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NUMERICAL KHATA

IT-17035

Root finding method:

(1) Bracketing method

(2) Opening method

$$\text{Error} = \left| \frac{\text{Present value} - \text{Old value}}{\text{Present value}} \times 100\% \right|$$

Bracketing methods

Bisection method: The bisection method in mathematics is a root finding method that repeatedly bisects an interval and then selects a subinterval in which a root must lie for further processing. It is a very simple and robust method, but it is relatively slow.

Algorithm:

Given a function $f(x)$ continuous on an interval $[a\ b]$ and $f(a) * f(b) < 0$

Do

step 1: root, $c = \frac{a+b}{2}$

step 2: if $f(a) * f(c) < 0$ then $b = c$

else $a = c$

same class

Example: ① Find out the root of $x^4 - x - 10 = 0$ for interval $[1.5 \quad 2]$ and iteration 5

Solution:

Iteration	a	b	c	f(a)	f(c)	f(a) * f(c)
1	1.5	2	1.75	-6.4375	-2.3711	15.2639
2	1.75	2	1.875	-2.3711	0.4846	-1.1491
3	1.75	1.875	1.8125	-2.3711	-1.0202	2.4191
4	1.8125	1.875	1.844	-1.0202	-0.2817	0.2874
5	1.844	1.875	1.8595	-0.2817	0.0965	-0.0272

So, $c = 1.8595$ is the root of $x^4 - x - 10 = 0$ for interval $[1.5 \quad 2]$

Bisection method with Error:

Given a function $f(x)$ continuous on an interval $[a \ b]$ and error, e_{abs}

Do

$$c = \frac{a+b}{2}$$

if $f(c) < 0$ then $a = c$

else $b = c$

error $= b - a$

if error $<= e_{abs}$ then stop

else goto Do

Example: 2

consider a function $f(x) = x^2 - 3$, $e_{abs} = 0.01$, $e_{step} = 0.01$ and start with the interval $[1 \ 2]$

[Here, e_{step} = current step − previous step]

Solution:

Iteration	a	b	c	f(c)	update a	update b	error
1	1	2	1.50	-0.75	1.50	2	0.50
2	1.50	2	1.75	0.0625	1.50	1.75	0.25
3	1.50	1.75	1.625	-0.3594	1.625	1.75	0.125
4	1.625	1.75	1.6875	-0.1523	1.6875	1.75	0.0625
5	1.6875	1.75	1.7188	-0.0459	1.7188	1.75	0.0312
6	1.7188	1.75	1.7344	0.0081	1.7188	1.7344	0.0156
7	1.7188	1.7344	1.7266	-0.0189	1.7266	1.7344	0.0078

So, $c = 1.7266$ is the root of $x^2 - 3 = 0$ for the interval $[1\ 2]$ with $e_{abs} = 0.01$.

Ans

class

Example: $f(x) = x^2 - 3 = 0$ $\quad [\overset{a}{1} \;\; \overset{b}{2}]$ $\quad$ 1 % $\quad$ 4 digit

step 1 : root $c = \dfrac{a+b}{2}$

step 2 : $f(a) * f(c) < 0$ then $b = c$

else $a = c$

Solution:

iteration বা step	a	b	c	f(a)	f(c)	f(a)*f(c)	error
1	1	2	1.5	−2	−0.75	>0	100 %
2	1.5	2	1.75	−0.75	0.0625	<0	14 %
3	1.5	1.75	1.6250	−0.75	−0.3593	>0	7.7 %
4	1.625	1.75	1.6875	−0.3593	−0.1523	>0	3.7 %
5	1.6875	1.75	1.7188	−0.1523	−0.0458	>0	1.82 %
6	1.7188	1.75	1.7344	−0.0458	0.0081	<0	0.9 %

Therefore, $c = 1.7344$ is the root of $x^2 - 3 = 0$ for the interval $[1 \;\; 2]$ with error 1 %. Ans

১ম step এ error $= \dfrac{1.5-0}{1.5} \times 100\%$
$= 100\%$

initially $a = 1$, $b = 2$
$c = 0$

Falseposition method: The falseposition method or regula falsi method is a term for problem solving methods in arithmetic, algebra and calculus. In simple terms, these methods begin by attempting to evaluate a problem using test ('false') values for the variables and then adjust the values accordingly.

Algorithm:

$$c = \frac{a * f(b) - b * f(a)}{f(b) - f(a)}$$

i) If $f(c) = 0$ then halt, we have found root.

ii) If $f(a)$ and $f(c)$ have opposite sign then root must lie on $[a, c]$, so assigns

$step_{error} = b - c$ and $b = c$

else $f(c)$ and $f(b)$ have opposite sign then root must lie on $[c\ b]$, so assigns

$step_{error} = c - a$ and $a = c$

class

Example 1: $f(x) = x^2 - 3 = 0$ $\quad \begin{array}{cc} a & b \\ \left[1 \right. & \left. 2\right] \end{array}$ $\quad 1\%$ $\quad$ 4 digit

step 1: root, $c = \dfrac{a * f(b) - b * f(a)}{f(b) - f(a)}$

step 2: if $f(a) * f(c) < 0$ then $b = c$

else $a = c$

solution:

step	a	b	f(a)	f(b)	c	f(c)	f(a)*f(c)	error
1	1	2	-2	1	1.6667	-0.2221	>0	100 %
2	1.6667	2	-0.2221	1	1.7273	-0.0164	>0	3.5 %
3	1.7273	2	-0.0164	1	1.7317	-0.0012	>0	0.25 %

So $c = 1.7317$ is the root of $x^2 - 3 = 0$ for the interval $[1 \;\; 2]$ with $e_{abs} = 0.01$.

Ans

Example 2: consider the function $f(x) = x^2 - 3$.

$e_{abs} = 0.01$, $e_{step} = 0.01$ (error) and start with the interval $[1 \ 2]$.

Solution:

× $c - a$

✓ $b - c$

Iteration	a	b	f(a) ✓	f(b) ×	c	f(c) × ✓	step error	update a	update b
1	1	2	-2	1	1.667	-0.221	0.667	1.667	2
2	1.667	2	-0.221	1	1.7273	-0.016	0.0603	1.7273	2
3	1.7273	2	-0.016	1	1.7315	-0.0019	0.0042	1.7315	2

So, $c = 1.7315$ is the root of $x^2 - 3 = 0$ for the interval $[1 \ 2]$ with $e_{abs} = 0.01$.

Ans

Example 3: Finding the root of $f(x) = e^{-x}(3.2\sin x - 0.5\cos x)$ on the interval [3 4] this time with $e_{step} = 0.001$ and $e_{abs} = 0.001$

Solution:

iteration	a	b	f(a)	f(b)	c	f(c)	error step	update a	update b	f(a)*f(c)
1	3	4	0.047	-.038	3.553	-.024	.447	3	3.553	< 0
2	3	3.553	0.047	-.024	3.366	-.007	.187	3	3.366	
3	3	3.666	0.047	-.007	3.319	-.002	.047	3	3.319	
4	3	3.319	0.047	-.002	3.306	-.001	.013	3	3.306	
5	3	3.306	0.047	-.001	3.299	-.0002	.007	3	3.299	
6	3	3.299	.047	-.0002	3.297	-.00004	.002	3	3.297	
7	3	3.297	.047	-.00004	3.296	-.00007	.001	3	3.296	

So $c = 3.296$ is the root of this function for the interval [3 4] with $e_{step} = 0.001$ and $e_{abs} = 0.001$.

Ans

N.B: Radian mode এ calculation করতে হবে।

Example 4: Find the root of $f(x) = x^3 - 3$ with falseposition method starting with the interval [1 2] and use e_{abs} & $e_{step} = 0.1$ and five decimal of accuracy.

Solution:

Iteration	a	b	f(a)	f(b)	c	f(c)	step error	update a	update b	f(a) * f(c)
1	1	2	-2	5	1.28571	-.87465	.28571	1.28571	2	>0
2	1.28571	2	-.87465	5	1.39206	-.302420	.10635	1.39206	2	
3	1.39206	2	-.30242	5	1.42673	-.09580	.03467	1.42673	2	

So $c = 1.42673$ is the root of $x^3 - 3$ for the interval [1 2] with $e_{abs} = 0.1$

Ans

Example 5 :

Find the root of $f(x) = x^2 - 10$ with the falseposition method with the interval [3 4] and e_{step}, e_{abs} = five decimal accuracy.

solution:

iteration	a	b	f(a)	f(b)	c	f(c)	step error	update a	update b	f(a) * f(c)
1	3	4	-1	6	3.14286	-.12243	.14286	3.14286	4	>0
2	3.14286	4	-.12243	6	3.16000	-.0144	.01714	3.16000	4	
3	3.16000	4	-.0144	6	3.16201	-.00169	.00201	3.16201	4	
4	3.16201	4	-.00169	6	3.16225	-.00017	.00024	3.16225	4	
5	3.16225	4	-.00017	6	3.16227	-.00004	.00002	3.16227	4	
6	3.16227	4	-.00004	6	3.16228	.00001	.83773	3.16227	3.16228	
7	3.16227	3.16228	-.00004	.00001	3.16228	.00001	.00000	3.16227	3.16228	<0

So $c = 3.16228$ is the root of $x^2 - 10 = 0$ for the interval [3 4] with $e_{abs} = .00001 \ldots$

Ans

~~Note~~

Example 6: $f(x) = xe^{x} - 1$, $a = 0$, $b = 1$

Solution:

Step	a	b	f(a)	f(b)	c	f(c)	f(a) * f(b)	error
1	0	1	-1	1.7183	.3679		< 0	
2	0	.3679	-1	-.4685	.6922		> 0	46.85%
3	.6922	.3679	.3830	-.4685	.5463		< 0	26.69%
4	.6922	.5463	.3830	-.0566	.5651		< 0	3.33%

So $c = 0.5651$ is the root of $f(x) = xe^{x} - 1$ in the initial guesses of $a = 0$, $b = 1$

Newton Raphson method: is a way to quickly find a good approximation for the root of a real valued function.

Open method :

* Newton - Raphson method :

steps to find root :

1) check if the given function is differentiable or not. If the function is not differentiable Newton-Raphson method cannot be applied.

2) Find the first derivatives $f'(x)$ of the given function $f(x)$.

3) Take an initial guess root of the function say x_1.

4) Use Newton's iteration formula to get new better approximate of the root, say x_2.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

5) Repeat the process for $x_3, x_4 \ldots$ till the actual root of the function is obtained, fulfilling the tolerance of error.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$\text{error} = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| * 100\ \%$$

class

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এই নিয়মে math আপ্রকার করছি

Example: Given, $f(x) = e^{-x} - x$, $x_0 = 0$, 1 %, 4 digit, Find the root using Newton-Raphson method.

solution: Given,

$$f(x) = e^{-x} - x$$

$$\therefore f'(x) = -e^{-x} - 1$$

iteration	X_i	$f(x_i)$	$f'(x_i)$	X_{i+1}	error
0	0	1	-2	0.5	100 %
1	0.5	0.1065	-1.6065	0.5662	11.7 %
2	0.5663	0.0013	-1.5676	$\boxed{0.5671}$	0.1462 %

0.5671 is the root of this function.

Ans

$$\text{Error} = \left| \frac{X_{i+1} - X_i}{X_{i+1}} \right| \times 100\ \%$$

$$X_{i+1} = X_i - \frac{f(x_i)}{f'(x_i)}$$

$$= 0 - \frac{1}{-2}$$

$$= 0.5$$

Secant method:

The secant method is an algorithm used to approximate the roots of a given function f.

This method requires the selection of two initial approximations x_0 & x_1.

$$\text{error} = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100\ \%$$

$$X_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

class

$$f'(x_i) = \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$

$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

class

Example:

Given, $f(x) = e^{-x} - x$, $x_{i-1} = 0$, $x_0 = 1$, 1%,

Find the root using secant method.

Solution: Given,

$$f(x) = e^{-x} - x$$

$$x_{i-1} = 0$$

$$x_0 = 1$$

iteration	x_{i-1}	x_i	$f(x_{i-1})$	$f(x_i)$	x_{i+1}	error
0	0	1	1	-0.6321	0.6127	63.21%
1	1	0.6127	-0.6321	-0.0708	0.5638	8.67%
2	0.6127	0.5638	-0.0708	0.0052	0.5671	0.59%

0.5671 is the root of this function. Ans

Modified Secant method:

Modified secant method is a much better approximation because it uses one point and the derivative is found by using another point some small distance, δ.

Here, δ = a small perturbation function

$$x_{i+1} = x_i - \frac{\delta x_i \cdot f(x_i)}{f(x_i + \delta x_i) - f(x_i)}$$

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class 4

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$f'(x_i) = \frac{f(x_i + \delta x_i) - f(x_i)}{\delta x_i}$$

$$\therefore x_{i+1} = x_i - \frac{\delta x_i \cdot f(x_i)}{f(x_i + \delta x_i) - f(x_i)}$$

class

Example:

Given, $f(x) = e^{-x} - x$, $x_0 = 1$, 1 %, $\delta = 0.01$.

Find the root using modified secant method.

solution: Given,

$$f(x) = e^{-x} - x, \quad \delta = 0.01, \quad x_0 = 1$$

iteration	x_i	δx_i	$x_i + \delta x_i$	$f(x_i)$	$f(x_i + \delta x_i)$	x_{i+1}	error
0	1	0.01	1.01	-0.6321	-0.6458	0.5386	85.66%
1	0.5386	0.0054	0.5440	0.0450	0.0364	0.5669	4.98%
2	0.5669	0.0057	0.5726	0.0004	-0.0085	0.5672	0.045%

$\therefore$ 0.5672 is the root of this function. Ans

Given, $f(x) = x^{10} - 1 = 0$, $x_0 = 0$, 1 %, 4 digit

Find the root using Newton-Raphson method.

solution: Given,

$$f(x) = x^{10} - 1$$

$$\therefore f'(x) = 10x^9$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$\text{error} = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100\ \%$$

iteration	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}	error
0	0	-1	0	0	

$$x_{i+1} = 0 - \frac{-1}{0}$$

Ans

Multiple Equation Solve

Gauss - Seidal Method:

Ex-1:

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \quad \text{—— (i)}$$

$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3 \quad \text{—— (ii)}$$

$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4 \quad \text{—— (iii)}$$

where initial value of $x_1 = x_2 = x_3 = 0$, error = 1 %. Find solution of this equation.

Solve:

From (i) $$x_1 = \frac{7.85 + 0.1x_2 + 0.2x_3}{3} \quad \text{—— (iv)}$$

From (ii) $$x_2 = \frac{-19.3 - 0.1x_1 + 0.3x_3}{7} \quad \text{—— (v)}$$

From (iii) $$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10} \quad \text{—— (vi)}$$

Step 1: $x_1 = x_2 = x_3 = 0$

$$x_1 = \frac{7.85 + 0 + 0}{3} = 2.6167$$

$$x_2 = \frac{-19.3 - 0.1(2.6167) + 0}{7} = -2.7945$$

$$x_3 = \frac{71.4 - 0.3(2.6167) + 0.2(-2.7945)}{10} = 7.0056$$

error,

$$x_1 = \left| \frac{2.9906 - 2.6167}{2.9906} \right| \times 100\% = 12.5\%$$

$$x_2 = \left| \frac{-2.4996 + 2.7945}{-2.4996} \right| \times 100\% = 11.8\%$$

$$x_3 = \left| \frac{7.0003 - 7.0056}{7.0003} \right| \times 100\% = 0.08\%$$

Step 3: $x_1 = 2.9906$, $x_2 = -2.4996$, $x_3 = 7.0003$

$$x_1 = \frac{7.85 + 0.1(-2.4996) + 0.2(7.0003)}{3} = 3$$

$$x_2 = \frac{-19.3 - 0.1(3) + 0.3(7.0003)}{7} = -2.50$$

$$x_3 = \frac{71.4 - 0.3(3) + 0.2(-2.50)}{10} = 7$$

Previous $[x_1 \quad x_2 \quad x_3] = [2.9906 \quad -2.4996 \quad 7.0003]$

present $[x_1 \quad x_2 \quad x_3] = [3 \quad -2.50 \quad 7]$

$\therefore$ error,

$$x_1 = \left| \frac{3 - 2.9906}{3} \right| \times 100\% = 0.31\%$$

$$x_2 = \left| \frac{-2.50 + 2.4996}{-2.50} \right| \times 100\% = 0.016\%$$

$$x_3 = \left| \frac{7 - 7.0003}{7} \right| \times 100\% = 0.0043\%$$

$< 1\%$

Now the solution is

$$x_1 = 3, \quad x_2 = -2.50, \quad x_3 = 7$$

Ans

Gauss - Elimination method:

(I) Each step we vanish one parameter until we have one parameter (forward elimination)

(II) Finally we putting values in previous eqn and determined all solution of this eqn (back substitution).

Ex-1:

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \quad \text{——— (i)}$$

$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3 \quad \text{——— (ii)}$$

$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4 \quad \text{——— (iii)}$$

solve:

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \quad \text{——— (iv)}$$

$$-210.1x_2 + 8.8x_3 = 586.85 \quad \text{——— (v)} \left[(i) - (ii) \times 30\right]$$

$$1.9x_2 - 100.2x_3 = -706.15 \quad \text{——— (vi)} \left[(i) - (iii) \times 10\right]$$

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \quad \text{——— (vii)}$$

$$-210.1x_2 + 8.8x_3 = 586.85 \quad \text{——— (viii)}$$

$$\cancel{-21035.3x_3} \; -21035.3x_3 = -147247.1 \quad \text{——— (ix)} \left[(v) \times 1.9 + (vi) \times -210.1\right]$$

From eqn (ix), we get

$$x_3 = \frac{-147247.1}{-21035.3} = 7$$

Now putting the value of x_3 in eqn (viii), we get

$$-210.1x_2 + 8.8 \times 7 = 586.85$$

$$\Rightarrow \quad x_2 = \frac{586.85 - (8.8 \times 7)}{-210.1}$$

$$= -2.5$$

Again, putting the value of x_2, x_3 in eqn (vii), we get

$$3x_1 - 0.1(-2.5) - 0.2 \times 7 = 7.85$$

$$\Rightarrow \quad x_1 = \frac{7.85 + (0.1)(-2.5)) + (0.2 \times 7)}{3}$$

$$= 3$$

$\therefore$ now the solution of this eqn is,

$$x_1 = 3$$

$$x_2 = -2.5$$

$$x_3 = 7$$

Ans

Gauss – Jordan method: A system with the same solution as the original one if

(i) $R_i \leftrightarrow R_j$ means: interchange row i and row j.

