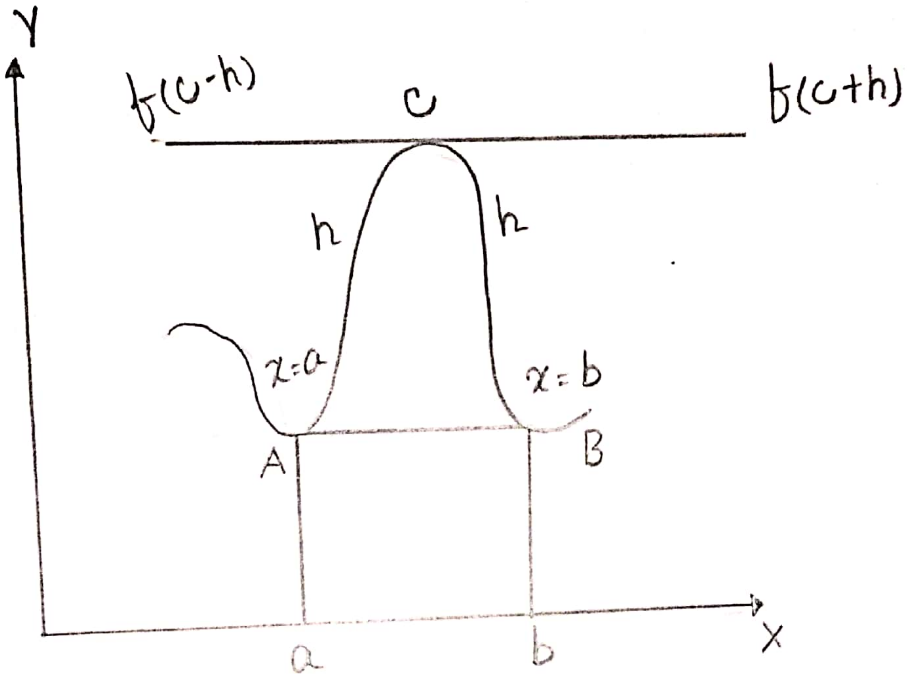
ROLL'S THEOREM



statement:

If $f(x)$ be a function such that

(i) $f(x)$ is continuous in $a \leq x \leq b$.

(ii) $f'(x)$ is exist for every point in the interval $a < x < b$.

(iii) $f(a) = f(b)$

Then $\exists$ at least a point c such that $f'(c) = 0$ where $a < c < b$.

$\boxed{Q}$

Discuss the applications of Roll's theorem to the function $f(x) = x^{2/3}$ in $(-1, 1)$.

Sol^n:

Given that,

$$f(x) = x^{2/3}$$

$$f'(x) = \frac{2}{3} x^{2/3 - 1}$$

$$= \frac{2}{3} x^{-1/3}$$

Now,

$$\lim_{x \to 0} f'(x)$$

$$= \lim_{x \to 0} \frac{2}{3} (x^{-1/3}) = \infty$$

R.H.D

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{(h^{2/3} - 0)}{h} = \infty$$

Again,

L.H.D.

$$\lim_{h\to 0^-} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{(-h)^{2/3}-0}{-h}$$

$$= -\alpha$$

$\because$ R.H.D $\neq$ L.H.D

$f'(x)$ doesnot exist in $(-1,1)$. Hence Roll's theorem is not applicable.

$\boxed{Q}$ Discuss the applications of Roll's theorem to the functi $f(x)=x^2$ in $(-1,1)$

Soln:

Given that,

$$f(x)=x^2$$

$$f(-1)=(-1)^2=1$$
$$f(1)=1^2=1$$

$f(-1) = f(1)$

R.H.D.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2hx + h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2x+h)}{h}$$

$$= \lim_{h \to 0} (2x+h)$$

$$= 2x$$

Again,

L.H.D. $$\lim_{h \to 0} \frac{f(x-h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x-h)^2 - x^2}{-h}$$

$$= \lim_{h \to 0} \frac{-h(2x-h)}{-h}$$

$$= \lim_{h \to 0} (2x-h)$$

$$= 2x$$

Here,

R.H.D = L.H.D

$f'(x)$ exists $\forall$ values of x in $(-1,1)$ Also $f(x)$ is continuous in the interval $-1 \leq x < 1$. as $f(x)$ is differentiable $\forall$ values of x in $(-1,1)$.

Hence Roll's theorem exists.

So,

$$f'(x) = 0$$

$$2x = 0$$

$x = 0$ which lies in $(-1,1)$

MEAN VALUE THEOREM

Statement:

If $f(x)$ be a function such that

(i) $f(x)$ is continuous in the closed interval $a \leq x \leq b$.

