Real Analysis 1

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Contents

Chapter 1

Introduction

1.1 Table Of Symbols

Symbol	Meaning	When It is used	Example
$\{\ldots\}$	Set notation	Used for creating a group	$A = \{a_1, a_2\}$
$\forall$	For all	Used for all elements in a group	$\forall$ students in the 11th grade
$\in$	Element of	Used to indicate a member is in a set	$\forall$ students $\in$ the 11th grade
$\setminus$	Without	Used to exclude an element from a set	Group of students $\setminus$ Romeo
$\mid$	Such that	Used when making conditions in a group	$A = \{x \mid x > 10\}$

Symbol	Meaning	When It is used	Example
$\subset$	Subset	Indicates that a set is a subset of another	$\mathbb{N} \subset \mathbb{Z}$
$\rightarrow$ / $\implies$	Implies	Used to show implication or cause	90% on exam $\implies A^+$
$\iff$	If and only if	Used when we say an argument happens $\iff$ another argument happens	n is even $\iff n = 2k$
$\exists$	There exists	Used when we say that something exists	$\exists n \in \mathbb{N}$ such that $n^2 > 4$

1.2 Groups

1.2.1 Definition

A group is represented by curly braces, with its elements separated by commas:

$$A = \{a_1, a_2, a_3, \dots, a_n\}$$

where A is the set name, and each a_i for $1 \leq i \leq n$ is an element of A.

Set names are typically capital letters, while elements are lowercase.

1.2.2 Examples of Groups

$$\{3, 7\} = \{7, 3\} = \{7, 7, 3\} = \{3, 3, 7\}$$

In set theory, order and repetition do not matter.

Examples:

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$$

$$\mathbb{Q} = \left\{\frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{Z}\backslash\{0\}\right\}$$

$$\mathbb{R} = \{x \mid -\infty < x < \infty\}$$

Example of Usage: $\forall x \in A,\ x > 10$. This means that every element in set A (let's name it x) is greater than 10.

$\exists\, y \in B$ such that y is odd, this means that exists an element in B, let's name it y, odd.

Whenever you see these symbols written in this order I want you to perceive it like this, because a lot of students when they begin to study **Infinitesimal Calculus** read symbols without understanding them.

1.2.3 Logical Operators and Proofs

Question 1

Given: n is odd. Prove that n^2 is odd.

Proof. Every odd number can be expressed as $2k + 1$, where $k \in \mathbb{N}$.

Let $n = 2k + 1$.

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \end{aligned}$$

This is clearly an even number $+1$, hence n^2 is odd. $\square$

Question 2

Given a number n, if n^2 is even, prove that n is even.

In the previous question, we proved that if n is odd, n^2 is odd.

Important to note: if $a \implies b$, then not $b \implies$ not a (this is logically equivalent).

Proof. n odd $\implies n^2$ is odd (Proved above).

This is equivalent to: n^2 is even $\implies n$ is even. □

1.2.4 Proving $\sqrt{2}$ is Irrational

We need to prove that $\sqrt{2} \notin \mathbb{Q}$.

Proof. Assume $\sqrt{2} = \frac{m}{n}$, where m and $n \neq 0$ are integers, and the fraction $\frac{m}{n}$ is at it's simplest form.

Squaring both sides:

$$2 = \frac{m^2}{n^2} \implies m^2 = 2n^2$$

Since m^2 is even, m must be even. Let $m = 2k$. Substitute into the equation:

$$(2k)^2 = 2n^2 \implies 4k^2 = 2n^2 \implies n^2 = 2k^2$$

Thus, n^2 is also even, so n is even. This contradicts our assumption that the fraction $\frac{m}{n}$ is at it's simplest form, so $\sqrt{2} \notin \mathbb{Q}$. □

1.3 Density of Rational Numbers

$\forall\, x < y \in \mathbb{R},\ \exists\, q \in \mathbb{Q}$ such that $x < q < y$

Proof.

$$x < y \Longrightarrow 0 < y - x$$

Then we can get the inverse of $y - x$ which is $\frac{1}{y-x}$

Based on the **Axiom Of Archimedes** that states: $\forall r \in \mathbb{R}\, \exists n \in \mathbb{N}$ such that $n > r$

$$\exists n \in \mathbb{N}\ s.t \quad n > \frac{1}{y - x} \iff ny - nx > 1$$

The distance between two real numbers is bigger than 1, that means: $\exists\, m \in \mathbb{N}$ such that $nx < m < ny$

$$nx < m < ny \iff x < \frac{m}{n} < y$$

$\square$

1.4 Absolute Value

1.4.1 Definition

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

1.4.2 Features without Proof

1. $|x| \geq 0$
2. $|x| \geq x$, $|x| \geq -x$
3. $|x| = |-x|$
4. $|x| = 0 \iff x = 0$
5. $|xy| = |x|\,|y|$
6. $|x+y| \leq |x| + |y|$ **Traingle Inequality 1(*)**
7. $\Big||x| - |y|\Big| \leq \Big|x - y\Big|$ **Traingle Inequality 2**
8. $|x| < M \iff -M < x < M$ **(*)**

1.5 Intervals

$$(a, b) = \{x \mid a < x < b\}$$

All numbers between a and b. Similarly:

$$[a, b] = \{x \mid a \leq x \leq b\}, \quad [a, b) = \{x \mid a \leq x < b\}, \quad (a, b] = \{x \mid a < x \leq b\}$$

$$(a, \infty) = \{x \mid a < x\}, \quad (-\infty, a) = \{x \mid x < a\}$$

1.5.1 Neighborhood of a Point

Given a point $x_0 \in \mathbb{R}$ and a number $\varepsilon > 0$, the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ is called the ε-neighborhood of x_0. **Example:** The interval $(3.5, 4.5)$ is the 0.5-neighborhood of 4.

1.5.2 More Features of Absolute Value

Claim: $|x - x_0| < \varepsilon \iff x \in (x_0 - \varepsilon, x_0 + \varepsilon)$

Proof. Given: $|x - x_0| < \varepsilon$

$$\iff -\varepsilon < x - x_0 < \varepsilon \quad \textbf{Feature 8}$$

$$\iff x_0 - \varepsilon < x < x_0 + \varepsilon$$

Which is the open interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ based of the definition of the open interval □

This means whenever we see the term $|x - x_0| < \varepsilon$ it is the same as saying $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ and vice versa.

1.6 Punctured Environment

A **punctured neighborhood** is the same as a normal neighborhood of a point, except that the point itself is excluded from the interval.

For a point $x_0 \in \mathbb{R}$ and a number $\varepsilon > 0$, the punctured neighborhood is denoted as:

$$(x_0 - \varepsilon, x_0 + \varepsilon) \backslash \{x_0\}$$

Example: For $x_0 = 4$ and $\varepsilon = 0.5$, the punctured neighborhood is:

$$(3.5, 4.5) \backslash \{4\}$$

In this case, the interval is the same as the neighborhood of 4, but the point 4 itself is excluded.

1.6.1 Bounded Real Groups

Group Bounded From Above

Given a group of **Real Elements** $A \subset \mathbb{R}$

A is bounded from above $\iff \exists\, M \in \mathbb{R}$ such that $\forall a \in A \; a \leq M$

Translation In English

A is bounded from above **If And Only If** exists a real number M such that for every element in A , M is bigger or equal to every element in A

Group Bounded From Below

A is bounded from above $\iff \exists\, m \in \mathbb{R}$ such that $\forall a \in A \; a \geq m$

Bounded Group

A is bounded $\iff$ $\exists\ m, M \in \mathbb{R}$ such that $\forall\ a \in A\ m < a < M$

A group is bounded $\iff$ it is bounded from above and below

Question:

A is bounded $\iff \exists\ K \in \mathbb{R}$ such that $\forall x \in A\ |x| \leq K$

This is an $\iff$ question, **If And Only If** question, which means that we have to proof both sides.

Proof. $\Longleftarrow$

Given: $\forall x \in A \quad |x| \leq K$

$$\Longrightarrow -K \leq x \leq K \quad \textbf{Feature 8}$$

$\forall x \in A\ -K \leq x \leq K$ which means that the group A is bounded

Now the other side of the proof

$\Longrightarrow$

Given: A is bounded

$$\Longrightarrow \exists\ m, M \in \mathbb{R} \quad \text{such that } \forall x \in A \quad m \leq x \leq M$$

Let $K = max\{|m|, |M|\}$

$M \leq |M|$ **Feature 2** $\Longrightarrow M \leq |M| \leq K \Longrightarrow M \leq K$

$-m \leq |m|$ **Feature 2** $\Longrightarrow\ m \geq -|m| \Longrightarrow -K \leq -|m| \leq m \Longrightarrow -K \leq m$

$$\Longrightarrow \forall x \in A \quad -K \leq m \leq x$$

And

$$\Longrightarrow \forall x \in A \quad x \leq M \leq K$$

$$\Longrightarrow \forall x \in A - K \leq m \leq x \leq M \leq K$$

$$\Longrightarrow -K \leq x \leq K$$

$$\Longrightarrow |x| \leq k \quad \textbf{Feature 8}$$

$\square$

Examples :

1. $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ bounded from below, and 1 is it's lower bound, on the other hand it is not bounded from above.

2. $(1, 4)$ is a bounded group, where 1 is it's lower bound, and 4 is it's upper bound.

1.7 Supermum and Infimum

1.7.1 Supermum

S is called the **Supermum** of group $A \iff$ it is the least upper bound of group A.

Example For Intuition

If we look at the open interval $B = (1, 4)$, 5 is an upper bound of B, 6 is an upper bound of B , 4.31 is an upper bound of B ...

4 is also an upper bound of B, but it is the least upper bound, meaning if we pick a number less than 4, let's say 3.7, then there exists an element in B such that that element is bigger than 3.7.

Understanding The Supermum

We know for a number to be called a **Supermum** of A.

First, it has to be an upper bound of A, meaning $\forall a \in A \quad a \leq S$

Second, it has to be the least upper bound, meaning as we discussed in the **Example**, If we pick a number lower than it, there exists an element in A that is bigger than that number.

How do we write it in maths terms/language?

We know that if we subtract the least upper bound from any positive number, then it's gonna be smaller than the least upper bound.

If we use our example, We picked 3.7, which is $4 - 0.3$, if we pick 3.9 it is $4 - 0.1$

which are not upper bounds.

