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Unit 2 Specialist Mathematics Test 4 2022
Matrices, Transformations and Trigonometry

Student name: MARK KING Teacher name: ANNE SERKI

Task type: Response

Time allowed for this task: 40 minutes

Number of questions: 11

Materials required: Formula Sheet

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: 1 A4 Page of Notes (Double Sided), NO CALCULATOR ALLOWED

Marks available: 38 marks

Task weighting: 10 %

Formula sheet provided: Yes

Note: All part questions worth more than 2 marks require working to obtain full marks.

Question 1 **[2 Marks]**

The table below shows information about two matrices, A and B

Matrix	Size	Rule
A	2×2	$a_{ij} = 2i + j$
B	2×2	$b_{ij} = i - j$

The element in row i and column j of matrix A is a_{ij}
The element in row i and column j of matrix B is b_{ij}

Calculate $A + B$

Solution
$A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $A + B = \begin{bmatrix} \mathbf{3} & \mathbf{3} \\ \mathbf{6} & \mathbf{6} \end{bmatrix}$
Marking key/Mathematical behaviours
✓ correct matrices A and B ✓ Add matrices together correctly (1 mark for A+B = 3i)

Question 2 **[2 Marks]**

Calculate $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \mathrm{L} \\ \mathrm{E} \\ \mathrm{A} \\ \mathrm{P} \\ \mathrm{S} \end{bmatrix} =$

Solution
$\begin{bmatrix} \mathbf{P} \\ \mathbf{A} \\ \mathbf{L} \\ \mathbf{E} \\ \mathbf{S} \end{bmatrix}$
Marking key/Mathematical behaviours
✓ correct size (5 x 1) ✓ correct matrix

Question 3 **[2 Marks]**

Write the single matrix that corresponds to a dilation by a factor of 2 from the *x*-axis followed by a rotation 90˚ degrees clockwise about the origin.

Solution
$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{2} \\ \mathbf{-1} & \mathbf{0} \end{bmatrix}$ *many incorrect answers of* $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-2} & \mathbf{0} \end{bmatrix}$
Marking key/Mathematical behaviours
✓ correct transformation matrices ✓ correct order and final answer (incorrect answer is 1 out of 2 if working shown)

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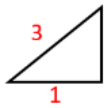
Question 4 **[2 Marks]**

Let $\sec(x) = 3$, where $\frac{3\pi}{2} \leq x \leq 2\pi$. Calculate the exact value of $\cot(x)$

Solution
$\cos x = \frac{1}{3}$ $\cot x = \frac{\text{adj}}{\text{opp}} = -\frac{1}{\sqrt{8}} = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4}$  $\sqrt{9-1} = 2\sqrt{2}$
Marking key/Mathematical behaviours
✓ Correct use of Pythagoras to get unknown side or sin(x) ✓ Correct final answer (doesn't need to be simplified) (-1 if the wrong sign)

Question 5 **[3 Marks]**

Use matrices to solve the pair of the simultaneous equations below:

$3x - 2y = 10$

$2x - 5y = 7$

Solution
$\begin{bmatrix} 3 & -2 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -5 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 7 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{11} \begin{bmatrix} -5 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 7 \end{bmatrix} = -\frac{1}{11} \begin{bmatrix} -36 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{36}{11} \\ -\frac{1}{11} \end{bmatrix}$ $\boldsymbol{x = \frac{36}{11}, y = -\frac{1}{11}}$
Marking key/Mathematical behaviours
✓ Converted to matrix form ✓ Correct re-arranging ✓ Correct answer (accept in matrix form if $\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{11} \begin{bmatrix} -36 \\ 1 \end{bmatrix}$)

Question 6 **[2 Marks]**

For which value(s) of a will the simultaneous equations below have a unique solution.

