Binomial Theorem Cheatsheet (Exercise 8.3, Class 11)

1. Binomial Series Formula

For $(1+x)^n$, where n is a negative integer or fraction:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Condition of Validity: $|x| < 1$ or as specified for modified forms.

Example: Expand $(1-x)^{1/2}$ up to 4 terms.

$$(1-x)^{1/2} = 1 + \frac{1}{2}(-x) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(-x)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(-x)^3$$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \cdots$$

Valid if $|x| < 1$.

2. General Form for $(a+bx)^n$

Rewrite as:

$$(a+bx)^n = a^n \left(1 + \frac{bx}{a}\right)^n$$

Apply binomial series to $\left(1 + \frac{bx}{a}\right)^n$.

Example: Expand $(4-3x)^{1/2}$ up to 4 terms.

$$(4-3x)^{1/2} = 4^{1/2}\left(1 - \frac{3x}{4}\right)^{1/2} = 2\left[1 + \frac{1}{2}\left(-\frac{3x}{4}\right) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\left(-\frac{3x}{4}\right)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}\left(-\frac{3x}{4}\right)^3\right]$$

$$= 2 - \frac{3x}{4} - \frac{9x^2}{64} - \frac{27x^3}{512} + \cdots$$

Valid if $\left|\frac{3x}{4}\right| < 1 \implies |x| < \frac{4}{3}$.

3. Product of Binomial Expansions

For expressions like $\frac{(a+bx)^p}{(c+dx)^q}$, expand each term separately:

$$(a+bx)^p(c+dx)^{-q}$$

Multiply the series and collect terms up to the required degree.

Example: Expand $\frac{(1-x)^{-1}}{(1+x)^2}$ up to 4 terms.

$$(1-x)^{-1}(1+x)^{-2} = \left(1+x+x^2+x^3\right)\left(1-2x+3x^2-4x^3\right)$$

$$= 1 - x + 2x^2 - 2x^3 + \cdots$$

Valid if $|x| < 1$.

4. Approximations (Neglecting Higher Powers)

When x is small, neglect x^2 or x^3 and higher powers for approximations.

Example: Show $\frac{1-x}{\sqrt{1-x}} \approx 1 - \frac{3}{2}x$ (neglect x^2 and higher).

$$(1-x)(1-x)^{-1/2} = (1-x)\left(1+\frac{1}{2}x\right) = 1 + \frac{1}{2}x - x = 1 - \frac{3}{2}x$$

5. Coefficient of x^n

Use general term: $T_{r+1} = \binom{n}{r}x^r$, or expand and multiply for composite expressions.

Example: Coefficient of x^n in $\frac{(1+x)^2}{(1-x)^2}$.

$$(1+x)^2(1-x)^{-2} = (1+2x+x^2)(1+2x+3x^2+\cdots+(n+1)x^n)$$

Coefficient of x^n: $(n+1) + 2n + (n-1) = 4n$.

6. Numerical Approximations

Use binomial expansion to approximate roots or powers.

Example: Find $\sqrt{99}$ to three decimal places.

$$\sqrt{99} = (100-1)^{1/2} = 10\left(1 - \frac{1}{100}\right)^{1/2} \approx 10\left(1 - \frac{1}{200} - \frac{1}{8000}\right) = 9.950$$

7. Series Identification

Identify series as $(1+x)^n$ by comparing terms to find n and x, then compute the sum.

Example: Sum $1 - \frac{1}{2}\left(\frac{1}{4}\right) + \frac{1\cdot 3}{2!\cdot 4}\left(\frac{1}{4}\right)^2 - \cdots$.

$$n = -\frac{1}{2}, x = \frac{1}{4} \implies \left(1 + \frac{1}{4}\right)^{-1/2} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}.$$

8. Proofs with Small Differences

For $x \approx 1$ or small differences, use $x = 1 + h$ and neglect higher powers.

Example: Prove $px^p - qx^q \approx (p-q)x^{p+q}$ if $x \approx 1$.

$$x = 1 + h, \quad px^p - qx^q \approx p(1+ph) - q(1+qh) = (p-q)(1+(p+q)h).$$