CT # What is FFT (Fast Fourier Transform)?

⇒ FFT is an efficient algorithm used to compute the discrete Fourier Transform. Simply, it converts a signal from the time domain into the frequency domain quickly.

The DFT equation which is used by FFT is:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

where, $k = 0, 1, 2, \ldots, n$

$x(n)$ = input sequence

$X(k)$ = frequency domain output.

N = numbers of samples.

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CT # Why FFT is needed?

⇒ The DFT converts a signal from the time domain into the frequency domain, but it is computationally expensive because,

DFT complexity is $O(N^2)$ and

FFT complexity is $O(N \log N)$

So, FFT makes frequency analysis much faster, especially for large datasets.

DFT directly uses the equation:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{j 2\pi kn/N}$$

But FFT computes the same result using fewer calculations by exploiting: symmetry, periodicity and divide and conquer approach.

Thus, the time complexity is reduced, and for this reason FFT is used and it is needed.

(C) # Describe the divide and conquer approach for FFT algorithm.

=> The Fast Fourier Transform (FFT) efficiently computes the Discrete Fourier Transform (DFT) by applying a divide and conquer strategy. Instead of directly evaluating an N-point DFT with $O(N^2)$ complexity. FFT recursively reducing the total computation to $O(N \log N)$.

Divide:

Given an N-point input sequence $x(n)$:

$x(n)$; $n = 0, 1, 2, \ldots, N-1$

The sequence is divided into N/2 point subsequences.

- Even-indexed element : $x_e(m) = x(2m)$
- Odd-indexed element : $x_o(m) = x(2m+1)$

where $m = 0, 1, \ldots, N/2 - 1$

Conquer:

Compute the N/2 point DFT of.

the even indexed sequence and the odd indexed sequence. This process is applied recursively until only 2 point DFTs remain, which is very simple to compute.

Here, the $x_e(m)$ produce $X_e(k)$ and

the $x_o(m)$ produce $X_o(k)$

Combine (Butterfly operation).

The smaller DFT results are combined using twiddle factors; which are roots of unity.

$$W_N^k = e^{-j2\pi k/N}$$

The final N-Point DFT is obtained as:

$$X(k) = X_e(k) + W_N^k X_o(k)$$

$$X(k+N/2) = X_e(k) - W_N^k X_o(k)$$

for $k = 0, 1, \ldots\ldots, N/2 - 1$

Effect gain:

- Direct DFT : $O(N^2)$
- FFT (Divide & conquer) : $O(N \log N)$

What IIR (Infinite Impulse Response) Filter ?

→ An IIR filter is a ~~large~~ type of digital filter that uses feedback, meaning its output depends on past inputs/current inputs and past outputs, allowing it to create sharp frequency response with fewer calculations than FIR filters.

General IIR filter equation is:

$$y(n) = \sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k)$$

Where,

$x(n)$ = input signal

$y(n)$ = output signal

b_k, a_k = filter co-efficient.

Application:

→ audio processing

→ Communication systems

→ Control systems.

8 # What is FIR filter?

⇒ A Finite Impulse Response filter is a type of digital filter that ~~response~~ produces an output based on a finite number of past input samples, meaning its impulse response becomes zero after a finite duration and does not use feedback.

General FIR filter equation:

$$y(n) = \sum_{k=0}^{M} b_k \, x(n-k)$$

where,

$x(n)$ = input signal

$y(n)$ = output signal

b_k = filter co efficient

M = filter order

① # Make a comparison amoung IIR and FIR filter.

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Feature	FIR	IIR
Impulse Response	Finite duration	Infinite duration
Feedback	No feedback	Uses feedback
Stability	Always stable	May be stable
phase Response	Exactly linear	usually non-linear
Computational complexity	higher	lower
Memory requirement	More memory	less memory
Filter Equation	$y(n) = \sum_{k=0}^{m} b_k x(n-k)$	$y(n) = \sum_{k=0}^{m} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k)$
Design complexity	Easier	Complex
Typical Applications	Audio, image, biomedical signals.	Control systems, real time processing.

(P) # Explain Radix-2 DIT FFT Algorithm and draw the butterfly diagram for 8 point DIT FFT.

