

Big-Oh Notation

Let f and g be functions from positive numbers to positive numbers. $f(n)$ is $O(g(n))$ if there are positive constants C and k such that:

$$f(n) \leq C g(n) \text{ whenever } n > k$$

$$f(n) \text{ is } O(g(n)) \equiv \exists C \exists k \forall n (n > k \rightarrow f(n) \leq C g(n))$$

To prove big-Oh, choose values for C and k and prove $n > k$ implies $f(n) \leq C g(n)$.

Standard Method to Prove Big-Oh

1. Choose $k = 1$.
2. Assuming $n > 1$, find/derive a C such that

$$\frac{f(n)}{g(n)} \leq \frac{C g(n)}{g(n)} = C$$

This shows that $n > 1$ implies $f(n) \leq C g(n)$.
Keep in mind:

- $n > 1$ implies $1 < n$, $n < n^2$, $n^2 < n^3$, ...
- “Increase” numerator to “simplify” fraction.

Proving Big-Oh: Example 1

Show that $f(n) = n^2 + 2n + 1$ is $O(n^2)$.

Choose $k = 1$.

Assuming $n > 1$, then

$$\frac{f(n)}{g(n)} = \frac{n^2 + 2n + 1}{n^2} < \frac{n^2 + 2n^2 + n^2}{n^2} = 4$$

Choose $C = 4$. Note that $2n < 2n^2$ and $1 < n^2$.

Thus, $n^2 + 2n + 1$ is $O(n^2)$ because $n^2 + 2n + 1 \leq 4n^2$ whenever $n > 1$.

Proving Big-Oh: Example 2

Show that $f(n) = 3n + 7$ is $O(n)$.

Choose $k = 1$.

Assuming $n > 1$, then

$$\frac{f(n)}{g(n)} = \frac{3n + 7}{n} < \frac{3n + 7n}{n} = \frac{10n}{n} = 10$$

Choose $C = 10$. Note that $7 < 7n$.

Thus, $3n + 7$ is $O(n)$ because $3n + 7 \leq 10n$ whenever $n > 1$.

Proving Big-Oh: Example 3

Show that $f(n) = (n + 1)^3$ is $O(n^3)$.

Choose $k = 1$.

Assuming $n > 1$, then

$$\frac{f(n)}{g(n)} = \frac{(n + 1)^3}{n^3} < \frac{(n + n)^3}{n^3} = \frac{8n^3}{n^3} = 8$$

Choose $C = 8$. Note that $n + 1 < n + n$ and $(n + n)^3 = (2n)^3 = 8n^3$. Thus, $(n + 1)^3$ is $O(n^3)$ because $(n + 1)^3 \leq 8n^3$ whenever $n > 1$.

Proving Big-Oh: Example 4

Show that $f(n) = \sum_{i=1}^n i$ is $O(n^2)$.

Choose $k = 1$.

Assuming $n > 1$, then

$$\frac{f(n)}{g(n)} = \frac{\sum_{i=1}^n i}{n^2} \leq \frac{\sum_{i=1}^n n}{n^2} = \frac{n^2}{n^2} = 1$$

Choose $C = 1$. Note that $i \leq n$ because n is the upper limit. Thus, $\sum_{i=1}^n i$ is $O(n^2)$ because $\sum_{i=1}^n i \leq n^2$ whenever $n > 1$.