6.3000: Signal Processing

Continuous-Time Fourier Transform

- Definition
- Examples
- Properties

Quiz 1: October 3, 2-4pm, room 50-340 (Walker).

- Closed book except for one page of notes (8.5"x11" both sides).
- No electronic devices. (No headphones, cellphones, calculators, ...)
- Coverage up to and including classes on September 21 and HW 3.

We have posted a practice quiz as a study aid for the upcoming quiz 1.

- Your solutions will not be submitted or counted in your grade.
- Solutions will be posted on Friday.

There is no HW 4.

If you have personal or medical difficulties, please contact S^3 and/or `6.3000-instructors@mit.edu` for accommodations.

September 26, 2023

From Periodic to Aperiodic

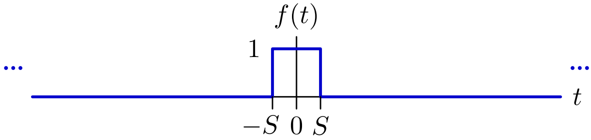
We have been focusing on frequency representations of **periodic** signals:
e.g., sounds, waves, music, ...

However, most real-world signals are not periodic.
None are truly periodic since they do not have infinite duration!

Today: generalizing Fourier representations to include aperiodic signals.

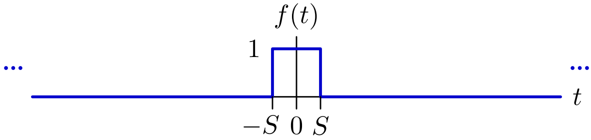
Fourier Representations of Aperiodic Signals

How can we represent an aperiodic signal as a sum of sinusoids?



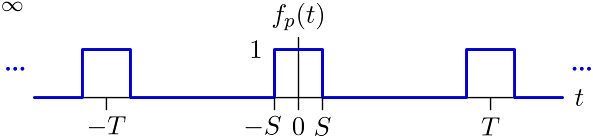
Fourier Representations of Aperiodic Signals

How can we represent an aperiodic signal as a sum of sinusoids?



Strategy: make a periodic version of $f(t)$ by summing shifted copies:

$$f_p(t) = \sum_{m=-\infty}^{\infty} f(t - mT)$$



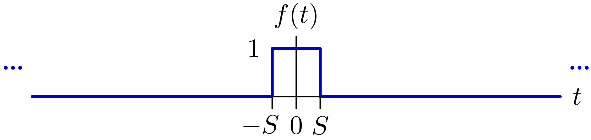
Since $f_p(t)$ is periodic, it has a Fourier series (which depends on T).

Find Fourier series coefficients $F_p[k]$ and take the limit of $F_p[k]$ as $T \to \infty$.

As $T \to \infty$, $f_p(t) \to f(t)$ and Fourier series will approach Fourier transform.

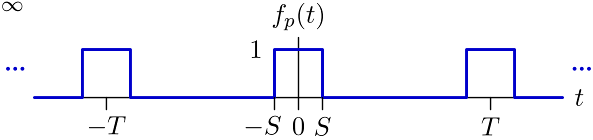
Fourier Representations of Aperiodic Signals

Example.



Strategy: make a periodic version of $f(t)$ by summing shifted copies:

$$f_p(t) = \sum_{m=-\infty}^{\infty} f(t - mT)$$



Calculate the Fourier series coefficients $F_p[k]$:

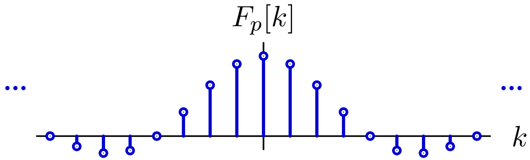
$$F_p[k] = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}\,dt = \frac{1}{T}\left.\frac{e^{-j\frac{2\pi}{T}kt}}{-j\frac{2\pi k}{T}}\right|_{-S}^{S} = \frac{2\sin\left(\frac{2\pi k}{T}S\right)}{T\left(\frac{2\pi k}{T}\right)}$$

