Lec:15

$$y = g(x)$$

$$E[y] = \sum_{x \in S_x} g(x)\, P_x(x)$$

$$E[y] = \sum_{y \in S_y} p\, P_y(y)$$

$$P_V(0) = \begin{cases} \frac{1}{7} & v = -3, -2, -1, 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

$$Y = \frac{V^2}{2}$$

$$E[Y] = \frac{1}{7}\left[\frac{9}{2} + \frac{4}{2} + \frac{1}{2} + 0 + \frac{1}{2} + \frac{4}{2} + \frac{9}{2}\right]$$

$$E[Y] = \frac{1}{7}[14]$$

$$\boxed{E[Y] = \mu_y = 2}$$

$$P_Y(y) = \begin{cases} 1/7 & y = 0 \\ 2/7 & y = 1/2, 2, 9/2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

$$E[Y] = 0\left(\frac{1}{7}\right) + \frac{2}{7}\left[\frac{1}{2} + \frac{4}{2} + \frac{9}{2}\right]$$

$$E[Y] = 0 + \frac{2}{7}\left[\frac{14}{2}\right]$$

$$\boxed{E[Y] = 2}$$

Variance:

$$VAR[Y] = E[(Y - \mu_Y)^2]$$

$$VAR[Y] = \sum (g(x) - \mu_Y)^2 \times P_X(x).$$

Theorem:

$$Y = aX + b.$$

$$E[Y] = E[aX + b]$$
$$E[Y] = aE[X] + b$$

$$E[b] = b.$$

Theorem:

$$Y = aX + b$$

$$VAR[Y] = VAR[aX + b]$$
$$VAR[Y] = a^2 \, VAR[X]$$

Cumulative Distribution Function: (CDF)

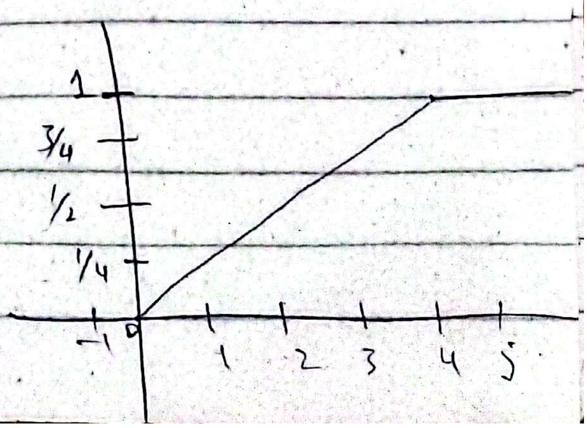
$$F_X(x) = P[X \le x].$$

(a) $F_X(-\infty) = 0$
(b) $F_X(+\infty) = 1$
(c) $P[x_1 \le X \le x_2] = F_X(x_2) - F_X(x_1)$

Quiz:

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y/4 & 0 \le y \le 4 \\ 1 & y > 4 \end{cases}$$

y	$y/4$
1	$1/4$
2	$1/2$
3	$3/4$
4	1



(a) $P[Y \le -1] = 0$

(b) $P[Y \le 1] = F_Y(1)$

$= 1/4$

(c) $P[2 < Y \le 3] = F_Y(3) - F_Y(2)$

$= \frac{3}{4} - \frac{2}{4}$

$= \frac{1}{4}$

(d) $P[Y > 1.5]$

$= 1 - P[Y \le 1.5]$

$= 1 - \frac{1.5}{4}$

$= 1 - \frac{15}{40}$

$= \frac{25}{40}$

$= \frac{5}{8}$

Probability Density function:
(PDF)

$$f_X(x) = \frac{d}{dx} F_X(x)$$

$$P[x_1 < X \le x_1 + \Delta] = \frac{F_X(x_1+\Delta) - F_X(x)\Delta}{\Delta}$$

(a) $f_X(x) \geq 0 \quad \forall x$

(b) $F_X(x) = \int f_X(u)\, du$

(c) $\int_{-\infty}^{+\infty} f_X(x)\, dx = 1$

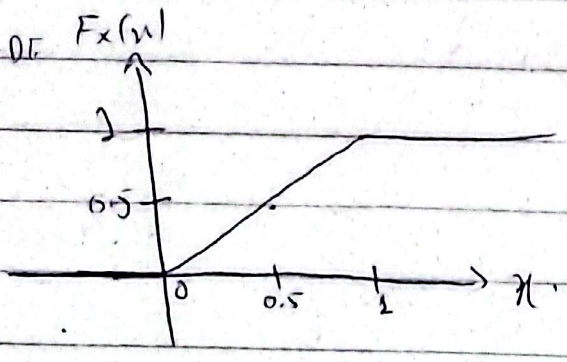
Lec : 16

$$F_X(x) = \begin{cases} 0 & x < 0, \\ x & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

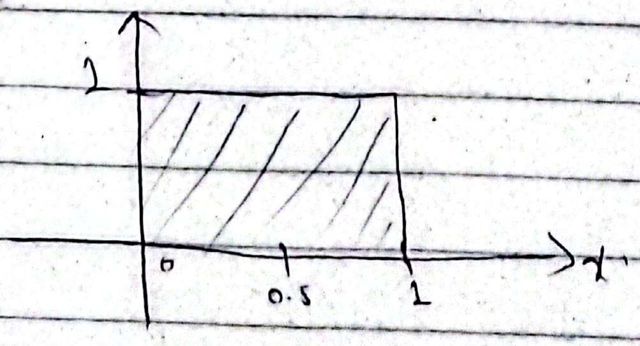
$$f_X(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x < 1, \\ 0 & x \ge 1. \end{cases}$$

$$f_X(x) = \begin{cases} 1 & 0 \le x < 1, \\ 0 & \text{otherwise}. \end{cases}$$

CDF



PDF



•) $\left\{ \frac{1}{4} < x \leq \frac{3}{4} \right\}$

with CDF

$$P\left[\frac{1}{4} < x \leq \frac{3}{4}\right] = F_X\left(\frac{3}{4}\right) - F_X\left(\frac{1}{4}\right)$$

$$= \frac{3}{4} - \frac{1}{4} = \frac{2}{4}$$

$$= \frac{1}{2}$$

with PDF:

$$P[1/4 < x \leq 3/4] = \int_{1/4}^{3/4} (1)\, dx$$

$$= x \Big|_{1/4}^{3/4} = \frac{3}{4} - \frac{1}{4}$$

$$= \frac{1}{2}$$

$$P[1/4 < x \leq 3/4] = (2)\left[\frac{3}{4} - \frac{1}{4}\right]$$

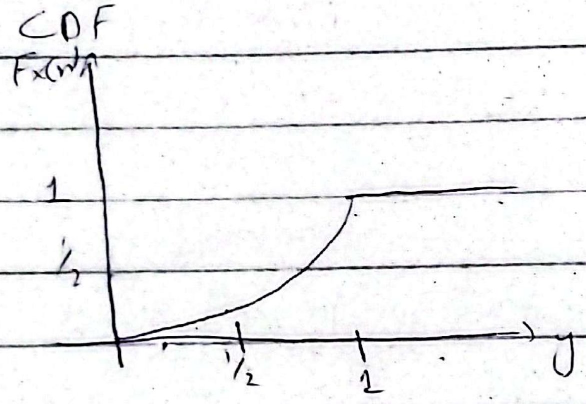
$$= \frac{1}{2}$$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y^3 & 0 \le y < 1 \\ 1 & y > 1 \end{cases}$$

(a) The PDF of Y:

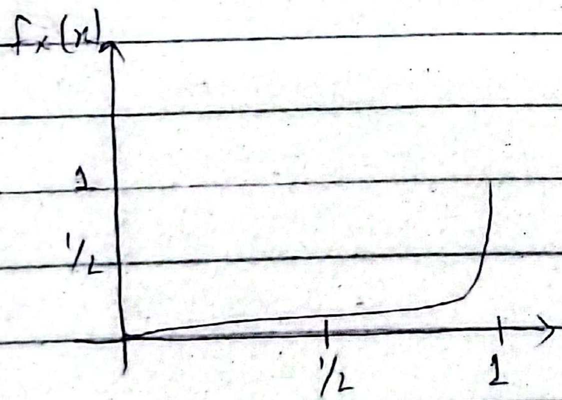
(b) $P[1/4 < Y \le 3/4]$

Sol:



(a)

$$f_Y(y) = \begin{cases} 3y^2 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$



$$= \int_0^1 3y^2 \, dy$$

$$= 1$$

1 (b)

with CDF

$$P\left[\frac{1}{4} < Y \leq \frac{3}{4}\right] = F_Y\left(\frac{3}{4}\right) - F_Y\left(\frac{1}{4}\right)$$

$$= \left(\frac{3}{4}\right)^3 - \left(\frac{1}{4}\right)^3$$

$$= \frac{27}{64} - \frac{1}{64}$$

$$= \frac{26}{64}$$

$$= \frac{13}{32}$$

with PDF

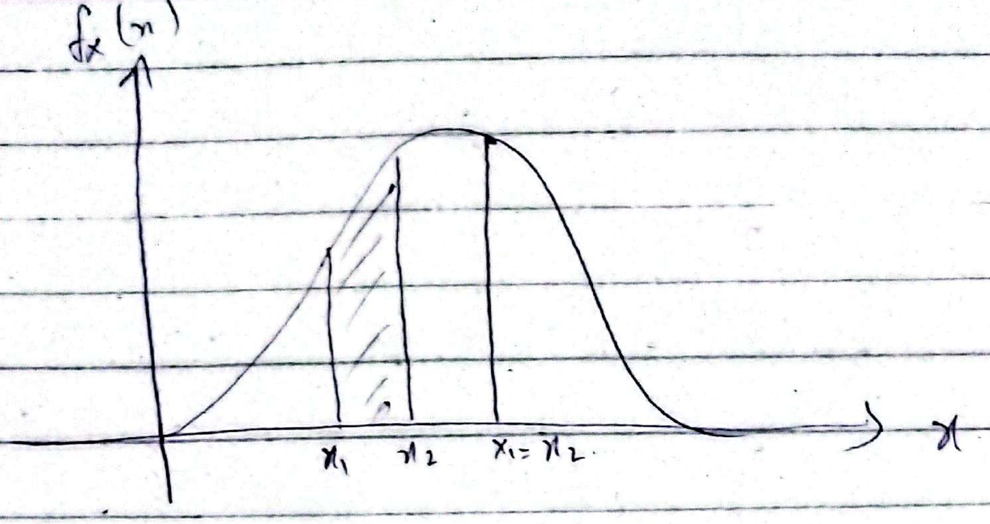
$$P[1/4 < x \leq 3/4] = 3\int_{1/4}^{3/4} y^2 \, dy.$$

$$= \frac{3}{3} \left| y^3 \right|_{1/4}^{3/4}$$

$$= \left[\left(\frac{3}{4}\right)^3 - \left(\frac{1}{4}\right)^3\right]$$

$$= \frac{13}{32}$$

Lec : 17



$$\int_{-\infty}^{+\infty} f_X(x)\, dx = 1.$$

$$P[x_1 \leq X \leq x_2] = \int_{x_1}^{x_2} f_X(x)\, dx = F_X(x_2) - F_X(x_1)$$

Quiz: 3.2

$$f_X(x) = \begin{cases} c x e^{-x/2} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) The constant "c"

$$\int_{-\infty}^{+\infty} f_X(x)\, dx = 1.$$

$$\int_{0}^{\infty} c x e^{-x/2}\, dx = 1$$

Integration By Parts:

$$\therefore \int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du$$

$u = cx \rightarrow du = c\,dx.$

$dv = e^{-x/2}dx \rightarrow v = \int e^{-x/2}\,dx$

$v = -2e^{-x/2}$

$$\int_0^{\infty} cxe^{-x/2}dx = \underbrace{(cx)(-2e^{-x/2})\Big|_0^{\infty}}_{\text{L'Hopital's Rule.}} - \int_0^{\infty} -2e^{-x/2}\,x\,c\,dx.$$

$$\lim_{x\to\infty} -2cxe^{-x/2}$$

$$= \lim_{x\to\infty} \frac{-2cx}{e^{x/2}}.$$

$$= \lim_{x\to\infty} \frac{-2c\,\frac{dx}{dx}}{\frac{d}{dx}e^{x/2}}.$$

$$= \lim_{x\to\infty} \frac{-2c}{\frac{1}{2}e^{x/2}} = 0$$

Now

$$c\int_0^{\infty} xe^{-x/2}dx = 2c\int_0^{\infty} e^{-x/2}dx$$

$$= 2c(-2)\left|e^{-x/2}\right|_0^{\infty}$$

$$= -4c\,[0-1]$$

$$4c = 1$$

$$\boxed{c = \frac{1}{4}}$$

Now

$$f_X(x) = \begin{cases} \frac{x}{4} e^{-x/2} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(b) CDF $F_X(x)$

(c) $P[0 \leq x \leq 4]$

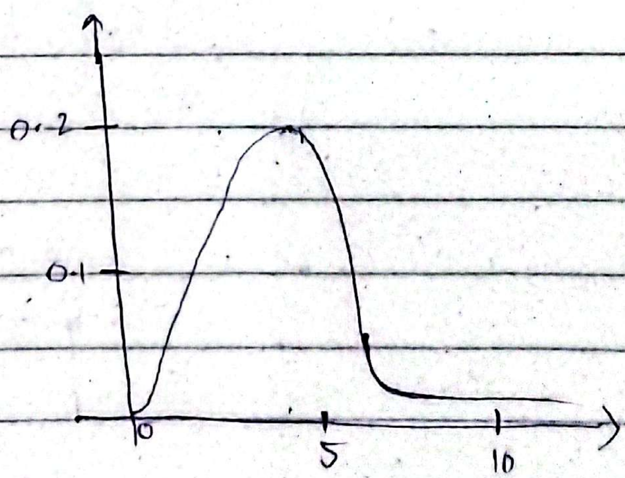
(d) $P[-2 \leq X \leq 2]$.

