6.3000: Signal Processing

Discrete-Time Fourier Transform

- Definition
- Examples
- Properties
- Relations between Fourier series and transforms (DT and CT)

Quiz 1: October 3, 2-4pm, room 50-340 (Walker).

- Closed book except for one page of notes (8.5"x11" both sides).
- No electronic devices. (No headphones, cellphones, calculators, ...)
- Coverage up to and including classes on September 21 and HW 3.

We have posted a practice quiz as a study aid for the upcoming quiz 1.

- Your solutions will not be submitted or counted in your grade.
- Solutions will be posted on Friday.

There is no HW 4.

If you have personal or medical difficulties, please contact S^3 and/or `6.3000-instructors@mit.edu` for accommodations.

September 28, 2023

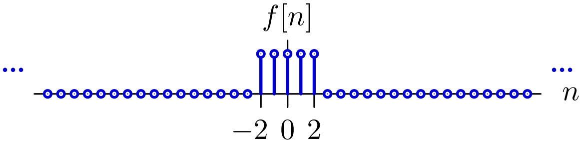
From Periodic to Aperiodic

Last time: representing arbitrary (aperiodic) CT signals as sums of sinusoidal components using the continuous-time Fourier transform.

Today: generalize the Fourier transform idea to discrete-time signals.

Fourier Representions of Aperiodic Signals

How can we represent an aperiodic signal as a sum of sinusoids?



Fourier Representations of Aperiodic Signals

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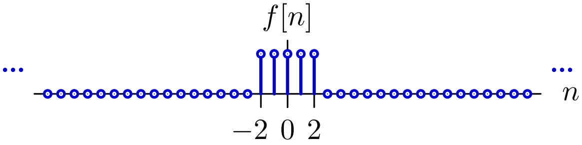
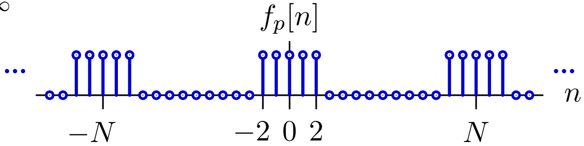


Strategy: make a periodic version of $f[n]$ by summing shifted copies:

$$f_p[n] = \sum_{m=-\infty}^{\infty} f[n - mN]$$



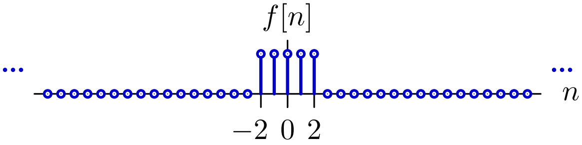
Since $f_p[n]$ is periodic, it has a Fourier series (which depends on N).

Find Fourier series coefficients $F_p[k]$ and take the limit of $F_p[k]$ as $N \to \infty$.

As $N \to \infty$, $f_p[n] \to f[n]$, and Fourier series will approach Fourier transform.

Fourier Representations of Aperiodic Signals

Example.



Strategy: make a periodic version of $f[n]$ by summing shifted copies:

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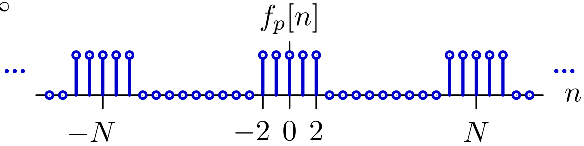

Calculate the Fourier series coefficients $F_p[k]$:

$$F_p[k] = \frac{1}{N} \sum_{n=<N>} f_p[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} + \frac{2}{N} \cos\frac{2\pi k}{N} + \frac{2}{N} \cos\frac{4\pi k}{N}$$

Fourier Representions of Aperiodic Signals

Calculate the Fourier series coefficients $F_p[k]$:

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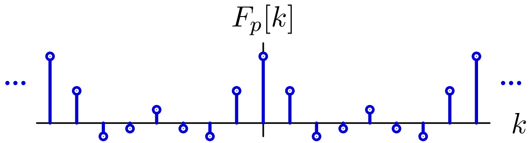
Plot the resulting Fourier coefficients for $N=8$.



What happens if you double the period N?

