Exercise 3.3 Cheatsheet: Matrices & Determinants

Q.1: Evaluate 3x3 Determinants Use co-factor expansion along row/column with most zeros or simple numbers. For $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$, expand along R_1:

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

Results:

- (i) $\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix} = 1$ (R_1 expansion)
- (ii) $\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix} = 10$ (R_1 expansion)
- (iii) $\begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix} = -9$ (R_1 expansion)
- (iv) $\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix} = 9al^2$ (R_1 or $C_1 + C_2 + C_3$, factor $3a$)
- (v) $\begin{vmatrix} 1 & 2 & -2 \\ -1 & 1 & -3 \\ 2 & 4 & -1 \end{vmatrix} = 9$ (R_1 expansion)
- (vi) $\begin{vmatrix} 2a & a & a \\ b & 2b & b \\ c & c & 2c \end{vmatrix} = 4abc$ (R_1 expansion)

Q.2: Show Det = 0 (No Expansion) Use row/column operations to create identical or proportional rows/columns (det = 0).

- (i) $\begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = 0$ (C_2 - C_1, C_3 - C_1 $\to$ C_2 = C_3)
- (ii) $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix} = 0$ (C_2 + C_3 $\to$ C_1 = C_2)
- (iii) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$ (C_2 - C_1, C_3 - C_1 $\to$ C_2 = C_3)

Q.3: Prove Determinant Identities Use properties (e.g., det(AB) = det(A)det(B)), row/column operations, or expansion.

- (i) $\begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{23}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$ (R_1 expansion)
- (ii) $\begin{vmatrix} 2 & 3 & 0 \\ 3 & 9 & 6 \\ 2 & 15 & 1 \end{vmatrix} = 9\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix}$ (Factor 3 from C_2, R_2)
- (iii) $\begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix} = l^2(3a+l)$ (R_1 + R_2 + R_3, factor $3a+l$)
- (iv) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$ ($C_1 \cdot x$, $C_2 \cdot y$, $C_3 \cdot z$, adjust R_3)
- (v) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$ (C_2 - C_3, R_1 expansion)
- (vi) $\begin{vmatrix} b & -1 & a \\ a & b & 0 \\ 1 & a & b \end{vmatrix} = a^3 + b^3$ (R_1 expansion)
- (vii) $\begin{vmatrix} r\cos\phi & 1 & -\sin\phi \\ 0 & 1 & 0 \\ r\sin\phi & 0 & \cos\phi \end{vmatrix} = r$ (R_1 expansion, $\cos^2\phi + \sin^2\phi = 1$)
- (viii) $\begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$ (C_1 + C_2, factor $a+b+c$)
- (ix) $\begin{vmatrix} a+\lambda & b & c \\ a & b+\lambda & c \\ a & b & c+\lambda \end{vmatrix} = \lambda^2(a+b+c+\lambda)$ (C_1 + (C_2 + C_3), factor $a+b+c+\lambda$)
- (x) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$ (C_2 - C_1, C_3 - C_1, R_1 expansion)
- (xi) $\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$ (C_1 + C_2, factor $a+b+c$)

Q.4–Q.17: Advanced Matrix Problems

- **Q.9**: For $A = \begin{vmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{vmatrix}$, $|AA^t| = 0$ (R$_1$ = R$_4$ after operations), $|A^tA| = 45$ (2x2 det).
- **Q.10**: $|kA| = k^3|A|$ for 3x3 matrix (factor k from each row).
- **Q.11**: Singular if $|A| = 0$. (i) $\lambda = 3$, (ii) $\lambda = 4$ (expand, solve for λ).
- **Q.12**: Non-singular if $|A| \neq 0$. (i) $|A| = 24$, non-singular; (ii) $|A| = 0$, singular; (iii) $|A| = 90$, non-singular.
- **Q.13**: Inverse: $A^{-1} = \frac{\text{adj } A}{|A|}$. For $A = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}$, $|A| = 5$, $A^{-1} = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ -1 & \frac{8}{5} & \frac{1}{5} \end{vmatrix}$, verify $A^{-1}A = I_3$.
- **Q.14–Q.17**: Matrix identities:
  - $(AB)^{-1} = B^{-1}A^{-1}$ (verify by computing inverses).
  - $(AB)^t = B^tA^t$, $(A^{-1})^t = (A^t)^{-1}$ (transpose properties).
  - $(A^{-1})^{-1} = A$ (inverse of inverse).

Key Strategies

• *Cofactor Expansion*: Choose row/column with most zeros to simplify. • *Row/Column Operations*: Add/subtract multiples (no det change); scale row by k (det scales by k). • *Common Factors*: Factor scalars from rows/columns; det(kA) = k^n det(A) for $n \times n$ matrix. • *Identical Rows/Columns*: If two rows/columns are identical, det = 0. • *Singular Matrix*: $|A| = 0 \implies$ singular; $|A| \neq 0 \implies$ non-singular. • *Inverse*: $A^{-1} = \frac{\text{adj } A}{|A|}$, adj A is transpose of cofactor matrix.