Fourier Representations of Aperiodic Signals

Calculate the Fourier series coefficients $F_p[k]$:

$$F_p[k] = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}\,dt = \frac{1}{T}\left.\frac{e^{-j\frac{2\pi}{T}kt}}{-j\frac{2\pi k}{T}}\right|_{-S}^{S} = \frac{2\sin\left(\frac{2\pi k}{T}S\right)}{T\left(\frac{2\pi k}{T}\right)}$$

Plot the resulting Fourier coefficients when $S{=}1$ and $T{=}8$.



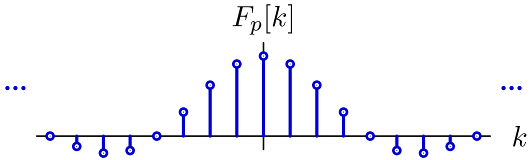
What happens if you double the period T?

Fourier Representations of Aperiodic Signals

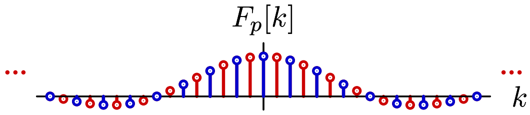
Calculate the Fourier series coefficients $F_p[k]$:

$$F_p[k] = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}\,dt = \frac{1}{T}\left.\frac{e^{-j\frac{2\pi}{T}kt}}{-j\frac{2\pi k}{T}}\right|_{-S}^{S} = \frac{2\sin\left(\frac{2\pi k}{T}S\right)}{T\left(\frac{2\pi k}{T}\right)}$$

Plot the resulting Fourier coefficients when S=1 and T=8.



What happens if you double the period T? Plot with S=1 and T=16.



There are twice as many samples per period of the sin function. (The red samples are at new intermediate frequencies.) The amplitude is halved.

Fourier Representations of Aperiodic Signals

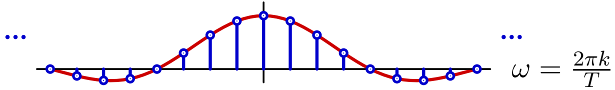
Define a new function $F(\omega)$ where $\omega = k\omega_o = 2\pi k/T$.

$$TF_p[k] = \frac{2\sin\left(\frac{2\pi k}{T}S\right)}{\frac{2\pi k}{T}} = 2\,\frac{\sin(\omega S)}{\omega}\bigg|_{\omega=\frac{2\pi k}{T}} = F(\omega)\big|_{\omega=\frac{2\pi k}{T}}$$

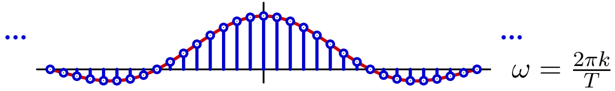
Then $TF_p[k]$ represents samples of $F(\omega)$ with increasing resolution in ω.

$$TF_p[k] = F(\omega)$$

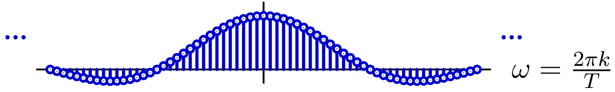
S=1 and T=8:



S=1 and T=16:



S=1 and T=32:



The discrete function $TF_p[k]$ is a sampled version of the function $F(\omega)$.

Fourier Representations of Aperiodic Signals

From $f(t)$ to $F(\omega)$:

$$f(t) \underbrace{\rightarrow}_{\substack{\text{periodic}\\\text{extension}}} f_p(t) \underbrace{\rightarrow}_{\substack{\text{Fourier}\\\text{series}}} F_p[k] \underbrace{\rightarrow}_{\text{interpolation}} F(\omega)$$

The limiting behaviors as $T \to \infty$ define the Fourier transform:

$$\begin{aligned} F(\omega) &= \lim_{T\to\infty} T F_p[k]\Big|_{k=\frac{\omega}{\omega_o}=\frac{T}{2\pi}\omega} \\ &= \lim_{T\to\infty} T\left[\frac{1}{T}\int_T f_p(t)e^{-j\frac{2\pi k}{T}t}dt\right]_{k=\frac{T}{2\pi}\omega} \\ &= \lim_{T\to\infty} \int_T f_p(t)e^{-j\omega t}dt \end{aligned}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

This **analysis equation** defines the Fourier transform.

