WACE
Revision Series

ACADEMIC TASK FORCE

MATHEMATICS METHODS

ATAR Course

Year 12
Units 3 and 4

Australian Curriculum

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01 Exponential Functions

Calculator Free

1. [3 marks: 1, 2]

Consider $y = e^{x+1}$.

(a) State the equation of the horizontal asymptote of this curve.

$y = 0$ ✓

(b) Find the point of intersection of this curve with the line $y = \frac{1}{e}$.

$e^{x+1} = e^{-1}$ ✓
$x + 1 = -1$
$x = -2$
Hence, $(-2,\ e^{-1})$. ✓

2. [6 marks: 1, 1, 2, 2]

Consider $y = e^{-2x} - 1$.

(a) State the equation of the horizontal asymptote of this curve.

$y = -1$ ✓

(b) Find the coordinates of the y-intercept of this curve.

$(0, 0)$ ✓

(c) Find the point of intersection of this curve with the line $y = e^4 - 1$.

$e^4 - 1 = e^{-2x} - 1$ ✓
$x = -2$
Hence, $(-2, e^4 - 1)$ ✓

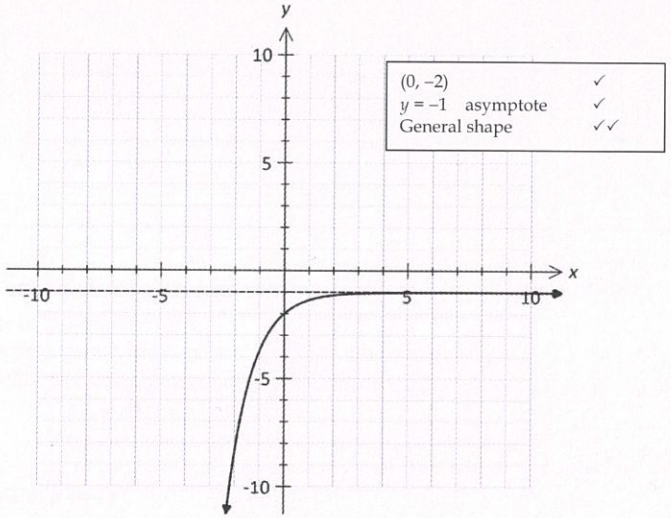
(d) Find the point of intersection of this curve with the curve $y = e^{x-1} - 1$.

$e^{x-1} - 1 = e^{-2x} - 1$
$x - 1 = -2x$
$x = \frac{1}{3}$ ✓
Hence, $(\frac{1}{3},\ e^{-2/3} - 1)$ ✓

Calculator Free

3. [4 marks] [TISC]

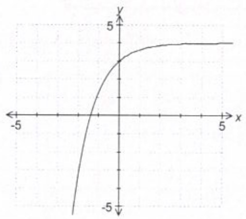
Sketch the graph of $y = -e^{-x} - 1$.
Indicate clearly the intercepts and asymptotes where they exist.



4. [4 marks: 2, 2]

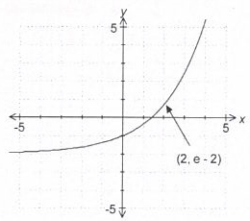
The graphs of $y = Ae^{kx} + B$ are sketched below. Find the equation of these curves.

(a)



$y = 4 - e^{-x}$ ✓✓

(b)

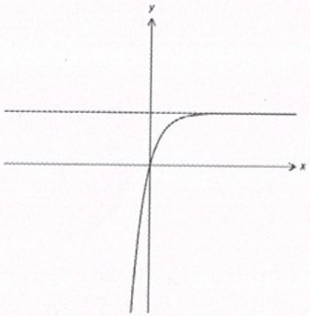


$y = e^{x/2} - 2$ ✓✓

Calculator Free

5. [6 marks: 3, 3] [TISC]

The diagram below shows the sketch of $y = a + b e^{kx}$ where a, b and k are constants.



(a) Complete the table below, indicating whether the constants a, b and k have positive or negative values.

Constant	Positive or negative value
a	positive
b	negative
k	negative

1 mark each

(b) Given that the curve passes through the origin and $b = k$, suggest one possible set of numerical values for a, b and k.

$y = a + b e^{kx}$
$0 = a + b \Rightarrow b = -a$

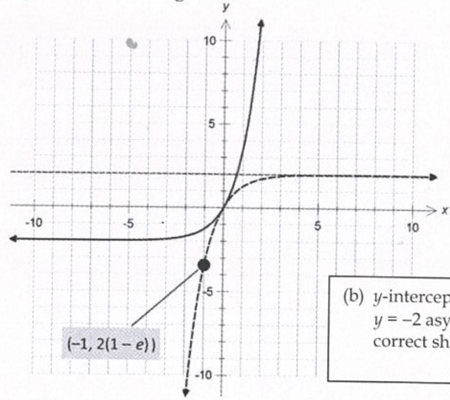
Hence, one possible set is $a = 1$, $b = -1$, $k = -1$

$a > 0$, ✓
$b = -a$ ✓
$k = -a$ ✓

Calculator Free

6. [6 marks: 3, 3]

The sketch of $y = A e^{kx} + B$ is given below.



(b) y-intercept (0, 0) ✓
$y = -2$ asymptote ✓
correct shape ✓

(a) Find A, B and k.

$A = -2$ $B = 2$ $k = -1$ ✓✓✓

(b) On the diagram given above, sketch $y = -A e^{-kx} - B$
Indicate clearly the y-intercept and asymptotes, if any.

Calculator Assumed

02 Logarithms

Calculator Assumed

7. [4 marks: 3, 1]

(a) Use your CAS calculator to determine exactly $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{kx}\right)^x$ for $k = 1, 2$ & 3.

$k=1,\ \lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x = e$ ✓

$k=2,\ \lim\limits_{x\to\infty}\left(1+\dfrac{1}{2x}\right)^x = e^{1/2}$ ✓

$k=3,\ \lim\limits_{x\to\infty}\left(1+\dfrac{1}{3x}\right)^x = e^{1/3}$ ✓

(b) Use your results in (a) to suggest the exact value of $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{kx}\right)^x$ where k is a positive integer.

$\lim\limits_{x\to\infty}\left(1+\dfrac{1}{kx}\right)^x = e^{1/k}$ ✓

02 Logarithms

Calculator Free

1. [3 marks]

Rewrite $3\log x - 2\log y + \log x^2$ as a single logarithmic term.

$3\log x - 2\log y + \log x^2 \equiv \log x^3 - \log y^2 + \log x^2$ ✓✓

$\equiv \log \dfrac{x^3x^2}{y^2} \equiv \log \dfrac{x^5}{y^2}$ ✓

2. [9 marks: 2, 3, 4]

Express in its simplest form :

(a) $\dfrac{\log_4 16}{\log_4 64}$

$\dfrac{\log_4 16}{\log_4 64} \equiv \dfrac{2\log_4 4}{3\log_4 4} \equiv \dfrac{2}{3}$ ✓✓

(b) $2\log 3 + \frac{1}{2}\log 5 - \log 4^{-1}$

$2\log 3 + \frac{1}{2}\log 5 - \log 4^{-1} \equiv \log 3^2 + \log 5^{1/2} + \log 4$ ✓

$\equiv \log(9 \times 4 \times \sqrt{5})$ ✓

$\equiv \log(36\sqrt{5})$ ✓

(c) $2 + \log_5 4 - \log_5 20$

$2 + \log_5 4 - \log_5 20 \equiv \log_5 25 + \log_5 4 - \log_5 20$ ✓✓

$\equiv \log_5 \dfrac{25 \times 4}{20}$ ✓

$\equiv \log_5 5$

$\equiv 1$ ✓

Calculator Free

3. [12 marks: 3, 3, 3, 3]

Given that $p = \log_5 2$ and $q = \log_5 6$, find in terms of p and q :

(a) $\log_5 12$

$$\log_5 12 = \log_5 (2 \times 6) = \log_5 2 + \log_5 6 \quad ✓✓$$
$$= p + q \quad ✓$$

(b) $\log_5 3$

$$\log_5 3 = \log_5 \left(\tfrac{6}{2}\right) = \log_5 6 - \log_5 2 \quad ✓✓$$
$$= q - p \quad ✓$$

(c) $\log_5 24$

$$\log_5 24 = \log_5 (6 \times 4)$$
$$= \log_5 6 + 2\log_5 2 \quad ✓✓$$
$$= q + 2p \quad ✓$$

(d) $\log_5 60$

$$\log_5 60 = \log_5 (2 \times 5 \times 6)$$
$$= \log_5 2 + \log_5 5 + \log_5 6 \quad ✓✓$$
$$= 1 + p + q \quad ✓$$

4. [15 marks: 3, 3, 3, 3, 3]

Given that $p = \log_2 3$ and $q = \log_2 7$, find in terms of p and q :

(a) $\log_2 21$

$$\log_2 21 = \log_2 (3 \times 7) = \log_2 3 + \log_2 7 \quad ✓✓$$
$$= p + q \quad ✓$$

(b) $\log_2 49$

$$\log_2 49 = \log_2 (7^2) = 2\log_2 7 \quad ✓✓$$
$$= 2q \quad ✓$$

(c) $\log_2 6$

$$\log_2 6 = \log_2 (2 \times 3) = \log_2 2 + \log_2 3 \quad ✓✓$$
$$= 1 + p \quad ✓$$

(d) $\log_2 1.5$

$$\log_2 1.5 = \log_2 \left(\tfrac{3}{2}\right) = \log_2 3 - \log_2 2 \quad ✓✓$$
$$= p - 1 \quad ✓$$

(e) $\log_2 42$

$$\log_2 42 = \log_2 (2 \times 3 \times 7) = \log_2 2 + \log_2 3 + \log_2 7 \quad ✓✓$$
$$= 1 + p + q \quad ✓$$

Calculator Free

5. [5 marks: 2, 3]

Given that $\log_4 (x^2 + 7) = 2$:

(a) rewrite this equation in exponential form (without involving logarithms)

$$\log_4 (x^2 + 7) = 2 \Rightarrow (x^2 + 7) = 4^2 \quad ✓$$
$$x^2 = 9 \quad ✓$$

(b) solve for x.

$$x^2 = 9 \Rightarrow x = \pm 3 \quad ✓$$

Check: $x = 3$, $\log_4 (3^2 + 7) = 2$ (consistent)

$x = -3$, $\log_4 [(-3)^2 + 7] = 2$ (consistent) ✓

Hence, solution is $x = \pm 3$ ✓

6. [3 marks]

Use the rules of logarithms to solve for x in $\log_2 10 + \log_2 x = 2$

Rewrite equation as: $\log_2 10x = 2$ ✓

$$10x = 2^2 \quad ✓$$
$$x = \frac{2}{5} \quad ✓$$

7. [4 marks]

Solve for x in $\log_x 4 - \log_x 3 = -2$

Rewrite equation as: $\log_x \frac{4}{3} = -2$ ✓

$$\frac{4}{3} = x^{-2} \quad ✓$$
$$x = \pm \frac{\sqrt{3}}{2} \quad ✓$$

But $x > 0$, hence, $x = \dfrac{\sqrt{3}}{2}$ ✓

(e) $\log_2 42$

$\log_2 42 = \log_2 (2 \times 3 \times 7) = \log_2 2 + \log_2 3 + \log_2 7$ ✓✓
$= 1 + p + q$ ✓

Calculator Free

8. [6 marks: 2, 2, 2]

Use an algebraic method to solve for t.

(a) $3^{2t+1} = 81$

$3^{2t+1} = 3^4 \Rightarrow 2t + 1 = 4$ ✓
Hence, $t = \frac{3}{2}$ ✓

(b) $4^{1-t} = 32$

$(2^2)^{1-t} = 2^5 \Rightarrow 2 - 2t = 5$ ✓
Hence, $t = -\frac{3}{2}$ ✓

(c) $5^{2+t} = \frac{1}{125}$

$5^{2+t} = 5^{-3}$
$2 + t = -3$ ✓
Hence, $t = -5$. ✓

9. [8 marks: 2, 3, 3]

Use common logarithms to solve for t where appropriate.

(a) $3^{-0.05t} = 5$

Take log on both sides:
$\log 3^{-0.05t} = \log 5$
$\Rightarrow -0.05t \log 3 = \log 5$ ✓
$t = \frac{\log 5}{-0.05 \log 3}$ ✓
$= \frac{-20 \log 5}{\log 3}$

$x = \pm \frac{\sqrt{3}}{2}$ ✓

But $x > 0$, hence, $x = \frac{\sqrt{3}}{2}$ ✓

Calculator Free

9. (b) $10 \times 2^{3t} = 30$

Rewrite as: $2^{3t} = 3$ ✓
Take log on both sides:
$\log (2^{3t}) = \log 3$
$3t \log 2 = \log 3$ ✓
Hence, $t = \frac{\log 3}{3 \log 2}$ ✓

(c) $3^t = 5^{1+2t}$

Take log on both sides:
$\log 3^t = \log 5^{1+2t}$
$\Rightarrow t \log 3 = (1 + 2t) \log 5$ ✓✓
$t (\log 3 - 2 \log 5) = \log 5$
$t = \frac{\log 5}{\log 3 - 2 \log 5}$ ✓

10. [11 marks: 3, 3, 5]

Use common logarithms to solve for t where appropriate.

(a) $5^t \times 25^{t-1} = 0.04$

Rewrite equation as: $5^t \times (5^2)^{t-1} = 5^{-2}$ ✓
$5^{t+2t-2} = 5^{-2}$
Hence, $3t - 2 = -2 \Rightarrow t = 0$ ✓✓

(b) $\frac{2^{2t+1}}{2^{1-t}} = 5$

Rewrite equation as: $2^{2t+1-(1-t)} = 5$ ✓
$2^{3t} = 5$
Hence, $3t \log 2 = \log 5$ ✓
$\Rightarrow t = \frac{\log 5}{3 \log 2}$ ✓

Calculator Free

10. (c) $(3^{2t+1}) - 8(3^t) - 3 = 0$

Rewrite equation as: $3\,(3^t)^2 - 8(3^t) - 3 = 0$ ✓

Let $y = 3^t$.

Hence, equation becomes:

$3(y)^2 - 8(y) - 3 = 0$ ✓

$(3y+1)(y-3) = 0$

Hence, $y = -\frac{1}{3}$ or 3 ✓

Therefore: $3^t = -\frac{1}{3}$ or $3^t = 3$

For $3^t = -\frac{1}{3}$, there is no solution. ✓

For $3^t = 3$, $t = 1$ ✓

Hence, solution is $t = 1$.

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11. [5 marks]

Use common logarithms to solve for x in $2(3^{2x}) + 5(3^x) - 3 = 0$.

Rewrite equation as: $2(3^x)^2 + 5(3^x) - 3 = 0$ ✓

Let $y = 3^x$.

Hence, equation becomes:

$2(y)^2 + 5(y) - 3 = 0$ ✓

$(2y-1)(y+3) = 0$

Hence, $y = \frac{1}{2}$ or -3 ✓

Therefore: $3^x = \frac{1}{2}$ or $3^x = -3$

For $3^x = \frac{1}{2}$, $x \log 3 = \log \frac{1}{2} \Rightarrow x = -\frac{\log 2}{\log 3}$ ✓

For $3^x = -3$, there is no solution for x. ✓

Hence, solution is $x = -\frac{\log 2}{\log 3}$.

Calculator Assumed

12. [9 marks: 3, 4, 2]

(a) Given that $\log 3 = p$, find t in terms of p if $3^{-0.1t} = 10$.

$-0.1t \log 3 = \log 10$ ✓

$t = \dfrac{\log 10}{-0.1 \log 3}$ ✓

$= \dfrac{-10}{p}$ ✓

(b) Given that $\log 1.05 = q$, find t in terms of q if $25 \times 1.05^{2t+1} = 2500$.

$1.05^{2t+1} = 100$ ✓

$(2t+1)\log 1.05 = \log 100$ ✓

$(2t+1) = \dfrac{\log 100}{\log 1.05}$

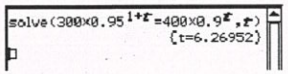
$2t + 1 = \dfrac{2}{q}$ ✓

$t = \dfrac{1}{2}\left(\dfrac{2}{q} - 1\right)$ ✓

(c) Solve for t if $300 \times 0.95^{1+t} = 400 \times 0.9^t$.
Give your answer accurate to 4 significant figures

$t = 6.26952$ ✓

$= 6.270$ (4 significant figures) ✓



Calculator Assumed

03 Natural Logarithms

Hence, solution is $x = -\frac{\log 2}{\log 3}$.

Calculator Assumed

13. [11 marks: 3, 3, 3, 2]

(a) Rewrite the expression $G = \log_2 10\,000$ in exponential form. Hence, find G.

$2^G = 10\,000$ ✓
$G \log 2 = \log 10\,000$ ✓
$G = 13.2877$ ✓

(b) Two students Alex and Ben play a game where Alex chooses a whole number between 1 and N inclusive. Ben needs to guess the number that Alex has chosen. Using the binary search algorithm the maximum number of guesses Ben needs to correctly guess the number Alex has chosen is $\log_2 N$. Using the binary search algorithm, what is the maximum number of guesses Ben needs to guess the number Alex has chosen if:

(i) N = 32

$G = \log_2 32$ ✓
$= \log_2 2^5$ ✓
$= 5$ ✓

(ii) N = 1024

$G = \log_2 1024$ ✓
$= \log_2 2^{10}$ ✓
$= 10$ ✓

(iii) N = 10 000

$G = \log_2 10\,000$
$= 13.2877$ ✓
$= 14$ ✓

03 Natural Logarithms

Calculator Free

1. [7 marks: 1, 2, 4]

Solve *exactly* for x:

(a) $\ln x = 5$

$x = e^5$ ✓

(b) $\ln(4x - 2) = -1$

$4x - 2 = e^{-1}$ ✓
$\Rightarrow x = \frac{1}{4}\left(2 + \frac{1}{e}\right)$ ✓

(c) $2(\ln x)^2 - 5\ln x + 2 = 0$

$(2\ln x - 1)(\ln x - 2) = 0$ ✓
$\Rightarrow \ln x = \frac{1}{2}$ or $\ln x = 2$ ✓
Hence, $x = e^{\frac{1}{2}}$ or e^2 ✓✓

2. [5 marks: 2, 3]

Solve *exactly* for x:

(a) $e^{0.05x} = 20$

$0.05x = \ln 20$ ✓
$\Rightarrow x = 20 \ln 20$ ✓

(b) $100\, e^{-0.02x} = 40$

$e^{-0.02x} = \frac{2}{5}$ ✓
$\Rightarrow -0.02x = \ln \frac{2}{5}$
Hence, $x = -50 \ln \frac{2}{5}$ ✓✓

Calculator Free

3. [4 marks]

Solve exactly for x in the equation $400\,e^{0.03x} = 500\,e^{0.01x}$

$$\frac{e^{0.03x}}{e^{0.01x}} = \frac{500}{400} \quad \checkmark$$

$$e^{0.02x} = \frac{5}{4} \quad \checkmark$$

$$0.02x = \ln\frac{5}{4} \quad \checkmark$$

$$\Rightarrow\ x = 50\ln\frac{5}{4} \quad \checkmark$$

4. [11 marks: 2, 4, 5]

Solve *exactly* for t.

(a) $t\,(e^t - 2) = 0$

$t = 0$ or $e^t = 2 \Rightarrow t = 0$ or $\ln 2$ ✓✓

(b) $e^{2t} - 3e^t + 2 = 0$

Rewrite as $(e^t)^2 - 3e^t + 2 = 0$ ✓
Let $y = e^t$.
Hence, equation becomes: $y^2 - 3y + 2 = 0$
$y = 1$ or 2 ✓

Hence, $e^t = 1$ or $e^t = 2 \Rightarrow t = 0$ or $\ln 2$ ✓✓

(c) $e^t - 6e^{-t} + 1 = 0$

Rewrite as $(e^t) - \dfrac{6}{e^t} + 1 = 0$

$\times e^t \quad (e^t)^2 + e^t - 6 = 0$ ✓✓
Let $y = e^t$.
Hence, equation becomes: $y^2 + y - 6 = 0$
$y = 2$ or -3 ✓

Hence, $e^t = 2$ or $e^t = -3 \Rightarrow t = \ln 2$ (reject $e^t = -3$ as $e^t > 0$) ✓✓

Calculator Assumed

5. [7 marks: 1, 4, 2]

The number of bacteria in a controlled culture is modelled by $P = \dfrac{10\,000}{10e^{-t} + 40}$

where t is time in hours.

(a) Find the initial population size.

$$P(0) = \frac{10\,000}{10 + 40} = 200 \quad \checkmark$$

(b) Use logarithms to find *exactly* when the population size will be 240.

$$\frac{10\,000}{10e^{-t} + 40} = 240$$

$$10000 = 240\,(10e^{-t} + 40) \quad \checkmark$$

$$10e^{-t} + 40 = \frac{125}{3} \quad \checkmark$$

$$e^{-t} = \frac{1}{6} \ \Rightarrow\ t = \ln 6 \quad \checkmark\checkmark$$

(c) Describe the population size for large values of t.

As $t \to \infty$ $\quad P \to \dfrac{10\,000}{0 + 40} = 250$ ✓

Hence, for large values of t, the number of bacteria never exceeds 250. ✓

$y = 2$ or -3 ✓

Hence, $e^t = 2$ or $e^t = -3 \Rightarrow t = \ln 2$ (reject $e^t = -3$ as $e^t > 0$) ✓✓

Calculator Assumed

6. [7 marks: 2, 2, 3,]

A translucent plastic sheet reduces the intensity of light that passes through it. The intensity of light after passing through x identical sheets placed adjacent to each other is given by $I = I_0\, e^{kx}$.

(a) Each plastic sheet reduces the intensity of light passing through it by 60%.
(i) Find the value of k correct to four significant figures.

$0.4I_0 = I_0 e^k$ ✓
$k = \ln 0.4 = -0.9163$ ✓

(ii) How many sheets would be required to reduce the intensity of light passing through these sheets by at least 99%?

$0.01I_0 < I_0 e^{-0.9163x}$ ✓
$x > 5.026$
Hence, at least 6 sheets. ✓

(b) Five sheets are required to reduce the light intensity by 20%. Find the percentage reduction of light intensity by one sheet.

$0.8I_0 = I_0 e^{5k}$ ✓
$k = -0.04463$
When $x = 1$, $I = I_0 e^{-0.04463}$ ✓
$= 0.9564I_0$
Hence, % reduction per sheet = 4.4% ✓

...ence, for large values of t, the number of bacteria never exceeds 250. ✓

04 Derivatives: First Principles

Calculator Free

1. [3 marks]

Use first principles to determine the derivative of $y = 5x^2$ with respect to x.

Let $f(x) = 5x^2 = 5 \times x^2$

$$\frac{dy}{dx} = 5 \times \lim_{h\to 0}\left[\frac{(x+h)^2 - x^2}{h}\right] \quad ✓$$

$$= 5 \times \lim_{h\to 0}\left[\frac{x^2 + 2xh + h^2 - x^2}{h}\right] \quad ✓$$

$$= 5 \times 2x = 10x \quad ✓$$

2. [4 marks]

Use first principles to determine the derivative of $y = \frac{1}{x^2}$ with respect to x.

Let $f(x) = \frac{1}{x^2}$.

$$\frac{dy}{dx} = \lim_{h\to 0}\left[\frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\left(\frac{x^2-(x+h)^2}{x^2(x+h)^2}\right)}{h}\right] \quad ✓$$

$$= \lim_{h\to 0}\left[\frac{\left(\frac{-2xh-h^2}{x^2(x+h)^2}\right)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{-2xh-h^2}{x^2(x+h)^2 h}\right] \quad ✓$$

$$= \lim_{h\to 0}\left[\frac{-2x-h}{x^2(x+h)^2}\right] \quad ✓$$

$$= \frac{-2x}{x^4} = \frac{-2}{x^3} \quad ✓$$

Calculator Free

3. [2 marks]

Use an appropriate derivative to evaluate $\lim\limits_{h \to 0}\left[\dfrac{\sqrt{2(x+h)}-\sqrt{2x}}{h}\right]$.

$$\lim_{h \to 0}\left[\frac{\sqrt{2(x+h)}-\sqrt{2x}}{h}\right] = \frac{d}{dx}\left(\sqrt{2x}\right) = \frac{1}{\sqrt{2x}} \qquad ✓✓$$

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4. [3 marks]

Evaluate $\lim\limits_{h \to 0}\left[\dfrac{\ln(3+h)-\ln 3}{h}\right]$ giving your answer in exact form.
Show clearly how you obtained your answer.

$$\lim_{h \to 0}\left[\frac{\ln(3+h)-\ln 3}{h}\right] = \frac{d}{dx}\ln x\Big|_{x=3} \qquad ✓$$
$$= \frac{1}{x}\Big|_{x=3} \qquad ✓$$
$$= \frac{1}{3} \qquad ✓$$

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5. [4 marks]

Evaluate $\lim\limits_{h \to 0}\left[\dfrac{\left(1+\sqrt{5+h}\right)^2-\left(1+\sqrt{5}\right)^2}{h}\right]$ giving your answer in exact form.
Show clearly how you obtained your answer.

$$\lim_{h \to 0}\left[\frac{\left(1+\sqrt{5+h}\right)^2-\left(1+\sqrt{5}\right)^2}{h}\right] = \frac{d}{dx}\left(1+\sqrt{x}\right)^2\Big|_{x=5} \qquad ✓$$
$$= 2\left(1+\sqrt{x}\right)\times\frac{1}{2\sqrt{x}}\Big|_{x=5} \qquad ✓$$
$$= \left(1+\sqrt{5}\right)\times\frac{1}{\sqrt{5}} \qquad ✓$$
$$= \frac{1+\sqrt{5}}{\sqrt{5}} \qquad ✓$$

Calculator Free

2. [2 marks]

05 Differentiation I (Chain Rule)

Calculator Free

1. [10 marks: 1, 2, 2, 2, 3]

Differentiate with respect to x. Leave answers with positive indices.

(a) $4x^3 + 2x^2 - 3x + 5$

Derivative $= 12x^2 + 4x - 3$ ✓

(b) $\dfrac{2}{3x^2} - x - 1$

Derivative $= \dfrac{-4}{3x^3} - 1$ ✓✓

(c) $\dfrac{-1}{2x^3} + 5\sqrt{x}$

Derivative $= \dfrac{3}{2x^4} + \dfrac{5}{2\sqrt{x}}$ ✓✓

(d) $\dfrac{1}{2\sqrt{x}} + \dfrac{5\sqrt{x}}{2}$

Derivative $= \dfrac{-1}{4x^{3/2}} + \dfrac{5}{4\sqrt{x}}$ ✓✓

(e) $\dfrac{\sqrt{x}-x^4}{3x}$

$$\frac{\sqrt{x}-x^4}{3x} = \frac{\sqrt{x}}{3x} - \frac{x^4}{3x} = \frac{x^{-1/2}}{3} - \frac{x^3}{3} \qquad ✓✓$$

Derivative $= \dfrac{-1}{6\sqrt{x^3}} - x^2$

Calculator Free

5. [15 marks: 2, 3, 3, 3, 4]

$$= \frac{1+\sqrt{5}}{\sqrt{5}} \quad ✓$$

Derivative $= \frac{-1}{6\sqrt{x^3}} - x^2$

Calculator Free

2. [2 marks]

Find the gradient of the curve $y = x^2 + 2\sqrt{x} + 1$ at the point where $x = 1$.

$y' = 2x + \frac{1}{\sqrt{x}}$ ✓

$y'(1) = 3$ ✓

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3. [5 marks]

Find the equation of the tangent to the curve $y = \frac{x^2 - x^3}{x^4}$ at the point where $x = -1$.

$y = \frac{x^2 - x^3}{x^4} = x^{-2} - x^{-1}$ ✓

$y' = \frac{-2}{x^3} + \frac{1}{x^2}$ ✓

$y'(-1) = 3$ ✓

When $x = -1$, $y = 2$.

Equation of tangent is $y - 2 = 3(x + 1)$

$y = 3x + 5$ ✓✓

---

4. [5 marks]

Find the coordinates of the point(s) on the curve $y = \frac{1}{x} + x$ with a gradient of 0.

$y' = \frac{-1}{x^2} + 1$ ✓

Gradient = 0 $\Rightarrow \frac{-1}{x^2} + 1 = 0$ ✓

$x^2 = 1$

$\Rightarrow x = -1$ or 1. ✓

Hence, $(-1, -2)$ and $(1, 2)$. ✓✓

Calculator Free

5. [15 marks: 2, 3, 3, 3, 4]

Differentiate with respect to x (do not simplify):

(a) $(1 - 2x^2)^5$

Derivative $= 5 \times (1 - 2x^2)^4 \times (-4x)$ ✓✓

(b) $\frac{1}{(2x+1)^5}$

Derivative $= (-5) \times (2x + 1)^{-6} \times (2)$ ✓ ✓ ✓

(c) $\frac{1}{(1+4x)^2} + \frac{2}{3(x^2-2)^4}$

Derivative $= (-2) \times (1 + 4x)^{-3} \times (4)$ ✓

$+ (-4) \times \frac{2}{3}(x^2 - 2)^{-5} \times 2x$ ✓✓

(d) $\frac{2}{\sqrt{1-x}} + \sqrt{(2x + 1)}$

Derivative $= \frac{-1}{2} \times 2(1-x)^{-3/2} \times (-1) + \frac{1}{2} \times (2x+1)^{-1/2} \times (2)$ ✓✓ ✓

(e) $\frac{2}{\sqrt[3]{1-3x^2}} - \frac{\sqrt{1+x^4}}{3}$

Derivative $= \frac{-1}{3} \times 2(1-3x^2)^{-4/3} \times (-6x)$ ✓✓

$- \frac{1}{2} \times \frac{1}{3}(1+x^4)^{-1/2} \times (4x^3)$ ✓✓

Calculator Free

6. [3 marks]

Find the gradient of the curve $y = (2 - \sqrt{x})^3$ at the point where $x = 4$.

$y' = 3 \times (2 - \sqrt{x})^2 \times \left(\frac{-1}{2\sqrt{x}}\right)$ ✓✓

$y'(4) = 0$ ✓

7. [5 marks]

Find the equation of the tangent to the curve $y = \left(1 - \frac{1}{x}\right)^3$ at the point where $x = -1$.

$y' = 3 \times \left(1 - \frac{1}{x}\right)^2 \times \left(\frac{1}{x^2}\right)$ ✓✓

$y'(-1) = 12$ ✓

When $x = -1$, $y = 8$.

Equation of tangent is $y - 8 = 12(x + 1)$

$y = 12x + 20.$ ✓✓

8. [7 marks]

The gradient of the curve with equation $y = \dfrac{1}{ax^2 + bx + 5}$ at the point $(1, \frac{1}{3})$ is 0.

Find a and b.

Rewrite equation as $y = (ax^2 + bx + 5)^{-1}$.

Gradient function $y' = -(ax^2 + bx + 5)^{-2} \times (2ax + b)$

$= \dfrac{-(2ax + b)}{(ax^2 + bx + 5)^2}$ ✓

On the curve, when $x = 1$, $y = \frac{1}{3}$. $\Rightarrow \frac{1}{3} = \frac{1}{a + b + 5}$ ✓

$a + b + 5 = 3 \Rightarrow a + b = -2$ (I) ✓

Gradient at $x = 1$ is 0. $\Rightarrow 0 = \dfrac{-(2a + b)}{(a + b + 5)^2}$ ✓

$2a + b = 0$ (II) ✓

Solve I and II simultaneously, $a = 2$, $b = -4$. ✓✓

Calculator Assumed

9. [7 marks: 4, 3] [TISC]

Given that $y = \dfrac{4}{(1+x)^3}$ where $x = f(u)$ and $\dfrac{dx}{du} = -1$ for all values of u.

(a) Use the chain rule to determine $\dfrac{dy}{du}$ when $y = \dfrac{1}{2}$.

$\dfrac{dy}{dx} = \dfrac{-12}{(1+x)^4}$ ✓

But:

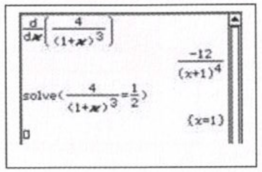
$\dfrac{dy}{du} = \dfrac{dy}{dx} \times \dfrac{dx}{du}$

$= \dfrac{-12}{(1+x)^4} \times \dfrac{dx}{du}$ ✓

When $y = \frac{1}{2}$, $x = 1$. ✓

Hence, $\dfrac{dy}{du} = \dfrac{-12}{(1+1)^4} \times -1$

$= \dfrac{3}{4}$ ✓



(b) If $u = g(t)$, use the chain rule to determine $\dfrac{dy}{dt}$ when $x = 0$ and $\dfrac{du}{dt} = 2$.

$\dfrac{dy}{dt} = \dfrac{dy}{du} \times \dfrac{du}{dt}$ ✓

When $x = 0$, $\dfrac{dy}{du} = 12.$ ✓

Hence, $\dfrac{dy}{dt} = 12 \times 2 = 24$ ✓

OR

$\dfrac{dy}{dt} = \dfrac{dy}{dx} \times \dfrac{dx}{du} \times \dfrac{du}{dt}$ ✓

$= \dfrac{-12}{(1+x)^4} \times -1 \times \dfrac{du}{dt}$ ✓

When $x = 0$:

$\dfrac{dy}{dt} = \dfrac{-12}{(1+0)^4} \times -1 \times 2 = 24$ ✓

06 Differentiation II (Product & Quotient Rules)

06 Differentiation II (Product & Quotient Rules)

Calculator Free

1. [12 marks: 2, 3, 3, 4]

Differentiate with respect to x (do not simplify):

(a) $(1 + x^2)(1 - 2x)$

Derivative $= 2x(1 - 2x) + (1 + x^2)(-2)$ ✓✓

(b) $2x^3(1 - x^2)^4$

Derivative $= 6x^2(1 - x^2)^4 + 2x^3 \times 4(1 - x^2)^3 \times (-2x)$
✓ ✓ ✓

(c) $\sqrt{2x}\,(1 + x)^2$

Derivative $= \dfrac{\sqrt{2}}{2\sqrt{x}} \times (1 + x)^2 + \sqrt{2x} \times 2 \times (1 + x)$
✓ ✓ ✓

(d) $x^3\sqrt{2 + 3x^2}$

Derivative $= 3x^2 \times \sqrt{2 + 3x^2} + x^3 \times \dfrac{1}{2} \times \dfrac{6x}{\sqrt{2 + 3x^2}}$
✓ ✓✓ ✓

Calculator Free

2. [12 marks: 2, 3, 3, 4]

Differentiate with respect to t (do not simplify):

(a) $\dfrac{2 - t}{(1 + 2t)}$

Derivative $= \dfrac{(-1)(1 + 2t) - (2 - t)(2)}{(1 + 2t)^2}$ ✓✓

(b) $\dfrac{1 - 4t}{(1 - 2t)^2}$

Derivative $= \dfrac{(-4)(1 - 2t)^2 - (1 - 4t) \times 2(1 - 2t) \times (-2)}{(1 - 2t)^4}$
✓ ✓ ✓

(c) $\dfrac{-t}{\sqrt{2t - 3}}$

Derivative $= \dfrac{-1 \times \sqrt{2t - 3} - (-t) \times \frac{1}{2}(2t - 3)^{-\frac{1}{2}} \times 2}{2t - 3}$
✓ ✓ ✓

(d) $\sqrt{\dfrac{1 + 2t}{1 - 2t}}$

$\sqrt{\dfrac{1 + 2t}{1 - 2t}} = \dfrac{(1 + 2t)^{\frac{1}{2}}}{(1 - 2t)^{\frac{1}{2}}}$ ✓

Derivative

$= \dfrac{\frac{1}{2} \times 2 \times (1 + 2t)^{-\frac{1}{2}}(1 - 2t)^{\frac{1}{2}} - (1 + 2t)^{\frac{1}{2}} \times \frac{1}{2} \times (-2) \times (1 - 2t)^{-\frac{1}{2}}}{(1 - 2t)}$
✓ ✓ ✓

Calculator Free

3. [4 marks]

Find the gradient of the tangent to the curve $y = x^2\sqrt{1-x}$ at the point where $x = -3$.

$$y' = 2x \times (1-x)^{\frac{1}{2}} + x^2 \times \tfrac{1}{2} \times (1-x)^{-\frac{1}{2}} \times (-1) \quad ✓✓✓$$

$$y'(-3) = \frac{-57}{4} \quad ✓$$

---

4. [5 marks]

Find the equation of the tangent to the curve $y = 2x\,(1 + \sqrt{x})^3$ at the point where $x = 1$.

$$y' = 2 \times (1 + \sqrt{x})^3 + 2x \times 3 \times (1 + \sqrt{x})^2 \times \frac{1}{2\sqrt{x}} \quad ✓✓$$

$y'(1) = 28$ ✓

When $x = 1$, $y = 16$.

Equation of tangent is $y - 16 = 28(x - 1)$

$y = 28x - 12$ ✓✓

---

5. [5 marks]

A curve has equation $y = (x^2 - 1)(1 + x)^3$. Show that $y' = (5x - 3)(1 + x)^3$.
Hence, find the x-coordinates of the point(s) where the gradient of the curve is 0.

$$\begin{aligned} y' &= 2x\,(1 + x)^3 + (x^2 - 1) \times 3\,(1 + x)^2 && ✓ \\ &= 2x\,(1 + x)^3 + 3(x^2 - 1)(1 + x)^2 \\ &= (1 + x)^2\,[2x\,(1 + x) + 3\,(x^2 - 1)] && ✓ \\ &= (1 + x)^2\,[5x^2 + 2x - 3] \\ &= (5x - 3)(1 + x)^3. \end{aligned}$$

When $y' = 0$; $\quad (5x - 3)(1 + x)^3 = 0$ ✓

$x = -1, \frac{3}{5}$ ✓✓

Calculator Free

6. [10 marks: 2, 2, 6]

A curve has equation $y = (ax + b)(x - 1)^2$. The curve has a turning point at $(-1, -16)$.

(a) Show that $a - b = 4$.

When $x = -1$, $y = -16$.

$\Rightarrow -16 = (-a + b) \times 4$ ✓

$a - b = 4$ ✓

(b) Find the gradient function in terms of a and b.

$$y' = a \times (x - 1)^2 + 2\,(ax + b) \times (x - 1) \quad ✓✓$$

(c) Find a and b.

Since $(-1, -16)$ is a turning point,
when $x = -1$, $y' = 0$. ✓

$\Rightarrow a \times (-1 - 1)^2 + 2\,(-a + b) \times (-1 - 1) = 0$ ✓

$4a - 4\,(-a + b) = 0$

$\Rightarrow b = 2a$ ✓

But from (a), $a - b = 4$.

Hence, $a - 2a = 4$ ✓

$a = -4$ ✓

$b = -8$ ✓

Calculator Free

07 Differentiation III (Graphs)

Calculator Free

7. [7 marks]

Find the coordinates of the point(s) on the curve $y = \dfrac{2x}{1-x}$ at which the tangents are parallel to the line $2y = x - 1$.

$$y' = \frac{2(1-x) - 2x(-1)}{(1-x)^2} \quad ✓$$
$$= \frac{2}{(1-x)^2} \quad ✓$$

Line $2y = x - 1$ has gradient $\frac{1}{2}$. ✓

Hence, $\dfrac{2}{(1-x)^2} = \dfrac{1}{2}$ ✓

$\Rightarrow$ $(1-x)^2 = 4$

$x = -1$ or 3 ✓

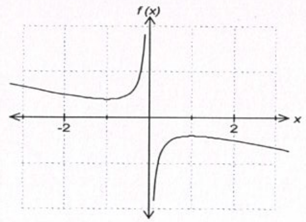
Therefore, $(-1, -1)$ and $(3, -3)$. ✓✓

07 Differentiation III (Graphs)

Calculator Free

1. [4 marks: 2, 2]

The graph of $y = f(x)$ is given below.



(a) Find the x-coordinate of the point(s) where the gradient of the curve is 0.

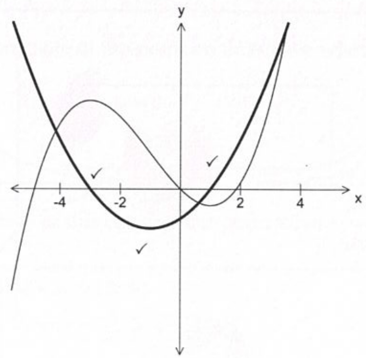
$x = -1, 1$ ✓✓

(b) For what values of x is the gradient of the curve negative?

$x < -1, x > 1$ ✓✓

2. [5 marks: 2, 3]

The graph of $y = f(x)$ is given below.



(a) For what values of x is the gradient positive?

$x < -3, x > 1$ ✓✓

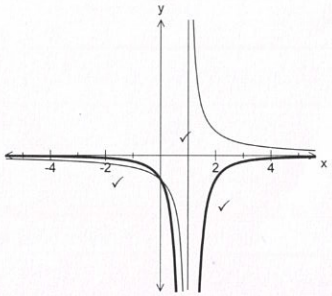
(b) Sketch on the same axes, a possible graph of $y = f'(x)$.

