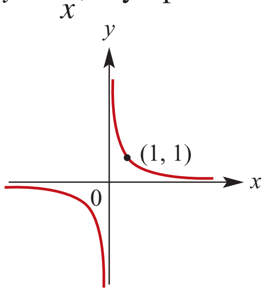
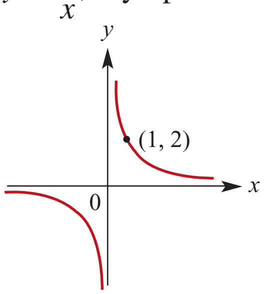
Chapter 4 – A gallery of graphs

Solutions to Exercise 4A

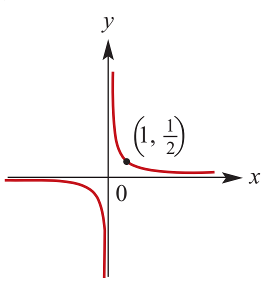
1 **a** $y = \frac{1}{x}$; asymptotes at $x = 0$ and $y = 0$



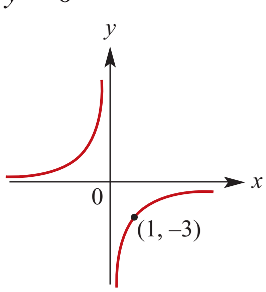
b $y = \frac{2}{x}$; asymptotes at $x = 0$ and $y = 0$



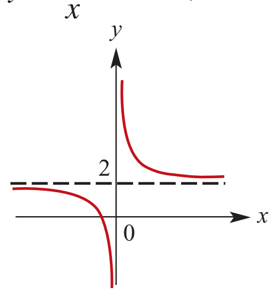
c $y = \frac{1}{2x}$; asymptotes at $x = 0$ and $y = 0$



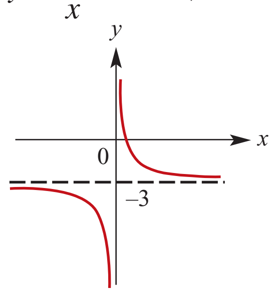
d $y = -\frac{3}{x}$; asymptotes at $x = 0$ and $y = 0$



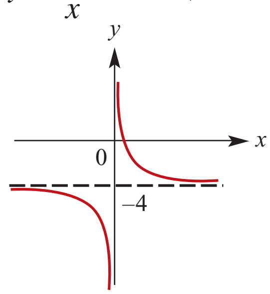
e $y = \frac{1}{x} + 2$; asymptotes at $x = 0$ and $y = 2$ x-intercept where $y = \frac{1}{x} + 2 = 0, \therefore x = -\frac{1}{2}$



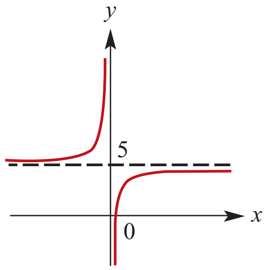
f $y = \frac{1}{x} - 3$; asymptotes at $x = 0$ and $y = -3$ x-intercept where $y = \frac{1}{x} - 2 = 0, \therefore x = \frac{1}{3}$



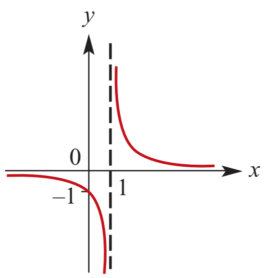
g $y = \frac{2}{x} - 4$; asymptotes at $x = 0$ and $y = -4$ x-intercept where $y = \frac{2}{x} - 4 = 0, \therefore x = \frac{1}{2}$



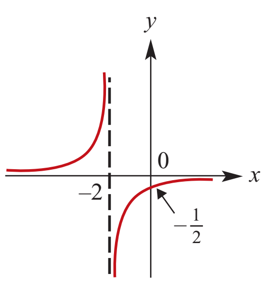
h $y = -\dfrac{1}{2x} + 5$;
asymptotes at $x = 0$ and $y = 5$
x-intercept where
$y = -\dfrac{1}{2x} + 5 = 0. \therefore x = 0.1$



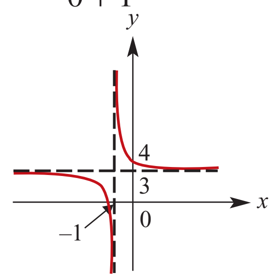
i $y = \dfrac{1}{x-1}$; asymptotes at $x = 1$ and $y = 0$
y-intercept where $y = \dfrac{1}{0-1} = -1$



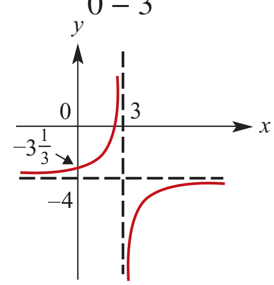
j $y = -\dfrac{1}{x+2}$;
Asymptotes at $x = -2$ and $y = 0$
y-intercept where $y = -\dfrac{1}{0+2} = -\dfrac{1}{2}$



k $y = \dfrac{1}{x+1} + 3$;
asymptotes at $x = -1$ and $y = 3$
x-intercept where
$y = \dfrac{1}{x+1} + 3 = 0$
$\therefore \dfrac{1}{x+1} = -3$
$\therefore x + 1 = -\dfrac{1}{3}$
$\therefore x = -\dfrac{4}{3}$
y-intercept where
$y = \dfrac{1}{0+1} + 3 = 4$

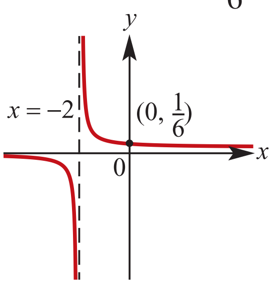


l $y = -\dfrac{2}{x-3} - 4$;
asymptotes at $x = 3$ and $y = -4$
x-intercept where $y = -\dfrac{2}{x-3} - 4 = 0$
$\therefore \dfrac{2}{x-3} = -4$
$\therefore x - 3 = -\dfrac{1}{2}$
$\therefore x = \dfrac{5}{2}$
y-intercept where
$y = -\dfrac{2}{0-3} - 4 = -\dfrac{10}{3}$

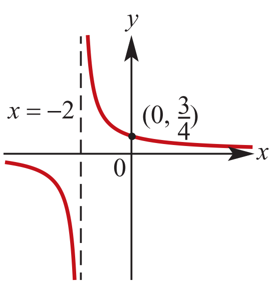


2 Answers given in question 1

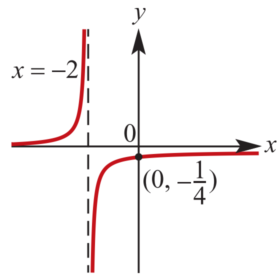
3 a The graph of $y = \dfrac{1}{3x+6}$ can be obtained by translating the graph of $y = \dfrac{1}{3x}$ two units to the left.
The equations of the asymptotes are $x = -2$ and $y = 0$.
When $x = 0, y = \dfrac{1}{6}$



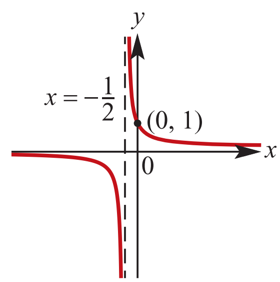
b The graph of $y = \dfrac{3}{2x+4}$ can be obtained by translating the graph of $y = \dfrac{3}{2x}$ two units to the left.
The equations of the asymptotes are $x = -2$ and $y = 0$.
When $x = 0, y = \dfrac{3}{4}$



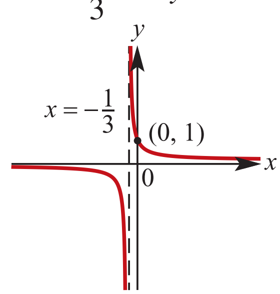
c The graph of $y = \dfrac{-1}{2x+4}$ can be obtained by translating the graph of $y = \dfrac{-1}{2x}$ two units to the left.
The equations of the asymptotes are $x = -2$ and $y = 0$.
When $x = 0, y = -\dfrac{1}{4}$



d The graph of $y = \dfrac{1}{2x+1}$ can be obtained by translating the graph of $y = \dfrac{1}{2x}$ half a unit to the left.
The equations of the asymptotes are $x = -\dfrac{1}{2}$ and $y = 0$.
When $x = 0, y = 1$



4 a $\dfrac{1}{3x+1} = \dfrac{1}{3(x+\frac{1}{3})}$.
Translate the graph of $y = \dfrac{1}{3x}$ one third of a unit to the left.
When $x = 0, y = 1$
Theequations of the asymptotes are $x = -\dfrac{1}{3}$ and $y = 0$



b Translate the graph of $y = \dfrac{1}{3x+1}$ one unit in the negative direcion of the y-axis.

When $x = 0, y = 0$

When $y = 0, \dfrac{1}{3x+1} - 1 = 0$

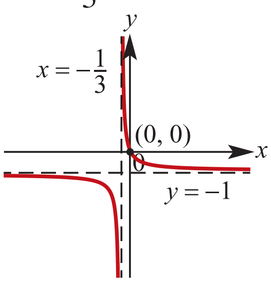
$$\frac{1}{3x+1} - 1 = 0$$

$$\frac{1}{3x+1} = 1$$

$$3x + 1 = 1$$

$$x = 0$$

The equations of the asymptotes are $x = -\frac{1}{3}$ and $y = -1$.



c Reflect the graph of $y = \dfrac{1}{3x+1}$ in the x axis and translate the image, $y = -\dfrac{1}{3x+1}$ one unit in the negative direcion of the y-axis.

When $x = 0, y = -2$

When $y = 0, -\dfrac{1}{3x+1} - 1 = 0$

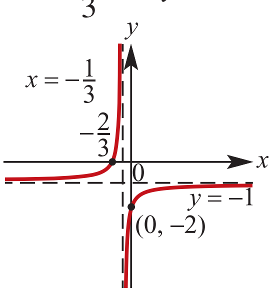
$$-\frac{1}{3x+1} - 1 = 0$$

$$-\frac{1}{3x+1} = 1$$

$$3x + 1 = -1$$

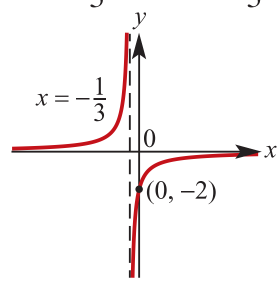
$$x = -\frac{2}{3}$$

The equation of the asymptotes are $x = -\frac{1}{3}$ and $y = -1$.



d Reflect the graph of $y = \dfrac{2}{3x+1}$ in the x axis. When $x = 0, y = -2$

The equation of the asymptotes are $x = -\frac{1}{3}$ and $y = -\frac{1}{3}$.



e Translate the graph of $y = \dfrac{-2}{3x+1}$ four units in the negative direcion of the y-axis.

When $x = 0, y = -6$

When $y = 0, -\dfrac{2}{3x+1} - 4 = 0$

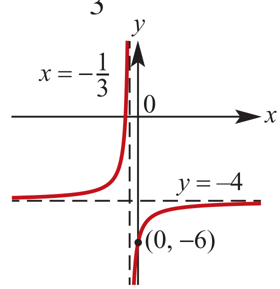
$$-\frac{2}{3x+1} - 4 = 0$$

$$-\frac{1}{3x+1} = 2$$

$$6x + 2 = -1$$

$$x = -\frac{1}{2}$$

The equation of the asymptotes are $x = -\frac{1}{3}$ and $y = -4$.



