

0-1 Knapsack problem

0-1 Knapsack problem

Given a knapsack with maximum capacity W , and a set S consisting of n items

Each item i has some weight w_i and benefit value b_i (all w_i , b_i and W are integer values)

Problem: How to pack the knapsack to achieve maximum total value of packed items?

0-1 Knapsack problem

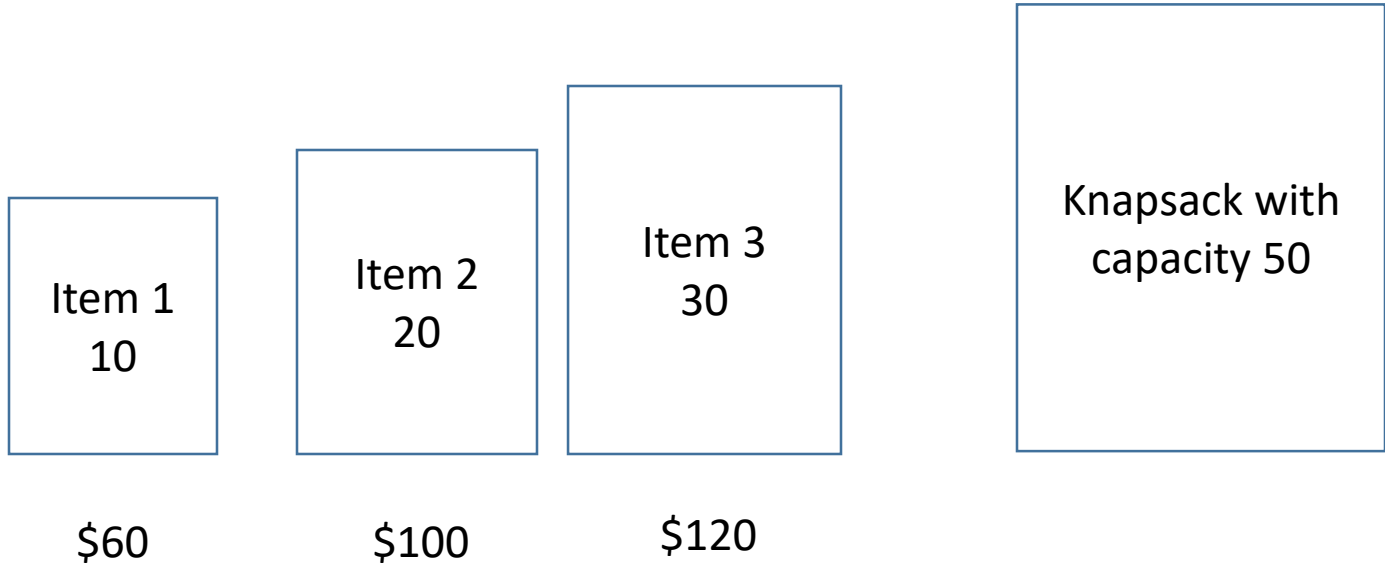
Problem, in other words, is to find

$$\max \sum_{i \in T} b_i \text{ subject to } \sum_{i \in T} w_i \leq W$$

The problem is called a “0-1” problem, because each item must be entirely accepted or rejected.

Just another version of this problem is the “*Fractional Knapsack Problem*”, where we can take fractions of items.

0-1 Knapsack problem (Example)



0-1 Knapsack problem: brute- force approach

Let's first solve this problem with a straightforward algorithm

Since there are n items, how many possible combination of items are possible?

It is the same problem as determining all possible subsets of a set.

0-1 Knapsack problem: brute- force approach

Since there are n items, there are 2^n possible combinations of items.

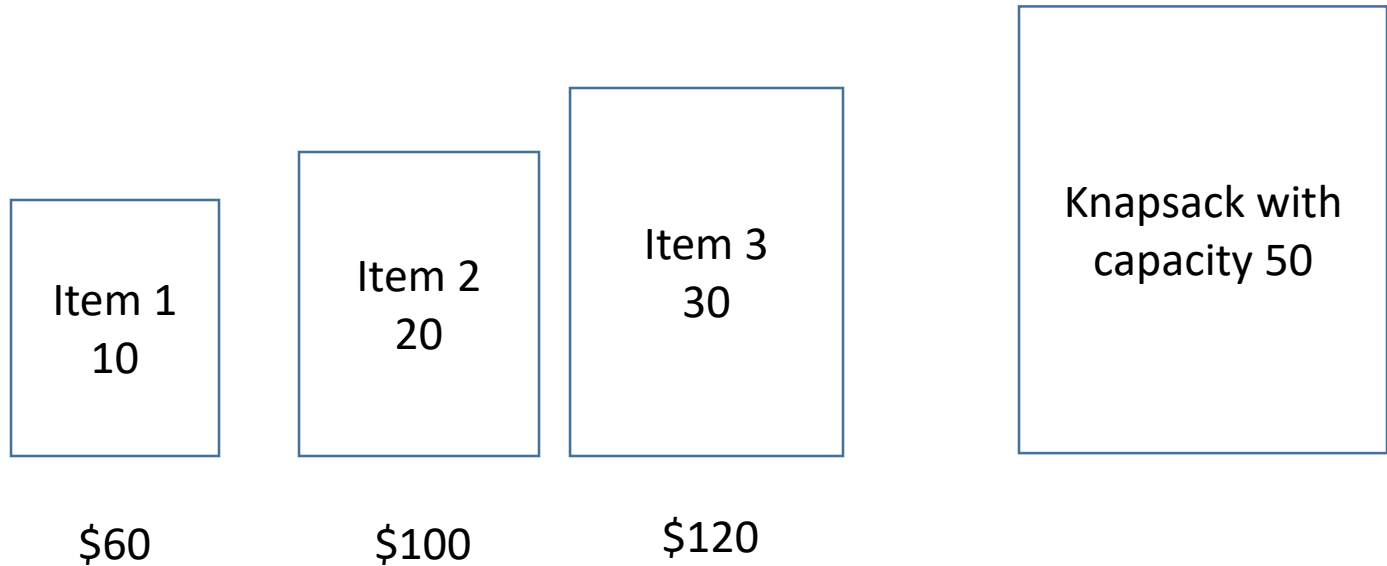
We go through all combinations and find the one with the most total value and with total weight less or equal to W

Running time will be $O(2^n)$

Does any greedy strategy work for 0-1 knapsack problem?

