Question 1 [6 marks]

Given $f(x) = \dfrac{1}{x}$, $g(x) = \sqrt{x+2}$ and $h(x) = x^2$.

Determine:

(a) $f \circ g\,(x)$ [1]

$= f(\sqrt{x+2})$

$= \dfrac{1}{\sqrt{x+2}}$

(b) $(g \circ h)(1)$ [2]

$g \circ h(x) = g(x^2)$

$= \sqrt{x^2+2}$

$g \circ h(1) = \sqrt{1+2}$

$= \sqrt{3}$

(c) the domain and range of $f \circ g(x)$. [3]

$R_g = \{ g : g \geqslant -2,\ g \in \mathbb{R} \}$

$D_f = \{ x : x \neq 0,\ x \in \mathbb{R} \}$

$R_g \not\subset D_f$

$\sqrt{x+2} \neq 0$

$\Rightarrow x \neq -2$

$\therefore D_{fg} = \{ x : x > -2,\ x \in \mathbb{R} \}$

$R_{fg} = \{ y : y > 0,\ y \in \mathbb{R} \}$

Question 2 [6 marks]

Differentiate the following, without simplifying:

(a) $y = 4x^5 + 3\sqrt{x}$ $= 4x^5 + 3x^{1/2}$ [2]

$\frac{dy}{dx} = 20x^4 + \frac{3}{2}x^{-1/2}$

$= 20x^4 + \frac{3}{2\sqrt{x}}$

(b) $y = (5e^{2x} + 1)^3$ [2]

$\frac{dy}{dx} = 3(5e^{2x}+1)^2 \cdot \frac{d}{dx}(5e^{2x}+1)$

$= 3(5e^{2x}+1)^2 \cdot 10e^{2x}$

$= 30e^{2x}(5e^{2x}+1)^2$

(c) $y = \dfrac{x^2+3}{2x-1}$ [2]

$\frac{dy}{dx} = \frac{(2x-1)\cdot 2x - (x^2+3)\cdot 2}{(2x-1)^2}$

$= \frac{4x^2 - 2x - 2x^2 - 6}{(2x-1)^2}$

$= \frac{2(x^2-x-3)}{(2x-1)^2}$

Question 3 [5 marks]

The probabilities of two events A and B are such that

$P(A) = 0.6$, $P(A \cup B) = 0.8$ and $P(B \mid A) = 0.5$.

(a) Find

(i) $P(A \cap B)$ [2]

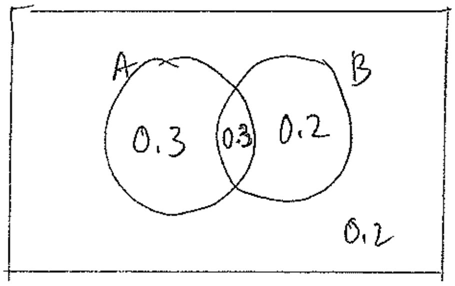
$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$0.5 = \frac{P(A \cap B)}{0.6}$$

$$P(A \cap B) = 0.3$$

(ii) $P(B)$ [1]

$P(B) = 0.5$



(b) Are the events A and B independent? Justify your answer. [2]

$P(B|A) = 0.5$

$P(B) = 0.5 = P(B|A)$

yes A & B are indep. events.

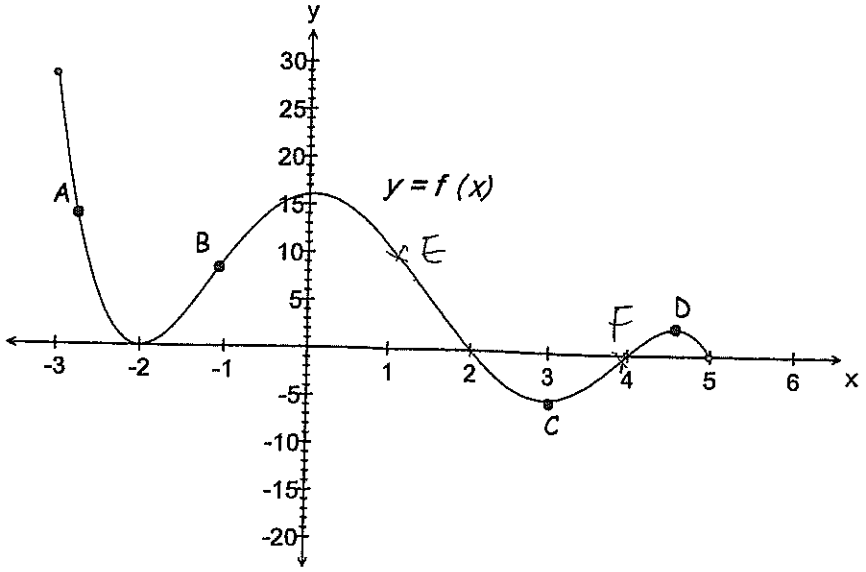
OR $P(A \cap B) = 0.3$

$P(A) \times P(B) = 0.6 \times 0.5$

$= 0.3$

Question 4 [9 marks]

The graph of the function y = f(x) on the interval $-3 \le x \le 5$ is given below.



