Exercise 3.1: Matrices and Determinants Cheat Sheet

Definitions

- **Matrix**: Rectangular array of numbers in brackets, e.g., $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$.
- **Order**: $m \times n$ for m rows, n columns, e.g., $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ is 2×3.
- **Equal Matrices**: Same order, corresponding elements equal, e.g., $\begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix}$ implies $x = 1$, $y = 2$.
- **Transpose**: Interchange rows and columns, denoted A^t, e.g., if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $A^t = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

Matrix Operations

- **Addition**: Same order, add corresponding elements:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

- **Subtraction**: Same order, subtract corresponding elements:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

- **Scalar Multiplication**: Multiply each element by scalar k:

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

- **Matrix Multiplication**: $A_{m\times n} \cdot B_{n\times p}$ yields $C_{m\times p}$. Element $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$, e.g.:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Key Properties

- **Addition/Subtraction**: Requires same order.
- **Multiplication**: Number of columns of A = number of rows of B.
- **Scalar Multiplication**:

$$\lambda(\mu A) = (\lambda\mu)A, \quad (\lambda+\mu)A = \lambda A+\mu A, \quad \lambda(A+B) = \lambda A$$

- **Transpose**: $(A+B)^t = A^t + B^t$.
- **Matrix Equations**: For $AX = B$, if A is invertible, $X = A^{-1}B$. For $XA = B$, $X = BA^{-1}$.

Examples

1. **Scalar and Addition** (Q1-like):

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}, \quad 4A-3A = \begin{bmatrix} 8 & 12 \\ 4 & 20 \end{bmatrix} - \begin{bmatrix} 6 & 9 \\ 3 & 15 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} =$$

2. **Matrix Addition/Subtraction** (Q4-like):

$$A = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 3 \\ 1 & -1 \end{bmatrix}, \quad 4A-3B = \begin{bmatrix} -4 & 8 \\ 4 & 0 \end{bmatrix} -$$

3. **Equal Matrices** (Q5-like):

$$\begin{bmatrix} 2 & x \\ y & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & x \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$$

Solve: $2+2 = 4$, $2x = -2 \implies x = -1$, $y+4 = 6 \implies y = 2$.

4. **Matrix Power** (Q9-like):

$$A = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

5. **Matrix Equation** (Q12-like):

$$X \begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 12 & 3 \end{bmatrix}$$

Compute A^{-1} where $|A| = 9$, adj $A = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$, so $A^{-1} = \frac{1}{9}\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$. Then:

$$X = \begin{bmatrix} -1 & 5 \\ 12 & 3 \end{bmatrix} \cdot \frac{1}{9}\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

6. **3x3 Multiplication** (Q14-like):

$$\begin{bmatrix} r\cos\phi & 0 & -\sin\phi \\ 0 & r & 0 \\ r\sin\phi & 0 & \cos\phi \end{bmatrix} \cdot \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -r\sin\phi & 0 & r\cos\phi \end{bmatrix} = r\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Uses identity: $\cos^2\phi + \sin^2\phi = 1$.

Tips

- Check matrix orders before operations.
- For inverses, ensure $|A| \neq 0$.
- In equal matrices, solve element-wise equations.
- For matrix powers, compute step-by-step (e.g., $A^4 = A^2 \cdot A^2$).
- Verify solutions by substituting back into equations.