(ii) αR_i means: Replace row i with α times row i.

(iii) $R_i + \alpha R_j$ means: Replace row i with the sum of row i and α times row j.

To solve Linear equations described in the following steps:

step 1: write the ~~aug~~ augmented matrix of the system.

step 2: use row operations to transform the augmented matrix in the form described below which is called reduced row echelon form (RREF).

(i) The rows (if any) consisting entirely of zeroes are grouped together at the bottom of the matrix.

(ii) In each row that does not consist entirely of zeros, the leftmost non-zero element is a 1 (called leading 1 or pivot)

(iii) Each column that contains a leading 1 has zeros in all other entries.

(iv) The leading 1 in any row is to the left of any leading 1's in the rows below it.

step 3: Stop process in step 2 if we obtain a row whose elements are all zeros except the last one on the right. In that case, the system is inconsistent and has no solutions. Otherwise, finish step 2 and read the solutions of the system from the final matrix.

Ex-1: Solve the following system by using the Gauss-Jordan Elimination method/Gauss jordan method:

$$x + y + z = 5$$
$$2x + 3y + 5z = 8$$
$$4x + 5z = 2$$

Solve: The augmented matrix of the system is:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

we will now perform row operations until we obtain a matrix in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow{R_3 - 4R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right] \xrightarrow{R_3 + 4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right] \xrightarrow{\frac{R_3}{13}}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_2 - 3R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_1 - R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

From this final matrix, we can read the solution of the system.

It is

$$x = 3$$
$$y = 4$$
$$z = -2$$

Ans

Curve fitting:

The process of finding the equation of the curve of best fit, which may be most suitable for predicting the unknown values, is known as curve fitting.

(i) Linear Regression:

A straight line, $y = mx + c$

$$m = \frac{\overset{\text{number of point}}{n} \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2}$$

$$c = \bar{y} - m\bar{x}$$

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Ex-1: Fit a straight line to the following set of data

x_i	1	2	3	4	5	6	7
y_i	0.5	2.5	2	4	3.5	6.0	5.5

Ans:

The various summations are given as follows:

x_i	y_i	$x_i y_i$	x_i^2
1	0.5	0.5	1
2	2.5	5	4
3	2	6	9
4	4	16	16
5	3.5	17.5	25
6	6.0	36	36
7	5.5	38.5	49
$\Sigma x_i = 28$	$\Sigma y_i = 24$	$\Sigma x_i y_i = 119.5$	$\Sigma x_i^2 = 140$

$$m = \frac{n \Sigma x_i y_i - \Sigma x_i \Sigma y_i}{n \Sigma x_i^2 - (\Sigma x_i)^2}$$

$$= \frac{(7 \times 119.5) - (28 \times 24)}{(7 \times 140) - (28)^2}$$

$$= 0.8393$$

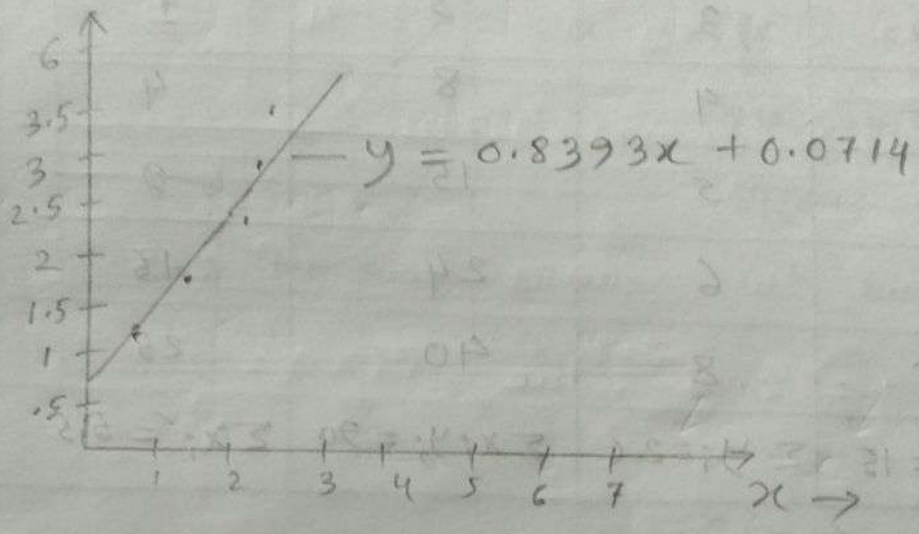
$$c = \bar{y} - m\bar{x}$$

$$= \frac{24}{7} - 0.8393 \times \frac{28}{7}$$

$$= 0.0714$$

Therefore, the linear equation is

$$y = 0.8393x + 0.0714$$ Ans



Ex-2: Fit a straight line to the following set of data

x	1	2	3	4	5
y	3	4	5	6	8

Ans: The various summations are given as follows:

x_i	y_i	$x_i y_i$	x_i^2
1	3	3	1
2	4	8	4
3	5	15	9
4	6	24	16
5	8	40	25
$\Sigma x_i = 15$	$\Sigma y_i = 26$	$\Sigma x_i y_i = 90$	$\Sigma x_i^2 = 55$

$$\therefore m = \frac{n \Sigma x_i y_i - \Sigma x_i \Sigma y_i}{n \Sigma x_i^2 - (\Sigma x_i)^2}$$

$$= \frac{(5 \times 90) - (15 \times 26)}{(5 \times 55) - (15)^2}$$

$$= 1.20$$

$$\therefore c = \bar{y} - m\bar{x}$$

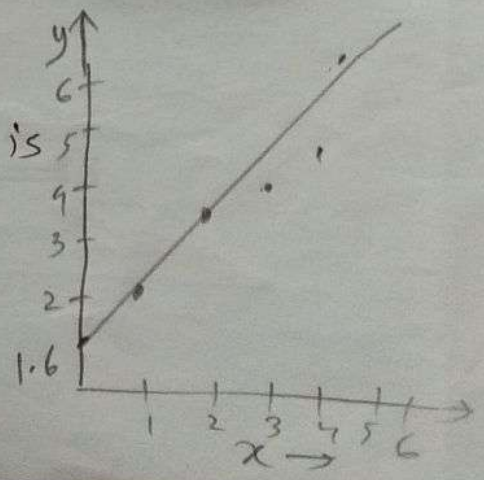
$$= \frac{26}{5} - \left(1.20 \times \frac{15}{5}\right)$$

$$= 1.60$$

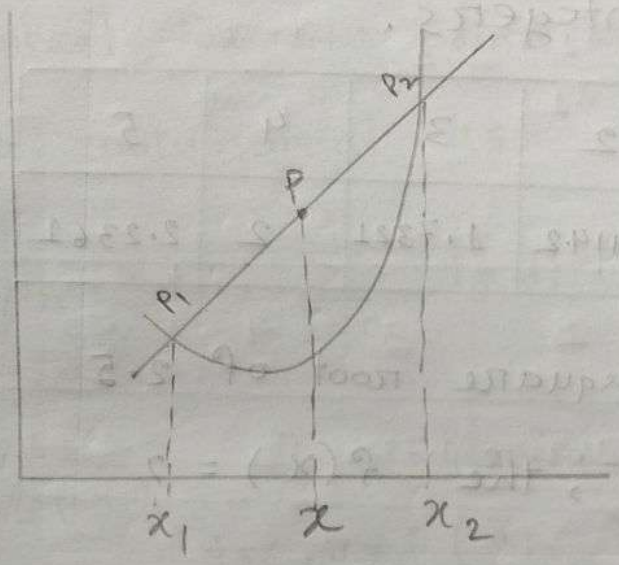
$\therefore$ Therefore, the linear eqn is

$$y = 1.2x + 1.6$$

Ans



(ii) Linear interpolation:



P_1 থেকে P_2 এর ভিতরের যেকোন point এই সূত্র দিয়ে বের করা যাবে।

$$* \; f(x) = f(x_1) + (x - x_1)\frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{—} (1)$$

this eqn is known as linear interpolation eqn. Here $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$ represents the slope of the line. Further, note the similarity of eqn (1) with Newton form of polynomial of first order

$$c_1 = x_1$$

$$a_0 = f(x_1)$$

$$a_1 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The coefficient a_1 represent the first derivative of the function.

Ex-1 : The table below gives square roots for integers.

x	1	2	3	4	5
$f(x)$	1	1.4142	1.7321	2	2.2361

Determine the square root of 2.5

or, $x = 2.5$, the $f(x) = ?$

solve:

The given value of 2.5 lies between the points 2 and 3. Therefore,

$x_1 = 2$, $f(x_1) = 1.4142$

$x_2 = 3$, $f(x_2) = 1.7321$

Then

$$f(2.5) = 1.4142 + (2.5 - 2)\frac{1.7321 - 1.4142}{3 - 2}$$

$$= 1.5732$$

Ans

(iii) Lagrange interpolation method:

In general, for $n+1$ points we have n^{th} degree polynomial as

$$P_n(x) = \sum_{i=0}^{n} f_i \, l_i(x) \quad \text{where} \qquad \text{——— (1)}$$

$$l_i(x) = \prod_{j=0,\, j\neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)} = \left(\frac{x-x_0}{x_i-x_0}\right)\left(\frac{x-x_1}{x_i-x_1}\right)\cdots \left(\frac{x_i-x_{i-1}}{x_i-x_{i-1}}\right)\left(\frac{x-x_{i+1}}{x_i-x_{i+1}}\right)\cdots\left(\frac{x-x_n}{x_i-x_n}\right)$$

Eqn (1) is called the Lagrange interpolation polynomial. The polynomials $l_i(x)$ are known as Lagrange basis polynomials.

Observe that

$$l_i(x_j) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

Now consider the case $n = 1$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1}$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0}$$

Therefore,

$$P_1(x) = f_0 \frac{x - x_1}{x_0 - x_1} + f_1 \frac{x - x_0}{x_1 - x_0}$$

1st order

Ex-1:

x	1	2	3	4	5
$f(x)$	1	1.4142	1.7321	2	2.2361

Find the square root of 2.5 using the second order Lagrange interpolation polynomial.

or, $x = 2.5$, $f(x) = ?$ using Lagrange

solve:

Let us consider the following three points:

$x_1 = 2$, $f_1 = 1.4142$

$x_2 = 3$, $f_2 = 1.7321$

$x_3 = 4$, $f_3 = 2$

for $x = 2.5$, we have

$$l_0(2.5) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{(2.5-3)(2.5-4)}{(2.0-3)(2-4)} = 0.3750$$

$$l_1(2.5) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} = \frac{(2.5-2)(2.5-4)}{(3-4)(3-2)} = 0.7500$$

$$l_2(2.5) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \frac{(2.5-2)(2.5-3)}{(4-2)(4-3)} = -0.125$$

$$\therefore P_2(2.5) = 0.3750 \times 1.4142 + 0.7500 \times 1.7321$$

2nd order

$$+ 2 \times (-0.125)$$

$$= 0.5303 + 1.2991 - 0.250$$

$$= 1.5794$$

Ans

2nd order

$$\frac{x-x_1}{x_n-x_1}\cdot\frac{x-x_2}{x_n-x_2}\cdot\frac{x-x_3}{x_n-x_3}\cdot\frac{x-x_4}{x_n-x_4}\cdot\frac{x-x_5}{x_n-x_5}\cdots$$

((1) বাদ দিয়ে l_1 (2) বাদ দিয়ে l_2 (3) বাদ দিয়ে l_3)

Ex-2 :

	x_1	x_2	x_3	x_4
i	0	1	2	3
x_i	0	1	2	3
$e^{x_i}-1$	0	1.7183	6.3891	19.0855

Find the Lagrange interpolation polynomial to fit the following data. Use the polynomial to estimate the value of $e^{1.5}$.

solve:

$$P(x) = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} f(x_1) +$$

$$\frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} f(x_2) +$$

$$\frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} f(x_3) + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} f(x_4)$$

$$= \frac{(x-1)(x-2)(x-3)}{(0-1)(0-1)(0-3)} \times 0 + \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} \times 1.7183$$

$$+ \frac{(x-0)(x-2)(x-3)}{(2-0)(2-1)(2-3)} \times 6.3891 + \frac{(x-0)(x-2)(x-3)}{(3-0)(3-1)(3-2)} \times 19.0855$$

$$\cancel{= 0 + \frac{x^3-6x^2+11x-6}{-3} +}$$

$$= 0 + \frac{x^3-5x^2+6x}{2} \times 1.7183 + \frac{x^3-4x^2+3x}{-2} \times 6.3891$$

$$+ \frac{x^3-3x^2+2x}{6} \times 19.0855$$

$$\therefore P(1.5) = \frac{(1.5)^3-5(1.5)^2+6(1.5)}{2} \times 1.7183 + \frac{(1.5)^3-4(1.5)^2+3(1.5)}{-2} \times 6.3891$$

$$+ \frac{(1.5)^3-3(1.5)^2+2(1.5)}{6} \times 19.0855$$

$$= -14.4982 + 32.3448 - 20.2783$$

$$= \boxed{-2.4317}$$

#

		x_1	x_2	x_3
x_i	0	1	2	3
$e^{x_i}-1$	0	1.7183	6.3891	19.0855
$p(x)$ e^{x_i}	1	2.7183	7.3891	20.0855

use lagrange find $e^{1.5}$? / $x = 1.5$, $e^{1.5} = ?$

$$P_2(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} f(x_1) + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} f(x_2)$$

$$+ \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$P_2(1.5) = \frac{(1.5-2)(1.5-3)}{(1-2)(1-3)} \times 2.7183 + \frac{(1.5-1)(1.5-3)}{(2-1)(2-3)} \times 7.3891$$

$$+ \frac{(1.5-1)(1.5-2)}{(3-1)(3-2)} \times 20.0855$$

$$= 1.0194 + 5.5418 - 2.5107$$

$$= \boxed{4.0505}$$

Ans

ঠিক আসে পরীক্ষায় 4.36 আসে

(iv) Newton interpolation polynomial:

The Newton form of polynomial is

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots$$
$$\cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

where the interpolation point $x_0, x_1 \cdots x_{n-1}$ act as centres.

Here,

$$a_0 = f[x_0]$$

$$a_1 = f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$a_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$\therefore f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Here, a_1 represents the first divided difference and a_2 the second divided difference and so on.

Ex-1: Estimate $\log 2.5$ using second order Newton interpolation polynomial.

i	0	1	2	3
x_i	x_0 1	x_1 2	x_2 3	4
$\log x_i$	0	0.3010	0.4771	0.6021

Solve: Second order polynomials require only three data points. We use the first three points.

$$a_0 = f[x_0] = 0$$

$$a_1 = f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{0.3010 - 0}{2 - 1} = 0.3010$$

b c b a ab

$$a_2 = f[\overset{a}{x_0}, \overset{b}{x_1}, \overset{c}{x_2}] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{0.4771 - 0.3010}{3 - 2} = 0.1761$$

$$\therefore a_2 = \frac{0.1761 - 0.3010}{3 - 1} = -0.06245$$

$$f[a, b, c] = \frac{f[b, c] - f[a, b]}{c - a}$$

$$\therefore P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

$$= 0 + 0.3010(x-1) + (-0.06245)(x-1)(x-2)$$

$$\therefore P_2(2.5) = 0.3010 \times (2.5-1) + (-0.06245)(2.5-1)(2.5-2)$$

$$= 0.4515 - 0.0468$$

$$= 0.4047$$ → ঠিক হচ্ছে (করেছ ans ও 0.4047)

Ans

Another try :

Here we use

$x_1 = 2$; $f(x_1) = 0.3010$

$x_2 = 3$; $f(x_2) = 0.4771$

$x_3 = 4$; $f(x_3) = 0.6021$

সিরিয়াল but মধ্যে (মধ্যে)

$$a_1 = f[x_1] = 0.3010$$

$$a_2 = f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{0.4771 - 0.3010}{3 - 2}$$

$$= 0.1761$$

$$a_3 = f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$$

$$f[x_2, x_3] = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{0.6021 - 0.4771}{4 - 3} = 0.125$$

$$\therefore a_3 = \frac{0.125 - 0.1761}{4 - 2} = -0.0256$$

$$\therefore P_2(x) = a_1 + a_2(x - x_1) + a_3(x - x_1)(x - x_2)$$

$$\therefore P_2(2.5) = 0.3010 + 0.1761(2.5 - 2) + (-0.0256)(2.5 - 2)(2.5 - 3)$$

$$= 0.3010 + 0.08805 + 0.0064$$

$$= 0.39545$$

Ans

Polynomial problem

Divided difference table:

(i) Backward difference:

$$P_n(s) = f_n + s\,\nabla f_n + \frac{s(s+1)}{2!}\nabla^2 f_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 f_n + \frac{s(s+1)(s+2)(s+3)}{4!}\nabla^4 f_n + \cdots\cdots$$

$$s = \frac{x - x_n}{h}$$

x = Find x

x_n = last value of x

h = the difference of x / step size

for back forward → ∇ (gradient)

Ex-1: Given some values →

θ	10	20	30	40	50
$\sin\theta$	0.1736	0.3420	0.5000	0.6428	0.7660

Find the value of $\sin 25$ using backward & forward difference?

