CSE 221: Algorithms

Heapsort

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References

1 T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms, Second Edition*. The MIT Press, September 2001.

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Contents

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Heapsort

- $O(n \lg n)$ in the worst case – like *merge sort*.

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Heapsort

- $O(n \lg n)$ in the worst case – like *merge sort*.
- Sorts *in place* – like *insertion sort*.
- Combines the best of both algorithms.
- Uses a data structure called the heap, which is also extensively used in other applications.

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Max vs. Min Heap

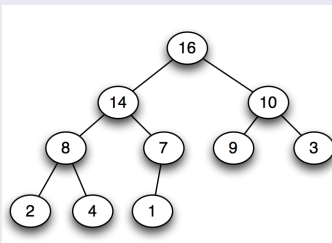Unless explicitly stated as max heap or min heap, heap means max heap in this course.

Heap-ordered tree

Definition

A binary tree is heap-ordered if:

1 the value at a node is $\geq$ the value at each of its children.
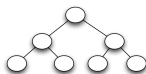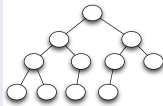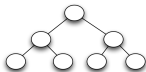
Example of (max) heap

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1. the value at a node is $\geq$ the value at each of its children.

2. the tree is almost-complete.
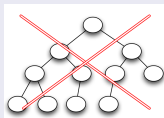
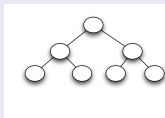### Example of complete tree (or *not*)

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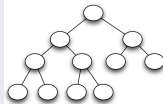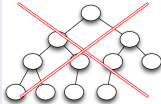
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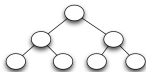
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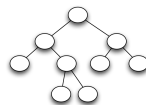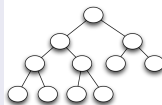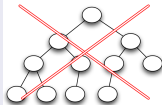
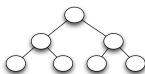### Example of complete tree (or *not*)

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2. the tree is almost-complete. Height of tree is $\Theta(\lg n)$.

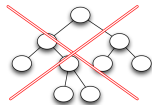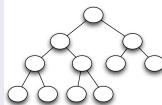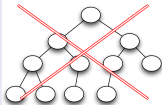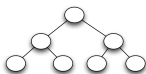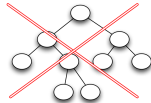### Example of complete tree (or *not*)
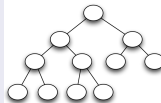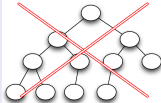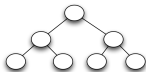
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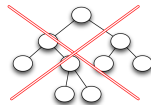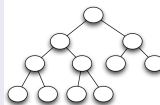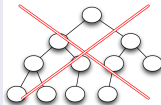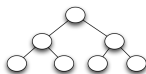
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- A heap of height h has the most number of elements if the tree is complete, so n equals the sum of nodes at each level.

$$n \leq 2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^h$$
$$= \frac{2^{h+1} - 1}{2 - 1}$$
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- $2^h \leq n \leq 2^{h+1} - 1 < 2^{h+1} \quad \Rightarrow \quad h \leq \lg n < h + 1$. Since h is an integer, $h = \lfloor \lg n \rfloor = \Theta(\lg n)$.

