Chapter 14 – Exponential functions and logarithms

Solutions to Exercise 14A

1 **a** $x^2x^3 = x^{2+3} = x^5$

b $2(x^3x^4)4 = 8x^{3+4} = 8x^7$

c $x^5 \div x^3 = x^{5-3} = x^2$

d $4x^6 \div 2x^3 = 2x^{6-3} = 2x^3$

e $(a^3)^2 = a^{2\times3} = a^6$

f $(2^3)^2 = 2^{3\times2} = 2^6$

g $(xy)^2 = x^2y^2$

h $(x^2y^3)^2 = (x^2)^2(y^3)^2$
$= x^{2\times2}y^{3\times2} = x^4y^6$

i $\left(\dfrac{x}{y}\right)^3 = \dfrac{x^3}{y^3}$

j $\left(\dfrac{x^3}{y^2}\right)^2 = \dfrac{(x^3)^2}{(y^2)^2}$
$= \dfrac{x^{3\times2}}{y^{2\times2}} = \dfrac{x^6}{y^4}$

2 **a** $3^5 \times 3^{12} = 3^{5+12} = 3^{17}$

b $x^3y^2 \times x^4y^3 = x^{3+4}y^{2+3} = x^7y^5$

c $3^{x+1} \times 3^{3x+2} = 3^{x+1+3x+2} = 3^{4x+3}$

d $5a^3b^2 \times 6a^2b^4 = 30a^{3+2}b^{2+4} = 30a^5b^6$

3 **a** $\dfrac{x^5y^2}{x^3y} = x^{5-3}y^{2-1} = x^2y$

b $\dfrac{b^{5x} \times b^{2x+1}}{b^{3x}} = b^{5x+2x+1-3x} = b^{4x+1}$

c $\dfrac{8a^2b \times 3a^5b^2}{6a^2b^2} = 4a^{2+5-2}b^{1+2-2} = 4a^5b$

4 **a** $7^{-2} = \dfrac{1}{7^2} = \dfrac{1}{49}$

b $\left(\dfrac{1}{4}\right)^{-3} = 4^3 = 64$

c $\left(\dfrac{5}{2}\right)^{-3} = \left(\dfrac{2}{5}\right)^3 = \dfrac{8}{125}$

5 **a** $(b^5)^2 = b^{10}$

b $\left(\left(\dfrac{1}{3}\right)^{-2}\right)^3 = \left(\dfrac{1}{3}\right)^{-6} = 3^6 = 729$

c $(b^5)^2 \times (b^2)^{-3} = b^{10} \times b^{-6} = b^4$

6 **a** $(3a^4b^3)^3 \times (4a^2b^4)^{-2}$
$= 27a^{12}b^9 \times 4^{-2}a^{-4}b^{-8}$
$= \frac{27}{16}a^8b$

b $\left(\dfrac{5a^3b^3}{ab^2c^2}\right)^3 \div (a^2b^{-1}c)^3$
$= \left(5a^2bc^{-2}\right)^3 \times a^{-6}b^3c^{-3}$
$= 125a^6b^3c^{-6} \times a^{-6}b^3c^{-3}$
$= 125b^6c^{-9}$
$= \dfrac{125b^6}{c^9}$

7 **a** $(-2)^6 = 64$

b $(-3a)^3 = -27a^3$

c $(-2a)^5 \times 3a^{-2} = -32a^5 \times 3a^{-2}$
$= -96a^3$

8 **a** $36^n \times 12^{-2n} = 2^{-2n}$

b $\dfrac{2^{-3} \times 8^4 \times 32^{-3}}{4^{-4} \times 2^{-2}} = 2^4$

c $\dfrac{5^{2n} \times 10^n}{8^n \times 5^n} = \dfrac{5^{2n}}{2^{2n}}$

9 **a** $x^3x^4x^2 = x^{3+4+2} = x^9$

b $2^4 4^3 8^2 = 2^4 2^6 2^6$
$= 2^{4+6+6} = 2^{16}$

c $3^4 9^2 27^3 = 3^4 3^4 3^9$
$= 3^{4+4+9} = 3^{17}$

d $(q^2p)^3(qp^3)^2 = q^6p^3q^2p^6$
$= q^{6+2}p^{3+6} = q^8p^9$

e $a^2b^{-3}(a^3b^2)^3 = a^2b^{-3}a^9b^6$
$= a^{2+9}b^{6-3} = a^{11}b^3$

f $(2x^3)^2(4x^4)^3 = 2^2x^{3x2}4^3x^{3x4}$
$= 2^2 2^6 x^6 x^{12} = 2^8x^{18}$

g $m^3p^2(m^2n^3)^4(p^{-2})^2 = m^3p^2m^8n^{12}p^{-4}$
$= m^{11}n^{12}p^{-2}$

h $2^3a^3b^2(2a^{-1}b^2)^{-2} = 2^3a^3b^2 2^{-2}a^2b^{-4}$
$= 2a^5b^{-2}$

10 **a** $\dfrac{x^3y^5}{xy^2} = x^{3-1}y^{5-2} = x^2y^3$

b $\dfrac{16a^5b4a^4b^3}{8ab} = \dfrac{64}{8}a^{5+4-1}b^{1+3-1}$
$= 8a^8b^3$

c $\dfrac{(-2xy)^2 2(x^2y)^3}{8(xy)^3} = \dfrac{4x^2y^2 2x^6y^3}{8x^3y^3}$
$= \dfrac{8}{8}x^{2+6-3}y^{2+3-3}$
$= x^5y^2$

d $\dfrac{(-3x^2y^3)^2}{(2xy)^3}\dfrac{4x^4y^3}{(xy)^3} = \dfrac{9x^4y^6}{8x^3y^3}\dfrac{4x^4y^3}{x^3y^3}$
$= \dfrac{9}{2}x^{4+4-3-3}y^{6+3-3-3}$
$= \dfrac{9x^2y^3}{2}$

11 **a**
$m^3n^2p^{-2}(mn^2p)^{-3} = m^3n^2p^{-2}\, m^{-3}n^{-6}p^{-3}$
$= m^{3-3}n^{2-6}p^{-2-3}$
$= n^{-4}p^{-5} = \dfrac{1}{n^4p^5}$

b
$\dfrac{x^3yz^{-2}2(x^3y^{-2}z)^2}{xyz^{-1}} = \dfrac{2x^3yz^{-2}x^6y^{-4}z^2}{xyz^{-1}}$
$= 2x^{3+6-1}y^{1-4-1}z^{-2+2+1}$
$= 2x^8y^{-4}z = \dfrac{2x^8z}{y^4}$

c $\dfrac{a^2b(ab^{-2})^{-3}}{(a^{-2}b^{-1})^{-2}} = \dfrac{a^2ba^{-3}b^6}{a^4b^2}$
$= a^{2-3-4}b^{1+6-2}$
$= a^{-5}b^5 = \dfrac{b^5}{a^5}$

d $\dfrac{a^2b^3c^{-4}}{a^{-1}b^2c^{-3}} = a^{2+1}b^{3-2}c^{3-4}$
$= \dfrac{a^3b}{c}$

e $\dfrac{a^{2n-1}b^3c^{1-n}}{a^{n-3}b^{2-n}c^{2-2n}}$
$= a^{2n-1-n+3}b^{3-2+n}c^{1-n-2+2n}$
$= a^{n+2}b^{n+1}c^{n-1}$

12 a $3^{4n}9^{2n}27^{3n} = 3^{4n}3^{4n}3^{9n}$
$= 3^{17n}$

b $\dfrac{2^n 8^{n+1}}{32^n} = \dfrac{2^n 2^{3n+3}}{2^{5n}}$
$= 2^{n+3n+3-5n} = 2^{3-n}$

c $\dfrac{3^{n-1}9^{2n-3}}{6^2 3^{n+2}} = \dfrac{3^{n-1}3^{4n-6}}{6^2 3^{n+2}}$
$= \dfrac{3^{4n-9}}{36} = \dfrac{3^{4n-11}}{2^2}$

d $\dfrac{2^{2n}9^{2n-1}}{6^{n-1}} = \dfrac{2^{2n}3^{4n-2}}{6^{n-1}}$
$= \dfrac{2^{2n}3^{4n-2}}{2^{n-1}3^{n-1}}$
$= 2^{2n-n+1}3^{4n-2-n+1}$
$= 2^{n+1}3^{3n-1}$

e $\dfrac{25^{2n}5^{n-1}}{5^{2n+1}} = \dfrac{5^{4n}5^{n-1}}{5^{2n+1}}$
$= 5^{4n+n-1-2n-1} = 5^{3n-2}$

f $\dfrac{6^{x-3}4^x}{3^{x+1}} = \dfrac{3^{x-3}2^{x-3}2^{2x}}{3^{x+1}}$
$= 3^{x-3-x-1}2^{x-3+2x}$
$= 2^{3x-3}3^{-4}$

g $\dfrac{6^{2n}9^3}{27^n 8^n 16^n} = \dfrac{3^{2n}2^{2n}3^6}{3^{3n}2^{3n}2^{4n}}$
$= 3^{2n+6-3n}2^{2n-3n-4n}$
$= 3^{6-n}2^{-5n}$

h $\dfrac{3^{n-2}9^{n+1}}{27^{n-1}} = \dfrac{3^{n-2}3^{2n+2}}{3^{3n-3}}$
$= 3^{n-2+2n+2-3n+3}$
$= 3^3 = 27$

i $\dfrac{8 2^5 3^7}{9\, 2^7 81} = \dfrac{2^3 2^5 3^7}{3^2 2^7 3^4}$
$= 2^{3+5-7}3^{7-2-4}$
$= (2)(3) = 6$

13 a $\dfrac{(8^3)^4}{(2^{12})^2} = \dfrac{2^{36}}{2^{24}}$
$= 2^{36-24}$
$= 2^{12} = 4096$

b $\dfrac{(125)^3}{(25)^2} = \dfrac{5^9}{5^4}$
$= 5^{9-4}$
$= 5^5 = 3125$

c $\dfrac{(81)^4 \div (27^3)}{9^2} = \dfrac{3^{16} \div 3^9}{3^4}$
$= \dfrac{3^{16} \div 3^9}{3^4}$
$= 3^{16-9-4}$
$= 3^3 = 27$

Solutions to Exercise 14B

1 **a** $125^{\frac{2}{3}} = 5^2 = 25$

b $243^{\frac{3}{5}} = 3^3 = 27$

c $81^{-\frac{1}{2}} = \dfrac{1}{\sqrt{81}} = \dfrac{1}{9}$

d $64^{\frac{2}{3}} = 4^2 = 16$

e $\left(\dfrac{1}{8}\right)^{\frac{1}{3}} = \dfrac{1}{2}$

f $32^{-\frac{2}{5}} = \dfrac{1}{32^{\frac{2}{5}}}$
$= \dfrac{1}{2^2} = \dfrac{1}{4}$

g $125^{-\frac{2}{3}} = \dfrac{1}{125^{\frac{2}{3}}}$
$= \dfrac{1}{5^2} = \dfrac{1}{25}$

h $32^{\frac{4}{5}} = 2^4 = 16$

i $1000^{\frac{4}{3}} = \dfrac{1}{100^{\frac{4}{3}}}$
$= \dfrac{1}{10^4} = \dfrac{1}{10\,000}$

j $10\,000^{\frac{3}{4}} = 10^3 = 1000$

k $81^{\frac{3}{4}} = 3^3 = 27$

l $\left(\dfrac{27}{125}\right)^{\frac{1}{3}} = \left(\dfrac{3}{5}\right)^{\frac{3}{3}} = \dfrac{3}{5}$

m $(-8)^{\frac{1}{3}} = -2$

n $(125)^{-\frac{4}{3}} = \left(\dfrac{1}{5}\right)^4 = \dfrac{1}{625}$

o $(-32)^{\frac{4}{5}} = (-2)^4 = 16$

p $\left(\dfrac{1}{49}\right)^{-\frac{3}{2}} = 7^3 = 343$

2 **a** $(a^2b)^{\frac{1}{3}} \div \sqrt{ab^3} = \dfrac{a^{\frac{2}{3}}b^{\frac{1}{3}}}{a^{\frac{1}{2}}b^{\frac{3}{2}}}$
$= a^{\frac{2}{3}-\frac{1}{2}}b^{\frac{1}{3}-\frac{3}{2}}$
$= a^{\frac{1}{6}}b^{-\frac{7}{6}}$

b $= a^{-6}b^3b^{\frac{3}{2}}$
$= a^{-6}b^{3+\frac{3}{2}}b^{\frac{3}{2}} = a^{-6}b^{\frac{9}{2}}$

c
$\left(45^{\frac{1}{3}}\right) \div \left(9^{\frac{3}{4}}15^{\frac{3}{2}}\right) = \left(3^{\frac{2}{3}}5^{\frac{1}{3}}\right) \div \left(3^{\frac{3}{2}}3^{\frac{3}{2}}5^{\frac{3}{2}}\right)$
$= 3^{\frac{2}{3}-\frac{3}{2}-\frac{3}{2}}5^{\frac{1}{3}-\frac{3}{2}}$
$= 3^{-\frac{7}{3}}5^{-\frac{7}{6}}$

d $2^{\frac{3}{2}}4^{-\frac{1}{4}}16^{-\frac{3}{4}} = 2^{\frac{3}{2}}2^{-\frac{1}{2}}2^{-3}$
$= 2^{\frac{3}{2}-\frac{1}{2}-3} = 2^{-2} = \dfrac{1}{4}$

e
$\left(\dfrac{x^3y^{-2}}{3^{-3}y^{-3}}\right)^{-2} \div \left(\dfrac{3^{-3}x^{-2}y}{x^4y^{-2}}\right)^2 = \left(\dfrac{x^{-6}y^4}{3^6y^6}\right)\left(\dfrac{x^8y^{-4}}{3^{-6}x^{-4}y^2}\right)$
$= 3^{6-6}x^{-6+8+4}y^{4-4-6-2}$
$= x^6y^{-8}$

f $\left((a^2)^{\frac{1}{5}}\right)^{\frac{3}{2}}\left((a^5)^{\frac{1}{3}}\right)^{\frac{1}{5}} = a^{\frac{2}{5}\frac{3}{2}}a^{\frac{5}{3}\frac{1}{5}}$
$= a^{\frac{3}{5}}a^{\frac{1}{3}}$
$= a^{\frac{3}{5}+\frac{1}{3}} = a^{\frac{14}{15}}$

