WEEK 2:

OPERATIONS ON SIGNALS

(DEPENDENT VARIABLE)

Dr. Arsla Khan

Operations on Signals

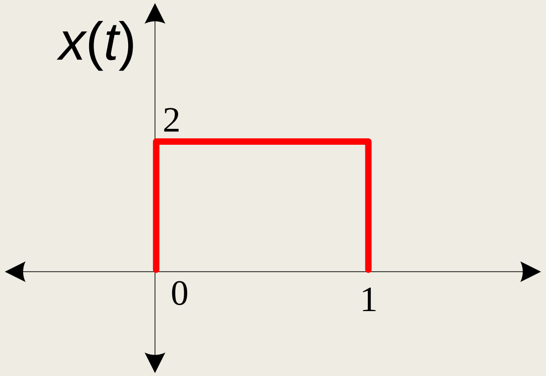
- Operations with respect to x-axis (Time axis) / Transformations on the independent variable
  - Time Shifting $x(t-k), x[n-k]$
  - Time Reversal/Folding/Flipping $x(-t), x[-n]$
  - Time Scaling $x(\alpha t), x[\alpha n]$

- Operations with respect to y-axis (Amplitude) / Transformations on the dependent variable
  - *Amplitude Scaling*
  - *Addition and Subtraction*
  - *Multiplication and Division*
  - *Differentiation and Integration*

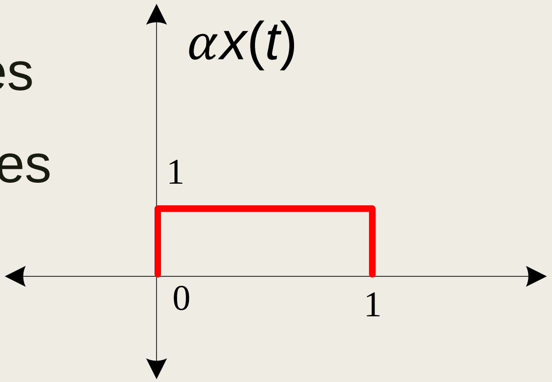
Links for Video Lectures

- **Addition of CT signals**
- https://www.youtube.com/watch?v=TmkTwJT79yc&list=PLBlnK6fEyqRhG6s3jYIU48CqsT5cyiDTO&index=3
- **Multiplication of CT signals**
- https://www.youtube.com/watch?v=jPCgU4ghB8Q&list=PLBlnK6fEyqRhG6s3jYIU48CqsT5cyiDTO&index=4
- **Amplitude Scaling of CT signals**
- https://www.youtube.com/watch?v=sTHbXeiAB_c&list=PLBlnK6fEyqRhG6s3jYIU48CqsT5cyiDTO&index=6
- **Amplitude Shifting of CT signals**
- https://www.youtube.com/watch?v=sTHbXeiAB_c&list=PLBlnK6fEyqRhG6s3jYIU48CqsT5cyiDTO&index=6

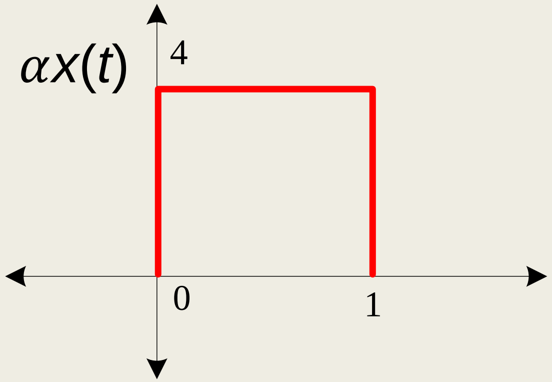
1) Amplitude Scaling



- It either increases or decreases the amplitude of the signal
    - $x(t) \rightarrow \alpha x(t)$
- If $\alpha > 1$, amplitude of the signal increases
- If $\alpha < 1$, amplitude of the signal decreases



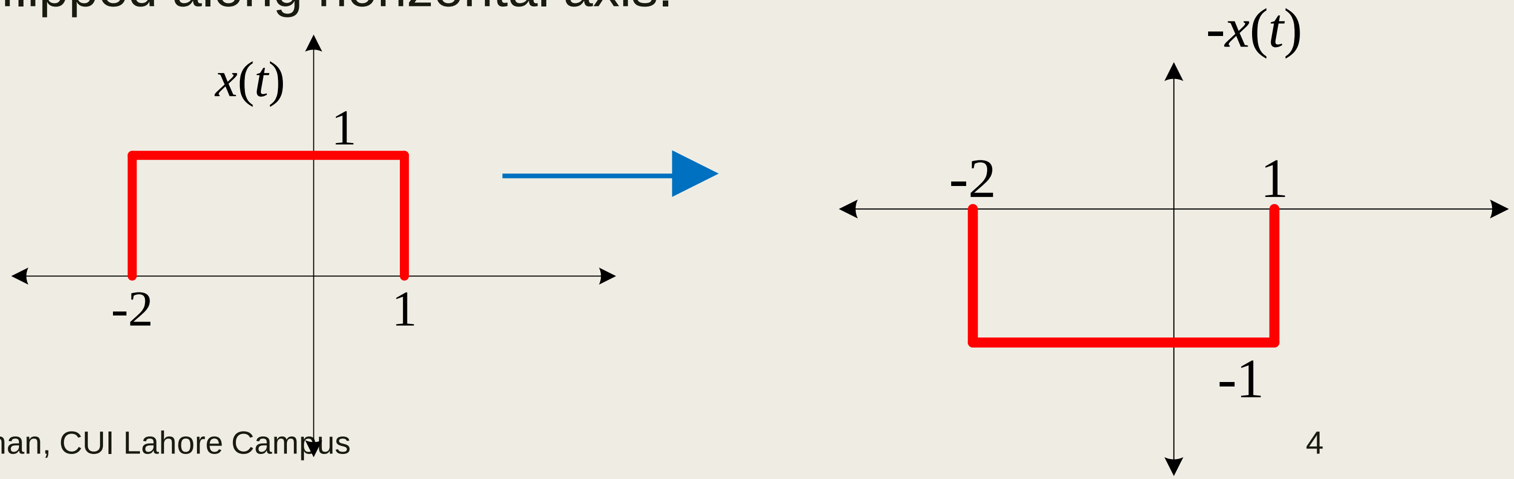
$\alpha = 0.5$



$\alpha = 2$

2) Amplitude Flipping

- Amplitude of the signal is flipped along horizontal axis.
    - $x(t) \rightarrow -x(t)$



3) Addition and Subtraction

- For CTS
  - $y(t) = x_1(t) + x_2(t)$
- For DTS
  - $y[n] = x_1[n] + x_2[n]$

4) Multiplication and Division

- For CTS
  - $y(t) = x_1(t).x_2(t)$ → Multiplication
  - $y(t) = x_1(t)/x_2(t)$ → Division
- For DTS
  - $y[n] = x_1[n].x_2[n]$ → Multiplication
  - $y[n] = x_1[n]/x_2[n]$ → Division

5) Differentiation and Integration → Only for CTS

- $y(t) = \frac{d}{dt}x(t)$
- $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

6) Difference and Accumulation → Only for DTS

- $y[n] = x[n] - x[n-1]$
- $y[n] = \sum_{k=-\infty}^{n} x[n]$

Operations	Operations w.r.t Time axis (x-axis)	Operations w.r.t Amplitude axis (y-axis)
Shifting	<u>**Time Shifting**</u> $x(t-k) \rightarrow Delay$ $x(t+k) \rightarrow Advance$	<u>**Amplitude Shifting**</u> $x(t)+k$ $x(t)-k$
Flipping/Folding/Reversal	<u>**Time Folding**</u> $x(-t)$	<u>**Amplitude Folding**</u> $-x(t)$
Scaling	<u>**Time Scaling**</u> $x(\alpha t)$	<u>**Amplitude Scaling**</u> $\alpha x(t)$

Operations w.r.t y-axis (amplitude) for DTS

- Operations w.r.t y-axis i.e. amplitude axis for DTS are performed sample by sample basis.

Exp 1: Plot

$(i)\ y_1[n] = x_1[n] + x_2[n]$

$(ii)\ y_2[n] = 2x_1[n]$

$(iii)\ y_3[n] = x_1[n]x_2[n]$

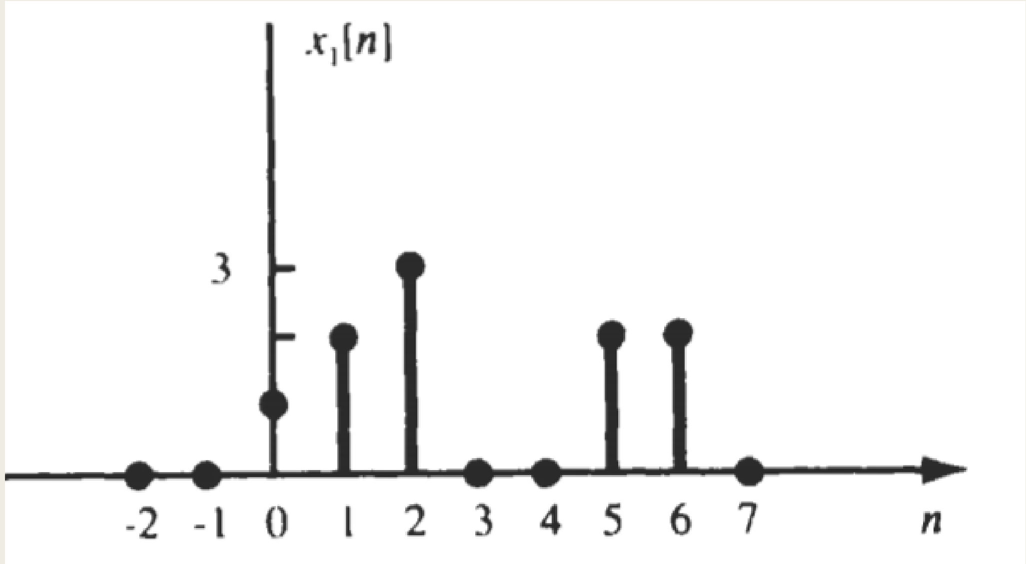
- **How to write DTS ???**

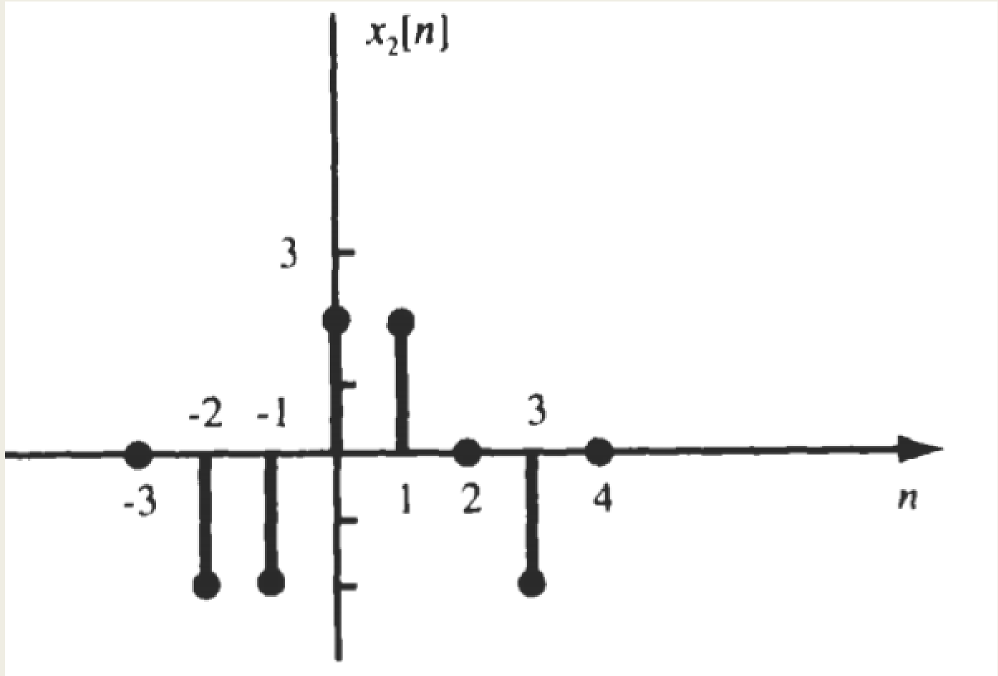
- $x_1[n] = \{0,0,\underset{\uparrow}{1},2,3,0,0,2,2,0\}$

  Position of sample at zeroth place

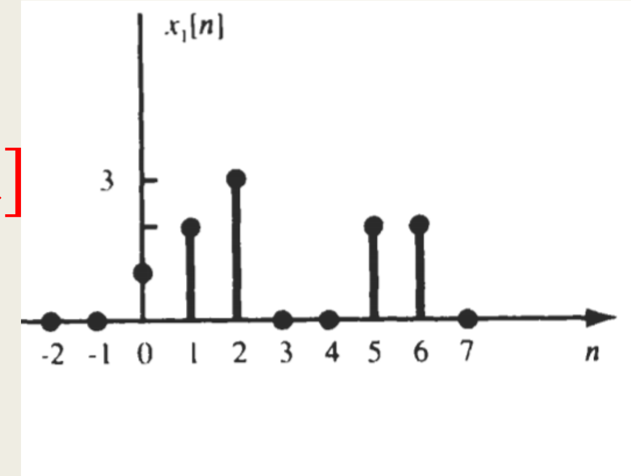
- $x_2[n] = \{0,-2,-2,\underset{\uparrow}{2},2,0,-2,0\}$