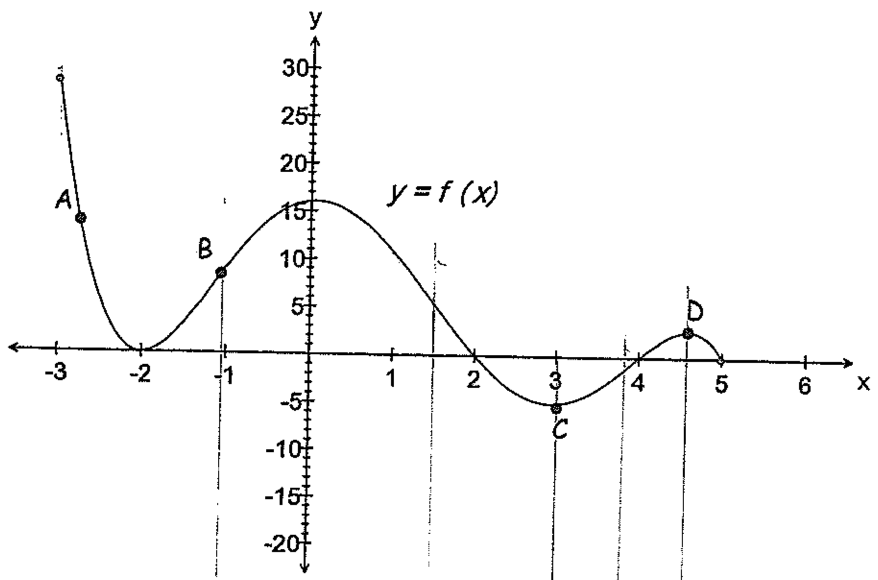
(a) A, B, C and D are four points on the graph of y = *f(x)*. Determine whether the first and second derivatives are positive, negative or equal to zero at these points. Record your answers in the table below. [4]

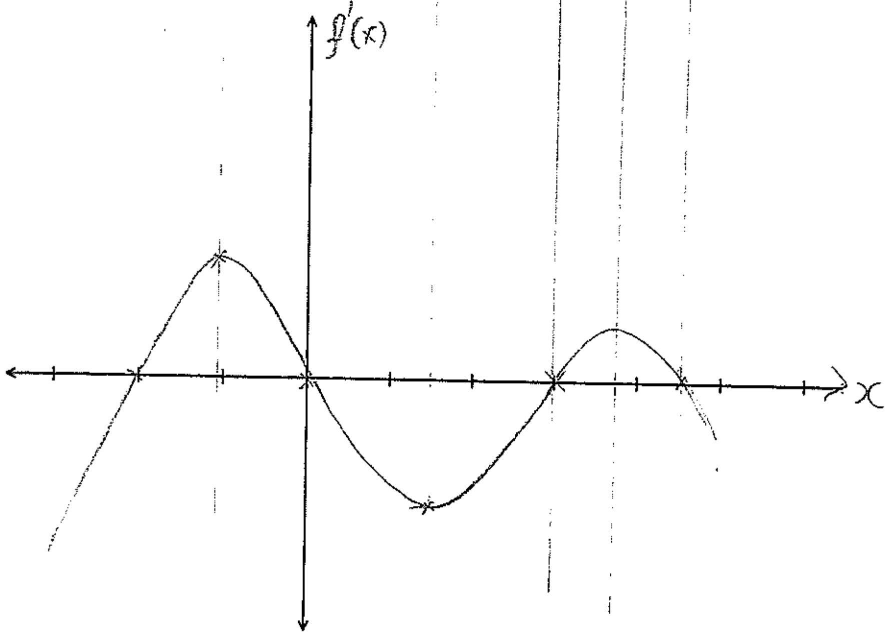
Point	f'	f''
A	−	+
B	+	0
C	0	+
D	0	−

(b) Indicate on the graph of y = *f(x)* above, the two other points of inflection, and label them as E and F. [1]

(c) Sketch the graph of $y = f'(x)$ on the axes provided below the graph of $y = f(x)$.