3 **a** $(2x-1)\sqrt{2x-1} = (2x-1)^{1+\frac{1}{2}}$

$= (2x-1)^{\frac{3}{2}}$

b $(x-1)^2\sqrt{x-1} = (x-1)^{2+\frac{1}{2}}$

$= (x-1)^{\frac{5}{2}}$

c $(x^2+1)\sqrt{x^2+1} = (x^2+1)^{1+\frac{1}{2}}$

$= (x^2+1)^{\frac{3}{2}}$

d $(x-1)^3\sqrt{(x-1)} = (x-1)^{3+\frac{1}{2}}$

$= (x-1)^{\frac{7}{2}}$

e $\dfrac{1}{\sqrt{x-1}} + \sqrt{x-1} = \dfrac{1+x-1}{\sqrt{x-1}}$

$= x(x-1)^{-\frac{1}{2}}$

f $(5x^2+1)(5x^2+1)^{\frac{1}{3}} = (5x^2+1)^{1+\frac{1}{3}}$

$= (5x^2+1)^{\frac{4}{3}}$

Solutions to Exercise 14C

1 a $47.8 = 4.78 \times 10^1 = 4.78 \times 10$

b $6728 = 6.728 \times 10^3$

c $79.23 = 7.923 \times 10^1 = 7.923 \times 10$

d $43\,580 = 4.358 \times 10^4$

e $0.0023 = 2.3 \times 10^{-3}$

f $0.000\,000\,56 = 5.6 \times 10^{-7}$

g $12.000\,34 = 1.2000\,34 \times 10^1$
$= 1.2000\,34 \times 10$

h Fifty million $= 50\,000\,000$
$= 5.0 \times 10^7$

i $23\,000\,000\,000 = 2.3 \times 10^{10}$

j $0.000\,000\,0013 = 1.3 \times 10^{-9}$

k 165 thousand $= 165\,000$
$= 1.65 \times 10^5$

l $0.000\,014\,567 = 1.4567 \times 10^{-5}$

2 a The decimal point moves 8 places to the right $= 1.0 \times 10^{-8}$

b The decimal point moves 24 places to the right $= 1.67 \times 10^{-24}$

c The decimal point moves 5 places to the right $= 5.0 \times 10^{-5}$

d The decimal point moves 3 places to the left $= 1.853\,18 \times 10^3$

e The decimal point moves 12 places to the left $= 9.461 \times 10^{12}$

f The decimal point moves 10 places to the right $= 2.998 \times 10^{10}$

3 a The decimal point move 13 places to the right $= 81\,280\,000\,000\,000$

b The decimal point move 8 places to the right $= 270\,000\,000$

c The decimal point move 13 places to the left $= 0.000\,000\,000\,000\,28$

4 a $456.89 \approx 4.569 \times 10^2$
(4 significant figures)

b $34567.23 \approx 3.5 \times 10^4$
(2 significant figures)

c $5679.087 \approx 5.6791 \times 10^3$
(5 significant figures)

d $0.04536 \approx 4.5 \times 10^{-2}$
(2 significant figures)

e $0.09045 \approx 9.0 \times 10^{-2}$
(2 significant figures)

f $4568.234 \approx 4.5682 \times 10^3$
(5 significant figures)

5 a
$$\frac{324\,000 \times 0.000\,000\,7}{4000}$$
$$= \frac{3.24 \times 10^5 \times 7 \times 10^{-7}}{4 \times 10^3}$$
$$= \frac{3.24 \times 7}{4} \times 10^{5+-7-3}$$
$$= 5.67 \times 10^{-5}$$
$$= 0.0000567$$

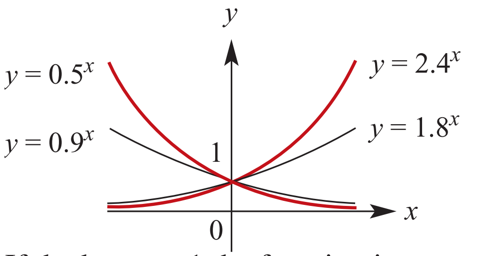
b $$\frac{5\,240\,000 \times 0.8}{42\,000\,000}$$
$$= \frac{5.24 \times 10^6 \times 8 \times 10^{-1}}{4.2 \times 10^7}$$
$$= \frac{41.92 \times 10^5}{4.2 \times 10^7}$$
$$= \frac{4192 \times 10^3}{42\,000 \times 10^3}$$
$$= \frac{4192}{42\,000} = \frac{262}{2625}$$

6 a $$\frac{\sqrt[3]{a}}{b^4} = \frac{\sqrt[3]{2 \times 10^9}}{3.215^4}$$
$$= \frac{\sqrt[3]{2} \times \sqrt[3]{10^9}}{106.8375\ldots}$$
$$= \frac{1.2599\ldots \times 10^3}{106.8375\ldots}$$
$$= 0.011\,792\ldots \times 10^3 \approx 11.8$$

b
$$\frac{\sqrt[4]{a}}{4b^4} = \frac{\sqrt[4]{2 \times 10^{12}}}{4 \times 0.05^4}$$
$$= \frac{\sqrt[4]{2} \times \sqrt[4]{10^{12}}}{4 \times 0.000\,006\,25}$$
$$= \frac{1.189\,2\ldots \times 10^3}{4 \times 6.25 \times 10^{-6}}$$
$$= 0.047\,568\ldots \times 10^9 \approx 4.76 \times 10^7$$

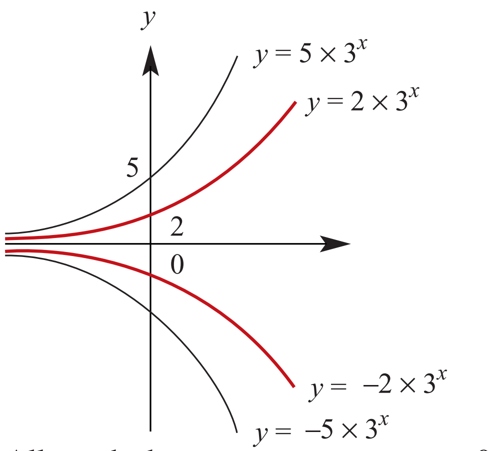
Solutions to Exercise 14D

1



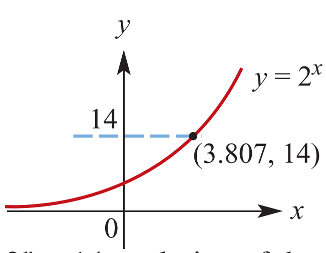
If the bases > 1 the function is increasing; if < 1 they are decreasing.

2



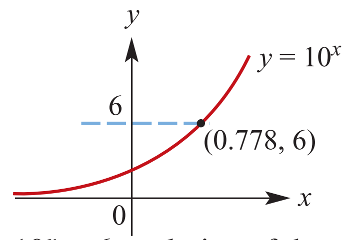
All graphs have an asymptote at $y = 0$. The y-intercepts are wherever the constant is in front of the exponential, however, at $2, -2, 5$ and -5.
The negative values are also below the axis instead of above.

3 $y = 2^x$ for $x \in [-4, 4]$:



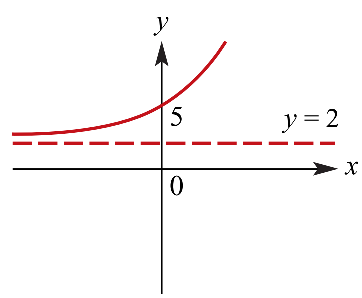
$2^x = 14$: solution of the equation is where the graph cuts the line $y = 14$, i.e. $x = 3.807$

4 $y = 10^x$; $x \in [-0.4, 0.8]$



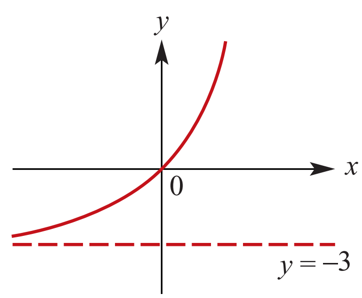
$10^x = 6$: solution of the equation is where the graph cuts the line $y = 6$, i.e. $x = 0.778$

5 **a** $f\colon R \to R; f(x) = 3(2^x) + 2$



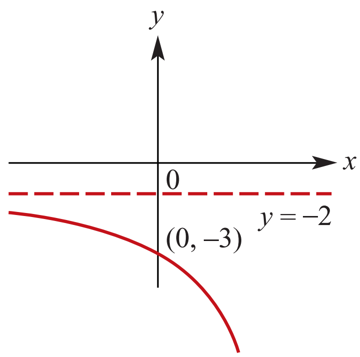
Asymptote at $y = 2$,
y-axis intercept at $(0, 5)$,
range $= (2, \infty)$

b $f\colon R \to R; f(x) = 3(2^x) - 3$



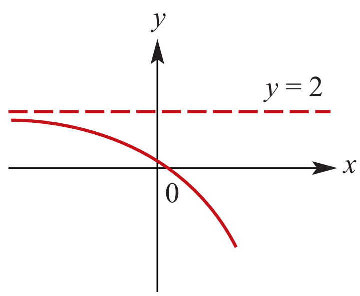
Asymptote at $y = -3$,
y-axis intercept at $(0, 0)$,
range $= (-3, \infty)$

c $f\colon R \to R; f(x) = -3^x - 2$



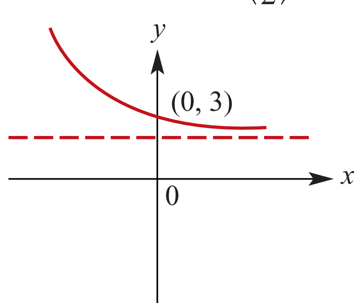
Asymptote at $y = -2$,
y-axis intercept at $(0, -3)$,
range $= (-\infty,\ -2)$

d $f : R \to R;\ f(x) = -2(3^x) + 2$



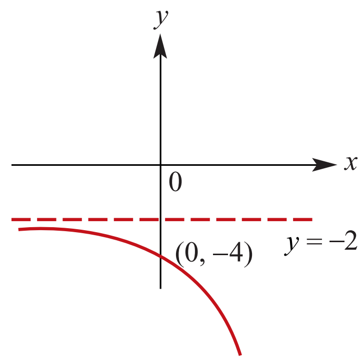
Asymptote at $y = 2$,
y-axis intercept at $(0, 0)$,
range $= (-\infty, 2)$

e $f : R \to R;\ f(x) = \left(\dfrac{1}{2}\right)^x + 2$



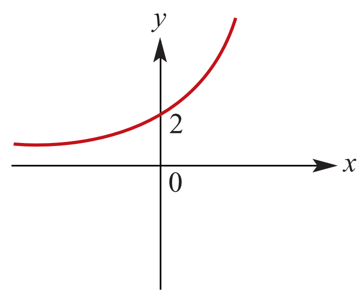
Asymptote at $y = 2$,
y-axis intercept at $(0, 3)$,
range $= (2, \infty)$

f $f : R \to R;\ f(x) = -2(3^x) - 2$



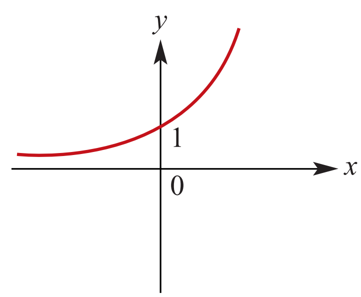
Asymptote at $y = -2$,
y-axis intercept at $(0, -4)$,
range $= (-\infty, -2)$

