6.3000: Signal Processing

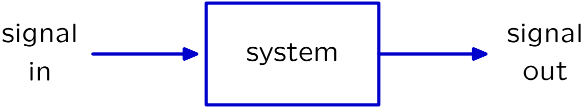
Frequency Response and Filtering

- Discrete-Time Frequency Response
- Continuous-Time Frequency Response

October 17, 2023

Context: The System Abstraction

Describe a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



This abstraction is particularly powerful for **linear and time-invariant** systems, which are both **prevalent** and **mathematically tractable**.

We previously studied representations based on difference/differential equations and on convolution:

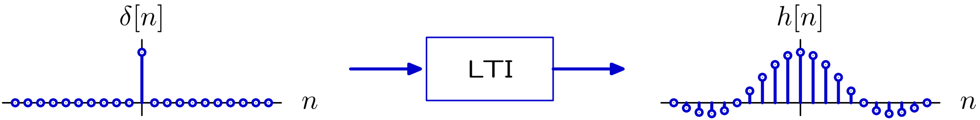
- **Difference/Differential Eq:** algebraic input/output **constraint** √
- **Convolution:** represent system by **unit-sample/impulse response** √
- **Filter:** represent a system by its **frequency response**

Today: representations based on the system's frequency response.

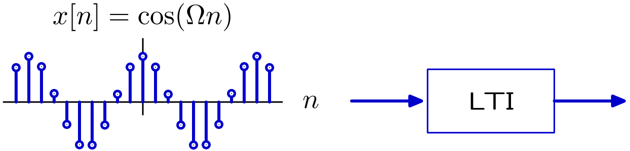
Frequency Response

Use convolution to characterize the frequency response of a system.

The response of an LTI system to the unit-sample signal $\delta[n]$ is the unit-sample response $h[n]$.



The response $y[n]$ to a sinusoid $x[n] = \cos(\Omega n)$ is $y[n] = (x * h)[n]$.



$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Response to Sinusoids

Determining response to a sinusoid by direct application of convolution.

$$x[n]=\cos(\Omega n) \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]=(x*h)[n]$$

$$\begin{aligned}
y[n] &= \sum_{m=-\infty}^{\infty} x[n-m]h[m] \\
&= \sum_{m=-\infty}^{\infty} \cos(\Omega(n-m))h[m] \\
&= \sum_{m=-\infty}^{\infty} h[m]\cos(\Omega n-\phi[m]) \\
&= A\cos(\Omega n-\phi)
\end{aligned}$$

Each cosine in the weighted sum has the same frequency Ω as the input.
But each term in the sum has a different amplitude $h[m]$ and phase $\phi[m]$.
We'd like to find the net amplitude A and net phase ϕ.

Response to Complex Exponentials

Using complex exponentials is easier than using trigonometric functions.

$$x[n]=e^{j\Omega n} \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]=(x*h)[n]$$

$$y[n]=(x*h)[n]=\sum_{m=-\infty}^{\infty}x[n-m]h[m]=\sum_{m=-\infty}^{\infty}e^{j\Omega(n-m)}h[m]$$

$$=e^{j\Omega n}\sum_{m=-\infty}^{\infty}h[m]e^{-j\Omega m}=H(\Omega)\,e^{j\Omega n}$$

The response to a complex exponential is a complex exponential with the **same frequency (Ω)** but with amplitude and phase given by $H(\Omega)$.

$$e^{j\Omega n} \longrightarrow \boxed{\text{LTI}} \longrightarrow H(\Omega)e^{j\Omega n}$$

The map for how a system modifies the amplitude and phase of a complex exponential input is the **Fourier transform of the unit-sample response.**

Responses to Sinusoids

The response to a sinusoid follows directly from Euler's formula.

$$x[n] = \cos(\Omega n) \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = A\cos(\Omega n - \phi)$$

$$\begin{aligned}
e^{j\Omega n} &\to H(\Omega)e^{j\Omega n} \\
e^{-j\Omega n} &\to H^*(\Omega)e^{-j\Omega n} \\
\cos(\Omega n) &\to \tfrac{1}{2}\left(H(\Omega)e^{j\Omega n} + H^*(\Omega)e^{-j\Omega n}\right) \\
&\to \text{Re}(H(\Omega)e^{j\Omega n}) \\
&\to \text{Re}\left(|H(\Omega)|e^{j\angle H(\Omega)}e^{j\Omega n}\right) \\
&\to |H(\Omega)|\,\text{Re}\left(e^{j(\angle H(\Omega)+\Omega n)}\right) \\
&\to |H(\Omega)|\cos\left(\Omega n + \angle H(\Omega)\right)
\end{aligned}$$

The response of an LTI system to a cosine input is a cosine output with amplitude equal to the magnitude of $H(\Omega)$ and phase given by $\angle H(\Omega)$.

