

The Inverse Z-Transform

$$x[n] = \frac{1}{2\pi j} \int_{|z|=r} X(z) z^{n-1} dz$$

Example 10.9

Given that $X(z) = \frac{3 - \frac{5}{6} z^{-1}}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{3} z^{-1})}$, $|z| > \frac{1}{3}$

Remember

before partial fraction, function $X(z)$ should be in proper fraction. In this example, it is in proper form.

$$\begin{aligned} \frac{3 - \frac{5}{6} z^{-1}}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{3} z^{-1})} &= \frac{A}{1 - \frac{1}{4} z^{-1}} + \frac{B}{1 - \frac{1}{3} z^{-1}} \\ &= \frac{1}{1 - \frac{1}{4} z^{-1}} + \frac{2}{1 - \frac{1}{3} z^{-1}} \quad \left(\begin{array}{l} A=1 \\ B=2 \end{array} \right) \text{ by partial fraction} \end{aligned}$$

Since $X(z)$ with ROC $|z| > \frac{1}{3}$ & the fact $\frac{1}{3} > \frac{1}{4}$, so both are RSS. Therefore

$$\frac{1}{1 - \frac{1}{4} z^{-1}} \xrightarrow{z^{-1}} \left(\frac{1}{4}\right)^n u[n] \text{ with ROC: } |z| > \frac{1}{4}$$

$$\frac{1}{1 - \frac{1}{3} z^{-1}} \xrightarrow{z^{-1}} \left(\frac{1}{3}\right)^n u[n] \text{ with ROC: } |z| > \frac{1}{3}$$

so

$$x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n] \rightarrow \text{Unique solution for given ROC } |z| > \frac{1}{3}$$

Example 10.10

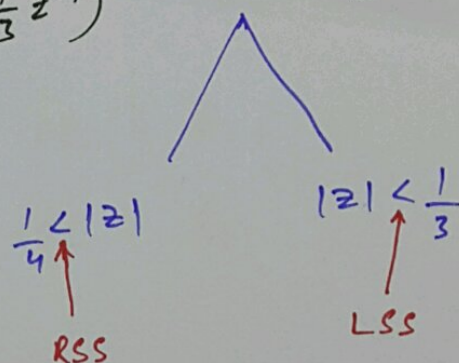
$$X(z) = \frac{3 - \frac{5}{6} z^{-1}}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{3} z^{-1})}, \quad \frac{1}{4} < |z| < \frac{1}{3}$$

$$X(z) = \frac{1}{1 - \frac{1}{4} z^{-1}} + \frac{2}{1 - \frac{1}{3} z^{-1}}$$

$$\frac{1}{1 - \frac{1}{4} z^{-1}}, \quad |z| > \frac{1}{4} \xrightarrow[\text{RSS}]{z^{-1}} \left(\frac{1}{4}\right)^n u[n]$$

$$\frac{1}{1 - \frac{1}{3} z^{-1}}, \quad |z| < \frac{1}{3} \xrightarrow[\text{LSS}]{z^{-1}} -\left(\frac{1}{3}\right)^n u[-n-1]$$

Overall \Rightarrow
$$x[n] = \left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[-n-1]$$

Example 10.11

$$X(z) = \frac{3 - \frac{5}{6} z^{-1}}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{3} z^{-1})}, \quad |z| < \frac{1}{4}$$

$$X(z) = \frac{1}{1 - \frac{1}{4} z^{-1}} + \frac{2}{1 - \frac{1}{3} z^{-1}}$$

$$\frac{1}{1 - \frac{1}{4} z^{-1}} \xrightarrow[\text{LSS}]{z^{-1}} -\left(\frac{1}{4}\right)^n u[-n-1]$$

$$\frac{1}{1 - \frac{1}{3} z^{-1}} \xrightarrow[\text{LSS}]{z^{-1}} -\left(\frac{1}{3}\right)^n u[-n-1]$$

Overall \Rightarrow

$$x[n] = -\left(\frac{1}{4}\right)^n u[-n-1] - \left(\frac{1}{3}\right)^n u[-n-1]$$

$P_1 = \frac{1}{4}$
 $P_2 = \frac{1}{3}$
 since $\frac{1}{4} < \frac{1}{3}$
 so both LSS

NOTE: If ROC is not provided, then there can be more than 1 solutions. In that case, you need to find all possible $x[n]$ for given $X(z)$.