Heap – array representation of heap-ordered tree
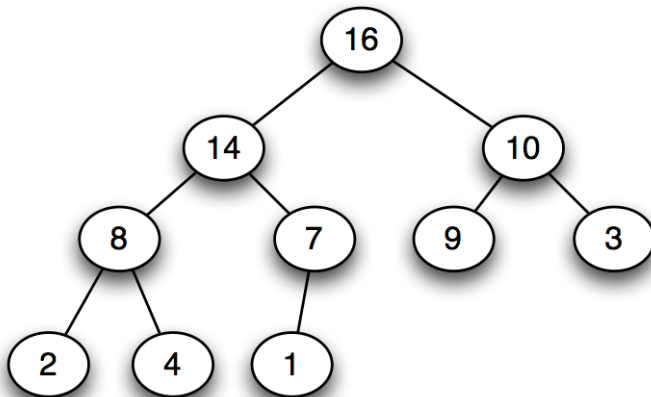
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1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1
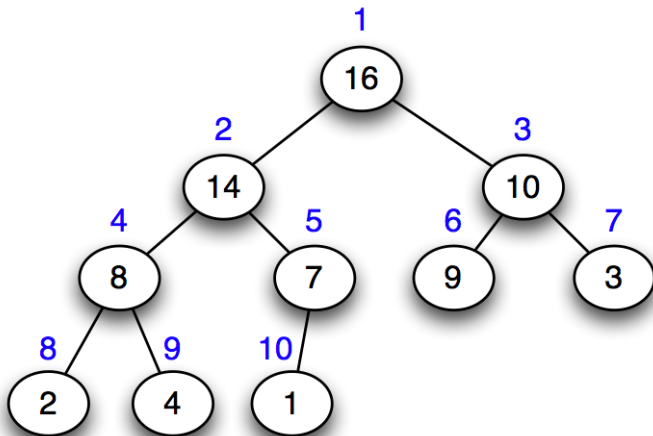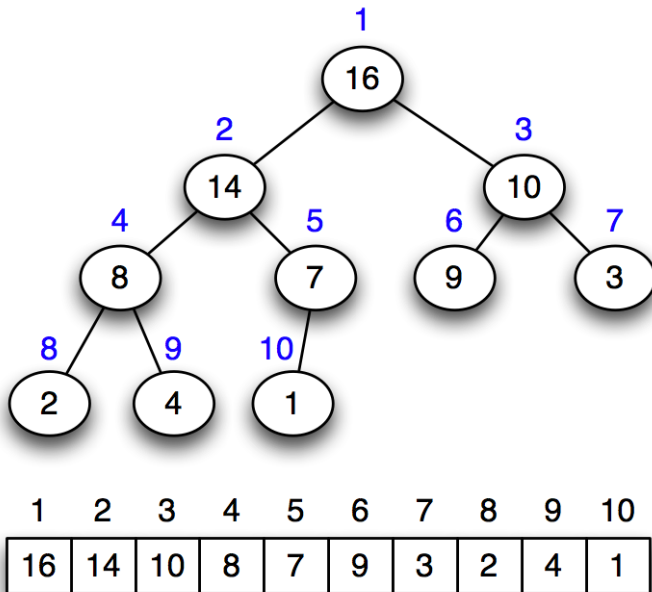
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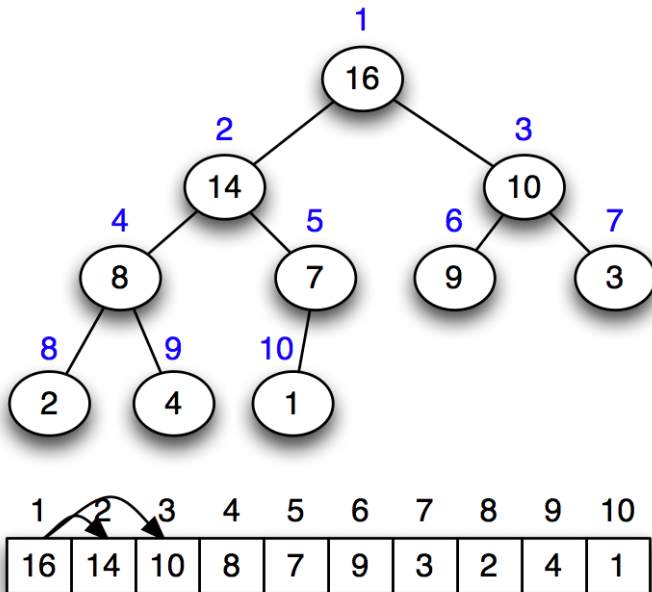
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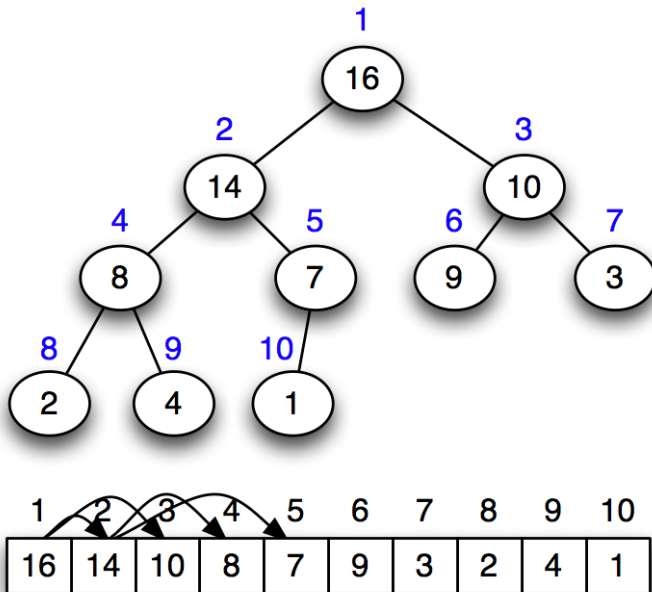
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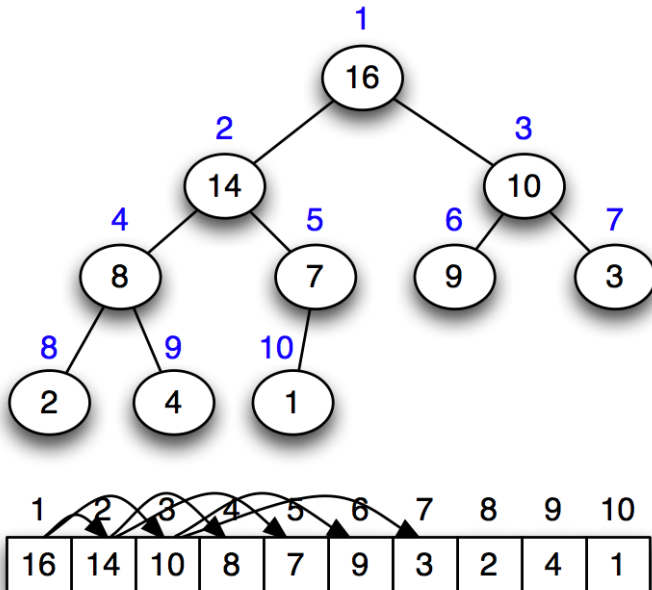
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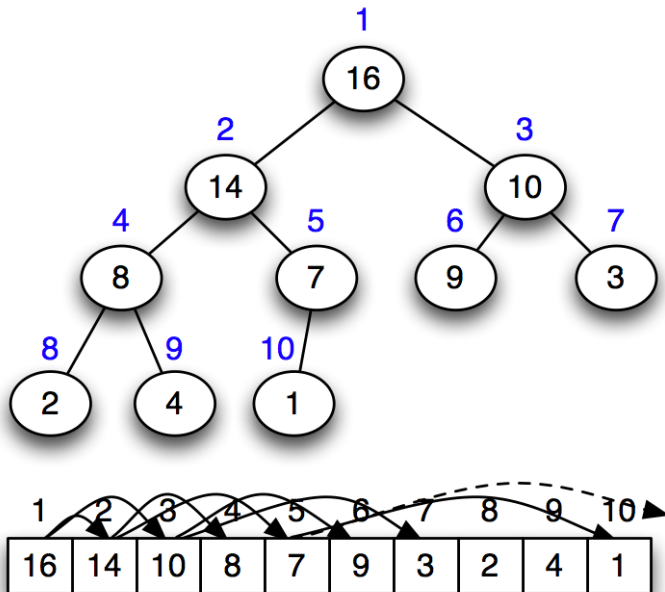
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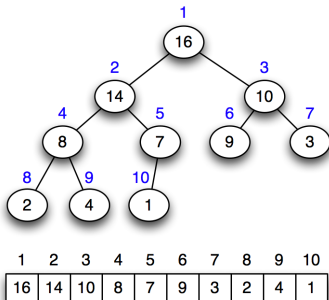
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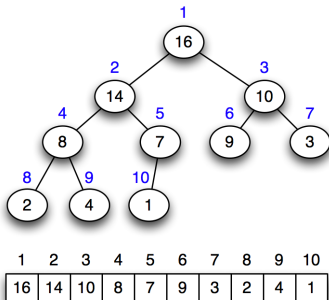
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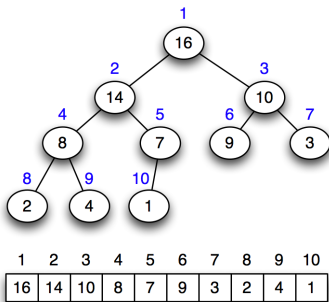
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Question

What if $\text{PARENT}(i) < 1$?

