GRAPHS AND NETWORKS

Euler's Rule

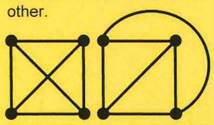
If a graph is planar, then:

$\boxed{V - E + F = 2}$

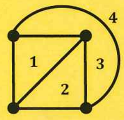
V: Number of vertices
E: Number of edges
F: Number of faces/regions

Planar Graphs

A graph is planar if it can be redrawn in such a way that no edges cross over each other.



Note: When counting faces in a planar graph, outside region counts as 1 face.



Including the outside face, total number of faces is 4. From Euler's rule, $4 - 6 + 4 = 2$, hence graph is planar.

Graph and Network Terminology

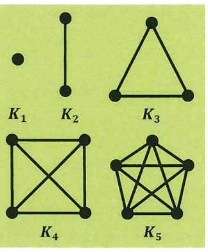
Loop: an edge that joins a vertex to itself (e.g. A and F).
Multiple Edges: two or more edges that have the same start and end vertices (e.g. B to C have two edges between them).
Isolated Vertex: a disconnected vertex in the graph (e.g. G).
Bridge: connects parts of a graph that would otherwise result in an isolated vertex or vertices (e.g. A to B and E to F).
Degree: Number of edges connected to a vertex. Loops are counted twice (e.g. $deg\ A = 3$, $deg\ C = 4$, $deg\ G = 0$).
Subgraph: A graph that has vertices and edges that are a subset of a larger graph (e.g. $D - C - E$ is a subgraph).
Simple Graph: a graph that has no loops or multiple edges.
Directed Graph (Digraph): a graph where all edges are directed from one vertex to another (shown by an arrow).
Weighted Graph: a graph whose edges has been assigned weights (weights are listed next to each edge in the graph).
Connected Graph: a graph with a path between every vertex.

Complete Graph (K_n)

A graph with n vertices in which every vertex is connected to every other vertex by one edge.

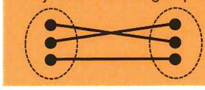
$\boxed{\#\, Edges\ in\ K_n = n(n-1)/2}$

Number of edges follow the triangular number sequence: 0, 1, 3, 6, 10, 15, 21, 28... (i.e. difference between each number is 1, 2, 3, 4, 5, 6, 7...)



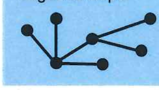
Bipartite Graph

A graph that has 2 sets of vertices. Any edges can only connect the 2 groups.



Tree

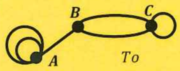
A connected graph that has no multiple edges or loops.



Adjacency Matrix

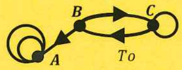
Matrix that shows how many times each vertex is connected (adjacent) to another vertex by a single path. Loops count as 1 edge in an adjacency matrix.

In an undirected graph, the matrix is symmetrical along the diagonal:



$$\begin{array}{cc} & \text{To} \\ & \begin{array}{ccc} A & B & C \end{array} \\ \text{From} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \end{array}$$

In a directed graph, the matrix is *not* symmetrical along the diagonal:



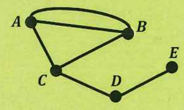
$$\begin{array}{cc} & \text{To} \\ & \begin{array}{ccc} A & B & C \end{array} \\ \text{From} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

Tips to Find Shortest Path Between Two Vertices

Tip 1: Find all possible paths between each of the two vertices and test each path individually.
Tip 2: Where there are multiple edges, ignore the higher weighted edges.
Tip 3: Sometimes the shortest path doesn't mean the least amount of edges used; check all options.

Transition Matrix

Any entries of 0 in an adjacency matrix to the n^{th} power means it's impossible to move to and from the points corresponding to that entry in n steps.



1 Step: adjacency matrix M

$$M = \begin{bmatrix} 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

It is *impossible* to go from A to A, B to B, C to C, D to A, D to B, D to D, E to A, E to B, E to C or E to E in 1 step.

An example of a 1 step option is that it is *possible* to go from A to B in 1 step in 2 different ways.

2 Step: adjacency matrix M^2

$$M^2 = \begin{bmatrix} 5 & 1 & 2 & 1 & 0 \\ 1 & 5 & 2 & 1 & 0 \\ 2 & 2 & 3 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

It is *impossible* to go from D to C, E to A, E to B or E to D in 2 steps.

An example of a 2 step option is that it is *possible* to go from A to A in 2 steps in 5 different ways.

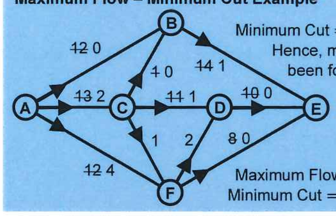
1 or 2 Step: adjacency matrix $M + M^2$

$$M + M^2 = \begin{bmatrix} 5 & 3 & 3 & 1 & 0 \\ 3 & 5 & 3 & 1 & 0 \\ 3 & 3 & 3 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

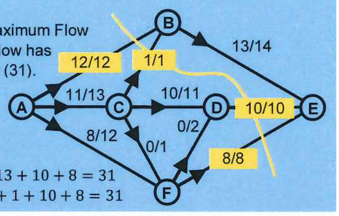
It is *impossible* to go from E to A or E to B in 1 or 2 steps.

An example of a 1 or 2 step options is that it is *possible* to go from A to C in 1 or 2 steps in 3 different ways.

Maximum Flow – Minimum Cut Example



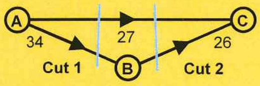
Minimum Cut = Maximum Flow
Hence, max flow has been found (31).



