Semester 1 2016

PENRHOS COLLEGE

STRIVE FOR THE HIGHEST

Year 12 Mathematics Specialist Unit 3

Test 1 - Complex numbers and functions

Calculator Free

Time: 30 minutes Marks: 34

Name: _____

Equipment Allowed: **Formula sheet, ruler, pens, pencils, eraser.**

SHOW ALL APPROPRIATE WORKING IN THE SPACES PROVIDED.

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1. (4 marks)

(a) Given $z = \sqrt{3} + i$ evaluate z^6, giving the answer in Cartesian form.

[2]

(b) Solve $x^2 - 6x + 12 = 0$ for $x \in \mathrm{Im}$, in exact form.

[2]

2. (4 marks)

If $f(x)=\frac{3}{x}$ and $g(x)=2x-1$ find:

(a) (in terms of x)

(i) $fog(x)$

(ii) $gof(x)$

[2]

(b) State the natural domain and range of each function in (a) above

[2]

3. (5 marks)

(a) State the natural domain and corresponding range of the function $f(x)=\frac{1}{\sqrt{x+3}}+4$.

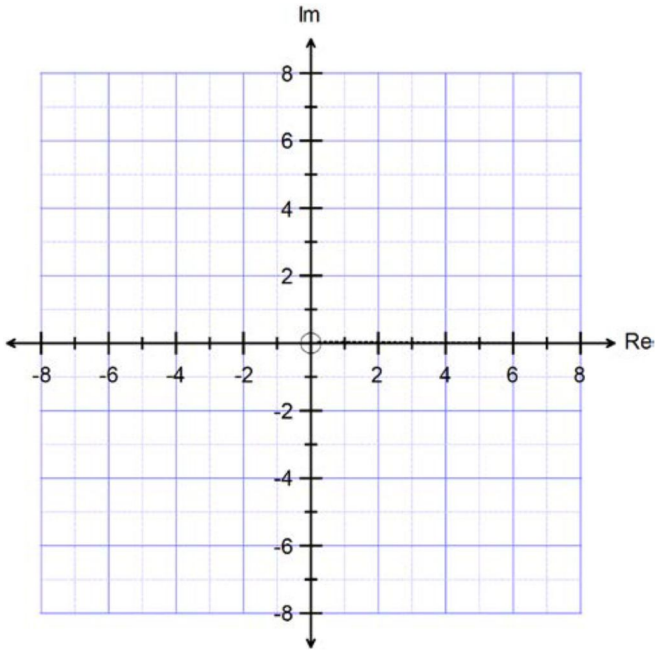
[1]

(b) Find an expression for $f^{-1}(x)$, the inverse of $f(x)$ and state the domain and range of $f^{-1}(x)$.

[4]

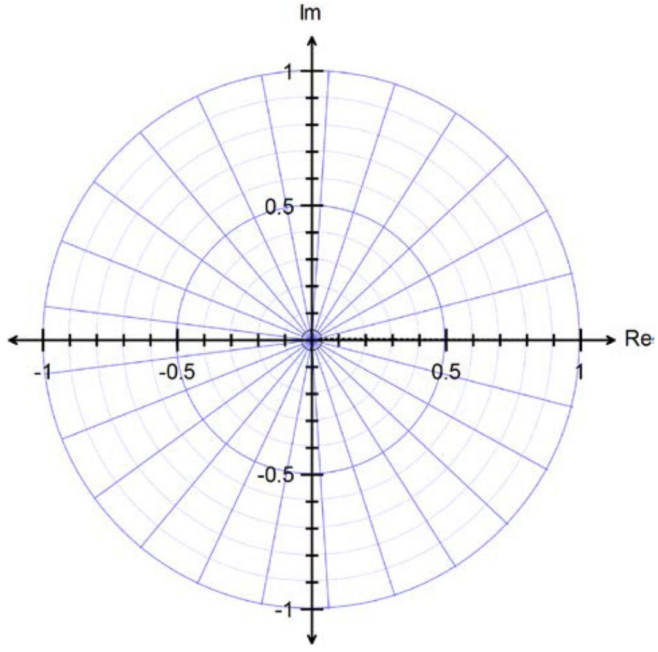
4. (4 marks)

Sketch the set of points defined by $|z-(2+3i)|=\sqrt{13}$.



