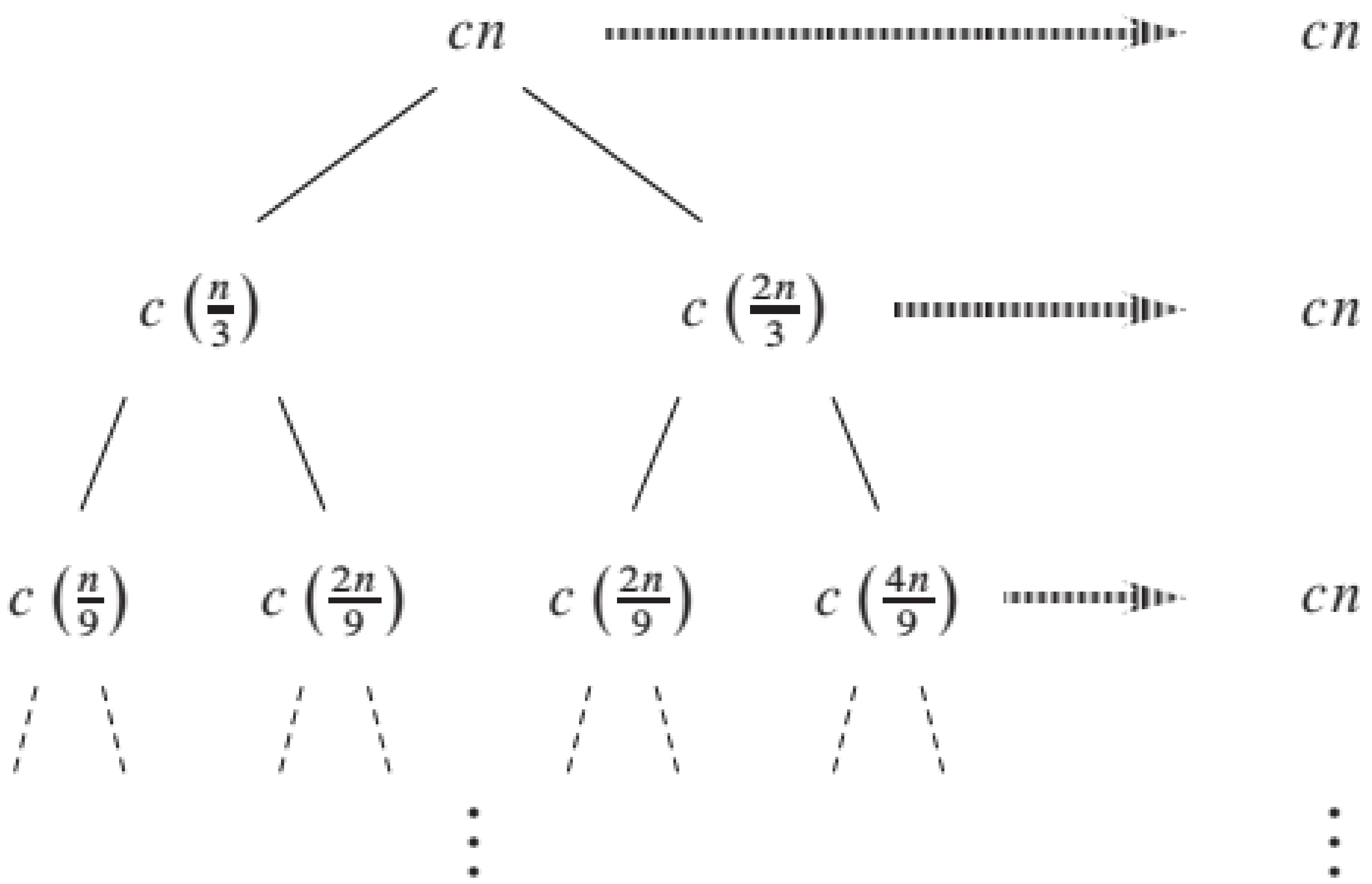
Solving Recurrences

Lecture 4

$$T(n) = T(n/3) + T(2n/3) + cn.$$

$$\frac{n}{\frac{3}{2}} = \frac{2n}{3}$$

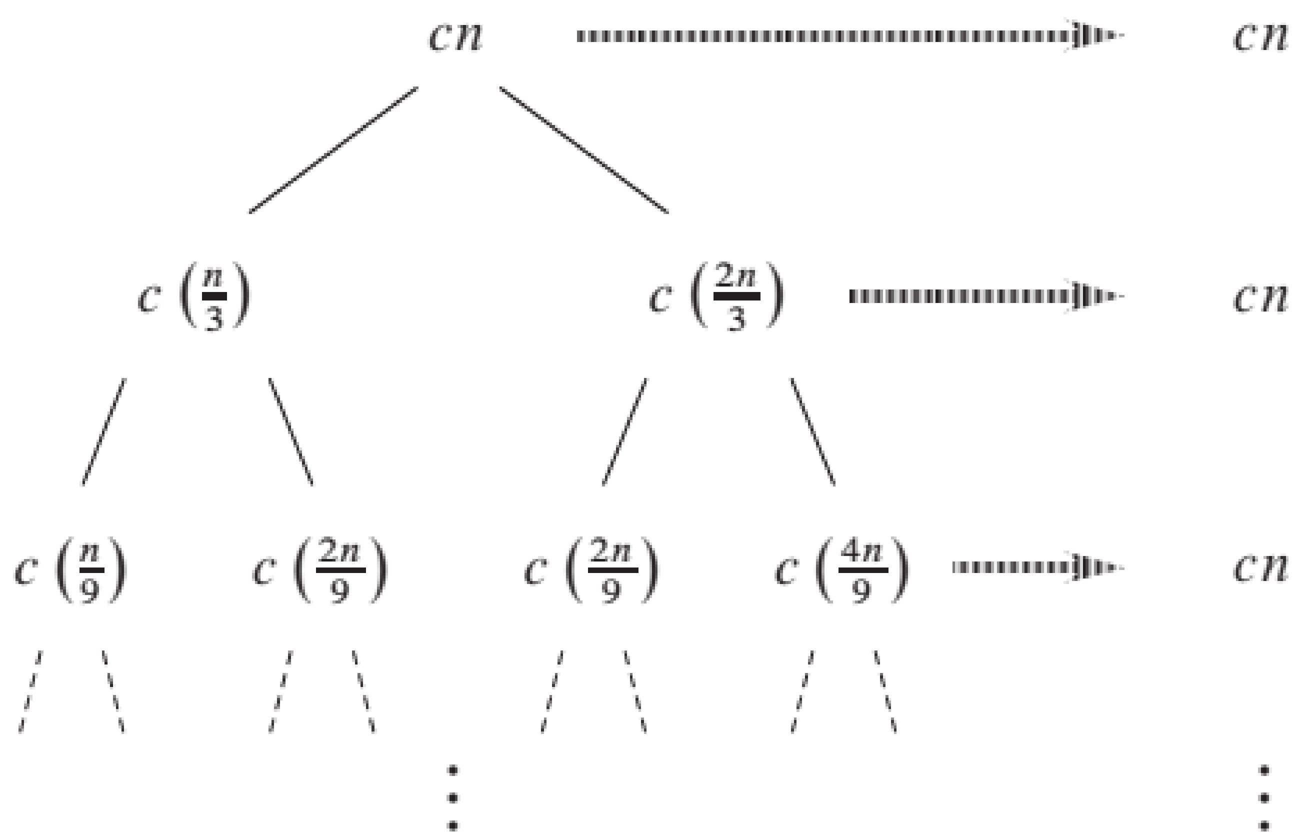
$$T(n) = T(n/3) + T(2n/3) + cn.$$



$$T(n) = T(n/3) + T(2n/3) + cn.$$

- What is height of tree?