Lec: 18

~~(b) CDF:~~ (a)

$$f_X(x) = \begin{cases} \frac{1}{4} x e^{-x/2} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Sketch



x	$f_X(x)$
0	
2	
4	
6	
8	
10	

(b) CDF: $F_x(x)$

$$F_x(x) = \int_{-\infty}^{x} f_x(y)\, dy$$

$$= \int_{0}^{x} f_x(y)\, dy$$

5609948

$$= \int_{0}^{x} \frac{y}{4} e^{-y/2}\, dy$$

$$\because \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$u = \frac{y}{4}, \quad du = \frac{1}{4} dy$$

$$dv = e^{-y/2} dy$$

$$v = \int e^{-y/2} dy$$

$$v = -2e^{-y/2}$$

$$F_x(x) = \left(\frac{y}{4}\right)\left(-2e^{-y/2}\right)\Big|_0^x - \frac{(-2)}{4}\int_0^x e^{-y/2}\, dy$$

$$F_x(x) = -\frac{1}{2}\left[x e^{-x/2} - 0\right] + \frac{1}{2}(-2)\, e^{-y/2}\Big|_0^x$$

$$F_X(x) = -\frac{1}{2} x e^{-x/2} - \left[e^{-x/2} - 1 \right]$$

$$F_X(x) = -\frac{1}{2} x e^{-x/2} - e^{-x/2} + 1$$

$$F_X(x) = 1 - e^{-x/2} \left(\frac{1}{2} x + 1 \right)$$

$$F_X(x) = \begin{cases} 1 - e^{-x/2} \left(\frac{x}{2} + 1 \right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(c) $P[0 \le x \le 4] = F_X(4) - F_X(0)$

$= \left[1 - e^{-4/2}\left(\frac{4}{2} + 1\right)\right] - \left[1 - e^{-0/2}\left(\frac{0}{2} + 1\right)\right]$

$= 1 - e^{-2}(3) - [1 - 1(1)]$

$= \cancel{1 - 3e^{-2}}$

$$\boxed{= 1 - 3e^{-2}}$$

(d) $P[-2 \le X \le 2]$

$$= \int_{-2}^{2} \frac{1}{4} x e^{-x/2}\, dx$$

$$= \int_{0}^{2} \frac{1}{4} x e^{-x/2}\, dx$$

Alternate Method:

$P[-2 \le X \le 2] = F_X(2) - F_X(-2)$

$= 1 - e^{-1}(1+1) - 0$

$$\boxed{= 1 - 2e^{-1}}$$

Expected Value:

Discrete:

$$E[x] = \sum_{x_i \in R_x} x_i P_x(x_i)$$

Continous:

$$E[x] = \int_{-\infty}^{+\infty} x f_x(x)\, dx$$

$$E[x] = \int_0^{\infty} x \left(\frac{1}{4} x e^{-x/2}\right) dx$$

$$E[x] = \frac{1}{4} \int_0^{\infty} x^2 e^{-x/2}\, dx$$

Example:

$$f_x(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[x] = \int_{-\infty}^{+\infty} x f_x(x)\, dx$$

$$E[x] = \int_0^1 x\, dx$$

$$E[x] = \frac{1}{2} x^2 \Big|_0^1$$

$$\boxed{E[x] = \frac{1}{2}}$$

Example:

$$f_Y(y) = \begin{cases} 3y^2 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

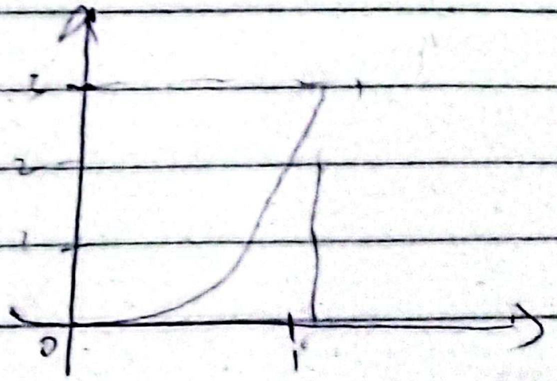
$$E[Y] = \int_0^1 y \, 3y^2 \, dy$$

$$= \int_0^1 y \, 3y^2 \, dy.$$

$$= 3\int_0^1 y^3 \, dy.$$

$$= 3 \left. \frac{y^4}{4} \right|_0^1$$

$$\mu_Y: \boxed{E[Y] = \frac{3}{4}}$$



y	$f_Y(y)$
0	0
1/4	3/16
1/2	3/4
3/4	9/16
1	3

$$W = g(x)$$

$$E[W] = E[g(x)] = \int_{-\infty}^{+\infty} g(x)\, f_x(x)\, dx.$$

Let

$$W = g(x) = x^2.$$

$$E[W] = E[g(x)] = E[x^2].$$

$$= \int_{-\infty}^{+\infty} x^2 f_x(x)\, dx.$$

Variance:

$$VAR[x] = \sigma_x^2 = E[(x - \mu_x)^2].$$

$$= E[x^2] - (E[x])^2.$$

Previous Qu

$$E[x^2] = \int_{-\infty}^{+\infty} x^2 f_x(x)\, dx$$

$$= \int_0^1 x^2\, dx,$$

$$= \frac{x^3}{3}\Big|_0^1$$

$$\boxed{E[x^2] = \frac{1}{3}}$$

$$VAR = \frac{1}{3} - \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{3} - \frac{1}{4}$$

$$\boxed{VAR = \frac{1}{12}}$$

Now $\sigma_x = \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}}$

~~Now the Qu~~

$E[y^2] = \int_0^1 y^2\, 3y^2\, dy$

$= 3\int_0^1 y^4\, dy$

$= \cancel{3}\frac{y^5}{\cancel{5}\,2}\Big|_0^1$

$= \frac{\cancel{3}}{\cancel{2}}$

$= \frac{3}{5}$

$VAR = \frac{3}{5} - \left[\frac{3}{4}\right]^2$

$= \frac{3}{5} - \frac{9}{16}$

$\boxed{VAR\ \sigma_y^2 = \frac{3}{80}}$

$\sigma_Y = \sqrt{\frac{3}{80}}$

Theorems

(a) $E[X - \mu_x] = 0$

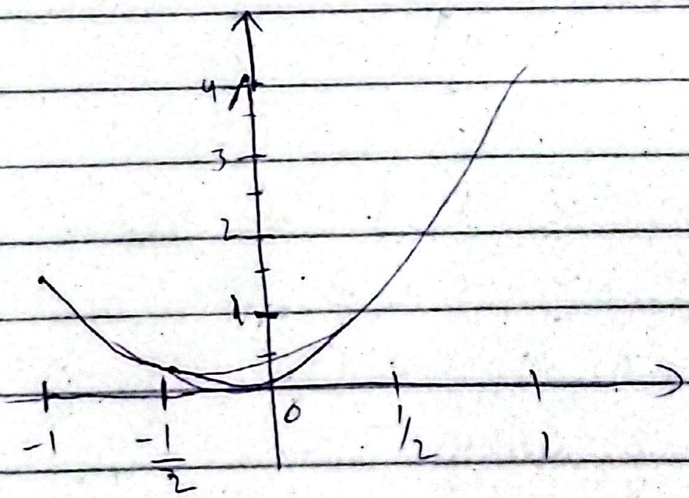
(b) $E[ax + b] = aE[x] + b$.

(c) $VAR[x] = E[x^2] - (E[x])^2$.

(d) $VAR[ax+b] = a^2 VAR[x]$.

Quiz 3.3

$$F_x(y) = \begin{cases} 3y^2/2 & -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$



X	$f_Y(y)$
-1	$+3/2$
$-1/2$	$3/8$
0	0
$1/2$	$3/8$
1	$3/2$.

(1) $E[Y]$

$$E[Y] = \int_{-\infty}^{+\infty} y f_Y(y)\,dy$$

$$= \int_{-1}^{1} y \frac{3y^2}{2}$$

$$= \frac{3}{2} \int_{-1}^{1} y^3$$

$$= \frac{3}{4} \times \left. \frac{y^4}{4} \right|_{-1}^{1}$$

$$= \frac{3}{4} [1 - 1]$$

$$E[Y] = 0$$

(2) $$E[Y^2] = \int_{-1}^{1} y^2 \frac{3y^2}{2}$$

$$= \frac{3}{2} \int_{-1}^{1} y^4$$

$$= \frac{3}{2} \times \left. \frac{y^5}{5} \right|_{-1}^{1}$$

$$E[Y^2] = \frac{3}{5}$$

(3) $\boxed{\text{VAR.} = \frac{3}{5}}$

(4) $\sigma_x = \sqrt{\frac{3}{5}}$

Lec : 19

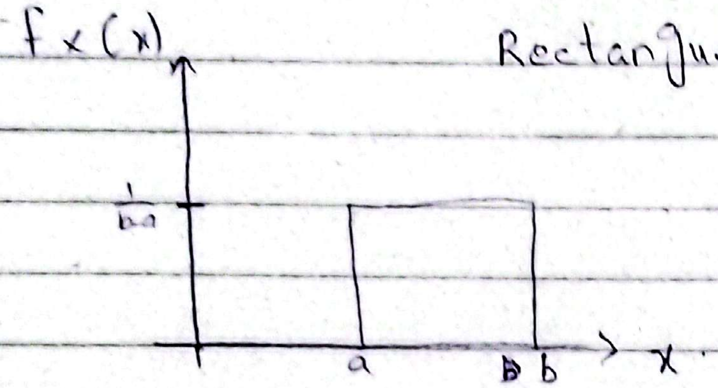
Continous Uniform Distribution:

$X \sim \text{uniform}(a, b)$.