[3]





(d) State the coordinates of the global maximum and global minimum points of $y = f(x)$ on the interval $-3 \le x \le 5$.

[1]

Global max $(-3, \approx 29)$

Global min $(3, -5)$

Question 5 [7 marks]

(a) Determine the following indefinite integrals: (answer with positive indices)

(i) $\int\left(5x^6 - \frac{4}{x^3} - 1\right)dx$ $= \int(5x^6 - 4x^{-3} - 1)\,dx$ [2]

$= \frac{5x^7}{7} - \frac{4}{-2x^2} - x + c$

$= \frac{5x^7}{7} + \frac{2}{x^2} - x + c$

(ii) $\int 5x(3x^2+2)^4\,dx$ [2]

$= \frac{5}{6}\int 6x(3x^2+2)^4\,dx$

$= \frac{5}{6}\,\frac{(3x^2+2)^5}{5} + c$

$= \frac{1}{6}(3x^2+2)^5 + c$

(b) Evaluate $\int_0^2 \left(\frac{1}{e^{5x}}\right)dx$ and give your answer in terms of e. [3]

$= \int_0^2 e^{-5x}\,dx$

$= \left[\frac{e^{-5x}}{-5}\right]_0^2$

$= \left[-\frac{1}{5e^{5x}}\right]_0^2$

$= \frac{1}{5}\left[-\frac{1}{e^{10}} - \left(-\frac{1}{e^0}\right)\right] = \frac{1}{5}\left[1 - \frac{1}{e^{10}}\right] = \frac{e^{10}-1}{5e^{10}}$

Question 6 [4 marks]

Solve the system of equations

$$2x - 2y - 2z = 0$$

$$x - y - z = 0 \quad \text{—— (1)}$$
$$x + 2y + z = 1 \quad \text{—— (2)}$$
$$3x + y - 2z = 8 \quad \text{—— (3)}$$

(1) + (2) $\Rightarrow$ $2x + y = 1$ —— (4)

(3) − 2(1) $\Rightarrow$ $x + 3y = 8$ —— (5)

(4) − 2(5) $\Rightarrow$ $-5y = -15$

$\underline{y = 3}$

from (5) $x = 8 - 3(3)$

$\underline{x = -1}$

from (1) $z = x - y$

$= -1 - 3$

$\underline{z = -4}$

$\therefore$ $x = -1$

$y = 3$

$z = -4$

Question 7 [3 marks]

Solve the inequality $5 \geq \dfrac{4x}{x-2}$ where $x \neq 2$.

