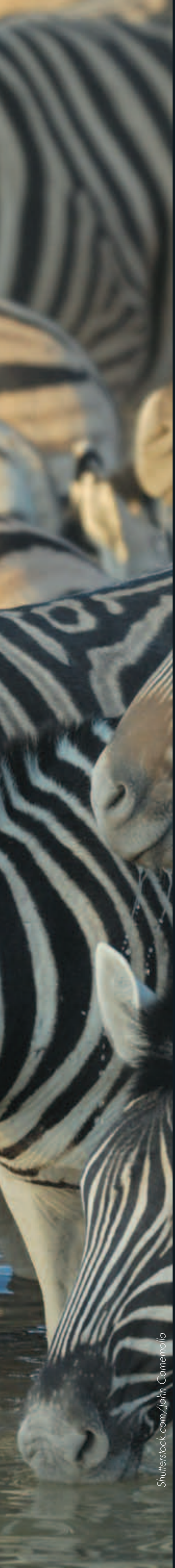


Shutterstock.com/John Carnemolla

3

TERMINOLOGY

acceleration
concave downwards
concave upwards
concavity
displacement
horizontal inflection
increments formula
inflection
local maximum
local minimum
optimisation
second derivative
stationary point
turning point
velocity

FURTHER DIFFERENTIATION AND APPLICATIONS

APPLICATIONS OF DERIVATIVES

Prior learning

THE SECOND DERIVATIVE AND APPLICATIONS OF DIFFERENTIATION

- use the increments formula: $\delta y \cong \frac{dy}{dx} \times \delta x$ to estimate the change in the dependent variable y resulting from changes in the independent variable x (ACMMM107)
- understand the concept of the second derivative as the rate of change of the first derivative function (ACMMM108)
- recognise acceleration as the second derivative of position with respect to time (ACMMM109)
- understand the concepts of concavity and points of inflection and their relationship with the second derivative (ACMMM110)
- understand and use the second derivative test for finding local maxima and minima (ACMMM111)
- sketch the graph of a function using first and second derivatives to locate stationary points and points of inflection (ACMMM112)
- solve optimisation problems from a wide variety of fields using first and second derivatives. (ACMMM113) AC

3.01 THE INCREMENTS FORMULA

Consider the function $y = f(x) = 3x^2 - 5x + 3$.

When $x = 2$, $y = 3 \times 2^2 - 5 \times 2 + 3 = 5$, so it goes through the point (2, 5).

$\frac{dy}{dx} = 6x - 5$ and the derivative at $x = 2$ is $f'(2) = 6 \times 2 - 5 = 7$.

What is the effect of a small change in the value of x, say $\delta x = 0.01$?

$f(2.01) = 3 \times (2.01)^2 - 5 \times 2.01 + 3 = 5.0703$, and $f(1.99) = 4.9303$.

The change in y, dy is about $0.07 = 7 \times 0.01$, so $\delta y \approx \frac{dy}{dx} \times \delta x$.

This is true for any *small* change in the independent variable of a function.

IMPORTANT

The change in a function is approximately equal to the product of the derivative and the change in the independent variable, provided the change is small:

This can be written as the **increments formula** $\delta y \cong \frac{dy}{dx} \times \delta x$.

You can show this as follows for a general function $y = f(x)$. Consider the small change δx and the corresponding change of the function, δy from point P to point Q. The diagram is shown on the right.

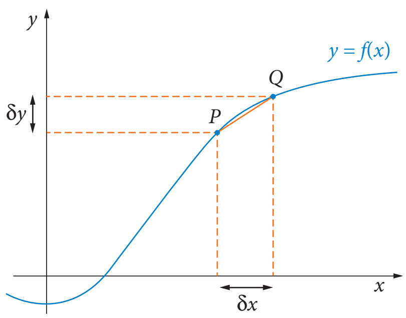
The slope of the **secant** from P to Q is given by $m = \frac{\delta y}{\delta x}$.

You can see this is almost the same as the slope of the curve between P and Q.

But the slope of a curve at any point is the value of the derivative at the point, so $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$.

Multiplying by δx gives the formula: $\delta y \cong \frac{dy}{dx} \times \delta x$.

Clearly, the smaller the value of δx, the more exact the formula will be.



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Example 1

Differentiate $\sqrt{x}$ with respect to x and use the result to find an approximate value for $\sqrt{103}$.

Solution

Write $\sqrt{x}$ as a power. $f(x)=\sqrt{x}=x^{\frac{1}{2}}$

Differentiate. $f'(x)=\frac{1}{2}x^{-\frac{1}{2}}$

Find a close easy value. $f(100)=\sqrt{100}=10$

Find the derivative as well. $f'(100)=\frac{1}{2}(100)^{-\frac{1}{2}}=0.05$

Apply the increments formula. $\delta y\approx\frac{dy}{dx}\times\delta x$

Substitute $\delta x = 3$. $=0.05\times 3$

$=0.15$

Find $\sqrt{103}$. $\sqrt{103}\approx f(100)+\frac{dy}{dx}\times 3$

Substitute and evaluate. $= 10 + 0.15$

$= 10.15$

Write the solution. $\sqrt{103}$ is about 10.15.

The actual $\sqrt{103}\approx 10.14889157...$ By using the increments formula you have approximated its value correct to 4 significant figures!

Example 2

The cost function to manufacture pocket radios is given by

$$C(x)=300+2x+\sqrt{x}\text{ dollars}$$

where $C(x)$ is the production cost for a week in which x radios are produced. Normal production is 10 000 radios per week. Find a linear function that approximates the production costs near 10 000 radios. Use the function to find the approximate production cost if the production increases to 10 200 radios.

Solution

Write the cost function. $C(x) = 300 + 2x + \sqrt{x}$

Calculate $C(10\,000)$. $C(10\,000) = 300 + 2 \times 10\,000 + \sqrt{10\,000}$
$= 20\,400$

Differentiate $C(x)$. $C'(x) = 2 + \frac{1}{2}x^{-\frac{1}{2}}$

Calculate $C'(10\,000)$. $C'(10\,000) = 2 + \frac{1}{2}(10000)^{-\frac{1}{2}}$
$= 2.005$

Write the increments formula. $\delta C \approx \frac{dC}{dx} \times \delta x$

Substitute values for 10 200. $= 2.005 \times 200$
$= 401$

Calculate $C(10\,200)$. $C \approx C(10\,000) + \delta C$
$= 20\,400 + 401$
$= 20\,801$

Round and write the result. 10 200 radios will cost about \$20 800 to make.

You can calculate the **percentage change** (or the percentage error) in y by expressing the change δy as a percentage, so percentage change (error) $= \frac{\delta y}{y} \times 100\%$, where $\delta y \approx \frac{dy}{dx} \times \delta x$.

Example 3

Find the percentage error made in the volume of a hot-air balloon of diameter 30 m if no allowance was made for the stretching of the fabric, resulting in a 2% error in the diameter.

123RF/Phaitoon Sutunyawatchai

Solution

Write the rule for volume of a sphere. $V = \frac{4\pi r^3}{3}$

Differentiate. $\frac{dV}{dr} = 4\pi r^2$

Write the increments formula. $\delta V \approx \frac{dV}{dr} \times \delta r$

Write as a percentage. % change of $V = \frac{\delta V}{V} \times 100\%$

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Substitute for δV and V.	$\approx \dfrac{4\pi r^2 \delta r}{\frac{4}{3}\pi r^3} \times 100\%$
Simplify.	$= 3 \times \dfrac{\delta r}{r} \times 100\%$
But $\dfrac{\delta r}{r} \times 100\%$ = % change of r.	= 3 × % change of r
But % change of r = % change of diameter.	= 3 × 2% = 6%
Write the answer.	The percentage error in volume is about 6%.

EXERCISE 3.01 The increments formula

WS Approximation with derivatives

Concepts and techniques

1 Example 1 Use derivatives to find an approximation for $\sqrt{85}$.

2 Use derivatives and the value of $\sqrt[3]{125}$ to find an approximation for $\sqrt[3]{130}$.

3 Use derivatives to find an approximate value for:

a $\sqrt{50}$ b $\sqrt[4]{85}$ c $\sqrt{2536}$ d $\sqrt[5]{250}$ e $\sqrt[6]{70}$

4 Find the value of $64^{\frac{2}{3}}$ and use derivatives to find an approximation for $67^{\frac{2}{3}}$.

5 Use the value of 10^7 to find an approximation for 10.06^7.

6 Use derivatives to find an approximation for 4.05^4. Compare your approximation with the exact value.

Reasoning and communication

7 Example 2 The cost function for a manufacturer of CD racks in dollars is given by

$$C(x) = 4000 + 2.1x + 0.01x^{\frac{3}{2}}$$

where x is the number of racks produced in a week. Use derivatives to estimate the change in production cost if production is increased from 5000 to 5100 racks per week.

123RF/Harald Biebel

8 Example 3 The radius of a sphere is measured to be 5 m ± 10 cm. Find the approximate percentage error of the calculated volume of the sphere.

9 The equation of a curve is $y = 2x^3 - 3x^2 + 4x - 1$.

a Find $\frac{dy}{dx}$.

b Find the value of $\frac{dy}{dx}$ when $x = 3$.

c Find the approximate change in y as x increases from $x = 3$ to $x = 3.02$.

10 Given that the equation of a curve is $y = 4x^3 - 3x$, find the approximate increase in y as x increases from 2 to 2.03.

11 A spherical ball of radius 12 cm is pumped with air. Find an approximation of the increase in volume of the ball when the radius increases by 0.05 cm.

12 A manufacturer of soft drink cans makes the cans 6 cm in diameter to hold 375 mL.

a What is the height of a can?

b Find an expression for the error in volume caused by a small error in the height of δx.

c Find an expression for the error in volume caused by a samll error of δy in the diameter.

d Find the error in volume caused by an error of 1 mm in the height.

e Find the error in volume caused by an error of 2 mm in the diameter.

13 The gravitational acceleration, g, can be determined by timing a pendulum. If a pendulum of length l has a period of T s, then $g = \frac{4\pi^2 l}{T^2}$. A 2 m pendulum is timed to take 57 s for 20 swings.

a Calculate the value of g from the data.

b Find an expression for the approximate error in g for an error of δt in the timing of 20 swings.

c Calculate the possible error in g if the timing was made 'to the nearest second'.

14 The side of a square is measured to be 1 m ± 1 mm. Use derivatives to find the approximate error of the calculated area of the square.

15 The sales manager of a car yard estimates that his staff will sell 80 cars next month. Bonuses and other incentives mean that the profit function for car sales is given by

$$P(n) = 2000n + 10n^2$$

in dollars, where n is the number of cars sold in the month. Use derivatives to find the error in the profit if the manager's estimate of sales is out by each of the following amounts.

a 5% b 8% c 10%

Alamy/Bert Klassen

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16 The edge of a cube is measured to be 17 cm. What is the approximate percentage error in the calculation of the volume of the cube if there is a 2% error in the measurement of the side?

17 The area of a circle is to be calculated using the measured length of its radius. It is necessary that the area of the circle be calculated with at most 2% error. What is the approximate maximum allowable percentage error that may be made in measuring the radius?

3.02 THE SECOND DERIVATIVE

You know that the derivative of $5x^8$ is $40x^7$. The derivative of $40x^7$ is $280x^6$. So the derivative of the derivative of $5x^8$ is $280x^6$. If you did this again, you would have the derivative of the derivative of the derivative of $5x^8$, which is $1680x^5$. You can keep differentiating derivatives many times. To avoid getting lost, we say the second derivative of $5x^8$ is $280x^6$ and the third derivative of $5x^8$ is $1680x^5$.