Lets find out!

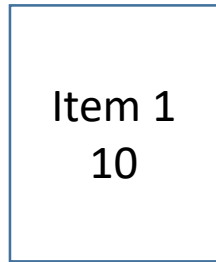
0-1 Knapsack problem (Example 1)



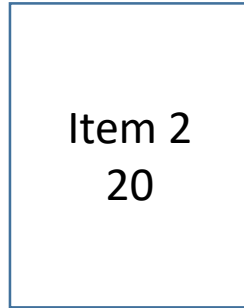
Value per weight: Item 1 (6) Item 2 (5) Item 3 (4)

Lets try value per weight greedy strategy that we used for fractional knapsack

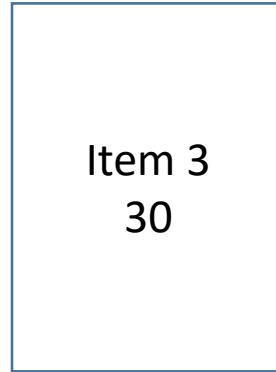
0-1 Knapsack (Counter Example for Greedy Algorithm)



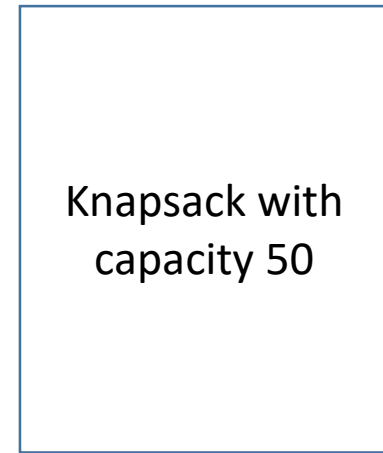
\$60



\$100

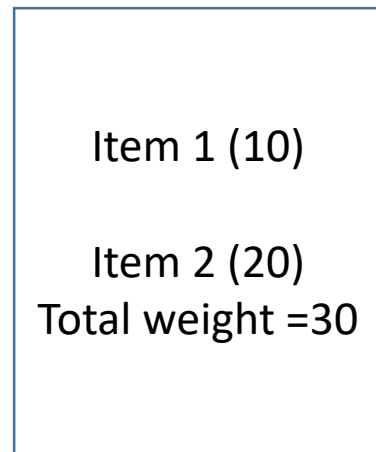


\$120



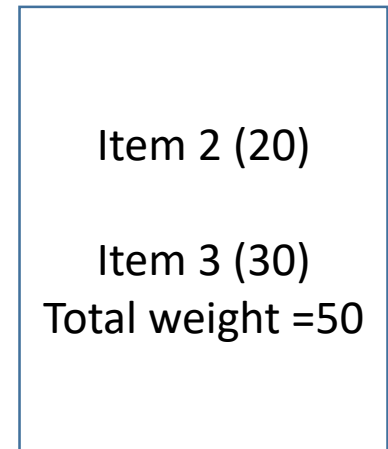
Value per weight: Item 1 (6) Item 2 (5) Item 3 (4)

