Calculus

Calculus: is the branch of Mathematics concerned with describing the precise way in which changes in one variable relate to changes in another.

Variable: a quantity ~~wh~~ or symbol which can take various values.

(I) Independent Variable: can take any arbitrary value.

(II) Dependent " : its ~~value~~ assumes its value as a result of 2nd variable.

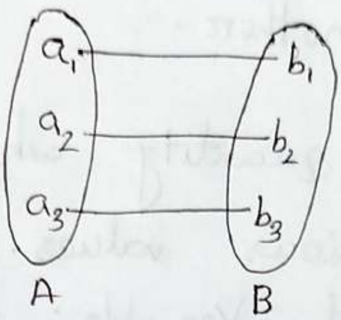
Relation: Let A and B be two non-empty set. If each element of A is related to one or more elements of B, then it is called a relation.

Function:

Let A and B be two non-empty set. If each element of A is related to a unique element of B, then the relation is called a function from the set A into B. It is denoted by $f: A \rightarrow B$.

Domain : If $f: A \rightarrow B$ is a function, then the set A is called domain of the function (The set of all possible inputs).

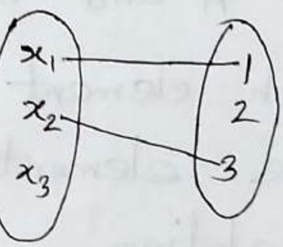
Example:



Domain $= \{a_1, a_2, a_3\}$

Range : The set of the images of the elements of A is called the range of the function.

Example:



Range $= \{1, 3\}$

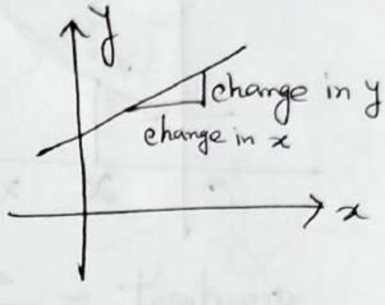
*

Differential Calculus

Differentiation:

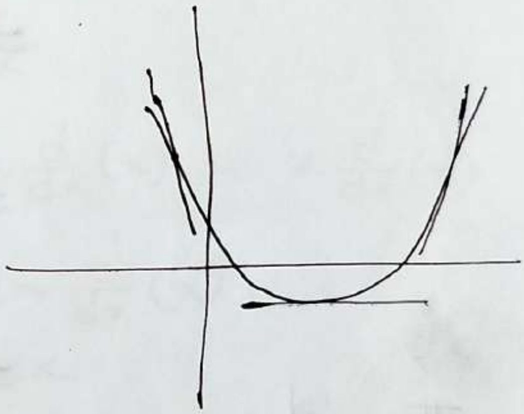
1. Finding the rate of change — Gradient.

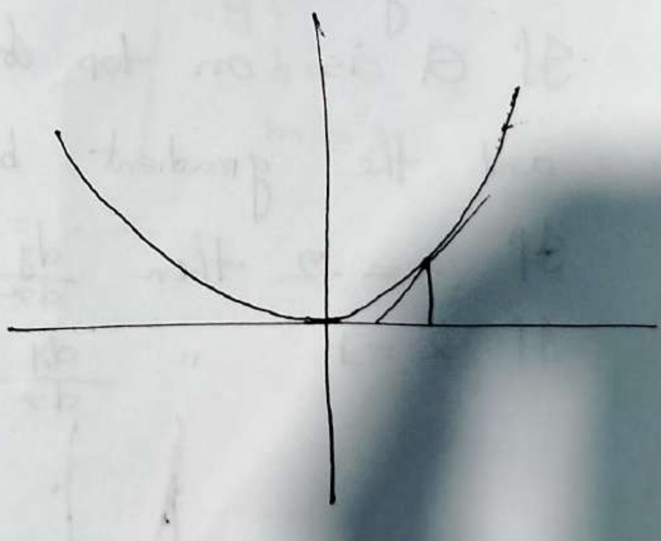
$$\text{gradient} = \frac{\Delta y}{\Delta x}$$

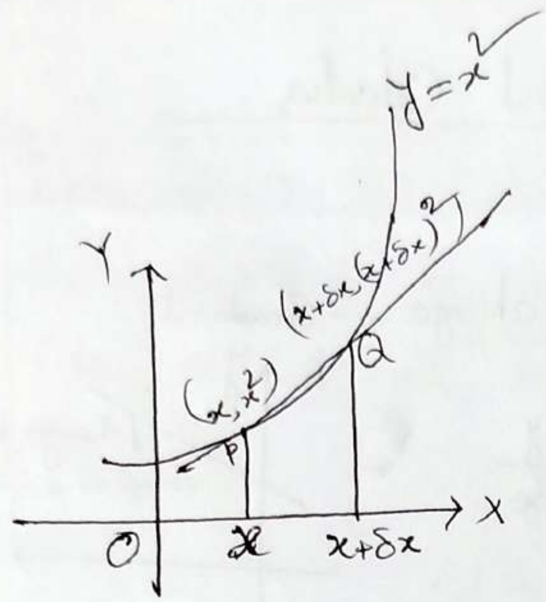


Δ means change in

★ Gradient is constantly changing
- depends on where you are
- equation in x







At P, $y=x^2$

At Q, $y=(x+\delta x)^2$

$$\text{gradient} = \frac{\Delta y}{\Delta x} = \frac{(x+\delta x)^2 - x^2}{x+\delta x - x}$$