IMPORTANT

Derivatives can themselves be differentiated to give **higher derivatives**.

The derivative of a function $y = f(x)$, the **first derivative**, is written as $f'(x)$, $f^{(1)}(x)$ or $\frac{dy}{dx}$.

The derivative of the derivative, the **second derivative**, is written as $f''(x)$, $f^{(2)}(x)$ or $\frac{d^2y}{dx^2}$.

The derivative of $f'(x)$ is written as $f''(x)$, and so on.

The brackets around the derivative number are usually omitted as in $f^2(x)$, $f^3(x)$, …

Example 4

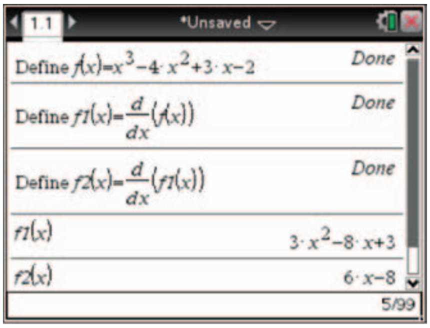
Find the first two derivatives of $f(x) = x^3 - 4x^2 + 3x - 2$ and evaluate $f''(-3)$.

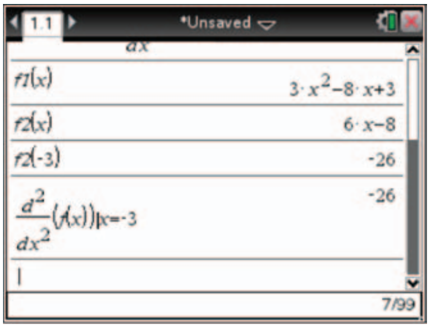
Solution

Write the function.	$f(x) = x^3 - 4x^2 + 3x - 2$
Differentiate.	$f'(x) = 3x^2 - 8x + 3$
Differentiate the first derivative.	$f''(x) = 6x - 8$
Substitute $x = -3$.	$f''(-3) = 6 \times (-3) - 8 = -26$

TI-Nspire CAS

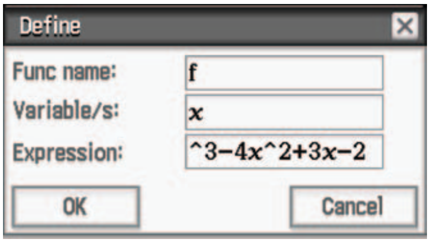
Use a Calculator page.
Use menu, 1: Actions and 1: Define or type Define to define the function.
Then define the derivative using menu, 4: Calculus and 1: Derivative to find $f'(x)$.
Do it again to find $f''(x)$.
Type $f1(x)$ and $f2(x)$ to see the results.
Type $f2(-3)$ to find $f''(-3)$.
You can also use menu, 4: Calculus and 2: Derivative at a point with the variable x, the value -3 and the 2nd derivative.

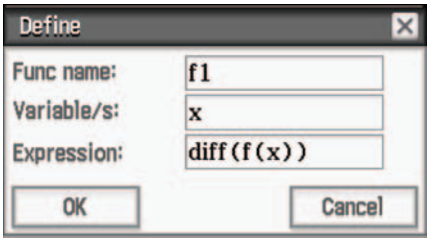




ClassPad

Use the Main menu.
Tap Interactive and Define. Fill in the box as shown.
Repeat for the second function, as shown. Use Math2 and tap $\frac{d}{d\square}\square$, which appears as diff.
The third function, f2(x), $f''(x)$, is similarly defined as the derivative of f1(x).
Do it again to find $f''(x)$.
Type f2(–3) to find $f''(-3)$, or use diff(f(x),x,2,–3).





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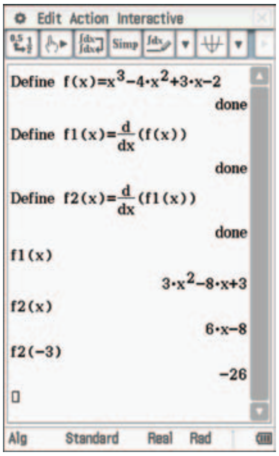
Type f1(x), f2(x) and f2(−3) to find the derivatives and the value of the second derivative when $x = -3$.



Write the answers.

$f'(x) = 3x^2 - 8x + 3, f'(x) = 3x - 8$ and $f''(-3) = -26$

Example 5

Find the second derivative of $y = 2 \sin(3x)$.

Solution

Write the function. $y = 2 \sin(3x)$

Differentiate.
$$\frac{dy}{dx} = [2 \cos(3x)] \times 3$$
$$= 6 \cos(3x)$$

Differentiate again.
$$\frac{d^2y}{dx^2} = 6 \times [-\sin(3x)] \times 3$$
$$= -18 \sin(3x)$$

Derivatives have many practical applications. The motion of a particle is one of these applications. You should remember that **velocity** and **acceleration** are both derivatives.

Higher derivatives

IMPORTANT

The **position** of a particle relative to a fixed point (the origin) is usually shown as x.

A **displacement** is a change of position, usually shown as s.

Velocity, v, is the rate of change of displacement or position, so $v = \frac{dx}{dt} = \frac{ds}{dt}$.

Acceleration, a, is the rate of change of velocity, so $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \frac{d^2s}{dt^2}$.

Example 6

The position of a particle is given by $x = 2t^3 - 5t$ cm, where t is time in seconds.

a Find the velocity of the particle after 3 s.

b Find the initial acceleration.

c Find the acceleration after 2 s.

d At what time will the acceleration reach 30 m/s^2?

Solution

a Write the function.	$x = 2t^3 - 5t$
Differentiate to find the velocity.	$v = \frac{dx}{dt} = 6t^2 - 5$
Substitute $t = 3$.	$v(3) = 6 \times 3^2 - 5 = 49$
Write the answer.	The velocity after 3 s is 49 cm/s.
b Differentiate again to find the acceleration.	$a = \frac{d^2x}{dt^2} = 12t$
Substitute $t = 0$ for the initial value.	$a(0) = 12 \times 0 = 0$
Write the answer.	The initial acceleration is 0 m/s^2.
c Substitute $t = 2$ into a.	$a(2) = 12 \times 2 = 24$
Write the answer.	The acceleration after 2 s is 24 cm/s^2.
d Substitute $a = 30$	$30 = 12t$
Solve to find t.	$t = 2.5$
Write the answer.	The acceleration is 30 cm/s^2 after 2.5 s.

EXERCISE 3.02 The second derivative

Concepts and techniques

1 Example 4 Find the first two derivatives of $x^7 - 2x^5 + x^4 - x - 3$.

2 If $f(x) = x^9 - 5$, find $f''(x)$.

3 Find $f'(x)$ and $f''(x)$ if $f(x) = 2x^5 - x^3 + 1$.

4 Find the second derivative of $y = x^7 - 2x^5 + 4x^4 - 7$.

5 Example 5 Find the second derivative of $y = 5\cos(2x)$.

6 CAS Differentiate the following twice.

a $y = 2x^2 - 3x + 3$ **b** $y = x^{-4}$

7 Find $f'(1)$ and $f''(-2)$ given $f(x) = 3t^4 - 2t^3 + 5t - 4$.

8 CAS If $f(x) = x^4 - x^3 + 2x^2 - 5x - 1$, find $f'(-1)$ and $f''(2)$.

9 If $g(x) = \sqrt{x}$ find $g''(4)$.

10 Given $h = 5t^3 - 2t^2 + t + 5$, find $\frac{d^2h}{dt^2}$ when $t = 1$.

11 Find $f'(x)$ and $f''(x)$ if $f(x) = \sqrt{2-x}$.

12 Find the first and second derivatives of $f(x) = \frac{x+5}{3x-1}$.

13 Find $\frac{d^2v}{dt^2}$ if $v = (t + 3)(2t - 1)^2$.

Reasoning and communication

14 Find the value of x for which $\frac{d^2y}{dx^2} = 3$, given $y = 3x^3 - 2x^2 + 5x$.

15 Find all values of x for which $f''(x) > 0$, given $f(x) = 2x^3 - x^2 + x + 9$.

16 Differentiate $y = (4 \sin(x) - 2)^5$ twice.

17 Find $f'(x)$ and $f''(x)$ if $f(x) = 2 \sin\left(\frac{x}{2}\right) - 3\cos^2(x) + 1$.

18 Find the value of b in $y = bx^3 - 2x^2 + 5x + 4$ if $\frac{d^2y}{dx^2} = -2$ when $x = \frac{1}{2}$.

19 Find the value of b if $f(x) = 5bx^2 - 4x^3$ and $f''(-1) = -3$.

20 Example 6 The displacement of an object is given by $x = t^3 - 7t + 4$, where t is in seconds and x is in metres. Find the velocity and acceleration of the object after 3 s.

21 The position of an object is given by $x(t) = t^3 - 6t^2 + 8t + 5$, where $x(t)$ is in metres and t is in seconds.
- a Find the velocity after 2 s.
- b Find the velocity after 4 s.
- c Find the acceleration after 2 s.
- d Find the acceleration after 5 s.

22 The displacement of an object is given by $d(t) = 7t^2 - 2t^3 + 3t + 3$, where $d(t)$ is in metres and t is in seconds.
- a Find the velocity after 1 s.
- b Find the velocity after 3 s.
- c Find the acceleration after 1 s.
- d Find the acceleration after 3 s.

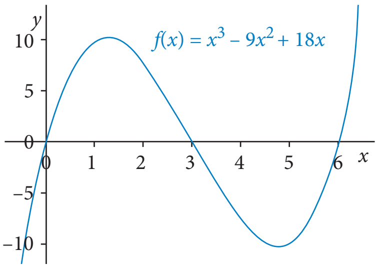
23 An object is travelling along a straight line. At t seconds, its displacement is given by the formula $x = t^3 + 6t^2 - 2t + 1$ m.
- a Find the equations of its velocity and acceleration.
- b What will its displacement be after 5 s?
- c What will its velocity be after 5 s?
- d Find its acceleration after 5 s.

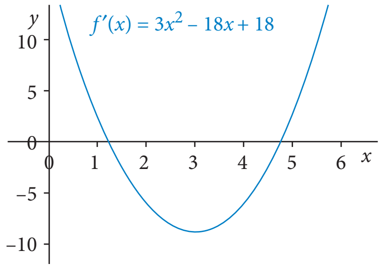
24 The displacement of a particle is given by $s = ut + \frac{1}{2}gt^2$, where $u = 2$ m/s and $g = -10$ m/s^2.
- a Find an expression for the velocity of the particle.
- b Find the velocity after 10 s.
- c Show that the acceleration is equal to g.

25 The displacement for an object is given by $s = \frac{2t-5}{3t+1}$, where s is in metres and t is in seconds. Find the equations for velocity and acceleration.

3.03 THE SECOND DERIVATIVE AND CONCAVITY

You can use extra information from the second derivative to help sketch the graph of a function. Look at the graphs of $f(x) = x^3 - 9x^2 + 18x$ and $f'(x) = 3x^2 - 18x + 18$.





The graph of $f(x)$ is shaped like a mound to the left of $x = 3$ and the graph $f'(x)$ is decreasing.

The graph of $f(x)$ is shaped like a bowl to the right of $x = 3$ and the graph $f'(x)$ is increasing.

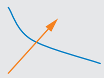
The slope of $f'(x)$ is negative below $x = 3$ and positive above $x = 3$.

This means that $f''(x) = 6x - 18$ is negative below $x = 3$ and positive above $x = 3$. It changes sign at $x = 3$, so the gradient of $f'(x)$ changes direction.

You can always use the sign of $f''(x)$ to tell if $f(x)$ is shaped like a mound or bowl.