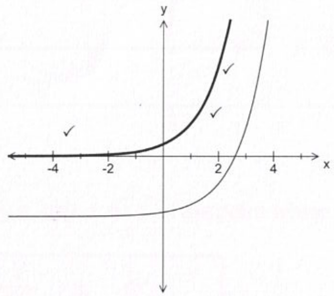
Calculator Free

3. [6 marks: 3, 3]

The graph of $y = f(x)$ is given below. Sketch on the same axes, a possible graph of $y = f'(x)$.

(a) (b)



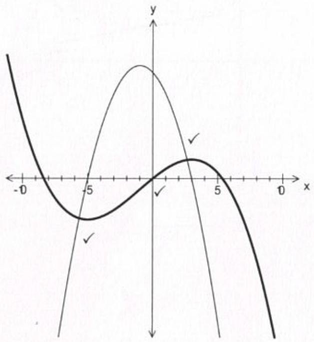


4. [6 marks: 2, 1, 3]

The graph of $y = f'(x)$ is given below.

(a) State the x-coordinate of the point(s) where the gradient of $y = f(x)$ is zero.

$x = -5, 3$ ✓✓



(b) State the x-coordinate of the minimum turning point on $y = f(x)$.

$x = -5$ ✓

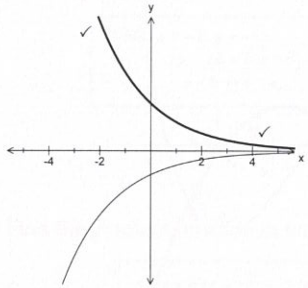
(c) Sketch on the same axes a possible graph of $y = f(x)$.

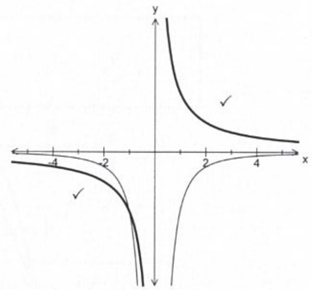
Calculator Free

5. [4 marks: 2, 2]

The graph of $y = f'(x)$ is given below.
Sketch on the same axes, a possible graph of $y = f(x)$.

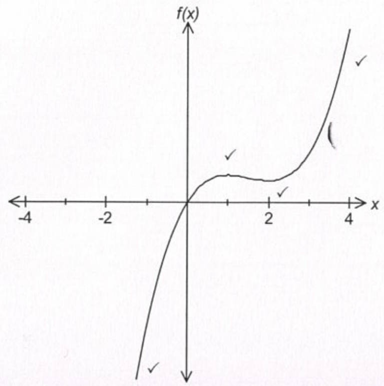
(a) (b)





6. [4 marks]

The curve $y = f(x)$ cuts the x-axis at the origin and nowhere else. $\frac{dy}{dx} = 0$ at $x = 1$ and $x = 2$. $\frac{dy}{dx} < 0$ only for $1 < x < 2$. Give a possible sketch of $y = f(x)$.



08 Differentiation IV (Miscellaneous)

Calculator Assumed

08 Differentiation IV (Miscellaneous)

Calculator Free

1. [9 marks: 1, 2, 3, 3]

Differentiate with respect to x. Do not simplify your answer.

(a) $y = (1 + 2e)^3$

$y' = 0$ ✓

(b) $y = \dfrac{3}{2\sqrt{x+2}}$

$y' = \frac{3}{2} \times \frac{-1}{2} \times (x+2)^{-3/2}$ ✓✓

(c) $y = (1-x)^3\left(1+\dfrac{2}{x}\right)^2$

$y' = \left(3 \times (1-x)^2 \times (-1)\right)\left(1+\frac{2}{x}\right)^2 + (1-x)^3\left(2 \times \frac{-2}{x^2} \times \left(1+\frac{2}{x}\right)^1\right)$
✓ ✓ ✓

(d) $y = \dfrac{1+\sqrt{x}}{1-\sqrt{2x}}$

$y' = \dfrac{(1-\sqrt{2x})(\frac{1}{2\sqrt{x}}) - (1+\sqrt{x})\left(\frac{-\sqrt{2}}{2\sqrt{x}}\right)}{(1-\sqrt{2x})^2}$ ✓✓✓

Calculator Assumed

2. [11 marks: 5, 2, 2, 2]

A curve has equation $y = x^3 + 5x^2 + 3x + 2$.

(a) Use calculus to find all points on the curve where the gradient is 0.

$y' = 3x^2 + 10x + 3$ ✓
When $y' = 0$, $3x^2 + 10x + 3 = 0$ ✓
$(3x + 1)(x + 3) = 0$
$x = \frac{-1}{3}, -3$. ✓

Hence, $(\frac{-1}{3}, \frac{41}{27})$ and $(-3, 11)$. ✓✓

(b) Find $\dfrac{d^2y}{dx^2}$.

$y'' = 6x + 10$ ✓✓

(c) Find the x-coordinate of the point on this curve where $\dfrac{d^2y}{dx^2} = 0$.

$6x + 10 = 0$ ✓
$x = \frac{-5}{3}$ ✓

(d) Find the gradient of this curve at the point where $\dfrac{d^2y}{dx^2} = 0$.

$y' = 3x^2 + 10x + 3$
When $x = \frac{-5}{3}$, $y' = \frac{-16}{3}$. ✓✓

Calculator Assumed

3. [4 marks]

Use calculus to find the equation of the tangent to the curve $y = x^3\left(1+\frac{1}{\sqrt{x}}\right)^2$ at the point (1, 4).

$$y' = 3x^2\left(1+\frac{1}{\sqrt{x}}\right)^2 + x^3\left[2\times\frac{-1}{2x^{3/2}}\times\left(1+\frac{1}{\sqrt{x}}\right)\right] \quad ✓✓$$

$y'(1) = 10$ ✓

Hence, equation of tangent is $y - 4 = 10(x-1)$

$y = 10x - 6.$ ✓

4. [6 marks: 2, 4]

A curve has equation $y = (x+1)^2(x-2)$.

(a) Find the x-intercepts of the curve.

$x = -1, 2$ ✓✓

(b) Use derivatives to find the equation of the tangents to this curve at each of the x-intercepts

$y' = 2(x+1)(x-2) + (x+1)^2$ ✓

At $x = -1$:

$y'(-1) = 0.$ $\Rightarrow$ Equation of tangent is $y = 0$. ✓

At $x = 2$:

$y'(2) = 9.$ ✓

Hence, equation of tangent is $y - 0 = 9(x-2)$

$y = 9x - 18.$ ✓

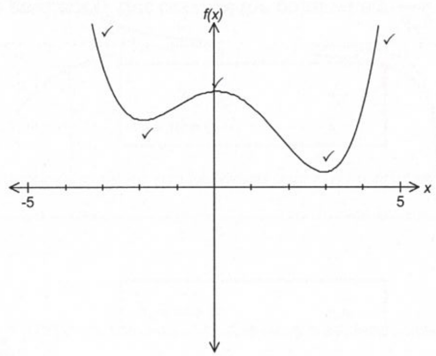
Calculator Assumed

5. [5 marks]

The graph of $y = f(x)$ has the following properties:

- $y \geq 0$ for all x
- $\frac{dy}{dx} = 0$ for $x = -2, 0, 3$
- $\frac{dy}{dx} \geq 0$ for $-2 \leq x \leq 0$ and $x \geq 3$

Sketch a possible graph for $y = f(x)$.



Calculator Assumed

09 Differentiation of Exponential Functions

Calculator Assumed

6. [8 marks]

The graph of $y = \frac{ax+b}{cx+d}$,where a, b, c and d are non-zero constants, intersects the x-axis at the point where $x = \frac{-1}{2}$. It also intersects the y-axis at the point where $y = 1$. At the point it intersects the y-axis, it is also parallel to the line $y = 3x + 10$. Find the values of a, b, c and d.

$y = \frac{ax+b}{cx+d}$

When $y = 0$, $x = \frac{-1}{2}$.

$\Rightarrow \frac{-a}{2} + b = 0 \quad \Rightarrow a = 2b$ ✓✓

When $x = 0$, $y = 1$.

$\Rightarrow \frac{b}{d} = 1 \quad \Rightarrow d = b$ ✓

Hence, $y = \frac{2bx+b}{cx+b}$.

$y' = \frac{(2b)(cx+b) - (2bx+b)(c)}{(cx+b)^2}$ ✓

When $x = 0$, $y' = 3$.

$\Rightarrow \frac{2b^2 - bc}{b^2} = 3 \quad \Rightarrow b(b+c) = 0$ ✓

$\Rightarrow c = -b$ (Reject $b = 0$) ✓

Hence, $y = \frac{2bx+b}{-bx+b} = \frac{2x+1}{-x+1}$

Therefore, $a = 2k$, $b = k$, $c = -k$ and $d = k$ where k is a real number. ✓✓

09 Differentiation of Exponential Functions

Calculator Free

1. [9 marks: 1, 2, 2, 2, 2]

Find $\frac{dy}{dx}$.

(a) $y = 5e^{3x}$

$\frac{dy}{dx} = 15e^{3x}$ ✓

(b) $y = \frac{4}{5e^x}$

$y = \frac{4}{5}e^{-x}$ ✓

$\frac{dy}{dx} = \frac{-4}{5}e^{-x}$ ✓

(c) $y = e^x + \frac{1}{2e^{2x}}$

$y = e^x + \frac{1}{2}e^{-2x}$ ✓

$\frac{dy}{dx} = e^x + -e^{-2x}$ ✓

(d) $y = \frac{e^{2x} + e^x}{e^{3x}}$

$y = e^{-x} + e^{-2x}$ ✓

$\frac{dy}{dx} = -e^{-x} - 2e^{-2x}$ ✓

(e) $y = e^{2x^2 + 3x}$

$\frac{dy}{dx} = (4x+3)e^{2x^2+3x}$ ✓✓

Calculator Free

2. [11 marks: 2, 3, 3, 3]

Find $\dfrac{dy}{dx}$.

(a) $y = (e^2 + e^{2x})^3$

$$\frac{dy}{dx} = 3\,(e^2 + e^{2x})^2 \times 2\,e^{2x} \quad ✓✓$$

(b) $y = x^4 e^{x^2}$

$$\frac{dy}{dx} = \underset{✓}{4x^3 e^{x^2}} + \underset{✓}{x^4 \times} \underset{✓}{\left(2x\, e^{x^2}\right)}$$

(c) $y = \dfrac{e^{2x}}{1+x^2}$

$$\frac{dy}{dx} = \frac{(1+x^2)(2e^{2x}) - 2xe^{2x}}{(1+x^2)^2} \quad ✓ ✓ ✓$$

(d) $y = \dfrac{x^2 - e^{2x}}{2e^x}$

$$\frac{dy}{dx} = \frac{2e^x(2x - 2e^{2x}) - 2e^x(x^2 - e^{2x})}{(2e^x)^2} \quad ✓ ✓ ✓$$

Calculator Free

3. [4 marks]

Find the equation of the tangent to the curve $y = -x\,e^{2x}$ at $x = \dfrac{-1}{2}$.

$\dfrac{dy}{dx} = -e^{2x} + -2x\,e^{2x}$ ✓

When $x = \dfrac{-1}{2}$, $y = \dfrac{1}{2e}$ and $\dfrac{dy}{dx} = 0$. ✓✓

Hence, equation of tangent is $y = \dfrac{1}{2e}$. ✓

4. [5 marks]

Given that $y = x^2 e^x$, prove that $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2e^x(1 + x) = 0$

$y = x^2 e^x$

$\dfrac{dy}{dx} = 2x\,e^x + x^2 e^x$ ✓

$\dfrac{d^2y}{dx^2} = (2e^x + 2x\,e^x) + (2x\,e^x + x^2 e^x)$ ✓

$= 4x\,e^x + x^2 e^x + 2e^x$ ✓

LHS $= (4x\,e^x + x^2 e^x + 2e^x) - (2x\,e^x + x^2 e^x) - 2e^x(1 + x)$ ✓✓

$= 0$

$=$ RHS

10 Differentiation of Logarithmic Functions

Calculator Free

10 Differentiation of Logarithmic Functions

Calculator Free

1. [12 marks: 2, 2, 2, 2, 2, 2]

Differentiate with respect to x:

(a) $y = \ln(1 + x^3)$

$$y' = \frac{3x^2}{1+x^3} \quad ✓✓$$

(b) $y = \ln x^4$

Rewrite as $y = 4 \ln x$

$$y' = \frac{4}{x} \quad ✓✓$$

(c) $y = \ln(1 - 2x)^3$

Rewrite as $y = 3 \ln(1 - 2x)$

$$y' = 3\left[\frac{-2}{1-2x}\right] \quad ✓✓$$

$$= \frac{-6}{1-2x}$$

(d) $y = \ln\sqrt{x^2 + 2x}$

Rewrite as $y = ½ \ln(x^2 + 2x)$

$$y' = \frac{1}{2}\left[\frac{2x+2}{x^2+2x}\right] \quad ✓✓$$

$$= \frac{x+1}{x^2+2x}$$

(e) $y = \ln(1 + e^{2x})$

$$y' = \frac{2e^{2x}}{1+e^{2x}} \quad ✓✓$$

(f) $y = \log_2(1+x)$

Rewrite as $y = \dfrac{\ln(1+x)}{\ln 2}$ ✓

$$y' = \frac{1}{\ln 2\,(1+x)} \quad ✓$$

Calculator Free

2. [9 marks: 3, 3, 3]

Differentiate with respect to x:

(a) $y = \ln[(x + 1)^2(2x - 1)^3]$

Rewrite as $y = \ln(x + 1)^2 + \ln(2x - 1)^3$

$$y = 2\ln(x+1) + 3\ln(2x-1) \quad ✓$$

$$y' = \frac{2}{x+1} + 3\left[\frac{2}{2x-1}\right] \quad ✓✓$$

$$= \frac{2}{x+1} + \frac{6}{2x-1}$$

(b) $y = \ln\left[\dfrac{1+2x}{1+x^2}\right]$

Rewrite as $y = \ln(1 + 2x) - \ln(1 + x^2)$ ✓

$$y' = \frac{2}{1+2x} - \frac{2x}{1+x^2} \quad ✓✓$$

(c) $y = \ln[x^2\sqrt{(3 - 2x)}]$

Rewrite as $y = \ln x^2 + \ln\sqrt{(3 - 2x)}$

$$y = 2\ln x + \frac{1}{2}\ln(3-2x) \quad ✓$$

$$y' = \frac{2}{x} + \frac{1}{2}\left[\frac{-2}{3-2x}\right] \quad ✓✓$$

$$= \frac{2}{x} - \frac{1}{3-2x}$$

Calculator Free

3. [9 marks: 3, 3, 3]

Differentiate with respect to x:

(a) $y = x^2 \ln(1 - x^3)$

$$y' = 2x \ln(1 - x^3) + x^2 \left[\frac{-3x^2}{1 - x^3}\right] \quad \checkmark\checkmark\checkmark$$
$$= 2x \ln(1 - x^3) - \frac{3x^4}{1 - x^3}$$

(b) $y = (1 + e^x) \ln(x^2 + x - 1)$

$$y' = e^x \ln(x^2 + x - 1) + (1 + e^x)\left[\frac{2x + 1}{x^2 + x - 1}\right] \quad \checkmark\checkmark\checkmark$$

(c) $y = \dfrac{\ln(x)}{x}$

$$y' = \frac{x\left(\frac{1}{x}\right) - \ln x}{x^2} \quad \checkmark\checkmark\checkmark$$
$$= \frac{1 - \ln x}{x^2}$$

Calculator Free

4. [2 marks]

Find the gradient of the curve $y = \ln(2x - 5)$ at the point where $x = 3$.

Gradient function $y' = \dfrac{2}{2x - 5}$ ✓

When $x = 3$, $y' = 2$. ✓

5. [4 marks]

Find the coordinates of the point on $y = \ln(1 + 2x)$ where the gradient of the curve is 2.

Gradient function $y' = \dfrac{2}{1 + 2x}$. ✓

Gradient of curve $= 2 \Rightarrow y' = 2$.

Hence, $\dfrac{2}{1 + 2x} = 2 \Rightarrow 1 + 2x = 1 \Rightarrow x = 0$ ✓✓

When $x = 0$, $y = 0$. Hence, the required point is $(0, 0)$. ✓

6. [6 marks]

Find the coordinates of the point(s) on the curve $y = x^2 \ln(x)$ where the gradient of the curve is zero.

Gradient function $y' = 2x \ln x + x^2 \times \dfrac{1}{x}$ ✓

$= 2x \ln x + x$

$= x(2 \ln x + 1)$ ✓

When gradient $= 0$, $y' = 0$.

$\Rightarrow x = 0$ or $\ln x = -\dfrac{1}{2}$ ✓✓

Reject $x = 0$ as $x > 0$. Hence, $x = e^{-\frac{1}{2}}$ ✓

Hence, required point is $\left(e^{-\frac{1}{2}}, -\dfrac{1}{2e}\right)$. ✓

11 Differentiation of Trigonometric Functions

Calculator Free

1. [12 marks: 6, 6]

Use first principles and the results $\lim\limits_{t\to 0}\left(\dfrac{\sin t}{t}\right)$ and/or $\lim\limits_{t\to 0}\left(\dfrac{1-\cos t}{t}\right)$ to determine the derivative with respect to x for:

(a) $f(x) = \sin 2x$

$$f'(x) = \lim_{h\to 0}\left[\frac{\sin 2(x+h) - \sin 2x}{h}\right] \quad ✓$$

$$= \lim_{h\to 0}\left[\frac{\sin 2x\cos 2h \;+\; \cos 2x\sin 2h \;-\; \sin 2x}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\sin 2x(\cos 2h - 1) \;+\; \cos 2x\sin 2h}{h}\right] \quad ✓$$

$$= \lim_{h\to 0}\left[\sin 2x\left(\frac{\cos 2h - 1}{h}\right) \;+\; \cos 2x\left(\frac{\sin 2h}{h}\right)\right] \quad ✓$$

$$= \sin 2x \times \lim_{h\to 0} 2\left(\frac{\cos 2h - 1}{2h}\right) + \cos 2x \lim_{h\to 0} 2\left(\frac{\sin 2h}{2h}\right) \quad ✓✓$$

$$= 2\cos 2x \quad ✓$$

(b) $f(x) = \cos 0.5x$.

$$f'(x) = \lim_{h\to 0}\left[\frac{\cos 0.5(x+h) - \cos 0.5x}{h}\right] \quad ✓$$

$$= \lim_{h\to 0}\left[\frac{\cos 0.5x\cos 0.5h \;-\; \sin 0.5x\sin 0.5h \;-\; \cos 0.5x}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\cos 0.5x(\cos 0.5h - 1) \;-\; \sin 0.5x\sin 0.5h}{h}\right] \quad ✓$$

$$= \lim_{h\to 0}\left[\cos 0.5x\left(\frac{\cos 0.5h - 1}{h}\right) \;-\; \sin 0.5x\left(\frac{\sin 0.5h}{h}\right)\right] \quad ✓$$

$$= \cos 0.5x \times \lim_{h\to 0} 0.5\left(\frac{\cos 0.5h - 1}{0.5h}\right) - \sin 0.5x \lim_{h\to 0} 0.5\left(\frac{\sin 0.5h}{0.5h}\right) \quad ✓✓$$

$$= -0.5\sin 0.5x \quad ✓$$

Calculator Free

2. [12 marks: 1, 2, 2, 3, 2, 2]

Find $\dfrac{dy}{dx}$ for each of the following. You do not need to simplify your answer.

(a) $y = \cos\dfrac{\pi}{4}$

$$\frac{dy}{dx} = 0 \quad ✓$$

(b) $y = \sin(1+2x)$

$$\frac{dy}{dx} = 2\cos(1+2x) \quad ✓✓$$

(c) $y = \sin\left(1 + \dfrac{1}{x}\right)$

$$\frac{dy}{dx} = \left(-\frac{1}{x^2}\right)\cos\left(1+\frac{1}{x}\right) \quad ✓✓$$

(d) $y = \cos(1-2x)^3$

$$\frac{dy}{dx} = -\sin(1-2x)^3 \times 3(1-2x)^2 \times (-2) \quad ✓✓✓$$

(e) $y = \tan x^2$

$$\frac{dy}{dx} = 2x\sec^2 x^2 \;\text{ or }\; \frac{2x}{\cos^2(x^2)} \quad ✓✓$$

(f) $y = \dfrac{\sin x + \cos x}{\cos x}$

Rewrite $y = \tan x + 1$ ✓

$$\frac{dy}{dx} = \sec^2 x \;\text{ or }\; \frac{1}{\cos^2(x)} \quad ✓$$

Calculator Free

3. [13 marks: 2, 3, 3, 3, 2]

Find $\frac{dy}{dt}$ for each of the following. You do not need to simplify your answer.

(a) $y = (1 - \cos t)^3$

$\frac{dy}{dt} = 3(1 - \cos t)^2 \times \sin t$ ✓✓

(b) $y = \tan^3(2t)$

$\frac{dy}{dt} = 3 \times [\tan^2(2t)] \times [\sec^2(2t)] \times 2$
✓ ✓ ✓

(c) $y = \sin^2(1 + \sqrt{t})$

$\frac{dy}{dt} = 2\sin(1 + \sqrt{t}) \times \cos(1 + \sqrt{t}) \times \frac{1}{2\sqrt{t}}$
✓ ✓ ✓

(d) $y = \sqrt{\cos(e^{2t})}$

✓ ✓
$\frac{dy}{dt} = \frac{-\sin(e^{2t}) \times (2e^{2t})}{2\sqrt{\cos(e^{2t})}}$
✓

(e) $y = e^{\sin 3t}$

$\frac{dy}{dt} = e^{\sin 3t} \times 3\cos 3t$
✓ ✓

Calculator Free

4. [15 marks: 2, 3, 4, 3, 3]

Find $\frac{dy}{dx}$ for each of the following. You do not need to simplify your answer.

(a) $y = x^2 \cos(1 - x)$

$\frac{dy}{dx} = 2x\cos(1 - x) + x^2 \sin(1 - x)$ ✓✓

(b) $y = x \sin x^2$

$\frac{dy}{dx} = \sin x^2 + x\cos x^2 \times 2x$
✓ ✓ ✓

(c) $y = (1 + 2x)^3 \tan(1 - \sqrt{x})$

$\frac{dy}{dx} = 6(1 + 2x)^2 \tan(1 - \sqrt{x}) + (1 + 2x)^3 [\sec^2(1 - \sqrt{x})] \times -\frac{1}{2\sqrt{x}}$
✓ ✓ ✓ ✓

(d) $y = x^2 \ln(\cos x)$

$\frac{dy}{dx} = 2x \ln(\cos x) + x^2 \times \frac{-\sin x}{\cos x}$
✓ ✓ ✓

(e) $y = \cos 2x \sin(1 + 3x)$

$\frac{dy}{dx} = -2\sin 2x \sin(1 + 3x) + \cos 2x \cos(1 + 3x) \times 3$
✓ ✓ ✓

Mathematics Methods Units 3 & 4 Revision Series 11 Differentiation of Trigonometric Functions

Calculator Free

Mathematics Methods Units 3 & 4 Revision Series 11 Differentiation of Trigonometric Functions

Calculator Free

5. [15 marks: 3, 3, 3, 3, 3]

Find $\frac{dy}{dt}$ for each of the following. You do not need to simplify your answer.

(a) $y = \dfrac{t^2}{\sin(1+2t)}$

$$\frac{dy}{dt} = \frac{2t\sin(1+2t) - 2t^2\cos(1+2t)}{\sin^2(1+2t)}$$ ✓ ✓ ✓

(b) $y = \dfrac{\cos^2 t}{t}$

$$\frac{dy}{dt} = \frac{-2t\cos t\sin t - \cos^2 t}{t^2}$$ ✓ ✓ ✓

(c) $y = \dfrac{e^t}{\tan t}$

$$\frac{dy}{dt} = \frac{e^t\tan t - e^t\sec^2 t}{\tan^2 t}$$ ✓ ✓ ✓

(d) $y = \dfrac{\sin t}{\cos^2 t}$

$$\frac{dy}{dt} = \frac{\cos^3 t - \sin t \times 2\cos t \times (-\sin t)}{\cos^4 t}$$ ✓ ✓ ✓

(e) $y = \dfrac{e^{\sin t}}{\ln \sin t}$

$$\frac{dy}{dt} = \frac{\cos t\, e^{\sin t}(\ln \sin t) - e^{\sin t} \times \frac{\cos t}{\sin t}}{(\ln \sin t)^2}$$ ✓ ✓ ✓

Calculator Free

6. [2 marks]

Find the gradient of the curve $y = \tan x$ at the point where $x = \frac{\pi}{4}$.

Gradient function $y' = \dfrac{1}{\cos^2 x}$ ✓

When $x = \frac{\pi}{4}$, $y' = 2$. ✓

7. [4 marks]

Find the coordinates of the point on $y = \sin^2 x$ for $0 \le x \le 2\pi$ where the gradient of the curve is 1.

Gradient function $y' = 2\sin x\cos x$. ✓

Gradient of curve = 1 $\Rightarrow y' = 1$.

Hence, $2\sin x\cos x = 1 \Rightarrow \sin 2x = 1$

$2x = \frac{\pi}{2}, \frac{5\pi}{2}$ ✓

$x = \frac{\pi}{4}, \frac{5\pi}{4}$ ✓

When $x = \frac{\pi}{4}$, $y = \frac{1}{2}$. When $x = = \frac{5\pi}{4}$, $y = \frac{1}{2}$.

Hence: $(\frac{\pi}{4}, \frac{1}{2})$ and $(\frac{5\pi}{4}, \frac{1}{2})$. ✓

8. [3 marks]

Find the equation of the tangent to the curve $y = x + \cos x$ at the point $(\pi, \pi - 1)$.

Gradient function $y' = 1 + \sin x$ ✓

At the point $(\pi, \pi - 1)$, $y' = 1$ ✓

Hence, equation of tangent is

$y - (\pi - 1) = (x - \pi)$

$y = x - 1$ ✓

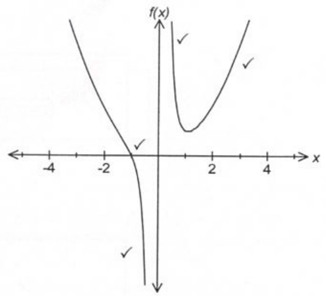
12 Stationary & Inflection Points

Calculator Free

1. [4 marks]

Sketch the graph of $y = f(x)$ given the following properties:

- $\frac{dy}{dx} < 0$ for $x < 0$ and $0 < x < 1$
- $\frac{dy}{dx} = 0$ for $x = 1$
- $\frac{dy}{dx} > 0$ for $x > 1$
- $y = \frac{d^2y}{dx^2} = 0$ for $x = -1$



2. [8 marks]

Consider the curve with equation $y = x^2 + \frac{1}{x^2}$. Use Calculus to find all the stationary points on this curve. State the nature of each stationary point.

Gradient function $y' = 2x - 2x^{-3}$ ✓
At stationary points, $y' = 0. \Rightarrow 2x - 2x^{-3} = 0$ ✓
$x = \frac{1}{x^3}$
$x^4 = 1$
Hence, $x = \pm 1.$ ✓

$y'' = 2 + 6x^{-4}$ ✓

When $x = 1$, $y = 2$ and $y'' > 0$. ✓
Hence $(1, 2)$ is a minimum point. ✓

When $x = -1$, $y = 2$ and $y'' > 0$. ✓
Hence $(-1, 2)$ is a minimum point. ✓

Calculator Free

3. [7 marks]

Use Calculus to find the exact coordinates of the turning point(s) on the curve $y = xe^{0.05x}$. State the nature of the turning point(s).

Gradient function $y' = e^{0.05x} + 0.05x\, e^{0.05x}$ ✓
$= e^{0.05x}(1 + 0.05x)$ ✓

For turning points, $y' = 0. \Rightarrow 1 + 0.05x = 0$ ✓
$x = -20$ ✓

When $x = -20$, $y = -\frac{20}{e}$. ✓

	20^-	20	20^+
y'	–	0	+

✓

Hence, $(-20, -\frac{20}{e})$ is a minimum point. ✓

4. [7 marks]

Use Calculus to determine the coordinates of the point(s) of inflection of the curve with equation $y = x(1 - x)^3$.

Gradient function $y' = (1-x)^3 + x \times 3(1-x)^2 \times -1$
$= (1-x)^2(1-4x)$ ✓

Second derivative $y'' = 2(1-x) \times (-1)(1-4x) + (1-x)^2 \times -4$
$= -2(1-x)(3-6x)$ ✓

For inflection points; $y'' = 0 \Rightarrow x = 1, \frac{1}{2}$ ✓✓

$y'' = -12x^2 + 18x - 6 \Rightarrow y''' = -24x + 18$ ✓
When $x = 1$, $y''' \neq 0$. Also, when $x = \frac{1}{2}$, $y''' \neq 0$. ✓
Hence, $(1, 0)$ and $(\frac{1}{2}, \frac{1}{16})$ are inflection points. ✓

Mathematics Methods Units 3 & 4 Revision Series 12 Stationary & Inflection Points

Mathematics Methods Units 3 & 4 Revision Series 12 Stationary & Inflection Points

Calculator Free

5. [8 marks]

Consider the curve with equation $y = ax^3 + bx^2 + cx + 1$.
The curve has a turning point at $(-1, \frac{11}{3})$. The curve also has an inflection point at $x = 2$. Find a, b and c. Show clearly how you obtained your answer.

$\frac{dy}{dx} = 3ax^2 + 2bx + c$ ✓

$\frac{d^2y}{dx^2} = 6ax + 2b$ ✓

When $x = 2$, $\frac{d^2y}{dx^2} = 0 \Rightarrow 12a + 2b = 0 \quad b = -6a$ ✓

When $x = -1$, $y = \frac{11}{3}$. $\quad -a + b - c + 1 = \frac{11}{3}$

$-7a - c = \frac{8}{3}$ I ✓

When $x = -1$, $\frac{dy}{dx} = 0 \Rightarrow 3a - 2b + c = 0$

$15a + c = 0$ II ✓

Solve I and II simultaneously: $a = \frac{1}{3}$, $c = -5$ ✓✓

$b = -2$ ✓

6. [10 marks]

A curve has equation $y = ax^3 + bx^2 + cx + d$. The curve has an inflection point at $x = -2$, a turning point at $x = 1$, a y-intercept at $(0, -33)$ and a tangent with equation $y = -24x - 37$ at $x = -1$. Find the values of a, b, c and d. Show clearly how you obtained your answer.

y-intercept at $(0, -33) \Rightarrow d = -33$ ✓

$\frac{dy}{dx} = 3ax^2 + 2bx + c$ ✓

$\frac{d^2y}{dx^2} = 6ax + 2b$ ✓

Inflection Point at $x = -2 \Rightarrow -12a + 2b = 0$ ✓

$b = 6a$

Hence, $\frac{dy}{dx} = 3ax^2 + 12ax + c$ ✓

Turning Point at $x = 1 \Rightarrow 15a + c = 0$ ✓

Gradient at $x = 1$ is $-24 \Rightarrow -9a + c = -24$ ✓

$a = 1$ ✓

$c = -15$ ✓

$b = 6$ ✓

Calculator Free

7. [8 marks: 2, 2, 4]

The points A, B, C, D, E, F and G are points on the graph of a continuous function $y = f(x)$. The table below shows the sign of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at these points.

y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ have zero values at only the points indicated in this table.

Point	A	B	C	D	E	F	G
x	–2	–1	0	1	2	3	4
y	–	0	+	+	+	0	–
dy/dx	+	+	0	–	–	0	–
d^2y/dx^2	–	–	–	0	+	0	–

(a) Identify the maximum point on this graph. Justify your answer.

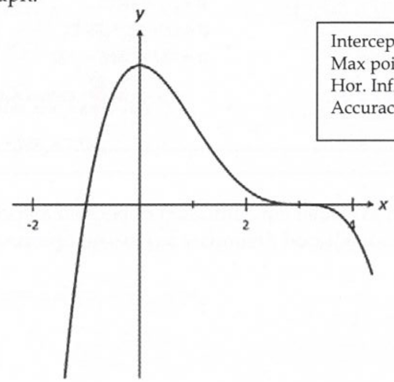
Maximum point is C. ✓

At C, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$. ✓

(b) Identify the inflection point(s) if any on this graph.

Inflection Points: D & F. ✓✓

(c) Sketch this graph.



Intercepts: ✓
Max point: ✓
Hor. Inflection Point: ✓
Accuracy: ✓

Calculator Free

8. [7 marks: 5, 2] [TISC]

Consider the curve with equation $y = \dfrac{2x^2 - 1}{1 - x^2}$. This curve has one stationary point.

(a) Find the coordinates of this stationary point.

$$\frac{dy}{dx} = \frac{(1-x^2)(4x) - (2x^2-1)(-2x)}{(1-x^2)^2} \quad ✓✓$$
$$= \frac{4x - 4x^3 + 4x^3 - 2x}{(1-x^2)^2}$$
$$= \frac{2x}{(1-x^2)^2}$$

For stationary points: $\dfrac{dy}{dx} = 0$.

$$\frac{2x}{(1-x^2)^2} = 0 \quad ✓$$
$$x = 0 \quad ✓$$

Hence, $(0, -1)$. ✓

(b) Use an appropriate test to determine if this stationary point is a minimum point, maximum point or an inflection point.

Use the sign test:

$\left.\dfrac{dy}{dx}\right|_{x=0^-} < 0$ and $\left.\dfrac{dy}{dx}\right|_{x=0^+} > 0$. ✓

Therefore, stationary point is a minimum point. ✓

Calculator Free

9. [12 marks: 5, 4, 3] [TISC]

A curve has equation $y = 3x^4 - 8x^3 + 6x^2$.

(a) Determine the coordinates of the stationary point(s) for this curve.
Use an appropriate method to determine the nature of the stationary point(s).

$\dfrac{dy}{dx} = 12x^3 - 24x^2 + 12x$ ✓

For stationary points, $\dfrac{dy}{dx} = 0$.

Hence:
$$12x^3 - 24x^2 + 12x = 0 \quad ✓$$
$$12x(x^2 - 2x + 1) = 0$$
$$12x(x-1)^2 = 0$$
$$\Rightarrow x = 0, 1 \quad ✓$$

When $x = 0$, $y = 0$.

x	0^-	0	0^+
$\frac{dy}{dx}$	$-$	0	$+$

Hence, $(0, 0)$ is a local minimum point. ✓

When $x = 1$, $y = 1$.

x	1^-	1	1^+
$\frac{dy}{dx}$	$+$	0	$+$

Hence, $(1, 1)$ is a horizontal inflection point. ✓

Calculator Free

Calculator Free

9. (b) The curve has two inflection points, one of which is located at $x = 1$. Determine the coordinates of the second inflection point.

$$\frac{d^2y}{dx^2} = 36x^2 - 48x + 12 \quad ✓$$

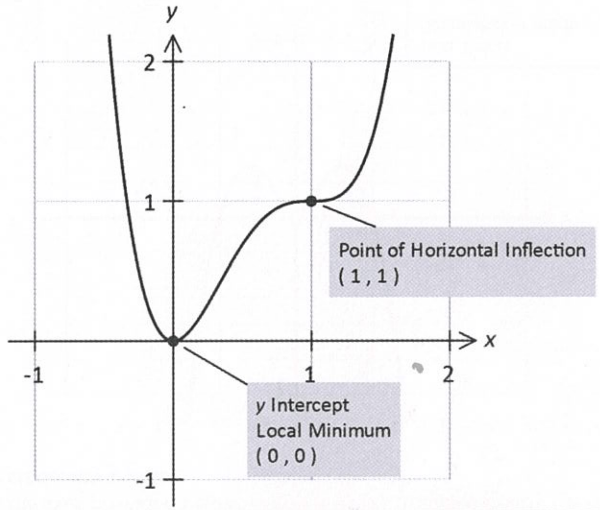
For inflection points, $\frac{d^2y}{dx^2} = 0$.

Hence:

$$36x^2 - 48x + 12 = 0 \quad ✓$$
$$12(3x^2 - 4x + 1) = 0$$
$$12(3x - 1)(x - 1) = 0$$
$$\Rightarrow x = \frac{1}{3}, 1 \quad ✓$$

Hence second inflection point is at $(\frac{1}{3}, \frac{11}{27})$. ✓

(c) The curve has an x-intercept at $x = 0$ and no other x-intercepts. On the axes provided below, sketch this curve.



Correct shape. ✓
Min, at (0, 0) ✓
Horizontal inflection at (1, 1). ✓

Calculator Assumed

10. [10 marks: 8, 2]

Given that $f(x) = 4x^2 \ln x^2$.

(a) Use calculus techniques to find the exact coordinates of the stationary and inflection point(s) on the curve $y = f(x)$.

$y = 4x^2 \ln x^2$

$$\frac{dy}{dx} = 8x \ln x^2 + 4x^2 \times \frac{2}{x} = 8x \ln x^2 + 8x \quad ✓$$

For stationary points:

$8x \ln x^2 + 8x = 0$

$$x = 0, \pm e^{-\frac{1}{2}} \quad ✓$$

But $x \neq 0$.

Hence, stationary points are $(\pm e^{-\frac{1}{2}}, -4e^{-1})$. ✓✓

$$\frac{d^2y}{dx^2} = 8 \ln x^2 + 24 \quad ✓$$

For inflection points:

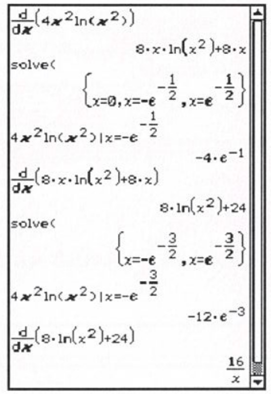
$8 \ln x^2 + 24 = 0$:

$$x = \pm e^{-\frac{3}{2}} \quad ✓$$

$$\frac{d^3y}{dx^3} = \frac{16}{x}$$

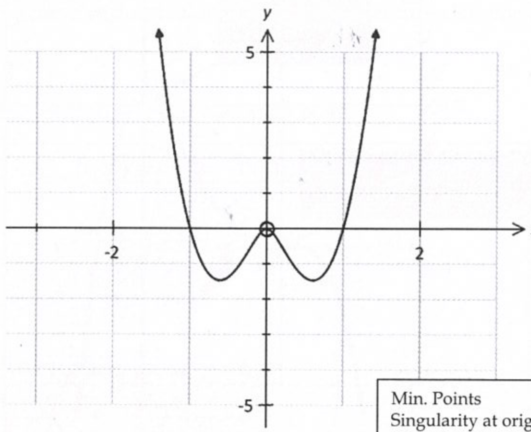
$$x = \pm e^{-\frac{3}{2}}, \frac{d^3y}{dx^3} \neq 0 \quad ✓$$

Hence, inflection points are $(\pm e^{-\frac{3}{2}}, -12e^{-3})$. ✓



Calculator Assumed

10. (b) On the axes provided below, sketch $y = f(x)$. Indicate clearly the location of the stationary points.



Min. Points	✓
Singularity at origin	✓

Calculator Assumed

11. [11 marks: 8, 3]

Consider the curve with equation $y = e^{\cos x}$ for $0 \le x \le 2\pi$.

(a) Use a calculus method to determine the coordinates of the stationary points on this curve. Use the second derivative test to determine the nature of these stationary points.

$y = e^{\cos x}$

$\frac{dy}{dx} = -\sin x\, e^{\cos x}$ ✓

For stationary points:

$-\sin x\, e^{\cos x} = 0$ ✓

$x = 0, \pi, 2\pi$ ✓

Hence, stationary points are:

$(0, e)$, $(\pi, \frac{1}{e})$ and $(2\pi, e)$ ✓

$\frac{d^2y}{dx^2} = \sin^2 x\, e^{\cos x} - \cos x\, e^{\cos x}$ ✓

For $(0, e)$, $\frac{d^2y}{dx^2} = -e < 0$.

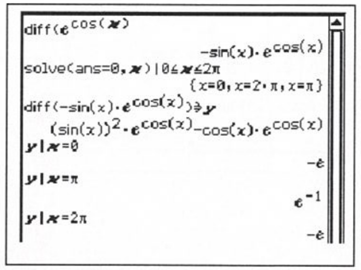
Hence, $(0, e)$ is a maximum point. ✓

For $(\pi, \frac{1}{e})$, $\frac{d^2y}{dx^2} = \frac{1}{e} > 0$.

Hence, $(\pi, \frac{1}{e})$ is a minimum point. ✓

For $(2\pi, e)$, $\frac{d^2y}{dx^2} = -e < 0$.

