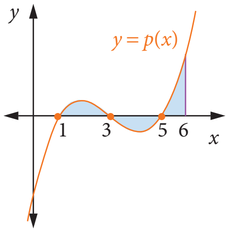

7 Find the area between the x-axis and each of the following curves for the domain given.

a $y = x^2 - 5x + 8$ from $x = 1$ to $x = 4$

b $f(x) = 15 + 8x - 6x^2$ between $x = -1$ and $x = 2$

c $y = 4x^3 - 3x^2 + 6x - 2$ between $x = 1$ and $x = 3$

d $f(x) = 6x^3 + 8x^2 - 2x + 8$ from $x = -1$ to $x = 2$

e $y = 7 - 6x^3$ from $x = -2$ to $x = 1$

8 Find the area enclosed by $x = 2$, $x = -2$, the x-axis and the curve $f(x) = e^x - e^{-x}$.

9 Find the area under the curve $y = (2x+1)^{-\frac{1}{3}}$ from $x = 0$ to $x = 13$.

10 Find the physical area enclosed by:

a $y = 2x^2 + 3x - 35$, $x = 2$, $x = 5$ and $y = 0$

b $y = 7x^2 + 19x - 6$, $x = -5$, $x = -2$ and the x-axis.

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11 Example 11 Find the area enclosed by:

a $f(x) = 3x^2 - 12$ and the x-axis

b $y = 3x^2 + 7x - 6$ and the x-axis

c $f(x) = 4x^2 - 16x + 15$ and the x-axis.

12 Find the physical area cut off:

a $y = (5 - x)(x + 1)(x - 3)$ by the x-axis

b $y = (x + 2)^2(x - 1)(x + 4)$ by the x-axis.

13 Find the area enclosed by $y = \sin(3x)$ and the x-axis between $x = 0$ and $x = 2\pi$.

14 Find the area enclosed by $y = e^{2x} + e^{-2x}$, $y = 0$, $x = -1.5$ and $x = 1.5$.

15 Find the area enclosed between the curve $y = \dfrac{2}{(x-3)^2}$, the x-axis and the lines $x = 0$ and $x = 1$.

16 Find the area enclosed between the curve $y = \sqrt{4 - x^2}$, the x-axis and the y-axis in the first quadrant.

17 Find the exact area enclosed between the curve $y = e^{4x-3}$, the x-axis and the lines $x = 0$ and $x = 1$.

18 Find the area enclosed between the curve $y = x + e^{-x}$, the x-axis and the lines $x = 0$ and $x = 2$ correct to 2 decimal places.

19 Find the exact area bounded by the curve $y = \cos(3x)$, the x-axis and the lines $x = 0$ and $x = \dfrac{\pi}{12}$.

20 Example 12 CAS Find an approximation for the area cut off:

a $y = 6x - 4 - x^2$ by the x-axis

b $y = 5x^2 - x^3 - 2x - 8$ by the x-axis

c $y = x^3 + 3x^2 - 10x - 3$ by the x-axis.

Reasoning and communication

21 Find the physical area enclosed by:

a $y = x^2 - x - 6$, $x = 1$, $x = 5$ and $y = 0$

b $y = 2x^2 + 7x - 30$, $x = -2$, $x = 4$ and the x-axis.

22 Find the area enclosed by each of these curves and the x-axis.

a $y = x^2 - 7x + 10$ b $y = 2x^2 + 13x + 15$

23 Find the physical area cut off:

a $y = (x + 2)(x - 2)(x - 4)$ by the x-axis

b $y = (2x + 7)(x + 1)(3 - x)$ by the x-axis

c $y = (x - 2)(x - 3)^2(x - 4)(x - 6)$ by the x-axis.

24 The marginal profit (the rate of change of profit) of a manufacturer of fencing systems is given by $M(n) = 400(1 - 4e^{-0.015n})$, where n is the number of systems sold and $M(n)$ is in dollars. Use integration to find the profit made when 500 systems are sold.

25 The force required to compress a powerful spring is given by $F = 600\,000x$ newtons, where x is the amount of compression (in metres). Use the integral $\int F dx$ to find the energy stored in the spring when it is compressed by 5 cm. Note that if the compression is in metres and the force is in newtons (N), the energy obtained is in joules (J).

6.05 AREAS BETWEEN CURVES

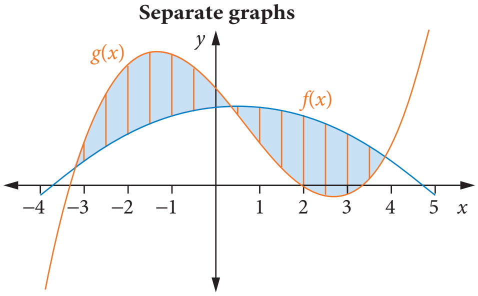
The area between a curve and the x-axis is sometimes called the *area under the curve*.

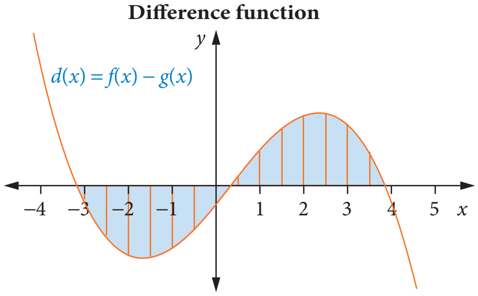
Sometimes, however, the area *between two curves* or enclosed by two curves is required. For an enclosed area, it may be necessary to find the intersection points of the curves. The area can be calculated in two ways:

- as the difference between the areas under the two functions; or
- as the area under the **difference function**, because
  $$\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$$

The intersections of the functions correspond to the zeros of the difference function.

The diagram below shows this geometrically using area strips.





In problems of this type it is a good idea to sketch the functions or plot them on your CAS calculator to make sure that you can see the area you are calculating. You need only do enough to find the information you need, which would not usually include stationary points.

Example 13

Find the area cut off $f(x) = x^2 - 6x + 13$ by $x - y + 3 = 0$.

Solution

$f(x) = x^2 - 6x + 13$ is a quadratic function with a minimum. The zeros are not obvious, so create a table to find some points.

$f(x) = x^2 - 6x + 13$

x	−1	1	3	5	7
$f(x)$	20	8	4	8	20

Find the x-intercept of $x - y + 3 = 0$.

At $y = 0$, $x = -3$ so the x-intercept is $(-3, 0)$.

Find the y-intercept of $x - y + 3 = 0$.

At $x = 0$, $y = 3$ so the y-intercept is $(0, 3)$.

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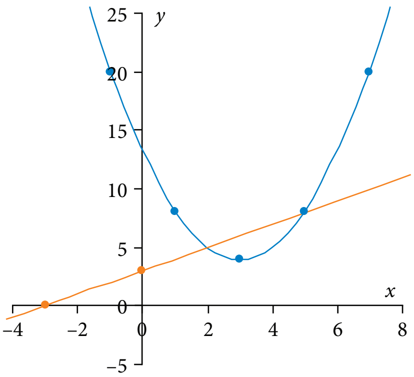
Sketch the functions.



State the result.

The lines apparently intersect.

The lines look as if they intersect at $x = 2$ and $x = 5$. Test to check.

$f(2) = 5$ and $2 - 5 = 3 = 0$ so one intersection is at (2, 5). The other is at (5, 8).