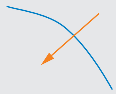
IMPORTANT

- A function is **concave upwards** (or **convex downwards**) where the second derivative is positive. An arrow drawn through the curve towards the centre of curvature will point upwards. It is sometimes called **cup** shaped.



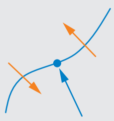
Concave upwards

- A function is **concave downwards** (or **convex upwards**) where the second derivative is negative. An arrow drawn through the curve towards the centre of curvature will point downwards. It is sometimes called **cap** shaped.



Concave downwards

- A **point of inflection** is a point where the concavity of the function changes.



Point of inflection

- A **horizontal point of inflection** is both a stationary point and a point of inflection.

Horizontal point of inflection

A point where $f''(x) = 0$ could be a minimum, a maximum or a point of inflection. It may or may not be a stationary point.

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Example 7

Find the values of x for which the curve $f(x) = 2x^3 - 7x^2 - 5x + 4$ is concave downwards.

Solution

Write the function.	$f(x) = 2x^3 - 7x^2 - 5x + 4$
Find the derivative.	$f'(x) = 6x^2 - 14x - 5$
Find the second derivative.	$f''(x) = 12x - 14$
State the condition for concavity.	$f''(x) < 0$ for concave downwards.
Substitute.	$12x - 14 < 0$
Solve the inequality.	$x < 1\frac{1}{6}$
Write the answer.	$f(x) = 2x^3 - 7x^2 - 5x + 4$ is concave downwards for $x < 1\frac{1}{6}$.

Example 8

Does $y = x^4$ have a point of inflection?

Solution

Write the function.	$y = x^4$
Find the derivative.	$\frac{dy}{dx} = 4x^3$
Find the second derivative.	$\frac{d^2y}{dx^2} = 12x^2$
State the condition.	$\frac{d^2y}{dx^2} = 0$ at points of inflection.
Find where the second derivative is zero.	$12x^2 = 0$ only at $x = 0$.
State the meaning.	There is a possible point of inflection at $x = 0$.
Check the signs of the second derivative before and after $x = 0$ to determine concavity.	(see table below)
Make a conclusion.	$\frac{d^2y}{dx^2}$ does not change sign at $x = 0$, so it is not a point of inflection.
Write the answer.	$y = x^4$ does not have a point of inflection.

x	−1	0	1
$\frac{d^2y}{dx^2}$	12	0	12

Example 9

Find any points of inflection on the curve $y = x^3 - 6x^2 + 5x + 9$.

Solution

Find the derivative.

$\frac{dy}{dx} = 3x^2 - 12x + 5$

Find the second derivative.

$\frac{d^2y}{dx^2} = 6x - 12$

State the condition.

$\frac{d^2y}{dx^2} = 0$ at points of inflection.

Determine where the second derivative is zero.

$6x - 12 = 0$ at $x = 2$.

State the meaning.

There is a possible point of inflection at $x = 2$

Check the signs of the second derivative before and after $x = 2$ to determine concavity.

x	1	2	3
$\frac{d^2y}{dx^2}$	−6	0	6

Make a conclusion.

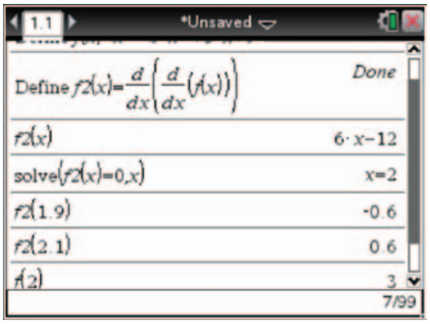
$\frac{d^2y}{dx^2}$ changes sign at $x = 2$, so there is a point of inflection at $x = 2$.

Find the point.

At $x = 2$, $y = 2^3 - 6 \times 2^2 + 5 \times 2 + 9 = 3$

TI-Nspire CAS

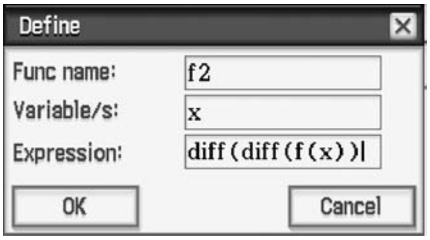
Use a calculator page.
Define the function and find the second derivative. Solve to find the values for which $f2(x) = 0$. Find the point.



ClassPad

Method 1

Use the Main menu.
Define the function and find the second derivative. If you use the interactive menu for define, you will have to take the derivative twice (see right).



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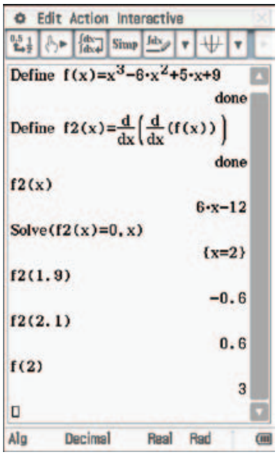
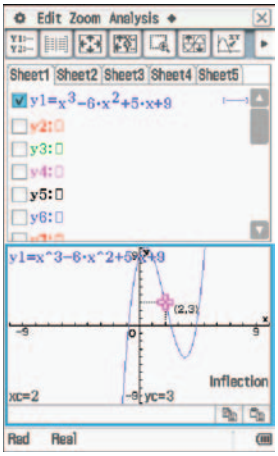
Alternatively, use Define from the Catalog, or type it in, and use }.
Solve to find the values for which $f2(x) = 0$.
Find the point.



Method 2
Use the Graph&Table menu to sketch the graph. You may need to move the graph around and use View Window to ensure that it fits.
Tap Analysis and Inflection.
The inflection is identified as (2, 3).



Answer the question.

Since the concavity changes at $x = 2$, there is a point of inflection at (2, 3).

EXERCISE 3.03 The second derivative and concavity

WS
Concavity

Concepts and techniques

1 Example 7 For what values of x is the curve $y = x^3 + x^2 - 2x - 1$ concave upwards?

2 Find all values of x for which the curve $y = (x - 3)^3$ is concave downwards.

3 Prove that the curve $y = 8 - 6x - 4x^2$ is always concave downwards.

4 Show that the curve $y = x^2$ is always concave upwards.

5 Find the domain over which the curve $f(x) = x^3 - 7x^2 + 1$ is concave downwards.

6 Find all values of x for which the function $f(x) = x^4 + 2x^3 - 12x^2 + 12x - 1$ is concave downwards.

7 Example 8 Determine whether there are any points of inflection on these curves.
a $y = x^6$ b $y = x^7$ c $y = x^5$ d $y = x^9$ e $y = x^{12}$

8 **Example 9** Find any points of inflection on the curve $g(x) = x^3 - 3x^2 + 2x + 9$.

9 Find the points of inflection on the curve $y = x^4 - 6x^2 + 12x - 24$.

10 **CAS** Find any points of inflection on the curve $y = x^4 - 8x^3 + 24x^2 - 4x - 9$

11 **CAS** For the function $f(x) = 3x^5 - 10x^3 + 7$, find any points of inflection.

Reasoning and communication

12 Sketch a curve that is always concave up.

13 Sketch a curve where $f''(x) < 0$ for $x < 1$ and $f''(x) > 0$ for $x > 1$.

14 Show that $f(x) = \dfrac{2}{x^2}$ is concave upwards for all $x \neq 0$.

15 a Show that the curve $y = x^4 + 12x^2 - 20x + 3$ has no points of inflection.
   b Describe the concavity of the curve.

16 If $y = ax^3 - 12x^2 + 3x - 5$ has a point of inflection at $x = 2$, evaluate a.

17 Evaluate p if $f(x) = x^4 - 6px^2 - 20x + 11$ has a point of inflection at $x = -2$.

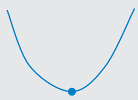
18 The curve $y = 2ax^4 + 4bx^3 - 72x^2 + 4x - 3$ has points of inflection at $x = 2$ and $x = -1$. Find the values of a and b.

3.04 THE SECOND DERIVATIVE TEST

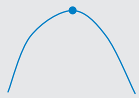
You can tell whether a **stationary point** is a minimum, maximum or point of inflection using the second derivative. You still need to find them using $f'(x) = 0$ first.

IMPORTANT

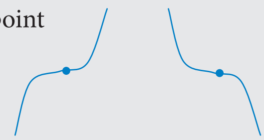
- If $f'(x) = 0$ and $f''(x) > 0$, the stationary point is concave upwards, so it is a local minimum.



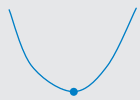
- If $f'(x) = 0$ and $f''(x) < 0$, the stationary point is concave downwards, so it is a local maximum.



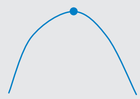
- If $f'(x) = 0$, $f''(x) = 0$, and concavity changes, there is a horizontal point of inflection.



- If $f'(x) = 0$, $f''(x) = 0$ and $f''(x) > 0$ around the value, there is a local minimum.



- If $f'(x) = 0$, $f''(x) = 0$ and $f''(x) < 0$ around the value, there is a local maximum.



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Example 10

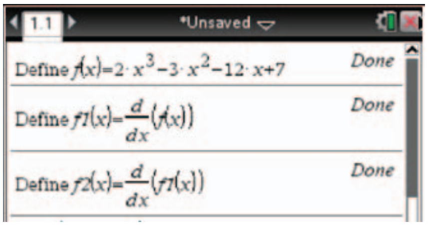
Find the stationary points on the curve $f(x) = 2x^3 - 3x^2 - 12x + 7$ and determine their nature.

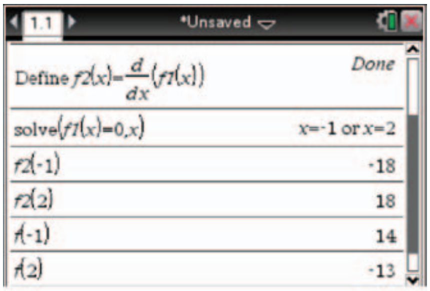
Solution

Find the derivative.	$f'(x) = 6x^2 - 6x - 12$
Find the stationary points.	$6x^2 - 6x - 12 = 0$
Factorise the LHS.	$6(x + 1)(x - 2) = 0$
Solve.	$x = -1$ or $x = 2$
Find the values.	$f(-1) = 14$ and $f(2) = -13$
State the result.	The stationary points are $(-1, 14)$ and $(2, -13)$.
Now find the second derivative.	$f''(x) = 12x - 6$
Find the value at $x = -1$.	$f''(-1) = -18$
Write the conclusion.	$f(x)$ is concave down, so $(-1, 14)$ is a local maximum.
Find the value at $x = 2$.	$f''(2) = 18$
Write the conclusion.	$f(x)$ is concave up, so $(2, -13)$ is a local minimum.

TI-Nspire CAS

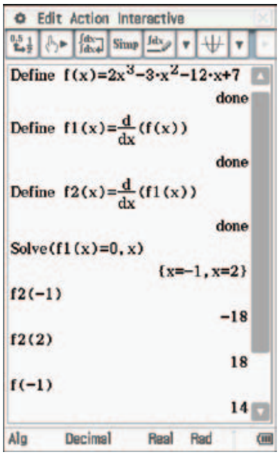
Use a calculator page.
Define the function and its derivatives. Solve $f1(x) = 0$ to find the stationary points. Find the signs of $f2(x)$ at those points.
Find the values as well.





ClassPad

Use the Main menu.
Define the function and its derivatives. Solve $f1(x) = 0$ to find the stationary points. Find the signs of $f2(x)$ at those points.
Find the values as well.
Alternatively, draw the graph and search for maximum and minimum values, and points of inflection. Any points of inflection must be checked to ensure that the gradient is zero.



