Cheat Sheet: Conic Sections - Exercise 6.2 (Chapter 6, Mathematics Part-II, Class 12)

This cheat sheet summarizes key concepts and techniques from Exercise 6.2, Chapter 6: Conic Sections, focusing on tangents, normals, point positions, tangent lengths, chord lengths, intersection points, and chord of contact for circles. Examples are drawn from the exercise to clarify high-difficulty applications.

1. Key Formulas

- **Circle General Form**: $x^2 + y^2 + 2gx + 2fy + c = 0$

    - Center: $(-g, -f)$
    - Radius: $\sqrt{g^2 + f^2 - c}$

- **Tangent at Point** (x_1, y_1):

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

- **Normal at Point** (x_1, y_1):

$$(y - y_1)(x_1 + g) = (x - x_1)(y_1 + f)$$

  Note: Normal passes through the center.

- **Position of Point** (x_1, y_1):

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \begin{cases} > 0 & \text{(outside)} \\ = 0 & \text{(on)} \\ < 0 & \text{(inside)} \end{cases}$$

- **Length of Tangent** from (x_1, y_1):

$$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

- **Chord of Contact** from (x_1, y_1):

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

- **Tangent with Slope** m:

$$y = mx \pm r\sqrt{1 + m^2}, \quad r = \text{radius}$$

2. Key Problem Types and Techniques

1. **Tangent and Normal at a Point (Q.1, Q.2)**:
   - *Example (Q.1i)*: Circle $x^2 + y^2 = 25$, point $(4, 3)$.
     - General form: $x^2 + y^2 - 25 = 0$, $g = 0$, $f = 0$, $c = -25$.
     - Tangent: $4x + 3y - 25 = 0$.
     - Normal: $3x - 4y = 0$.
   - *Example (Q.2)*: Circle $4x^2 + 4y^2 - 16x + 24y - 117 = 0$, points with $x = -4$.
     - Divide by 4: $x^2 + y^2 - 4x + 6y - \frac{117}{4} = 0$, $g = -2$, $f = 3$, $c = -\frac{117}{4}$.
     - Find points: Substitute $x = -4$, solve $4y^2 + 24y + 11 = 0$, get $y = -\frac{11}{2}, -\frac{1}{2}$.
     - Tangents: At $(-4, -\frac{1}{2})$, $24x - 10y + 91 = 0$; at $(-4, -\frac{11}{2})$, $24x + 10y + 151 = 0$.
     - Normals: At $(-4, -\frac{1}{2})$, $5x + 12y + 26 = 0$; at $(-4, -\frac{11}{2})$, $5x - 12y - 46 = 0$.
   - *Parametric Point (Q.1b)*: Point $(5 \cos \theta, 5 \sin \theta)$.
     - Tangent: $x \cos \theta + y \sin \theta - 5 = 0$.
     - Normal: $x \sin \theta - y \cos \theta = 0$.

2. **Position of a Point (Q.3)**:
   - *Example (Q.3i)*: Circle $x^2 + y^2 = 81$, point $(5, 6)$.
     - Substitute: $5^2 + 6^2 - 81 = -20 < 0$, inside.
   - *Example (Q.3ii)*: Circle $2x^2 + 2y^2 + 12x - 8y + 1 = 0$, point $(5, 6)$.
     - Substitute: $2(5^2) + 2(6^2) + 12(5) - 8(6) + 1 = 135 > 0$, outside.

3. **Length of Tangent (Q.4)**:
   - *Example*: Circle $5x^2 + 5y^2 - 10x + 15y - 131 = 0$, point $(-5, 4)$.
     - Divide by 5: $x^2 + y^2 - 2x + 3y - \frac{131}{5} = 0$.
     - Length: $\sqrt{(-5)^2 + 4^2 - 2(-5) + 3(4) - \frac{131}{5}} = \sqrt{\frac{184}{5}}$.

4. **Chord Length (Q.5)**:
   - *Example*: Circle $x^2 + y^2 = 26$, line $2x + 3y = 13$.
     - Solve: Substitute $y = \frac{13 - 2x}{3}$, get $x^2 - 4x - 5 = 0$, points $(5, 1)$, $(-1, 5)$.
     - Length: $\sqrt{(5 - (-1))^2 + (1 - 5)^2} = 2\sqrt{13}$.

5. **Intersection Points (Q.6)**:
   - *Example*: Circle $x^2 + y^2 - 2x - 2y - 39 = 0$, line $x + 2y = 6$.
     - Solve: Substitute $x = 6 - 2y$, get $5y^2 - 22y - 15 = 0$, points $(-4, 5)$, $\left(\frac{36}{5}, -\frac{3}{5}\right)$.

6. **Tangents with Given Slope (Q.7)**:
   - *Parallel to Line (Q.7i)*: Circle $x^2 + y^2 = 2$, parallel to $x - 2y + 1 = 0$.
     - Slope $m = \frac{1}{2}$, radius $r = \sqrt{2}$.

– Tangent: $y = \frac{1}{2}x \pm \frac{\sqrt{10}}{2}$, or $x - 2y \pm \sqrt{10} = 0$.

- *Perpendicular to Line (Q.7ii)*: Perpendicular to $3x + 2y = 6$.
  – Slope $m = \frac{2}{3}$, tangents: $2x - 3y \pm \sqrt{26} = 0$.

7. **Tangents from a Point (Q.8, Q.11)**:

- *Example (Q.8i)*: Circle $x^2 + y^2 = 16$, point $(0, 5)$.
  – Points of tangency: $\left(\pm\frac{12}{5}, \frac{16}{5}\right)$.
  – Tangents: $15x + 20y = 100$, $15x - 20y + 100 = 0$.
- *Example (Q.11)*: Circle $(x + 1)^2 + (y - 2)^2 = 26$, point $(-7, -2)$.
  – Points: $(-2, -3)$, $(-6, 3)$.
  – Tangents: $x + 5y + 17 = 0$, $5x - y + 33 = 0$.

8. **Chord of Contact (Q.9)**:

- *Example*: Circle $2x^2 + 2y^2 - 8x + 12y + 21 = 0$, point $(4, 5)$.
  – Divide by 2: $x^2 + y^2 - 4x + 6y + \frac{21}{2} = 0$.
  – Chord: $4x + 16y + 35 = 0$.

3. Common Pitfalls

- **Tangent/Normal Equations**: Ensure correct substitution of g, f, c (e.g., Q.1i: avoid sign errors in $4x + 3y - 25 = 0$).
- **Point Position**: Check sign of expression (Q.3: negative = inside, positive = outside).
- **Quadratic Solving**: For intersection points or tangency, solve quadratics carefully (Q.5, Q.6).
- **Tangent Slope**: Use condition $c^2 = r^2(1 + m^2)$ correctly (Q.7).
- **Tangents from Point**: Two tangents exist; solve perpendicularity condition accurately (Q.8, Q.11).
- **Chord of Contact**: Same formula as tangent at a point, but applied to external point (Q.9).