Christ Church Grammar School

Mathematics Department
Year 11 Mathematics Methods

Semester 2 Examination, 2019

Question/Answer booklet

MATHEMATICS METHODS UNITS 1 AND 2
Section One: Calculator-free

SOLUTIONS

Student name: ______________________________

Time allowed for this section

Reading time before commencing work:	five minutes
Working time:	fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Christ Church Grammar School assessments are detailed in the Reporting and Assessment Policy. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answer to the specific question asked and to follow any instructions that are specified to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

Working time: 50 minutes.

Question 1 **(5 marks)**

Determine the gradient of the curve $y = x^2 - 4x - 60$ at the point(s) where it crosses the x-axis.

Solution
$(x+6)(x-10)=0$ $x=-6, x=10$ $\frac{dy}{dx} = 2x - 4$ $x=-6, \frac{dy}{dx} = -16$ $x = 10, \frac{dy}{dx} = 16$ At $(-6, 0)$ gradient is -16 and at $(10, 0)$ gradient is 16.
Specific behaviours
✓ factorises quadratic ✓ determines roots ✓ derivative of quadratic ✓ one point and gradient ✓ second point and gradient

Question 2 **(4 marks)**

The line segment between the points $A(-1,-2)$ and $B(-1,8)$ is the diameter of a circle.

Determine the equation of the circle in the form $x^2+ax+y^2+by=c$, where a, b and c are constants.

Solution
Centre: $\left(-1,\frac{-2+8}{2}\right)=(-1,3)$ Radius: $r=8-3=5$ Equation: $(x+1)^2+(y-3)^2=5^2$ $x^2+2x+1+y^2-6y+9=25$ $x^2+2x+y^2-6y=15$
Specific behaviours
✓ centre ✓ radius ✓ factored equation ✓ correct equation

Question 3 **(9 marks)**

(a) Solve the following exponential equations. (2 marks)

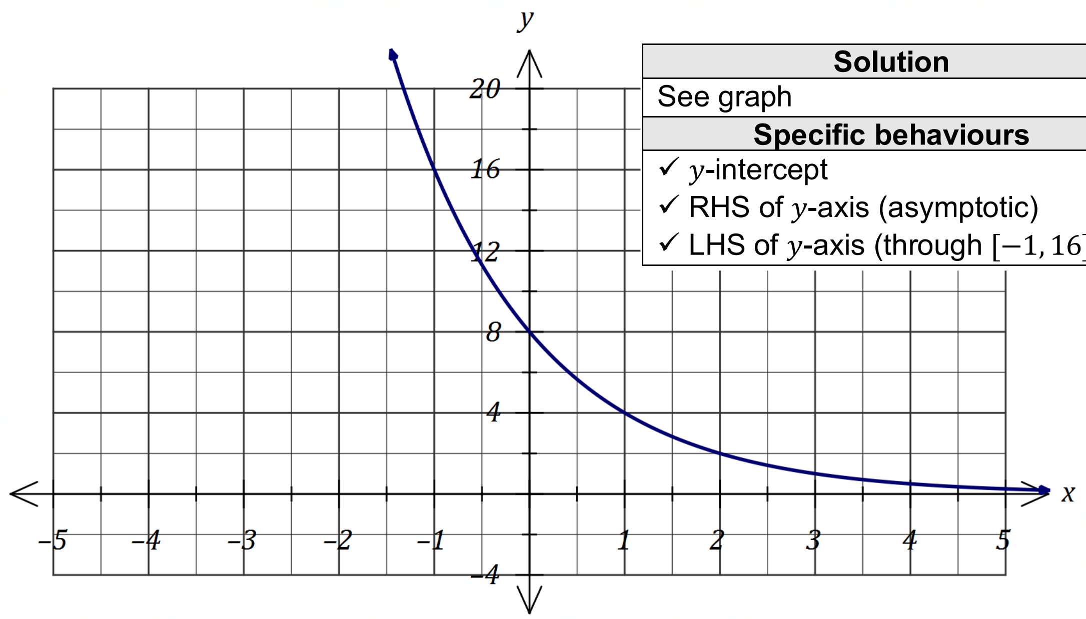
(i) $25^x = \dfrac{\sqrt{5}}{125}$

Solution
$(5^2)^x = 5^{0.5} \times 5^{-3}$ $5^{2x} = 5^{-2.5}$ $2x = -2.5$ $x = -1.25 = -\dfrac{5}{4}$
Specific behaviours
✓ simplifies equation in base 5 ✓ correct solution

(ii) $\sqrt{9^{x^2+1}} = 3^{x+3}$ (4 marks)

Solution
$(3^{2x^2+2})^{\frac{1}{2}} = 3^{x+3}$ $3^{x^2+1} = 3^{x+3}$ $x^2+1 = x+3$ $x^2 - x - 2 = 0$ $(x+1)(x-2) = 0$ $x = -1 \; or \; 2$
Specific behaviours
✓ writes LHS in base 3 ✓ simplifies index in LHS ✓ generates quadratic equation ✓ correct solution

(c) Sketch the graph of $y = 2^{(3-x)}$ on the axes below. (3 marks)



Solution
See graph
Specific behaviours
✓ y-intercept ✓ RHS of y-axis (asymptotic) ✓ LHS of y-axis (through $[-1, 16]$)

Question 4 **(7 marks)**

(a) A small body A is moving along a straight line so that at any time t seconds, its displacement relative to a fixed point O on the line is given by $x = 2t^3 - 9t^2 + 1$ cm.

