PERTH MODERN SCHOOL Exceptional schooling. Exceptional students. **Independent Public School**	Year 12 Methods TEST 2 Monday 8 April 2019 TIME: 45 minutes working No notes allowed **Calculator Assumed** 43 marks 5 Questions

Name: _Solutions_ _____   Teacher: _____

Note: All part questions worth more than 2 marks require working to obtain full marks.

Question 1 (7 marks)

(a) Differentiate $\frac{x}{e^x}$ and simplify your answer if necessary. (3 marks)

$$\frac{d}{dx}\frac{x}{e^x} = \frac{1 \times e^x - e^x \cdot x}{(e^x)^2}$$

✓✓ Uses quotient rule (−1 per error)

$$= \frac{e^x(1-x)}{e^x \cdot e^x}$$

$$= \frac{1-x}{e^x}$$

✓ simplifies $\frac{dy}{dx}$

(b) Using your result from (a) above and **without the use** of a classpad, show how to determine the definite integral $\int_0^1 \frac{1-x}{2e^x}\,dx$. (4 marks)

$$\int_0^1 \frac{1-x}{2e^x} = \frac{1}{2}\int_0^1 \frac{1-x}{e^x}$$

✓ Takes out $\frac{1}{2}$.

$$= \frac{1}{2}\int_0^1 \left(\frac{d}{dx}\frac{x}{e^x}\right)$$

$$= \frac{1}{2}\left[\frac{x}{e^x}\right]_0^1$$

✓ Uses F.T.C.

$$= \frac{1}{2}\left[\frac{1}{e} - \frac{0}{e^0}\right]$$

✓ Substitution

$$= \frac{1}{2e}$$

✓ Correct answer.

Question 2   (8 marks)
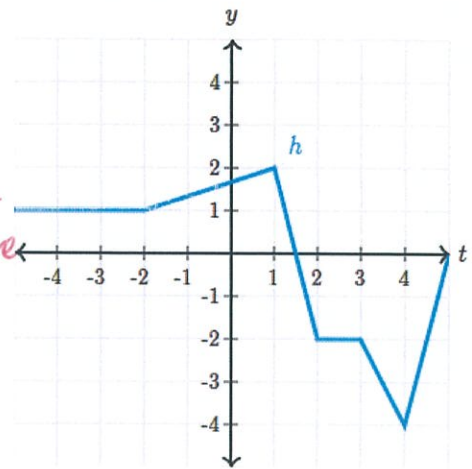
The graph of $h(x)$ is shown on the right.

(a) Evaluate the following definite integrals

(i) $\int_{-2}^{2} h(x)\, dx$ (2 marks)

$$= \left(\frac{3}{2} + 3\right) + \left(\frac{1}{4}\right) - \left(\frac{1}{4}\right)$$ ✓ Uses area under the curve

$$= 4.5 \quad \text{or} \quad \frac{9}{2}$$ ✓ correct answer

(ii) $\int_{-2}^{2} h'(x)\, dx$ (2 marks)

$$= h(2) - h(-2)$$ ✓   Uses F. T. C.

$$= (-2) - (1)$$

$$= -3$$ ✓   Correct answer

(b) Determine the area bounded by the graph of $h'(x)$ and the x axis between $x = -2$ and $x = 2$. Justify your answer.   (4 marks)

$$A = \int_{-2}^{2} |f'(x)|\, dx$$ ✓ writes the expression for area in terms of absolute value

where $f'(x) > 0$ for $-2 \le x \le 1$ ✓ determines the interval

$f'(x) < 0$ for $1 \le x \le 2$ where $f'(x)$ is "+" & "-"

So $A = \left| \int_{-2}^{1} f'(x)\, dx \right| + \left| \int_{1}^{2} f'(x)\, dx \right|$ ✓ breaks the integral over the correct intervals

$$= \left| f(1) - f(-2) \right| + \left| f(2) - f(1) \right|$$

$$= \left| 2 - 1 \right| + \left| (-2) - 2 \right|$$

$$= \quad 1 \quad + \quad 4$$

$$= \quad 5$$ ✓ calculates the correct area
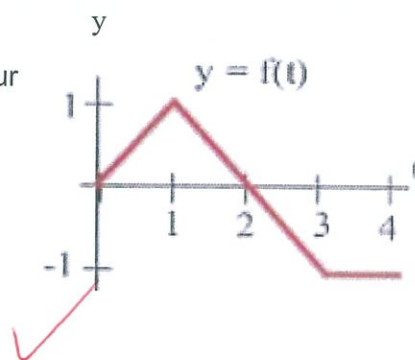
Question 3   **(10 marks)**

$F(x) = \int_0^x f(t)\, dt$ for $f(x)$ in the picture on the right.

(a) Determine the value of x for a maximum of $F(x)$. Briefly explain your
   reasons.   (3 marks)

Uses F.T.C to determine $F'(x) = f(x)$ ✓

Optimal value occurs at horizontal intercept ✓

Selects $x = 2$ as $F''(x) = f'(x) < 0$ ∴ local max ✓

<u>OR</u>

Area under $f(t)$ is positive for $1 \le t \le 2$ ✓

After $t = 2$, area change is negative.
   Selects $t = 2$ as maximum area ✓

(b) Evaluate $F(3)$.   (2 marks)

$F(3) = \int_0^3 f(t)\, dt$

$= \int_0^2 f(t)\, dt + \int_2^3 f(t)\, dt$ ✓ breaks the integrals over
                                              the correct intervals

$= 1 \qquad\qquad + \left(-\frac{1}{2}\right)$

$= \frac{1}{2}$ ✓ Calculates the correct
                                        area.