For any positive number let's call it ε, if we subtract it from our **Supermum**, then there exists an element in our group that is bigger than the result of subtracting the **Supermum** from a positive number .

$\forall\, \varepsilon > 0\ \exists\, a \in A$ such that $\quad S - \varepsilon < a$

Definition Of The Supermum

S is the **Supermum** of group $A \iff$

1. $\forall\, a \in A \quad a \leq S$
2. $\forall\, \varepsilon > 0\ \exists\, a \in A$ such that $S - \varepsilon < a$

Given a group A, If these rules apply on a number S, then it is the **Supermum** of group A.

1.7.2 Infimum

The **Infimum** is the biggest lower bound

Example

If we look at the open interval $(1, 4)$, 1 is the least upper bound of the group $(1, 4)$.

Definition Of The Infimum

I is the **Infimum** of group $A \iff$

1. $\forall\, a \in A \quad I \leq a$

2. $\forall \varepsilon > 0 \quad \exists\, a \in A$ such that $a < I + \varepsilon$

1.8 Axiom Of Completeness

For all **Bounded** from above , **Not Empty** groups of real numbers exists a **Supermum** for that group.

For all **Bounded** from below , **Not Empty** groups of real numbers exists an **Infimum** for that group.

This is really important, because now we know that forevery bounded none empty group of real numbers exists a **Supermum** and an **Infimum**

If we look at the group $A = \{x \in \mathbb{Q} \mid x < \sqrt{2}\}$, it does not have a **Supermum** in $\mathbb{Q}$, because $\sqrt{2}$ as we proved is not rationl meaning $\notin \mathbb{Q}$.

1.9 Mathematical Induction

1.9.1 Introduction

We say that the claim $P(x)$ is true $\forall\ x \in A = \{a_1, a_2, a_3, ..., a_n\} \iff P(a_1), P(a_2), P(a_3), \ldots, P(a_n)$ are all true.

For example, we want to prove that the claim: $1 + 2 + 3 + \ldots + n = \frac{(1+n) \cdot n}{2}$ is true $\forall\ n \in \{1, 2, 3, 4, 5, 6, 7\}$

We can do this by checking if the claim $P(n):\ (1 + 2 + 3 + \ldots + n = \frac{(n+1) \cdot n}{2})$ is true, by checking each element in the set $\{1, 2, 3, \ldots, 8\}$ manually.

Meaning if we substitute 1 in the claim $1 = \frac{(1+1) \cdot 1}{2}$ then we can see that the claim is true for $n = 1$ or $P(1)$ is true.

$P(2) : (1 + 2) = \frac{(1+2) \cdot 2}{2}$ which is true.

$P(3) : (1 + 2 + 3) = \frac{(1+3) \cdot 3}{2}$ which is also true.

We can do this manually for each element in the set, but i'm not going to, because at this point I think you get the idea.

Now I want you to prove that the claim above is true $\forall\ n \in \mathbb{N}$.

Meaning we want to prove that $P(1), P(2), P(3), \ldots$ are all true.

We can't do this manually because there are infinitely many elements in $\mathbb{N}$

1.9.2 How Mathematical Induction Work

If we know that $P(1)$ is true, and we can prove that if given any natural number (let's name it) k, $P(k)$ is true and $P(k + 1)$ is true, then $\forall\ n \in \mathbb{N}\ P(n)$ is true.

Why does this work, If we know that $P(1)$ is true, then $P(2)$ is true, because we proved that if $\forall\, k \in \mathbb{N}\ P(k)$ is true $\Longrightarrow P(k+1)$ is true.

same goes for $P(3)$,

$P(2)$ is true $\Longrightarrow P(3)$ is true.

$P(3)$ is true $\Longrightarrow P(4)$ is true.

$P(4)$ is true $\Longrightarrow P(5)$ is true.

$$P(1) \Longrightarrow P(2) \Longrightarrow P(3) \Longrightarrow P(4) \Longrightarrow P(5) \Longrightarrow P(6) \ldots$$

As you can see by doing this we can prove for any natural number that a claim is true.

1.9.3 Mathematical Induction Formal Proof

You need to prove two things in order to prove that a claim for all natural numbers is true using Mathematical Induction.

1. Induction Base: you want to prove manually that $P(1)$ is true.
2. Induction Assumption: you want to prove that given any natural number $k, P(k)$ is true that $P(k+1)$ is true.

Question 1

We want to prove that

$$1+2+3+\ldots+n=\frac{(1+n)\cdot n}{2} \qquad \forall\, n\in\mathbb{N}$$

Step 1: Induction Base

$$1=\frac{1+1}{2}\Longrightarrow P(1)\text{ is true .}$$

Step 2: Induction Assumption

Let's assume that given any number $k\in\mathbb{N}$ that $P(k)$ is true.

$$\Longrightarrow 1+2+3+\ldots+k=\frac{(1+k)\cdot k}{2}$$

Remember we want to prove that $P(k+1)$ using the Induction Assumption.

Meaning that:

$$1+2+3+\ldots+k+(k+1)=\frac{(1+(k+1))\cdot(k+1)}{2}=\frac{(2+k)\cdot(k+1)}{2}$$

We know that:

$$1+2+3+\ldots+k=\frac{(1+k)\cdot k}{2}$$

Adding $(k+1)$ to both sides.

$$(1+2+3+\ldots+k)+(k+1)=\frac{(1+k)\cdot k}{2}+(k+1)=\frac{(1+k)\cdot k}{2}+\frac{2\cdot(k+1)}{2}$$

$\frac{k+1}{2}$ is a common factor

$$1+2+3+\ldots+k+(k+1)=\frac{(k+1)}{2}\cdot(k+2)=\frac{(k+1)(k+2)}{2}$$

And with that we proved that: $1+2+3+\ldots+n=\frac{(1+n)\cdot n}{2} \qquad \forall\, n\in\mathbb{N}$

Definition Of A Finite Set

A finite set A means that the set A has a finite amount of elements.

$|A|$ gives us how many elements are in a set.

Elements In Sets Are Always Real Numbers In This Book.

Question 2

Given a finite set of real numbers $A=\{a_1,a_2,a_3,\ldots,a_n\}$, we want to prove using Mathematical Induction that it has a maximum and a minimum.

Another way to rephrase this question is to prove that $\forall\, n\in\mathbb{N}$ such that $|A|=n$, A has a maximum and a minimum.

Step 1: Induction Base

$A_1=\{a_1\}\Longrightarrow max\{A_1\}=a_1,\ min\{A_1\}=a_1\Longrightarrow P(1)$ is true.

Step 2: Induction Assumption

Let's assume that given any natural number k that $P(k)$ is true.

Meaning:

$$A_k = \{a_1, a_2, a_3, \ldots, a_k\} \text{ has a max } M_1 \text{ and a minimum } m_1$$

$max\{A_1\} = M_1,\ min\{A_1\} = m_1$

We want to prove that for all sets of real numbers A_{k+1} such that $|A_{k+1}| = k+1$ A_{k+1} has a maximum and a minimum.

$$A_{k+1} = \{a_1, a_2, a_3, \ldots, a_k, a_{k+1}\}$$

From the Induction Assumption we know that the elements $\{a_1, a_2, a_3, \ldots, a_k\}$ have a maximum M_1, and a minimum m_1.

$$max\{A_{k+1}\} = max\{M_1, a_{k+1}\}, \qquad min\{A_{k+1}\} = min\{m_1, a_{k+1}\}$$

With that we proved that a finite set of real numbers always has a minimum and a maximum.

Chapter 2

Sequences

2.1 What is a Sequence

2.1.1 Definition

A sequence is a list of numbers in a specified order. The different numbers occurring in a sequence are called the terms of the sequence.

$$a_1, a_2, a_3, a_4 \ldots, a_n$$

Where the first term of the sequence is a_1, second term is a_2, n'th term is a_n. We write a sequence using these notations:

$$a_n,\ (a_n),\ \{a_n\},\ \{a_n\}_{n=1}^{\infty}$$

Examples:

1. $a_n = n = 1, 2, 3, 4, \ldots$

2. $a_n = n^2 = 1, 4, 9, 16, \ldots$

3. $a_n = (-1)^{n+1} = 1, -1, 1, -1, \ldots$

4. $a_n = \frac{1}{n} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

5. $a_n = \frac{(-1)^{n+1}}{n} = 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots$

6. $a_n = 1$ if n is even, n if n is odd $= 1, 1, 3, 1, 5, 1, 7. \ldots$

7. $a_n = 2, 3, 5, 7, 11, 13, \ldots$ this is the sequence for prime numbers, it does not have a formula!

2.2 Limit Of A Sequence

2.2.1 Definition

L is the limit of a sequence a_n, or a_n approaches L , $\lim\limits_{n\to\infty} a_n = L$, $a_n \to L \iff$

$\forall\, \varepsilon > 0\,, \exists\, N \in \mathbb{R}$, such that $\forall\, n > N \Longrightarrow |a_n - L| < \varepsilon$

Some people might be reading this and thinking what is all of this, so i'm gonna translate it for you. For any positive number let's call it ε, there exists a number let's call it N such that for any index that is bigger than that number (N) the following happens $\Longrightarrow |a_n - L| < \varepsilon$, this is a literal translation from maths to english.

