Infinite Integration

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \;;\; m,n>0$$

which is the 1st Eulerian equation.

$\Gamma n \to$ gamma n

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx \;;\; n>0 \;;$$

which is the 2nd Eulerian equation.

1) Prove that $\Gamma(n+1) = n!$ / $\Gamma(1) = 0!$ / 1

Sol^n:

We know that,

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx$$

$$\Rightarrow \Gamma(n+1) = \int_0^\infty x^{n+1-1} e^{-x}\,dx$$

$$= \int_0^\infty x^n e^{-x}\,dx$$

{integration UV}

$$= -x^n e^{-x}\Big|_0^\infty - \int_0^\infty n x^{n-1}(-e^{-x})\,dx$$

$$= 0 + n\int_0^\infty x^{n-1} e^{-x}\,dx$$

$$= n\,\Gamma(n)$$

Replacing n by $n-1, n-2, n-3 \dots 1$

$$\Gamma(n) = (n-1)\,\Gamma(n-1)$$

$$\Gamma(n-1) = (n-2)\,\Gamma(n-2)$$

$$\Gamma(n-2) = (n-3)\,\Gamma(n-3)$$

$$\therefore \Gamma(n+1) = n(n-1)(n-2)\dots 2\cdot 1 = n!$$ [Proved]

if $n=0$ then, we have $\Gamma 1 = 0!$

Math 2: Show that, $\beta(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n}$

or, the relationship between gamma and beta function.

Proof:

We know that,

$$\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx$$

Let, $x = yz$ where z is constant.

$$dx = z\, dy$$

when, $x = 0$ then $y = 0$ — lower limit

" $x = \infty$ then $y = \infty$ — upper

$$\therefore \Gamma n = \int_0^\infty (yz)^{n-1} e^{-yz} z \cdot dy$$

$$\Rightarrow \Gamma n = \int_0^\infty y^{n-1} z^n e^{-yz} dy \quad \cdots\cdots (i)$$

$$\Rightarrow \frac{\Gamma n}{z^n} = \int_0^\infty y^{n-1} e^{-yz} dy \quad \cdots\cdots (ii)$$

Multiplying both sides of eq^n (i) by $z^{m-1} e^{-z}$ we get,

$$\Gamma n \cdot z^{m-1} e^{-z} = z^{m+n-1} \cdot e^{-z} \int y^{n-1} e^{-yz} \, dy$$

Taking integration above $0 \to \infty$ with respect to z

$$\therefore \underbrace{\Gamma n \int_0^\infty z^{m-1} e^{-z} \, dz}_{\Gamma m} = \int_0^\infty \int_0^\infty z^{m+n-1} e^{-(1+y)z} \cdot y^{n-1} \, dz \, dy$$

$$\Rightarrow \Gamma n \cdot \Gamma m = \int_0^\infty \frac{\Gamma m+n}{(1+y)^{m+n}} y^{n-1} \, dy \quad [\text{using eq}^n \ 2]$$

$$\Rightarrow \frac{\Gamma n \cdot \Gamma m}{\Gamma m+n} = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} \, dy \quad \cdots\cdots (3)$$

Again, we know that,

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$ ☆ Memorise

Let,

$$x = \frac{1}{1+y}$$ when $x=0$ then $y = \infty$

$$\Rightarrow dx = \frac{-1}{(1+y)^2} dy$$ " $x = 1$ " $y = 0$

Now,

$$\beta(m,n) = -\int_{\infty}^{0}\left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}\frac{1}{(1+y)^2}\,dy$$

$$= \int_{0}^{\infty}\frac{1}{(1+y)^{m-1}}\left(\frac{y}{1+y}\right)^{n-1}\frac{1}{(1+y)^2}\,dy$$

$$= \int_{0}^{\infty}\frac{y^{n-1}}{(1+y)^{m-1+n-1+2}}\,dy$$

$$= \int_{0}^{\infty}\frac{y^{n-1}}{(1+y)^{m+n}}\,dy \text{ ----- } (4)$$

From eq^n (3) and (4) we get,

$$\beta(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n}$$

[Showed]

since the right hand side of both beta and gamma function is equal, we can say that

$$\beta(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n}$$

3. Prove that $\Gamma n = \frac{1}{n} \int_0^\infty e^{-y^{\frac{1}{n}}} dy$

Proof:

We know from the definition of Gamma function,

$$\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx, \; n > 0 \;\ldots\ldots (i)$$

Let,

$$x^n = y$$

$$\Rightarrow n x^{n-1} dx = dy$$

$$\Rightarrow x^{n-1} dx = \frac{1}{n} dy$$

when $x = 0$; then $y = 0$

" $x = \infty$, " $y = \infty$

Now from eqn (1) we get,

$$\Gamma n = \int_0^\infty e^{-y^{\frac{1}{n}}} \cdot \frac{1}{n} dy \quad \left[\because x^n = y \Rightarrow x = y^{\frac{1}{n}}\right]$$

$$= \frac{1}{n} \int_0^\infty e^{-y^{\frac{1}{n}}} dy$$ → Direct এই line এ না যেয়ে আরেকটু elaborate করে দিতে হবে (1 more line)

