Signal Analysis and Transmission

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Signals

Signal: A signal is a function of one or more independent variables which contains a set of information or data. Some examples of signals are:

- A telephone or Television signals
- Radio signal
- Computer signal. Etc.

Signal Size: The size of any entity is a number that indicates the largeness or strength of that entity. For example, the size of human being can be measured by finding volume of person's

$$V = \pi \int_0^H r^2(h)dh$$

where, V is the volume of person

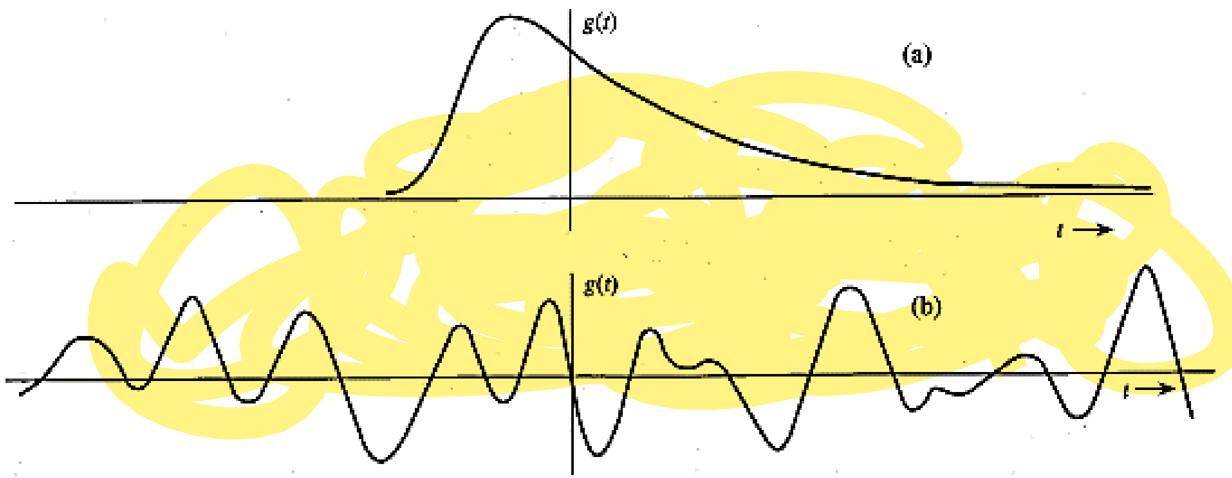
H is the height of a person

r is the radius of a person (if we consider a person as cylinder) for aperiodic

Signals

Signal Energy:

- Area under a signal g(t) is its size
- Signal size takes two values amplitude and duration
- This measuring approach is defective for large signals having positive and negative positions. So, positive portion is cancelled by negative portion. This can be solved by calculating area under $g^2(t)$.



$$Signal\ Energy, E_g = \int_{-\infty}^{\infty} g^2(t)dt,$$ For a real signal

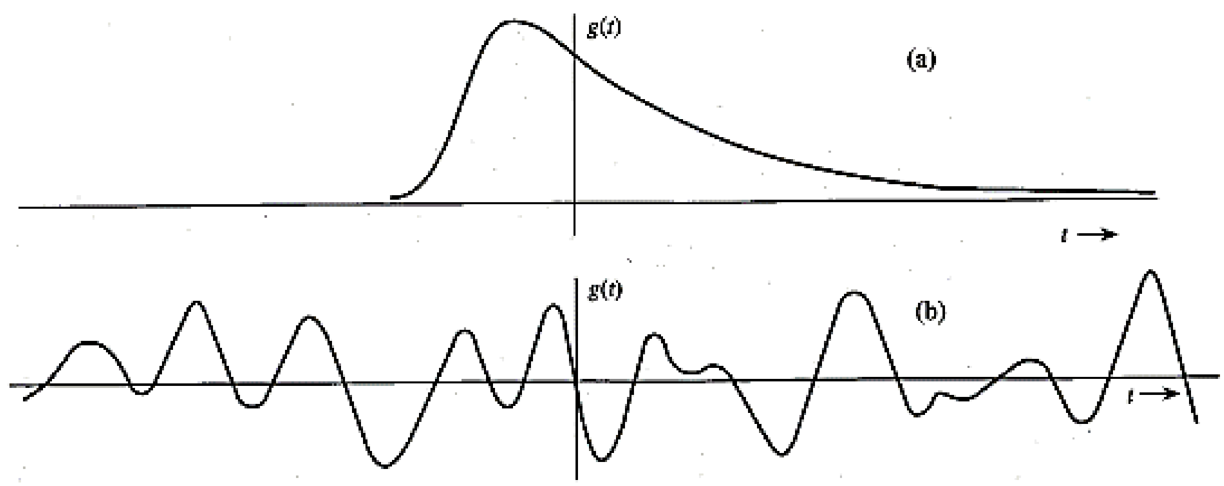
$$Signal\ Energy, E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt,$$ For a complex signal

for periodic

Signals

Signal Power:

- The signal size to be meaningful if the energy is finite.
- The condition for energy to be finite is amplitude→0 as $|t| = \infty$.
- If amplitude of *g(t)* does not →0 as $|t| = \infty$, the signal energy is infinite. In this case, more meaningful measure of signal size is time average of the energy, which is average power, P_g.

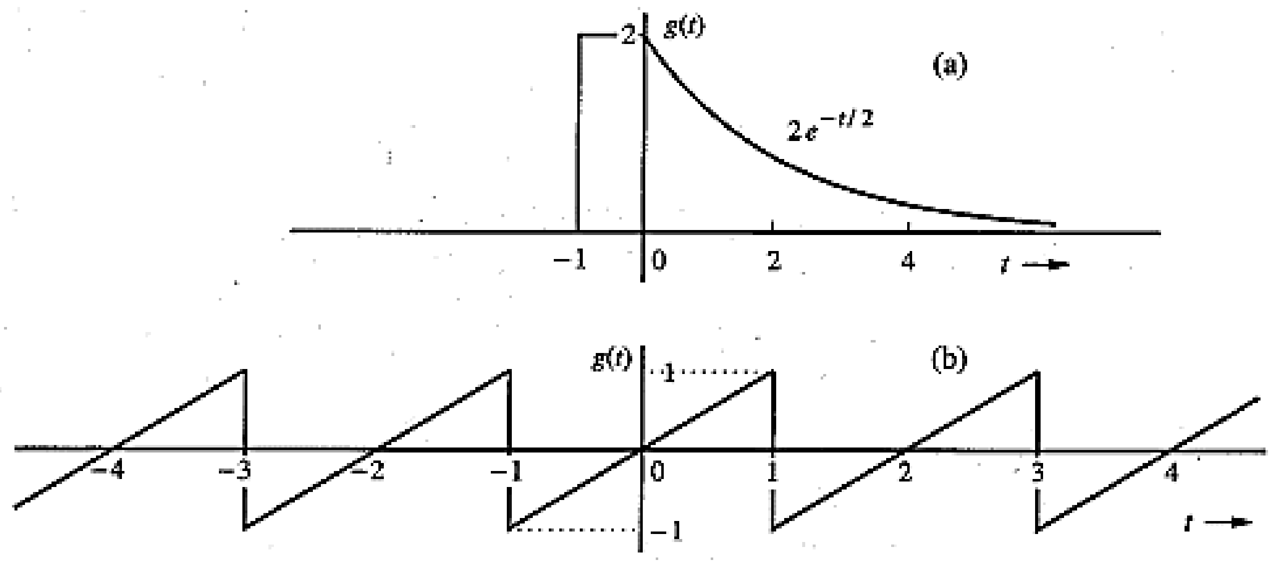


$$Signal\ Power, P_g = \int_{-T/2}^{T/2} g^2(t)dt,$$ For a real signal

$$Signal\ Power, P_g = \int_{-T/2}^{T/2} |g(t)|^2 dt,$$ For a complex signal

Example

Determine the suitable measures of the signal in the following figure:



For a figure (a), amplitude→0 as $|t| = \infty$. Therefore, suitable measure of this signal is its energy, E_g.

$$E_g = \int_{-\infty}^{\infty} g^2(t)dt = \int_{-1}^{0} 2^2 dt + \int_{0}^{\infty} 4e^{-t} dt = 4 + 4 = 8$$

For a figure (b), amplitude does not→0 as $|t| = \infty$. Therefore, suitable measure of this signal is its power, P_g.

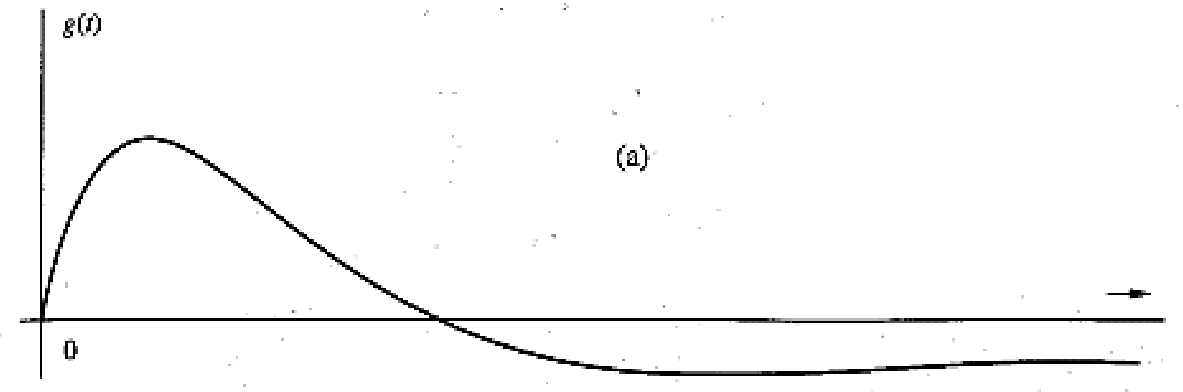
$$P_g = \int_{-T/2}^{T/2} g^2(t)dt = \int_{-1}^{1} t^2 dt = \frac{1}{3}$$

Signals

Signal Classification: Signals are classified into the following categories:

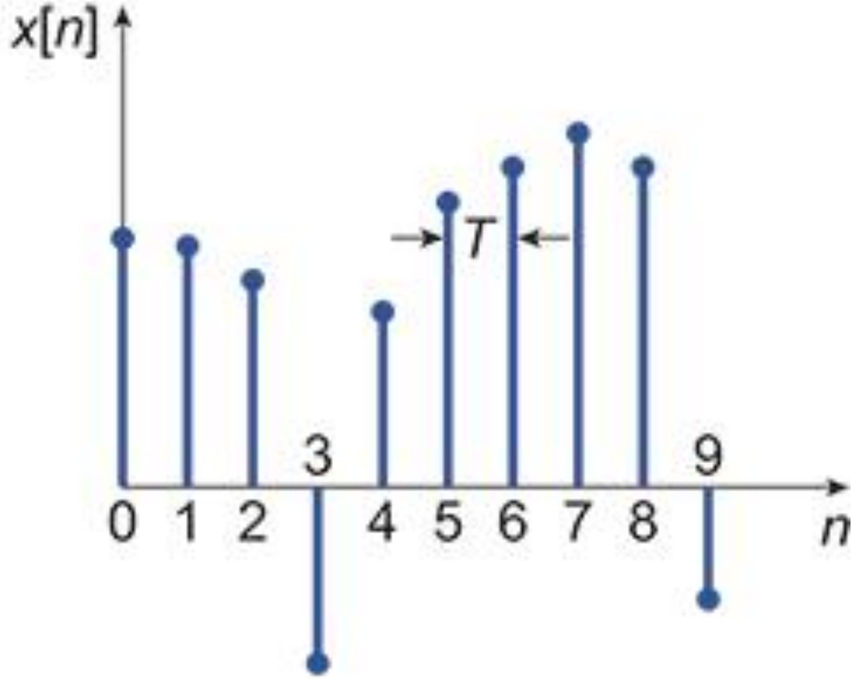
- Continuous Time and Discrete Time Signals
- Analog and Digital Signals
- Deterministic and Non-deterministic Signals
- Even and Odd Signals
- Periodic and Aperiodic Signals
- Energy and Power Signals
- Real and Imaginary Signals

Continuous-time Signal: A signal is said to be continuous when it is defined for all instants of time.



Signals

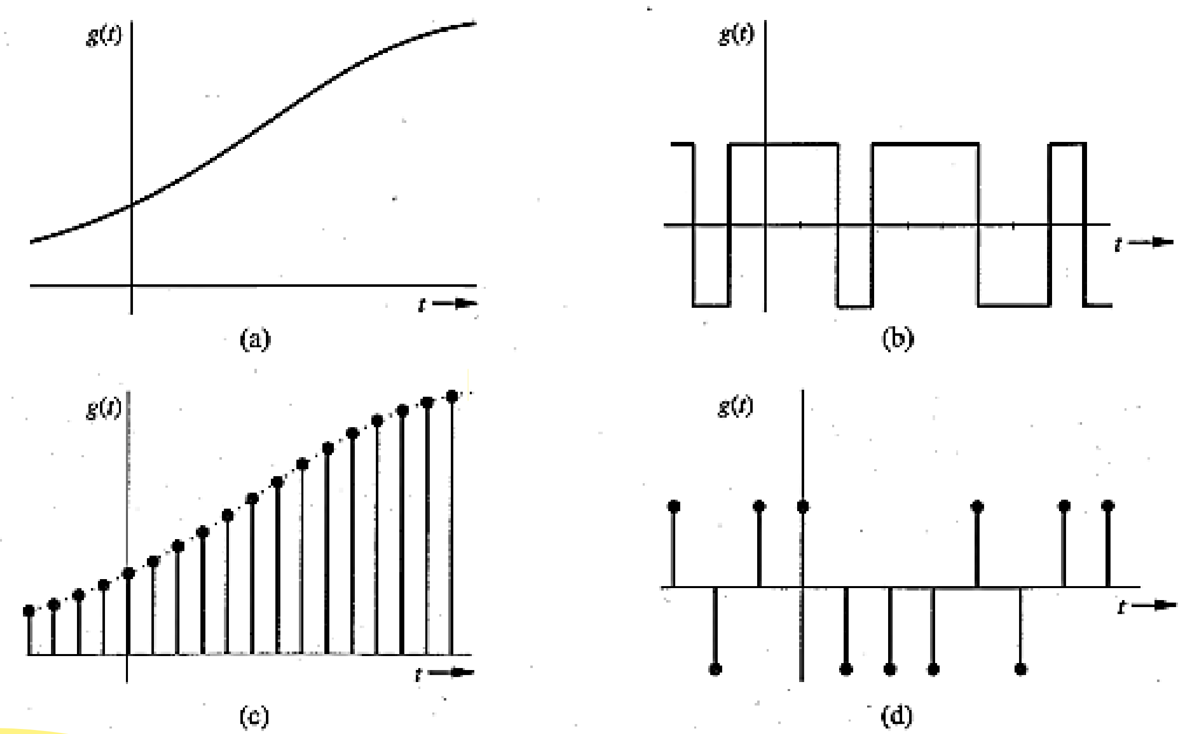
Discrete-time Signal: A signal is said to be discrete when it is defined at only discrete instants of time.



