Exercise-I

Q # 01

For the given matrix / vector pairs Compute the following quantities
a_{ii}, $a_{ij}a_{ij}$, $a_{ij}a_{jk}$, $a_{ij}b_j$, $a_{ij}b_ib_j$, b_ib_j, b_ib_i

1(a)

$$a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \quad b_i = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$a_{ii} = a_{11} + a_{22} + a_{33}$$
$$= 1 + 4 + 1 = 6 \text{ (scalar)}$$

$$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$$
$$= 1 + 1 + 1 + 0 + 16 + 4 + 0 + 1 + 1$$
$$= 25 \text{ (scalar)}$$

$$a_{ij}a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 & 4 \\ 0 & 18 & 10 \\ 0 & 5 & 3 \end{bmatrix} \text{ (matrix)}$$

$$a_{ij}b_j = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \text{ (vector)}$$

$$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$$

$$= 1+0+2+0+0+0+0+0+4$$

$$= 7 \text{ (scalar)}$$

$$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} \text{ (matrix)}$$

$$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 1+0+4 = 5 \text{ (scalar)}$$

(b)

$$a_{ij} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}, \quad b_i = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$a_{ii} = a_{11} + a_{22} + a_{33} = 1+2+2 = 5 \text{ (scalar)}$$

$$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$$

$$= 1+4+0+0+4+1+0+16+4$$

$$a_{ij}a_{ij} = 30 \text{ (scalar)}$$

$$a_{ij}a_{jk} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 & 2 \\ 0 & 8 & 4 \\ 0 & 16 & 8 \end{bmatrix} \text{ (matrix)}$$

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3$$

$$= \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix} \text{ (vector)}$$

$$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$$
$$= 4+4+0+0+2+1+0+4+2 = 17 \text{ (scalar)}$$

$$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \text{ (matrix)}$$

$$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3$$
$$= 4+1+1 = 6 \text{ (scalar)}$$

(c)

$$a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}, \quad b_i = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$a_{ii} = a_{11} + a_{22} + + a_{33}$$
$$= 1 + 0 + 4$$
$$= 5 \text{ (scalar)}$$

$$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$$
$$= 1+1+1+1+0+4+0+1+16$$
$$= 25 \text{ (scalar)}$$

$$a_{ij}a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 2 & 7 \\ 1 & 3 & 9 \\ 1 & 4 & 18 \end{bmatrix} \text{ (matrix)}$$

$$a_{ij} b_j = a_{i1} b_1 + a_{i2} b_2 + a_{i3} b_3$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$a_{ij} b_j = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{(vector)}$$

$$a_{ij} b_i b_j = a_{11} b_1 b_1 + a_{12} b_1 b_2 + a_{13} b_1 b_3 + a_{21} b_2 b_1 + a_{22} b_2 b_2 + a_{23} b_2 b_3 + a_{31} b_3 b_1 + a_{32} b_3 b_2 + a_{33} b_3 b_3$$

$$= 1 + 1 + 0 + 1 + 0 + 0 + 0 + 0 + 0$$

$$= 3 \text{ (scalar)}$$

$$b_i b_j = \begin{bmatrix} b_1 b_1 & b_1 b_2 & b_1 b_3 \\ b_2 b_1 & b_2 b_2 & b_2 b_3 \\ b_3 b_1 & b_3 b_2 & b_3 b_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{(matrix)}$$

$$b_i b_i = b_1 b_1 + b_2 b_2 + b_3 b_3$$

$$= 1 + 1 + 0$$

$$= 2 \text{ (scalar)}$$

Q # 02

Use decomposition result to express a_{ij} from exercise 1-1 in terms of sum of symmetric and antisymmetric matrices.

(a)

$$a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 1 \\ 1 & 2 & 1 \end{bmatrix}\right) + \frac{1}{2}\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 1 \\ 1 & 2 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate conditions.

(b)

$$a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}\right) + \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}$$

clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate conditions.

(c)

$$a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 2 & 4 \end{bmatrix}\right) + \frac{1}{2}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 2 & 4 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate conditions.

