Cheat Sheet: Conic Sections - Exercise 6.3 (Chapter 6, Mathematics Part-II, Class 12)

This cheat sheet summarizes key concepts from Exercise 6.3, Chapter 6: Conic Sections, focusing on circle properties (normals, tangents, circumcenter, mean proportional) and parabolas (definitions, standard forms). Examples are drawn from the exercise to clarify high-difficulty applications.

1. Circle Properties

- **Circle General Form**: $x^2 + y^2 + 2gx + 2fy + c = 0$
  - Center: $(-g, -f)$
  - Radius: $\sqrt{g^2 + f^2 - c}$
- **Tangent at Point** (x_1, y_1):

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

- **Normal at Point** (x_1, y_1): Passes through center $(-g, -f)$.

$$(y - y_1)(x_1 + g) = (x - x_1)(y_1 + f)$$

- **Chord of Contact** from (x_1, y_1):

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

2. Key Circle Theorems

1. **Normals Pass Through Center (Q.1)**:
   - *Proof*: Circle $x^2 + y^2 = r^2$, point (x_1, y_1).
     - Tangent slope: $\frac{dy}{dx} = -\frac{x_1}{y_1}$.
     - Normal slope: $\frac{y_1}{x_1}$.
     - Normal equation: $y - y_1 = \frac{y_1}{x_1}(x - x_1)$, simplifies to $x_1 y = y_1 x$.
     - Center $(0, 0)$ satisfies: $x_1 \cdot 0 = y_1 \cdot 0$.
2. **Line from Center Perpendicular to Tangent (Q.2)**:
   - *Proof*: Circle $x^2 + y^2 = r^2$, point (x_1, y_1).
     - Tangent slope: $-\frac{x_1}{y_1}$.
     - Perpendicular slope: $\frac{y_1}{x_1}$.

- Line from center $(0,0)$: $y = \frac{y_1}{x_1}x$, passes through (x_1, y_1).

3. **Midpoint of Hypotenuse as Circumcenter (Q.3)**:
   - *Proof*: Right triangle at $O(0,0)$, vertices $A(a,0)$, $B(0,b)$.
     - Midpoint of hypotenuse AB: $C\left(\frac{a}{2}, \frac{b}{2}\right)$.
     - Distances: $|CA| = |CB| = |CO| = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$.
     - Equal distances confirm C is circumcenter.
4. **Perpendicular from Point to Diameter (Q.4)**:
   - *Proof*: Circle $x^2 + y^2 = r^2$, point $P(a,b)$, diameter $AB(-r,0)$ to $(r,0)$.
     - Perpendicular at $Q(a,0)$.
     - Lengths: $|PQ| = b$, $|AQ| = r + a$, $|QB| = r - a$.
     - Mean proportional: $|AQ| \cdot |QB| = (r+a)(r-a) = r^2 - a^2 = b^2 = |PQ|^2$.
5. **Chord of Contact (Q.9 from Ex. 6.2)**:
   - *Example*: Circle $2x^2 + 2y^2 - 8x + 12y + 21 = 0$, point $(4,5)$.
     - Divide by 2: $x^2 + y^2 - 4x + 6y + \frac{21}{2} = 0$, $g = -2$, $f = 3$, $c = \frac{21}{2}$.
     - Chord: $4x + 5y - 2(x+4) + 3(y+5) + \frac{21}{2} = 0$, simplifies to $4x + 16y + 35 = 0$.

3. Parabola Introduction

- **Definition**: Set of points $P(x,y)$ where distance from focus F equals distance from directrix L ($|PF| = |PM|$).
- **Standard Forms** (vertex at origin):
  - $y^2 = 4ax$ (opens right)
  - $y^2 = -4ax$ (opens left)
  - $x^2 = 4ay$ (opens up)
  - $x^2 = -4ay$ (opens down)
- **Vertex at** (h,k):
  - $(y-k)^2 = 4a(x-h)$
  - $(y-k)^2 = -4a(x-h)$
  - $(x-h)^2 = 4a(y-k)$
  - $(x-h)^2 = -4a(y-k)$
- **Key Features**:
  - *Axis*: Line through focus, perpendicular to directrix.
  - *Vertex*: Intersection of axis and parabola.
  - *Chord*: Line joining two points on parabola.
  - *Focal Chord*: Chord through focus.

  - *Latus Rectum*: Focal chord perpendicular to axis.

4. Common Pitfalls

- **Normal Proof**: Ensure normal slope is reciprocal of tangent slope (Q.1).
- **Perpendicular Line**: Verify line from center passes through tangency point (Q.2).
- **Circumcenter**: Use midpoint formula correctly; check distances (Q.3).
- **Mean Proportional**: Confirm $|AQ| \cdot |QB| = |PQ|^2$ using circle equation (Q.4).
- **Parabola Forms**: Distinguish between vertex at origin and shifted vertex.