$$= \frac{x^2 + 2x\delta x + (\delta x)^2 - x^2}{\delta x}$$

$$= \frac{\delta x\,(2x+\delta x)}{\delta x}$$

$$= 2x + \delta x$$

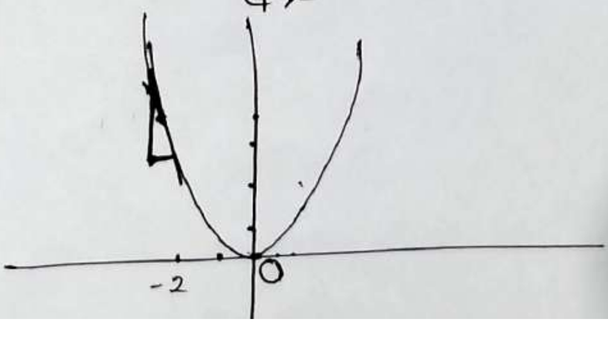
The closer Q is to P the more accurate the gradient.

If Q is on top of P then $\delta x = 0$

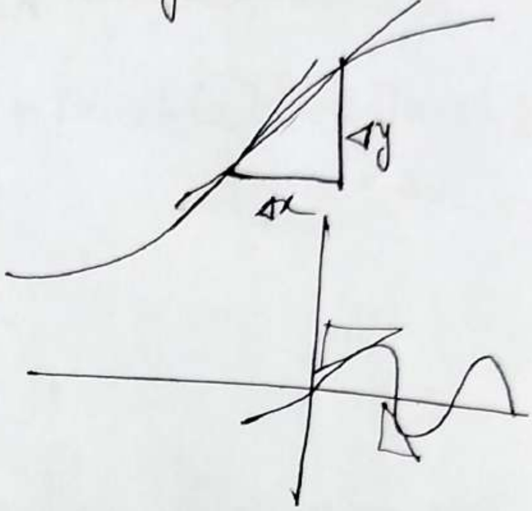
and the gradient becomes $\frac{dy}{dx} = 2x$

If $x = -2$ then $\frac{dy}{dx} = +4$

If $x = 1$ " $\frac{dy}{dx} = 2$



Slope of Chord = $\frac{\Delta y}{\Delta x}$ small change



* If function is linear the derivative is constant

$y =$

$$\frac{d}{dx}(y+z) = \frac{dy}{dx} + \frac{dz}{dx}$$

$$\frac{d}{dx}(yz) = y\frac{dz}{dx} + z\frac{dy}{dx}$$

Exa: $\frac{d}{dx}(x^2) = \frac{d}{dx}(xx) = x\frac{d}{dx}(x) + x\frac{d}{dx}(x)$

$$= 2x\frac{d}{dx}(x)$$

Again,

$$= 2x$$

$$\frac{d}{dx}(mx+b) = m$$

$$\frac{d}{dx}(x) = 1$$

If
$b = 0$
$m = 1$

$$\therefore \frac{d}{dx}(x^2) = 2x$$

Q. Find the domain and range of $y = 2x+9$, $y = \frac{x-1}{x+3}$

SUBJECT

Domain - Range

Find the Domain & Range of the followings:

(i) $f(x) = \frac{x}{|x|}$ (ii) $f(x) = |x| + |x-1|$ (iii) $f(x) = |x-1| - |x-2|$

(iv) $f(x) = |x+1| + |x| + |x-1|$

(i) $\underline{\text{Sol}^n}$: Given that,

$$f(x) = \frac{x}{|x|}$$

When $x = 0$, than the function $f(x)$ can not define at $x = 0$.

So,

$f(x)$ can be define for all real values of x, except $x = 0$.

$\therefore$ Domain, $D_f = \mathbb{R} - \{0\}$ (Ans.)

For $x \geq 0$ then $f(x) = \frac{x}{x} = 1$

and $x < 0$ " $f(x) = \frac{x}{-x} = -1$

$\therefore$ Range, $R_f = \{-1, 1\}$ (Ans.)

(ii) Given that,

$$f(x) = |x| + |x-1|$$

$$= \begin{cases} -x - (x-1) & \text{when } x < 0 \\ x - (x-1) & \text{when } 0 \leq x < 1 \\ x + x - 1 & \text{" } x \geq 1 \end{cases}$$

$$= \begin{cases} -2x+1 & \text{when } x<0 \\ 1 & \text{" } 0\le x<1 \\ 2x-1 & \text{" } x\geq 1 \end{cases}$$

$\therefore D_f = x<0 \cup 0\le x<1 \cup x\geq 1$

$= (-\infty, 0) \cup [0, 1) \cup [1, \infty)$

$= (-\infty, \infty)$

Again, for $x<0$, $f(x) = \begin{cases} \infty \\ 1 \end{cases}$

$= (1, \infty)$

" $0\le x<1$, $f(x) = 1$

$= (1)$

Also for $x\geq 1$, $f(x) = \begin{cases} 1 \\ \infty \end{cases}$

$= [1, \infty)$

$\therefore R_f = (1, \infty) \cup (1) \cup [1, \infty)$

$= [1, \infty)$ (Ans.)