$$5(x-2)^2 \geqslant \frac{4x}{x-2}(x-2)^2$$

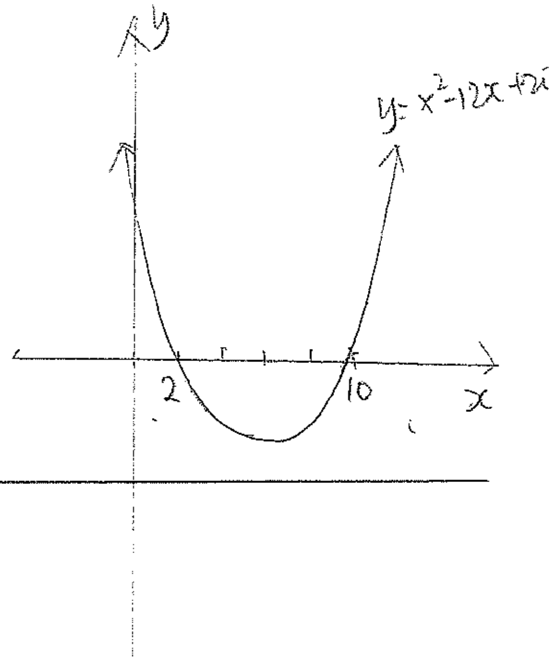
$$5(x^2-4x+4) \geqslant 4x(x-2)$$

$$5x^2-20x+20 \geqslant 4x^2-8x$$

$$x^2-12x+20 \geqslant 0$$

$$(x-10)(x-2) \geqslant 0$$

$$x \geqslant 10 \quad \text{or} \quad x < 2$$



OR

$$\frac{4x}{x-2} - 5 \leq 0$$

$$\frac{4x-5(x-2)}{(x-2)} \leq 0$$

$$\frac{-x+10}{x-2} \leq 0$$

Critical values at $x=10$ & $x=2$

sign test : $-$ (2) $+$ (10) $-$

$\therefore x < 2$ or $x \geqslant 10$

END OF SECTION ONE

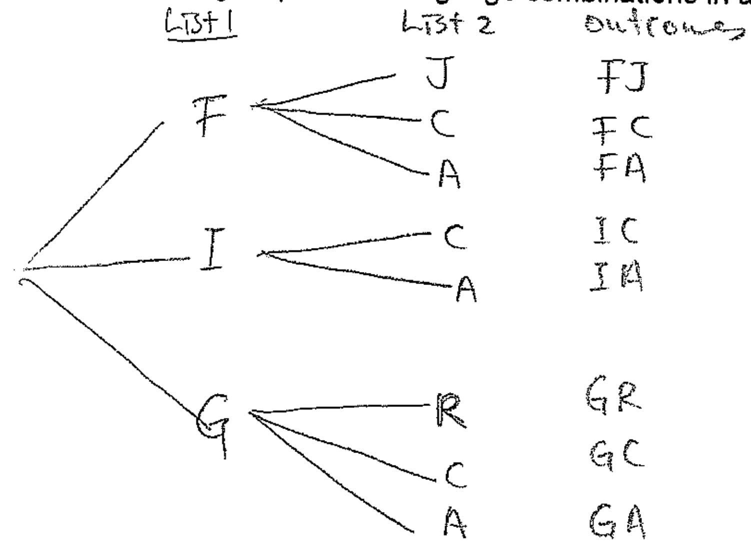
Question 8 [7 marks]

Ashleigh wanted to choose to study two languages from the table below. Due to timetable clashes she could only choose from List 1 and List 2 in the following way:

List 1	French(F), Italian(I), German(G)
List 2	Russian(R), Japanese(J), Chinese(C), Arabic(A)

She must first choose one language from List 1 and another language from List 2.
She cannot study Japanese unless she also studies French.
She cannot study Russian unless she also studies German.

(a) Display Ashleigh's possible language combinations in a tree diagram. [2]



(b) How many possible combinations are there for Ashleigh? [1]

8

(c) If each combination is equally likely, what is the probability that Ashleigh

(i) will choose French and Japanese? [1]

$\frac{1}{8}$

(ii) will not choose German? [1]

$\frac{5}{8}$

(iii) will choose Chinese given that she has chosen French? [2]

$\frac{1}{3}$

Question 9 **[7 marks]**

At the beginning of a mice plague there were 15 mice in a farm shed and the number of mice, N, increased continuously at a daily rate equal to 8% of the existing number of mice.

(a) Write an equation for the rate of change of the number of mice. [1]

$$\frac{dN}{dt} = 0.08N$$

(b) Find the number of mice in the farm shed after one week. [1]

$N = 15e^{0.08t}$

when $t = 7$ $N = 15e^{0.08(7)}$

$\simeq 26$

$\simeq 26$ mice after 1 week

(c) When will the population of mice double? [2]

solve $30 = 15e^{0.08t}$

$t = 8.66$ days

pop^n will double after 8.66 days or on the 9th day

(d) The number of mice peaked at 368. From then on the number of mice decreased continuously at a daily rate equal to 20% of the number present. On which day of the plague did the number of mice return to 15? [3]

solve $368 = 15e^{0.08t}$

$t = 40$

after 40 days $N = 368e^{-0.2t}$

solve $15 = 368e^{-0.2t}$

$t = 16.0$

$\therefore$ the no of mice return to 15 on the 57th day

Question 10 [3 marks]

(a) Evaluate $\int_{-1}^{3}(2x+1)\ dx$ [1]

$$= \left[\frac{2x^2}{2} + x\right]_{-1}^{3} = \left[x^2 + x\right]_{-1}^{3}$$

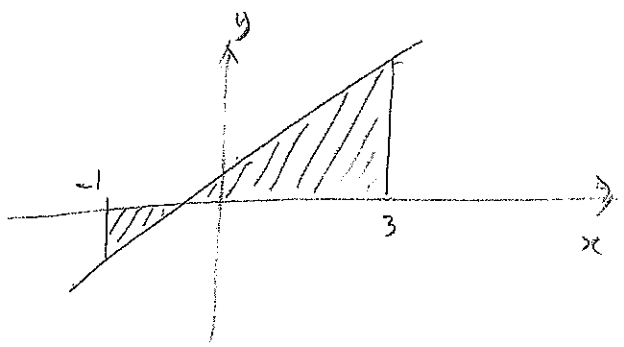
$$= (3^2+3) - ((-1)^2 - 1)$$

$$= 12$$

(b) Comment on whether the following statement is true or false.
The integral in (a) gives the area between the line y = 2x + 1 and the x-axis, from x = -1 to x = 3. [2]

False. The area under $y = 2x+1$ from $x=-1$ to $x=3$ is

$$\int_{-1}^{3} |2x+1|\ dx = 12.5 \text{ units}^2$$



Question 11 [10 marks]

Four boys and three girls, Adam, Bob, Charles, David, Eva, Felicity and Grace, stand in a row for a photo.