$5x + ay = 20$

$2ax + 4y = -13$

Solution
$20 - 2a^2 \neq 0$ $\boldsymbol{a \neq \sqrt{10} \text{ or } -\sqrt{10}}$
Marking key/Mathematical behaviours
✓ Correct equation for determinant equals zero ✓ Correct answers (both)

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Question 7 **[3 Marks]**

If $A = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix}$, find $\boldsymbol{X}$ such that $A\boldsymbol{X} + B = C$

Solution
AX = C – B X = A^{-1} (C – B) $A^{-1} = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{10}\begin{bmatrix} 6 & -4 \\ -2 & 3 \end{bmatrix}$ X = A^{-1}(C – B) = $\frac{1}{10}\begin{bmatrix} 6 & -4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 1 & -4 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 2 & -2 \\ 1 & -6 \end{bmatrix} = \begin{bmatrix} 0.2 & -0.2 \\ 0.1 & -0.6 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{1}{10} & -\frac{3}{5} \end{bmatrix}$
Marking key/Mathematical behaviours
✓ Rearranges equation correctly to solve for X ✓ Correct calculation of A^{-1} ✓ Correct calculation of X (can be in un-simplified fraction or decimal form)

Question 8 **[3 Marks]**

The matrix $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ transforms the point $(3, 4)$ to the point $(11, 10)$. Find the values of a and b.

Solution
$\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \end{bmatrix}$ $\begin{bmatrix} 3a + 4b \\ 3b + 4a \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \end{bmatrix}$ 3a + 4b = 11 x 4   12a + 16b = 44 4a + 3b = 10 x -3   -12a – 9b = -30 7b = 14   ➔ **b = 2, a = 1**
Marking key/Mathematical behaviours
✓ Sets up transformation in Matrix form ✓ Converts to simultaneous equations ✓ Solves simultaneous equations (any way) stating values of a and b clearly

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Question 9 **[7 Marks = 2, 3, 2]**

If $\cos(A) = -\frac{3}{5}$ where $\frac{\pi}{2} \le A \le \pi$ and $\sin(B) = -\frac{5}{13}$ where $\frac{3\pi}{2} \le B \le 2\pi$, calculate the following as exact values.

a) $\cot(A)$

Solution
$sin^2(A) = 1 - cos^2(A) = 1 - \left(\frac{-3}{5}\right)^2 = \frac{16}{25}$ $\quad sin(A) = \frac{4}{5}$ (or use triangle)
$cot(A) = cos(A) \div sin(A) = \frac{-3}{5} \div \frac{4}{5} = \frac{-3}{5} \times \frac{5}{4} = -\mathbf{\frac{3}{4}}$
Marking key/Mathematical behaviours
✓ Correct use of Pythagoras to find unknown side or cos(A)
✓ Calculate value and sign of cot(A)

b) $\tan(2B)$

Solution
$cos^2(B) = 1 - sin(B) = 1 - \left(\frac{-5}{13}\right)^2 = \frac{144}{169}$ $\quad cos(B) = \frac{12}{13}$
$tan(B) = sin(B) \div cos(B) = \frac{-5}{13} \div \frac{12}{13} = \frac{-5}{13} \times \frac{13}{12} = -\frac{5}{12}$
$tan(2B) = \frac{2tan(B)}{1 - tan^2(B)} = \frac{2\left(-\frac{5}{12}\right)}{1 - \left(-\frac{5}{12}\right)^2} = -\frac{10}{12} \div \frac{119}{144} = -\frac{10}{12} \times \frac{144}{119} = -\mathbf{\frac{120}{119}}$
Marking key/Mathematical behaviours
✓ Calculate value of cos(B), then tan(B)
✓ Correctly substitute into double angle formula to calculate tan(2B)
✓ Correct final answer (doesn't need to be simplified)

c) $cos\ (A + B)$

Solution
$cos(A + B) = cos(A)cos(B) - sin(A)sin(B)$
$cos(A + B) = \frac{-3}{5} \times \frac{12}{13} - \frac{4}{5} \times -\frac{5}{13} = -\frac{36}{65} + \frac{20}{65} = -\mathbf{\frac{16}{65}}$
Marking key/Mathematical behaviours
✓ Correct substitution into cos (A+B) equation
✓ Correct calculation and simplification

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Question 10 **[6 Marks = 3, 3]**

Prove the following identities.