6 **a** $y = 2(5^x)$



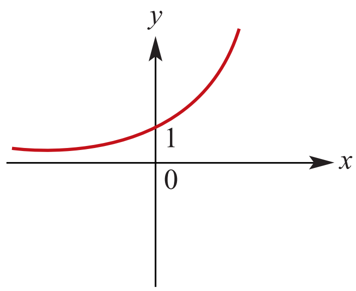
Asymptote at $y = 0$,
y-axis intercept at $(0, 2)$,
range $= (0, \infty)$

b $y = 3^{3x}$



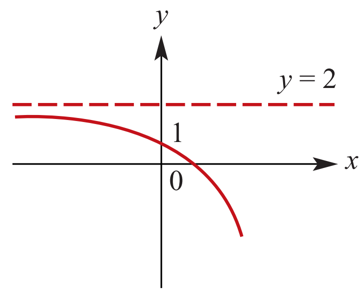
Asymptote at $y = 0$,
y-axis intercept at $(0, 1)$,
range $= (0, \infty)$

c $y = 5^{\frac{x}{2}}$



Asymptote at $y = 0$,
y-axis intercept at $(0, 1)$,
range $= (0, \infty)$

d $y = -3(2^x) + 2$



Asymptote at $y = 2$,
y-axis intercept at $(0, 1)$,
range $= (-\infty, 2)$

Solutions to Exercise 14E

1 a $3^x = 27 = 3^3, \therefore\ x = 3$

b $4^x = 64 = 4^3, \therefore\ x = 3$

c $49^x = 7 = 49^{\frac{1}{2}}, \therefore\ x = \dfrac{1}{2}$

d $16^x = 8, \therefore\ 2^{4x} = 2^3$
$\therefore\ 4x = 3, \therefore\ x = \dfrac{3}{4}$

e $125^x = 5, \therefore\ 5^{3x} = 5$
$\therefore\ 3x = 1, \therefore\ x = \dfrac{1}{3}$

f $5^x = 625 = 5^4, \therefore x = 4$

g $16^x = 256 = 16^2, \therefore\ x = 2$

h $4^{-x} = \dfrac{1}{64}, \therefore\ 4^x = 64$
$\therefore\ 4^x = 4^3, \therefore\ x = 3$

i $5^{-x} = \dfrac{1}{125}, \therefore\ 5^x = 125$
$\therefore\ 5^x = 5^3, \therefore\ x = 3$

2 a
$$5^n 25^{2n-1} = 125$$
$$\therefore\ 5^n 5^{4n-2} = 5^3$$
$$5^{5n-2} = 5^3$$
$$5n - 2 = 3, \therefore\ n = 1$$

b
$$3^{2n-4} = 1$$
$$\therefore 3^{2n-4} = 3^0$$
$$2n - 4 = 0, \therefore\ n = 2$$

c
$$3^{2n-1} = \frac{1}{81}$$
$$\therefore\ 3^{2n-1} = 3^{-4}$$
$$2n - 1 = -4, \therefore\ n = -\frac{3}{2}$$

d
$$\frac{3^{n-2}}{9^{1-n}} = 1$$
$$\therefore\ 3^{n-2} = 9^{1-n}$$
$$3^{n-2} = 3^{2(1-n)}$$
$$n - 2 = 2 - 2n$$
$$3n = 4,\ n = \frac{4}{3}$$

e
$$3^{3n} 9^{-2n+1} = 27$$
$$\therefore 3^{3n} 3^{2-4n} = 3^3$$
$$3^{3n+2-4n} = 3^3$$
$$2 - n = 3, \therefore\ n = -1$$

f
$$2^{-3n} 4^{2n-2} = 16$$
$$\therefore\ 2^{-3n} 2^{4n-4} = 2^4$$
$$2^{4n-3n-4} = 2^4$$
$$n - 4 = 4, \therefore\ n = 8$$

g
$$2^{n-6} = 8^{2-n} = 2^{6-3n}$$
$$\therefore\ n - 6 = 6 - 3n$$
$$4n = 12, \therefore\ n = 3$$

h
$$9^{3n+3} = 27^{n-2}$$
$$\therefore 3^{6n+6} = 3^{3n-6}$$
$$6n + 6 = 3n - 6$$
$$3n = -12, \therefore\ n = -4$$

i
$$4^{n+1} = 8^{n-2}$$
$$\therefore\ 2^{2n+2} = 2^{3n-6}$$
$$2n + 2 = 3n - 6, n = 8$$

j $32^{2n+1} = 8^{4n-1}$

$\therefore\ 2^{10n+5} = 2^{12n-3}$

$10n + 5 = 12n - 3$

$2n = 8,\ \therefore\ n = 4$

k $25^{n+1} = 5 \times 390\,625$

$\therefore 25^{n+1} = (25)^{\frac{1}{2}}(25)^4 = 25^{\frac{9}{2}}$

$n + 1 = \dfrac{9}{2},\ \therefore\ n = \dfrac{7}{2} = 3\dfrac{1}{2}$

l $125^{4-n} = 5^{6-2n}$

$\therefore\ 5^{12-3n} = 5^{6-2n}$

$12 - 3n = 6 - 2n,\ \therefore\ n = 6$

m $4^{2-n} = \dfrac{1}{2048}$

$\therefore\ 2^{4-2n} = 2^{-11}$

$4 - 2n = -11$

$2n = 15, \therefore\ n = \dfrac{15}{2}$

3 a $2^{x-1}4^{2x+1} = 32$

$\therefore\ 2^{x-1}2^{4x+2} = 2^5$

$2^{x-1+4x+2} = 2^5$

$5x + 1 = 5, \therefore\ x = \dfrac{4}{5}$

b $3^{2x-1}9^x = 243$

$\therefore\ 3^{2x-1}3^{2x} = 3^5$

$3^{2x-1+2x} = 3^5$

$4x - 1 = 5$

$4x = 6,\ \therefore\ x = \dfrac{3}{2}$

c $(27\ 3^x)^2 = 27^x 3^{\frac{1}{2}}$

$\therefore\ (3^3 3^x)^2 = 3^{3x}3^{\frac{1}{2}}$

$3^{6+2x} = 3^{3x+\dfrac{1}{2}}$

$2x + 6 = 3x + \dfrac{1}{2}, \therefore\ x = \dfrac{11}{2} = 5\dfrac{1}{2}$

4 a $4(2^{2x}) = 8(2^x) - 4,\ A = 2^x$

$\therefore\ 4A^2 = 8A - 4$

$A^2 - 2A + 1 = 0$

$(A - 1)^2 = 0$

$A = 2^x = 1,\ \therefore\ x = 0$

b $8(2^{2x}) - 10(2^x) + 2 = 0,\ A = 2^x$

$\therefore\quad 8A^2 - 10A + 2 = 0$

$4A^2 - 5A + 1 = 0$

$(4A - 1)(A - 1) = 0$

$A = 2^x = \dfrac{1}{4}, 1$

$\therefore\quad x = -2, 0$

c $3(2^{2x}) - 18(2^x) + 24 = 0,\ A = 2^x$

$\therefore\quad 3A^2 - 18A + 24 = 0$

$A^2 - 6A + 8 = 0$

$(A - 2)(A - 4) = 0$

$A = 2^x = 2, 4$

$\therefore\quad x = 1, 2$

d $9^x - 4(3^x) + 3 = 0,\ A = 3^x$

$\therefore\ (A - 1)(A - 3) = 0$

$A = 3^x = 1, 3$

$\therefore\quad x = 0, 1$

5 **a** $2^x = 5, \therefore x = 2.32$

b $4^x = 6, \therefore x = 1.29$

c $10^x = 18, \therefore x = 1.26$

d $10^x = 56, \therefore x = 1.75$

6 **a** $7^x > 49, \therefore 7^x > 7^2$

$\therefore x > 2$

b $8^x > 2, \therefore 2^{3x} > 2^1$

$3x > 1, \therefore x > \dfrac{1}{3}$

c $25^x \le 5, \therefore 5^{2x} \le 5^1$

$2x \le 1, \therefore x \le \dfrac{1}{2}$

d $3^{x+1} < 81, \therefore 3^{x+1} < 3^4$

$x + 1 < 4, \therefore x < 3$

e $9^{2x+1} < 243, \therefore 3^{4x+2} < 3^5$

$4x + 2 < 5$

$4x < 3, \therefore x < \dfrac{3}{4}$

f $4^{2x+1} > 64, \therefore 4^{2x+1} > 4^3$

$2x + 1 > 3, \therefore x > 1$

g $3^{2x-2} \le 81, \therefore 3^{2x-2} \le 3^4$

$2x - 2 \le 4, \therefore x \le 3$

Solutions to Exercise 14F

1 **a** $\log_2 128 = 7$

b $\log_3 81 = 4$

c $\log_5 125 = 3$

d $\log_{10} 0.1 = -1$

2 **a** $\log_2 10 + \log_2 a = \log_2 10a$

b $\log_{10} 5 + \log_{10} 2 = \log_{10} 10 = 1$

c $\log_2 9 - \log_2 4 = \log_2\left(\frac{9}{4}\right)$

d $\log_2 10 - \log_2 5 = \log_2\left(\frac{10}{5}\right)$

$= \log_2 2 = 1$

e $\log_2 a^3 = 3\log_2 a$

f $\log_2 8^3 = 3\log_2 8 = 9$

g $\log_5\left(\frac{1}{6}\right) = -\log_5 6$

h $\log_5\left(\frac{1}{25}\right) = -\log_5 25 = -2$

3 **a** $\log_3 27 = \log_3 3^3$

$= 3\log_3 3 = 3$

b $\log_5 625 = \log_5 5^4$

$= 4\log_5 5 = 4$

c $\log_2\left(\frac{1}{128}\right) = \log_2 2^{-7}$

$= -7\log_2 2 = -7$

d $\log_4\left(\frac{1}{64}\right) = \log_4 4^{-3}$

$= -3\log_4 4 = -3$

e $\log_x\ x^4 = 4\log x\ x = 4$

f $\log_2 0.125 = -\log_2 8$

$= -3\log_2 2 = -3$

g $\log_{10} 10000 = \log_{10} 10^4$

$= 4\log_{10} 10 = 4$

h $\log_{10} 0.000001 = \log_{10} 10^{-6}$

$= -6\log_{10} 10 = -6$

i $-3\log_5 125 = -3\log_5 5^3$

$= -9\log_5 5 = -9$

j $-4\log_{16} 2 = -\log_{16} 16 = -1$

k $2\log_3 9 = 4\log_3 3 = 4$

l $-4\log_{16} 4 = -2\log_{16} 16 = -2$

4 **a**

$\frac{1}{2}\log_{10} 16 + 2\log_{10} 5 = \log_{10}(\sqrt{16}\,(5^2))$

$= \log_{10} 100 = 2$

b $\log_2 16 + \log_2 8 = \log_2 2^4 + \log_2 2^3$

$= 4 + 3 = 7$

c $\log_2 128 + \log_3 45 - \log_3 5$

$= \log_2 2^7 + \log_3 5(3^2) - \log_3 5$

$= 7 + 2\log_3 3 + \log_3 5 - \log_3 5$

$= 7 + 2 = 9$

d $$\begin{aligned}\log_4 32 - \log_9 27 &= \log_4 2^5 - \log_9 3^3\\ &= \log_4 4^2 - \log_9 9^2\\ &= \frac{5}{2} - \frac{3}{2} = 1\end{aligned}$$

e

$$\begin{aligned}\log_b b^3 - \log_b \sqrt{b} &= \log_b b^3 - \log_b\left(b^{\frac{1}{2}}\right)\\ &= 3 - \frac{1}{2} = \frac{5}{2}\end{aligned}$$

f

$$\begin{aligned}2\log_x a + \log_x a^3 &= 2\log_x a + 3\log_x a\\ &= 5\log_x a\\ &= \log_x a^5\end{aligned}$$

g

$$\begin{aligned}x\log_2 8 + \log_2(8^{1-x}) &= \log_2 8^x + \log_2(8^{1-x})\\ &= \log_2(8^{x+1-x})\\ &= \log_x\ 8 = 3\end{aligned}$$

h $$\begin{aligned}\frac{3}{2}\log_a a - \log_a \sqrt{a} &= \frac{3}{2} - \log_a\left(a^{\frac{1}{2}}\right)\\ &= \frac{3}{2} - \frac{1}{2} = 1\end{aligned}$$

