Cheat Sheet: Conic Sections - Exercise 6.1 (Chapter 6, Mathematics Part-II, Class 12)

This cheat sheet summarizes key concepts and techniques from Exercise 6.1, Chapter 6: Conic Sections, focusing on circles. It includes definitions, equations, and step-by-step methods for solving problems related to finding circle equations, centers, radii, tangents, and circle relationships (external/internal touching). Examples are drawn from the exercise to clarify high-difficulty applications.

1. Key Definitions

- **Conic Section**: Curves obtained by intersecting a plane with a right circular cone.
- **Circle**: Set of all points in a plane equidistant from a fixed point (center). Distance from center to any point is the radius.
- **Point Circle**: A circle with radius $r = 0$, reducing to a single point.

2. Equations of a Circle

- **Standard Form**: For center (h, k) and radius r:

  $$(x - h)^2 + (y - k)^2 = r^2$$

  If center is at origin $(0, 0)$:

  $$x^2 + y^2 = r^2$$

- **General Form**:

  $$x^2 + y^2 + 2gx + 2fy + c = 0$$

  Center: $(-g, -f)$, *Radius*: $\sqrt{g^2 + f^2 - c}$ *Note*: If $g^2 + f^2 - c < 0$, no real circle exists; if $= 0$, it's a point circle.

- **Parametric Form**: For radius r:

  $$x = r\cos\theta, \quad y = r\sin\theta$$

3. Key Problem Types and Techniques

1. **Finding Equation of a Circle**
   - *Given Center and Radius (Q.1a)*:
     - Example: Center $(5, -2)$, radius 4.
     - Use standard form: $(x - 5)^2 + (y + 2)^2 = 16$.

    - Expand to general form: $x^2 + y^2 - 10x + 4y + 13 = 0$.
- *Given Diameter Endpoints (Q.1c)*:
    - Example: Diameter endpoints $(-3, 2)$, $(5, -6)$.
    - Center = Midpoint: $\left(\frac{-3+5}{2}, \frac{2-6}{2}\right) = (1, -2)$.
    - Radius = Distance from center to endpoint: $\sqrt{(1+3)^2 + (-2-2)^2} = \sqrt{32}$.
    - Equation: $(x-1)^2 + (y+2)^2 = 32$, or $x^2 + y^2 - 2x + 4y - 27 = 0$.
- *Given Three Points (Q.3a)*:
    - Example: Points $A(4, 5)$, $B(-4, -3)$, $C(8, -3)$.
    - Use general form: $x^2 + y^2 + 2gx + 2fy + c = 0$.
    - Substitute points to get equations:

$$41 + 8g + 10f + c = 0, \quad 25 - 8g - 6f + c = 0, \quad 73 + 16g - 6f + c = 0$$

    - Solve system: $g = -2$, $f = 1$, $c = -25$.
    - Equation: $x^2 + y^2 - 4x + 2y - 25 = 0$.

2. **Finding Center and Radius (Q.2a)**:
    - Example: $x^2 + y^2 + 12x - 10y = 0$.
    - Compare with general form: $2g = 12$, $2f = -10$, $c = 0$.
    - Center: $(-g, -f) = (-6, 5)$.
    - Radius: $\sqrt{g^2 + f^2 - c} = \sqrt{36 + 25} = \sqrt{61}$.
3. **Circle with Center on a Line and Passing Through Points (Q.4a)**:
    - Example: Passes through $A(3, -1)$, $B(0, 1)$, center on $4x - 3y - 3 = 0$.
    - General form: $x^2 + y^2 + 2gx + 2fy + c = 0$.
    - Substitute points: $10 + 6g - 2f + c = 0$, $1 + 2f + c = 0$.
    - Center condition: $4(-g) - 3(-f) - 3 = 0 \Rightarrow -4g + 3f - 3 = 0$.
    - Solve: $g = -\frac{15}{2}$, $f = -9$, $c = 17$.
    - Equation: $x^2 + y^2 - 15x - 18y + 17 = 0$.
4. **Circle Tangent to Axes (Q.5)**:
    - Example: Radius a, tangent to both axes in 2nd quadrant.
    - Center: $(-a, a)$, radius a.
    - Equation: $(x+a)^2 + (y-a)^2 = a^2 \Rightarrow x^2 + y^2 + 2ax - 2ay + a^2 = 0$.
5. **Proving Tangency (Q.6)**:
    - Example: Lines $3x - 2y = 0$, $2x + 3y - 13 = 0$ tangent to $x^2 + y^2 + 6x - 4y = 0$.
    - Circle center: $(-3, 2)$, radius: $\sqrt{9+4} = \sqrt{13}$.
    - Distance from center to line: $\frac{|3(-3) - 2(2)|}{\sqrt{9+4}} = \sqrt{13}$, equals radius.

- Repeat for second line: $\frac{|2(-3)+3(2)-13|}{\sqrt{4+9}} = \sqrt{13}$. Both are tangents.

6. **Circles Touching Externally/Internally (Q.7, Q.8)**:
   - *External Touching (Q.7)*: Circles $x^2 + y^2 + 2x - 2y - 7 = 0$, $x^2 + y^2 - 6x + 4y + 9 = 0$.
     - Centers: $(-1, 1)$, $(3, -2)$; Radii: 3, 2.
     - Distance between centers: $\sqrt{(3+1)^2 + (-2-1)^2} = 5$.
     - Check: $r_1 + r_2 = 3 + 2 = 5$. Touches externally.
   - *Internal Touching (Q.8)*: Circles $x^2 + y^2 + 2x - 8 = 0$, $x^2 + y^2 - 6x + 6y - 46 = 0$.
     - Centers: $(-1, 0)$, $(3, -3)$; Radii: 3, 8.
     - Distance: $\sqrt{(3+1)^2 + (-3-0)^2} = 5$.
     - Check: $r_2 - r_1 = 8 - 3 = 5$. Touches internally.

7. **Circle Tangent to a Line at a Point (Q.9)**:
   - Example: Radius 2, tangent to $x - y - 4 = 0$ at $(1, -3)$.
   - Standard form: $(x - h)^2 + (y - k)^2 = 4$.
   - Point condition: $(1 - h)^2 + (-3 - k)^2 = 4$.
   - Tangent condition: Slope of line $x - y - 4 = 0$ is 1. Slope of radius to $(1, -3)$: $\frac{k+3}{h-1}$. Perpendicular: $\frac{k+3}{h-1} = -1 \Rightarrow k = -h - 2$.
   - Solve: $h = 1 \pm \sqrt{2}$, $k = -3 \mp \sqrt{2}$.
   - Equations: $(x-(1+\sqrt{2}))^2+(y-(-3-\sqrt{2}))^2 = 4$, $(x-(1-\sqrt{2}))^2+(y-(-3+\sqrt{2}))^2 = 4$.

4. Common Pitfalls

- **General Form Conversion**: Ensure correct expansion from standard to general form (e.g., Q.1a: $(x - 5)^2 + (y + 2)^2 = 16 \Rightarrow x^2 + y^2 - 10x + 4y + 13 = 0$).
- **Radius Check**: If $g^2 + f^2 - c < 0$, no real circle (e.g., Q.2c: radius $= 0$, point circle).
- **System of Equations**: When solving for g, f, c in Q.3, ensure consistent elimination (e.g., Q.3a: subtract equations correctly).
- **Tangency**: Verify distance from center to line equals radius (Q.6). Use absolute value in distance formula.
- **Two Circles**: Tangent conditions may yield two solutions (Q.9, Q.4b). Check both for validity.