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Vector and Scalars

Chapter-1

Scalar Quantity: A physical quantity which possesses magnitude only but is not related to any definite direction in space is known as a scalar quantity or simply a scalar.

Example: Volume, area, mass, time, density, length, temperature work potential, any real number.

Vector Quantity: A physical quantity which possesses magnitude and is associated with a direction in space is known as a vector quantity or simply a vector.

Example: Displacement, velocity, force, acceleration.

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Free Vector: When we are at liberty to choose the origin of a vector at any point, then it is said to be a free vector.

Collinear Vector: Collinear vectors are two or more vectors which are parallel to the same line irrespective of their magnitudes and direction.

Coplanar Vector: The vectors which are parallel to the same plane or lie on the same ~~place~~ plane are said to be coplanar.

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Zero / Null vector: A null vector is a vector that has a magnitude equal to zero and is directionless.

Unit vector: A vector whose modulus is unity is called a unit vector.

Suppose A is any vector with length $|A| > 0$. Then $A/|A|$ is a unit vector. Denoted by a which has the same direction as A. Then $a = \frac{A}{|A|}$.

Scalar field: If two each point (x, y, z) of a region R in a space there corresponds a number or scalar $\varphi(x, y, z)$, then φ is called a scalar function of the position or a scalar point function.

Example: The temperature at any point of earth's surface

[A scalar field φ, which is independent of time, is called a stationary or steady-state scalar field.]

Vector field: If two each point (x, y, z) of a region R in a space there corresponds a vector $V(x, y, z)$, then V is called a vector function of the position or a vector point function.

Example: Velocity, acceleration at any point within a moving fluid.

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[A vector field V, which is independent of time is called a stationary or steady-state vector field.

Laws of vector Algebra:

Suppose A, B, C are vectors and m and n are scalars. Then the following laws hold:

Vector Addition

1. $(A+B)+C = A+(B+C) \rightarrow$ Associative Law for Addition.
2. $0+A = A+0 = A \rightarrow$ Existence of zero element.
3. $A+(-A) = (-A)+A = 0 \rightarrow$ Existence of negatives.
4. $A+B = B+A \longrightarrow$ Commutative Law for Addition.

Scalar multiplication.

5. $m(A+B) = mA+mB \rightarrow$ Distributive Law
6. $(m+n)A = mA+nA \rightarrow$ Distributive Law
7. $m(nA) = (mn)A \rightarrow$ Associative Law
8. $1(A) = A \rightarrow$ Unit Multiplication
9. $A+C = B+C \rightarrow A = B \rightarrow$ Cancellation Law

i) For any scalar m and zero vector 0, we have $m0 = 0$

ii) For any vector A and scalar 0, we have $0A = 0$

iii) If $mA = 0$, then ~~m0 or~~ $m = 0$ or $A = 0$

iv) For any vector A and scalar m, we have

$$(-m)A = m(-A) = -(mA).$$

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Linear Combination: A vector R is said to be a linear combination of the given vectors A, B, C ---- etc, if there exists a system of scalars x, y, z such that $\underline{r} = x\underline{a} + y\underline{b} + z\underline{c} + \cdots$

Linearly dependent vector: The system of n vectors $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$ is said to be linearly dependent if there exist $x_1, x_2, \ldots, x_n$ not all zero such that $x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = 0$

Linearly Independent vector: The system of n vectors $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$ is said to be linearly independent if there exists scalars $x_1, x_2, \ldots, x_n$ all zero.

Resultant vector: $R = R_1 + R_2$

Condition:

If determinant is zero $(=0)$, it is linearly dependent.

If determinant is not zero $(\neq 0)$, it is linearly independent.

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

if the value of this is zero, then the vectors are linearly dependent. otherwise, linearly independent.

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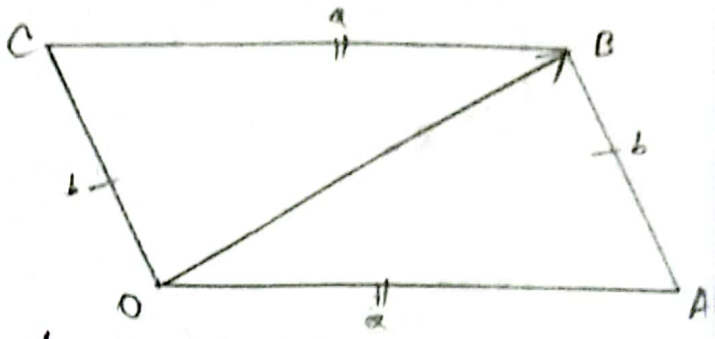
Show that addition of vectors is commutative, that is,

$\underline{a} + \underline{b} = \underline{b} + \underline{a}$.



The commutative law of vector addition is normally represented that, $\underline{a} + \underline{b} = \underline{b} + \underline{a}$

where $\underline{a}$ and $\underline{b}$ are two vectors. If $\underline{OA} = \underline{a}$ and $\underline{AB} = \underline{b}$, then $\underline{OB} = \underline{a} + \underline{b}$

Now, completing the parallelogram whose two sides are OC and CB. Since, the opposite sides of a parallelogram are equal and parallel.

Therefore, we say that,

$$\underline{OA} = \underline{CB} = \underline{a}$$

$$\underline{OC} = \underline{AB} = \underline{b}$$

$$\therefore \underline{OA} + \underline{AB} = \underline{OB} = \underline{OC} + \underline{CB}$$

$$\therefore \underline{a} + \underline{b} = \underline{b} + \underline{a}.$$

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Find a unit vector u parallel to the resultant R of vectors $r_1 = 2i + 4j - 5k$ and $r_2 = -i - 2j + 3k$.

$\Rightarrow$ Resultant, $R = r_1 + r_2 = (2i + 4j - 5k) + (-i - 2j + 3k)$

$= i + 2j - 2k.$

Also,
Magnitude of $R = |R| = |i + 2j - 2k| = \sqrt{(1)^2 + (2)^2 + (-2)^2}$

$= 3$

Then u is equal to $R/|R|$. That is,

$u = R/|R| = (i + 2j - 2k)/3$

$= (1/3)i + (2/3)j - (2/3)k$

Check: $|(1/3)i + (2/3)j - (2/3)k| =$

$= \sqrt{(1/3)^2 + (2/3)^2 + (-2/3)^2}$

$= 1$

(Ans)

Suppose, $r_1 = 2i - j + k$, $r_2 = i - 3j - 2k$, $r_3 = -2i + j - 3k$.
Write $r_4 = i + 3j + 2k$ as a linear combination of r_1, r_2, r_3; that is, find scalars a, b, c such that $r_4 = ar_1 + br_2 + cr_3$.

$\Rightarrow$ We require

$i + 3j + 2k = a(2i - j + k) + b(i - 3j - 2k) + c(-2i + j - 3k)$

$= (2a + b - 2c)i + (-a - 3b + c)j + (a - 2b - 3c)k$

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Since i, j, k are non-coplanar, we set corresponding coefficients equal to each other obtaining $2a+b-2c=1$, $-a+3b+c=3$, $a-2b-3c=2$.

solving, $a=-2$, $b=1$, $c=-2$. Thus $r_4 = -2r_1 + r_2 - 2r_3$

The vector r_4 is said to be linearly dependent on r_1, r_2 and r_3; in the other words r_1, r_2, r_3 and r_4 constitute a linearly dependent set of vectors. On the other hand, any three of these vectors are linearly independent.

Suppose a, b, c are non-coplanar vectors. Determine whether the following vectors are linearly independent or linearly dependent:

$r_1 = 2a-3b+c$, $r_2 = 3a-5b+2c$, $r_3 = 4a-5b+c$

$\Rightarrow$ Given that,

$$r_1 = 2a-3b+c$$

$$r_2 = 3a-5b+2c$$

$$r_3 = 4a-5b+c$$

$\therefore r_3 = x r_1 + y r_2$

$\Rightarrow 4a-5b+c = x(2a-3b+c) + y(3a-5b+2c)$

$= (2x+3y)a + (-3x-5y)b + (x+2y)c$

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Since a, b, c are non-Coplanar vectors, we get

$$2x + 3y = 4 \quad \cdots\cdots (i)$$

$$-3x - 5y = -5 \quad \cdots\cdots (ii)$$

$$x + 2y = 1 \quad \cdots\cdots (iii)$$

Subtracing 2 times (iii) from (i), we get

$$\begin{array}{rcrcl} 2x & + & 3y & = & 4 \\ 2x & + & 4y & = & 2 \\ (-) & & (-) & & (-) \\ \hline & & -y & = & 2 \end{array}$$

$$\therefore y = -2$$

From (iii), we get

$$x + 2\cdot(-2) = 1$$

$$\Rightarrow x - 4 = 1$$

$$\therefore x = 5$$

Thus $r_3 = 5r_1 - 2r_2$

Therefore, the vectors are linearly dependent.

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In each case, determine whether the vectors are linearly independent or linearly dependent:

a) $A = 2i + j - 3k$
$B = i - 4k$
$C = 4i + 3j - k$

b) $A = i - 3j + 2k$,
$B = 2i - 4j - k$,
$C = 3i + 2j - k$

$\Rightarrow$ a) Given that,

$$\vec{A} = 2\hat{i} + \hat{j} - 3\hat{k}$$
$$\vec{B} = \hat{i} - 4\hat{k}$$
$$\vec{C} = 4\hat{i} + 3\hat{j} - \hat{k}$$

$\therefore \vec{C} = a\vec{A} + b\vec{B}$

$\Rightarrow 4\hat{i} + 3\hat{j} - \hat{k} = a(2\hat{i} + \hat{j} - 3\hat{k}) + b(\hat{i} - 4\hat{k})$

$= (2a+b)\hat{i} + a\hat{j} + (-3a - 4b)\hat{k}$

Since $\hat{i}, \hat{j}, \hat{k}$ are non-collinear, we get

$2a + b = 4$ ..... (i)

$a = 3$ ..... (ii)

$-3a - 4b = -1$ ..... (iii)

From (i), we get

$2 \cdot 3 + b = 4$

$\Rightarrow 6 + b = 4$

$\therefore b = -2$

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Thus, $\vec{C} = 3\vec{A} - 2\vec{B}$

Therefore, the vectors are linearly dependent.

