LEC 2
ON
FOURIER SERIES

Dr. Arsla Khan

Transformation Methods for CT signals

- A real time naturally available signal is in the form of *time domain*
- However, analysis of a signal is far more convenient in *frequency domain.*
- A signal in its original form can not be defined in terms of frequency. Hence, by transformation methods, a time domain signal can be represented with respect to frequency components.

- The three important transformation methods available in practice for a continuous time domain signals are as follows
    - Fourier Series
    - Fourier Transform
    - Laplace Transform

Fourier Series

- The representation of signals over a certain interval of time in terms of the linear combination of the orthogonal functions is called Fourier Series.

- Fourier Series expansion is used for periodic signals to expand them in terms of their harmonics which are sinusoidal and orthogonal to each other.

- Two important classes of Fourier series methods are primarily available. They are as follows;
    - Exponential Fourier Series
        - If orthogonal functions are exponential functions (written in form of $e^{j\omega t}$)
    - Trigonometric Fourier Series
        - If orthogonal functions are trigonometric functions (written in the form of $\sin nx$, $\cos mx$)

- Orthogonal Functions: Two functions (signals) f and g are said to be orthogonal when their dot product is zero.
  - $z(t) = f(t)g(t) = 0$

- Periodic Signal: Signal which repeats itself after a fixed interval of time is called periodic signal.
  - $x(t) = x(t + T)$

- Harmonics: A **harmonic** is a signal or wave whose frequency is an integer multiple of the fundamental frequency.
  - Suppose f_o is fundamental frequency of any signal. Thus, $2f_o$, $4f_o$, $6f_o$ etc. are even harmonics of the signal while $3f_o$, $5f_o$, $7f_o$ etc. are odd harmonics of the signal

Derivative of Complex Exponential Fourier Series

- A complex exponential signal can be written as,

$$x(t) = ae^{j\omega_o t} = ae^{j2\pi f_o t}$$

- If we take the signal at various intervals of the fundamental period or at various harmonics, then:

- the constant signal or dc signal is given by $:x_o(t) = a_o e^{j2\pi(0)t} = a_o$
- The first harmonic of the signal is given by $:x_1(t) = a_1 e^{j2\pi(1f_o)t} = a_1 e^{j2\pi f_o t} = a_1 e^{j\omega_o t}$
- The second harmonic of the signal is given by $:x_2(t) = a_2 e^{j2\pi(2f_o)t} = a_2 e^{j(2\omega_o)t}$
- The third harmonic of the signal is given by $:x_3(t) = a_3 e^{j2\pi(3f_o)t} = a_3 e^{j(3\omega_o)t}$
- Similarly for *k*th harmonic, $x_k(t) = a_k e^{j2\pi(kf_o)t} = a_k e^{j(k\omega_o)t}$

- Thus, the linear combination of these complex exponential can be written as;
- $x(t) = x_o(t) + x_1(t) + x_2(t), \ldots x_k(t)$
- $x(t) = a_o + a_1 e^{j\omega_o t} + a_2 e^{j(2\omega_o)t} + \cdots a_k e^{j(k\omega_o)t}$

  - $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(k\omega_o)t} \qquad (i)$

- Eq. (i) is also known as synthesis equations. Here k is an integer value whereas a_k is the weight of each harmonic and is known as Fourier series coefficients or spectral coefficients.

- **Determination of Fourier series is nothing but determination of <u>Fourier series coefficients.</u>**

How to find Fourier Series (FS) Coefficients

- $a_o = \frac{1}{T_o}\int_0^{T_o} x(t)dt \qquad (ii)$

- $a_k = \frac{1}{T_o}\int_0^{T_o} x(t)e^{-j(k\omega_o)t}dt \qquad (iii)$

- Eq. (ii) and (iii) are known as Analysis Equation

Exp 1: (Exp 3.3): Write FS expansion of the signal

$$x(t) = \sin \omega_o t$$

- **Solution:** To find the FS expansion, we need to determine the FS coefficients i.e. a_o and a_k

- Method 1: Use analysis equation to find the coefficients
- $a_o = \frac{1}{T_o} \int_0^{T_o} x(t)dt$ and $a_k = \frac{1}{T_o} \int_0^{T_o} x(t)e^{-j(k\omega_o)t}dt$

- Method 2: It is a simple case. We can expand the sinusoidal signal into exponentials.
- $x(t) = \sin \omega_o t = \frac{e^{j\omega_o t} - e^{-j\omega_o t}}{2j}$
- $x(t) = \frac{e^{j\omega_o t}}{2j} + \frac{e^{-j\omega_o t}}{-2j}$

- $x(t) = \frac{1}{2j} e^{j\omega_o t} + \frac{1}{-2j} e^{-j\omega_o t}$
- By inspection, we can find that when
- $k = 1, \quad e^{jk\omega_o t} \rightarrow e^{j\omega_o t}$
- $k = -1, \quad e^{jk\omega_o t} \rightarrow e^{-j\omega_o t}$

- **Therefore,**
- $\boldsymbol{a_1 = \frac{1}{2j}}$ **and** $\boldsymbol{a_{-1} = -\frac{1}{2j}}$
- $\boldsymbol{a_k = 0 \quad when\ k \neq +1\ and\ -1}$

Exp 2: Write FS expansion of the signal

$$x(t) = \sin\left(2t + \frac{\pi}{3}\right)$$

- Solution: It is a simple case. We can expand the sinusoidal signal into exponentials.
- $x(t) = \sin\left(2t + \frac{\pi}{3}\right) = \dfrac{e^{j\left(2t+\frac{\pi}{3}\right)} - e^{-j\left(2t+\frac{\pi}{3}\right)}}{2j}$
- $x(t) = \left(\frac{1}{2j}\right) e^{j\left(2t+\frac{\pi}{3}\right)} + \left(-\frac{1}{2j}\right) e^{-j\left(2t+\frac{\pi}{3}\right)}$
- $x(t) = \left(\frac{1}{2j} e^{j\frac{\pi}{3}}\right) e^{j(2t)} + \left(-\frac{1}{2j} e^{-j\frac{\pi}{3}}\right) e^{-j(2t)}$
- By inspection, we can find that when
- $k = 2, ,$ $\quad e^{jk\omega_o t} \rightarrow e^{j(2t)} \quad k(1)t = 2t$
- $k = -2,$

- Therefore,
- $a_2 = \left(\frac{1}{2j} e^{j\frac{\pi}{3}}\right)$ and $a_{-2} = \left(-\frac{1}{2j} e^{j\frac{\pi}{3}}\right)$
- $a_k = 0 \quad when\ k \neq +2\ and\ -2$

Exp 3: Write FS expansion of the signal

$$x(t) = \cos 3t + \sin 6t$$

- $x(t) = \cos 3t + \sin 6t = \dfrac{e^{j(3t)} + e^{-j(3t)}}{2} + \dfrac{e^{j(6t)} - e^{-j(6t)}}{2j}$
- $x(t) = \frac{1}{2} e^{j(3t)} + \frac{1}{2} e^{-j(3t)} + \frac{1}{2j} e^{j(6t)} + \frac{1}{-2j} e^{-j(6t)}$
- By inspection, we can find that when $k(1)t = 6t$
- $k = 3,\ k = -3,\ k = 6,\ k = -6,$
- Therefore,
- $\boldsymbol{a_3 = \left(\frac{1}{2}\right), a_{-3} = \left(\frac{1}{2}\right), a_6 = \left(\frac{1}{2j}\right), a_{-6} = \left(\frac{1}{-2j}\right)}$
- $\boldsymbol{a_k = 0 \quad when\ k \neq |3|\ and\ |6|}$

Exp 4: (Exp 3.4): Write FS expansion of the given signal and also draw their magnitude and phase spectrum

$$x(t) = 1 + \sin \omega_o t + 2\cos \omega_o t + \cos\left(2\omega_o t + \frac{\pi}{4}\right)$$

- $x(t) = 1 + \left(\frac{1}{2j}\right)\left[e^{j(\omega_o t)} - e^{-j(\omega_o t)}\right] + 2\left(\frac{1}{2}\right)\left[e^{j(\omega_o t)} + e^{-j(\omega_o t)}\right] + \left(\frac{1}{2}\right)\left[e^{j\left(2\omega_o t + \frac{\pi}{4}\right)} + e^{-j\left(2\omega_o t + \frac{\pi}{4}\right)}\right]$

- $x(t) = 1 + \left(\frac{1}{2j}\right)\left[e^{j(\omega_o t)} - e^{-j(\omega_o t)}\right] + (1)\left[e^{j(\omega_o t)} + e^{-j(\omega_o t)}\right] + \left(\frac{1}{2}e^{j\left(\frac{\pi}{4}\right)}\right) e^{j(2\omega_o t)} + \left(\frac{1}{2}e^{-j\left(\frac{\pi}{4}\right)}\right) e^{-j(2\omega_o t)}$
- Collect terms with similar exponential powers

- $x(t) = 1 + \left(1 + \frac{1}{2j}\right) e^{j(\omega_o t)} + \left(1 - \frac{1}{2j}\right) e^{-j(\omega_o t)} + \left(\frac{1}{2}e^{j\left(\frac{\pi}{4}\right)}\right) e^{j(2\omega_o t)} + \left(\frac{1}{2}e^{-j\left(\frac{\pi}{4}\right)}\right) e^{-j(2\omega_o t)}$
- $a_o = 1, a_1 = \left(1 + \frac{1}{2j}\right), a_{-1} = \left(1 - \frac{1}{2j}\right), a_2 = \left(\frac{1}{2}e^{j\left(\frac{\pi}{4}\right)}\right), a_{-2} = \left(\frac{1}{2}e^{-j\left(\frac{\pi}{4}\right)}\right)$

- $a_o = 1$
- $a_{+1} = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j$
- $a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j$
- $a_2 = \left(\frac{1}{2}e^{j\left(\frac{\pi}{4}\right)}\right) = \frac{1}{2}(\cos\frac{\pi}{4} + jsin\frac{\pi}{4}) = \frac{1}{2}(\cos 45^o + jsin45^o) = \frac{1}{2}\left(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4}(1 + \mathrm{j})$
- $a_2 = \frac{\sqrt{2}}{4}(1 + \mathrm{j})$
- $a_{-2} = \left(\frac{1}{2}e^{-j\left(\frac{\pi}{4}\right)}\right) = \frac{1}{2}(\cos\frac{\pi}{4} - jsin\frac{\pi}{4}) = \frac{1}{2}(\cos 45^o - jsin45^o) = \frac{1}{2}\left(\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4}(1 - \mathrm{j})$
- $a_{-2} = \frac{\sqrt{2}}{4}(1 - \mathrm{j})$
- $a_k = 0 \;\; for\ |k| > 2$