(ii) $f'(x)$ exists in the open interval $a < x < b$ then $\exists$ at least one point c such that

$$f(b) - f(a) = b-a \quad f'(c) \text{ where } a \leq c \leq b.$$

Proof:

Let,

$$\phi(x) = f(x) - Ax \quad \text{———} (1)$$

be an auxiliary function

A = constant so.

Let,

$$\phi(b) = \phi(a)$$

i.e. $f(b) - Ab = f(a) - A(a)$ [using eq^n(1)]

$$\Rightarrow f(b) - f(a) = Ab - Aa$$

$$A = \frac{f(b) - f(a)}{b-a} \quad \text{———} (II)$$

putting the value of A in eq^n (I)

$$\phi(x) = f(x) - \left\{ \frac{f(b) - f(a)}{b-a} \right\} x \quad \cdots\cdots (III)$$

Eq^n shows that $\phi(x)$ is continuous in $a \leq x \leq b$

differentiable in $a < x < b$ and $\phi(a) = \phi(b)$

All the condition of Roll's theorem exists. So,

$\phi'(c) = 0 \quad , a < c < b$

Now, from eq^n (I)

$$\phi'(x) = f'(x) - A$$

$$\phi'(c) = f'(c) - A$$

Here, $f'(c) = \dfrac{f(b) - f(a)}{b-a}$

$$\Rightarrow (b-a) f'(c) = f(b) - f(a)$$

$$\Rightarrow f(b) - f(a) = (b-a) f'(c)$$

(showed)

$\boxed{Q}$

Verify the mean value theorem in the interval $(0,4)$ for the function

$$f(x) = (x-1)(x-2)(x-3)$$

Sol^n:

Given that,

$$f(x) = (x-1)(x-2)(x-3)$$

$$= (x^2-3x+2)(x-3)$$

$$= x^3-3x^2+2x-3x^2+9x-6$$

$$f(x) = x^3-6x^2+11x-6$$

$$f'(x) = 3x^2-12x+11$$

Here,

$a=0$

$b=4$

$f(0) = -6$ and

$$f(4) = 4^3-6.4^2.11.4-6$$

$$=6$$

We know from mean value theorem,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$= \frac{f(4)-f(0)}{4-0}$$

$$= \frac{6-(-6)}{4-0} = 3$$

$$\Rightarrow 3c^2 - 12c + 11 = 3$$

$$3c^2 - 12c + 8 = 0$$

$$c = \frac{-(-12) \pm \sqrt{(-12)^2 - 4.3.8}}{2.3}$$

$$= \frac{12 \pm \sqrt{144-96}}{2.3}$$

$$= \frac{12 \pm \sqrt{144-96}}{6}$$

$$= 2 \pm \frac{\sqrt{48}}{6}$$

since both these values of c lies in (0,4) and hence mean value theorem is varified for the given function f(x) in (0,4)

$\boxed{Q}$

Verify the mean value theorem in the interval $(0, \frac{1}{2})$ for the function,

$$f(x) = x(x-1)(x-2)$$

Sol^n:

Given that,

$$f(x) = x(x-1)(x-2)$$

$$= (x^2 - x)(x-2)$$

$$= x^3 - 2x^2 - x^2 + 2x$$

$$f'(x) = 3x^2 - 6x + 2$$

Here,

$$a = 0$$

$$b = \frac{1}{2}$$

$$f(0) = 0$$

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 3 \cdot \left(\frac{1}{2}\right)^2 + 2 \cdot \frac{1}{2}$$

$$= \frac{1}{8} - \frac{3}{4} + 1$$

$$= \frac{1-6+8}{8} = \frac{3}{8}$$

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we know from mean value theorem

$$f'(c) = \frac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0}$$

$$= \frac{\frac{3}{8} - 0}{1/2}$$

$$= \frac{3}{8} \times 2$$

$$= \frac{3}{4}$$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{3}{4}$$

$$\Rightarrow 12c^2 - 24c + 8 = 3$$

$$12c^2 - 24c + 5 = 0$$

$$c = \frac{-(-24) \pm \sqrt{(-24)^2 - 4.12.5}}{2.12}$$

$$= \frac{24 \pm \sqrt{336}}{24}$$

$$c_1 = \frac{24 + \sqrt{336}}{24} \qquad c_2 = \frac{24 - \sqrt{336}}{24}$$

taking the negative sign we get

$c = 1 - \frac{\sqrt{336}}{24}$ which lies in the in the interval $(0, \frac{1}{2})$ and hence the mean value theorem varified in $(0, \frac{1}{2})$ for the given function.