ACADEMIC TASK FORCE

Creelman

Exam Questions

MATHEMATICS METHODS

ATAR Course Units 3 and 4

2017 Edition

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Contents

Unit 3 Content

An understanding of the Year 11 content is assumed knowledge for students in Year 12. It is recommended that students studying Unit 3 and Unit 4 have completed Unit 1 and Unit 2. The examinable content for Unit 3 is as follows:

Topic 3.1: Further differentiation and applications (20 hours)

Exponential functions

3.1.1 estimate the limit of $\frac{a^h - 1}{h}$ as $h \to 0$, using technology, for various values of $a > 0$

3.1.2 identify that e is the unique number a for which the above limit is 1

3.1.3 establish and use the formula $\frac{d}{dx}(e^x) = e^x$

3.1.4 use exponential functions of the form Ae^{kx} and their derivatives to solve practical problems

Trigonometric functions

3.1.5 establish the formulas $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x)$ by graphical treatment, numerical estimations of the limits, and informal proofs based on geometric constructions

3.1.6 use trigonometric functions and their derivatives to solve practical problems

Differentiation rules

3.1.7 examine and use the product and quotient rules

3.1.8 examine the notion of composition of functions and use the chain rule for determining the derivatives of composite functions

3.1.9 apply the product, quotient and chain rule to differentiate functions such as xe^x, $\tan x$, $\frac{1}{x^n}$, $x \sin x$, $e^{-x} \sin x$ and $f(ax - b)$

The second derivative and applications of differentiation

3.1.10 use the increments formula: $\delta y \approx \frac{dy}{dx} \times \delta x$ to estimate the change in the dependent variable y resulting from changes in the independent variable x

3.1.11 apply the concept of the second derivative as the rate of change of the first derivative function

3.1.12 identify acceleration as the second derivative of position with respect to time

3.1.13 examine the concepts of concavity and points of inflection and their relationship with the second derivative

3.1.14 apply the second derivative test for determining local maxima and minima

3.1.15 sketch the graph of a function using first and second derivatives to locate stationary points and points of inflection

3.1.16 solve optimisation problems from a wide variety of fields using first and second derivatives

Topic 3.2: Integrals (20 hours)

Anti-differentiation

3.2.1 identify anti-differentiation as the reverse of differentiation

3.2.2 use the notation $\int f(x)dx$ for anti-derivatives or indefinite integrals

3.2.3 establish and use the formula $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$ for $n \neq -1$

3.2.4 establish and use the formula $\int e^x dx = e^x + c$

3.2.5 establish and use the formulas $\int \sin x\, dx = -\cos x + c$ and $\int \cos x\, dx = \sin x + c$

3.2.6 identify and use linearity of anti-differentiation

3.2.7 determine indefinite integrals of the form $\int f(ax - b)dx$

3.2.8 identify families of curves with the same derivative function

3.2.9 determine $f(x)$, given $f'(x)$ and an initial condition $f(a) = b$

Definite integrals

3.2.10 examine the area problem and use sums of the form $\sum_i f(x_i)\ \delta x_i$ to estimate the area under the curve $y = f(x)$

3.2.11 identify the definite integral $\int_a^b f(x)\,dx$ as a limit of sums of the form $\sum_i f(x_i)\ \delta x_i$

3.2.12 interpret the definite integral $\int_a^b f(x)\,dx$ as area under the curve $y = f(x)$ if $f(x) > 0$

3.2.13 interpret $\int_a^b f(x)\,dx$ as a sum of signed areas

3.2.14 apply the additivity and linearity of definite integrals

Fundamental theorem

3.2.15 examine the concept of the signed area function $F(x) = \int_a^x f(t)\,dt$

3.2.16 apply the theorem: $F'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$ and illustrate its proof geometrically

3.2.17 develop the formula $\int_a^b f'(x)\,dx = f(b) - f(a)$ and use it to calculate definite integrals

Applications of integration

3.2.18 calculate total change by integrating instantaneous or marginal rate of change

3.2.19 calculate the area under a curve

3.2.20 calculate the area between curves

3.2.21 determine displacement given velocity in linear motion problems

3.2.22 determine positions given linear acceleration and initial values of position and velocity.

Topic 3.3: Discrete random variables (15 hours)

General discrete random variables

3.3.1 develop the concepts of a discrete random variable and its associated probability function, and their use in modelling data

3.3.2 use relative frequencies obtained from data to obtain point estimates of probabilities associated with a discrete random variable

3.3.3 identify uniform discrete random variables and use them to model random phenomena with equally likely outcomes

3.3.4 examine simple examples of non-uniform discrete random variables

3.3.5 identify the mean or expected value of a discrete random variable as a measurement of centre, and evaluate it in simple cases

3.3.6 identify the variance and standard deviation of a discrete random variable as measures of spread, and evaluate them using technology

3.3.7 examine the effects of linear changes of scale and origin on the mean and the standard deviation

3.3.8 use discrete random variables and associated probabilities to solve practical problems

Bernoulli distributions

3.3.9 use a Bernoulli random variable as a model for two-outcome situations

3.3.10 identify contexts suitable for modelling by Bernoulli random variables

3.3.11 determine the mean p and variance $p(1 - p)$ of the Bernoulli distribution with parameter p

3.3.12 use Bernoulli random variables and associated probabilities to model data and solve practical problems

Binomial distributions

3.3.13 examine the concept of Bernoulli trials and the concept of a binomial random variable as the number of 'successes' in n independent Bernoulli trials, with the same probability of success p in each trial

3.3.14 identify contexts suitable for modelling by binomial random variables

3.3.15 determine and use the probabilities $P(X = x) \binom{n}{x} p^x(1 - p)^{n-x}$ associated with the binomial distribution with parameters n and p; note the mean np and variance $np(1 - p)$ of a binomial distribution

3.3.16 use binomial distributions and associated probabilities to solve practical problems

Unit 4 Content

This unit builds on the content covered in Unit 3. The examinable content for Unit 4 is as follows:

Topic 4.1: The logarithmic function (18 hours)

Logarithmic functions

4.1.1 define logarithms as indices: $a^x = b$ is equivalent to $x = \log_a b$ i.e. $a^{\log_a b} = b$

4.1.2 establish and use the algebraic properties of logarithms

4.1.3 examine the inverse relationship between logarithms and exponentials: $y = a^x$ is equivalent to $x = \log_a y$

4.1.4 interpret and use logarithmic scales

4.1.5 solve equations involving indices using logarithms

4.1.6 identify the qualitative features of the graph of $y = \log_a x (a > 1)$, including asymptotes, and of its translations $y = \log_a x + b$ and $y = \log_a (x - c)$

4.1.7 solve simple equations involving logarithmic functions algebraically and graphically

4.1.8 identify contexts suitable for modelling by logarithmic functions and use them to solve practical problems

Calculus of the natural logarithmic function

4.1.9 define the natural logarithm $\ln x = \log_e x$

4.1.10 examine and use the inverse relationship of the functions $y = e^x$ and $y = \ln x$

4.1.11 establish and use the formula $\frac{d}{dx}(\ln x) = \frac{1}{x}$

4.1.12 establish and use the formula $\int \frac{1}{x} dx = \ln x + c$, for $x > 0$

4.1.13 find derivatives of the form $\frac{d}{dx}(\ln f(x))$ and integrals of the form $\int \frac{f'(x)}{f(x)} dx, f(x) > 0$

4.1.14 use logarithmic functions and their derivatives to solve practical problems

Topic 4.2: Continuous random variables and the normal distribution (15 hours)

General continuous random variables

4.2.1 use relative frequencies and histograms obtained from data to estimate probabilities associated with a continuous random variable

4.2.2 examine the concepts of a probability density function, cumulative distribution function, and probabilities associated with a continuous random variable given by integrals; examine simple types of continuous random variables and use them in appropriate contexts

4.2.3 identify the expected value, variance and standard deviation of a continuous random variable and evaluate them using technology

4.2.4 examine the effects of linear changes of scale and origin on the mean and the standard deviation

Normal distributions

4.2.5 identify contexts, such as naturally occurring variation, that are suitable for modelling by normal random variables

4.2.6 identify features of the graph of the probability density function of the normal distribution with mean and standard deviation and the use of the standard normal distribution

4.2.7 calculate probabilities and quantiles associated with a given normal distribution using technology, and use these to solve practical problems

Topic 4.3: Interval estimates for proportions (22 hours)

Random sampling

4.3.1 examine the concept of a random sample

4.3.2 discuss sources of bias in samples, and procedures to ensure randomness

4.3.3 use graphical displays of simulated data to investigate the variability of random samples from various types of distributions, including uniform, normal and Bernoulli

Sample proportions

4.3.4 examine the concept of the sample proportion $\hat{p}$ as a random variable whose value varies between samples, and the formulas for the mean p and standard deviation $\sqrt{\frac{p(1-p)}{n}}$ of the sample proportion $\hat{p}$

4.3.5 examine the approximate normality of the distribution of $\hat{p}$ for large samples

4.3.6 simulate repeated random sampling, for a variety of values of p and a range of sample sizes, to illustrate the distribution of $\hat{p}$ and the approximate standard normality of $\frac{\hat{p}-p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}$ where the closeness of the approximation depends on both n and p

Confidence intervals for proportions

4.3.7 examine the concept of an interval estimate for a parameter associated with a random variable

4.3.8 use the approximate confidence interval $\left(\hat{p}-z\sqrt{\left(\frac{\hat{p}(1-\hat{p})}{n}\right)}, \hat{p}+z\sqrt{\left(\frac{\hat{p}(1-\hat{p})}{n}\right)}\right)$ as an interval estimate for p, where z is the appropriate quantile for the standard normal distribution

4.3.9 define the approximate margin of error $E = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ and understand the trade-off between margin of error and level of confidence

4.3.10 use simulation to illustrate variations in confidence intervals between samples and to show that most, but not all, confidence intervals contain p

Techniques of Differentiation

Chapter 1

1. **[7 marks]** *(CA 2008:01a,b,c)*

Differentiate the following with respect to x. Simplify your answers.

(a) $\sqrt{x^3+1}$ [2]

$y = (x^3+1)^{1/2}$

$y' = \frac{1}{2}(x^3+1)^{-1/2} \cdot 3x^2$

$= \dfrac{3x^2}{2\sqrt{(x^3+1)}}$

(b) $\dfrac{\sin x}{\cos x + 2}$ [3]

$y' = \dfrac{(\cos x + 2)\cos x - \sin x(-\sin x)}{(\cos x + 2)^2}$

$= \dfrac{\cos^2 x + 2\cos x + \sin^2 x}{(\cos x + 2)^2}$

$= \dfrac{1 + 2\cos x}{(\cos x + 2)^2}$

(c) $-3e^{-4x}$ [2]

$y' = 12e^{-4x}$

2. **[5 marks]** *(CA 2009:02a)*

Find $\frac{dy}{dx}$ given that

(a) $y = e^{3x+1}\cos(4x)$ SIMPLIFY YOUR ANSWER [3]

$y' = 3e^{3x+1}(\cos 4x) + -4\sin 4x\, e^{3x+1}$

$= e^{3x+1}(3\cos 4x - 4\sin 4x)$

(b) $y = \dfrac{\sqrt{x}}{3x+2}$ DO NOT SIMPLIFY YOUR ANSWER [2]

$= \dfrac{x^{0.5}}{3x+2}$

$y' = \dfrac{0.5x^{-0.5}(3x+2) - 3x^{0.5}}{(3x+2)^2}$

3. [7 marks] *(3ABMAS 2010:CF2a,b,c)*

Determine $\dfrac{dy}{dx}$ for each of the following, simplifying your answers:

(a) $y = (7x - 2)^6$ [2]

$y' = 6(7x - 2) \cdot 7$

$= 42(7x - 2)$

(b) $y = 2xe^{3x}$ [2]

$y' = 6xe^{3x} + 2e^{3x}$

(c) $y = \dfrac{5x}{4 + 9x}$ [3]

$$y' = \frac{5(4 + 9x) - 9(5x)}{(4 + 9x)^2}$$

$$= \frac{20 + 45x - 45x}{(4 + 9x)^2}$$

$$= \frac{20}{(4 + 9x)^2}$$

2. **[5 marks]** *(CA 2009:02a)*

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$y' = 3e^{3x+1}(\cos 4x) + -4\sin 4x\, e^{3x+1}$

$= e^{3x+1}(3\cos 4x - 4\sin 4x)$

(b) $y = \frac{\sqrt{x}}{3x+2}$ DO NOT SIMPLIFY YOUR ANSWER [2]

$= \frac{x^{0.5}}{3x+2}$

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$$= \frac{20}{(4+9x)^2}$$

4. [4 marks] *(3CDMAT 2013:CF3)*

Let $f(x) = \frac{1}{x^2} + \frac{e^{2x}}{2}$.

x^{-2} $-x^{-3}$ $-2x^{-3}$

Determine the second derivative $f''(x)$.

$f'(x) = \frac{-2}{x^3} + e^{2x}$

$f''(x) = \frac{6}{x^4} + 2e^{2x}$

5. **[6 marks]** *(3CDMAT 2013:CF4a,b)*

Let $f(x) = (x - 1)(x^2 - 16)$.

(a) Show that $f'(x) = (3x - 8)(x + 2)$. [3]

$f'(x) = (x^2 - 16) + 2x(x - 1)$

$= x^2 - 16 + 2x^2 - 2x$

$= 3x^2 - 2x - 16$

$= (3x - 8)(x + 2)$

(b) Determine the equation of the tangent to the graph of $f(x)$ at the point where $x = 3$. [3]

when $x = 3$

$f'(3) = (9 - 8)(3 + 2)$

$= 5$

$f(3) = (2)(9 - 16) = -14$

$(5, -14)$ $m = 5$ $y = 5x + c$

$-14 = 5(5) + c$

$c = -39$

$y = 5x - 39$

Chapter

2 Growth and Decay with Exponential Functions

1. **[5 marks]** *(3CDMAT 2012:CA11)*

Iodine-131 is present in radioactive waste from the nuclear power industry.

It has a half-life of eight days. This means that every eight days, one half of the iodine-131 decays to a form that is not radioactive.

This decay can be represented by the equation $N = N_0e^{kt}$,

where N = amount of iodine-131 present after t days, and
N_0 = amount of iodine-131 present initially.

(a) Determine the value of k correct to **three (3)** decimal places. [3]

(b) If 125 milligrams of iodine-131 are considered to be safe, how many days will it take for 88 grams of iodine-131 to decay to a safe amount? [2]

2. **[13 marks]** *(3CDMAT 2013:CA11)*

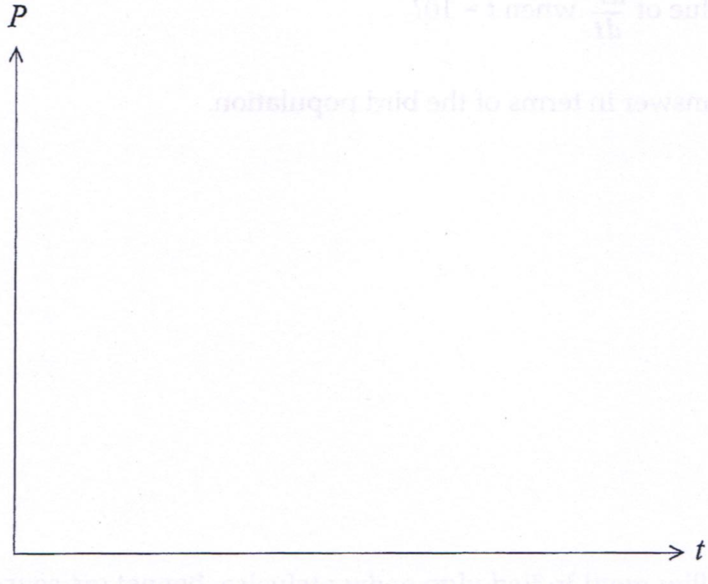
The size of a population of birds is changing according to the rule $\frac{dP}{dt} = -0.08P$, (i.e. $P = P_0e^{-kt}$) where P is the number of birds in the population, and t is the time in years from the initial population measurement.

There are initially 1000 birds in the population.

(a) Describe the type of relationship between P and t. [2]

(b) State an equation for P in terms of t. [1]

(c) Sketch the graph of P against t on the axes below. [3]



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2. (cont)

(d) Determine

(i) the number of birds in the population after 10 years. [1]

(ii) the number of years, to one decimal place, after which there are 800 birds in the population. [3]

(e) What is the value of $\frac{dP}{dt}$ when $t = 10$?

Interpret this answer in terms of the bird population. [3]

3. **[7 marks]** *(3ABMAS 2014:CA18)*

A botanist has found that the rate of the loss of leaves from a particular rainforest tree is proportional to the number of leaves on the tree.

If the initial number of leaves on the tree is L_0, then the number of these leaves that remain after time t years is

$$L(t) = L_0e^{-kt}$$

where k is a positive constant.

(a) The botanist wishes to study lichens that grow on the leaves of this tree. From past observations, he has found that of 100 leaves tagged, 80 remain after three years. Use this information to determine the value of the constant k. [2]

(b) Using your result from part (a), calculate the number of leaves that the botanist should initially tag if he needs to have 200 leaves to study in eight years' time. [3]

(c) If initially N leaves are tagged, calculate when only half of them will remain on the tree. Give your answer correct to two significant figures. [2]

4. [6 marks] *(3CDMAS 2015:CA19a,b,c,d)*

A type of lichen, L_1, is flat and circular, and grows continuously, always maintaining its circular shape with a fixed centre.

At any time t months, the instantaneous rate of growth of its surface area, A cm^2, is $0.2A$.
So $A = A_0e^{0.2t}$

(a) Write the specific formula for A, given the surface area of L_1 is 2 cm^2 when $t = 0$. [1]

(b) Find the surface area of L_1 after three months. [1]

Another type of lichen L_2 grows in such a way that its rate of increase of surface area is given by $\frac{dB}{dt} = 0.4\,B$. So $B = B_0e^{0.4t}$

(c) Write the specific formula for B, given that the surface area of L_2 is 1.5 cm^2 when $t = 1$. [2]

(d) When, to the nearest month, will L_1 and L_2 have the same surface areas? [2]

5. [6 marks] *(MMETH 2016S:CA10)*

Certain medical tests require the patient to be injected with a solution containing 0.5 micrograms (μg) of the radioactive material Technetium-99. This material decays according to the rule:

$T = T_0 e^{-0.1155t}$ where t is the time (in hours) from injection.

(a) What is the value of T_0? [1]

(b) What is the half-life of Technetium-99? [2]

(c) After how long is the amount of Technetium-99 left in the patient's system less than 1% of the initial amount? Give your answer to the nearest hours. [3]

6. **[7 marks]** *(MMETH 2016:CA09)*

Fermium-257 is a radioactive substance whose decay rate can be modelled by the formula $P = P_0e^{kt}$, where P is the mass in grams and t is measured in days and P_0 = original amount and k is a constant. The time taken to decay to half of the original amount is known as half-life. The half-life of Fermium-257 is 100.5 days.

(a) Determine the value of k to three significant figures. [3]

(b) How many days will it take for 100 grams of the substance to first decay below five grams? [2]

(c) Determine the rate of change of the amount of Fermium on the day found in part (b). [2]

Calculus and Trigonometric Functions

Chapter 3

1. *(Projected)*

Given that x is in radians and that $x = \frac{\pi}{180} \cdot \theta$ and $y = \sin x$

(a) Find $\frac{dy}{d\theta}$ using the chain-rule $\frac{dy}{d\theta} = \frac{dy}{dx} \cdot \frac{dx}{d\theta}$ and express your answer in terms of θ.

$\frac{dx}{d\theta} = \frac{\pi}{180}$

$\frac{dy}{dx} = \cos x$

$\frac{dy}{d\theta} = \frac{\pi}{180} \cos x$

$x = \frac{\pi}{180}\theta$

$\frac{dy}{d\theta} = \frac{\pi}{180} \cos\left(\frac{\pi}{180}\theta\right)$

(b) Use your answer in (a) to find $\frac{dy}{d\theta}$ when $\theta = 60°$.

when $\theta = 60$

$= \frac{\pi}{180} \cos\left(\frac{\pi}{180} \times 60\right)$

2. **[6 marks]** *(CA 2003:04)*

Find the equation of the line that is tangent to the graph of

$$y = 3\sin 2x - \cos 2x$$

at the point on the graph where $x = \dfrac{\pi}{4}$.

3. **[4 marks]** *(CA 2007:01b)*

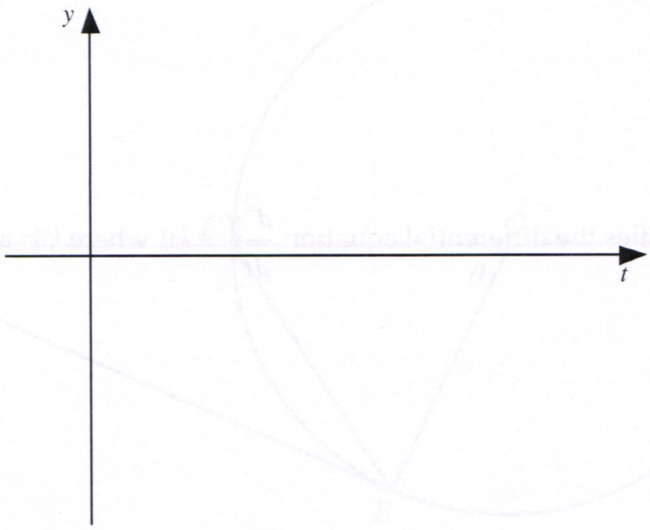
Find the equation of the tangent to the curve $y = 3\sin\left(\dfrac{4\pi x}{3}\right)$ at $x = 3$.

4. [10 marks] *(CA 2008:15a,b,c,d,e)*

The height $y(t)$ metres of a sinusoidal wave at time t seconds is given by

$$y(t) = 4\sin(3t + 1).$$

(a) Sketch the graph of $y(t)$ over the time interval $0 \le t \le 5$. [2]



(b) Write down the amplitude and the period of the wave. [2]

(c) Find, exactly, the first time after $t = 0$ when the height of the wave is 2 metres. [3]

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4. (cont)

(d) Find the maximum rate of increase of the height of the wave. [1]

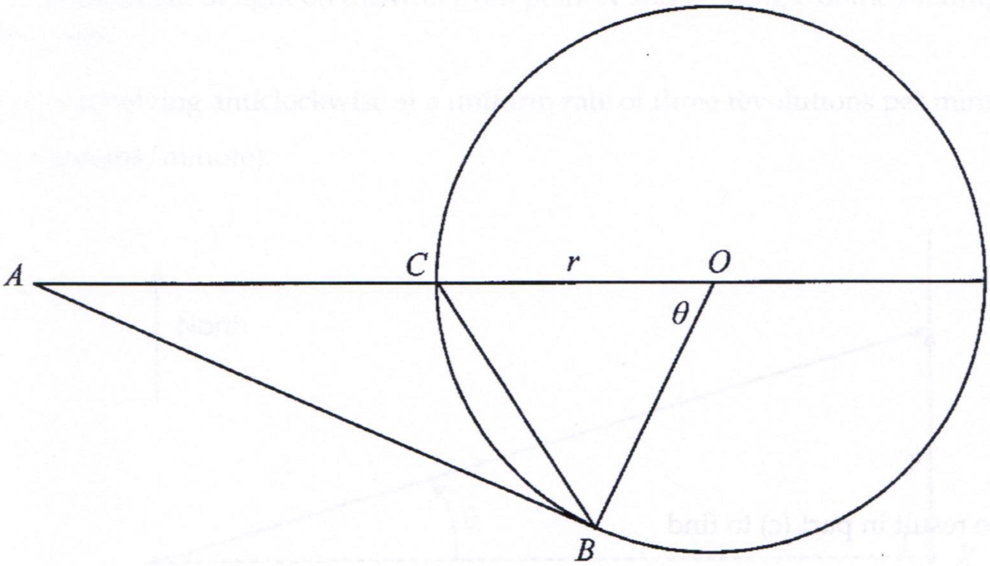
The function $y(t)$ satisfies the differential equation $\frac{d^2y}{dt^2} = ky$, where k is a constant.

(e) Find k. [2]

5. **[9 marks]** *(3CDMAS 2015:CF7)*

The diagram below shows a tangent AB to the circle with centre O and radius r.

Let the size of $\angle BOA$ be θ radians where $0 \leq \theta \leq \frac{\pi}{2}$.



(a) Determine the area of $\triangle BOC$ in terms of r and θ. [1]

(b) Determine the area of $\triangle AOB$ in terms of r and θ. [2]

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5. (cont)

(c) Given that $Q(\theta) = \dfrac{\text{Area of } \Delta BOC}{\text{Area of } \Delta AOB}$

Show that this quotient simplifies to $Q(\theta) = \cos\theta$ [2]

(d) Use the result in part (c) to find

(i) the angle θ at which the area of $\triangle BOC$ is exactly half the area of $\triangle AOB$. [2]

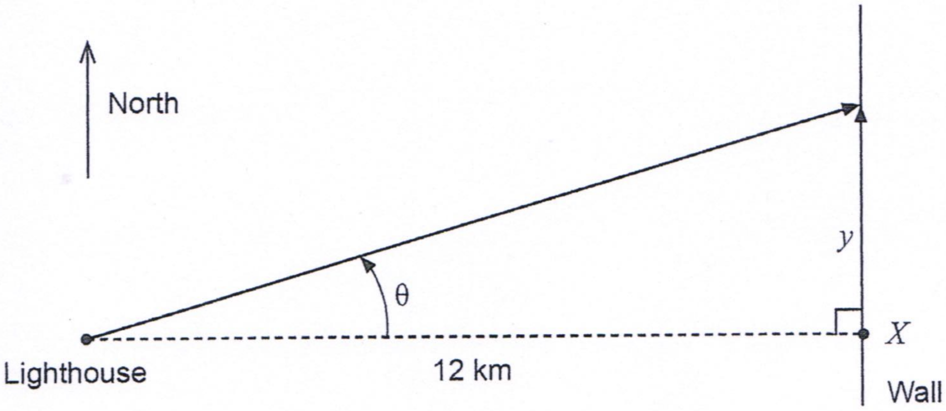
(ii) the angle θ at which the rate of change of $Q(\theta)$ is decreasing at 0.5. (ignore units for rate) [2]

6. **[6 marks]** *(MMETH 2016:CA21)*

A lighthouse is situated 12 km away from the shoreline, opposite point X as seen in the diagram below. A long brick wall is placed along the shoreline and at night the light from the lighthouse can be seen moving along this wall.

Let y = displacement of light on the wall from point X and θ = angle of the rotating light from the lighthouse.

The light is revolving anticlockwise at a uniform rate of three revolutions per minute ($\frac{d\theta}{dt}$ = 6π radians/minute).