OR, $\theta = 25$, $\sin\theta = ?$

Solve:

For Backward difference:

	θ	$\sin\theta$	∇f_n	$\nabla^2 f_n$	$\nabla^3 f_n$	$\nabla^4 f_n$
x_0	10	0.1736				
			0.1684			
	20	0.3420		-0.0104		
			0.1580		-0.0048	
	30	0.5000		-0.0152		-0.0004
			0.1428		-0.0044	
	40	0.6428		-0.0196		
			0.1232			
x_n	50	0.7660 $\downarrow f_n$				

Here, $x_n = \theta_n = 50$ Given $x = \theta = 25$

$h = (20 - 10) = 10$

$$\therefore s = \frac{x - x_n}{h} = \frac{25 - 50}{10} = -2.5$$

$$\therefore P_4(s) = f_n + s\nabla f_n + \frac{s(s+1)}{2!}\nabla^2 f_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 f_n + \frac{s(s+1)(s+2)(s+3)}{4!}\nabla^4 f_n$$

$$\therefore P_4(-2.5) = 0.7660 + (-2.5)(0.1232)$$

$$+ \frac{(-2.5)(-1.5)(-0.0196)}{2} + \frac{(-2.5)(-1.5)(-0.5)(0.0044)}{6}$$

$$= + \frac{(-2.5)(-1.5)(-0.5)(0.5)(-0.0004)}{24}$$

$$= 0.7660 - 0.308 - 0.0368$$

$$- 0.0014 + 0.00002$$

$$= 0.76601 - 0.3462$$

$$= 0.41981$$

$$= 0.4200$$

For forward difference: $s = \frac{x - x_0}{h}$

$$P_n(s) = f_0 + s\,\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0$$

$$+ \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0 + \cdots\cdots$$

For Forward difference:

θ	$\sin\theta$	Δf_n	$\Delta^2 f_n$	$\Delta^3 f_n$	$\Delta^4 f_n$
10	0.1736 f_0				
		0.1684			
20	0.3420		-0.0104		
		0.1580		0.0048	
30	0.5000		-0.0152		-0.0004
		0.1428		0.0044	
40	0.6428		-0.0196		
		0.1232			
50	0.7660				

Here,

$x_0 = \theta_0 = 10$

$h = 10$

Given,

$x = \theta = 25$

$$\therefore s = \frac{x - x_0}{h} = \frac{25-10}{10} = 1.5$$

$$\therefore P_4(s) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0 + \frac{s(s-1)(s-2)(s-3)}{4!}\Delta^4 f_0$$

$$\therefore P_4(1.5) = 0.1736 + (1.5)(0.1684) + \frac{(1.5)(0.5)(-0.0104)}{2} + \frac{1.5(0.5)(-0.5)(0.0048)}{6} + \frac{(1.5)(0.5)(-0.5)(-1.5)(-0.0004)}{24}$$

$= 0.1736 + 0.2526 - 0.0039 - 0.0003$

$- 0.0000094$

$= 0.4262 - 0.00421$

$= 0.4219906$

$= 0.4220$

Ans

Example: Given that,

$y(0) = 1$, $y(1) = 0$, $y(2) = 1$, $y(3) = 10$,

$y(4) = ?$

solve: forward difference:

x	y	Δf_n	$\Delta^2 f_n$	$\Delta^3 f_n$
0	$1 \to y_0$			
		-1		
1	0		2	
		1		6
2	1		8	
		9		
3	$10 \to y_n$			

using $h = 1 - 0 = 1$

~~backward~~, forward, $s = \frac{x - x_0}{h} = \frac{4 - 0}{1} = 4$

We know,

$$y(x) = y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0$$

$$\therefore y(4) = 1 + 4(-1) + \frac{4(4-1)}{2!}\cdot 2 + \frac{4(4-1)(4-2)}{3!}\,6$$

$$= 1 - 4 + 12 + 24$$

$$= 33$$ Ans

For Backward difference:

$$h = (3-2) = 1$$

$$s = \frac{x - x_n}{h} = \frac{4-3}{1} = 1$$

We know,

$$y(x) = y_n + s\nabla y_n + \frac{s(s+1)}{2!}\nabla^2 y_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 f_n$$

$$\therefore y(4) = 10 + 1\times 9 + \frac{1\times 2}{2}\times 8 + \frac{1\times 2\times 3}{6}\cdot 6$$

$$= 10 + 9 + 8 + 6$$

$$= 33$$

Ans

Example: Given some value,

x	1891	1901	1911	1921	1931
y	46	66	81	93	101

Find the value of 1895, using forward and backward difference.

Solve: For forward difference:

x	y	Δy_n	$\Delta^2 y_n$	$\Delta^3 y_n$	$\Delta^4 y_n$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

Here, $h = (1901 - 1891) = 10$

$$s = \frac{x - x_0}{h} = \frac{1895 - 1891}{10} = \frac{4}{10} = 0.4$$

$$y(x) = y_0 + s\,\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0$$

$$+ \frac{s(s-1)(s-2)(s-3)}{4!}\Delta^4 y_0$$

$$\therefore y(1895) = 46 + (0.4)(20) + \frac{(0.4)(0.4-1)}{2!}(-5) +$$

$$\frac{0.4(0.4-1)(0.4-2)}{3!}(2) + \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{4!}(-3)$$

$$= 46 + 8 + 0.6 + 0.128 + 0.1248$$

$$= 54.8528$$ Ans

For backward difference:

$$h = 1931 - 1921 = 10$$

$$s = \frac{x - x_n}{h} = \frac{1895 - 1931}{10} = -3.6$$

$$y(x) = y_n + s\,\nabla y_n + \frac{s(s+1)}{2!}\nabla^2 y_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 y_n$$

$$+ \frac{s(s+1)(s+2)(s+3)}{4!}\nabla^4 y_n$$

$$y(1895) = 101 + (-3.6)8 + \frac{(-3.6)(-3.6+1)}{2!}(-4) +$$

$$\frac{(-3.6)(-3.6+1)(-3.6+2)}{3!}(-1) + \frac{(-3.6)(-3.6+1)(-3.6+2)(-3.6+3)}{24}(-3)$$

$$= 101 - 28.8 - 18.72 + 2.496 - 1.1232$$

$$= 54.8528$$ Ans

Numerical Differentiation:

(i) Forward difference:

- $$f'(x) = \frac{f(x+h) - f(x)}{h}$$

(ii) Backward difference:

- $$f'(x) = \frac{f(x) - f(x-h)}{h}$$

(iii) central difference:

- $$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

348 ন: 11.1 # $f(x) = x^2$ at $x = 1$. Find the value of $f'(x)$ when $h = 0.2, 0.1, 0.01$.

Solve: For Forward,:

Given,

$$f(x) = x^2$$

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

↳ when $h = 0.2$, then

$$f'(1) = \frac{f(1+0.2) - f(1)}{0.2}$$

$$= \frac{1.44 - 1}{0.2}$$

$$= 2.2$$

$f(x) = x^2$

$f'(x) = 2x$

$f'(1) = 2 \cdot 1$
$= 2$

$$\therefore \text{Error} = \left| \frac{2.2 - 2}{2.2} \right| \times 100\%$$

$$= 9.1\ \%$$

↳ when $h = 0.1$, then

$$f'(1) = \frac{f(1+0.1) - f(1)}{0.1}$$

$$= \frac{(1.1)^2 - 1^2}{0.1}$$

$$= 2.1$$

$$\therefore \text{Error} = \left| \frac{2.1 - 2}{2.1} \right| \times 100\ \%$$

$$= 4.76\ \%$$

↳ when $h = 0.01$, then

$$f'(1) = \frac{f(1+0.01) - f(1)}{0.01}$$

$$= \frac{(1.01)^2 - 1^2}{0.01}$$

$$= 2.01$$

$$\therefore \text{Error} = \left| \frac{2.01 - 2}{2.01} \right| \times 100\ \%$$

$$= 0.5\ \%$$

যেহেতু error কম, সুতরাং accurate বেশি।

Note that the correct answer is 2. The derivative approximation approches the exact value as h decreases.

Ex : Estimate the approximate derivative of $f(x) = x^2$ at $x = 1$ for $h = 0.2, 0.1,$ $\boxed{0.05}$ and 0.01 using the first order backward difference formula.

Solve:

For Backward:

$$f'(x) = \frac{f(x) - f(x-h)}{h}$$

Given,

$$f(x) = x^2$$

$$f'(x) = 2x$$

$$f'(1) = 2 \times 1 = 2$$

when $h = 0.2$, then

$$f'(1) = \frac{f(1) - f(1-0.2)}{0.2}$$

$$= \frac{1^2 - (0.8)^2}{0.2}$$

$$= 1.8$$

$$\therefore \text{Error} = \left| \frac{1.8 - 2}{1.8} \right| \times 100\%$$

$$= 11.11\%$$

↳ when $h = 0.1$, then

$$f'(1) = \frac{f(1) - f(1-0.1)}{0.1}$$

$$= \frac{1^2 - (0.9)^2}{0.1}$$

$$= 1.9$$

$$\text{Error} = \left| \frac{1.9 - 2}{1.9} \right| \times 100\ \%$$

$$= 5.26\ \%$$

↳ when $h = 0.01$, then

$$f'(1) = \frac{f(1) - f(1-0.01)}{0.01}$$

$$= \frac{1^2 - (0.99)^2}{0.01}$$

$$= 1.99$$

$$\therefore \text{Error} = \left| \frac{1.99 - 2}{1.99} \right| \times 100\ \%$$

$$= 0.5\ \%$$

Ex: Estimate approximate derivative of $f(x) = x^2$ at $x = 1$ for $h = 0.2, 0.1$ & 0.01 using the first order central difference formula.

Solve: For central:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

Given, $f(x) = x^2$

$$f'(x) = 2x$$

$$f'(1) = 2 \cdot 1 = 2$$

↳ when $h = 0.2$, then

$$f'(1) = \frac{f(1+0.2) - f(1-0.2)}{2 \times 0.2}$$

$$= \frac{(1.2)^2 - (0.8)^2}{0.4}$$

$$= 2$$

$$\text{Error} = \left| \frac{2-2}{2} \right| \times 100\ \%$$

$$= 0\ \%$$

↳ when $h = 0.1$, then

$$f'(1) = \frac{f(1+0.1) - f(1-0.1)}{2 \times 0.1}$$

$$= \frac{(1.1)^2 - (0.9)^2}{0.2}$$

$$= 2$$

$$\therefore \text{ERROR} = \left| \frac{2-2}{2} \right| \times 100\ \%$$

$$= 0\ \%$$

↳ when $h = 0.01$, then

$$f'(1) = \frac{f(1+0.01) - f(1-0.01)}{2 \times 0.01}$$

$$= \frac{(1.01)^2 - (0.99)^2}{0.02}$$

$$= 2$$

$$\therefore \text{ERROR} = \left| \frac{2-2}{2} \right| \times 100\ \%$$

$$= 0\ \%$$

# #

for Forward:

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h}$$

for backward:

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h}$$

for central:

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$$

Ex: $f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$ with initial $x = 0.5$ & $h = 0.5$

Solve: Given that,

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

$$f'(x) = -0.4x^3 - 0.45x^2 - x - 0.25$$

$x_{i-1} = 0$

$x_i = 0.5$

$x_{i+1} = 1.0$

$f(x_{i-1}) = 1.2$

$f(x_i) = 0.925$ $\therefore f'(x_i) = -0.9125$

$f(x_{i+1}) = 0.2$

For forward,

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h}$$

$$= \frac{0.2 - 0.925}{0.5}$$

$$= -1.45$$

$$\text{error} = \left| \frac{-0.9125 + 1.45}{-0.9125} \right| \times 100\ \%$$

$$= 58.9\ \%$$

For Backward,

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h}$$

$$= \frac{0.925 - 1.2}{0.5}$$

$$= -0.55$$

$$\text{error} = \left| \frac{-0.9125 + 0.55}{-0.9125} \right| \times 100\ \%$$

$$= 39.73\ \%$$

For central,

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$$

$$= \frac{0.2 - 1.2}{1}$$

$$= -1$$

$$\text{error} = \left| \frac{-0.9125 + 1}{-0.9125} \right| \times 100\ \%$$

$$= 9.58\ \%$$

Ex: $f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$

with initial $x = 0.5$ and $h = 0.25$

Solve: Given that,

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

$$f'(x) = -0.4x^3 - 0.45x^2 - x - 0.25$$

$$x_{i-1} = 0.25$$

$$x_i = 0.5$$

$$x_{i+1} = 0.75$$

$$f(x_{i-1}) = 1.104$$

$$f(x_i) = 0.925 \qquad f'(x_i) = -0.9125$$

$$f(x_{i+1}) = 0.636$$

For forward,

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h}$$

$$= \frac{0.636 - 0.925}{0.25} = -1.156$$

$$\text{error} = \left| \frac{-0.9125 + 1.156}{-0.9125} \right| \times 100\ \%$$

$$= 26.68\ \%$$

For backward,

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h}$$

$$= \frac{0.925 - 1.104}{0.25}$$

$$= -0.716$$

$$\text{error} = \left| \frac{-0.9125 + 0.716}{-0.9125} \right| \times 100\ \%$$

$$= 21.53\ \%$$

For central

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$$

$$= \frac{0.636 - 1.104}{0.5}$$

$$= -0.936$$

$$\text{error} = \left| \frac{-0.9125 + 0.936}{-0.9125} \right| \times 100\ \%$$

$$= 2.57\ \%$$

For forward

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2}$$

For backward

$$f''(x_i) = \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2}$$

for central

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$$

Ex: $f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$ with initial $x = 0.5$ & $h = 0.25$

Solve: Given that

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

$$f'(x) = -0.4x^3 - 0.45x^2 - x - 0.25$$

$$f''(x) = -1.2x^2 - 0.90x - 1$$

$$\therefore f''(0.5) = -1.75$$

$x_{i-2} = 0$

$x_{i-1} = 0.25$

$x_i = 0.5$

$x_{i+1} = 0.75$

$x_{i+2} = 1.00$

$\therefore f(x_{i-2}) = 1.2$

$\therefore f(x_{i-1}) = 1.104$

$f(x_i) = 0.925$

$f(x_{i+1}) = 0.636$

$f(x_{i+2}) = 0.2$

for forward,

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2}$$

$$= \frac{0.2 - 2 * 0.636 + 0.925}{(0.25)^2}$$

$$= -2.352$$

$$\text{error} = \left| \frac{-1.75 + 2.352}{-1.75} \right| \times 100\%$$

$$= 34.4\%$$

for backward,

$$f''(x_i) = \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2}$$

$$= \frac{0.925 - 2 * 1.104 + 1.2}{(0.25)^2}$$

$$= -1.328$$

$$\text{error} = \left| \frac{-1.75 + 1.328}{-1.75} \right| \times 100\ \%$$

$$= 24.11\ \%$$

for central,

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$$

$$= \frac{0.636 - 2 * 0.925 + 1.104}{(0.25)^2}$$

$$= -1.76$$

$$\text{error} = \left| \frac{-1.75 + 1.76}{-1.75} \right| \times 100\ \%$$

$$= 0.57\ \%$$

Richardson Extrapolation

যেকোন eqn এর জন্য Richardson formula,

$$f'(x) = \frac{D(rh) - r^2 D(h)}{1 - r^2}$$

Here, r = Richardson co-efficient

$$D(h) = \frac{f(x+h) - f(x-h)}{2h}$$ [From central difference]

$$D(rh) = \frac{f(x+rh) - f(x-rh)}{2hr}$$

Given

x	-0.5	-0.25	0	0.25	0.5	0.75	1.0	1.25	1.5
$f(x) = e^x$	0.6065	0.7788	1.0000	1.2840	1.6487	2.1170	2.7183	3.4903	4.4817

Find the value of $f'(x_i)$ at $x = 0.5$ using Richardson extrapolation with $h = 0.5$ and $r = 1/2$.

