Cheatsheet: Roots of Unity and Polynomials (Exercise 4.4)

Class 11 Mathematics (Chapter 4)

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Overview

Exercise 4.4 focuses on cube and fourth roots of unity, their properties, and solving polynomial equations reducible to quadratic forms. It also introduces polynomial functions, the Remainder Theorem, and the Factor Theorem (PDF pp.257–269).

> **Note**
>
> Use properties like $\omega^3 = 1$ and $1+\omega+\omega^2 = 0$ to simplify expressions involving cube roots of unity.

Cube Roots of Unity

Definition The cube roots of unity satisfy $x^3 = 1$. Solving $x^3 - 1 = 0$:

$$(x-1)(x^2+x+1) = 0 \implies x = 1 \text{ or } x = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = \omega, \omega^2$$

Cube roots: $1, \omega = \frac{-1+i\sqrt{3}}{2}, \omega^2 = \frac{-1-i\sqrt{3}}{2}$ (p.258).

Properties

(i) Each complex cube root is the square of the other: $\omega^2 = (\omega^2)^2$, and vice versa.

(ii) Sum: $1 + \omega + \omega^2 = 0$.

(iii) Product: $1 \cdot \omega \cdot \omega^2 = \omega^3 = 1$.

(iv) For any integer n, ω^n is one of $1, \omega, \omega^2$, e.g., $\omega^4 = \omega$, $\omega^{-5} = \omega$ (p.258).

Example Find cube roots of 8 (Q.1, p.259):

$$x^3 = 8 \implies x^3-8 = 0 \implies (x-2)(x^2+2x+4) = 0 \implies x = 2 \text{ or } x = \frac{-2 \pm 2i\sqrt{3}}{2} = 2\omega, 2\omega^2$$

Cube roots: $2, 2\omega, 2\omega^2$.

Fourth Roots of Unity

Definition The fourth roots of unity satisfy $x^4 = 1$. Solving $x^4 - 1 = 0$:

$$(x^2-1)(x^2+1) = 0 \implies x^2 = 1 \text{ or } x^2 = -1 \implies x = \pm 1, \pm i$$

Fourth roots: $1, -1, i, -i$ (p.259).

Properties

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(i) Sum: $1 + (-1) + i + (-i) = 0$.

(ii) Real roots ($1, -1$) are additive inverses: $1 + (-1) = 0$.

(iii) Complex roots ($i, -i$) are conjugates.

(iv) Product: $1 \cdot (-1) \cdot i \cdot (-i) = -1$ (p.259).

Example Find fourth roots of 16 (Q.7, p.267):

$$x^4 = 16 \implies x^4 - 16 = 0 \implies (x^2 - 4)(x^2 + 4) = 0 \implies x = \pm 2, \pm 2i$$

Fourth roots: $2, -2, 2i, -2i$.

Polynomial Equations

Solving Higher-Degree Equations Equations like $x^4 = k$ or $x^5 = k$ can be factored into quadratics or linear terms using roots of unity (Q.8, p.268).
Example Solve $2x^4 - 32 = 0$ (Q.8(i), p.268):

$$x^4 = 16 \implies (x^2 - 4)(x^2 + 4) = 0 \implies x^2 = 4 \text{ or } x^2 = -4 \implies x = \pm 2, \pm 2i$$

Solution set: $\{\pm 2, \pm 2i\}$.

Polynomial Theorems

Remainder Theorem If a polynomial $f(x)$ of degree $n \geq 1$ is divided by $x - a$, the remainder is $f(a)$ (p.269).
Factor Theorem $x - a$ is a factor of $f(x)$ if and only if $f(a) = 0$ (p.269).

Tip

Use the sum and product properties of roots of unity to simplify complex expressions, e.g., $\omega + \omega^2 = -1$.

Key Reminders

- Cube roots of -1: $-1, \frac{1 \pm i\sqrt{3}}{2}$ (Q.5, p.265).
- Use $\omega^3 = 1$ to reduce powers, e.g., $\omega^{28} = \omega$ (Q.2, p.262).
- Factorize polynomials using difference of cubes or squares to find roots.
- Verify solutions for polynomial equations to ensure correctness.

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