Maximum Flow = $13 + 10 + 8 = 31$
Minimum Cut = $12 + 1 + 10 + 8 = 31$

Alternate Maximum Flow – Minimum Cut Method

Step 1: Determine the value of all possible cuts.
Step 2: Find the value of the smallest cut; this is the maximum flow through the network.



Cut 1 = $34 + 27 = 61$ and Cut 2 = $27 + 26 = 53$.
Cut 2 is the smaller cut, hence 53 is maximum flow.

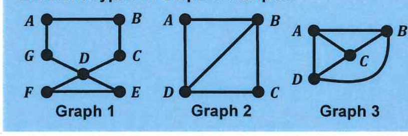
Maximum Flow - Minimum Cut Theorem

- A cut is a line drawn through a number of edges which stops all flow from start to finish.
- Value of a cut is the total flow of the edges cut.
- If there is an edge flows toward the start of the network (source) rather than toward the finish (sink), it can be included in a cut and its flow can be treated as 0.

To determine the minimum cut, find the max flow through the network and create a cut that is as close to the start (source) as possible that passes through all edges with full flow.

$\boxed{Maximum\ Flow = Minimum\ Cut}$

Different Types of Graphs Examples



Using Graph 1, give an example of a:
- Eulerian Circuit: ABCDEFDGA

Using Graph 2, give an example of a:
- Semi-Eulerian Trail: BDABCD

Using Graph 3, give an example of a:
- Hamiltonian Cycle: ABCDA
- Semi-Hamiltonian Path: ABDC

Different Types of Walks and Graphs

Rule	Open/Closed	Name	Also Called	Edges	Vertices
Uses only some vertices or edges in a graph	Open	Path	-	Can't Repeat	Can't Repeat
	Closed	Path	Cycle*	Can't Repeat	Can't Repeat
	Open	Trail	-	Can't Repeat	May Repeat
	Closed	Trail	Circuit	Can't Repeat	May Repeat
Rule	**Type**	**Name**	**Open/Closed**	**Edges**	**Vertices**
Uses all edges or vertices in a graph	Eulerian**	Circuit	Closed	All Edges in Graph Once	May Repeat
	Semi-Eulerian***	Trail	Open		
	Hamiltonian	Cycle	Closed	May Repeat	All Vertices in Graph Once
	Semi-Hamiltonian	Path	Open		

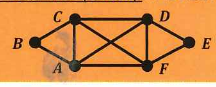
*Cycle: there is an exception; the starting and finishing vertex is allowed to repeat whilst all other vertices cannot.

**Eulerian: a graph is eulerian if every vertex has an even degree (the circuit starts and ends on any vertex).

***Semi-Eulerian: a graph is semi-eulerian if it has one pair of vertices with an odd degree (the trail starts and ends on either one of these two vertices).

Different Types of Walks Examples

Using the graph, give an example of a:
- Open Walk: BACAFDE
- Closed Walk: DCFACD
- Open Path: BADEF
- Closed Path (Cycle): BCDEFAB
- Open Trail: BCDFCA
- Closed Trail (Circuit): FACFDEF



Walks (Open and Closed)

Walk: alternating sequence of vertices/edges in a graph. Length of a walk is the number of edges used.
Open Walk: a walk that starts/ends on two different vertices.
Closed Walk: a walk that starts/ends on the same vertex.

Minimum Completion Time and Critical Path Analysis

Minimum Completion Time: the minimum time it takes for all activities in a project to be completed.

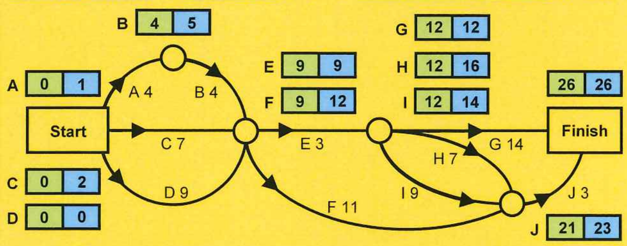
Critical Path: the sequence of activities that have the longest duration and dictates the minimum completion time.

Labelling Activities in Network: **Activity** | **EST** | **LST** |

Minimum Completion Time Example

Activity	Duration	Predecessors
A	4 hours	None
B	4 hours	A
C	7 hours	None
D	9 hours	None
E	3 hours	B, C, D
F	11 hours	B, C, D
G	14 hours	E
H	7 hours	E
I	9 hours	E
J	3 hours	H, I, F

Number of Activities: 10 Critical Path: D - E – G Minimum Completion Time: 26 hours



Earliest Starting Time (EST)

The earliest time an activity can be commenced given any predecessors; found by forward scanning:

Step 1: EST for the starting activities is 0.

Step 2: To find the EST for the other activities add the EST from the previous activity to the activity duration (e.g. EST of activity H is 12 as EST of activity E is 9 and duration for activity E is 3, hence $9 + 3 = 12$).

If there are multiple activities feeding into another activity (such as Activity E whose predecessors are activities B, C and D), choose the highest duration of those activities (i.e. Activity D with duration of 9).

Step 3: Continue this process (moving from left to right through the network) until you get to the finish.

Latest Starting Time (LST)

The latest time an activity can be delayed without changing the critical path; found by backward scanning:

Step 1: Set LST equal to the EST of the finishing time.

Step 2: Using the LST of the finish time, work backwards through the network by subtracting the activity duration from the LST of the previous activity (e.g. duration of Activity J is 3 and LST of Activity J is $26 - 3 = 23$).

If there are multiple activities feeding into another activity (such as Activity E whose predecessors are activities G, H and I), choose the lowest LST's of those activities to subtract from (i.e. Activity G with LST of 12).

Step 3: Continue this process (moving from right to left through the network) until you get to the start.