$X \sim U(a, b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Rectangular Distribution



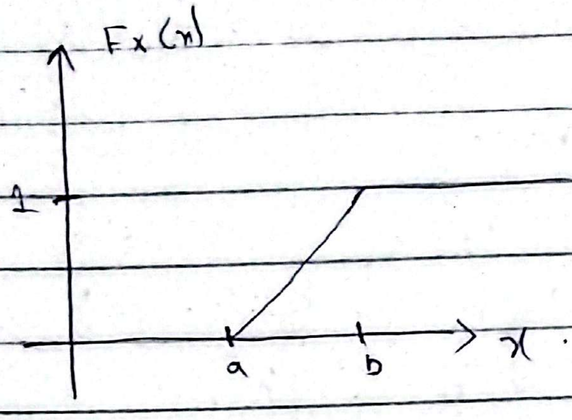
Now CDF:

$$F_X(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x \leq b \\ 1 & x > b \end{cases}$$

$$F_X(x) = \int_{-\infty}^{+\infty} f_X(y)\, dy = \int_a^x \frac{1}{b-a}\, dy.$$

$$= \frac{1}{b-a} \left| y \right|_a^x$$

$$= \frac{x-a}{b-a}$$



Expected val

$$E[x] = \int_{-\infty}^{+\infty} x\, f_x(x)\, dx.$$

$$E[x] = \frac{b+a}{2}$$

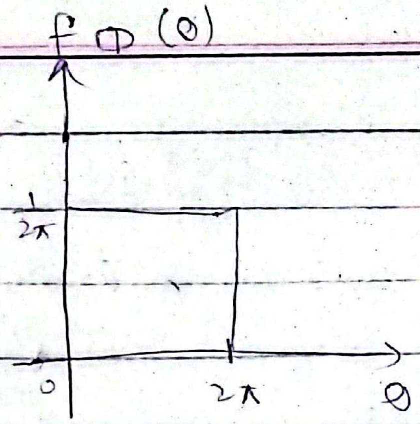
Now

$$VAR[x] = \frac{(b-a)^2}{12}.$$

Example:

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 < \theta \leq 2\pi, \\ 0 & \text{otherwise.} \end{cases}$$

$a = 0, \quad b = 2\pi$



Now CDF

$$F_\Phi(\theta) = \frac{\theta - 0}{2\pi - 0} = \frac{\theta}{2\pi}$$

$$F_\Phi(\theta) = \begin{cases} 0 & \theta < 0 \\ \frac{\theta}{2\pi} & 0 < \theta < 2\pi \\ 1 & \theta \geq 2\pi \end{cases}$$

$$E[x] = \frac{b+a}{2} \Rightarrow \frac{2\pi + 0}{2}$$

$$\boxed{E[x] = \pi}$$

$$VAR = \cancel{\frac{2\pi + 0}{12}} = \cancel{\frac{\pi}{6}} \quad \frac{(2\pi - 0)^2}{12}$$

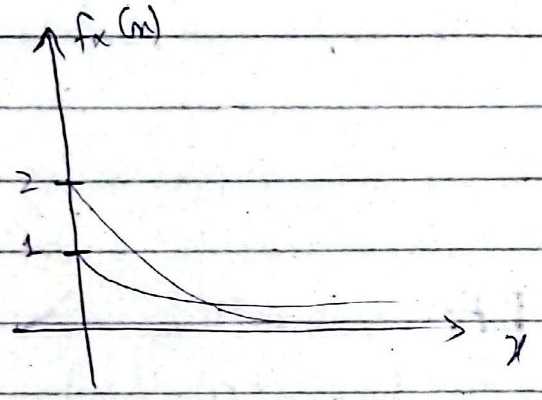
$$= \frac{4\pi^2}{12}$$

$$\boxed{VAR = \frac{\pi^2}{3}} \qquad \sigma_\Phi = \frac{\pi}{\sqrt{3}}$$

Exponential Distribution:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$\lambda > 0$ and is called rate Parameter.



Exponential Distribution is used to model the time elapsed between events.

E.g.

(a) How long a Piece of machinary will work.

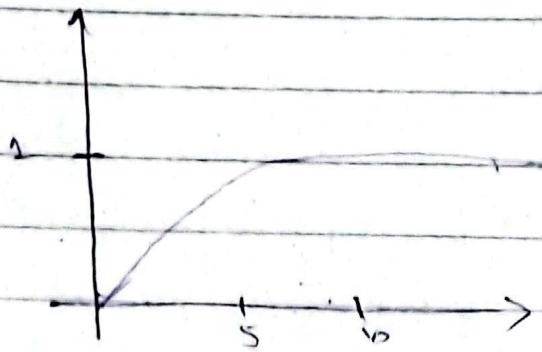
(b) How much time will elapse before flood in a region.

(c) How long will it take before you receive the next Phone call.

$7/4 \div 2$

Example: 3-12

$$F_T(t) = \begin{cases} 1 - e^{-t/3} & t \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$



(a) PDF:

$$f_T(t) = \frac{d}{dt} F_T(t)$$

$$f_T(t) = \frac{1}{3} e^{-t/3}$$

$$f_T(t) = \begin{cases} \frac{1}{3} e^{-t/3} & t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\boxed{\lambda = \frac{1}{3}}$$

~~$f_T(t)$~~

(b) $P[2 \le t < 4]$

$$P[2 \le t < 4] = F_T(4) - F_T(2)$$

$$= 1 - e^{-4/3} - (1 - e^{-2/3})$$

$$= e^{-2/3} - e^{-4/3}$$

$$\boxed{P[2 \le t < 4] = 0.2498}$$

(a) $$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) $$E[X] = \frac{1}{\lambda}$$

(c) $$VAR[X] = \frac{1}{\lambda^2}$$

(d) The nth moment is:

$$E[X^n] = \frac{n!}{\lambda^n}$$

Lec: 20

Example: 3.13

CDF.

$$F_T(t) = \begin{cases} 1 - e^{-t/3} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

PDF.

$$f_T(t) = \begin{cases} \frac{1}{3} e^{-t/3} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) $E[T]$

$$\mu_T = E[T] = \frac{1}{\lambda} = \frac{1}{1/3} = 3$$

(b)

$$\sigma_T^2 \quad VAR[T] = \frac{1}{(1/3)^2} = 9$$

$$\sigma_T = 3$$

Alternatively:

$$E[T] = \int_{-\infty}^{\infty} t \, f_T(t) \, dt$$

$$E[T] = \frac{1}{3} \int_0^{\infty} t \, e^{-t/3} \, dt = 3$$

$$VAR = \sigma_T^2 = E[T^2] - \{E[T]\}^2$$

$$E[T^2] = \frac{1}{3}\int_0^\infty t^2 e^{-t/3}\,dt = 18$$

(C) $[\mu_T - \sigma_T < T \leq \mu_T + \sigma_T]$

$[3 - 3 < T \leq 3 + 3]$

$[0 < T \leq 6] = F_T(6) - F_T(0)$

$= 1 - e^{-2} - [1 - e^{0}]$

$= 1 - e^{-2}$

$\boxed{[0 < T \leq 6] = 0.8647}$

Independent and identically Distributed Random variable.

Erlang distribution.

Quiz 3.4

$E[x] = 3$

$VAR[x] = 9$

(a) Exponential PDF:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[x] = \frac{1}{\lambda} = 3 \Rightarrow \lambda = \frac{1}{3}$$

$$f_X[x] = \begin{cases} \frac{1}{3} e^{-x/3} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) Uniform: - PDF

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x < b \\ 0 & \text{otherwise} \end{cases}$$

$$E[x] = \frac{b+a}{2} = 3$$

$$VAR[x] = \frac{(b-a)^2}{12} = 9.$$

$$b + a = 6.$$

$$(b-a)^2 = 108.$$

$$a = 8.195 , \quad b = -2.195$$

$$a = -2.195 , \quad b = 8.195 .$$

$$(\text{as } b > a)$$

$$f_x(x) = \frac{1}{8.195 + 2.195} = \frac{1}{10.39}.$$

$$f_x(x) = \begin{cases} \frac{1}{10.39} & -2.195 < x < 8.195 \\ 0 & \text{otherwise.} \end{cases}$$

Gaussian Random Variable.
Normal Random variable.

$X \sim \text{Gaussian}(\mu, \sigma)$
$X \sim \text{Normal}(\mu, \sigma^2)$

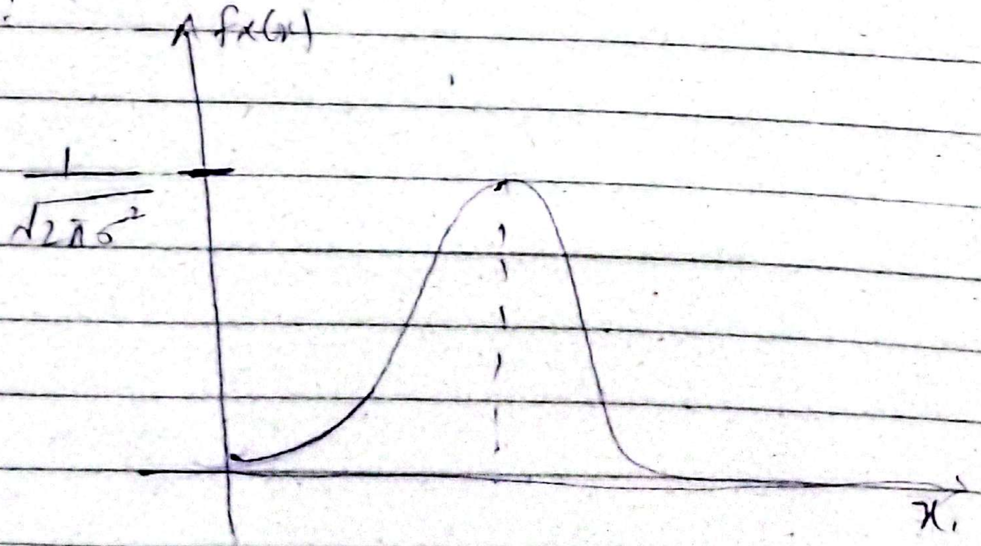
PDF.

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$E[X] = \mu$
$VAR[X] = \sigma^2$

Plot:



Theorem:

A linear transformation of Normal random variable Produces another normal random variable.

$$Y = aX + b$$

$$\mu_y = a\,\mu_x + b$$

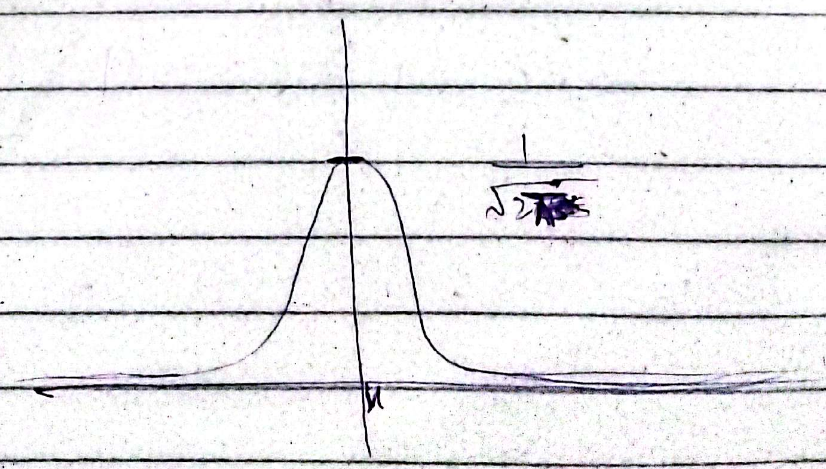
$$\sigma_y = a\,\sigma_x$$

$$\mu = 0, \quad \sigma^2 = 1$$

then we call the Gaussian distribution as

Standard Normal Distribution

Plot:



$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad (1)$$

Relation:

$$f_X(x) = \frac{1}{\sigma} \varphi\left(\frac{x-\mu}{\sigma}\right)$$

Let $z = \frac{x-\mu}{\sigma}$ in (1)

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$P[a < X \leq b] = F_X(b) - F_X(a)$$
$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

$$P[z_1 < Z \leq z_2] = \Phi(z_2) - \Phi(z_1).$$

$P[a < x \leq b] = P[z_1 < Z \leq Z_2]$.

where:

$$z_1 = \frac{a - \mu}{\sigma}$$

$$z_2 = \frac{b - \mu}{\sigma}$$

Example:

$x = 48$.

Gaussian $(61, 10)$

$\sigma = 10$

$\mu = 61$.