Fourier Representions of Aperiodic Signals

Calculate the Fourier series coefficients $F_p[k]$:

$$F_p[k] = \frac{1}{N} \sum_{n=\langle N \rangle} f_p[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} + \frac{2}{N}\cos\frac{2\pi k}{N} + \frac{2}{N}\cos\frac{4\pi k}{N}$$

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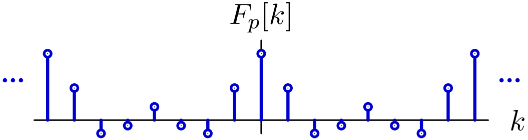
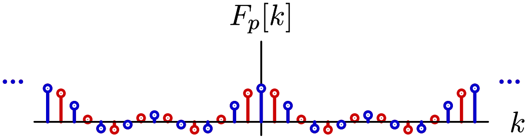


What happens if you double the period N? Make a plot for $N=16$.



There are now twice as many samples per period. (The red samples are at new intermediate frequencies.) The amplitude is halved.

Fourier Representions of Aperiodic Signals

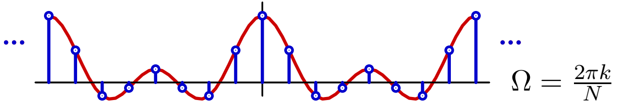
Define a new function $F(\Omega) = NF_p[k]$ where $\Omega = k\Omega_o = 2\pi k/N$.

$$NF_p[k] = 1 + 2\cos\frac{2\pi k}{N} + 2\cos\frac{4\pi k}{N} = 1 + 2\cos(\Omega) + 2\cos(2\Omega) = F(\Omega)\Big|_{\Omega=\frac{2\pi k}{N}}$$

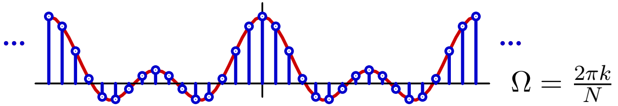
Then $NF_p[k]$ represents samples of $F(\Omega)$ with increasing resolution in Ω.

$$NF_p[k] = F(\Omega)$$

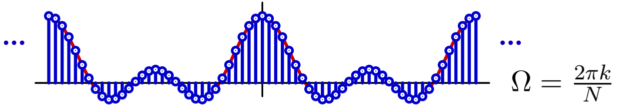
N=8:



N=16:



N=32:



The discrete function $NF_p[k]$ is a sampled version of the function $F(\Omega)$.

Fourier Representations of Aperiodic Signals

From $f(t)$ to $F(\omega)$:

$$f[n] \underbrace{\rightarrow}_{\substack{\text{periodic}\\ \text{extension}}} f_p[n] \underbrace{\rightarrow}_{\substack{\text{Fourier}\\ \text{series}}} F_p[k] \underbrace{\rightarrow}_{\text{interpolation}} F(\Omega)$$

The limiting behaviors as $N \to \infty$ define the Fourier transform:

$$\begin{aligned}
F(\Omega) &= \lim_{N\to\infty} N F_p[k]\Big|_{k=\frac{N}{2\pi}\Omega} \\
&= \lim_{N\to\infty} N \left[\frac{1}{N} \sum_{n=<N>} f_p[n] e^{-j\frac{2\pi k}{N}n}\right]_{k=\frac{N}{2\pi}\Omega} \\
&= \lim_{N\to\infty} \sum_{n=<N>} f_p[n] e^{-j\Omega n} \\
F(\Omega) &= \sum_{n=-\infty}^{\infty} f[n] e^{-j\Omega n}
\end{aligned}$$

This **analysis equation** defines the Fourier transform.

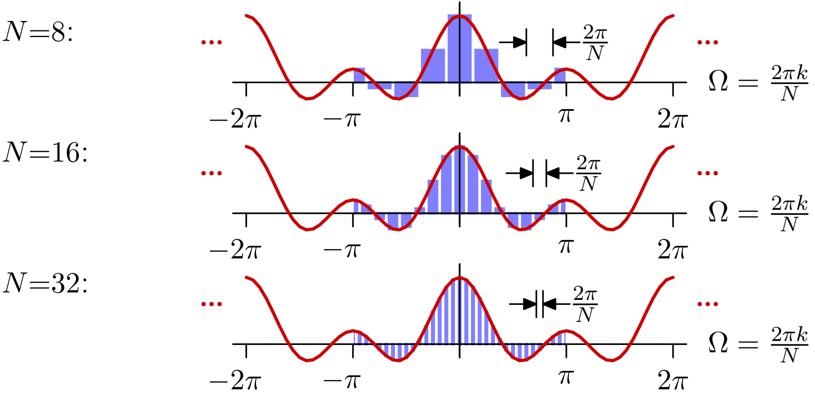
Fourier Representations of Aperiodic Signals

