Cheatsheet: Systems of Linear Equations (Exercise 3.5)

Class 11 Mathematics (Chapter 3)

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1. Cramer's Rule (Q1)

Definition

Solve $AX = B$ where A is an $n \times n$ coefficient matrix, X is the variable vector, and B is the constant vector. If $|A| \neq 0$, a unique solution exists.

Formula

For a 3x3 system:

$$x_i = \frac{|A_i|}{|A|}, \quad i = 1, 2, 3$$

where A_i is A with the i-th column replaced by B.

Steps

1. Compute $|A|$. If $|A| \neq 0$, proceed.
2. Form A_1, A_2, A_3 by replacing columns of A with B.
3. Compute $|A_1|, |A_2|, |A_3|$.
4. Calculate $x_i = \frac{|A_i|}{|A|}$.

Example

For system (Q1(i)):

$$\begin{aligned} 2x + 2y + z &= 13 \\ 3x - 2y - 2z &= 1 \\ 5x + y - 3z &= 2 \end{aligned}$$

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{bmatrix}, \quad |A| = 27$$

Solution: $x = 1, y = 0, z = 1$.

2. Matrix Inversion Method (Q2)

Definition

Solve $AX = B$ using $X = A^{-1}B$, where A^{-1} exists if $|A| \neq 0$.

Formula

$$A^{-1} = \frac{\text{Adj } A}{|A|}, \quad \text{Adj } A = (\text{cofactor matrix})^t$$

Cofactor $A_{ij} = (-1)^{i+j} \cdot$ minor.

Steps

1. Compute $|A|$. If $|A| \neq 0$, proceed.
2. Find cofactors A_{ij}.
3. Form Adj A by transposing cofactor matrix.
4. Compute $A^{-1} = \frac{\text{Adj } A}{|A|}$.
5. Calculate $X = A^{-1}B$.

Example

For system (Q2(i)):

$$\begin{aligned} x - 2y + z &= -1 \\ 3x + y - 2z &= 4 \\ y - z &= 1 \end{aligned}$$

$$A^{-1} = -\frac{1}{2} \begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -3 & 7 \end{bmatrix}$$

Solution: $x = 1, y = 1, z = 0$.

3. Row Reduction Method (Q3)

Echelon Form

- Leading entry in each non-zero row is 1.
- Zeros before leading 1 increase in successive rows.

Reduced Echelon Form

- Echelon form + column of leading 1 has zeros elsewhere.

Steps

1. Form augmented matrix $[A|B]$.
2. Use row operations (swap, scale, add/subtract) to reach echelon or reduced echelon form.
3. For echelon: Back-substitute to solve.
4. For reduced echelon: Read solutions directly.

Example

For system (Q3(i)):

$$\begin{aligned} x_1 - 2x_2 - 2x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 1 \\ 5x_1 - 4x_2 - 3x_3 &= 1 \end{aligned}$$

Augmented matrix:

$$\begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 2 & 3 & 1 & : & 1 \\ 5 & -4 & -3 & : & 1 \end{bmatrix}$$

Solution: $x_1 = \frac{11}{19}, x_2 = -\frac{9}{19}, x_3 = \frac{24}{19}$.

4. Homogeneous Systems (Q4)

Definition

Solve $AX = 0$. If $|A| = 0$, non-trivial solutions exist (infinite solutions).

Steps

1. Compute $|A|$. If $|A| = 0$, proceed.
2. Use two equations to form ratios:
$$\frac{x_1}{\det(M_1)} = \frac{-x_2}{\det(M_2)} = \frac{x_3}{\det(M_3)} = t$$
where M_i are 2x2 submatrices.
3. Express x_1, x_2, x_3 in terms of t.

Example

For system (Q4(i)):

$$\begin{aligned} x + 2y - 2z &= 0 \\ 2x + y + 5z &= 0 \\ 5x + 4y + 8z &= 0 \end{aligned}$$

$$|A| = 0, \quad \frac{x}{-4} = \frac{y}{3} = \frac{z}{1} = t$$

Solution: $x = -4t, y = 3t, z = t$.

5. Non-Trivial Solutions with Parameter λ (Q5)

Steps

1. Set $|A| = 0$ to find λ.
2. For each λ, solve the homogeneous system as in Q4.

Example

For system (Q5(i)):

$$\begin{aligned} x + y + z &= 0 \\ 2x + y - \lambda z &= 0 \\ x + 2y - 2z &= 0 \end{aligned}$$

$$|A| = 0 \implies \lambda = -5$$

Solution: $x = -4t, y = 3t, z = t$.

6. Non-Unique Solutions with Parameter λ (Q6)

Steps

1. Set $|A| = 0$ to find λ.
2. For each λ, reduce augmented matrix $[A|B]$ to echelon form.
3. Assign free variables and solve.

Example

For system (Q6):

$$\begin{aligned} x_1 + 4x_2 + \lambda x_3 &= 2 \\ 2x_1 + x_2 - 2x_3 &= 11 \\ 3x_1 + 2x_2 - 2x_3 &= 16 \end{aligned}$$

$$|A| = 0 \implies \lambda = 6$$

Solution: $x_1 = 2t + 6, x_2 = -2t - 1, x_3 = t$.