ELEC 3224 — Stochastic Control and Estimation

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Introduction

- **Assumption:** Full state is available for measurement.
- Noisy state equation

$$\dot{X} = AX + Bu + B_w w$$

$w(t)$ — **normally distributed white noise.**

- $$E[X(0)] = m_o, \; E[X(0)X^T(0)] = \Sigma_X(0)$$

Introduction

- A real measure of performance **with z denoting the output**

$$\begin{aligned} J &= E\left[\int_0^{t_f} z^T(t)W(t)z(t)dt\right] \\ &= \int_0^{t_f} E\left[z^T(t)W(t)z(t)\right]dt \end{aligned}$$

- For finite t_f, **constant** W and stationary random inputs, this integral is **proportional** to

$$J = E\left[z^T(t)Wz(t)\right]$$

Introduction

- The **cost can be computed by noticing that J is scalar** and $J = \mathrm{trace}\{J\}$.
-
$$J = \mathrm{trace}\{E\left[z^T(t)Wz(t)\right]\}$$
- The **trace** of a **square** matrix is **the sum of the elements on its diagonal.**
- **Fact:** The trace is **invariant** under cyclic permutation, i.e.,

$$J = \mathrm{trace}\{WE\left[z(t)z^T(t)\right]\} = \mathrm{trace}\{W\Sigma_z\}$$

Introduction

- $$\Sigma_z = \int_0^\infty \int_0^\infty g_{cl}(\tau_1) R_w(\tau_1 - \tau_2) g_{cl}^T(\tau_2) d\tau_1 d\tau_2$$
- where g_{cl} is the **impulse response of the closed-loop system from input w to output z.**
- **Assumption:** w is white noise with spectral density S_w.
- Then
$$\Sigma_z = \int_0^\infty g_{cl}(\tau) S_w g_{cl}^T(\tau)\, d\tau$$

Introduction

- With output equation $z = C_z X$, Σ_z can be computed from Σ_x as

$$\Sigma_z = C_z E\left[x(t)x^T(t)\right] C_z^T = C_z \Sigma_x C_z^T$$

- In this case

$$J = \mathrm{trace}\{W C_z \Sigma_x C_z^T\}$$

- where

$$A_{cl}\Sigma_x + \Sigma_x A_{cl}^T + B_{cl} S_w B_{cl}^T = 0$$

- Matlab function for finding Σ_x: lyap.

LQG Control

- Particular interest in extending the LQR to the stochastic case. No particular problems in doing this for the deterministic case because the system was assumed to be stable and the only disturbance was the state initial condition $X(0)$ whose contribution $\rightarrow 0$ as $t \rightarrow \infty$. Hence J_{LQR} is finite.
- A **similar line of reasoning is not possible in the stochastic case as there is continuing driving noise and so even the optimised cost**$\rightarrow \infty$.
- Instead, use **a time-averaged cost**

$$J = \lim_{t_f \rightarrow \infty} E\left[\frac{1}{t_f}\int_0^{t_f}(X^T(\tau)QX(\tau) + u^T(\tau)Ru(\tau))d\tau\right]$$

LQG Control

- In the time-invariant case then for a closed-lop stable system, $X(t)$ and $U(t)$ are stationary random processes and then

$$J = \lim_{t \to \infty} E\left[X^T(t)QX(t) + u^T(t)Ru(t)\right]$$

- $Q \succ 0$, $R \succ 0$ provides **the steady-state mean-square response.**
- This form can be used to study controller performance.
- Control law

$$u = -KX$$

LQG Control

- In this case, it can be shown that

$$J = \mathrm{trace}\{(Q + K^T R K)\Sigma_{X,ss}\}$$

- where $\Sigma_{X,ss}$ solves

$$(A - BK)\Sigma_{X,ss} + \Sigma_{X,ss}(A - BK)^T + B_w S_w B_w^T = 0$$

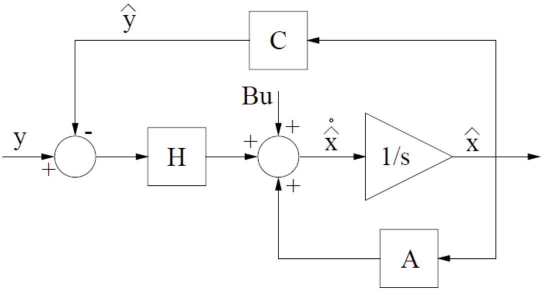
Kalman Filtering

- In the deterministic case, the state estimator has the form

$$\dot{\hat{X}} = A\hat{X} + Bu + H(y - C\hat{X})$$

- also $\hat{y} = C\hat{X}$ – **given the statistical properties of system disturbances and sensor noise, the Kalman filter design gives an optimal** H.
- See the next figure.

Kalman Filtering



Kalman Filtering

- System state-space model

$$\begin{aligned}\dot{X} &= AX + Bu + w_1 \\ y &= CX + w_2\end{aligned}$$

- w_1 and w_2 – **random input signals.**
- w_1 — disturbance on the state dynamics.
- w_2 — sensor noise — corrupts y.
- **Assumptions:** w_1 and w_2 are both unbiased white noise and all channels are uncorrelated.
- Hence the following hold.

