Random variable and probability distribution

* **Discrete random variable:** A random variable on Ω is defined to be a discrete random variable if it takes at most a countable number of values.

Continous random variable: A random variable X is said to be continous if it can take all possible values within certain limits.

Probability function: If X is a discrete random variable with possible values $x_1, x_2, \dots x_n$ and $f(x_1), f(x_2), \dots, f(x_n)$ be their corresponding probabilities, then the set of ordered pair $(x, f(x))$ is called the probability function or probability mass function of the discrete random variable X. Symbolically

$$f(x_i) = \begin{cases} P[X = x_i] & \text{if } X = x_i,\ i = 1, 2, \dots n \\ 0 & \text{if } X \neq x_i \end{cases}$$

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Properties of probability function:-

If $f(x)$ is a probability function of ~~the~~ a discrete random variable X, then $f(x)$ satisfy the following two properties:

1. $0 \leq f(x) \leq 1$
2. $\Sigma f(x_i) = 1$

Example: Let X be a random variable with probability function defined by $f(-2) = \frac{1}{10}$, $f(0) = \frac{2}{10}$, $f(4) = \frac{4}{10}$ and $f(11) = \frac{3}{10}$.

Find i) $P[-2 \leq X < 4]$ (ii) $P[X > 0]$ and (iii) $P[X \leq 4]$

Soln: Here the random variable X takes the values $-2, 0, 4$ and 11 with respective probabilities $\frac{1}{10}, \frac{2}{10}, \frac{4}{10}$ and $\frac{3}{10}$.

Values of $X : x$	-2	0	4	11
$f(x)$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{4}{10}$	$\frac{3}{10}$

i) $P[-2 \leq X < 4] = P[X = -2] + P[X = 0]$

$= f(-2) + f(0)$

$= \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$

(ii) $P[X>0] = P[X=4] + P[X=11]$

$= f(4) + f(11)$

$= \frac{4}{10} + \frac{3}{10} = \frac{7}{10}$

(iii) $P[X \le 4] = P[X=-2] + P[X=0] + P[X=4]$

$= f(-2) + f(0) + f(4)$

$= \frac{1}{10} + \frac{2}{10} + \frac{4}{10}$

$= \frac{7}{10}$

Probability density function:

If X is a continous random variable, then the function $f(x)$, defined on the continous sample space Ω is called probability density function.

Probability density function $f(x)$ satisfy the following properties:

i. $\int_R f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1$

ii) The probability of any event A defined by $X < x$ is $P[A] = P[X=x] = \int_{-\infty}^{x} f(x)\,dx$

and if B is an event defined by $a \le X \le b$, then

$P[B] = P[a \le x \le b] = \int_a^b f(x)\,dx.$

Example: Let X be a continuous random variable, with probability density function

$$f(x) = \begin{cases} \frac{2}{3}x, & \text{if } 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $P[X \le 1.2]$, $P[X > 1.2]$, $P[1.2 < X \le 1.6]$, $P[X = 1.5]$

~~$P[X = 1.5]$ and, $P[X \le .2]$, $P[X > 3]$~~

Soln:

$$P[X \le 1.2] = \int_1^{1.2} f(x)\,dx$$

$$= \int_1^{1.2} \frac{2}{3}x\,dx$$

$$= \frac{2}{3}\left[\frac{x^2}{2}\right]_1^{1.2}$$

$$= \frac{2}{3} \times \frac{1}{2}\left[1.2^2 - 1^2\right]$$

$$= .1467$$

$$P[X > 1.2] = \int_{1.2}^{2} f(x)\,dx = \int_{1.2}^{2} \frac{2}{3}x\,dx$$

$$= \frac{2}{3}\left[\frac{x^2}{2}\right]_{1.2}^{2}$$

$$= \frac{1}{3}\left[2^2 - 1.2^2\right] = .8533$$

iii) $P[1.2 \le X \le 1.6] = \int_{1.2}^{1.6} \frac{2}{3} x \, dx = \frac{2}{3}\left[\frac{x^2}{2}\right]_{1.2}^{1.6}$

$= \frac{2}{3} \times \frac{1}{2}\left[1.6^2 - 1.2^2\right]$

$= .3733$

~~iv) $P[X \le 2]$.~~

iv) $P[X = 1.5] = 0$

Since continous random variable cannot take isolated value.