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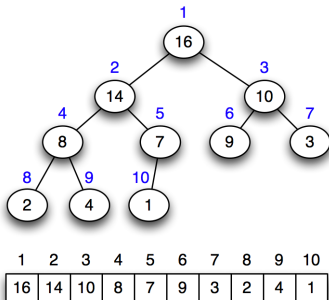
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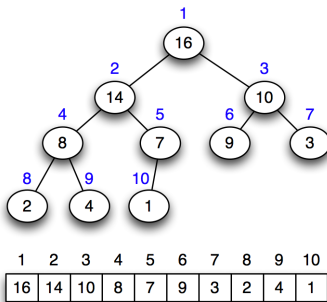
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What if $\textsc{left}(i) > n$?

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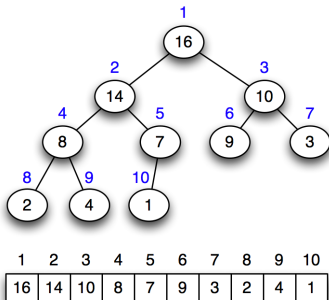
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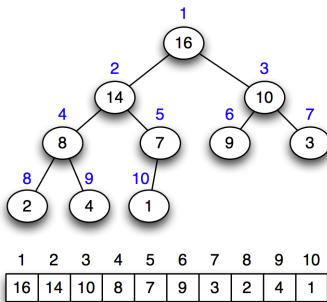
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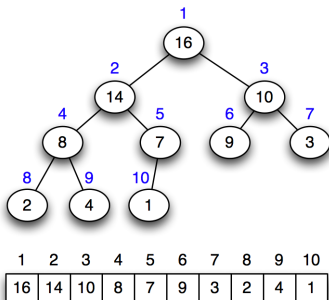
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Lemma

All nodes $i > \lfloor length[A]/2 \rfloor$ (or equivalently, $i > \lfloor heap\text{-}size[A]/2 \rfloor$) are leaf nodes.

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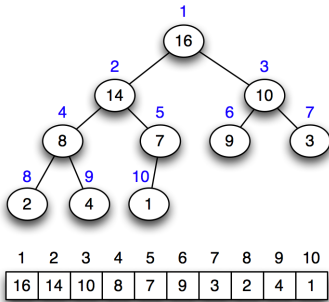
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Definition (Heap property)

Heap property: For every node i other than the root,

$$A[\text{PARENT}(i)] \geq A[i].$$

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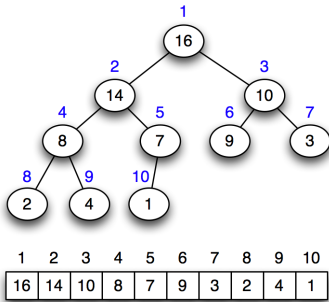
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Question

Why do we insist that a heap-ordered tree be a complete binary tree? (Hint: draw the array representation of a tree that is not complete and see the *gaps*).

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6. HEAP-EXTRACT-MAX(A) – Extract the largest element from heap A.

Example of MAX-HEAPIFY ("sink") operation
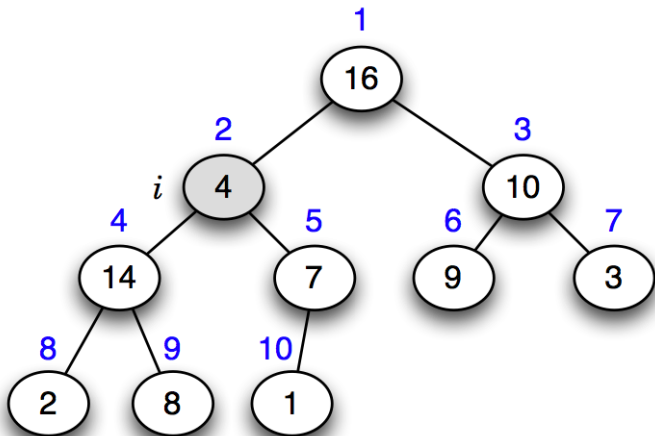
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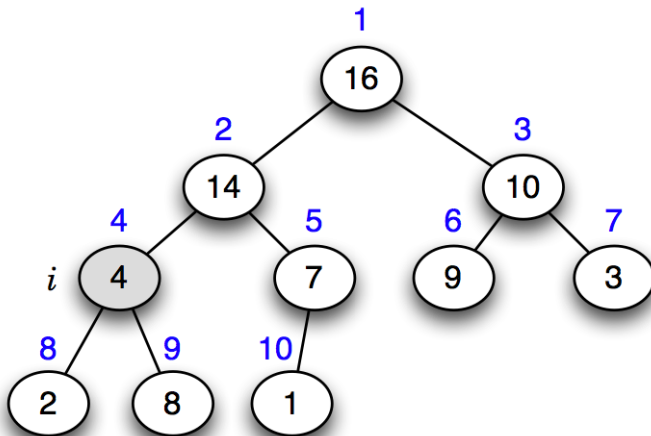
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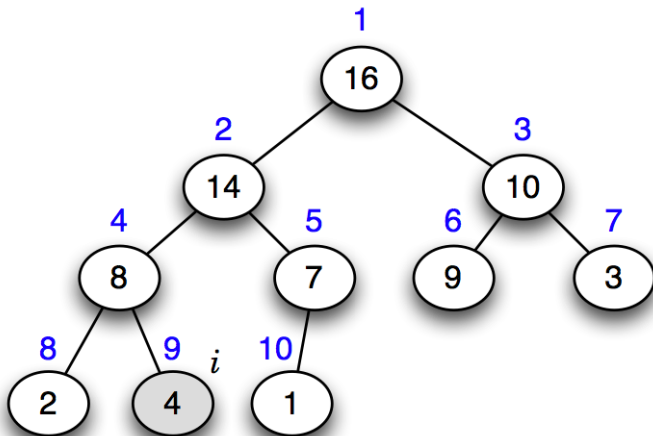
MAX-HEAPIFY algorithm

MAX-HEAPIFY(A, i)

  1 $l \leftarrow left(i)$
  2 $r \leftarrow right(i)$
  3 **if** $l \leq heap\text{-}size[A]$ and $A[l] > A[i]$
  4   **then** $largest \leftarrow l$
  5   **else** $largest \leftarrow i$
  6 **if** $r \leq heap\text{-}size[A]$ and $A[r] > A[largest]$
  7   **then** $largest \leftarrow r$
  8 **if** $largest \neq i$
  9   **then** exchange $A[i] \leftrightarrow A[largest]$
 10   MAX-HEAPIFY($A, largest$)