Slack / Float of an Activity

$\boxed{Slack = LST - EST}$

If the slack of an activity is equal to 0, then that activity is on the Critical Path.

By how much can Activity H be lengthened without changing the critical path?

$16 - 12 = 4\ hours.$

By how much can Activity G be lengthened without changing the critical path?

Activity G is on the critical path, $12 - 12 = 0$ hours slack.

If Activity I is shortened to 7 hours, how much slack does activity J have?

Shortening Activity I to 7 hours changes the EST of Activity J from 21 to 20 (as Activity F is now the next highest completion time of 20 hours as $9 + 11 = 20$). Hence $EST\ Activity\ J = 20$ and $Slack = 23 - 20 = 3\ hours.$

If Activity G is shortened to 10 hours, what is the new critical path and minimum completion time?

The new critical path is D – E – I – J with a minimum completion time of $9 + 3 + 9 + 3 = 24$ hours.

GRAPHS AND NETWORKS

Hungarian Algorithm

Step 1: Subtract the smallest entry in each row from all entries in its row.
Step 2: Subtract the smallest entry in each column from all the entries in its column.
Step 3: Draw lines through appropriate rows and columns so that all the zero entries of the matrix are covered and the minimum number of lines is used.
Step 4: If the minimum number of covering lines is equal to the number of rows in the matrix, go to step 6. If the number of covering lines is less than the number of rows in the matrix, go to step 5.
Step 5: Determine the smallest entry not covered by any line. Subtract this entry from each uncovered row and then add it to each covered column. Return to step 3.
Step 6: Select zero entries in each column of the matrix so that other zero entries are not in its row; match with original.

Hungarian Algorithm Example

$$\begin{array}{c|ccc} & D & E & F \\ \hline A & 8 & 8 & 6 \\ B & 2 & 3 & 7 \\ C & 4 & 9 & 5 \end{array}$$

Note: you may have to create cost matrix from a bipartite graph.

Step 1: subtract smallest entry in each row from each row.

$$\begin{bmatrix} 8 & 8 & \underline{6} \\ \underline{2} & 3 & 7 \\ \underline{4} & 9 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 5 \\ 0 & 5 & 1 \end{bmatrix}$$

Step 2: subtract smallest entry in each column from each column.

$$\begin{bmatrix} 2 & 2 & \underline{0} \\ \underline{0} & \underline{1} & 5 \\ \underline{0} & 5 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 5 \\ 0 & 4 & 1 \end{bmatrix}$$

Step 3: Draw a minimum number of lines to cover all zero entries.

$$\begin{bmatrix} 2 & 1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{5} \\ \cancel{0} & \cancel{4} & \cancel{1} \end{bmatrix}$$

Note: double check that you are using the least amount of lines.

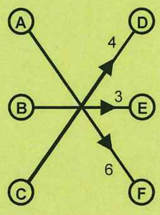
Step 4: there are 3 lines drawn in step 3 which is equal to number rows in the cost matrix (which is 3) go to step 6.

Step 6: Select a zero entry in each row so that it's not in the same column; match with original;

$$\begin{bmatrix} 2 & 1 & \boxed{0} \\ 0 & \boxed{0} & 5 \\ \boxed{0} & 4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 8 & 8 & \boxed{6} \\ 2 & \boxed{3} & 7 \\ \boxed{4} & 9 & 5 \end{bmatrix}$$

Minimum cost is: $4 + 3 + 6 = 13$

A is matched with F
B is matched with E
C is matched with D



Minimum Cost Assignment
Use the Hungarian Algorithm.

Maximum Cost Assignment
Subtract every entry in the cost matrix from the largest entry.

Then use the Hungarian Algorithm on the new matrix.

$$\begin{bmatrix} 1 & 4 & 5 \\ 5 & 7 & 6 \\ 5 & \underline{8} & \underline{8} \end{bmatrix} \rightarrow \begin{bmatrix} 7 & 4 & 3 \\ 3 & 1 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

In the example above, subtract all entries in the matrix from 8.

 # Rows Does Not Equal # Columns in the Cost Matrix
Add a "dummy" row or column of zeros to the cost matrix so that the number of rows equals the number of columns.

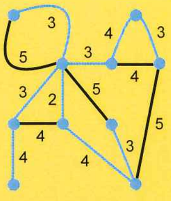
Then proceed to use the Hungarian Algorithm on the new matrix.

$$\begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 2 \\ \underline{0} & \underline{0} & \underline{0} \end{bmatrix}$$

Note: any matching involving the dummy row or column of zeroes is to be ignored in the final answer.

Prim's Algorithm on a Graph (Minimum Spanning Tree)

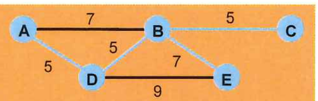
Step 1: Create a tree by selecting a random vertex from the graph.
Step 2: Grow the tree by selecting the closest vertex not yet in the tree. If there is a tie between two or more vertices, pick one at random.
Step 3: Repeat Step 2 until all vertices of the graph are selected such that no loops are created.