4. Prove that, $\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx = \frac{\Gamma{n+1}}{(1+m)^{n+1}}$

Proof: Let,

$$I = \int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx \quad \text{-----(i)}$$

and $x = e^{-z}$

$\Rightarrow dx = -e^{-z} dz$

$\therefore \frac{1}{x} = e^{z}$

$\Rightarrow \ln \frac{1}{x} = \ln e^{z}$

$\Rightarrow \ln \frac{1}{x} = z$

when $x = 0$ then $z = \infty$

" $x = 1$ " $z = 0$

Now, from eq^n (i) we get,

$$I = -\int_\infty^0 \left(e^{-z}\right)^m z^n \cdot e^{-z} dz$$

$$= \int_0^\infty z^n e^{-(1+m)z} dz$$

Again, Let, $(1+m)z = u$

$\Rightarrow z = \frac{1}{1+m} u$

$\Rightarrow dz = \frac{du}{1+m}$

$$\therefore I = \int_0^\infty \left(\frac{u}{1+m}\right)^n e^{-u} \frac{du}{1+m}$$

$$\Rightarrow \int_0^\infty x^m \left(\log\left(\frac{1}{x}\right)^n\right) dx = \frac{1}{(1+m)^{n+1}} \int_0^\infty u^n e^{-u} du$$

$$= \frac{1}{(1+m)^{n+1}} \underbrace{\int_0^\infty u^{n+1-1} e^{-u} du}_{\Gamma n+1}$$

$$= \frac{1}{(1+m)^{n+1}} \cdot \Gamma n+1$$

$$= \frac{\Gamma n+1}{(1+m)^{n+1}}$$

[proved]

5. Prove that, $\int_0^\infty \frac{\cos x}{x^n}\,dx = \frac{\pi}{2\Gamma n \cos\frac{n\pi}{2}}$; $0<n<1$

Proof:

We know from Gamma function,

$$\Gamma n = \int_0^\infty u^{n-1} e^{-u}\,du \;;\; n>0$$

Let, $u = xt$ where x is constant.

$\Rightarrow du = x\,dt$

when $u = 0$ then $t = 0$

" $u = \infty$ " $t = \infty$

$$\therefore \Gamma n = \int_0^\infty (xt)^{n-1} e^{-xt}\, x\,dt$$

$$\Gamma n = x^{n-1+1} \int_0^\infty t^{n-1} e^{-xt}\,dt$$

$$\therefore \frac{1}{x^n} = \frac{1}{\Gamma n} \int_0^\infty t^{n-1} e^{-xt}\,dt$$

Now, multiplying both sides with $\cos x$

$$\frac{\cos x}{x^n} = \frac{\cos x}{\Gamma n} \int_0^\infty t^{n-1} e^{-xt}\,dt$$

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Taking integration as $0 \to \infty$; with respect to x;

$$\int_0^\infty \frac{\cos x}{x^n}\,dx = \frac{1}{\Gamma n}\int_0^\infty \cos x\,dx \int_0^\infty t^{n-1} e^{-xt}\,dt$$

$$\Rightarrow \int_0^\infty \frac{\cos x}{x^n}\,dx = \frac{1}{\Gamma n}\int_0^\infty \left\{ \int_0^\infty e^{-xt}\cos x\,dx \right\} t^{n-1}\,d$$

$$= \frac{1}{\Gamma n}\int_0^\infty e^{-xt}\,\frac{(-t\cos x+\sin x)}{t^2+1^2}\Bigg|_0^\infty t^{n-1}\,dt$$

$$= \frac{1}{\Gamma n}\int_0^\infty \left[\frac{0-e^{-0}(-t+0)}{t^2+1}\right] t^{n-1}\,dt$$

$$= \frac{1}{\Gamma n}\int_0^\infty \frac{t^{1+n-1}}{t^2+1}\,dt$$

$$= \frac{1}{\Gamma n}\int_0^\infty \frac{t^n}{1+t^2}\,dt$$

Let,
$t = \tan\theta$

$\therefore dt = \sec^2\theta\,d\theta$

where $t = 0$ then $\theta = 0$

" $t = \infty$ " $\theta = \frac{\pi}{2}$

$$\therefore \int_0^\infty \frac{\cos x}{x^n}\,dx = \frac{1}{\Gamma n}\int_0^{\frac{\pi}{2}} \frac{\tan^n\theta \sec^2\theta\, d\theta}{1+\tan^2\theta}$$

$$= \frac{1}{\Gamma n}\int_0^{\frac{\pi}{2}} \frac{\tan^n\theta \sec^2\theta\, d\theta}{\sec^2\theta}$$

