Partial Fractions Cheatsheet - Exercise 5.4 (Class 11 Mathematics)

Prepared for Entry Test Preparation

1. Concept of Partial Fractions

Partial fractions decompose a rational function $\frac{P(x)}{Q(x)}$ (where the degree of $P(x) < Q(x)$) into simpler fractions. If the degree of $P(x) \geq Q(x)$, perform polynomial division first to obtain a quotient and a proper fraction.

Key Rule: The denominator $Q(x)$ is factored into linear and/or irreducible quadratic factors, and the partial fraction form is set based on these factors. Exercise 5.4 focuses on repeated quadratic factors and combinations with linear factors.

2. Types of Denominator Factors and Corresponding Partial Fraction Forms

Denominator Factor	Partial Fraction Form	Example Denominator	Partial Fraction Setup
Linear: $(x-a)$	$\frac{A}{x-a}$	$(x-1)$	$\frac{A}{x-1}$
Repeated Linear: $(x-a)^n$	$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}$	$(x+1)^2$	$\frac{A}{x+1} + \frac{B}{(x+1)^2}$
Irreducible Quadratic: (x^2+bx+c)	$\frac{Ax+B}{x^2+bx+c}$	(x^2+1)	$\frac{Ax+B}{x^2+1}$
Repeated Quadratic: $(x^2+bx+c)^n$	$\frac{A_1x+B_1}{x^2+bx+c} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(x^2+bx+c)^n}$	$(x^2+x+1)^2$	$\frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2}$

3. Steps to Resolve into Partial Fractions

1. **Factor the Denominator**: Express $Q(x)$ as a product of linear and/or irreducible quadratic factors, including repeated factors.
2. **Set Up Partial Fractions**: Write the partial fraction form with appropriate numerators for each factor (constant for linear, linear for quadratic).
3. **Clear Denominator**: Multiply both sides by the denominator to obtain a polynomial equation.
4. **Solve for Constants**:
   - *Method 1: Substitution*: Substitute roots of linear factors (e.g., $x = a$ for $(x-a)$) to find constants.

   - *Method 2: Equate Coefficients*: Expand the right-hand side and equate coefficients of corresponding powers of x.
5. **Write Final Form**: Substitute constants back into the partial fraction setup.

4. Special Case: Improper Fractions

If the degree of $P(x) \geq Q(x)$, divide $P(x)$ by $Q(x)$ to get:

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where $S(x)$ is the quotient and $R(x)$ is the remainder (degree of $R(x) < Q(x)$). Then, resolve $\frac{R(x)}{Q(x)}$ into partial fractions.

5. Examples from Exercise 5.4

Example 1: Repeated Quadratic Factors

Problem: $\frac{x^3+2x+2}{(x^2+x+1)^2}$

- **Setup**: $\frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2}$
- **Solve**:
  - Equate coefficients of x^3: $A = 1$.
  - Equate coefficients of x^2: $A + B = 0 \implies B = -1$.
  - Equate coefficients of x: $A + B + C = 2 \implies C = 2$.
  - Equate constant terms: $B + D = 2 \implies D = 3$.
- **Result**: $\frac{x-1}{x^2+x+1} + \frac{2x+3}{(x^2+x+1)^2}$

Example 2: Linear and Repeated Quadratic Factors

Problem: $\frac{x^2}{(x^2+1)^2(x-1)}$

- **Setup**: $\frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$
- **Solve**:
  - Put $x = 1$: $1 = A(2)^2 \implies A = \frac{1}{4}$.
  - Equate coefficients of x^4: $A + B = 0 \implies B = -\frac{1}{4}$.
  - Equate coefficients of x^3: $-B + C = 0 \implies C = -\frac{1}{4}$.
  - Equate coefficients of x^2: $2A + B - C + D = 1 \implies D = \frac{1}{2}$.
  - Equate coefficients of x: $-B + C - D + E = 0 \implies E = \frac{1}{2}$.
- **Result**: $\frac{1}{4(x-1)} - \frac{x+1}{4(x^2+1)} + \frac{x+2}{2(x^2+1)^2}$

Example 3: Multiple Linear and Repeated Quadratic Factors

Problem: $\frac{8x^2}{(1-x)(1+x)(x^2+1)^2}$

- **Setup**: $\frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$
- **Solve**:
  - Put $x = 1$: $8 = A(2)(4) \implies A = 1$.
  - Put $x = -1$: $8 = B(2)(4) \implies B = 1$.
  - Equate coefficients of x^5: $A - B - C = 0 \implies C = 0$.
  - Equate coefficients of x^4: $A + B - D = 0 \implies D = 2$.
  - Equate coefficients of x^3: $2A - 2B - E = 0 \implies E = 0$.
- **Result**: $\frac{1}{1-x} + \frac{1}{1+x} + \frac{2}{x^2+1} - \frac{4}{(x^2+1)^2}$

6. Key Formulas

- Linear factor $(x - a)$: $\frac{A}{x-a}$.
- Repeated linear factor $(x - a)^n$: $\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}$.
- Irreducible quadratic $(x^2 + bx + c)$: $\frac{Ax+B}{x^2+bx+c}$.
- Repeated quadratic $(x^2 + bx + c)^n$: $\frac{A_1x+B_1}{x^2+bx+c} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(x^2+bx+c)^n}$.