The **synthesis equation** follows from piecewise constant approximation.

$$f[n] = \lim_{N\to\infty} f_p[n] = \lim_{N\to\infty} \sum_{k=\langle N\rangle} F_p[k] e^{j\frac{2\pi}{N}kn}$$

$$= \lim_{N\to\infty} \left(\frac{1}{2\pi}\right) \sum_{k=\langle N\rangle} N F_p[k] e^{j\frac{2\pi}{N}kn} \left(\frac{2\pi}{N}\right) = \frac{1}{2\pi}\int_{2\pi} F(\Omega) e^{j\Omega n} d\Omega$$

$$N F_p[k] = F(\Omega)$$



Fourier Transform relation: $f[n] \overset{\mathrm{FT}}{\Longrightarrow} F(\Omega)$

Fourier Series and Fourier Transform

Fourier series and transforms are similar:
both represent signals by their frequency content.

Discrete-Time Fourier Series

$$F[k] = F[k+N] = \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk\Omega_o n}$$

analysis equation

$$f[n] = f[n+N] = \sum_{k=\langle N \rangle} F[k] e^{jk\Omega_o n}$$

synthesis equation

where $\Omega_o = \frac{2\pi}{N}$

Discrete-Time Fourier Transform

$$F(\Omega) = F(\Omega+2\pi) = \sum_{n=-\infty}^{\infty} f[n] e^{-j\Omega n}$$

analysis equation

$$f[n] = \frac{1}{2\pi} \int_{2\pi} F(\Omega) e^{j\Omega n}\, d\Omega$$

synthesis equation

Fourier Series and Fourier Transform

All of the information in a periodic signal is contained in one period.

The information in an aperiodic signal can spread across all time.

Discrete-Time Fourier Series

$$F[k] = F[k+N] = \frac{1}{N}\sum_{n=\langle N\rangle} f[n]e^{-jk\Omega_o n}$$ **analysis equation**

$$f[n] = f[n+N] = \sum_{k=\langle N\rangle} F[k]e^{jk\Omega_o n}$$ **synthesis equation**

where $\Omega_o = \frac{2\pi}{N}$

Discrete-Time Fourier Transform

$$F(\Omega) = F(\Omega+2\pi) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\Omega n}$$ **analysis equation**

$$f[n] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega n}\,d\Omega$$ **synthesis equation**

Fourier Series and Fourier Transform

Periodic signals can be synthesized from a discrete set of k harmonics. Aperiodic signals generally require a continuous set of frequencies Ω.

Discrete-Time Fourier Series

$$F[k] = F[k+N] = \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk\Omega_o n}$$

analysis equation

$$f[n] = f[n+N] = \sum_{k=\langle N \rangle} F[k] e^{jk\Omega_o n}$$

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where $\Omega_o = \frac{2\pi}{N}$

Discrete-Time Fourier Transform

$$F(\Omega) = F(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} f[n] e^{-j\Omega n}$$

analysis equation

$$f[n] = \frac{1}{2\pi} \int_{2\pi} F(\Omega) e^{j\Omega n} \, d\Omega$$

synthesis equation

Fourier Series and Fourier Transform

Harmonic frequencies $k\Omega_o$ are samples of continuous frequency Ω.

Discrete-Time Fourier Series

$$F[k] = F[k+N] = \frac{1}{N}\sum_{n=\langle N\rangle} f[n]e^{-jk\Omega_o n}$$

analysis equation

$$f[n] = f[n+N] = \sum_{k=\langle N\rangle} F[k]e^{jk\Omega_o n}$$

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where $\Omega_o = \frac{2\pi}{N}$

Discrete-Time Fourier Transform

$$F(\Omega) = F(\Omega+2\pi) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\Omega n}$$

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$$f[n] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega n}\,d\Omega$$

synthesis equation

CT and DT Fourier Transforms

DT frequencies alias because adding 2π to Ω does not change $e^{-j\Omega n}$.

