6.3000: Signal Processing

Systems

- System Abstraction
- Linearity and Time Invariance

Results for Quiz 1 have been posted.

HW 5 has been posted.

There will not be a HW 6
(because of the quiz this week and the student holiday next week).

There will not be a Lab 5 or Lab 6.

HW 5 will be due on Monday, October 16, at 10pm.

October 5, 2023

Points, 10-Point Scale, and Letter Grade

Grading Procedure:

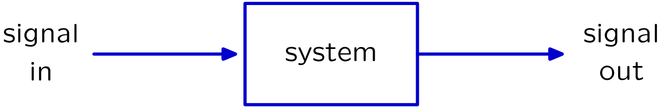
- We grade the exams on a **point** basis (out of 70 points for Quiz 1).
- We convert the point score into a **10-point** scale using MIT's definitions of letter grades.
- Your final score in 6.3000 will be a **weighted sum of your 10-point scores** for homeworks, labs, quizzes, and final exam.

total points	10-point score	letter grade
100%	10	
		A
A/B boundary	9	
		B
B/C boundary	8	
		C
C/D boundary	7	
		D
D/F boundary	6	
		F
0%	0	

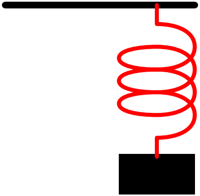
The goal of this scheme is to be transparent about your grade status.

From Signals to Systems: The System Abstraction

Represent a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



Example: Mass and Spring



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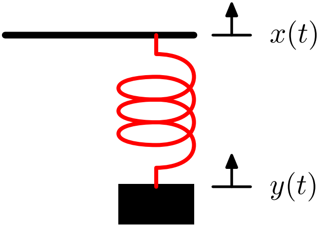

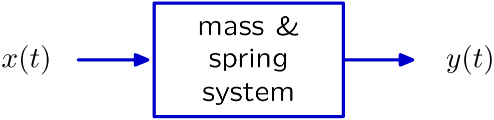

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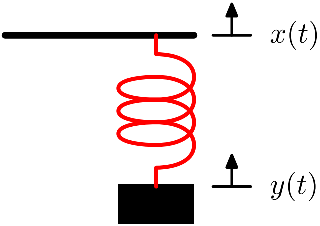
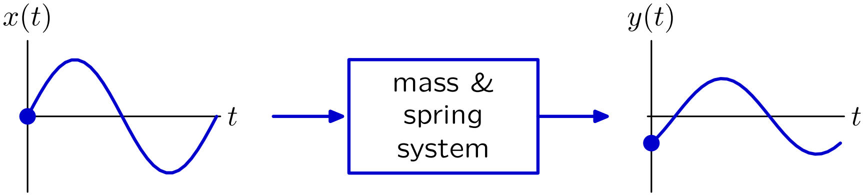
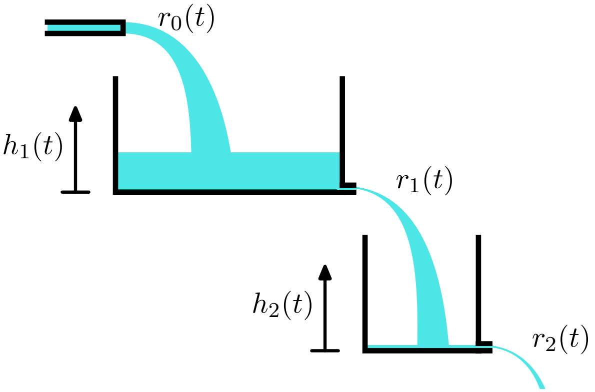
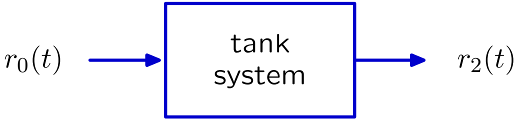