Fourier Representations of Aperiodic Signals

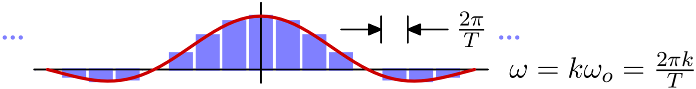
The **synthesis equation** follows from piecewise constant approximation.

$$f(t) = \lim_{T\to\infty} f_p(t) = \lim_{T\to\infty} \sum_{k=-\infty}^{\infty} F_p[k] e^{j\frac{2\pi}{T}kt}$$

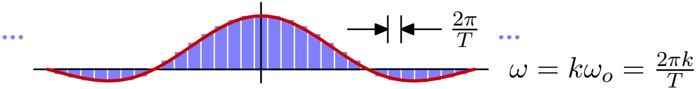
$$= \lim_{T\to\infty} \left(\frac{1}{2\pi}\right) \sum_{k=-\infty}^{\infty} T F_p[k] e^{j\frac{2\pi}{T}kt} \left(\frac{2\pi}{T}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

$$T F_p[k] = F(\omega)$$

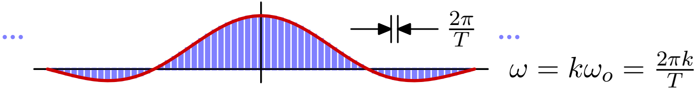
S=1 and T=8:



S=1 and T=16:



S=1 and T=32:



Fourier Transform relation: $f(t) \overset{\text{FT}}{\Longrightarrow} F(\omega)$

Continuous-Time Fourier Representations

Fourier series and transforms are similar:
both represent signals by their frequency content.

Continuous-Time Fourier Series

$$F[k] = \frac{1}{T}\int_T f(t)e^{-jk\omega_o t}dt$$

analysis equation

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} F[k]e^{jk\omega_o t}$$

synthesis equation

where $\omega_o = \frac{2\pi}{T}$

Continuous-Time Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

analysis equation

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

synthesis equation

Continuous-Time Fourier Representations

All of the information in a periodic signal is contained in one period.
The information in an aperiodic signal is spread across all time.

Continuous-Time Fourier Series

$$F[k] = \frac{1}{T}\int_T f(t)e^{-jk\omega_o t}dt$$ **analysis equation**

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} F[k]e^{jk\omega_o t}$$ **synthesis equation**

where $\omega_o = \frac{2\pi}{T}$

Continuous-Time Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$ **analysis equation**

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$ **synthesis equation**

Continuous-Time Fourier Representations

Periodic signals can be synthesized from a discrete set of harmonics.
Aperiodic signals generally require all possible frequencies.

Continuous-Time Fourier Series

$$F[k] = \frac{1}{T}\int_T f(t)e^{-jk\omega_o t}dt$$

analysis equation

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} F[k]e^{jk\omega_o t}$$

synthesis equation

where $\omega_o = \frac{2\pi}{T}$

Continuous-Time Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

analysis equation

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

synthesis equation

Continuous-Time Fourier Representations

Harmonic frequencies $k\omega_o$ are samples of continuous frequency ω.

