Math

Symmetric Matrix:

A square matrix $A = [a_{ij}]$ is called symmetric provided $a_{ij} = a_{ji}$, for all values of i and j.

Example:

$$A = \begin{bmatrix} 1 & -3 & 5 \\ -3 & 2 & 7 \\ 5 & 7 & 3 \end{bmatrix} \quad \therefore A^T = \begin{bmatrix} 1 & -3 & 5 \\ -3 & 2 & 7 \\ 5 & 7 & 3 \end{bmatrix}$$

$\therefore$ A is symmetric (if $A^T = A$)

Skew-Symmetric Matrix:

A square matrix $A = [a_{ij}]$ is called skew-symmetric provided $a_{ij} = -a_{ji}$, for all values of i and j.

Example:

$$A = \begin{bmatrix} 0 & 5 & 2 \\ -5 & 0 & 7 \\ -2 & -7 & 0 \end{bmatrix} \quad \therefore A^T = \begin{bmatrix} 0 & -5 & -2 \\ 5 & 0 & -7 \\ 2 & 7 & 0 \end{bmatrix}$$

$\therefore$ Since $A^T = -A$, Thus The matrix A is skew-symmetric.

** **Theorem:**

Every square matrix can be uniquely expressed as the sum of a symmetric and a skew symmetric matrices.

Proof:

Let A be a square matrix. Then we can write

$$A = \frac{1}{2}A + \frac{1}{2}A$$

$$= \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right) \quad \text{--- (1)}$$

Since $\frac{1}{2}A$, $\frac{1}{2}A^T$ and conformable for addition,

A being a square matrix

Now,

$$\left\{\frac{1}{2}\left(A + A^T\right)\right\}^T = \frac{1}{2}\left(A + A^T\right)$$

$$= \frac{1}{2}\left\{A^T + \left(A^T\right)^T\right\}$$

$$= \frac{1}{2}\left(A^T + A\right)$$

$$= \frac{1}{2}\left(A + A^T\right)$$

Therefore, $\left\{\frac{1}{2}\left(A + A^T\right)\right\}^T = \frac{1}{2}\left(A + A^T\right)$ ... (2)

Therefore, by definition $\frac{1}{2}(A + A^T)$ is a symmetric matrix.

Again, $\left\{\frac{1}{2}(A - A^T)\right\}^T = \frac{1}{2}(A - A^T)^T$

$$= \frac{1}{2}\left\{A + (-1)A^T\right\}$$

$$= \frac{1}{2}\left\{A^T + (-1)(A^T)^T\right\}$$

$$= \frac{1}{2}\left\{A^T + (-1)A\right\}$$

$$= \frac{1}{2}\left\{A^T - A\right\}$$

$$= -\frac{1}{2}(A - A^T)$$

Therefore, $\frac{1}{2}(A - A^T)^T = -\frac{1}{2}(A - A^T)$ ..... (3)

Therefore, by definition $\frac{1}{2}(A - A^T)$ is a skew symmetric.

Hence from equation (1), (2) and (3), we find that The matrix A can be expressed as the sum of a symmetric and a skew-symmetric matrices.

To prove that the representation ① is unique

Let $A = A_1 + A_2$ ... ... ④

where A_1 is symmetric and A_2 is skew-symmetric.

Then $A_1 = A_1^T$ ... ... ⑤

$A_2 = -A_2$ ... ... ⑥

From eqn ④, we have,

$$A^T = (A_1 + A_2)^T$$

$$= A_1^T + A_2^T$$

$$= A_1 - A_2 \quad \text{[By eqns ⑤ and ⑥]}$$

$$\therefore A = A_1 - A_2 \quad \text{... ⑦}$$

Adding and subtracting eqns ④ and ⑦, we get,

$$A + A^T = 2A_1$$

$$\Rightarrow A_1 = \frac{1}{2}(A + A^T)$$

$$A - A^T = 2A_2$$

$$\Rightarrow A_2 = \frac{1}{2}(A - A^T)$$

From equation ④, we get,

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T),$$ which is the same as equation ①

Hence, the representation is unique.

** Problem:

Find the symmetric and skew-symmetric part of the matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix}$

Sol^n: Given that,

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix}$$

$$(A + A^T) = \begin{bmatrix} 2 & 8 & 7 \\ 8 & 16 & 6 \\ 7 & 6 & 14 \end{bmatrix}$$

$$\therefore \frac{1}{2}(A+A^T) = \begin{bmatrix} 1 & 4 & 7/2 \\ 4 & 8 & 3 \\ 7/2 & 3 & 7 \end{bmatrix}$$

$$\therefore (A-A^T) = \begin{bmatrix} 0 & -4 & -1 \\ 4 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix}$$

Theorem :

If A and B are both skew-symmetric matrices of same order such that $AB = BA$, then show that AB is symmetric.

Sol^n :

Given that,

A and B are both skew-symmetric matrices

Then,

$$A^T = -A \Rightarrow A = -A^T$$

$$B^T = -B \Rightarrow B = -B^T$$

$$\therefore AB = BA$$

$$= (-B^T)(-A^T)$$

$= B^T A^T$

$= (AB)^T$

$\therefore (AB)^T = AB$

Thus the matrix AB is symmetric.