Find the area by subtracting the integrals of the lower from the higher.

$$A = \int_2^5 (x+3)\,dx - \int_2^5 (x^2 - 6x + 13)\,dx$$

Integrate.

$$= \left[\frac{x^2}{2} + 3x\right]_2^5 - \left[\frac{x^3}{3} - 3x^2 + 13x\right]_2^5$$

Evaluate.

$$= 19\frac{1}{2} - 15$$

$$= 4\frac{1}{2}$$

Write the answer.

The area cut off $f(x) = x^2 - 6x + 13$ by $x - y + 3 = 0$ is $4\frac{1}{2}$ square units.

While you could establish the intersections in Example **13** by checking your observations, this is not usually enough. It is usually quicker to use the difference function than to find intersections anyway.

Example 14

Find the area between the curves $f(x) = 12 - x^2$ and $g(x) = x^2 + 1$ and the lines $x = -1$ and $x = 2$.

Solution

Find the difference function.

$$d(x) = x^2 + 1 - (12 - x^2)$$

Simplify.

$$= 2x^2 - 11$$

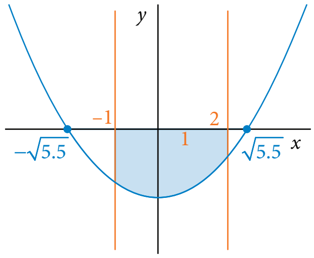
Find the zeros.

$$2x^2 - 11 = 0$$

Write the zeros.

$$x = \pm\sqrt{5.5} \approx \pm 2.35$$

Sketch the graph and lines.
Shade the area needed.



Write the necessary integral. $\int_{-1}^{2}(2x^2 - 11)dx$

Integrate. $= \left[\frac{2x^3}{3} - 11x\right]_{-1}^{2}$

Evaluate. $= -16\frac{2}{3} - 10\frac{1}{3} = -27$

Write the answer in the form of the question. The area between $f(x) = 12 - x^2$, $g(x) = x^2 + 1$, $x = -1$ and $x = 2$ is 27 square units.

A CAS calculator is useful when the functions are more complex.

Example 15

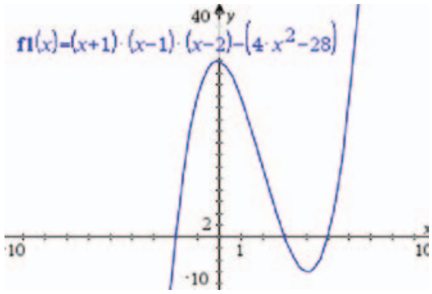
CAS Find the area between the curves $f(x) = (x + 1)(x - 1)(x - 2)$ and $g(x) = 4x^2 - 28$.

Solution

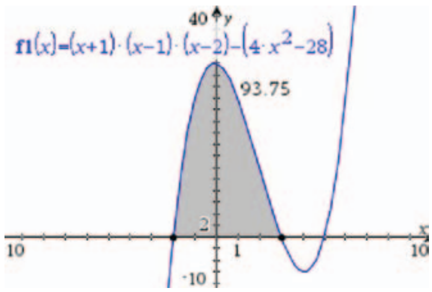
Find the difference function. $d(x) = (x + 1)(x - 1)(x - 2) - (4x^2 - 28)$

TI-Nspire CAS

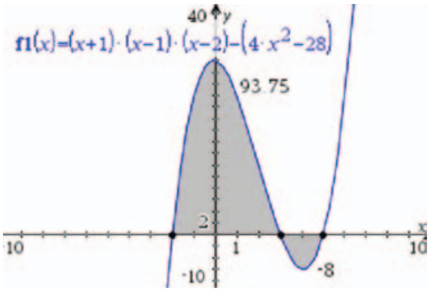
Use a Graph page.
Draw the graph of the difference function and adjust the scales. You don't need to simplify the function.



To find the first area, use menu, 6: Analyse Graph and 7: Integral. For the lower bound, move the pointer to the first intersection and press enter when the words 'intersection point' are displayed. Do the same for the upper bound.

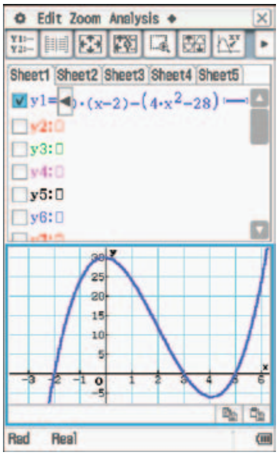


Use the same procedure for the second area.

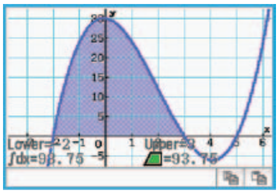


ClassPad

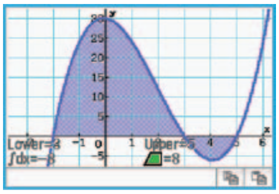
Use the Graph&Table menu.
Draw the graph of the difference function and adjust the scales using View Window.
You don't need to simplify the function.



Tap Analysis, G-Solve, Integral and ∫dx Root. Choose lower and upper bounds and find the first area.



Use the same procedure for the second area.



Calculate the total physical area.

$$\text{Area} = 93.75 + 8$$
$$= 101.75$$

Write the answer.

The area between the curves $f(x) = (x + 1)(x - 1)(x - 2)$ and $g(x) = 4x^2 - 28$ is $101\frac{3}{4}$ square units.

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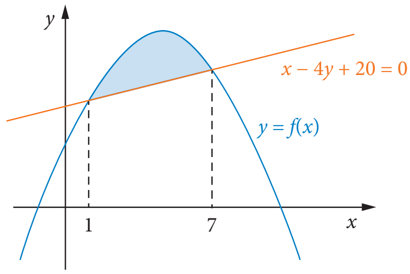
EXERCISE 6.05 Areas between curves

