6.3000: Signal Processing

Discrete-Time Fourier Series

- Fourier series representations for discrete-time signals
- Comparison of Fourier series for CT and DT signals
- Properties of DT Fourier series
- Applications of Fourier analysis

September 21, 2023

Recall: Continuous-Time Fourier Series

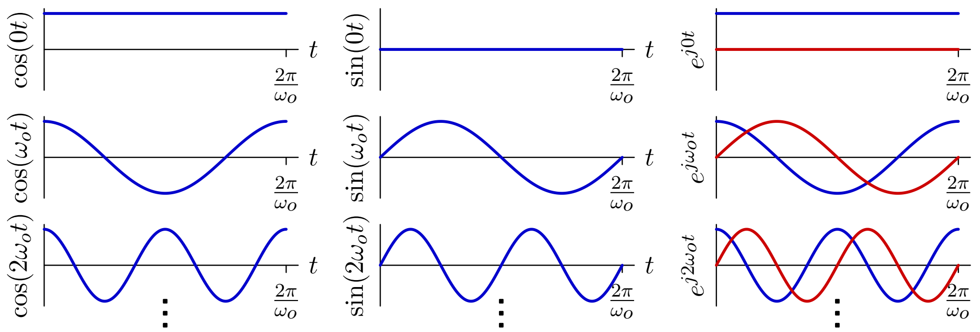
Only **periodic** signals can be represented by Fourier series.

$$f(t) = f(t+T) \;=\; c_0 + \sum_{k=1}^{\infty} c_k \cos k\omega_o t + \sum_{k=1}^{\infty} d_k \sin k\omega_o t \;=\; \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

where $\omega_o = \frac{2\pi}{T}$ represents the fundamental frequency.

Real-valued basis functions

Complex basis functions



What is the equivalent constraint for discrete-time signals?

Check Yourself

What is the fundamental (shortest) period of each of the following DT signals?

1. $f_1[n] = \cos\left(\frac{\pi n}{12}\right)$
2. $f_2[n] = \cos\left(\frac{\pi n}{12}\right) + 3\cos\left(\frac{\pi n}{15}\right)$
3. $f_3[n] = \cos(n)$

Check Yourself

$$f_1[n] = \cos\left(\frac{\pi n}{12}\right) = f_1[n+N] = \cos\left(\frac{\pi n}{12} + \frac{\pi N}{12}\right)$$

then

$$\frac{\pi N}{12} = 2\pi m$$

where both N and m are integers. Solving, we find

$$N = \frac{24\pi m}{\pi}$$

which is 24 if $m = 1$. Therefore $N = 24$.

Check Yourself

Similarly if

$$f_2[n] = \cos\left(\frac{\pi n}{12}\right) + 3\cos\left(\frac{\pi n}{15}\right) = f_2[n+N]$$

then

$$\frac{\pi N}{12} = 2\pi m_1 \quad \rightarrow N = \frac{24\pi m_1}{\pi}$$

and

$$\frac{\pi N}{15} = 2\pi m_2 \quad \rightarrow N = \frac{30\pi m_2}{\pi}$$

for integers m_1 and m_2.

We seek the smallest possible value of N so

$$N = 24m_1 = 2 \times 2 \times 2 \times 3 \times m_1 = 30m_2 = 2 \times 3 \times 5 \times m_2$$

and the smallest possible N is $2 \times 2 \times 2 \times 3 \times 5 = 120$.

Check Yourself

If

$$f_3[n] = \cos(n) = f_3[n+N]$$

then

$$N = 2\pi m$$

where both N and m are integers.

This is not possible since π is not rational.

Therefore $f_3[n]$ is not periodic in n.

Check Yourself

What is the fundamental (shortest) period of each of the following DT signals?

1. $f_1[n] = \cos\left(\frac{\pi n}{12}\right)$ 24
2. $f_2[n] = \cos\left(\frac{\pi n}{12}\right) + 3\cos\left(\frac{\pi n}{15}\right)$ 120
3. $f_3[n] = \cos(n)$ ∞

Check Yourself

What is the fundamental (shortest) period of each of the following DT signals?

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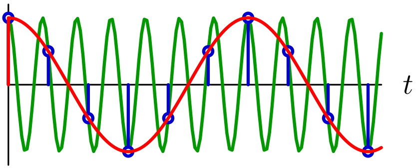
2. $f_2[n] = \cos\left(\frac{\pi n}{12}\right) + 3\cos\left(\frac{\pi n}{15}\right)$ 120
3. $f_3[n] = \cos(n)$ ∞

The period of a periodic DT signal **must be an integer**.