$$(2/3)^k n = 1$$



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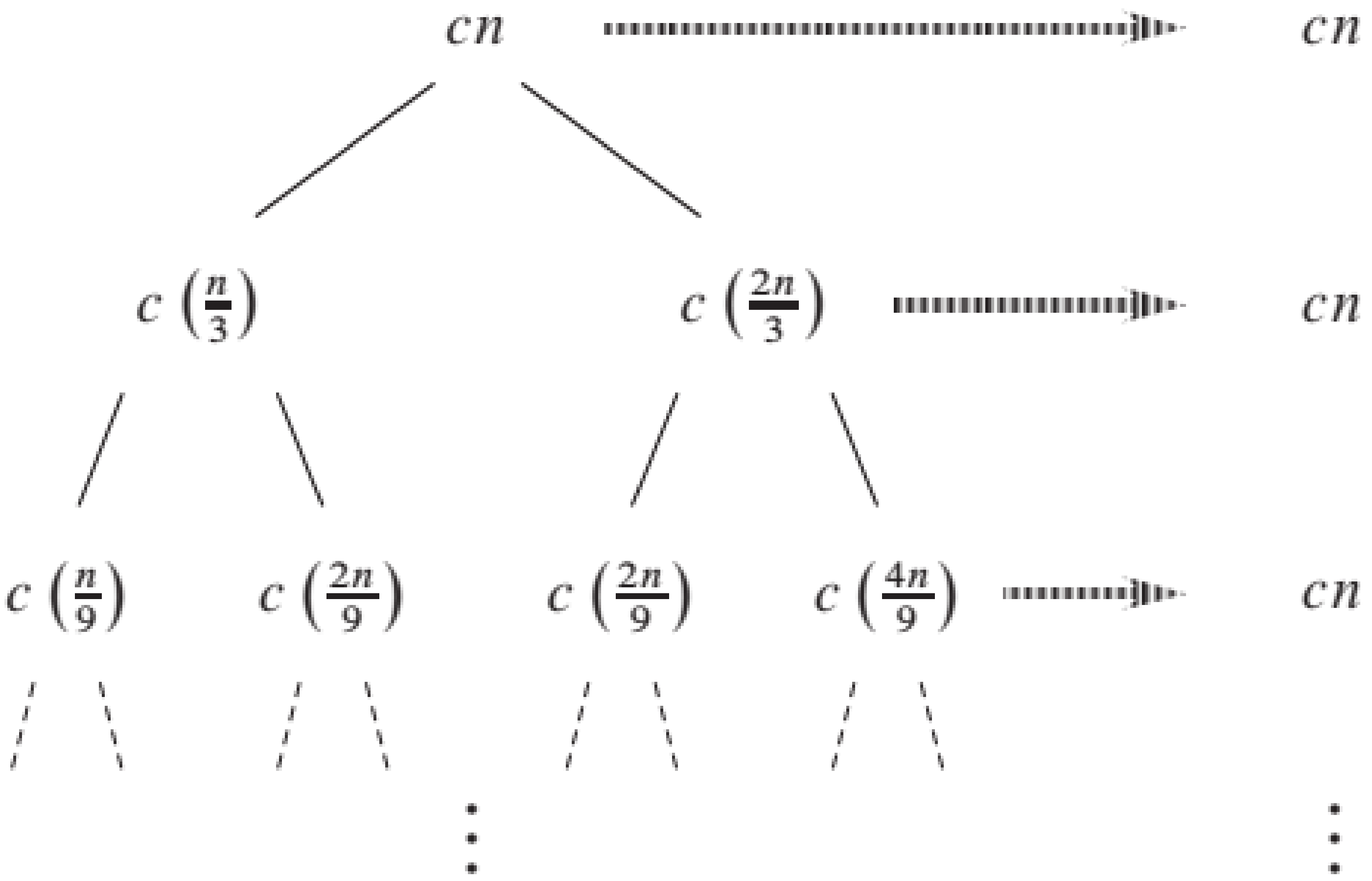


- What is height of tree?