Q #03

If a_{ij} is symmetric and b_{ij} is antisymmetric prove in general that the product $a_{ij}b_{ij}$ is zero.

$$a_{ij}b_{ij} = -a_{ji}b_{ji} = -a_{ij}b_{ij}$$

$$\Rightarrow 2a_{ij}b_{ij} = 0 \Rightarrow a_{ij}b_{ij} = 0$$

From Ex 1-2(a)

$$a_{(ij)}a_{[ij]} = \frac{1}{4}tr\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T\right) = 0$$

From ex 1-2(b)

$$a_{(ij)}a_{[ij]} = \frac{1}{4}tr\left(\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}^T\right) = 0$$

From ex 1-2(c)

$$a_{(ij)}a_{[ij]} = \frac{1}{4}tr\left(\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T\right) = 0$$

Q # 04

Explicitly verify the following properties of kronecker delta:-

$$\delta_{ij} a_j = a_i$$
$$\delta_{ij} a_{jk} = a_{ik}$$

$$\delta_{ij} a_j = \delta_{i1} a_1 + \delta_{i2} a_2 + \delta_{i3} a_3$$

$$= \begin{bmatrix} \delta_{11} a_1 + \delta_{12} a_2 + \delta_{13} a_3 \\ \delta_{21} a_1 + \delta_{22} a_2 + \delta_{23} a_3 \\ \delta_{31} a_1 + \delta_{32} a_2 + \delta_{33} a_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_i$$

$$\delta_{ij} a_{jk} = \begin{bmatrix} \delta_{11} a_{11} + \delta_{12} a_{21} + \delta_{13} a_{31} + \delta_{11} a_{12} + \delta_{12} a_{22} + \\ \delta_{13} a_{32} + \delta_{11} a_{13} + \delta_{12} a_{23} + \delta_{13} a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Q # 05

Formally expand the expression (1.3.4) for determinant and justify that either index notation form yields a result that matches the traditional form of $\det[a_{ij}]$.

$$\det(a_{ij}) = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k}$$

$$= \varepsilon_{123} a_{11} a_{22} a_{33} + \varepsilon_{231} a_{12} a_{23} a_{31} + \varepsilon_{312} a_{13} a_{21} a_{32} + \varepsilon_{321} a_{13} a_{22} a_{31} + \varepsilon_{132} a_{11} a_{23} a_{32} + \varepsilon_{213} a_{12} a_{21} a_{33}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Q # 06

Determine the components of vector b_i and matrix a_{ij} given in Exercise 1-1 in a new coordinate system found through a rotation of $45°$ about the x_1-axis.

$45°$ rotation about x_1-axis

$$\Rightarrow Q_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

From Ex # 1-1 (a)

$$b'_i = Q_{ij}b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$a'_{ij} = Q_{ip}Q_{jp}a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & \sqrt{2} & 0 \\ 0 & 4 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

From Ex # 1-1 (b)

$$b_i' = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2} \\ 0 \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jp} a_{pq}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & \sqrt{2} & -\sqrt{2} \\ 0 & \frac{9}{2} & -\frac{3}{2} \\ 0 & \frac{3}{2} & -1/2 \end{bmatrix}$$

From Ex # 1-1 (c)

$$b_i' = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jp} a_{pq}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2}/2 & 7/2 & 5/2 \\ -\sqrt{2}/2 & 3/2 & 1/2 \end{bmatrix}$$

Q # 07

Consider the two dimensional coordinate transformation shown in Fig 1-7. Show that the transformation matrix for this case is given by

$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

If $b_i = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, $a_{ij} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ are components of a first and 2nd order tensor.

$$Q_{ij} = \begin{bmatrix} \cos(x_1', x_1) & \cos(x_1', x_2) \\ \cos(x_2', x_1) & \cos(x_2', x_2) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \cos(90^\circ - \theta) \\ \cos(90^\circ + \theta) & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$b_i' = Q_{ij} b_j = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$= \begin{bmatrix} b_1\cos\theta + b_2\sin\theta \\ -b_1\sin\theta + b_2\cos\theta \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jp} Q_{pq} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T$$

$$= \begin{bmatrix} a_{11}\cos^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta + a_{22}\sin^2\theta & a_{12}\cos^2\theta - (a_{11}-a_{22})\sin\theta\cos\theta - a_{21}\sin^2\theta \\ a_{21}\cos^2\theta - (a_{11}-a_{22})\sin\theta\cos\theta - a_{21}\sin^2\theta & a_{11}\sin^2\theta - (a_{12}+a_{21})\sin\theta\cos\theta + a_{22}\cos^2\theta \end{bmatrix}$$