(a) How many different arrangements are possible? [1]

$$^7P_7 = 7! = 5040$$

(b) How many different arrangements have all the girls together? [1]

$$5! \times 3! = 720$$

(c) What is the probability of the arrangement if

(i) boys and girls must alternate? [2]

$$\underset{4}{B}\ \underset{3}{G}\ \underset{3}{B}\ \underset{2}{G}\ \underset{2}{B}\ \underset{1}{G}\ \underset{1}{B}$$

P(BG alternate)

$$= \frac{4! \times 3!}{7!} = \frac{1}{35}$$

(ii) Charles and Bob refuse to stand together? [2]

$$P(\text{Charles \& Bob are together}) = \frac{6! \times 2}{7!} = \frac{2}{7}$$

$$P(\text{Charles \& Bob are not together}) = 1 - \frac{2}{7} = \frac{5}{7}$$

(d) A group of four people is to be selected from seven students above.
What is the probability that

(i) the four selected will contain all three girls? [2]

$$\frac{^3C_3 \cdot {^4C_1}}{^7C_4} = \frac{4}{35}$$

(ii) the four selected will contain at most two boys given that at least one girl will be selected? [2]

$$n(\text{at least 1 girl}) = {^3C_1}\cdot{^4C_3} + {^3C_2}\cdot{^4C_2} + {^3C_3}\cdot{^4C_1} = 12+18+4 = 34$$

$$n(\text{at most 2 boys \& at least one girl}) = {^3C_2}\cdot{^4C_2} + {^3C_3}\cdot{^4C_1} = 18+4=22$$

$$\therefore P_r = \frac{22}{34} = \frac{11}{17}$$

Question 12 [7 marks]

(a) Use calculus methods, showing all working clearly, to determine the **exact** coordinates of any stationary points and point of inflection on the curve

$$y = (3 - x)\,e^x$$

[5]

$$\frac{dy}{dx} = 2e^x - xe^x$$

For stationary points ; solve: $\frac{dy}{dx} = 0 \Rightarrow x = 2$ & $y = e^2$

$$\frac{d^2y}{dx^2} = -xe^x + e^x$$

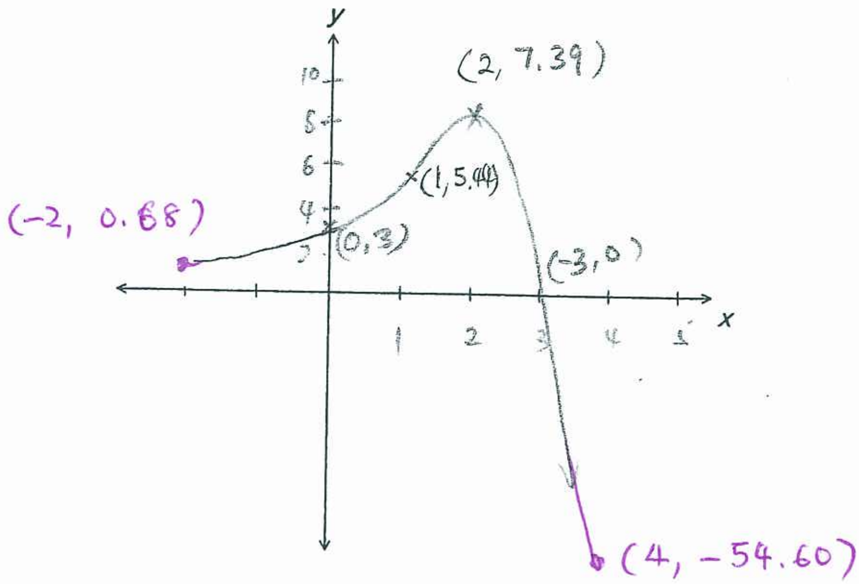
at $x = 2$, $\frac{d^2y}{dx^2} = -e^2 < 0$ $\therefore$ y is max at $(2, e^2)$

For when $\frac{d^2y}{dx^2} = 0$ $x = 1$ & $y = 2e$

$\frac{d^3y}{dx^3} = -xe^x$ at $x = 1$ $\frac{d^3y}{dx^3} = -e \neq 0$

$\therefore$ $(1, 2e)$ is a point of inflection.

