Exercise 2.6: Relations, Functions, and Inverses Cheatsheet

1. Binary Relation

Definition: A binary relation from set A to set B is any subset of the Cartesian product $A \times B$. If $A = B$, it's a relation on A. It's a set of ordered pairs (x, y), where $x \in A$, $y \in B$.
Analogy: Like a friendship list pairing students from Group A with friends in Group B.
Example (Q.1, Exercise 2.6):
- Set: $A = \{1, 2, 3, 4\}$.
- Cartesian Product: $A \times A = \{(1, 1), (1, 2), \ldots, (4, 4)\}$ (16 pairs).
- Relation: $r_1 = \{(x, y) \mid y = x\} = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$.
- **Explanation**: Pairs each element with itself.

2. Domain and Range

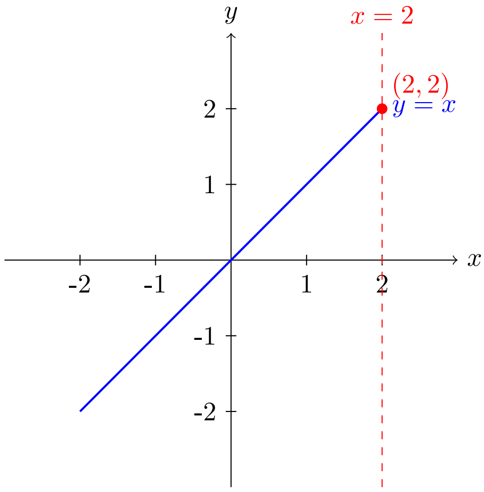
Definition:
- **Domain**: Set of all first elements (x) in the relation's ordered pairs.
- **Range**: Set of all second elements (y) in the ordered pairs.
Analogy: In a dance, domain is students from Group A who danced; range is their partners from Group B.
Example (Q.1, Exercise 2.6):
- Relation: $r_3 = \{(x, y) \mid x + y < 5\} = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)\}$.
- **Domain**: $\{1, 2, 3\}$ (first elements).
- **Range**: $\{1, 2, 3\}$ (second elements).
- **Explanation**: Not all elements of $A = \{1, 2, 3, 4\}$ (e.g., 4) are used.

3. Function

Definition: A relation $f : A \to B$ where each element in A maps to exactly one element in B.
Analogy: A vending machine where each button (input) gives exactly one item (output).
Example (Q.2, Exercise 2.6):
- Relation: $r_1 = \{(x, y) \mid y = x\}$ on $\mathbb{R}$.
- **Why a Function?**: Each x maps to one $y = x$ (e.g., $x = 2 \to y = 2$).
- Compare: $r_3 = \{(x, y) \mid x + y < 5\}$ is not a function (e.g., $x = 1 \to y = 2, 3$).

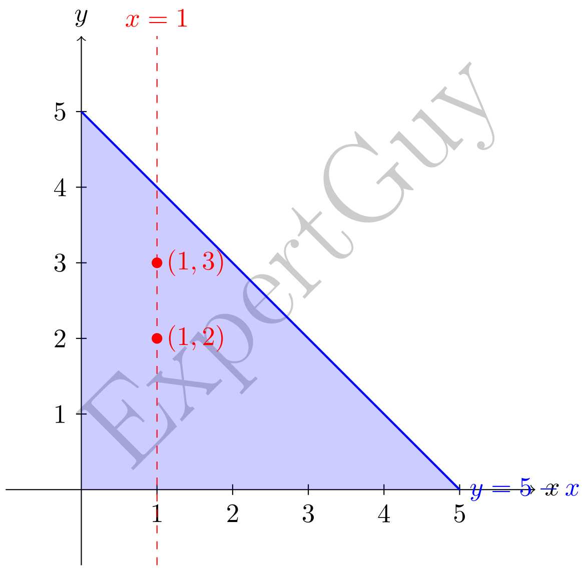
4. Vertical Line Test

Definition: A relation is a function if any vertical line intersects its graph at most one point, ensuring each x maps to one y.
Analogy: A vending machine's log where each button press yields one item.
Example 1: $r_1 = \{(x, y) \mid y = x\}$.
- **Graph**: Line $y = x$.
- **Test**: Vertical line at $x = 2$ intersects at $(2, 2)$, one point.
- **Conclusion**: Function.
Visualization:



Example 2: $r_3 = \{(x, y) \mid x + y < 5\}$.
- **Graph**: Region below $y = 5 - x$.
- **Test**: Vertical line at $x = 1$ intersects all points $(1, y)$, $y < 4$, multiple points.
- **Conclusion**: Not a function.