Example 11

Find any stationary points on the curve $y = 2x^5 - 3$ and determine their nature.

Solution

Write down the function.	$y = 2x^5 - 3$
Differentiate.	$\frac{dy}{dx} = 10x^4$
Find the stationary points.	$10x^4 = 0$
Solve.	$x = 0$ only, so there is one stationary point
Find the second derivative.	$\frac{d^2y}{dx^2} = 40x^3$
Find the value at $x = 0$.	$f''(0) = 0$
Write a conclusion.	It could be any kind of stationary point at $x = 0$
Check concavity near $x = 0$.	

x	-1	0	1
$\frac{d^2y}{dx^2}$	-40	0	40

Write a conclusion.	$f(x)$ changes from concave up to concave down.
Find the point.	$f(0) = 3$
State the result.	There is only one stationary point, $(0, -3)$, which is a horizontal point of inflection (on a rising curve).

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EXERCISE 3.04 The second derivative test

Concepts and techniques

1 Example 10 Find any stationary points on the curve $y = x^2 - 2x + 1$ and determine their nature.

2 Find any stationary points on $y = 3x^4 + 1$ and determine what type they are.

3 Show that $y = 3x^2 - 12x + 7$ has only one stationary point and that it is a minimum.

4 Determine the stationary point on the curve $y = x - x^2$ and show that it is a maximum.

5 Example 11 Find any stationary points on the curve $f(x) = 2x^3 - 5$ and determine their nature.

6 Does the function $f(x) = 3x^5 + 8$ have a stationary point? If it does, determine its nature.

7 Find any stationary points on the curve $f(x) = 2x^3 + 15x^2 + 36x - 50$ and determine their nature.

8 Find the stationary points on the curve $y = 3x^4 - 4x^3 - 12x^2 + 1$ and determine their nature.

9 CAS Find any stationary points on the curve $y = (4x^2 - 1)^4$ and determine their nature.

10 CAS Find any stationary points on the curve $y = 2x^3 - 27x^2 + 120x$ and find their types.

11 CAS Find any stationary points on the curve $y = (x-3)\sqrt{4-x}$ and determine their nature.

12 CAS Find any stationary points on the curve $y = x^4 + 8x^3 + 16x^2 - 1$ and determine their nature.

Reasoning and communication

13 The curve $y = ax^2 - 4x + 1$ has a stationary point where $x = -3$.

a Find the value of a.

b Hence, or otherwise, determine the nature of the stationary point.

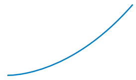
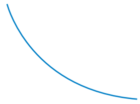
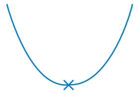
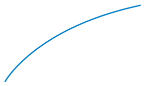
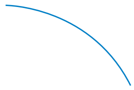
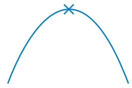
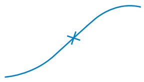
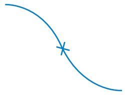
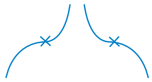
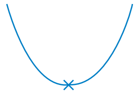
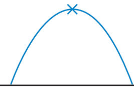
14 The curve $y = x^3 - mx^2 + 8x - 7$ has a stationary point at $x = -1$. Find the value of m.

15 The curve $y = ax^3 + bx^2 - x + 5$ has a point of inflection at $(1, -2)$. Find the values of a and b.

3.05 GRAPH SKETCHING

If you combine the information from the first and second derivatives, this will tell you about the shape of the curve. The first derivative tells you if the curve is increasing, decreasing or stationary, while at the same time, the second derivative tells you if the curve is concave upwards, downwards or stationary.

Here is a summary of the shape of a curve given the first and second derivatives.

	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} = 0$
$\frac{d^2y}{dx^2} > 0$			
$\frac{d^2y}{dx^2} < 0$			
$\frac{d^2y}{dx^2} = 0$ and concavity changes			
$\frac{d^2y}{dx^2} = 0$ and it is positive nearby			
$\frac{d^2y}{dx^2} = 0$ and it is negative nearby			

Example 12

$f(2) = -1$, $f'(2) > 0$ and $f''(2) < 0$. Sketch the graph near $x = 2$, showing its shape at this point.

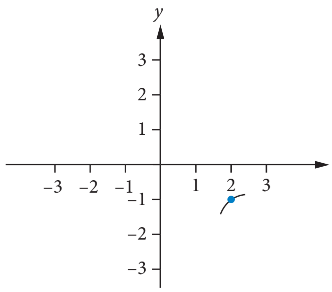
Solution

Interpret $f(2) = -1$. $f(2) = -1$ means $(2, -1)$ is on the curve.

Interpret $f'(2) > 0$. $f'(2) > 0$ means the curve is increasing.

Interpret $f''(2) < 0$. $f''(2) < 0$ means the curve is concave downwards.

Sketch the graph near $(2, -1)$.



Remember that the first derivative is both the gradient and the rate of change of the function.

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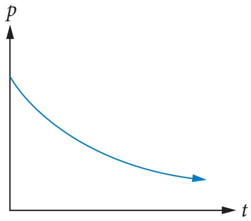
Example 13

The curve below shows the number of unemployed people P over time t months.



a State the signs of $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$.

b How is the number of unemployed people changing over time?

c Is the rate of change of unemployment increasing or decreasing?

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Solution

a	The curve is decreasing.	$\frac{dP}{dt} < 0$
	The curve is concave upwards.	$\frac{d^2P}{dt^2} > 0$
b	The curve is decreasing.	The number of unemployed people is decreasing.
c	The curve is concave upwards, the gradient is increasing.	The rate of change of unemployment is increasing (becoming less negative).

You can use the information from the first and second derivatives to work out the shape of a curve at different points. This will help you to sketch the graph.

Example 14

Find any stationary points and points of inflection on $f(x) = x^3 - 3x^2 - 9x + 1$ and hence sketch the curve.

Solution

Write the function. $f(x) = x^3 - 3x^2 - 9x + 1$

Differentiate. $f'(x) = 3x^2 - 6x - 9$

Find any stationary points. $3x^2 - 6x - 9 = 0$

Factorise. $3(x - 3)(x + 1) = 0$

Solve for x. $x = -1$ or $x = 3$

Find the values at −1 and 3. $f(-1) = 6$ and $f(3) = -26$

State the result. $(-1, 6)$ and $(3, -26)$ are stationary points.

Find the second derivative. $f''(x) = 6x - 6$

Find the values at −1 and 3. $f''(-1) = -12 < 0$ and $f''(3) = 12 > 0$

State the result. $(-1, 6)$ is a local maximum and $(3, -26)$ is a local minimum.

Find the points of inflection. $f''(x) = 0$ at points of inflection.

Write an equation. $6x - 6 = 0$

Solve and write a conclusion. $x = 1$, so there could be a point of inflection at $x = 1$

Check the concavity near $x = 1$.

x	0	1	2
$\dfrac{d^2y}{dx^2}$	−6	0	6

Write a conclusion. The concavity changes, so there is a point of inflection.

Find the value at $x = 1$. $f(1) = -10$

State the result. $(1, -10)$ is a point of inflection.

Find the y-intercept. $f(0) = 1$

State the dominant term. The dominant term is x^3.

State the result. As $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to -\infty$

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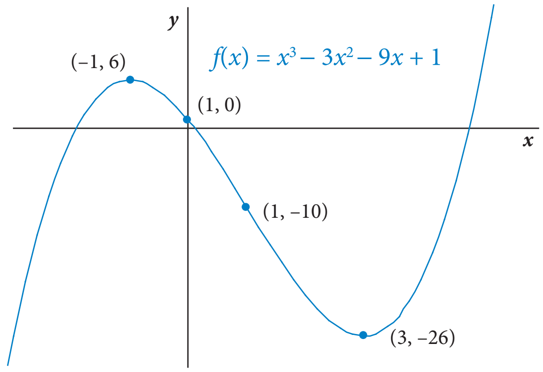
Consider the zeros.

The zeros are difficult to calculate and can be left out in this case.

Sketch the graph using all the information.



EXERCISE 3.05 Graph sketching

Curve sketching with derivatives

Concepts and techniques

1 Example 12 Draw a diagram to show the shape of each curve:

a $f'(x) < 0$ and $f''(x) < 0$

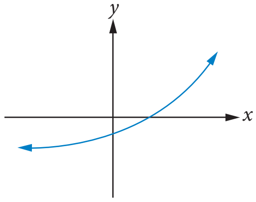
b $f'(x) > 0$ and $f''(x) < 0$

c $f'(x) < 0$ and $f''(x) > 0$

d $f'(x) > 0$ and $f''(x) > 0$

2 For each curve, describe the sign of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

a



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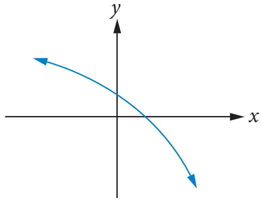
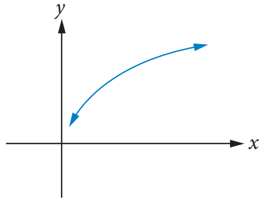


c



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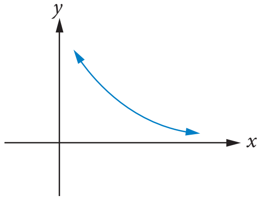
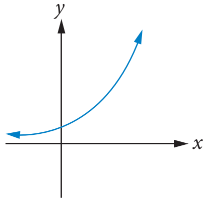


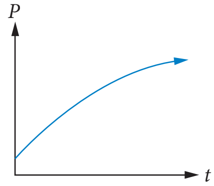
e



3 Example 13 The curve below shows the population of a colony of seals.

a Describe the sign of the first and second derivatives.

b How is the population rate changing?



4 Inflation is increasing, but the rate of increase is slowing. Draw a graph to show this trend.

Reasoning and communication

5 The size of classes at a local TAFE college is decreasing and the rate at which this is happening is decreasing. Draw a graph to show this.

6 As an iceblock melts, the rate at which it melts increases. Draw a graph to show this information.

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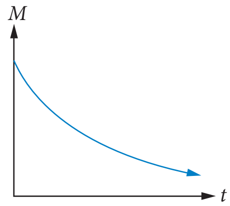
7 The graph shows the decay of a radioactive substance. State the signs of $\frac{dM}{dt}$ and $\frac{d^2M}{dt^2}$ and describe what it means in terms of the decay.



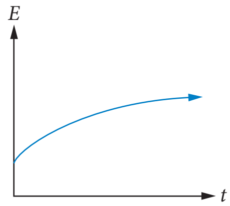
8 The population P of fish in a certain lake was studied over time, and at the start the number of fish was 2500.

a During the study, $\frac{dP}{dt} < 0$. What does this say about the number of fish during the study?

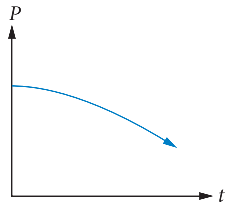
b If at the same time, $\frac{d^2P}{dt^2} > 0$, what can you say about the population rate?

c Sketch the graph of the population P against t.

9 The graph shows the level of education of youths in a certain rural area over the past 100 years.
Describe how the level of education has changed over this period of time. Include a mention about the rate of change.



10 The graph shows the number of students in a high school over several years.
Describe how the school population is changing over time, including the rate of change.



11 Example 14 Find the stationary point on the curve $f(x) = x^2 - 3x - 4$ and determine its type. Find the intercepts on the axes and sketch the curve.

12 Sketch $y = 6 - 2x - x^2$, showing the stationary point.

13 Find the stationary point on the curve $y = (x - 1)^3$ and determine its nature. Hence sketch the curve.

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14 Sketch $y = x^4 + 3$, showing any stationary points.