5. (3 marks)

Show all solutions in the Argand plane below to $z^5=1$.



6. (8 marks)

Given $H(z) = z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8$

(a) Evaluate $H(i)$, $H(-i)$ and $H(2)$.

[3]

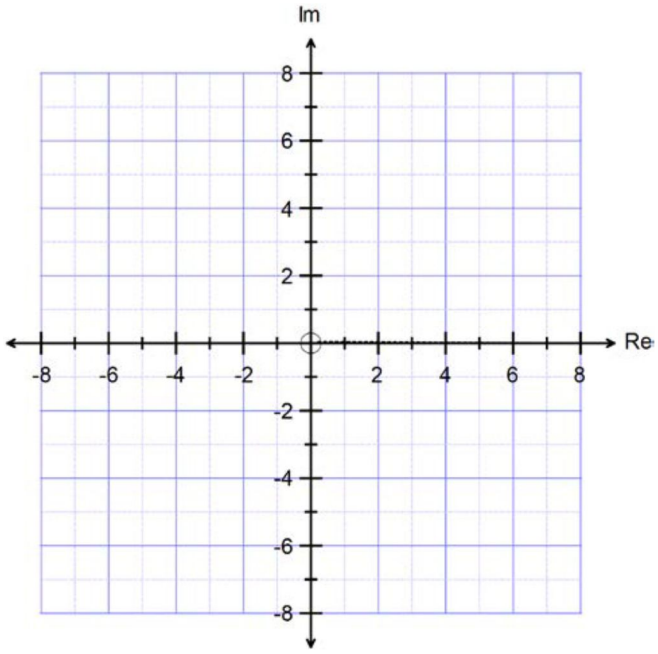
(b) Hence, find all roots of the equation $z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8 = 0$.

[5]

7. (2 marks)

Sketch on an Argand Diagram the locus of the point $z = x + yi$ satisfying the following conditions:

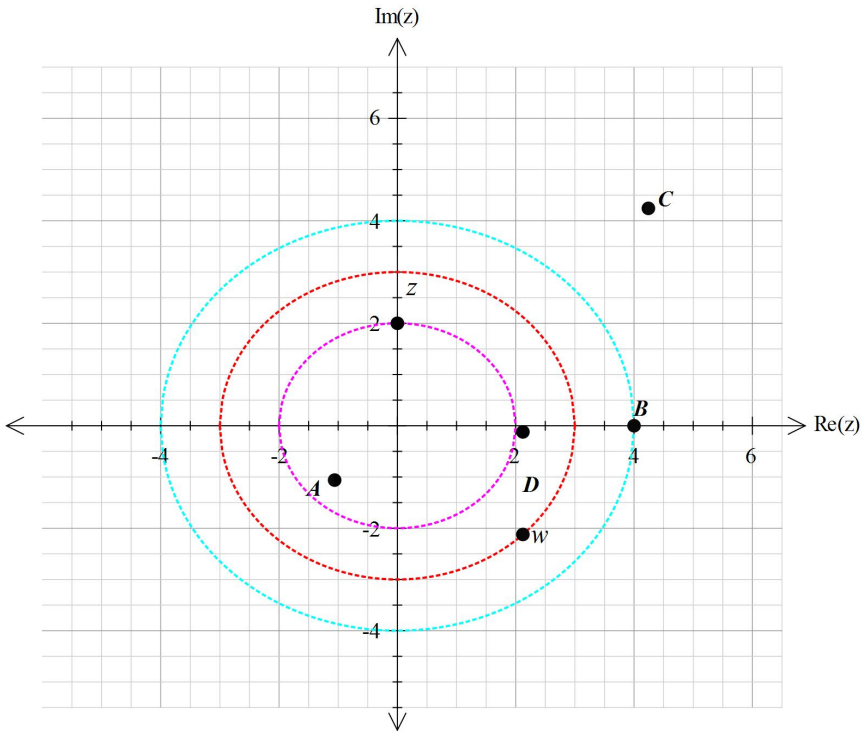
$$\frac{\pi}{4} \leq \arg(z) \leq \frac{3\pi}{4} \text{ and } 4 \leq |z| \leq 6$$



8. (4 marks)

Given the position of *z* and *w* on the Argand diagram below. Label each of the points A, B, C and D using one the following options for each point.

$w + z$ $\quad wz$ $\quad z\bar{z}$ $\quad \frac{w}{z}$ $\quad z^2$



Semester 1 2016

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Year 12 Mathematics Specialist Unit 3

Test 1 - Complex numbers and functions

Calculator Assumed

Time: 20 minutes Marks: 20

Name: _____

Equipment Allowed: **Formula sheet, one page of A4 notes back and front, CAS calculator and/or scientific calculator, pens, pencils, ruler, eraser.**

SHOW ALL APPROPRIATE WORKING IN THE SPACES PROVIDED.

9. (8 marks)

(a) Expand and simplify the expression $F(\theta) = (\cos\theta + i\sin\theta)^5$.

[2]

(b) Hence, express the Re(F) in terms of $\cos\theta$.

[2]

(c) Use Re(F) to solve the equation $16x^5 - 20x^3 + 5x - 1 = 0$ and express the solutions in trigonometric form.

[4]

10. (8 marks)

Given $z = \cos\theta + i\sin\theta$:

(a) Express $\dfrac{\left(z - \dfrac{1}{z}\right)}{i\left(z + \dfrac{1}{z}\right)}$ in trigonometric form.

[4]

(b) Show $z^2 + \dfrac{1}{z^2} = 2\cos 2\theta$ and hence prove $\cos 2\theta = 2\cos^2\theta - 1$.

[4]

11. (4 marks)

A university environmental team are studying the cost to remove contaminated soil. The function $C = \frac{15p}{1.1 - p}$ models the estimated cost (C in 1000s) to remove the percentage (p as a decimal) of the contaminated soil.

(a) State a reasonable domain and range for the function.

[2]

(b) Use a graph of the function, or otherwise, to find the percentage of the contaminated soil that could be removed on a budget of $75 000.

[2]

Semester 1 2016

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Test 1 - Complex numbers and functions

Calculator Free

Time: 30 minutes

Marks: 34

Name: Solutions.

Equipment Allowed: **Formula sheet, ruler, pens, pencils, eraser.**

SHOW ALL APPROPRIATE WORKING IN THE SPACES PROVIDED.

1. (4 marks)

(a) Given $z = \sqrt{3} + i$ evaluate z^6, giving the answer in Cartesian form.

[2]

$z = 2\,\text{cis}\,\frac{\pi}{6}$ $\qquad z^6 = \left(2\,\text{cis}\,\frac{\pi}{6}\right)^6$

$= 64\,\text{cis}\,\pi$ ✓

$\therefore \quad z^6 = -64$ ✓

(b) Solve $x^2 - 6x + 12 = 0$ for $x \in \text{Im}$, in exact form.

[2]

$x = \frac{6 \pm \sqrt{6^2 - 4 \times 1 \times 12}}{2 \times 1}$ ✓

$= \frac{6 \pm \sqrt{-12}}{2}$

$= 3 \pm \sqrt{3}\,i$ ✓

2. (4 marks)

If $f(x)=\frac{3}{x}$ and $g(x)=2x-1$ find:

(a) (in terms of x)

(i) $fog(x) = \frac{3}{2x-1}$ ✓

(ii) $gof(x) = 2\left(\frac{3}{x}\right)-1$

$= \frac{6}{x}-1$ ✓ [2]

(b) State the natural domain and range of each function in (a) above

(i) fog

D: $x \neq \frac{1}{2}$ ✓

R: $y \neq 0$

gof

D: $x \neq 0$ ✓

R: $y \neq -1$.

(-1 per error) [2]

3. (5 marks)

(a) State the natural domain and corresponding range of the function $f(x)=\frac{1}{\sqrt{x+3}}+4$.

D: $x+3>0$

$x>-3$

R: $y>4$ ✓ [1]

(b) Find an expression for $f^{-1}(x)$, the inverse of $f(x)$ and state the domain and range of $f^{-1}(x)$.

