CSE 221: Algorithms

Heapsort

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References

1. T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms, Second Edition*. The MIT Press, September 2001.

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Heapsort

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Heapsort

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- Sorts *in place* – like *insertion sort*.
- Combines the best of both algorithms.
- Uses a data structure called the heap, which is also extensively used in other applications.

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Max vs. Min Heap

Unless explicitly stated as max heap or min heap, heap means max heap in this course.

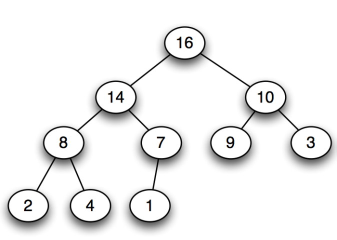
Heap-ordered tree

Definition

A binary tree is heap-ordered if:

1. the value at a node is $\geq$ the value at each of its children.

Example of (max) heap



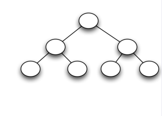
Heap-ordered tree

Definition

A binary tree is heap-ordered if:

1. the value at a node is $\geq$ the value at each of its children.
2. the tree is almost-complete.

Example of complete tree (or *not*)



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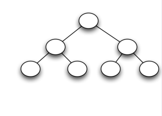

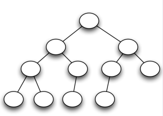

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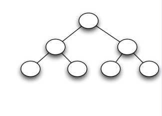

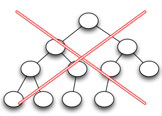

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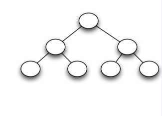

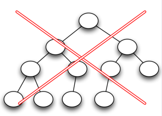

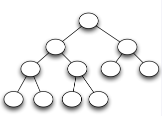

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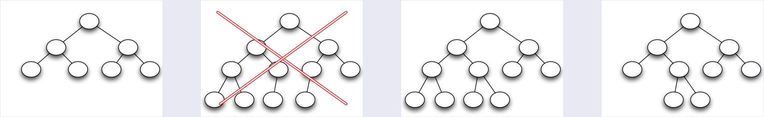

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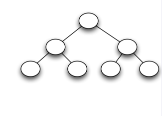
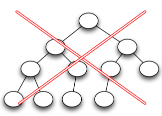
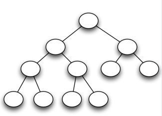
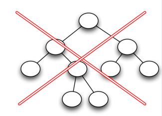
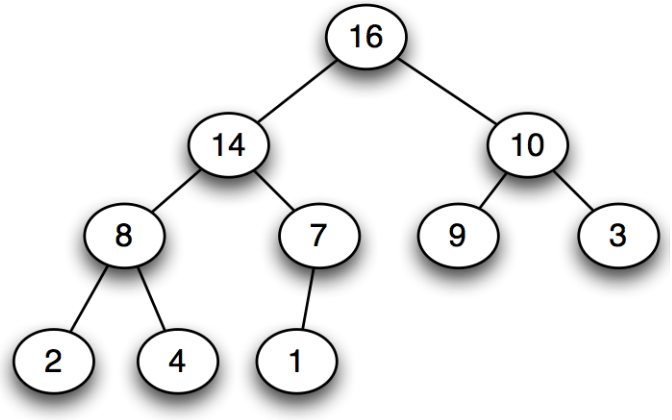





Heap – array representation of heap-ordered tree



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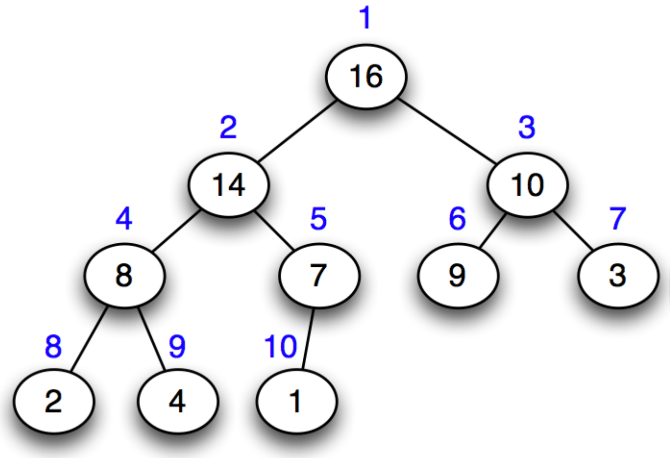