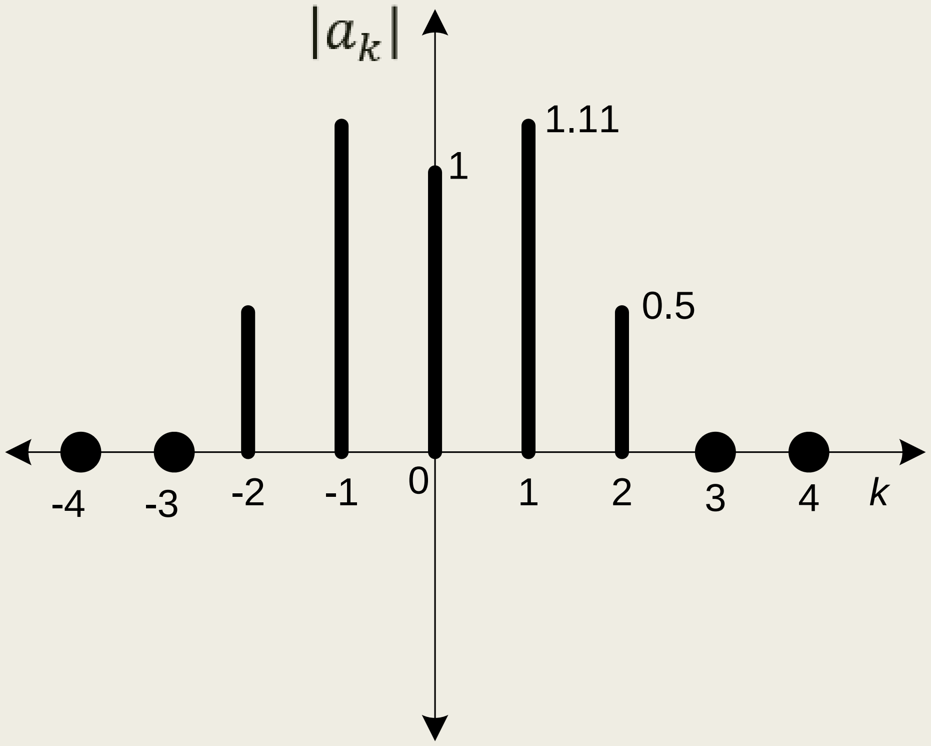
Magnitude and Phase Spectrum

- Magnitude Spectrum $|a_k| = \sqrt{Re^2 + Img^2}$

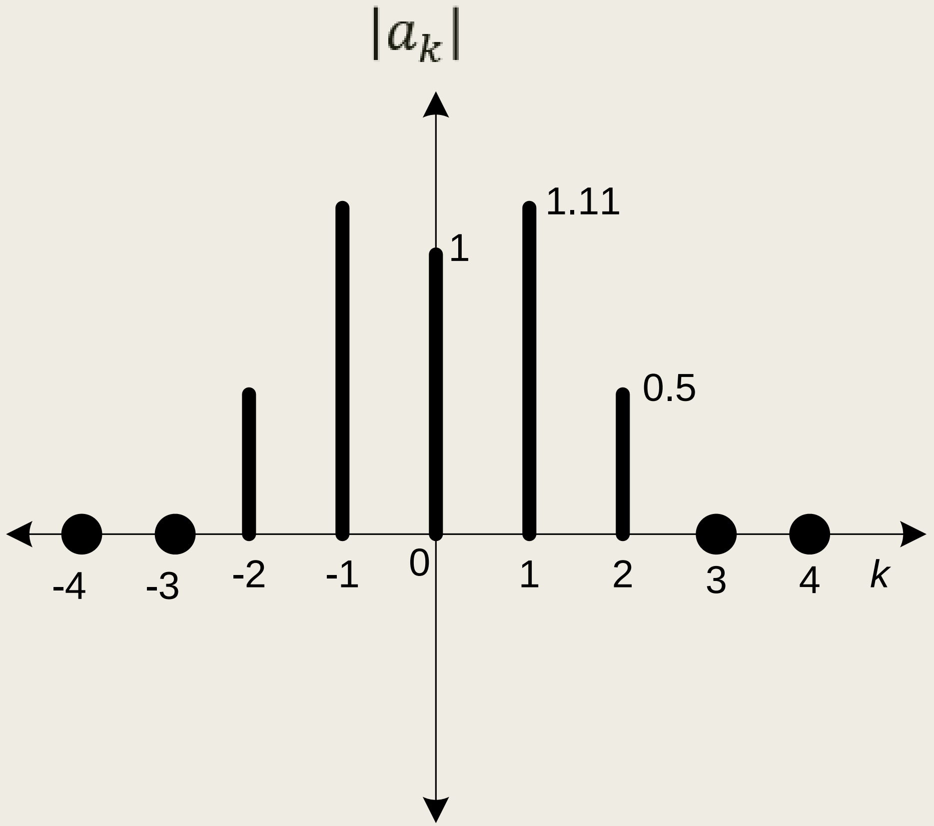
- Phase Spectrum $< a_k = tan^{-1}\left(\frac{Img}{Re}\right)$

- Magnitude Spectrum Now for different values of k
- $|a_o| = \sqrt{1^2 + 0^2} = 1$
- $|a_1| = \sqrt{1^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{1 + \frac{1}{4}} = \sqrt{5/4} = 1.11$
- $|a_{-1}| = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \sqrt{1 + \frac{1}{4}} = \sqrt{5/4} = 1.11$