⇒ The Radix-2 DIT FFT algorithm efficiently computes the DFT by recursively splitting an N-point time-domain signal into two N/2 point sequence: one with even indexed samples and one with odd indexed, then combining their DFT's. The divide and conquer approach using butterfly structure and bit-reversal input order, dramatically reduces computation from $O(N^2)$ to $O(N \log N)$.

Concept:

Radix-2: The input sequence is split into 2-sub sequences at each stage, requiring N to be a power of 2.

Decimal In Time (DIT): The time domain signal $x(n)$ is decimated into even $x(2n)$ and odd $x(2n+1)$ samples, reducing the problem size.

Butterfly operation:

For inputs a and b:
$$y_1 = a + W_N^k b$$
$$y_2 = a - W_N^k b$$

Each stage consists of multiple butterfly operations.

Bit reversal: The input sequence is recorded by reversing the bits of its index.

Number of stages:

Number of stages $= \log_2 N$

when, $N = 8 \longrightarrow 3$ stages

$N = 16 \rightarrow 4$ stages

Complexity:

For multiplications : $N/2 \log N$ and

for additions : $N \log_2 N$

Much faster than direct DFT

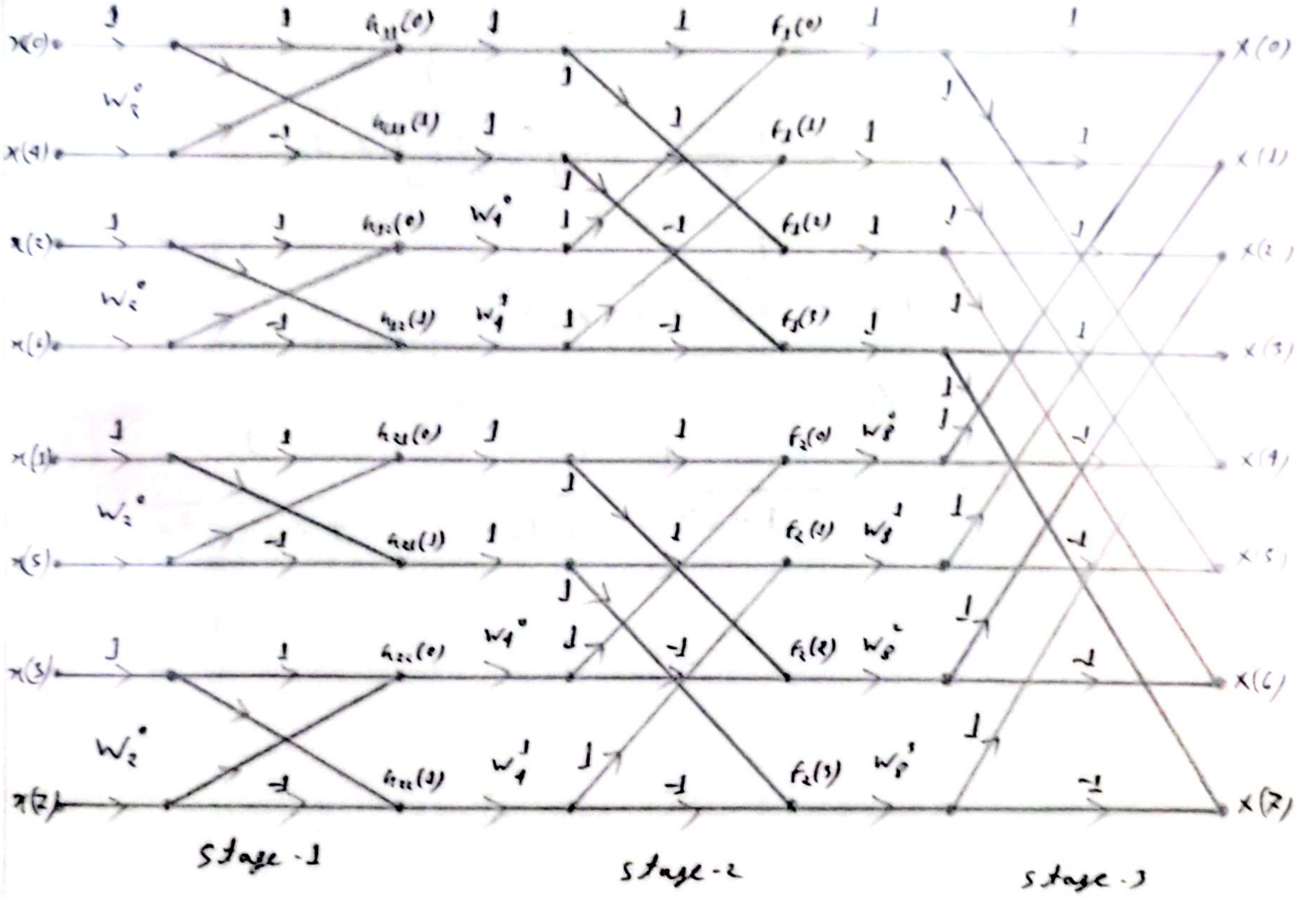

Fig: The signal flow graph or butterfly diagram for 8-point radix-2 DIT FFT.

Chapter-8

Example 8.1

We know that the mapping formula for the backward difference for the derivative is given by

$$s = \frac{1 - z^{-1}}{T}$$

For the given analog filter function $H_a(s) = \frac{2}{s+3}$, the corresponding digital filter function is:

$$H(z) = H_a(s)\Big|_{s = \frac{1-z^{-1}}{T}} = \frac{2}{\frac{(1 - z^{-1})}{T} + 3}$$

$$= \frac{2T}{1 - z^{-1} + 3T}$$