Q # 08

Show that second order tensor $a\delta_{ij}$ where a is an arbitrary constant retains its form under any transformation Q_{ij}.

$$a'\delta'_{ij} = Q_{ip}Q_{jq}\delta_{pq}$$
$$= a\,Q_{ip}Q_{jp}$$
$$a'\delta'_{ij} = a\,\delta_{ij}$$

Q # 09

The most general form of a fourth order isotropic tensor can be expressed by

$$\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ij}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

verify that this form remains the same under the general transformation given by $(1\cdot 5\cdot 1)_s$.

$$\alpha\delta'_{ij}\delta'_{kl} + \beta\delta'_{ik}\delta'_{jl} + \gamma\delta'_{il}\delta'_{jk} = Q_{im}Q_{jn}Q_{kp}Q_{lq}(\alpha\delta_{mn}\delta_{pq} + \beta\delta_{mp}\delta_{nq} + \gamma\delta_{mq}\delta_{np})$$

$$= \alpha Q_{im}Q_{jm}Q_{kp}Q_{lp} + \beta Q_{im}Q_{jn}Q_{km}Q_{ln} + \gamma Q_{im}Q_{jn}Q_{kn}Q_{lm}$$

$$= \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

$\sin\theta$] T
$\cos\theta$]
$-a_{22}\sin\theta\cos\theta$
$-\theta$
$\theta -$
$\sin\theta\cos\theta + a_{22}\cos^2\theta$

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Q #10

For the fourth order isotropic tensor given in Ex # 1-9, show that if $\beta = \gamma$ then the tensor will have the following symmetry

$$C_{ijkl} = C_{klij}$$

$$C_{ijkl} = \alpha \delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

$$= \alpha\delta_{ij}\cdot\delta_{kl} + \beta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$= \alpha\delta_{kl}\delta_{ij} + \beta(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li})$$

$$C_{ijkl} = C_{klij}$$

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Q no 11

Sol:

If $a = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$I_a = a_{ii} = \lambda_1 + \lambda_2 + \lambda_3$

$II_a = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} + \begin{vmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \end{vmatrix}$

$= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$

$III_a = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} = \lambda_1\lambda_2\lambda_3$

Q No 12 (a)

Sol:

$a_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow I_a = -1$

$II_a = -2$ & $III_a = 0$

Characteristic Eqn is

$-\lambda^3 - \lambda^2 + 2\lambda = 0 \Rightarrow \lambda(\lambda^2 + \lambda - 2) = 0$

$\lambda(\lambda + 2)(\lambda - 1) = 0$

So, roots

$\lambda_1 = 0$, $\lambda_2 = -2$ & $\lambda_3 = 1$

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$\lambda_1 = 0$ Case: $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0$

$-n_1^{(1)} + n_2^{(1)} = 0$

$n_3^{(1)} = 0 \quad \Rightarrow n_1^{(1)} = +n_2^{(1)} = \pm\sqrt{2}/2, \; n^{(1)} = \pm(\sqrt{2}/2)$

$n_1^{(1)2} + n_2^{(1)2} + n_3^{(1)2} = 1$

$(+1, 1, 0)$

$\lambda_2 = -2$ Case: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0$

$+n_1^{(1)} + n_2^{(1)} = 0$

$n_3^{(1)} = 0 \quad \Rightarrow n_1 = n_2 = \pm\sqrt{2}/\sqrt{2} \Rightarrow n^{(2)} = \pm(\sqrt{2}/2)$

$n_1^{(1)2} + n_2^{(1)2} + n_3^{(1)2} = 1$

$(-1, 1, 0)$

$\lambda_3 = 1$ Case: $\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$

$-2n_1^{(3)} + n_2^{(3)} = 0$

$n_1^{(3)} - 2n_2^{(3)} = 0 \quad \Rightarrow n_1 = n_2 = 0, \; n_3^{(3)} = 1 \Rightarrow n^{(3)} = \pm(0, 0, 1)$

$n_1^{(3)2} + n_2^{(3)2} + n_3^{(3)2} = 1$

The rotation matrix is given by

$$Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$ and

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$$a'_{ij} = Q_{ip} Q_{jp} a_{pq}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q NO 12 (b)