Continuous-Time Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$ **analysis equation**

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$ **synthesis equation**

Discrete-Time Fourier Transform

$$F(\Omega) = F(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} f[n] e^{-j\Omega n}$$ **analysis equation**

$$f[n] = \frac{1}{2\pi} \int_{2\pi} F(\Omega) e^{j\Omega n} d\Omega$$ **synthesis equation**

CT and DT Fourier Transforms

DT frequencies alias because adding 2π to Ω does not change $e^{-j\Omega n}$.
Since $F(\Omega)$ is periodic in 2π, we need only integrate $d\Omega$ over a 2π interval.

Continuous-Time Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

analysis equation

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

synthesis equation

Discrete-Time Fourier Transform

$$F(\Omega) = F(\Omega+2\pi) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\Omega n}$$

analysis equation

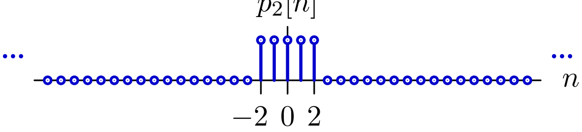
$$f[n] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega n}\,d\Omega$$

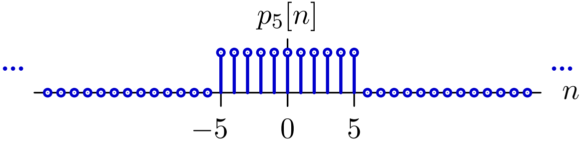
synthesis equation

Examples of Fourier Transforms

Find the Fourier Transform (FT) of a rectangular pulse of width $2S+1$:

$$p_S[n] = \begin{cases} 1 & -S \le N \le S \\ 0 & \text{otherwise} \end{cases}$$





$$P_S(\Omega) = \sum_{n=-\infty}^{\infty} p_S[n]e^{-j\Omega n} = \sum_{n=-S}^{S} e^{-j\Omega n}$$

$$= e^{j\Omega S} + e^{j\Omega(S-1)} + \cdots + 1 + \cdots + e^{-j\Omega(S-1)} + e^{-j\Omega S}$$

Close the sum to better identify trends across S.

Working with Sums

Closed form sums of geometric sequences.

$$A = \sum_{n=0}^{N-1} \alpha^n$$

If the series has finite length (here N terms), it will converge for finite α.

$$\begin{aligned}
A &= 1 + \alpha + \alpha^2 + \cdots + \alpha^{N-1} \\
\alpha A &= \phantom{1 +{}} \alpha + \alpha^2 + \cdots + \alpha^{N-1} + \alpha^N \\
A - \alpha A &= 1 \phantom{{}+ \alpha + \alpha^2 + \cdots + \alpha^{N-1}} - \alpha^N \\
A &= \begin{cases} \dfrac{1-\alpha^N}{1-\alpha} & \text{if } \alpha \neq 1 \\ N & \text{if } \alpha = 1 \end{cases}
\end{aligned}$$

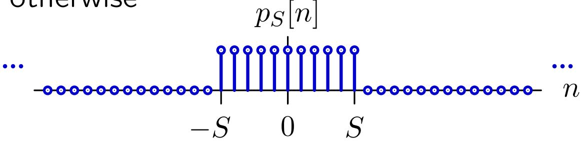
If the series has infinite length, it will converge if $|\alpha| < 1$.

$$\sum_{n=0}^{\infty} \alpha^n = \lim_{N\to\infty} \sum_{n=0}^{N-1} \alpha^n = \lim_{N\to\infty} \frac{1-\alpha^N}{1-\alpha} = \frac{1}{1-\alpha} \quad \text{if } |\alpha| < 1$$