Mathematical expectation:

Mathematical expectation of a discrete random variable: If x is a discrete random variable which take take different possible values $x_1, x_2, \ldots x_n$ with corresponding probabilities $f(x_1), f(x_2) \ldots f(x_n)$ then the mathematical expectation of the random variable x denoted by μ is defined by

$$\mu = E[X] = \sum_{i=1}^{n} x_i f(x_i)$$

here μ = mean

Example: Suppose that X is a discrete random variable which can take values $-2, 0, 3$ and 5 with probabilities $P[X=-2] = .1$, $P[X=0] = .3$, $P[X=3] = .4$ and $P[X=5] = .2$. Find the mean of X. or ~~mathematical expectation of x~~ expected value of X.

$$\text{Mean} = \mu = E[X]$$

$$= \sum x_i f(x_i) = x_1 f(x_1) + x_2 f(x_2) + x_3 f(x_3) + x_4 f(x_4)$$

$$= -2 \times .1 + 0 \times .3 + 3 \times .4 + 5 \times (.2)$$

$$= -2 \times .1 + 0 \times .3 + 3 \times .4 + 5 \times .2$$

$$= -.2 + 1.2 + 1.0 = 2$$

therefore the mean of the random variable X is 2.

Mathematical expectation of ~~a~~ a continous random variable:

If X is a continous random variable with probability density function $f(x)$, then mathematical expectation of X is defined by

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

Example: Let X be a random variable with density function

$$f(x) = \begin{cases} \frac{1}{9}(4x - x^2), & \text{for } 1 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$$

Find $E[X]$, $E[X^2]$ and $E[X-1]$

Soln: ~~Let X be a random variable with density function~~

~~$f(x) =$~~

By definition, mean is

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_1^4 x \frac{1}{9}(4x - x^2)\, dx$$

$$= \frac{1}{9} \int_1^4 (4x^2 - x^3)\, dx$$

$$= \frac{1}{9} \left[\frac{4x^3}{3} - \frac{x^4}{4} \right]_1^4$$

$$= \frac{81}{36} = 2.25$$

$$E[x^2] = \cancel{\frac{1}{9}\int_1^4 x} \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_1^4 x^2 \frac{1}{9}(4x - x^2)\, dx$$

$$= \frac{1}{9}\int_1^4 (4x^3 - x^4)\, dx$$

$$= \frac{1}{9}\left[\frac{4x^4}{4} - \frac{x^5}{5}\right]_1^4$$

$$= \frac{252}{45} = 5.6$$

$$E[X-1] = E[X] - E[1]$$
$$= E[X] - 1$$
$$= 2.25 - 1$$
$$= 1.25$$

$$E[X-a] = E[X] - a$$

Binomial distribution: A discrete random variable X is said to have a binomial distribution if its probability function is given by

$$f(x; n, p) = \begin{cases} \binom{n}{x} p^x q^{n-x}; & \text{for } x = 0, 1, 2, \cdots n \\ 0 & \text{otherwise.} \end{cases} \quad \text{—(i)}$$

where the two parameters n and p satisfy $0 \le p \le 1$ and n is a positive integer.

A distribution defined by the probability function of X given in (i) is called binomial distribution.

pf $p+q=1$

Example: In a community, the probability that a newly born child will be boy is $\frac{2}{5}$. Among the 4 newly born children in that community, what is the probability that

a) All the four boys

b) at least two boys

c) no boys

d) exactly one boys

e) at most two boys.

Soln: let, a newly born child is a boy as a success in a in a bernoulli trial.

i.e $\cancel{P(X}$ $p = \frac{2}{5}$

$q = 1 - \frac{2}{5} = \frac{5-2}{5} = \frac{3}{5}$

$p + q = 1$

$\therefore \cancel{f(x,}$ here $n = 4$, $X = 0, 1, 2, 3, 4$

$$f(x, n, p) = \binom{n}{c_x} p^x q^{n-x}$$

$$f(x; 4, \tfrac{2}{5}) = \binom{4}{c_x}\left(\frac{2}{5}\right)^x \left(\frac{3}{5}\right)^{4-x}$$

a) $P[\text{all the boys}] = P[X = 4]$

$$= \binom{4}{c_4}$$

$$= \cancel{\binom{4}{c_x}}$$

$$= \binom{4}{c_4}\left(\frac{2}{5}\right)^4 \left(\frac{3}{5}\right)^{4-4}$$

$$= \left(\frac{2}{5}\right)^4 = \frac{16}{625}$$

b) $P[\text{at least two boys}] = P[X \geq 2]$

$$= 1 - P[X < 2]$$

$$= 1 - \{P[X=0] + P[X=1]\}$$

$$= 1 - \left\{ {}^4c_0 \left(\frac{2}{5}\right)^0 \left(\frac{3}{5}\right)^{4-0} + {}^4c_1 \left(\frac{2}{5}\right)^{4-1} \left(\frac{3}{5}\right)^{4-1} \right\}$$