Sol: $a_{ij} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow I_a = -4$

$II_a = 3$, $III_a = 0$

Characteristic Eq. $-\lambda^3 - 4\lambda^2 - 3\lambda = 0$

$\lambda(\lambda^2 + 4\lambda + 3) = 0 \Rightarrow \lambda(\lambda+3)(\lambda+1) = 0$

Roots $\Rightarrow \lambda_1 = -3$, $\lambda_2 = -1$, $\lambda_3 = 0$

$\lambda_1 = -3$ Case: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix}$

$n_1^{(1)} + n_2^{(1)} = 0$

$n_3^{(1)} = 0$ $\Rightarrow n_1^{(1)} = -n_2^{(1)} = \pm\sqrt{2}/2$, $n^{(1)} = \pm(\sqrt{2}/2)$

$(1, 1, 0)$

$n_1^{(1)^2} + n_2^{(1)^2} + n_3^{(1)^2} = 1$

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$\lambda_2 = -1$ Case: $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$

$-n_1^{(2)} + n_2^{(2)} = 0$

$n_3^{(2)} = 0 \Rightarrow n_1 = n_2 = \pm\sqrt{2}/2 \Rightarrow n^{(2)} = \pm(\sqrt{2}/2)$

${n_1^{(2)}}^2 + {n_2^{(2)}}^2 + {n_3^{(2)}}^2 = 1$ $(1, 1, 0)$

$\lambda_3 = 0$ Case: $\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$

$-2n_1^{(3)} + n_2^{(3)} = 0$

$n_1^{(3)} - 2n_2^{(3)} = 0 \Rightarrow n_1 = n_2 = 0,\ n_3^{(3)} = 1 \Rightarrow n^3 = \pm(0, 0, 1)$

${n_1^{(3)}}^2 + {n_2^{(3)}}^2 + {n_3^{(3)}}^2 = 1$

The rotation Matrix

$$Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$

$$a'_{ij} = Q_{ip} Q_{jp} a_{pq}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$$

$$a'_{ij} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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Q No 14 (a)

$\vec{v} = x_1\vec{e}_1 + x_1x_2\vec{e}_2 + 2x_1x_2x_3\vec{e}_3$

Sol: $\nabla \cdot v = v_{1,1} + v_{2,2} + v_{3,3}$

$= 1 + x_1 + 2x_1x_2$

$$\nabla \times \vec{v} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1 & x_1x_2 & 2x_1x_2x_3 \end{vmatrix}$$

$= 2x_1x_3\vec{e}_1 + 2x_2 2x_3\vec{e}_2 + \cancel{2}\,x_2\vec{e}_3$

$\nabla^2\vec{v} = 0\vec{e}_1 + 0\vec{e}_2 + 0\vec{e}_3 = 0$

$$\nabla\vec{v} = \begin{bmatrix} 1 & 0 & 0 \\ x_2 & x_1 & 0 \\ 2x_2x_3 & 2x_1x_3 & 2x_1x_2 \end{bmatrix}$$

$= tr\,\nabla(\vec{v}) = 1 + x_1\ 2x_1x_2$

Q No 14 (b)

$\vec{v} = x_2^2\vec{e}_1 + 2x_2x_3e_2 + 4x_1^2e_3$

$\nabla \cdot \vec{v} = v_{1,1} + v_{2,2} + v_{3,3} = 0 + 2x_3 + 0$

$$\nabla \times \vec{v} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_2^2 & 2x_2x_3 & 4x_1^2 \end{bmatrix} = -2x_2\vec{e}_1 - 8x_1\vec{e}_2 - 2x_2\vec{e}_3$$

$\nabla^2\vec{v} = 2\vec{e}_1 + 0\vec{e}_2 + 8\vec{e}_3 = 0$

$$\nabla\vec{v} = \begin{bmatrix} 0 & 2x_2 & 0 \\ 0 & 2x_3 & 2x_2 \\ 8x_1 & 0 & 0 \end{bmatrix}, \quad tr(\nabla\vec{v}) = 3x_3$$