(i) Determine the velocity of A when $t = 1$. (2 marks)

Solution
$v = \frac{dx}{dt} = 6t^2 - 18t$ $v(10) = 6(1)^2 - 18(1)$ $= -12 \text{ cm/s}$
Specific behaviours
✓ expression for velocity ✓ correct velocity

(ii) Determine the displacement of A relative to O at the instant(s) that it is stationary. (3 marks)

Solution
$6t^2 - 18t = 0$ $6t(t-3) = 0$ $t = 0, t = 3$ $x(0) = 1 \text{ cm}, \qquad x(3) = -26 \text{ cm}$
Specific behaviours
✓ factorises velocity ✓ one correct displacement ✓ both correct displacement

(b) A small body B has velocity given by $v = 6t^2 - 4t - 2$ cm/s and when $t = 2$ it has a displacement of 6 cm relative to O.

Determine an expression for the displacement of B relative to O at any time t. (2 marks)

Solution
$\frac{dx}{dt} = 6t^2 - 4t - 2$ $x = 2t^3 - 2t^2 - 2t + c$ $c = 6 - (16 - 8 - 4) = 2$ $x = 2t^3 - 2t^2 - 2t + 2$
Specific behaviours
✓ antidifferentiates ✓ correct expression

Question 5 **(8 marks)**

(a) Solve the following equations.

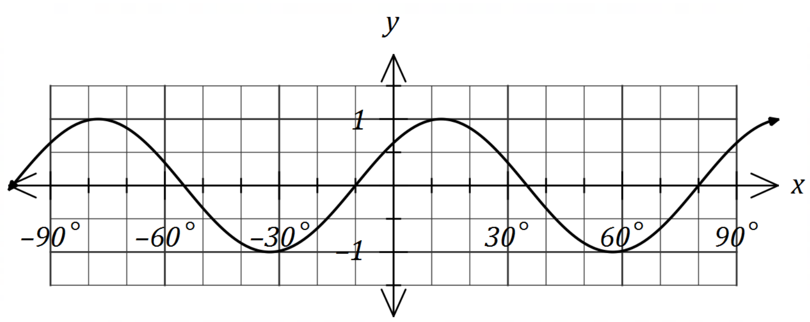
(i) $\tan(2x) = -\sqrt{3},\ 0 \le x \le \pi.$ (2 marks)

Solution
$$2x = \frac{2\pi}{3}, \frac{5\pi}{3}$$ $$x = \frac{\pi}{3}, \frac{5\pi}{6}$$
Specific behaviours
✓ one correct solution in radians ✓ both correct solutions in radians

(ii) $2\cos(x - 60°) = \sqrt{3} + \cos x,\ 0° \le x \le 360°.$ (4 marks)

Solution
$$2(\cos x \cos 60° + \sin x \sin 60°) = \sqrt{3} + \cos x$$ $$2\left(\cos x\left(\frac{1}{2}\right) + \sin x\left(\frac{\sqrt{3}}{2}\right)\right) = \sqrt{3} + \cos x$$ $$\sin x = 1$$ $$x = 90°$$
Specific behaviours
✓ uses angle difference identity ✓ substitutes exact values ✓ simplifies equation ✓ correct solution in degrees

(b) The graph of $y = \sin(ax + b)$ is shown below, where a and b are positive constants.



Determine the minimum possible value of each of the constants. (2 marks)

Solution
Period of $90° \Rightarrow a = 360° \div 90° = 4$ $y = \sin(4(x + 10)) = \sin(4x + 40°)$ $a = 4, \quad b = 40°$
Specific behaviours
✓ value of a ✓ value of b

Question 6 **(6 marks)**

(a) Expand $(x-3)^3$. (2 marks)

Solution
$(x-3)^3 = x^3 + 3x^2(-3) + 3x(-3)^2 + (-3)^3$ $= x^3 - 9x^2 + 27x - 27$
Specific behaviours
✓ correct coefficients   ✓ correct expansion

(b) Hence, or otherwise, determine the equation of the tangent to the curve $y = (x-3)^3$ at the point where $x = -1$. (4 marks)

Solution
$\frac{dy}{dx} = 3x^2 - 18x + 27$ When $x = -1$ $y = (-4)^3 = -64$ $\frac{dy}{dx} = 3 + 18 + 27 = 48$ Hence equation of tangent is $y + 64 = 48(x+1)$ thus $y = 48x - 16$
Specific behaviours
✓ derivative   ✓ y-coordinate   ✓ gradient   ✓ equation of tangent (any form)

Question 7 **(6 marks)**

Determine the coordinates of all stationary points of the curve $y = x^4 + 4x^2 - 12x + 20$.