Hence, $(2\pi, e)$ is a maximum point. ✓



Mathematics Methods Units 3 & 4 Revision Series *12 Stationary & Inflection Points*

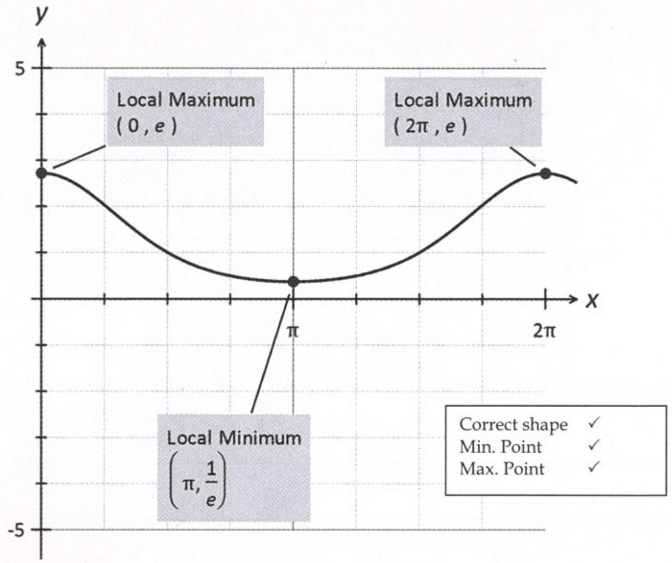
Calculator Assumed

Mathematics Methods Units 3 & 4 Revision Series *12 Stationary & Inflection Points*

Calculator Assumed

Calculator Assumed

11\. (b) On the axes provided below, sketch $y = f(x)$. Indicate clearly the location of the stationary points.



Calculator Assumed

12\. [9 marks: 4, 5] [TISC]

Consider the curve with equation $y = \dfrac{1}{\sqrt{1+e^x}}$.

(a) Use a calculus method to show that the curve has no stationary points.

$$\frac{dy}{dx} = \frac{-e^x}{2(e^x+1)^{\frac{3}{2}}} \qquad ✓✓$$

For stationary points, $\dfrac{dy}{dx} = 0$.

$$\Rightarrow \frac{-e^x}{2(e^x+1)^{\frac{3}{2}}} = 0 \qquad ✓$$

$$e^x = 0$$

But $e^x > 0$ for all real x. ✓

Hence, curve does not have any stationary points.

(b) Given that the curve has one inflection point, use derivatives to determine the coordinates of the inflection point.

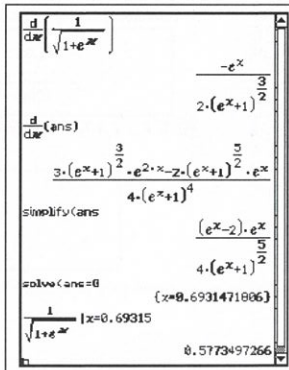
$$\frac{d^2y}{dx^2} = \frac{(e^x-2)e^x}{4(e^x+1)^{\frac{5}{2}}} \qquad ✓✓$$

For inflection points, $\dfrac{d^2y}{dx^2} = 0$.

$$\Rightarrow \frac{(e^x-2)e^x}{4(e^x+1)^{\frac{5}{2}}} = 0 \qquad ✓$$

$$x = 0.69315 \qquad ✓$$

Hence, inflection point has coordinates (0.69, 0.58) ✓



13 Rates of Change

Calculator Assumed

1. [4 marks: 1, 1, 2]

Given that, $A = 2t^2 - 5t + 3$, find:

(a) an expression for the instantaneous rate of change of A with respect to t

$\frac{dA}{dt} = 4t - 5$ ✓

(b) the instantaneous rate of change of A with respect to t when $t = 0$

When $t = 0$, $\frac{dA}{dt} = -5$ ✓

(c) t when the instantaneous rate of change of A with respect to t is 3.

When $\frac{dA}{dt} = 3$, $4t - 5 = 3$ ✓

$\Rightarrow t = 2$ ✓

2. [6 marks: 3, 2, 2]

Given that, $P = \sqrt{(1 + t^2)}$, find:

(a) the instantaneous rate of change of P with respect to t when $t = 0$ and $t = 4$

$P'(t) = \frac{t}{\sqrt{1+t^2}}$ ✓

Hence, $P'(0) = 0$ ✓

and $P'(4) = 0.9701$ ✓

(b) the average rate of change of P with respect to t for $0 \le t \le 4$

Average Rate of change $= \frac{P(4) - P(0)}{4}$ ✓

$= 0.7808$ ✓

(c) Compare your answers in (a) and (b) and comment on them.

The rates in (a) refer to the rate at a particular instant in time whereas the average rate refers to the rate across the time interval. ✓ ✓

Calculator Assumed

3. [7 marks: 3, 2, 2]

Given that, $P = 100\,e^{0.08t}$, find:

(a) the instantaneous rate of change of P with respect to t when $t = 3$ and $t = 10$

$P'(t) = 8\,e^{0.08t}$ ✓

Hence, $P'(3) = 10.17$ ✓

and $P'(10) = 17.80$ ✓

(b) the average rate of change of P with respect to t for $3 \le t \le 10$.

Average Rate of change $= \frac{P(10) - P(3)}{7}$ ✓

$= 13.63$ ✓

(c) Interpret your answers in (a) and (b).

The rates in (a) refer to the rate at a particular instant in time whereas the average rate refers to the rate across the time interval. ✓ ✓

4. [7 marks: 3, 2, 2]

Given that, $Q = 100\,t\,e^{-0.5t}$, find:

(a) using derivatives, the instantaneous rate of change of Q with respect to t when $t = 1$ and $t = 5$

$Q'(t) = 100\,e^{-0.5t} - 50\,t\,e^{-0.5t}$, ✓

Hence, $Q'(1) = 30.3265$ ✓

and $Q'(5) = -12.3127$ ✓

(b) the average change in Q for $1 \le t \le 5$

Average Rate of change $= \frac{Q(5) - Q(1)}{4}$ ✓

$= -4.90$ ✓

(c) the value of t when the instantaneous rate of change of Q with respect to t is zero.

$100\,e^{-0.5t} - 50\,t\,e^{-0.5t} = 0$ ✓

$t = 2$ ✓

Mathematics Methods Units 3 & 4 Revision Series 13 Rates of Change

Calculator Assumed

Mathematics Methods Units 3 & 4 Revision Series 13 Rates of Change

Calculator Assumed

Calculator Assumed

5. [5 marks: 2, 3]

Given that, $N = \frac{2t}{5+t}$ where $t \geq 0$, find, showing the use of derivatives:

(a) the instantaneous rate of change of N with respect to t when $t = 0$

$N'(t) = 10/(5+t)^2$, ✓
Hence, $N'(0) = 0.4$. ✓

(b) the value of t when the instantaneous rate of change of N with respect to t is 0.1.

When $N'(t) = 0.1 \Rightarrow 10/(5+t)^2 = 0.1$ ✓
$(5+t)^2 = 100$ ✓
$\Rightarrow t = 5$ (reject −15) ✓

6. [5 marks: 2, 1, 2]

Given that, $A = 50(1+t)^2e^{-t}$ for $t \geq 0$, find:

(a) an expression for the instantaneous rate of change of A with respect to t

$A'(t) = -(50t^2 - 50)e^{-t}$ ✓✓

(b) the instantaneous rate of change of A with respect to t when $t = 1$

$A'(1) = 0$ ✓

(c) the average rate of change of A in the first second.

Average Rate of change $= \frac{A(1) - A(0)}{1}$ ✓
$= 23.58$ ✓

Calculator Assumed

7. [7 marks: 1, 2, 2, 2]

The price of a listed share C cents, is modelled by $C = 75\sqrt{1+0.8t}$, $t \geq 0$, where t is the number of years after 2000.

(a) Find the per unit cost in 2000.

$C(0) = 75$ ✓

(b) Find the average rate of cost rise between 2000 and 2010.

Average Rate of change $= \frac{C(10) - C(0)}{10}$ ✓
$= 15$ ✓

(c) Find using derivatives, the instantaneous rate of cost rise in 2005

$C'(t) = \frac{30\sqrt{5}}{\sqrt{4t+5}}$ ✓
Hence, $C'(5) = 6\sqrt{5}$. ✓

(d) Find when the instantaneous rate of cost rise is 10 cents per year.

$C'(t) = 10 \Rightarrow \frac{30\sqrt{5}}{\sqrt{4t+5}} = 10$ ✓
Hence, $t = 10$. ✓

Calculator Assumed

8. [13 marks: 2, 3, 3, 2, 3]

The water level at a jetty is modelled by $h = 3 + \cos\left(\frac{\pi t}{12}\right)$, $0 \le t \le 24$, where h metres is the depth of water measured from the river bed at time t hours after 6 am.

(a) Find the water level at 7 am and 10 am.

At 7 am, $t = 1$: $h(1) = 3.9659 \approx 3.97$ m ✓
At 10 am, $t = 4$: $h(4) = 3.5$ m ✓

(b) Find the average rate of change of the water depth between 7 am and 10 am.

$$\text{Average rate of change} = \frac{h(4) - h(1)}{4 - 1}$$
$$= \frac{3.5 - 3.9659}{3} \quad ✓$$
$$= -0.1553 \quad ✓$$

Water depth is decreasing at a rate of 0.15 m/hr ✓

(c) Find the rate of change of the water depth at 10 am.

$$\frac{dh}{dt} = -\frac{\pi}{12}\sin\left(\frac{\pi t}{12}\right) \quad ✓$$

At 10 am, $t = 4$: $\frac{dh}{dt} = -0.2267$. ✓

Water depth is decreasing at a rate of 0.23 m/hr ✓

(d) Comment on your answers in (b) and (c).

Answer in (c) describes the rate of change specifically at 10 am while answer in (d) describes the rate of change between 7 am and 10 am. ✓✓

(e) Find when the water level is increasing at a rate of 0.1 metres per hour.

$$\frac{dh}{dt} = -\frac{\pi}{12}\sin\left(\frac{\pi t}{12}\right) = 0.1$$
$$t = 13.4971,\ 22.5029 \quad ✓$$

That is at 7.30 pm and 4.30 am ✓✓

Calculator Assumed

9. [12 marks: 4, 3, 2, 3]

The mass of an object being printed by a 3D-printer is given by $M = ln\,(1 + t^3)$ g for $0 \le t \le 10$ minutes.

(a) Find the average rate of change of mass of the object during the first 5 seconds and the second 5 seconds.

1st 5 seconds: $\text{Average rate of change} = \frac{M(5) - M(0)}{5 - 0}$
$$= \frac{4.8363 - 0}{5} \quad ✓$$
$$= 0.9673 \approx 0.97 \text{ g/minute} \quad ✓$$

2nd 5 seconds: $\text{Average rate of change} = \frac{M(10) - M(5)}{10 - 5}$
$$= \frac{6.9088 - 4.8363}{5} \quad ✓$$
$$= 0.4145 \approx 0.42 \text{ g/minute} \quad ✓$$

(b) Find the rate of change of mass at $t = 5$ and $t = 10$ minutes.

$$\frac{dM}{dt} = \frac{3t^2}{1 + t^3} \quad ✓$$

For $t = 5$: $\frac{dM}{dt} = 0.5952 \approx 0.60$ ✓

For $t = 10$: $\frac{dM}{dt} = 0.2997 \approx 0.30$ ✓

(c) Comment on your answers in (a) and (b).

The average rate of change and instantaneous rates of change are declining for increasing values of t. ✓✓

(d) Use an analytical method to determine the values of $0 \le t \le 10$ for which the rate of change of mass is decreasing.

$$\frac{d^2M}{dt^2} = \frac{6t - 3t^4}{(1 + t^3)^2} \quad ✓$$
$$\frac{6t - 3t^4}{(1 + t^3)^2} < 0 \quad ✓$$
$$1.26 \le t \le 10 \quad ✓$$

$\text{diff}\left(\frac{3\cdot x^2}{x^3+1}\right)$ $\quad \frac{-(3\cdot x^4 - 6\cdot x)}{(x^3+1)^2}$

$\text{solve}\left(\frac{-(3\cdot x^4 - 6\cdot x)}{(x^3+1)^2} < 0, x\right) \mid 0 \le x \le 10$

$\{x > 1.25992105, x < 0 \text{ and } x \ne -1\}$

Mathematics Methods Units 3 & 4 Revision Series — *14 Optimisation*

$t = 13.4971, 22.5029$ ✓

That is at 7.30 pm and 4.30 am ✓✓

$1.26 \le t \le 10$ ✓

{x>1.25992105,x<0 and x≠-1}

14 Optimisation

Calculator Assumed

1. [10 marks: 6, 2, 2]

The cost per hour , \$C, of operating a truck travelling at a constant speed of v kmh^{-1} is modelled by $C = \dfrac{(v-20)^3}{5000} + \dfrac{400}{v} + 200$ where $v > 0$. The truck has a speed limit of 80 kmh^{-1}.

(a) Use Calculus to find the speed the truck should travel on for the hourly cost to be minimized. Find the minimum hourly cost.

$$C = \frac{(v-20)^3}{5000} + \frac{400}{v} + 200$$

$$C'(v) = \frac{3(v-20)^2}{5000} - \frac{400}{v^2} \quad ✓$$

$C'(v) = 0 \Rightarrow v = 40.2737$ (reject -20.2737) ✓

$$C''(v) = \frac{6(v-20)}{5000} - \frac{800}{v^3} \quad ✓$$

$C''(40.2737) > 0$. ✓

Hence, C has a local minimum when $v = 40.3$ kmh^{-1}. ✓

Minimum cost C(40.2737) = \$211.60. ✓

(b) Find the difference in hourly cost if the truck were to be travelling at its posted speed limit.

C(40.2737) = \$211.60.

C(80) = \$ 248.20 ✓

Hence, difference in cost = \$36.60 ✓

(c) Suggest a reason why the Company that owns this truck may not operate it at the speed that achieves the minimum hourly cost.

Optimum speed is half the maximum speed; hence jobs will take twice as long. In that same time, twice as many jobs may be completed, possibly generating greater profits. ✓✓

Calculator Assumed

2. [7 marks: 1, 1, 1, 4]

The concentration (C mg/mL) of a drug in a person's bloodstream is modelled by $C = 15te^{-0.5t}$, where t is time in hours after the drug is administered.

(a) Find the concentration after 2 hours.

$C = 15te^{-0.5t}$

$C(2) = 11.0364$ mg/mL ✓

(b) Find an expression for the rate of change of concentration.

$C'(t) = 15\,e^{-0.5t} - 7.5t\,e^{-0.5t}$ ✓

(c) Find the rate of change of concentration after 10 hours.

$C'(10) = -0.4043$ ✓

(d) Use Calculus to find the maximum concentration of the drug. State when this occurs.

$C'(t) = 0 \Rightarrow 15\,e^{-0.5t} - 7.5t\,e^{-0.5t} = 0$ ✓

$t = 2$

$C''(t) = -7.5\,e^{-0.5t} - (7.5\,e^{-0.5t} - 3.75t\,e^{-0.5t})$

$= -15\,e^{-0.5t} + 3.75t\,e^{-0.5t}$ ✓

$C''(2) < 0$ ✓

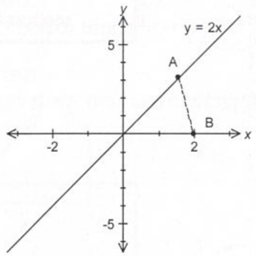
Hence, maximum for C is C(2) = 11.04 mg/mL ✓

Calculator Assumed

3. [7 marks: 1, 2, 4]

Point A with coordinates (h, k) lies on the line $y = 2x$.

(a) Explain clearly why $k = 2h$.



Since (h, k) lies on $y = 2x$, when $x = h$, $y = k \Rightarrow k = 2h$ ✓

(b) Find in terms of h, d, the distance between A and the point B with coordinates (2, 0).

$$d = \sqrt{(h-2)^2 + (k-0)^2} \quad ✓$$
$$= \sqrt{(h-2)^2 + (2h-0)^2}$$
$$= \sqrt{5h^2 - 4h + 4} \quad ✓$$

(c) Use Calculus to find the coordinates of the point on $y = 2x$ that is closest to B.

$$d = \sqrt{5h^2 - 4h + 4}$$
$$d'(h) = \frac{5h-2}{\sqrt{5h^2-4h+4}} \quad ✓$$
$$d'(h) = 0 \Rightarrow h = 0.4 \quad ✓$$

When $h = 0.4$:

h	0.4^-	0.4	0.4^+
$d'(h)$	–	0	+

✓

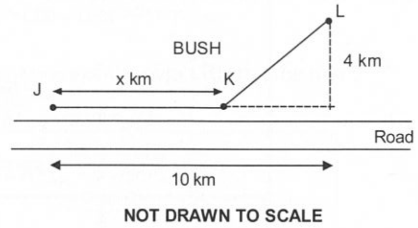
Hence, d is minimum when $h = 0.4$.

Hence, (0.4, 1.8) is closest to B. ✓

Calculator Assumed

4. [6 marks: 1, 2, 3]

A fibre optic cable is to be laid from J to L. It costs $500 per km to lay the cable alongside the road and $800 per km to lay it across the bush. K is a point x km from J along the road. The cable will be laid alongside the road from J to K and across the bush from K to L.



(a) Show that the distance between K and L is $\sqrt{x^2 - 20x + 116}$ km.

$$KL = \sqrt{(10-x)^2 + (4)^2} \quad ✓$$
$$= \sqrt{x^2 - 20x + 116}$$

(b) Find the total cost for laying the cable from J to L via K (as described).

Cost $C = 500x + 800\sqrt{x^2 - 20x + 116}$ ✓✓

(c) Use an analytical method to find x so that this cost is minimized.

$$C'(x) = 500 + \frac{800(x-10)}{\sqrt{x^2-20x+116}} \quad ✓$$
$$C'(x) = 0 \Rightarrow x = 6.7974 \quad ✓$$

When $x = 6.7974$:

x	6.7974^-	6.7974	6.7974^+
$C'(x)$	–	0	+

✓

Hence C is minimised when $x = 6.80$ km.

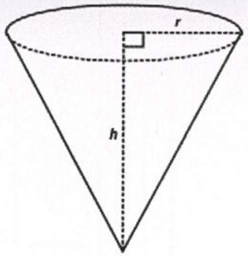
Calculator Assumed

5. [8 marks: 3, 5]

The accompanying diagram shows an inverted cone of height h cm and base radius r cm.

The volume of the cone is fixed at $\frac{\pi}{3}$ m^3.

The curved surface area of the cone is given by $A = \pi r\sqrt{h^2 + r^2}$.

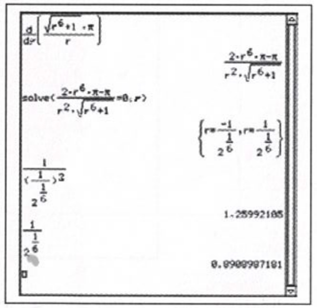


(a) Show that $A = \frac{\pi}{r}\sqrt{1+r^6}$.

Volume of cone $\frac{1}{3}\pi r^2 h = \frac{\pi}{3}$	✓
$h = \frac{1}{r^2}$	✓
Hence, $A = \pi r\sqrt{h^2+r^2}$	
$= \pi r\sqrt{\left(\frac{1}{r^2}\right)^2+r^2}$	✓
$= \frac{\pi\sqrt{1+r^6}}{r}$	

(b) Use a calculus method to find the value of r and h that minimises A.

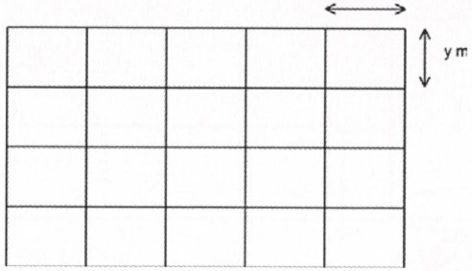
$A = \frac{\pi\sqrt{1+r^6}}{r}$	
$\frac{dA}{dr} = \pi\left(\frac{2r^6-1}{r^2\sqrt{1+r^6}}\right)$	✓✓
For max/min values: $\frac{dA}{dr} = 0$	
$\pi\left(\frac{2r^6-1}{r^2\sqrt{1+r^6}}\right) = 0$	✓
$r = \frac{1}{2^{\frac{1}{6}}} = 0.89$ m	✓
$h = 2^{\frac{1}{3}} = 1.26$ m	✓



Calculator Assumed

6. [11 marks: 3, 8]

The rectangular backyard of area 100 m^2 is to be divided into 20 equal rectangles as shown below. The boundaries are to be marked with single length red ribbons.



(a) Find a rule for y in terms of x.

Total Area covered = $20xy$	
Hence $20xy = 100$	✓✓
$y = \frac{5}{x}$	✓

(b) Use a calculus method to find the exact length and width of each rectangle that will minimize the total length of ribbon used.

Total Perimeter $P = 25x + 24y$	✓
$P = 25x + \frac{120}{x}$	✓
$\frac{dP}{dx} = 25 - \frac{120}{x^2}$	✓
For max/min values: $25 - \frac{120}{x^2} = 0$	✓
$x = \frac{2\sqrt{30}}{5}$ reject $-\frac{2\sqrt{30}}{5}$	✓
$\frac{d^2P}{dx^2} = \frac{240}{x^3}$	
When $x = \frac{2\sqrt{30}}{5}$, $\frac{d^2P}{dx^2} > 0$.	✓
Hence, P is minimised when $x = \frac{2\sqrt{30}}{5}$ and $y = \frac{5\sqrt{30}}{12}$	✓✓

Calculator Assumed

7. [7 marks]

The length of a closed box is twice its width, x cm. The volume of the closed box is 5 000 m^3. Use Calculus to determine the dimensions of the box for its surface area to be minimized.

Let height of box be h.

Volume $2x \times x \times h = 5000 \Rightarrow h = 2500/x^2$ ✓

Surface Area A $= 2(2x^2) + 2(xh) + 2(2xh)$

$= 4x^2 + 6x \times 2500/x^2$

$= 4x^2 + 15\,000/x$ ✓

$A'(x) = 8x - 15\,000/x^2$ ✓

$A'(x) = 0 \Rightarrow x = 12.3311$ ✓

$A''(x) = 8 + 30\,000/x^3$, $A''(12.3311) > 0.$ ✓

Hence, A is minimised when $x = 12.33$

Dimensions of box is $12.33 \times 24.66 \times 16.44$ cm. ✓✓

8. [6 marks]

A closed cylindrical can of diameter $2r$ is to have a volume of 50π cm^3. Use Calculus to find the dimensions of this can if it is to have minimum surface area. Ignore the thickness of the material used to make the can.

Let height of cylinder be h.

Volume $\pi \times r^2 \times h = 50\pi \Rightarrow h = 50/r^2$ ✓

Surface Area A $= 2(\pi r^2) + 2\pi rh$

$= 2\pi r^2 + 2\pi r \times 50/r^2$

$= 2\pi r^2 + 100\pi/r$ ✓

$A'(x) = 4\pi r - 100/r^2$ ✓

$A'(x) = 0 \Rightarrow x = 2.9240$ ✓

$A''(x) = 4\pi + 200/r^3$, $A''(2.9240) > 0.$ ✓

Hence, A is minimised when $x = 2.9240$

Dimensions of cylinder is

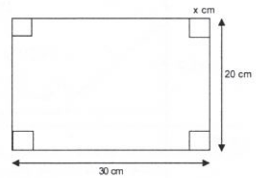
base radius = 2.9 cm and height = 5.8 cm. ✓

Mathematics Methods Units 3 & 4 Revision Series *14 Optimisation*

Calculator Assumed

9. [7 marks]

A rectangular sheet of cardboard measuring 20 cm by 30 cm is to be used to make an open box. A square of width x cm is to be removed from each corner of the sheet to form the net of the box. The net is then folded up to form the box.



Use Calculus to find the dimensions of the box with the largest possible volume. Give this volume.

Length of box $= 30 - 2x$

Width of box $= 20 - 2x$

Height of box $= x$ ✓

Hence, volume $V = x\,(20 - 2x)\,(30 - 2x)$

$= 4x^3 - 100x^2 + 600x$ ✓

$V'(x) = 12x^2 - 200x + 600$

$V'(x) = 0 \Rightarrow x = 3.9237$ (reject 12.7429 as this makes the width < 0) ✓

$V''(x) = 24x - 200$, $V''(3.9237) < 0.$ ✓

Hence, V has a maximum when height = 3.93 cm, width = 12.15 cm and length = 22.14 cm ✓✓

Maximum value for $V = 3.93 \times 12.15 \times 22.14 \approx 1057$ cm^3 ✓

Mathematics Methods Units 3 & 4 Revision Series *14 Optimisation*

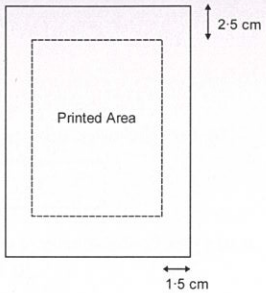
Calculator Assumed

Mathematics Methods Units 3 & 4 Revision Series *14 Optimisation*

Calculator Assumed

10. [8 marks]

A printed poster using a minimal area of cardboard is to be designed. The printed area must be 500 cm^2. The top and bottom margins must be 2.5 cm each. The left and right margins must be 1.5 cm each. Use Calculus to determine the optimal dimensions of the poster. Give the width, height and total area of the optimal poster.



Let width of printed area = x
Let length of printed area = y

Area of printed area $A = xy$
But $xy = 500$. $\Rightarrow y = 500/x$ ✓

Width of poster = $x + (2 \times 1.5) = x + 3$ ✓
Length of poster = $y + (2 \times 2.5) = 500/x + 5$ ✓

Area of poster, $A(x) = (x + 3)(500/x + 5)$
$= 515 + 5x + 1500/x$ ✓

$A'(x) = 5 - 1500/x^2$ ✓
$A'(x) = 0 \Rightarrow 5x^2 = 1500$
$x = 17.3205$

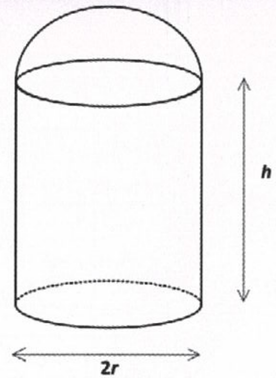
$A''(x) = 3000/x^3$, $A''(17.3205) > 0$. ✓

Hence, A has a minimum when width = 17.32 + 3 = 20.32 cm and length = 28.87 + 5 = 33.87 cm. ✓
Minimum value for A = 20.32 × 33.87 ≈ 688 cm^2 ✓

Calculator Assumed

11. [10 marks: 3, 3, 4] [TISC]

A container consists of a cylinder of height h cm and base radius r cm with a hemispherical cap sitting on top of the cylinder. The container has a fixed volume of 360π cm^3.



(a) Show that $h = \dfrac{360}{r^2} - \dfrac{2r}{3}$.

$\text{Volume} = \pi r^2 h + \dfrac{1}{2} \times \dfrac{4\pi r^3}{3}$ ✓

$= \pi r^2 h + \dfrac{2\pi r^3}{3}$

But Volume = 360π.
Hence:

$\pi r^2 h + \dfrac{2\pi r^3}{3} = 360\pi$ ✓✓

$h = \dfrac{360}{r^2} - \dfrac{2r}{3}$

solve$\left(\pi r^2 h + \dfrac{2\pi r^3}{3} = 360\pi, h\right)$
$\left\{h = \dfrac{-2\cdot r}{3} + \dfrac{360}{r^2}\right\}$

(b) Show that the total external surface area of the container is given by
$S = \dfrac{5\pi r^2}{3} + \dfrac{720\pi}{r}$.

$S = 2\pi rh + \pi r^2 + \dfrac{1}{2} \times 4\pi r^2$ ✓

$= 3\pi r^2 + 2\pi r \times \left(\dfrac{360}{r^2} - \dfrac{2r}{3}\right)$ ✓✓

$= \dfrac{5\pi r^2}{3} + \dfrac{720\pi}{r}$.

combine$(2\pi r \times h + \pi r^2 + 2\pi r^2) \mid h = \dfrac{-2\cdot r}{3} + \dfrac{360}{r^2}$
$3\cdot r^2\cdot \pi - 2\cdot r\cdot \left(\dfrac{2\cdot r}{3} - \dfrac{360}{r^2}\right)\cdot \pi$
simplify(
$\dfrac{5\cdot r^2\cdot \pi}{3} + \dfrac{720\cdot \pi}{r}$

Calculator Assumed

11. (c) Use a calculus method to determine the minimum surface area of the container. Give your answer in exact form.

$$S = \frac{5\pi r^2}{3} + \frac{720\pi}{r}$$

$$\frac{ds}{dr} = \frac{10\pi r}{3} - \frac{720\pi}{r^2} \quad ✓$$

$$\frac{ds}{dr} = 0 \Rightarrow r = 6 \quad ✓✓$$

Hence: $S_{min} = 180\pi$ ✓

$\frac{d}{dr}\left(\frac{5\cdot r^2\cdot\pi}{3}+\frac{720\cdot\pi}{r}\right)$

$\frac{10\cdot r^3\cdot\pi-2160\cdot\pi}{3\cdot r^2}$

solve(ans=0,r)

$\{r=6\}$

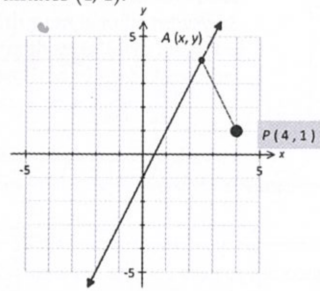
$\frac{5\cdot r^2\cdot\pi}{3}+\frac{720\cdot\pi}{r}|r=6$

$180\cdot\pi$

Calculator Assumed

12. [8 marks: 2, 6] [TISC]

The point A (x, y) lies on the line with equation $y = 2x - 1$.
The point P has coordinates $(4, 1)$.



(a) Show that the distance between A and P is given by $s = \sqrt{5x^2 - 16x + 20}$.

Distance $s = \sqrt{(x-4)^2 + (y-1)^2}$ ✓
But $y = 2x - 1$.
Hence $s = \sqrt{(x-4)^2 + (2x-1-1)^2}$ ✓
$= \sqrt{5x^2 - 16x + 20}$

(b) Use a calculus/algebraic method to find the shortest distance and the longest distance between the point P and the line $y = 2x - 1$ for $1 \le x \le 5$.

Distance $s = \sqrt{5x^2 - 16x + 20}$

$$\frac{ds}{dt} = \frac{5x - 8}{\sqrt{5x^2 - 16x + 20}}$$

$$\frac{ds}{dt} = 0 \Rightarrow x = \frac{8}{5}$$

$s\left(\frac{8}{5}\right) = 2.68$

$s(1) = 3$

$s(5) = 8.06$

Hence, shortest distance = 2.68
longest distance = 8.06

- Obtain derivative ✓
- Set derivative = 0 ✓
- relative extremum ✓
- Test end points ✓
- Correct answers ✓✓

$\sqrt{(x-4)^2+(y-1)^2}\ |y=2x-1$

$\sqrt{5\cdot x^2-16\cdot x+20}$

$\frac{d}{dx}(\text{ans})$

$\frac{5\cdot x-8}{\sqrt{5\cdot x^2-16\cdot x+20}}$

solve(

$\left\{x=\frac{8}{5}\right\}$

$\sqrt{5\cdot x^2-16\cdot x+20}\ |x=\frac{8}{5}$

2.683281573

$\sqrt{5\cdot x^2-16\cdot x+20}\ |x=1$

3

$\sqrt{5\cdot x^2-16\cdot x+20}\ |x=5$

8.062257748

15 Incremental Change & Marginal Rates

Calculator Assumed

1. [6 marks: 1, 3, 2]

Consider $y = -2x^2 + 3x - 1$.

(a) Find, the change in y (4 decimal places) when x changes from 1.00 to 1.01.

$y(1.01) - y(1.00) = -0.0102 - 0 = -0.0102$ ✓

(b) Use the method of incremental change to find the approximate change in y (4 decimal places) when x changes from 1.00 to 1.01.

$\frac{dy}{dx} = -4x + 3$ ✓

$\delta y \approx (-4x + 3)\,\delta x$ ✓

$x = 1, \delta x = 0.01 \Rightarrow \delta y \approx -0.0100.$ ✓

(c) Find the percentage difference between the actual and the approximated change.

Percentage difference $= \frac{-0.0102 - (-0.01)}{-0.0102} \times 100$ ✓

$= 1.96\%$ ✓

---

2. [3 marks]

Consider $y = \frac{x}{1+e^x}$.

Use the method of incremental change to find the approximate change in y (4 significant places) when x changes from 1.00 to 0.99.

$\frac{dy}{dx} = \frac{1+e^x - xe^x}{(1+e^x)^2}$ ✓

$\delta y \approx \frac{1+e^x - xe^x}{(1+e^x)^2} \times \delta x$ ✓

$x = 1, \delta x = -0.01 \Rightarrow \delta y \approx -0.000\,7233.$ ✓

- Correct answers ✓✓

Calculator Assumed

3. [8 marks: 4, 4]

A curve has equation $y = 2x^3 + 3x^2 - 12x$.

(a) Use the method of small increments to estimate the change in y when x changes from: (i) 2.00 to 1.99 (ii) 2.00 to 2.01

$\frac{dy}{dx} = 6x^2 + 6x - 12$ ✓

$\delta y \approx (6x^2 + 6x - 12) \times \delta x$ ✓

(i)

When x changes from 2.00 to 1.99, $\delta x = -0.01$.

$\delta y \approx 24 \times -0.01$

≈ -0.24 ✓

(ii)

When x changes from 2.00 to 2.01, $\delta x = 0.01$.

$\delta y \approx 24 \times 0.01$

≈ 0.24 ✓

(b) Use your answer in (a) to estimate the value of y when $x = 1.99$ and $x = 2.01$.

When $x = 2, y = 4$.

When $x = 1.99$:

$y = 4 + \delta y$

$\approx 4 - 0.24$ ✓

≈ 3.76 ✓

When $x = 2.01$:

$y = 4 + \delta y$

$\approx 4 + 0.24$ ✓

≈ 4.24 ✓

Calculator Assumed

4. [8 marks: 5, 3]

(a) Use the method of small changes to find the approximate change in the radius of a spherical balloon corresponding to a change in its volume from 500 cm³ to 499 cm³.

> Volume $V = \frac{4}{3}\pi r^3$.
>
> $\frac{dV}{dr} = 4\pi r^2$. ✓
>
> $\delta V \approx 4\pi r^2\, \delta r$. ✓
>
> When $V = 500$, $\delta V = -1$, $r = \sqrt[3]{\frac{375}{\pi}}$. ✓
>
> Hence, $-1 = 4\pi\left(\sqrt[3]{\frac{375}{\pi}}\right)^2 \times \delta r$ ✓
>
> $\Rightarrow\ \delta r \approx -0.0033$ cm. ✓

(b) Use your answer in (a) to find the approximate change in the surface area of a spherical balloon corresponding to a change in its volume from 500 cm³ to 499 cm³.

> Surface area $A = 4\pi r^2$.
>
> $\frac{dA}{dr} = 8\pi r$. ✓
>
> $\delta A \approx 8\pi r\, \delta r$. ✓
>
> $r = \sqrt[3]{\frac{375}{\pi}}$ and $\delta r = -0.0033$
>
> $\Rightarrow\ \delta A \approx -0.41$ cm^2 ✓

Calculator Assumed

5. [10 marks: 2, 1, 2, 2, 3]

Mathcom sells each "Template X" for \$30. The cost of producing x items is given by $C(x) = \frac{80x}{(x+1)} + 0.04x^2 + 500$.

(a) Find an expression for the profit $P(x)$ corresponding to the manufacture and sale of x items.

> Profit = Revenue – Cost
>
> $= 30x - [\frac{80x}{(x+1)} + 0.04x^2 + 500]$ ✓
>
> $= -0.04x^2 + 30x - \frac{80x}{(x+1)} - 500$ ✓

(b) Find an expression $P'(x)$.

> $P'(x) = -0.08x + 30 - \frac{80}{(x+1)^2}$ ✓

(c) Find $P'(100)$. Interpret this value.

> $P'(100) = \$21.99$ ✓
>
> This is the profit associated with the sale of the 101st item. ✓

(d) Find the average profit per item associated with the manufacture and sale of 100 items.

> Average profit $= \frac{P(100)}{100}$ ✓
>
> $= \frac{2\,020.79}{100} = \$20.21$ ✓

(e) Find how many items were manufactured and sold if the profit associated with the sale of the next item is approximately \$20.

> $-0.08x + 30 - \frac{80}{(x+1)^2} = 20$ ✓
>
> $x = -3.8$ (reject), 1.8, and 124.9 ✓
>
> Reject $x = 1.8$ as $P(1) < 0$ and $P(2) < 0$.
>
> $P(125) - P(124) \approx \$20.03$
>
> Hence, 125 items. ✓

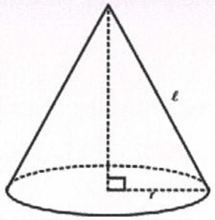
Mathematics Methods Units 3 & 4 Revision Series *15 Incremental Change & Marginal Rates*

Mathematics Methods Units 3 & 4 Revision Series *15 Incremental Change & Marginal Rates*

Calculator Assumed

6. [6 marks]

Use a calculus method to find the approximate percentage change in the curved surface area of a solid cone of fixed height 20 cm corresponding to a 1% increase in its base radius from 10 cm.
[Hint: The curved surface area of a cone of base radius r, and slant edge l is given by πrl.]



$$A = \pi rl$$
$$= \pi r \times \sqrt{r^2+400} \quad ✓$$
$$\frac{dA}{dr} = \frac{2\pi r^2+400\pi}{\sqrt{r^2+400}} \quad ✓$$

Hence $\delta A = \dfrac{2\pi r^2+400\pi}{\sqrt{r^2+400}} \times \delta r$ ✓

$$\frac{\delta A}{A} = \frac{\left(\dfrac{2\pi r^2+400\pi}{\sqrt{r^2+400}}\right)\delta r}{\pi r\sqrt{r^2+400}}$$
$$= \frac{(2\pi r^2+400\pi)\delta r}{\pi r(r^2+400)}$$
$$= \frac{(2\pi r^2+400\pi)\delta r}{\pi r(r^2+400)} \times \frac{\delta r}{r} \quad ✓$$

When $r = 10$, $\dfrac{\delta r}{r} = 0.01$:

$$\frac{\delta A}{A} = \frac{(2\pi 10^2+400\pi)}{10\pi(10^2+400)} \times 0.01 \quad ✓$$
$$= 0.0012 \quad ✓$$

Calculator Assumed

7. [7 marks: 4, 3] [TISC]

A curve has equation $y = x^n$ where n is a rational number.
As x increases by δx, the corresponding increase in y is δy.

(a) Use the incremental formula to show that if x changes by k %, then y changes by $n \times k$ %.

$$\frac{dy}{dx} = nx^{n-1} \quad ✓$$
$$\Rightarrow \delta y \approx nx^{n-1}\delta x$$
$$\frac{\delta y}{y} \approx \frac{nx^{n-1}\delta x}{x^n} \quad ✓$$
$$\approx n\frac{\delta x}{x} \quad ✓$$

But $\dfrac{\delta x}{x} = k$ %.

Hence, $\dfrac{\delta y}{y} \approx n \times k$ %. ✓

Hence % change in $y \approx n \times k$ %.

(b) Show how you can use the result in (a) to estimate the percentage error in calculating the volume of a cube if the instrument for measuring the side has an error of 2%.

Volume of cube of side length x
is given by $V = x^3$. ✓

Using the result in (a):
% error in $V = 3 \times$ % error in x ✓
$V = 3 \times 2\%$
$= 6\%$ ✓

Calculator Assumed

8. [9 marks: 3, 3, 3] [TISC]

A circle has radius r metres.

(a) The circumference of the circle is increasing at a rate of π metres per hour.

(i) Find the rate with which the radius of the circle is changing when the radius of the circle is 10 metres.

Circumference of circle $P = 2\pi r$

$$\frac{dP}{dt} = 2\pi \frac{dr}{dt} \quad ✓$$

$$\pi = 2\pi \frac{dr}{dt} \quad ✓$$

$$\frac{dr}{dt} = \frac{1}{2} \text{ metres per hour.} \quad ✓$$

(ii) Find the rate with which the area of the circle is changing when the radius of the circle is 10 metres.

Area of circle $A = \pi r^2$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \quad ✓$$

$$= 2\pi \times 10 \times \frac{1}{2} \quad ✓$$

$$= 10\pi \text{ metres}^2 \text{ per hour.} \quad ✓$$

(b) Use the incremental method (method of small changes) to find the approximate change in the area of the circle when the radius of the circle changed from 10 metres to 10.1 metres.

Area of circle $A = \pi r^2$

$$\frac{dA}{dr} = 2\pi r \quad ✓$$

$$\delta A = 2\pi r\, \delta r \quad ✓$$

$$= 2\pi \times 10 \times 0.1$$

$$= 2\pi \text{ metres}^2 \quad ✓$$

Calculator Assumed

9. [7 marks: 4, 3] [TISC]

Let $A = \dfrac{1}{\sqrt{x+1}}$ where $x = f(t) \geq 0$ for time $t \geq 0$.