(a) Show that $\frac{dy}{d\theta} = \frac{12}{\cos^2\theta}$. [3]

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6. (cont)

(b) Determine the velocity, in kilometres per minute, of the light on the wall when the light is 5 km north of point *X*. [3]

(Hint: $\frac{dy}{dt} = \frac{dy}{d\theta} \times \frac{d\theta}{dt}$)

Curve Sketching

Chapter
4

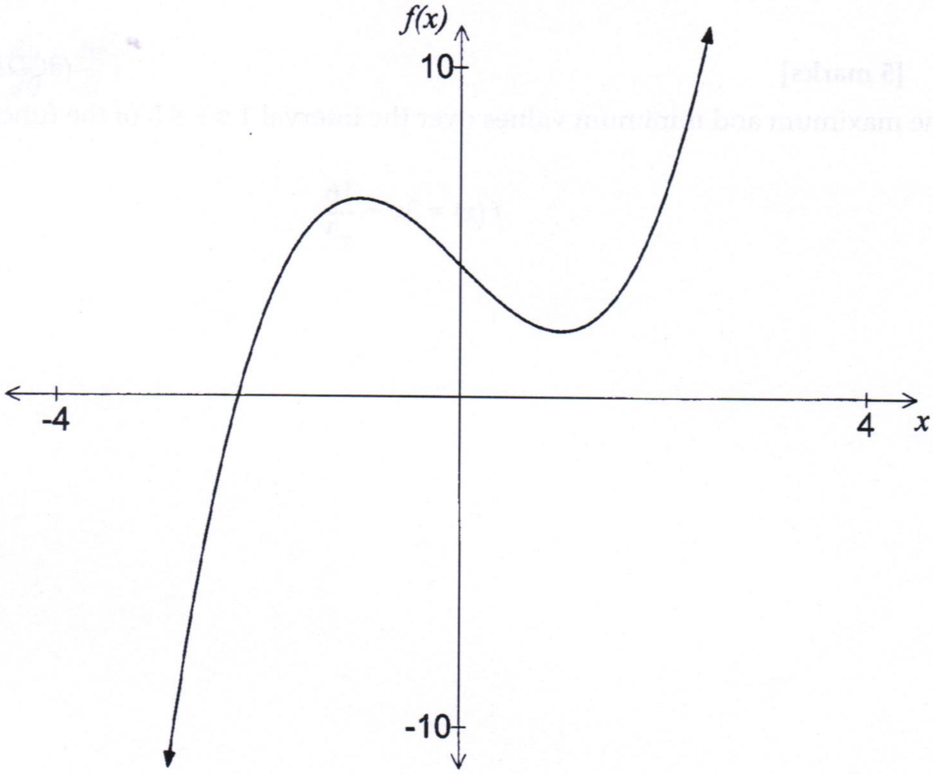
1. **[5 marks]** *(3CDMAT 2010:CF3)*

Find the maximum and minimum values over the interval $1 \le x \le 5$ of the function

$$f(x) = 3x + \frac{16}{x^3}$$

2. **[5 marks]** *(3CDMAT 2010:CF7a,b)*

The graph of $y = f(x) = x^3 - 3x + 4$ is shown below.



(a) Determine the coordinates of the turning points of the function f. [3]

(b) For what values of x is it true that $f'(x) < 0$ and $f''(x) > 0$? [2]

3. **[4 marks]** *(3CDMAT 2011:CF2)*

Calculate the maximum and minimum values of $x^2(6 - x)$ in the interval $1 \le x \le 5$.

4. [3 marks] *(3CDMAT 2011:CF6)*

The cubic polynomial $p(x) = ax + bx^2 + cx^3$ has the following properties:

- $p(3) = 135$
- $p(x)$ has a turning point at $x = 6$
- $p(x)$ has a point of inflection at $x = 2$

Explain why the constants a, b and c satisfy the simultaneous equations:

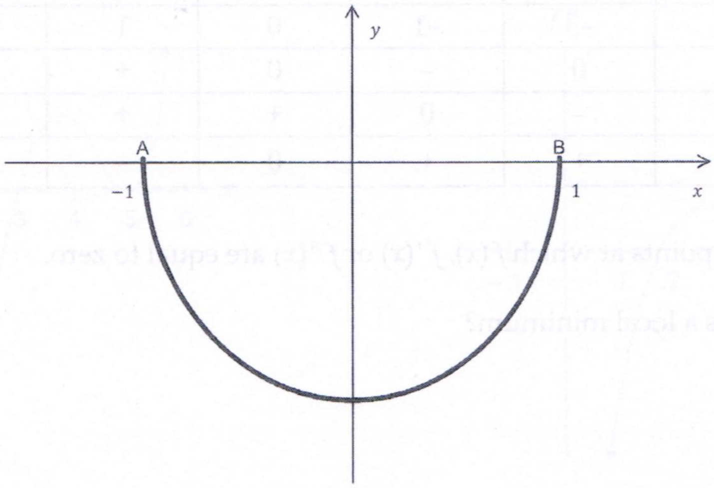
$a + 12b + 108c = 0$, $b + 6c = 0$ and $a + 3b + 9c = 45$.

5. **[4 marks]** *(3CDMAT 2011:CA18a,b)*

A cable hanging between two points $A(-1, 0)$ and $B(1, 0)$ lies on the curve

$$y = e^{cx} - d + e^{-cx},$$

where c and d are positive constants.



(a) Show that $d = e^{c} + e^{-c}$. [1]

(b) Use calculus to show that the lowest point on the cable occurs where it crosses the y-axis, that is, where $x = 0$. [3]

6. [7 marks] *(3CDMAT 2012:CF3)*

Let A, B, C, D, E, F and G be points on the graph of a continuous function $f(x)$.
The table below contains information about the sign of $f(x)$, $f'(x)$ and $f''(x)$ at these points.

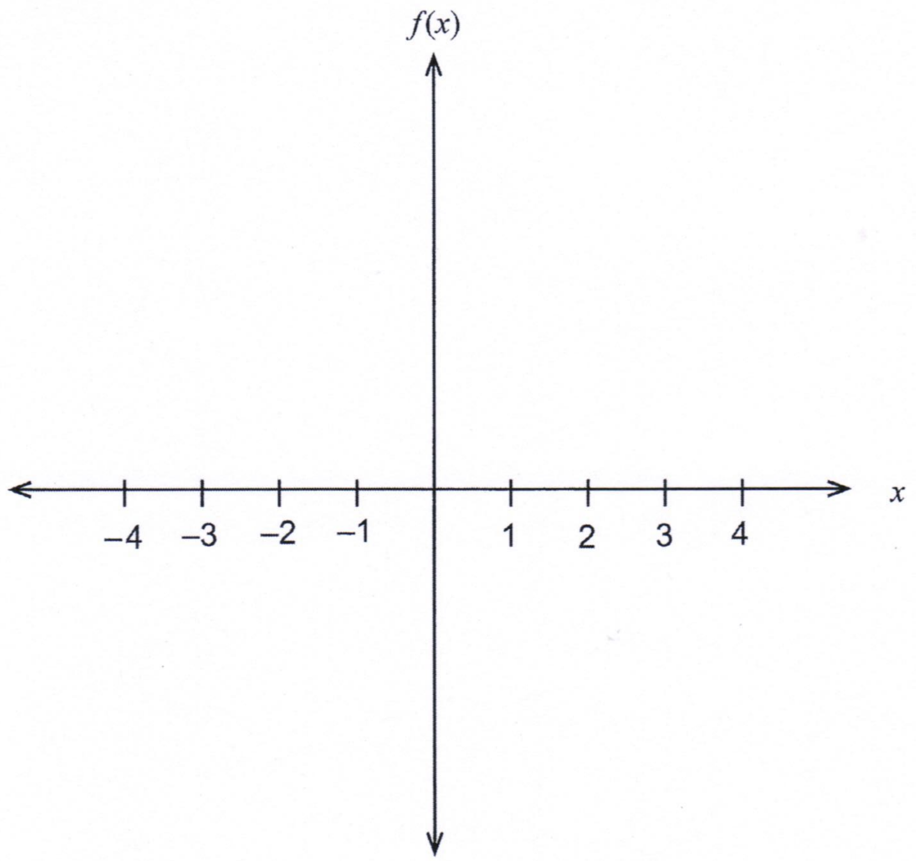
Point	**A**	**B**	**C**	**D**	**E**	**F**	**G**
x	–4	–3	–1	0	1	2	4
$f(x)$	+	0	–	0	+	+	+
$f'(x)$	–	–	0	+	+	0	+
$f''(x)$	+	+	+	0	–	0	+

There are no other points at which $f(x)$, $f'(x)$ or $f''(x)$ are equal to zero.

(a) Which point is a local minimum? [1]

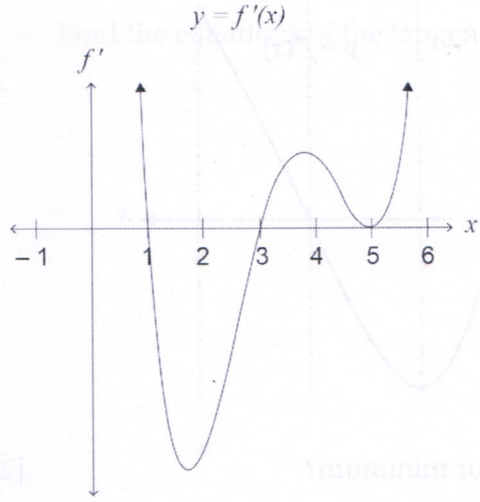
(b) Describe the nature of the graph at point F. [2]

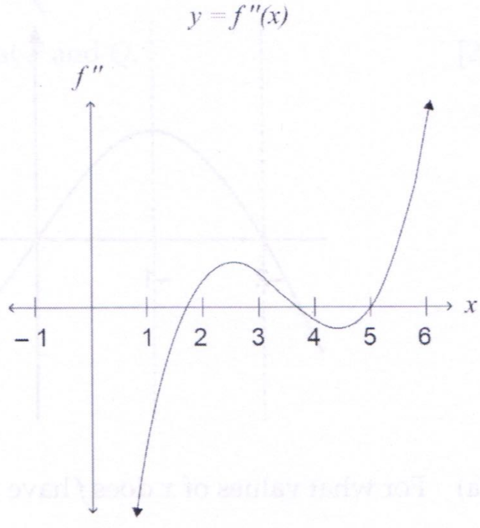
(c) Sketch the function on the axes below. [4]



7. [4 marks] *(3CDMAT 2014:CF7)*

The graphs of $y = f\,'(x)$ and $y = f\,''(x)$ of a function $y = f(x)$ are shown below. The function $y = f(x)$ passes through points (1, 4), (3, –2) and (5, 1).



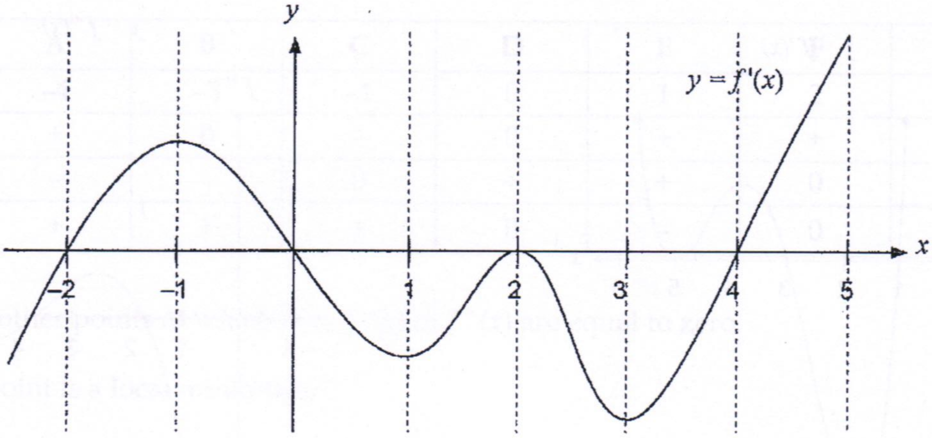


(a) Determine the coordinates of the maximum and minimum points of $y = f(x)$. [2]

(b) Determine whether there exists a horizontal point of inflection. Give reasons. [2]

8. [9 marks] *(3CDMAT 2015:CF7)*

The figure below shows the graph of the derivative f' of a function f.

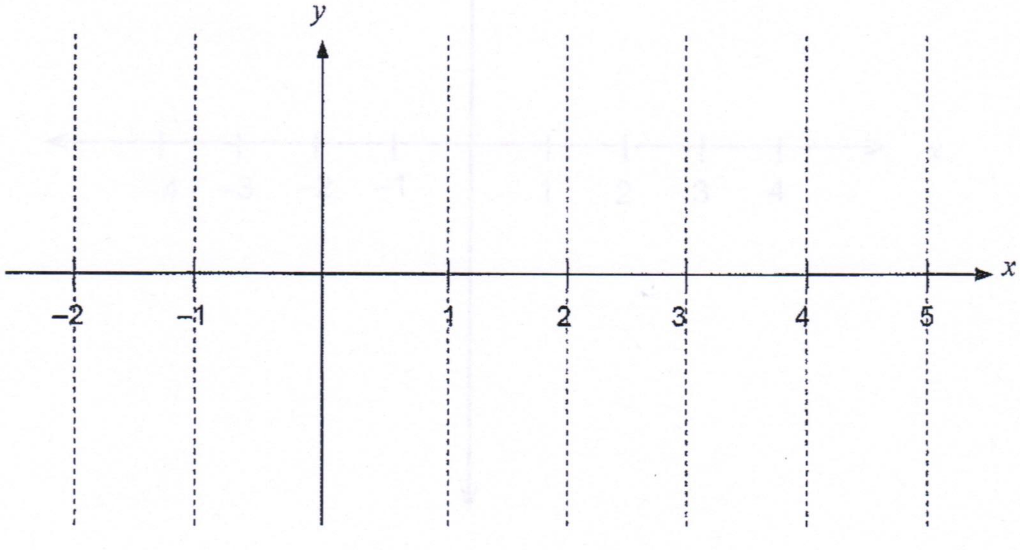


(a) For what values of x does f have a local maximum or minimum? [2]

(b) For what values of x does f have an inflection point? [2]

(c) Does f have a horizontal point of inflection? Explain. [2]

(d) On the axis below, sketch the graph of f''. [3]



9. **[4 marks]** *(3CDMAT 2015:CA11)*

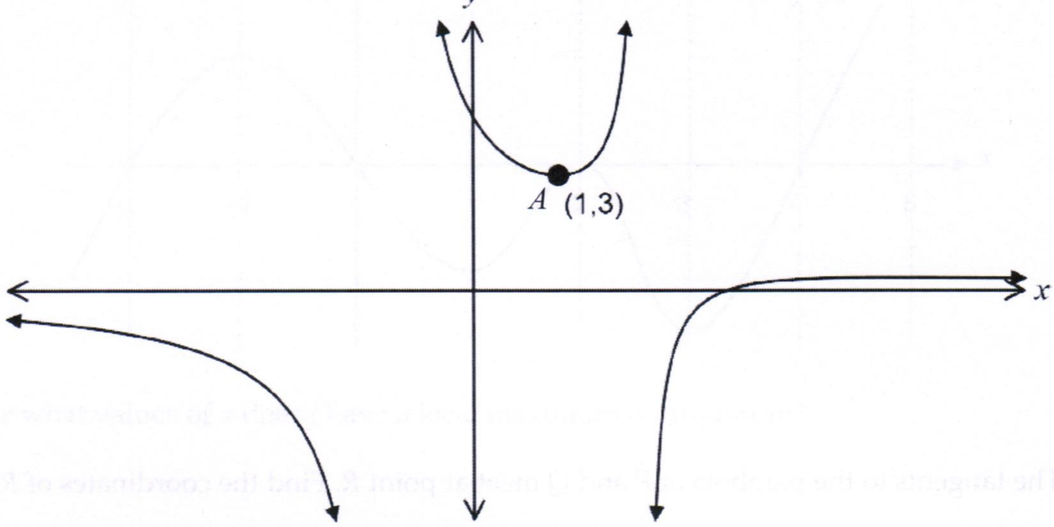
The points $P(-2, 1)$ and $Q(6, 9)$ lie on the parabola $y = \frac{x^2}{4}$.

(a) Find the equations of the tangents to the parabola at P and Q. [2]

(b) The tangents to the parabola at P and Q meet at point R. Find the coordinates of R. [2]

10. [6 marks] *(MMETH 2016S:CF8)*

Consider the graph of $f(x) = \dfrac{3x-9}{x^2 - x - 2}$ shown below with a local minimum at A (1, 3).



(a) Show that $f'(x) = \dfrac{-3(x-1)(x-5)}{(x^2 - x - 2)^2}$ [3]

(b) Hence or otherwise determine the coordinates of the local maximum value of $f(x)$. [3]

11. **[7 marks]** *(MMETH 2016:CF03)*

Consider the function $f(x) = \dfrac{(x-1)^2}{e^x}$.

(a) Show that the first derivative is $f'(x) = \dfrac{-x^2+4x-3}{e^x}$ [2]

(b) Use your result from part (a) to explain why there are stationary points at $x = 1$ and $x = 3$. [2]

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11. (cont)

It can be shown that the second derivative is $f''(x) = \dfrac{x^2 - 6x + 7}{e^x}$.

(c) Use the second derivative to describe the type of stationary points at $x = 1$ and $x = 3$. [3]

Small Changes Incremental Formula

Chapter **5**

1. *(Projected)*

A spherical balloon has volume $V = \dfrac{4\pi r^3}{3}$, where r is its radius.

(a) Determine an expression for $\dfrac{dV}{dr}$.

(b) Using the formula $\delta V \approx \dfrac{dV}{dr} \cdot \delta r$, find the increase in the balloon's volume when its radius increases from 10 to 10.1 cm. Give your answer to the nearest cm^3.

2. *(Projected)*

The volume V of blood flowing through an artery per unit time can be modelled by the formula $V = kr^4$, where r is the radius of the artery and k is a constant.

(a) Find $\frac{dV}{dr}$.

(b) Use the incremental formula to estimate the increase in the blood flow when the radius of the artery increases from 1 mm to 1.2 mm. Use $k = 0.95$.

(c) Compare your answer from part (b) to the true value of the increase in blood flow using $V = kr^4$ and comment why the incremental formula may not be a good estimate.

3. *(Projected)*

A spherical balloon has surface area $S = 4\pi r^2$, where r is its radius.

(a) Determine an expression for $\frac{dS}{dr}$.

(b) Using the formula $\delta S \approx \frac{dS}{dr} \cdot \delta r$, find the approximate percentage increase in the balloon's surface area when its radius increases by 1%.

4. **[4 marks]** *(3CDMAT 2014:CF5)*

Given that $y = x^{\frac{1}{3}}$ use $x = 1000$ and the increments formula $\delta y \approx \frac{dy}{dx}\,\delta x$ to determine an approximate value for $\sqrt[3]{1006}$.

5. **[3 marks]** *(3CDMAT 2015:CA17b)*

A pendulum consists of a bob connected to a rope of length ℓ metres, where ℓ is a function of time t.

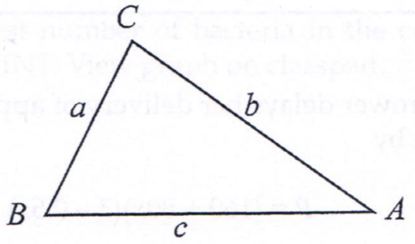
The time T seconds taken for a complete swing (back and forth) is given by the formula

$$T = 2\pi\sqrt{\frac{\ell}{10}}.$$

Use the increments formula $\delta T \approx \frac{dT}{d\ell}\,\delta\ell$ to determine the approximate percentage change in T if ℓ changes by 2% (that is, $\frac{\delta\ell}{\ell} = 0.02$).

6. [3 marks] *(MMETH 2016:CA11)*

The area of a triangle can be found by the formula: $Area = \dfrac{ab\sin C}{2}$.



Using the incremental formula, determine the approximate change in area of an equilateral triangle, with each side of 10 cm, when each side increases by 0.1 cm.

Chapter
6 Optimisation

1. *(Projected:CF)*

For each week an apple grower delays her delivery of apples to the market, her profit from selling the apples is given by

$$P = (160 + 80x)(3 - 0.5x)$$

where P is the profit in \$ after a delay of x weeks.

Use Calculus techniques to find how many weeks the apple grower should delay in order to maximise profits and find this maximum profit.

2. **[6 marks]** *(CA 2009:10a,b)*

The number of bacteria in a culture at time t hours is given by

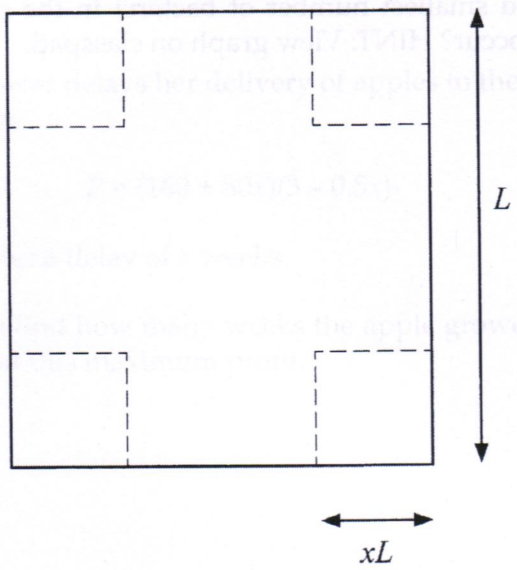
$$N = 500(10 + te^{-\frac{t}{10}}), \text{ for } t \geq 0.$$

(a) Find the greatest and smallest number of bacteria in the culture during the first two days. When do they occur? HINT: View graph on classpad. [3]

(b) When is the number of bacteria decreasing most rapidly? Justify your answer. [3]

3. **[7 marks]** *(3CDMAT 2012:CA19)*

A square sheet of metal has sides of fixed length L cm.
A tray is constructed by cutting smaller square pieces out of the corners of the metal sheet and folding up the sides. Each of the pieces has side length xL cm.



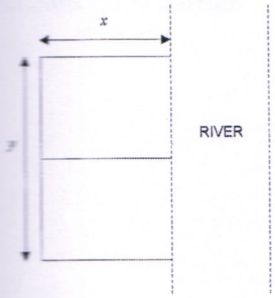
(a) Show that the volume of the tray is given by $V = L^3(x - 4x^2 + 4x^3)$ cm^3. [3]

(b) What is the maximum possible volume of the tray, in terms of L? [4]

4. **[7 marks]** *(3CDMAT 2013:CA10)*

A farmer has $1500 available to build an E-shaped fence along a straight river so as to create two identical rectangular pastures.

The materials for the side parallel to the river cost $6 per metre and the materials for the three sides perpendicular to the river cost $5 per metre.



Each of the sides perpendicular to the river is x metres long, and the side parallel to the river is y metres long.

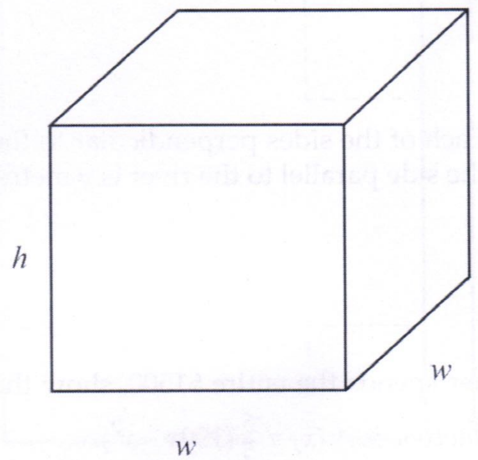
(a) Assuming that the farmer spends the entire $1500, show that the total area $A(x)$ of the two pastures, in square metres, is $A(x) = \frac{5}{2}(100x - x^2)$. [3]

(b) Use calculus methods to determine the dimensions of the fence that maximise the total area, and state this area. [4]

5. [7 marks] *(3CDMAT 2014:CA16)*

A closed box is constructed with a square base. Exactly 10 square metres of material is to be used in the construction of the box, without wastage.

Let h = height of the box, w = width of box = length of box.



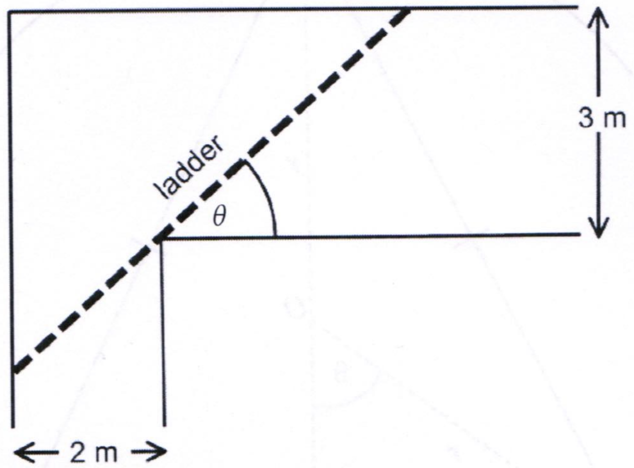
(a) Show that $5 = w^2 + 2wh$. [2]

(b) By using calculus, determine the maximum volume of the box and state the dimensions required to achieve this maximum. [5]

6. [7 marks] *(MMETH 2016S:CA19)*

Two corridors meet at right angles and are 3 m and 2 m wide respectively. The angle between the wall and the ladder is marked on the diagram as θ.

A ladder (of negligible width) is to be carried horizontally along this L-shaped space by two workers. The workers need to know the length of the longest ladder that can be carried around this corner.

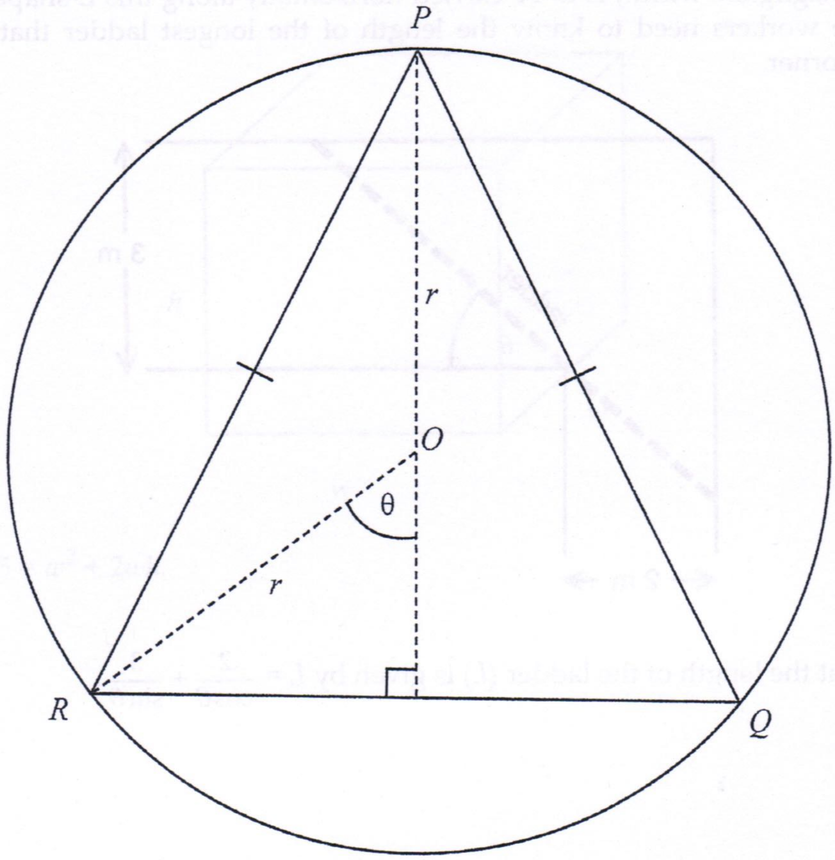


(a) Show that the length of the ladder (L) is given by $L = \dfrac{2}{\cos\theta} + \dfrac{3}{\sin\theta}$. [3]

(b) The workers' ladder is 6.5 m long. Will they be able to carry their ladder along this L-shaped space? Justify your answer. [4]

7. [7 marks] *(MMETH 2016:CF08)*

An isosceles triangle $\triangle PQR$ is inscribed inside a circle of fixed radius r and centre O. Let θ be defined as in the diagram below.