$$(2/3)^k n = 1$$

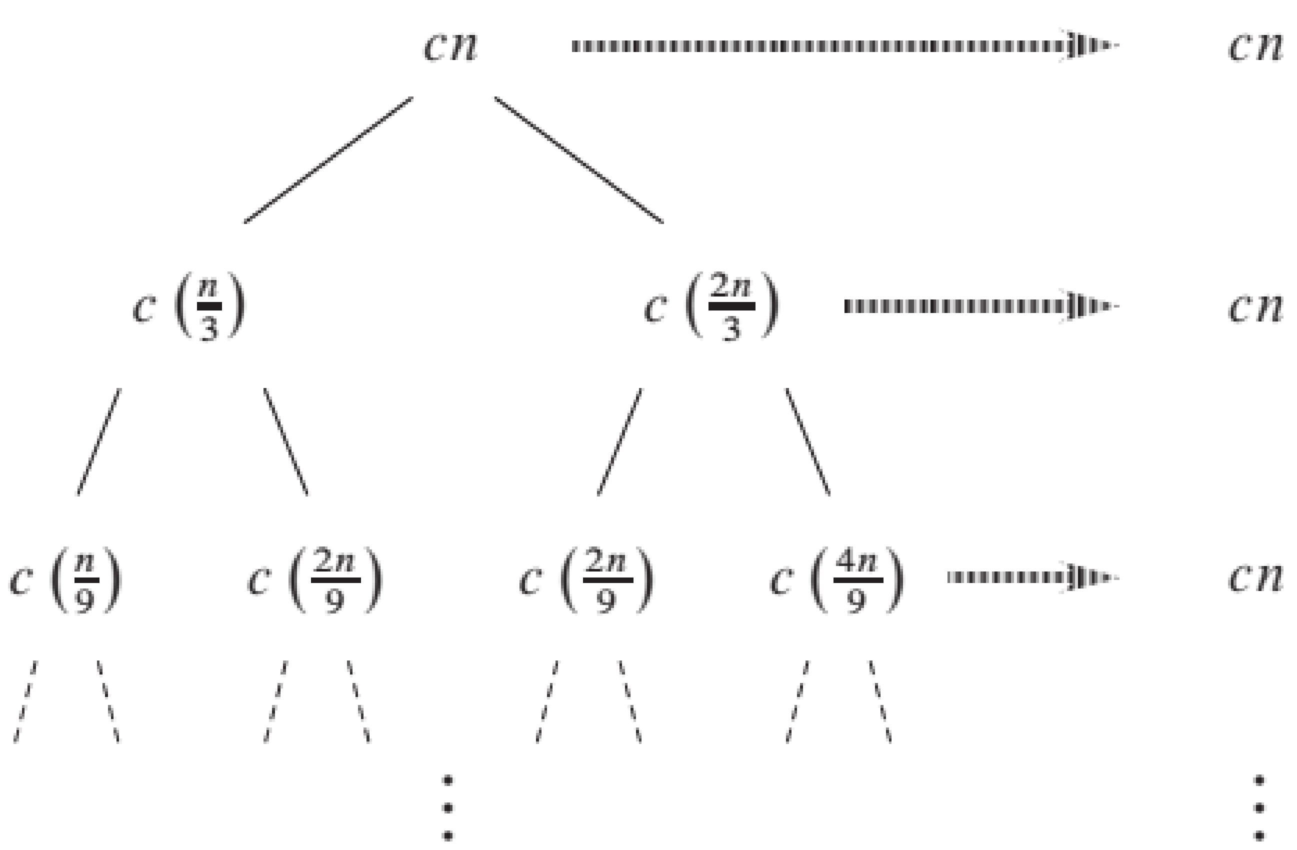
$$n = 3/2^k$$

$$\log_{3/2} n = \log_{3/2} 3/2^k$$

$$k = \log_{3/2} n$$

$$T(n) = T(n/3) + T(2n/3) + cn.$$

- What is height of tree?
- N is being divided by 3/2 at every level so height is $\log_{3/2} n$