Value per weight
Greedy Solution



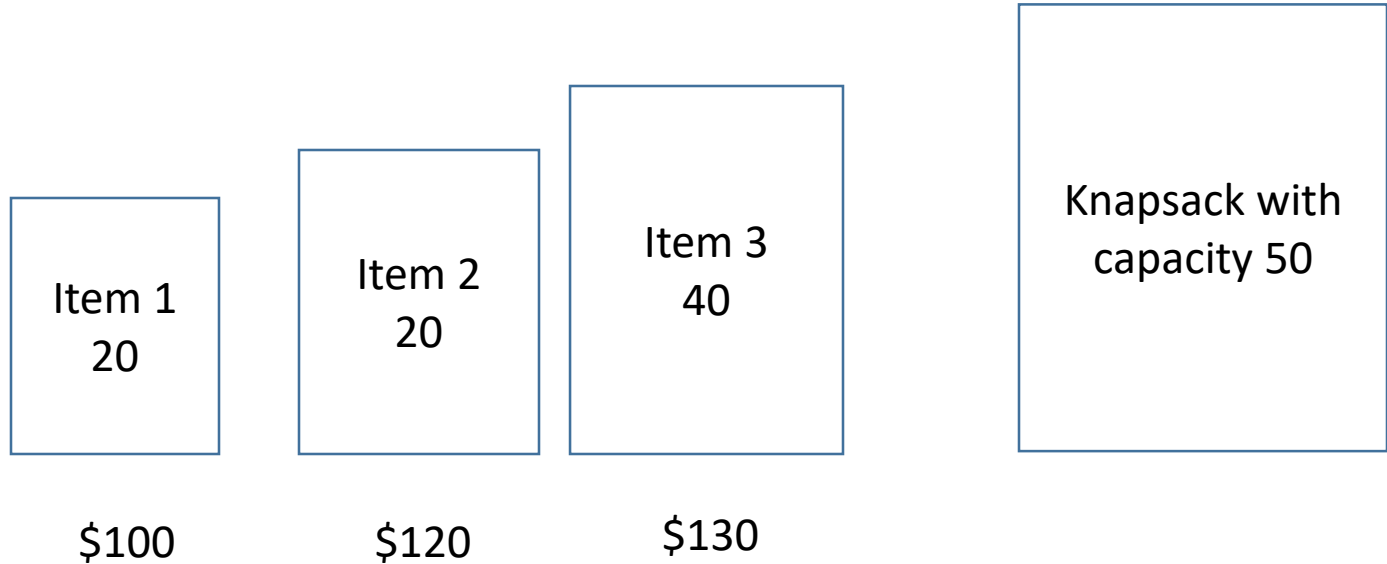
Total Value = \$160

Optimal
Solution



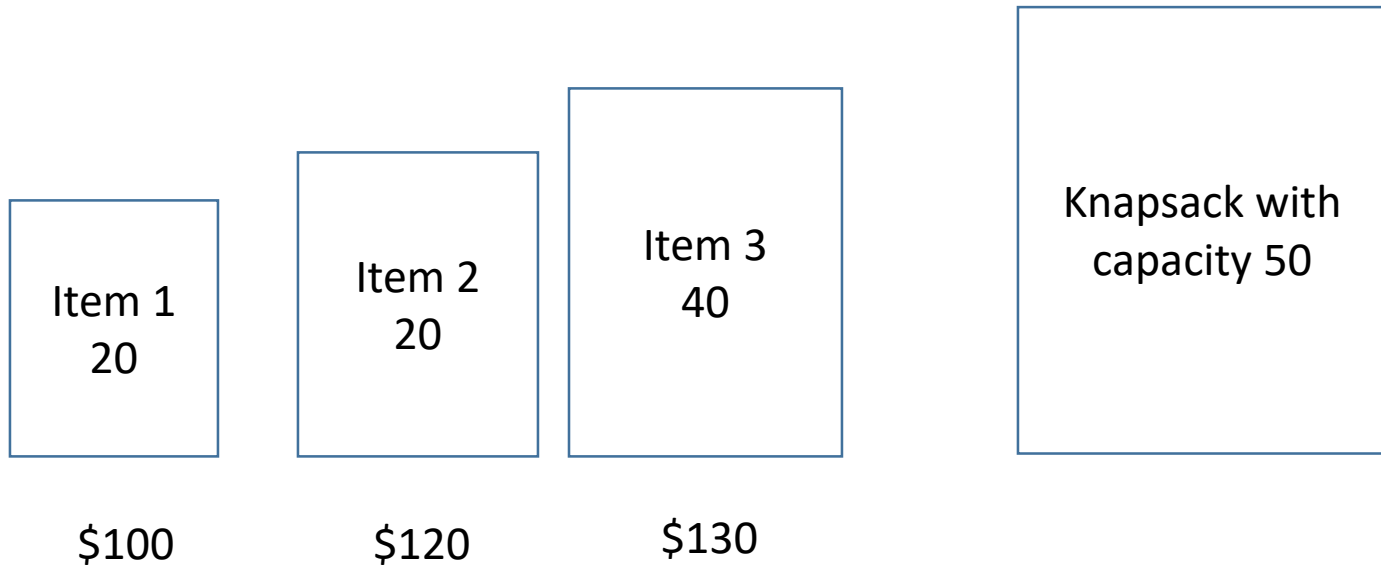
Total Value = \$220

0-1 Knapsack problem (Example 2)

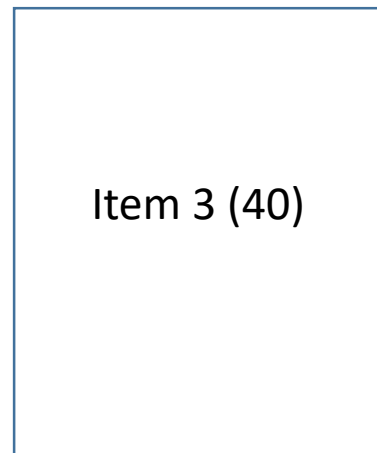


Lets try **greedy algorithm**: Select item with **maximum value** first.

0-1 Knapsack problem (Example 2)