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Q NO 15

Sol:

$$a_i = -\frac{1}{2}\varepsilon_{ijk}a_{jk}$$

$$\varepsilon_{imn}a_i = \frac{-1}{2}\varepsilon_{ijk}\varepsilon_{imn}a_{jk} = \frac{-1}{2}\begin{vmatrix} \delta_{ii} & \delta_{im} & \delta_{in} \\ \delta_{ji} & \delta_{jm} & \delta_{jn} \\ \delta_{ki} & \delta_{km} & \delta_{kn} \end{vmatrix} a_{jk}$$

$$= -\frac{1}{2}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})a_{jk}$$

$$= -\frac{1}{2}(a_{mn} - a_{nm}) = \frac{-1}{2}(a_{mn} + a_{mn})$$

$$= -a_{mn} \qquad \therefore a_{jk} = -\varepsilon_{ijk}a_i$$

Q NO 16 (a)

Sol:

$$\nabla(\phi\psi) = (\phi\psi)_{,k} = \phi\psi_{,k} + \phi_{,k}\psi$$

$$= \nabla\phi\psi + \phi\nabla\psi$$

$$\nabla^2(\phi\psi) = (\phi\psi)_{,kk} = (\phi\psi_{,k} + \phi_{,k}\psi)_{,k}$$

$$= \phi\psi_{,kk} + \phi_{,k}\psi_{,k} + \phi_{,k}\psi_{,k} + \phi_{,kk}\psi$$

$$= \phi_{,kk}\psi + \phi\psi_{,kk} + 2\phi_{,k}\psi_{,k}$$

$$= (\nabla^2\phi)\psi + \phi(\nabla^2\psi) + 2\nabla\phi\cdot\nabla\psi$$

$$\nabla\cdot(\phi\vec{U}) = (\phi U_k)_{,k} \Rightarrow \phi U_{k,k} + \phi_{,k}U_k$$

$$= \nabla\phi\cdot\vec{U} + \phi(\nabla\cdot\vec{U})$$

δ_{in}
δ_{jn} | g_{jk}
δ_{kn}

$n)$

$_{,kk}\Psi$

Q No 16 (c)

$$\nabla\cdot(\nabla\times\vec{U}) = (\varepsilon_{ijk} U_{k,j})_{,i} = \varepsilon_{ijk} U_{k,ji} = 0$$

$$\nabla\times(\nabla\times\vec{U}) = \varepsilon_{mni}(\varepsilon_{ijk} U_{k,j})_{,n}$$

$$= \varepsilon_{imn}\varepsilon_{ijk} U_{k,jn}$$

$$= (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}) U_{k,jn}$$

$$= U_{n,nm} - U_{m,nm}$$

$$= \nabla(\nabla\cdot\vec{U}) - \nabla^2\vec{U}$$

$$\vec{U}\times(\nabla\times\vec{U}) = \varepsilon_{ijk} U_j (\varepsilon_{kmn} U_{n,m})$$

$$= \varepsilon_{kij}\varepsilon_{kmn} U_j U_{n,m}$$

$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) U_j U_{n,m}$$

$$= U_n U_{n,j} - U_m U_{i,m}$$

$$= \frac{1}{2}\nabla(\vec{U}\cdot\vec{U}) - \vec{U}\cdot\nabla\vec{U}.$$

Q No 17

Sol:-

Cylinderical coordinates : $\xi^1 = r,\ \xi^2 = \theta$
$\xi^3 = z$

$$(ds)^2 = (dr)^2 + (r\,d\theta)^2 + (dz)^2 \Rightarrow h_1 = 1,\ h_2 = r$$
$$h_3 = 1$$

$$\hat{e}_r = \cos\theta\, e_1 + \sin\theta\, e_2,\quad \hat{e}_\theta = -\sin\theta\, e_1 + \cos\theta\, e_2$$
$$\hat{e}_z = e_3$$

$$\frac{\partial \hat{e}_r}{\partial\theta} = \hat{e}_\theta,\quad \frac{\partial \hat{e}_\theta}{\partial\theta} = -\hat{e}_r$$

$$\frac{\partial \hat{e}_r}{\partial r} = \frac{\partial \hat{e}_\theta}{\partial r} = \frac{\partial \hat{e}_z}{\partial\theta} = \frac{\partial \hat{e}_z}{\partial z} = 0$$