Kalman Filtering

-

$$
\begin{aligned}
E(w_1(t_1)w_1^T(t_2)) &= V_1\delta(t_1 - t_2) \\
E(w_2(t_1)w_2^T(t_2)) &= V_2\delta(t_1 - t_2) \\
E(w_1(t_1)w_2^T(t_2)) &= 0
\end{aligned}
$$

- $\delta(t)$ – impulse (or delta) function.
- V_1, V_2 — **diagonal matrices** of intensities.
- **Estimation error**

$$e = X - \hat{X}$$

Kalman Filtering

- Also the **estimation error** $e = X - \hat{X}$ satisfies
- $$\dot{e} = (A - HC)e + (V_1 - HV_2)$$
- The eigenvalues of $A - HC$ determine the stability properties of the estimator error dynamics. The second term $(V_1 + HV_2)$ can be considered **as an external input.**

Kalman Filtering

- The **Kalman filter design gives the H that minimises the scalar cost function**

$$J = E(e^T W e)$$

- where $W \succ 0$ is a **weighting matrix.**
- A related matrix is the **symmetric error covariance** given by

$$P = E(ee^T)$$

- We consider the steady-state solution.

Kalman Filtering

- The steady-state solution is

$$H = PC^T V_2^{-1}$$

where P is the solution of

$$0 = AP + PA^T + V_1 - PC^T V_2^{-1} CP$$

- The Kalman filter is **guaranteed** to give stable nominal dynamics $A - HC$ provided the **system is observable.** (Dual to the stability property of the LQR solution.)
- The matrix P is known as **the error covariance matrix.**
- The qualitative dependence of the estimator gain H on the other parameters is easily seen.

Kalman Filtering

- A large uncertainty P creates a large H – places emphasis on the corrective action of the filter.
- A small disturbance (intensity) V_1 and large sensor noise (intensity) V_2 creates a small H, weighting the model dynamics $A\hat{X} + Bu$ more.
- A large disturbance (intensity) V_1 and small sensor noise (intensity) V_2 creates a large H, hence the filter's correction is dominant.
- **An optimal output feedback controller is formed by combining the Kalman filter and an LQR full state feedback gain.**
- **This is commonly known as Linear Quadratic Gaussian design or LQG.**

Kalman Filtering

- **System**

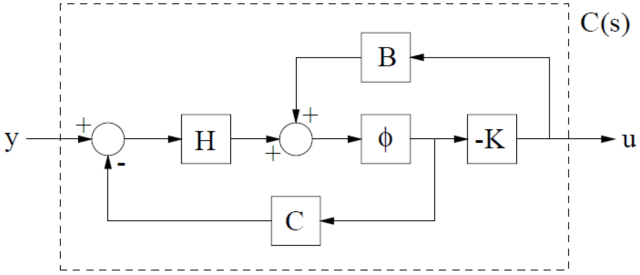
$$\begin{aligned} \dot{X} &= AX + Bu + w_1 \\ y &= CX + w_2 \end{aligned}$$

- The Kalman filter and controller gain K are combined as

$$\begin{aligned} \dot{\hat{X}} &= A\hat{X} + Bu + H(y - C\hat{X}) \\ u &= -K\hat{X} \end{aligned}$$

- See next figure.

Kalman Filtering



$$\phi = (sI-A)^{-1}$$

Kalman Filtering

- **The separation principle holds here as well,** i.e., the eigenvalues of the nominal closed-loop system are the union of those for $A - HC$ and $A - BK$.
- The Kalman filter is extensively used in Guidance and Navigation Systems.

Extended Kalman Filtering

- The derivation of the Kalman filter is based on a **model for linear dynamics, or, more accurately, a linear Gaussian model.**
- **Fact;** In many applications, including in aerospace and hence guidance and control, the dynamics are **nonlinear.**
- One option in such cases is to use an **extended Kalman filter (EKF).**
- The EKF does not assume linear Gaussian models, but does assume Gaussian noise. Also it uses local linear approximations of the model dynamics to retain the efficiency of the Kalman filter setting.

Kalman Filtering — Example

- Consider an unstable first order system

$$\begin{aligned} \dot{x} &= x + u + w_1 \\ y &= x + w_2 \end{aligned}$$

- The uncorrelated noise signals w_1 and w_2 are white noise signals with intensities V_1 and V_2.
- In this case the Riccati equation is

$$2P + V_1 - \frac{P^2}{V_2} = 0$$

Kalman Filtering — Example

- This last equation has the positive solution

$$P = V_2 + V_2\sqrt{1 + \frac{V_1}{V_2}}$$

- **In this case, the Kalman filter gain is only a function of the ratio $\beta = \frac{V_1}{V_2}$.**
- The Kalman filter gain is

$$H = \frac{P}{V2} = 1 + \sqrt{1 + \beta}$$

- The Kalman filter error dynamics are

$$\begin{aligned} \dot{\tilde{x}} &= (A - HC)\tilde{x} + w_1 - Kw_2 \\ &= -\sqrt{1+\beta}\tilde{x} + w_1 - (1 + \sqrt{1+\beta})w_2 \end{aligned}$$

Kalman Filtering — Example

- The pole of the Kalman filter pole is at $-\sqrt{1+\beta}$. Hence as β increases, the pole moves to the left on the real axis. Consequently the Kalman filter very much 'trusts' the measurements. Conversely, if $\beta \to 0$, the pole of the Kalman filter approaches -1, i.e., the Kalman filter pole is as fast as the system pole. In this case the filter 'trusts' the system model much more than the measurements.