Concepts and techniques

WS Calculating areas between curves

1 Example 13 What expression is used to find the shaded area on the right?



A $\int_1^7 f(x)dx + \int_1^7 (4x-5)dx$

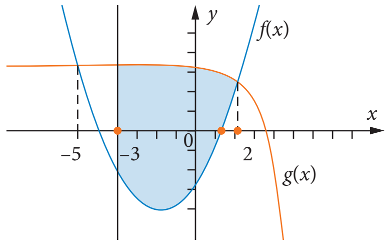
B $\int_1^7 f(x)dx - \int_1^7 (4x-5)dx$

C $\int_1^7 f(x)dx - \int_1^7 (\frac{1}{4}x+5)dx$

D $\int_1^7 f(x)dx - \int_1^7 (\frac{1}{4}x-5)dx$

E $\int_1^7 f(x)dx - \int_1^7 (x+20)dx$

2 Example 14 The area bounded by the curves $f(x)$, $g(x)$ and the line $x = -3$ is equal to:



A $\int_{-3}^2 g(x)dx + \int_{-3}^2 f(x)dx$

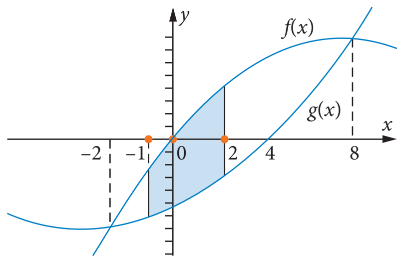
B $\int_{-3}^2 [g(x) - f(x)]dx$

C $\int_{-3}^2 [g(x) + f(x)]dx$

D $\int_{-5}^2 [g(x) - f(x)]dx$

E $\int_{-5}^2 f(x)dx - \int_{-5}^2 g(x)dx$

3 Which one of the following is equal to the shaded area shown?



A $\int_{-2}^8 f(x)dx - \int_{-2}^8 g(x)dx$

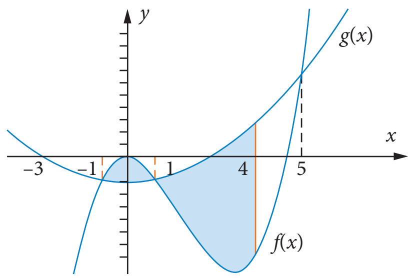
B $\int_{-1}^2 [f(x) + g(x)]dx$

C $\int_{-1}^2 f(x)dx + \int_{-1}^2 g(x)dx$

D $\int_{-1}^2 f(x)dx - \int_{-1}^2 g(x)dx$

E $\int_{-2}^8 f(x)dx + \int_{-2}^8 g(x)dx$

4 The shaded area shown here is equal to:



A $\int_{-1}^1 [f(x) - g(x)]dx + \int_1^4 [g(x) - f(x)]dx$

B $\int_{-1}^4 [g(x) - f(x)]dx$

C $\int_{-1}^1 [f(x) - g(x)]dx + \int_1^4 [f(x) - g(x)]dx$

D $\int_{-1}^5 [g(x) - f(x)]dx$

E $\int_{-1}^1 [f(x) - g(x)]dx - \int_1^4 [g(x) - f(x)]dx$

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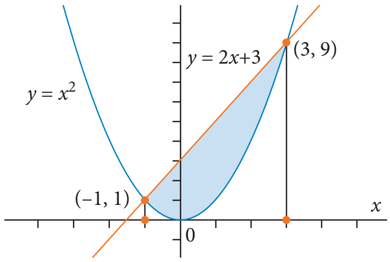
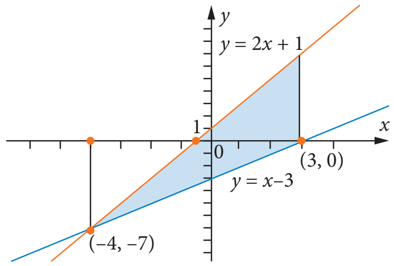
5 Find the area enclosed between the curve $y = x^2$ and the line $y = x + 6$.

6 Example 13 Calculate each of the shaded areas shown below.

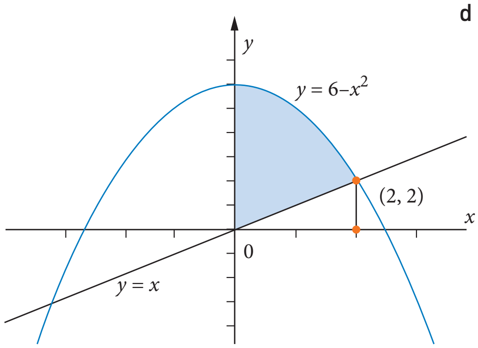
a



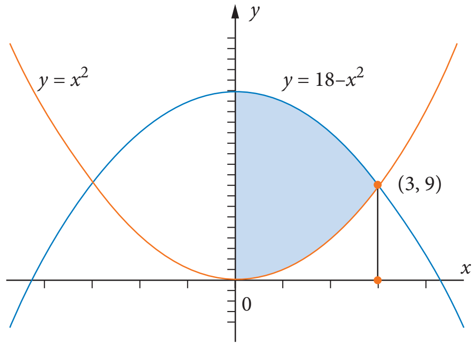
b



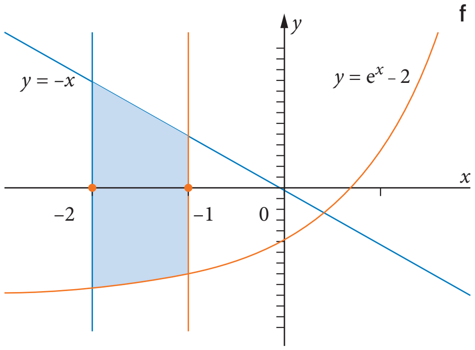
c



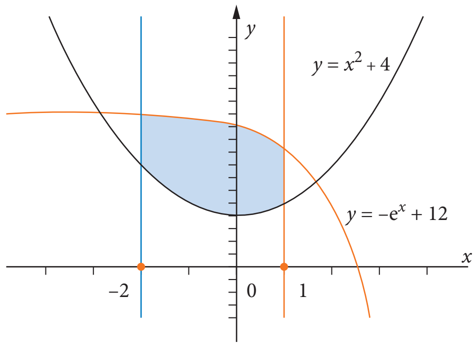
d



e



f



7 Find the area enclosed between the line $y = 2$ and the curve $y = x^2 + 1$.

8 Find the area bounded by the curve $y = 9 - x^2$ and the line $y = 5$.

9 Find the area enclosed between the curve $y = x^2$ and the line $y = -6x + 16$.

10 Find the area cut off:

a $y = 2x^2 - 12x + 20$ by $2x + y = 12$

b $f(x) = 2x + 8 - x^2$ by $y = 2x - 1$

c $f(x) = 4x^2 + 24x + 26$ by $y = 4x + 26$

d $f(x) = 4x^2 + 4x + 5$ by $y = 5 - 2x$

e $y = 4x^2 + 12x + 8$ by $y = 4x + 13$.

11 Find the area enclosed by the curves $y = x^2$ and $y = x^3$.

12 Example 14 Find the area bounded by the curves $y = 1 - x^2$ and $y = x^2 - 1$.

13 Example 15 CAS Find the area enclosed by the curves $y = (x - 1)^2$ and $y = (x + 1)^2$ and the x-axis.

14 CAS Find the area enclosed by the curve $y = x^3$, the x-axis and the line $y = -3x + 4$.

15 CAS Find the area enclosed by the curves $y = (x - 2)^2$ and $y = (x - 4)^2$, and the x-axis.

16 Find the exact area enclosed by the curve $y = \sqrt{4 - x^2}$ and the line $x - y + 2 = 0$.

17 CAS Find the areas between the following curves and lines.

a $f(x) = 3 - x^3$, $g(x) = x^2 - 9$, $x = -3$ and $x = 1$

b $f(x) = 2x^2 - 8$, $g(x) = x^2 - 2x + 6$, $x = -4$ and $x = 2$

c $f(x) = (x + 2)(x - 2)^2$ and $y = x + 2$

d $y = (x + 1)(x - 1)(x - 2)$ and $y = 2 + 5x - x^2$

e $y = 3x^2 + 2x - 21$ and $y = (x + 1)(x + 3)(x - 3)$

Reasoning and communication

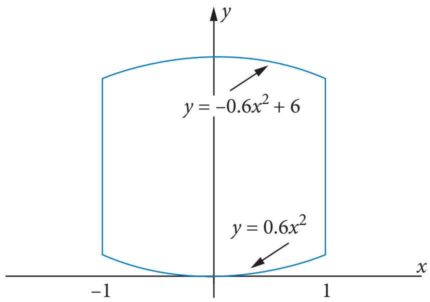
18 Find the area enclosed by the curves $y = x^2$ and $x = y^2$.

19 A cam shaft has a cross-section that is a good approximation to the area between the curves

$$y = 0.75x^2 - 3.75x + 6 \text{ and } y = 5x - 1 - x^2$$

where x is in centimetres. The shaft is 32 cm long. The shaft is made of high-tensile steel with a density of 7920 kg/m^3. Find the mass of the shaft.

20 The diagram shown on the right represents a piece of glass that has been cut from a rectangular piece of glass measuring 6 m by 2 m. All dimensions are shown in metres. What is the area of glass that was discarded from the rectangular sheet?