Convolution in Time Equivalent to Multiplication in Frequency

The response of an LTI system can be computed by convolution.

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = (x * h)[n]$$

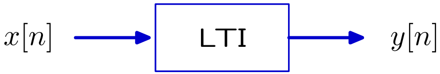
$$\begin{aligned}
Y(\Omega) &= \sum_n y[n] e^{-j\Omega n} \\
&= \sum_n \left(\sum_m x[m] h[n-m] \right) e^{-j\Omega n} \\
&= \sum_m x[m] \sum_n h[n-m] e^{-j\Omega n} \\
&= \sum_m x[m] \sum_l h[l] e^{-j\Omega(l+m)} \quad \text{where } l = n - m \\
&= \left(\sum_m x[m] e^{-j\Omega m} \right) \left(\sum_l h[l] e^{-j\Omega l} \right) = X(\Omega) H(\Omega)
\end{aligned}$$

$$X(\Omega) \longrightarrow \boxed{H(\Omega)} \longrightarrow Y(\Omega) = X(\Omega) H(\Omega)$$

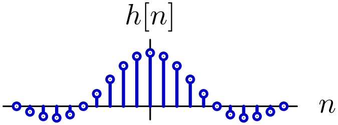
The frequency response $H(\Omega)$ relates the Fourier transform of the input signal to the Fourier transform of the output signal.

Unit-Sample Response and Frequency Response

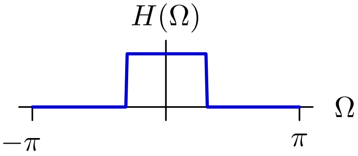
Two complete representations for linear, time-invariant systems.



Unit-Sample Response: responses across time for a unit-sample input.



Frequency Response: responses across frequencies for sinusoidal inputs.



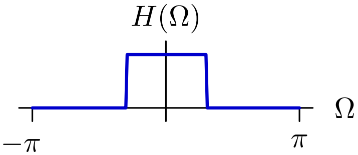
The **frequency response** is Fourier transform of **unit-sample response**!

Intuitive View of Frequency Response

The frequency response can be an **insightful** description of a system.

Example:

A low-pass filter passes frequencies near 0 and rejects those near π.



Very natural way to describe audio components:

- microphones
- loudspeakers
- audio equalizers

Many other examples in last half of 6.3000.

Example

Find the frequency response of a causal system described by the following:

$$y[n] - \alpha y[n-1] = x[n]$$

A causal system is one in which an input at time $n = n_0$ cannot affect the output at times that are less than $n = n_0$.

Example

Find the frequency response of a causal system described by the following:

$$y[n] - \alpha y[n-1] = x[n]$$

Method 1:

Find the unit-sample response and take its Fourier transform.

$$h[n] - \alpha h[n-1] = \delta[n]$$

Solve the difference equation for $h[n]$.

$$h[n] = \delta[n] + \alpha h[n-1]$$

Causality $\rightarrow h[-1] = 0$

$$\begin{aligned}
h[0] &= \delta[0] + \alpha h[-1] = 1 \\
h[1] &= \delta[1] + \alpha h[0] = \alpha \\
h[2] &= \delta[2] + \alpha h[1] = \alpha^2 \\
h[3] &= \delta[3] + \alpha h[2] = \alpha^3
\end{aligned}$$

$$h[n] = \alpha^n u[n]$$

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\Omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(\alpha e^{-j\Omega}\right)^n = \frac{1}{1-\alpha e^{-j\Omega}}$$

Example

Find the frequency response of a system described by the following:

$$y[n] - \alpha y[n-1] = x[n]$$

Method 2:

Find the response to $e^{j\Omega n}$ directly.

$$x[n] = e^{j\Omega n}$$

Because the system is linear and time-invariant, the output will have the same frequency as the input, but possibly different amplitude and phase.

$$y[n] = H(\Omega)e^{j\Omega n}$$

$$y[n-1] = H(\Omega)e^{j\Omega(n-1)} = H(\Omega)e^{-j\Omega}e^{j\Omega n}$$

Substitute into the difference equation.

$$H(\Omega)e^{j\Omega n} - \alpha H(\Omega)e^{-j\Omega}e^{j\Omega n} = H(\Omega)(1-\alpha e^{-j\Omega})e^{j\Omega n} = e^{j\Omega n}$$

Since $e^{j\Omega n}$ is never 0, we can divide it out.

$$H(\Omega) = \frac{1}{1 - \alpha e^{-j\Omega}}$$

Same answer as method 1.

Example

Find the frequency response of a system described by the following:

$$y[n] - \alpha y[n-1] = x[n]$$

Method 3:

Take the Fourier transform of the difference equation.

$$Y(\Omega) - \alpha e^{-j\Omega} Y(\Omega) = X(\Omega)$$

Solve for $Y(\Omega)$.

$$Y(\Omega) = \frac{1}{1 - \alpha e^{-j\Omega}} X(\Omega)$$

Since $Y(\Omega) = H(\Omega)X(\Omega)$,

$$H(\Omega) = \frac{1}{1 - \alpha e^{-j\Omega}}$$

Same answer as methods 1 and 2.

Check Yourself

Plot the frequency response.

$$H(\Omega) = \frac{1}{1-\alpha e^{-j\Omega}}$$

Assume $0 \leq \alpha \leq 1$.

Which of the following describes the frequency response?

- Low baseband frequencies are amplified.
- High baseband frequencies are amplified.
- All baseband frequencies are amplified.
- Low baseband frequencies are delayed.
- High baseband frequencies are delayed.

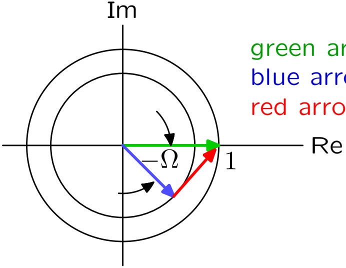
Example

Plot the frequency response.

$$H(\Omega) = \frac{1}{1-\alpha e^{-j\Omega}}$$

Note that denominator is the difference of 2 complex numbers.

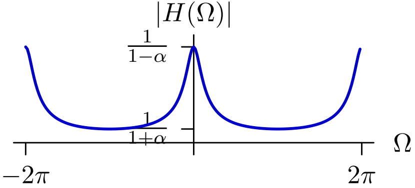
If $0 < \alpha < 1$:

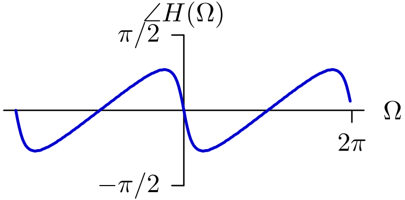


green arrow: 1

blue arrow: $\alpha e^{-j\Omega}$

red arrow: $1-\alpha e^{-j\Omega}$





Check Yourself

Plot the frequency response.

$$H(\Omega) = \frac{1}{1-\alpha e^{-j\Omega}}$$

Assume $0 \le \alpha \le 1$.

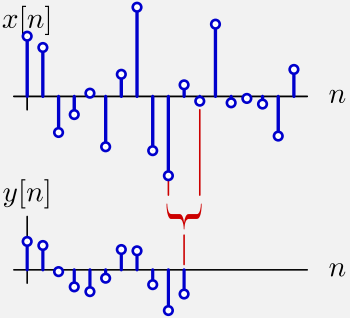
Which of the following describes the frequency response? [1,4,5]

1. Low baseband frequencies are amplified. $\surd$
2. High baseband frequencies are amplified.
3. All baseband frequencies are amplified.
4. Low baseband frequencies are delayed. $\surd$
5. High baseband frequencies are delayed. $\surd$

Check Yourself

Find the frequency response of a three-point averager:

$$y[n] = \frac{1}{3}\Big(x[n-1] + x[n] + x[n+1]\Big)$$



Can we think of this as a low-pass filter?

Does it pass low frequences and block high frequencies?

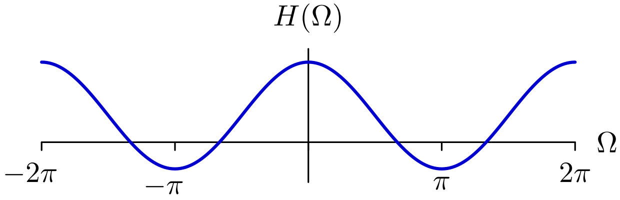
Check Yourself

Find the frequency response of a three-point averager:

$$y[n] = \frac{1}{3}\Big(x[n-1] + x[n] + x[n+1]\Big)$$

$$h[n] = \frac{1}{3}\Big(\delta[n-1] + \delta[n] + \delta[n+1]\Big) = \begin{cases} 1/3 & \text{if } -1 \le n \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} = \frac{1}{3}e^{j\Omega} + \frac{1}{3} + \frac{1}{3}e^{-j\Omega} = \frac{1}{3}\Big(1 + 2\cos(\Omega)\Big)$$



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Check Yourself

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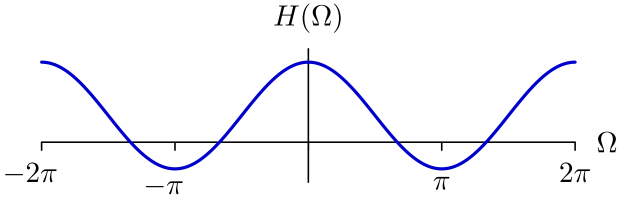


Can we think of this as a low-pass filter?