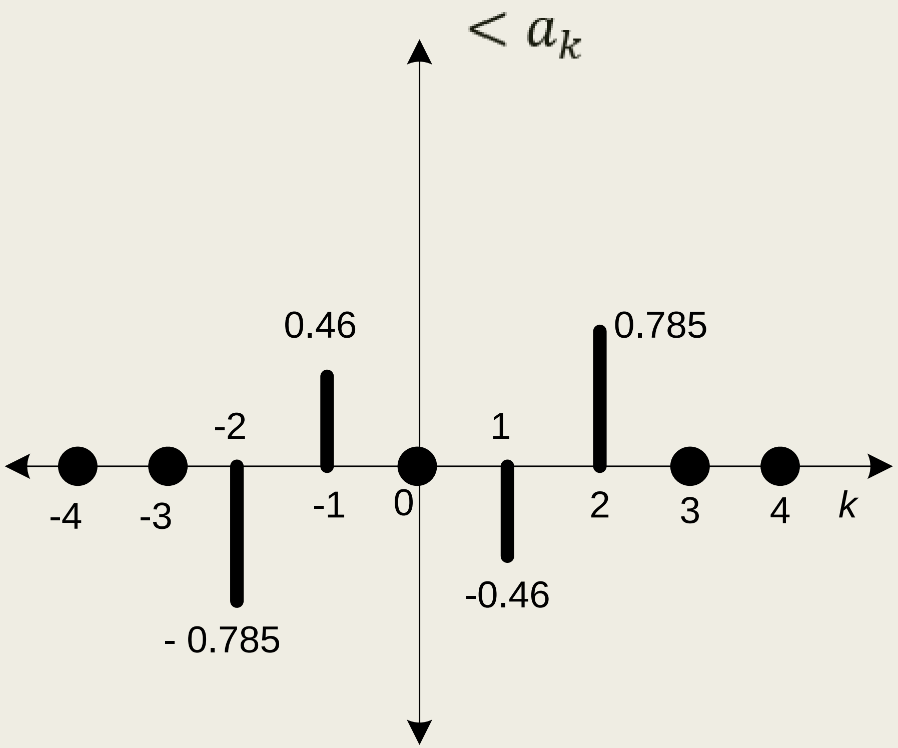
- $|a_2| = \sqrt{\left(\frac{\sqrt{2}}{4}\right)^2 + \left(\frac{\sqrt{2}}{4}\right)^2} = \sqrt{\frac{2}{16} + \frac{2}{16}} = \sqrt{4/16} = \frac{2}{4} = 0.5$
- $|a_{-2}| = \sqrt{\left(\frac{\sqrt{2}}{4}\right)^2 + \left(-\frac{\sqrt{2}}{4}\right)^2} = \sqrt{\frac{2}{16} + \frac{2}{16}} = \sqrt{4/16} = \frac{2}{4} = 0.5$



- Phase Spectrum
- $a_o = tan^{-1}\left(\frac{Img}{Re}\right) = tan^{-1}\left(\frac{0}{1}\right) = tan^{-1}(0) = 0$
- $a_1 = tan^{-1}\left(\frac{Img}{Re}\right) = tan^{-1}\left(\frac{-1/2}{1}\right) = tan^{-1}(-0.5) = -0.46$
- $a_{-1} = tan^{-1}\left(\frac{Img}{Re}\right) = tan^{-1}\left(\frac{1/2}{1}\right) = tan^{-1}(0.5) = 0.46$
- $a_2 = tan^{-1}\left(\frac{Img}{Re}\right) = tan^{-1}\left(\frac{\frac{\sqrt{2}}{4}}{\frac{\sqrt{2}}{4}}\right) = tan^{-1}(1) = \frac{\pi}{4} = 0.785$
- $a_{-2} = tan^{-1}\left(\frac{Img}{Re}\right) = tan^{-1}\left(\frac{-\frac{\sqrt{2}}{4}}{\frac{\sqrt{2}}{4}}\right) = tan^{-1}(-1) = -\frac{\pi}{4} = -0.785$



Magnitude Spectrum follows even symmetry



Phase Spectrum follows odd symmetry

Thank You !!!