CSE 221: Algorithms

Divide and Conquer

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References

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Divide-and-Conquer design strategy

1. *Divide* the problem (instance) into subproblems.
2. *Conquer* the subproblems by solving these recursively.
3. *Combine* the solutions to the subproblems.

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Example (of D&C strategy)

1. Binary search – divide the problem into half, and recursively search the appropriate 1 subproblem.
2. Mergesort – divide the problem into half, and recursively sort 2 subproblems, and then merge the results into a complete sorted sequence.
3. Computing x^n, computing fibonacci numbers, multiplying matrices (using *Strassen's algorithm*), etc.

Binary search

The problem

Find an element in a sorted array:

1. *Divide:* Check the middle element.
2. *Conquer:* Recursively search 1 subarray.
3. *Combine:* Trivial.

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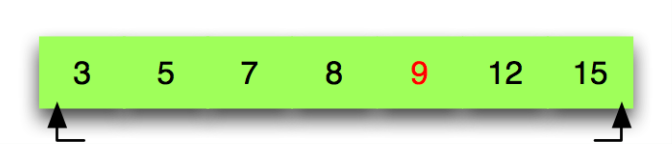
Example (Searching for 9)



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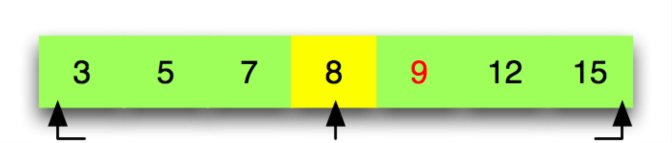

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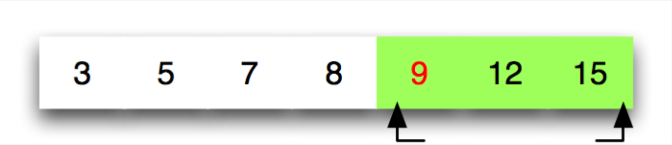

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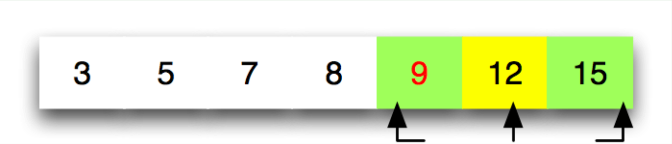

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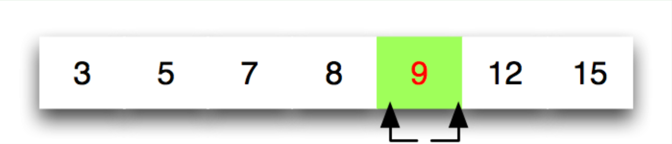

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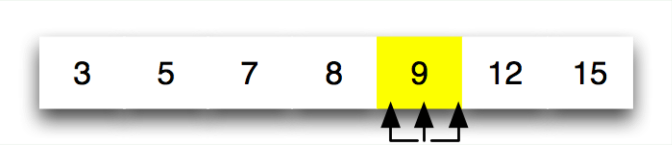

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Recurrence for binary search

$$T(n) = 1\,T(n/2) + \Theta(1)$$

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Analysis

$n^{\log_b a} = n^{\log_2 1} = n^0 = 1 \Rightarrow \text{CASE 2} \; (k = 0)$

$$\Rightarrow \boxed{T(n) = \Theta(n^{\log_b a} \lg^{k+1} n) = \Theta(\lg n)}$$

Merge sort

The problem

Find an element in a sorted array:

1. *Divide:* Trivial.
2. *Conquer:* Recursively sort 2 subarrays.
3. *Combine:* Merge the sorted subarrays in $\Theta(n)$ time.

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Key subroutine

MERGE – to merge two sorted arrays in linear-time.