Does it pass low frequencies and block high frequencies? No

Is there a simple way to make an "averager" that blocks high frequencies?

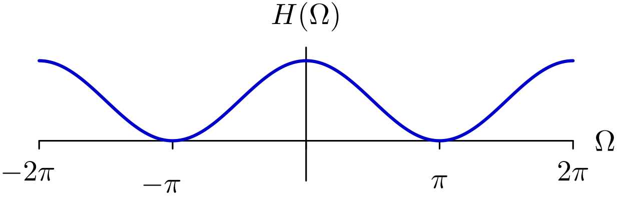
Check Yourself

Find the frequency response of a three-point averager:

$$y[n] = \frac{1}{4}\Big(x[n-1] + 2x[n] + x[n+1]\Big)$$

$$h[n] = \frac{1}{4}\Big(\delta[n-1] + 2\delta[n] + \delta[n+1]\Big)$$

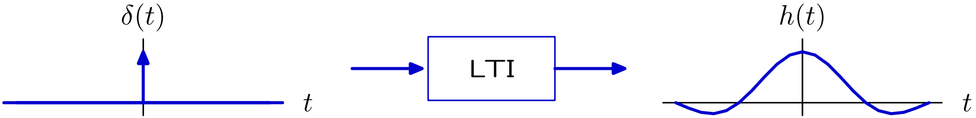
$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} = \frac{1}{2}\Big(1 + \cos(\Omega)\Big)$$



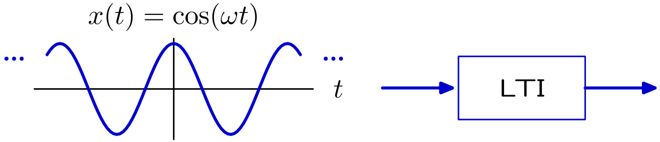
Frequency Response of a Continuous-Time System

Use convolution to characterize the frequency response of a system.

The response of a CT LTI system to the Dirac delta function $\delta(t)$ is the impulse response $h(t)$.



The response $y(t)$ to a sinusoid $x(t) = \cos(\omega t)$ is $y(t) = (x * h)(t)$.



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Frequency Response

Use complex exponentials to characterize the frequency response.

$$x(t) = e^{j\omega t} \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = (x * h)(t)$$

$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} e^{j\omega(t-\tau)}h(\tau)d\tau$$

$$= e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau = H(\omega)\,e^{j\omega t}$$

The response to a complex exponential is a complex exponential with the **same frequency (ω)** but with amplitude and phase given by $H(\omega)$.

$$e^{j\omega t} \longrightarrow \boxed{\text{LTI}} \longrightarrow H(\omega)e^{j\omega t}$$

The map for how a system modifies the amplitude and phase of a complex exponential input is the **Fourier transform of the impulse response.**

Example

Find the frequency response of a system described by the following:

$$y(t) + \alpha \frac{dy(t)}{dt} = 2x(t)$$

Example

Find the frequency response of a system described by the following:

$$y(t) + \alpha \frac{dy(t)}{dt} = 2x(t)$$

Method 1:

Find the response to $e^{j\omega t}$ directly.

$$x(t) = e^{j\omega t}$$

Because the system is linear and time-invariant, the output will have the same frequency as the input, but possibly different amplitude and phase.

$$y(t) = H(\omega)e^{j\omega t}$$

$$\frac{dy(t)}{dt} = j\omega H(\omega)e^{j\omega t}$$

Substitute into the differential equation.

$$H(\omega)e^{j\omega t} + j\omega\alpha H(\omega)e^{j\omega t} = (1 + j\omega\alpha)H(\omega)e^{j\omega t} = 2e^{j\omega t}$$

Since $e^{j\omega t}$ is never 0, we can divide it out.

$$H(\omega) = \frac{2}{1 + j\omega\alpha}$$

Example

Find the frequency response of a system described by the following:

$$y(t) + \alpha \frac{dy(t)}{dt} = 2x(t)$$

Method 2:

Take the Fourier transform of the differential equation.

$$Y(\omega) + j\omega\alpha Y(\omega) = 2X(\omega)$$

Solve for $Y(\omega)$.

$$Y(\omega) = \frac{1}{1 + j\omega\alpha} 2X(\omega)$$

Since $Y(\omega) = H(\omega)X(\omega)$,

$$H(\omega) = \frac{2}{1 + j\omega\alpha}$$

Same answer as method 1.