5 **a** $\log_3 9 = x$

$x = \log_3 3^2 = 2$

b $\log_3 x = 3$

$x = 3^3, \therefore\ x = 27$

c $\log_5 x = -3$

$x = 5^{-3}, \therefore\ x = \dfrac{1}{125}$

d $$\begin{aligned}\log_{10} x &= \log_{10} 4 + \log_{10} 2\\ \log_{10} x &= \log_{10} 8\\ \therefore\quad x &= 8\end{aligned}$$

e

$$\begin{aligned}\log_{10} 2 + \log_{10} 5 + \log_{10} x - \log_{10} 3 &= 2\\ \log_{10}\left(\frac{10x}{3}\right) &= 2\\ \frac{10x}{3} &= 10^2\\ \therefore\quad x &= 30\end{aligned}$$

f $$\begin{aligned}\log_{10} x &= \frac{1}{2}\log_{10} 36 - 2\log_{10} 3\\ \log_{10} x &= \log_{10}\ \sqrt{36} - \log_{10} 3^2\\ \log_{10} x &= \log_{10}\frac{6}{9}\\ \therefore\quad x &= \frac{2}{3}\end{aligned}$$

g $\log_x 64 = 2$

$64 = x^2$

$x^2 = 64, \therefore\ x = 8$

(no negative solutions for log base)

h $\log_5(2x - 3) = 3$

$2x - 3 = 5^3$

$2x - 3 = 125, \therefore\ x = 64$

i $\log_5(x + 2) - \log_3 2 = 1$

$\log_3 \dfrac{x+2}{2} = 1$

$\dfrac{x+2}{2} = 3^1$

$\dfrac{x+2}{2} = 3$

$x + 2 = 6, \therefore\ x = 4$

j $\log_x 0.01 = -2$

$0.01 = x^{-2}$

$x^{-2} = 0.01$

$x^2 = 100, \therefore\ x = 10$

6 a $\log_x\left(\frac{1}{25}\right) = -2$

$\log_x 25 = 2$

$25 = x^2$

$x^2 = 25, \therefore\ x = 5$

(No negative solutions for log base)

b $\log_4(2x-1) = 3$

$2x - 1 = 4^3$

$2x - 1 = 64, \therefore\ x = \frac{65}{2} = 32.5$

c $\log_4(3x+2) - \log_4 6 = 1$

$\log_4 \frac{x+2}{6} = 1$

$\frac{x+2}{6} = 4^1$

$\frac{x+2}{6} = 4$

$x + 2 = 24, \therefore\ x = 22$

d $\log_4(3x+4) + \log_4 16 = 5$

$\log_4(3x+4) + 2 = 5$

$\log_4(3x+4) = 3$

$3x + 4 = 4^3$

$3x + 4 = 64, \therefore\ x = 20$

e $\log_3(x^2 - 3x - 1) = 0$

$x^2 - 3x - 1 = 1$

$x^2 - 3x - 2 = 0$

$\therefore\ x = \frac{3 \pm \sqrt{17}}{2}$

f $\log_3(x^2 - 3x + 1) = 0$

$x^2 - 3x + 1 = 1$

$x^2 - 3x = 0$

$x(x-3) = 0,\ x = 0, 3$

7 $\log_{10} x = a; \log_{10} y = c$:

$\log_{10}\left(\frac{100x^3y^{-\frac{1}{2}}}{y^2}\right) = \log_{10}\left(100x^3y^{-\frac{5}{2}}\right)$

$= \log_{10}(100x^3) + \log_{10}(y^{-\frac{5}{2}})$

$= \log_{10}(100) + 3\log_{10} x - \frac{5}{2}\log_{10} y$

$= 3a - \frac{5c}{2} + 2$

8 $\log_{10}\frac{ab^2}{c} + \log_{10}\frac{c^2}{ab} - \log_{10}(bc)$

$= \log_{10}\left(\frac{ab^2}{c}\right)\left(\frac{c^2}{ab}\right) - \log_{10}(bc)$

$= \log_{10}(bc) - \log_{10}(bc)$

$= \log_{10}\left(\frac{bc}{bc}\right) = \log_{10} 1 = 0$

9

$$\log_a\left(\frac{11}{3}\right) + \log_a\left(\frac{490}{297}\right) - 2\log_a\left(\frac{7}{9}\right) = \log_a(k)$$

$$\log_a\left(\frac{11}{3}\right)\left(\frac{490}{297}\right) - 2\log_a\left(\frac{7}{9}\right) = \log_a(k)$$

$$\log_a\left(\frac{490}{81}\right) - \log_a\left(\frac{7}{9}\right)^2 = \log_a(k)$$

$$\log_a 10 + \log_a 1 = \log_a(k)$$

$$\log_a 10 = \log_a(k)$$

$$\therefore\ k = 10$$

10 a $\log_{10}(x^2 - 2x + 8) = 2\log_{10} x$

$\log_{10}(x^2 - 2x + 8) = \log_{10} x^2$

$x^2 - 2x + 8 = x^2$

$-2x + 8 = 0, \therefore\ x = 4$

b

$$\begin{aligned}
\log_{10}(5x) - \log_{10}(3-2x) &= 1\\
\log_{10}\left(\frac{5x}{3-2x}\right) &= 1\\
\left(\frac{5x}{3-2x}\right) &= 10^1\\
5x &= 10(3-2x)\\
x &= 2(3-2x)\\
5x &= 6\\
\therefore \qquad x &= \frac{6}{5}
\end{aligned}$$

c $3\log_{10}(x-1) = \log_{10} 8$

$3\log_{10}(x-1) = 3\log_{10} 2$

$x - 1 = 2, \therefore\ x = 3$

d

$$\begin{aligned}
\log_{10}(20x) - \log_{10}(x-8) &= 2\\
\log_{10}\left(\frac{20x}{x-8}\right) &= 2\\
\left(\frac{20x}{x-8}\right) &= 10^2\\
20x &= 100(x-8)\\
x &= 5x - 40\\
4x &= 40\\
\therefore \qquad x &= 10
\end{aligned}$$

e
$$\begin{aligned}
\text{LHS} &= 2\log_{10} 5 + \log_{10}(x+1)\\
&= \log_{10} 5^2 + \log_{10}(x+1)\\
&= \log_{10} 25(x+1)
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= 1 + \log_{10}(2x+7)\\
&= \log_{10} 10 + \log_{10}(2x+7)\\
&= \log_{10} 10(2x+7)
\end{aligned}$$

$$\begin{aligned}
\therefore 25(x+1) &= 10(2x+7)\\
5x + 5 &= 4x + 14\\
x &= 9
\end{aligned}$$

f
$$\begin{aligned}
\text{LHS} &= 1 + 2\log_{10}(x+1)\\
&= \log_{10} 10 + \log_{10}(x+1)^2\\
&= \log_{10} 10(x+1)^2
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= \log_{10}(2x+1) + \log_{10}(5x+8)\\
&= \log_{10}(2x+1)(5x+8)
\end{aligned}$$

$$\begin{aligned}
\therefore \qquad 10(x+1)^2 &= (2x+1)(5x+8)\\
10x^2 + 20x + 10 &= 10x^2 + 21x + 8\\
20x + 10 &= 21x + 8\\
x &= 2
\end{aligned}$$

Solutions to Exercise 14G

1 a $2^x = 7$

$\therefore\ x = \dfrac{\log 7}{\log 2} = 2.81$

b $2^x = 0.4$

$\therefore\ x = \dfrac{\log 0.4}{\log 2} = -1.32$

c $3^x = 14$

$\therefore\ x = \dfrac{\log 14}{\log 3} = 2.40$

d $4^x = 3$

$\therefore\ x = \dfrac{\log 3}{\log 4} = 0.79$

e $2^{-x} = 6$

$\therefore\ x = -\dfrac{\log 6}{\log 2} = -2.58$

f $0.3^x = 2$

$\therefore\ x = \dfrac{\log 2}{\log 0.3} = -0.58$

2 a

$$\begin{aligned} 5^{2x-1} &= 90 \\ \therefore\ (2x-1) &= \log_5 90 \\ 2x &= \log_5(90) + 1 \\ x &= \frac{1}{2}(\log_5(90) + 1) \\ x &= 1.90 \end{aligned}$$

b

$$\begin{aligned} 3^{x-1} &= 10 \\ \therefore\ (x-1)\log 3 &= \log 10 \\ (x-1) &= \frac{\log 10}{\log 3} \\ x - 1 &= 2.10 \\ x &= 3.10 \end{aligned}$$

or

$$\begin{aligned} 3^{x-1} &= 10 \\ \therefore\ (x-1) &= \log_3(10) \\ x &= \log_3(10) + 1 \\ x &= 3.10 \end{aligned}$$

c

$$\begin{aligned} 0.2^{x+1} &= 0.6 \\ \therefore\ (x+1)\log 0.2 &= \log 0.6 \\ (x+1) &= \frac{\log 0.6}{\log 0.2} \\ x + 1 &= 0.32 \\ x &= -0.68 \end{aligned}$$

3 a $2^x > 8, \quad \therefore\ 2^x > 2^3$

$\therefore\ x > 3$

b $3^x < 5, \quad \therefore\ x\log 3 < \log 5$

$\therefore\ x < \dfrac{\log 5}{\log 3} < 1.46$

c

$0.3^x > 4, \quad \therefore\ x\log 0.3 < \log 4$

$x < \dfrac{\log 4}{\log 0.3}$

$\therefore\ x < \dfrac{\log 4}{\log 0.3} < -1.15$

d

$$3^{x-1} \le 7, \quad \therefore \quad (x-1)\log 3 \le \log 7$$

$$(x-1) \le \frac{\log 7}{\log 3}$$

$$(x-1) \le \frac{\log 7}{\log 3} = 1.77$$

$$\therefore \quad x \le 2.77$$

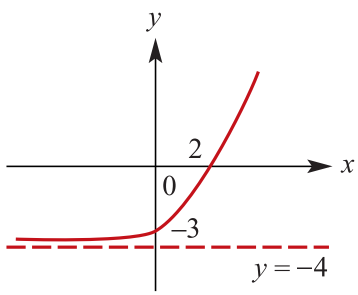
e $0.4^x \le 0.3, \quad \therefore \quad x \le 2.77$

$$\therefore \quad x \ge \frac{\log 0.3}{\log 0.4} \ge 1.31$$

4 a $f(x) = 2^x - 4$

Asymptote at $y = -4$,

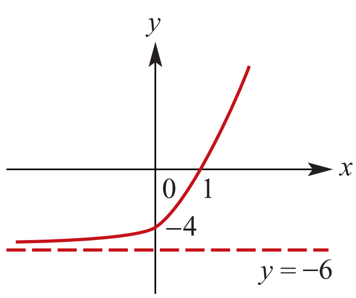
axis intercepts at $(0, -3)$ and $(2, 0)$