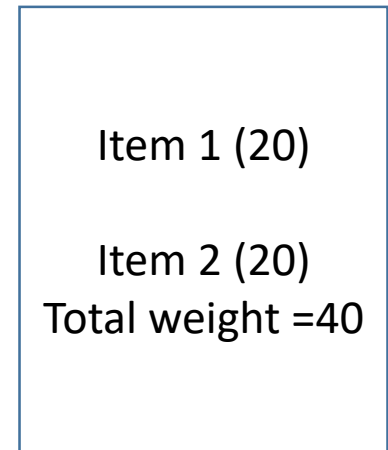


Maximum value first
Greedy Solution



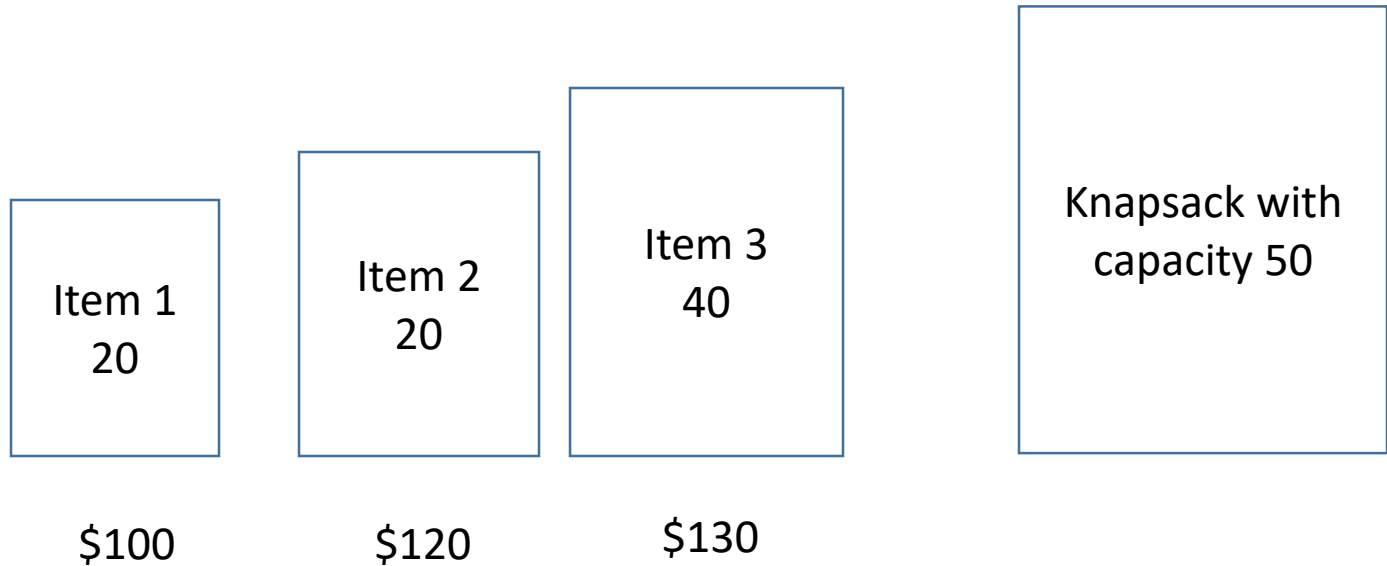
Total Value = \$130

Optimal
Solution



Total Value = \$220

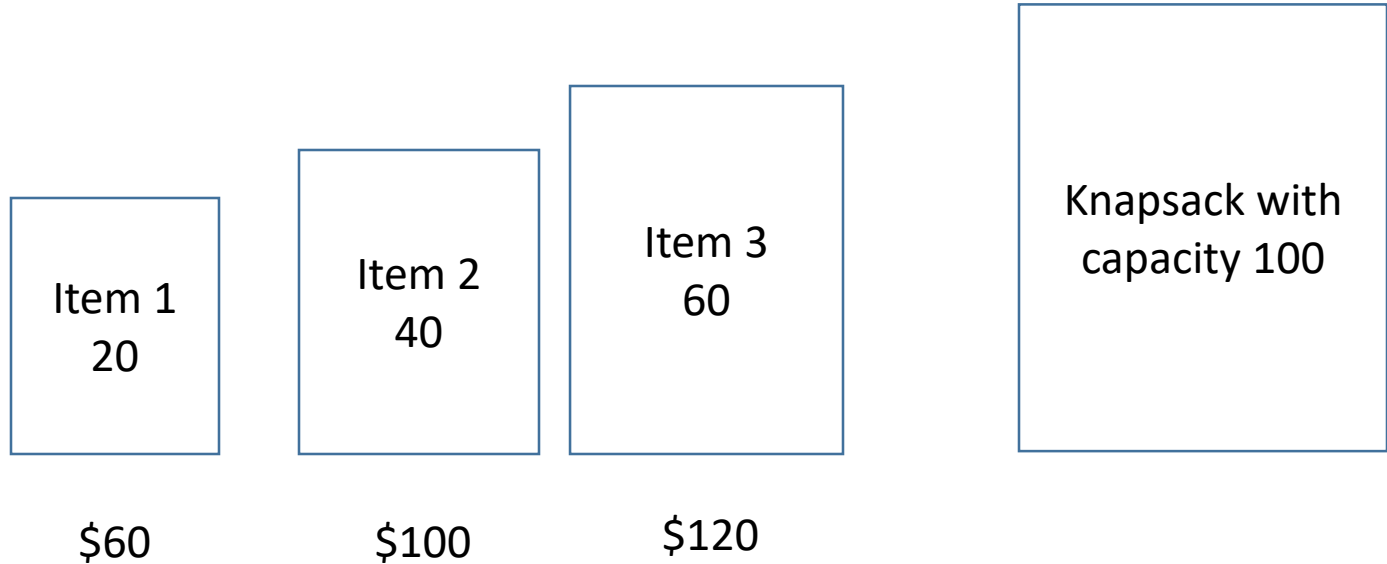
0-1 Knapsack problem (Example 2)



Lets try another greedy algorithm: Select item with minimum weight first.

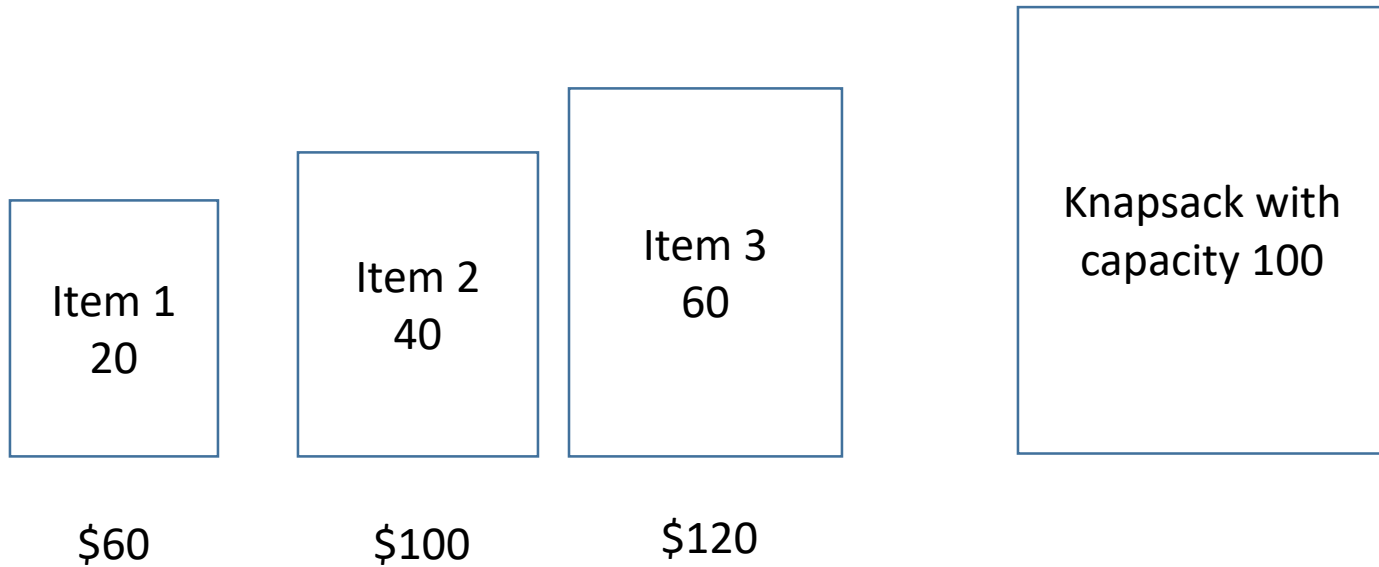
It will give optimal solution in this example with value \$220