(a) Show that the area A of the triangle $\triangle PQR$ is given by $A = r^2 \sin\theta\,(1 + \cos\theta)$. [2]

7. (cont)

(b) Using calculus, determine the value of θ that maximises the area A of the inscribed triangle. State this area in terms of r exactly. Justify your answer.
(Hint: you may need the identity $\sin^2 x = 1 - \cos^2 x$ in your working.) [5]

Chapter 7 Techniques of Integration

1. *(Projected)*

Evaluate the following integral, showing sufficient working to justify your answer.

$$\int \left(e^{2t} - 2 + \frac{1}{e^{2t}}\right) dt$$

2. [3 marks] *(3CDMAT 2010:CF5a)*

Evaluate $\int_1^3 (x^3 - 1)\, dx$

3. [2 marks] *(3CDMAT 2011:CF5a)*

Evaluate $\int_{-0.5}^{0} 3(1-x)^2 \; dx$

4. [6 marks] *(3CDMAT 2012:CF5b,c)*

(a) If $\frac{dy}{dx} = \frac{2}{x^2} + 4x$, and $y = 3$ when $x = 2$, determine the value of y when $x = 5$. [3]

(b) Evaluate $\int_1^2 \frac{d}{dx}\left(\frac{x^3}{x^2+1}\right) dx$. [3]

5. **[9 marks]** *(3CDMAT 2014:CF1)*

Evaluate the following:

(a) $\int_0^3 (6x^2 + 2x + 1)\, dx$ [3]

(b) $\int_1^2 \frac{d}{dx}\left(\frac{x^5}{x^2 + 1}\right) dx$ [3]

(c) $\frac{d}{dx} \int_4^{x^2} \frac{2}{3t^3 - 1}\, dt.$ [3]

6. [5 marks] *(3CDMAS 2015:CF6)*

(a) Determine $\frac{d}{dx}(xe^{2x})$. [2]

(b) Hence evaluate exactly $\int_0^1 2xe^{2x}\,dx$. [3]

7. **[5 marks]** *(3CDMAT 2015:CF6)*

The function $f(x)$ has the following properties:

- $f(x)$ is defined for all real numbers
- $\int_{-\infty}^{x} f(t)dt = f(x)$
- $f(0) = 1$.

(a) Determine a function $f(x)$ that satisfies all of the above properties. (Hint: consider the derivative of $f(x)$.) [3]

(b) Is the function $f(x)$ above unique? Justify your answer. [2]

8. [5 marks] *(MMETH 2016:CF02)*

(a) Determine $\dfrac{d}{dx}\,(2xe^{2x})$. [2]

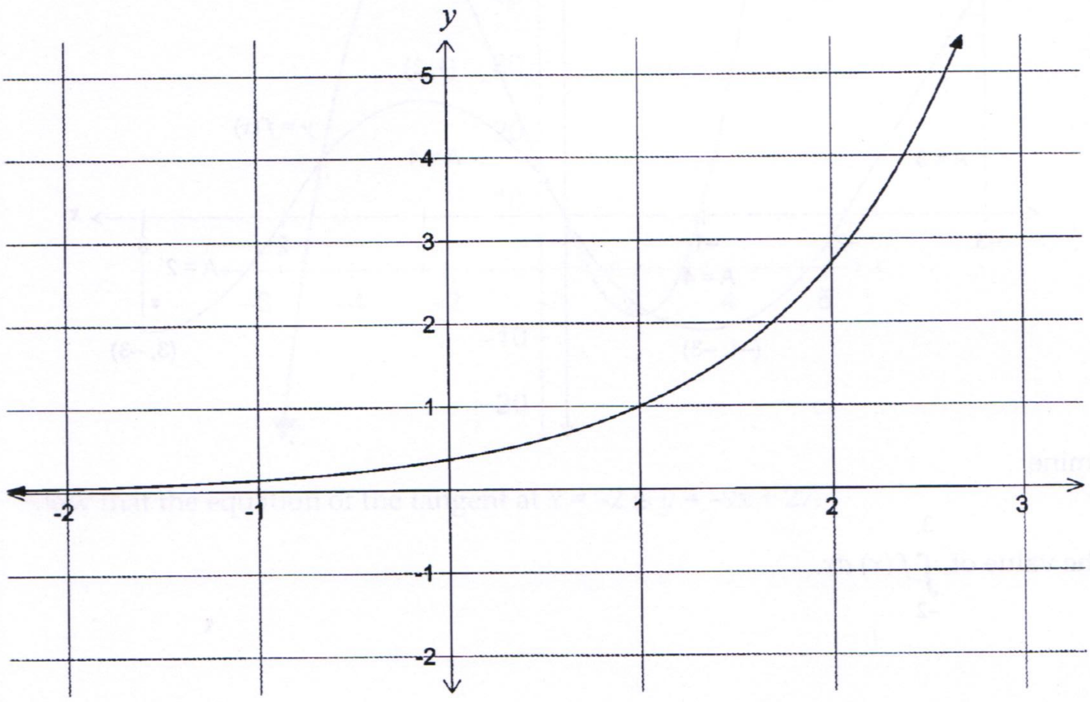
(b) Use your answer in part (a) to determine $\int 4xe^{2x}dx$. [3]

Area under the Curve

Chapter **8**

1. [6 marks] *(3CDMAS 2010:CF4)*

The graph of $y = e^{(x-1)}$ is shown below.



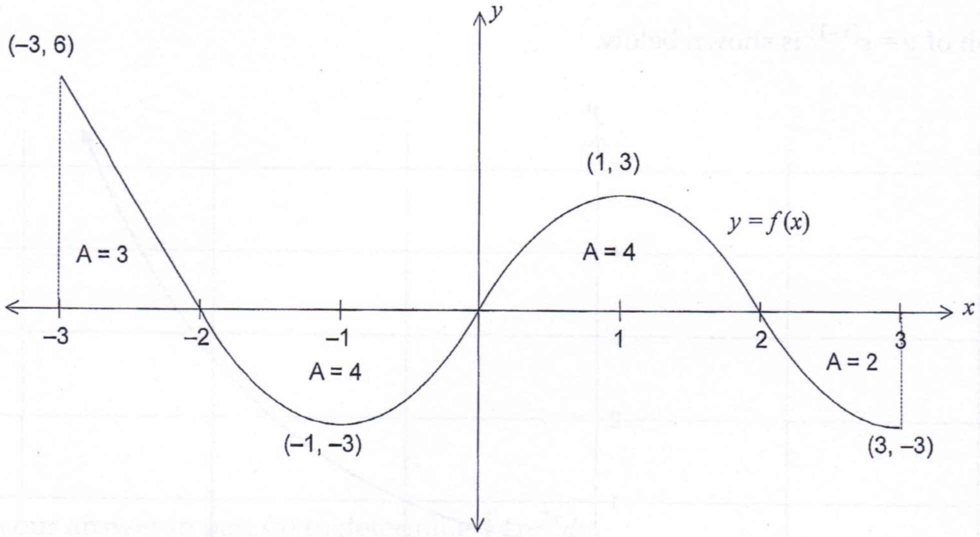
(a) Calculate the exact area between the graph $y = e^{(x-1)}$ and the x-axis between $x = 1$ and $x = 2$. [2]

(b) Calculate the exact area between the graphs of $y = e^{(x-1)}$, $y = 2 - x$ and the two axes. [4]

2. **[7 marks]** *(3CDMAT 2013:CF7a,b,d)*

The graph of the function $f(x)$ is shown below for $-3 \le x \le 3$.

The areas enclosed between the graph, the x-axis and the lines $x = -3$ and $x = 3$ are marked in the appropriate regions.



Determine:

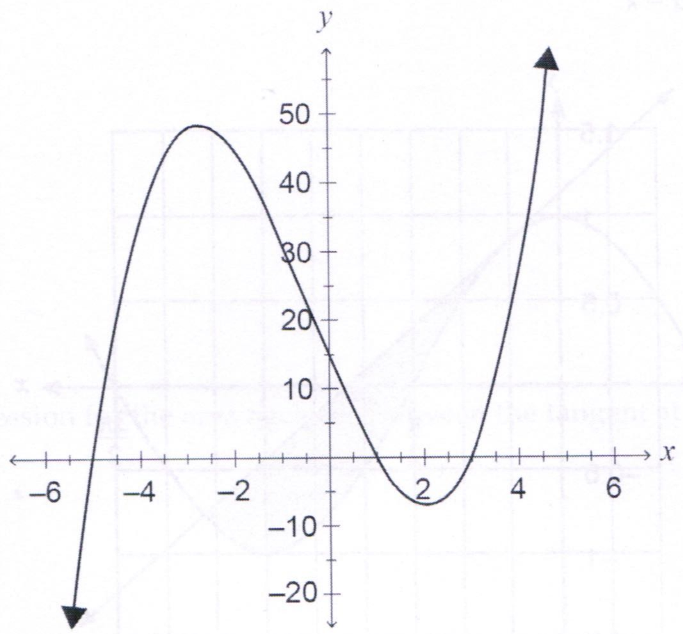
(a) the value of $\int_{-2}^{3} f(x)\,dx$. [2]

(b) the area enclosed between the graph of $f(x)$ and the x-axis, from $x = -2$ to $x = 3$. [2]

(c) the value of $\int_{0}^{2} (x - f(x))\,dx$. [3]

3. [7 marks] *(3CDMAT 2014:CA15)*

Consider the curve defined by the rule $y = x^3 + x^2 - 17x + 15$ shown below.

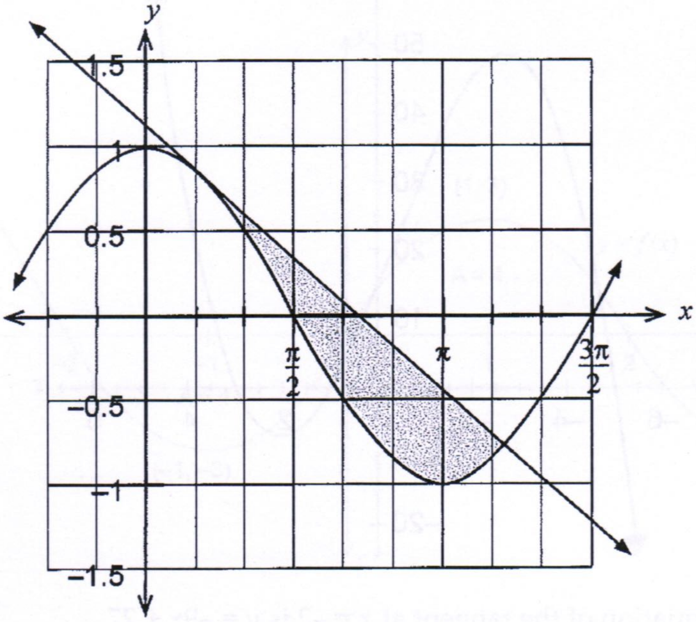


(a) Show that the equation of the tangent at $x = -2$ is $y = -9x + 27$. [3]

(b) Determine the area enclosed between the curve and the tangent at $x = -2$. [4]

4. [8 marks] *(3CDMAS 2015:CA10)*

The tangent to the graph of $y = \cos x$ is drawn at $x = \frac{\pi}{6}$. This tangent intersects the graph of $y = \cos x$ again at $x = k$.



(a) Determine the equation of the tangent at $x = \frac{\pi}{6}$.

(State the y intercept in the equation to 4 d.p.'s) [3]

4. (cont)

(b) Determine the value of k, correct to three decimal places. [2]

(c) Write an expression for the area enclosed between the tangent at $x = \frac{\pi}{6}$ and the curve $y = \cos x$. [2]

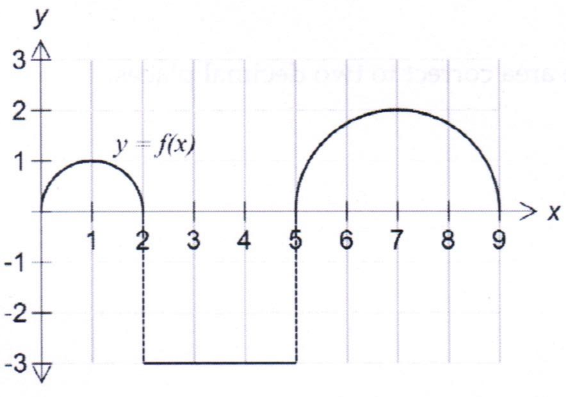
(d) Hence evaluate the area correct to two decimal places. [1]

5. **[4 marks]** *(3CDMAT 2015:CA13)*

The area bound by the parabola $y = 6x^2 - 6x$, the x-axis and the lines $x = 1$ and $x = c$, $(c > 1)$, is equal to 1 unit^2. Find the value of the constant.

6. **[4 marks]** *(MMETH 2016S:CF1)*

Use the graph of $y = f(x)$ to calculate the following definite integrals.



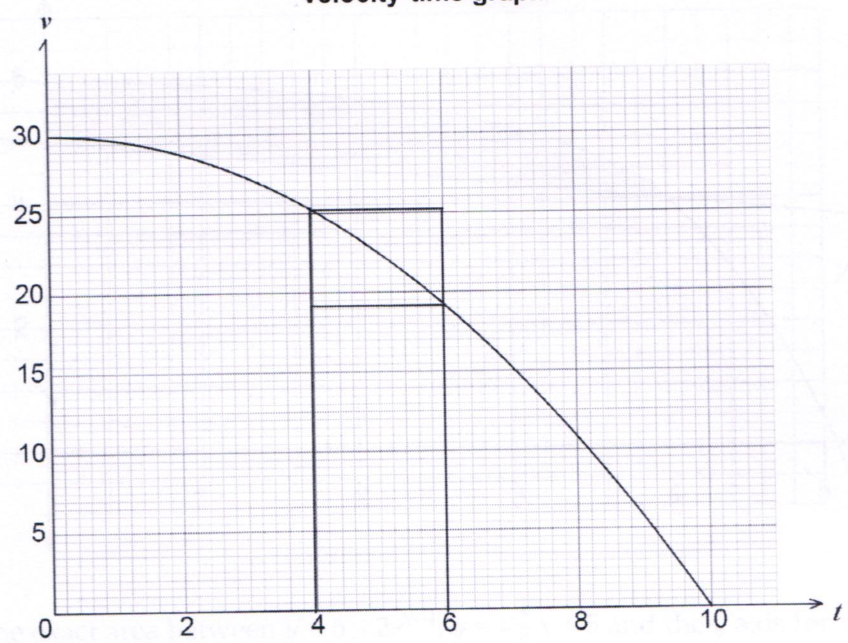
(a) $\int_0^5 f(x)dx$ [2]

(b) $\int_0^9 f(x)dx$ [2]

7. **[6 marks]** *(MMETH 2016S:CA15)*

A train is travelling at 30 metres per second when the brakes are applied. The velocity of the train is given by the equation $v = 30 - 0.3t^2$, where t represents the time in seconds after the brakes are applied.

Velocity-time graph



The area under a velocity-time graph gives the total distance travelled for a particular time period.

(a) Complete the tables below and estimate the distance travelled by the train during the first six seconds by calculating the mean of the areas of the circumscribed and inscribed rectangles. (The rectangles for the 4–6 seconds interval are shown on the grid above.)

Time (t)	0	2	4	6
Velocity (v)		28.8		19.2

Rectangle	0–2	2–4	4–6	Total
Circumscribed area			50.4	
Inscribed area			38.4	

Estimate of total distance travelled ____________________ metres. [5]

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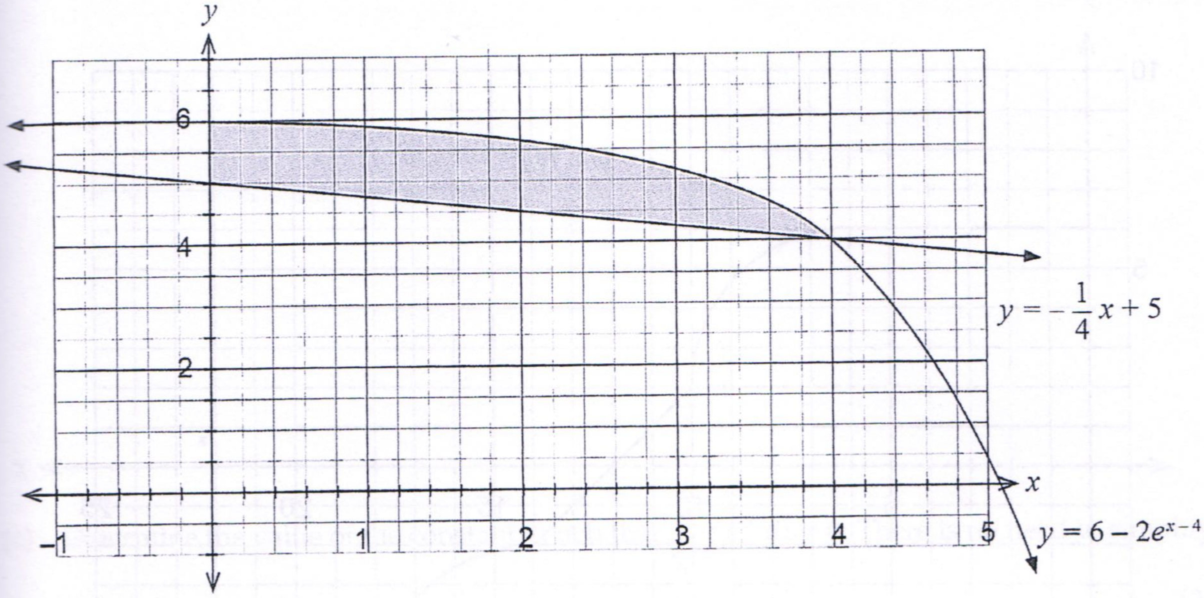
7. (cont)

(b) Describe how you could better estimate the distance travelled by the train during the first six seconds than by the method used in part (a). [1]

8. **[4 marks]** *(MMETH 2016:CF06)*

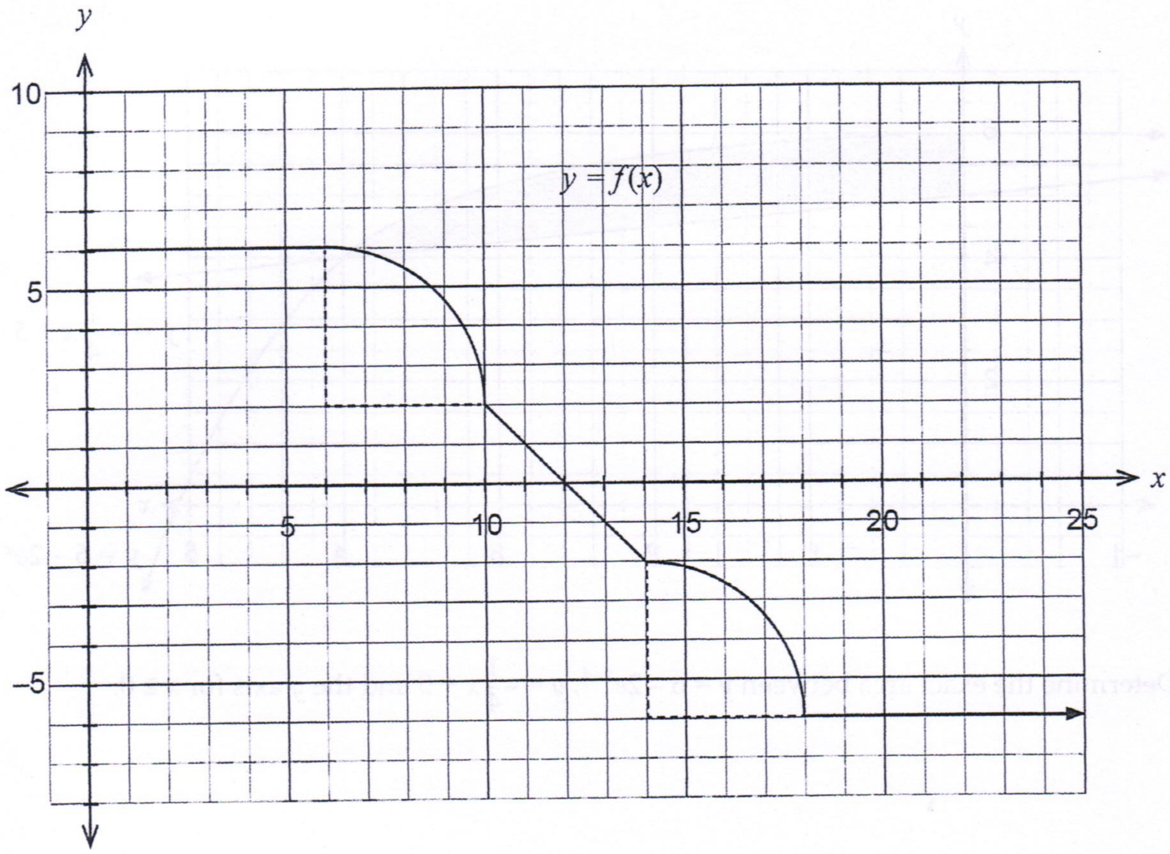
The graphs $y = 6 - 2e^{x-4}$ and $y = -\frac{1}{4}x + 5$ intersect at $x = 4$ for $x \geq 0$.



Determine the exact area between $y = 6 - 2e^{x-4}$, $y = -\frac{1}{4}x + 5$ and the y axis for $x \geq 0$.

9. [7 marks] *(MMETH 2016:CF07)*

Consider the graph $y = f(x)$. Both arcs have a radius of four units.



Using the graph of $y = f(x)$, $x \geq 0$, evaluate exactly the following integrals.

(a) $\displaystyle\int_0^{12} f(x)\,dx$ [3]

9. (cont)

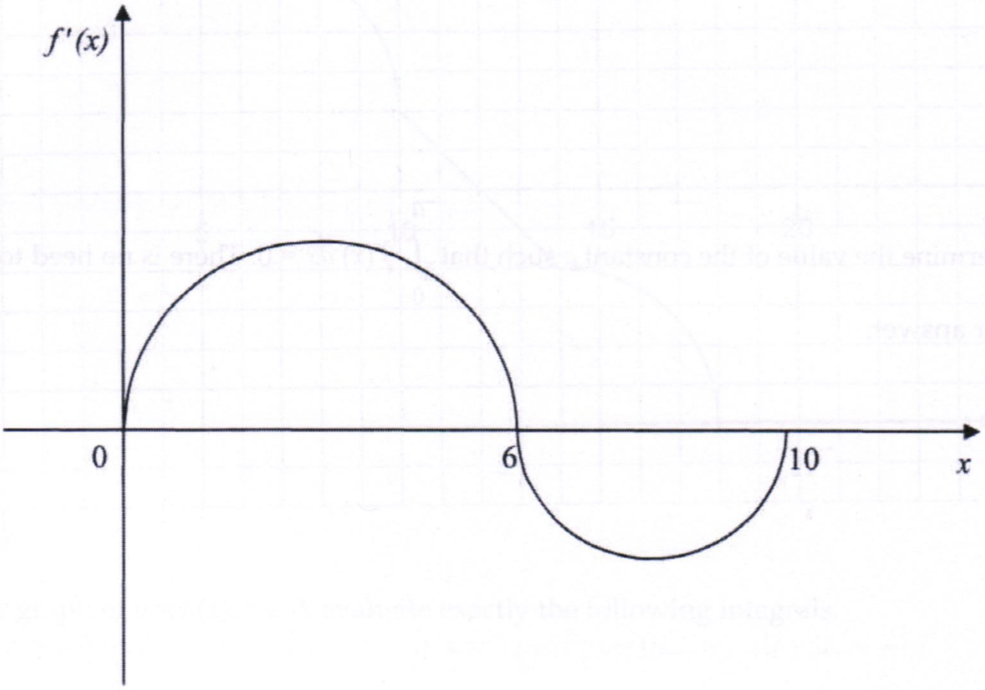
(b) $\int_{0}^{18} f(x)\, dx$ [2]

(c) Determine the value of the constant a such that $\int_{0}^{a} f(x)\, dx = 0$. There is no need to simplify your answer. [2]

Chapter 9 Total Change from Rate of Change

1. *(Projected)*

After x minutes, the volume of water in a tank is $f(x)$. The tank initially had 1 Litre of water and its rate of change of volume, $f'(x)$, is shown below and consists of two semi-circular arcs. What are the maximum and minimum volumes of water in the tank over the first ten minutes?



2. [4 marks] *(CA 1995:11)*

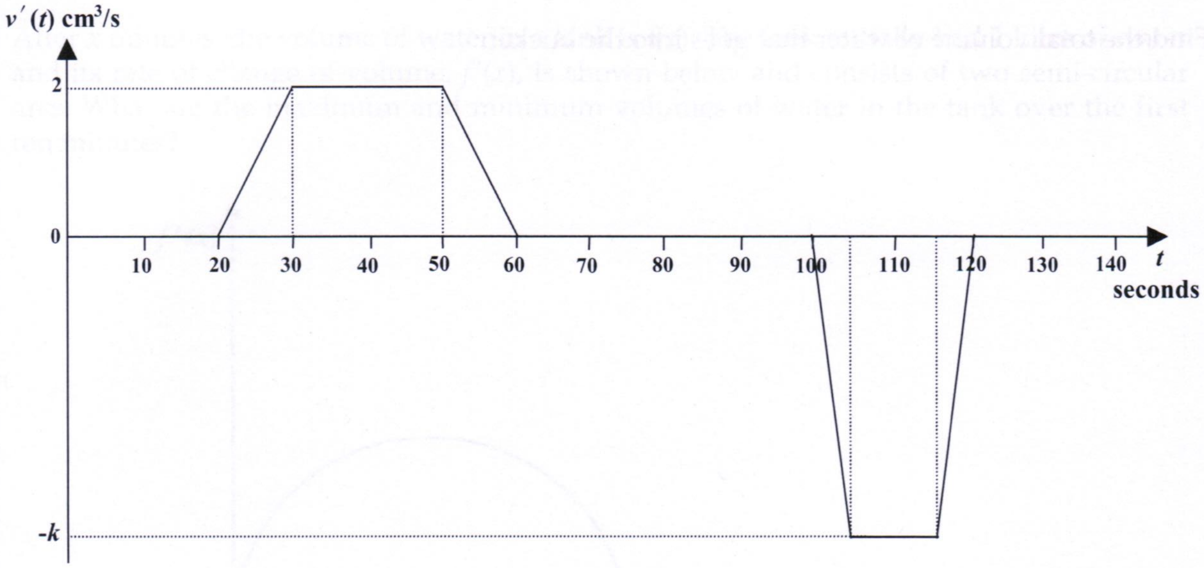
Water flows from a tap into a bucket at a rate of $f(t)$ litres per second where time t is measured in seconds and $f(t) = \begin{cases} 0.04(t-10)^2 & \text{for } 0 \le t \le 10 \\ 0 & \text{for } t > 10 \end{cases}$

Find the total volume of water that goes into the bucket.