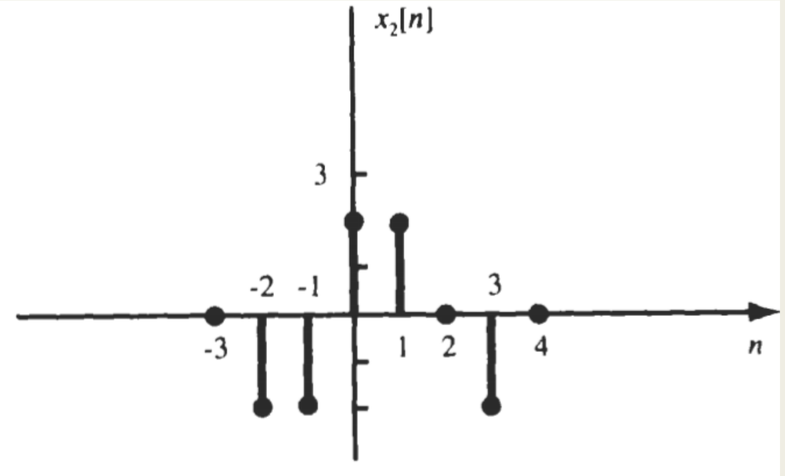
  Position of sample at zeroth place



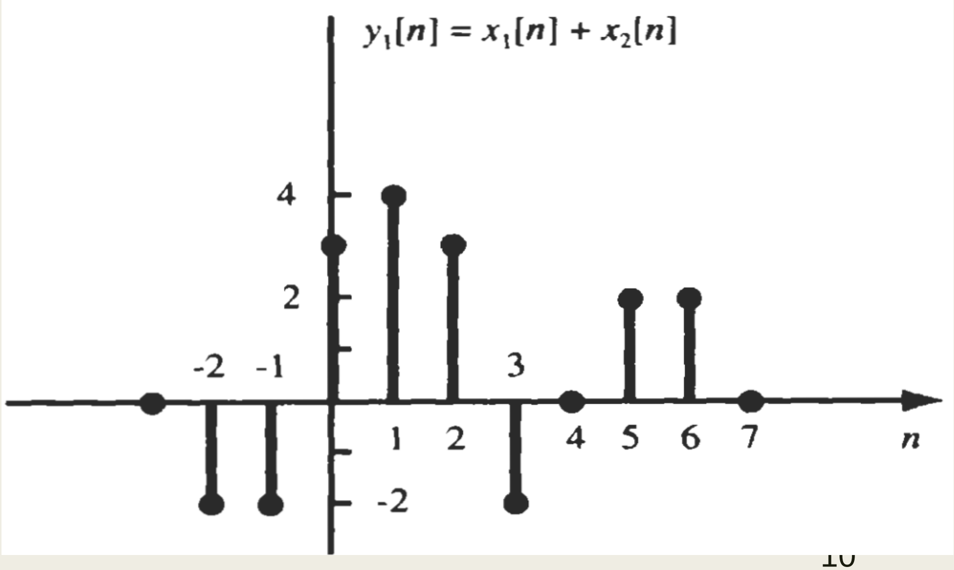


(i) $y_1[n] = x_1[n] + x_2[n]$





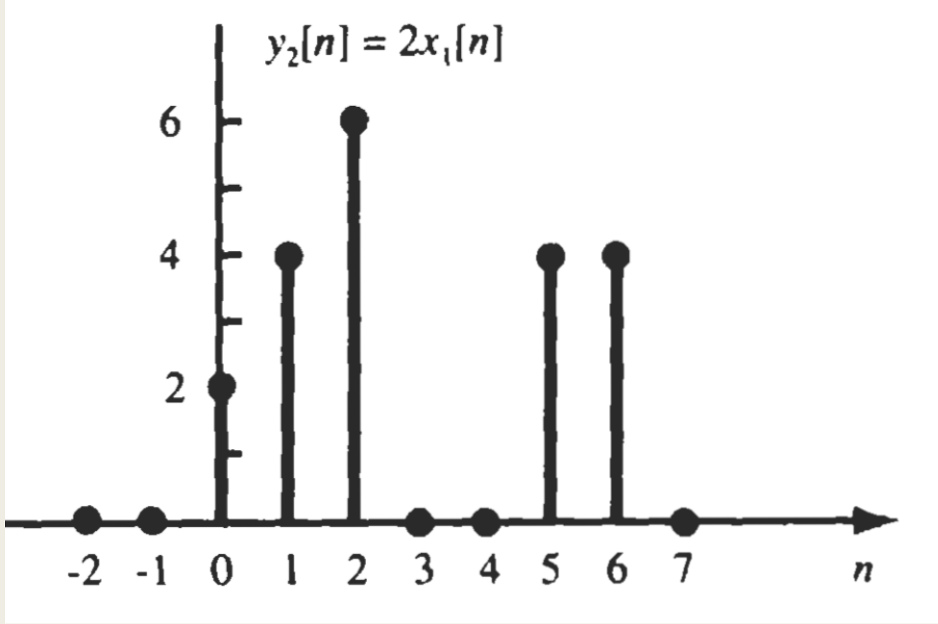
n	-3	-2	-1	0	1	2	3	4	5	6	7
$x_1[n]$	0	0	0	1	2	3	0	0	2	2	0
$x_2[n]$	0	-2	-2	2	2	0	-2	0	0	0	0
$y_1[n]$	0	-2	-2	3	4	3	-2	0	2	2	0



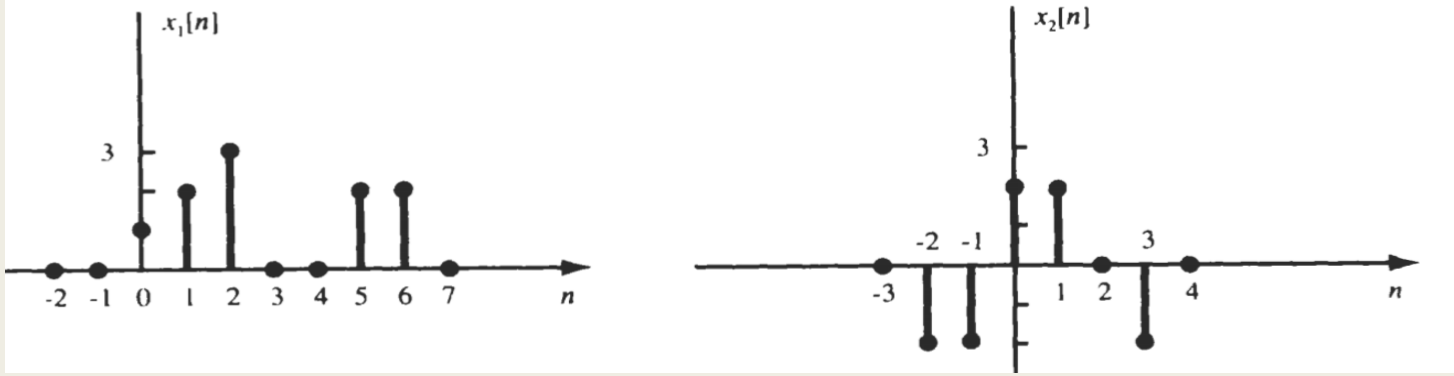
(ii) $y_2[n] = 2x_1[n]$

(ii) $y_2[n] = 2x_1[n]$

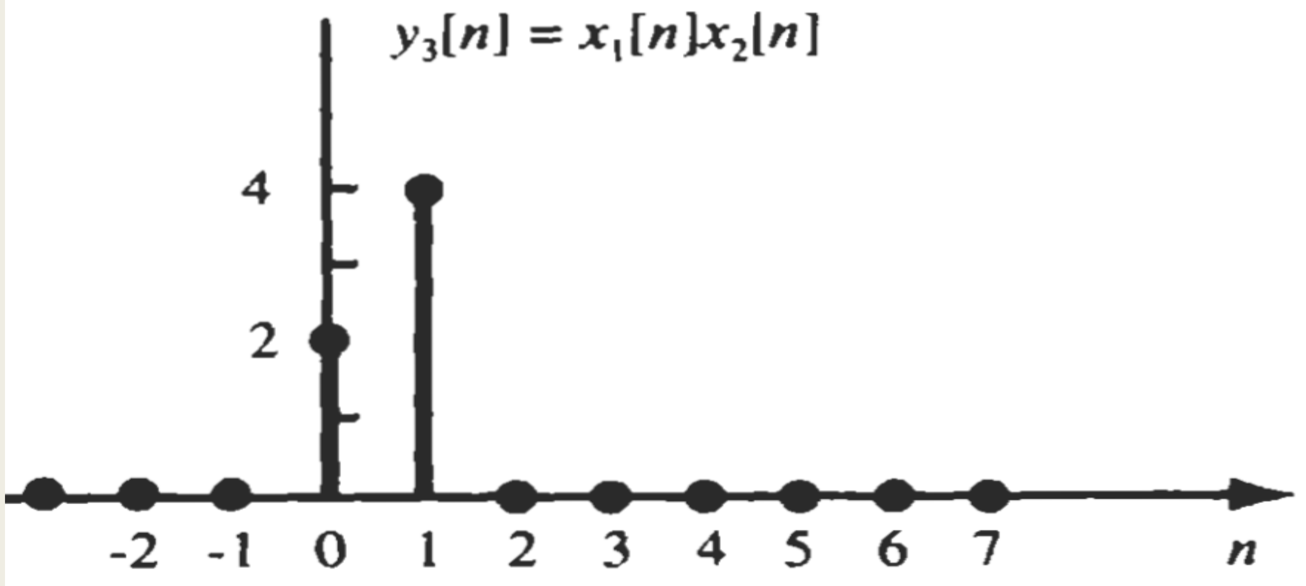
n	-2	-1	0	1	2	3	4	5	6	7
$x_1[n]$	0	0	1	2	3	0	0	2	2	0
	Multiple each sample with 2									
$y_2[n]$	0	0	2	4	6	0	0	4	4	0



(iii) $y_3[n] = x_1[n]x_2[n]$



n	-3	-2	-1	0	1	2	3	4	5	6	7
$x_1[n]$	0	0	0	1	2	3	0	0	2	2	0
$x_2[n]$	0	-2	-2	2	2	0	-2	0	0	0	0
$y_3[n]$	0	0	0	2	4	0	0	0	0	0	0

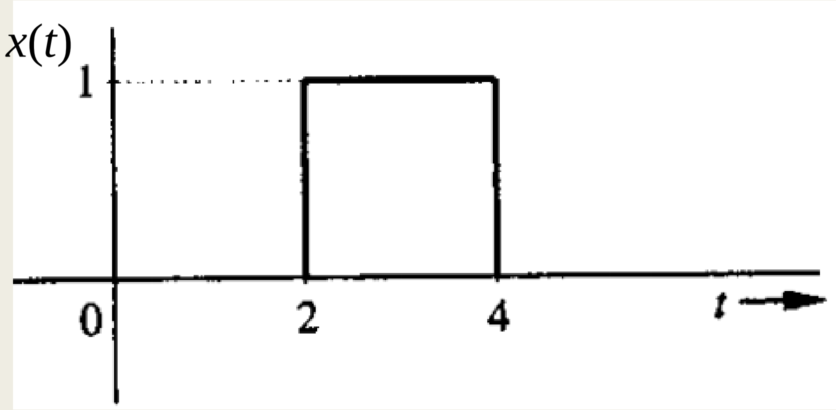


Operations w.r.t y-axis for CTS

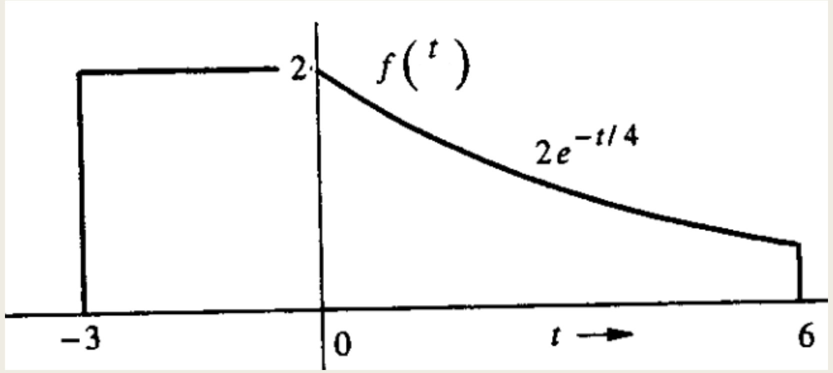
- It is preferable to write mathematical representation of signals before applying operations on CTS w.r.t y-axis.

