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School Curriculum and Standards Authority

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Mathematics: Specialist

Stage 3C/3D Standards Guide

Exemplification of Standards through the 2011 WACE Examination

2012/2980

CONTENTS

Introductory notes for Mathematics: Specialist Stage 3C/3D Standards Guide 2011

What are the 'standards' and how were they developed?

Standards describe the kinds of qualities seen in candidate responses in WACE examination conditions. In late 2011, WACE (written) examination scripts for Mathematics: Specialist Stage 3C/3D were analysed by teacher expert panels who identified the qualities of candidates' scripts at each of five performance bands: 'excellent achievement', 'high achievement', 'satisfactory achievement', 'limited achievement' and 'inadequate achievement'. WACE Course scores were reported against these performance bands.

The band descriptions for Mathematics: Specialist Stage 3C/3D are provided in Appendix 1.

What do standards tell us?

The standards described through the band descriptions tell us, in general terms, how students need to be performing if they wish to achieve a particular 'standard'. To get a clearer picture of what each standard means, teachers and students can refer to the candidate responses provided. This will help students see what they need to do to improve and help them understand how their work compares with the standards. Standards can also assist teachers in providing students with feedback about their work and see how they might need to modify their teaching.

What is provided in this Standards Guide?

There are five main components in this standards guide:

1. questions from the examination paper
2. the marking key for each question
3. candidate responses and annotated marker notes
4. keywords and examination statistics such as the highest and lowest marks achieved, mean, standard deviation, etc
5. examiner comments.

What standards have been exemplified in this guide?

Sample candidate responses which illustrate 'excellent' and 'satisfactory' performance have been included in this guide, along with marker annotations. In most cases, 'excellent' responses received full marks or close to full marks. If there were no responses judged to be 'excellent', a 'high achievement' response sample may be provided. Judgements about the standard illustrated in a candidate response must also take into account the difficulty of the question. It should also be remembered that overall judgements about standards are best made with reference to a range of performances across a range of assessment types and conditions.

How well did this examination 'target' the ability of candidates?

Rasch analysis of raw examination marks achieved by candidates enables us to provide estimates of question difficulty and student ability, on the same scale. From this relationship, we are able to evaluate how well the questions in this examination were broadly targeted to candidates' abilities.

Table 1 in Appendix 2 provides estimates of the difficulty of each question. Graph 2 (where provided) in Appendix 2 shows the distribution of the student ability and question difficulty. Graph 3 (where provided) shows the distribution of the student ability and item thresholds. Explanatory notes for these graphs are also provided in Appendix 2.

Other points to consider when viewing this guide

Use of half marks
Examination items are marked in whole numbers. Half marks occurring in this guide are a result of averaging the whole number marks from each of two markers.

Section statistics and marks weightings
Section statistics for the highest mark achieved, lowest mark achieved, mean and standard deviation are based on weighted section total marks. Raw mark totals are provided for each section. The raw marks distribution and the weighted total marks distribution is provided on the following page.

Examination standards for 2011 WACE examinations
The analysis of written examination scripts was used to determine performance band descriptions for 2011.

Marks distribution for this examination

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	40	$33\frac{1}{3}\%$
Section Two: Calculator-assumed	13	13	100	80	$66\frac{2}{3}\%$
			Total	120	100

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Mathematics: Specialist
Stage 3C/3D

Section One: Calculator-free

40 marks

Note:
Raw section total marks = 40

Weighted section total marks = $\mathbf{33\frac{1}{3}}$%

Weighted section statistics

Statistics ID = MAS3CD-S01
Number of attempts = 1396
Highest mark achieved = 33.33
Lowest mark achieved = 0.00
Mean = 18.42
Standard deviation = 6.62
Correlation between section and exam = 0.93

This section has **seven (7)** questions

Working time: 50 minutes

Examiners' comments for this section

Calculator-free allowed candidates to demonstrate that they had been well-prepared to answer routine problems from the various strands of the course. This year a completely non-standard question was included in the calculator-free part of the paper and many found this problem to be very challenging. Candidates also found the non-standard questions in the calculator-assumed section to be difficult but these successfully discriminated between the most able candidates and hence served their intended purpose. It is pleasing to report that a number of candidates achieved near perfect answers to the whole paper and a few scripts only failed to be worthy of full marks owing to a few very minor errors.
Common weaknesses that continue to be evident in examination scripts include:

1. Graphs and diagrams. The general standard of presentation of drawings was low, though perhaps a little better than in previous years. Whether a question asks for a graph or for a sketch, it is expected that the result should be neat, should show suitable scales on the axes and should clearly include obvious key points. It is also advised that where appropriate, diagrams should be used to illustrate the information in a question. On this examination, this point was most clearly demonstrated in question 14 (vectors). Far too many candidates attempted to solve this question without any reference to a diagram and these efforts inevitably made little meaningful progress.

2. CAS Calculators. The examiners were pleased to find that candidates showed improved skills with their calculators. Some questions are included in the examination to allow candidates to demonstrate their knowledge of CAS technology. However, in many cases, there was little evidence that they were able to take advantage of it. A good example of this is in question 15 where very few candidates were able to write the binomial expansion using the capability of the calculator to avoid large quantities of intricate algebra.

3. Basic mathematical accuracy. The importance of accuracy in reading the details of the question and in completing the steps of the solution cannot be over-emphasised. This year there seemed to be many transcription errors in copying information provided within the question and a plethora of careless, elementary arithmetical and algebraic errors. Often these mistakes meant the question could not be completed and as a result many marks were lost unnecessarily.

Examiners' comments for this section (continued)

4. Reasoning. Candidates need to be aware that if asked to prove a given, stated result then the examiners are expecting to see an answer containing a high degree of rigour. Too often proofs and arguments were presented with poor or inappropriate nomenclature and these solutions tended to be less than convincing. Equally, when using the CAS to simplify algebra, to differentiate and so on, it is not sufficient to state that the calculator verifies that the given statement is true. The best example of this type of problem was provided by question 18 (c); many candidates merely stated that the function $P(t)$ satisfies the given differential equation 'by CAS'.

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Question

Question 1
(5 marks)

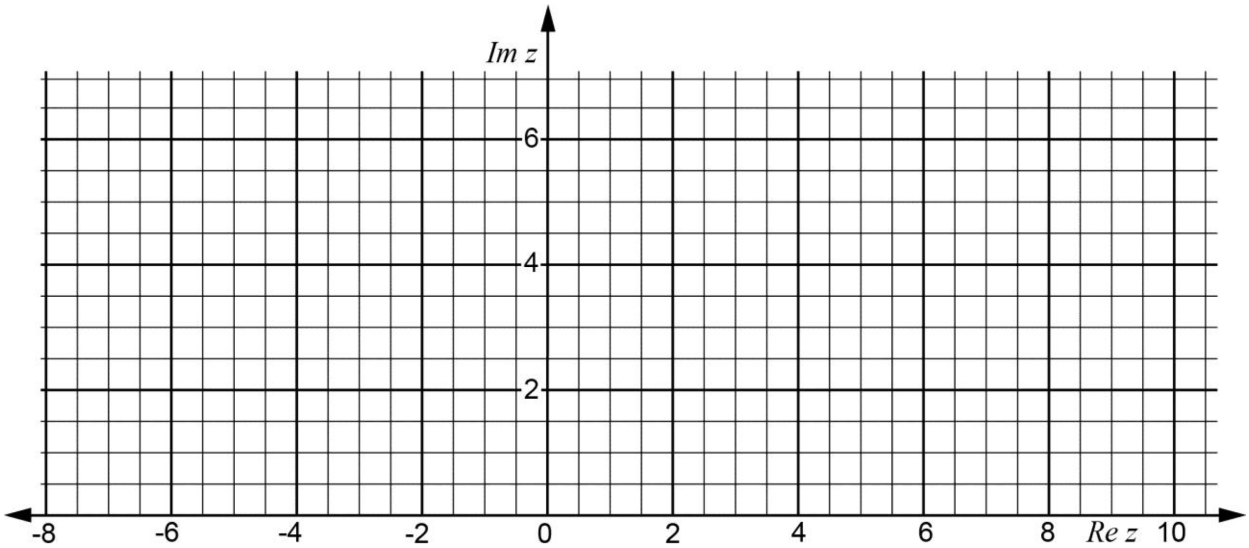
Question statistics

Statistics ID = MAS3CD-1
Number of attempts = 1387
Highest mark achieved = 5.00
Lowest mark achieved = 0.00
Mean = 3.86
Standard deviation = 1.39
Question difficulty = N/A
Correlation between question and section = 0.65

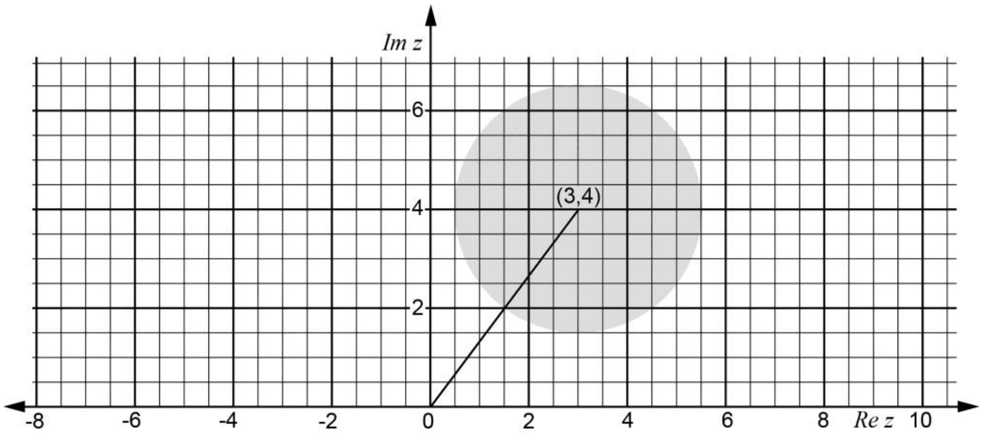
1(a)

Sketch, on the complex plane below, the region defined by $\left|z-3-4i\right| \leq \frac{5}{2}$.

(3 marks)



Marking key

Solution

Specific behaviours
✓ correctly shades a circular disk ✓ gives correct coordinates for the centre of the circle and ✓ circumference passes through at least three of: (3, 1.5), (5.5,4), (3, 6.5) and (0.5, 5)

Keywords

Complex numbers

Question statistics

Statistics ID = MAS3CD-2
Number of attempts = 1384
Highest mark achieved = 3.00
Lowest mark achieved = 0.00
Mean = 2.55
Standard deviation = 0.77
Question difficulty = Easy
Correlation between question and section = 0.48

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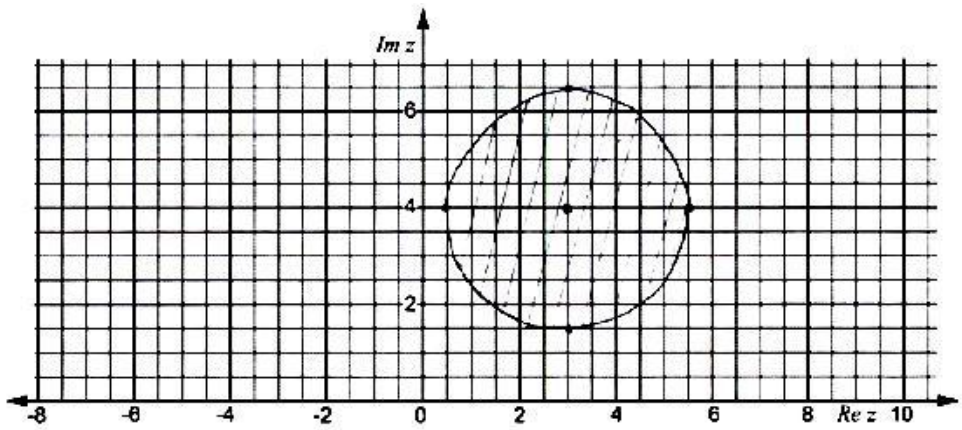
Candidate responses

1(a)

Sketch, on the complex plane below, the region defined by $|z-3-4i|\leq \frac{5}{2}$.

(3 marks)



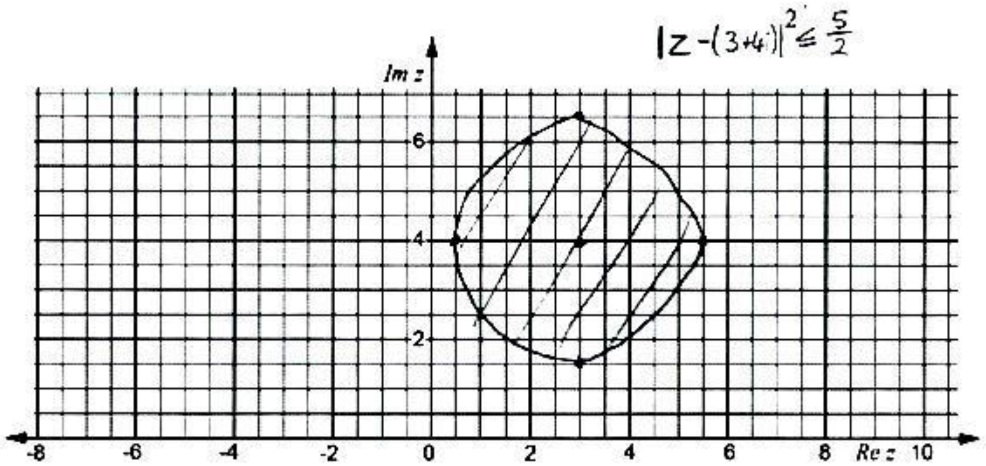
Notes

Excellent response
3/3 marks

Neatly sketches the shaded circular disk.

Places the centre of the circle in the correct position.

Ensures that the circumference of the circle is carefully drawn and passes through the correct points.



Satisfactory response
2/3 marks

Sketches the shaded circular disk.

Places the centre of the circle in the correct position.

The circumference of the circle is poorly drawn.

Question

1(b)

For the region in (a), find the maximum value of $|z|$.
(2 marks)

Marking key

Solution
Maximum value of $\lvert z\rvert$ = the distance from the origin to the centre + the length of the radius i.e. Maximum value of $\lvert z\rvert = 5 + \frac{5}{2} = \frac{15}{2}$
Specific behaviours
✓ calculates the distance from the origin to the centre of the circle ✓ states the correct answer Note ✓ shows understanding of process but wrong answer

Keywords

Complex numbers, Absolute value

Question statistics

Statistics ID = MAS3CD-3
Number of attempts = 1340
Highest mark achieved = 2.00
Lowest mark achieved = 0.00
Mean = 1.36
Standard deviation = 0.90
Question difficulty = Moderate
Correlation between question and section = 0.56

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Candidate responses

1(b)

For the region in (a), find the maximum value of $|z|$.
(2 marks)

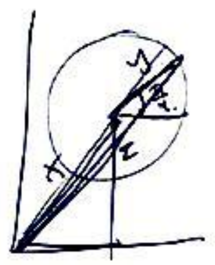
$$|z| = |3+4i| + \frac{5}{2}$$
$$= \sqrt{3^2+4^2} + \frac{5}{2}$$
$$= 5 + \frac{5}{2}$$
$$= \frac{15}{2}$$

Notes

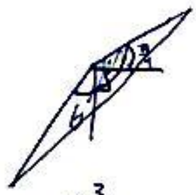
Excellent response
2/2 marks

States that the maximum distance is equal to the distance of the centre of the circle from the origin, plus the length of the radius.

Correctly calculates the answer.



$|x| = \sqrt{3^2+4^2}$
$= 5$ unit.
$|y| =$ radius $= \frac{5}{2}$



$\theta = \tan^{-1}(\frac{3}{4})$.

$$|z| = \sqrt{5^2 + (\tfrac{5}{2})^2 - 2(5)(\tfrac{5}{2})\cos(\tan^{-1}(\tfrac{3}{4}))}$$
$$= \sqrt{25 + \tfrac{25}{2} - 25\cos(\tan^{-1}(\tfrac{3}{4}))}$$
$$= \sqrt{37.5 - 25\cos(\tan^{-1}(\tfrac{3}{4}))}$$

Satisfactory response
1/2 marks

Calculates the centre of the circle from the origin, plus the length of the radius.

Does not identify that the maximum distance between the origin and the required point is the sum of the above, hence uses the cosine rule to solve.

Examiners' comments

This question was generally well answered. Most candidates recognised the required region to be a circular disk, but the standard of the sketches was often poor. Some placed the centre of the circle in the wrong quadrant and others thought the region to be semi-circular. Many struggled with part (b) and the overall standard of explanations was not good.

Question

Question 2

Use proof by exhaustion to prove that no square number ends in 8.
(3 marks)

Marking key

Solution

x	0	1	2	3	4	5	6	7	8	9
x^2	0	1	4	9	16	25	36	49	64	81

No number from 0 to 9 inclusive has a square that ends in 8. Any number greater than 9 has a units digit equal to one of those shown in the first line of the table. When squared, the end digit will be the same as the corresponding end digit in the second line of the table. Thus no square number ends in 8.

Specific behaviours

- ✓ demonstrates that no number from 0 to 9 inclusive has a square which ends in 8
- ✓ $(x+10)^2$ indicates the place value of each group
- ✓ the units digit is always filled by the x^2 unit

Or

- ✓ calculates squares from 0 to 9 or 1 to 10
- ✓ calculates squares from 10 to 19 (or 11 to 20) and compares end-digits with above
- ✓ correctly explains why the pattern continues

Keywords

Mathematical proof

Question statistics

Statistics ID = MAS3CD-4
Number of attempts = 1318
Highest mark achieved = 3.00
Lowest mark achieved = 0.00
Mean = 1.19
Standard deviation = 1.08
Question difficulty = Moderate
Correlation between question and section = 0.37

Candidate responses

Question 2

Use proof by exhaustion to prove that no square number ends in 8.
(3 marks)

Let $A = 10n+k$ be the original number; where $n \in \mathbb{Z}$, $k \in \{0, 1, 2, \cdots, 9\}$

$\therefore A^2 = (10n)(10n) + 20nk + k^2$

$= 100n^2 + 20nk + k^2$

$= 10(10n^2 + 2nk) + k^2$

$\therefore 10(10n^2+2nk)$ ends in 0.

Thus A^2 ends in k^2 (units digit of k^2)

$\therefore k^2 \in \{0^2, 1^2, 2^2 \ldots 9^2\}$

i.e $k^2 \in \{0, 1, 4, 9, 16, 25, 36, 49, 64, 81\}$

Thus None of these numbers ends in 8.

Hence by exhaustion; there is no square number that ends in 8.

Notes

Excellent response
3/3 marks

Clearly defines the number to be squared which ends with the digits k = 1 to 9 and expresses the square in these terms.

Shows that the square number will contain k^2 in the units column.

States that k^2 will not contain an 8 from the set given and hence proves the proposition.

Candidate responses (continued)

Note: the last digit of a product is equal to the multiplication of the last digits of the numbers being multiplied.

the last digit of product:

$\therefore$ $1 \times 1 = 1$ | $\neq 8$
$2 \times 2 = 4$ | $\neq 8$
$3 \times 3 = 9$ | $\neq 8$
~~4 × 4 = 16~~
~~5 × 5 =~~
$4 \times 4 = 16$ | $\neq 8$
$5 \times 5 = 25$ | $\neq 8$
$6 \times 6 = 36$ | $\neq 8$
$7 \times 7 = 49$ | $\neq 8$
$8 \times 8 = 64$ | $\neq 8$
$9 \times 9 = 81$ | $\neq 8$
$10 \times 10 = 100$ | $\neq 8$.

$\therefore$ no square number ends in 8.

Notes

Satisfactory response 2/3 marks

Clearly states that the last number of the product will end with the last digits of the numbers to be multiplied $k = 1$ to 9 and expresses the square in these terms.

Defines the number to be squared which ends with the digits $k = 1$ to 9 and shows that they do not contain an 8.

Examiners' comments

This question was answered poorly and there was generally insufficient justification to award full marks to all but a small handful of answers. Most candidates managed to show that the squares of the numbers 1–9 did not end with an 8, although several forgot to square zero. A number of candidates tried to consider even and odd numbers separately. They were usually able to establish that the square of an odd number couldn't end with an 8, but were unable to argue adequately the case for even numbers. Others ignored the instruction to use exhaustion.

Question

Question 3
(6 marks)

Question statistics

Statistics ID = MAS3CD-5
Number of attempts = 1392
Highest mark achieved = 6.00
Lowest mark achieved = 0.00
Mean = 3.82
Standard deviation = 2.08
Question difficulty = N/A
Correlation between question and section = 0.78

Determine the following integrals:

3(a)

$$\int \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta+1}\,d\theta$$

(3 marks)

Marking key

Solution
$\int \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta+1}\,d\theta = \frac{1}{2}\int \frac{\sin\theta}{\cos\theta+1}\,d\theta$ Hence $\int \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta+1}\,d\theta = -\frac{1}{2}\ln\lvert\cos\theta+1\rvert + c$ Or $\int \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta+1}\,d\theta = \frac{1}{2}\int \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\,d\theta$ i.e. $\int \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta+1}\,d\theta = \int \frac{1}{u}\,du$ where $u=\sin\frac{\theta}{2}$ Hence $\int \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta+1}\,d\theta = \ln\lvert u\rvert + c = \ln\left\lvert\sin\frac{\theta}{2}\right\rvert + c$
Specific behaviours
✓ replaces $\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{1}{2}\sin\theta$ ✓ gives logarithmic solution ✓ provides correct coefficient -½, including c Or ✓ replaces $\cos\theta+1=\sin^2\frac{\theta}{2}$ ✓ gives logarithmic solution ✓ provides correct solution, including c

Keywords

Trigonometric expressions, Integration

Question statistics

Statistics ID = MAS3CD-6
Number of attempts = 1295
Highest mark achieved = 3.00
Lowest mark achieved = 0.00
Mean = 1.79
Standard deviation = 1.20
Question difficulty = Moderate
Correlation between question and section = 0.64

Candidate responses

3(a)

$$\int \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta+1}\,d\theta$$

(3 marks)

(a) $\displaystyle\int \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta+1}\,d\theta$

$\displaystyle = -\frac{1}{2}\int \frac{-\sin\theta}{\cos\theta+1}\,d\theta$

$\displaystyle = -\frac{1}{2}\ln\left|\cos\theta+1\right| + c$

Notes

Excellent response
3/3 marks

Uses the half angle identity to convert the integrand to the form $\int \frac{f'(\theta)}{f(\theta)}d\theta$.