Solve: Given,

$x = 0.5$

$r = 0.5$

$h = 0.5$

$$D(0.5) = \frac{f(0.5+0.5) - f(0.5-0.5)}{2 \times 0.5}$$

$$= \frac{f(1) - f(0)}{1}$$

$$= \frac{2.7183 - 1.0000}{1}$$

$$= 1.7183$$

$$D(0.5 \times 0.5) = \frac{f(0.5+0.25) - f(0.5-0.25)}{2 \times 0.25}$$

$$= \frac{2.1170 - 1.2840}{0.50}$$

$$= 1.666$$

$$\therefore f'(x) = \frac{D(rh) - r^2 D(h)}{1 - r^2}$$

$$\therefore f'(0.5) = \frac{1.666 - (0.5)^2 \times 1.7183}{1 - (0.5)^2}$$

$$= 1.6486$$ Ans

$$\boxed{\begin{aligned} f(x) &= e^x \\ &= e^{0.5} \qquad \therefore \text{error} = 0.0001 \\ &= 1.6487 \end{aligned}}$$

Numerical Integration

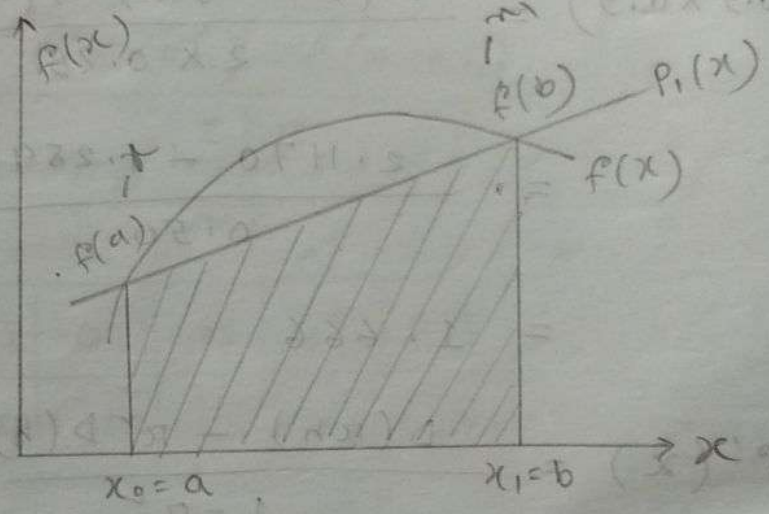
① Trapezoidal rule:

$$I = \int_a^b f(x)\, dx$$

$$I = \frac{(b-a)}{2}\left[f(a) + f(b)\right]$$

* Prove that,

Integral, I = width * average height



x_0 , x_1 $\cancel{x}/2$

$x - x_0$

$x_1 - x_0$

$$\therefore \frac{f_1(x) - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\Rightarrow f_1(x) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \cdot (x - x_0) + f(x_0)$$

$$\Rightarrow f_1(x) = \frac{f(b) - f(a)}{b - a} \cdot (x - a) + f(a)$$

putting,
$x_1 = b$
$x_0 = a$

$$\therefore I = \int_a^b f_1(x)\, dx$$

$$= \int_a^b \left[\frac{f(b)-f(a)}{b-a} \cdot (x-a) + f(a) \right] dx$$

$$= \int_a^b \left[\frac{x f(b) - x f(a) - a f(b) + \cancel{a f(a)} + b f(a) - \cancel{a f(a)}}{b-a} \right] dx$$

$$= \int_a^b \left[\frac{b f(a) - a f(b)}{b-a} + \frac{x\{f(b)-f(a)\}}{b-a} \right] dx$$

integrating

$$= \left[\frac{b f(a) - a f(b)}{b-a} x + \frac{f(b)-f(a)}{b-a} \cdot \frac{x^2}{2} \right]_a^b$$

$$= \frac{b f(a) - a f(b)}{\cancel{(b-a)}} \cdot \cancel{(b-a)} + \frac{f(b)-f(a)}{b-a} \cdot \left(\frac{b^2-a^2}{2} \right)$$

$$= b f(a) - a f(b) + \frac{b f(b) - b f(a) + a f(b) - a f(a)}{2}$$

$$= \frac{2 b f(a) - 2 a f(b) + b f(b) - b f(a) + a f(b) - a f(a)}{2}$$

$$= \frac{b f(a) - a f(b) + b f(b) - a f(a)}{2}$$

$$= \frac{f(a)(b-a) + f(b)(b-a)}{2}$$

$$= \frac{(b-a)\{f(a)+f(b)\}}{2}$$

$$= (b-a) * \frac{f(a)+f(b)}{2}$$

$$= \text{width} \cdot \text{average height}$$ [proved]

$\boxed{\text{Ex-12.1 :}}$

Evaluate the integral

$$I = \int_a^b (x^3+1)\,dx$$

for the intervals ⓐ $(1, 2)$ & ⓑ $(1, 1.5)$

Also estimate truncation error in each case and compare the results with the exact answer.

Ans: (a) Given,

$$a = 1$$
$$b = 2$$

$$\therefore I_{exact} = \int_1^2 \overbrace{(x^3+1)}^{f(x)}\,dx$$

$$= \left[\frac{x^4}{4} + x\right]_1^2$$

$$= \left(\frac{2^4}{4} + 2\right) - \left(\frac{1^4}{4} + 1\right)$$

$$= 4.75$$

$$I_{true} = \frac{(b-a)}{2}\left[f(a) + f(b)\right]$$

$$= \frac{2-1}{2}\left[f(1) + f(2)\right]$$

$$= \frac{1}{2}\left[2 + 9\right]$$

$$= 5.5$$

$$\therefore \text{ERROR} = \left|\frac{I_{true} - I_{exact}}{I_{true}}\right| \times 100\%$$

$$= \left|\frac{5.5 - 4.75}{5.5}\right| \times 100\%$$

$$= 13.64\%$$

(b) Given,

$a = 1$

$b = 1.5$

$$\therefore I_{exact} = \int_1^{1.5} (x^3 + 1)\, dx$$

$$= \left[\frac{x^4}{4} + x\right]_1^{1.5}$$

$$= \left\{ \frac{(1.5)^4}{4} + 1.5 \right\} - \left\{ \frac{1}{4} + 1 \right\}$$

$$= 1.5156$$

$$I_{true} = \frac{(b-a)}{2} \left[f(a) + f(b) \right]$$

$$= \frac{1.5-1}{2} \left[f(1) + f(1.5) \right]$$

$$= 0.25 \left[2 + 4.375 \right]$$

$$= 1.5938$$

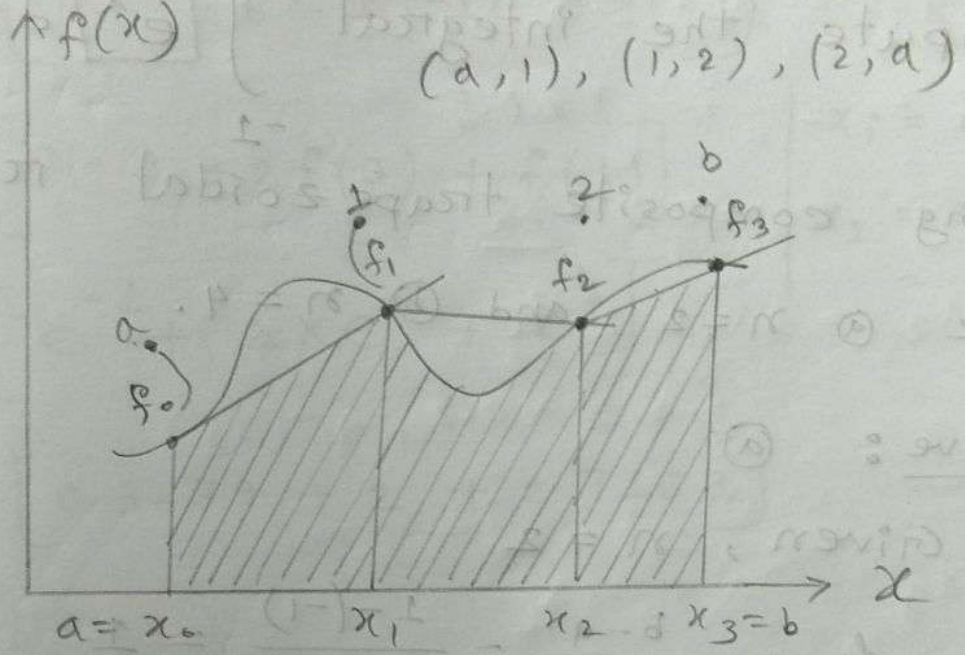
$$\text{Error} = \left| \frac{1.5938 - 1.5156}{1.5938} \right| \times 100\ \%$$

$$= 4.91\ \%$$

—

* যখন integration এর পরিমাণ কম, তখন error ও কম হবে।

Composite Trapezoidal rule:



$$h = \frac{b-a}{n}$$

$n = 2 \rightarrow$ 2টি point আছে

$n = 4 \rightarrow$ " 4টি " "

$$x_i = a + ih \quad , \quad i = 0, 1, \ldots n$$

$$\text{Integral}, \; I = \frac{h}{2}\left[f_0 + 2\sum_{i=1}^{n-1} f_i + f_n\right]$$

Ex: 12.2

compute the integral $\int_{-1}^{1} \boxed{e^x}\, dx$ $f(x)$

using composite trapezoidal rule

for ⓐ $n = 2$ and ⓑ $n = 4$.

Solve: ⓐ

Given, $n = 2$

$$h = \frac{b-a}{n} = \frac{1-(-1)}{2} = 1$$

$$I_{exact} = \int_{-1}^{1} e^x\, dx$$

$$= \left[e^x\right]_{-1}^{1}$$

$$= \left[e^1 - e^{-1}\right]$$

$$= 2.3504$$

f_i মানে $f(x_i)$
মানে
$f(a + ih)$

$x_i = a + ih$

$$I_{ct} = \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f_i + f(b)\right]$$

$$= \frac{1}{2}\left[f(-1) + 2\sum_{i=1}^{1} f_i + f(1)\right]$$

$$= \frac{1}{2}\left[f(-1) + 2f(x_1) + f(1)\right]$$

$$= \frac{1}{2}\left[f(-1) + 2f(0) + f(1)\right]$$

$x_i = a + ih$

$x_1 = -1 + 1 \cdot 1$
$= 0$

$$= \frac{1}{2}\left[e^{-1} + 2 \times e^{0} + e^{1}\right]$$

$$= \frac{1}{2}\left[0.3679 + 2 + 2.7183\right]$$

$$= 2.5431$$

$$\therefore \text{ERROR} = \left|\frac{I_{cr} - I_{exact}}{I_{cr}}\right| \times 100\ \%$$

$$= 7.58\ \%$$

(b) Given,

$n = 4$

$$h = \frac{b-a}{n} = \frac{1-(-1)}{4} = 0.5$$

$$\therefore I_{exact} = \int_{-1}^{1} e^{x}\, dx$$

$$= \left[e^{x}\right]_{-1}^{1}$$

$$= \left[e^{1} - e^{-1}\right] = 2.3504$$

I $x_1 = a + 1 \cdot h$

$= -1 + 1 \times 0.5$

$= -0.5$

$x_2 = a + 2 \cdot h$

$= -1 + 2 \times 0.5$

$= 0$

$x_3 = a + 3 \cdot h$

$= -1 + 3 \times 0.5$

$= 0.5$

$$\therefore I_{ct} = \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f_i + f(b)\right]$$

$$= \frac{0.5}{2}\left[f(-1) + 2\left\{f(x_1) + f(x_2) + f(x_3)\right\} + \cancel{f(b)}\ f(1)\right]$$

$$= 0.25\left[e^{-1} + 2\left\{f(-0.5) + f(0) + f(0.5)\right\} + e^{1}\right]$$

$$= 0.25\left[0.3679 + 2\left\{e^{-0.5} + e^{0} + e^{0.5}\right\} + 2.7183\right]$$

$$= 0.25\left[0.3679 + 6.5105 + 2.7183\right]$$

$$= 2.3992$$

$$\therefore \text{Error} = \left| \frac{2.3992 - 2.3504}{2.3992} \right| \times 100\%$$

$$= 2.03\%$$

* এখানে point যত বাড়বে error এর পরিমাণ তত কমবে।

Simpson's $\frac{1}{3}$ rule:

$$I = \frac{b-a}{6}\left[f(a) + 4f(x_i) + f(b)\right]$$

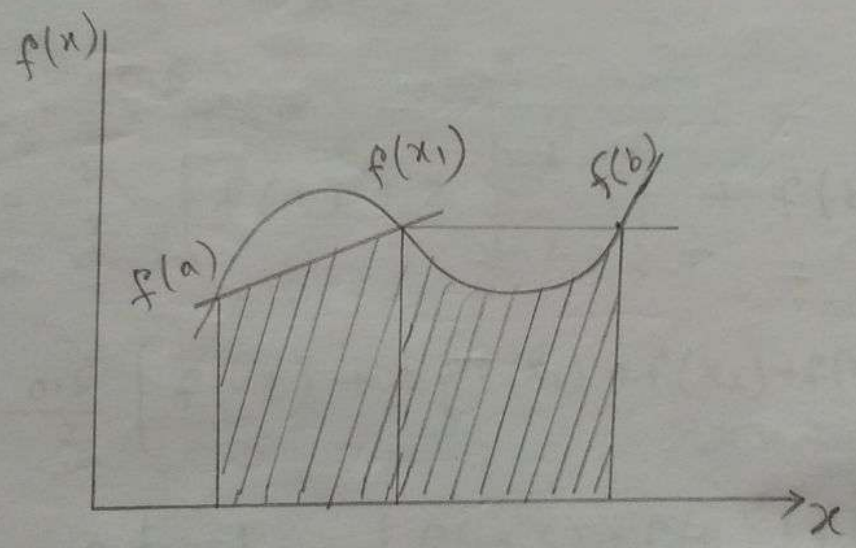
OR

$$I = \frac{h}{3}\left[f(a) + 4f(x_i) + f(b)\right]$$

বা $4f(x_1)$

$$h = \frac{b-a}{2}$$

$$x_i = a + ih$$



Ex - 12.3 : Evaluate the following integrals using simpson's $\frac{1}{3}$ rule.

sin (a) $\int_{-1}^{1} e^{x}\, dx$ (b) $\int_{0}^{\pi/2} \sqrt{\sin x}\, dx$

Solve: (a) $\int_{-1}^{1} e^{x}\, dx$

Given, $a = -1$

$b = 1$

$$h = \frac{b-a}{2} = \frac{1-(-1)}{2} = 1$$

$$I_{exact} = \int_{-1}^{1} e^{x}\, dx$$

$$= \left[e^{x}\right]_{-1}^{1}$$

$$= \left[e^{1} - e^{-1}\right]$$

$$= 2.3504$$

$$I_s = \frac{h}{3}\left[f(a) + 4f(x_1) + f(b)\right]$$

$$= \frac{1}{3}\left[f(-1) + 4f(0) + f(1)\right]$$

$$= \frac{1}{3}\left[e^{-1} + 4 \times e^{0} + e^{1}\right]$$

$$= \frac{1}{3} \times 7.0862$$

$$= 2.3621$$

$x_1 = \frac{a+b}{2}$

$= \frac{-1+1}{2}$

$= 0$

or

$x_1 = a + 1 \times h$

$= -1 + 1 \times 1$

$= 0$

$$\therefore \text{Error} = \left|\frac{I_s - I_{exact}}{I_s}\right| \times 100\%$$

$$= \left|\frac{2.3621 - 2.3504}{2.3621}\right| \times 100\%$$

$$= 0.5\ \%$$

(b) Given,

$$\int_0^{\pi} \sqrt{\sin x}\,dx = 2.39628$$

$a = 0$

$b = \pi/2$

$$h = \frac{b-a}{2} = \frac{\pi/2 - 0}{2} = \frac{\pi}{4} = 0.7854$$

$$I_{exact} = \int_0^{\pi/2} \sqrt{\sin(x)}\,dx = \pi/4 = \boxed{1.19816}99 = 1.17338$$

$$I_s = \frac{h}{3}\left[f(a) + 4f(x_1) + f(b)\right]$$

$$= \frac{\pi}{12}\left[f(0) + 4f\left(\frac{\pi}{4}\right) + f(\pi/2)\right]$$

$x_1 = a + ih$
$= 0 + 1 \times \frac{\pi}{4}$
$= \pi/4$

$$= 0.2618\,(0 + 3.3636 + 1)$$

$$= 1.14238$$

$$\therefore \text{Error} = \left|\frac{1.1424 - \pi/4}{1.1424}\right| \times 100\%$$

$$= 31.25\ \%$$

Simpson's $\frac{3}{8}$ rule:

Ex: Find the integral of the function

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$ where $a = 0$ & $b = 0.8$

Solve: We know that,

$$I_{exact} = \int_a^b f(x)\,dx$$

$$x_1 = \frac{b+a}{2} = \frac{0.8}{2} = 0.4$$

$$= \int_0^{0.8} f(x)\,dx$$

1.6405

$$= 1.640533 \rightarrow \text{exact value}$$

$$I_s = \frac{(b-a)}{6}\left[f(a) + 4f(x_1) + f(b)\right]$$

$$= \frac{(0.8-0)}{6}\left[f(0) + 4f(0.4) + f(0.8)\right]$$

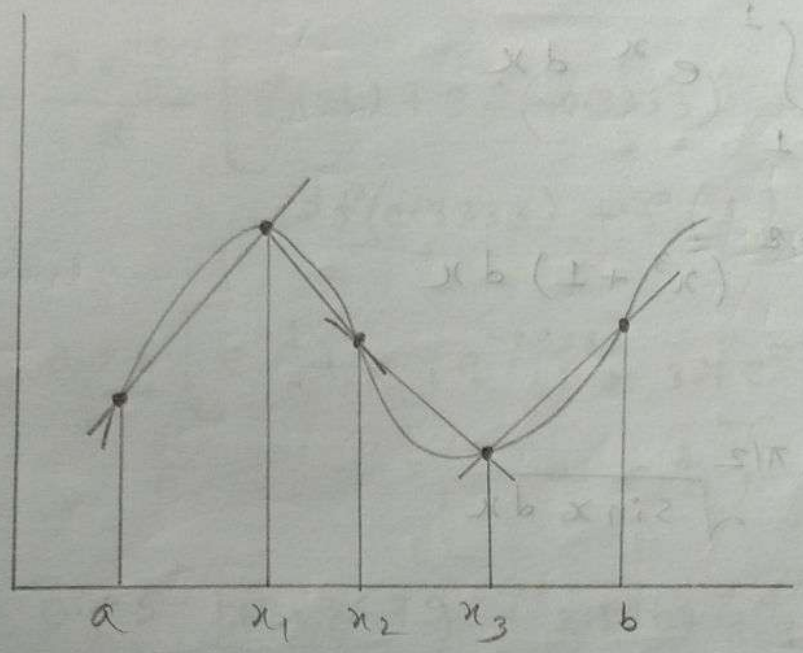
$$= \frac{0.8}{6}\left[0.2 + 9.824 + 0.232\right]$$

$$= 1.367$$

$$\therefore \text{Error} = \left|\frac{1.640533 - 1.367}{1.640533}\right| \times 100\%$$

$$= 16.67\%$$

Simpson's $\frac{3}{8}$ rule:



$$* \; I = \frac{3h}{8}\left[f(a) + 3f(x_1) + 3f(x_2) + f(b)\right]$$

$$h = \frac{b-a}{3}$$

$$x_i = a + ih$$

$$x_1 = a + 1 \cdot h$$

$$x_2 = a + 2 \cdot h$$

Ex: Use Simpson's $\frac{3}{8}$ rule to evaluate

Sin (a) $\int_{-1}^{1} e^{x}\, dx$

(b) $\int_{1}^{2} (x^3 + 1)\, dx$

(c) $\int_{0}^{\pi/2} \sqrt{\sin x}\, dx$

Solve:

(a) $\int_{-1}^{1} e^{x}\, dx$

Given, $a = -1$

$b = 1$

$h = \frac{1-(-1)}{3} = \frac{2}{3}$

$I_{exact} = \int_{-1}^{1} e^{x}\, dx$

$= \left[e^{x} \right]_{-1}^{1}$

$= e^{1} - e^{-1}$

$= 2.3504$

$$I_s = \frac{3h}{8}\left[f(a) + 3f(x_1) + 3f(x_2) + f(b)\right]$$

$$= \frac{3 \times \frac{2}{3}}{8}\left[f(-1) + 3f(-0.3333) + 3f(0.3333) + f(1)\right]$$

$$= 0.25\left[e^{-1} + 3 \times e^{-0.3333} + 3 \times e^{0.3333} + e^{1}\right]$$

$$= 0.25\left[0.3679 + 2.1497 + 4.1867 + 2.7183\right]$$

$$= 2.3556$$

$x_1 = a + 1 \cdot h$
$= -1 + 1 \times \frac{2}{3}$
$= -0.3333$
$x_2 = a + 2 \cdot h$
$= -1 + 2 \times \frac{2}{3}$
$= 0.3333$

$$\therefore \text{Error} = \left|\frac{2.3556 - 2.3504}{2.3556}\right| \times 100\ \%$$

$$= 0.2\ \%$$

(b) $\int_1^2 \overbrace{(x^3+1)}^{f(x)}\, dx$

Given, $a = 1$

$b = 2$

$$h = \frac{2-1}{3} = \frac{1}{3}$$

$$I_{exact} = \int_1^2 (x^3+1)\, dx$$

$$= \left[\frac{x^4}{4} + x\right]_1^2$$

$$= \left\{\frac{(2)^4}{4} + 2\right\} - \left\{\frac{(1)^4}{4} + 1\right\}$$

$$= 4.75$$

$$x_1 = a + 1 \cdot h$$

$$= 1 + 1 \cdot \frac{1}{3} = \frac{4}{3}$$

$$x_2 = a + 2 \cdot h = 1 + 2 \times \frac{1}{3} = \frac{5}{3}$$

$$I_s = \frac{3h}{8}\left[f(a) + 3f(x_1) + 3f(x_2) + f(b)\right]$$

$$= \frac{3 \times \frac{1}{3}}{8}\left[f(1) + 3 \times f\left(\frac{4}{3}\right) + 3 \times f\left(\frac{5}{3}\right) + f(2)\right]$$

$$= \frac{1}{8}\left[2 + 10.1111 + 16.8889 + 9\right]$$

$$= 4.75$$

$$\therefore \text{Error} = \left| \frac{4.75 - 4.75}{4.75} \right| \times 100\%$$

$$= 0\%$$

(c) $\int_0^{\pi/2} \sqrt{\sin x}\, dx$

Given,

$a = 0$

$b = \frac{\pi}{2}$

$h = \frac{\pi/2 - 0}{3} = \frac{\pi}{6}$

$I_{exact} = \int_0^{\pi/2} \sqrt{\sin x}\, dx = \frac{\pi}{4} = 1.17338$

$= 1.1981699$

$x_1 = a + 1 \cdot h = 0 + 1 \times \frac{\pi}{6} = \frac{\pi}{6}$

$x_2 = a + 2 \times h = 0 + 2 \times \frac{\pi}{6} = \frac{\pi}{3}$

$\frac{\pi}{6} = \frac{\pi}{6} \times \frac{180}{\pi}$

$= 30^\circ$

$3\sqrt{\sin 30}$

$= 2.12132$

$$I_s = \frac{3h}{8}\left[f(a) + 3f(x_1) + 3f(x_2) + f(b)\right]$$

$$= \frac{3 \times \frac{\pi}{6}}{8}\left[f(0) + 3 \times f\left(\frac{\pi}{6}\right) + 3 \times f\left(\frac{\pi}{3}\right) + f\left(\frac{\pi}{2}\right)\right]$$

$$= \frac{\pi}{16}\left[0 + 2.12132 + 2.79181 + 1.0\right]$$

$$= 1.16104$$

$$\text{Error} = \left| \frac{1.16104 - 0.7854}{1.16104} \right| \times 100\%$$

$$= 32.35 \%$$

Ex: Find the integral of the function $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ where $a = 0$ & $b = 0.8$

Solve: we know that,

$$I = \int_a^b f(x)\,dx$$

$$= \int_0^{0.8} f(x)\,dx$$

$$h = \frac{b-a}{3} = \frac{0.8}{3} = 0.267$$

$$= 1.64053 \rightarrow \text{exact value}$$

Again,

$$I = \frac{3h}{8}\left[f(a) + 3f(x_1) + 3f(x_2) + f(b)\right]$$

$$= \frac{3 \times 0.267}{8}\left[f(0) + 3 \times f(.267) + 3 \times f(.534) + f(0.8)\right]$$

$= 0.1\left[0.2 + 4.30 + 10.47 + 0.232\right]$

$= 1.5202$

$$\text{error} = \left|\frac{1.640533 - 1.5202}{1.640533}\right| \times 100\%$$

$= 7.33\%$

$x_1 = a + 1 \cdot h$
$= 0 + 1 \times .267$
$= 0.267$

$x_2 = a + 2 \cdot h$
$= 0 + 2 \times 0.267$
$= 0.534$

[এই method use করে Integral এর মান real মানের কাছাকাছি পাওয়া যায়। তাই error কম পাওয়া যায়]

Higher Order Rules

Boole's Rule :

* $$I_b = \frac{2h}{45}\left[7f(a) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(b)\right]$$

$$h = \frac{b-a}{4}$$

$$x = a + ih$$

Ex: Use Boole's five point formula to compute

(a) $\int_{-1}^{1} e^{x}\, dx$

(b) $\int_{1}^{2} (x^3+1)\, dx$

(c) $\int_{0}^{\pi/2} \sqrt{\sin x}\, dx$

solve: (a) $\int_{-1}^{1} e^{x}\, dx$

Given; $a = -1$

$b = 1$

$h = \frac{1-(-1)}{4} = 0.5$

$I_{exact} = \int_{-1}^{1} e^{x}\, dx$

$= \left[e^{x}\right]_{-1}^{1}$

$= e^{1} - e^{-1}$

$= 2.3504$

$$x_1 = a + 1 \cdot h = -1 + 1 \times 0.5 = -0.5$$

$$x_2 = a + 2 \cdot h = -1 + 2 \times 0.5 = 0$$

$$x_3 = a + 3 \cdot h = -1 + 3 \times 0.5 = 0.5$$

$$I_b = \frac{2h}{45}\left[7 f(a) + 32 f(x_1) + 12 f(x_2) + 32 f(x_3) + 7 f(b)\right]$$

$$= \frac{2 \times 0.5}{45}\left[7 \times f(-1) + 32 \times f(-0.5) + 12 \times f(0) + 32 \times f(0.5) + 7 \times f(1)\right]$$

$$= \frac{1}{45}\left[7 \times e^{-1} + 32 \times e^{-0.5} + 12 \times e^{0} + 32 \times e^{0.5} + 7 \times e^{1}\right]$$

$$= \frac{1}{45}\left[2.5752 + 19.40898 + 12 + 52.7591 + 19.02797\right]$$

$$= 2.3505$$

$$\therefore \text{Error} = \left|\frac{2.3505 - 2.3504}{2.3505}\right| \times 100\ \%$$

$$= 0.003\ \%$$

(b) $\int_1^2 (x^3 + 1)\, dx$

Ans: Given,

$a = 1$

$b = 2$

$h = \frac{b-a}{4} = \frac{2-1}{4} = 0.25$

$I_{exact} = \int_1^2 (x^3 + 1)\, dx$

$= \left[\frac{x^4}{4} + x\right]_1^2$

$= \left(\frac{2^4}{4} + 2\right) - \left(\frac{1^4}{4} + 1\right)$

$= 6 - \frac{5}{4}$

$= \frac{19}{4} = 4.75$

$x_1 = a + 1 \times h = 1 + 1 \times 0.25 = 1.25$

$x_2 = a + 2 \times h = 1 + 2 \times 0.25 = 1.5$

$x_3 = a + 3 \times h = 1 + 3 \times 0.25 = 1.75$

$$I_b = \frac{2h}{45}\left[7f(a) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(b)\right]$$

$$= \frac{2\times 0.25}{45}\left[7\times f(1) + 32f(1.25) + 12f(1.5) + 32f(1.75) + 7\times f(2)\right]$$

$$= \frac{0.5}{45}\left[14 + 94.5 + 52.5 + 203.5 + 63\right]$$

$$= 4.75$$

$$\therefore \text{Error} = \left|\frac{I_b - I_{exact}}{I_b}\right| \times 100\ \%$$

$$= \left|\frac{4.75 - 4.75}{4.75}\right| \times 100\ \%$$

$$= 0\ \%$$ Ans

(c) $\int_0^{\pi/2} \sqrt{\sin x}\, dx$ 387 ন:

Ans: Given,

$a = 0$

$b = \frac{\pi}{2}$

$h = \frac{\pi/2 - 0}{4} = \frac{\pi}{8}$

$$I_{exact} = \int_0^{\pi/2} \sqrt{\sin x}\, dx = \frac{\pi}{4} = 1.1981699$$

$x_1 = a + 1 \times h = 0 + 1 \times \frac{\pi}{8} = \frac{\pi}{8}$

$x_2 = a + 2 \times h = 0 + 2 \times \frac{\pi}{8} = \frac{\pi}{4}$

$x_3 = a + 3 \times h = 0 + 3 \times \frac{\pi}{8} = \frac{3\pi}{8}$

$$\therefore I_b = \frac{2h}{45}\left[7f(a) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(b)\right]$$

$$= \frac{2 \times \frac{\pi}{8}}{45}\left[7 \times f(0) + 32f(\pi/8) + 12f\left(\frac{\pi}{4}\right) + 32f\left(\frac{3\pi}{8}\right) + 7f\left(\frac{\pi}{2}\right)\right]$$

$$= \frac{\pi}{180}\left[0 + 32 \times 0.61861 + 12 \times 0.84090 + 32 \times 0.96119 + 7 \times 1\right]$$

$$= 1.18062$$

$$\therefore \text{Error} = \left|\frac{I_b - I_{exact}}{I_b}\right| \times 100\%$$

$$\cancel{= \left|\frac{1.1981699 - 1.18062}{1.1981699}\right| \times 100\%}$$

$$= \left|\frac{1.18062 - 1.1981699}{1.18062}\right| \times 100\%$$

$$= 0.97\%$$

Gaussian Quadrature formula / Gauss Legendre formula:

(i) If f is continuous on $[-1, 1]$ then

$$\int_{-1}^{1} f(x)\,dx = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$$

(ii) If a & b different values, then

$$\int_{a}^{b} f(x) = \int_{-1}^{1} f(p+qy)\cdot q\,dy$$

$$= q\int_{-1}^{1} f(p+qy)\,dy$$

$$x = p + qy$$

$$dx = q\,dy$$

$$p = \frac{b+a}{2}, \quad q = \frac{b-a}{2}$$

$$\boxed{\begin{aligned} &\int_{a}^{b} f(x)\,dx = c\int_{-1}^{1} g(z)\,dz \\ &x = \frac{b-a}{2}z + \frac{b+a}{2} \\ &I_g = \frac{b-a}{2}\left[w_1 g(z_1) + w_2 g(z_2)\right] \end{aligned}}$$

Exam এ এই প্রমাণ দিতে আসবে।

Prove that, $I = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

Proof:

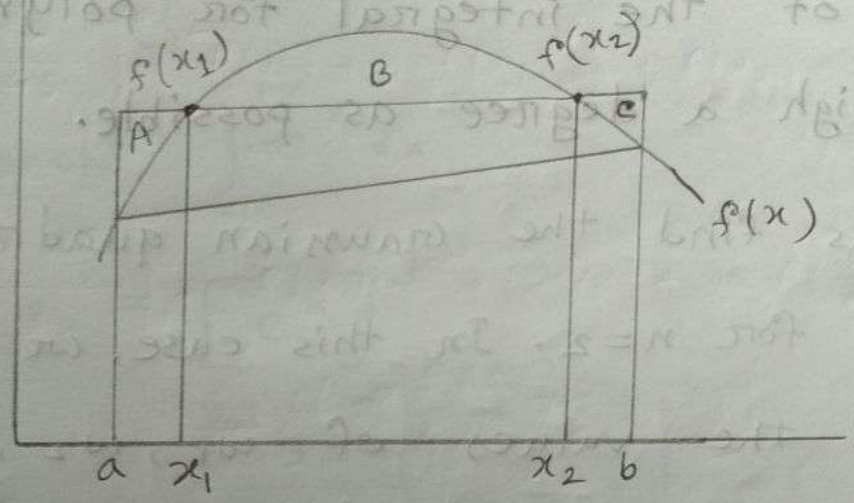


fig: Gaussian rule

↳ The method of implementing the strategy of finding appropriate values of x_i and ω_i and obtaining the integral of $f(x)$ is called Gaussian integration or quadrature.

↳ Gauss integration assumes an approximation of the form

$$I_g = \int_{-1}^{1} f(x)\, dx = \sum_{i=1}^{n} \omega_i f(x_i) \quad \text{——— (1)}$$

↳ Eqn ① contains $2n$ unknowns to be determined. These unknowns can be determined using the condition given in the integration formula of eqn ①. This should give the exact value of the integral for polynomials of as high a degree as possible.

↳ Let us find the Gaussian quadrature formula for $n=2$. In this case, we need to find the values of ω_1, ω_2, x_1 and x_2. Let us assume that the integral will be exact up to cubic polynomials. This implies that the functions 1, x, x^2 and x^3 can be numerically integrated to obtain exact results.

$$\omega_1 + \omega_2 = \int_{-1}^{1} dx = 2$$

$$\omega_1 x_1 + \omega_2 x_2 = \int_{-1}^{1} x\, dx = 0$$

①

4 পৃঃ পর প্রমাণের বাকিটুকু লিখছি দেখ

Ex: compute $\int_{-1}^{1} e^{x}\,dx$ using two-point Gauss-Legendre formula.

Solve:

$$\int_{-1}^{1} e^{x}\,dx$$

$$I_{exact} = \int_{-1}^{1} e^{x}\,dx$$

$$= \left[e^{x}\right]_{-1}^{1}$$

$$= 2.3504$$

$$I_G = \int_{-1}^{1} e^{x}\,dx$$

$$= f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= e^{-\frac{1}{\sqrt{3}}} + e^{\frac{1}{\sqrt{3}}}$$

$$= 2.3427$$

$$\therefore \text{Error} = \left|\frac{2.3427 - 2.3504}{2.3427}\right| \times 100\ \%$$

$$= 0.33\ \%$$

Ex: Compute the integral

$$I = \int_{-2}^{2} e^{-x/2}\, dx$$

using Gaussian two point formula.

