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Chapter-3

Z-transform

The z-transform is a mathematical transformation used to represent a discrete-time signal in the complex frequency domain.

If $x(n)$ is a discrete-time signal, then its z-transform is defined as:

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}}$$

where,
$x(n)$ = discrete time signal
z = complex variable ($z = re^{j\omega}$)
$X(z)$ = z-transform of ~~x(n)~~ $x(n)$

z-transform may be one-side (unilateral) or two-sided (bilateral)

The one-sided or unilateral Z-transform is defined as:

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

Region of convergence (ROC):

The Region of Convergence (ROC) is the region in the z-plane for which the z-transform summation converges to a finite value.

For the signal $x(n) = a^n u(n)$

the z-transform is $X(z) = \dfrac{1}{1 - a z^{-1}}$

and the ROC is $|z| > |a|$

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Advantages of z-transform:

1) The z-transform converts the difference equations of a discrete-time system into linear algebraic equations so that the analysis becomes easy and simple

2) Convolution in time domain is converted into multiplication in z-domain.

3) z-transform exists for most of the signals for which Discrete-Time Fourier Transform (DTFT) does not exist.

4) Also since the Fourier transform is nothing but z-transform evaluated along the unit circle in the z-plane, the frequency response can be determined.

> Convolution করতে Time lage But z transform এ Just গুণ করলেই হয়ে যায়.
> $y(n) = x(n) * h(n)$ কষিন $Y(z) = X(z)H(z)$ হবে
> DTFT exist না করলেও কেবল বাস signal এ z-transform exist করে
> Fourier Transform z-transform এর এক special case
> → যুক্ত z-plane এ
> → শুধু Unit circle এর উপর থাকা z-plane এ

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$x(n) = \{\underset{\uparrow}{1}, 2, 3, 4, 0, 2\}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\therefore X(z) = x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + x(5)z$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-5}$$

$x(n) = \{1, 2, \underset{\uparrow}{3}, 4, 0, 2\}$

$$\therefore X(z) = x(-2)z^{2} + x(-1)z + x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$

$$= z^{2} + 2z + 3 + 4z^{-1} + 2z^{-3}$$

Relation Between DTFT and Z-transform

The Z-transform of a discrete-time signal $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where z is a complex variable given by

$z = re^{j\omega}$ where r is the radius of a circle

The Discrete-Time Fourier Transform (DTFT) is defined as

$$X(e^{j\omega}) \text{ or } X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

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$$\therefore X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)(re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [x(n) r^{-n}] e^{-j\omega n}$$

Example 3.1 Prove that, for causal sequences, the ROC is the exterior of a circle of radius r.

Proof:

For a causal sequence,

$$x(n) = 0 \quad \text{for } n < 0$$

The z-transform is

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

For convergence of the z-transform,

$$\sum_{n=0}^{\infty} |x(n) z^{-n}| < \infty$$

Consider a causal sequence,

$$x(n) = r^n u(n)$$

$$\therefore X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} r^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} r^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (r z^{-1})^n$$

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The above summation converges for

$$|rz^{-1}| < 1, \text{ i.e. for } |z| > r$$

Hence, for the causal sequences, the ROC is the exterior of a circle of radius r.

Example 3.6

The given sequence values are:

$$x(0) = 1,\ x(1) = 0,\ x(2) = -2,\ x(3) = 3,\ x(4) = 5 \text{ and } x(5) = 4.$$

we know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\therefore X(z) = x(0)z^{0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + x(5)z^{-5}$$

$$\therefore Z[x(n)] = X(z) = 1 - 2z^{-2} + 3z^{-3} + 5z^{-4} + 4z^{-5}$$

The $X(z)$ converges for all values of z except at $z=0$

$$\therefore \text{ROC}: |z| > 0$$

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Properties of Z-Transform

Linearity Property:

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The linearity property of Z-transform states that, the Z-transform of a weighted sum of two signals is equal to the sum of individual Z-transforms.

That is, the linearity property states that

If $x_1(n) \xleftrightarrow{ZT} X_1(z)$, with ROC $= R_1$

and $x_2(n) \xleftrightarrow{ZT} X_2(z)$, with ROC $= R_2$

Then $a\,x_1(n) + b\,x_2(n) \xleftrightarrow{ZT} a\,X_1(z) + b\,X_2(z)$, with ROC $= R_1 \cap R_2$

Proof:

We know that

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$\therefore Z[a\,x_1(n) + b\,x_2(n)] = \sum_{n=-\infty}^{\infty} [a\,x_1(n) + b\,x_2(n)]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a\,x_1(n)\, z^{-n} + \sum_{n=-\infty}^{\infty} b\,x_2(n)\, z^{-n}$$

$$= a\sum_{n=-\infty}^{\infty} x_1(n)\, z^{-n} + b\sum_{n=-\infty}^{\infty} x_2(n)\, z^{-n}$$

$$= a\,X_1(z) + b\,X_2(z)\ ;\ \text{ROC}: R_1 \cap R_2$$

$$\therefore \boxed{a\,x_1(n) + b\,x_2(n) \xleftrightarrow{ZT} a\,X_1(z) + b\,X_2(z)}$$

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The ROC for the Z-transform of a sum of sequences is equal to the intersection of the ROCs of the individual transforms.

V.I

Time Shifting Property:

The time shifting property of Z-transform states that

If $x(n) \xleftrightarrow{ZT} X(z)$, with zero initial conditions with ROC = R

Then $x(n-m) \xleftrightarrow{ZT} z^{-m} X(z)$

with ROC = R except for the possible addition or deletion of the origin or infinity.

Multiplication by an Exponential Sequence property:

The multiplication by an exponential sequence property of Z-transform states that

If $x(n) \xleftrightarrow{ZT} X(z)$ with ROC = R

Then $a^n x(n) \xleftrightarrow{ZT} X\left(\frac{z}{a}\right)$ with ROC = $|a|R$

where a is a complex number.

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V.I

Time Reversal Property:

The time reversal property of Z-transform states that

If $x(n) \xleftrightarrow{ZT} X(Z)$, with ROC = R

Then $x(-n) \xleftrightarrow{ZT} X\left(\frac{1}{Z}\right)$, with ROC = $\frac{1}{R}$

Time Expansion Property:

The time expansion property of Z-transform states that

If $x(n) \xleftrightarrow{ZT} X(Z)$, with ROC = R

Then $x_k(n) \xleftrightarrow{ZT} X(Z^k)$, with ROC = $R^{1/k}$

where $x_k(n) = x\left(\frac{n}{k}\right)$, if n is an integer multiple of k

$= 0$, otherwise

V.I.

Multiplication by n or Differentiation in Z-domain Property:

The multiplication by n or differentiation in Z-domain property of Z-transform states that

If $x(n) \xleftrightarrow{ZT} X(Z)$, with ROC = R

The $n\,x(n) \xleftrightarrow{ZT} -Z\frac{d}{dZ}X(Z)$, with ROC = R

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C.T

V.V.V.I.

Convolution Property:

The convolution property of z-transform states that z-transform of the convolution of two signals is equal to the multiplication of their z-transform, i.e.

If $x_1(n) \xleftrightarrow{ZT} X_1(z)$, with ROC $= R_1$

and $x_2(n) \xleftrightarrow{ZT} X_2(z)$, with ROC $= R_2$

Then $x_1(n) * x_2(n) \xleftrightarrow{ZT} X_1(z) X_2(z)$, with ROC $= R_1 \cap R_2$

C.T

V.V.V.I.