Continuous-Time Fourier Series

$$F[k] = \frac{1}{T}\int_T f(t)e^{-jk\omega_O t}dt$$

analysis equation

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} F[k]e^{jk\omega_O t}$$

synthesis equation

where $\omega_o = \frac{2\pi}{T}$

Continuous-Time Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

analysis equation

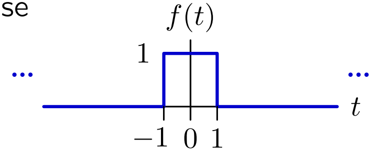
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

synthesis equation

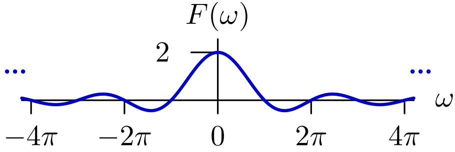
Example

Find the Fourier Transform (FT) of a rectangular pulse:

$$f(t) = \begin{cases} 1 & -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-1}^{1} e^{-j\omega t} dt = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-1}^{1} = 2 \frac{\sin \omega}{\omega}$$



$F(\omega)$ provides a recipe for constructing $f(t)$ from sinusoidal components:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

A square pulse contains (almost) all frequencies ω (missing just π, 2π, ...).

Properties of Fourier Transforms

Fourier transforms offer an alternative view of an **aperiodic** signal.

A signal and its Fourier transform contain exactly the same information, but some information is more easily seen in one domain than in the other.

There are many **properties** of Fourier transforms. These properties summarize systematic relations between time and frequency representations.

Properties of Fourier Transforms

Time delay maps to linear phase delay of the Fourier transform.

If $\qquad f(t) \overset{\text{FT}}{\Longrightarrow} F(\omega)$

then $\qquad f(t-\tau) \overset{\text{FT}}{\Longrightarrow} e^{-j\omega\tau}F(\omega)$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$G(\omega) = \int_{-\infty}^{\infty} f(t-\tau)e^{-j\omega t}dt$$

Let $u = t - \tau$ (and therefore $du = dt$ since τ is a constant)

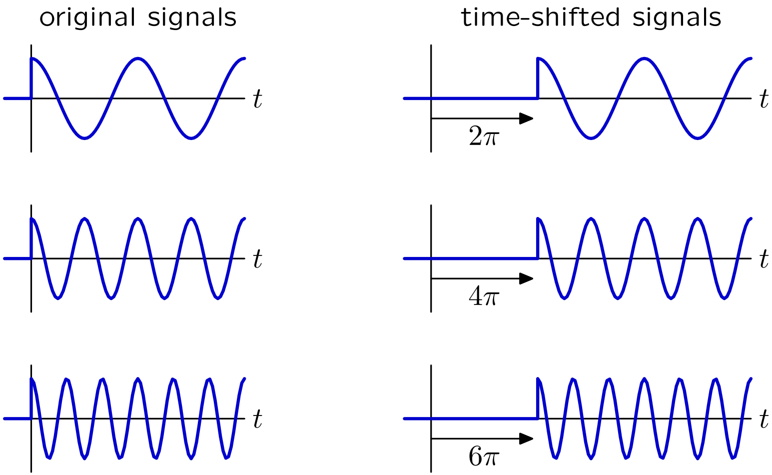
$$G(\omega) = \int_{-\infty}^{\infty} f(u)e^{-j\omega(u+\tau)}du = e^{-j\omega\tau}\int_{-\infty}^{\infty} f(u)e^{-j\omega u}du = e^{-j\omega\tau}F(\omega)$$

The angle of $e^{-j\omega\tau} = -\omega\tau$.

Why does time delay change phase by an amount proportional to frequency?

Properties of Fourier Transforms

Why does time delay change phase by an amount proportional to frequency?



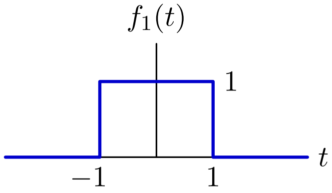
Doubling the frequency of a sinusoid doubles the change in phase associated with a given time delay.

Properties of Fourier Transforms

Scaling time.