$(AB)^T = A^T B^T$ ✗

$(AB)^T = B^T A^T$ ✓

□ **Orthogonal matrix:**

A square matrix A is said to be an orthogonal matrix if $AA^T = A^T A = I$, where I is the Identity matrix of the same order of A.

Example: Every identity matrix is orthogonal.

Problem:

Prove that the following matrices are orthogonal.

① $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

② $A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

① Given that,

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore AA^T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta+\sin^2\theta & \cos\theta\sin\theta-\cos\theta\sin\theta & 0 \\ \sin\theta\cos\theta-\sin\theta\cos\theta & \sin^2\theta+\cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem :

Prove that the following matrices are orthogonal.

① $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

② $A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

$$\therefore A^T A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $AA^T = A^T A = I$, Thus the matrix A is orthogonal.

Complex conjugate of a matrix:

The matrix obtained from any given matrix A of order $m \times n$ with complex elements a_{ij} by replacing its elements by the corresponding conjugate numbers is called the complex conjugate of A denoted by $\bar{A}$.

Example:

$$A = \begin{bmatrix} 1+i & 2+3i \\ 2 & 2i \end{bmatrix}$$

$$\therefore \bar{A} = \begin{bmatrix} 1-i & 2-3i \\ 2 & -2i \end{bmatrix}$$ complex conjugate of A.

Transpose conjugate of a matrix:

The Transpose of conjugate of a matrix A that is $(\bar{A})^t$ is defined as transposed conjugate or tranjugate A and is denoted by $(\bar{A})^t$.

$$A = \begin{bmatrix} 1+i & 2+3i \\ 2 & 2i \end{bmatrix}$$

$\therefore$ Conjugate of A, $\bar{A} = \begin{bmatrix} 1-i & 2-3i \\ 2 & -2i \end{bmatrix}$

$$\therefore (\bar{A})^t = \begin{bmatrix} 1-i & 2 \\ 2-3i & -2i \end{bmatrix}$$

* **Theorem:** Show that, $(\bar{A})^t = A^t$

*** **Hermitian matrix:**

A square matrix A such that $(\bar{A})^t = A$ is called Hermitian that is the matrix $[a_{ij}]$ is Hermitian provided that $a_{ij} = \bar{a}_{ji}$ for all values of i and j.

Example:

Is $A = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3-i \\ -2-5i & 3+i & 5 \end{bmatrix}$ a Hermitian matrix?

Sol^n: Given that,

Conjugate of A is, $\bar{A} = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3+i \\ -2+5i & 3-i & 5 \end{bmatrix}$

$\therefore (\bar{A})^t = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3-i \\ -2-5i & 3+i & 5 \end{bmatrix} = A$

$$\therefore (\bar{A})^t = \begin{bmatrix} 3 & 7+4i & -2+5i \\ 7+4i & -2 & 3-i \\ -2-5i & 3+i & 5 \end{bmatrix}$$

Since $(\bar{A})^t = A$, Thus the given matrix A is Hermitian.

Skew-Hermitian Matrix:

A square matrix A such that $(\bar{A})^t = -A$ is called skew-Hermitian that is the matrix $[a_{ij}]$ is skew-Hermitian provided $a_{ij} = -\bar{a}_{ji}$ for all values of i and j.

Example:

Show that the matrix $A = \begin{bmatrix} 0 & 2-3i & -2-i \\ -2-3i & 0 & 3+4i \\ 2-i & 3+4i & 0 \end{bmatrix}$ is skew-Hermitian.

Theorem:

The diagonal elements of a Hermitian matrix are necessarily real.

$$a_{ij} = \bar{a}_{ji} \quad \text{——} \quad (1)$$

Singular and Non-singular matrices:

Let D be the determinant of the square matrix A, then if $D=0$, the matrix A is called singular matrix and if $D \neq 0$, the matrix A is called the non-singular matrix.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

$$D = |A| = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix}$$

$$= 6-6 = 0$$

Inverse matrix:

A square matrix A is invertible if there exists a unique matrix B such that $AB = BA = I$, where I is the unit matrix. We call such a matrix B the inverse of A and is generally denoted by A^{-1}.

Example:

Find the inverse of the matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$

Also show that $AA^{-1} = A^{-1}A = I$, where I is the unit matrix.

Given that:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$$

Let D be the determinant of the matrix A.

$$\therefore D = \begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix}$$

$$= 14 - 12$$

$$= 2 \neq 0$$

Since $D = 2 \neq 0$, Thus the matrix A is non-singular and A^{-1} exists.

Now the cofactors of D are $A_{11} = (-1)^2 \cdot 7$

$$= 7$$

$$A_{12} = (-1)^3 \cdot 4$$

$$= -4$$

$$A_{21} = (-1)^{2+1} \cdot 3$$

$$= -3$$

$$A_{22} = (-1)^{2+2} \cdot 2$$

$$= 2$$

$\therefore$ The cofactor matrix of $A = \begin{bmatrix} 7 & -4 \\ -3 & 2 \end{bmatrix}$

$\therefore \operatorname{adj} A = \begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix}$

Given that:

$$A^{-1} = \frac{1}{|D|} \cdot \operatorname{adj} A$$

$$= \frac{1}{2} \begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix}$$

Let D be the determinant of the matrix A.

$\therefore D = \begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix}$

$= 14 - 12$

$= 2 \neq 0$

Since $D = 2 \neq 0$, Thus the matrix A is non-singular and A^{-1} exists.

Now the cofactors of A are

$A_{11} = (-1)^{1+1} \cdot 7 = 7$

$A_{12} = (-1)^{1+2} \cdot 4 = -4$

$A_{21} = (-1)^{2+1} \cdot 3 = -3$

$A_{22} = (-1)^{2+2} \cdot 2 = 2$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix}$$

Also show that $AA^{-1} = A^{-1}A = I$ where I is the unit matrix.

$$AA^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 14-12 & -6+6 \\ 28-28 & -12+14 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^{-1}A = \frac{1}{2} \begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 14-12 & 21-21 \\ -8+8 & -12+14 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore AA^{-1} = A^{-1}A = I$

H.W

Problem :

1. Find the inverse of $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$

2. Show that $(A^{-1})^t = (A^t)^{-1}$ for the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Sol^n:

1. Let $D = \begin{vmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \end{vmatrix}$

$$= 2(-4+1) + 1(2-1) - 1(-1+2)$$

$$= -6 \neq 0$$

Since, $D = -6 \neq 0$

Thus, the matrix is non-singular and A^{-1} exists.

Now cofactors of D are,

$$A_{11} = \begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix} = -4+1 = -3$$

$$A_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -(2-1) = -1$$

$$A_{13} = \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} = 1$$

$$A_{21} = -\begin{vmatrix} -1 & -1 \\ -1 & 2 \end{vmatrix} = -(-2-1) = 3$$

$$A_{22} = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 5$$

$$A_{23} = -\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = 1$$

$$A_{31} = \begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} = -3$$

$$A_{32} = -\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = -3$$

$$A_{33} = \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} = -3$$

The cofactor matrix of A,

$$A = \begin{bmatrix} -3 & -1 & 1 \\ 3 & 5 & 1 \\ -3 & -3 & -3 \end{bmatrix}$$

$$\text{adj. } A = \begin{bmatrix} -3 & 3 & -3 \\ -1 & 5 & -3 \\ 1 & 1 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \cdot \text{adj } A$$

$$= \frac{1}{-6} \begin{bmatrix} -3 & 3 & -3 \\ -1 & 5 & -3 \\ 1 & 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & -\frac{5}{6} & \frac{1}{2} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{2} \end{bmatrix}$$

Ans

2. Given that,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Let D be the determinant of the matrix A.

$$\therefore D = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$= 4 - 6$$

$$= -2 \neq 0$$

Since $D = -2 \neq 0$, Thus the matrix A is non-singular and A^{-1} exists.

Now the cofactors of D are

$$A_{11} = (-1)^2 \cdot 4 = 4$$

$$A_{12} = (-1)^3 \cdot 3 = -3$$

$$A_{21} = (-1)^3 \cdot 4 = -2$$

$$A_{22} = (-1)^4 \cdot 1 = 1$$

$\therefore$ The cofactor matrix of A $= \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

$$\therefore \text{adj. } A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|D|} \cdot \text{adj } A$$

$$= -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

$$\therefore (A^{-1})^t = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

(A) Let D be the determinant of the matrix A^t.

Now the cofactors of D are

$$A_{11} = (-1)^2 \cdot 4 = 4$$

$$A_{12} = (-1)^3 \cdot 2 = -2$$

$$A_{21} = (-1)^3 \cdot 3 = -3$$

$$A_{22} = (-1)^4 \cdot 1 = 1$$

$\therefore$ The cofactor matrix of A^t is

$$\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\therefore \text{adj. } A = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|D|} \cdot \text{adj. } A$$

$$= -\frac{1}{2} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}$$

2. Given that

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Let D be the determinant of the matrix A.

$$\therefore D = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$= 4 - 6$$

$$= -2 \neq 0.$$

Thus the matrix A is non-singular and A^{-1} exists.

The cofactors matrix of A^t =

$$\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\text{adj}\cdot A^t = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

$$\therefore (A^t)^{-1} = \frac{1}{D}\cdot \text{adj}\cdot A^t$$

$$= \frac{1}{-2}\cdot\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}$$

$$(A^{-1})^t = (A^t)^{-1}$$

Problem solution No-02 (3rd class)

Given that,

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$A^T = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$AA^T = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4} & \frac{1}{4}-\frac{1}{4}+\frac{1}{4}-\frac{1}{4} & \frac{1}{4}-\frac{1}{4}-\frac{1}{4}+\frac{1}{4} & \frac{1}{4}-\frac{1}{4}-\frac{1}{4}+\frac{1}{4} \\ \frac{1}{4}-\frac{1}{4}+\frac{1}{4}-\frac{1}{4} & \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4} & \frac{1}{4}-\frac{1}{4}-\frac{1}{4}+\frac{1}{4} & \frac{1}{4}-\frac{1}{4}+\frac{1}{4}-\frac{1}{4} \\ \frac{1}{4}+\frac{1}{4}-\frac{1}{4}-\frac{1}{4} & \frac{1}{4}-\frac{1}{4}-\frac{1}{4}+\frac{1}{4} & \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4} & \frac{1}{4}-\frac{1}{4}+\frac{1}{4}-\frac{1}{4} \\ \frac{1}{4}-\frac{1}{4}-\frac{1}{4}+\frac{1}{4} & \frac{1}{4}-\frac{1}{4}+\frac{1}{4}-\frac{1}{4} & \frac{1}{4}-\frac{1}{4}+\frac{1}{4}-\frac{1}{4} & \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

$$A^T A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Since $AA^T = A^T A = I$, Thus the matrix A is orthogonal.