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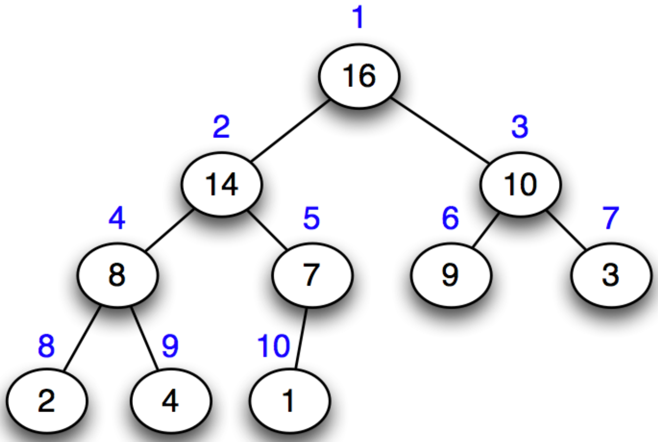

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16	14	10	8	7	9	3	2	4	1

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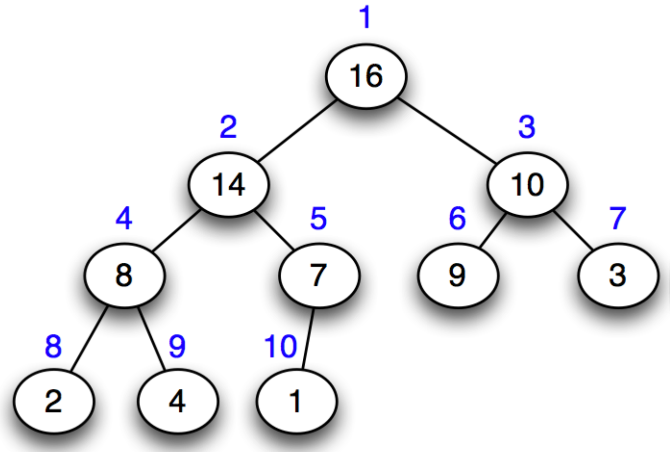

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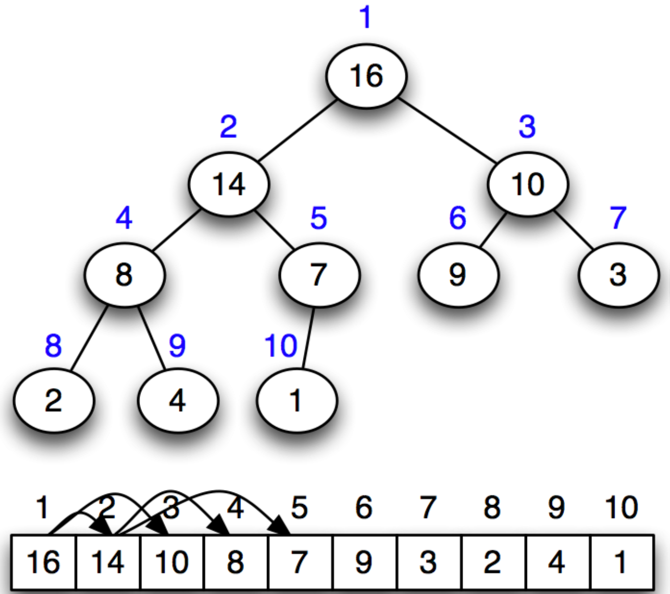

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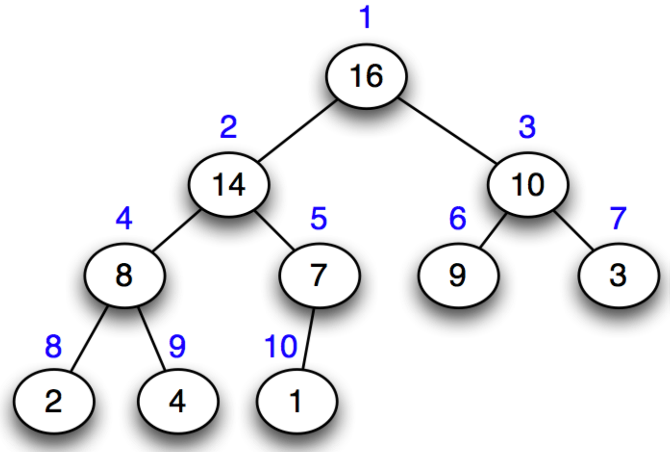