Minimum Weight: 29

Prim's Algorithm on a Distance Matrix (Minimum Spanning Tree)

Step 1: Select a random vertex, delete its row and mark its column.
Step 2: Scan all marked columns for the lowest non-zero entry and circle that entry. If there is a tie, pick an entry at random.
Step 3: Delete the row containing the circled entry and mark its column.
Step 4: Repeat steps 2 and 3 until all rows in the matrix are deleted.



$$\begin{array}{c|ccccc} & \overset{*}{A} & \overset{*}{B} & \overset{*}{C} & \overset{*}{D} & \overset{*}{E} \\ \hline \cancel{A} & 0 & 7 & 0 & 5 & 0 \\ \cancel{B} & 7 & 0 & 5 & 5 & 7 \\ \cancel{C} & 0 & 5 & 0 & 0 & 0 \\ \cancel{D} & 5 & 5 & 0 & 0 & 9 \\ \cancel{E} & 0 & 7 & 0 & 9 & 0 \end{array}$$

Selected Arcs:
AD / DB / BE / BC

Minimum Weight:
22

BIVARIATE DATA

Types of Variables

- **Numerical or Quantitative** → Discrete, Continuous
- **Categorical or Qualitative** → Ordinal, Nominal

Numerical or Quantitative: have values that describe a measurable quantity as a number, like 'how many' or 'how much'.
Discrete: can take whole values (e.g. number of children or number of cars).
Continuous: can take any value (e.g. height, time and temperature).
Categorical or Qualitative: have values that describe a 'quality' or 'characteristic' of data.
Ordinal: observations that can logically ordered or ranked (e.g. academic grades such as A, B, C, D or clothing sizes such as small, medium, large).
Nominal: observations that cannot be ordered logically (e.g. eye colour, brand, gender, religion).

Response and Explanatory Variables

Response Variable	Explanatory Variable
Dependent Variable	Independent Variable
Vertical Axis (y-axis)	Horizontal Axis (x-axis)

Explanatory Variable causes the Response Variable (e.g. being immunized causes resistance to disease or number of books read causes reading speed to be low, medium or high).

Pearson's Correlation Coefficient (r)
The value r such that $-1 \leq r \leq 1$ measures the direction and strength of a linear relationship between two variables.

Coefficient of Determination (r^2)
The value r^2 such that $0 \leq r^2 \leq 1$ shows the percentage of the variation in the response variable with the variation in the explanatory variable. It shows what percent of the data that is the closest to the line of best fit (i.e. if $r^2 = 0.85$, then 85% of the data is close to the line of best fit). Also, r^2 is equal to Pearson's Correlation Coefficient squared.

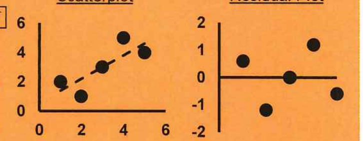
Least-Squares Line/Line of Best Fit ($y = ax + b$)
A linear equation that summarises the relationship between two variables where a is the gradient of the line (calculated by $a = rise/run$) and b is the y-intercept.

Describing a Scatterplot
Form: The type of pattern that the points follow (i.e. linear or non-linear).
Direction: What direction the points tend towards (i.e. positive or negative).
Strength: How closely the points follow a linear pattern (i.e. perfect, strong, moderate, weak or no relationship).

Value of r	Form	Direction	Strength
$r = 1$	Linear	Positive	Perfect
$0.75 \leq r < 1$	Linear	Positive	Strong
$0.5 \leq r < 0.75$	Linear	Positive	Moderate
$0.25 \leq r < 0.5$	Linear	Positive	Weak
$-0.25 \leq r < 0.25$	None	None	None
$-0.5 \leq r < -0.25$	Linear	Negative	Weak
$-0.75 \leq r < -0.5$	Linear	Negative	Moderate
$-1 < r < -0.75$	Linear	Negative	Strong
$r = -1$	Linear	Negative	Perfect

Residuals
Residual formula: $\boxed{e = y - \hat{y}}$

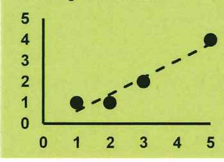
e: is the residual
y: is the observed value (y co-ordinate from the data)
$\hat{y}$: is the predicted value (substitute x co-ordinate into line of best fit equation)



x	y	$\hat{y}$	e
1	2	1.4	0.6
2	1	2.2	-1.2
3	3	3	0
4	5	3.8	1.2
5	4	4.6	-0.6

Step 1: Create a scatterplot and determine the correlation coefficient and the line of best fit (i.e. line of best fit is $y = 0.8x + 0.6$ and $r = 0.8$).
Step 2: determine the residual using the formula $e = y - \hat{y}$ and create a residual plot.
Step 3: analyse residual plot (i.e. random pattern indicates linear model is a good fit).

Interpolation
Using the line of best fit to predict values that lie within the range of the data.



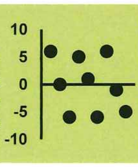
Line of best fit: $y = 0.8x - 0.2$

Estimating y when $x = 4$ can be determined by substituting $x = 4$ the line of best fit. This is considered interpolation as $x = 4$ is within the range of x values (0 – 5).

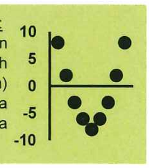
$y = 0.8(4) - 0.2 = 3$
$\therefore$ (4 , 3) is the interpolated point.

Extrapolation
Using the line of best fit to predict values that lie outside the range of the original data. Not recommended as the nature of the data beyond what was recorded is unknown (especially if the correlation coefficient is weak). Using the graph on the left, estimating the value of y when $x = 10$ is considered extrapolation as $x =$ 10 lies outside the range of x values (0 – 5).
$y = 0.8(10) - 0.2 = 7.8$
$\therefore$ (10 , 7.8) is the extrapolated point.

Residual Plots
Random Pattern:
A random pattern in a residual plot indicates that the data is a good fit for a linear model.

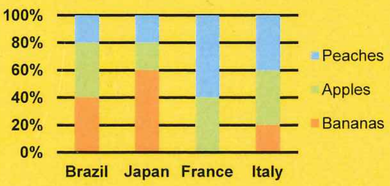


Non-Random Pattern:
A non-random pattern in a residual plot (such as a U-shaped pattern) indicates that the data is not a good fit for a linear model.



