Rossmoyne Senior High School

Semester One Examination, 2018

Question/Answer booklet

MATHEMATICS METHODS UNIT 3

Section One: Calculator-free

SOLUTIONS

Name: MARKING GUIDE.

Teacher's Name: ____________________

Time allowed for this section

Reading time before commencing work:	five minutes
Working time:	fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet

Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	12	12	100	81	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

2 marks MAX for correct answer only.
-1 for incorrect rounding OVERALL
-1 for missing/incorrect units OVERALL
(See marking guide for the questions where units would be penalised)

Question 1 **(6 marks)**

(a) Solve exactly for x in the following: (3 marks)

$$10e^{x} = 2e^{2.1x}$$

ALTERNATE SOLUTION #1

$\ln 5e^{x} = \ln e^{2.1x}$ ✓

$\ln 5 + \ln e^{x} = \ln e^{2.1x}$ ✓

$\ln 5 + x = 2.1x$

$\ln 5 = 1.1x$

$x = \dfrac{\ln 5}{1.1}$ ✓

Solution
$5e^{x} = e^{2.1x}$ $5 = \dfrac{e^{2.1x}}{e^{x}}$ $5 = e^{1.1x}$ $\ln(5) = 1.1x$ $x = \dfrac{10\ln(5)}{11}$ OR $\dfrac{\ln 5}{1.1}$
Specific behaviours
✓ Uses index laws to simplify exponential terms ✓ Takes ln on both sides ✓ Solves exactly for x

ALTERNATIVE SOLUTION #2:

$\log_a 5e^{x} = \log_a e^{2.1x}$ ✓

$\log_a 5e^{x} + \log_a e^{x} = \log_a e^{2.1x}$

$-x\log_a e + 2.1x\log_a e = \log_a 5$ ✓

$1.1x\log_a e = \log_a 5$

$x = \dfrac{\log_a 5}{1.1\log_a e}$ ✓

where a = base number used

(b) If $log\ a = x^2$ and $log\ b = \frac{x^2}{2}$ determine $log\frac{\sqrt{a}}{b^2}$ (3 marks)

Solution
$\log\dfrac{\sqrt{a}}{b^2} = \log a - \log b^2$ $= \dfrac{1}{2}\log a - 2\log b$ $= \dfrac{1}{2}x^2 - \dfrac{2x^2}{2}$ $= -\dfrac{1}{2}x^2$
Specific behaviours
✓ Rewrites the expression using log laws ✓ Substitutes $\log a$ and $\log b$ into the expression ✓ Rearranges

Question 2 **(6 marks)**

Determine $\frac{dy}{dx}$ for the following, simplifying each answer.

(a) $y = \sqrt{8x+1}$. (2 marks)

Solution
$y = (8x+1)^{\frac{1}{2}}$ $\frac{dy}{dx} = \frac{1}{2}(8)(8x+1)^{-\frac{1}{2}}$ $= \frac{4}{\sqrt{8x+1}}$
Specific behaviours
✓ indicates use of chain rule ✓ correct derivative, simplified

← 1 mark penalty for not simplifying here.

(b) $y = 2x^5\cos(5x)$. (2 marks)

Solution
$\frac{dy}{dx} = 10x^4\cos(5x) + 2x^5(-5)\sin(5x)$ $= 10x^4\cos(5x) - 10x^5\sin(5x)$
Specific behaviours
✓ indicates use of product rule ✓ correct derivative of $\cos(5x)$

If not simplified do not penalise. Follow the behaviours

(c) $y = \int_x^3 t(1-t^2)^3\ dt$. (2 marks)

Solution
$\frac{dy}{dx} = -\int_3^x t(1-t^2)^3\ dt$ $= -x(1-x^2)^3$
Specific behaviours
✓ reverse limits ✓ correct derivative

Question 3 **(4 marks)**

A particle travels in a straight line so that its distance x cm from a fixed point O on the line after t seconds is given by

$$x = \frac{2t^3}{3t+1}, t \geq 0.$$

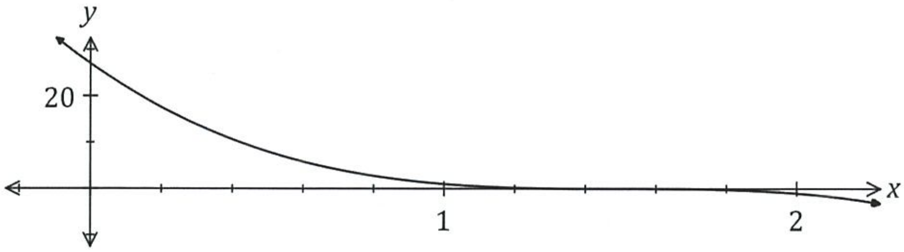
Calculate the velocity of the particle when $t = 1$.