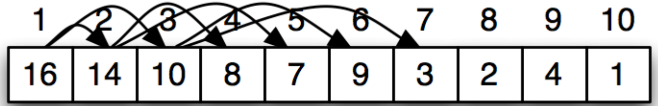

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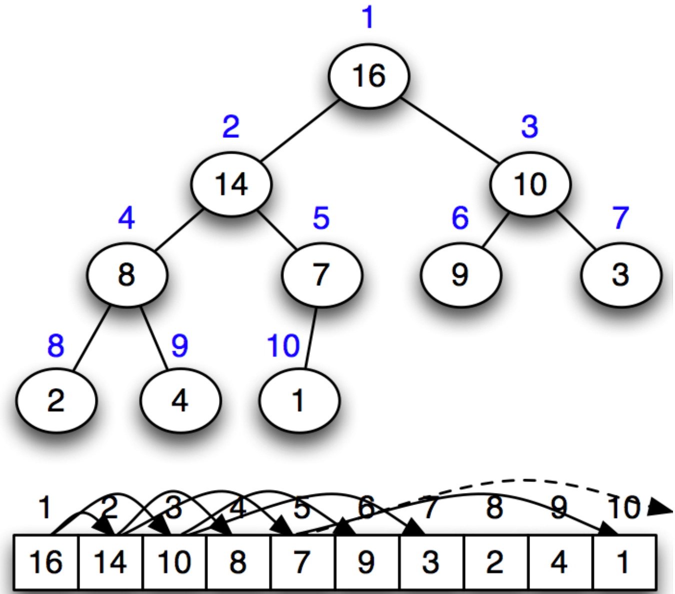
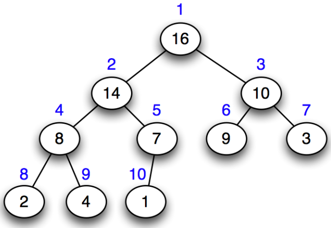


Heap – accessing parent and children



1	2	3	4	5	6	7	8	9	10
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MAXIMUM(A)

  return $A[1]$

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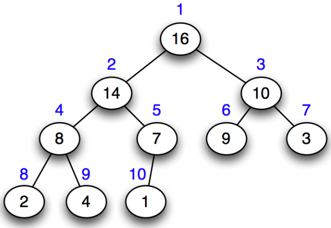

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$\text{PARENT}(i)$

return $\lfloor i/2 \rfloor$

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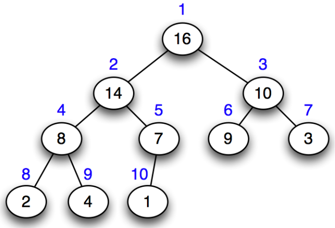

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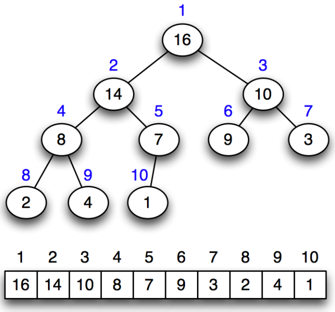

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Question

What if LEFT$(i) > n$?

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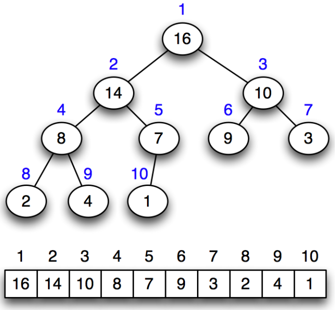

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LEFT(i)

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RIGHT(i)

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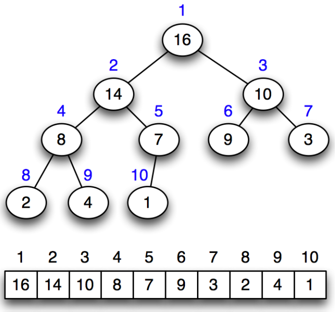

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$\text{LEFT}(i)$
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$\text{RIGHT}(i)$
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Question

What if $\text{RIGHT}(i) > n$?

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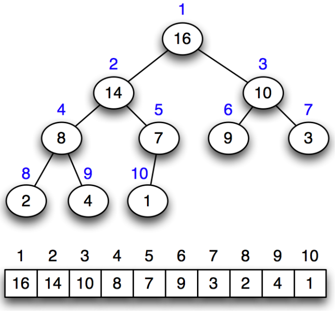

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return $\lfloor i/2 \rfloor$

$\text{LEFT}(i)$

return $2i$

$\text{RIGHT}(i)$

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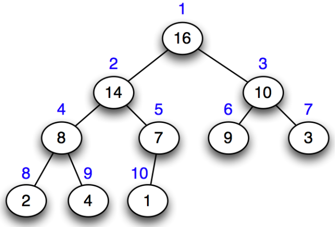
Lemma

All nodes $i > \lfloor length[A]/2 \rfloor$ (or equivalently, $i > \lfloor heap\text{-}size[A]/2 \rfloor$) are leaf nodes.