Solution
$\frac{dy}{dx} = 4x^3 + 8x - 12$
$4x^3 + 8x - 12 = 0$ $x^3 + 2x - 3 = 0$
By inspection (factor theorem), $x = 1$ is a solution.
$x^3 + 2x - 3 = (x - 1)(x^2 + ax + 3)$
From x^2 coefficient: $-1 + a = 0 \Rightarrow a = 1$
$x^2 + x + 3 = 0$
$b^2 - 4ac = 1 - 4(1)(3) = -11 \Rightarrow \text{No solutions}$
$y = 1 + 4 - 12 + 20 = 13$
Hence just one stationary point at $(1, 13)$.
Specific behaviours
✓ derivative ✓ equates derivative to 0 ✓ one solution by inspection ✓ factorises derivative ✓ indicates quadratic factor has no roots ✓ states coordinates of stationary point

Question 8 **(7 marks)**

An arithmetic sequence has a recursive definition given by $T_{n+1} = T_n + d,\ T_1 = a$. It has fourth term of 50 and tenth term of 20.

(a) Determine the value of the constant a and the constant d. (2 marks)

Solution
$(10-4)d = 20-50$ $6d = -30$ $d = -5$ $a = 50 - 3(-5) = 65$
Specific behaviours
✓ value of d ✓ value of a

(b) Determine T_{2019}. (2 marks)

Solution
$T_n = 65 + (n-1)(-5)$ $T_{301} = 65 + 2018(-5)$ $= -10\,025$
Specific behaviours
✓ indicates rule for general term ✓ correct value

(c) The sum of the first m terms of the sequence is 350. Determine the value(s) of the integer constant m. (3 marks)

Solution
$350 = \frac{m}{2}\left(2(65) + (m-1)(-5)\right)$ $700 = m(130 - 5m + 5)$ $5m^2 - 135m + 700 = 0$ $m^2 - 27m + 140 = 0$ $(m-7)(m-20) = 0$ $m = 7, \qquad m = 20$
Specific behaviours
✓ substitutes into sum formula ✓ simplifies and equates quadratic to zero ✓ both correct solutions

End of questions

Christ Church Grammar School

Mathematics Department
Year 11 Mathematics Methods

Semester 2 Examination, 2019

Question/Answer booklet

MATHEMATICS METHODS UNITS 1 AND 2
Section Two: Calculator-assumed

SOLUTIONS

Student name: ______________________________________

Time allowed for this section

Reading time before commencing work:	ten minutes
Working time:	one hundred minutes

Materials required/recommended for this section

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To be provided by the candidate

Standard items:	pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters
Special items:	drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

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Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

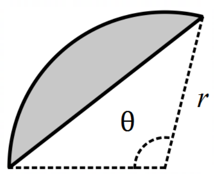
Working time: 100 minutes.

Question 9 **(6 marks)**

(a) Convert $108°$ to an exact radian measure. (1 mark)

Solution
$108 \times \frac{\pi}{180} = \frac{3\pi}{5}$
Specific behaviours
✓ correct value

(b) A segment of a circle of radius 28 cm is shown below, where $\theta = 108°$.



(i) Determine the area of the segment. (2 marks)

Solution
$A = \frac{1}{2}(28)^2\left(\frac{3\pi}{5} - \sin\frac{3\pi}{5}\right)$ $\approx 366.09 \text{ cm}^2$
Specific behaviours
✓ indicates correct use of formula   ✓ correct area

(ii) Determine the perimeter of the segment. (3 marks)

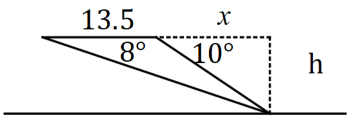
Solution
Arc length is L and chord length is C. $L = 28 \times \frac{3\pi}{5} \approx 52.78$ $C^2 = 28^2 + 28^2 - 2(28)(28)\cos 108°$ or $C = 2rsin(\frac{\theta}{2})$ $C \approx 45.3$ $P \approx 52.78 + 45.3$ $\approx 98.08 \text{ cm}$ Alternatively: $C = 2Rsin\frac{\theta}{2} = 45.3$
Specific behaviours
✓ arc length   ✓ uses cosine rule (or $C = 2rsin(\frac{\theta}{2})$) to calculate chord length   ✓ correct perimeter

Question 10 **(7 marks)**

A drone is flying in a straight line and at a constant height h m above a level pitch towards a thin goal post. It maintains a constant speed of 4.5 ms^{-1}.

Initially, the angle of depression from the drone to the base of the post is $8°$. Exactly 3 seconds later this angle has increased to $10°$.

(a) Sketch a diagram to show the two angles of depression from the drone to the base of the post. (1 mark)

Solution

Specific behaviours
✓ sketch with angles

(b) Determine, showing all working, the value of h and calculate the time after leaving its initial position that the drone will collide with the post. (6 marks)

Solution

$$d = 4.5 \times 3 = 13.5$$

$$\tan 8° = \frac{h}{x + 13.5}, \tan 10° = \frac{h}{x}$$

$$(x + 13.5)\tan 8° = x \tan 10° \Rightarrow x = 53.018$$

$$h = 53.018 \times \tan 10° = 9.35 \text{ m}$$

$$t = \frac{13.5 + 53}{4.5} = 14.8 \text{ s}$$

Specific behaviours

✓ calculates distance travelled
✓ writes equation using trig
✓ writes second equation using trig
✓ solves equations
✓ states h
✓ states time

Question 11 **(8 marks)**

From a random survey of telephone usage in 320 households it was found that 48 households had access to a mobile phone but not a landline, 268 households had access to a landline and 188 more households had access to a mobile phone than did not.