$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{48 - 61}{10}$$

$$= \frac{-15}{10}$$

$$\boxed{z = -1.5}$$

Theorem:

$$\Phi(-z) = 1 - \Phi(z).$$

Now:

$$X \sim \text{Gaussian}(61, 10)$$

$$P[X \le 46].$$

$$\mu = 61, \quad \sigma = 10$$

$$z = \frac{46 - 61}{10} = -1.5$$

$$P[X \le 46] = F_X(46) = \Phi(-1.5)$$

$$= 1 - \Phi(1.5)$$

$$= 1 - 0.9332$$

$$\boxed{P[X \le 46] = 0.0668}$$

$$P[51 < X \le 71] = F_X(71) - F_X(51).$$

$$z_1 = \frac{51 - 61}{10} = -1$$

$$z_2 = \frac{71 - 61}{10} = 1$$

From Tables

Therefore:

$P[51 < X \leq 71] = P[-1 < Z \leq 1]$

$= \Phi(1) - \Phi(-1).$

$= \Phi(1) - [1 - \Phi(1)].$

$= 2\Phi(1) - 1.$

$= 2 \times 0.8413 - 1$

$\boxed{P[51 < X \leq 71] = 0.6826}$

Lec 21

$\Phi(2.98) = 0.99856$
$\Phi(2.99) = 0.99861$
$\Phi(3) = 0.9987$
$\Phi(4) = 0.9999768$

Standard Normal Complementary CDF.
→ Q-Function.

$$Q(z) = 1 - \Phi(z)$$

$$Q(z) = P[Z > z]$$

$$P[Z > z] = 1 - P[Z \leq z]$$

$$Q(z) = 1 - \Phi(z).$$

Example: 3.18

$$Q\left(\sqrt{\frac{\gamma}{2}}\right)$$

γ = SNR

$Q(z) < 10^{-6}$
$z \geq 4.76$

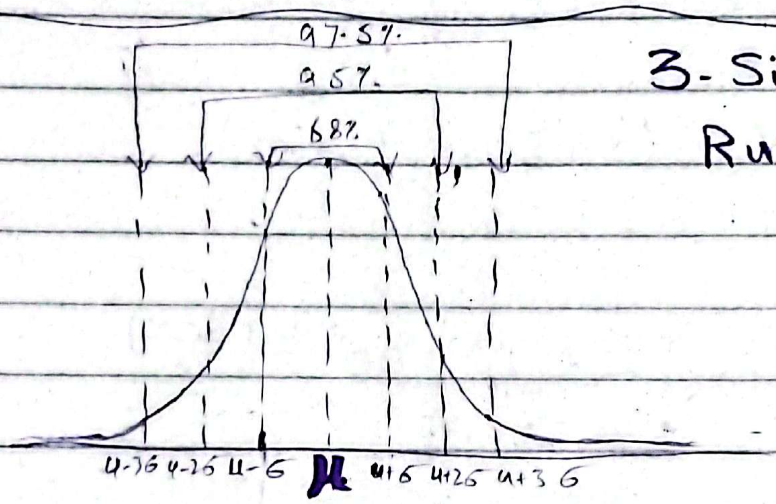
$$\sqrt{\frac{\gamma}{2}} = 4.76$$

$$\frac{\gamma}{2} = 22.6576$$

$$\gamma = 45.3152.$$

Therefore

$P[E] < 10^{-6}$ when $\gamma \geq 45.3152.$



3- Sigma Rule

68% values are within 1σ of mean

95% values are within 2σ of mean

97.5% values are within 3σ of mean

Example:

$X \sim \text{Gaussian}(0, 1)$

$\mu = 0, \quad \sigma = 1$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Peak value of $f_X(x) = \frac{1}{2\pi} = 0.3989 \simeq 0.4$

$Y \sim \text{Gaussian}(0, 2)$

$\mu = 0, \quad \sigma = 2.$

$$f_Y(y) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-0}{2}\right)^2}$$

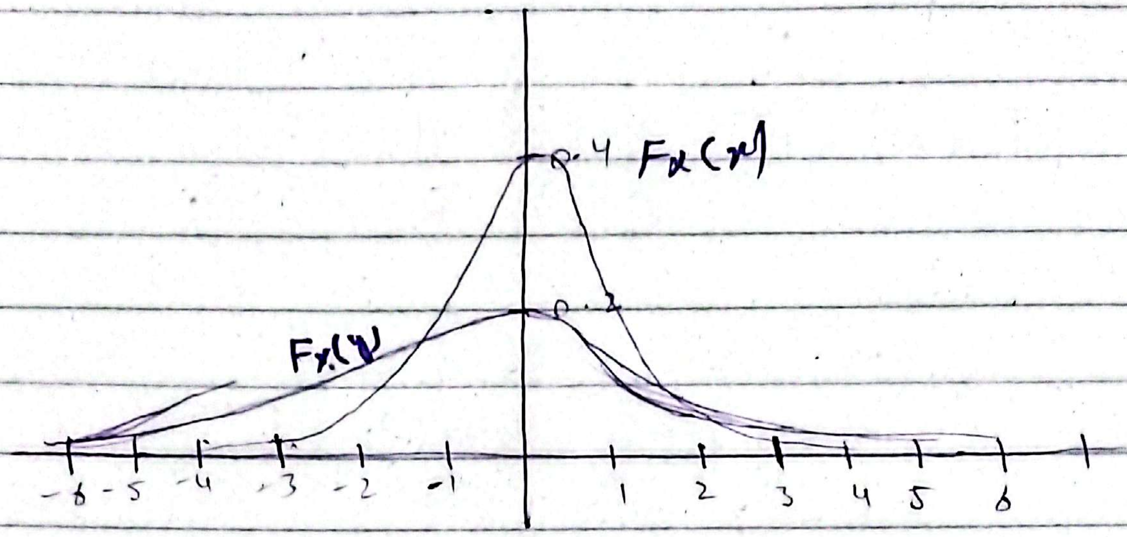
$$f_Y(y) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{y^2}{8}}$$

Peak value of $f_Y(y) = \frac{1}{2\sqrt{2\pi}}$

$= 0.1995$

$\simeq 0.2$

(a) Sketch:



(b) $P[-1 < X \le 1] = P[-1 < Z \le 1]$

$= \Phi(1) - \Phi(-1)$

$= \Phi(1) - [1 - \Phi(1)]$

$= 2\Phi(1) - 1$

$= 2 \times 0.8413$ (from Table)

$= 0.6826.$

(c) $P[-1 < Y \le 1] = F_Y(1) - F_Y(-1)$

$$z_1 = \frac{-1-0}{2} = -\frac{1}{2}$$

$$z_2 = \frac{1-0}{2} = \frac{1}{2}$$

$$\boxed{Z = \frac{X - \mu}{\sigma}}$$

$$P[-1 < Y \le 1] = \Phi\left(\tfrac{1}{2}\right) - \Phi\left(-\tfrac{1}{2}\right)$$
$$= \Phi\left(\tfrac{1}{2}\right) - \left[1 - \Phi\left(\tfrac{1}{2}\right)\right].$$

$$= 2\Phi\left(\tfrac{1}{2}\right) - 1$$
$$= (2 \times 0.6915) - 1$$
$$= 0.383.$$

(d) $P[X > 3.5] = P[Z > 3.5]$
$$= Q(3.5).$$
$$= 2.33 \times 10^{-4}.$$

(E) $P[Y > 3.5] =$

$$Z = \frac{3.5 - 0}{2} = 1.75$$

$$P[Y > 3.5] = Q(1.75) = 1 - \Phi(1.75)$$
$$= 1 - 0.9599.$$

$$P[Y > 3.5] = 0.0401$$

$$Y = g(x)$$

Example:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \geq 1 \end{cases}$$

$$Y = g(x) = 100X$$

$$F_Y(y) = P[Y \le y] = [100X \le y].$$

$$F_Y(y) = \left[X \le \frac{y}{100}\right] = F_X\left(\frac{y}{100}\right)$$

$$F_Y(y) = \begin{cases} 0 & y/100 < 0 \\ y/100 & 0 \le y/100 < 1 \\ 1 & y/100 \geq 1 \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y/100 & 0 \le y < 100 \\ 1 & y \geq 100 \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

PDF:

$$f_Y(y) = \begin{cases} 100 & 0 \le y < 10 \\ 0 & \text{otherwise} \end{cases}$$

$Y \sim$ Uniform $(0, 100)$.

PDF of FX:

$$f_X(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 < x < 1 \\ 0 & x \ge 1 \end{cases}$$

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$X \sim$ uniform $(0, 1)$.

Theorem:

If $Y = aX$, where $a > 0$,
then Y has the CDF

$$F_Y(y) = F_X\left(\frac{y}{a}\right)$$

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y}{a}\right)$$

Lec: 22

Pairs of Random Variables:

Joint CDF of X, Y.

$$F_{X,Y}(x,y) = P[X \le x, Y \le y]$$

$$F_X(x) = P[X \le x] = P[X \le x, Y \le \infty]$$
$$= \lim_{y \to \infty} F_{X,Y}(x,y) = F_{X,Y}(x,\infty)$$

Theorem:

(a) $0 \le F_{X,Y}(x,y) \le 1.$

(b) $F_X(x) = F_{X,Y}(x, \infty)$

(c) $F_Y(y) = F_{X,Y}(\infty, y)$

(d) $F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = 0$

(e) If $x \le x_1$ and $y \le y_1$ then

$$F_{X,Y}(x,y) \le F_{X,Y}(x_1, y_1).$$

(e) $F_{X,Y}(\infty, \infty) = 1$.

Joint PMF.

$P_{X,Y}(x, y) = P[X = x, Y = y]$

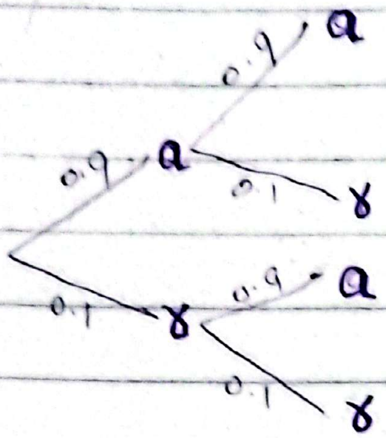
Example: 4-1

$p = 0.9$

Accept $= a$
Reject $= r$
$P[a] = 0.9$
$P[r] = 1 - P[a] = 1 - 0.9$
$= 0.1$

No. of acceptable circuits $= X$
No. of successful tests
Before First Reject $= Y$.

$S = \{aa, ar, ra, rr\}$



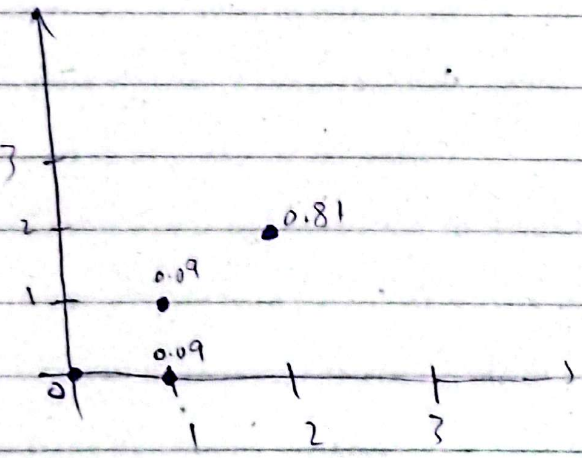
$P[aa] = 0.81$ $\quad X=2, Y=2$

$P[a\gamma] = 0.09$ $\quad X=1, Y=1$

$P[\gamma a] = 0.09$ $\quad X=1\ Y=0$

$P[\gamma\gamma] = 0.01$ $\quad X=0, Y=0$

S	aa	aγ	γa	γγ
P(·)	0.81	0.09	0.09	0.01
X	2	1	1	0
Y	2	1	0	0



	$Y=0$	$Y=1$	$Y=2$	$P_X(x)$
$X=0$	0.01	0	0	0.01
$X=1$	0.09	0.09	0	0.18
$X=2$	0	0	0.81	0.81
$P_Y(y)$	0.10	0.09	0.81	1

$$P_{X,Y}(x,y) = \begin{cases} 0.81 & x=2,\ y=2 \\ 0.09 & x=1,\ y=1 \\ 0.09 & x=1,\ y=0 \\ 0.01 & x=0,\ y=0 \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{x \in S_X} \sum_{y \in S_X} P_{X,Y}(x,y) = 1.$$

$$P_X(x) = \begin{cases} 0.01 & x = 0 \\ 0.18 & x = 1 \\ 0.81 & x = 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$P_Y(y) = \begin{cases} 0.10 & y = 0 \\ 0.09 & y = 1 \\ 0.81 & y = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.2

$P[B] = P_{X,Y}(2,2) + P_{X,Y}(1,1) + P_{X,Y}(0,0)$

$= 0.81 + 0.09 + 0.1$

$P[B] = 0.91$

Quiz 4.2

$P_{Q,G}(q,g)$	$g=0$	$g=1$	$g=2$	$g=3$
$q=0$	0.06	0.18	0.24	0.12
$q=1$	0.04	0.12	0.16	0.08
~~$q=2$~~				
g				

Joint PDF.