□ Problem :

(i) Show that $(AB)^{-1} = B^{-1}A^{-1}$, where $A = \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix}$

and $B = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$

(ii) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ by adjoint matrix.

(iii) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ by using Linear system of equations.

* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad A^{-1} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Given that,

$A = \begin{bmatrix} \quad \end{bmatrix}$

$AX = B$, where $A = \begin{bmatrix} \quad \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} a \\ b \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x+2y \\ 3x+4y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \therefore x+2y = a, \quad \therefore 3x+4y = b$$

$$x = +2a + b$$
$$y = -3a/2 - b/2$$

$$AX = B$$
$$\Rightarrow X = A^{-1}B$$
$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

(iii) Solⁿ:

Given that,

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$$

Let D be the determinant of the matrix A.

$$\therefore D = |A| = \begin{vmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{vmatrix}$$

$$= 1(-8-3) + 2(2+4)$$

Since $D \neq 0$, Thus the matrix A is non-singular and hence A^{-1} exists.

Now, we consider the Linear system

$$Ax = B,$$

where $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x+2z \\ 2x-y+3z \\ 4x+y+8z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Equating the correspoing elements, we obtain,

$$x + 2z = a$$

$$2x - y + 3z = b$$

$$4x + y + 8z = c$$

Solving the above eq^{n}s, we get,

$$x = -11a + 2b + 2c \quad \ldots\ldots (1)$$

$$y = -4a + 0.b + c \quad \ldots\ldots (2)$$

$$z = 6a - b - c \quad \ldots\ldots (3)$$

Putting the eq^{n}s (1), (2) and (3) in matrix form as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \ldots\ldots (4)$$

Again,

$$AX = B$$

$$\Rightarrow X = A^{-1}B \quad \ldots\ldots (5)$$

Comparing eq^{n}s (4) and (5) we get,

$$A^{-1} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}. \quad \text{(Ans)}$$

□ $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ by using canonical form find inverse (A^{-1}).

or, find inverse by Augmented matrix.

1st class

Matrices

Matrix: A system of any mn numbers arranged in a rectangular array of m rows and n columns is called a matrix of order $m \times n$ or an $m \times n$ matrix.

For example:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & \cdots & a_{mn} \end{bmatrix}$$

is a matrix of order $m \times n$

Column matrix, row matrix, null matrix, square, unit, diagonal, rectangular matrix.

Transpose of a matrix:

Diagonal matrix : A square matrix which is both upper and lower triangular is called a Diagonal matrix.

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a diagonal matrix of order 3×3

Scaler matrix: If in a square matrix A all the diagonal elements are equal to a (where $a \neq 0$) and all the remaining element are equal to 0 then it is called are scaler matrix.

Ex: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is a scaler matrix of order 3×3

Commutative matrix : If A and B are to square matrices such that $AB = BA$ then A and B are called commutative matrices.

If $AB = -BA$, The matrices A and B are said to be anti-commute.

Transposed matrix:

The matrix of order $n \times m$ obtained by inter changing the rows and columns of a matrix of order $m \times n$ is called the Transposed matrix of A and is denoted by A' or A^t or A^T.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\therefore A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Idempotent matrix:

A square matrix A is called Idempotent provided is satisfies the relation $A^2 = A$

Ex: If a unit matrix is Idempotent

(57 Page) 1, 2, 3, 4, 7 Ex:

Since $A^2 = A$ Thus the Given matrix A is Idempotent.

Exam এ আসবে

If A and B are Idempotent, then show that A+B will be idempotent if and only if (iff): $AB = BA = O$.

Example-01:

Show that the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent

Solve:

$$A^2 = A \cdot A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \cdot \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2-4 & -4-6+8 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= A$$

Hence the matrix A is idempotent.

Example-2:

If A and B are idempotent matrices, then show that AB is idempotent if A and B commute

Solve

If A is idempotent, then $A^2 = A$ and if B is idempotent, then $B^2 = B$

If A and B commute, then $AB = BA$

$$(AB)^2 = (AB)(AB)$$

$$= A(BA)B \quad [\text{by associative law}]$$

$$= A(AB)B \quad [A \text{ and } B \text{ commute}]$$

$$= (AA)(BB) \quad [\text{by associative law}]$$

$$= A^2B^2$$

$$= AB \quad [A \text{ and } B \text{ are idempotent}]$$

Example-3:

If A is an idempotent matrix, then the matrix $B = I - A$ is idempotent and $AB = 0 = BA$

Solve:

If A is an idempotent, then $A^2 = A$

$B = I - A$

$\Rightarrow B^2 = (I-A)^2$

$= (I-A)(I-A)$

$= I^2 - IA - AI + A^2$

$= I - A - A + A^2$

$= I - 2A + A$

$= I - A$

$= B$

Hence the matrix B is idempotent.