Heap – accessing parent and children



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$\text{RIGHT}(i)$

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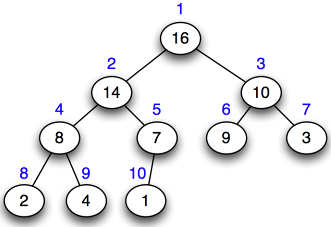
Definition (Heap property)

Heap property: For every node i other than the root,

$$A[\text{PARENT}(i)] \geq A[i].$$

Heap – accessing parent and children



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Question

Why do we insist that a heap-ordered tree be a complete binary tree? (Hint: draw the array representation of a tree that is not complete and see the *gaps*).

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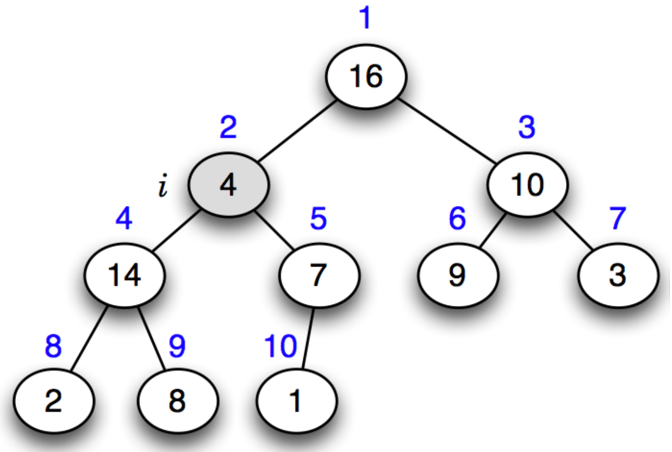
6. HEAP-EXTRACT-MAX(A) – Extract the largest element from heap A.

Example of MAX-HEAPIFY ("sink") operation



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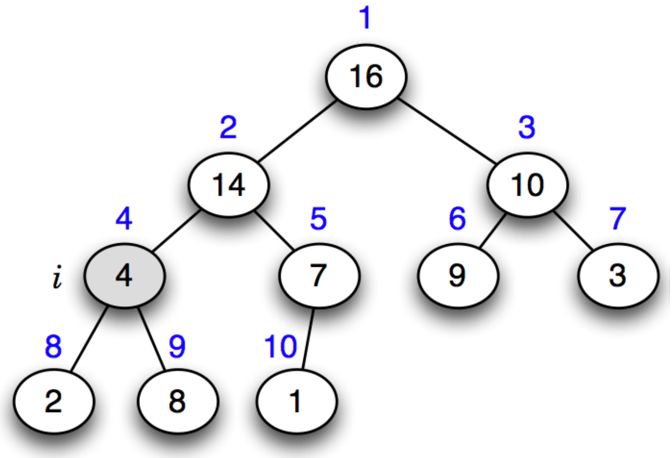

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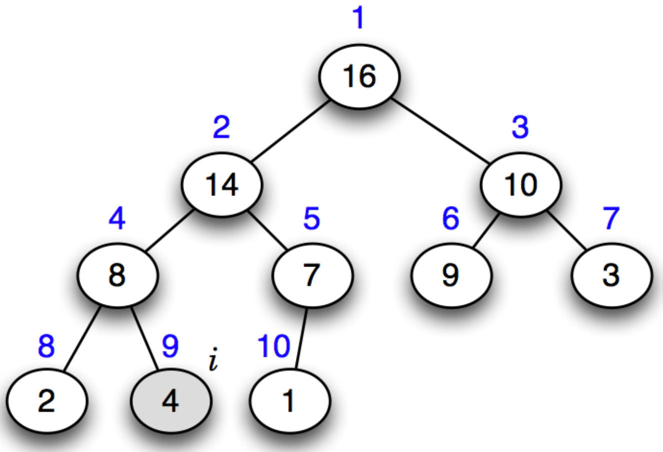

MAX-HEAPIFY algorithm

MAX-HEAPIFY(A, i)

$l \leftarrow left(i)$
$r \leftarrow right(i)$
if $l \leq heap\text{-}size[A]$ and $A[l] > A[i]$
  then $largest \leftarrow l$
  else $largest \leftarrow i$
if $r \leq heap\text{-}size[A]$ and $A[r] > A[largest]$
  then $largest \leftarrow r$
if $largest \neq i$
  then exchange $A[i] \leftrightarrow A[largest]$
    MAX-HEAPIFY$(A, largest)$