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\, dv\, du.$$

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y}.$$

$$P[a < X \leq b] = F_X(b) - F_X(a)$$

$$P[a < X \leq b,\ c < Y \leq d].$$

$$P[x_1 < X \leq x_2,\ y_1 < Y \leq y_2] = \cancel{F_{X,Y}(x_2, y_2)}$$
$$= F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1)$$

Theorems

(a) $f_{X,Y}(x,y) \geq 0 \quad \forall\, x, y.$

(b) $$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = 1.$$

Example: 4.4

$$f_{X,Y}(x,y) = \begin{cases} c & 0 \leq x \leq 5,\ 0 \leq y \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

(a) $c = ?$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = 1.$$

$$\int_{x=0}^{5}\int_{y=0}^{3} c\,dy\,dx = 1.$$

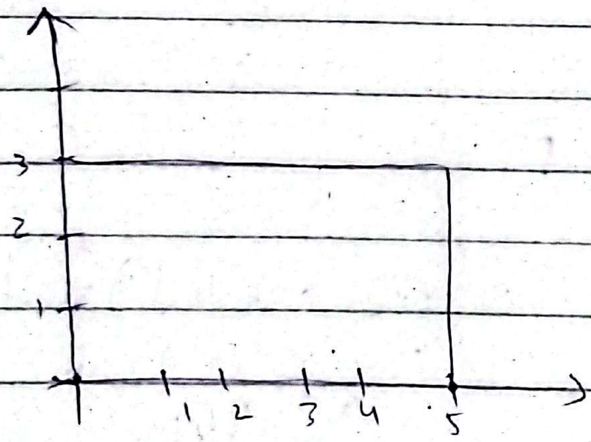
$$c\int_{x=0}^{5} |y|_0^3\,dx.$$

$$= 3c \left| x \right|_{0}^{5}$$

$$= 15c = 1.$$

$$\boxed{c = \frac{1}{15}}$$

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{15} & 0 \le x \le 5,\ 0 \le y \le 3 \\ 0 & \end{cases}$$



(b) $P[A] = P[2 \le x < 3,\ 1 \le y < 3]$.

$$P[A] = \int_{x=2}^{3} \int_{y=1}^{3} \frac{1}{15}\, dy\, dx.$$

$$P[A] = \frac{1}{15}\int_{x=1} [y]_1^3 \, dx$$

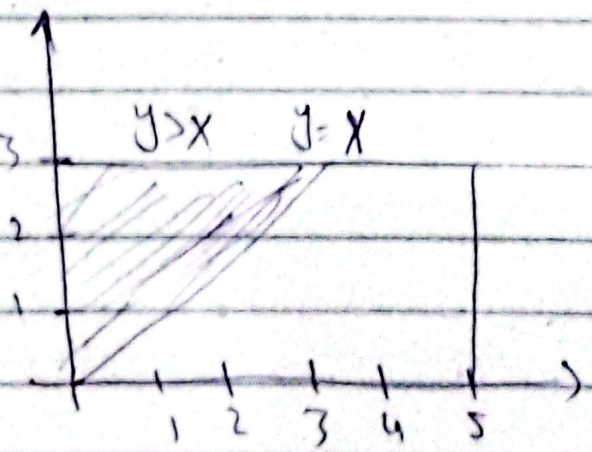
$$= \frac{2}{15}[x]_1^3$$

$$\boxed{P[A] = \frac{2}{15}}$$

Example: 4.6

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{15} & 0 \le x < 5,\ 0 \le y < 3 \\ 0 & \text{otherwise} \end{cases}$$

$$P[A] = P[Y > X].$$



$$P[A] = \int_{y=0}^{3}\int_{x=0}^{y} \frac{1}{15} \, dx \, dy$$

$$P[A] = \frac{1}{15}\int_{y=0}^{3} |\cancel{x}|_0^y$$

$$\cancel{= \frac{1}{15}\int_0^3 y \, dy}$$

$$\cancel{= \frac{1}{15} |y|_0^3}$$

$$\cancel{= \frac{3}{15}}$$

$$\cancel{\boxed{P[A] = \frac{1}{5}}}$$

$$\cancel{= \frac{1}{30} |y^2|_0^3}$$

$$= \frac{1}{15}\int_0^3 y \, dy$$

$$= \frac{1}{30} |y^2|_0^3$$

$$= \frac{9}{30} = \boxed{P[A] = \frac{3}{10}}$$

Lec: 23

$$f_{X,Y}(x,y) = \begin{cases} cxy & 0 \le x \le 1 ; 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Constant c:

$$\int_{x=0}^{1}\int_{y=0}^{2} cxy \, dy \, dx$$

$$\int_{x=0}^{1} cx \frac{y^2}{2} \Big|_0^2 \, dx$$

$$\int_{x=0}^{1} 2cx \, dx$$

$$\Rightarrow 2c \frac{x^2}{2} \Big|_0^1$$

$$\Rightarrow \frac{2c}{2} = 1$$

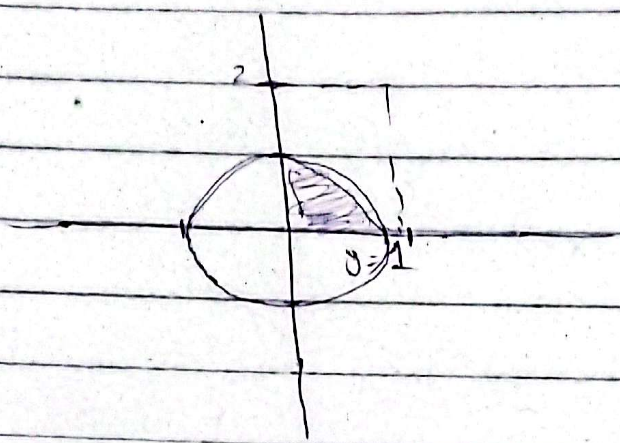
$$\Rightarrow \boxed{c = 1}$$

(b) ~~$P[A] = P[x^2 + y^2 \le 1]$~~

$$f_{X,Y}(x,y) = \begin{cases} xy & 0 \le x \le 1, 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

1b) $P[A] = P[x^2 + y^2 \le 1]$

$x^2 + y^2 = 1$ is circle.



$y^2 = 1 - x^2$

$y = \pm\sqrt{1-x^2}$

$y = +\sqrt{1-x^2}$ $\quad \therefore y = 1$

$$P[A] = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} xy \, dy \, dx.$$

coma ✗
Cama ✓

Marginal PDF.

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x, y)\, dy$$

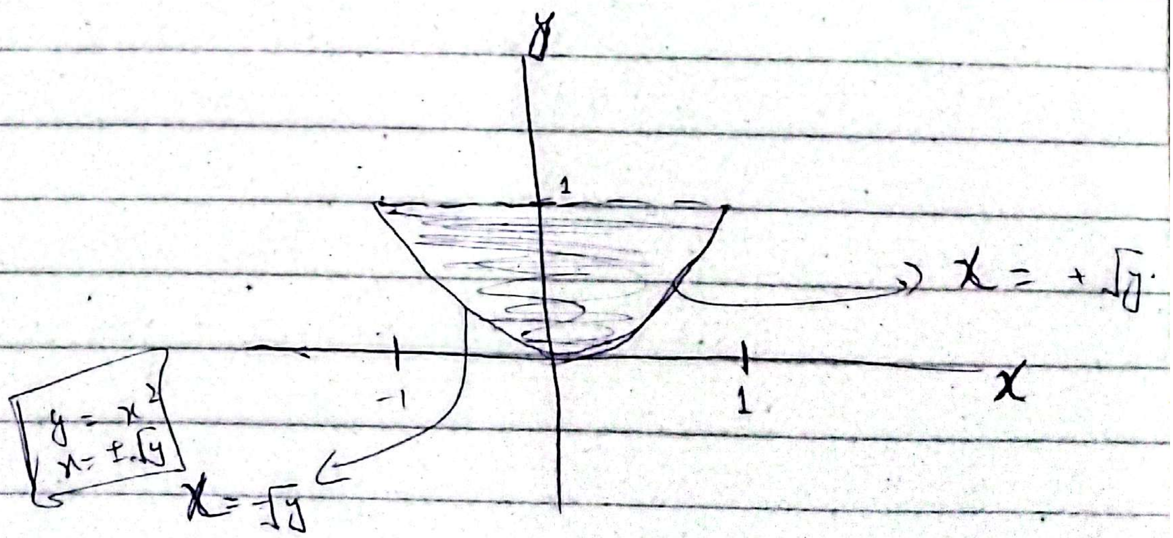
$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x, y)\, dx$$

CDF:

f

Example 4.7

$$f_{X,Y}(x, y) = \begin{cases} 5y/4 & -1 \le x \le 1,\ x^2 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$



$$f_X(x) = \int_{y=x^2}^{1} \frac{5}{4} y \, dy$$

$$= \frac{5}{8} \left| y^2 \right|_{x^2}^{1}$$

$$f_X(x) = \frac{5}{8} (1 - x^4)$$

Therefore

$$f_X(x) = \begin{cases} \frac{5}{8}(1-x^4) & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{5}{4} y \, dx$$

$$= \frac{5}{4} y \cdot \left| x \right|_{-\sqrt{y}}^{\sqrt{y}}$$

$$= \frac{5}{4} y (\sqrt{y} + \sqrt{y})$$

$$= \frac{5}{4} y \times 2\sqrt{y}$$

$$= \frac{5}{2} y \, dy$$

$$f_Y(y) = \frac{5}{2} y^{3/2}$$

$$f_Y(y) = \begin{cases} \frac{5}{2} y^{3/2} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$W = g(X, Y)$$

$$E[X] = \int_{-\infty}^{+\infty} x \, f_X(x) \, dx$$

continuous $\leftarrow$ $$E[W] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y) \, f_{X,Y}(x, y) \, dx \, dy$$

Discrete $$E[W] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x, y) \, P_{X,Y}(x, y)$$

$$E[X] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x f_{X,Y}(x, y) \, dx \, dy$$

$$E[Y] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y \, f_{X,Y}(x,y) \, dx \, dy.$$

$$Var[X] = E[X^2] - (E[X])^2$$

$$E[X^2] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 \, f_{X,Y}(x,y) \, dx \, dy.$$

$$E[Y^2] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y^2 \, f_{X,Y}(x,y) \, dx \, dy$$

Theorem:

$$\cancel{E[g_1(X,Y) + g_2(X,Y)] = E[g_1(X,Y)] + E[g}$$

$$E[g_1(X,Y) + g_2(X,Y) \ldots g_n(X,Y)].$$
$$= E[g_1(X,Y)] + E[g_2(X,Y)] + \cdots + E[g_n(X,Y)]$$

Theorem:

$$E[X+Y] = E[X] + E[Y].$$

Theorem:

$$Var[X+Y] = Var[X] + Var[Y] + 2E[(X-\mu_x)(Y-\mu_y)]$$

$$Var[X+Y] = \sigma_x^2 + \sigma_y^2 + 2\sigma_{x,y}^2$$

Co-Variance:

For two random variables X and Y the covariance is defined as:

$$Cov[X,Y] = \sigma_{x,y} = E[(X-\mu_x)(Y-\mu_y)]$$

$$Cov[X,Y] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x-\mu_x)(y-\mu_y) f_{x,y}(x,y)\, dx\, dy.$$

$$Cov[X,Y] = \sum_{y\in S_y}\sum_{x\in S_x} (x-\mu_x)(y-\mu_y) f_{x,y}(x,y)$$

Corelation:

correlation of Rvs X and Y

$$r_{x,y} = E[XY].$$

Theorem:

(a) $\text{cov}[x, y] = r_{xy} - \mu_x \mu_y$

$\text{cov}[x, y] = E[xy] - E[x]E[y]$

(b) If $X = Y$

$\text{Cov}[x, y] = E[x^2] - (E[x])^2$

$\cancel{\text{Var}} = \text{Var}[x] = \text{Var}[y].$

Example:

$P_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$	$P_X(x)$
$x=0$	0.01	0	0	0.01
$x=1$	0.09	0.09	0	0.18
$x=2$	0	0	0.81	0.81
$P_Y(y)$	0.10	0.09	0.81	

(c) r_{XY}

$$r_{XY} = E[XY] = \sum_{x=0}^{2} \sum_{y=0}^{2} xy\, P_{X,Y}(x,y).$$

$$= (0)(0)(0.01) + (0)(1)(0) + (0)(2)(0)$$
$$+ (1)(0)(0.09) + (1)(1)(0.09) + (1)(2)(0)$$
$$+ (2)(0)(0) + (2)(1)(0) + (2)(6)(0.81)$$

$$r_{XY} = 0.09 + 3.24$$
$$= 3.33$$

$$\boxed{r_{XY} = E[XY] = 3.33}$$

$$E[X] = (0)(0.01) + (1)(0.18) + (2)(0.81)$$
$$E[X] = 1.80$$

$$E[Y] = (0)(0.10) + (1)(0.09) + (2)(0.81)$$
$$E[Y] = 1.71$$

$$\mathrm{cov}[X,Y] = \sigma_{XY} = r_{XY} - \mu_X \mu_Y$$

$$= 3.33 - (1.80)(1.71)$$
$$= 0.252$$

$$\boxed{\mathrm{Cov}[X,Y] = 0.252}$$

CamScanner

Lec: 24

Correlation:

$$r_{xy} = E[xy]$$

Covariance:

$$cov[x,y] = E[xy] - E[x]$$
$$= r_{xy} - \mu_x \mu_y.$$

Orthogonal Random variable:

$$r_{xy} = 0$$
$$E[xy] = 0$$

Uncorrelated Random variable:

$$cov[xy] = 0.$$
$$\sigma_{xy} = E[xy] - E[x]E[y] = 0.$$
$$- E[xy] = E[x]E[y].$$

Correlation Coefficient.

$$\rho_{xy} = \frac{cov[x,y]}{\sqrt{VAR[x]\,VAR[y]}}$$

$$= \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

$$-1 \leq \rho_{xy} \leq +1.$$

Quiz 4.7

(A) → Discrete

$P_{L,T}(l,t)$	$t=40s$	$t=60s$	$P_L(l)$
$l=1P$	0.15	0.1	0.25
$l=2P$	0.30	0.2	0.50
$l=3P$	0.15	0.1	0.25
$P_T(t)$	0.60	0.40	1

$$P_L(l) = \begin{cases} 0.25 & l = 1, 3 \\ 0.50 & l = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$P_T(t) = \begin{cases} 0.60 & t = 40, \\ 0.40 & t = 60, \\ 0 & \text{otherwise} \end{cases}$$

(1)

$E[L] = 2$ Pages.