- Mathematical Definition of Signals
    - CTS are defined in the form of their ranges

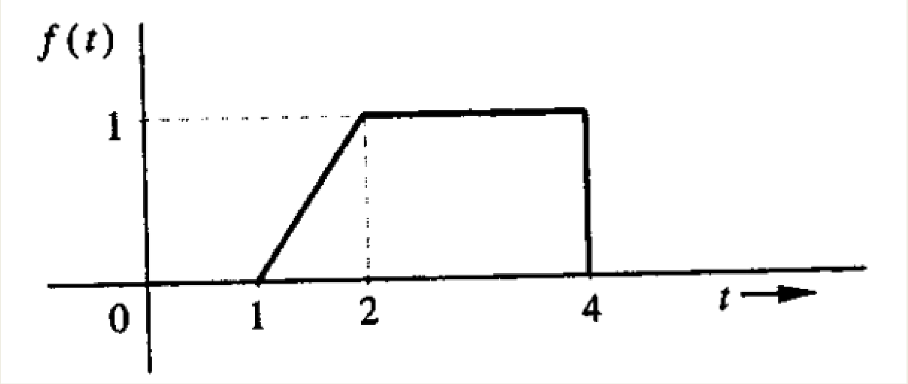
- $x(t) = \begin{cases} 1 & 2 \le t \le 4 \\ 0 & Otherwise \end{cases}$



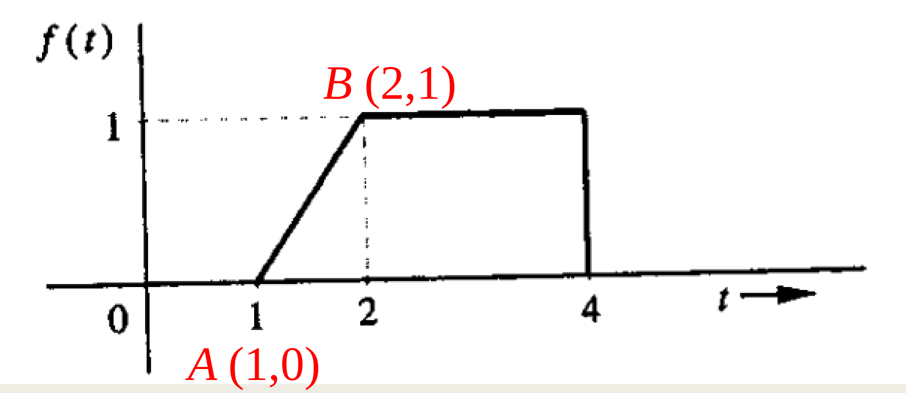
- $f(t) = \begin{cases} 2 & -3 \le t \le 0 \\ 2e^{-t/4} & 0 < t \le 6 \\ 0 & Otherwise \end{cases}$



- $f(t) = \begin{cases} ??? & 1 \le t \le 2 \\ 1 & 2 < t \le 4 \\ 0 & Otherwise \end{cases}$



How to define ramp signal?

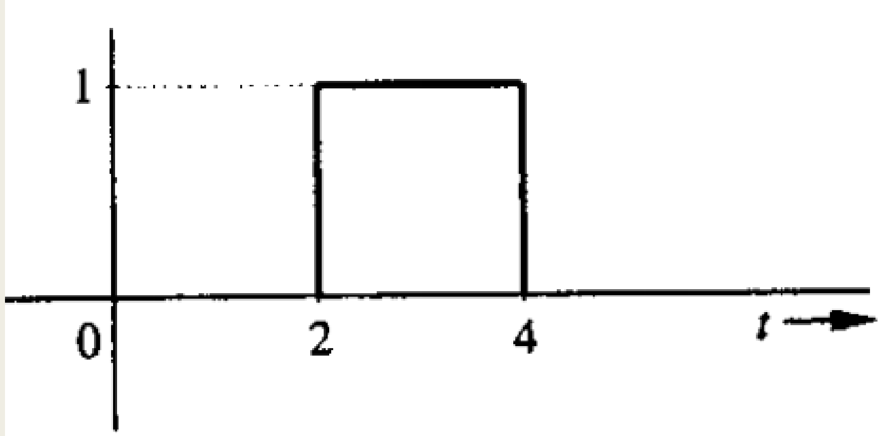


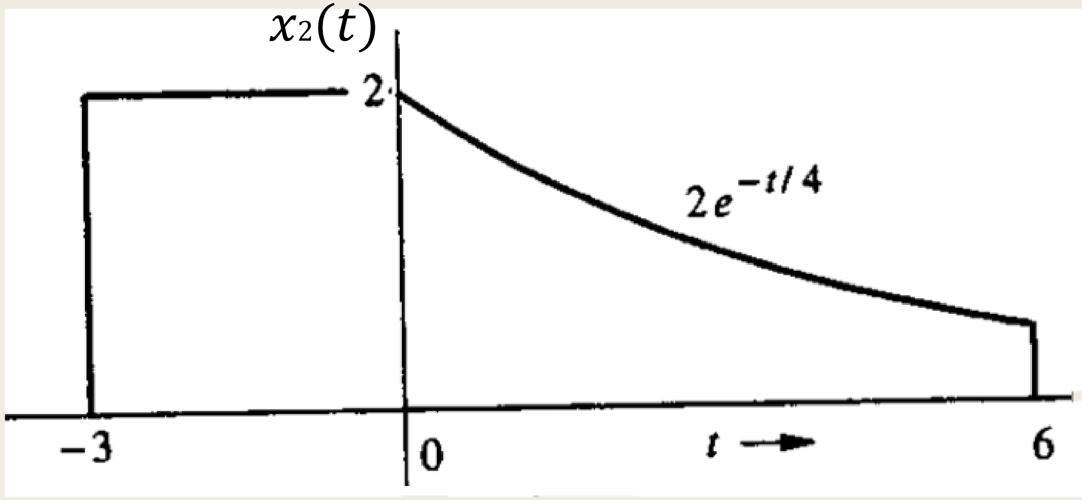
- Define points in which ramp exists
  - Point A (x_1, y_1) = (1,0)
  - Point B (x_2, y_2) = (2,1)
- Write equation for line and put values
  - $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$
  - $\frac{y-0}{1-0} = \frac{x-1}{2-1}$
  - $y = x - 1$
- Replace x with t and y with $f(t)$
  - $\therefore f(t) = t - 1$

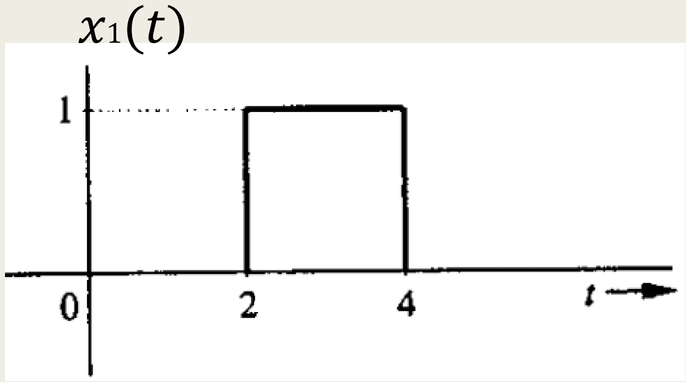
$$f(t) = \begin{cases} t-1 & 1 < t < 2 \\ 1 & 2 < t < 4 \\ 0 & Otherwise \end{cases}$$

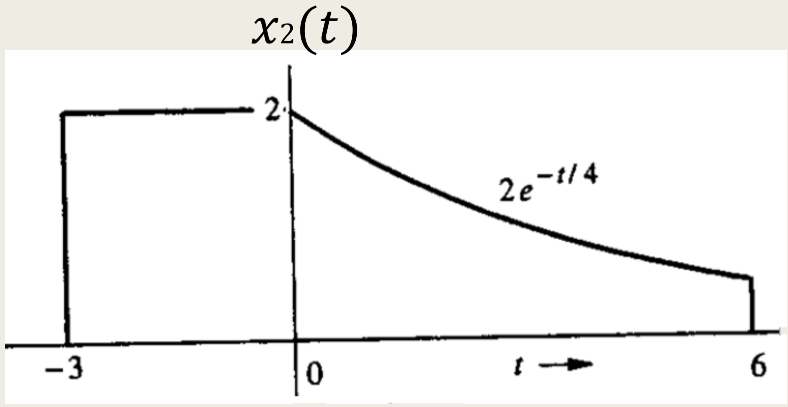
Exp 2: Find and draw waveform for $y_1(t) = x_1(t) + x_2(t)$

$x_1(t)$









$$x_1(t) = \begin{cases} 1 & 2 < t < 4 \\ 0 & Otherwise \end{cases}$$

$$x_2(t) = \begin{cases} 2 & -3 < t < 0 \\ 2e^{-t/4} & 0 < t < 6 \\ 0 & Otherwise \end{cases}$$

- First, define the signal in mathematical form
- Now check their ranges. Start from $-\infty$ and goes to $+\infty$
- $y_1(t) = \begin{cases} 0 + 2 = 2 & -3 < t < 0 \\ 0 + 2e^{-t/4} = 2e^{-t/4} & 0 < t < 2 \\ 1 + 2e^{-t/4} & 2 < t < 4 \\ 0 + 2e^{-t/4} = 2e^{-t/4} & 4 < t < 6 \end{cases}$

Now you can draw it

ELEMENTARY SIGNALS
+
DIFFERENTIATION OPERATION

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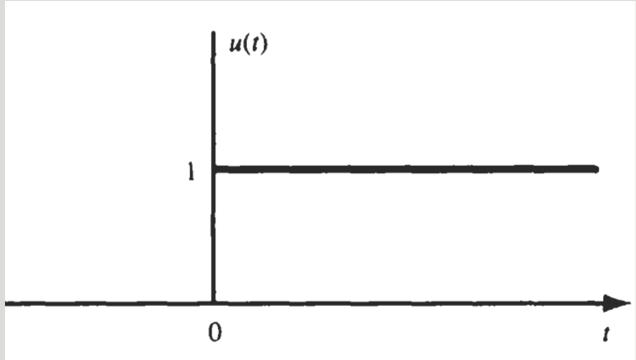
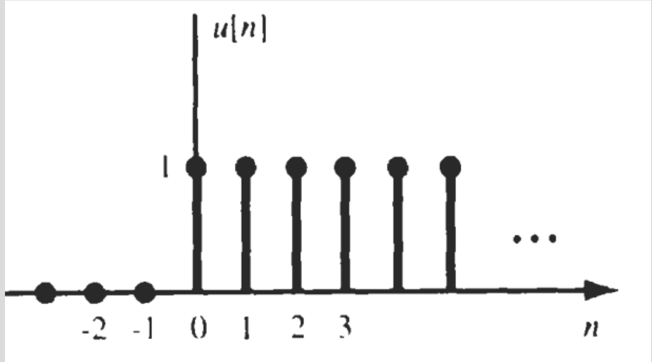
Basic / Elementary Signals

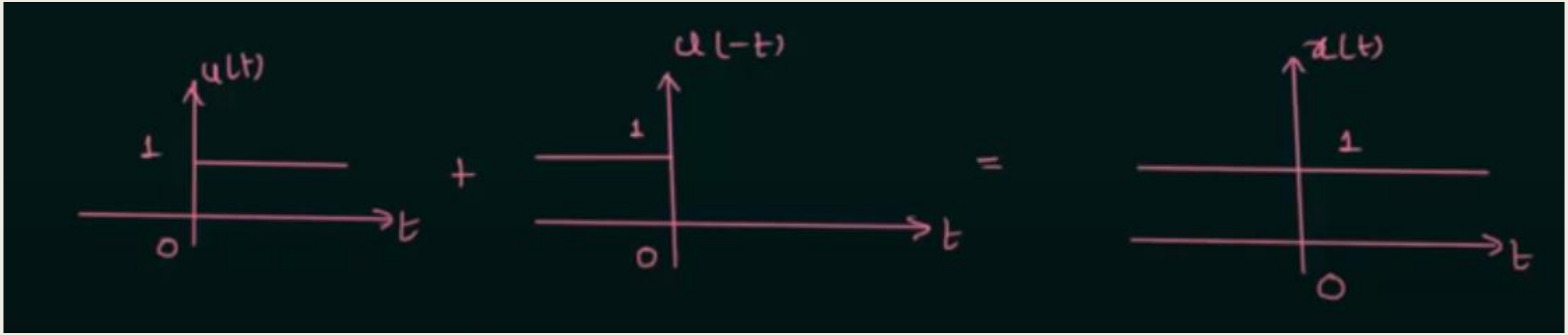
- Standard signals are used for the analysis of systems. These signals are;
  - Unit step function
  - Unit impulse or Delta function
  - Unit ramp function
  - Complex exponential function
  - Sinusoidal function

1. Unit Step function $(u(t)\ or\ u[n])$

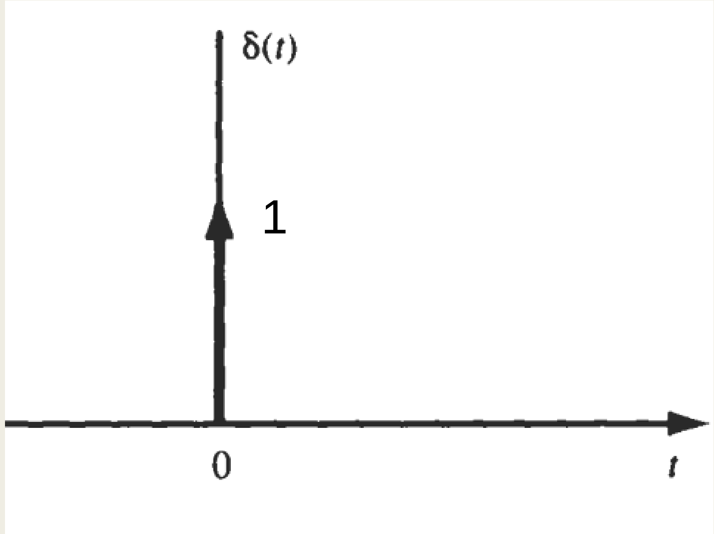
Parameter	CT unit step signal $u(t)$	DT unit step signal $u[n]$
Definition	The unit step signal has amplitude of ‘1’ for positive values of time and it has amplitude of ‘0’ for negative values of time.	
Mathematical representation	$u(t) = \begin{cases} 1 & for\ t \geq 0 \\ 0 & for\ t < 0 \end{cases}$	$u[n] = \begin{cases} 1 & for\ n \geq 0 \\ 0 & for\ n < 0 \end{cases}$ or $u[n] = \{\ldots, 0,0,0,\underset{\uparrow}{1},1,1,1, \ldots\}$
Waveform		
Significance	DT unit step signal is sampled version of CT unit step signal	

Plot $x(t)=u(t)+u(-t)$???

