The z-transform

The z-transform is used for the analysis of discrete-time signal and systems.

$$x[n] \underset{}{\overset{Z}{\rightleftharpoons}} X(z)$$

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}} \quad \text{———} \quad (1)$$

Relation between z-transform and DT Fourier Transform

complex variable → $Z = re^{jw}$ (in polar form)

- r = radius
- w = angular frequency

putting in eq (1) ;

$$X(re^{jw}) = \sum_{n=-\infty}^{\infty} x[n] (re^{jw})^{-n}$$

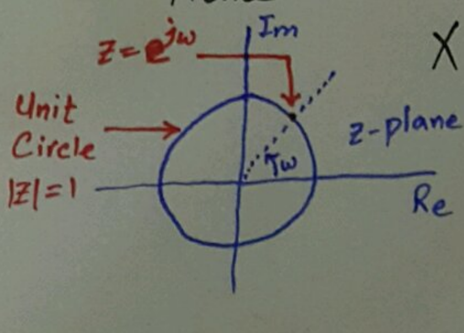
$$X(re^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, r^{-n} e^{-jwn}$$

with $r = 1$;

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn} \quad \text{(DTFT)}$$

This is fourier transform of a discrete-time signal.

Hence



$$X(z)\Big|_{|Z|=1} = X(e^{jw})$$

Z-transform is the General transform & DTFT is a special form of Z-transform with $|Z| = 1$.
(DTFT exist on unit circle of Z-tran)

Types of z-transform

Unilateral (One-sided)

$$X(z) = \sum_{n=0}^{+\infty} x[n]\, z^{-n}$$

Bilateral (two-sided)

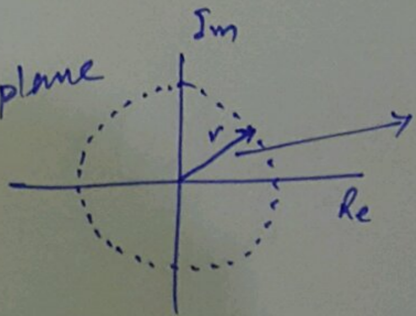
$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n}$$

⇒ Like Laplace Transform, the z-transform is defined with its Region of Convergence (ROC).

⇒ The ROC of the z-transform of $x[n]$ consists of the values of z for which $x[n]\, r^{-n}$ is absolutely summable:

$$\sum_{n=-\infty}^{+\infty} |x[n]|\, r^{-n} < \infty$$

→ The convergence is dependant only on $r = |z|$ and not ω. Therefore, for any value of z where z-transform converges, the ROC will contain all values of ω making ROC a ring.



One value of r, all values of ω makes ROC a ring

Example 10.1

$x[n] = a^n u[n]$ → RSS

$0 < a < 1$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} a^n u[n] \cdot z^{-n}$$

$$X(z) = \sum_{n=0}^{+\infty} a^n z^{-n}$$

$$X(z) = \sum_{n=0}^{+\infty} (az^{-1})^n$$

$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

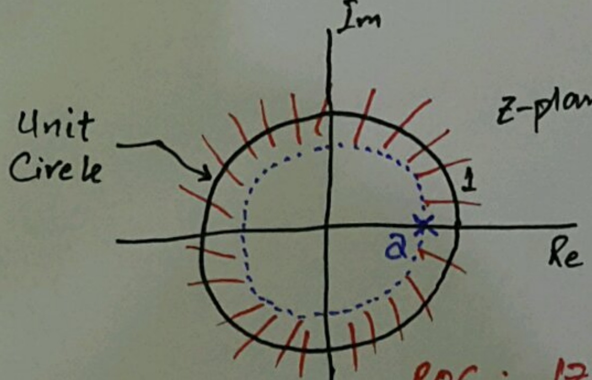
pole $\to z - a = 0$, $z = a$

for Convergence:

$$|1 - az^{-1}| > 0 \qquad |z-a| > 0$$

$$1 > |az^{-1}| \qquad |z| > |a| \text{ (ROC)}$$

Right Sided Sequence



ROC is outside of the pole.

ROC : $|z| > |a|$

Example 10.2

$x[n] = -a^n u[-n-1]$ → LSS

$0 < a < 1$

$$X(z) = \sum_{n=-\infty}^{+\infty} -a^n u[-n-1]\, z^{-n}$$

$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$X(z) = -\sum_{n=1}^{+\infty} a^{-n} z^n$$

$$X(z) = -\sum_{n=1}^{+\infty} (a^{-1}z)^n$$

$$X(z) = (a^{-1}z)^0 - (a^{-1}z)^0 - \sum_{n=1}^{\infty} (a^{-1}z)^n$$

$$X(z) = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

$$X(z) = 1 - \frac{1}{1-a^{-1}z}$$

$$= 1 - \frac{1}{1-a^{-1}z} = \frac{1-a^{-1}z-1}{1-a^{-1}z} = -\frac{a^{-1}z}{1-a^{-1}z} = -\frac{1}{\frac{1}{a^{-1}z}-1} = -\frac{1}{az^{-1}-1} = -\frac{1}{-1(1-az^{-1})} = \frac{1}{1-az^{-1}}$$

$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

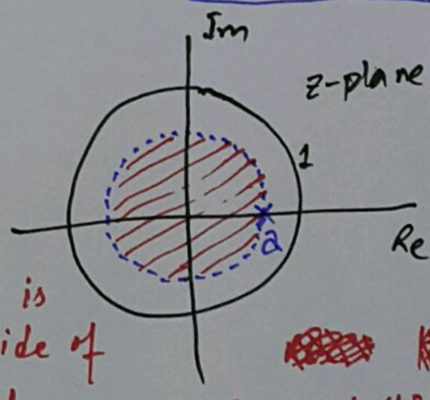
$z - a = 0$ ← pole, $z = a$

ROC

$|z-a| < 0$

$|z| < |a|$

Left Sided Sequence



ROC is inside of the pole.