21 The bed of a river can be approximated by the part of the function

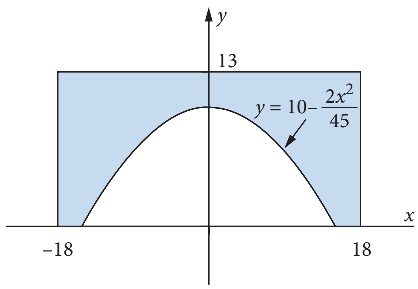
$$d = -0.007x(0.45x - 11)^2$$

cut off by the x-axis, where x is the distance in metres from one bank at water level. During normal conditions, the speed of the water is 30 cm/s. Find the amount of water in litres that flows past a point on the riverbank in half an hour.

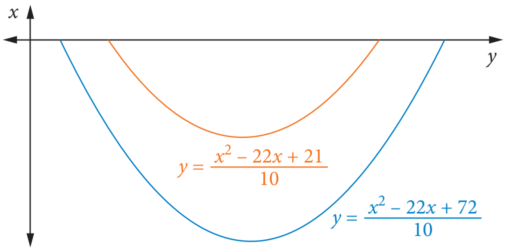
22 The diagram on the right represents a cross-section of a concrete bridge. All dimensions are shown in metres.

a What is the area of the cross-section?

b If the bridge is 25 m wide, what is the volume of concrete it contains?



23 A sweeping 'circular' driveway actually has two parabolas as its edges to allow room for parking near the house. The x-axis is the edge of the roadway, the driveway lies between the curves $y = \dfrac{x^2 - 22x + 21}{10}$ and $y = \dfrac{x^2 - 22x + 72}{10}$ (in metres)



Find the area of this driveway and hence the cost of concreting to a depth of 15 cm, with concrete costing $350/m^3.

6.06 TOTAL CHANGE

You have previously seen that the fundamental theorem of calculus states that:

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is a primitive function of $f(x)$.

This can be restated a slightly different way using the fact that $f(x)$ is the derivative of $F(x)$.

Using, $f(x) = F'(x)$, we can state the fundamental theorem of calculus as:

$$\int_a^b F'(x)dx = F(b) - F(a)$$

If we consider $F'(x)$ as the rate of change of a quantity, then $\int_a^b F'(x)dx$ is the total or net change in the quantity as x changes from a to b.

This can be stated as the total (or net) change theorem.

IMPORTANT

Total change theorem

The definite integral of the rate of change of a quantity, $F'(x)$, gives the total change in that quantity.

$\int_a^b F'(x)dx = F(b) - F(a)$

Example 16

The rate of change of temperature in degrees Celsius per minute for a cup of coffee is given by:

$$F'(t) = 10e^{-0.25t}$$

where t is in minutes.

What is the total change in the coffee temperature between $t = 1$ and $t = 8$?

Solution

Write the integral of the rate of change function.	$\int_1^8 F'(t)dt = \int_1^8 10e^{-0.25t}(t)dt$
Integrate.	$= \left[-40e^{-0.25t}\right]_1^8$
Evaluate.	$= -40e^{-2} - (-40e^{-0.25})$
Factorise.	$= 40(e^{-0.25} - e^{-2})$
Evaluate.	$= 40(0.7788\ldots - 0.1353\ldots)$ ≈ 25.7
State the result.	The total temperature change of the coffee between $t = 1$ and $t = 8$ is about 25.7°C.

Example 17

Following the Second World War, there was a significant increase in the birth rates among the western countries. If it is assumed that the rate of births in millions of babies per year for the post war years is approximated by:

$$B'(t) = 2t + 5 \text{ for } 0 \le t \le 15$$

a how many babies were born in the first 15 years after the war?

b how long did it take for the number of babies born after the war to reach 104 million?

Solution

a Write the integral of the birth rate function, $B'(t)$.	$\int_0^{15} B'(t)dt = \int_0^{15} (2t+5)dt$
Integrate.	$= \left[t^2 + 5t\right]_0^{15}$
Evaluate.	$= 300$
State the result.	300 million babies were born in the first 15 years after the war.

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b We want to know the value of t when the total number of babies born was 104 million.

Let T be the number of years after the war for 104 million babies to be born.

Write the integral. $\int_0^T (2t+5)\,dt = 104$

Integrate. $\left[t^2+5t\right]_0^T = 104$

Evaluate. $T^2 + 5T = 104$

Rearrange. $T^2 + 5T - 104 = 0$

Factorise. $(T-8)(T+13) = 0$

Solve. $T = 8$ or $T = -13$.

Disregard any inapplicable solutions. $T = -13$ is a false solution.

State the answer. It took 8 years for 104 million babies to be born after the war.

In business and economics, **marginal** cost, profit, revenue, etc. is defined as the cost, profit or revenue for the last item or unit. It is approximated by the rate of change, so the derivative is taken to be the marginal quantity. This means that the total cost, profit, revenue, etc. is the integral of the marginal amount or marginal rate of change.

Example 18

A factory manufactures precision high-pressure valves. The marginal cost, $C'(x)$, and marginal revenue, $R'(x)$ are given below.

$$C'(x) = \frac{6}{5}x + 5$$

$$R'(x) = 6x - \frac{3}{5}x^2 + 62$$

where the marginal cost and marginal revenue are both measured in dollars.

Calculate the total profit from the production and sale of the first 20 valves.

Solution

Find the total cost for the production of the first 20 valves.

$$\int_0^{20} C'(x)\,dx = \int_0^{20}\left(\frac{6}{5}x+5\right)dx$$

Integrate.

$$= \left[\frac{6x^2}{5\times 2}+5x\right]_0^{20}$$

$$= \left[\frac{3}{5}x^2+5x\right]_0^{20}$$

Evaluate.

$$= \frac{3\times 20^2}{5}+100-0$$

$$= 340$$

Find the total revenue for the sale of the first 20 valves.	$\int_0^{20} R'(x)dx = \int_0^{20}\left(6x - \frac{3}{5}x^2 + 62\right)dx$
Integrate.	$= \left[\frac{6x^2}{2} - \frac{3x^3}{5\times 3} + 62x\right]_0^{20}$
	$= \left[3x^2 - \frac{1}{5}x^3 + 62x\right]_0^{20}$
Evaluate at the limits.	$= 3\times 20^2 - \frac{20^3}{5} + 62\times 20 - 0$
	$= 840$
Use the rule for total profit.	$P(x) = R(x) - C(x)$
Evaluate.	$= 840 - 340$ $= 500$
State the answer.	The total profit from the production and sale of 20 valves is \$500.

In the previous example you could have integrated the difference function $P'(x) = R'(x) - C'(x)$ to calculate the total profit instead of calculating the total revenue and cost.

EXERCISE 6.06 Total change

WS Total change

Concepts and techniques

1 Example 16 Oil is leaking from a tank. The rate of leakage (in litres/hour) is given by:

$$\text{Rate of leakage} = 1000e^{-0.1t}$$

where t is in hours.
The total change in the volume of oil in the tank (in L) after the first 5 hours is:

A $1000e^{-0.5}$ B $\int_0^5 1000e^{-0.1t}\,dt$ C 606
D $\int_1^5 1000e^{-0.1t}\,dt$ E $1000e^{-0.5} - 1000e^0$

2 Example 17 The rate of increase in the population of lizards in a closed environment is given by:

$$P'(t) = 6 + \sqrt{10t}$$

where t is in days. The total change in the population of lizards after the first 10 days is:

A 16 B $\left[6+\sqrt{10t}\right]_0^{10}$ C $\int\left(6+\sqrt{10t}\right)dt$
D $\left[6t + \frac{2\sqrt{10}\,t^{\frac{3}{2}}}{3}\right]_0^{10}$ E $\left[6t + (10t)^{\frac{3}{2}}\right]_0^{10}$

3 If $H'(t)$ is the rate of change in the height of a conical pile of fertiliser measured in centimetres per hour, what does $\int_0^{10} H'(t)dt$ represent?