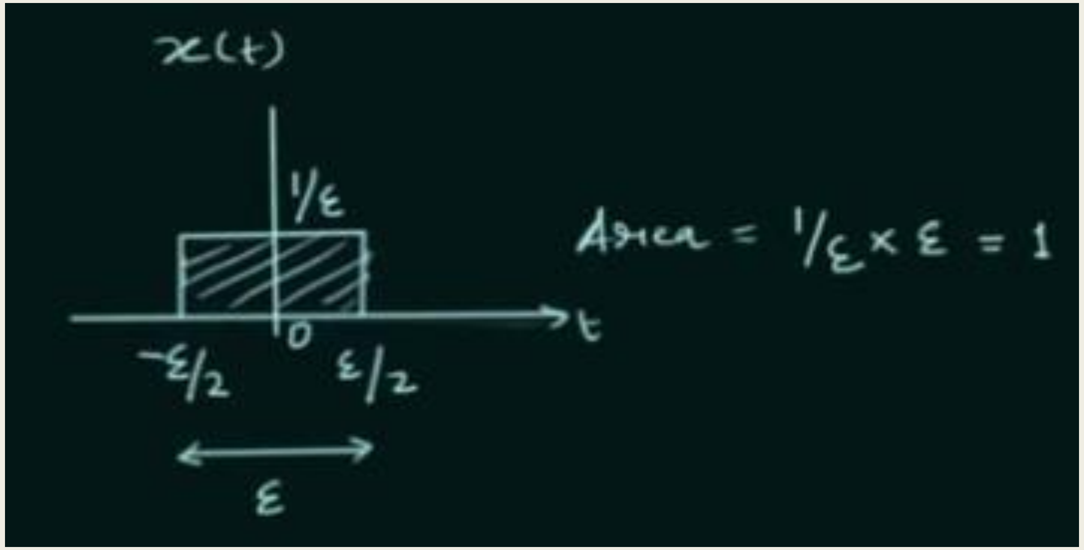
$x(t)$

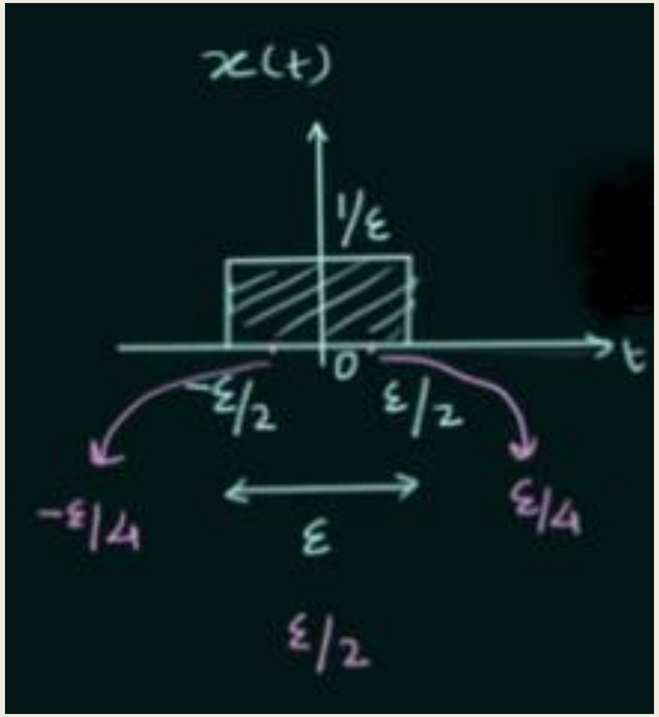


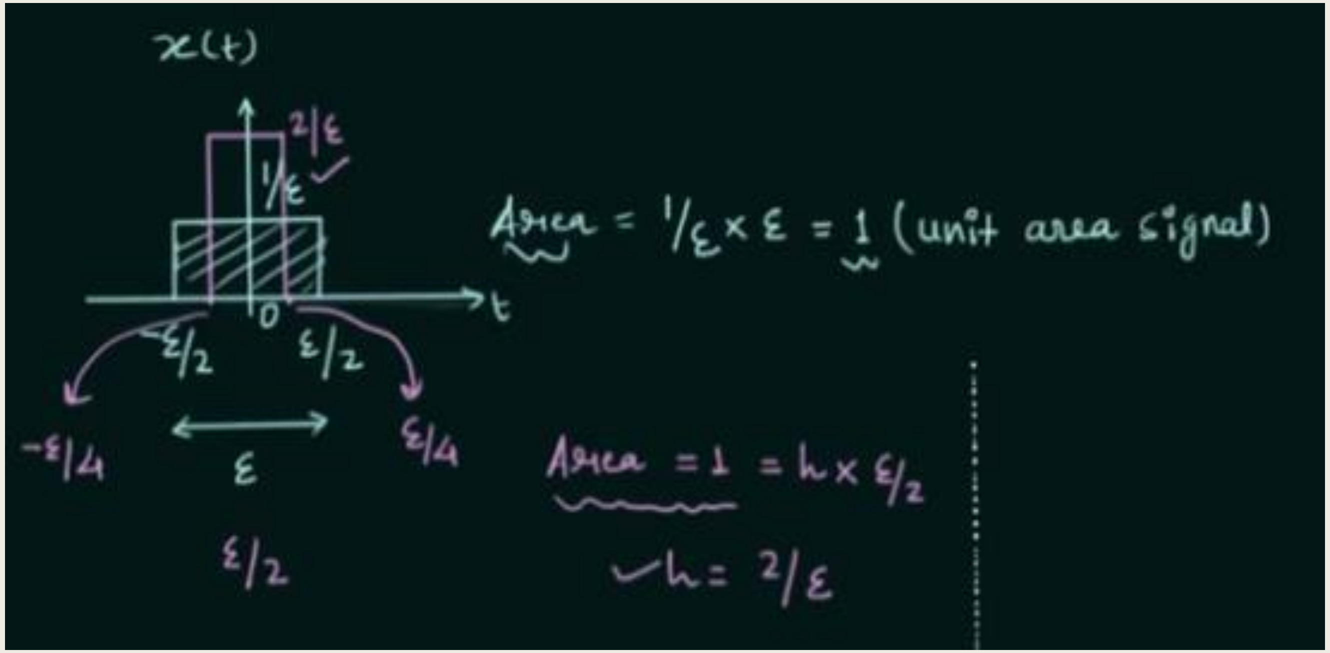
2. Unit Impulse Signal



- Continuous Time Unit Impulse Signal is $\boldsymbol{\delta(t)}$
- It is also known as dirac delta
- It is defined as "Area under unit impulse is '1' as its width approaches zero. Thus, it has zero value everywhere except $t = 0$"
- Thus, coefficient with $\delta(t)$ shows its strength or area not amplitude
- $$\delta(t) = \begin{cases} \int_{-\infty}^{\infty} \delta(t)dt = 1 & for\ t = 0 \\ 0 & for\ t \neq 0 \end{cases}$$

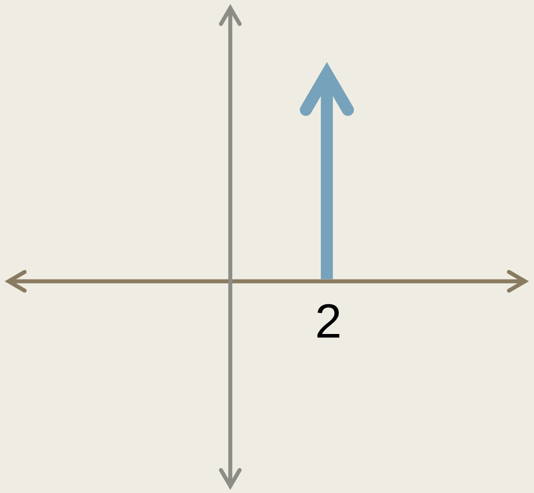

x(t)
1/ε
Area = 1/ε × ε = 1
t
0
-ε/2
ε/2
ε


x(t)
1/ε
t
0
-ε/2
ε/2
-ε/4
ε
ε/4
ε/2

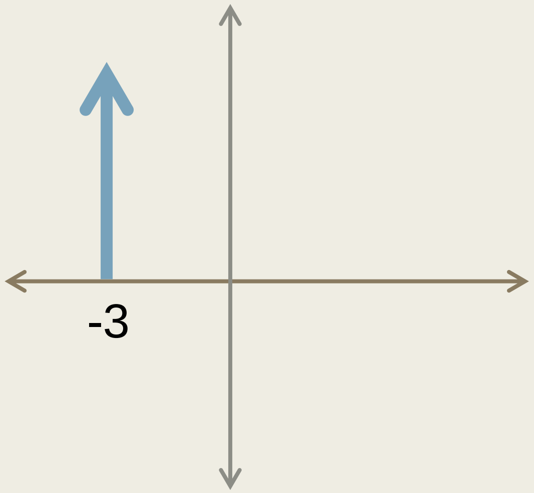

x(t)
2/ε
1/ε
Area = 1/ε × ε = 1 (unit area signal)
t
0
-ε/2
ε/2
-ε/4
ε
ε/4
Area = 1 = h × ε/2
ε/2
h = 2/ε

A) Time Shifting

i) $\delta(t-2)$

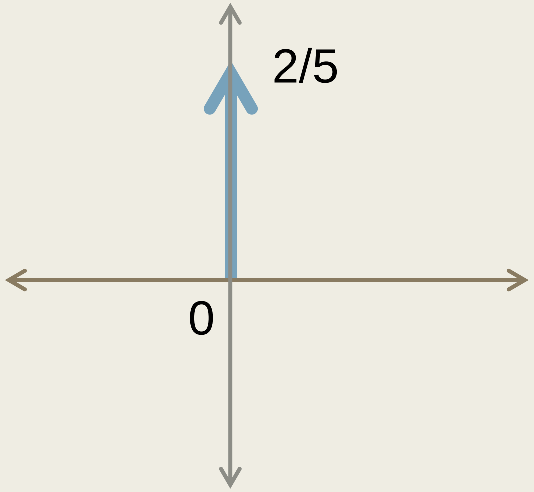


ii) $\delta(t+3)$



B) Amplitude Scaling

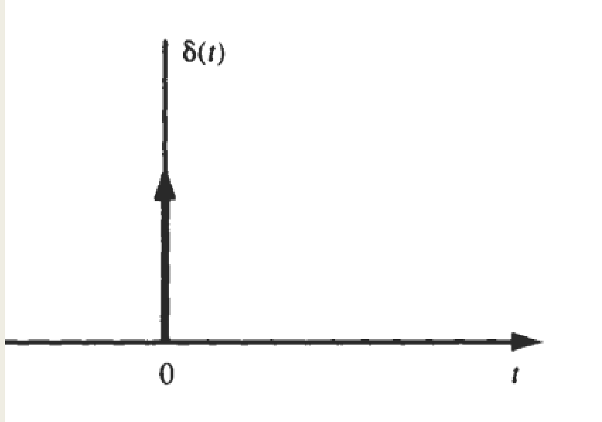
iii) $\frac{2}{5}\delta(t)$



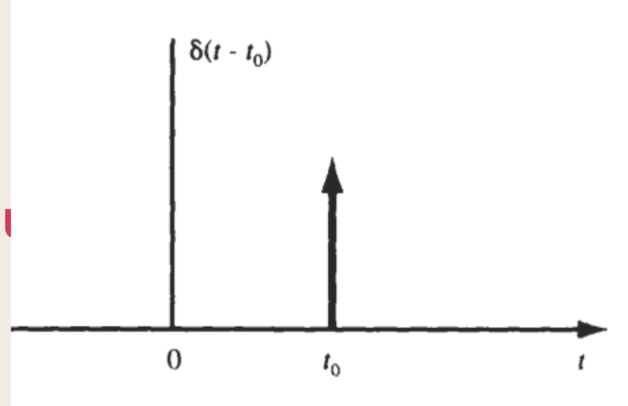
C) Time Scaling

- $\delta(at) = \frac{1}{|a|}\delta(t)$

Properties of CT Unit Impulse or Delta function $\delta(t)$



- 1) Integrating a unit impulse function results in '1'
  - $\int_{-\infty}^{+\infty} \delta(t)dt = 1$
  - $\int_{-\infty}^{+\infty} A\delta(t)dt = A$
- 2) The scaled version of $\delta(at)$ is
  - $\delta(at) = \frac{1}{|a|}\delta(t)$
- 3) The flipped version of $\delta(t)$ is
  - $\delta(-t) = \delta(t)$
- 4) **When an arbitrary function $f(t)$ is multiplied by a shifted impu function, the product is given by;**
  - $\int_{-\infty}^{+\infty} f(t)\delta(t)dt = f(t)|_{t=0}\delta(t)$
  - $\int_{-\infty}^{+\infty} f(t)\delta(t-t_o)dt = f(t_o) = f(t)|_{t=t_o}\delta(t-t_o)$