Or,

$$\text{Determinant} = \begin{vmatrix} 2 & 1 & -3 \\ 1 & 0 & -4 \\ 4 & 3 & -1 \end{vmatrix}$$

$$\cancel{= 2(0-12) - (1+16) - 3(-3-0)}$$

$$= 2(0+12) - 1(-1+16) - 3(3-0)$$

$$= 24 - 15 - 9$$

$$= 0$$

So, the vectors are linearly dependent.

b) Given that,

$$\vec{A} = \hat{i} - 3\hat{j} + 2\hat{k}$$

$$\vec{B} = 2\hat{i} - 4\hat{j} - \hat{k}$$

$$\vec{C} = 3\hat{i} + 2\hat{j} - \hat{k}$$

$$\therefore \vec{C} = a\vec{A} + b\vec{B}$$

$$\Rightarrow 3\hat{i} + 2\hat{j} - \hat{k} = a(\hat{i} - 3\hat{j} + 2\hat{k}) + b(2\hat{i} - 4\hat{j} - \hat{k})$$

$$= (a+2b)\hat{i} + (-3a-4b)\hat{j} + (2a-b)\hat{k}$$

Since $\hat{i}, \hat{j}, \hat{k}$ are non-collinear, we have

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$a + 2b = 3$ ..... (i)

$-3a - 4b = 2$ ..... (ii)

$2a - b = -1$ ..... (iii)

(i) + 2 × (iii)

$$\begin{array}{r} a + 2b = 3 \\ 4a - 2b = -2 \\ \hline 5a \quad = 1 \end{array}$$

$\therefore a = \frac{1}{5}$

From (i),

$\frac{1}{5} + 2b = 3$

$\Rightarrow 1 + 10b = 15$

$\Rightarrow 10b = 14$

$\therefore b = \frac{14}{10} = \frac{7}{5}$

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Chapter - 2

Dot or Scalar ~~Pr~~ Product:

The dot or scalar product of two vectors $\vec{A}$ and $\vec{B}$, denoted by $\vec{A}\cdot\vec{B}$ is defined as the product of the magnitudes of A and B and the cosine of the angle θ between them.

$$\vec{A}\cdot\vec{B} = |A||B|\cos\theta, \quad 0 \le \theta \le \pi$$

where $\vec{A}\cdot\vec{B}$ is a scalar and not a vector.

Proposition: Suppose $\vec{A}, \vec{B}, \vec{C}$ are vectors and m is scalar.

i) $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ → Commutative Law for Dot products

ii) $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$ → Distributive Law

iii) $m(\vec{A}\cdot\vec{B}) = (m\vec{A})\cdot\vec{B} = \vec{A}\cdot(m\cdot\vec{B}) = (\vec{A}\cdot\vec{B})m$

iv) $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0, \quad \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

v) If $\vec{A}\cdot\vec{B} = 0$ and $\vec{A}$ and $\vec{B}$ are not null vectors, then $\vec{A}$ and $\vec{B}$ are perpendicular.

$\hat{i}, \hat{j}, \hat{k}$ → unit vectors.

Given that,

$$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$

$$\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$$

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$\therefore \vec{A}\cdot\vec{B} = (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})\cdot(B_1\hat{i} + B_2\hat{j} + B_3\hat{k})$

$= A_1B_1 + A_2B_2 + A_3B_3$

and $\vec{A}\cdot\vec{A} = A_1^2 + A_2^2 + A_3^2$.

Cross product:

C The cross product of two vectors $\vec{A}$ and $\vec{B}$ is a vector $\vec{C} = \vec{A}\times\vec{B}$, defined as the magnitude of $\vec{C} = \vec{A}\times\vec{B}$ is equal to the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the sine of the angle θ between them.

$$\vec{A}\times\vec{B} = |A|\,|B|\sin\theta\, u, \quad 0 \le \theta \le \pi$$

where u is a unit vector indicating the direction $\vec{A}\times\vec{B}$. [$\vec{A}$, $\vec{B}$ and u form a right-handed system]

proposition: suppose $\vec{A}, \vec{B}, \vec{C}$ are vectors and m is scalar.

i) $\vec{A}\times\vec{B} = -(\vec{B}\times\vec{A})$ → Commutative law for cross Product

ii) $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$ → Distributive Law

iii) $m(\vec{A}\times\vec{B}) = (m\vec{A})\times\vec{B} = \vec{A}\times(m\vec{B}) = (\vec{A}\times\vec{B})m$

iv) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$, $\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$

v) If $\vec{A}\times\vec{B} = 0$ and $\vec{A}$ and $\vec{B}$ are not null vectors, then $\vec{A}$ and $\vec{B}$ are parallel.

vi) The magnitude of $\vec{A}\times\vec{B}$ is the same as the area of a parallelogram with sides $\vec{A}\times\vec{B}$.

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$\hat{i}, \hat{j}, \hat{k} \rightarrow$ unit vectors.

Given that,

$$\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$$

$$\vec{B} = B_1 \hat{i} + B_2 \hat{j} + B_3 \hat{k}$$

$$\therefore \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= \begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix} \hat{i} - \begin{vmatrix} A_1 & A_3 \\ B_1 & B_3 \end{vmatrix} \hat{j} + \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix} \hat{k}$$

Triple Products:

Dot and cross multiplication of three vectors $\vec{A}, \vec{B}$ and $\vec{C}$ may produce meaningful products, called triple products of the form $(\vec{A} \cdot \vec{B})\vec{C}$, $\vec{A} \times (\vec{B} \times \vec{C})$ and ~~A~~ $\vec{A} \cdot (\vec{B} \times \vec{C})$.

Preposition: Suppose $\vec{A}$, $\vec{B}$ and $\vec{C}$ are vectors and m is a ~~sto~~ scalar.

i) In general, $(\vec{A} \cdot \vec{B}) \cdot \vec{C} \neq \vec{A} (\vec{B} \cdot \vec{C})$

ii) $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) =$ volume of a parallelepiped having $\vec{A}$, $\vec{B}$ and $\vec{C}$ as edges, or the negative of this volume,

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according as $\vec{A}$, $\vec{B}$ and $\vec{C}$ do or do not form a right handed system.

iii) In general, $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$

(Associative Law for Cross Products Fails)

iv) $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

$(\vec{A} \times \vec{B}) \times \vec{C} = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{B} \cdot \vec{C})\vec{A}$

Given that,

$$\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$$
$$\vec{B} = B_1 \hat{i} + B_2 \hat{j} + B_3 \hat{k}$$
$$\vec{C} = C_1 \hat{i} + C_2 \hat{j} + C_3 \hat{k}$$

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \hat{i}(B_2C_3 - B_3C_2) - \hat{j}(B_1C_3 - B_3C_1) + \hat{k}(B_1C_2 - B_2C_1)$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = A_1(B_2C_3 - B_3C_2) - A_2(B_1C_3 - B_3C_1) + A_3(B_1C_2 - B_2C_1)$$

$$\therefore \vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$= -\begin{vmatrix} B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \vec{B} \cdot (\vec{C} \times \vec{A})$$

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$$= -\begin{vmatrix} C_1 & C_2 & C_3 \\ B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= \vec{C} \cdot (\vec{A} \times \vec{B})$$

$$\therefore \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

[proved]

Suppose $\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$ and $\vec{B} = B_1 \hat{i} + B_2 \hat{j} + B_3 \hat{k}$. prove that $\vec{A} \cdot \vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3$.

$\Rightarrow$ Given that,

$$\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$$

$$\vec{B} = B_1 \hat{i} + B_2 \hat{j} + B_3 \hat{k}$$

Since $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ and all other dot products are zero,

Therefore, we have

$$\vec{A} \cdot \vec{B} = (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}) \cdot (B_1 \hat{i} + B_2 \hat{j} + B_3 \hat{k})$$

$$= A_1 \hat{i} \cdot (B_1 \hat{i} + B_2 \hat{j} + B_3 \hat{k}) + A_2 \hat{j} \cdot (B_1 \hat{i} + B_2 \hat{j} + B_3 \hat{k})$$

$$+ A_2 \hat{k} \cdot (B_1 \hat{i} + B_2 \hat{j} + B_3 \hat{k})$$

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$= A_1 B_1 \hat{i}\cdot\hat{i} + A_1 B_2 \hat{i}\cdot\hat{j} + A_1 B_3 \hat{i}\hat{k} + A_2 B_1 \hat{i}\cdot\hat{j} + A_2 B_2 \hat{j}\cdot\hat{j}$

$+ A_2 B_3 \hat{j}\cdot\hat{k} + A_3 B_1 \hat{k}\cdot\hat{i} + A_3 B_2 \hat{k}\cdot\hat{j} + A_3 B_3 \hat{k}\cdot\hat{k}$

$= A_1 B_1 + A_2 B_2 + A_3 B_3$

$\therefore \vec{A}\cdot\vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3$ [proved]

Suppose $A\cdot B = 0$ and A and B are not zero. Show that A is perpendicular to B.

CT

$\Rightarrow$ If $\vec{A}\cdot\vec{B} = |A||B|\cos\theta = 0$

then, $\cos\theta = 0$

$\Rightarrow \cos 90^\circ = 0$

$\therefore \theta = 90^\circ$

Conversely, if $\theta = 90^\circ$, $A\cdot B = 0$

$\therefore \vec{A}$ is perpendicular to $\vec{B}$ [showed]

Find the angle between $\vec{A} = 2\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{B} = 7\hat{j} + 24\hat{k}$.

$\Rightarrow$ we know,

$\vec{A}\cdot\vec{B} = |A||B|\cos\theta$

Here, $|A| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3$

$|B| = \sqrt{7^2 + 24^2} = 25$

$\therefore \vec{A}\cdot\vec{B} = (2\hat{i} + 2\hat{j} - \hat{k})\cdot(7\hat{i} + 24\hat{k})$

$= 14 + 0 - 24$

$= -10$

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$$\therefore \cos\theta = \frac{\vec{A}\cdot\vec{B}}{|A|\,|B|} = \frac{-10}{3\times 25} = -\frac{2}{15} = -0.13333$$

$\therefore \theta = 98^\circ$ (approximately).

Determine the value of α so that $\vec{A} = 2\hat{i} + \alpha\hat{j} + \hat{k}$ and $\vec{B} = \hat{i} + 3\hat{j} - 8\hat{k}$ are perpendicular.