3. [10 marks] *(CA 2002:08)*

A clown entertains an audience by inflating an empty balloon and, later, lets all the air out. $v(t)$ is the volume of the balloon in cm^3 at time t seconds.

Consider the graph of the derivative $v'(t)$.



(a) After how many seconds does the clown begin to inflate the balloon? [1]

(b) During which period is the rate of change of the volume the greatest? [2]

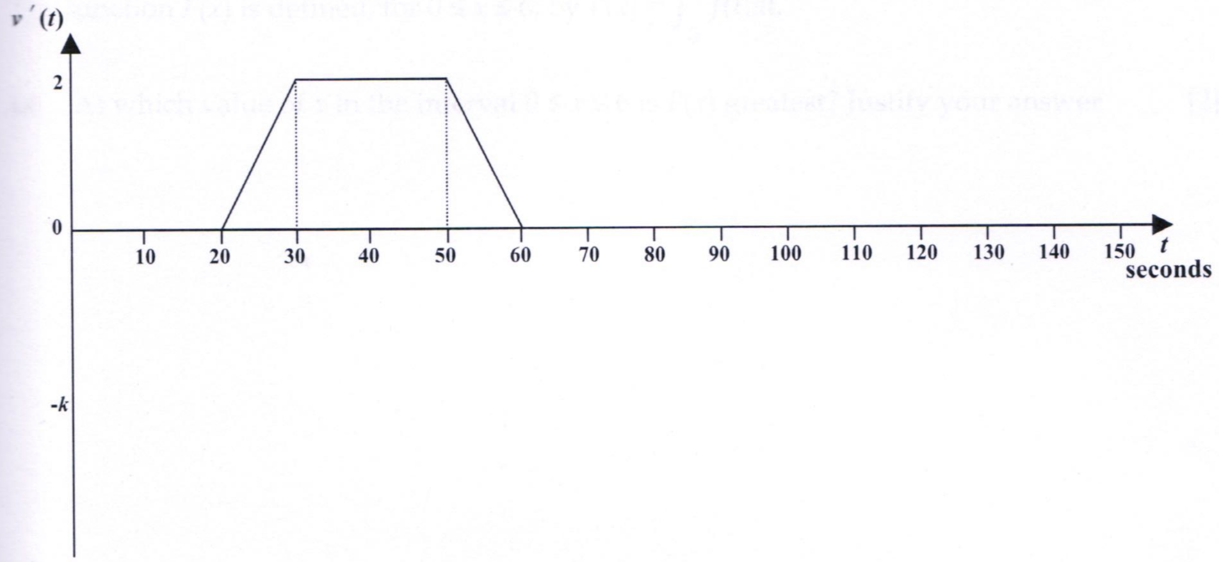
(c) For how many seconds did the clown inflate the balloon? [1]

(d) Find the value of k on the vertical axis of the graph above. Justify your answer by showing the appropriate calculations. [3]

3. (cont)

(e) Instead of letting the air go out all at once, the clown could have opened the balloon at $t = 100$ seconds for a period of ten seconds, then kept it closed for ten seconds, then opened for ten seconds, and so on, so that the balloon is completely empty at exactly $t = 150$ seconds.

Illustrate on the following diagram what the graph of $v'(t)$ could have looked like in that situation. Briefly justify your drawing. [3]



4. **[4 marks]** *(CA 2004:07)*

The function $F(x)$ is defined for all $x \geq 0$ by the integral

$$F(x) = \int_0^x cos(t^2)\,dt$$

Use the incremental formula $\delta y \approx \frac{dy}{dx}\delta x$ to estimate the change in $F(x)$ as x changes from 5 to 5.01.

5. [6 marks] *(3CDMAT 2012:CF8)*

A continuous function $f(x)$ is increasing on the interval $0 < x < 3$ and decreasing on the interval $3 < x < 6$. Some of its values are given in the table below.

x	0	1	2	3	4	5	6
$f(x)$	5	16	27	32	25	0	–49

The function $F(x)$ is defined, for $0 \le x \le 6$, by $F(x) = \int_0^x f(t)dt$.

(a) At which value of x in the interval $0 \le x \le 6$ is $F(x)$ greatest? Justify your answer. [2]

(b) At which value of x in the interval $0 \le x \le 6$ is $F'(x)$ greatest? Justify your answer. [2]

(c) Use the values of $f(x)$ in the table to show that $48 \le F(3) \le 75$. [2]

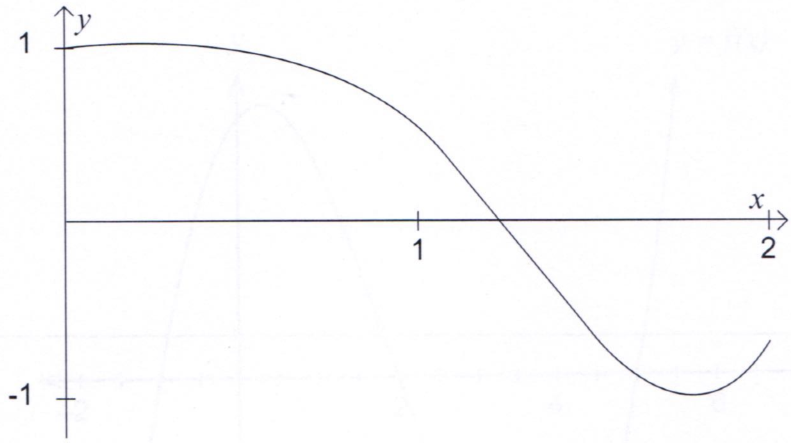
6. [5 marks] *(MMETH 2016S:CF3)*

A curve has a gradient function $\dfrac{dA}{dt} = 60 - 3at^2$, where a is a constant.

Given that the curve has a maximum turning point when $t = 2$ and passes through the point (1, 62), determine the equation of the curve.

7. **[10 marks]** *(MMETH 2016S:CA20)*

The graph of the function $f(x) = \cos x^2$ for $0 \le x \le 2$ is provided below.



(a) The function $A(x)$ is defined as $A(x) = \int_0^x f(t)\,dt$, for $0 \le x \le 2$. [3]

Determine the value of x when $A(x)$ starts to decrease.

(b) Complete the table below. [2]

x	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$A(x)$	0.200	0.399		0.768	0.905	0.974				0.461

(c) On the axes below, plot the values from the table in part (b), and hence sketch the graph of $A(x)$ for $0 \le x \le 2$. [2]

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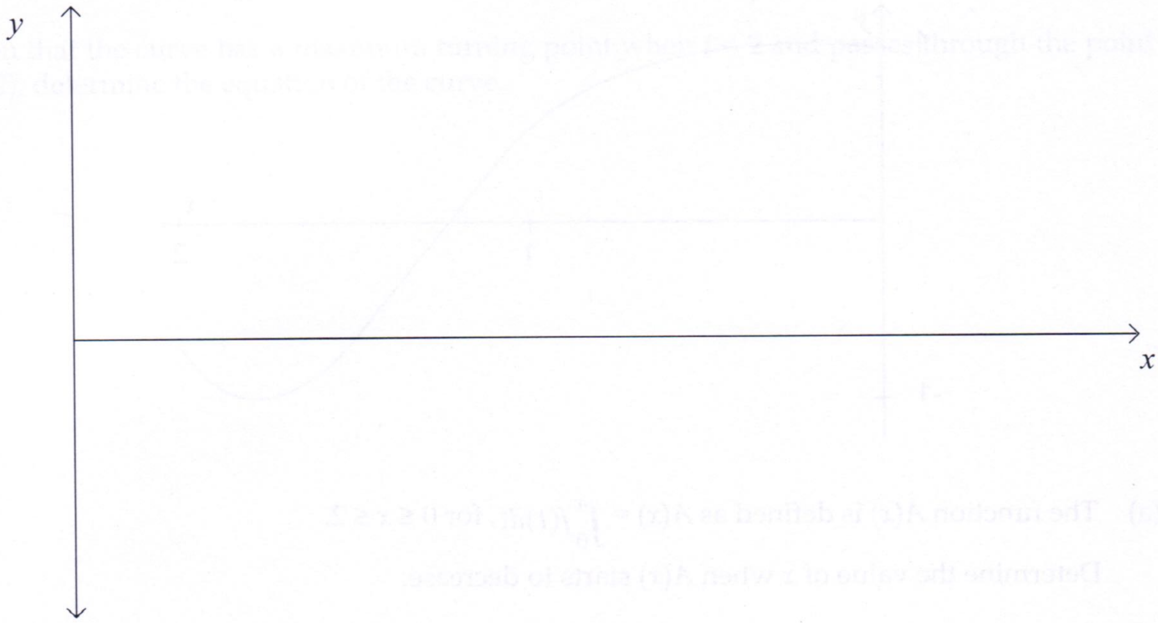
7. (cont)

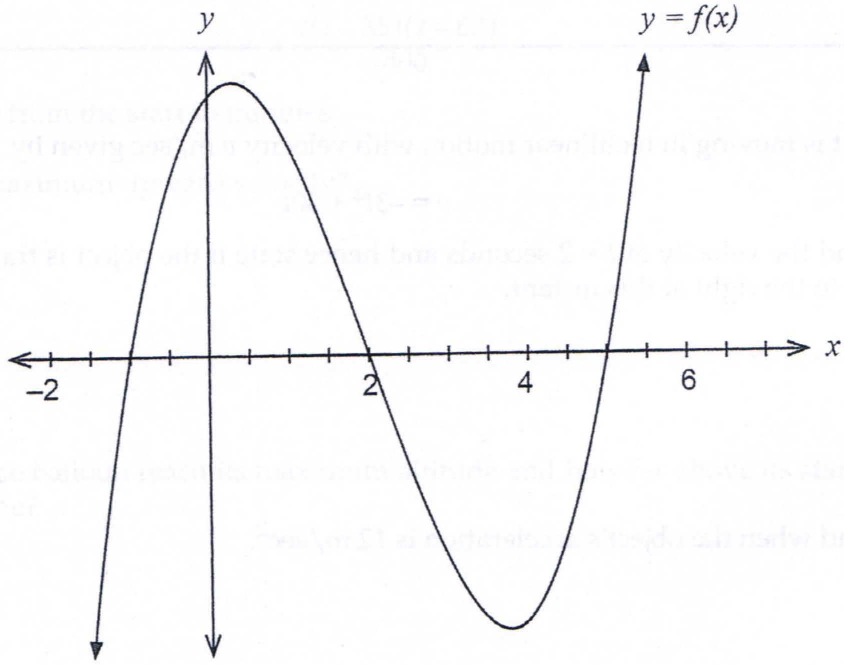
(d) Use your graph from part (c) to sketch the graph of $A'(x)$ on the axes below. [2]



(e) Based on your observations of the graphs in this question, make a conjecture about the defining rule for $A'(x)$. [1]

8. **[6 marks]** *(MMETH 2016:CF05)*

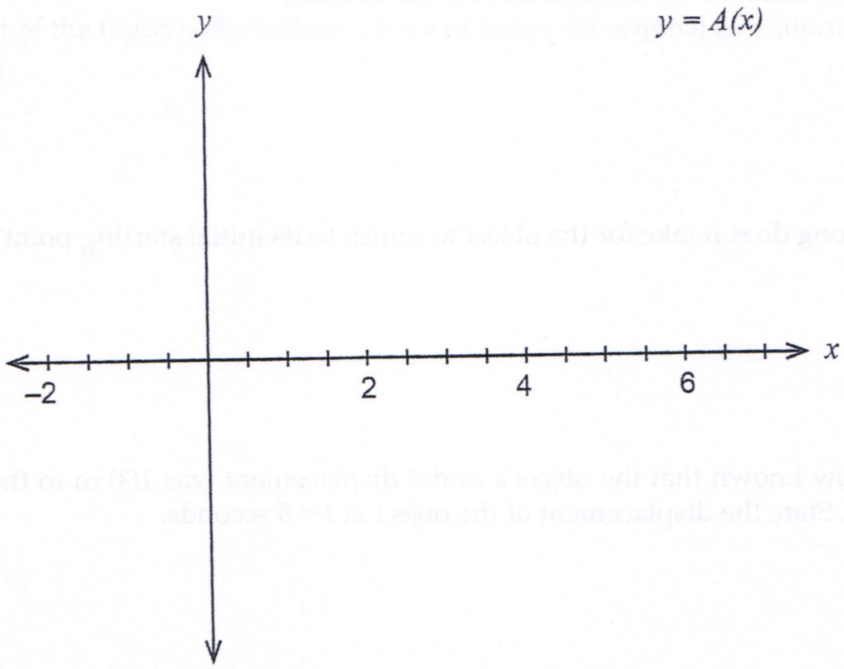
Consider the graph of $y = f(x)$ which is drawn below.



Let $A(x)$ be defined by the integral $A(x) = \int_{-1}^{x} f(t)\,dt$ for $-1 \le x \le 6$.

It is known that $A(2) = 15$, $A(5) = 0$ and $A(6) = 8$.

Sketch on the axes below the function $A(x)$ for $-1 \le x \le 6$ labelling clearly key features such as x intercepts, turning points and inflection points if any.



Chapter 10 Rectilinear Motion

1. *(Projected)*

An object is moving in rectilinear motion with velocity v m/sec given by

$$v = -3t^2 + 30t$$

(a) Find the velocity at $t = 2$ seconds and hence state if the object is travelling to the left or to the right at this instant.

(b) Find when the object's acceleration is 12 m/sec^2.

(c) When does the particle stop and change direction?

(d) Find the distance travelled in the first 12 seconds.

(e) How long does it take for the object to return to its initial starting point?

(f) It is now known that the object's initial displacement was 100 m to the right of the origin. State the displacement of the object at $t = 5$ seconds.

2. [9 marks] *(CA 2000:18)*

A hot air balloon begins a 60 minute flight by rising upwards from the side of a hill. Its vertical velocity v (metres per minute) is given by:

$$v = \frac{t(t-35)(t-60)}{1400}$$

where t is the time from the start in minutes.

(a) What is the maximum upward velocity? [2]

(b) When does the balloon reach its maximum altitude and how far above its starting point is it at that time? [4]

(c) At the end of the flight is the balloon above or below its original elevation? Explain your reasoning. [3]

3. [5 marks] *(3CDMAT 2011:CA14)*

During a volcanic eruption a rock is ejected from the top of the volcano. The rock rises upward and then falls onto a flat plain 1500 metres below the top of the volcano. During its flight, the vertical velocity of the rock, v m/s, is given by

$$v = 160 - 9.8t$$

where t seconds is the time after the ejection of the rock.

(a) How high does the rock rise above the top of the volcano? [3]

(b) How long does it take for the rock to reach the plain below? [2]

(3CDMAT 2012:CA14a,b,c)

4. **[8 marks]**

The velocity of a robotic engine moving on a monorail is given by

$v = 3t^2 - 12t + 9$ metres per second, where t = time in seconds.

Determine

(a) the acceleration after 4 seconds. [2]

(b) how far the engine is from its starting point after 10 seconds. [3]

(c) the total distance travelled by the engine in the first 10 seconds. [3]

5. [5 marks] *(3CDMAT 2014:CA12)*

A bicycle is travelling at a constant speed of 20 kilometres per hour.

(a) Determine the distance, in metres, that the bicycle travels in one second. [1]

When the brakes of the bicycle are applied, this results in a deceleration (negative acceleration) of 10 metres per second squared. Let t represent the time, in seconds, from when the brakes are initially applied.

(b) State the velocity function of the bicycle, in metres per second, in terms of t after the brakes are applied. [2]

(c) How far will the bicycle travel while braking before it stops? [2]

6. [10 marks] *(3CDMAT 2015:CA19)*

A monorail services two mining towns A and B on a straight line from a depot in the city. At time $t = 0$, the monorail passes through the depot with velocity $v = 6t^2 - 60t + 126$, with velocity in km/h and time in hours. The monorail is on a test run and will travel through Town A without stopping, moving on to Town B, then returning to Town A and then Town B for a second time, without stopping.

(a) Determine the displacement function x from the depot, in terms of t. [2]

(b) Determine the times that the monorail will stop at Towns A and B. [3]

(c) What is the distance between the two towns? [2]

(d) Determine the distance travelled and the time taken when the monorail enters Town B for the second time. [3]

7. [9 marks] *(MMETH 2016S:CF7)*

A particle moves in a straight line according to the function $f(x) = e^{\sin x}$, $x \geq 0$, where x is in seconds, and $f(x)$ is the displacement in metres.

(a) Determine the velocity function for this particle. [3]

(b) Determine the rate of change of the velocity at any time, $x \geq 0$ seconds. [3]

(c) Evaluate exactly $\int_0^{\frac{\pi}{2}} f'(x)(dx)$. [2]

(d) Interpret the answer to part (c) in terms of the context of the particle moving according to the function $f(x) = e^{\sin x}$, $x \geq 0$ seconds. [1]

8. **[8 marks]** *(MMETH 2016:CF04)*

The displacement x micrometres at time t seconds of a magnetic particle on a long straight superconductor is given by the rule $x = 5 \sin 3t$.

(a) Determine the velocity of the particle when $t = \frac{\pi}{2}$. [3]

(b) Determine the rate of change of the velocity when $t = \frac{\pi}{2}$. [3]

Let v = velocity of the particle at t seconds.

(c) Determine $\int_0^{\frac{\pi}{2}} \frac{dv}{dt}\,dt$. [2]

9. [8 marks] *(MMETH 2016:CA19)*

The displacement in centimetres of a particle from the point O in a straight line is given by $x(t) = \frac{1}{3}\left(\frac{t}{2} - 4\right)^2 - 2$ for $0 \le t \le 10$, where t is measured in seconds.

Calculate the:

(a) time(s) that the particle is at rest. [2]

(b) displacement of the particle during the fifth second. [2]

9. (cont)

(c) maximum speed of the particle and the time when this occurs. [2]

(d) total distance travelled in the first 10 seconds. [2]

Chapter

11 Discrete Probability Distributions

1. *(Projected)*

Given the discrete uniform probability distribution of a random variable X is given by

$$P(X = x) = \begin{cases} \dfrac{1}{10} & x = 1, 2, 3, \ldots, n \\ 0 & \text{elsewhere} \end{cases}$$

(a) State the value of n.

$A = b \times h$

$1 = \frac{1}{10} \times n$

$n = 10$

(b) Find k if $P(2 < X < k) = 0.4$.

(c) State the mean of the distribution.

2. [8 marks] *(AM 2004:04)*

Two spinners each have five equally likely outcomes: 1, 2, 2, 3 and 3. Let S denote the sum of the results when these two spinners are spun.

(a) Give the possible values of S. [1]

2, 3, 4, 5, 6

(b) Use the table below to give the probability associated with each value s of S. [5]

	1	2	2	3	3
1	2	3	3	4	4
2	3	4	4	5	5
2	3	4	4	5	5
3	4	5	5	6	6
3	4	5	5	6	6

s	2	3	4	5	6			
$P(S = s)$	1/25	4/25	8/25	8/25	4/25			

(c) Explain why the table in (b) is the probability distribution for a discrete random variable. [2]

① each probability is between 1 and 0
② sum of all the probabilities is 1
③ all probabilitis are integers, the variable is discrete

3. **[2 marks]** *(AM 2007:08a)*

Calculate the value of a for the following probability distribution:

x	1	2	3	4
$P(X = x)$	a	$2a$	$3a$	$4a$

$1(a) + 2(2a) + 3(3a) + 4(4a) = 1$

$a + 4a + 9a + 16a = 1$

$20a = 1$

$a = \frac{1}{20}$

4. **[2 marks]** *(AM 2008:16a)*

The probability distribution for the discrete random variable V is defined by

$$P(V = v) = \frac{k}{v+1} \quad \text{for} \quad v \in \{0,1,2,3,4\}.$$

Find the value of k.

$1 = \frac{k}{1} + \frac{k}{2} + \frac{k}{3} + \frac{k}{4} + \frac{k}{5}$

$k =$

5. [10 marks] *(MMETH 2016S:CA14)*

(a) The discrete random variable X has the following probability distribution:

x	1	2	3	4	5
$P(X = x)$	0.1	a	0.3	0.25	b

(i) Determine the values of a and b if the expected value, $E(X) = 3.3$. [3]

$3.3 = 0.1 + 2a + 0.9 + 1 + 5b$

$1.3 = 2a + 5b$ — ①

$1 = 0.1 + a + 0.3 + 0.25 + b$

$0.35 = a + b$ — ②

$a = 0.15 \quad b = 0.2$

(ii) Determine the variance, $Var(x)$. [2]

$Var(x) = \left[\frac{1^2(0.1) + 2^2(0.15) + 3^2(0.3) + 4^2(0.25) + 5^2(0.2)}{5}\right] - 3.3^2$

$= 1.23^2$

$= 1.513$

(iii) State the value of $Var(X + 5)$. [1]

$Var(x + 5) = 1.513$

(iv) State the value of $Var(2X + 5)$. [1]

$Var(2x + 5) = 2^2(2x + 5)$

$= 2^2 \times 1.513$

$= 6.052$

(b) Daniel has been offered a sales position at a car yard. His weekly pay will comprise two components, a retainer of \$250 and a commission of \$400 for each new car sold. The following table shows the probability of his selling specific numbers of cars each week.

n	0	1	2	3	4
$P(N = n)$	0.3	0.4	0.25	0.04	0.01

Calculate Daniel's expected weekly pay. [3]

$\bar{x} = 0(0.3) + 1(0.4) + 2(0.25) + 3(0.04) + 4(0.01)$

$= 1.06$

Weekly pay $= 250 + 1.06 \times 400$

$= 674$

6. [6 marks] *(MMETH 2016:CA15)*

A tetrahedral die has the numbers 1 to 4 on each face. When thrown, each side is equally likely to land facedown. Let X be defined as the sum of the numbers on the facedown side when the die is thrown twice.

(a) Complete the following table. [1]

		Roll two			
	Sum of two rolls	**1**	**2**	**3**	**4**
Roll one	1	1 + 1 = 2	3		
	2	3			
	3		5		
	4				

(b) (i) Hence, or otherwise, complete the probability distribution of X, which is given by the following table. [1]

x	2	3	4	5	6	7	8
$P(X = x)$	$\frac{1}{16}$						$\frac{1}{16}$

(ii) Calculate the probability of obtaining a sum of five or less. [2]

(iii) Determine the mean and standard deviation for X. [2]

7. [7 marks] *(MMETH 2016:CA17)*

A school has analysed the examination scores for all its Year 12 students taking Methods as a subject. Let X = the examination percentage scores of all the Methods Year 12 students at the school. The school found that the mean was 75 with a standard deviation of 22.

Determine the following.

(a) $E(X + 5)$ [1]

(b) $Var\,(25 - 2X)$ [2]

The school has decided to scale the results using the transformation $Y = aX + b$ where a and b are constants and Y = the scaled percentage scores. The aim is to change the mean to 60 and the standard deviation to 15.

(c) Determine the values of a and b. [4]

Chapter 12 The Binomial Distribution

1. **[5 marks]** *(AM 2006:07a,b,c)*

Twelve percent of the population is left-handed. What is the probability that in a randomly selected group of four people

(a) all are left-handed? [2]

(b) there are exactly three right-handed people? [1]

(c) there are more left-handed people than right-handed people? [2]

2. [3 marks] *(AM 2006:16a)*

Police set up a roadblock to check cars for seatbelt use. From experience, they estimate that 80% of drivers wear seat belts. Use this estimate to answer the following questions:

What is the probability that the second unbelted driver is in the 8th car stopped?

3. [9 marks] *(3CDMAT 2012:CA18)*

A new treatment for back pain is being tested.

A trial group consists of 100 randomly chosen patients with back pain. There is a 25% chance that any one of these patients will report an improvement after one month if no treatment is given.

Let X denote the number of patients who will report an improvement after one month, assuming that no treatment is given.

(a) Is the random variable X discrete or continuous? [1]

(b) State the probability distribution of X. [2]

(c) Calculate the mean and standard deviation of X. [2]

(d) What is the probability that 35 or more of the patients in the trial group will report an improvement after one month, assuming that no treatment is given? [2]

(e) Now suppose that each patient in the trial group is given the new treatment and that 35 of them report an improvement after one month. Is this strong evidence that the treatment is effective? Justify your answer. [2]

4. **[5 marks]** *(MMETH 2016S:CF4)*

Harry fires an arrow at a target n times. The probability, p, of Harry hitting the target is constant and all shots are independent.

Let X be the number of times Harry hits the target in the n attempts.

The mean of X is 32 and the standard deviation is 4.

(a) State the distribution of X. [1]

(b) Determine n and p. [4]

5. [6 marks] *(MMETH 2016:CA20a,b)*

A chocolate factory produces chocolates of which 80% are pink. Each box of chocolates contains exactly 30 pieces.

(a) Identify the probability distribution of X = the number of pink chocolates in a single box and also give the mean and standard deviation. [3]

(b) Determine the probability, to three decimal places, that there are at least 27 pink chocolates in a randomly selected box. [3]

Logarithms

Chapter

13

(3ABMAS 2011:CF1)

1. **[5 marks]**

Let $\log_{10} 2 = x$ and $\log_{10} 3 = y$. Express each of the following in terms of x and y.

(a) $\log_{10} 6$ [1]

(b) $\log_{10} 0.6$ [1]

(c) $\log_{10} 45$ [3]

2. **[2 marks]** *(3ABMAS 2012:CF6)*

If $\log_a y = 1 + 2\log_a x$, express y in terms of x.

3. [6 marks] *(3ABMAS 2013:CA8)*

Given that $\log_a 4 = x$ and $\log_a 5 = y$,

(a) write expressions, in terms of x and y, for:

(i) $\log_a 0.8$. [2]

(ii) $\log_a 100$. [2]

(b) Evaluate exactly a^{3x}. [2]

4. [6 marks] *(3ABMAS 2014:CA9)*

Given that $a^x = 4$ and $a^y = 7$,

(a) evaluate a^{x+y}. [2]

(b) write an expression, in terms of x and/or y, for

(i) $\log_a\left(\frac{7}{4}\right)$. [2]

(ii) $\log_a(49a^2)$. [2]

5. **[6 marks]** *(3ABMAS 2014:CA15)*

The loudness of a sound L, measured in decibels (dB), is given by:

$$L = 10\log\left(\frac{I}{I_0}\right)$$

where I = the intensity of a sound (in watts per square metre) and
where I_0 = the intensity of the threshold of hearing (in watts per square metre).

The threshold of hearing is the intensity that is just audible to the human ear. It is known that $I_0 = 10^{-12}$ watts per square metre.

(a) Determine the loudness of a sound, measured in decibels, that is at the threshold of hearing. [2]

At a house party, an amplifier is set at a level so that it is radiating 5 watts of sound power from the speakers. It is found that the sound intensity, I, of this music varies with the distance d from the speakers according to the relationship:

$$I = \frac{5}{\pi d^3}$$

where d = the distance from the speakers in metres.

(b) Calculate the loudness of the party music at a distance of 10 metres, correct to the nearest decibel. [2]

The party music is considered to be at an acceptable level for the neighbours if the loudness level is below conversation level (60 dB).

