Cheatsheet: Radical Equations (Exercise 4.3)

Class 11 Mathematics (Chapter 4)

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Overview

Radical equations involve square roots of variable expressions and can often be reduced to quadratic equations. Solutions must be verified to eliminate extraneous roots introduced during squaring (PDF pp.240–257).

Warning

Always check solutions by substituting back into the original equation, as squaring may introduce extraneous roots.

Type I: Equations of the Form $l(ax^2+bx)+m\sqrt{ax^2+bx+c}=k$

Concept Substitute the radical term, e.g., $\sqrt{ax^2+bx+c}=y$, to eliminate the square root and form a quadratic equation.

Steps

1. Set $\sqrt{ax^2+bx+c}=y$, so $ax^2+bx+c=y^2$.
2. Rewrite the original equation using y and solve the resulting quadratic in y.
3. Substitute y back to solve for x using the quadratic formula or factoring.
4. Verify solutions to exclude extraneous roots.

Example Solve $3x^2+2x-\sqrt{3x^2+2x-1}=3$ (Q.1, p.241):

$$\text{Let } \sqrt{3x^2+2x-1}=y \implies 3x^2+2x-1=y^2 \implies 3x^2+2x=y^2+1$$
$$y^2+1-y=3 \implies y^2-y-2=0 \implies (y-2)(y+1)=0 \implies y=2,-1$$
$$y=2: \quad 3x^2+2x-5=0 \implies x=1,-\frac{5}{3}$$
$$y=-1: \quad 3x^2+2x-2=0 \implies x=\frac{-1\pm\sqrt{7}}{3}$$
$$\text{Check: } x=1,-\frac{5}{3} \text{ satisfy; } \frac{-1\pm\sqrt{7}}{3} \text{ are extraneous.}$$

Solution set: $\left\{-\frac{5}{3},1\right\}$.

Type II: Equations of the Form $\sqrt{x+a}+\sqrt{x+b}=\sqrt{x+c}$

Concept Square both sides to eliminate radicals, isolate remaining radicals, and square again if necessary to obtain a quadratic equation.

Steps

1. Square both sides to expand the equation.
2. Isolate the remaining radical term.
3. Square again to eliminate the radical, forming a quadratic in x.
4. Solve for x and verify solutions.

Example Solve $\sqrt{2x+8}+\sqrt{x+5}=7$ (Q.3, p.244):

$$(\sqrt{2x+8}+\sqrt{x+5})^2 = 49 \implies 2x+8+x+5+2\sqrt{(2x+8)(x+5)} = 49$$
$$2\sqrt{2x^2+18x+40} = 36-3x \implies \sqrt{2x^2+18x+40} = \frac{36-3x}{2}$$
$$4(2x^2+18x+40) = (36-3x)^2 \implies x^2-288x+1136=0 \implies x = 4, 284$$
$$\text{Check: } x=4 \text{ satisfies; } x=284 \text{ is extraneous.}$$

Solution set: $\{4\}$.

Tip

Ensure the expressions under square roots are non-negative when checking solutions.

Type III: Equations of the Form $\sqrt{ax^2+bx+c}+\sqrt{px^2+qx+r}=\sqrt{lx^2+mx+n}$

Concept Factor out common terms (if possible) or use substitutions like $\sqrt{ax^2+bx+c}=a$, $\sqrt{px^2+qx+r}=b$, and solve using a^2-b^2 and $a+b$.

Steps

1. Factorize to identify common terms, e.g., $\sqrt{x-1}$.
2. Set the factored equation to zero or use substitutions $a=\sqrt{ax^2+bx+c}$, $b=\sqrt{px^2+qx+r}$.
3. Solve the resulting quadratic equations and verify solutions.

Example Solve $\sqrt{3x^2-5x+2}+\sqrt{6x^2-11x+5}=\sqrt{5x^2-9x+4}$ (Q.9, p.251):

$$\sqrt{(x-1)(3x-2)}+\sqrt{(x-1)(6x-5)} = \sqrt{(x-1)(5x-4)}$$
$$\sqrt{x-1}[\sqrt{3x-2}+\sqrt{6x-5}-\sqrt{5x-4}] = 0$$
$$x=1 \text{ or } \sqrt{3x-2}+\sqrt{6x-5} = \sqrt{5x-4}$$
$$9x-7+2\sqrt{(3x-2)(6x-5)} = 5x-4 \implies 56x^2-84x+31=0$$
$$x = \frac{21\pm\sqrt{7}}{28}, 1 \text{ (extraneous check confirms } x=1\text{).}$$

Solution set: $\{1\}$.

Type IV: Equations of the Form $\sqrt{ax^2+bx+c}+\sqrt{px^2+qx+r}=mx+n$

Concept Substitute the square root terms or move one radical to the other side, square both sides, and simplify to a quadratic equation.
Steps

1. Isolate one radical or use substitutions like $\sqrt{ax^2+bx+c} = a$, $\sqrt{px^2+qx+r} = b$.
2. Use $a^2 - b^2 = (a-b)(a+b)$ to form equations.
3. Solve for x and verify to exclude extraneous roots.

Example Solve $\sqrt{5x^2+7x+2} - \sqrt{4x^2+7x+18} = x-4$ (Q.12, p.256):

$$\begin{aligned} \text{Let } a = \sqrt{5x^2+7x+2}, b = \sqrt{4x^2+7x+18} &\implies a-b = x-4 \\ a^2-b^2 = x^2-16 \implies a+b = x+4 &\implies a = x \\ 5x^2+7x+2 = x^2 \implies 4x^2+7x+2 = 0 &\implies x = \frac{-7 \pm \sqrt{17}}{8} \\ \text{Check: Both solutions are extraneous.} \end{aligned}$$

Solution set: $\emptyset$.

Note

For Type IV, the substitution $a = x$ often simplifies the equation significantly.

Key Reminders

- Always verify solutions by substituting back into the original equation.
- Check domain constraints (e.g., non-negative expressions under square roots).
- Use substitutions to simplify complex radical expressions.
- Be cautious with squaring, as it may introduce extraneous roots.