Stacked Column Graph
Data "stacked" on top of each other and totals to 100% on the y-axis. Below are the results of a survey taken from 4 countries that shows their preferred fruit.

To determine what percentage of each fruit that a country likes, find how wide each coloured column is by comparing it to the y-axis.



Breakdown of Percentages by Country:

Country	Peaches	Apples	Bananas
Brazil	20%	40%	40%
Japan	20%	20%	60%
France	60%	40%	0%
Italy	40%	40%	20%

Two-Way Table
Displays data between two variables. Below is a two-way table showing the popularity of apples, bananas and peaches among males and females:

Fruit	Male	Female	Total
Apple	20	40	60
Banana	90	110	200
Peach	50	70	120
Total	160	220	380

What % of apples are liked by males?

$$\frac{total\ likes\ of\ apples\ by\ males}{total\ apples} = \frac{20}{60} = 33.33\%$$

What % of males or females don't like peaches?

$$\frac{total\ likes\ of\ bananas\ and\ apples}{total\ males\ and\ females} = \frac{260}{380} = 68.42\%$$

Construct a table of percentages:

Fruit	Male	Female	Total
Apple	5.26%	10.53%	15.79%
Banana	23.68%	28.95%	52.63%
Peach	13.16%	18.42%	31.58%
Total	42.11%	57.89%	100%

Correlation Does Not Imply Causation
If two variables have a strong correlation between them, it does not necessarily mean that one variable causes the other variable in reality (e.g. if the variables *ice cream sales* and *number of deaths due to drowning* have a strong positive correlation coefficient of 0.9, it doesn't mean the two variables have a strong observable relationship in real life).

Causes of Incorrect Calculations of Pearson's Correlation Coefficient

- Coincidence: it could be a coincidence that data collected has a strong correlation (i.e. there is always the possibility that the data collected showed a strong correlation by random chance). To reduce the chance of a coincidence occurring, more data needs to be collected (at least 25 results).
- Confounding: a third variable that was failed to be taken into account had an influence between the two variables being tested (i.e. *ice cream sales* are impacted by another variable; the time of year, which will have an effect on the *number of deaths due to drowning* in the summer months).

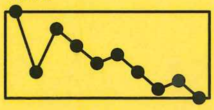
It also works in reverse; just because two variables have a weak correlation, due to coincidence and confounding, the two variables may in fact have a strong observable relationship in reality.

TIME SERIES

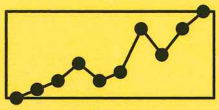
Time Series
Displays time (x-axis) and another variable (y-axis) such as cost, sales and rainfall. Time series can be described in three different ways:

- Trend
- Seasonal Pattern
- Cyclic/Irregular Pattern

Trend

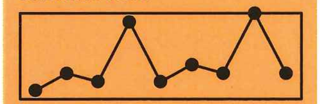


Negative
Downward Trend



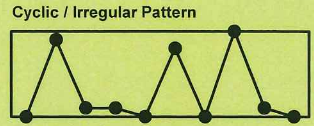
Positive
Upward Trend

Seasonal Pattern



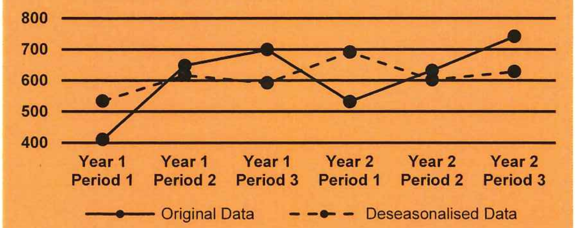
Seasonal Pattern with a pattern of 4 (i.e. 4PTMA is appropriate).

Cyclic / Irregular Pattern



Cyclic Pattern with significant peaks and troughs at irregular intervals.

Deseasonalising Data
Data adjusted that smooths the data due to seasonal fluctuations (e.g. reduce the fluctuations in apple picking between summer and winter).

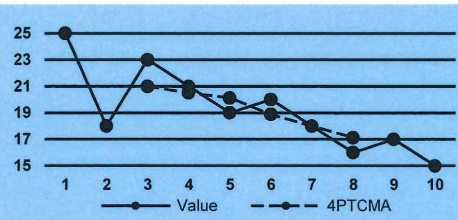


Step 1: Determine the average of the non-seasons (rows) and round answers to 4 decimal places. Non-seasons are commonly weeks, months or years.
Step 2: Divide the original data respectively by the average of the non-seasons (rows) found in step 1 and round answers to 4 decimal places.
Step 3: Determine the average of the seasons (columns) found in step 2 and round answers to 4 decimal places. This is called a seasonal index or seasonal indices. Seasons are commonly days, seasons or time periods.
Step 4: Divide the original data respectively by the seasonal indices found in step 3 and round to the nearest whole number. This is the deseasonalised data which can then be graphed as above.