Examples of Fourier Transforms

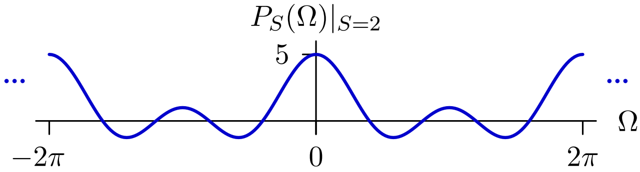
Find the Fourier Transform (FT) of a rectangular pulse of width $2S+1$:

$$p_S[n] = \begin{cases} 1 & -S \le n \le S \\ 0 & \text{otherwise} \end{cases}$$



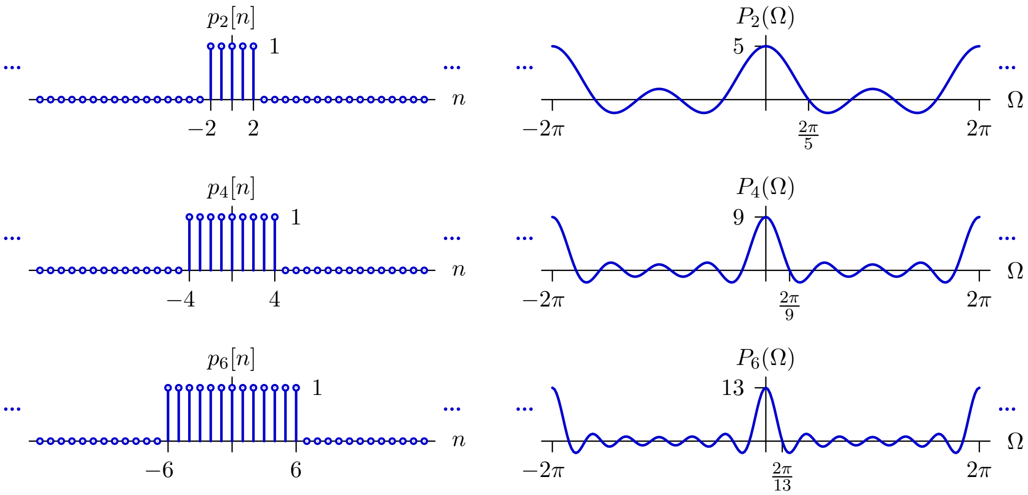
$$P_S(\Omega) = \sum_{n=-S}^{S} e^{-j\Omega n} = \left(e^{j\Omega S}\right) \sum_{n=0}^{2S} e^{-j\Omega n} = \left(\frac{e^{j\Omega(S+\frac{1}{2})}}{e^{j\Omega/2}}\right) \frac{1 - e^{-j\Omega(2S+1)}}{1 - e^{-j\Omega}}$$

$$= \left(\frac{e^{j\Omega(S+\frac{1}{2})} - e^{-j\Omega(S+\frac{1}{2})}}{e^{j\Omega/2} - e^{-j\Omega/2}}\right) = \frac{\sin\left(\Omega(S+\frac{1}{2})\right)}{\sin\left(\Omega/2\right)}$$



Examples of Fourier Transforms

Compare Fourier transforms of pulses with different widths.

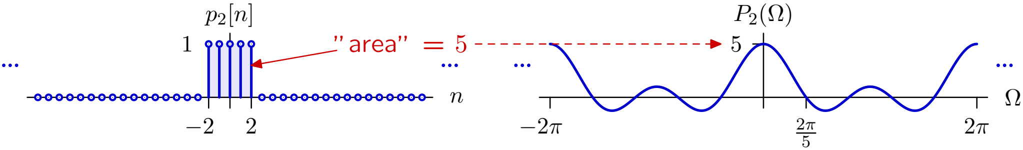


As the function widens in n the Fourier transform narrows in Ω.

Areas

Similar to CT, the value of $F(\Omega)$ at $\Omega = 0$ is the sum of $f[n]$ over time t.