If $T = 1s$, $H(z) = \frac{2}{1 - z^{-1} + 3 \cdot 1} = \frac{2}{4 - z^{-1}}$

(Ans.)

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Example 8.2

The mapping formula for the backward difference by the derivative is:

$$s = \frac{1 - z^{-1}}{T}$$

Therefore, for the given $H_a(s)$, the corresponding digital function is:

$$H(z) = H_a(s)\Big|_{s = \frac{1-z^{-1}}{T}} = \frac{9}{\left[\frac{1-z^{-1}}{T}\right]^2 + 9}$$

$$= \frac{4T^2}{1 - 2z^{-1} + z^{-2} + 9T^2}$$

If $T = 1s$, $H(z) = \frac{4}{1 - 2z^{-1} + z^{-2} + 9} = \frac{4}{10 - 2z^{-1} + z^{-2}}$ (Ans.)

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The mapping formula for the backward difference by the derivative is:

$$s = \frac{1 - z^{-1}}{T}$$

Therefore, for the given $H_a(s)$, the corresponding digital filter function is

$$H(z) = H_a(s)\Big|_{s = \frac{1-z^{-1}}{T}} = \frac{1}{\left[\frac{1-z^{-1}}{T}\right]^2 + 16}$$

$$= \frac{T^2}{1 - 2z^{-1} + z^{-2} + 16\cancel{z}T^2}$$

If $T = 1$ sec, then

$$H(z) = \frac{1^2}{1 - 2z^{-1} + z^{2} + 16} = \frac{1}{17 - 2z^{-1} + z^{2}}$$

Example 8.3

For the given $H_a(s)$, the system function of the corresponding digital filter is:

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{T}} = \frac{3}{(s+0.5)^2+16}\Big|_{s=\frac{1-z^{-1}}{T}}$$

$$= \frac{3}{\left[\frac{1-z^{-1}}{T}+0.5\right]^2+16} = \frac{3T^2}{\left[(1+0.5T)-z^{-1}\right]^2+16T^2}$$

$$= \frac{3T^2}{(1+0.5T)^2+z^{-2}-2(1+0.5T)z^{-1}+16T^2}$$

If $T=1s$, then

$$H(z) = \frac{3}{2.25+z^{-2}-3z^{-1}+16} = \frac{3}{18.25-3z^{-1}+z^{-2}}$$

Example 8.4:

Given, $H_a(s) = \dfrac{2}{(s+1)(s+3)}$

Using partial fractions, $H_a(s)$ can be expressed as:

$$H_a(s) = \frac{A}{s+1} + \frac{B}{s+3}$$

$$A = (s+1)H_a(s)\Big|_{s=-1} = \frac{2}{s+3}\Big|_{s=-1} = 1$$

$$B = (s+3)H_a(s)\Big|_{s=-3} = \frac{2}{s+1}\Big|_{s=-3} = -1$$

$$\therefore H_a(s) = \frac{1}{s+1} - \frac{1}{s+3} = \frac{1}{s-(-1)} - \frac{1}{(s-(-3)}$$

By impulse invariant transformation, we know that,

$$\frac{A_i}{s - P_i} \xrightarrow[\text{(is transformed to)}]{} \frac{A_i}{1 - e^{P_i T} z^{-1}}$$

Here $H_a(s)$ has two poles and $P_1 = -1$ and $P_2 = -3$

Therefore, the system function of the digital filter is:

$$H(z) = \frac{1}{1 - e^{P_1 T} z^{-1}} - \frac{1}{1 - e^{P_2 T} z^{-1}}$$

$$= \frac{1}{1 - e^{-T} z^{-1}} - \frac{1}{1 - e^{-3T} z^{-1}}$$

a) when $T = 1s$,

$$H(z) = \frac{1}{1 - e^{-1} z^{-1}} - \frac{1}{1 - e^{-3} z^{-1}}$$

$$= \frac{1}{1 - 0.3678 z^{-1}} - \frac{1}{1 - 0.0497 z^{-1}}$$

$$= \frac{(1 - 0.04972 z^{-1}) - (1 - 0.3678 z^{-1})}{(1 - 0.3678 z^{-1})(1 - 0.0497 z^{-1})}$$

$$= \frac{0.3181 z^{-1}}{1 - 0.4175 z^{-1} + 0.0182 z^{-2}}$$

b) when $T = 0.5s$,

$$H_z = \frac{1}{1 - e^{-0.5} z^{-1}} - \frac{1}{1 - e^{-3 \times 0.5} z^{-1}}$$

$$= \frac{1}{1 - 0.606 z^{-1}} - \frac{1}{1 - 0.223 z^{-1}}$$

$$= \frac{(1 - 0.223 z^{-1}) - (1 - 0.606 z^{-1})}{(1 - 0.606 z^{-1})(1 - 0.223 z^{-1})}$$

$$= \frac{0.383 z^{-1}}{1 - 0.829 z^{-1} + 0.135 z^{-2}}$$

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Example 8.5:

Observe that the given system function of the analog filter is of the standard form $H_a(s) = \dfrac{s+a}{(s+a)^2+b^2}$, where we are given $a = 0.1$ and $b = 3$.

By the impulse invariant transformation, we know that

$$\frac{s+a}{(s+a)^2+b^2} \xrightarrow[\text{(is transformed to)}]{} \frac{1 - e^{-aT}(\cos bT)z^{-1}}{1 - 2e^{-aT}(\cos bT)z^{-1} + e^{-2aT}z^{-2}}$$

Therefore, for the give $H_a(s)$, we can write the system function of the digital filter:

$$H(z) = \frac{1 - e^{-0.1T}(\cos 3T)z^{-1}}{1 - 2e^{-0.1T}(\cos 3T)z^{-1} + e^{-2(0.1)T}z^{-2}}$$

Assuming $T = 1s$, we have

$$H(z) = \frac{1 - e^{-0.1}(\cos 3)z^{-1}}{1 - 2e^{-0.1}(\cos 3)z^{-1} + e^{-0.2}z^{-2}}$$

$$= \frac{1 - 0.9048(-0.9899)z^{-1}}{1-2(0.9048)(-0.9899)z^{-1} + 0.8187z^{-2}}$$

$$= \frac{1 + 0.8956z^{-1}}{1 + 1.7913z^{-1} + 0.8187z^{-2}}$$

$$\frac{s+0.2}{(s+0.2)^2+9} \qquad \therefore \; a = 0.2, \quad b = 3$$

$$\therefore H(z) = \frac{1 - e^{-0.2T}(\cos 3T)\, z^{-1}}{1 - 2e^{-0.2T}(\cos 3T)\, z^{-1} + e^{-2\times 0.2T}\, z^{-2}}$$