$VAR[L] = E[L^2] - (E[L])^2$

$= 4.5 - (2)^2$

$\boxed{VAR[L] = 0.5}$

(2) $E[T]$ and $VAR[T]$:

$E[T] = 48$ second.

$VAR[T] = E[T^2] - (E[T])^2$

$= 2400 - (48)^2$

$$\boxed{VAR[T] = 96}$$

(3) $r_{L,T} = E[LT]$.

$= \sum_{l=1}^{3} \sum_{t=40,60} l \times t \times P_{LT}(l,t)$

$r_{L,T} = (1)(40)(0.15) + (1)(60)(0.1)$
$+ (2)(40)(0.30) + (2)(60)(0.2)$
$+ (3)(40)(0.15) + (3)(60)(0.1)$

$$\boxed{r_{L,T} = 96} \rightarrow E[LT].$$

(4) $COV[L,T]$.

$COV[L,T] = \sigma_{L,T} = E[LT] - E[L]E[T]$

$= 96 - (2 \times 48)$

$$\boxed{COV[LT] = 0}$$

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(5) $$\rho_{L,T} = \frac{\text{COV}[L,T]}{\sqrt{\text{VAR}[L]\ \text{VAR}[T]}} = 0$$

$$\boxed{\rho_{L,T} = 0}$$

(B) ⇒ Contions.

$$f_{X,Y}(x,y) = \begin{cases} xy & 0 \le x \le 1,\ 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

~~(1) E[X], VAR[X]~~.

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy$$

$$f_X(x) = \int_{0}^{2} xy\, dy$$

$$= \frac{1}{2} x \left. y^2 \right|_0^2$$

$$\boxed{f_X(x) = 2x}$$

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{x=0}^{1} xy\,dx$$

$$= \frac{1}{2} y \left[x^2\right]_0^1$$

$$f_Y(y) = \frac{1}{2} y$$

$$f_Y(y) = \begin{cases} xy & 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

4) $E[X]$, $VAR[X]$

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx$$

$$E[X] = \int_0^1 (x)(2x)\,dx$$

$$E[X] = \frac{2}{3} \left| x^3 \right|_0^1$$

$$\boxed{E[X] = \frac{2}{3}}$$

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Alternatively:

$$E[Y] = \int_{y=-\infty}^{+\infty} \int_{x=-\infty}^{+\infty} x \, f_{X,Y}(x,y) \, dx \, dy$$

$$E[X] = \int_{y=0}^{2} \int_{x=0}^{1} x \, (x y) \, dx \, dy.$$

$$VAR[X] = E[X^2] - (E[X])^2.$$

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x) \, dx$$

$$= \int_{x=0}^{1} x^2 \, 2x \, dx$$

$$E[X^2] = \frac{1}{2}$$

So,

$$VAR[X] = \frac{1}{2} - \left(\frac{2}{3}\right)^2$$

$$\boxed{VAR[X] = \frac{1}{18}}$$

(2) $E[Y]$, $VAR[Y]$

$$E[Y] = \int_{-\infty}^{+\infty} y\, f_Y(y)\, dy$$

$$= \int_{y=0}^{2} y \left(\frac{y}{2}\right) dy$$

$$\boxed{E[Y] = \frac{4}{3}}$$

$$VAR[Y] = E[Y^2] - (E[Y])^2$$

$$E[Y^2] = \int_{-\infty}^{+\infty} y^2\, f_X(y)\, dy$$

$$= \int_{0}^{2} y^2 \left(\frac{y}{2}\right) dy$$

$$E[Y^2] = 2$$

$$VAR[Y] = 2 - \left(\frac{4}{3}\right)^2$$

$$\boxed{VAR[Y] = \frac{2}{9}}$$

(3) $r_{X,Y} = E[XY]$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f_{X,Y}(x,y)\, dx\, dy.$$

$$= \int_{y=0}^{2}\int_{x=0}^{1} (xy)^2\, dx\, dy.$$

$$= \int_{y=0}^{2} \frac{y^2}{3} \left| x^3 \right|_0^1 dy$$

$$= \frac{1}{3}\int_{y=0}^{2} y^2\, dy.$$

$$= \frac{1}{9}\left| y^3 \right|_0^2$$

$$\boxed{r_{X,Y} = \frac{8}{9}} = E[XY].$$

(4) $\mathrm{cov}[X,Y]$.

$$\mathrm{cov}[X,Y] = r_{XY} - \mu_X \mu_Y$$

$$= \frac{8}{9} - \left(\frac{2}{3}\right)\left(\frac{4}{3}\right) = 0.$$

$$\boxed{\mathrm{cov}[X,Y] = 0}$$

15) ρ_{XY}.

$$\rho_{XY} = \frac{\mathrm{Cov}[X, Y]}{\sqrt{\mathrm{VAR}[X]\,\mathrm{VAR}[Y]}} = 0$$

Important.

Independent Random Variables.

$$P[AB] = P[A] \times P[B].$$

Random variables X and Y are independent if and only if

Discrete $P_{X,Y}(x,y) = P_X(x) \times P_Y(y)$

Continuous $f_{X,Y}(x, y) = f_X(x) \times f_Y(y)$

Observe

$P_{X|Y}(x|y) = P_X(x)$, $P_{Y|X}(y|x) = P_Y(y)$

$f_{X|Y}(x|y) = f_X(x)$, $f_{Y|X}(y|x) = f_Y(y)$

continuous

Examples:

$$f_{X,Y}(x,y) = \begin{cases} 4xy & 0 \le x < 1, 0 < y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent ?

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy$$

$$= 4 \int_{y=0}^{1} xy\, dy$$

$$= \frac{4x}{2} \left| y^2 \right|_0^1$$

$$f_X(x) = 2x$$

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dx$$

$$= 4 \int_{=0}^{1} xy\, du$$

$$= \frac{4}{2} y \left| x^2 \right|_0^1$$

$$f_Y(y) = 2y$$

$$f_Y(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

We observe that

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

$$4xy = (2x)(2y)$$

$$4xy = 4xy$$

Therefore X and Y are independent.

Theorems:

For independent random variables X and Y.

(a) $E[g(x)\,h(y)] = E[g(x)]\,E[h(y)]$.

(b) $Cov[X,Y] = \sigma_{XY} = 0$

(c) $E[XY] = E[X]\,E[Y]$.

(d) ~~$VAR[X+Y] =$~~

(d) $VAR[X+Y] = VAR[X] + VAR[Y] + 2Cov[X,Y]$

As $Cov[X,Y] = 0$

$VAR[X+Y] = VAR[X] + VAR[Y]$.

(e) $E[X \mid Y = y] = E[X]$.

(f) $E[Y \mid X = x] = E[Y]$.

while $cov[X,Y] = 0$ is a necessary property for independence, but it is not sufficient

Example 4.25.

$P_{X,Y}(x,y)$	$y=-1$	$y=0$	$y=1$	$P_X(x)$
$x=-1$	0	0.25	0	0.25
$x=1$	0.25	0.25	0.25	0.75
$P_Y(y)$	0.25	0.50	0.25	1

(1) Are X and Y independent.

$$P_{X,Y}(x,y) = P_X(x)\, P_Y(y)$$

$P_X(1) = 0.75$

$P_Y(1) = 0.25$

$P_{X,Y}(1,1) = 0.25$

$P_X(1)\, P_Y(1) = 0.75 \times 0.25$

$= 0.1875$

$\neq P_{X,Y}(1,1)$

Hence

X and Y are not independent

(2) Are X and Y uncorrelated?

$$\text{cov}[X,Y] = E[XY] - E[X]E[Y]$$

$$E[XY] = \sum_{y=-1}^{1} \sum_{x=-1}^{1} xy \cdot P_{X,Y}(x,y)$$

$$\boxed{E[XY] = 0}$$

$$E[X] = 0.50$$
$$E[Y] = 0$$

$$\text{Cov}[X,Y] = 0 - (0.50)(0)$$
$$\boxed{\text{Cov}[X,Y] = 0}$$

X and Y are uncorrelated.

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Theorem:

$X_1, X_2, \dots X_n$

$W = X_1 + X_2 + \dots X_n$

$$E[W] = E[X_1] + E[X_2] + \dots E[X_n].$$

$$VAR[W] = \sum_{i=1}^{n} VAR[X_i] + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} COV[X_i, X_j]$$

if $X_1, X_2, \dots, X_n$ are uncorrelated

$$VAR[W] = \sum_{i=1}^{n} VAR[X_i] = VAR[X_1] + VAR[X_2] \dots + VAR[X_n].$$

$$\mathcal{L}[f_1(x) * f_2(x)] = F_1(s) F_2(s).$$

$$f_1(x) * f_2(x) = \mathcal{L}^{-1}[F_1(s) F_2(s)].$$

1) characteristic Function.

2) Moment Generating Function (MGF)

3) Probability Generating Function (PGF)

Moment Generating Function:

$$\Phi_x(s) = E[e^{sx}]$$

Continous Random variable

$$\Phi_x(s) = \int_{-\infty}^{+\infty} e^{sx} f_x(x)\, dx$$

Discrete Random variable.

$$\Phi_x(s) = \sum_{x \in S_x} e^{sx_i} P_x(x).$$

Theorem:

→ $E[x^n] = \frac{d^n}{ds^n} \Phi_x(s) \Big|_{s=0}$

•) $\Phi_x(s) \big|_{s=0} = \Phi_x(0) = 1.$

•) $Y = X_1 + Y_2.$

$\Phi_y(s) = \Phi_{x_1}(s)\, \Phi_{x_2}(s).$

Example

Derive MGF of exponential Random Variable.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\Phi_x(s) = E\left[e^{sx}\right].$$

$$\Phi_x(s) = \int_{-\infty}^{+\infty} f_x(x)\, dx.$$

$$= \int_{0}^{\infty} e^{sx}\left(\lambda e^{-\lambda x}\right) dx.$$

$$= \lambda \int_{0}^{\infty} e^{-(\lambda - s)x}\, dx.$$

$$= \frac{-\lambda}{\lambda - s} \left| e^{-(\lambda - s)} \right|_{0}^{\infty}$$

$$= \frac{-\lambda}{\lambda - s} \left[\frac{1}{e^{\infty}} - e^{0} \right]$$

$$\Phi(s) = \frac{-\lambda}{\lambda - s} \left[0 - 1 \right]$$

$$\boxed{\Phi_x(s) = \frac{\lambda}{\lambda - s}}$$

Let's find nth moment

$$E[x^n] = \frac{d^n}{ds^n} \Phi_x(s) \Big|_{s=0}$$

$$- \frac{d}{ds} \Phi_x(s) = \lambda(\lambda - s)^{-1}$$

$$= -\lambda(\lambda - s)^{-2} (-1)$$

$$\frac{d}{ds} \Phi_x(s) = \frac{\lambda}{(\lambda - s)^2} = \lambda(\lambda - s)^{-2}$$

$$\frac{d^2}{ds^2} \Phi_x(s) = -2\lambda(\lambda - s)^{-3} (-1)$$

$$= \frac{2\lambda}{(\lambda - s)^3}$$

$$\frac{d^n}{ds^n} \Phi_x(s) = \frac{n!}{(\lambda - s)^{n+1}}$$

Now

$$E[x] = \frac{\lambda}{(\lambda - s)^2} \Big|_{s=0} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}.$$

$$E[x^2] = \frac{2\lambda}{(\lambda-s)^3}\bigg|_{s=0} = \frac{2\lambda}{\lambda^3}$$

$$E[x^2] = \frac{2}{\lambda^2}$$

$$E[x^n] = \frac{n!\,\lambda}{(\lambda-s)^{n+1}}\bigg|_{s=0}$$

$$\boxed{E[x^n] = \frac{n!}{\lambda^n}}$$

Theorem:

For a set of independent random variables $x_1, x_2, \ldots, x_n$.

