Partial Fractions Cheatsheet - Exercise 5.3 (Class 11 Mathematics)

Prepared for Entry Test Preparation

1. Concept of Partial Fractions

Partial fractions decompose a rational function $\frac{P(x)}{Q(x)}$ (where the degree of $P(x) < Q(x)$) into simpler fractions. If the degree of $P(x) \geq Q(x)$, perform polynomial division first.

Key Rule: The denominator $Q(x)$ is factored into linear and/or irreducible quadratic factors, and the partial fraction form is set based on these factors.

2. Types of Denominator Factors and Corresponding Partial Fraction Forms

Denominator Factor	Partial Fraction Form	Example Denominator	Partial Fraction Setup
Linear: $(x-a)$	$\frac{A}{x-a}$	$(x-1)$	$\frac{A}{x-1}$
Repeated Linear: $(x-a)^n$	$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}$	$(x+2)^2$	$\frac{A}{x+2} + \frac{B}{(x+2)^2}$
Irreducible Quadratic: (x^2+bx+c)	$\frac{Ax+B}{x^2+bx+c}$	(x^2+1)	$\frac{Ax+B}{x^2+1}$

3. Steps to Resolve into Partial Fractions

1. **Factor the Denominator**: Express $Q(x)$ as a product of linear and/or irreducible quadratic factors.
2. **Set Up Partial Fractions**: Based on the factor type, write the partial fraction form with unknown constants (e.g., A, B, C).
3. **Clear Denominator**: Multiply both sides by the denominator to get a polynomial equation.
4. **Solve for Constants**:
   - *Method 1: Substitution*: Substitute roots of linear factors (e.g., $x = a$ for $(x-a)$) to find constants.
   - *Method 2: Equate Coefficients*: Expand the right-hand side and equate coefficients of corresponding powers of x.
5. **Write Final Form**: Substitute constants back into the partial fraction setup.

4. Special Case: Improper Fractions

If the degree of $P(x) \geq Q(x)$, divide $P(x)$ by $Q(x)$ to get:

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where $S(x)$ is the quotient and $R(x)$ is the remainder (degree of $R(x) < Q(x)$). Then, resolve $\frac{R(x)}{Q(x)}$ into partial fractions.

Example: For $\frac{x^4}{1-x^4}$:

- Rewrite: $\frac{x^4}{1-x^4} = \frac{-x^4}{x^4-1}$.
- Divide: $\frac{-x^4}{x^4-1} = -1 - \frac{1}{x^4-1}$.
- Resolve $\frac{1}{x^4-1} = \frac{1}{(x+1)(x-1)(x^2+1)}$ into:

$$\frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x^2+1)}$$

- Final form: $-1 + \frac{1}{4(x+1)} - \frac{1}{4(x-1)} + \frac{1}{2(x^2+1)}$.

5. Examples from Exercise 5.3

Example 1: Linear and Quadratic Factors

Problem: $\frac{9x-7}{(x^2+1)(x+3)}$

- **Setup**: $\frac{Ax+B}{x^2+1} + \frac{C}{x+3}$
- **Solve**:
  - Put $x = -3$: $9(-3) - 7 = C(9+1) \implies -34 = 10C \implies C = -\frac{17}{5}$.
  - Equate coefficients of x^2: $A + C = 0 \implies A = \frac{17}{5}$.
  - Equate coefficients of x: $3A + B = 9 \implies B = -\frac{6}{5}$.
- **Result**: $\frac{17x-6}{5(x^2+1)} - \frac{17}{5(x+3)}$

Example 2: Repeated Linear Factors

Problem: $\frac{1}{(x-1)^2(x^2+2)}$

- **Setup**: $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+2}$
- **Solve**:
  - Put $x = 1$: $1 = B(1+2) \implies B = \frac{1}{3}$.
  - Equate coefficients of x^3: $A + C = 0 \implies A = -C$.

  - Solve system: $A = -\frac{2}{9}$, $C = \frac{2}{9}$, $D = -\frac{1}{9}$.
- **Result**: $\frac{-2}{9(x-1)} + \frac{1}{3(x-1)^2} + \frac{2x-1}{9(x^2+2)}$

Example 3: Multiple Linear and Quadratic Factors

Problem: $\frac{x^2+2x+2}{(x^2+3)(x+1)(x-1)}$

- **Setup**: $\frac{Ax+B}{x^2+3} + \frac{C}{x+1} + \frac{D}{x-1}$
- **Solve**:
  - Put $x = -1$: $1 = C(1+3)(-2) \implies C = -\frac{1}{8}$.
  - Put $x = 1$: $5 = D(1+3)(2) \implies D = \frac{5}{8}$.
  - Equate coefficients: $A = -\frac{1}{2}$, $B = \frac{1}{4}$.
- **Result**: $\frac{1-2x}{4(x^2+3)} - \frac{1}{8(x+1)} + \frac{5}{8(x-1)}$

6. Key Formulas

- For linear factor $(x-a)$: Partial fraction is $\frac{A}{x-a}$.
- For repeated linear factor $(x-a)^n$: Partial fractions are $\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}$.
- For irreducible quadratic (x^2+bx+c): Partial fraction is $\frac{Ax+B}{x^2+bx+c}$.
- Polynomial division: $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$.