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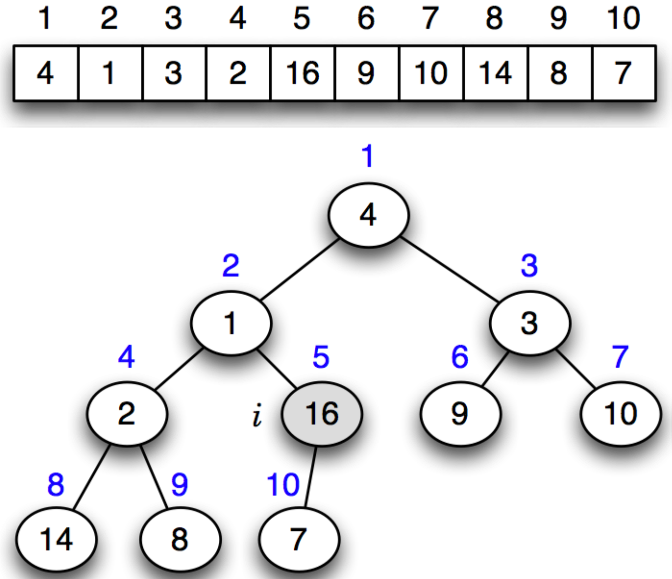
Analysis – second way

The running time of MAX-HEAPIFY on a node of height h is $T(n) = O(h) = O(\lg n)$.

Example of BUILD-MAX-HEAP ("heapify") operation



Example of BUILD-MAX-HEAP ("heapify") operation

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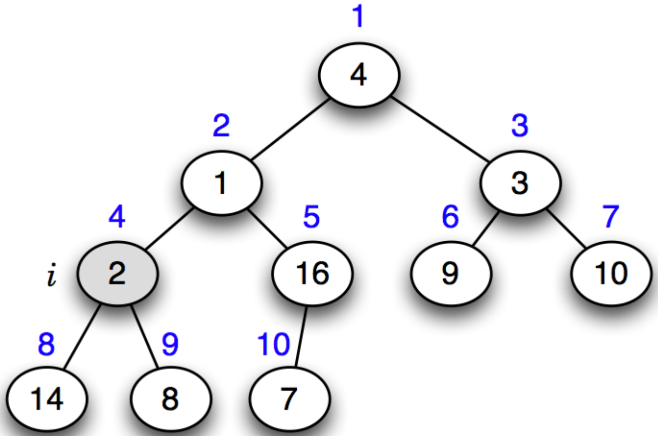
4	1	3	2	16	9	10	14	8	7



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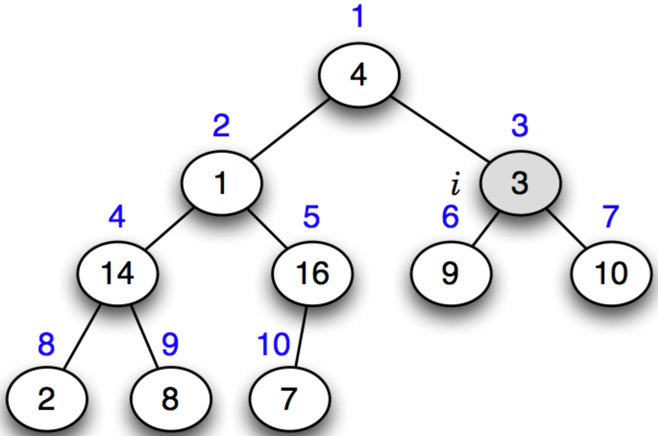

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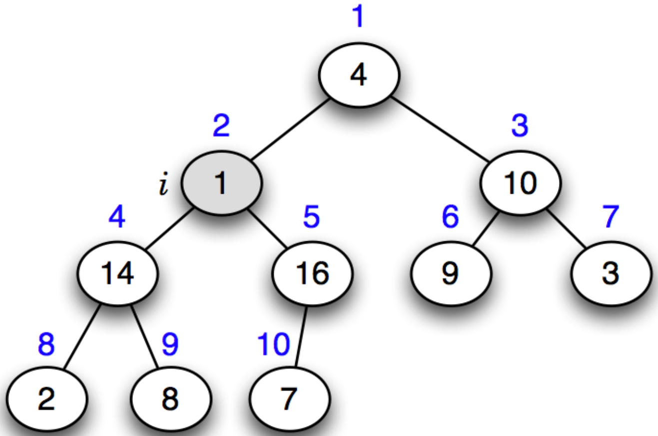

