WESLEY COLLEGE
SOUTH PERTH

Semester Two Examination, 2011

Question/Answer Booklet

MATHEMATICS

3C/3D

Section Two:

Calculator-assumed

Student Name: Solutions

Time allowed for this section

Reading time before commencing work:	Ten (10) minutes
Working time for this section:	One hundred (100) minutes

Material required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet

Formula Sheet

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, eraser, correction fluid, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this course.

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	8	8	50	40	33 1/3
Section Two: Calculator-assumed	13	13	100	80	66 2/3
				120	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2011*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

3. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil** except in diagrams.

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Section Two: Calculator-assumed (80 Marks)

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the space provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

The working time for this section is 100 minutes.

Question 9 **(4 marks)**

Entrants in a schools' whitewater race could choose from three types of boats: single kayaks, double kayaks or 3-man canoes.

(A single kayak has one person in it and a double kayak has 2 occupants.)

Entry fees were calculated at \$5 per boat and \$2 per person, to a maximum of \$10 per boat.

Last year, there were 37 boats and 63 people in the race for a total entry fee of \$304.

Set up and solve a system of three equations to find the number of each type of boat in the race.

x singles y doubles z 3-man

cost \$7 \$9 \$10.

$$\left.\begin{aligned} x+y+z &= 37 \\ x+2y+3z &= 63 \\ 7x+9y+10z &= 304 \end{aligned}\right\}$$

$x = 18$ singles

$y = 12$ doubles

$z = 7$ 3-man

(by calc).

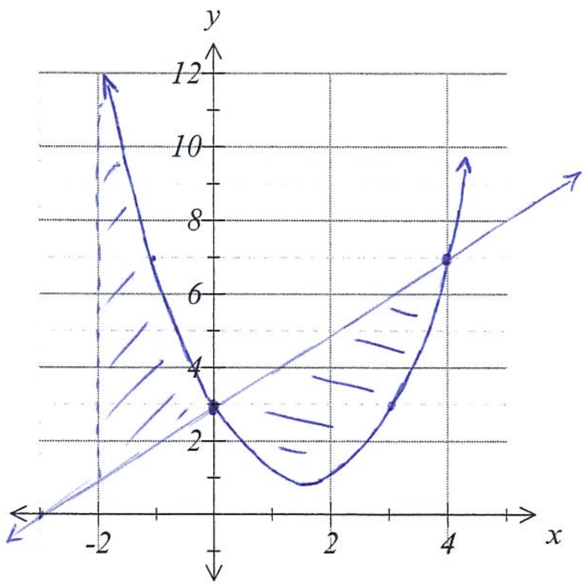
✓✓ set up

✓✓ solve.

[4]

Question 10 **(8 marks)**

(a) Draw the graphs of $y = x^2 - 3x + 3$ and $y = x + 3$ in the interval $-2 \le x \le 4$ on the given axes.



[2]

(b) Find the area contained between the curve and the line, showing all necessary calculations you use.

$$\int_0^4 \left((x+3) - (x^2 - 3x + 3)\right) dx = \frac{32}{3} \quad (\text{or } 10.\dot{6})$$ ✓ ✓

sq. units

[2]

(c) Find $\int_{-2}^{4} (4x - x^2)\, dx$ and interpret the result in terms of the graphs drawn above.

$$\int_{-2}^{4} (4x - x^2)\, dx = 0.$$ ✓

(shaded ⫽)

Area enclosed between $x = -2 \rightarrow x = 0$, is equal to area enclosed between $x = 0 \rightarrow x = 4$. (shaded ≡) ✓

[2]

(d) Determine the volume of revolution when the area bounded by curves in the first quadrant is rotated about the x-axis

$$V = \pi \int_0^4 \left((x+3)^2 - (x^2 - 3x + 3)^2\right) dx$$ ✓

$$= \frac{1088\pi}{15} \quad (\text{or } 227.87 \text{ (2dp)})$$ ✓

cubic units

[2]

Question 11 **(6 marks)**

A computer screen saver program generates a coloured region of random shape and size. This region then expands until it fills the screen. A new region of a different colour is then formed.

The program is written so that the rate at which the area of the region increases is proportional to its current area.

(a) Write an equation involving $\frac{dA}{dt}$ and show that $A = A_0 e^{kt}$ satisfies this equation where A_0 is the initial area of the region in cm^2 and k is a constant.