(a) Use the incremental method (method of small changes) to find the approximate change in A when x changes from 3 to 2.99.

$$\delta A \approx \left(-\frac{1}{2(x+1)^{\frac{3}{2}}} \right) \times \delta x \quad ✓✓$$

$$\approx \left(-\frac{1}{16}\right) \times (-0.01) \quad ✓$$

$$\approx 0.000\,625 \quad ✓$$

(b) Find the rate of change of A when $x = 3$, given that when $x = 3$, $\dfrac{dx}{dt} = 4$.

$$\frac{dA}{dx} = \left(-\frac{1}{2(x+1)^{\frac{3}{2}}} \right)$$

$$\Rightarrow \frac{dA}{dt} = \left(-\frac{1}{2(x+1)^{\frac{3}{2}}} \right) \times \frac{dx}{dt} \quad ✓$$

When $x = 3$, $\dfrac{dx}{dt} = 4$.

Hence: $$\frac{dA}{dt} = \left(-\frac{1}{16}\right) \times 4 \quad ✓$$

$$= -\frac{1}{4} \quad ✓$$

16 Exponential Growth & Decay

Calculator Assumed

1. [12 marks: 1, 1, 2, 3, 5]

The amount (A g) of radioactive substance R remaining after t days is given by $A = 800\,e^{-0.04t}$.

(a) What is the initial amount of substance present?

800 g ✓

(b) How much is left after 7 days?

$A(7) = 800\,e^{-0.04(7)} = 604.6$ g ✓

(c) How much has decayed after 14 days?

$A(14) = 800\,e^{-0.04(14)} = 456.97$ ✓
Hence, amount left = 800 – 456.97 = 343.0 g ✓

(d) Find the half-life of this radioactive substance.

When $A = 400$, $800\,e^{-0.04t} = 400$ ✓✓
$e^{-0.04t} = 0.5$
$t = 17.33$ ✓
Hence, the half-life is 17.33 days.

(e) Find the rate of decay when 100 g of the substance is left.

Rate of decay $\dfrac{dA}{dt} = -32\,e^{-0.04t}$. ✓

When $A = 100$, $800\,e^{-0.04t} = 100$ ✓
$e^{-0.04t} = \dfrac{1}{8}$ ✓

Hence, when $A = 100$, $\dfrac{dA}{dt} = -32 \times \dfrac{1}{8}$ ✓
$= -4$ g/day ✓

Calculator Assumed

2. [13 marks: 4, 3, 3, 3]

The instantaneous rate of population growth is proportional to its population size. The population grew from an initial 10 000 to 15 000 in 8 years.

(a) Find an expression for the population size at time t years.

$\dfrac{dP}{dt} = kP \Rightarrow P = 10\,000\,e^{kt}$ ✓
When $t = 8$, $P = 15\,000$.
$\Rightarrow 10\,000\,e^{8k} = 15\,000$ ✓
$k = 0.05068$ ✓

Hence, $P = 10\,000\,e^{0.05068t}$ ✓

(b) Find the time taken for the population to double its size.

When $P = 20\,000$,
$10\,000\,e^{0.05068t} = 20\,000$ ✓✓
$t = 13.68$

Hence, the doubling-time is 13.68 years. ✓

(c) Find the instantaneous rate of population growth when $t = 8$.

Rate of decay $\dfrac{dP}{dt} = 506.83\,e^{0.05068t}$. ✓
When $t = 8$, $\dfrac{dP}{dt} = 506.83\,e^{0.05068(8)}$
≈ 760 persons per year ✓✓

(d) Find when the instantaneous rate of population growth is 1000 persons/year.

When $\dfrac{dP}{dt} = 1000$,
$506.83\,e^{0.05068t} = 1000$ ✓✓

$t = 13.4$ years ✓

Calculator Assumed

3. [5 marks]

A radioactive substance has a half-life of 50 days. After 20 days, only 30g were left. Assume that the radioactive substance decays exponentially.
Find the initial amount of substance.

> Let $A = A_0 e^{kt}$
> When $t = 50$, $A = 0.5A_0$. ✓
> $\Rightarrow \quad A_0 e^{50k} = 0.5 A_0$
> $e^{50k} = 0.5$ ✓
> $k = -0.01386$ ✓
>
> Hence $\quad A = A_0 e^{-0.01386t}$
>
> When $t = 20$, $A = 30$.
> $\Rightarrow \quad A_0 e^{-0.01386(20)} = 30$ ✓
> $A_0 = 39.6$ g ✓

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4. [5 marks] [TISC]

A batch of cattle grain is found to be contaminated with radioactive substance X. The radioactive substance X decays exponentially with a half-life of ten days. The amount of radioactive substance X was found to be ten times the maximum permitted level. How long should be grain be stored before it is fed to the cattle? Justify your answer.

> Let $A(t)$: Amount of substance X left after t days.
> Then, $\quad A(t) = A(0)\, e^{kt}$
> Half-life of 10 days $\Rightarrow \quad e^{10k} = 0.5$ ✓
> $k = -0.069315$ ✓
>
> Let maximum permitted level be M.
> $\Rightarrow A(0) = 10M$ ✓
> $10M\, e^{-0.069315t} < M$ ✓
> $t > 33.2$ days
> Hence, at least 34 days. ✓

Calculator Assumed

5. [12 marks: 2, 4, 2, 2, 2]

Two heated objects, P and Q are placed one at each end of a long laboratory bench. The temperature (in degrees Celsius) of P and Q, t minutes after being placed on the bench top are given by $\theta_P = 18 + 75\, e^{-0.09t}$ and $\theta_Q = 18 + 60\, e^{-0.01t}$ respectively.

(a) Find when the two objects have the same temperature.

> $18 + 75\, e^{-0.09t} = 18 + 60\, e^{-0.01t}$ ✓
> $t = 2.79$ minutes ✓

(b) Find when the two objects are losing heat at the same rate.

> For P: $\dfrac{d\theta}{dt} = -6.75\, e^{-0.09t}$ ✓
> For Q: $\dfrac{d\theta}{dt} = -0.6\, e^{-0.01t}$ ✓
> When $-6.75\, e^{-0.09t} = -0.6\, e^{-0.01t}$ ✓
> $t = 30.25$ minutes ✓

(c) Find the time taken for the temperature of each object to reach 25 C.

> For P: $25 = 18 + 75\, e^{-0.09t} \Rightarrow t = 26.4$ minutes ✓
> For Q: $25 = 18 + 60\, e^{-0.01t} \Rightarrow t = 214.8$ minutes ✓

(d) Which object is losing heat at a faster rate? Justify your answer.

> P is losing heat at a faster rate as it reaches 25° C more quickly than Q. ✓✓

(e) Find the temperature of each body for large values of t.

> For P: As $t \to \infty$, its temperature approaches 18°C. ✓
> For Q: As $t \to \infty$, its temperature approaches 18°C. ✓

Calculator Assumed

6. [8 marks: 2, 3, 3]

The mass (M milligrams) of a slow growing tumour grows exponentially according to the formula $M = A e^{kt}$ where t is time in weeks. The mass of the tumour doubles every 100 weeks.

(a) Find k to four significant figures.

$2 = e^{100k}$ ✓
$k = 0.006931$ ✓

(b) A 25 year old patient was diagnosed with a similar tumour. The mass of the tumour was estimated to be 120 milligrams. The tumour is operable only when it reaches a mass of 200 milligrams.

(i) When will the tumour be operable?

$200 = 120\, e^{0.006931t}$ ✓✓
$t = 73.7$ weeks
That is after another ≈ 74 weeks ✓

(ii) Estimate how long the tumour has been growing in the patient's body. Explain clearly how you arrived at your answer.

For t = 25 years = 1300 weeks:
$120 = A\, e^{0.006931(1300)}$ ✓
$A \approx 0.015$ mg ✓

Hence, the patient was probably born with it! ✓

Calculator Assumed

7. [7 marks: 1, 2, 2, 2] [TISC]

Fifty litres of a swimming pool chemical is accidentally spilled into a swimming pool. The amount of chemical (in litres) left in the pool t days after the spill is given by

$$A = A_0\, e^{kt}.$$

(a) State the value of A_0.

$A_0 = 50$ litres ✓

(b) Determine the value of k (to four significant figures) if 80% of the chemical disappears after 10 days.

$A = 50\, e^{kt}$
When $t = 10$, $A = 50 \times 0.2 = 10$
$10 = 50\, e^{10k}$ ✓
$k = -0.1609438$
$= -0.1609$ ✓

(c) Find the rate with which the amount of chemical left in the pool is changing when 80% of the chemical has disappeared.

$\frac{dA}{dt} = -0.1609A$ ✓

When $A = 10$, $\frac{dA}{dt} = -0.1609 \times 10 = -1.6$ litres/day ✓

(d) The swimming pool is safe for use if the amount of chemical in the pool is between 1 litre and 5 litres inclusive. To one decimal place, for what value(s) of t will the pool be safe for use?

Safe when $1 \le A \le 5$.
Hence, $1 \le 50\, e^{-0.1609t} \le 5$ ✓
$\Rightarrow 14.3 \le t \le 24.3$ ✓

Calculator Assumed

8. [6 marks: 3, 2, 1]

It is known that the amount of chemical X (in mg) left unabsorbed in a blood stream after t hours is given by $U = 100e^{-0.05t}$.

(a) Show that the rate of change of U with respect to time t is proportional to the amount of substance X remaining.

$$U = 100e^{-0.05t} \Rightarrow \frac{dU}{dt} = -0.05 \times 100e^{-0.05t} \quad \checkmark$$
$$= -0.05 \times U \quad \checkmark$$
$$\text{Hence,} \quad \frac{dU}{dt} \propto U. \quad \checkmark$$

(b) Find the time taken for 90% of the initial amount of X to be absorbed by the bloodstream. Give your answer to the nearest hour.

$$0.1 = 100e^{-0.05t} \quad \checkmark$$
$$t = 138.1551$$
$$= 138 \text{ hours.} \quad \checkmark$$

(c) Find an expression that describes the amount of chemical X absorbed by the bloodstream after t hours.

$$\text{Amount absorbed} = 100 - 100e^{-0.05t} \quad \checkmark$$

Calculator Assumed

9. [7 marks: 2, 3, 2] [TISC]

The length of a vine (L cm) grows according to the formula

$$L = 10\,e^{0.03t} \quad \text{where } t \geq 10 \text{ days.}$$

The vine is trimmed (cut) each time its length exceeds 5 metres.
Each time it is trimmed to a length of approximately 2 metres.
Assume that the vine grows at the same rate.
[A vine is a type of plant e.g. grape vine.]

(a) Find when the vine is trimmed for the first time.

$$10\,e^{0.03t} = 500 \quad \checkmark$$
$$t = 130.4 \approx 130 \text{ days} \quad \checkmark$$

(b) Find when the vine is trimmed for the second time.

$$\text{New formula } L = 200\,e^{0.03t} \quad \checkmark$$
$$200\,e^{0.03t} = 500 \quad \checkmark$$
$$t = 30.5 \approx 31 \text{ days} \quad \checkmark$$

(c) To what length should the vine be trimmed so that it is trimmed once every 60 days. The vine is trimmed each time its length exceeds 5 m. Give your answer to the nearest cm.

$$\text{New formula } L = A\,e^{0.03t}$$
$$\text{When } t = 60,\ L = 500$$
$$A\,e^{0.03 \times 60} = 500 \quad \checkmark$$
$$A = 82.6 \approx 83 \text{ cm} \quad \checkmark$$

17 Anti-Differentiation

Calculator Free

1. [13 marks: 2, 2, 2, 2, 2, 3]

Find the anti-derivative of each of the following:

(a) $\frac{3x^2}{4} + 4x + 5$

Anti-derivative = $\frac{x^3}{4} + 2x^2 + 5x + C$ ✓✓

(b) $\frac{1}{x^3} - \frac{2}{3x^2}$

Anti-derivative = $\frac{-1}{2x^2} + \frac{2}{3x} + C$ ✓✓

(c) $(1 + 4x)^5$

Anti-derivative = $\frac{(1+4x)^6}{24} + C$ ✓✓

(d) $\frac{1}{2\sqrt{x}}$

$\frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$ ✓

Anti-derivative = $\sqrt{x} + C$ ✓

(e) $\sqrt{x}(x^2 + 1)$

$\sqrt{x}(x^2+1) = x^{5/2} + x^{1/2}$ ✓

Anti-derivative = $\frac{2}{7}x^{7/2} + \frac{2}{3}x^{3/2} + C$ ✓

(f) $(2x - \frac{3}{x})^2$

$(2x - \frac{3}{x})^2 = 4x^2 - 12 + 9x^{-2}$ ✓

Anti-derivative = $\frac{4x^3}{3} - 12x - \frac{9}{x} + C$ ✓✓

Calculator Free

2. [17 marks: 3, 2, 3, 3, 3, 3]

Find:

(a) $\int \frac{3}{2}(\sqrt{x}+1)^2 \; dx$

$$\int \frac{3}{2}(\sqrt{x}+1)^2 \; dx = \frac{3}{2}\int x + 2x^{\frac{1}{2}} + 1 \; dx \quad ✓$$
$$= \frac{3}{2}\left(\frac{x^2}{2} + \frac{4x^{3/2}}{3} + x\right) + C \quad ✓✓$$

(b) $\int \frac{x^4 + x^3}{x} dx$

$$\int \frac{x^4+x^3}{x} dx = \int x^3 + x^2 \; dx \quad ✓$$
$$= \frac{x^4}{4} + \frac{x^3}{3} + C \quad ✓$$

(c) $\int \frac{x - 2x^3}{3x^5} dx$

$$\int \frac{x-2x^3}{3x^5} dx = \int \frac{x^{-4}}{3} + \frac{2x^{-2}}{3} dx \quad ✓$$
$$= \frac{-1}{9x^3} - \frac{2}{3x} + C \quad ✓✓$$

(d) $\int \sqrt{1-2x} \; dx$

$$\int \sqrt{1-2x} \; dx = \int (1-2x)^{\frac{1}{2}} \; dx \quad ✓$$
$$= \frac{-(1-2x)^{3/2}}{3} + C \quad ✓✓$$

(e) $\int \frac{4}{(3x-2)^2} dx$

$$\int \frac{4}{(3x-2)^2} dx = \int 4(3x-2)^{-2} \; dx \quad ✓$$
$$= \frac{-4}{3(3x-2)} + C \quad ✓✓$$

(f) $\int \frac{4}{3\sqrt{x-1}} dx$

$$\int \frac{4}{3\sqrt{x-1}} dx = \int \frac{4(x-1)^{-1/2}}{3} dx \quad ✓$$
$$= \frac{8(x-1)^{1/2}}{3} + C \quad ✓✓$$

Calculator Free

3. [12 marks: 2, 2, 2, 2, 2, 2]

Find:

(a) $\int 2x(1-x^2)^3 \; dx$

$$\int 2x(1-x^2)^3 \; dx = -\int -2x(1-x^2)^3 \; dx = \frac{-(1-x^2)^4}{4} + C \quad ✓✓$$

(b) $\int \frac{x}{(1-x^2)^4} dx$

$$\int \frac{x}{(1-x^2)^4} dx = -\frac{1}{2}\int -2x(1-x^2)^{-4} \; dx = \frac{1}{6(1-x^2)^3} + C \quad ✓✓$$

(c) $\int x\sqrt{1-2x^2} \; dx$

$$\int x\sqrt{1-2x^2} \; dx = \frac{-1}{4}\int -4x(1-2x^2)^{\frac{1}{2}} \; dx = \frac{-(1-2x^2)^{\frac{3}{2}}}{6} + C \quad ✓✓$$

(d) $\int \frac{-5x}{\sqrt{1+x^2}} dx$

$$\int \frac{-5x}{\sqrt{1+x^2}} dx = -\frac{5}{2}\int 2x(1+x^2)^{-\frac{1}{2}} \; dx = -5(1+x^2)^{\frac{1}{2}} + C \quad ✓✓$$

(e) $\int \frac{1}{x^2}\left(1-\frac{1}{x}\right)^3 dx$

$$\int \frac{1}{x^2}\left(1-\frac{1}{x}\right)^3 dx = \frac{1}{4}\left(1-\frac{1}{x}\right)^4 \quad ✓✓$$

(f) $\int \frac{1}{\sqrt{x}}(2+\sqrt{x})^4 \; dx$

$$\int \frac{1}{\sqrt{x}}(2+\sqrt{x})^4 \; dx = 2\int \frac{1}{2\sqrt{x}}(2+\sqrt{x})^4 \; dx = \frac{2(2+\sqrt{x})^5}{5} + C \quad ✓✓$$

Calculator Free

4. [12 marks: 1, 1, 2, 2, 2, 2, 2]

Find:

(a) $\int e^{0.05x} \; dx$

$$\int e^{0.05x} \; dx = 20e^{0.05x} + C \quad ✓$$

(b) $\int e^{-3x} \; dx$

$$\int e^{-3x} \; dx = \frac{e^{-3x}}{-3} + C \quad ✓$$

(c) $\int \frac{1}{2e^{2x}} \; dx$

$$\int \frac{e^{-2x}}{2} \; dx = \frac{e^{-2x}}{-4} + C \quad ✓✓$$

(d) $\int (1+e^{2x})^2 \; dx$

$$\int (1+e^{2x})^2 \; dx = \int 1 + 2e^{2x} + e^{4x} \; dx \quad ✓$$
$$= x + e^{2x} + \frac{e^{4x}}{4} + C \quad ✓$$

(e) $\int x\,e^{-x^2} \; dx$

$$\int x\,e^{-x^2} \; dx = -\frac{1}{2}\int -2x\,e^{-x^2} \; dx \quad ✓$$
$$= -\frac{1}{2}e^{-x^2} + C \quad ✓$$

(f) $\int x^2 e^{2x^3} \; dx$

$$\int x^2 e^{2x^3} \; dx = \frac{1}{6}\int 6x^2 e^{2x^3} \; dx \quad ✓$$
$$= \frac{1}{6}e^{2x^3} + C \quad ✓$$

(g) $\int e^{-x}(1+e^{-x})^3 \; dx$

$$\int e^{-x}(1+e^{-x})^3 \; dx = -\int -e^{-x}(1+e^{-x})^3 \; dx \quad ✓$$
$$= \frac{-(1+e^{-x})^4}{4} + C \quad ✓$$

Mathematics Methods Units 3 & 4 Revision Series — 17 Anti-Differentiation

Calculator Free

Mathematics Methods Units 3 & 4 Revision Series — 17 Anti-Differentiation

Calculator Free

5. [16 marks: 2, 2, 2, 2, 2, 3, 3]

Determine:

(a) $\int \cos 2x \ dx$

$$\int \cos 2x \ dx = \frac{\sin 2x}{2} + C \qquad ✓✓$$

(b) $\int \sin\left(\frac{x}{2}\right) dx$

$$\int \sin\left(\frac{x}{2}\right) dx = -2\cos\left(\frac{x}{2}\right) + C \qquad ✓✓$$

(c) $\int 3\cos(1-4x) \ dx$

$$\int 3\cos(1-4x) \ dx = \frac{3\sin(1-4x)}{-4} + C \qquad ✓✓$$

(d) $\int x\sin(x^2) \ dx$

$$\int x\sin(x^2) \ dx = \frac{-\cos(x^2)}{2} + C \qquad ✓✓$$

(e) $\int 3\cos x\,(\sin x)^4 \ dx$

$$\int 3\cos x\,(\sin x)^4 \ dx = \frac{3(\sin x)^5}{5} + C \qquad ✓✓$$

(f) $\int \frac{2\sin x}{5(\cos x)^3} \ dx$

$$\int \frac{2\sin x}{5(\cos x)^3} \ dx = \frac{3}{5} \times \frac{(\cos x)^{-2}}{2} + C \qquad ✓✓✓$$

(g) $\int -3\cos 2x\sqrt{1+\sin 2x} \ dx$

$$\int -3\cos 2x\sqrt{1+\sin 2x} \ dx = \frac{-3}{2} \times \frac{(1+\sin 2x)^{\frac{3}{2}}}{\frac{3}{2}} + C \qquad ✓✓✓$$

Calculator Free

6. [11 marks: 2, 2, 3, 4]

Determine:

(a) $\int \frac{-4}{3+2x} \ dx$

$$\int \frac{-4}{3+2x} \ dx = -2\int \frac{2}{3+2x} \ dx \qquad ✓$$
$$= -2 \ ln\,|3+2x| + C \qquad ✓$$

(b) $\int \frac{5x}{1+3x^2} \ dx$

$$\int \frac{5x}{1+3x^2} \ dx = \frac{5}{6}\int \frac{6x}{1+3x^2} \ dx \qquad ✓$$
$$= \frac{5}{6} \ ln\,|1+3x^2| + C \qquad ✓$$

(c) $\int \left(1-\frac{1}{x}\right)^2 \ dx$

$$\int \left(1-\frac{1}{x}\right)^2 dx = \int 1 - \frac{2}{x} + \frac{1}{x^2} \ dx \qquad ✓$$
$$= x - 2 \ ln\,|x| - \frac{1}{x} + C \qquad ✓✓$$

(d) $\int \frac{(x+2)^2}{x} \ dx$

$$\int \frac{(x+2)^2}{x} \ dx = \int \frac{(x^2+4x+4)}{x} \ dx \qquad ✓$$
$$= \int x + 4 + \frac{4}{x} \ dx \qquad ✓$$
$$= \frac{x^2}{2} + 4x + 4 \ ln\,|x| + C \qquad ✓✓$$

Calculator Free

7. [10 marks: 2, 3, 2, 3]

Determine:

(a) $\int \frac{5e^{-x}}{3+4e^{-x}}\ dx$

$$\int \frac{5e^{-x}}{3+4e^{-x}}\ dx = \frac{5}{-4}\int \frac{-4e^{-x}}{3+4e^{-x}}\ dx \quad ✓$$
$$= -\frac{5}{4}\ ln\ |3+4e^{-x}| + C \quad ✓$$

(b) $\int \tan 2x\ dx$

$$\int \tan 2x\ dx = \int \frac{\sin 2x}{\cos 2x}\ dx \quad ✓$$
$$= \frac{1}{-2}\int \frac{-2\sin 2x}{\cos 2x}\ dx \quad ✓$$
$$= -\frac{1}{2}\ ln\ |\cos 2x| + C \quad ✓$$

(c) $\int \frac{\cos \pi x}{1+\sin \pi x}\ dx$

$$\int \frac{\cos \pi x}{1+\sin \pi x}\ dx = \frac{1}{\pi}\int \frac{\pi \cos \pi x}{1+\sin \pi x}\ dx \quad ✓$$
$$= \frac{1}{\pi}\ ln\ |1+\sin \pi x| + C \quad ✓$$

(d) $\int \frac{4\sin 2x}{1-3\sin^2 x}\ dx$

$$\int \frac{4\sin 2x}{1-3\sin^2 x}\ dx = \int \frac{8\sin x \cos x}{1-3\sin^2 x}\ dx \quad ✓$$
$$= \frac{8}{-6}\int \frac{-6\sin x \cos x}{1-3\sin^2 x}\ dx \quad ✓$$
$$= -\frac{4}{3}\ ln\ |1-3\sin^2 x| + C \quad ✓$$

Calculator Free

8. [3 marks]

Find $f(x)$ if $f'(x) = \frac{1}{\sqrt{x}} + 4x - 1$ and $f(9) = 150$.

$$f(x) = \int x^{-\frac{1}{2}} + 4x - 1\ dx$$
$$= 2x^{\frac{1}{2}} + 2x^2 - x + C \quad ✓$$

When $f(9) = 150$:
$\Rightarrow 150 = 6 + 162 - 9 + C \Rightarrow C = -9$
Hence, $f(x) = 2x^{\frac{1}{2}} + 2x^2 - x - 9$ ✓✓

9. [4 marks]

Find $f(x)$ if $f'(x) = 2x^2 - 3x + a$ where a is a constant and $f(0) = -2$ and $f(-1) = -4$.

$$f(x) = \int 2x^2 - 3x + a\ dx$$
$$= \frac{2x^3}{3} - \frac{3x^2}{2} + ax + C \quad ✓$$

$f(0) = -2 \Rightarrow -2 = C$ ✓

Hence, $f(x) = \frac{2x^3}{3} - \frac{3x^2}{2} + ax - 2$

$f(-1) = -4 \Rightarrow \frac{-2}{3} - \frac{3}{2} - a - 2 = -4 \Rightarrow a = \frac{-1}{6}$ ✓

Hence, $f(x) = \frac{2x^3}{3} - \frac{3x^2}{2} - \frac{x}{6} - 2$ ✓

10. [4 marks]

The gradient function of a curve is given by $g(x) = \frac{1}{(1-2x)^2}$. Find the equation of the curve given that the curve has a horizontal asymptote with equation $y = -4$.

$$y = \int (1-2x)^{-2}\ dx \quad ✓$$
$$= \frac{(1-2x)^{-1}}{(-1)(-2)} + C \quad ✓$$
$$= \frac{1}{2(1-2x)} + C$$

Since horizontal asymptote is $y = -4$, $C = -4$.
Hence $y = \frac{1}{2(1-2x)} - 4$ ✓✓

Mathematics Methods Units 3 & 4 Revision Series 17 Anti-Differentiation
Mathematics Methods Units 3 & 4 Revision Series 17 Anti-Differentiation

Calculator Free

11. [3 marks]

Given that $f'(x) = 2e^{-2x}$, find $f(x)$ if $f(0) = 2$.

$$f(x) = \int 2e^{-2x}\,dx$$
$$= -e^{-2x} + C \quad ✓$$
$$f(0) = 2 \Rightarrow 2 = -1 + C \Rightarrow C = 3 \quad ✓$$
Hence, $f(x) = -e^{-2x} + 3$. ✓

12. [4 marks]

The gradient function of a curve is given by $\dfrac{dy}{dx} = x - \dfrac{e^x}{2}$. Find the equation of this curve given that it passes through the point (0, –2).

$$y = \int x - \frac{e^x}{2}\,dx$$
$$= \frac{x^2}{2} - \frac{e^x}{2} + C \quad ✓$$
$$x = 0, y = -2 \Rightarrow -2 = -\frac{1}{2} + C \Rightarrow C = -\frac{3}{2} \quad ✓✓$$
Hence, $y = \dfrac{x^2}{2} - \dfrac{e^x}{2} - \dfrac{3}{2}$ ✓

13. [4 marks]

The gradient function of a curve is given by $\dfrac{dy}{dx} = x(1+0.5x^2)^5$. Find the equation of this curve given that it passes through the point (0, 1)

$$y = \int x(1+0.5x^2)^5\,dx$$
$$= \frac{(1+0.5x^2)^6}{6} + C \quad ✓✓$$
$$x = 0, y = 1 \Rightarrow 1 = \frac{1}{6} + C \Rightarrow C = \frac{5}{6} \quad ✓$$
Hence, $y = \dfrac{(1+0.5x^2)^6}{6} + \dfrac{5}{6}$. ✓

Since horizontal asymptote is $y = -4$, $C = -4$.

Hence $y = \dfrac{1}{2(1-2x)} - 4$ ✓✓

Calculator Free

14. [4 marks]

The gradient function of a curve is given by $\dfrac{dy}{dx} = \dfrac{x}{1+x^2}$. Find the equation of this curve given that it passes through the point (0, 1).

$$y = \int \frac{x}{1+x^2}\,dx$$
$$= \frac{1}{2}\int \frac{2x}{1+x^2}\,dx \quad ✓$$
$$= \frac{1}{2}\ln|1+x^2| + C \quad ✓$$
When $x = 0, y = 1 \Rightarrow C = 1$ ✓

Hence, $y = \dfrac{1}{2}\ln|1+x^2| + 1$ ✓

15. [5 marks]

The gradient function of a curve is given by $\dfrac{dy}{dx} = a + b\cos 2x$. The curve has a stationary point at (0, 0). Find the equation of this curve.

Stationary point at (0, 0):
$$\Rightarrow a + b\cos 0 = 0 \quad ✓$$
$$b = -a \quad ✓$$
$$y = \int a - a\cos 2x\,dx$$
$$= ax - \frac{a}{2}\sin 2x + C \quad ✓$$
When $x = 0, y = 0: \Rightarrow C = 0$

Hence, $y = ax - \dfrac{a}{2}\sin 2x$ for $a \neq 0, a \in R$ ✓✓

Calculator Free

16. [9 marks: 2, 4, 3]

(a) Given that $y = x\,e^x$, find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = x\,e^x + e^x \quad ✓✓$$

(b) Use your answer in (a) to find $\int x\,e^x\,dx$.

Reversing the result in (a):

$$\int x\,e^x + e^x\,dx = x\,e^x + C \quad ✓$$
$$\int x\,e^x\,dx + \int e^x\,dx = x\,e^x + C \quad ✓$$
$$\int x\,e^x\,dx + e^x = x\,e^x + C \quad ✓$$
Hence, $\int x\,e^x\,dx = x\,e^x - e^x + C$ ✓

(c) The gradient function of a curve is given by $\dfrac{dy}{dx} = x\,e^x$. Find the equation of this curve given that when $x = 0$, $y = 2$.

$y = x\,e^x - e^x + C$ ✓

$x = 0, y = 2 \Rightarrow 2 = -1 + C \Rightarrow C = 3$ ✓

Hence, $y = x\,e^x - e^x + 3$ ✓

18 The Fundamental Theorem of Calculus

Calculator Free

1. [10 marks: 3, 3, 4]

Evaluate each of the following definite integrals by first determining the associated indefinite integral:

(a) $\int_{-1}^{1} x^3 + 2x + 1\,dx$

$$\int_{-1}^{1} x^3 + 2x + 1\,dx = \left[\frac{x^4}{4} + x^2 + x\right]_{-1}^{1} \quad ✓$$
$$= \frac{9}{4} - \frac{1}{4} = 2 \quad ✓✓$$

(b) $\int_{0}^{1} \dfrac{3x^2}{4} + \sqrt{x} - 2\,dx$

$$\int_{0}^{1} \frac{3x^2}{4} + \sqrt{x} - 2\,dx = \left[\frac{x^3}{4} + \frac{2x^{3/2}}{3} - 2x\right]_{0}^{1} \quad ✓$$
$$= -\frac{13}{12} - 0 = -\frac{13}{12} \quad ✓✓$$

(c) $\int_{1}^{4} \dfrac{2}{x^3} - \dfrac{1}{2\sqrt{x}}\,dx$

$$\int_{1}^{4} 2x^{-3} - \frac{x^{-1/2}}{2}\,dx = \left[\frac{-1}{x^2} - x^{1/2}\right]_{1}^{4} \quad ✓✓$$
$$= -\frac{33}{16} - (-2)$$
$$= -\frac{1}{16} \quad ✓✓$$

Calculator Assumed

Calculator Assumed

2. [9 marks: 3, 3, 3]

Evaluate each of the following definite integrals by first determining the associated indefinite integral:

(a) $\displaystyle\int_0^2 (1-4x)^3\,dx$

$$\int_0^2 (1-4x)^3\,dx = \left[\frac{(1-4x)^4}{-16}\right]_0^2 \quad ✓✓$$
$$= -150 \quad ✓$$

(b) $\displaystyle\int_0^5 \sqrt{(4+x)^3}\,dx$

$$\int_0^5 (4+x)^{3/2}\,dx = \left[\frac{2(4+x)^{5/2}}{5}\right]_0^5 \quad ✓✓$$
$$= \frac{422}{5} \quad ✓$$

(c) $\displaystyle\int_{-\frac{1}{2}}^0 \frac{-3}{\sqrt{(1-6x)}}\,dx$

$$\int_{-\frac{1}{2}}^0 -3(1-6x)^{-1/2}\,dx = \left[(1-6x)^{1/2}\right]_{-1/2}^0 \quad ✓✓$$
$$= -1 \quad ✓$$

Calculator Assumed

3. [9 marks: 3, 3, 3]

Evaluate each of the following definite integrals by first determining the associated indefinite integral:

(a) $\displaystyle\int_{-1}^1 (1-x)(1+x^2)\,dx$

$$\int_{-1}^1 (1-x)(1+x^2)\,dx = \left[\frac{-x^4}{4}+\frac{x^3}{3}-\frac{x^2}{2}+x\right]_{-1}^1 \quad ✓✓$$
$$= \frac{8}{3} \quad ✓$$

(b) $\displaystyle\int_1^4 \frac{\sqrt{x}+x^2}{x}\,dx$

$$\int_1^4 \frac{\sqrt{x}+x^2}{x}\,dx = \left[\frac{x^2}{2}+2x^{1/2}\right]_1^4 \quad ✓✓$$
$$= \frac{19}{2} \quad ✓$$

(c) $\displaystyle\int_4^9 \frac{9x^4-7x^3}{\sqrt{x}}\,dx$

$$\int_4^9 \frac{9x^4-7x^3}{\sqrt{x}}\,dx = \left[2x^{9/2}-2x^{7/2}\right]_4^9 \quad ✓✓$$
$$= 34\ 224 \quad ✓$$

Calculator Free

4. [7 marks: 3, 4]

Use Calculus to find the exact value of each of the following:

(a) $\int_0^2 3e^{\frac{x}{2}}\,dx$

$$\int_0^2 3e^{\frac{x}{2}}\,dx = \left[6e^{\frac{x}{2}}\right]_0^2 \quad ✓$$
$$= 6e - 6 \quad ✓✓$$

(b) $\int_0^1 (1+e^x)^2\,dx$

$$\int_0^1 (1+e^x)^2\,dx = \int_0^1 (1+2e^x+e^{2x})\,dx \quad ✓$$
$$= \left[x+2e^x+\frac{e^{2x}}{2}\right]_0^1 \quad ✓$$
$$= \left(1+2e+\frac{e^2}{2}\right)-\frac{5}{2} \quad ✓$$
$$= 2e+\frac{e^2}{2}-\frac{3}{2} \quad ✓$$

5. [4 marks: 2, 2]

Find $\frac{dy}{dx}$:

(a) $y=\int_1^x \sqrt{1+t^3}\,dt$

$$\frac{dy}{dx}=\sqrt{1+x^3} \quad ✓✓$$

(b) $y=\int_1^{x^2} \frac{1}{\sqrt{1+t^2}}\,dt$

$$\frac{dy}{dx}=\frac{2x}{\sqrt{1+x^4}} \quad ✓✓$$

Calculator Free

6. [11 marks: 2, 2, 2, 2, 3]

Find $\frac{dy}{dx}$:

(a) $y=\int_1^{e^x} (1+\sqrt{t})^5\,dt$

$$\frac{dy}{dx}=\left(1+\sqrt{e^x}\right)^5 e^x \quad ✓✓$$

(b) $y=\int_1^{e^{2x}} \frac{1}{\sqrt{1+t}}\,dt$

$$\frac{dy}{dx}=\frac{2e^{2x}}{\sqrt{1+e^{2x}}} \quad ✓✓$$

(c) $y=\int_1^{2x} e^{t^2}\,dt$

$$\frac{dy}{dx}=2e^{4x^2} \quad ✓✓$$

(d) $y=\int_1^{1+x^2} \frac{1}{\sqrt{1+e^t}}\,dt$

$$\frac{dy}{dx}=\frac{2x}{\sqrt{1+e^{1+x^2}}} \quad ✓✓$$

(e) $y=\int_{e^{2x}}^1 e^t\,dt$

$$\frac{dy}{dx}=-e^{e^{2x}}\times 2e^{2x} \quad ✓✓✓$$

Mathematics Methods Units 3 & 4 Revision Series 18 The Fundamental Theorem of Calculus

Calculator Free

7. [11 marks: 2, 2, 2, 3, 2]

Determine $\frac{dy}{dx}$:

(a) $y = \int_{1}^{\cos x} 1+t^2 \, dt$

$$\boxed{\frac{dy}{dx} = (-\sin x)\,(1+\cos^2 x) \quad ✓✓}$$

(b) $y = \int_{1}^{\tan x} e^{t^2} \, dt$

$$\boxed{\frac{dy}{dx} = (\sec^2 x)\, e^{\tan^2 x} \quad ✓✓}$$

(c) $y = \int_{1}^{\sin 2x} \sin^2(1+t) \, dt$

$$\boxed{\frac{dy}{dx} = (2\cos 2x)\sin^2(1+\sin 2x) \quad ✓✓}$$

(d) $y = \int_{\cos^2 x}^{0} \tan(\pi+u) \, du$

$$\boxed{\frac{dy}{dx} = (2\sin x \cos x)\tan(\pi+\cos^2 x) \quad ✓✓✓}$$

(e) $y = \int_{0}^{e^{\sin x}} \sqrt{1+u} \, du$

$$\boxed{\frac{dy}{dx} = (\cos x)\, e^{\sin x}\sqrt{1+e^{\sin x}} \quad ✓✓}$$

Calculator Free

8. [12 marks: 2, 2, 2, 3, 3]

Determine $\frac{dy}{dx}$:

(a) $y = \int_{1}^{\ln x} \ln(2+3t) \, dt$

$$\boxed{\frac{dy}{dx} = \frac{\ln[2+3\ln(x)]}{x} \quad ✓✓}$$

(b) $y = \int_{1}^{\ln x} \frac{1}{t} \, dt$

$$\boxed{\frac{dy}{dx} = \frac{1}{x\ln(x)} \quad ✓✓}$$

(c) $y = \int_{1}^{\ln \sin(x)} \sqrt{t} \, dt$

$$\boxed{\frac{dy}{dx} = \frac{\cos(x)}{\sin(x)} \times \sqrt{\ln \sin(x)} \quad ✓✓}$$

(d) $y = \int_{1}^{\ln(1+e^x)} (1-t)^2 \, dt$

$$\boxed{\frac{dy}{dx} = \frac{e^x}{1+e^x} \times [1-(\ln(1+e^x)]^2 \quad ✓✓✓}$$

(e) $y = \int_{x}^{\ln(x)} t^2 \, dt$

$$\boxed{\frac{dy}{dx} = \frac{[\ln(x)]^2}{x} - x^2 \quad ✓✓✓}$$

Calculator Free

9. [4 marks: 2, 2]

Evaluate:

(a) $\int_1^2 \frac{d}{dx}(1+x)\ dx$

$$\int_1^2 \frac{d}{dx}(1+x)\ dx = [1+x]_1^2 \quad \checkmark$$
$$= 3-2=1 \quad \checkmark$$

(b) $\int_0^2 \frac{d}{dx}\left(\frac{1+x}{1+x^2}\right) dx$

$$\int_0^2 \frac{d}{dx}\left(\frac{1+x}{1+x^2}\right) dx = \left[\frac{1+x}{1+x^2}\right]_0^2 \quad \checkmark$$
$$= \frac{3}{5} - 1 = -\frac{2}{5} \quad \checkmark$$

---

10. [4 marks]

A curve has equation given by $y = \int_0^{x^2} \sqrt{1+u^2}\ du$. When $x = 1$, $\frac{dx}{dt} = -1$.

Find $\frac{dy}{dt}$ when $x = 1$.

$$\frac{dy}{dx} = 2x\sqrt{1+x^4} \quad \checkmark\checkmark$$

Using the Chain rule: $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$$= 2x\sqrt{1+x^4} \times \frac{dx}{dt} \quad \checkmark$$

When $x = 1$, $\frac{dy}{dt} = 2\sqrt{2} \times -1$

$$= -2\sqrt{2} \quad \checkmark$$

Calculator Free

11. [7 marks: 2, 5]

A curve has equation given by $y = \int_0^{x^2} (t-4)^3\ dt$.

(a) Find the gradient of the curve at the point where $x = 1$.

$$\frac{dy}{dx} = (x^2-4)^3 \times 2x \quad \checkmark$$

When $x = 1$: $\frac{dy}{dx} = -54$ $\checkmark$

(b) Find x-coordinate of the turning points of the curve. Identify the nature of these points.

For turning points $\frac{dy}{dx} = 0$:

$x = 0, \pm 2$ $\checkmark\checkmark$

For $x = 0^-$, $\frac{dy}{dx} > 0$

$x = 0^+$, $\frac{dy}{dx} < 0$.

Hence there is a maximum point at $x = 0$. $\checkmark$

For $x = 2^-$, $\frac{dy}{dx} < 0$

$x = 2^+$, $\frac{dy}{dx} > 0$.

Hence there is a minimum point at $x = 2$. $\checkmark$

For $x = -2^-$, $\frac{dy}{dx} < 0$

$x = -2^+$, $\frac{dy}{dx} > 0$.

Hence there is a minimum point at $x = -2$. $\checkmark$

Calculator Assumed

19 Net Change

Calculator Assumed

12. [6 marks: 3, 3]

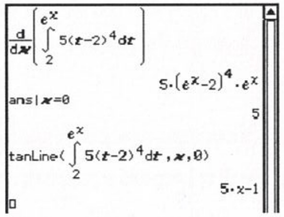
A curve has equation given by $y = \int_2^{e^x} 5(t-2)^4\, dt$.

