CSE 221: Algorithms

Graph Algorithms

Mumit Khan

Computer Science and Engineering
BRAC University

References

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2. Jon Kleinberg and Éva Tardos, *Algorithm Design*. Pearson Education, 2006.
3. M. Goodrich and R. Tamassia, *Algorithm Design*. John-Wiley and Sons. 2002.

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Introduction to graph algorithms

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Contents

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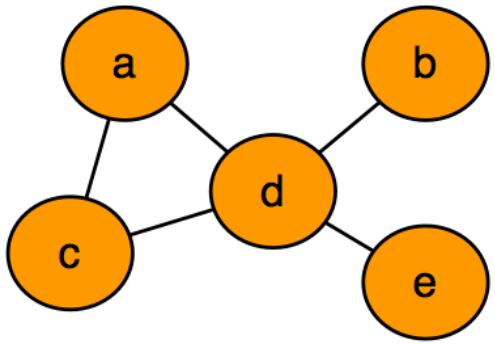

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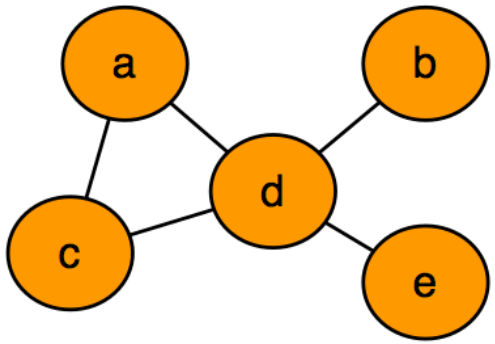

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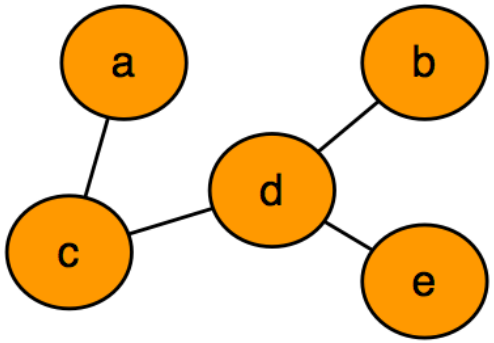

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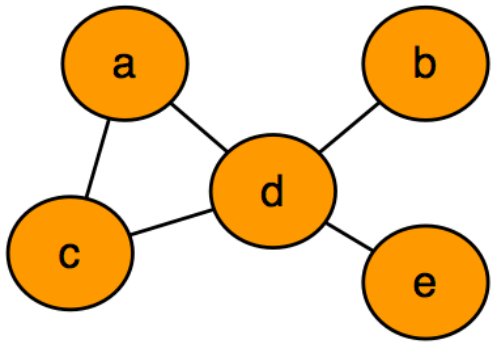

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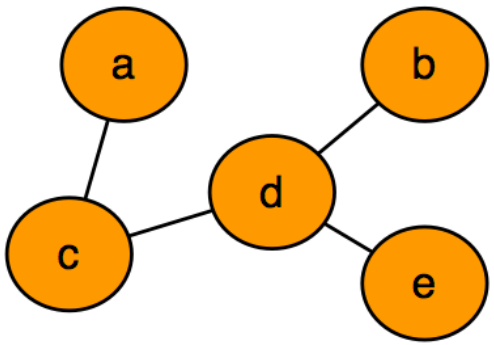

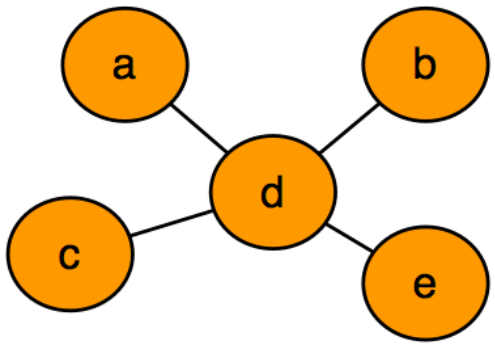

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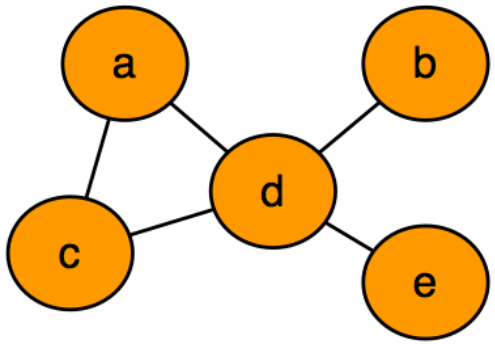

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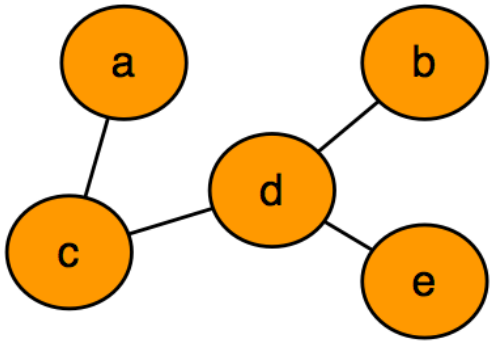

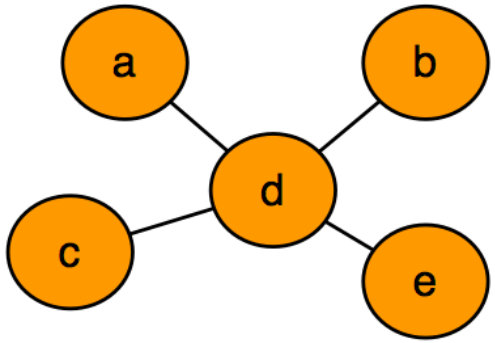

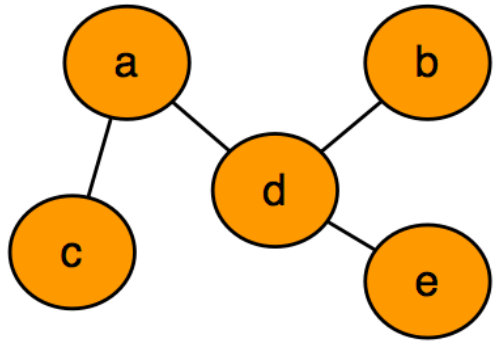

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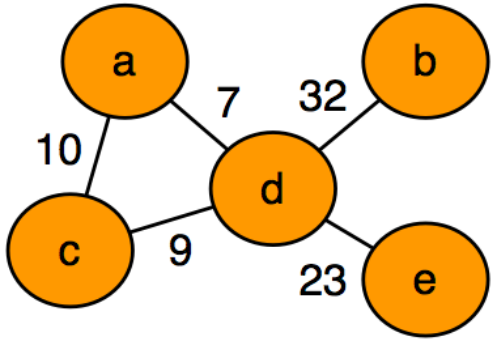

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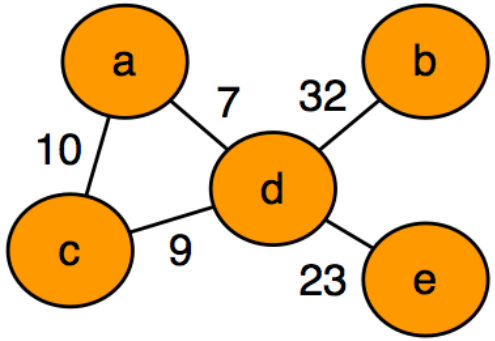
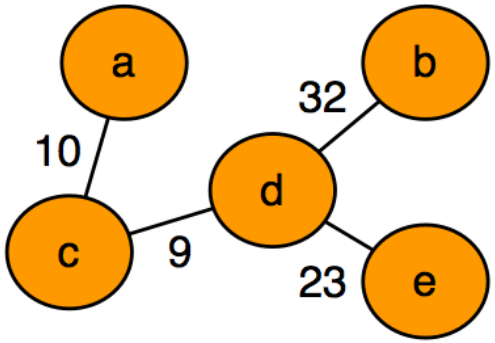
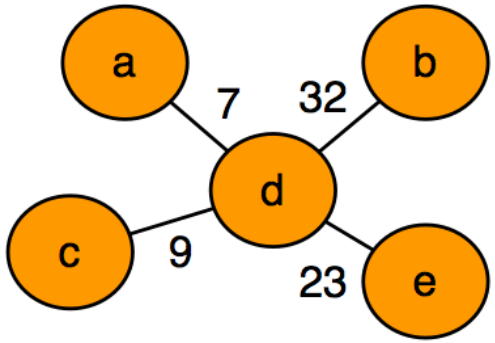


The spanning trees (with associated total weights) are:



$\sum_{e \in T} w(e) = 74$



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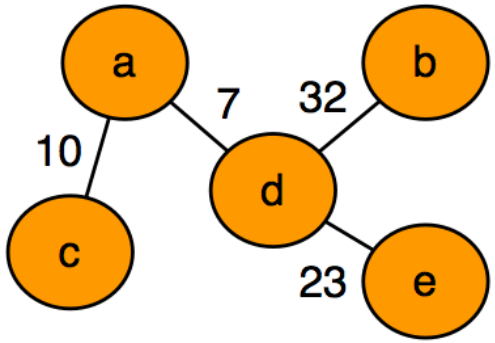

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Minimum-cost Spanning Tree (MST)

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The minimum-cost spanning tree of a graph A spanning tree T of a undirected graph $G = (V, E)$ is a minimum-cost spanning tree of G if the total weight $w(T) = \sum_{(u,v)\in T} w(u, v)$ is minimized.

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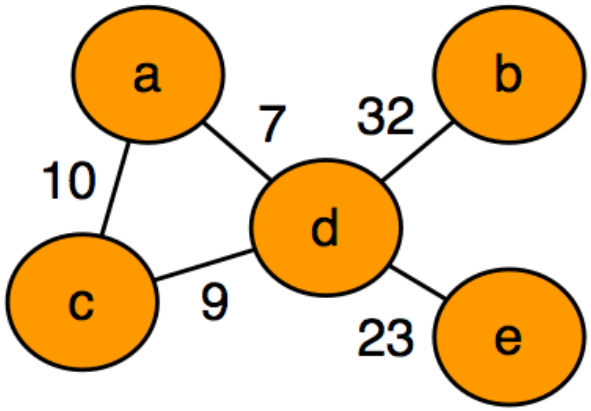

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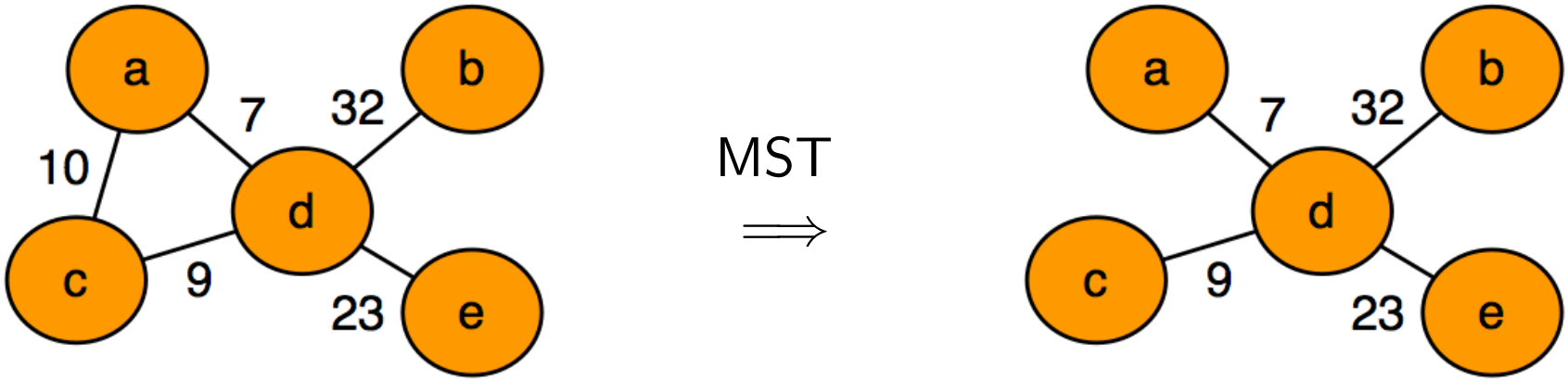

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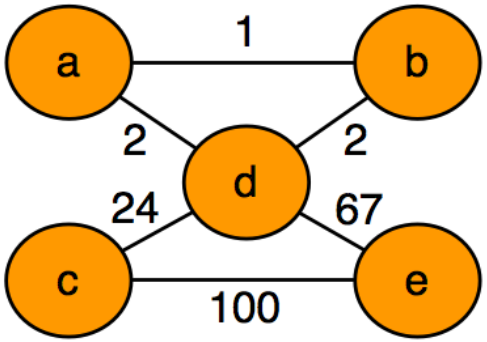

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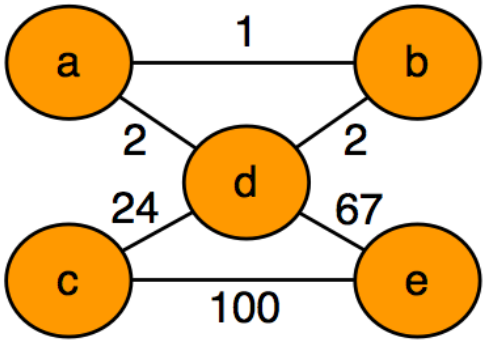

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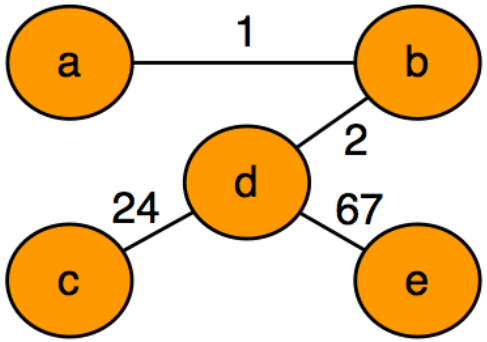

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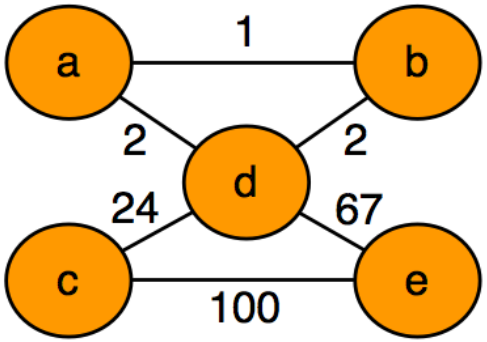

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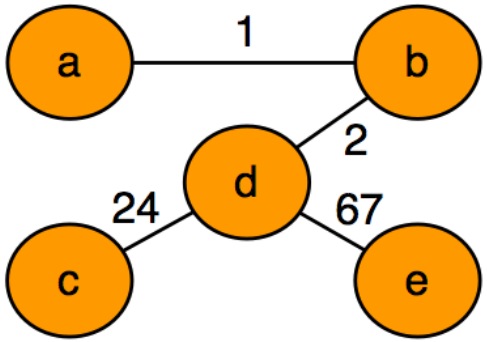

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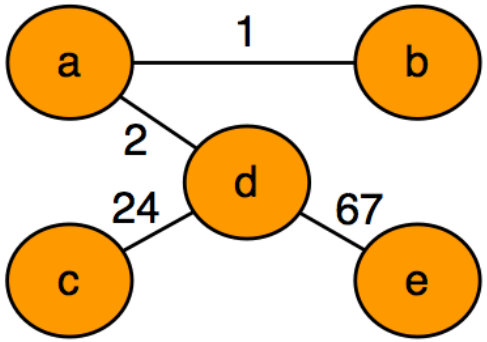

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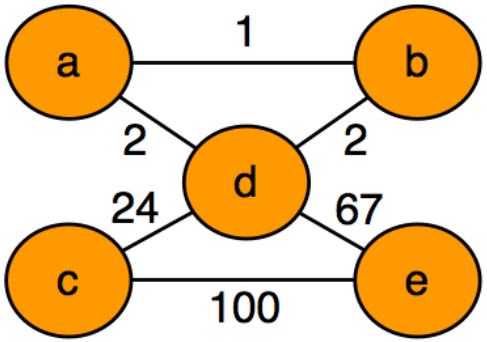

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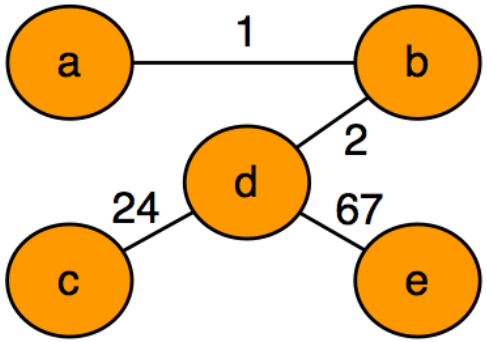

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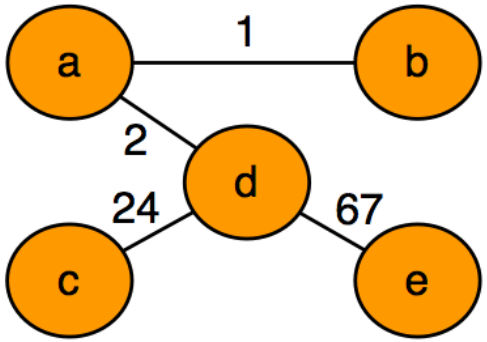

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Key observation

However, if the weights are all distinct (i.e., $w(u_i, v_j) \neq w(u_k, v_l)$ unless $i = k$ and $j = l$), then it is indeed unique.

Computing an MST

- We grow the tree one edge at a time, starting with a graph $G' = (V, \emptyset)$.
- At each step, add a new safe edge, ensuring that it does not create a cycle (why?).
- If adding an edge guarantees that the tree after each step is a subset of some MST, then the final result will be an MST.

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Key question

How do we pick the next safe edge?
Which algorithm design strategy does this question remind you of?