Let $W = x_1 + x_2 + \cdots + x_n$.

$$\Phi_W(s) = \Phi_{x_1}(s)\,\Phi_{x_2}(s) \cdots \Phi_{x_n}(s)$$

When $x_1, x_2, \ldots, x_n$ are independent and identically distributed then.

$$\therefore \Phi_{x_1}(s) = \Phi_{x_2}(s) = \cdots = \Phi_{x_n}(s)$$

$$\Phi_w(s) = [\Phi_x(s)]^n$$

Example:

$$P_J(j) = \begin{cases} 0.2 & j = 1, \\ 0.6 & j = 2, \\ 0.2 & j = 3, \\ 0 & \text{otherwise,} \end{cases}$$

$$P_K(k) = \begin{cases} 0.5 & k = -1, \\ 0.5 & k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

(a) MGF of $M = J + K$

$$\Phi_x(s) = E[e^{sx}]$$

$$\Phi_x(s) = \sum_{x_i \in S_x} e^{sx_i} P_x(x_i)$$

$$\Phi_J(s) = \sum_{j \in S_J} e^{sj} P_J(j)$$

$$\Phi_J(s) = 0.2e^{s} + 0.6e^{2s} + 0.2e^{3s}$$

$$\Phi_K(s) = 0.5e^{-s} + 0.5e^{s}$$

$$m = j + k$$

$$\Phi_M(s) = \Phi_J(s) \cdot \Phi_K(s)$$

$$= (0.2e^{s} + 0.6e^{2s} + 0.2e^{3s}) \times (0.5e^{-s} + 0.5e^{s})$$

$$= 0.1 + 0.1e^{2s} + 0.3e^{s} + 0.3e^{3s} + 0.1e^{2s} + 0.1e^{4s}$$

$$\Phi_M(s) = 0.1 + 0.3e^{s} + 0.2e^{2s} + 0.3e^{3s} + 0.1e^{4s}$$

(b) $E[M^3]$

$$\frac{d}{ds}\Phi_M(s) = 0.3e^{s} + 0.4e^{2s} + 0.9e^{3s} + 0.4e^{4s}$$

$$\frac{d^2}{ds^2}\Phi_M(s) = 0.3e^{s} + 0.8e^{2s} + 2.7e^{3s} + 1.6e^{4s}$$

$$\frac{d^3}{ds^3}\Phi_m(s) = 0.3e^{s} + 1.6e^{2s} + 8.1e^{3s} + 6.4e^{4s}$$

$$E[M^3] = \frac{d^3}{ds^3}\Phi_m(s)\Big|_{s=0}$$

$$= 0.3 + 1.6 + 8.1 + 6.4$$

$$\boxed{E[M^3] = 16.4}$$

(c) $P_M(m)$

for $m=0$, 0.1
$m=1$, 0.3
$m=2$, 0.2
$m=3$, 0.3
$m=4$, 0.1

$$P_m(m) = \begin{cases} 0.1 & m = 0, 4 \\ 0.3 & m = 1, 2 \\ 0.2 & m = 2 \\ 0 & \text{otherwise} \end{cases}$$

Continued from Lec 25

(c) continued:

$$E[M^3] = (0)^3 \times 0.1 + (4)^3 \times 0.1 + (1)^3 \times 0.3 + (3)^3 (0.3) + 2^3 (0.2)$$

$$\boxed{E[M^3] = 18.4}$$

Theorem:

Let $X_1, X_2 \ldots X_n$ be a collection of IID Random variables, each with MGF $\Phi_X(s)$.
Let N be a non-negative integer valued random variable that is independent of $X_1, X_2 \ldots X_n$.

The random sum

$$R = X_1 + X_2 + \cdots + X_n$$

has the MGF

$$\Phi_R(s) = \Phi_N(\ln \Phi_X(s))$$

As — (Bernoulli dist)

Central limit Theorem.

Given n independent random variables X_i

Let

$$X = X_1 + X_2 + \cdots X_n = \sum_{i=1}^{n} X_i$$

X is a random variable with mean μ

$$\mu = \mu_1 + \mu_2 + \cdots \mu_n$$

and variance σ^2

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + \cdots \sigma_n^2$$

According to the central limit Theorem, as n increases, the distribution $F_X(x)$ of X approaches a normal distribution with the same mean and variance.

$$F_X(x) \simeq \Phi\left(\frac{x - \mu}{\sigma}\right)$$

Imp

If x_i are continuous type.

$$f_x(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Imp If x_i are IID.

$$E[x] = \mu = n\mu_x$$

$$VAR[x] = \sigma^2 = n\sigma_x^2$$

$$\sigma = \sqrt{n}\,\sigma_x$$

$$z = \frac{x-\mu}{\sigma} = \frac{\sum_{i=1}^{n} x_i - n\mu_x}{\sqrt{n}\,\sigma_x}$$

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$$Z = \frac{\sum_{i=1}^{n} X_i - n\mu_x}{\sqrt{n}\,\sigma_x}$$

Example: 6.14

Let x_i be one bit which is 0 or 1

$X_i \sim \text{Bernoulli}(0.5)$
$p = 0.5$

$E[X_i] = p = 0.5$
$VAR[X_i] = p(1-p)$
$= 0.5(1-0.5)$
$\sigma_x^2 = VAR[X_i] = 0.25$

No. of bits $= n = 10^6 = 1000000$
$W = X_1 + X_2 + \cdots + X_n = \sum_{i=1}^{n=10^6} X_i$

As X_i are IID, therefore
$\mu = E[W] = n\mu_x$
$\mu = 10^6 \times 0.5$
$= 5 \times 10^5$
$\mu = 500000$

$$\sigma^2 = VAR[W] = n\,VAR[X_i]$$
$$= 10^6 \times 0.25$$
$$\sigma^2 = 250000$$

$$\sigma = \sqrt{250000} = 500$$

Now: $P[W \geq 502000]$

$$Z = \frac{W - \mu}{\sigma}$$

$$= \frac{502000 - 500000}{500}$$

$$Z = 4$$

$$P[W \geq 502000] = 1 - \Phi(4) = Q(4).$$

$$P[W \geq 502000] = 3.17 \times 10^{-5}$$

Example: 6.15

$$499000 \leq W \leq 501000$$
$$P[A] = P[499000 \leq W \leq 501000].$$
$$= F_W(501000) - F_W(499000)$$

$$= \Phi\left(\frac{501000 - 500000}{500}\right) - \Phi\left(\frac{499000 - 500000}{5}\right)$$

$= \Phi(2) - \Phi(-2)$

$= \Phi(2) - [1 - \Phi(2)]$

$= 2\Phi(2) - 1$

$P[A] = 2 \times 0.97725 - 1$

$\boxed{P[A] = 0.9545}$

Chapter 7 . CLO3

Sample mean:

$$M_n(X) = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

Theorem:

$$E[M_n(X)] = E[X].$$

$$VAR[M_n(X)] = \frac{VAR[X]}{n}.$$

Markov Inequality: (tells upper Bound)

For a random variable X such that $(X \geq 0)$ $P[X < 0] = 0$, and a constant $K > 0$,

$$P[X \geq K] \leq \frac{E[X]}{K} = \frac{\mu_X}{K}$$

$\boxed{f_X(x) = 0 \text{ for } x < 0}$

Example.

$E[X] = 5.5$ feet.

$K = 11$ feet.

$$P[X \geq 11] \leq \frac{5.5}{11} = \frac{1}{2}$$

$$P[X \geq 11] \leq \frac{1}{2}$$

Markov Inequality $\rightarrow$ Upper Bound
Loose Bound.

Chebyshev Inequality.

For an arbitrary random variable X, and constant $K > 0$,

$$P[|X - \mu_X| \geq K] \leq \frac{VAR[X]}{K^2} = \frac{\sigma^2}{K^2} \quad (1)$$

Let:

$K = n\sigma$ in (1).

$$P[|X - \mu_X| \geq n\sigma] \leq \frac{\sigma^2}{n^2 \sigma^2} = \frac{1}{n^2}$$

$$P[|X - \mu_X| \geq n\sigma] \leq \frac{1}{n^2} \quad (2)$$

multiply (1) by "-1"

$$-P[|X-\mu_x| \geq K] \geq -\frac{\sigma^2}{K^2}$$

Add 1 to both sides.

$$1-P[|X-\mu_x| \geq K] \geq 1-\frac{\sigma^2}{K^2}$$

However:

$$1-P[Y \geq y] = P[Y < y]$$

$$P[|X-\mu_x| < K] \geq 1-\frac{\sigma^2}{K^2} \quad (3)$$

$$P[|X-\mu_x| < n\sigma] \geq 1-\frac{1}{n^2} \quad (4)$$

$|X-\mu_x| \geq K$

$\pm(X-\mu_x) \geq K$

$X-\mu_x \geq K$ and $-(X-\mu_x) \geq K$

$(X-\mu_x) \geq K$ implies

$\mu_x+K \leq X \leq \mu_x-K$.

$$P[\mu_x + K < X \; \mu_x - K] \leq \frac{\sigma^2}{K^2} \quad \text{— ① (B)}$$

$$P[\mu_x + n\sigma \leq X < \mu_x - n\sigma] \leq \frac{1}{n^2} \quad \text{— ② (B)}$$

$$P[\mu_x - K \leq X < \mu_x + K] > 1 - \frac{\sigma^2}{K^2} \quad \text{— ③ (B)}$$

$$P[\mu_x - n\sigma \leq X < \mu_x + n\sigma] > 1 - \frac{1}{n^2} \quad \text{— 4 (B)}$$

Example.

$$E[X] = 5.5 \text{ feet}$$

$$K = 11 \text{ feet}$$

$$\sigma = 1$$

$$P[X \geq 11] = P[X - \mu_x \geq 11 - \mu_x]$$

$$= P[|X - \mu_x| \geq 11 - 5.5]$$

$$= P[|X - \mu_x| \geq 5.5]$$

According to Chebyshev Inequality.

$$P[|x - \mu_x| \geq k] \leq \frac{\sigma^2}{k^2}.$$

$$P[x \geq 11] = P[|x - \mu_x| \geq 5.5] \leq \frac{1}{(5.5)^2}$$

$$= 0.033058.$$

$$\boxed{P[x \geq 11] \leq 0.033088}$$

Chebyshev $\rightarrow$ upper Bound

Tight Bound.

Example:

$\mu = 75$, $\sigma = 3$.

At Least what percent of the students have marks between 66 and 84.

$$\mu - 3\sigma = 75 - 3 \times 3 = 75 - 9.$$

$$\boxed{\mu - 3\sigma = 66}$$

$$\mu + 3\sigma = 75 + 9$$

$$\boxed{\mu + 3\sigma = 84}$$

$$P[66 \le X \le 84] = P[\mu - 3\sigma \le X \le \mu + 3\sigma]$$

$$= P[|X - \mu_X| < n\sigma] \ge 1 - \frac{1}{n^2}$$

$$= 1 - \frac{1}{9} = \frac{8}{9}.$$

$$\therefore P[|X - \mu_X| < 3\sigma] \ge \frac{8}{9}$$

$$= 88.89\,\%.$$

Indicator Random variable

$$X_A = \begin{cases} 1 & \text{If event } A \text{ occurs} \\ 0 & \text{otherwise.} \end{cases}$$

$X_A \sim \text{Bernoulli}(p)$.

Suppose:

Probability of event $A = P[A]$.

$$P_{X_A}(x_A) = \begin{cases} P[A] & x_A = 1, \\ 0 & \text{otherwise} \end{cases}$$

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Indicator Random Variable:

Let

Event : A.

Probability : P[A].