(b)

Signals

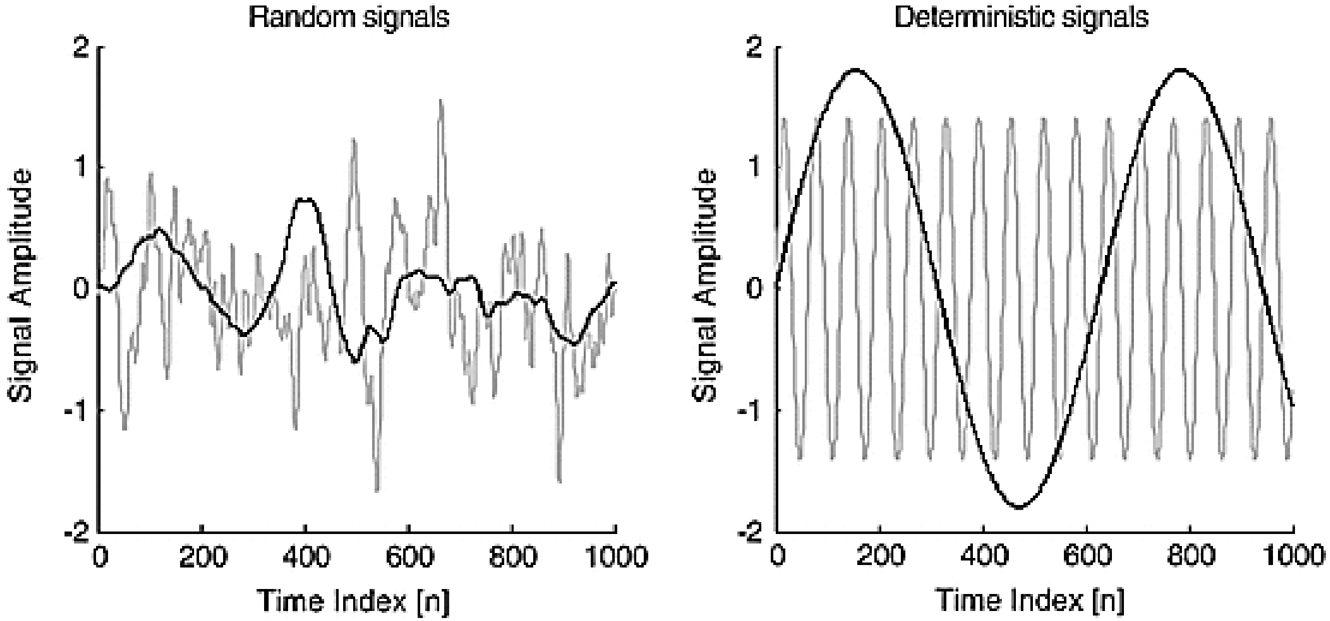
Analog Signals: A signal whose amplitude can take on any value in a continuous range is an analog signal. This means that an analog signal amplitude can take an infinite number of values.



Digital Signals: A digital signal has only finite a finite number of values.

Signals

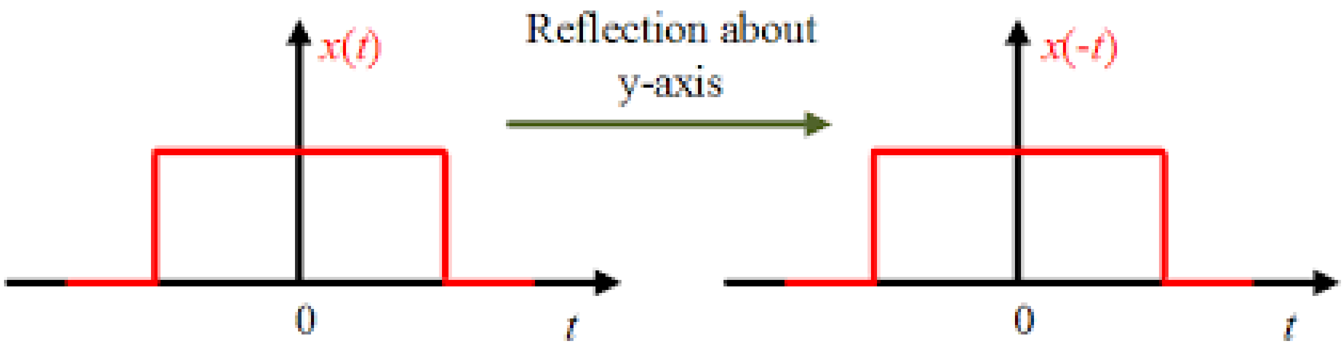
Deterministic Signal: A signal is said to be deterministic if there is no uncertainty with respect to its value at any instant of time. Or, signals which can be defined exactly by a mathematical formula are known as deterministic signals

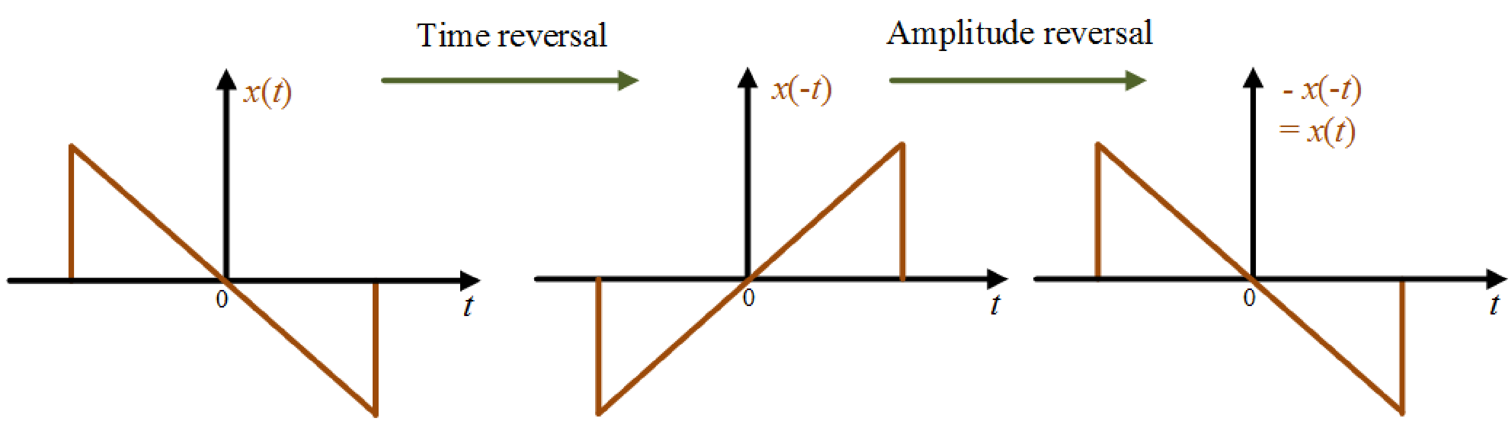


Non-deterministic Signal: A signal is said to be non-deterministic if there is uncertainty with respect to its value at some instant of time. Non-deterministic signals are random in nature hence they are called random signals.

Signals

Even Signal: A signal is said to even signal when it satisfies the condition *x(t) = x(-t).*

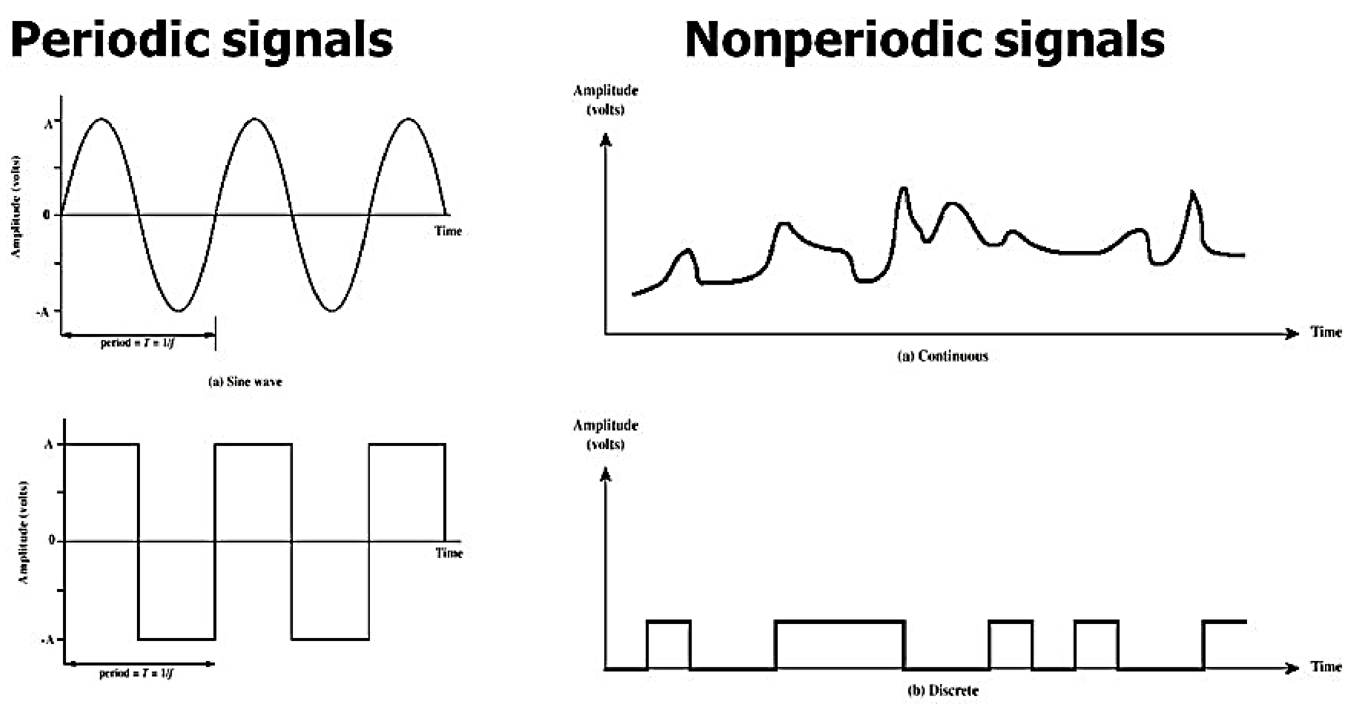




Odd Signal: A signal is said to be odd when it satisfies the condition *x(t) = -x(-t).*

Signals

Periodic Signal: A signal is said to be periodic if it satisfies the condition x(t) = x(t + T).



Aperiodic Signal: A signal is said to be periodic if it does not repeat.

Signals

Real Signal: A signal is said to be real when it satisfies the condition $x(t) = x^*(t)$

Imaginary Signal: A signal is said to be odd when it satisfies the condition $x(t) = -x^*(t)$

Example:

If $x(t)= 3$ then $x^*(t)=3^*=3$ here $x(t)$ is a real signal.

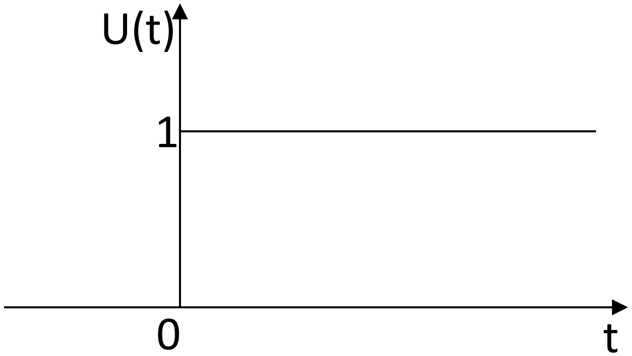
If $x(t)= 3j$ then $x^*(t)=3j^* = -3j = -x(t)$ hence $x(t)$ is a odd signal.

Note: For a real signal, imaginary part should be zero. Similarly for an imaginary signal, real part should be zero

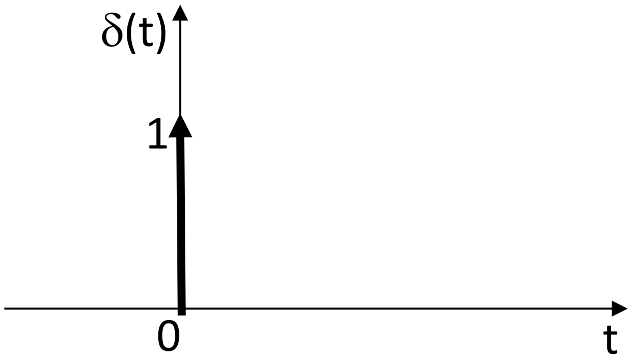
Singularity Functions

Unit Step Function: The unit step function is exist only for positive side and zero for negative side. It is denoted by *u(t).*

$$u(t) = \begin{cases} 0 \; for \; t < 0 \\ 1 \; for \; t > 0 \end{cases}$$



Unit Impulse Function: It is one of the most used elementary function used in the analysis of communication system, which is denoted by $\delta(t)$.