Prim's algorithm to compute an MST

MST-PRIM(G, w, r)

```
 1  for each u ∈ V[G]
 2        do key[u] ← ∞
 3           π[u] ← NIL
 4  key[r] ← 0
 5  Q ← V[G]
 6  while Q ≠ ∅
 7        do u ← EXTRACT-MIN(Q)
 8           for each v ∈ Adj[u]
 9                 do if v ∈ Q and w(u, v) < key[v]
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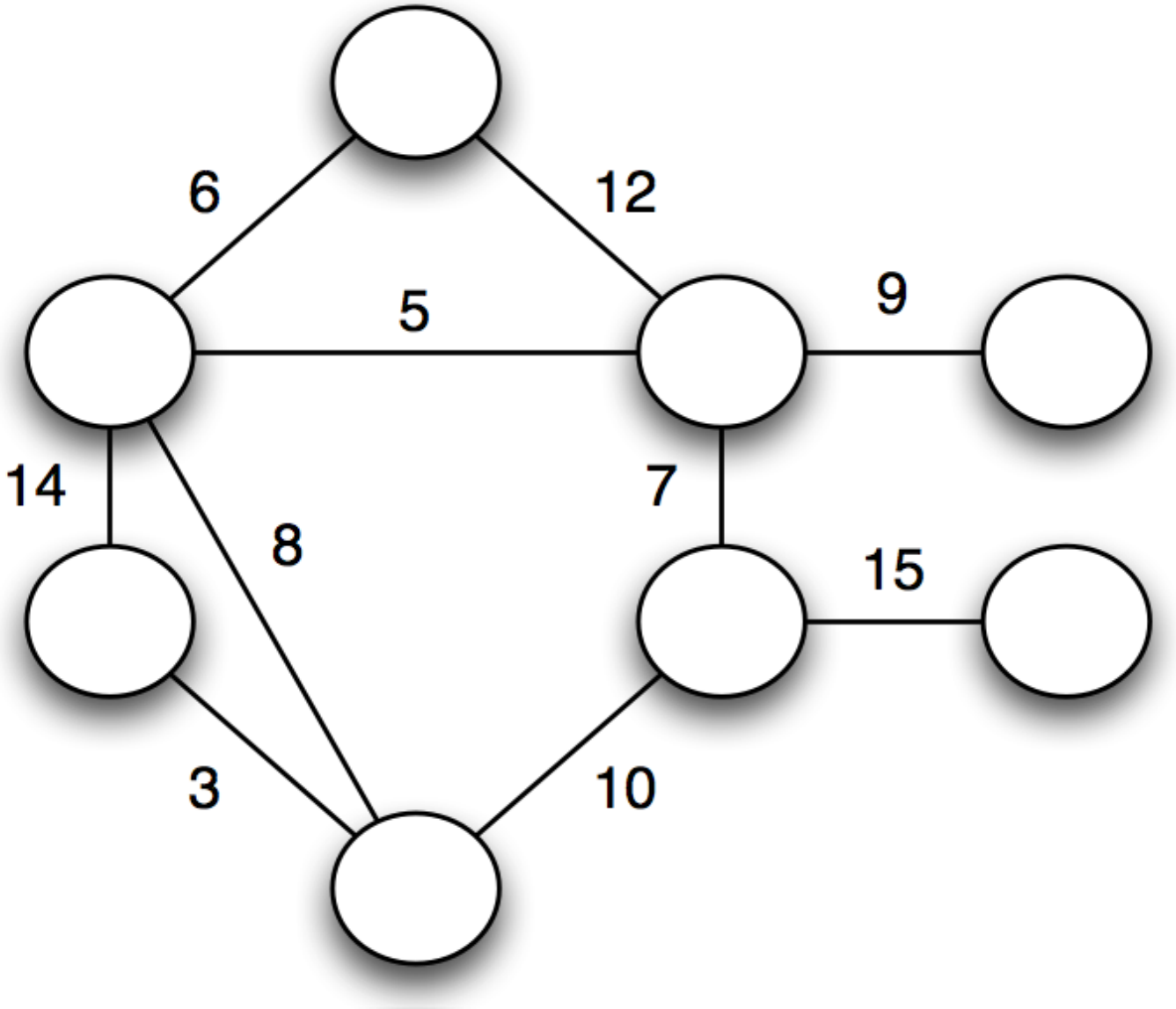
Running time

$O(V \lg V + E \lg V) = O(E \lg V)$

Prim's algorithm in action



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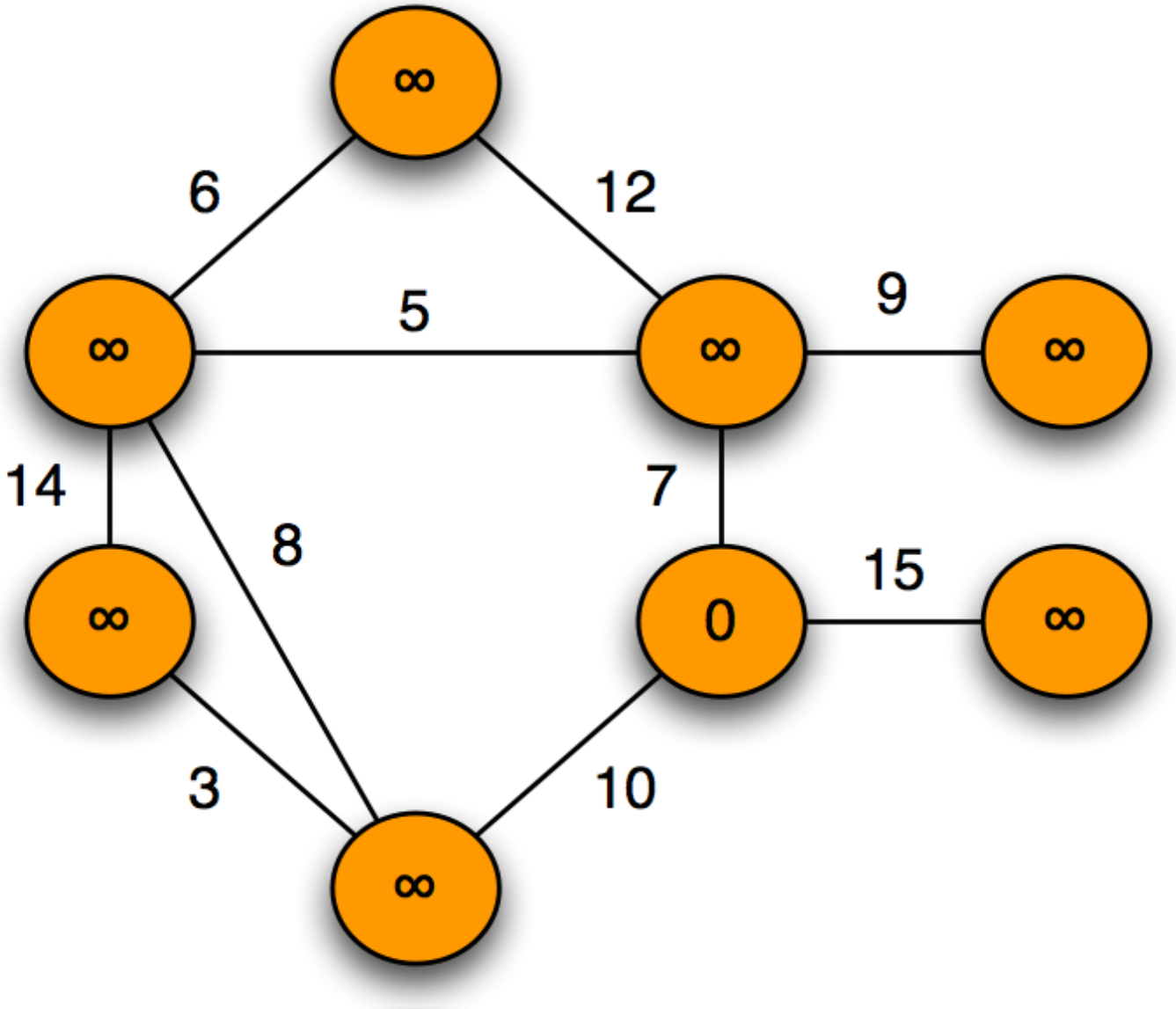

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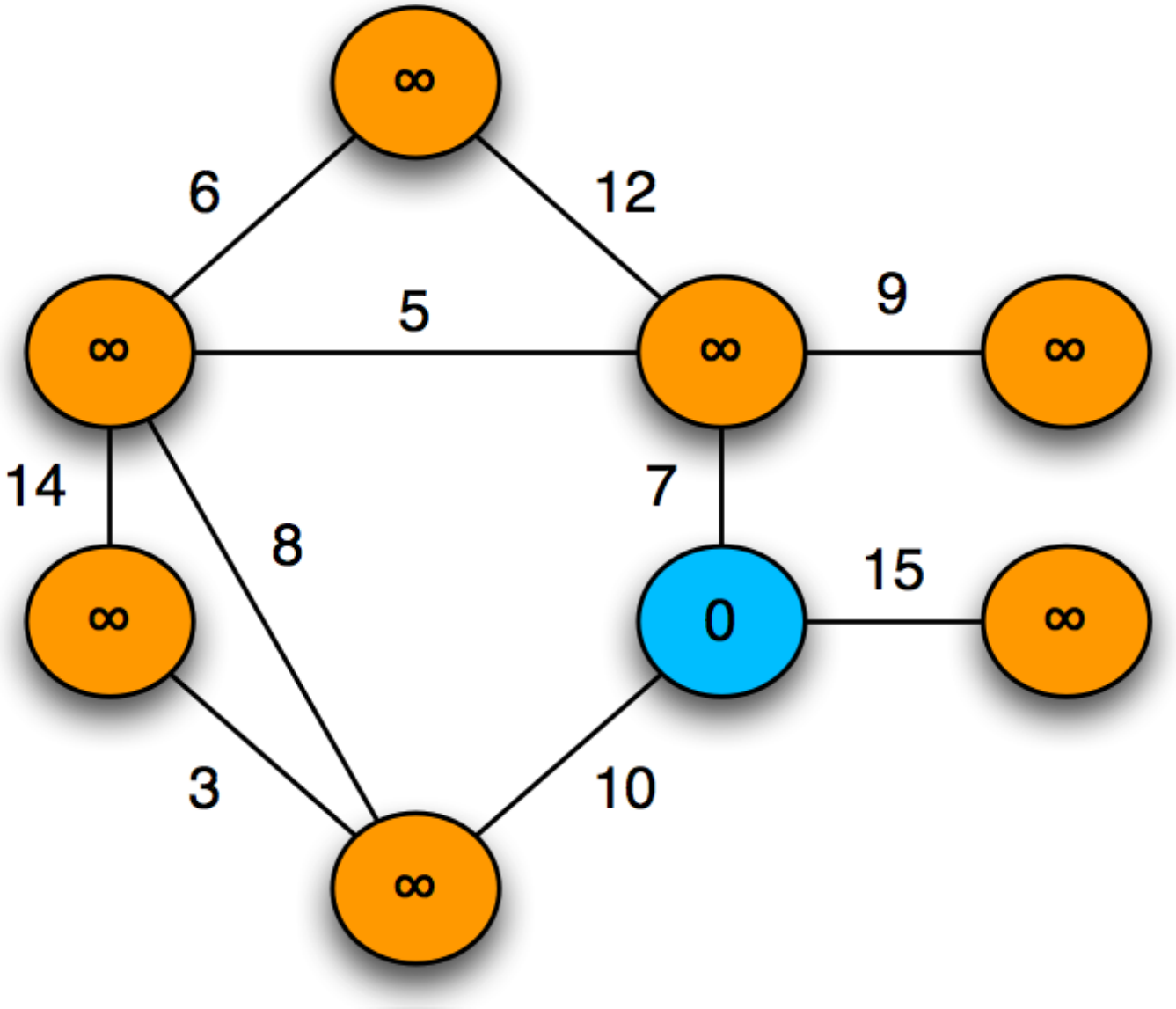

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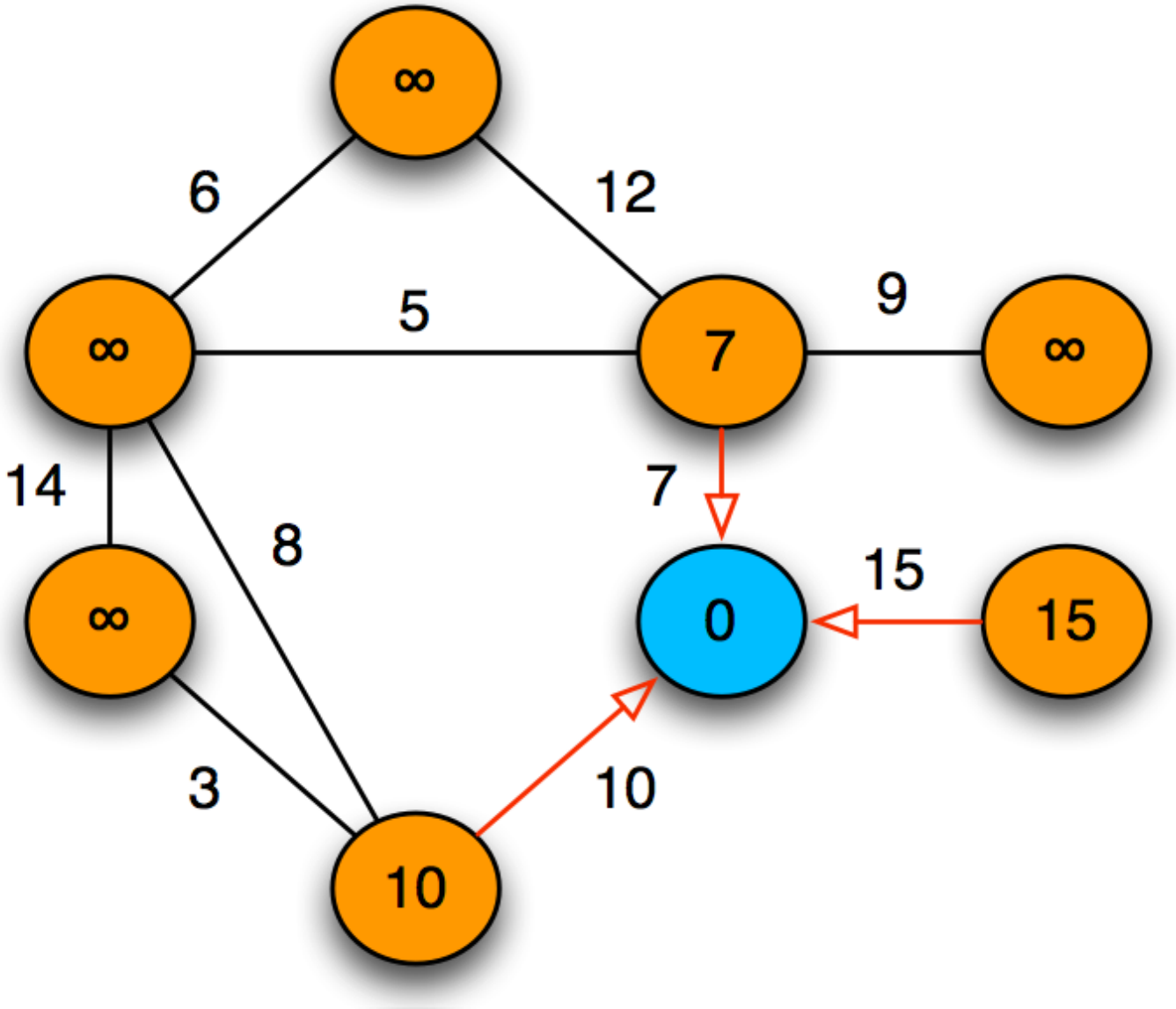

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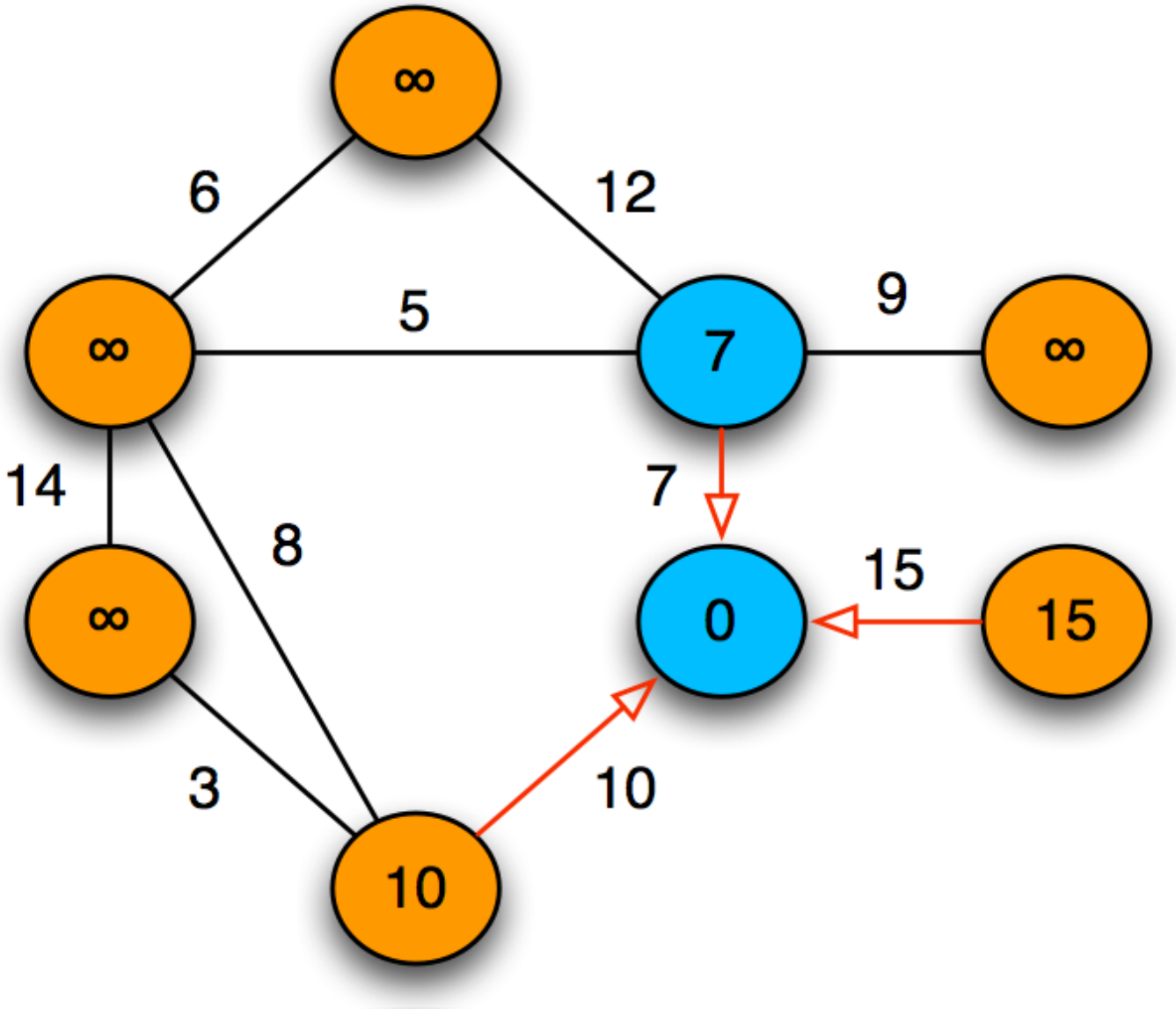

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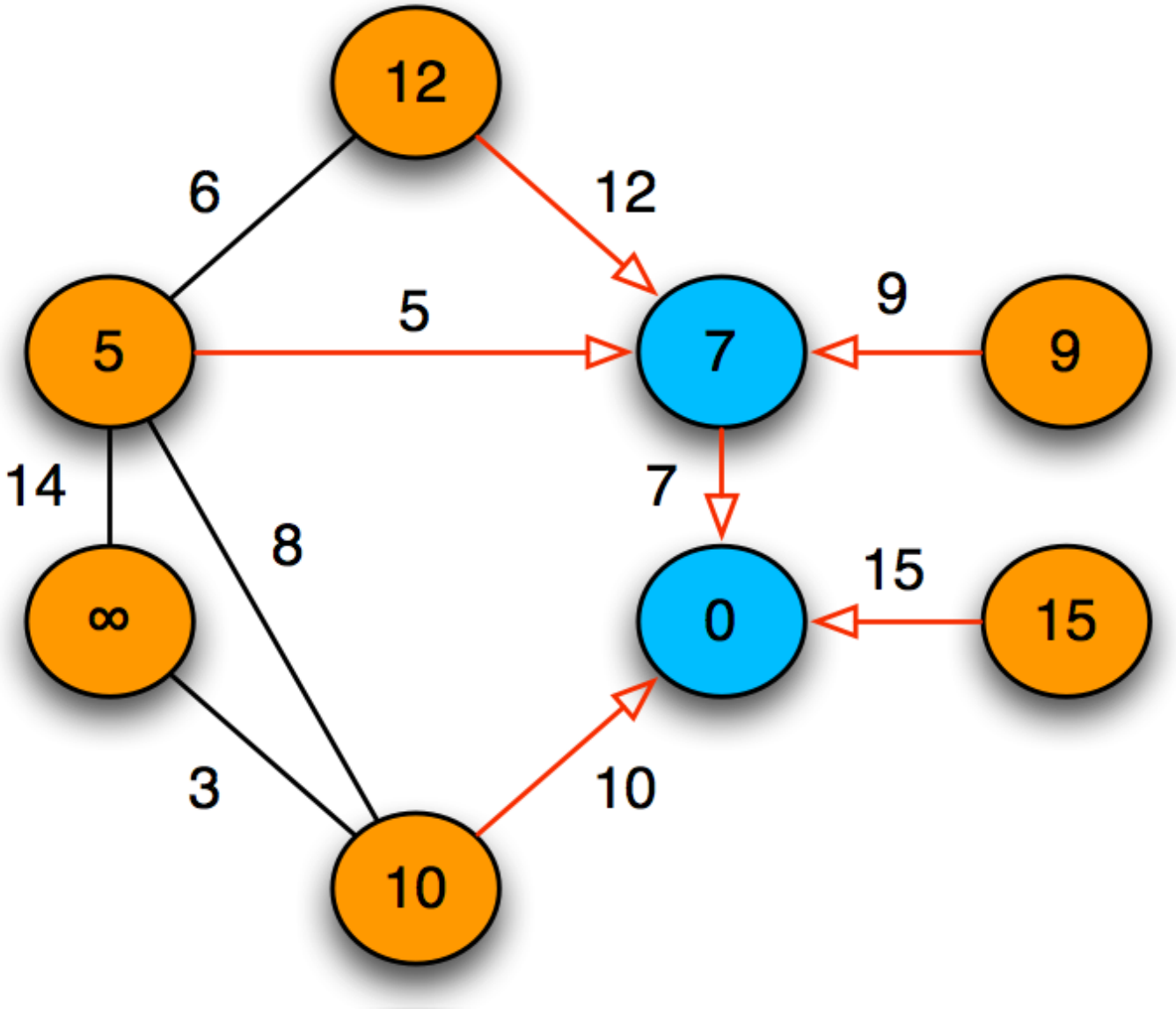