$AB = A(I-A)$

$= AI - A \cdot A$

$= A - A^2$

$= A - A$

$= 0$

also, $BA = (I-A)A$

$= IA - A \cdot A$

$= A - A^2$

$= A - A$

$= 0$

$\therefore AB = BA = 0$.

$\therefore$ B is an idempotent.

Example-5

If A and B are idempotent matrices, then $A+B$ will be idempotent if and only if $AB = BA = O$.

Solve:

If A is idempotent, then $A^2 = A$ and if B is idempotent, then $B^2 = B$

given that,

$$AB = BA = O$$

$$(A+B)^2 = (A+B)(A+B)$$

$$= A \cdot A + AB + BA + B \cdot B$$

$$= A^2 + AB + AB + B^2$$

$$= A^2 + O + O + B^2$$

$$= A + B$$

Hence, $A+B$ is an idempotent matrix.

Again, if $A+B$ is an idempotent matrix, then

$$(A+B)^2 = A+B$$

$$\Rightarrow (A+B)(A+B) = A+B$$

$$\Rightarrow A^2 + AB + BA + B^2 = A+B$$

$$\Rightarrow A + AB + BA + B = A+B$$

$$\Rightarrow AB + BA = O$$

$$\Rightarrow AB = -BA$$

P.T.O

Also,

$AB = A^2B$

$= AABB$

$= A(AB)$

$= A(-BA)$

$= -ABA$

$= -(AB)A$

$= -(-BA)A$

$= BAA$

$= BA^2$

$= BA$

$\therefore AB + AB = 0$

$\Rightarrow 2AB = 0$

$\Rightarrow AB = 0$

$\therefore BA = 0$

$\therefore AB = BA = 0$

Hence $A+B$ is idempotent if and only if $AB = BA = 0$.

6th class

Solve the following linear system of equations with the help of matrices:-

① $2x+y=1$
$x-2y=3$

② $x+2y+3z=14$
$3x+y+2z=11$
$2x+3y+z=11$

③ $x+y+z=6$
$x-y+z=2$
$2x+y-z=1$

Given linear system eqns is:

$$x+y+z=6$$
$$x-y+z=2$$
$$2x+y-z=1$$

The linear system of linear eqns can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \quad \text{①}$$

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$

$\therefore$ Eq^n ② $\Rightarrow Ax = B$

$\Rightarrow (A^{-1}A)x = A^{-1}B$

$\Rightarrow Ix = A^{-1}B$ [where I is the unit matrix]

$\Rightarrow X = A^{-1}B$ ... ②

Let D be the determinant of the matrix A.

$$\therefore D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= 1(1-1) - 1(-1-2) + 1(1+2)$$

$$= -1(-3) + 3$$

$$= 3+3$$

$$= 6 \neq 0$$

Since $D = 6 \neq 0$, $\therefore$ Thus the matrix A is non-singular and hence A^{-1} exists.

Now, the cofactors of D are,

$A_{11} = (-1)^2(1-1) = 0$ $\quad A_{22} = -3$

$A_{12} = (-1)^3 \cdot (-3) = 3$ $\quad A_{23} = 1$

$A_{13} = (-1)^4 \cdot (1+2) = 3$ $\quad A_{31} = 2$

$A_{21} = (-1)^3 \cdot (-1-1) = 2$ $\quad A_{32} = 0$

$A_{33} = -2$

$\therefore$ The cofactor matrix of $A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 1 \\ 2 & 0 & -2 \end{bmatrix}$

$$\text{adj} \cdot A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|D|} \cdot \text{adj} \cdot A$$

$$= \frac{1}{6} \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

From eq^n ② we get,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 0 \\ 18 \\ 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore x = 1, \; y = 2, \; z = 3$

characteristic eqn of matrix (1 set qstion 2b)

characteristic matrix :

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$$

Let $A = [a_{ij}]$ be an $n \times n$ matrix

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{bmatrix}$$

characteristic polynomial

$\therefore$ The characteristic polynomial of the matrix A is, $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix}$

$$= 2 - \lambda - 2\lambda + \lambda^2 - 6$$

$= \lambda^2 - 3\lambda - 4$

characteristic equation

$|A - \lambda I| = 0$

$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$

$\Rightarrow \lambda^2 - 4\lambda + \lambda - 4 = 0$

$\Rightarrow \lambda(\lambda - 4) + 1(\lambda - 4) = 0$

$\Rightarrow (\lambda - 4)(\lambda + 1) = 0$

$\therefore \lambda = -1, 4$ (Eigen values)

□ Find the Eigen values of the matrix:

$$A = \begin{bmatrix} -3 & 2 & 2 \\ -6 & 5 & 2 \\ -7 & 4 & 3 \end{bmatrix}$$

Given matrix

$A = [\ ]$

The characteristic matrix of A

$\therefore$ The characteristic eq^n of the given matrix A

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -3-\lambda & 2 & 2 \\ -6 & 5-\lambda & 2 \\ -7 & 4 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-3-\lambda)\{(5-\lambda)(4-\lambda)-8\} - 2\{-6(4-\lambda)+14\} + 2\{(-24+7(5-\lambda)\}$$

$$\Rightarrow (-3-\lambda)(20-5\lambda-4\lambda+\lambda^2-8) - 2(-24+6\lambda+14) + 2(-24+35-7\lambda)$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\Rightarrow \lambda^3 - \lambda^2 - 5\lambda^2 + 5\lambda + 6\lambda - 6 = 0$$

$$\Rightarrow \lambda^2(\lambda-1) - 5\lambda(\lambda-1) + 6(\lambda-1) = 0$$

$$\Rightarrow (\lambda^2 - 5\lambda + 6)(\lambda - 1) = 0$$

$$\Rightarrow (\lambda^2 - 3\lambda - 2\lambda + 6)(\lambda - 1) = 0$$

$$\Rightarrow \{\lambda(\lambda-3) - 2(\lambda-3)\}(\lambda-1) = 0$$

$$\Rightarrow (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$$\therefore \lambda = 1, 2, 3$$