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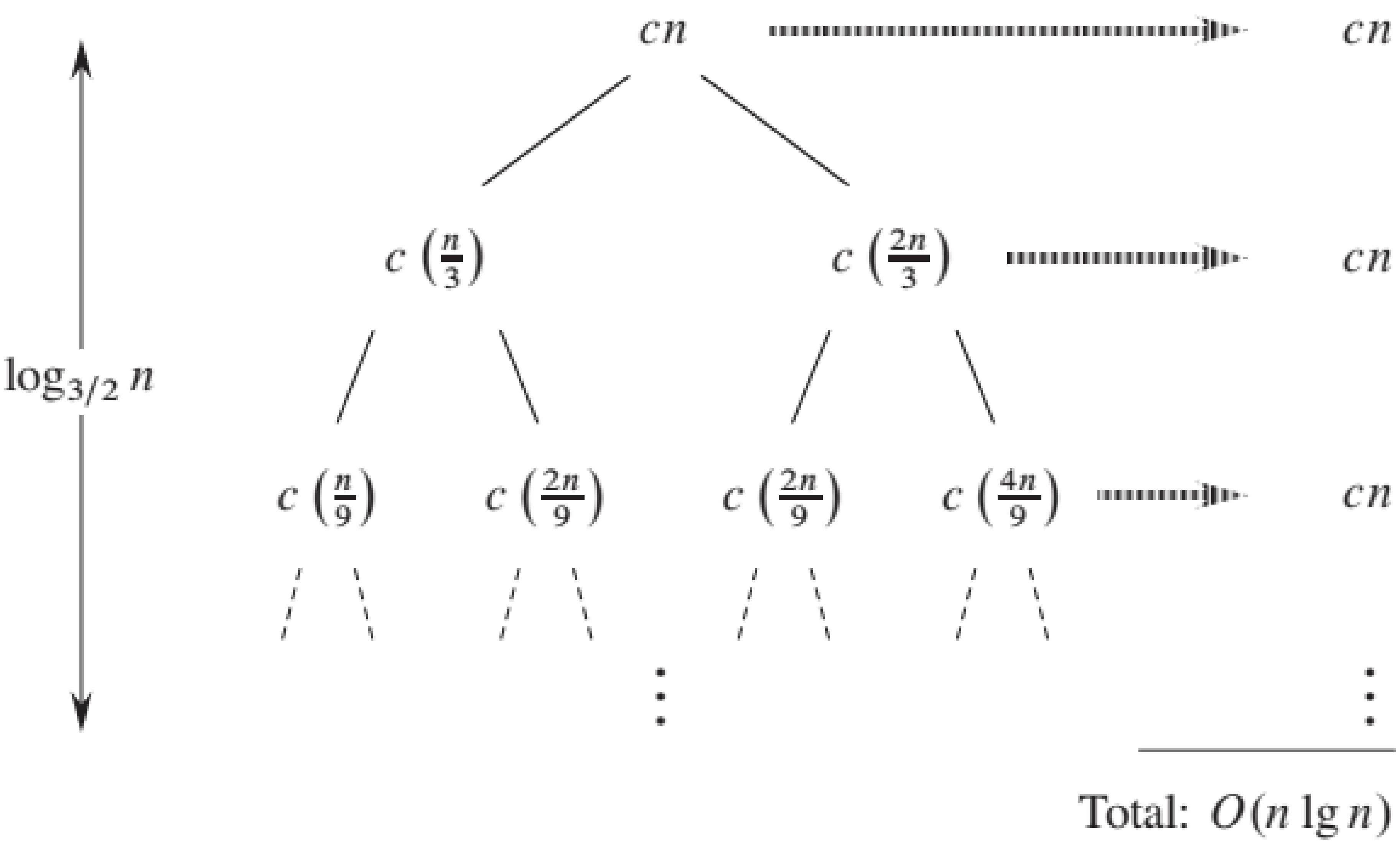