Moving Averages (Even Seasonal Pattern)

Period	Value	4PTMA	4PTCMA
1	25		
1.5			
2	18		
2.5		21.75	
3	23		21.00
3.5		20.25 **(A)**	
4	21		20.50
4.5		20.75	
5	19		20.13 **(B)**
5.5		19.50	
6	20		18.88
6.5		18.25	
7	18		18.00
7.5		17.75	
8	16 **(C)**		17.13
8.5		16.50	
9	17		
9.5			
10	15		

4PTMA → 4 Point Moving Average
4PTCMA → 4 Point Centered Moving Average



Calculate the value of (A):
As this is a 4PTMA, find the 4 numbers in the value column that has **(A)** vertically halfway:

$$\frac{18+23+21+19}{4} = 20.25 = A$$

Calculate the value of (B):
To find a 4PTCMA value, average the 2 closest to it from the 4PTMA column:

$$\frac{20.75+19.50}{2} = 20.13 = B$$

Calculate the value of (C):
As this is a 4PTMA, find a group of 4 numbers that include **(C)**. Then find the 4PTMA for that group and solve for C:

$$\frac{20+18+C+17}{4} = 17.75$$

$$\frac{55+C}{4} = 17.75$$

$$55 + C = 17.75 \times 4$$

$$55 + C = 71 \rightarrow C = 16$$

Example Deseasonalising Data (Shown Above)

Year	Period 1	Period 2	Period 3
Year 1	411	648	699
Year 2	532	632	741

Step 1: Average the non-seasons (i.e. years)

Average Year 1	$\frac{411+648+699}{3} = 586$
Average Year 2	$\frac{532+632+741}{3} = 635$

Step 2: Divide original data respectively by Step 1

Year	Period 1	Period 2	Period 3
Year 1	$\frac{411}{586} = 0.7014$	$\frac{648}{586} = 1.1058$	$\frac{699}{586} = 1.1928$
Year 2	$\frac{532}{635} = 0.8378$	$\frac{632}{635} = 0.9953$	$\frac{741}{635} = 1.1669$

Step 3: Average the seasons in Step 2 (i.e. periods)

Average Period 1	$\frac{0.7014+0.8378}{2} = 0.7696$
Average Period 2	$\frac{1.1058+0.9953}{2} = 1.0505$
Average Period 3	$\frac{1.1928+1.1669}{2} = 1.1799$

Step 4: Divide original data respectively by Step 3

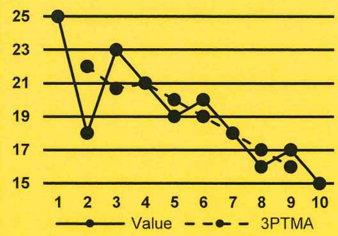
Year	Period 1	Period 2	Period 3
Year 1	$\frac{411}{0.7696} = 534$	$\frac{648}{1.0505} = 617$	$\frac{699}{1.1799} = 592$
Year 2	$\frac{532}{0.7696} = 691$	$\frac{632}{1.0505} = 602$	$\frac{741}{1.1799} = 628$

For example, the deseasonalised data for Period 2 of Year 1 is 617 and for Period 3 of Year 2 is 628.

Moving Averages (Odd Seasonal Pattern)

3PTMA → 3 Point Moving Average

Period	Value	3PTMA
1	25	
2	18	22
3	23	20.67
4	21	21 **(D)**
5	19	20
6	20	19
7	18 **(E)**	18
8	16	17
9	17	16
10	15	



Calculate the value of (D):
As this is a 3PTMA, find the 3 numbers in the value column that has **(D)** vertically halfway:

$$\frac{23+21+19}{3} = 21 = D$$

Calculate the value of (E):
As this is a 3PTMA, find a group of 3 numbers that include **(E)**. Then find the 3PTMA for that group and solve for E:

$$\frac{20+E+16}{3} = 18$$

$$\frac{36+E}{3} = 18$$

$$36 + E = 18 \times 3$$

$$36 + E = 54$$

$$E = 54 - 36 = 18$$

Properties of Seasonal Indicies / Seasonal Index
Percentage Property: Converting seasonal indicies to percentages indicates performance above or below average for that season (e.g. $1.0505 \rightarrow 105.05\% \rightarrow 5.05\%$ above average for Period 2 and $0.7696 \rightarrow 76.96\% \rightarrow 23.04\%$ below average for Period 1).
Additive Property: The sum of the seasonal Indicies equals the number of seasons in the data (i.e. $0.7696 + 1.0505 + 1.1799 = 3$).

Shortcut to Calculating 4PTCMA Values
Determine 4PTCMA values by using the formula:

$$\boxed{4PTCMA = \frac{(0.5 \times a) + b + c + d + (0.5 \times e)}{4}}$$

Where a, b, c, d and e are the 5 successive data points associated with the 4PTCMA that you are looking for.

From the 4PTMA example above, to determine **(B)** using the formula:

$$\frac{(0.5 \times 23) + 21 + 19 + 20 + (0.5 \times 18)}{4} = 20.13$$

The 5 numbers used in the example (23, 21, 19, 20 and 18) are the 5 numbers that are found in the value column and are centered around 20.13 **(B)** in the 4PTCMA column.

YOUR NOTES AND EXAMPLES

SEQUENCES AND FINANCE

Arithmetic and Geometric Sequences Formulae

Type	Explicit	Recursive	Sum of Series	Sum to Infinity
Arithmetic (+ or −)	$T_n = a + (n-1)d$	$T_{n+1} = T_n + d$, $T_1 = a$	$S_n = \frac{n}{2}(2a + (n-1)d)$	$S_\infty = \infty$ or $-\infty$
Geometric (× or ÷)	$T_n = ar^{n-1}$	$T_{n+1} = T_n \times r$, $T_1 = a$	$S_n = \frac{a(1-r^n)}{1-r}$	$S_\infty = \frac{a}{1-r}$

T_n: n^{th} term in the sequence
a: first term in the sequence (i.e. T_1)
r: common ratio between terms
d: common difference between terms
S_n: sum of the first n terms in the sequence
S_∞: sum of all possible terms in the sequence

Growth or Decay Sequences Formulae

Type	Explicit	Recursive
Growth (+)	$P_t = a(1+r)^t$	$P_{t+1} = (1+r)P_t$, $P_1 = a$
Decay (−)	$P_t = a(1-r)^t$	$P_{t+1} = (1-r)P_t$, $P_1 = a$

r: rate of growth or decay (as a decimal)
P_t: population at time t
a: initial amount (i.e. P_1)
t: time in years

Arithmetic Sequence Examples

Some values of an arithmetic sequence are shown in the table below:

n	4	5	6	7
T_n	21.5	24.2	26.9	29.6

Find the explicit rule for the n^{th} term.