State and prove Cayley-Hamilton Theorem :

Statement : Every matrix is zero of its characteristic polynomial equation or, every matrix satisfies its characteristic equation.

V.V.I □ Verify Cayley-Hamilton theorem for the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

Soln : Given that,

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

The characteristic matrix of A is $A - \lambda I = \begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix}$

$\therefore$ The characteristic equation of the matrix A is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 3 - 4\lambda + \lambda^2 - 8 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda - 5 = 0 \quad \ldots\ldots \text{①}$$

$$\Rightarrow \lambda(\lambda - 5) + 1(\lambda - 5) = 0$$

$$\Rightarrow (\lambda - 5)(\lambda + 1) = 0$$

$\therefore \lambda = 5, -1$.

$\therefore$ Eigen values $-1, 5$.

According to the Cayley-Hamilton theorem,

we have to show that $A^2 - 4A - 5I = 0$; where 0 is the null matrix.

Also, $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$$\therefore A^2 = A \cdot A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+8 & 4+12 \\ 2+6 & 8+9 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

$$\therefore A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Thus every square matrix satisfies its characteristic equation.

extra :

Find the inverse by using cayley - Hamilton theorem

Verify Caley Hamilton Theorem for the matrix

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Also Find A^{-2}.

25-09-23

Eigenvalues and Eigenvectors: (Definition exm ला आहे)

If A is an $n \times n$ matrix, Then a non-zero vectors v in R^n is called an eigenvectors of A is AV is a scaler multiple of v, That is $A\underline{V} = \lambda \underline{V}$ for some scaler λ. The scaler λ is called an eigenvalues of A and V is said to be an eigenvector of A corresponding to λ.

$\boxed{A\underline{V} = \lambda \underline{V}}$.

$\underline{V}$ = vector
A = matrix
λ = Eigenvalues

□ (i) Find the Eigenvalues and associated eigenvectors of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ in field $\mathbb{R}$.

(ii) Find the Eigenvales and associated eigenvectors of the matrix $A = \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}$ in the field $\mathbb{R}$. Also find an invertiable matrix P such that $P^{-1}AP$ is diagonal.

Solve

(ii) Given that,

$$A = \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}$$

The characteristic matrix of A is $A - \lambda I$

$$= \begin{bmatrix} 1-\lambda & 4 \\ 9 & 1-\lambda \end{bmatrix}$$

characteristic equation of A is,

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = |A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ 9 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 - 36 = 0 \quad \text{①} \ldots\ldots$$

$$\Rightarrow 1 - 2\lambda + \lambda^2 - 36 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 35 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 5\lambda - 35 = 0$$

$$\Rightarrow \lambda(\lambda - 7) + 5(\lambda - 7) = 0$$

$$\Rightarrow (\lambda - 7)(\lambda + 5) = 0$$

$$\therefore \lambda = 7, -5$$

which are the eigenvalues of the given matrix A.

Let $\underline{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ is the eigenvector of the matrix A corresponding to the eigenvalue λ.

Then by the definition we have

$$(\lambda I - A)\underline{v} = 0, \text{ where } 0 \text{ is a null matrix}$$

$$\Rightarrow \left[\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 4 \\ 9 & 1 \end{pmatrix}\right] \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} \lambda - 1 & -4 \\ -9 & \lambda - 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \ldots\ldots (1)$$

Now for $\lambda = -5$, eqn (1) $\Rightarrow$

$$\begin{pmatrix} -6 & -4 \\ -9 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -6x - 4y \\ -9x - 6y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which are the eigenvalues of the given matrix A.

Let $\underline{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ is the eigenvectors of the matrix A corresponding to the eigenvalue λ.

Then by the definition we have

$$(\lambda I - A)\underline{v} = 0, \text{ where } 0 \text{ is a null matrix}$$

$$\Rightarrow \left[\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 4 \\ 9 & 1 \end{pmatrix}\right] \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} \lambda - 1 & -4 \\ -9 & \lambda - 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \ldots\ldots (1)$$

Now for $\lambda = -5$, eqn (1) $\Rightarrow$

$$\begin{pmatrix} -6 & -4 \\ -9 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -6x - 4y \\ -9x - 6y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow -6x - 4y = 0$

$\Rightarrow -9x - 6y = 0$

$\Rightarrow 3x + 2y = 0$

$3x + 2y = 0$

$\Rightarrow 3x + 2y = 0 \quad \ldots \quad \ldots \quad (2)$

Now it is clear that $x = 2$ and $y = -3$ is a solution of equation (2)

Therefore, $\underline{V}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -5$

when $\lambda = 7$, then eq^n (1) $\Rightarrow$

$$\begin{pmatrix} 6 & -4 \\ -9 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow 6x - 4y = 0$

$-9x + 6y = 0$

$\Rightarrow 3x - 2y = 0$

$3x - 2y = 0$

$\Rightarrow 3x - 2y = 0 \quad \ldots\ldots\ldots \quad (3)$

Now it is clear that $x = 2$, $y = 3$ is a solution of the equation given by eq^n (3)

Therefore, $V_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 7$.