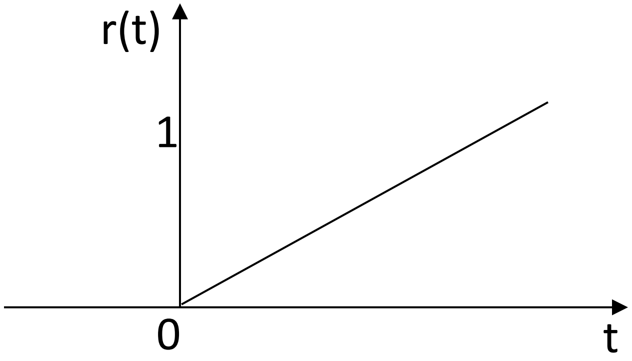
$$\delta(t) = 0 \qquad\qquad \mathrm{t} \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$ when t=0

Singularity Functions

Unit Ramp Function: It is a function which starts at *t=0* and increases linearly with time. It is denoted by *r(t).*

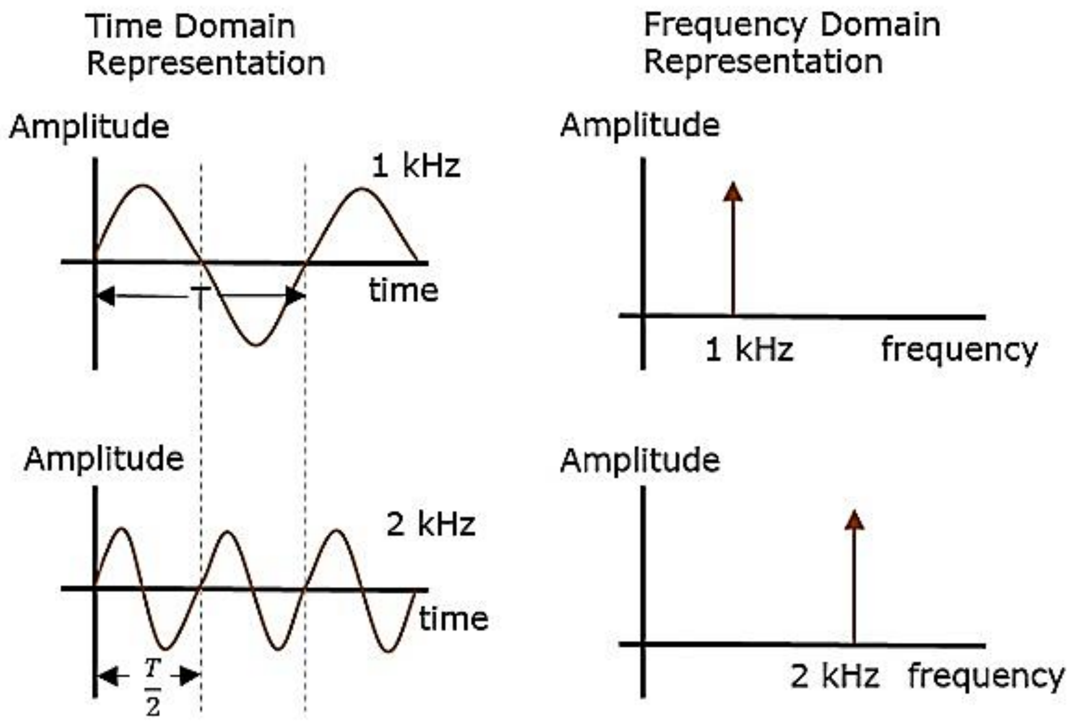
$$r(t) = \begin{cases} 0 \; for \; t \leq 0 \\ t \; for \; t \geq 0 \end{cases}$$



Representation of Signals

Time domain representation: In the time domain representation, a signal is time varying quantity.

to find amplitute



Frequency domain representation: In the frequency domain representation, a signal is represented by its frequency spectrum. It is also called line spectrum.

to find bandwith

How to plot line spectrum

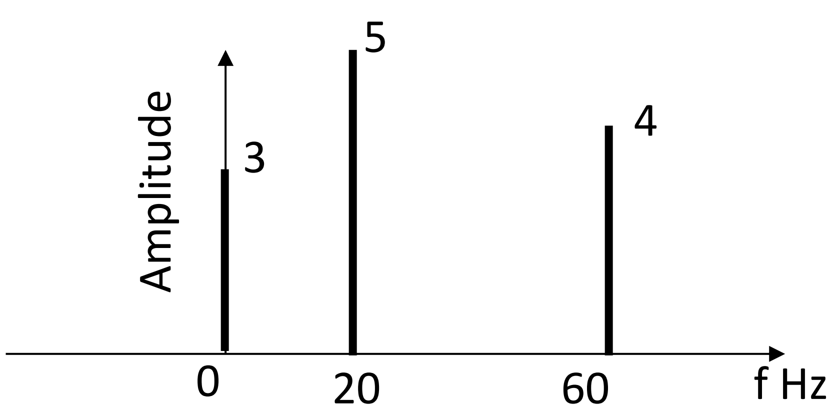
Sketch the line spectrum of the following signal:

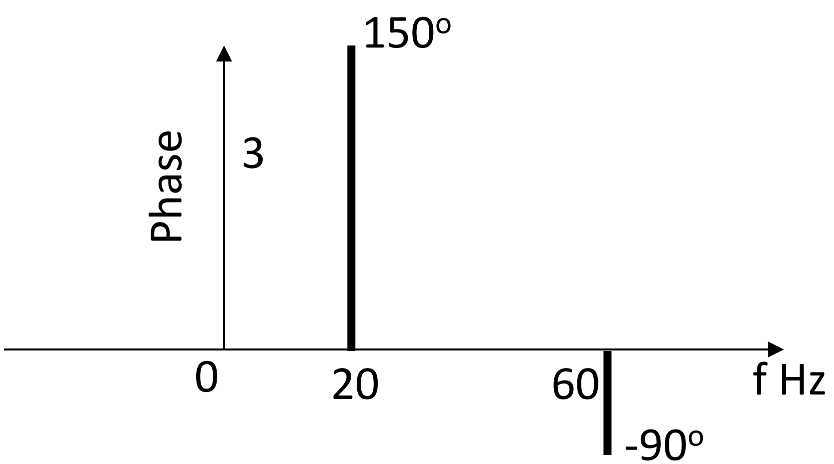
$$g(t) = 3 - 5\cos(40\pi t - 30°) + 4\sin(120\pi t)$$

Solution: The above equation can be re-written as

$$g(t) = 3 + 5\cos(40\pi t + 150°) + 4\cos(120\pi t - 90°)$$

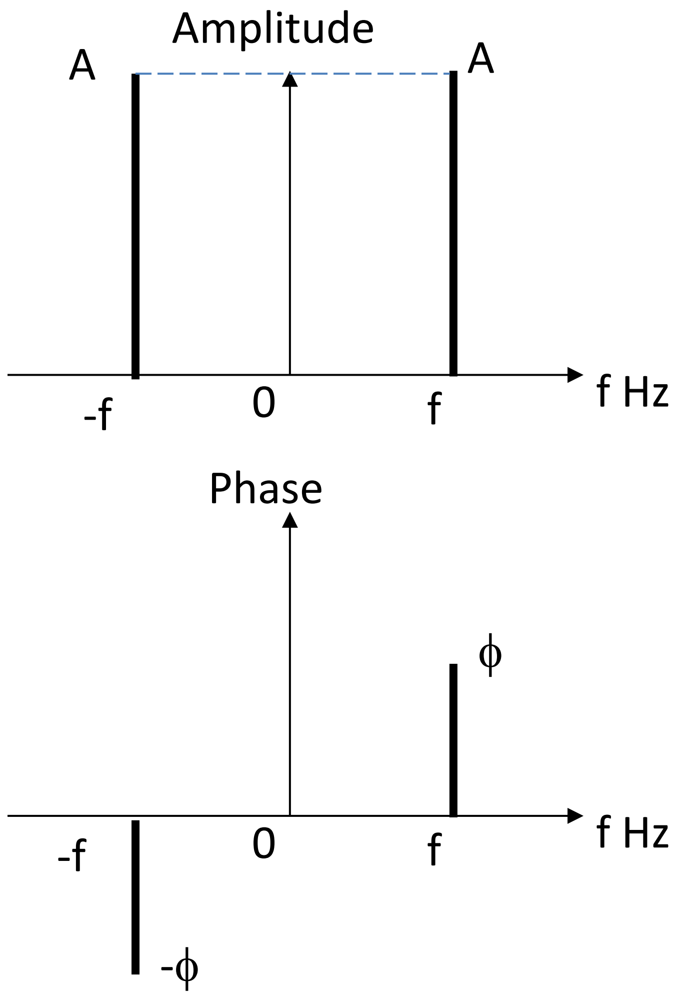
S. No.	Term	Amplitude	Frequency	Phase
1.	$3cos2\pi 0t$	3 V	0 Hz	0^o
2.	$5\cos(40\pi t + 150°)$	5 V	20 Hz	150^0
3.	$4\cos(120\pi t - 90°)$	4 V	60 Hz	-90^o





Double Sided Line Spectrum

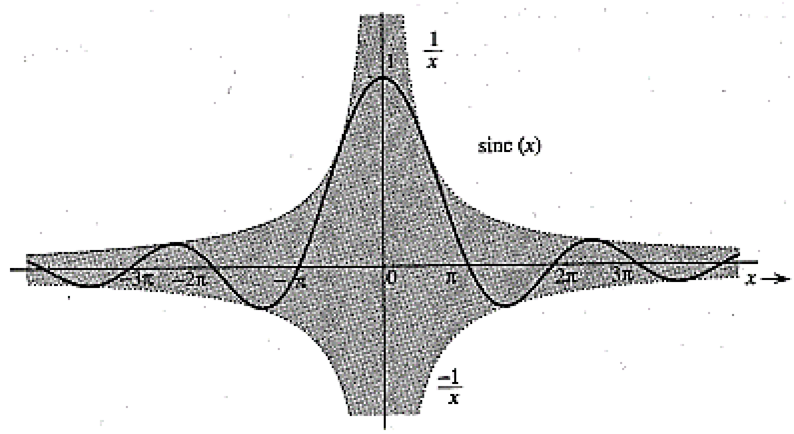
If spectrum is represented for both the positive and negative frequency is called double sided line spectrum.



Filtering or Interpolating function

- The function $\frac{sinx}{x}$ is the sine over argument function denoted by $Sinc(x)$.
- It is also known as filtering or interpolating function.
- $Sinc(x)$ is an even function of *x.*
- $Sinc(x) = 0$ when $sinx = 0$ except $x = 0$, where it is intermediate. That is $Sinc(x) = 0$ for $x = \pm\pi, \pm2\pi, \pm3\pi \ldots\ldots$
- $Sinc(x)$ is the product of an oscillating signal $sinx$ (of period 2π) and a monotonically decreasing function $\frac{1}{x}$. Therefore, $sincx$ exhibits sinusoidal oscillation of period 2π, with amplitude decreasing continuously as $\frac{1}{x}$.

$$\text{sinc}(x) \equiv \begin{cases} 1 & \text{for } x = 0 \\ \dfrac{\sin x}{x} & \text{otherwise,} \end{cases}$$



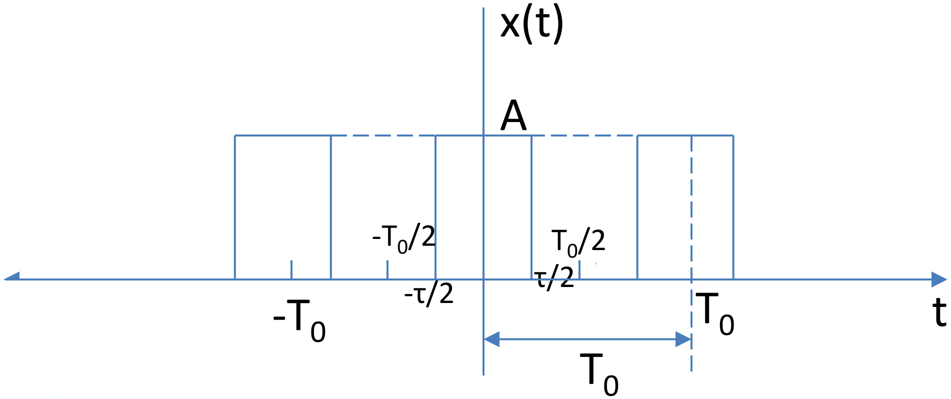
Fourier Series

Fourier series are important for the following reasons:

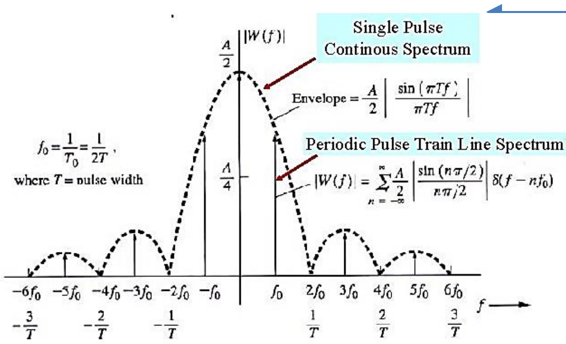
- How many frequency components are present in the signal
- Their amplitudes
- Their relative phase difference between these frequency component.
- Fourier series is used for periodic signals

Obtain the Fourier series for the following rectangular pulse train

Solution:

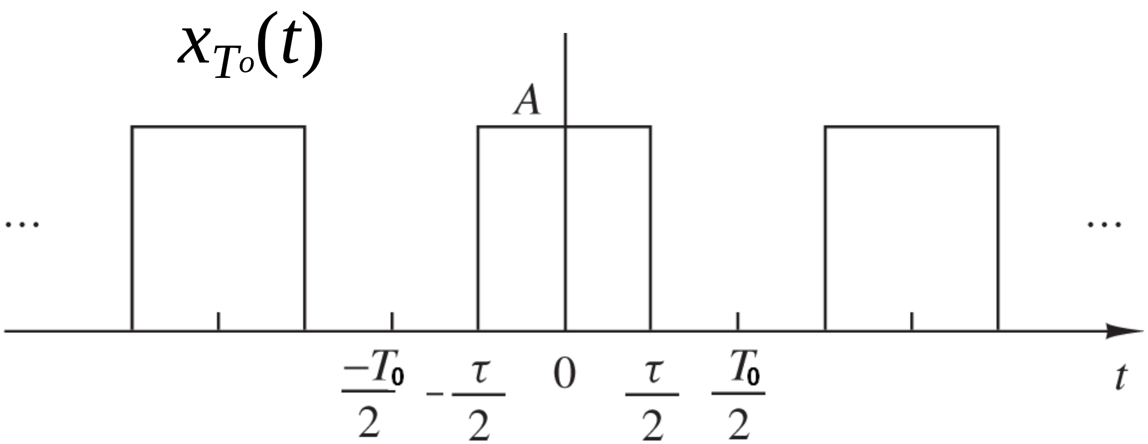


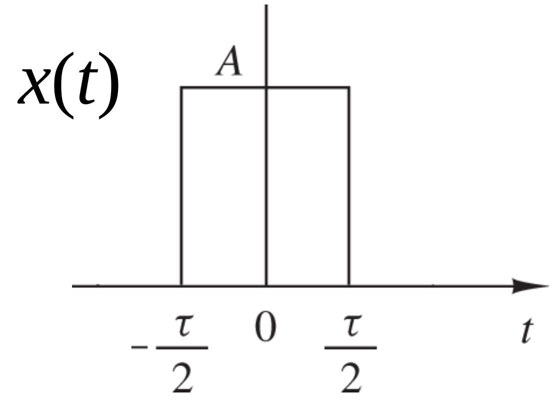
$$x(t) = \sum_{n=-\infty}^{n=\infty} \frac{\tau A}{T_0} sinc\left[\frac{n\tau}{T_0}\right] e^{j2\pi nt/T_0}$$



(b) Magnitude Spectrum

Link between FT and FS

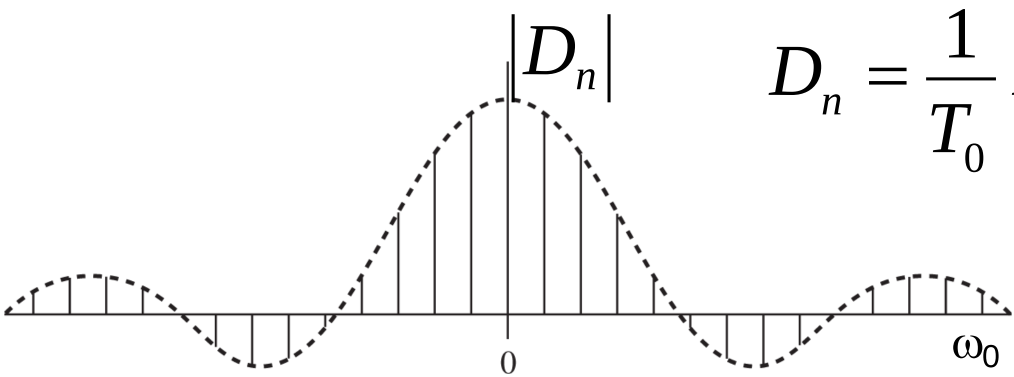




$$x(t) = \lim_{T_0 \to \infty} x_{T_0}(t)$$

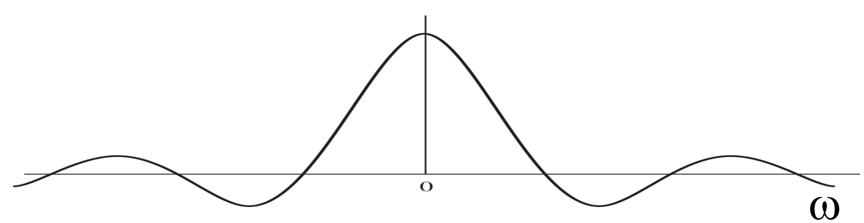
$$T_0 \to \infty \quad \Rightarrow\Rightarrow \quad \omega_0 \to 0$$

As T_0 gets larger and larger the fundamental frequency ω_0 gets smaller and smaller so the spectrum becomes continuous.



$$D_n = \frac{1}{T_0} X(n\omega_0)$$

$|X(\omega)|$



Fourier Transform (FT)

Fourier transform are important for the following reasons:

- The non-periodic signals which extend from $-\infty$ to ∞ can be represented easily using FT.
- It is used to transform from the time domain to frequency domain.