$\Rightarrow$ $\vec{A}$ and $\vec{B}$ are perpendicular when $\vec{A}\cdot\vec{B} = 0$

Thus,

$$\vec{A}\cdot\vec{B} = (2\hat{i} + \alpha\hat{j} + \hat{k})\cdot(\hat{i} + 3\hat{j} - 8\hat{k})$$
$$= 2 + 3\alpha - 8$$
$$= 3\alpha - 6$$

As, $\vec{A}\cdot\vec{B} = 0$

$\therefore 3\alpha - 6 = 0$

$\therefore \alpha = 2$

$\therefore$ The value of α is 2.

Find the projection of the vector $\vec{A} = \hat{i} - 2\hat{j} + 3\hat{k}$ on the vector $\vec{B} = \hat{i} + 2\hat{j} + 2\hat{k}$.

$\Rightarrow$ A unit vector in the direction of $\vec{B}$ is,

$$\hat{b} = \frac{\vec{B}}{|\vec{B}|} = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}$$

$$= \frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}.$$

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The projection of $\vec{A}$ on vector $\vec{B}$ is,

$$\vec{A}\cdot\hat{b} = (\hat{i}-2\hat{j}+3\hat{k})\cdot\left(\tfrac{1}{3}\hat{i}+\tfrac{2}{3}\hat{j}+\tfrac{2}{3}\hat{k}\right)$$

$$= \frac{1}{3} - \frac{4}{3} + 2$$

$$= \frac{1-4+6}{3} = \frac{3}{3} = 1$$

Find the work done in moving an object along a vector $\vec{r} = 3\hat{i}+\hat{j}-5\hat{k}$ if the applied force is $F = 2\hat{i}-\hat{j}-\hat{k}$.

$\Rightarrow$ work done $= Fr\cos\theta$

$$= \vec{F}\cdot\vec{r}$$

$$= (2\hat{i}-\hat{j}-\hat{k})\cdot(3\hat{i}+\hat{j}-5\hat{k})$$

$$= 6 - 1 + \cancel{10}\ 5$$

$$= 10$$

Suppose $\vec{A}\times\vec{B} = 0$ and $\vec{A}$ and $\vec{B}$ are not zero. Show that $\vec{A}$ is parallel to $\vec{B}$.

$\Rightarrow$ Since $\vec{A}\times\vec{B} = |A|\,|B|\sin\theta\, u = 0$

$$\Rightarrow \sin\theta = 0$$

$$\Rightarrow \theta = 0^\circ \text{ or } 180^\circ$$

So, the angle between $\vec{A}$ and $\vec{B}$ is 0° or 180°.

$\therefore$ $\vec{A}$ is parallel to $\vec{B}$.

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Determine a unit vector perpendicular to the plane of $\vec{A} = 2\hat{i} - 6\hat{j} - 3\hat{k}$ and $\vec{B} = 4\hat{i} + 3\hat{j} - \hat{k}$.

$\Rightarrow \vec{A} \times \vec{B}$ is a vector perpendicular to the plane of $\vec{A}$ and $\vec{B}$.

$$\therefore \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} = \hat{i}(6+9) - \hat{j}(-2+12) + \hat{k}(6+24)$$

$$= 15\hat{i} - 10\hat{j} + 30\hat{k}$$

A unit vector parallel to $\vec{A} \times \vec{B}$ is,

$$\hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \frac{15\hat{i} - 10\hat{j} + 30\hat{k}}{\sqrt{15^2 + (-10)^2 + (30)^2}}$$

~~A unit vector parallel to $\vec{A} \times \vec{B}$ is~~,

$$= \frac{15\hat{i} - 10\hat{j} + 30\hat{k}}{35}$$

$$= \frac{3}{7}\hat{i} - \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k}$$

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Find the acute angles that the line joining the points $(1, -3, 2)$ and $(3, -5, 1)$ makes with the coordinate axes.

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$\Rightarrow$ Given that,

The two points $(1,-3,2)$ and $(3,-5,1)$

Vector joining the two points,

$$\vec{A} = (3-1)\hat{i} + (-5+3)\hat{j} + (1-2)\hat{k}$$

$$= 2\hat{i} - 2\hat{j} - \hat{k}$$

Let α, β, γ be the angles which vector $\vec{A}$ makes with the positive x, y, z axes, respectively.

$$\therefore \vec{A}\cdot\hat{i} = |\vec{A}|\,|\hat{i}| \cos\alpha$$

$$\Rightarrow (2\hat{i} - 2\hat{j} - \hat{k})\cdot\hat{i} = \sqrt{2^2+(-2)^2+(-1)^2}\cdot 1 \cos\alpha$$

$$\Rightarrow 2 = 3\cos\alpha$$

$$\therefore \alpha = \cos^{-1}\left(\tfrac{2}{3}\right)$$

Similarly,

$$\vec{A}\cdot\hat{j} = |\vec{A}|\,|\hat{j}| \cos\beta$$

$$\Rightarrow (2\hat{i} - 2\hat{j} - \hat{k})\cdot\hat{j} = \sqrt{2^2+(-2)^2+(-1)^2}\cdot 1\cdot \cos\beta$$

$$\Rightarrow -2 = 3\cos\beta$$

$$\therefore \beta = \cos^{-1}\left(-\tfrac{2}{3}\right)$$

Similarly,

$$\vec{A}\cdot\hat{k} = |\vec{A}|\,|\hat{k}| \cos\gamma$$

$$\Rightarrow (2\hat{i} - 2\hat{j} - \hat{k})\cdot\hat{k} = \sqrt{2^2+(-2)^2+(-1)^2}\cdot 1\cdot \cos\gamma$$

$$\Rightarrow -1 = 3\cos\alpha$$

$$\therefore \gamma = \cos^{-1}\left(-\tfrac{1}{3}\right)$$

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The cosines of α, β, γ are called direction cosines of the vector $\vec{A}$.

Find the projection of the vector A on the vector B:

a) $\vec{A} = 2\hat{i} - 3\hat{j} + 6\hat{k}$ and $\vec{B} = \hat{i} + 2\hat{j} + 2\hat{k}$.

→ Given that,

$$\vec{A} = 2\hat{i} - 3\hat{j} + 6\hat{k}$$
$$\vec{B} = \hat{i} + 2\hat{j} + 2\hat{k}$$

A unit vector in the direction of $\vec{B}$ is,

$$\hat{n} = \frac{\vec{B}}{|\vec{B}|} = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

The projection of the vector $\vec{A}$ on vector $\vec{B}$ is,

$$\vec{A} \cdot \hat{n}$$

$$= (2\hat{i} - 3\hat{j} + 6\hat{k}) \cdot \left(\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$$

$$= \frac{2}{3} - 2 + 4$$

$$= \frac{2 + 6}{3}$$

$$= \frac{8}{3}$$

//

Find the projection of the vector $\vec{A} = 4\hat{i} - 3\hat{j} + \hat{k}$ on the line passing through the points $(2, 3, -1)$ and $(-2, -4, 3)$.

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⇒ Given that,

$$\vec{A} = 4\hat{i} - 3\hat{j} + \hat{k}$$

The two points $P(2,3,-1)$ and $Q(-2,-4,3)$

$$\therefore \vec{PQ} = (-2-2)\hat{i} + (-4-3)\hat{j} + (3+1)\hat{k}$$

$$= -4\hat{i} - 7\hat{j} + 4\hat{k}$$

A unit vector in the direction of $\vec{PQ}$ is,

$$\hat{n} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{-4\hat{i} - 7\hat{j} + 4\hat{k}}{\sqrt{(-4)^2 + (-7)^2 + 4^2}} = -\frac{4}{9}\hat{i} - \frac{7}{9}\hat{j} + \frac{4}{9}\hat{k}.$$

The projection of the vector $\vec{A}$ on $\vec{PQ}$ is,

$$\vec{A} \cdot \hat{n} = (4\hat{i} - 3\hat{j} + \hat{k}) \cdot \left(-\frac{4}{9}\hat{i} - \frac{7}{9}\hat{j} + \frac{4}{9}\hat{k}\right)$$

$$= -\frac{16}{9} + \frac{7}{3} + \frac{4}{3}$$

$$= \frac{-16 + 21 + 4}{9}$$

$$= \frac{9}{9}$$

$$= 1$$

Find a unit vector perpendicular to both vector A and vector B where:

a) $\vec{A} = 4\hat{i} - \hat{j} + 3\hat{k}$
$\vec{B} = -2\hat{i} + \hat{j} - 2\hat{k}$

b) $\vec{A} = 6\hat{i} + 22\hat{j} - 5\hat{k}$
$\vec{B} = \hat{i} + 6\hat{j} - 2\hat{k}$.

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⇒ a) Given that,

$$\vec{A} = 4\hat{i} - \hat{j} + 3\hat{k}$$

$$\vec{B} = -2\hat{i} + \hat{j} - 2\hat{k}$$

$\therefore$ $\vec{A} \times \vec{B}$ is a vector perpendicular to both vector $\vec{A}$ and $\vec{B}$.

$$\therefore \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} = \hat{i}(2-3) - \hat{j}(-8+6) + \hat{k}(4-2)$$

$$= -\hat{i} + 2\hat{j} + 2\hat{k}$$

A unit vector parallel to $\vec{A} \times \vec{B}$ is,

$$\hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \frac{-\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{(-1)^2 + 2^2 + 2^2}}$$

$$= -\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

b) Given that,

$$\vec{A} = 6\hat{i} + 22\hat{j} - 5\hat{k}$$

$$\vec{B} = \hat{i} + 6\hat{j} - 2\hat{k}$$

$\therefore$ $\vec{A} \times \vec{B}$ is a vector perpendicular to both vector $\vec{A}$ and $\vec{B}$

$$\therefore \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 22 & -5 \\ 1 & 6 & -2 \end{vmatrix} = \hat{i}(-44+30) - \hat{j}(-12+5) + \hat{k}(36-22)$$

$$= -14\hat{i} + 7\hat{j} + 14\hat{k}$$

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A unit vector parallel to $\vec{A} \times \vec{B}$ is,

$$\hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \frac{-14\hat{i} + 7\hat{j} + 14\hat{k}}{\pm\sqrt{(-14)^2 + 7^2 + 14^2}}$$

$$= \frac{-14\hat{i} + 7\hat{j} + 14\hat{k}}{+21}$$

$$= \frac{\pm(2\hat{i} - \hat{j} - 2\hat{k})}{3}$$

//

Q1 # Find a unit vector parallel to the xy-plane and perpendicular to the vector $4\hat{i} - 3\hat{j} + \hat{k}$.