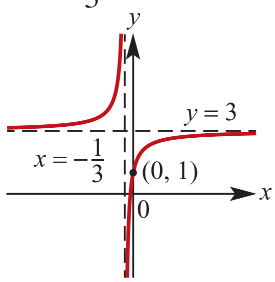
f Translate the graph of $y = \dfrac{-2}{3x+1}$ three units in the positive direcion of the y-axis.

When $x = 0, y = 1$

When $y = 0, -\dfrac{2}{3x+1} + 3 = 0$

$$-\frac{2}{3x+1}+3=0$$
$$\frac{2}{3x+1}=3$$
$$9x+3=2$$
$$x=-\frac{1}{9}$$

The equation of the asymptotes are $x=-\frac{1}{3}$ and $y=3$.



g $\dfrac{2}{3x+2}=\dfrac{2}{3(x+\frac{2}{3})}$

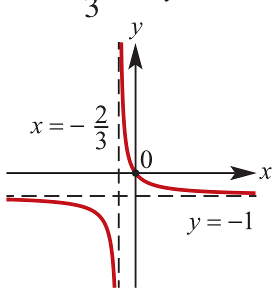
Translate the graph of $y=\dfrac{2}{3x}$ two thirds units in the negative direcion of the x-axis and one unit in the negative direction of the y-axis.

When $x=0, y=0$

When $y=0, \dfrac{2}{3x+2}-1=0$

$$\frac{2}{3x+2}-1=0$$
$$\frac{2}{3x+2}=1$$
$$3x+2=2$$
$$x=0$$

The equation of the asymptotes are $x=-\frac{2}{3}$ and $y=-1$.



h $\dfrac{3}{3x+4}=\dfrac{3}{3(x+\frac{4}{3})}=\dfrac{1}{3x+\frac{4}{3}}$

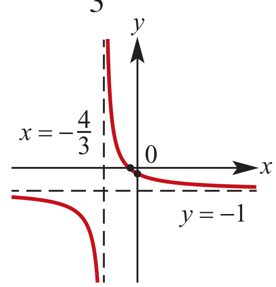
Translate the graph of $y=\dfrac{1}{x}$ four thirds units in the negative direcion of the x-axis and one unit in the negative direction of the y-axis.

When $x=0, y=-\frac{1}{4}$

When $y=0, \dfrac{3}{3x+4}-1=0$

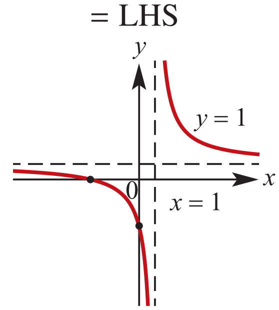
$$\frac{3}{3x+4}-1=0$$
$$\frac{3}{3x+4}=1$$
$$3x+4=3$$
$$x=-\frac{1}{3}$$

The equation of the asymptotes are $x=-\frac{4}{3}$ and $y=-1$.

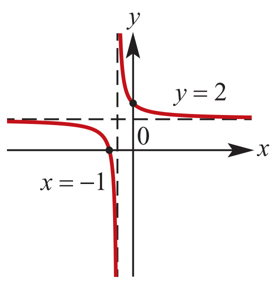


5 RHS $=\dfrac{4}{x-1}+1$

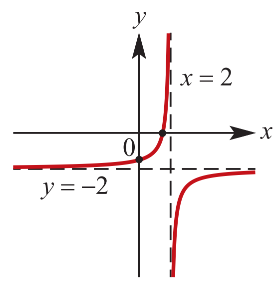
$$=\frac{4+x-1}{x-1}$$
$$=\frac{x+3}{x-1}$$
$$=\text{LHS}$$



6
$$\begin{aligned}\text{RHS} &= \frac{1}{x+1} + 2\\ &= \frac{1 + 2(x+1)}{x+1}\\ &= \frac{2x+3}{x+1}\\ &= \text{LHS}\end{aligned}$$

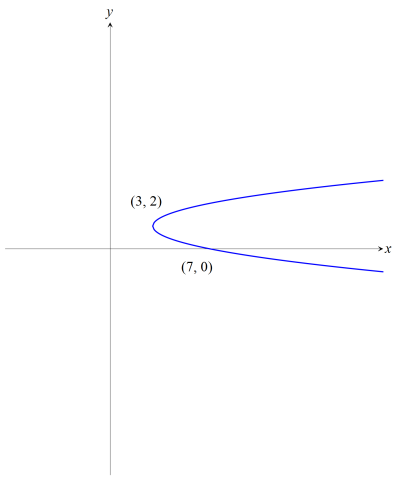


7
$$\begin{aligned}\text{RHS} &= -\frac{1}{x-2} - 2\\ &= \frac{-1 - 2(x-2)}{x-2}\\ &= \frac{-1 - 2x + 4}{x-2}\\ &= \frac{3 - 2x}{x-2}\\ &= \text{LHS}\end{aligned}$$

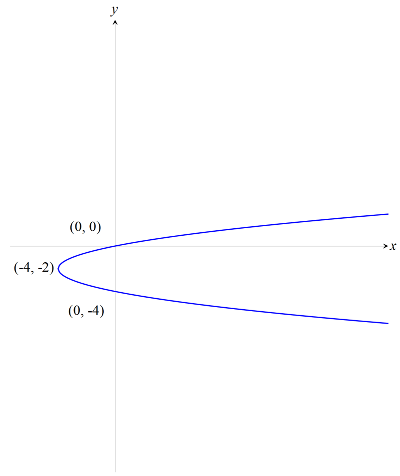


Solutions to Exercise 4B

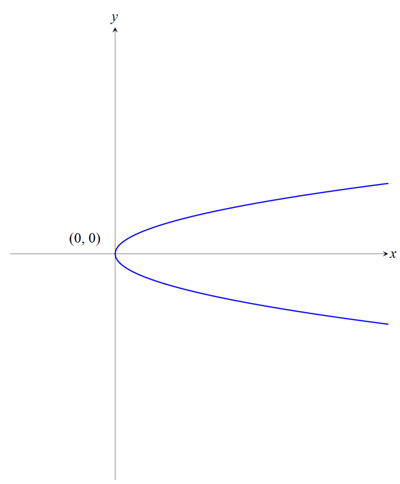
1 a The graph of $y^2 = x$ is translated 3 units right and 2 units up. The vertex is at $(3, 2)$.
To find the y-axis intercepts, let $x = 0$:
$(y-2)^2 = -3$ which means that there is no y-intercept.
To find the x-axis intercept, let $y = 0$:
$(-2)^2 = x - 3 \Rightarrow x = 7$



b The graph of $y^2 = x$ is translated 4 units left and 2 units down. The vertex is at $(-4, -2)$.
To find the y-axis intercepts, let $x = 0$:
$(y+2)^2 = 4 \Rightarrow y = -4 \text{ or } 0$
To find the x-axis intercept, let $y = 0$:
$(y+2)^2 = x + 4 \Rightarrow x = 0$



c The graph of $y^2 = x$ is dilated by a factor of $\frac{1}{2}$ from the y-axis. The vertex is at $(0, 0)$.
To find the y-axis intercepts, let $x = 0$:
$y^2 = 2(0) \Rightarrow y = 0$
To find the x-axis intercept, let $y = 0$:
$(0)^2 = 2x \Rightarrow x = 0$



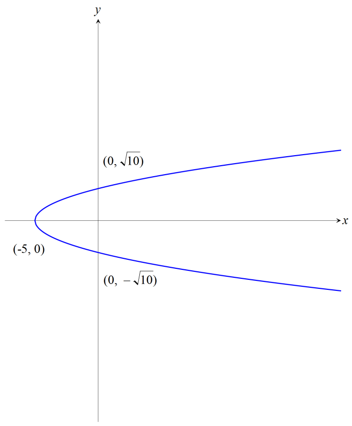
d The graph of $y^2 = x$ is dilated by a factor of $\frac{1}{2}$ from the y-axis and then translated 5 units left. The vertex is at $(-5, 0)$.

To find the y-axis intercepts, let $x = 0$:

$y^2 = 2(5) \Rightarrow y = \pm\sqrt{10}$

To find the x-axis intercept, let $y = 0$:

$0^2 = 2(x + 5) \Rightarrow x = -5$



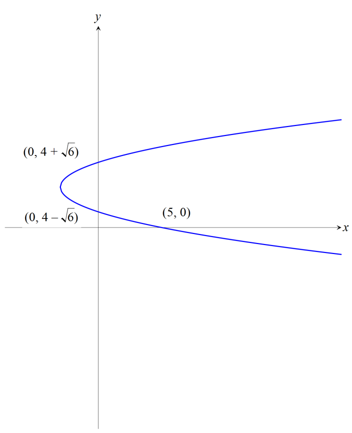
e The graph of $y^2 = x$ is dilated by a factor of $\frac{1}{2}$ from the y-axis, translated 3 units left and four units up. The vertex is at $(-3, 4)$.

To find the y-axis intercepts, let $x = 0$:

$(y - 4)^2 = 2(3) \Rightarrow y = 4 \pm \sqrt{6}$

To find the x-axis intercept, let $y = 0$:

$(-4)^2 = 2(x + 3) \Rightarrow x = 5$



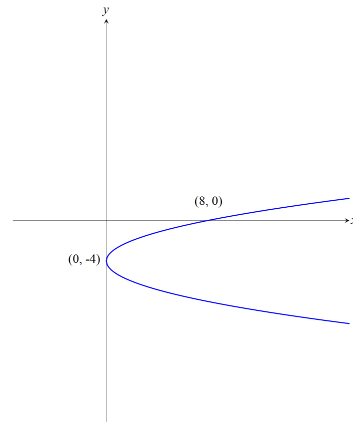
f The graph of $y^2 = x$ is dilated by a factor of $\frac{1}{2}$ from the y-axis and translated 4 units down. The vertex is at $(0, -4)$.

To find the y-axis intercepts, let $x = 0$:

$(y + 4)^2 = 2(0) \Rightarrow y = -4$

To find the x-axis intercept, let $y = 0$:

$(y + 4)^2 = 2x \Rightarrow x = 8$



g Factorise the right hand side first:

$(y + 3)^2 = 2(x - 2)$

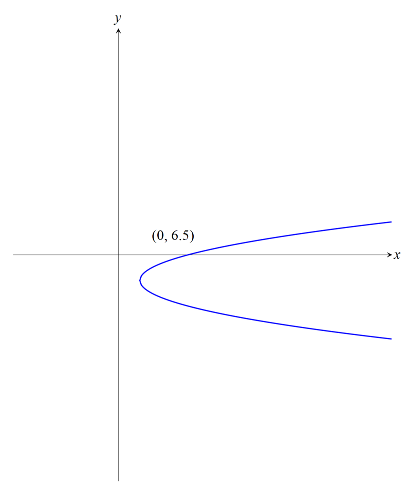
The graph of $y^2 = x$ is dilated by a factor of $\frac{1}{2}$ from the y-axis, translated 2 units right and 3 units down. The vertex is at $(2, -3)$.