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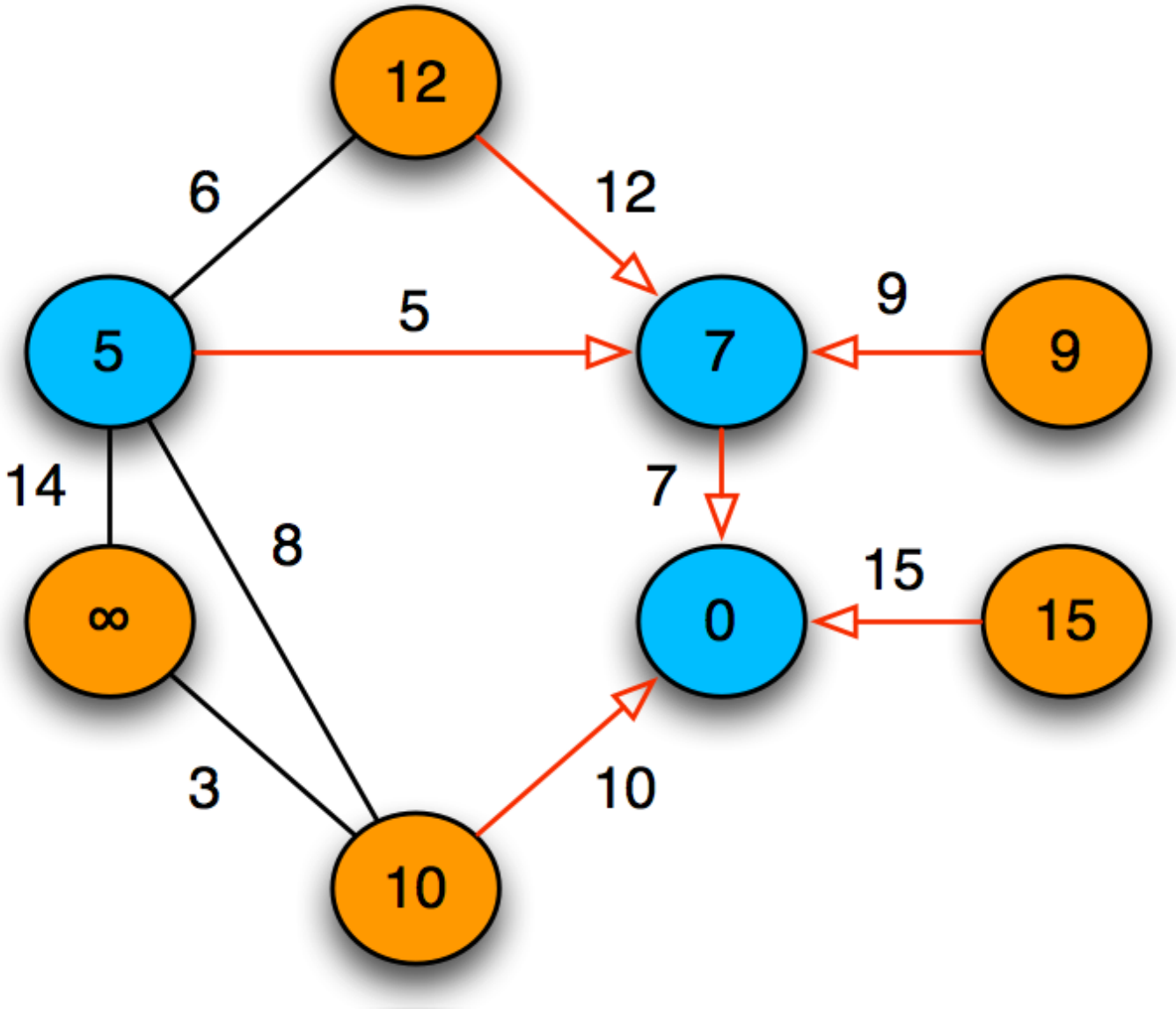

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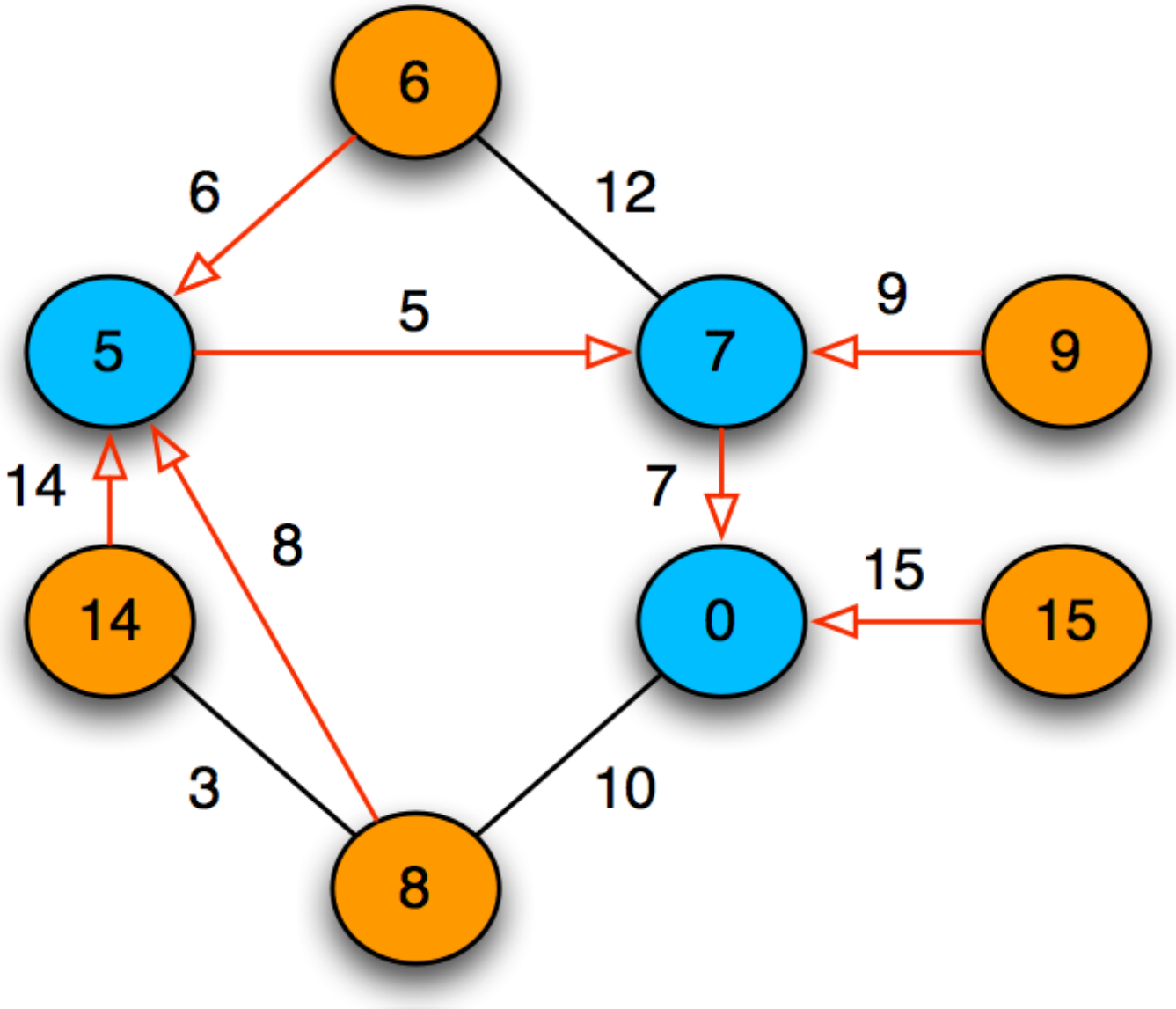

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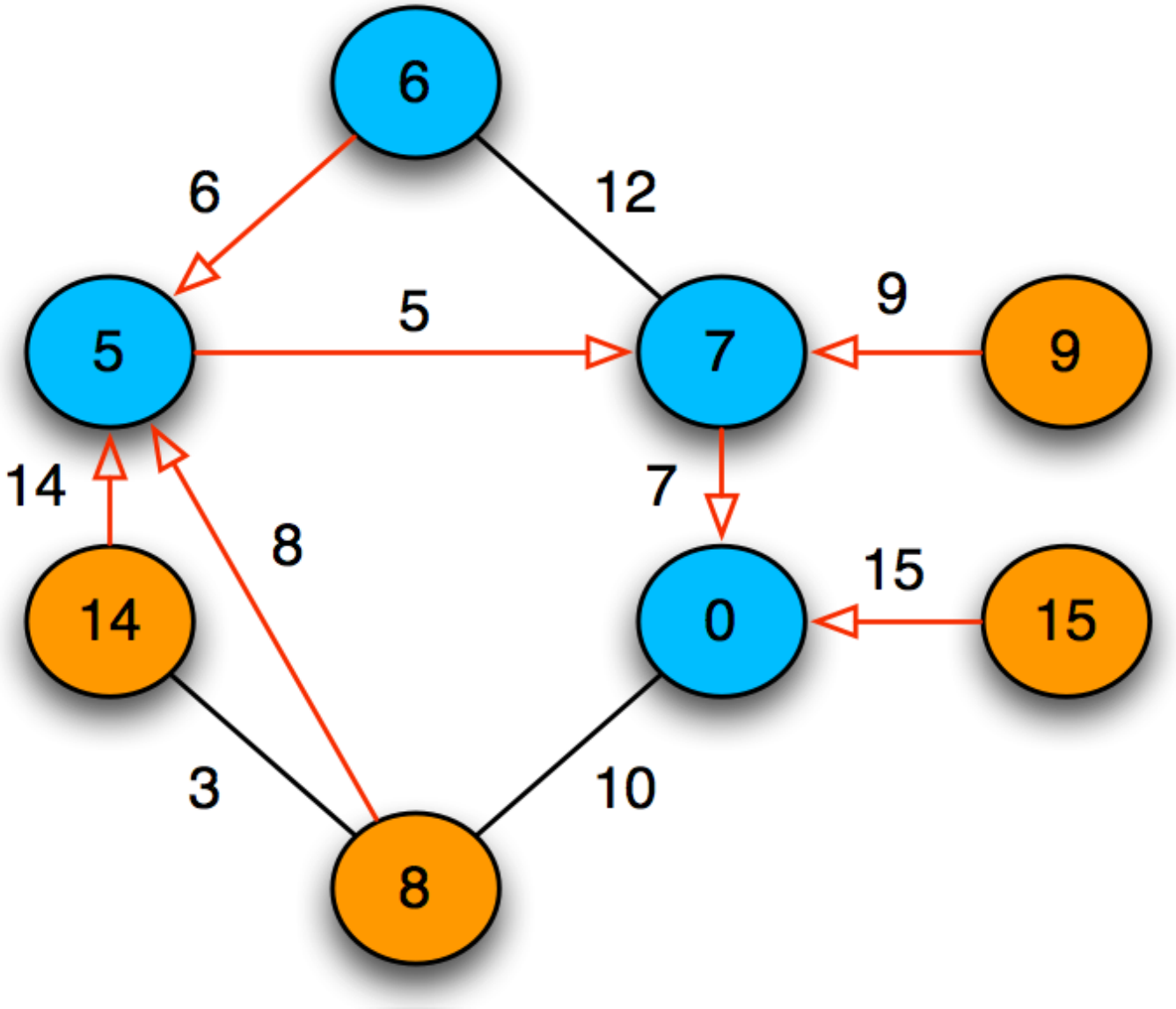

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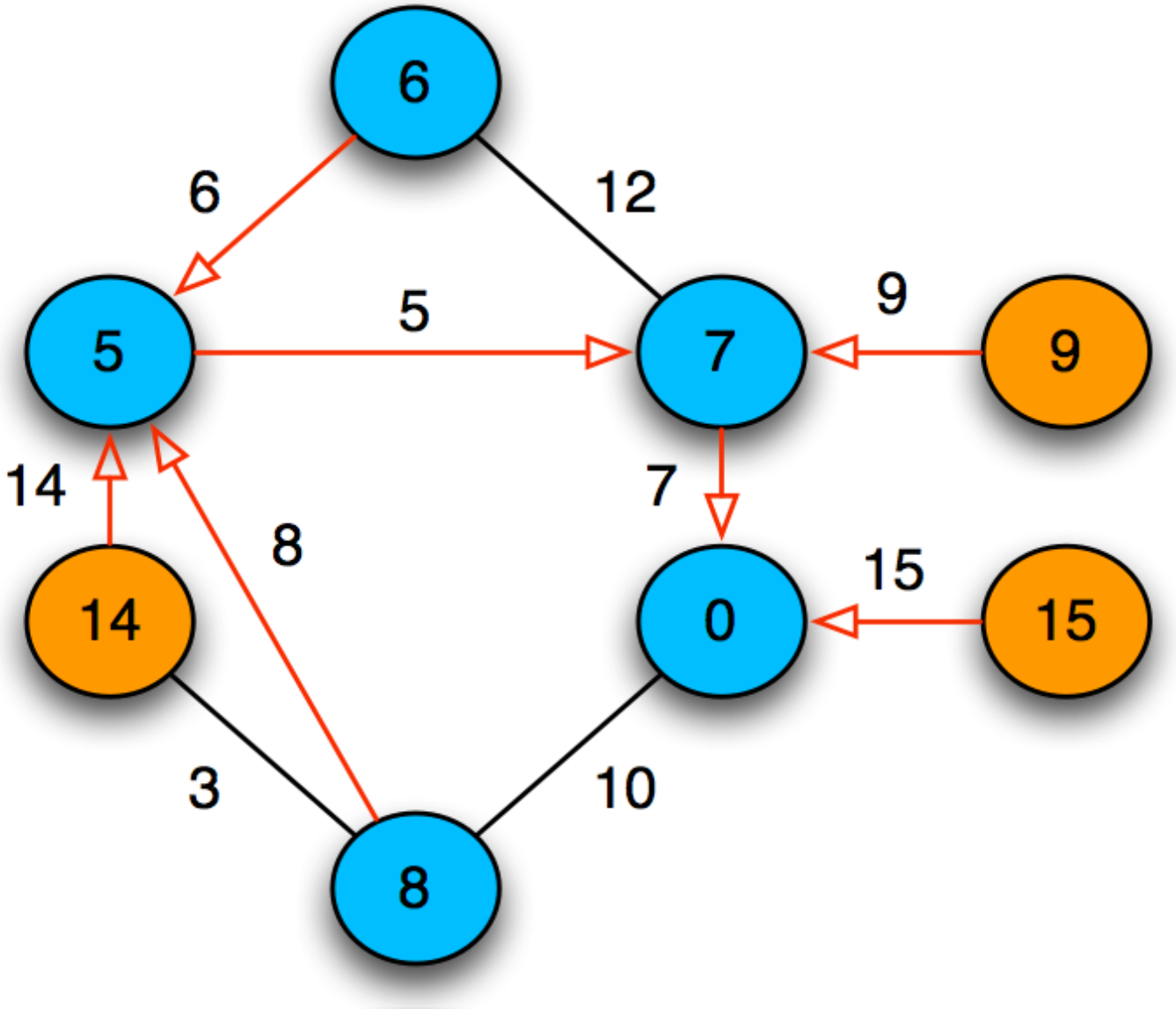

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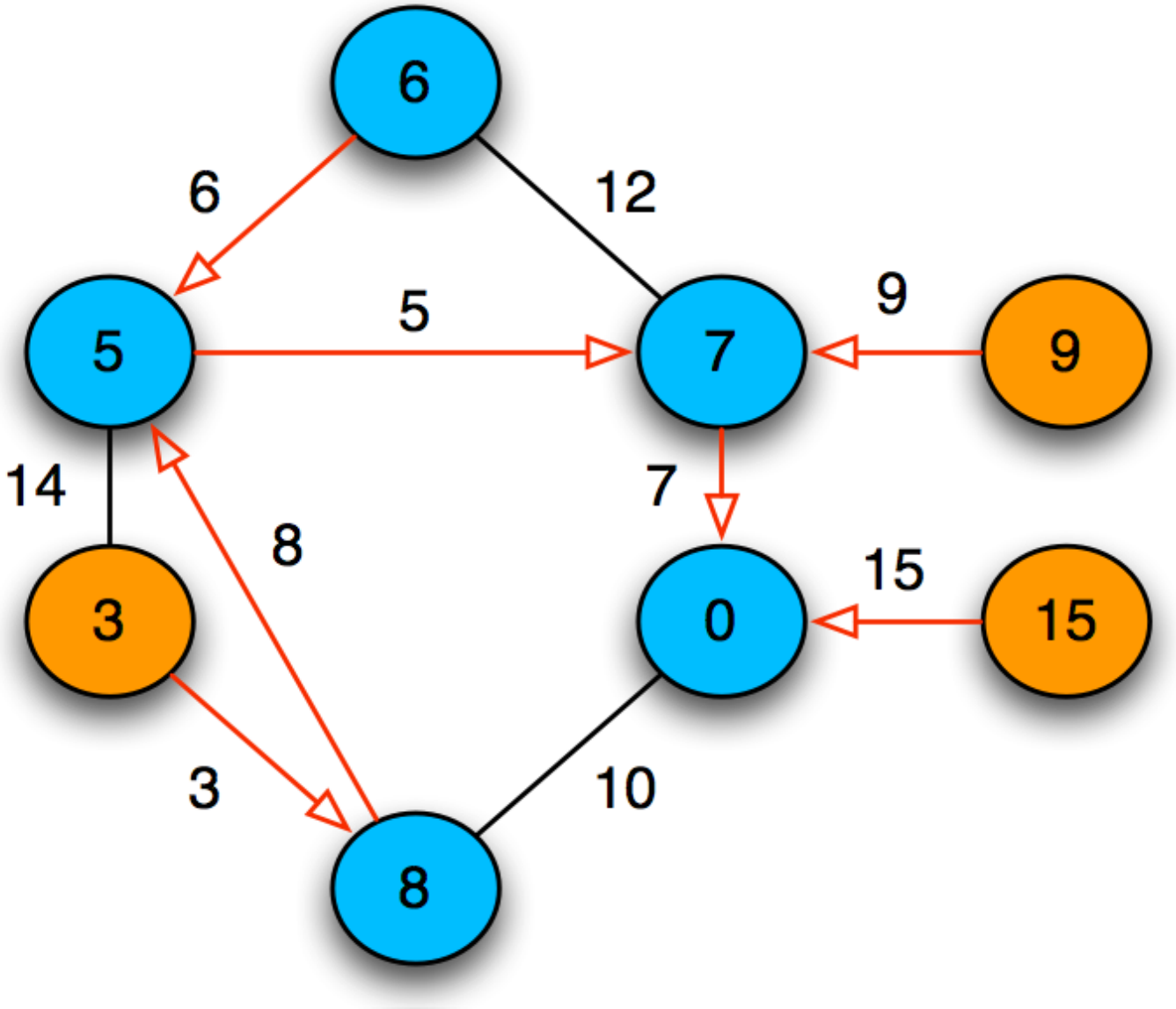

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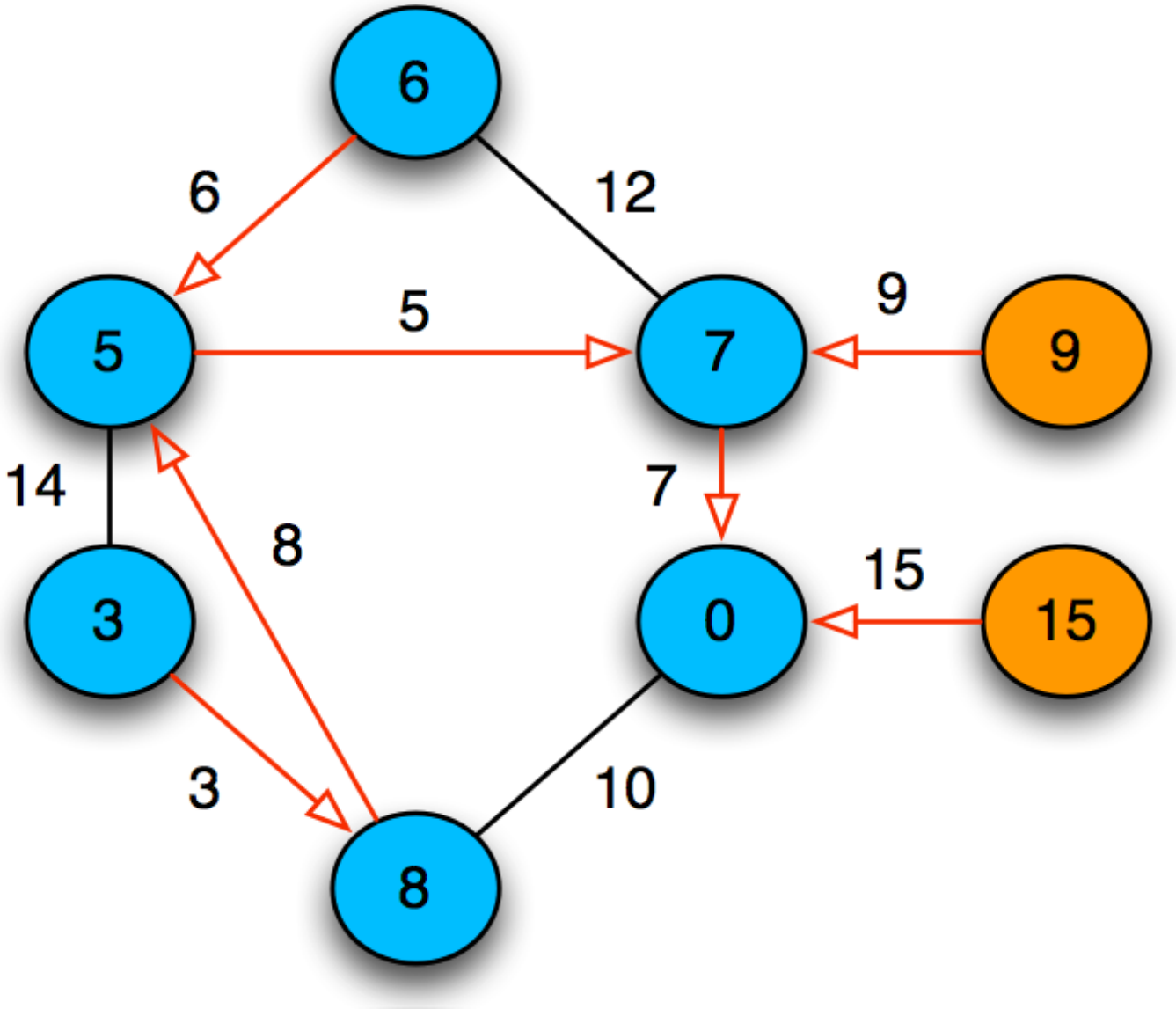

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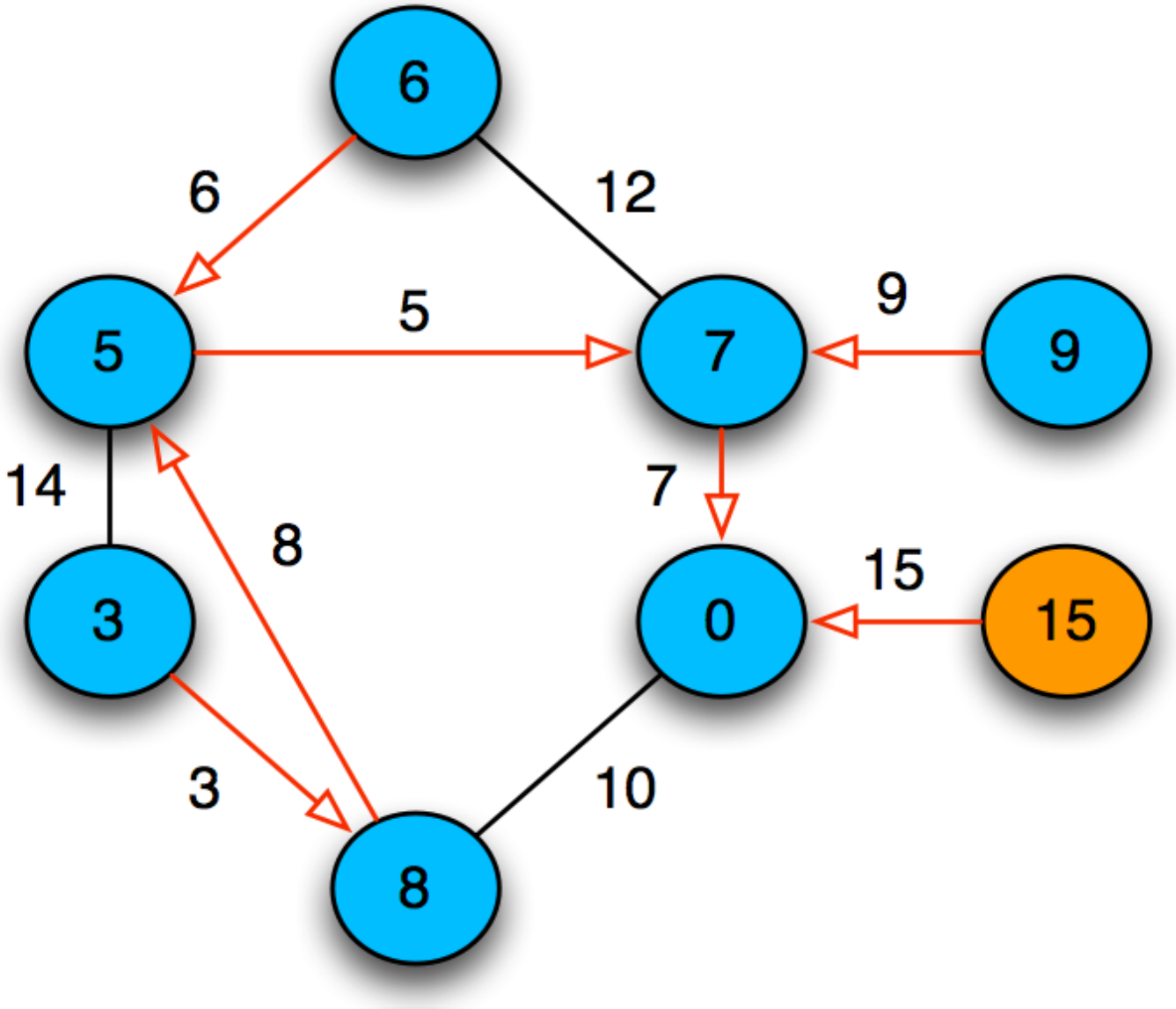
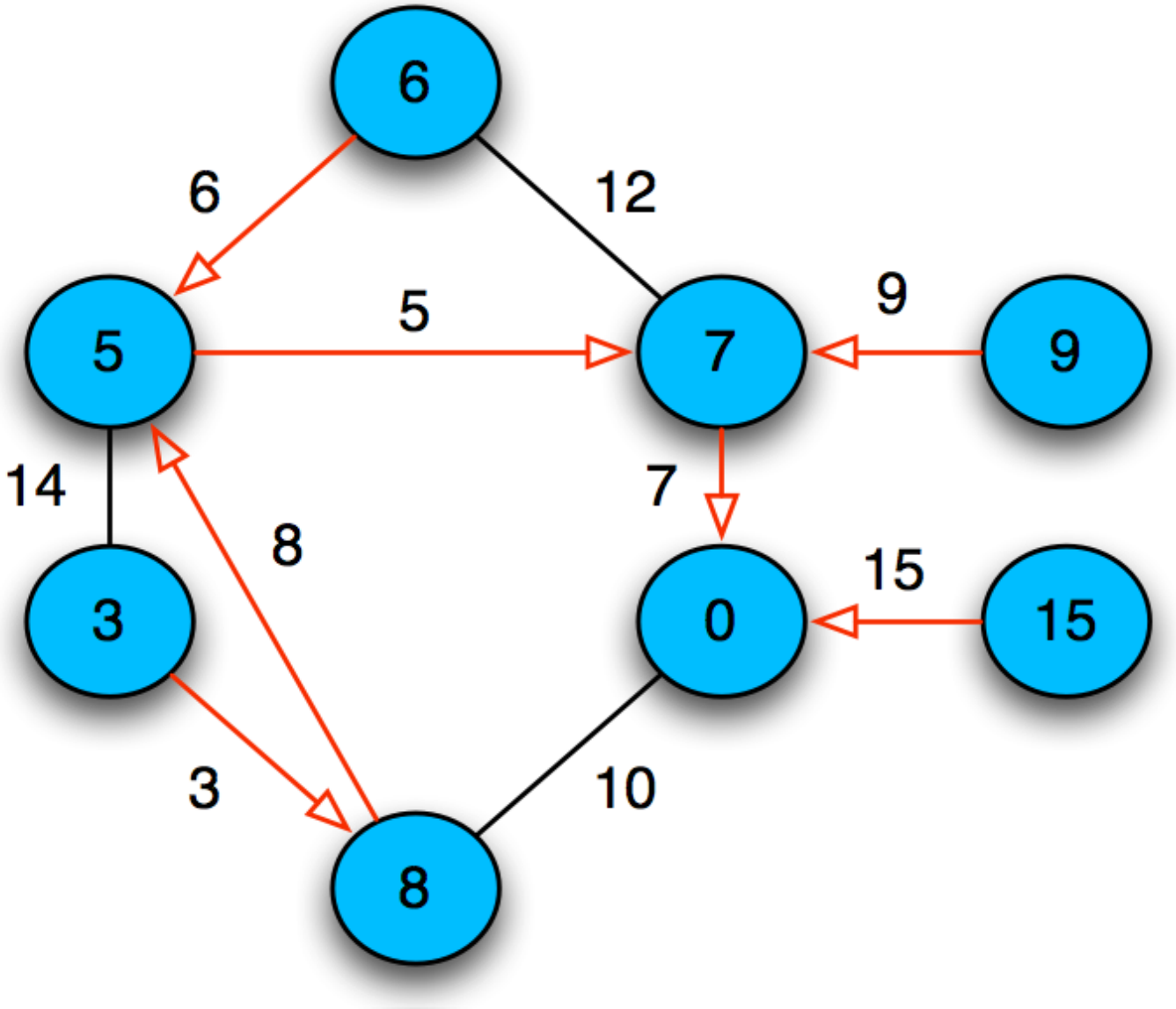
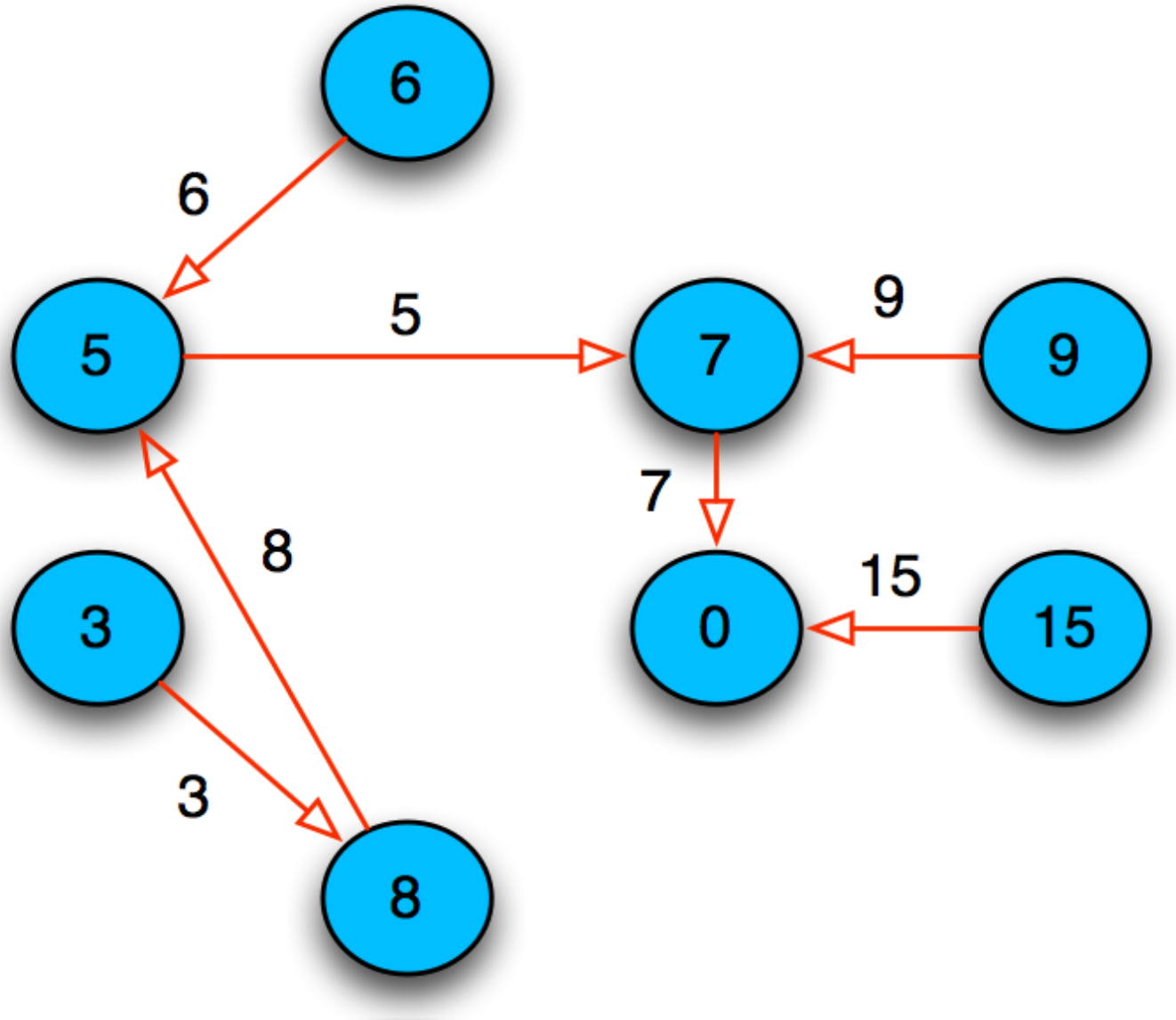


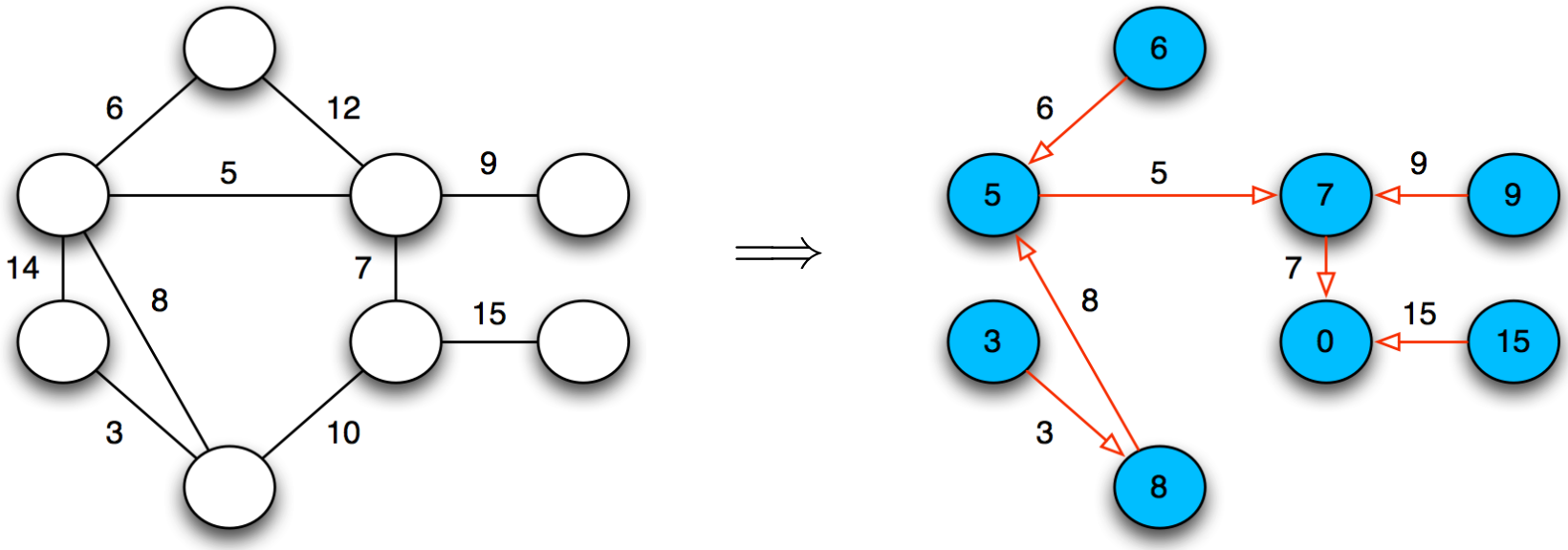
Prim's algorithm in action



Prim's algorithm in action



Prim's algorithm in action



Kruskal's algorithm to compute an MST

MST-KRUSKAL(G, w)

