Exercise 6.9: Conic Sections Cheatsheet

Axis Rotation (Remove xy-Term)

Transform using rotation angle θ: $x = X\cos\theta - Y\sin\theta$, $y = X\sin\theta + Y\cos\theta$ Angle θ where: $\tan 2\theta = \frac{2h}{a-b}$ (for $ax^2 + 2hxy + by^2 + \cdots = 0$)

Steps: 1. Compute $\tan 2\theta$ using coefficients a, h, b. 2. Solve for θ and find $\cos\theta$, $\sin\theta$. 3. Substitute into transformation equations. 4. Simplify to remove XY-term.

Examples: - $5x^2 - 6xy + 5y^2 - 8 = 0$: $\tan 2\theta = \frac{-6}{5-5} = \infty \to \theta = 45^\circ$ $\cos 45^\circ = \sin 45^\circ = \frac{\sqrt{2}}{2}$ Transformed: $X^2 + 4Y^2 - 4 = 0$ (Ellipse) - $4x^2 - 4xy + y^2 - 6 = 0$: $a = 4,\ 2h = -4,\ b = 1$ $\tan 2\theta = \frac{-4}{4-1} = \frac{-4}{3}$ $2\tan^2\theta - 3\tan\theta - 2 = 0 \to \tan\theta = 2$ $\cos\theta = \frac{1}{\sqrt{5}}$, $\sin\theta = \frac{2}{\sqrt{5}}$ Transformed: $Y^2 = \frac{6}{5}$ (Parabola)

Identify Conic and Elements

- **Ellipse**: $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$ Center: Solve $X = 0,\ Y = 0$ for (x, y). Foci: $(\pm ae, 0)$, $e = \sqrt{1 - \frac{b^2}{a^2}}$. - **Parabola**: $Y^2 = 4aX$ Vertex: $X = 0, Y = 0 \to (x, y)$. Focus: $(a, 0)$. Axis: $Y = 0 \to x + y = 0$.

Examples: - $x^2 - 2xy + y^2 - 8x - 8y = 0$: $\theta = 45^\circ$, Transformed: $Y^2 - 4\sqrt{2}X = 0$ Conic: Parabola, Vertex $(0,0)$, Focus $(\sqrt{2}, 0)$. - $x^2 + xy + y^2 - 4 = 0$: $\theta = 45^\circ$, Transformed: $\frac{X^2}{8} + \frac{Y^2}{8} = 3$ Conic: Ellipse, Center $(0,0)$.

Tangent to Conic

For $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at (x_1, y_1): $\frac{dy}{dx} = \frac{-(ax_1 + hy_1 + g)}{-(hx_1 + by_1 + f)}$ Tangent: $y - y_1 = m(x - x_1)$.

Examples: - $3x^2 - 7y^2 + 2x - y - 48 = 0$ at $(4, 1)$: $m = \frac{6(4)+2}{14(1)+1} = \frac{26}{15}$ Tangent: $26x - 15y - 89 = 0$. - $x^2 + 5xy - 4y^2 + 4 = 0$ at $(0, -1)$: $m = \frac{5}{8}$ Tangent: $5x - 8y - 8 = 0$.

Pair of Straight Lines

For $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$: Check determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$. Factorize to find line equations.

Example: - $10xy + 8x - 15y - 12 = 0$: Determinant $= 0$, Factor: $(5y + 4)(2x - 3) = 0$ Lines: $5y + 4 = 0$, $2x - 3 = 0$.