$T = 1$,

$$H(z) = \frac{1 - e^{-0.2}(\cos 3)\, z^{-1}}{1 - 2e^{-0.2}(\cos 3)\, z^{-1} + e^{-0.4}\, z^{-2}}$$

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$$= \frac{1 - 0.8187(-0.9899)\, z^{-1}}{1 - 1.6375(-0.9899)\, z^{-1} + 0.6703\, z^{-2}}$$

$$= \frac{1 + 0.8104\, z^{-1}}{1 + 1.6209\, z^{-1} + 0.6703\, z^{-2}}$$

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Requirements for converting a stable Analog Filter into a stable Digital Filter.

→ To obtain a stable digital filter from a stable analog filter, the following requirements must be statisfied:

1. Stability Mapping: The left half of the s-plane must be mapped inside the unit circle of the z plane to ensure digital stabitity.

2. Preservation of Frequency Characteristics: The important frequency characteristics such as passband and stopband should be preserved after transformation.

3. One-to-One Mapping: The transformation method should be provide a unique mapping between analog frequency and digital frequency to avoid distortion.

4. Avoidance of Aliasing: The conversion method should prevent aliasing effects during transformation.

Chapter 9

Example 9.1

The length of the filter is 7. Therefore, for linear phase,

$$\alpha = \frac{N-1}{2} = \frac{7-1}{2} = 3$$

The condition for symmetry when N is odd, is $h(n) = h(N-1-n)$

Therefore, the filter coefficients are $h(0) = h(6)$, $h(1) = h(5)$, $h(2) = h(4)$ and $h(3)$.

Therefore,

$$\sum_{n=0}^{N-1} h(n)\sin(\alpha - n)\omega = \sum_{n=0}^{6} h(n)\sin(3-n)\omega$$

$$= h(0)\sin 3\omega + h(1)\sin 2\omega + h(2)\sin\omega + h(3)\sin 0 + h(4)\sin(-\omega) + h(5)\sin(-2\omega) + h(6)\sin(-3\omega)$$

$$= 0$$

Hence, the equation $\sum_{n=0}^{N-1} h(n)\sin(\alpha - n)\omega = 0$ is satisfied.

Example 9.2

The transfer function of the filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)h^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8}$$

The phase delay $\alpha = \frac{N-1}{2} = \frac{9-1}{2} = 4$. Since $\alpha = 4$, the transfer function can be expressed as.

$$H(z) = z^{-4}\left[h(0)z^{4} + h(1)z^{3} + h(2)h^{2} + h(3)z^{1} + h(4)z^{0} + h(5)z^{-1} + h(6)z^{-2} + h(7)z^{-3} + h(8)z^{-4}\right]$$

Since $h(n) = h(N-1-n)$

$$H(z) = z^{-4}\left[h(0)(z^{4}+z^{-4}) + h(1)(z^{3}+z^{-3}) + h(2)(z^{2}+z^{-2}) + h(3)(z+z^{-1}) + h(4)\right]$$

The frequency response is obtained by replacing z with $e^{j\omega}$.

$$H(\omega) = e^{-j4\omega}\Big[h(0)\left[e^{j4\omega}+e^{-j4\omega}\right] + h(1)\left[e^{j3\omega}+e^{-j3\omega}\right] + h(2)\left[e^{j2\omega}+e^{-j2\omega}\right] + h(3)\left[e^{j\omega}+e^{-j\omega}\right] + h(4)\Big]$$

$$= e^{-j4\omega}\left[h(4) + 2\sum_{n=0}^{3} h(n)\cos(4-n)\omega\right]$$

$$= e^{-j4\omega}|H(\omega)|$$

where $|H(\omega)|$ is the magnitude response and $\theta(\omega) = -5\omega$ is the phase response. The plane delay τ_p and group delay τ_g are given by

$$\tau_p = -\frac{\theta(\omega)}{\omega} = 5 \quad \text{and} \quad \tau_g = -\frac{d(\theta(\omega))}{d\omega} = -\frac{d(-5\omega)}{d\omega} = 5$$

Thus, the phase delay and the group delay are the same and constants.