```
1  A ← ∅
2  for each vertex v ∈ V[G]
3        do MAKE-SET(v)
4  sort the edges of E into non-decreasing order by weight w
5  for each edge (u, v) ∈ E, taken in non-decreasing order by weight
6        do if FIND-SET(u) ≠ FIND-SET(v)
7              then A ← A ∪ {(u, v)}
8                   UNION(u, v)
9  return A
```

Kruskal's algorithm to compute an MST

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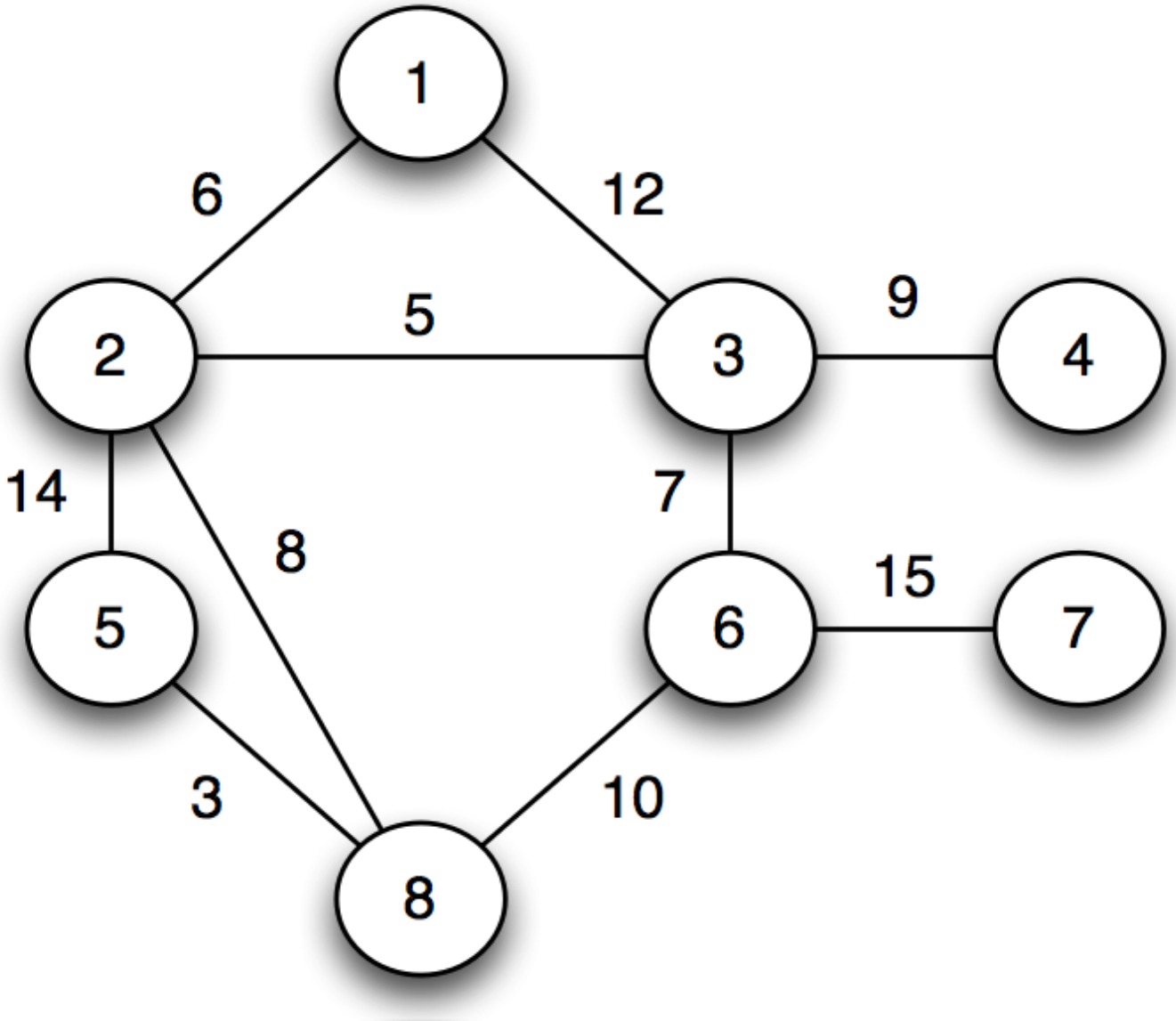
8                   UNION(u, v)
9  return A
```

Running time

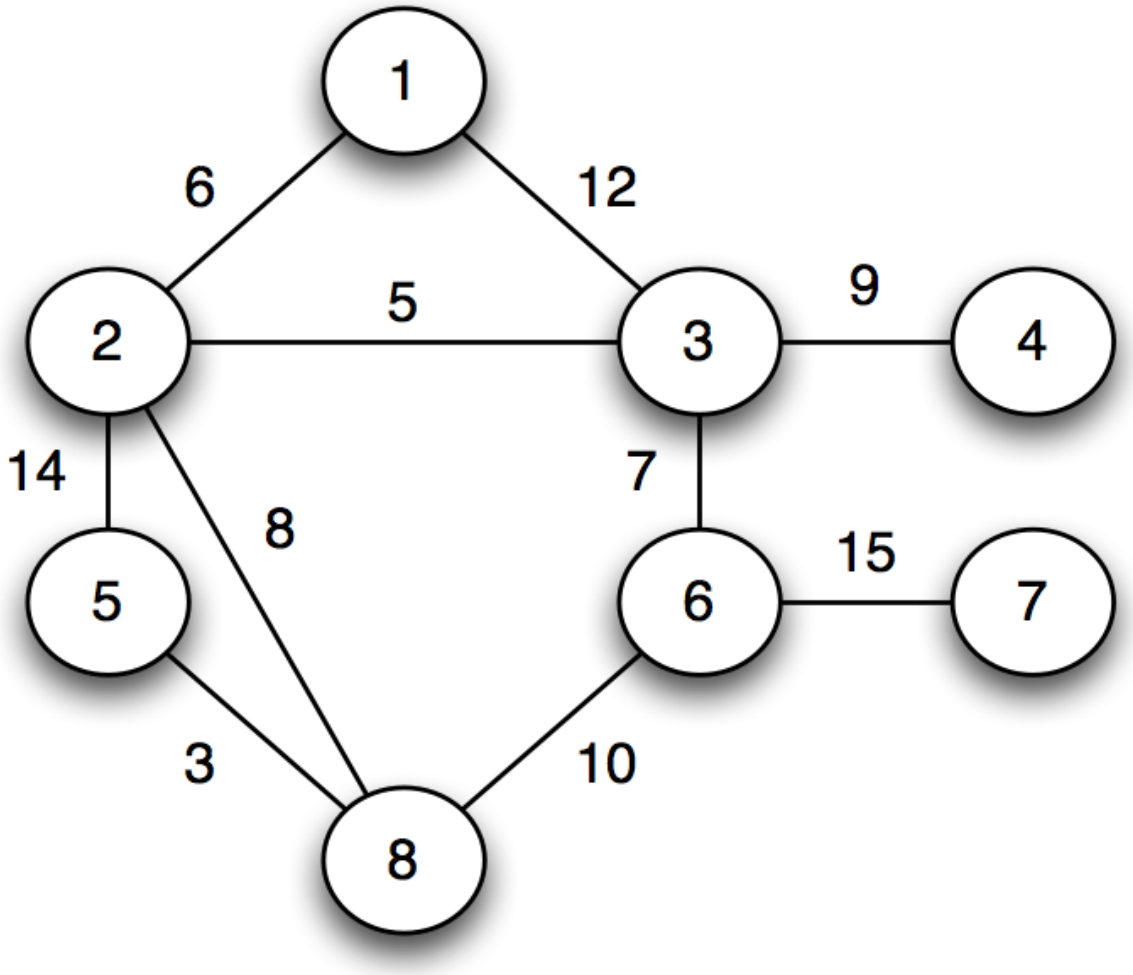
$O(E \lg E)$

Kruskal's algorithm in action



Kruskal's algorithm in action

$w(u,v)$	(u,v)
3	(5,8)
5	(2,3)
6	(1,2)
7	(3,6)
8	(2,8)
9	(3,4)
10	(6,8)
12	(1,3)
14	(2,5)
15	(6,7)



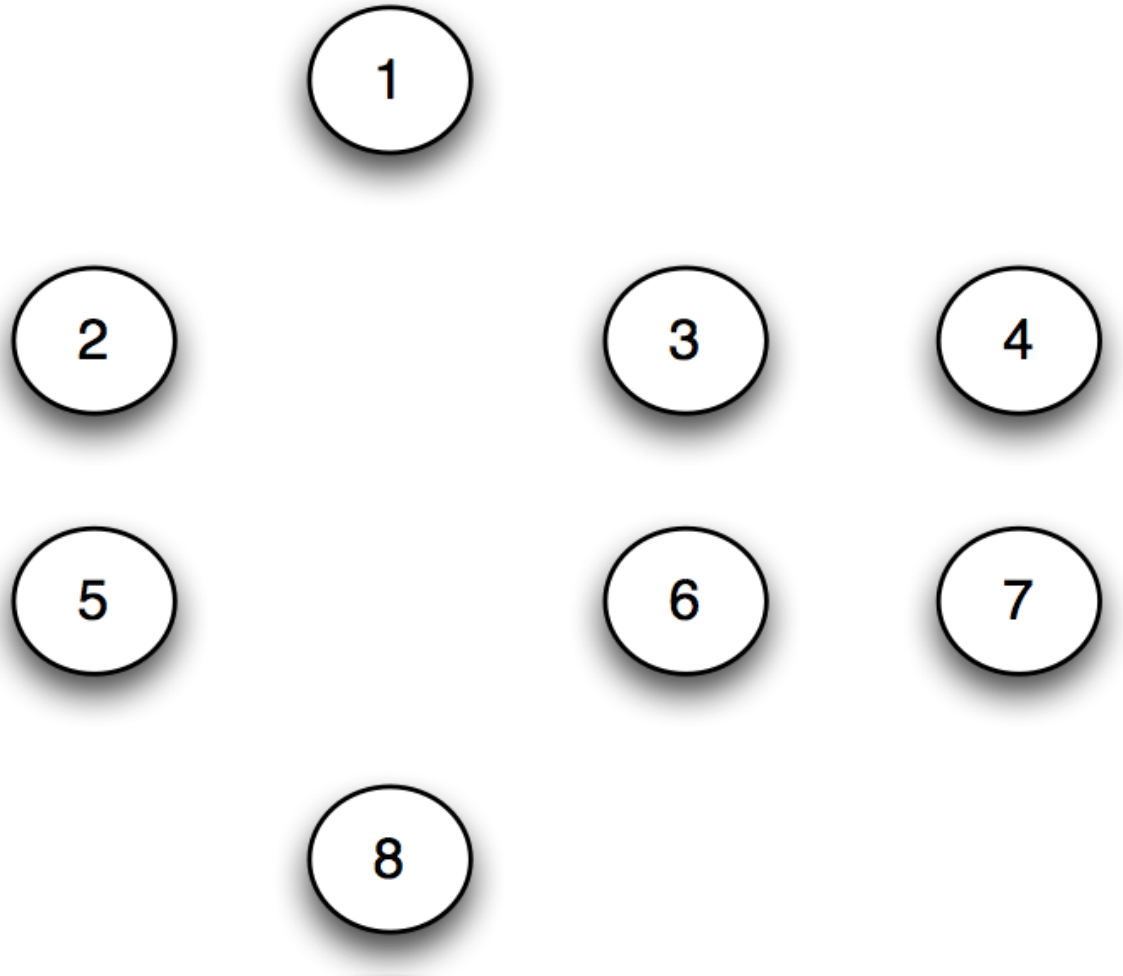
$|V| = 8$

$|E| = 10$

$|T| = 0$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
3	(5,8)
5	(2,3)
6	(1,2)
7	(3,6)
8	(2,8)
9	(3,4)
10	(6,8)
12	(1,3)
14	(2,5)
15	(6,7)



$|V| = 8$

$|E| = 10$

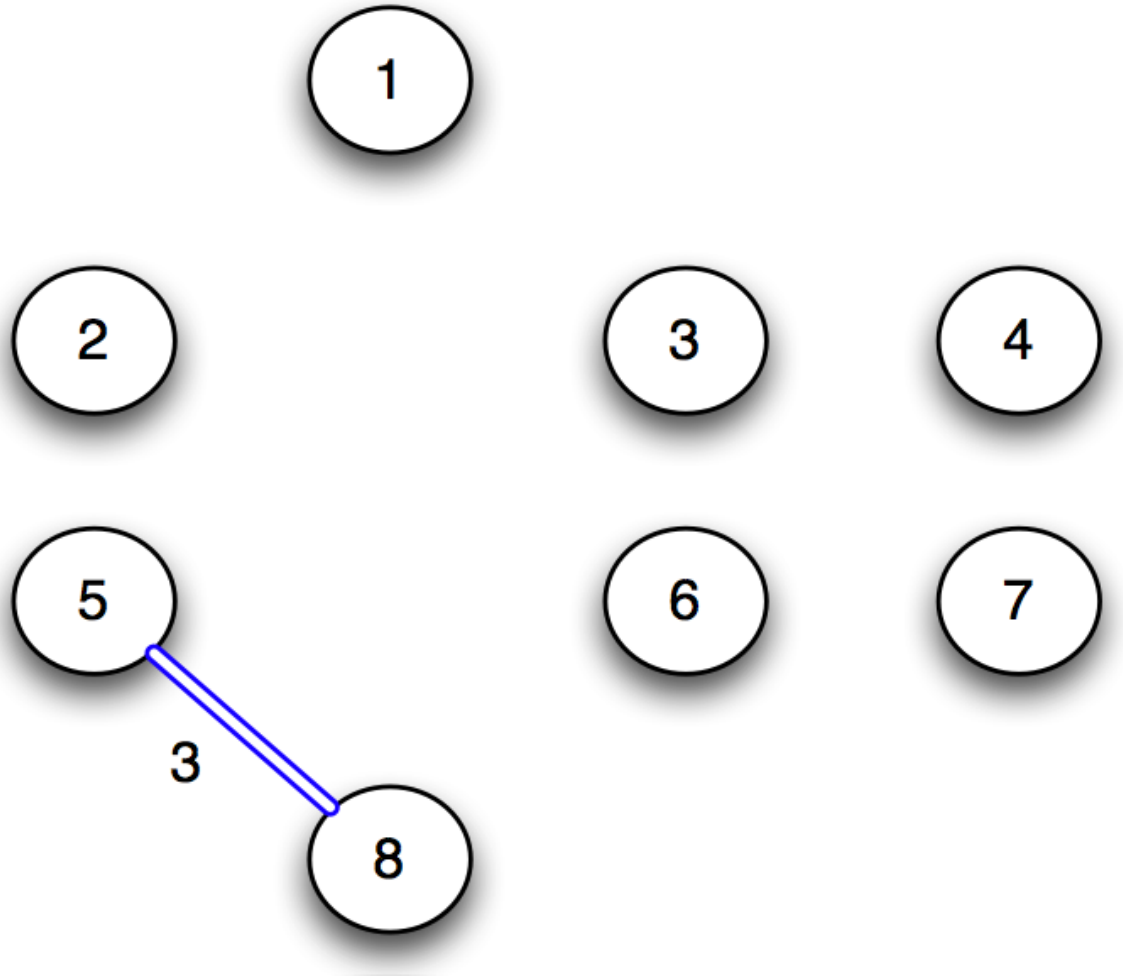
$|T| = 0$

Vertex sets:

{1},{2},{3},{4},{5},{6},{7},{8}

Kruskal's algorithm in action

$w(u, v)$	(u, v)
3	(5,8)
5	(2,3)
6	(1,2)
7	(3,6)
8	(2,8)
9	(3,4)
10	(6,8)
12	(1,3)
14	(2,5)
15	(6,7)



$|V| = 8$

$|E| = 10$

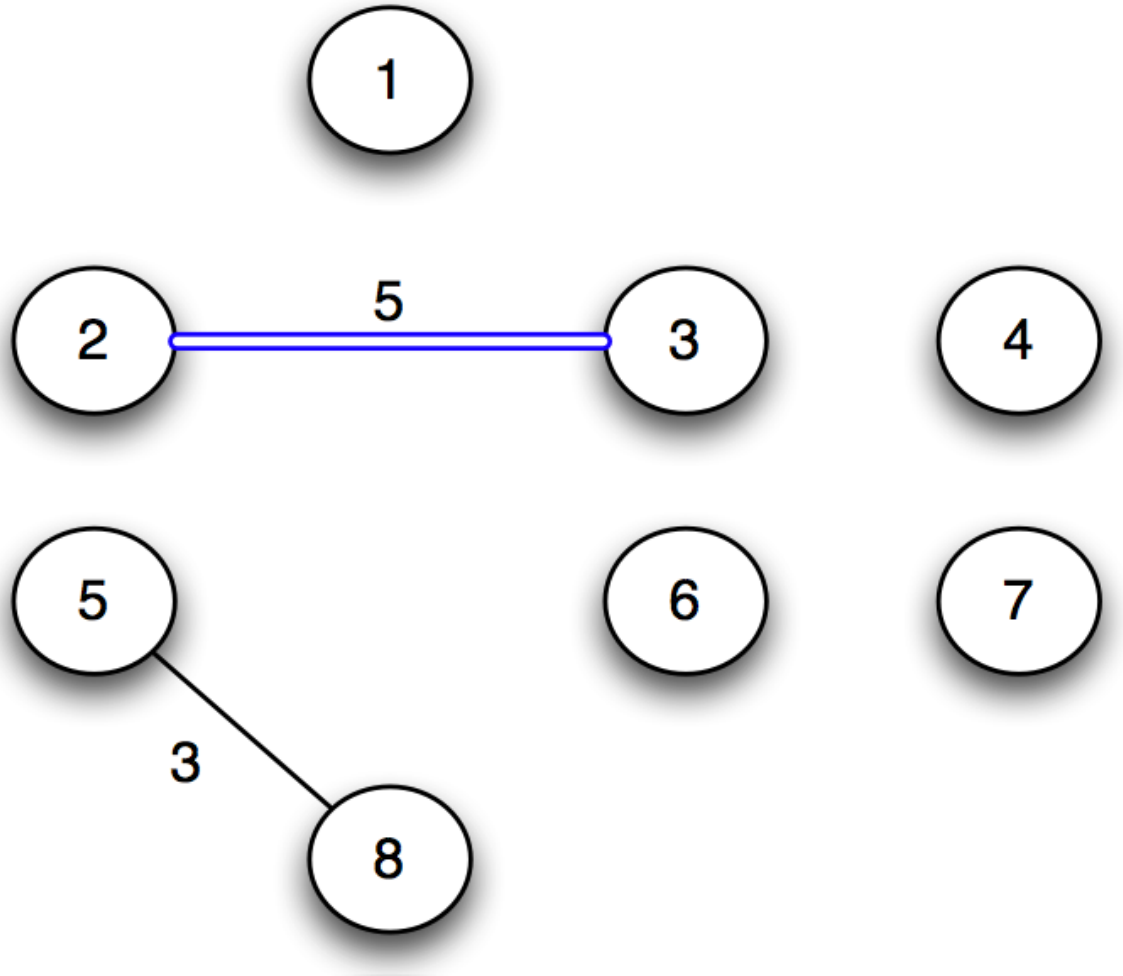
$|T| =$ 1

Vertex sets:

$\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\} \Longrightarrow \{1\},\{2\},\{3\},\{4\},$ {5, 8} $,\{6\},\{7\}$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
✓ 3	(5,8)
5	(2,3)
6	(1,2)
7	(3,6)
8	(2,8)
9	(3,4)
10	(6,8)
12	(1,3)
14	(2,5)
15	(6,7)



$|V| = 8$

$|E| = 10$

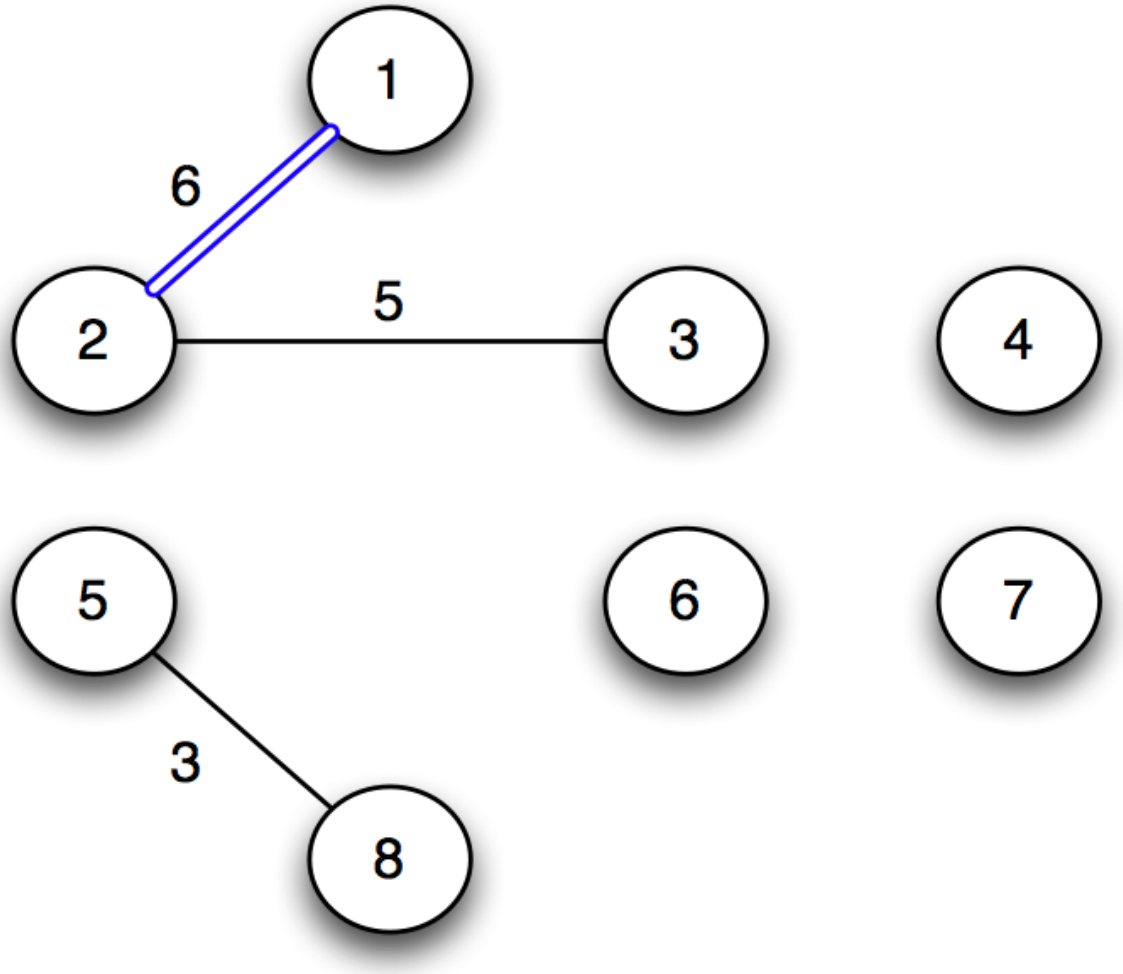
$|T| =$ 2

Vertex sets:

$\{1\},\{2\},\{3\},\{4\},\{5, 8\},\{6\},\{7\} \Longrightarrow \{1\},$ {2, 3} $,\{4\},\{5, 8\},\{6\},\{7\}$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
✓ 3	(5,8)
✓ 5	(2,3)
6	(1,2)
7	(3,6)
8	(2,8)
9	(3,4)
10	(6,8)
12	(1,3)
14	(2,5)
15	(6,7)



$|V| = 8$

$|E| = 10$

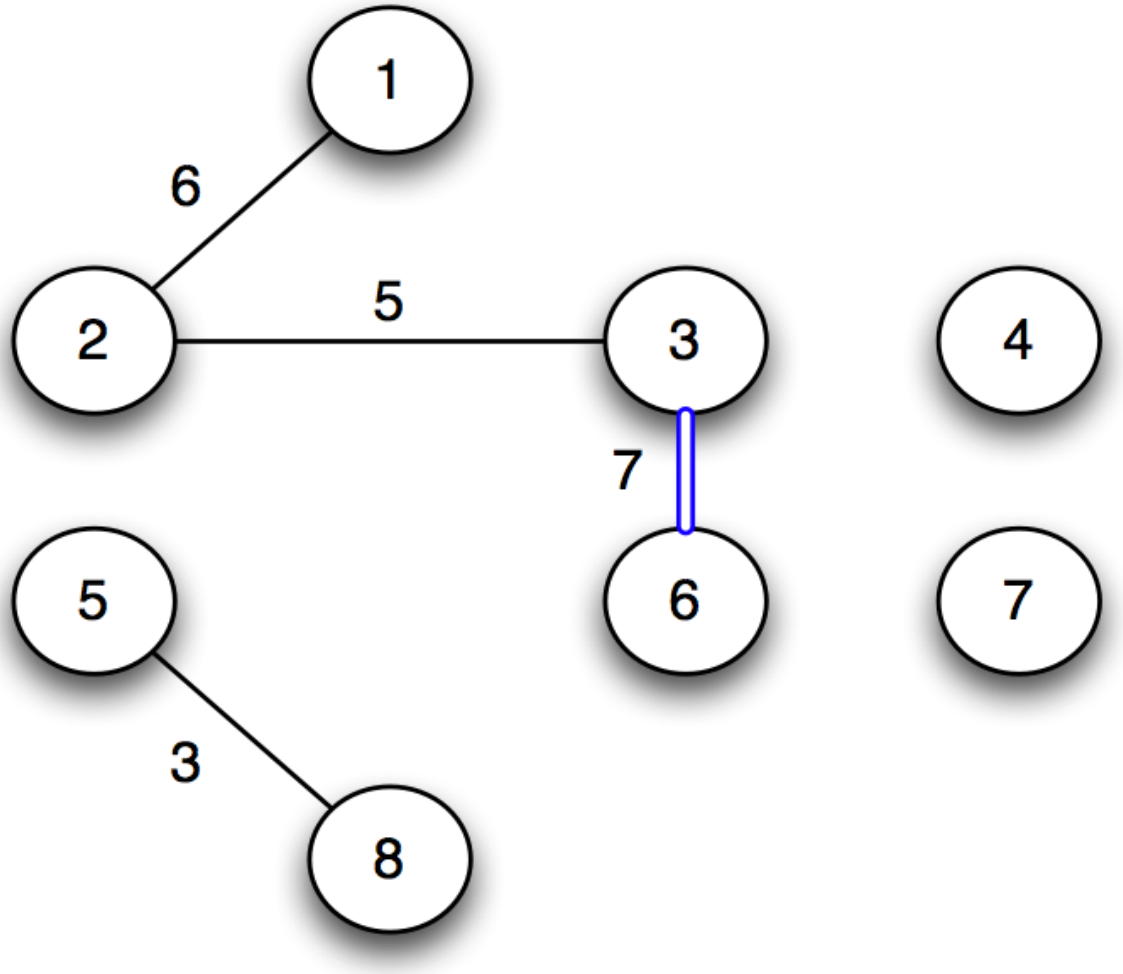
$|T| = \boxed{3}$

Vertex sets:

$\{1\},\{2, 3\},\{4\},\{5, 8\},\{6\},\{7\} \Longrightarrow \boxed{\{1, 2, 3\}},\{4\},\{5, 8\},\{6\},\{7\}$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
✓ 3	(5,8)
✓ 5	(2,3)
✓ 6	(1,2)
7	(3,6)
8	(2,8)
9	(3,4)
10	(6,8)
12	(1,3)
14	(2,5)
15	(6,7)



$|V| = 8$

$|E| = 10$

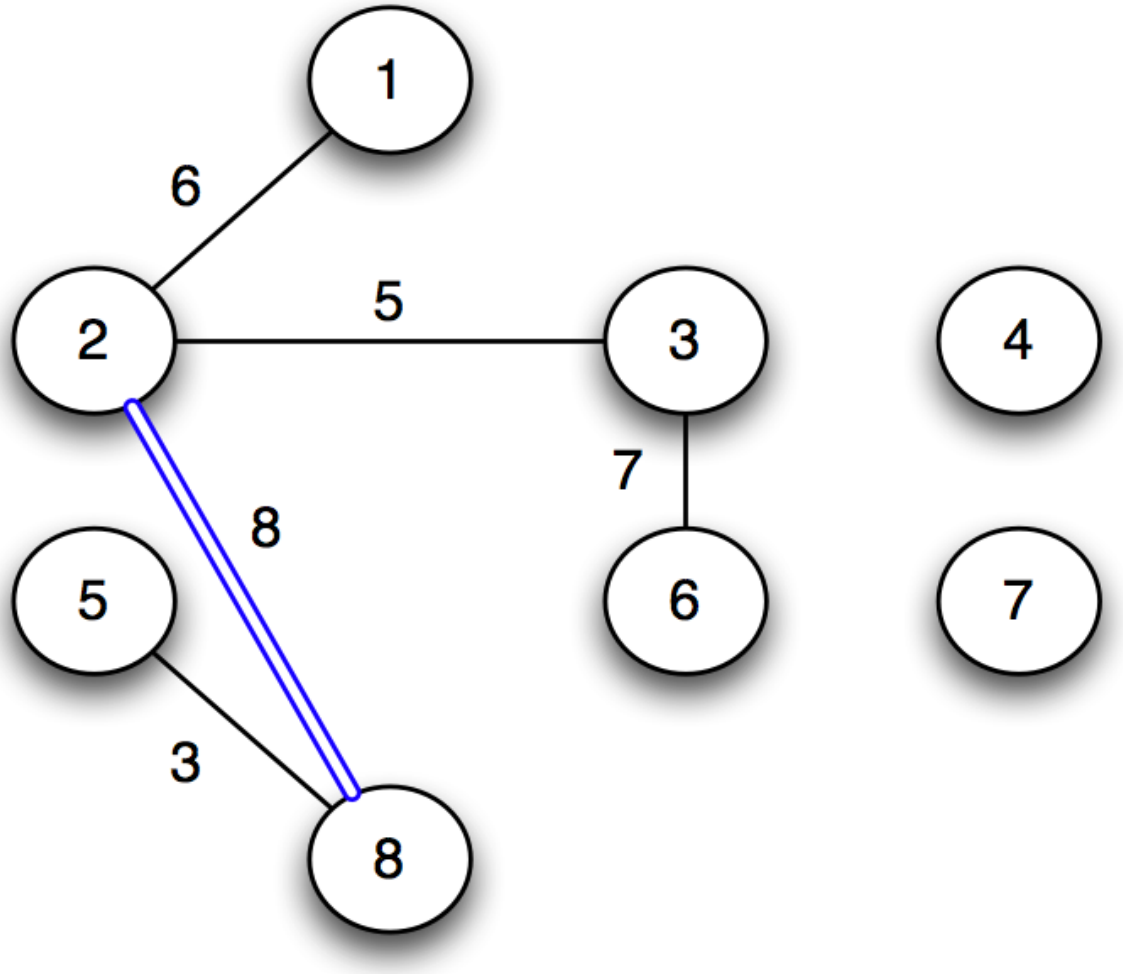
$|T| = 4$

Vertex sets:

$\{1, 2, 3\}, \{4\}, \{5, 8\}, \{6\}, \{7\} \Longrightarrow \{1, 2, 3, 6\}, \{4\}, \{5, 8\}, \{7\}$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
✓ 3	(5,8)
✓ 5	(2,3)
✓ 6	(1,2)
✓ 7	(3,6)
8	(2,8)
9	(3,4)
10	(6,8)
12	(1,3)
14	(2,5)
15	(6,7)



$|V| = 8$

$|E| = 10$

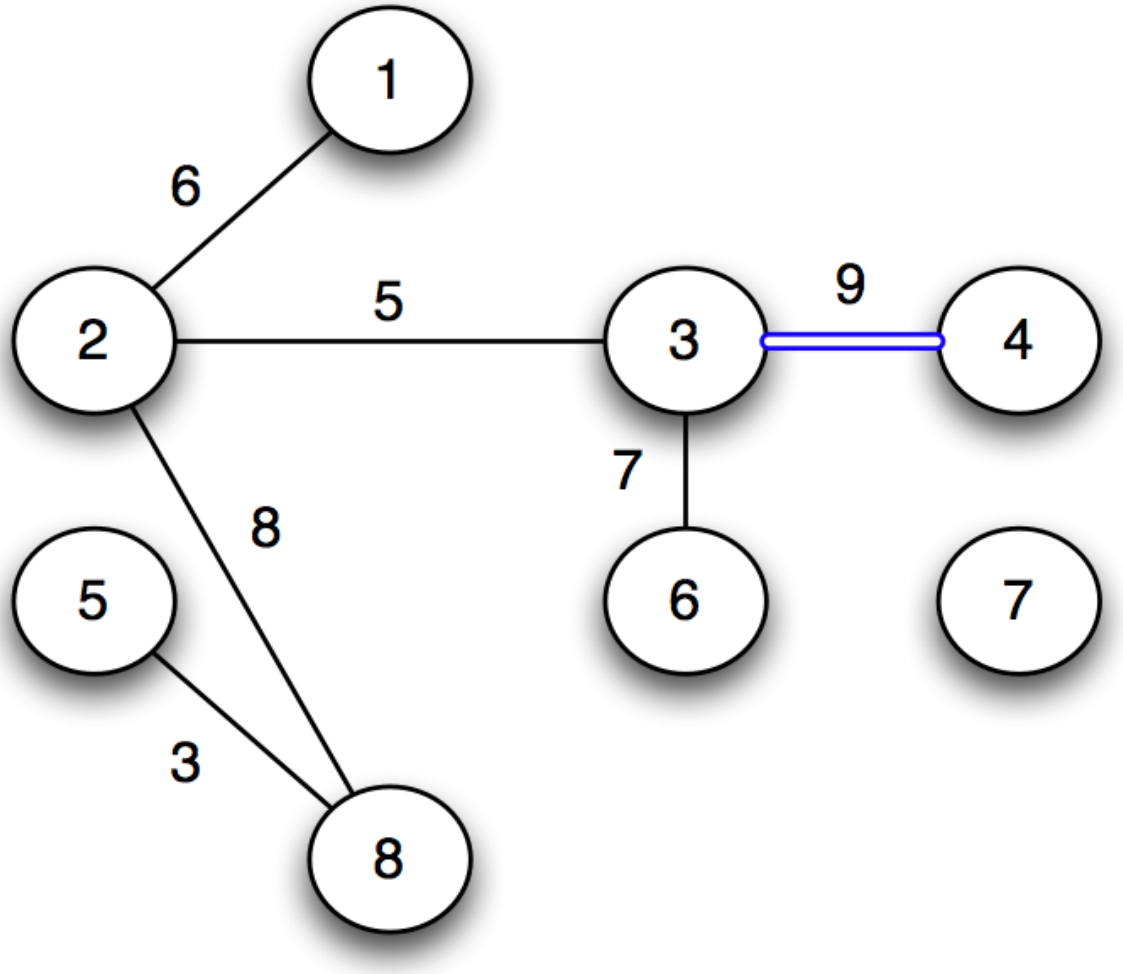
$|T| = \boxed{5}$

Vertex sets:

$\{1, 2, 3, 6\}, \{4\}, \{5, 8\}, \{7\} \Longrightarrow \boxed{\{1, 2, 3, 5, 6, 8\}}, \{4\}, \{7\}$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
✓ 3	(5,8)
✓ 5	(2,3)
✓ 6	(1,2)
✓ 7	(3,6)
✓ 8	(2,8)
9	(3,4)
10	(6,8)
12	(1,3)
14	(2,5)
15	(6,7)



