Exercise 2.8: Groups and Algebraic Structures Cheatsheet

1. Group

Definition: A set G with operation $*$ satisfying:
- **Closure**: $a * b \in G$.
- **Associativity**: $(a * b) * c = a * (b * c)$.
- **Identity**: Exists $e \in G$: $a * e = e * a = a$.
- **Inverse**: For each $a \in G$, exists $a^{-1} \in G$: $a * a^{-1} = a^{-1} * a = e$.

Formulas:
- Closure: $a * b \in G$.
- Associativity: $(a * b) * c = a * (b * c)$.
- Identity: $a * e = e * a = a$.
- Inverse: $a * a^{-1} = a^{-1} * a = e$.

Analogy: A team where tasks stay within the team, grouping doesn't matter, a neutral task exists, and tasks are reversible.
Example (Q.1): Set $G = \{0, 1\}$ with $\oplus$:

$\oplus$	0	1
0	0	1
1	1	0

- Closure: All entries in G.
- Associativity: Holds (e.g., $(0 \oplus 1) \oplus 1 = 0 \oplus (1 \oplus 1)$).
- Identity: 0 ($1 \oplus 0 = 1$).
- Inverse: $1^{-1} = 1$ ($1 \oplus 1 = 0$).

2. Abelian Group

Definition: A group where $a * b = b * a$.
Formula: $a * b = b * a$.
Analogy: A team where collaboration order doesn't matter.
Example (Q.2): Set $\{0, 1, 2, 3\}$ with $\oplus$ (addition modulo 4):

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

- Commutative: Symmetric table.
- Group properties: Identity 0, inverses ($1^{-1} = 3$).

3. Semigroup

Definition: A set with a closed, associative operation.
Formulas:

- Closure: $a * b \in G$.
- Associativity: $(a * b) * c = a * (b * c)$.

Analogy: A club where interactions stay within and grouping is consistent.
Example (Q.9): Set $\{a, b, c\}$ with $*$:

$*$	a	b	c
a	c	a	b
b	a	b	c
c	b	c	a

- Closure: All entries in set.
- Associativity: Ensured by completing table.

4. Monoid

Definition: A semigroup with an identity element.
Formula: $a * e = e * a = a$.
Analogy: A club with a neutral member.
Example (Q.8): Power set $P(S)$ with intersection ($\cap$):
- Closure: $A \cap B \in P(S)$.
- Associativity: $A \cap (B \cap C) = (A \cap B) \cap C$.
- Identity: S ($A \cap S = A$).

5. Group Properties Over Specific Sets

Definition: Verifying if a set with an operation forms a group.
Analogy: Auditing a system for all group rules.
Example (Q.3):
- $\mathbb{Q}$ with $+$: Group (identity 0, inverses $-a$).
- $\mathbb{Q}$ with $\times$: Not a group (no inverse for 0).
- $\mathbb{Q}^+$ with $\times$: Group (identity 1, inverses $\frac{1}{a}$).

6. Solving Equations in Groups

Definition: Solve $a * x = b$ or $x * a = b$ in a group.
Formulas:
- $a * x = b \implies x = a^{-1} * b$.
- $x * a = b \implies x = b * a^{-1}$.

Analogy: Solving a puzzle by reversing operations.
Example (Q.6):
- $a * x = b$: $x = a^{-1} * b$.
- $x * a = b$: $x = b * a^{-1}$.

7. Non-Abelian Group

Definition: A group where $a * b \neq b * a$ for some a, b.
Analogy: A team where task order matters.
Example (Q.10): 2×2 non-singular matrices over reals with multiplication:
- Non-commutative: $A \cdot B \neq B \cdot A$.
- Group: Closure, associativity, identity I_2, inverses.

8. Special Sets and Operations

Definition: Analyzing unique sets for group properties.
Analogy: Checking a special system for group structure.
Example (Q.5): Set $\{1, \omega, \omega^2\}$, $\omega^3 = 1$, with multiplication:

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

- Abelian group: Commutative, identity 1, inverses ($\omega^{-1} = \omega^2$).
Example (Q.7): Set $\{a + \sqrt{3}b \mid a, b \in \mathbb{Q}\}$ with addition:
- Abelian group: Identity $0 + \sqrt{3}0$, inverses $-(a + \sqrt{3}b)$.