$x = \frac{1}{\sqrt{y+3}} + 4$ ✓ [4]

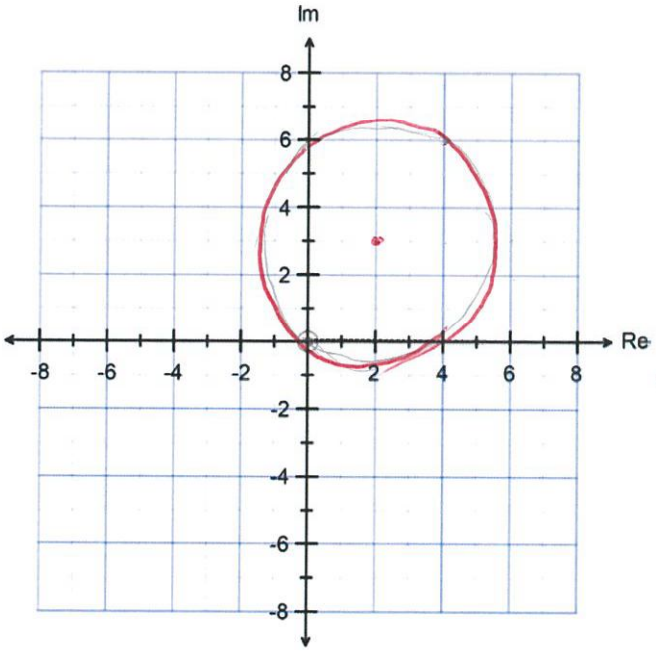
$\frac{1}{(x-4)^2} - 3 = y$ ✓

D: $x>4$ ✓

R: $y>-3$ ✓

4. (4 marks)

Sketch the set of points defined by $|z-(2+3i)|=\sqrt{13}$.



Circle ✓

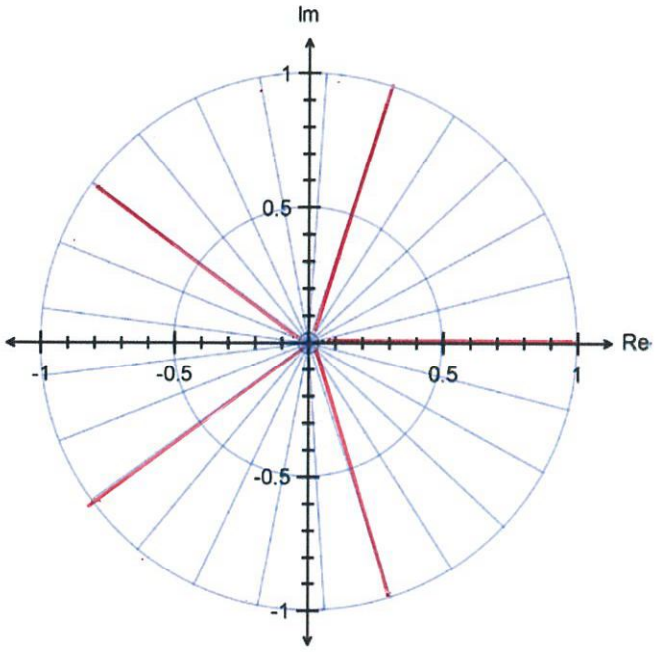
centre (2, 3) ✓

radian $\sqrt{13}$ ✓

through the point (0,0) ✓ or similar.

5. (3 marks)

Show all solutions in the Argand plane below to $z^5=1$.



✓ first solution $z_1 = 1$

✓ solutions every $\frac{2\pi}{5}$

✓ all solutions correct.

6. (8 marks)

Given $H(z) = z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8$

(a) Evaluate $H(i)$, $H(-i)$ and $H(2)$. [3]

$H(i) = 0$ ✓

$H(-i) = 0$ ✓

$H(2) = 0$ ✓

(b) Hence, find all roots of the equation $z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8 = 0$. [5]

From above $(z+i)$, $(z-i)$, $(z-2)$ are all factors. ✓

and from $(z+i)(z-i) = z^2+1$ is also a factor.