0-1 Knapsack problem (Example 3)

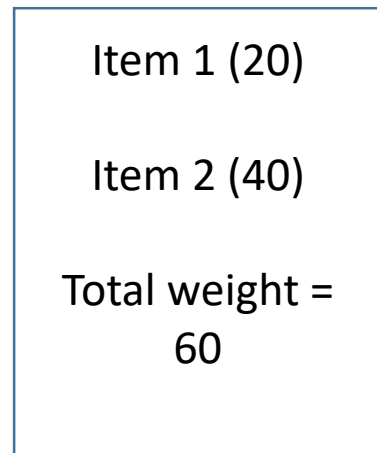


Lets try **greedy algorithm**: Select item with **minimum weight** first.

0-1 Knapsack problem (Example 3)

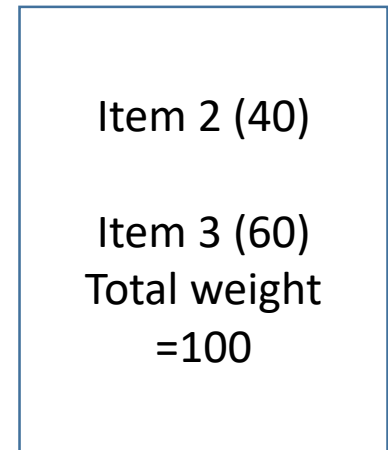


Minimum weight first
Greedy Solution



Total Value = \$160

Optimal
Solution



Total Value = \$220

No greedy strategy guarantees optimal solution for 0-1 Knapsack problem

- We have seen counter examples for three greedy algorithms in previous slides

0-1 Knapsack problem

Can we do better than exponential time brute force?

Yes, with an algorithm based on dynamic programming

We need to carefully identify the sub problems

Let's try this:

If items are labeled $1..n$, then a sub problem would be to find an optimal solution for

$S_k = \{items\ labeled\ 1, 2, .. k\}$

Defining a Subproblem

If items are labeled $1..n$, then a subproblem would be to find an optimal solution for S_k
 $= \{items\ labeled\ 1, 2, .. k\}$

This is a valid subproblem definition.

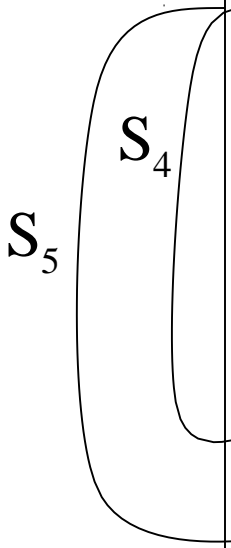
The question is: can we describe the final solution (S_n) in terms of subproblems (S_k)?

Unfortunately, we can't do that.

Explanation follows....

Defining a Subproblem

Max weight: $W = 20$



Item #	Weight w_i	Benefit b_i
1	2	3
2	3	4
3	4	5
4	5	8
5	9	10

Suppose original problem has 5 items S_5 and we define a sub problem with first 4 items S_4

Defining a Subproblem

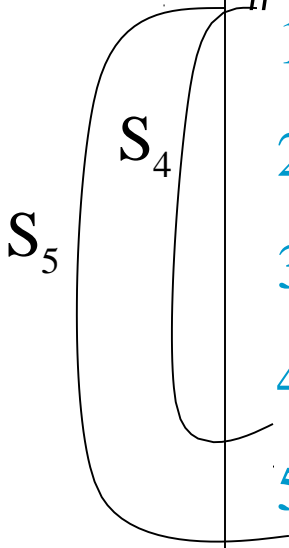
Max weight: $W = 20$

Optimal solution For S_5 :

$w_1=2$	$w_3=4$	$w_4=5$	$w_5=9$
$b_1=3$	$b_3=5$	$b_4=8$	$b_5=10$

Total weight: 20
total benefit: 26

Item #	Weight	Benefit
	w_i	b_i
1	2	3
2	3	4
3	4	5
4	5	8
5	9	10



Does the optimal solution for S_5 contain optimal solution for sub problem S_4 ?