Exp 4.1: Evaluate i) $\int_{-\infty}^{+\infty} e^{-t}\delta(2t-2)dt$ ii) $\int_{-5}^{-2} e^{-t}\delta(2t-2)dt$

(i) $\int_{-\infty}^{+\infty} e^{-t}\delta(2t-2)dt$

- $\delta(2t-2) = \delta[2(t-1)] = \frac{1}{2}\delta(t-1)$
- $=\int_{-\infty}^{+\infty} e^{-t}\frac{1}{2}\delta(t-1)dt$
- $= \frac{1}{2}\int_{-\infty}^{+\infty} e^{-t}\delta(t-1)dt$
- $= \frac{1}{2}e^{-t}\big|_{t=1}$
- $= \frac{1}{2}e^{-1}$

ii) $\int_{-5}^{-2} e^{-t}\delta(2t-2)dt$

= 0

*Unit impulse should be present between the limits of integration

Exp 4.2: Evaluate the following integrals

(a) $\displaystyle\int_{-1}^{1} (3t^2 + 1)\delta(t)\, dt$

(b) $\displaystyle\int_{1}^{2} (3t^2 + 1)\delta(t)\, dt$

(c) $\displaystyle\int_{-\infty}^{\infty} (t^2 + \cos \pi t)\, \delta(t - 1)\, dt$

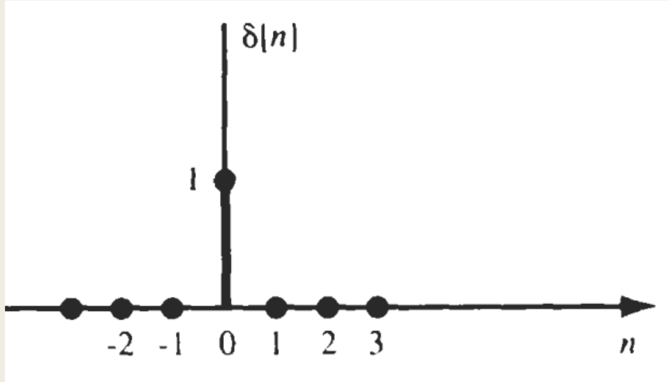
(d) $\displaystyle\int_{-\infty}^{\infty} e^{-t}\delta(2t - 2)\, dt$

Solution:

- **(a)** $$\int_{-1}^{1}(3t^2+1)\delta(t)\,dt=(3t^2+1)\big|_{t=0}=1$$
- **(b)** $$\int_{1}^{2}(3t^2+1)\delta(t)\,dt=0$$
- **(c)** $$\int_{-\infty}^{\infty}(t^2+\cos\pi t)\delta(t-1)\,dt=(t^2+\cos\pi t)\big|_{t=1}$$ $$=1+\cos\pi=1-1=0$$
- **(d)** $$\int_{-\infty}^{\infty}e^{-t}\delta(2t-2)\,dt=\int_{-\infty}^{\infty}e^{-t}\delta[2(t-1)]\,dt$$ $$=\int_{-\infty}^{\infty}e^{-t}\frac{1}{|2|}\delta(t-1)\,dt=\frac{1}{2}e^{-t}\Big|_{t=1}=\frac{1}{2e}$$

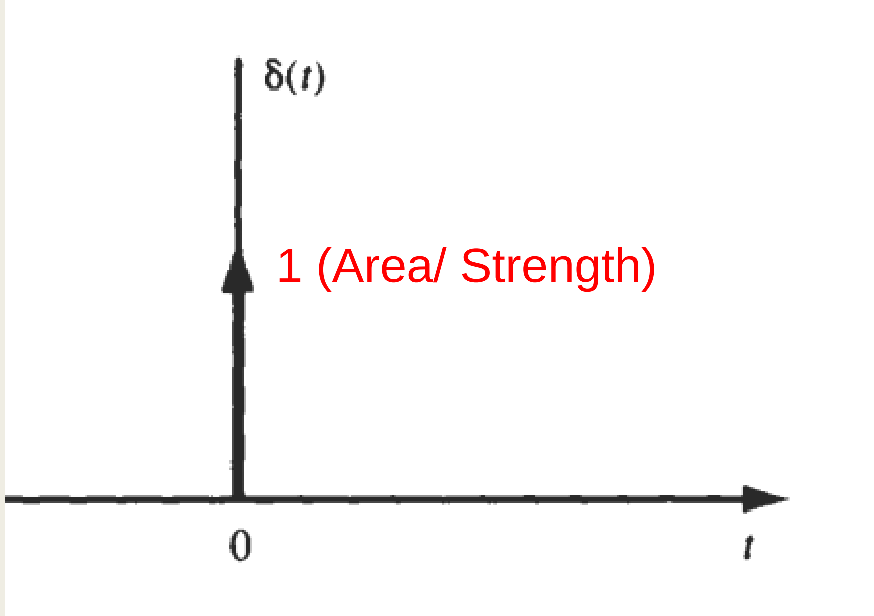
DT Unit Sample Signal/ Unit Impulse Sequence $\delta[n]$

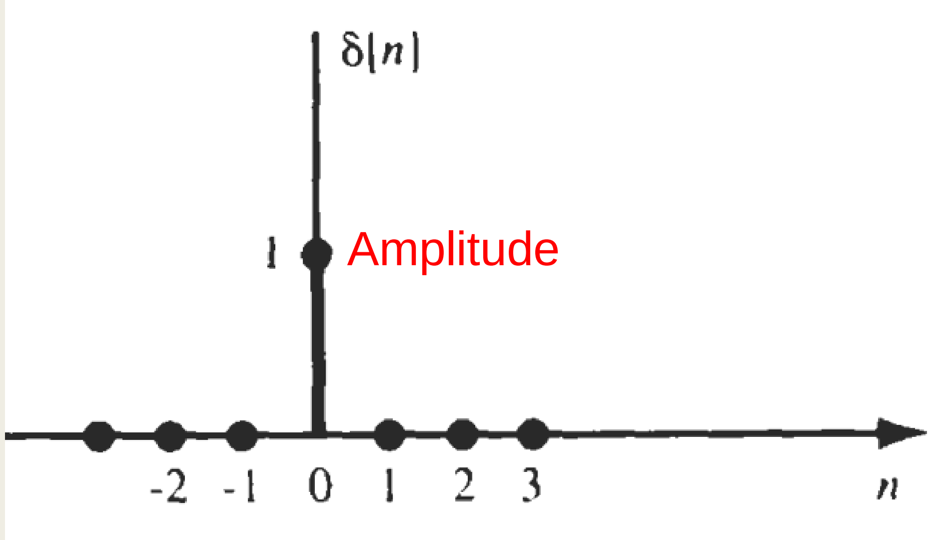
- Amplitude of unit sample is '1' at $n = 0$ and it has zero value at all other values of n

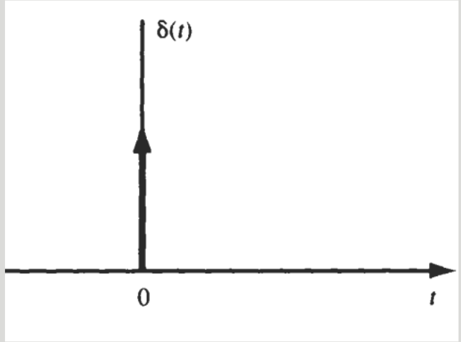
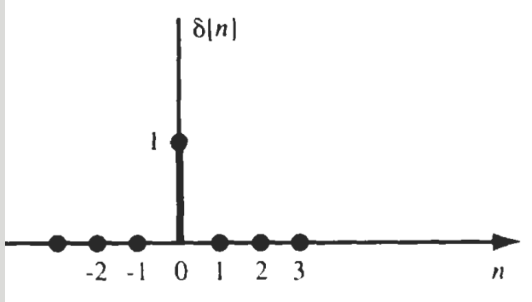


- $\delta[n] = \begin{cases} 1 & for\ n = 0 \\ 0 & for\ n \neq 0 \end{cases}$ or $\delta[n] = \{\dots, 0,0,0,\underset{\blacktriangle}{1},0,0,0, \dots\}$

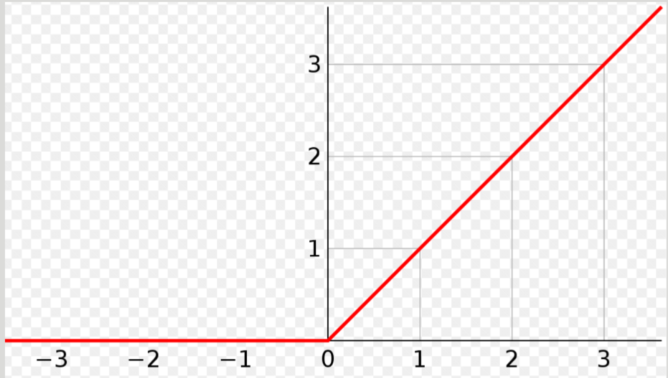
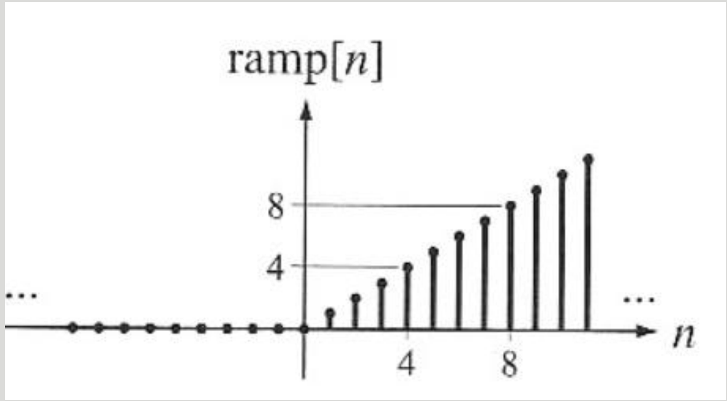
- $\delta[n]$ is not the sampled version of $\delta(t)$.
- *The main difference is Area under $\delta(t) = 1$ while Amplitude of $\delta[n] = 1$*


δ(t)
1 (Area/ Strength)
0
t


δ[n]
1
Amplitude
-2
-1
0
1
2
3
n

Parameter	CT unit impulse signal $\delta(t)$	DT unit sample signal $\delta[n]$
Definition	Area under unit impulse approaches ‘1’ as its width approaches zero. Thus, it has zero value everywhere except $t = 0$	Amplitude of unit sample is ‘1’ at $n = 0$ and it has zero value at all other values of n.
Mathematical representation	$\delta(t) = \begin{cases} \infty & for\ t = 0 \\ 0 & for\ t \neq 0 \end{cases}$ $\int_{-\infty}^{\infty} \delta(t)dt = 1$ when $t \rightarrow 0$ $\delta(t) = 0$ for $t \neq 0$	$\delta[n] = \begin{cases} 1 & for\ n = 0 \\ 0 & for\ n \neq 0 \end{cases}$ or $\delta[n] = \{\dots, 0,0,0,\underset{\uparrow}{1},0,0,0, \dots\}$
Waveform		
Significance	$\delta[n]$ is not the sampled version of $\delta(t)$. *The main difference is Area under $\delta(t) = 1$ while Amplitude of $\delta[n] = 1$*	

3. Unit Ramp function

Parameter	CT unit impulse signal $r(t)$	DT unit sample signal $r[n]$
Definition	It is linearly growing function for positive values of time.	
Mathematical representation	$r(t) = \begin{cases} t & for\ t \geq 0 \\ 0 & for\ t < 0 \end{cases}$	$r[n] = \begin{cases} n & for\ n \geq 0 \\ 0 & for\ n < 0 \end{cases}$
Waveform		
Significance	Ramp function indicates linear function	