Gives the correct logarithmic solution.

Follows through correctly with the correct coefficient of -½ and a constant of integration c.

No candidate response is provided.

Satisfactory response

Question

3(b)

$$\int \cos^3 x\, dx$$

(3 marks)

Marking key

Solution
$\int \cos^3 x\, dx = \int \left(1 - \sin^2 x\right) \cos x\, dx$ i.e. $\int \cos^3 x\, dx = \int \cos x - \sin^2 x \cos x\, dx$ Let $u = \sin x \Rightarrow \int \cos x\, dx - \int u^2 du$ Hence $\int \cos^3 x\, dx = \sin x - \frac{\sin^3 x}{3} + c$ Or $\int \cos^3 x\, dx = \frac{1}{4} \int \left(\cos 3x + 3\cos x\right) dx$ Hence $\int \cos^3 x\, dx = \frac{1}{4}\left(\frac{\sin 3x}{3} + 3\sin x\right) + c$
Specific behaviours
✓ recognises that $\cos^3 x$ may be replaced by $\left(1-\sin^2 x\right)\cos x$ ✓ uses the substitution $u = \sin x$ or by inspection   ✓ integrates correctly   Or   ✓ recognises that $\cos^3 x$ may be replaced by $\frac{1}{4}\left(\cos 3x + 3\cos x\right)$ ✓✓ Integrates correctly for each term

Keywords

Trigonometric expressions, Integration

Question statistics

Statistics ID = MAS3CD-7
Number of attempts = 1388
Highest mark achieved = 3.00
Lowest mark achieved = 0.00
Mean = 2.16
Standard deviation = 1.21
Question difficulty = Moderate
Correlation between question and section = 0.65

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Candidate responses

3(b)

$$\int \cos^3 x \, dx$$

(3 marks)

(b) $\int \cos^3 x \, dx$

$= \int \cos^2 x \cos x \, dx$

$= \int (1-\sin^2 x)\cos x \, dx$

$= \int \cos x - \sin^2 x \cos x \, dx$

$= \sin x - \frac{1}{3}\sin^3 x + c$

$= \int \cos^2 x \cos x$

$= \int (1-\sin^2 x)\cos x$

$= \int \cos x - \sin^2 x \cos x$

$= \sin x - \sin^3 x + c$

Notes

Excellent response
3/3 marks

Replaces $\cos^3 x$ with $\left(1-\sin^2 x\right)\cos x$ to give two integrals.

Integrates both terms correctly.

Follows through correctly with the correct coefficients and includes a constant of integration c.

Satisfactory response
2/3 marks

Replaces $\cos^3 x$ with $\left(1-\sin^2 x\right)\cos x$ to give two integrals.

Fails to integrate both terms correctly as does not follow through correctly with the correct coefficient. Includes a constant of integration c.

Examiners' comments

Part (a) was answered reasonably well, but many candidates did not realise that the numerator of the integrand was just a poorly-disguised double angle formula. Others were hampered due to errors in their algebraic manipulation. Part (b) was answered better. This was a fairly standard type of question and most candidates knew how to tackle the question. A few responses by candidates became entangled within the algebra but these were in the distinct minority.

Question

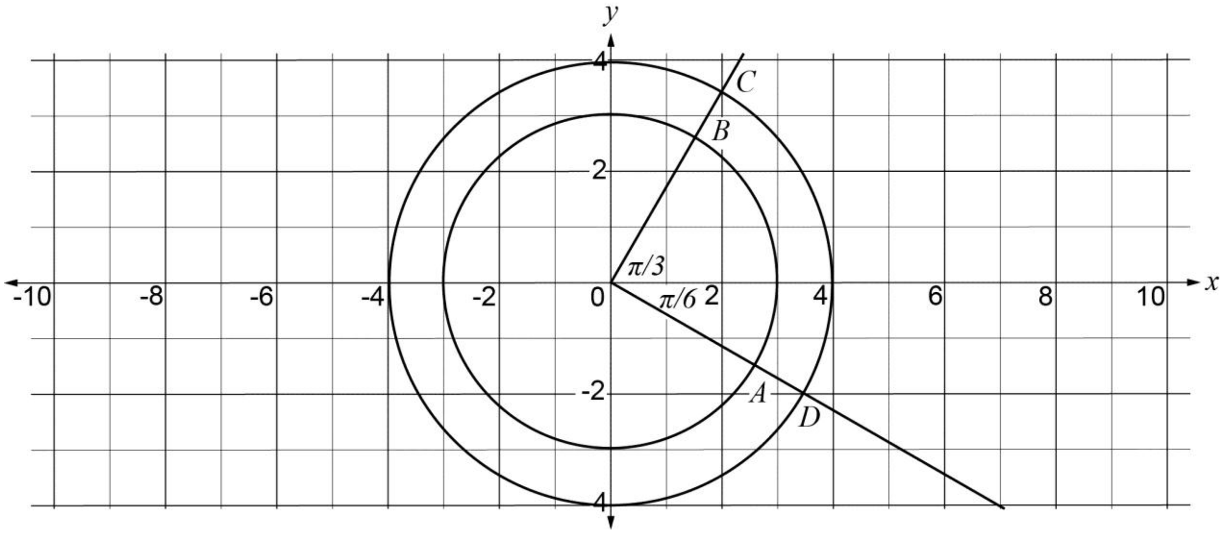
Question 4
(6 marks)

Question statistics

Statistics ID = MAS3CD-8
Number of attempts = 1387
Highest mark achieved = 6.00
Lowest mark achieved = 0.00
Mean = 3.81
Standard deviation = 1.56
Question difficulty = N/A
Correlation between question and section = 0.70

4(a)

Use polar inequalities to describe the region bounded by the minor arcs AB and CD and the straight lines BC and AD in the diagram below.
(2 marks)



Marking key

Solution
$3 \le r \le 4$ and $-\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$
Specific behaviours
✓ identifies inequalities with both radii using polar notations
✓ identifies inequalities with both angles using polar notations
* Note
✓ for correct radii and angles using other notation

Keywords

Polar coordinates, Cartesian planes

Question statistics

Statistics ID = MAS3CD-9
Number of attempts = 1333
Highest mark achieved = 2.00
Lowest mark achieved = 0.00
Mean = 1.09
Standard deviation = 0.72
Question difficulty = Moderate
Correlation between question and section = 0.49

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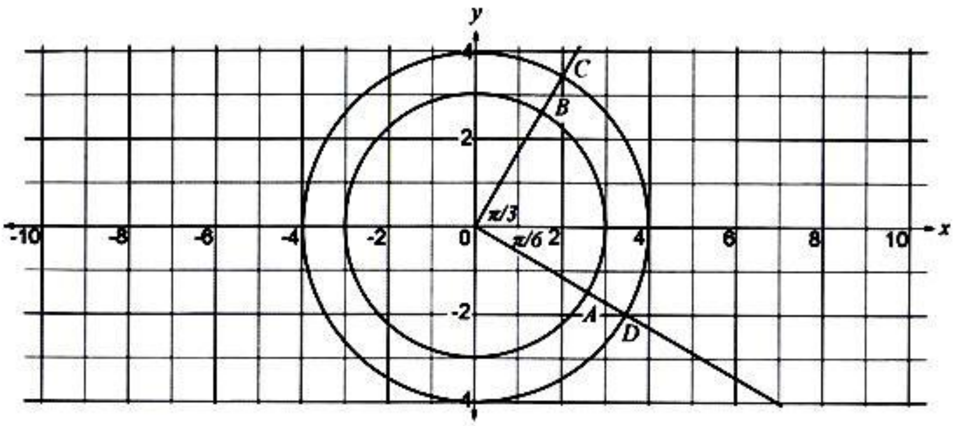
Candidate responses

4(a)

Use polar inequalities to describe the region bounded by the minor arcs *AB* and *CD* and the straight lines *BC* and *AD* in the diagram below.
(2 marks)

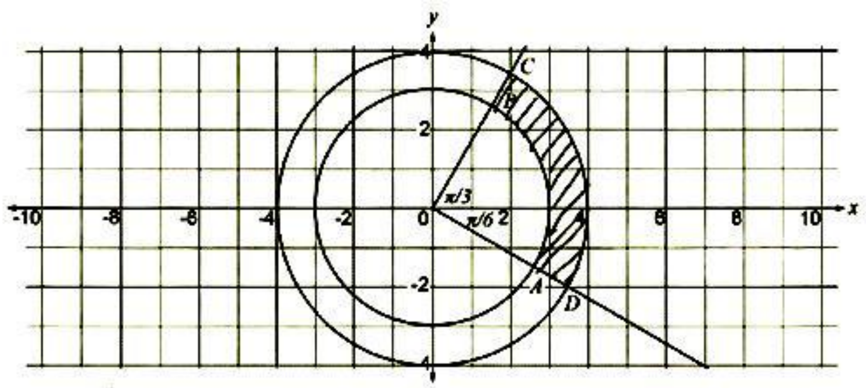


$$-\frac{\pi}{6} \le \theta \le \frac{\pi}{3} \quad , \quad 3 \le r \le 4$$

Notes

Excellent response
2/2 marks

States the correct radius and angle boundaries.
Uses polar notation.



$$\left\{ z : -\frac{\pi}{6} \le \arg(z) \le \frac{\pi}{3} \ \& \ 3 \le |z| \le 4 \right\}$$

Satisfactory response
1/2 marks

States the correct radius and angle boundaries, but does not use polar notation.

Question

4(b)

If the graph of $r = k\theta,\ k > 0$, passes through A, find a possible value for k.
(2 marks)

Marking key

Solution
A may be described as the point with polar coordinates $\left(3,\ \frac{11\pi}{6}\right)$.   For this value of θ, $3 = \frac{11k\pi}{6}$ i.e. $k = \frac{18}{11\pi}$
Specific behaviours
✓ identifies possible coordinates for A ✓ solves for k

Keywords

Polar coordinates

Question statistics

Statistics ID = MAS3CD-10
Number of attempts = 1359
Highest mark achieved = 2.00
Lowest mark achieved = 0.00
Mean = 1.15
Standard deviation = 0.75
Question difficulty = Moderate
Correlation between question and section = 0.48

Candidate responses

4(b)

If the graph of $r = k\theta,\ k > 0$, passes through A, find a possible value for k.
(2 marks)

for A : $3 = \frac{11\pi}{6}k \rightarrow k = \frac{18}{11\pi}$

$\theta = -\frac{\pi}{6} \quad r = 3$

$\therefore 3 = k\left(-\frac{\pi}{6}\right)$

$k = -\frac{18}{\pi}$

Notes

Excellent response
2/2 marks

Identifies a correct polar coordinate for point A.

Solves for k.

Satisfactory response
1/2 marks

Identifies correct polar coordinates for A.

Solves for k, but ignores the domain and provides a negative solution for k.

Question

4(c)

Find the distance between B and D.
(2 marks)

Marking key

Solution
$\left\lvert\overrightarrow{OB}\right\rvert = 3$; $\left\lvert\overrightarrow{OD}\right\rvert = 4$ and $\angle BOD = \frac{\pi}{2}$ Hence $\left\lvert\overrightarrow{BD}\right\rvert = 5$ units
Specific behaviours
✓ identifies the measurements of triangle OBD ✓ solves for $\left\lvert\overrightarrow{BD}\right\rvert$

Or

Solution
B has Cartesian coordinates $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$; D has Cartesian coordinates $\left(2\sqrt{3}, -2\right)$ Hence $\lvert BD\rvert = \sqrt{\left(\frac{3}{2} - 2\sqrt{3}\right)^2 + \left(\frac{3\sqrt{3}}{2} + 2\right)^2} = \sqrt{25} = 5$
Specific behaviours
✓ Identifies the Cartesian coordinates of B and D ✓ Solves for $\left\lvert\overrightarrow{BD}\right\rvert$

Keywords

Polar coordinates, Cartesian planes

Question statistics

Statistics ID = MAS3CD-11
Number of attempts = 1356
Highest mark achieved = 2.00
Lowest mark achieved = 0.00
Mean = 1.67
Standard deviation = 0.63
Question difficulty = Easy
Correlation between question and section = 0.43

Candidate responses

4(c)

Find the distance between B and D.
(2 marks)

$$BD = \sqrt{3^2 + 4^2 - 2 \times 3 \times 4 \cos\left(\frac{\pi}{2}\right)}$$

$$= \sqrt{9 + 16 - 24 \times 0}$$

$$= \sqrt{25}$$

$$= 5 \text{ units}$$

Notes

Excellent response
2/2 marks

Identifies the measurements of triangle OBD.

Correctly applies the cosine rule to determine the distance between B and D.

$B - \left(3, \frac{\pi}{3}\right)$ $D - \left(4, -\frac{\pi}{6}\right)$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$a^2 = 3^2 + 4^2 - 2 \times 3 \times 4 \cos\frac{\pi}{6}$$

$$a^2 = 25 - 24 \times \frac{\sqrt{3}}{2}$$

$$a^2 = 25 - 12\sqrt{3}$$

$$a = \sqrt{25 - 12\sqrt{3}}$$

Excellent response
1/2 marks

Identifies the measurements of triangle OBD, but uses an incorrect size for the included angle in the cosine rule.

Follows through to solve for the distance between B and D.

Examiners' comments

Too many candidates lost marks due to adopting complex number notation. The question clearly asked for regions defined in terms of polar co-ordinates but the overwhelming majority of answers were expressed in terms of $|z|$ and/or arg(z). Part (b) was not particularly well answered; as many ignored the information that $k > 0$ and hence used a negative value for the polar angle. Several could not find the polar co-ordinates of A properly. Part (c) was well answered though many did not realise that the requisite triangle is right-angled and so a simple application of Pythagoras was all that was required.

Question

Question 5
(6 marks)

Question statistics

Statistics ID = MAS3CD-12
Number of attempts = 1391
Highest mark achieved = 6.00
Lowest mark achieved = 0.00
Mean = 3.86
Standard deviation = 1.72
Question difficulty = N/A
Correlation between question and section = 0.68

5(a)

Solve the equation $X\begin{bmatrix} 2 & -2 \\ -7 & 4 \end{bmatrix} + X = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$ for the 2×2 matrix X.
(4 marks)

Marking key

Solution

$$X\begin{bmatrix} 2 & -2 \\ -7 & 4 \end{bmatrix} + X = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

$$X\left(\begin{bmatrix} 2 & -2 \\ -7 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

$$X\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}^{-1}$$

$$X = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \times \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$$

$$X = \begin{bmatrix} 19 & 8 \\ -8 & -3 \end{bmatrix}$$

Or

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then $\begin{bmatrix} 3a-7b & -2a+5b \\ 3c-7d & -2c+5d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$

Solve the two pairs of simultaneous equations to find

$a = 19,\ b = 8,\ c = -8,\ d = -3$

Specific behaviours

✓ recognises that $X = X\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and simplifies the LS

✓✓ post multiplies both sides of the equation by $\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}^{-1}$, where $\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$

✓ solves for X

Or

✓ substitutes $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and simplifies the LS

✓✓ solves one pair of simultaneous equations

✓ solves the second pair of simultaneous equations

Keywords

Matrices

Question statistics

Statistics ID = MAS3CD-13
Number of attempts = 1387
Highest mark achieved = 4.00
Lowest mark achieved = 0.00
Mean = 3.06
Standard deviation = 1.26
Question difficulty = Moderate
Correlation between question and section = 0.59

Candidate responses

5(a)

Solve the equation $X\begin{bmatrix} 2 & -2 \\ -7 & 4 \end{bmatrix} + X = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$ for the 2×2 matrix X.
(4 marks)

$$X\left(\begin{bmatrix} 2 & -2 \\ -7 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

$$X\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}^{-1} \qquad \frac{1}{\det} = \frac{1}{1}$$

$$X = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} 19 & 8 \\ -8 & -3 \end{bmatrix}$$

Notes

Excellent response
4/4 marks

Simplifies the left-hand side of the equation to give a single matrix times X.

Uses inverse matrix method to solve for X.

Carries through the calculations accurately.

$$x\left(\begin{pmatrix} 2 & -2 \\ -7 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix}^{-1}$$

$$= \frac{1}{15-14}\begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$$

$$= \frac{1}{1}\begin{pmatrix} 19 & 8 \\ -8 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 19 & 8 \\ -8 & -3 \end{pmatrix}$$

Satisfactory response
2/4 marks

Simplifies the left-hand side of the equation to give a single matrix times X.

Uses inverse matrix method to solve for X.

Makes errors with the follow through calculations.

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Question

5(b)

If A is a square matrix satisfying $A^2 - 2A + I = 0$, where I is the 2×2 identity matrix, determine an expression for A^{-1} in terms of A and I.
(2 marks)

Marking key

Solution
$A^2 - 2A + I = 0$ $I = 2A - A^2$ $A^{-1} = 2I - A$
Specific behaviours
✓ rearranges the equation to express I in terms of A ✓ multiplies both sides by A^{-1} and simplifies the RHS to establish the result

Or

Solution
$A^2 - 2A + I = 0$ $A - 2I + A^{-1} = 0$ multiply both sides by A^{-1} Hence $A^{-1} = 2I - A$
Specific behaviours
✓ multiplies both sides by A^{-1} ✓ rearranges the equation to express A^{-1} in terms of A and I

Keywords

Matrices

Question statistics

Statistics ID = MAS3CD-14
Number of attempts = 1336
Highest mark achieved = 2.00
Lowest mark achieved = 0.00
Mean = 0.84
Standard deviation = 0.89
Question difficulty = Moderate
Correlation between question and section = 0.45

Candidate responses

5(b)

If A is a square matrix satisfying $A^2 - 2A + I = 0$, where I is the 2×2 identity matrix, determine an expression for A^{-1} in terms of A and I.
(2 marks)

$$A^2 - 2A + I = 0$$
$$A^2 - 2A = -I$$
$$A(A - 2I) = -I$$
$$(A - 2I) = A^{-1}(-I)$$
$$A - 2I = -A^{-1}$$
$$\therefore \quad A^{-1} = -A + 2I$$

Notes

Excellent response
2/2 marks

Uses the inverse A^{-1} on both sides of the equation.

Rearranges the expression to give inverse A^{-1} in terms of A and I.

$$A^2 - 2A = -I$$
$$A(A-2) = -I$$
$$A - 2 = A^{-1}(-I)$$
$$A^{-1} = 2 - A$$

Satisfactory response
1/2 marks

Factorises the term on the left-side of the equation, but writes 2 instead of $2I$ which gives the wrong expression.

Uses the inverse A^{-1} on both sides of the equation, but carries through the mistake from the earlier line.

Examiners' comments

A disappointingly large number of candidates seemed unsure how to tackle either of the two types of problem presented here. Some solved part (a) by factorising the left side, post-multiplying and then determining an inverse matrix while others solved a pair of simultaneous equations for the elements of *X*. Too many could not multiply 2 × 2 matrices properly; others treated the matrix *X* as a scalar and there were many instances of sloppy arithmetic. A surprising number of candidates seemed to think that the identity matrix consisted of four 1's.

Many candidates could only make progress on part (b) by assuming that matrix manipulation obeys exactly the same rules as equivalent non-matrix algebra. This led to a variety of completely nonsensical answers, including some incorporating the square root of a matrix and/or the division by a matrix. Others tried to add matrix and scalar quantities together and others assumed that if the product of two matrices is zero then one of them must be zero. This tended to give the impression that the majority of candidates had little appreciation of the particular properties of matrices.

Question

Question 6

Evaluate exactly: $\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt$

(5 marks)

Marking key

Solution

$$\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt$$

Let $u = 2t^3 + t + 1$

Then $\frac{du}{dt} = 6t^2 + 1$

When $t = 0,\ u = 1; \quad t = 10,\ u = 2011$

Hence $\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt = \int_1^{2011} \frac{1}{\sqrt{u}}\,du$

Hence $\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt = \left[2\sqrt{u}\right]_1^{2011} = 2\left(\sqrt{2011}-1\right)$

Or $\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt = \int_{x=0}^{x=10} \frac{1}{\sqrt{u}}\,du$

Hence $\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt = \left[2\sqrt{u}\right]_{t=0}^{t=10} = 2\left[\sqrt{2t^3+t+1}\right]_0^{10} = 2\left(\sqrt{2011}-1\right)$

Specific behaviours

✓ recognises the format $\int \frac{f'(x)}{f(x)}\,dx$

✓ correctly rewrites the integral in terms of *u*

✓ correctly substitutes the new limits of integration
(or replaces *u* with $2t^3 + t + 1$ after integration)

✓ integrates correctly

✓ solves exactly

Or

Marking key (continued)

Solution
$\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt$ Let $u=\sqrt{\left(2t^3+t+1\right)}$ Then $\frac{du}{dt}=\frac{6t^2+1}{2\sqrt{\left(2t^3+t+1\right)}}$ When $t=0,\ u=1;\quad t=10,\ u=\sqrt{2011}$ Hence $\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt=2\int_1^{\sqrt{2011}} du$ Hence $\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt=\left[2u\right]_1^{\sqrt{2011}}=2\left(\sqrt{2011}-1\right)$ Or $\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt=2\int_{x=0}^{x=10} du$ Hence $\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt=\left[2u\right]_{t=0}^{t=10}=2\left[\sqrt{2t^3+t+1}\right]_0^{10}=2\left(\sqrt{2011}-1\right)$
Specific behaviours
✓ recognises the format $\int \frac{f'(x)}{f(x)}\,dx$ ✓ correctly rewrites the integral in terms of u ✓ correctly substitutes the new limits of integration (or replaces u with $\sqrt{2t^3+t+1}$ after integration) ✓ integrates correctly ✓ solves exactly

Keywords

Integration (Functions)

Question statistics

Statistics ID = MAS3CD-15
Number of attempts = 1387
Highest mark achieved = 5.00
Lowest mark achieved = 0.00
Mean = 3.86
Standard deviation = 1.62
Question difficulty = Moderate
Correlation between question and section = 0.61

Candidate responses

Question 6

Evaluate exactly: $\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt$

(5 marks)

Evaluate exactly: $\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt$

$$= \int_0^{10} (6t^2+1)\left[2t^3+t+1\right]^{-\frac{1}{2}} dt$$

$$= \left[\frac{(2t^3+t+1)^{1/2}}{1/2}\right]_0^{10}$$

which is of the form $f'(x)\left[f(x)\right]^n$

$$= 2\left(2(10)^3+10+1\right)^{1/2} - 2\left(2(0)^3+0+1\right)^{1/2}$$

$$= 2(2011)^{1/2} - 2$$

$$= 2\left(\sqrt{2011}-1\right)$$

Notes

Excellent response
5/5 marks

Recognises the format $\int f(u)\,.\,u'(t)\,dt$ where $u = 2t^3+t+1$.