$$z^2+1 \overline{)\, z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8}$$

$$= z^3 - 2z^2 + 4z - 8$$

$\therefore (z^2+1)(z^3 - 2z^2 + 4z - 8)$ ✓

$(z-2)$ is also a factor

so

$$z-2 \overline{)\, z^3 - 2z^2 + 4z - 8}$$

$$= z^2 + 4$$

$\therefore z^5 - 2z^4 + 5z^2 - 10z^2 + 4z - 8$

$= (z^2+1)(z-2)(z^2+4)$ ✓

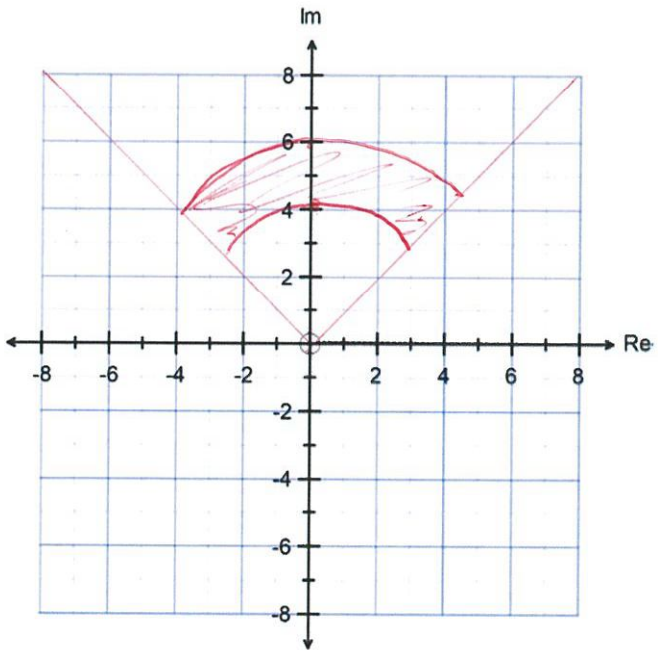
ie $(z-i)(z-i)(z-2)(z-2i)(z+2i) = 0$

$\therefore z = \pm i, 2, \pm 2i$ ✓✓

7. (2 marks)

Sketch on an Argand Diagram the locus of the point $z = x + yi$ satisfying the following conditions:

$$\frac{\pi}{4} \leq \arg(z) \leq \frac{3\pi}{4} \text{ and } 4 \leq |z| \leq 6$$

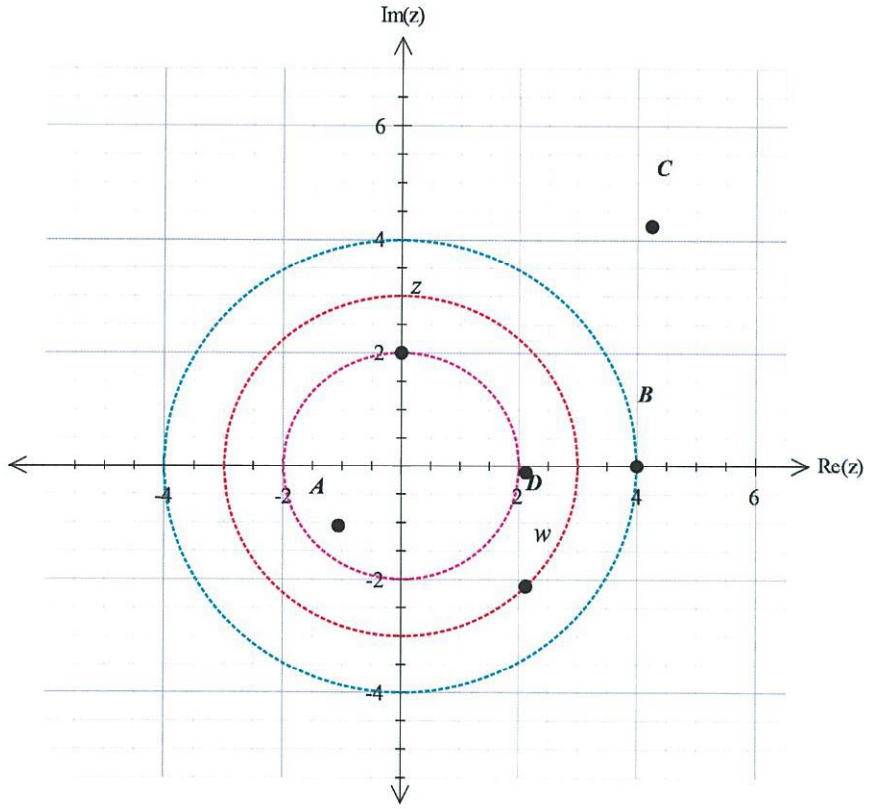


√√ (+1 per error)

8. (4 marks)

Given the position of *z* and *w* on the Argand diagram below. Label each of the points A, B, C and D using one the following options for each point.