(c) At what distance, correct to the nearest metre, will the party music be considered to be acceptable? [2]

6. **[7 marks]** *(MMETH 2016S:CF2)*

(a) Solve, exactly, each of the follow equations.

(i) $\log_x 4 = 2$ [2]

(ii) $e^{2x} = 5$ [2]

(b) If $\log a + \log a^2 + \log a^3 + \ldots + \log a^{50} = k \log a$, determine k. [3]

7. [5 marks] *(MMETH 2016S:CA13)*

The decibel scale for sound, measured in decibels (dB), is defined as: $D = 20\log_{10}\left(\frac{P}{P_{ref}}\right)$, where P is the pressure of the sound being measured and P_{ref} is a fixed reference pressure.

(a) What is the decibel measure for a sound with pressure $2P_{ref}$? [1]

(b) The sound produced by a symphony orchestra measures 120 dB, while that of a rock concert measures 150 dB. How many times greater is the sound pressure of the rock concert than that of the orchestra? [4]

8. [5 marks] *(MMETH 2016:CF01)*

(a) Given that $\log_8 x = 2$ and $\log_2 y = 5$, evaluate $x - y$. [2]

(b) Express y in terms of x given that $\log_2 (x + y) + 2 = \log_2 (x - 2y)$. [3]

9. **[3 marks]** *(MMETH 2016:CA12)*

The Richter magnitude, M, of an earthquake is determined from the logarithm of the amplitude, A, of waves recorded by seismographs.

$M = \log_{10}\dfrac{A}{A_0}$, where A_0 is a reference value.

An earthquake in a town in New Zealand in November 2015 was estimated at 5.5 on the Richter scale, while the earthquake just north of Hayman Island measured 3.4 on the same scale. How many times larger was the amplitude of the waves in New Zealand compared to those at Hayman Island?

Chapter

14 Calculus and Natural Logarithms

1. *(Projected:CF)*

(a) Find the exact area between the curve $y = \frac{1}{x}$ and the x-axis between $x = 2$ and $x = 8$.

(b) Consider the function: $f(x) = \ln(x + 1) - 2$

(i) State the coordinates of the y intercept

(ii) State the equation of the vertical asymptote of the graph of $f(x)$

(iii) Find x when $f(x) = 0$

(iv) Give the equation of the function $g(x)$ which is the result of the graph of $f(x)$ being translated 2 units to the right then 3 units up.

2. [3 marks] *(CA 2006:10b)*

Show that $\int_0^1 \frac{x}{1+x^2}\,dx = \ln\sqrt{2}$.

3. [5 marks] *(CA 2008:08c)*

Show that $\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{2+\sin\theta}\,d\theta = \ln\frac{3}{2}$.

4. [3 marks] *(3ABMAS 2010:CF2d)*

Determine $\frac{dy}{dx}$ for the function below, simplifying your answer.

$$y = \ln x^3 - \ln(7-4x)$$

5. [6 marks] *(3ABMAS 2012:CA11)*

The approximate apparent magnitudes of two heavenly bodies are listed in the table below:

Heavenly body	Apparent magnitude m
Sirius	-1.5
Antares	1

The ratio of brightness (or intensity) $\frac{I_A}{I_B}$ of two objects A and B, of apparent magnitudes m_A and m_B respectively, satisfies the equation

$$\log_e\left(\frac{I_A}{I_B}\right) = m_B - m_A .$$

(a) Determine the ratio of brightness of Sirius to Antares $\frac{I_S}{I_A}$.

State answer to the nearest integer. [3]

(b) If the ratio $\frac{I_{Jupiter}}{I_{Sirius}}$ is $\sqrt{e}$, determine the apparent magnitude of Jupiter. [3]

6. [9 marks] *(3ABMAS 2013:CF3)*

Evaluate the following derivatives:

(a) $g'(0)$ where $g(x) = \ln(2x + 1)$. [3]

(b) $h'(1)$ where $h(x) = 4x\,e^{2x}$. [3]

(c) $q'(e)$ where $q(x) = \dfrac{x}{\ln x}$. [3]

7. [3 marks] *(3ABMAS 2014:CF4b)*

Given that $g(x) = \dfrac{\ln x}{x}$, evaluate $g'(1)$.

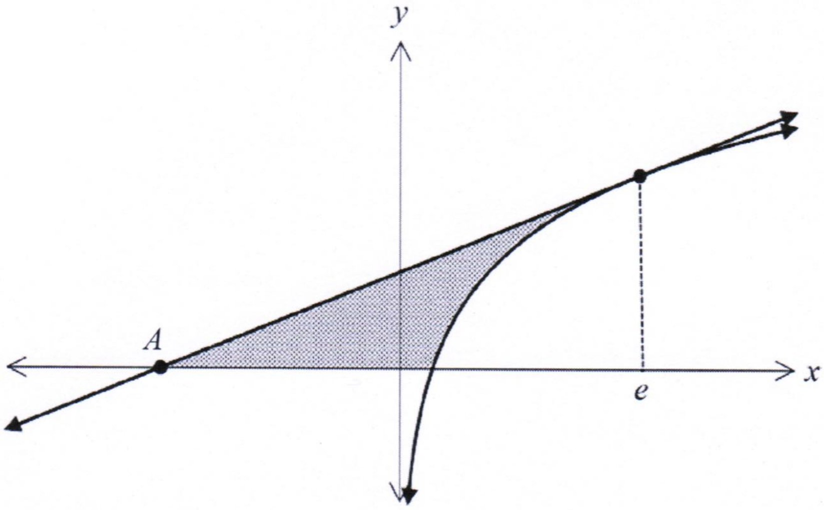
(MMETH 2016S:CF6a)

8. [3 marks]

Given $f'(x) = x^2\ln(2x + 1)$, determine $f''(x)$. Do not simplify.

9. [9 marks] *(MMETH 2016S:CA11)*

The diagram below shows the graph of the function $f(x) = \ln x + 1$ and a linear function $g(x)$, which is a tangent to $f(x)$. When $x = e$, $g(x) = f(x)$.



(a) Determine $g(x)$, the equation of the tangent. [3]

(b) Determine the exact coordinates of A, the point where $g(x)$ intersects the x-axis. [1]

(c) Verify that $f(x)$ cuts the x-axis at the point $\left(\frac{1}{e}, 0\right)$. [1]

9. (cont)

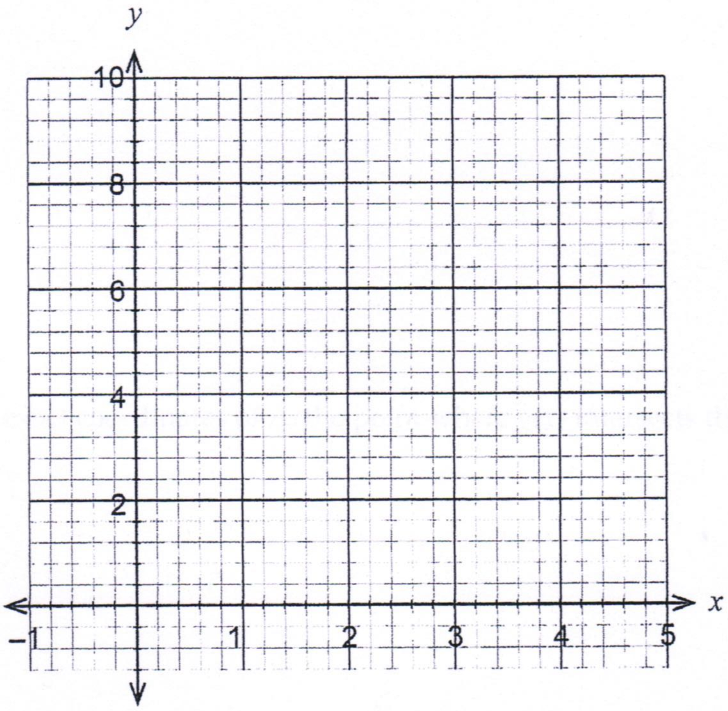
(d) Determine the area of the shaded region enclosed by $f(x)$, $g(x)$ and the x-axis. [4]

10. **[10 marks]** *(MMETH 2016:CA13)*

(a) Determine $\dfrac{d}{dx}(x^2 \ln x)$. [2]

(b) Using your answer from part (a), show that the graph of $y = x^2 \ln x$ has only one stationary point. [3]

(c) Sketch the graph of $y = x^2 \ln x$, showing all features. [3]



(d) Calculate the area bounded by the graph of $y = x^2 \ln x$, the x axis, $x = 1$ and $x = e$. [2]

Continuous Probability Distributions

Chapter

15

1. *(Projected:CF)*

The waiting times at a Perth railway station vary between five and 13 minutes and are distributed uniformly.

(a) What is the probability that Jim waits more than 10 minutes.

(b) Jim has observed that 60% of the time the train arrives before T minutes. Determine the value of T.

2. **[6 marks]** *(AM 2000:22)*

The time taken for an individual to respond to a particular stimulus is anywhere between one and twenty seven seconds. The probability that an individual takes t seconds to respond can be obtained from the following probability density function, f.

$$f(t) = \begin{cases} \dfrac{(t-1)^2}{2}e^{-(t-1)} & 1 \le t \le 27 \\ 0 & \text{elsewhere} \end{cases}$$

Calculate the probability that an individual responds in

(a) at most 15 seconds, [2]

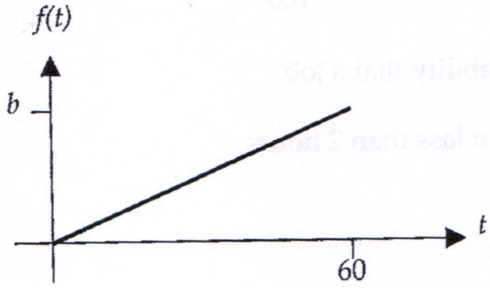
(b) at least 7 seconds, [2]

(c) between 2.6 and 9 seconds. [2]

3. [11 marks] *(AM 2001:11)*

Lee never arrives at school before 8.00 am and never arrives after 9.00 am. The probability distribution function for her time, t, of arrival at school is given below, with constant b and t being measured as minutes after 8.00 am.

$$f(t) = \frac{b}{60}t$$



(a) What is the value of b? [2]

(b) Calculate the probabilty that Lee arrives at school

(i) before 8.30 am, [2]

(ii) after 8.40 am. [2]

(c) Given that Lee arrives at school after 8.30 am, what is the probability that she arrives after 8.40 am? [2]

(d) Find the time, T, to the nearest minute, such that the probability that Lee arrives at school before T am is the same as the probability that she arrives at school after T am. [3]

4. [5 marks] *(AM 2002:12)*

A particular type of job can take between 1 and 5 hours to complete. The time taken to complete a job has an associated probability distribution function given by

$$f(t) = \frac{3}{160}(t + 2)(t - 1)(5 - t) \qquad 1 \le t \le 5.$$

Calculate the probability that a job

(a) is completed in less than 2 hours, [2]

(b) is not completed within 4 hours, [2]

(c) takes exactly 3 hours to complete. [1]

5. [6 marks] *(AM 2003:15a,b,d)*

For each of the following, state whether it represents the probability distribution of a discrete random variable or the probability of a continuous random variable or neither. Briefly justify each of your answers.

(a) $f(x) = \begin{cases} \frac{1}{4}x & 0 \le x \le 2 \\ \frac{1}{2} & 2 < x \le 3 \\ 0 & \text{elsewhere} \end{cases}$ [2]

(b) [2]

x	0	1	2	3	4
$P(X = x)$	0.10	0.15	0.20	0.25	0.30

(c) $f(x) = \begin{cases} 2x^3 - 6x + 3.5 & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$ [2]

6. **[5 marks]** *(MMETH 2016S:CF5)*

The continuous random variable X is defined by the probability density function

$$f(x) = \begin{cases} \dfrac{q}{x} & 1 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}.$$

(a) Determine the exact value of q. [3]

(b) Determine $P\,(2 < X < 3)$. [2]

7. **[4 marks]** *(MMETH 2016S:CA16)*

Roland spends X hours writing poetry during the day.

The probability distribution of X is given by:

$$f(x) = \begin{cases} 2(1-x) & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}.$$

(a) Evaluate $E(X)$, the expected value of X, to the nearest minute. [2]

(b) Determine the variance of X. [2]

8. [10 marks] *(MMETH 2016:CA16)*

An automated milk bottling machine fills bottles uniformly to between 247 ml and 255 ml. The label on the bottle states that it holds 250 ml.

(a) Determine the probability that a bottle selected randomly from the conveyor belt of this machine contains less than the labelled amount. [3]

(b) Calculate the mean and standard deviation of the amount of milk in the bottles. [4]

A worker selects bottles from the conveyor belt, one at a time.

(c) Determine the probability that it takes the selection of 15 bottles before five bottles containing less than the labelled amount have been selected. [3]

The Normal Distribution

Chapter

16

1. **[4 marks]** *(AM 2000:04)*

Susie bought a 3 hour videotape to record a film to be shown on TV. The brand of videotape she chose has a length of tape that is normally distributed with a mean of 185 minutes and a standard deviation of 2 minutes.

(a) What percentage of this brand of videotape have lengths less than three hours? [2]

(b) Including all the credits, the film that Susie wants to record is three hours and seven minutes long. What is the probability that Susie can get all of the film and credits on her tape? [2]

2. **[5 marks]** *(AM 2001:03)*

A teacher travels to school each day by car. She sets out from her home each day at 8 am exactly. The time taken for her journey can be considered to be normally distributed with a mean of 15 minutes and a standard deviation of 2 minutes.

(a) Calculate the probability that her journey takes

(i) exactly 15 minutes, [1]

(ii) more than 18 minutes. [2]

(b) Find the time such that there is a 90% chance that the teacher will arrive before this time. [2]

3. [6 marks] *(AM 2002:10)*

Flour from a mill is poured into '1 kg' bags for sale to the public. In fact the weight of flour in the bags is normally distributed with a mean of 1005 g and a standard deviation of 4 g.

(a) What is the probability that a randomly selected bag of flour is under the marked weight? [2]

(b) What is the weight that is exceeded by only 10% of the bags? [2]

(c) The mill owners decide that no more than 1% of the bags should be underweight. They increase the mean weight of the bags without changing the standard deviation. What should be the mean weight of the flour (to the nearest 0.1 g) to ensure that no more than 1% of bags are underweight? [2]

4. [12 marks] *(AM 2003:13)*

Cans of soft drinks are advertised as containing 375 mL. However, there have been complaints for one particular brand that several cans contained less than 375 mL causing the manufacturer to investigate the settings on the machines that fill the cans. It turns out that the amount the machines are filling the cans with follows a normal distribution with a mean of 378 mL and a standard deviation of 2 mL.

In answering each of the following questions indicate clearly what distribution you are using.

(a) What is the probability that a can will contain exactly 375 mL? [2]

(b) What is the probability that a can contains less than the advertised amount of drink? [3]

(c) If cans are sold in boxes of 24, what is the probability of finding at least two cans in the box with less than the advertised amount of drink? [3]

(d) The manufacturer knows sufficient statistics to realise that if the machines are set to fill the same average amount but the exact amount in each can is more consistent (ie the standard deviation is smaller) there will be less chance of cans being under-filled. What standard deviation (correct to 2 dp) would result in at most 1 in a 1000 cans being underfilled? Show all working. [4]

5. [3 marks] *(AM 2004:17)*

The weight of packets of a new brand of snack food is normally distributed with a mean weight of 30 grams and a standard deviation of 4 grams. The packets are advertised as containing 24 grams. The company making and packaging the snack food knows that it will have problems with the consumer protection group if the packets weigh less than the advertised 24 grams.

What is the probability that a randomly chosen packet of the snack food will

(a) weigh exactly 24 grams, [1]

(b) be underweight? [2]

6. **[12 marks]** *(MMETH 2016S:CA12)*

Rebecca sells potatoes at her organic fruit and vegetable shop that have weights normally distributed with a mean of 230 g and a standard deviation of 5 g.

(a) Determine the probability that one of Rebecca's potatoes, selected at random, will weigh between 223 g and 235 g. [1]

(b) Five percent of Rebecca's potatoes weigh less than w g. Determine w to the nearest gram. [2]

(c) A customer buys twelve potatoes.

(i) Determine the probability that all twelve potatoes weigh between 223 g and 235 g. [2]

(ii) If the customer is selecting the twelve potatoes one at a time, determine the probability that it takes the selection of eight potatoes before six potatoes weighing between 223 g and 235 g have been selected. [3]

6. (cont)

Rebecca also sells oranges. The weights of these oranges are also normally distributed. It is known that 5% of the oranges weigh less than 153 g while 12% of the oranges weigh more than 210 g.

(d) Determine the mean and standard deviation of the weights of the oranges. [4]

7. [6 marks] *(MMETH 2016:CA18)*

The waiting times at a Perth Airport departure lounge have been found to be normally distributed. It is observed that passengers wait for less than 55 minutes, 5% of the time, while there is a 13% chance that the waiting times will be greater than 100 minutes.

(a) Determine the mean and standard deviation for the waiting times at Perth Airport departure lounge. [5]

(b) Determine the probability that the waiting time will be between 75 and 90 minutes. [1]

Sample Proportions and Confidence Intervals

Chapter **17**

(Projected)

1.

(a) A survey of 1000 people asks if speeding fines are too harsh. It was found that 387 out of the 1000 people thought they were too harsh. Find the 95% confidence interval for the population proportion and interpret your answer.

(b) A survey is to be carried out with the aim of having the 90% confidence interval on the population proportion having a margin of error of 0.025. Using the sample proportion as 0.5, find the sample size.

2. *(Projected)*

A fair die is rolled 600 times and a 6 was rolled on 90 occasions.

(a) What is the value of p, the population proportion of the number of 6's?

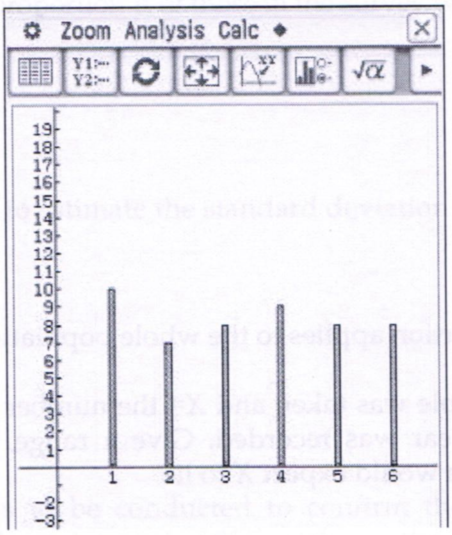
(b) What is the value of $\hat{p}$, the sample proportion of the number of 6's?

(c) Calculate the mean and standard deviation of $\hat{p}$ for such samples of 600 rolls.

3. [5 marks] *(MMETH 2016S:CF9)*

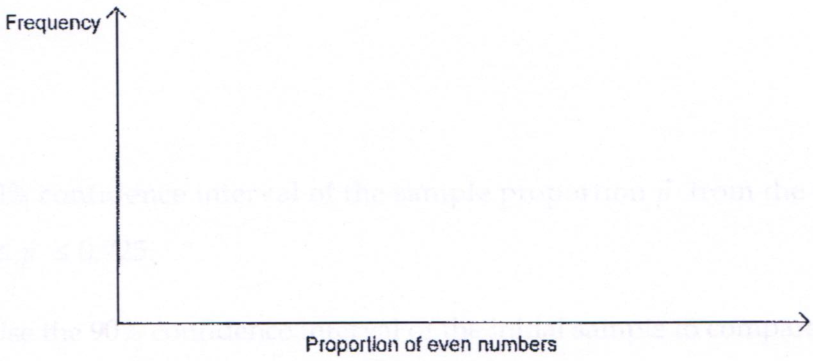
The graph on the calculator screen shot below shows the results of a simulation of the tossing of a standard six-sided die, 50 times.

Simulated results of 50 tosses of a standard six-sided die



This simulation is repeated another 100 times.

(a) Draw a frequency graph to illustrate the likely distribution of the proportion of even numbers obtained from the 100 simulations. [2]



(b) Comment on the key features of your graph, showing the results of the 100 simulations compared with the graph of the single simulated results in part (a). [3]

4. [8 marks] *(MMETH 2016S:CA17)*

A random sample of 100 people indicated that 19% had taken a plane flight in the last year.

(a) Determine a 90% confidence interval for the proportion of the population that had taken a plane flight in the last year. [2]

Assume the 19% sample proportion applies to the whole population.

(b) A new sample of 200 people was taken and X= the number of people who had taken a plane flight in the last year was recorded. Give a range, using the 90% confidence interval, within which you would expect X to lie. [1]

(c) Determine the probability that in a random sample of 120 people, the number who had taken a plane flight in the last year was greater than 26. [3]

(d) If seven surveys were taken and for each a 95% confidence interval for P was calculated, determine the probability that at least four of the intervals included the true value of P. [2]

5. [10 marks] *(MMETH 2016S:CA18)*

A random survey was conducted to estimate the proportion of mobile phone users who favoured smart phones over standard phones. It was found that 283 out of 412 people surveyed preferred a smart phone.

(a) Determine the sample proportion $\hat{p}$ of those in the survey who preferred a smart phone. [1]

(b) Use the survey results to estimate the standard deviation of $\hat{p}$. [2]

(c) A follow-up survey is to be conducted to confirm the results of the initial survey. Working with a confidence interval of 95%, estimate the sample size necessary to ensure margin of error of at most 4%. [3]

The 90% confidence interval of the sample proportion $\hat{p}$ from the initial survey is $0.649 \le \hat{p} \le 0.725$.

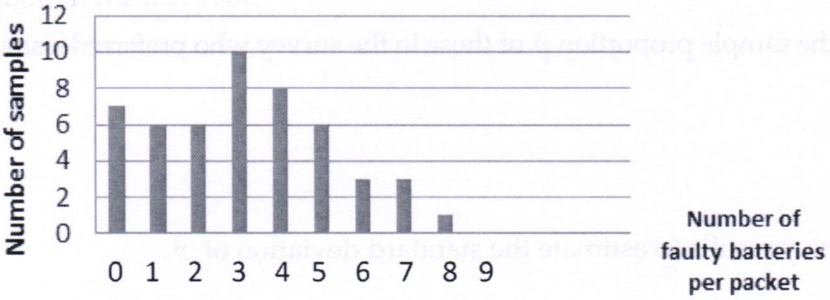
(d) Use the 90% confidence interval of the initial sample to compare the following samples:

(i) A random sample of 365 people at a shopping centre found that 258 had a preference for a smart phone. [2]

(ii) A random sample of 78 people at a retirement village found that 32 had a preference for a smart phone. [2]

6. [5 marks] *(MMETH 2016S:CA21)*

The graph below shows the number of faulty batteries per packet of 50 AAA batteries, when 50 packets are sampled at random.



(a) Identify the type of distribution of X = the number of faulty batteries per packet of 50 AAA batteries. [1]

A manufacturer of AAA batteries assumes that 99% of the batteries produced are fault-free. Ten samples of 50 packets of 50 AAA batteries are selected at random and tested. The number of faulty batteries in each of the 10 random samples is shown below.

Sample	1	2	3	4	5	6	7	8	9	10
Number of faulty batteries	34	28	22	28	28	30	22	28	28	30

(b) Using the assumption that 99% of batteries are fault free calculate the 95% confidence interval for the proportion of faulty batteries expected when sampling. [3]

(c) Decide which of the samples, if any lie outside the 95% confidence level. [1]

7. [12 marks] *(MMETH 2016:CA10)*

A survey in Western Australia was conducted on the popularity of a calculator known as Type A. Out of 1450 Year 12 students, the survey found that 986 students used the Type A calculator.

Determine the following.

(a) A 90% confidence interval, to three decimal places, for the proportion of Western Australian Year 12 students who use the Type A calculator. What assumption was made in calculating this interval? [3]

(b) The margin of error in this confidence interval. [2]

CONTINUED NEXT PAGE

7. (cont)

Another three surveys of Year 12 students were conducted on the use of Type A calculators across Australia.

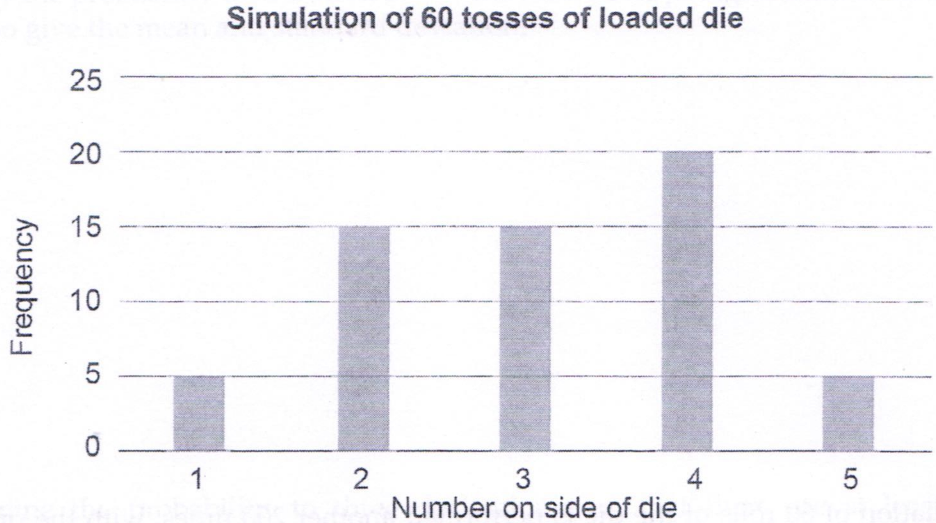
Survey 2	Survey 3	Survey 4
Type A usage 1772 out of 3221 Year 12 students	Type A usage 1021 out of 1566 Year 12 students	Type A usage 2203 out of 3221 Year 12 students

(c) Determine which of these surveys were more likely to have been taken outside of Western Australia. Justify your answer(s). [3]

(d) Using the sample proportion of the survey at the start of the question, determine a sample size that will halve the margin of error for the proportion of Western Australian Year 12 students who use the Type A calculator, with a confidence of 90%. [4]

8. [9 marks] *(MMETH 2016:CA14)*

The simulation of a loaded (unfair) five-sided die rolled 60 times is recorded with the following results.



(a) Calculate the proportion of prime numbers recorded in this simulation. [2]

(b) Determine the mean and standard deviation for the sample proportion of prime numbers in 60 tosses, using the results above. [2]

CONTINUED NEXT PAGE

8. (cont)

(c) It has been decided to create a confidence interval for the proportion of prime numbers using the simulation results on the previous page. The level of confidence will be chosen from 90% or 95%. Explain which level of confidence will give the smallest margin of error. State this margin of error. [3]

This simulation of 60 rolls of the die is performed another 200 times, with the proportion of prime numbers recorded each time and graphed.

(d) Comment briefly on the key features of this graph. [2]

(MMETH 2016:CA20)

9. [14 marks]

A chocolate factory produces chocolates of which 80% are pink. Each box of chocolates contains exactly 30 pieces.

(a) Identify the probability distribution of X = the number of pink chocolates in a single box and also give the mean and standard deviation. [3]

(b) Determine the probability, to three decimal places, that there are at least 27 pink chocolates in a randomly selected box. [3]

Quality Control collects samples sizes of 20 boxes and counts the number of pink chocolates in total.