Combined with aliasing, this constraint drastically reduces both the number of possible DT series and the complexity of each of those series.

Aliasing

Recall that the same sequence of samples can result when a CT sinusoid is sampled at integer multiples of the sampling interval Δ.



Samples (blue) of the original high-frequency signal (green) could just as easily have come from a much lower frequency signal (red).

Discrete-Time Sinusoids

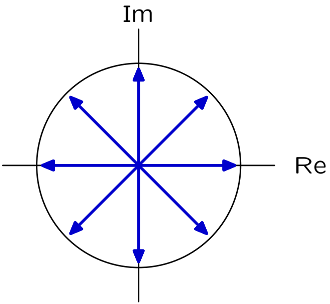
There are (only) N distinct complex exponentials with integer period N.

If $f[n] = e^{j\Omega n}$ is periodic in N then

$$f[n] = e^{j\Omega n} = f[n{+}N] = e^{j\Omega(n+N)} = e^{j\Omega n} e^{j\Omega N}$$

and $e^{j\Omega N}$ must be 1. Therefore $e^{j\Omega}$ must be one of the N^{th} roots of 1.

Example: $N = 8$



There are only 8 distinct complex exponentials with period $N = 8$:

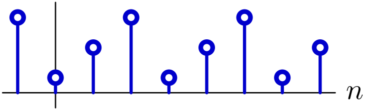
$$e^{j0\pi/4},\quad e^{j1\pi/4},\quad e^{j2\pi/4},\quad e^{j3\pi/4},\quad e^{j4\pi/4},\quad e^{j5\pi/4},\quad e^{j6\pi/4},\quad e^{j7\pi/4}.$$

There are an infinite number of complex exponentials with period T in CT!

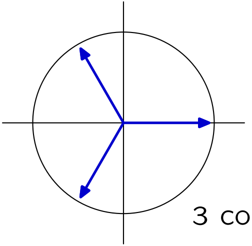
Discrete-Time Sinusoids

There are N distinct complex exponentials with period N.

Example: periodic in N=3

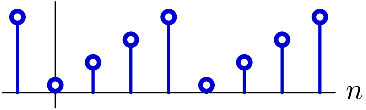


3 samples repeated in time

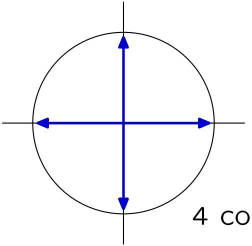


3 complex exponentials

Example: periodic in N=4



4 samples repeated in time



4 complex exponentials

If a DT signal is periodic with period N,
then its Fourier series will contain just N terms.

Discrete Time Fourier Series

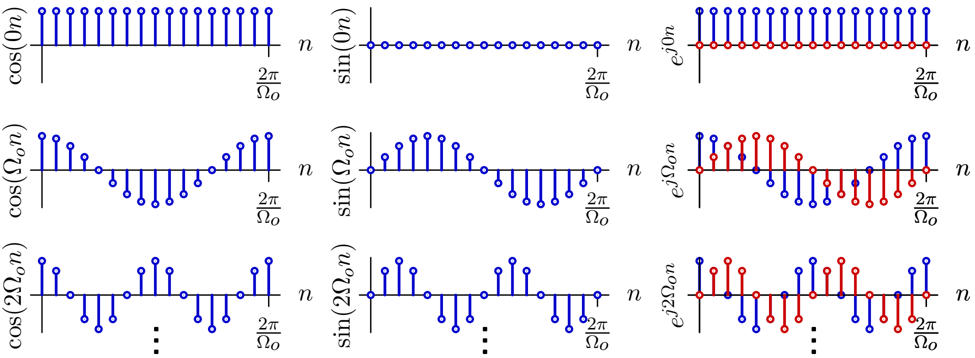
A DT Fourier Series has just N harmonic frequencies $k\Omega_o$.

$$f[n] = f[n+N] = \sum_{k=\langle N \rangle} c_k \cos(k\Omega_o n) + \sum_{k=\langle N \rangle} d_k \sin(k\Omega_o n) = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_o n}$$

where Ω_o represents the fundamental frequency **(radians/sample)**.
Otherwise, DT Fourier series are similar to CT Fourier series.

Real-valued basis functions

Complex basis functions



Recall: Continuous-Time Fourier Series

We found the Fourier series coefficients using two key insights.

1. Multiplying complex harmonics of ω_o yields a complex harmonic of ω_o:

$$e^{jk\omega_o t} \times e^{jl\omega_o t} = e^{j(k+l)\omega_o t}$$

2. Integrating a complex harmonic over a period T yields zero unless the harmonic is at DC:

$$\int_{t_0}^{t_0+T} e^{jk\omega_o t} dt \equiv \int_T e^{jk\omega_o t} dt = \begin{cases} T & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$
$$= T\delta[k]$$

where $\delta[k]$ is the Kronecker delta function

$$\delta[k] = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow$ Fourier components are **orthogonal**.

Discrete-Time Fourier Series

The same two key insights apply to DT analysis.

1. Multiplying complex **DT** harmonics of Ω_o yields a new harmonic of Ω_o:

$$e^{jk\Omega_o n} \times e^{jl\Omega_o n} = e^{j(k+l)\Omega_o n}$$

2. **Summing** a complex harmonic over a period N is zero unless the harmonic is at DC:

$$\sum_{n=n_0}^{n_0+N-1} e^{jk\Omega_o n} \equiv \sum_{n=\langle N\rangle} e^{jk\Omega_o n} = \begin{cases} N & \text{if } k=0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

$$= N\delta[k]$$

$\rightarrow$ **DT** Fourier components are **orthogonal**.

Recall: Continuous-Time Fourier Series

Use orthogonality to find the Fourier series coefficients.

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

Multiply $f(t)$ by the complex conjugate of the basis function of interest, and then integrate over T.

$$\begin{aligned}
\int_T f(t) e^{-jl\omega_o t} dt &= \int_T \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t} \right) e^{-jl\omega_o t} dt \\
&= \sum_{k=-\infty}^{\infty} a_k \int_T e^{j(k-l)\omega_o t} dt \\
&= \sum_{k=-\infty}^{\infty} a_k T \delta[k-l] = a_l T
\end{aligned}$$

Solving for a_l and then substituting k for l yields

$$a_k = \frac{1}{T} \int_T f(t) e^{-jk\omega_o t} dt$$

Discrete-Time Fourier Series

Using orthogonality to find the DT Fourier series coefficients.

$$f[n] = f[n+N] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_o n}$$

Multiply $f[n]$ by the complex conjugate of the basis function of interest, and then sum over N.

$$\begin{aligned}
\sum_{n=\langle N\rangle} f[n]e^{-jl\Omega_o n} &= \sum_{n=\langle N\rangle} \left(\sum_{k=\langle N\rangle} a_k e^{jk\Omega_o n} \right) e^{-jl\Omega_o n} \\
&= \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-l)\Omega_o n} \\
&= \sum_{k=\langle N\rangle} a_k N\delta[k-l] = a_l N
\end{aligned}$$

Solving for a_l and then substituting k for l yields

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} f[n]e^{-jk\Omega_o n}$$

Fourier Series Summary

CT and DT Fourier series are similar, but DT Fourier series require just N components while CT Fourier series require an infinite number.

Continuous-Time Fourier Series

$$a_k = \frac{1}{T}\int_T f(t)e^{-jk\omega_o t}dt$$

analysis equation

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

synthesis equation

where $\omega_o = \frac{2\pi}{T}$

Discrete-Time Fourier Series

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} f[n]e^{-jk\Omega_o n}$$

analysis equation

$$f[n] = f[n+N] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_o n}$$

synthesis equation

where $\Omega_o = \frac{2\pi}{N}$

Properties of Discrete-Time Fourier Series

Operations on the time representation of a signal can often be interpreted as equivalent (but easier) operations on the series coefficients.

We will discuss four (of many) properties of Fourier series.

- linearity
- time shift
- time reversal
- conjugate symmetry

Linearity

The Fourier series coefficients of a linear combination of two signals is the linear combination of their Fourier series coefficients.

Let

$$f[n] = af_1[n] + bf_2[n] \quad \text{where} \quad f_1[n] = f_1[n+N] \text{ and } f_2[n] = f_2[n+N]$$

then the Fourier series coefficents for $f[n]$ are given by

$$F[k] = \frac{1}{N}\sum_{n=\langle N\rangle} f[n]e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=\langle N\rangle}\Big(af_1[n] + bf_2[n]\Big)e^{-jk\frac{2\pi}{N}n}$$

$$= \underbrace{a\,\frac{1}{N}\sum_{n=\langle N\rangle} f_1[n]e^{-jk\frac{2\pi}{N}n}}_{F_1[k]} + \underbrace{b\,\frac{1}{N}\sum_{n=\langle N\rangle} f_2[n]e^{-jk\frac{2\pi}{N}n}}_{F_2[k]}$$

$$= aF_1[k] + bF_2[k]$$

where $F_1[k]$ and $F_2[k]$ are Fourier series coefficients for $f_1[n]$ and $f_2[n]$.

Time Shift

Shifting time changes the phases of a signal's Fourier coefficients.

Let

$$g[n] = f[n-n_0] \quad \text{where} \quad f[n] = f[n+N]$$

If

$$F[k] = \frac{1}{N}\sum_{n=\langle N\rangle} f[n]e^{-jk\frac{2\pi}{N}n}$$

then

$$G[k] = \frac{1}{N}\sum_{n=\langle N\rangle} g[n]\,e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=\langle N\rangle} f[n-n_0]\,e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N}\sum_{m=\langle N\rangle} f[m]\,e^{-jk\frac{2\pi}{N}(m+n_0)} \quad \text{where} \quad m = n-n_0$$

$$= e^{-jk\frac{2\pi}{N}n_0}\;\frac{1}{N}\sum_{m=\langle N\rangle} f[m]\,e^{-jk\frac{2\pi}{N}m} = e^{-jk\frac{2\pi}{N}n_0}\,F[k]$$

Time Reversal

Reversing time reverses frequency.

Let

$$g[n] = f[-n] \quad \text{where} \quad f[n] = f[n+N]$$

If

$$F[k] = \frac{1}{N} \sum_{n=\langle N\rangle} f[n] e^{-jk\frac{2\pi}{N}n}$$

then

$$G[k] = \frac{1}{N} \sum_{n=\langle N\rangle} g[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N} \sum_{n=\langle N\rangle} f[-n] e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N} \sum_{m=\langle N\rangle} f[m] e^{+jk\frac{2\pi}{N}m} \quad \text{where} \quad m = -n$$

$$= F[-k]$$

Conjugate Symmetry

If $f[n]$ is real-valued, then its Fourier coefficients have conjugate symmetry.

If $f[n]$ is real-valued, then $f[n] = f^*[n]$.

$$\begin{aligned}
F[k] &= \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk\frac{2\pi}{N}n} \\
F^*[k] &= \frac{1}{N} \sum_{n=\langle N \rangle} f^*[n] e^{jk\frac{2\pi}{N}n} \\
&= \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{jk\frac{2\pi}{N}n} \\
&= F[-k]
\end{aligned}$$

Applications of Fourier Series

Signal processing is **widely used** in science and engineering to ...

- **model** some aspect of the world,
- **analyze** the model, and
- **interpret** results to gain a new or better understanding.

$$
\begin{tikzcd}
\text{model} \arrow[rr, "\text{analyze}", "\text{(math, computation)}"'] & & \text{result} \arrow[d, "\text{interpret results}"] \\
\text{world} \arrow[u, "\text{make model}"] & & \text{new understanding}
\end{tikzcd}
$$

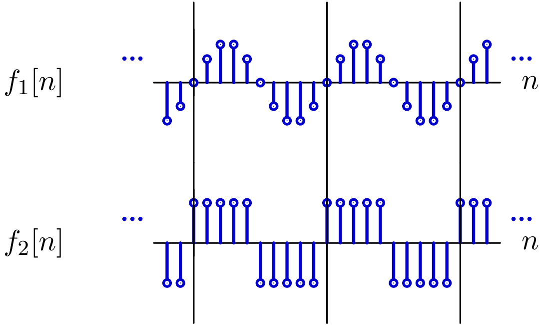
We previously touched on applications in physics, including the wave equation and how it leads directly to Fourier analysis.

Applications of Fourier analysis in **hearing**.

Applications of Fourier Analysis in Hearing

What determines the pitch of a sound? This seemingly simple question has evoked debate (sometimes fierce) for more than 150 years.

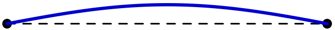
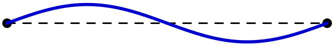
Compare two periodic signals with the same period, each played with 4000 samples per second



Different sounds, same pitch. We would like to understand why.