Example: Tanks





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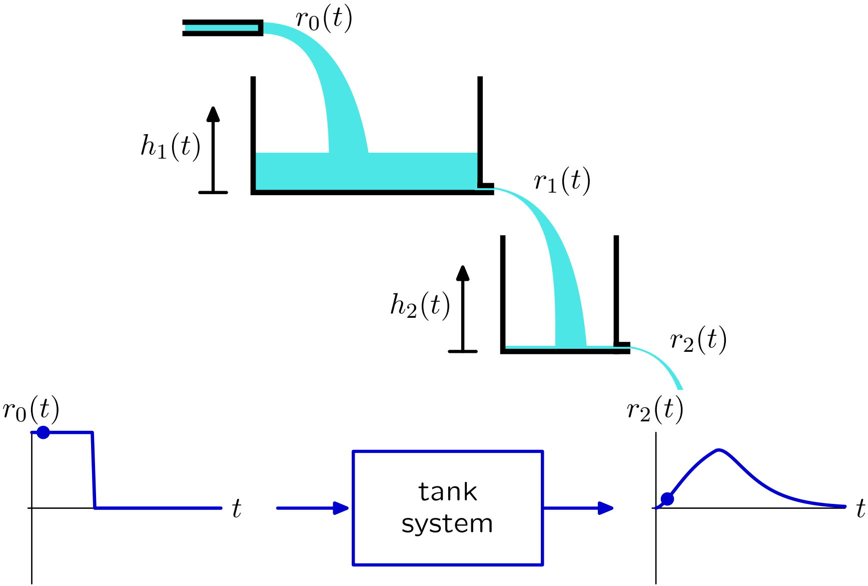
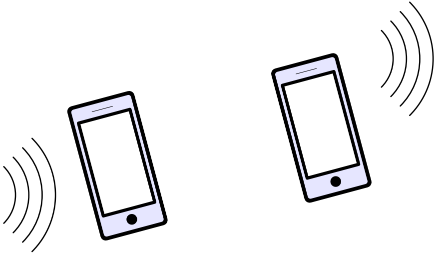