Proof:

We know that

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

Let $x(n) = x_1(n) * x_2(n)$

We have

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\therefore Z[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \right] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\, z^{-(n-k)}\, z^{-k}$$

Interchanging the order of summations,

$$X(z) = \sum_{k=-\infty}^{\infty} x_1(k) z^{-k} \sum_{n=-\infty}^{\infty} x_2(n-k) z^{-(n-k)}$$

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$\therefore X(z) = X_1(z) X_2(z)$

$$\boxed{\therefore x_1(n) * x_2(n) \xleftrightarrow{ZT} X_1(z) X_2(z); \; ROC = R_1 \cap R_2}$$

The Multiplication Property or Complex Convolution property:

If $x_1(n) \xleftrightarrow{ZT} X_1(z)$ and $x_2(n) \xleftrightarrow{ZT} X_2(z)$

Then $x_1(n) x_2(n) \xleftrightarrow{ZT} \frac{1}{2\pi j} \oint_c X_1(v) X_2\left(\frac{z}{v}\right) v^{-1} dv$

V.I. Correlation property:

The correlation property of Z-transform states that

If $x_1(n) \xleftrightarrow{ZT} X_1(z)$ and $x_2(n) \xleftrightarrow{ZT} X_2(z)$

Then $R_{12}(n) = x_1(n) \circledast x_2(n) \xleftrightarrow{ZT} X_1(z) X_2(z^{-1})$

Sequence $x(n)$	Z-transform	ROC
$a^n u(n)$	$\frac{z}{z-a} = \frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$a^{n-1} u(n-1)$	$\frac{1}{z-a}$	
$u(n)$	$\frac{z}{z-1}$	
$n\, u(n-1)$	$\frac{z}{(z-1)^2}$	

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Properties of ROC:

- ROC is a circular region (ring) in the z-plane centered at the origin.
- ROC does not contain any poles.
- For causal (right-sided) sequence, ROC is outside the outermost pole.
- For anti-causal (left-sided) sequence, ROC is inside the innermost pole.
- For two-sided sequence, ROC lies between two poles.
- For finite-duration sequence, ROC is the entire z-plane except possibly $z=0$ or $z=\infty$.
- ROC must be a connected region.

Example 3.17

Find the convolution of the sequences

$$x_1(n) = \left(\frac{1}{2}\right)^n u(n) \text{ and } x_2(n) = \left(\frac{1}{3}\right)^{n-2} u(n-2)$$

using

a) Convolution property of Z-transforms and

b) Time domain method.

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=> a) Given,

$$x_1(n) = \left(\frac{1}{2}\right)^n u(n) \quad \text{and} \quad x_2(n) = \left(\frac{1}{3}\right)^{n-2} u(n-2)$$

$$\therefore X_1(z) = Z\left[\left(\frac{1}{2}\right)^n u(n)\right] = \frac{1}{1-(1/2)z^{-1}} = \frac{z}{z-(1/2)} \; ; \; ROC, |z| > \frac{1}{2}$$

$$\text{and } X_2(z) = Z\left[\left(\frac{1}{3}\right)^{n-2} u(n-2)\right] = z^{-2} Z\left[\left(\frac{1}{3}\right)^n u(n)\right]$$

$$= z^{-2} \frac{1}{1-(1/3)z^{-1}} = z^{-2} \cdot \frac{z}{z-(1/3)}$$

$$= \frac{z^{-1}}{z-(1/3)} \; ; \; ROC, |z| > \frac{1}{3}$$

we know that,

$$x(n) = x_1(n) * x_2(n)$$

$$\therefore Z[x(n)] = X(z) = Z[x_1(n) * x_2(n)] = X_1(z) X_2(z)$$

$$= \frac{z}{z-(1/2)} \cdot \frac{z^{-1}}{z-(1/3)}$$

$$= \frac{1}{[z-(1/2)][z-(1/3)]}$$

$$\Rightarrow Z[\cancel{x_1(n) * x_2(n)}\; x(n)] = 6\left[\frac{1}{z-(1/2)} - \frac{1}{z-(1/3)}\right]$$ [inverse operator / partial fraction]

$$\therefore x(n) = z^{-1}\left[\frac{1}{z-\frac{1}{2}} - \frac{1}{z-\frac{1}{3}}\right] 6$$

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Taking inverse z-transform

$$\boxed{\frac{1}{z-a} \xleftrightarrow{z^{-1}} a^{n-1}\, u(n-1)}$$

$$\therefore x(n) = 6\left[\left(\frac{1}{2}\right)^{n-1} u(n-1) - \left(\frac{1}{3}\right)^{n-1} u(n-1)\right]$$

b) $$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

$$= \sum_{k=-\infty}^{n-2} \left(\frac{1}{2}\right)^{k} u(k) \left(\frac{1}{3}\right)^{n-2-k} u(n-2-k)$$

$$= \sum_{k=0}^{n-2} \left(\frac{1}{2}\right)^{k} \left(\frac{1}{3}\right)^{n-2-k}$$

$$= \sum_{k=0}^{n-2} \left(\frac{1}{2}\right)^{k} \left(\frac{1}{3}\right)^{n} \left(\frac{1}{3}\right)^{-2} \left(\frac{1}{3}\right)^{-k}$$

$$\boxed{\sum_{k=0}^{N} r^k = \frac{1-r^{N+1}}{1-r}}$$

$$= 9\left(\frac{1}{3}\right)^{n} \sum_{k=0}^{n-2} \left(\frac{3}{2}\right)^{k} = 9\left(\frac{1}{3}\right)^{n}\left[\frac{1-(3/2)^{n-1}}{1-(3/2)}\right]$$

$$= 9\left(\frac{1}{3}\right)^{n}\left[\frac{1-(3/2)^{n-1}}{\frac{2-3}{2}}\right] = -18\left(\frac{1}{3}\right)^{n}\left[1-\left(\frac{3}{2}\right)^{n-1}\right]$$

$$= -18\left(\frac{1}{3}\right)^{n} + 18\left(\frac{1}{3}\right)^{n}\left(\frac{3}{2}\right)^{n-1}$$

$$= -18\,\frac{2}{3}\left(\frac{1}{3}\right)^{n} + 18\left(\frac{1}{3}\right)^{n}\left(\frac{3}{2}\right)^{n}\left(\frac{3}{2}\right)^{-1}$$

$$= -6\left(\frac{1}{3}\right)^{n-1} + 18\left(\frac{1}{3}\right)^{n} \cdot 3^{n} \cdot \frac{1}{3} \cdot \left(\frac{1}{2}\right)^{n} \cdot \left(\frac{1}{2}\right)^{-1}$$

$$= -6\left(\frac{1}{3}\right)^{n-1} + 6\left(\frac{1}{2}\right)^{n-1}$$

$$= 6\left[\left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{3}\right)^{n-1}\right] u(n-1)$$

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Example 3.18

Determine the cross correlation sequence $r_{x_1x_2}(l)$ of the sequences

$x_1(n) = (1, 2, 3, 4)$ and $x_2(n) = (4, 3, 2, 1)$

$\Rightarrow$ The cross correlation sequence can be obtained using the correlation property of z-transforms.

Therefore, for the given $x_1(n)$ and $x_2(n)$, we have

$$X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

and $$X_2(z) = 4 + 3z^{-1} + 2z^{-2} + z^{-3}$$

Thus, we have $X_2(z^{-1}) = 4 + 3z + 2z^2 + z^3$

Now $R_{x_1x_2}(z) = X_1(z)\, X_2(z^{-1})$

$$= (1 + 2z^{-1} + 3z^{-2} + 4z^{-3})(4 + 3z + 2z^2 + z^3)$$

$$= 4 + 3z + 2z^2 + z^3 + 8z^{-1} + 6 + 4z + 2z^2 + 12z^{-2} + 9z^{-1} + 6 + 3z + 16z^{-3} + 12z^{-2} + 8z^{-1} + 4$$

$$= 1z^3 + 4z^2 + 10z + 20 + 25z^{-1} + 24z^{-2} + 16z^{-3}$$

Therefore, $r_{x_1x_2}(l) = z^{-1}[R_{x_1x_2}(z)] = \{1, 4, 10, \underset{\uparrow}{20}, 25, 24, 16\}$

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Inverse z-Transform

The process of finding the time domain signal $x(n)$ from its z-transform $X(z)$ is called the inverse z-transform which is denoted as:

$$x(n) = z^{-1}[X(z)]$$

(P) 4 methods to find the inverse z-transform.

