Cheatsheet: Quadratic Equations and Roots (Exercise 4.6)

Class 11 Mathematics (Chapter 4)

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Overview

Exercise 4.6 focuses on manipulating the roots of quadratic equations, forming new equations with transformed roots, proving identities involving roots, and finding conditions for specific root relationships.

> **Note**
>
> Use the sum and product of roots to simplify expressions and form new equations efficiently.

Roots of a Quadratic Equation

Concept For a quadratic equation $ax^2 + bx + c = 0$ with roots α, β:

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}$$

Key Identities

- $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
- $\alpha^3 + \beta^3 = (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta]$
- $\alpha^4 + \beta^4 = [(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2$
- $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$
- $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$

Example For $3x^2 - 2x + 4 = 0$, find $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$:

$$\alpha + \beta = \frac{2}{3}, \quad \alpha\beta = \frac{4}{3}$$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2}, \quad \alpha^2 + \beta^2 = \left(\frac{2}{3}\right)^2 - 2 \cdot \frac{4}{3} = \frac{4}{9} - \frac{8}{3} = \frac{-20}{9}$$

$$\frac{\frac{-20}{9}}{\left(\frac{4}{3}\right)^2} = \frac{-20}{9} \cdot \frac{9}{16} = -\frac{5}{4}$$

Forming Equations with Transformed Roots

Concept To form a quadratic equation with roots $f(\alpha), f(\beta)$, compute:

$$\text{Sum} = f(\alpha) + f(\beta), \quad \text{Product} = f(\alpha)f(\beta)$$

The equation is:

$$x^2 - (\text{sum})x + \text{product} = 0$$

Steps

1. Use $\alpha+\beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$.
2. Express the new sum and product in terms of $\alpha+\beta$, $\alpha\beta$.
3. Substitute and simplify to form the equation.

Example For $5x^2 - x - 2 = 0$, form the equation with roots $\frac{3}{\alpha}, \frac{3}{\beta}$:

$$\alpha+\beta = \frac{1}{5}, \quad \alpha\beta = -\frac{2}{5}$$

$$\text{Sum} = \frac{3}{\alpha} + \frac{3}{\beta} = \frac{3(\alpha+\beta)}{\alpha\beta} = \frac{3\cdot\frac{1}{5}}{-\frac{2}{5}} = -\frac{3}{2}$$

$$\text{Product} = \frac{3}{\alpha}\cdot\frac{3}{\beta} = \frac{9}{\alpha\beta} = \frac{9}{-\frac{2}{5}} = -\frac{45}{2}$$

$$x^2 - \left(-\frac{3}{2}\right)x - \frac{45}{2} = 0 \implies 2x^2 + 3x - 45 = 0$$

Proving Identities Involving Roots

Concept Use the sum and product of roots to prove identities or relationships.

Example For $x^2 - px - p - c = 0$, prove $(1+\alpha)(1+\beta) = 1 - c$:

$$\alpha+\beta = p, \quad \alpha\beta = -p - c$$

$$(1+\alpha)(1+\beta) = 1 + \alpha + \beta + \alpha\beta = 1 + p + (-p - c) = 1 - c$$

Conditions for Specific Root Relationships

Concept Determine coefficients p, q in $x^2 + px + q = 0$ such that roots satisfy given conditions (e.g., one root is double, square, or inverse of the other).

Examples

- **One root double the other**: Roots $\alpha, 2\alpha$:

$$\alpha + 2\alpha = -p \implies 3\alpha = -p, \quad \alpha\cdot 2\alpha = q \implies 2\alpha^2 = q$$

$$\alpha = -\frac{p}{3}, \quad 2\left(-\frac{p}{3}\right)^2 = q \implies 2p^2 = 9q$$

- **Roots differ by 1**: Roots $\alpha, \alpha - 1$:

$$2\alpha - 1 = p, \quad \alpha(\alpha - 1) = q \implies p^2 = 4q + 1$$

- **Additive inverse**: Roots $\alpha, -\alpha$:

$$\alpha - \alpha = -p \implies p = 0, \quad \alpha\cdot(-\alpha) = q \implies q = -\alpha^2$$

Tip

Simplify fractions and verify calculations to avoid errors in root transformations.

Key Reminders

- Always express higher powers or reciprocals of roots using $\alpha + \beta$, $\alpha\beta$.
- For complex transformations (e.g., $\frac{1-\alpha}{1+\alpha}$), compute sum and product carefully.
- Check conditions for real or complex roots when dealing with differences like $\alpha^2 - \beta^2$.
- Use algebraic identities to simplify expressions (e.g., $\alpha^3 + \beta^3$).