Solve: Here, $a = -2$

$b = 2$

$$I_{exact} = \int_{-2}^{2} e^{-x/2}\, dx$$

$$= \left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}} \right]_{-2}^{2}$$

$$= \left[\frac{e^{-\frac{2}{2}}}{-1/2} - \frac{e^{-\frac{-2}{2}}}{-1/2} \right]$$

$$= -0.7357588 - (-5.436563)$$

$$= 4.700804775$$

Again,

$$p = \frac{b+a}{2} = \frac{2-2}{2} = 0$$

$$q = \frac{b-a}{2} = \frac{2+2}{2} = 2$$

$$\therefore x = p + qy = 0 + 2y = 2y$$

$$\therefore f(2y) = e^{-\frac{2y}{2}} = e^{-y}$$

We know,

$$\int_a^b f(x)\,dx = 2\int_{-1}^{1} f(p+qy)\,dy$$

$$\therefore \int_{-2}^{2} e^{-\frac{x}{2}}\,dx = 2\int_{-1}^{1} e^{-y}\,dy$$

$$= 2\left[f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= 2\left[0.56138 + 1.781312\right.$$

$$= 4.6853922$$

$$\therefore \text{Error} = \left|\frac{4.6853922 - 4.7008048}{4.6853922}\right| \times 100\%$$

$$= 0.328\ \%$$

Ex: Compute the integral

$$I = \int_2^4 e^x\,dx$$

using Gaussian two point formula

Solve: Given, $a = 2$, $b = 4$

$$I_{exact} = \int_2^4 e^x\,dx$$

$$= \left[e^x\right]_2^4 = \left[e^4 - e^2\right] = 47.2091$$

Now,

$$p = \frac{b+a}{2} = \frac{4+2}{2} = 3$$

$$q = \frac{b-a}{2} = \frac{4-2}{2} = 1$$

$$x = p + qy = 3 + y$$

$$\therefore f(3+y) = e^{3+y}$$

We know,

$$\int_a^b f(x)\,dx = q\int_{-1}^{1} f(p+qy)\,dy$$

$$\therefore \int_2^4 e^x\,dx = 1\int_{-1}^{1} f(3+y)\,dy$$

$$= \int_{-1}^{1} e^{3+y}\,dy$$

$$= f\left(-\frac{1}{\sqrt{3}}\right) + f\left(1/\sqrt{3}\right)$$

$$= 11.2757 + 35.7786$$

$$= 47.0543$$

$$\therefore \text{Error} = \left|\frac{47.0543 - 47.2091}{47.0543}\right| \times 100\%$$

$$= 0.33\ \%$$

Ex: Use the two point Gauss-Legendre rule to approximate the result,

$$f(x) = 5x^3 - 3x^2 + 2x + 1 \quad ; \quad -1 \le x \le 1$$

Solve:

$$f\left(\frac{1}{\sqrt{3}}\right) = 5\left(\frac{1}{\sqrt{3}}\right)^3 - 3\left(\frac{1}{\sqrt{3}}\right)^2 + 2\left(\frac{1}{\sqrt{3}}\right) + 1$$

$$= 2.117$$

$$f\left(-\frac{1}{\sqrt{3}}\right) = 5\left(-\frac{1}{\sqrt{3}}\right)^3 - 3\left(-\frac{1}{\sqrt{3}}\right)^2 + 2\left(-\frac{1}{\sqrt{3}}\right) + 1$$

$$= -2.1169$$

$$\therefore \int_{-1}^{1} f(x)\,dx = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$$

$$= 2.117 - 2.1169$$

$$= 0.0001$$

$$\approx 0$$

Ans

M

Ex: If $f(x) = 2x^3 - 3x^2 + 4x - 5$;

$-2 \le x \le 4$

Ans:

Here,

$a = -2$

$b = 4$

$\therefore p = \frac{b+a}{2} = \frac{4-2}{2} = 1$

$q = \frac{b-a}{2} = \frac{4+2}{2} = 3$

$\therefore x = p + qy = 1 + 3y$

$\therefore f(1+3y) = 2(1+3y)^3 - 3(1+3y)^2 + 4(1+3y) - 5$

$= 2(1 + 9y + 27y^2 + 27y^3) - 3(1 + 6y + 9y)$
$+ 4 + 12y - 5$

$= 2 + 18y + 54y^2 + 54y^3 - 3 - 18y - 27y^2$
$+ 4y + 12y - 5$

$= 54y^3 + 27y^2 + 12y - 2$

We know,

$$\int_a^b f(x)\,dx = q\int_{-1}^{1} f(p+qy)\,dy$$

$$\therefore \int_{-2}^{4} f(x)\,dx = 3\int_{-1}^{1} (54y^3+27y^2+12y-2)\,dy$$

$$= 3\left[24.32 - 10.32\right]$$

$$= 3\times 14$$

$$= 42$$ Ans

Gaussian Quadrature formula:

$$\omega_1 x_1^2 + \omega_2 x_2^2 = \int_{-1}^{1} x^2 \, dx = \frac{2}{3}$$

$$\omega_1 x_1^3 + \omega_2 x_2^3 = \int_{-1}^{1} x^3 \, dx = 0$$

Solving these simultaneous equations, we obtain

$$\omega_1 = \omega_2 = 1$$

$$x_1 = -\frac{1}{\sqrt{3}} = -0.5113502$$

$$x_2 = \frac{1}{\sqrt{3}} = 0.5773502$$

Thus, we have the Gaussian quadrature formula for $n = 2$ as:

$$\int_{-1}^{1} f(x) \, dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \quad \text{——— (11)}$$

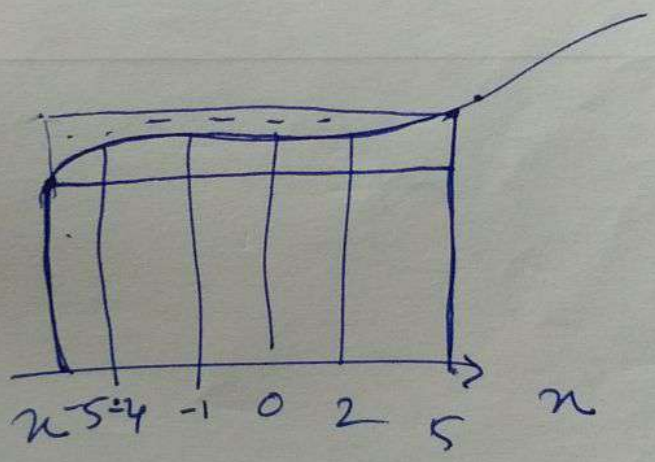
This formula will give correct value for integral of $f(x)$ in

the range $(-1, 1)$ for any ~~fun~~ function up to third order. Eqn (11) is also known as Gauss - Legendre formula.

—

$$* \int_{-5}^{5} \underline{\underline{f(x)}}\, dx \qquad + C$$

$=$



$$= \int_{-5}^{5}$$

$10x = -$

Gaussian Quadrature formula:

Ordinary Differential Equations

1.

Picard's Method:

$$y^{i+1} = y_0 + \int_{x_0}^{x} f(x, y^i)\, dx$$

no. of iteration

This eqn is known as Picard's method.

Ex-13.3: Solve the following equations by Picards method

(i) $\underbrace{y'(x)}_{f(x,y)} = x^2 + y^2$; $y(\overset{x_0}{0}) = \overset{y_0}{0}$

(ii) $y'(x) = xe^y$; $y(0) = 0$

and estimate $y(0.1)$, $y(0.2)$ and $y(1)$.

Ans: (i) Given,

$$y'(x) = x^2 + y^2 \quad ; \quad y(0) = 0$$

and $y_0 = 0$ and $x_0 = 0$

$\frac{dy}{dx}$ বা y'
বা y_1

$$\therefore y^1 = y_0 + \int_{x_0}^{x} f(x, y^0)\, dx$$

$$= 0 + \int_0^x f(x, 0)\, dx$$

$$= 0 + \int_0^x (x^2 + 0^2)\, dx$$

$$= \left[\frac{x^3}{3}\right]_0^x$$

$$= \frac{x^3}{3}$$

$$\therefore y^2 = y_0 + \int_{x_0}^{x} f(x, y^1)\, dx$$

$$= 0 + \int_0^x x^2 + \frac{x^6}{9}\, dx$$

$$= \left[\frac{x^3}{3} + \frac{x^7}{63}\right]_0^x$$

$$= \frac{x^3}{3} + \frac{x^7}{63}$$

This process can be continued further although it may be a difficult task.

If we stop at y^2, then

$$y(x) = \frac{x^3}{3} + \frac{x^7}{63}$$

$$\therefore y(0.1) = \frac{(0.1)^3}{3} + \frac{(0.1)^7}{63}$$

$$= 0.0003333 + 0.00000001$$

$$= 0.000333$$

$$\therefore y(0.2) = \frac{(0.2)^3}{3} + \frac{(0.2)^7}{63}$$

$$= 0.0026667 + 0.000000203$$

$$= 0.0026669$$

$$\therefore y(1) = \frac{(1)^3}{3} + \frac{(1)^7}{63}$$

$$= 0.3492063$$

(ii) Given,

$$y'(x) = xe^{y} \quad ; \quad y(0) = 0$$

$$\therefore y_0 = 0 \quad , \quad x_0 = 0$$

$$\therefore y^1 = y_0 + \int_{x_0}^{x} f(x, y^0)\, dx$$

$$= 0 + \int_{0}^{x} x e^{0}\, dx$$

$$= \left[\frac{x^2}{2}\right]_0^x$$

$$= \frac{x^2}{2}$$

$$\therefore y^2 = y_0 + \int_{x_0}^{x} f(x, y^1)\, dx$$

$$= 0 + \int_{0}^{x} f\left(x, e^{\frac{x^2}{2}}\right) dx$$

$$= \int_{0}^{x} x\, e^{\frac{x^2}{2}}\, dx$$

$$\boxed{= \left[\frac{x^2}{2} \cdot \frac{e^{\frac{x^2}{2}}}{\frac{1}{2}}\right]_0^x}$$

$$= \int_0^{\frac{x^2}{2}} e^z \, dz$$

$$= \left[e^z \right]_0^{\frac{x^2}{2}}$$

$$= e^{\frac{x^2}{2}} - 1$$

Let,

$\frac{x^2}{2} = z$

বা, $\frac{1}{2} \cdot x^2 = z$

বা, $\frac{1}{\cancel{2}} \cdot \cancel{2} x \, dx = dz$

$\therefore x \, dx = dz$

x	z
0	0
0	$\frac{x^2}{2}$

Note that further integration will become more difficult and even impossible. Now, Let us assume

$$y(x) = e^{\frac{x^2}{2}} - 1$$

$\therefore y(0.1) = 0.0050125$

$\therefore y(0.2) = 0.0202013$

$\therefore y(1) = 0.6487213$

Ans

এই পেইজ আসবে না, তবুও লিখতেছি

We know that,

the exact soln of $y'(x) = xe^{y}$ is

$$y(x) = -\ln\left(1 - \frac{x^2}{2}\right)$$

$\therefore y(0.1)_{exact} = 0.0050125$

$\therefore y(0.2)_{exact} = 0.0202027$

$\therefore y(1)_{exact} = 0.6933147$

Note that the error increases when $(x - x_0)$ increases. Better accuracy can be achieved by using initial values i.e. $y(0.1)$.

2. Euler's method:

* Define Euler method. Derive the equation of Euler's method.

Ans: **Euler's method:** Euler method is a numerical method which is used to solve ordinary differential equation as the form of

$$\frac{dy}{dx} = f(x, y) \quad \text{with} \quad y(x_0) = y_0$$

Derivation:

Consider the differential equation,

$$\frac{dy}{dx} = f(x, y) \quad \text{——— (i)} \quad \text{with}$$

$$y(x_0) = y_0 \quad \text{——— (ii)}$$

suppose we want to solve the eqn (i) for y at

$$x_{i+1} = x_i + h \quad \text{where } i = 0, 1, 2, \ldots$$

By integrating eqn ①, we get

$$\int_{y_0}^{y_1} dy = \int_{x_0}^{x_1} f(x, y)\, dx$$

$$\Rightarrow (y_1 - y_0) = (x_1 - x_0) f(x, y)$$

$$\Rightarrow y_1 = y_0 + (x_1 - x_0) f(x, y)$$

$$\therefore y_1 = y_0 + h f(x, y) \quad [\because x_1 = x_0 + h]$$

Assuming $f(x, y) = f(x_0, y_0)$ in $x_0 \le x \le x_1$

$$\therefore y_1 = y_0 + h f(x_0, y_0)$$

For the range, $x_1 \le x \le x_2$

$$y_2 = y_1 + h f(x_1, y_1)$$

For the range, $x_2 \le x \le x_3$

$$\therefore y_3 = y_2 + h f(x_2, y_2)$$

$$\therefore y_{n+1} = y_n + h f(x_n, y_n)$$

This formula is known as Euler's formula.

Example: Using Euler's method integrate the equation $\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$ from $x = 0$ to $x = 2$ with step size 0.5 and also compare the result with the exact solution. The initial condition at $x = 0$ is $y = 1$ or $y(0) = 1$.

solution: Given that,

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5 \quad \text{——— (I)}$$

$x_0 = 0$, $y_0 = 1$ and $h = 0.5$

Exact solution:

Integrating eqn (I), we get

$$y = -2 \cdot \frac{x^4}{4} + 12 \cdot \frac{x^3}{3} - 20 \cdot \frac{x^2}{2} + 8.5x + c$$

$$\Rightarrow y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + c \quad \text{——— (II)}$$

using initial condition, we obtain

$$1 = -0 + 0 - 0 + 0 + c$$

$$\therefore c = 1$$

Putting the value of c in eqn (II),

we get

$$y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1 \quad \text{——— (III)}$$

From Euler's eqn, we know that

$$y_{i+1} = y_i + h f(x_i, y_i)$$

<u>step 0 :</u> when $i = 0$

$$\therefore y_1 = y_0 + h f(x_0, y_0)$$

$$= 1 + 0.5 f(0, 1)$$

$$= 1 + 0.5 \times 8.5$$

$$= 5.25$$

$$y_1 = y(x_1) = y(x_0 + h) = y(0 + 0.5)$$

$$= y(0.5)$$

$$\therefore y(0.5) = 5.25$$

error = Actual value − calculated value

$= 3.21875 - 5.25$

$= -2.03125$

<u>step 1:</u> when $i = 1$

$y_2 = y_1 + h f(x_1, y_1)$

$= 5.25 + 0.5 \times f(0.5, 5.25)$

$= 5.25 + 0.5 \times 1.25$

$= 5.875$

$\therefore y_2 = y(x_2) = y(x_1 + h) = y(0.5 + 0.5) = y(1)$

$\therefore y(1) = 5.875$

$\therefore$ error = Actual value − calculated value

$= 3 - 5.875$

$= -2.875$

step 2: when $n = 2$

$$y_3 = y_2 + h f(x_2, y_2)$$

$$= 5.875 + 0.5 \times f(1, 5.875)$$

$$= 5.875 + 0.5 \times (-1.5)$$

$$= 5.125$$

$$\therefore y_3 = y(x_3) = y(x_2 + h) = y(1 + 0.5) = y(1.5)$$

$$\therefore y(1.5) = 5.125$$

error = Actual value − calculated value

$$= 2.21875 - 5.125$$

$$= -2.90625$$

Example: Using Euler's method integrate the equation

$$\frac{dy}{dx} + 0.4y = 3e^{-x}$$

From $x = 0$ to $x = 3$ with stepsize 3 and also compare the result with the exact solution. The initial condition at $x = 0$ is $y = 5$ or $y(0) = 5$.

Sol^n : Given that,

$$\frac{dy}{dx} + 0.4y = 3e^{-x} \quad \text{——— (I)}$$

$$\Rightarrow \frac{dy}{dx} = 3e^{-x} - 0.4y \quad \text{——— (II)}$$

$x_0 = 0$, $y_0 = 5$ and $h = 3$

Exact solution:

I.F of eqn (I), we get

$$I.F. = e^{\int P\,dx} = e^{\int 0.4\,dx} = e^{0.4x}$$

Multiplying eqn (I), by $e^{0.4x}$, we get

$$y \cdot e^{0.4x} = \int 3e^{-x} \cdot e^{0.4x}\,dx$$

$$\Rightarrow y e^{0.4x} = 3\int e^{-0.6x}\,dx$$

$$\Rightarrow y e^{0.4x} = \frac{3}{-0.6} e^{-0.6x} + c$$

$$\Rightarrow y e^{0.4x} = -5e^{-0.6x} + c$$

$$\Rightarrow y = \frac{-5e^{-0.6x}}{e^{0.4x}} + \frac{c}{e^{0.4x}}$$

$$\therefore y = -5e^{-x} + ce^{-0.4x} \quad \text{——— (III)}$$

Using initial condition, we obtain

$$5 = -5e^{-0} + ce^{-0.4\times 0}$$

$$\Rightarrow 5 = -5 + c$$

$$\therefore c = 10$$

Putting the value of c in eqn (III),

we get

$$y = -5e^{-x} + 10e^{-0.4x} \quad \text{—— (iv)}$$

From Euler's eqn, we know that

$$y_{i+1} = y_i + hf(x_i, y_i)$$

<u>step 0:</u> when $i = 0$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 5 + 3 \times f(0, 5)$$

$$= 5 + 3 \times (3e^{-0} - 0.4 \times 5)$$

[From (II)]

$$= 5 + 3(3-2)$$

$$= 8$$

$$y_1 = y(x_1) = y(x_0 + h) = y(0+3) = y(3)$$

$$\therefore y(3) = 8$$

$\therefore$ error = Actual value − calculated value

$= 2.763 - 8$

$= -5.237$

class :

Euler's method :

$$y_{i+1} = y_i + h f(x_i, y_i)$$

420 ~~n~~:

Example - 13.4: Given the equation

10 20 30

h = step size

$= 20 - 10$

$= 10$

$$\frac{dy}{dx} = 3x^2 + 1 \quad \text{with} \quad y(1) = 2$$

estimate $y(2)$ by Euler's method using (i) $h = 0.5$ and (ii) $h = 0.25$

Ans: (i)

Given,

$$\underbrace{\frac{dy}{dx}}_{f(x_i, y_i)} = 3x^2 + 1 \quad ; \quad y(2) = ?$$

1	1.5	2

$h = 0.5$

$y(1) = 2$

$\therefore y(1.5) = y(1) + 0.5 \times [3(1)^2 + 1]$

$= 2 + 2$

$= 4$

$\therefore y(2) = y(1.5) + 0.5 \times [3(1.5)^2 + 1]$

$= 4 + 3.875$

$= 7.875$

Ans

original value:

$\int 3x^2 + 1$

$\cancel{3} \cdot \frac{x^3}{\cancel{3}} + x$

when $x = 2$

$(2)^3 + 2 = 10$

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$1 \rightarrow 1.25 \rightarrow 1.50 \rightarrow 1.75 \rightarrow 2$

(11) Given,

$$\frac{dy}{dx} = 3x^2 + 1 \quad ; \quad y(2) = ?$$

$h = 0.25$

$y(\overset{x}{1}) = 2$

$\therefore y(1.25) = y(1) + 0.25\left[3(1)^2 + 1\right]$

$= 2 + 1$

$= 3$

$\therefore y(1.50) = y(1.25) + 0.25\left[3(1.25)^2 + 1\right]$

$= 3 + 1.42188$

$= 4.42188$

$\therefore y(1.75) = y(1.50) + 0.25\left[3 \times (1.50)^2 + 1\right]$

$= 4.42188 + 1.9375$

$= 6.3594$

$\therefore y(2) = y(1.75) + 0.25\left[3 \times (1.75)^2 + 1\right]$

$= 6.3594 + 2.5469$

$= 8.9063$

Ans

Ex: Given the equation

$$y'(x) = \frac{2y}{x} \quad \text{with } y(1) = 2$$

estimate $y(2)$ using Euler's method for $h = 0.25$

Ans: Given,

$$h = 0.25$$

$$y(\overset{x}{1}) = \overset{y}{2}$$

$$\therefore y(1.25) = y(1) + 0.25\left[\frac{2 \times 2}{1}\right]$$

$$= 2 + 1$$

$$= 3$$

$$\therefore y(1.5) = y(1.25) + 0.25\left[\frac{2 \times 3}{1.25}\right]$$

$$= 3 + 1.2$$

$$= 4.2$$

$$\therefore y(1.75) = y(1.5) + 0.25\left[\frac{2 \times 4.2}{1.5}\right]$$

$$= 4.2 + 1.4$$

$$= 5.6$$

$$\therefore y(2) = y(1.75) + 0.25\left[\frac{2 \times 5.6}{1.75}\right]$$

$$= 5.6 + 1.6$$

$$= 7.2 \quad \text{Ans}$$

3. Heun's Method:

What is Heun's method? Find out the predictor and corrector eqn by using this eqn. method.