(a) Complete the missing entries in the table below. (3 marks)

	Mobile	No mobile	Total
Landline	**206**	**62**	**268**
No landline	48	**4**	**52**
Total	**254**	**66**	320

Solution
See table $x + (x + 188) = 320 \Rightarrow x = 66$
Specific behaviours
✓ totals column; ✓ totals row; ✓ rest of table

(b) If one household is randomly selected from those surveyed, determine the probability that

(i) it had access to a landline. (1 mark)

Solution
$P(L) = 268 \div 320 \approx 0.8375$
Specific behaviours
✓ correct probability

(ii) it had no access to a mobile phone given that it had access to a landline. (1 mark)

Solution
$P(\overline{M}\|L) = 62 \div 258 \approx 0.2313$
Specific behaviours
✓ correct probability

(iii) it had access to a landline given that it no access to a mobile phone. (1 mark)

Solution
$P(L\|\overline{M}) = 62 \div 66 \approx 0.9394$
Specific behaviours
✓ correct probability

(c) Comment on the possible independence of households having access to a mobile phone and households having access to a landline. Justify your comment. (2 marks)

Solution
The events are not independent as $P(L) \neq P(L\|\overline{M})$ (or since $P(L) \times P(M) \neq P(L \cap M)$)
Specific behaviours
✓ states not independent ✓ justifies by comparing probabilities

Question 12 **(10 marks)**

When a manufacturer makes x litres of a chemical using process X, the cost in dollars per litre $C(x)$ varies according to the rule

$$C(x) = \frac{240}{x+15}, \qquad 5 \le x \le 45.$$

(a) Determine

(i) the cost per litre when 35 L is made. (1 mark)

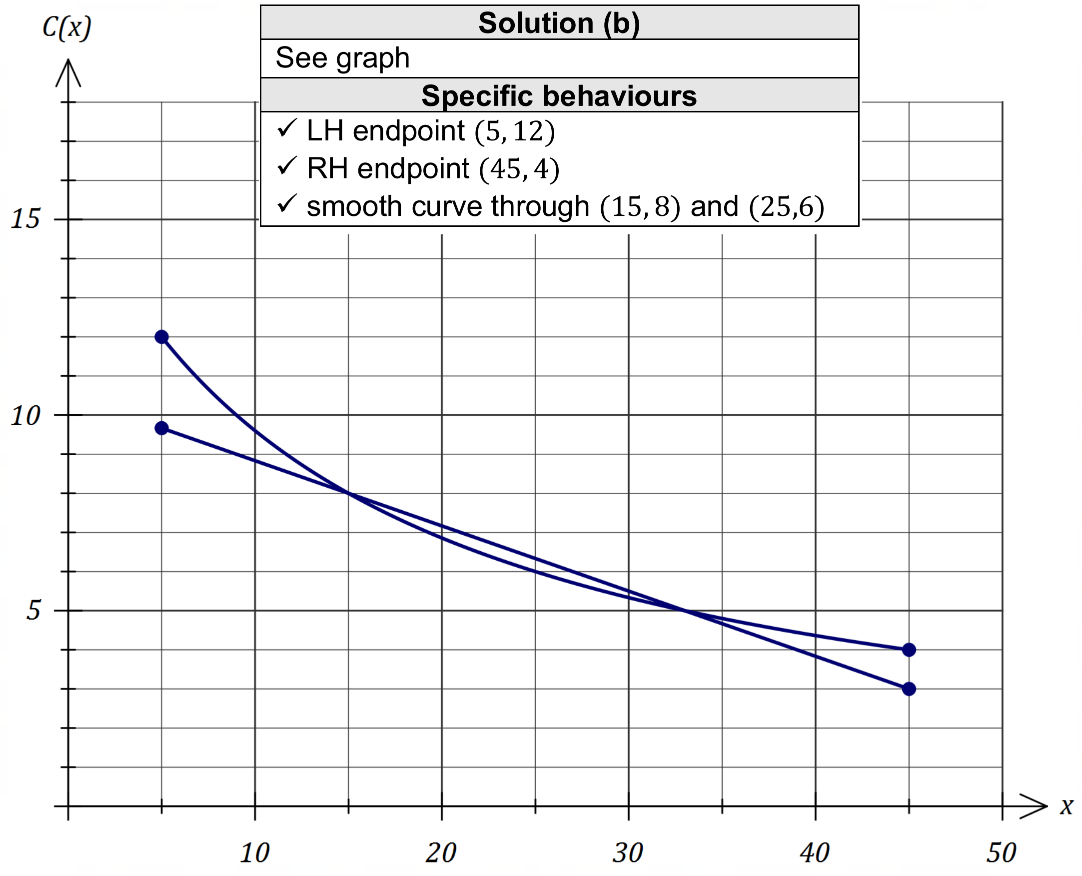
Solution
$C(35) = 4.8\ \$/\mathrm{L}$
Specific behaviours
✓ correct cost per litre

(ii) the total cost of making 17 L of the chemical. (2 marks)

Solution
$C(17) = 7.5$ $T = 7.5 \times 17 = \$127.50$
Specific behaviours
✓ cost per litre ✓ correct total cost

(b) Graph the cost per litre over the given domain on the axes below. (3 marks)