(a) Use a calculus method to find the equation of the tangent to the curve at the point (0, –1).

$\frac{dy}{dx} = 5e^x(e^x - 2)^4$ ✓

When $x = 0$, $\frac{dy}{dx} = 5$ ✓

Hence, tangent has equation $y = 5x - 1$ ✓



(b) Find the x-coordinate of the stationary point on this curve. Determine the nature of this point

$\frac{dy}{dx} = 5e^x(e^x - 2)^4$

$\frac{dy}{dx} = 0 \Rightarrow x = \ln 2 \approx 0.6931$ ✓

For $x < 0.6931$ $\frac{dy}{dx} > 0$

For $x > 0.6931$ $\frac{dy}{dx} > 0$ ✓

Hence, $x = \ln 2 \approx 0.6931$ is the x-coordinate of a horizontal inflection point. ✓

19 Net Change

Calculator Assumed

1. [5 marks: 1, 3, 1]

Given that $\frac{dV}{dt} = (t-2)^2 + 1$, find:

(a) the instantaneous rate of change of V with respect to t when $t = 4$

When $t = 4$, $\frac{dV}{dt} = 5$ ✓

(b) the net change in V when t changes from $t = 1$ to $t = 4$

Net Change $= \int_1^4 (t-2)^2 + 1\, dt$ ✓✓

$= 6.$ ✓

(c) the average rate of change of V in the interval $1 \le t \le 4$ seconds.

Average Change $= \frac{6}{3} = 2$ ✓

2. [3 marks]

The rate of change of concentration of a chemical is given by $\frac{dC}{dt} = e^{-2t} - 2$ mg/L per week. Find the net change in concentration over the first week.

Net Change $= \int_0^1 e^{-2t} - 2\, dt$ ✓✓

$= -1.57$ mg/L. ✓

Calculator Assumed

3. [5 marks: 3, 2]

The marginal cost for producing x hundred units of a product is given by $\dfrac{dC}{dx} = 0.08x$, where \$C is the cost of producing x hundred items of the product.

(a) Find the cost of producing 100 of these items if the fixed cost is \$2000.

$C = \int 0.08x\,dx = 0.04x^2 + K$ ✓

$C(0) = 2000 \Rightarrow K = 2000$ ✓

Hence, $C = 0.04x^2 + 2000$

$\Rightarrow C(100) = \$2400$ ✓

(b) Find the net change in cost if the number of items produced is changed from 1000 to 2000. Justify your answer.

Net change in cost $= \int_{1000}^{2000} 0.08x\,dx$ ✓

$= 120\,000$ ✓

4. [5 marks: 3, 2]

The marginal profit associated with the sale of x items of a product is given by $\dfrac{dP}{dx} = -0.000\,81x^2 + 0.4x - 5.4$, where \$P is the profit associated with the sale of x units of this product.

(a) Given that there is a loss of \$500 if no items are sold, find the profit associated with the sale of 50 items.

$P = \int -0.000\,81x^2 + 0.4x - 5.4\,dx$

$= -0.000\,27x^3 + 0.2x^2 - 5.4x + K$ ✓

$P(0) = -500 \Rightarrow K = -500$ ✓

Hence, $P = -0.000\,27x^3 + 0.2x^2 - 5.4x - 500$

$\Rightarrow P(50) = -\$303.75$ ✓

(b) Find the net change in profit if the number of items sold is changed from 50 to 350.

Net Change in $P = \int_{50}^{350} -0.000\,81x^2 + 0.4x - 5.4\,dx$ ✓

$= \$10\,837.50$ ✓

Calculator Assumed

5. [5 marks: 2, 3]

Given that $P = (t-2)(t-4)$ cm where t is time in seconds.

(a) Find using Calculus, the instantaneous rate of change of P when $t = 5$ s.

$\dfrac{dP}{dt} = 2t - 6$ ✓

When $t = 5$, $\dfrac{dP}{dt} = 4\text{ cms}^{-1}$. ✓

(c) Find the average rate of change of P over the interval $0 \le t \le 5$ s.

Net Change $= \int_0^5 2t - 6\,dt$ ✓

$= -5$ cm. ✓

Hence, average rate of change $= -5/5 = -1\text{ cms}^{-1}$. ✓

6. [4 marks: 2, 2]

The instantaneous rate with which the temperature, θ degrees Celsius, of a body changes with respect to time, t minutes, is modelled by $\dfrac{d\theta}{dt} = -5 + 0.1t$.

(a) Find when the minimum temperature of the body occurred.

$\dfrac{d\theta}{dt} = 0 \Rightarrow t = 50$ ✓

$\dfrac{d^2\theta}{dt^2} = 0.1$ ✓

$\Rightarrow$ Min. temperature occurs when $t = 50$ minutes.

(b) Find the net change in temperature in the 10th minute.

Net change in the 5th minute $= \int_9^{10} \dfrac{d\theta}{dt}\,dt$ ✓

$= \int_9^{10} -5 + 0.1t\,dt$

$= -\dfrac{81}{40} = -4.05$ ✓

Mathematics Methods Units 3 & 4 Revision Series *19 Net Change*

Mathematics Methods Units 3 & 4 Revision Series *19 Net Change*

Calculator Assumed

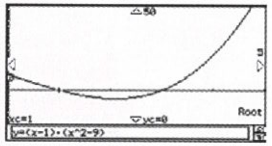
7. [7 marks: 2, 2, 3]

The instantaneous rate with which the volume of water in a holding tank changes with time, is modelled by $\frac{dV}{dt} = (t-1)(t^2-9)$, for $0 \le t \le 5$, where V is the volume of water in the tank in kL and t is time in hours.

(a) Find the interval of time during which water is flowing out of the tank. Justify your answer.

$\frac{dV}{dt} < 0$ ✓ for $1 < t < 3$

Hence, the interval $1 < t < 3$ hours. ✓



(b) Find the amount of water that has flowed out of the tank.

Amount of water that has flowed out

$= \left| \int_1^3 (t-1)(t^2-9)\, dt \right|$ ✓

$= \frac{20}{3}$ kL ✓

(c) Find the amount of water that has flowed into the tank.

Amount of water that has flowed in

$= \int_0^1 (t-1)(t^2-9)\, dt + \int_3^5 (t-1)(t^2-9)\, dt$ ✓✓

$= \frac{53}{12} + \frac{148}{3} = \frac{215}{4}$ kL ✓

Calculator Assumed

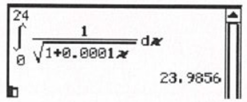
8. [7 marks: 2, 3, 2]

The instantaneous rate with which the number of computers infected with a virus at time t hours is modelled by $\frac{dN}{dt} = \frac{1}{\sqrt{1+0.0001t}}$ where $N(t)$ is the number of computers already infected with the virus.

(a) Find the net change in the number of computers infected with the virus within the first day.

$\Delta N = \int_0^{24} \frac{1}{\sqrt{1+0.0001t}}\, dt$ ✓

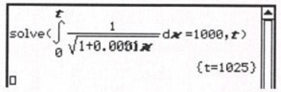
≈ 24 computers ✓



(b) How long will it take to infect 1000 computers?

$\int_0^k \frac{1}{\sqrt{1+0.0001t}}\, dt = 1000$ ✓✓

$k = 1025$ hours ✓



(c) Determine with reasons if this mathematical model is a reasonable model.

No. ✓

The number of computers infected increases without limit! ✓

Calculator Assumed

9. [10 marks: 2, 2, 2, 2, 2]

The rate of population change of a bacteria culture is modelled by $\frac{dP}{dt} = 100\,e^{-0.01t}$ where t is time in hours.

(a) Find the initial rate of population change.

$\left.\frac{dP}{dt}\right|_{t=0} = 100$ ✓

Initial rate of change = increasing at rate 100 per hour. ✓

(b) Describe the rate of change for large values of t.

As $t \to \infty$, $\frac{dP}{dt} \to 0$ ✓

For large values of t, the population size stops growing. ✓

(c) Calculate the net population change in the first 10 hours.

$\Delta P = \int_0^{10} 100\,e^{-0.01t}\,dt$ ✓

$= 951.63 \approx 951$ ✓

(d) Calculate the net population change in the first 2 000 hours.

$\Delta P = \int_0^{2000} 100\,e^{-0.01t}\,dt$ ✓

$\approx 10\,000$ ✓

(e) Given that the initial population was 100, find the maximum population size. Show clearly how you obtained your answer.

Maximum population size = Initial size + ΔP(for large t) ✓

= 100 + 10 000

= 10 100 ✓

Calculator Assumed

10. [8 marks: 2, 3, 3]

The change in altitude of a balloon is modelled by $\frac{dh}{dt} = \frac{1}{2+t}$ where h metres is the altitude of the balloon at time t seconds.

(a) Find the height increase of the balloon in the 5th second.

$\Delta h = \int_4^5 \frac{1}{2+t}\,dt$ ✓

$= 0.1542 \approx 0.15$ metres ✓

(b) Find the average rate of height increase in the first 10 seconds.

$\Delta h = \int_0^{10} \frac{1}{2+t}\,dt$ ✓

$= 1.7918$ ✓

Average height increase $= \frac{1.7918}{10}$

≈ 0.18 metres/second ✓

(c) The initial height of the balloon was 2 metres. Find when the height of the balloon first exceeds 5 metres.

$\Delta h = 3$ ✓

$\int_0^T \frac{1}{2+t}\,dt = 3$ ✓

$T = 38.17$ (reject -42.17)

Hence, after 38.2 seconds. ✓

$\text{solve}\left(\int_0^x \frac{1}{2+t}\,dt = 3, x\right)$

{x=-42.17107385,x=38.17107385}

Calculator Assumed

11. [10 marks: 3, 3, 3, 1]

The rate of change of pressure acting on an object is modelled by
$\dfrac{dP}{dt} = 4\cos\left(\dfrac{\pi t}{12}\right)$ where P (kilopascals) is the pressure at time t hours.

(a) Find the net change in pressure in the interval $0 \le t \le 12$.

$$\Delta P = \int_0^{12} 4\cos\left(\frac{\pi t}{12}\right)\, dt \quad ✓$$
$$= 0 \quad ✓$$

Net change is zero

(b) Find the net change in pressure in the interval $6 \le t \le 12$.

$$\Delta P = \int_6^{12} 4\cos\left(\frac{\pi t}{12}\right)\, dt \quad ✓$$
$$= -15.2789 \quad ✓$$

Net change is a decrease in pressure of 15.3 kilopascals. ✓

(c) The net increase in pressure in the interval $0 \le t \le T$ is 10 kilopascals. Find T given that $T \le 24$.

$$\Delta P = \int_0^{T} 4\cos\left(\frac{\pi t}{12}\right)\, dt = 10 \quad ✓$$
$T = 2.7254$ or 9.2746 ✓✓

$\text{solve}\left(\int_0^x 4\cos\left(\frac{\pi t}{12}\right)dt = 10, x\right) \mid 0 \le x \le 24$

$\{x = 9.2746, x = 2.7254\}$

(d) Comment on your answer in part (c).

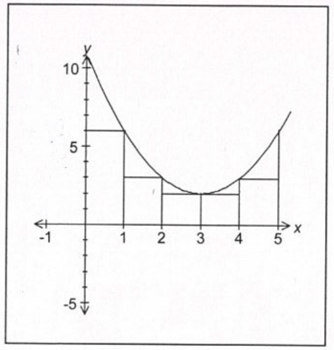
Net change for is 10 kilopascals
for $0 \le t \le 2.73$ hours and $0 \le t \le 9.27$ hours.
Hence, for $2.73 \le t \le 9.27$ hours the net change is 0. ✓

20 Area Under a Curve

Calculator Assumed

1. [10 marks: 4, 4, 2]

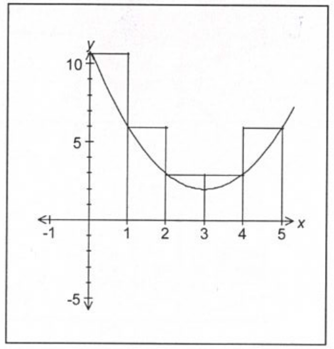
The graph of $y = 2 + (x-3)^2$ is shown in the accompanying diagram.



(a) Estimate the area of the region trapped between this curve, the x-axis and the lines $x = 0$ and $x = 5$ using inscribed rectangles of uniform width 1.

Let $f(x) = 2 + (x-3)^2$
$A = [1 \times f(1)] + [1 \times f(2)] + [1 \times f(3)] + [1 \times f(3)] + [1 \times f(4)]$ ✓✓✓
$= 6 + 3 + 2 + 2 + 3$
$= 16$ ✓

(b) Estimate the area of the region trapped between this curve, the x-axis and the lines $x = 0$ and $x = 5$ using circumscribed rectangles of uniform width 1.



Let $f(x) = 2 + (x-3)^2$
$A = [1 \times f(0)] + [1 \times f(1)] + [1 \times f(2)] + [1 \times f(4)] + [1 \times f(5)]$ ✓✓✓
$= 11 + 6 + 3 + 3 + 6$
$= 29$ ✓

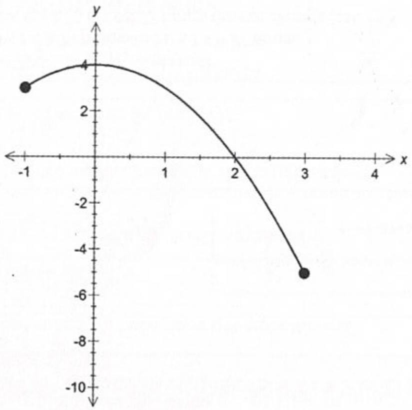
(c) Use your answer in (a) and (b) to estimate to one decimal place the area of the region trapped between this curve, the x-axis and the lines $x = 0$ and $x = 5$

From (a) and (b), $16 \le A \le 29$.
Hence $A \approx \frac{16 + 29}{2} = 22.5$ ✓✓

Calculator Assumed

2. [8 marks: 3, 3, 2]

The graph of $y = 4 - x^2$ is shown in the accompanying diagram.



(a) Estimate the area of the region trapped between this curve and the lines $x = 1$ and $x = 3$ using *inscribed* rectangles of uniform width 0.5. Give your answer to one decimal place.

$$\begin{aligned} \text{Area} &= 0.5 \times (4 - 1.5^2) + 0.5 \times (4 - 2.5^2) \quad ✓✓ \\ &= 0.875 + 1.125 \\ &= 2 \quad ✓ \end{aligned}$$

(b) Estimate the area of the region trapped between this curve and the lines $x = 1$ and $x = 3$ using *circumscribed* rectangles of uniform width 0.5. Give your answer to one decimal place.

$$\begin{aligned} \text{Area} &= 0.5 \times (4 - 1^2) + 0.5 \times (4 - 1.5^2) \quad ✓ \\ &\quad + 0.5 \times (4 - 2.5^2) + 0.5 \times (4 - 3^2) \quad ✓ \\ &= 2.375 + 3.625 \\ &= 6 \quad ✓ \end{aligned}$$

(c) Use your answer in (a) and (b) to estimate to one decimal place the area of the region trapped between this curve and the lines $x = 1$ and $x = 3$.

$$\text{Estimate} = \frac{2+6}{2} = 4 \quad ✓✓$$

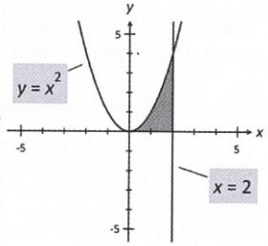
Calculator Free

3. [10 marks: 3, 4, 3]

Use calculus to find the exact area of the shaded region:

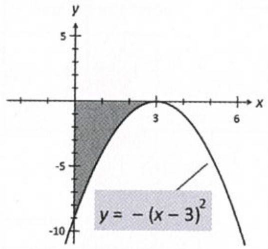
(a)

$$\begin{aligned} \text{Area} &= \int_0^2 x^2 \, dx \quad ✓ \\ &= \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3} \quad ✓✓ \end{aligned}$$



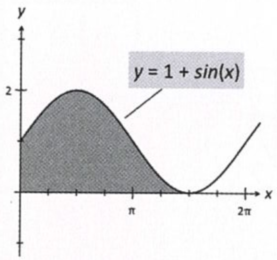
(b)

$$\begin{aligned} \text{Area} &= -\int_0^3 -(x-3)^2 \, dx \quad ✓✓ \\ &= \left[\frac{(x-3)^3}{3}\right]_0^3 \quad ✓ \\ &= 9 \quad ✓ \end{aligned}$$



(c)

$$\begin{aligned} \text{Area} &= \int_0^{3\pi/2} 1 + \sin(x) \, dx \quad ✓ \\ &= \left[x - \cos(x)\right]_0^{\frac{3\pi}{2}} \quad ✓ \\ &= 1 + \frac{3\pi}{2} \quad ✓ \end{aligned}$$



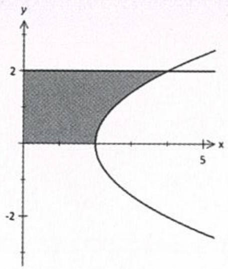
Mathematics Methods Units 3 & 4 Revision Series *20 Area Under a Curve*

Mathematics Methods Units 3 & 4 Revision Series *20 Area Under a Curve*

Calculator Free

4. [5 marks]

The shaded region shown in the accompanying diagram is trapped by the x-axis, the y-axis, the line $y = 2$ and the curve with equation $y^2 = 2x - 4$. Find the area of the shaded region.



$y = 2$ and $y^2 = 2x - 4$ intersect at:

$4 = 2x - 4 \Rightarrow x = 4$ ✓

$$\text{Area} = 4 \times 2 - \int_2^4 \sqrt{2x-4}\, dx \quad ✓✓$$

$$= 8 - \left[\frac{2(2x-4)^{\frac{3}{2}}}{3(2)}\right]_2^4 \quad ✓$$

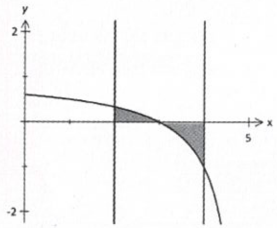
$$= 8 - \frac{1}{3}\left[(8-4)^{\frac{3}{2}}\right]$$

$$= 8 - \frac{8}{3} = \frac{16}{3} \quad ✓$$

5. [6 marks]

The shaded region shown in the accompanying diagram is trapped by the lines $x = 2$, $x = 4$ and the curve with equation $y = \dfrac{2}{x-5} + 1$.

Find the area of the shaded region.



The given curve cuts the x-axis at $x = 3$. ✓

$$\text{Area} = \int_2^3 \frac{2}{x-5} + 1\ dx - \int_3^4 \frac{2}{x-5} + 1\ dx \quad ✓✓$$

$$= \left[2\ln(x-5) + x\right]_2^3 - \left[2\ln(x-5) + x\right]_3^4 \quad ✓$$

$$= 1 + \ln\frac{4}{9} - (1 - 2\ln 2) \quad ✓$$

$$= 2\ln\frac{4}{3} \quad ✓$$

Calculator Assumed

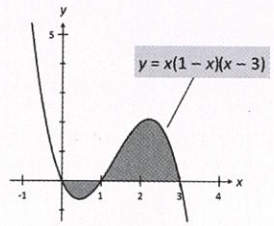
6. [14 marks: 3, 3, 4, 4]

Use an appropriate method to find the area of the shaded region. Show clearly how you obtained your answer.

(a)

$$\text{Area} = \int_0^3 \left| x(1-x)(x-3) \right| dx \quad ✓✓$$

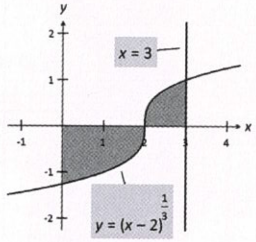
$$= \frac{37}{12} \text{ (or 3.08)} \quad ✓$$



(b)

$$\text{Area} = \int_0^3 \left| (x-2)^{\frac{1}{3}} \right| dx \quad ✓✓$$

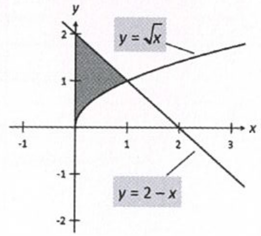
$$= 2.64 \quad ✓$$



(c)

$$\text{Area} = \int_0^1 (2-x) - x^{\frac{1}{2}}\ dx \quad +✓✓✓$$

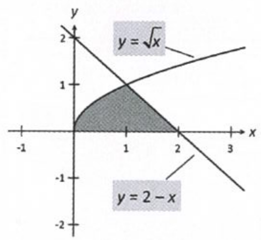
$$= \frac{5}{6} \quad ✓$$



(d)

$$\text{Area} = \left[\int_0^1 x^{\frac{1}{2}}\ dx\right] + \left[\frac{1}{2} \times 1 \times 1\right] \quad ✓✓$$

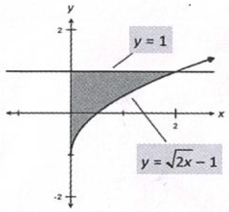
$$= \frac{2}{3} + \frac{1}{2} = \frac{7}{6} \quad ✓✓$$



Calculator Assumed

7. [7 marks: 2, 3, 2]

Consider the curve with equation $y = \sqrt{2x} - 1$.



(a) Rewrite the equation of the curve with y as the independent variable.

$$\sqrt{2x} = y + 1$$
$$x = \frac{1}{2}(y+1)^2 \quad ✓✓$$

(b) Let R represent the region trapped between the curve with equation $y = \sqrt{2x} - 1$, the y-axis and the line $y = 1$.

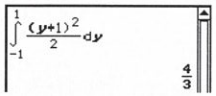
(i) Use your answer in (a) and an integral to express the area of R.

$y = \sqrt{2x} + 1$ intersects the y-axis at $y = -1$. ✓

$$\text{Area} = \int_{-1}^{1} \frac{1}{2}(y+1)^2 \, dy \quad ✓✓$$

(ii) Use your answer above to determine the exact area of R.

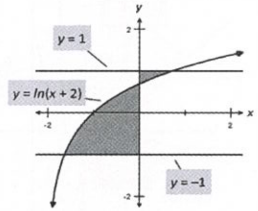
$$\text{Area} = \int_{-1}^{1} \frac{1}{2}(y+1)^2 \, dy$$
$$= \frac{4}{3} \quad ✓✓$$



Calculator Assumed

8. [8 marks: 2, 4, 2]

Consider the curve with equation $y = ln\,(x + 2)$.



(a) Rewrite the equation of the curve with y as the independent variable.

$$ln\,(x+2) = y$$
$$x + 2 = e^y \quad ✓$$
$$x = e^y - 2 \quad ✓$$

(b) Let R represent the region trapped between the curve with equation $y = ln\,(x + 2)$, the y-axis and the lines $y = -1$ and $y = 1$.

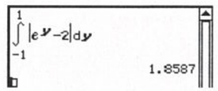
(i) Use your answer in (a) and integrals to express the area of R.

$y = ln\,(x + 2)$ intersects the y-axis at $y = ln\,2$. ✓

$$\text{Area} = -\int_{-1}^{ln\,2} e^y - 2 \, dy + \int_{ln\,2}^{1} e^y - 2 \, dy \quad ✓✓✓$$

(ii) Use your answer above to determine the area of R. Give your answer to two decimal places.

$$\text{Area} = -\int_{-1}^{ln\,2} e^y - 2 \, dy + \int_{ln\,2}^{1} e^y - 2 \, dy$$
$$= 1.7542 + 0.1046 \quad ✓$$
$$\approx 1.86 \quad ✓$$



Mathematics Methods Units 3 & 4 Revision Series — 20 Area Under a Curve
Mathematics Methods Units 3 & 4 Revision Series — 20 Area Under a Curve

Calculator Assumed

9. [7 marks: 2, 2, 3]

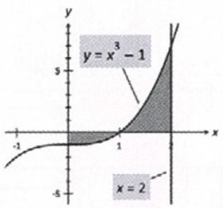
(a) Use Calculus to evaluate $\int_0^2 x^3 - 1 \, dx$.

$$\text{Integral} = \left[\frac{x^4}{4} - x\right]_0^2 \quad ✓$$
$$= 2 \quad ✓$$

(b) Explain clearly why the value of $\int_0^2 x^3 - 1 \, dx$ does not refer to the area of the region trapped between the curve $y = x^3 - 1$, the x-axis, the y-axis and the line $x = 2$.

Required region consists of a sub-region above and a sub-region below the x-axis. ✓

$\int_0^2 x^3 - 1 \, dx$ only gives the area if the curve is completely above the x-axis for $0 \le x \le 2$. ✓



(c) Hence, explain clearly how you would calculate the area of the region trapped between the curve $y = x^3 - 1$, the x-axis, the y-axis and the line $x = 2$.

Use Area $= \int_0^2 \left| x^3 - 1 \right| dx$ and evaluate this integral using a CAS/Graphics calculator definite integral routine. ✓✓✓

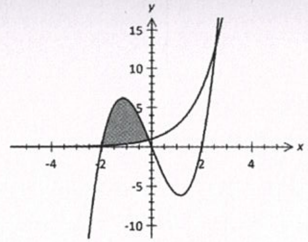
Alternatively, use calculus to evaluate:

$$\text{Area} = -\int_0^1 x^3 - 1 \, dx + \int_1^2 x^3 - 1 \, dx. \quad ✓✓✓$$

Calculator Assumed

10. [8 marks: 4, 4]

The accompanying diagram shows the graphs of $y = e^x$ and $y = 2x(x^2 - 4)$.



(a) Write an integral that can be used to determine the area of the shaded region. Hence, find the area of this region.

For $x \le 0$, the two curves intersect at $x = -1.99141$ and $x = -0.112097$ ✓

$$\text{Area} = \int_{-1.99141}^{-0.112097} 2x(x^2 - 4) - e^x \, dx \quad ✓✓$$
$$= 7.19178 \approx 7.19 \text{ units}^2 \quad ✓$$

(b) Write but do not evaluate, an expression involving integrals that can be used to determine the area of the region trapped between the two curves, for $x \ge 0$.

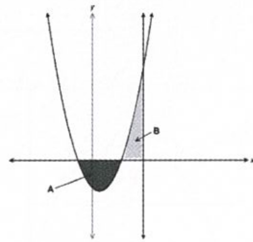
For $x \ge 0$, the two curves intersect at $x = 2.552433$ and $x = 5.890915$ ✓✓

$$\text{Required Area} = \int_0^{2.552433} e^x - 2x(x^2 - 4) \, dx + \int_{2.552433}^{5.890915} 2x(x^2 - 4) - e^x \, dx$$
✓ ✓

11. [6 marks]

The accompanying diagram shows the graph of $y = (x + 1)(x - 2)$. The region bounded by the curve and the x-axis is denoted A. The region bounded by the curve, the positive x-axis and the line $x = k$ is denoted B.

Find k if the area of A = area of B.



$$\text{Area of A} = -\int_{-1}^{2} (x+1)(x-2) \, dx = \frac{9}{2} \quad ✓✓$$

$$\text{Area of B} = \int_2^k (x+1)(x-2) \, dx$$
$$= \frac{k^3}{3} - \frac{k^2}{2} - 2k + \frac{10}{3} \quad ✓✓$$

$$\text{Hence, } \frac{k^3}{3} - \frac{k^2}{2} - 2k + \frac{10}{3} = \frac{9}{2} \quad ✓$$

$$k = \frac{7}{2} \quad (\text{reject } k = -1) \quad ✓$$

Calculator Free

12. [9 marks: 2, 2, 3, 2]

The function $y = f(x)$ is continuous for all real values of x and $f(x) \geq 0$ for $1 \leq x \leq 4$. It is known that $\int_1^4 f(x)\,dx = \text{A}$ and $\int_4^6 f(x)\,dx = -\text{B}$ here A and B are positive real numbers. Find, with reasons, in terms of A and/or B where appropriate:

(a) the area of the region trapped between the curve $y = 2f(x)$, the x-axis and the lines $x = 1$ and $x = 4$.

$$\text{Area} = \int_1^4 2f(x)\,dx \quad ✓$$
$$= 2\int_1^4 f(x)\,dx$$
$$= 2\text{A} \quad ✓$$

(b) $\int_1^6 f(x)\,dx$

$$\int_1^6 f(x)\,dx = \int_1^4 f(x)\,dx + \int_4^6 f(x)\,dx \quad ✓$$
$$= \text{A} - \text{B} \quad ✓$$

(c) $\int_4^6 2x - f(x)\,dx$

$$\int_4^6 2x - f(x)\,dx = \int_4^6 2x\,dx - \int_4^6 f(x)\,dx \quad ✓$$
$$= \left[x^2\right]_4^6 - (-\text{B}) \quad ✓$$
$$= 20 + \text{B} \quad ✓$$

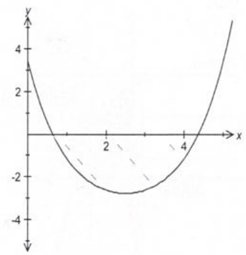
(d) $\int_{-1}^{-4} f(-x)\,dx$

$$\int_{-1}^{-4} f(-x)\,dx = -\int_{-4}^{-1} f(-x)\,dx \quad ✓$$
$$= -\int_1^4 f(x)\,dx$$
$$= -\text{A} \quad ✓$$

Calculator Assumed

13. [12 marks: 2, 5, 5]

A 10-metre portion of an irrigation channel is of uniform cross-section. The cross-section is modelled by the equation $y = e^{x-3} + e^{-x+2} - 4$ where x is measured in metres. The top edge of the channel is modelled by the line $y = 0$.



(a) For what values of x is the model of the cross-section valid?

Model is valid for $y \leq 0$.
Hence, $0.6375 \leq x \leq 4.3625$ ✓✓

(b) Use Calculus to find the depth of the deepest point of this portion of the channel.

$y' = e^{x-3} - e^{-x+2}$ ✓

At the deepest point, $y' = 0$.
$\Rightarrow e^{x-3} - e^{-x+2} = 0$ ✓
$\Rightarrow x = 2.5$ ✓

$y'' = e^{x-3} + e^{-x+2}$
When $x = 2.5$, $y'' > 0$. ✓

Hence, channel is deepest when $x = 2.5$ m
with depth $y = e^{-0.5} + e^{-0.5} - 4 = 2.79$ metres. ✓

(c) Find the maximum capacity of this portion of the channel. [$1\text{ m}^3 = 1$ kL]

$$\text{Cross-sectional area} = \int_{0.6375}^{4.3625} \left| e^{x-3} + e^{-x+2} - 4 \right| dt \quad ✓✓$$
$$= 7.2765\text{ m}^2 \quad ✓$$

Hence, maximum capacity $= 7.2765 \times 10$ ✓
$= 72.765\text{ m}^3$
$= 72.77$ kL ✓

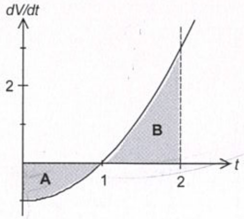
Mathematics Methods Units 3 & 4 Revision Series — *20 Area Under a Curve*

Mathematics Methods Units 3 & 4 Revision Series — *20 Area Under a Curve*

Calculator Assumed

14. [9 marks: 1, 2, 2, 2, 2]

The instantaneous rate with which the amount of fuel, V litres, in a holding tank, changes with respect to time t minutes, is modelled by $\frac{dV}{dt} = t^2 - 1$. The sketch of $\frac{dV}{dt}$ against t is shown in the accompanying diagram.



(a) Explain what happens at $t = 1$ minute.

> The amount of fuel in the tank is at minimum level (locally). ✓

(b) Find the area of region A and interpret your answer.

> Area of A = $-\int_0^1 t^2 - 1\, dt = 2/3$ ✓
> This is the amount of fuel (L) drawn out from the tank in the first minute. ✓

(c) Find the area of region B and interpret your answer.

> Area of A = $\int_1^2 t^2 - 1\, dt = 4/3$ ✓
> This is the amount of fuel (L) entering the tank in the second minute. ✓

(d) Find the amount of fuel in the tank after 2 minutes, if initially there were 5 litres in the tank.

> Amount of fuel = $5 - 2/3 + 4/3$ ✓
> $= 17/3$ L ✓

(e) Use the information in (d) to find the average rate of change of the amount of fluid in the first 2 minutes.

> Average rate of change = $\frac{17/3 - 5}{2}$ ✓
> $= 1/3$ L /minute ✓

Calculator Assumed

15. [10 marks: 3, 5, 2]

Let $F(x) = \int_0^x 3t^2 - 12t + 9\, dt$ where for $1 \le x \le 5$.

(a) Calculate $F(1)$ and $F(4)$.

> $F(x) = \int_0^x 3t^2 - 12t + 9\, dt = x^3 - 6x^2 + 9x$ ✓
> $F(1) = 4$ ✓
> $F(5) = 20$ ✓

(b) Find the local minimum and maximum points on the curve $y = F(t)$.

> $F'(x) = 3x^2 - 12x + 9$ ✓
> $F'(x) = 0 \Rightarrow x = 1, 3$ ✓
>
> $F''(x) = 6x - 12$ ✓
>
> When $x = 1$, $F(1) = 4$ and $F''(1) = -6 < 0$
> Hence. (1, 4) is a local maximum point. ✓
>
> When $x = 3$, $F(3) = 0$ and $F''(1) = 6 > 0$
> Hence. (3, 0) is a local minimum point. ✓

(c) Determine the global minimum and global maximum values for $F(x)$.

> End points: $F(1) = 4$
> $F(5) = 20$
>
> Local Minimum (3, 0)
> Local Maximum (1, 4).
>
> Hence, global minimum for $F(x)$ is 0 ✓
> global maximum for $F(x)$ is 20. ✓

21 Rectilinear Motion

Calculator Assumed

1. [11 marks: 4, 3, 4]

The displacement of a body moving along a straight line is given by $s = -t^3 + at^2 + bt + 3$ metres where t is time in seconds. The initial velocity of the body is 5 ms^{-1}. The body is momentarily at rest when $t = 1$ second.

(a) Find the values of a and b.

Velocity $v = \dfrac{ds}{dt} = -3t^2 + 2at + b$ ✓
When $t = 0$, $v = 5$. $\Rightarrow$ $b = 5$ ✓

Hence, $v = -3t^2 + 2at + 5$
Body is at rest when $t = 1$. $\Rightarrow -3 + 2a + 5 = 0$ ✓
Hence, $a = -1$ ✓

(b) Find when the body changes direction.

When body changes direction,
$v = 0$;
$\Rightarrow -3t^2 - 2t + 5 = 0$ ✓
$t = 1$ second (reject $\frac{-5}{3}$) ✓
$v(1^-) > 0$ and $v(1^+) < 0$, hence $t = 1$ second ✓

(c) Find the instantaneous speed at $t = 5$ seconds and the average speed in the first 5 seconds.

Instantaneous speed at $t = 5$,
$v(5) = -80$ ms^{-1}. ✓

Distance travelled in the first 5 seconds
$= \int_0^5 \left|-3t^2 - 2t + 5\right| dt = 131$ m ✓✓

Hence, average speed = 26.2 ms^{-1}. ✓

Calculator Assumed

2. [7 marks: 1, 3, 1, 2]

The displacement of a body at time t seconds is given by $s = 4t + \dfrac{1}{1+t}$ metres.

(a) Find an expression for the velocity of the body at time t seconds.

Velocity $v = \dfrac{ds}{dt} = 4 - \dfrac{1}{(1+t)^2}$ ✓

(b) Show that the body is never stationary.

For the body to be stationary,
$4 - \dfrac{1}{(1+t)^2} = 0$ ✓
$t = -\frac{1}{2}$ or $-\frac{3}{2}$ ✓
But time $t \geq 0$. ✓
Hence, the body is never stationary.

(c) Find an expression for the acceleration at time t seconds.

Acceleration $a = \dfrac{dv}{dt} = \dfrac{2}{(1+t)^3}$ ✓

(d) Describe the motion of the body for large values of t.

$v = 4 - \dfrac{1}{(1+t)^2}$.
For large values of t, $\dfrac{1}{(1+t)^2} \to 0$. ✓
Hence, $v \to 4$.
That is, for large values of t the body moves with a constant velocity of 4 ms^{-1}. ✓

Mathematics Methods Units 3 & 4 Revision Series 21 Rectilinear Motion
Mathematics Methods Units 3 & 4 Revision Series 21 Rectilinear Motion

Calculator Assumed

3. [13 marks: 2, 5, 2, 4]

The displacement (s metres) of particle P, t seconds after passing a fixed point O is given by $s = 10\,t\,e^{-t} - 1$.

(a) Find an expression for the velocity at time t seconds and hence the velocity at $t = 2$ seconds.

> Velocity $v = \frac{ds}{dt} = 10\,e^{-t}(1-t)$ ✓
>
> $v(2) = -1.35\ \text{ms}^{-1}$. ✓

(b) Use Calculus to find the maximum displacement of P in the first two seconds.

> For local maximum: $v = \frac{ds}{dt} = 0$. ✓
>
> $\Rightarrow 1 - t = 0 \ \Rightarrow\ t = 1$ ✓
>
> At $t = 1$;

	1^-	1	1^+
v	+	0	–

> Hence, local maximum occurs at $t = 1$. ✓
>
> s(0) = –1, s(2) = 1.71, s(1) = 2.68 ✓
>
> Hence, maximum displacement for P (in the first 2 seconds) is 2.68 m ✓

(c) Find the acceleration of P at maximum displacement.

> Acceleration $a = \frac{dv}{dt} = -20\,e^{-t} + 10\,t\,e^{-t}$ ✓
>
> When $t = 1$, $a = -3.68\ \text{ms}^{-2}$ ✓

(d) Find the average speed in the first 2 seconds.

> Distance travelled in the first 2 seconds
>
> $= \int_0^2 \left|10e^{-t}(1-t)\right| dt = 4.6509$ m ✓✓✓
>
> Hence, average speed = $2.33\ \text{ms}^{-1}$. ✓

Calculator Assumed

4. [8 marks: 2, 3, 3]

The velocity (ms^{-1}) of a particle moving along a straight line is given by $v = 5\,e^{-t} - 1$, where t is time in seconds. The displacement of the particle from a fixed point O at time $t = 1$ second is 5 metres.

(a) Find the displacement (exact answer) of the particle at time t.

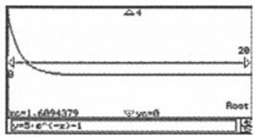
> Displacement $x = \int 5\,e^{-t} - 1\ dt$
>
> $= -5e^{-t} - t + C$ ✓
>
> When $t = 1$, $x = 5 \ \Rightarrow\ C = 6 + \frac{5}{e}$ ✓
>
> Hence, $x = -5e^{-t} - t + 6 + \frac{5}{e}$.

(b) Find when and where the particle reversed its direction of motion.

> $v = 0 \ \Rightarrow\ 5e^{-t} - 1 = 0$ ✓
>
> $t = 1.6094$ ✓
>
> $x(1.6094) = 5.23$ m ✓

(c) Find the distance travelled in the first 20 seconds.

> Distance travelled $= \int_0^{1.6094} 5\,e^{-t} - 1\ dt + \left|\int_{1.6094}^{20} 5\,e^{-t} - 1\ dt\right|$ ✓✓
>
> $= 2.3906 + 17.3906$
>
> $= 19.7811$ m ✓



Calculator Assumed

5. [6 marks: 2, 2, 2] [TISC]

The velocity v cms^{-1}, of a particle P moving in a straight line at a point x cm from the origin is given by the equation $v^2 = -\int x\, dx$. P starts from the origin with a velocity of 10 cms^{-1}.

(a) Show that $v^2 = 100 - \dfrac{x^2}{2}$.

$$v^2 = -\frac{x^2}{2} + C \quad ✓$$

When $x = 0$, $v = 10$:

$$C = 100 \quad ✓$$

Hence, $v^2 = 100 - \dfrac{x^2}{2}$

(b) Find where P is instantaneously at rest.

$$v = 0 \Rightarrow 100 - \frac{x^2}{2} = 0$$
$$x = \pm 10\sqrt{2} \text{ cm} \quad ✓✓$$

(c) Find the maximum speed of P and state where it occurs.

When $x = 0$, $v^2 = 100$

$v = \pm 10$ cms^{-1}.

Hence, max $v = 10$ cms^{-1} at $x = 0$. ✓✓

Calculator Assumed

6. [9 marks: 2, 3, 4]

The acceleration (ms^{-2}) of a particle moving along a straight line is given by $a = -4\cos 2t$, where t is time in seconds. At $t = 0$, the velocity of the particle is 0 ms^{-1} and the displacement of the particle is 1 m.

(a) Find an expression for the velocity of the particle at any time t.

$$v = \int -4\cos 2t\, dt$$
$$= -2\sin 2t + C \quad ✓$$

When $t = 0$, $v = 0$. $\Rightarrow C = 0$

Hence, $v = -2\sin 2t$. ✓

(b) Find an expression for the displacement of the particle at any time t.

Displacement $x = \int -2\sin 2t\, dt$

$= \cos 2t + K$ ✓

When $t = 0$, $x = 1$. $\Rightarrow K = 0$ ✓

Hence, $x = \cos 2t$. ✓

(c) Find the average speed in the first π seconds.