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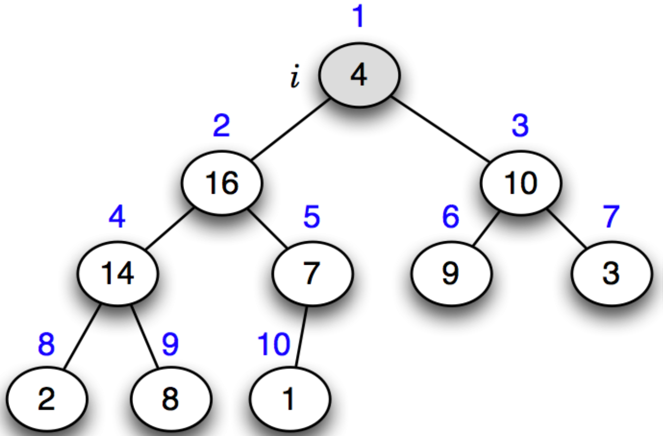

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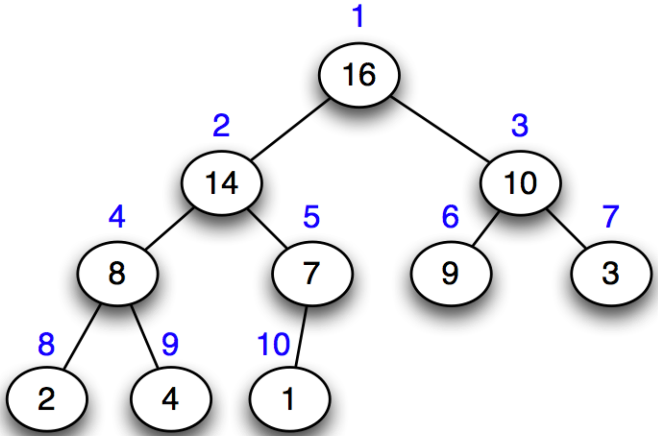


BUILD-MAX-HEAP algorithm

BUILD-MAX-HEAP(A)

$heap\text{-}size[A]] \leftarrow length[A]$

for $i \leftarrow \lfloor length[A]/2 \rfloor$ **downto** 1

  do MAX-HEAPIFY(A, i)

BUILD-MAX-HEAP algorithm

BUILD-MAX-HEAP(A)

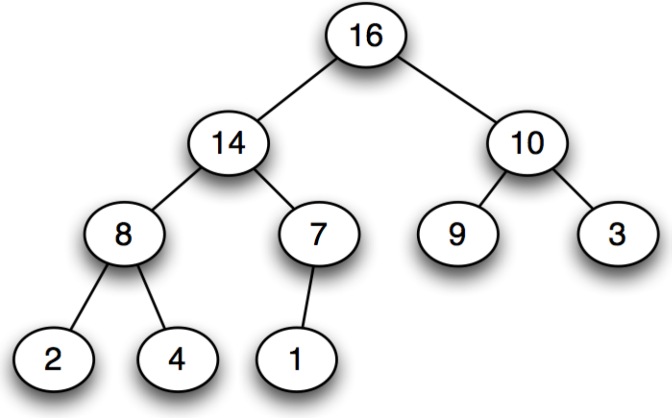
1 $heap\text{-}size[A]] \leftarrow length[A]$
2 **for** $i \leftarrow \lfloor length[A]/2 \rfloor$ **downto** 1
3 **do** MAX-HEAPIFY(A, i)

Analysis

$T(n) = O(n)$ (see textbook for details)

Contents

Heap use - heapsort



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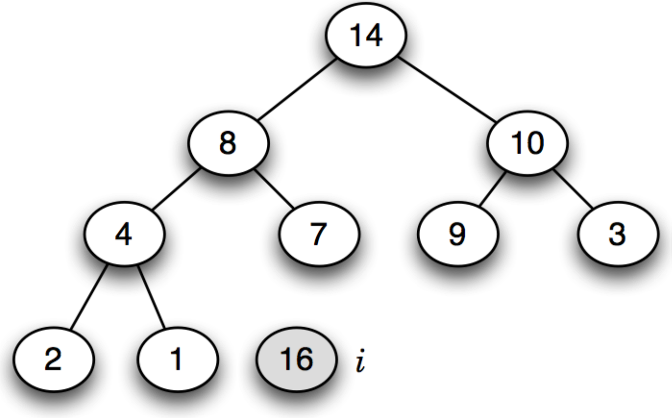