Solution (b)
See graph
Specific behaviours
✓ LH endpoint $(5, 12)$ ✓ RH endpoint $(45, 4)$ ✓ smooth curve through $(15, 8)$ and $(25,6)$



(c) State the range of $C(x)$. (1 mark)

Solution
$4 \leq C(x) \leq 12$
Specific behaviours
✓ correct range

(d) When the manufacturer uses process Z, the cost in dollars per litre $K(x)$ is modelled by

$$K(x) = 10.5 - \frac{x}{6}, \qquad 5 \leq x \leq 45.$$

(i) Add this function $K(x)$ to the graph. (1 mark)

Solution
See graph for line.
Specific behaviours
✓ ruled line through $(15, 8)$ and $(33, 5)$

(ii) determine the production quantities for which process X is cheaper than process Z. (2 marks)

Solution
Process X is cheaper than Z for $15 < x < 33$ litres.
Specific behaviours
✓ correct bounds ✓ does not include bounds in answer

Question 13 **(5 marks)**

A geometric sequence has a second term of -8.4 and a sum to infinity of 15.

Determine the sum of the first 4 terms of the sequence.

Solution

$$ar = -8.4, \qquad \frac{a}{1-r} = 15$$

$$\left.\begin{cases} a \times r = -8.4 \\ \dfrac{a}{1-r} = 15 \end{cases}\right|_{a,\,r}$$

$$\left\{\left\{a=-6,\ r=\frac{7}{5}\right\},\ \left\{a=21,\ r=-\frac{2}{5}\right\}\right\}$$

Solving simultaneously gives $a = 21, r = -0.4$

(ignore $r = 1.4$ since $|r| < 1$ for sum to infinity)

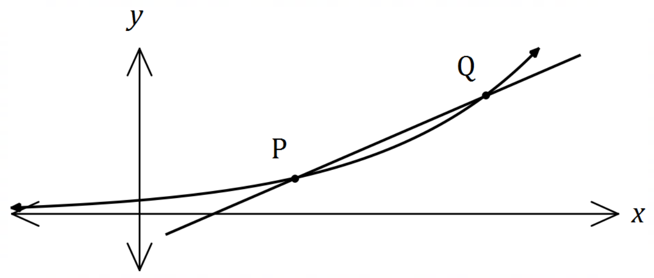
$$S_4 = \frac{21(1-(-0.4)^4)}{1-(-0.4)}$$
$$= \frac{1827}{125} = 14.616$$

Specific behaviours

✓ equation using T_2
✓ equation using S_∞
✓ solves for a and r
✓ discards invalid solution
✓ calculates S_4

Question 14 **(7 marks)**

The graph of $y = f(x)$ is shown below, where $f(x) = 4^x$, together with the secant to the curve through the points P and Q.



P has coordinates $(1, 4)$ and Q has coordinates $\left(1 + h, f(1 + h)\right)$ where $0 < h \leq 1$.

(a) Complete the second column in the table below, rounding values to 4 decimal places where necessary. (4 marks)

h	$\dfrac{f(1+h) - f(1)}{h}$
1	12
0.1	5.9479
0.01	5.5838
0.001	5.5490

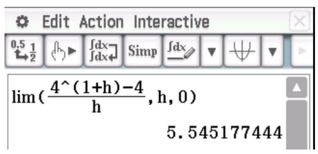
Solution
See table
Specific behaviours
✓ one correct value
✓ three correct values
✓ all correct
✓ last 3 all to 4 dp

(b) Name the feature of the graph above that the values you calculated in part (a) represent. (1 mark)

Solution
Values are gradient of secant PQ as Q moves closer to P.
Specific behaviours
✓ indicates gradient of secant

(c) Determine an estimate, correct to 3 decimal places, for the value that $\dfrac{f(1+h) - f(1)}{h}$ approaches as h becomes closer and closer to 0 and state what this value represents. (2 marks)

Solution
Value approaches 5.545 (3 dp). Value is gradient of curve at P.
Specific behaviours
✓ correct value
✓ states value approaches gradient at point



Question 15 **(7 marks)**

The amount of water in a tank, W litres, varies with time t, in minutes, and can be modelled by the equation $W = 200 - 185(1.2)^{-t}$, $t \geq 0$.