(c) Determine the value of x for a maximum of $F'(x)$. Briefly explain your reasons.   (3 marks)

$F'(x) = \frac{d}{dx}\int_0^x f(t)\, dt$ ✓ differentiates both sides

$= f(x)$ ✓ Uses F.T.C.

maximum $f(1) = 1$

∴ $x = 1$ ✓ state x.

(d) Evaluate $F'(4)$.   (2 marks)

$F'(4) = f(4)$ ✓ Uses F.T.C

$= -1$ ✓ Correct answer

Question 4   (10 marks)

A new substance labelled **XX** is found to decay by the rule $N = 1200e^{-0.116t}$, where N equals the mass of the substance in kilograms at time t minutes.

Determine the following:

a) the initial mass of **XX**.   (1 mark)

$N = 1200 \times e^0 = 1200 \text{ kg}$ ✓ (correct answer with *units*).

b) the time taken for half of the mass to decay away to the nearest minute.   (3 marks)

$1200 e^{-0.116t} = 600$ ✓

$e^{-0.116t} = \dfrac{1}{2}$ Recognises $e^{-0.116t}$ is half

$t = 5.9754 \approx 6 \text{ min}.$ ✓ correct answer

~~safe~~   ~~less~~   ✓ correct rounding

The radiation is ~~dangerous~~ to humans when the rate of decay is ~~greater~~ than 100kg per minute.

~~at~~

c) Determine ~~after~~ what time the radiation will be safe for humans.   (3 marks)

$N'(t) = -0.116 \times 1200 e^{-0.116t}$ ✓ determines $N'(t)$

$= -100$ ✓ equates $N'(t) = -100$

$t \approx 2.85 \text{ minutes}.$ ✓ correct value for t.

A different substance **YY** has a rate of decay given by $\dfrac{dN}{dt} = -50e^{-0.447t}$, where N equals the mass of the substance in kilograms at time t minutes.

d) Determine the total change in the mass from $t = 3$ to $t = 7$ minutes.   (3 marks)

$\displaystyle\int_3^7 -50e^{-0.447t} \, dt$ ✓ Uses correct integral
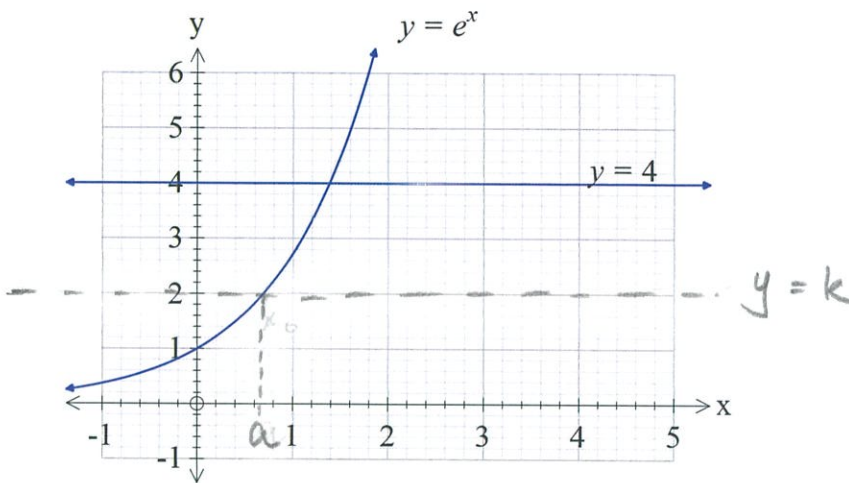✓ Uses correct limits 3 & 7

$= -24.36 \text{ kg}.$ ✓ Evaluates correct answer.

Question 5   (8 **marks**)

a) Using the solve facility on your classpad, determine to 2 decimal places the x value where the two graphs above intersect.   (2 marks)

$$y = e^x = 4 \qquad x = 1.39 \quad \checkmark\checkmark \quad \text{correct answer}$$

b) Determine to two decimal places the area bounded by $y = e^x, y = 4$ and the y axis.
(3 marks)

$$\int_0^{1.39}(4 - e^x)\, dx \qquad \checkmark \quad \text{correct boundary points}$$

$$\checkmark \quad \text{correct difference of functions}$$

$$= 2.55 \qquad \checkmark \quad \text{correct value for area.}$$

c) Let $y = k$ where $1 \le k \le 4$, determine the value of k, to two decimal places, such that the area between $y = 4, y = k, y = e^x$ and the y axis equals 1.5 sq units.   (3 marks)

let a be the x coordinate for the point of intersection between $y=k$ and $y =4$

$$\int_0^a \left(e^a - e^x\right) = 2.55 - 1.5 = 1.05$$

$\checkmark$ Sets up the integral using difference of area with correct limits

$$\left[e^a x - e^x\right]_0^a = 1.05$$

$$\left(e^a \cdot a - e^a\right) - \left(e^a \cdot 0 - e^0\right) = 1.05$$

$$e^a(a-1) + 1 = 1.05$$

$$a = 1.018 \quad \checkmark \quad \text{Solves for } a$$

$$k = e^a = 2.77 \quad \checkmark \quad \text{Subs. for } k.$$