4 If $B'(t)$ represents the growth rate of the number of bacteria in a culture, where t is the time in hours, what does $\int_2^8 B'(t)dt$ represent?

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5 $P'(t)$ is the rate of growth of a population of kangaroos in an isolated area, measured in kangaroos per year. There were 150 kangaroos in the population in the year 2013 ($t = 0$). Write an integral expression that represents the population in 2017.

Shutterstock.com/Jacqui Martin

6 Fluid leaks from a storage facility at the rate:

$$F'(t) = 3500e^{-0.4t} \text{ litres/day}$$

where t is measured in days.

a How much fluid will leak in the first 5 days ($0 \leq t \leq 5$)?

b How much fluid will leak in the next 5 days ($5 \leq t \leq 10$)?

c Why does less fluid leak from the facility in the second five-day period?

7 The population of rabbits in a certain confined location is known to increase according to the function:

$$P'(t) = 4t + 1$$

where $P'(t)$ is measured in hundreds of rabbits per month and t is measured in months. The measurement of the population commences at $t = 0$.

a What is the total change in the population in the first 10 months after measuring commenced?

b How long did it take for the increase in the population of rabbits to reach 2100?

8 Example 18 The marginal cost function (in thousands of dollars) for manufacturing x components per year is given by:

$$C'(x) = 25 - \frac{1}{2}x$$

Calculate the total cost for the production of the first 50 components.

9 The marginal revenue function (in thousands of dollars) for the sale of x hundred units of a product is given by:

$$R'(x) = 12 - 3x^2 + 4x$$

Find the total revenue function and calculate the total revenue from the sale of the first 400 units.

Reasoning and communication

10 Water flows into a tank at a rate given by:

$$W'(t) = \frac{1}{75}(20t - t^2 + 600)$$

where $W'(t)$ is measured in L/hour and t is in hours. Initially there are 200 L of water in the tank. How many litres of water are in the tank after 24 hours?

11 A company manufactures toy cars. It is known that the marginal cost and revenue functions can be approximated as follows.

$$R'(x) = 10 - 0.002x$$
$$C'(x) = 2$$

There are fixed costs of \$7000 regardless of the number of toy cars produced.
Determine the total cost and total revenue functions and calculate the total profit for the first 1000 toy cars produced.

12 A manufacturer produces components. The marginal cost (\$) of producing x units is:

$$C'(x) = 5 + 16x - 3x^2$$

If the total cost of producing 5 units is \$500, find the total cost function.

13 The marginal cost, $C'(x)$ measured in dollars, of a certain product is known to be a constant multiple of the number of units (x) produced. The manufacturing process involves a fixed cost of \$5000 regardless of the number of units produced. If the cost of producing 24 units is \$5144, find the total cost function.

6.07 APPLICATION OF INTEGRATION TO MOTION

You have already studied the relationships between position, velocity and acceleration in differential calculus. You should remember that if x, v and a are the position, velocity and acceleration of an object, then $v = \frac{dx}{dt}$ and $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$. You can reverse these to obtain the following.

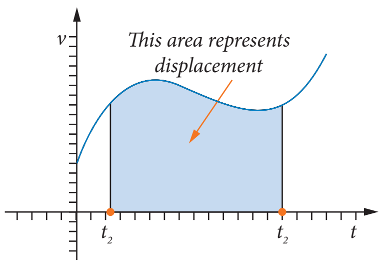
IMPORTANT

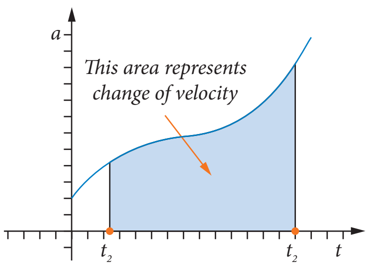
If x, v and a are the position, velocity and acceleration of an object then

$$x = \int v\,dt \qquad \text{and} \qquad v = \int a\,dt.$$

The displacement (change of position) between times T_1 and T_2 is given by $\int_{T_1}^{T_2} v\,dt$ and the change of velocity is given by $\int_{T_1}^{T_2} a\,dt$.

Using the fact that the definite integral is an area under a curve, you can interpret displacement and velocity using graphs as shown below.





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Example 19

The velocity of a vehicle increases from rest to 80 km/h in 30 seconds under constant acceleration. Find the displacement in metres of the vehicle during this time.

Solution

Express the velocity in m/s.

$$80\text{ km/h} = \frac{80\times1000}{60\times60}\text{ m/s}$$
$$= 22\tfrac{2}{9}\text{ m/s}$$

Calculate the acceleration.

$$a = \frac{22\frac{2}{9}}{30}\text{ m/s}^2$$
$$= \frac{22}{27}\text{ m/s}^2$$

Find the velocity at time t.

$$v = \int_0^t \frac{22}{27}\,dt$$
$$= \frac{22}{27}t$$

Integrate to find the displacement.

$$\Delta x = \int_0^{30} \frac{22}{27}t\,dt$$

Integrate.

$$= \left[\frac{11}{27}t^2\right]_0^{30}$$

Evaluate.

$$= \frac{1100}{3}$$
$$= 333.333\ldots\text{ m}$$

State the result.

The displacement is about 333 m.

Many situations involve motion subject to gravity. In these cases, acceleration due to gravity must be taken into account. For practical purposes, acceleration due to gravity (g) is taken to be 9.8 m/s^2 acting towards the centre of the Earth. This means that if you consider upwards as positive, then, $g = -9.8$ m/s^2, but if downwards is positive, then $g = 9.8$ m/s^2.

Getty Images

Example 20

An object is propelled vertically upwards from 2 m above the ground at 25 m/s.

a Find the velocity after 2 seconds.

b When does the object first come to rest?

c Find the position of the object after 3 s.

Solution

a Find the change of velocity, with up as the positive direction.

$$\text{Change of velocity} = \int_0^2 -9.8\,dt = -19.6$$

Find the velocity.

$$v = 25 + -19.6 = 5.4 \text{ m/s}$$

State the result.

After 2 seconds, the object has a velocity of 5.4 m/s upwards.

b Write an expression for v at time t.

$$v(t) = 25 + \int_0^t -9.8\,dt = 25 - 9.8t$$

Find when the velocity is 0.

$$25 - 9.8t = 0$$

Solve for t.

$$t = \frac{25}{9.8} \approx 2.55 \text{ s}$$

State the answer.

The object comes to rest after about 2.55 s.

c Find the displacement.

$$\text{Displacement} = \int_0^3 (25 - 9.8t)\,dt$$

Find the integral.

$$= \left[25t - 4.9t^2\right]_0^3$$

Evaluate.

$$= 30.9 \text{ m}$$

Find the position.

$$\text{Position} = 2 + 30.9 = 32.9$$

State the answer.

After 3 seconds the object is 32.9 m above the ground.

When air resistance is taken into account, the acceleration of objects falling under gravity is not constant. In fact, they eventually reach a terminal velocity where the forces of gravity and air friction are balanced. For a human body, the terminal velocity is about 200 km/h (56 m/s). Parachutes are designed to reduce the terminal velocity to about 8 m/s. The equations for velocity and displacement under these conditions are exponential functions.