15 Find the stationary point on the curve $y = x^5$ and show that it is a point of inflection. Hence sketch the curve.

16 Sketch $f(x) = x^7$.

17 Identify any stationary points and points of inflection on $y = 2x^3 - 9x^2 - 24x + 30$ and sketch its graph.

18 a Determine any stationary points on the curve $y = x^3 + 6x^2 - 7$.
b Find any points of inflection on the curve.
c Sketch the curve.

19 Find any stationary points and points of inflection on the curve $y = x^3 - 6x^2 + 3$ and hence sketch the curve.

20 Find any stationary points and points of inflection on the curve $y = 2 + 9x - 3x^2 - x^3$. Hence sketch the curve.

21 Sketch the function $f(x) = 3x^4 + 4x^3 - 12x^2 - 1$, showing all stationary points.

22 Find the stationary points on the curve $y = (x - 4)(x + 2)^2$ and hence sketch the curve.

23 Find all stationary points and inflections on the curve $y = (2x + 1)(x - 2)^4$. Sketch the curve.

3.06 OPTIMISATION

You looked at problems involving **optimisation** on an interval in Year 11 and found that, even though there is no one method you can use to solve these problems, there are general guidelines that can be helpful.

IMPORTANT

Solving optimisation problems

1 Read the problem slowly and carefully and think about the given facts.
2 Represent the unknown quantities as variables.
3 If possible, draw a sketch or diagram and label it, showing variables.
4 Make a list of known facts together with any relationships involving the variables as equations.
5 Determine the variable to be optimised (maximised or minimised) and express this variable as a function of one of the other variables. Using the relationships you have, eliminate all but one of the other variables.
6 Find the stationary points of the function obtained and test each one to see whether it is a maximum or minimum.
7 If you still can't solve the problem, don't give up. Look for another variable or relationship or redraw the diagram. Keep trying!

Example 15

The product of two positive numbers is 40. Find the numbers such that the sum of twice one number and 5 times the other is a minimum.

Solution

There are three quantities involved.	Let the numbers be a, b and m, where m is the 'sum'.
Write the relationships.	$ab = 40$, $m = 2a + 5b$, $a, b > 0$
Identify what has to be optimised/minimised.	m is to be minimised
Write b in terms of a (say).	$b = \frac{40}{a}$
Write m in terms of a.	$m = 2a + 5 \times \frac{40}{a}$
Simplify. Use index form.	$= 2a + 200a^{-1}$
Differentiate.	$\frac{dm}{da} = 2 - 200a^{-2}$
Find stationary points.	$2 - 200a^{-2} = 0$
Rearrange and simplify.	$a^2 = 100$
Solve.	$a = -10$ or $a = 10$
Use $a > 0$.	Since $a > 0$, $a = 10$ is the only possibility.
Find the second derivative.	$\frac{d^2m}{da^2} = 400a^{-3}$
Find the value at $a = 10$.	$\left.\frac{d^2m}{da^2}\right\|_{a=10} = 0.4 > 0$
Identify the stationary point.	m is a minimum at $a = 10$ as it is concave down.
Find the value of b.	$b = \frac{40}{a} = \frac{40}{10} = 4$
State the result.	The required numbers are 10 and 4.

Now you will look at a more practical example.

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Example 16

A farmer is growing field tomatoes for juicing. The more plants that are crowded into a field, the more competition there is for light and nutrients, and so the smaller the mass of fruit from each plant. From previous experience, it is known that when there are n plants on a planting bed of area 100 m^2, the average mass of fruit from each plant is given by

$$m = 40 - 0.05n, \text{ where } m \text{ is in kilograms.}$$

What is the maximum total quantity of fruit, and what planting density is needed?

Shutterstock.com/Federico Rostagno

Solution

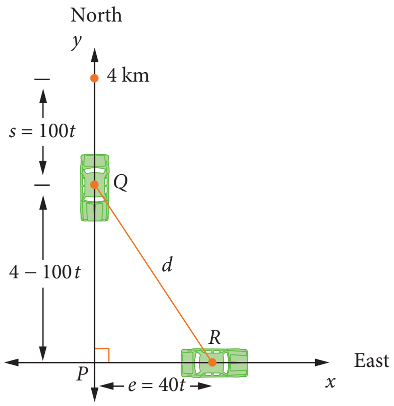
What extra variables are needed?	Let T be the mass of tomatoes from 100 m^2.
Write a relationship for T.	$T = n \times m$
Write in terms of one variable.	$= n(40 - 0.05n)$
Expand.	$= 40n - 0.05n^2$
Identify what has to be optimised.	T is to be maximised.
Find the derivative of T.	$T'(n) = 40 - 0.1n$
Find any stationary points.	$40 - 0.1n = 0$
Solve.	$n = 400$
State a conclusion.	The only possible stationary point is at $n = 400$.
Find the second derivative.	$T''(n) = -0.1 < 0$
Identify the stationary point.	There is a maximum at $n = 400$
Calculate the maximum.	$T(400) = 400(40 - 0.05 \times 400) = 8000$
State the final result.	The maximum total mass of fruit will be 8000 kg/100 m^2 from 400 plants/100 m^2.

In some questions, the equation is given as part of the question. However, you might have to work out an equation first. This is usually the hardest part.

Example 17

A north–south highway intersects an east–west highway at a point P. A vehicle crosses P at 10:00 a.m., travelling east at a constant speed of 40 km/h. At the same instant, another vehicle is 4 km north of P, travelling south at 100 km/h. Find the time at which the vehicles are closest to each other and estimate the minimum distance between them.

Solution

Choose some possible variables.	Let hours after 10 a.m. be t, the distances travelled be e and s respectively, and the distance between them be d.
Write the relationships.	$e = 40t,\ s = 100t$
Draw a diagram showing the positions at time t.	
Use Pythagoras to find d.	$d^2 = (4 - 100t)^2 + (40t)^2$
Simplify.	$= 16 - 800t + 10\,000t^2 + 1600t^2$
Collect terms.	$= 11\,600t^2 - 800t + 16$
Identify what has to be optimised.	d, and so d^2 has to be minimised. Let $f(t) = d^2$
Write $f(t)$.	$f(t) = 11\,600t^2 - 800t + 16$
Differentiate.	$f'(t) = 23\,200t - 800$
Find any stationary points.	$23\,200t - 800 = 0$
Solve.	$t = \frac{800}{23\,200} = \frac{1}{29} \approx 0.0345$
Write a conclusion.	The only stationary point is at $t = \frac{1}{29}$
Find the second derivative.	$f''(t) = 23\,200 > 0$
Identify the stationary point.	There is a minimum at $t = \frac{1}{29}$ hr $\approx$ 2.1 min
Calculate the value at $t = \frac{1}{29}$.	$f\left(\frac{1}{29}\right) \approx 2.2069$
Find the square root.	$d = \sqrt{f\left(\frac{1}{29}\right)} \approx 1.49$
State the result.	The minimum distance is about 1.5 km and this is at just after 10:02 a.m.

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In problems like Example **17** you should use the most accurate values on your calculator until you get to the end of the problem. It is best to use your CAS calculator for this purpose.

INVESTIGATION Heron's problem

One of the first non-trivial optimisation problems, 'The Shortest Path', was solved by Heron of Alexandria, who lived about 10–75 CE. The following is a problem which uses this theory.

One boundary of a farm is a straight river bank, and on the farm stands a house, with a shed some distance away; each is sited away from the river bank. Each morning the farmer takes a bucket from his house to the river, fills it with water, and carries the water to the shed.

Find the position on the river bank that will allow him to walk the shortest distance from house to river to shed. Further, describe how the farmer could solve the problem on the ground with the aid of a few stakes for sighting.

EXERCISE 3.06 Optimisation

Optimisation

Concepts and techniques

1 Example 15 Find two positive numbers a and b whose product is 27 and for which the value of $3a + 4b$ is the least.

2 Find two numbers whose sum is 25 and whose product is a maximum.

3 Find two numbers whose difference is 40 and whose product is the least.

4 The sum of two numbers is 32. Find the numbers if the sum of their squares is a minimum.

5 P is a movable point on the line $y = 7 - x$. Find the coordinates of P when it is closest to the origin.

6 Find the point (x, y) on the graph of $y = \sqrt{x}$ that is nearest to the point $(4, 0)$.

Reasoning and communication

7 For the function $f(x) = \dfrac{6}{x^2 + 3}$ find the equation of the tangent with:

a the maximum slope

b the minimum slope.

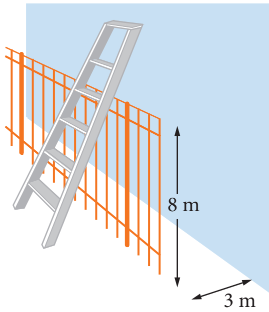
8 Example 16 A farmer is stocking a pond with fish as a supplementary form of income. The more fish there are in the pond, the more competition there will be for the available food supply, and so the more slowly the fish will gain weight. From previous experience, it is known that when there are n fish per unit area of water, the average weight gain of each fish (in grams) during the growing season is given by $w(n) = 600 - 30n$.

What value of n will lead to the maximum total production of weight in the fish?

9 Example 17 Two catamarans are 6 km apart in open water and the first is at a bearing of 045° from the second. The first begins moving directly south at 12 km/h and the second travels directly east at 18 km/h. What is the minimum distance between the catamarans?

10 A small boat moving at v km/h uses fuel at a rate that is approximated by the function $q = 8 + \frac{v^2}{50}$, where q is measured in litres/hour. Determine the speed of the boat at which the amount of fuel used for any given journey is the least.

11 An 8 m high fence stands 3 m from a large vertical wall. Find the length of the shortest ladder that will reach over the fence to the wall behind, as shown in the diagram.



12 An electric train uses power at the rate of 250 000 + v^3 J/h, where v is the speed of the train in kilometres/hour, and the speed it travels between stations ranges from 20–80 km/h. Find the speed at which the train should travel to minimise the use of electricity between stations.

13 An Australian test cricket player strikes the ball so that its equation of motion is given by $y = 1.4 + x - 0.004x^2$, where y is the height (m) reached by the ball and x is the horizontal distance (m) travelled by the ball. What is the greatest height reached by the ball?

14 During the course of an epidemic, the proportion of the population infected t months after the epidemic began is given by $p = \frac{t^2}{5(1+t^2)^2}$.

a Find the maximum proportion of the population that becomes infected.

b Find the time at which the proportion infected is increasing most rapidly.

15 An ironman has drifted 1 km along the beach from the finish of the swim stage. He is 400 m from shore and can swim at 4 km/h and run at 12 km/h. At what angle to the shore should he swim to reach the finish as quickly as possible?

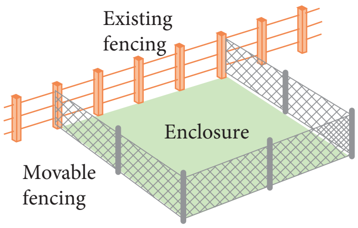
AAP Images/PR Images

3.07 OPTIMISATION IN AREA AND VOLUME

Many processes, especially manufacturing, involve the optimisation of physical quantities such as area and volume. For instance, a container manufacturer will probably want to minimise the surface area of a container designed to hold a specific quantity. This will allow the most number of containers to be produced from each unit of raw materials. When solving problems involving area and volume, you should take particular care in drawing the diagram. These problems will involve some basic two-dimensional figures and three-dimensional shapes and so you will need to know the area, surface area and volume formulas for them.