Visualization:

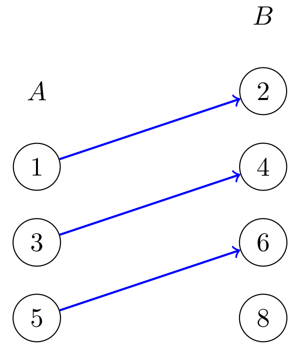


5. Types of Functions

5.1 Into Function

Definition: Range is a proper subset of B (Range$(f) \subset B$). Some elements in B are not outputs.
Analogy: A vending machine where some items can't be dispensed.
Example (Q.3, Exercise 2.6):
- Function: $f = \{(1, 2), (3, 4), (5, 6)\}$, $A = \{1, 3, 5\}$, $B = \{2, 4, 6, 8\}$.
- **Domain**: $\{1, 3, 5\}$.
- **Range**: $\{2, 4, 6\}$.
- **Why Into?**: Range $\neq B$ (8 not an output).

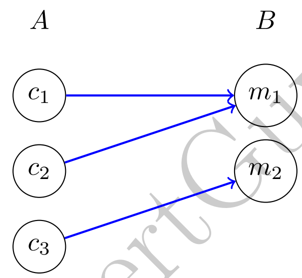
Visualization:



5.2 Onto (Surjective) Function

Definition: Range equals B (Range(f) $= B$). Every element in B is an output.
Analogy: Every item in the vending machine can be dispensed.
Example (Q.3, Exercise 2.6):
- Function: $f = \{(c_1, m_1), (c_2, m_1), (c_3, m_2)\}$, $A = \{c_1, c_2, c_3\}$, $B = \{m_1, m_2\}$.
- **Domain**: $\{c_1, c_2, c_3\}$.
- **Range**: $\{m_1, m_2\}$.
- **Why Onto?**: Range $= B$.

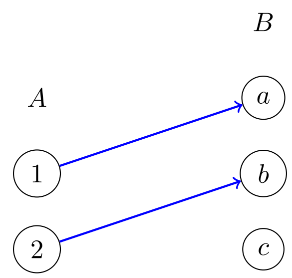
Visualization:



5.3 Injective (One-to-One) Function

Definition: No two distinct elements in A map to the same element in B. Range has no repeated elements.
Analogy: Each vending machine item comes from a unique button.
Example (Q.3, Exercise 2.6):
- Function: $f = \{(1, a), (2, b)\}$, $A = \{1, 2\}$, $B = \{a, b, c\}$.
- **Domain**: $\{1, 2\}$.
- **Range**: $\{a, b\}$.
- **Why Injective?**: Each output from a unique input.
- **Why Into?**: Range $\neq B$ (no output to c).

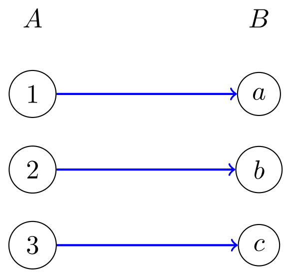
Visualization:



5.4 Bijective (One-to-One and Onto) Function

Definition: Both injective (no repeated outputs) and onto (range $= B$). A one-to-one correspondence.
Analogy: Each vending machine item has exactly one button, covering all items.
Example (Q.3, Exercise 2.6):
- Function: $f = \{(1, a), (2, b), (3, c)\}$, $A = \{1, 2, 3\}$, $B = \{a, b, c\}$.
- **Domain**: $\{1, 2, 3\}$.

- **Range**: $\{a, b, c\}$.
- **Why Bijective?**: Injective (unique outputs) and onto (range $= B$).
Visualization:



6. Inverse of a Relation/Function

Definition: The inverse of a relation $R \subseteq A \times B$ is $R^{-1} \subseteq B \times A$, obtained by swapping pairs: if $(x, y) \in R$, then $(y, x) \in R^{-1}$. For R^{-1} to be a function, each element in the domain of R^{-1} (range of R) must map to exactly one element (i.e., R must be injective).
Analogy: Like reversing a vending machine: items (outputs) become buttons, and buttons (inputs) become items.
Example (Q.4, Exercise 2.6):
- Relation: $r = \{(2, 1), (3, 2), (4, 3), (5, 4), (6, 5)\}$, $A = \{2, 3, 4, 5, 6\}$, $B = \{1, 2, 3, 4, 5\}$.
- **Domain**: $\{2, 3, 4, 5, 6\}$.
- **Range**: $\{1, 2, 3, 4, 5\}$.
- **Is r a Function?**: Yes, each input maps to one output.
- **Inverse**: $r^{-1} = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$.
- **Is r^{-1} a Function?**: Yes, each input (1, 2, 3, 4, 5) maps to one output (injective r).
Visualization: