Question 1 **(6 marks)**

$$\frac{x-5}{x^2-5x+6} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$$, where A and B are real numbers.

(a) Determine the values of A and B. (3 marks)

$$\begin{aligned} x-5 &= A(x-3) + B(x-2) \\ &= Ax - 3A + Bx - 2B \\ &= (A+B)x - (3A+2B) \end{aligned}$$

✓ set up equation

$$\therefore \begin{cases} A+B = 1 \\ 3A+2B = 5 \end{cases}$$

✓ equate coefficients

$$\begin{cases} 3A+3B = 3 \\ 3A+2B = 5 \end{cases}$$

$$\therefore B = -2 \ , A = 3$$

✓ both correct

(b) Hence, or otherwise, evaluate $\displaystyle\int_0^1 \frac{x-5}{x^2-5x+6}\,dx$. (3 marks)

$$\int_0^1 \frac{x-5}{x^2-5x+6}\,dx = \int_0^1 \left(\frac{3}{x-2} - \frac{2}{x-3}\right)dx$$

$$= \Big[3\ln|x-2| - 2\ln|x-3|\Big]_0^1$$

✓ integrate using $\ln|x|$

$$= 3\ln 1 - 2\ln 2 - 3\ln 2 + 2\ln 3$$

✓ substitute limits

$$= 2\ln 3 - 5\ln 2$$

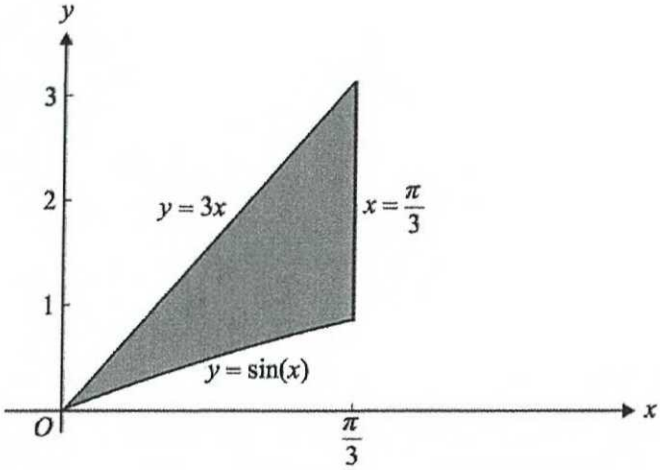
✓ simplify

$$= \ln 9 - \ln 32$$

$$= \ln\left(\frac{9}{32}\right)$$

Question 2 **(6 marks)**

The shaded region below is enclosed by the graph of $y = \sin(x)$ and the lines $y = 3x$ and $x = \frac{\pi}{3}$.



The region is rotated about the x-axis.
Determine the volume of the resulting solid of revolution.

$$\text{Volume} = \pi\int_0^{\frac{\pi}{3}} (3x)^2\,dx - \pi\int_0^{\frac{\pi}{3}} (\sin x)^2\,dx$$ ✓ πr^2 ✓ $(3x)^2 - (\sin x)^2$

$$= \pi\int_0^{\frac{\pi}{3}} \left(9x^2 - \frac{1-\cos(2x)}{2}\right)dx$$ ✓ $\frac{1-\cos(2x)}{2}$

$$= \pi\left[3x^3 - \frac{1}{2}x + \frac{\sin(2x)}{4}\right]_0^{\frac{\pi}{3}}$$ ✓ integrate

$$= \pi\left(3\left(\frac{\pi}{3}\right)^3 - \frac{\pi}{6} + \frac{\left(\frac{\sqrt{3}}{2}\right)}{4}\right)$$ ✓ substitute limits

$$= \frac{\pi^4}{9} - \frac{\pi^2}{6} + \frac{\sqrt{3}\pi}{8}$$ ✓

Question 3 **(3 marks)**

The position vector of a moving particle is given by $\mathbf{r}(t) = \sin\left(\frac{t}{3}\right)\mathbf{i} + \frac{1}{2}\sin\left(\frac{2t}{3}\right)\mathbf{j}, t \geq 0.$

Determine the cartesian equation of the path followed by the particle.

$x = \sin\left(\frac{t}{3}\right)$

$y = \frac{1}{2}\sin\left(\frac{2t}{3}\right)$

$\therefore \quad y = \frac{1}{2}\left(2\sin\left(\frac{t}{3}\right)\cos\left(\frac{t}{3}\right)\right)$ ✓ use double-angle formula

$\therefore \quad y^2 = \left(\sin\left(\frac{t}{3}\right)\right)^2\left(\cos\left(\frac{t}{3}\right)\right)^2$

$\therefore \quad y^2 = \left(\sin\left(\frac{t}{3}\right)\right)^2\left(1 - \sin^2\left(\frac{t}{3}\right)\right)$ ✓ use Pythagorean identity

$\therefore \quad y^2 = x^2\left(1 - x^2\right)$ ✓

End of questions

Question 4 **(6 marks)**

With **i** and **j** horizontal and vertical unit vectors respectively, a particle moves with a constant acceleration of 2**j** m/s^2.

Let t, $t \geq 0$, be the time in seconds. Initially, i.e. when $t = 0$, the position vector of the body is $(2\mathbf{i} + 5\mathbf{j})$ m, and its velocity vector is $(3\mathbf{i} - 7\mathbf{j})$ m/s.

Determine the cartesian equation of the path of the particle.

$$\underset{\sim}{a}(t) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\Downarrow \quad \underset{\sim}{v}(t) = \begin{pmatrix} c_1 \\ 2t + c_2 \end{pmatrix} \quad ✓$$

$$\text{At } t=0, \ \underset{\sim}{v}(0) = \begin{pmatrix} 3 \\ -7 \end{pmatrix}, \ \text{so } \underset{\sim}{v}(t) = \begin{pmatrix} 3 \\ 2t-7 \end{pmatrix} \quad ✓$$

$$\underset{\sim}{v}(t) = \begin{pmatrix} 3 \\ 2t-7 \end{pmatrix}$$

$$\Downarrow \quad \underset{\sim}{r}(t) = \begin{pmatrix} 3t + c_3 \\ t^2 - 7t + c_4 \end{pmatrix} \quad ✓$$

$$\text{At } t=0, \ \underset{\sim}{r}(0) = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \ \text{so } \underset{\sim}{r}(t) = \begin{pmatrix} 3t+2 \\ t^2 - 7t + 5 \end{pmatrix} \quad ✓$$

$$\therefore \begin{cases} x = 3t + 2 \\ y = t^2 - 7t + 5 \end{cases} \qquad t = \frac{x-2}{3} \quad ✓$$

$$\therefore \quad y = \left(\frac{x-2}{3}\right)^2 - 7\left(\frac{x-2}{3}\right) + 5 \quad ✓$$

$$9y = x^2 - 4x + 4 - 21x + 42 + 45$$

$$9y = x^2 - 25x + 91$$

Question 5 **(11 marks)**

The position vector $\mathbf{r}(t)$, from the origin O, of a drone t seconds after leaving the ground is given by

$$\mathbf{r}(t) = \left(50 + 25\cos\left(\frac{\pi t}{30}\right)\right)\mathbf{i} + \left(50 + 25\sin\left(\frac{\pi t}{30}\right)\right)\mathbf{j} + \frac{2t}{5}\mathbf{k}$$

where $\mathbf{i}$ is a unit vector to the east, $\mathbf{j}$ is a unit vector to the north and $\mathbf{k}$ is a unit vector vertically up. Displacement components are measured in metres.