2.2.2 Understanding The Limit

Given any positive number ε, there exists a number N such that for all indexes bigger than N all the terms of the sequence apply this:

$$|a_n - L| < \varepsilon$$

Which also means:

$$-\varepsilon < a_n - L < \varepsilon \quad \textbf{Feature 8}$$

$$\Longleftrightarrow$$

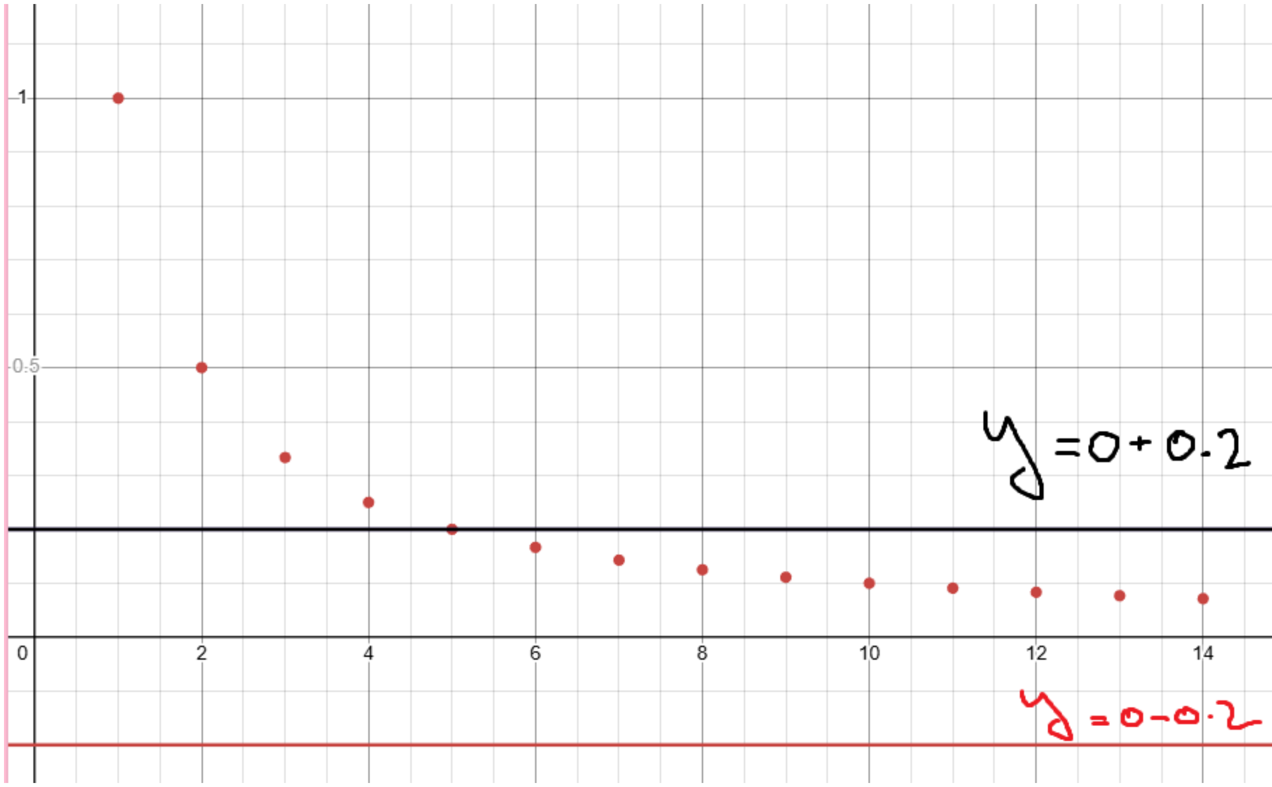
$$L - \varepsilon < a_n < L + \varepsilon$$

which means after a certain number N all the terms of the sequence are going to be located between $L - \varepsilon$ and $L + \varepsilon$ or, $\forall\, \varepsilon > 0\; \exists\, N$, such that $\forall\, n > N \Longrightarrow a_n \in (L - \varepsilon, L + \varepsilon)$

2.2.3 Visually Seeing The Limit

Examples:

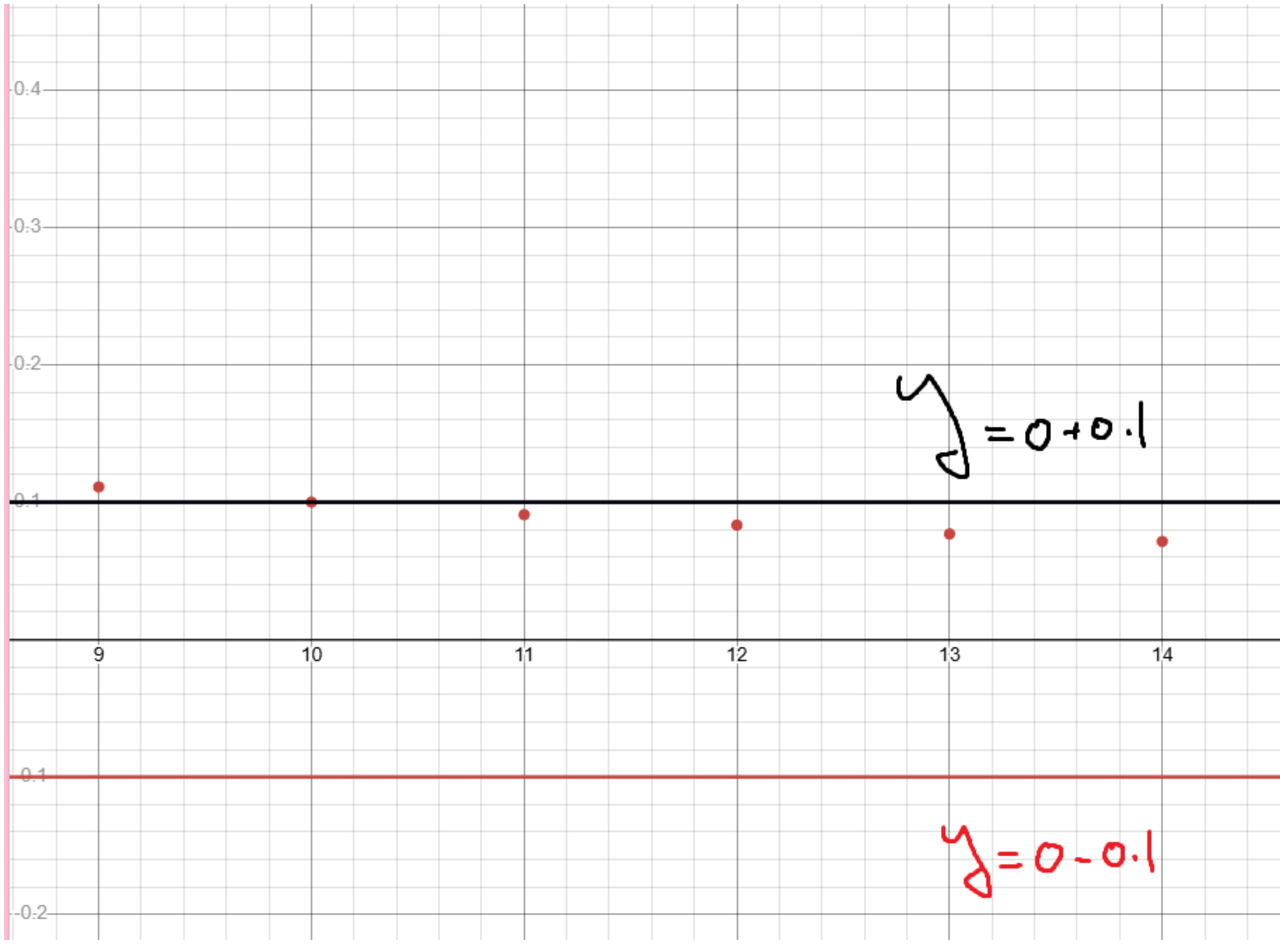
If we look at the sequence $a_n = \frac{1}{n} = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ we can tell that it approaches 0, because the fractions are getting smaller and smaller each term.



Given an $\varepsilon = 0.2 \ \exists \ N = 5$, such that for all indexes bigger than 5

$\implies 0 - 0.2 < a_n < 0 + 0.2$

The definition of a limit states, that given any positive number ε, therefor we can pick any positive number ε no matter how big and small it is, and there is always going to be a number N such that for all the indexes bigger than N such that $a_n \in (0 - \varepsilon, 0 + \varepsilon)$



Given an $\varepsilon = 0.1$ $\exists$ $N = 10$, such that for all indexes bigger than 10

$\implies 0 - 0.1 < a_n < 0 + 0.1$

2.2.4 How To Prove The Limit Of A Sequence Is L

In order to prove $\lim_{n\to\infty} a_n = L$, a sequence a_n approaches L, we first have to understand how the definition of a limit works.

In basic terms:

For any positive number let's name it ε, exists a number let's name it $N \ldots$

Meaning we have to prove exists a number N such that for all the indexes(n) bigger than $N \implies$

$$|a_n - L| < \varepsilon$$

We prove something exists finding out it's value, meaning we have to find the value of N.

Here is how i like to prove these type of questions, which are to prove that a sequence a_n approaches L, or for short $\lim_{n\to\infty} a_n = L,\ a_n \to L$

Question 1: Proving $\lim_{n\to\infty} \frac{1}{n} = 0$

At first i like to write down the definition of an approaching sequence (Almost)

$\forall\ \varepsilon > 0,\ \exists\ N = \square$, such that $\forall\ n > N \implies$

as you can see so far all good, but i did not write what N is equal to, i leave it blank, because i do not know it's value yet.

just so you remember, we want to because prove:

$\forall\ \varepsilon > 0,\ \exists\ N$, such that $\forall\ n > N \implies$

$$|a_n - L| = |\frac{1}{n} - 0| < \varepsilon$$

In the definition we say that N exists, but we don't know what it is in this scenario, that's what we're going to find.

$\forall\, \varepsilon > 0,\ \exists\, N = \square$, such that $\forall\, n > N \implies$

$$|\frac{1}{n} - 0| = \frac{1}{n} < \frac{1}{N}$$

Because we're looking for all $n > N$, meaning we're looking at terms that are bigger than N, if $n > N$, then $\frac{1}{n} < \frac{1}{N}$

now if $\frac{1}{N} = \varepsilon$ then $\frac{1}{n} < \varepsilon$ and $|\frac{1}{n} - 0| < \varepsilon$.

$\frac{1}{N} = \varepsilon \iff N = \frac{1}{\varepsilon}$, now sub in $N = \frac{1}{\varepsilon}$, here's what the reader is gonna see.

$\forall\, \varepsilon > 0,\ \exists\, N = \frac{1}{\varepsilon}$, such that $\forall\, n > N \implies$

$$|\frac{1}{n} - 0| = \frac{1}{n} < \frac{1}{N} = \varepsilon$$

Question 2: Proving $\lim\limits_{n\to\infty} c = c$

Given a sequence $a_n = c,\ c \in \mathbb{R}$ a constant sequence, we need to prove that $\lim\limits_{n\to\infty} a_n = c$.

as we said we first write the definition while leaving N blank to fill it in when we find it.

$\forall\, \varepsilon > 0,\ \exists\, N = \square$, such that $\forall\, n > N \implies$

$$|c - c| = 0 < \varepsilon$$

for any input(n), therefor N can be any number, let's pick 1 here's what the reader is gonna see after you fill in for $N = 1$

$\forall\, \varepsilon > 0,\ \exists\, N = 1$, such that $\forall\, n > N \Longrightarrow$

$$|c - c| = 0 < \varepsilon$$

Question 3: Proving $\lim\limits_{n\to\infty} \dfrac{3n^2 - 5n + 4}{n^2 + 4n - 5} = 3$

Given the sequence $a_n = \frac{3n^2-5n+4}{n^2+4n-5}$, we need to prove that $\lim\limits_{n\to\infty} a_n = 3$

Again, we start by writing the definition of a limit while leaving N blank to fill it in when we find it.

$\forall\, \varepsilon > 0\ \exists\, N = \square$, such that $\forall\, n > N \Longrightarrow$

1st Step: Simplify The Expression

$$|\frac{3n^2 - 5n + 4}{n^2 + 4n - 5} - 3| = |\frac{3n^2 - 5n + 4}{n^2 + 4n - 5} - 3 \cdot \frac{n^2 + 4n - 5}{n^2 + 4n - 5}| = |\frac{3n^2 - 5n + 4 - 3n^2 - 12n + 15}{n^2 + 4n - 5}|$$

$$|\frac{-17n + 19}{n^2 + 4n - 5}| = |\frac{-1 \cdot (17n - 19)}{n^2 + 4n - 5}| = |-1| \cdot |\frac{17n - 19}{n^2 + 4n - 5}| = |\frac{17n - 19}{n^2 + 4n - 5}| \quad \textbf{(Feature 5)}$$

2nd Step: Get Rid Of The Absolute Value

We get rid of the Absolute Value by making whatever is inside the Absolute Value positive, **Reminder**: $|a| = a \iff a \geq 0$

If we look at a fraction $\frac{a}{b},\ b \neq 0$, if a gets bigger, then the fractions it self gets bigger.

For all $n > 2 \Longrightarrow 17n - 19 > 0 \Longrightarrow |17n - 19| = 17n - 19 < 17n$

$$\Longrightarrow |\frac{17n - 19}{n^2 + 4n - 5}| = \frac{17n - 19}{|n^2 + 4n - 5|} < \frac{17n}{|n^2 + 4n - 5|}$$

For all $n > 2$ the denominator $n^2 + 4n - 5 > 0 \Longrightarrow |n^2 + 4n - 5| = n^2 + 4n - 5$

$\forall\, n > 2 \Longrightarrow 4n - 5 > 0 \iff n^2 + 4n - 5 > n^2 \iff \frac{1}{n^2+4n-5} < \frac{1}{n^2}$

$$\frac{17n}{|n^2 + 4n - 5|} < \frac{17n}{n^2} = \frac{17}{n}$$

Where are looking for all the indexes $(n) > N \iff \frac{1}{n} < \frac{1}{N}$

$$\frac{17}{n} < \frac{17}{N}$$

If $\frac{17}{N} = \varepsilon$ then the inital term $|\frac{3n^2-5n+4}{n^2+4n-5} - 3| < \varepsilon \Longrightarrow \frac{17}{N} = \varepsilon \iff N = \frac{17}{\varepsilon}$

$$\frac{17}{n} < \frac{17}{N} < \varepsilon$$

3rd Step: Picking N

All of this works if all the indexes are bigger than 2, so we can't say that $N = \frac{17}{\varepsilon}$ is a valid answer, because we're saying that this statment works for every positive number ε, some values for $\frac{17}{\varepsilon}$ are less than 1.

There is a solution for this.

$\forall\, \varepsilon > 0,\ \exists\, N = max\{\frac{17}{\varepsilon}, 2\}$, such that $\forall\, n > N \Longrightarrow$

$$|\frac{3n^2 - 5n + 4}{n^2 + 4n - 5} - 3| < \ldots < \frac{17}{N} = \varepsilon$$

If $\frac{17}{\varepsilon} > 2$ then if $N = \frac{17}{\varepsilon} \Longrightarrow |\frac{3n^2-5n+4}{n^2+4n-5} - 3| < \varepsilon$

Otherwise, meaning if $\frac{17}{\varepsilon} \leq 2 \iff \frac{17}{2} \leq \varepsilon$ and for $N = 2 \Longrightarrow \frac{17}{N} = \frac{17}{2} \leq \varepsilon$

2.3 The Limit Of A Sequence Is Unique

2.3.1 Proof In Plain English

If a sequence has a limit, then it is **Unique**. Before we got to the mathematical proof, i'm gonna explain in plain english using the knowledge we know so far to prove how the limit of a sequence is unique.

Given a sequence $a_n \to L$, Using the definition of a limit: $\forall\ \varepsilon > 0,\ \exists\ \mathbb{N} > 0$, such that for all indexes(n) bigger than $N \implies$

$$|a_n - L| < \varepsilon \iff a_n \in (L - \varepsilon, L + \varepsilon) \equiv L - \varepsilon < a_n < L + \varepsilon$$

Meaning after a certain point N every term of the sequence a_n is in

$$|a_n - L| < \varepsilon \ \equiv\ L - \varepsilon < a_n < L + \varepsilon$$

If there are two limits $(L, K), L < K$, then the sequence is gonna be approaching two limits L and K.

First Limit

$a_n \to K \implies \forall\ \varepsilon > 0, \exists\ N_1$, such that $\forall n > N_1 \implies$

$$|a_n - K| < \varepsilon \iff K - \varepsilon < a_n < K + \varepsilon$$

Second Limit

$a_n \to L \implies \forall\, \varepsilon > 0, \exists\, N_2$, such that $\forall n > N_2 \implies$

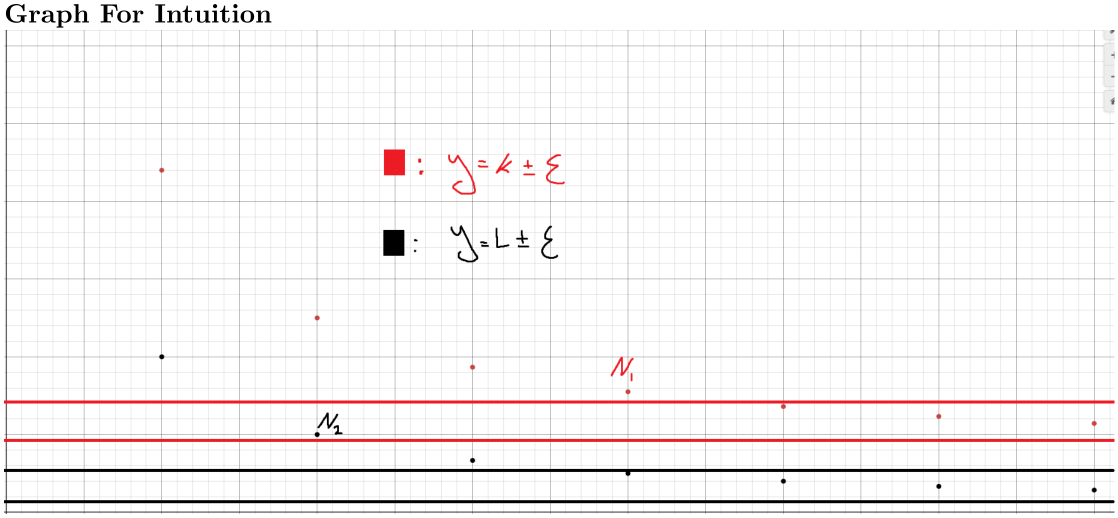
$$|a_n - L| < \varepsilon \iff L - \varepsilon < a_n < L + \varepsilon$$

The Definition Of A Limit states that $\forall\, \varepsilon > 0\; \exists\, N$, such that $\forall\, n > N \implies$

$$|b_n - B| < \varepsilon$$

To remind you it simply means, giving a any positive number **We Named It** ε,... $\implies |b_n - B| < \varepsilon$, this means that it must work for $1, 2, \frac{1}{2}, 0.1292$ as well, as long as it is a positive number. If we pick a small enough number(ε), we can prove that the terms of the sequence a_n are gonna be located in two completely different places, which is not possible.

Graph For Intuition



2.3.2 Mathematical Proof

Given: $a_n \to K \implies$

$\forall\, \varepsilon > 0,\ \exists\, N_1$, such that $\forall\, n > N_1 \implies$

$$|a_n - K| < \varepsilon\ \equiv K - \varepsilon < a_n < K + \varepsilon$$

Given: $a_n \to L \implies$

$\forall\, \varepsilon > 0,\ \exists\, N_2$, such that $\forall\, n > N_2 \implies$

$$|a_n - L| < \varepsilon\ \equiv L - \varepsilon < a_n < L + \varepsilon$$

Becuase it works for every positive number ε then it must work for $\frac{K-L}{3}$

$K > L \iff K - L > 0 \iff \frac{K-L}{3} > 0$

For $\varepsilon = \frac{K-L}{3}\ \exists\, N = max\{N_1, N_2\}$ such that $\forall\, n > N \implies$

$$K - \varepsilon < a_n < K + \varepsilon\ \equiv\ K - \frac{K-L}{3} < a_n < K + \frac{K-L}{3}$$

$$\frac{3K}{3} - \frac{K-L}{3} < a_n < \frac{3K}{3} + \frac{K-L}{3}\ \equiv\ \frac{2K+L}{3} < a_n < \frac{4K-L}{3}$$

$$\frac{2K+L}{3} < a_n < \frac{4K-L}{3}$$

We know that this statment is true, because if we're looking for all indexes bigger than $max\{N_1, N_2\}$ then we're looking for all indexes bigger than N_1. At the same time, we know that if we're looking at all the indexes bigger than

$max\{N_1, N_2\}$, then we're looking for all the indexes bigger than $N_2 \Longrightarrow$

$$L - \frac{K-L}{3} < a_n < L + \frac{K-L}{3} \;\equiv\; \frac{3L}{3} - \frac{K-L}{3} < a_n < \frac{3L}{3} + \frac{K-L}{3}$$

$$\frac{4L-K}{3} < a_n < \frac{K+2L}{3}$$

Given that $L < K \Longrightarrow \frac{K+2L}{3} < \frac{2K+L}{3} \Longrightarrow$

$$\frac{4L-K}{3} < a_n < \frac{K+2L}{3} < \frac{2K+L}{3} < a_n < \frac{4K-L}{3}$$

Meaning for all the indexes bigger than N the terms of the sequence are located in two different places, which is a contradiction.

Note: This is not an easy proof to grasp and understand, if you did not understand it at first, give it some time, go over it a couple of time, maybe sleep on it, but rest assured when you're reviewing the material in a couple of weeks, this proof is gonna be as smooth as butter.

2.4 Bounded Sequences

2.4.1 Definitions

1. A sequence a_n is bounded from above $\iff \exists\ M \in \mathbb{R}$, such that $\forall n \implies a_n \leq M$
2. A sequence a_n is bounded from below $\iff \exists\ m \in \mathbb{R}$, such that $\forall n \implies m \leq a_n$
3. A sequence a_n is bounded $\iff$ it is bounded from above and below

Practice Question

a_n is bounded $\iff \exists\ K \in \mathbb{R}$ such that $\forall\ n \implies |a_n| \leq K$

This is an identical question to the question in the **Groups** section that you should be able to solve on your own.

2.4.2 Equivalent Definition For The Limit Of A Sequence

$\forall\ \varepsilon > 0,\ \exists\ N \in \mathbb{N}$, such that $\forall n > N \implies$

$$|a_n - L| < \varepsilon$$

In the previous definition states that exists a real number N, but in this one it states that exists a natural number N, they're two different definition, we want to prove that they are equivalent.

Floor/Ceiling Functions

$$\lfloor x \rfloor = max\{m \in \mathbb{Z} \mid m \leq x\}$$

$$\lceil x \rceil = min\{m \in \mathbb{Z} \mid m \geq x\}$$

The Floor of x, $\lfloor x \rfloor$ returns the closest integer to x from below

The Ceiling of x, $\lceil x \rceil$ returns the closest integer to x from above

Examples:

$$\lfloor 5.4 \rfloor = 5,\ \lfloor -3.2 \rfloor = -4,\ \lfloor 3 \rfloor = 3$$

$$\lceil 5.4 \rceil = 6,\ \lceil -3.2 \rceil = -3,\ \lceil 3 \rceil = 3$$

Features of $\lfloor x \rfloor,\ \lceil x \rceil$

$$\lfloor x \rfloor \leq x \qquad x \leq \lceil x \rceil$$

If we look at any real number $N \in \mathbb{R},\ N \leq \lceil N \rceil,\ \lceil N \rceil \in \mathbb{N}$ Using the definition of a limit we know:

$\forall\, \varepsilon > 0\,, \exists\, N \in \mathbb{R}$, such that $\forall\, n > N \implies$

$$|a_n - L| < \varepsilon$$

$\forall\ n > \lceil N \rceil$, meaning we're looking at the indexes bigger than $\lceil N \rceil \implies N \leq \lceil N \rceil < n$, meaning we're also looking at the indexes bigger than $N \implies$

$$|a_n - L| < \varepsilon$$

Theorem:

$a_n \to L \implies a_n$ is bounded

Proof. $a_n \to L \implies$ for $\varepsilon = 1$, $\exists\ N \in \mathbb{N}$, such that $\forall n > N \implies$

$$|a_n - L| < 1 \iff L - 1 < a_n < L + 1$$

Since the sequence approaches L this means that for any positive number ε, $\exists N \ldots$

Therefor i can pick any positive number i want, and it is still going to work, because it works with any positive number.

Note: You can pick any positive number instead of 1 and the proof still works the same.

If we look at the terms of the sequence:

$$a_1, a_2, a_3, \ldots, a_N, a_{N+1}, \ldots, a_n$$

We know for a fact that forall indexes after N the sequence is bounded because of the definition of the limit $\forall n > N \implies L - 1 < a_n < L + 1$

That leaves us with the first N terms, if we look at the group $\{a_1, a_2, \ldots, a_N\}$, it is finite group of real numbers, meaning it has a maximum and a minimum.

$$K = max\{a_1, a_2, \ldots, a_N\},\ k = min\{a_1, a_2, \ldots, a_N\}$$

$\forall\ 1 \le n \le N \Longrightarrow k \le a_n \le K$, although the first N terms are bounded, and all the terms after Nth term are bounded as well, this does not prove that the sequence is bounded.

Reminder:

A sequence is bounded $\iff \exists\ m, M$, such that $\forall n \Longrightarrow m \le a_n \le M$

Let $M = max\{K, L+1\}$, $m = min\{k, L-1\} \Longrightarrow$

$$\forall\ n \quad m \le a_n \le M$$

$\square$

2.5 Limit Theorems

2.5.1 Limit Arithmetics

Given: $a_n \to L,\ b_n \to K \Longrightarrow$

1. $\forall c \in \mathbb{R} \Longrightarrow c \cdot a_n \to c \cdot L$
2. $a_n + b_n \to L + K$
3. $a_n \cdot b_n \to L \cdot K$

4. If $k \neq 0, \forall\, n\; b_n \neq 0 \implies \frac{a_n}{b_n} \to \frac{L}{K}$

5. $a_n \to L \implies |a_n| \to |L|$

6. $a_n \to 0 \iff |a_n| \to 0$

7. $a_n \to L,\; L \geq 0,\; a_n \geq 0\; \forall\, n \implies \sqrt{a_n} \to \sqrt{L}$

Proof 1

We want to prove that $c \cdot a_n \to c \cdot L$ meaning:

We want to prove $\forall\, \varepsilon > 0,\; \exists\, N \in \mathbb{R}$, such that $\forall n > N \implies$

$$|c \cdot a_n - c \cdot L| = |c \cdot (a_n - L)| = |c| \cdot |a_n - L| < \varepsilon$$

$$\iff |a_n - L| < \frac{\varepsilon}{|c|} \quad \text{For } c \neq 0$$

$a_n \to L \implies$ For $\frac{\varepsilon}{|c|}$, $\exists\, N_1 \in \mathbb{R}$, such that $\forall n > N \implies$

$$|a_n - L| < \frac{\varepsilon}{|c|}$$

$\exists\, N = N_1$ such that $\forall n > N_1 \implies$

$$|c \cdot a_n - c \cdot L| < \varepsilon$$

Remember the definition of a limit works for any positive number, we just named it ε that's why we say $\forall\, \varepsilon > 0$, as you can see $\frac{\varepsilon}{|c|}$ is a positive number, this is why we can pick it to be our positive number.

The theorem works for $c \neq 0$ although it states that $\forall\, c \in \mathbb{R}$, meaning we have to prove that it works for $c = 0$.

For $c = 0,\ 0 \cdot a_n = 0$, the sequence $0 \cdot a_n$ is a constant sequence and we proved

$\forall\, c \in \mathbb{R}\ \lim\limits_{n\to\infty} c = c \Longrightarrow 0 \cdot a_n \to 0$

Proof 2

We want to prove $a_n + b_n \to L + K$

We want to prove that $\forall\, \varepsilon > 0,\ \exists N \in \mathbb{R},$ such that $\forall\, n > N \Longrightarrow$

$$|(a_n + b_n) - (L + K)| = |(a_n - L) + (b_n - K)|$$

$$|(a_n - L) + (b_n - K)| \leq |a_n - L| + |b_n - K| < \varepsilon \quad \textbf{Feature 6}$$

$$\Longleftrightarrow\ |a_n - L| < \frac{\varepsilon}{2} \text{ And } |b_n - K| < \frac{\varepsilon}{2}$$

$a_n \to L \Longrightarrow$ For any positive number the definition of a limit works on it.

For $\frac{\varepsilon}{2}$ $\exists\, N_1,$ such that $\forall n > N_1 \Longrightarrow$

$$|a_n - L| < \frac{\varepsilon}{2}$$

$b_n \to K \Longrightarrow$ For any positive number the definition of a limit works on it.

For $\frac{\varepsilon}{2}$ $\exists\, N_2,$ such that $\forall n > N_2 \Longrightarrow$

$$|b_n - K| < \frac{\varepsilon}{2}$$

For $N = max\{N_1, N_2\}$, $\forall\, n > N$ for all indexes bigger than N are also bigger or equal to N_1 and $N_2 \Longrightarrow$

$$|a_n - L| < \frac{\varepsilon}{2} \quad \text{And} \quad |b_n - K| < \frac{\varepsilon}{2}$$

Proof 3

We want to prove that $a_n \cdot b_n \to L \cdot K$

We want to prove that $\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{R},$ such that $\forall\, n > N \Longrightarrow$

$$|a_n \cdot b_n - L \cdot K| = |a_n b_n - a_n K + a_n K - LK|$$

$$|a_n b_n - a_n K + a_n K - LK| = |a_n(b_n - K) + K(a_n - L)|$$

$$|a_n(b_n - K) + K(a_n - L)| \leq |a_n(b_n - K)| + |K(a_n - L)| \quad \textbf{Feature 6}$$

$$|a_n(b_n - K)| + |K(a_n - L)| < \varepsilon$$

We want to prove that $|a_n(b_n - K)| + |K(a_n - L)| < \varepsilon$ in order to do so we have to prove that $|a_n(b_n - K)| < \frac{\varepsilon}{2}$ and $|K(a_n - L)| < \frac{\varepsilon}{2}$

a_n is convergent, that means that it is bounded, we proved earlier that every convergent sequence is bounded.

$\Longrightarrow \exists\, M \in \mathbb{R}$ such that $|a_n| < M$ and $|K| < M$ **You'll see why later**.

$$\Longrightarrow |a_n(b_n - K)| \leq |M(b_n - K)| = |M| \cdot |b_n - K|$$

$$|M| \cdot |b_n - K| < \frac{\varepsilon}{2} \iff |b_n - K| < \frac{\varepsilon}{2M}$$

This is what we want to prove, important to note that $\frac{\varepsilon}{2M}$ is positive.

$b_n \to K$, Which is why we can say for $\frac{\varepsilon}{2M}$, $\exists\, N_1 \in \mathbb{R}$, such that $\forall\, n > N_1 \implies$

$$|b_n - k| < \frac{\varepsilon}{2M}$$

$$|K(a_n - L)| = |K| \cdot |a_n - L|$$

$$|K| \cdot |a_n - L| < \frac{\varepsilon}{2} \iff |M| \cdot |a_n - L| < \frac{\varepsilon}{2}$$

$$|M| \cdot |a_n - L| < \varepsilon \iff |a_n - L| < \frac{\varepsilon}{2M}$$

We could've continued to prove this theory without getting rid of the K, but if we haven't, we would've had to prove the case were $K = 0$

What would've happened:

$$|K| \cdot |a_n - L| < \frac{\varepsilon}{2} \iff |a_n - L| < \frac{\varepsilon}{2K}$$

Which is true for $K \neq 0$, then we would have another proof for when $K = 0$, now if we get a number M which bigger than $|K|$, then this number M is bigger than 0, because $0 \leq |K| < M \implies 0 < M$

We want to prove that $|a_n - L| < \frac{\varepsilon}{2M}$, important to note that $\frac{\varepsilon}{2M}$ is positive, $a_n \to L$, Which is why we can say

For $\frac{\varepsilon}{2M}$, $\exists\, N_2 \in \mathbb{R},$ such that $\forall n > N_2 \Longrightarrow$

$$|a_n - L| < \frac{\varepsilon}{2M}$$

$\forall\, \varepsilon > 0,\ \exists\, N = max\{N_1, N_2\}$, such that $\forall\, n > N \Longrightarrow$

$$|a_n b_n - LK| < \varepsilon$$

Proof 5

We want to prove that $\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{R},$ such that $\forall\, n > N \Longrightarrow$

$$\Big||a_n| - |L|\Big| \leq \Big|a_n - L\Big| < \varepsilon \quad \textbf{Feature 7}$$

We want to prove that $|a_n - L| < \varepsilon$

$a_n \to L \Longrightarrow \forall\, \varepsilon > 0,\ \exists N \in \mathbb{R},$ such that $\forall n > N \Longrightarrow$

$$|a_n - L| < \varepsilon$$

Proof 7

We want to prove $\sqrt{a_n} \to \sqrt{L}$, given that $L \geq 0,\ a_n \geq 0\ \forall\, n$

We want to prove that $\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{R},$ such that $\forall\, n > N \Longrightarrow$

$$|\sqrt{a_n} - \sqrt{L}| = \Big|\frac{(\sqrt{a_n} - \sqrt{L}) \cdot (\sqrt{a_n} + \sqrt{L})}{\sqrt{a_n} + \sqrt{L}}\Big| = \Big|\frac{a_n - L}{\sqrt{a_n} + \sqrt{L}}\Big|$$

$a_n \geq 0 \; \forall \; n \Longrightarrow \sqrt{a_n} + \sqrt{L} \leq \sqrt{L} \iff \frac{1}{\sqrt{a_n}+\sqrt{L}} \leq \frac{1}{\sqrt{L}} \; \forall \; L \neq 0$

$$\Big|\frac{a_n - L}{\sqrt{a_n} + \sqrt{L}}\Big| \leq \Big|\frac{a_n - L}{\sqrt{L}}\Big| = \frac{|a_n - L|}{\sqrt{L}}$$

$$\frac{|a_n - L|}{\sqrt{L}} < \varepsilon \iff |a_n - L| < \varepsilon\sqrt{L}$$

For $\varepsilon\sqrt{L}$, $\exists \; N \in \mathbb{R}$, such that $\forall \; n > N \Longrightarrow$

$$|a_n - L| < \varepsilon\sqrt{L}$$

We have to prove when $L = 0$, we have to prove that $\forall \; \varepsilon > 0$, $\exists \; N \in \mathbb{R}$, such that $\forall \; n > N \Longrightarrow$

$$|\sqrt{a_n} - \sqrt{0}| = |\sqrt{a_n}| = \sqrt{a_n}$$

$$\sqrt{a_n} < \varepsilon \iff a_n < \varepsilon^2$$

For ε^2, $\exists \; N \in \mathbb{R}$, such that $\forall n > N \Longrightarrow$

$$|a_n - 0| = a_n < \varepsilon^2$$

2.5.2 The 0 - Bounded Theorem

Given: $a_n \to 0,\ b_n$ is bounded $\Longrightarrow a_n b_n \to 0$

Proof. We want to prove $\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{R},$ such that $\forall\, n > N \Longrightarrow$

$$|a_n b_n - 0| = |a_n b_n| = |a_n| \cdot |b_n| < \varepsilon$$

b_n is bounded $\Longrightarrow$

$$\exists\, M \in \mathbb{R}, \text{ such that } |b_n| < M$$

$$\Longrightarrow |a_n| \cdot |b_n| \leq M \cdot |a_n|$$

$$M|a_n| < \varepsilon \iff |a_n| < \frac{\varepsilon}{M}$$

Meaning we need to prove that $|a_n| < \frac{\varepsilon}{M}$, since $a_n \to 0$, for $\frac{\varepsilon}{M}$, $\exists\, N \in \mathbb{R},$ such that $\forall\, n > N \Longrightarrow$

$$|a_n - 0| = |a_n| < \frac{\varepsilon}{M}$$

□

2.5.3 Sandwich Theorem

Given: $\forall\, n \quad a_n \leq b_n \leq c_n,$ and $a_n \to L,\ c_n \to L \Longrightarrow b_n \to L$

$b_n \to L \iff \forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{R},$ such that $\forall\, n > N \Longrightarrow$

$$|b_n - L| < \varepsilon$$

This is what it means for a sequence to convergent to L, which is what we want to prove.

$a_n \to L \implies$

$\forall\, \varepsilon > 0,\ \exists\, N_1 \in \mathbb{R},\ \text{ such that } \forall\, n > N \implies$

$$|a_n - L| < \varepsilon \iff L - \varepsilon < a_n < L + \varepsilon$$

$c_n \to L \implies$

$\forall\, \varepsilon > 0,\ \exists\, N_2 \in \mathbb{R},\ \text{ such that } \forall\, n > N \implies$

$$|c_n - L| < \varepsilon \iff L - \varepsilon < c_n < L + \varepsilon$$

$\forall\, n > N_1 \implies$

$$L - \varepsilon < a_n \leq b_n$$

$\forall\, n > N_2 \implies$

$$b_n \leq c_n < L + \varepsilon$$

$\exists\, N = max\{N_1, N_2\},\ \text{ such that } \forall\, n > N \implies$

$$L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$$

$$\implies L - \varepsilon < b_n < L + \varepsilon$$

2.5.4 More Limit Thoerems

Theroem 1

$a_n \to L > 0 \Longrightarrow \exists\, N$, such that $\forall n > N \Longrightarrow a_n > 0$

$$a_n \to L \Longrightarrow \forall\, \varepsilon > 0,\ \exists\, N_1, \text{ such that } \forall\, n > N \Longrightarrow$$

$$|a_n - L| < \varepsilon \iff L - \varepsilon < a_n < L + \varepsilon$$

$L > 0 \Longrightarrow \frac{L}{2} > 0$, For $\varepsilon = \frac{L}{2}$, $\exists\, N_1$, such that $\forall n > N_1 \Longrightarrow$

$$L - \frac{L}{2} < a_n < L + \frac{L}{2}$$

$$0 < \frac{L}{2} < a_n < \frac{3L}{2} \Longrightarrow a_n > 0$$

Theorem 2

$a_n \to L,\ b_n \to K,\ \forall\, n \in \mathbb{N} \Longrightarrow a_n \geq b_n \Longrightarrow L \geq K$

Given two real numbers L, K that we know nothing about, we know that one of two things is true:

1. $L \geq K$
2. $L < K$

If we are able to prove that the statment $L < K$ is false, then we prove that $L \geq K$.

We're going to do a prove by contradiction, by assuming that something is true, then coming to a contradiction about what we know for a fact is true.

Let's assume that $L < K$

$a_n \to L \implies \forall\, \varepsilon > 0, \exists\, N$, such that $\forall\, n > N \implies$

$$L - \varepsilon < a_n < L + \varepsilon$$

$b_n \to L \implies \forall\, \varepsilon > 0, \exists\, N$, such that $\forall\, n > N \implies$

$$L - \varepsilon < b_n < L + \varepsilon$$

Given: $K - L > 0 \iff \frac{K-L}{3} > 0 \implies$ For $\varepsilon = \frac{K-L}{3}$ $\exists\, N_1$ such that $\forall\, n > N_1 \implies$

$$L - \frac{K-L}{3} < a_n < \text{Ł} + \frac{K-L}{3}$$

$$a_n < \frac{K+2L}{3}$$

$\exists\, N_2$ such that $\forall\, n > N_2 \implies$

$$K - \frac{K-L}{3} < b_n < K + \frac{K-L}{3}$$

$$\frac{2K+L}{3} < b_n$$

$K > L \Longrightarrow \frac{K+2L}{3} < \frac{2K+L}{3} \Longrightarrow$ For $N = max\{N_1, N_2\}$ and $\forall\ n > N \Longrightarrow$

$$a_n < \frac{K+2L}{3} < \frac{2K+L}{3} < b_n$$

Which is a contradiction to the statment that $a_n \geq b_n \forall\ n \in \mathbb{N}$, meaning that the statment $L < K$ is not true. that means that the statment $L \geq K$ is true.

2.5.5 Harmonic, Geometric, Arithmetic Mean

Definitions

Arithmetic Mean

Given a set of real numbers $\{x_1, x_2, x_3, \ldots, x_n\}$, the arithmetic mean of this set is:

$$\frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

Geometric Mean

Given a set of positive numbers $\{y_1, y_2, y_3, \ldots, y_n\}$, the geometric mean of this set is:

$$\sqrt[n]{y_1 \cdot y_2 \cdot y_3 \cdot\ \ldots\ \cdot y_n}$$

Harmonic Mean

Given a set of positive numbers $\{z_1, z_2, z_3, \ldots, z_n\}$, the harmonic mean of this set is:

$$\frac{n}{\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\ldots+\frac{1}{z_n}}$$

Inequalities

$$\text{Harmonic Mean } \leq \text{Geometric Mean } \leq \text{Arithmetic Mean}$$

In Short:

HM GM AM Inequalities

Given a set of positive numbers $\{x_1, x_2, x_3, \ldots, x_n\}$

$$\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\ldots+\frac{1}{x_n}} \leq \sqrt[n]{x_1 \cdot x_2 \cdot x_3 \cdot \ \ldots \ \cdot x_n} \leq \frac{x_1+x_2+x_3+\ldots+x_n}{n}$$

GM AM Inequality Proof

We will prove this inequality using induction.

Base Case:

For $n = 1$ $\frac{x_1}{1} = {x_1}^1$ we're more interested when $n = 2$

For $n = 2$

$$\sqrt[2]{x_1 \cdot x_2} \leq \frac{x_1+x_2}{2}$$

$$\Updownarrow$$

$$x_1 \cdot x_2 \leq \frac{(x_1+x_2)^2}{4} = \frac{{x_1}^2+2x_1x_2+{x_2}^2}{4}$$

$$\Updownarrow$$

$$4x_1x_2 \leq {x_1}^2 + 2x_1x_2 + {x_2}^2$$

$$\Updownarrow$$

$$0 \leq {x_1}^2 - 2x_1x_2 + {x_2}^2$$

$(x_1 - x_2)^2 = {x_1}^2 - 2x_1x_2 + {x_2}^2$

$$\Updownarrow$$

$$0 \leq (x_1 - x_2)^2$$

Which is always true.