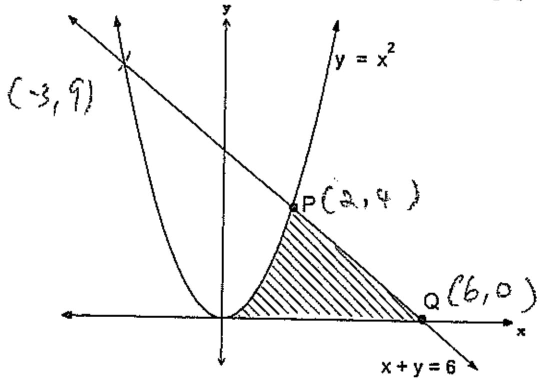
(b) Sketch the curve $y = (3 - x)\,e^x$, for $-2 \le x \le 4$, showing clearly all the main features where necessary, round all values to two decimal places. [2]



Question 13 [7 marks]

The diagram below shows a sketch of the curve $y = x^2$ and the line $x + y = 6$. Find

(a) the coordinates of points P and Q. [2]



P (2, 4)
Q (6, 0)

(b) the area bounded by the curve $y = x^2$, the line $x + y = 6$ and the x-axis (i.e the area of the shaded region). For full marks to be awarded for this part of the question, you will have to state the integrals involved in working out the area. [3]

$$A = \int_0^2 x^2\,dx + \tfrac{1}{2} \times 4 \times 4$$

$$= \left[\frac{x^3}{3}\right]_0^2 + 8$$

$$= \frac{8}{3} + 8 = 10\tfrac{2}{3} \text{ unit}^2$$

OR $A = \int_0^2 x^2\,dx + \int_2^6 (6-x)\,dx$

$$= \left[\frac{x^3}{3}\right]_0^2 + \left[6x - \frac{x^2}{2}\right]_2^6$$

$$= \frac{8}{3} + (18 - 10)$$

$$= 10\tfrac{2}{3} \text{ unit}^2$$

(c) the area enclosed by the curve $y = x^2$ and the line $x + y = 6$. [2]

$$A = \int_{-3}^{2} (6 - x - x^2)\,dx$$

$$= \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2}$$

$$= \frac{125}{6} = 20\tfrac{5}{6} \text{ unit}^2$$

Question 14 [6 marks]

(a) If $y = kx^3$ for some constant k, use the incremental formula to estimate the percentage increase in y required to yield a 1.5% increase in x. [3]

$y = kx^3$

$\frac{dy}{dx} = 3kx^2$

now $\frac{\delta y}{\delta x} \approx \frac{dy}{dx} \Rightarrow \delta y \approx \frac{dy}{dx} \cdot \delta x$

$\frac{\delta x}{x} \times 100 = 1.5$

$\delta x = \frac{1.5x}{100}$

$= 0.015x$

$\delta y = 3kx^2 \cdot 0.015x$

$= 0.045\,kx^3$

$\therefore \frac{\delta y}{y} \times 100 = \frac{0.045\,kx^3}{kx^3} \times 100 = 4.5\%$

(b) A company sells goods such that its revenue, in dollars, from selling x items is given by the equation,

$$R(x) = 5x\,(20x - x^2)$$

(i) Determine the marginal revenue, when x = 10. [2]

$R'(x) = \frac{dR}{dx} = 200x - 15x^2$

$R'(10) = 2000 - 1500 = \$500/\text{unit}$

(ii) What does this represent? [1]

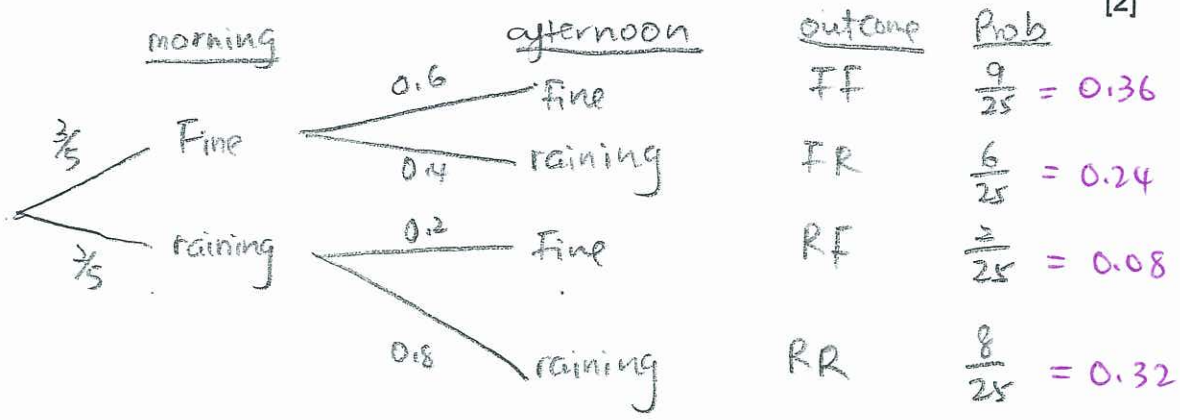
The revenue taken in by selling one more item at the stage where 10 items have been sold.