b $f(x) = 2(3^x) - 6$

Asymptote at $y = -6$,

axis intercepts at $(0, -4)$ and $(1, 0)$

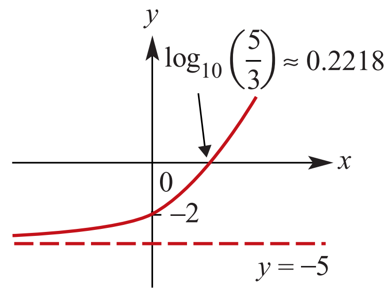


c $f(x) = 3(10^x) - 5$

Asymptote at $y = -5$,

axis intercepts at $(0, -2)$ and

$\left(\log_{10}\left(\frac{5}{3}\right), 0\right)$

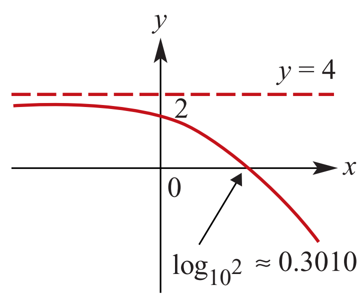


d $f(x) = -2(10^x) + 4$

Asymptote at $y = 4$,

axis intercepts at $(0, 2)$ and

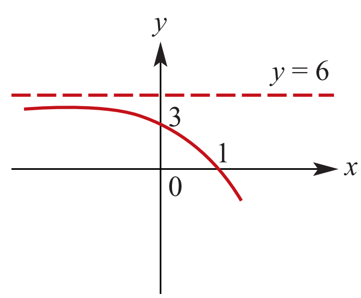
$(\log_{10} 2, 0)$



e $f(x) = -3(2^x) + 6$

Asymptote at $y = 6$,

axis intercepts at $(0, 3)$ and $(1, 0)$

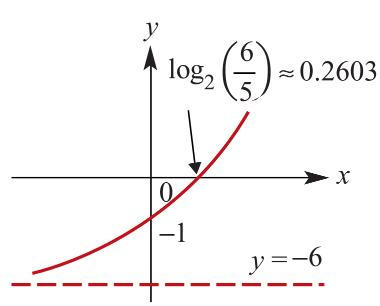


f $f(x) = 5(2^x) - 6$

Asymptote at $y = -6$,

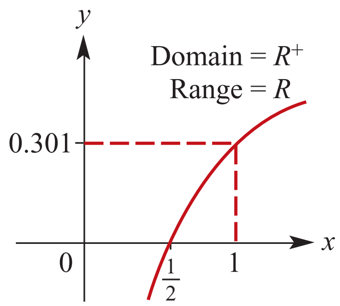
axis intercepts at $(0, -1)$ and

$(\log_2 1.2, 0)$

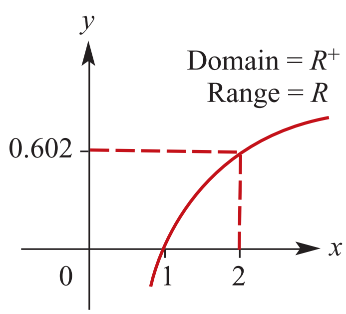


Solutions to Exercise 14H

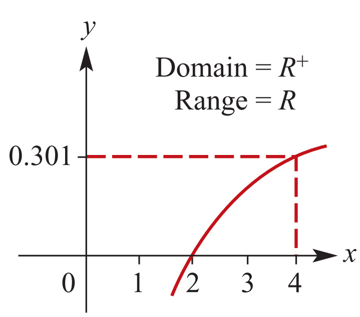
1 a $y = \log_{10}(2x)$; domain $(0, \infty)$, range R, x-intercept $\left(\frac{1}{2}, 0\right)$



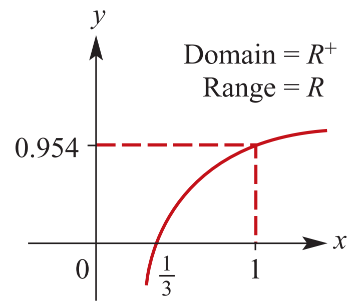
b $y = 2\log_{10} x$; domain $(0, \infty)$, range R, x-intercept $(1, 0)$



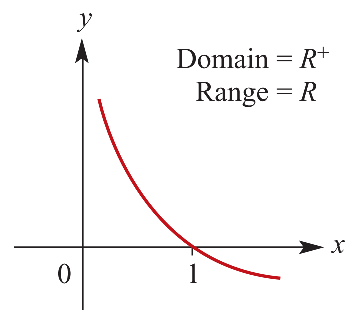
c $y = \log_{10}\left(\frac{x}{2}\right)$; domain $(0, \infty)$ range R, x-intercept $(2, 0)$



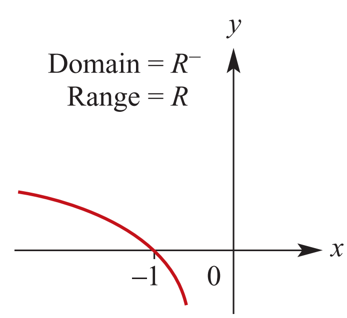
d $y = 2\log_{10}(3x)$; domain $(0, \infty)$, range R, x-intercept $\left(\frac{1}{3}, 0\right)$



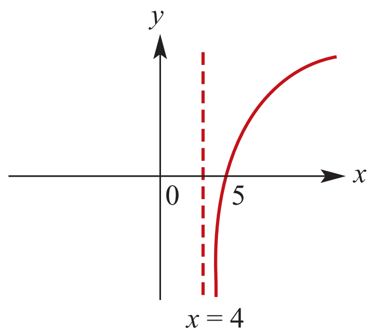
e $y = -\log_{10} x$; domain $(0, \infty)$, range R, x-intercept $(1, 0)$



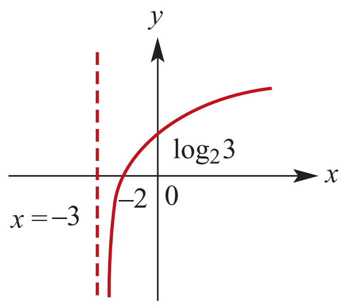
f $y = \log_{10}(-x)$ domain $(-\infty, 0)$, range R, x-intercept $(-1, 0)$



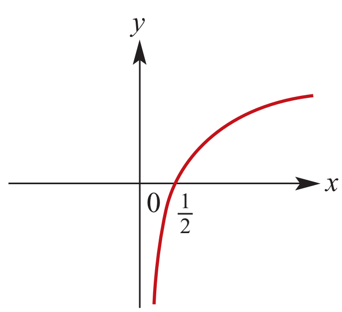
2 a $f(x) = \log_2(x - 4)$
Domain $(4, \infty)$, asymptote $x = 4$, x-intercept at (5,0)



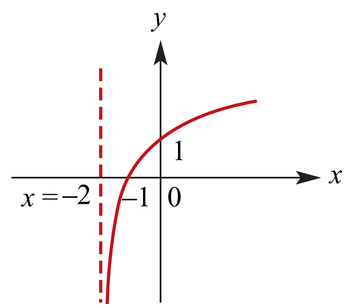
b $f(x) = \log_2(x + 3)$
Domain $(-3, \infty)$, asymptote $x = -3$, x-intercept at $(-2, 0)$, y-intercept at $(0, \log_2 3)$



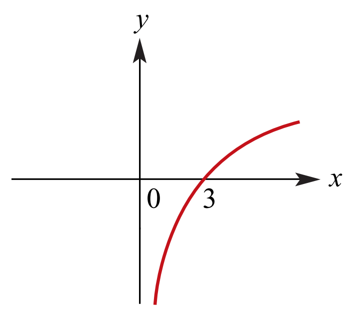
c $f(x) = \log_2(2x)$
Domain $(0, \infty)$, asymptote $x = 0$, x-intercept at $\left(\frac{1}{2}, 0\right)$



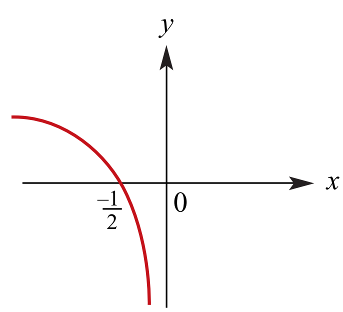
d $f(x) = \log_2(x + 2)$
Domain $(-2, \infty)$, asymptote $x = -2$, x-intercept at $(-1, 0)$, y-intercept at $(0, 1)$



e $f(x) = \log_2\left(\frac{x}{3}\right)$
Domain $(0, \infty)$, asymptote $x = 0$, x-intercept at $(3, 0)$



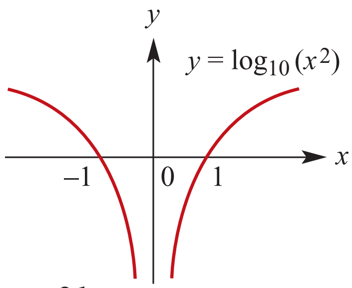
f $f(x) = \log_2(-2x)$
Domain $(-\infty, 0)$, asymptote $x = 0$, x-intercept at $\left(-\frac{1}{2}, 0\right)$



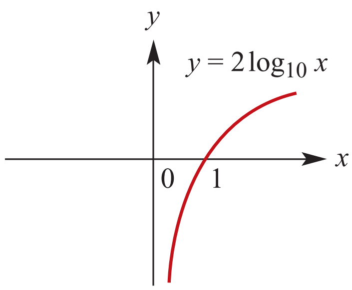
3 **a** $2^{-x} = x, \therefore\ x = 0.64$

b $\log_{10}(x) + x = 0, \therefore\ x = 0.40$

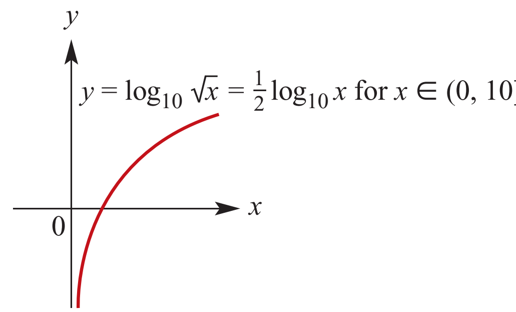
4 $y = \log_{10}(x^2)$;
$x \in [-10, 10],\ x \neq 0$



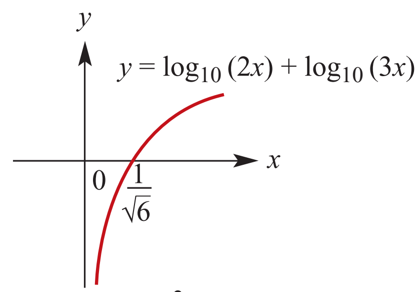
$y = 2\log_{10} x$;
$x \in [-10, 10],\ x \neq 0$



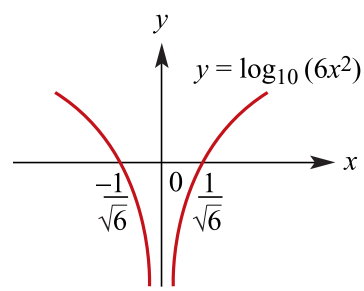
5 $y = \log_{10} \sqrt{x}$;
$x \in (0, 10],\ x \neq 0$
$y = \dfrac{1}{2}\log_{10} x$;
$x \in (0, 10],\ x \neq 0$



6 $y = \log_{10}(2x) + \log_{10}(3x)$



$y = \log_{10}(6x^2)$



Solutions to Exercise 14I

1 Let N be the number of bacteria at time t minutes.

a $N = 1000 \times 2^{\frac{t}{15}}$

b $10\,000 = 1000 \times 2^{\frac{t}{15}}$

$$10 = 2^{\frac{t}{15}}$$

$$\frac{t}{15} = \log_2 10$$

$$t = 49.8289...$$

$$t \approx 50.$$

It will take approximately 50 minutes

2 Choose $A(t) = A_0 \times 10^{-kt}$ as the model where $A_0 = 10$ is the original amount and t is the time in years.

First find k:

$$5 = 10 \times 10^{-24\,000k}$$

$$\log_{10} \frac{1}{2} = -24\,000k$$

$$k = -\frac{1}{24\,000} \log_{10} \frac{1}{2}k \quad = 1.254296... \times 10^{-5}$$

If $A(t) = 1$

$$1 = 10 \times 10^{-kt}$$

$$0.1 = 10^{-kt}$$

$$\therefore kt = 1$$

$$\therefore t = \frac{1}{1.254296 \times 10^{-5}}$$

$$t \approx 79\,726.$$

It will take 79 726 years for there to be 10% of the original.

3 Choose $A(t) = A_0 \times 10^{-kt}$ as the model where A_0 is the original amount and t is the time in years.

First find k:

$$\frac{1}{2}A_0 = A_0 \times 10^{-5730k}$$

$$\log_{10}\frac{1}{2} = -5730k$$

$$k = -\frac{1}{5730}\log_{10}\frac{1}{2}$$

$$k = 5.2535...... \times 10^{-5}$$

When $A(t) = 0.4A_0$

$$0.4A_0 = A_0 \times 10^{-kt}$$

$$0.4 = 10^{-kt}$$

$$\therefore kt = \log_{10} 0.4$$

$$\therefore t = \frac{1}{5.2535... \times 10^{-5}} \times \log_{10} 0.4$$

$$t \approx 7575$$

It is approximately 7575 years old.

4 $P(h) = 1000 \times 10^{-0.0542h}$

a $P(5) = 1000 \times 10^{-0.0542\times 5}$

$$= 535.303\ldots$$

$P(h) \approx 535$millibars

b If $P(h) = 400$

Then $400 = 1000 \times 10^{-0.05428h}$

$$\frac{2}{5} = 10^{-0.05428h}$$

$$\log 10\left(\frac{2}{5}\right) = -0.05428h$$

$h \approx 7331$ metres correct to the nearest metre

5 $N(t) = 500\,000(1.1)^t$ where $N(t)$ is the number of bacteria at time t

$$4\,000\,000 = 500\,000(1.1)^t$$

$$8 = 1.1^t$$

$$t = 21.817\ldots$$

The number will exceed 4 million bacteria after 22 hours.

6 $T = T_0 10^{-kt}$

When $t = 0, T = 100$. Therefore $T_0 = 100$

We have $T = 100 \times 10^{-kt}$
When $t = 5, T = 40$
$\therefore 40 = 100 \times 10^{-5k}$

$$\frac{2}{5} = 10^{-5k}$$

$$k = -\frac{1}{5}\log 10\frac{2}{5}$$

$$k = 0.07958\ldots$$

When $t = 15$
$T = 100 \times 10^{-15k} \quad = 6.4$
The temperature is 6.4°C after 15 minutes.

7 $\quad A(t) = 0.9174^t$

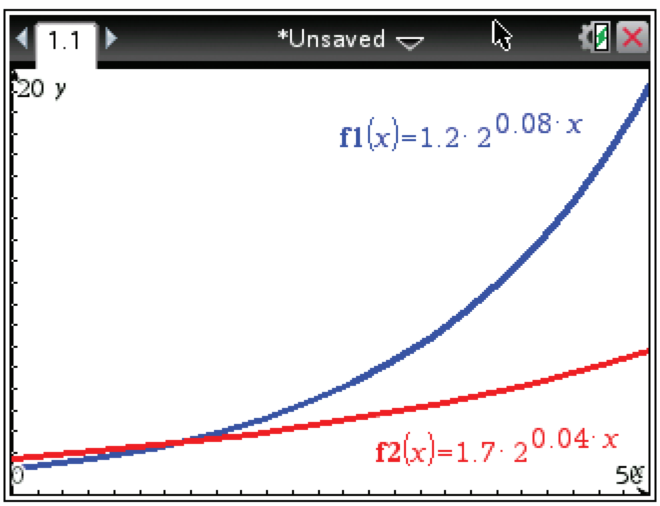
When $A(t) = 0.2$

$$0.2 = 0.9174^t$$

$$t = 18.668\ldots$$

$t > 18.668\ldots$

8 **a**



b **i**

$$p = q$$

$$\Leftrightarrow \quad 2^{0.04t} = \frac{17}{12}$$

$$\therefore t = 12.56 \qquad \text{(mid 1962)}$$

ii Solve the equation $p = 2q$

i.e.
$$1.2 \times 2^{0.08t} = 2(1.7 \times 2^{0.04t})$$
$$\frac{6}{17} \times 2^{0.04t} = 1$$
$$2^{0.04t} = \frac{17}{6}$$
$$t = 37.56 \qquad \text{(mid 1987)}$$

9 a We can write

$$a \times b^1 = 15 \qquad (1)$$
$$a \times b^4 = 1875 \qquad (2)$$

Dividing equation (2) by equation (1) gives $b^3 = 125$. Thus $b = 5$, and substituting into equation (1) gives $a = 3$.
$\therefore y = 3 \times 5^x$

b We can write

$$a \times b^2 = 1 \qquad (1)$$
$$a \times b^5 = \frac{1}{8} \qquad (2)$$

Dividing equation (2) by equation (1) gives $b^3 = \frac{1}{8}$. Thus $b = \frac{1}{2}$, and substituting into equation (1) gives $a = 4$.
$\therefore y = 4 \times (\frac{1}{2})^x$

c We can write

$$a \times b^1 = \frac{15}{2} \qquad (1)$$
$$a \times b^{\frac{1}{2}} = \frac{5\sqrt{6}}{2} \qquad (2)$$

Dividing equation (2) by equation (1) gives $b^{-\frac{1}{2}} = \frac{\sqrt{6}}{3}$. Thus $b = \frac{3}{2}$, and substituting into equation (1) gives $a = 5$.
$y = 5 \times (\frac{3}{2})^x$

10 $S = 5 \times 10^{-kt}$

a
$$\begin{aligned} S &= 3.2 \text{ when } t = 2 \\ 3.2 &= 5 \times 10^{-2k} \\ 0.64 &= 10^{-2k} \\ k &= -\frac{1}{2}\log_{10} 0.64 \\ &= 0.0969\ldots \end{aligned}$$

b When $S = 1$
$$\begin{aligned} 1 &= 5 \times 10^{-0.9969\ldots t} \\ 10^{(-0.0969\ldots)t} &= 0.2 \\ (-0.0969\ldots)t &= \log_{10} 0.2 \\ t &= 7.212... \end{aligned}$$
There will be 1 kg of sugar remaining after approximatel 7.21 hours

11 a When $t = 0, N = 1000$
$$\begin{aligned} N &= ab^t \\ 1000 &= ab^0 \\ a &= 1000 \end{aligned}$$
When $t = 5, N = 10\,000$
$$\begin{aligned} \therefore 10 &= b^5 \\ \therefore b &= 10^{\frac{1}{5}} \\ \therefore N &= 1000 \times 10^{\frac{t}{5}} \end{aligned}$$

b When $N = 5000$
$$\begin{aligned} 5 &= 10^{\frac{t}{5}} \\ \frac{t}{5} &= \log_{10} 5 \\ t &= 5\log_{10} 5 \\ &\approx 3.4948 \text{ hours} \\ &= 210 \text{ minutes} \end{aligned}$$

c When $N = 1\,000\,000$

$$\begin{aligned} 1000 &= 10^{\frac{t}{5}} \\ \frac{t}{5} &= \log_{10} 1000 \\ t &= 5 \times 3 \\ &= 15 \text{ hours} \end{aligned}$$

d $N(12) = 1000 \times 10^{\frac{12}{5}} \approx 251188.64$

12 We can write

$a \times 10^{2k} = 6 \qquad (1)$

$a \times 10^{5k} = 20 \qquad (2)$

Dividing equation (2) by equation (1) gives $10^{3k} = \frac{10}{3}$. Thus $k = \frac{1}{3}\log_{10}\frac{10}{3}$, and substituting into equation (1) gives $a = 6 \times \left(\frac{10}{3}\right)^{-\frac{2}{3}}$.

13 Use two points, say $(0, 1.5)$ and $(10, 0.006)$ to find $y = ab^x$.

$$\begin{array}{ll} \text{at } (0, 1.5) & 1.5 = a \times b^0 \\ \therefore & 1.5 = a \\ \therefore & y = 1.5b^x \\ \text{at } (10, 0.006) & 0.006 = 1.5b^{10} \\ \therefore & b^{10} = \dfrac{0.006}{1.5} \\ & \phantom{b^{10}} = 0.004 \\ \therefore & b = (0.004)^{\frac{1}{10}} \approx 0.5757 \\ \therefore & y = 1.5 \times 0.58^x \end{array}$$

If CAS is used with exponential regression, $a = 1.5$ and $b = 0.575$, so $y = 1.5(0.575)^x$

14 Use two points, say (0, 2.5) and (8, 27.56) to find $p = ab^t$.

$$
\begin{aligned}
&\text{at } (0, 2.5) && 2.5 = a \times b^0 \\
&\therefore && 2.5 = a \\
&\therefore && p = 2.5b^t \\
&\text{at } (8, 27.56) && 27.56 = 2.5b^8 \\
&\therefore && b^8 = \frac{27.56}{2.5} \\
& && = 11.024 \\
&\therefore && b = (11.024)^{\frac{1}{8}} \\
& && \approx 1.3499 \\
&\therefore && p = 2.5 \times 1.35^t
\end{aligned}
$$

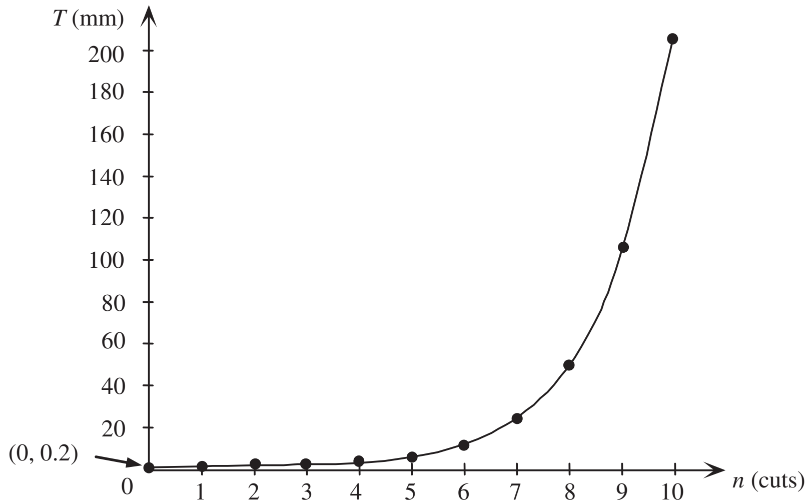
If CAS is used with exponential regression, $a = 1.5$ and $b = 0.575$, so $y = 1.5(0.575)^x$

15 **a**

Cuts, n	Sheets	Total thickness, T (mm)
0	1	0.2
1	2	0.4
2	4	0.8
3	8	1.6
4	16	3.2
5	32	6.4
6	64	12.8
7	128	25.6
8	256	51.2
9	512	102.4
10	1024	204.8

b $T = 0.2 \times 2^n$

c



d When $n = 30$, $T = 0.2 \times 2^{30}$

$= 214\ 748\ 364.8$

Total thickness is 214 748364.8 mm = 214 748.4m

16 $d = d_0(10^{mt})$

$d(1) = 52; d(3) = 80$

$\therefore\ d_0(10^m) = 52; d_0(10^{3m}) = 80$

Take $\log_{10}$ both equations:

(1): $\log_{10} d_0 + m \log_{10} 10 = \log_{10} 52$

$\therefore\ \log_{10} d_0 + m = \log_{10} 52$

(2): $\log_{10} d_0 + 3m \log_{10} 10 = \log_{10} 80$

$\therefore\ \log_{10} d_0 + 3m = \log_{10} 80$

(2)–(1) gives

$$2m = \log_{10}\left(\frac{80}{52}\right)$$

$$\therefore\ m = \frac{1}{2}\log_{10}\left(\frac{20}{13}\right) = 0.0935$$

Substitute into **(1):**

$$\log_{10} d_0 = \log_{10} 52 - 0.0935$$

$$= \log_{10} 52 - \log_{10}(10^{0.0935})$$

$$= \log_{10}\left(\frac{52}{1.240}\right) = \log_{10} 41.92$$

$\therefore\ d_0 = 41.92$ cm

Solutions to Review: Short-answer questions

1 **a** $\dfrac{a^6}{a^2} = a^{6-2} = a^4$

b $\dfrac{b^8}{b^{10}} = b^{8-10}$

$= b^{-2} = \dfrac{1}{b^2}$

c $\dfrac{m^3n^4}{m^5n^6} = m^{3-5}n^{4-6}$

$= m^{-2}n^{-2} = \dfrac{1}{m^2n^2}$

d $\dfrac{a^3b^2}{(a^*b^2)^4} = \dfrac{a^3b^2}{a^4b^8}$

$= a^{3-4}b^{2-8} = \dfrac{1}{ab^6}$

e $\dfrac{6a^8}{4a^2} = \left(\dfrac{6}{4}\right)a^{8-2} = \dfrac{3a^6}{2}$

f $\dfrac{10a^7}{6a^9} = \left(\dfrac{10}{6}\right)a^{7-9} = \dfrac{5}{3a^2}$

g $\dfrac{8(a^3)^2}{(2a)^3} = \dfrac{8a^6}{8a^3}$

$= a^{6-3} = a^3$

h $\dfrac{m^{-1}n^2}{(mn^{-2})^3} = \dfrac{m^{-1}n^2}{m^3n^{-6}}$

$= m^{-1-3}n^{2+6} = \dfrac{n^8}{m^4}$

i $({}^2p^{-1}q^{-2}) = p^{-2}q^{-4} = \dfrac{1}{p^2q^4}$

j $\dfrac{(2a^{-4})^3}{5a^{-1}} = \dfrac{8a^{-12}}{5a^{-1}}$

$= \dfrac{8a^{1-12}}{5} = \dfrac{8}{5a^{11}}$

k $\dfrac{6a^{-1}}{3a^{-2}} = \left(\dfrac{6}{3}\right)a^{-1+2} = 2a$

l $\dfrac{a^4 + a^8}{a^2} = \dfrac{a^2(a^2 + a^6)}{a^2}$

$= a^2 + a^6$

m $\dfrac{a^4 + a^8}{a^2} = \dfrac{a^4}{a^2}(1 + a^4)$

$= a^2(1 + a^4) = a^2 + a^6$

2 $32 \times 10^{11} \times 12 \times 10^{-5}$

$= (32 \times 12) \times 10^{11-5}$

$= 384 \times 10^6$

$= 3.84 \times 10^8$

3 1 L (1000 mL) of blood contains 5×10^{12} red blood cells so 500 mL of blood contains 2.5×10^{12} red blood cells.
Thus, the time required is equal to
$\dfrac{2.5 \times 10^{12}}{2.5 \times 10^6} = 1.0 \times 10^6$ seconds.

4 $\dfrac{1.5 \times 10^8}{3 \times 10^6} = 0.5 \times 10^2$

$= 50$

The Sun is 50 times further from Earth than the comet.