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a) Power series method or long division method.

b) Partial fraction expansion method.

c) Complex inversion integral method or the residue method.

d) Convolution integral method.

Long Division Method

We expand $X(z)$ into a power series of z^{-1}

Because, $X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots$

So, the coefficients of z^{-n} directly gives $x(n)$.

$$X(z) = \frac{N(z)}{D(z)}$$

If ROC is $|z| > |a|$, expand in descending power of z^{-1} (Causal)

If ROC is $|z| < |a|$, expand in ascending power of z (Anticausal)

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Example 3.22:

Find the ~~inverse~~ inverse Z-transform of

$$X(z) = z^3 + 2z^2 + z + 1 - 2z^{-1} - 3z^{-2} + 4z^{-3}$$

⇒ we know that,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \ldots + x(-3) z^3 + x(-2) z^2 + x(-1) z + x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + \ldots$$

Comparing this $X(z)$ with given $X(z)$, we have

$$x(n) = \{1, 2, 1, \underset{\uparrow}{1}, -2, -3, 4\}$$

Alternatively, taking inverse Z-transform of $X(z)$, we have

$$x(n) = \delta(n+3) + 2\delta(n+2) + \delta(n+1) + \delta(n) - 2\delta(n-1) - 3\delta(n-2) + 4\delta(n-3)$$

CT. Example 3.23

Using long division determine the ...

1. State and Prove the convolution property of Z-transform. 6 m of
2. Using long division, determine the inverse Z-transform for the following function: 6

$$X(z) = \frac{z^2 + 2Z}{z^3 - 3z^2 + 4z + 1}$$

3. A causal LTI system is defined by the difference equation: 8

$$2y(n) - y(n-2) = x(n-1) + 3x(n-2) + 2x(n-3)$$

Find the transfer function and frequency response of the system.

... in $|z| > 1$, $x(n)$ must be a causal sequence.

For getting a causal sequence, the $N(z)$ and $D(z)$ of $X(z)$ must be put either in descending power of z or in ascending power of z^{-1} before performing

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Long division.

In the given $X(z)$ both $N(z)$ and $D(z)$ are already in decending powers of z.

$$
\begin{array}{r|l}
 & z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \\
\hline
z^3 - 3z^2 + 4z + 1 & z^2 + 2z \\
 & z^2 - 3z + 4 + z^{-1} \\
\hline
 & 5z - 4 - z^{-1} \\
 & 5z - 15 + 20z^{-1} + 5z^{-2} \\
\hline
 & \cancel{19 - 19z^{-1}} \\
 & 11 - 21z^{-1} - 5z^{-2} \\
 & 11 - 33z^{-1} + 44z^{-2} + 11z^{-3} \\
\hline
 & 12z^{-1} - 49z^{-2} - 11z^{-3} \\
 & 12z^{-1} - 36z^{-2} + 48z^{-3} + 12z^{-4} \\
\hline
 & -13z^{-2} - 59z^{-3} - 12z^{-4}
\end{array}
$$

$\therefore X(z) = z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \ldots$

$\therefore x(n) = \{0, 1, 5, 11, 12, -13, \ldots\}$

Writing $N(z)$ and $D(z)$ of $X(z)$ in ascending ~~order~~ power of z^{-1}, we have

$$X(z) = \frac{N(z)}{D(z)} = \frac{z^2 + 2z}{z^3 - 3z^2 + 4z + 1} = \frac{z^{-1} + 2z^{-2}}{1 - 3z^{-1} + 4z^{-2} + z^{-3}}$$

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$$
\begin{array}{r|l}
 & z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \\
\hline
1 - 3z^{-1} + 4z^{-2} + z^{-3} & z^{-1} + 2z^{-2} \\
 & z^{-1} - 3z^{-2} + 4z^{-3} + z^{-4} \\
\hline
 & 5z^{-2} - 4z^{-3} - z^{-4} \\
 & 5z^{-2} - 15z^{-3} + 20z^{-4} + 5z^{-5} \\
\hline
 & 11z^{-3} - 21z^{-4} - 5z^{-5} \\
 & 11z^{-3} - 33z^{-4} + 44z^{-5} + 11z^{-6} \\
\hline
 & 12z^{-4} - 49z^{-5} - 11z^{-6} \\
 & 12z^{-4} - 36z^{-5} + 48z^{-6} + 12z^{-7} \\
\hline
 & -13z^{-5} - 59z^{-6} - 12z^{-7}
\end{array}
$$

$\therefore X(z) = z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \ldots$

$\therefore x(n) = \{0, 1, 5, 11, 12, -13, \ldots\}$

Example 3.30

Using long division, determine the inverse z-transform of

$$X(z) = \frac{z^2 + z + 2}{z^3 - 2z^2 + 3z + 4} \; ; \; ROC; \; |z| < 1$$

$\Rightarrow$ Since ROC is $|z| < 1$, $x(n)$ must be a non-causal sequence. For getting a non-causal sequence, the $N(z)$ and $D(z)$ must be put either in ascending power of z or in descending powers of z^{-1} before performing long division.

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For ascending power of z,

$$X(z) = \frac{z^2+z+2}{z^3-2z^2+3z+4} = \frac{2+z+z^2}{4+3z-2z^2+z^3}$$

$$\begin{array}{r|l}
 & \frac{1}{2} - \frac{1}{8}z + \frac{19}{32}z^2 - \frac{81}{128}z^3 + \frac{411}{512}z^4 \\
\hline
4+3z-2z^2+z^3 & 2+z+z^2 \\
 & 2+\frac{3}{2}z-z^2+\frac{1}{2}z^3 \\
\hline
 & -\frac{1}{2}z+2z^2-\frac{1}{2}z^3 \\
 & -\frac{1}{2}z-\frac{3}{8}z^2+\frac{1}{4}z^3-\frac{1}{8}z^4 \\
\hline
 & \frac{19}{8}z^2-\frac{3}{4}z^3+\frac{1}{8}z^4 \\
 & \frac{19}{8}z^2+\frac{57}{32}z^3-\frac{19}{16}z^4+\frac{19}{32}z^5 \\
\hline
 & -\frac{81}{32}z^3+\frac{21}{16}z^4-\frac{19}{32}z^5 \\
 & -\frac{81}{32}z^3-\frac{243}{128}z^4+\frac{81}{64}z^5-\frac{81}{128}z^6 \\
\hline
 & \frac{411}{128}z^4-129z^5+\frac{81}{128}z^6
\end{array}$$

$$\therefore X(z) = \frac{1}{2} - \frac{1}{8}z + \frac{19}{32}z^2 - \frac{81}{128}z^3 + \frac{411}{512}z^4 + \cdots$$

$$\therefore x(n) = \left\{ \cdots\cdots, \frac{411}{512}, -\frac{81}{128}, \frac{19}{32}, -\frac{1}{8}, \underset{\uparrow}{\frac{1}{2}} \right\}$$

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Transform Analysis of LTI System:

In an LTI (Linear Time-Invariant) system, input and output are related by convolution:

$$y(n) = x(n) * h(n)$$

Taking Z-transform on both sides,

$$Y(z) = X(z)\,H(z)$$

So in Z-domain,

→ Convolution becomes multiplication

→ System analysis becomes easy.

