Exercise 3.2: Matrices and Determinants Cheat Sheet

Definitions

- **Field**: Non-empty set F with addition (abelian group), multiplication (abelian group excluding 0), and right distributive law, e.g., $\mathbb{R}$, $\mathbb{C}$.
- **Identity Matrix**: I_n with 1s on diagonal, 0s elsewhere, e.g., $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- **Determinant**: For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $|A| = ad - bc$.
- **Adjoint**: For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, adj $A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- **Inverse**: For A with $|A| \neq 0$, $A^{-1} = \frac{1}{|A|}\text{adj } A$, satisfies $A^{-1}A = I$.
- **Matrix Equation**: Form $AX = B$ or $kX - mA = B$, solved using inverses or algebra.

Key Properties

- **Identity**: For $A_{m\times n}$, $I_m A = A$ and $AI_n = A$.
- **Inverse**: $A^{-1}A = I_n = AA^{-1}$ if $|A| \neq 0$.
- **Subtraction**: $A - B$ requires same order, subtract corresponding elements.
- **Associative**: $(AB)C = A(BC)$ for compatible matrices.
- **Distributive**: $(A + B)C = AC + BC$ for compatible matrices.
- **Non-commutative**: Generally, $AB \neq BA$, so $(A + B)^2 \neq A^2 + 2AB + B^2$, $(A - B)^2 \neq A^2 - 2AB + B^2$, $(A + B)(A - B) \neq A^2 - B^2$.
- **Transpose Products**: For $A_{m\times n}$, AA^t is $m \times m$, A^tA is $n \times n$.

Formulas

- **Determinant**: $|A| = ad - bc$ for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
- **Inverse**: $A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- **System of Equations**: $AX = B \implies X = A^{-1}B$, where $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.
- **Matrix Equation**: $kX - mA = B \implies X = \frac{1}{k}(mA + B)$.
- **Matrix Equation with Multiplication**: $CA = D \implies A = C^{-1}D$ if $|C| \neq 0$.

Examples

1. **Identity Matrix** (Q1-like): For $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$, $I_3 A = A$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

2. **Inverse** (Q2-like): For $A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$, $|A| = 3 \cdot 1 - (-1) \cdot 2 = 5$, adj $A = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$, so:

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}.$$

3. **System of Equations** (Q3-like): Solve $2x - 3y = 5$, $5x + y = 4$:

$$\begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

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$|A| = 2 \cdot 1 - (-3) \cdot 5 = 17$, adj $A = \begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}$, $A^{-1} = \frac{1}{17} \begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}$. Then:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

4. **Subtraction** (Q4-like): For $A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$:

$$A - B = \begin{bmatrix} 1-2 & -1-1 \\ 3-1 & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}.$$

5. **Associative Property** (Q5-like): For $A = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}$, $B = \begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}$, $C = \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$, $(AB)C = A(BC)$ holds (verified by matrix multiplication).

6. **Matrix Equation** (Q8-like): Solve $3X - 2A = B$, where $A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 5 & 4 \end{bmatrix}$:

$$3X = 2\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix}, \quad X = \frac{1}{3}\begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

7. **Matrix Equation with Multiplication** (Q9-like): Solve $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$:

$$|C| = 4{\cdot}2 - 3{\cdot}2 = 2, \quad \text{adj } C = \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}, \quad C^{-1} = \frac{1}{2}\Bigg[$$

$$A = \frac{1}{2}\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -7 \\ 3 & 9 \end{bmatrix}.$$

Tips

- Check $|A| \neq 0$ before computing inverse.
- Verify matrix orders for multiplication and subtraction.
- For $AX = B$, use $X = A^{-1}B$; for $CA = D$, use $A = C^{-1}D$.
- Non-commutative: $AB \neq BA$, so check order in properties like $(A+B)^2$.
- Simplify complex numbers using $i^2 = -1$ (e.g., Q2, Q5).

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