Suppose that P is the matrix which has the above two eigenvectors ~~of~~ as columns.

$$\therefore P = \begin{pmatrix} 2 & 2 \\ -3 & 3 \end{pmatrix}$$

~~One can easily~~

Now we can easily find that the inverse of P is

$$P^{-1} = \frac{1}{12} \begin{pmatrix} 3 & -2 \\ 3 & 2 \end{pmatrix}$$

$$\therefore P^{-1}AP = \frac{1}{12}\begin{pmatrix} 3 & -2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 9 & 0 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ -3 & 3 \end{pmatrix}$$

$$= \frac{1}{12}\begin{pmatrix} 3 & -2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} -10 & 14 \\ 15 & 21 \end{pmatrix}$$

$$= \frac{1}{12}\begin{pmatrix} -60 & 0 \\ 0 & 84 \end{pmatrix}$$

□ **Problem:** $A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$ এর জন্য উপরের সব কাজ

Soln: Given that,

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$$

The characteristic matrix of A is $A - \lambda I$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{bmatrix}$$

characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \{(2-\lambda)(1-\lambda)\} - 6 = 0$$

$$\Rightarrow 2 - 2\lambda - \lambda + \lambda^2 - 6 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + \lambda - 4 = 0$$

$$\Rightarrow \lambda(\lambda - 4) + 1(\lambda - 4) = 0$$

$$\Rightarrow (\lambda - 4)(\lambda + 1) = 0$$

$$\therefore \lambda = 4, -1$$

which are the eigenvalues of the given matrix A

Let $\underline{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ is the eigenvector of the matrix A corresponding to the eigenvalue λ.

Then by the definition we have

$$(\lambda I - A)\underline{v} = 0 \quad , \text{ where } 0 \text{ is a null matrix.}$$

$$\Rightarrow \left[\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\right] \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda - 2 & -3 \\ -2 & \lambda - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \ldots (1)$$

Now for $\lambda = 4$, eq^n (1) $\Rightarrow$

$$\begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x-3y \\ -2x+3y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $2x-3y=0$

$-(2x-3y)=0$

$\Rightarrow$ $2x-3y=0$

$2x-3y=0$

$\therefore 2x-3y=0$ ——— (ii)

Now it is clear that $x=3$ and $y=2$ is a solution of eqn (2)

Therefore, $v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 4$.

when, $\lambda = -1$ then eqn (1)

$$\begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $-3x - 3y = 0$

$-2x - 2y = 0$

$\Rightarrow$ $x + y = 0$

$x + y = 0$

$x + y = 0$ .... (III)

Now it is clear that $x = -1$ and $y = 1$ is a solution of eqn (3)

Therefore, $\underline{V_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = -1$.

suppose that P is the matrix which has above two eigenvectors as columns

$$\therefore P = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$$

Now we can easily find that the inverse of P is

$$P^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$$

$$\therefore P^{-1}AP = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 4 & 4 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 20 & 0 \\ 0 & -5 \end{pmatrix}$$

(আজকের ক্লাস থেকে ১০ মার্কস থাকবে) 01-10-23

Problem:

Find the eigenvalues and corresponding eigen-vectors of the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$ in field $\mathbb{R}$

Soln:

Given that,

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

characteristic equation of the matrix A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)\{(4-\lambda)(3-\lambda) - 2\} - 1(6 - 2\lambda - 2) + 1(2 - 4 + \lambda) = 0$$

$$\Rightarrow (3-\lambda)(12 - 4\lambda - 3\lambda + \lambda^2 - 2) - 6 + 2\lambda + 2 + 2 - 4 + \lambda = 0$$

$$\Rightarrow (3-\lambda)(\lambda^2 - 7\lambda + 10) + 3\lambda - 6 = 0$$

$$\Rightarrow 3\lambda^2 - 21\lambda + 30 - \lambda^3 + 7\lambda^2 - 10\lambda + 3\lambda - 6$$

$$\Rightarrow 10\lambda^2 - 28\lambda - \lambda^3 + 24 = 0$$

$$\Rightarrow -(\lambda^3 - 10\lambda^2 + 28\lambda - 24) = 0$$

$$\Rightarrow \lambda^3 - 10\lambda^2 + 28\lambda - 24 = 0$$

$$\Rightarrow \lambda^2(\lambda - 6) - 4\lambda(\lambda - 6) + 4(\lambda - 6) = 0$$

$$\Rightarrow (\lambda^2 - 6)(\lambda^2 - 4\lambda + 4) = 0$$

$\Rightarrow (\lambda-6)(\lambda-2)^2 = 0$

$\therefore \lambda = 2, 2, 6$

$\therefore$ The eigen values of the matrix A are

$2, 2, 6$.

~~The~~ By the definition of eigen vector we have,

$(\lambda I - A)\underline{v} = 0 \ldots (1)$ where $\underline{v}$ is the eigenvalue associated with λ and 0 is the null matrix.

