Inverse Trigonometric Functions Cheatsheet
Exercise 13.2

1 Key Identities

1.1 Sum and Difference Formulas

- **Arcsine**: $\sin^{-1} A + \sin^{-1} B = \sin^{-1}\left(A\sqrt{1-B^2} + B\sqrt{1-A^2}\right)$

$$\text{Result in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

- **Arctangent**: $\tan^{-1} A + \tan^{-1} B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$, $AB < 1$

$$\text{Result in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

- **Arccosine**: $\cos^{-1} A + \cos^{-1} B = \cos^{-1}\left(AB - \sqrt{(1-A^2)(1-B^2)}\right)$

1.2 Double-Angle Formulas

- $\sin 2\theta = 2\sin\theta\cos\theta$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta$
- $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

1.3 General Identities

- $\cos(\sin^{-1} x) = \sqrt{1-x^2}$
- $\sin(2\cos^{-1} x) = 2x\sqrt{1-x^2}$
- $\cos(2\sin^{-1} x) = 1 - 2x^2$
- $\tan^{-1}(-x) = -\tan^{-1} x$
- $\sin^{-1}(-x) = -\sin^{-1} x$
- $\cos^{-1}(-x) = \pi - \cos^{-1} x$
- $\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}$

2 Key Techniques

2.1 Proving Sum/Difference Identities

1. Let $\sin^{-1} A = \alpha$, $\sin^{-1} B = \beta$, compute $\sin(\alpha+\beta)$ or $\cos(\alpha-\beta)$.

2. Use right triangle: For $\tan^{-1}\frac{a}{b}$, opposite a, adjacent b, hypotenuse $\sqrt{a^2+b^2}$.
3. Example: Q.1: $\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{7}{25}=\cos^{-1}\frac{253}{325}$.

$$\cos\left(\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{7}{25}\right)=\frac{12}{13}\cdot\frac{24}{25}-\frac{5}{13}\cdot\frac{7}{25}=\frac{253}{325}.$$

2.2 Proving Double-Angle Identities

1. Let $\cos^{-1}x=\theta$, use $\sin 2\theta=2\sin\theta\cos\theta$.
2. Example: Q.4: $\tan^{-1}\frac{120}{119}=2\cos^{-1}\frac{12}{13}$.

$$\tan 2\theta=\frac{2\cdot\frac{5}{12}}{1-\left(\frac{5}{12}\right)^2}=\frac{120}{119}.$$

2.3 General Identities

1. Substitute $\sin^{-1}x=\alpha$, compute trigonometric function.
2. Example: Q.13: $\cos(\sin^{-1}x)=\sqrt{1-x^2}$.

$$\sin\alpha=x,\ \cos\alpha=\sqrt{1-x^2}.$$

2.4 Evaluating Functions

1. For $x=\sin^{-1}a$, compute all trigonometric ratios.
2. Example: Q.20: $x=\sin^{-1}\frac{1}{2}\implies \sin x=\frac{1}{2},\cos x=\frac{\sqrt{3}}{2},\tan x=\frac{1}{\sqrt{3}},\csc x=2,\sec x=\frac{2}{\sqrt{3}},\cot x=\sqrt{3}$.

3 Common Values

- $\sin^{-1}\frac{1}{2}=\frac{\pi}{6}$, $\sin^{-1}1=\frac{\pi}{2}$.
- $\cos^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{6}$, $\cos^{-1}\frac{1}{2}=\frac{\pi}{3}$.
- $\tan^{-1}1=\frac{\pi}{4}$, $\tan^{-1}\frac{1}{\sqrt{3}}=\frac{\pi}{6}$.
- $\sin\frac{\pi}{6}=\frac{1}{2}$, $\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$, $\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$.

4 Problem Types

- **Sum/Difference Identities**: Prove sums equal to inverse function (e.g., Q.6: $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{77}{85}$).

$$\sin^{-1}\left(\frac{3}{5}\cdot\frac{15}{17}+\frac{8}{17}\cdot\frac{4}{5}\right)=\sin^{-1}\frac{77}{85}.$$

- **Double-Angle Identities**: Relate multiple angles to inverse (e.g., Q.3: $2\tan^{-1}\frac{2}{3} = \sin^{-1}\frac{12}{13}$).
$$\sin 2\theta = 2 \cdot \frac{2}{\sqrt{13}} \cdot \frac{3}{\sqrt{13}} = \frac{12}{13}.$$
- **General Identities**: Prove trigonometric identities (e.g., Q.19: $\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}$).
$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{x}{\sqrt{1-x^2}}.$$
- **Evaluation**: Compute trigonometric functions (e.g., Q.20: $x = \sin^{-1}\frac{1}{2}$).

5 Tips and Tricks

- Check domain: Ensure $|x| \leq 1$ for $\sin^{-1} x$, $AB < 1$ for $\tan^{-1} A + \tan^{-1} B$.
- Simplify fractions: Rationalize denominators (e.g., $\frac{2}{\sqrt{13}}$).
- Use identities: $\cos(\sin^{-1} x) = \sqrt{1-x^2}$, $\sin 2\theta = 2\sin\theta\cos\theta$.
- Right triangle: For $\tan^{-1}\frac{a}{b}$, use opposite a, adjacent b.
- Verify signs: Ensure result matches principal range (e.g., $\sin^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$).
- Combine terms: Simplify numerators/denominators before taking inverse (e.g., Q.9: $\frac{513-88}{209+216} = 1$).

6 Applications

- **Physics**: Angle sums in optics or mechanics.
- **Calculus**: Simplify integrals with inverse functions.
- **Geometry**: Compute angles in complex figures.