To find the y-axis intercepts, let $x = 0$:

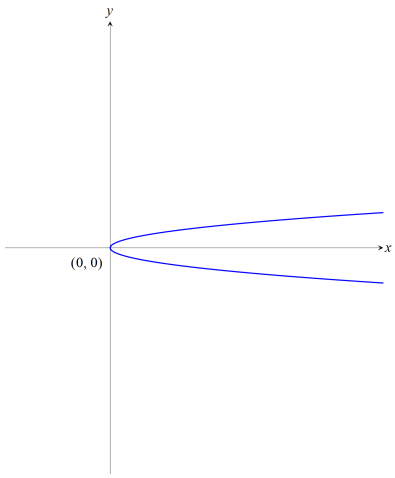
$(y + 3)^2 = 2(-2)$ which means that there is no y-intercept.

To find the x-axis intercept, let $y = 0$:

$(3)^2 = 2(x - 2) \Rightarrow y = 6.5$



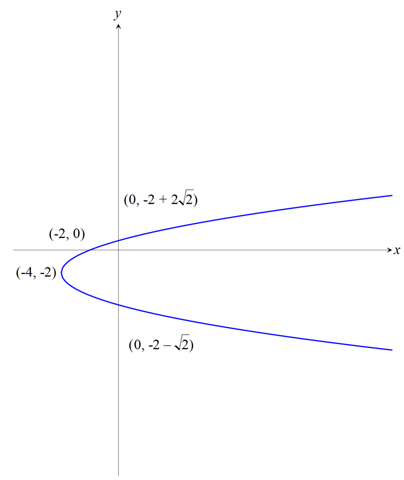
h The graph of $y^2 = x$ is dilated by a factor of 2 from the y-axis. The vertex is at $(0, 0)$.
To find the y-axis intercepts, let $x = 0$:
$y^2 = \frac{0}{2} \Rightarrow y = 0$
To find the x-axis intercept, let $y = 0$:
$0^2 = \frac{x}{2} \Rightarrow x = 0$



i Complete the square on the left hand side and then factorise the right hand side:

$$y^2 + 4y = 2x + 4$$
$$y^2 + 4y + 4 = 2x + 8$$
$$(y + 2)^2 = 2(x + 4)$$

The graph of $y^2 = x$ is dilated by a factor of $\frac{1}{2}$ from the y-axis, translated 4 units left and 2 units down. The vertex is at $(-4, -2)$.
To find the y-axis intercepts, let $x = 0$:
$(y + 2)^2 = 2(4) \Rightarrow y = -2 \pm 2\sqrt{2}$
To find the x-axis intercept, let $y = 0$:
$(2)^2 = 2(x + 4) \Rightarrow x = -2$



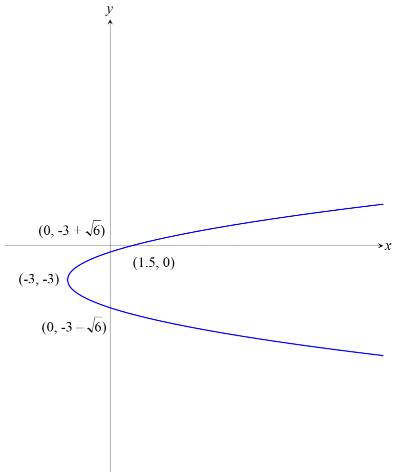
j Move all the y terms to the left hand side and all the x terms to the right hand side.

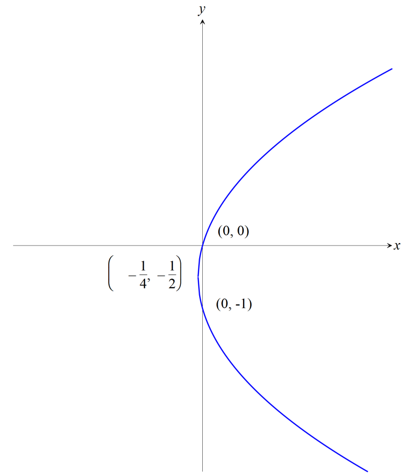
$$y^2 + 6y - 2x + 3 = 0$$
$$y^2 + 6y = 2x - 3$$

Complete the square on the left hand side and then factorise the right hand side:

$$y^2 + 6y = 2x - 3$$
$$y^2 + 6y + 9 = 2x + 6$$
$$(y + 3)^2 = 2(x + 3)$$

The graph of $y^2 = x$ is dilated by a factor of $\frac{1}{2}$ from the y-axis, translated 3 units left and 3 units down. The vertex is at $(-3, -3)$.
To find the y-axis intercepts, let $x = 0$:
$(y + 3)^2 = 2(3) \Rightarrow y = -3 \pm \sqrt{6}$
To find the x-axis intercept, let $y = 0$:
$(3)^2 = 2(x + 3) \Rightarrow x = 1.5$





k Move all the y terms to the left hand side and all the x terms to the right hand side.

$y^2 + y - x = 0$

$y^2 + y = x$

Complete the square on the left hand side.

$$y^2 + y = x$$

$$y^2 + y + \left(\frac{1}{2}\right)^2 = x + \left(\frac{1}{2}\right)^2$$

$$\left(y + \frac{1}{2}\right)^2 = x + \frac{1}{4}$$

The graph of $y^2 = x$ is translated $\frac{1}{4}$ units left and $\frac{1}{2}$ units down. The vertex is at $\left(-\frac{1}{4}, -\frac{1}{2}\right)$.

To find the y-axis intercepts, let $x = 0$:

$$\left(y + \frac{1}{2}\right)^2 = \frac{1}{4} \Rightarrow y = 0 \text{ or } -1$$

To find the x-axis intercept, let $y = 0$:

$$\left(\frac{1}{2}\right)^2 = x + \frac{1}{4} \Rightarrow x = 0$$

l Move all the y terms to the left hand side and all the x terms to the right hand side.

$y^2 + 7y - 5x + 3 = 0$

$y^2 + 7y = 5x - 3$

Complete the square on the left hand side and then factorise the right hand side:

$$y^2 + 7y = 5x - 3$$

$$y^2 + 7y + \left(\frac{7}{2}\right)^2 = 5x + \frac{37}{4}$$

$$\left(y + \frac{7}{2}\right)^2 = 5\left(x + \frac{37}{20}\right)$$

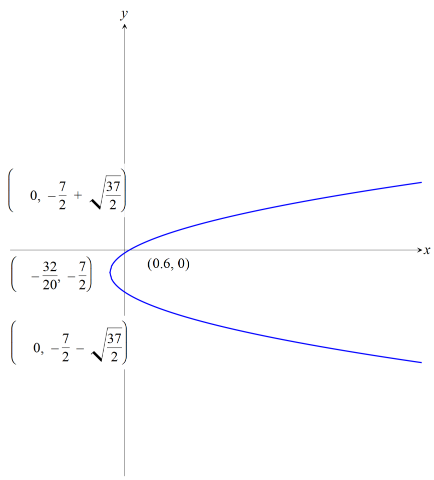
The graph of $y^2 = x$ is dilated by a factor of $\frac{1}{5}$ from the y-axis, translated $\frac{37}{20}$ units left and $\frac{7}{2}$ units down. The vertex is at $\left(-\frac{37}{20}, -\frac{7}{2}\right)$.

To find the y-axis intercepts, let $x = 0$:

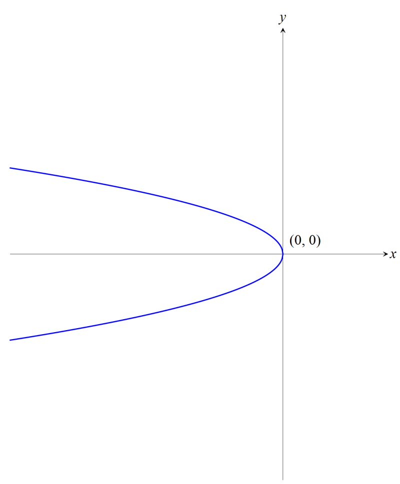
$$\left(y + \frac{7}{2}\right)^2 = 5\left(\frac{37}{20}\right) \Rightarrow y = -\frac{7}{2} \pm \frac{\sqrt{37}}{2}$$

To find the x-axis intercept, let $y = 0$:

$$\left(\frac{7}{2}\right)^2 = 5\left(x + \frac{37}{20}\right) \Rightarrow x = \frac{3}{5}$$



m The graph of $y^2 = x$ is reflected in the y axis. The vertex is at $(0, 0)$.
To find the y-axis intercepts, let $x = 0$:
$y^2 = 0 \Rightarrow y = 0$
To find the x-axis intercept, let $y = 0$:
$0^2 = x \Rightarrow x = 0$



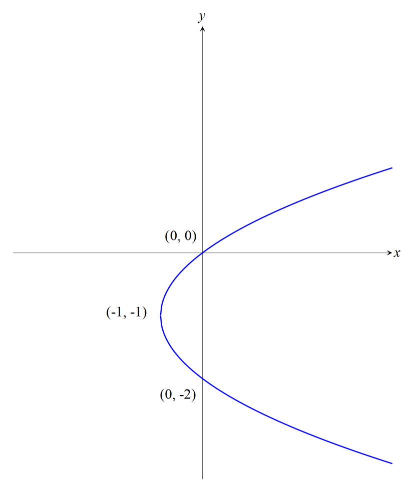
n Move all the y terms to the left hand side and all the x terms to the right hand side.

$$y^2 + 2y - x = 0$$
$$y^2 + 2y = x$$

Complete the square on the left hand side.

$$y^2 + 2y = x$$
$$y^2 + 2y + 1 = x + 1$$
$$(y + 1)^2 = x + 1$$

The graph of $y^2 = x$ is translated 1 unit left and 1 unit down. The vertex is at $(-1, -1)$.
To find the y-axis intercepts, let $x = 0$:
$(y + 1)^2 = 0 + 1 \Rightarrow x = 0 \text{ or } -2$
To find the x-axis intercept, let $y = 0$:
$(1)^2 = x + 1 \Rightarrow x = 0$

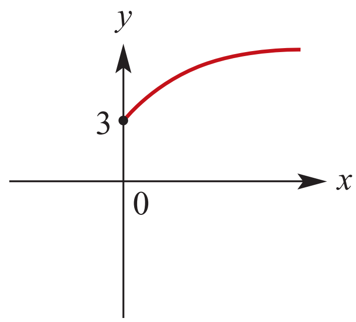


Solutions to Exercise 4C

1 a $y = 2\sqrt{x} + 3$; $\{x\colon x \geq 0\}$

No x-intercept; y-intercept at (0,3)

y is defined for $\{y\colon y \geq 3\}$

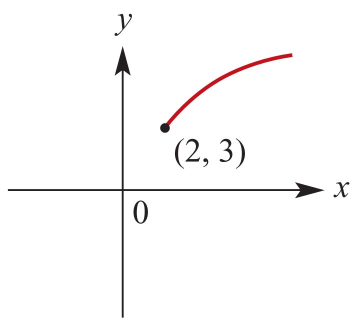


b $y = \sqrt{x-2} + 3$; $\{x\colon x \geq 2\}$

No axis intercepts

y is defined for $\{y\colon y \geq 3\}$

Starting point at (2,3)



c $y = \sqrt{x-2} - 3$; $\{x\colon x \geq 2\}$

x intercept where $\sqrt{x-2} - 3 = 0$

$\therefore \sqrt{x-2} = 3$

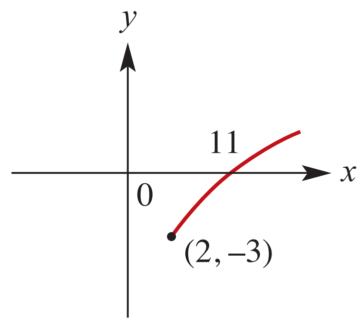
$\therefore x - 2 = 9$

$\therefore x = 11$

x-intercept at (11,0); no y-intercept.