Relationship between the Signals

1. Relationship between Unit step and Unit ramp signal

- The unit ramp function is defined as,
  - $r(t) = \begin{cases} t & for\ t \geq 0 \\ 0 & for\ t < 0 \end{cases}$
- Differentiating w.r.t 't' gives
- $\frac{d}{dt} r(t) = \begin{cases} \frac{d}{dt}(t) & for\ t \geq 0 \\ 0 & for\ t < 0 \end{cases} \quad = \begin{cases} 1 & for\ t \geq 0 \\ 0 & for\ t < 0 \end{cases} \quad = u(t)$
- $\boxed{\therefore \frac{d}{dt} r(t) = u(t) \quad \text{or} \quad r(t) = \int u(t)dt}$

2. Relationship between Unit step and Unit Impulse signal

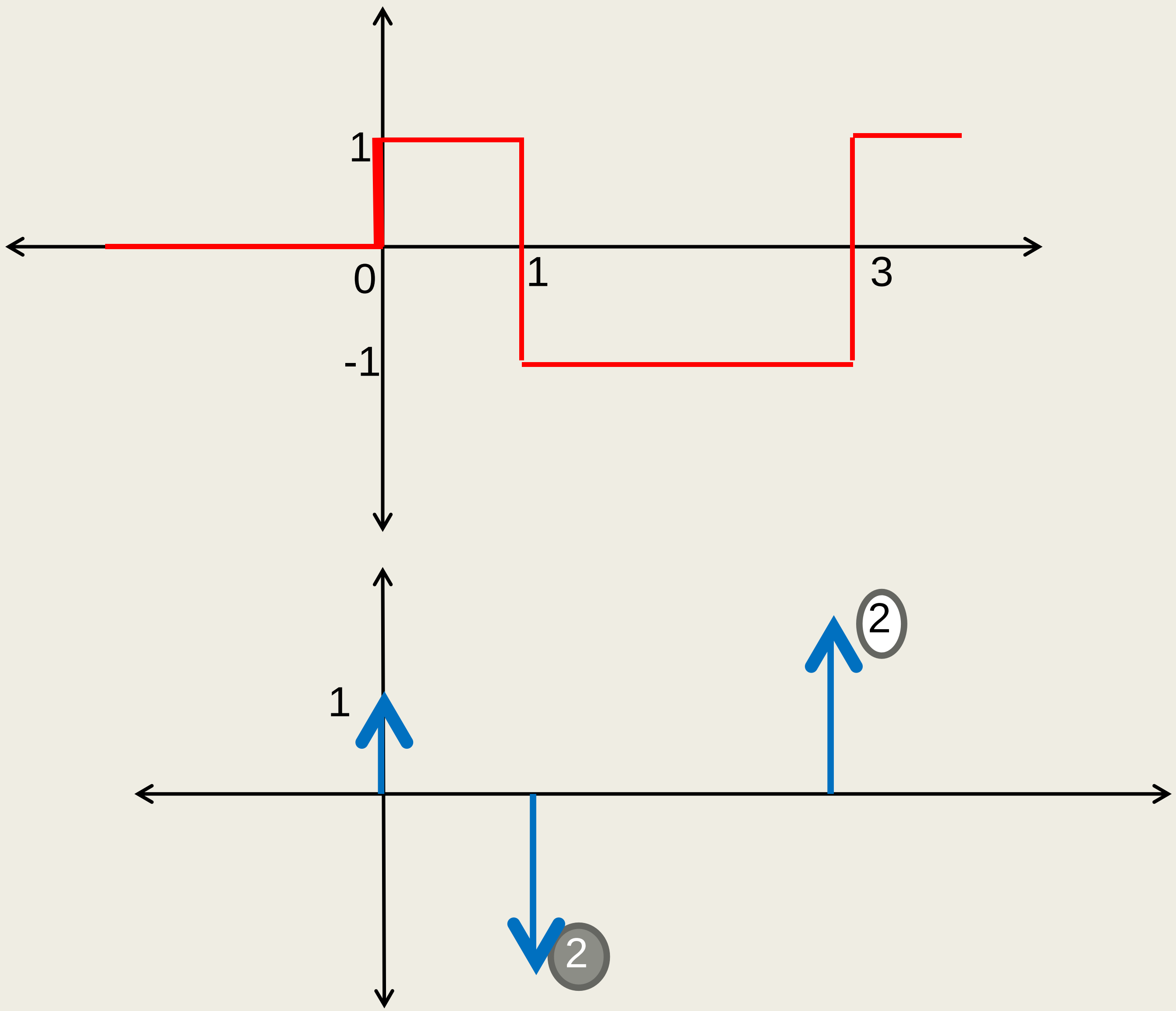
- $\frac{d}{dt} u(t) = \delta(t)$

- or $u(t) = \int \delta(t)dt$

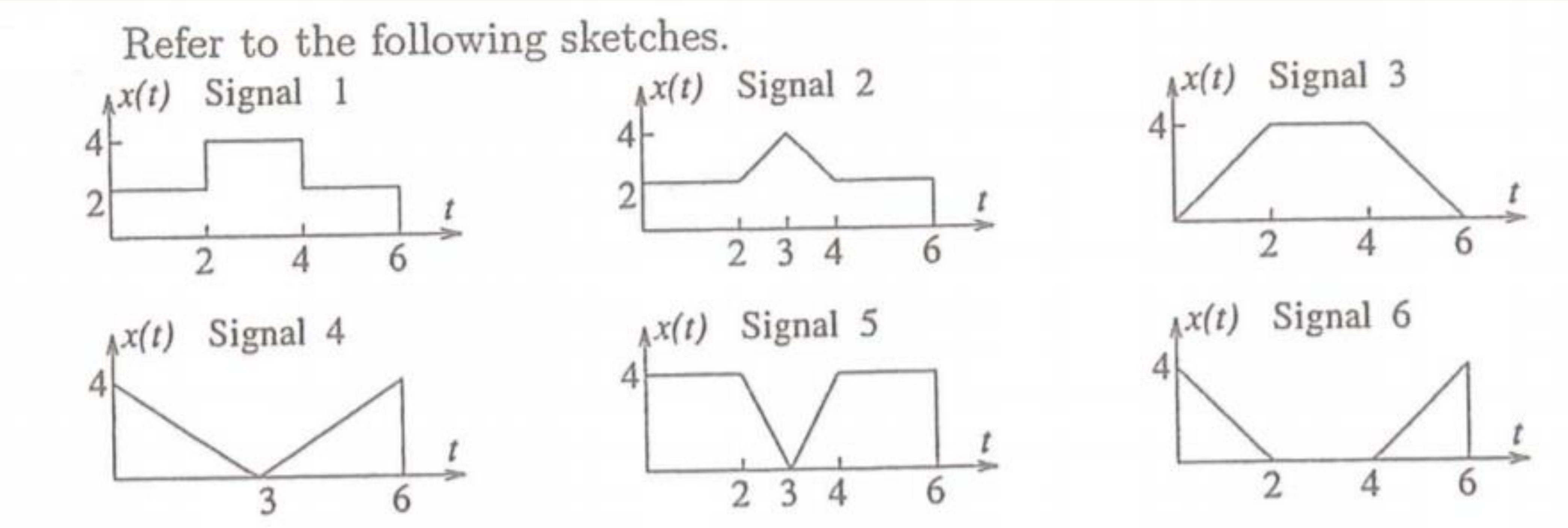
P.P 4.1:

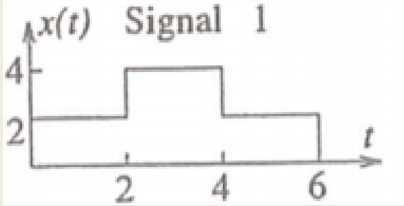
How can we write $\delta[n]$ in terms of $u[n]$. Also write $u[n]$ in terms of $\delta[n]$

Exp 4.3: Draw waveform for the differentiated signal (*)

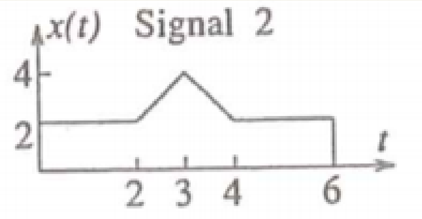


Exp 4.4: Draw waveform for the differentiated version of signals from 1 to 6

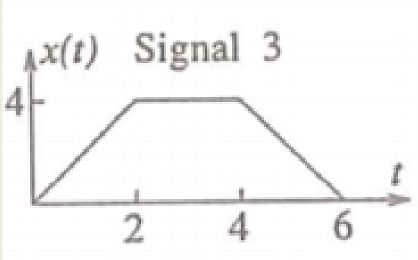




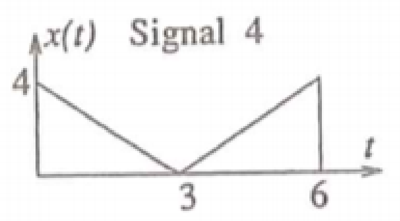
$$(\text{Signal 1:}) \quad x(t) = \begin{cases} 2 & 0 < t < 2 \\ 4 & 2 < t < 4 \\ 2 & 4 < t < 6 \\ 0 & \text{elsewhere} \end{cases}$$



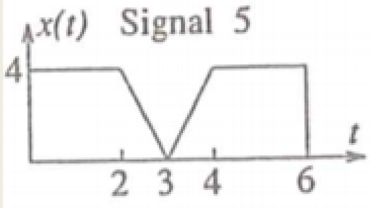
$$(\text{Signal 2:}) \quad x(t) = \begin{cases} 2 & 0 < t \le 2 \\ 2t-2 & 2 \le t \le 3 \\ -2+10 & 3 \le t \le 4 \\ 2 & 4 \le t < 6 \\ 0 & \text{elsewhere} \end{cases}$$



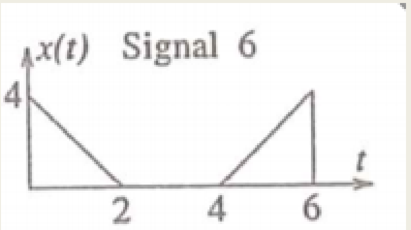
$$(\text{Signal 3:}) \quad x(t) = \begin{cases} 2t & 0 \le t \le 2 \\ 4 & 2 \le t \le 4 \\ -2t+12 & 4 \le t \le 6 \\ 0 & \text{elsewhere} \end{cases}$$



$$(\text{Signal 4:}) \quad x(t) = \begin{cases} -2t+4 & 0 < t \le 2 \\ 2t-4 & 2 \le t < 4 \\ 0 & \text{elsewhere} \end{cases}$$

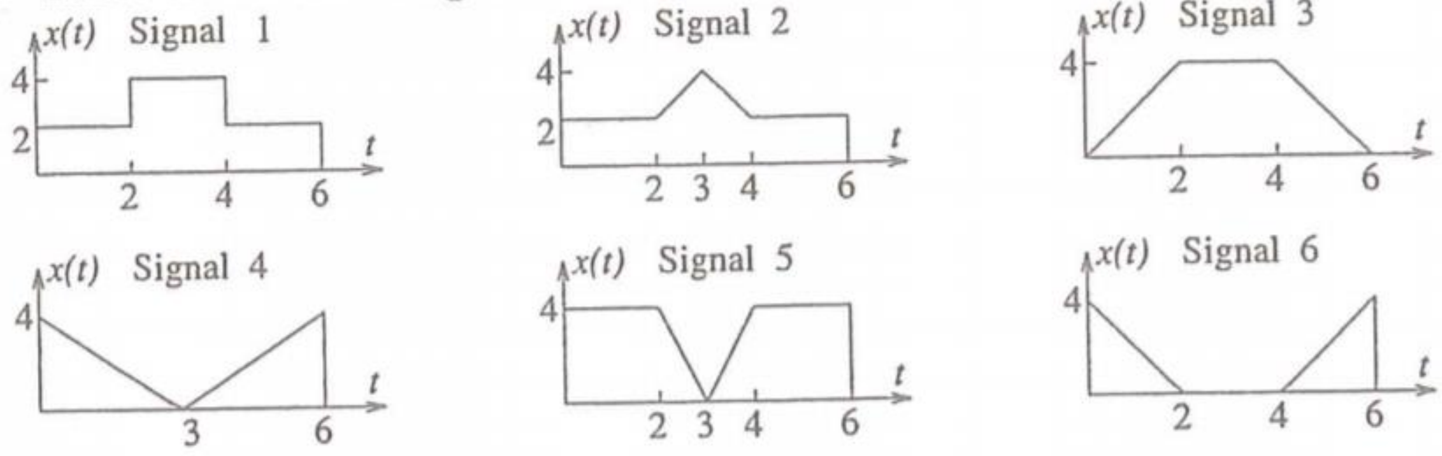


$$(\text{Signal 5:}) \quad x(t) = \begin{cases} 4 & 0 < t \le 2 \\ -4t+12 & 2 \le t \le 3 \\ 4t-12 & 3 \le t \le 4 \\ 4 & 4 \le t < 6 \\ 0 & \text{elsewhere} \end{cases}$$