Example 18

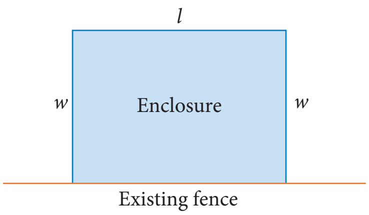
A cattle property manager wants to construct a temporary rectangular enclosure using 200 m of movable fencing and an existing property fence as shown here. The manager wants to place the movable fencing so that the enclosure has the greatest possible area. Calculate the maximum area and state the dimensions for this area.



Solution

Choose the variables. Let the length, width and area be l, w, and A.

Draw a diagram and label it.



Write the relationships. $2w + l = 200$, $A = lw$

What has to be optimised? The area A has to be maximised.

Change to a single variable. $l = 200 - 2w$

Write the area. $A = (200 - 2w) \times w$

Expand. $= 200w - 2w^2$

Differentiate to find the critical numbers. $A'(w) = 200 - 4w$

Find any stationary points. $200 - 4w = 0$

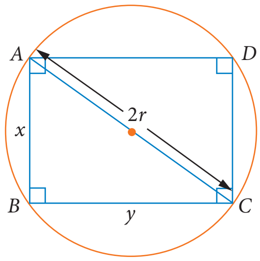
Solve.	$w = 50$
Write a conclusion.	There is only 1 stationary point, at $w = 50$.
Find the second derivative.	$A''(w) = -4 < 0$
Write a conclusion.	There is a maximum at $w = 50$.
Find the length.	Length $= 200 - 2 \times 50$ $= 100$
Find the area.	$A = 50 \times 100$ $= 5000$
State the result.	The maximum area is 5000 m^2 when the enclosure is 50 m $\times$ 100 m. The long side is parallel to the existing fence.

Example 19

Find the area of the largest rectangle that can be enclosed in a circle of radius r.

Solution

The largest rectangle must have corners that touch the circle. Draw the diagram and name the corners A, B, C and D.
The diagram shows the names chosen for the variables. The diagonal will be a diameter because the angle subtended is a right angle.



State what has to be optimised.	You want to maximise the area of the rectangle.
Write an equation for the area of the rectangle.	Let the area $= A = xy$
Write a relationship using Pythagoras' theorem.	$(2r)^2 = x^2 + y^2$
Write in terms of, say, x.	$y^2 = 4r^2 - x^2$
Consider r as fixed for any circle.	$y = \sqrt{4r^2 - x^2}$
Write the area in terms of x only.	$A = x\sqrt{4r^2 - x^2}$
Simplify by considering the square.	A is maximised when A^2 is maximised.
Write A^2 as a function.	$f(x) = A^2$ $= x^2(4r^2 - x^2)$
Expand.	$= 4r^2x^2 - x^4$
Find the derivative.	$f'(x) = 8r^2x - 4x^3$, since r is a constant.

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Find any stationary points.	$8r^2x - 4x^3 = 0$
Factorise.	$4x(2r^2 - x^2) = 0$
Use the difference of squares.	$4x\left(\sqrt{2}r - x\right)\left(\sqrt{2}r + x\right) = 0$
Solve.	$x = 0,\ x = \sqrt{2}r$ or $x = -\sqrt{2}r$
x is the length of a side.	$x = \sqrt{2}r$ is the only possibility as x is a length.
Find the second derivative.	$f''(x) = 8r^2 - 12x^2$
Check the sign at $x = \sqrt{2}r$.	$f''\left(\sqrt{2}r\right) = 8r^2 - 12\left(\sqrt{2}r\right)^2$
Simplify.	$= 8r^2 - 12 \times 2r^2$ $= -16r^2 < 0$
Identify the stationary point.	There is a maximum at $x = \sqrt{2}r$
Calculate the maximum area.	$A = \sqrt{2}r\sqrt{4r^2 - \left(\sqrt{2}r\right)^2}$
Simplify.	$= \sqrt{2}r\sqrt{4r^2 - 2r^2}$ $= \sqrt{2}r\sqrt{2}r$ $= 2r^2$
State the result.	The largest rectangle that can be enclosed in a circle of radius r has an area of $2r^2$.

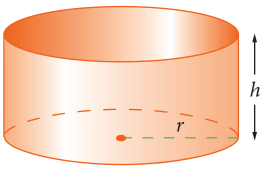
In practical situations like those in Example **19**, you can often use practical considerations to discard some of the solutions that arise from the mathematical model. Some people call these false solutions.

Example 20

A cylindrical metal can is open at the top. It must be designed to have a capacity of 1078 cm^3. The outside of the can has to be galvanised, and the manufacturer wants the surface area of the can to be minimised in order to save money. Find the dimensions of the can.

Solution

Draw a diagram and label the obvious variables, in this case the height h and the radius r of the can.



State what has to be optimised.	The external surface area A is to be minimised.
Write the relationships.	$A = 2\pi rh + \pi r^2$ and $V = 1078 = \pi r^2 h$
Use the volume to express h in terms of r.	$h = \dfrac{1078}{\pi r^2}$
Express the surface area in terms of r.	$A = 2\pi r\dfrac{1078}{\pi r^2} + \pi r^2$

Simplify.	$A(r) = \pi r^2 + \dfrac{2156}{r}$ $= \pi r^2 + 2156r^{-1}$
Differentiate.	$A'(r) = 2\pi r - 2156r^{-2}$
Find any stationary points.	$2\pi r - 2156r^{-2} = 0$
Multiply by r^2.	$2\pi r^3 - 2156 = 0$
Solve.	$r = \sqrt[3]{\dfrac{2156}{2\pi}} = \sqrt[3]{\dfrac{1078}{\pi}} = 7.000\,939\ldots$
Find the second derivative of A.	$A''(r) = 2\pi + 4312r^{-3}$
Find the value at the stationary point.	$A''\left(\sqrt[3]{\dfrac{1078}{\pi}}\right) = 2\pi + 4312\left(\sqrt[3]{\dfrac{1078}{\pi}}\right)^{-3}$
Simplify.	$= 6\pi > 0$
Identify the stationary point.	There is a minimum when $r \approx 7$.
Find the height.	$h = \dfrac{1078}{\pi}r^{-2} = \dfrac{1078}{\pi}\left(\sqrt[3]{\dfrac{1078}{\pi}}\right)^{-2} = \sqrt[3]{\dfrac{1078}{\pi}} = r$
State the result.	A 1078 cm^3 cylinder has minimal surface area when its height and radius are equal, about 7 cm.

Notice that Example **20** almost proves that an open cylinder of fixed volume has the minimal surface area when the height and radius are equal. Of course, the result for a closed cylinder would be different.

WS
Optimisation – Area and volume

EXERCISE 3.07 Optimisation in area and volume

Reasoning and communication

1 Example 18 A rectangular paddock shares one side with an existing paddock, so it requires no fence on that side. There are only 2000 m of fencing material available to fence the remaining sides. Find the maximum possible area and dimensions of the paddock.

2 Example 19 A right circular cone is machined from a solid sphere of radius 30 cm. Find the ratio of the volume of the cone to the volume of the sphere when the volume of the cone is a maximum.

3 A strip of galvanised iron is 24 cm wide. The strip of iron is to be used to form a length of guttering for a house. The guttering is going to have a rectangular cross-section and be open at the top so that water can flow from the roof of the house into the guttering. What are the dimensions of the cross-section of the guttering if it is to hold the maximum volume of water?

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4 Example 20 A manufacturer produces open-topped rectangular boxes, each with a square base and a volume of 500 cm^3. What are the dimensions of a box if the least amount of material is to be used in its construction?

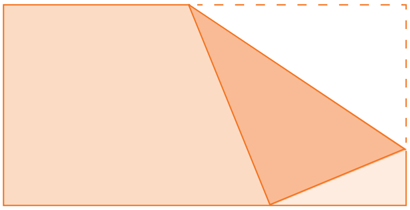
Shutterstock.com/Quang Ho

5 A metal box has square ends, rectangular sides and bottom and is open at the top. What is the least area of metal that is required to construct the box if it must have a volume of 36 m^3?

6 A right circular cone has a slant edge measuring 9 cm. What is the height of the cone that will have the greatest volume?

7 A rectangular sheet of cardboard measuring 16 cm by 10 cm is to be formed into an open rectangular box by cutting out identical squares from each corner and folding up the sides. What size must the squares be for the box to have the maximum volume?

8 A rectangular piece of paper measures 12 cm by 6 cm. One corner of the sheet of paper is folded up to just reach the opposite side as shown below. What is the minimum length of the resulting crease in the paper?



9 An opening in a wall is to be made in the shape of a semicircle over a rectangle. The distance around the edge of the opening must measure 12 m. What are the dimensions of the opening that will result in the greatest area?

10 What are the dimensions of the cone of greatest volume that can just fit inside a sphere of radius 2 m?

11 A window consists of a rectangular sheet of glass surmounted by a semicircular sheet as shown in the photograph. Calculate the maximum area obtainable if the perimeter of the window is fixed at 6 m.

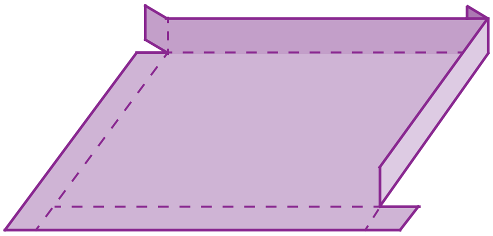
12 Find the greatest volume of a rectangular box with a square base if the sum of the height and the side of the base must not exceed 30 cm.

Thinkstock/photoncatcher

13 Postal cylinders are made according to the rule that sum of the length and circumference not be more than 150 cm. What are the dimensions of the cylinder with the largest volume?

14 A settling pond is to be constructed to hold 324 m^3 of waste. The settling pond has a square base and four vertical sides, all made of concrete, and a square top made of steel. If the steel costs twice as much per unit area as the concrete, determine the dimensions of the pond that will minimise the cost of construction.

15 Cardboard trays are made from rectangular pieces of cardboard with a length to width ratio of 3 : 2. The trays are made as shown, by cutting one side of a square corner, then bending and gluing it to make the tray. What dimensions will the trays with the largest volume have if they are made from a cardboard rectangle with length:



a 20 cm? b 30 cm? c 50 cm?

3.08 OPTIMISATION IN BUSINESS

If the management of a business knows how their profit depends on some variable, then they can choose the value that makes the profit as large as possible. You can set up revenue, cost and profit as functions for this purpose.

Example 21

A small manufacturing firm can sell everything it produces at a price of \$8 per item. It costs $C(n) = 100\,000 + 8n - 0.0156n^2 + 2 \times 10^{-6}n^3$ to produce n items per week (in dollars).

What should the weekly production target be in order to maximise profits?

Solution

Write a function for profit.	$P(n) = 8n - (100\,000 + 8n - 0.0156n^2 + 2 \times 10^{-6}n^3)$
Simplify.	$= 0.0156n^2 - 100\,000 - 2 \times 10^{-6}n^3$
Find the first derivative.	$P'(n) = 0.0312n - 6 \times 10^{-6}n^2$
Find any stationary points.	$0.0312n - 6 \times 10^{-6}n^2 = 0$
Factorise.	$n(0.0312 - 6 \times 10^{-6}n) = 0$
Solve.	$n = 0$ or $n = 5200$
Find the second derivative.	$P''(n) = 0.0312 - 1.2 \times 10^{-5}n$

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Check the relevant values.	$P''(0) = 0.0312 > 0$ and $P''(5200) = -0.0312 < 0$
State the maximum and minimum points.	A minimum is at $n = 0$ and a maximum $n = 5200$
Calculate the maximum profit.	$P(5200) = 0.0156 \times 5200^2 - 100\,000 - 2 \times 10^{-6} \times 5200^3$
Simplify.	$= 40\,608$
State the result.	The maximum profit of \$40 608 is achieved with a weekly production of 5200 items.