Correctly calculates the integral in terms of *t*.

Accurately applies the limits of integration.

Simplifies the answer as an exact value.

Candidate responses (continued)

Evaluate exactly: $\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt$

$y = 2t^3 + t + 1$

$y' = 6t^2 + 1$

$\int_0^{10} (6t^2+1)(2t^3+t+1)^{0.5}\,dt$

$= \left[\frac{(2t^3+t+1)^{3/2}}{3/2}\right]_0^{10}$

$= \left[\frac{2(2t^3+t+1)^{3/2}}{3}\right]_0^{10}$

$= \frac{2(2(10)^3+10+1)^{\frac{3}{2}}}{3} - \frac{2(2(0)+0+1)^{\frac{3}{2}}}{3}$

$= \frac{2(2000+10+1)^{\frac{3}{2}}}{3} - \frac{2\ (1)^{\frac{3}{2}}}{3}$

$= \frac{2(2011)^{\frac{3}{2}} - 2}{3} = \frac{2[(2011)^{\frac{3}{2}} - 1]}{3}$

Notes

Satisfactory response
3/5 marks

Recognises the format $\int f(u)\,.\,u'(t)\,dt$ where $u = 2t^3 + t + 1$.

Omits a negative sign and carries this through the rest of the calculations.

Accurately applies the limits of integration.

Simplifies the answer as an exact value.

Examiners' comments

Most candidates recognised the connection between the numerator and the denominator, but several were stumped by the problem of identifying the appropriate anti-derivative. A popular choice was to claim that the anti-derivative was a logarithmic function. However, overall the majority realised what was required to answer the question, though some were then let down by faulty reasoning, transcription errors or careless arithmetic.

Question

Question 7
(9 marks)

Consider the integrals $I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}dx$ and $J = \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}dx$.

Question statistics

Statistics ID = MAS3CD-16
Number of attempts = 1378
Highest mark achieved = 9.00
Lowest mark achieved = 0.00
Mean = 1.90
Standard deviation = 2.32
Question difficulty = N/A
Correlation between question and section = 0.71

7(a)

Use the substitution $u = a - x$ to show that $I = J$.
(3 marks)

Marking key

Solution

$$I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}dx$$

Let $u = a - x$

Then $du = -dx$; when $x = 0,\ u = a$; when $x = a,\ u = 0$

Then $I = \int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}(-du)$

i.e. $I = \int_0^a \frac{f(a-u)}{f(a-u)+f(u)}du = \int_0^a \frac{f(a-x)}{f(a-x)+f(x)}dx = J$

Specific behaviours

✓ correctly rewrites I new limits

✓ correctly rewrites integrand in terms of u

✓ recognises that $\int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}(-du) = \int_0^a \frac{f(a-x)}{f(a-x)+f(x)}dx = J$

Keywords

Integration (Functions)

Question statistics

Statistics ID = MAS3CD-17
Number of attempts = 1369
Highest mark achieved = 3.00
Lowest mark achieved = 0.00
Mean = 0.80
Standard deviation = 1.01
Question difficulty = Difficult
Correlation between question and section = 0.59

Candidate responses

7(a)

Use the substitution $u = a - x$ to show that $I = J$.
(3 marks)

$x = a \quad u = 0$

$x = 0 \quad u = a$

$$I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx$$

$$= \int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}(-du)$$

$$= -\int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}\,du$$

$$= \int_0^a \frac{f(a-u)}{f(a-u)+f(u)}\,du$$

$$= \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx$$ change variable from u to x.

$$= J$$

Notes

Excellent response
3/3 marks

Applies the substitution correctly and includes the limits in the process.

Correctly rewrites the integrand in terms of *u.*

States that the variables *x* and *u* can be swapped and hence proves the terms are equal.

Let $u = a - x \quad x = a - u \quad \frac{dx}{du} = -1$

$u = a - a = 0, \quad u = a - 0 = a$

$$I = \int_a^0 \frac{F(a-u)}{F(a-u)+f(u)}(-1)\,du = -\int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}\,du$$

$$J = \int_a^0 \frac{f(u)}{f(a-u)+f(u)}(-1)\,du = -\int_a^0 \frac{f(u)}{f(a-u)+f(u)}\,du$$

Satisfactory response
2/3 marks

Uses the substitution to rewrite the new integrand in terms of *u* and changes the limits.

Fails to swap the limits by multiplying by the negative one.

Does not state that the variables *x* and *u* can be swapped and hence fails to prove the terms are equal.

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Question

7(b)

By considering $I + J$, or otherwise, evaluate I in terms of a .
(2 marks)

Marking key

Solution
$I+J=\int_0^a \frac{f(x)}{f(x)+f(a-x)}dx+\int_0^a \frac{f(a-x)}{f(x)+f(a-x)}dx=\int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)}dx$ Hence, since $I=J$ $\quad 2I=\int_0^a dx=a$ i.e. $\quad I=\frac{a}{2}$
Specific behaviours
✓ correctly simplifies to find $2I=\int_0^a dx$ ✓ integrates correctly

Keywords

Integration (Functions)

Question statistics

Statistics ID = MAS3CD-18
Number of attempts = 1105
Highest mark achieved = 2.00
Lowest mark achieved = 0.00
Mean = 0.63
Standard deviation = 0.85
Question difficulty = Difficult
Correlation between question and section = 0.55

Candidate responses

7(b)

By considering $I + J$, or otherwise, evaluate I in terms of a .
(2 marks)

$$I + J = \int_0^a \frac{f(x)}{f(x)+f(a-x)} + \frac{f(a-x)}{f(x)+f(a-x)}\,dx$$

$$= \int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)}\,dx$$

$$= \int_0^a 1\,dx$$

$$= [x]_0^a$$

$$= a - 0$$

$$= a$$

Since $I = J$

$$\therefore \quad I + I = a$$

$$I = \frac{1}{2}a$$

Notes

Excellent response
2/2 marks

Clearly sets out the addition of I and J and follows through with the simplification.

Integrates to get the correct solution.

Candidate responses (continued)

$I + J$

$$= \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx + \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx$$

$y: x^2$

$$= \int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)}\,dx$$

$$= \int_0^a dx$$

$$= [x]_0^a$$

$$= a$$

Notes

Satisfactory response
1/2 marks

Clearly sets out the addition of I and J and follows through with the simplification.

Integrates to get the correct solution, but does not state that the expression is equal to $2I$ and hence misses the solution.

Question

7(c)

Use the result from (b) to evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin\left(x+\frac{\pi}{4}\right)}dx$.
(4 marks)

Marking key

Solution
Let $I=\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin\left(x+\frac{\pi}{4}\right)}dx=\int_0^{\frac{\pi}{2}} \frac{\sqrt{2}\sin x}{\sin x+\cos x}dx$ i.e. $I=\sqrt{2}\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\sin\left(\frac{\pi}{2}-x\right)}dx$ (since $\sin\left(\frac{\pi}{2}-x\right)=\cos x$) i.e. $I=\sqrt{2}\int_0^a \frac{f(x)}{f(x)+f(a-x)}dx$, where $f(x)=\sin x$ and $a=\frac{\pi}{2}$ Hence, using the result from (b), $I=\frac{\sqrt{2}\,\pi}{4}$
Specific behaviours
✓ correctly expands and simplifies $\sin\left(x+\frac{\pi}{4}\right)$ ✓ recognises that $\cos x=\sin\left(\frac{\pi}{2}-x\right)$ and hence that ✓ the integral matches the pattern for $I=\int_0^a \frac{f(x)}{f(x)+f(a-x)}dx$ ✓ states the value of I.

Keywords

Integration (Functions)

Question statistics

Statistics ID = MAS3CD-19
Number of attempts = 967
Highest mark achieved = 4.00
Lowest mark achieved = 0.00
Mean = 0.85
Standard deviation = 1.24
Question difficulty = Difficult
Correlation between question and section = 0.59

Candidate responses

7(c)

Use the result from (b) to evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin\left(x+\frac{\pi}{4}\right)}dx$.
(4 marks)

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin\left(x+\frac{\pi}{4}\right)} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}} dx$$

$$= \int_0^{\pi/2} \frac{\sin x}{\sin x \frac{1}{\sqrt{2}} + \cos x \frac{1}{\sqrt{2}}} dx$$

$$= \int_0^{\pi/2} \frac{\sin x}{\frac{1}{\sqrt{2}}(\sin x + \cos x)} dx$$

$$= \sqrt{2} \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx$$

$$= \sqrt{2} \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \sin\left(\frac{\pi}{2} - x\right)} dx$$

$$= \sqrt{2}\left(\frac{\frac{\pi}{2}}{2}\right)$$

$$= \frac{\sqrt{2}\pi}{4}$$

Notes

Excellent response
4/4 marks

Correctly expands and simplifies the term $\sin\left(x+\frac{\pi}{4}\right)$ in the denominator.

Substitutes $\sin\left(\frac{\pi}{2}-x\right)$ for $\cos x$.

Uses the solution from part (b) to substitute for the integral which equals $\frac{a}{2}$.

States the correct value for I.

Candidate responses (continued)

$$\sin\left(x+\frac{\pi}{4}\right) = \sin x \cdot \cos\left(\frac{\pi}{4}\right) + \cos x \sin\left(\frac{\pi}{4}\right)$$
$$= \frac{\sqrt{2}}{2}\left(\sin x + \cos(x)\right)$$

$$\int_0^{\pi/2} \frac{2\sin x}{\sqrt{2}(\sin x) + \sqrt{2}\cos(x)}\,dx \qquad a = \frac{\pi}{2} \qquad f(a-x)$$

$$\int_0^{\frac{\pi}{2}} \frac{2\sin(x)}{\sqrt{2}\sin x + \sqrt{2}\cos x}\,dx + \int_0^{\pi/2} \frac{2\cos(x)}{\sqrt{2}\sin x + \sqrt{2}\cos(x)}$$

$$= \int_0^{\pi/2} \frac{\cancel{2}^{\sqrt{2}}\cancel{(\sin(x)+\cos(x))}}{\cancel{\sqrt{2}}\cancel{(\sin(x)+\cos(x))}}\,dx = \int_0^{\pi/2} \sqrt{2}\,dx$$

$$f(a-x) = \sin\left(\frac{\pi}{2} - x\right)$$
$$= \sin\left(\frac{\pi}{2}\right)\cos(x) - \cos\left(\frac{\pi}{2}\right)\cdot\sin(x)$$
$$= \cos(x)$$

$$= \left[\sqrt{2}\,x\right]_0^{\pi/2}$$

$$= \frac{\sqrt{2}\cdot\pi}{2}$$

Notes

Satisfactory response
2/4 marks

Correctly expands and simplifies the term $\sin\left(x+\frac{\pi}{4}\right)$ in the denominator.

Adds the second term equal to J and substitutes $\cos x$ for $\sin\left(\frac{\pi}{2}-x\right)$ which equals $2I = 2J$.

Simplifies the integrand correctly to get the correct integral, but does not state that the expression is equal to $2I$ and hence gives the wrong solution.

Examiners' comments

This was undoubtedly the hardest question on the paper, but nevertheless, the responses were still generally poor. The responses to part (a) indicated clearly that the overall understanding of integration by substitution is not good. Many were confused with the new variable and left expressions consisting of a mixture of x's and u's; others made the substitution properly, but did not consider changing the limits on the integral. Of those who could change the integrand and limits correctly, the majority did not recognise the role of a dummy variable within a definite integral. In part (b) although several candidates could calculate $I + J$ correctly, they did not realise how part (a) enabled them to deduce the value of I alone.

Very few candidates continued on to part (c). However, those who did, produced a significant proportion of correct and excellent answers. Some did not manage to complete part (c) because they failed to see the connection between the given integral and the results of part (b).

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Mathematics: Specialist
Stage 3C/3D

Section Two: Calculator-assumed

80 marks

Note:
Raw section total marks = 80

Weighted section total marks = $66\frac{2}{3}\%$

Weighted section statistics

Statistics ID = MAS3CD-S02
Number of attempts = 1395
Highest mark achieved = 65.83
Lowest mark achieved = 0.00
Mean = 36.11
Standard deviation = 11.74
Correlation between section and exam = 0.98

This section has **thirteen (13)** questions.

Working time: 100 minutes.

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Question

Question 8
(5 marks)

Radium decays at a rate proportional to its present mass; that is, if $Q(t)$ is the mass of radium present at time t, then $\frac{dQ}{dt} = k \cdot Q$

It takes 1600 years for any mass of radium to reduce by half.

Question statistics

Statistics ID = MAS3CD-20
Number of attempts = 1377
Highest mark achieved = 5.00
Lowest mark achieved = 0.00
Mean = 4.45
Standard deviation = 1.06
Question difficulty = N/A
Correlation between question and section = 0.48

8(a)

Find the value of k.
(3 marks)

Marking key

Solution
$Q(t) = Ae^{kt}$ $\frac{1}{2} = e^{1600k}$ Hence k = -0.000433 (Accept $\frac{-2\log^{2}}{1600}$)
Writes the specific behaviours
✓ writes the exponential decay equation ✓ writes an equation for the half-life of radium ✓ solves for k

Keywords

Radioactive decay

Question statistics

Statistics ID = MAS3CD-21
Number of attempts = 1375
Highest mark achieved = 3.00
Lowest mark achieved = 0.00
Mean = 2.76
Standard deviation = 0.60
Question difficulty = Easy
Correlation between question and section = 0.38

Candidate responses

8(a)

Find the value of k.
(3 marks)

$$Q = Q_0 e^{kt}$$

$$\frac{1}{2} = e^{1600\cancel{t}k} \qquad k = -4.33 \times 10^{-4}$$

Excellent response
3/3 marks

Correctly writes the decay equation from the given differential equation.

Uses the half-life equation to get an expression with k.

Solves correctly for k.

$$\frac{dQ}{dt} = kQ$$

$$\frac{1}{Q}\, dQ = k\, dt.$$

$$\ln Q = -kt + c.$$

$$Q = e^{-kt+c}$$

$$Q = Ae^{-kt.}$$

$$0.5A = Ae^{-kt}$$

$$0.5 = e^{-k(1600)}$$

$$\ln 0.5 = -1600k.$$

$$K = 0.00043322$$

Satisfactory response
2/3 marks

Writes the decay equation from the given differential equation but includes a negative sign.

Uses the half-life equation to get an expression with k.

Solves for k but gives the wrong sign due to previous error.

Question

8(b)

A factory site is contaminated with radium. The mass of radium on the site is currently five times the safe level. How many years will it be before the mass of radium reaches the safe level?
(2 marks)

Marking key

Solution
Let S be the safe level of radium. Then the initial value satisfies $A = 5S$ i.e. $\frac{1}{5} = e^{\frac{\ln 0.5}{1600}t}$ t = 3715 (Accept 3715 or 3716) It will be 3716 years before the site is safe Or $\frac{1}{5} = e^{-0.000433t}$ t = 3716.95 (Accept 3716 or 3717) It will be 3717 years before the site is safe
Specific behaviours
✓ correctly expresses A in terms of S (or correct ratio) ✓ solves for t

Keywords

Radioactive decay

Question statistics

Statistics ID = MAS3CD-22
Number of attempts = 1334
Highest mark achieved = 2.00
Lowest mark achieved = 0.00
Mean = 1.75
Standard deviation = 0.56
Question difficulty = Easy
Correlation between question and section = 0.36

Candidate responses

8(b)

A factory site is contaminated with radium. The mass of radium on the site is currently five times the safe level. How many years will it be before the mass of radium reaches the safe level?
(2 marks)

$$\frac{1}{5}=e^{-4.33\times10^{-4}t}$$

$t = 3715$ years

Notes

Excellent response
2/2 marks

Correctly sets up the new equation as

$$\frac{1}{5}=e^{-4.33\times10^{-4}t}$$

Solves for *t.*

$$\frac{1}{5}=e^{-0.0004t}$$

$t = 4023$ years

Satisfactory response
1/2 marks

Correctly sets up the new equation as

$$\frac{1}{5}=e^{-0.0004t}$$

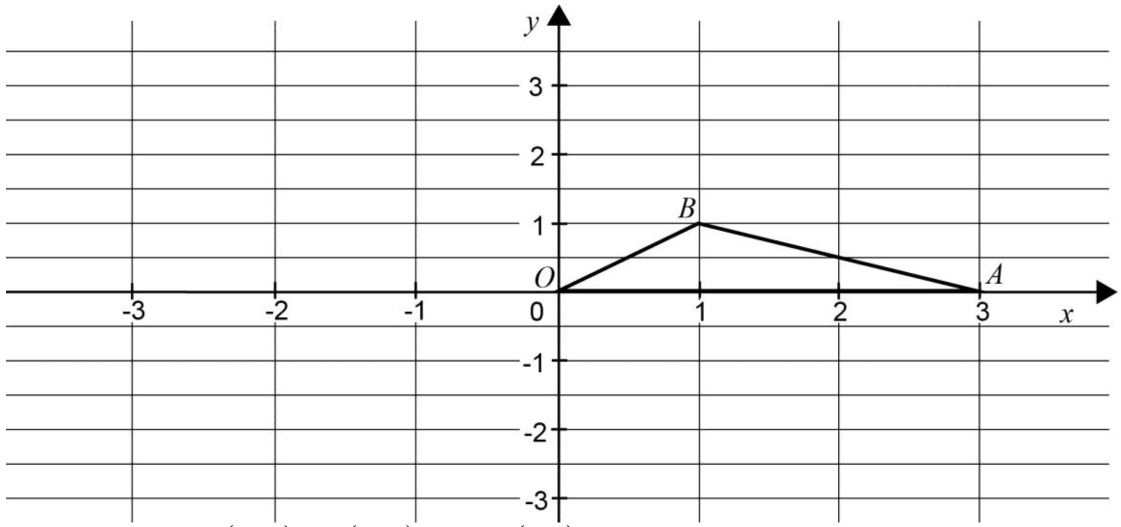
Solves for *t* but rounding the constant from part (a) makes the answer inaccurate.

Examiners' comments

This was a routine question and candidates generally answered it very well. Most of the few problems arose from rounding the value of k too severely. In these types of questions at least three significant figures ought to be retained if subsequent calculations are required. Some used a defining equation containing a negative sign which led to a positive value of k. In part (b) some found the elapsed time to be negative as a result of trying to solve an incorrect problem, but then conveniently managed to 'lose' the minus sign in their answer.

Question

Question 9
(4 marks)



A triangle has vertices $O(0, 0)$, $A(3, 0)$ and $B(1, 1)$, as shown in the diagram above.

Question statistics

Statistics ID = MAS3CD-23
Number of attempts = 1388
Highest mark achieved = 4.00
Lowest mark achieved = 0.00
Mean = 2.79
Standard deviation = 1.29
Question difficulty = N/A
Correlation between question and section = 0.58

9(a)

Write down the matrix that rotates triangle OAB through 90° clockwise about the origin.
(1 mark)

Marking key

Solution
Required matrix is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
Specific behaviours
✓ correctly identifies the 2×2 rotation matrix

Keywords

Rotation

Question statistics

Statistics ID = MAS3CD-24
Number of attempts = 1387
Highest mark achieved = 1.00
Lowest mark achieved = 0.00
Mean = 0.87
Standard deviation = 0.34
Question difficulty = Easy
Correlation between question and section = 0.33

Candidate responses

9(a)

Write down the matrix that rotates triangle OAB through 90° clockwise about the origin.
(1 mark)

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Notes

Excellent response
1/1 mark

Correctly states the rotation matrix.

Question

9(b)

If triangle OAB is transformed by a dilation about the origin of scale factor k ($k > 0$), determine the matrix that will create an image of area 24 square units.
(3 marks)

Marking key

Solution
Area of triangle OAB = 1.5 square units Area of new triangle is 16 times the area of triangle OAB. i.e. $det\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = 16$ Hence, required matrix is $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$
Calculates the ratio specific behaviours
✓ calculates the ratio between the areas of shapes before and after dilation ✓ correctly states the dilation matrix in terms of k ✓ solves for k

Or

Solution
Dilation matrix is $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ Coordinates of dilated triangle are $(0, 0)$, $(3k, 0)$, (k, k) Hence area of dilated triangle = $\frac{1}{2} \times 3k \times k = \frac{3}{2}k^2$ Hence $\frac{3}{2}k^2 = 24$ i.e. $k = 4$
Specific behaviours
✓ correctly states the dilation matrix in terms of k ✓ uses the new coordinates to determine the area of the dilated triangle in terms of k ✓ solves for k

Keywords

Dilation

Question statistics

Statistics ID = MAS3CD-25
Number of attempts = 1339
Highest mark achieved = 3.00
Lowest mark achieved = 0.00
Mean = 2.00
Standard deviation = 1.14
Question difficulty = Moderate
Correlation between question and section = 0.51

Candidate responses

9(b)

If triangle OAB is transformed by a dilation about the origin of scale factor k ($k > 0$), determine the matrix that will create an image of area 24 square units.
(3 marks)

$\text{Area}_{\text{original}} = 1.5 \quad \therefore \ |\det| = \frac{24}{1.5}$

$= 16$

$\therefore \ k^2 = 16$

$k = 4$

$\therefore$ matrix is $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

Notes

Excellent response
3/3 marks

Calculates the ratio between the areas of the object before and after the dilation.