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Example 21

The velocity of a particle moving in a straight line is given by:

$$\frac{dx}{dt} = 20 - 8e^{-0.4t}$$

Calculate the total distance travelled by the particle in the first 3 seconds.

Solution

State the equation for velocity. $\frac{dx}{dt} = 20 - 8e^{-0.4}$

Integrate to find displacement. $x = \int_0^3 (20 - 8e^{-0.4t})\,dt$

Find the integral. $= \left[20t - \frac{8e^{-0.4t}}{-0.4}\right]_0^3$

Simplify. $= \left[20t + 20e^{-0.4t}\right]_0^3$

Evaluate. $= 60 + 20e^{-1.2}$

≈ 66

State the answer. The displacement of the particle after 3 s is about 66 m.

EXERCISE 6.07 Application of integration to motion

Displacement, velocity and acceleration

Concepts and techniques

1 Example 19 If $v(t)$ is the velocity of a particle moving along the x-axis, measured in metres per second, what does $\int_2^{10} v(t)\,dt$ represent?

2 An object is initially at rest. It is then subject to a constant acceleration of 6 m/s^2. Calculate its displacement after 4.1 s.

3 The velocity (cm/s) of a particle moving horizontally to the right of the origin is given by $v = 3t^2 + 2t + 1$. If the particle is initially 2 cm to the left of the origin, find the displacement of the particle after 5 s and its position relative to its initial position.

4 The acceleration of a particle is given by $a = -9 \sin(3t)$ cm/s^2. If the initial velocity is 5 cm/s and the particle is 3 cm to the left of the origin, find its exact position after π s.

5 The velocity of an object is given by $v = 4\cos(2t)$ m/s. If the object is 3 m to the right of the origin after π s, find the exact:

a position after $\frac{\pi}{6}$ s

b acceleration after $\frac{\pi}{6}$ s

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6 Example 20 An object is propelled vertically upwards with an initial velocity of 20 m/s from the top of a cliff that is 300 m high. Consider the height relative to the bottom of the cliff.

a Find its height after 5 seconds.

b What is the velocity of the object after 5 seconds?

c When does the object reach its greatest height?

7 A ball is thrown vertically upwards from the ground at 14 m/s.

a How high does it go?

b What is the time taken for the ball to hit the ground?

c What is the velocity of the ball when it hits the ground?

8 A stone is dropped from a bridge over a river. The splash from the stone hitting the river is heard 2.5 seconds after it is dropped. How high is the bridge, correct to two decimal places?

9 A rocket is launched with an initial velocity of 2.1×10^3 m/s upwards. How high does it rise before returning to the ground?

10 Example 21 A particle accelerates according to the equation $a = -e^{2t}$ (cm/s^2). If the particle is initially at rest, find its displacement after 4 seconds, correct to the nearest centimetre.

11 The acceleration (m/s^2) of an object is given by $a = e^{3t}$. The particle is initially at the origin with velocity −2 m/s. Find its displacement after 3 seconds, correct to 3 significant figures.

12 The acceleration of a particle is given by $a = 25e^{5t}$ m/s^2 and its velocity is 5 m/s initially. If its initial position is 1 m to the right of the origin:

a find its velocity after 9 seconds

b find its position after 6 seconds.

Reasoning and communication

13 A manilla folder falls from an aircraft and has a velocity given by

$$v(t) = \frac{g}{2}(1 - e^{-2t})$$

How far does it fall in the first 100 s?

14 A person can jump 2 m high on Earth.

a What is the initial velocity of the jumper?

b How high could the same person jump on the moon if the acceleration due to gravity on the moon is 1.6m/s^2, assuming they started with the same velocity?

15 The velocity of an object is given by $v = t^2(t^3 + 1)$ cm/s and the object is initially 2 cm to the right of the origin.

a Find its acceleration after 1 second.

b Find the position of the object after 2 seconds.

16 A weight at the end of a spring accelerates according to the equation:

$$a = \cos^2\left(t + \frac{\pi}{4}\right) - \sin^2\left(t + \frac{\pi}{4}\right)$$

Initially the weight is at rest at the origin.

a Find the velocity of the weight after $\frac{\pi}{2}$ s.

b Find its displacement after $\frac{\pi}{4}$ s.

c Find the times at which the weight has a velocity of $-\frac{1}{2}$ cm/s.

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CHAPTER SUMMARY
APPLICATIONS OF INTEGRATION

6

- The **indefinite integral** of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$ and is written as $\int f(x)dx$. It is usually written as $\int f(x)dx = F(x) + c$, where c is an arbitrary constant called the **constant of integration**, because indefinite integrals differ only by a constant.
- An indefinite integral is also called an **antiderivative** or a **primitive**.
- The indefinite integrals of basic functions are:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$$
$$\int kx^n\,dx = \frac{kx^{n+1}}{n+1} + c \quad (n \neq -1)$$
$$\int \sin(x)dx = -\cos(x) + c$$
$$\int \sin(ax+b)\,dx = -\frac{\cos(ax+b)}{a} + c$$
$$\int \cos(x)\,dx = \sin(x) + c \qquad \int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a} + c$$
$$\int e^x\,dx = e^x + c \qquad \int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + c$$
$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \quad (n \neq -1)$$
$$\int kf'(x)dx = kf(x) + c$$

- The **linearity of integration** refers to the properties below.

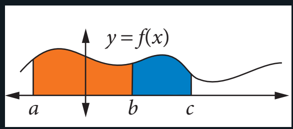
Linear combination: $\int [af(x)+bg(x)]dx = a\int f(x)dx + b\int g(x)dx$

Constant multiple: $\int kf(x)dx = k\int f(x)dx$

Sum of functions: $\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$

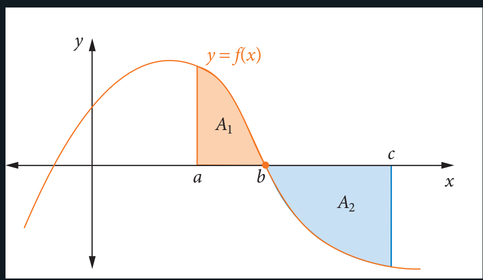
Difference of functions: $\int [f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$

- The substitution of different values for the constant of integration leads to the formation of a **family of functions** that are vertical translations of each other.
- The properties of definite integrals include:



$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$
$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$
$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$
$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$
$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

- The **signed** or **algebraic area** between a curve and the x-axis is given by the integral. It is the difference between the areas above and below the x-axis.
- The **physical area** between a curve and the x-axis is always positive. Areas above and below the x-axis must be calculated separately and negative areas are subtracted.



- The area can be calculated in two ways:
  - as the difference between the areas under the two functions:
    $$\int_a^b f(x)dx - \int_a^b g(x)dx$$
  - as the area under the **difference function**:
    $$\int_a^b [f(x)-g(x)]dx$$
- The **total change theorem** states that the definite integral of the rate of change of a quantity, $F'(x)$, gives the total change in that quantity.
  $$\int_a^b F'(x)dx = F(b)-F(a)$$
- The **marginal** value of a quantity such as cost, revenue or profit is the cost, profit or revenue for the last item or unit. It is normally taken as being the rate of change of the quantity.
- If x, v and a are the position, velocity and acceleration of an object, then $x = \int v\,dt$ and $v = \int a\,dt$.
- The displacement (change of position) between times T_1 and T_2 is given by $\int_{T_1}^{T_2} v\,dt$ and the change of velocity is $\int_{T_1}^{T_2} a\,dt$
  $\int_{T_1}^{T_2} v(t)dt = [x(t)]_{T_1}^{T_2} = x(T_2) - x(T_1)$

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CHAPTER REVIEW 6
APPLICATIONS OF INTEGRATION

Multiple choice

1 Example 1 The function $f(x)$ such that $f'(x) = 15x^2 - 3$ and $f(x) = 6$ when $x = 1$ is:

A $5x^3 - 3x$ B $5x^3 - 3x - 1$ C $30x$

D $\frac{15x^3}{2} - 3x^2 + 6$ E $5x^3 - 3x + 4$

2 Example 2 $\int 12x\sqrt{x}\,dx =$

A $6x\sqrt{x} + c$ B $4x\sqrt{x} + c$ C $2.4x\sqrt{x} + c$

D $8x^2\sqrt{x} + c$ E $4.8x^2\sqrt{x} + c$

3 Example 4 $\int (3x^3 - 5x + 2)dx$ is equal to:

A $\int 3x^3\,dx - \int 5x\,dx + \int 2\,dx$ B $3x^3 - 5x + \int 2\,dx$ C $\int (3x^3 - 5x)dx + 2$

D $\int 3x^3 dx - 5x + 2$ E $3x^3 - \int (5x + 2)dx$

4 Example 5 Given that $\frac{dy}{dx} = \frac{mx^3}{2} + 3x$ and $y = 4$ when $x = 0$, an expression for y is:

A $\frac{mx^4}{4} + 3x^2 + 1$ B $2mx^4 + 3x^2 + 1$ C $\frac{mx^4}{8} + \frac{3x^2}{2} + 4$

D $2mx^4 + \frac{3x^2}{2} - x - 2$ E $ax^3 - ax + 2a$

5 Example 6 $\int x^3(4x + 3)dx$ is equal to:

A $\int (4x^4 + 3x^3)dx$ B $\int x^3\,dx \int (4x + 3)dx$ C $(4x + 3)\int x^3\,dx$

D $x^3 \int (4x + 3)dx$ E $\int (x^3 + 4x + 3)dx$

6 Example 9 $\int_0^{\frac{\pi}{6}} \cos(3x)dx =$

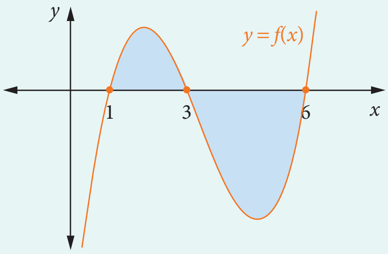
A -1 B $-\frac{1}{3}$ C 0 D $\frac{1}{3}$ E 1

7 Example 10 The algebraic area (signed area) between the graph of $f(x)$, the x-axis and the lines $x = -3$ and $x = 2$ is equal to:

A $\int_2^{-3} f(x)dx$ B $\int_{-2}^{-3} f(x)dx$ C $\int_{-3}^{0} f(x)dx - \int_0^2 f(x)dx$

D $\int_{-3}^{2} f(x)dx$ E $-\int_{-3}^{0} f(x)dx + \int_0^2 f(x)dx$

8 Example 11 The shaded area shown here is equal to:



A $\int_1^6 f(x)dx$

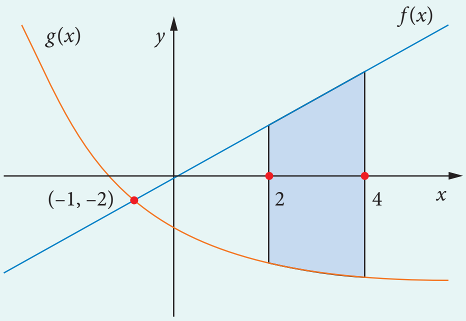
B $-\int_1^6 f(x)dx$

C $\int_1^3 f(x)dx - \int_3^6 f(x)dx$

D $\int_1^3 f(x)dx + \int_3^6 f(x)dx$

E $\int_1^6 f(x)dx - \int_3^6 f(x)dx$

9 Example 13 Which one of the following is equal to the shaded area shown?



A $\int_{-1}^4 f(x)dx - \int_{-1}^4 g(x)dx$

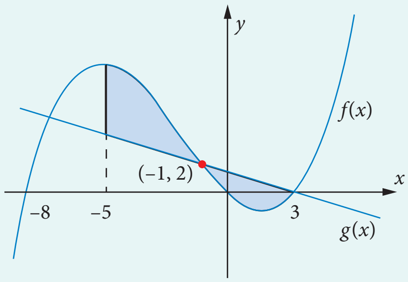
B $\int_2^4 [f(x) - g(x)]dx$

C $\int_2^4 [f(x) + g(x)]dx$

D $\int_2^4 f(x)dx + \int_2^4 g(x)dx$

E $\int_2^4 [g(x) - f(x)]dx$

10 Example 13 The shaded area shown here is equal to:



A $\int_{-5}^{-1} [g(x) - f(x)]dx + \int_{-1}^3 [f(x) - g(x)]dx$

B $\int_{-5}^3 [f(x) - g(x)]dx$

C $\int_{-5}^{-1} [f(x) - g(x)]dx + \int_{-1}^3 [g(x) - f(x)]dx$

D $\int_{-8}^3 [g(x) - f(x)]dx$

E $\int_{-5}^{-1} [f(x) - g(x)]dx - \int_{-1}^3 [g(x) - f(x)]dx$

11 Example 16 Water is flowing out of a tank. The flow rate (in litres/hour) is given by:

$$R'(t) = 100e^{-0.2t}$$

where t is in hours.

The total change in the volume of water in the tank (in L) after the first 3 hours is:

A $\int_0^3 100e^{-0.2t}\,dt$ B $500e^{-0.6} - 500e^0$ C 274

D $500e^{-0.6}$ E $\int_1^3 100e^{-0.2t}\,dt$

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Short answer

12 Example 1–3 Find the indefinite integral of each of the following, giving answers with positive indices.

a $\int (y^3 - 3y^2 + 4y + 1)dy$ b $\int -n^{-2}\,dn$ c $\int \frac{-2}{x^2}dx$

d $\int (9x^2 - 2)(3x^3 - 2x + 4)dx$ e $\int \sin(x)dx$ f $\int -3\cos(6x)dx$

g $\int -5\sin(10x)dx$ h $\int e^{3t}\,dt$ i $\int \frac{3}{e^{2x}}dx$

j $4(x-5)^{-3}$ k $\frac{1}{3(2x+7)^4}$ l $\sqrt{4x+7}$

13 Example 4 Find each indefinite integral.

a $\int (x^4 + 7)dx$ b $\int (5x^4 - 2x^3 + 4x)dx$ c $\int (6x^3 - 8x^2 - 3)dx$

14 Example 5 Find y in terms of x if $\frac{dy}{dx} = 6x - 4$ and $y = 22$ when $x = -2$.

15 Example 5 Find the equation of the curve $f(x)$ given that $f'(x) = \frac{3}{2\sqrt{x}}$ and the curve passes through the point (1, 5).