Fourier transform of a function x(t) can be expressed as

$$X(\omega) = F[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Inverse Fourier transform of a function can be expressed as

$$x(t) = F^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

Fourier Transform (FT)

Find the Fourier transform of $e^{-at}u(t)$.

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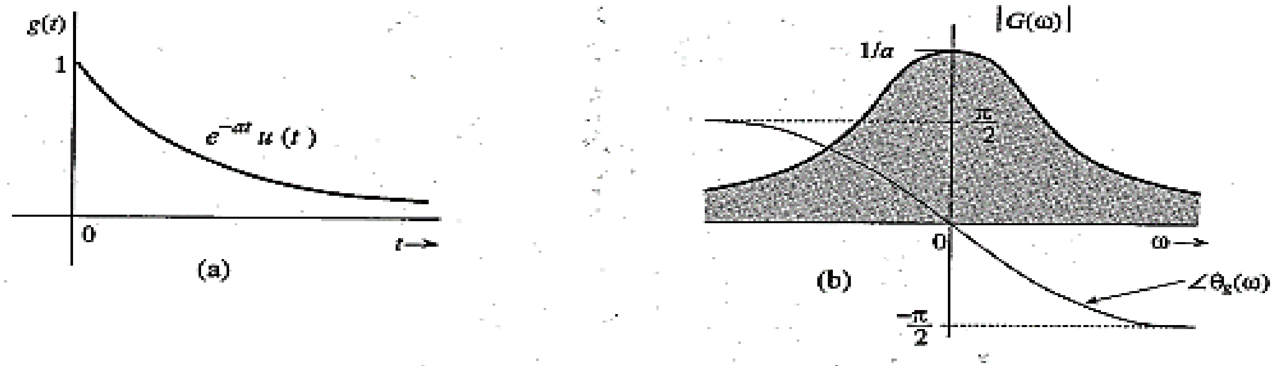


$e^{-at}u(t)$ and its Fourier spectra.

By definition [Eq. (3.8a)],

$$G(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt = \int_0^{\infty} e^{-(a+j\omega)t}\,dt = \frac{-1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty}$$

But $|e^{-j\omega t}| = 1$. Therefore, as $t \to \infty$, $e^{-(a+j\omega)t} = e^{-at}e^{-j\omega t} = 0$ if $a > 0$. Therefore,

$$G(\omega) = \frac{1}{a+j\omega} \qquad a > 0$$

Expressing $a + j\omega$ in the polar form as $\sqrt{a^2+\omega^2}\, e^{j\tan^{-1}\left(\frac{\omega}{a}\right)}$, we obtain

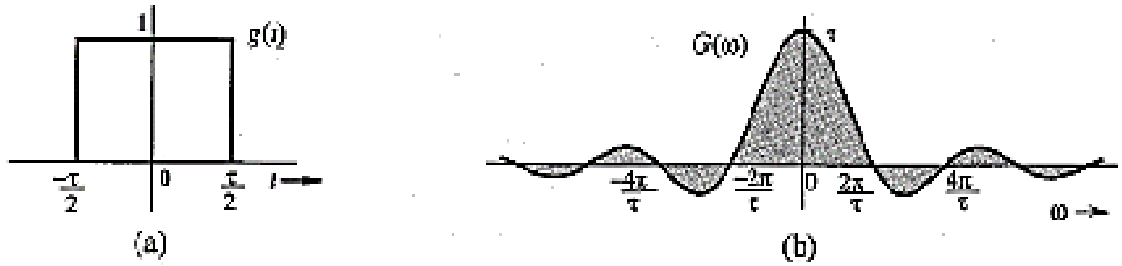
$$G(\omega) = \frac{1}{\sqrt{a^2+\omega^2}}e^{-j\tan^{-1}\left(\frac{\omega}{a}\right)}$$

Therefore,

$$|G(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \qquad \text{and} \qquad \theta_g(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

Fourier Transform (FT)

Find the Fourier transform of $g(t) = rec(\frac{t}{\tau})$



Gate pulse and its Fourier spectrum.

We have

$$G(\omega) = \int_{-\infty}^{\infty} \text{rect}\left(\frac{t}{\tau}\right) e^{-j\omega t} dt$$

Since rect $(t/\tau) = 1$ for $|t| < \tau/2$, and since it is zero for $|t| > \tau/2$,

$$G(\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt$$

$$= -\frac{1}{j\omega}(e^{-j\omega\tau/2} - e^{j\omega\tau/2}) = \frac{2\sin(\omega\tau/2)}{\omega}$$

$$= \tau\frac{\sin(\omega\tau/2)}{(\omega\tau/2)} = \tau \text{ sinc}\left(\frac{\omega\tau}{2}\right)$$

Therefore,

$$\text{rect}\left(\frac{t}{\tau}\right) \Longleftrightarrow \tau \text{sinc}\left(\frac{\omega\tau}{2}\right)$$

Fourier Transform (FT)

Find the Fourier transform of the unit impulse $\delta(t) = 1$

If it is given that, $g(t) = \delta(t)$

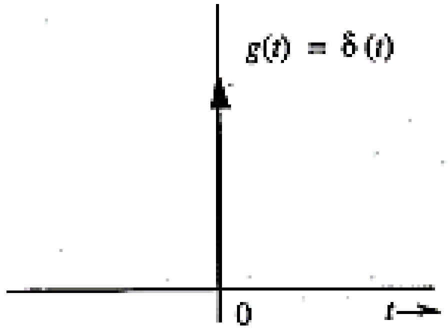
Then, from the definition of Fourier transform, we have,

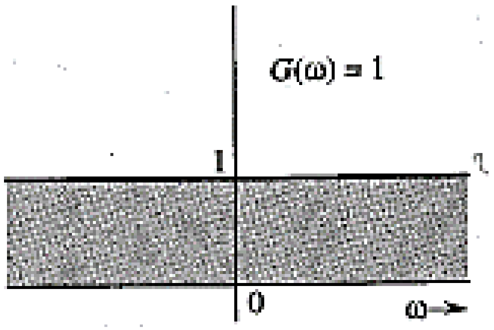
$$G(\omega) = F[g(t)] = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} 1.e^{-j\omega t}dt = e^{-j\omega t}\Big|_{t=0} = 1$$

So, $F[\delta(t)] = 1 \; or \; \delta(t) \leftrightarrow 1$





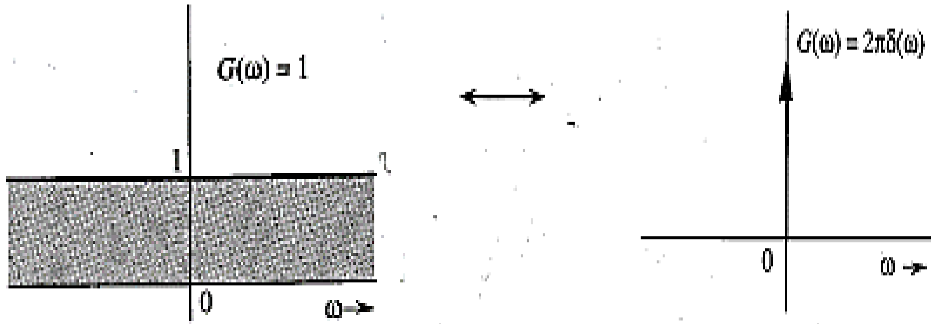
That is, *the Fourier transform of a unit impulse function is unity.*

Fourier Transform (FT)

Find the inverse Fourier transform of the unit impulse $\delta(\omega)$.

Inverse Fourier transform of a function can be expressed as

$$g(t) = F^{-1}[G(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega$$



The inverse Fourier transform of G(ω)=1 is determined through inverse Fourier transform of impulse function [δ(ω)].

$$F^{-1}[G(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 1.\, e^{j\omega t}d\omega = \frac{1}{2\pi}.\, e^{j\omega t}\Big|_{\omega=0} = \frac{1}{2\pi}$$

$$F^{-1}[\delta(\omega)] = \frac{1}{2\pi},\ F^{-1}[2\pi\delta(\omega)] = 1,\ 1 \leftrightarrow 2\pi\delta(\omega).$$

Fourier Transform (FT)

Find the inverse Fourier transform of the unit impulse $\boldsymbol{\delta(\omega-\omega_0)}$.

Inverse Fourier transform of a function can be expressed as

$$F^{-1}[\delta(\omega-\omega_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}d\omega$$

Let, $\omega' = \omega-\omega_0$, $d\acute{\omega} = d\omega$ and $\omega = \acute{\omega}+\omega_0$

$$F^{-1}[\delta(\acute{\omega})] = \frac{1}{2\pi}e^{j\omega_0 t}\int_{-\infty}^{\infty}\delta(\acute{\omega})e^{j\acute{\omega}t}d\acute{\omega}$$

$$=> F^{-1}[\delta(\acute{\omega})] = \frac{1}{2\pi}e^{j\omega_0 t}$$

$$=> F^{-1}[2\pi\delta(\acute{\omega})] = e^{j\omega_0 t}$$

$$=> F^{-1}\,[2\pi\delta(\omega-\omega_0)] = e^{j\omega_0 t}$$

Hence, the Fourier transform of the complex exponential function is given by,

$$\boldsymbol{e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0)}$$

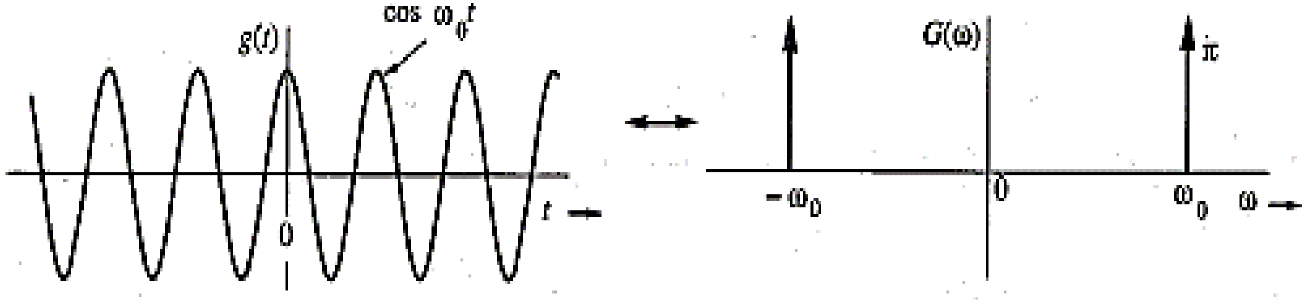
Fourier Transform (FT)

Find the Fourier transform of the everlasting sinusoid $\boldsymbol{cos\omega_0 t}$.

We can write that, $cos\omega_0 t = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$ since, $e^{i\theta} = cos\theta + isin\theta$

Fourier transform of $cos\omega_0 t$ is

$$cos\omega_0 t \leftrightarrow \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)], \qquad \boldsymbol{e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)}$$

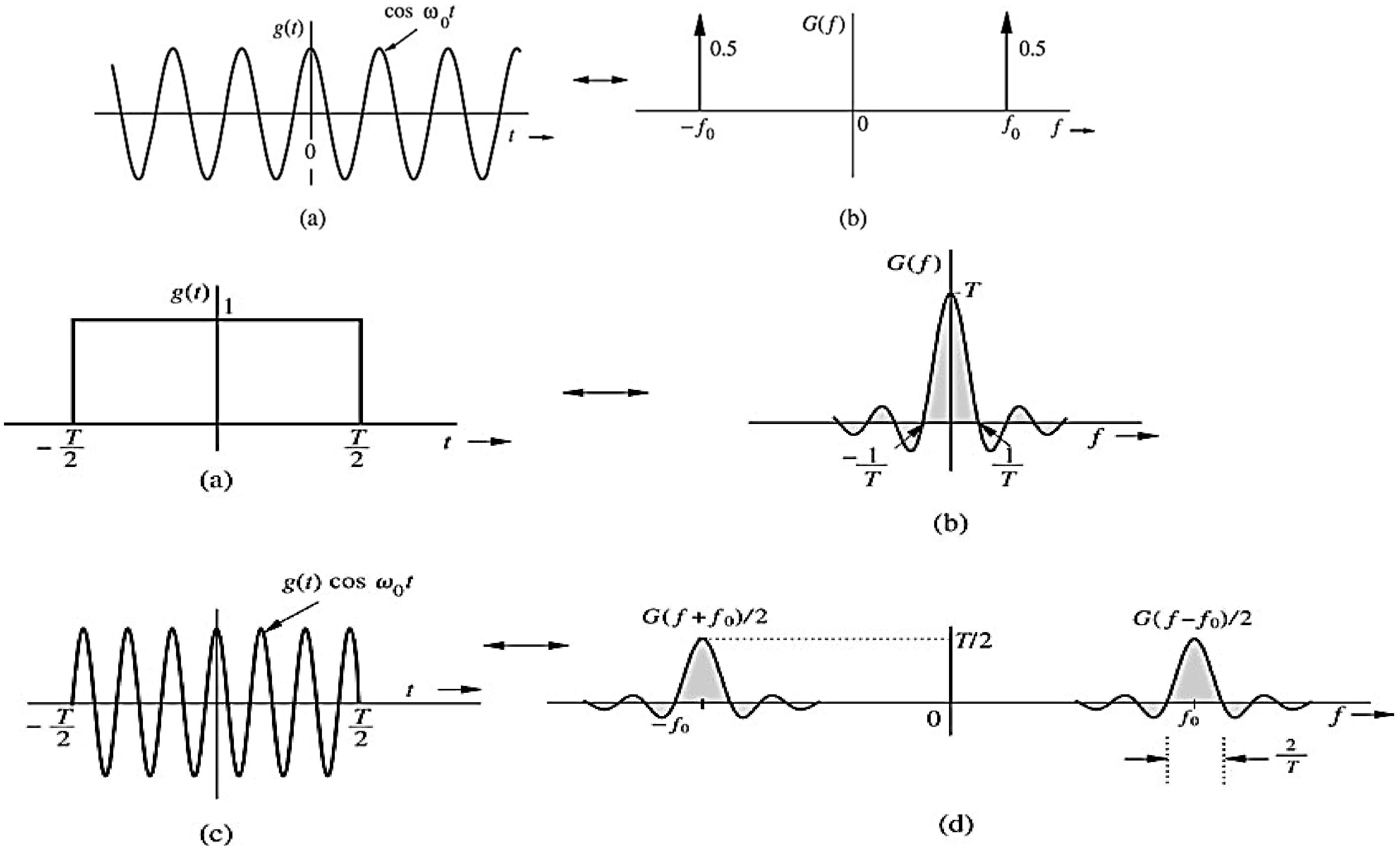


The spectrum of $cos\omega_0 t$ consists of two impulses at ω_0 and $-\omega_0$.