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MAX-HEAPIFY(A, i)

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1   l ← left(i)
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3   if l ≤ heap-size[A] and A[l] > A[i]
4       then largest ← l
5       else largest ← i
6   if r ≤ heap-size[A] and A[r] > A[largest]
7       then largest ← r
8   if largest ≠ i
9       then exchange A[i] ↔ A[largest]
10           MAX-HEAPIFY(A, largest)
```

Analysis – first way

Since the children's subtrees each have at most size of $2n/3$ (when the last row is exactly half full), we have

$$T(n) \leq T(2n/3) + \Theta(1).$$

According to case 2 of the Master theorem, $T(n) = O(\lg n)$.

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10   MAX-HEAPIFY$(A, largest)$

Analysis – second way

The running time of MAX-HEAPIFY on a node of height h is
$T(n) = O(h) = O(\lg n)$.

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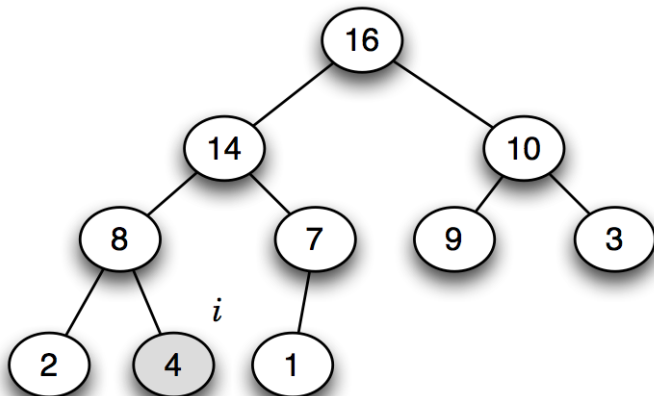
Example of HEAP-INCREASE-KEY ("swim") operation
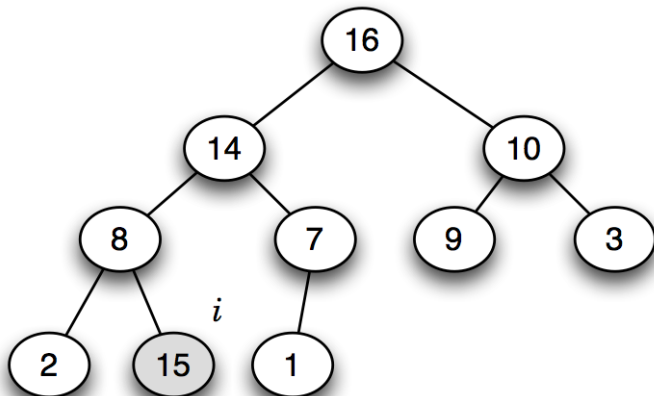
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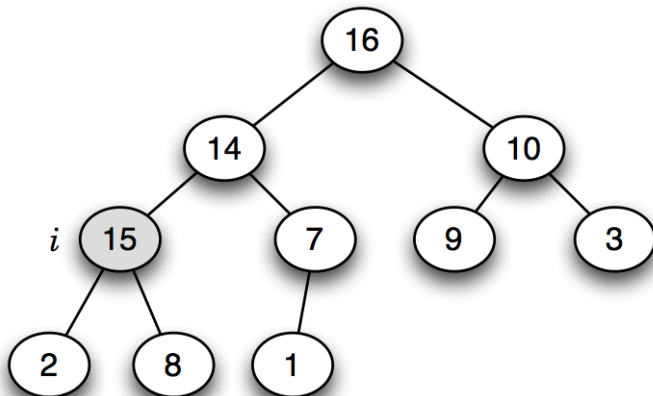
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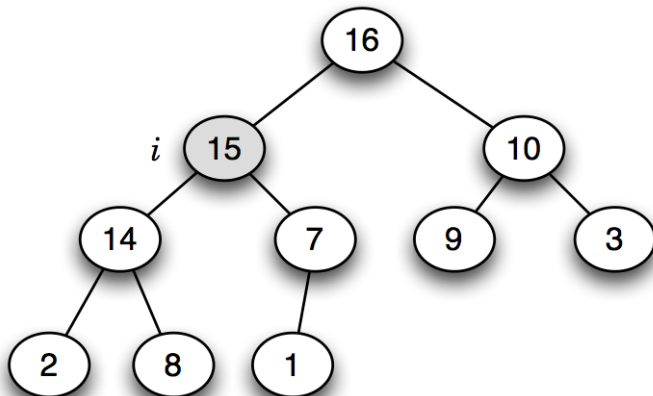
HEAP-INCREASE-KEY algorithm

HEAP-INCREASE-KEY(A, i, key)

1 **if** $key < A[i]$
2   **then error** "new key is smaller than current key"
3 $A[i] \leftarrow key$
4 **while** $i > 1$ and $A[\text{PARENT}(i)] < A[i]$
5   **do** exchange $A[i] \leftrightarrow A[parent(i)]$
6            $i \leftarrow \text{PARENT}(i)$

HEAP-INCREASE-KEY algorithm

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Analysis