Need to determine a and d:
Calculating a: $a = 21.5 - (3 \times 2.7) = 13.4$
Calculating d: $d = 24.2 - 21.5 = 2.7$
Substitute values into $T_n = a + (n-1)d$
Hence, $T_n = 13.4 + (n-1) \times 2.7$

Find the recursive rule for the $(n+1)^{th}$ term.

From above, $a = 13.4$ and $r = 2.7$
Substitute values into $T_{n+1} = T_n + d,\ T_1 = a$
Hence, $T_{n+1} = T_n + 2.7,\ T_1 = 13.4$

Geometric Sequence Examples

Some values of a geometric sequence are shown in the table below:

n	3	4	5	6
T_n	0.5	2	8	32

Find the explicit rule for the n^{th} term.

$T_3 = ar^{3-1} = \frac{1}{2} \ldots$ *Equation* 1
$T_4 = ar^{4-1} = 2 \ldots$ *Equation* 2
Solve for a and r: $a = 0.03125$ and $r = 4$
Substitute into $T_n = ar^{n-1}$
Hence, $T_n = 0.03125 \times 4^{n-1}$

Find the recursive rule for the $(n+1)^{th}$ term.

From above, $a = 0.3125$ and $r = 4$
Substitute values into $T_{n+1} = T_n \times r,\ T_1 = a$
Hence, $T_{n+1} = 4T_n,\ T_1 = 0.03125$

Recurrence Relation Example

A recurrence relation is defined as:
$T_{n+1} = aT_n + b$ for some value of a and b.
Find the recurrence relation of a sequence where the first three terms are 3, 4 and 7.

$$\boxed{3} \xrightarrow{\times a + b} \boxed{4} \xrightarrow{\times a + b} \boxed{7}$$

From diagram above, create two equations that links T_1 with T_2 and T_2 with T_3.
$T_2 = aT_1 + b \rightarrow 4 = 3a + b \ldots$ *Equation* 1
$T_3 = aT_2 + b \rightarrow 7 = 4a + b \ldots$ *Equation* 2

Using ClassPad, solve *Equation* 1 and *Equation* 2 to find a and b: $a = 3$ and $b = -5$
Substitute into $T_{n+1} = aT_n + b,\ T_1 = 3$
Hence $T_{n+1} = 3T_n - 5,\ T_1 = 3$

Bouncing Ball Formulae

Below are shortcut formulae for the geometric sequence that models a ball dropped from an initial height a bouncing at r% efficiency.

Ball height after n^{th} bounce:
$\boxed{Height = ar^n}$

Total vertical distance travelled (S_∞):
$\boxed{Distance = a\left(\frac{1+r}{1-r}\right)}$

Vertical distance travelled up to n^{th} bounce:
$\boxed{Distance = a\left(\frac{1+r-2r^n}{1-r}\right)}$

r: bounce common ratio (as a decimal)
a: drop height
n: number of bounces

Long Term Steady State Solution

Two methods to find steady state solution:
- Substitute T_{n+1} and T_n with T and solve for T.
- Using ClassPad Sequences App, find a term for a large value of n (e.g. T_{50}) and look for a consistency.

Find the long term steady state solution for the sequence $T_{n+1} = 0.8T_n + 24,\ T_1 = 196$
- Solving: $T = 0.8T + 24$ gives $T = 120$
- ClassPad Sequences App: $T_{30} = 120.0941$ and $T_{50} = 120.0011$ which approaches 120

Simple Interest Formulae

$\boxed{I = PRT}$ $\boxed{A = I + P}$

A: amount (principal plus interest)
P: principal (starting amount)
I: total amount of interest
R: interest rate (as a decimal)
T: time in years

Simple Interest Example

Noah purchased an iPhone worth $600 using his credit card that charges 19.8% p.a. simple interest on the 30th of March. He paid the account on the 11th of April.

What is that total interest that was charged?

$I = PRT = 600 \times 0.198 \times \frac{13}{365} = \$4.23$

What is the total amount Noah paid for the iPhone?

$A = I + P = 4.23 + 600 = \$604.23$

Effective Annual Rate

Effective annual rate of interest converts $i\%\ p.a.$ compounding n times per year to $i\%\ p.a.$ compounding annually.

$$\boxed{i_{effective} = \left(1 + \frac{i}{n}\right)^n - 1}$$

$i_{effective}$: effective annual rate of interest (as a decimal)
i: annual interest rate (as a decimal)
n: number of times per year that interest is compounded

Frequency of Compounding Interest

The more times interest compounds per year, the more interest is earned. The higher the value of n, the higher the effective annual rate of interest. There is diminishing returns on interest gained as n increases.

n	i	Converting to effective annual rate	$i_{effective}$
Yearly (1)	5%		5%
Half-Yearly (2)	5%		5.062%
Quarterly (4)	5%		5.095%
Monthly (12)	5%		5.116%
Fortnightly (26)	5%		5.122%
Weekly (52)	5%		5.125%
Daily (365)	5%		5.127%

Compound Interest Formulae

$\boxed{A = P\left(1 + \frac{r}{n}\right)^{nt}}$ $\boxed{I = A - P}$

A: amount (principal plus interest)
P: principal (starting amount)
I: total amount of interest
r: annual interest rate (as a decimal)
n: number of times interest is compounded per year
t: time in years

Compound Interest Recurrence Relation

$$\boxed{A_{n+1} = \left(1 + \frac{i}{n}\right)A_n + r,\ A_0 = P}$$

i: interest rate (as a decimal)
n: number of times interest is compounded per year
r: regular payments (for investments, r is positive and for loans and annuities, r is negative)
P: principal (initial amount)

Compound Interest Table Form

Investment: Lucas invests $1,000 into an account that pays 12% p.a. compounding monthly and makes monthly deposits of $200.

