Trigonometric Identities Cheatsheet - Exercise 10.3

1. Double-Angle Identities

1.1 Key Formulas

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$
$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{\sin 2\alpha}{\cos 2\alpha}$$

Example: If $\sin\alpha = \frac{12}{13}$, $0 < \alpha < \frac{\pi}{2}$, find $\sin 2\alpha$, $\cos 2\alpha$, $\tan 2\alpha$.

$$\cos\alpha = \sqrt{1 - \frac{144}{169}} = \frac{5}{13}$$

$$\sin 2\alpha = 2\cdot\frac{12}{13}\cdot\frac{5}{13} = \frac{120}{169}, \quad \cos 2\alpha = \frac{25}{169} - \frac{144}{169} = -\frac{119}{169}, \quad \tan 2\alpha = \frac{\frac{120}{169}}{-\frac{119}{169}} = -\frac{120}{119}$$

2. Half-Angle Identities

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}, \quad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$
$$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha}$$

Example: Prove $\frac{1-\cos\alpha}{\sin\alpha} = \tan\frac{\alpha}{2}$.

$$\frac{1-\cos\alpha}{\sin\alpha} = \frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \tan\frac{\alpha}{2}$$

3. Triple-Angle Identities

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

Example: Prove $\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 2$.

$$\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = \frac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\sin\theta\cos\theta} = \frac{\sin 2\theta}{\sin\theta\cos\theta} = \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = 2$$

4. Cotangent and Secant Identities

$$\cot 2\alpha = \frac{\cos 2\alpha}{\sin 2\alpha}, \quad \sec 2\alpha = \frac{1}{\cos 2\alpha}$$
$$\cot\alpha - \tan\alpha = 2\cot 2\alpha$$

Example: Prove $\cot\alpha - \tan\alpha = 2\cot 2\alpha$.

$$2\cot 2\alpha = 2\frac{\cos 2\alpha}{\sin 2\alpha} = \frac{2(\cos^2\alpha - \sin^2\alpha)}{2\sin\alpha\cos\alpha} = \frac{\cos^2\alpha}{\sin\alpha\cos\alpha} - \frac{\sin^2\alpha}{\sin\alpha\cos\alpha} = \cot\alpha - \tan\alpha$$

5. Specific Angle Evaluations

Use identities to find exact values for angles like 18°, 36°, 54°, 72°, 144°.

$$\sin 18^\circ = \frac{\sqrt{5}-1}{4}, \quad \cos 18^\circ = \sqrt{\frac{10+2\sqrt{5}}{16}}$$
$$\cos 36^\circ = \frac{\sqrt{5}+1}{4}, \quad \sin 36^\circ = \sqrt{\frac{10-2\sqrt{5}}{16}}$$
$$\sin 54^\circ = \frac{\sqrt{5}+1}{4}, \quad \cos 54^\circ = \sqrt{\frac{10-2\sqrt{5}}{16}}$$
$$\sin 72^\circ = \sqrt{\frac{10+2\sqrt{5}}{16}}, \cos 72^\circ = \frac{\sqrt{5}-1}{4}$$
$$\sin 144^\circ = \sqrt{\frac{10-2\sqrt{5}}{16}}, \quad \cos 144^\circ = -\frac{\sqrt{5}+1}{4}$$

Example: Prove $\cos 36^\circ \cos 72^\circ \cos 108^\circ \cos 144^\circ = \frac{1}{16}$.

$$\cos 108^\circ = \cos(180^\circ - 72^\circ) = -\cos 72^\circ$$

$$\cos 36^\circ \cos 72^\circ (-\cos 72^\circ) \cos 144^\circ = \left(\frac{\sqrt{5}+1}{4}\right)\left(\frac{\sqrt{5}-1}{4}\right)\left(-\frac{\sqrt{5}-1}{4}\right)\left(-\frac{\sqrt{5}+1}{4}\right) = \left(\frac{4}{16}\right)^2 = \frac{1}{16}$$

6. Power Reduction

Reduce higher powers of trigonometric functions to first powers of multiple angles.

$$\sin^4\theta = \left(\frac{1-\cos 2\theta}{2}\right)^2 = \frac{3 - 4\cos 2\theta + \cos 4\theta}{8}$$

Example: Reduce $\sin^4\theta$.

$$\sin^4\theta = \frac{1 - 2\cos 2\theta + \frac{1+\cos 4\theta}{2}}{4} = \frac{3 - 4\cos 2\theta + \cos 4\theta}{8}$$

7. Advanced Identities

$$\frac{\sin 2\alpha}{1+\cos 2\alpha} = \tan\alpha, \quad \frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha} = \sec 2\alpha - \tan 2\alpha$$
$$\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}} = \frac{\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}}, \quad \frac{\cos 3\theta}{\cos\theta} + \frac{\sin 3\theta}{\sin\theta} = 4\cos 2\theta$$

Example: Prove $\frac{\sin 2\theta \sin 2\theta}{\cos\theta + \cos 3\theta} = \tan\theta \tan 2\theta$.

$$\frac{2\sin\theta\sin 2\theta}{\cos\theta + (4\cos^3\theta - 3\cos\theta)} = \frac{2\sin\theta\sin 2\theta}{2\cos\theta(2\cos^2\theta - 1)} = \frac{\sin 2\theta}{\cos 2\theta}\cdot\frac{\sin\theta}{\cos\theta} = \tan 2\theta\tan\theta$$

8. Applications

- **Physics:** Double-angle identities simplify wave equations.
- **Engineering:** Power reduction aids in signal processing.