(iii) Given that

$$f(x) = |x-1| - |x-2|$$

Here

$$f(x) = |x-1| - |x-2|$$

$$= \begin{cases} -(x-1) + (x-2) & \text{when } x < 1 \\ (x-1) + (x-2) & " \ 1 \le x < 2 \\ x-1-(x-2) & " \ x \geq 2 \end{cases}$$

$$= \begin{cases} -1 & \text{when } x < 1 \\ 2x-3 & " \ 1 \le x < 2 \\ 1 & " \ x \geq 2 \end{cases}$$

$\therefore$ Domain, $D_f = x<1 \cup 1 \le x < 2 \cup x \geq 2$

$$= (-\infty, 1) \cup [1, 2) \cup [2, \infty)$$

$$= (-\infty, \infty) = \mathbb{R}$$

Again, for $x < 1$,

$$f(x) = -1 = (-1)$$

For $1 \le x < 2$, $f(x) = \begin{cases} -1 \\ 1 \end{cases}$

$$= [-1, 1)$$

And, for $x \geq 2$, $f(x) = 1$
$= (1)$

$\therefore$ Range, $R_f = (-1) \cup [-1, 1) \cup (1)$
$= [-1, 1)$

Let,
* $y = f(x) = \frac{1}{x-1}$ ---(i)

When $x = 1$, then the function $f(x)$ can not define at $x = 1$.
So, $f(x)$ can be define for all real values of x, adcept $x = 0$.

$\therefore$ Domain, $D_f = \mathbb{R} - \{1\}$ (An.)

Now, from eqn (i) we get,

$y = \frac{1}{x-1}$

$\Rightarrow x - 1 = \frac{1}{y}$

$\Rightarrow x = 1 + \frac{1}{y}$

$\Rightarrow x = \frac{y+1}{y}$ ---(ii)

We can see from eqn (ii) the given function can be define $\forall$ real values of y except $y = 0$.

$\therefore$ Range, $R_f = \mathbb{R} - \{0\}$ (An.)

Graph

Draw a graph of the followings :-

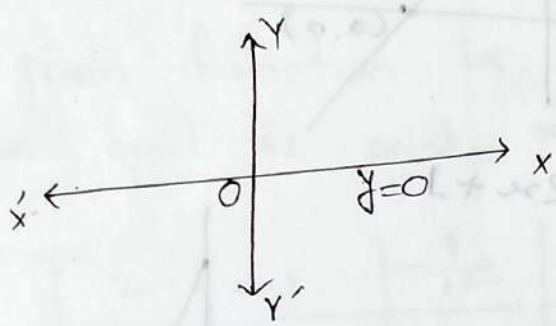
(I) $y = 0$. (II) $x = 0$ (III) $y = 3x$ (IV) $\frac{x}{a} + \frac{y}{b} = 1$

(V) $f(x) = -2x + 1$ (VI) $f(x) = x^2$ (VII) $y = -x^2$ (VIII) $y = \sqrt{x}$

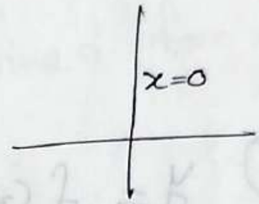
(IX) $y = \frac{1}{x}$ (X) $y = \sin x$ (XI) $y = \cos x$

Solⁿ:

(I)



(II)

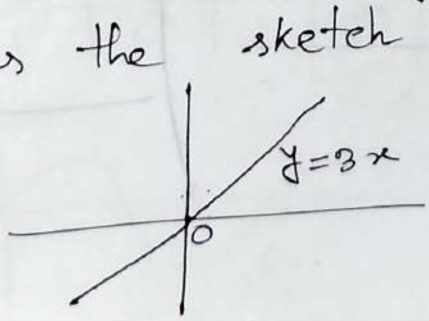


(III) $y = 3x$, $y = mx$

To obtain the graph, it is necessary to plot some points. There some points on the following table

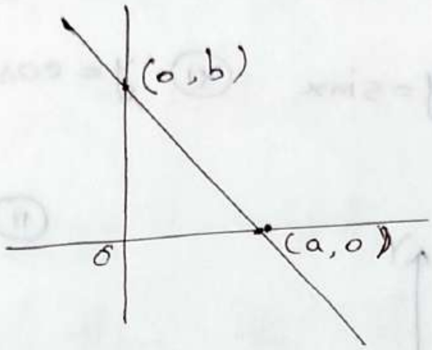
x	0	1	2	−1	−2
y	0	3	6	−3	−6

~~Plotting~~ Here, $(0, 0)$, $(1, 3)$, $(2, 6)$, $(-1, -3)$, $(-2, 6)$ points are on the graph. Plotting the points gives us the sketch in figue.



(IV) $\frac{x}{a} + \frac{y}{b} = 1$

Here, $(a, 0)$ and $(0, b)$ are two points on the graph.

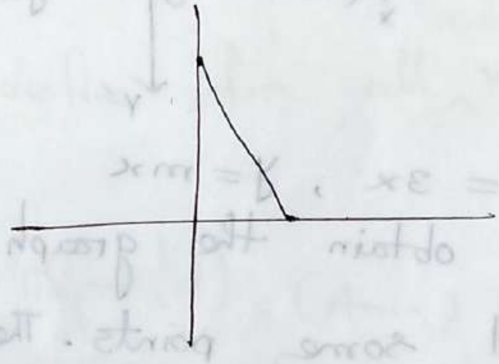


(V) $y = f(x) = -2x + 1$

$\Rightarrow \quad 2x + y = 1$

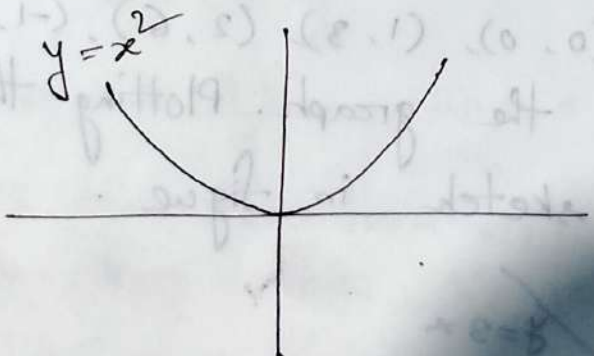
$\Rightarrow \quad \frac{x}{\frac{1}{2}} + \frac{y}{1} = 1$