⇒ ~~Then~~ Any vector parallel to xy plane is $a\hat{i} + b\hat{j}$.

If this vector is perpendicular to $4\hat{i} - 3\hat{j} + \hat{k}$,

then, $(a\hat{i} + b\hat{j})(4\hat{i} - 3\hat{j} + \hat{k}) = 0$

$\Rightarrow 4a - 3b = 0$

$\Rightarrow b = \frac{4a}{3}$

Therefore, the vector parallel to the xy plane is,

$$a\hat{i} + \frac{4}{3}a\hat{j}$$

Required unit vector is $= \pm \dfrac{a\hat{i} + \frac{4}{3}a\hat{j}}{\sqrt{a^2 + (\frac{4}{3}a)^2}}$

$$= \pm \frac{a\hat{i} + \frac{4}{3}a\hat{j}}{a\sqrt{1 + \frac{16}{9}}}$$

$$= \pm \frac{\hat{i} + \frac{4}{3}\hat{j}}{\frac{5}{3}}$$

$$= \pm \frac{(3\hat{i} + 4\hat{j})}{5}$$

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Show that A, B, and C are ~~not~~ mutually orthogonal unit vectors where:

$\vec{A} = (2\hat{i} - 2\hat{j} + \hat{k})/3$, $\vec{B} = (\hat{i} + 2\hat{j} + 2\hat{k})/3$

and $\vec{C} = (2\hat{i} + \hat{j} - 2\hat{k})/3$

$\Rightarrow \vec{A}\cdot\vec{B} = \left(\frac{2}{3}\cdot\frac{1}{3}\right) + \left(-\frac{2}{3}\cdot\frac{2}{3}\right) + \left(\frac{1}{3}\cdot\frac{2}{3}\right)$

$= \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0.$

$\vec{B}\cdot\vec{C} = \left(\frac{1}{3}\cdot\frac{2}{3}\right) + \left(\frac{2}{3}\cdot\frac{1}{3}\right) + \left(\frac{2}{3}\cdot\left(-\frac{2}{3}\right)\right)$

$= \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$

$\vec{C}\cdot\vec{A} = \left(\frac{2}{3}\cdot\frac{2}{3}\right) + \left(-\frac{2}{3}\cdot\frac{1}{3}\right) + \left(-\frac{1}{3}\cdot\left(-\frac{2}{3}\right)\right)$

$= \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$

Since, $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{C} = \vec{C}\cdot\vec{A} = 0$, so, $\vec{A}$, $\vec{B}$ and $\vec{C}$ are mutually orthogonal unit vectors. ~~where~~

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Find the work done in moving an object along a straight line from (3, 2, −1) to (2, −1, 4) in a force field given by $\vec{F} = 4\hat{i} - 3\hat{j} + 2\hat{k}$.

$\Rightarrow$ A vector joining the two points is,

$(2-3)\hat{i} + (-1-2)\hat{j} + (4+1)\hat{k}$

$= -\hat{i} - 3\hat{j} + 5\hat{k}$

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Here $\vec{F} = 4\hat{i} - 3\hat{j} + 2\hat{k}$

$\therefore$ work done $= |\vec{F}||\vec{r}| \cos\theta$

$= \vec{F} \cdot \vec{r}$

$= -4 + 9 + 10$

$= 15$ //

Find the area of a parallelogram having diagonals $\vec{A} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{B} = \hat{i} - 3\hat{j} - 4\hat{k}$.

$\Rightarrow$ Given that, The ~~gia~~ diagonals,

$\vec{A} = 3\hat{i} + \hat{j} - 2\hat{k}$

$\vec{B} = \hat{i} - 3\hat{j} - 4\hat{k}$

Area of a parallelogram $= \frac{1}{2} |\vec{A} \times \vec{B}|$

Here,

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & -4 \end{vmatrix} = \hat{i}(-4-6) - \hat{j}(-12+2) + \hat{k}(-9-1)$$

$$= -10\hat{i} + 10\hat{j} - 10\hat{k}$$

$\therefore |\vec{A} \times \vec{B}| = \sqrt{(-10)^2 + 10^2 + (-10)^2} = \sqrt{300} = 10\sqrt{3}$

$\therefore$ Area of a parallelogram $= \frac{1}{2} \times 10\sqrt{3}$

$= 5\sqrt{3}$ //

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Suppose $\vec{A} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\vec{B} = \hat{i} - 2\hat{j} + \hat{k}$. Find a vector of magnitude 5 perpendicular to both A and B.

$\Rightarrow$ Given that,

$$\vec{A} = 2\hat{i} + \hat{j} - 3\hat{k}$$

$$\vec{B} = \hat{i} - 2\hat{j} + \hat{k}$$

A vector perpendicular to Both $\vec{A}$ and $\vec{B}$ is,

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -3 \\ 1 & -2 & 1 \end{vmatrix} = \hat{i}(1-6) - \hat{j}(2+3) + \hat{k}(-4-1)$$

$$= -5\hat{i} - 5\hat{j} - 5\hat{k}$$

$$= -5(\hat{i} + \hat{j} + \hat{k})$$

$$\therefore |\vec{A} \times \vec{B}| = \sqrt{(-5)^2 + (-5)^2 + (-5)^2} = \sqrt{75} = 5\sqrt{3}$$

A vector of magnitude 5 perpendicular to both $\vec{A}$ and $\vec{B}$ is, $= \pm 5 \dfrac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$

$$= \pm 5 \cdot \frac{5(\hat{i} + \hat{j} + \hat{k})}{5\sqrt{3}}$$

$$= \pm \frac{5}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$$

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~~2~~ Find the constant a so that the following vectors are coplanar: $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} - 3\hat{k}$, $3\hat{i} + a\hat{j} + 5\hat{k}$.

$\Rightarrow$ Let,

$$\vec{A} = 2\hat{i} - \hat{j} + \hat{k}$$

$$\vec{B} = \hat{i} + 2\hat{j} - 3\hat{k}$$

$$\vec{C} = 3\hat{i} + a\hat{j} + 5\hat{k}$$

We know,

The vectors $\vec{A}, \vec{B}, \vec{C}$ are coplanar if $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$

Now, $\vec{B} \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 3 & a & 5 \end{vmatrix} = \hat{i}(10+3a) - \hat{j}(5+9) + \hat{k}(a-6)$

$= \hat{i}(10+3a) - 14\hat{j} + \hat{k}(a-6)$

$\therefore \vec{A} \cdot (\vec{B} \times \vec{C}) = 2(10+3a) + 14 + (a-6)$

$= 20 + 6a + 14 + a - 6$

$= 7a + 28$

According to the condition of coplanar, we have

$7a + 28 = 0$

$\therefore a = -4.$

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Chapter-3

A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = -t - 5$ where t is the time. Find the components of it's velocity and acceleration at time $t = 1$ in the direction $\hat{i} - 2\hat{j} + 2\hat{k}$.

$\Rightarrow$ velocity $= \frac{dr}{dt} = \frac{d}{dt}\left[(2t^2)\hat{i} + (t^2 - 4t)\hat{j} + (-t-5)\hat{k}\right]$

$= (4t)\hat{i} + (2t-4)\hat{j} - \hat{k}$

$= 4\hat{i} - 2\hat{j} - \hat{k}$ at $t = 1$

Unit vector in direction $\hat{i} - 2\hat{j} + 2\hat{k}$ is $\frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{1^2 + (-2)^2 + 2^2}}$

$= \frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

Then the component in the given direction is

$(4\hat{i} - 2\hat{j} - \hat{k}) \cdot \left(\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = 2$

Acceleration $= \frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right)$

$= \frac{d}{dt}\left[(4t)\hat{i} + (2t-4)\hat{j} - k\right]$

$= 4\hat{i} + 2\hat{j}$.

Then the component of the acceleration in the given direction is $(4\hat{i} + 2\hat{j}) \cdot \left(\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = 0$

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a) Find the unit tangent vector to any point on the curve $x = t^2 - t$, $y = 4t - 3$, $z = 2t^2 - 8t$.

b) Determine the unit tangent at the point where $t = 2$.

⇒ a) A tangent to the curve at any point is

$$\frac{dr}{dt} = \frac{d}{dt}\left[(t^2 - t)\hat{i} + (4t-3)\hat{j} + (2t^2 - 8t)k\right]$$

$$= (2t-1)\hat{i} + 4\hat{j} + (4t-8)\hat{k}.$$

The magnitude of the vector is $|dr/dt|$

$$= \left[(2t-1)^2 + (4)^2 + (4t-8)^2\right]^{1/2}$$

Then the required unit tangent vector is,

$$T = \left[(2t-1)\hat{i} + 4\hat{j} + (4t-8)k\right] \Big/ \left[(2t-1)^2 + (4)^2 + (4t-8)^2\right]^{1/2}$$

Note that,

Since $|dr/dt| = ds/dt$, we have

$$T = \frac{dr/dt}{ds/dt} = \frac{dr}{ds}.$$

b) At $t = 2$, the unit tangent vector is

$$T = \frac{3i + 4j}{\sqrt{3^2 + 4^2 + 0^2}} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}$$

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A particle moves so that its position vector is given by $\vec{r} = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$ where ω is a constant. show that

a) the velocity v of the particle is perpendicular to r

b) the acceleration a is directed toward the origin and has magnitude proportional to the distance from the origin, c) $r \times v$ = a constant vector.

a) $$v = \frac{dr}{dt} = -\omega \sin\omega t\,\hat{i} + \omega \cos\omega t\,\hat{j}$$

Then

$$\vec{r}\cdot\vec{v} = [\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}]\cdot[-\omega \sin\omega t\,\hat{i} + \omega \cos\omega t\,\hat{j}]$$

$$= (\cos\omega t)(-\omega \sin\omega t) + (\sin\omega t)(\omega \cos\omega t) = 0$$

and r and v are perpendicular.

b) $$\frac{d^2r}{dt^2} = \frac{dv}{dt} = -\omega^2 \cos\omega t\,\hat{i} - \omega^2 \sin\omega t\,\hat{j}$$

$$= -\omega^2[\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}].$$

$$= -\omega^2 r$$

Then the acceleration is opposite to the direction of r, that is, it is directed toward the origin. Its magnitude is proportional to $|r|$, which is the distance from the origin.