Indicator RV : X_A

$$P_{X_A}(x_A) = \begin{cases} 1 - P[A] & x_A = 0, \\ P[A] & x_A = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\hat{P}[A] = \frac{X_1 + X_2 + X_3 + \cdots X_n}{n}$$

↳ Relative Frequency Notion

- Point Estimate
- Interval Estimate

Properties of Point Estimate:

1 Bias:

Actual Parameter = r

$E[\hat{R}]$

Estimated Parameter = $\hat{R}$

$$\text{Bias} = E[\hat{R}] - \gamma$$

In case, the estimator is unbiased

$$\text{Bias} = E[\hat{R}] - \gamma = 0$$

$$\therefore \boxed{E[\hat{R}] = \gamma}$$

2 Consistency:

The sequence of estimates $\hat{R}_1, \hat{R}_2, \ldots$ of Parameter γ is Consistent if for any $\epsilon > 0$,

$$\lim_{n \to \infty} P\left[|\hat{R}_n - \gamma| \geq \epsilon \right] = 0$$

$$\therefore |\hat{R}_n - \gamma| \leq \epsilon$$

$$\hat{R}_n \simeq \gamma$$

3. Accuracy:

↳ By **Mean square Error:**

The MSE of Estimator $\hat{R}$ of γ is

$$e = E[(R - \gamma)^2] \quad \text{--- (1)}$$

If $\hat{R}$ is unbiased estimator of r, then,

$$E[\hat{R}] = r$$

$$r = E[\hat{R}] \quad \text{--- (2)}$$

Put in (1)

$$e = E[(\hat{R} - E[\hat{R}])^2] \quad \text{--- (3)}$$

It is basically the variance ~~of~~.

Root Mean Square Error. (RMSE).

$$RMSE = \sqrt{e} = \sqrt{E[(\hat{R} - r)^2]}$$

If $\hat{R}$ is unbiased.

$$RMSE = \sqrt{e} = \sqrt{E[(\hat{R} - E[\hat{R}])^2]}$$

↳ standard Error.

Theorem:

The sample mean $M_n(x)$ is an unbiased estimate of $E[x]$.

Theorem:

The sample mean estimator $M_n(x)$ has mean square error.

$$e_n = E[(M_n(x) - E[x])^2]$$
$$= VAR[M_n(x)]$$
$$= \frac{VAR[X]}{n}$$

Example:

No. of (independent) Trials = n.
Estimator $\hat{P}_n[A]$
$P[A]$

$\sqrt{e} < 0.1$.

$$P_n[A] = M_n(x_A) = \frac{X_{A_1} + X_{A_2} + \cdots + X_{A_n}}{n}$$

MSE of $M_n(x_A)$ is given by

$$e_n = E[(m_n(X_A) - E[m_n(X_A)])^2].$$

But $E[m_n(X_A)] = E[X_A]$.

as it is unbiased estimator

Therefore.

$$e_n = E[(m_n(X_A) - E[X_A])^2] = VAR[m_n(X_A)]$$

$$= \frac{VAR[X_A]}{n}.$$

$$P_{X_A}(x_A) = \begin{cases} 1-P[A] & x_A = 0, \\ P[A] & x_A = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$E[X_A] = 0 \times (1-P[A]) + 1 \times P[A] = P[A].$$

$$E[X_A^2] = 0 + 1^2 \times P[A] = P[A].$$

$$VAR[X_A] = E[X_A^2] - (E[X_A])^2.$$

$$= P[A] - (P[A])^2.$$

$$VAR[X_A] = P[A](1-P[A])$$

$$e_n = \frac{VAR[X_A]}{n} = \frac{P[A](1-P[A])}{n}$$

$$\sqrt{e_n} < 0.1$$
$$e_n < 0.01$$

$$\frac{P[A]\,(1-P[A])}{n} < 0.01$$

$$P[A] = 1 - P[A]$$
$$2P[A] = 1$$
$$P[A] = \frac{1}{2} = 0.5$$

or

Let $P[A] = x$

$$P[A]\,(1-P[A]) = x(1-x)$$
$$f = x - x^2$$
$$\frac{df}{dx} = 1 - 2x = 0$$
$$x = \frac{1}{2} = P[A]$$

$$\frac{d^2f}{dx^2} = -2 < 0$$

Now

$$\frac{(0.5)(0.5)}{n} < 0.01$$

$$\frac{0.25}{n} < 0.01$$

$$n > \frac{0.25}{0.01} = 25$$

$$\boxed{n \geq 25}$$

Confidence Intervals.

→ Confidence Interval.

→ Confidence coefficient.

$$P[|Y-\mu_Y| \geq c] \leq \frac{VAR[Y]}{c^2}.$$

or

$$P[|Y-\mu_Y| < c] \geq 1 - \frac{VAR[Y]}{c^2}.$$

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Confidence Interval:

a> Confidence Interval.
b> Confidence Coefficient.

Theorem:

According to Chebyshev Inequality

$$P[|Y-\mu_y| \geq c] \leq \frac{VAR[Y]}{c^2}$$

OR $$P[|X-\mu_y| < c] \geq 1 - \frac{VAR[Y]}{c^2}.$$

So, therefore, for any constant $c>0$

(a) $P[|M_n(x)-\mu_x| \geq c] \leq \frac{VAR[M_n(x)]}{c^2} = \frac{VAR[X]}{nc^2} = \alpha$

(b) $P[|M_n(X)-\mu_x| < c] \geq 1-\frac{VAR[M_n(X)]}{c^2} = 1-\frac{VAR[X]}{nc^2}$

$= \boxed{1-\alpha}$

confidence coeffi

$\mu_x - c < M_n(x) < \mu_x + c.$

$\Rightarrow \mu_x + c - (\mu_x - c)$

$\Rightarrow 2c$ Confidence Interval

$P[|m_n(x) - \mu_x| < C] \geq 1 - \alpha$

states that the Probability that the sample mean is in the confidence ~~Decan~~ interval $2c$ is at least $1 - \alpha$.

Example:

Length of Interval $= 2C = 0.02$.

$$C = \frac{0.02}{2} = 0.01$$

$1 - \alpha = 0.999$

According to chebyshev Inequality

$$P[|\hat{P}_n(A) - P[A]| < C] \geq 1 - \frac{VAR[\hat{P}_n(A)]}{C^2} \quad (1)$$

However, $\hat{P}(A) = m_n(X_A)$.

Therefore

$$VAR[\hat{P}_n(A)] = VAR[m_n(X_A)] = \frac{VAR[X_A]}{n}$$

$X_A \sim$ Bernoulli

But: $VAR[X_A] = P[A](1 - P[A])$

So,

$$VAR[\hat{P}_n(A)] = \frac{P[A](1-P[A])}{n}$$

~~Therefore Eq (1) Becomes~~

Therefore Eq (1) Becomes.

$$P[|\hat{P}_n(A) - P[A]| < c] \geq 1 - \frac{P[A](1-P[A])}{nc^2}$$

$$= 1 - \alpha.$$

Hence

$$\cancel{1 -} \frac{P[A](1-P[A])}{nc^2} = 1 - \alpha = 0.999$$

Let $p[A] = x$:

$\cancel{1 - x(1-x)}$

$$P[A](1-P[A]) = x(1-x)$$

$$= x - x^2.$$

$$\frac{d}{dx}(x - x^2) = 0$$

$$1 - 2x = 0$$

$$x = \frac{1}{2}$$

$$P[A] = 0.5$$

$$\frac{d^2}{d^x x^2} = -2 < 0.$$

$$P[A](1-P[A]) = 0.5(0.5) = 0.25 = \frac{1}{4}$$

$$1 - \frac{1}{4nc^2} = 0.999.$$

$$\frac{1}{4nc^2} = 1 - 0.999$$

$$\frac{1}{4nc^2} = 0.001$$

$$4nc^2 = 1000$$

$$n = \frac{1000}{4(0.01)^2}$$

$$n = 2.5 \times 10^6$$

$$\boxed{n \geq 2.5 \times 10^6}$$

Example:

$$X_i = b + Z_i$$

s.d of $Z_i = 1$

$$2c = 0.2 = 0.1$$

$$1 - d = 0.99$$

$$P[|M_n(X) - b| < 0.1] \geq 1 - \frac{VAR[M_n(X)]}{(0.1)^2}$$

$$= 1 - \frac{VAR[X_i]}{n^3(0.01)^2} = 1 - d$$

As b and z_i are independent,

However:

$$VAR[x_i] = VAR[b + z_i] = VAR[b] + VAR[z_i]$$

$$VAR[b] = 0 \text{, as constant}$$

$$VAR[x_i] = VAR[z_i] = 1.$$

Hence

$$P[M_n(x) - 0.1 < b < M_n(x) + 0.1] \geq 1 - \frac{1}{n(0.1)^2} = 0.99.$$

$$1 - \frac{1}{n \times 0.01} = 0.99$$

$$\frac{100}{n} = 1 - 0.99 = 0.01.$$

$$n = 10,000$$

$$\boxed{n \geq 10,000}$$

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Theorem;

Let X be a Gaussian (μ, σ) random variable. A confidence interval estimate of μ of the form

$$|M_n(x) - \mu| \leq c.$$

$$M_n(x) - c \leq \mu \leq M_n(x) + c.$$

has confidence coefficien $1-\alpha$,

where

$$\frac{\alpha}{2} = Q\left(\frac{c\sqrt{n}}{\sigma}\right) = 1 - \Phi\left(\frac{c\sqrt{n}}{\sigma}\right)$$

$$\because Q(z) = 1 - \Phi(z)$$

Example:

$$2c = 0.2$$

$$c = 0.1$$

$$VAR[X_i] = VAR[Z_i] = 1.$$

$$X_i = b + Z_i.$$

$$VAR[X_i] = VAR[b] + VAR[Z_i].$$

$$VAR[X_i] = VAR[Z_i].$$

$$\sigma = 1.$$

z_i is a Gaussian random variable

$$\frac{\alpha}{2} = Q\left(\frac{c\sqrt{n}}{\sigma}\right)$$

$$\frac{\alpha}{2} = Q\left(\frac{0.1\sqrt{n}}{1}\right)$$

$$\frac{\alpha}{2} = Q\left(\frac{\sqrt{n}}{10}\right) = 1 - \Phi\left(\frac{\sqrt{n}}{10}\right)$$

But

$$1 - \alpha = 0.99 \quad \text{—— (1)}$$

$$\alpha = 0.01$$

$$\frac{\alpha}{2} = 0.005 \quad \text{—— (2)}$$

Comparing (1) and (2)

$$Q\left(\frac{\sqrt{n}}{10}\right) = 1 - \Phi\left(\frac{\sqrt{n}}{10}\right) = 0.005$$

$$\Phi\left(\frac{\sqrt{n}}{10}\right) = 0.995$$

From table

$Q(z) \simeq 0.995$ when

$$z = 2.58$$

i.e $\frac{\sqrt{n}}{10} = 2.58$

$\sqrt{n} = 25.8$

$n = 666$

$n \geq 666$

Example:

$1-\alpha = 0.99$

$\alpha = 0.01$

$\sigma_Y^2 = 10 \Rightarrow \sigma = \sqrt{10}$

$n = 100$

$M_{100}(Y) = 33.2$

Now,

$P[|M_n(Y) - \mu| < c] \geq 1-\alpha$

OR:

$P[M_n(Y) - c \leq \mu \leq M_n(Y) + c] \geq 1-\alpha$

where,

$\frac{\alpha}{2} = Q\left(\frac{c\sqrt{n}}{\sigma}\right) = 1 - \Phi\left(\frac{c\sqrt{n}}{\sigma}\right)$

$\frac{0.01}{2} = Q\left(\frac{c \times \sqrt{100}}{\sqrt{10}}\right) = 1 - \Phi\left(\frac{c \times \sqrt{100}}{\sqrt{10}}\right)$

CLO 1; Before mids

$0.005 = 1 - \Phi(\sqrt{10}\,c)$

$\Phi(\sqrt{10}\,c) = 0.995$

$\sqrt{10}\quad c = 2.58$

$c = \dfrac{2.58}{\sqrt{10}}$

$\boxed{c = 0.8159}$

Hence;

$Mn(Y) - c = 33.2 - 0.8159 = 32.3841$

$Mn(Y) + c = 33.2 + 0.8159 = 34.0159$

Hence our interval estimate of the expected value μ is

$32.3841 \leq \mu \leq 34.0159$

with confidence 0.99.

There is 0.99 probability that μ lies within 33.2 ± 0.8159 i.e between 32.3841 and 34.0159.