Example: Amplitude Modulation

Example: Find the FT for the signal

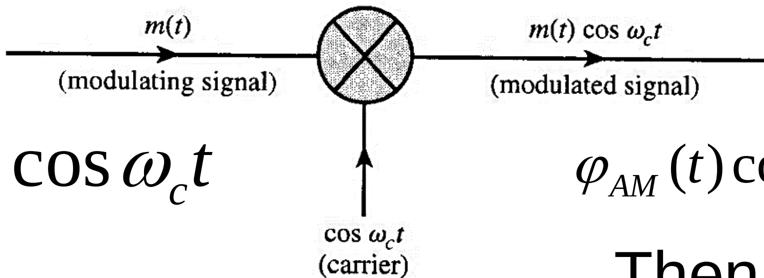
$$x(t) = rect(t/4)\cos 10t$$



Amplitude Modulation

Modulation

$$\varphi_{AM}(t) = m(t)\cos\omega_c t$$

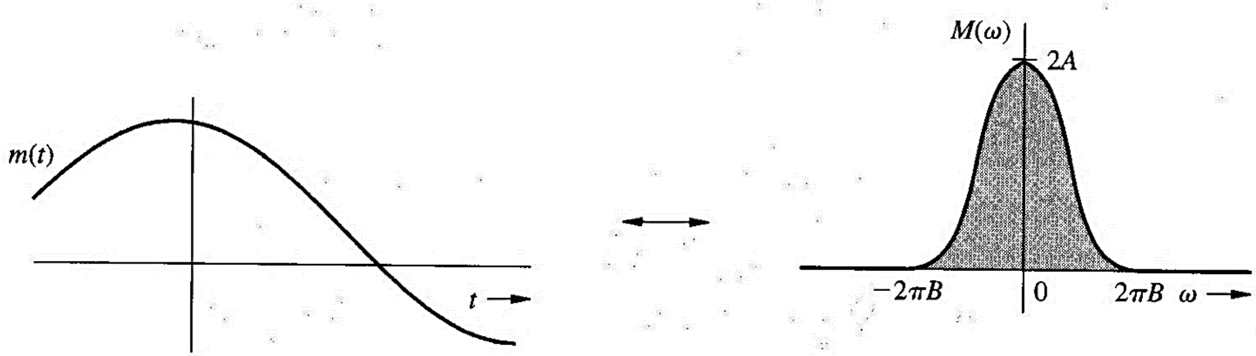


(a)

Demodulation

$$\varphi_{AM}(t)\cos^2\omega_c t = 0.5m(t)[1+\cos 2\omega_c t]$$

Then lowpass filtering



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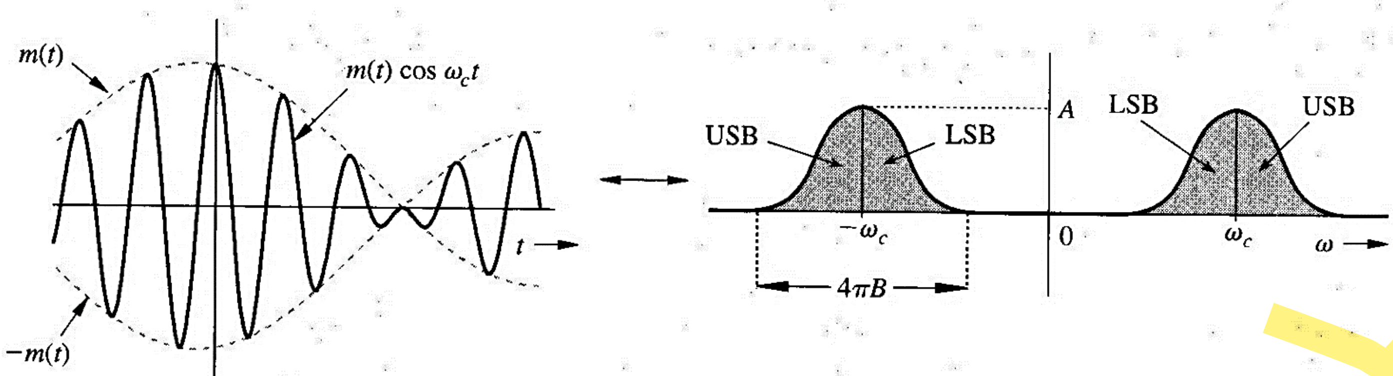

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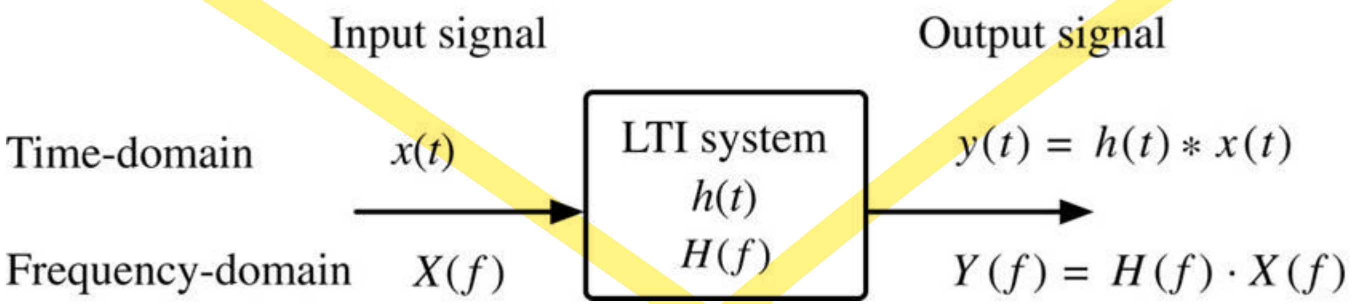
Signal Transmission Through a Linear System



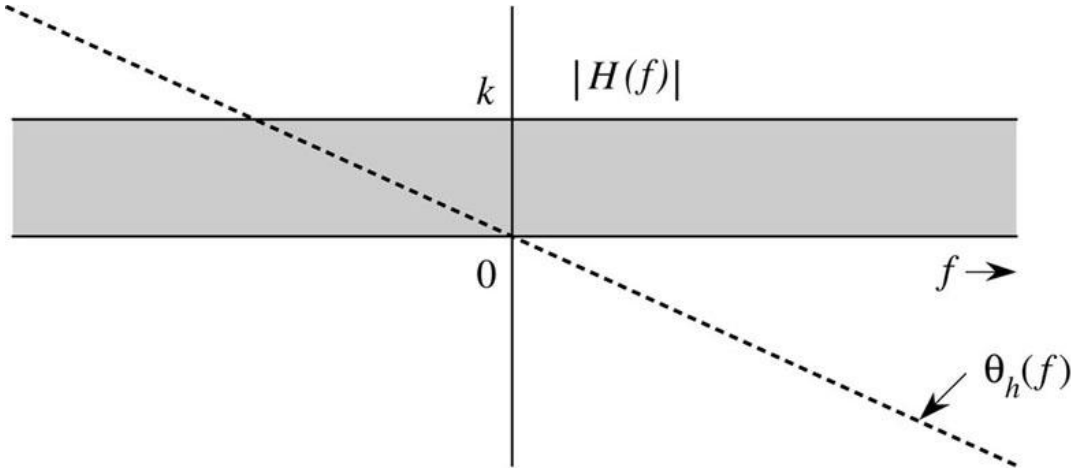
$$X(f) = |X(f)|e^{j\theta_x(f)}$$

$$H(f) = |H(f)|e^{j\theta_h(f)}$$

$$|Y(f)|e^{j\theta_y(f)} = |X(f)||H(f)|e^{j[\theta_h(f)+\theta_x(f)]}$$

Distortionless Transmission (System)

$$y(t) = k.x(t - t_d)$$



$$Y(f) = kX(f)\, e^{-j2\pi f t_d}$$

$$H(f) = ke^{-j2\pi f t_d}$$

$$|H(f)| = k \qquad \theta(f) = -2\pi f t_d$$

Slope is constant for distortionless system

$$t_d(f) = -\frac{1}{2\pi}\frac{d\theta}{df}$$ group delay

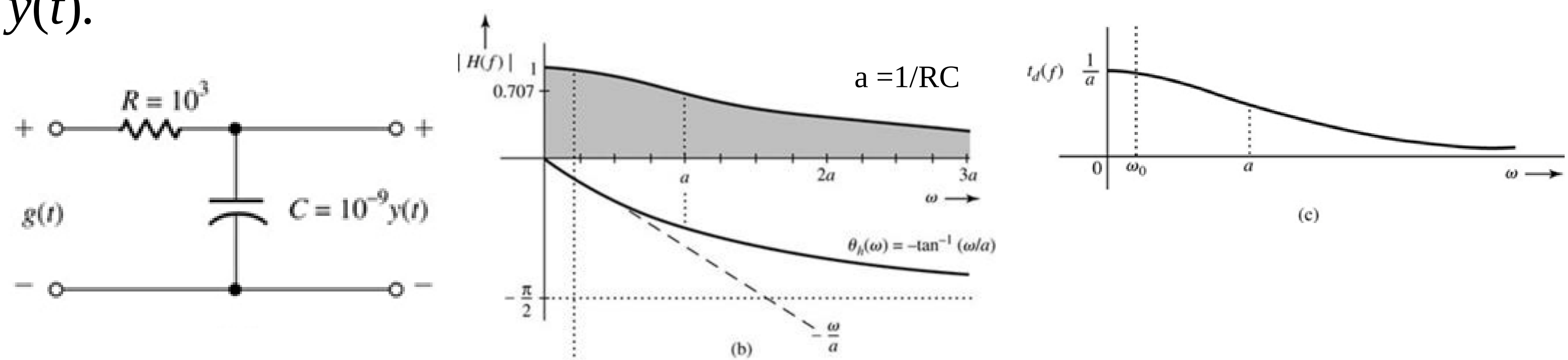
$$x(t) = 3\cos(2\pi f_1 t) + 5\cos(2\pi f_2 t)$$

$$y(t) = x(t - t_d) = 3\cos(2\pi f_1(t - t_d)) + 5\cos(2\pi f_2(t - t_d))$$

$$y(t) = 3\cos(2\pi f_1 t - 2\pi f_1 t_d) + 5\cos(2\pi f_2 t - 2\pi f_2 t_d)$$

Example 3.16

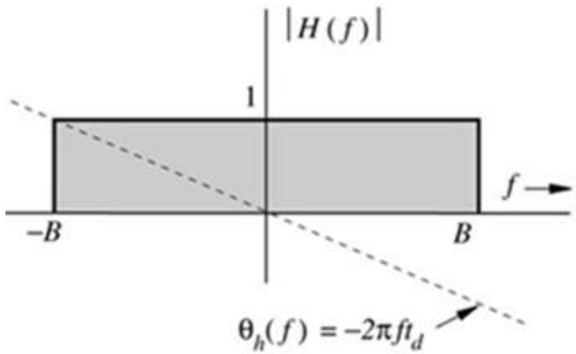
A transmission medium is modeled by a simple RC low-pass filter shown below. If $g(t)$ and $y(t)$ are the input and the output, respectively to the circuit, determine the transfer function $H(f)$, $\theta_h(f)$, and $t_d(f)$. For distortionless transmission through this filter, what is the requirement on the bandwidth of $g(t)$ if amplitude response variation within 2% and time delay variation within 5% are tolerable? What is the transmission delay? Find the output $y(t)$.