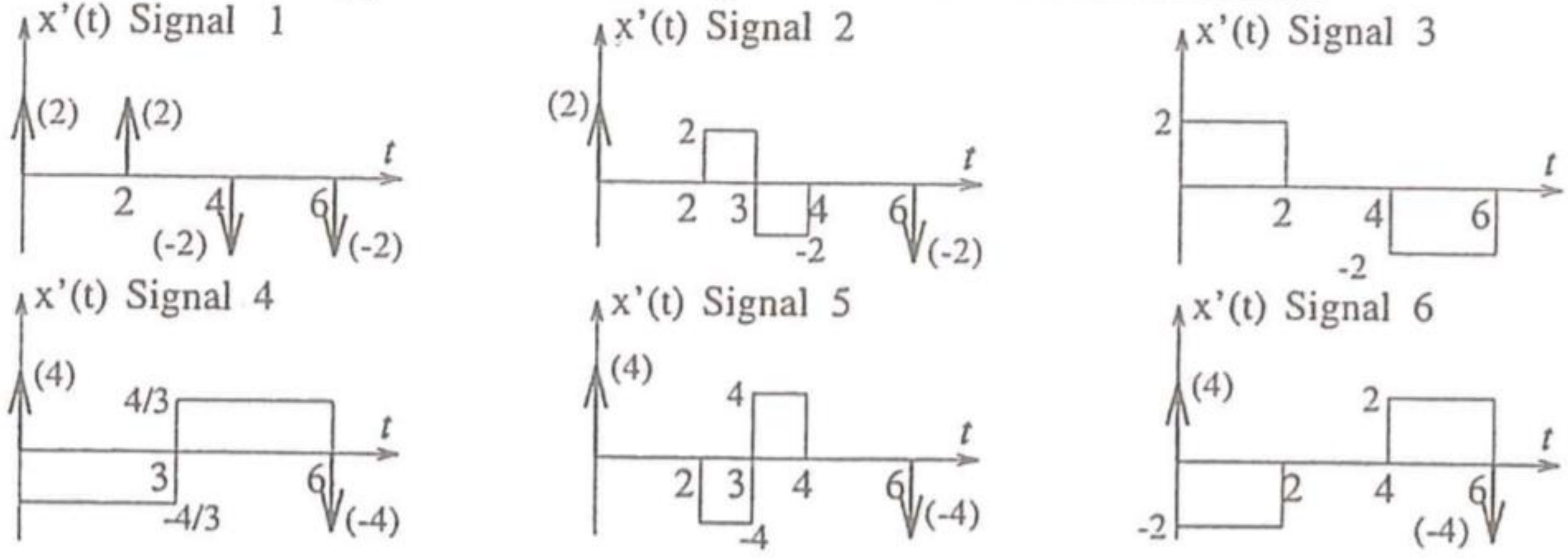


$$(\text{Signal 6:}) \quad x(t) = \begin{cases} -2t+4 & 0 < t \le 2 \\ 2t-8 & 4 \le t < 6 \\ 0 & \text{elsewhere} \end{cases}$$

Refer to the following sketches.

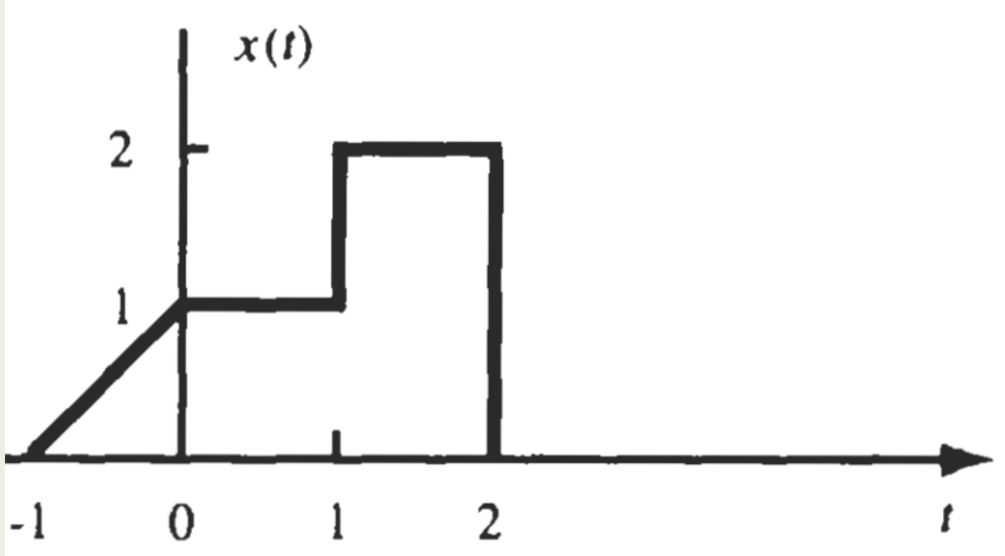


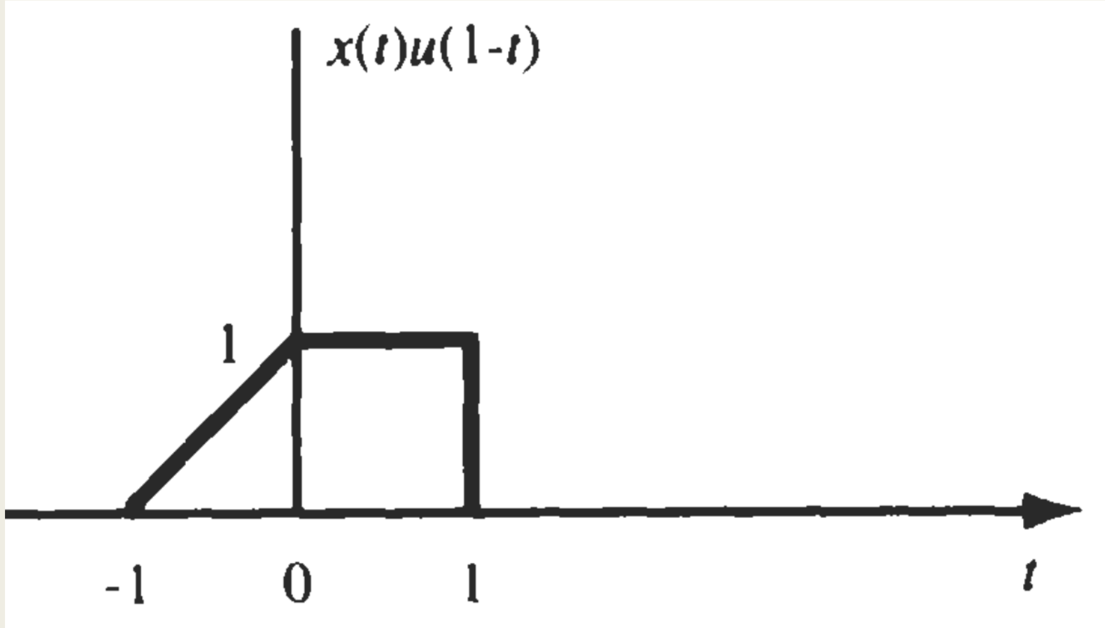
The first derivative $x'(t)$ will contain impulses at the discontinuities:

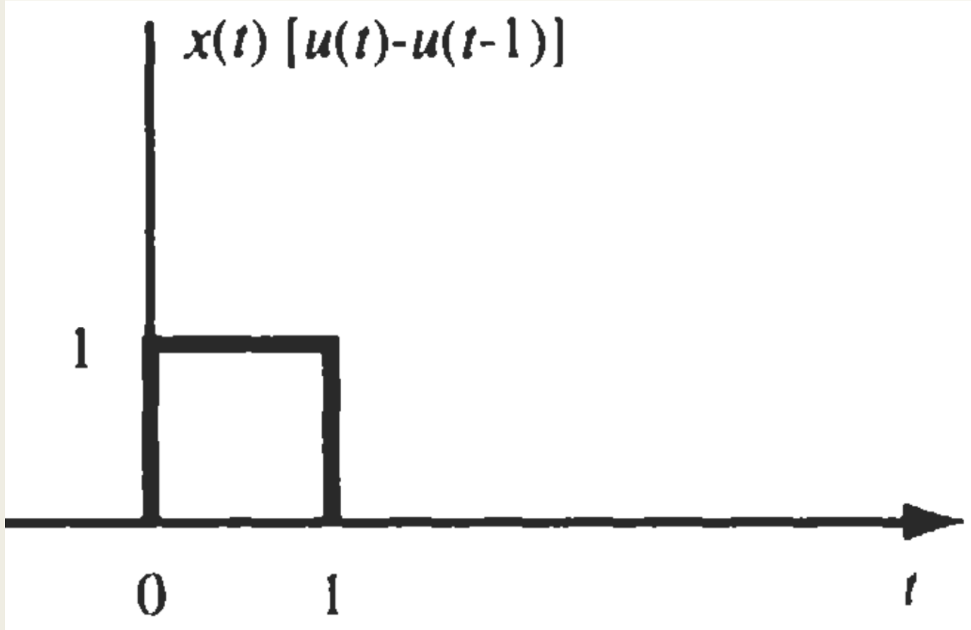


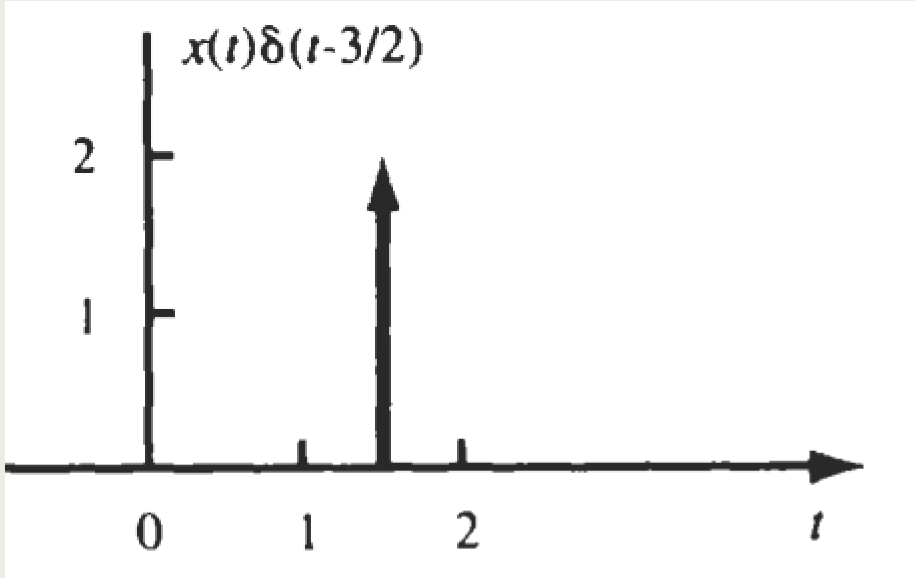
Exp 4.5: A CTS $x(t)$ is shown in Fig. Sketch and label each of the following signals