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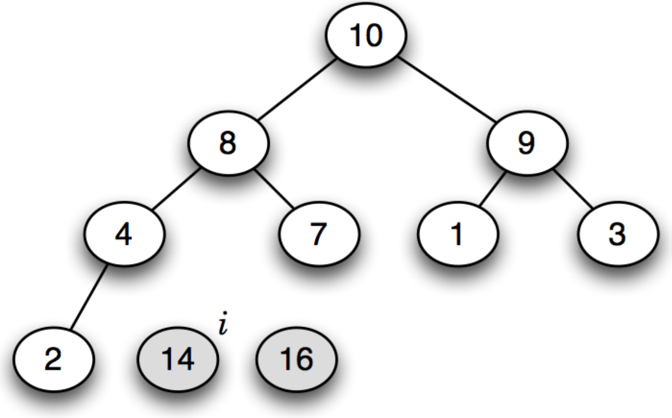

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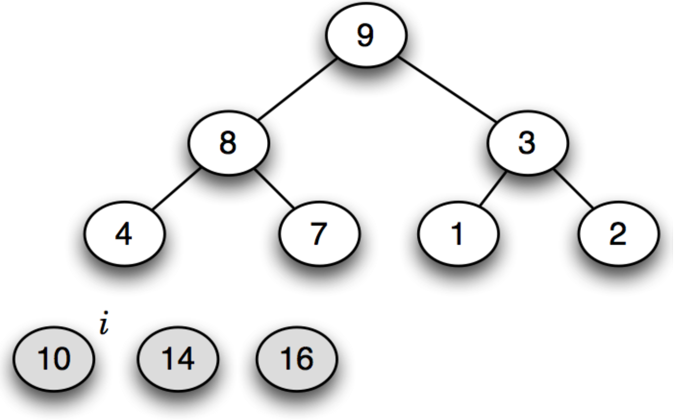

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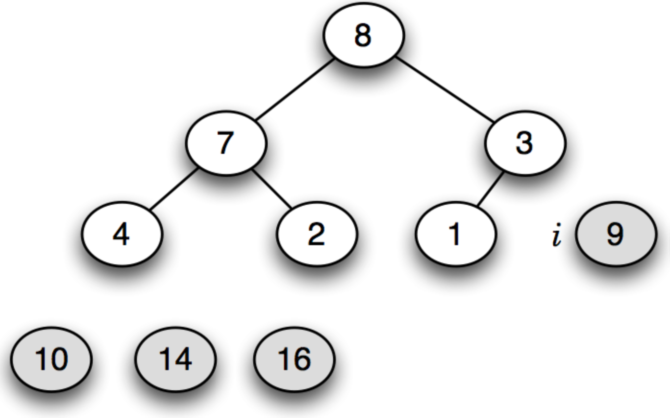

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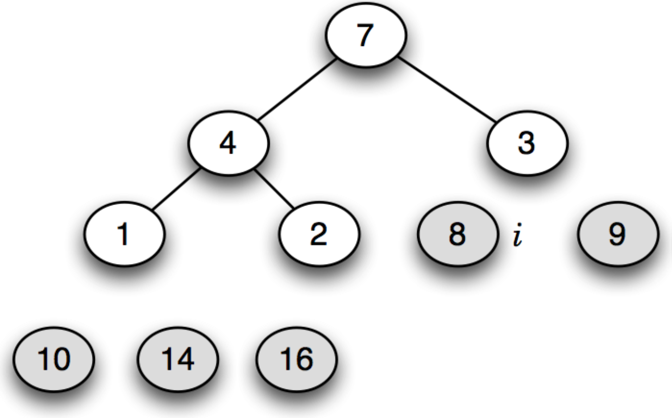

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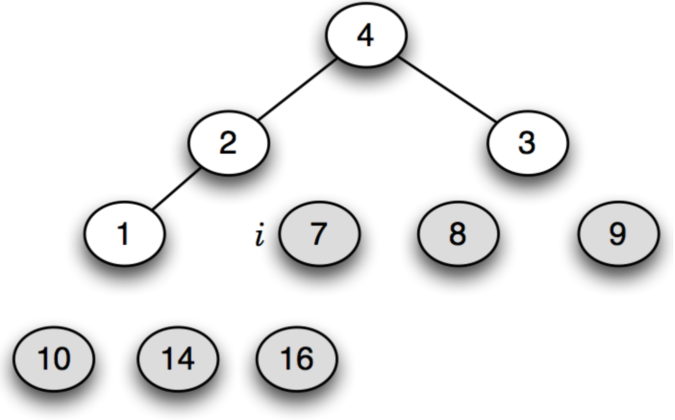

