Triangle Radii Cheatsheet
Exercise 12.8

1 Triangle Radii Fundamentals

1.1 Definition and Notation

In $\triangle ABC$, angles α, β, γ are opposite sides a, b, c. Key radii:

- **Inradius** (r): Radius of incircle.
- **Circumradius** (R): Radius of circumcircle.
- **Exradii** (r_1, r_2, r_3): Radii of excircles opposite vertices A, B, C.
- **Semi-perimeter**: $S = \frac{a+b+c}{2}$.
- **Area**: $\Delta = \sqrt{S(S-a)(S-b)(S-c)}$.

1.2 Key Formulas

- **Radii**:

$$r = \frac{\Delta}{S}, \quad R = \frac{abc}{4\Delta}, \quad r_1 = \frac{\Delta}{S-a}, \quad r_2 = \frac{\Delta}{S-b}, \quad r_3 = \frac{\Delta}{S-c}$$

- **Half-Angle Identities**:

$$\sin\frac{\alpha}{2} = \sqrt{\frac{(S-b)(S-c)}{bc}}, \quad \cos\frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}}, \quad \tan\frac{\alpha}{2} = \sqrt{\frac{(S-b)(S-c)}{S(S-a)}}$$

(similar for β, γ).

- **Key Identities**:

$$r = 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}, \quad S = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$r = a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2} = b\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}\sec\frac{\beta}{2} = c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2}$$

$$r_1 = 4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}, \quad r_2 = 4R\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}, \quad r_3 = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}$$

$$r_1 = S\tan\frac{\alpha}{2}, \quad r_2 = S\tan\frac{\beta}{2}, \quad r_3 = S\tan\frac{\gamma}{2}$$

$$r_1r_2 + r_2r_3 + r_3r_1 = S^2, \quad r_1r_2r_3 = \Delta^2, \quad r_1 + r_2 + r_3 - r = 4R$$

$$\Delta = r^2\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}, \quad r = S\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}, \quad \Delta = 4Rr\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$\frac{1}{2rR} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}, \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

$$r = \frac{a \sin\frac{\beta}{2} \sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}} = \frac{b \sin\frac{\alpha}{2} \sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}} = \frac{c \sin\frac{\alpha}{2} \sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}$$

$$abc(\sin\alpha + \sin\beta + \sin\gamma) = 4\Delta S, \quad (r_1 + r_2)\tan\frac{\gamma}{2} = c, \quad (r_3 - r)\cot\frac{\gamma}{2} = c$$

2 Proving Identities

2.1 Steps for Identities

1. Substitute definitions: $r = \frac{\Delta}{S}$, $R = \frac{abc}{4\Delta}$, $r_1 = \frac{\Delta}{S-a}$, etc.
2. Use half-angle identities: $\sin\frac{\alpha}{2} = \sqrt{\frac{(S-b)(S-c)}{bc}}$, etc.
3. Simplify square roots and fractions to show L.H.S. = R.H.S.
4. For equilateral triangles, set $a = b = c$, compute $\Delta = \frac{\sqrt{3}a^2}{4}$, and simplify ratios.

2.2 Steps for Numerical Calculations

1. Compute semi-perimeter: $S = \frac{a+b+c}{2}$.
2. Calculate area: $\Delta = \sqrt{S(S-a)(S-b)(S-c)}$.
3. Compute radii: $r = \frac{\Delta}{S}$, $R = \frac{abc}{4\Delta}$, $r_1 = \frac{\Delta}{S-a}$, etc.
4. Report results exact or to three decimal places.

2.3 Example: Numerical Calculation

Given: $a = 13, b = 14, c = 15$.

$$S = \frac{13+14+15}{2} = 21, \quad \Delta = \sqrt{21 \cdot (21-13) \cdot (21-14) \cdot (21-15)} = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$$

$$R = \frac{13 \cdot 14 \cdot 15}{4 \cdot 84} = 8.125, \quad r = \frac{84}{21} = 4, \quad r_1 = \frac{84}{8} = 10.5, \quad r_2 = \frac{84}{7} = 12, \quad r_3 = \frac{84}{6} = 14$$

3 Problem Types

- **Prove Radii Identities**: Use half-angle formulas and simplify (e.g., $r = 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$).

  $$\text{E.g., Q.1(i): } 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = \frac{abc}{4\Delta} \cdot \sqrt{\frac{(S-b)(S-c)}{bc}} \cdots = \frac{\Delta}{S} = r.$$

- **Prove Exradii Identities**: Similar approach (e.g., $r_1 = S\tan\frac{\alpha}{2}$).

  $$\text{E.g., Q.4(i): } S\tan\frac{\alpha}{2} = S\sqrt{\frac{(S-b)(S-c)}{S(S-a)}} = \frac{\Delta}{S-a} = r_1.$$

- **Prove Combined Relations**: Manipulate products/sums (e.g., $r_1 r_2 r_3 = \Delta^2$).

$$\text{E.g., Q.5(ii): } r_1 r_2 r_3 = \frac{\Delta}{S-a} \cdot \frac{\Delta}{S-b} \cdot \frac{\Delta}{S-c} = \frac{\Delta^3}{\Delta} = \Delta^2.$$

- **Numerical Calculations**: Compute radii for given sides (e.g., $a = 13, b = 14, c = 15 \implies R = 8.125, r = 4$).
- **Equilateral Triangle Ratios**: Show $r : R : r_1 = 1 : 2 : 3$ or $r : R : r_1 : r_2 : r_3 = 1 : 2 : 3 : 3 : 3$.

$$\text{E.g., Q.7: } r = \frac{\sqrt{3}a}{6}, R = \frac{a}{\sqrt{3}}, r_1 = \frac{\sqrt{3}a}{2} \implies 1 : 2 : 3.$$

4 Tips and Tricks

- Verify triangle inequality ($a + b > c$) before calculations.
- Use $\Delta = \sqrt{S(S-a)(S-b)(S-c)}$ for area in proofs.
- Simplify fractions by canceling abc or Δ terms.
- For equilateral triangles, use $\Delta = \frac{\sqrt{3}a^2}{4}$, $\alpha = \beta = \gamma = 60°$.
- Check units: r, R, r_1, r_2, r_3 in same units as sides.
- Use exact values for standard angles (e.g., $\sin 30° = 0.5$).

5 Applications

- **Surveying**: Calculate incircle/excircle properties for land plots.
- **Engineering**: Analyze triangular structures using circumradius.
- **Trigonometry**: Derive advanced geometric relationships.