Merge sort in action

1	2	3	4	5	6	7	8
9	4	5	2	1	7	4	6

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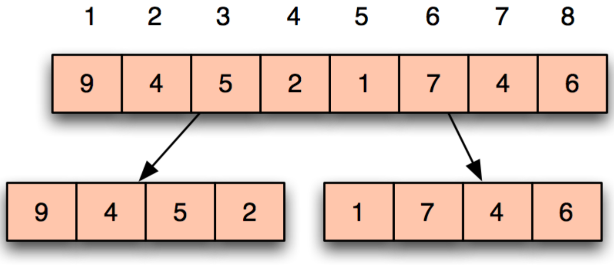

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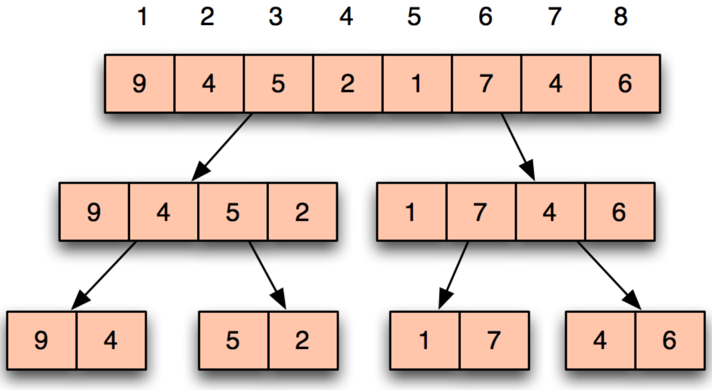

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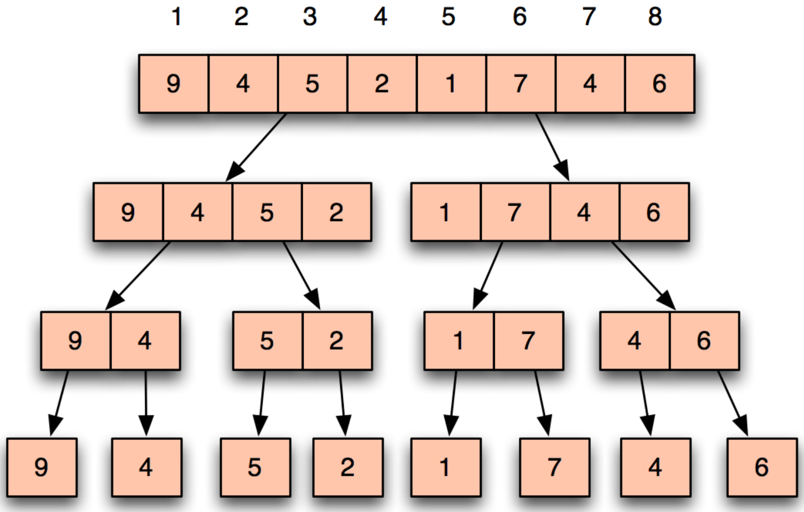

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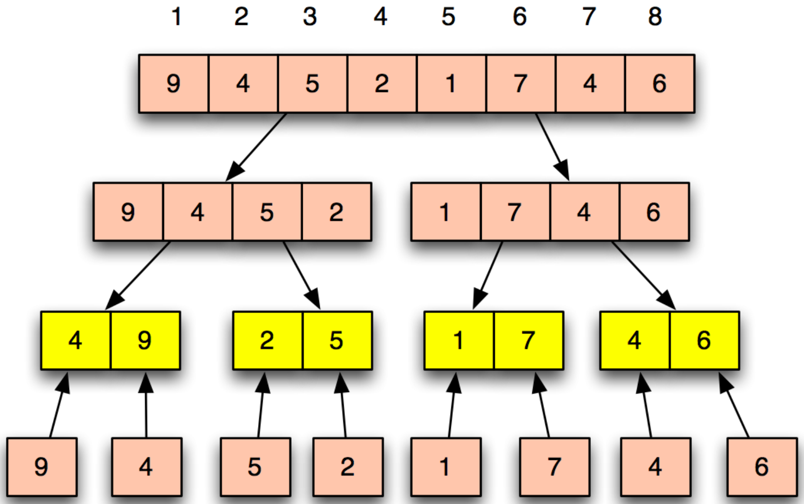