OR the revenue for selling the 11th item is $500.

Question 15 [6 marks]

The weather in Melbourne is difficult to predict in Autumn. On average, 3 mornings out of 5 are fine. If it is fine in the morning, there is a 60% chance that it will be fine in the afternoon. However if it is raining in the morning, there is only 20% chance that it will be fine in the afternoon.

(a) Display the above information in a tree diagram, indicating clearly the probabilities for all possible outcomes. [2]



(b) Find the probability that on a randomly selected Autumn day next year, the weather in Melbourne will be

(i) fine in the morning and raining in the afternoon. [1]

$$\frac{3}{5} \times 0.4 = \frac{6}{25} = 0.24$$

(ii) fine in the afternoon. [1]

$$\frac{9}{25} + \frac{2}{25} = \frac{11}{25} = 0.44$$

Sanne travelled by plane from Perth to Melbourne, arriving on an Autumn afternoon and it was raining.

(c) Given this, what is the probability that it was fine in Melbourne that morning. [2]

P(raining in the afternoo) $= \frac{6}{25} + \frac{8}{25} = \frac{14}{25}$

P(fine in the morning & rain afternoo) $= \frac{6}{25}$

$$P(F|R) = \frac{\frac{6}{25}}{\frac{14}{25}} = \frac{6}{14} = \frac{3}{7}$$

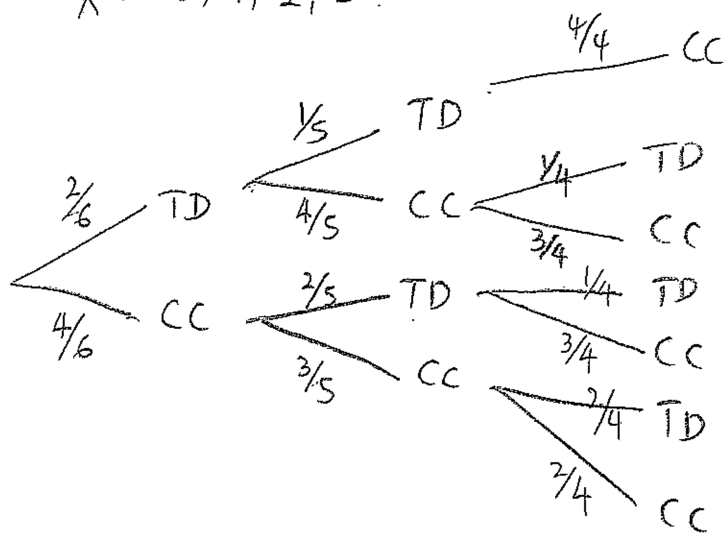
Question 16 [4 marks]

A box has 6 chocolates inside of it. There are two turkish delights and four coconut creams. Three chocolates are selected from the bag, one after the other, and eaten after each selection.

(a) Draw a probability distribution table, for the random variable X, the number of coconut creams selected in this process. [3]

X ~ the number of coconut creams chocolate

$X = 0, 1, 2, 3.$



outcome	Prob
(TD TD CC)	$\frac{8}{24}$
(TD CC TD)	$\frac{8}{24}$
(TD CC CC)	$\frac{24}{120}$
(CC TD TD)	$\frac{8}{120}$
(CC TD CC)	$\frac{24}{120}$
(CC CC TD)	$\frac{24}{120}$
(CC CC CC)	$\frac{24}{120}$

x	0	1	2	3
$P(X=x)$	0	$\frac{24}{120}=\frac{1}{5}$	$\frac{72}{120}=\frac{3}{5}$	$\frac{24}{120}=\frac{1}{5}$

✓✓

(b) Find the mean value of X. [1]

$$\text{mean} = 0 + 1 \times \frac{24}{120} + 2 \times \frac{72}{120} + 3 \times \frac{24}{120}$$

$$= 2$$ ✓

Question 17 [5 marks]

The probability distribution function for a discrete random variable X is given by

$$P(X = x) = \begin{cases} k(5 - x); & x = 1, 2, 3, 4 \text{ or } 5 \\ 0\ ; & \text{else where} \end{cases}$$