Let's assume that for any natural number $k \geq 2$ that the claim, we want to prove that the claim is also true for $2k$, Meaning:

$$\sqrt[2k]{x_1 \cdot x_2 \cdot x_3 \cdot \ \ldots \ x_{2k}} \leq \frac{x_1 + x_2 + x_3 + \ldots + x_{2k}}{2k}$$

Using our induction assumption that:

$$\sqrt[k]{x_1 \cdot x_2 \cdot x_3 \cdot \ \ldots \ x_k} \leq \frac{x_1 + x_2 + x_3 + \ldots + x_k}{k}$$

We can see that:

$$\frac{x_1 + x_2 + x_3 + \ldots + x_{2k}}{2k} = \frac{\frac{x_1+x_2+x_3+\ldots+x_k}{k} + \frac{x_{k+1}+x_{k+2}+x_{k+3}+\ldots+x_{2k}}{k}}{2}$$

$$\frac{\frac{x_1+x_2+x_3+\ldots+x_k}{k}+\frac{x_{k+1}+x_{k+2}+x_{k+3}+\ldots+x_{2k}}{k}}{2} \geq \frac{\sqrt[k]{x_1\cdot x_2\cdot x_3\ \cdot\ x_k}+\sqrt[k]{x_{k+1}\cdot x_{k+2}\cdot x_{k+3}\cdot\ldots x_{2k}}}{2}$$

$$\frac{\sqrt[k]{x_1\cdot x_2\cdot x_3\ \cdot\ x_k}+\sqrt[k]{x_{k+1}\cdot x_{k+2}\cdot x_{k+3}\cdot\ldots x_{2k}}}{2} \geq \sqrt{\sqrt[k]{x_1\cdot x_2\cdot x_3\ \cdot\ x_k}\cdot\sqrt[k]{x_{k+1}\cdot x_{k+2}\cdot x_{k+3}\cdot\ldots x_{2k}}}$$

$$\sqrt{\sqrt[k]{x_1\cdot x_2\cdot x_3\ \cdot\ x_k}\cdot\sqrt[k]{x_{k+1}\cdot x_{k+2}\cdot x_{k+3}\cdot\ldots x_{2k}}} = \sqrt{\sqrt[k]{x_1\cdot x_2\cdot x_3\cdot\ldots\cdot x_k\cdot\ldots x_{2k}}}$$

$$\sqrt{\sqrt[k]{x_1\cdot x_2\cdot x_3\cdot\ldots\cdot x_k\cdot\ldots x_{2k}}} = \sqrt[2k]{x_1\cdot x_2\cdot x_3\cdot\ldots\cdot x_k\cdot\ldots x_{2k}}$$

We proved that if we know that $P(k)$ is true, then $P(2k)$ is true.

Meaning: $P(2)$ is true (We Proved It In Base Case), then $P(4)$ is also true, then $P(8)$ is true, then $P(16)$ is also true ..., it is not quite a full proof because what about $P(3)$ and $P(5)$, $P(6)\ldots$, our second step will fill in the gaps.

We're still on our induction assumption that $P(k)$ is true.

we want to prove that $P(k-1)$ is true.

From our induction assumption we know that:

$$\frac{x_1+x_2+x_3+\ldots+x_k}{k} \geq \sqrt{x_1\cdot x_2\cdot x_3\cdot\ \ldots\ \cdot x_k}$$

Note that this inequality is true when x_n is any positive number, meaning i can pick any specified number instead of x_n, and this inequality will still be true.

Let $x_k = \frac{x_1+x_2+x_3+\ldots x_{k-1}}{k-1}$

$$\frac{x_1+x_2+x_3+\ldots+x_k}{k} = \frac{x_1+x_2+x_3+\ldots+\frac{x_1+x_2+x_3+\ldots x_{k-1}}{k-1}}{k}$$

$$\frac{x_1+x_2+x_3+\ldots+x_{k-1}+\frac{x_1+x_2+x_3+\ldots x_{k-1}}{k-1}}{k} \geq \sqrt[k]{x_1\cdot x_2\cdot x_3\cdot\ \ldots\ \cdot\frac{x_1+x_2+x_3+\ldots x_{k-1}}{k-1}}$$

$$\frac{x_1 + x_2 + x_3 + \ldots + x_{k-1}}{k-1} \leq x_1 + x_2 + x_3 + \ldots + x_{k-1}$$

$$\Longrightarrow \frac{x_1 + x_2 + x_3 + \ldots + x_{k-1} + \frac{x_1+x_2+x_3+\ldots x_{k-1}}{k-1}}{k} \leq \frac{x_1 + x_2 + x_3 + \ldots + x_{k-1}}{k-1}$$

$$\frac{x_1 + x_2 + x_3 + \ldots + x_{k-1}}{k-1} \geq \sqrt[k]{x_1 \cdot x_2 \cdot x_3 \cdot \ \ldots \ \cdot \frac{x_1 + x_2 + x_3 + \ldots x_{k-1}}{k-1}}$$

Power both sides by k

$$(\frac{x_1 + x_2 + x_3 + \ldots + x_{k-1}}{k-1})^k \geq x_1 \cdot x_2 \cdot x_3 \cdot \ \ldots \ \cdot \frac{x_1 + x_2 + x_3 + \ldots x_{k-1}}{k-1}$$

Dividing both sides by $\frac{x_1+x_2+x_3+\ldots+x_{k-1}}{k-1}$

$$\frac{(\frac{x_1+x_2+x_3+\ldots+x_{k-1}}{k-1})^k}{\frac{x_1+x_2+x_3+\ldots+x_{k-1}}{k-1}} \geq \frac{x_1 \cdot x_2 \cdot x_3 \cdot \ \ldots \ \cdot \frac{x_1+x_2+x_3+\ldots x_{k-1}}{k-1}}{\frac{x_1+x_2+x_3+\ldots x_{k-1}}{k-1}}$$

$$(\frac{x_1 + x_2 + x_3 + \ldots + x_{k-1}}{k-1})^{k-1} \geq x_1 \cdot x_2 \cdot x_3 \cdot \ \ldots \ \cdot x_{k-1}$$

k'th root both sides

$$\frac{x_1 + x_2 + x_3 + \ldots + x_{k-1}}{k-1} \geq \sqrt[k-1]{x_1 \cdot x_2 \cdot x_3 \cdot \ \ldots \ \cdot x_{k-1}}$$

Which is what we wanted to prove.

If we know that $P(k)$ is true, then $P(k-1)$ is true.

Meaning that $P(2)$ is true, then $P(4)$ is true based on our previous proof,$P(4)$ is true, then $P(3)$ is true, then $P(6)$ is true, then $P(5)$ is true, then $P(10)$ is true, then $P(9)$ is true, and if $P(4)$ is true, then $P(8)$ is true and $P(7)$ is true.

Meaning for all natural numbers the inequality above is true.

HM GM Inequality Proof

Just to remember that the Harmonic Mean is:

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \ldots + \frac{1}{x_n}}$$

We want to prove that:

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \ldots + \frac{1}{x_n}} \leq \sqrt[n]{x_1 \cdot x_2 \cdot x_3 \cdot \ \ldots \ \cdot x_n}$$

For any positive number x_n, $\frac{1}{x_n}$, $\forall\ n \in \mathbb{N}$ is also positive, let's define a_n to be the fraction $\frac{1}{x_n}$.

We know:

$$\sqrt[n]{a_1 \cdot a_2 \cdot a_3 \cdot \ \ldots \ \cdot a_n} \leq \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}$$

Let's take the inverse of both sides

$$\frac{1}{\sqrt[n]{a_1 \cdot a_2 \cdot a_3 \cdot \ \ldots \ \cdot a_n}} \geq \frac{n}{a_1 + a_2 + a_3 + \ldots + a_n}$$

Substitute $\frac{1}{x_n}$ for a_n

$$\frac{1}{\sqrt[n]{\frac{1}{x_1} \cdot \frac{1}{x_2} \cdot \frac{1}{x_3} \cdot \ \ldots \ \cdot \frac{1}{x_n}}} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \ldots + \frac{1}{x_n}}$$

$$\frac{1}{\sqrt[n]{\frac{1}{x_1}\cdot\frac{1}{x_2}\cdot\frac{1}{x_3}\cdot\ \ldots\ \cdot\frac{1}{x_n}}} = \frac{1}{\sqrt[-n]{x_1\cdot x_2\cdot x_3\cdot\ \ldots\ \cdot x_n}} = \sqrt[n]{x_1\cdot x_2\cdot x_3\cdot\ \ldots\ \cdot x_n}$$

$$\sqrt[n]{x_1\cdot x_2\cdot x_3\cdot\ \ldots\ \cdot x_n} \geq \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\ldots+\frac{1}{x_n}}$$

With that we proved that the Harmonic Mean $\leq$ the Geometric Mean.