A node may move all the way from a leaf node to the root because of increased value, so $T(n) = O(h) = O(\lg n)$.

Example of MAX-HEAP-INSERT operation
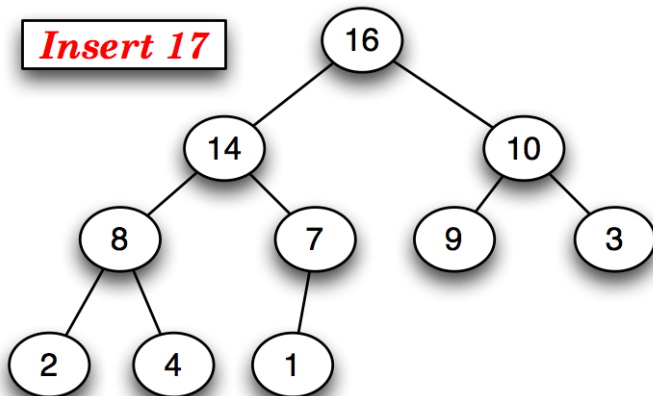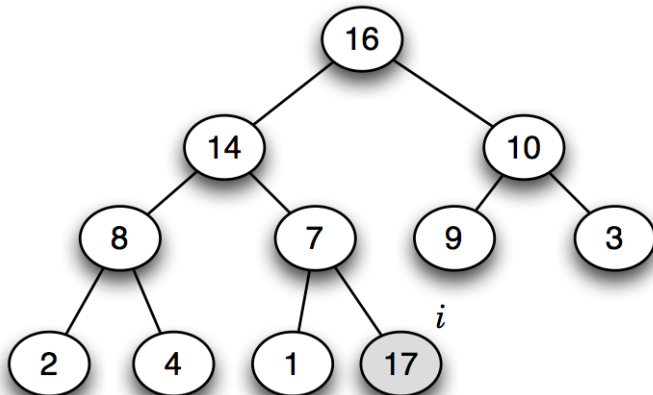
Insert 17

Example of MAX-HEAP-INSERT operation

Example of MAX-HEAP-INSERT operation
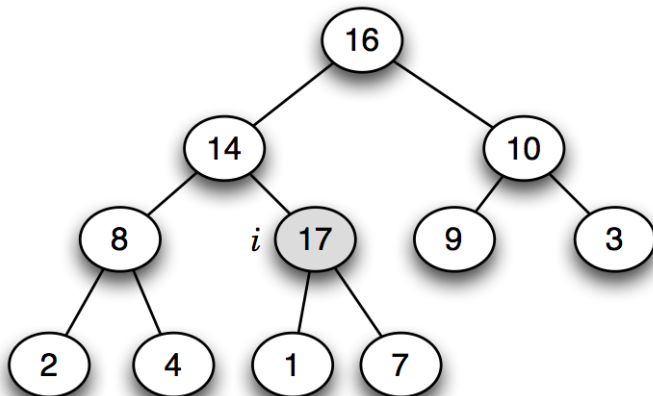
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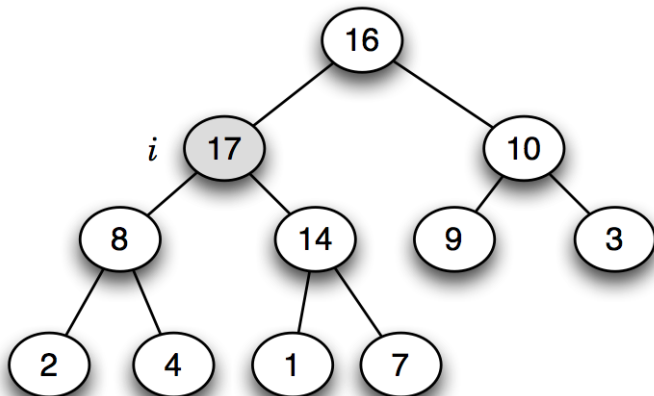
Example of MAX-HEAP-INSERT operation
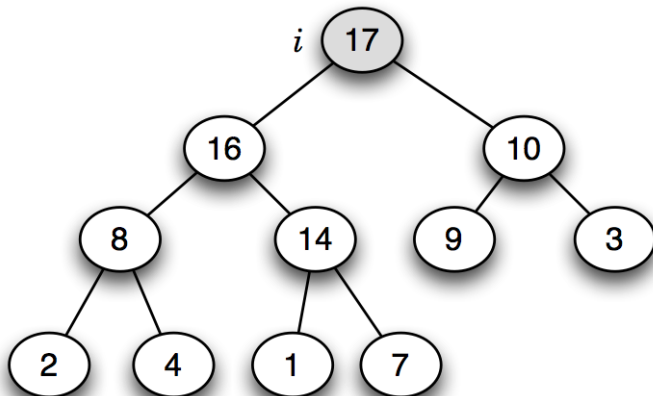
MAX-HEAP-INSERT algorithm

MAX-HEAP-INSERT(A, key)

1   $heap\text{-}size[A] \leftarrow heap\text{-}size[A] + 1$
2   $A[heap\text{-}size[A]] \leftarrow key$
3   $i \leftarrow heap\text{-}size[A]$
4   **while** $i > 1$ and $A[\text{PARENT}(i)] < A[i]$
5   **do** exchange $A[i] \leftrightarrow A[parent(i)]$
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MAX-HEAP-INSERT algorithm

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Can also be done using HEAP-INCREASE-KEY.

MAX-HEAP-INSERT(A, key)

1 $heap\text{-}size[A] \leftarrow heap\text{-}size[A] + 1$
2 $A[heap\text{-}size[A]] \leftarrow -\infty$
3 HEAP-INCREASE-KEY(A, $heap\text{-}size[A]$, key)

MAX-HEAP-INSERT algorithm

MAX-HEAP-INSERT(A, key)

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3 $i \leftarrow heap\text{-}size[A]$
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Analysis

$T(n) = O(h) = O(\lg n)$.

Simple BUILD-MAX-HEAP algorithm

BUILD-MAX-HEAP'(A)

1 *heap-size*[A] ← 1
2 **for** i ← 2 **to** *length*[A]
3   **do** MAX-HEAP-INSERT(A, $A[i]$)

Simple BUILD-MAX-HEAP algorithm

BUILD-MAX-HEAP'(A)

1 *heap-size*[A] $\leftarrow$ 1
2 **for** $i \leftarrow 2$ **to** *length*[A]
3   **do** MAX-HEAP-INSERT($A, A[i]$)

Analysis