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$$

$$\Delta f = \hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$$

$$\cancel{\Delta f = \hat{e}_r \frac{\partial f}{\partial \theta}}$$

$$\nabla \cdot u = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla \times u = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right)\hat{e}_z$$

Q No 18

Sol:-

Spherical Coordinates: $\xi^1 = R, \xi^2 = \phi, \xi^3 = \theta$

$x^1 = \xi^1 \sin\xi^2 \cos\xi^3, \ x^2 = \xi^1 \sin\xi^2 \sin\xi^3,$

$x^3 = \xi^1 \cos\xi^2$

scale factors.

$$(h_1)^2 = \frac{\partial x^k}{\partial \xi^1}\frac{\partial x^k}{\partial \xi^1} = (\sin\phi\cos\theta)^2 + (\sin\phi\cos\theta)^2 + \cos^2\phi$$

$$(h_1)^2 = 1$$

$$h_1 = 1$$

$$(h_2)^2 = \frac{\partial x^k}{\partial \xi^2}\frac{\partial x^k}{\partial \xi^2} = R^2 \Rightarrow h_2 = R$$

$$(h_3)^2 = \frac{\partial x^k}{\partial \xi^3}\frac{\partial x^k}{\partial \xi^3} = R^2\sin^2\phi \Rightarrow h_3\ R\sin\phi$$

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Unit vectors

$\hat{e}_R = \cos\theta \sin\phi e_1 + \sin\theta \sin\phi e_2 + \cos\phi e_3$

$\hat{e}_\phi = \cos\theta \cos\phi e_1 + \sin\theta \cos\phi e_2 - \sin\phi e_3$

$\hat{e}_\theta = -\sin\theta e_1 + \cos\theta e_2$

$\frac{\partial \hat{e}_R}{\partial R} = 0, \quad \frac{\partial \hat{e}_\phi}{\partial \phi} = \hat{e}_\phi, \quad \frac{\partial \hat{e}_R}{\partial \theta} = \sin\phi \hat{e}_\theta$

$\frac{\partial \hat{e}_\phi}{\partial R} = 0, \quad \frac{\partial \hat{e}_\phi}{\partial \phi} = -\hat{e}_\phi, \quad \frac{\partial \hat{e}_\phi}{\partial \theta} = \cos\phi \hat{e}_\theta$

$\frac{\partial \hat{e}_\theta}{\partial R} = 0, \quad \frac{\partial \hat{e}_\theta}{\partial \phi} = 0, \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\cos\phi \hat{e}_\phi$

Using $(1.9.12) - (1.9.16) \Rightarrow$

$$\nabla = \hat{e}_R \frac{\partial}{\partial R} + \hat{e}_\phi \frac{1}{R}\frac{\partial}{\partial \phi} + \hat{e}_\theta \frac{1}{R\sin\phi}\frac{\partial}{\partial \theta}$$

$$\nabla f = \hat{e}_R \frac{\partial f}{\partial R} + \hat{e}_\phi \frac{1}{R}\frac{\partial f}{\partial \phi} + \hat{e}_z \frac{1}{R\sin\phi}\frac{\partial f}{\partial \theta}$$

$$\nabla \cdot u = \frac{1}{R^2\sin\phi}\frac{\partial}{\partial R}(R^2 \sin\phi\, u_R) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}(R\sin\phi\, u_\phi)$$

$$+ \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \theta}(R u_\theta)$$

$$\nabla \cdot u = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 u_R) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi\, u_\phi) +$$

$$\frac{1}{R\sin\phi}\frac{\partial}{\partial \theta}(u_\theta)$$

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$$\nabla^2 f = \frac{1}{R^2\sin\phi}\frac{\partial}{\partial R}\left(R^2\sin\phi\frac{\partial f}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right)$$

$$+ \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial f}{\partial\theta}\right)$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial f}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right) +$$

$$\frac{1}{R^2\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2}$$

$$\nabla\times U = \left(\frac{1}{R^2\sin\phi}\left[\frac{\partial}{\partial\phi}(R\sin\phi\, U_\theta) - \frac{\partial}{\partial\theta}(RU_\phi)\right]\right)\hat{e}_R +$$

$$\left(\frac{1}{R\sin\phi}\left[\frac{\partial}{\partial\theta}(U_R) - \frac{\partial}{\partial R}(R\sin\phi\, U_\theta)\right]\right)\hat{e}_\phi +$$

