Cheatsheet: Equations Reducible to Quadratic Form (Exercise 4.2)

Class 11 Mathematics (Chapter 4)

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Overview

Equations that appear higher-degree or non-quadratic can often be reduced to the quadratic form $ay^2 + by + c = 0$ using appropriate substitutions. Five types are covered in Exercise 4.2 (PDF pp.216–240).

> **Note**
>
> Always verify solutions by substituting back into the original equation, especially for equations with negative or fractional exponents.

Type I: Equations of the Form $ax^{2n} + bx^n + c = 0$

Concept Substitute $x^n = y$ to reduce to $y^2 + by + c = 0$. Solve for y, then find x using $x^n = y$.

Steps

1. Identify the exponent n and set $x^n = y$.
2. Rewrite the equation in terms of y.
3. Solve the quadratic equation in y using factorization or the quadratic formula.
4. Solve for x by taking the n-th root of y.

Example Solve $x^4 - 6x^2 + 8 = 0$:

$$\begin{aligned}
x^4 - 6x^2 + 8 &= 0 \\
\text{Let } x^2 = y \implies y^2 - 6y + 8 &= 0 \\
(y-4)(y-2) &= 0 \implies y = 4, 2 \\
x^2 = 4 \implies x = \pm 2, \quad x^2 = 2 &\implies x = \pm\sqrt{2}
\end{aligned}$$

Solution set: $\{\pm\sqrt{2}, \pm 2\}$.

Type II: Equations of the Form $(x+a)(x+b)(x+c)(x+d) = k$, where $a + b = c + d$

Concept Pair terms such that $(x + a)(x + d)$ and $(x + b)(x + c)$ share a common quadratic form. Substitute to reduce to a quadratic equation.

Steps

1. Identify pairs where $a + d = b + c$.
2. Group as $[(x + a)(x + d)][(x + b)(x + c)] = k$.

3. Simplify each pair to $x^2 + (a+d)x + ad$, set equal to y.
4. Solve the resulting quadratic in y, then solve for x.

Example Solve $(x-5)(x-7)(x+6)(x+4) - 504 = 0$:

$$\begin{aligned}(-5+4) = (-7+6) &= -1\\ [(x-5)(x+4)][(x-7)(x+6)] &= 504\\ (x^2-x-20)(x^2-x-42) &= 504\\ \text{Let } x^2-x=y \implies (y-20)(y-42) &= 504\\ y^2-62y+336 = 0 &\implies (y-56)(y-6)=0\\ y = 56, 6 &\implies x^2-x-56=0, \quad x^2-x-6=0\\ x = 8, -7, &\quad x = 3, -2\end{aligned}$$

Solution set: $\{-7, -2, 3, 8\}$.

> **Tip**
>
> Check the condition $a+b = c+d$ carefully to pair terms correctly.

Type III: Exponential Equations

Concept Equations with variables in exponents, e.g., $a^{2x} + ba^x + c = 0$. Substitute $a^x = y$ to form a quadratic in y.
Steps

1. Rewrite exponents to share a common base.
2. Set $a^x = y$ and form a quadratic in y.
3. Solve for y, then solve $a^x = y$ using logarithms or exponent rules.

Example Solve $4 \cdot 2^{2x+1} - 9 \cdot 2^x + 1 = 0$:

$$\begin{aligned}8 \cdot (2^x)^2 - 9 \cdot 2^x + 1 &= 0\\ \text{Let } 2^x = y \implies 8y^2 - 9y + 1 &= 0\\ (8y-1)(y-1) = 0 &\implies y = \frac{1}{8}, 1\\ 2^x = \frac{1}{8} = 2^{-3} \implies x = -3, &\quad 2^x = 1 \implies x = 0\end{aligned}$$

Solution set: $\{-3, 0\}$.

Type IV: Reciprocal Equations

Concept Equations unchanged when x is replaced by $\frac{1}{x}$, e.g., $ax^4+bx^3+cx^2+bx+a = 0$. Use substitution $x + \frac{1}{x} = y$ or divide by x^2.
Steps

1. Divide by x^2 to group terms like $x^2 + \frac{1}{x^2}$ and $x + \frac{1}{x}$.

2. Set $x + \frac{1}{x} = y$, note $x^2 + \frac{1}{x^2} = y^2 - 2$.
3. Solve the quadratic in y, then solve for x.

Example Solve $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$:

$$6x^2 - 35x + 62 - \frac{35}{x} + \frac{6}{x^2} = 0$$
$$6\left(x^2 + \frac{1}{x^2}\right) - 35\left(x + \frac{1}{x}\right) + 62 = 0$$
$$\text{Let } x + \frac{1}{x} = y, \quad x^2 + \frac{1}{x^2} = y^2 - 2$$
$$6(y^2 - 2) - 35y + 62 = 0 \implies 6y^2 - 35y + 50 = 0$$
$$(3y - 10)(2y - 5) = 0 \implies y = \frac{10}{3}, \frac{5}{2}$$
$$x + \frac{1}{x} = \frac{10}{3} \implies x = 3, \frac{1}{3}, \quad x + \frac{1}{x} = \frac{5}{2} \implies x = 2, \frac{1}{2}$$

Solution set: $\left\{\frac{1}{3}, \frac{1}{2}, 2, 3\right\}$.

Tip

Use $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$ to simplify expressions.

Type V: Radical Equations

Concept Equations with radical expressions, e.g., $l(ax^2+bx)+m\sqrt{ax^2+bx+c} = 0$. Substitute the radical term to eliminate square roots.

Steps

1. Set the radical expression, e.g., $\sqrt{ax^2+bx+c} = y$.
2. Express the quadratic term in terms of y^2.
3. Form and solve a quadratic in y, then solve for x.
4. Check for extraneous roots by substituting back.

Example Solve $3x^2 + 2x - \sqrt{3x^2 + 2x - 1} = 3$:

$$\text{Let } \sqrt{3x^2 + 2x - 1} = y \implies 3x^2 + 2x - 1 = y^2$$
$$3x^2 + 2x = y^2 + 1$$
$$y^2 + 1 - y = 3 \implies y^2 - y - 2 = 0$$
$$(y - 2)(y + 1) = 0 \implies y = 2, -1$$
$$3x^2 + 2x - 5 = 0 \implies x = 1, -\frac{5}{3}, \quad \text{No solution for } y = -1$$

Solution set: $\left\{-\frac{5}{3}, 1\right\}$.

Warning

Radical equations may introduce extraneous roots. Always verify solutions.

Key Reminders

- For Type I, ensure the correct root (e.g., square or cube root) is applied.
- For Type II, verify the pairing condition $a + b = c + d$.
- For Type III, check the base of exponents for consistency.
- For Type IV, confirm the equation's symmetry for reciprocal properties.
- For Type V, exclude negative y values for square roots and check for extraneous roots.