Example: Cell Phone System

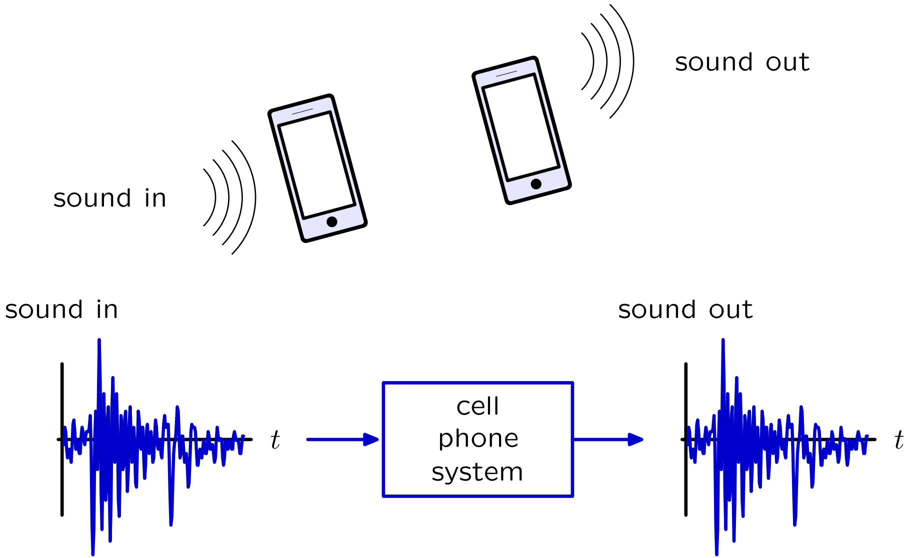
sound out



sound in

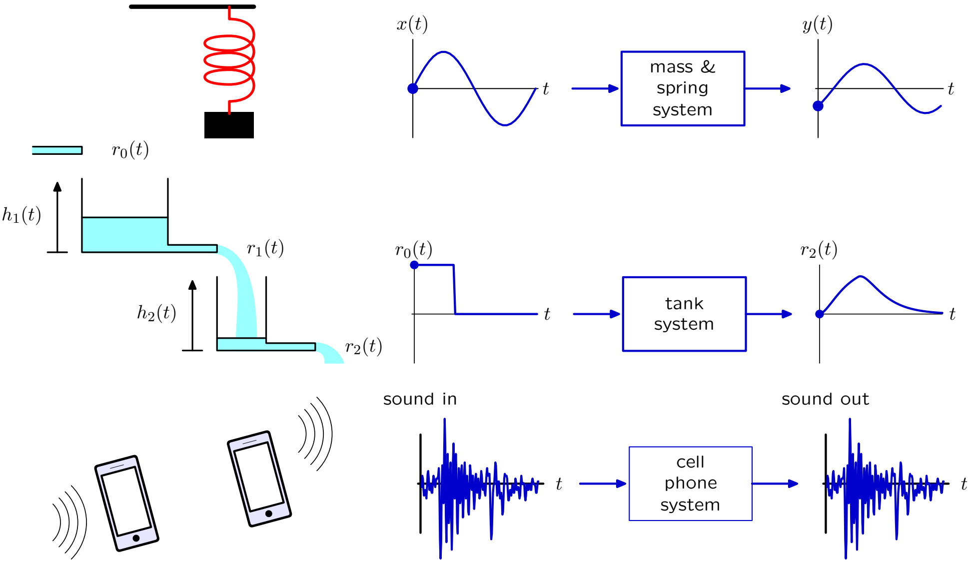
sound in → cell phone system → sound out

Example: Cell Phone System



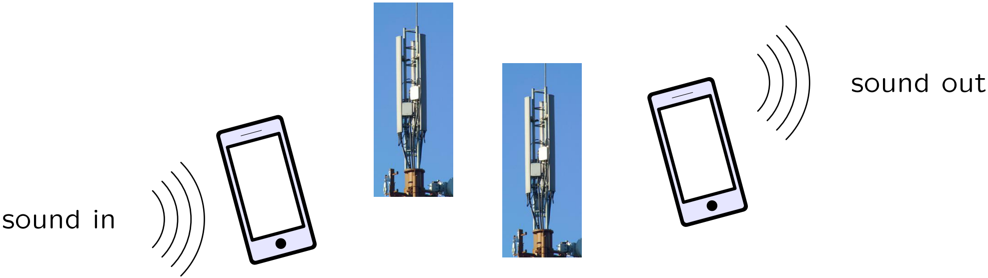
Signals and Systems: Widely Applicable

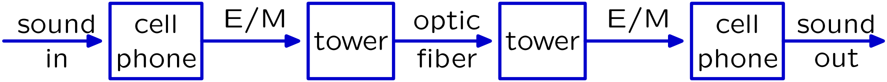
The Signals and Systems approach has broad application: electrical, mechanical, optical, acoustic, biological, financial, ...



Signals and Systems: Modular

The representation does not depend upon the physical substrate.



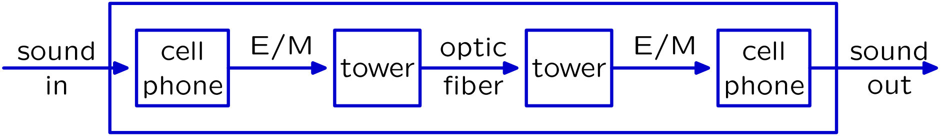


focuses on the flow of **information**, abstracts away everything else

Signals and Systems: Hierarchical

Representations of component systems are easily combined.

Example: cascade of component systems

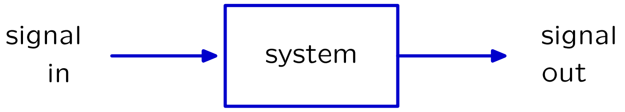


Composite system

Component and composite systems have the same form, and are analyzed with same methods.

System Abstraction

The system abstraction builds on and extends our work with signals.



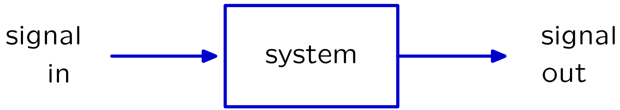
The remainder of this subject will focus on systems:

- **audio:** equalization, noise reduction, reverberation reduction, echo cancellation, pitch shift (auto-tune)
- **image:** smoothing, edge enhancement, unsharp masking, feature detection
- **video:** image stabilization, motion magnification

Each of these important areas builds directly on our work with signals.

System Abstraction

The system abstraction builds on and extends our work with signals.