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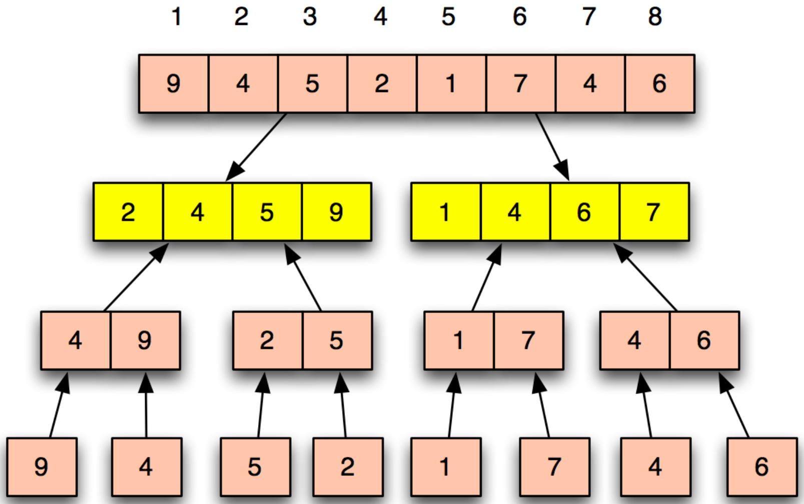

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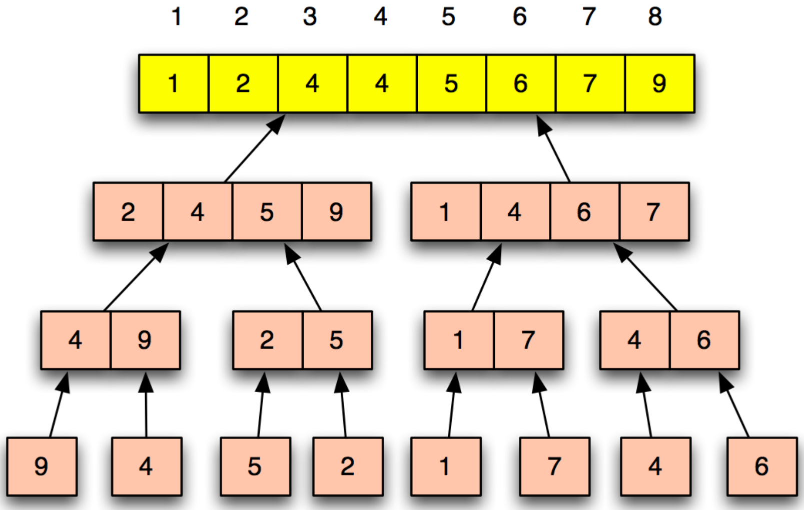

Merging in $\Theta(n)$ time

2 7 13 20 | 1 9 11 12

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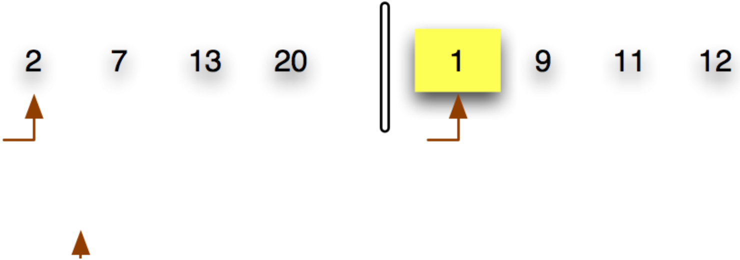