$$\left(\frac{1}{R}\frac{\partial}{\partial R}\left[(RU_\phi) - \frac{\partial}{\partial\phi}(U_R)\right]\right)\hat{e}_\theta$$

$$= \left[\frac{1}{R\sin\phi}\left(\frac{\partial}{\partial\phi}(\sin\phi\, U_\theta) - \frac{\partial U_\phi}{\partial\theta}\right)\right]\hat{e}_R +$$

$$\left[\frac{1}{R\sin\phi}\frac{\partial U_R}{\partial\theta} - \frac{1}{R}\frac{\partial}{\partial R}(RU_\theta)\right]\hat{e}_\phi +$$

$$\left[\frac{1}{R}\left(\frac{\partial}{\partial R}(RU_\phi) - \frac{\partial U_R}{\partial\phi}\right)\right]\hat{e}_\theta$$

Question

Transform strain-displacement from cartesian to cylindrical coordinates

cylindrical coordinates

$$U_x = U_r\cos\theta - U_\theta \sin\theta$$

$$U_y = U_r\sin\theta + U_\theta\cos\theta$$

$$U_z = u_z$$

Derivative of $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

where $r = \sqrt{x^2 + y^2}$

$\theta = \arctan\left(\frac{y}{x}\right)$ is given by

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

It follows that

$$\frac{\partial^2}{\partial x^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial^2\theta} + \left(\cos\theta\frac{\partial}{\partial r}\right)\left(-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)$$

$$- \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\cos\theta\frac{\partial}{\partial r}$$

$$= \cos^2(\theta)\frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} - \cos\theta\sin\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} - \cos\theta\sin\theta\left[\frac{-1}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right]$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial\theta\partial r}$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}$$

$$- \frac{1}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial r\partial\theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial\theta\partial r} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r}$$

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$$= \cos^2\theta \frac{\partial^2}{\partial r^2} + \sin^2\theta \left[\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right] +$$

$$2\sin\theta\cos\theta \left(\frac{1}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial r \partial\theta} \right)$$

Likewise

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta \frac{\partial^2}{\partial r^2} + \cos^2\theta \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial r^2}{\partial \theta^2} \right)$$

$$- 2\sin\theta\cos\theta \left(\frac{1}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta} \right)$$

$$e_{xx} = \frac{\partial U_x}{\partial x} = \cos\theta \frac{\partial}{\partial r}\left(U_r\cos\theta - U_\theta \sin\theta \right)$$

$$- \frac{\sin\theta}{r}\left(\frac{\partial}{\partial\theta}\left(U_r\cos\theta - U_\theta\sin\theta \right) \right)$$

$$= \frac{\partial U_r}{\partial r}\cos^2\theta - \frac{\partial U}{\partial r}\theta \sin\theta\cos\theta - \frac{2U_r}{2\theta}\frac{\sin\theta\cos\theta}{r}$$

$$+ \frac{U_r}{r}\sin^2\theta + \frac{\partial U_\theta}{\partial\theta}\frac{\sin^2\theta}{r} + \frac{U_\theta}{r}\sin\theta\cos\theta$$

$$= 2U_r\cos^2\theta + \left(\frac{U_\theta}{r} - \frac{\partial U_\theta}{\partial r} - \frac{1}{r}\frac{\partial U_r}{\partial\theta} \right)\sin\theta\cos\theta$$

$$+ \left(\frac{U_r}{r} + \frac{1}{r}\frac{\partial U_\theta}{\partial\theta} \right)\sin^2\theta$$

$$e_{yy} = \frac{\partial U_y}{\partial y} = \sin\theta\frac{\partial}{\partial r}\left(U_r\sin\theta + U_\theta\cos\theta \right) +$$

$$\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(U_r\sin\theta + U_\theta\cos\theta \right)$$

$$e_{xy} = r\left(\frac{\partial x}{\partial y} + \frac{\partial y}{\partial x} \right)$$

Thus

$$e_{rr} = \frac{\partial U_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\left(U_r + \frac{\partial U_\theta}{\partial\theta} \right)$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial U_r}{\partial\theta} + \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r} \right)$$

and

$$e_{zz} = \frac{\partial U_z}{\partial z}$$