In complicated cases, check your results using a graphics or CAS calculator by drawing a graph of the function to be optimised.

Example 22

A printer has a contract to print 20 000 copies of a one-page catalogue. It costs \$800/h to run the printing press, which will produce 120 impressions per hour. The printer can run up to 50 plates at a time, and each impression produces x copies, where x is the number of plates used. Each plate costs \$200 to set up. How many plates should be made so that the job is done the most economically?

Solution

Decide on the variable names.	Let printing time = t and cost = C.
State what has to be optimised.	C has to be minimised.
Write the rate of production.	Copies produced = $120x$ per hour.
Write the relationships.	$120x \times t = 20\,000$ and $C = 200x + 800t$
Write t in terms of x only.	$t = \dfrac{20\,000}{120x} = \dfrac{500}{3x}$
Write C in terms of x only.	$C(x) = 200x + 800 \times \dfrac{500}{3x}$
Simplify.	$= 200x + \dfrac{400\,000}{3x}$
	$= 200x + \dfrac{400\,000}{3}x^{-1}$
Differentiate.	$C'(x) = 200 - \dfrac{400\,000}{3}x^{-2}$
Find any stationary points.	$200 - \dfrac{400\,000}{3}x^{-2} = 0$
Multiply by $3x^2$.	$600x^2 - 400\,000 = 0$
Solve.	$x = \pm\sqrt{\dfrac{2000}{3}} \approx \pm 25.82$
Use common sense.	Only $x = +\sqrt{\dfrac{2000}{3}}$ is valid for the number of plates.
Find the second derivative.	$C''(x) = \dfrac{800\,000}{3}x^{-3}$

Find the value at $x = \sqrt{\frac{2000}{3}}$.

$C''\left(\sqrt{\frac{2000}{3}}\right) = \frac{800\,000}{3} \times \left(\sqrt{\frac{2000}{3}}\right)^{-3} > 0$

Write a conclusion.

There is a minimum at $x = \sqrt{\frac{2000}{3}} \approx 25.82$

State the possible results.

The answer has to be an integer, so it's either 25 or 26.

Check the first value.

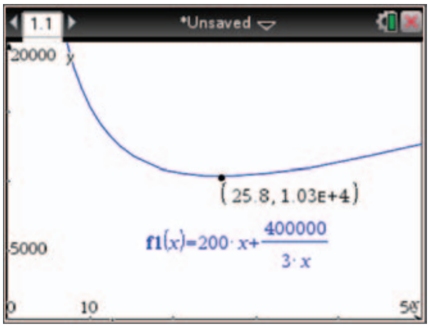
$C(25) = 200 \times 25 + \frac{400\,000}{3 \times 25} \approx 10\,333.33$

Check the cost for 26.

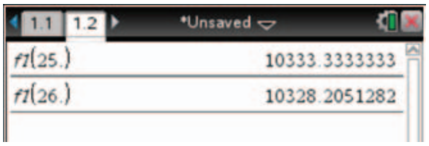
$C(26) = 200 \times 26 + \frac{400\,000}{3 \times 26} \approx 10\,328.21$

TI-Nspire CAS

Use a graph page.
Draw the graph and find the minimum using b, 6: analyse Graph and 2: Minimum.

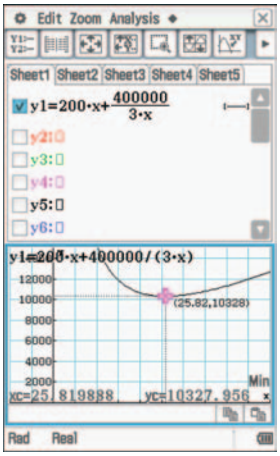


You can find the values at 25 and 26 by adding a Calculator page.

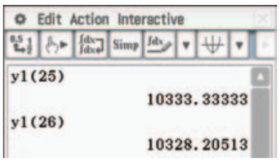


ClassPad

Use the Graph and Table menu.
Draw the graph and find the minimum by tapping Analysis, G-Solve and Min.



You can check the values at 25 and 26 by going to the Main menu.



State the result.

The job will be done most economically for \$10 328.21 when 26 plates are made.

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INVESTIGATION Fun rides

Flash owns a 'fun ride' that he sets up at country shows. Each year he brings his ride to the Moomba Festival. The ride has the capacity to take up to 90 people, but Flash knows from experience that he only gets about a third of the ride's capacity when he charges his normal price of $10 for a ride. He also knows that each time he runs the ride it costs the same amount regardless of how many people have bought tickets. When Flash reduces the price of the ride, he attracts more customers. He wants to make the ride more profitable and has an idea that if he gets the price right, he can achieve this. He has learnt that charging too little means that he loses money.

Recently, Flash has noticed that rides costing only $5 are usually full, and he is sure that his ride would be full every time if he dropped his price to that figure.

123RF/windu

1 Assuming that the number of customers will increase in direct proportion to the price drop, decide what price would give him the most profit.

2 How much better off would Flash be each time he runs the ride at the new fare compared with the takings at the fare of $10?

3 Investigate other methods of approaching this problem by considering different assumptions.

EXERCISE 3.08 Optimisation in business

WS Optimisation – Cost and revenue

Reasoning and communication

1 Example 21 The Western Star Novelty Company produces x tonnes of silver per month at a total cost given by

$$C(x) = 2000 - 75x - 5x^2 + \frac{x^3}{3}$$

where C is in dollars. Find the production level for which costs are minimised.

2 Example 21 A machine shop can produce 30 items a day on one machine. The shop can use more than one machine for the job, but if it does, for technical reasons, each machine cannot then work to full capacity. In fact, production per machine is slowed by a factor equal to $\frac{1}{10}$ of the square of the number of additional machines put on the job. That is, if x additional machines are put on the job, the number of items produced by each machine is $30 - \frac{x^2}{10}$. How many machines should be used to achieve the maximum production?

3 A firm sells all the units that it produces at \$4 per unit. The firm's total cost C for producing x units is given in dollars by

$$C(x) = 50 + 1.3x + 0.001x^2$$

a Write an expression for total profit as a function of x.

b How many items should be produced so that the profit is a maximum?

4 For a particular item, the price at which x items are sold is given by the equation

$$p = 5 - 0.001x \text{ for } 0 < x < 5000$$

The cost of production is given by

$$C(x) = 2800 + x$$

a Find the value of x that maximises the revenue.

b Find the value of x that maximises the profit.

5 The cost (in dollars) of producing x plastic mounts is

$$C(x) = 4000 - 3x + 10^{-3}x^2$$

a Find the value of x that minimises production costs.

b If the plastic mounts are sold for \$4 each, find the value of x that maximises profit and calculate the maximum profit.

6 A manufacturer can sell x clock radios per week at d dollars per item, where $x = 600 - 3d$. Production costs (in dollars) are

$$C(x) = 600 + 10x + \frac{1}{2}x^2$$

How should the clock radios be priced in order to maximise profits?

7 The weekly cost (in dollars) of producing x ironing boards is given by

$$C(x) = 40\,000 - 30x + 10^{-2}x^2$$

a Find the value of x that minimises production costs.

b If the ironing boards are sold for \$40 each, find the value of x that maximises profit and calculate the maximum profit.

Alamy/Andreas von Einsiedel

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8 The total weekly cost (in dollars) of producing carpet is found to be approximated by the function

$$C(x) = 100 + 28x - 5x^2 + \frac{x^3}{3}$$

where x is the number of rolls of carpet. The price (in dollars) at which carpet rolls are sold is given by $p = 5000 - 5x$ for $0 < x < 1000$.

A tax of \$222 per roll is imposed by the government. The manufacturer adds this to the production costs. The manufacturer sells to a number of different outlets and has no problem getting rid of however much is produced.

a Change the manufacturer's cost function to include the government tax.
b State the revenue function for selling x rolls of carpet.
c Write the profit function.
d Work out the number of rolls of carpet that need to be produced each week in order to maximise profit.


Dreamstime.com/Ragne Kabanova

9 By reducing the number of counter staff, a fast-food outlet can reduce its labour costs, but can also expect to lose business because of customer dissatisfaction at having to wait. Assume that the rate of pay for counter staff is \$80 per day and that the loss of profit from having only n counter staff is $\frac{5000}{n+1}$ dollars per day. Calculate the value of n that minimises the sum of the loss plus the wage cost.

10 A parcel delivery firm makes regular trips between Brisbane and Ipswich. The accountant, who also happens to be a good mathematician, has calculated that fuel and other running costs are approximated by the function

$$F(v) = 2v^{\frac{3}{2}} + 59 \text{ cents/trip}$$

where v = average speed in km/h. The cost of paying a driver is given by

$$D(v) = \frac{1.5 \times 10^5}{v} + 2000 \text{ cents/trip}$$

Find the most economical speed for the trip and the cost of a trip from Brisbane to Ipswich at this speed.

3.09 GENERAL OPTIMISATION PROBLEMS

Optimisation theory is versatile and can be used to solve many complicated problems in a variety of situations. You have looked specifically at area and volume problems and the solutions to business problems. In this section you will look at the use of optimisation to solve problems in other fields. The methods you use to solve the problems remain the same.

Example 23

A tennis ball follows a path defined by $h = -t^2 + 6t + 2.5$ and $d = 14t$, where h is the height of the ball in metres, d is the horizontal distance in metres and t is the time in seconds. What is the maximum height reached by the tennis ball?

Thinkstock/nickp37

Solution

Write the relevant function.	$h = -t^2 + 6t + 2.5$
Differentiate.	$h' = -2t + 6$
Find any stationary points.	$-2t + 6 = 0$
Solve.	$t = 3$
Find the second derivative.	$h'' = -2 < 0$
Write down the nature of the stationary point.	There is a maximum at $t = 3$.
Substitute into the height.	$h(3) = -3^2 + 6 \times 3 + 2.5$ $= 11.5$
State the result.	The maximum height reached is 11.5 metres.

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Example 24

The water bug population of a lake is dependent upon the water temperature. The number of water bugs is about $N = -T^3 + 13.5T^2 + 1740T + 2575$, where T is the water temperature in °C for $20° \le T \le 30°$. Find the maximum water bug population.

Thinkstock/thomasmales

Solution

Write the function.	$N = -T^3 + 13.5T^2 + 1740T + 2575$
Differentiate.	$N' = -3T^2 + 27T + 1740$
Find any stationary points.	$-3T^2 + 27T + 1740 = 0$
Simplify to solve.	$T^2 - 9T - 580 = 0$
Factorise the quadratic.	$(T + 20)(T - 29) = 0$
Complete the solution.	$T = -20$ or $T = 29$
Discard the false answer.	$T = 29$ only as -20 is outside the range.
Find the second derivative.	$N'' = -6T + 27$
Find the value at $T = 29$.	$N''(29) = -6 \times 29 + 27$ $= -147 < 0$
State the nature of the stationary point.	There is a maximum at $T = 29°$
Find the maximum population.	$T(29) = -(29)^3 + 13.5 \times 29^2 + 1740 \times 29 + 2575$
Simplify.	$= 39\,999.5$
State the result.	The maximum population is about 40 000 at 29°C.

WS

Optimisation – Mixed problems

EXERCISE 3.09 General optimisation problems

Reasoning and communication

1 Example 23 A projectile is fired into the air and its height in metres is given by $h = 40t - 5t^2 + 4$, where t is in seconds. What is the maximum height that the projectile reaches?

2 The displacement in cm after time t seconds of a particle moving in a straight line is given by $x = 2 + 3t - t^2$. What is its maximum displacement?