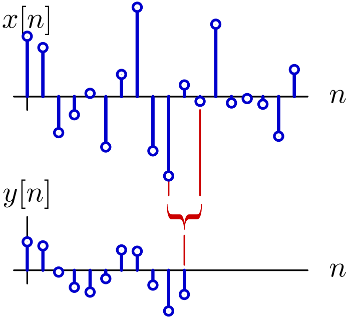
We will look at three different representations for systems:

- **Difference Equation:** algebraic **constraint** on samples
- **Convolution:** represent a system by its **unit-sample response**
- **Filter:** represent a system by its **frequency response**

Example: Three-Point Averaging

The output at time n is average of inputs at times $n-1$, n, and $n+1$.

$$y[n] = \frac{1}{3}\Big(x[n-1] + x[n] + x[n+1]\Big)$$



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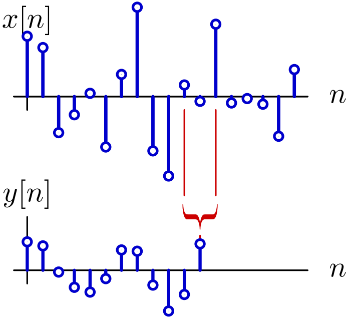

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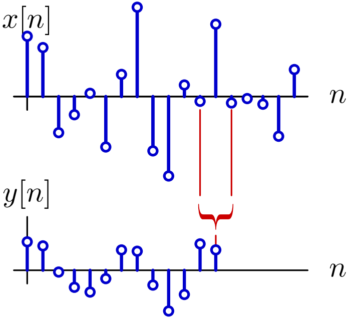

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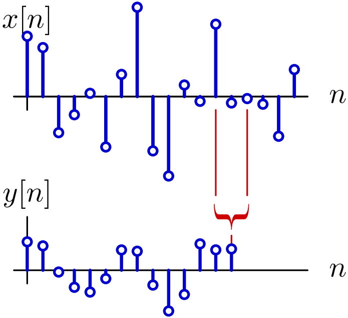

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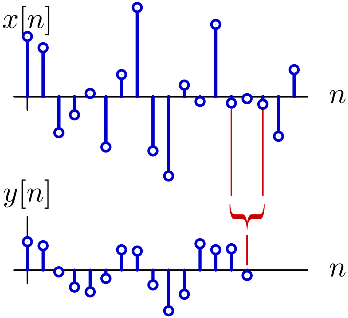

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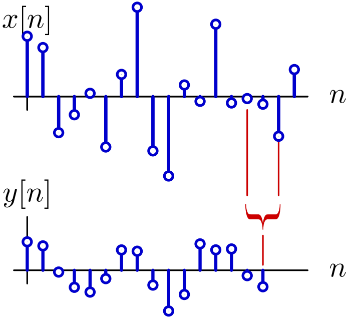
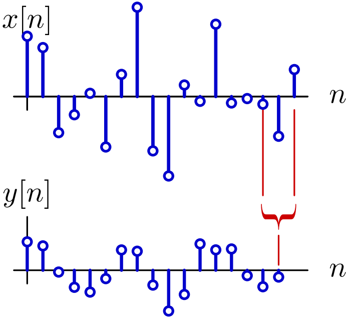


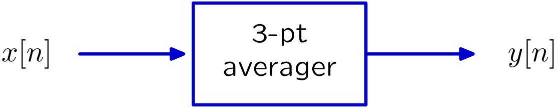
Example: Three-Point Averaging

The output at time n is average of inputs at times $n-1$, n, and $n+1$.

$$y[n] = \frac{1}{3}\Big(x[n-1] + x[n] + x[n+1]\Big)$$



Think of this process as a system with input $x[n]$ and output $y[n]$.



Properties of Systems

We will focus primarily on systems that have two important properties:

- **linearity**
- **time invariance**

Such systems are both useful and mathematically tractable.

Additivity

A system is additive if its response to a **sum of signals** is equal to the **sum of the responses** to each signal taken one at a time.

Given

$$x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n]$$

and

$$x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_2[n]$$

the **system is additive** if

$$x_1[n] + x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n] + y_2[n]$$

for all possible inputs and all times n.

Additivity

Example: The three-point averager is additive.