Solution
$v = \dfrac{6t^2(3t+1) - 2t^3(3)}{(3t+1)^2}$ $= \dfrac{12t^3 + 6t^2}{(3t+1)^2}$ $v(1) = \dfrac{12(1)^3 + 6(1)^2}{(3(1)+1)^2}$ $= \dfrac{9}{8}$ cm/s
Specific behaviours
✓ correct form of quotient rule   ✓ simplifies expression for v ✓ substitutes   ✓ determines velocity at the given time

Units are deducted here.

Question 4 **(8 marks)**

The graph of $y = (3 - 2x)^3$ is shown below.

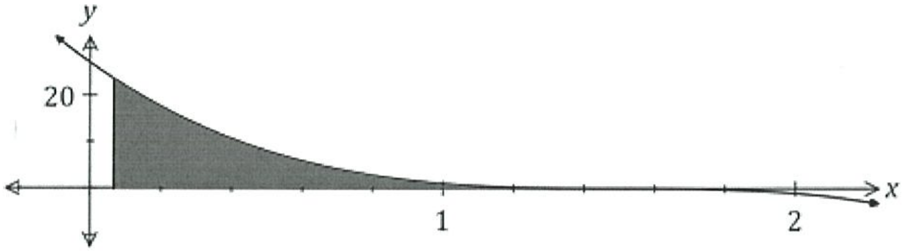


(a) Determine the area of the region enclosed by the curve and the coordinates axes. (4 marks)

Solution
$3 - 2x = 0 \Rightarrow x = 1.5$ $A = \int_0^{1.5} (3-2x)^3\,dx$ $= \left[\frac{(3-2x)^4}{-8}\right]_0^{1.5}$ $= (0) - \left(\frac{81}{-8}\right)$ $= \frac{81}{8}$ sq units
Specific behaviours
✓ writes integral with limits   ✓ antidifferentiates   ✓ expression with both limits substituted   ✓ correct area

Units are <u>not</u> deducted here.

(b) Given that the area of the region bounded by the curve, the x-axis and the line $x = k$ is 8 square units, determine the value of k, where $0 < k < 1.5$. (4 marks)



Solution
$A = \int_k^{1.5} (3-2x)^3\,dx \Rightarrow 8 = \left[\frac{(3-2x)^4}{-8}\right]_k^{1.5}$ $8 = (0) - \left(\frac{(3-2k)^4}{-8}\right)$ $(3-2k)^4 = 64$ $3 - 2k = \sqrt{8}$ $2k = 3 - 2\sqrt{2}$ $k = \frac{3}{2} - \sqrt{2}$ OR $\frac{3-\sqrt[4]{64}}{2}$
Specific behaviours
✓ sets up the appropriate integral ✓ writes an equation with the antiderivative ✓ substitutes limits ✓ value of k

1 mark penalty if students had $\frac{3}{2} \pm \sqrt{2}$.
$\frac{3}{2} + \sqrt{2}$ would exceed the boundaries for k.

Question 5 **(5 marks)**

(a) Determine $\frac{d}{dx}(x\,ln\,3x)$ (2 marks)

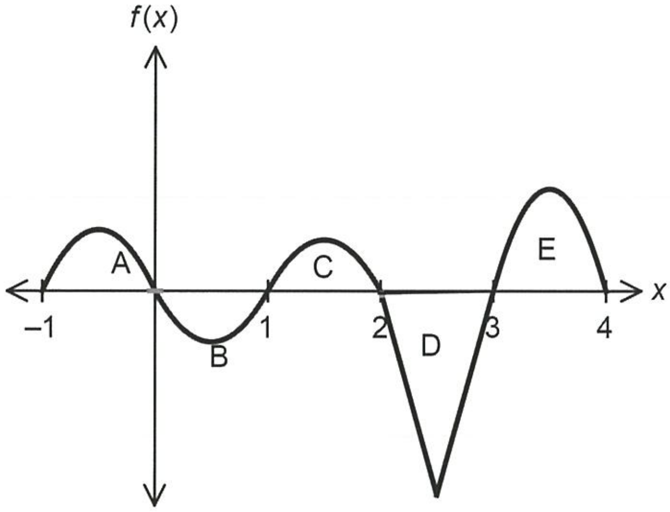
Solution
$$\frac{d}{dx}(x\ln 3x) = 1 \times \ln 3x + x \times \frac{1}{x}$$ $$= \ln 3x + 1$$ $\ln 3x + \left(\frac{1}{x} \times x\right)$ is ok
Specific behaviours
✓ uses the product rule ✓ determines the derivative

(b) Hence determine $\int ln\,3x\,dx$ (3 marks)

Solution
$$\int \frac{d}{dx}(x\ln 3x)\,dx = \int \ln 3x\,dx + \int 1\,dx$$ $$\int \ln 3x\,dx = \int \frac{d}{dx}(x\ln 3x)\,dx - \int 1\,dx$$ $$\int \ln 3x\,dx = x\ln 3x - x + c$$
Specific behaviours
✓ Applies the linearity of integration ✓ Uses Fundamental Theorem of Calculus ✓ Determines the integral with a constant

Question 6 **(6 marks)**

Consider the graph of $y = f(x)$ for $-1 \le x \le 4$.