There are $n - 1$ calls to MAX-HEAP-INSERT, each taking $O(\lg n)$ time, so $T(n) = O(n \lg n)$.

Simple BUILD-MAX-HEAP algorithm

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1   $heap\text{-}size[A] \leftarrow 1$
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Analysis

There are $n - 1$ calls to MAX-HEAP-INSERT, each taking $O(\lg n)$
time, so $T(n) = O(n \lg n)$.

Better way?

A better way is to build up the heap from the smaller trees. See
next.

Example of BUILD-MAX-HEAP ("heapify") operation

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BUILD-MAX-HEAP algorithm

BUILD-MAX-HEAP(A)

1    $heap\text{-}size[A]] \leftarrow length[A]$
2   **for** $i \leftarrow \lfloor length[A]/2 \rfloor$ **downto** 1
3   **do** MAX-HEAPIFY(A, i)

BUILD-MAX-HEAP algorithm

BUILD-MAX-HEAP(A)

1   $heap\text{-}size[A]] \leftarrow length[A]$
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3   **do** MAX-HEAPIFY(A, i)

Analysis

$T(n) = O(n)$ (see textbook for details)

Contents
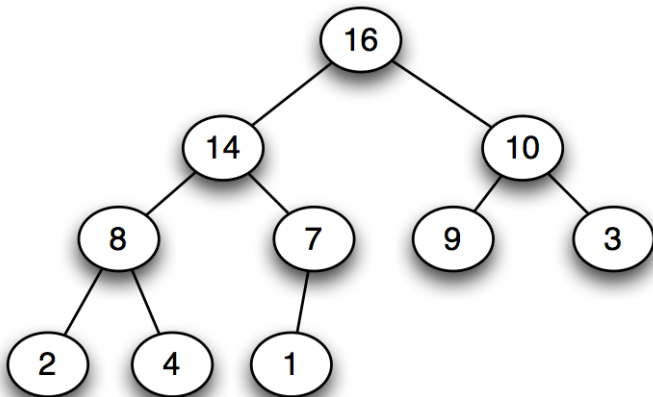
Heap use - heapsort
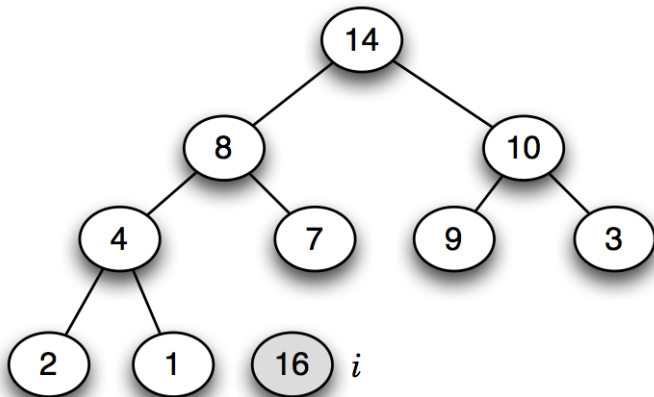
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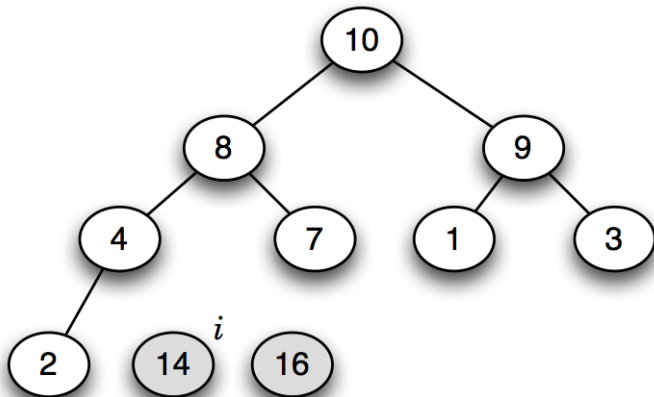
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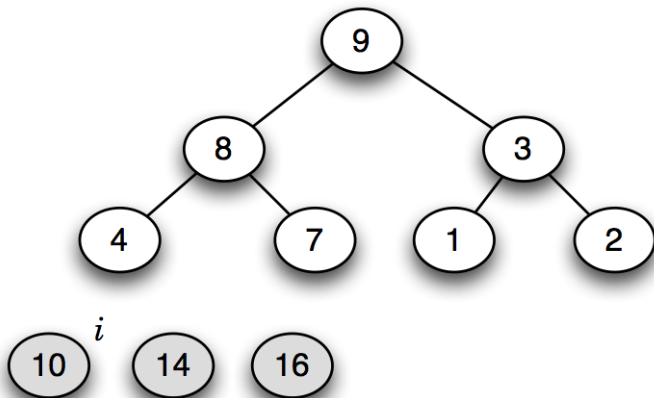
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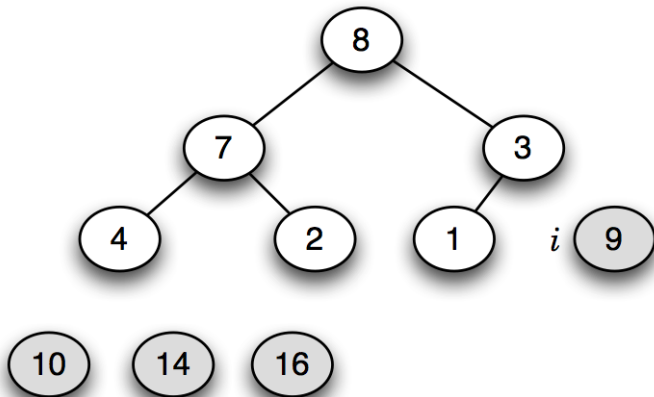
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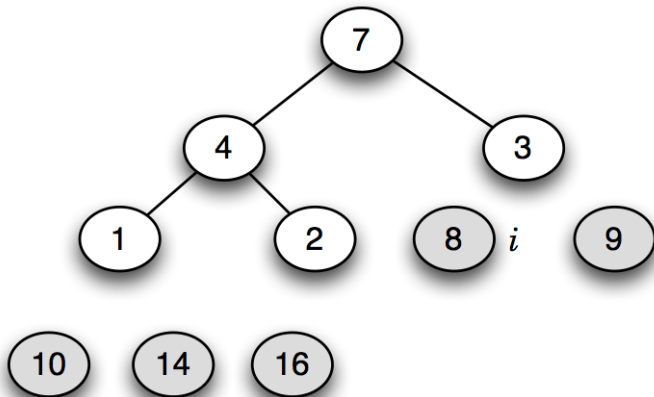
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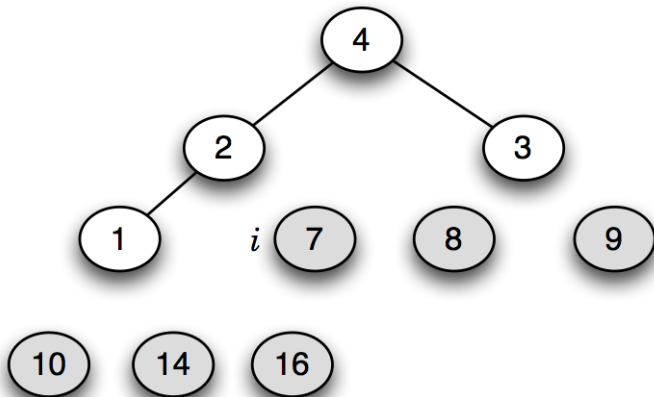
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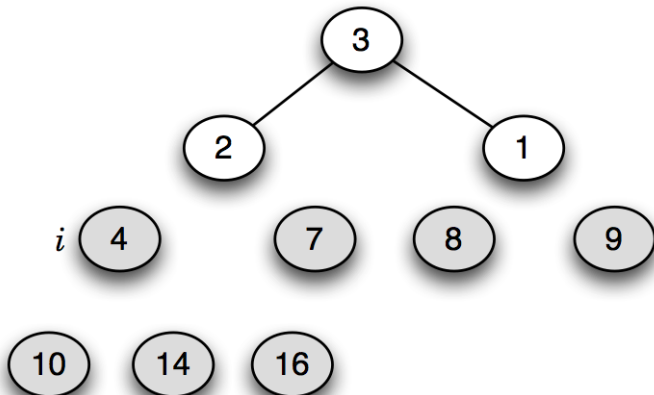
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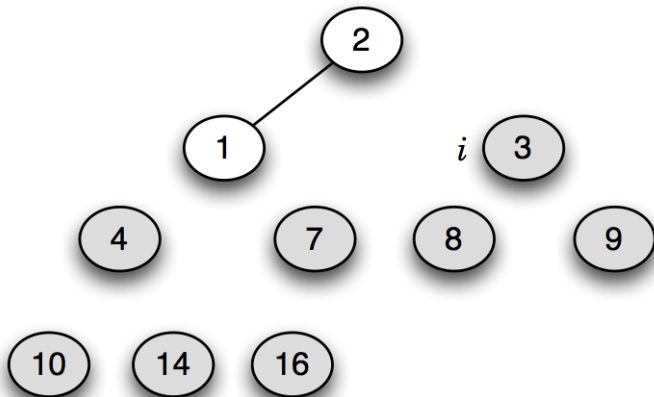
