CSE 221: Algorithms

Sorting lower bounds and Linear time sorting

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References

1. T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms, Second Edition*. The MIT Press, September 2001.
2. Erik Demaine and Charles Leiserson, *6.046J Introduction to Algorithms*. MIT OpenCourseWare, Fall 2005. Available from: `ocw.mit.edu/OcwWeb/Electrical-Engineering-and-Computer-Science/6-046JFall-2005/CourseHome/index.htm`

Last modified: February 25, 2013

Contents

1 Sorting lower bounds
- What's the best we can do?
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2 Sorting in linear time
- Counting sort
- Radix sort
- Conclusion

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What's the best we can do?

- Bubble, selection, insertion, quicksort . . . $O(n^2)$
- Heapsort, mergesort . . . $O(n \lg n)$

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Can a sorting algorithm do better than $O(n \lg n)$ in the worst-case?

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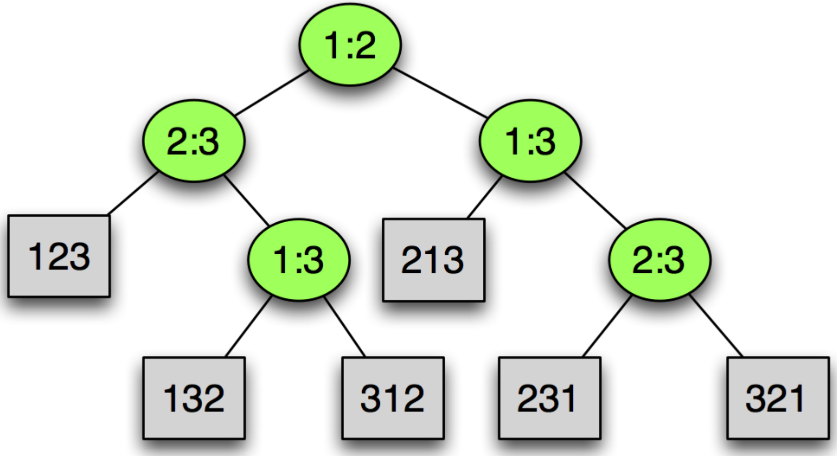
We can use a decision tree to answer this question.

Decision tree for comparison based sorting

- A model to represent any comparison based sorting algorithm
- For any comparison based sorting algorithm, if detailed operations are ignored and only the sequence of comparisons among the pair of elements is focused, then, for a input sequence, it follows a path in this tree from root to leaf to output correct solution
- Each node of this tree is labeled as (i, j) which indicates comparison between element i and element j
- If $i \leq j$ then left subtree is gives subsequent comparisons. Otherwise, right subtree is followed
- Each leaf represents a permutation of the input sequence

Decision tree for comparison based sorting

Sequence $A = \langle 9, 4, 6 \rangle$



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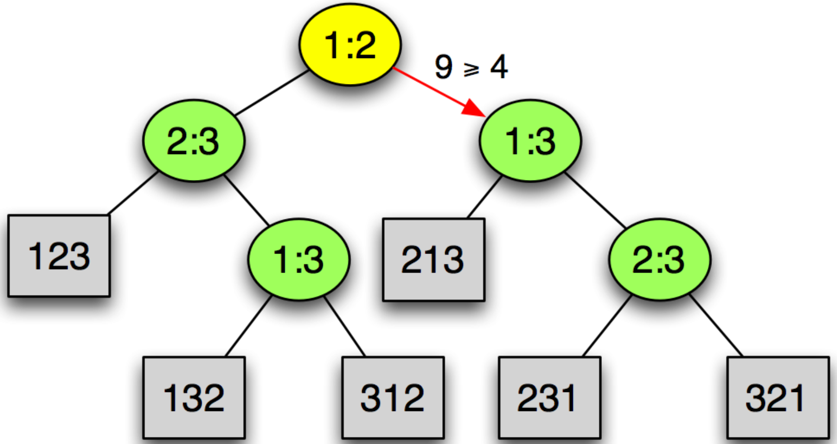

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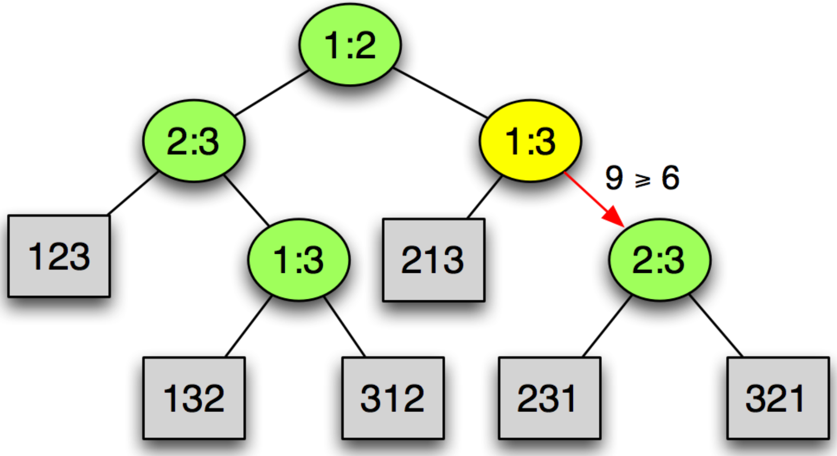

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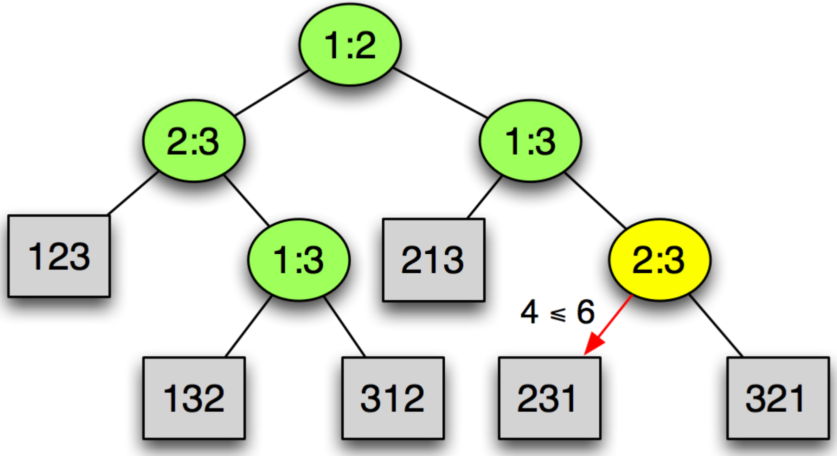
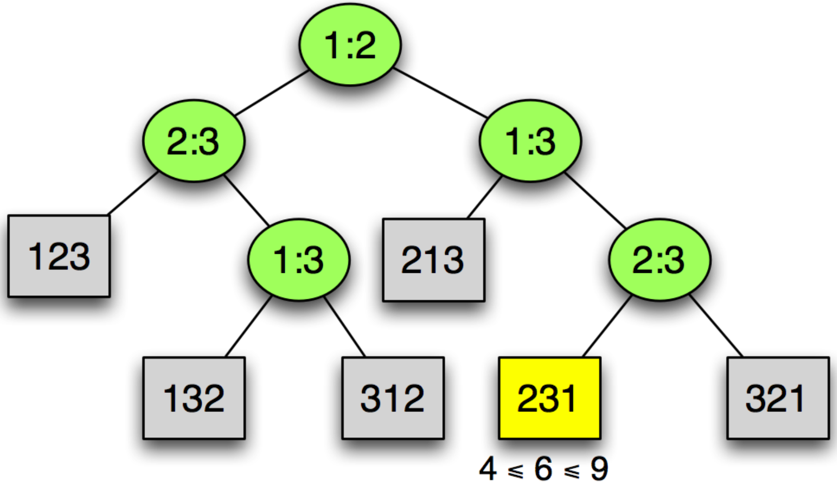


Decision tree for comparison based sorting

Sequence $A = \langle 9, 4, 6 \rangle$ $\Longrightarrow$ Sequence $B = \langle 4, 6, 9 \rangle$



Lower bound on comparison based sorting

Question

What is the best that we can do with comparison-based sorting?

- $n!$ possible permutations,that means $n!$ leaves and one of which is the sorted sequence.
- Minimum number of comparisons is the path from root of the decision tree to one of the $n!$ leaves.
- Each path is not of the same depth
- Path having maximum depth represents worst case
- Path having smallest depth represents best case
- We will prove that worst case cannot be less than $O(n \lg n)$

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Theorem

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- Let, Maximum height is h
- Complete binary tree having height h has 2^h leaves
- But decision tree might not be a complete binary tree so it can have less than 2^h leaves.
- *Number* of leaves $\leq 2^h$

$2^h \geq n!$ (number of leaves$=n!$)
$\lg 2^h \geq \lg n!$
$h \geq \lg n!$
(Note: by Stirling's approximation $n! = (n/e)^n$, where e is Euler's constant.)
$h \geq \lg((n/e)^n)$
$h \geq (n \lg n - n \lg e)$
$h \geq (n \lg n)$
$h = \Omega(n \lg n)$

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Sorting in linear time: Counting sort

Counting sort: No comparisons between elements.

- **Input**: $A[1\,..\,n]$, where $A[j] \in \{1, 2, \ldots, k\}$.
- **Output**: $B[1\,..\,n]$, sorted.
- **Auxiliary storage**: $C[1\,..\,k]$.

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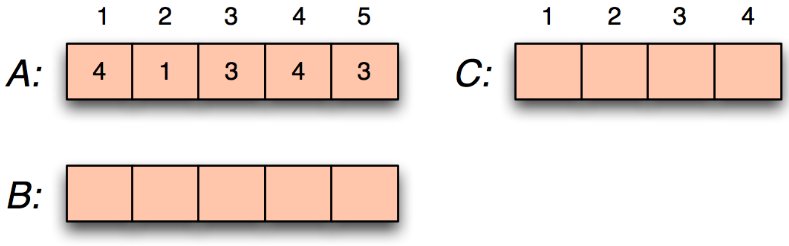
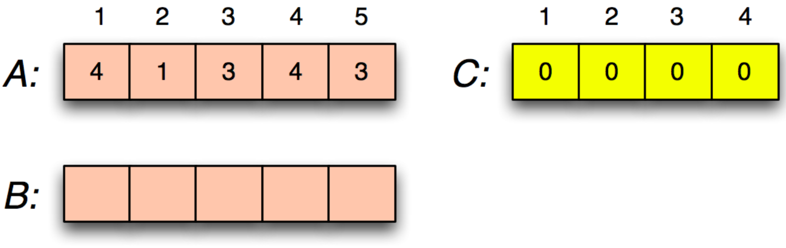
Counting sort algorithm

```
1  for $i \leftarrow 1$ to $k$
2          do $C[i] \leftarrow 0$
3  for $j \leftarrow 1$ to $n$
4          do $C[A[j]] \leftarrow C[A[j]] + 1$            $\triangleright\ C[i] = |\{key = i\}|$
5  for $i \leftarrow 2$ to $k$
6          do $C[i] \leftarrow C[i] + C[i-1]$             $\triangleright\ C[i] = |\{key \leq i\}|$
7  for $j \leftarrow n$ downto $1$
8          do $B[C[A[j]]] \leftarrow A[j]$
9             $C[A[j]] \leftarrow C[A[j]] - 1$
```

Counting sort example



Counting sort example: loop 1



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for $i \leftarrow 1$ to $k$
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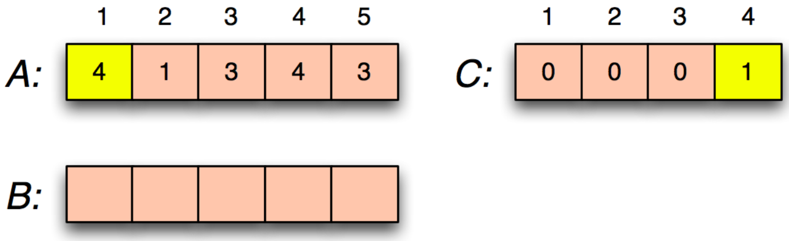
Counting sort example: loop 2



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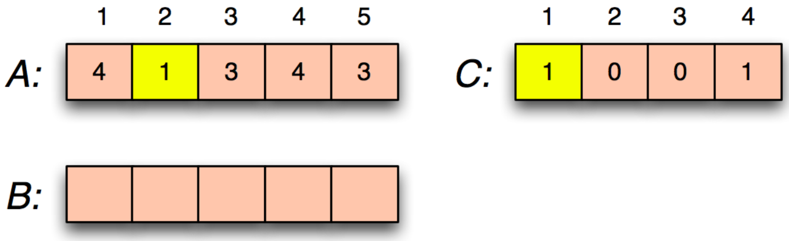

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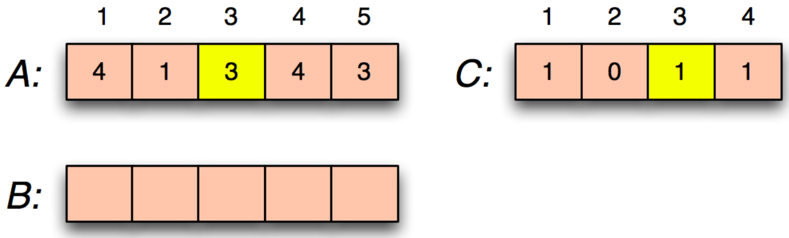

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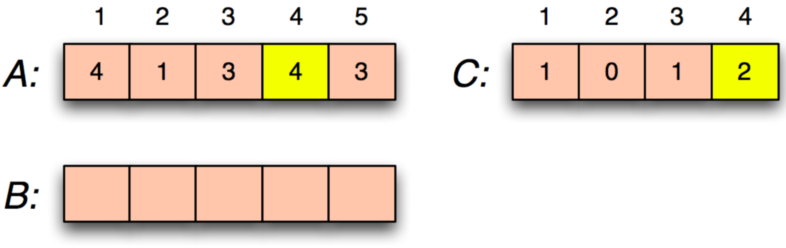

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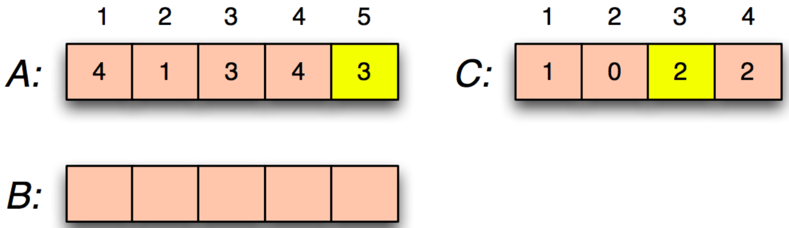

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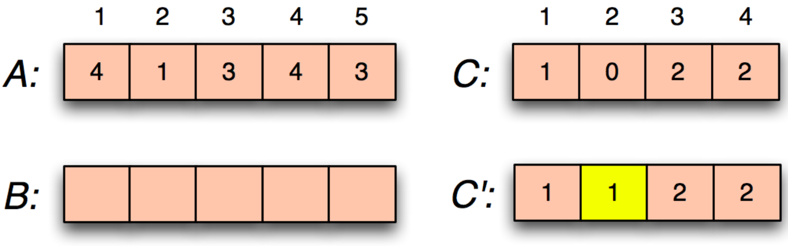

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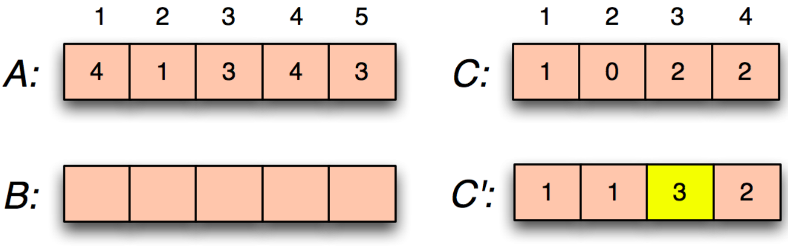

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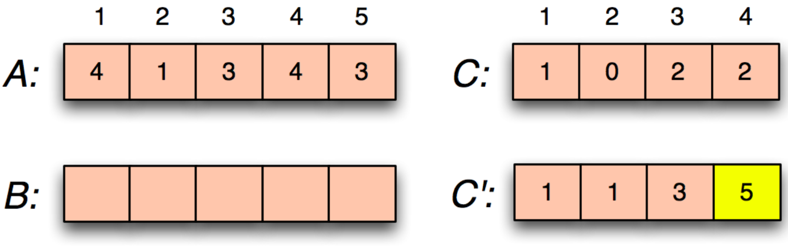

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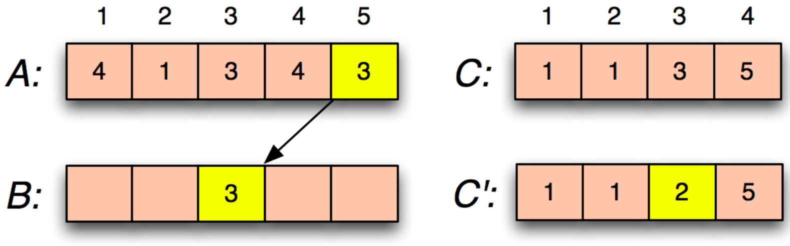

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for $j \leftarrow n$ downto 1
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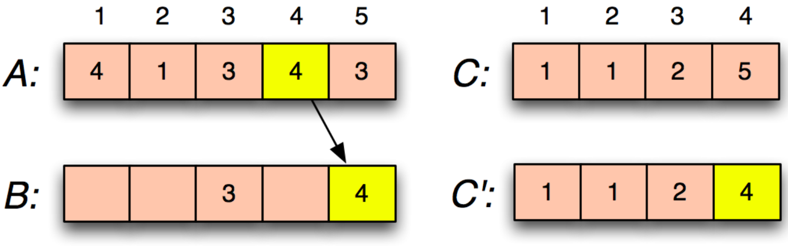

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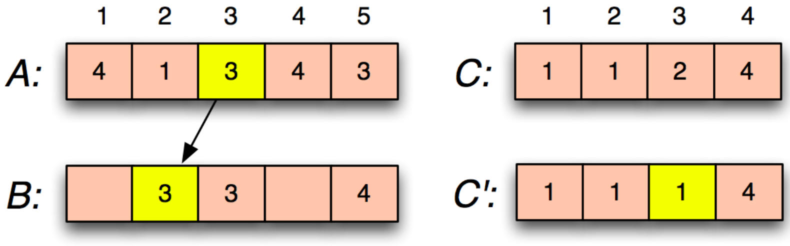

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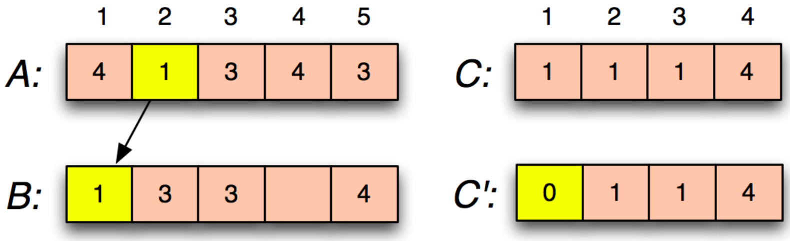

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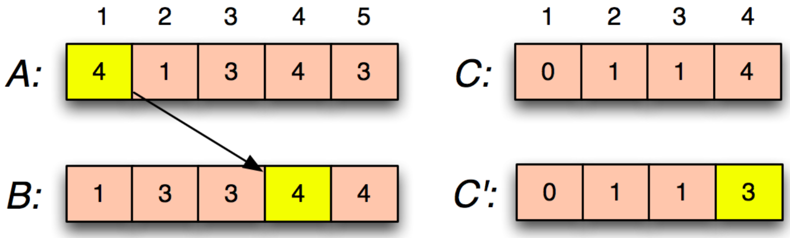

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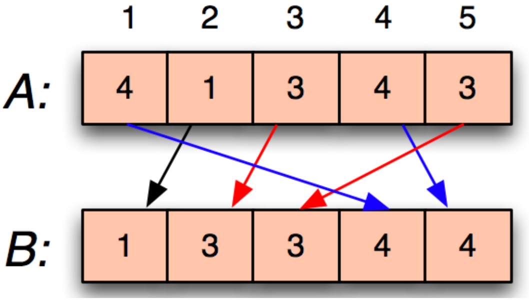

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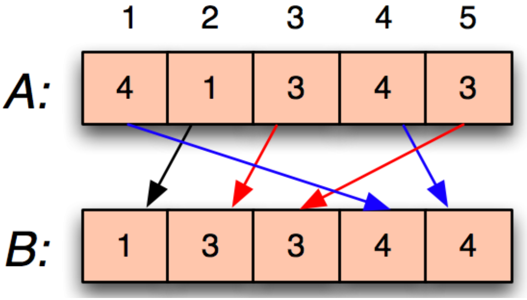

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- Would it still be stable if we had used **for** $j \leftarrow 1$ **to** n instead of **for** $j \leftarrow n$ **downto** 1?

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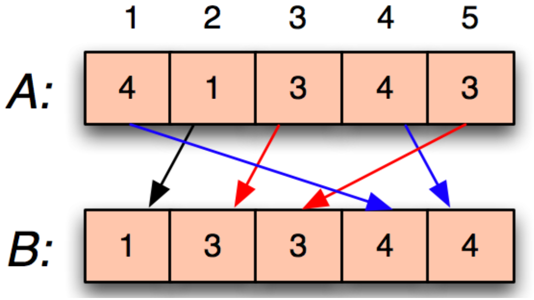

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- Would it still be stable if we had used **for** $j \leftarrow 1$ **to** n instead of **for** $j \leftarrow n$ **downto** 1?
- What other sort algorithms that you’ve seen so far are stable?

Counting sort complexity

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$\Theta(n + k)$

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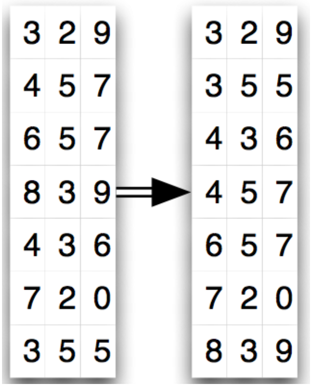
- But didn't we just prove that sorting takes $\Omega(n \lg n)$ time?
- So what's wrong with this picture?

And the answer is ...

- The $\Omega(n \lg n)$ is for comparison sorting.
- Counting sort is **not** a comparison sort.
- In fact, counting sort does not use a single comparison.

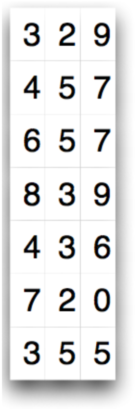
Radix sort



Radix sort basics

- Digit by digit sort.
- Can be either *most-significant* digit first, or *least-significant* digit first.
- A good way is to stably sort *least-significant* digit first.

Radix sort in action



3	2	9
4	5	7
6	5	7
8	3	9
4	3	6
7	2	0
3	5	5

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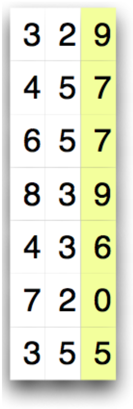

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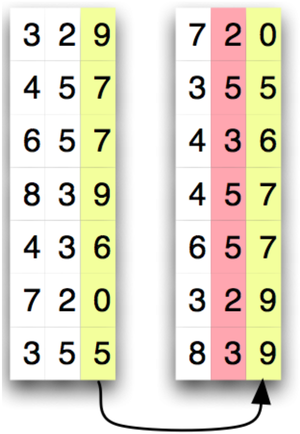