Meaning:

$$\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\ldots+\frac{1}{x_n}} \leq \sqrt[n]{x_1\cdot x_2\cdot x_3\cdot\ \ldots\ \cdot x_n} \leq \frac{x_1+x_2+x_3+\ldots x_n}{n}$$

2.5.6 Even More Limit Theorems

Given $a_n \to L,\ a_n > 0\ \forall\ n \in \mathbb{N}$ We want to prove the following statments:

1. The sequence of Arithmetic Mean $x_n = \frac{a_1+a_2+a_3+\ldots+a_n}{n} \to L$
2. The sequence of Geometric Mean $y_n = \sqrt[n]{a_1\cdot a_2\cdot a_3\cdot\ \ldots\ \cdot a_n} \to L$
3. The sequence of Harmonic Mean $z_n = \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\ldots+\frac{1}{a_n}} \to L$

Proof 1

$a_n \to L \iff \forall\ \varepsilon > 0,\ \exists N_1 \in \mathbb{N}$ such that $\forall\ n > N \Longrightarrow$

$$|a_n - L| < \frac{\varepsilon}{2}$$

$$\left|\frac{a_1+a_2+a_3+\ldots+a_n}{n} - L\right| = \left|\frac{a_1+a_2+a_3+\ldots+a_n-nL}{n}\right|$$

$$\left|\frac{a_1+a_2+a_3+\ldots+a_n-nL}{n}\right| = \left|\frac{(a_1-L)+(a_2-L)+(a_3-L)+\ldots+(a_{N_1}-L)+\ldots+(a_n-L)}{n}\right|$$

$$=$$

$$\left|\frac{(a_1 - L) + (a_2 - L) + (a_3 - L) + \ldots + (a_{N_1} - L)}{n} + \frac{(a_{N_1+1} - L) + (a_{N_1+2} - L) + (a_{N_1+3} - L) + \ldots + (a_n - L}{n}\right.$$

$$\leq$$

$$\left|\frac{(a_1 - L) + (a_2 - L) + (a_3 - L) + \ldots + (a_{N_1} - L)}{n}\right| + \left|\frac{(a_{N_1+1} - L) + (a_{N_1+2} - L) + (a_{N_1+3} - L) + \ldots + (a_n -}{n}\right.$$

Let $a_1 + a_2 + a_3 + \ldots + a_{N_1} - N_1 \cdot L = t$ which is a constant, because we know what the values of $a_1, \ldots a_{N_1}$ are, and N_1 and L

If we look at the sequence $\frac{t}{n}$ we can see that it approaches 0, therefor $\forall\, \varepsilon > 0,\ \exists N_2$ such that $\forall\, n > N_2 \Longrightarrow$

$$\left|\frac{(a_1 - L) + (a_2 - L) + (a_3 - L) + (a_{N_1} - L)}{n}\right| = \left|\frac{t}{n}\right| < \frac{\varepsilon}{2}$$

We know that $\forall\, n > N_1 \Longrightarrow |a_n - L| < \varepsilon$

$$\left|\frac{(a_{N_1+1} - L) + (a_{N_1+2} - L) + (a_{N_1+3} - L) + \ldots + (a_n - L)}{n}\right| \leq \left|a_{N_1+1} - L\right| + \left|a_{N_1+2} - L\right| + \ldots + \left|a_n - L\right| \cdot \frac{n - N_1}{n}$$

$$\frac{n - N_1}{n} < \frac{n}{n} = 1$$

$$\left|a_{N_1+1} - L\right| + \left|a_{N_1+2} - L\right| + \ldots + \left|a_n - L\right| \cdot \frac{n - N_1}{n} < \frac{\varepsilon}{2} \cdot \frac{n - N_1}{n} < \frac{\varepsilon}{2}$$

For $N = max\{N_1, N_2\}$

$$|x_n - L| < \varepsilon$$

Proof 3

Let $c_n = \frac{1}{a_n} \implies c_n \to L$

$$\text{Let } t_n = \frac{c_1 + c_2 + c_3 + \ldots + c_n}{n} \to \frac{1}{L}$$

Let $z_n = \frac{1}{t_n} \implies z_n \to L$

$$z_n = \frac{n}{c_1 + c_2 + c_3 + \ldots + c_n} = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n}} \to L$$

Proof 2

We know about the HM GM AM Inequalities

$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 \cdot a_2 \cdot a_3 \cdot \ \ldots \ a_n} \leq \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}$$

We just proved these two limits

$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n}} \to L$$

$$\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \to L$$

Using Sandwich Theorem

$$L \leq \sqrt[n]{a_1 \cdot a_2 \cdot a_3 \cdot \ \ldots \ a_n} \leq L$$

$\implies \sqrt[n]{a_1 \cdot a_2 \cdot a_3 \cdot \ \ldots \ a_n} \to L$

Ratio Test Theorem

If $a_n > 0 \; \forall \; n \in \mathbb{N}$, $\lim\limits_{n\to\infty} \dfrac{a_n}{a_{n-1}} = L \Longrightarrow \sqrt[n]{a_n} = L$

Important to note that we don't know that $a_n \to L$

Let's define a_0 to be 1

$$a_n = \frac{a_1}{a_0} \cdot \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdot \frac{a_4}{a_3} \cdot \; \ldots \; \cdot \frac{a_n}{a_{n-1}}$$

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{a_1}{a_0} \cdot \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdot \frac{a_4}{a_3} \cdot \; \ldots \; \cdot \frac{a_n}{a_{n-1}}}$$

Let $b_n = \frac{a_n}{a_{n-1}}$, we know that $b_n \to L$

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{a_1}{a_0} \cdot \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdot \frac{a_4}{a_3} \cdot \; \ldots \; \cdot \frac{a_n}{a_{n-1}}} = \sqrt[n]{b_1 \cdot b_2 \cdot b_3 \cdot \; \ldots \; b_n}$$

We just proved that if $b_n > 0 \; \forall \; n \in N$ and $b_n \to L \Longrightarrow \sqrt[n]{b_1 \cdot b_2 \cdot b_3 \cdot \; \ldots \; b_n} \to L$

$$\sqrt[n]{a_n} = \sqrt[n]{b_1 \cdot b_2 \cdot b_3 \cdot \; \ldots \; b_n} \to L$$

Theorem Applications

We want to prove that given $c > 0$ $\sqrt[n]{c} \to 1$

$a_n = c \Longrightarrow \lim\limits_{n\to\infty} \dfrac{a_n}{a_{n-1}} = \dfrac{c}{c} = 1 \Longrightarrow \sqrt[n]{a_n} = 1 = \sqrt[n]{c}$

With the same concept we find the limit for the sequence $\sqrt[n]{n}$, let $a_n = n$

$\lim\limits_{n\to\infty} \dfrac{a_n}{a_{n-1}} = \dfrac{n}{n-1} \to 1 \Longrightarrow \sqrt[n]{n} = 1$

2.6 Approaching ∞ and $-\infty$

2.6.1 Understanding The Definition

We say that a sequence approaches ∞ if and only if we pick any real number, there exists a point such that for all the indexes bigger than that point all the terms of the sequence are bigger than that number.

2.6.2 The Definition

$a_n \to \infty \iff \forall\, M \in \mathbb{R},\ \exists\, N \in \mathbb{R}$, such that $\forall n > N \Longrightarrow$

$$a_n > M$$

Proving that the sequence $a_n = n \to \infty$

$\forall\, M \in \mathbb{R},\ \exists\, N = \square$, such that $\forall n > N \Longrightarrow$

$$n > N = M$$

2.6.3 Equivalent Definition

$a_n \to \infty \iff \forall\, M > 0,\ \exists\, N \in \mathbb{R}$, such that $\forall\, n > N \Longrightarrow$

$$a_n > M$$

We can't just say that this is an equivalent definition for a sequence approaching infinity, we have to prove it.

Meaning we have to prove:

$\forall\ M \in \mathbb{R},\ \exists\ N \in \mathbb{R}$, such that $\forall n > N \Longrightarrow a_n > M \iff a_n \to \infty \iff$ $\forall\ M > 0,\ \exists\ N \in \mathbb{R}$, such that $\forall\ n > N \Longrightarrow a_n > M$

$\Longrightarrow$

If we know that the definition works $\forall\ M \in \mathbb{R}$, then it works $\forall\ M > 0$

$\Longleftarrow$

If we know that $\forall\ M > 0,\ \exists\ \mathbb{N} \in \mathbb{R}$, such that $\forall\ n > N \Longrightarrow$

$$a_n > M > 0$$

then it works $\forall\ M \leq 0$

You'll see why this is important.

2.6.4 Definitions For Approaching $-\infty$

Similarly $a_n \to -\infty \iff \forall\ M \in \mathbb{R}$, or $\forall\ M > 0,\ \exists\ N \in \mathbb{R}$, such that $\forall\ n > N \Longrightarrow$

$$a_n < M$$

2.6.5 Proving $2^n \to \infty$

$\forall\, M \in \mathbb{R},\ \exists\, N = \square$, such that $\forall\, n > N \implies$

$$2^n > 2^N = M$$

meaning that $N = \log_2 M$, the problem with this statment is that M could be any number including negative numbers, which is a problem because $\log_2 x$ is defined when $x > 0$, this is we another definition for approaching infinity, $\forall\, M > 0$.

This is also true $\forall\, a > 1$ meaning $a^n \to \infty$ **Prove As Practice**

2.6.6 Infinity Limit Theorems

$a_n \to \infty, a_n \neq 0\ \forall\, n \implies \frac{1}{a_n} \to 0$ We want to prove that $\forall\, \varepsilon > 0\ \exists\, N$, such that $\forall\, n > N \implies$

$$|\frac{1}{a_n} - 0| = |\frac{1}{a_n}| < \varepsilon$$

$$\Updownarrow$$

$$|a_n| > \frac{1}{\varepsilon}$$

$a_n \to \infty \implies$ For $M = \frac{1}{\varepsilon}$, $\exists\, N_1$, such that $\forall\, n > N_1$

$$a_n > \frac{1}{\varepsilon}$$

For $N = N_1$, such that $\forall\, n > N_1 \Longrightarrow$

$$a_n > \frac{1}{\varepsilon} \Longrightarrow |a_n| > \frac{1}{\varepsilon}$$

Given: $a_n \to 0, a_n > 0\, \forall\, n \Longrightarrow \frac{1}{a_n} \to \infty$

We want to prove that $\forall\, M > 0,\ \exists\, N$, such that $\forall\, n > N \Longrightarrow$

$$\frac{1}{a_n} > M \iff a_n < \frac{1}{M}$$

$a_n \to 0 \Longrightarrow$ For $\varepsilon = \frac{1}{M} \Longrightarrow \exists\, N_1$, such that $\forall\, n > N_1 \Longrightarrow$

$$|a_n - 0| = |a_n| = a_n < \frac{1}{M}$$

Given $|q| < 1 \Longrightarrow |q|^n \to 0$

If $q = 0$, then $0^n \to 0$

Otherwise, for $q \neq 0 \Longrightarrow$ if $|q| < 1 \Longrightarrow \frac{1}{|q|} > 1$.

Let's define $a = \frac{1}{|q|} \Longrightarrow a > 1 \Longrightarrow a^n \to \infty \Longrightarrow \frac{1}{a_n} = |q|^n \to 0$

The Pizza Theroem

Given: $a_n \leq b_n\, \forall\, n,\ a_n \to \infty \Longrightarrow b_n \to \infty$ $a_n \to \infty \Longrightarrow \forall\, M \in \mathbb{R},\ \exists\, N_1$, such that $\forall\, n > N_1 \Longrightarrow$

$$a_n > M$$

Given: $a_n \leq b_n \rightarrow$

$$M < a_n \leq b_n \Longrightarrow M < b_n$$

Meaning that $b_n \rightarrow \infty$