(c) Determine a 95% confidence interval for the proportion of pink chocolates in a sample of 20 boxes, using the assumption that 80% of chocolates in the sample are pink. [2]

CONTINUED NEXT PAGE

9. (cont)

(d) Quality Control collects three samples and determines a 95% confidence interval each time. Determine the probability that only one of these intervals will not contain the true value 0.8 of the proportion of pink chocolates. [2]

(e) Using your 95% confidence interval in part (c), determine the range in which the expected number of pink chocolates in a sample of 20 boxes would lie. [2]

Quality Control counted the number of pink chocolates in five samples as shown below.

Sample	**1**	**2**	**3**	**4**	**5**
Number of pink chocolates	433	463	482	473	566

(f) Decide which samples lie outside the 95% confidence interval, if any. Justify. [2]

Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS METHODS
ATAR COURSE

FORMULA SHEET

2016

Measurement

Circle:	$C = 2\pi r = \pi D$, where C is the circumference, r is the radius and D is the diameter $A = \pi r^2$, where A is the area
Triangle:	$A = \frac{1}{2}bh$, where b is the base and h is the perpendicular height
Parallelogram:	$A = bh$
Trapezium:	$A = \frac{1}{2}(a + b)h$, where a and b are the lengths of the parallel sides
Prism:	$V = Ah$, where V is the volume and A is the area of the base
Pyramid:	$V = \frac{1}{3}Ah$
Cylinder:	$S = 2\pi rh + 2\pi r^2$, where S is the total surface area $V = \pi r^2 h$
Cone:	$S = \pi rs + \pi r^2$, where s is the slant height $V = \frac{1}{3}\pi r^2 h$
Sphere:	$S = 4\pi r^2$ $V = \frac{4}{3}\pi r^3$

Exponentials

Index laws: For $a, b > 0$ and m, n real,

$a^m b^m = (a\,b)^m$ $\quad a^m a^n = a^{m+n}$ $\quad (a^m)^n = a^{mn}$

$a^{-m} = \dfrac{1}{a^m}$ $\quad \dfrac{a^m}{a^n} = a^{m-n}$ $\quad a^0 = 1$

For $a > 0$ and m an integer and n a positive integer, $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

See next page

Logarithms

For a, b, y, m and n positive real and k real:

$$1 = a^0 \Leftrightarrow \log_a 1 = 0 \qquad y = a^x \Leftrightarrow \log_a y = x$$

$$\log_a mn = \log_a m + \log_a n \qquad a = a^1 \Leftrightarrow \log_a a = 1$$

$$\log_e x = \ln x \qquad \log_a(m^k) = k \log_a m$$

Calculus

Differentiation: If $f(x) = y$ then $f'(x) = \dfrac{dy}{dx}$ If $f(x) = \ln x$ then $f'(x) = \dfrac{1}{x}$

If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ If $f(x) = \sin x$ then $f'(x) = \cos x$

If $f(x) = e^x$ then $f'(x) = e^x$ If $f(x) = \cos x$ then $f'(x) = -\sin x$

Product rule: If $y = f(x)\, g(x)$ then $y' = f'(x)\, g(x) + f(x)\, g'(x)$ or If $y = uv$ then $\dfrac{dy}{dx} = \dfrac{du}{dx} v + u\dfrac{dv}{dx}$

Quotient rule: If $y = \dfrac{f(x)}{g(x)}$ then $y' = \dfrac{f'(x)\, g(x) - f(x)\, g'(x)}{(g(x))^2}$ or If $y = \dfrac{u}{v}$ then $\dfrac{dy}{dx} = \dfrac{\frac{du}{dx} v - u \frac{dv}{dx}}{v^2}$

Chain rule: If $y = f(g(x))$ then $y' = f'(g(x))\, g'(x)$ or If $y = f(u)$ and $u = g(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

Powers: $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c,\ n \neq -1$

Exponentials: $\int e^x dx = e^x + c$

Natural logarithm: $\int \dfrac{1}{x}\, dx = \ln x + c$ and $\int \dfrac{f'(x)}{f(x)}\, dx = \ln f(x) + c$

Trigonometry: $\int \sin x\, dx = -\cos x + c$ and $\int \cos x\, dx = \sin x + c$

Fundamental Theorem of Calculus: $\dfrac{d}{dx}\left(\int_a^x f(t)\, dt\right) = f(x)$ and $\int_a^b f'(x)\, dx = f(b) - f(a)$

Incremental formula: $\delta y \simeq \dfrac{dy}{dx} \delta x$

Exponential growth and decay: If $\dfrac{dy}{dt} = ky$, then $y = Ae^{kt}$

See next page

Random variables, distributions, probability and proportions

Probability: For any event A and its complement $\bar{A}$, and event B

$P(A) + P(\bar{A}) = 1$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A)\,P(B|A) = P(B)\,P(A|B)$

In a Bernoulli trial: $\bar{x}$ is the sample proportion $\hat{p}$,

Mean $\mu = p$ and standard deviation $\sigma = \sqrt{p(1-p)}$

In a binomial distribution:

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

Mean $\mu = np$ and standard deviation $\sigma = \sqrt{np(1-p)}$

Expected value: If X is a discrete random variable,

$E(x) = \sum p_i x_i$, where x_i are the possible values of X and $p_i = P(X = x_i)$

If X is a continuous random variable,

$E(x) = \int_{-\infty}^{\infty} x p(x) dx$, where $p(x)$ is the probability density function of X.

Variance: If X is a discrete random variable,

$Var(x) = \sum p_i (x_i - \mu)^2$, where $\mu = E(X)$ is the expected value

If X is a continuous random variable,

$Var(x) = \int_{-\infty}^{\infty} (x - \mu)^2 p(x) dx$.

A confidence interval for the proportion, p, of a population is:

$$\left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \quad , \quad \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right)$$

where $\hat{p}$ is the sample mean,
n is the sample size and
z is the cut-off value on the standard normal distribution corresponding to the confidence level.

Margin of error: $E = z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$ is the half-width of the confidence interval

Note: Any additional formulas identified by the examination panel as necessary will be included in the body of the particular question.

Solutions

Chapter 1: Techniques of Differentiation

1. (CA 2008:01a,b,c)

(a) $\frac{d}{dx}(x^3+1)^{\frac{1}{2}} = \frac{1}{2}(x^3+1)^{-\frac{1}{2}} \times (3x^2)$

$= \frac{3x^2}{2\sqrt{x^3+1}}$

(b) $\frac{d}{dx}\left(\frac{\sin x}{\cos x + 2}\right) = \frac{(\cos x + 2)\cos x - \sin x(-\sin x)}{(\cos x + 2)^2}$

$= \frac{\cos^2 x + 2\cos x + \sin^2 x}{(\cos x + 2)^2} = \frac{1 + 2\cos x}{(\cos x + 2)^2}$

(c) $12e^{-4x}$

2. (CA 2009:02a)

(a) $\frac{dy}{dx} = 3\cos(4x)\, e^{3x+1} - 4\sin(4x)e^{3x+1}$

(b) $\frac{dy}{dx} = \frac{0.5x^{-0.5}(3x+2) - 3\sqrt{x}}{(3x+2)^2}$

3. (3ABMAS 2010:CF2a,b,c)

(a) $\frac{dy}{dx} = 6(7x-2)^5 \times 7$

i.e. $\frac{dy}{dx} = 42(7x-2)^5$

(b) $\frac{dy}{dx} = 2e^{3x} + 6xe^{3x}$

(c) $\frac{dy}{dx} = \frac{5(4+9x) - 5x \times 9}{(4+9x)^2}$

i.e. $\frac{dy}{dx} = \frac{20 + 45x - 45x}{(4+9x)^2}$

i.e. $\frac{dy}{dx} = \frac{20}{(4+9x)^2}$

4. (3CDMAT 2013:CF3)

$f'(x) = -2x^{-3} + e^{2x}$

$\Rightarrow f''(x) = 6x^{-4} + 2e^{2x} = \frac{6}{x^4} + 2e^{2x}$

5. (3CDMAT 2013:CF4a,b)

(a) $f'(x) = (x^2-16) + (x-1)(2x) = 3x^2 - 2x - 16 = (3x-8)(x+2)$

(b) $f'(3) = (3\times 3 - 8)(3+2) = 5$

$f(3) = (3-1)(9-16) = -14$

$(5, -14)$ and $m = 5 \Rightarrow y = 5x - 29$

Chapter 2: Growth and Decay with Exponential Functions

1. (3CDMAT 2012:CA11)

(a) $0.5 = e^{8k} \Rightarrow k = -0.087$

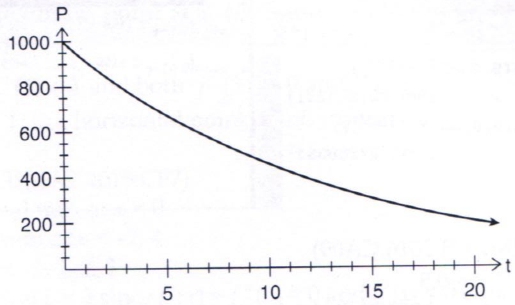
(b) $125 = 88\,000e^{-0.087T} \Rightarrow T = 75.7 \approx 76$ days.

2. (3CDMAT 2013:CA11)

(a) Exponential decay

(b) $P = 1000e^{-0.08t}$

(c)



(d) (i) $P(10) = 1000e^{-0.08\times 10} \approx 449$

(ii) $800 = 1000e^{-0.08t} \Rightarrow t = 2.8$ years.

(e) $\left.\frac{dP}{dt}\right|_{t=10} = -0.08 \times 449 = -35.9$

This means the population of the birds at this time is decreasing at approximately 36 birds per year.

3. (3ABMAS 2014:CA18)

(a) $80 = 100 \times e^{-3k}$

$e^{-3k} = \frac{4}{5}$

$k = -\frac{1}{3}\ln\left(\frac{4}{5}\right) \approx 0.074$

(b) When $t = 8$ the botanist requires 200 leaves then the initial number L_0 must satisfy $200 = L_0e^{-8k}$

so $L_0 = 200e^{8k} \approx 200e^{0.5950\ldots} \approx 362.6$

The botanist should tag approximately 363 leaves.

(c) $\frac{N}{2} = N\, e^{-kt}$

i.e. $e^{-kt} = \frac{1}{2}$

$-kt = \ln\frac{1}{2}$

$\therefore\ t = -\frac{1}{k}\ln\frac{1}{2}$, OR $t = \frac{-\ln\frac{1}{2}}{-\frac{1}{3}\ln\frac{4}{5}}$, OR $t = \frac{-\ln\frac{1}{2}}{0.074}$

$\therefore\ t = 9.31885\ldots = 9.3$ (to two significant figures)

9.3 years will have elapsed.

4. (3CDMAS 2015:CA19a,b,c,d)

(a) $\frac{dA}{dt} = 0.2A$

$A = 2e^{0.2t}\ A_0 = 2$

(b) $A = 3.644\ \text{cm}^2$

(c) $B = 1.0055e^{0.4t}$

(d) $2e^{0.2t} = 1.0055e^{0.4t}$

Solving for $t = 3.438$

approx 3 months

5. (MMETH 2016S:CA10)

(a) $0.5\ \mu g$

(b) $0.5 = e^{-0.1155t}$

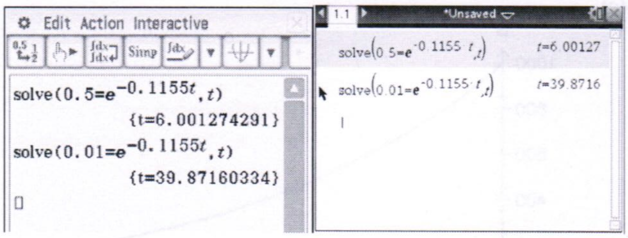
$t = 6$ The half-life of Technetium-99 is 6 hours.

(c) $0.01T_0 = T_0e^{-0.1155t}$

$0.01 = e^{-0.1155t}$

$t = 39.9$h

i.e. 40 hours



6. (MMETH 2016:CA09)

(a) $e^{k(100.5)} = 0.5$

$k = -0.00690$ (3 sig. figures as req'd)

(b) $100e^{-0.000690t} = 5$

$t = 434.4$ days

(c) $P'(434.4) = -0.0345$ g/day

Chapter 3: Calculus and Trigonometric Functions

1. (Projected)

(a) $\frac{\pi}{180}\cos\left(\frac{\pi}{180}\theta\right)$

(b) $\frac{\pi}{180}\cos\left(\frac{\pi}{180}\cdot 60°\right) = \frac{\pi}{360}$

2. (CA2003:04)

To find the tangent we need to find the gradient and the point of contact with the curve.

$y = 3\sin 2x - \cos 2x$ when $x = \frac{\pi}{4}$

$y' = 6\cos 2x + 2\sin 2x$ $\quad y = 3\sin\frac{\pi}{2} - \cos\frac{\pi}{2}$

when $x = \frac{\pi}{4}$ $\quad y = 3$

$y' = 6\cos\frac{\pi}{2} + 2\sin\frac{\pi}{2}$

$= 2$

Eqn of the tangent therefore is given by:

$(y - 3) = 2\left(x - \frac{\pi}{4}\right)$ or $y = 2x + \left(3 - \frac{\pi}{2}\right)$

3. (CA2007:01b)

$y = 3\sin\left(\frac{4\pi x}{3}\right)$

$\frac{dy}{dx} = 3 \cdot \frac{4\pi}{3}\cos\left(\frac{4\pi x}{3}\right)$

at $x = 3$ $\quad \frac{dy}{dx} = 4\pi\cos(4\pi)$

$= 4\pi$

at $x = 3$ $\quad y = 3\sin(4\pi)$

$= 0$

Tangent: $\quad y = mx + c$

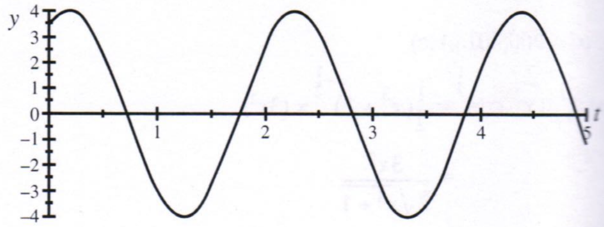
$m = 4\pi$

at $(3, 0)$ $\quad 0 = 4\pi \cdot 3 + c$

$\therefore y = 4\pi x - 12\pi$

4. (CA 2008:15a,b,c,d,e)

(a)



Shape

Phase

(b) Amplitude 4

Period $\frac{2\pi}{3}$ sec

(c) $\sin(3t + 1) = \frac{1}{2}$

$3t + 1 = \frac{5\pi}{6}$ (ignore $\frac{\pi}{6}$ as $\frac{\pi}{6} - 1 < 0$)

$t = \frac{\frac{5\pi}{6} - 1}{3}$

$t = \frac{5\pi}{18} - \frac{1}{3}$

(d) $\frac{dy}{dt} = 4\cos(3t + 1) \times 3$

Max value of $\frac{dy}{dx} = 12 \times 1 = 12$ m/s

(e) $\frac{d^2y}{dt^2} = 12 \times -\sin(3t + 1) \times 3$

$= -9 \times 3\sin(3t + 1)$

$= -9y \Rightarrow k = -9$

5. (3CDMAS 2015:CF7)

(a) Area $= 0.5r^2\sin\theta$

(b) $AB = r\tan\theta$

Area $= 0.5\,(r\tan\theta)\,.\,r$

$= 0.5r^2\tan\theta$

(c) $Q(\theta) = \frac{0.5r^2\sin\theta}{0.5r^2\tan\theta} = \cos\theta$

(d) (i) $\cos\theta = 0.5 \rightarrow \theta = \frac{\pi}{3}$ (ii) $-\sin\theta = -0.5 \rightarrow \theta = \frac{\pi}{6}$.

6. (MMETH 2016:CA21)

(a) $\tan\theta = \frac{y}{12}$

$y = 12\tan\theta$ or use $y = \frac{12\sin\theta}{\cos\theta}$ and use quotient rule

$y = 12\sec^2\theta$

$y = \frac{12}{\cos^2\theta}$

(b) $\frac{dy}{dt} = \frac{12}{\cos^2\theta} \times 6\pi$

$x = 5 \quad \tan\theta = \frac{5}{12} \quad \theta = 22.62°$ (0.395 rads)

$\frac{dy}{dt} = \frac{12}{\cos^2(22.62)°} \times 6\pi$

$= 265.465$ km/min

Chapter 4: Curve Sketching

1. (3CDMAT 2010:CF3)
Since $f(x)$ is differentiable then the extreme values occur at the stationary points or the end points.

$f'(x) = 3 - \dfrac{48}{x^4} = 0 \Rightarrow x = 2$ over the given domain

$f(1) = 19, f(2) = 8$ and $f(5) = 15\dfrac{16}{125} \Rightarrow f_{\max} = 19$ and $f_{\min} = 8$

2. (3CDMAT 2010:CF7a,b)
(a) $f'(x) = 3(x^2 - 1) = 0 \Rightarrow x = \pm 1$
$f(1) = 2$ and $f(-1) = 6 \Rightarrow$ Stationary points are $(1, 2)$ and $(-1, 6)$
(b) $f'(x) = 3(x^2 - 1) < 0 \Rightarrow -1 < x < 1$
$f''(x) = 6x > 0 \Rightarrow x > 0$
So the solution is $0 < x < 1$

3. (3CDMAT 2011:CF2)
Since $y = x^2(6 - x)$ is a continuous function, the max/min values of the function on a given interval will occur at the end points of the interval or at turning points of the function.
Stationary points of $y = 6x^2 - x^3$ occur where $\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = 0 \Rightarrow 12x - 3x^2 = 0 \Rightarrow x = 0$ or $x = 4$

Reject $x = 0$ as it is not in the interval $1 \le x \le 5$
When $x = 4$, $y = 32$
Check values of y at interval end points: When $x = 5$, $y = 25$ and when $x = 1$, $y = 5$
Thus the maximum value of the function is 32 and the minimum value is 5.

4. (3CDMAT 2011:CF6)
$p'(x) = a + 2bx + 3cx^2$
$p'(6) = 0 \Rightarrow a + 12b + 108c = 0 \ldots (1)$
$p''(x) = 2b + 6cx$
$p''(2) = 0 \Rightarrow 2b + 12c = 0 \Rightarrow b + 6c = 0 \ldots (2)$
$p(3) = 135 \Rightarrow 3a + 9b + 27c = 135 \Rightarrow a + 3b + 9c = 45 \ldots (3)$

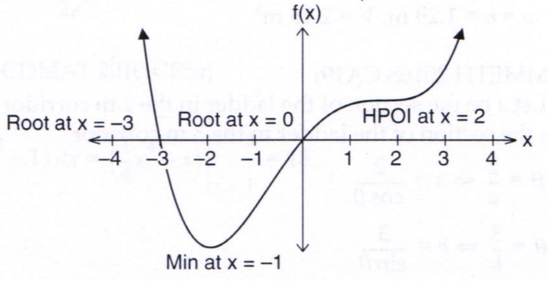
5. (3CDMAT 2011:CA18a,b)
(a) At $x = 1$, $y = 0 \Rightarrow 0 = e^c - d + e^{-c} \Rightarrow d = e^c + e^{-c}$
(b) At the lowest point of the cable, $\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = ce^{cx} - ce^{-cx} = 0 \Rightarrow c(e^{cx} - e^{-cx}) = 0 \Rightarrow e^{cx} = e^{-cx} \Rightarrow e^{2cx} = 1$

$\Rightarrow e^{2cx} = e^0 \Rightarrow 2cx = 0 \Rightarrow x = 0$

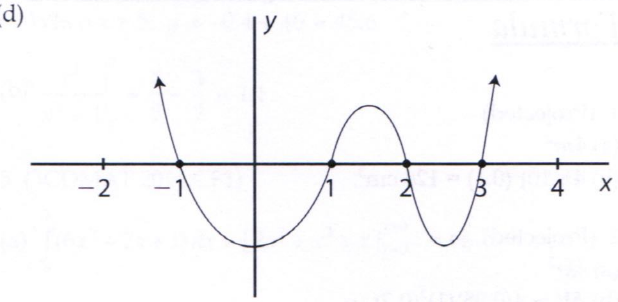
$\dfrac{d^2x}{dy^2} = c^2e^{cx} + c^2e^{-cx} \Rightarrow \left.\dfrac{d^2x}{dy^2}\right|_{x=0} = 2c^2 > 0 \Rightarrow$ min TP at $x = 0$

6. (3CDMAT 2012:CF3)
(a) Point C
(b) Horizontal POI increasing either side, and above the x-axis.
(c)



7. (3CDMAT 2014:CF7)
(a) Maximum point is at (1, 4) and minimum point is at (3, –2).
(b) $f'(5) = 0$ and both $f'(5^-) > 0$ and $f'(5^+) > 0$
$\Rightarrow$ (5, 1) is a horizontal point of inflection.

8. (3CDMAT 2015:CF7)
(a) local max at $x = 0$
local min at $x = -2, 4$
(b) $x = -1, 1, 2, 3$
(c) Yes at $x = 2$ since $f'(x) = f''(x) = 0$ and $f'(x) < 0$ either side of $x = 2$
(d)



9. (3CDMAT 2015:CA11)
(a) $\dfrac{dy}{dx} = \dfrac{x}{2}$
$m = y'(-2) = -1$
$m = y'(6) = 3$
$y = -x - 1$
$y = 3x - 9$
(b) Use simultaneous equations on calculator R is (2, –3)

10. (MMETH 2016S:CF8)

(a) $f'(x) = \dfrac{3(x^2 - x - 2) - (3x - 9)(2x - 1)}{(x^2 - x - 2)^2}$

$= \dfrac{3x^2 - 3x - 6 - (6x^2 - 21x + 9)}{(x^2 - x - 2)^2}$

$= \dfrac{-3x^2 + 18x - 15}{(x^2 - x - 2)^2}$

$= \dfrac{-3(x - 1)(x - 5)}{(x^2 - x - 2)^2}$

(b) $\dfrac{-3(x - 1)(x - 5)}{(x^2 - x - 2)^2} = 0$

$(x - 1)(x - 5) = 0$

$x = 1, 5$

$\therefore$ maximum at $x = 5$ and maximum value of $f(x)$ is $\dfrac{1}{3}$

11. (MMETH 2016:CF03)

(a) $f'(x) = \dfrac{e^x 2(x - 1) - (x - 1)^2 e^x}{e^{2x}}$

$= \dfrac{e^x(x - 1)(2 - (x - 1))}{e^{2x}}$

$= \dfrac{e^x(x - 1)(3 - x)}{e^{2x}}$

$= \dfrac{-(x - 1)(x - 3)}{e^x}$

$$= \frac{-x^2 + 4x - 3}{e^x}$$

(b) $f'(x) = 0 = (x-1)(x-3)$
So $f'(1) = f'(3) = 0$
(c) $f''(1) = \frac{2}{e}$ + so local min.

$f''(3) = -\frac{2}{e^3}$ - so local max.

Chapter 5: Small Changes Incremental Formula

1. (Projected)
(a) $4\pi r^2$
(b) $4\pi(10)^2(0.1) = 126$ cm^3.

2. (Projected)
(a) $4kr^3$
(b) $\delta V \approx 4(0.95)(1)^3(0.2)$
$= 0.76$
(c) True Increase = $0.95(1.2)^4 - 0.95(1)^4$
$= 1.01992$
Big difference of $1.01992 - 0.76 = 0.25992$
Incremental 'formula' only good for small changes.
1 mm → 1.2 mm is 20% which is not a small change.

3. (Projected)
(a) $8\pi r$
(b) $\frac{\delta V}{V} \approx 8\pi r \cdot \frac{\delta r}{4\pi r^2}$

$= 2 \cdot \frac{\delta r}{r}$
$= 2(1\%)$
$= 2\%$

4. (3CDMAT 2014:CF5)

$$\Delta y \simeq \frac{1}{3x^{\frac{2}{3}}} \times \Delta x \Rightarrow \Delta y \simeq \frac{1}{3 \times 10^2} \times 6 = 0.02$$

$$\Rightarrow \sqrt[3]{1006} \simeq 10.02$$

5. (3CDMAT 2015:CA17b)

$$\frac{\delta T}{T} \approx \frac{dT}{dl} \times \frac{\delta l}{T}$$

$$= \pi \times \left(\frac{l}{10}\right)^{-\frac{1}{2}} \times \frac{1}{10} \times \frac{\delta l}{2\pi\sqrt{\frac{l}{10}}}$$

$= 0.5\,\delta l$
$= 1\%$

6. (MMETH 2016:CA11)

$$A = 0.5\ell^2 \sin 60° = \frac{\sqrt{3}}{4}\ell^2$$

$$8A \approx \frac{2\sqrt{3}}{4}\ell \cdot \delta\ell$$

$$= \frac{2\sqrt{3}}{4}(10)(0.1)$$

$= 0.866$ cm^2

Chapter 6: Optimisation

1. (Projected:CF)
$P = 480 + 160x - 40x^2$
$p' = 160 - 80x = 0$
$x = 2 \quad P = 640 \quad p''(2) = -80 \therefore$ max.

2. (CA 2009:10a,b)
(a) Greatest = 6839 @ $t = 10$ hours
Least = 5000 @ $t = 0$ hours

(b) @ $t = 20$, $\frac{dN}{dt} = -67.66$

$\therefore \frac{dN}{dt}$ is min at $t = 20$ hours

3. (3CDMAT 2012:CA19)
(a) $V = (L - 2xL)^2 \times xL = L^3(1 - 4x + 4x^2) \times x$
$= L^3(x - 4x^2 + 4x^3)$
(b) $\frac{dV}{dx} = L^3(1 - 8x + 12x^2) = 0 \Rightarrow x = \frac{1}{6}, \frac{1}{2}$

$$\frac{d^2V}{dx^2} = 24x - 8 \Big|_{x=\frac{1}{6}} < 0 \Rightarrow \text{Maximum}$$

$$\Rightarrow V_{MAX} \text{ occurs at } x = \frac{1}{6} \Rightarrow V_{MAX} = \frac{2L^3}{27}$$

4. (3CDMAT 2013:CA10)
(a) Cost equation is: $1500 = 6y + 5(3x) \Rightarrow y = \frac{5}{2}(100 - x)$

Area is given by: $A = xy = x \times \frac{5}{2}(100 - 2x) = \frac{5}{2}(100x - 2x^2)$

(b) $\frac{dA}{dx} = \frac{5}{2}(100 - 2x) = 0 \Rightarrow x = 50, y = 125$

So the fence will be 50 m wide and 125 m long, giving an Area of 6250 m^2.