5 **a** $2^x = 7, \therefore\ x = \log_2 7$

b $2^{2x} = 7, 2x = \log_2 7$

$\therefore x = \dfrac{1}{2}\log_2 7$

c $10^x = 2, \therefore\ x = \log_{10} 2$

d $10^x = 3.6, \therefore\ x = \log_{10} 3.6$

e $10^x = 110, \therefore\ x = \log_{10} 110$ (or $1 + \log_{10} 11$)

f $10^x = 1010, \therefore\ x = \log_{10} 1010$ (or $1 + \log_{10} 101$)

g $2^{5x} = 100, \therefore\ 5x = \log_2 100$
$\therefore\ x = \dfrac{1}{5}\log_2 100$

h $2^x = 0.1, \therefore\ x = \log_2 0.1$
$= -\log_2 10$

6 **a** $\log_2 64 = \log_2 2^6$
$= 6\log_2 2 = 6$

b $\log_{10} 10^7 = 7\log_{10} 10 = 7$

c $\log_a a^2 = 2\log_a a = 2$

d $\log_4 1 = 0$ by definition

e $\log_3 27 = \log_3 3^3$
$= 3\log_3 3 = 3$

f $\log_2 \dfrac{1}{4} = \log_2 2^{-2}$
$= -2\log_2 2 = -2$

g $\log_{10} 0.001 = \log_{10} 10^{-3}$
$= -3\log_{10} 10 = -3$

h $\log_2 16 = \log_2 2^4$
$= 4\log_2 2 = 4$

7 **a** $\log_{10} 2 + \log_{10} 3 = \log_{10}(2 \times 3) = \log_{10} 6$

b $\log_{10} 4 + 2\log_{10} 3 - \log_{10} 6$
$= \log_{10} 4 + \log_{10}(3^2) - \log_{10} 6$
$= \log_{10} \dfrac{4(3^2)}{6} = \log_{10} 6$

c
$2\log_{10} a - \log_{10} b = \log_{10} a^2 - \log_{10} b$
$= \log_{10}\left(\dfrac{a^2}{b}\right)$

d
$2\log_{10} a - 3 - \log_{10} 25$
$= \log_{10} a^2 - \log_{10} 25 - \log_{10} 10^3$
$= \log_{10}\left(\dfrac{a^2}{25000}\right)$

e $\log_{10} x + \log_{10} y - \log_{10} x = \log_{10} y$

f $2\log_{10} a + 3\log_{10} b - \log_{10} c$
$= \log_{10} a^2 + \log_{10} b^3 - \log_{10} c$
$= \log_{10}\left(\dfrac{a^2b^3}{c}\right)$

8 **a** $3^x(3^x - 27) = 0$
$3^x = 27, \therefore\ x = 3$
($3^x \neq 0$ for any real x)

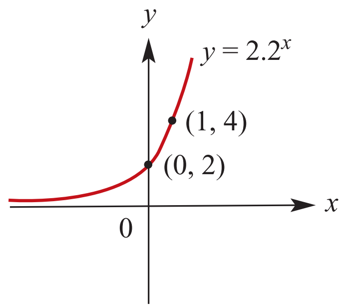
b $(2^x - 8)(2^x - 1) = 0$
$2^x = 1, 8, \therefore\ x = 0, 3$

c $2^{2x} - 2^{x+1} = 0$
$(2^x)(2^x - 2) = 0$
$2^x = 2, \therefore\ x = 1$
($2^x \neq 0$ for any real x)

d $2^{2x} - 12(2^x) + 32 = 0$
$(2^x - 8)(2^x - 4) = 0$
$2^x = 4, 8, \therefore\ x = 2, 3$

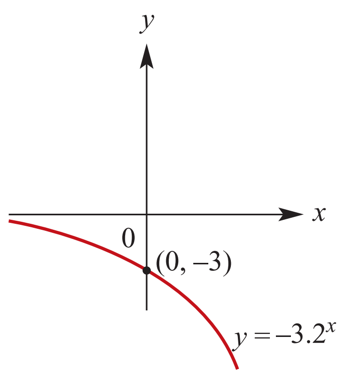
9 a $y = 2 \times 2^x$

Asymptote at $y = 0$, y-intercept at $(0, 1)$



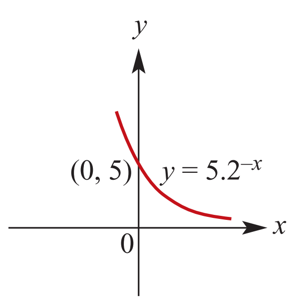
b $y = -3 \times 2^x$

Asymptote at $y = 0$, y-intercept at $(0, 1)$



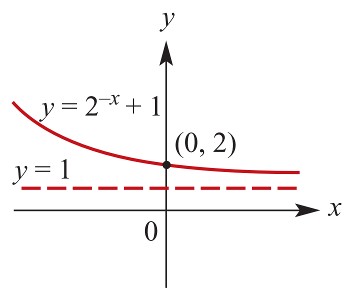
c $y = 5 \times 2^{-x}$

Asymptote at $y = 0$, y-intercept at $(0, 1)$



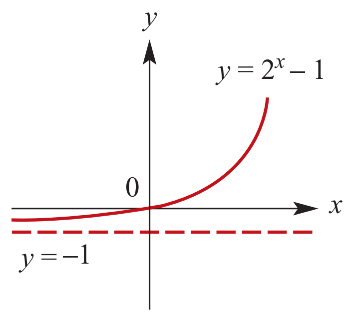
d $y = 2^{-x} + 1$

Asymptote at $y = 1$, y-intercept at $(0, 2)$



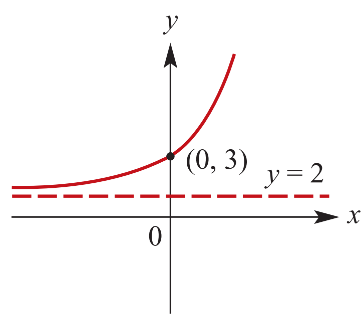
e $y = 2^x - 1$

Asymptote at $y = -1$, y-intercept at $(0, 0)$



f $y = 2^x + 2$

Asymptote at $y = 2$, y-intercept at $(0, 3)$



10

$$\begin{aligned}
\log_{10} x + \log_{10} 2x - \log_{10}(x+1) &= 0 \\
\therefore \quad \log_{10} \frac{2x^2}{x+1} &= 0 \\
\frac{2x^2}{x+1} &= 1 \\
2x^2 &= x + 1 \\
2x^2 - x - 1 &= 0 \\
(2x+1)(x-1) &= 0
\end{aligned}$$

$\therefore\ x = -\frac{1}{2}, 1$

Since $\log x$ is not defined for $x \le 0,\ x = 1$

11 **a** $2(4^{a+1}) = 16^{2a}$

$\therefore\ 4^{\frac{1}{2}}(4^{a+1}) = 4^{4a}$

$4^{a+\frac{3}{2}} = 4^{4a}$

$a + \frac{3}{2} = 4a$

$3a = \frac{3}{2},\ \therefore\ a = \frac{1}{2}$

b $\log_2 y^2 = 4 + \log_2(y + 5)$

$\therefore\ \log_2 y^2 - \log_2(y + 5) = 4$

$\log_2\left(\frac{y^2}{y+5}\right) = 4$

$\frac{y^2}{y+5} = 2^4$

$y^2 = 16y + 80$

$y^2 - 16y - 80 = 0$

$(y - 20)(y + 4) = 0$

$\therefore\ y = -4, 20$

(Both solutions must be included here, because the only domain restriction is that $y > -5$)

Solutions to Review: Multiple-choice questions

1 C $\dfrac{8x^3}{4x^{-3}} = \dfrac{8}{4}x^{3+3} = 2x^6$

2 A $\dfrac{a^2b}{(2ab^2)^3} \div \dfrac{ab}{16a^0} = \dfrac{a^2b}{8a^3b^6}\dfrac{16}{ab}$

$$= \frac{16}{8}a^{2-3-1}b^{1-6-1}$$
$$= 2a^{-2}b^{-6}$$
$$= \frac{2}{a^2b^6}$$

3 C The range of $y = 3 \times 2^x$ is $(0, \infty)$ but $f(x) = 3(2^x) - 1$ is translated 1 unit down

$\therefore$ range $= (-1, \infty)$

4 A $\log_{10}(x-2) - 3\log_{10} 2x = 1 - \log_{10} y$

$$\therefore \log_{10}\frac{x-2}{(2x)^3} + \log_{10} y = 1$$
$$\log_{10}\frac{y(x-2)}{8x^3} = 1$$
$$\frac{y(x-2)}{8x^3} = 10$$
$$\therefore y = \frac{80x^3}{x-2}$$

5 B $5(2^{5x}) = 10, \therefore 2^{5x} = 2^1$

$$\therefore 5x = 1, \therefore x = \frac{1}{5}$$

6 A The vertical asymptote of $y = \log x$ is at $x = 0$. Here $5x = 0$ so $x = 0$. (y-direction translations don't affect the vertical asymptote.)

7 A $f(x) = 2^{ax} + b;\ a, b > 0$
Function must be increasing, with a horizontal asymptote at $y = b$ which the graph approaches at large negative values of x, and there will be no x-intercept because $b > 0$

8 A Vertical asymptote, hence log or hyperbola. But B and C both have a vertical asymptote $x = -b$.

9 A $\dfrac{2mh}{(3mh^2)^3} \div \dfrac{mh}{81m^2} = \dfrac{2mh}{27m^3h^6}\dfrac{81m^2}{mh}$

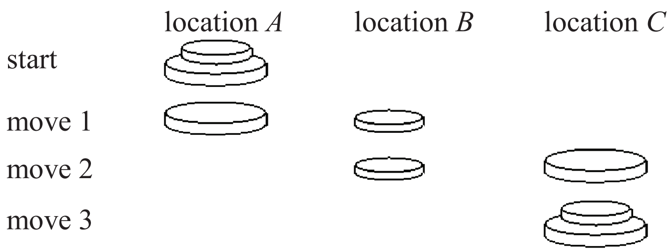
$$= 6m^{1+2-3-1}h^{1-6-1}$$
$$= 6m^{-1}h^{-6}$$
$$= \frac{6}{mh^6}$$

Solutions to Review: Extended-response questions

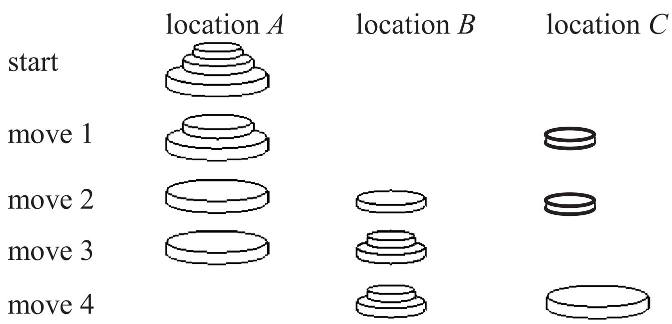
1 a

Number of discs, n	0	1	2	3	4
Minimum no. of moves, M	0	1	3	7	15

For two discs, the following procedure may be used.



For three discs, the procedure is as follows.



Now the problem reduces to taking the two discs from B to C, i.e. three more moves (using the technique for two discs).

$$\therefore \quad \text{total number of moves} = 3 + 4 = 7$$

This procedure can be generalised for n discs.

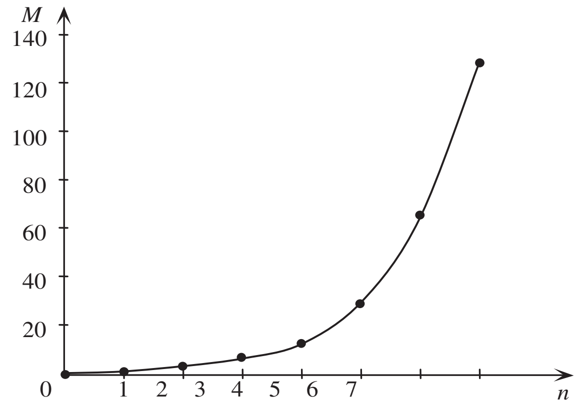
- The top $n - 1$ discs can be moved from A to B in $2^{n-1} - 1$ moves.
- The remaining bottom disc can be moved from A to C.
- The $n - 1$ discs on B can be moved to C in $2^{n-1} - 1$ moves.

$$\begin{aligned}\therefore \quad \text{total number of moves} &= 2^{n-1} - 1 + 1 + 2^{n-1} - 1 \\ &= 2 \times 2^{n-1} - 1 \\ &= 2^n - 1\end{aligned}$$

b $M = 2^n - 1$

Number of discs, n	0	5	6	7
Minimum no. of moves, M	0	31	63	127

c



d Let the top disc be called D_1, the next D_2, then D_3 and so on to nth disc, D_n.
For 3 discs, D_1 moves 4 times, D_2 2 times and D_3 once.
For 4 discs, D_1 moves 8 times, D_2 4 times, D_3 2 times and D_4 once.
For n discs, D_1 moves 2^{n-1} times, D_2 2^{n-2} times, ..., D_n 2^0 times.