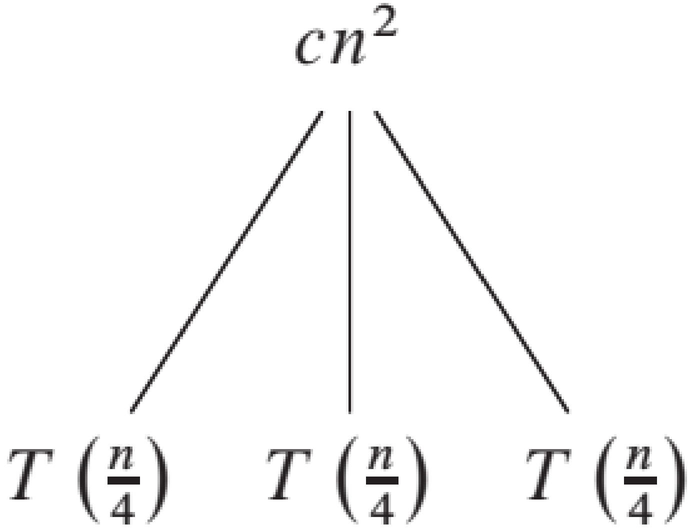
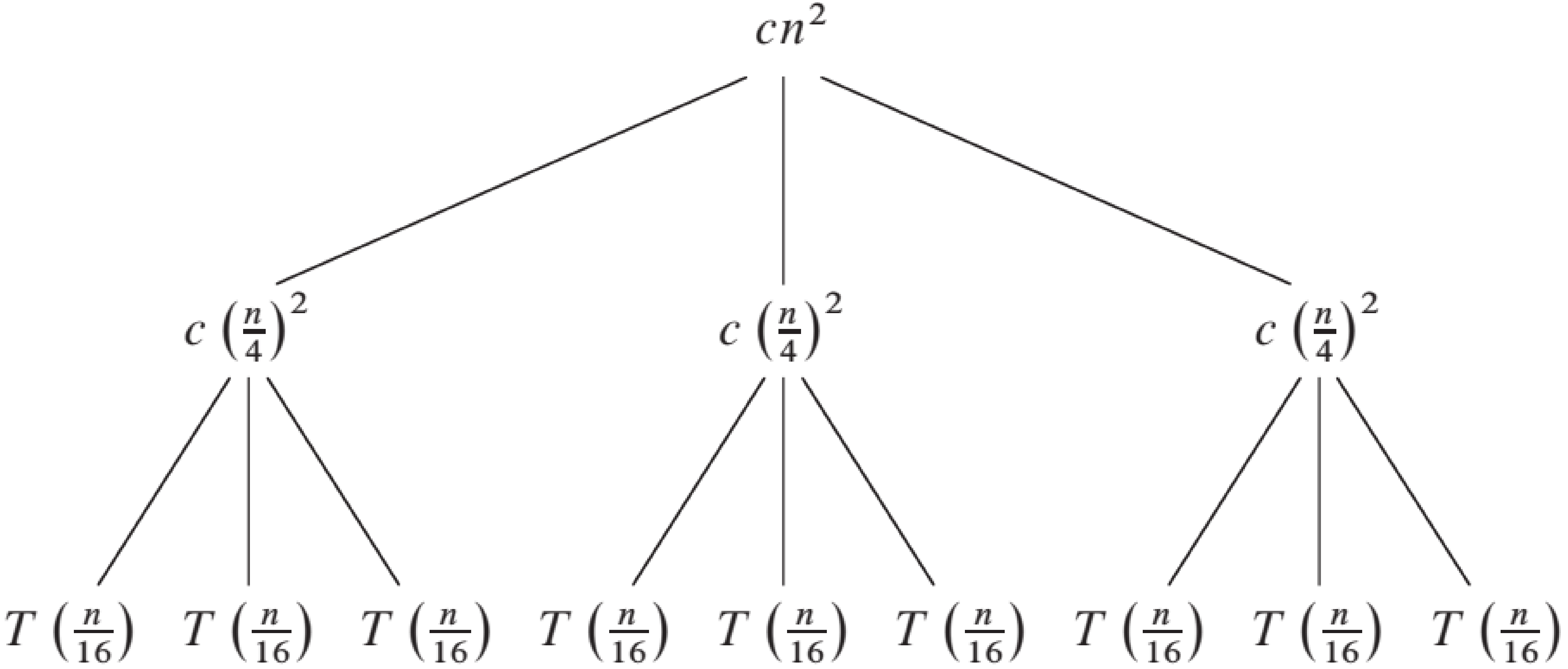


