SUGGESTED SOLUTIONS

Question 1 **(4 marks)**

Point A has position vector $\begin{pmatrix} 3 \\ -3 \\ -12 \end{pmatrix}$ and point B has position vector $\begin{pmatrix} 11 \\ 1 \\ 4 \end{pmatrix}$.

Determine the position vector of the point P that divides AB internally in the ratio 3:5.

$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{3}{8}\left(\overrightarrow{AB}\right)$$ $\frac{3}{8}$ ✓

$$= \begin{pmatrix} 3 \\ -3 \\ -12 \end{pmatrix} + \frac{3}{8}\begin{pmatrix} 11 - 3 \\ 1 - (-3) \\ 4 - (-12) \end{pmatrix}$$ $\overrightarrow{OA}$ ✓

$$= \begin{pmatrix} 3 \\ -3 \\ -12 \end{pmatrix} + \begin{pmatrix} 3 \\ 1.5 \\ 6 \end{pmatrix}$$ $+ \frac{3}{8}\overrightarrow{AB}$ ✓

$$= \begin{pmatrix} 6 \\ -1.5 \\ -6 \end{pmatrix}$$ ✓

Question 2 **(6 marks)**

Consider the three points $P(1,3,0)$, $Q(3,4,-3)$ and $R(3,6,2)$.
The three points are on the plane Π.

Determine the cartesian equation of the plane Π.

$$\vec{PQ} = \begin{pmatrix} 3 - 1 \\ 4 - 3 \\ -3 - 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$ ✓

$$\vec{PR} = \begin{pmatrix} 3 - 1 \\ 6 - 3 \\ 2 - 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$ ✓

$$\vec{n} = \vec{PQ} \times \vec{PR}$$

$$= \begin{vmatrix} i & j & k \\ 2 & 1 & -3 \\ 2 & 3 & 2 \end{vmatrix}$$

$$= \begin{pmatrix} \begin{vmatrix} 1 & -3 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & -3 \\ 2 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} \end{pmatrix}$$ ✓ Method

$$= \begin{pmatrix} 11 \\ -10 \\ 4 \end{pmatrix}$$ ✓ Answer

$\therefore \quad 11x - 10y + 4z = \alpha$ ✓

sub in $(1,3,0)$:

$11 - 30 + 0 = \alpha \quad , \quad \alpha = -19$

$\therefore \quad 11x - 10y + 4z = -19$ ✓

Question 3 **(5 marks)**

As part of a product recall, a shop removed all sizes of a variety of soup from its shelves. The soup was sold in 300 mL, 500 mL and 800 mL sizes for \$4.50, \$6.00 and \$7.50 respectively. The total volume of soup in all 42 cans removed was 25 L and the value of these cans was to \$267.

If x 300 mL cans, y 500 mL cans and z 800 mL cans were removed, then some of the above information can be expressed by the equations $3x + 4y + 5z = 178$ and $x + y + z = 42$.

Write down a third equation from the information and use it to find how many of each size of can were removed.

$4.5x + 6y + 7.5z = 267$

$\Downarrow$

$3x + 4y + 5z = 178$

$\therefore$ The third equation is

$0.3x + 0.5y + 0.8z = 25$ ✓

$\therefore 3x + 5y + 8z = 250$

✓ Method

$$\begin{cases} 3x + 4y + 5z = 178 \\ 3x + 5y + 8z = 250 \\ x + y + z = 42 \end{cases}$$

$$\begin{cases} y + 2z = 52 \\ 2y + 5z = 124 \\ x + y + z = 42 \end{cases}$$

✓ 2 vars.

$$\left.\begin{aligned} 2y + 4z = 104 \\ 2y + 5z = 124 \end{aligned}\right\}$$

$\therefore z = 20$ ✓

$y = 12$

$x = 10$ ✓

$\therefore$ 10 300 mL cans, 12 500 mL cans and 20 800 mL cans were removed.

End of questions

Question 4 **(8 marks)**

Let Π be the plane given by the cartesian equation $4x - 3y + z = 42$.

(a) Determine a vector equation for the plane in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$ (5 marks)

Three position vectors on the plane:

$$\vec{OA} = \begin{pmatrix} 0 \\ 0 \\ 42 \end{pmatrix}, \quad \vec{OB} = \begin{pmatrix} 0 \\ -14 \\ 0 \end{pmatrix}, \quad \vec{OC} = \begin{pmatrix} 10.5 \\ 0 \\ 0 \end{pmatrix}$$

✓ 3 points

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{OA} + \lambda \vec{AB} + \mu \vec{AC}$$

✓ vector in plane

$$= \begin{pmatrix} 0 \\ 0 \\ 42 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -14 \\ -42 \end{pmatrix} + \mu \begin{pmatrix} 10.5 \\ 0 \\ -42 \end{pmatrix}$$

Answers

✓ ✓ ✓

Let L be the line given by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$.

