Q. $f(x) = x + x^2$ for $-\pi < x < \pi$. Find the Fourier expression of $f(x)$

Sol^n:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$\therefore\ a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{x} - (1+2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1+2\pi)\left(\frac{(-1)^n}{n^2}\right) - (1-2\pi)\frac{(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\cdot\frac{(-1)^n 4\pi}{n^2}$$

$$= \frac{4(-1)^n}{n^2}$$

Again $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} (x+x^2)\sin nx\,dx$

$$= \frac{1}{\pi}\left[(x+x^2)\left(-\frac{\cos nx}{n}\right)-(1+2x)\left(-\frac{\sin nx}{n^2}\right)+2\,\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(\pi+\pi^2)\frac{(-1)^n}{n}+2\cdot\frac{(-1)^n}{n^3}+(-\pi+\pi^2)\left(\frac{(-1)^n}{n}\right)-2\frac{(-1)^n}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-2\pi\,\frac{(-1)^n}{n}\right]$$

$$= -\frac{2(-1)^n}{n}$$

$$\therefore f(x) = \frac{\pi^2}{3}+\sum_{n=0}^{\infty}\left(\frac{4(-1)^n}{n^2}\cos nx-\frac{2(-1)^n}{n}\sin nx\right)$$

$$\therefore f(x) = \frac{\pi^2}{3}+\sum_{n=}^{\infty}\left(\frac{4(-1)^n}{n^2}\cos nx-\frac{2(-1)^n}{n}\sin nx\right)$$

(Ans:)

Q. Find the fourier series of $f(x) = \pi - x$ for $0<x<2\pi$

Solⁿ:
$$a_0 = \frac{1}{\pi}\int_0^{2\pi}(\pi-x)\,dx$$

$$= \frac{1}{\pi}\left[-\frac{x^2}{2}\right]_0^{2\pi}$$

$$\therefore a_0 = -\pi$$

Now, $a_n = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)\cos nx\,dx$

$$= \frac{1}{\pi}\left[(\pi-x)\frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 - \frac{1}{n^2} + \frac{1}{n^2}\right]$$

$\therefore a_n = 0$

Again, $b_n = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)\sin nx\,dx$

$$= \frac{1}{\pi}\left[(\pi-x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-(\pi-x)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-(\pi-2\pi)\frac{1}{n} - 0 + \pi\frac{1}{n} - 0\right]$$

$$= \frac{2}{n}$$

$$\therefore f(x) = -\frac{\pi}{2} + \sum_{n=0}^{\infty}\left(\frac{2}{n}\sin nx\right)$$

Ans:

Find the fourier series to represent, $f(x) = x - x^2$ from $x = -\pi$ to π and show that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \cdots$

Soln: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\,dx$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= -\frac{2\pi^2}{3}$$

Now, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\,dx$

$$= \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} - (1 - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\frac{-\sin nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} + (1 - 2x)\frac{\cos nx}{n^2} + 2\frac{\sin nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[0 + (1 - 2\pi)\frac{(-1)^n}{n^2} + 0 - 0 - (1 + 2\pi)\frac{(-1)^n}{n^2} - 0\right]$$

$$a_n = -4\pi\frac{(-1)^n}{n^2}$$

Again, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin nx\,dx$

$$= \frac{1}{\pi}\left[(x - x^2)\left(\frac{-\cos nx}{n}\right) - (1 - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x^2-x)\frac{\cos nx}{n} + (1-2x)\frac{\sin nx}{n^2} - 2\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi^2-\pi)\frac{(-1)^n}{n} + 0 - 2\frac{(-1)^n}{n^3} - (\pi^2+\pi)\frac{(-1)^n}{n} + 2\frac{(-1)^n}{n^3}\right]$$

$$= \frac{1}{\pi}\left[\pi^2\frac{(-1)^n}{n} - \pi\cdot\frac{(-1)^n}{n} - \pi^2\frac{(-1)^n}{n} - \pi\frac{(-1)^n}{n}\right]$$

$$= \frac{1}{\pi}\left[-2\pi\frac{(-1)^n}{n}\right]$$

$$= -2\frac{(-1)^n}{n}$$

$$\therefore f(x) = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left(-4\frac{(-1)^n}{n^2}\cos nx - 2\frac{(-1)^n}{n}\sin nx\right)$$

(Ans:)

Now, $f(0) = 0^2 - 0 = 0$

$$\Rightarrow 0 = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty} -4\frac{(-1)^n}{n^2}$$

$$\Rightarrow \frac{\pi^2}{12} = \sum_{n=0}^{\infty}\frac{-(-1)^n}{n^2}$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