y is defined for $\{y\colon y \geq -3\}$

Starting point at $(2, -3)$

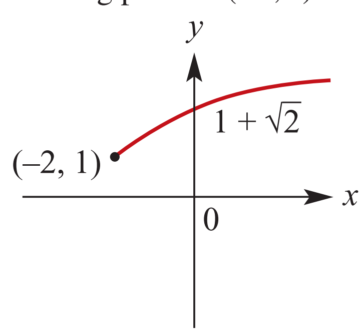


d $y = \sqrt{x+2} + 1$; $\{x\colon x \geq -2\}$

x-intercept at $(1 + \sqrt{2}, 0)$; no y-intercept.

y is defined for $\{y\colon y \geq 1\}$

Starting point at $(-2, 1)$



e $y = -\sqrt{x+2} + 3$; $\{x\colon x \geq -2\}$

y-intercept at $(3 - \sqrt{2}, 0)$

x-intercept where $-\sqrt{x+2} + 3 = 0$

$\therefore \sqrt{x+2} = 3$

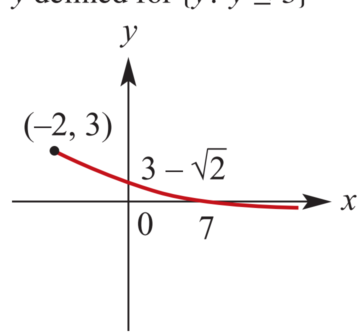
$\therefore x + 2 = 9$

$\therefore x = 7$

x-intercept at $(7, 0)$

Starting point at $(-2, 3)$

y defined for $\{y\colon y \leq 3\}$



f $y = 2\sqrt{x+2} - 3$; $\{x \colon x \geq -2\}$

y-intercept at $(3 - 2\sqrt{2}, 0)$

x-intercept where $2\sqrt{x+2} - 3 = 0$

$\therefore \sqrt{x+2} = \dfrac{3}{2}$

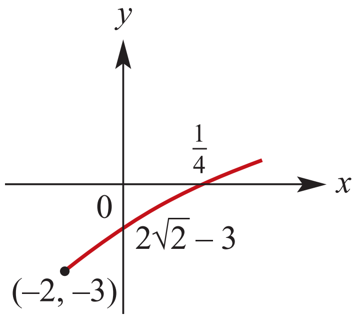
$\therefore x + 2 = \dfrac{9}{4}$

$\therefore x = \dfrac{1}{4}$

x-intercept at $\left(\dfrac{1}{4}, 0\right)$

Starting point at $(-2, -3)$

y defined for $\{y \colon y \geq -3\}$



2 a $y = -\sqrt{x-2} + 3$; $\{x \colon x \geq 2\}$

No y-intercept;

x-intercept where $-\sqrt{x-2} + 3 = 0$

$\therefore \sqrt{x-2} = 3$

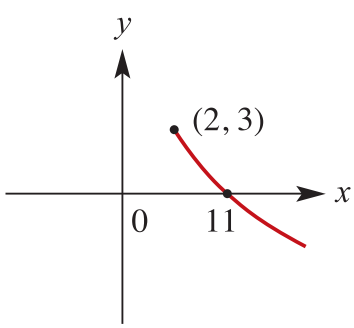
$\therefore x - 2 = 9$

$\therefore x = 11$

x intercept at $(11,0)$

Starting point at $(2,3)$

y defined for $\{y \colon y \leq 3\}$



b $y = \sqrt{-(x-4)} - 2$; $\{x \colon x \leq 4\}$

y-intercept at $(0, 0)$

x-intercept where $\sqrt{-(x-4)} - 2 = 0$

$\therefore \sqrt{-(x-4)} = 2$

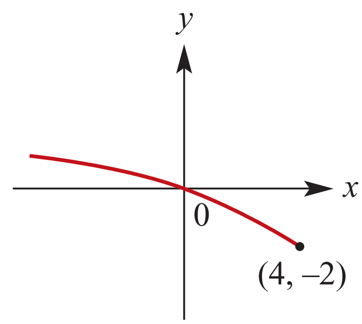
$\therefore -(x-4) = 4$

$\therefore x = 0$

x-intercept at $(0, 0)$

Starting point at $(4, -2)$

y defined for $\{y \colon y \geq -2\}$

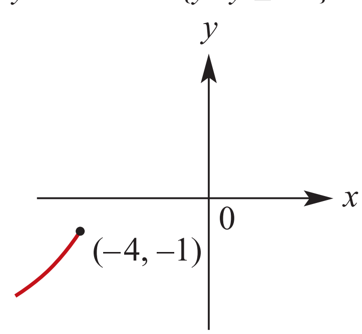


c $y = -2\sqrt{-(x+4)} - 1$; $\{x \colon x \leq -4\}$

No axis intercepts.

Starting point at $(-4, -1)$

y defined for $\{y \colon y \leq -1\}$



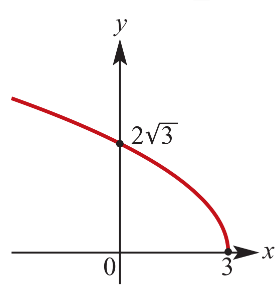
d $y = 2\sqrt{3-x}$

When $x = 0$, $y = 2\sqrt{3}$

When $y = 0$, $x = 3$

Starting point at $(3, 0)$

Defined for $x \leq 3$ and then $y \geq 0$

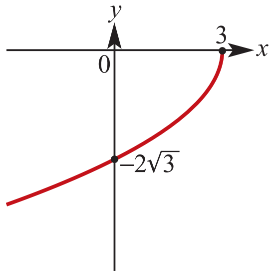


e $y = -2\sqrt{3 - x}$

When $x = 0, y = -2\sqrt{3}$

When $y = 0, x = 3$

Starting point at $(3, 0)$ Defined for $x \le 3$ and then $y \le 0$



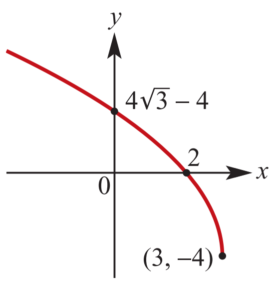
f $y = 4\sqrt{3 - x} - 4$

When $x = 0, y = 4\sqrt{3} - 4$

When $y = 0, x = 2$

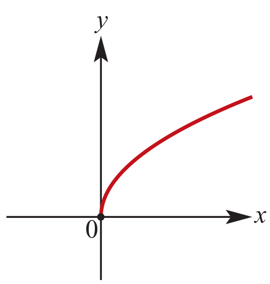
Starting point at $(3, -4)$

Defined for $x \le 3$ and then $y \ge -4$



3 a $y = \sqrt{3x}; \quad x \ge 0$

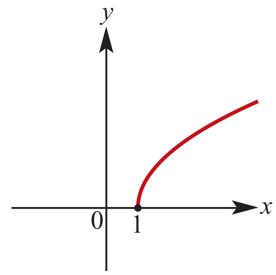
y-values $y \ge 0$



b $y = \sqrt{3(x - 1)}; \quad x \ge 1$

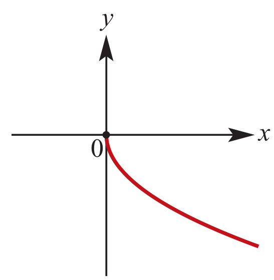
Graph of $y = \sqrt{3x}$ translated 1 unit in the positive direction of the x- axis.

x-axis intercept is $(1, 0)$ y-values $y \ge 0$



c $y = -\sqrt{2x} \quad x \ge 0$

y-values $y \le 0$



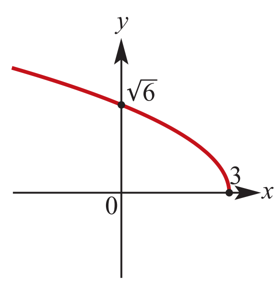
d $y = -\sqrt{2(3 - x)} \quad x \le 3$

The graph of $y = -\sqrt{-2x}$ translated 3 units in the positive direction of the x- axis

x-axis intercept $(3, 0)$

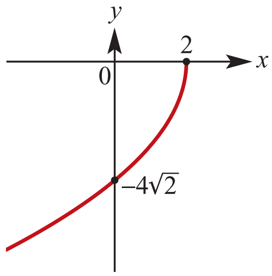
y -axis intercept $(0, \sqrt{6})$

y-values $y \ge 0$



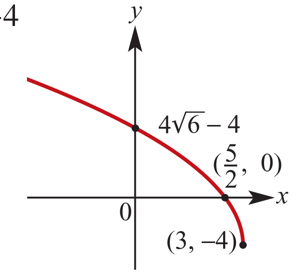
e $y = -2\sqrt{4(2-x)}$ $\quad x \le 2$

$y \le 0$



f $y = 4\sqrt{2(3-x)} - 4$ $\quad x \le 3$

$y \ge -4$



Solutions to Exercise 4D

1 a $C(0,0),\ r = 3 \therefore\ x^2 + y^2 = 9$

b $C(0,0),\ r = 4 \therefore\ x^2 + y^2 = 16$

c $C(1,3),\ r = 5$
$\therefore (x-1)^2 + (y-3)^2 = 25$

d $C(2,-4),\ r = 3$
$\therefore (x-2)^2 + (y+4)^2 = 9$

e $C(-3,4),\ r = \dfrac{5}{2}$
$\therefore (x+3)^2 + (y-4)^2 = \dfrac{25}{4}$

f $C(-5,-6),\ r = 4.6$
$\therefore (x+5)^2 + (y+6)^2 = 4.6^2$

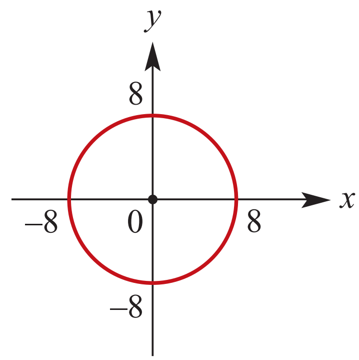
2 a $(x-1)^2 + (y-3)^2 = 4$
$C(1,3),\ r = \sqrt{4} = 2$

b $(x-2)^2 + (y+4)^2 = 5$
$C(2,-4),\ r = \sqrt{5}$

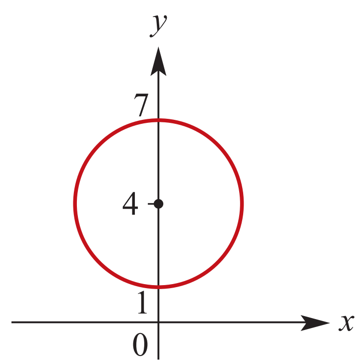
c $(x+3)^2 + (y-2)^2 = 9$
$C(-3,2),\ r = \sqrt{9} = 3$

d $(x+5)^2 + (y-4)^2 = 8$
$C(-5,4),\ r = \sqrt{8} = 2\sqrt{2}$

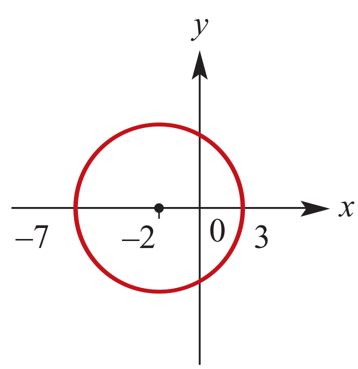
3 a $x^2 + y^2 = 64$
x-intercepts at $(\pm 8, 0)$
y-intercepts at $(0,\ \pm 8)$