5. (3CDMAT 2014:CA16)
(a) $2w^2 + 4hw = 10 \Rightarrow w^2 + 2hw = 5$
(b) $V = w^2h = \frac{w^2(5 - w^2)}{2w} = \frac{(5w - w^3)}{2}$

$$\frac{dV}{dw} = \frac{(5 - 3w^2)}{2} = 0 \Rightarrow w = \pm\frac{\sqrt{5}}{\sqrt{3}}$$

$$\frac{d^2V}{dw^2} = -6w\Big|_{w=\frac{\sqrt{5}}{\sqrt{3}}} < 0 \Rightarrow \text{Maximum.}$$

Thus, at the maximum volume, $w = \frac{\sqrt{5}}{\sqrt{3}}$ m, $h = \frac{\sqrt{5}}{\sqrt{3}}$ m and

$$V = \left(\frac{\sqrt{5}}{\sqrt{3}}\right)^3 \text{ m}^3$$

OR $w = h = 1.29$ m, $V = 2.15$ m^3

6. (MMETH 2016S:CA19)
(a) Let a be the section of the ladder in the 2 m corridor and b be the section of the ladder in the 3 m corridor

$$\cos\theta = \frac{2}{a} \Rightarrow a = \frac{2}{\cos\theta}$$

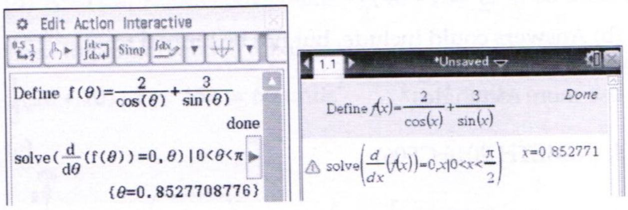
$$\sin\theta = \frac{3}{b} \Rightarrow b = \frac{3}{\sin\theta}$$

$L = a + b = \dfrac{2}{\cos\theta} + \dfrac{3}{\sin\theta}$

(b) The maximum length of the ladder occurs when $\dfrac{dL}{d\theta} = 0$.

$L = \dfrac{2}{\cos\theta} + \dfrac{3}{\sin\theta}$

Solve $\left(\dfrac{d}{d\theta}(f(0) = 0, 10° \le \theta \le 90°)\right)$



$\theta = 48.86°$ ($\theta = 0.8528$)

Hence maximum length, $L = \dfrac{2}{\cos 48.86°} + \dfrac{3}{\sin 48.86°}$
$= 7.0235$

$\therefore$ The worker will be able to carry a 6.5 m long ladder around the corner of the corridors.

7. (MMETH 2016:CF08)
(a) Let M be the midpoint of RQ
$RM = r\sin\theta$ so $RQ = 2r\sin\theta$
$OM = r\cos\theta$ so $PM = r + r\cos\theta$
Area of $\triangle PQR = 0.5 \times RQ \times PM$
$= 0.5 \times 2r\sin\theta \times (r + r\cos\theta)$
$= r^2\sin\theta\,(1 + \cos\theta)$

(b) $\dfrac{dA}{d\theta} = r^2(\cos\theta + \cos^2\theta - \sin^2\theta)$
$= r^2(2\cos^2\theta + \cos\theta - 1)$
$= r^2(2\cos\theta - 1)(\cos\theta + 1)$

For max $\dfrac{dA}{d\theta} = 0$

So $\cos\theta = \dfrac{1}{2}$, $\cos\theta = -1$

$\theta = \dfrac{\pi}{3}$ $\theta = \pi$ (reject)

$A''\left(\dfrac{\pi}{3}\right) = -$ so maximum

$A = \dfrac{\sqrt{3}}{2}\, r^2\left(1 + \dfrac{1}{2}\right)$

$= \dfrac{3\sqrt{3}}{4}\, r^2$

Chapter 7: Techniques of Integration

1. (Projected)
$\dfrac{e^{2t}}{2} - 2t - \dfrac{1}{2e^{2t}} + c$

2. (3CDMAT 2010:CF5a)
$\displaystyle\int_1^3 (x^3 - 1)\,dx = \left(\frac{1}{4}x^4 - x\right)\Big|_{x=1}^{x=3} = 18$

3. (3CDMAT 2011:CF5a)
$\displaystyle\int_{-0.5}^{0} 3\,(1-x)^2\,dx = -3\int_{-0.5}^{0} -1\,(1-x)^2\,dx$

$= -3\left[\dfrac{(1-x)^3}{3}\right]_{-0.5}^{0} = -1(1^3 - 1.5^3) = \dfrac{19}{8}$ (or 2.375)

4. (3CDMAT 2012:CF5b,c)

(a) $\dfrac{dy}{dx} = 2x^{-2} + 4x \Rightarrow y = -\dfrac{2}{x} + 2x^2 + c$

$(2, 3) \Rightarrow 3 = -1 + 8 + c \Rightarrow c = -4$

$\Rightarrow$ When $x = 5$, $y = -0.4 + 46 = 45.6$

(b) $\left.\dfrac{x^3}{x^2 + 1}\right|_1^2 = \dfrac{8}{5} - \dfrac{1}{2} = 1.1$

5. (3CDMAT 2014:CF1)

(a) $\displaystyle\int_0^3 (6x^2 + 2x + 1)\,dx = [2x^3 + x^2 + x]_{x=0}^{x=3} = 66$

(b) $\displaystyle\int_1^2 \frac{d}{dx}\left(\frac{x^5}{x^2+1}\right)dx = \left[\frac{x^5}{x^2+1}\right]_{x=1}^{x=2} = \frac{32}{5} - \frac{1}{2} = 5.9$

(c) $\displaystyle\frac{d}{dx}\int_4^{x^2} \frac{2}{3t^3 - 1}\,dt = \frac{2 \times 2x}{3x^6 - 1} = \frac{4x}{3x^6 - 1}$

6. (3CDMAS 2015:CF6)
(a) $e^{2x} + 2xe^{2x}$
(b) Since

$\dfrac{d}{dx}(xe^{2x}) = e^{2x} + 2xe^{2x}$

$\displaystyle\int_0^1 \frac{d}{dx}(xe^{2x}) = \int_0^1 (e^{2x} + 2xe^{2x})\,dx$

$\displaystyle\int_0^1 2xe^{2x}\,dx = \int_0^1 \frac{d}{dx}(xe^{2x})\,dx - \int_0^1 xe^{2x}\,dx$

$= [xe^{2x}]_0^1 - [e^{2x} \div 2]_0^1$

$= \dfrac{e^2}{2} + \dfrac{1}{2}$

7. (3CDMAT 2015:CF6)

(a) $f'(x) = \dfrac{d}{dx}\displaystyle\int_{-\infty}^{x} f(t)dt$

$f'(x) = f(x)$
$\therefore f(x) = Ae^x$ $f(0) = 1$
$\therefore f(x) = 1e^x = e^x$
(b) Yes unique.
Since $f(0) = 1$ gives only 1 A value and solution.

8. (MMETH 2016:CF02)
(a) $4xe^{2x} + 2e^{2x}$

(b) $\displaystyle\int \frac{d}{dx}(2xe^{2x}) = \int 4xe^{2x}\,dx + \int 2e^{2x}\,dx$

$\displaystyle\int 4xe^{2x} = 2xe^{2x} - e^{2x} + c$

Chapter 8: Area under the Curve

1. (3CDMAS 2010:CF4)

(a) $\int_1^2 e^{x-1} = [e^{x-1}]_1^2 = e - 1$

(b) $e^{x-1} = 2 - x$ at $x = 1$

$\Rightarrow$ Area $= \int_0^1 e^{x-1}dx + \int_1^2 (2-x)dx$

$= e^{x-1}\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^2$

$= \frac{3}{2} - \frac{1}{e}$

2. (3CDMAT 2013:CF7a,b,d)

(a) $\int_{-2}^{3} f(x)dx = -4 + 4 - 2 = -2$

(b) Area $= 4 + 4 + 2 = 10$

(c) $\int_0^2 (x - f(x))dx = \int_0^2 x dx - \int_0^2 f(x)dx = 2 - 4 = -2$

3. (3CDMAT 2014:CA15)

(a) $y' = 3x^2 + 2x - 17 \Rightarrow$ gradient at $x = -2$ is -9.

Also, $y(-2) = 45$. Thus tangent has equation $y = -9x + c$ and passes through $(-2, 45) \Rightarrow$ Tangent is $y = -9x + 27$

(b) Tangent intersects curve at $x = -2$ and 3.

$\Rightarrow$ Area $= \int_{-2}^{3} (-9x + 27) - (x^3 + x^2 - 17x + 15)\, dx = 52.083$ units squared.

4. (3CDMAS 2015:CA10)

(a) $y = -0.5x + 1.1278$

(b) 3.817

(c) $\int_{\frac{\pi}{6}}^{3.817} \left[\frac{-x}{2} + \frac{\pi}{12} + \frac{\sqrt{3}}{2} - \cos x\right] dx$

(d) 1.27 units^2

5. (3CDMAT 2015:CA13)

On CAS solve.

Solver $\int_1^c (6x^2 - 6x)\, dx = 1$

$c = 1.5$ (reject $c = 0$)

6. (MMETH 2016S:CF1)

(a) $\int_0^5 f(x)\, dx$ = Area of semicircle – area of square

$= \frac{\pi}{2} - 9$

(b) $\frac{\pi}{2} - 9 + \frac{4\pi}{2} = \frac{5\pi}{2} - 9$

7. (MMETH 2016S:CA15)

(a)

Time (t)	0	2	4	6
Velocity (v)	**30**	28.8	**25.2**	19.2

Rectangle	0–2	2–4	4–6	Total
Circumscribed area	**60**	**57.6**	50.4	**168**
Inscribed area	**57.6**	**50.4**	38.4	**146.4**

Estimate of total distance travelled: 157.2 m

$\therefore$Mean $= \frac{314.4}{2} = 157.2$ m

(b) Answers could include, but not be limited to:
use smaller rectangles
use more rectangles

8. (MMETH 2016:CF06)

Area $= \int_0^4 6 - 2e^{x-4} - \left(-\frac{1}{4}x + 5\right) dx$

$= \left[-2e^{x-4} + \frac{x^2}{8} + x\right]_0^4$

$= (-2 + 2 + 4) - (-2e^{-4})$

$= 4 + \frac{2}{e^4}$

9. (MMETH 2016:CF07)

(a) $(6 \times 6) + (4 \times 2) + (0.5 \times 2^2) + \frac{\pi \times 4^2}{4}$
$= 4\pi + 46$

(b) $4\pi + 46 - \left(\frac{1}{2} \times 2^2 + 4 \times 6 - \frac{\pi \times 4^2}{4}\right)$
$= 8\pi + 20$

(c) $6(\alpha - 18) + 26 - 4\pi = 4\pi + 46$

$\alpha = \frac{20 + 8\pi}{6} + 18$

$= 21\frac{1}{3} + \frac{4\pi}{3}$

Chapter 9: Total Change from Rate of Change

1. (Projected)

The graph of $f(x)$ indicates that the function is increasing on the interval $0 \le x \le 6$ and decreasing on the interval $6 \le x \le 10$.
A maximum turning point exists at $x = 6$.
The maximum value is given by:

$f(x) = \int_0^6 f'(x)\, dx + f(0)$ *ie* the area of the semi-circle +1

$= \frac{1}{2}\pi(3)^2 + 1$

$= 1 + \frac{9\pi}{2}$

The minimum value of the function must be at one of the end points of the interval $0 \le x \le 10$.

Because the magnitude $\int_0^6 f'(x)\, dx >$ the magnitude of $\int_6^{10} f'(x)\, dx, f(10) > f(0)$.

Hence the minimum value of $f(x)$ on $0 \le x \le 10$ is 1.

2. (CA 1995:11)

$\frac{40}{3}$ litres = $13\frac{1}{3}$ litres

3. (CA 2002:08)

(a) 20 seconds

(b) $30 < t < 50$ (Here $v'(t)$ is a maximum.)

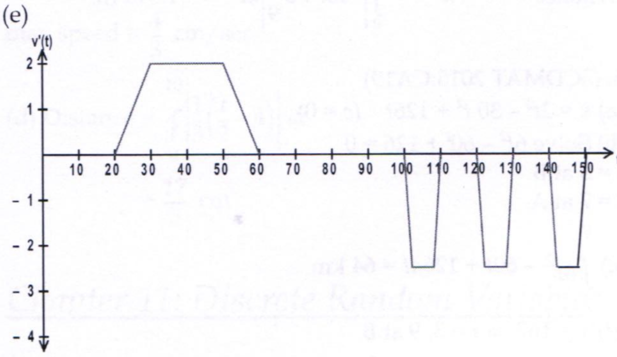
(c) 40 seconds

(d) Area trapezium above t-axis $= \frac{1}{2}(40 + 20) \cdot 2 = 60$ units^2.

Area trapezium above t-axis =

$\frac{1}{2}(20 + 10) \cdot k = 15k = 60$ units^2 $\therefore k = 4$ units.

(e)



The above diagram is a possible solution.

(i) There are three (3) regions under the t-axis with the same time interval.

(ii) The total area of these three regions is equal to the area of the region above the t-axis.

4. (CA 2004:07)

$$F(x) = \int_0^x \cos(t^2)\ dt$$

$F'(x) = \cos(x^2)$.

$\delta F \cong \frac{dF}{dt} \cdot \delta x$ where $\delta x = 0.01$

$= \cos(x^2) \cdot \delta x$

$= \cos(5^2) \cdot 0.01$

$= \cos(25) \cdot 0.01$

$\delta F \cong 0.00991$

5. (3CDMAT 2012:CF8)

(a) $x = 5$ $\because$ on the interval $0 \le x \le 5$, $f(x) \ge 0$ $\forall$ x.

(b) $F'(x) = \frac{d}{dx}\int_0^x f(t)dt = f(x) \Rightarrow F'(x)$ is a maximum (32) when $x = 3$.

(c) Inscribed area using rectangles = 5 + 16 + 27 = 48

Circumscribed area using rectangles = 16 + 27 + 32 = 75

$\Rightarrow 48 \le F(x) \le 75$

6. (MMETH 2016S:CF3)

$\frac{dA}{dt} = 60 - 3at^2$

At $t = 2$, $\frac{dA}{dt} = 0 = 60 - 12a$

$\therefore a = 5$

Hence $A = 60t - 5t^3 + c$

Substituting (1, 62) into the equation

$62 = 60 - 5 + c$

$\therefore c = 7$

So $A = 60t - 5t^3 + 7$

7. (MMETH 2016S:CA20)

(a) $A(x)$ starts to decrease at the point where $f(x) = 0$.

i.e. where $\cos x^2 = 0$, $0 \le x \le 2$.

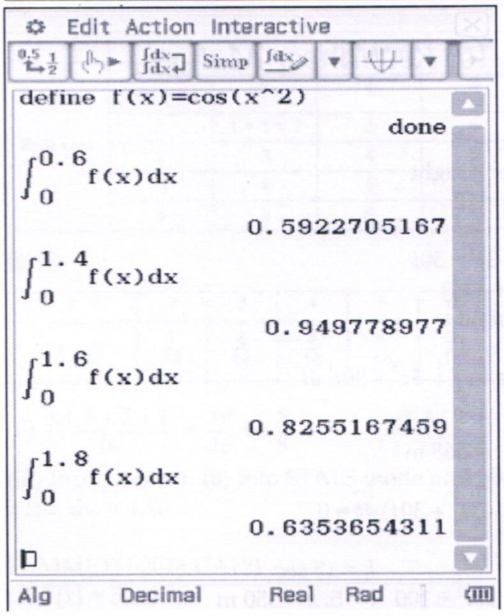
$\Rightarrow x^2 = \frac{\pi}{2}$

$\Rightarrow x = \sqrt{\frac{\pi}{2}}$

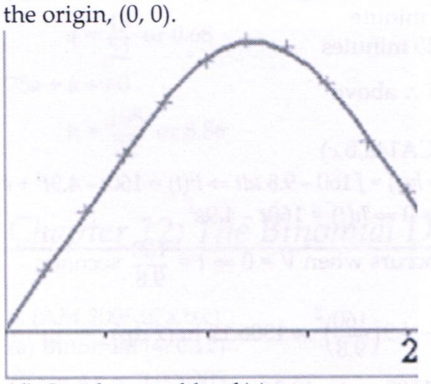
$\Rightarrow x \approx 1.253$

(b)

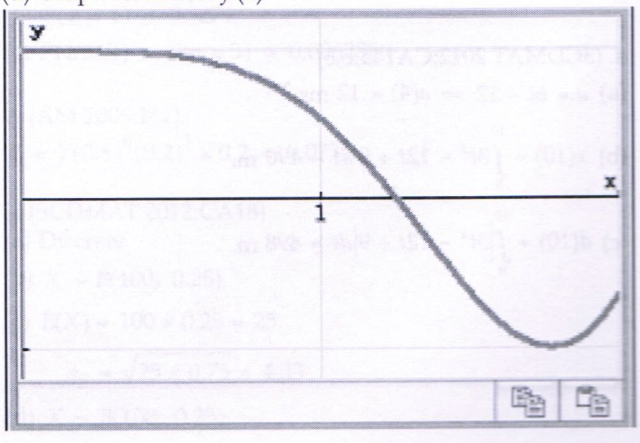
x	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$A(x)$	0.200	0.399	**0.592**	0.768	0.905	0.974	**0.950**	**0.826**	**0.635**	0.461



(c) Plot of points, jointed by curve. Should pass through the origin, (0, 0).

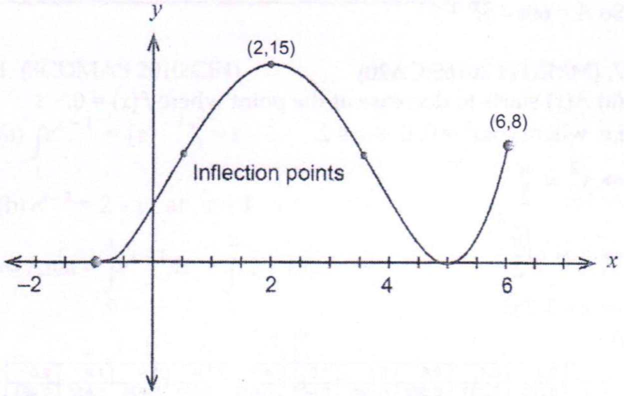


(d) Graph resembles $f(x)$



(e) $A'(x) = f(x)$

8. (MMETH 2016:CF05)



Chapter 10: Rectilinear Motion

1. (Projected)
(a) 48 m/sec + right
(b) $-6t + 30 = 12$
$t = 3$ sec
(c) $v = 0 = -3t^2 + 30t$
$0 = -3t\,(t - 10)$
$t = 10$ seconds
(d) Distance = $\int_0^{12} \left|-3t^2 + 30t\right| dt$
= 568 m
(e) Solve $\int_0^k (-3t^2 + 30t)\, dt = 0$
$k = 15$ sec
(f) $x = -t^3 + 15t^2 + 100 \quad t = 5,\ x = 350$ m

2. (CA 2000:18)
(a) 9.66 metres per minute
(b) 216.93 m after 35 minutes
(c) $\left|\int_{35}^{60} v\,dt\right| = 88.35 \therefore$ above

3. (3CDMAT 2011:CA14a,b,c)
(a) $h(t) = \int v(t)\, dt \Rightarrow h(t) = \int 160 - 9.8\, t dt \Rightarrow h(t) = 160t - 4.9t^2 + c$
At $t = 0,\ h = 0 \Rightarrow c = 0 \Rightarrow h(t) = 160t - 4.9t^2$
Maximum height occurs when $V = 0 \Rightarrow t = \dfrac{160}{9.8}$ seconds
$\Rightarrow h_{\max} = 160\left(\dfrac{160}{9.8}\right) - 4.9\left(\dfrac{160}{9.8}\right)^2 \approx 1306.12$ m (2 dp)
(b) $160t - 4.9t^2 = -1500 \Rightarrow t \approx 40.26$ s (2 dp)

4. (3CDMAT 2012:CA14a,b,c)
(a) $a = 6t - 12 \Rightarrow a(4) = 12$ ms^{-2}
(b) $x(10) = \int_0^{10} 3t^2 - 12t + 9\, dt = 490$ m.
(c) $d(10) = \int_0^{10} \left|3t^2 - 12t + 9\right| dt = 498$ m.

5. (3CDMAT 2014:CA12)
(a) 20 km/h = $\dfrac{20000}{3600} = 5\dfrac{5}{9}$ m/s
$\Rightarrow$ distance travelled in 1 second is $5\dfrac{5}{9}$ m.
(b) $\dfrac{dv}{dt} = -10 \Rightarrow v = -10t + 5\dfrac{5}{9}$
(c) When $v = 0$ then $-10t + 5\dfrac{5}{9} = 0 \Rightarrow t = \dfrac{5}{9}$ s.
Distance = $x = \int_0^{\frac{5}{9}} \left|-10t + 5\dfrac{5}{9}\right| dt = 1.543$ m.

6. (3CDMAT 2015:CA19)
(a) $x = 2t^3 - 30t^2 + 126t \quad (c = 0)$
(b) Solve $6t^2 - 60^t + 126 = 0$
$t = 3$ at B
$t = 7$ at A
(c) $\int_3^7 6t^2 - 60t + 126 dt = 64$ km
(d) $x = 162 \Rightarrow t = 3, 9$ at B
$\therefore$ Distance travelled = $\int_0^9 \left|6t^2 - 60t + 126\right| dt$
= 290 km

7. (MMETH 2016S:CF7)
(a) Velocity = $x'(t)$
$= \cos t \times e^{\sin t}$
(b) Rate of change of velocity = $x''(t)$
$= -\sin t \times (e)^{\sin t} + (\cos t)^2 \times (e)^{\sin t}$
(c) $\int_0^{\frac{\pi}{2}} x'(t)\, dt = [x(t)]_0^{\frac{\pi}{2}}$
$= (e)^{\sin(\frac{\pi}{2})} - (e)^{\sin 0}$
$= e - 1$
(d) $\int_0^{\frac{\pi}{2}} x'(t)\, dt$ = the change in displacement of the particle between 0 and $\dfrac{\pi}{2}$ seconds.

8. (MMETH 2016:CF04)
(a) $v = 15 \cos 3t$
$v = 0$ mic/sec
(b) $\dfrac{dv}{dt} = -45 \sin 3t$
= 45 mic/sec^2
(c) $\int_0^{\frac{\pi}{2}} -45 \sin 3t\, dt$
$= [15 \cos 3t]_0^{\frac{\pi}{2}}$
= −15 mic/sec

9. (MMETH 2016:CA19)

(a) $\frac{1}{3}\left(\frac{t}{2}-4\right) = 0$

$t = 8$ sec

(b) $= \int_4^5 \frac{1}{3}\left(\frac{t}{2}-4\right) dt$

$= -0.58\dot{3}$ cm

(c) $v = \frac{1}{3}\left(\frac{t}{2}-4\right)$

at $t = 0$ $v = -\frac{4}{3}$

max speed $= \frac{4}{3}$ cm/sec

(d) Distance $= \int_0^{10}\left|\frac{1}{3}\left(\frac{t}{2}-4\right)\right| dt$

$= \frac{17}{3}$ cm

Chapter 11: Discrete Random Variables

1. (Projected)
(a) 10 (b) $k = 7$ (c) 5.5

2. (AM 2004:04)
(a) S = {2, 3, 4, 5, 6}
(b)

	1	2	2	3	3
1	2	3	3	4	4
2	3	4	4	5	5
2	3	4	4	5	5
3	4	5	5	6	6
3	4	5	5	6	6

s	2	3	4	5	6
$P(S = s)$	$\frac{1}{25}$	$\frac{4}{25}$	$\frac{8}{25}$	$\frac{8}{25}$	$\frac{4}{25}$

(c)
(1) Each probability is between 1 and 0.
(2) The sum of the individual probabilities is 1.
(3) The variable s only has discrete (integer) values.

3. (AM 2007:08a)
$a + 2a + 3a + 4a = 1$

$\therefore a = \frac{1}{10}$

4. (AM 2008:16a)

$\frac{k}{1}+\frac{k}{2}+\frac{k}{3}+\frac{k}{4}+\frac{k}{5} = 1$

$\Rightarrow k\left(\frac{137}{60}\right) = 1$

$\Rightarrow k = \frac{60}{137} = 0.4380.$

5. (MMETH 2016S:CA14)
(a)
(i) $0.1 + 2a + 0.9 + 1 + 5b = 3.3$
$0.1 + a + 0.3 + 0.25 + b = 1$
$a = 0.15$ $b = 0.2$
(ii) $1.23^2 = 1.51$
(iii) $Var(X + 5) = Var(X)$
$= 1.51$
(iv) $Var(2X + 5) = 2^2(Var(X))$
$= 2^2 \times 1.51 = 6.04$
(b) $E(N) = 0 + 0.4 + 0.5 + 0.12 + 0.04 = 1.06$
Expected weekly pay $= 250 + 1.06 \times 400 = \$674$

6. (MMETH 2016:CA15)
(a)

		Roll two			
	Sum of two rolls	1	2	3	4
Roll one	1	1 + 1 = 2	3	4	5
	2	3	4	5	6
	3	4	5	6	7
	4	5	6	7	8

(b) (i)

x	2	3	4	5	6	7	8
$P(X = x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

(ii) $\frac{4+3+2+1}{16} = \frac{10}{16}$ or $\frac{5}{8}$

(iii) Input table in (b) into STATS mode and obtain mean = 5 std. dv. = 1.58

7. (MMETH 2016:CA17)
(a) $E(x) + 5 = 80$
(b) 2^2 Var(x) = 88
(c) $22a = 15$

$a = \frac{15}{22}$ or 0.68

$75a + b = 60$

$b = \frac{195}{22}$ or 8.86

Chapter 12: The Binomial Distribution

1. (AM 2006:07a,b,c)
(a) Binomial (4, 0.12)
$P(x = 4) = 0.000207$
(b) $P(x = 1) = 0.3271$
(c) $P(L > R) = P(x \geq 3) = 0.0062899$

2. (AM 2006:16a)
$P = 7(0.8)^6(0.2)^1 \times 0.2 = 0.0734$

3. (3CDMAT 2012:CA18)
(a) Discrete
(b) $X \sim B(100, 0.25)$
(c) $E(X) = 100 \times 0.25 = 25$

$s_X = \sqrt{25 \times 0.75} = 4.33$

(d) $X \sim B(100, 0.25)$
$P(X \geq 35) = 0.0164$

(e) Yes, evidence that treatment is effective. 35% improved with this treatment, while only 25% improved without it. There is only 1.64% chance that this improvement would have occurred without the treatment.

