Question 1 **(6 marks)**

$$\frac{3x^2+2x+5}{(x+1)(x^2+2)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+2)}$$, where A, B and C are real numbers.

(a) Determine the values of A, B and C. (4 marks)

$$3x^2+2x+5 = (x^2+2)A + (Bx+C)(x+1)$$
$$= Ax^2 + 2A + Bx^2 + Bx + Cx + C$$
$$= (A+B)x^2 + (B+C)x + (2A+C)$$

✓ set up equation

$$\therefore \begin{cases} A + B = 3 \\ B + C = 2 \\ 2A + C = 5 \end{cases}$$

✓ compare coefficients

$$\therefore \begin{cases} A - C = 1 \\ 2A + C = 5 \end{cases}$$

✓ Attempts to solve

$\therefore \quad 3A = 6$

$\therefore \quad A = 2$
$B = 1$
$C = 1$

✓ All three correct

(b) Hence, given that $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right), a \neq 0,$

determine $\int \frac{3x^2+2x+5}{(x+1)(x^2+2)}\,dx$ (2 marks)

$$\int \left(\frac{2}{x+1} + \frac{x}{x^2+2} + \frac{1}{x^2+2}\right) dx$$

$$= 2\ln|x+1| + \frac{1}{2}\ln(x^2+2) + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

✓ for any two correct

✓ for all correct

Question 2 **(5 marks)**

A particle with position vector $\mathbf{r}(t)$ (in metres) moves such that its velocity (in metres per second) at time t seconds, $t \geq 0$, is given by

$$\mathbf{v}(t) = -4\sin(2t)\,\mathbf{i} + 6\cos(2t)\,\mathbf{j}$$

(a) Given that $\mathbf{r}(0) = 2\mathbf{i}$, determine an expression for the position of the particle at time t. (3 marks)

$$r(t) = \begin{pmatrix} 2\cos(2t) + c_1 \\ 3\sin(2t) + c_2 \end{pmatrix}$$ ✓

$$r(0) = \begin{pmatrix} 2 + c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \text{ so } c_1 = c_2 = 0$$ ✓

$$\therefore\ r(t) = \begin{pmatrix} 2\cos(2t) \\ 3\sin(2t) \end{pmatrix}$$ ✓

(b) Determine the cartesian equation of the path followed by the particle. (2 marks)

$$\begin{cases} x = 2\cos(2t) \\ y = 3\sin(2t) \end{cases}$$

$$\begin{cases} x^2 = 4\cos^2(2t) \\ y^2 = 9\sin^2(2t) \end{cases}$$ ✓ squaring

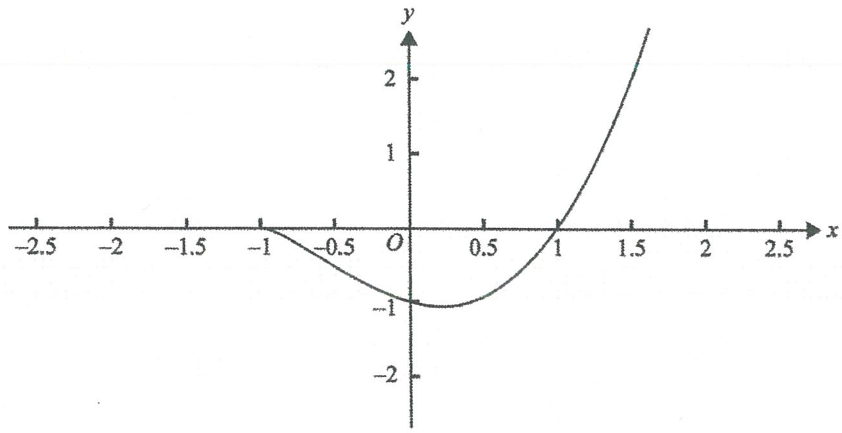
$$\therefore\ 9x^2 + 4y^2 = 36\left(\cos^2(2t) + \sin^2(2t)\right)$$

$$\therefore\ 9x^2 + 4y^2 = 36$$ ✓

$$\therefore\ \frac{x^2}{4} + \frac{y^2}{9} = 1$$

Question 3 **(4 marks)**

Part of the graph with equation $y = (x^2 - 1)\sqrt{x+1}$ is shown below.



Determine the area of the region bounded by the curve and the x-axis.