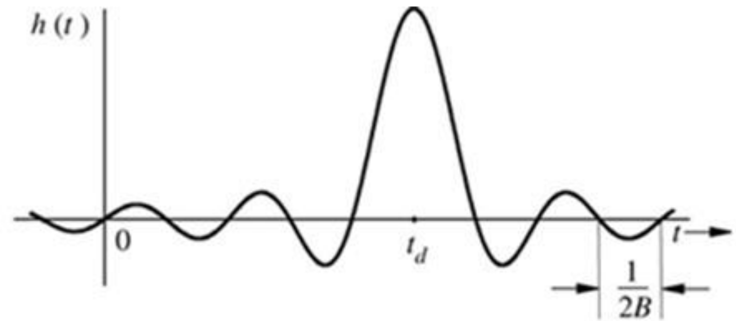


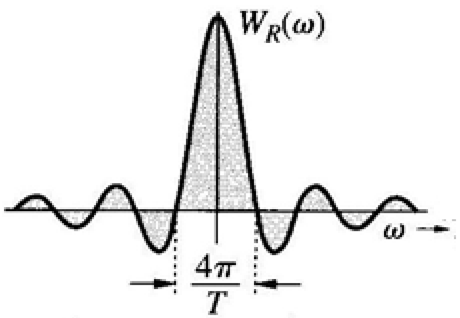
Answer: f_o = 32.31 kHz

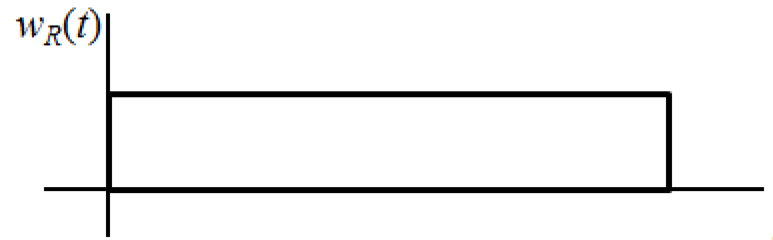
$$\frac{d}{dx}(tan^{-1}ax) = \frac{a}{1+a^2x^2}$$

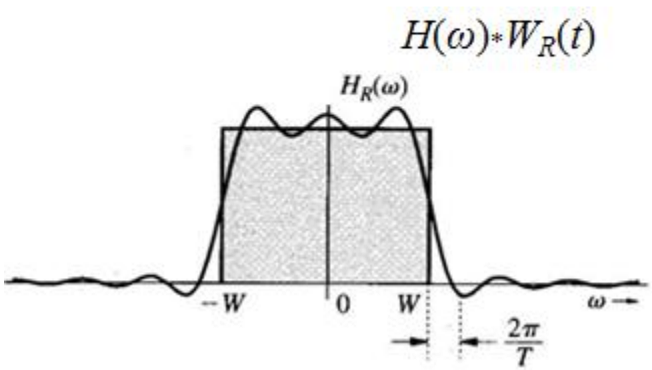
Ideal Versus Practical Filters

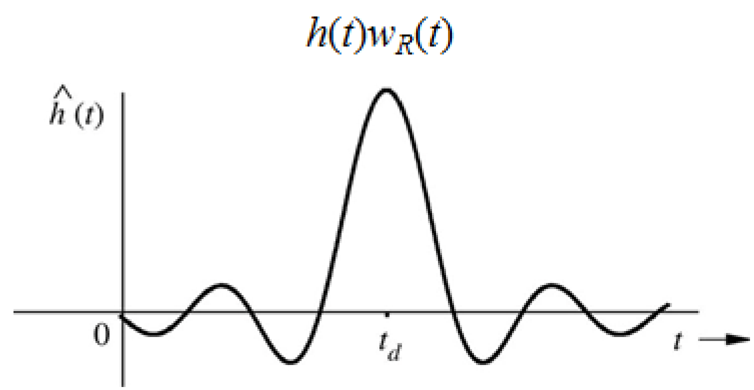




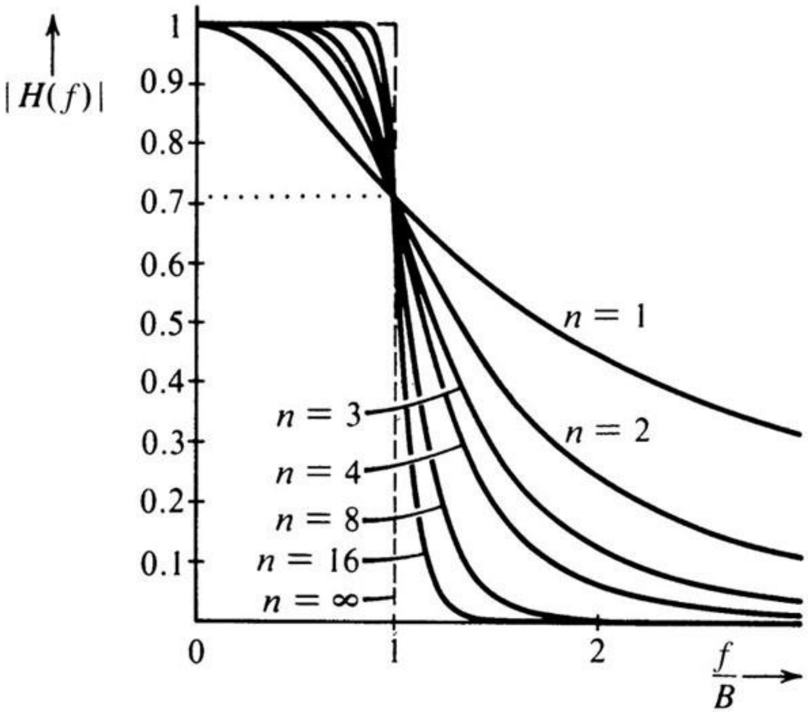


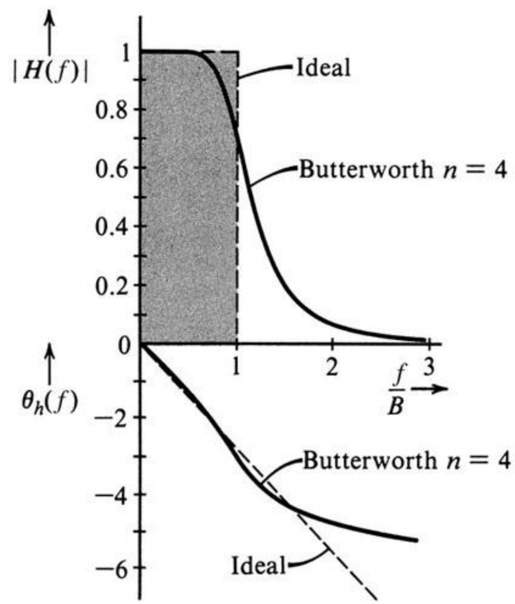


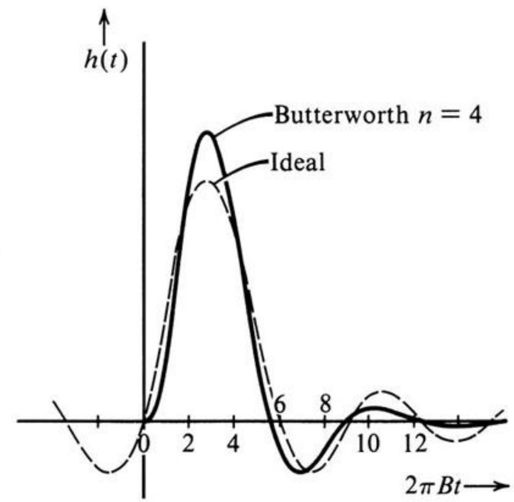




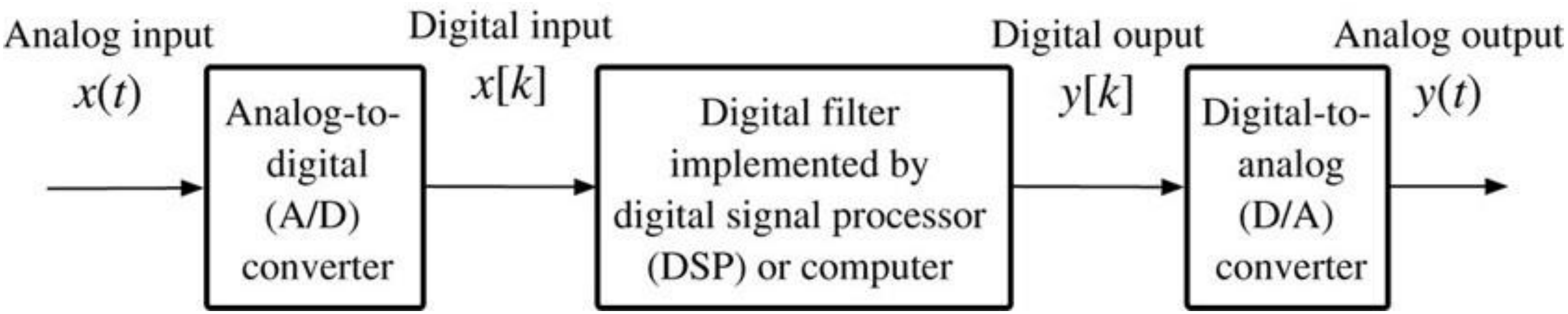
Ideal Versus Practical Filters







Digital Filter



Signal Distortion Over a Communication Channel

1. Linear Distortion
2. Channel Nonlinearities
3. Multipath Effects
4. Fading Channels

   - Channel fading vary with time. To overcome this distortion is to use automatic gain control (AGC)

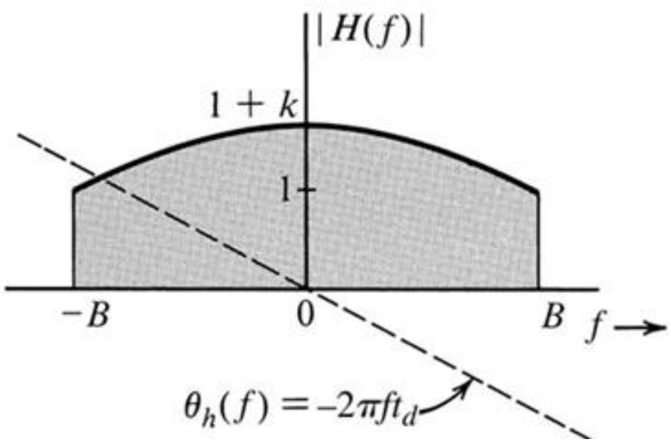
Linear Distortion

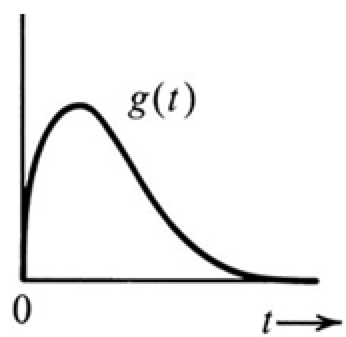
Channel causes magnitude distortion, phase distortion, or both.

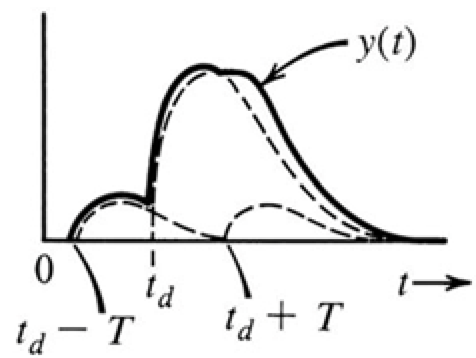
Example: A channel is modeled by a low-pass filter with transfer function $H(f)$ give by

$$H(f) = \begin{cases} (1 + k\cos 2\pi f T)e^{-j2\pi f t_d} & |f| < B \\ 0 & |f| > B \end{cases}$$

A pulse $g(t)$ band-limited to B Hz is applied at the input of this filter. Find the output $y(t)$.







Nonlinear Distortion

$$y(t) = f(g(t))$$

$f(g)$ can be expanded by Maclaurin series

$$y(t) = a_0 + a_1 g(t) + a_2 g^2(t) + \cdots + a_k g^k(t)$$

If the bandwidth of $g(t)$ is B Hz then the bandwidth of $y(t)$ is kB Hz.

Example: The input $x(t)$ and the output $y(t)$ of a certain nonlinear channel are related as

$$y(t) = x(t) + 0.000158\, x^2(t)$$

Find the output signal $y(t)$ and its spectrum $\mathrm{Y(f)}$ if the input signal is $x(t) = 2000\ sinc(2000\pi t)$. Verify that the bandwidth of the output signal is twice that of the input signal. This is the result of signal squaring. Can the signal $x(t)$ be recovered (without distortion) from the output $y(t)$?

Continue Example

What is the consequence with this type of distortion if two signals are transmitted in adjacent bands?

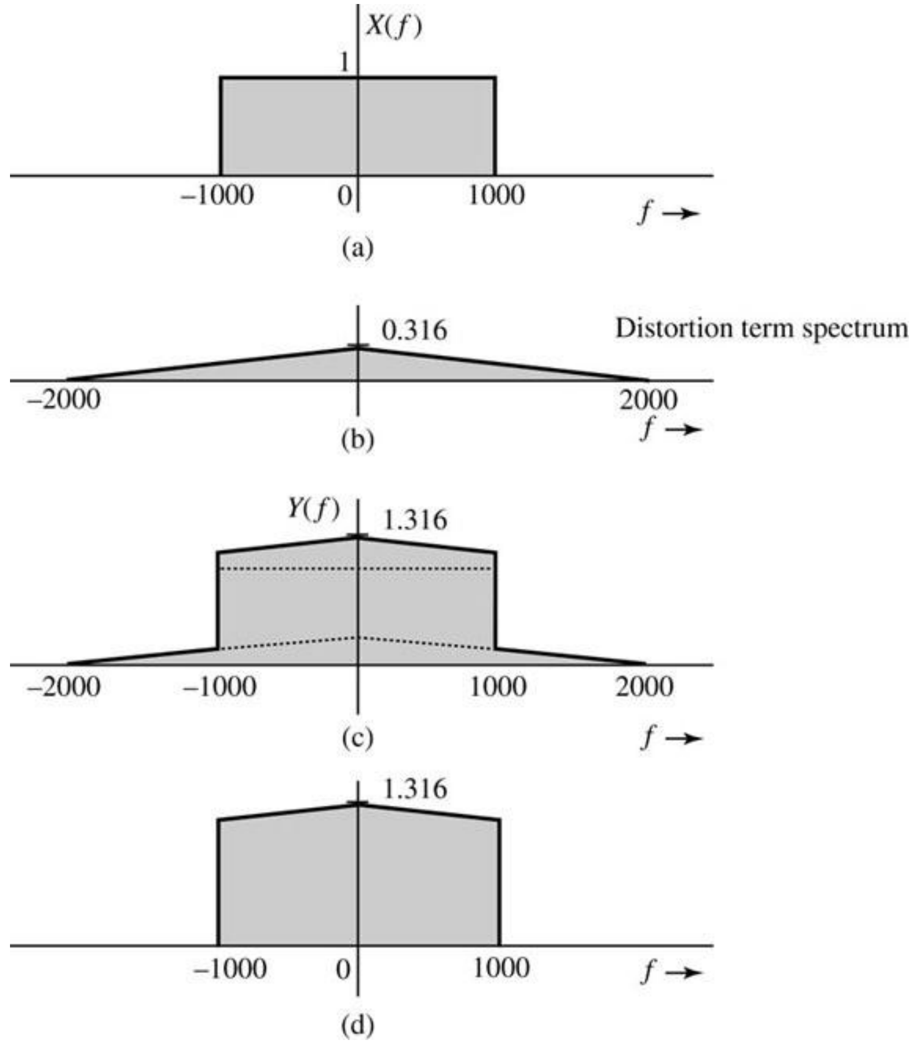
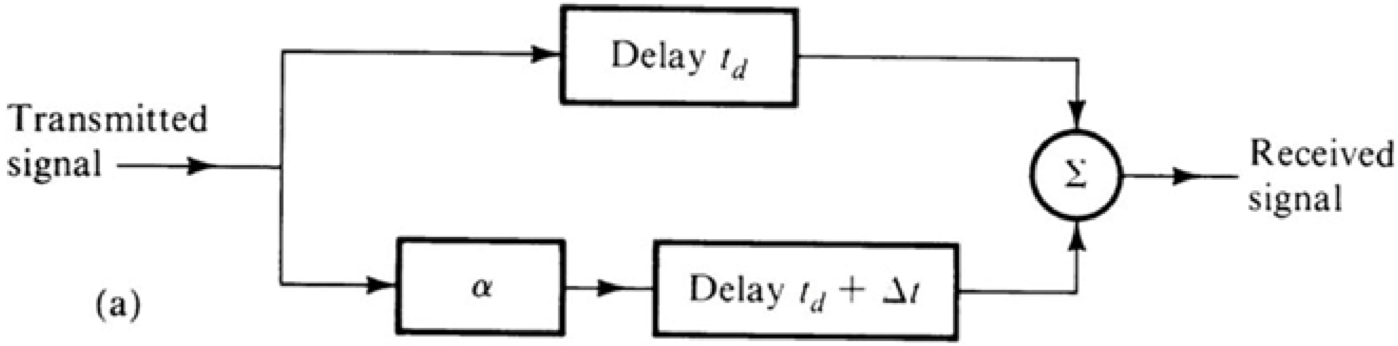
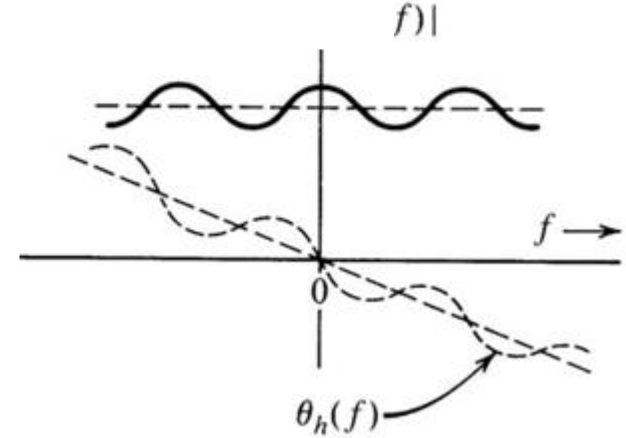


Figure 3.36 Signal distortion caused by nonlinear operation: (a) desired (input) signal spectrum; (b) spectrum of the unwanted signal (distortion) in the received signal; (c) spectrum of the received signal; (d) spectrum of the received signal after low-pass filtering.