(a) $x(t)u(1-t)$; (b) $x(t)[u(t)-u(t-1)]$; (c) $x(t)\delta(t-\frac{3}{2})$



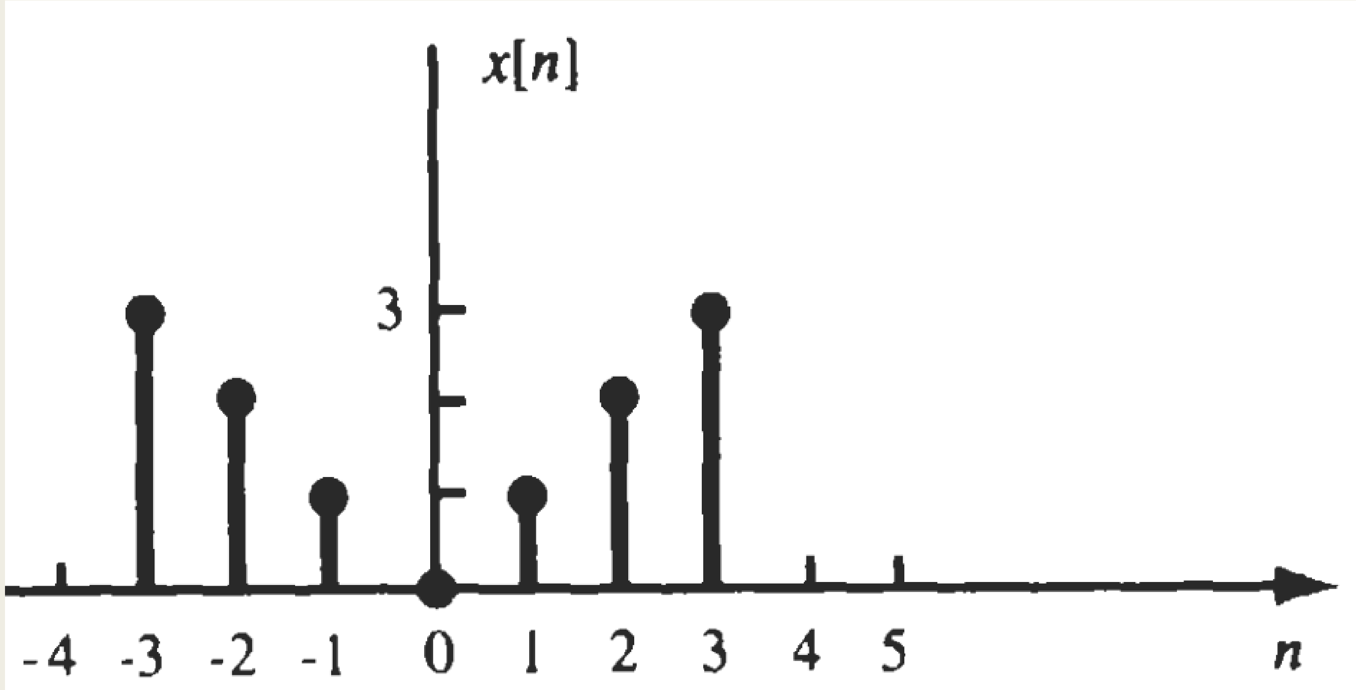

x(t)u(1-t)
1
-1
0
1
t

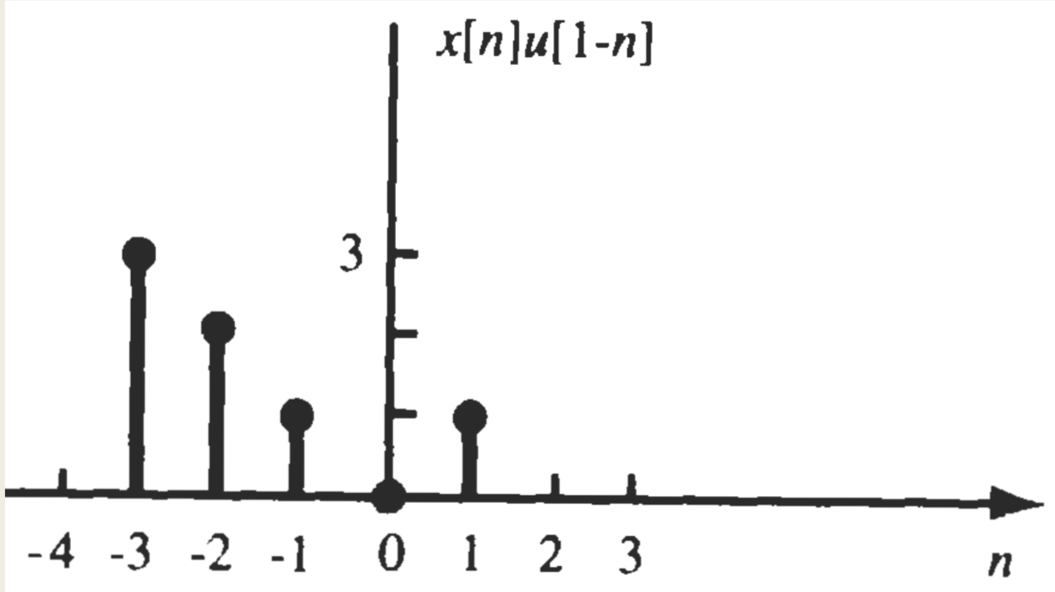

x(t) [u(t)-u(t-1)]
1
0
1
t

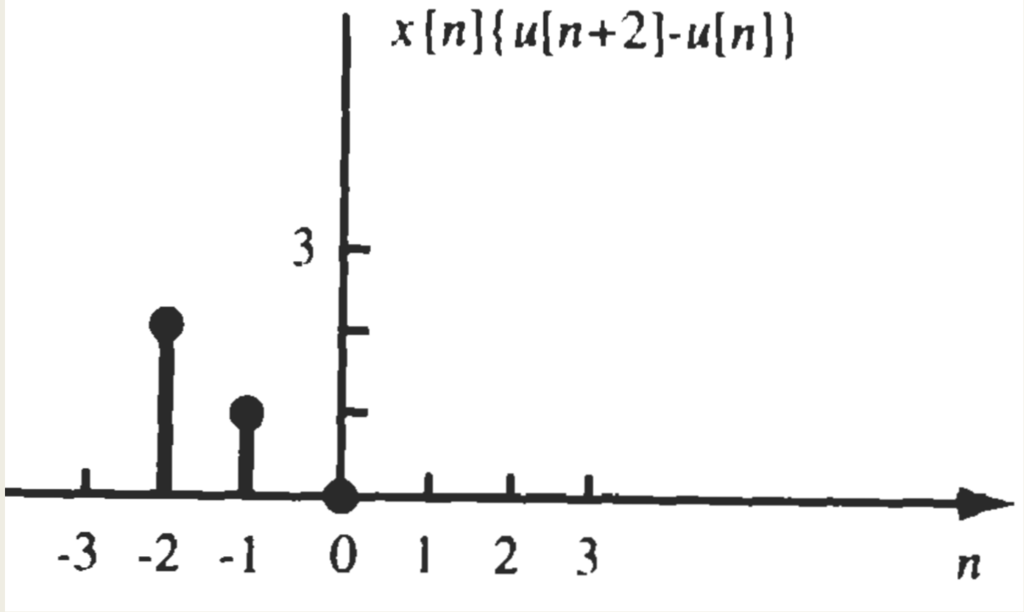

x(t)δ(t-3/2)
2
1
0
1
2
t

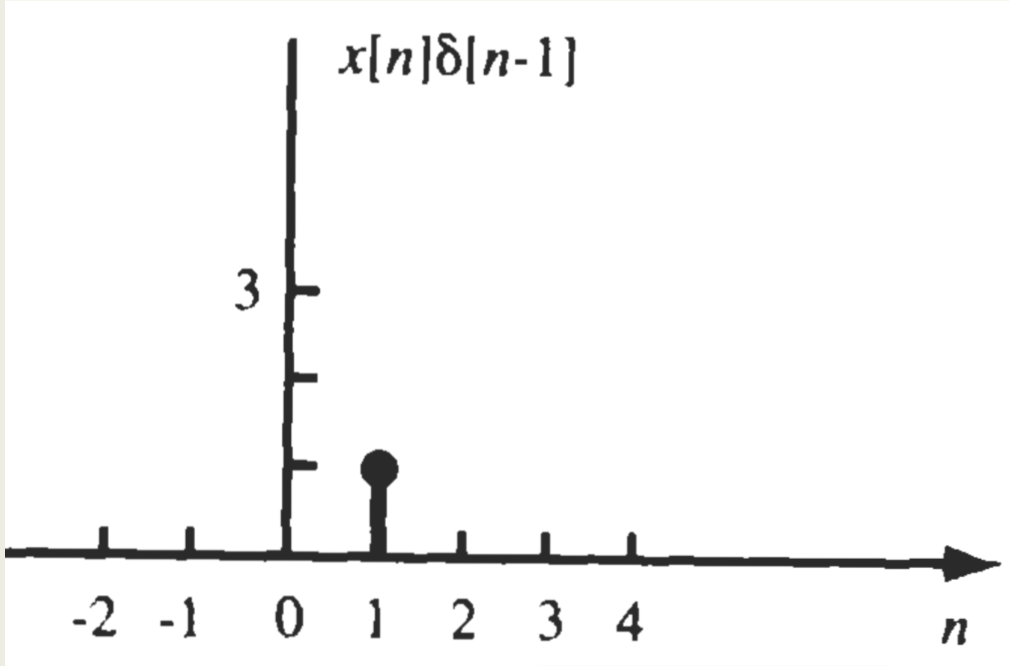
Exp 4.6: A DTS x[n] is shown in Fig. Sketch and label each of the following signals

$(a)\ x[n]u[1-n];\ (b)\ x[n]\{u[n+2]-u[n]\};\ (c)\ x[n]\delta[n-1]$

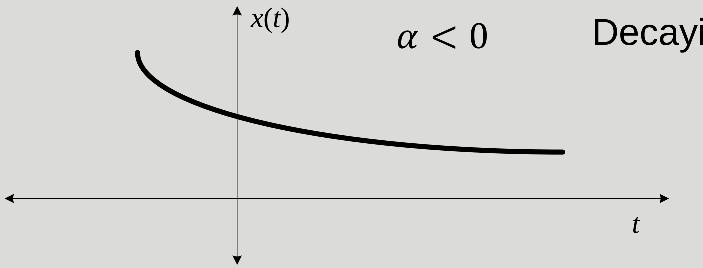
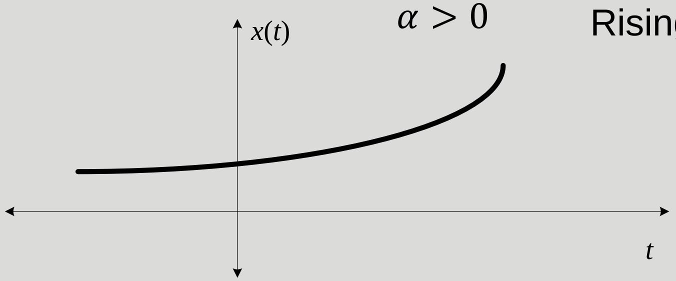
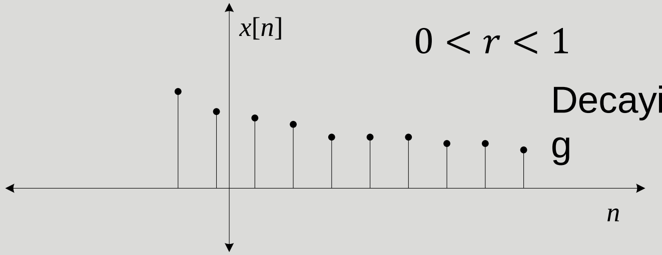
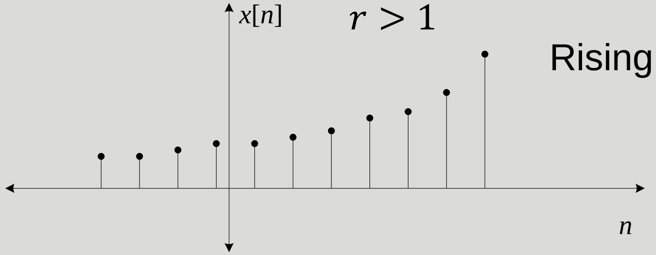



x[n]u[1-n]
3
-4 -3 -2 -1 0 1 2 3
n


x{n]{u[n+2]-u[n]}
3
-3 -2 -1 0 1 2 3
n


x[n]δ[n-1]
3
-2 -1 0 1 2 3 4
n

4. Real Exponential Signal

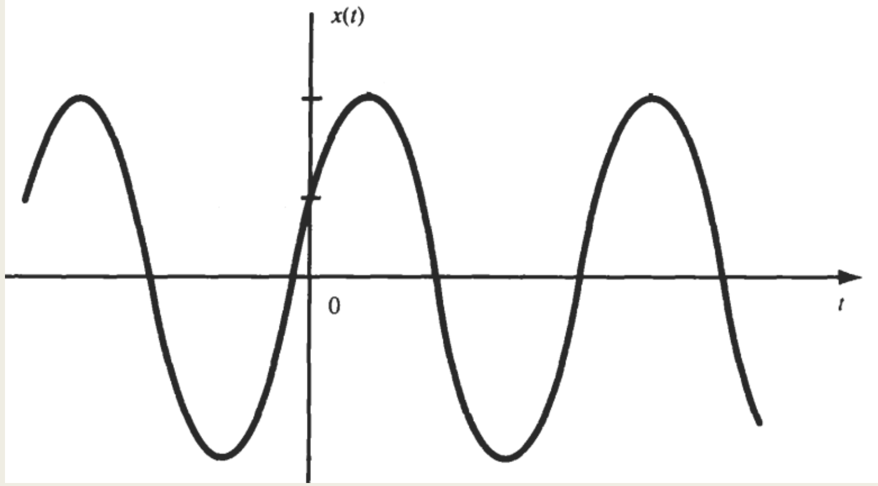
Parameter	**CT real exponential signals**	**DT real exponential signals**
Definition	It is exponentially growing or decaying signal	
Mathematical representation	$x(t) = be^{\alpha t}$ b and α are real	$x[n] = br^{n}$ If $r = e^{\alpha}$ $x[n] = be^{\alpha n}$ b and α are real
Waveform	 	 

5. Complex Exponential Signal

- When exponent is purely imaginary, then signal is said to be complex exponential
- It is given as
  - CT: $x(t) = e^{j\omega t}$
  - DT: $x[n] = e^{j\omega n}$

6. Sinusoidal Signal

- It is given as
  - CT: $x(t) = \cos(\omega t + \phi)$
  - DT: $x[n] = \cos(\omega n + \phi)$



P.P 4.2: Evaluate the following integrals

(a) $\int_{-1}^{8}[u(t+3)-2\delta(t)u(t)]dt$

(b) $\int_{1/2}^{5/2}\delta(3t)dt$

- Solution:
- (a) Ans: 7

P.P 4.3: Draw waveforms of the following

(a) $f_1(t) = 3u(t-1)$
(b) $f_2(t) = u(2-t)$
(c) $f(t) = f_1(t)f_2(t)$

Thank You !!!