$(\frac{1}{2}, 0), \quad (0, 1)$



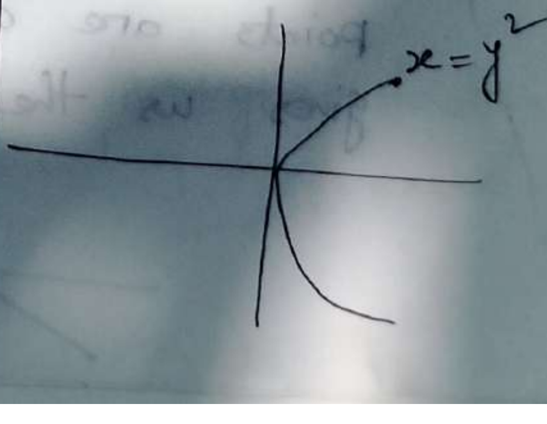
(VI) $y = x^2$

x	0	-1	-2	1	2
y	0	1	4	1	4



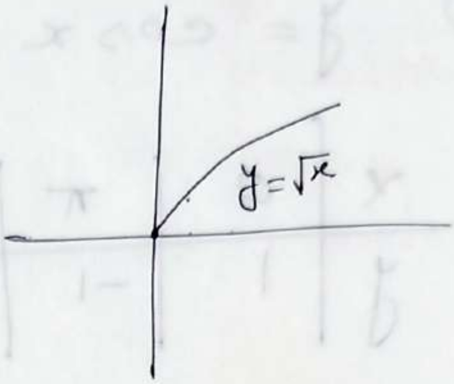
$x = y^2$

x	
y	



(VII) $y = \sqrt{x}$

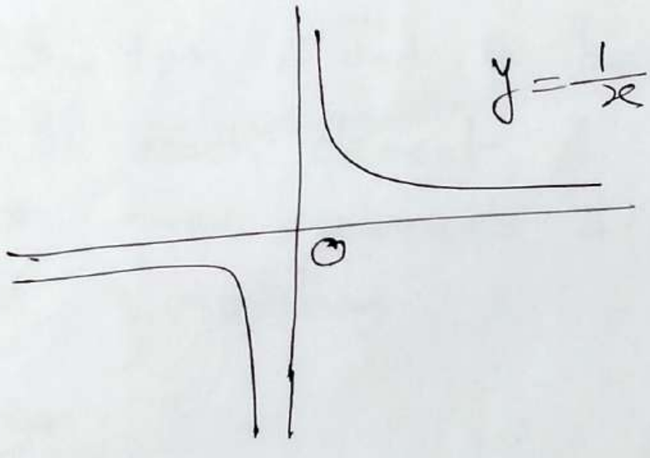
x	4	0	1	9
y	2	0	1	3



** (VIII) $y = \frac{1}{x}$

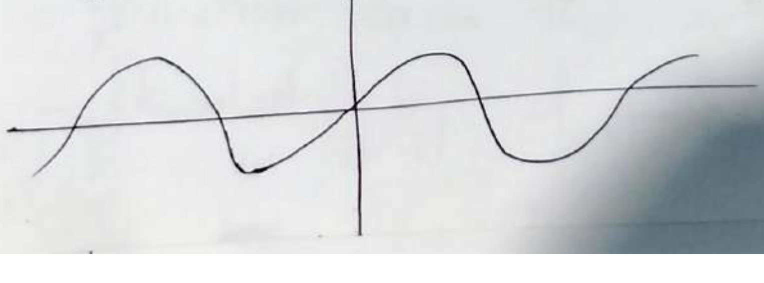
since given function is undefined for $x=0$,
here we omit the point $x=0$.

x	1	2	-1	$-\frac{1}{2}$	-2
y	1	$\frac{1}{2}$	-1	-2	$-\frac{1}{2}$



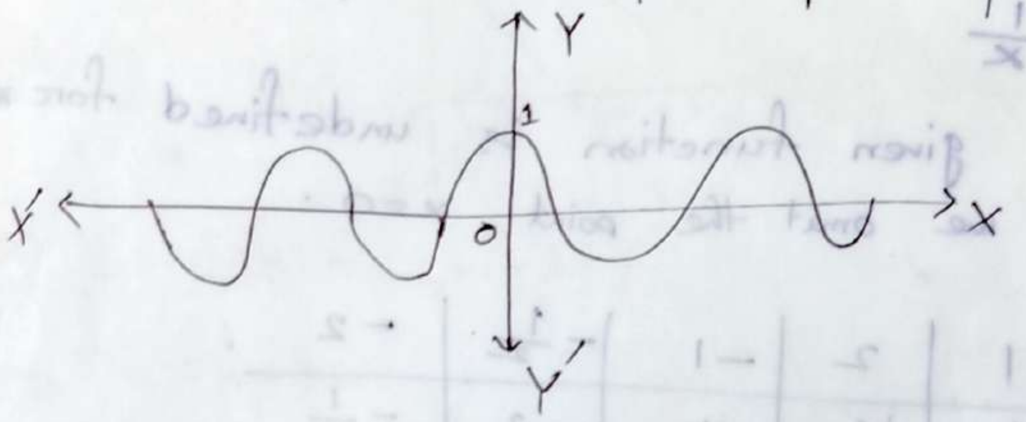
(IX) $y = \sin x$

x	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π
y	0	-1	0	1	0



(x) $y = \cos x$

x	0	π	$\frac{\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	$\frac{3\pi}{2}$	2π
y	1	-1	0	-1	0	0	1