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c) $r \times v = [\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}] \times [-\omega \sin\omega t\,\hat{i} + \omega \cos\omega t\,\hat{j}]$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\omega t & \sin\omega t & 0 \\ -\omega\sin\omega t & \omega\cos\omega t & 0 \end{vmatrix}$$

$$= \omega(\cos^2\omega t + \sin^2\omega t)\,\hat{k}$$

$$= \omega\hat{k}, \text{ a constant vector.}$$

physically, the motion is that of a particle moving on the circumference of a circle with constant angular speed ω. The acceleration, directed toward the center of the circle, is the centripetal acceleration. //

Find a unit tangent vector to any point on the curve $x = a\cos\omega t$, $y = a\sin\omega t$, $z = bt$ where a, b, ω are constant.

$\Rightarrow$ The position vector r is, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$= a\cos\omega t\,\hat{i} + a\sin\omega t\,\hat{j} + bt\hat{k}$$

$\therefore$ Tangent vector,

$$\frac{dr}{dt} = -a\omega\sin\omega t\,\hat{i} + a\omega\cos\omega t\,\hat{j} + b\hat{k}$$

$\therefore$ Unit vector of the tangent vector is,

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~~# Find a unit tangent vector to any point on the curve x~~

$$= \frac{-a\omega \sin\omega t\,\hat{i} + a\omega\cos\omega t\,\hat{j} + b\hat{k}}{\sqrt{(-a\omega\sin\omega t)^2 + (a\omega\cos\omega t)^2 + b^2}}$$

$$= \frac{-a\omega\sin\omega t\,\hat{i} + a\omega\cos\omega t\,\hat{j} + b\hat{k}}{\sqrt{a^2\omega^2 + b^2}}$$

Suppose $\vec{A} = \sin u\,\hat{i} + \cos u\,\hat{j} + u\hat{k}$, $\vec{B} = \cos u\,\hat{i} - \sin u\,\hat{j} - 3\hat{k}$ and $\vec{C} = 2\hat{i} + 3\hat{j} - \hat{k}$. Find $\frac{d}{du}(A \times (B \times C))$ at $u = 0$.

$\Rightarrow$ Here,

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos u & -\sin u & -3 \\ 2 & 3 & -1 \end{vmatrix} = \hat{i}(\sin u + 9) - \hat{j}(-\cos u + 6) + \hat{k}(3\cos u + 2\sin u)$$

$$= \hat{i}(\sin u + 9) + \hat{j}(\cos u - 6) + \hat{k}(3\cos u + 2\sin u).$$

Again,

$$A \times (B \times C) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sin u & \cos u & u \\ \sin u + 9 & \cos u - 6 & 3\cos u + 2\sin u \end{vmatrix}$$

$$= \hat{i}(3\cos^2 u + 2\sin u\cos u - u\cos u + 6u) +$$

$$\hat{j}(u\sin u + 9u - 3\sin u\cos u - 2\sin^2 u) +$$

$$\hat{k}(\sin u\cos u - 6\sin u - \sin u\cos u - 9\cos u)$$

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$= \hat{i}(3\cos^2 u + \sin 2u - u\cos u + 6u) +$

$\hat{j}(u\sin u - 2\sin^2 u + 9u - \frac{3}{2}\sin 2u) +$

$k(-6\sin u - 9\cos u)$

$\therefore \frac{d}{du}(A\times(B\times C)) = (-6\cos u\sin u + 2\cos 2u + u\sin u - \cos u + 6)\hat{i}$

$+ (u\cos u + \sin u - 4\sin u\cos u + 9 - 3\cos 2u)\hat{j} + (-6\cos u +$

$9\sin u)k$

At $u = 0$,

$\frac{d}{du}(A\times(B\times C)) = (0 + 2 + 0 - 1 + 6)\hat{i} + (0 + 0 + 9 - 3)\hat{j} +$

$(-6 + 0)\hat{k}$

$= 7\hat{i} + 6\hat{j} - 6\hat{k}.$

Suppose $A(t) = 3t^2\hat{i} - (t+4)\hat{j} + (t^2 - 2t)k$ and $B(t) = \sin t\,\hat{i} + 3e^{-t}\hat{j} - 3\cos t\,k$. Find $\frac{d^2}{dt^2}(A\times B)$ at $t=0$.

$\Rightarrow$ Here,

$$A\times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3t^2 & -(t+4) & t^2 - 2t \\ \sin t & 3e^{-t} & -3\cos t \end{vmatrix}$$

$= \hat{i}(3t\cos t + 12\cos t - 3t^2e^{-t} + 6te^{-t}) +$

$\hat{j}(t^2\sin t - 2t\sin t + 9t^2\cos t) + \hat{k}(9t^2e + t\sin t + 4\sin t)$

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$$\therefore \frac{d}{dt}(A\times B) = \hat{i}\left(-3t\sin t + 3\cos t - 12\sin t - 6te^{-t} + 3t^2e^{-t} + 6e^{-t} - 6te^{-t}\right) + \hat{j}\left(2t\sin t + t^2\cos t - 2\sin t - 2t\cos t + 18t\cos t - 9t^2\sin t\right) + \hat{k}\left(18te^{-t} - 9t^2e^{-t} + t\cos t + \sin t + 4\cos t\right)$$

$$= \hat{i}\left(-3t\sin t + 3\cos t - 12\sin t - 12te^{-t} + 3t^2e^{-t} + 6e^{-t}\right) + \hat{j}\left(2t\sin t + t^2\cos t - 2\sin t + 16t\cos t - 9t^2\sin t\right) + \hat{k}\left(18te^{-t} - 9t^2e^{-t} + t\cos t + \sin t + 4\cos t\right)$$

$$\therefore \frac{d^2}{dt^2}(A\times B) = \hat{i}\left(-3\sin t - 3t\cos t - 3\sin t - 12\cos t - 12e^{-t} + 12te^{-t} + 6te^{-t} - 3t^2e^{-t} - 6e^{-t}\right) + \hat{j}\left(2t\cos t + 2\sin t + 2t\cos t - t^2\sin t - 2\cos t + 16\cos t - 16t\sin t - 18t\sin t - 9t^2\cos t\right) + \hat{k}\left(18e^{-t} - 18te^{-t} - 18te^{-t} + 9t^2e^{-t} + \cos t - t\sin t + \cos t - 4\sin t\right)$$

At $t = 0$,

$$\frac{d^2}{dt^2}(A\times B) = \hat{i}(-12-12-6) + \hat{j}(-2+16) + \hat{k}(18+1+1)$$

$$= -30\hat{i} + 14\hat{j} + 20\hat{k}.$$

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Show that $r = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$, where c_1 and c_2 are constant vectors, is a solution of the differential equation $\frac{d^2r}{dt^2} + 2 \cdot \frac{dr}{dt} + 5r = 0$.

$\Rightarrow$ Given that,

$$r = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$$

$$\therefore \frac{dr}{dt} = -c_1 \cos 2t \cdot e^{-t} - c_1 e^{-t} \cdot 2 \sin 2t + c_2 e^{-t} \cdot 2 \cos 2t - c_2 \sin 2t \cdot e^{-t}$$

$$\therefore \frac{d^2r}{dt^2} = c_1 \cos 2t \cdot e^{-t} + c_1 e^{-t} \cdot 2 \sin 2t + 2 c_1 \sin 2t\, e^{-t} - 2 c_1 e^{-t} \cdot 2 \cos 2t - 2 c_2 \cos 2t \cdot e^{-t} - 2 c_2 e^{-t} \cdot 2 \sin 2t + c_2 \sin 2t \cdot e^{-t} - c_2 e^{-t} \cdot 2 \cos 2t$$

$$= 4 c_1 e^{-t} \sin 2t - 3 c_1 e^{-t} \cos 2t - 4 c_2 e^{-t} \cos 2t - 3 c_2 e^{-t} \sin 2t.$$

$$\therefore \frac{d^2r}{dt^2} + 2 \cdot \frac{dr}{dt} + 5r$$

$$= 4 c_1 e^{-t} \sin 2t - 3 c_1 e^{-1} \cos 2t - 4 c_2 e^{-t} \cos 2t \cancel{-3 c_2} - 3 c_2 e^{-t} \sin 2t - 2 c_1 e^{-t} \cos 2t - 4 c_1 e^{-t} \sin 2t + 4 c_2 e^{-t} \cos 2t - 2 c_2 e^{-t} \sin 2t + 5 c_1 e^{-t} \cos 2t + 5 c_2 e^{-t} \sin 2t.$$

$$= 0.$$

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Chapter-1

Gradient (vector field):

Let $\phi(x,y,z)$ be a scalar f defined and differentiable at each point (x,y,z) in a certain region of space. [That is, ϕ defines a differentiable scalar field]. Then the gradient of ϕ (phi), written $\nabla\phi$ or grad ϕ is defined as follows:

$$\nabla\phi = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\phi$$

$$= \frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k}$$

Gradient $(\nabla\phi)$ defines a vector field.

Example:

Suppose $\phi(x,y,z) = 3xy^3 - y^2z^2$. Find $\nabla\phi$ (or grad ϕ) at the point $P(1,1,2)$

$$\nabla\phi = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)(3xy^3 - y^2z^2)$$

$$= 3y^3\hat{i} + (9xy^2 - 2yz^2)\hat{j} + (-2y^2z)\hat{k}$$

$$\nabla\phi(1,1,2) = 3\cdot 1^3\cdot\hat{i} + (9\cdot 1\cdot 1^2 - 2\cdot 1\cdot 2^2)\hat{j} + (-2\cdot 1^2\cdot 2)\hat{k}$$

$$= 3\hat{i} + \hat{j} - 4\hat{k}$$

Directional Derivatives:

Let a scalar function $\phi = \phi(x,y,z)$. Then the directional derivative of ϕ in the direction of a vector A is denoted by $D_A(\phi)$.