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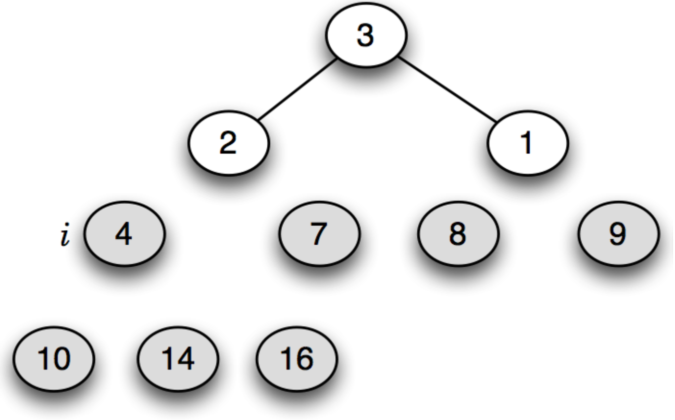

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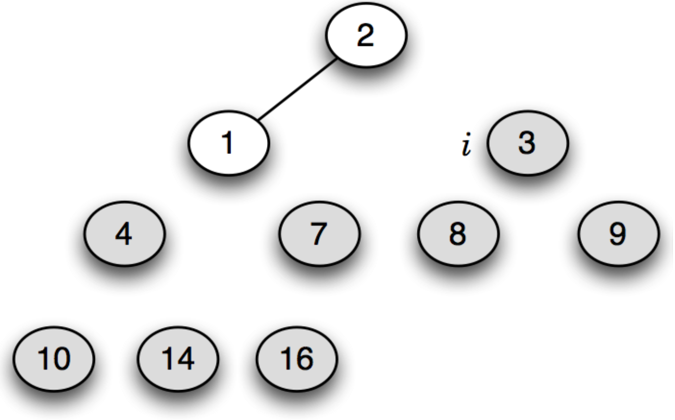

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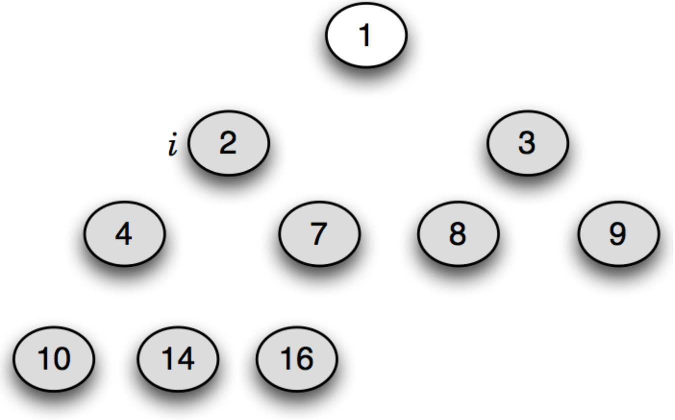

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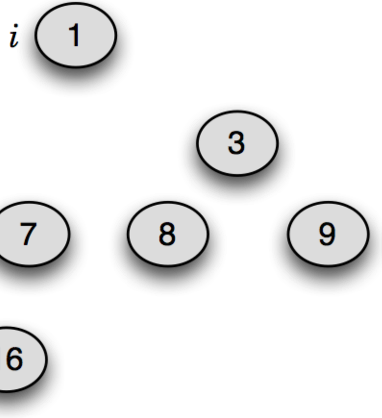

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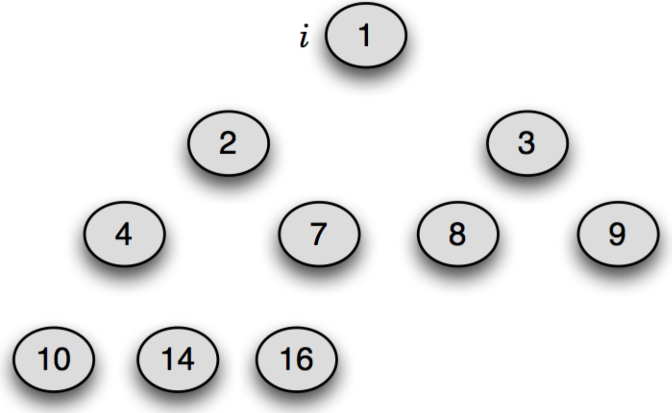

1	2	3	4	5	6	7	8	9	10
1	2	3	4	7	8	9	10	14	16

HEAPSORT algorithm

HEAPSORT(A)

	cost	*times*
BUILD-MAX-HEAP(A)	$\Theta(n)$	1
for $i \leftarrow$ *length*$[A]$ **downto** 2	$\Theta(1)$	n
do exchange $A[1] \leftrightarrow A[i]$	$\Theta(1)$	$n-1$
heap-size$[A] \leftarrow$ *heap-size*$[A] - 1$	$\Theta(1)$	$n-1$
MAX-HEAPIFY$(A, 1)$	$\Theta(\lg n)$	$n-1$

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Worst-case analysis

$T(n) = \Theta(n \lg n)$