Trigonometric Identities Cheatsheet - Exercise 10.4

1. Product-to-Sum Identities

1.1 Key Formulas

$$\begin{aligned}
2\sin\alpha\cos\beta &= \sin(\alpha+\beta)+\sin(\alpha-\beta)\\
2\cos\alpha\sin\beta &= \sin(\alpha+\beta)-\sin(\alpha-\beta)\\
2\cos\alpha\cos\beta &= \cos(\alpha+\beta)+\cos(\alpha-\beta)\\
-2\sin\alpha\sin\beta &= \cos(\alpha+\beta)-\cos(\alpha-\beta)
\end{aligned}$$

Example: Express $2\sin 3\theta\cos\theta$ as a sum.

$$2\sin 3\theta\cos\theta = \sin(3\theta+\theta)+\sin(3\theta-\theta) = \sin 4\theta+\sin 2\theta$$

Note: Multiply and divide by 2 when needed, e.g., $\sin\alpha\cos\beta = \frac{1}{2}[2\sin\alpha\cos\beta]$.

2. Sum-to-Product Identities

$$\begin{aligned}
\sin P+\sin Q &= 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)\\
\sin P-\sin Q &= 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)\\
\cos P+\cos Q &= 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)\\
\cos P-\cos Q &= -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)
\end{aligned}$$

Example: Express $\sin 5\theta+\sin 3\theta$ as a product.

$$\sin 5\theta+\sin 3\theta = 2\sin\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right) = 2\sin 4\theta\cos\theta$$

3. Proving Identities

Use product-to-sum and sum-to-product identities to simplify and prove trigonometric equalities. **Example:** Prove $\frac{\sin 3x-\sin x}{\cos x-\cos 3x} = \cot 2x$.

$$\frac{\sin 3x-\sin x}{\cos x-\cos 3x} = \frac{2\cos\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)}{-2\sin\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)} = \frac{2\cos 2x\sin x}{2\sin 2x\sin x} = \frac{\cos 2x}{\sin 2x} = \cot 2x$$

4. Product of Multiple Trigonometric Functions

For products of multiple sines or cosines, iteratively apply product-to-sum identities. **Example:** Prove $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{16}$.

$$\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{2} \cos 40^\circ \cos 20^\circ \cos 80^\circ$$

$$= \frac{1}{4}[\cos 60^\circ + \cos 20^\circ] \cos 80^\circ = \frac{1}{4} \left[\frac{1}{2} + \cos 20^\circ\right] \cos 80^\circ$$

$$= \frac{1}{8}[\cos 80^\circ + \cos 100^\circ + \cos 60^\circ] = \frac{1}{8} \cdot \frac{1}{2} = \frac{1}{16}$$

5. Specific Angle Applications

Use known values (e.g., $\cos 60^\circ = \frac{1}{2}$, $\cos 90^\circ = 0$) to simplify expressions. **Example:** Prove $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$.

$$\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 2 \cos 80^\circ \cos 60^\circ + \cos 100^\circ$$

$$= \cos 80^\circ + \cos 100^\circ = 2 \cos 90^\circ \cos 10^\circ = 0$$

6. Advanced Identities

$$\frac{\sin 8x + \sin 2x}{\cos 8x + \cos 2x} = \tan 5x$$

$$\frac{\sin \alpha - \sin \beta}{\sin \alpha + \sin \beta} = \tan \left(\frac{\alpha - \beta}{2}\right) \cot \left(\frac{\alpha + \beta}{2}\right)$$

$$\frac{\sin \theta + \sin 3\theta + \sin 5\theta + \sin 7\theta}{\cos \theta + \cos 3\theta + \cos 5\theta + \cos 7\theta} = \tan 4\theta$$

Example: Prove $\sin\left(\frac{\pi}{4} - \theta\right) \sin\left(\frac{\pi}{4} + \theta\right) = \frac{1}{2} \cos 2\theta$.

$$\sin\left(\frac{\pi}{4} - \theta\right) \sin\left(\frac{\pi}{4} + \theta\right) = \left(\frac{\sqrt{2}}{2} \cos \theta - \frac{\sqrt{2}}{2} \sin \theta\right) \left(\frac{\sqrt{2}}{2} \cos \theta + \frac{\sqrt{2}}{2} \sin \theta\right) = \frac{1}{2}(\cos^2 \theta - \sin^2 \theta) = \frac{1}{2} \cos 2$$

7. Applications

- **Physics:** Sum-to-product identities simplify wave interference equations.
- **Signal Processing:** Product-to-sum identities aid in frequency analysis.