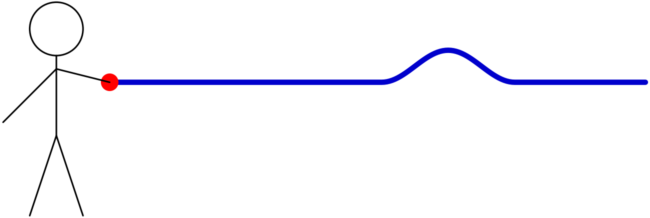
Pitch Experiments

Early experiments were based on stringed instruments and tubes, which were known to produce not just a fundamental but also harmonic overtones.

dominant frequency	dominant mode
fundamental	
2nd harmonic	
3rd harmonic	

Physical Example: Vibrating String

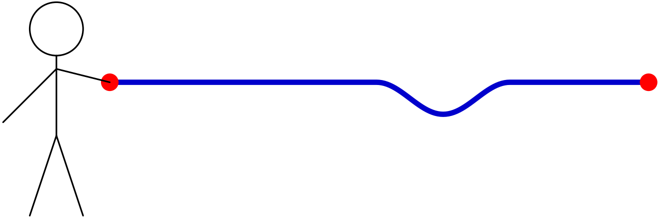
A taut string supports wave motion.



The speed of the wave depends on the tension on and mass of the string.

Physical Example: Vibrating String

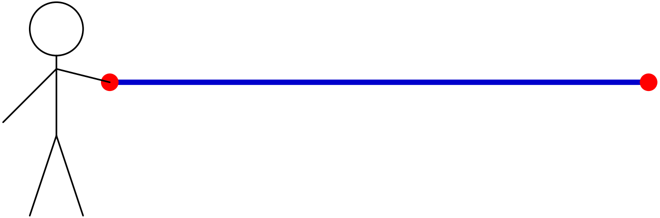
The wave will reflect off a rigid boundary.



The amplitude of the reflected wave is opposite that of the incident wave.

Physical Example: Vibrating String

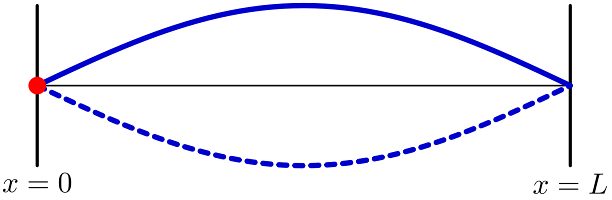
Reflections can interfere with excitations.



The interference can be constructive or destructive depending on the frequency of the excitation.

Physical Example: Vibrating String

We get constructive interference if round-trip travel time equals the period.

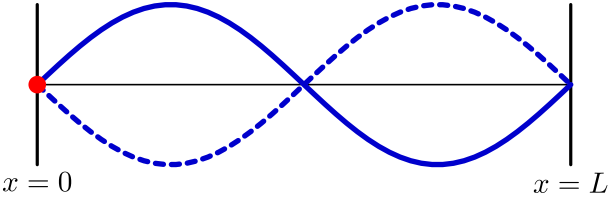


Round-trip travel time $= \dfrac{2L}{v} = T$

$$\omega_o = \frac{2\pi}{T} = \frac{2\pi}{2L/v} = \frac{\pi v}{L}$$

Physical Example: Vibrating String

We also get constructive interference if round-trip travel time is $2T$.

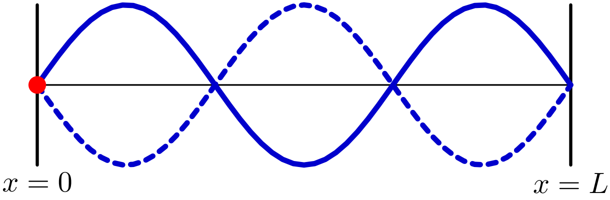


Round-trip travel time $= \dfrac{2L}{v} = 2T$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{L/v} = \frac{2\pi v}{L} = 2\omega_o$$

Physical Example: Vibrating String

In fact, we also get constructive interference if round-trip travel time is kT.