If

$$
\begin{aligned}
x_1[n] &\to y_1[n] = \tfrac{1}{3}\Big(x_1[n-1] + x_1[n] + x_1[n+1]\Big) \\
x_2[n] &\to y_2[n] = \tfrac{1}{3}\Big(x_2[n-1] + x_2[n] + x_2[n+1]\Big)
\end{aligned}
$$

and

$$x_3[n] = x_1[n] + x_2[n]$$

then

$$
\begin{aligned}
x_3[n] &\to \tfrac{1}{3}\Big(x_3[n-1] + x_3[n] + x_3[n+1]\Big) \\
x_1[n] + x_2[n] &\to \tfrac{1}{3}\Big((x_1[n-1] + x_2[n-1]) + (x_1[n] + x_2[n]) + (x_1[n+1] + x_2[n+1])\Big) \\
&= \tfrac{1}{3}\Big(x_1[n-1] + x_1[n] + x_1[n+1]\Big) + \tfrac{1}{3}\Big(x_2[n-1] + x_2[n] + x_2[n+1]\Big) \\
&= y_1[n] + y_2[n]
\end{aligned}
$$

Homogeneity

A system is homogeneous if multiplying its input signal by a constant multiplies the output signal by the same constant.

Given

$$x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n]$$

the **system is homogeneous** if

$$\alpha x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow \alpha y_1[n]$$

for all α and all possible inputs and all times n.

Homogeneity

Example: The three-point averager is homogeneous.

If

$$x_1[n] \to y_1[n] = \tfrac{1}{3}\Big(x_1[n{-}1] + x_1[n] + x_1[n{+}1]\Big)$$

then

$$\begin{aligned}\alpha x_1[n] \to \tfrac{1}{3}\Big(\alpha x_1[n{-}1] + \alpha x_1[n] + \alpha x_1[n+1]\Big)\\ = \alpha\tfrac{1}{3}\Big(x_1[n{-}1] + x_1[n] + x_1[n{+}1]\Big)\\ = \alpha y_1[n]\end{aligned}$$

Linearity

A system is linear if its response to a **weighted sum of input signals** is equal to the **weighted sum of its responses** to each of the input signals.

Given

$$x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n]$$

and

$$x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_2[n]$$

the **system is linear** if

$$\alpha x_1[n] + \beta x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow \alpha y_1[n] + \beta y_2[n]$$

for all α and β and all possible inputs and all times n.

A system is linear if it is both additive and homogeneous.

Time-Invariance

A system is time-invariant if delaying the input signal simply delays the output signal by the same amount of time.

Given

$$x[n] \longrightarrow \boxed{\text{system}} \longrightarrow y[n]$$

the **system is time invariant** if

$$x[n-n_0] \longrightarrow \boxed{\text{system}} \longrightarrow y[n-n_0]$$

for all n_0 and for all possible inputs and all times n.

Time-Invariance

Example: The three-point averager is time invariant.

If

$$x[n] \to y[n] = \frac{1}{3}\Big(x[n-1] + x[n] + x[n+1]\Big)$$

and

$$x_1[n] = x[n - n_o]$$

then

$$x_1[n] \to \frac{1}{3}\Big(x_1[n-1] + x_1[n] + x_1[n+1]\Big)$$

$$x[n-n_o] \to \frac{1}{3}\Big(x[n-n_o-1] + x[n-n_o] + x[n-n_o+1]\Big)$$

$$= y[n-n_o]$$

Representing Systems with Difference Equations

Consider a system represented by the following difference equation:

$$y[n] = x[n] + x[n-1]$$

for all n.

Is this system **linear**?

Representing Systems with Difference Equations

Consider a system represented by the following difference equation:

$$y[n] = x[n] + x[n-1]$$

for all n.

Is this system **linear**?

Assume that $x_1[n] \rightarrow y_1[n]$. Then $y_1[n] = x_1[n] + x_1[n-1]$.

Assume that $x_2[n] \rightarrow y_2[n]$. Then $y_2[n] = x_2[n] + x_2[n-1]$.

Let $x_3[n] = \alpha x_1[n] + \beta x_2[n]$.