Ans:

Heun's method: If we determine the slope of two derivatives for the interval - one at the initial point and another at the end point, then the two derivatives are then arranged to obtain an improved estimate of the slope for the entire interval. This is called Heun's method.

$$y'_{i+1} = f(x_{i+1}, y_{i+1}) \quad \text{--- (iii)}$$

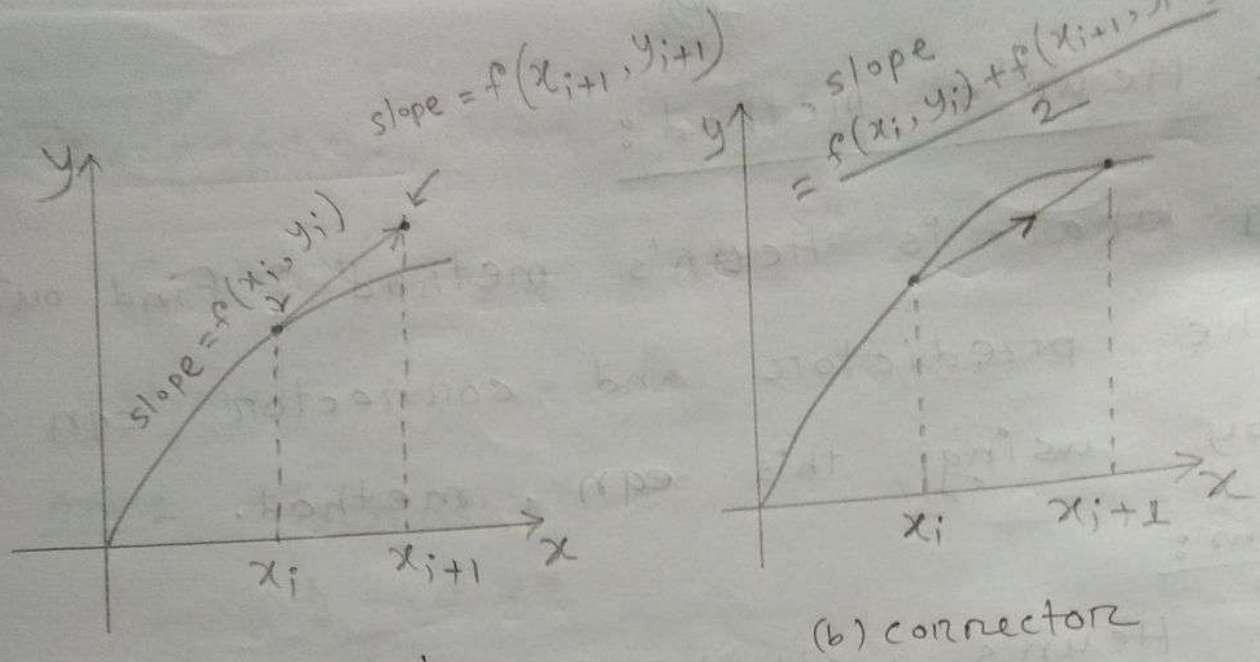
Combining eqn (i) & eqn (iii), we get

$$y' = \frac{y'_i + y'_{i+1}}{2} = \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1})}{2}$$

From Euler's method,

$$y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1})}{2} h$$

which is called a corrector equation.



(a) Predictor (b) corrector

fig: Graphical depiction of Heun's method

From Ordinary differential equation we get the slope at the begining of an interval.

$$y_i' = f(x_i, y_i) \quad \text{——— (I)}$$

By using Euler's method we get from eqn (I), predicton

$$y_{i+1} = y_i + f(x_i, y_i)h \quad \text{——— (II)}$$

From eqn (II) we can estimate y_{i+1} but it is not final answer. This

Example: Using Heun's method integrate $\frac{dy}{dx} = 4e^{0.8x} - 0.5y$ from $x = 0$ to $x = 4$ with a stepsize 1. The initial condition at $x = 0$, $y = 2$.

Soln: Given that

$$\frac{dy}{dx} = 4e^{0.8x} - 0.5y \quad \text{—— (I)}$$

$$\therefore \frac{dy}{dx} + 0.5y = 4e^{0.8x} \quad \text{—— (II)}$$

and $x = 0$, $y = 2$, $h = 1$

Exact soln: From (II), we get

$$I.F, \; e^{\int p\,dx} = e^{\int 0.5\,dx} = e^{0.5x}$$

multiplying eqn (II) by $e^{0.5x}$, we get

$$y e^{0.5x} = 4\int e^{0.8x} \cdot e^{0.5x}\,dx \text{ we get}$$

$$\Rightarrow y e^{0.5x} = 4\int e^{1.3x}\,dx$$

$$\Rightarrow y e^{0.5x} = \frac{4}{1.3} e^{1.3x} + c$$

$$\Rightarrow y e^{0.5x} = 3.08 e^{1.3x} + c$$

$$\therefore y = 3.08 e^{0.8x} + c e^{-0.5x} \quad \text{—— (III)}$$

Using initial condition, we get

$$2 = 3.08 + c$$

$$\therefore c = -1.08$$

Putting the value of c in eqn (III), we get

$$y = 3.08 e^{0.8x} - 1.08 e^{-0.5x}$$

<u>Step 0:</u> when $i = 0$

$$y_{i+1} = y_i + f(x_i, y_i) h$$

$$\Rightarrow y_1 = y_0 + f(x_0, y_0) h$$

$$= 2 + f(0, 2) \times 1$$

$$= 2 + 3 \times 1$$

$$= 5$$

and $x_{i+1} = x_i + h$

$$\Rightarrow x_1 = x_0 + h = 0 + 1 = 1$$

$$\therefore y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1})}{2} \times h$$

$$\therefore y_1 = y_0 + \frac{f(x_0, y_0) + f(x_1, y_1)}{2} \times h$$

$$= 2 + \frac{3 + f(1, 5)}{2} \times 1$$

$$= 2 + \frac{3 + 6.4022}{2} \times 1$$

$$= 6.701082$$

$$\therefore y_1 = y(x_1) = y(x_0 + h) = y(0+1) = y(1)$$

$$\therefore y(1) = 6.701082$$

<u>step 1:</u> when $i = 1$

$$y_{i+1} = y_i + f(x_i, y_i)\, h$$

$$\therefore y_2 = y_1 + f(x_1, y_1)\, h$$

$$= 6.701082 + f(1, 6.701082) \times 1$$

$$= 6.701082 + 5.552$$

$$= 12.253$$

and $x_{i+1} = x_i + h$

$\therefore x_2 = x_1 + h = 1 + 1 = 2$

$$\therefore y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1})}{2} h$$

$$\therefore y_2 = y_1 + \frac{f(x_1, y_1) + f(x_2, y_2)}{2} h$$

$$= 6.701082 + \frac{5.552 + f(2, 12.253)}{2} 1$$

$$= 16.31989$$

Ans

class

Heun's method :

$$y_{i+1} = \underset{\text{initial}}{y_i} + \left(\frac{m_1 + m_2}{2}\right) h$$

$$m_1 = y'(x_i) = f(x_i, y_i)$$

$$m_2 = y'(x_{i+1}) = f(x_{i+1}, y_{i+1})$$

Eulers method ↳ eqn থেকে পাওয়া m_2 কে কাজে লাগে।

427 page $1 - 1.25 - 1.50 - 1.75 - 2$

Ex-13.6 : Given the equation

$$y'(x) = \frac{2y}{x} \quad \text{with } y(1) = 2$$

estimate $y(2)$ using Heun's method using $h = 0.25$ and compare the results with exact answers.

Ans: Heun's method:

$y(1) = 2$ ($1 \to x$, $2 \to y_i$ → initial মান)

1.25 iteration 1:

$$m_1 = \frac{2y}{x} = \frac{2 \times 2}{1} = 4$$

$$y_e(1.25) = \overset{y_i}{2} + \overset{h}{0.25} \times \overset{m_1}{4}$$

$$= 3$$

Euler

$$y_{i+1}^{e} = y_i + f(x_i, y_i) \times h$$

$$= y_i + m_1 \times h$$

$$m_2 = \frac{2 \times 3}{1.25} = 4.8$$

$$\therefore y(1.25) = y_i + \left(\frac{m_1 + m_2}{2}\right) \times h$$

(y_i → initial মান)

$$= 2 + \left(\frac{4 + 4.8}{2}\right) \times 0.25$$

$$= 3.1$$

iteration 2: $y(1.25) = 3.1$

$$m_1 = \frac{2y}{x} = \frac{2 \times 3.1}{1.25} = 4.96$$

$$y_e(1.5) = 3.1 + 4.96 \times 0.25 \quad \Big| \; y^e_{i+1} = y_i + m_1 \times h$$

$$= 4.34$$

$$m_2 = \frac{2 \times 4.34}{1.5} = \cancel{4.8}\ 5.79$$

$$\therefore y(1.5) = 3.1 + \left(\frac{4.96 + \overset{5.79}{\cancel{4.8}}}{2}\right) \times 0.25$$

$$= 4.44$$

iteration 3: $y(1.5) = 4.44$

$$m_1 = \frac{2 \times 4.44}{1.5} = 5.92$$

$$\therefore y_e(1.75) = 4.44 + 5.92 \times 0.25$$

$$= 5.92$$

$$m_2 = \frac{2 \times 5.92}{1.75} = 6.77$$

$$\therefore y(1.75) = 4.44 + \left(\frac{5.92 + 6.77}{2}\right) \times 0.25$$

$$= 6.03$$

iteration 4: $y(1.75) = 6.03$

$$m_1 = \frac{2 \times 6.03}{1.75} = 6.89$$

$$y_e(2) = 6.03 + 6.89 \times 0.25$$

$$= 7.75$$

$$\therefore y(2) = m_2 = \frac{2 \times 7.75}{2} = 7.75$$

$$\therefore y(2) = 6.03 + \left(\frac{6.89 + 7.75}{2}\right) \times 0.25$$

$$= 7.86$$

Exact solution of the eqn $y'(x) = \frac{2y}{x}$

with $y(1) = 2$ is obtained as

$$y(x) = 2x^2$$

The exact values of $y(x)$ and the estimated values by both the methods are tabulated below:

x	Euler's	Heun's	Original/ analytical
1	2	2	2
1.25	3	3.1	3.125
1.50	4.2	4.44	4.50
1.75	5.6	6.03	6.125
2	7.2	7.86	8

$\therefore$ It is clear that Heun's method provides better results compared to Euler's method. Ans

$$\frac{dy}{dx} = \frac{2y}{x}$$

$$\Rightarrow x\,dy = 2y\,dx$$

$$\Rightarrow \frac{dy}{y} = 2\frac{dx}{x}$$

$$\Rightarrow \ln y = 2\ln x + \ln c$$

$$\Rightarrow \ln y = \ln x^2 + \ln c$$

$$\therefore y = cx^2$$

4. Polygon's method:

SIR :

$$m_1 = f(x_i, y_i)$$

$$m_2 = f\left(x_i + \frac{h}{2}, \; y_i + \frac{m_1 h}{2}\right)$$

$$y_{i+1} = y_i + \left(\frac{m_1 + 2m_2}{3}\right) h$$

बरे 433 र:

$$m_1 = f(x_i, y_i)$$

$$m_2 = f\left(x_i + \frac{h}{2}, \; y_i + \frac{m_1 h}{2}\right)$$

$$y_{i+1} = y_i + m_2 h$$

Ex-13.7 :

$$y'(x) = \frac{2y}{x} \quad ; \quad y(1) = 2$$

$$y(2) = ? \qquad h = 0.25$$

using Polygon's method.

Ans :

step 1 : $y(1) = 2$

$$m_1 = f(x_i, y_i) = \frac{2 \times 2}{1} = 4$$

$$m_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{m_1 h}{2}\right)$$

$$= f(1 + 0.125, 2 + 0.5)$$

$$= 4.44$$

$$\therefore y(1.25) = y_i + m_2 h$$

$$= 2 + 4.44 \times 0.25$$

$$= 3.11$$

step 2 :

$$m_1 = f(x_i, y_i) = \frac{2 \times 3.11}{1.25} = 4.976$$

$$m_2 = f\left(1.25 + 0.125, 3.11 + \frac{4.976 \times .25}{2}\right)$$

$$= f(1.375, 3.732)$$

$$= 5.428$$

$$y(1.250) = y_i + m_2 h$$

$$= 3.11 + 5.428 \times 0.25$$

$$= 4.47$$

~~$y(1.75) =$~~

step 3:

$$m_1 = f(x_i, y_i) = \frac{2 \times 4.47}{1.50} = 5.96$$

$$m_2 = f\left(1.50 + 0.125,\ 4.47 + \frac{5.96 \times 0.25}{2}\right)$$

$$= f(1.625,\ 5.215)$$

$$= 6.418$$

$$y(1.75) = y_i + m_2 h$$

$$= 4.47 + 6.418 \times 0.25$$

$$= 6.075$$

step 4:

$$m_1 = f(x_i, y_i) = \frac{2 \times 6.075}{1.75} = 6.94$$

$$m_2 = f\left(1.75 + 0.125,\ 6.075 + \frac{6.94 \times 0.25}{2}\right)$$

$$= f(1.875,\ 6.943)$$

$$= 7.405$$

$\therefore y(2) = y_i + m_2 h$

$= 6.075 + 7.405 \times 0.25$

$= 7.93$

$\therefore$ Polygon's method yields better results compared to the Heun's method.

Another নিয়ম:

$y'(x) = \frac{2y}{x}$; $y(1) = 2$

$y(2) = ?$ $h = 0.25$

Step 1: $y(1) = 2$

$m_1 = 4$

$m_2 = 4.44$

$y(1.25) = y_i + \left(\frac{m_1 + 2m_2}{3}\right) h$

$= 2 + \left(\frac{4 + 2 \times 4.44}{3}\right) \times 0.25$

$= 3.07$

step 2:

$$m_1 = f(x_i, y_i) = \frac{2 \times 3.07}{1.25} = 4.917$$

$$m_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{m_1 h}{2}\right)$$

$$= f\left(1.25 + 0.125, 3.07 + \frac{4.917 \times 0.25}{2}\right)$$

$$= f(1.375, 3.685)$$

$$= 5.36$$

$$y(1.50) = 3.07 + \left(\frac{4.917 + 2 \times 5.36}{3}\right) 0.25$$

$$= 4.37$$

step 3:

$$m_1 = f(x_i, y_i) = \frac{2 \times 4.37}{1.50} = 5.831$$

$$m_2 = f\left(1.50 + 0.125, 4.37 + \frac{5.831 \times 0.25}{2}\right)$$

$$= f(1.625, 5.099)$$

$$= 6.28$$

$$y(1.75) = 4.37 + \left(\frac{5.831 + 2 \times 6.28}{3}\right) \times 0.25$$

$$= 4.37 + 1.532$$

$$= 5.902$$

step 4:

$$m_1 = f(x_i, y_i) = \frac{2 \times 5.902}{1.75} = 6.74$$

$$m_2 = f\left(1.75 + 0.125,\ 5.902 + \frac{6.74 \times 0.25}{2}\right)$$

$$= f(1.875,\ 6.745)$$

$$= 7.194$$

$$y(2) = 5.902 + \left(\frac{6.74 + 2 \times 7.194}{3}\right) \times 0.25$$

$$= 7.66$$

5. Runge Kutta method:

1st order $\longrightarrow$ Euler's method

2nd " $\longrightarrow$ Heun's " $y_{i+1} = y_i + \left(\frac{m_1 + m_2}{2}\right)h$

3rd " $\longrightarrow$ Polygon " $y_{i+1} = y_i + \left(\frac{m_1 + 2m_2}{3}\right)h$

4th " $\longrightarrow$ Higher Order

4th Order Runge-Kutta Method:

$$m_1 = f(x_i, y_i)$$

$$m_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{m_1 h}{2}\right)$$

$$m_3 = f\left(x_i + \frac{h}{2}, y_i + \frac{m_2 h}{2}\right)$$

$$m_4 = f(x_i + h, y_i + m_3 h)$$

$$y_{i+1} = \left(y_i + \left(\frac{m_1 + 2m_2 + 2m_3 + m_4}{6}\right)h\right)$$

Ex - 18.8 : Use the classical RK method to estimate $y(0.4)$ when

$y'(x) = x^2 + y^2$ with $y(0) = 0$

Assume $h = 0.2$ $\quad 0 \text{ — } 0.2 \text{ — } 0.4$

Ans:

Iteration 1: $f(x, y) = x^2 + y^2$; $y(0) = 0$

$$m_1 = f(x_i, y_i)$$
$$= f(0, 0)$$
$$= 0$$

$$m_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{m_1 h}{2}\right)$$
$$= f\left(0 + 0.1, 0 + \frac{0 \times 0.2}{2}\right)$$
$$= f(0.1, 0)$$
$$= 0.01$$

$$m_3 = f\left(x_i + \frac{h}{2}, y_i + \frac{m_2 h}{2}\right)$$
$$= f\left(0 + \frac{0.2}{2}, 0 + \frac{0.01 \times 0.2}{2}\right)$$
$$= f(0.1, 0.001)$$
$$= 0.01$$

$$m_4 = f(x_i + h,\ y_i + m_3 h)$$

$$= f(0 + 0.2,\ 0 + 0.01 \times 0.2)$$

$$= f(0.2,\ 0.002)$$

$$= 0.04$$

$$\therefore y(0.2) = y_i + \left(\frac{m_1 + 2m_2 + 2m_3 + m_4}{6}\right) h$$

$$= 0 + \left(\frac{0 + 2 \times 0.01 + 2 \times 0.01 + 0.04}{6}\right) 0.2$$

$$= 0.002667$$

<u>itercation 2:</u>

$$m_1 = f(x_i, y_i) = f(0.2,\ 0.002667)$$
$$= 0.04$$

$$m_2 = f\left(x_i + \frac{h}{2},\ y_i + \frac{m_1 h}{2}\right)$$

$$= f\left(0.2 + 0.1,\ 0.002667 + \frac{0.04 \times 0.2}{2}\right)$$

$$= f(0.3,\ 0.006667)$$

$$= 0.090044$$

$$m_3 = f\left(x_i + \frac{h}{2},\ y_i + \frac{m_2 h}{2}\right)$$

$$= f\left(0.2 + 0.1,\ 0.002667 + \frac{0.090044 \times 0.2}{2}\right)$$

$$= f(0.3,\ 0.01167)$$

$$= 0.090136$$

$$m_4 = f(x_i + h,\ y_i + m_3 h)$$

$$= f(0.2 + 0.2,\ 0.002667 + 0.090136 \times 0.2)$$

$$= f(0.4,\ 0.02069)$$

$$= 0.160428$$

$$\therefore y(0.4) = y_i + \left(\frac{m_1 + 2m_2 + 2m_3 + m_4}{6}\right)h$$

$$= 0.002667 + \left(\frac{0.04 + 2 \times 0.090044 + 2 \times 0.090136 + 0.160428}{6}\right) \times 0.2$$

$$= 0.021360$$

The exact answer is 0.021359

Ans

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এই ম্যাথটার পরে বিস্তারিত করেছি (2 পৃ: পর)

Ex - 13.14 : solve the following eqn for $y(0.2)$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6x$$

given $y(0) = 0$, $y'(0) = 1$. Use Heun's method.