PROPERTIES OF Z-TRANSFORM

① Linearity

$$\text{if } x_1[n] \xleftrightarrow{Z} X_1(z), \text{ ROC} = R_1$$

$$\& x_2[n] \xleftrightarrow{Z} X_2(z), \text{ ROC} = R_2$$

then

$$a x_1[n] + b x_2[n] \xleftrightarrow{Z} a_1 X_1(z) + b X_2(z), \text{ ROC} = R_1 \cap R_2$$

② Time Shifting

$$x[n] \xleftrightarrow{Z} X(z), \text{ ROC} = R$$

$$x[n-n_0] \xleftrightarrow{Z} z^{-n_0} X(z), \text{ ROC} = R$$

other than pole-zero
cancellation possibility

③ Scaling in z-domain

$$x[n] \xleftrightarrow{Z} X(z), \text{ ROC} = R$$

$$z_0^n x[n] \xleftrightarrow{Z} X\left(\frac{z}{z_0}\right), \text{ ROC} = |z_0| R$$

$$\text{if } z_0^n = r_0^n e^{j\omega_0 n}$$

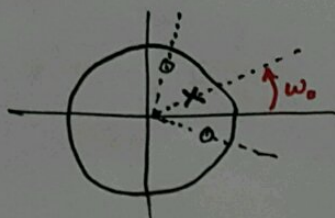
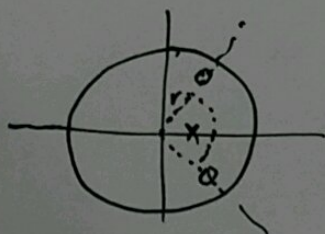
for $r_0 < 1$, ROC compress

for $r_0 > 1$, ROC expands

$$\& r_0 = 1 \Rightarrow z_0^n = e^{j\omega_0 n}, \text{ so}$$

$$e^{j\omega_0 n} \cdot x[n] \xleftrightarrow{Z} X\left(e^{j\omega_0} \frac{z}{z_0}\right), \text{ ROC} = R$$

Shift of pole location due to ω_0

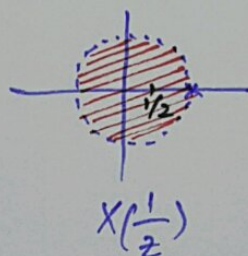
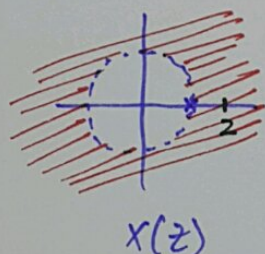


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④ Time Reversal

$$x[n] \xleftrightarrow{z} X(z), \quad \text{ROC} = R \rightarrow (\text{If } R_{SS})$$

$$x[-n] \xleftrightarrow{z} X\left(\frac{1}{z}\right), \quad \text{ROC} = \frac{1}{R} \quad \begin{array}{l} \text{Inverted} \\ \text{ROC} \\ \downarrow \\ \text{LSS} \end{array}$$



⑤ The Convolution Property

$$x_1[n] \xleftrightarrow{z} X_1(z), \quad \text{ROC} = R_1$$

$$x_2[n] \xleftrightarrow{z} X_2(z), \quad \text{ROC} = R_2$$

then

$$x_1[n] * x_2[n] \xleftrightarrow{z} X_1(z) X_2(z), \quad \text{ROC} = R_1 \cap R_2$$

⑥ Differentiation in z-Domain

$$x[n] \xleftrightarrow{z} X(z), \quad \text{ROC} = R$$

$$n x[n] \xleftrightarrow{z} -z \frac{d}{dz} X(z), \quad \text{ROC} = R$$

Example 10.18

$$x[n] = ? \quad \text{for} \quad X(z) = \frac{a z^{-1}}{(1 - a z^{-1})^2}, \quad |z| > |a|$$

$$\text{Since} \quad a^n u[n] = \frac{1}{1 - a z^{-1}}, \quad |z| > |a|$$

$$n a^n u[n] = -z \frac{d}{dz} \left(\frac{1}{1 - a z^{-1}} \right) = -z \cdot \frac{-a/z^2}{(1 - a z^{-1})^2} = \frac{+ a z^{-1}}{(1 - a z^{-1})^2}$$

so

$$n a^n u[n] \xleftrightarrow{z} \frac{a z^{-1}}{(1 - a z^{-1})^2}, \quad |z| > |a|$$