From the definition of the system,

$$\begin{aligned}
y_3[n] &= x_3[n] + x_3[n-1] \\
&= \alpha x_1[n] + \beta x_2[n] + \alpha x_1[n-1] + \beta x_2[n-1] \\
&= \alpha x_1[n] + \alpha x_1[n-1] + \beta x_2[n] + \beta x_2[n-1] \\
&= \alpha(x_1[n] + x_1[n-1]) + \beta(x_2[n] + x_2[n-1]) \\
&= \alpha y_1[n] + \beta y_2[n]
\end{aligned}$$

Therefore the system is **linear.**

Representing Systems with Difference Equations

Determining linearity from a difference equation representation.

Example 2.

$$y[n] = x[n] \times x[n-1]$$

for all n.

Is this system **linear**?

Representing Systems with Difference Equations

Determining linearity from a difference equation representation.

Example 2.

$$y[n] = x[n] \times x[n-1]$$

for all n.

Is this system **linear**?

Assume that $x_1[n] \rightarrow y_1[n]$. Then $y_1[n] = x_1[n] \times x_1[n-1]$.

Find the response $y_2[n]$ when $x_2[n] = \alpha x_1[n]$:

$$\begin{aligned} y_2[n] &= x_2[n] \times x_2[n-1] \\ &= \alpha x_1[n] \times \alpha x_1[n-1] \\ &= \alpha^2 x_1[n] \times x_1[n-1] \\ &= \alpha^2 y_1[n] \end{aligned}$$

Multiplying input $x_1[n]$ by α does **not** multiply the output $y_1[n]$ by α. It multiplies $y_1[n]$ by α^2 !

Therefore the system is **neither homogeneous not linear**.

Representing Systems with Difference Equations

Determining linearity from a difference equation representation.

Example 3:

$$y[n] = nx[n]$$

for all n.

Is the system **linear**?

Representing Systems with Difference Equations

Determining linearity from a difference equation representation.

Example 3:

$$y[n] = nx[n]$$

for all n.

Is the system **linear**?

Let $x[n] = \alpha x_1[n] + \beta x_2[n]$.

Then

$$\begin{aligned} y[n] &= n(\alpha x_1[n] + \beta x_2[n]) \\ &= \alpha n x_1[n] + \beta n x_2[n] \\ &= \alpha y_1[n] + \beta y_2[n] \end{aligned}$$

Therefore the system is **linear.**

Representing Systems with Difference Equations

Determining time invariance from a difference equation.

Example 3.

$$y[n] = nx[n]$$

for all n.

Is the system **time-invariant**?

Representing Systems with Difference Equations

Determining time invariance from a difference equation.

Example 3.

$$y[n] = nx[n]$$

for all n.

Is the system **time-invariant**?

If time-invariant, delaying input by 1 should delay output by 1.
Let $x_1[n]$ represent a delayed version of the input.

$$x_1[n] = x[n-1]$$

The corresponding output $y_1[n]$ is given by

$$y_1[n] = nx_1[n] = nx[n-1]$$

This is not the same as delaying the original output:

$$y[n-1] = (n-1)x[n-1]$$

Since $y_1[n] \neq y[n-1]$, the system is **not time-invariant.**

Check Yourself

Assume that a system can be represented by a linear difference equation with constant coefficients.

$$\sum_l c_l y[n-l] = \sum_m d_m x[n-m]$$

Is such a system linear?

Is such a system time invariant?

Linear Difference Equations with Constant Coefficients

If a discrete-time system can be described by a linear difference equation with constant coefficients, then the system is linear and time-invariant.

General form:

$$\sum_l c_l y[n-l] = \sum_m d_m x[n-m]$$

Additivity: output of sum is sum of outputs

$$\sum_l c_l (y_1[n-l] + y_2[n-l]) = \sum_m d_m (x_1[n-m] + x_2[n-m]) \qquad \surd$$

Homogeneity: scaling an input scales its output

$$\sum_l \alpha c_l y[n-l] = \sum_m \alpha d_m x[n-m] \qquad \surd$$

Time invariance: delaying an input delays its output

$$\sum_l c_l y[(n-n_0)-l] = \sum_m d_m x[(n-n_0)-m] \qquad \surd$$

Check Yourself

Consider a system that is defined by

$$y[n] = x[n] + 1$$

Is this system linear?

Is this system time invariant?

Check Yourself

Consider a system that is defined by

$$y[n] = x[n] + 1$$

This system is **not linear**.

It is neither homogeneous nor additive.

This system is **time invariant**.

