ELEC 3224 — Ballistic Dynamics

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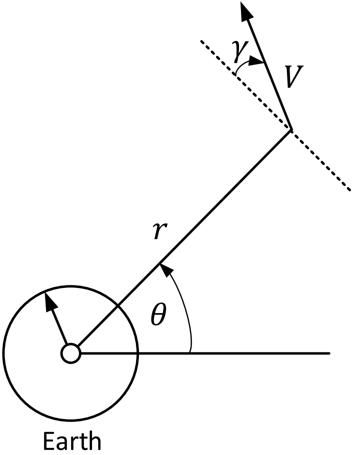
Introduction

- **Definition:** The path of an unpowered object moving only under the influence of gravity and possibly atmospheric friction and with its surface providing no significant lift to alter the course of flight.
- For speeds below 500 ft/sec (152.4 m/sec), altitude less than 100,000 ft (30.48 km) and ranges less than 100 nautical miles, a flat-earth, constant gravity model is 'reasonable'.
- In this section, the scenario shown in the next figure is considered.

Problem Setup



Analysis

- Write the position in polar co-ordinates as

$$\underline{r}(t) = r(t)e^{j\Theta(t)}$$

- For ease of notation the dependence on t is dropped from variables from this point onwards.
- **Velocity**

$$\dot{\underline{r}} = \dot{r}e^{j\Theta} + jr\dot{\Theta}e^{j\Theta}$$

- Acceleration

$$\begin{aligned} \ddot{r} &= \ddot{r}e^{j\Theta} + j\dot{r}\dot{\Theta}e^{j\Theta} + j\dot{r}\dot{\Theta}e^{j\Theta} \\ &+ jr\ddot{\Theta}e^{j\Theta} + j^2r\dot{\theta}^2e^{j\Theta} \end{aligned}$$

Analysis

- Hence
$$\ddot{r} = (\ddot{r} - r\dot{\Theta}^2)e^{j\Theta} + (r\ddot{\Theta} + +2\dot{r}\dot{\Theta})je^{j\Theta}$$
- Treating the object as a point mass (m) in free motion gives
$$m(\ddot{r} - r\dot{\Theta}^2) = -mg_{local} \qquad (1)$$
- and
$$m(r\ddot{\Theta} + 2\dot{r}\dot{\Theta}) = 0 \qquad (2)$$
- g_{local}, via
$$Gravity = \frac{GM_{earth}m}{r^2}$$

Analysis

- M_{earth} is the mass of the earth, and G is the gravitation constant.
- At $r = a$ (g acceleration due to gravity)

$$\frac{GM_{earth}m}{a^2} = mg$$

- Hence

$$GM_{earth} = ga^2$$

- and therefore

$$mg_{local} = \frac{ga^2m}{r^2}$$

Analysis

- Equation (1) now becomes

$$\ddot{r} - r\dot{\Theta}^2 = -\frac{ga^2}{r^2} \tag{3}$$

- and Equation (2) becomes

$$r\ddot{\Theta} + 2\dot{r}\dot{\Theta} = 0 \tag{4}$$

- Suppose that at $t = 0$, $r = r_0$ (the radial distance) and the tangent velocity (see the figure earlier) is $V\cos\gamma$ (V and γ are constants).

Analysis

- On multiplying (4) by r it follows that

$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \qquad (5)$$

- Integrating (5) and using the initial conditions goves

$$r^2\dot{\theta} = C$$

- At $t = 0$, $r = r_0$ and $r_0\dot{\Theta} = V\cos\gamma$. Hence

$$r^2\dot{\Theta} = r_0 V\cos\gamma \qquad (6)$$

t

Analysis

- Equations (3) and (6) specify the **free ballistic** motion of a body from specified initial conditions.
- These are **nonlinear differential equations** but a change of variable can be used to make them integrable.
- The route is by the substitution $u = \frac{1}{r}$.
- Using (6)

$$\frac{du}{dt} = \frac{du}{d\Theta} \cdot \frac{d\Theta}{dt} = \frac{r_0 V \cos\gamma}{r^2} \frac{du}{d\Theta} \tag{7}$$

Analysis

- Alternatively
$$\frac{du}{dt} = \frac{du}{dr}\cdot\frac{dr}{dt} = -\frac{1}{r^2}\frac{dr}{dt} \tag{8}$$
- Equating (7) and (8) gives
$$\frac{dr}{dt} = -r_0 V\cos\gamma\frac{du}{d\Theta} \tag{9}$$
- Let
$$z = \frac{dr}{dt}$$

Analysis

- $$\begin{aligned} \frac{d^2 r}{dt^2} &= \frac{dz}{dt} = \frac{dz}{d\Theta} \cdot \frac{d\Theta}{dt} \\ &= \frac{r_0 V \cos\gamma}{r^2} \frac{dz}{d\Theta} \end{aligned}$$

- and (using (9))

$$\frac{dz}{d\Theta} = \frac{d}{d\Theta}\left[\frac{dr}{dt}\right] = -r_0 V \cos\gamma \frac{d^2 u}{d\Theta^2}$$

Analysis

- Hence

$$\frac{d^2 r}{dt^2} = -\frac{[r_0 V \cos\gamma]^2}{r^2} \frac{d^2 u}{d\Theta^2} \tag{10}$$

- Substituting (10) and (6) into (3) gives (work the details)

$$\frac{d^2 u}{d\Theta^2} + \frac{1}{r} = \frac{ga^2}{[r_0 V \cos\gamma]^2}$$

- Since $u = \frac{1}{r}$ and introduce

$$\lambda = \frac{r_0 V^2}{ga^2}$$

Analysis

- Then
$$\frac{d^2u}{d\Theta^2} + u = \frac{1}{\lambda r_0 \cos^2\gamma} \tag{11}$$
- This is a **linear** second order differential equation in u $(= \frac{1}{r})$ and independent variable Θ.
- Consider the case of $\gamma = 0$ and boundary conditions
$$u(\Theta = 0) = \frac{1}{r_0}, \;\; \frac{du}{d\Theta}(0) = 0$$
(where the last initial condition is from (9)).
- Then it can be shown that
$$r = \frac{\lambda r_0}{(\lambda - 1)\cos\Theta + 1} \tag{12}$$

Analysis

- **Key point:** The solution depends on λ.
- Consider the case when $\lambda = 1$. Then $r = r_0$ and is a **circular orbit.**
- For $0 < \lambda < 2$, the denominator in (12) is **never zero** and r is **periodic.**
- The solution in this case is an **elliptical path but not necessarily an orbit.**
- **Fact:** If λ is too small (e.g., $\lambda = 0.5$), an elliptical path that **intersects the earth** results.

Analysis

- If $\lambda = 2$
$$r = \frac{2r_0}{1 + \cos\Theta}$$
and $r \to \infty$.
- This is a **parabolic trajectory.**
- If $\lambda > 2$, again $r \to \infty$ and is a **hyperbolic trajectory.**