Give your answer in the form $\frac{a\sqrt{b}}{c}$, where a, b and c are integers.

$$\int_{-1}^{1} (x^2-1)\sqrt{x+1}\,dx$$

Subsitution:

$t = x+1$ ✓ correct substitution $t = x+1$

$\frac{dt}{dx} = 1$

$x = t-1$ ✓ translate all to t variable

$$= \int_0^2 \left((t-1)^2 - 1\right)\sqrt{t}\,dt$$

$$= \int_0^2 (t^2 - 2t)\,t^{\frac{1}{2}}\,dt$$

$$= \int_0^2 \left(t^{\frac{5}{2}} - 2t^{\frac{3}{2}}\right) dt$$

$$= \left[\frac{2}{7}t^{\frac{7}{2}} - \frac{4}{5}t^{\frac{5}{2}}\right]_0^2$$ ✓ correct antiderivative

$$= \frac{2}{7}2^{\frac{7}{2}} - \frac{4}{5}2^{\frac{5}{2}}$$

$$= \frac{2}{7}\times 8\sqrt{2} - \frac{4}{5}\times 4\sqrt{2}$$

$$= \left(\frac{5\times 16 - 7\times 16}{35}\right)\sqrt{2}$$

$$= -\frac{32}{35}\sqrt{2}$$

$\therefore$ Area $= \frac{32}{35}\sqrt{2}$ ✓

End of questions

Question 4 **(4 marks)**

Let S be the curve in the cartesian plane defined by $\mathbf{r}(t) = (2 - t)\,\mathbf{i} + (3 + t^2)\,\mathbf{j}$.
Let T be the curve in the cartesian plane defined by $y = 6x - 10$.

Determine the coordinates of the points in $S \cap T$, the intersection of S and T.

$S:\quad \begin{cases} x = 2 - t \\ y = 3 + t^2 \end{cases}$

$\therefore\quad y = 3 + (x-2)^2$ ✓ substitution

$= 3 + x^2 - 4x + 4$

$= x^2 - 4x + 7$

$S \cap T:\quad x^2 - 4x + 7 = 6x - 10$ ✓ equating

$x^2 - 10x + 17 = 0$

using CAS:

$x = -2\sqrt{2} + 5$ or $x = 2\sqrt{2} + 5$ ✓ solve for x

I.e. S and T intersect at $(-2\sqrt{2} + 5, -12\sqrt{2} + 20)$

and $(2\sqrt{2} + 5, 12\sqrt{2} + 20)$

✓ two correct coordinates

✓ all four correct

Question 5 **(8 marks)**

Consider the function $f: [0, \infty) \to \mathbb{R}$, where $f(x) = \dfrac{6x\sqrt{x}}{3x^2 + 1}$

The graph of f is rotated about the x-axis between $x = 0$ and $x = \frac{1}{\sqrt{3}}$ to form a solid of revolution with volume V.

(a) Show that $V = 2\pi \displaystyle\int_0^{\frac{1}{\sqrt{3}}} \frac{18x^3}{(3x^2+1)^2}\, dx$ (1 mark)

$$V = \pi \int_0^{\frac{1}{\sqrt{3}}} (f(x))^2\, dx = \pi \int_0^{\frac{1}{\sqrt{3}}} \frac{36x^3}{(3x^2+1)^2}\, dx = 2\pi \int_0^{\frac{1}{\sqrt{3}}} \frac{18x^3}{(3x^2+1)^2}\, dx \quad \checkmark$$

(b) Use the substitution $u = 3x^2 + 1$ to express V in the form $2\pi \displaystyle\int_a^b \left(\frac{c}{u} + \frac{d}{u^2}\right) du$ (4 marks)

$$2\pi \int_0^{\frac{1}{\sqrt{3}}} \frac{18x^3}{(3x^2+1)^2}\, dx$$

$$\begin{cases} u = 3x^2 + 1 \\ \dfrac{du}{dx} = 6x \end{cases} \qquad \therefore 3x^2 = u - 1 \quad \checkmark \qquad \therefore du = 6x\, dx \quad \checkmark$$

$$= 2\pi \int_1^2 \frac{u-1}{u^2}\, du$$

$$= 2\pi \int_1^2 \left(\frac{1}{u} - \frac{1}{u^2}\right) du$$

✓ correct limits

✓ correct c and d

(c) **Hence**, by using an appropriate antiderivative, determine the exact value of V. (3 marks)

$$2\pi \int_1^2 \left(\frac{1}{u} - \frac{1}{u^2} \right) du$$

$$= 2\pi \left[\ln u + \frac{1}{u} \right]_1^2$$ ✓ correct antiderivative

$$= 2\pi \left(\left(\ln 2 + \frac{1}{2} \right) - \left(\ln 1 + \frac{1}{1} \right) \right)$$ ✓ correct substitution

$$= 2\pi \left(\ln 2 - \frac{1}{2} \right)$$ ✓ correct answer

Question 6 **(11 marks)**

The position (in metres) of a projectile at time t seconds, $t \geq 0$, is given by

$$\mathbf{r}(t) = 400t\,\mathbf{i} + (500t - 5t^2)\,\mathbf{j}$$

The projectile is fired from a point on the ground.