$T(n) = 3T(n/4) + cn^2$



cn^2
$T\left(\frac{n}{4}\right)$ $T\left(\frac{n}{4}\right)$ $T\left(\frac{n}{4}\right)$

$$T(n) = 3T(n/4) + cn^2$$


cn^2
$c\left(\frac{n}{4}\right)^2$ $c\left(\frac{n}{4}\right)^2$ $c\left(\frac{n}{4}\right)^2$
$T\left(\frac{n}{16}\right)$ $T\left(\frac{n}{16}\right)$ $T\left(\frac{n}{16}\right)$ $T\left(\frac{n}{16}\right)$ $T\left(\frac{n}{16}\right)$ $T\left(\frac{n}{16}\right)$ $T\left(\frac{n}{16}\right)$ $T\left(\frac{n}{16}\right)$ $T\left(\frac{n}{16}\right)$

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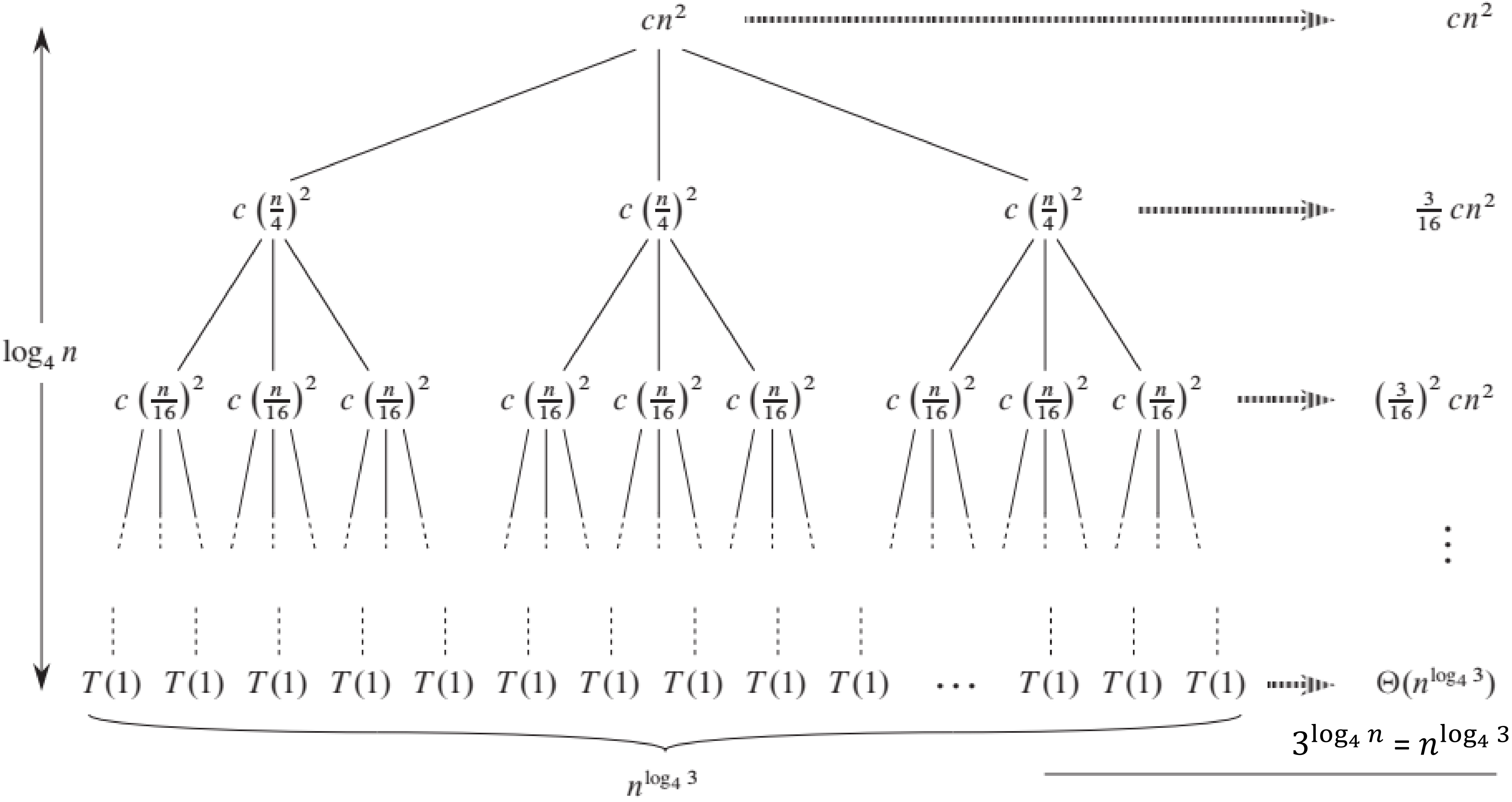
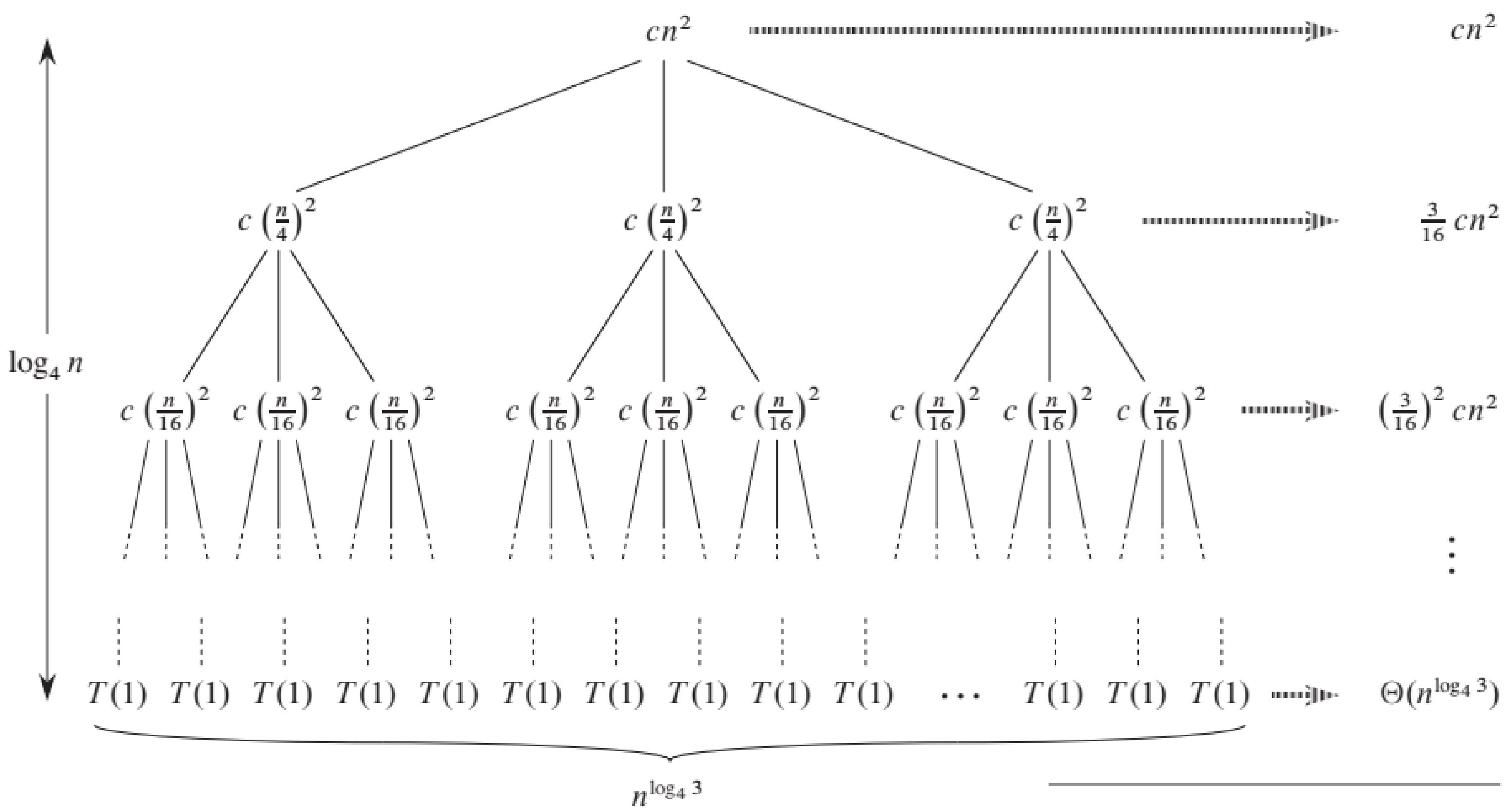