ROC : $|z| < |a|$

Example 10.3

$$x[n] = 7\left(\tfrac{1}{3}\right)^n u[n] - 6\left(\tfrac{1}{2}\right)^n u[n]$$

Solving for z-transform:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n}$$

$$X(z) = 7\sum_{n=-\infty}^{+\infty} \left(\tfrac{1}{3}\right)^n u[n]\, z^{-n} - 6\sum_{n=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^n u[n]\, z^{-n}$$

$$X(z) = 7\sum_{n=0}^{+\infty} \left(\tfrac{1}{3} z^{-1}\right)^n - 6\sum_{n=0}^{+\infty} \left(\tfrac{1}{2} z^{-1}\right)^n$$

$$X(z) = \frac{7}{1-\frac{1}{3}z^{-1}} \; (1) - \frac{6}{1-\frac{1}{2}z^{-1}} \; (2)$$

zero at $z=0$ zero at $z = 3/2$

$$X(z) = \frac{7z}{z-\frac{1}{3}} - \frac{6z}{z-\frac{1}{2}} = \frac{z\left(z-\frac{3}{2}\right)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)} \; (3)$$

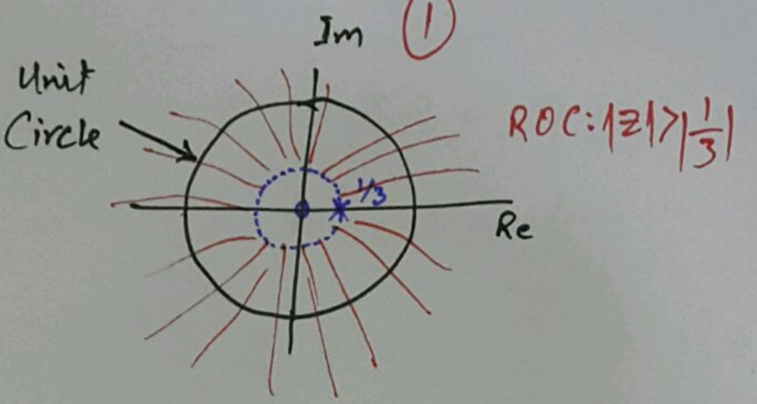
pole $z = \frac{1}{3}$ pole $z = \frac{1}{2}$

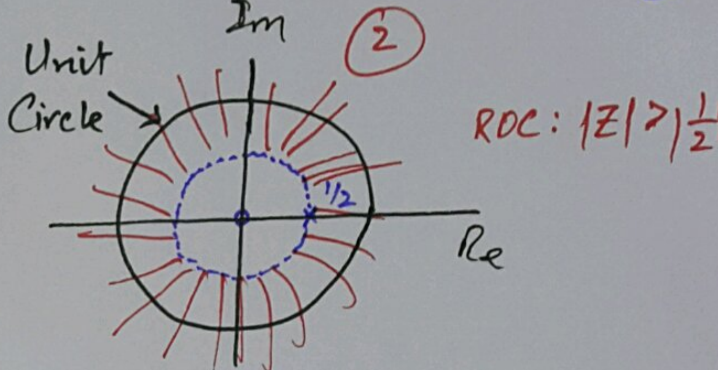
(1) $\left(\tfrac{1}{3}\right)^n u[n] \overset{z}{\longleftrightarrow} \dfrac{1}{1-\frac{1}{3}z^{-1}}$, $|z| > \frac{1}{3}$ (ROC 1)

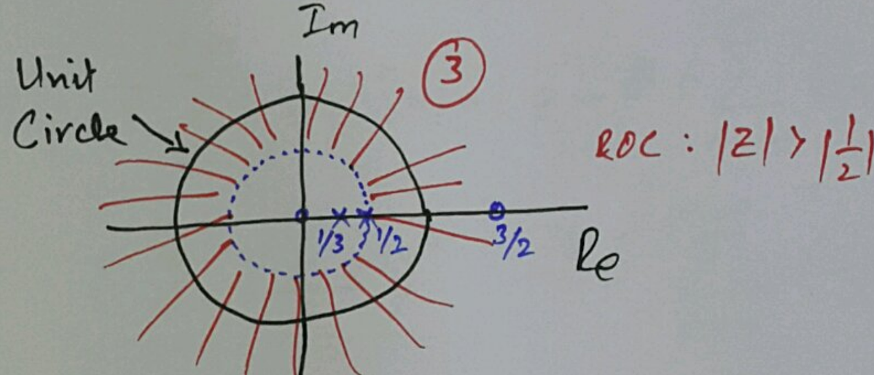
(2) $\left(\tfrac{1}{2}\right)^n u[n] \overset{z}{\longleftrightarrow} \dfrac{1}{1-\frac{1}{2}z^{-1}}$, $|z| > 1/2$ (ROC 2)

(3) $7\left(\tfrac{1}{3}\right)^n u[n] - 6\left(\tfrac{1}{2}\right)^n u[n] \overset{z}{\longleftrightarrow} \dfrac{7}{1-\frac{1}{3}z^{-1}} - \dfrac{6}{1-\frac{1}{2}z^{-1}}$, $|z| > \frac{1}{2}$

(ROC 1) $\cap$ (ROC 2)







<u>Note</u> :- On pole-zero plot, always plot unit circle with solid line ——

& plot ROC circle with dotted line ....

H.W : Example 10.4 where the poles are complex and therefore define with e^{jw} where at angular freq. $\omega = \frac{\pi}{4}$.