(a) Find the time taken for the projectile to reach the ground again. (2 marks)

$500t - 5t^2 = 0$ ✓

$5t(100 - t) = 0$

$t = 0, 100$

$\therefore$ 100 seconds ✓

(b) Determine the speed at which the projectile hits the ground. (3 marks)

$\dot{r}(t) = \begin{pmatrix} 400 \\ 500 - 10t \end{pmatrix}$ ✓

$\dot{r}(100) = \begin{pmatrix} 400 \\ -500 \end{pmatrix}$ ✓

$\therefore \left| \begin{pmatrix} 400 \\ -500 \end{pmatrix} \right| \approx 640.31$ m/s ✓

(c) Determine the maximum height of the projectile. (2 marks)

$500 - 10t = 0$ ✓

$t = 50$

$\therefore 500 \times 50 - 5 \times 50^2 = 12500$ m ✓

The distance travelled by the projectile between times $t = a$ and $t = b$ is given by

$$\int_a^b |\mathbf{v}(t)|\, dt$$

where $|\mathbf{v}(t)|$ is the speed of the projectile at time t.

(d) Find the time taken from when the projectile is fired from a point on the ground to it has completed 80% of the total distance travelled by the projectile. (4 marks)

$$v(t) = \begin{pmatrix} 400 \\ 500 - 10t \end{pmatrix}$$

$$|v(t)| = \sqrt{400^2 + (500 - 10t)^2} \quad ✓$$

$$\text{total distance travelled} = \int_0^{100} \sqrt{400^2 + (500 - 10t)^2}\, dt$$

$$\approx 48777 \text{ m} \quad ✓$$

$\therefore$ 80% of total distance travelled ≈ 39022 m

Solve using CAS: $\displaystyle\int_0^T \sqrt{400^2 + (500-10t)^2}\, dt = 39022$ ✓

$$\Downarrow$$

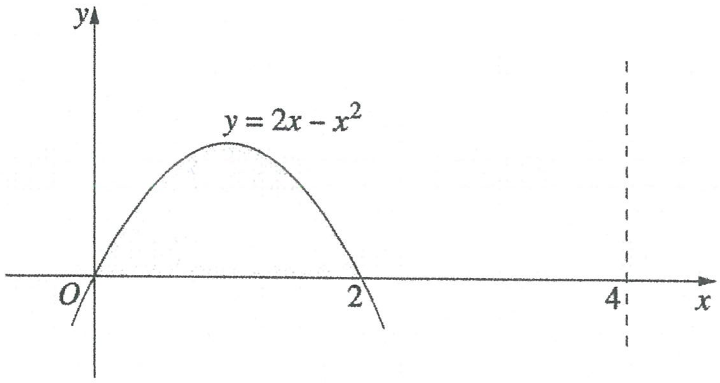
$$T \approx 83.1$$

$\therefore$ Approximately 83.1 seconds. ✓

Question 7 **(6 marks)**

The shaded region in the diagram below is bounded by the x-axis and the curve $y = 2x - x^2$.

The shaded region is rotated about the line $x = 4$ to form a solid.

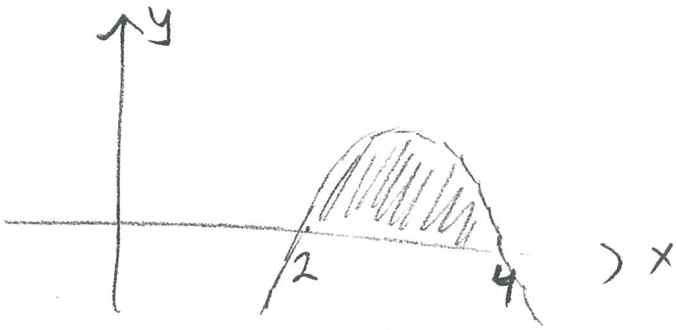


Determine the volume of the solid.

Let $f(x) = 2x - x^2$

The question is then equivalent to rotating $f(x-2)$ around the y-axis:

✓ translation



$$f(x-2) = 2(x-2) - (x-2)^2$$
$$= -x^2 + 6x - 8$$

✓ calculation

$$\therefore \text{Volume} = \int_2^4 (2\pi x)\, y \, dx$$

✓ limits
✓ $(2\pi x)$
✓ y

$$= 2\pi \int_2^4 (-x^3 + 6x^2 - 8x)\, dx$$
$$= 2\pi \times 4$$
$$= 8\pi$$

✓

End of questions