$|V| = 8$

$|E| = 10$

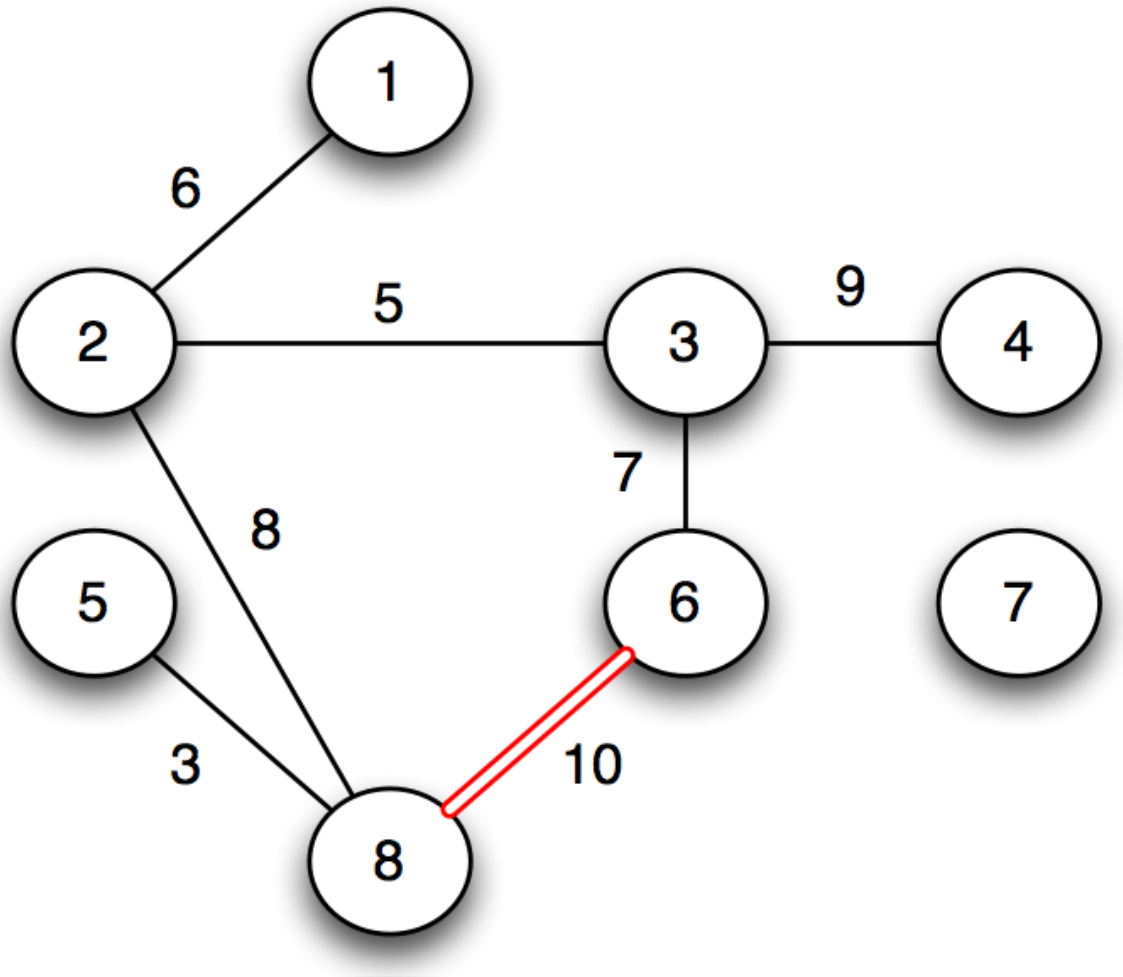
$|T| = 6$

Vertex sets:

$\{1, 2, 3, 5, 6, 8\}, \{4\}, \{7\} \Longrightarrow \{1, 2, 3, 4, 5, 6, 8\}, \{7\}$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
✓ 3	(5,8)
✓ 5	(2,3)
✓ 6	(1,2)
✓ 7	(3,6)
✓ 8	(2,8)
✓ 9	(3,4)
10	(6,8)
12	(1,3)
14	(2,5)
15	(6,7)



$|V| = 8$

$|E| = 10$

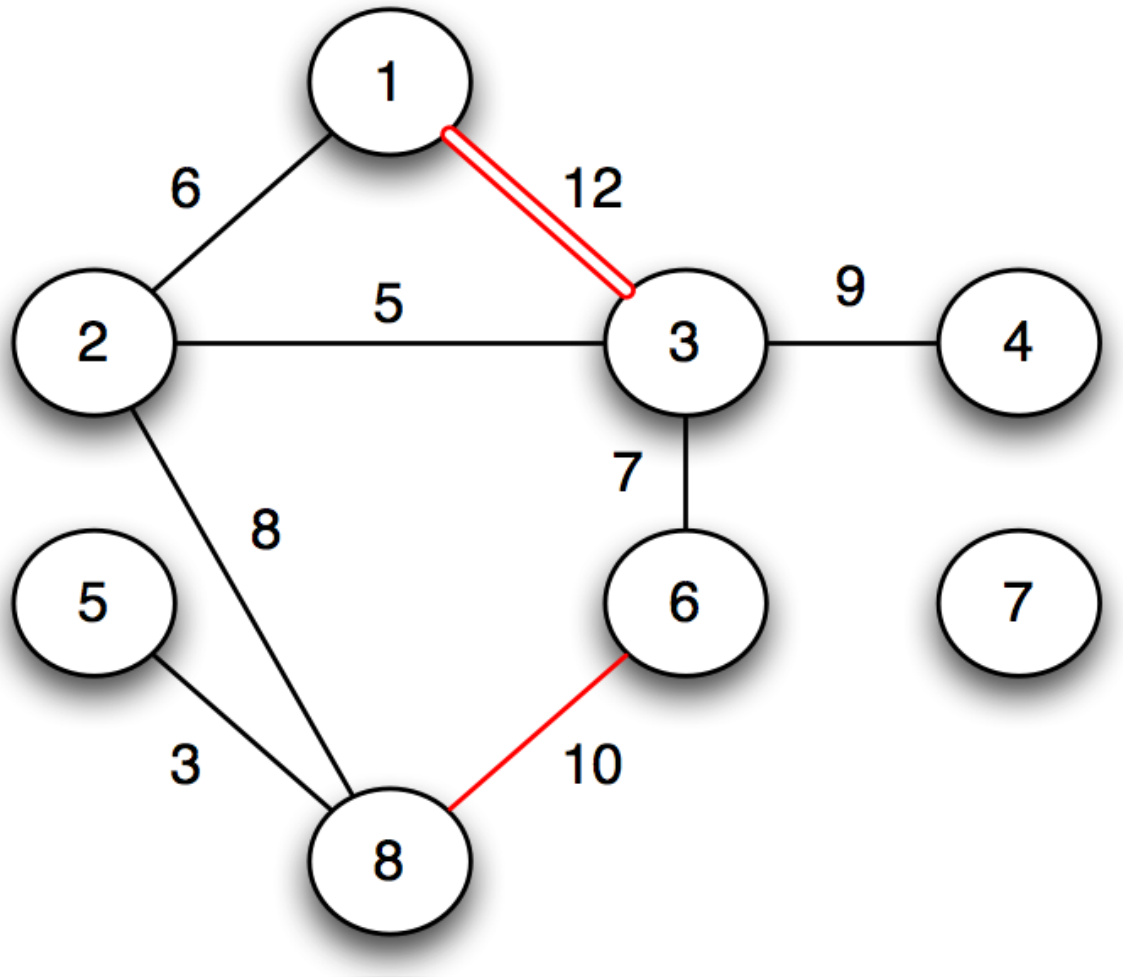
$|T| = 6$

Vertex sets:

$\{1, 2, 3, 4, 5, 6, 8\}, \{7\} \Longrightarrow \{1, 2, 3, 4, 5, 6, 8\}, \{7\}$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
✓ 3	(5,8)
✓ 5	(2,3)
✓ 6	(1,2)
✓ 7	(3,6)
✓ 8	(2,8)
✓ 9	(3,4)
× 10	(6,8)
12	(1,3)
14	(2,5)
15	(6,7)



$|V| = 8$

$|E| = 10$

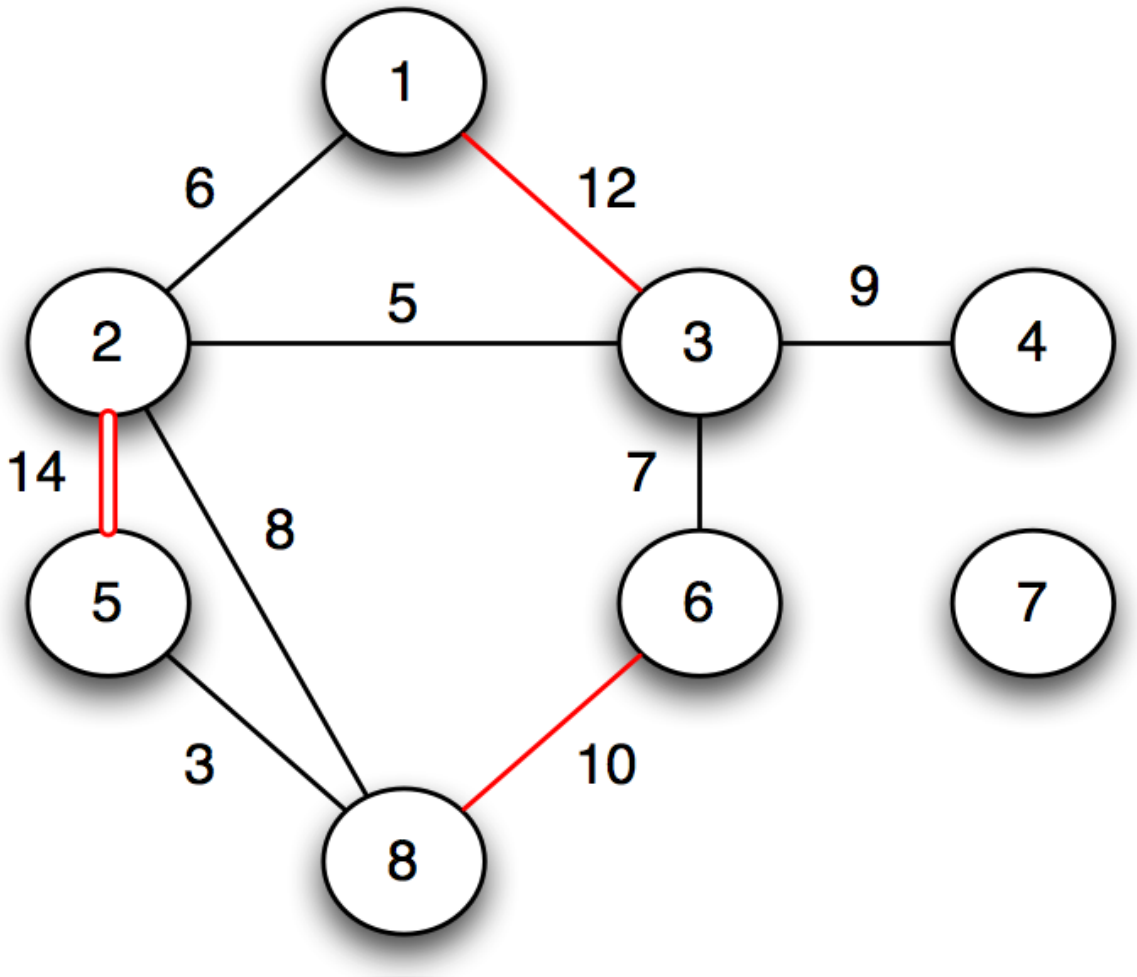
$|T| = \boxed{6}$

Vertex sets:

$\{1, 2, 3, 4, 5, 6, 8\}, \{7\} \Longrightarrow \{1, 2, 3, 4, 5, 6, 8\}, \{7\}$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
✓ 3	(5,8)
✓ 5	(2,3)
✓ 6	(1,2)
✓ 7	(3,6)
✓ 8	(2,8)
✓ 9	(3,4)
× 10	(6,8)
× 12	(1,3)
14	(2,5)
15	(6,7)



$|V| = 8$

$|E| = 10$

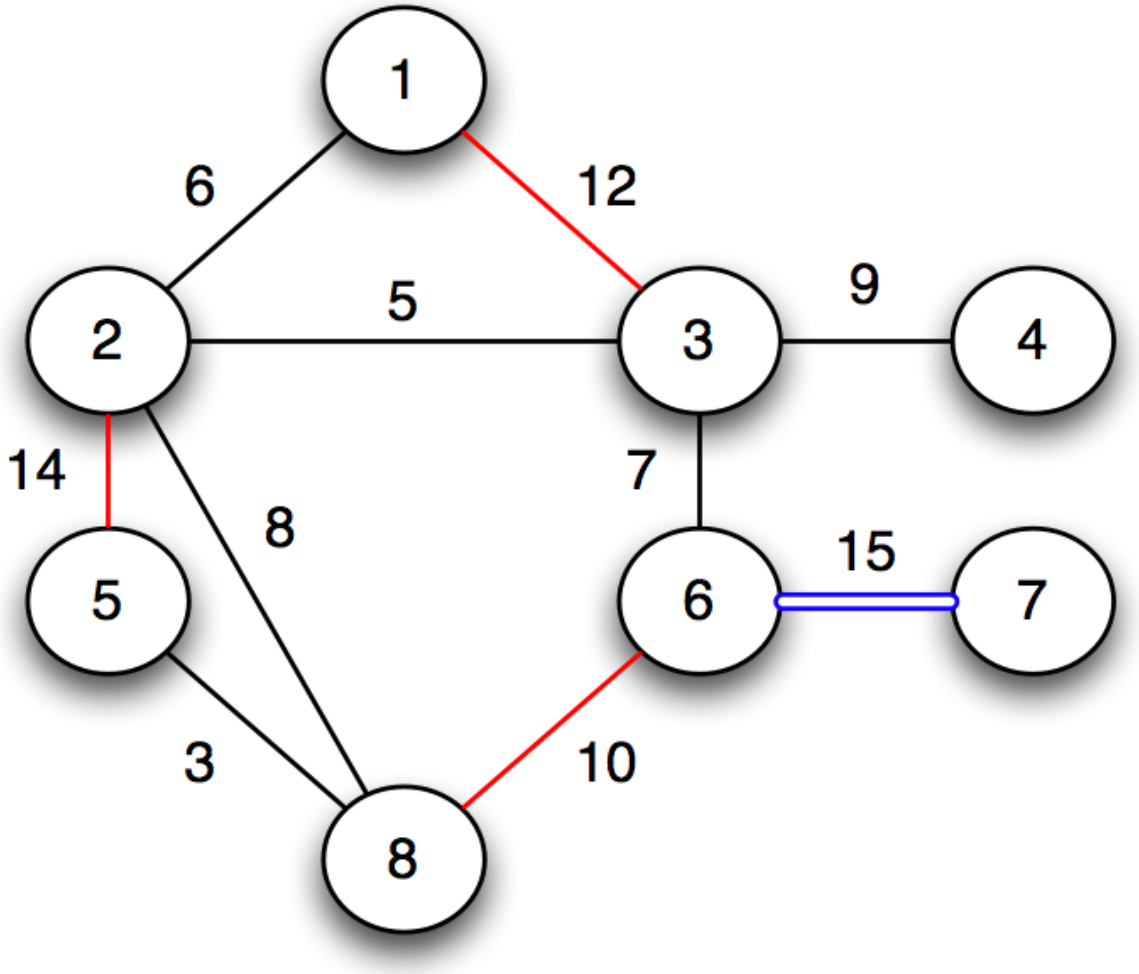
$|T| = 6$

Vertex sets:

$\{1, 2, 3, 4, 5, 6, 8\}, \{7\} \Longrightarrow \{1, 2, 3, 4, 5, 6, 8\}, \{7\}$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
✓ 3	(5,8)
✓ 5	(2,3)
✓ 6	(1,2)
✓ 7	(3,6)
✓ 8	(2,8)
✓ 9	(3,4)
× 10	(6,8)
× 12	(1,3)
× 14	(2,5)
15	(6,7)



$|V| = 8$

$|E| = 10$

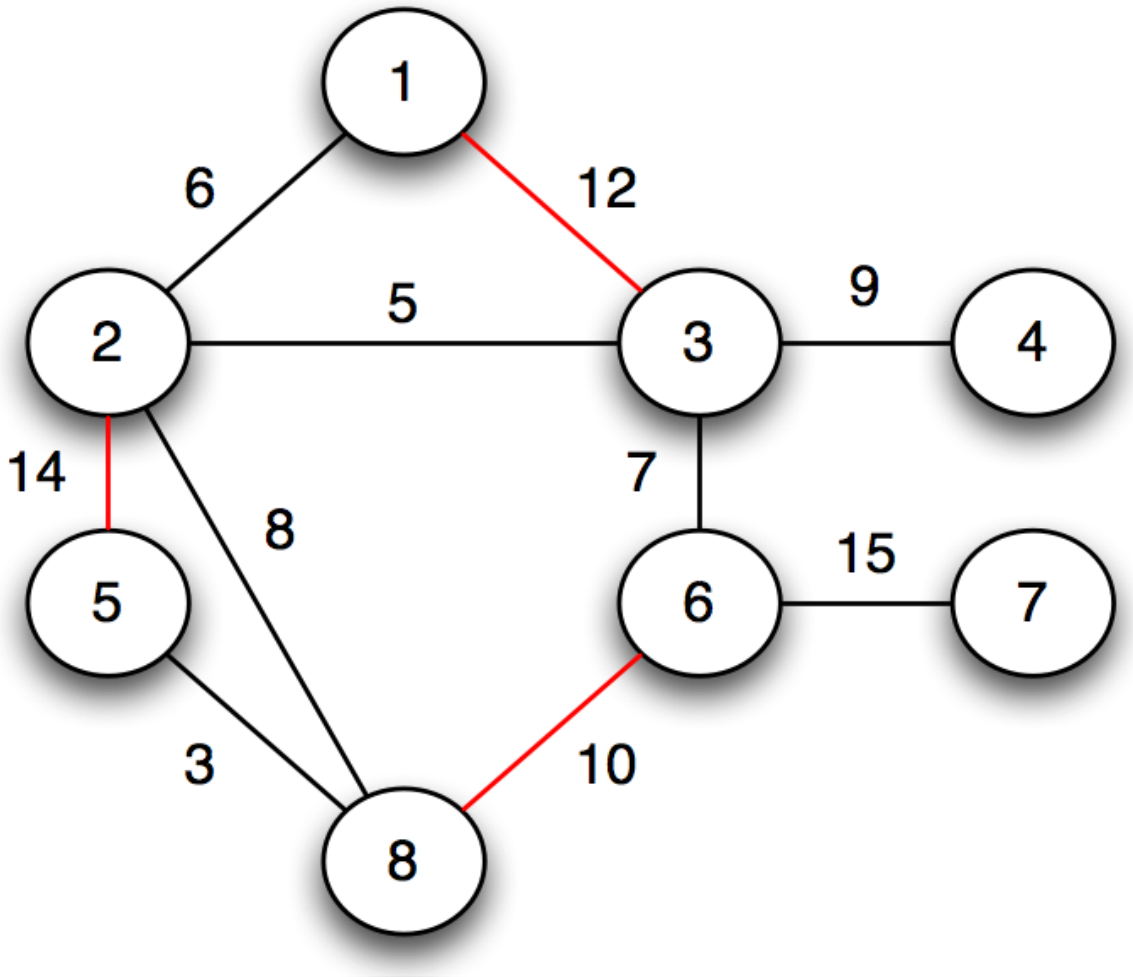
$|T| = \boxed{7}$

Vertex sets:

$\{1, 2, 3, 4, 5, 6, 8\}, \{7\} \Longrightarrow \boxed{\{1, 2, 3, 4, 5, 6, 7, 8\}}$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
✓ 3	(5,8)
✓ 5	(2,3)
✓ 6	(1,2)
✓ 7	(3,6)
✓ 8	(2,8)
✓ 9	(3,4)
× 10	(6,8)
× 12	(1,3)
× 14	(2,5)
✓ 15	(6,7)



$|V| = 8$

$|E| = 10$

$|T| = 7$

Vertex sets:

$\{1, 2, 3, 4, 5, 6, 7, 8\}$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
✓ 3	(5,8)
✓ 5	(2,3)
✓ 6	(1,2)
✓ 7	(3,6)
✓ 8	(2,8)
✓ 9	(3,4)
× 10	(6,8)
× 12	(1,3)
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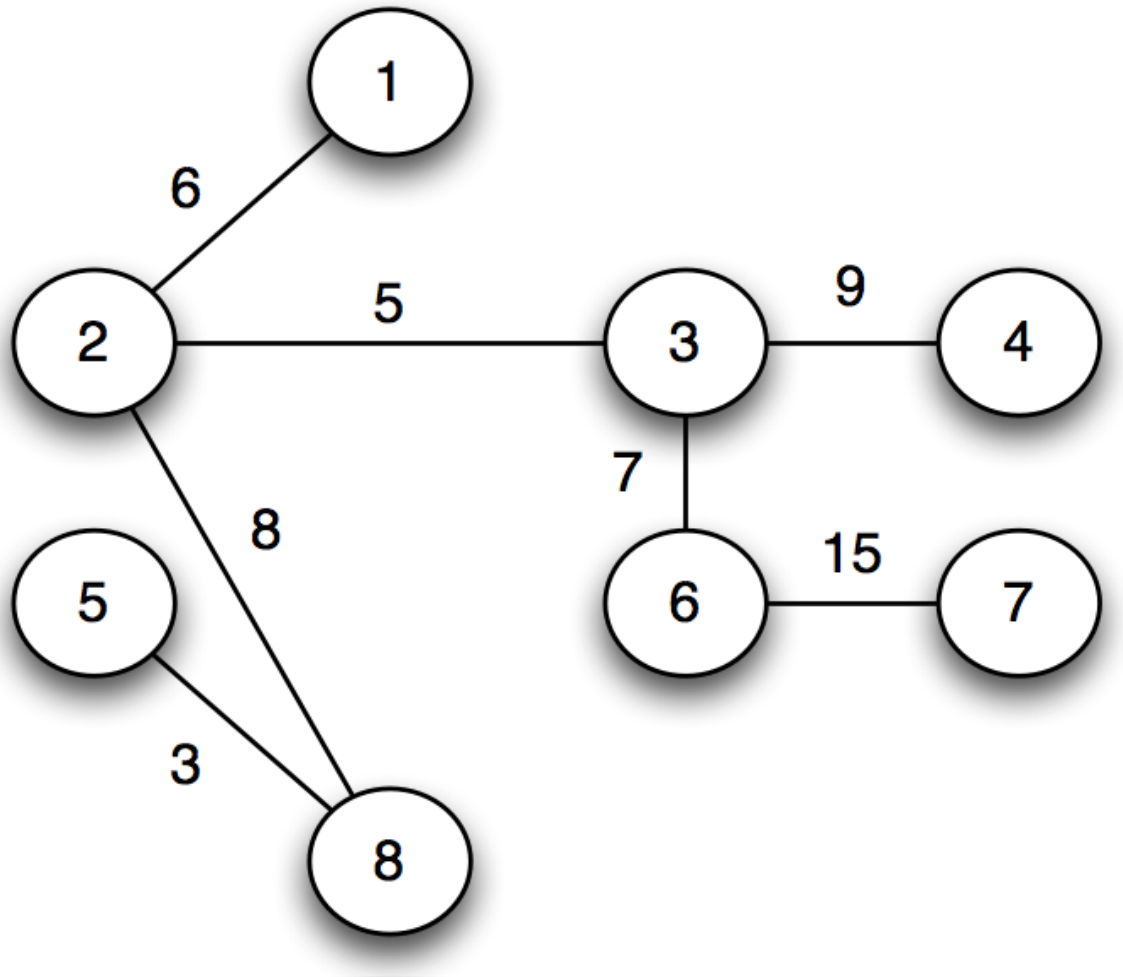