Solves for k.

States the correct dilation matrix.

$\therefore M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

but $|M| = 24$

$\therefore \ k^2 = 24$

$k = 2\sqrt{6}$

$\therefore \ m = \begin{pmatrix} 2\sqrt{6} & 0 \\ 0 & 2\sqrt{6} \end{pmatrix}$

Satisfactory response
2/3 marks

Miscalculates the ratio between the areas of the object before and after the dilation.

Solves for k.

Follows through with a calculation for the dilation matrix.

Examiners' comments

Candidates made very few mistakes in part (a). Part (b) was answered less well however most knew that the ratio of the area of the image to the area of the original figure is the determinant of the transforming matrix. Common mistakes included forgetting to take the square root of the determinant to isolate k and, more worryingly, a significant minority incorrectly calculated the area of the triangle *OAB* to be twice its actual value.

Question

Question 10
(8 marks)

Two radio-controlled model planes take off at the same time from two different positions and with constant velocities. Model A leaves from the point with position vector $(-3\boldsymbol{i}-7\boldsymbol{j})$ metres and has velocity $(5\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k})$ m/s; model B leaves from the point with position vector $(7\boldsymbol{i}-\boldsymbol{j}-8\boldsymbol{k})$ metres and has velocity $(3\boldsymbol{i}-4\boldsymbol{j}+6\boldsymbol{k})$ m/s.

Question statistics

Statistics ID = MAS3CD-26
Number of attempts = 1381
Highest mark achieved = 8.00
Lowest mark achieved = 0.00
Mean = 5.88
Standard deviation = 2.29
Question difficulty = N/A
Correlation between question and section = 0.66

10(a)

Find the distance between the two model planes after 1 second of flight.
(3 marks)

Marking key

Solution
$\boldsymbol{r}_A=-3\boldsymbol{i}-7\boldsymbol{j}+t(5\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k})$; $\boldsymbol{r}_B=7\boldsymbol{i}-\boldsymbol{j}-8\boldsymbol{k}+t(3\boldsymbol{i}-4\boldsymbol{j}+6\boldsymbol{k})$ i.e. $\boldsymbol{r}_A=(5t-3)\boldsymbol{i}+(-t-7)\boldsymbol{j}+(2t)\boldsymbol{k}$; $\boldsymbol{r}_B=(3t+7)\boldsymbol{i}+(-4t-1)\boldsymbol{j}+(6t-8)\boldsymbol{k}$ When $t=1$, $\boldsymbol{r}_A=2\boldsymbol{i}-8\boldsymbol{j}+2\boldsymbol{k}$; $\boldsymbol{r}_B=10\boldsymbol{i}-5\boldsymbol{j}-2\boldsymbol{k}$ Hence ${}_A\boldsymbol{r}_B=-8\boldsymbol{i}-3\boldsymbol{j}+4\boldsymbol{k}$ Hence distance between the two planes = norm $[-8,\ -3,\ 4]$ = 9.43 metres
Specific behaviours
✓ correctly determines $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ ✓ determines ${}_A\boldsymbol{r}_B$ when $t=1$ ✓ finds the required distance

Keywords

Vectors (Geometry)

Question statistics

Statistics ID = MAS3CD-27
Number of attempts = 1380
Highest mark achieved = 3.00
Lowest mark achieved = 0.00
Mean = 2.55
Standard deviation = 0.74
Question difficulty = Easy
Correlation between question and section = 0.42

Candidate responses

10(a)

Find the distance between the two model planes after 1 second of flight.
(3 marks)

$$r_A = \begin{pmatrix} -3 \\ -7 \\ 0 \end{pmatrix} + t\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$$

$$r_B = \begin{pmatrix} 7 \\ -1 \\ -8 \end{pmatrix} + t\begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix}$$

$$t = 1 \Rightarrow \quad r_A = \begin{pmatrix} 2 \\ -8 \\ 2 \end{pmatrix} \qquad r_B = \begin{pmatrix} 10 \\ -5 \\ -2 \end{pmatrix}$$

$$r_A - r_B = \begin{pmatrix} -8 \\ -3 \\ 4 \end{pmatrix} \quad |r_A - r_B| = 9.43 \text{ m}$$

$$\therefore 9.43 \text{ m}$$

Notes

Excellent response
3/3 marks

Correctly defines position vectors, $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$.

Calculates ${}_A\boldsymbol{r}_B$ when $t = 1$.

Calculates the required distance.

$$A = \begin{pmatrix} -3 \\ -7 \\ 0 \end{pmatrix} + t\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 7 \\ -1 \\ -8 \end{pmatrix} + t\begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix}$$

at $t = 1$,

$$A = \begin{pmatrix} -3 \\ -7 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \\ 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 7 \\ -1 \\ -8 \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \\ -8 \end{pmatrix}$$

$$|B - A| = \left|\begin{pmatrix} 8 \\ 3 \\ -10 \end{pmatrix}\right| = \sqrt{8^2 + 3^2 + 10^2} = \sqrt{173} \text{ m}$$

Satisfactory response
2/3 marks

Correctly defines position vectors, $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$.

Calculates ${}_A\boldsymbol{r}_B$ when $t = 1$, but makes a numerical error.

Follows through to calculate the distance.

Question

10(b)

Find:

(i) the shortest distance between the two model planes.

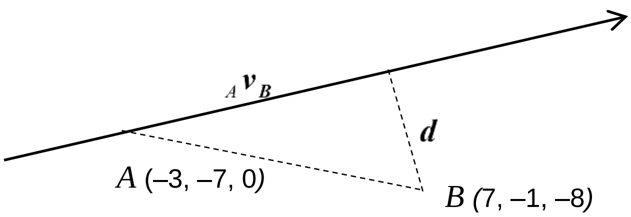
(ii) the time when this occurs.

(5 marks)

Marking key

Solution

(i) and (ii)



$$\boldsymbol{d} = \overrightarrow{BA} + \boldsymbol{t}_{\boldsymbol{A}}\boldsymbol{v}_{\boldsymbol{B}} = -3\boldsymbol{i} - 7\boldsymbol{j} - (7\boldsymbol{i} - \boldsymbol{j} - 8\boldsymbol{k}) + \boldsymbol{t}(2\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k})$$

i.e. $\boldsymbol{d} = (2\boldsymbol{t} - 10)\boldsymbol{i} + (3\boldsymbol{t} - 6)\boldsymbol{j} + (-4\boldsymbol{t} + 8)\boldsymbol{k}$

$\boldsymbol{d} \bullet {}_A\boldsymbol{v}_{\boldsymbol{B}} = 0$

i.e. $((2\boldsymbol{t} - 10)\boldsymbol{i} = (3\boldsymbol{t} - 6)\boldsymbol{j} + (-4\boldsymbol{t} + 8)\boldsymbol{k}) \bullet (2\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}) = 0$

i.e. $t = \frac{70}{29} = 2.41$ seconds

and $|\boldsymbol{d}| = 5.57$ metres

Or $d = (2t - 10)\boldsymbol{i} + (3t - 6)\boldsymbol{j} + (-4t + 8)\boldsymbol{k}$

Hence $|\boldsymbol{d}| = \sqrt{(2\boldsymbol{t} - 10)^2 + (3\boldsymbol{t} - 6)^2 + (-4\boldsymbol{t} + 8)^2}$

Use a calculator to find the minimum value of $|\boldsymbol{d}| = 5.57$ metres

and the value of *t* for which the minimum occurs:

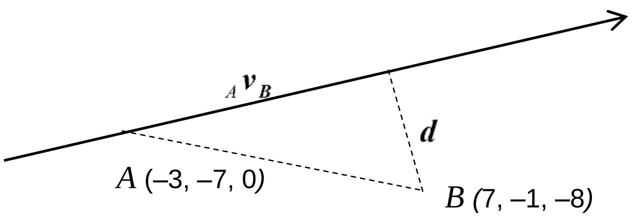
i.e. $t = \frac{70}{29} = 2.41$ seconds

Specific behaviours

- ✓ expresses the general distance between the two planes at time *t* as $\boldsymbol{d} = \overrightarrow{BA} + t\ {}_A\boldsymbol{v}_{\boldsymbol{B}}$
- ✓ expresses, $\boldsymbol{d}$ in terms of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and *t*
- ✓ either determines $\boldsymbol{d} \bullet {}_A\boldsymbol{v}_{\boldsymbol{B}} = 0$ or $|\boldsymbol{d}|$
- ✓ solves for the minimum value of $|\boldsymbol{d}|$
- ✓ solves for the corresponding value of *t*

Or

Marking key (continued)

Solution
(i) and (ii)  Angle between $\overrightarrow{AB}$ and ${}_A\boldsymbol{v}_B$ from CAS is angle ([10, 6, –8] , [2, 3, –4]) = 23.20°. Hence $\lvert\boldsymbol{d}\rvert = \lvert\overrightarrow{AB}\rvert \times \sin 23.20^{\circ} = 5.57$ m Also $t \times {}_A\boldsymbol{v}_B = \lvert\overrightarrow{AB}\rvert \times \cos 23.20^{\circ} = 13.00$ m Hence $t = \dfrac{13.00}{norm[2, 3, -4]} = 2.41$ seconds

Specific behaviours
✓ draws a right triangle with A, B, ${}_A\boldsymbol{v}_B$ and $\boldsymbol{d}$ shown
✓ determines the angle between the vectors $\overrightarrow{AB}$ and ${}_A\boldsymbol{v}_B$
✓ uses the right triangle to determine the length of $\boldsymbol{d}$
✓ uses the right triangle to determine the length of $t \times {}_A\boldsymbol{v}_B$
✓ solves for t

Keywords

Distance

Question statistics

Statistics ID = MAS3CD-28
Number of attempts = 1322
Highest mark achieved = 5.00
Lowest mark achieved = 0.00
Mean = 3.48
Standard deviation = 1.84
Question difficulty = Moderate
Correlation between question and section = 0.60

Candidate responses

10(b)

Find:

(i) the shortest distance between the two model planes.

(ii) the time when this occurs.

(5 marks)

$${}_A r_B = r_A - r_B = \begin{pmatrix} -10 \\ -6 \\ 8 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$$

$$|r_A - r_B| = \sqrt{(-10+2t)^2 + (3t-6)^2 + (8-4t)^2}$$

$$= \sqrt{29t^2 - 140t + 200}$$

$$= \sqrt{29(t - \tfrac{70}{29})^2 + 31.03} \geq 5.57.$$

$\therefore$ 5.57 m

$$t = \frac{70}{29} = 2.41 \text{ s}$$

$\therefore$ 2.41 s.

Notes

Excellent response
5/5 marks

Expresses the displacement between the two model planes in terms of *t*.

$${}_A\boldsymbol{r}_B = \overrightarrow{BA} + t\,{}_A\mathbf{v}_B$$

Expresses, the displacement ${}_A\mathbf{r}_B$ in terms of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and t.

Calculates the magnitude of the displacement vector $|{}_A\boldsymbol{r}_B|$.

Solves for the minimum value for $|{}_A\boldsymbol{r}_B|$. Solves for the corresponding value for *t.*

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Candidate responses (continued)

$${}_A r_B = {}_B r_A$$
$$= \begin{pmatrix} 7+3t \\ -1-t \\ -8 \end{pmatrix} - \begin{pmatrix} -3+5t \\ -7+t \\ 2t \end{pmatrix}$$
$$= \begin{pmatrix} 10-2t \\ 6-2t \\ -8-2t \end{pmatrix}$$

$${}_A v_B = v_B - v_A$$
$$= \begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix}$$

shortest distance when ${}_A r_B \cdot {}_A v_B = 0$

$$\begin{pmatrix} 10-2t \\ 6-3t \\ -8-2t \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix} = 0$$
$$-2(10-2t)-3(6-3t)+4(-8-2t)=0$$

solving gives $t=20$. hence

$t=20$

$${}_A r_B = \begin{pmatrix} 10-40 \\ 6-60 \\ -8-20 \end{pmatrix}$$
$$= \begin{pmatrix} -30 \\ -54 \\ -28 \end{pmatrix}$$
$$|{}_A r_B| = \sqrt{30^2+54^2+28^2}$$
$$= \sqrt{4600}$$
$$= 67.823\text{m}$$

Notes

Satisfactory response
3/5 marks

Expresses the displacement between the two model planes in terms of t ${}_A\boldsymbol{r}_B = \boldsymbol{r}_B - \boldsymbol{r}_A$, which is the opposite of the correct answer.

Writes the wrong vector for $\mathbf{r}_B$ by omitting the term in *t.*

Expresses, the displacement ${}_A\mathbf{v}_B$ in terms of i, j. and k.

Uses the dot product $({}_A\mathbf{r}_B)\cdot({}_A\mathbf{v}_B) = 0$ to solve for t.

Calculates the magnitude of the displacement vector for the given value of *t.*

Examiners' comments

Most candidates performed well on part (a) which seemed to be a familiar type of question. Most errors arose due to transcription or arithmetical slips although some assumed that the distance between the two planes was just the difference in their individual distances from the origin.

There were three main approaches to part (b): using the dot product, minimising the length of the displacement vector or graphing this length as a function of time. A few candidates just gave the solution to the problem without any indication of the method adopted. As the question was worth more than 2 marks, this was deemed to be insufficient working. It is not acceptable working simply to state that an e-activity on the CAS was used.

Question

Question 11
(4 marks)

The triangle ABC has vertices $A(2, 1, 0)$, $B(3, -3, 3)$ and $C(5, 0, 4)$.

Question statistics

Statistics ID = MAS3CD-29
Number of attempts = 1375
Highest mark achieved = 4.00
Lowest mark achieved = 0.00
Mean = 2.66
Standard deviation = 1.30
Question difficulty = N/A
Correlation between question and section = 0.54

11(a)

Find the size of $\angle ABC$ correct to the nearest degree.
(2 marks)

Marking key

Solution
$\overrightarrow{BA} = -\boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k}$; $\overrightarrow{BC} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ Hence angle ([–1, 4, –3] , [2 ,3, 1]) = $\angle ABC = 68^\circ$ (using a CAS)
Specific behaviours
✓ determines the components of vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$ ✓ calculates the required angle

Keywords

Acute angled triangles, Cartesian planes

Question statistics

Statistics ID = MAS3CD-30
Number of attempts = 1370
Highest mark achieved = 2.00
Lowest mark achieved = 0.00
Mean = 1.29
Standard deviation = 0.76
Question difficulty = Moderate
Correlation between question and section = 0.35

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Candidate responses

11(a)

Find the size of $\angle ABC$ correct to the nearest degree.
(2 marks)

$\overrightarrow{AB} = \langle 1, -4, 3 \rangle$

$\overrightarrow{CB} = \langle -2, -3, -1 \rangle$

$\therefore \angle ABC = 68^\circ$ (0 d.p.)

Notes

Excellent response
2/2 marks

Correctly gives vectors $\overrightarrow{AB}$ and $\overrightarrow{CB}$ in component form.

Calculates the angle correctly to the nearest degree.

$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$ $\overrightarrow{BC} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

Angle between $\overrightarrow{AB}$ and $\overrightarrow{BC}$, $\angle ABC \simeq 112^\circ$.

Satisfactory response
1/2 marks

Correctly gives vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ in component form.

Gives the supplementary angle.

Question

11(b)

Given that the vector $(-13\boldsymbol{i}+5\boldsymbol{j}+11\boldsymbol{k})$ is perpendicular to the plane which contains the triangle ABC, find the vector equation of this plane.
(2 marks)

Marking key

Solution
Vector equation of the plane is $(\boldsymbol{r}-(2\boldsymbol{i}+\boldsymbol{j}))\bullet(-13\boldsymbol{i}+5\boldsymbol{j}+11\boldsymbol{k})=0$ Or $(\boldsymbol{r}-(3\boldsymbol{i}-3\boldsymbol{j}+3\boldsymbol{k}))\bullet(-13\boldsymbol{i}+5\boldsymbol{j}+11\boldsymbol{k})=0$ Or $(\boldsymbol{r}-(5\boldsymbol{i}+4\boldsymbol{k}))\bullet(-13\boldsymbol{i}+5\boldsymbol{j}+11\boldsymbol{k})=0$ Or $\boldsymbol{r}\bullet(-13\boldsymbol{i}+5\boldsymbol{j}+11\boldsymbol{k})=-21$
Specific behaviours
✓ determines a general vector in the plane ✓ correctly determines the plane equation

Keywords

Plane figures

Question statistics

Statistics ID = MAS3CD-31
Number of attempts = 1306
Highest mark achieved = 2.00
Lowest mark achieved = 0.00
Mean = 1.44
Standard deviation = 0.86
Question difficulty = Moderate
Correlation between question and section = 0.42

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Candidate responses

11(b)

Given that the vector $(-13\boldsymbol{i}+5\boldsymbol{j}+11\boldsymbol{k})$ is perpendicular to the plane which contains the triangle ABC, find the vector equation of this plane.
(2 marks)

$$\begin{pmatrix}2\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-13\\5\\11\end{pmatrix} = -26 + 5 = -21$$

$$\therefore \quad \underset{\sim}{r} \cdot \begin{pmatrix}-13\\5\\11\end{pmatrix} = -21$$

Notes

Excellent response
2/2 marks

Uses the vector normal to the given plane to define its vector equation.

Carries through the calculation accurately and uses the correct conventions in the answer.

$$r\cdot\begin{pmatrix}-13\\5\\11\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-13\\5\\11\end{pmatrix}$$

$$r\cdot\begin{pmatrix}-13\\5\\11\end{pmatrix} = -26+5+11 = -10.$$

$$\therefore \ r\cdot\begin{pmatrix}-13\\5\\11\end{pmatrix} = -10.$$

Satisfactory response
1/2 marks

Uses the vector normal to the given plane to define its vector equation.

Makes a calculation error with the dot product giving the wrong constant c in the answer $\mathbf{r}\bullet\mathbf{n}=c$.

Examiners' comments

Many candidates found the supplement of the required angle in part (a) as they did not appreciate the significance of the point *B* being in the middle of the specified angle *ABC*. Some candidates did not utilise the capabilities of the CAS so spent a long time in calculations for a question worth only a couple of marks. Responses to part (b) were of very variable quality; a few did not use the normal to the plane but instead used one point and two parallel vectors to deduce the required equation.

Question

Question 12
(6 marks)

Three dry-cleaning companies, A, B and C compete for business. Each year A loses 40% of its customers to B and 20% to C; B loses 30% to A and 50% to C; C loses 60% to A and 10% to B.

Question statistics

Statistics ID = MAS3CD-32
Number of attempts = 1391
Highest mark achieved = 6.00
Lowest mark achieved = 0.00
Mean = 5.11
Standard deviation = 1.32
Question difficulty = N/A
Correlation between question and section = 0.49

12(a)

Complete the following transition matrix.
(2 marks)

1.
2.
3.

$$\text{To}\quad \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \text{From} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.4 & \_\_\_\_ & \_\_\_\_ \\ 0.4 & \_\_\_\_ & \_\_\_\_ \\ 0.2 & \_\_\_\_ & \_\_\_\_ \end{bmatrix} \end{array}$$

Marking key

Solution
$\begin{bmatrix} 0.4 & 0.3 & 0.6 \\ 0.4 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}$
Specific behaviours
✓✓ correctly completes the transition matrix Or ✓ partially correct (at least four(4) correct)

Keywords

Matrices

Question statistics

Statistics ID = MAS3CD-33
Number of attempts = 1391
Highest mark achieved = 2.00
Lowest mark achieved = 0.00
Mean = 1.96
Standard deviation = 0.22
Question difficulty = Very easy
Correlation between question and section = 0.14

Candidate responses

12(a)

Complete the following transition matrix.
(2 marks)

$$\begin{array}{ccc} & & \text{From} \\ & & \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ & \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} & \begin{bmatrix} 0.4 & \underline{0.3} & \underline{0.6} \\ 0.4 & \underline{0.2} & \underline{0.1} \\ 0.2 & \underline{0.5} & \underline{0.3} \end{bmatrix} \\ \text{To} & & \end{array} = T.$$

$$L = \begin{array}{ccc} & & \text{From} \\ & & \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ & \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} & \begin{bmatrix} 0.4 & \underline{0.3} & \underline{0.6} \\ 0.4 & \underline{0.3} & \underline{0.1} \\ 0.2 & \underline{0.5} & \underline{0.3} \end{bmatrix} \\ \text{To} & & \end{array}$$

Notes

Excellent response
2/2 marks

Correctly completes all elements of the transition matrix.

Satisfactory response
1/2 marks

Copies an incorrect element into the transition matrix.

Question

12(b)

At the end of 2011, company A will have 80% of market share, while B and C will have 10% each. What will be the market share of each company at the end of 2012?
(2 marks)

Marking key

Solution
$\begin{bmatrix} 0.4 & 0.3 & 0.6 \\ 0.4 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.3 \end{bmatrix} \times \begin{bmatrix} 80 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 41 \\ 35 \\ 24 \end{bmatrix}$ Hence A will have 41%, B will have 35% and C will have 24%.
Specific behaviours
✓ sets up column matrix for market share ✓ accurately multiplies transition matrix with market share matrix

Keywords

Matrices

Question statistics

Statistics ID = MAS3CD-34
Number of attempts = 1315
Highest mark achieved = 2.00
Lowest mark achieved = 0.00
Mean = 1.76
Standard deviation = 0.60
Question difficulty = Easy
Correlation between question and section = 0.33

Candidate responses

12(b)

At the end of 2011, company A will have 80% of market share, while B and C will have 10% each. What will be the market share of each company at the end of 2012?
(2 marks)

$$T \cdot \begin{bmatrix} 0.80 \\ 0.1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0.41 \\ 0.35 \\ 0.24 \end{bmatrix}$$

So:

A	B	C
41%	35%	24%

Notes

Excellent response
2/2 marks

Defines the column matrix containing the market share for each company.