$w+z$ wz $z\bar{z}$ $\frac{w}{z}$ z^2



A: $\frac{w}{z}$

B: $z\bar{z}$

C: wz

D: $w+z$

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SHOW ALL APPROPRIATE WORKING IN THE SPACES PROVIDED.

9. (8 marks)

(a) Expand and simplify the expression $F(\theta) = (\cos\theta + i\sin\theta)^5$. [2]

$F(\theta) = (\cos\theta + i\sin\theta)^5$ ✓

$= \cos^5\theta + 5\cos\theta\sin^4\theta - 10\cos^3\theta\sin^2\theta)$ ✓

$+ i(\sin^5\theta + 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta)$

(b) Hence, express the Re(F) in terms of $\cos\theta$. [2]

$Re(F) = \cos^5\theta + 5\cos\theta(1-\cos^2\theta)^2 - 10\cos^3\theta(1-\cos^2\theta)$

$= \cos^5\theta + 5\cos\theta(1-2\cos^2\theta+\cos^4\theta) - 10\cos^3\theta + 10\cos^5\theta$ ✓

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ ✓

(c) Use Re(F) to solve the equation $16x^5 - 20x^3 + 5x - 1 = 0$ and express the solutions in trigonometric form. [4]

$F(\theta) = (\cos\theta + i\sin\theta)^5$

$= \cos 5\theta + i\sin 5\theta$

$Re(F) = \cos 5\theta$ ✓

$\therefore \cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

$0 = 16x^5 - 20x^3 + 5x - 1$

$1 = 16x^5 - 20x^3 + 5x - 1$ ✓

$\therefore \cos 5\theta = 1$

$5\theta = 0, 2\pi, \ldots$ ✓

$\therefore x = \cos\left(\frac{2\pi n}{5}\right) \quad n = 0, 1, 2, 3, 4$ ✓

10. (8 marks)

Given $z = \cos\theta + i\sin\theta$:

$z = \cos\theta + i\sin\theta$

$z^{-1} = \frac{1}{z} = \cos\theta - i\sin\theta$

(a) Express $\dfrac{\left(z - \frac{1}{z}\right)}{i\left(z + \frac{1}{z}\right)}$ in trigonometric form.

[4]

$= \dfrac{(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)}{i\,((\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta))}$ ✓ ✓

$= \dfrac{2i\sin\theta}{2i\cos\theta}$ ✓

$= \tan\theta$ ✓

(b) Show $z^2 + \frac{1}{z^2} = 2\cos 2\theta$ and hence prove $\cos 2\theta = 2\cos^2\theta - 1$.

[4]

$z^2 + \frac{1}{z^2} = \cos 2\theta + i\sin 2\theta + \cos 2\theta - i\sin 2\theta$

$= 2\cos 2\theta$ ✓

$z^2 + \frac{1}{z^2} = (\cos\theta + i\sin\theta)^2 + (\cos\theta - i\sin\theta)^2$

$= \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta + \cos^2\theta - 2i\sin\theta\cos\theta - \sin^2\theta$

$= 2\cos^2\theta - 2\sin^2\theta$ ✓

$= 2\cos^2\theta - 2(1 - \cos^2\theta)$

$= 4\cos^2\theta - 2$

$= 2(2\cos^2\theta - 1)$ ✓

$\therefore \quad 2\cos 2\theta = 2(2\cos^2\theta - 1)$ ✓

$\cos 2\theta = 2\cos^2\theta - 1$ ✓

11. (4 marks)

A university environmental team are studying the cost to remove contaminated soil. The function $C = \frac{15p}{1.1 - p}$ models the estimated cost (C in 1000s) to remove the percentage (p as a decimal) of the contaminated soil.

(a) State a reasonable domain and range for the function.

[2]

D: $0 \leq p \leq 1$ ✓

R: $0 \leq C \leq 150$ ✓

(b) Use a graph of the function, or otherwise, to find the percentage of the contaminated soil that could be removed on a budget of $75 000.

[2]

$p = 0.91\dot{6}$ ✓

$\therefore$ 92% of the contaminated soil. ✓