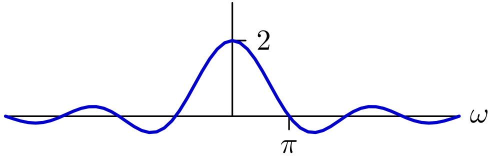
Consider the following signal and its Fourier transform.

Time representation:



Frequency representation:

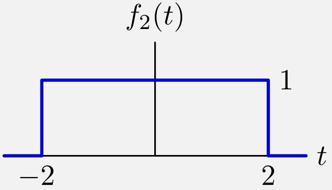
$$F_1(\omega) = \frac{2\sin\omega}{\omega}$$

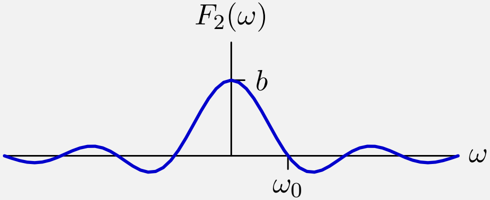


How would these functions scale if time were stretched?

Check Yourself

Signal $f_2(t)$ and its Fourier transform $F_2(\omega)$ are shown below.





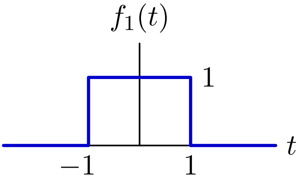
Which of the following is true?

1. $b = 2$ and $\omega_0 = \pi/2$
2. $b = 2$ and $\omega_0 = 2\pi$
3. $b = 4$ and $\omega_0 = \pi/2$
4. $b = 4$ and $\omega_0 = 2\pi$
5. none of the above

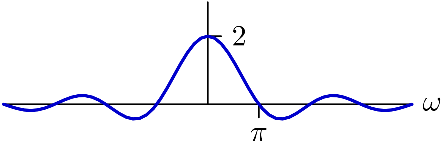
Check Yourself

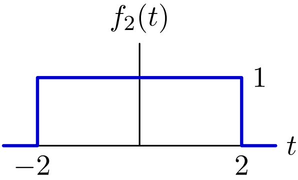
Find the Fourier transform.

$$F_2(\omega) = \int_{-2}^{2} e^{-j\omega t}dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-2}^{2} = \frac{2\sin 2\omega}{\omega} = \frac{4\sin 2\omega}{2\omega}$$

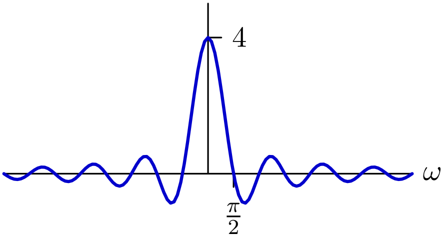


$$F_1(\omega) = \frac{2\sin\omega}{\omega}$$





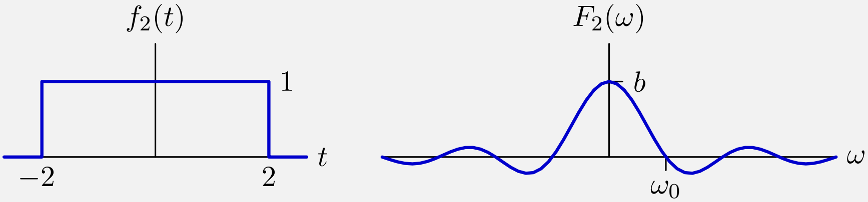
$$F_2(\omega) = \frac{4\sin 2\omega}{2\omega}$$



Stretching time compresses frequency.

Check Yourself

Signal $f_2(t)$ and its Fourier transform $F_2(\omega)$ are shown below.



Which of the following is true? 3

1. $b = 2$ and $\omega_0 = \pi/2$
2. $b = 2$ and $\omega_0 = 2\pi$
3. $b = 4$ and $\omega_0 = \pi/2$
4. $b = 4$ and $\omega_0 = 2\pi$
5. none of the above

Properties of Fourier Transforms

Find a general scaling rule.