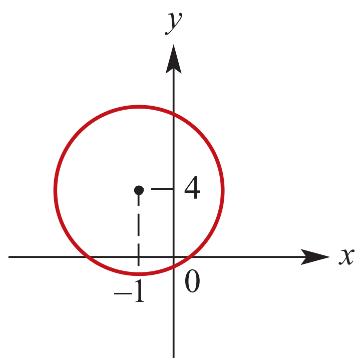
b $x^2 + (y-4)^2 = 9$
No x-intercepts,
y-intercepts at (0,1) and (0,7)



c $(x+2)^2 + y^2 = 25$
x-intercepts at (3,0) and $(-7,0)$

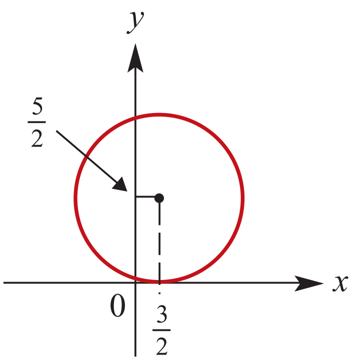


d $(x+1)^2 + (y-4)^2 - 169 = 0$
Centre at $(-1,4)$, radius 13



e $(2x-3)^2 + (2y-5)^2 = 36$

Centre at $\left(\frac{3}{2}, \frac{5}{2}\right)$, radius $\frac{5}{2}$

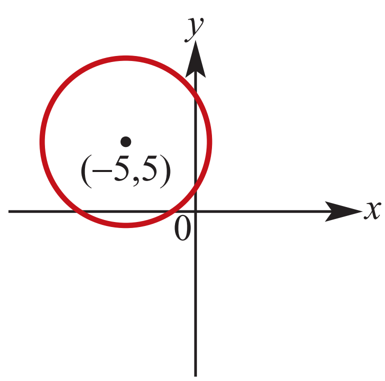


f $(x+5)^2 + (y-5)^2 = 36$

Centre at $(-5, 5)$, radius 6

x-intercepts at $(-5+\sqrt{11}, 0)$ and $(-5-\sqrt{11}, 0)$

y-intercepts at $(5+\sqrt{11}, 0)$ and $(5-\sqrt{11}, 0)$



4 a $x^2 + y^2 - 6y - 16 = 0$

$\therefore\ x^2 + y^2 - 6y + 9 - 9 - 16 = 0$

$\therefore\ x^2 + (y-3)^2 - 25 = 0$

$\therefore\ x^2 + (y-3)^2 = 25$

$C(0, 3),\ r = \sqrt{25} = 5$

b $x^2 + y^2 - 8x + 12y + 10 = 0$

$\therefore\ x^2 - 8x + 16 + y^2 + 12y + 36 = 42$

$\therefore\ (x-4)^2 + (y+6)^2 = 25$

$C(4, -6),\ r = \sqrt{42}$

c $x^2 + y^2 - 6x + 4y + 9 = 0$

$\therefore\ x^2 - 6x + y^2 + 4y + 9 = 0$

$\therefore\ x^2 - 6x + 9 + y^2 + 4y + 4 - 4 = 0$

$\therefore\ (x-3)^2 + (y+2)^2 = 0$

$C(3, -2),\ r = \sqrt{4} = 2$

d $x^2 + y^2 + 4x - 6y - 12 = 0$

$\therefore\ x^2 + 4x + 4 + y^2 - 6y + 9 - 12 - 4 - 9 = 0$

$\therefore\ (x+2)^2 + (y-3)^2 - 25 = 0$

$\therefore\ (x+2)^2 + (y-3)^2 = 25$

$C(-2, 3),\ r = \sqrt{25} = 5$

e $x^2 + y^2 - 8x + 4y + 1 = 0$

$\therefore\ x^2 - 8x + 16 + y^2 + 4y + 4 + 1 - 20 = 0$

$\therefore\ (x-4)^2 + (y+2)^2 = 19$

$C(4, -2),\ r = \sqrt{19}$

f $x^2 + y^2 - x + 4y + 2 = 0$

$\therefore\ x^2 - x + \frac{1}{4} + y^2 + 4y + 4 = 2 + \frac{1}{4}$

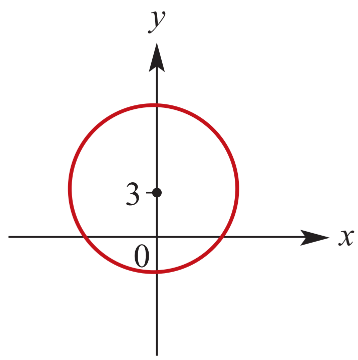
$\therefore\ (x - \frac{1}{2})^2 + (y+2)^2 = \frac{9}{4}$

$C(\frac{1}{2}, -2),\ r = \frac{3}{2}$

5 a $x^2 + y^2 - 6y - 16 = 0$

$\therefore x^2 + (y - 3)^2 = 25$

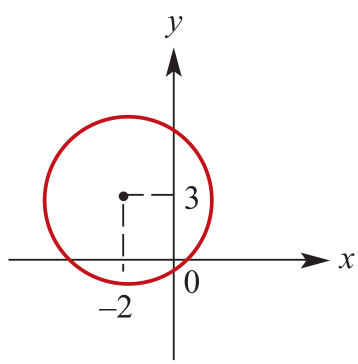
Centre at (0,3), radius 5



b $x^2 + y^2 + 4x - 6y - 3 = 0$

$\therefore (x + 2)^2 + (y - 3)^2 = 16$

Centre at $(-2, 3)$, radius 4

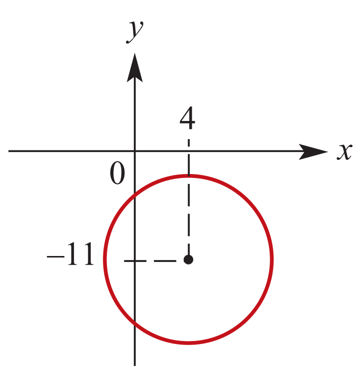


c $x^2 + y^2 - 8x + 22y + 27 = 0$

$\therefore (x - 4)^2 + (y + 11)^2 = 110$

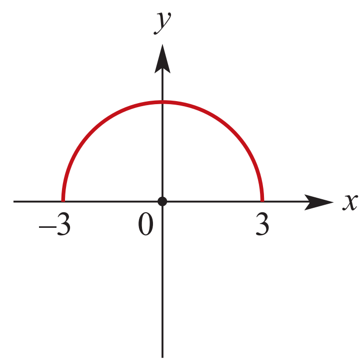
No axis intercepts

Centre (4,-11), radius $\sqrt{110}$

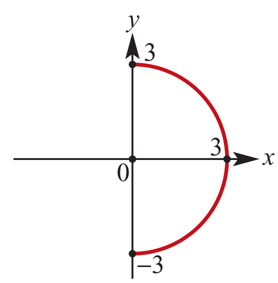


6 a $y = +\sqrt{9 - x^2}$

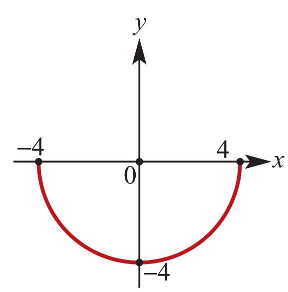
Starting points at $(\pm 3, 0)$



b

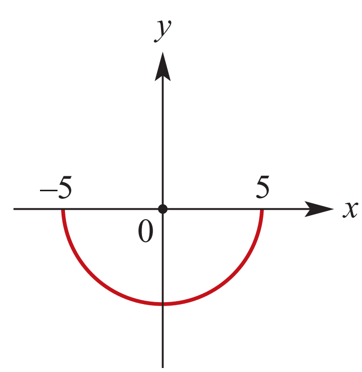


c

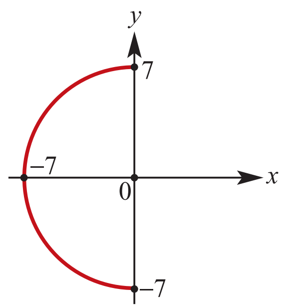


d $y = -\sqrt{25 - x^2}$

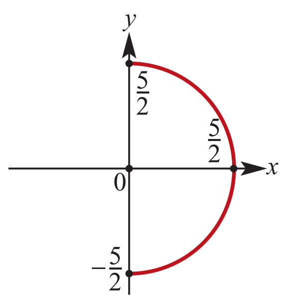
Starting points at $(\pm 5, 0)$



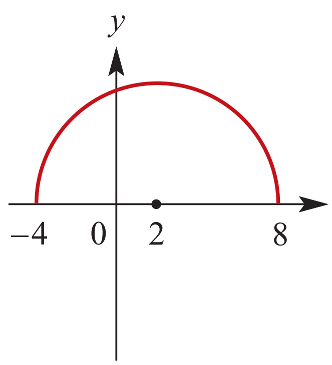
e



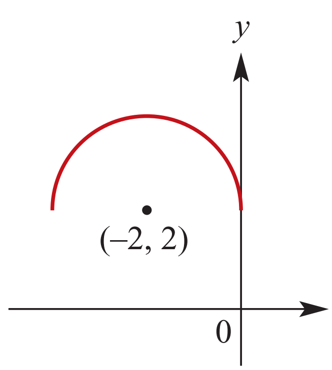
f



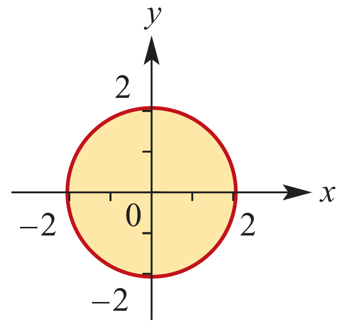
7 **a** $y = \sqrt{36 - (x - 2)^2}$
Starting points at (–4, 0) and (8,0)
Centre at (2,0)



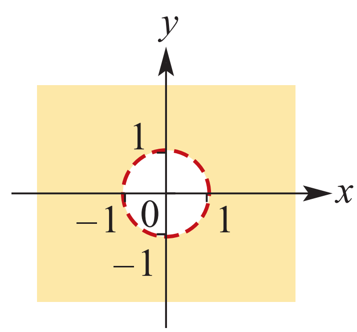
b $y - 2 = \sqrt{4 - (x + 2)^2}$
Starting points at (–4, 2) and (0,2)
Centre at (–2, 2)