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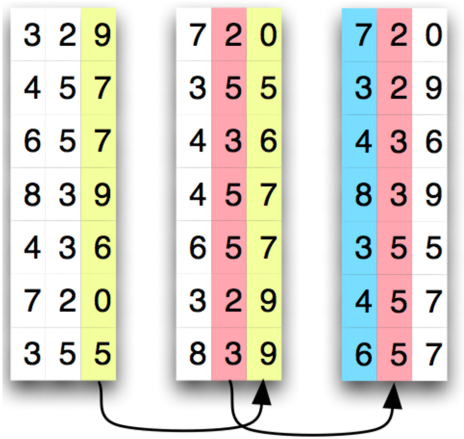

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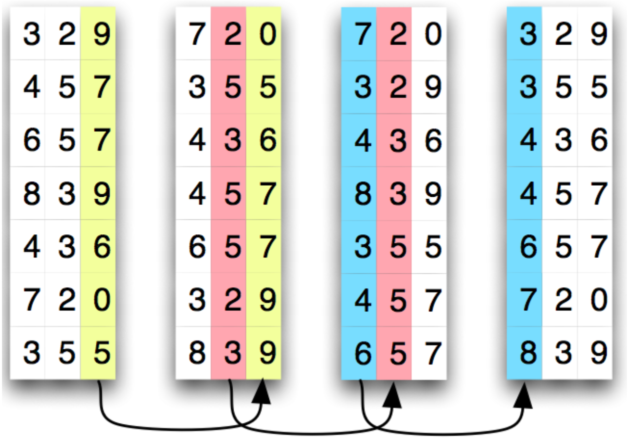
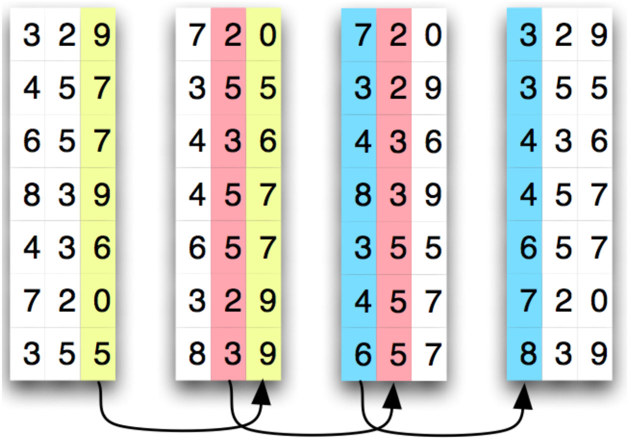


Radix sort in action



Analysis: For numbers in the range $[0 \mathinner{..} n^d - 1]$, radix sort runs in $\Theta(dn)$ time.

Bucket Sort

Assumption:

- Input elements are ‘distributed uniformly’ over known range, e.g. fractions in [0,1)
- or, integers in the range $[1, \ldots, m]$

Idea:

- Divide the interval to k buckets
- Distribute the inputs into k buckets
- Sort the numbers in each bucket
- Scan sorted buckets and combine them to produce the output array

Bucket Sort

Bucket Sort algorithm

```
1  $n \leftarrow$ length of $A[\dots]$ // input array
2  $k \leftarrow$ number of buckets
3  **for** $i \leftarrow 1$ **to** $n$
4          **do** insert $A[i]$ into list $B[\lfloor (kA[i])/m \rfloor]$
5  **for** $j \leftarrow 1$ **to** $k$
6          **do** sort list $B[j]$ using $Insertion - sort$
7  Concatenate the lists $B[1], B[2], \dots, B[k]$ together in order
```

Bucket Sort

- First 'for-loop' takes $O(n)$ time
- **Average case**: Line:5 takes $O(k)$ time in total. Line:6 takes $O(k * n/k) = O(n)$ in average case based on the assumption that the input elements are uniformly distributed over the range [1 m] or [0 1). So overall complexity becomes $O(n + k + n) = O(n)$
- **Worst case**: Worst case occurs when input elements does not satisfy the assumption that means they are not uniformly distributed over the range and all of them goes into one bucket. So, sorting elements at that bucket takes $O(n^2)$ in worst case. That means second 'for-loop' cause $O(n^2)$. Thus Bucket Sort takes $O(n + n^2) = O(n^2)$ in worst case.

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- Why do we recommend sorting *least-significant* digits first in radix sort?