Heap use - heapsort
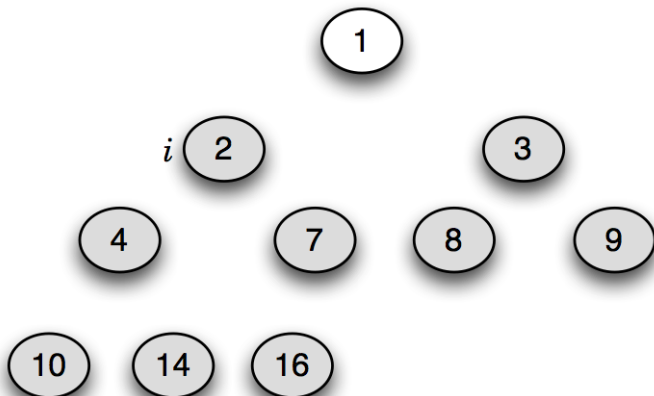
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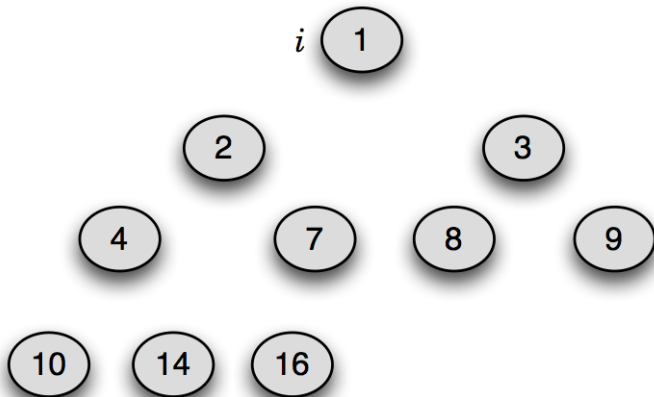
Heap use - heapsort

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HEAPSORT algorithm

HEAPSORT(A)

		cost	times
1	BUILD-MAX-HEAP(A)	$\Theta(n)$	1
2	**for** $i \leftarrow length[A]$ **downto** 2	$\Theta(1)$	n
3	**do** exchange $A[1] \leftrightarrow A[i]$	$\Theta(1)$	$n-1$
4	$heap\text{-}size[A] \leftarrow heap\text{-}size[A] - 1$	$\Theta(1)$	$n-1$
5	MAX-HEAPIFY($A, 1$)	$\Theta(\lg n)$	$n-1$

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HEAPSORT algorithm

$\textsc{heapsort}(A)$

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Worst-case analysis

$T(n) = \Theta(n \lg n)$

Contents

Heap use - priority queue

Definition (Priority Queue)

A priority queue is a data structure for maintaining a set S of elements, each with an associated value called a *key*. A max-priority queue supports the following operations.

1. $\text{INSERT}(S, x)$ – inserts the element x into the set S. This operation could be written as $S \leftarrow S \cup \{x\}$.

2. $\text{MAXIMUM}(S)$ – returns the element of S with the largest key.

3. $\text{EXTRACT-MAX}(S)$ – removes and returns the element of S with the largest key.

4. $\text{INCREASE-KEY}(S, x, k)$ – increases the value of element x's key to k. Assume $k \geq x$'s current value.

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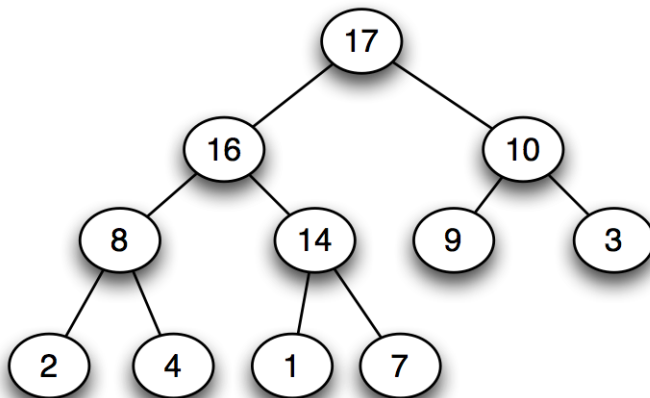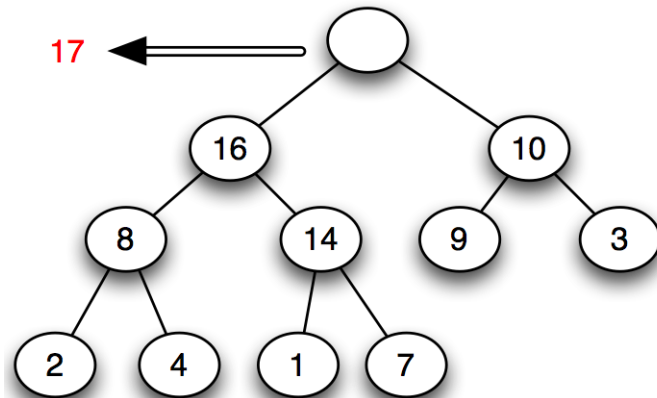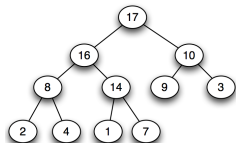- A FIFO queue is a priority queue where the priority is inversely proportional to time of arrival.