Distortion Caused by Multipath Effects





$$H(f) = e^{-j2\pi f t_d} + \alpha e^{-j2\pi f(t_d+\Delta t)}$$

$$H(f) = e^{-j2\pi f t_d}(1 + \alpha e^{-j2\pi f \Delta t})$$

$$H(f) = e^{-j2\pi f t_d}(1 + \alpha\, cos2\pi f \Delta t - j\alpha\, sin2\pi f \Delta t)$$

$$H(f) = \sqrt{1 + \alpha^2 + 2\alpha\, cos2\pi f \Delta t}\; exp\left[-j\left(2\pi f t_d + tan^{-1}\frac{\alpha\, sin2\pi f \Delta t}{1 + \alpha\, cos2\pi f \Delta t}\right)\right]$$

Common distortion in this type of channel is frequency selective fading

Energy and Energy Spectral Density

$$E_g = \int_{-\infty}^{\infty} g(t)g^*(t)dt$$

Energy in the time domain

$$E_g = \int_{-\infty}^{\infty} |G(f)|^2 df$$

Energy in the frequency domain

Energy spectral density (ESD), $\Psi_g(f)$, is the energy per unit bandwidth (in hertz) of the spectral components of $g(t)$ centered at frequency f.

$$\Psi_g(f) = |G(f)|^2$$

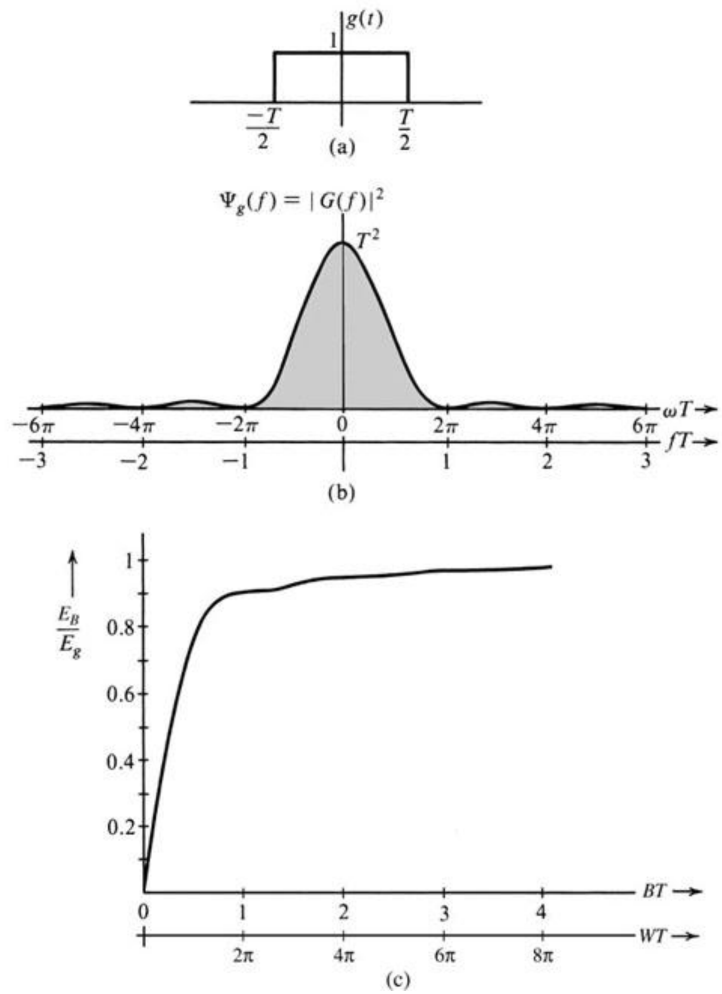
The ESD of the system's output in term of the input ESD is

$$\Psi_x(f) \longrightarrow \boxed{H(f)} \longrightarrow$$

$$\Psi_y(f) = |H(f)|^2 \Psi_x(f)$$

Essential Bandwidth of a Signal

Estimate the essential bandwidth of a rectangular pulse $g(t) = \Pi(t/T)$, where the essential bandwidth must contain at least 90% of the pulse energy.



$$E_g = \int_{-\infty}^{\infty} g^2(t)dt = \int_{-T/2}^{T/2} dt = T$$

$$E_B = \int_{-B}^{B} T^2 sinc^2(\pi f T)df = 0.9E_g$$

Solve the above equation numerically to find B

Energy of Modulated Signals

The modulated signal appears more energetic than the signal $g(t)$ but its energy is half of the energy of the signal $g(t)$. Why?

$$\varphi(t) = g(t)\, cos2\pi f_0 t$$

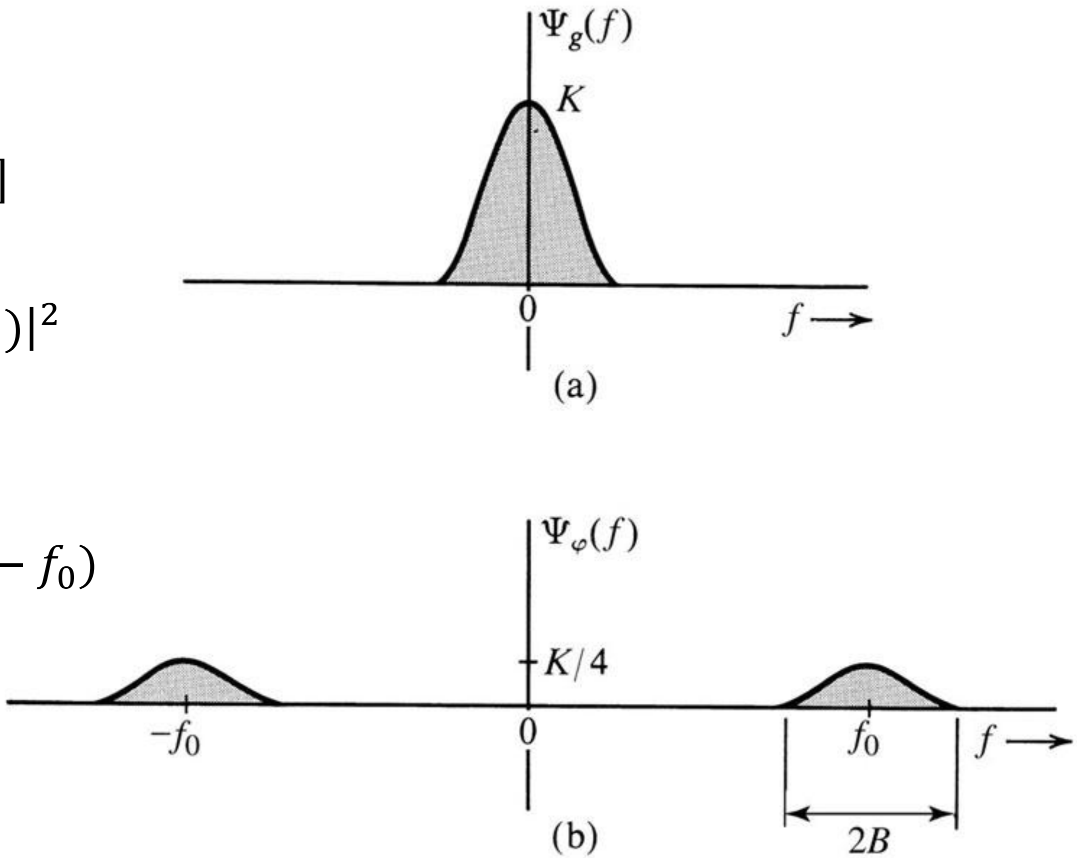
$$\Phi(f) = \frac{1}{2}[G(f + f_0) + G(f - f_0)]$$

$$\Psi_\varphi(f) = \frac{1}{4}|G(f + f_0) + G(f - f_0)|^2$$

If $f_0 > 2B$ then

$$\Psi_\varphi(f) = \frac{1}{4}\Psi_g(f + f_0) + \frac{1}{4}\Psi_g(f - f_0)$$

$$E_\varphi = \frac{1}{2}E_g$$



Time Autocorrelation Function and Energy Spectral Density

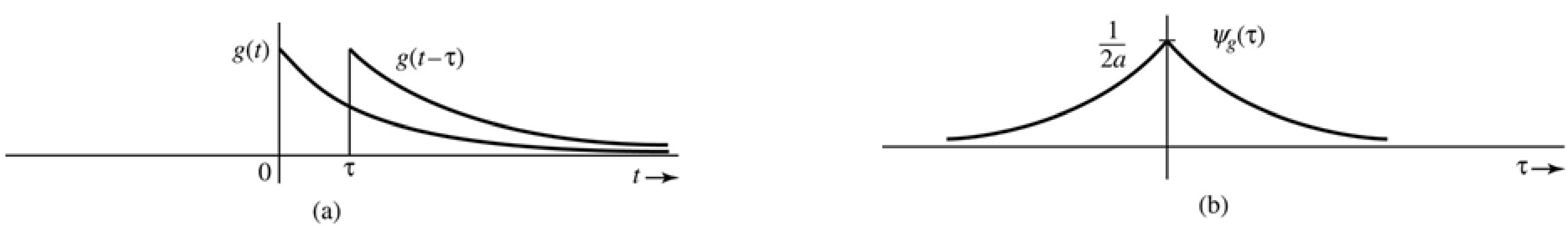
The autocorrelation $\psi_g(\tau)$ of a signal $g(t)$ and its ESD $\Psi_g(f)$ form a Fourier transform pair, that is

$$\psi_g(\tau) \underset{FT\ and\ IFT}{\Longleftrightarrow} \Psi_g(f)$$

Note: the autocorrelation of $g(t)$ is the convolution of $g(t)$ with $g(-t)$.

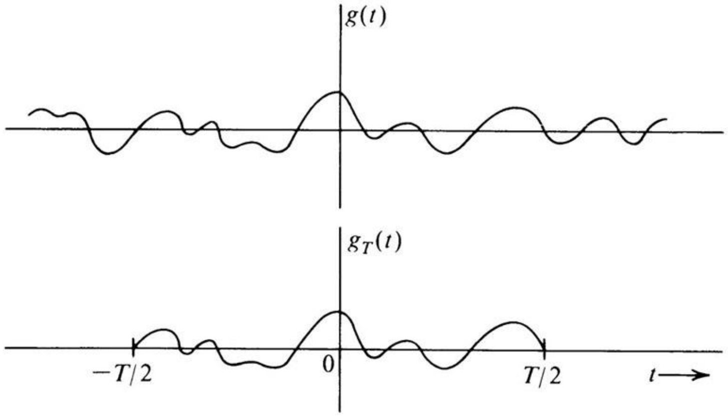
Example: Find the time autocorrelation function of the signal $g(t) = e^{-at}u(t)$, and from it determine the ESD of $g(t)$.

$$\psi_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t-\tau)dt$$



Signal Power and Power Spectral Density

Power P_g of the signal $g(t)$



$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t)g^*(t)dt$$

$$P_g = \lim_{T\to\infty} \frac{E_{gT}}{T}$$

Power spectral density $S_g(f)$ of the signal $g(t)$

$$S_g(f) = \lim_{T\to\infty} \frac{|G_T(f)|^2}{T}$$

$$P_g = \int_{-\infty}^{\infty} S_g(f)df = 2\int_{0}^{\infty} S_g(f)df$$

$$S_x(f) \longrightarrow \boxed{H(f)} \longrightarrow$$

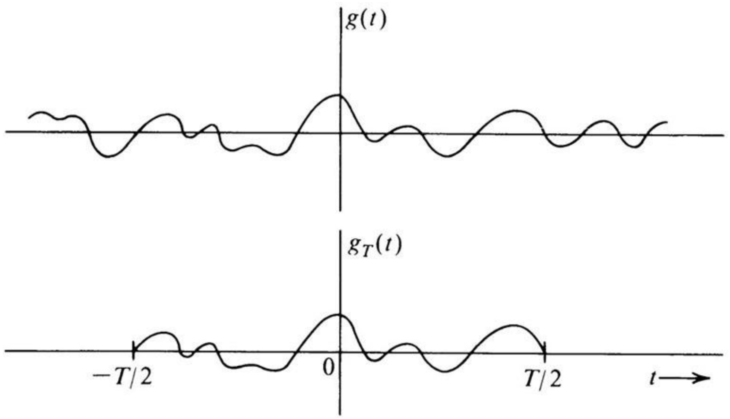
$$S_y(f) = |H(f)|^2 S_x(f)$$

Time Autocorrelation Function of Power Signals

Time autocorrelation $\mathcal{R}_g(\tau)$ of a power signal $g(t)$

$$\mathcal{R}_g(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t)g(t-\tau)dt$$

$$\mathcal{R}_g(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} g_T(t)g_T(t+\tau)dt$$



$$\mathcal{R}_g(\tau) = \lim_{T\to\infty} \frac{\psi_{gT}(\tau)}{T}$$

$$\mathcal{R}_g(\tau) \underset{FT\ and\ IFT}{\Longleftrightarrow} S_g(f)$$

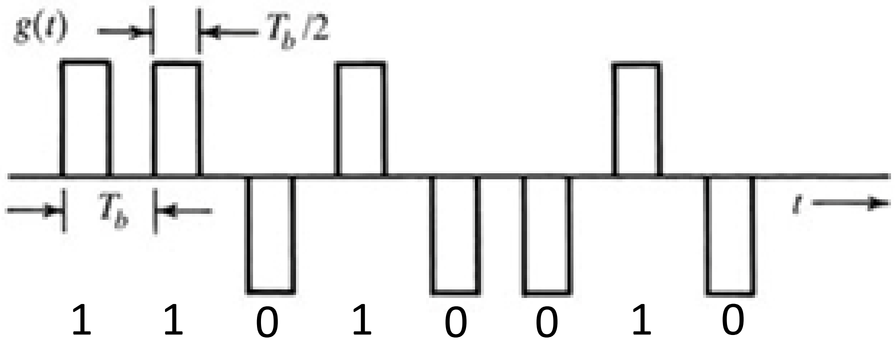
Autocorrelation a Powerful Tool

If the energy or power spectral density can be found by the Fourier transform of the signal $g(t)$, then why do we need to find the time autocorrelation?