$$\text{Round-trip travel time} = \frac{2L}{v} = kT$$

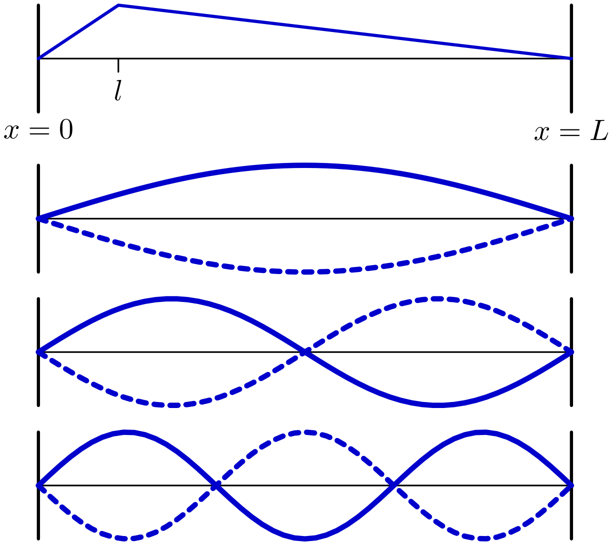
$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2L/kv} = \frac{k\pi v}{L} = k\omega_o$$

Only certain frequencies (harmonics of $\omega_o = \pi v/L$) persist.

This is the basis of stringed instruments.

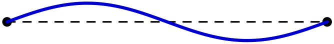
Physical Example: Vibrating String

More complicated motions can be expressed as a sum of normal modes using Fourier series. Here the string is "plucked" at $x = l$.



Pitch Experiments

Early experiments were based on stringed instruments and tubes, which were known to produce not just a fundamental but also harmonic overtones.

dominant frequency	dominant mode
fundamental	
2nd harmonic	
3rd harmonic	

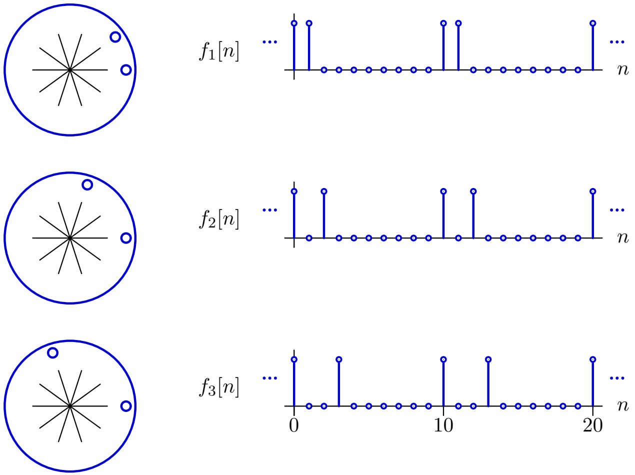
Although different sources produced different mixtures of harmonics, it was difficult to separate effects of one harmonic from those of others.

A breakthrough occured with the work of Seebeck who used sirens to generate more complicated sounds.

Very clever experiment, but very controversial interpretations.

Sirens

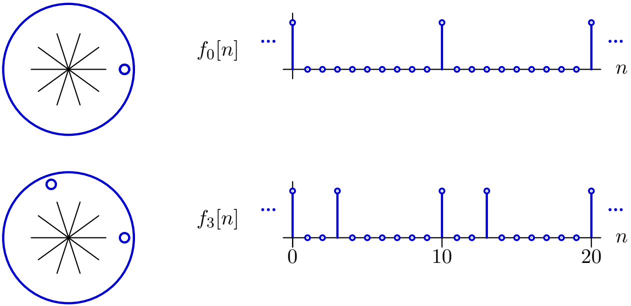
Seebeck used a siren to generate more complicated sounds (circa 1841) by passing a jet of compressed air through holes in a spinning disk.



The pattern of holes determined the pattern of pulses in each period. The speed of spinning controlled the number of periods per second.

Sirens

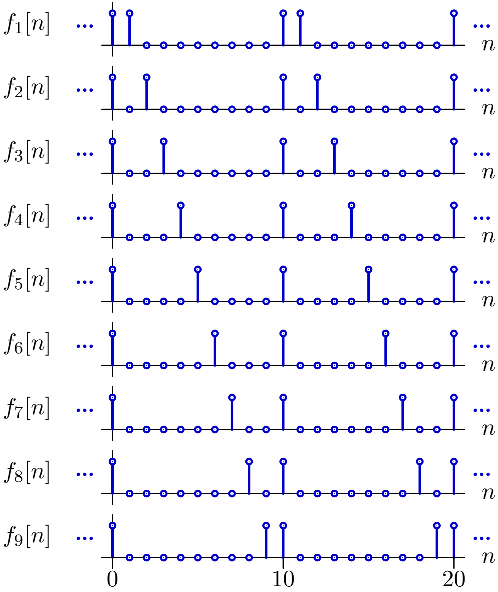
Strangely, adding a second hole per period didn't seem to affect the pitch.