Distance travelled $= \displaystyle\int_0^{\pi} \left| -2\sin 2t \right| dt$ ✓✓

$= 4$ m ✓

Average speed $= \dfrac{4}{\pi} \approx 1.27$ ms^{-1} ✓

Mathematics Methods Units 3 & 4 Revision Series 21 Rectilinear Motion

Mathematics Methods Units 3 & 4 Revision Series 21 Rectilinear Motion

Calculator Assumed

7. [10 marks: 1, 6, 3]

A particle starts off from a fixed point O with an acceleration (mms^{-2}) of $a = mt - 24$, where t is time in seconds. The particle travels in a straight line and returns to O at $t = 4$ seconds and has a change of displacement of –9 mm in the third second (it moves in the same direction during this time).

(a) Find in terms of m an expression for the velocity of the particle at any time t.

> Velocity $v = \int mt - 24 \; dt$
> $= \dfrac{mt^2}{2} - 24t + C$ ✓

(b) Find the displacement of the particle at any time t.

> Displacement $x = \dfrac{mt^3}{6} - 12t^2 + Ct + K$ ✓
>
> When $t = 0, x = 0$. $\Rightarrow$ $K = 0$.
> Hence, $x = \dfrac{mt^3}{6} - 12t^2 + Ct$ ✓
>
> When $t = 4, x = 0$. $\Rightarrow$ $\dfrac{64m}{6} + 4C = 192$ (I) ✓
>
> Also, $x(3) - x(2) = -9$. $\Rightarrow$ $\dfrac{19m}{6} + C = 51$ (II) ✓
>
> Solve I and II simultaneously, $m = 6, C = 32$
>
> Hence, $x = t^3 - 12t^2 + 32t$ ✓✓

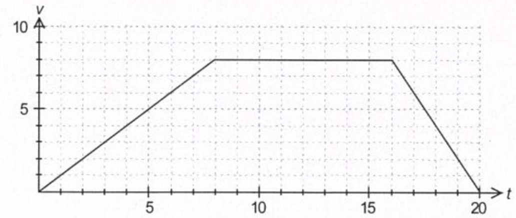
(c) Find when the particle is at O the third time (if it does).

> When it is at O, $x = 0$.
> $\Rightarrow$ $t^3 - 12t^2 + 32t = 0$ ✓
> $t = 0, 4, 8$ ✓
> Hence, the particle is at O for the third time at t = 8 seconds. ✓

Calculator Assumed

8. [10 marks: 2, 2, 2, 4]

The velocity-time graph of a particle moving in a straight line is given below. Velocity v is in ms^{-1} and time t in seconds.



(a) Find an expression for the velocity of the particle for the first 8 seconds.

> For the first 8 seconds, the velocity of the particle is modelled by a straight line passing through the origin.
>
> Gradient of line = 1. ✓
> Hence velocity $v = t$. ✓

(b) Hence, or otherwise, find the distance travelled in the first 8 seconds.

> Distance = Area under the v–t curve for $0 \le t \le 8$
> $= \dfrac{1}{2} \times 8 \times 8$ (Area of triangle) ✓
> $= 32$ m ✓

(c) Find the change in displacement between $t = 0$ and $t = 20$ seconds.

> Change in displacement
> = Area under the v–t curve for $0 \le t \le 20$
> $= \dfrac{1}{2} \times (20 + 8) \times 8$ (Area of trapezium) ✓
> $= 112$ m ✓

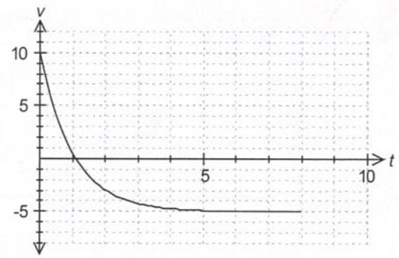
(d) Find t when the particle was 64m away from its starting position.

> At $t = 8$, it is 32 m away from its starting position. ✓
> For $8 \le t \le 16$, it is travelling at a constant 8 ms^{-1}.
> Hence, it covers the remaining 32 m in 4 seconds. ✓✓
>
> Hence, particle is 64 m away from its starting position at $t = 8 + 4 = 12$ seconds. ✓

Calculator Assumed

9. [11 marks: 2, 3, 3, 3]

A particle P moving in a straight line, starts off from a fixed point O with velocity v_0 ms^{-1}. Its velocity at any time t is given by $v = 15e^{-t} - k$ ms^{-1}, where k is a constant. The velocity-time graph of P is given in the accompanying diagram. Velocity v is in ms^{-1} and time t in seconds.



(a) Find v_0 and k.

> $v_0 = 10$ ms^{-1} ✓
> Hence, $k = 5$. ✓

(b) Find the time (correct to 2 decimal places) when P reverses direction.

> $v = 15e^{-t} - 5$
> When it reverses direction, $v = 0$.
> $\Rightarrow \; v = 15e^{-t} - 5 = 0$ ✓
> $t = 1.10$ seconds. ✓
> $v(1.1^-) > 0$ and $v(1.1^+) < 0$, hence $t = 1.1$ seconds ✓

(c) Find the displacement of P at the time it reversed its direction.

> Displacement $x = \int 15e^{-t} - 5 \; dt$
> $= -15e^{-t} - 5t + C$ ✓
>
> When $t = 0, x = 0. \Rightarrow C = 15$
> Hence, $x = -15e^{-t} - 5t + 15$ ✓
>
> At $t = 1.10$, $x = 4.51$ m ✓

(d) Find the average speed in the first 8 seconds.

> Distance travelled $= \int_0^8 \left|15e^{-t} - 5\right| dt$ ✓
> $= 34.0189$ m ✓
> Hence, average speed $= \dfrac{34.0189}{8}$
> $= 4.25$ ms^{-1} ✓

Calculator Assumed

10. [11 marks: 3, 3, 2, 3] [TISC]

The acceleration a ms^{-2}, of a particle P moving in a straight line at time t seconds is given by $a = mt + n$.

(a) Find the change in velocity in the first two seconds.

> $v = \int mt + n \; dt$
> $= \dfrac{mt^2}{2} + nt + k$ ✓
> $v(0) = k$
> $v(2) = 2m + 2n + k$ ✓
> Change in velocity $= v(2) - v(0)$
> $= 2(m + n)$ ✓

The average acceleration during the first two seconds is 1 ms^{-2} and the initial velocity of the particle is 4 ms^{-1}.

(b) Show that $v = \dfrac{1}{2}(1 - n)\,t^2 + nt + 4$.

> Average acceleration $= \dfrac{v(2) - v(0)}{2 - 0}$ ✓
> $= m + n = 1$
> $m = 1 - n$ ✓
> Hence, $v = \dfrac{(1-n)\,t^2}{2} + nt + k$
>
> $v(0) = 4 \;\Rightarrow\; k = 4.$
> $v = \dfrac{(1-n)\,t^2}{2} + nt + 4$ ✓

(c) If in addition, the instantaneous acceleration of the particle at $t = 2$ is -1 ms^{-2}, show that $v = -t^2 + 3t + 4$.

> $t = 2, a = -1 \;\Rightarrow\; 2m + n = -1$
> But from (b) $m + n = 1$
> Hence, $m = -2$ ✓
> and $n = 3$ ✓
> Hence, $v = -t^2 + 3t + 4$

Mathematics Methods Units 3 & 4 Revision Series *21 Rectilinear Motion*

Mathematics Methods Units 3 & 4 Revision Series *21 Rectilinear Motion*

Calculator Assumed

10. (d) Find the total distance travelled from the moment the particle starts travelling to before the particle changes direction.

$v = -t^2 + 3t + 4$
Particle changes direction when $v = 0$:
$t = 4$ (reject $t = -1$) ✓

Distance travelled $= \int_0^4 -t^2 + 3t + 4 \, dt$ ✓

$= \frac{56}{3}$ metres. ✓

Calculator Assumed

11. [7 marks: 3, 3, 1] [TISC]

A particle P travels along the x-axis. Its acceleration at time t seconds is given by $a = -0.01\, e^{-0.01t}$ cm s^{-1}. The particle starts from the point with coordinates (0, 0) with an initial velocity of 2 cm s^{-1}.

(a) Find the velocity of P after 1 minute.

velocity $v = \int -0.01\, e^{-0.01t} \, dt$ ✓
$= e^{-0.01t} + C$
$v(0) = 2 \Rightarrow C = 1$
Hence $v = e^{-0.01t} + 1$ ✓
$v(60) = 1.55$ cm s^{-1}. ✓

(b) After 1 minute, P is located at the point B. Find the coordinates of B.

Displacement $s = \int e^{-0.01t} + 1 \, dt$ ✓
$= -100\, e^{-0.01t} + t + K$
$s(0) = 0 \Rightarrow K = 100$
Hence $s = -100\, e^{-0.01t} + t + 100$ ✓
$s(60) = 105.12$ cm.

Coordinates of B (105.12, 0). ✓

(c) What is the long term velocity of P?

$v = e^{-0.01t} + 1$
As $t \to \infty, v \to 1$.
Hence, long term velocity of P is 1 cms^{-1}. ✓

Calculator Assumed

12. [11 marks: 4, 7]

A particle P travels along the x-axis and its acceleration at time t seconds is given by $\dfrac{d^2x}{dt^2} = pt^2 + qt + r$ cms^{-2} where p, q and r are constants. The particle starts from the point K with coordinates (8, 0) with velocity -12 cms^{-1}. The particle changes direction at the same point when $t = 1$ and $t = 2$ seconds.

(a) Show that its displacement at time t seconds is given by

$$x = \frac{pt^4}{12} + \frac{qt^3}{6} + \frac{rt^2}{2} - 12t + 8.$$

$$\frac{d^2x}{dt^2} = pt^2 + qt + r$$

$$\frac{dx}{dt} = \frac{pt^3}{3} + \frac{qt^2}{2} + rt - 12 \qquad ✓✓$$

$$x = \frac{pt^4}{12} + \frac{qt^3}{6} + \frac{rt^2}{2} - 12t + 8 \qquad ✓✓$$

(b) Find the values of p, q and r.

$v(1) = 0$

$$\Rightarrow \frac{p}{3} + \frac{q}{2} + r - 12 = 0 \qquad \text{I} \qquad ✓$$

$v(2) = 0$

$$\Rightarrow \frac{8p}{3} + 2q + 2r - 12 = 0 \qquad \text{II} \qquad ✓$$

$x(1) = x(2)$

$$\Rightarrow \frac{p}{12} + \frac{q}{6} + \frac{r}{2} - 12 + 8 = \frac{16p}{12} + \frac{8q}{6} + 2r - 24 + 8 \qquad \text{III} \qquad ✓✓$$

$$p = 12,\ q = -36,\ r = 26 \qquad ✓✓✓$$

$$\left\{\begin{array}{l} \frac{x}{3}+\frac{y}{2}+z-12=0 \\ \frac{8x}{3}+2y+2z-12=0 \\ \frac{x}{12}+\frac{y}{6}+\frac{z}{2}-12+8=\frac{16x}{12}+\frac{8y}{6}+2z-24+8 \end{array}\right|_{x,y,z}$$

$$\{x=12, y=-36, z=26\}$$

22 Discrete Random Variables I

Calculator Free

1. [4 marks: 2, 2]

Determine with reasons if each of the following functions are probability distribution functions for discrete random variables.

(a) $f(k) = 1/5$ for $k = 0, 1, 2, 3, 4, 5$

$\Sigma f(k) = 6/5 \neq 1.$ ✓
Hence, $f(k)$ is not a pdf. ✓

(b) $f(k) = k/5$ for $k = -2, -1, 0, 1, 2, 3$

$f(-2) = -2/5 < 0.$ ✓
Hence, $f(k)$ is not a pdf. ✓

2. [7 marks]

Verify that the table below describes the probability distribution function of a discrete random variable X. Determine the mean and variance for X.

x	-5	-4	-3	-2	-1
$f(x)$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{3}$

$\Sigma f(k) = 1.$ ✓
Also $0 \le f(x) \le 1$ for all x. ✓
Hence, $f(k)$ is a pdf. ✓

$$\text{Mean E(X)} = -\frac{5}{12} - \frac{4}{6} - \frac{3}{4} - \frac{2}{6} - \frac{1}{3} = -\frac{5}{2} \qquad ✓$$

$$\text{E}(X^2) = \frac{25}{12} + \frac{16}{6} + \frac{9}{4} + \frac{4}{6} + \frac{1}{3} = 8 \qquad ✓$$

$$\text{Var(X)} = 8 - \left(\frac{-5}{2}\right)^2 = \frac{7}{4} \qquad ✓✓$$

Mathematics Methods Units 3 & 4 Revision Series 22 Discrete Random Variables I

Calculator Free

3. [10 marks: 3, 3, 2, 2]

X is a Discrete Random Variable with probability distribution defined as follows:

$$P(X=x)=\begin{cases} x\times P(X=x+1) & x=1,2,3 \\ k & x=4 \end{cases}.$$

(a) Find the value of k.

x	1	2	3	4
$P(X=n)$	$6k$	$6k$	$3k$	k

Hence, $6k+6k+3k+k=1$ ✓✓

$k=\frac{1}{16}$ ✓

(b) Find $P(X\le 3 \mid X>1)$.

$P(X\le 3\mid X>1)=\frac{P(1<X\le 3)}{P(X>1)}$ ✓✓

$=\frac{9k}{10k}=\frac{9}{10}$ ✓

(c) Determine the mean for X.

Mean for $X = 6k+12k+9k+4k$ ✓

$=31k=\frac{31}{16}$ ✓

(d) Show that the variance for X is $\frac{207}{256}$.

$E(X^2)=6k+24k+27k+16k$

$=73k=\frac{73}{16}$ ✓

$\mathrm{Var}(X)=\frac{73}{16}-\left(\frac{31}{16}\right)^2$ ✓

$=\frac{73}{16}-\frac{961}{256}$

$=\frac{207}{256}$

Calculator Free

4. [6 marks: 3, 3] [TISC]

The table below describes the probability distribution for a discrete random variable X.

X	1	2	3	4	5
$P(X=x)$	0.1	p	q	0.2	0.2

(a) Determine the values of p and q if $P(X\le 2)=0.25$.

$P(X\le 2)=0.1+p=0.25$

$p=0.15$ ✓

$\sum P(X=x)=1$

$\Rightarrow 0.1+0.15+q+0.2+0.2=1$ ✓

$q=0.35$ ✓

(b) Determine the values of p and q if $P(X\ge 3\mid X\le 4)=\frac{5}{8}$.

$P(X\ge 3\mid X\le 4)=\frac{5}{8}$

$\Rightarrow \frac{P(3\le X\le 4)}{P(X\le 4)}=\frac{5}{8}$

$\frac{q+0.2}{0.8}=\frac{5}{8}$ ✓

$q=0.3$ ✓

$\sum P(X=x)=1$

$\Rightarrow 0.1+p+0.3+0.2+0.2=1$

$p=0.2$ ✓

Calculator Assumed

5. [6 marks]

Verify that the function $f(x) = \dfrac{\binom{15}{x}\binom{5}{5-x}}{\binom{20}{5}}$ for $x = 0, 1, 2, 3, 4, 5$

may be used as the probability distribution function of a discrete random variable X. Determine the exact mean and variance for X.

$\sum f(k) = 1.$ ✓

Also $0 \le f(x) \le 1$ for all x. ✓

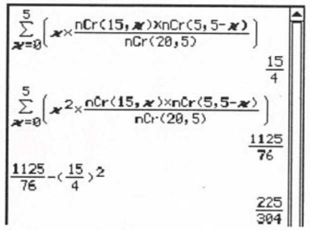
Hence, $f(k)$ is a pdf. ✓

$$\text{Mean } E(X) = \sum_{x=0}^{x=5} x \times \frac{\binom{15}{x}\binom{5}{5-x}}{\binom{20}{5}}$$

$$= \frac{15}{4} \quad ✓$$

$$E(X^2) = \sum_{x=0}^{x=5} x^2 \times \frac{\binom{15}{x}\binom{5}{5-x}}{\binom{20}{5}} = \frac{1125}{76} \quad ✓$$

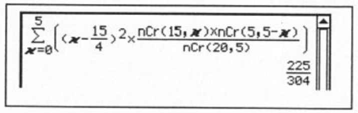
$$\text{Var}(X) = \frac{1125}{76} - \left(\frac{15}{4}\right)^2 = \frac{225}{304} \quad ✓$$



Or

$$\text{Var}(X) = \sum_{x=0}^{x=5} \left(x - \tfrac{15}{4}\right)^2 \times \frac{\binom{15}{x}\binom{5}{5-x}}{\binom{20}{5}} \quad ✓$$

$$= \frac{225}{304} \quad ✓$$



Calculator Assumed

6. [9 marks: 3, 3, 3]

A discrete random variable X has probability distribution

$$P(X = k) = \begin{cases} \dfrac{x}{4} & x = 1, 2 \\ \dfrac{x}{k} & x = 3, 4, 5 \end{cases}.$$

(a) Find the value of k.

x	1	2	3	4	5
$P(X = k)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{k}$	$\frac{4}{k}$	$\frac{5}{k}$

Hence, $\frac{1}{4} + \frac{2}{4} + \frac{3}{k} + \frac{4}{k} + \frac{5}{k} = 1$ ✓✓

$k = 48.$ ✓

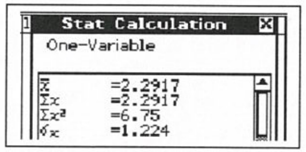
(b) Determine the mean and variance of X.

Mean $E(X) = \frac{55}{24} \approx 2.2917$ ✓

$\text{Var}(X) = \frac{27}{4} - \left(\frac{55}{24}\right)^2 = \frac{863}{576} \approx 1.4982$ ✓

OR

$\text{Var}(X) = 1.224^2 \approx 1.4982$ ✓



(c) Find $P(X \le 4 \mid X > 2)$.

$$P(X \le 4 \mid X > 2) = \frac{P(2 < X \le 4)}{P(X > 2)} \quad ✓✓$$

$$= \frac{7/48}{1/4} = \frac{7}{12} \quad ✓$$

Mathematics Methods Units 3 & 4 Revision Series *22 Discrete Random Variables I*

Calculator Assumed

7. [9 marks: 4, 2, 3] [TISC]

Consider a probability distribution for a discrete random variable, X, such that

$$P(X \le x) = \frac{x(1+x)}{30}$$, where x is a positive integer.

(a) Complete the following table:

x	1	2	3	4	5
$P(X \le x)$	$\frac{2}{30}$	$\frac{6}{30}$	$\frac{12}{30}$	$\frac{20}{30}$	$\frac{30}{30}$
$P(X = x)$	$\frac{2}{30}$	$\frac{4}{30}$	$\frac{6}{30}$	$\frac{8}{30}$	$\frac{10}{30}$

−1 per error ✓✓✓✓

(b) Explain clearly why the maximum value for x is 5.

As $P(X \le 5) = 30/30 = 1$, ✓
X cannot exceed 5. ✓

(c) Find $P(X = 5 \mid X \ge 3)$.

$$P(X = 5 \mid X \ge 3) = \frac{P(X=5)}{P(X \ge 3)} \quad ✓$$
$$= \frac{10/30}{24/30} \quad ✓$$
$$= \frac{5}{12} \quad ✓$$

Calculator Assumed

8. [7 marks: 2, 2, 3] [TISC]

(a) The table below shows the values taken by a function $f(x)$.

x	−1	0	1	0.5
$f(x)$	0.2	0.6	0.1	0.1

Peter argues that $f(x)$ cannot be a probability distribution function of a discrete random variable as x has a negative value. Comment on his answer.

x can be negative but $f(x)$ cannot be negative.
$\Sigma f(k) = 1$. ✓
Also $0 \le f(x) \le 1$ for all x. ✓
Hence, $f(k)$ is a pdf, that is, Peter is wrong.

(b) The table below shows the values taken by a function $f(x)$.

x	0.0	0.5	1.0	1.5	2.0
$f(x)$	0.2	0.5	a	b	0.1

(i) Under what conditions can $f(x)$ represent the probability distribution function of a discrete random variable?

$a + b = 0.2$ ✓
and $0 < a < 1$ and $0 < b < 1$ ✓

(ii) If $f(x)$ is the probability distribution function of a discrete random variable X, find the values of a and b given that $P(X = 1.0) = 2 \times P(X = 1.5)$.

$a = 2b$ ✓
But $a + b = 0.2$
Hence, $3b = 0.2$
$b = \frac{1}{15}$ ✓
$a = \frac{2}{15}$ ✓

Calculator Assumed

9. [9 marks: 1, 2, 3, 3] [TISC]

X is a discrete random variable with probability distribution function

$P(X = x) = \frac{1}{6}$ for $x = 1, 2, 3, 4, 5, 6$.

Y is discrete random variable with probability distribution function

$P(Y = y) = \frac{1}{10}$ for $y = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.

It is known that the variables X and Y are independent.

(a) Find $P(X \le 4)$

$P(X \le 4) = 4 \times \frac{1}{6} = \frac{2}{3}$ ✓

(b) Find P(X = 5 and Y = 5).

$P(X = 5 \text{ and } Y = 5) = \frac{1}{6} \times \frac{1}{10}$ ✓

$= \frac{1}{60}$ ✓

(c) Find P(X = 5 or Y = 5).

$P(X = 5 \text{ or } Y = 5) = P(X = 5) + P(Y = 5) - P(X = 5 \text{ and } Y = 5)$

$= \frac{1}{6} + \frac{1}{10} - \frac{1}{60}$ ✓✓

$= \frac{1}{4}$ ✓

(d) Find P(X + Y = 12). Show clearly how you obtained your answer.

X	Y	Prob.
6	6	1/60
5	7	1/60
4	8	1/60
3	9	1/60
2	10	1/60

✓✓

Hence, $P(X + Y = 12) = \frac{1}{60} \times 5$

$= \frac{1}{12}$ ✓

Calculator Assumed

10. [10 marks: 3, 7]

The probability distribution function for a random variable X is given by

$p(x) = \frac{x}{15}$ for $x = 1, 2, 3, 4, 5$. The random variable $Y = 10 - 2X$.

(a) Calculate $P(2 \le Y \le 6)$.

$P(2 \le Y \le 6) = P(2 \le 10 - 2X \le 6)$

$= P(2 \le X \le 4)$ ✓✓

$= \frac{2}{15} + \frac{3}{15} + \frac{4}{15}$

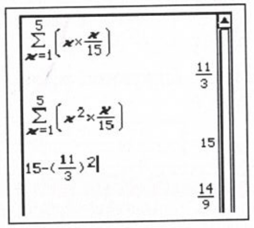
$= \frac{3}{5}$ ✓

(b) Calculate the mean and standard deviation for Y.

$E(X) = \sum_{x=1}^{x=5}\left(x \times \frac{x}{15}\right) = \frac{11}{3}$ ✓

$E(X^2) = \sum_{x=1}^{x=5}\left(x^2 \times \frac{x}{15}\right) = 15$

$Var(X) = 15 - \left(\frac{11}{3}\right)^2 = \frac{14}{9}$ ✓✓



$E(Y) = E(10 - 2X)$

$= 10 - 2E(X)$

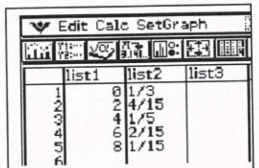
$= 10 - 2 \times \frac{11}{3} = \frac{8}{3}$ ✓✓

$Var(Y) = Var(10 - 2X)$

$= 4Var(X)$

$= 4 \times \frac{14}{9} = \frac{56}{9}$ ✓

Standard deviation for $Y = \frac{2\sqrt{14}}{3}$ ✓



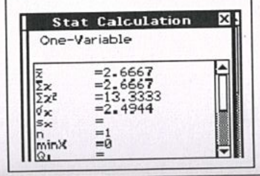
OR

$Y = 10 - 2X \Rightarrow X = \frac{10 - Y}{2}$. ✓

$\Rightarrow p(y) = \frac{10 - y}{30}$ for $y = 0, 2, 4, 6, 8$. ✓✓✓

$E(Y) = \frac{8}{3}$ ✓✓

Standard deviation for $Y \approx 2.4944$ ✓



Calculator Assumed

11. [6 marks]

The random variable X has mean 100 and standard deviation 4. The random variable $Y = aX + b$. Find a and b if the mean and standard deviation for Y are 90 and 6 respectively.

$\text{Var}(Y) = a^2\text{Var}(X)$

$36 = 16a^2$ ✓

$a = \pm\dfrac{3}{2}$ ✓

For $a = \dfrac{3}{2}$:

$E(Y) = \dfrac{3}{2}E(X) + b$

$\Rightarrow 150 + b = 90$

$b = -60$

Hence: $a = \dfrac{3}{2},\ b = -60$ ✓✓

For $a = -\dfrac{3}{2}$:

$\Rightarrow -150 + b = 90$

$b = 240$

Hence: $a = -\dfrac{3}{2},\ b = 240$ ✓✓

Calculator Assumed

12. [10 marks: 4, 6]

The probability distribution function of the discrete random variable X is given by $p(x) = kx^2$ for $x = 1, 2, 3, 4, \ldots, n$.

(a) Given that the expected value of X and the variance for X are $\dfrac{45}{11}$ and $\dfrac{644}{605}$ respectively, determine the expected value and variance of the discrete random variable Y where $Y = 2X$.

$E(Y) = E(2X) = 2E(X)$ ✓

$= 2 \times \dfrac{45}{11} = \dfrac{90}{11}$ ✓

$\text{Var}(Y) = \text{Var}(2X) = 4\text{Var}(X)$ ✓

$= 4 \times \dfrac{644}{605} = \dfrac{2576}{605}$ ✓

(b) Given that the expected value of X is $\dfrac{234}{25}$, use your CAS calculator to find two expressions involving k and n and hence determine the values of k and n.

$\displaystyle\sum_{x=1}^{x=n} kx^2 = 1$

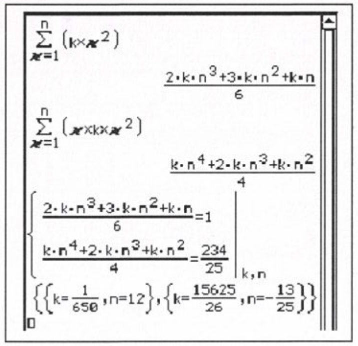
$\Rightarrow \dfrac{2kn^3 + 3kn^2 + kn}{6} = 1$ (I) ✓✓

$\displaystyle\sum_{x=1}^{x=n} x \times kx^2 = \dfrac{234}{25}$

$\Rightarrow \dfrac{kn^4 + 2kn^3 + kn^2}{4} = \dfrac{234}{25}$ (II) ✓✓

Solve I & II simultaneously,

$k = \dfrac{1}{650},\ n = 12$ ✓✓



23 Discrete Random Variables II

Calculator Free

1. [7 marks: 2, 2, 1, 2]

The table below shows the projected returns for every \$100 000 in an investment scheme and the accompanying probabilities.

Returns	–\$20 000	–\$10 000	k	\$20 000	\$50 000
Probability	0.01	p	0.2	0.27	0.02

(a) Determine the value of p.

$0.01 + p + 0.2 + 0.27 + 0.02 = 1$ ✓
$p = 0.5$ ✓

(b) Find the mean return per \$100 000 in terms of k.

$E(X) = -20\,000 \times 0.01 + -10\,000 \times 0.5 + 0.2k + 20\,000 \times 0.27 + 50\,000 \times 0.02$
$= 1\,200 + 0.2k$ ✓✓

(c) Find the mean profit per \$100 000 if k = \$5 000.

Mean profit = $1\,200 + 0.2 \times 5000$
= \$2 200. ✓

(d) Find the value(s) of k if the mean profit per \$100 000 must exceed \$5000.

$E(X) > 5\,000$
$1\,200 + 0.2k > 5\,000$ ✓
$k > 19\,000$ ✓

Calculator Assumed

2. [14 marks: 1, 5, 3, 5]

At an agricultural fair, a games stall operator offers prizes worth \$20, \$5, and \$1 for one attempt at a particular game. The probabilities of winning these prizes are respectively 0.001, 0.01 and 0.5.

(a) Find the probability of not winning a prize.

P(not winning a prize) = $1 - 0.001 - 0.01 - 0.5$
= 0.489 ✓

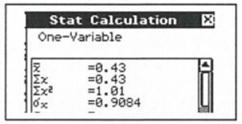
(b) If each game costs \$1, find the expected profit per game and the accompanying standard deviation for the games stall operator.

Let X: Profit per game for the operator. ✓
Hence, the probability distribution for X is:

x	–19	–4	0	1
$P(X = x)$	0.001	0.01	0.5	0.489

✓✓

Expected profit = \$0.43 ✓
Standard deviation = \$0.91 ✓



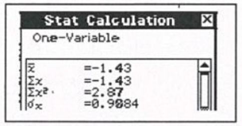
(c) Vegas played 50 games at \$2 for a game. Find her expected profit/loss.

Let X: Profit per game for the Vegas.
Hence, the probability distribution for X is:

x	18	3	–1	–2
$P(X = x)$	0.001	0.01	0.5	0.489

✓

Expected profit per game = –\$1.43 ✓
Total Loss = 1.43×50 = \$71.5 ✓



Mathematics Methods Units 3 & 4 Revision Series 23 Discrete Random Variables II
Mathematics Methods Units 3 & 4 Revision Series 23 Discrete Random Variables II

Calculator Assumed

2. (d) The games stall operator made a profit of \$193 from 100 games. How much did he charge per game?

Let the price per game be $\$k$.
Let X: Profit per game for the operator.
Hence, the probability distribution for X is:

x	$k-20$	$k-5$	$k-1$	k
$P(X=x)$	0.001	0.01	0.5	0.489

✓

Given $E(X) = \frac{193}{100} = 1.93.$ ✓

But $E(X) = 0.001(k-20) + 0.01(k-5) + 0.5(k-1) + 0.489k$ ✓

Hence, $0.001(k-20) + 0.01(k-5) + 0.5(k-1) + 0.489k = 1.93$ ✓

$\Rightarrow k = 2.50$

Hence, cost per game = \$2.50. ✓

Calculator Assumed

3. [12 marks: 6, 6]

A random number generator generates whole numbers randomly between 1 and n. Define X: the number generated by the random number generator.

(a) Given that the mean number generated is 5 000, determine the standard deviation for X.

Probability distribution function for X is $p(x) = \frac{1}{n}$ ✓

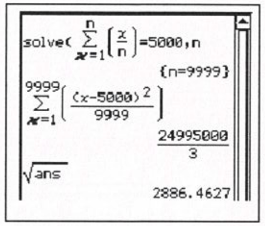
Hence: $\sum_{x=1}^{x=n} \left(\frac{x}{n}\right) = 5\,000$ ✓

$n = 9\,999$ ✓

$\text{Var}(X) = \sum_{x=1}^{x=9999} \left(\frac{(x-5000)^2}{9999}\right)$ ✓

$= \frac{24\,995\,000}{3}$ ✓

Standard deviation for X ≈ 2886.46 ✓



(b) Find the mean for X if the variance for X is 20 750.

Probability distribution function for X is $p(x) = \frac{1}{n}$

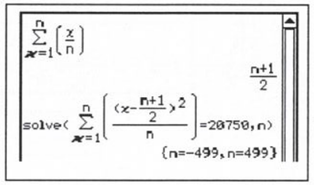
Hence: $E(X) = \sum_{x=1}^{x=n} \left(\frac{x}{n}\right)$ ✓

$= \frac{n+1}{2}$ ✓

$\text{Var}(X) = \sum_{x=1}^{x=n} \left(\frac{(x-\frac{n+1}{2})^2}{n}\right) = 20\,750$ ✓✓

$\Rightarrow n = 499$ ✓

Therefore, $E(X) = \frac{499+1}{2} = 250$ ✓



Calculator Assumed

4. [8 marks: 4, 2, 2]

A committee of five students is to be selected from a group of five female students and five male students.
Define X: Number of female students in this committee.

(a) Find the probability distribution for X.

X = 0, 1, 2, 3, 4, 5 ✓

$P(X = 0) = P(\text{0 female \& 5 males}) = \dfrac{\binom{5}{0}\binom{5}{5}}{\binom{10}{5}} = \dfrac{1}{252}$

$P(X = 1) = P(\text{1 female \& 4 males}) = \dfrac{\binom{5}{1}\binom{5}{4}}{\binom{10}{5}} = \dfrac{25}{252}$

$P(X = 2) = P(\text{2 females \& 3 males}) = \dfrac{\binom{5}{2}\binom{5}{3}}{\binom{10}{5}} = \dfrac{100}{252}$

$P(X = 3) = P(\text{3 females \& 2 males}) = \dfrac{\binom{5}{3}\binom{5}{2}}{\binom{10}{5}} = \dfrac{100}{252}$

$P(X = 4) = P(\text{4 females \& 1 male}) = \dfrac{\binom{5}{4}\binom{5}{1}}{\binom{10}{5}} = \dfrac{25}{252}$

$P(X = 5) = P(\text{5 females \& 0 male}) = \dfrac{\binom{5}{5}\binom{5}{0}}{\binom{10}{5}} = \dfrac{1}{252}$

✓✓✓ –1 per error

(b) Find the probability that there are at least as many females as males in the committee.

$P(X \geq 3) = P(X = 3) + P(X = 4) + P(X = 5)$

$= \dfrac{100}{252} + \dfrac{25}{252} + \dfrac{1}{252}$ ✓

$= \dfrac{126}{252} = \dfrac{1}{2}$ ✓

(c) Find the expected number of females in the committee.

Expected number of females = $E(X) = \dfrac{5}{2}$ ✓✓

Calculator Assumed

5. [10 marks: 1, 1, 2, 2, 4]

It is known that 0.5% of USB Drives are defective. USB drives are randomly picked from a large carton.

(a) Find the probability that the second USB drive picked is defective.

Prob. = 0.005 ✓

(b) Find the probability that the first defective drive is the 3rd drive picked.

Prob. = $0.995^2 \times 0.005 = 0.004950$ ✓

(c) Find the probability the first defective drive is the 11th drive picked.

Prob. = $0.995^{10} \times 0.005 = 0.004756$ ✓✓

(d) Write an expression for the probability that first defective USB drive is the nth drive picked.

Prob. = $0.995^{n-1} \times 0.005$ ✓✓

(e) Define X: No of USB drives that need to be selected to pick the first defective drive. Determine with reasons if X is a discrete random variable with an appropriate probability distribution function.

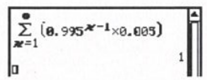
X = 1, 2, 3, …… ✓

$P(X = x) = 0.995^{x-1} \times 0.005$

Clearly $0 < P(X = x) < 1$ for all x. ✓

$\sum_{1}^{\infty} 0.995^{x-1} \times 0.005 = 1.$ ✓

Hence, X is a discrete random variable. ✓



Mathematics Methods Units 3 & 4 Revision Series — *23 Discrete Random Variables II*
Mathematics Methods Units 3 & 4 Revision Series — *23 Discrete Random Variables II*

Calculator Assumed

6. [10 marks: 2, 2, 6]

It is known that 65% of students at a certain college are foreign born. Students are randomly chosen from this college.

(a) Find the probability that the second foreign born student is the third student selected.

Prob. = $(2 \times 0.65 \times 0.35) \times 0.65$	✓
$= 0.29575$	✓

(b) Find the probability that 4 students need to be picked before picking the second foreign born student.

Prob. = $(3 \times 0.65 \times 0.35^2) \times 0.65$	✓
$= 0.1553$	✓

(c) Define X: No of students that need to be selected before picking the second foreign born student. Determine with reasons if X is a discrete random variable with an appropriate probability distribution function..

X = 2, 3, 4, 5, ……	✓
$P(X = x) = (x-1) \times 0.65 \times 0.35^{x-2} \times 0.65$	✓✓
Clearly $0 < P(X = x) < 1$ for all x.	✓
$\sum_{1}^{\infty}(x-1) \times 0.35^{x-2} \times 0.65^2 = 1.$	✓
Hence, X is a discrete random variable.	✓

Calculator Assumed

7. [9 marks: 4, 5]

It is known that $100p\%$ of homes in a certain city have broadband internet connections.

(a) Define X: No. of homes with broadband internet connections out of n homes selected. Determine with reasons if X is a discrete random variable.

$X = 0, 1, 2, 3, \ldots, n$	✓
$P(X = x) = {}^nC_x \times p^x \times (1-p)^{n-x}$	
Clearly $0 < P(X = x) < 1$ for all x.	✓
$\sum_{0}^{n} {}^nC_x \times p^x \times (1-p)^{n-x} = 1.$	✓
Hence, X is a discrete random variable.	✓

(b) Define X: Out of 10 homes selected, the only home with broadband internet connection is the xth home selected, where $x \leq 10$. For $p = 0.2$, determine with reasons if X is a discrete random variable.

$X = 1, 2, 3, \ldots, 10$	✓
$P(X = x) = 0.2 \times 0.8^9$	✓
Clearly $0 < P(X = x) < 1$ for all x.	✓
$0.2 \times 0.8^9 \times 10 = 0.2684 \neq 1$	✓
Hence, X is *not* a discrete random variable.	✓

Calculator Assumed

8. [10 marks: 2, 2, 3, 1]

To win a prize, customers of a certain well known brand of soft-drink are required to collect bottle caps imprinted with certain letters of the alphabet. The inside of each bottle cap is imprinted with exactly one of the letters C, E, K or O. 90% of caps are imprinted with the letter C, 5% with the letter E, 4.95% with the letter O and the rest with the letter K. To win a prize, customers are required to present exactly four caps with imprints of the letters C, E, K and O.

Define X: No. of bottles required to collect all four required caps in a minimum of x bottles purchased.

(a) State the range for X.

X = 4, 5, 6, 7, ….. ✓✓

(b) Find, showing clearly your reasoning, P(X = 4).

P(X = 4) = 0.9 × 0.05 × 0.0495 × 0.0005 × 4! ✓✓
= 0.000 02673 ✓

(c) If 200 000 such caps were used for this promotion, what is the minimum number of prize winners. Explain why there may possibly be more winners.

No. = 0.000 02673 × 200 000 ✓
= 5.3 ≈ 5 ✓
Other winners from purchasing more than 4 bottles. ✓

(d) Given that X = 5, give three other sequences of collections of bottle caps other than CCKEO.

CCEOK, CCOEK, CCOKE ✓

Calculator Assumed

9. [8 marks: 4, 4]

A box contains 5 red and 15 green balls.

(a) Four balls are chosen *with* replacement from this box.
Define X: No. of red balls chosen.
Determine with reasons if X is a discrete random variable with a clearly stated probability distribution function..

X = 0, 1, 2, 3, 4.

$P(X = x) = {}^4C_x\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{4-x}$ ✓

Clearly $0 < P(X = x) < 1$ for all x. ✓

$\sum_0^4 {}^4C_x\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{4-x} = 1.$ ✓

Hence, X is a discrete random variable. ✓

(b) Four balls are chosen *without* replacement from this box.
Define X: No. of red balls chosen.
Determine with reasons if X is a discrete random variable with a clearly stated probability distribution function..

X = 0, 1, 2, 3, 4.

$P(X = x) = \frac{{}^5C_x\,{}^{15}C_{4-x}}{{}^{20}C_4}$ ✓

Clearly $0 < P(X = x) < 1$ for all x. ✓

$\sum_0^4 \left[\frac{{}^5C_x\,{}^{15}C_{4-x}}{{}^{20}C_4}\right] = 1.$ ✓

Hence, X is a discrete random variable. ✓

Mathematics Methods Units 3 & 4 Revision Series *23 Discrete Random Variables II*

Calculator Assumed

10. [8 marks: 5, 3]

A box contains b blue and w white discs.

(a) n discs are randomly chosen from this box *with* replacement.
Define X: No. of white discs selected.
X is a discrete random variable. Find the probability distribution for X. Determine the expected number of white discs selected.

> Probability Distribution for X:
>
> $P(X = x) = {}^{n}C_x \left(\frac{w}{b+w}\right)^x \left(\frac{b}{b+w}\right)^{n-x}$ ✓✓
>
> where $x = 0, 1, 2, 3, \ldots n$. ✓
>
> Clearly , X is a Binomial Variable with parameters n & $\left(\frac{w}{b+w}\right)$ ✓
>
> Hence, expected No. $= n\left(\frac{w}{b+w}\right)$ ✓

(b) n discs are randomly chosen from this box *without* replacement.
Define X: No. of white discs selected.
X is a discrete random variable. Find the probability distribution for X.

> Probability Distribution for X:
>
> $P(X = x) = \frac{{}^{w}C_x \, {}^{b}C_{n-x}}{{}^{b+w}C_n}$ ✓✓
>
> where $x = 0, 1, 2, 3, \ldots n$. ✓

24 The Binomial Distribution

Calculator Free

1. [8 marks: 2, 2, 2, 2]

40% of students in a certain school are short-sighted.

