Lec #7. 1

# The Continuous time Fourier Transform

## Domains.

| Time Domain | Frequency Domain. |
|---|---|
| $x(t)$ | $X(j\omega)$ |

Time → frequency (Fourier Transform)

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt.$$

↓

Book equation (4.9)

Time ← Frequency (Inverse Fourier Transform)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

↓

Book equation (4.8)

| Fourier Series | Fourier Transform. |
|---|---|
| → Periodic Signals. | → Periodic/A-periodic Signals. |
| → $a_k$ (Fourier Series Coefficients) | → $X(j\omega)$ |
| → $k$ is Present | → No $k$. |

## 4.1 Representation of A-periodic Signals:
## (The Continuous Time Fourier Transform)

**Example 4.1**

$$x(t) = e^{-at}\, u(t) \qquad a > 0$$

→ Find $X(j\omega)$

signal is A-periodic

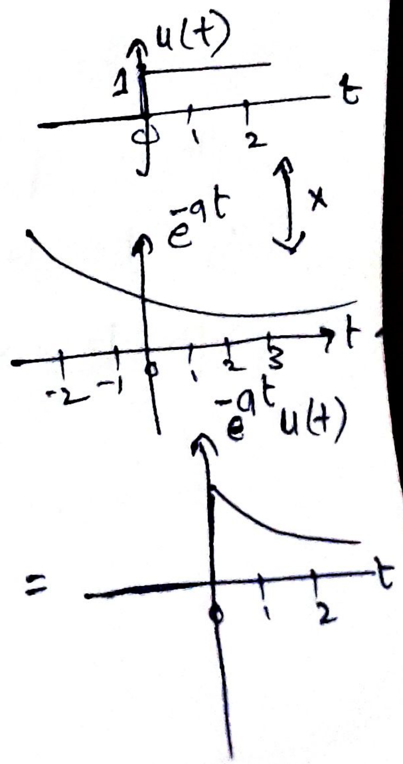
In order to find the Fourier Transform $X(j\omega)$, of $x(t)$, using formula:

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt.$$

$$X(j\omega) = \int_{-\infty}^{+\infty} \left(e^{-at} u(t)\right) e^{-j\omega t}\, dt.$$

$$X(j\omega) = \int_{-\infty \to 0}^{+\infty} e^{-at}\, e^{-j\omega t}\, dt.$$

$-\infty \to 0$ because of $u(t)$



$$X(j\omega) = \int_{0}^{+\infty} e^{-at} e^{-j\omega t} \, dt.$$

$$= \int_{0}^{+\infty} e^{-t(a+j\omega)} \, dt. = \left. \frac{e^{-t(a+j\omega)}}{-(a+j\omega)} \right|_{0}^{+\infty}$$

$$= \frac{-1}{a+j\omega} \left[ e^{-\infty} - e^{0} \right]$$

$$= -\frac{1}{a+j\omega} \left[ 0 - 1 \right] = \frac{1}{a+j\omega} \quad a>0$$

$$\boxed{X(j\omega) = \frac{1}{a+j\omega} \quad a>0}$$

$$\boxed{\begin{aligned} e^{-\infty} &= 0 \\ e^{+\infty} &= \infty \\ e^{0} &= 1 \end{aligned}}$$

Now plotting the spectrums. The spectrums of Fourier Transform is also called Continuous Spectrum.

$$X(j\omega) = \frac{1}{a+j\omega} \quad a>0$$

Multiply and dividing by Conjugates, we get:-

$$X(j\omega) = \frac{1}{a+j\omega} \times \frac{a-j\omega}{a-j\omega}$$

$$= \frac{a-j\omega}{(a)^2 - (j\omega)^2} = \frac{a-j\omega}{a^2 - (j^2\omega^2)} \qquad j^2=-1$$

$$= \frac{a-j\omega}{a^2+\omega^2} = \underbrace{\frac{a}{a^2+\omega^2}}_{\text{Real part}} - j\underbrace{\frac{\omega}{a^2+\omega^2}}_{\text{imaginary part.}}$$