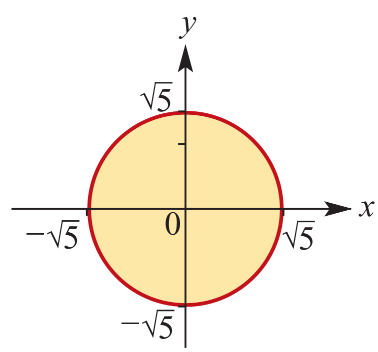
8 **a** $x^2 + y^2 \le 4$



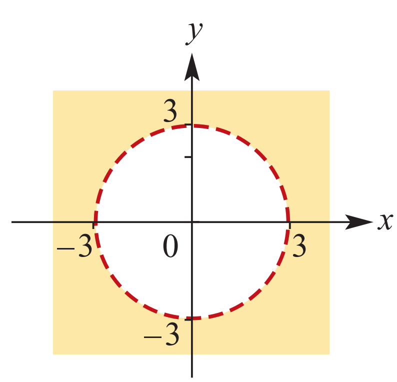
b $x^2 + y^2 > 1$



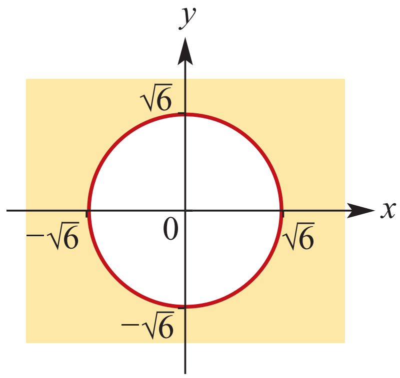
c $x^2 + y^2 \le 5$



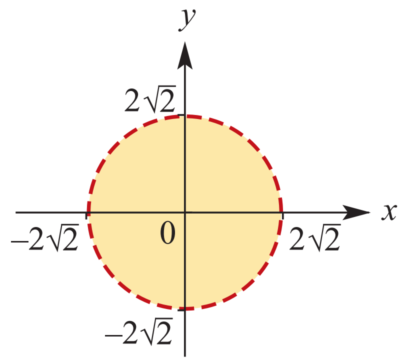
d $x^2 + y^2 > 9$



e $x^2 + y^2 \ge 6$



f $x^2 + y^2 < 8$



Solutions to Exercise 4E

1 $y = \frac{a}{x} + 3$
Passes through $(1, 8)$
$\therefore 8 = a + 3$
$\therefore a = 5$
$\therefore y = \frac{5}{x} + 3$

2 $h = 3, k = 4 \therefore y = \frac{a}{x-3} + 4$
Passes through $(0, 6)$
Therefore, $6 = \frac{a}{-3} + 4$
$\therefore a = -6$
$\therefore y = -\frac{6}{x-3} + 4$

3 $y = \frac{a}{x} + k$
When $x = 1, y = 8$
$\therefore 8 = a + k \ldots (1)$
When $x = -1, y = 7$
$\therefore 7 = -a + k \ldots (2)$
Add (1) and (2)
$15 = 2k$
$k = \frac{15}{2}$ and $a = \frac{1}{2}$

4 $h = 2, k = -4 \therefore y = \frac{a}{x-2} - 4$
Passes through $(0, 4)$
Therefore, $4 = \frac{a}{-2} - 4$
$\therefore a = -16$
$\therefore y = -\frac{16}{x-2} - 4$

5 $y = a\sqrt{x}$
When $x = 2, y = 8$
$\therefore 8 = a\sqrt{2}$
$\therefore a = \frac{8}{\sqrt{2}} = 4\sqrt{2}$

6 $y = a\sqrt{x-h}$
When $x = 1, y = 2$
$\therefore 2 = a\sqrt{1-h} \ldots (1)$
When $x = 10, y = 4$
$\therefore 4 = a\sqrt{10-h} \ldots \ (2)$
Divide (2) by (1)
$2 = \frac{\sqrt{10-h}}{\sqrt{1-h}}$
$2\sqrt{1-h} = \sqrt{10-h}$
Square both sides.
$4(1-h) = 10-h$
$4 - 4h = 10 - h$
$3h = -6$
$h = -2$
Substitute in (1)
$2 = a\sqrt{3}$
$a = \frac{2\sqrt{3}}{3}$

7 A circle with centre (2,1) has equation:
$(x-2)^2 + (y-1)^2 = a^2$
If it passes through $(4, -3)$, then:
$(4\ -2)^2 + (-3-1)^2 = a^2$
$\therefore 4 + 16 = a^2, \therefore a = \pm 2\sqrt{5}$
$(x-2)^2 + (y-1)^2 = 20$

8 Circle centre $(-2, 3)$ has equation of the form
$(x+2)^2 + (y-3)^2 = r^2$
Circle passes through $(-3, 3)$
Therefore
$(-3+2)^2 + (3-3)^2 = r^2$
$\therefore r = 1 \therefore (x+2)^2 + (y-3)^2 = 1$
Note: Centre $(-2, 3)$ and passing through $(-3, 3)$ immediately gives you $r = 1$. Think of the horizontal diameter.

9 Again using the simple approach.

The diameter through the circle with entre $(-2, 3)$ and passing through $(2, 3)$ tells us that the radius is 4. Hence the equation is $(x+2)^2 + (y-3)^2 = 16$

10 A circle with centre $(2, -3)$ has equation:
$(x-2)^2 + (y+3)^2 = a^2$
If it touches the x-axis, then it must be at (2,0):
$\therefore (0+3)^2 = a^2, \therefore a = \pm 3$
$(x-2)^2 + (y+3)^2 = 9$

11 A circle with centre on the line $y = 4$ has equation: $(x-b)^2 + (y-4)^2 = a^2$
If it passes through (2,0) and (6,0) then
$(2 - b)^2 + (0-4)^2 = a^2 \ldots \mathbf{(1)}$
$(6 - b)^2 + (0-4)^2 = a^2 \ldots \mathbf{(2)}$
(2) – **(1)** gives $(6-b)^2 - (2-b)^2 = 0$
$\therefore (36 - 12b + b^2) = (4 - 4b + b^2)$
$\therefore \quad 32 - 12b = -4b$
$\therefore \quad 8b = 32, \therefore b = 4$
Substitute into **(1)**:
$(2-4)^2 + (0-4)^2 = a^2$
$\therefore \quad 4 + 16 = a = 20$
$(x-4)^2 + (y-4)^2 = 20$

12 It touches the x- axis and has radius 5. Let$(a, 5)$ be the centre. It is easy to show it cannot be $(a, -5)$ if it goes through $(0, 8)$.
We also know that $a^2 + (5-8)^2 = 25$.
$\therefore a = 4$ or $a = -4$
The circle has equation
$(x-4)^2 + (y-5)^2 = 25$ or
$(x+4)^2 + (y-5)^2 = 25$

13 The circle has equation of the form
$x^2 + y^2 + bx + cy + d = 0$
When $x = 2, y = 0$
$$4 + 2b + d = 0 \ldots (1)$$
When $x = -4, y = 0$
$$16 - 4b + d = 0 \ldots (2)$$
When $x = 0, y = 2$
$$4 + 2c + d = 0 \ldots (3)$$
From (1) and (3)
we see that $c = b$.
Subtract (1) from (2)
$$12 - 6b = 0$$
$$\therefore b = 2 \text{ and } c = 2$$
$$\therefore d = -8$$
The equation is $x^2 + y^2 + 2x + 2y = 8$
or $(x+1)^2 + (y+1)^2 = 10$

14 **a** $(x-2)^2 + (y+2)^2 = 49$

b $y = 3\sqrt{x-1} - 2$

c $y = \dfrac{1}{x-2} + 2$

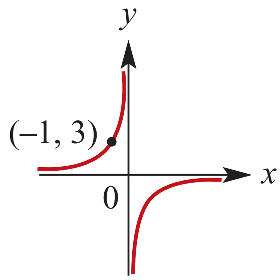
d $y = -\dfrac{2}{x-1} - 2$

e $y = \sqrt{2-x} + 1$

f $(y+2)^2 = 2x + 9$

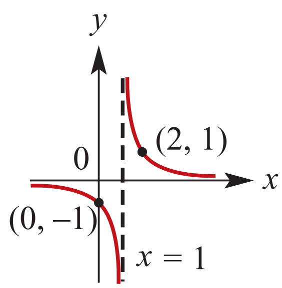
Solutions to Review: Short-answer questions

1 **a** $y = -\dfrac{3}{x}$; asymptotes at $x = 0,\ y = 0$



b $y = \dfrac{1}{x-1}$; asymptotes at $x = 1,\ y = 0$

y-intercept at $(0, -1)$

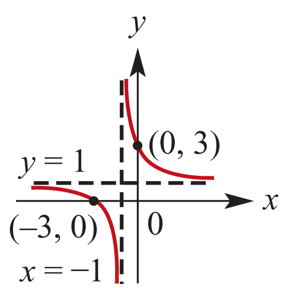


c $y = \dfrac{2}{x+1} + 1$;

asymptotes at $x = -1,\ y = 1$

x-intercept at $(-3, 0)$

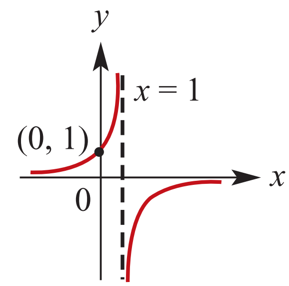
y-intercept at $(0,3)$



d $y = -\dfrac{1}{x-1}$; asymptotes at

$x = 1,\ y = 0$

y-intercept at $(0, 1)$



e $y = \dfrac{4}{2-x} + 3$; asymptotes at

$x = 2,\ y = 3$

x-intercept: $\dfrac{4}{2-x} + 3 = 0$

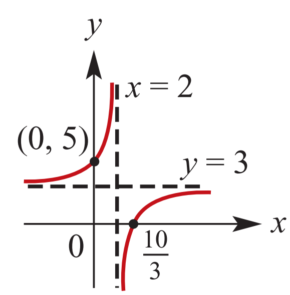
$\therefore \dfrac{4}{2-x} = -3$

$\therefore 4 = -3(2 - x)$

$\therefore 4 = 3x - 6 \therefore x = \dfrac{10}{3}$

x-intercept at $\left(\dfrac{10}{3}, 0\right)$

y-intercept at $(0,5)$



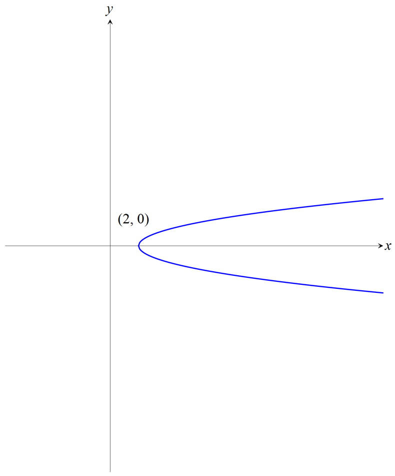
f The graph of $y^2 = x$ is translated 2 units right. The vertex is at $(-2, 0)$.

To find the y-axis intercepts, let $x = 0$:

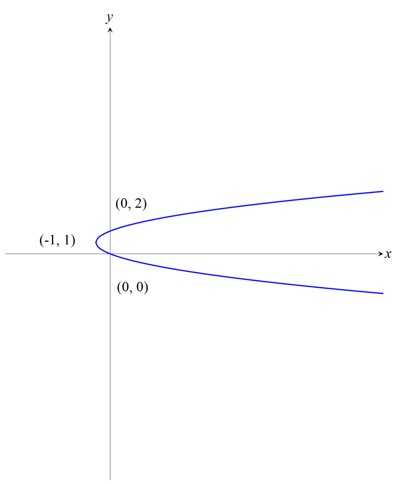
$y^2 = -2$ which means that there is no y-intercept.

To find the x-axis intercept, let $y = 0$:

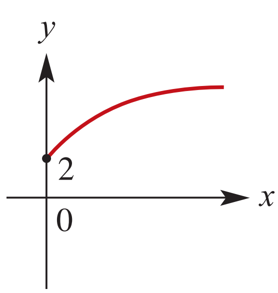
$0^2 = x - 2 \Rightarrow x = 2$



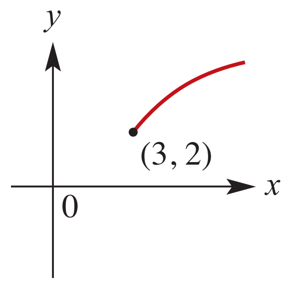
g The graph of $y^2 = x$ is translated 1 unit left and 1 units up. The vertex is at $(-1, 1)$.
To find the y-axis intercepts, let $x = 0$:
$(y - 1)^2 = 1 \Rightarrow y = 0 \text{ or } 2$
To find the x-axis intercept, let $y = 0$:
$(-1)^2 = x + 1 \Rightarrow x = 0$



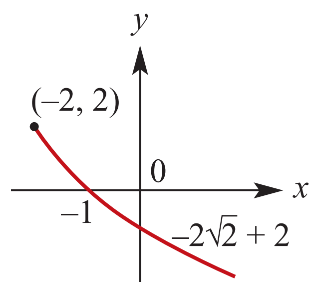
h $y = 2\sqrt{x} + 2$
y-intercept at (0,2)



i $y = 2\sqrt{x - 3} + 2$
Starting point at (3,2)



j $y = -2\sqrt{x + 2} + 2$
Starting point at $(-2, 2)$
x-intercept: $-2\sqrt{x+2} + 2 = 0$
$\therefore \sqrt{x+2} = 1$
$\therefore x + 2 = 1, \therefore x = -1$
x-intercept at $(-1, 0)$
y-intercept at $(0, 2 - 2\sqrt{2})$



2 **a** $x^2 + y^2 - 6x + 4y - 12 = 0$
$\therefore x^2 - 6x + 9 + y^2 + 4y + 4 - 12 = 13$
$\therefore (x-3)^2 + (y+2)^2 = 5^2$

b $x^2 + y^2 - 3x + 5y - 4 = 0$
$\therefore x^2 - 3x + \dfrac{9}{4} + y^2 + 5y + \dfrac{25}{4} = 4 + \dfrac{34}{4}$
$\therefore \left(x - \dfrac{3}{2}\right)^2 + \left(y + \dfrac{5}{2}\right)^2 = \dfrac{50}{4}$
$\therefore \left(x - \dfrac{3}{2}\right)^2 + \left(y + \dfrac{5}{2}\right)^2 = \left(\dfrac{5\sqrt{2}}{2}\right)^2$
$\therefore \left(x - \dfrac{3}{2}\right)^2 + \left(y + \dfrac{5}{2}\right)^2 = \dfrac{25}{4}$

c $2x^2 + 2y^2 - x + y - 4 = 0$
$\therefore x^2 + y^2 - \dfrac{x}{2} + \dfrac{y}{2} = 2$
$\therefore x^2 - \dfrac{x}{2} + \dfrac{1}{16} + y^2 + \dfrac{y}{2} + \dfrac{1}{16} = 2 + \dfrac{1}{8}$

$\therefore \left(x - \frac{1}{4}\right)^2 + \left(y + \frac{1}{4}\right)^2 = \frac{17}{8}$

d $x^2 + y^2 + 4x - 6y = 0$
$\therefore x^2 + 4x + 4 + y^2 - 6y + 9 = 13$
$\therefore (x + 2)^2 + (y - 3)^2 = (\sqrt{13})^2$

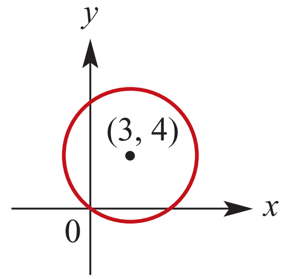
e $x^2 + y^2 = 6(x + y)$
$\therefore x^2 - 6x + 9 + y^2 - 6y + 9 = 18$
$\therefore (x - 3)^2 + (y - 3)^2 =$
$(\sqrt{18})^2 = (3\sqrt{2})^2$

f $x^2 + y^2 = 4x - 6y$
$\therefore x^2 - 4x + 4 + y^2 + 6y + 9 = 13$
$\therefore (x - 2)^2 + (y + 3)^2 = (\sqrt{13})^2$

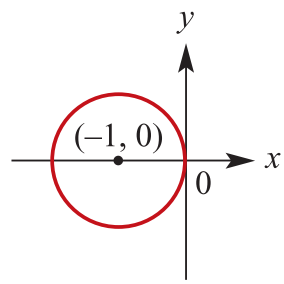
3 $x^2 + y^2 - 4x + 6y = 14$
$\therefore x^2 - 4x + 4 + y^2 + 6y + 9 = 14 + 13$
$\therefore (x - 2)^2 + (y + 3)^2 = 27$
A diameter must pass through the centre of the circle at $(2, -3)$.
A line connecting $(0,0)$ with $(2, -3)$ has gradient $= -\frac{3}{2}$, hence the equation
$y = -\frac{3x}{2}$ or $3x + 2y = 0$

4 $x^2 + y^2 - 3x + 2y = 26$
$\therefore x^2 - 3x + \frac{9}{4} + y^2 + 2y + 1 = 26 + \frac{13}{4}$
$\therefore \left(x - \frac{3}{2}\right)^2 + (y + 1)^2 = \frac{117}{4}$
Centre of circle is at $\left(\frac{3}{2}, -1\right)$.
Diameter of the circle which cuts the x-axis at 45° has gradient = 1.
$\therefore y - (-1) = x - \frac{3}{2}$
$y = x - \frac{5}{2}$
Or $2x + 2y = 1$ if the diameter goes below the axis at an angle of $-45°$. In this case the gradient $= -1$.

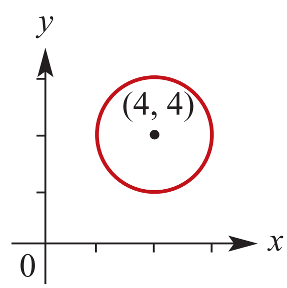
5 a $C(3, 4),\ r = 5$
$\therefore (x - 3)^2 + (y - 4)^2 = 25$



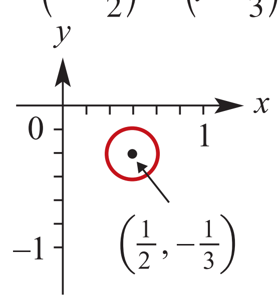
b $C(-1, 0),\ r = 1$
$\therefore (x + 1)^2 + y^2 = 1$



c $C(4, 4),\ r = 2$
$\therefore (x - 4)^2 + (y - 4)^2 = 4$



d $C\left(\frac{1}{2}, -\frac{1}{3}\right),\ r = \frac{1}{6}$
$\therefore \left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{3}\right)^2 = \frac{1}{36}$



6

$$x^2 + y^2 + 4x - 6y = 23$$
$$x^2 + 4x + 4 + y^2 - 6y + 9 = 23 + 4 + 9$$
$$(x + 2)^2 + (y - 3)^2 = 36$$

Centre: $(-2, 3)$ Radius: 6

7 $x^2 + y^2 - 2x - 4y = 20$

$\therefore x^2 - 2x + 1 + y^2 - 4y + 4 = 25$

$\therefore (x - 1)^2 + (y - 2)^2 = 5^2$

Length cut off on the x-axis and y-axis = distance between x- and y-intercepts:

$y = 0$: $\therefore (x - 1)^2 + (0 - 2)^2 = 5^2$

$\therefore (x - 1)^2 = 21, \therefore x = 1 \pm \sqrt{21}$

x-axis length $= 2\sqrt{21}$

$x = 0$: $\therefore (0 - 1)^2 + (y - 2)^2 = 5^2$

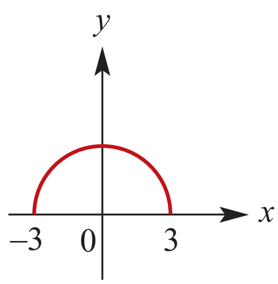
$\therefore (y - 2)^2 = 24, \therefore y = 2 \pm 2\sqrt{6}$

y-axis length $= 4\sqrt{6}$

8 **a** $y = \sqrt{9 - x^2}$

x-intercepts at $(-3, 0)$ and $(3,0)$

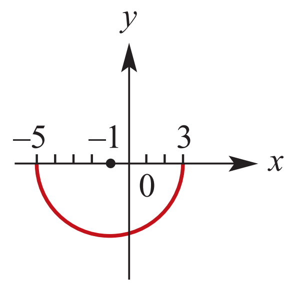
y-intercept at $(0,3)$



b $y = -\sqrt{16 - (x + 1)^2}$

Starting points at $(-5, 0)$ and $(3,0)$

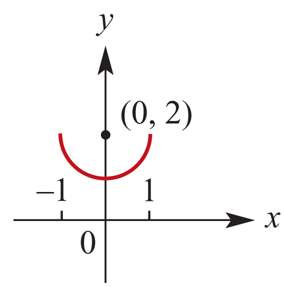
y-intercept at $(0, -\sqrt{15})$, centre at $(-1, 0)$



c $y - 2 = -\sqrt{1 - x^2}$

No x-intercepts, y-intercept at $(0, 1)$

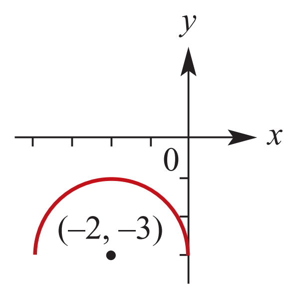
Centre at $(0,2)$



d $y + 3 = \sqrt{4 - (x + 2)^2}$

No x-intercepts, y-intercept at $(0, -3)$

Centre at $(-2, 3)$



Solutions to Review: Multiple-choice questions

1 E $(x-a)^2 + (y-b)^2 = 36$
Centre on the x-axis so $b = 0$
Using (6,6): $(6 - a)^2 + 6^2 = 36$
$\therefore a = 6$

2 B $y = 5 - \dfrac{1}{3x-5}$ has asymptotes at $y = 5$ and $3x - 5 = 0$
$\therefore x = \dfrac{5}{3}$

3 E $y = \dfrac{5}{x} + 3$
If $x = \dfrac{a}{2}$,
$y = \dfrac{5}{(a/2)} + 3 = \dfrac{10}{a} + 3$

4 A $(x-a)^2 + (y-b)^2 = c^2$
y-axis is an axis of symmetry so $a = 0$ Using (0,0):
$(-b)^2 = c^2, \therefore b = c; c > 0$
Using (0,4):
$(4 - b)^2 = b^2, \therefore b = 2$
$\therefore x^2 + (y-2)^2 = 4$

5 A $y = \dfrac{2}{x+2} - 4$ has asymptotes at $y = -4$ and $x = -2$

6 D $(x-5)^2 + (y+2)^2 = 9$
$C(5, -2),\ r = \sqrt{9} = 3$

7 D $y = -2\sqrt{x} + 3; x \geq 0$
$-2\sqrt{x} \leq 0 \therefore y \leq 3$

8 C Circle end points at $(-2, 8)$ and (6,8);
centre is at $x = \dfrac{-2+6}{2} = 2$ and $y = 8$.
Radius $= 6 - 2 = 4$
$(x-2)^2 + (y-8)^2 = 4^2$

9 E Only $y^2 = 16 - x^2$ has the general form with $(x-a)^2 + (y-b)^2 = c^2$

10 B **A** is a full circle, **B** is correct, **C** isn't circular, and **D** and **E** are negatives.

Solutions to Review: Extended-response questions

1 **a** The circle has centre (10, 0) and radius 5 and therefore has the equation $(x-10)^2 + y^2 = 25$.

b The line with equation $y = mx$ meets the circle with equation $(x-10)^2 + y^2 = 25$.