- A LIFO stack is a priority queue where the priority is proportional to time of arrival.

Example of HEAP-EXTRACT-MAX operation

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Example of HEAP-EXTRACT-MAX operation

Example of HEAP-EXTRACT-MAX operation

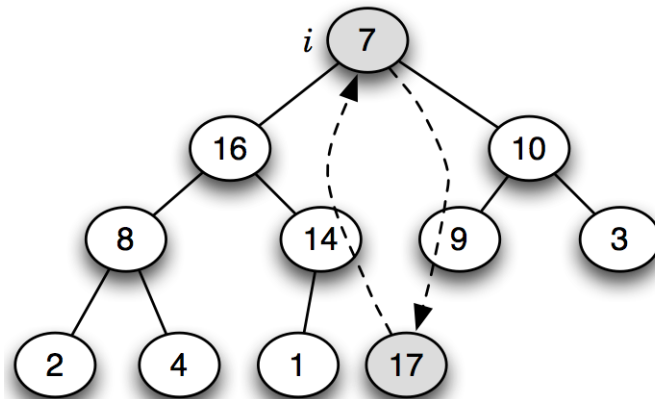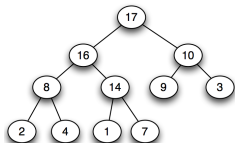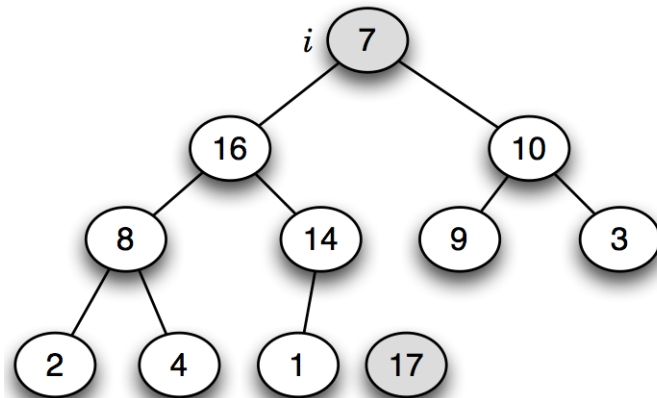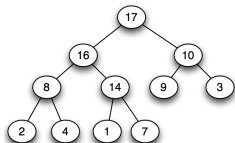## Example of HEAP-EXTRACT-MAX operation
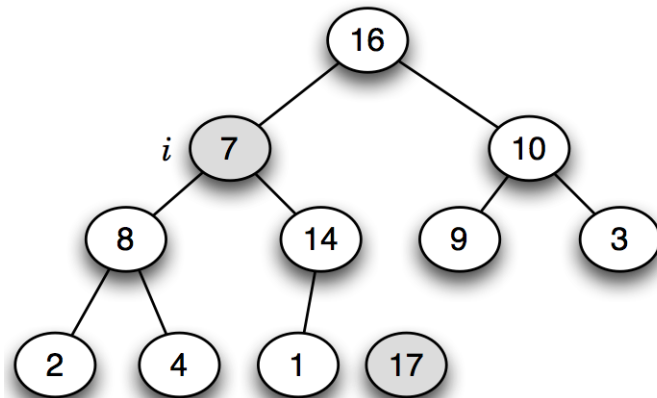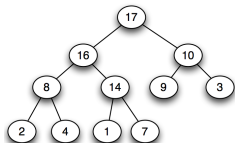
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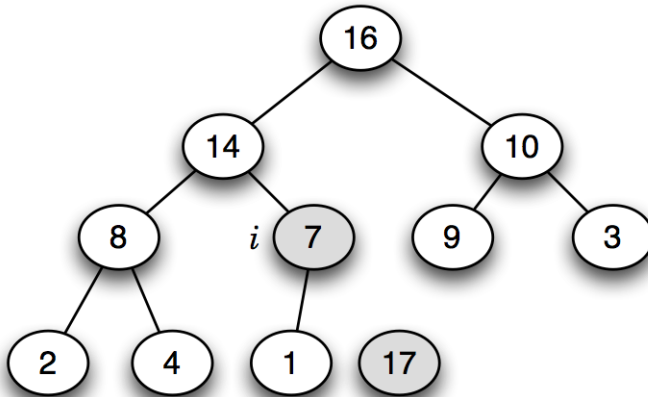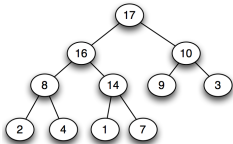
Example of HEAP-EXTRACT-MAX operation

HEAP-EXTRACT-MAX algorithm

EXTRACT-MAX(A)

		cost	times
1	**if** *heap-size*[A] < 1	$\Theta(1)$	1
2	**then error** "heap underflow"	$\Theta(1)$	1
3	*max* ← A[1]	$\Theta(1)$	1
4	A[1] ← A[*heap-size*[A]]	$\Theta(1)$	1
5	*heap-size*[A] ← *heap-size*[A] − 1	$\Theta(1)$	1
6	MAX-HEAPIFY(A, 1)	$\Theta(\lg n)$	1

HEAP-EXTRACT-MAX algorithm

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6	MAX-HEAPIFY($A, 1$)	$\Theta(\lg n)$	1

Worst-case analysis

$T(n) = \Theta(\lg n)$

Conclusion

- Heap plays a very important role in many algorithms, either used directly or as part of a priority queue implementation.
- If the size of a queue is known in advance, then an array representation (using a fixed size array) provides compact storage coupled with fast operations.
- Even if the size of the heap is not known in advance, "intelligent" resizing can still provide good benefits.
- Heapsort is a natural application of Heap with two very important properties – $\Theta(n \lg n)$ complexity, and in-place sorting.