Accurately multiplies by the transition matrix to give the percentage market share for each company.

$$\begin{bmatrix} 0.4 & 0.3 & 0.6 \\ 0.4 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.3 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0.41 \\ 0.35 \\ 0.25 \end{bmatrix}$$

At end of 2012
A has 41% of market share
B has 35% of market share
C has 25% of market share.

Satisfactory response
1/2 marks

Sets up the matrix calculation correctly, but fails to round one of the results correctly.

Question

12(c)

If these conditions remain unchanged, what will be the long-term percentage market share for each company, correct to **one (1)** decimal place?
(2 marks)

Marking key

Solution
$\begin{bmatrix} 0.4 & 0.3 & 0.6 \\ 0.4 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}^{20} \times \begin{bmatrix} 80 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 43.6 \\ 25.6 \\ 30.8 \end{bmatrix}$ Hence A will have 43.6%, B will have 25.6% and C will have 30.8%.
Specific behaviours
✓ chooses a suitably large index for the transition matrix to ensure stability   ✓ states the value of each company’s share

Keywords

Matrices

Question statistics

Statistics ID = MAS3CD-35
Number of attempts = 1295
Highest mark achieved = 2.00
Lowest mark achieved = 0.00
Mean = 1.60
Standard deviation = 0.65
Question difficulty = Easy
Correlation between question and section = 0.35

Candidate responses

12(c)

If these conditions remain unchanged, what will be the long-term percentage market share for each company, correct to **one (1)** decimal place?
(2 marks)

$$T^{60} \cdot \begin{bmatrix} 0.7 \\ 0.1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0.436 \\ 0.256 \\ 0.308 \end{bmatrix}$$

$$T^{70} \cdot \begin{bmatrix} 0.7 \\ 0.1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0.436 \\ 0.256 \\ 0.308 \end{bmatrix}$$

So: <u>A</u> 43.6% <u>B</u> 25.6% <u>C</u> 30.8%

Notes

Excellent response
2/2 marks

Chooses two large index values to use with the transition matrix, showing that the market share has stabilised in the long term.

Correctly states the market share for each company.

$$L^{20} \cdot \begin{bmatrix} 0.8 \\ 0.1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0. \\ \\ \end{bmatrix}$$

A = 6.2 B =

Satisfactory response
1/2 marks

Chooses a large index value to use with the transition matrix, but fails to follow through with the final calculation.

Examiners' comments

This was one of the easiest questions on the examination and answered very well as evidenced by the very high mean mark. All but a handful completed the matrix in part (a). Most of the errors in part (b) arose through writing the market share as a row vector rather than a column vector and then pre-multiplying with it. This led to a total market share in excess of 100% but this did not seem to trouble any candidates. More candidates attempted part (c) than part (b) and most realised that the question demanded the matrix be raised to a sufficiently high power. Several spoiled their work by failing to identify the market share for each company or to give it in the specified percentage format.

Question

Question 13
(6 marks)

An engine piston undergoes simple harmonic motion that can be described by the differential equation $\frac{d^2x}{dt^2} = -9x$, here x m is the displacement of the piston from its mean position at t seconds.

Question statistics

Statistics ID = MAS3CD-36
Number of attempts = 1380
Highest mark achieved = 6.00
Lowest mark achieved = 0.00
Mean = 3.44
Standard deviation = 1.96
Question difficulty = N/A
Correlation between question and section = 0.59

13(a)

Write down the period of the motion.
(1 mark)

Marking key

Solution
$n^2 = 9$ where n is the angular velocity Hence the period of motion is $\frac{2\pi}{3}$ seconds
Specific behaviours
✓ correctly defines the period

Keywords

Periodic functions

Question statistics

Statistics ID = MAS3CD-37
Number of attempts = 1376
Highest mark achieved = 1.00
Lowest mark achieved = 0.00
Mean = 0.88
Standard deviation = 0.32
Question difficulty = Easy
Correlation between question and section = 0.36

Candidate responses

13(a)

Write down the period of the motion.
(1 mark)

$$T = \frac{2\pi}{3}$$

Period is $\frac{2\pi}{3}$ seconds.

Notes

Excellent response
1/1 mark

States the correct time period of the motion.

Question

13(b)

If the maximum speed of the piston is 5 m/s, find the amplitude of the motion.
(2 marks)

Marking key

Solution
$v_{\max} = An$ where A is the amplitude Hence $A = \frac{5}{3}$ metres
Specific behaviours
✓ uses the equation $v_{\max} = An$ or $v^2 = n^2(A^2 - x^2)$ at $x = 0$ ✓ correctly solves for A

Keywords

Amplitude

Question statistics

Statistics ID = MAS3CD-38
Number of attempts = 1342
Highest mark achieved = 2.00
Lowest mark achieved = 0.00
Mean = 1.45
Standard deviation = 0.81
Question difficulty = Moderate
Correlation between question and section = 0.45

Candidate responses

13(b)

If the maximum speed of the piston is 5 m/s, find the amplitude of the motion.
(2 marks)

$V_{max} = |ka|$

$\Rightarrow a = \frac{5}{3}$

The amplitude is $\frac{5}{3}$ units.

Notes

Excellent response
2/2 marks

Uses the relationship $v_{\max} = |ka|$ to solve for a.

Follows through and calculates the correct answer.

Max $\dot{x} = 5$ m/s

~~ ~~ x

$V_{max} = |k \times a|$

$5 = \frac{22}{3} a.$

$a = \frac{15}{22}$ m/s

Satisfactory response
1/2 marks

Uses the relationship $v_{\max} = |ka|$ to solve for a.

Mistakenly substitutes for T instead of k when solving for a.

Question

13(c)

The amplitude and period of the motion are now changed, but the piston still undergoes simple harmonic motion. These new readings are taken:

when x = 1 m, speed $=\sqrt{60}$ m/s; when x = 3 m, speed $=\sqrt{28}$ m/s

Find the new exact values for:

(i) the period.
(ii) the amplitude.

(3 marks)

Marking key

Solution
$v^2 = n^2\left(A^2 - x^2\right)$ Hence: $60 = n^2\left(A^2 - 1\right)$ and $28 = n^2\left(A^2 - 9\right)$ Solving gives n = 2 and A = 4 Hence: (i) period = π seconds (ii) amplitude = 4 metres
Specific behaviours
✓ correctly uses the equation $v^2 = n^2\left(A^2 - x^2\right)$ ✓ uses a CAS to solve for n ✓ uses a CAS to solve for A

Keywords

Periodic functions, Amplitude

Question statistics

Statistics ID = MAS3CD-39
Number of attempts = 1192
Highest mark achieved = 3.00
Lowest mark achieved = 0.00
Mean = 1.32
Standard deviation = 1.38
Question difficulty = Moderate
Correlation between question and section = 0.43

Candidate responses

13(c)

The amplitude and period of the motion are now changed, but the piston still undergoes simple harmonic motion. These new readings are taken:

when $x = 1$ m, speed $= \sqrt{60}$ m/s; when $x = 3$ m, speed $= \sqrt{28}$ m/s

Find the new exact values for:

(i) the period.
(ii) the amplitude.

(3 marks)

~~$x = a\sin(kt)$~~
~~$v = ka\sin(kt)$~~
~~when $x = 1$, $v = \sqrt{60}$~~
~~$1 = a\sin(kt)$~~
~~$\sqrt{60} =$~~

$v^2 = k^2(a^2 - x^2)$.

at $x = 1$, $v = \sqrt{60}$

$\Rightarrow 60 = k^2(a^2 - 1) \Rightarrow k^2 = \frac{60}{a^2-1}$.

at $x = 3$, $v = \sqrt{28}$

$\Rightarrow 28 = k^2(a^2 - 9) \Rightarrow k^2 = \frac{28}{a^2-9}$.

$\frac{60}{a^2-1} = \frac{28}{a^2-9}$

$60(a^2-9) = 28(a^2-1)$

$\Rightarrow a = \sqrt{\frac{60 \times 9 - 28}{32}}$

$= 4$.

$\Rightarrow k = 2$

$\Rightarrow T = \frac{2\pi}{2} = \pi \Rightarrow$ Period is π second

(ii) the amplitude.

As shown above, the amplitude is 4 m.

Notes

Excellent response
3/3 marks

Uses the relationship $v^2 = k^2\left(a^2 - x^2\right)$,
to develop two equations for solving a and k.

Solves for the period T.

Solves for a.

Candidate responses (continued)

(i) **the period.**

$T = \frac{2\pi}{k}$ $\quad x = a\sin(k\theta + \mu)$

$\dot{x} = ak\cos(k\theta + \mu)$

$v^2 = k^2(a^2 - x^2)$

Solve simultaneously $(\sqrt{60})^2 = k^2(a^2 - 1)$ & $(\sqrt{48})^2 = k^2(a^2 - 3)$

$k = 4$, $a = \frac{\sqrt{19}}{2}$

~~T~~ period $= \frac{2\pi}{k}$

$= \frac{2\pi}{4}$

$= \frac{\pi}{2}$ s

(ii) **the amplitude.**

refer to (i)

$a = \frac{\sqrt{19}}{2}$

Notes

Satisfactory response
2/3 marks

Uses the relationship
$v^2 = k^2\left(a^2 - x^2\right)$
to develop two equations for solving a and k.

Fails to solve correctly for the period T and amplitude a.

Examiners' comments

Some candidates did not attempt any part of the problem. Most who tackled part (a) did so correctly; however there were some long drawn out methods for part (b). A common mistake here was to use the period calculated in part (a) as the value of 'n' rather than 3. Some candidates could write down the required amplitude directly, some started with the displacement/time expression and differentiated while others knew of the existence of a displacement/speed relationship. It was the candidate in the latter category who had most success with part (c). Those who derived the requisite pair of simultaneous equations solved them either by hand or by using their CAS facility. Those who used the technology tended to make fewer mistakes. A number of candidates did not discount the negative solutions and quoted amplitudes and/or periods less than zero. A few candidates attempted to solve part (c) starting from the displacement/time function, but of these the majority were tripped up because they did not realise that different values of the time t should be assumed for the two sets of information provided, thus giving rise to four sets of equations in four unknowns.

Question

Question 14

The points P, Q and R are such that $\overrightarrow{PQ} = 5\boldsymbol{i}$ and $\overrightarrow{PR} = \boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$

Find the vector $\overrightarrow{RM}$ that is parallel to $\overrightarrow{PQ}$ and such that the size of $\angle RQM$ is 90º.
(5 marks)

Marking key

Solution
Let $\overrightarrow{RM} = \lambda\boldsymbol{i}$ for some real number λ $\overrightarrow{QM} = \overrightarrow{QP} + \overrightarrow{PR} + \overrightarrow{RM} = -5\boldsymbol{i} + \boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k} + \lambda\boldsymbol{i} = (-4+\lambda)\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$ If angle RQM is 90º, then $\overrightarrow{QM} \bullet \overrightarrow{QR} = 0$ i.e. $((-4+\lambda)\ \boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}) \bullet (-4\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}) = 0$ i.e. $\lambda = 9$ so $\overrightarrow{RM} = 9\boldsymbol{i}$
Specific behaviours
✓ uses parallelism to define $\overrightarrow{RM}$ ✓ expresses $\overrightarrow{QM}$ in terms of $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{RM}$ ✓ simplifies in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ ✓ equates the dot product of perpendicular vectors to zero   ✓ solves for λ and hence $\overrightarrow{RM}$

Keywords

Vectors (Geometry)

Question statistics

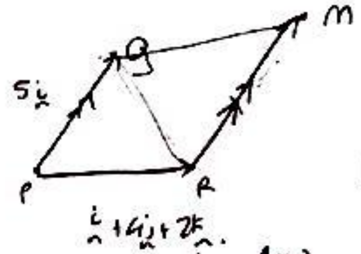
Statistics ID = MAS3CD-40
Number of attempts = 1346
Highest mark achieved = 5.00
Lowest mark achieved = 0.00
Mean = 1.93
Standard deviation = 1.85
Question difficulty = Moderate
Correlation between question and section = 0.61

Candidate responses

Question 14

The points P, Q and R are such that $\overrightarrow{PQ} = 5\boldsymbol{i}$ and $\overrightarrow{PR} = \boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$

Find the vector $\overrightarrow{RM}$ that is parallel to $\overrightarrow{PQ}$ and such that the size of $\angle RQM$ is 90º.
(5 marks)



$\overrightarrow{RM} = k\,\underset{\sim}{i}.$

$$\overrightarrow{RQ} = \overrightarrow{PQ} - \overrightarrow{PR}.$$
$$= \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ -2 \end{pmatrix}.$$

$$\overrightarrow{RQ} \cdot \overrightarrow{QM} = 0.$$
$$\overrightarrow{RQ} \cdot \left(\overrightarrow{RM} - \overrightarrow{RQ}\right) = 0.$$
$$\begin{pmatrix} 4 \\ -4 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} k-4 \\ 4 \\ 2 \end{pmatrix} = 0.$$

$$4k - 16 - 16 - 4 = 0.$$
$$4K = 36$$
$$K = 9$$

$$\Rightarrow \overrightarrow{RM} = 9\,\underset{\sim}{i}.$$

Notes

Excellent response
5/5 marks

Defines $\overrightarrow{RM}$ as a scale factor of $\overrightarrow{PQ}$ since they are parallel.

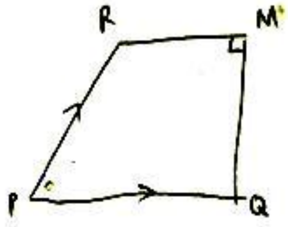
Draws a neat diagram which helps define the various vectors.

Expresses $\overrightarrow{QM}$ and $\overrightarrow{RQ}$ in i, j and k component form.

Uses the dot product $\overrightarrow{QM} \cdot \overrightarrow{RQ} = 0$ to give perpendicularity.

Solves for k and hence $\overrightarrow{RM}$.

Candidate responses (continued)



$\vec{RM} \parallel \vec{PQ} \Rightarrow \vec{RM} = k\vec{PQ} = k\begin{pmatrix}5\\0\\0\end{pmatrix}$

$\vec{QM} = \vec{QP} + \vec{PR} + \vec{RM}$

$= -\begin{pmatrix}5\\0\\0\end{pmatrix} + \begin{pmatrix}1\\4\\2\end{pmatrix} + k\begin{pmatrix}5\\0\\0\end{pmatrix}$

$= \begin{pmatrix}5k-4\\4\\2\end{pmatrix}$

$\vec{RM} \cdot \vec{QM} = 0$

$k\begin{pmatrix}5\\0\\0\end{pmatrix} \cdot \begin{pmatrix}5k-4\\4\\2\end{pmatrix} = 0$

$5(5k-4) = 0$

$k = \frac{4}{5}$

$\therefore \vec{RM} = \frac{4}{5}\begin{pmatrix}5\\0\\0\end{pmatrix} = 4\hat{i}$

Notes

Satisfactory response
3/5 marks

Defines $\overrightarrow{RM}$ as a scale factor of $\overrightarrow{PQ}$ since they are parallel.

Draws a diagram with the right angle at M which is incorrect.

Follows through and expresses $\overrightarrow{QM}$ and $\overrightarrow{RM}$ in i, j and k component form.

Uses the dot product $\overrightarrow{QM} \bullet \overrightarrow{RM} = 0$ to give perpendicularity.

Solves for k and hence $\overrightarrow{RM}$.

Examiners' comments

This question was most successfully tackled by those candidates who drew a simple sketch of the problem. Very few attempts without diagrams made much useful progress. Some candidates over-complicated the problem by initially assuming that the vector *RM* had non-zero components in all three directions and others got lost in the process of expressing *QM* in terms of the other vectors.

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Question

Question 15
(5 marks)

Question statistics

Statistics ID = MAS3CD-41
Number of attempts = 1349
Highest mark achieved = 5.00
Lowest mark achieved = 0.00
Mean = 2.84
Standard deviation = 1.47
Question difficulty = N/A
Correlation between question and section = 0.53

15(a)

Use Euler's formula $\left(e^{ix} = \cos x + i\sin x\right)$ to show that $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$.
(3 marks)

Marking key

Solution
$\dfrac{e^{ix} - e^{-ix}}{2i} = \dfrac{(\cos x + i\sin x) - (\cos x - i\sin x)}{2i}$ i.e. $\dfrac{e^{ix} - e^{-ix}}{2i} = \dfrac{2i\sin x}{2i} = \sin x$
Specific behaviours
✓ rewrites e^{ix} as $\cos x + i\sin x$ ✓ rewrites e^{-ix} as $\cos x - i\sin x$ ✓ correctly simplifies

Keywords

Complex numbers

Question statistics

Statistics ID = MAS3CD-42
Number of attempts = 1332
Highest mark achieved = 3.00
Lowest mark achieved = 0.00
Mean = 2.23
Standard deviation = 1.14
Question difficulty = Moderate
Correlation between question and section = 0.37

Candidate responses

15(a)

Use Euler's formula $\left(e^{ix} = \cos x + i \sin x\right)$ to show that $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$.
(3 marks)

$$e^{ix} - e^{-ix} = (\cos x + i \sin x) - (\cos(-x) + i \sin(-x))$$
$$e^{ix} - e^{-ix} = (\cos x + i \sin x) - (\cos x - i \sin x)$$
$$e^{ix} - e^{-ix} = 2i \sin x$$
$$\therefore \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\frac{e^{ix} - e^{-ix}}{2i} = \frac{\operatorname{cis} x - \operatorname{cis}(-x)}{2 \operatorname{cis} \pi/2}$$
$$= i \sin x - i \sin -x$$
$$= \frac{2i \sin x}{2i}$$
$$= \sin x$$

Notes

Excellent response
3/3 marks

Rewrites $e^{ix} = \cos x + i \sin x$ and $e^{-ix} = \cos x - i \sin x$..

Adds both terms and correctly simplifies.

Satisfactory response
2/3 marks

Rewrites $e^{ix} = \operatorname{c} isx$ and $e^{-ix} = \operatorname{c} i \operatorname{s}(-ix)$, which should be expanded as part of the proof.

Adds both terms and correctly simplifies.

Question

15(b)

Expand $\left(\dfrac{e^{ix}-e^{-ix}}{2i}\right)^5$ to obtain an expression for $\sin^5 x$ in terms of $\sin x$, $\sin 3x$ and $\sin 5x$.

(2 marks)

Marking key

Solution
cexpand$\left(\dfrac{e^{ix}-e^{-ix}}{2i}\right)^5 = \dfrac{5\sin x}{8} - \dfrac{5\sin 3x}{16} + \dfrac{\sin 5x}{16}$ i.e. $\sin^5 x = \dfrac{5\sin x}{8} - \dfrac{5\sin 3x}{16} + \dfrac{\sin 5x}{16}$ Note: There will be students who initially expand the bracket to get: $\dfrac{1}{32i}\left(e^{5ix}-5e^{3ix}+10e^{ix}-10e^{-ix}+5e^{-3ix}-e^{-5ix}\right)$ and continue the expansion with $\sin x$ which is acceptable if correct.
Specific behaviours
✓ rewrites $\sin^5 x$ as $\left(\dfrac{e^{ix}-e^{-ix}}{2i}\right)^5$ ✓ uses a CAS calculator to expand $\left(\dfrac{e^{ix}-e^{-ix}}{2i}\right)^5$ to give the required result

Keywords

Complex numbers, Circular functions

Question statistics

Statistics ID = MAS3CD-43
Number of attempts = 1193
Highest mark achieved = 2.00
Lowest mark achieved = 0.00
Mean = 0.72
Standard deviation = 0.72
Question difficulty = Difficult
Correlation between question and section = 0.44

Candidate responses

15(b)

Expand $\left(\dfrac{e^{ix}-e^{-ix}}{2i}\right)^5$ to obtain an expression for $\sin^5 x$ in terms of $\sin x$, $\sin 3x$ and $\sin 5x$.
(2 marks)

$$\sin^5 x = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^5$$

$$= \left(\frac{1}{2i}\right)^5 \left(e^{i5x} - 5e^{i3x} + 10e^{ix} - 10e^{-ix} + 5e^{-i3x} - e^{-i5x}\right)$$

$$= \frac{1}{32i}\left[\left(e^{i5x} - e^{-i5x}\right) - 5\left(e^{i3x} - e^{-i3x}\right) + 10\left(e^{ix} - e^{-ix}\right)\right]$$

$$= \frac{1}{16}\left[\left(\frac{e^{i5x}-e^{-i5x}}{2i}\right) - 5\left(\frac{e^{i3x}-e^{-i3x}}{2i}\right) + 10\left(\frac{e^{ix}-e^{-ix}}{2i}\right)\right]$$

$$= \frac{1}{16}\sin 5x - \frac{5}{16}\sin 3x + \frac{5}{8}\sin x$$

Notes

Excellent response
2/2 marks

Expresses $\sin^5 x = \left(\dfrac{e^{ix}-e^{-ix}}{2i}\right)^5$.

Expands and simplifies the right-hand side of the equation above to obtain the correct terms.

$$\sin^5 x = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^5 = \frac{1}{32i}\left(e^{5ix} - 5e^{4ix}e^{-ix} + 10e^{3ix}e^{-2ix} - 10e^{2ix}e^{-3ix} + 5e^{-4ix}e^{ix} - e^{-5ix}\right)$$

$$= \frac{1}{32i}\left(e^{5ix} - 5e^{3ix} + 10e^{ix} - 10e^{-ix} + 5e^{-3ix} - e^{-5ix}\right)$$

$$= \frac{1}{32i}\left(i\sin 5x - 5\sin 3x\, i + 10\sin x\, i\right)$$

$$= \frac{1}{32}\left(\sin 5x - 5\sin 3x + 10\sin x\right)$$

Satisfactory response
1/2 marks

Expresses $\sin^5 x = \left(\dfrac{e^{ix}-e^{-ix}}{2i}\right)^5$.