Given $\frac{dA}{dt} = kA$

$A = A_0 e^{kt} \Rightarrow \frac{dA}{dt} = A_0 k e^{kt} = kA$

shown ✓

[2]

Given that, once formed, the area of a region increases by 50% in 0.4 seconds

(b) Show clearly that $k = 1.014$ correct to 4 significant figures

$A = 1.5A_0$ at $t = 0.4$

$\Rightarrow 1.5 = e^{0.4k}$ ✓

$\Rightarrow k = 1.01366277$ (calc)

$\simeq 1.014$ (4sf) ✓

[2]

A coloured region of area 3.6 cm^2 is generated on a screen measuring 24cm by 32cm.

(c) Find, in seconds correct to 1 decimal place, how long it takes for the region to fill the screen.

solve $(24 \times 32 = 3.6 e^{kt}, t$ ✓ using k above

$\Rightarrow t = 5.29$

$\simeq 5.3$ sec. ✓

[2]

Question 12 **(5 marks)**

The rate of change of the value of the Australian dollar (AUD), measured in terms of the US dollar, at time t months, has been modelled by economists as $\frac{dV}{dt} = t\left(\sqrt{t} - 2\right)$ for $t \geq 0$.

The present value is \$0.94 US.

(a) Estimate

(i) The value of the Australian dollar in 3 months

$V = \frac{2t^{5/2}}{5} - t^2 + c \qquad V(0) = 94 \Rightarrow c = 94$ ✓

$\Rightarrow V = 94 + \frac{2t^5}{5} - t^2 \Big|_{t=3} = 91.24¢$

ie/ \$0.91 US ✓

[2]

(ii) The value of the Australian dollar in 8 months

$V = 94 + \frac{2t^5}{5} - t^2 \Big|_{t=8} = 102.41¢$

ie/ \$1.02 US ✓

[1]

(b) When does the dollar reach parity? (i.e. \$1 AUD = \$1 US)

✓

solve $\left(100 = 94 + \frac{2t^5}{5} - t^2, t\right.$

$\Rightarrow t = 7.612$ ✓

ie/ in 7.6 months

[2]

Question 13 **(7 marks)**

Australian government legislation states that products containing more than 9.5 grams per kg of genetically modified material must be identified as genetically modified.

It has been discovered that a canola crop contains genetically modified canola. The grower needs to know whether or not the crop must be identified as genetically modified.

The crop is divided into sections; these sections, taken collectively, can be considered to be a population.

Tests are undertaken on a randomly chosen sample of fifty sections and X, the amount of genetically modified canola per kg in each section is measured. The mean amount of genetically modified canola in the fifty sections is 9.21 grams per kg.

(a) Calculate a 95% confidence interval for the amount of genetically modified canola in the crop. Assume that the population has a standard deviation of 0.74 grams per kg.

$$\mu = \bar{x} \pm 1.96 \times \frac{0.74}{\sqrt{50}} = 9.21 \pm 0.205 \quad ✓$$

$$9.005 \le \mu \le 9.415 \text{ g/kg.} \quad ✓$$

[2]

(b) The grower claims that the crop will not need to be identified as genetically modified. Do you think this claim is justified? Explain.

Yes, ✓ 9.5g/Kg outside the above interval. ✓

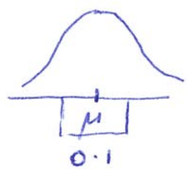
or $p(x > 9.5) = \text{normCdf}\left(9.5, \infty, 9.21, \frac{0.74}{\sqrt{50}}\right) = 2.79 \times 10^{-3}$

is v. small.

[2]

(c) How many sections of the crop need to be sampled to obtain a 95% confidence interval with a width of no more than 0.1 grams per kg?