Amplitude spectrum:- $|X(j\omega)|$

$$|X(j\omega)| = \sqrt{\left(\frac{a}{a^2+\omega^2}\right)^2 + \left(\frac{-\omega}{a^2+\omega^2}\right)^2}$$

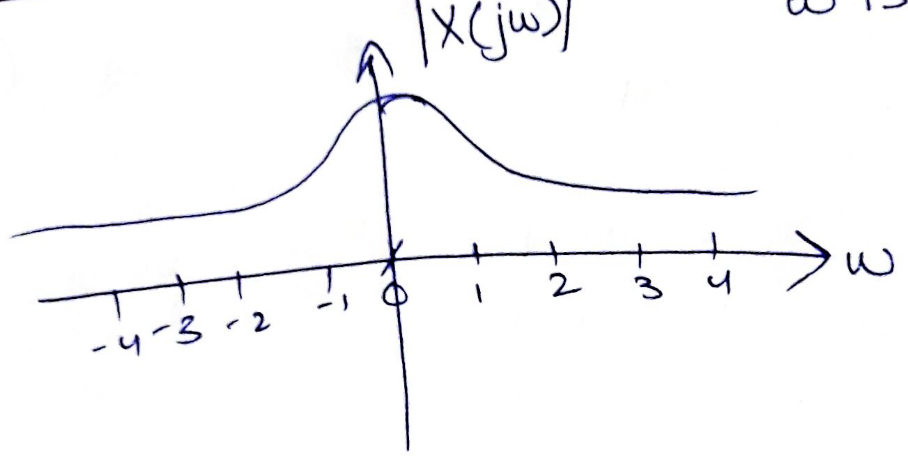
$$= \sqrt{\frac{a^2}{(a^2+\omega^2)^2} + \frac{\omega^2}{(a^2+\omega^2)^2}}$$

(5)

$$|X(j\omega)| = \sqrt{\frac{\cancel{a^2+\omega^2}}{(a^2+\omega^2)^{\cancel{2}}}}$$

$$\boxed{|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}}$$

here $a > 0$ and $\omega$ is variable.




Similarly Phase spectrum $\angle X(j\omega)$

$$\angle X(j\omega) = \tan^{-1}\frac{I}{R}.$$

$$= \tan^{-1}\frac{-\omega/\cancel{(a^2+\omega^2)}}{a/\cancel{(a^2+\omega^2)}}$$

$$\boxed{\angle X(j\omega) = \left(-\tan^{-1}\omega/a\right)}$$

$a > 0$ and $\omega$ is variable

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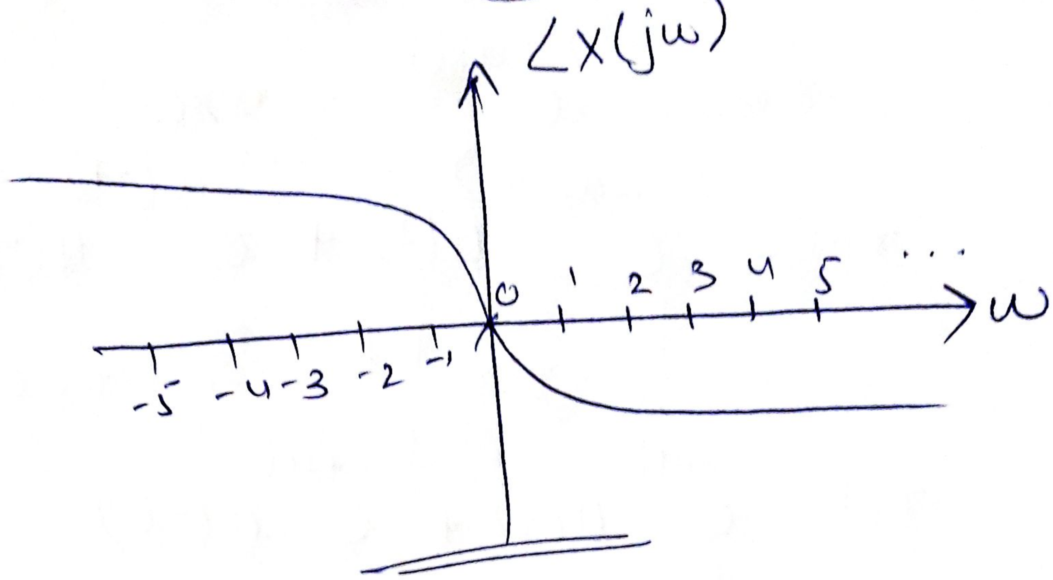