(a) Determine amount of water in the tank

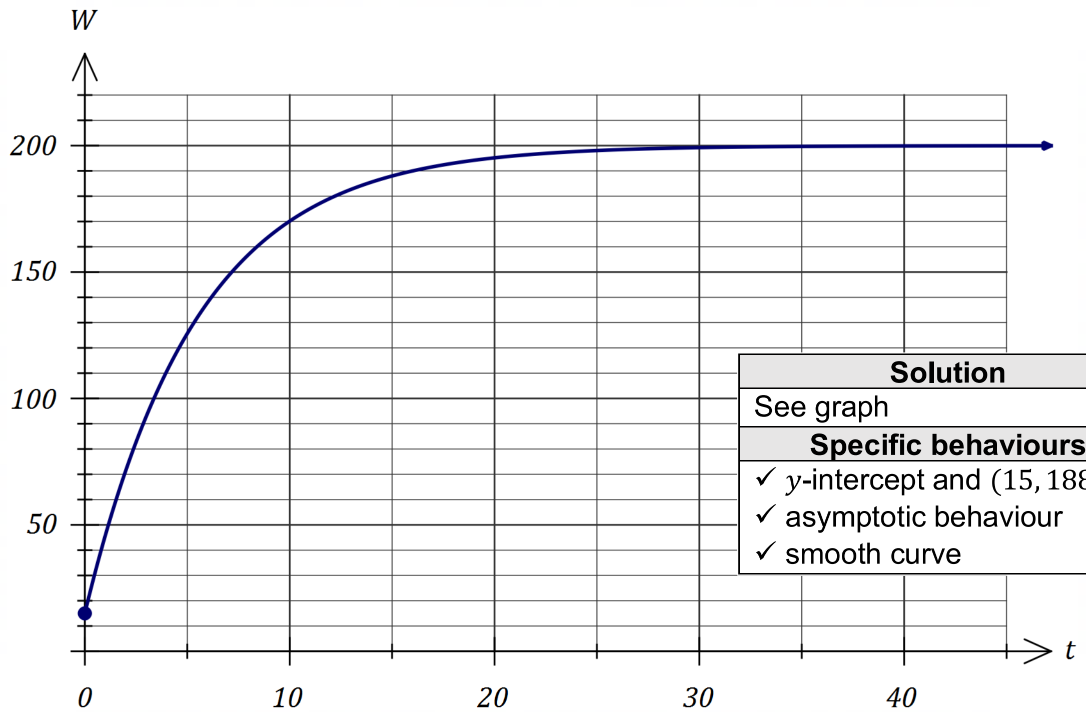
(i) initially. (1 mark)

Solution
$W(0) = 15\text{ L}$
Specific behaviours
✓ correct value

(ii) after 15 minutes. (1 mark)

Solution
$W(15) = 188\text{ L}$
Specific behaviours
✓ correct value

(b) Graph W against t for $0 \leq t \leq 45$ on the axes below. (3 marks)



Solution
See graph
Specific behaviours
✓ y-intercept and $(15, 188)$ ✓ asymptotic behaviour ✓ smooth curve

(c) Over time, the amount of water in the tank approaches v litres. State the value of v and determine the time at which the amount of water in the tank reaches 99% of this value. (2 marks)

Solution
$v = 200\text{ L}$ $W = 0.99(200) \Rightarrow t = 24.8\text{ minutes}$
Specific behaviours
✓ correct value of v ✓ correct time

Question 16 **(7 marks)**

When a patient takes a painkilling drug A, the probability that they experience some side effects is known to be 0.1.

(a) A doctor prescribes drug A to two unrelated patients. Determine the probability that

(i) neither patient experiences some side effects. (1 mark)

Solution
$P = (0.9)^2 = 0.81$
Specific behaviours
✓ correct probability

(ii) one patient experiences some side effects and the other does not. (2 marks)

Solution
$P = 0.1 \times 0.9 \times 2$ $= 0.18$
Specific behaviours
✓ calculates $p(1-p)$ ✓ doubles to obtain correct probability

Other painkilling drugs are available. Of those who take drug A, 88% of patients who suffer some side effects will switch to another drug whereas no patient who has no side effects will switch.

(b) The doctor prescribes drug A to a patient. Determine the probability that the patient does not switch to another drug. (2 marks)

Solution
$P = 0.9 + 0.1 \times 0.12$ $= 0.9 + 0.012$ $= 0.912$
Specific behaviours
✓ probability of side effect and does not switch ✓ correct probability

(c) The doctor prescribes drug A to three unrelated patients. Determine the probability that at least one of these patients switch to another drug. (2 marks)

Solution
$P(\text{none}) = 0.912^3$ ≈ 0.7586 $P = 1 - 0.7586$ ≈ 0.2414
Specific behaviours
✓ probability none switch ✓ correct probability

Question 17 **(12 marks)**

The function f is defined by $f(x) = x^3 + ax^2 + bx + c$, where a, b and c are constants.

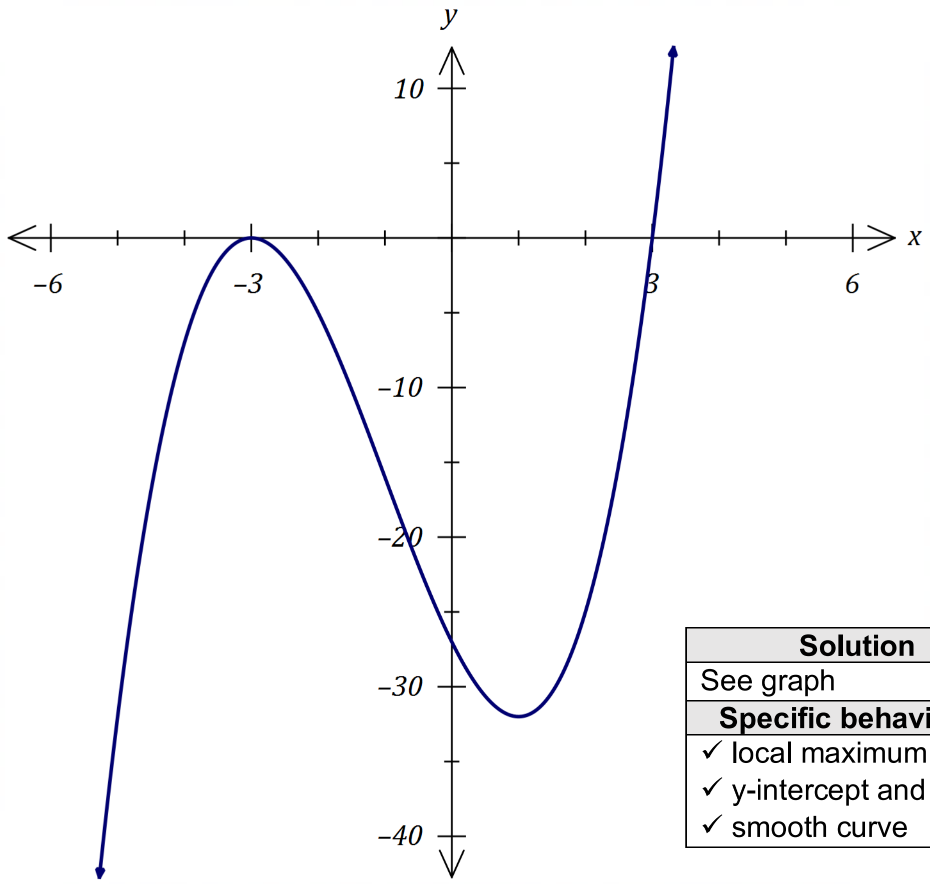
The graph of $y = f(x)$ has the following features:

- passes through $(0, -27)$ and $(3, 0)$
- has a local maximum at $(-3, 0)$

(a) Determine the value of a, the value of b and the value of c. (3 marks)

Solution
$f(x) = (x-3)(x+3)^2$ $= x^3 + 3x^2 - 9x - 27$ $a = 3, \quad b = -9, \quad c = -27$
Specific behaviours
✓ writes in factored form ✓ expands ✓ states all three values

(b) Sketch the graph of $y = f(x)$ on the axes below. (3 marks)



Solution
See graph
Specific behaviours
✓ local maximum ✓ y-intercept and root ✓ smooth curve

(c) Use a calculus method to determine the exact coordinates of the local minimum of the graph of $y = f(x)$. (3 marks)

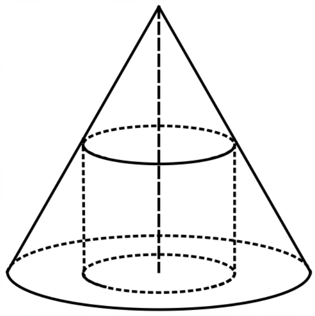
Solution
$f'(x) = 3x^2 + 6x - 9$ $f'(x) = 0 \Rightarrow x = -3, 1$ $f(1) = -32$ Local minimum at $(1, -32)$
Specific behaviours
✓ shows $f'(x)$ ✓ shows $f'(x) = 0$ and solutions ✓ correct coordinates

(d) Determine the coordinates of the point where the tangent to $y = f(x)$ at $(0, -27)$ intersects the curve $y = f(x)$, other than at the point of tangency. (3 marks)

Solution
$f'(0) = -9$ Tangent: $y = -9x - 27$ $x^3 + 3x^2 - 9x - 27 = -9x - 27$ $x = 0, x = -3$ Intersects at $(-3, 0)$
Specific behaviours
✓ equation of tangent ✓ equates tangent to curve and solves ✓ correct coordinates

Question 18 **(7 marks)**

A right circular cone of base radius 10 cm and height 25 cm stands on a horizontal surface. A cylinder of radius x cm and volume V cm^3 stands inside the cone with its axis coincident with that of the cone and such that the cylinder touches the curved surface of the cone as shown.



(a) Show that the volume of the cylinder, $V = 25\pi x^2 - 2.5\pi x^3$. (3 marks)

Solution
From similar triangles $$\frac{h}{10-x} = \frac{25}{10} \Rightarrow h = 25 - 2.5x$$ Hence $$V = \pi r^2 h$$ $$V = \pi x^2(25 - 2.5x)$$ $$= 25\pi x^2 - 2.5\pi x^3$$
Specific behaviours
✓ relation between x and h using similar triangles ✓ expresses h in terms of x ✓ substitutes into cylinder volume formula

(b) Given that x can vary, use a calculus method to determine the maximum value of V. (4 marks)

Solution
$$\frac{dV}{dx} = 50\pi x - 7.5\pi x^2$$ $$\frac{dV}{dx} = 0 \text{ when } x = 0, x = \frac{20}{3}$$ $$x = 0 \Rightarrow V = 0 \text{ (minimum)}$$ $$x = \frac{20}{3} \Rightarrow V = \frac{10\,000\pi}{27} \approx 1164 \text{ cm}^3 \text{ (maximum)}$$
Specific behaviours
✓ derivative ✓ equates derivative to 0 ✓ solves for x ✓ states maximum volume

Question 19 **(8 marks)**

Two events A and B are such that $P(A \cap \bar{B}) = x, P(A) = 0.2$ and $P(\bar{A} \cap B) = 0.6$.