(Showed)

Q. Find a Fourier series to represent $f(x) = x\sin x$, for $0 < x < 2\pi$

Soln: $a_0 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\, dx$

$= \frac{1}{\pi}\left[-x\cos x + \sin x\right]_0^{2\pi}$

$\therefore a_0 = -2$

Now,

$a_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\, dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(x+nx) + \sin(x-nx)\right]dx$

$= \frac{1}{2\pi}\int_0^{2\pi}\{x\sin(x+nx) + x\sin(x-nx)\}dx$

Now, $\int_0^{2\pi} x\cdot\sin(1+n)x\, dx$

$= \left[x\frac{-\cos(1+n)x}{1+n} - \frac{-\sin(n+1)}{(1+n)^2}\right]_0^{2\pi}$

$= -\frac{2\pi}{1+n}$

Similarly, $\int_0^{2\pi} x\sin(1-n)x\, dx = -\frac{2\pi}{1-n}$

$$\therefore a_n = \frac{1}{2\pi}\left(-\frac{2\pi}{1+n} - \frac{2\pi}{1-n}\right)$$

$$= \frac{1}{n-1} - \frac{1}{1+n}$$

Again,

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\left[\cos(nx - x) - \cos(x + nx)\right] dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \{x \cos(n-1)x - x\cos(1+n)x\} dx$$

Now, $\int_0^{2\pi} x \cos(n-1)x \, dx$

$$= \left[x \cdot \frac{\sin(n-1)x}{n-1} + \frac{\cos(n-1)x}{(n-1)^2}\right]_0^{2\pi}$$

$$= 0 + \frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{1}{(n-1)^2}$$

$$\therefore \quad = \frac{\cos(2(n-1)\pi) - 1}{(n-1)^2}$$

Ans Similarly,

$$\int_0^{2\pi} x \cos(n+1)x\,dx$$

$$= \frac{\cos(2(n+1)\pi)-1}{(n+1)^2}$$

$$\therefore b_n = \frac{1}{2\pi}\left\{\frac{\cos(2(n-1)\pi)-1}{(n-1)^2} - \frac{\cos(2(n+1)\pi)-1}{(n+1)^2}\right\}$$

$$\therefore f(x) = -1 + \sum_{n=1}^{\infty}\left[\left(\frac{1}{n-1} - \frac{1}{1+n}\right)\cos nx + \frac{1}{2\pi}\left\{\frac{\cos(2(n-1)\pi)-1}{(n-1)^2} - \frac{\cos(2(n+1)\pi)-1}{(n+1)^2}\right\}\sin nx\right]$$

(Ans)

[1] Find the fourier series for $f(x)$, if $f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$

Soln:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} -\pi\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx$$

$$= -\left[x\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= -\pi + \frac{\pi}{2}$$

$$\therefore a_0 = -\frac{\pi}{2}$$

Now, $a_n = \frac{1}{\pi}\int_{-\pi}^{0} -\pi\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$

$$= -\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= 0 + \frac{1}{\pi}\left[0 + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$\therefore a_n = \frac{1}{\pi n^2}\left\{(-1)^n - 1\right\}$$

Again $b_n = \int_{-\pi}^{0} -\pi\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx$

$$= -\left[-\frac{\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{0}^{\pi}$$

$$= \frac{1}{n} - \frac{(-1)^n}{n} + \frac{1}{\pi}\left[-\frac{\pi(-1)^n}{n} + 0 + 0 - 0\right]$$

$$= \frac{1}{n} - \frac{(-1)^n}{n} - \frac{(-1)^n}{n}$$

$$= \frac{1}{n} - \frac{2(-1)^n}{n}$$

$$\therefore b_n = \frac{1}{n}\left\{1 - 2(-1)^n\right\}$$

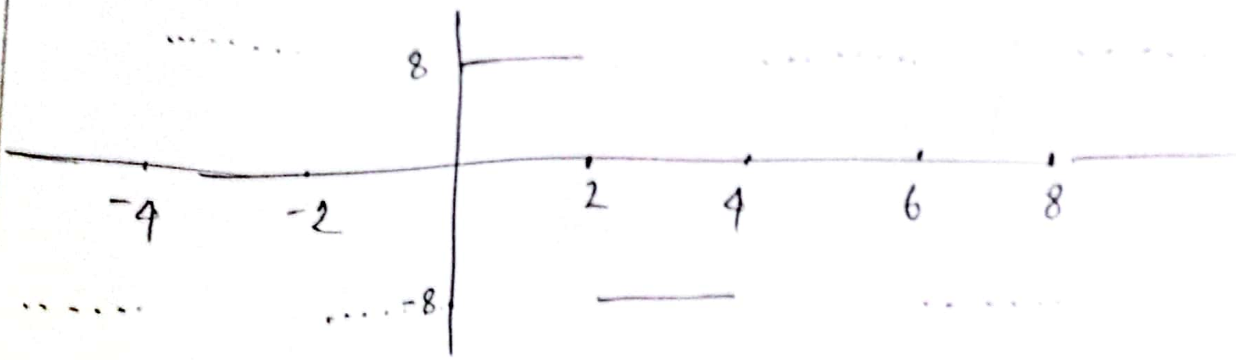
$$\therefore f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{\pi n^2}\left\{(-1)^n - 1\right\}\cos nx + \frac{1}{n}\left\{1 - 2(-1)^n\right\}\sin nx\right]$$

Ans:

2.34 (a)

Soln:

$$f(x) = \begin{cases} 8 & 0<x<2 \\ -8 & 2<x<4 \end{cases}$$



$$a_0 = \frac{1}{2}\int_0^2 8\,dx + \frac{1}{2}\int_2^4 -8\,dx$$

$$= 4[x]_0^2 - 4[x]_2^4$$

$$= 8 - 4\times 2$$

$$= 0$$

Now,

$$a_n = \frac{1}{2}\int_0^2 8\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_2^4 -8\cos\frac{n\pi x}{2}\,dx$$

$$= 4\left[\sin\frac{n\pi x}{2}\cdot\frac{2}{n\pi}\right]_0^2 - 4\left[\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]_2^4$$

$\therefore a_n = 0$

Again, $b_n = \frac{1}{2}\int_0^2 8\sin\frac{n\pi x}{2}\,\partial x + \frac{1}{2}\int_2^4 -8\sin\frac{n\pi x}{2}\,\partial x$

$= 4\left[-\frac{\cos\frac{n\pi x}{2}\cdot\frac{2}{n\pi}}{}\right]_0^2 - 4\left[-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_2^4$

$= 4\left[-\frac{2}{n\pi}(-1)^n + \frac{2}{n\pi}\right] - 4\left[-\frac{2}{n\pi} + \frac{2}{n\pi}(-1)^n\right]$

$= -\frac{16}{n\pi}(-1)^n + \frac{16}{n\pi}$

$\therefore b_n = \frac{16}{n\pi}\left\{1-(-1)^n\right\}$

$\therefore f(x) = \sum_{n=1}^{\infty}\frac{16}{n\pi}\left\{1-(-1)^n\right\}\sin\frac{n\pi x}{2}$

Ans:

2.34 (c) $f(x) = 4x \quad , \quad 0 < x < 10 \quad$ period 10

Sol^n:

$$a_0 = \frac{1}{5}\int_0^{10} 4x\,dx$$

$$= \frac{4}{5}\left[\frac{x^2}{2}\right]_0^{10}$$

$$= 40$$

Now,

$$a_n = \frac{1}{5}\int_0^{10} 4x \cos\frac{n\pi x}{5}\,dx$$

$$= \frac{4}{5}\left[x \sin\frac{n\pi x}{5}\cdot\frac{5}{n\pi} + \frac{25}{n^2\pi^2}\cos\frac{n\pi x}{5}\right]_0^{10}$$

$$= \frac{4}{5}\left[0 + \frac{25}{n^2\pi^2} - \frac{25}{n^2\pi^2}\right]$$

$$= 0$$

Again,

$$b_n = \frac{1}{5}\int_0^{10} 4x \sin\frac{n\pi x}{5}\,dx$$

$$= \frac{4}{5}\left[-\frac{5x}{n\pi}\cos\frac{n\pi x}{5} + \frac{25}{n^2\pi^2}\sin\frac{n\pi x}{5}\right]_0^{10}$$

$$= \frac{4}{5}\left[-10\frac{5}{n\pi} + 0 + 0 - 0\right]$$

$$= -\frac{40}{n\pi}$$

$$\therefore f(x) = 20 + \sum_{n=1}^{\infty} -\frac{40}{n\pi}\sin\frac{n\pi x}{5}$$

(Ans:)

Expand $f(x) = \begin{cases} 2-x & 0<x<4 \\ x-6 & 4<x<8 \end{cases}$ in a Fourier series of period 8.

Sol^n:

$$a_0 = \frac{1}{4}\int_0^4 (2-x)\,dx + \frac{1}{4}\int_4^8 (x-6)\,dx$$

$$= \frac{1}{4}\left[2x - \frac{x^2}{2}\right]_0^4 + \frac{1}{4}\left[\frac{x^2}{2} - 6x\right]_4^8$$

$$= \frac{1}{4}\left[8 - \frac{16}{2}\right] + \frac{1}{4}\left[\frac{64}{2} - 48 - \frac{16}{2} + 24\right]$$

$$= 0$$

Now, $a_n = \frac{1}{4}\int_0^4 (2-x)\cos\frac{n\pi x}{4}\,dx + \frac{1}{4}\int_4^8 (x-6)\cos\frac{n\pi x}{4}\,dx$

$$= \frac{1}{4}\left[\frac{4}{n\pi}(2-x)\sin\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}(-1)\cos\frac{n\pi x}{4}\right]_0^4$$

$$+ \frac{1}{4}\left[\frac{4}{n\pi}(x-6)\sin\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4}\right]_4^8$$

$$= \frac{1}{4}\left[0 - \frac{16}{n^2\pi^2}(-1)^n + \frac{16}{n^2\pi^2}\right] + \frac{1}{4}\left[\frac{16}{n^2\pi^2} - \frac{16}{n^2\pi^2}(-1)^n\right]$$

$$= -\frac{16}{n^2\pi^2}(-1)^n + \frac{16}{n^2\pi^2}$$

$$\therefore a_n = \frac{16}{n^2\pi^2}\{1 - (-1)^n\}$$

$$b_n = \frac{1}{4}\int_0^4 (2-x)\sin\frac{n\pi x}{4}dx + \frac{1}{4}\int_4^8 (x-6)\sin\frac{n\pi x}{4}dx$$

$$= \frac{1}{4}\left[-\frac{4}{n\pi}(2-x)\cos\frac{n\pi x}{4} + (-1)\sin\frac{n\pi x}{4}\cdot\frac{16}{n^2\pi^2}\right]_0^4 +$$

$$\frac{1}{4}\left[-\frac{4}{n\pi}(x-6)\cos\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\sin\frac{n\pi x}{4}\right]_4^8$$

$$= \frac{1}{4}\left[2\cdot\frac{4}{n\pi}(-1)^n - 0 + \frac{4}{n\pi}\right] + \frac{1}{4}\left[-2\cdot\frac{4}{n\pi} + (-2)\frac{4}{n\pi}(-1)^n\right]$$

$$= \frac{1}{4}\left[\frac{8}{n\pi}(-1)^n + \frac{4}{n\pi}\right] + \frac{1}{4}\left[-\frac{8}{n\pi} - \frac{8}{n\pi}(-1)^n\right]$$

$$= \frac{2}{n\pi}(-1)^n + \frac{1}{n\pi} - \frac{2}{n\pi} - \frac{2}{n\pi}(-1)^n$$

$$= \frac{1}{n\pi} - \frac{2}{n\pi}$$

$$\therefore b_n = -\frac{1}{n\pi}$$

$$\therefore f(x) = \sum_{n=1}^{\infty}\left[\frac{16}{n^2\pi^2}\{1-(-1)^n\}\cos\frac{n\pi x}{4} - \frac{1}{n\pi}\sin\frac{n\pi x}{4}\right]$$

Ans:

CamScanner

Q. Obtain a fourier expression for $f(x) = x^3$ for $-\pi < x < \pi$

Soln: $f(x) = x^3$

let $x = 1$ then $f(1) = 1$

$x = -1$ then $f(-1) = -1$

$\therefore f(x) = x^3$ is an odd function

$\therefore a_0 = a_n = 0$

$$\therefore b_n = \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx\, dx$$

$$= \frac{2}{\pi}\left[-x^3 \frac{\cos nx}{n} + 3x^2 \frac{\sin nx}{n^2} - 6x \frac{\cos nx}{n^3} + 6\frac{\sin nx}{n^4}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi^3}{n}(-1)^n - \frac{6\pi}{n^3}(-1)^n\right]$$

$$= -\frac{2}{n}(-1)^n\left(\pi^2 + \frac{6}{n}\right)$$

$$\therefore f(x) = \sum_{n=1}^{\infty}\left\{-\frac{2}{n}(-1)^n\left(\pi^2 + \frac{6}{n}\right)\sin nx\right\}$$

(Ans:)