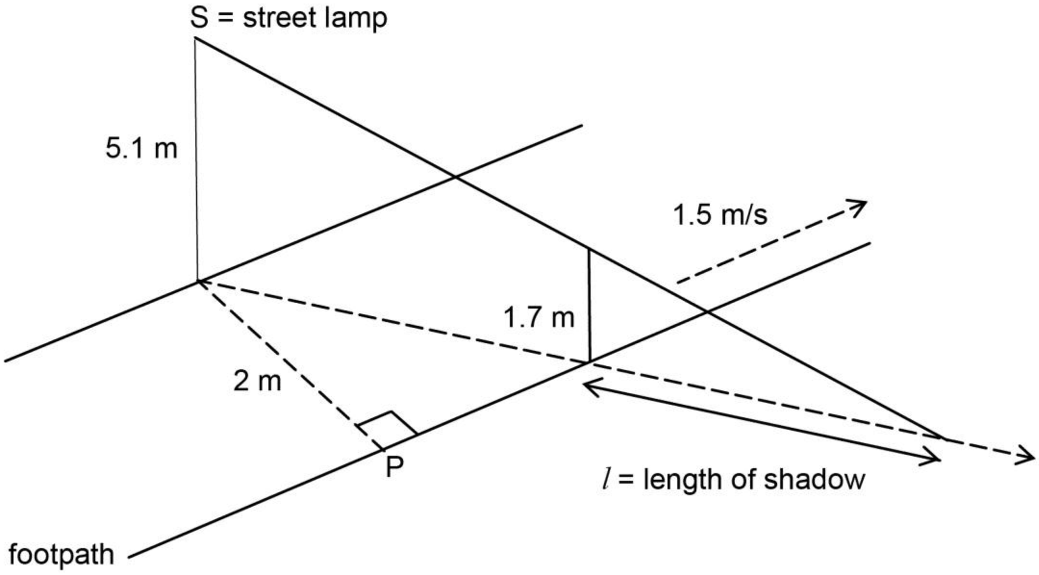
Expands and simplifies the right-hand side of the equation above, but makes an error with the factor of $2i$.

Examiners' comments

Part (a) was generally well answered although several did not know the required form of exp($-ix$). On the other hand, part (b) was rather poorly answered largely because candidates did not realise that the CAS could be used to expand the binomial. Those who attempted to do the expansion by hand usually became overwhelmed by the algebra and could not simplify properly at the end. This was a further example of where the mark allocation for the questions should have been a hint that complicated algebraic manipulation was not expected.

Question

Question 16
(6 marks)



In the diagram above, P is the initial position of a boy, of height 1.7 metres, who is walking along a straight footpath in the direction shown.

S is the position of a street lamp of height of 5.1 metres; its base is 2 metres from P.

The street lamp will cast a moving shadow of the boy as he continues to walk along the footpath at 1.5 m/s.

Question statistics

Statistics ID = MAS3CD-44
Number of attempts = 1318
Highest mark achieved = 6.00
Lowest mark achieved = 0.00
Mean = 4.41
Standard deviation = 1.86
Question difficulty = N/A
Correlation between question and section = 0.66

16(a)

If x metres is the distance walked by the boy, show that the length (l metres) of the boy's shadow is $l = \frac{1}{2}\sqrt{4+x^2}$
(3 marks)

Marking key

Solution
The hypotenuse of the triangle right-angled at P is $\sqrt{4+x^2}$ Then $\frac{1.7}{l}=\frac{5.1}{l+\sqrt{4+x^2}}$ (using similar triangles) i.e. $l=\frac{1}{2}\sqrt{4+x^2}$
Specific behaviours
✓ expresses the hypotenuse of the right triangle in terms of x ✓ uses similar triangles to determine an equation in x and l ✓ simplifies correctly to express l in terms of x

Keywords

Rate of change

Question statistics

Statistics ID = MAS3CD-45
Number of attempts = 1251
Highest mark achieved = 3.00
Lowest mark achieved = 0.00
Mean = 2.27
Standard deviation = 1.12
Question difficulty = Moderate
Correlation between question and section = 0.55

Candidate responses

16(a)

If x metres is the distance walked by the boy, show that the length (l metres) of the boy's shadow is $l = \frac{1}{2}\sqrt{4 + x^2}$
(3 marks)

$$d = \sqrt{2^2 + x^2} = \sqrt{4+x^2}$$

$$l/(d+l) = 1.7/5.1$$

$$l/(\sqrt{4+x^2} + l) = 1/3$$

$$\therefore 3l = \sqrt{4+x^2} + l$$

$$2l = \sqrt{4+x^2}$$

$$l = \tfrac{1}{2}\sqrt{4+x^2}$$

Notes

Excellent response
3/3 marks

Correctly uses the information given to get an expression for the hypotenuse opposite P in terms of x.

Uses similar triangles to get an equation with l and x.

Simplifies correctly to express l in terms of x.

by similar triangles $\quad \dfrac{5.1}{y+l} = \dfrac{1.7}{l}$

$$y = \sqrt{4+x^2}$$

$$\frac{y+l}{5.1} = \frac{l}{1.7}$$

$$1.7(y+l) = 5.1l$$

$$5.1y = 3.4l$$

$$l = \frac{3.4}{5.1}\sqrt{4+x^2}$$

$$= \frac{2}{3}\sqrt{4+x^2}$$

there seems to be a mistake in the paper. $\frac{1}{2}$ should be $\frac{2}{3}$.

Satisfactory response
2/3 marks

Uses the given information correctly to get an expression for the hypotenuse opposite y in terms of x.

Uses similar triangles to get an equation with l and y.

Simplifies incorrectly to express l in terms of x.

Question

16(b)

Find the rate of change, in m/s, of the length of the boy's shadow after 5 seconds.
(3 marks)

Marking key

Solution
$l = \frac{1}{2}\sqrt{4+x^2}$ Hence $\frac{dl}{dt} = \frac{dl}{dx} \times \frac{dx}{dt} = \frac{x}{2\sqrt{4+x^2}} \times \frac{dx}{dt}$ i.e. $\frac{dl}{dt} = \frac{3x}{4\sqrt{4+x^2}}$ since $\frac{dx}{dt} = 1.5$ When t = 5, x = 7.5 Hence $\frac{dl}{dt} = \frac{3 \times 7.5}{4\sqrt{4+7.5^2}} = 0.72$ m/s
Specific behaviours
✓ differentiates l with respect to x ✓ uses chain rule with $\frac{dx}{dt} = 1.5$ ✓ carries through calculation accurately

Keywords

Rate of change

Question statistics

Statistics ID = MAS3CD-46
Number of attempts = 1266
Highest mark achieved = 3.00
Lowest mark achieved = 0.00
Mean = 2.35
Standard deviation = 0.90
Question difficulty = Moderate
Correlation between question and section = 0.51

Candidate responses

16(b)

Find the rate of change, in m/s, of the length of the boy's shadow after 5 seconds.
(3 marks)

At $t = 5$,

$x = 1.5 \times 5 = 7.5\text{m}$

$\therefore \frac{d}{dt}(l) = \frac{d}{dt}\left(\frac{1}{2}\sqrt{4+x^2}\right)$

$\therefore \frac{dl}{dt} = \frac{1}{4\sqrt{4+x^2}} \cdot 2x \cdot \frac{dx}{dt}$

$= \frac{1}{4\sqrt{4+7.5^2}} \times 15 \cdot (1.5)$

$= \cancel{1.66\ ms^{-1}\ (2.d.p)}$

$= 0.72\ ms^{-1}\ (2.d.p)$

Notes

Excellent response
3/3 marks

Uses implicit differentiation to obtain $\frac{dl}{dt}$.

Substitutes the correct values for x and $\frac{dx}{dt}$ into $\frac{dl}{dt}$.

Carries through the calculation accurately.

$\frac{dl}{dt} = \frac{dl}{dx} \cdot \frac{dx}{dt}$

$= \frac{1}{2\sqrt{4+x^2}} \cdot 1.5$

$x = 1.5t$

$t=5,\ x = 1.5 \times 5$
$= 7.5m$

$\therefore \frac{dl}{dt} = \frac{3}{4\sqrt{4+7.5^2}}$
$= 0.097\ m/s$

Satisfactory response
2/3 marks

Uses the chain rule $\frac{dl}{dt} = \frac{dl}{dx} \cdot \frac{dx}{dt}$.

Incorrectly differentiates l with respect to x to obtain $\frac{dl}{dx}$.

Carries through the calculation without any further mistakes.

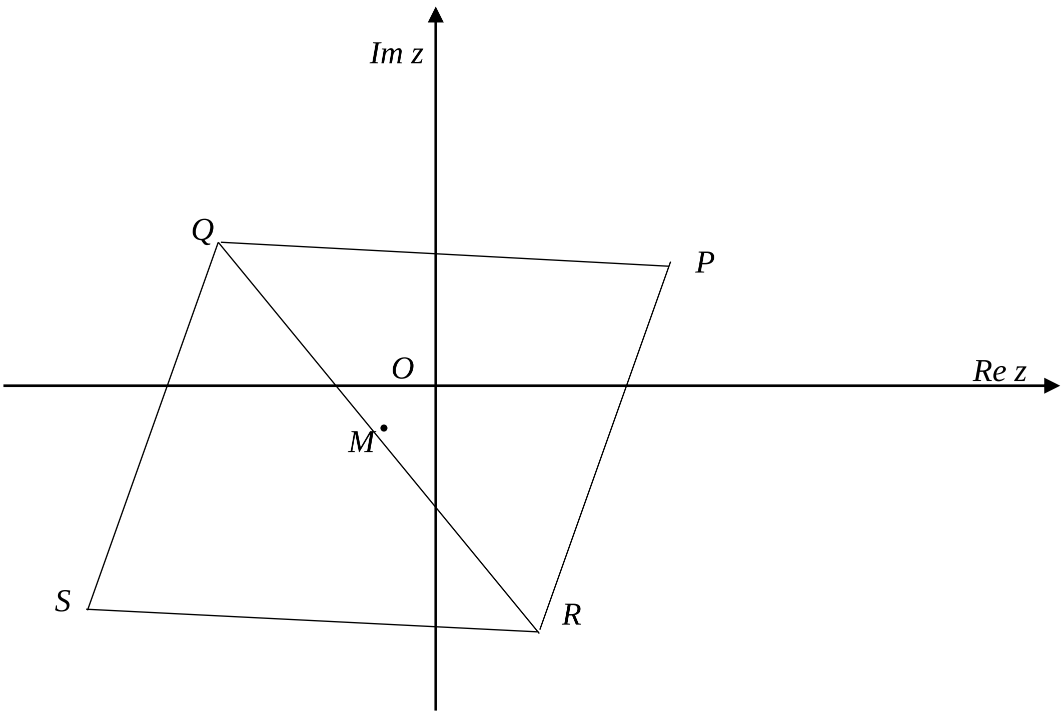
Examiners' comments

This question was generally well answered with most candidates using similar triangles to deduce the required expression for the length of the shadow. In part (b) the application of the chain rule was well answered although a few spoiled their answers by incorrectly assuming that $x = 5$ and not $t = 5$. A few circumvented the need for the chain rule by immediately substituting the result $x = 1.5t$ to obtain a formula for the required length in terms of t.

Question

Question 17
(9 marks)

The point P on the Argand diagram below represents the complex number z. The points Q and R represent the points wz and $\bar{w}z$ respectively, where $w = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. The point M is the midpoint of QR.
(The diagram is not drawn to scale.)



Question statistics

Statistics ID = MAS3CD-47
Number of attempts = 1341
Highest mark achieved = 9.00
Lowest mark achieved = 0.00
Mean = 3.22
Standard deviation = 2.03
Question difficulty = N/A
Correlation between question and section = 0.69

17(a)

If $z = r\,cis(\theta)$, find wz and $\bar{w}z$ in polar form.
(2 marks)

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Marking key

Solution
$wz = rcis\left(\theta + \frac{2\pi}{3}\right)$ $\bar{w}z = rcis\left(\theta - \frac{2\pi}{3}\right)$
Specific behaviours
✓ writes wz in polar form   ✓ writes $\bar{w}z$ in polar form

Keywords

Complex numbers, Polar coordinates

Question statistics

Statistics ID = MAS3CD-48
Number of attempts = 1333
Highest mark achieved = 2.00
Lowest mark achieved = 0.00
Mean = 1.67
Standard deviation = 0.66
Question difficulty = Easy
Correlation between question and section = 0.45

Candidate responses

17(a)

If $z = r\,cis(\theta)$, find wz and $\bar{w}z$ in polar form.
(2 marks)

$$w = cis\,\frac{2\pi}{3}$$

$$wz = cis\,\frac{2\pi}{3} \cdot r\,cis\,\theta$$

$$= r\,cis\left(\theta + \frac{2\pi}{3}\right)$$

$$\bar{w}z = cis\left(-\frac{2\pi}{3}\right) r\,cis\,\theta$$

$$= r\,cis\left(\theta - \frac{2\pi}{3}\right)$$

Notes

Excellent response
2/2 marks

Writes wz and $\bar{w}z$ in polar form, obeying the correct conventions for notation.

$$w = cis\,\frac{2\pi}{3}$$

$$wz = cis\,\frac{2\pi}{3} \times r\,cis\,\theta$$

$$wz = r\,cis\left(\theta + \frac{2\pi}{3}\right)$$

$$\bar{w} = cis\left(\frac{2\pi}{3} - \frac{\pi}{2}\right)$$

$$= cis\left(\frac{4\pi}{6} - \frac{3\pi}{6}\right)$$

$$= cis\left(\frac{\pi}{6}\right)$$

$$\bar{w}z = cis\left(\pi/6\right) \times r\,cis\,\theta$$

$$= r\,cis\left(\theta + \pi/6\right)$$

Satisfactory response
1/2 marks

Writes wz in polar form, obeying the correct conventions for notation.

However, writes the wrong expression for $\bar{w}z$.

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Question

17(b)

Hence explain why $\left|\overrightarrow{OP}\right| = \left|\overrightarrow{OQ}\right| = \left|\overrightarrow{OR}\right|$.
(2 marks)

Marking key

Solution
$\overrightarrow{OP} = rcis(\theta)$ $\overrightarrow{OQ} = rcis\left(\theta + \frac{2\pi}{3}\right)$ $\overrightarrow{QR} = rcis\left(\theta - \frac{2\pi}{3}\right)$ $\therefore \left|\overrightarrow{OP}\right| = \left|\overrightarrow{OQ}\right| = \left|\overrightarrow{OR}\right| = r$
Specific behaviours
✓ expresses the vectors as complex numbers   ✓ states $\operatorname{mod} z = \operatorname{mod} wz = \operatorname{mod} \overline{w}z = r$

Keywords

Complex numbers, Polar coordinates

Question statistics

Statistics ID = MAS3CD-49
Number of attempts = 1165
Highest mark achieved = 2.00
Lowest mark achieved = 0.00
Mean = 1.08
Standard deviation = 0.84
Question difficulty = Moderate
Correlation between question and section = 0.51

Candidate responses

17(b)

Hence explain why $|\overrightarrow{OP}| = |\overrightarrow{OQ}| = |\overrightarrow{OR}|$.
(2 marks)

$\overrightarrow{OP} = r\text{cis}\,\theta \Rightarrow |\overrightarrow{OP}| = r$

$\overrightarrow{OQ} = wz$
$= r\text{cis}(\theta + \frac{2\pi}{3}) \Rightarrow |\overrightarrow{OQ}| = r$

$\overrightarrow{OR} = \bar{w}z$
$= r\text{cis}(\theta - \frac{2\pi}{3}) \Rightarrow |\overrightarrow{OR}| = r$

$\therefore |\overrightarrow{OP}| = |\overrightarrow{OQ}| = |\overrightarrow{OR}| = r$

ie $|\overrightarrow{OP}| = |\overrightarrow{OQ}| = |\overrightarrow{OR}|$ QED

Notes

Excellent response
2/2 marks

Expresses the vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{QR}$ as polar complex numbers.

Shows the magnitude of these three complex numbers and hence equal to each other.

Candidate responses (continued)

$\vec{OP}$, $O\vec{Q}$ and $\vec{OR}$ all have
modulus, r
Hence $|\vec{OP}| = |\vec{OQ}| = |O\vec{R}|$

Notes

Satisfactory response
1/2 marks

States that the vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{QR}$ have a modulus = r, but does not explain why.

Question

17(c)

Show that the complex number representing M is $-\frac{1}{2}z$.

(2 marks)

Marking key

Solution
$OM = \frac{1}{2}\overrightarrow{OQ}+\frac{1}{2}\overrightarrow{OR}$ $= \frac{1}{2}\left(wz+\overline{w}z\right)$ $= \frac{1}{2}z\left(cis\frac{2\pi}{3}+cis\left(\frac{-2\pi}{3}\right)\right)$ $= \frac{1}{2}z\left(2\cos\frac{2\pi}{3}\right)$ $= -\frac{1}{2}z$
Specific behaviours
✓ correctly defines $\overrightarrow{OM}$ as vector terms ✓ simplifies the expression using polar form

Keywords

Complex numbers, Polar coordinates

Question statistics

Statistics ID = MAS3CD-50
Number of attempts = 1009
Highest mark achieved = 2.00
Lowest mark achieved = 0.00
Mean = 0.30
Standard deviation = 0.57
Question difficulty = Difficult
Correlation between question and section = 0.44

Candidate responses

17(c)

Show that the complex number representing M is $-\dfrac{1}{2}z$.

(2 marks)

$$\overrightarrow{OM} = \overrightarrow{OQ} + \overrightarrow{QM}$$
$$= \overrightarrow{OQ} + \tfrac{1}{2}\overrightarrow{QR}$$
$$= \overrightarrow{OQ} + \tfrac{1}{2}(\overrightarrow{OR} - \overrightarrow{OQ})$$
$$= \tfrac{1}{2}\overrightarrow{OQ} + \tfrac{1}{2}\overrightarrow{OR}$$
$$= \tfrac{1}{2}(\overrightarrow{OQ} + \overrightarrow{OR})$$
$$= \tfrac{1}{2}(wz + \bar{w}z)$$
$$= \tfrac{1}{2}z(w + \bar{w})$$
$$= \tfrac{1}{2}z\left(\cos\tfrac{2\pi}{3} + i\sin\tfrac{2\pi}{3} + \cos\left(\tfrac{-2\pi}{3}\right) + i\sin\left(-\tfrac{2\pi}{3}\right)\right)$$
$$= \tfrac{1}{2}z\left(-\tfrac{1}{2} + \left(-\tfrac{1}{2}\right)\right)$$
$$= -\tfrac{1}{2}z$$

$\therefore$ M is $-\tfrac{1}{2}z$

Notes

Excellent response
2/2 marks

Expresses the vector $\overrightarrow{OM}$ in terms of $\overrightarrow{OQ}$ and $\overrightarrow{QR}$ in polar complex number form.

Simplifies the expression on the left-hand side to give the required result.

Candidate responses (continued)

$$w z + \bar{w} z = r\cos\left(\frac{2\pi}{3} + \theta\right) + r\cos\left(\theta - \frac{2\pi}{3}\right)$$

$$= r\left(\cos\left(\frac{2\pi}{3} + \theta\right) + i\sin\left(\frac{2\pi}{3} + \theta\right)\right) + r\left(\cos\left(\theta - \frac{2\pi}{3}\right) + i\sin\left(\theta - \frac{2\pi}{3}\right)\right)$$

$$M = \frac{1}{2}(w z + \bar{w} z)$$

$$= \frac{1}{2} z (w + \bar{w})$$

$$= -\frac{1}{2} z$$

Notes

Satisfactory response
1/2 marks

Fails to express $\overrightarrow{OM}$ in vector terms.

Assumes that $w + \bar{w} = -1$, without showing it.

Question

17(d)

The point S is chosen so that $PQSR$ is a parallelogram. Find the complex number represented by S in terms of z.
(3 marks)

Marking key

Solution
$\overrightarrow{OS} = \overrightarrow{OM} + \overrightarrow{MS}$ $= \overrightarrow{OM} - \overrightarrow{MP}$ $= \overrightarrow{OM} - \left(\overrightarrow{MO} + \overrightarrow{OP}\right)$ $= 2\overrightarrow{OM} - \overrightarrow{OP}$ $= 2\left(-\frac{1}{2}z\right) - z$ $= -2z$
Specific behaviours
✓ correctly defines $\overrightarrow{OS}$ in vector terms ✓ converts from vector terms to complex numbers ✓ correctly simplifies

Keywords

Complex numbers, Polar coordinates

Question statistics

Statistics ID = MAS3CD-51
Number of attempts = 757
Highest mark achieved = 3.00
Lowest mark achieved = 0.00
Mean = 0.71
Standard deviation = 1.09
Question difficulty = Difficult
Correlation between question and section = 0.45

Candidate responses

17(d)

The point S is chosen so that $PQSR$ is a parallelogram. Find the complex number represented by S in terms of z.
(3 marks)

$\overrightarrow{QP} = -wz + z$

$\therefore SR = -wz + z$

$SR = \cancel{\ } - \overrightarrow{OS} + \overrightarrow{OR}$

$\overrightarrow{OS} = \overrightarrow{OR} - \overrightarrow{SR}$

$= \bar{w}z - (-wz + z)$

$= \bar{w}z + wz - z$

$= z(\bar{w} + w - 1)$

$= z(-1-1) = -2z$

$\therefore$ S is the complex number $-2z$

Notes

Excellent response
3/3 marks

Uses $\overrightarrow{SR} = \overrightarrow{QP}$ since they are opposite sides of the parallelogram.

Correctly defines $\overrightarrow{OS} = \overrightarrow{OR} - \overrightarrow{SR}$.

Writes the left-hand side of the equation in terms of z and w.

Simplifies to get the required answer.

~~___~~ equal lengths of $\overline{MP}$ & $\overline{MS}$

$|MP| = 1.5|z|$

$MS = 1.5|z|$

$= 1.5r$

$OS = 2\cdot$

$S = -2z$

Satisfactory response
2/3 marks

Uses the magnitudes of the complex numbers instead of the vectors.

Swaps from r to z in the last step.

Examiners' comments

Many candidates answered this question poorly because they failed to make the fundamental connection between complex numbers and vectors. Too many left elements of the question unanswered. Part (a) was generally well done although in some cases the nomenclature was casual at best. Part (b) produced a number of 'proofs', most of which were distinguished by a lack of logic although there were a few very good expositions. Again, part c) generated a number of impressive answers although in the main the notation used for vectors was extremely poor. Problems requiring a proof or the derivation of a given answer generally need to be answered rigorously. Not many candidates continued on to Part (d) and most of these answers were of a low standard.