Letting $a = A/|A|$, the unit vector in the direction of A,

$$D_A(\phi) = \nabla\phi \cdot A.$$

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Divergence (Scalar field):

Suppose $V(x,y,z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ is defined and differentiable at each point (x,y,z). [That is, V defines a differentiable vector field.] Then the divergence of V, written $\nabla \cdot V$ or div V is defined as follows:

$$\nabla \cdot V = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right) \cdot (V_1\hat{i} + V_2\hat{j} + V_3\hat{k})$$

$$= \frac{\delta V_1}{\delta x} + \frac{\delta V_2}{\delta y} + \frac{\delta V_3}{\delta z}$$

Although V is a vector, $\nabla \cdot V$ is a scalar.

i) $\nabla \cdot V$ represents volume

ii) $\nabla \cdot V > 0$ expand

iii) $\nabla \cdot V < 0$ compressed

iv) $\nabla \cdot V = 0$ Solenoidal

Note

i) $\nabla \cdot (\nabla \times A) = 0$

ii) $\nabla \times (\nabla \phi) = 0$

Example:

Suppose $A = x^2z^2\hat{i} - 2y^2z^2\hat{j} + xy^2z\hat{k}$. Find $\nabla \cdot A$ (or div A) at the point $P(1,-1,1)$.

$$\nabla \cdot A = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)(x^2z^2\hat{i} - 2y^2z^2\hat{j} + xy^2z\hat{k})$$

$$= 2xz^2 - 4yz^2 + xy^2$$

At the point $P(1,-1,1)$

$$\nabla \cdot A = 2 \cdot 1 \cdot 1^2 - 4 \cdot (-1) \cdot 1^2 + 1 \cdot (-1)^2 = 7$$

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Curl: (vector field)

Suppose, $V(x,y,z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ is a differentiable vector field. Then the curl or rotation of V, written $\nabla \times V$, curl V or rot V, is defined as follows:

$$\nabla \times V = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right) \times (V_1\hat{i} + V_2\hat{j} + V_3\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

Note

i) $\nabla \times V = 0$ irrotational

ii) $\nabla \times V \neq 0$ rotational

$$= \hat{i}\left(\frac{\delta V_3}{\delta y} - \frac{\delta V_2}{\delta z}\right) - \hat{j}\left(\frac{\delta V_3}{\delta x} - \frac{\delta V_1}{\delta z}\right) + \hat{k}\left(\frac{\delta V_2}{\delta x} - \frac{\delta V_1}{\delta y}\right)$$

Example:

Suppose, $A = x^2z^2\hat{i} - 2y^2z^2\hat{j} + xy^2z\hat{k}$. Find $\nabla \times A$ or (curl A) at the point $(1,-1,1)$

$$\nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x^2z^2 & -2y^2z^2 & xy^2z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\delta}{\delta y}(xy^2z) + \frac{\delta}{\delta z}(2y^2z^2)\right] + \hat{j}\left[\frac{\delta}{\delta x}(xy^2z) - \frac{\delta}{\delta z}(x^2z^2)\right]$$

$$+ \hat{k}\left[\frac{\delta}{\delta x}(-2y^2z^2) - \frac{\delta}{\delta y}(x^2z^2)\right]$$

$$= \hat{i}(2xyz^2 + 4y^2z) + \hat{j}(2x^2z - y^2z) + \hat{k}(0-0)$$

$$= (2xyz^2 + 4y^2z)\hat{i} + (2x^2z - y^2z)\hat{j}$$

At $P(1,-1,1) = (-2+4)\hat{i} + (2-1)\hat{j} = 2\hat{i} + \hat{j}$

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4.1 Suppose $\phi(x,y,z) = 3x^2y - y^2z^2$. Find $\nabla\phi$ at the point $(1,-2,-1)$

$$\nabla\phi = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)(3x^2y - y^3z^2)$$

$$= \hat{i}\left[\frac{\delta}{\delta x}(3x^2y - y^3z^2)\right] + \hat{j}\left[\frac{\delta}{\delta y}(3x^2y - y^3z^2)\right] + \hat{k}\left[\frac{\delta}{\delta z}(3x^2y - y^3z^2)\right]$$

$$= 6xy\hat{i} + (3x^2 - 3y^2z^2)\hat{j} - 2y^3z\hat{k}$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k} \rightarrow \text{At point } (1,-2,-1)$$

4.4 Show that, $\nabla r^n = n r^{n-2} \cdot r$

$$\nabla r^n = \nabla\left(\sqrt{x^2+y^2+z^2}\right)^n = \nabla(x^2+y^2+z^2)^{n/2}$$

$$= \hat{i}\frac{\delta}{\delta x}\{(x^2+y^2+z^2)^{n/2}\} + \hat{j}\frac{\delta}{\delta y}\{(x^2+y^2+z^2)^{n/2}\} + \hat{k}\frac{\delta}{\delta z}\{(x^2+y^2+z^2)^{n/2}\}$$

$$= \hat{i}\left\{\frac{n}{2}(x^2+y^2+z^2)^{n/2-1} \cdot 2x\right\} + \hat{j}\left\{\frac{n}{2}(x^2+y^2+z^2)^{n/2-1} \cdot 2y\right\} + \hat{k}\left\{\frac{n}{2}(x^2+y^2+z^2)^{n/2-1} \cdot 2z\right\}$$

$$= \frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= n(r^2)^{n/2-1} \cdot r = n r^{n-2} r$$

4.6 Find a unit normal to the surface $-x^2yz^2 + 2xy^2z = 1$ at the point $(1,1,1)$.

Let $\phi = -x^2yz^2 + 2xy^2z$. $\nabla\phi(1,1,1)$ is normal to the surface $-x^2yz^2 + 2xy^2z = 1$ at the point $(1,1,1)$. Hence,

$$\frac{\nabla\phi(1,1,1)}{|\nabla\phi(1,1,1)|} \text{ will surface}$$

$$\nabla\phi = (-2xyz^2)\hat{i} + (-x^2z^2 + 4xyz)\hat{j} + (-2x^2yz + 2xy^2)\hat{k}$$

Then $\nabla\phi(1,1,1) = 3j$

$|\nabla\phi(1,1,1)| = 3$

Thus at point $P(1,1,1) = \frac{3j}{3} = j$ is a unit normal to $-x^2yz^2 + 2xy^2z = 1$

4.10 Let $\phi = x^2yz - 4xyz^2$. Find the directional derivative of ϕ at $P(1,3,1)$ in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$.

First find $\nabla\phi = (2xyz - 4yz^2)\hat{i} + (x^2z - 4xz^2)\hat{j} + (x^2y - 8xyz)\hat{k}$

Then $\nabla\phi(1,3,1) = -6\hat{i} - 3\hat{j} - 21\hat{k}$. The unit vector in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$ is

$$a = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{(2)^2 + (-1)^2 + (-2)^2}} = \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$$

Thus the required directional derivative is:

$$\nabla\phi(1,3,1)\cdot a = (-6\hat{i} - 3\hat{j} - 21\hat{k})\cdot\left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right)$$

$$= -4 + 1 + 14 = 11$$

4.11 Let $\phi = x^2y^3z^6$ (a) In what directions from the point $P(1,1,1)$ is the directional derivative of ϕ a maximum?
(b) What is the magnitude of this maximum?

$$\nabla\phi = \nabla(x^2y^3z^6)$$

$$= 2xy^3z^6\hat{i} + 3x^2y^2z^6\hat{j} + 6x^2y^3z^5\hat{k}$$

$$\nabla\phi(1,1,1) = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

(a) The directional derivative is a maximum in the direction $\nabla\phi(1,1,1) = 2\hat{i} + 3\hat{j} + 6\hat{k}$

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(b) The magnitude of this maxium is $|\nabla\phi(1,1,1)| =$

$$\sqrt{2^2+3^2+6^2} = 7$$

4.22

Determine the constant a so that the following vector is solenoidal.

$$V = (-4x-6y+3z)\hat{i} + (-2x+y-5z)\hat{j} + (5x+6y+az)\hat{k}$$

A vector V is solenoidal if its divergence is zero:

$$\nabla\cdot V = \frac{\delta}{\delta x}(-4x-6y+3z) + \frac{\delta}{\delta y}(-2x+y-5z) + \frac{\delta}{\delta z}(5x+6y+az)$$

$$\Rightarrow \nabla\cdot V = -4+1+a$$

$$\Rightarrow \nabla\cdot V = -3+a$$

Then $\nabla\cdot V = 0$

$$\Rightarrow -3+a = 0$$

$$\therefore a = 3$$

4.23 $A = x^2z^2\hat{i} - 2y^2z^2\hat{j} + xy^2z\hat{k}$. Find $\nabla\times A$ (or curl A) at the point $P(1,-1,1)$

$$\nabla\times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x^2z^2 & -2y^2z^2 & xy^2z \end{vmatrix} = \left[\frac{\delta}{\delta y}(xy^2z) - \frac{\delta}{\delta z}(-2y^2z^2)\right]\hat{i} - \left[\frac{\delta}{\delta x}(xy^2z) - \frac{\delta}{\delta z}(x^2z^2)\right]\hat{j} + \hat{k}\left[\frac{\delta}{\delta x}(-2y^2z^2) - \frac{\delta}{\delta z}(x^2z^2)\right]$$

$$= (2xyz+4yz^2)\hat{i} - (y^2z - 2x^2z)\hat{j}$$

Thus $\nabla\times A$ at point P is $= 2\hat{i} + \hat{j}$

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4.32 a) Find constants a, b and c so that

$$V = (-4x-3y+az)\hat{i} + (bx+3y+5z)\hat{j} + (4x+cy+3z)\hat{k}$$

is irrotational.

b) show that V can be expressed as the gradient of a scalar function.

a) curl $V = \nabla \times V$

$$\nabla \times V = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ -4x-3y+az & bx+3y+5z & 4x+cy+3z \end{vmatrix}$$

$$= (c-5)\hat{i} - (4-a)\hat{j} + (b+3)\hat{k}$$

This equals the zero vector when $a=4, b=-3, c=5$

so $V = (-4x-3y+4z)\hat{i} + (-3x+3y+5z)\hat{j} + (4x+5y+3z)\hat{k}$

b) Assume $V = \nabla\phi = \frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k}$

$$\frac{\delta\phi}{\delta x} = -4x-3y+4z \quad \text{—(i)}$$

$$\frac{\delta\phi}{\delta y} = -3x+3y+5z \quad \text{—(ii)}$$

$$\frac{\delta\phi}{\delta z} = 4x+5y+3z \quad \text{—(iii)}$$

Integrating (1) partially with respect to x keeping y and z constant we obtain:

$$\phi = -2x^2-3xy+4xz+f(y,z) \quad \text{—(iv)}$$

Similarly we obtain from (2) and (3)

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$$\phi = -3xy + \frac{3}{2}y^2 + 5yz + g(x,z) \quad \text{--- (v)}$$

$$\phi = 4xz + 5yz + \frac{3}{2}z^2 + h(x,y) \quad \text{--- (vi)}$$

Comparison of (iv), (v) and (vi) shows that there will be a common value of ϕ if we choose:

$$f(y,z) = \frac{3}{2}y^2 + 5yz + \frac{3}{2}z^2$$

$$g(x,z) = -2x^2 + 4xz + \frac{3}{2}z^2 \qquad h(x,y) = -2x^2 - 3xy + \frac{3}{2}y^2$$

So that,

$$\phi = -2x^2 + \frac{3}{2}y^2 + \frac{3}{2}z^2 - 3xy + 4xz + 5yz$$

4.58 Find a unit vector that is perpendicular to the surface of the paraboloid of revolution $z = x^2 + y^2$ at the point $(1,2,5)$.