(a) Calculate the value of k. [2]

$$k(4) + k(3) + k(2) + k(1) + k(0) = 1$$

$$10k = 1$$

$$k = \frac{1}{10}$$

(b) Find

(i) $P(2 \le X < 4)$ [1]

$$= P(X=2) + P(X=3)$$

$$= \frac{3}{10} + \frac{2}{10} = \frac{1}{2}$$

(ii) $P(2 \le X < 4 \mid X \ge 3)$ [2]

$$= \frac{P(3 \le X < 4)}{P(X \ge 3)}$$

$$= \frac{P(X=3)}{P(X=3) + P(X=4) + P(X=5)}$$

$$= \frac{\frac{2}{10}}{\frac{2}{10} + \frac{1}{10}}$$

$$= \frac{\frac{2}{10}}{\frac{3}{10}} = \frac{2}{3}$$

Question 18 [8 marks]

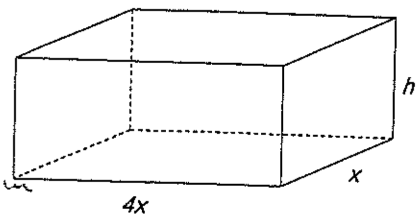
The diagram below shows a framework of a rectangular prism made using a 60 cm piece of wire. The length of the base is four times the width, x cm, of the prism.

(a) Show that the height, h of the prism in terms of x, is (15 – 5x) cm. [2]

$4h + 4x + 4(4x) = 60$

$4h = 60 - 20x$

$h = \frac{60-20x}{4} = 15 - 5x$ cm



(b) Find an expression for the volume enclosed by the framework in terms of x. [1]

$V = 4x \cdot x \cdot h = 4x^2(15-5x) = 60x^2 - 20x^3$

(c) Using calculus methods and showing full reasoning,

(i) find the dimensions of the framework for maximum volume. [4]

$V = 4x^2(15-5x) = 60x^2 - 20x^3$

$\frac{dV}{dx} = 120x - 60x^2$ when $\frac{dV}{dx} = 0$

$60x(2-x) = 0 \therefore \cancel{x=0}$ or $x = 2$

$\frac{d^2V}{dx^2} = 120 - 120x$ at $x = 2$ $\frac{d^2V}{dx^2} = -120 < 0$

$\therefore V$ is max when $x = 2$ cm

and from $h = 15 - 5x = 15 - 10 = 5$ cm

$\therefore$ length = 8 cm, width = 2 cm height = 5 cm

(ii) determine the maximum volume. [1]

max $V = 8 \times 2 \times 5 = 80$ cm^3

Question 19 [5 marks]

(a) If $y = (x-3)\,e^{-2x}$, show using calculus techniques, that the exact value of the gradient of the tangent line to this curve at $x = 1$ is $\frac{5}{e^2}$. [3]

$y = (x-3)e^{-2x}$

$\frac{dy}{dx} = (x-3)e^{-2x}(-2) + e^{-2x}$

$= -(2x-7)e^{-2x}$

at $x = 1$ $\frac{dy}{dx} = -(2-7)e^{-2}$

$= \frac{5}{e^2}$

(b) Find the equation of the tangent to the curve $y = \frac{4}{(x-5)^3}$ at the point where $x = 4$. [2]

$y = \frac{4}{(x-5)^3}$

$\frac{dy}{dx} = \frac{-12}{(x-5)^4}$

When $x = 4$ $\frac{dy}{dx} = \frac{-12}{(-1)^4} = -12$

$y = \frac{4}{(4-5)^3} = -4$

$\therefore$ eq^n of the tangent at $(4, -4)$ is

$y + 4 = -12(x-4)$

$y = -12x + 48 - 4$

$\underline{y = -12x + 44}$

Question 20 [5 marks]

The gradient function of a curve is given by $\frac{dy}{dx} = \frac{k}{x^2} + 1$ where k is a constant

(a) Find the equation of this curve given that when $x = 1$, $\frac{dy}{dx} = 3$ and $y = 3$. [4]

$\frac{dy}{dx} = \frac{k}{x^2} + 1$ when $x = 1$, $\frac{dy}{dx} = 3$

$3 = k + 1$

$\Rightarrow k = 2$

$\int \frac{dy}{dx} \cdot dx = \int \frac{k}{x^2} + 1 \; dx$

$y = \int (2x^{-2} + 1)\, dx$

$y = \frac{2x^{-1}}{-1} + x + c$

$y = -\frac{2}{x} + x + c$ when $x = 1$ $y = 3$

$3 = -2 + 1 + c$

$c = 4$

$\therefore y = -\frac{2}{x} + x + 4$

(b) Find the value of y when $x = 2$. [1]

$y = -\frac{2}{2} + 2 + 4 = 5$

END OF SECTION TWO