Check Yourself

Consider a system whose output $y[n]$ is related to its input $x[n]$ as follows:

$$x[n] \quad \rightarrow \quad y[n] = \begin{cases} x[n] & \text{if } x[0] \neq x[1] \\ 0 & \text{otherwise} \end{cases}$$

Is this system homogeneous?

Is this system additive?

Is this system linear?

Check Yourself

Consider a system whose output $y[n]$ is related to its input $x[n]$ as follows:

$$x[n] \quad \rightarrow \quad y[n] = \begin{cases} x[n] & \text{if } x[0] \neq x[1] \\ 0 & \text{otherwise} \end{cases}$$

Is this system **homogeneous**?

$$\alpha x[n] \quad \rightarrow \quad \begin{cases} \alpha x[n] & \text{if } \alpha x[0] \neq \alpha x[1] \\ 0 & \text{otherwise} \end{cases}$$

If $\alpha = 0$:

$$\alpha x[n] = 0 \quad \rightarrow \quad \begin{cases} 0 & \text{if } 0 \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad = \quad 0$$

If $\alpha \neq 0$:

$$\alpha x[n] \quad \rightarrow \quad \begin{cases} \alpha x[n] & \text{if } x[0] \neq x[1] \\ 0 & \text{otherwise} \end{cases} \quad = \quad \alpha y[n]$$

In either case, $\alpha x[n] \rightarrow \alpha y[n]$ so **the system is homogeneous**.

Check Yourself

Consider a system whose output $y[n]$ is related to its input $x[n]$ as follows:

$$x[n] \quad \rightarrow \quad y[n] = \begin{cases} x[n] & \text{if } x[0] \neq x[1] \\ 0 & \text{otherwise} \end{cases}$$

Is this system homogeneous? YES

Is this system **additive**?

The response to $x_1[n] = \delta[n]$ will be $y_1[n] = \delta[n]$, and
The response to $x_2[n] = \delta[n-1]$ will be $y_2[n] = \delta[n-1]$.

But the response to $x_1[n] + x_2[n]$ is 0, which is not $y_1[n] + y_2[n]$.

Therefore **the system is NOT additive**.

Check Yourself

Consider a system whose output $y[n]$ is related to its input $x[n]$ as follows:

$$x[n] \quad \rightarrow \quad y[n] = \begin{cases} x[n] & \text{if } x[0] \neq x[1] \\ 0 & \text{otherwise} \end{cases}$$

Is this system homogeneous? YES

Is this system additive? NO

Is this system **linear**? NO (because it is not additive).

Check Yourself

Consider a system whose output $y[n]$ is the complex conjugate of its input.

Is this system homogeneous?

Is this system additive?

Is this system linear?

Check Yourself

Consider a system whose output $y[n]$ is the complex conjugate of its input.

Is this system homogeneous?

$$x[n] \rightarrow y[n] = x^*[n]$$

$$cx[n] \rightarrow (cx[n])^* = c^*x^*[n] \neq cy[n] = cx^*[n] \text{ unless } \operatorname{Im}(c) = 0$$

Therefore **the system is not homogeneous**.

Check Yourself

Consider a system whose output $y[n]$ is the complex conjugate of its input.

Is this system homogeneous? NO

Is this system additive?

If $x_1[n] \rightarrow y_1[n]$ and $x_2[n] \rightarrow y_2[n]$, then $x_1[n] + y_1[n] \rightarrow x_1^*[n] + x_2^*[n] = y_1[n] + y_2[n]$

Therefore **the system is additive**.

Check Yourself

Consider a system whose output $y[n]$ is the complex conjugate of its input.

Is this system homogeneous? NO

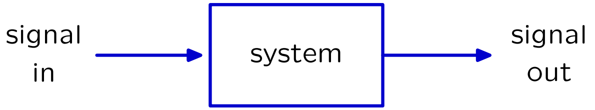
Is this system additive? YES

Is this system linear? NO (because it is not homogeneous).

Summary: System Abstraction

The system abstraction builds on and extends our work with signals.

Goal: characterize a **system** to better understand the relation between two signals.



Three representations for systems:

- **Difference Equation:** algebraic **constraint** on samples $\surd$
- **Convolution:** represent a system by its **unit-sample response**
- **Filter:** represent a system by its **frequency response**