Cheat Sheet: Solutions of Trigonometric Equations (Chapter 14)

This cheat sheet summarizes key concepts and methods for solving trigonometric equations from Exercise 14.1, Chapter 14, Mathematics (Part-I). It covers linear, quadratic, and multiple-angle equations, quadrant-based solutions, and general forms, with examples from Q.1–Q.20. All angles are in radians, and solutions are exact.

1. Key Concepts

- **Trigonometric Equations**: Equations with at least one trigonometric function (e.g., $\sin x = \frac{2}{5}$, $\sec x = \tan x$).
- **Principal Solutions**: Solutions in $[0, 2\pi]$ or principal range (e.g., $\sin^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$).
- **General Solutions**: Account for periodicity: $\sin x, \cos x$ (period 2π), $\tan x, \cot x$ (period π).
- **Quadrant Rules**: Use ASTC (All, Sin, Tan, Cos) to determine where functions are positive/negative.
- **Reference Angle**: Angle θ where $\sin\theta = |\sin x|$, $\cos\theta = |\cos x|$, etc.

2. Standard Angles and Values

- $\sin\frac{\pi}{6} = \frac{1}{2}$, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$
- $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{\pi}{3} = \frac{1}{2}$, $\tan\frac{\pi}{3} = \sqrt{3}$
- $\sin\frac{\pi}{2} = 1$, $\cos\frac{\pi}{2} = 0$, $\tan\frac{\pi}{2} = \text{undefined}$

3. Solution Methods

- **Linear Equations (e.g., Q.1)**: Solve $\sin x = k$, $\cos x = k$, etc.
  - Example: $\sin x = -\frac{\sqrt{3}}{2}$ (Q.1i)
  - Reference angle: $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$
  - Negative in III, IV quadrants: $x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$, $x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$
  - General: $x = \pi + \frac{\pi}{3} + 2n\pi$, $x = 2\pi - \frac{\pi}{3} + 2n\pi$
- **Quadratic Equations (e.g., Q.3, Q.8)**: Convert to quadratic form in $\sin x$, $\cos x$, or $\sec x$.
  - Example: $4\sin^2\theta - 8\cos\theta + 1 = 0$ (Q.8)
  - Use $\sin^2\theta = 1 - \cos^2\theta$: $4(1 - \cos^2\theta) - 8\cos\theta + 1 = 0$
  - Solve: $4\cos^2\theta + 8\cos\theta - 5 = 0 \implies \cos\theta = \frac{1}{2}$
  - Solutions: $\theta = \frac{\pi}{3} + 2n\pi$, $\theta = \frac{5\pi}{3} + 2n\pi$
- **Multiple-Angle Equations (e.g., Q.10–20)**: Use identities like $\sin 2x = 2\sin x\cos x$, $\cos 2x = \cos^2 x - \sin^2 x$.
  - Example: $\sin 3x + \sin 2x + \sin x = 0$ (Q.16)
  - Use: $\sin 3x + \sin x = 2\sin 2x\cos x$
  - Factor: $2\sin 2x\cos x + \sin 2x = \sin 2x(2\cos x + 1) = 0$
  - Solve: $\sin 2x = 0 \implies x = \frac{n\pi}{2}$, $2\cos x + 1 = 0 \implies x = \frac{2\pi}{3} + 2n\pi, \frac{4\pi}{3} + 2n\pi$

4. Key Identities

- $\sin^2 x + \cos^2 x = 1$
- $\sec^2 x = 1 + \tan^2 x$, $\csc^2 x = 1 + \cot^2 x$
- Sum-to-product: $\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$
- Difference: $\sin a - \sin b = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$
- $\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$
- Double-angle: $\sin 2x = 2\sin x\cos x$, $\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1$
- Triple-angle: $\sin 3x = 3\sin x - 4\sin^3 x$, $\cos 3x = 4\cos^3 x - 3\cos x$

5. General Solution Forms

- $\sin x = 0 \implies x = n\pi$
- $\cos x = 0 \implies x = (2n+1)\frac{\pi}{2}$
- $\tan x = 0 \implies x = n\pi$
- $\sin x = \sin\alpha \implies x = \alpha + 2n\pi, (-1)^n\alpha + n\pi$
- $\cos x = \cos\alpha \implies x = \alpha + 2n\pi, -\alpha + 2n\pi$
- $\tan x = \tan\alpha \implies x = \alpha + n\pi$
- For kx: Adjust period (e.g., $\sin kx = 0 \implies kx = n\pi \implies x = \frac{n\pi}{k}$)

6. Common Mistakes

- Forgetting all quadrants (e.g., Q.1: $\sin x = -\frac{\sqrt{3}}{2}$ needs III, IV quadrants).
- Incorrect period: $\sin x, \cos x$ period is 2π, $\tan x, \cot x$ period is π.
- Ignoring invalid solutions (e.g., Q.8: $\cos\theta = -\frac{5}{2}$ is impossible).
- Not factoring correctly (e.g., Q.16: Factor $\sin 2x(2\cos x + 1) = 0$).

7. Example Solutions

- **Q.9**: $\sqrt{3}\tan x - \sec x - 1 = 0$
  - Convert: $\sqrt{3}\tan x = 1 + \sec x$, square both sides.
  - Use $\tan^2 x = \sec^2 x - 1$, simplify to $\sec^2 x - \sec x - 2 = 0$.
  - Solve: $\sec x = 2 \implies x = \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi$; $\sec x = -1 \implies x = \pi + 2n\pi$.
- **Q.19**: $\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta = 0$
  - Group: $(\sin 7\theta + \sin\theta) + (\sin 5\theta + \sin 3\theta) = 0$.
  - Use sum-to-product: $2\sin 4\theta\cos 3\theta + 2\sin 4\theta\cos\theta = 0$.
  - Factor: $\sin 4\theta(\cos 3\theta + \cos\theta) = 0$.
  - Solve: $\sin 4\theta = 0 \implies \theta = \frac{n\pi}{4}$, $\cos 2\theta\cos\theta = 0 \implies \theta = (2n+1)\frac{\pi}{4}, (2n+1)\frac{\pi}{2}$.