Now let $\underline{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is the eigenvector corresponding to λ.

$\therefore$ Eq^n (1) $\Rightarrow$

$$\Rightarrow \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \lambda-3 & -1 & -1 \\ -2 & \lambda-4 & -2 \\ -1 & -1 & \lambda-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where $\lambda = 2$, eq^n ② $\Rightarrow$

$$\begin{bmatrix} -1 & -1 & -1 \\ -2 & -2 & -2 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x-y-z=0$

$-2x-2y-2z=0$

$-x-y-z=0$

$\Rightarrow \quad x+y+z=0$

$x+y+z=0$

$x+y+z=0$

$\therefore x+y+z=0$ ... ... ②

Here the number of independent variables is two.

$\therefore$ Two independent eigenvectors exists for $\lambda = 2$.

Let $y=a$, $z=b \Rightarrow x=-a-b$; $a, b \in \mathbb{R}$; $a \neq 0$, $b \neq 0$.

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -a-b \\ a \\ b \end{bmatrix} = -a \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} - b \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\therefore$ Eigenvectors corresponding to $\lambda = 2$ are

$$\underline{V}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \underline{V}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and all eigenvalues are $\begin{bmatrix} -a-b \\ a \\ b \end{bmatrix}$; $a, b \in \mathbb{R}$; $a \neq 0$, $b \neq 0$

when $\lambda = 6$, eq^n ② $\Rightarrow$

$$\begin{bmatrix} \lambda-3 & -1 & -1 \\ -2 & \lambda-4 & -2 \\ -1 & -1 & \lambda-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \dots \dots ②$$

$$\Rightarrow \begin{bmatrix} 3 & -1 & -1 \\ -2 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x-y-z \\ -2x+2y-2z \\ -x-y+3z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

[তয় টাকে উপরে আনলে এমন হচ্ছে (3)
⇒ সমন, পরের 2টা একইরকম করে
$y-2z=0$ পেয়েছি]

$$\Rightarrow \quad x+y-3z=0$$
$$y-2z=0$$

Here the number of independent variable is 1.

$\therefore$ let $z=a \Rightarrow y=2a, \; x=a$

when $\lambda=6$, eq^n ② $\Rightarrow$

$\therefore$ The independent eigenvalue corresponding to $\lambda=6$ is

$$V_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

and all eigenvalues are

$$\begin{bmatrix} a \\ 2a \\ a \end{bmatrix} ; \; a \in \mathbb{R}, a \neq 0.$$

(Exam এসাইন)

* For what values of a, b, c the matrix

$$\begin{bmatrix} 2 & a-2b+2c & 2a+b+c \\ 3 & 5 & a+c \\ 0 & -2 & 7 \end{bmatrix}$$ is symmetric.

Solⁿ:

Let $A = \begin{bmatrix} 2 & a-2b+2c & 2a+b+c \\ 3 & 5 & a+c \\ 0 & -2 & 7 \end{bmatrix}$

Since the given matrix is symmetric,

$$A^T = A$$

$$\therefore A^T = \begin{bmatrix} 2 & 3 & 0 \\ a-2b+2c & 5 & -2 \\ 2a+b+c & a+c & 7 \end{bmatrix}$$

Now,

$$\begin{bmatrix} 2 & 3 & 0 \\ a-2b+2c & 5 & -2 \\ 2a+b+c & a+c & 7 \end{bmatrix} = \begin{bmatrix} 2 & a-2b+2c & 2a+b+c \\ 3 & 5 & a+c \\ 0 & -2 & 7 \end{bmatrix}$$

Equating the corresponding elements from both sides we get

$$a-2b+2c = 3 \quad \ldots \ldots \quad (1)$$

$$2a+b+c = 0 \quad \ldots \ldots \quad (2)$$

$$a+c = -2 \quad \ldots \ldots \quad (3)$$

- সমীকরণ থেকে a, b, c এর মান বের করতে হবে।

Echelon form : (½ set Q)

Define Rank of a matrix? Q.

Equivalence matrix

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix}$$

Reduced echelon এ 0 এর পরে সংখ্যা 1 থাকবে

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2' = R_1 - 2R_2, \quad R_3' = 3R_1 - 2R_3$$

∴ **Rank of a matrix:** Let A be an $m \times n$ matrix and let A_R be the row echelon form of A then the rank of the matrix A is the number

of non-zero rows of A_R and is denoted by $\rho(A)$.

* Find the rank of the matrix A

$$A = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$$

Soln:

Given that,

$A = (\quad)$

Reduce the given matrix to row a echelon form by means of the elementary row Transformations and join sucessive matrices by the equivalance sign $\sim$, we get

$$A = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & -1 & -2 & -1 & 1 \\ 0 & 3 & 6 & 3 & -3 \\ 0 & 1 & 2 & 1 & -1 \end{bmatrix} \quad \begin{array}{l} R_2' = R_1 - R_2 \\ R_3' = 2R_1 - R_3 \\ R_4' = 3R_1 - R_4 \end{array}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & -1 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_3'' = 3R_2' + R_3' \\ R_4'' = R_2' + R_4' \end{array}$$

which shows that the number of non-zero rows in echelon form is 2.

$\therefore$ The rank of A is $\rho(A) = 2$