It is known that:

- $\int_{-1}^{1} f(x)dx = 0$
- Areas C, D and E are 1, 5 and 4 units^2 respectively.
- When $x = 1.5$, $f(x) = 1$ and when $x = 3.5$, $f(x) = 2$

Units are not deducted here.

(a) Determine:

(i) $\int_{-1}^{4} f(x)dx$ (2 marks)

Solution
$$\int_{-1}^{4} f(x)dx = 1 - 5 + 4 = 0$$
Specific behaviours
✓ Uses signed area ✓ Determines the value of the integral

(ii) the area enclosed by the graph of $f(x)$ and the x-axis between 0 and 4 given that Area A = 3 units^2 (2 marks)

Solution
$3 + 1 + 5 + 4 = 13\ units^2$
Specific behaviours
✓ Recognises area must be positive ✓ Determines the required area

(b) Determine the value of $\int_{1.5}^{3.5} 2f'(x)dx$ (2 marks)

Solution
$$2\int_{1.5}^{3.5} f'(x)dx = 2[f(3.5) - f(1.5)] = 2(2 - 1) = 2$$
Specific behaviours
✓ Applies Fundamental Theorem ✓ Determines the required value

Question 7 **(12 marks)**

The function g is such that $g'(x) = ax^2 - 12x + b$, it has a point of inflection at $(1, -11)$ and a stationary point when $x = -1$

(a) Determine the values of $\boldsymbol{a}$ and $\boldsymbol{b}$. (4 marks)

Solution
$g''(x) = 2ax - 12$ $g''(1) = 2a - 12 = 0 \Rightarrow a = 6$ $g'(-1) = 0 \Rightarrow 6(-1)^2 - 12(-1) + b = 0$ $b = -18$
Specific behaviours
✓ Determines the second derivative ✓ value of a ✓ Uses the stationary point for the first derivative ✓ value of b

(b) Determine $g(x)$. (3 marks)

Solution
$g'(x) = 6x^2 - 12x - 18$ $g(x) = 2x^3 - 6x^2 - 18x + c$ $g(1) = -11 \Rightarrow 2 - 6 - 18 + c = -11$ $c = 11$ $g(x) = 2x^3 - 6x^2 - 18x + 11$
Specific behaviours
✓ Determines the antiderivative of $g'(x)$ ✓ Uses the point of inflection to find c ✓ Determines $g(x)$

(c) Determine the coordinates and nature of all the stationary points in $g(x)$ (5 marks)

Solution
$g'(x) = 6x^2 - 12x - 18$ $0 = 6(x-3)(x+1)$ $x = -1, x = 3$ $g''(-1) = -24, \quad \therefore Maximum$ $g''(3) = 24, \quad \therefore Minimum$ $g(-1) = 2(-1)^3 - 6(-1)^2 - 18(-1) + 11 = 21$ $g(3) = 2(3)^3 - 6(3)^2 - 18(3) + 11 = -43$ $\therefore Max\ T.P.\ at\ (-1, 21)\ and\ Min\ T.P.\ at\ (3, -43)$
Specific behaviours
✓ Determines the x-values of the stationary points ✓ Uses the second derivative to determine their concavity ✓ Determines the maximum ✓ Determines the minimum ✓ Determines the location of both points

} If this step is not included but the steps below have been written with the Max & Min identified give full marks.

Question 8 **(5 marks)**

The height, in metres, of a lift above the ground t seconds after it starts moving is given by

$$h = 4\cos^2\left(\frac{t}{7}\right).$$

Use the increments formula to estimate the change in height of the lift from $t = \frac{7\pi}{4}$ to $t = \frac{88\pi}{50}$.

Solution
$\frac{dh}{dt} = 4 \times 2 \times \cos\left(\frac{t}{7}\right) \times \frac{d}{dt}\left(\cos\left(\frac{t}{7}\right)\right)$ $= -\frac{8}{7}\cos\left(\frac{t}{7}\right)\sin\left(\frac{t}{7}\right)$ $\delta t = \frac{88\pi}{50} - \frac{7\pi}{4} = \frac{\pi}{100}$ $\delta h \approx -\frac{8}{7}\cos\left(\frac{7\pi}{4\times 7}\right)\sin\left(\frac{7\pi}{4\times 7}\right) \times \frac{\pi}{100}$ $\approx -\frac{8}{7} \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} \times \frac{\pi}{100}$ $\approx -\frac{\pi}{175}$ m
Specific behaviours
✓ correctly uses chain rule ✓ correct derivative ✓ increment of time ✓ substitutes correctly into increments formula ✓ fully simplifies

Units are deducted here.

Supplementary page

Question number: _________