Pitch should be different if it is determined by the intervals between pulses.

Sirens

There was one very interesting exception.



But hearing this exception required precise alignment of the siren's holes.

Sirens and Controversy

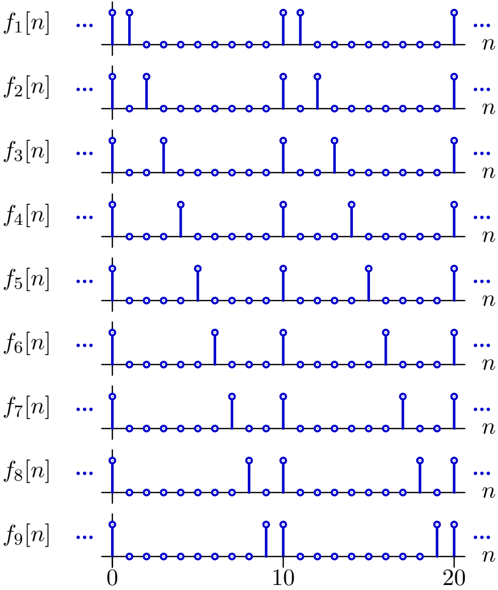
Seebeck interpreted his results in terms of the intervals between the holes. He held that pitch results from **timing** with some intervals being more important than others. As the lengths of the two intervals in his experiment converged, the pitch favored what had been the second harmonic and that frequency increasingly dominated.

Georg Ohm (already known for his work on electrical conduction) interpreted Seebeck's results using Fourier's recently described series. He held that the pulses generated by a siren contained a **fundamental** and **harmonics** that were physically present just as much as they are in a stringed instrument.

A bitter controversy ensued.

Fourier Interpretation

To understand Ohm's argument, compute the Fourier series for the siren's sound.



Fourier Interpretation

Find the k^{th} coefficient of the i^{th} signal.

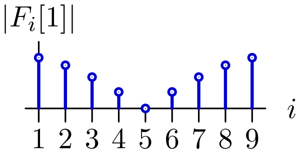
$$F_i[k] = \frac{1}{N}\sum_{n=\langle N\rangle} f_i[n]\, e^{-j\frac{2\pi k}{N}n} = \frac{1}{10}\sum_{n=0}^{9} f_i[n]\, e^{-j\frac{2\pi k}{10}n} = \frac{1}{10}\left(1+e^{-j\frac{2\pi k}{10}i}\right)$$

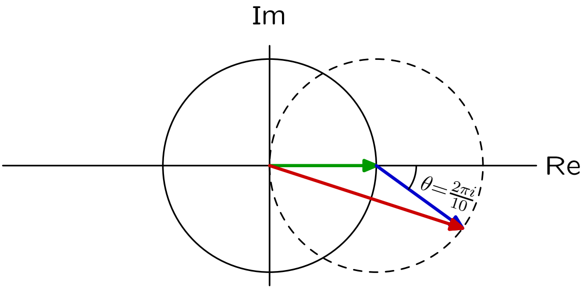
DC: the $k{=}0$ term is $2/10$ for all i