Three discs	D_1	D_2	D_3
Times moved	4	2	1

Four discs	D_1	D_2	D_3	D_4
Times moved	8	4	2	1

n discs	D_1	D_2	D_3	...	D_{n-1}	D_n
Times moved	2^{n-1}	2^{n-2}	2^{n-3}		2^1	2^0

Note: For n discs, total number of moves $= 1 + 2 + 4 + \ldots + 2^{n-1}$

$$= \frac{1(2^n - 1)}{2 - 1} = 2^n - 1$$

2 $$2187 = 9 \times 9 \times 9 \times 3 = 9^3 \times 3^1$$

This gives 3 switches of Type 1 and 1 switch of Type 2.
However, if n of Type 1 and $n + 1$ of Type 2 are used, there needs to be one more 3 than the number of 9s in the factorisation.

$$2187 = 9 \times 9 \times 3 \times 3 \times 3 = 9^2 \times 3^3$$

Two switches of Type 1 and three of Type 2 are needed. Hence, $n = 2$.

3 **a** $F = \dfrac{6.67 \times 10^{-11} \times 200 \times 200}{12^2}$

$= 1.8528 \times 10^{-8}$

$= 1.9 \times 10^{-8}$ (to 2 s.f.)

b $$F = \frac{6.67 \times 10^{-11} m_1 m_2}{r^2}$$

$$\therefore\ Fr^2 = 6.67 \times 10^{-11} m_1 m_2$$

$$\therefore\ m_1 = \frac{Fr^2}{6.67 \times 10^{-11} m_2}$$

c Using the formula in part **b**, substitute in $F = 2.4 \times 10^4$, $r = 6.4 \times 10^6$ and $m_2 = 1500$:

$$\begin{aligned} m_1 &= \frac{Fr^2}{6.67 \times 10^{-11} m_2} \\ &= \frac{2.4 \times 10^{-11} \times \left(6.4 \times 10^6\right)^2}{6.67 \times 10^{-11} \times 1500} \\ &= 9.8 \times 10^{24} \text{ kg (to 2 s.f.)} \end{aligned}$$

4 a $$\left(\frac{1}{8}\right)^n = \left(\left(\frac{1}{2}\right)^3\right)^n = \left(\frac{1}{2}\right)^{3n}$$

b $$\begin{aligned} \left(\frac{1}{4}\right)^{n-1}\left(\frac{1}{2}\right)^{3n} &= \left(\left(\frac{1}{2}\right)^2\right)^{n-1}\left(\frac{1}{2}\right)^{3n} \\ &= \left(\frac{1}{2}\right)^{2(n-1)}\left(\frac{1}{2}\right)^{3n} \\ &= \left(\frac{1}{2}\right)^{2n-2}\left(\frac{1}{2}\right)^{3n} = \left(\frac{1}{2}\right)^{5n-2} \end{aligned}$$

c $$\left(\frac{1}{2}\right)^{n-3}\left(\frac{1}{2}\right)^{5n-2} = \left(\frac{1}{2}\right)^{6n-5}$$

Now, $$\left(\frac{1}{2}\right)^{6n-5} = \frac{1}{8192} = \frac{1}{2^{13}} = \left(\frac{1}{2}\right)^{13}$$

$$\therefore\quad 6n - 5 = 13$$

$$\therefore\quad 6n = 18 \quad \therefore\ n = 3$$

5 a

Times used	1	2	3	n
Caffeine remaining	$729\left(\frac{1}{4}\right)^1$	$729\left(\frac{1}{4}\right)^2$	$729\left(\frac{1}{4}\right)^3$	$729\left(\frac{1}{4}\right)^n$

b

Times used	1	2	3	n
Tannin remaining	$128\left(\frac{1}{2}\right)^1$	$128\left(\frac{1}{2}\right)^2$	$128\left(\frac{1}{2}\right)^3$	$128\left(\frac{1}{2}\right)^n$

c Can be re-used if amount of tannin ≤ 3× amount of caffeine.

i.e. $$128\left(\frac{1}{2}\right)^n \le 3 \times 729\left(\frac{1}{4}\right)^n$$

$$\Leftrightarrow \quad 128\left(\frac{1}{2}\right)^n \le 2187\left(\frac{1}{2}\right)^{2n}$$

$$\Leftrightarrow \quad \frac{128}{2187} \le \frac{\left(\frac{1}{2}\right)^{2n}}{\left(\frac{1}{2}\right)^n}$$

$$\Leftrightarrow \quad \frac{128}{2187} \le \left(\frac{1}{2}\right)^n$$

$$\Leftrightarrow \quad \log_{10}\left(\frac{128}{2187}\right) \le \log_{10}\left(\frac{1}{2}\right)^n$$

$$\Leftrightarrow \quad \log_{10}\left(\frac{128}{2187}\right) \le n\log_{10}\left(\frac{1}{2}\right)$$

$$\Leftrightarrow \quad \frac{\log_{10}\left(\frac{128}{2187}\right)}{\log_{10}\left(\frac{1}{2}\right)} \ge n \text{ as } \log_{10}\left(\frac{1}{2}\right) < 0$$

$$\therefore \quad n \le 4.09$$

Hence, the tea leaves can be re-used 4 times.

6 **a** Brightness Batch 1 after n years = $15(0.95)^n$
Brightness of Batch 2 after n years = $20(0.94)^n$

b Let n be the number of years until brightness is the same.

$$\begin{aligned}
15(0.95)^{n+1} &= 20(0.94)^n \\
\frac{(0.95)^{n+1}}{(0.94)^n} &= \frac{20}{15} \\
\log_{10}\left(\frac{(0.95)^{n+1}}{(0.94)^n}\right) &= \log_{10}\left(\frac{4}{3}\right) \\
\therefore \quad \log_{10}(0.95)^{n+1} - \log_{10}(0.94)^n &= \log_{10}\left(\frac{4}{3}\right) \\
(n+1)\log_{10}(0.95) - n\log_{10}(0.94) &= \log_{10}\left(\frac{4}{3}\right) \\
n\log_{10}(0.95) + \log_{10}(0.95) - n\log_{10}(0.94) &= \log_{10}\left(\frac{4}{3}\right) \\
n(\log_{10}(0.95) - \log_{10}(0.94)) &= \log_{10}\left(\frac{4}{3}\right) - \log_{10}(0.95) \\
n\log_{10}\left(\frac{0.95}{0.94}\right) &= \log_{10}\left(\frac{4}{3\times 0.95}\right) \\
n &= \frac{\log_{10}\left(\frac{400}{285}\right)}{\log_{10}\left(\frac{95}{94}\right)} \\
&= 32.033
\end{aligned}$$

Hence, the brightness is the same early in the 33rd year (i.e. after about 32 years).

7 Let W be the number of wildebeest and n the number of years.

Then $\quad W = 700(1.03)^n$

Let Z be the number of zebras.

Then
$$\begin{aligned}
Z &= (0.96)^n \times 1850 \\
&= 1850(0.96)^n
\end{aligned}$$

a
$$\begin{aligned}
(0.96)^n \times 1850 &= 700(1.03)^n \\
\frac{1850}{700} &= \left(\frac{1.03}{0.96}\right)^n \\
\frac{37}{14} &= \left(\frac{103}{96}\right)^n \\
\therefore \quad n &= 13.81
\end{aligned}$$

After 13.81 years, the number of wildebeest exceeds the number of zebras.

b Let A be the number of antelopes.

$$A = 1000 + 50n$$

The number of antelopes is greater than the number of zebras when

$$1000 + 50n > 1850(0.96)^n$$

From a CAS calculator,

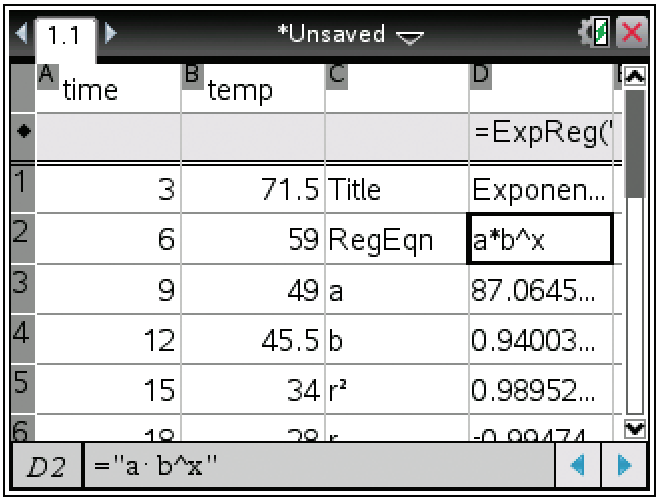
$$1000 + 50n > 1850(0.96)^n \text{ for } n > 7.38$$

After 7.38 years, the number of antelopes exceeds the number of zebras.

8 a TI: Type the given data into a new Lists & Spreadsheet application. Call column A *time*, and column B *temp*

Press **Menu → 4:Statistics → 1:Stat Calculations → A:Exponential Regression**

Set **X List** to **time** and **Y List** to **temp** then ENTER



CP: Open the Statistics application. Type the time data into list1 and the temperature data into list2

Tap **Calc → abExponential Reg** and set **XList** to **list1** and **YList** to **list2**

The values of a and b are given as $a = 87.06$ and $b = 0.94$, correct to 2 decimal places,

$$\therefore \quad T = 87.06 \times 0.94^t$$

b **i** When $t = 0$, $T = 87.06°\text{C}$

ii When $t = 25$, $T = 18.56°\text{C}$

c $(12, 45.5)$ is the reading which appears to be incorrect.

Re-calculating gives $a = 85.724\ldots$ and $b = 0.9400$

$$\therefore \quad T = 85.72 \times 0.94^t$$

d **i** When $t = 0$, $T = 85.72°\text{C}$

ii When $t = 12$, $T = 40.82°\text{C}$

e When $T = 15$, $t = 28.19$ min

9 **a** At $(1, 1)$ $1 = a \times b^1$

$\therefore$ $1 = ab$ (1)

At $(2, 5)$ $5 = a \times b^2$ (2)

Divide (2) by (1) $5 = b$

Substitute $b = 5$ into (1) $1 = a \times 5$

$\therefore$ $a = \dfrac{1}{5} = 0.2$

$\therefore$ $a = 0.2$, $b = 5$

b Let $b^x = 10^z$

i By the definition of logarithm:

$$z = \log_{10} b^x$$

$$\therefore \quad = x \log_{10} b$$

ii

$$y = a \times 10^{kx}$$

$$= a \times b^x \quad \text{where } b^x = 10^{kx}$$

From **b i**, $b^x = 10^{kx}$ can be rewritten

$$kx = x \log_{10} b$$

$$\therefore \quad k = \log_{10} b$$

From **a**, $a = 0.2$ and $b = 5$, $\therefore$ $k = \log_{10} 5$

10 **a** Use two points, say $(0, 2)$ and $(10, 200)$ to find $y = ab^x$.

At $(0, 2)$ $\quad 2 = a \times b^0$

$\therefore \quad 2 = a$

$\therefore \quad y = 2b^x$

At$(10, 200)$ $\quad 200 = 2b^{10}$

$\therefore \quad b^{10} = \dfrac{200}{2} = 100$

$\therefore \quad b = (100)^{\frac{1}{10}}$

$= 1.5849$ (correct to 4 decimal places)

$\therefore \quad y = 2 \times 1.5849^x$

Using CAS regression $y = 2 \times 1.585^x$

b From Question 9, $\quad k = \log_{10} b$

and from part **a**, $\quad a = 2$ and $b = (100)^{\frac{1}{10}}$

$\therefore \quad k = \log_{10}(100)^{\frac{1}{10}}$

$= \dfrac{1}{10}\log_{10} 100$

$= \dfrac{1}{10} \times 2 = \dfrac{1}{5}$

$\therefore \quad y = 2 \times 10^{\frac{x}{5}} = 2 \times 10^{0.2x}$

c $\quad y = 2 \times 10^{\frac{x}{5}}$

can be written $\quad \dfrac{y}{2} = 10^{\frac{x}{5}}$

By definition of logarithms:

$\dfrac{x}{5} = \log_{10}\left(\dfrac{y}{2}\right)$

$x = 5\log_{10}\left(\dfrac{y}{2}\right)$