ICT

Mawlana Bhashani Science and Technology University
Department of Mathematics
Class Test No

Name
Year
Course Title
Session

Limit

Def^n: If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x\to a} f(x) = L$$

This expression can also be written as

$$f(x) \to L \text{ as } x \to a.$$

Q. If $f(x) = \frac{x^2-5x+6}{x-3}$ then show that the limit exist at point $x=3$.

Sol^n: Here,

$$f(x) = \frac{x^2-5x+6}{x-3} \quad \text{--- (1)}$$

$$R.H.L. = \lim_{x\to 3^+} f(x)$$

$$= \lim_{x\to 3^+} \frac{x^2-5x+6}{x-3}$$

$$= \lim_{x\to 3^+} \frac{x^2-3x-2x+6}{x-3}$$

$$= \lim_{x\to 3^+} \frac{(x-3)(x-2)}{(x-3)}$$

$$= x-2 = 3-2 = 1$$

$$L.H.L. = \lim_{x \to 3^-} f(x)$$

$$= \lim_{x \to 3^-} \frac{x^2-5x+6}{x-3}$$

$$= \lim_{x \to 3^-} \frac{(x-3)(x-2)}{(x-3)}$$

$$= 3-2$$

$$= 1$$

Since $L.H.L. = R.H.L.$

$\therefore$ The limit exists at $x=3$.

$$\therefore \lim_{x \to 3} f(x) = 1.$$

Q. $f(x) = \dfrac{x^2-4}{x-2}$ at $x=2$

Q. If $f(x) = \begin{cases} 1+2x & \text{when } -\frac{1}{2} \le x < 0 \\ 1-2x & \text{"} \quad 0 \le x < \frac{1}{2} \\ -1+2x & \text{"} \quad x > \frac{1}{2} \end{cases}$

then find the value of $\lim_{x \to 0} f(x)$ and $\lim_{x \to \frac{1}{2}} f(x)$ if $\exists$ limit at $x=0$ and $x=\frac{1}{2}$.

Sol:

Given that

$f(x) = \Bigg\{$

for $x=0$,

$L.H.L. = \lim_{x\to 0^-} f(x)$

$= \lim_{x\to 0^-} (1+2x)$

$= 1+0$

$= 1$

$R.H.L. = \lim_{x\to 0^+} f(x)$

$= \lim_{x\to 0^+} (1-2x)$

$= 1-0$

$= 1$

Since $L.H.L. = R.H.L$

$\exists$ limit at $x=0$

$\therefore \lim_{x\to 0} f(x) = 1$ (Ans.)

Again,

for $x = \frac{1}{2}$

$L.H.L. = \lim_{x\to \frac{1}{2}^-} f(x)$

$= \lim_{x\to \frac{1}{2}^-} (1-2x)$

$= 1 - 2\cdot\frac{1}{2}$

$= 0$

$R.H.L. = \lim_{x\to \frac{1}{2}^+} f(x)$

$= \lim_{x\to \frac{1}{2}^+} (-1+2x)$

$= -1 + 2\cdot\frac{1}{2}$

$= 0$

Since $L.H.L. = R.H.L.$ for $x = \frac{1}{2}$, so $\exists$ limit at $x=\frac{1}{2}$.

$\therefore \lim_{x\to \frac{1}{2}} f(x) = 0$ (Ans)

$$R.H.L. = \lim_{x \to 0^+} f(x)$$

$$= \lim_{x \to 0^+} \frac{3x + |x|}{7x - 5|x|}$$

$$= \lim_{x \to 0^+} \frac{3x + x}{7x - 5x}$$

$$= \lim_{x \to 0^+} \frac{4x}{2x}$$

$$= 2$$

$$L\cancel{R}.H.L. = \lim_{x \to 0^-} f(x)$$

$$= \lim_{x \to 0^-} \cancel{f(x)} \frac{3x + |x|}{7x - 5|x|}$$

$$= \lim_{x \to 0^-} \frac{3x - x}{7x + 5x}$$

$$= \lim_{x \to 0^-} \frac{2x}{12x}$$

$$= \frac{1}{6}$$

Since $\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$

$\therefore$ There exists no value of $\lim_{x \to 0} f(x)$.

ICT

Differential Calculus

Mawlana Bhashani Science and Technology Univers

Department Of Mathematics

Class Test No-

Name:

Year

Course Title:

Session:

Roll No:

Semester:

Course Code:

Date:

Successive Differentiation

If $y = f(x)$ be a function.

$\frac{dy}{dx} = y_1 = f'(x)$, $y_{11} = f''(x)$, ..., $y_n = f^n(x) = \frac{d^n y}{dx^n} = \left(\frac{d}{dx}\right)^n y = D^n y$

① Find the n^{th} derivative of $y = x^n$.

Given that,

$y = x^n$

$\Rightarrow y_1 = \frac{dy}{dx} = \frac{d}{dx}(x^n) = n x^{n-1}$

$\Rightarrow \frac{d^2 y}{dx^2} = y_2 = \frac{d}{dx}(n x^{n-1}) = n(n-1)x^{n-2}$

$\therefore y_3 = n(n-1)(n-2)x^{n-3}$

$y_n = n(n-1)(n-2)\ldots\{n-(n-1)\}x^{n-n}$

$= n(n-1)(n-2)\ldots 3\cdot 2\cdot 1\, x^0$

$\therefore y_n = n!$ (Ans.)

(11) $y = (ax+b)^m$, $m > n$. find y_n. Also for $m = n$ (iii)

$y_1 = am(ax+b)^{m-1}$

$y_2 = m(m-1)a^2(ax+b)^{m-2}$

$y_3 = m(m-1)(m-2)a^3(ax+b)^{m-3}$

$\vdots$

$y_n = m(m-1)(m-2)\cdots\{m-(n-1)\}a^n(ax+b)^{m-n}$

$$= \frac{m(m-1)(m-2)\cdots(m-n+1)(m-n)\cdots 3.2.1\,(ax+b)^{m-n}a^n}{(m-n)\cdots 3.2.1}$$

$$= \frac{m!\,a^n(ax+b)^{m-n}}{(m-n)!} \text{ (Ans.)}$$

Also if $m = n$ then

$$y_n = \frac{n!\,a^n(ax+b)^{n-n}}{(n-n)!}$$

$= n!\,a^n$ (Ans.)

(iii) Find the nth derivative of $y = e^{ax+b}$.

$y = e^{ax+b}$

$y_1 = \frac{d}{dx}\left(e^{ax+b}\right)$

$= a e^{ax+b}$

$y_2 = a^2 e^{ax+b}$

$y_3 = a^3 e^{ax+b}$

$\vdots$

$y_n = a^n e^{ax+b}$

(iv) $y = \frac{x}{x^2-1}$ then find y_n.

$\Rightarrow y = \frac{x}{(x+1)(x-1)}$

$= \frac{-1}{(x+1)(-1-1)} + \frac{1}{(1+1)(x-1)}$

$= \frac{-1}{-2(x+1)} + \frac{1}{2(x-1)}$

$= \frac{1}{2}\left[\frac{1}{x+1} + \frac{1}{x-1}\right]$

$= \frac{1}{2}\left[(x+1)^{-1} + (x-1)^{-1}\right]$

SUBJECT: NO DATE / / SAT SUN MON TUE WED THU FRI

$$y_1 = \frac{1}{2}\left[(-1)(x+1)^{-2} + (-1)(x-1)^{-2}\right]$$

$$y_2 = \frac{1}{2}\left[(-1)(-2)(x+1)^{-3} + (-1)(-2)(x-1)^{-3}\right]$$

$$y_3 = \frac{1}{2}\left[(-1)(-2)(-3)(x+1)^{-4} + (-1)(-2)(-3)(x-1)^{-4}\right]$$

. . . .

. . . .

$$y_n = \frac{1}{2}\left[(-1)(-2)(-3)\cdots\cdots(-n)(x+1)^{-(n+1)} + (-1)(-2)(-3)\cdots\cdots(-n)(x-1)^{-(n+1)}\right]$$

$$= \frac{(-1)^n}{2}\, n!\left[\frac{1}{(x+1)^{n+1}} + \frac{1}{(x-1)^{n+1}}\right]$$

(Ans.)

(iv)

Leibnitz's theorem

The nth derivative of the product of two fs.

Statement: If u and v be two functions of x possessing derivatives of the n^{th} order, then

$$(uv)_n = u_n + {}^nc_1 u_{n-1} v_1 + {}^nc_2 u_{n-2} v_2 + \cdots + {}^nc_r u_{n-r} v_r$$
$$+ \cdots + {}^nc_n u v_n .$$

Ex. If $y = \cos(m \sin^{-1}x)$ then show that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$$ and

hence find $y_n(0)$.

Soln: We have

$$y = \cos(m \sin^{-1}x) \quad \text{—— (i)}$$

$$\therefore y_1 = \sin(m \sin^{-1}x) \cdot m \frac{1}{\sqrt{1-x^2}} \quad \text{—— (ii)}$$

$$\Rightarrow (1-x^2) y_1^2 = m^2 \sin^2(m \sin^{-1}x)$$

$$= m^2 [1 - \cos^2(m \sin^{-1}x)]$$

$$= m^2 [1 - y^2]$$

Diff. w.r.to. x

$$(1-x^2) 2y_1 y_2 - 2xy_1^2 = -2m^2 y y_1$$

$$\Rightarrow (1-x^2)\, \cancel{2y_1}\, y_2 - xy_1 = -m^2 y$$

$$\Rightarrow (1-x^2) y_2 - xy_1 + m^2 y = 0 \quad \text{——— (III)}$$

Applying Leibnitz's theorem,

$$(1-x^2) y_{n+2} + {}^nc_1 \cdot -2x \cdot y_{n+1} + {}^nc_2 \cdot -2 \cdot y_n - xy_{n+1} -$$

$$ {}^nc_1 \cdot y_n + m^2 y_n = 0$$

$$\Rightarrow (1-x^2) y_{n+2} - 2nx\, y_{n+1} - x y_{n+1} - 2 \cdot \frac{n(n-1)}{2!} y_n$$

$$- n y_n + m^2 y_n = 0$$

$$\Rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2 - n) y_n - n y_n + m^2 y_n = 0$$

$$- (n^2 - n + n - m^2) y_n = 0$$

$$\Rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} + (m^2 - n^2) y_n = 0 \quad \text{——— (IV)}$$