$$F_i[0] = \frac{1}{10}\left(1+e^{-j\frac{2\pi 0}{10}i}\right) = \frac{2}{10}$$

Fundamental: $k{=}1$ term

$$F_i[1] = \frac{1}{10}\left(1+e^{-j\frac{2\pi}{10}i}\right)$$

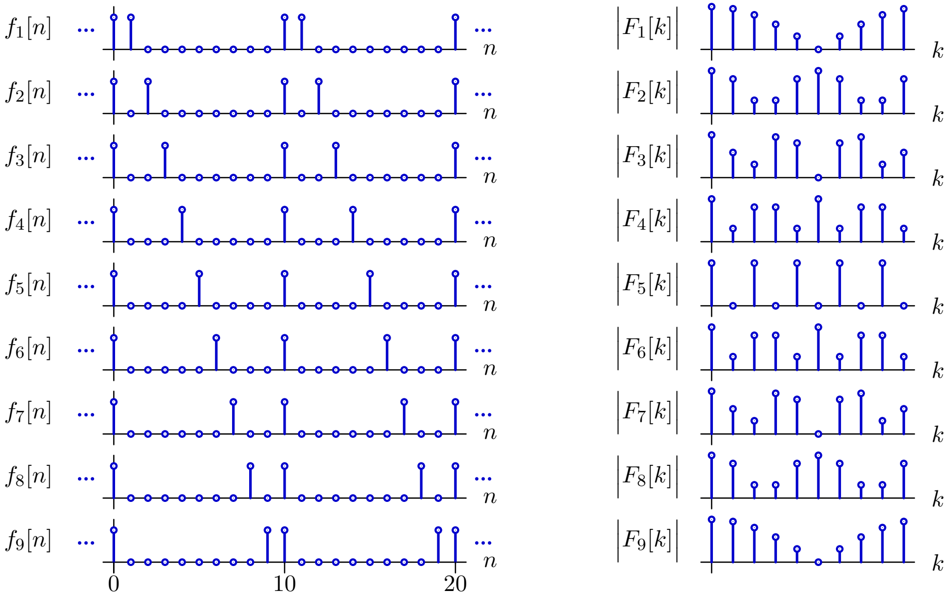




Notice that $f_5[n]$ has no fundamental component!

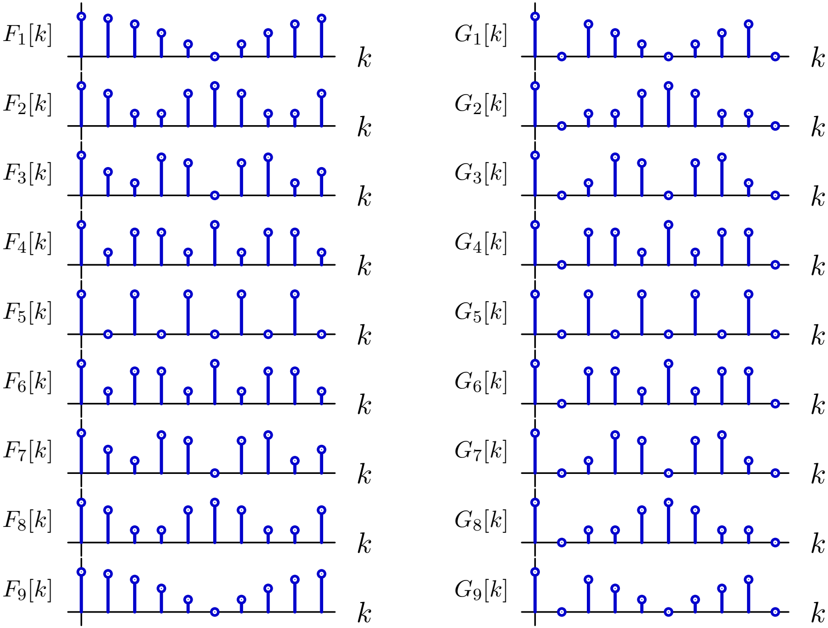
Fourier Series

Notice that $f_5[n]$ has no fundamental component!



Fourier Series With and Without the Fundamental

Resynthesize each waveform without its fundamental component.



Although perception of the fundamental is weakened, it is not gone!

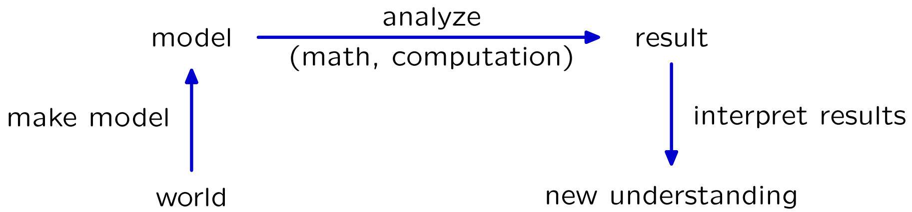
Summary

Seebeck designed an extremely clever **experiment** to test pitch perception.

Ohm analyzed an important **theory** (from Fourier) and argued that harmonics are present even in the pulsatile sounds generated by a siren.

Neither Seebeck nor Ohm could convincingly account for experimental results that demonstrated the dominance of the fundamental, even when it was weak or missing.

Progress in understanding the "missing fundamental" awaited Helmholtz, who demonstrated the importance of "combination tones" in the ear.



Summary

Today we focused on discrete-time Fourier analysis.

- We developed Fourier series for discrete-time signals.
- We compared Fourier series for CT and DT signals.
- We looked at four (of many) properties of DT Fourier series.
- We looked briefly at applications of Fourier analysis in hearing.

Next time: Fourier analysis of aperiodic signals (CT and DT).