Month (n)	Amount @ Start (A_n)	Interest ($A_n \times \frac{i}{r}$)	Deposit ($+r$)	Amount @ End (A_{n+1})
1	$1,000	+ $10	+ $200	$1,210.00
2	$1,210.00	+ $12.10	+ $200	$1,422.10
3	$1,422.10	+ $14.22	+ $200	$1,636.32

Loan: Sophia borrows $25,000 at 4% p.a. compounding weekly and makes weekly payments of $3,000 to pay off the loan.

Week (n)	Amount @ Start (A_n)	Interest ($A_n \times \frac{i}{r}$)	Payment ($-r$)	Amount @ End (A_{n+1})
1	$25,000	+ $19.23	- $3,000	$22,019.23
2	$22,019.23	+ $16.94	- $3,000	$19,036.17
3	$19,036.17	+ $14.64	- $3,000	$16,050.81

Annuity: Charlotte invests $1,000 to buy an annuity that pays $200 per year at 7% p.a. compounding annually.

Year (n)	Amount @ Start (A_n)	Interest ($A_n \times \frac{i}{r}$)	Withdraw ($-r$)	Amount @ End (A_{n+1})
1	$1,000	+ $70	- $200	$870.00
2	$870.00	+ $60.90	- $200	$730.90
3	$730.90	+ $51.16	- $200	$582.06

Compound Interest Example

Oliver borrowed $50,000 and makes monthly repayments of $1,120 to pay off the loan. Interest is 12% p.a. compounding monthly.

Find the recurrence relation that shows amount owing.

$A_{n+1} = \left(1 + \frac{0.12}{12}\right)A_n - 1120,\ A_0 = 50000$

How much does Oliver still owe after two years?

$A_{24} = \$33{,}276.45$

How much interest is charged during this period?

To calculate total interest, use formula: $I = A - P$
Total paid off loan $= 50000 - 33276.45 = \$16{,}723.55$
Total repayments $= 1120 \times 24 = \$26{,}880$
Total interest $= 26880 - 16723.55 = \$10{,}156.45$

Compound Interest Increasing Payments Example

Isaac deposits $300,000 into an account that earns interest at 8% p.a. compounded annually, withdrawing $37,500 at the end of the first year and the withdrawal amount increasing by 3% each year.

Find the recurrence relation that shows amount owing.

$A_{n+1} = 1.08A_n - 37500(1.03)^n,\ A_0 = 300000$

What is the final withdrawal amount?

Account reaches 0 in the 11^{th} year and final withdrawal is equal to $1.08A_{10}$ which is $1.08 \times 36421.04 = \$39{,}334.72$

ClassPad Compound Interest Variables

N	Number of time periods
I%	Annual interest rate (as a whole number)
PV	Present value
PMT	Regular payment amount
FV	Future value
P/Y	Number of payments per year
C/Y	Number of times interest is compounded per year

Compound vs. Simple Interest

Simple interest has a linear pattern (meaning that interest is constant overtime).

Compound interest has an exponential pattern (meaning that interest increases overtime).

Loans

Borrowing a sum of money that needs to be paid back in full.

PV	Positive Value
PMT	Negative Value
FV	0

Investments

Investments are a deposit that grows over time due to interest, making regular contributions.

PV	Negative Value
PMT	Negative Value
FV	Positive Value

Annuities

Investment that pays all of it out over time through regular intervals.

PV	Negative Value
PMT	Positive Value
FV	0

Perpetuities

Investing enough money to be able to "live off interest" and have the initial investment never deplete.

$$Q = PE$$

Q: annual withdrawal amount
P: principal (initial investment)
E: effective annual rate of interest (as a decimal)

ClassPad Compound Interest Examples

Jackson borrows $20,000 at 12% p.a. compounding monthly. He pays $350 every month to pay off the loan. How much would he still owe after 5 years of making payments?

N	60
I%	12
PV	20000
PMT	-350
FV	-7749.55
P/Y	12
C/Y	12

Emily borrows $25,000 at a rate of 12% p.a. compounding half-yearly. Her loan needs to be repaid in 4 years. What are Emily's half-yearly repayments?

N	8
I%	12
PV	25000
PMT	-4025.90
FV	0
P/Y	4
C/Y	4

James borrows $50,000 and is to be fully repaid in monthly repayments of $485.60 for 12 years. If interest is compounded monthly, determine the annual rate of interest.

N	144
I%	5.91%
PV	50000
PMT	-485.60
FV	0
P/Y	12
C/Y	12

Lily invests $10,000 at 7% p.a. compounding half-yearly. Lily wants her account to reach $50,000 in 10 years. How much does she need to deposit every six months?

N	20
I%	7
PV	-10000
PMT	-1064.44
FV	50000
P/Y	2
C/Y	2

Lachlan invests $2,000 and adds $200 to his account every quarter. Interest rate is 3.2% p.a. compounding quarterly. Determine how much is in his account in 5 years.

N	20
I%	3.2
PV	-2000
PMT	-200
FV	6664.63
P/Y	4
C/Y	4

Grace invests $700,000 to buy an annuity that pays $50,000 at 5.4% p.a. compounding annually. How many years will Grace be able to withdraw money?

N	26.82
I%	5.4
PV	-700000
PMT	50000
FV	0
P/Y	1
C/Y	1