1st order y_{10}, y_0, x_0

2nd order $y_{20} \rightarrow$ 2nd order y_0, x_0

Ans: Given,

$$\frac{d^2y}{dx^2} = 6x + 3y - 2\frac{dy}{dx}$$

Let, $y = y_1$,

$$\frac{dy}{dx} = y_2$$

Then $\frac{dy_1}{dx} = y_2$, $y_{10} = 0$

$$\frac{dy_2}{dx} = 6x + 3y_1 - 2y_2, \quad y_{20} = 1$$

$$m_1 = f(x_1, y_1)$$
$$m_2 = y_{10} + h m_1 \quad (1)$$

Let, $h = 0.2$

$m_1(1) = y_{20} = 1$

$m_1(2) = 6x_0 + 3y_{10} - 2y_{20} = -2$

$m_2(1) = y_{20} + h \cdot m_1(2)$

$= 1 + 0.2 \times (-2)$

$= 0.6$

$m_2(2) = 6(0.2) + 3[0 + 0.2 \times 1] - 2 \times 0.6$

$= 1.2 + 0.6 - 1.2$

$= 0.6$

$\therefore m_1(1) = \dfrac{1 + 0.6}{2} = 0.8$

$\therefore m_2(2) = \dfrac{-2 + 0.6}{2} = -0.7$

$\therefore y_1(0.2) = y_1(0) + 0.2\, m(1)$

$= 0 + 0.2 \times 0.8$

$= 0.16$

$\therefore y_2(0.2) = y_2(0) + 0.2\, m(2)$

$= 1 + (0.2)(-0.7)$

$= 0.86$

$\therefore y(x)$ at $x = 0.2 = 0.16$ Ans

$\therefore y'(x)$ at $x = 0.2 = 0.86$ Ans

OR, একই প্রশ্ন $y(0) = 0$ $y'(0) = 1$

y_{10} $y_{20} = 1$

Ans:

Given,

$$\frac{d^2y}{dx^2} = 6x + 3y - 2\frac{dy}{dx}$$

Let, $y = y_1$, $\frac{dy}{dx} = y_2$

$$\therefore \frac{dy_1}{dx} = y_2 \; ; \; y_{10} = 0$$

$$\frac{dy_2}{dx} = 6x + 3y_1 - 2y_2 ; \quad y_{20} = 1$$

Let, $h = 0.2$

$$m_1(1) = y_{20} = 1$$

$$m_1(2) = 6x_0 + 3y_{10} - 2y_{20}$$

$$= 6 \times 0 + 3 \times 0 - 2 \times 1$$

$$= -2 \qquad y_{i+1} = y_i + hm_1$$

$$m_2(1) = y_{20} + h\,m_1(2)$$

initial

আগেরটার

$$= 1 + 0.2(-2)$$

$$= 0.6$$

$$m_2(2) = 6x_2 + 3y_1 - 2y_2$$

$$= 6$$

Higher order Equations:

① $\frac{dy_1}{dx} = f_1(x, y_1, y_2)$, $y_1(x_0) = y_{10}$

② $\frac{dy_2}{dx} = f_2(x, y_1, y_2)$, $y_2(x_0) = y_{20}$

Ans:

$$m_1(1) = f_1(x_0, y_{10}, y_{20})$$

$$m_1(2) = f_2(x_0, y_{10}, y_{20})$$

$$m_2(1) = f_1(x_0 + h, y_{10} + h\, m_1(1), y_{20} + h\, m_1(2))$$

$$m_2(2) = f_2(x_0 + h, y_{10} + h\, m_1(1), y_{20} + h\, m_1(2))$$

$$m(1) = \frac{m_1(1) + m_2(1)}{2}$$

$$m(2) = \frac{m_1(2) + m_2(2)}{2}$$

$$y_1(x_1) = y_1(1) = y_{10} + m(1)h$$

$$y_2(x_1) = y_2(1) = y_{20} + m(2)h$$

457 Q:

13.14

Solve the following eqn for $y(0.2)$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6x$$

Given $y(0) = 0$, $y'(0) = 1$. Use Heun's method

Ans: Given,

$$\frac{d^2y}{dx^2} = 6x + 3y - 2\frac{dy}{dx}$$

Let, $y = y_1$ $\therefore \frac{dy}{dx} = y_2$

$$\therefore \underbrace{\frac{dy_1}{dx}}_{f_1(x_0, y_{10}, y_{20})} = y_2 \quad ; \quad y_{10} = 0$$

$$\therefore \underbrace{\frac{dy_2}{dx}}_{f_2(x_0, y_{10}, y_{20})} = 6x + 3y_1 - 2y_2 \quad ; \quad y_{20} = 1$$

Let $h = 0.2$

$$m_1(1) = f_1(x_0, y_{10}, y_{20})$$

$$= y_{20}$$

$$= 1$$

$$m_1(2) = f_2(x_0, y_{10}, y_{20})$$

$$= 6x_0 + 3y_{10} - 2y_{20}$$

$$= 6 \times 0 + 3 \times 0 - 2 \times 1$$

$$= -2$$

$m_1 = f(x_i, y_i)$

$y_{i+1} = y_i + h m_2$

Again,

$$m_2(1) = f_1\left(x_0 + h,\ y_{10} + h\, m_1(1),\ y_{20} + h\, m_1(2)\right)$$

$$= y_{20} + h\, m_1(2)$$

$$= 1 + 0.2 \times (-2)$$

$$= 0.6$$

$$m_2(2) = f_2\left(x_0 + h,\ y_{10} + h\, m_1(1),\ y_{20} + h\, m_1(2)\right)$$

$$= 6 \times (0.2) + 3\left[0 + 0.2 \times 1\right] - 2 \times 0.6$$

$$= 1.2 + 0.6 - 1.2$$

$$= 0.6$$

$$\therefore m(1) = \frac{m_1(1) + m_2(1)}{2} = \frac{1 + 0.6}{2} = 0.8$$

$$\therefore m(2) = \frac{m_1(2) + m_2(2)}{2} = \frac{-2 + 0.6}{2} = -0.7$$

$$\therefore y_1(0.2) = y_{10} + m(1)h$$
$$= 0 + 0.8 \times 0.2$$
$$= 0.16$$

$$\therefore y_2(0.2) = y_{20} + m(2)h$$
$$= 1 + (-0.7) \times 0.2$$
$$= 0.86$$

$\therefore y(x)$ at $x = 0.2 = 0.16$

$\therefore y'(x)$ at $x = 0.2 = 0.86$

Balagurushamy :

(49)

461 P2:

21 (c) $\frac{dy}{dx} = x + y$

$\Rightarrow y'(x) = x + y \quad ; \quad y(0) = 1$

and $y_0 = 1, \; x_0 = 0$

$y(0.25) = ? \qquad y(0.5) = ?$

$$\therefore y^1 = y_0 + \int_{x_0}^{x} f(x, y^i)\, dx$$

$$= 1 + \int_0^x f(x, y^0)\, dx$$

$$= 1 + \int_0^x x + 1 \; dx$$

$$= 1 + \left[\frac{x^2}{2} + x\right]_0^x$$

$$= 1 + \frac{x^2}{2} + x$$

$$y^2 = y_0 + \int_{x_0}^{x} f(x, y')\, dx$$

$$= 1 + \int_0^x (x + y')\, dx$$

$$= 1 + \int_0^x x + 1 + \frac{x^2}{2} + x \quad dx$$

$$= 1 + \left[\frac{x^2}{2} + x + \frac{x^3}{6} + \frac{x^2}{2}\right]_0^x$$

$$= 1 + \left[\cancel{2} \cdot \frac{x^2}{\cancel{2}} + x + \frac{x^3}{6}\right]_0^x$$

$$= 1 + x^2 + x + \frac{x^3}{6}$$

If we stop at y^2, then

$$y(x) = 1 + x^2 + x + \frac{x^3}{6}$$

$$\therefore y(0.25) = 1 + (0.25)^2 + 0.25 + \frac{(0.25)^3}{6}$$

$$= 1.315104$$

$$\therefore y(0.50) = 1 + (0.50)^2 + 0.50 + \frac{(0.50)^3}{6}$$

$$= 1.7708$$

(e) $\frac{dy}{dx} = \frac{x}{y}$

$\Rightarrow \quad y'(x) = \frac{x}{y} \quad ; \quad y(0) = 1$

$x_0 = 0 \quad$ and $\quad y_0 = 1$

$y(0.25) = ? \qquad y(0.50) = ?$

$$y^1 = y_0 + \int_0^x f(x, y^0)\, dx$$

$$= 1 + \int_0^x \frac{x}{y^0}\, dx$$

$$= 1 + \int_0^x x\, dx$$

$$= 1 + \frac{x^2}{2}$$

$$y^2 = y_0 + \int_0^x f(x, y^1)\, dx$$

$$= 1 + \int_0^x \frac{x}{1 + \frac{x^2}{2}}\, dx$$

$$= 1 + \int_0^x x\left(1 + x^2/2\right)^{-1} dx$$

$$= 1 + x\int_0^x \left(1 + \frac{x^2}{2}\right)^{-1} dx - \int_0^x \frac{\left(1 + \frac{x^2}{2}\right)^{-1+1}}{-1+1}\, dx$$

$$= 1 + x\, \frac{\left(1 + \frac{x^2}{2}\right)^{-1+1}}{-1+1}$$

461 R:

(53)

3| (a) $y'(x) = x^2 + y^2$ $\quad y(0) = 2$

$y(1) = ?$ $\quad h = 0.5$

iteration 1: when $i = 0$ $\quad 0 - 0.5 \quad 1$

$$y(0.5) = y_i + h f(x_i, y_i)$$
$$= 2 + 0.5 \times [0^2 + 2^2]$$
$$= 4$$

iteration 2: when $i = 1$

$$y(1) = y(0.5) + h f(x_i, y_i)$$
$$= 4 + 0.5 [0.5^2 + 4^2]$$
$$= 12.125$$

Ans

5| $y'(x) = 2xy \quad ; \quad y(0) = 1$

$y(1) = ? \qquad h = 0.5$

<u>iteration 1</u> :

$m_1 = 2 \times 0 \times 1 = 0$

$y_e(0.5) = y_0 + 0.5 \times 0 = 1$

$m_2 = 2 \times 1 \times 0.5 = 1$

$y(0.5) = y_i + \left(\frac{m_1 + m}{2}\right) h$

$= 1 + \left(\frac{0+1}{2}\right) \times 0.5$

$= 1.25$

<u>iteration 2</u> :

$m_1 = 2 \times 0.5 \times 1.25 = 1.25$

$y_e(1.00) = y_i + h m_1$

$= 1 + 0.5 \times 1.25$

$= 1.625$

$m_2 = 2 \times 1 \times 1.625 = 3.25$

$$y(1) = y_i + \left(\frac{m_1 + m}{2}\right) h$$

$$= 1.25 + \left(\frac{1.25 + 3.25}{2}\right) \times 0.5$$

$$= 2.375$$

8| (a) $\frac{dy}{dx} = x + y \quad ; \quad y(0) = 1$

$y(0.5) = ? \qquad h = 0.25$

$\boxed{\text{Ans}}$ <u>iteration 1</u> : $f(x_i, y_i) = x + y \quad ; \quad y(0) = 1$

$$m_1 = x + y = 0 + 1 = 1$$

$$m_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{m_1 h}{2}\right)$$

$$= f(0 + 0.125, 1 + 0.125)$$

$$= 1.25$$

$$m_3 = f\left(x_i + \frac{h}{2}, y_i + \frac{m_2 h}{2}\right)$$

$$= f\left(0.125, 1 + \frac{1.25 \times .25}{2}\right)$$

$$= f(0.125, 1.15625)$$

$$= 1.28125$$

$$m_4 = (x_i + h \;,\; y_i + m_3 h)$$

$$= (0 + 0.25 \;,\; 1 + (1.28125 \times 0.25))$$

$$= 1.57031$$

$$y(0.25) = y_i + \left(\frac{m_1 + 2m_2 + 2m_3 + m_4}{6}\right) h$$

$$= 1 + \left(\frac{1 + 2 \times 1.25 + 2 \times 1.28125 + 1.57031}{6}\right) 0.25$$

$$= 1.318034$$

<u>iteration 2</u> :

$$m_1 = f(x_i, y_i) = 1.56803$$

$$m_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{m_1 h}{2}\right)$$

$$= f(0.25 + 0.125 \;,\; 1.318034 + 0.196003)$$

$$= 1.88904$$

$$m_3 = f\left(x_i + \frac{h}{2}, y_i + \frac{m_2 h}{2}\right)$$

$$= f\left(0.25 + 0.125, 1.318034 + \frac{1.88904 \times 0.25}{2}\right)$$

$$= f(0.375, 1.5542)$$

$$= 1.9292$$

$$m_4 = f(x_i + h, y_i + m_3 h)$$

$$= f(0.25 + 0.25, 1.318034 + 0.48229)$$

$$= 2.30032$$

$$y(0.50) = y_i + \left(\frac{m_1 + 2m_2 + 2m_3 + m_4}{6}\right)h$$

$$= \cancel{1.56803}\ 1.318034 + \left(\frac{1.56803 + 2\times 1.88904 + 2\times 1.9292 + 2.30032}{6}\right)\cdot 3$$

$$= 1.797402$$ Ans

13| $10\frac{\partial^2 y}{\partial x^2} + \left(\frac{\partial y}{\partial x}\right)^2 + 6x = 0$,

$y(0) = 1$, $y'(0) = 0$

$y(0.2) = ?$

Ans $\frac{\partial^2 y}{\partial x^2} + \frac{1}{10}\left(\frac{\partial y}{\partial x}\right)^2 + \frac{3}{5}x = 0$

$\therefore \frac{\partial^2 y}{\partial x^2} = -\frac{3}{5}x - \frac{1}{10}\left(\frac{\partial y}{\partial x}\right)^2$

Let, $y = y_1$ $\therefore \frac{\partial y}{\partial x} = y_2$

$m_1(1) \rightarrow \therefore \frac{\partial y_1}{\partial x} = y_2$ $y_{10} = 1$

$m_2(2) \rightarrow \therefore \frac{\partial y_2}{\partial x^2} = -\frac{3}{5}x - \frac{1}{10}(y_2)^2$; $y_{20} = 0$

Let $h = 0.2$

$m_1(1) = f_1(x_0, y_{10}, y_{20})$

$= y_{20}$

$= 0$

$$m_1(2) = f_2(x_0, y_{10}, y_{20})$$

$$= -\frac{3}{5}x_0 - \frac{1}{10}(y_{20})$$

$$= -\frac{3}{5} \times 0 - \frac{1}{10} \times 0$$

$$= 0$$

Again,

$$m_2(1) = f_1(x_0 + h, y_{10} + h\,m_1(1), y_{20} + h\,m_1(2))$$

$$= y_{20} + h\,m_1(2)$$

$$= 0 + 0.2 \times 0$$

$$= 0$$

$$m_2(2) = f_2(x_0 + h, y_{10} + h\,m_1(1), y_{20} + h\,m_2(2))$$

$$= -\frac{3}{5}(x_0 + h) - \frac{1}{10}\left[y_{20} + h\,m_2(2)\right]$$

$$= -\frac{3}{5} \times 0.2 - \frac{1}{10} \times 0$$

$$= -0.12$$

$$m(1) = \frac{m_1(1) + m_2(1)}{2} = \frac{0+0}{2} = 0$$

$$m(2) = \frac{m_1(2) + m_2(2)}{2} = \frac{0 + (-0.12)}{2} = -0.0$$

$$\therefore y_1(0.2) = y_{10} + m(1)h$$

$$= 1 + 0 \times 0.2$$

$$= 1$$ Ans

$$\therefore y_2(0.2) = y_{20} + m(2)h$$

$$= 0 + (-0.06) \times 0.2$$

$$= -0.012$$ Ans