Check Yourself

Plot the frequency response of the following system:

$$y(t) + \alpha \frac{dy(t)}{dt} = 2x(t)$$

Which of the following describes the frequency response?

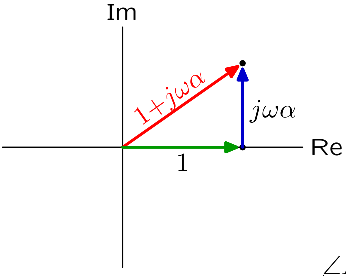
- Low frequencies are attenuated.
- High frequencies are attenuated.
- All frequencies are attenuated.
- Low frequencies are delayed.
- High frequencies are delayed.

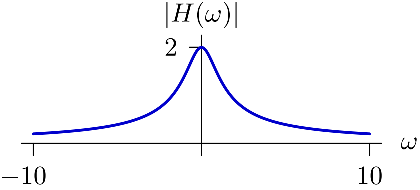
Example

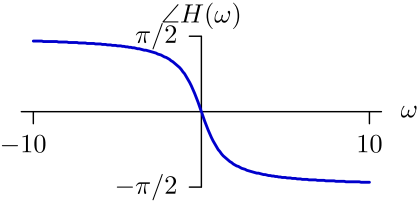
Plot the frequency response of the following system:

$$H(\omega) = \frac{2}{1 + j\omega\alpha}$$

Note that denominator is sum of 2 complex numbers.







Amplifies low frequencies, attenuates high frequencies, adds phase delay.

Check Yourself

Plot the frequency response of the following system:

$$y(t) + \alpha \frac{dy(t)}{dt} = 2x(t)$$

Which of the following describes the frequency response? [2,4,5]

- Low frequencies are attenuated.
- High frequencies are attenuated. $\surd$
- All frequencies are attenuated.
- Low frequencies are delayed. $\surd$
- High frequencies are delayed. $\surd$

Check Yourself

Find the frequency response of a rectangular box averager:

$$y(t) = \frac{1}{2}\int_{t-1}^{t+1} x(\tau)d\tau$$

(This CT averager is analogous to the three-point averager in DT.)

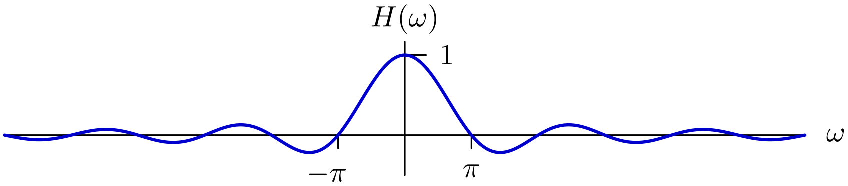
Check Yourself

Find the frequency response of a rectangular box averager:

$$y(t)=\frac{1}{2}\int_{t-1}^{t+1}x(\tau)d\tau$$

$$h(t)=\frac{1}{2}\int_{t-1}^{t+1}\delta(\tau)d\tau=\begin{cases}\frac{1}{2} & \text{if } -1<t<1\\ 0 & \text{otherwise}\end{cases}$$

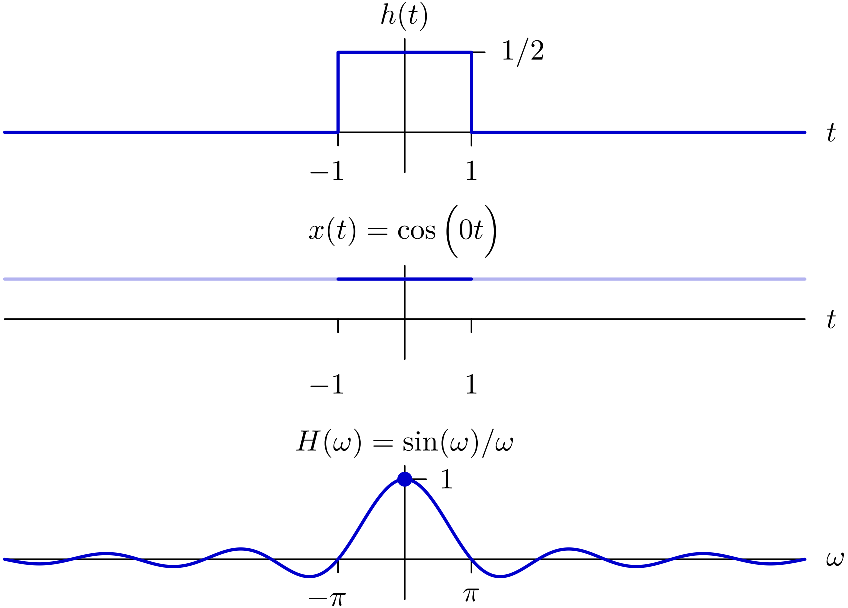
$$H(\omega)=\int_{-\infty}^{\infty}h(t)e^{-j\omega t}dt=\frac{1}{2}\int_{-1}^{1}e^{-j\omega t}dt=\frac{\sin(\omega)}{\omega}$$



As with the three-point averager, high frequencies are attenuated relative to low frequencies, and there is a sign flip for certain frequencies.