Question

Question 18
(8 marks)

A model for a population, P , of numbats is

$P = \dfrac{900}{3+2e^{-t/4}}$, where *t* is the time in years from today.

Question statistics

Statistics ID = MAS3CD-52
Number of attempts = 1383
Highest mark achieved = 8.00
Lowest mark achieved = 0.00
Mean = 3.83
Standard deviation = 1.78
Question difficulty = N/A
Correlation between question and section = 0.66

18(a)

What is the population today?
(1 mark)

Marking key

Solution
Population today = $\dfrac{900}{3+2} = 180$
Specific behaviours
✓ sets $t = 0$ and solves for P

Keywords

Exponential functions

Question statistics

Statistics ID = MAS3CD-53
Number of attempts = 1382
Highest mark achieved = 1.00
Lowest mark achieved = 0.00
Mean = 0.91
Standard deviation = 0.29
Question difficulty = Very easy
Correlation between question and section = 0.24

Candidate responses

18(a)

What is the population today?
(1 mark)

$t = 0.\quad P = \dfrac{900}{3+2\cdot 1} = 180$ numbats.

Notes

Excellent response
1/1 mark

Substitutes $t = 0$ into the population equation and solves for P.

Question

18(b)

What does the model predict that the eventual population will be?
(1 mark)

Marking key

Solution
Eventual population = $\frac{900}{3} = 300$
Specific behaviours
✓ lets $t \to \infty$ and solves for P

Keywords

Exponential functions

Question statistics

Statistics ID = MAS3CD-54
Number of attempts = 1337
Highest mark achieved = 1.00
Lowest mark achieved = 0.00
Mean = 0.86
Standard deviation = 0.35
Question difficulty = Easy
Correlation between question and section = 0.26

Candidate responses

18(b)

What does the model predict that the eventual population will be?
(1 mark)

$\lim_{t \to \infty} e^{-t/4} = 0.$

$\lim_{t \to 0} \Rightarrow$ Eventual population $= \dfrac{900}{3+2\cdot 0}$

$= \dfrac{900}{3}$

$= 300$ numbats.

Notes

Excellent response
1/1 mark

Uses the limit $t \to \infty$ and hence solves for P.

Question

18(c)

By first expressing $e^{-t/4}$ in terms of P, or otherwise, show that P satisfies the differential equation
$\frac{dP}{dt} = \frac{P}{4}\left(1 - \frac{P}{300}\right)$.

(4 marks)

Marking key

Solution
$e^{-t/4} = \frac{450}{P} - \frac{3}{2}$ Hence $-\frac{e^{-t/4}}{4} = -\frac{450}{P^2} \times \frac{dP}{dt}$ i.e. $\frac{1}{4} \times \left(\frac{450}{P} - \frac{3}{2}\right) = \frac{450}{P^2} \times \frac{dP}{dt}$ Hence $\frac{dP}{dt} = \frac{P^2}{4 \times 450} \times \left(\frac{450}{P} - \frac{3}{2}\right)$ i.e. $\frac{dP}{dt} = \frac{P}{4}\left(1 - \frac{P}{300}\right)$
Specific behaviours
✓ correctly rearranges the equation   ✓ differentiates $\left(e^{-t/4} = \frac{450}{P} - \frac{3}{2}\right)$ implicitly with respect to t ✓ substitutes $\left(e^{-t/4} = \frac{450}{P} - \frac{3}{2}\right)$ to give an equation for $\frac{dP}{dt}$ involving P only   ✓ rearranges and simplifies

Keywords

Differentiation

Question statistics

Statistics ID = MAS3CD-55
Number of attempts = 1333
Highest mark achieved = 4.00
Lowest mark achieved = 0.00
Mean = 1.46
Standard deviation = 1.19
Question difficulty = Moderate
Correlation between question and section = 0.53

Candidate responses

18(c)

By first expressing $e^{-t/4}$ in terms of P, or otherwise, show that P satisfies the differential equation $\frac{dP}{dt}=\frac{P}{4}\left(1-\frac{P}{300}\right)$.

(4 marks)

$P(3+2e^{-t/4}) = 900.$

$2e^{-t/4} = \frac{900}{P} - 3.$

$e^{-t/4} = \frac{450}{P} - 1.5.$

Diff. wrt t $\quad -\frac{1}{4}e^{-t/4} = -450p^{-2}\cdot\frac{dp}{dt}$

$\frac{dp}{dt} = \frac{+\frac{1}{4}e^{-t/4}\cdot p^2}{+450}$

$= \frac{\frac{1}{4}\cdot\left(\frac{450}{p}-1.5\right)p^2}{450}$

$= \frac{p}{4}\left(1 - \frac{1.5}{450}p\right)$

$= \frac{p}{4}\left(1 - \frac{p}{300}\right).$

Notes

Excellent response
4/4 marks

Rearranges the given population equation with the term $e^{\frac{-t}{4}}$ on the left-hand side.

Differentiates the expression $\left(e^{\frac{-t}{4}} = \frac{450}{P^2} - \frac{3}{2}\right)$ implicitly with respect to t.

Substitutes the expression $\frac{450}{P^2} - \frac{3}{2}$ for $e^{\frac{-t}{4}}$, to give an expression in terms of P.

Rearranges and simplifies the expression $\frac{dp}{dt}$ into the required form.

Candidate responses (continued)

$$P(3+2e^{-t/4}) = 900$$

$$e^{-t/4} = \frac{\frac{900}{P}-3}{2} = \frac{\frac{900-3P}{P}}{2}$$

$$= \frac{900-3P}{2P}$$

$$e^{-t/4} = 450P^{-1} - 1.5$$

$$\cancel{\ln\left|\frac{450}{P}-1.5\right| = -\frac{t}{4}} \qquad e^{t/4} = \frac{2P}{900-3P}$$

$$\frac{dP}{dt} = \frac{450e^{\frac{t}{4}}}{\left(3e^{\frac{t}{4}}+2\right)^2}$$

$$= \frac{450 \times \frac{1}{450\left(\frac{1}{P}\right)-1.5}}{\left[3\left(\frac{1}{450\left(\frac{1}{P}\right)-1.5}\right)+2\right]^2}$$

$\rightarrow$ simplify

$$\frac{dP}{dt} = \frac{P}{4}\left(1-\frac{P}{300}\right)$$

Notes

Satisfactory response
2/4 marks

Rearranges the given population equation with the term $e^{\frac{-t}{4}}$ on the left-hand side.

Attempts to differentiate the expression $\left(P = \frac{900}{3+2e^{\frac{-t}{4}}}\right)$ with respect to *t.*

Gets confused with the manipulation of the algebra.

Question

18(d)

What is the instantaneous percentage annual rate of growth today?
(2 marks)

Marking key

Solution
The instantaneous rate of change today = $\frac{dP}{dt}$ when t = 0 and P = 180 i.e. $\left.\frac{dP}{dt}\right\|_{t=0} = 18$ Hence the instantaneous percentage rate of growth today = $\frac{18}{180} \times 100 = 10\%$
Specific behaviours
✓ calculates the instantaneous rate of change today ✓ calculates the required percentage

Keywords

Differentiation

Question statistics

Statistics ID = MAS3CD-56
Number of attempts = 1085
Highest mark achieved = 2.00
Lowest mark achieved = 0.00
Mean = 0.87
Standard deviation = 0.72
Question difficulty = Moderate
Correlation between question and section = 0.38

Candidate responses

18(d)

What is the instantaneous percentage annual rate of growth today?
(2 marks)

$$\frac{dP}{dt} = \frac{180}{4} \cdot \left(1 - \frac{180}{300}\right)$$
$$= 18.$$

$$\% \text{ Growth} = \frac{18}{180} \times 100$$
$$= 10\%.$$

Notes

Excellent response
2/2 marks

Uses the previous result $\frac{dp}{dt} = \frac{P}{4}\left(1 - \frac{P}{300}\right)$, and P =180 to get the change.

$$\frac{dP}{dt} = \frac{180}{4}\left(1 - \frac{180}{300}\right)$$

$$\frac{dP}{dt} = 18\%$$

Satisfactory response
1/2 marks

Uses the previous result $\frac{dp}{dt} = \frac{P}{4}\left(1 - \frac{P}{300}\right)$ and P =180, to get the rate of change.

Does not calculate percentage change as required, but uses the actual change as a percentage.

Examiners' comments

This question was not answered as well as expected. The first (easy) two parts were well answered, although too many tackled the limiting process in part (b) by calculating a series of values and then attempting to draw a conclusion. Many candidates ignored the hint given in part (c) and the majority relied on the CAS to effect a solution. In the majority of these cases insufficient explanation of the processes undertaken was given, resulting in a deduction of marks. Others attempted to integrate the original statement of $\frac{dP}{dt}$ using the CAS. Part (d) was answered poorly given that it was a routine application of instantaneous rate of change; too many converted their answer to the derivative at t = 0 directly to a percentage without realising an additional step was required.

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Question

Question 19
(7 marks)

Let $f(n) = 3^{n+2} + (-1)^n \times, 2^n$ for all positive integers n.

Question statistics

Statistics ID = MAS3CD-57
Number of attempts = 1368
Highest mark achieved = 7.00
Lowest mark achieved = 0.00
Mean = 2.53
Standard deviation = 1.91
Question difficulty = N/A
Correlation between question and section = 0.66

19(a)

Show that $2f(n+1) - f(n)$ is divisible by 5.
(2 marks)

Marking key

Solution
$2f(n+1) - f(n) = 2\left(3^{n+3} + (-1)^{n+1} \times 2^{n+1}\right) - 3^{n+2} - (-1)^n \times 2^n$ Using a CAS the RHS simplifies to $45 \times 3^n - 5 \times (-2)^n$ Hence result
Specific behaviours
✓ correctly expands $2f(n+1) - f(n)$ ✓ simplifies to correct term with factor of 5

Keywords

Mathematical induction

Question statistics

Statistics ID = MAS3CD-58
Number of attempts = 1345
Highest mark achieved = 2.00
Lowest mark achieved = 0.00
Mean = 0.78
Standard deviation = 0.89
Question difficulty = Moderate
Correlation between question and section = 0.52

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Candidate responses

19(a)

Show that $2f(n+1) - f(n)$ is divisible by 5.
(2 marks)

$2f(n+1) - f(n) = 2(3^{n+3} + (-1)^{n+1} \times 2^{n+1}) - (3^{n+2} + (-1)^n \times 2^n)$

$\cancel{= 2\times 3^3 \times 3^n + (-1)(-1)^n \times 2 \times 2^n - 3^2 \times 3^n - (-1)^n \times 2^n)}$

$= 2\times 3^{n+3} + 2(-1)^{n+1} \times 2^{n+1} - 3^{n+2} - (-1)^n \times 2^n$

$2\times 3\times 3^{n+2} \cancel{+2(} -4(-1)^n \times 2^n - \cancel{3}3^{n+2} - (-1)^n \times 2^n$

$\cancel{6(3^{n+2})} -$

$3^{n+2}(2\times 3 - 1) - (-1)^n \times 2^n (4+1)$

$= 3^{n+2} \times 5 - (-1)^n \times 2^n \times 5$

$= 5(3^{n+2} - (-1)^n \times 2^n)$

$= 5k$

$\therefore$ divisible by 5

Notes

Excellent response
2/2 marks

Writes the expansion for $2f(n+1) - f(n)$, in index form.

Simplifies and gathers terms to show a common factor of 5.

Candidate responses (continued)

$$2f(n+1) - f(n) = 2\times 3^{n+3} + 2(-2)^{n+1} - 3^{n+2} - (-2)^{n}$$
$$= (2\times 3^{n+3} - 3^{n+2}) - (2(-2)^{n+1} - (-2)^{n})$$
$$= (6\times 3^{n+2} - 3^{n+2}) - (-4\times(-2)^{n} - (-2)^{n})$$
$$= 5\times 3^{n+2} + 5(-2)^{n}$$

Which is divisible by 5.

Notes

Satisfactory response
1/2 marks

Writes the expansion for $2f(n+1)-f(n)$, in index form.

Makes an error with a negative term which is carried through.

Simplifies and gathers terms to show a common factor of 5.

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Question

19(b)

Hence, or otherwise, prove by induction that $f(n)$ is divisible by 5.
(5 marks)

Marking key

Solution
Let $P(n)$ be the statement $f(n) = 3^{n+2} + (-1)^n \times 2^n = 5s$ for some integer s.
$P(1)$ is true because $3^3 - 2 = 25 = 5 \times 5$.
Assume $P(k+1)$ is true.
i.e. Assume $f(k) = 5w$ for some integer w.
Consider $P(k+1)$.
Required to show that $f(k+1) = 5p$ for some integer p.
From part (i), $2f(k+1) - f(k) = 5t$ for some integer t.
Hence $2f(k+1) = f(k) + 5t = 5w + 5t = 5(w+t)$ using the induction assumption
Hence $f(k+1) = 5p$ for some integer p, since 2 is not divisible by 5
Thus if $P(k)$ is true, then $P(k+1)$ is also true.
But $P(1)$ is true.
Hence $P(n)$ is true for all $n \geq 1$

Specific behaviours	
✓	shows that $P(1)$ is true
✓	states the induction assumption
✓	shows that $2f(k+1)$ is divisible by 5 if $P(k)$ is true
✓	justifies that this proves that $f(k+1)$ is divisible by 5 if $P(k)$ is true
✓	makes a final statement which explains why this is a valid proof by induction

Keywords

Mathematical induction

Question statistics

Statistics ID = MAS3CD-59
Number of attempts = 1228
Highest mark achieved = 5.00
Lowest mark achieved = 0.00
Mean = 1.97
Standard deviation = 1.30
Question difficulty = Moderate
Correlation between question and section = 0.58

Candidate responses

19(b)

Hence, or otherwise, prove by induction that $f(n)$ is divisible by 5.
(5 marks)

Assume $f(n)$ divisible by 5

$3^{n+\cancel{3}2} + (-1)^n \times 2^n = 5(k)$

Consider $f(n+1)$

$3^{n+3} + (-1)^{n+1} \times 2^{n+1} = 3 \times 3^{n+2} - \cancel{3}2(-1)^n \times 2^n$

$= (5-2)3^{n+2} - 2(-1)^n \times 2^n$

$5 \times 3^{n+2} - 2 \times 3^{n+2} - 2 \times (-1)^n \times 2^n$

$5 \times 3^{n+2} - 2(f(n))$

$= 5 \times 3^{n+2} - 2 \times 5k$

$= 5(3^{n+2} - 2k)$

$\therefore$ if true for n, true for $n+1$

let $n=1$ $3^3 + (-1)^1 \times 2^1$

$= 27 - 2$

$= 25$

$= 5 \times 5$

$\therefore$ true for $n=1$

$\therefore$ true for $n=2$, $n=3 \ldots$

true $\forall n \in \mathbb{Z}^+$

QED

Notes

Excellent response
5/5 marks

Shows that the assertion is true for $n = 1$.

Assumes the assertion is true for $n = k \Rightarrow f(k) = 5w$.

Uses the expression $2f(k+1) - f(k)$ from part (a), to prove $f(k+1)$ is divisible by 5.

Makes the final induction statement.

Candidate responses (continued)

$f(x) = 3^{n+2} + (-2)^k$

For $x = 1$

$3^{1+2} + (-2)^1 = 25$ which is divisk

Assum th $f(k)$ is divisible by five

To prove that $f(k+1)$ is divisible by five

$f(k+1) = 3^{k+3} + (-2)^{k+1}$

$= 3 \times 3^{k+2} + -2\,(-2)^k$

subtract $5 \times 3^{k+2}$ (divisible by five)

$= -2 \times 3^{k+2} - 2\,(-2)^k$

Notes

Satisfactory response
3/5 marks

Shows that the assertion is true for $n = 1$.

Assumes that the assertion is true for $n = k$.

Attempts to prove that $f(k+1)$ is divisible by 5; given that $f(k)$ is divisible by 5, but the logic is flawed.

Makes the final induction statement.

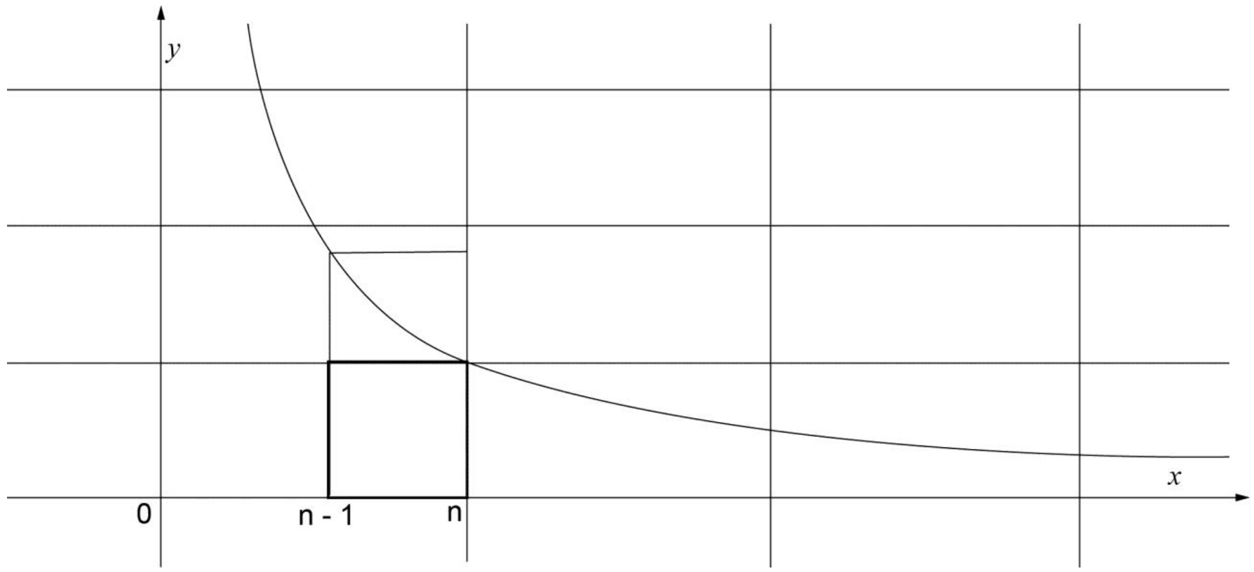
Examiners' comments

This question caused many candidates a good deal of difficulty. Few used their calculator to simplify the expression in part (a) and preferred to proceed manually. Unfortunately this often ended in confusion after some elementary algebraic error. Even given the answer to part (a), many candidates struggled to provide a convincing induction argument for part (b). This was due in large part by a refusal to use the result of part (a) even though this virtually removed the need to do any more algebra.

Question

Question 20
(7 marks)

Let n be a positive integer greater than 1. The area of the region under the curve $y = \frac{1}{x}$ from $x = n-1$ to $x = n$ lies between the areas of the two rectangles, as shown in the diagram.



Question statistics

Statistics ID = MAS3CD-60
Number of attempts = 1186
Highest mark achieved = 7.00
Lowest mark achieved = 0.00
Mean = 1.27
Standard deviation = 1.94
Question difficulty = N/A
Correlation between question and section = 0.60

20(a)

Use the diagram to show that $e^{-\frac{n}{n-1}} < \left(1 - \frac{1}{n}\right)^n < e^{-1}$.

(6 marks)

Marking key

Solution
Area of the larger rectangle is $\frac{1}{n-1}$ sq units; area of the smaller rectangle is $\frac{1}{n}$ sq units Hence $\frac{1}{n} < \int_{n-1}^{n} \frac{1}{x}\,dx < \frac{1}{n-1}$ i.e. $\frac{1}{n} < [\ln x]_{n-1}^{n} < \frac{1}{n-1}$ i.e. $\frac{1}{n} < \ln\left(\frac{n}{n-1}\right) < \frac{1}{n-1}$ Hence $e^{\frac{1}{n}} < \frac{n}{n-1} < e^{\frac{1}{n-1}}$ Hence $\frac{1}{e^{\frac{1}{n-1}}} < \frac{n-1}{n} < \frac{1}{e^{\frac{1}{n}}}$ Hence $\frac{1}{e^{\frac{n}{n-1}}} < \left(\frac{n-1}{n}\right)^{n} < \frac{1}{e^{\frac{n}{n}}}$ Hence $e^{-\frac{n}{n-1}} < \left(1-\frac{1}{n}\right)^{n} < e^{-1}$
Specific behaviours
✓ identifies that $\int_{n-1}^{n} \frac{1}{x}\,dx$ lies between the areas of the two rectangles ✓ integrates and simplifies to establish $\frac{1}{n} < \ln\left(\frac{n}{n-1}\right) < \frac{1}{n-1}$ ✓ uses the inverse relationship between $\ln x$ and e^x ✓ inverts the fractions ✓ recognises the need to reverse the order of the inequalities ✓ raises each term to the power n

Keywords

Integration (Functions)

Question statistics

Statistics ID = MAS3CD-61
Number of attempts = 972
Highest mark achieved = 6.00
Lowest mark achieved = 0.00
Mean = 1.05
Standard deviation = 1.87
Question difficulty = Difficult
Correlation between question and section = 0.58

Candidate responses

20(a)

Use the diagram to show that $e^{-\frac{n}{n-1}} < \left(1-\frac{1}{n}\right)^n < e^{-1}$.

(6 marks)

A under the curve $= \int_{n-1}^{n} \frac{1}{x}\,dx = -\ln(n-1) + \ln(n)$

A rectangle below the curve $= \frac{1}{n}$.