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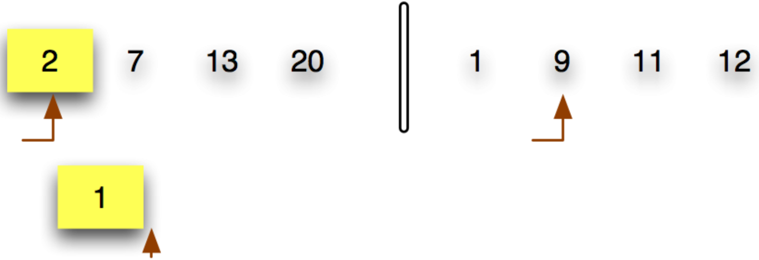

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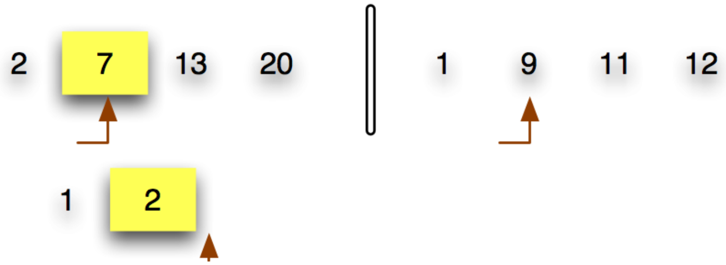

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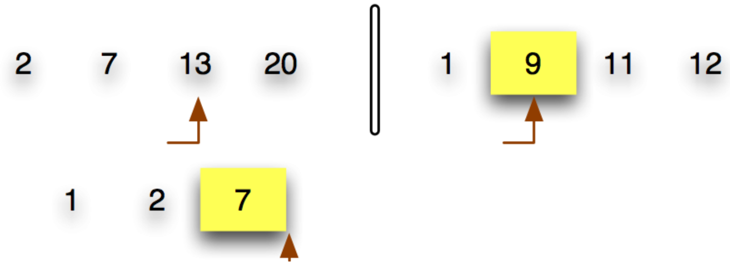

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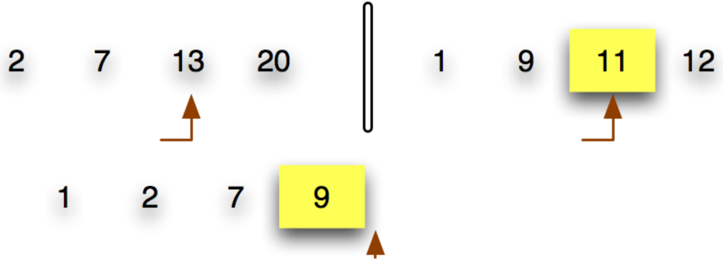

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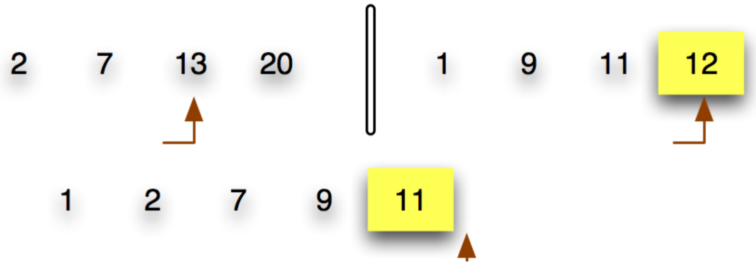

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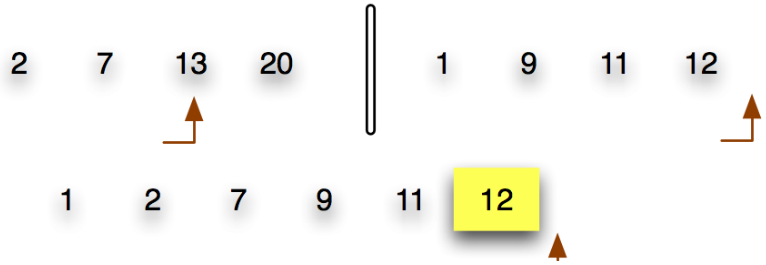