$$\therefore \cancel{(1-x^2) y_{n+2} - (2n+1) x y_{n+1} + (m^2 - n^2) y_n = 0}$$

Putting $x = 0$, we get

$$y_{n+2}(0) = (n^2 - m^2) y_n(0) \quad \text{——— (V)}$$

From eq^n (i), (ii) and (iii), we get

$y(0) = 1$, $y_1(0) = 0$, $y_2(0) = m^2 y(0) = m^2$

Putting $n = 1, 2, 3, 4, \ldots$ in

$$y_3(0) = (1^2 - m^2) y_1(0) = 0$$

$$y_4(0) = (2^2 - m^2) y_2(0)$$

$$= m^2 (2^2 - m^2)$$

$$y_5(0) = 0$$

$$y_6(0) = (4^2 - m^2) y_4(0)$$

$$= m^2 (2^2 - m^2)(4^2 - m^2)$$

In general,

$y_n(0) = 0$ if n is odd.

$= m^2 (2^2 - m^2)(4^2 - m^2) \ldots \{(n-2)^2 - m^2\}$

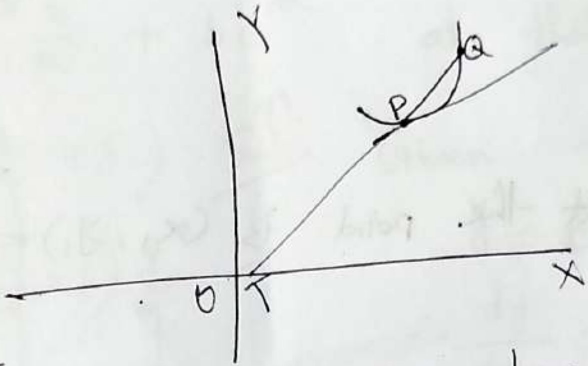
if n is even.

ICT

Tangent — Normal

$$Y - y = \frac{\delta y}{\delta x}(X - x)$$

Tangent :



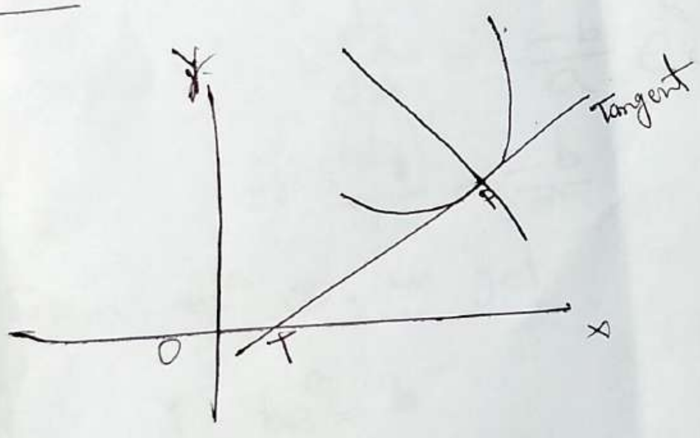
$y = f(x)$

$P(x, y)$

$Q(x+\delta x,\ y+\delta y)$ Two points

PT = tangent

Normal:



$y = f(x)$

If $y = f(x)$, at (x, y) point the eqn of normal is

$$X - x + \frac{dy}{dx}(Y - y) = 0$$

Ex. 1 Find the eq^n of the tangent and the normal at any point ~~(x, y)~~ $x=1$ of the ~~ff~~ curve ~~f(x)~~ $y = x^2+2x+3$.

Given that,

$$y = x^2+2x+3 \quad \text{------} \quad (1)$$

When $x=1$ then from eq^n (1), we get

$$y = 1^2+2\cdot 1+3$$

$$\Rightarrow y = 6$$

Therefore ~~at (1, 6) point~~ the point is $(x_1, y_1) = (1, 6)$

Now,

$$\frac{dy}{dx} = 2x+2$$

At $(1, 6)$ point $\frac{dy}{dx} = 2\cdot 1+2 = 4$

The eq^n of tangent is,

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$\Rightarrow y - 6 = 4(x-1)$$

$$\Rightarrow y - 6 = 4x - 4$$

$$\Rightarrow 4x - y + 2 = 0 \quad \text{(Ans.)}$$

Again, the eq^n of normal is.

$$x - x_1 + \frac{dy}{dx}(y - y_1) = 0$$

$$\Rightarrow x - 1 + 4(y-6) = 0$$

$$\Rightarrow x - 1 + 4y - 24 = 0$$

$$\Rightarrow x + 4y - 25 = 0$$

(Ans.)

$y = be^{-x/a} \Rightarrow \frac{dy}{dx} = \frac{-b}{a} e^{-x/a}$

$y - y_1 = \frac{dy}{dx}(x - x_1)$

$\Rightarrow y - b = \frac{-b}{a} e^{-x/a}(x - a)$

$\Rightarrow y - b = -be^{-x/a}\frac{x}{a} + be^{-x/a}$

$\Rightarrow y - b = y\left(\frac{x}{a} + 1\right)$

$\Rightarrow \frac{y}{b} - 1 = \left(\frac{x}{a} + 1\right)\frac{y}{b}$

$\Rightarrow \frac{y}{b} - 1 = \frac{xy}{ab} + \frac{y}{b}$

Q. Show that the curve $y = be^{-x/a}$ intersect the y axis at which point, ~~find~~ the eqn of tangent will be $\frac{x}{a} + \frac{y}{b} = 1$ at that point.

