Inverse Trigonometric Functions Cheatsheet
Exercise 13.1

1 Definitions and Ranges

1.1 Inverse Trigonometric Functions

In $\triangle ABC$, inverse functions are defined with restricted domains to be one-to-one:

- **Inverse Sine**: $y = \sin^{-1} x \iff x = \sin y$

$$\text{Domain: } [-1, 1], \quad \text{Range: } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

- **Inverse Cosine**: $y = \cos^{-1} x \iff x = \cos y$

$$\text{Domain: } [-1, 1], \quad \text{Range: } [0, \pi]$$

- **Inverse Tangent**: $y = \tan^{-1} x \iff x = \tan y$

$$\text{Domain: } (-\infty, \infty), \quad \text{Range: } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

- **Inverse Cotangent**: $y = \cot^{-1} x \iff x = \cot y$

$$\text{Domain: } (-\infty, \infty), \quad \text{Range: } (0, \pi)$$

- **Inverse Secant**: $y = \sec^{-1} x \iff x = \sec y$

$$\text{Domain: } |x| \geq 1, \quad \text{Range: } [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$$

- **Inverse Cosecant**: $y = \csc^{-1} x \iff x = \csc y$

$$\text{Domain: } |x| \geq 1, \quad \text{Range: } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$$

2 Key Techniques

2.1 Evaluating Inverse Functions

1. Let $y = f^{-1}(x)$, find y such that $f(y) = x$ and y is in the principal range.
2. Example: For $\sin^{-1}(-1)$, find $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ where $\sin y = -1$. Thus, $y = -\frac{\pi}{2}$.

2.2 Proving Equalities

1. Convert one side to a trigonometric function of the other (e.g., for $\tan^{-1}\frac{5}{12} = \sin^{-1}\frac{5}{13}$, show $\sin\left(\tan^{-1}\frac{5}{12}\right) = \frac{5}{13}$).
2. Use right triangle: For $\tan^{-1}\frac{a}{b}$, assume opposite a, adjacent b, hypotenuse $\sqrt{a^2+b^2}$.

2.3 Evaluating Composite Expressions

1. Find inner inverse function: $y = f^{-1}(x)$, so $f(y) = x$.
2. Compute outer function: Use identities (e.g., $\cos\left(\sin^{-1}x\right) = \sqrt{1-x^2}$).
3. Example: For $\cos\left(\sin^{-1}\frac{1}{\sqrt{2}}\right)$, let $y = \sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$, then $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

3 Common Values

- $\sin^{-1}1 = \frac{\pi}{2}$, $\sin^{-1}(-1) = -\frac{\pi}{2}$, $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$, $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$.
- $\cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$, $\cos^{-1}\frac{1}{2} = \frac{\pi}{3}$.
- $\tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$, $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$, $\tan^{-1}(-1) = -\frac{\pi}{4}$.
- $\cot^{-1}(-1) = \frac{3\pi}{4}$.
- $\csc^{-1}\left(-\frac{2}{\sqrt{3}}\right) = -\frac{\pi}{3}$.

4 Problem Types

- **Evaluate Inverse Functions**: Find principal angle (e.g., Q.1: $\sin^{-1}(1) = \frac{\pi}{2}$).

$$\text{E.g.,}\ \csc^{-1}\left(-\frac{2}{\sqrt{3}}\right) \implies \csc y = -\frac{2}{\sqrt{3}},\ \sin y = -\frac{\sqrt{3}}{2},\ y = -\frac{\pi}{3}.$$

- **Prove Equalities**: Show equivalence (e.g., Q.2: $\tan^{-1}\frac{5}{12} = \sin^{-1}\frac{5}{13}$).

$$\text{Let}\ \theta = \tan^{-1}\frac{5}{12},\ \tan\theta = \frac{5}{12},\ \sin\theta = \frac{5}{\sqrt{169}} = \frac{5}{13}.$$

- **Composite Expressions**: Evaluate nested functions (e.g., Q.3: $\cos\left(\sin^{-1}\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$).

$$\text{Let}\ y = \sin^{-1}\frac{1}{\sqrt{2}},\ \sin y = \frac{1}{\sqrt{2}},\ \cos y = \sqrt{1-\frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

5 Tips and Tricks

- Check domain/range: Ensure x is valid (e.g., $|x| \le 1$ for $\sin^{-1}x$).
- Use identities: $\cos\left(\sin^{-1}x\right) = \sqrt{1-x^2}$, $\sin\left(\cos^{-1}x\right) = \sqrt{1-x^2}$.

- Negative arguments: $\sin^{-1}(-x) = -\sin^{-1} x$, $\tan^{-1}(-x) = -\tan^{-1} x$.
- Simplify fractions: Rationalize denominators (e.g., $\frac{2}{\sqrt{3}}$).
- Right triangle for Q.2: For $\tan^{-1} \frac{a}{b}$, use opposite a, adjacent b.
- Verify signs: Ensure angle sign matches quadrant (e.g., $\tan^{-1}(-1) = -\frac{\pi}{4}$).

6 Applications

- **Physics**: Compute angles in projectile motion.
- **Calculus**: Simplify derivatives/integrals of inverse functions.
- **Geometry**: Solve for angles in triangles.