This process is called transform analysis of LTI systems.

System Function and Impulse Response:

The system function is defined as the ratio of output transform to input transform.

$$H(z) = \frac{Y(z)}{X(z)}$$

→ It completely describes the behavior of an LTI system.

→ Also ~~kno~~ called transfer function.

→ Contains ~~poles and~~ poles and zeros.

→ Determines ~~pole~~ stability and causality.

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If the input $x(n)$ is an impulse sequence, then $X(z) = 1$. So, $Y(z) = H(z)$. So the transfer function is also defined as the z-transform of the impulse response of the system.

- The poles of the system are defined as the values of z for which the system function $H(z)$ is infinity.
- The zeros of the system are the values of z for which the system function $H(z)$ is zero.

Relationship between Transfer function and Difference Equation

In terms of a difference equation, an nth order discrete-time LTI system is specified as:

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

Expanding it, we have

$$a_0 y(n) + a_1 y(n-1) + a_2 y(n-2) + \cdots + a_N y(n-N) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \cdots + b_M x(n-M)$$

Taking z-transform on both sides and neglecting the initial conditions, we obtain

$$a_0 Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \cdots + a_N z^{-N} Y(z) \cancel{= b_0 Y(z) + b_1 z^{-1} Y(z)}$$

$$= b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \cdots + b_M z^{-M} X(z)$$

$$\Rightarrow Y(z)\left[a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}\right] = X(z)\left[b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}\right]$$

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$$\therefore \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}$$

$$= \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Now, $Y(z)/X(z) = H(z)$ is called the transfer function of the system or the system function. The frequency responce of a system is obtained by substituting $z = e^{j\omega}$ in $H(z)$.

Example 13.43

A causal system is represented by

$$H(z) = \frac{z+2}{2z^2 - 3z + 4}$$

Find the difference equation and the frequency response of the system.

$\Rightarrow$ Given, $H(z) = \dfrac{z+2}{2z^2 - 3z + 4}$

As the system is causal, $H(z)$ is ~~represented~~ expressed in negative powers of z.

উপরের নিচের 1st check highest power. Then divide it by all

$$\therefore H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{z+2}{2z^2 - 3z + 4} = \frac{z^{-1} + 2z^{-2}}{2 - 3z^{-1} + 4z^{-2}}$$

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$$\Rightarrow 2Y(z) - 3z^{-1}Y(z) + 4z^{-2}Y(z) = z^{-1}X(z) + 2z^{-2}X(z)$$

Taking inverse z-transform on both the sides, we have

$$2y(n) - 3y(n-1) + 4y(n-2) = x(n-1) + 2x(n-2)$$

which is the required difference equation.

putting $z = e^{j\omega}$ in $H(z)$, we get the frequency response $H(\omega)$ of the system.

$$H(\omega) = \frac{z+2}{2z^2-3z+4}\Big|_{z=e^{j\omega}}$$

$$= \frac{e^{j\omega}+2}{2e^{j2\omega}-3e^{j\omega}+4}$$

$$\boxed{\begin{aligned} e^{j\omega} &= \cos\omega + j\sin\omega \\ e^{j2\omega} &= \cos 2\omega + j\sin 2\omega \end{aligned}}$$

$$= \frac{2+\cos\omega + j\sin\omega}{4+2\cos 2\omega + 2j\sin 2\omega - 3\cos\omega - 3j\sin\omega}$$

$$= \frac{2+\cos\omega + j\sin\omega}{4+(2\cos 2\omega - 3\cos\omega) + j(2\sin 2\omega - 3\sin\omega)}$$

Example 3.60

Determine the impulse response and step response of the causal system given below and discuss on stability

$$y(n) + y(n-1) - 2y(n-2) = x(n-1) + 2x(n-2)$$

$\Rightarrow$ The given difference equation is

$$y(n) + y(n-1) - 2y(n-2) = x(n-1) + 2x(n-2)$$

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Taking z-transform on both sides, we have

$$Y(z) + z^{-1} Y(z) - 2 z^{-2} Y(z) = z^{-1} X(z) + 2 z^{-2} X(z)$$

$$\Rightarrow Y(z) \left[1 + z^{-1} - 2 z^{-2} \right] = X(z) \left[z^{-1} + 2 z^{-2} \right]$$

The system transfer function $H(z)$ is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} + 2 z^{-2}}{1 + z^{-1} - 2 z^{-2}}$$

$$= \frac{z + 2}{z^2 + z - 2}$$

$$= \frac{z + 2}{(z+2)(z-1)} = \frac{1}{z-1}$$

Taking inverse z-transform, the impulse response of the system is

$$h(n) = u(n-1)$$

$$\frac{1}{z-a} \xleftrightarrow{z^{-1}} a^{n-1} u(n-1)$$

For step response, $x(n) = u(n)$

$$\therefore X(z) = \frac{z}{z-1}$$

$\therefore$ The output $Y(z) = X(z) H(z) = \frac{z}{z-1} \cdot \frac{1}{z-1} = \frac{z}{(z-1)^2}$

Taking inverse z transform on both the sides, the step response is

$$y(n) = n u(n-1)$$

For stability:
→ ROC must include unit circle
→ pole is at $z = 1$
Since poles lies on unit circle
→ system is not stable

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C.T Example 3.69 A Causal LTI system is defined by the difference equation

$$2y(n) - y(n-2) = x(n-1) + 3x(n-2) + 2x(n-3)$$

Find the transfer function, frequency response, magnitude response and phase response of the system.

$\Rightarrow$ Given, $2y(n) - y(n-2) = x(n-1) + 3x(n-2) + 2x(n-3)$

Taking Z-transform on both sides and neglecting initial conditions, we have

$$2Y(z) - z^{-2}Y(z) = z^{-1}X(z) + 3z^{-2}X(z) + 2z^{-3}X(z)$$

$$\Rightarrow Y(z)\left[2 - z^{-2}\right] = X(z)\left[z^{-1} + 3z^{-2} + 2z^{-3}\right]$$

$\therefore$ The transfer function of the system

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} + 3z^{-2} + 2z^{-3}}{2 - z^{-2}} = \frac{z^2 + 3z + 2}{2z^3 - z}$$

The frequency response of the system

$$H(\omega) = \frac{e^{j2\omega} + 3e^{j\omega} + 2}{2e^{j3\omega} - e^{j\omega}}$$

The magnitude response of the system

$$|H(\omega)| = \left|\frac{(\cos 2\omega + 3\cos\omega + 2) + j(\sin 2\omega + 3\sin\omega)}{(2\cos 3\omega - \cos\omega) + j(2\sin 3\omega - \sin\omega)}\right|$$

$$= \sqrt{\frac{(\cos 2\omega + 3\cos\omega + 2)^2 + (\sin 2\omega + 3\sin\omega)^2}{(2\cos 3\omega - \cos\omega)^2 + (2\sin 3\omega - \sin\omega)^2}}$$

$|a + jb| = \sqrt{a^2 + b^2}$

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$H(\omega) = \frac{N(\omega)}{D(\omega)}$ $\angle H(\omega) = \angle N(\omega) - \angle D(\omega)$

phase response

$$\angle H(\omega) = \tan^{-1}\frac{\sin 2\omega + 3\sin\omega}{\cos 2\omega + 3\sin\omega + 2} - \tan^{-1}\frac{2\sin 3\omega - \sin\omega}{2\cos 3\omega - \cos\omega}$$

Example 3.21 Find the response of the time-invariant system with impulse response $h(n) = \{1, 2, 1, -1\}$ to an input signal $x(n) = \{1, 2, 3, 6\}$.