(P.T.O)

$$= 1 - \left\{\left(\frac{3}{5}\right)^4 + 4\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^3\right]$$

$$= 1 - \left\{\frac{81}{625} + \frac{216}{625}\right\}$$

$$= 1 - \frac{297}{625} = \frac{328}{625}$$

c) $P[\text{no boys}] = P[X=0]$

$$= \left({}^4C_0\right)\left(\frac{2}{5}\right)^0\left(\frac{3}{5}\right)^{4-0}$$

$$= \left(\frac{3}{5}\right)^4 = \frac{81}{625}$$

d) $P[\text{exactly one boys}] = P[X=1]$

$$= \left({}^nC_x \cdot p^x q^{n-x}\right)$$

$$= \left\{{}^4C_1\left(\frac{2}{5}\right)^1\left(\frac{3}{5}\right)^{4-1}\right\}$$

$$= 4 \times \frac{2}{5} \times \left(\frac{3}{5}\right)^4 = \frac{216}{625}$$

(e) $P[\text{at most two boys}] = P[X \le 2]$

$$= P[X=0] + P[X=1] + P[X=2]$$

$$= {}^4C_0\left(\frac{2}{5}\right)^0\left(\frac{3}{5}\right)^{4-0} + {}^4C_1\left(\frac{2}{5}\right)^1\left(\frac{3}{5}\right)^{4-1}$$

$$+ {}^4C_2\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^{4-2}$$

$$= \left(\frac{3}{5}\right)^4 + 4\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^3 + 6\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^2$$

$$= \frac{81}{625} + \frac{216}{625} + \frac{216}{625} = \frac{513}{625}$$

Poisson distribution:

A discrete random variable X is said to have a poisson distribution if its probability function is

$$f(x;\lambda) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!} & ; x = 0, 1, \ldots \infty \\ 0 & \text{otherwise} \end{cases}$$

where $e = 2.718$ and λ is the parameter of the distribution which is the mean number of success.

Example: The average number of traffic accidents on a certain section of highway is to per week. Assume that the number of accidents follows a poisson distribution with $\lambda = 2$.

i) Find the probability of no accident on this section of highway during 1-week period.

ii) Find the probability of at most three accidents on this section of highway during a 2-week period.

Soln:

the average number of accident per week is $\lambda = 2$

$\therefore$ i) The probability of no accidents

$$P[X=0] = f(0,2) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \frac{e^{-2} 2^0}{0!} = e^{-2} = .135335$$

(ii) During a 2 week period, the average number of accidents on this section of highway is $2 \times 2 = 4$.

The probability of at most three accidents during a 2 week periods is

$$P[X \leq 3] = P[x=0] + P[x=1] + P[x=2] + P[X=3]$$

$$= f(0,4) + f(1,4) + f(2,4) + f(3,4)$$

$$= \frac{e^{-4} 4^0}{0!} + \frac{e^{-4} 4^1}{1!} + \frac{e^{-4} 4^2}{2!} + \frac{e^{-4} 4^3}{3!}$$

$$= .0183 + 0.0732 + 0.1465 + .1953$$

$$= .433471$$

Application of poisson distribution:

i) The number of cars passing through a certain street in time t.

ii) Number of suicides reported in a particular day

iii) Number of printing mistake at each page of a book.

Normal distribution:

A continous random variable x is said to have a normal distribution if its density function is given by

$$f(x, \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} ; -\alpha < x < \alpha$$

where the parameters μ and σ^2 satisfy $-\alpha < \mu < \alpha$ and $\sigma^2 > 0$.

Some important characteristic of the normal distribution.

i) The curve of the distribution is symmetrical at point $x = \mu$

ii) It is bell shaped.

iii) Since the distribution is symmetrical about the point $x = \mu$, it devides the whole distribution into equal part.

iv) It's mean, median and mode are equal.

v) For normal distribution, skewness and kurtosis are $B_1 = 0$ and $B_2 = 3$.

* The Philips manufacturing electric bulbs that have a length of life that is normally distributed with mean equal 800 hours and standard deviation of 40 hours. Find the probability that

a) A bulb burns between 778 and 834 hours.

b) life length will be more than 787 hours.

c) " " " " less " 835 "

Soln: X = the length of life of bulbs

standard normal variate is

$$z = \frac{X-\mu}{\sigma}$$

μ = mean = 800 hours

σ = standard deviation = 40

i) $P[778 \le x \le 834] = P\left[\frac{778-\mu}{\sigma} \le \frac{x-\mu}{\sigma} \le \frac{834-\mu}{\sigma}\right]$

$= P\left[\frac{778-800}{40} \le z \le \frac{834-800}{40}\right]$

$= P[-.55 \le z \le .85]$

$= \cancel{P[.85]}$

$P[z \le .85] - P[z \le -.55]$

$= .8023 - .29116$

$= .5111$

ii) $P[X \geq 787] = 1 - P[X \leq 787]$

$= 1 - P\left[\frac{X-\mu}{\sigma} \leq \frac{787-\mu}{\sigma}\right]$

$= 1 - P\left[Z \leq \frac{787-800}{40}\right]$

$= 1 - P[Z \leq -.325]$

$= 1 - .3725$

$= .627$

iii) $P[X \leq 835] = P\left[\frac{X-\mu}{\sigma} \leq \frac{835-\mu}{\sigma}\right]$

$= P\left[Z \leq \frac{835-800}{40}\right]$

$= P[Z \leq .875]$

$= .8092$