$|V| = 8$

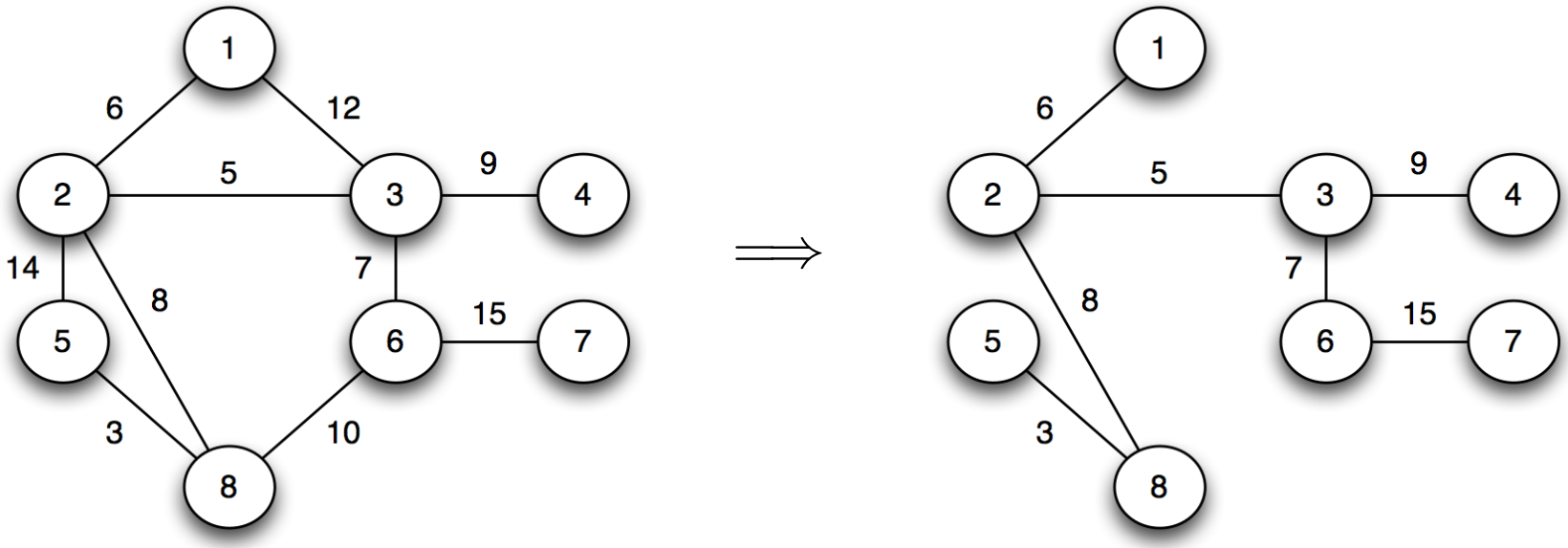
$|E| = 10$

$|T| = \boxed{7}$

Vertex sets:

$\{1, 2, 3, 4, 5, 6, 7, 8\}$

Kruskal's algorithm in action



Contents

Dijkstra's algorithm for SSSP

DIJKSTRA(G, s)

```
 1  for each v ∈ V[G]
 2        do d[v] ← ∞
 3           π[v] ← NIL
 4  d[s] ← 0
 5  S ← ∅
 6  Q ← V[G]
 7  while Q ≠ ∅
 8        do u ← EXTRACT-MIN(Q)
 9           S ← S ∪ {u}
10           for each vertex v ∈ Adj[u]
11                do if d[v] > d[u] + w(u, v)
12                      then d[v] ← d[u] + w(u, v)
13                           π[v] ← u
```

Dijkstra's algorithm for SSSP

DIJKSTRA(G, s)