$\Rightarrow$ Let $y(n)$ be the response or output of the system. We know that the response is given by the convolution of input $x(n)$ and impulse response $h(n)$.

$$\therefore y(n) = x(n) * h(n)$$

$$\therefore Y(z) = X(z)H(z)$$

Given,

$$x(n) = \{1, 2, 3, 6\} = \delta(n) + 2\delta(n-1) + 3\delta(n-2) + 6\delta(n-3)$$

$$\therefore X(z) = 1 + 2z^{-1} + 3z^{-2} + 6z^{-3}$$

$$h(n) = \{1, 2, 1, -1\} = \delta(n) + 2\delta(n-1) + \delta(n-2) - \delta(n-3)$$

$$\therefore H(z) = 1 + 2z^{-1} + z^{-2} - z^{-3}$$

$$\therefore Y(z) = X(z)H(z)$$

$$= (1 + 2z^{-1} + 3z^{-2} + 6z^{-3})(1 + 2z^{-1} + z^{-2} - z^{-3})$$

$$= 1 + 2z^{-1} + z^{-2} - z^{-3} + 2z^{-1} + 4z^{-2} + 2z^{-3} - 2z^{-4} + 3z^{-2} + 6z^{-3} + 3z^{-4} - 3z^{-5} + 6z^{-3} + 12z^{-4} + 6z^{-5} - 6z^{-6}$$

$$= 1 + 4z^{-1} + 8z^{-2} + 13z^{-3} + 13z^{-4} + 3z^{-5} - 6z^{-6}$$

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Taking inverse z-transform on both sides, we have;

$$y(n) = \delta(n) + 4\delta(n-1) + 8\delta(n-2) + 13\delta(n-3) + 13\delta(n-4) + 3\delta(n-5) - 6\delta(n-6)$$

$$\therefore y(n) = \{1, 4, 8, 13, 13, 3, -6\}$$

Chapter - 4

System Realization

System realization is the process of implementing a system in terms of its input-output relationship using basic building block like adders, multipliers, and delay elements.

Adder: An adder is used to add two or more signals. The output of adder is equal to the sum of all incoming ~~singals~~ signals.

Constant multiplier: A constant multiplier is used to multiply the signals by a constant. The output of the multiplier is equal to the product of the input signal and the constant of the multiplier.

Unit delay element: A unit delay element is used to delay the signal passing through it by one sampling time.

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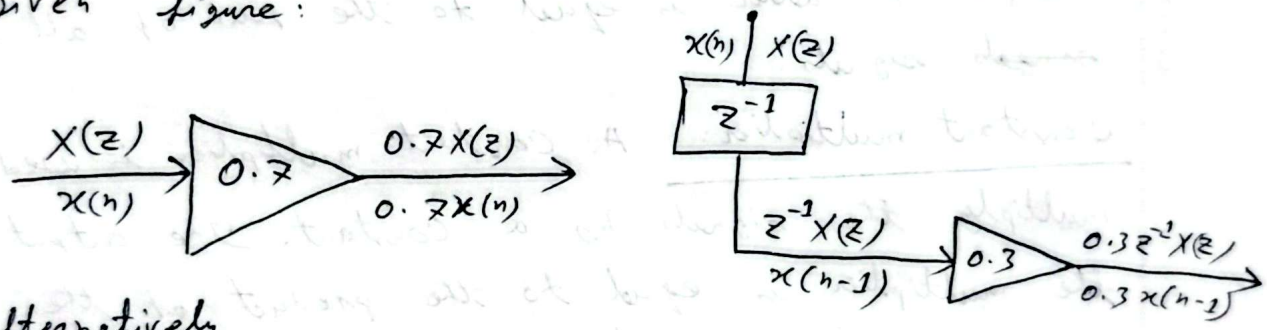
Ex 4.1 Construct the block diagram for the discrete-time system whose input-output relations are described by the following difference equation:

a) $y(n) = 0.7x(n) + 0.3x(n-1)$

b) $y(n) = 0.5\, y(n-1) + 0.8\, x(n) + 0.4\, x(n-1)$

⇒ Given $y(n) = 0.7x(n) + 0.3x(n-1)$

The system may be ~~rel~~ realized by ~~th~~ using the difference equation directly or by using the z-transformed version of that. The individual terms of the given difference equation are $0.7x(n)$ and $0.3x(n-1)$. They are represented by the basic elements as the given figure:



Alternatively,

Taking z-transform on both sides of the given difference equation, we have

$$Y(z) = 0.7X(z) + 0.3z^{-1}X(z)$$

The individual terms of the above equation are: $0.7X(z)$ and $0.3z^{-1}X(z)$.

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The input to the system is $X(z)$ [or $x(n)$] and the output of the system is $Y(z)$ [or $y(n)$]. The above elements are connected as shown in the below figure to get the output $Y(z)$ [or $y(n)$].

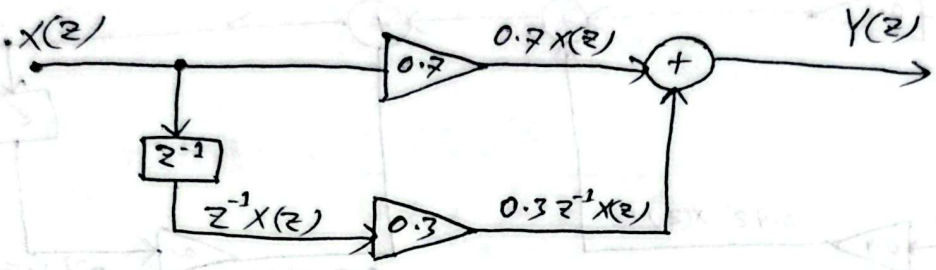

Fig: Realization of system described by $y(n) = 0.7x(n) + 0.3x(n-1)$

b) Given $y(n) = 0.5y(n-1) + 0.8x(n) + 0.4x(n-1)$

The individual terms of the above equations are $0.5y(n-1)$, $0.8x(n)$, $0.4x(n-1)$. Alternatively, taking Z-transform on both sides of the given difference equation, we have

$$Y(z) = 0.5z^{-1}Y(z) + 0.8X(z) + 0.4z^{-1}X(z)$$

The individual terms of the above equation are $0.5z^{-1}Y(z)$, $0.8X(z)$ and $0.4z^{-1}X(z)$. They are represented by the basic elements as shown in Figure 1.

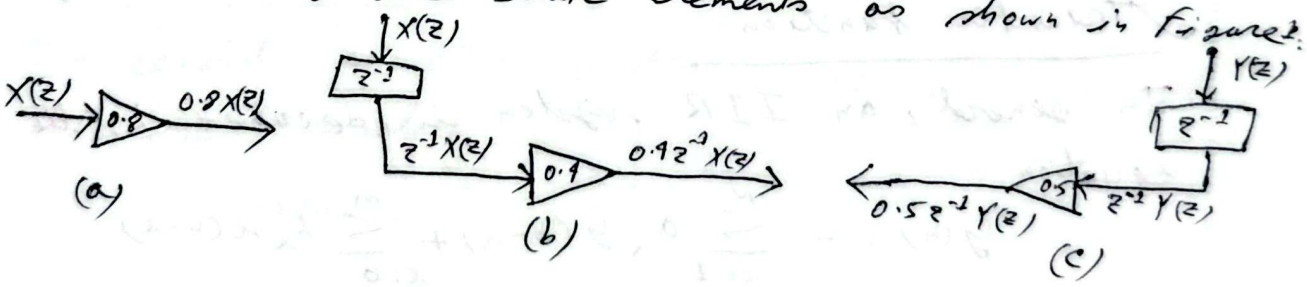

Figure 1: Block diagram representation of (a) $0.8X(z)$, (b) $0.4z^{-1}X(z)$ and (c) $0.5z^{-1}Y(z)$

The input to the system is $X(z)$ and the output is $Y(z)$. The above elements are connected as shown in Figure-2 to get the output $Y(z)$.

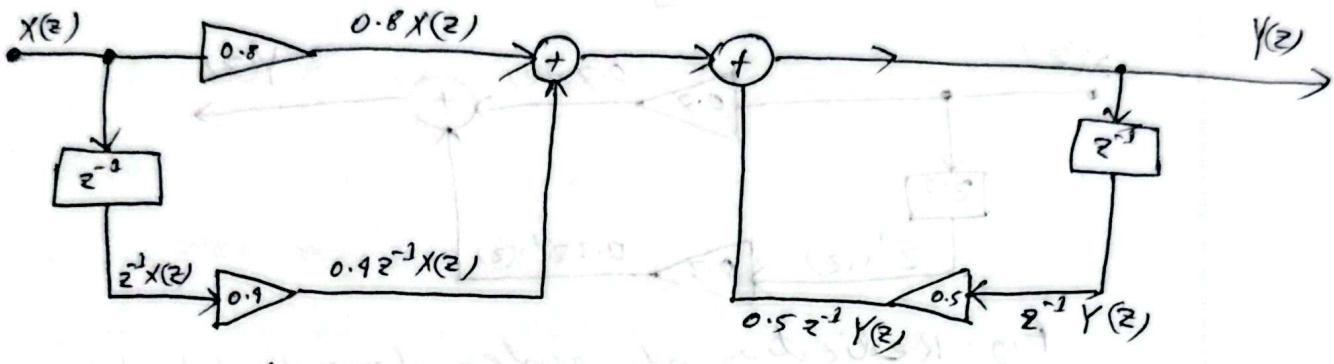

Figure-2: Realization of the system described by ~~y~~

$$y(n) = 0.5\,y(n-1) + 0.8\,x(n) + 0.9\,x(n-1)$$

Structures for Realization of IIR Systems:

IIR systems are systems whose impulse response has infinite number of samples. They are designed by using all the samples of the infinite duration impulse response. The convolution formula for IIR system is given by

$$y(n) = \sum_{k=0}^{\infty} h(k)\,x(n-k)$$

Transfer function:

In general, an IIR system is described by the difference equation

$$y(n) = -\sum_{k=1}^{N} a_k\,y(n-k) + \sum_{k=0}^{M} b_k\,x(n-k)$$

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$$\Rightarrow Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Rightarrow Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Rightarrow Y(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\therefore \frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

Structures For Realization of FIR Systems:

For FIR (Finit duration Impulse Response) systems are the systems whose impulse response consists of finite number of samples. They are designed by using only a finite number of samples of the infinite duration impulse response. The convolution formula for FIR system is given by

$$y(n) = \sum_{k=0}^{N-1} h(k)\, x(n-k)$$

Transfer function:

In general, an FIR system is described by the difference equation

$$y(n) = \sum_{k=0}^{N-1} b_k x(n-k)$$

$$\Rightarrow Y(z) = \sum_{k=0}^{N-1} b_k z^{-k} X(z)$$

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$$\therefore \frac{Y(z)}{X(z)} = H(z) = \sum_{k=0}^{N-1} b_k z^{-k}$$

$$= b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{N-1} z^{-(N-1)}$$

Chapter-5

Discrete-time Fourier Transform

The Discrete time Fourier Transform (DTFT) is used to convert a discrete-time signal from the time domain ~~signal~~ in to the frequency domain representation.

The DTFT of $x(n)$ is defined as:

$$F[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

The inverse DTFT of $X(\omega)$ is defined as:

$$F^{-1}[X(\omega)] = x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$$x(n) \xleftrightarrow{FT} X(\omega)$$

Existence of DTFT:

The Fourier transform exists for a discrete-time sequence $x(n)$ if and only if the sequence is absolutely summable, i.e. the sequence has to satisfy the condition:

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

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The DTFT doesnot exist for the sequences that are growing exponentially (ex. $a^n u(n)$, $a>1$) since they are not absolutely summable.

The difference between Fourier transforms of a discrete-time signal and ~~alan~~ analog signal are as follows:

1. The Fourier transform of analog signals consists of a spectrum with a frequency range $-\infty$ to ∞. ~~Fo~~ But the Fourier transform of discrete-time signals is unique in the frequency range $-\pi$ to π.
2. since the analog signals are continous, the ~~to~~ Fourier transform of analog signals involves integration, but the Fourier transform of discrete-time signals involves summation because the signals are discrete.

Relation between Z-transform and Fourier transform:

The z-transform of a discrete sequence $x(n)$ is defined as:

$$X(z) = \sum_{k=-\infty}^{\infty} x(n) z^{-n}$$; where z is a complex variable.

The Fourier transform of a discrete-time sequence $x(n)$ is defined as:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

The Fourier transform is a special case of the z-transform.

If we put $z = re^{j\omega}$

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$$\therefore X(z) = \sum_{n=-\infty}^{\infty} x(n)(r e^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[x(n) r^{-n}\right] e^{-j\omega n}$$

when $r = 1$,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = X(e^{j\omega}) = X(\omega)$$

Therefore, DTFT is the z-transform evaluated on the unit circle.

Example 5.1 Find the DTFT of the following sequence:

a) $\delta(n)$ b) $u(n)$

c) $\delta(n-m)$ d) $u(n-m)$

e) $a^n u(n)$ f) $-a^n u(n-1)$

g) $\delta(n+3) - \delta(n-3)$ h) $u(n+3) - u(n-3)$

$\Rightarrow$ a) Given,

$$x(n) = \delta(n)$$

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

$$X(\omega) = F\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n) e^{-j\omega n}\Big|_{n=0} = 1$$

$$\therefore F\{\delta(n)\} = 1$$

$$\boxed{\delta(n) \xleftrightarrow{FT} 1}$$

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b) Given $x(n) = u_n$

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$X(\omega) = F\{u(n)\} = \sum_{n=-\infty}^{\infty} u(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (1)\, e^{-j\omega n} = \frac{1}{1 - e^{-j\omega}}$$

$$\therefore F\{u(n)\} = \frac{1}{1 - e^{-j\omega}}$$

$$\boxed{u(n) \xleftrightarrow{FT} \frac{1}{1 - e^{-j\omega}}}$$

c) Given $x(n) = \delta(n-m)$

$$\delta(n-m) = \begin{cases} 1 & \text{for } n = m \\ 0 & \text{for } n \neq m \end{cases}$$

$$X(\omega) = F\{\delta(n-m)\} = \sum_{n=-\infty}^{\infty} \delta(n-m)\, e^{-j\omega n} = e^{-j\omega n}\Big|_{n=m} = e^{-j\omega m}$$

$$\therefore F\{\delta(n-m)\} = e^{-j\omega m}$$

$$\boxed{\delta(n-m) \xleftrightarrow{FT} e^{-j\omega m}}$$

d) Given, $x(n) = u(n-m)$

$$u(n-m) = \begin{cases} 1 & \text{for } n \geq m \\ 0 & \text{for } n < m \end{cases}$$

$$X(\omega) = F\{u(n-m)\} = \sum_{n=-\infty}^{\infty} u(n-m)\, e^{-j\omega n} = \sum_{n=m}^{\infty} (1)\, e^{-j\omega n}$$

$$= e^{-j\omega m} + e^{-j\omega(m+1)} + e^{-j\omega(m+2)} + \cdots$$

$$= e^{-j\omega m}\left(1 + e^{-j\omega} + e^{-j\omega 2\omega} + \cdots\right)$$