Let $f_2(t) = f_1(at)$ where $a > 0$.

$$F_2(\omega) = \int_{-\infty}^{\infty} f_1(at) e^{-j\omega t} dt$$

Let $\tau = at$. Then $d\tau = a\,dt$.

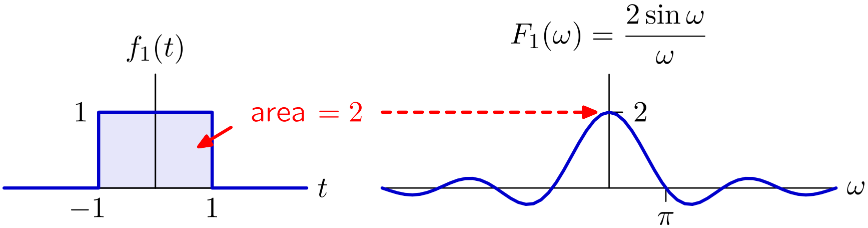
$$F_2(\omega) = \int_{-\infty}^{\infty} f_1(\tau) e^{-j\omega\tau/a}\ \frac{1}{a}\,d\tau = \frac{1}{a} F_1\left(\frac{\omega}{a}\right)$$

Stretching time compresses frequency and increases amplitude (preserving area).

Area Properties

The value of $F(\omega)$ at $\omega = 0$ is the integral of $f(t)$ over time t.

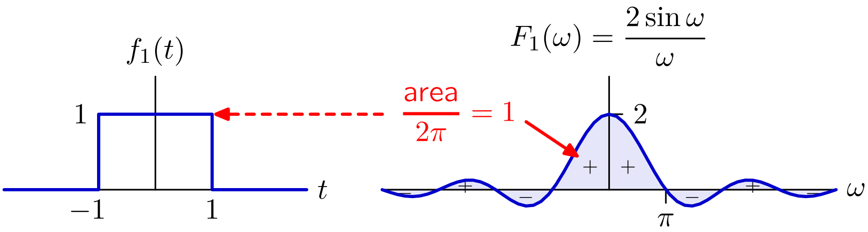
$$F(\omega)\big|_{\omega=0} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(t)e^{-j0t}dt = \int_{-\infty}^{\infty} f(t)\,dt$$



Areas

The value of $f(0)$ is the integral of $F(\omega)$ divided by 2π.

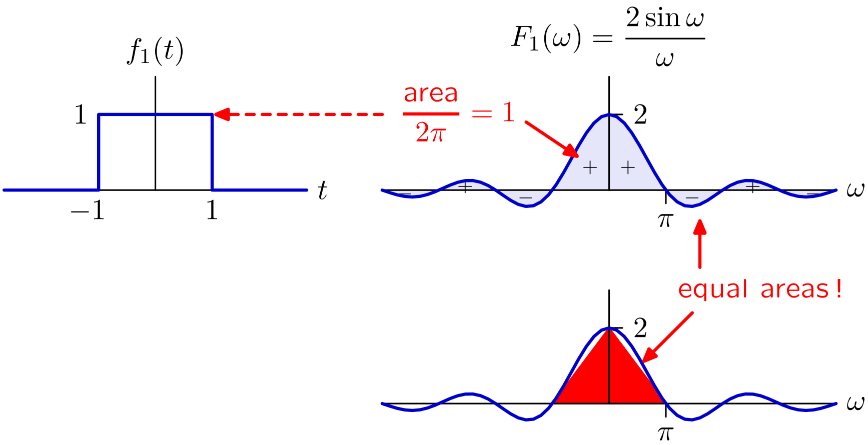
$$f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, d\omega$$



