Trigonometric Identities Cheatsheet - Exercise 10.2

1. Angle Sum and Difference Identities

1.1 Key Identities

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \\
\tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \\
\cot(\alpha+\beta) &= \frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta} \\
\cot(\alpha-\beta) &= \frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}
\end{aligned}$$

Example: Prove $\sin(180^\circ+\theta) = -\sin\theta$.

$$\sin(180^\circ+\theta) = \sin 180^\circ\cos\theta + \cos 180^\circ\sin\theta = 0\cdot\cos\theta + (-1)\cdot\sin\theta = -\sin\theta$$

2. Specific Angle Transformations

$$\begin{aligned}
\sin(180^\circ+\theta) &= -\sin\theta \\
\cos(180^\circ+\theta) &= -\cos\theta \\
\tan(180^\circ+\theta) &= \tan\theta \\
\cos(360^\circ-\theta) &= \cos\theta \\
\cos(270^\circ+\theta) &= \sin\theta \\
\sin(270^\circ+\theta) &= -\cos\theta \\
\tan(270^\circ-\theta) &= \cot\theta \\
\cos(\theta-180^\circ) &= -\cos\theta
\end{aligned}$$

Example: Prove $\tan(270^\circ-\theta) = \cot\theta$.

$$\tan(270^\circ-\theta) = \frac{\sin(270^\circ-\theta)}{\cos(270^\circ-\theta)} = \frac{\sin 270^\circ\cos\theta - \cos 270^\circ\sin\theta}{\cos 270^\circ\cos\theta + \sin 270^\circ\sin\theta} = \frac{-\cos\theta}{-\sin\theta} = \cot\theta$$

3. Evaluating Specific Angles

Use angle sum/difference identities to find exact values.

$$
\begin{aligned}
\sin 15^\circ &= \sin(60^\circ - 45^\circ) = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} \\
\cos 15^\circ &= \cos(60^\circ - 45^\circ) = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{1+\sqrt{3}}{2\sqrt{2}} \\
\tan 15^\circ &= \tan(60^\circ - 45^\circ) = \frac{\sqrt{3}-1}{1+\sqrt{3}} \\
\sin 105^\circ &= \sin(60^\circ + 45^\circ) = \frac{\sqrt{3}+1}{2\sqrt{2}} \\
\cos 105^\circ &= \cos(60^\circ + 45^\circ) = \frac{1-\sqrt{3}}{2\sqrt{2}} \\
\tan 105^\circ &= \tan(60^\circ + 45^\circ) = \frac{\sqrt{3}+1}{1-\sqrt{3}}
\end{aligned}
$$

Example: Find $\sin 105^\circ$.

$$
\sin 105^\circ = \sin(60^\circ + 45^\circ) = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}
$$

4. Sum and Difference of Cotangents

$$
\begin{aligned}
\cot(\alpha+\beta) &= \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta} \\
\cot(\alpha-\beta) &= \frac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}
\end{aligned}
$$

Example: Prove $\cot(\alpha-\beta) = \frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$.

$$
\cot(\alpha-\beta) = \frac{\cos(\alpha-\beta)}{\sin(\alpha-\beta)} = \frac{\cos\alpha\cos\beta + \sin\alpha\sin\beta}{\sin\alpha\cos\beta - \cos\alpha\sin\beta} = \frac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}
$$

5. Triangle Angle Identities

For angles α, β, γ in a triangle ($\alpha + \beta + \gamma = 180^\circ$):

$$
\begin{gathered}
\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2} \\
\cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha = 1
\end{gathered}
$$

Example: Prove $\cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha = 1$.

$$
\alpha + \beta = 180^\circ - \gamma \implies \tan(\alpha+\beta) = -\tan\gamma \implies \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = -\tan\gamma
$$

$$
\frac{\cot\alpha + \cot\beta}{\cot\alpha\cot\beta - 1} = \frac{-1}{\cot\gamma} \implies \cot\alpha\cot\gamma + \cot\beta\cot\gamma + \cot\alpha\cot\beta = 1
$$

6. Expressing Linear Combinations

Express $a \sin\theta + b\cos\theta$ as $r\sin(\theta+\phi)$.

$$r = \sqrt{a^2+b^2}, \quad \cos\phi = \frac{a}{r}, \quad \sin\phi = \frac{b}{r}, \quad \tan\phi = \frac{b}{a}$$

Example: Express $12\sin\theta + 5\cos\theta$.

$$r = \sqrt{12^2+5^2} = \sqrt{169} = 13, \quad \tan\phi = \frac{5}{12} \implies 12\sin\theta + 5\cos\theta = 13\sin\left(\theta + \tan^{-1}\frac{5}{12}\right)$$

7. Quadrant-Based Calculations

Use quadrant information to determine signs of trigonometric functions.

$$\begin{aligned}
&\text{1st Quadrant: All positive}\\
&\text{2nd Quadrant: } \sin,\csc \text{ positive; } \cos,\sec,\tan,\cot \text{ negative}\\
&\text{3rd Quadrant: } \tan,\cot \text{ positive; } \sin,\cos,\csc,\sec \text{ negative}\\
&\text{4th Quadrant: } \cos,\sec \text{ positive; } \sin,\csc,\tan,\cot \text{ negative}
\end{aligned}$$

Example: If $\sin\alpha = \frac{4}{5}$, $\frac{\pi}{2} < \alpha < \pi$, find $\sin(\alpha-\beta)$.

$$\cos\alpha = -\sqrt{1-\frac{16}{25}} = -\frac{3}{5}, \quad \sin\beta = \frac{12}{13}, \quad \cos\beta = -\frac{5}{13}$$

$$\sin(\alpha-\beta) = \frac{4}{5}\cdot\frac{-5}{13} - \left(-\frac{3}{5}\right)\cdot\frac{12}{13} = \frac{-20}{65} + \frac{36}{65} = \frac{16}{65}$$

8. Applications

- **Physics:** Use angle sum identities in wave superposition.
- **Engineering:** Apply linear combination forms in signal analysis.