Five students from this school are randomly selected. The random variable

$$S_i = \begin{cases} 1 & \text{if student } i \text{ is short-sighted (success)} \\ 0 & \text{if student } i \text{ is not short-sighted (failure)} \end{cases} \quad \text{for } i = 1, 2, 3, 4, 5.$$

(a) State the probability distribution function for S_i.

> $P(S_i = n) = \begin{cases} 0.4 & n=1 \\ 0 & n=0 \end{cases}$ ✓✓

(b) Determine the mean and variance for S_i.

> $E(S_i) = 0.4$ ✓
>
> $Var(S_i) = 0.4 \times 0.6 = 0.24$ ✓

Define the random variable $X = S_1 + S_2 + S_3 + S_4 + S_5$.

(c) State the probability distribution function for X.

> $P(X = x) = {}^{5}C_x (0.4)^x (0.6)^{5-x}$ for $x = 0, 1, 2, 3, 4, 5$ ✓✓

(d) State the mean and variance for X.

> $E(X) = 5 \times 0.4 = 2$ ✓
>
> $Var(X) = 5 \times 0.4 \times 0.6 = 1.2$ ✓

Calculator Assumed

2. [12 marks: 4, 3, 5]

10 students in a class of 25 are short-sighted. 5 students are randomly selected from this class. The random variable

$$S_i = \begin{cases} 1 & \text{if student } i \text{ is short-sighted (success)} \\ 0 & \text{if student } i \text{ is not short-sighted (failure)} \end{cases} \quad \text{for } i = 1, 2, 3, 4, 5.$$

Define the random variable $X = S_1 + S_2 + S_3 + S_4 + S_5$.

(a) Use the variables S_i to verify that X is not a binomial variable.

> $P(S_1 = 1) = \frac{10}{25}$.
>
> $P(S_2 = 1 \mid S_1 = 1) = \frac{9}{24}$
>
> $P(S_2 = 1 \mid S_1 = 0) = \frac{10}{24}$ ✓
>
> Hence, $P(S_2 = 1 \mid S_1 = 1) \neq P(S_2 = 1 \mid S_1 = 0)$ ✓
>
> That is, the variables S_1 and S_2 are not independent. ✓
>
> Hence, $X = S_1 + S_2 + S_3 + S_4 + S_5$
> is not the sum of independent Bernoulli variables. ✓
> Therefore X cannot be a binomial variable.

(b) Determine the probability distribution function for X.

> $$P(X = x) = \frac{{}^{10}C_x \, {}^{15}C_{5-x}}{{}^{25}C_5} \quad \text{for } x = 0, 1, 2, 3, 4, 5$$
> ✓✓ ✓

(c) Calculate the mean and variance for X.

> $$E(X) = \sum_{x=0}^{x=5} \left(x \times \frac{{}^{10}C_x \, {}^{15}C_{5-x}}{{}^{25}C_5} \right)$$ ✓
>
> $= 2$ ✓
>
> $$\text{Var}(X) = \sum_{x=0}^{x=5} \left((x-2)^2 \times \frac{{}^{10}C_x \, {}^{15}C_{5-x}}{{}^{25}C_5} \right)$$ ✓✓
>
> $= 1$ ✓
>
> $$\sum_{x=0}^{5} \left(x \times \frac{\text{nCr}(10, x) \times \text{nCr}(15, 5-x)}{\text{nCr}(25, 5)} \right) \qquad 2$$
>
> $$\sum_{x=0}^{5} \left((x-2)^2 \times \frac{\text{nCr}(10, x) \times \text{nCr}(15, 5-x)}{\text{nCr}(25, 5)} \right) \qquad 1$$

Calculator Free

3. [3 marks] [TISC]

X is a Binomial variable with parameters n and $p = \frac{1}{2}$.

Find in terms of n, $P(X > 1)$.

> $P(X > 1) = 1 - P(X = 0) - P(X = 1)$ ✓
>
> $= 1 - \left(\frac{1}{2}\right)^n - \binom{n}{1}\left(\frac{1}{2}\right)^n$ ✓
>
> $= 1 - \left(\frac{1}{2}\right)^n - n\left(\frac{1}{2}\right)^n$ ✓
>
> $= 1 - (n+1)\left(\frac{1}{2}\right)^n$.

---

4. [6 marks: 1, 2, 2, 1] [TISC]

The probability that John is late for school on any school day is 0.4 and it is independent of other days. For a school week of five days, write expressions, but <u>*do not evaluate*</u>, for:

(a) the probability that he is late every day

> P(late every day) = 0.4^5 ✓

(b) the probability that he is late on exactly two days

> P(late exactly two days) = $\binom{5}{2} \times 0.4^2 \times 0.6^3$ ✓✓

(c) the probability that he is late at least once

> P(late at least once) = $1 - 0.6^5$ ✓✓

(d) the mean number of days he will be late

> Mean = $5 \times 0.4 = 2$ days ✓

Calculator Assumed

5. [9 marks: 5, 2, 2] [TISC]

A computer firm receives a shipment of 10 CD-ROM Drives from a manufacturer. The drives are of the same model but made in different factories. Four of the drives were made in a factory in Malaysia and the rest were made in a factory in Singapore. A computer technician chooses 5 of these drives without replacement. Define the random variable X as "the number of Malaysian made drives chosen".

(a) Complete the table below for the probabilities for X. Leave your answers in fraction form.

x	$P(X = x)$
0	$\dfrac{\binom{6}{5}\binom{4}{0}}{\binom{10}{5}} = \dfrac{6}{252}$ ✓
1	$\dfrac{\binom{6}{4}\binom{4}{1}}{\binom{10}{5}} = \dfrac{60}{252}$ ✓
2	$\dfrac{\binom{6}{3}\binom{4}{2}}{\binom{10}{5}} = \dfrac{120}{252}$ ✓
3	$\dfrac{\binom{6}{2}\binom{4}{3}}{\binom{10}{5}} = \dfrac{60}{252}$ ✓
4	$\dfrac{\binom{6}{1}\binom{4}{4}}{\binom{10}{5}} = \dfrac{6}{252}$ ✓

$\text{seq}\left(\dfrac{\text{nCr}(6,5-x)\cdot\text{nCr}(4,x)}{\text{nCr}(10,5)}, x, 0, 4, 1\right)$

$\left\{\dfrac{1}{42}, \dfrac{5}{21}, \dfrac{10}{21}, \dfrac{5}{21}, \dfrac{1}{42}\right\}$

(b) In general, the CD-ROM Drive manufacturer makes 40% of its drives in Malaysia. A sample of 10 drives from this manufacturer was randomly chosen. Find the probability that:

(i) exactly 4 of these drives were made in Malaysia

> X is Binomial with $n = 10$ and $p = 0.4$ ✓
> $P(X = 4) = 0.2508$ ✓

(ii) not more than 4 of these drives were made in Malaysia.

> X is Binomial with $n = 10$ and $p = 0.4$
> $P(X \le 4) = 0.6331$ ✓✓

Calculator Assumed

6. [8 marks: 3, 2, 3]

On average, 20% of teachers in a particular state have previously been treated for work related depression. The mathematics department in a particular school in this state has 11 staff members (all teachers!).

(a) Find the expected number of depressed teachers and its expected standard deviation. Justify your answer.

> Let X: No. of depressed teachers out of 11
> $X \sim B(n = 11, p = 0.2)$ ✓
> Expected No. $= np = 11 \times 0.2 = 2.2$ ✓
> Std. Deviation $= \sqrt{(npq)} = 1.3266$ ✓

(b) Find the probability that there are no more than five staff members in this department that have previously been treated for work related stress.

> $P(X \le 5) = 0.9883$ ✓✓

(c) Find the probability that there are exactly two staff members who have previously been treated for work related stress given that there are no more than five of them.

$$P(X = 2 \mid X \le 5) = \frac{P(X = 2 \cap X \le 5)}{P(X \le 5)}$$
$$= \frac{0.29528}{0.98835} \quad ✓✓$$
$$= 0.2988 \quad ✓$$

Calculator Assumed

7. [12 marks: 3, 1, 3, 2, 3]

It is known that the probability of finding a defective biro in a large consignment of biros is 0.05. These biros are packed and sold in packs of 12 biros each.

(a) Find the expected number of defective biros in any given pack and its associated standard deviation.

> Let X: No. of defective biros out of 12.
> $X \sim B(12, 0.05)$ ✓
> Expected No. $= 12 \times 0.05 = 0.6$ ✓
> Std. Deviation $= 0.7550$ ✓

(b) Find the probability that a randomly chosen pack has exactly two defective biros.

> $P(X = 2) = 0.09879$ ✓

(c) Given that a randomly chosen pack has no more than 4 defective biros, find the probability that the pack has at least two defective biros.

> $P(X \geq 2 \mid X \leq 4) = \dfrac{P(2 \leq X \leq 4)}{P(X \leq 4)}$
> $= \dfrac{0.11818}{0.99982}$ ✓✓
> $= 0.1182$ ✓

(d) Find the probability that there were no more than 4 defective biros each in two randomly chosen packs.

> $P(X \leq 4) \times P(X \leq 4) = 0.99982^2$ ✓
> $= 0.9996$ ✓

(e) Find the probability that in a box of 50 such packs (of 12 biros each), no more than 10 of these packs had exactly two defective biros each.

> Let X: No. of packs out of 50 with 2 defective biros.
> $X \sim B(50, 0.09879)$ ✓✓
> $P(X \leq 10) = 0.9914$ ✓

Calculator Assumed

8. [11 marks: 4, 4, 3] [TISC]

60% of students in a school own at least one Apple® device.

(a) Ten students were randomly chosen. Find the probability that more than six of these students own at least one Apple® device. Show how you obtained your answer.

> Define X: No. of students out of 10 who own at least one Apple® device.
> $X \sim B(n = 10, p = 0.6)$
> $P(X > 6) = P(7 \leq X \leq 10) = 0.38228$
>
> - Random variable used declared in words ✓
> - Distribution identified as binomial with correct parameters ✓
> - $P(X \geq 7)$ or $P(7 \leq X \leq 10)$ ✓
> - Correct probability ✓

(b) n students were randomly chosen. Let X: Number of students out of n students who own at least one Apple® device. The mean for X is μ and the accompanying standard deviation is $\dfrac{4\sqrt{15}}{5}$. Find n and μ.

> Define X: No. of students out of n who own at least one Apple® device.
> $X \sim B(n, p = 0.6)$ ✓
> Standard deviation for X $= \sqrt{n \times 0.6 \times 0.4} = \dfrac{4\sqrt{15}}{5}$ ✓
> $n = 40$ ✓
> Mean $\mu = 40 \times 0.6 = 24$ ✓

(c) n students were randomly chosen. The probability that all of these students chosen owned at least one Apple® device is 0.01. Find n.

> Define X: No. of students out of n who own at least one Apple® device.
> $X \sim B(n, p = 0.6)$ ✓
> $P(X = n) = 0.6^n = 0.01$ ✓
> $n = 9$ ✓
> (accept $n = 10$)

Calculator Assumed

9. [9 marks: 3, 3, 3] [TISC]

80% of all cats in a certain city have been bio-chipped.

(a) Fifty cats were randomly selected.

(i) Calculate the probability that at least 38 but no more than 42 cats have been bio-chipped. Show clearly how you obtained your answer.

> Define X: No. of cats have been bio-chipped out of 50. ✓
> $X \sim B(50, 0.8)$ ✓
> $P(38 \le X \le 42) = 0.6235$ ✓

(ii) Calculate the probability that at least 38 cats have been bio-chipped given that no more than 42 have been bio-chipped.

> $P(X \ge 38 \mid X \le 42) = \dfrac{P(38 \le X \le 42)}{P(X \le 42)}$ ✓✓
> $= 0.7702$ ✓

(b) In a sample of 20 cats from this city, 15 were found to have been bio-chipped, while the other 5 have not been bio-chipped.
Five cats were chosen from this sample.
Calculate the probability that all five cats have been bio-chipped.

> $P(\text{5 cats with bio-chips}) = \dfrac{\binom{15}{5}\binom{5}{0}}{\binom{20}{5}}$ ✓✓
> $= 0.1937$ ✓

Calculator Assumed

10. [11 marks: 2, 1, 3, 4] [TISC]

It is estimated that 35% of students at a school come from a Non-English Speaking Background.
Clearly stating the probability distribution you are using, estimate the probability that, in a class of twenty students from this school,

(a) exactly five of them are from a Non-English Speaking Background.

> Let X: No. of NESB students out of 20.
> $X \sim B(20, 0.35)$ ✓
> $P(X = 5) = 0.12720$ ✓

(b) more than ten of them are from a Non-English Speaking Background.

> $P(X > 10) = 0.05317$ ✓

In five classes of 20 students each,

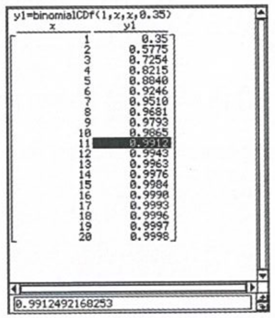
(c) estimate the probability that at least two of these classes each contain more than ten students from Non-English Speaking backgrounds.

> Let Y: No. of classes out of 5 classes with more than 10 NESB students each.
> $Y \sim B(5, 0.05317)$ ✓✓
> $P(Y \ge 2) = 0.02538$ ✓

A sample of n students from this school was chosen.
The probability that at least one of the students selected is from a Non-English Speaking Background is greater than 0.99.

(d) Find the minimum value of n. Show all working.

> Let W: No. of NESB students out of n.
> $W \sim B(n, 0.35)$
> $P(W \ge 1) > 0.99$ ✓
> $\Rightarrow P(W = 0) < 0.01$
> ${}^{n}C_0 \times 0.35^0 \times 0.65^n < 0.01$
> $0.65^n < 0.01$ ✓
> $n > 10.69$ ✓
> $n = 11$ ✓
>
> OR From CAS calculator table
> $n = 11$



y1=binomialCDf(1,x,x,0.35)

x	y1
1	0.35
2	0.5775
3	0.7254
4	0.8215
5	0.8840
6	0.9246
7	0.9510
8	0.9681
9	0.9793
10	0.9865
11	0.9912
12	0.9943
13	0.9963
14	0.9976
15	0.9984
16	0.9990
17	0.9993
18	0.9996
19	0.9997
20	0.9998

0.9912492168253

Calculator Assumed

11. [14 marks: 4, 5, 5]

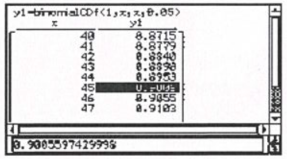
It is known that the probability of finding a defective biro in a large consignment of biros is p .

(a) Given that $p = 0.05$, find the minimum number of biros that need to be selected so that the probability that:

(i) at least one of them is defective is greater than 90%

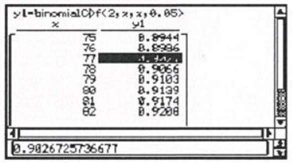
Let X: No. of defective biros out of n.
$X \sim B(n, 0.05)$ ✓

$P(X \geq 1) > 0.9$
$1 - P(X = 0) > 0.9 \Rightarrow P(X = 0) < 0.1$
${}^{n}C_0 \times 0.05^0 \times 0.95^n < 0.1$ ✓
$0.95^n < 0.1$
$\Rightarrow n > 44.9$ ✓
Hence, $n = 45$. ✓



(ii) more than one of them is defective is greater than 90%.

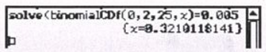
$P(X > 1) > 0.9$
$1 - P(X = 0) - P(X = 1) > 0.9$ ✓
$\Rightarrow P(X = 0) + P(X = 1) < 0.1$
${}^{n}C_0 \times 0.05^0 \times 0.95^n + {}^{n}C_1 \times 0.05 \times 0.95^{n-1} < 0.1$ ✓
$0.95^n + (0.05n \times 0.95^{n-1}) < 0.1$ ✓
Solve graphically, $\Rightarrow n > 76.3$ ✓
Hence, $n = 77$. ✓



(b) In a sample of 25 biros, what is the maximum value for p so that the probability that there are no more than two defective biros in the sample exceeds 0.5%? Justify your answer.

Let X: No. of defective biros out of 25.
$X \sim B(25, p)$ ✓
$P(X \leq 2) > 0.005$
$P(X = 0) + P(X = 1) + P(X = 2) > 0.005$ ✓
${}^{25}C_0 \times p^0 \times (1-p)^{25} + {}^{25}C_1 \times p \times (1-p)^{24} + {}^{25}C_2 \times p^2 \times (1-p)^{23} > 0.005$
$(1-p)^{25} + 25p(1-p)^{24} + 300p^2(1-p)^{23} > 0.005$ ✓
Solve graphically, $\Rightarrow 0 < p < 0.3210$ ✓✓

Or solve on CAS calculator:
$0 < p < 0.3210$



Calculator Assumed

12. [13 marks: 2, 2, 5, 4]

Police know that from long experience, on a particular stretch of road, 1 car in every 10 will exceed the speed limit. A radar trap is set on this stretch of road.

(a) Find the probability that the police will find that the first 5 cars will be within the limit and the sixth will be speeding.

Prob. $= 0.9^5 \times 0.1 = 0.059049$ ✓✓

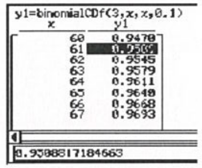
(b) Find the probability that of 6 cars passing this radar trap, exactly one will be speeding.

Prob. $= {}^{6}C_1 \times 0.9^5 \times 0.1 = 0.3543$ ✓✓

(c) n cars passed this speed trap. The probability that more than 2 cars were speeding was more than 95%. Find the least value of n.

Let X: No. of speeding cars out of n.
$X \sim B(n, 0.1)$ ✓

$P(X > 2) > 0.95$
$P(X = 0) + P(X = 1) + P(X = 2) < 0.05$
${}^{n}C_0 \times 0.1^0 \times 0.9^n + {}^{n}C_1 \times 0.1 \times 0.9^{n-1} + {}^{n}C_2 \times 0.1^2 \times 0.9^{n-2} < 0.05$ ✓
$0.9^n + (0.1n \times 0.9^{n-1}) + (0.01 \times \frac{n(n-1)}{2} \times 0.9^{n-2}) < 0.05$ ✓
Solve graphically, $\Rightarrow n > 60.8$ ✓
Hence, $n = 61$. ✓



(d) On another stretch of road, $100p$ cars out of 100 will have speeds exceeding the speed limit. In a sample of 20 cars, find the range of values for p such that the probability of less than 1 speeding car is exceeds 1%.

Let X: No. of speeding cars out of 20.
$X \sim B(20, p)$ ✓

$P(X < 1) > 0.01$
$P(X = 0) > 0.01$
${}^{20}C_0 \times p^0 \times (1-p)^{20} > 0.01$ ✓
$(1-p)^{20} > 0.01$ ✓
Solve graphically, $\Rightarrow 0 < p < 0.2057$ ✓

Mathematics Methods Units 3 & 4 Revision Series *24 The Binomial Distribution*

Calculator Assumed

13. [12 marks: 4, 2, 2, 4]

X is a binomial variable with parameters n and p.

(a) Find n and p if the mean for X and its standard deviation are $\frac{85}{4}$ and $\frac{\sqrt{51}}{4}$ respectively.

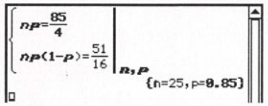
$X \sim B(n, p)$.

$np = \frac{85}{4}$ I ✓

$np(1-p) = \frac{51}{16}$ II ✓

Solve simultaneously:

$\Rightarrow n=25, p=0.85$ ✓✓



(b) For $n = 100$, $p = 0.8$, in each instance write algebraic expressions for evaluating each of the following probabilities before evaluating them.

(i) $P(X = 80)$

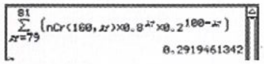
Prob. $= {}^{100}C_{80} \times 0.8^{80} \times 0.2^{20}$ ✓

$= 0.0993002$ ✓

(ii) $P(79 \le X \le 81)$.

$P(79 \le X \le 81) = \sum_{x=79}^{x=81} {}^{100}C_x \times 0.8^x \times 0.2^{100-x}$ ✓

$= 0.29195$ ✓



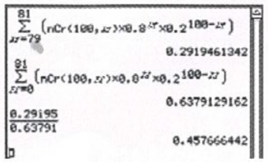
(iii) $P(X \ge 79 \mid X \le 81)$

$P(X \ge 79 \mid X \le 81) = \frac{P(79 \le X \le 81)}{P(X \le 81)}$

$= \frac{\sum_{x=79}^{x=81} {}^{100}C_x \times 0.8^x \times 0.2^{100-x}}{\sum_{x=0}^{x=81} {}^{100}C_x \times 0.8^x \times 0.2^{100-x}}$ ✓✓

$= \frac{0.29195}{0.63791}$ ✓

$= 0.4577$ ✓



Calculator Assumed

14. [8 marks: 5, 3]

It is estimated that a out of every 100 learners are successful in obtaining their driver's licence in their first attempt. On a given day there were n learners having their first attempt in getting their driver's licence.

(a) Given that the expected number of successful learners in the sample is $\frac{112}{5}$ and its variance is $\frac{112}{25}$. Find a and n.

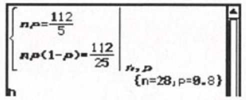
Let: X: No. of successful learners out of n.
$X \sim B(n, p)$.

$np = \frac{112}{5}$ I ✓

$np(1-p) = \frac{112}{25}$ II ✓

Solve simultaneously:

$\Rightarrow n = 28, p = 0.8$ ✓✓



Hence, $a = 80$. ✓

(b) Find the most likely number of successful learners in the sample. Justify your answer.

Let: X: No. of successful learners out of n.
$X \sim B(28, 0.8)$.

Expected number $= \frac{112}{5} = 22.4$ ✓

$P(X = 22) = 0.17791$
$P(X = 23) = 0.18565$
$P(X = 24) = 0.15470$ ✓

Hence, most likely no. = 23. ✓

25 Continuous Random Variables

Calculator Assumed

1. [7 marks: 2, 2, 3]

Determine with reasons if each of the following functions are probability density functions for continuous random variables:

(a) $f(x) = 4$ for $-\frac{1}{4} \le x \le 0$

$f(x) \ge 0$ for $\frac{-1}{4} \le x \le 0$

Also, $\int_{-1/4}^{0} 4\,dx = 1$. ✓(both conditions)

Hence, $f(x)$ is a pdf. ✓

(b) $f(x) = \begin{cases} \frac{2}{x} & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

$\int_0^2 \frac{2}{x}\,dx$ is undefined. ✓

Hence, $f(x)$ is not a pdf. ✓

(c) $f(x) = 0.1\,e^{-0.1x}$ for $x > 0$

$f(x) \ge 0$ for $x > 0$ ✓

Also, $\int_0^{\infty} 0.1\,e^{-0.1x}\,dx = 1$. ✓

Hence, $f(x)$ is a pdf. ✓

Calculator Free

2. [6 marks]

The random variable X has as its probability density function $f(x) = \frac{x}{2}$ for $0 \le x \le 2$. Determine the mean and variance for X.

$E(X) = \frac{1}{2}\int_0^2 x \times x\,dx$ ✓

$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{4}{3}$ ✓✓

$E(X^2) = \frac{1}{2}\int_0^2 x^2 \times x\,dx$

$= \frac{1}{2}\left[\frac{x^4}{4}\right]_0^2 = 2$ ✓

$\text{Var}(X) = 2 - \left(\frac{4}{3}\right)^2 = \frac{2}{9}$ ✓✓

3. [6 marks]

The random variable X has as its probability density function $f(x) = \frac{2(x+1)}{3}$ for $0 \le x \le 1$. Determine the mean and variance for X.

$E(X) = \frac{2}{3}\int_0^1 x \times (x+1)\,dx$ ✓

$= \frac{2}{3}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = \frac{5}{9}$ ✓✓

$E(X^2) = \frac{2}{3}\int_0^1 x^2 \times (x+1)\,dx$

$= \frac{2}{3}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 = \frac{7}{18}$ ✓

$\text{Var}(X) = \frac{7}{18} - \left(\frac{5}{9}\right)^2 = \frac{13}{162}$ ✓✓

Calculator Free

4. [4 marks] [TISC]

$F(x) = \int_0^x k(1-x^2)\,dx$ for $0 \le x \le 1$, is the cumulative distribution function for a continuous random variable. Find k.

$$\int_0^1 k(1-x^2)\,dx = 1 \quad ✓$$

$$k\left[x - \frac{x^3}{3}\right]_0^1 = 1 \quad ✓$$

$$k\left[1 - \frac{1}{3}\right] = 1 \quad ✓$$

$$k = \frac{3}{2} \quad ✓$$

5. [6 marks]

The random variable X has mean $\frac{2}{3}$ and probability density function $f(x) = kx$ for $0 \le x \le a$. Determine the values of k and a.

$$\int_0^a kx\,dx = 1 \quad ✓$$

$$\left[\frac{kx^2}{2}\right]_0^a = 1$$

$$ka^2 = 2 \quad \text{I} \quad ✓$$

$$E(X) = \int_0^a x \times kx\,dx = \frac{2}{3} \quad ✓$$

$$\left[\frac{kx^3}{3}\right]_0^a = \frac{2}{3}$$

$$ka^3 = 2 \quad \text{II} \quad ✓$$

Substitute I into II: $a = 1, k = 2$ ✓✓

Calculator Assumed

6. [13 marks: 3, 3, 4, 3]

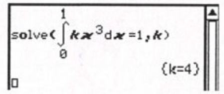
The random variable X has as its probability density function $f(x) = kx^3$ for $0 \le x \le 1$.

(a) Find k.

$$k\int_0^1 x^3\,dx = 1 \quad ✓$$

$$k \times \frac{1}{4} = 1 \quad ✓$$

$$\Rightarrow k = 4 \quad ✓$$



(b) Find $P(X \le 0.6 \mid X \le 0.8)$

$$P(X \le 0.6 \mid X \le 0.8) = \frac{P(X \le 0.6 \cap X \le 0.8)}{P(X \le 0.8)}$$

$$= \frac{P(X \le 0.6)}{P(X \le 0.8)} \quad ✓$$

$$= \frac{0.1296}{0.4096} \quad ✓$$

$$= 0.3164 \quad ✓$$

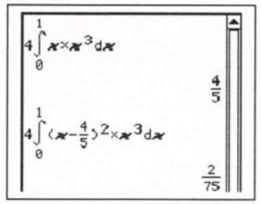
(c) Determine μ and σ^2, respectively the mean and variance for X.

$$\mu = 4\int_0^1 x \times x^3\,dx \quad ✓$$

$$= \frac{4}{5} \quad ✓$$

$$\sigma^2 = 4\int_0^1 \left(x - \frac{4}{5}\right)^2 \times x^3\,dx \quad ✓$$

$$= \frac{2}{75} \quad ✓$$



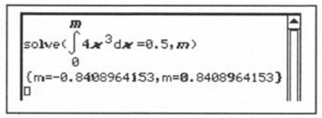
(d) Find m the median of X.

$$P(X \le m) = 0.5$$

$$\Rightarrow 4\int_0^m x^3\,dx = 0.5 \quad ✓$$

$$m^4 = 0.5 \quad ✓$$

$$\Rightarrow m = 0.8409 \quad ✓$$



Calculator Assumed

7. [10 marks: 3, 3, 2, 2]

The probability density function for a continuous random variable T is given by

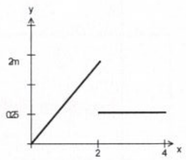
$$f(t) = \begin{cases} mt & 0 \le t \le 2 \\ \frac{1}{4} & 2 < t < 4 \end{cases}$$

(a) Find the value of m.

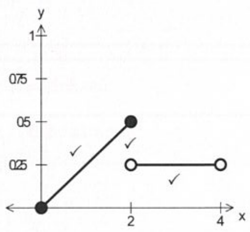
Area under curve = 1

$\frac{1}{2} \times 2 \times 2m + 2 \times \frac{1}{4} = 1$ ✓✓

$m = \frac{1}{4}$ ✓



(b) Sketch the graph of the probability density function of T.



(c) Find $P(T \le 1)$.

$P(T \le 1) = \frac{1}{2} \times 1 \times \frac{1}{4}$ ✓

$= \frac{1}{8}$ ✓

(d) Find the median of T.

Let median be m.

$\Rightarrow P(X \le m) = 0.5$

But $P(X \le 2) = 0.5$ ✓

Hence, median of T = 2. ✓

Calculator Assumed

8. [12 marks: 3, 2, 3, 4]

The probability density function of a random variable X is given by $f(x) = x^2 + ax$ for $0 < x < 1$.

(a) Find the value of a.

$\int_0^1 x^2\,dx + a\int_0^1 x\,dx = 1$ ✓

$\frac{1}{3} + \frac{a}{2} = 1$ ✓

$\Rightarrow a = \frac{4}{3}$ ✓

$\text{solve}\left(\int_0^1 x^2 + a x\,dx = 1, a\right)$

$\left\{a = \frac{4}{3}\right\}$

(b) Find $P(X > 0.5)$.

$P(X > 0.5) = \int_{0.5}^{1} x^2 + \frac{4x}{3}\,dx$ ✓

$= \frac{19}{24}$ ✓

(c) Find the value of k if $P(X \le k) = 0.9$.

$\int_0^k x^2 + \frac{4x}{3}\,dx = 0.9$ ✓

$\frac{k^3}{3} + \frac{2k^2}{3} = 0.9$ ✓

$\Rightarrow k = 0.9558$ ✓

$\text{solve}\left(\int_0^k x^2 + \frac{4x}{3}\,dx = 0.9, k\right)$

$\{k = 0.9558\}$

(d) Calculate the mean and variance for X.

$E(X) = \int_0^1 x \times \left(x^2 + \frac{4x}{3}\right) dx$ ✓

$= \frac{25}{36}$ ✓

$Var(X) = \int_0^1 \left(x - \frac{25}{36}\right)^2 \times \left(x^2 + \frac{4x}{3}\right) dx$ ✓

$= \frac{331}{6480}$ ✓

$\int_0^1 x \times \left(x^2 + \frac{4x}{3}\right) dx$

$\frac{25}{36}$

$\int_0^1 \left(x - \frac{25}{36}\right)^2 \times \left(x^2 + \frac{4x}{3}\right) dx$

$\frac{331}{6480}$

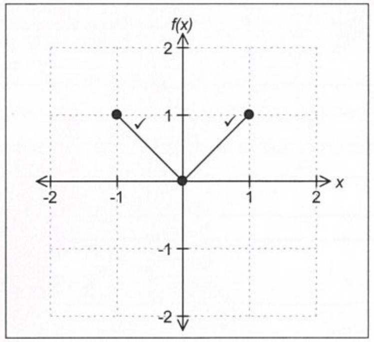
Calculator Assumed

9. [9 marks: 2, 1, 2, 3, 1] [TISC]

The probability density function of a continuous random variable X is given by:

$$f(x)=\begin{cases} -x & -1\le x\le 0 \\ x & 0<x\le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) On the axes below, sketch the graph of $f(x)$.



(b) Find P(X = 0).

P(X = 0) = 0. ✓

(c) Find P(–0.5 < X < 0.5).

P(–0.5 < X < 0.5) = (½ × 0.5 × 0.5) × 2 ✓ = 0.25 ✓

(d) Find a such that P(0 < X < a) = 0.25.

P(0 < X < a) = (½ × a × a) Hence, ½ × a × a = 0.25 ✓✓ a = 0.7071 ✓

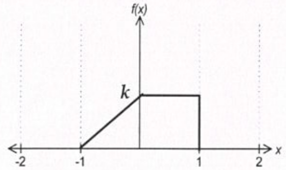
(e) Find the median for X.

Median = 0. ✓

Calculator Assumed

10. [9 marks: 2, 2, 3, 2] [TISC]

The graph of the probability density function of a continuous random variable X is shown in the accompanying diagram.



(a) Find the value of k.

Area under curve = 1 Hence, $(k \times 1) + \frac{1}{2} \times 1 \times k = 1$ ✓ $\Rightarrow k = \frac{2}{3}$ ✓

(b) Find P(X < 0.5).

$P(X < 0.5) = 1 - \frac{1}{2} \times \frac{2}{3} = \frac{2}{3}$ ✓✓

(c) Find P(X > 0 | X < 0.5).

$P(X>0 \mid X<0.5) = \frac{P(X>0 \cap X<0.5)}{P(X<0.5)}$ $= \frac{P(0<X<0.5)}{P(X<0.5)}$ ✓ $= \frac{1/3}{2/3} = \frac{1}{2}$ ✓✓

(d) Find a if $P(X > a) = \frac{1}{2}$.

Area of rectangle to the right of the line x = a is $(1 - a) \times \frac{2}{3}$ Hence, $(1-a) \times \frac{2}{3} = \frac{1}{2}$ ✓ $\Rightarrow a = \frac{1}{4}$ ✓

Calculator Assumed

11. [8 marks: 2, 3, 3] [TISC]

The probability density function of a continuous random variable X is given by $f(x) = k\left(\sqrt{4-x}\right)$ for $0 \le x \le 4$ where k is a real constant.

(a) Show that $k = \dfrac{3}{16}$.

$$k\int_0^4 \sqrt{4-x}\, dx = 1 \quad ✓$$

$$k \times \frac{16}{3} = 1 \quad ✓$$

$$\Rightarrow \quad k = \frac{3}{16}$$

```
solve(∫[0,4] k√(4−x) dx = 1, k)
                         {k=3/16}
```

(b) Find $P(X > 1 \mid X < 3)$.

$$P(X > 1 \mid X < 3) = \frac{P(X > 1 \cap X < 3)}{P(X < 3)}$$

$$= \frac{P(1 < X < 3)}{P(X < 3)} \quad ✓$$

$$= \frac{0.5245}{0.875} \quad ✓$$

$$= 0.5995 \quad ✓$$

(c) Find the median for X accurate to 2 decimal places. Justify your answer.

Let median be m.

$$\Rightarrow P(X \le m) = 0.5$$

$$\frac{3}{16}\int_0^m \sqrt{4-x}\, dx = 0.5 \quad ✓$$

$$\frac{3}{16} \times \left[\frac{2(4-x)^{3/2}}{-3}\right]_0^m = 0.5 \quad ✓$$

$$\frac{3}{16} \times \left[\frac{2(4-m)^{3/2}}{-3} - \frac{2(4)^{3/2}}{-3}\right] = 0.5$$

$$\Rightarrow \quad m = 1.48 \quad ✓$$

```
solve(∫[0,m] 3/16 √(4−x) dx = 0.5, m)
                         {m=1.4801579}
```

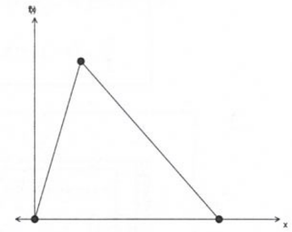

Calculator Assumed

12. [9 marks: 2, 2, 2, 3] [TISC]

The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} 0.5x & 0 \le x < 1 \\ \dfrac{-x}{6} + \dfrac{2}{3} & 1 \le x \le k \end{cases}$$

where k is a real number. The sketch of $y = f(x)$ is given in the accompanying diagram.



(a) Show that $k = 4$.

$$f(1) = \frac{1}{2} \quad ✓$$

Area under curve = 1.

$$\Rightarrow \frac{1}{2} \times \left(\frac{1}{2} \times k\right) = 1 \quad ✓$$

$$k = 4$$

(b) Find $P(X > 0.5)$.

$$P(X > 0.5) = 1 - \frac{1}{2} \times \left(\frac{1}{2} \times \frac{1}{4}\right) \quad ✓$$

$$= \frac{15}{16} \quad ✓$$

(c) Find $P(X \le 2 \mid X > 0.5)$.

$$P(X \le 2 \mid X > 0.5) = \frac{P(0.5 < X \le 2)}{P(X > 0.5)}$$

$$= \frac{1 - \frac{1}{2}\left(\frac{1}{2} \times \frac{1}{4}\right) - \frac{1}{2}\left(2 \times \frac{1}{3}\right)}{\frac{15}{16}} \quad ✓$$

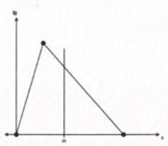
$$= \frac{29}{45} \quad ✓$$

(d) Find m, the median value of x, such that $P(X < m) = 0.5$.

$$P(X < m) = 0.5$$

$$\frac{1}{2} \times (4 - m)\left(\frac{2}{3} - \frac{m}{6}\right) = \frac{1}{2} \quad ✓✓$$

$$m = 1.5505 \quad ✓$$

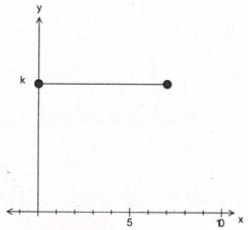


26 The Uniform Distribution

Calculator Free

1. [10 marks: 1, 2, 3, 4]

The sketch of the probability density function of a continuous random variable X is given in the accompanying diagram.



(a) Find k.

Area under curve = 1

$7 \times k = 1$

$\Rightarrow \ k = \frac{1}{7}$ ✓

(b) State the probability density function of the variable.

$f(x) = \frac{1}{7} \quad 0 \le x \le 7.$ ✓✓

(c) Find $P(X > 2 \mid X < 5)$.

$P(X > 2 \mid X < 5) = \frac{P(2 < X < 5)}{P(X < 5)}$ ✓✓

$= \frac{3}{5}$ ✓

(d) Find μ and σ are respectively the mean and standard deviation for X.

$\mu = \frac{7}{2}$ ✓

$E(X^2) = \int_0^7 \frac{x^2}{7}\,dx$

$= \frac{1}{7}\left[\frac{x^3}{3}\right]_0^7 = \frac{49}{3}$ ✓

$\sigma^2 = \frac{49}{3} - \left(\frac{7}{2}\right)^2 = \frac{49}{12}$ ✓

$\sigma = \frac{7\sqrt{3}}{6}$ ✓

Calculator Free

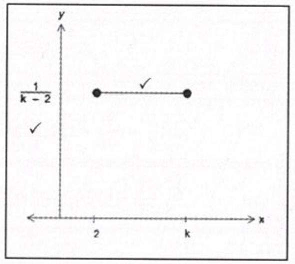
2. [10 marks: 2, 2, 3, 3]

The continuous random variable X is uniformly distributed over the interval $2 \le x \le k$.

(a) Find in terms of k, the probability distribution function for X.

$f(x) = \frac{1}{k-2} \quad 2 \le x \le k.$ ✓✓

(b) Sketch the probability distribution function for X.



(c) Find $P(X \le 10 \mid X \ge 4)$.

$P(X \le 10 \mid X \ge 4) = \frac{P(4 \le X \le 10)}{P(X \ge 4)}$ ✓

$= \frac{6}{k-4}$ ✓✓

It is known that $P(X \le 5) = 0.25$.

(d) Find the median for X. Justify your answer.

$P(X \le 5) = 0.25$

$\Rightarrow \ \frac{5-2}{k-2} = \frac{1}{4}$ ✓

$k = 14$ ✓

Hence, median = 8 ✓

Calculator Free

3. [4 marks]

X is uniformly distributed in the interval $a \le x \le b$. The probability density function of X is given by $f(x) = 0.1$ for $a \le x \le b$. Given that $P(X > 7 \mid X \le 8) = \frac{1}{3}$, find a and b. Show clearly how you obtained your answer.

$$P(X > 7 \mid X \le 8) = \frac{1}{3}$$
$$\Rightarrow \frac{P(7 < X \le 8)}{P(a < X \le 8)} = \frac{1}{3} \quad ✓$$
$$\frac{1 \times 0.1}{(8-a) \times 0.1} = \frac{1}{3} \quad ✓$$
$$\Rightarrow a = 5 \quad ✓$$
$$b = 15 \quad ✓$$

4. [4 marks]

X is uniformly distributed over the interval $57 \le x \le 67$.

(a) State the probability density function for X.

$$f(x) = \frac{1}{10} \text{ for } 57 \le x \le 67 \quad ✓✓$$

(b) 10 observations of X were taken. Find the probability that none of these observations exceed 58.

$$P(X \le 58) = \frac{1}{10} \quad ✓$$

Required Prob. $= 0.1^{10}$. ✓

Calculator Free

5 [9 marks: 2, 1, 2, 4]

The length of time Daniel is late to class may be modelled by a uniform distribution with a minimum late time of 5 minutes and a maximum late time of 25 minutes.
Define T: The length of time Daniel is late to class.

(a) Write the probability density function for T.

$$f(t) = \frac{1}{20} \quad 5 \le t \le 25 \quad ✓✓$$

(b) Find the probability that Daniel is exactly 15 minutes late.

Prob. $= 0$ ✓

(c) Find the probability the Daniel is no more than 20 minutes late given that he is at least 10 minutes late.

$$P(T \le 20 \mid T \ge 10) = \frac{10}{15} \quad ✓✓$$

(d) On a school week of five days, find the probability that Daniel is late by at least 15 minutes at least once.