Solve $1.96 \times \frac{0.74}{\sqrt{n}} \le 0.05$ ✓



$n \ge 841.46$ ✓

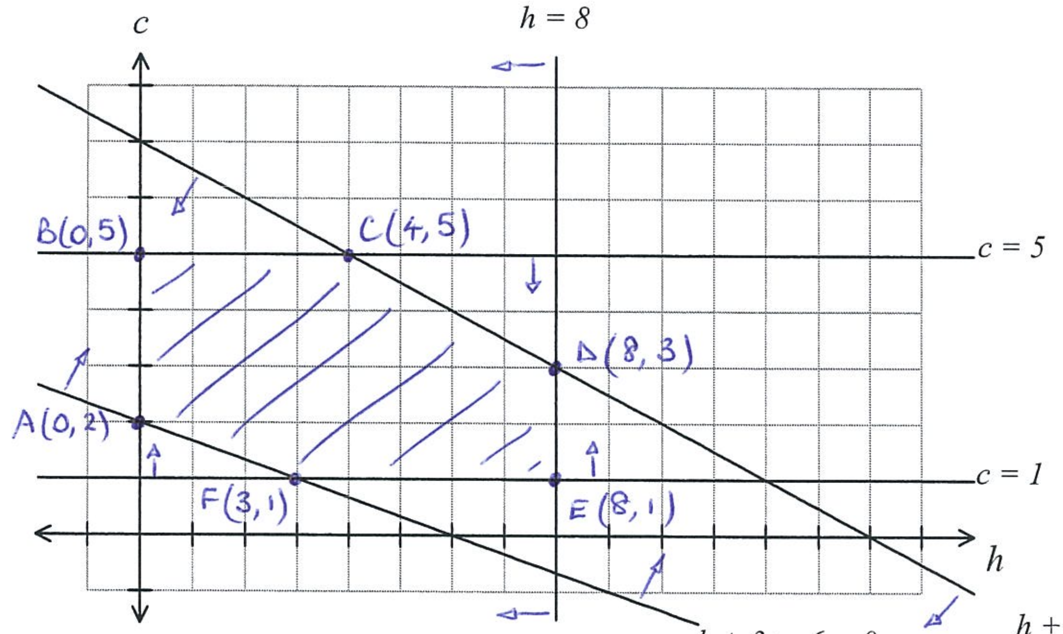
$\Rightarrow$ 842 or more ✓

[3]

Question 14 **(8 marks)**

A dance teacher is writing up a business plan for a dance studio she intends to start. She plans to teach Hip-hop and Contemporary. Each hip-hop class will involve 3 hours of tuition and produce an income of \$60. Each class of contemporary will involve 6 hours of tuition and produce an income of \$180. The teacher plans to work for up to 42 hours a week and would incur fixed costs (rent, power, etc.,) of \$360 per week. She plans to teach at least one contemporary class per week but also decides to teach no more than five contemporary classes and no more than eight hip-hop classes per week. The teacher also decides that her return (income – costs) must not be negative or she will not set up the business.

Let h denote the number of hip-hop classes and c the number of contemporary classes. The constraints have been drawn on the diagram below.



(profit)

$60h + 180c - 360 \geq 0$

$h + 3c - 6 \geq 0$

(time)

$3h + 6c \leq 42$

$h + 2c - 14 \leq 0$

(a) Shade the feasible region.

as above.

[2]

(b) How many hip-hop and contemporary classes should the dance teacher aim to have pupils for so that she can maximise her weekly profit? Show all working and state clearly your optimal solution and maximum value.

✓

	Maximise $P = 60h + 180c - 360$
A (0, 2)	0
B (0, 5)	540
C (4, 5)	780
D (8, 3)	660
E (8, 1)	300
F (3, 1)	0

not req'd

✓

Max profit of \$780 with 4 hip-hop & 5 contemporary classes ✓

[3]

(c) There are no dance classes in the school holidays and throughout the year some classes may need to be cancelled for other reasons. Consequently the dance teacher realises that if she is considering her average profit over the whole year, the more realistic income from hip=hop classes is \$50 per class. What is the lowest income per contemporary class that the teacher can receive so that the solution from part (b) is the only combination of classes that gives maximum profit per week?

New profit $= 50h + kc - 360$ ✓

Compare

$P(0,5) < P(4,5)$ & $P(4,5) > P(8,3)$.

$5k - \cancel{360} < 200 + 5k - \cancel{360}$ $\quad$ $200 + 5k - \cancel{360} > 400 + 3k - \cancel{360}$

no solution. $\quad$ $2k > 200$

$k > 100$.

✓

$\Rightarrow$ lowest income is \$100.01 ✓

[3]

Question 15 **(7 marks)**

12 magazines appear on one shelf of the newsstand of the airport newsagent. Of these magazines, 6 are printed monthly, 4 are printed fortnightly and 2 are printed weekly.

(a) In how many ways can the magazines be arranged on the shelf if all the monthly magazines must be kept together

✓ idea of !