16 Example 6 Find each indefinite integral.

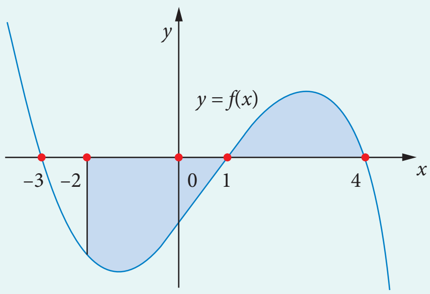
a $\int \frac{x^5 - 3x^3 + 7x}{x}dx$ b $\int (2-3x)^2\,dx$ c $\int \frac{3x^2 - 5x + 2}{\sqrt{x}}dx$

17 Examples 7–9 Evaluate the following.

a $\int_{-1}^{2} (12x^2 - 6x + 1)dx$ b $\int_{1}^{9} x^{\frac{1}{2}}\,dx$ c $\int_{5}^{10} \frac{dx}{(x-4)^2}$

d $\int_{0}^{\frac{\pi}{3}} \sin(3x)dx$ e $\int_{0}^{\frac{\pi}{4}} \sin\left(2x + \frac{\pi}{3}\right)dx$ f $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 12\cos(3x)dx$

18 Example 10 Express the shaded area in terms of definite integrals.



19 Example 10 Find the area under each of the following curves for the domain given.

a $y = x^2 - 7x - 8$ from $x = -2$ to $x = 6$

b $f(x) = e^x + e^{-x}$ from $x = -1$ and $x = 2$

c $y = (1-2x)^{\frac{1}{3}}$ from $x = -1$ to $x = 3$.

20 Example 11 Use your graphics calculator to find an approximation for the area cut off $y = x^3 - 3x^2 - 8x + 9$ by the x-axis.

21 Example 11 Find the area enclosed by $f(x) = 3(x - 2)^2$, $x = -1$ and $x = 1$.

22 Example 12 CAS Find the area under the curve $y = 8x^3 - 3x^2 + 6x$ between $x = 3$ and $x = 7$.

23 Example 12 CAS Find the physical area cut off $y = (x + 3)(x - 1)(x - 2)$ by the x-axis.

24 Example 13 Find the area enclosed between the curve $y = x^2 - 2x - 2$ and the line $y = x + 2$.

25 Example 14 Find the area enclosed between the curves $y = 3x^2 - 8x - 3$ and $y = 2x^2 - 5x + 7$.

26 Example 15 CAS Find the area between the curves $f(x) = e^x$, $g(x) = 16 - x^2$ between $x = -1$ and $x = 1$.

27 Example 16 If $C'(t)$ is the rate of temperature change of a hot liquid measured in °C per minute, what does $\int_0^5 C'(t)dt$ represent?

28 Example 16 Fluid leaks from a vessel at the rate:

$$V'(t) = 150e^{-0.2t} \text{ litres/hour}$$

where t is measured in hours.

a How much fluid will leak in the first 3 hours ($0 \le t \le 3$)?

b How much fluid will leak in the next 3 hours ($3 \le t \le 6$)?

29 Example 17 The population of mice in a closed habitat is known to increase according to the function:

$$P'(t) = \frac{t}{3} + 6$$

where $P'(t)$ is measured in hundreds of mice per month and t is measured in months. The measurement of the population commences at $t = 0$.

a What is the total change in the population in the first 3 months after measuring commenced?

b How long will it take for the increase in the population of mice to reach 4200?

30 Example 18 The marginal revenue function (in dollars) for the sale of x units of a product is given by:

$$R'(x) = 1500 - 3x^2 - 4x$$

Find the total revenue function and calculate the total revenue from the sale of the first 30 units.

31 Example 19 The acceleration (m/s^2) of a particle is given by $a = 6t - 12$. If the particle is initially at rest 2 m to the left of the origin, find the displacement of the particle after 5 s and its position relative to its initial position.

32 Example 20 An object is propelled vertically upwards with an initial velocity of 30 m/s.

a Find its height after 4 seconds.

b What is the velocity of the object after 5 seconds?

c When does the object reach its greatest height?

33 Example 21 A particle accelerates according to the equation $a = -20(1 + 2t)^2$ (cm/s^2), where t is in seconds. If the particle has an initial velocity of 30 cm/s, find an expression for the velocity as a function of time.

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Application

34 The diagram to the right shows the graph of the function $y = -e^{\frac{x}{4}}$. PQ is the perpendicular to the curve at the point P. Find:

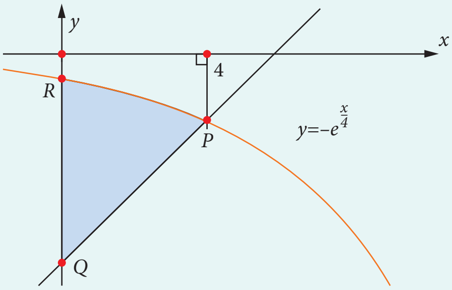
a the coordinates of P

b the equation of PQ

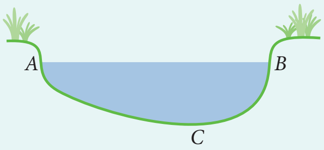
c the coordinates of Q

d the coordinates of R.

e the physical area bound by the curve and the normal and the y-axis.



35 The cross-section of a river bed is shown at right. The deepest part of the river is towards one side, at C, 6 m from bank B and 12 m from bank A. The water level is the line AB. If the origin of a coordinate system is placed at A, then the curve from A to C can be written as



$$y = \frac{x^2}{24} - x$$

and the curve from C to B can be written as

$$y = \frac{x^2}{6} - 4x + 18$$

where both x and y are in metres.

a Find the maximum depth of the river.

b Find the cross-sectional area of the river.

c If the water flows at an average rate of 1.4 m/s, how much water flows in 1 s?

d Find the amount of water that flows in a day.

Vertical levee banks that allow for a further height of water of 1.5 m are placed at A and B. During floods, the flow rate increases to 2.5 m/s and the water rises up the levees.

e Find the ratio of the flow of water in a flood to the flow at the height AB, if the flood reaches the top of the levee banks.

36 The shape of a *shade sail* is formed by three intersecting parabolas. The parabolas have equations

$$y = x^2 - 8x + 16,\ y = x^2 + 8x + 16 \text{ and } y = 2 - \frac{x^2}{8}$$

Find the area of the sail if the dimensions are in metres.

37 The acceleration of a particle is given by $a = 3t + 1$ m/s^2, where t is in seconds. If the particle is initially at the origin and moving with a velocity of 15 m/s:

a find its velocity after 3 s, correct to 1 decimal place

b show that the particle is never at rest.

38 Find the exact area enclosed between the curve $y = x\sqrt{x^2 - 1}$, the x-axis and the lines $x = 1$ and $x = 2$.

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39 A driveway sweeps from a narrow entrance to a double garage with its doors perpendicular to the road (the x-axis). The driveway can be modelled as the area between the positive axes and the curves

$$y = \frac{12 - x^2}{4} \text{ and } y = \frac{36 - x^2}{4} \text{ (in metres)}$$

Find the area of this driveway and the cost of concreting to a depth of 10 cm with concrete costing \$325/m^3.

40 A canvas sheet has two opposite edges sewn to allow steel poles to be run through the edges. The canvas is then held up flat by poles and guy ropes to make a shelter. Unfortunately, when it rains water pools on top of the shelter and stretches the canvas so that it sags in the middle. While the curve is actually a type of 3D catenary, the side-to-side cross-section can be approximated by the quadratic

$$y = \frac{x(4 - x)h}{400}$$

where h is the amount of sag in the centre, and all are in metres. The canvas will rip if there is more than 5 kg of water on top. The shelter is 4 m wide and 6 m long and water weighs 1000 kg/m^3. Find the maximum sag that can be allowed before the canvas rips.

41 A water trough is a prism with a parabolic cross-section. The trough is 3 m long and the cross-section corresponds to the part of

$$y = \frac{5x^2 - 400x + 350}{90}$$

that is below the x-axis, where the units are in centimetres. Find the area of the cross-section and the volume of water that can be contained in the trough.

Practice quiz

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