Trigonometric Identities Cheatsheet - Exercise 10.1

1. Fundamental Law of Trigonometry

1.1 Core Identity

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Deductions:

- $\cos\left(\frac{\pi}{2} - \beta\right) = \sin\beta$
- $\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin\alpha$
- $\sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha$
- $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
- $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
- $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
- $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

Example: Find $\sin 150^\circ$.

$$\sin 150^\circ = \sin(180^\circ - 30^\circ) = \sin 180^\circ \cos 30^\circ - \cos 180^\circ \sin 30^\circ = 0 \cdot \frac{\sqrt{3}}{2} - (-1) \cdot \frac{1}{2} = \frac{1}{2}$$

2. Double Angle Identities

$$\begin{aligned}
\sin 2\alpha &= 2\sin\alpha\cos\alpha \\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha \\
\tan 2\alpha &= \frac{2\tan\alpha}{1 - \tan^2\alpha}
\end{aligned}$$

Example: Verify $\cos 2 \cdot 30^\circ = 2\cos^2 30^\circ - 1$.

$$\begin{aligned}
\text{LHS} &= \cos 60^\circ = \frac{1}{2} \\
\text{RHS} &= 2\left(\frac{\sqrt{3}}{2}\right)^2 - 1 = 2 \cdot \frac{3}{4} - 1 = \frac{3}{2} - 1 = \frac{1}{2} \\
\text{LHS} &= \text{RHS}
\end{aligned}$$

3. Triple Angle Identities

$$\begin{aligned}
\sin 3\alpha &= 3\sin\alpha - 4\sin^3\alpha \\
\cos 3\alpha &= 4\cos^3\alpha - 3\cos\alpha \\
\tan 3\alpha &= \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}
\end{aligned}$$

Example: Find $\sin 3\cdot 30^\circ$.

$$\sin 3\cdot 30^\circ = \sin 90^\circ = 1; \quad 3\sin 30^\circ - 4\sin^3 30^\circ = 3\cdot\frac{1}{2} - 4\left(\frac{1}{2}\right)^3 = \frac{3}{2} - 4\cdot\frac{1}{8} = \frac{3}{2} - \frac{1}{2} = 1$$

4. Half Angle Identities

$$\begin{aligned}
\sin\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{2}} \\
\cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}} \\
\tan\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}
\end{aligned}$$

Sign Rule: Choose $\pm$ based on the quadrant of $\frac{\alpha}{2}$. **Example:** Find $\sin 15^\circ$.

$$\sin 15^\circ = \sin\frac{30^\circ}{2} = \sqrt{\frac{1-\cos 30^\circ}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

5. Sum, Difference, and Product Identities

$$\begin{aligned}
2\sin\alpha\cos\beta &= \sin(\alpha+\beta) + \sin(\alpha-\beta) \\
2\cos\alpha\sin\beta &= \sin(\alpha+\beta) - \sin(\alpha-\beta) \\
2\cos\alpha\cos\beta &= \cos(\alpha+\beta) + \cos(\alpha-\beta) \\
-2\sin\alpha\sin\beta &= \cos(\alpha+\beta) - \cos(\alpha-\beta) \\
\sin P + \sin Q &= 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2} \\
\sin P - \sin Q &= 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2} \\
\cos P + \cos Q &= 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2} \\
\cos P - \cos Q &= -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}
\end{aligned}$$

Example: Prove $\cos 306^\circ + \cos 234^\circ + \cos 162^\circ + \cos 18^\circ = 0$.

$$\begin{aligned}
\text{LHS} &= \cos 306^\circ + \cos 234^\circ + \cos 162^\circ + \cos 18^\circ \\
&= \cos(360^\circ - 54^\circ) + \cos(180^\circ + 54^\circ) + \cos(180^\circ - 18^\circ) + \cos 18^\circ \\
&= \cos 54^\circ - \cos 54^\circ - \cos 18^\circ + \cos 18^\circ = 0 = \text{RHS}
\end{aligned}$$

6. Evaluating Trigonometric Functions

Reduce angles to equivalent angles in $[0^\circ, 360^\circ)$.

$$\theta = k \cdot 360^\circ + \phi \quad \text{or} \quad \theta = k \cdot 2\pi + \phi$$

Quadrant Rules:

- $k \mod 4 = 0$: 1st or 4th quadrant
- $k \mod 4 = 1$: 2nd or 1st quadrant
- $k \mod 4 = 2$: 3rd or 2nd quadrant
- $k \mod 4 = 3$: 4th or 3rd quadrant

Example: Find $\sin(-780^\circ)$.

$$-780^\circ = -(2 \cdot 360^\circ + 60^\circ) = -720^\circ - 60^\circ; \quad \sin(-780^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$$

7. Triangle Angle Identities

For a triangle with angles α, β, γ, where $\alpha + \beta + \gamma = 180^\circ$:

$$\begin{aligned}
&\sin(\alpha + \beta) = \sin\gamma \\
&\cos(\alpha + \beta) = -\cos\gamma \\
&\cos\left(\frac{\alpha + \beta}{2}\right) = \sin\frac{\gamma}{2} \\
&\tan(\alpha + \beta) + \tan\gamma = 0
\end{aligned}$$

Example: Prove $\sin(\alpha + \beta) = \sin\gamma$.

$$\alpha + \beta = 180^\circ - \gamma; \quad \sin(\alpha + \beta) = \sin(180^\circ - \gamma) = \sin\gamma$$

8. Applications

- **Physics:** Use sum-to-product identities in wave interference.
- **Engineering:** Apply angle sum identities in signal processing.