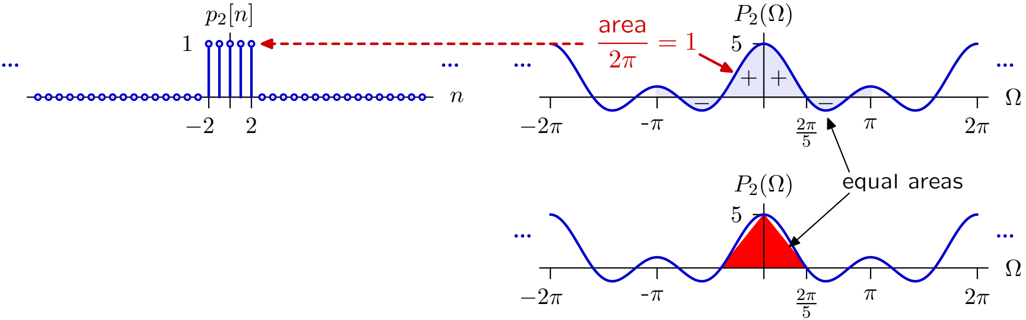
$$F(0) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} f[n]$$



Areas

The value of $f[0]$ is $\frac{1}{2\pi}$ times the integral of $F(\Omega)$ over $\Omega = [-\pi, \pi]$.

$$f[0] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)\, e^{j\Omega n} d\Omega = \frac{1}{2\pi}\int_{2\pi} F(\Omega)\, d\Omega$$



Very similar to analogous relations for CT Fourier transforms.

Extreme Cases

The Fourier transform of the shortest possible CT signal $f(t) = \delta(t)$ is the widest possible CT transform $F(\omega) = 1$.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t} = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega 0} = 1$$

A similar result holds in DT.

$$F(\Omega) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\Omega 0} = 1$$

Extreme Cases

The Fourier transform of the widest possible CT signal $f(t) = 1$ is the narrowest possible CT transform $F(\omega) = 2\pi\delta(\omega)$.[1]

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega)e^{-j\omega t}d\omega = \int_{-\infty}^{\infty}\delta(\omega)e^{-j0t}d\omega = 1$$

A similar result holds in DT.

$$f[n] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{-j\Omega n}d\Omega = \frac{1}{2\pi}\int_{2\pi} 2\pi\delta(\Omega)e^{-j\Omega n}d\Omega = \int_{2\pi}\delta(\Omega)e^{-j0n}d\Omega = 1$$

[1] the factor of 2π is needed to cancel the 2π in the synthesis equation.

More Math With Impulses

A similar construction reveals the transform of a complex exponential.

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_o)e^{j\omega t}d\omega \\ &= \int_{-\infty}^{\infty} \delta(\omega-\omega_o)e^{j\omega_o t}d\omega \\ &= e^{j\omega_o t}\int_{-\infty}^{\infty} \delta(\omega-\omega_o)d\omega \\ &= e^{j\omega_o t} \end{aligned}$$

Thus, the Fourier transform of a complex exponential is a delta function at the frequency of the complex exponential:

$$e^{j\omega_o t} \quad \stackrel{\text{CTFT}}{\Longrightarrow} \quad 2\pi\delta(\omega-\omega_o)$$

The impulse in frequency has infinite value at $\omega = \omega_o$ and is zero at all other frequencies.

More Math With Impulses

A similar construction reveals the transform of a complex exponential.

$$\begin{aligned}
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_o)e^{j\omega t}d\omega \\
&= \int_{-\infty}^{\infty} \delta(\omega-\omega_o)e^{j\omega_o t}d\omega \\
&= e^{j\omega_o t}\int_{-\infty}^{\infty} \delta(\omega-\omega_o)d\omega \\
&= e^{j\omega_o t}
\end{aligned}$$

It's important to notice that I worked backwards in this argument. I knew the general form of the result. The signal had to be infinite at $\omega-\omega_o$ and zero elsewhere – so I guessed that the transform would be a delta function.

Going forward would have required integrating an infinite signal. Closing such an integral is very difficult. Going backwards was much easier.

More Math With Impulses

A similar construction applies in DT.

$$\begin{aligned}
f[n] &= \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega n}d\Omega \\
&= \frac{1}{2\pi}\int_{2\pi} 2\pi\delta(\Omega-\Omega_o)e^{j\Omega n}d\Omega \\
&= \int_{2\pi} \delta(\Omega-\Omega_o)e^{j\Omega_o n}d\Omega \\
&= e^{j\Omega_o n}\int_{2\pi} \delta(\Omega-\Omega_o)d\Omega \\
&= e^{j\Omega_o n}
\end{aligned}$$

Thus, the Fourier transform of a complex exponential is a delta function at the frequency of the complex exponential:

$$e^{j\Omega_o n} \quad \overset{\text{DTFT}}{\Longrightarrow} \quad 2\pi\delta(\Omega-\Omega_o)$$

The impulse in frequency shows that the transform is infinite at $\Omega = \Omega_o$ and is zero at all other frequencies.