4. (MMETH 2016S:CF4)
(a) The distribution is binomial.
(b) $32 = np \quad 4 = \sqrt{np(1-p)}$
$4 = \sqrt{32(1-p)}$
$16 = 32(1-p)$
$p = \frac{1}{2}$
$32 = \frac{1}{2}n$
$\therefore n = 64$

5. (MMETH 2016:CA20)
(a) Binomial $n = 30 \quad p = 0.8$
Mean = 24 (np)
Std Dev. = 2.191 ($\sqrt{np(1-p)}$)
(b) 0.123

Chapter 13: Logarithms

1. (3ABMAS 2011:CF1)
(a) $\log_{10} 6 = \log(2 \times 3)$
$= \log 2 + \log 3 = x + y$
(b) $\log_{10} 0.6 = \log 6/10$
$= \log 6 - \log 10$
$= x + y - 1$
(c) $\log_{10} 45 = \log 90/2$
$= \log(3^2 \times 10) - \log 2$
$= 2\log 3 + \log 10 - \log 2$
$= 2y - x + 1$

2. (3ABMAS 2012:CF6)
$\log_a y = \log_a a + \log_a x^2$
$\log_a y = \log_a ax^2$
$y = ax^2$

3. (3ABMAS 2013:CA8)
(a) (i) $\log_a\left(\frac{4}{5}\right) = \log_a 4 - \log_a 5 = x - y$
(ii) $\log_a(4 \times 5^2) = \log_a 4 + 2\log_a 5 = x + 2y$
(b) $(a^x)^3 = (4)^3 = 64$

4. (3ABMAS 2014:CA9)
(a) $a^{x+y} = a^x \cdot a^y = 4(7) = 28$
(b)
(i) $\log_a\left(\frac{7}{4}\right) = \log_a 7 - \log_a 4$ As $a^y = 7$ then $y = \log_a 7$
$= y - x$ also as $a^x = 4$ then $x = \log_a 4$
(ii) $\log_a(49a^2) = \log_a 49 + \log_a a^2$
$= 2\log_a 7 + 2$
$= 2y + 2$

5. (3ABMAS 2014:CA15)
(a) $L = 10\log\frac{I_0}{I_0} = 10\log(1) = 0$ dB
(b)
$L = 10\log\left(\frac{5}{\pi(10)^3 \times 10^{-12}}\right)$
$= 10\log\left(\frac{5 \times 10^9}{\pi}\right) = 10\log(1.5915\ldots \times 10^9) = 92.018\ldots$ dB
Hence the party music has a loudness of 92 dB at a distance of 10 m.
(c) Require $L \le 60$ i.e. $122.018 - 30\log(d) \le 60$
Solving gives $d \ge 116.752$ ... metres
i.e. The music will be acceptable at a distance of 117 metres. (to the nearest metre)

6. (MMETH 2016S:CF2)
(a) (i) $\log_x 2^2 = 2$ or $x^2 = 4$ $x > 0$
$2\log_x 2 = 2 \Rightarrow x = 2$
$\therefore x = 2$
(ii) $\ln e^{2x} = \ln 5$
$2x = \ln 5$
$x = \frac{\ln 5}{2}$
(b) $\log a + \log a^2 + \log a^3 + \ldots + \log a^{50}$
$= 1\log a + 2\log a + 3\log a + \ldots + 50\log a$
$= (1 + 2 + 3 + \ldots + 50)\log a$
$= (25 \times 51)\log a$
$= 1275\log a$
$\therefore k = 1275$

7. (MMETH 2016S:CA13)
(a) $D = 20\log_{10}\frac{2P_{ref}}{P_{ref}} = 20\log_{10} 2 = 6.0$ dB
(b) Let P_r denote the pressure of the rock concert and P_o denote the pressure of the orchestra. Then $20\log_{10}\frac{P_r}{P_{ref}} = 150$ and $20\log_{10}\frac{P_o}{P_{ref}} = 120$. Subtracting the two gives $20\log_{10}\frac{P_r}{P_{ref}} - 20\log_{10}\frac{P_o}{P_{ref}} = 30$. Simplifying the LHS gives $20\log_{10}\frac{P_r}{P_o} = 30$ or $\log_{10}\frac{P_r}{P_o} = 1.5$. Then $\frac{P_r}{P_o} = 10^{1.5} = 31.6$.
The sound pressure of the rock concert is 31.6 times greater than the symphony orchestra.

8. (MMETH 2016:CF01)
(a) $8^2 - 2^5 = 32$
(b) $\log_2(x + y) + \log_2 4 = \log^2(x - 2y)$
$\log_2 4(x + y) = \log_2(x - 2y)$
$4x + 4y = x - 2y$
$y = -0.5x$

9. (MMETH 2016:CA12)

N Z $5.5 = \log_{10}\frac{A}{A_0}$ so $A = 10^{5.5}\,A_0$

HAYMAN $3.4 = \log_{10}\frac{A}{A_0}$ so $A = 10^{3.4}\,A_0$

$$\frac{10^{5.5}A_0}{10^{3.4}A_0} = 10^{2.1}$$

approx. 125.89 times larger.

Chapter 14: Calculus and Natural Logarithms

1. (Projected:CF)

(a) $\int_2^8 \frac{1}{x}\,dx = [\ln x]_2^8$

$= \ln 8 - \ln 2$

$= \ln \frac{8}{2}$

$= \ln 4$

(b) (i) (0, –2) (ii) $x = -1$ (iii) $x = e^2 - 1$ (iv) $g(x) = \ln(x-1) + 1$

2. (CA 2006:10b)

$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$

$= \frac{1}{2}[\ln|1+x^2|]$

$= \frac{1}{2}[\ln 2 - \ln 1]$

$= \frac{1}{2}\ln 2$

$= \ln 2^{\frac{1}{2}}$

$= \ln\sqrt{2}$

3. (CA 2008:08c)

$\int_0^{\pi/2} \frac{\cos\theta}{2+\sin\theta}\,d\theta$

$= \int_2^3 \frac{\cos\theta}{u} \times \frac{du}{\cos\theta}$

$= \int_2^3 \frac{1}{u}\,du$

$= [\ln u]_2^3$

$= \ln 3 - \ln 2$

$= \ln\frac{3}{2}$

Let $u = 2 + \sin\theta$

$\frac{du}{d\theta} = \cos\theta$

$d\theta = \frac{du}{\cos\theta}$

if $\theta = 0$ then $u = 2$

if $\theta = \frac{\pi}{2}$ then $u = 3$

4. (3ABMAS 2010:CF2d)

$y = 3\ln x - \ln(7-4x)$

Thus $\frac{dy}{dx} = \frac{3}{x} + \frac{4}{7-4x}$

5. (3ABMAS 2012:CA11)

(a) $\ln\left(\frac{I_S}{I_A}\right) = 1 - (-1.5) = 2.5$

Hence $\left(\frac{I_S}{I_A}\right) = e^{2.5} = 12.18$

= 12 to the nearest integer

(b) $\frac{I_J}{I_S} = \sqrt{e} = e^{0.5}$

Then $m_S - m_J = ln\,(e^{0.5}) = 0.5$

Hence $m_J = m_S - 0.5 = -2$

6. (3ABMAS 2013:CF3)

(a)

$g'(x) = \frac{1}{2x+1}\cdot 2 = \frac{1}{2x+1}$

$g'(0) = 2$

(b)

$h'(x) = 4x\,.\,e^{2x}.2 + e^{2x}\,.\,4 = 4e^{2x}\,[2x+1]$

$h'(1) = 4e^{2}.\,3 = 12e^2$

(c)

$q'(x) = \frac{\ln x\cdot(1) - x\cdot\left(\frac{1}{x}\right)}{(\ln x)^2}$

$= \frac{\ln x - 1}{(\ln x)^2}$

$q'(e) = \frac{\ln e - 1}{(\ln e)^2} = 0$

7. (3ABMAS 2014:CF4b)

If $g(x) = \frac{\ln x}{x}$ then $g'(x) = \frac{x\cdot\left(\frac{1}{x}\right) - (\ln x)\cdot 1}{x^2} = \frac{1-\ln x}{x^2}$.

Hence $g'(1) = \frac{1-\ln 1}{1} = 1 - 0 = 1$

8. (MMETH 2016S:CF6a)

$f''(x) = 2x\ln(2x+1) + x^2\,\frac{2}{2x+1}$

9. (MMETH 2016S:CA11)

(a) When $x = e$, $f(x) = \ln e + 1 = 2$

$\therefore$ $(e, 2)$ lies on the line

$f(x) = \ln x + 1$

$f'(x) = \frac{1}{x}$

or

Substituting $(e, 2)$ into $f'(x)$

$m = \frac{1}{e}$

i.e. $y = \frac{1}{e}x + c$

Substituting $(e, 2)$ into this equation

$c = 1$

$\therefore$ Equation of tangent is $y = \frac{1}{e}x + 1$

(b) $0 = \frac{1}{e}x + 1$

$x = -e$

$(-e, 0)$

(c) $f\left(\frac{1}{e}\right) = \ln\frac{1}{e} + 1$

$= -\ln e + 1$

$= 0$

$\therefore f(x)$ cuts the x-axis at $\left(\frac{1}{e}, 0\right)$

(d) Area $= \int_{-e}^{e}\left(\frac{x}{e} + 1\right) dx - \int_{\frac{1}{e}}^{e}(\ln x + 1)\, dx$

$= \frac{1}{2} 2e \times 2 - \int_{\frac{1}{e}}^{e}(\ln x + 1)\, dx$

$= 2.35$

10. (MMETH 2016:CA13)

(a) $x^2 \cdot \frac{1}{x} + 2x \cdot \ln x$

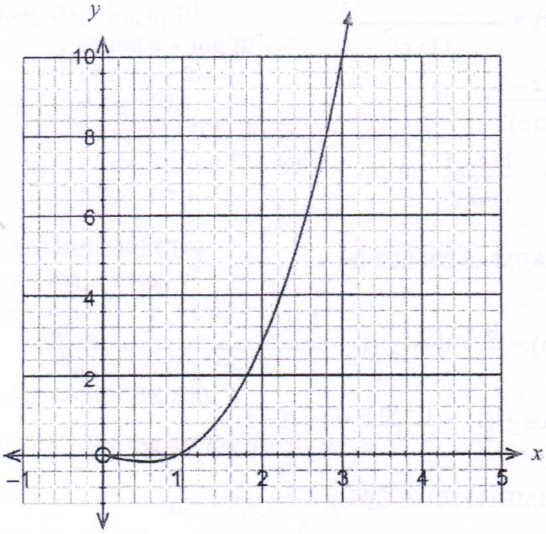
$= x + 2x\ln x$

(b) $\frac{dy}{dx} = x\,(1 + 2\ln x) = 0$

$\ln x = -\frac{1}{2}$

$x = e^{-\frac{1}{2}}$ is only solution ($x \neq 0$)

(c)



(d) Area $= \int_{1}^{e} x^2 \ln x$

$= 4.5746$ units^2

Chapter 15: Continuous Random Variables

1. (Projected:CF)

(a) Prob = Area of rectangle

$= (13 - 10) \times \frac{1}{13 - 5}$

$= \frac{3}{8}$

(b) $\frac{13 - T}{8} = 0.6$

$T = 9.8$ mins

2. (AM 2000:22)

(a) $\int_{1}^{15}(t-1)^2 e^{-(t-1)}\, dt = \underline{0.99991}$

(b) $\int_{7}^{27}(t-1)^2 e^{-(t-1)}\, dt = \underline{0.06197}$

(c) $\int_{2.6}^{9}(t-1)^2 e^{-(t-1)}\, dt = \underline{0.76960}$

3. (AM 2001:11)

(a) For any p.d.f. the total area under its curve over the interval in which it is defined must sum to 1

$\therefore \frac{1}{2} \times 60 \times b = 1 \qquad \therefore b = \frac{1}{30}$

(b)

(i) P (before 8:30 am) $= \frac{1}{2} \times 30 \times \frac{30}{1800} = \frac{1}{4} = 0.25$

(ii) P (after 8:40 am) $= 1 - P$ (before 8:40 am)

$= 1 - \left[\frac{1}{2} \times 40 \times \frac{40}{1800}\right]$

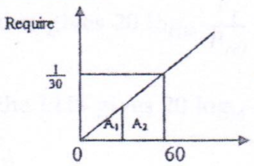
$= 1 - \frac{4}{9}$

$= \frac{5}{9}$

(c) P (after 8:40/after 8:30 am) $= \frac{\text{P(time > 8:40)}}{\text{P(time > 8:30)}} = \frac{\frac{5}{9}}{\frac{3}{4}} = \frac{20}{27}$

(d)

Method 1



$A_1 = A_2 \therefore \int_{0}^{k} \frac{1}{1800}$

$\Rightarrow k = 42.43$

Time required is 8:42 (to the nearest minute)

Method 2

P (time > t) = P (time < t) = 0.5 (where $T = 8\,.\,t$ am)

$\therefore \frac{1}{2} \times t \times \frac{t}{1800} = \frac{1}{2}$

$\therefore t^2 = 1800$

$t = 42.43$

Time required is 8:42 am (to the nearest minute)

4. (AM 2002:12)

Let t = time taken to complete the job

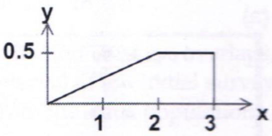
$\therefore$ Given $f(t) = \frac{3}{160}\,(t+2)(t-1)(5-t)$ for $1 \le t \le 5$

(a) $P(t < 2) = \int_{1}^{2} f(t)\, dt = 0.11406$

(b) $\int_{4}^{5} f(t)dt = 0.1984$

(c) $P(t = 3) = 0 \quad P(t = a)$ is 0 for a continuous random variable

5. (AM 2003:15a,b,d)
(a) Continuous
Area = $0.5 \times 2 \times 0.5 + 1 \times 0.5 = 1$



(b) Discrete
$\sum P(X = x) = 0.1 + 0.15 + 0.2 + 0.25 + 0.3 = 1$
(c) Neither. Continuous & integral = 1 BUT graph BELOW x-axis.

6. (MMETH 2016S:CF5)
(a) $\int_1^3 \frac{q}{x}\,dx = 1$

$[q \ln x]_1^3 = 1$

$q \ln 3 - q \ln 1 = 1$

$q = \frac{1}{\ln 3}$

(b) $\frac{1}{\ln 3}\int_2^3 \frac{1}{x}\,dx = \frac{1}{\ln 3}\,[\ln x]_2^3 = 1 - \frac{\ln 2}{\ln 3}$

7. (MMETH 2016S:CA16)
(a) $E(X) = \int_0^1 2x\,(1 - x)dx$

$= \frac{1}{3}$ h = 20 minutes

(b) $Var = \int_0^1 \left(x - \frac{1}{3}\right)^2 2(1 - x)dx$

$= \frac{1}{18}$

Alternative solution

$E(x^2) = 2\int_0^1 (x^2 - x^3)\,dx = \frac{1}{6}$

$Var = E(x^2) - (E(x))^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$

8. (MMETH 2016:CA16)
(a) continuous uniform distribution

$f(x) = \frac{1}{8}$

$P(x < 250) = (250 - 247) \times \frac{1}{8}$

$= \frac{3}{8}$

(b) Mean = 251 (midpoint)

VAR = $\int_{247}^{255} \frac{1}{8}(x - 251)^2\,dx = 5.333$

Std. Dev. = $\sqrt{5.333} = 2.31$

(c) Binomial $n = 14$ $p = \frac{3}{8}$

Prob = $P(x = 4) \times \frac{3}{8}$

= 0.0675

Chapter 16: The Normal Distribution

1. (AM 2000:04)
(a) Let x = the length of tape the cassette holds
$\therefore x \sim N(\mu = 185, \sigma^2 = 4)$
Hence $P(x > 180) = P(z > -2.5) = 0.0062096$
$\therefore$ Percentage less than 3 hours = 0.62% (2 dp)

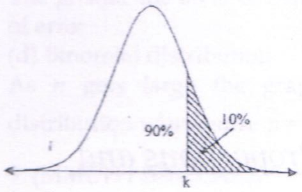
(b) $P(x > 187) = P(z > 1) = 0.15866$

2. (AM 2001:03)
(a) (i) Let X = the time taken for the journey
$X \sim N(\mu = 15, \sigma = 2)$
$P(X = 15) = 0$
$[P(X = x) = 0$ for a continuous random variable$]$
(ii) $P(X > 18) = P(Z > 1.5) = 0.06681$
(b) Let the time after 8 am be T minutes
$\therefore P(X < T) = 0.9$

$\therefore P(Z < \frac{T - 15}{2}) = P(Z < 1.2815)$

$\therefore T = 17.563$
Hence Time is 17 mins and 34 seconds past 8 am.

3. (AM 2002:10)
Let X = weight of the bags of flour in g.
$\therefore X \sim N(\mu = 1005, \sigma^2 = 16)$
(a) P (under market weight) = $P(X < 1000) = P(Z < -1.02) = 0.10564$
(b)



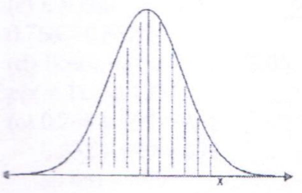
$P(x > k) = 0.10$
$\therefore P(Z > 1.2815) = 0.10$

Using $Z = \frac{x - \mu}{\sigma}$

$\Rightarrow 1.2815 = \frac{x - 1005}{4}$

$\therefore x = 1010.13$ (2 dp)

The calculator may be able to find the final result in an inverse normal algorithm.
(c)



(Weight marked x = 1000g)

P(Bag under weight) = 0.01
$P(X < 1000) = 0.01$
$\therefore P(Z < -2.3263) = 0.01$

$\therefore$ Using $z = \frac{x - \mu}{\sigma}$

$\therefore -2.3263 = \frac{1000 - \mu}{4}$

$\therefore \mu = 1009.3$ (1 dp)

4. (AM 2003:13)
This question is changing the type of random variables being used and thus students must state their random variables (and their parameters) to receive full marks.
(a) Let X = contents (in mls) of the can
$X \sim N(378, 2^2)$
$P(X = 375) = 0$ $[P(x = a)$ 0 for continuous distribution$]$
(b) $X \sim N(378, 2^2)$
$P(X < 375) = P(Z < -1.5) = 0.06681$ (5 dp)
(c) Let Y = no. cans in box with less than advertised amount
$Y \sim B(n = 24, p = 0.06681)$
$P(Y \geq 2) = 1 - P(Y \leq 1) = 1 - 0.51711 = 0.48289$ (5 dp)
(d) SD = $(375 - 378)/-3.0902 = 0.97$

5. (AM 2004:17)
Let X = Weight of the Packets in g
$X \sim N(\mu - 30, \sigma^2 = 16)$
(a) $P(X = 24) = 0$ (continuous r.v.)
(b) $P(X < 24) = P(Z \leq -115) = 0.066807$

6. (MMETH 2016S:CA12)
(a) $X \sim N(230, 5^2)$
$P(223 < x < 235) = 0.7606$
(b) $P(X < w) = 0.05$
$w \approx 221.8 = 222$ g to the nearest gram

(c) (i) $Y \sim bin(12, 0.7606)$
$P(y = 12) = 0.0375$
(ii) $W \sim bin(7, 0.7606)$
$P(w = 5) \times 0.7606 = 0.3064 \times 0.7606 = 0.2330$
(d) $Z \sim Norm(\mu, \sigma^2)$
$P(Z < z) = 0.05 \Rightarrow z = -1.6449$
$P(Z > z) = 0.12 \Rightarrow z = 1.1750$

$$z = \frac{x - \mu}{\sigma}$$

$$-1.6449 = \frac{153 - \mu}{\sigma} \text{ and } 1.1750 = \frac{210 - \mu}{\sigma}$$

$\mu = 186.2 \quad \sigma = 20.2$

7. (MMETH 2016:CA18)
(a) 5% has $Z = -1.645$ 13% has $Z = 1.126$

$$-1.645 = \frac{55 - \overline{X}}{S}$$

$$1.126 = \frac{100 - \overline{X}}{S}$$

$\overline{X} = 81.709$ mins
$S = 16.238$ mins
(b) 0.3554

Chapter 17: Sample Proportions and Confidence Intervals

1. (Projected)

(a) $0.387 \pm 1.960 \ \frac{\sqrt{0.387(1 - 0.387)}}{1000}$

$= 0.357 \rightarrow 0.417$

(b) $n > \left(\frac{Z}{\text{Error}}\right)^2 \cdot \hat{p}(1 - \hat{p})$

$n > \left(\frac{1.645}{0.025}\right)^2 (0.5)(0.5)$

$n = 1083$

2. (Projected)

(a) $p = \frac{1}{6}$

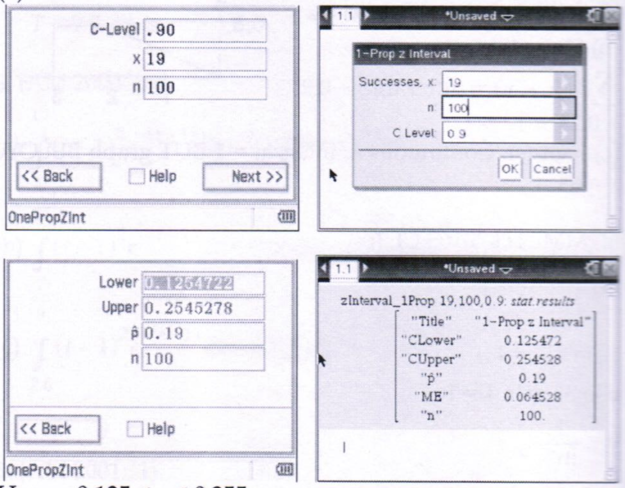
(b) $\hat{p} = \frac{90}{600} = \frac{3}{20}$ or 0.15

(c) mean of $\hat{p}$ is $\frac{1}{6}$

standard deviation $= \sqrt{\dfrac{\frac{1}{6}\left(1 - \frac{1}{6}\right)}{600}} = 0.0152$

3. (MMETH 2016S:CF9)
(a) The probability distribution is uniform: $p = 24/50$
(b) The graph in part (a) illustrates a typical result of the proportion of even numbers when the simulation is repeated 100 times. The distribution in part (b) should reflect a binomial distribution since we are counting how many even numbers (as opposed to odd numbers) occur per simulation.
This distribution tends towards a normal distribution as the number of simulations increases.
Hence the frequency distribution is roughly normal centred around $p = 0.5$.

4. (MMETH 2016S:CA17)
(a)



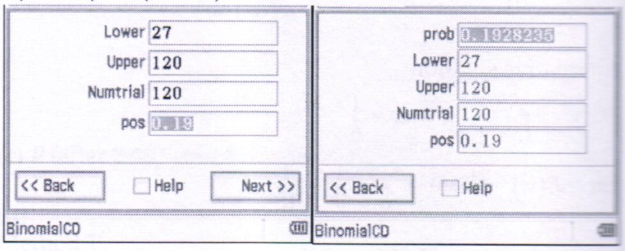
Hence, $0.125 \le p \le 0.255$
Alternative solution

$$p - z\sqrt{\frac{p(1-p)}{n}} \le p \le p + z\sqrt{\frac{p(1-p)}{n}}$$

$n = 100$, $z = 1.645$, $p = 0.19$

$$0.19 - 1.645\sqrt{\frac{0.19(1 - 0.19)}{100}} \le p \le 0.19 + 1.645\sqrt{\frac{0.19(1 - 0.19)}{100}}$$

$0.125 \le p \le 0.255$
(b) $100 \times 0.125 \le X \le 100 \times 0.254 \Rightarrow 13 \le X \le 25$
(c) The distribution is binomial with $p = 0.19$ and $n = 120$.
$P(X > 26) = P(X \ge 27)$, since n is discrete



Hence the required probability is 0.1928 (to four decimal places)
(d) $bin(7, 0.95) \Rightarrow P(4 \le x \le 7) = 0.9998$

5. (MMETH 2016S:CA18)

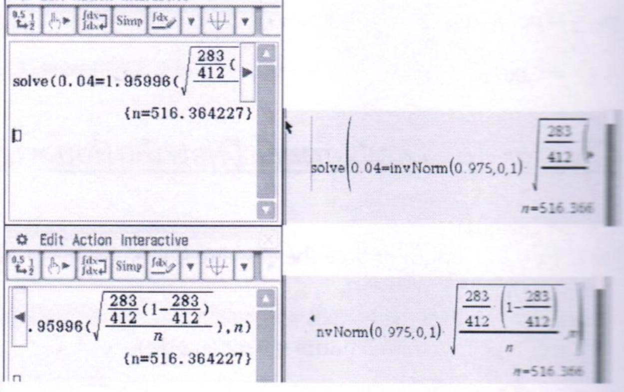
(a) $p = \frac{283}{412} = 0.6869$

(b) Standard deviation $= \sqrt{\dfrac{\frac{283}{412}\left(1 - \frac{283}{412}\right)}{412}} = 0.0228$

(c)

$$0.04 = 1.96\sqrt{\frac{\frac{283}{412}\left(1 - \frac{283}{412}\right)}{n}}$$

$n = 516.366$
$n \approx 517$



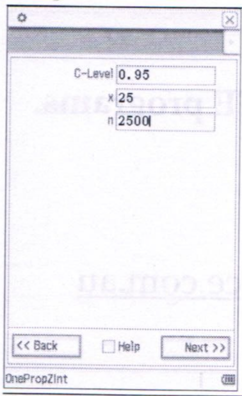
(d) (i) $p = \dfrac{258}{365} = 0.71$ and $0.668 \le p \le 0.746$

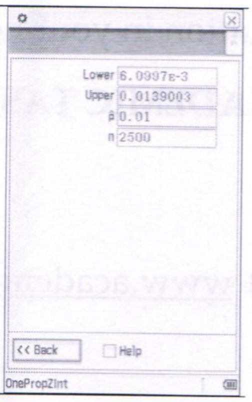
The confidence interval for this second survey overlaps, significantly, the 90% confidence interval of the initial survey so this indicates we are sampling from the same population.

(ii) $p = \dfrac{32}{78} = 0.41$ and $0.319 \le p \le 0.502$

The confidence interval for this sample is quite different than that of the original survey. While this could be a random outlier it is more likely to be a biased survey from inside the retirement village.

6. (MMETH 2016S:CA21)
(a) The batteries tested are either faulty or not faulty. Each test of a battery is a Bernoulli trial.
Hence the underlying distribution is binomial.
(b and c) Using 25 out of 2500 batteries being faulty. This gives us $p = 0.01$ and a 95% confidence interval of:
$0.0060997 \le p \le 0.0139003 \Rightarrow 15 \le n \le 35$ where n = the number of faulty batteries.
Sample 4 $n = 38$ lies outside the interval.





7. (MMETH 2016:CA10)

(a) $\hat{p} = 0.68$
$0.68 \pm 1.645\ (0.0123)$
$0.66 \le p \le 0.70$
(b) $(0.7 - 0.66) \div 2$
$= 0.02015$

(c) Survey 2 CI is 0.536 – 0.565
Survey 3 CI is 0.632 – 0.672
Survey 4 CI is 0.670 – 0.697
Survey 2 does not overlap with CI from (a)
(d) Use formula

$$n > \frac{Z^2}{T^2}\hat{p}(1-\hat{p})$$

$n > 5800.97$
$n = 5801$

8. (MMETH 2016:CA14)

(a) $\dfrac{15 + 15 + 5}{60} = \dfrac{35}{60}$

$= 0.58\dot{3}$

(b) Mean $\hat{p} = 0.58\dot{3}$
Std Dev. = 0.0637

(c) 90% CI is 0.4786 – 0.6880 ($\hat{p} = 0.58\dot{3}$)
90% has the smaller margin of error.
The greater the level of confidence the greater the margin of error
(d) Binomial distribution
As n gets large the graph tends towards a Normal distribution with centre $\hat{p} = 0.58\dot{3}$

9. (MMETH 2016:CA20)
(a) Binomial $n = 30 \quad p = 0.8$
Mean = 24 (np)
Std Dev. = 2.191 ($\sqrt{np(1-p)}$)
(b) 0.123
(c) $n = 600 \quad p = 0.8$
0.768 – 0.832
(d) Binomial $n = 3$, $p = 0.05$
$p(x = 1) = 0.1354$
(e) $0.768 \times 600 = 461$
$0.832 \times 600 = 499$
so 461 – 499
(f) Samples 1 and 5 are not in the range 461 – 499