3 A ball is rolled up a slope. Its position after time t seconds is given by $x = 15t - 3t^2$ m. How far up the slope will the ball roll before it starts to roll back down?

4 A skier decides to jump a ramp. Her height above the water, h m, t seconds after jumping, is given by $h = -4t^2 + 4t + 10$. Find the maximum height that she attains.

Getty Images/Donald Miralle

5 A toy rocket is launched from the top of an 8 metre high building so that its height, h, in metres, above the ground t seconds later is given by $h = 8 - 8t^2 + 32t$. What is the maximum height reached by the rocket?

6 A person standing on top of a 35 m high cliff throws a stone into the air. Its height after t seconds is given by $y = -5t^2 + 8t + 35$. What is the maximum height that it reaches?

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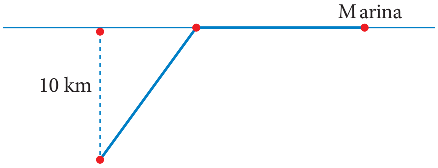
7 Example 24 A seal colony eats fish with a population dependent on the biomass of krill and other small organisms in the area. The density of these changes as the temperature of the water varies. The seal population, P, can be modelled as $P = -2\frac{2}{3}T^3 + 52T^2 - 176T + 382$, where T is the water temperature in °C for $-2° \leq T \leq 15°$. What will the maximum and minimum seal colony populations be, and at what temperatures will they occur? Hint: Check the population at −2°C and 15°C as well.

123RF/Elice Lin

8 A farmer estimates that if 75 pecan nut trees are planted per hectare, the average yield per tree will be 7 kg. For every tree less that he plants on the same acreage, the average yield per tree will increase by 0.2 kg per tree. How many trees per hectare should the farmer plant to maximize the total yield?

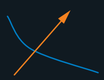
9 A man can row at 6 km/h and run at 8 km/h. He is 10 km out to sea and needs to get to a marina on the coast 18 km from the nearest point on the shore from his current position. Where should he land on the coast to get to the marina as soon as possible?



10 A 4WD driver broke down 15 km down a very winding track running away from the main road, but only 5 km from the road itself at its closest point. He needs to get to a service station on the main road, which is another 5 km down the road from the closest point. He can walk at 5 km/h on the road or track but only 3 km/h cross country. What should he do to get to the service station in the shortest possible time?

3 CHAPTER SUMMARY
APPLICATIONS OF DERIVATIVES

- For a small change δx, in the independent variable x, the corresponding change δy in the dependent variable y is given by the **increments formula** $\delta y \approx \frac{dy}{dx} \times \delta x$
- The **second derivative** can be found by differentiating a function twice. It is shown as $f''(x)$, $f^{(2)}(x)$, $f^2(x)$, y'', or $\frac{d^2y}{dx^2}$.
- The **position** of a particle relative to a fixed point (the origin) is usually shown as x.
- A **displacement** is a change of position, usually shown as s.
- **Velocity**, v, is the rate of change of displacement or position, so $v = \frac{dx}{dt} = \frac{ds}{dt}$.
- **Acceleration**, a, is the rate of change of velocity, so $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \frac{d^2s}{dt^2}$.
- The second derivative shows the change in the first derivative and the **concavity** of a curve.
    - If $f''(x) > 0$, then $f'(x)$ is increasing: the curve is **concave upwards** (**cup** shaped).



Concave upwards

    - If $f''(x) < 0$, then $f'(x)$ is decreasing: the curve is **concave downwards** (**cap** shaped).

Concave downwards

    - If $f''(x) = 0$, then $f'(x)$ is stationary

- A **point of inflection** is a point where the concavity of the function (sign of $f''(x)$) changes.

Point of inflection

- A **horizontal point of inflection** is a both a stationary point and a point of inflection.

Horizontal point of inflection

- For stationary points, $f'(x) = 0$. The second derivative test is used to determine the kind of stationary point.
    - If $f'(x) = 0$ and $f''(x) > 0$, the curve is concave up so there is a **local minimum**
    - If $f'(x) = 0$ and $f''(x) < 0$, the curve is concave down so there is a **local maximum**
    - If $f'(x) = 0$, $f''(x) = 0$ and concavity changes, there is a **point of horizontal inflection**
    - If $f'(x) = 0$, $f''(x) = 0$, concavity does not change and $f''(x) > 0$ on either side, then there is a **local minimum**
    - If $f'(x) = 0$, $f''(x) = 0$, concavity does not change and $f''(x) < 0$ on either side, then there is a **local maximum**

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- To solve **optimisation** problems, do the following.

  1 Read the problem carefully

  2 Represent the unknown quantities as variables

  3 If possible, sketch a diagram and label it, showing variables

  4 Make a list of known facts together with any relationships involving the variables as equations

  5 Determine the variable to be optimised (maximised or minimised) and express this variable as a function of *one* of the other variables. Use the relationships you have to eliminate all but one of the other variables.

  6 Find the stationary points of the function obtained and test each one to see whether it is a maximum or minimum.

3 CHAPTER REVIEW

APPLICATIONS OF DERIVATIVES

Multiple choice

1 Example 1 Using the derivative of $\sqrt{x}$, an approximation for $\sqrt{147}$ is:

A 12 B 12.03 C 12.125 D 12.3 E 13

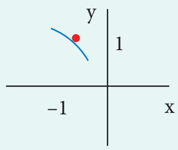
2 Example 5 If $y = 3\cos(4x)$, then $\dfrac{d^2y}{dx^2}$ equals

A $48\sin(4x)$ B $-48\cos(4x)$ C $-12\cos(4x)$

D $-12\sin(4x)$ E $48\cos(4x)$

3 Example 12 For a particular function, $f(-1) = 1$, $f'(-1) < 0$ and $f''(-1) < 0$. Which sketch of the curve at this point, shows its shape?

a



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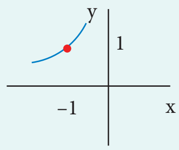

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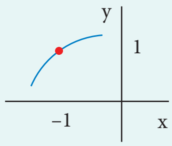

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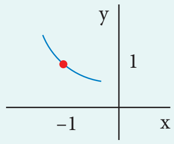

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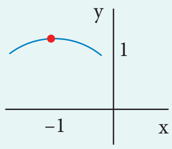

4 Example 15 Two numbers have a difference of 24. What are the two numbers if the addition of their product and their sum is a minimum?

A 13 and 37 B 11 and 35 C −13 and 11

D −11 and 13 E −13 and −37

5 Example 18 80 metres of fencing is available for enclosing a play area. What is the maximum area which can be enclosed?

A 80 m^2 B 90 m^2 C 200 m^2 D 400 m^2 E 800 m^2

Short answer

6 Example 1 Use the derivative of $x^{\frac{2}{5}}$ for $x = 32$ to find an approximation for:

a $33^{0.4}$ b $31.7^{0.4}$

7 Example 2 A spherical ball with a diameter of 12 cm is being pumped up. The volume is given by $V = \dfrac{4\pi r^3}{3}$. Find an approximation of the increase in volume when the radius expands by 0.04 cm.

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8 Example 3 A steel drum manufacturer makes 60 L drums that have a height : diameter ratio of 1.6.

a Find the dimensions of the drums.

b Find the approximate percentage error in volume, given that there is an error of x% in the height.

c Find the approximate percentage error in volume, given that there is an error of y% in the diameter.

d Find the approximate percentage error in volume if a tolerance of 3% is allowed in both the height and the diameter.

9 Example 4 Find the first two derivatives of:

a $y = 5x^6 - 3x^2 + x + 10$

b $f(t) = (2t + 9)^4$

c $f(n) = (3n - 1)^2(2n + 4)$

d $y = \dfrac{6x - 9}{3x - 1}$

10 Example 6 The position of a particle after t seconds is given by $x = t^3 - 12t^2 + 36t - 9$ cm.

a Find an expression for its velocity.

b Find an expression for its acceleration

c What is its acceleration after 2 seconds?

11 Example 7 Find all x-values for which the curve $y = 2x^3 - 7x^2 - 3x + 1$ is concave upwards.

12 Example 8 Does $f(x) = 4x^7$ have a point of inflection?

13 Example 9 Find any points of inflection on the curve $y = x^4 + 4x^3 - 48x^2 + 1$.

14 Example 10 For the following functions, find all the stationary points and determine their nature.

a $y = 3x^2 - 6x + 3$

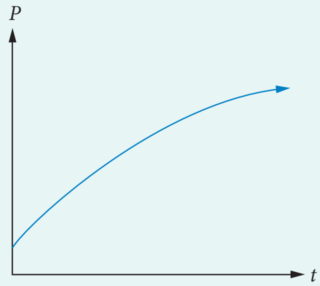
b $f(x) = 5x - x^2$

c $f(x) = 3x^4 - 4x^3 - 12x^2 + 7$

d $y = (2x - 1)^4$

15 Example 11 Find the stationary point on $y = 2x^3 - 1$ and determine its nature.

16 Example 13 The population P of possums on an island over time t years is shown on the graph.



a State the signs of $\dfrac{dP}{dt}$ and $\dfrac{d^2P}{dt^2}$.

b How is the number of possums changing over time?

c Is the rate of population growth increasing or decreasing?

17 Example 14 Sketch the graphs of the following functions, showing clearly any turning points or points of inflection.

a $y = 4x^2 - 9x + 5$

b $f(x) = 2x^3 - 6x$

c $f(x) = 2x^3 - 12x^2$

d $f(x) = x^3 - 2x^2 + x - 2$

e $f(x) = 3x^3 - x^2 - 7x - 3$

18 Example 15 Find two positive numbers, x and y, whose product is 48 such that $3x + y$ is a minimum.

Application

19 An avocado grower has 20 trees in his plot. Each tree produces 240 fruit. For each additional tree planted, the output per tree drops by 10 fruit. How many trees should be added to the existing plot in order to maximize the total output of the trees?

20 At 8 a.m. a ship going due west at 12 km/h is 10 km due north of a second ship going due north at 16 km/h.

a Find the time at which the ships are closest to each other.

b Find the minimum distance between them.

21 A block of land is bordered on one side by a straight stretch of river and on the other three sides by 640 metres of fencing. Find the dimensions of the block if its area is to be a maximum.

22 The area of a rectangle is128 cm^2. Find its dimensions so that the distance from one corner to the midpoint of a non-adjacent edge is a minimum.

23 A manufacturer wants to minimise the cost of materials used when making a closed rectangular box with a square base, which has a volume of 64 000 cm^3. Find the minimum total surface area of the box.

24 A manufacturer makes a batch of n items with the cost (in dollars) of each item being $C(n) = n^2 - 6n + 35$. The manufacturer sells the items for \$50 each. How many items should be produced in each batch to maximise profit?

25 The production cost of toy talking bears is given by $C(x) = 0.005x^2 - 2x + 250$, where x is the number of bears produced in a week. The number of bears that can be sold in a week is given by $x = 600 - 2d$, where d is the price in dollars of each bear.

a How many bears should be produced to make the greatest profit?

b What will the price of a bear then be?

c What will be the profit?

26 A skateboarder decides to do a ramp. The path of the jump can be approximated by $h = -3t^2 + 6t + 1$ and $d = 2t$, where h represents the height above the ground in metres, s is the horizontal displacement and t represents time after leaving the time ramp in seconds. Find the maximum height reached by the skateboarder.

Practice quiz

27 The population of locusts in an area changes as the temperature varies. In a certain area the population, P, can be modelled as $P = -\frac{1}{3}t^3 + 16.5t^2 + 70t + 5000$, where t is the air temperature in °C for $0° \le T \le 37°$. At what temperatures will the population of locust swarm be at its maximum size?

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