Check Yourself

Why is $H(\omega)$ positive for some frequencies and negative for others?



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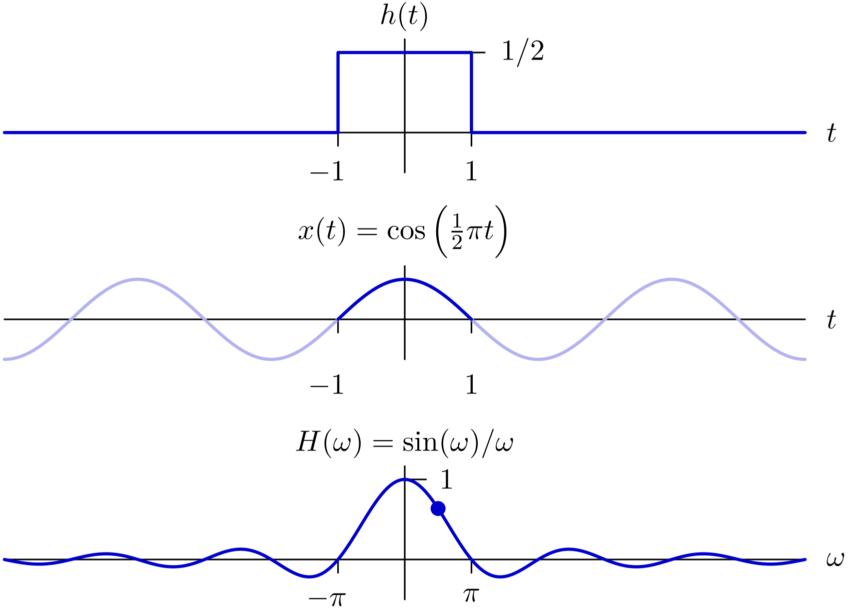

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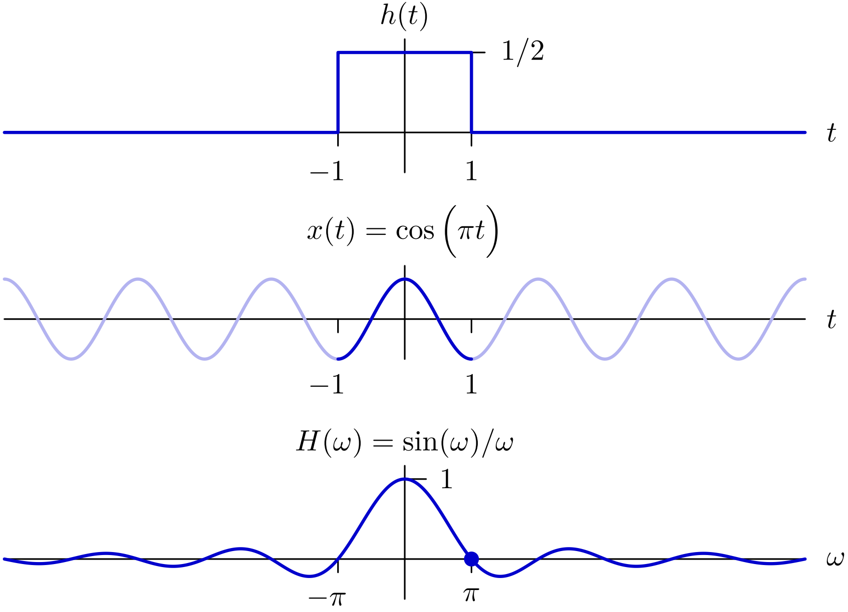

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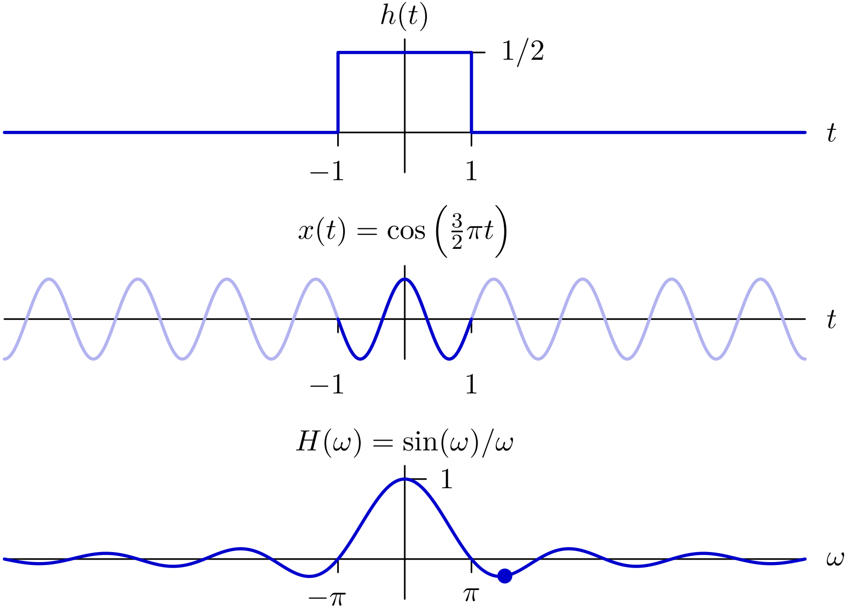

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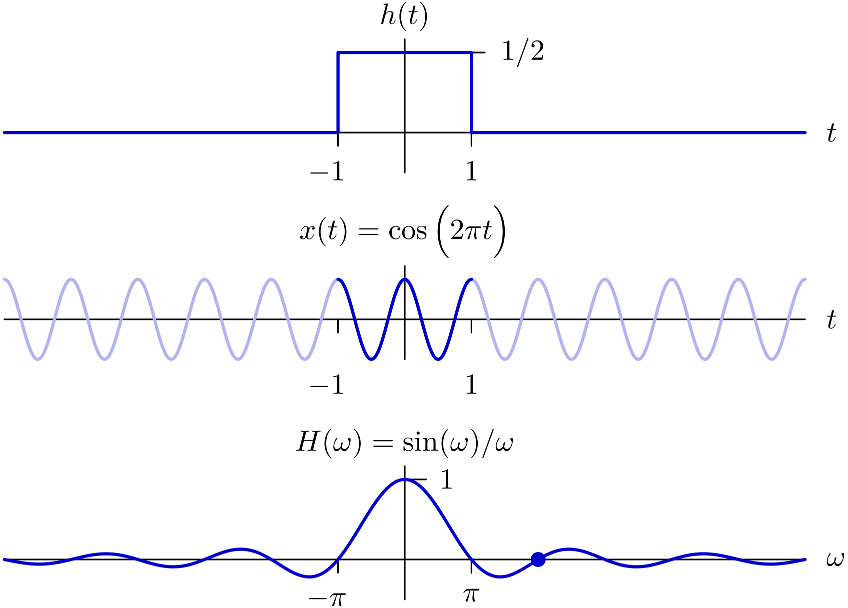


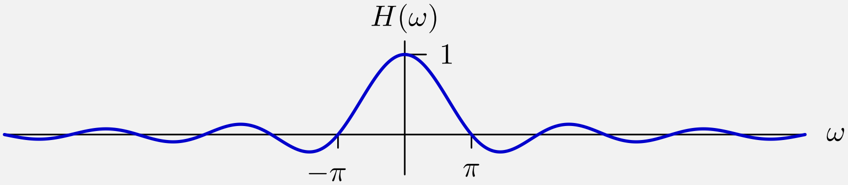
Check Yourself

Find the frequency response of a rectangular box averager:

$$y(t) = \frac{1}{2}\int_{t-1}^{t+1} x(\tau)d\tau$$

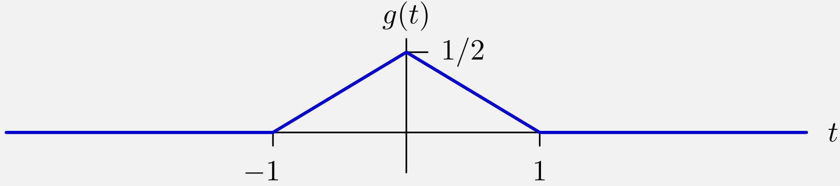
$$h(t) = \frac{1}{2}\int_{t-1}^{t+1} \delta(\tau)d\tau = \begin{cases} \frac{1}{2} & \text{if } -1 < \tau < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$H(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = \frac{1}{2}\int_{-1}^{1} e^{-j\omega t}dt = \frac{\sin(\omega)}{\omega}$$