$$= \frac{e^{-j\omega m}}{1 - e^{-j\omega}}$$

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$$\therefore F\{u(n-m)\} = \frac{e^{-j\omega m}}{1-e^{-j\omega}}$$

$$\boxed{u(n-m) \xleftrightarrow{FT} \frac{e^{-j\omega m}}{1-e^{-j\omega}}}$$

e) Given $x(n) = a^n u(n)$

$$X(\omega) = F\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n = \frac{1}{1-a e^{-j\omega}}$$

$$\therefore F\{a^n u(n)\} = \frac{1}{1-a e^{-j\omega}}$$

$$\boxed{a^n u(n) \xleftrightarrow{FT} \frac{1}{1-a e^{-j\omega}}}$$

f) Given $x(n) = -a^n u(-n-1)$

$$X(\omega) = F\{-a^n u(-n-1)\} = \sum_{n=-\infty}^{\infty} -a^n u(-n-1) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} -a^n e^{-j\omega n} = -\sum_{n=1}^{\infty} a^{-n} e^{j\omega n}$$

$$= -\sum_{n=1}^{\infty} (a^{-1} e^{j\omega})^n$$

$$= -\left[a^{-1} e^{j\omega} + (a^{-1} e^{j\omega})^2 + (a^{-1} e^{j\omega})^3 + \cdots\right]$$

$$= -a^{-1} e^{j\omega}\left[1 + (a^{-1} e^{j\omega}) + (a^{-1} e^{j\omega})^2 + \cdots\right]$$

$$= \frac{-a^{-1} e^{j\omega}}{1-a^{-1} e^{j\omega}}$$

$$\therefore F\{-a^n u(-n-1)\} = \frac{1}{1-a e^{-j\omega}}$$

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$$\boxed{-a^n u(n)(-u-1) \xleftrightarrow{FT} \frac{1}{1-ae^{-ju}}}$$

g) given $x(n) = \delta(n+3) - \delta(n-3)$

$$X(\omega) = F\{\delta(n+3) - \delta(n-3)\}$$

$$= \sum_{n=-\infty}^{\infty} \{\delta(n+3) - \delta(n-3)\} e^{-j\omega n}$$

$$= e^{-j\omega n}\Big|_{n=-3} - e^{-j\omega n}\Big|_{n=3}$$

$$= e^{j3\omega} - e^{-j3\omega}$$

$$= \cos 3\omega + j\sin 3\omega - (\cos 3\omega - j\sin 3\omega)$$

$$= 2j\sin 3\omega$$

h) Given $x(n) = u(n+3) - u(n-3)$

$$X(\omega) = F\{u(n+3) - u(n-3)\}$$

$$= \sum_{n=-\infty}^{\infty} \{u(n+3) - u(n-3)\} e^{-j\omega n}$$

$$= \sum_{n=-3}^{\infty} (1) e^{-j\omega n} - \sum_{n=3}^{\infty} (1) e^{-j\omega n}$$

$$= e^{-j3\omega} + e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega} + \cdots$$

$$- e^{-j3\omega} - e^{-j4\omega} - \cdots$$

$$= e^{j3\omega} + e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega}$$

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Example: 5.2) Find the DTFT of:

a1) $x(n) = \{1, -2, 2, 3\}$ (P) a2) $x(n) = \{2, -1, 3, 2\}$

b) $x(n) = 3^n u(n)$ c1) $x(n) = (0.5)^n u(n) + 2^n u(-n-1)$

(P) c2) $x(n) = (0.2)^n u(n) - 2^n u(-n-1)$ d) $x(n) = \left(\frac{1}{4}\right)^n u(n+1)$

e) $x(n) = \begin{cases} n, & -4 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$ f) $x(n) = \begin{cases} 1, & 0 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$

g) $x(n) = a^{|n|}$

⇒ a1) Given $x(n) = \{1, -2, 2, 3\}$

$$X(\omega) = F\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= x(0) + x(1) e^{-j\omega} + x(2) e^{-j2\omega} + x(3) e^{-j3\omega}$$

$$= 1 - 2e^{-j\omega} + 2e^{-j2\omega} + 3e^{-j3\omega}$$

(P) a2) Given $x(n) = \{2, -1, 3, 2\}$

$$X(\omega) = F\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= x(0) + x(1) e^{-j\omega} + x(2) e^{-j2\omega} + x(3) e^{-j3\omega}$$

$$= 2 - e^{-j\omega} + 3e^{-j2\omega} + 2e^{-j3\omega}$$

b) Given $x(n) = 3^n u(n)$. The given sequence is not absolutely summable. Therefore, its DTFT does not exists.

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Q1) Given, $x(n) = (0.5)^n u(n) + 2^n u(-n-1)$

$$X(\omega) = F\{x(n)\} = \sum_{n=-\infty}^{\infty} \{(0.5)^n u(n) + 2^n u(-n-1)\} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \{(0.5)^n u(n)\} e^{-j\omega n} + \sum_{n=-\infty}^{\infty} \{2^n u(-n-1)\} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.5)^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} 2^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.5 e^{-j\omega})^n + \sum_{n=1}^{\infty} (2^{-1} e^{j\omega})^n$$

$$= \frac{1}{1 - 0.5 e^{-j\omega}} + \frac{2^{-1} e^{j\omega}}{1 - 2^{-1} e^{j\omega}}$$

$$= \frac{1}{1 - 0.5 e^{-j\omega}} - \frac{1}{1 - 2 e^{-j\omega}}$$

$$\frac{\cancel{2^{-1} e^{j\omega}}}{1 - 2^{-1} e^{j\omega}}$$

$$= \frac{2^{-1}}{e^{-j\omega} - 2^{-1}}$$

$$= -\frac{2^{-1}}{2^{-1} - e^{-j\omega}}$$

$$= -\frac{1}{1 - 2 e^{-2\omega}}$$

(P) Q2) Given, $x(n) = (0.2)^n u(n) - 2^n u(-n-1)$

$$X(\omega) = F\{x(n)\} = \sum_{n=-\infty}^{\infty} \{(0.2)^n u(n) - 2^n u(-n-1)\} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \{(0.2)^n u(n)\} e^{-j\omega n} - \sum_{n=-\infty}^{\infty} \{2^n u(-n-1)\} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.2)^n e^{-j\omega n} - \sum_{n=-\infty}^{-1} 2^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.2 e^{-j\omega})^n - \sum_{n=1}^{\infty} (2^{-1} e^{j\omega})^n$$

$$= \frac{1}{1 - 0.2 e^{-j\omega}} - \frac{2^{-1} e^{j\omega}}{1 - 2^{-1} e^{j\omega}}$$

$$= \frac{1}{1 - 0.2 e^{-j\omega}} + \frac{1}{1 - 2 e^{-j\omega}}$$

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d) Given $x(n) = \left(\frac{1}{4}\right)^n u(n+1)$

$$X(\omega) = F\left\{\left(\frac{1}{4}\right)^n u(n+1)\right\} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n u(n+1)\, e^{-j\omega n}$$

$$= \sum_{n=-1}^{\infty} \left(\frac{1}{4}\right)^n e^{-j\omega n} = \sum_{n=-1}^{\infty} \left(\frac{1}{4} e^{-j\omega}\right)^n$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

$$= \left(\frac{1}{4} e^{-j\omega}\right)^{-1} \left[\sum_{n=0}^{\infty} \frac{1}{4} e^{-j\omega}\right]^n$$

$$= \left(\frac{1}{4} e^{-j\omega}\right)^{-1} \frac{1}{1 - \frac{1}{4} e^{-j\omega}} = \frac{4 e^{j\omega}}{1 - \frac{1}{4} e^{-j\omega}}$$

e) ~~Given $x(n) = \begin{cases} 1, & 0 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$~~