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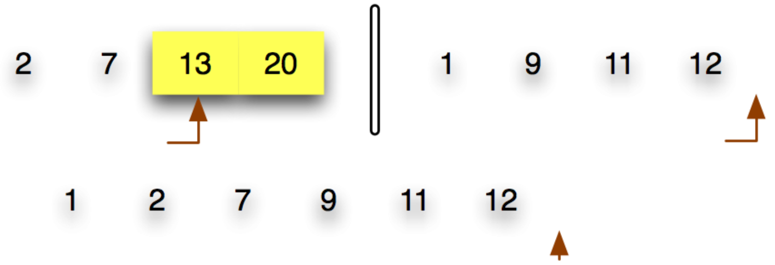

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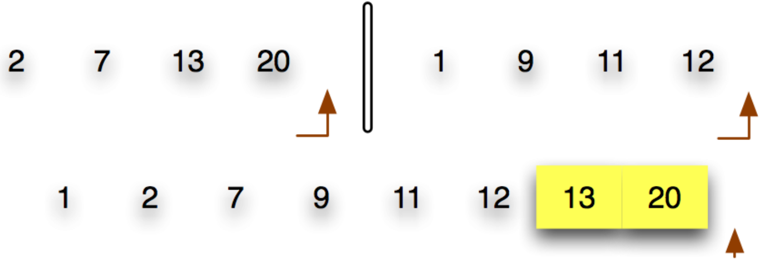

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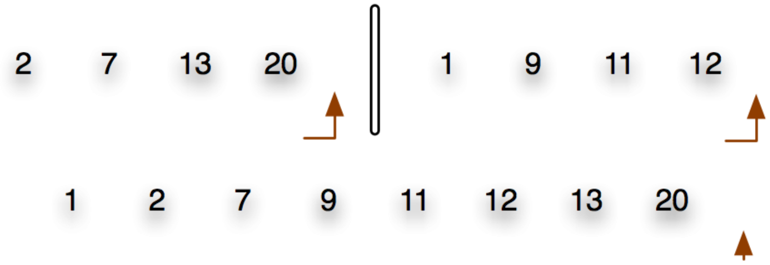

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A $\Theta(n)$ time merge algorithm

MERGE(A, B)

INPUT: Two sorted arrays A and B

OUTPUT: Returns C as the merged array

$\triangleright$ $n_1 = length[A]$, $n_2 = length[B]$, $n = n_1 + n_2$

Create $C[1\,..\,n]$

Initialize two indices to point to A and B

while A and B are not empty

    do Select the smaller of two and add to end of C

        Advance the index that points to the smaller one

if A or B is not empty

    then Copy the rest of the non-empty array to the end of C

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Issue: Out-of-place algorithm. Can it be made in-place?

Merge sort algorithm

MERGE-SORT(A) ▷ $A[1\,..\,n]$

if $n = 1$
    then return
    else ▷ recursively sort the two subarrays
        $A_1 = \text{MERGE-SORT}(A[1\,..\,\lceil n/2 \rceil])$
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Few notes on the algorithm

1. Dividing is trivial, but it's the merging that requires most time.
2. The merging algorithm presented here is an out-of-place algorithm, which will increase space complexity.

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Powering a number

Problem: Compute a^n, where $n \in \mathbb{N}$.
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$$T(n) = T(n/2) + \Theta(1) \Rightarrow \boxed{T(n) = \Theta(\lg n)}.$$

Conclusion

- Divide and Conquer is just one of several algorithm design strategies.
- Used by many of the commonly used algorithms
  - Binary search
  - Merge sort
  - Fast Fourier Transform (FFT)
  - Finding closest pair of points
  - Matrix multiplication (Strassen's algorithm)
  - Matrix inversion
  - Quicksort and (k^{th}) selection
  - ...
- Can be easily analyzed using recurrences
- Often leads to efficient algorithms