More Math With Impulses

The integral on Ω in DT covers just a 2π interval. That integral does not specify the Fourier transform outside that interval.

$$f[n] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega n}d\Omega = e^{j\Omega_o n}$$

What is the value of $F(\Omega)$ outside the 2π interval used in the integral?

More Math With Impulses

The integral on Ω in DT covers just a 2π interval. That integral does not specify the Fourier transform outside that interval.

$$f[n] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega n}d\Omega = e^{j\Omega_o n}$$

What is the value of $F(\Omega)$ outside the 2π interval used in the integral?

All DT Fourier transforms are periodic in 2π, as seen from the definition:

$$F(\Omega) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\Omega n}$$

The only function of Ω on the right-hand side is the complex exponential, and it is periodic in 2π (regardless of n). A sum of terms that are each periodic in 2π is periodic in 2π.

Thus the previous expression

$$e^{j\Omega_o n} \quad \overset{\text{DTFT}}{\Longrightarrow} \quad 2\pi\delta(\Omega-\Omega_o)$$

was incomplete. A better expression is the following:

$$e^{j\Omega_o n} \quad \overset{\text{DTFT}}{\Longrightarrow} \quad \sum_{m=-\infty}^{\infty} 2\pi\delta(\Omega-\Omega_o-2\pi m)$$

Relations Between Fourier Series and Fourier Transforms

Continuous Time:

$$e^{j\omega_o t} \quad \overset{\text{CTFT}}{\Longrightarrow} \quad 2\pi\delta(\omega-\omega_o)$$

$$f(t) = f(t+T) \quad \overset{\text{CTFS}}{\Longrightarrow} \quad F[k]$$

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} F[k]e^{j\omega_o t} \quad \overset{\text{CTFT}}{\Longrightarrow} \quad \sum_{k=-\infty}^{\infty} 2\pi F[k]\delta(\omega-\omega_o)$$

Discrete Time:

$$e^{j\Omega_o n} \quad \overset{\text{DTFT}}{\Longrightarrow} \quad \sum_{m=-\infty}^{\infty} 2\pi\delta(\Omega-\Omega_o-2\pi m)$$

$$f[n] = f[n+N] \quad \overset{\text{DTFS}}{\Longrightarrow} \quad F[k]$$

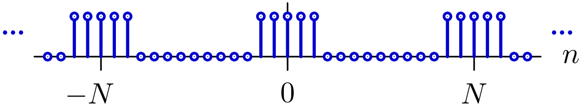
$$f[n] = f[n+N] = \sum_{k=\langle N\rangle} F[k]e^{j\Omega_o n} \quad \overset{\text{DTFT}}{\Longrightarrow} \quad \sum_{k=\langle N\rangle} \sum_{m=-\infty}^{\infty} 2\pi F[k]\delta(\Omega-\Omega_o-2\pi m)$$

Periodic DT signals that have Fourier series representations also have Fourier transform representations.

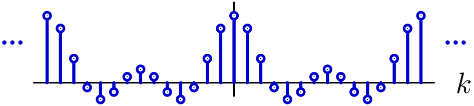
Relations Between Fourier Series and Fourier Transforms

Each Fourier series term is replaced by an impulse in the Fourier transform.

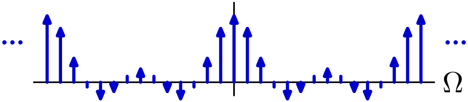
$$f[n] = f[n-N] \text{ for all integers } n$$



$$F[k] = F[k-N] \text{ for all integers } k$$



$$F(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi F[k]\delta\left(\Omega - k\frac{2\pi}{N}\right)$$



Periodic DT signals that have Fourier series representations also have Fourier transform representations.

Summary

Discrete-Time Fourier Transform

- Definition
- Examples
- Properties
- Relations between Fourier series and transforms (DT and CT)