A rectangle ~~above~~ above the curve $= \frac{1}{n-1}$

$$\frac{1}{n} < -\ln(n-1) + \ln(n) < \frac{1}{n-1}$$

$$\frac{1}{n} < \ln\left(\frac{n}{n-1}\right) < \frac{1}{n-1}$$

$$e^{\frac{1}{n}} < \frac{n}{n-1} < e^{\frac{1}{n-1}}$$

$$e < \left(\frac{n}{n-1}\right)^n < e^{\frac{n}{n-1}}$$

$$e^{\frac{-n}{n-1}} < \left(1+\frac{1}{n-1}\right)^{-n} < e^{-1}$$

$$e^{\frac{-n}{n-1}} < \left(1-\frac{1}{n}\right)^n < e^{-1}$$

Notes

Excellent response
6/6 marks

Integrates $\int_{n-1}^{n} \frac{1}{x} dx = \ln(n) - \ln(n-1)$.

Expresses the area of the two rectangles in relation to the area of the integral.

Establishes the relationship $\frac{1}{n} < \ln\left(\frac{n}{n-1}\right) < \frac{1}{n-1}$.

Expresses the relationship in exponential terms $e^{\frac{1}{n}} < \frac{n}{n-1} < e^{\frac{1}{n-1}}$.

Raises each term in the expression to the power of n.

Inverts the fraction and recognises the need to reverse the inequality signs.

Candidate responses (continued)

(a) Use the diagram to show that $e^{-\frac{n}{n-1}} < \left(1-\frac{1}{n}\right)^n < e^{-1}$.

area 1 = $\frac{1}{n}$

area under curve = $\int_{n-1}^{n} \frac{1}{x}\,dx$ area 2 = $\frac{1}{n-1}$

$= \left[\ln x\right]_{n-1}^{n}$

$= \ln \frac{n}{n-1}$

$\frac{1}{n} < \ln \frac{n}{n-1} < \frac{1}{n-1}$

$\therefore e^{\frac{1}{n}} < \frac{n}{n-1} < e^{\frac{1}{n-1}}$

Notes

Satisfactory response
3/6 marks

Integrates $\int_{n-1}^{n} \frac{1}{x}dx = \ln\left(\frac{n}{n-1}\right)$.

Expresses the area of the two rectangles in relation to the area of the integral.

Establishes the relationship $\frac{1}{n} < \ln\left(\frac{n}{n-1}\right) < \frac{1}{n-1}$.

Expresses the relationship in exponential terms $e^{\frac{1}{n}} < \frac{n}{n-1} < e^{\frac{1}{n-1}}$.

Question

20(b)

Hence deduce $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$.
(1 mark)

Marking key

Solution
$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^{-1}$
Specific behaviours
✓ uses the pinching theorem to establish the limit

Keywords

Limits (Functions)

Question statistics

Statistics ID = MAS3CD-62
Number of attempts = 1053
Highest mark achieved = 1.00
Lowest mark achieved = 0.00
Mean = 0.46
Standard deviation = 0.50
Question difficulty = Moderate
Correlation between question and section = 0.37

Candidate responses

20(b)

Hence deduce $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$.
(1 mark)

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^{-1}.$$

Notes

Excellent response
1/1 mark

States the correct answer for the limit.

Examiners' comments

This question was answered poorly. Candidates tended to ignore the initial information and concentrated solely on the area beneath the curve and then stopped. Those who did manage to determine the areas of the upper and lower rectangles made a start on the inequality and consequently acquired at least a few marks. A few tried to work backwards from the inequality, but there were some very good answers which clearly demonstrated the candidate's confidence in manipulating inequalities.

Appendix 1: Course achievement band descriptions

Excellent achievement (75 - 100)

- Can relate and apply all elements of the course: calculus, vectors, matrices and complex numbers.
- Solves complicated, unpractised problems requiring a synthesis of ideas from various strands.
- Derives logical solutions using high-level mathematical reasoning and presents them with correct use of syntax and notation.
- Applies calculator technology efficiently.

High achievement (65 - 74)

- Shows proficiency in the main components of the course and makes connections between these.
- Solves routine, well-practised problems adeptly and makes effective use of diagrams where relevant.
- Attempts complex problems with some success.
- Exhibits persistence, providing evidence of logical thinking and incorporation of relevant mathematical notation.
- Uses appropriate calculator technology in a variety of situations.

Satisfactory achievement (50 - 64)

- Solves most questions involving a familiar context, particularly those involving matrices and vectors.
- Demonstrates some skill in elementary calculus techniques.
- Selects the correct procedure in more challenging contexts; however, algebraic and computational errors often prevent the correct conclusions being drawn.
- Makes some use of appropriate calculator technology, but does not always discern when it is best used.

Limited achievement (35 - 49)

- Correctly applies skills and concepts from a very restricted array of components of the course.
- Achieves some success in problems where the structure is well-defined.
- Tackles parts of questions involving unfamiliar contexts, but with only partial progress towards completion.

Inadequate achievement (0 - 34)

- Displays progress in some routine and well-practised problems.
- Attempts fragments of some questions but rarely achieves success.
- Demonstrates a very restricted skills set and frequently makes elementary mistakes.

Cut points:
Excellent/High =72.12
High/Satisfactory = 62.13
Satisfactory/Limited = 45.59
Limited/Inadequate = 35.07

Appendix 2: Rasch analysis of examinations

Rasch analysis is used to test the reliability and validity of an examination. It produces numerical estimates of the ability of the students who sat the examination and the difficulty of each item in the examination. An ‘item’ is a scoring opportunity. It may be a whole question (e.g. a multiple-choice question) or, in the case of questions that are broken down into discrete elements, a part of a question or a sub-part of a question.

In Rasch analysis, the estimates of student ability and item difficulty are placed on a common measurement scale, like a ruler. Items are clustered into five bands: *Very easy*, *Easy*, *Moderate*, *Difficult* and *Very difficult*. Items that are less difficult to answer correctly are located to the left and items that are more difficult to answer correctly are located to the right. Similarly, using the same scale, less able students are located to the left and more able students are located to the right.

The boundary of the *Moderate* difficulty band is determined by the difficulty of the middle 68% of items, i.e. the difficulty is one standard deviation from the mean student location. The boundaries for the *Very easy–Easy* and the *Difficult–Very difficult* band are determined by reference to student abilities.

Less able students | More able students

Less difficult questions < - - - - - - - - o - - - - - - - - - o - - - - - - - - - o - - - - - - - - - o - - - - - - - - -> More difficult questions

Very easy | Easy | Moderate | Difficult | Very difficult

Table 1 on the following page provides the item difficulty analysis for the 2011 Stage 3C/3D WACE examination for Mathematics Specialist.

Notes

- N/A (in the Difficulty estimates in this Guide) indicates the item was not used in the Rasch analysis. This is because one or more items had too few responses (or no responses).
- As shown in Table 1, when a question consists of a number of items (e.g. in the case of questions (14-21), the difficulty estimate is given for each item, not for the question as a whole.
- In the Rasch model, the higher a student’s ability, relative to the difficulty of an item, the greater the chances are of that student scoring the correct answer. When a student’s *ability location* is equal to the *difficulty location* of an item, there is a 50/50 (or 0.5) probability of that student scoring the correct answer.
- The Rasch model is used in the analysis of data for NAPLAN and PISA and is also used in disciplines such as medicine and the social sciences.
- Location values for student difficulty and item ability are given as *logits* (a contraction of the phrase ‘log-odds units’). Because logit values are based on probability, they are also referred to as *estimates* of item difficulty and *estimates* of student ability.

Table 1 - Item difficulty analysis for the 2011 Stage 3C/3D Mathematics Specialist examination

Section	Question	Location	Difficulty
S01	1		N/A
S01	1a	-0.84	Easy
S01	1b	-0.07	Moderate
S01	2	0.73	Moderate
S01	3		N/A
S01	3a	0.25	Moderate
S01	3b	-0.21	Moderate
S01	4		N/A
S01	4a	0.26	Moderate
S01	4b	0.12	Moderate
S01	4c	-0.84	Easy
S01	5		N/A
S01	5a	-0.30	Moderate
S01	5b	0.68	Moderate
S01	6	-0.26	Moderate
S01	7		N/A
S01	7a	1.23	Difficult
S01	7b	1.20	Difficult
S01	7c	1.39	Difficult
S02	8		N/A
S02	8a	-1.19	Easy
S02	8b	-0.94	Easy
S02	9		N/A
S02	9a	-1.73	Easy
S02	9b	-0.09	Moderate
S02	10		N/A
S02	10a	-1.06	Easy
S02	10b	0.09	Moderate
S02	11		N/A
S02	11a	-0.15	Moderate
S02	11b	-0.12	Moderate
S02	12		N/A
S02	12a	-2.86	Very easy
S02	12b	-0.72	Easy
S02	12c	-0.51	Easy
S02	13		N/A
S02	13a	-1.86	Easy
S02	13b	-0.23	Moderate

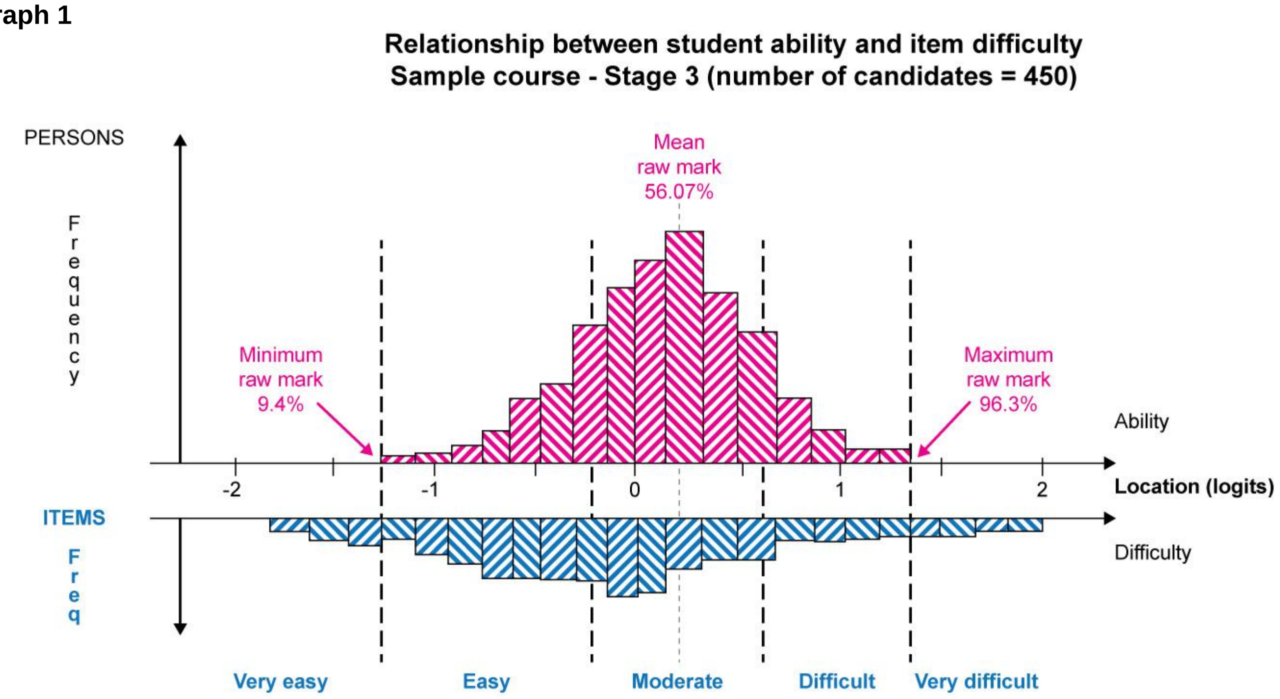
Table 1 - Item difficulty analysis for the 2011 Stage 3C/3D Mathematics Specialist examination (continued)

Section	Question	Location	Difficulty
S02	13c	0.66	Moderate
S02	14	0.63	Moderate
S02	15		N/A
S02	15a	-0.21	Moderate
S02	15b	1.21	Difficult
S02	16		N/A
S02	16a	-0.12	Moderate
S02	16b	-0.27	Moderate
S02	17		N/A
S02	17a	-0.71	Easy
S02	17b	0.57	Moderate
S02	17c	2.36	Difficult
S02	17d	1.52	Difficult
S02	18		N/A
S02	18a	-2.04	Very easy
S02	18b	-1.37	Easy
S02	18c	0.76	Moderate
S02	18d	1.06	Moderate
S02	19		N/A
S02	19a	0.77	Moderate
S02	19b	0.82	Moderate
S02	20		N/A
S02	20a	1.29	Difficult
S02	20b	1.10	Moderate

A walk-through of a graph of student ability and item difficulty

Graph 1 (for a sample Stage 3 WACE examination with 450 candidates) provides an example of how data from a Rasch analysis of student ability and item difficulty can be represented.

Graph 1



- The frequency distribution of estimates of student abilities is shown in the top half of the graph.
- The frequency distribution of estimates of item difficulties is shown in the bottom half of the graph.
- These two measures share a common horizontal scale showing locations, expressed as logits.
- Logit values do not relate directly to percentage marks; however, the percentage raw exam scores are represented on the graph, e.g. the maximum raw mark, the minimum raw mark and the mean raw mark of the examination.
- The relationship between the ability of students and the difficulty of items is such that
    - a student with an ability estimate *equal* to the difficulty of an item has a 50% chance of achieving the maximum available mark for the item
    - a student with an ability estimate *greater* than the difficulty of an item has more than a 50% chance of achieving the maximum available mark for the item
    - a student with an ability estimate *less* than the difficulty of an item has less than a 50% chance of achieving the maximum available mark for the item.
- Items of ‘average’ or ‘moderate’ difficulty are placed around the mean person ability; items of increasing difficulty are placed to the right and items of decreasing difficulty are placed to the left.

A good spread of student abilities and item difficulties – and some questions for discussion

Graph 1 presents one example of a good spread of student abilities and question difficulties resulting from a Rasch analysis of a fictional examination:

- the mean raw mark (56.07) is considered appropriate, within general statistical terms and in terms of the expectations for WACE examinations
  - *Discussion question: In terms of raw marks, has this been a difficult or easy examination?*
- the minimum examination mark is close to zero and the maximum is close to 100%.
  - *Discussion questions: What are the implications of having a range of raw marks from 0% to 100%? For example, is this useful for the purposes of discrimination?*
- the range of marks (9.4% to 96.3%) is appropriate.
  - *Discussion question: In terms of raw marks, has this examination efficiently discriminated among students, i.e. were some items too easy or too difficult for this cohort?*
- the distribution of item difficulties is good in relation to the distribution of student abilities.
  - *Discussion question: What implications are suggested when there are items with difficulty estimates greater than the maximum ability estimate and less than the minimum ability estimate?*

Why are two graphs provided for some examinations?

When an item is worth just 1 mark, it is known as a dichotomous item. When an item is worth more than 1 mark, it is known as a polytomous item.

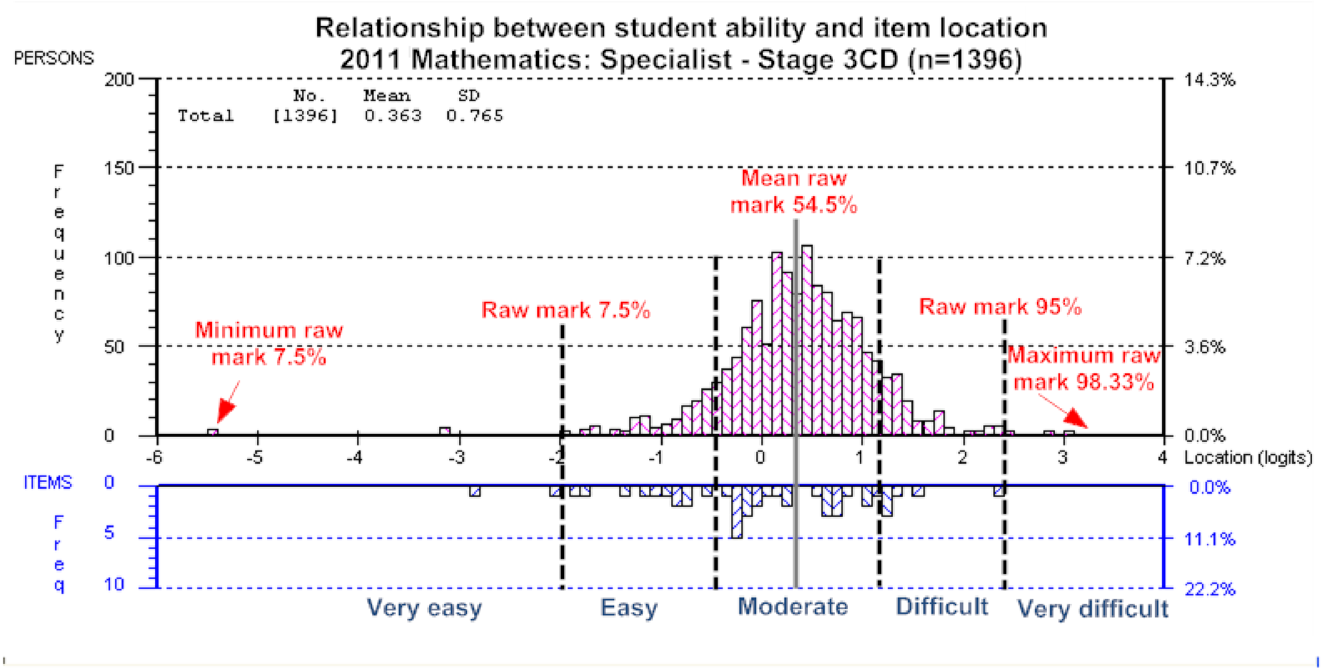
For polytomous items, the item difficulty is the average of the difficulties of achieving each mark allocated to the item. Misleading conclusions can sometimes be drawn from graphs of these data when there are gaps in the item difficulty distribution, e.g. there may appear to be not enough difficult items or not enough easy items.

In these instances, it can be useful to check the distribution of the difficulty estimates for achieving each individual mark (marks category). A Rasch analysis allows for graphing the difficulty in scoring each mark, or the *threshold* for moving from one mark to the next.

Where possible, therefore, *two* graphs are provided in the Standards Guides 2011: Examples are Graphs 2 and 3 below for the 2011 Stage 3C/3D Mathematics Specialist examination.

Graph 2 shows the item difficulty and the student abilities frequency distribution.

Graph 2 - Distribution between student ability and item difficulty

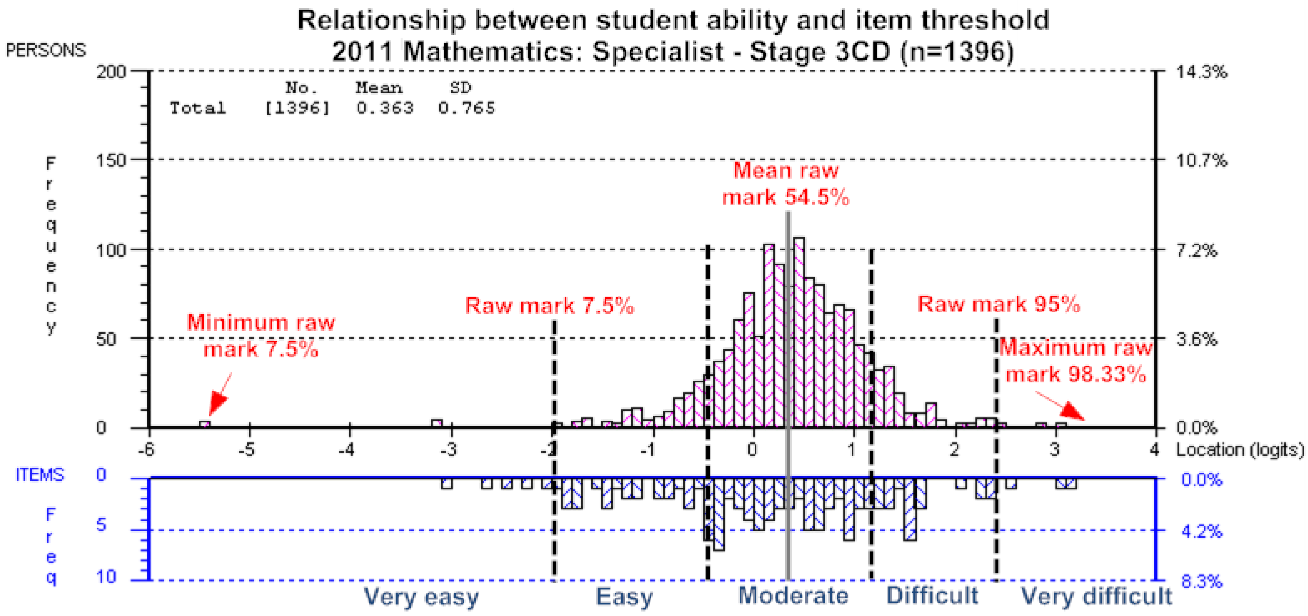


Notes

As the graph shows, there were no 'very difficult' items. However, although an item might not be 'difficult' or 'very difficult', it is possible that it was difficult to achieve the higher marks for that item. The difficulty of achieving each mark is called a 'threshold'. Therefore, for a better understanding of item difficulty, we need to analyse the distribution of person abilities and thresholds, as shown in Graph 3.

Graph 3 *(opposite)* shows the difficulty of achieving each category and the student abilities frequency distribution.

Graph 3 - Distribution between student ability and item thresholds difficulty



Notes

From the spread of item thresholds in this graph, we can see that there were items that had marks that were 'difficult' and 'very difficult' to achieve. The spread of the thresholds was quite good and the examination population was challenged and targeted well.

Some points to bear in mind for understanding examination analysis

- When evaluating the **range** (spread) of examination marks, consider the size of the cohort sitting the examination. A small cohort may involve a narrow range of student abilities.
- When evaluating the **mean** examination mark, consider the nature of the cohort sitting the examination. The examination difficulty may be appropriate for the cohort for which the course was designed, but the actual cohort may be weaker or stronger than expected.
- In these notes, the **difficulty** of the item refers to the *average* of the difficulties of acquiring each marks category for the item. For example, it may be very difficult to obtain a high mark for an item rated as being of 'moderate difficulty', if that item is worth a large number of marks. Conversely it may be very easy to obtain a low mark.
- Recommendations to remove items of a certain level of difficulty or easiness do not imply that these are poor items, but simply that there are too many items at the same level of difficulty.
- Recommendations to add more difficult items may result in a better discrimination among students.