X: No. of days late by $\ge$ 15 minutes out of 5. ✓

P(late $\ge$ 15 minutes) $= P(T \ge 15) = \frac{1}{2}$ ✓

Hence, $X \sim B(5, \frac{1}{2})$. ✓

$$P(X \ge 1) = 1 - P(X = 0)$$
$$= 1 - \left(\frac{1}{2}\right)^5 \quad ✓$$
$$= \frac{31}{32}$$

Mathematics Methods Units 3 & 4 Revision Series

Calculator Assumed

6. [9 marks: 2, 2, 5]

The length of the red cycle of a set of traffic lights is 90 seconds. Assume that vehicles arrive at the traffic lights randomly and independently of each other. Define the random variable T as the waiting time at the traffic lights.

(a) Describe the probability density function of T.

$f(t) = \frac{1}{90} \quad 0 \le t \le 90$. ✓✓

(b) Find the probability that a motorist has to wait less than 30 seconds at the lights.

$P(T < 30) = \frac{1}{3}$ ✓✓

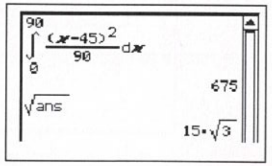
(c) Calculate $P(\mu - \sigma < T < \mu + \sigma)$ where μ and σ are respectively the mean and standard deviation for T.

$\mu = 45$ ✓

$\sigma^2 = \int_0^{90} (t-45)^2 \times \frac{1}{90}\, dt$

$= 675$ ✓

$\sigma = 15\sqrt{3}$ ✓



Hence,

$P(45 - 15\sqrt{3} < T < 45 + 15\sqrt{3}) = \frac{30\sqrt{3}}{90}$

$= \frac{\sqrt{3}}{3}$ ✓✓

Calculator Assumed

7. [9 marks: 2, 2, 2, 3]

An automatic filling machine fills and packs 1 kg packs of sugar. It is known that the machine can fill any pack with any amount of between 0 and 10 grams (inclusive) of extra sugar. Define M as the extra mass of sugar (in grams) fed into each bag.

(a) Find the probability density function for M.

$f(m) = \frac{1}{10} \quad 0 \le t \le 10$. ✓✓

(b) Find $P(3 \le M \le 7)$.

$P(3 \le M \le 7) = \frac{4}{10} = \frac{2}{5}$ ✓✓

(c) Find the probability of obtaining a 1 kg pack of sugar with mass of between 1.003 and 1.007 kg.

Required Prob. $= P(3 \le M \le 7) = \frac{2}{5}$ ✓✓

(d) The probability of obtaining a 1 kg pack of sugar with a mass of no more than α kg is 0.75. Find α.

Let the excess be k.

$P(M \le k) = 0.75$

Hence, $\frac{k}{10} = 0.75$ ✓

$k = 7.5$ g ✓

Therefore, $\alpha = 1.0075$ kg. ✓

27 The Normal Distribution

Calculator Assumed

1. [7 marks: 2, 2, 3]

The speed of vehicles passing a school zone each school-day is normally distributed with a mean of 35 km/h and a standard deviation of 3 km/h.

(a) Find the probability that a vehicle passing the school zone travels with a speed in excess of the mean speed by at least 2 standard deviations.

> $X \sim N(35, 3^2)$
> $P(X > 35 + 2\times 3) = P(X > 41)$ ✓
> $= 0.02275$ ✓

(b) 25% of vehicles passing the school zone travel in excess of a km/h. Find a.

> $P(X > a) = 0.25$ ✓
> $a = 37.0$ km/h ✓

```
invNormCDf("R",0.25,3,35)
                         37.02346925
solve(normCDf(a,∞,3,35)=0.25,a)
                       {a=37.02346925}
```

(c) On a certain morning, 30 vehicles were noted passing through the school zone. Find the probability that no more than 15 were travelling in excess of 35 km/h.

> X: No. of vehicles out of 30 with speeds > 35 km/h
> $X \sim B(30, 0.5)$ ✓✓
>
> $P(X \le 15) = 0.57223$ ✓

Calculator Assumed

2. [10 marks: 1, 1, 3, 2, 3] [TISC]

The life-span of a light bulb manufactured by GloWest is normally distributed with a mean of 800 hours and a standard deviation of 120 hours.

(a) Find the probability that a randomly chosen light bulb manufactured by GloWest has a life-span :

(i) of exactly 800 hours

> $X \sim N(800, 120^2)$
> $P(X = 800) = 0$ ✓

(ii) that exceeds 700 hours

> $P(X > 700) = 0.7977$ ✓

(iii) that is less than 900 hours given that it exceeds 700 hours.

> $$P(X < 900 \mid X > 700) = \frac{P(700 < X < 900)}{P(X > 700)}$$
> $$= \frac{0.5953}{0.7977} \quad ✓✓$$
> $$= 0.7463 \quad ✓$$

(b) Find the lifespan exceeded by 95% of all globes manufactured by GloWest.

> $P(X > k) = 0.95 \Rightarrow k = 602.6$ hours ✓✓

(c) Jan needs to calculate the probability that in the batch of 500 light bulbs from GloWest, there are at least 400 light bulbs with life-spans that exceed 700 hours. State what probability distribution(s) Jan should use and the values of the corresponding parameters. Calculate this probability.

> Y: No. of bulbs with life-spans that exceed 700 hours out of 5 00 bulbs.
> $Y \sim B(500, 0.7977)$ ✓✓
>
> $P(Y \ge 400) = 0.4755$ ✓

Calculator Assumed

3. [9 marks: 1, 2, 3, 3] [TISC]

Mr Green owns an environmentally friendly car called the eco-car. The fully charged battery of an eco-car, allows the driver to travel a certain distance, D km, before the battery needs recharging. This distance D is a normal variable with mean 120 km and standard deviation 12 km.

(a) Find the probability that the car will travel exactly 100 km before the battery needs recharging.

> $D \sim N(120, 12^2)$
> $P(D = 100) = 0$ ✓

Mr Green needs to drive from the town where he lives to visit his mother 150 km away. He starts off with a fully charged battery.

(b) What is the probability that Mr Green will be able to get to his mother without having to recharge the car battery along the way. Show clearly the probability you calculated.

> $P(D > 150) = 0.0062$ ✓✓

(c) Given that Mr Green was not able to get to his mother without having to recharge the car battery along the way, what is the probably that he got to within 10 km of his mother. Show clearly the probabilities you calculated.

$$P(D > 140 \mid D < 150) = \frac{P(140 < D < 150)}{P(D < 150)}$$
$$= \frac{0.04158}{0.99379} \quad ✓✓$$
$$= 0.04184 \quad ✓$$

(d) Mr Green attached a solar-powered booster to his car battery so that the mean for D is now μ with the standard deviation remaining at 12 km. Find μ given that the probability that he will be able to reach his mother on a fully charged battery without recharging along the way is 10%. Show clearly the different stages of your calculations.

$$P(D > 150) = 0.1$$
$$P\left(Z > \frac{150-\mu}{12}\right) = 0.1 \quad ✓$$
$$\frac{150-\mu}{12} = 1.28155 \quad ✓$$
$$\Rightarrow \quad \mu = 134.6 \text{ km} \quad ✓$$

```
solve(normCDf(150,∞,12,x)=0.1,x)
                {x=134.6213812}
```


Calculator Assumed

4. [11 marks: 2, 3, 2, 4]

An aircraft manufacturer has determined that the life of rivets in the construction of the fuselage of its aeroplanes is normally distributed with a mean life of 9 000 flying hours and a standard deviation of 450 hours. The rivets become weak due to metal fatigue.

(a) Find the probability that a randomly chosen rivet has a life exceeding 8 000 hours

> $X \sim N(9000, 450^2)$
> $P(L > 8000) = 0.98687$ ✓✓

(b) Find the probability that a randomly chosen rivet with a life exceeding 8 000 hours has a life not exceeding 10 000 hours

$$P(L < 10\,000 \mid L > 8000) = \frac{P(8000 < L < 10000)}{P(L > 8000)}$$
$$= \frac{0.97373}{0.98687} \quad ✓✓$$
$$= 0.9867 \quad ✓$$

(c) It is considered unsafe to allow a plane to fly if more than 5% of its rivets are weakened; but it is not possible to check all the rivets. The best strategy is simply to perform a major service, replacing all the rivets when the plane becomes unsafe. After how many flying hours should this be done.

> $P(L > k) = 0.05$ ✓
> $\Rightarrow \quad k = 9740$ hours ✓

(d) 2000 rivets were identified. Show clearly how the probability that at least 99% of them has a life exceeding 8 000 hours may be calculated. Find this probability.

> X: No. of rivets out of 2000 with life exceeding 8000
> $X \sim B(2000, 0.98687)$ ✓✓
> $P(X \geq 0.99 \times 2000) = P(X \geq 1980)$ ✓
> $= 0.1265$ ✓

Calculator Assumed

5. [10 marks: 1, 3, 3, 3] [TISC]

The annual rainfall (in mm) over a dam catchment area may be considered a normal variable with mean 900 mm and standard deviation 30 mm.

(a) Find the probability of the catchment area receiving an annual rainfall of less than 850 mm.

$X \sim N(900, 30^2)$
$P(X < 850) = 0.04779$ ✓

(b) Find the probability of the catchment area receiving an annual rainfall of no more than 850 mm given that it received no more than 900 mm.

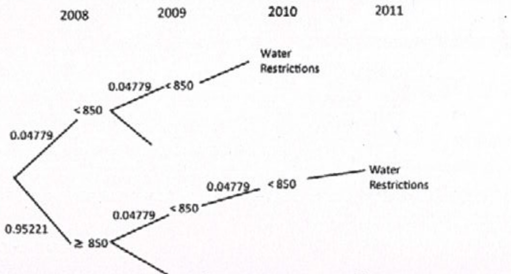
$$P(X \le 850 \mid X \le 900) = \frac{P(X \le 850)}{P(X \le 900)}$$
$$= \frac{0.04779}{0.5} \quad ✓✓$$
$$= 0.09558 \quad ✓$$

Water Restrictions are imposed in this area in the year following two consecutive years where the annual rainfall is less than 850 mm. Assume that the amount of annual rainfall is independent from year to year.
In 2007 there were no Water Restrictions imposed in this area.

(c) Use the given information to find the probability of Water Restrictions being imposed in 2010. Justify your answer.

Required Prob. $= 0.04779^2$ ✓✓
$= 0.00228$ ✓

(d) Use the given information to find the probability of Water Restrictions being imposed in 2010 or 2011. Justify your answer



Required Prob. $= 0.04779^2 + 0.95221 \times 0.04779^2$ ✓✓
$= 0.00446$ ✓

Calculator Assumed

6. [12 marks: 2, 3, 3, 4]

A poultry farm supplies processed chickens to a Fast Food Store. The mass of chickens supplied is normally distributed with mean 1.8 kg and standard deviation 100 g.

(a) Find the probability that a randomly chosen chicken has mass more than 2.0 kg.

Let X: mass of chicken
$X \sim N(1.8, 0.1^2)$ ✓
$P(X > 2) = 0.02275$ ✓

(b) Chickens that have mass less than 1.5 kg are rejected. In a delivery of 1 000 chickens, how many will be rejected? Justify your answer.

$P(X < 1.5) = 0.0013499$ ✓

Hence, no. rejected $= 0.0013499 \times 1\,000$ ✓
$= 1.35$
≈ 1 (accept 2) ✓

(c) A sample of fifty chickens were selected. Determine the probablity that no more than two of these chickens have mass more than 2.0 kg.

Let Y: No. of chickens with mass > 2.0 kg out of 50 ✓
P(mass > 2.0 kg) = 0.02275 (constant)
$Y \sim B(50, 0.02275)$ ✓
$P(Y \le 2) = 0.8948$ ✓

(d) To improve the consistency of the mass of the chickens supplied, the poultry farm wishes to reduce the probability of a chicken with mass more than 2.0 kg to 0.005. Keeping the mean mass unchanged, what should the new standard deviation for the mass be?
Give your answer to the nearest g.

Let W: mass of chicken
$W \sim N(1.8, \sigma^2)$
$P(X > 2) = 0.005$
$P\left(Z > \frac{2-1.8}{\sigma}\right) = 0.005$ ✓✓
Hence, $\frac{2-1.8}{\sigma} = 2.575829$ ✓
$\sigma = 0.07764$ kg ≈ 78 g ✓

```
solve(normCDf(2,∞,x,1.8)=0.005,x)
                {x=0.07764489663}
```

Calculator Assumed

7. [13 marks: 2, 2, 4, 5]

Janine and John run a fish farm. The mass (when mature) of a species of fish bred in the farm is normally distributed with a mean of 1.8 kg and a standard deviation of 100 g. In a certain season, 10 000 fish were harvested.

(a) Find the 95th percentile mass of this species of fish. Show clearly how you obtained your answer.

$X \sim N(1.8, 0.1^2)$
$P(X \le k) = 0.95$ ✓
$\Rightarrow k = 1.9644854 = 1.964$ kg ✓

(b) Estimate the number of fish with mass within 100g of the mean weight.

$P(1.7 \le X \le 1.9) = 0.6826894$ ✓

$N = 10\,000 \times 0.6826894 = 6827$ ✓

When the fish reaches maturity, the harvested fish are sold at the following prices.

Mass, m kg	Price/kg
$m \ge 2.0$	$30
$1.8 \le m < 2.0$	$40
$1.6 \le m < 1.8$	$25
$m < 1.6$	$10

(c) Estimate the revenue received from the sales of fish with mass in excess of 1.9 kg.

$P(1.9 < X < 2) = 0.1359051$ ✓
$P(X \ge 2) = 0.0227501$ ✓
Revenue = 1359 × $40 + 228 × $30 ✓
= $61 200 ✓

(d) Estimate the total revenue from this season's harvest.

$P(X \ge 2.0) = 0.0227501 \Rightarrow$ Revenue = 228 × $30 = $6840 ✓
$P(1.8 \le X < 2.0) = 0.4772498 \Rightarrow$ Revenue = 4772 × $40 = $190 880 ✓
$P(1.6 \le X < 1.8) = 0.4772498 \Rightarrow$ Revenue = 4772 × $25 = $119 300 ✓
$P(X < 1.6) = 0.0227501 \Rightarrow$ Revenue = 228 × $10 = $2280 ✓

Hence, total Revenue = $319 300 ✓

Calculator Assumed

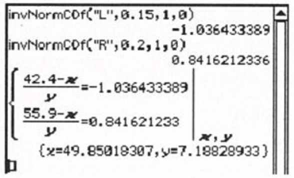
8. [6 marks]

The mass of a batch of free range eggs is normally distributed. It is known that 15% of the eggs have mass less than 42.4g and 20% of eggs have mass that exceed 55.9g. Find the mean and standard deviation of the mass of these eggs.

Let W: mass of eggs.
$W \sim N(\mu, \sigma^2)$
$P(W < 42.4) = 0.15$
$P(Z < \frac{42.4-\mu}{\sigma}) = 0.15$ ✓
$\frac{42.4-\mu}{\sigma} = -1.036433$ I ✓

$P(W > 55.9) = 0.20$
$P(Z > \frac{55.9-\mu}{\sigma}) = 0.20$ ✓
$\frac{55.9-\mu}{\sigma} = 0.8416212$ II ✓

Solve I & II simultaneously:
$\mu = 49.85$ g, $\sigma = 7.1883$ g ✓✓



Calculator Assumed

9. [11 marks: 1, 2, 6, 2] [TISC]

(a) The life span of a native toad (Species A) is normally distributed with mean 58 months and standard deviation 3 months.

(i) Find the probability of a toad having a life span less than 56 months.

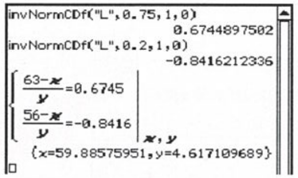
> $X \sim N(58, 3^2)$
> $P(X \le 56) = 0.2525$ ✓

(ii) A sample of 50 toads was selected. Find the probability no more than 10 of these toads have life spans less than 56 months.

> Y: No. of toads with life spans less than 56 months out of 50
> $Y \sim B(n = 50, p = 0.2525)$ ✓
>
> $P(Y \le 10) = 0.2493$ ✓

(b) The life span of a related species of toad (Species B) may be modelled by a normal distribution with mean μ months and standard deviation σ months. The 75th percentile life span is 63 months, while the 20th percentile lifespan is 56 months. Find to one decimal place, μ and σ.

> Let $W \sim N(\mu, \sigma^2)$
> $P(W \le 63) = 0.75$
> $P(Z \le \frac{63-\mu}{\sigma}) = 0.75$ ✓
> $\Rightarrow \frac{63-\mu}{\sigma} = 0.6745$ I ✓
>
> $P(W \le 56) = 0.20$
> $P(Z \le \frac{56-\mu}{\sigma}) = 0.20$ ✓
> $\Rightarrow \frac{56-\mu}{\sigma} = -0.8416$ II ✓
>
> Solve I & II: $\mu = 59.9$ months ✓
> $\sigma = 4.6$ months. ✓



(c) A toad is captured and is found to have a life span less than 56 months. Determine with reasons, whether this toad is more likely to be of Species A or B.

> For Species A, $P(X \le 56) = 0.2525$.
> For Species B, $P(W \le 56) = 0.20$. ✓
>
> Hence, this toad is more likely to be of Species A. ✓

Calculator Assumed

10. [9 marks: 2, 1, 2, 4] [TISC]

The 8 am bus arrives each week day at bus stop C anytime between 7.58 am and 8.08 am. If the bus arrives at C before 8 am, it cannot leave until 8 am. The bus is late if it arrives after 8.00 am.

(a) State the probability density function for *L*.

> *L* is uniformly distributed in the interval $-2 \le L \le 8$.
> Hence probability density function is:
> $f(L) = 0.1 \quad -2 \le L \le 8.$ ✓✓

(b) Find the probability that the bus arrives early at bus stop C.

> $P(-2 \le L < 0) = 0.2$ ✓

(c) Find the probability that the bus is no more than 5 minutes late given that it is late.

> $P(L \le 5 \mid L > 0) = \frac{P(0 < L \le 5)}{P(L > 0)}$ ✓
> $= \frac{5}{8}$ ✓

Debbie lives near bus stop C. The time she takes to walk to the bus stop C is a normal variable with mean 150 seconds and standard deviation 15 seconds. She leaves her home each day at 7.57 am.

(d) Find the probability that Debbie misses the bus given that the bus is early. Show clearly how you arrived at your answer.

> Let W: Time Debbie takes to walk to the bus stop.
> $W \sim N(150, 15^2)$ ✓
> P(Debbie arrives at C after 8 am) = $P(W \ge 180)$
> $= 0.02275$ ✓
> As the event "Debbie misses bus" and the event "bus is early" are independent, ✓
> Hence, P(Debbie misses bus | bus is early) = P(Debbie misses bus)
> $= 0.02275$ ✓

28 Sample Proportion

Calculator Free

1. [7 marks: 4, 3]

It is known that 10% of adult residents in a state are fluent in at least two languages. 200 samples each with 64 adult residents were randomly chosen and the proportions of those fluent in at least two languages calculated.

(a) Describe the *sampling distribution* of sample proportions of size 64 of adult residents fluent in at least two languages, stating its mean and standard deviation.

> As sample size $n = 64 > 30$,
> sample proportion $\hat{\pi}$ is approximately normally distributed ✓
> with mean $= \frac{1}{10}$ ✓ and standard deviation $= \sqrt{\dfrac{\frac{1}{10} \times \frac{9}{10}}{64}}$ ✓ $= \frac{3}{80}$ ✓

(b) Describe the *frequency distribution* of the 200 sample proportions of adult residents fluent in at least two languages, stating its mean and standard deviation.

> As the number of samples N = 200 is large,
> frequency distribution tends towards its sampling distribution.
> Hence, it is approximately normal ✓
> with mean $= \frac{1}{10}$ and standard deviation $= \frac{3}{80}$ ✓✓

Calculator Free

2. [8 marks: 1, 4, 3]

The mass of sugar dispensed by an automatic sugar dispenser is uniformly distributed over the interval 1.5 g to 2.5 g.

(a) Calculate the probability that in any use of the dispenser, the mass of sugar dispensed exceeds 2.4g.

> P(mass of sugar > 2.4) $= \frac{0.1}{1} = \frac{1}{10}$ ✓

The dispenser was used 36 times and the proportion of times the mass of sugar dispensed exceed 2.4 g recorded. This was repeated 100 times so that a collection of 100 sample proportions was obtained.

(b) Describe the *sampling distribution* of sample proportions of size 36 for the mass of sugar dispensed exceeding 2.5 g, stating its mean and standard deviation.

> As sample size $n = 36 > 30$,
> sample proportion $\hat{\pi}$ is approximately normally distributed ✓
> with mean $= \frac{1}{10}$ ✓
> and standard deviation $= \sqrt{\dfrac{\frac{1}{10} \times \frac{9}{10}}{36}} = \frac{1}{20}$ ✓✓

(c) Describe the *frequency distribution* of the 100 sample proportions of the mass of sugar dispensed exceeding 2.5 g, stating its mean and standard deviation.

> As the number of samples N = 100 is large,
> frequency distribution tends towards its sampling distribution.
> Hence, it is approximately normal ✓
> with mean $= \frac{1}{10}$ ✓
> and standard deviation $= \frac{1}{20}$ ✓

Calculator Assumed

3. [9 marks: 2, 3, 1, 3]

The waiting time at a Transport Licensing Centre is uniformly distributed over the interval 5 to 25 minutes.

(a) Find the probability that the waiting time for any customer is no more than 10 minutes.

> X: Waiting time for customers.
> X is uniformly distributed in the interval $5 \le x \le 25$.
> $P(X \le 10) = \frac{5}{20} = \frac{1}{4}$ ✓✓

In a review conducted on the queuing system used, the waiting times of samples of 50 customers each, were recorded.

(b) Describe the sampling distribution (size 50) of the proportion of customers with waiting times of no more than 10 minutes.

> As sample size $n = 50 > 30$,
> sample proportion $\hat{\pi}$ is approximately normally distributed ✓
> with mean $\mu = \frac{1}{4}$ ✓
> and standard deviation $= \sqrt{\frac{\frac{1}{4} \times \frac{3}{4}}{50}} = \frac{\sqrt{6}}{40}$. ✓

(c) Find the probability that a randomly chosen sample has a sample proportion of customers with waiting times of no more than 10 minutes that exceeds 0.3.

> $\hat{\pi} \sim N(\frac{1}{4}, \left(\frac{\sqrt{6}}{40}\right)^2)$.
> $P(\hat{\pi} \ge 0.3) = 0.2071$ ✓

(d) 40 samples each comprising 50 customers were chosen. Determine with reasons, the expected number of samples with sample proportions of customers with waiting times of no more than 10 minutes that exceeds 0.3.

> As the number of samples N = 40 is large, ✓
> frequency distribution of sample proportion tends towards its sampling distribution.
> Hence, frequency distribution is approximately $N(\frac{1}{4}, \left(\frac{\sqrt{6}}{40}\right)^2)$. ✓
> Therefore, expected number $\approx 0.2071 \times 40 \approx 8$ ✓

Calculator Assumed

4. [9 marks: 2, 3, 2, 2]

The mass of sugar in a 1 kg pack is normally distributed with mean 998 g with standard deviation 1 g.

(a) Find the probability that the mass of sugar in a randomly chosen pack exceeds 1 kg. Give your answer to 3 significant figures.

> Let W: mass of sugar in a 1 kg pack
> $W \sim N(998, 1^2)$. ✓
> $P(W > 1000) = 0.02275 \approx 0.0228$ ✓

Samples of size n packs, where $n > 50$, are selected and the proportion of packs with sugar mass exceeding 1 kg recorded.

(b) Describe the sampling distribution (size $n > 50$) of the proportion of packs with sugar mass exceeding 1 kg.

> As sample size $n > 50$,
> sample proportion $\hat{\pi}$ is approximately normally distributed ✓
> with mean $\mu = 0.0228$ ✓
> and standard deviation $= \sqrt{\frac{0.02275 \times (1 - 0.02275)}{n}} = \sqrt{\frac{0.0222}{n}}$. ✓

(c) For $n = 100$, calculate the probability that a randomly chosen sample has a sample proportion of packs with sugar mass exceeding 1kg of between 0.02 and 0.03.

> $\hat{\pi} \sim N(0.0228, \frac{0.0222}{100})$. ✓
> $P(0.02 \le \hat{\pi} \le 0.03) = 0.260$ ✓

(d) Determine the value of n if the standard deviation of the sampling distribution (size $n > 50$) of the proportion of packs with sugar mass exceeding 1 kg is not to exceed 0.01.

> $\sqrt{\frac{0.0222}{n}} \le 0.01$ ✓
> $n \ge 222$ ✓

Calculator Assumed

5. [9 marks: 4, 3, 2]

An unbiased six-sided die is rolled 80 times. This is repeated 150 times to form 150 samples each consisting of 80 rolls of the die. Event S is defined as the roll of the die producing a six.

(a) Calculate the probability that a randomly chosen sample has a sample proportion of event S that exceeds 15%.

> P(S occurring) = $\frac{1}{6}$
> As sample size $n = 50 > 30$,
> sample proportion $\hat{\pi}$ is approximately normally distributed ✓
> with mean $\mu = \frac{1}{6}$ ✓
> and standard deviation = $\sqrt{\dfrac{\frac{1}{6}\times\left(1-\frac{1}{6}\right)}{80}} = \frac{1}{24}$. ✓
> Hence, $P(\hat{\pi} > 0.15) = 0.6554$ ✓

(b) Estimate with reasons, the expected number of samples with sample proportions of event S that exceeds 15%.

> As the number of samples N = 150 is large, ✓
> frequency distribution of sample proportion tends towards its sampling distribution.
> Hence, frequency distribution is approximately $N\left(\frac{1}{6}, \left(\frac{1}{24}\right)^2\right)$. ✓
> Therefore, expected number ≈ 0.6554 × 150 ≈ 98 ✓

(c) In a separate experiment, the same die was rolled n times. Find n if the standard deviation of the sampling distribution of sample proportion of event E is not to exceed 0.04.

> $\sqrt{\dfrac{\frac{1}{6}\times\left(1-\frac{1}{6}\right)}{n}} \le 0.04$ ✓
> $n \ge 86.8$
> Integer $n \ge 87$ ✓

Calculator Assumed

6. [8 marks: 3, 5]

60% of vehicles arriving at a school entrance are classified as sport utility vehicles (SUVs).

(a) Calculate the probability that in a random sample of 50 cars arriving at the school entrance, exactly 30 are SUVs.

> P(30 SUVs out of 50) = $\binom{50}{30}\times 0.6^{30}\times 0.4^{20}$ ✓✓
> $= 0.11456 \approx 0.1146$ ✓

(b) Samples each comprising 50 vehicles arriving at the school entrance were taken and the number of samples with exactly 30 SUVs recorded. For a randomly chosen sample of 50 vehicles, estimate the probability that the sample proportion of exactly 30 SUVs does not exceed 12% given that it exceeds 11%.

> As sample size $n = 50 > 30$,
> sample proportion $\hat{\pi}$ is approximately normally distributed ✓
> with mean $\mu = 0.11456$ ✓
> and standard deviation = $\sqrt{\dfrac{0.11456\times(1-0.11456)}{50}} = 0.04504$ ✓
> Hence, $P(\hat{\pi} \le 0.12 \mid \hat{\pi} \ge 0.11) = \dfrac{P(0.11 \le \hat{\pi} \le 0.12)}{P(\hat{\pi} \ge 0.11)}$ ✓
> $= \dfrac{0.08839}{0.54032}$
> ≈ 0.1636 ✓

7. [3 marks]

It is known that π % of high school students carry school bags with masses exceeding 15 kg. Samples of 100 students are chosen. The sampling distribution for the sample proportion of students with school bags exceeding 15 kg has standard deviation $\frac{\sqrt{91}}{200}$. Find π.

> $\sqrt{\dfrac{\pi(1-\pi)}{100}} = \dfrac{\sqrt{91}}{200}$ ✓
> $\pi = 0.35$ or 0.65 ✓✓

```
solve(√(x(1-x)/100) = √91/200, x)
{x=0.35, x=0.65}
```

Calculator Assumed

8. [10 marks: 3, 4, 3]

A bag has 6 green marbles and 4 red marbles. Five marbles are drawn without replacement from this bag and the number of green marbles noted. This procedure is repeated 50 times to form a sample of 50 observations on X.
Let X: No. of green balls drawn
and let $\hat{\pi}$: The proportion of draws where $X \geq 3$. .

(a) Find the probability distribution for X

$$P(X = x) = \frac{\binom{6}{x}\binom{4}{5-x}}{\binom{10}{5}} \quad \text{for } x = 1, 2, 3, 4, 5$$
✓✓ ✓

(b) Determine the probability distribution for $\hat{\pi}$.

$$P(X \geq 3) = \sum_{x=3}^{x=5}\left[\frac{\binom{6}{x}\binom{4}{5-x}}{\binom{10}{5}}\right] = \frac{31}{42} \quad ✓$$

Hence,

$$\hat{\pi} \sim N\left(\frac{31}{42}, \frac{\frac{31}{42}\times\left(1-\frac{31}{42}\right)}{50}\right) \quad ✓✓$$

$$\sim N\left(\frac{31}{42}, \frac{341}{88\,200}\right) \quad ✓$$

$$\sum_{x=3}^{5}\left[\frac{nCr(6,x)\times nCr(4,5-x)}{nCr(10,5)}\right] \quad \frac{31}{42}$$

$$\frac{\frac{31}{42}\times(1-\frac{31}{42})}{50} \quad \frac{341}{88200}$$

(c) Calculate the probability that in 100 samples of size 50 each; at least 60 samples would have $\hat{\pi}$ with a value between 0.7 and 0.8.

$P(0.7 < \hat{\pi} < 0.8) = 0.57023$ ✓

Define Y: No. samples with $\hat{\pi}$ between 0.7 and 0.8
$Y \sim B(n = 100, p = 0.57023)$ ✓
$P(Y \geq 60) = 0.30978 \approx 0.3098$ ✓

Calculator Assumed

9. [9 marks: 2, 4, 3]

NyRopes manufactures high tensile nylon ropes. The continuous random variable X describes the rope length (m) between two consecutive kinks in the rope. The probability density function of X is given by $f(x) = 0.01\,e^{-0.01x}$, where $x > 0$.

(a) Find the probability that a randomly chosen piece of rope has a rope length of at least 50 m between consecutive kinks.

$$P(X \geq 50) = \int_{50}^{\infty} 0.01\,e^{-0.01x}\,dx \quad ✓$$
$$= 0.60653 \approx 0.6065 \quad ✓$$

Samples of 30 coils of nylon ropes were examined and $\hat{\pi}$ the proportion of ropes with rope length of at least 50 m between consecutive kinks recorded.

(b) Calculate the probability that a random sample of 30 coils of nylon ropes has a $\hat{\pi}$ value between 0.6 and 0.7.

As sample size $n = 30$,
sample proportion $\hat{\pi}$ is approximately normally distributed ✓
with mean $\mu = 0.60653$ ✓
and standard deviation $= \sqrt{\frac{0.60653\times(1-0.60653)}{30}} = 0.08919$ ✓

Hence, $P(0.6 < \hat{\pi} \leq 0.7) = 0.38186 \approx 0.3819$ ✓

(c) Determine the minimum number of coils of nylon rope per sample required so that the standard deviation for $\hat{\pi}$ is less than 0.08.

$$\sqrt{\frac{0.60653\times(1-0.60653)}{n}} \leq 0.08 \quad ✓$$
$$n \geq 37.29 \quad ✓$$
Hence, minimum number is 38. ✓

29 Point & Interval Estimates for π

Calculator Assumed

1. [9 marks: 1, 3, 3, 2]

The mass (nearest g) of 30 eggs from an egg farm is listed below.

65	66	64	68	65	67	66	64	69	65
66	68	65	67	66	64	70	68	65	67
66	67	63	65	67	68	64	68	67	66

(a) Use this sample to estimate π, the proportion of eggs with mass above 66 g.

> Sample proportion $\hat{\pi} = \frac{13}{30}$ ✓

(b) Use this sample to provide a 95% confidence interval for π, the proportion of eggs with mass above 66 kg.

> 95% confidence interval for π
>
> $\frac{13}{30} \pm 1.96 \times \sqrt{\frac{\frac{13}{30}\times\left(1-\frac{13}{30}\right)}{30}}$ ✓✓
>
> $\Rightarrow\ 0.26 \le \pi \le 0.61$ ✓

(c) In a second sample, 27 eggs out of 60 had mass above 66 kg.
Use the confidence interval in (b) to determine if eggs in the second sample have mass that are statistically different from those of the first sample.

> Sample proportion of second sample = $\frac{27}{60} = 0.45$ ✓
>
> This lies within the 95% confidence interval ✓
> for π from the first sample.
> Hence, it is not statistically different. ✓

(d) A third sample of eggs has a 95% confidence interval for π as $0.58 \le \pi \le 0.89$.
Determine with reasons if the mass of the third sample of eggs are statistically different from those of the first sample.

> There is just a slight overlap between ✓
> the two confidence intervals.
> Hence, the two samples are statistically different. ✓

Calculator Assumed

2. [12 marks: 1, 3, 3, 5]

The accompanying table shows the number of students achieving x correct responses in a standardised test consisting of 60 questions.
Let π be the true proportion of students achieving no more than 20 correct responses.

Number of correct responses, n	No. of Students
$1 \le t \le 10$	7
$11 \le t \le 20$	12
$21 \le t \le 30$	18
$31 \le t \le 40$	20
$41 \le t \le 50$	10
$51 \le t \le 60$	3

(a) Use this data to find a point estimate for π.

> Sample proportion $\hat{\pi} = \frac{19}{70}$ ✓

(b) Use this sample to provide a 90% confidence interval for π.

> 90% confidence interval for π
>
> $\frac{19}{70} \pm 1.645 \times \sqrt{\frac{\frac{19}{70}\times\left(1-\frac{19}{70}\right)}{70}}$ ✓✓
>
> $\Rightarrow\ 0.18 \le \pi \le 0.36$ ✓

(c) Use your answer in (a) to find the size of the next sample if the error margin for a 90% confidence interval for π, is no more than 0.05.

> $1.645 \times \sqrt{\frac{\frac{19}{70}\times\left(1-\frac{19}{70}\right)}{n}} \le 0.05$ ✓
>
> $\Rightarrow\ n \ge 214.1$ ✓
> Integer $n \ge 215$ ✓

(d) A third sample consisting of 100 students provided a confidence interval of $0.17 \le \pi \le 0.33$. Find the point estimate for π in this sample and the level of confidence for this interval.

> Sample proportion $\hat{\pi} = \frac{0.17 + 0.33}{2}$
> $= 0.25$ ✓
> Error = 0.33 – 0.25 = 0.08 ✓
>
> Hence, $z \times \sqrt{\frac{0.25\times(1-0.25)}{100}} = 0.08$ ✓
> $z = 1.84752$ ✓
> $P(-1.84752 \le z \le 1.84752) = 0.9353$
> Therefore, level of confidence ≈ 93.5% ✓

Calculator Assumed

3. [8 marks: 2, 5, 1]

To estimate the true proportion π of the residents of a certain city that agree that the international airport serving the city should be relocated, samples were taken and the proportion of those in agreement calculated.

(a) In one sample of 200 residents, the 99% confidence for π was $0.78 \le \pi \le 0.92$. How many in this sample were in agreement with the proposal?

> Sample proportion $\hat{\pi} = \dfrac{0.78 + 0.92}{2}$
> $= 0.85$ ✓
> Hence, number in agreement $= 0.85 \times 200 = 170$ ✓

(b) A second sample of 500 residents, the 90% confidence interval for π was $0.77 \le \pi \le 0.83$. Determine if the second sample is statistically different from the first sample.

> Sample proportion $\hat{\pi} = \dfrac{0.77 + 0.83}{2}$
> $= 0.80$ ✓
>
> 99% Confidence Interval for π is:
> $0.80 \pm 2.576 \times \sqrt{\dfrac{0.8 \times (1 - 0.8)}{500}}$ ✓
> $\Rightarrow 0.75 \le \pi \le 0.85$ ✓
> The 99% confidence interval for the second sample lies entirely within the 99% confidence interval for the first sample. ✓
> Hence, the two samples are not statistically different. ✓

(c) If 100 samples of 50 residents each were selected, and the associated 99% confidence intervals for π calculated in the same manner. How many of these confidence intervals would be expected to contain π?

> Expected number $= 100 \times 0.99$
> $= 99$ ✓

Calculator Assumed

4. [8 marks: 3, 1, 1, 3]

To estimate the true proportion π of the adults in a certain state who suffer from hay fever (allergic rhinitis), samples were taken and the proportion of those suffering from hay fever calculated.

A sample of 500 adults was taken and 105 were found to suffer from hay fever.

(a) Calculate a 95% confidence interval for π.

> 95% confidence interval for π
> $\dfrac{105}{500} \pm 1.96 \times \sqrt{\dfrac{\frac{105}{500} \times \left(1 - \frac{105}{500}\right)}{500}}$ ✓✓
> $\Rightarrow 0.17 \le \pi \le 0.25$ ✓

(b) Calculate a 99% confidence interval for π.

> 99% confidence interval for π
> $\dfrac{105}{500} \pm 2.576 \times \sqrt{\dfrac{\frac{105}{500} \times \left(1 - \frac{105}{500}\right)}{500}}$
> $\Rightarrow 0.16 \le \pi \le 0.26$ ✓

(c) Comment on width of the confidence intervals in (a) and (b).

> The higher the confidence level, the wider the interval. ✓

In a second sample of 500 adults, 126 were found to suffer from hay fever.

(d) Using your previously calculated confidence intervals, determine with reasons if the results of the second sample is statistically different from that of the first.

> Sample proportion for 2nd sample $\hat{\pi} = \dfrac{126}{500} = 0.252$ ✓
> This is outside the 95% confidence interval for π but within the 99% confidence interval for π. ✓
>
> Hence, not statistically different if the 99% confidence interval is used but statistically different if the 95% confidence interval is used. ✓

Calculator Assumed

5. [8 marks: 3, 2, 3]

To estimate the true proportion π of the 1 kg packets of sugar that are under the advertised weight, samples of 1 kg packets of sugar were examined.

In a sample of 50 packets of sugar 3 were found to be underweight.

(a) Use this sample to calculate a 90% confidence interval for π.

> 90% confidence interval for π
>
> $$\frac{3}{50} \pm 1.645 \times \sqrt{\frac{\frac{3}{50}\times\left(1-\frac{3}{50}\right)}{50}} \qquad ✓✓$$
>
> $\Rightarrow 0.0048 \le \pi \le 0.1152$ ✓

An additional 450 packets were added to the 50 packets to form a larger sample of 500 packets and 28 were found to be underweight.

(b) Use the larger sample to calculate a 90% confidence interval for π.

> 90% confidence interval for π
>
> $$\frac{28}{500} \pm 1.645 \times \sqrt{\frac{\frac{28}{500}\times\left(1-\frac{28}{500}\right)}{500}}$$
>
> $\Rightarrow 0.0391 \le \pi \le 0.0729$ ✓✓

(c) Determine with reasons which of the two confidence intervals would provide a statistically more reliable interval estimate for π.

> The larger the value of n, the closer the sampling distribution for $\hat{\pi}$ is to the normal distribution. ✓
> Hence, the second confidence interval would be more reliable as the size of the second sample is much larger. ✓✓

Calculator Assumed

6. [8 marks: 2, 3, 1, 2]

In a certain country the proportion of residents with type A blood is $\pi = 0.38$
Samples of 1000 residents are selected and the sample proportion $\hat{\pi}$ calculated.

(a) State the sampling distribution for $\hat{\pi}$.

> As sample size $n = 1000$ is large
>
> $\hat{\pi} \sim N\left(0.38, \left(\frac{0.38\times(1-0.38)}{1000}\right)\right)$. ✓
>
> $\hat{\pi} \sim N(0.38, 0.0002356)$ ✓

(b) Determine the interval $0.38 - k \le \hat{\pi} \le 0.38 + k$ such that $P(-k \le \hat{\pi} \le k) = 0.95$.

> $\hat{\pi} \sim N(0.38, 0.0002356)$
>
> $P(0.38 - k \le \hat{\pi} \le 0.38 + k) = 0.95$
>
> $\Rightarrow k = 0.030084$ ✓
>
> Hence, required interval is:
>
> $0.35 \le \hat{\pi} \le 0.41$ ✓✓
>
> ```
> solve(normCDf(0.38-x,0.38+x,√0.0002356 ,0.38)=0.95,x)
>                                        {x=0.03008401067}
> ```

In a sample of 1000 residents taken only from residents of ethnic group G, 312 were found with type A blood.

(c) Calculate a point estimate for the proportion of residents from G with type A blood.

> Sample proportion $\hat{\pi} = \frac{312}{1000}$
>
> $= 0.312$ ✓

(d) Determine with reasons if the proportion of residents from G with type A blood is significantly different from the overall population.

> Significantly different ✓
> as $\hat{\pi}$ for G = 0.312 is which is outside ✓
> the 95% variation interval for $\hat{\pi}$.