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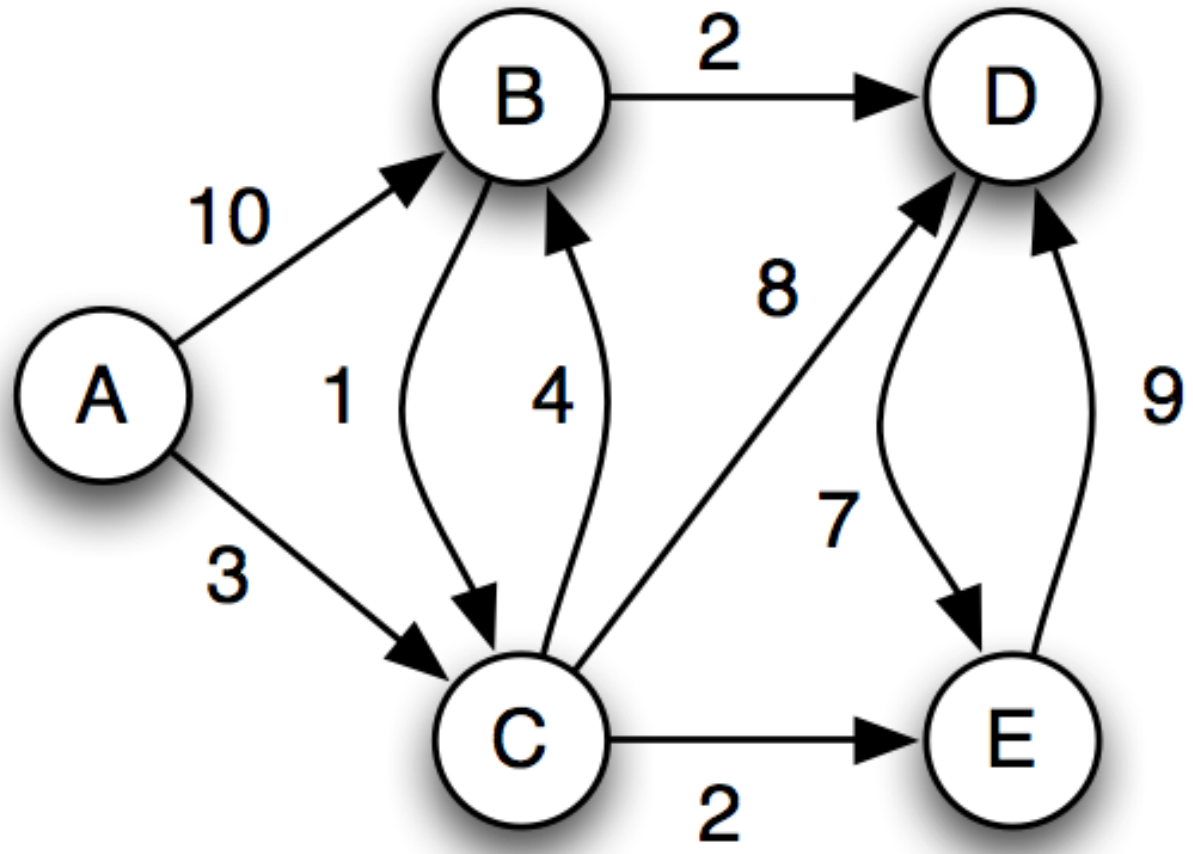
12                      then d[v] ← d[u] + w(u, v)
13                           π[v] ← u
```

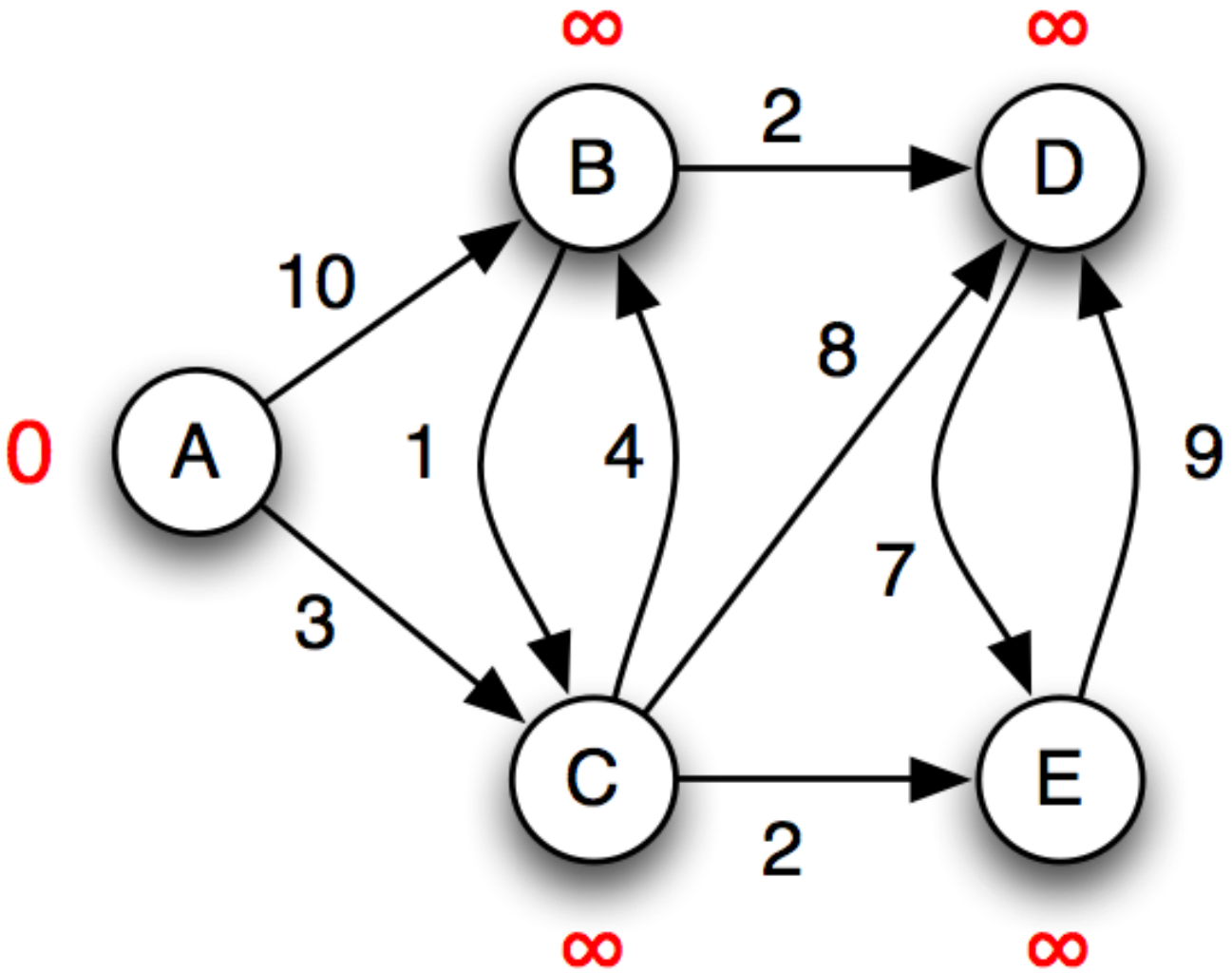
Running time

$O((V + E) \lg V)$

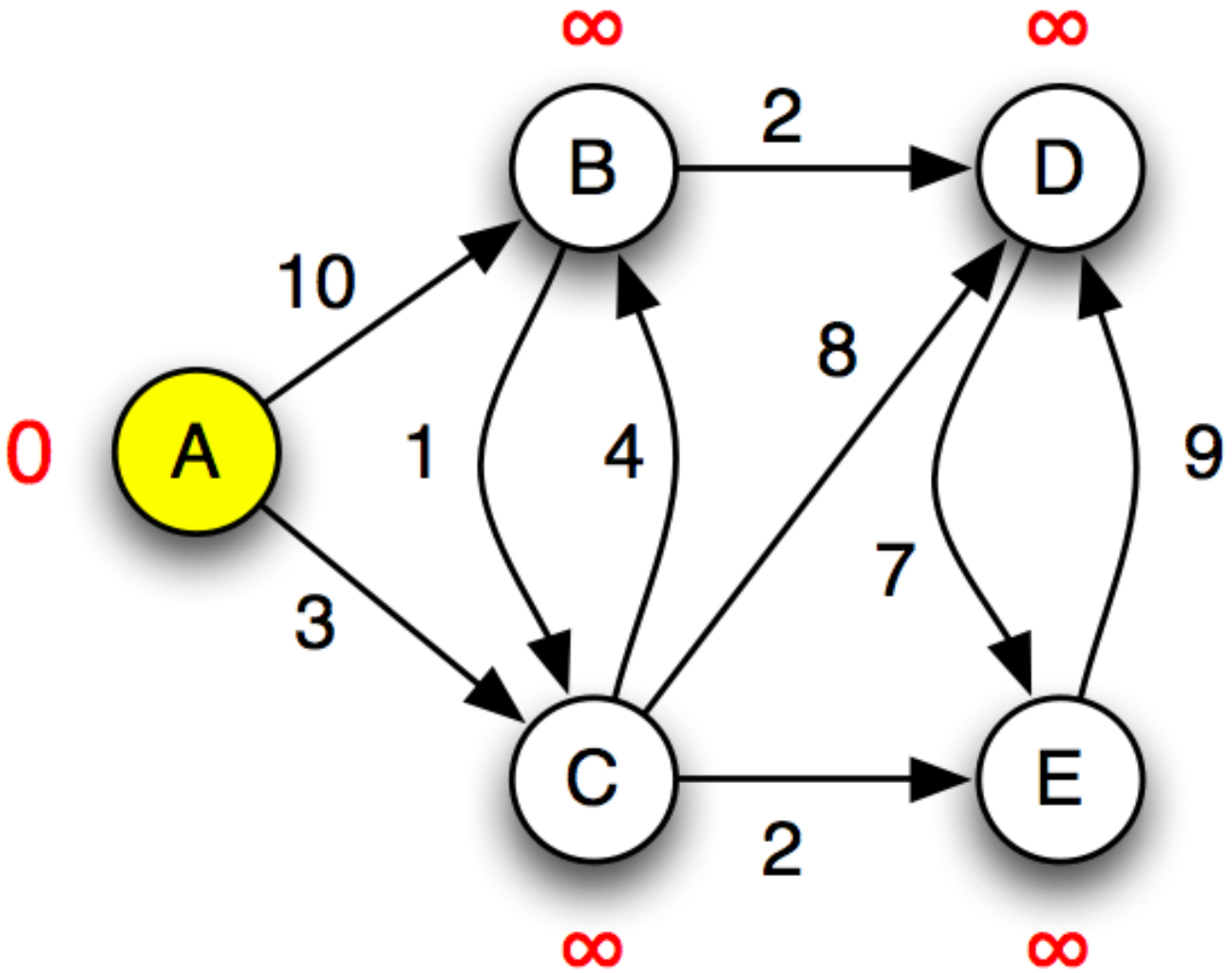
Dijkstra's SSSP algorithm in action



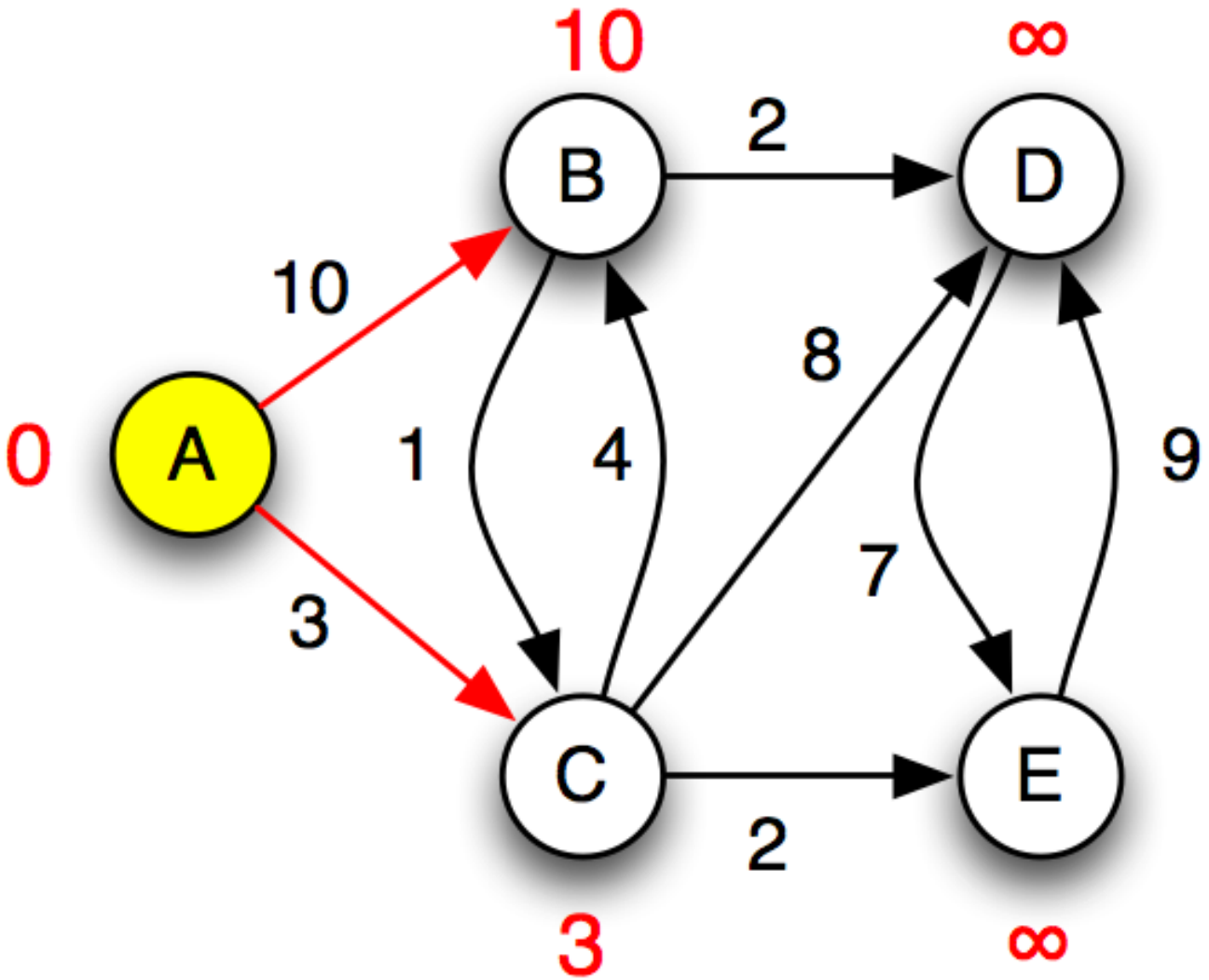
Dijkstra's SSSP algorithm in action



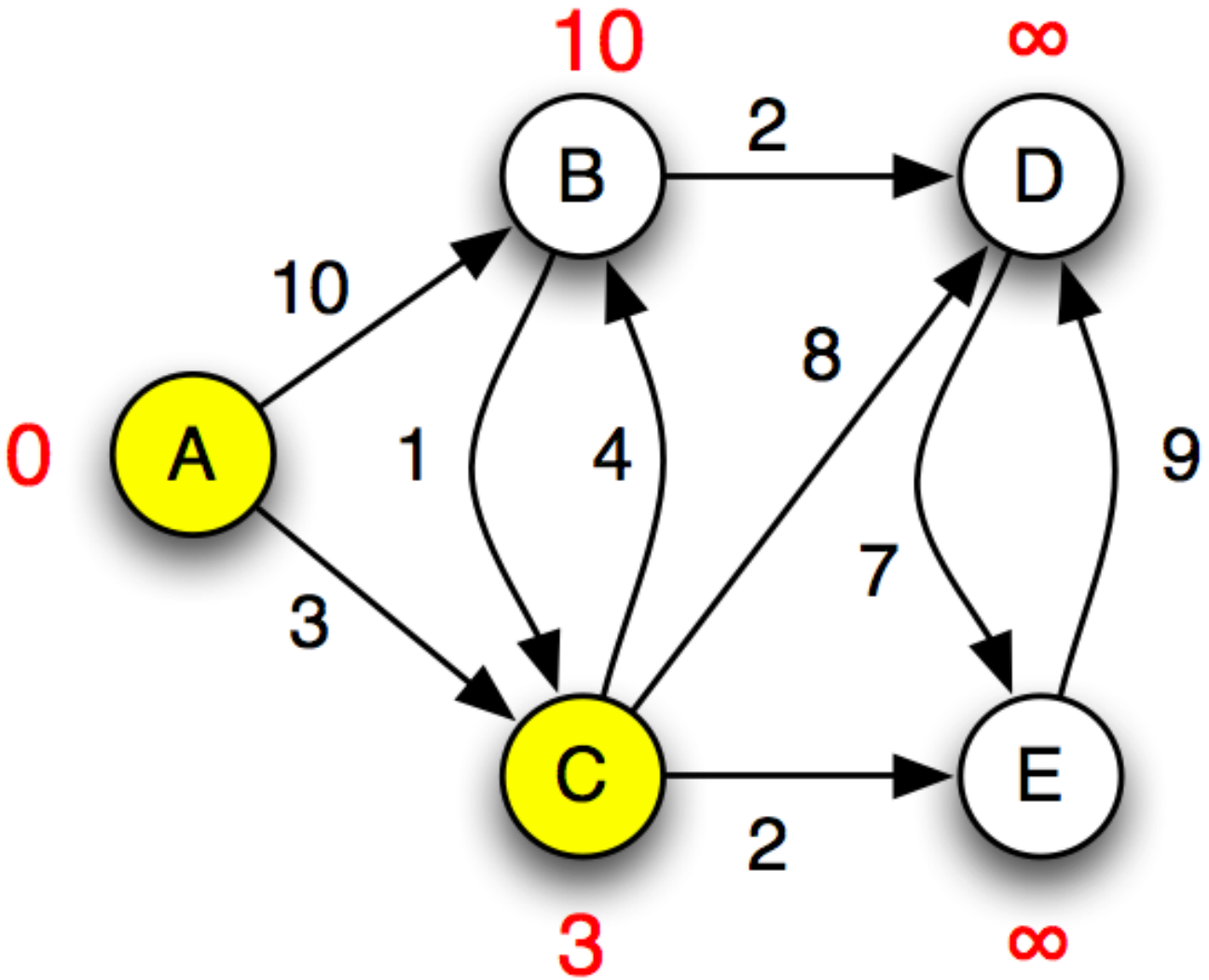
Dijkstra's SSSP algorithm in action



Dijkstra's SSSP algorithm in action



Dijkstra's SSSP algorithm in action



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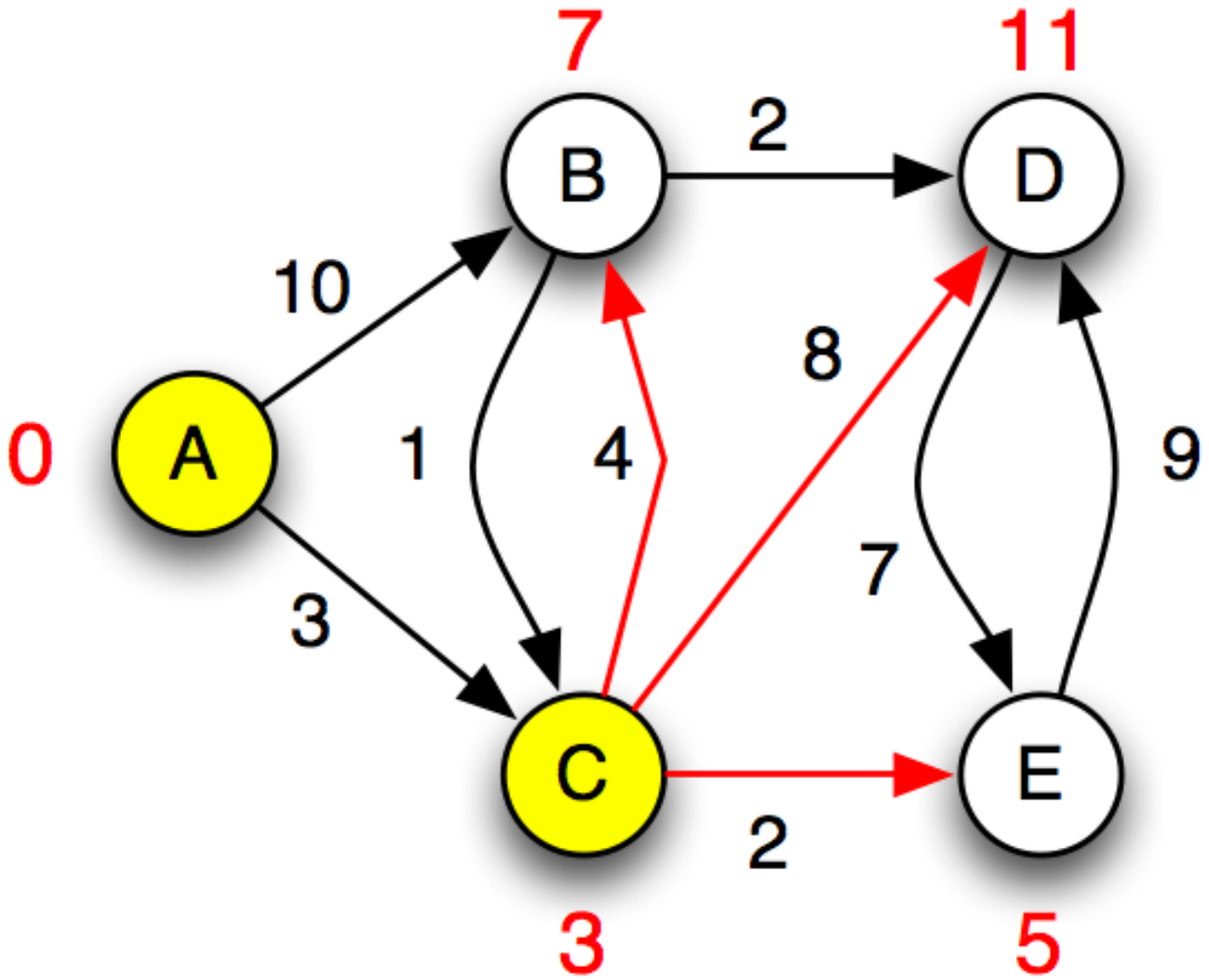
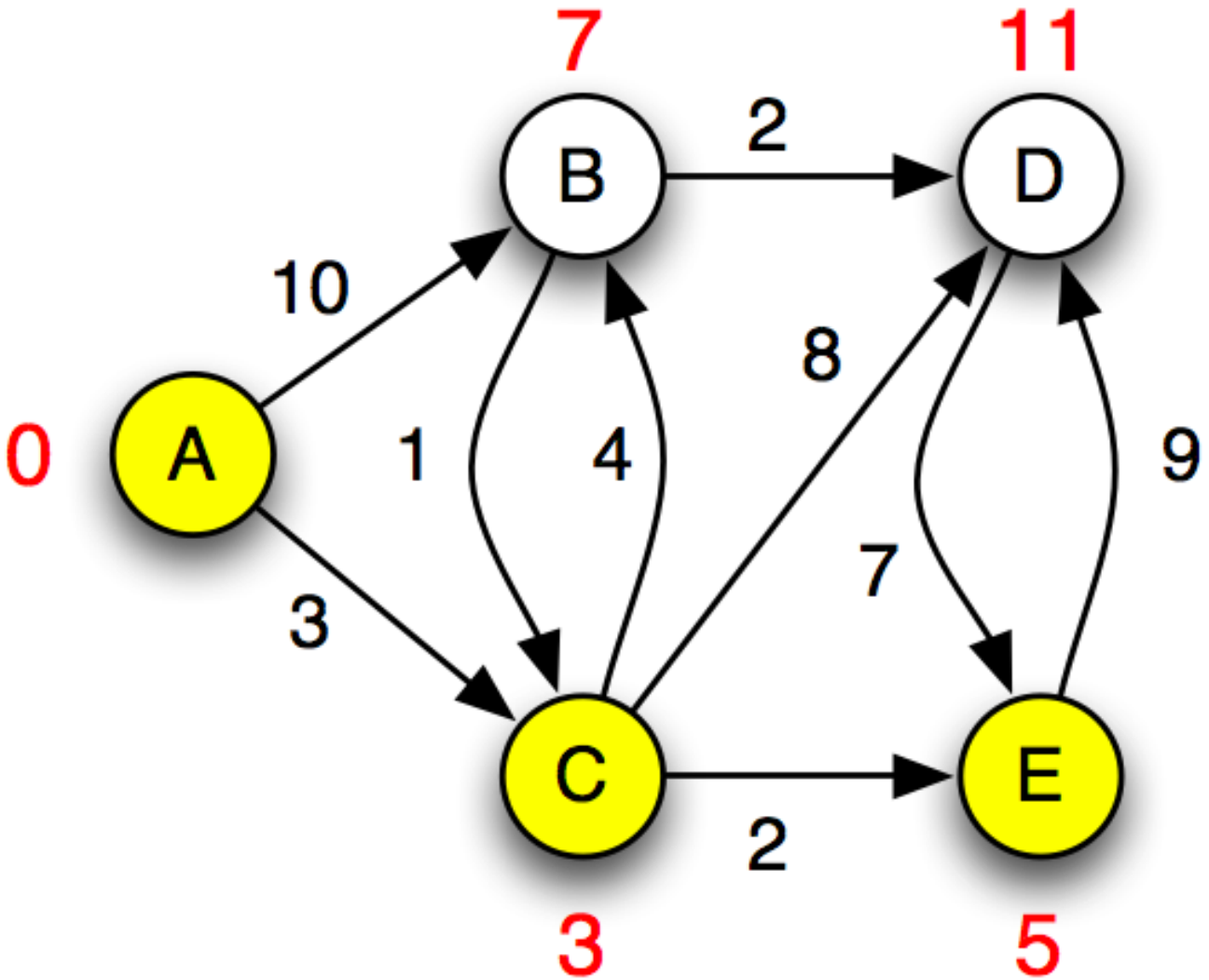
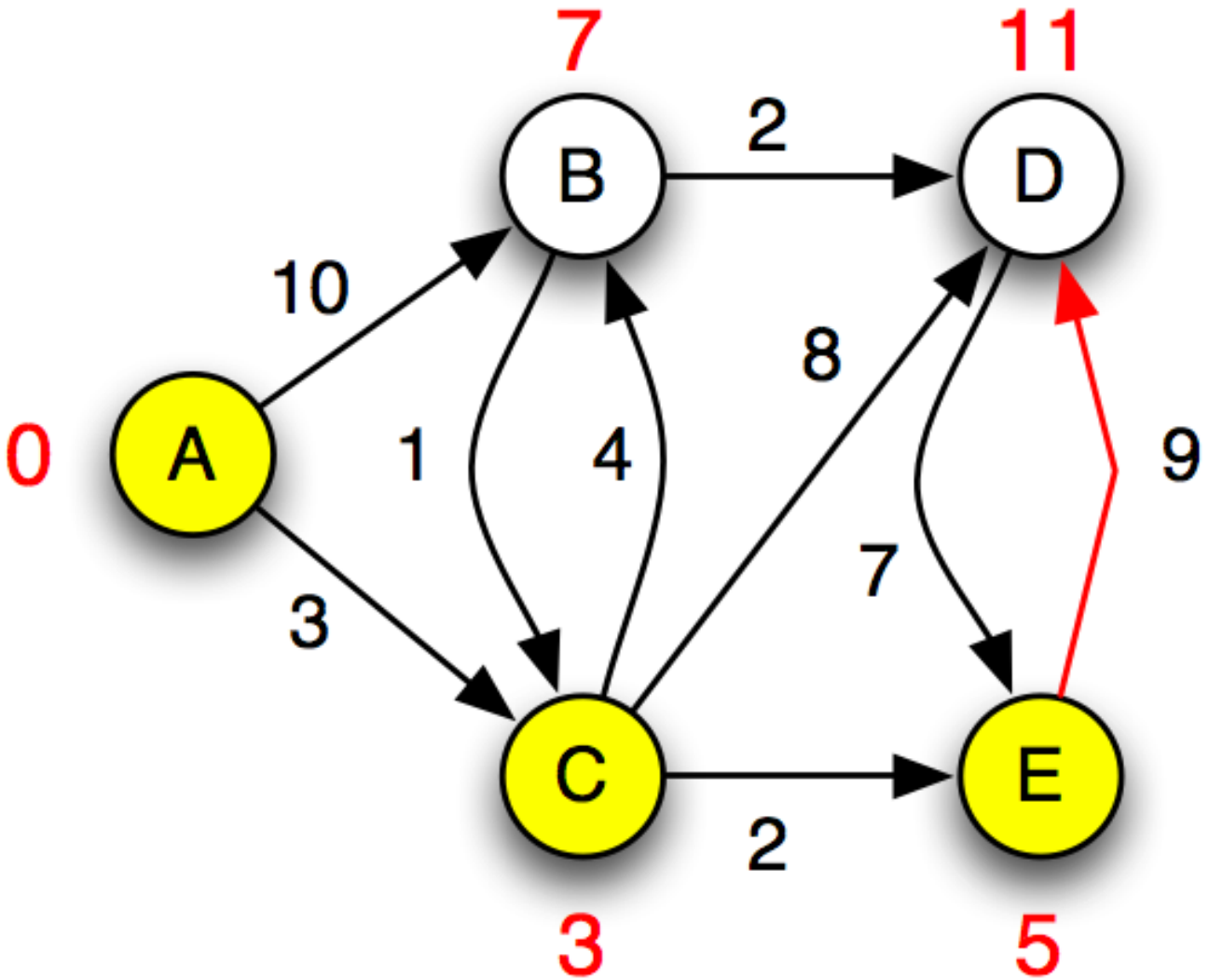


Dijkstra's SSSP algorithm in action



Dijkstra's SSSP algorithm in action



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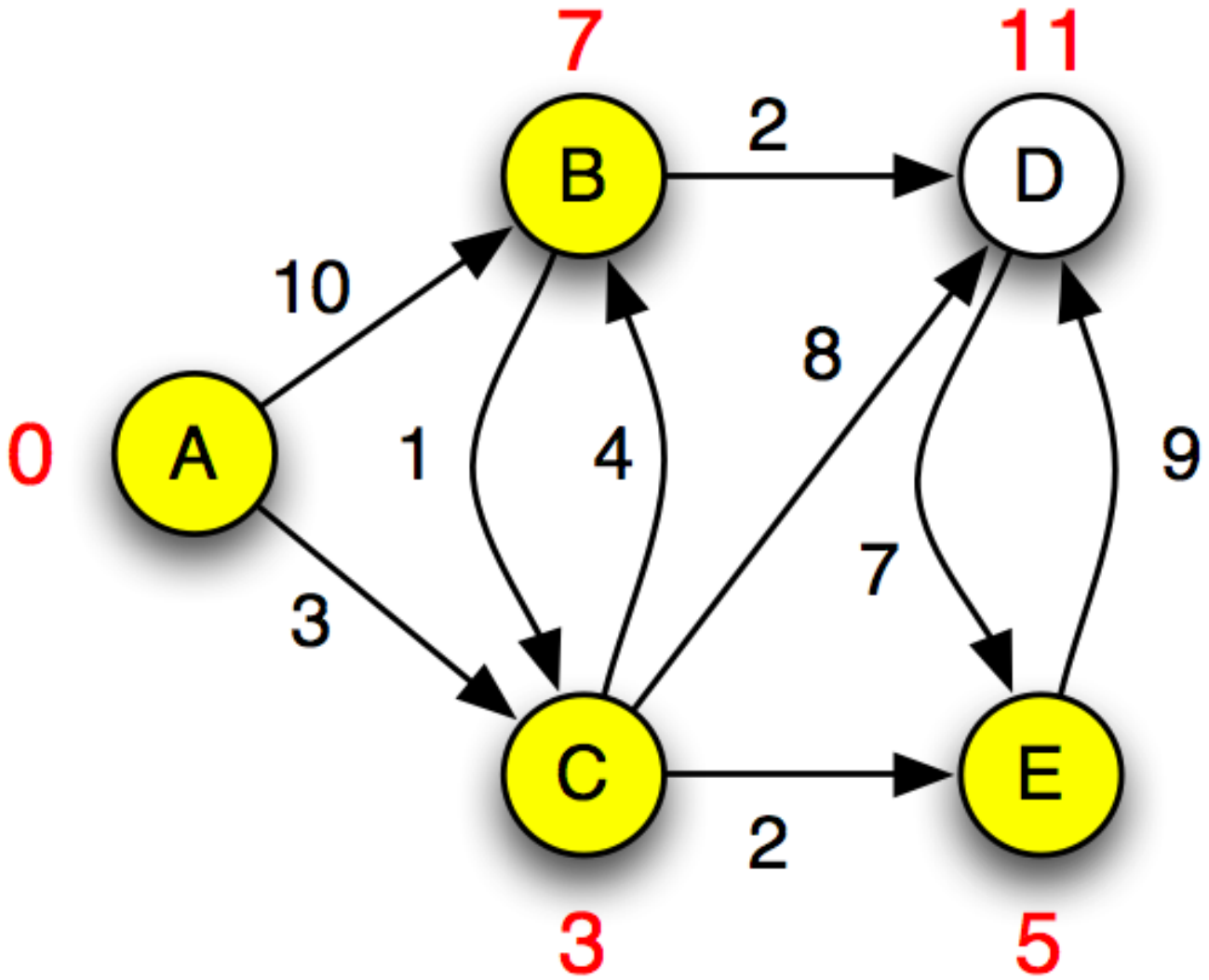

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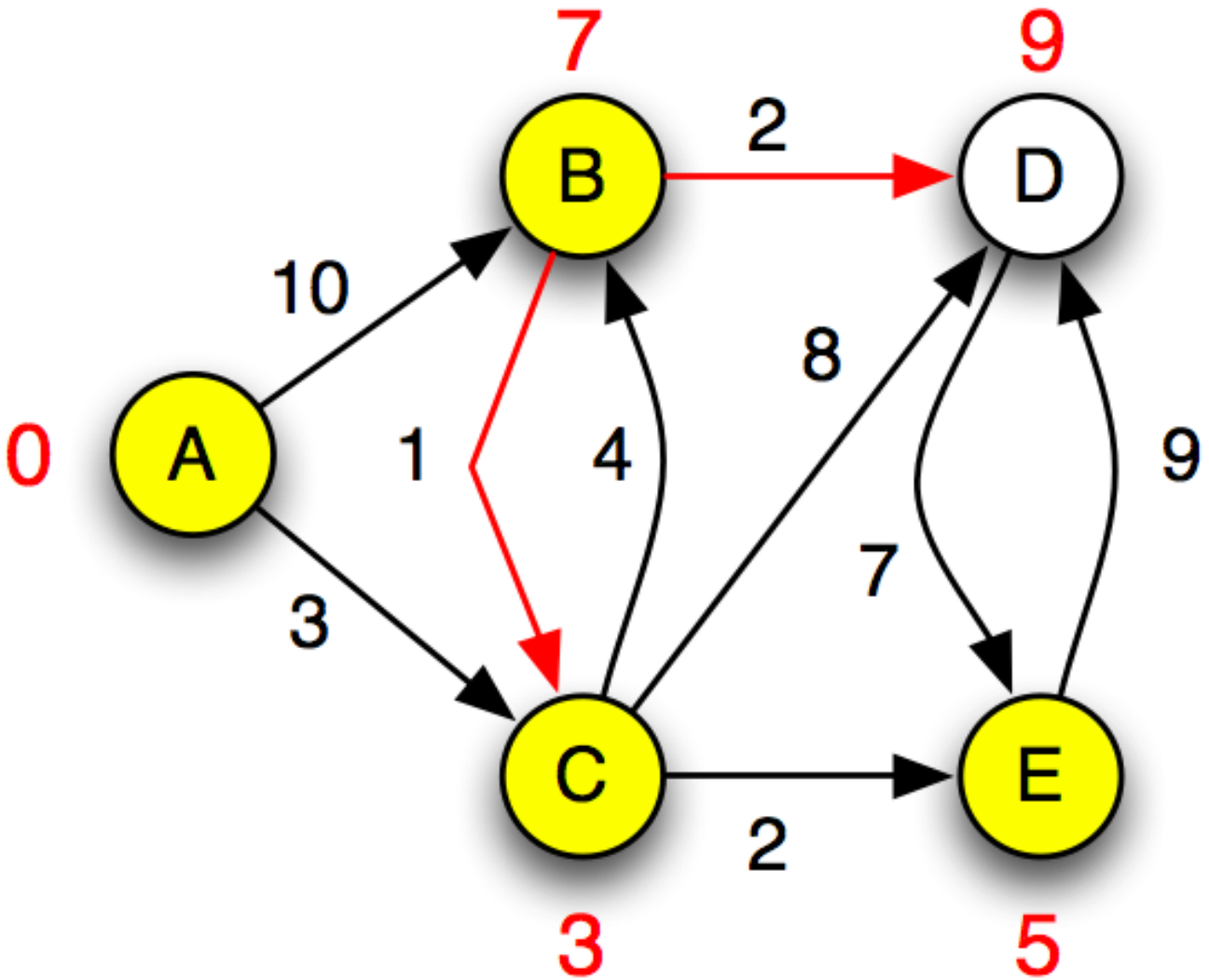

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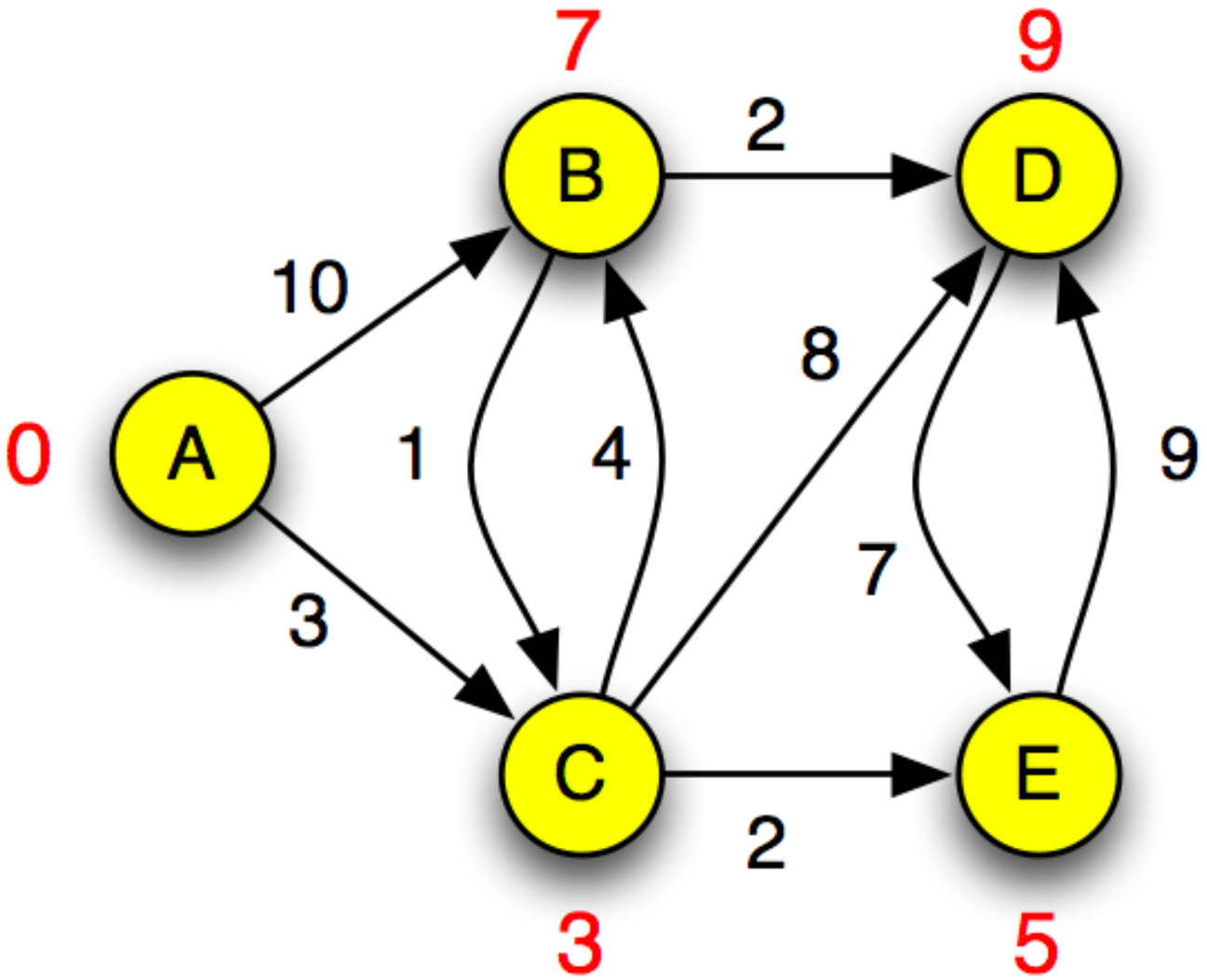
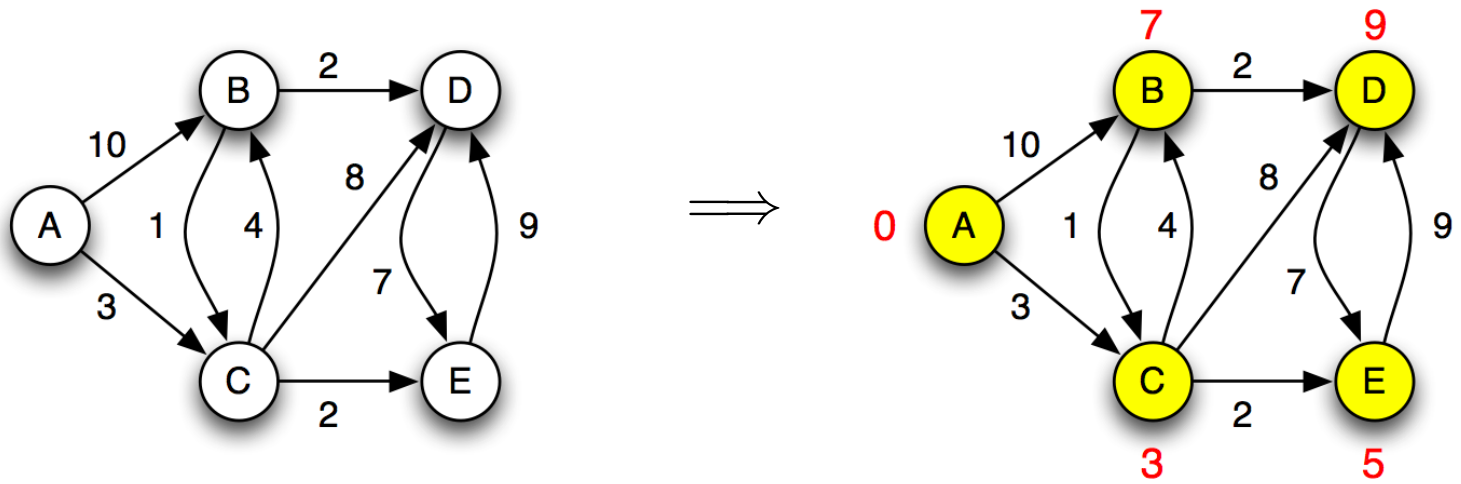


Dijkstra's SSSP algorithm in action



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