Soln:

For a surface defined implicity as $F(x,y,z) = 0$ the gradient vector at any point on the surface is perpendicular to the surface.

$$F(x,y,z) = z - x^2 - y^2$$

$$\nabla F = -2x\hat{i} - 2y\hat{j} + \hat{k}$$

At point $P(1,2,5)$ we get $= -2\hat{i} - 4\hat{j} + \hat{k}$

$$|\nabla F| = \sqrt{(-2)^2 + (-4)^2 + 1^2} = \sqrt{21}$$

The unit vector is $= \dfrac{\nabla F}{|\nabla F|}$

$$= \frac{-2\hat{i} - 4\hat{j} + \hat{k}}{\sqrt{21}}$$

$$= -\frac{2}{\sqrt{21}}\hat{i} - \frac{4}{\sqrt{21}}\hat{j} + \frac{1}{\sqrt{21}}\hat{k}$$

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4.60

Find an eqn for the tangent plane to the surface $xz^2+x^2y=z-1$ at the point $(1,-3,2)$

Let, $\phi = xz^2+x^2y-z+1=0$

$\nabla\phi = \left(\frac{\delta}{\delta x}\hat{i}+\frac{\delta}{\delta y}\hat{j}+\frac{\delta}{\delta z}\hat{k}\right)(xz^2+x^2y-z+1) = (z^2+2xy)\hat{i}+x^2\hat{j}+(2xz-1)\hat{k}$

Then a normal to the surface at the point $(1,-3,2)$ is:

$\nabla\phi = -2\hat{i}+\hat{j}+3\hat{k}$

The eqn of the tangent plane is:

$(r-r_0)\cdot N=0$

$\Rightarrow [(x\hat{i}+y\hat{j}+z\hat{k})-(\hat{i}-3\hat{j}+2\hat{k})]\cdot\nabla\phi=0$

$\Rightarrow [(x-1)\hat{i}+(y-3)\hat{j}+(z-2)\hat{k}]\cdot[-2\hat{i}+\hat{j}+3\hat{k}]=0$

$\Rightarrow -2(x-1)+(y-3)+3(z-2)=0$

$\Rightarrow -2x+2+y-3+3z-6=0$

$\therefore 2x-y-3z+1=0$

4.62 For what value of the constant a will the vector $\bar{A}=(axy-z^3)\hat{i}+(a-2)x^2\hat{j}+(1-a)xz^2\hat{k}$ have its curl identically equal to zero.

Soln:

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ axy-z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix}$$

$= \hat{i}(0-0)+\hat{j}[-3z^2-(1-a)z^2]+\hat{k}[2x(a-2)-ax]$

$= \hat{j}(-3z^2-z^2+az^2)+\hat{k}(2xa-4x-ax)$

$= \hat{j}(-4z^2+az^2)+\hat{k}(ax-4x)$

Since the curl is zero so,

$-4z^2+az^2=0$ $\qquad$ $ax-4x=0$

$\therefore a=4$ $\qquad$ $\therefore a=4$

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4.62 Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ in the direction $2\hat{i} - 3\hat{j} + 6\hat{k}$

Soln:

$$\nabla \times \phi = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)(4xz^3 - 3x^2y^2z)$$

$$= (4z^3 - 6xy^2z)\hat{i} + (-6x^2yz)\hat{j} + (12xz^2 - 3x^2y^2)\hat{k}$$

At point $(2,-1,2)$,

$$\nabla\phi = [4\cdot(2)^3 - 6\cdot 2\cdot(-1)^2\cdot 2]\hat{i} + [-6\cdot 2^2\cdot(-1)\cdot 2]\hat{j} + [12\cdot 2\cdot 2^2 - 3\cdot 2^2\cdot(-1)^2]\hat{k}$$

$$= 8\hat{i} + 48\hat{j} + 84\hat{k}$$

The unit vector in the direction of $2\hat{i} + 6\hat{k} - 3\hat{j}$ is

$$\hat{n} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{(2)^2 + (-3)^2 + 6^2}} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}$$

Thus the required directional derivative is:

$$\nabla\phi \cdot \hat{n} = (8\hat{i} + 48\hat{j} + 84\hat{k}) \cdot \left(\frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}\right)$$

$$= \frac{16 - 144 + 504}{7}$$

$$= \frac{520 - 144}{7}$$

$$= \frac{376}{7}$$

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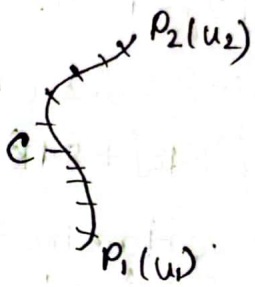
Chapter-5

Line Integral

Let, $\bar{r}(u) = x(u)\hat{i} + y(u)\hat{j} + z(u)\hat{k}$ where $\bar{r}(u)$ is the position vector of (x, y, z) defined a curve C joining two points P_1 and P_2 where $u = u_1$ and $u = u_2$ respectively.



Suppose the curve C is composed of a finite number of curves for each of which $\bar{r}(u)$ has continuous derivatives. Again Let $\bar{A}(x, y, z) = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ be a vector function of position defined and continuous along C. Then the line integral of $\bar{A}$ along C from P_1 to P_2 written as $\int_{P_1}^{P_2} \bar{A} \cdot d\bar{r}$ and is defined by

$$\int_{P_1}^{P_2} \bar{A} \cdot d\bar{r} = \int_C \bar{A} \cdot d\bar{r} = \int_C (A_1 dx + A_2 dy + A_3 dz)$$

The above integral is sometimes called the tangential component of $\bar{A}$ along C.

Conservative vector field:

If $\int_{P_1}^{P_2} \bar{A} \cdot d\bar{r}$ is independent of the path C joining any two points P_1 and P_2 in a region R, then $\bar{A}$ is called a conservative vector field and in this case, $\bar{A} = \nabla\phi$, where ϕ is known as a scaler potential.

In another way, a vector field $\bar{A}$ is conservative if $\nabla \times \bar{A} = 0$

$[\because \bar{A} = \nabla\phi]$

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6. Suppose, $\bar{A} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$. Evaluate $\int_C \bar{A}\cdot d\bar{r}$ from $(0,0,0)$ to $(1,1,1)$ along the following paths C:

a) $x=t,\ y=t^2,\ z=t^3$

b) the straight lines from $(0,0,0)$ to $(1,0,0)$ then to $(1,1,0)$ and then to $(1,1,1)$

c) the straight line joining $(0,0,0)$ and $(1,1,1)$.

= Let the position vector $\bar{r}$ be,

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\therefore d\bar{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$\therefore \int_C \bar{A}\cdot d\bar{r} = \int_C \left[(3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}\right]\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_C (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

a) If $x=t,\ y=t^2,\ z=t^3$ points $(0,0,0)$ and $(1,1,1)$ correspond to $t=0$ and $t=1$ respectively, Then

$$\int_C \bar{A}\cdot d\bar{r} = \int_{t=0}^{1} (3t^2+6t^2)\,dt - 14t^2\cdot t^3\,d(t^2) + 20\cdot t\cdot t^6\,d(t^3)$$

$$= \int_{t=0}^{1} 9t^2\,dt - 14t^5\cdot 2t\,dt + 20t^7\cdot 3t^2\,dt$$

$$= \int_{t=0}^{1} (9t^2 - 28t^6 + 60t^9)\,dt$$

$$= \left[9\cdot\frac{t^3}{3} - 28\cdot\frac{t^7}{7} + 60\cdot\frac{t^{10}}{10}\right]_0^1$$

$$= \left[3t^3 - 4t^7 + 6t^{10}\right]_0^1 = 3 - 4 + 6 = 9 - 4 = 5$$

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b) Along the straight line from $(0,0,0)$ to $(1,0,0)$ $y=0, z=0$ $dy=0$ $dz=0$ while x varies from 0 to 1. Then the integral over this part of the path is:

$$\int_{x=0}^{1} (3x^2+6.0)dx - 14.0.0.0 + 20x.0.0$$

$$= \int_{x=0}^{1} 3x^2 dx = \left[3.\frac{x^3}{3}\right]_0^1 = 1$$

Along the straight line from $(1,0,0)$ to $(1,1,0)$ $x=1, z=0$ $dx=0, dz=0$ while y varies from 0 to 1.

$$\int_{x=0}^{1} [3.(1)^2+6y].0 - 14.y.0.dy + 20.1.0.0$$

$$= \int_{y=0}^{1} (0-0+0) = 0$$

Along the straight line from $(1,1,0)$ to $(1,1,1)$ $x=1, y=1$ $dx=0$, $dy=0$ while z varies from 0 to 1.

$$\int_{x=0}^{1} (3.1^2+6.1)\,0 - 1.4.1.z.0 + 20.1.z^2 dz$$

$$= \left[20.\frac{z^3}{3}\right]_0^1 = \frac{20}{3}$$

Adding,

$$\int_C \bar{A}.d\bar{r} = 1+0+\frac{20}{3} = \frac{23}{3}$$

c) The straight line joining $(0,0,0)$ and $(1,1,1)$ is given in parametric from by $x=t$, $y=t$, $z=t$ Then

$$\int_C \bar{A}\cdot d\bar{r} = \int_C (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

$$= \int_{t=0}^{1} (3t^2+6t)\,dt - 14\cdot t\cdot t\cdot dt + 20\cdot t\cdot t^2\,dt$$

$$= \int_{t=0}^{1} (6t - 14t^2 + 20t^3)\,dt$$

$$= \left[6\cdot t^2/2 - 14\cdot\frac{t^3}{3} + 20\cdot\frac{t^4}{4}\right]_0^1$$

$$= 3\cdot 1 - \frac{14}{3}\cdot 1 + 5\cdot 1$$

$$= 8 - \frac{14}{3} = \frac{24-14}{3} = \frac{13}{3}$$

5.14 (a)

Suppose, $\bar{F}$ is a conservative vector field. Prove that, curl $\bar{F} = \nabla\times\bar{F} = 0$

$\Rightarrow$ If $\bar{F}$ is a conservative field, then $\bar{F} = \nabla\phi$

$$\text{curl } \bar{F} = \nabla\times\nabla\phi$$

$$= \nabla\times\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right]\hat{i} + \left[\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right)\right]\hat{j}$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\right]\hat{k}$$

$$= \left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right)\hat{i} + \left(\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial z\partial x}\right)\hat{j} + \left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y}\right)\hat{k}$$

$$= 0 = \nabla\times\vec{F}$$

$\therefore$ curl $\vec{F} = \nabla\times\nabla\phi = 0$

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5.12(a) Show that, $\bar{F} = (2xy+z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force field.