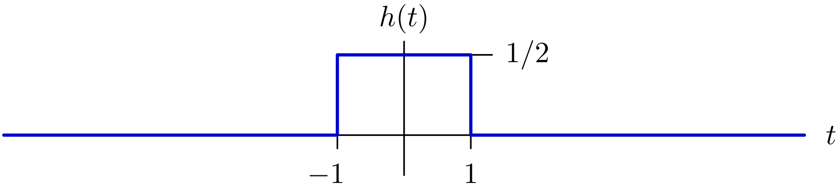
Check Yourself

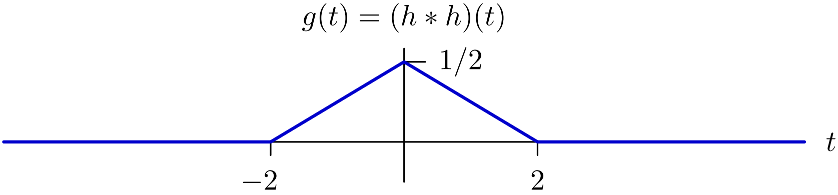
Find the frequency response of a triangular averager:



Check Yourself

The triangular averager $g(t)$ can be expressed as the cascade of two rectangular averagers $h(t)$.





Check Yourself

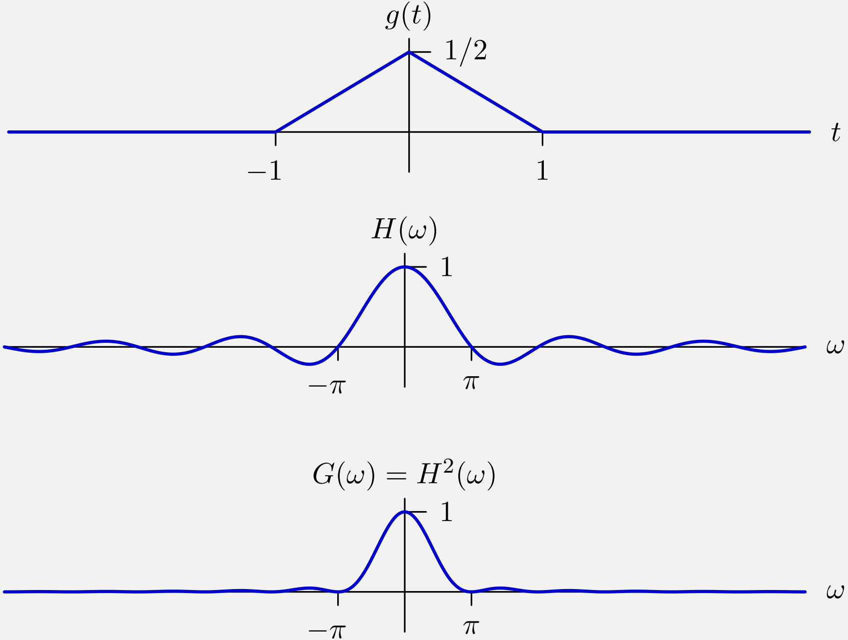
Convolution in time is equivalent to multiplication in frequency.

$$g(t) = (f * f)(t) = \int f(t-\tau)f(\tau)d\tau$$

$$\begin{aligned}
G(\omega) &= \int g(t)e^{-j\omega t}dt \\
&= \int_t \underbrace{\int_\tau f(t-\tau)f(\tau)d\tau}_{g(t)}\, e^{-j\omega t}dt \\
&= \int_\tau f(\tau) \underbrace{\int_t f(t-\tau)e^{-j\omega t}dt}_{e^{-j\omega\tau}F(\omega)}\, d\tau \\
&= F(\omega) \underbrace{\int_\tau f(\tau)e^{-j\omega\tau}d\tau}_{F(\omega)} \\
&= F^2(\omega)
\end{aligned}$$

Check Yourself

Find the frequency response of a triangular averager:

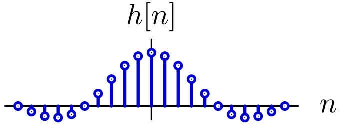


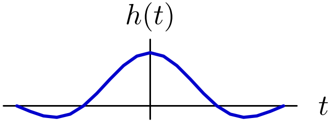
Time and Frequency Representations

Two complete representations for linear, time-invariant systems.

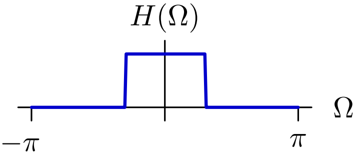
$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]$$

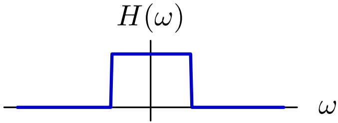
Convolution: responses calculated in the time domain.





Filtering: responses calculated in the frequency domain.





The representation in **frequency** is the Fourier transform of that in **time**!