Areas

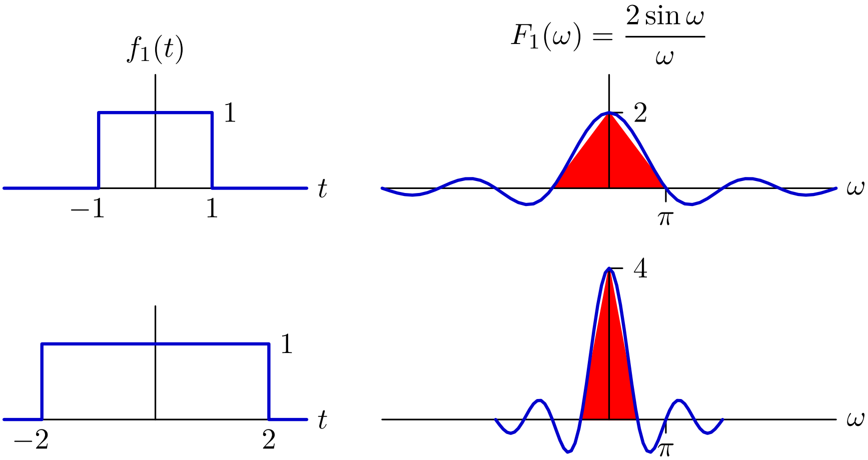
The value of $f(0)$ is the integral of $F(\omega)$ divided by 2π.

$$f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, d\omega$$



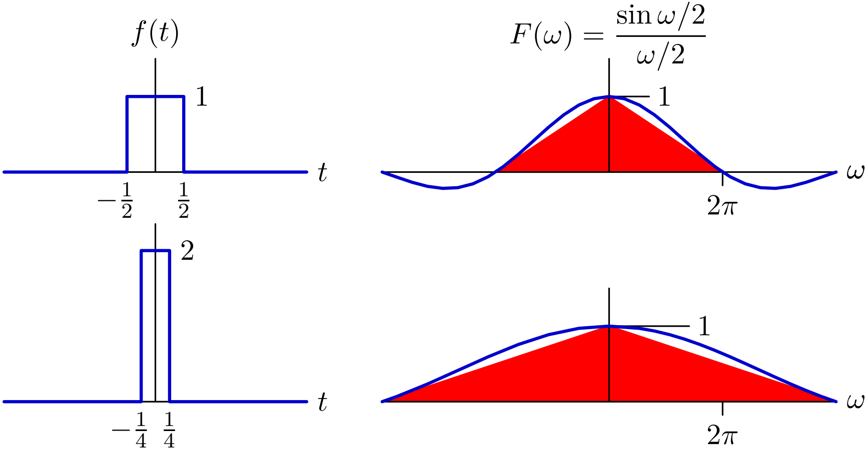
Stretching Time

Stretching time compresses frequency and increases amplitude (preserving area).



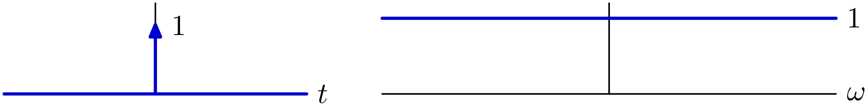
Compressing Time to the Limit

Alternatively, we could compress time while keeping area $= 1$.



In the limit, the pulse has zero width but area 1!

We represent this limit with the delta (or impulse) function: $\delta(t)$.



Math With Impulses

Although physically unrealizeable, the impulse (a.k.a. Dirac delta) function is useful as a mathematically tractable approximation to a very brief signal.

Example 1: Find the Fourier transform of a unit impulse function.

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

Since $\delta(t)$ is zero except near $t{=}0$, only values of $e^{-j\omega t}$ near $t{=}0$ are important. Because $e^{-j\omega t}$ is a smooth function of t, $e^{-j\omega t}$ can be replaced by $e^{-j\omega 0}$:

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega 0} dt = \int_{-\infty}^{\infty} \delta(t) dt = 1$$

This matches our previous result which was based explicitly on a limit. Here the limit is implicit.

Math With Impulses

Although physically unrealizeable, the impulse (a.k.a. Dirac delta) function is useful as a mathematically tractable approximation to a very brief signal.