$6! \times 7! = 3628800$

✓ ✓

[3]

A traveller rushes in to buy some reading material for a flight and randomly chooses 3 magazines. Of the 3 chosen magazines, what is the probability that

(b) none is a weekly

$p(\text{none weekly}) = \dfrac{{}^{10}C_3}{{}^{12}C_3} = \dfrac{6}{11} \quad (0.\overline{54})$ ✓ ✓

[2]

(c) at least two are monthly magazines

p(at least 2 monthly)

ie, chooses 2 or 3 monthly ✓

$= \dfrac{{}^6C_2 \times {}^6C_1}{{}^{12}C_3} + \dfrac{{}^6C_3}{{}^{12}C_3} = \dfrac{9}{22} + \dfrac{1}{11} = \dfrac{1}{2}$ ✓

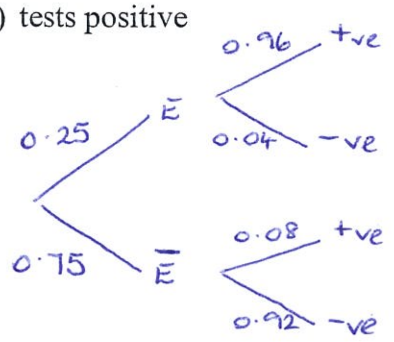
[2]

Question 16 **(7 marks)**

A pathology service performs blood tests to detect the presence of a certain type of enzyme E. For 4% of blood samples with enzyme E, the test suggests its absence (that is, it tests negative); while for 8% of samples without the enzyme, the test suggests its presence (that is, it tests positive). From past data, it is known that 25% of all samples received have the enzyme. Suppose one of the fresh samples is taken at random and tested for enzyme E.

Calculate the probability that the sample:

(a) tests positive



$p(\text{positive})$

$= 0.25 \times 0.96 + 0.75 \times 0.08$

$= 0.3$

[3]

(b) contains enzyme E given that the test is positive

$p(E \mid +ve) = \dfrac{0.25 \times 0.96}{0.3} = 0.8$

[2]

(c) tests positive <u>or</u> has the enzyme in it.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

$p(+ve \text{ or } E) = 0.3 + 0.25 - (0.25 \times 0.96)$

$= 0.31$

(or similar)

(or $0.25 + (0.75 \times 0.08) = 0.31$)

[2]

Question 17 **(8 marks)**

A student throws ten dice and records the number of sixes showing. The dice are fair, numbered 1 to 6 on the faces.

(a) Describe the distribution of the number of sixes obtained when ten dice are thrown.

Binomial $n = 10$ $p = \frac{1}{6}$. ✓✓

or $X \sim Bin(10, \frac{1}{6})$

[2]

(b) Find the mean and standard deviation of the distribution.

$\mu = np = 10 \times \frac{1}{6} = \frac{5}{3}$

$\sigma = \sqrt{npq} = \sqrt{\frac{5}{3} \times \frac{5}{6}} = 1.1785$ (4dp)

[2]

The student throws the ten dice 100 times, recording the number of sixes showing each time and then calculating the mean number of sixes.

(c) Describe, and justify, the distribution of these mean number of sixes

Normal distribution. ✓

Since $n = 100$ (ie, > 30) can apply CLT ✓

[2]

(d) Find the probability that the mean number of sixes is more than 1.8

$\bar{x} \sim N\left(\frac{5}{3}, \left(\frac{1.1785}{10}\right)^2\right)$

$p(\bar{x} > 1.8) = 0.1289$ (4sf) ✓✓

(or $p(\bar{x} > 1.8) = \text{norm}\left(1.8, \infty, \frac{5}{3}, \frac{1.1785}{10}\right)$)

[2]

Question 18 **(4 marks)**

The time T a planet takes to revolve about the Sun and the average distance r from the planet to the sun are related by the equation $T = k\, r^{\frac{3}{2}}$

If the Earth's distance from the Sun were increased by 1%, use an incremental technique to determine how much longer a year would become?

Justify your answer, giving your solution in days to 2 decimal places.

$\frac{\delta T}{T} \simeq \frac{3}{2} \frac{\delta r}{r} = 1.5\%$ ✓ ✓

$1.5\% \times 365.25 = 5.47875$ ✓

$\simeq 5.48$ days.

$\frac{\delta T}{T} \simeq \frac{\frac{dT}{dr} \times \delta r}{T}$

✓

$\frac{\delta T}{T} \simeq \frac{\frac{3}{2} k r^{1/2} \times \delta r}{k r^{3/2}} \simeq \frac{3}{2} \cdot \frac{\delta r}{r}$

[4]

Question 19 **(4 marks)**

Take a 2-digit integer, multiply the units digit by 4 and then add the 10s digit to produce another 2-digit integer. For example,

$$23 \quad \rightarrow \quad (3\times4)+2=14$$
$$15 \quad \rightarrow \quad (5\times4)+1=21$$

Some integers reproduce themselves:

$$13 \quad \rightarrow \quad (3\times4)+1=13$$

(a) Write down an algebraic expression for the result of applying this process to a 2-digit number *ab*.