Given $x(n) = \begin{cases} n, & -4 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$

$$X(\omega) = F\{x(n)\} = \sum_{n=-4}^{4} n\, e^{-j\omega n}$$

$$= -4e^{j4\omega} - 3e^{j3\omega} - 2e^{j2\omega} - e^{j\omega} + e^{-j\omega} + 2e^{-j2\omega} + 3e^{-j3\omega} + 4e^{-j4\omega}$$

$$= -4\cos 4\omega - 4j\sin 4\omega - 3\cos 3\omega - 3j\sin 3\omega - 2\cos 2\omega - 2j\sin 2\omega$$
$$- \cos\omega - j\sin\omega + \cos\omega - j\sin\omega + 2\cos 2\omega - 2j\sin 2\omega + 3\cos 3\omega$$
$$- 3\sin 3\omega + 4\cos 4\omega - 4j\sin 4\omega$$

$$= -2j\{4\sin 4\omega + 3\sin 3\omega + 2\sin 2\omega + \sin\omega\}$$

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1) Given $x(n) = \begin{cases} 1, & 0 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$

By definition of Fourier transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{0}^{3} (1) e^{-j\omega n}$$

$$= \frac{1 - e^{-j4\omega}}{1 - e^{-j\omega}} = \frac{1 - e^{-j2\omega} e^{-j2\omega}}{1 - e^{(-j\omega/2)} e^{(-j\omega/2)}}$$

$$\sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1 - r}$$

$$= \frac{e^{-j2\omega} \{e^{j2\omega} - e^{-j2\omega}\}}{e^{(-j\omega/2)} \{e^{(j\omega/2)} - e^{(-j\omega/2)}\}}$$

$$e^{j\theta} - e^{-j\theta} = 2j\sin\theta$$

$$= e^{-j2\omega + j\omega/2} \left[\frac{2j\sin 2\omega}{2j\sin(\omega/2)} \right]$$

$$= \frac{\sin 2\omega}{\sin \omega/2} e^{-j(3/2)\omega}$$

2) Given, $x(n) = a^{|n|}$

$$X(\omega) = F\{a^{|n|}\} e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

$$= \sum_{n=1}^{\infty} (a e^{j\omega})^n + \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

$$= \frac{a e^{j\omega}}{1 - a e^{j\omega}} + \frac{1}{1 - a e^{-j\omega}}$$

$$= \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$

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Inverse Discrete-Time Fourier Transform:

The Inverse Discrete time Fourier Transform is used to convert a signal from the frequency domain back to the time domain.

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Properties of Discrete-Time Fourier Transform:

Linearity Property

If $F[x_1(n)] = X_1(\omega)$ and $F[x_2(n)] = X_2(\omega)$

Then $F[a x_1(n) + b x_2(n)] = a X_1(\omega) + b X_2(\omega)$

Proof: $F\{a x_1(n) + b x_2(n)\} = \sum_{n=-\infty}^{\infty} [a x_1(n) + b x_2(n)] e^{-j\omega n}$

$$= \sum_{n=-\infty}^{\infty} a x_1(n) e^{-j\omega n} + \sum_{n=-\infty}^{\infty} b x_2(n) e^{-j\omega n}$$

$$= a X_1(\omega) + b X_2(\omega)$$

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Time shifting Property:

The time shifting property of DTFT states that

If $F[x(n)] = X(\omega)$

Then $F[x(n-m)] = e^{-j\omega m} X(\omega)$ where m is an integer

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Proof: $F\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(n-m)\, e^{-j\omega n}$

Let $n-m = p$

$\therefore\ n = p+m$

$\therefore\ F\{x(n-m)\} = \sum_{p=-\infty}^{\infty} x(p)\, e^{-j\omega p} = e^{-j\omega m} X(\omega)$

This result shows that the time shifting of a signal by m units does not change its amplitude spectrum but the phase spectrum is changed by $-\omega m$.

(P) V.V.I.

Frequency Shifting Property:

The frequency shifting property of DTFT states that

If $F\{x(n)\} = X(\omega)$

Then $F\{x(n)\, e^{j\omega_0 n}\} = X(\omega-\omega_0)$

Proof: $F\{x(n)\, e^{j\omega_0 n}\} = \sum_{n=-\infty}^{\infty} (x(n)\, e^{j\omega_0 n})\, e^{-j\omega n}$

$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega-\omega_0)n}$

$= X(\omega-\omega_0)$

This property is the dual of the time shifting property.

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Transfer Function:

Proof of Transfer function Using Inverse DTFT

If the input to the system is of the form $e^{j\omega n}$, then the output $y(n)$ is given by

$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)\, e^{j\omega(n-k)} = e^{j\omega n} H(\omega)$$

$$\Rightarrow y(n) = x(n) H(\omega)$$

we have $x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$

$$y(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega\, H(\omega)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, H(\omega)\, e^{j\omega n}\, d\omega$$

$$\Rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(\omega)\, e^{j\omega n}\, d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, H(\omega)\, e^{j\omega n}\, d\omega$$

$$\Rightarrow Y(\omega) = X(\omega)\, H(\omega)$$

$$\therefore H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

where $H(\omega)$ is known as the transfer function.

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Frequency Response of Discrete-Time Systems:

Proof: Let $h(n)$ be the impulse response of an LTI discrete system, and let the input $x(n)$ to the system be a complex exponential $e^{j\omega n}$.

The output of the system $y(n)$ can be obtained by using convolution sum.

$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

For $x(n) = e^{j\omega n}$,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, e^{j\omega(n-k)}$$

$$= e^{j\omega n} \sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega k}$$

$$= \underbrace{e^{j\omega n}}_{\text{Input}} \; \underbrace{H(\omega)}_{\text{Frequency response}}.$$

Example: 5.38 The output $y(n)$ for a linear shift-invariant system, with input $x(n)$ is given by

$$y(n) = x(n) - 2x(n-1) + x(n-2)$$

Determine the magnitude and phase response of the system.

$\Rightarrow$ Given, $y(n) = x(n) - 2x(n-1) + x(n-2)$

Taking Fourier transform on both sides, we have

$$Y(\omega) = X(\omega) - 2e^{-j\omega} X(\omega) + e^{-j2\omega} X(\omega)$$

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The transfer function of the system, i.e., the frequency response of the system is:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = 1 - 2e^{-j\omega} + e^{-j2\omega}$$

$$= 1 - 2[\cos\omega - j\sin\omega] + \cos 2\omega - j\sin 2\omega$$

$$= [1 - 2\cos\omega + \cos 2\omega] + j[2\sin\omega - \sin 2\omega]$$

The magnitude response of the system is:

$$|H(\omega)| = \sqrt{[1 - 2\cos\omega + \cos 2\omega]^2 + [2\sin\omega - \sin 2\omega]^2}$$

The phase response of the system is given as:

$$\angle H(\omega) = \tan^{-1}\left\{\frac{2\sin\omega - \sin 2\omega}{1 - 2\cos\omega + \cos 2\omega}\right\}$$

Chapter - 6

For a complex-valued sequence $x(n)$ of N points, the DFT may be expressed as

$$X_R(k) = \sum_{n=0}^{N-1}\left[x_R(n)\cos\frac{2\pi kn}{N} + x_I(n)\sin\frac{2\pi kn}{N}\right]$$

$$X_I(k) = \sum_{n=0}^{N-1}\left[x_R(n)\sin\frac{2\pi kn}{N} - x_I(n)\cos\frac{2\pi kn}{N}\right]$$

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