$3^{\log_4 n} = n^{\log_4 3}$



$$T(n) \quad = \quad cn^2 + \frac{3}{16}cn^2 + \left(\frac{3}{16}\right)^2 cn^2 + \cdots + \left(\frac{3}{16}\right)^{\log_4 n - 1} cn^2 + \Theta(n^{\log_4 3})$$

$$
\begin{aligned}
T(n) &= cn^2 + \frac{3}{16}cn^2 + \left(\frac{3}{16}\right)^2 cn^2 + \cdots + \left(\frac{3}{16}\right)^{\log_4 n - 1} cn^2 + \Theta(n^{\log_4 3}) \\
&= \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) \\
&= \frac{(3/16)^{\log_4 n} - 1}{(3/16) - 1} cn^2 + \Theta(n^{\log_4 3})
\end{aligned}
$$

Arithmetic series

The summation

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n ,$$

is an ***arithmetic series*** and has the value

$$\begin{aligned} \sum_{k=1}^{n} k &= \frac{1}{2} n(n+1) \\ &= \Theta(n^2) . \end{aligned}$$

Geometric series

For real $x \neq 1$, the summation

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$$

is a ***geometric*** or ***exponential series*** and has the value

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1} . \tag{A.5}$$

When the summation is infinite and $|x| < 1$, we have the infinite decreasing geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x} . \tag{A.6}$$

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) \\
&< \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) \\
&= \frac{1}{1 - (3/16)} cn^2 + \Theta(n^{\log_4 3}) \\
&= \frac{16}{13} cn^2 + \Theta(n^{\log_4 3}) \\
&= O(n^2) .
\end{aligned}
$$

$T(n) = 3T(4n/5) + \Theta(1)$

- Ans = $\Theta\ (n^{4.9})$

$T(n) = 2T(n-1) + \Theta(1)$

- Ans = $\Theta(2^n)$

$T(n) = T(n-1) + \Theta(n)$

- Ans = $\Theta(n^2)$