Example 4.2

$$x(t) = e^{-a|t|} \quad a > 0$$

Find $X(jw) = ?$

$|t|$ mean $\begin{cases} t > 0 \\ t < 0 \end{cases}$ (Both)

where $t > 0$ can also be represented using $u(t)$ and $t < 0$ using $u(-t)$

Always Remember

$$\boxed{\begin{array}{l} t > 0 \rightarrow u(t) \\ t < 0 \rightarrow u(-t) \end{array}}$$

and here $|t|$ has both

$$x(t) = e^{-a|t|} \qquad a > 0$$

$$x(t) = \underbrace{e^{-a(+t)} u(t)}_{t>0} + \underbrace{e^{-a(-t)} u(-t)}_{t<0}$$

$$x(t) = e^{-at} u(t) + e^{+at} u(-t)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt.$$

$$= \int_{-\infty}^{+\infty} \left[ e^{-at} u(t) + e^{+at} u(-t) \right] \cancel{dt}\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt + \int_{-\infty}^{0} e^{+at} e^{-j\omega t}\, dt.$$

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} = \frac{2a}{a^2 + \omega^2}$$

(Plz solve yourself and also find $|X(j\omega)|$ and $\angle X(j\omega)$ as shown previously.

4.3

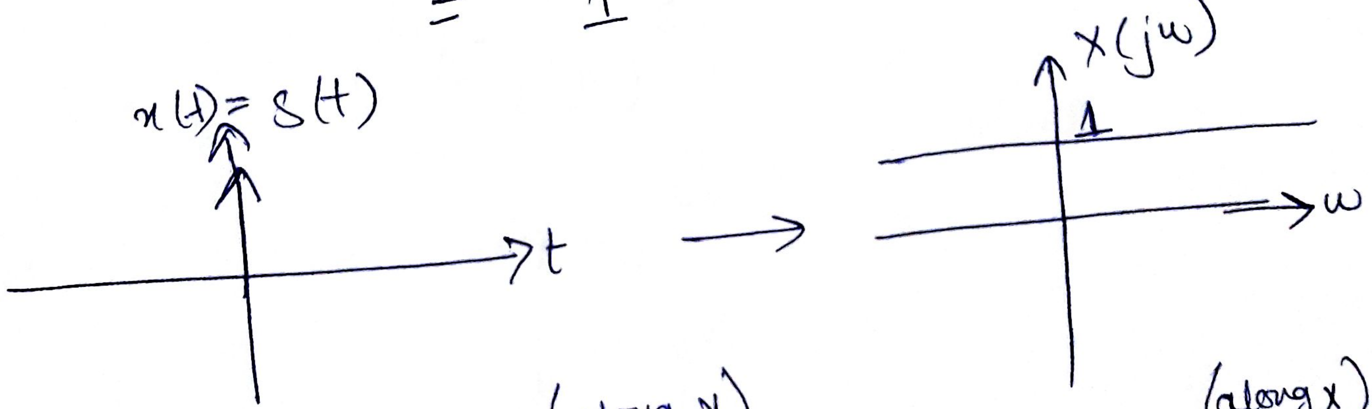
$$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{+\infty} \delta(t)\, e^{-j\omega t}\, dt$$

at $t=0$

$$= 1$$




(along y)
Infinite Amplitude $\longrightarrow$ Infinite width (along x)

(Time Domain) $\longleftrightarrow$ (F. Domain)

Inverse of eachother