Example 2: Find the function whose Fourier transform is an impulse.

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega) e^{j0t} d\omega = \frac{1}{2\pi}$$

$$1 \overset{\text{CTFT}}{\Longrightarrow} 2\pi\delta(\omega)$$

Notice the similarity to the previous result:

$$\delta(t) \overset{\text{CTFT}}{\Longrightarrow} 1$$

These relations are **duals** of each other.

- A constant in time consists of a single frequency at $\omega = 0$.
- An impulse in time contains components at all frequencies.

Math With Impulses

Although physically unrealizeable, the impulse (a.k.a. Dirac delta) function is useful as a mathematically tractable approximation to a very brief signal.

Example 3: Find the function whose Fourier transform is a shifted impulse.

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_o) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_o) e^{j\omega_o t} d\omega \\ &= \frac{1}{2\pi} e^{j\omega_o t} \int_{-\infty}^{\infty} \delta(\omega-\omega_o) d\omega \\ &= \frac{1}{2\pi} e^{j\omega_o t} \end{aligned}$$

$$e^{j\omega_o t} \quad \stackrel{\text{CTFT}}{\Longrightarrow} \quad 2\pi\delta(\omega-\omega_o)$$

Use this result to relate Fourier series to Fourier transforms.

Relation Between Fourier Series and Fourier Transforms

If a periodic signal $f(t) = f(t+T)$ has a Fourier series representation, then it can also be represented by an equivalent Fourier transform.

$$e^{j\omega_o t} \stackrel{\text{FT}}{\Longrightarrow} 2\pi\delta(\omega - \omega_o)$$

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} F[k] e^{j\frac{2\pi}{T}kt} \quad \stackrel{\text{CTFS}}{\longleftrightarrow} \quad F[k]$$

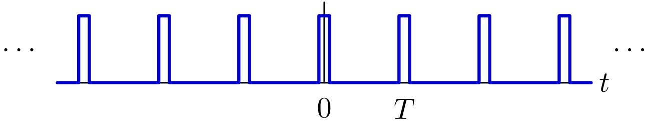
$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} F[k] e^{j\frac{2\pi}{T}kt} \quad \stackrel{\text{CTFT}}{\longleftrightarrow} \quad \sum_{k=-\infty}^{\infty} 2\pi F[k] \delta\left(\omega - \frac{2\pi}{T}k\right)$$

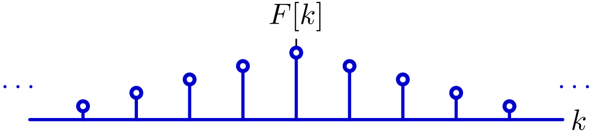
Each term in the Fourier series is replaced by an impulse in the Fourier transform.

Relation between Fourier Transform and Fourier Series

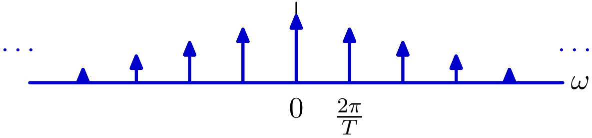
Each Fourier series term is replaced by an impulse in the Fourier transform.

$$f(t) = \sum_{m=-\infty}^{\infty} f(t - mT)$$





$$F(\omega) = \sum_{k=-\infty}^{\infty} 2\pi \, F[k] \, \delta(\omega - k\frac{2\pi}{T})$$



Summary

Fourier series and transforms are similar:
both represent signals by their frequency content.

Continuous-Time Fourier Series

$$F[k] = \frac{1}{T}\int_T f(t)e^{-jk\omega_o t}dt$$ **analysis equation**

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} F[k]e^{jk\omega_o t}$$ **synthesis equation**

where $\omega_o = \frac{2\pi}{T}$

Continuous-Time Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$ **analysis equation**

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$ **synthesis equation**

Next time: Fourier Transform for discrete-time signals.