$\bar{F} = \nabla \times \bar{F} = 0$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\delta}{\delta y}(3xz^2) - \frac{\delta}{\delta z}(x^2)\right] - \hat{j}\left[\frac{\delta}{\delta x}(3xz^2) - \frac{\delta}{\delta z}(2xy+z^3)\right]$$

$$+\hat{k}\left[\frac{\delta}{\delta x}(x^2) - \frac{\delta}{\delta y}(2xy+z^3)\right]$$

$$= \hat{i}(0-0) + \hat{j}(3z^2-3z^2) + \hat{k}(2x-2x)$$

$$= 0$$

curl $\bar{F} = \nabla \times \bar{F} = 0$

5.9 Suppose a force field is given by

$\vec{F} = (2x-y+z)\hat{i} + (x+y-z^2)\hat{j} + (3x-2y+4z)\hat{k}$

Find the work done in moving a particle once around a circle C in the xy plane with its centre at the origin and a radius of 3.

In the plane $z=0$

$\therefore \bar{F} = (2x-y)\hat{i} + (x+y)\hat{j} + (3x-2y)\hat{k}$

$d\bar{r} = dx\hat{i} + dy\hat{j}$ so the work done is

$$\int_C \bar{F}\cdot d\bar{r} = \int_C [(2x-y)\hat{i} + (x+y)\hat{j} + (3x-2y)\hat{k}]\cdot(dx\hat{i} + dy\hat{j})$$

$$= \int_C [(2x-y)dx + (x+y)dy]$$

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Here, $\bar{r} = x\hat{i} + y\hat{j} = r\cos t\,\hat{i} + r\sin t\,\hat{j}$

$= 3\cos t\,\hat{i} + 3\sin t\,\hat{j}$

choose the parametric equations of the circle as $x = 3\cos t$, $y = 3\sin$

where t varies from 0 to 2π. Then the line integral equals:

$$\int_{t=0}^{2\pi} (2\cdot 3\cos t - 3\sin t)\, d(3\cos t) + (3\cos t + 3\sin t)\, d(3\sin t)$$

$$= \int_{t=0}^{2\pi} (6\cos t - 3\sin t)(-3\sin t)\,dt + (3\cos t + 3\sin t)(3\cos t)\,dt$$

$$= \int_0^{2\pi} (-18\sin t\cos t + 9\sin^2 t + 9\cos^2 t + 9\sin t\cos t)\,dt$$

$$= \int_0^{2\pi} (9 - 9\sin t\cos t)\,dt = 9\int_0^{2\pi} dt - \int_0^{2\pi} \frac{9}{2}\cdot 2\sin t\cos t\,dt$$

$$= 9\cdot[t]_0^{2\pi} - \frac{9}{2}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi}$$

$$= 18\pi + \frac{9}{4}(\cos 4\pi - \cos 0) = 18\pi$$

Since the value of line integral is positive, so c has been traversed in the counter clockwise direction.

40. Find the work done in moving a particle in the force field $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along

a) the straight line from $(0,0,0)$ to $(2,1,3)$

b) the space curve $x = 2t^2$, $y = t$, $z = 4t^2 - 1$ from $t = 0$ to $t = 1$

c) the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$

a) $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$

$d\bar{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\therefore \int_c \vec{F}\cdot d\vec{r} = \int [3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}]\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

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$$= \int_C 3x^2 dx + (2xz - y)dy + z\,dz$$

Let, the straight line from $(0,0,0)$ to $(2,1,3)$ be given in parametric form by $x = 2t$, $y = t$, $z = 3t$. Then

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t=0}^{1} 3\cdot(2t)^2 \cdot d(2t) + (2\cdot 2t\cdot 3t - t)dt + 3t\cdot d(3t)$$

$$= \int_{t=0}^{1} 24t^2 dt + 12t^2 dt - dt + 9t\,dt$$

$$= 8 + 4 - \frac{1}{2} + \frac{9}{2} = 12 + \frac{8}{2} = 16$$

b) From (a), we get,

$$\int_C \vec{F}\cdot d\vec{r} = \int_C 3x^2 dx + (2xz - y)dy + z\,dz$$

If $x = 2t^2$, $y = t$, $z = 4t^2 - t$ from $t=0$ to $t=1$ then

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t=0}^{1} 3\cdot(2t^2)^2 \cdot d(2t^2) + \{2\cdot 2t^2\cdot(4t^2 - t) - t\}dt + (4t^2 - t)d(4t^2 - t)$$

$$= \int_{t=0}^{1} 4(12t^4 \cdot 4t)\,dt + (16t^4 - 4t^3 - t)dt + (4t^2 - t)(8t - 1)dt$$

$$= \int_{t=0}^{1} 48t^5 dt + (16t^4 - 4t^3 - t)dt + (32t^3 - 8t^2 - 4t^2 + t)dt$$

$$= \int_{t=0}^{1} (48t^5 + 16t^4 + 28t^3 - 12t^2)\,dt$$

$$= \left[48\cdot\frac{t^6}{6} + 16\cdot\frac{t^5}{5} + 28\cdot\frac{t^4}{4} - 12\cdot\frac{t^3}{3}\right]_0^1$$

$$= \left[8t^6 + \frac{16}{5}t^5 + 7t^4 - 4t^3\right]_0^1 = 11 + \frac{16}{5} = \frac{55+16}{5}$$

$$= \frac{71}{5}$$

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c) From (a) get,

$$\int_C \vec{F}\cdot d\vec{r} = \int_C 3x^2dx + (2xz-y)dy + z\,dz$$

putting $y = \frac{x^2}{4}$, $z = \frac{3x^3}{8}$ from $x=0$ to $x=2$ we get,

$$\int_C \vec{F}\cdot d\vec{r} = \int_{x=0}^{2} 3x^2dx + \left(2x\cdot\frac{3x^3}{8} - \frac{x^2}{4}\right)d\left(\frac{x^2}{4}\right) + \frac{3x^3}{8}\cdot d\left(\frac{3x^3}{8}\right)$$

$$= \int_{x=0}^{2} 3x^2dx + \left(\frac{3x^4-x^2}{4}\cdot\frac{1}{4}\cdot 2x\right)dx + \left(\frac{3x^3}{8}\cdot\frac{3}{8}\cdot 3x^2\right)dx$$

$$= \int_{x=0}^{2}\left(3x^2 + \frac{3x^5-x^3}{8} + \frac{27x^5}{64}\right)dx$$

$$= \left[3\cdot\frac{x^3}{3} + \frac{1}{8}\left(3\cdot\frac{x^6}{6} - \frac{x^4}{4}\right) + \frac{27}{64}\cdot\frac{x^6}{6}\right]_0^2$$

$$= \left[\frac{x^3}{1} + \frac{x^6}{16} - \frac{x^4}{32} + \frac{9}{64}\cdot\frac{x^6}{2}\right]_0^2 = 8 + \frac{64}{16} - \frac{16}{32} + \frac{9}{64}\cdot\frac{64}{2}$$

$$= 12 + \frac{8}{2} = 16$$

41. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = (x-3y)\hat{i} + (y-2x)\hat{j}$ and C is the closed curve in the xy-plane. $x = 2\cos t$, $y = 3\sin t$ from $t=0$ to $t=2\pi$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \left[(x-3y)\hat{i} + (y-2x)\hat{j}\right]\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_C (x-3y)dx + (y-2x)dy$$

putting $x = 2\cos t$, $y = 3\sin t$ from $t=0$ to $t=2\pi$ we get,

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t=0}^{2\pi} (2\cos t - 9\sin t)\,d(2\cos t) + (3\sin t - 4\cos t)\,d(3\sin t)$$

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$$= \int_{t=0}^{2\pi} (2\cos t - 9\sin t)\cdot(-2\sin t\, dt) + (3\sin t - 4\cos t)(3\cos t)\, dt$$

$$= \int_{t=0}^{2\pi} (18\sin^2 t - 4\sin t\cos t + 9\sin t\cos t - 12\cos^2 t)\, dt$$

$$= \int_{t=0}^{2\pi} \left[9\cdot(1-\cos 2t) + \frac{5}{2}\cdot\sin 2t - 6(1+\cos 2t)\right] dt$$

$$= \int_{t=0}^{2\pi} \left(9 - 9\cos 2t + \frac{5}{2}\sin 2t - 6 - 6\cos 2t\right) dt$$

$$= \int_{t=0}^{2\pi} \left(3 - 15\cos 2t + \frac{5}{2}\sin 2t\right) dt$$

$$= \left[3t - 15\cdot\frac{\sin 2t}{2} - \frac{5}{2}\cdot\frac{\cos 2t}{2}\right]_{t=0}^{2\pi}$$

$$= 3(2\pi - 0) - \frac{15}{2}(\sin 4\pi - \sin 0) - \frac{5}{4}(\cos 4\pi - \cos 0)$$

$= 6\pi - 0 - \frac{5}{4}(1-1) = 6\pi$ if C is traversed in the positive direction. If C is traversed in the negative (clockwise) direction, then the value will be -6π.

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