(a) Determine $P(A \cap B)$ when $x = 0.12$. (2 marks)

Solution
$P(A \cap B) = P(A) - P(A \cap \bar{B})$ $= 0.2 - 0.12$ $= 0.08$
Specific behaviours
✓ use of Venn diagram or other method ✓ correct probability

(b) Determine an expression for $P(A \cap B)$ in terms of x. (1 mark)

Solution
$P(A \cap B) = 0.2 - x$
Specific behaviours
✓ correct expression

(c) Determine the value of x when

(i) A and B are mutually exclusive. (1 mark)

Solution
$P(A \cap B) = 0.2 - x = 0 \Rightarrow x = 0.2$
Specific behaviours
✓ correct value

(ii) A and B are independent. (2 marks)

Solution
$P(A \cap B) = P(A) \times P(B)$ $0.2 - x = 0.2 \times (0.6 + 0.2 - x)$ $x = 0.05$
Specific behaviours
✓ uses rule for independence ✓ correct value

(iii) $P(A \mid B) = 0.04$. (2 marks)

Solution
$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ $0.04 = \dfrac{0.2 - x}{0.6 + 0.2 - x}$ $x = 0.175$
Specific behaviours
✓ uses conditional probability rule ✓ correct value

Question 20 **(8 marks)**

A fair six-sided dice numbered $1, 2, 3, 4, 5$ and 6 is thrown n times until it lands on a 6.

(a) Show that the probability that $n = 3$ is $\frac{25}{216}$. (1 mark)

Solution
$P(n=3) = \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{25}{216}$
Specific behaviours
✓ shows product of three fractions

(b) Determine the probability that $n = 5$. (1 mark)

Solution
$P(n=5) = \left(\frac{5}{6}\right)^4 \times \frac{1}{6} = \frac{625}{7776} \approx 0.0804$
Specific behaviours
✓ correct probability

(c) Write an expression in terms of n for the probability that the first 6 is thrown on the n^{th} throw and explain why the probabilities form a geometric sequence. (2 marks)

Solution
$P = \frac{1}{6}\left(\frac{5}{6}\right)^{n-1}$ The expression takes the form of the n^{th} term of a GP - $a(r)^{n-1}$ *or* There exists a common ratio $\left(\frac{5}{6}\right)$.
Specific behaviours
✓ correct expression   ✓ compares to general term of GP *or* states common ratio

(d) Determine the probability that the first 6 is thrown in 12 or less attempts. (2 marks)

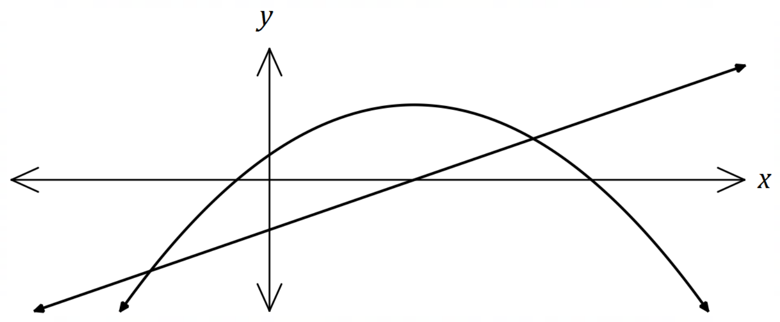
Solution
$S_{12} = \dfrac{\frac{1}{6}\left(1 - \left(\frac{5}{6}\right)^{12}\right)}{1 - \frac{5}{6}} \approx 0.8878$
Specific behaviours
✓ indicates use of sum formula   ✓ correct probability

(e) The probability that the first 6 is thrown in k or less attempts must be at least 99%. Determine the least value of integer k. (2 marks)

Solution
$0.99 = \dfrac{\frac{1}{6}\left(1 - \left(\frac{5}{6}\right)^{n}\right)}{1 - \frac{5}{6}} \Rightarrow n = 25.3$ $k = 26$
Specific behaviours
✓ solves for n ✓ correct value of k

Question 21 **(6 marks)**

The graphs of $y = f(x)$ and $y = g(x)$ are shown below where $f(x) = 1 + 4x - 2x^2$ and $g(x) = 2x + k$.



Determine the value(s) of the constant k so that the equation $f(x) = g(x)$ has

(a) one solution. (5 marks)

Solutions	
g must be a tangent to f: $f'(x) = 4 - 4x$ $= 2 \text{ when } x = \frac{1}{2}$ y-coordinate of point of tangency: $f\left(\frac{1}{2}\right) = 1 + 4\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)^2 = \frac{5}{2}$ Equation of tangent: $y - \frac{5}{2} = 2\left(x - \frac{1}{2}\right)$ $y = 2x + \frac{3}{2}$ Hence $k = \frac{3}{2} = 1.5$	$1 + 4x - 2x^2 = 2x + k$ $2x^2 - 2x + (k - 1) = 0$ For one solution, $\Delta = 0$ $b^2 - 4ac = 0$ $4 - 4(2)(k - 1) = 0$ $k - 1 = 0.5$ $\therefore k = 1.5$
Specific behaviours	
✓ indicates tangent required ✓ determines x-coordinate of point of tangency ✓ determines y-coordinate of point of tangency ✓ equation of tangent ✓ states correct value of k	✓ equates f(x) and g(x) ✓ generates quadratic equation with RHS being 0 ✓ states discriminant = 0 ✓ sets up equation in k ✓ solves for k

(b) no solutions. (1 mark)

Solution	
$k > 1.5$	$\Delta < 0$ $4 - 8(k - 1) < 0$ $\therefore k > 1.5$
Specific behaviours	
✓ correct inequality	

End of questions