(a) State the time, in seconds, required for the drone to gain an altitude of 60 m. (1 mark)

$\frac{2t}{5} = 60 \Rightarrow t = 150$ $\therefore$ 150 seconds ✓

(b) After how many seconds will the drone first be directly above the point of take-off? (1 mark)

$\frac{t}{30} = 2 \Rightarrow t = 60$ $\therefore$ 60 seconds ✓

(c) Show that the velocity of the drone is perpendicular to its acceleration. (4 marks)

$$\underset{\sim}{v}(t) = \begin{pmatrix} -25\left(\frac{\pi}{30}\right)\sin\left(\frac{\pi t}{30}\right) \\ 25\left(\frac{\pi}{30}\right)\cos\left(\frac{\pi t}{30}\right) \\ \frac{2}{5} \end{pmatrix}$$ ✓

$$\underset{\sim}{a}(t) = \begin{pmatrix} -25\left(\frac{\pi}{30}\right)^2\cos\left(\frac{\pi t}{30}\right) \\ -25\left(\frac{\pi}{30}\right)^2\sin\left(\frac{\pi t}{30}\right) \\ 0 \end{pmatrix}$$ ✓

$$\therefore \underset{\sim}{v}(t)\cdot\underset{\sim}{a}(t) = (25)^2\left(\frac{\pi}{30}\right)^3\sin\left(\frac{\pi t}{30}\right)\cos\left(\frac{\pi t}{30}\right) - (25)^2\left(\frac{\pi}{30}\right)^3\sin\left(\frac{\pi t}{30}\right)\cos\left(\frac{\pi t}{30}\right) = 0$$

✓ dot product
✓ equals zero

$\therefore$ $\underset{\sim}{v}(t)$ and $\underset{\sim}{a}(t)$ are perpendicular !

(d) Determine the speed of the drone. (2 marks)

$$|\underset{\sim}{v}(t)| = \sqrt{(-25)^2\left(\frac{\pi}{30}\right)^2\sin^2\left(\frac{\pi t}{30}\right) + (25)^2\left(\frac{\pi}{30}\right)^2\cos^2\left(\frac{\pi t}{30}\right) + \left(\frac{2}{5}\right)^2}$$

✓ use $|\underset{\sim}{v}(t)|$

$$= \sqrt{25^2\left(\frac{\pi}{30}\right)^2 + \left(\frac{2}{5}\right)^2}$$

$$\approx 2.65 \text{ m/s}$$ ✓

(e) A treetop has position vector $\mathbf{r}(t) = 60\,\mathbf{i} + 40\,\mathbf{j} + 8\,\mathbf{k}$.

Find the distance of the drone from the treetop after it has been travelling for 45 seconds. (3 marks)

$$\left|\begin{pmatrix} 50 + 25\cos\left(\frac{45\pi}{30}\right) - 60 \\ 50 + 25\sin\left(\frac{45\pi}{30}\right) - 40 \\ \frac{2(45)}{5} - 8 \end{pmatrix}\right|$$

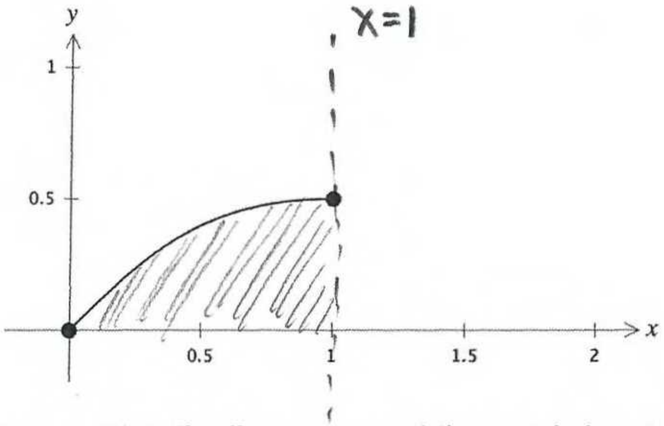
✓ correctly substitute $t = 45$
✓ difference vector