$ab \longrightarrow b\times 4 + a = a + 4b.$ ✓

[1]

(b) Noting that *ab* has a numerical value $10a + b$, show clearly that there are only two other 2-digit integers that reproduce themselves in this way.

$a + 4b = 10a + b$ ✓

$3b = 9a$

$b = 3a$ ✓

if $a = 1 \Rightarrow b = 3$

$a = 2 \quad b = 6$

$a = 3 \quad b = 9$

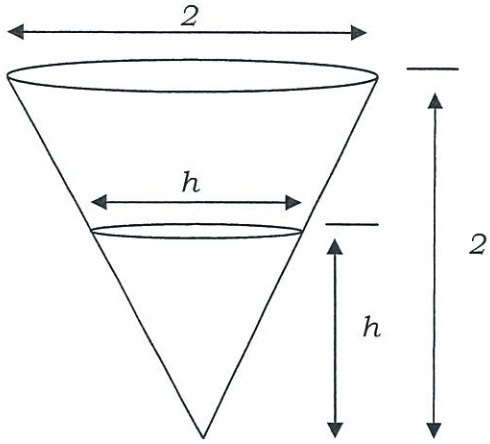
ie/ other 2-digit integers are 26 & 39. ✓

[3]

Question 20 **(5 marks)**

A container in the shape of a right circular cone with both height and diameter 2 metres (as shown in the diagram) is being filled with water at a rate of π m^3/minute

Find the rate of change of height h of the water when the container is one-eighth full (by volume).



$r = \frac{h}{2}$

$V = \frac{1}{3}\pi r^2 h$

$= \frac{\pi h^3}{12}$ ✓

If $\frac{\cancel{\pi} h^3}{12} = \frac{1}{8} \times \frac{1}{3} \times \cancel{\pi} \times 1^2 \times 2$ ✓

$h^3 = 1 \Rightarrow h = 1$ ✓

$\frac{dV}{dt} = \frac{\pi h^2}{4} \times \frac{dh}{dt}$ ✓

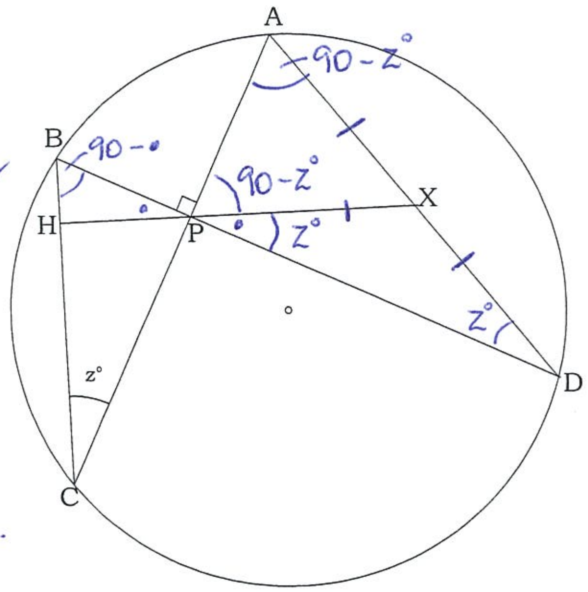
$\therefore \frac{dh}{dt} = \pi / (\pi/4) = 4$ m/min. ✓

[5]

Question 21 **(7 marks)**

The diagram below shows a circle with perpendicular chords AC and BD meeting at point P. Line segment HP is perpendicular to BC and $\measuredangle BCA = z°$.

HP is extended to meet AD at point X.



(a) Show $\measuredangle DPX = \measuredangle BCA$

$\measuredangle DPX = \measuredangle BPH$ (vert. opp) ✓

$\measuredangle BPH$ is complementary to $\measuredangle PBH$
(since ΔBHP is right angled) ✓

$\measuredangle BCA$ is also complementary to $\measuredangle PBH$ ✓

$\therefore \angle BCA = \angle BPH = \angle DPX$.

[3]

(b) Hence, or otherwise, show that triangle PAX is isosceles

$\angle BCA = \angle ADB$ (angles standing on the same arc)

$\Rightarrow PX = XD$. ✓

$\angle CBD = \angle CAD$. — " —

Also $\angle APX = 90 - z°$ since $\angle APD = 90°$ ✓

$\Rightarrow \Delta PAX$ is isosceles.

[2]

(c) Hence show that X is the midpoint of AD

From (b) $AX = PX$. (using ΔPAX) ✓

$PX = XD$ (using ΔPXD)

$PX = AX = XD \Rightarrow X$ is midpt of AD. ✓

[2]