a) $\cot\theta - \tan\theta = 2\cot(2\theta)$

Solution
LHS: $cot\theta - tan\theta = \frac{cos\theta}{sin\theta} - \frac{sin\theta}{cos\theta} = \frac{cos^2\theta - sin^2\theta}{sin\theta cos\theta} = \frac{cos2\theta}{\frac{1}{2}sin2\theta} = 2cot2\theta$ **Since LHS = RHS, it is proved that** $cot\theta - tan\theta = 2cot2\theta$
Marking key/Mathematical behaviours
✓ Correct substitution of $tan\theta = \frac{sin\theta}{cos\theta}$ and $cot\theta = \frac{cos\theta}{sin\theta}$ ✓ Correct substitution of double angle formulas and simplification.   ✓ Proof is very clear and finalised by saying equal to RHS or LHS

b) $\sin(2\theta) = \frac{2\tan\theta}{1+tan^2\theta}$

Solution
RHS: $\frac{2tan\theta}{1+tan^2\theta} = \frac{2sin\theta}{cos\theta} \div \left(\frac{cos^2\theta}{cos^2\theta} + \frac{sin^2\theta}{cos^2\theta}\right) = \frac{2sin\theta}{cos\theta} \div \left(\frac{cos^2\theta + sin^2\theta}{cos^2\theta}\right)$ $\frac{2sin\theta}{cos\theta} \div \left(\frac{1}{cos^2\theta}\right) = \frac{2sin\theta}{cos\theta} \times \frac{cos^2\theta}{1} = 2sin\theta cos\theta = sin2\theta$ **OR** RHS: $\frac{2tan\theta}{1+tan^2\theta} = \frac{2tan\theta}{sec^2\theta} = \frac{2sin\theta}{cos\theta} \times cos^2\theta = 2sin\theta cos\theta = sin2\theta$ **Since RHS = LHS, it is proved that** $sin2\theta = \frac{2tan\theta}{1+tan^2\theta}$
Marking key/Mathematical behaviours
✓ Correct use of $tan\theta = \frac{sin\theta}{cos\theta}$ and/or $1 + tan^2\theta = sec^2\theta$ into RHS   ✓ Correct use of double angle formula to convert to $sin2\theta$ ✓ Proof is very clear and finalised by saying equal to RHS or LHS

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Question 11 **[6 Marks = 3, 3]**

Solve each of the following equations for $0 \le x \le 2\pi$ (give exact values)

a) $\sin(3x)\cos(x) - \cos(3x)\sin(x) = \frac{\sqrt{3}}{2}$

Solution
$sin(3x)cos(x) - cos(3x)sin(x) = sin(3x - x) = sin(2x)$ $solve\ sin(2x) = \frac{\sqrt{3}}{2}$ $2x = \frac{\pi}{3}, \frac{2\pi}{3}$ $x = \frac{\pi}{6}, \frac{\pi}{3}$ **add period** $\frac{2\pi}{2} = \pi = \frac{6\pi}{6}$ **until out of range** $x = \frac{\pi}{6}, \frac{2\pi}{6}, \frac{7\pi}{6}, \frac{8\pi}{6} = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$
Marking key/Mathematical behaviours
✓ Simplify to sin(2x) ✓ Solve 2x correctly for at least 2 values ✓ Solve x correctly for all 4 values (don't need to be simplified)

b) $\sqrt{3}\sin x + \cos x = 1$

Solution
$\sqrt{3}sin(x) + 1cos(x) = 2\left(\frac{\sqrt{3}}{2}sin(x) + \frac{1}{2}cos(x)\right) = 2cos(x - A)$ $where\ \ sin(A) = \frac{\sqrt{3}}{2}\ and\ \ cos(A) = \frac{1}{2},\quad therefore\ A = \frac{\pi}{3}$ $2cos\left(x - \frac{\pi}{3}\right) = 1 \quad cos\left(x - \frac{\pi}{3}\right) = \frac{1}{2}$ **OR** $2sin\left(x + \frac{\pi}{6}\right) = 1 \quad sin\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$ $x - \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3} \quad x = \frac{2\pi}{3}, \frac{6\pi}{3}$ **OR** $x + \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6} \quad x = 0, \frac{2\pi}{3}$ $x = 0, \frac{2\pi}{3}, 2\pi$
Marking key/Mathematical behaviours
✓ Convert to an angle sum formula (either answer acceptable) ✓ Solve (x +/- A) correctly for 2 values ✓ Solve x correctly for all 3 values

END OF TEST

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Extra working space