Therefore x satisfies the equation $(x-10)^2 + (mx)^2 = 25$

Expanding and rearranging gives $x^2 - 20x + 100 + m^2x^2 = 25$

and therefore $(1 + m^2)x^2 - 20x + 75 = 0$

c The discriminant is

$$\begin{aligned}\Delta &= 400 - 4 \times 75 \times (1 + m^2)\\ &= 400 - 300(1 + m^2)\\ &= 100 - 300\,m^2\end{aligned}$$

As the line is a tangent to the circle, there is only one point of contact and hence only one solution to the equation obtained in part **b**. Therefore the discriminant = 0, which implies

$$m = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$$

d When $m^2 = \frac{1}{3}$, the equation $(1 + m^2)x^2 - 20x + 75 = 0$ becomes

$$\frac{4}{3}x^2 - 20x + 75 = 0$$

Multiplying both sides of the equation by 3 gives

$$4x^2 - 60x + 225 = 0$$

The left-hand side is a perfect square and hence

$$(2x - 15)^2 = 0$$

The solution is $x = \frac{15}{2}$

The y-coordinate is given by substituting into $y = mx = \pm\frac{\sqrt{3}}{3}x$.

$$\begin{aligned}y &= \pm\frac{\sqrt{3}}{3} \times \frac{15}{2}\\ &= \pm\frac{5\sqrt{3}}{2}\end{aligned}$$

The coordinates of P are $\left(\frac{15}{2}, \pm\frac{5\sqrt{3}}{2}\right)$.

e Distance of P from the origin $= \sqrt{\left(\frac{15}{2}\right)^2 + \left(\frac{5\sqrt{3}}{2}\right)^2}$

$$= \frac{1}{2}\sqrt{225 + 75}$$

$$= 5\sqrt{3}$$

2 a The circle has centre the origin and radius 4.
Hence the equation is $x^2 + y^2 = 16$.

b i The general form for a straight line is $y = mx + c$.
When $x = 8$, $y = 0$, hence $0 = 8m + c$ and $c = -8m$.
So the tangents have equations of the form $y = mx - 8m$

ii As in Question 1, consider when the line with equation $y = mx - 8m$ meets the circle $x^2 + y^2 = 16$.
Substitute for y: $(mx - 8m)^2 + x^2 = 16$
Expand and collect like terms to obtain

$$(m^2 + 1)x^2 - 16m^2 x + 64m^2 - 16 = 0$$

There will be a tangent when the discriminant is equal to 0, i.e. when there is only one solution.

$$\begin{aligned}\Delta &= 256m^4 - 4(m^2 + 1)(64m^2 - 16) \\ &= 256m^4 - 4(64m^4 + 48m - 16) \\ &= -4(48m^2 - 16) \\ &= 64(-3m^2 + 1)\end{aligned}$$

Thus there is a tangent if $3m^2 = 1$

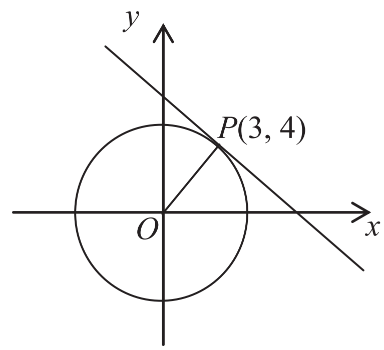
i.e. $m = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$

Since $y = mx - 8m$, is the equations of the tangents

$$y = \frac{\sqrt{3}}{3}x - \frac{8\sqrt{3}}{3}$$

and $$y = -\frac{\sqrt{3}}{3}x + \frac{8\sqrt{3}}{3}$$

3 **a** The gradient of $OP = \dfrac{4}{3}$.



b The tangent is perpendicular to the radius and therefore has gradient of $\dfrac{-3}{4}$.

c The equation of the tangent is given by
$y - 4 = -\dfrac{3}{4}(x - 3)$

Therefore $\quad 4y - 16 = -3x + 9$

and $\quad 4y + 3x = 25$

d The coordinates of A are $\left(\dfrac{25}{3}, 0\right)$ and the coordinates of B are $\left(0, \dfrac{25}{4}\right)$.

$$AB^2 = \left(\frac{25}{3}\right)^2 + \left(\frac{25}{4}\right)^2$$
$$= 625\left(\frac{1}{9} + \frac{1}{16}\right) = 625 \times \frac{25}{9 \times 16}$$

Therefore $\quad AB = \dfrac{25 \times 5}{12} = \dfrac{125}{12}$

4 **a** **i** The radius has gradient $\dfrac{y_1}{x_1}$.

ii The tangent is perpendicular to radius, so using $m_l,\ m_2 = -1$, it has the gradient $-\dfrac{x_1}{y_1}$.

b The equation of the tangent is given by $\quad y - y_1 = -\dfrac{x_1}{y_1}(x - x_1)$

Multiply both sides by y_1 $\quad yy_1 - y_1^2 = -x_1x + x_1^2$

Therefore $\quad yy_1 + x_1x = x_1^2 + y_1^2$

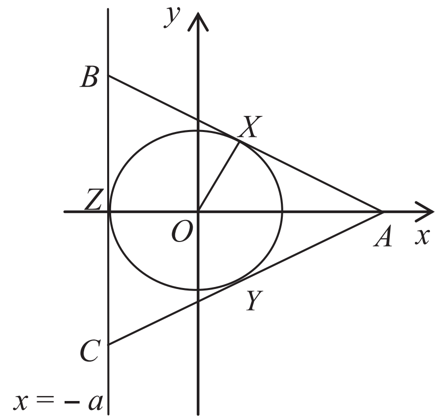
But $(x_1,\ y_1)$ is a point on the circle and hence $\quad x_1^2 + y_1^2 = a^2$

This gives $\quad x_1x + y_1y = a^2$

c If $x_1 = y_1$ and $a = 4$,then $x_1^2 + y_1^2 = a^2$ becomes $2x_1^2 = 16$.
Thus $x_1 = \pm 2\sqrt{2}$
The equations of the tangents are $\sqrt{2}x + \sqrt{2}y = 8$ and
$-\sqrt{2}x - \sqrt{2}y = 8$ or $\sqrt{2}x + \sqrt{2}y = -8$.

5 a Note that the triangle is equilateral, and that $AX = AY = XB = YC = CZ = ZB$, as these line segments are equal tangents from a point.

Let these equal lengths be b.



Then $\quad AZ^2 = AB^2 - BZ^2$

$$= 4b^2 - b^2 = 3b^2$$

(Pythagoras' theorem in triangle AZB)

Therefore $\quad AZ = \sqrt{3}b$

The gradient of line $BA = -\dfrac{1}{\sqrt{3}}$

and the gradient of $CA = \dfrac{1}{\sqrt{3}}$.

Note also that triangle BZA is similar to triangle AXO.

Therefore $\quad \dfrac{OX}{AX} = \dfrac{BZ}{AZ}$

Therefore $\quad \dfrac{a}{b} = \dfrac{b}{\sqrt{3}b} = \dfrac{1}{\sqrt{3}} \quad$ and $\quad b = \sqrt{3}a$

For line *BA*: when $x = -a$, $y = \sqrt{3}a$ and, using the form $y = mx + c$,

$$\sqrt{3}a = -\frac{1}{\sqrt{3}} \times -a + c$$

$$\therefore \quad c = \sqrt{3}a - \frac{a}{\sqrt{3}}$$

$$= \frac{2\sqrt{3}a}{3}$$

Therefore the equation of line BA is $\; y = -\dfrac{\sqrt{3}}{3}x + \dfrac{2\sqrt{3}a}{3}$

For line *CA*: $\quad y = \dfrac{\sqrt{3}}{3}x - \dfrac{2\sqrt{3}a}{3}$

b The circumcentre has centre O and radius OA. But, from the equation of BA, A has coordinates $(2a, 0)$. Hence the equation of the circle is $x^2 + y^2 = 4a^2$.

6 a If $(a,\ a)$ lies on the line $y = x$ and on the curve with equation $y = \sqrt{x - b} + c$,

then $\quad a = \sqrt{a - b} + c$

Subtract c from both sides and square $\quad (a - c)^2 = a - b$

Expand and rearrange $\quad a^2 - 2ac + c^2 = a - b$

$$a^2 - (2c + 1)a + c^2 + b = 0$$

b **i** The line meets the curve at one point if the discriminant of the quadratic in a is zero.

$$\begin{aligned}\Delta &= (2c+1)^2 - 4(c^2+b)\\ &= 4c^2 + 4c + 1 - 4c^2 - 4b\\ &= 4c - 4b + 1\end{aligned}$$

If the discriminant is zero, $c = \dfrac{4b-1}{4}$

ii Solving the equation $x = \sqrt{x} - \dfrac{1}{4}$ will give the required coordinates.

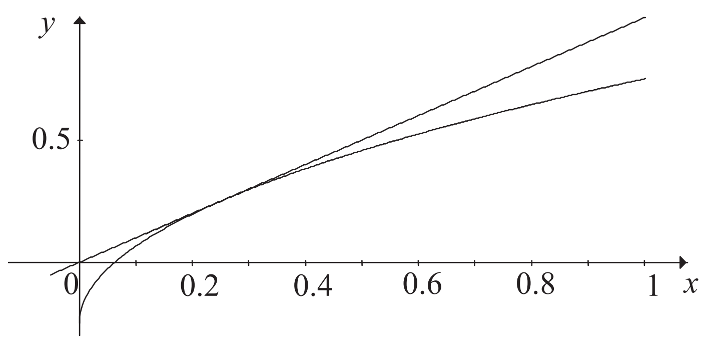
Squaring both sides of $x + \dfrac{1}{4} = \sqrt{x}$ gives $x^2 + \dfrac{1}{2}x + \dfrac{1}{16} = x$

and rearranging gives $x^2 - \dfrac{1}{2}x + \dfrac{1}{16} = 0$

$$\left(x - \frac{1}{4}\right)^2 = 0$$

Therefore $x = \dfrac{1}{4}$

and $y = \dfrac{1}{4}$



c **i** From the above, we know that the line with equation $y = x$ is tangent to the curve with equation $y = \sqrt{x} - \dfrac{1}{4}$.

Hence if $-\dfrac{1}{4} < k < 0$, the line will cross the curve twice.

ii If $k = 0$ or $k < -\dfrac{1}{4}$, the line will cross the curve once.

iii It will not meet the curve if $k > 0$.

7 **a** From the graphs of $y = kx$ and $y = \sqrt{x} - 1$, it is clear that if $k \le 0$, the line $y = kx$ can only cut the curve once.

For two solutions, consider the equation: $kx = \sqrt{x} - 1$

$$\begin{aligned}\Delta &= (2k - 1)^2 - 4k^2 \\ &= 4k^2 - 4k + 1 - 4k^2 \\ &= -4k + 1\end{aligned}$$

Thus for two solutions, $-4k + 1 > 0$, i.e. $k < \dfrac{1}{4}$, and $k > 0$.

Hence $0 < k < \dfrac{1}{4}$.

b There is one solution when $k = \dfrac{1}{4}$ or $k \le 0$.