(b) Determine the acute angle between the plane Π and the line L. (3 marks)

Angle between normal to the plane and the line

$$\theta = \cos^{-1}\left(\frac{\begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}}{\left|\begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}\right| \times \left|\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}\right|} \right) \approx 100.3^\circ$$

✓ $\cos^{-1}$

$\therefore$ Acute angle between Π and L is about

$90^\circ - 79.7^\circ = 10.3^\circ$ (1 dp.)

✓ ✓

Question 5 **(6 marks)**

Let S be the sphere defined by the equation $(x-4)^2 + (y+4)^2 + (z-3)^2 = 4$.
Let Π be the plane defined by the equation $2x - 2y + 5z = -8$.

The distance between a sphere and a plane that do not intersect is defined as the shortest distance between the two objects.

Determine the distance between S and Π.

Centre of $S = \begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix}$ ✓

Perpendicular line from Π to centre of S:

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}$ ✓

✓

$\therefore \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix} \left(\begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix} \right) = -8$

$\Downarrow$ ✓

$\lambda = \frac{-13}{11}$

✓

$\therefore$ Distance from S to Π $= \frac{13}{11} \left\| \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix} \right\| - 2$

$= \frac{13\sqrt{33}}{11} - 2$

≈ 4.79 ✓

Question 6 **(5 marks)**

Two gravity-defying drones, Drone A and Drone B, travel in space along straight lines.

Drone A starts at the point represented by the position vector $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ km.

Drone B starts at the point represented by the position vector $\begin{pmatrix} 5 \\ 28 \\ -6 \end{pmatrix}$ km.

Drone A has a velocity of $\begin{pmatrix} 7 \\ 10 \\ -3 \end{pmatrix}$ km/h and Drone B has a velocity of $\begin{pmatrix} 6 \\ 1 \\ -2 \end{pmatrix}$ km/h.

(a) Show that the two drones will collide. (3 marks)

$$\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + t\begin{pmatrix} 7 \\ 10 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 28 \\ -6 \end{pmatrix} + t\begin{pmatrix} 6 \\ 1 \\ -2 \end{pmatrix}$$

3 eqns ✓

$\Downarrow$

$t = 3$

$t=3$ ✓

$\therefore$ all three consistent! ✓

$\therefore$ The two drones will collide after 3 hours.

(b) Determine the distance travelled by Drone A until they collide. (2 marks)

$$\left| 3\begin{pmatrix} 7 \\ 10 \\ -3 \end{pmatrix} \right| = \sqrt{1422}$$

✓ correct dist

$\therefore$ The distance travelled is about 37.7 km ✓

Question 7 **(6 marks)**

Consider the following system of equations:

$$\begin{aligned} x + y + z &= 2 \\ x - y + 2z &= 7 \\ 3x - 3y + pz &= q \end{aligned}$$

Determine the possible values of p and q such that the system of equations has

(i) a unique solution,
(ii) no solution,
(iii) an infinite number of solutions.

$$\left|\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 7 \\ 3 & -3 & p & q \end{array}\right| \begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array}$$

$$\left|\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & 5 \\ 0 & 0 & p-6 & q-24 \end{array}\right| \begin{array}{l} \\ r_2' = r_2 - r_1 \\ r_3' = r_3 - 3r_2 \end{array}$$

✓ M
✓ M

$\therefore \ (p-6)z = q-24$, so

✓ Must show

✓

Unique solution for $p \neq 6$.

No solution for $p = 6$, $q \neq 24$. ✓

Infinite number of solutions for $p = 6$, $q = 24$. ✓

Question 8 **(5 marks)**

Find the equation of the line of intersection of the two planes whose equations are

$r \cdot \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} = 6$ and $r \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 4$ respectively.

$x + y - 3z = 6$ ①

$2x - y + z = 4$ ②

✓ cartesian

$3x - 2z = 10$ ① + ②

$\therefore\ 2z = 3x - 10$

$z = 1.5x - 5$

✓ Method (one variable)

Substitute $z = 1.5x - 5$ into ② :

$2x - y + (1.5x - 5) = 4$

$3.5x - y = 9$

$y = 3.5x - 9$

✓ Method (another variable)

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 3.5x - 9 \\ 1.5x - 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -9 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3.5 \\ 1.5 \end{pmatrix}$$

✓ ✓

End of questions