Exercise 2.7: Binary Operations and Groups Cheatsheet

1. Binary Operation

Definition: A function $*$ on a non-empty set G, assigning a unique element $a * b \in G$ to each pair $(a, b) \in G \times G$.
Formula: $(a, b) \mapsto a * b$.
Analogy: A recipe combining two ingredients into one dish.
Example (Q.1): Addition $(+)$ on integers $\mathbb{Z}$: $2 + 3 = 5 \in \mathbb{Z}$.

2. Properties of Binary Operations

Definition:
- **Closure**: $a * b \in S$.
- **Commutativity**: $a * b = b * a$.
- **Associativity**: $(a * b) * c = a * (b * c)$.
- **Identity**: Exists $e \in S$: $a * e = e * a = a$.
- **Inverse**: For each $a \in S$, exists $a' \in S$: $a * a' = a' * a = e$.
Formulas:
- Commutativity: $a * b = b * a$.
- Associativity: $a * (b * c) = (a * b) * c$.
- Identity: $a * e = e * a = a$.
- Inverse: $a * a' = a' * a = e$.
Analogy: A team game where results stay in play, order doesn't matter, grouping is flexible, a neutral player exists, and actions are reversible.
Example (Q.1, Integers with $+$):
- Closure: $2 + 3 = 5 \in \mathbb{Z}$.
- Commutativity: $2 + 3 = 3 + 2$.
- Associativity: $(1 + 2) + 3 = 1 + (2 + 3)$.
- Identity: 0 $(a + 0 = a)$.
- Inverse: $-a$ $(a + (-a) = 0)$.

3. Field Axioms

Definition: A set F is a field if:
- Abelian group under $+$.
- $F \setminus \{0\}$ is an Abelian group under $\times$.
- Distributive laws hold.
Formulas:
- Left Distributivity: $a \times (b + c) = (a \times b) + (a \times c)$.
- Right Distributivity: $(a + b) \times c = (a \times c) + (b \times c)$.
Analogy: A bank with balanced deposit (addition) and interest (multiplication) rules.
Example (Q.2): Real numbers $\mathbb{R}$ form a field. Complex numbers lack natural ordering (e.g., $2 + i$ vs. $3 - i$).

4. Residue Classes Modulo n

Definition: Set $\{0, 1, \ldots, n-1\}$ with operations modulo n.
Formula: $a * b = (a \cdot b) \mod n$.
Analogy: A circular track with n points, looping after n.
Example (Q.3, Multiplication Modulo 5):

$*$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

E.g., $2 * 3 = 6 \mod 5 = 1$.
Example (Q.4, Addition Modulo 4):

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

E.g., $2 + 3 = 5 \mod 4 = 1$.

5. Commutativity of Binary Operations

Definition: $a * b = b * a$.
Formula: $a * b = b * a$.
Analogy: A handshake where order doesn't matter.
Example (Q.5): Table (b) is commutative ($a * b = c$, $b * a = c$), table (a) is not ($a * b = c$, $b * a = b$).

6. Associativity of Binary Operations

Definition: $(a * b) * c = a * (b * c)$.
Formula: $(a * b) * c = a * (b * c)$.
Analogy: Stacking boxes where grouping doesn't change the stack.
Example (Q.6): Third row: $c * a = c$, $c * b = d$, $c * c = c$, $c * d = d$.

7. Groupoid

Definition: A set with a closed binary operation.
Analogy: A club where interactions stay within the club.
Example: Integers $\mathbb{Z}$ with subtraction ($a - b \in \mathbb{Z}$).

8. Semigroup

Definition: A set with a closed, associative binary operation.
Analogy: A team with consistent task combinations.
Example: Natural numbers $\mathbb{N}$ with addition ($+$).

9. Monoid

Definition: A semigroup with an identity element.
Analogy: A team with a neutral member.
Example: Whole numbers $\mathbb{W}$ with addition, identity 0.

10. Group

Definition: A monoid with inverses for all elements.
Analogy: A team where every action is reversible.
Example (Q.7): $\{0, 1, 2, 3\}$ with $+ \mod 4$:
- Identity: 0.
- Inverses: $0 \to 0$, $1 \to 3$, $2 \to 2$, $3 \to 1$.

11. Abelian Group

Definition: A group where the operation is commutative.
Analogy: A team where collaboration order doesn't matter.
Example: Integers $\mathbb{Z}$ with addition ($a + b = b + a$).