Defining a Subproblem

Optimal solution For S_5 :

$w_1=2$	$w_3=4$	$w_4=5$	$w_5=9$
$b_1=3$	$b_3=5$	$b_4=8$	$b_5=10$

Total weight: 20
total benefit: 26

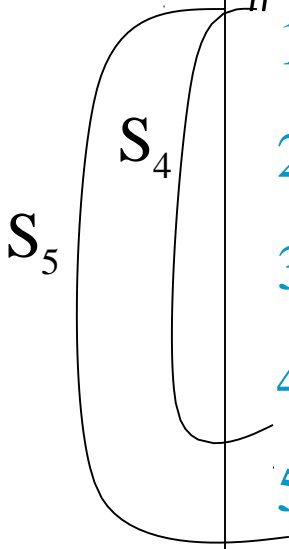
Optimal solution For S_4 :

$w_1=2$	$w_2=3$	$w_3=4$	$w_4=5$	
$b_1=3$	$b_2=4$	$b_3=5$	$b_4=8$	

Total weight: 14;
total benefit: 20

Max weight: $W = 20$

Item #	Weight w_i	Benefit b_i
1	2	3
2	3	4
3	4	5
4	5	8
5	9	10



**Solution for S_4 is
not part of the
solution for S_5 !!!**

Defining a Subproblem (continued)

As we have seen, the solution for S_4 is not part of the solution for S_5

So our definition of a subproblem is flawed and we need another one!

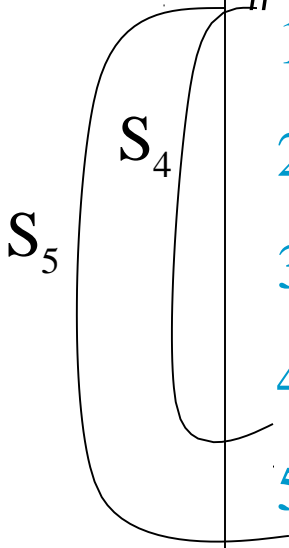
Let's add another parameter: w , which will represent the exact weight for each subset of items

Defining a Subproblem

Max weight: $W = 20$

Suppose original problem has 5 items and total weight 20, $S_{5,20}$ and we define a sub problem with first 4 items and how much weight?

If 5th item is part of optimal solution then the total weight for subproblem should be $20 - w_5 = 20 - 9 = 11$
subproblem = $S_{4,11}$,
if 5th item is not part of optimal solution then total capacity stays same so Subproblem = $S_{4,20}$



Item #	Weight w_i	Benefit b_i
1	2	3
2	3	4
3	4	5
4	5	8
5	9	10

Recursive Formula for subproblems

$$B[k, w] = \begin{cases} B[k-1, w] & \text{if } w_k > w \\ \max\{B[k-1, w], B[k-1, w - w_k] + b_k\} & \text{else} \end{cases}$$

It means, that the best subset of S_k that has total weight w is one of the two:

- 1) the best subset of S_{k-1} that has total weight w , **or**
- 2) the best subset of S_{k-1} that has total weight $w - w_k$ plus the item k

Recursive Formula

$$B[k, w] = \begin{cases} B[k-1, w] & \text{if } w_k > w \\ \max\{B[k-1, w], B[k-1, w - w_k] + b_k\} & \text{else} \end{cases}$$

The best subset of S_k that has the total weight w , either contains item k or not.

First case: $w_k > w$. Item k can't be part of the solution, since if it was, the total weight would be $> w$, which is unacceptable

Second case: $w_k \leq w$. Then the item k can be in the solution, and we choose the case with great value

0-1 Knapsack Algorithm

for $w = 0$ to W

$$B[0,w] = 0$$

for $i = 0$ to n

$$B[i,0] = 0$$

for $w = 0$ to W

if $w_i \leq w$ // item i can be part of the solution

if $b_i + B[i-1, w-w_i] > B[i-1, w]$

$$B[i, w] = b_i + B[i-1, w-w_i]$$

else

$$B[i, w] = B[i-1, w]$$

else $B[i, w] = B[i-1, w]$ // $w_i > w$

Running time

for w = 0 to W

$O(W)$

B[0,w] = 0

for i = 0 to n

Repeat n times

B[i,0] = 0

for w = 0 to W

$O(W)$

< the rest of the code >

What is the running time of this
algorithm $O(Wn)$

Example

Let's run our algorithm on the following data:

$n = 4$ (# of elements)

$W = 5$ (max weight)

Elements (weight, benefit):

(2,3), (3,4), (4,5), (5,6)

Example (2)

W	i	0	1	2	3	4
	0	0				
1	0					
2	0					
3	0					
4	0					
5	0					

for $w = 0$ to W

$$B[0,w] = 0$$

Example (3)

w	i	0	1	2	3	4
0		0	0	0	0	0
1		0				
2		0				
3		0				
4		0				
5		0				

for $i = 0$ to n
 $B[i,0] = 0$

Example (4)

	i	0	1	2	3	4
W						
0		0	0	0	0	0
1		0	0			
2		0				
3		0				
4		0				
5		0				

Items:
(Weight, benefit)

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

$i=1$

$b_i=3$

$w_i=2$

$w=1$

$w-w_i=-1$

if $w_i \leq w$ // item i can be part of the solution

if $b_i + B[i-1, w-w_i] > B[i-1, w]$

$B[i, w] = b_i + B[i-1, w-w_i]$

else

$B[i, w] = B[i-1, w]$

else $B[i, w] = B[i-1, w]$ // $w_i > w$

Example (5)

	i	0	1	2	3	4
W						
0		0	0	0	0	0
1		0	0			
2		0	3			
3		0				
4		0				
5		0				

$i=1$

$b_i=3$

$w_i=2$

$w=2$

$w-w_i=0$

Items:

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

if $w_i \leq w$ // item i can be part of the solution

if $b_i + B[i-1, w-w_i] > B[i-1, w]$

$B[i, w] = b_i + B[i-1, w-w_i]$

else

$B[i, w] = B[i-1, w]$

else $B[i, w] = B[i-1, w]$ // $w_i > w$

Example (6)

	i	0	1	2	3	4
W						
0		0	0	0	0	0
1		0	0			
2		0	3			
3		0	3			
4		0				
5		0				

$i=1$

$b_i=3$

$w_i=2$

$w=3$

$w-w_i=1$

Items:

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

if $w_i \leq w$ // item i can be part of the solution

if $b_i + B[i-1, w-w_i] > B[i-1, w]$

$B[i, w] = b_i + B[i-1, w-w_i]$

else

$B[i, w] = B[i-1, w]$

else $B[i, w] = B[i-1, w]$ // $w_i > w$

Example (7)

	i	0	1	2	3	4
W						
0		0	0	0	0	0
1		0	0			
2		0	3			
3		0	3			
4		0	3			
5		0				

$i=1$

$b_i=3$

$w_i=2$

$w=4$

$w-w_i=2$

Items:

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

if $w_i \leq w$ // item i can be part of the solution

if $b_i + B[i-1, w-w_i] > B[i-1, w]$

$B[i, w] = b_i + B[i-1, w-w_i]$

else

$B[i, w] = B[i-1, w]$

else $B[i, w] = B[i-1, w]$ // $w_i > w$

Example (8)

	i	0	1	2	3	4
W						
0		0	0	0	0	0
1		0	0			
2		0	3			
3		0	3			
4		0	3			
5		0	3			

$i=1$

$b_i=3$

$w_i=2$

$w=5$

$w-w_i=2$

Items:

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

if $w_i \leq w$ // item i can be part of the solution

if $b_i + B[i-1, w-w_i] > B[i-1, w]$

$B[i, w] = b_i + B[i-1, w-w_i]$

else

$B[i, w] = B[i-1, w]$

else $B[i, w] = B[i-1, w]$ // $w_i > w$

Example (9)

	i	0	1	2	3	4
W						
0		0	0	0	0	0
1		0	0	0		
2		0	3			
3		0	3			
4		0	3			
5		0	3			

$i=2$

$b_i=4$

$w_i=3$

$w=1$

$w-w_i=-2$

Items:

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

if $w_i \leq w$ // item i can be part of the solution

if $b_i + B[i-1, w-w_i] > B[i-1, w]$

$B[i, w] = b_i + B[i-1, w-w_i]$

else

$B[i, w] = B[i-1, w]$

else $B[i, w] = B[i-1, w]$ // $w_i > w$

Example (10)

	i	0	1	2	3	4
W						
0		0	0	0	0	0
1		0	0	0		
2		0	3	→ 3		
3		0	3			
4		0	3			
5		0	3			

$i=2$

$b_i=4$

$w_i=3$

$w=2$

$w-w_i=-1$

Items:

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

if $w_i \leq w$ // item i can be part of the solution

if $b_i + B[i-1, w-w_i] > B[i-1, w]$

$B[i, w] = b_i + B[i-1, w-w_i]$

else

$B[i, w] = B[i-1, w]$

else $B[i, w] = B[i-1, w]$ // $w_i > w$

Example (11)

	i	0	1	2	3	4
W						
0		0	0	0	0	0
1		0	0	0		
2		0	3	3		
3		0	3	4		
4		0	3			
5		0	3			

$i=2$

$b_i=4$

$w_i=3$

$w=3$

$w-w_i=0$

Items:

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

if $w_i \leq w$ // item i can be part of the solution

if $b_i + B[i-1, w-w_i] > B[i-1, w]$

$B[i, w] = b_i + B[i-1, w-w_i]$

else

$B[i, w] = B[i-1, w]$

else $B[i, w] = B[i-1, w]$ // $w_i > w$

Example (12)

	i	0	1	2	3	4
W						
0		0	0	0	0	0
1		0	0	0		
2		0	3	3		
3		0	3	4		
4		0	3	4		
5		0	3			

$i=2$

$b_i=4$

$w_i=3$

$w=4$

$w-w_i=1$

Items:

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

if $w_i \leq w$ // item i can be part of the solution

if $b_i + B[i-1, w-w_i] > B[i-1, w]$

$B[i, w] = b_i + B[i-1, w-w_i]$

else

$B[i, w] = B[i-1, w]$

else $B[i, w] = B[i-1, w]$ // $w_i > w$

Example (13)

	i	0	1	2	3	4
W						
0		0	0	0	0	0
1		0	0	0		
2		0	3	3		
3		0	3	4		
4		0	3	4		
5		0	3	7		

$i=2$

$b_i=4$

$w_i=3$

$w=5$

$w-w_i=2$

Items:

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

if $w_i \leq w$ // item i can be part of the solution

if $b_i + B[i-1, w-w_i] > B[i-1, w]$

$B[i, w] = b_i + B[i-1, w-w_i]$

else

$B[i, w] = B[i-1, w]$

else $B[i, w] = B[i-1, w]$ // $w_i > w$

Example (14)

	i	0	1	2	3	4
W						
0		0	0	0	0	0
1		0	0	0 → 0		
2		0	3	3 → 3		
3		0	3	4 → 4		
4		0	3	4		
5		0	3	7		

$i=3$

$b_i=5$

$w_i=4$

$w=1..3$

Items:

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

if $w_i \leq w$ // item i can be part of the solution

if $b_i + B[i-1, w-w_i] > B[i-1, w]$

$B[i, w] = b_i + B[i-1, w-w_i]$

else

$B[i, w] = B[i-1, w]$

else $B[i, w] = B[i-1, w]$ // $w_i > w$

Example (15)

	i	0	1	2	3	4
W						
0		0	0	0	0	0
1		0	0	0	0	
2		0	3	3	3	
3		0	3	4	4	
4		0	3	4	5	
5		0	3	7		

$i=3$

$b_i=5$

$w_i=4$

$w=4$

$w - w_i = 0$

Items:

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

if $w_i \leq w$ // item i can be part of the solution

if $b_i + B[i-1, w-w_i] > B[i-1, w]$

$B[i, w] = b_i + B[i-1, w - w_i]$

else

$B[i, w] = B[i-1, w]$

else $B[i, w] = B[i-1, w]$ // $w_i > w$

Example (15)

	i	0	1	2	3	4
W						
0		0	0	0	0	0
1		0	0	0	0	
2		0	3	3	3	
3		0	3	4	4	
4		0	3	4	5	
5		0	3	7	→ 7	

Items:

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

$i=3$

$b_i=5$

$w_i=4$

$w=5$

$w - w_i = 1$

if $w_i \leq w$ // item i can be part of the solution

if $b_i + B[i-1, w-w_i] > B[i-1, w]$

$B[i, w] = b_i + B[i-1, w - w_i]$

else

$B[i, w] = B[i-1, w]$

else $B[i, w] = B[i-1, w]$ // $w_i > w$

Example (16)

	i	0	1	2	3	4
W						
0		0	0	0	0	0
1		0	0	0	0 →	0
2		0	3	3	3 →	3
3		0	3	4	4 →	4
4		0	3	4	5 →	5
5		0	3	7	7	

$i=3$

$b_i=5$

$w_i=4$

$w=1..4$

Items:

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

if $w_i \leq w$ // item i can be part of the solution

if $b_i + B[i-1, w-w_i] > B[i-1, w]$

$B[i, w] = b_i + B[i-1, w-w_i]$

else

$B[i, w] = B[i-1, w]$

else $B[i, w] = B[i-1, w]$ // $w_i > w$

Example (17)

	i	0	1	2	3	4
W						
0		0	0	0	0	0
1		0	0	0	0	0
2		0	3	3	3	3
3		0	3	4	4	4
4		0	3	4	5	5
5		0	3	7	7	7

$i=3$

$b_i=5$

$w_i=4$

$w=5$

Items:

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

if $w_i \leq w$ // item i can be part of the solution

if $b_i + B[i-1, w-w_i] > B[i-1, w]$

$B[i, w] = b_i + B[i-1, w-w_i]$

else

$B[i, w] = B[i-1, w]$

else $B[i, w] = B[i-1, w]$ // $w_i > w$

Optimal Solution: Algorithm to Look Up the Table to Find the Selected Items

- When calculating the table, you are interested in $B[n][W]$ which is the maximum value obtained when selecting all n items with the weight limit W .
- If $B[n][W] = B[n - 1][W]$ then package n is not selected, you trace $B[n - 1][W]$.
- If $B[n][W] \neq B[n - 1][W]$, you notice that the optimal selection has the package n and trace $B[n - 1][W - w_n]$.

Example

	i	0	1	2	3	4
W						
0		0	0	0	0	0
1		0	0	0	0	0
2		0	3	3	3	3
3		0	3	4	4	4
4		0	3	4	5	5
5		0	3	7	7	7

Items:

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

Parameters to function call
will be n and W.

PrintSelectedItems(n, W)

PrintSelectedItems(i,w)

While $i > 0$

if $B[i-1,w] \neq B[i,w]$

Print "item i"

$w = w - w_i$

$i = i - 1$

Example

		i				
W		0	1	2	3	4
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	3	3	3	3	3
3	0	3	4	4	4	4
4	0	3	4	5	5	5
5	0	3	7	7	7	7

Items:

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

PrintSelectedItems(i,w)

While $i > 0$

if $B[i-1,w] \neq B[i,w]$

Print "item i"

$w = w - w_i$

$i = i - 1$

Example

		i				
W		0	1	2	3	4
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	3	3	3	3	3
3	0	3	4	4	4	4
4	0	3	4	5	5	5
5	0	3	7	7	7	7

Items:

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

PrintSelectedItems(i,w)

While $i > 0$

if $B[i-1,w] \neq B[i,w]$

Print "item i"

$w = w - w_i$

$i = i - 1$

Example

	i	0	1	2	3	4
W						
0		0	0	0	0	0
1		0	0	0	0	0
2		0	3	3	3	3
3		0	3	4	4	4
4		0	3	4	5	5
5		0	3	7	7	7

Items:

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

Item 2,

PrintSelectedItems(i,w)

While $i > 0$

if $B[i-1,w] \neq B[i,w]$

Print "item i"

$w = w - w_i$

$i = i - 1$

Example

	i	0	1	2	3	4
W						
0		0	0	0	0	0
1		0	0	0	0	0
2		0	3	3	3	3
3		0	3	4	4	4
4		0	3	4	5	5
5		0	3	7	7	7

Items:

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

Item 2, Item 1

PrintSelectedItems(i,w)

While $i > 0$

if $B[i-1, w] \neq B[i, w]$

Print "item i"

$w = w - w_i$

$i = i - 1$

items i $W=20$

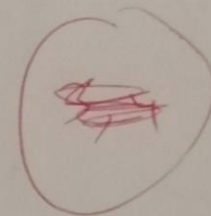
w_3	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	3	3	3	3	3
3	0	3	4	4	4	4
4	0	3	4	5	5	5
5	0	3	7	7	8	8
6	0	3	7	8	8	8
7	0	3	7	9	9	9
8	0	3	7	9	9	9
9	0	3	7	12	13	13
10	0	3	7	12	13	13
11	0	3	7	12	16	16
20	0	3	7	12	20	26

$(8-4=2)$
 $w-w_3$
 weight

$(11-5=6)$
 $w-w_4$

$(20-9=11)$
 $20-w_5=11$

Items i	Weight w_i	Benefit b_i
1	2	3
2	3	4
3	4	5
4	5	8
5	9	10



Items 1, 3, 4, 5

You cannot print items by only looking at last row