$$= 5\sqrt{17} \text{ m}$$

$$\approx 20.6 \text{ m}$$ ✓

Question 6 **(6 marks)**

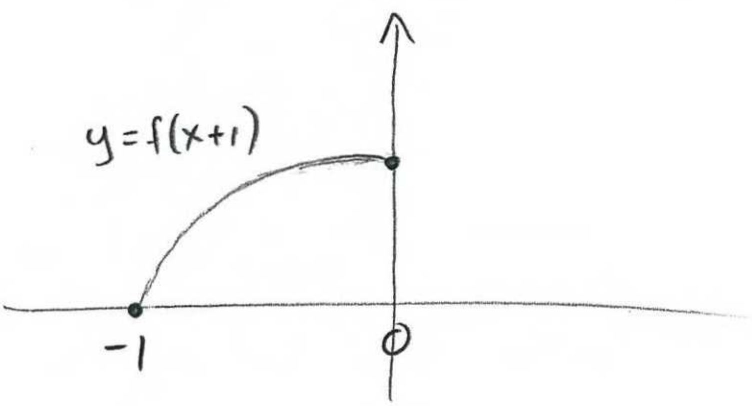
The diagram shows the graph of $f(x) = \dfrac{x}{1+x^2}$ for $0 \le x \le 1$.



The area bounded by $y = f(x)$, the line $x = 1$ and the x-axis is rotated about the line $x = 1$ to form a solid.

Determine the volume of the solid.

The graph of f will need to be translated 1 to the left:



$$\text{Volume} = 2\pi \int_{-1}^{0} |x|\, y \, dx$$

✓ $2\pi x y dx$
✓ use $|x|$
✓ lower limit $= -1$
✓ upper limit $= 0$

$$= 2\pi \int_{-1}^{0} \frac{|x|(x+1)}{1+(x+1)^2}\, dx$$

✓ use $y = f(x+1)$

$$\approx 0.83$$

✓

Question 7 **(7 marks)**

Let S be the curve in the cartesian plane defined by $\mathbf{r}(t) = \begin{pmatrix} 1+t \\ t^2-3 \end{pmatrix}$, $t \in \mathbb{R}$.

Let T be the curve in the cartesian plane defined by $\mathbf{r}(t) = \begin{pmatrix} 1+2\cos(t) \\ -4+3\sin(t) \end{pmatrix}$, $0 \le t \le \pi$.

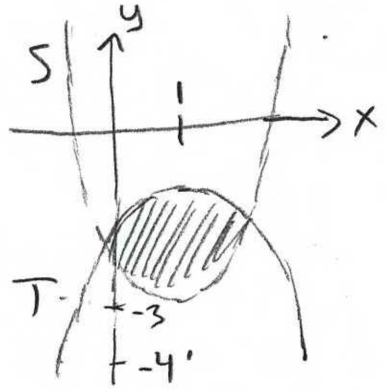
Calculate the area of the region bounded by the two curves.

$S: \begin{cases} x = 1+t \\ y = t^2-3 \end{cases}$ $\quad t = x-1$ ✓ find t

$\therefore\ y = (x-1)^2 - 3$ ✓

$T: \begin{cases} x = 1+2\cos t \\ y = -4+3\sin t \end{cases}$ $\quad \cos t = \left(\frac{x-1}{2}\right)$, $\sin t = \left(\frac{y+4}{3}\right)$ ✓ find $\cos t$ and $\sin t$

$\therefore\ \frac{(x-1)^2}{4} + \frac{(y+4)^2}{9} = 1$ ✓



Limits: $3\sqrt{1-\frac{(x-1)^2}{4}} - 4 = (x-1)^2 - 3 \Rightarrow x \approx \begin{cases} -0.1886 \\ 2.1886 \end{cases}$ ✓

$$\text{Area} \approx \int_{-0.1886}^{2.1886} \left(3\sqrt{1-\frac{(x-1)^2}{4}} - 4 - (x-1)^2 + 3\right) dx$$ ✓

≈ 3.19 ✓

End of questions