Sol: Given that

$$y = be^{-x/a} \quad \text{——— (1)}$$

$$\therefore \frac{dy}{dx} = \frac{-b}{a} e^{-x/a}$$

When the curve (1) intersect y-axis then $x = 0$

$$\therefore \frac{dy}{dx} = \frac{-b}{a} e^{0}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-b}{a}$$

$\therefore$ From eqn (1), we get

$$y = be^{0} = b$$

$\therefore$ The point is $(x_1, y_1) = (0, b)$

e eqn of tangent at $(0, b)$.

$$y - b = \frac{dy}{dx}(x - 0)$$

$$\Rightarrow y - b = \frac{-b}{a}x$$

$$\Rightarrow \frac{y}{b} - 1 = \frac{-x}{a}$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1$$

(Showed)

Ex: Find the eqn of tangent of the curve $xy^2 = 4(4-x)$ at point $(2, 2)$.

Sol:

~~Giv~~ Let,

$$f(x,y) = xy^2 - 4(4-x)$$

$$= xy^2 - 16 + 4x$$

$$f_x = y^2 + 4$$

$$f_y = 2xy.$$

At $(2,2)$ point

$$f_x = 2^2 + 4 = 8$$

$$f_y = 2 \cdot 2 \cdot 2 = 8$$

$\therefore$ The eqn of tangent at $(2,2)$ is

$$\cancel{(y-2) = \frac{dy}{dx}(x-2)}$$

$$\cancel{\Rightarrow (y-2)dx = dy(x-2)}$$

$$\cancel{\Rightarrow (y-2).}$$

$$(x-2)f_x + (y-2)f_y = 0$$

$$\Rightarrow (x-2)8 + (y-2)8 = 0$$

$$\Rightarrow 8x - 16 + 8y - 16 = 0$$

$$\Rightarrow x + y - 4 = 0$$

(Ans)

Maxima and Minima

1. Show that $x^5 - 5x^4 + 5x^2 - 1$ has a maximum value when $x=1$, a minimum value when $x=3$ and neither when $x=0$.

Sol:

Let,

$$f(x) = x^5 - 5x^4 + 5x^2 - 1$$

$$\therefore f'(x) = 5x^4 - 20x^3 + 10x$$

Now for maxima or minima

$$f'(x) = 0$$

$$\therefore 5x^4 - 20x^3 + 10x = 0$$

$$\Rightarrow 5x(x^3 - 4x^2 + 2) = 0$$

$$x = 0, \quad x^3 - 4x^2 + 2 = 0$$

$$f''(x) = 20x^3 - 60x^2 + 10$$

$$f''(1) = 20 - 60 + 10$$

$$= -30 \text{ (maxima)} < 0$$

$$f''(3) = 20 \cdot 3^3 - 60 \cdot 3^2 + 10 = 540 - 540 + 10$$

$$\cancel{= 160 - 240 + 10} = 10 > 0 \text{ (minimum)}$$

$$\cancel{= -20 \text{ (minima)} < 0}$$

$$\cancel{f''(0) = 10} \quad f(1) =$$

Ex Find the maximum and minimum values of polynomial f^n f is given by

Sol^n $f(x) = 8x^5 - 15x^4 + 10x^2$

$\therefore f'(x) = 40x^4 - 60x^3 + 20x$

For maxima and minima,

$f'(x) = 0$

$\Rightarrow 40x^4 - 60x^3 + 20x = 0$

$\Rightarrow 20x(2x^3 - 3x^2 + 1) = 0$

$2x^3 - 2x^2 - x^2 + x - x + 1 = 0$

$\Rightarrow 20x[2x^2(x-1) - x(x-1) - 1(x-1)] = 0$

At

$\Rightarrow 20x(x-1)(2x^2 - x - 1) = 0$

$2x^2 - 2x + x - 1 = 0$

$\Rightarrow 20x(x-1)[2x(x-1) + 1(x-1)] = 0$

$\therefore \Rightarrow 20x(x-1)(x-1)(2x+1) = 0$

$\Rightarrow 20x(x-1)^2(2x+1) = 0$

$x = 0, \quad x = 1, \quad x = \frac{-1}{2}$

$f''(x) = 160x^3 - 180x^2 + 20$

$f(0) = 0$ is a minima

$f''(0) = 20 > 0$

So we get minimum value ~~for~~ $x = 0$

$f''(1) = 160 - 180 + 20$

$= 0$

$\therefore$ $f(1)$ is neither a maximum nor a minimum value.

$$\therefore f''\left(-\frac{1}{2}\right) = 160\left(-\frac{1}{2}\right)^3 - 180\left(-\frac{1}{2}\right)^2 + 20$$

$$= -45 < 0$$

$\therefore$ $f(x)$ has maximum value for $x = -\frac{1}{2}$

$\therefore$ The maximum value is $f\left(-\frac{1}{2}\right) = 8\left(-\frac{1}{2}\right)^5 - 15\left(\frac{1}{2}\right)^4 + 10\left(\frac{1}{2}\right)^2$

$$= \frac{-\cancel{8}}{\cancel{32}_{4}} - \frac{15}{16} + \frac{\cancel{10}^{5}}{\cancel{4}_{2}}$$

$$= \frac{-4 - 15 + 40}{16}$$

$$= \frac{21}{16} \quad \text{(Ans.)}$$

E Q. Find the maxima and minima for the polynomial f^n given by

$$10x^6 - 24x^5 + 15x^4 - 40x^3 + 108$$

Sol: $\boxed{H.T}$

A