$$= \frac{1}{\Gamma n}\int_0^{\frac{\pi}{2}} \tan^n\theta\, d\theta$$

$$= \frac{1}{\Gamma n}\int_0^{\frac{\pi}{2}} \frac{\sin^n\theta}{\cos^n\theta}\, d\theta$$

$$= \frac{1}{\Gamma n}\int_0^{\frac{\pi}{2}} \sin^n\theta \cos^{-n}\theta\, d\theta$$

$$= \frac{1}{\Gamma n}\,\frac{\Gamma\frac{n+1}{2}\,\Gamma\frac{-n+1}{2}}{2\,\Gamma\frac{n-n+2}{2}}$$

$$= \frac{1}{\Gamma n}\,\frac{\Gamma\frac{n+1}{2}\cdot\Gamma\left(1-\frac{n+1}{2}\right)}{2\,\Gamma 1}$$

$$= \frac{1}{2\Gamma n}\cdot\frac{\cancel{n}\,\pi}{\sin\left(\frac{n+1}{2}\right)\pi}$$

$$= \frac{1}{2\Gamma n}\,\frac{p}{\sin\left(\frac{\pi}{2}+\frac{n\pi}{2}\right)}$$

$$= \frac{1}{2\Gamma n}\cdot\frac{\pi}{\cos\frac{n\pi}{2}}$$ [proved]

Question ta Sutro, Mukhosto korte hobe.

6. $$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx = \frac{\Gamma\frac{m+1}{2} \cdot \Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$$

Mukhosto korte hobe.

L.H.S. $$= \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$$

Let
$\sin^2 x = z$

$2\sin x.\cos x dx = dz$
$\Rightarrow \sin x \cos x \, dx = \frac{dz}{2}$

when $x = 0$ then $z = 0$
" $x = \frac{\pi}{2}$ " $z = 1$

$$= \int_0^{\frac{\pi}{2}} \sin^{m-1} x \cos^{n-1} x \sin x \cos x \, dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 x)^{\frac{m-1}{2}} \cdot (\cos^2 x)^{\frac{n-1}{2}} \sin x \cos x \, dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 x)^{\frac{m-1}{2}} (1 - \sin^2 x)^{\frac{n-1}{2}} \sin x \cos x \, dx \quad (1)$$

From (1):

$$= \int_0^1 z^{\frac{m-1}{2}} (1-z)^{\frac{n-1}{2}} \frac{dz}{2}$$

$$= \frac{1}{2} \int_0^1 z^{\frac{m+1}{2} - 1} (1-z)^{\frac{n+1}{2} - 1} dz$$

$$= \frac{1}{2} \beta \left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$$= \frac{\Gamma\frac{m+1}{2} \cdot \Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+1+n+1}{2}}$$

$$= \frac{\Gamma\frac{m+1}{2} \cdot \Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$$

[proved]

ii) $\int_0^{\frac{\pi}{2}} \sin^5 x \cos^6 x \, dx = ?$

Sol^n: We Know that,

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx = \frac{\Gamma\left(\frac{m+1}{2}\right) \cdot \Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}$$

Given that,

$m = 5$ and $n = 6$;

$$\therefore \int_0^{\frac{\pi}{2}} \sin^5 x \cos^6 x \, dx = \frac{\Gamma\left(\frac{5+1}{2}\right) \cdot \Gamma\left(\frac{6+1}{2}\right)}{2\,\Gamma\left(\frac{5+6+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{6}{2}\right) \Gamma\left(\frac{7}{2}\right)}{2\,\Gamma\left(\frac{13}{2}\right)}$$

$$\boxed{* \ \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}} \ **$$

$$= \frac{\Gamma 3 \cdot \Gamma\left(\frac{7}{2}\right)}{2\,\Gamma\left(\frac{13}{2}\right)}$$

$$= \frac{2 \cdot 1 \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)}{2 \cdot \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{8}{693}$$

8 $\int_0^{\frac{\pi}{2}} \sin^6 x \, dx = ?$

Sol^n: $\int_0^{\frac{\pi}{2}} \sin^m x \, dx = \dfrac{\Gamma\frac{m+1}{2} \cdot \Gamma\frac{1}{2}}{2\Gamma 2}$

Give that, $m = 6$; we get,

$$\therefore \int_0^{\frac{\pi}{2}} \sin^6 x \, dx = \frac{\Gamma\frac{6+1}{2} \cdot \Gamma\frac{1}{2}}{2\Gamma 2}$$

$$= \frac{\Gamma\frac{7}{2} \cdot \Gamma\frac{1}{2}}{2\Gamma 4}$$

$$= \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\frac{1}{2} \cdot \Gamma\frac{1}{2}}{2 \cdot 3 \cdot 2 \cdot 1 \cdot \Gamma 1}$$

$$= \frac{5 \cdot 3 \cdot (\sqrt{\pi})^2}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 2}$$

$$= \frac{5\pi}{32}$$