Ans: In communication field and in general the signal $g(t)$ is not deterministic and it is probabilistic function.

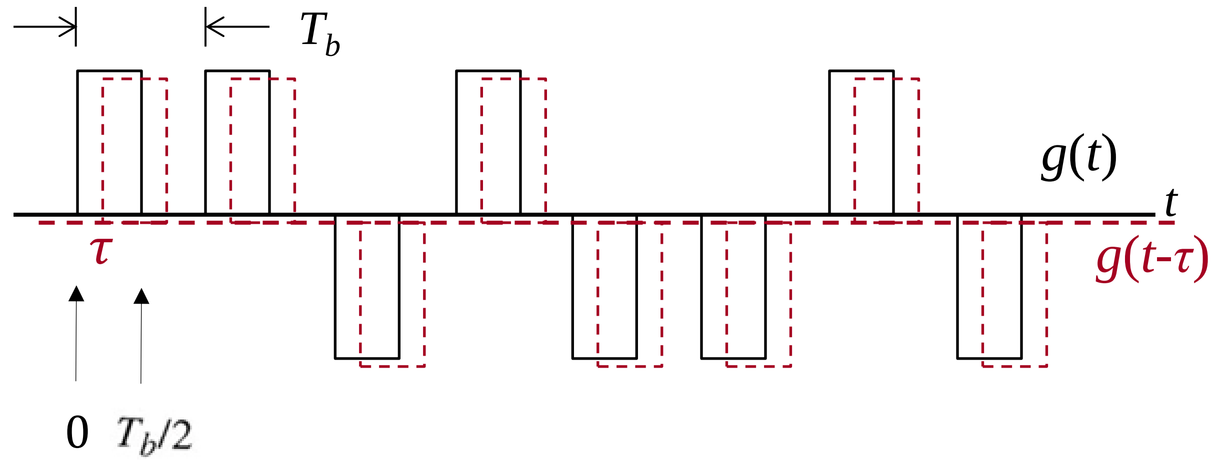
Example

A random binary pulse train $g(t)$. The pulse width is $T_b/2$, and one binary digit is transmitted every T_b seconds. A binary **1** is transmitted by positive pulse, and a binary **0** is transmitted by negative pulse. The two symbols are equally likely and occur randomly. Determine the PSD and the essential bandwidth of this signal.



Challenge: $g(t)$ is not deterministic and can not be expressed mathematically to find the Fourier transform and PSD. $g(t)$ is probabilistic (almost random) signal.

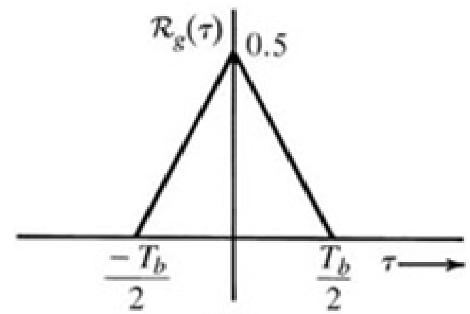
$$\mathcal{R}_g(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t)g(t-\tau)dt$$



For $0 < \tau < T_b/2$

Let $T = NT_b$ and solve the above integral.

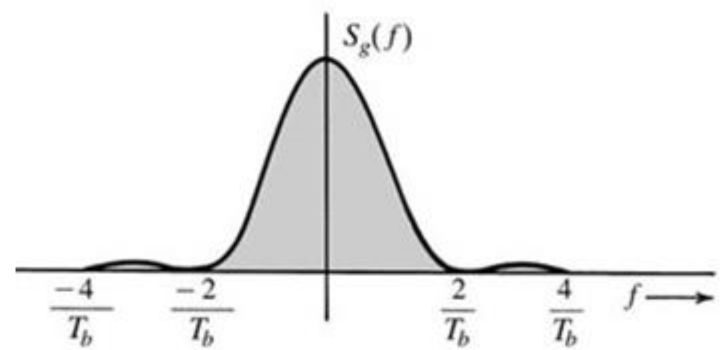
$$\mathcal{R}_g(\tau) = \lim_{N\to\infty} \frac{1}{NT_b}\left(\frac{T_b}{2} - \tau\right)N = \left(\frac{1}{2} - \frac{\tau}{T_b}\right)$$



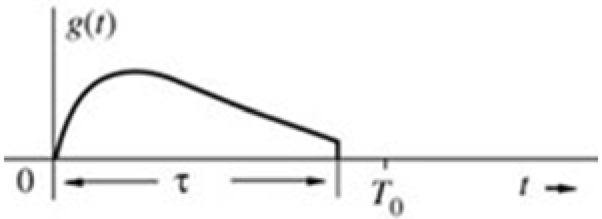
For $\tau > T_b/2$

$$\mathcal{R}_g(\tau) = 0$$

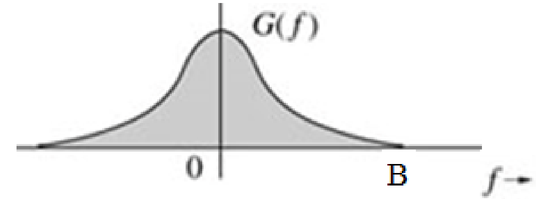
$$S_g(f) = \frac{T_b}{4}\,\mathrm{sinc}^2\left(\frac{\pi f T_b}{2}\right)$$

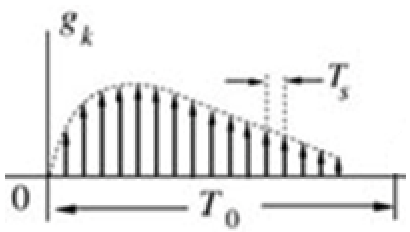


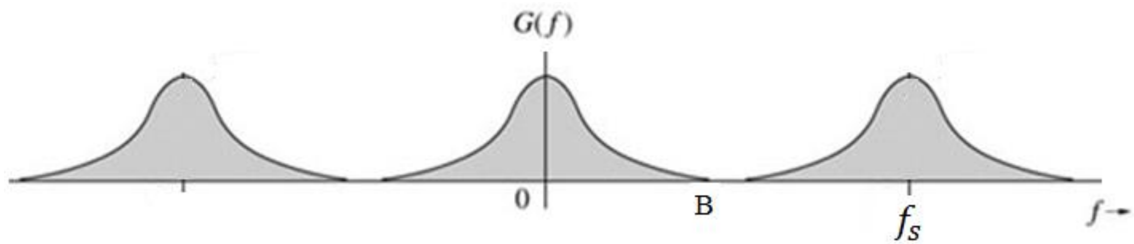
Discrete Fourier Transform (DFT, FFT)



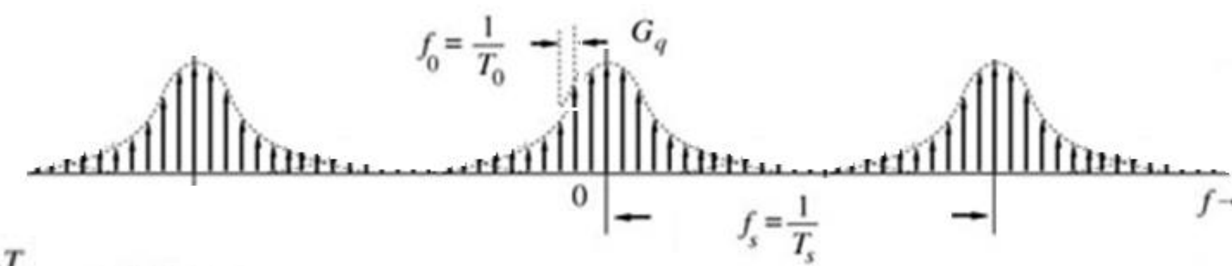
$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt$$

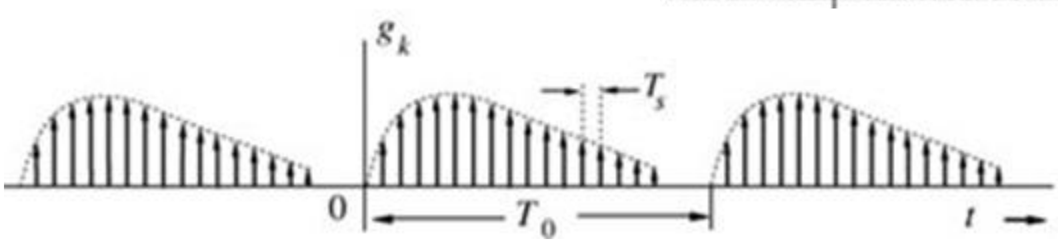






$$f_s = \frac{1}{T_s} \geq 2B$$





$$N_0 = \frac{T_0}{T_s}$$

$$G(qf_0) = \sum_{k=0}^{N_0-1} g(kT_s) e^{-j2\pi q f_0 k T_s}$$

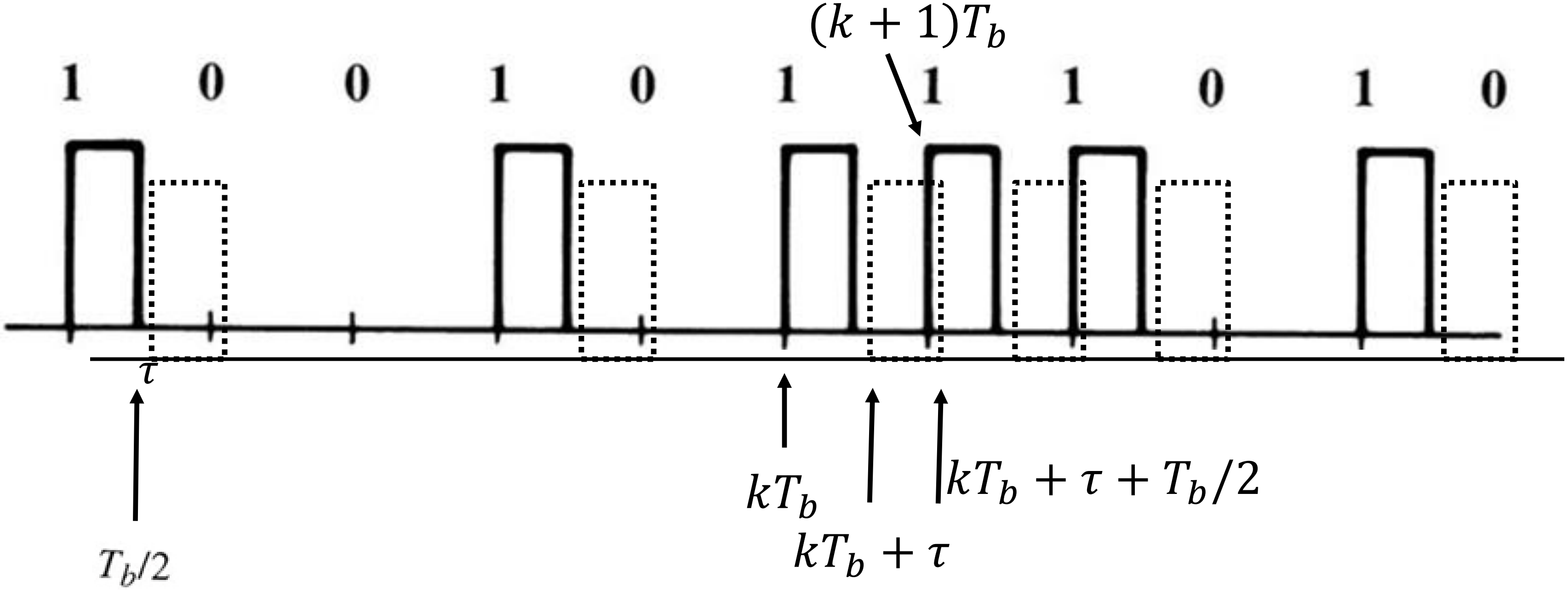
$$G_q = \sum_{k=0}^{N_0-1} g_k e^{-jq\Omega_0 k}$$

$$\Omega_0 = \frac{2\pi}{N_0}$$

Homework Problem

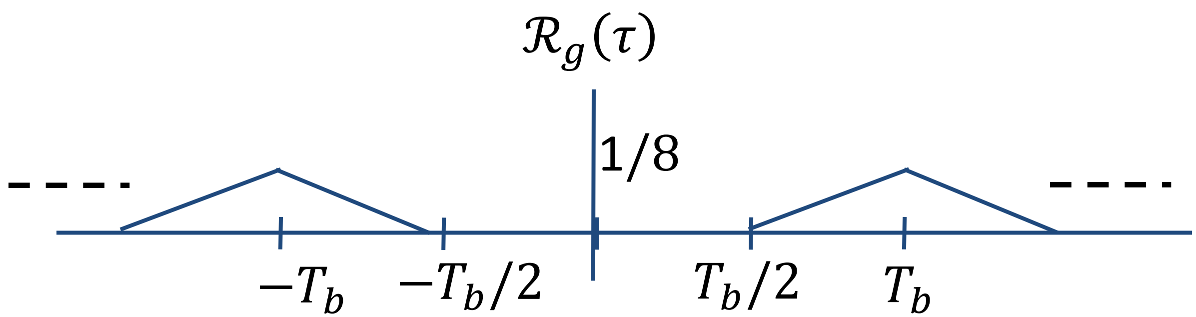
$$\mathcal{R}_g(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t)g(t-\tau)dt$$

$T_b/2 < \tau < T_b$



$$\mathcal{R}_g(\tau) = \lim_{N\to\infty} \frac{1}{NT_b} \left(kT_b + \tau + \frac{T_b}{2} - (k+1)T_b \right) \frac{N}{4}$$

$$\mathcal{R}_g(\tau) = \frac{1}{4}\left(\frac{\tau}{T_b} - \frac{1}{2}\right)$$



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