ELEC 3224 — Frequency Domain Control Design Part II

Professor Eric Rogers

School of Electronics and Computer Science
University of Southampton
etar@ecs.soton.ac.uk
Office: Building 1, Room 2037

PID Controllers

- An extensively used controller in many industries is Proportional plus Integral plus Derivative (PID) control.
- Deleting the D term gives PI control.
- Deleting the I term gives PD control.
- Deleting the D and I terms gives P control (proportional control).
- Control law in the time domain

$$u = k_p e + k_i \int_0^t e(\tau) d\tau + k_d \frac{de}{dt}$$

- e is the error signal, i.e.,

$$e = r - y$$

PID controllers

- r is the **reference signal** — represents **ideal behaviour from the controlled system.**
- There are a number of alternative formulations of a PID controller — also termed 'a three term' controller.
- The transfer-function of a PID controller is

$$C(s) = k_p + \frac{k_i}{s} + k_d s$$

- This controller has infinite gain at zero frequency, i.e., $C(0) = \infty$. Hence

$$G_{yr} = \frac{PC}{1 + PC}$$

is equal to 1 in this case.

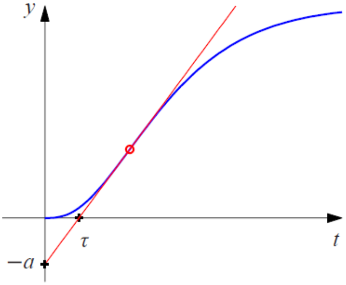
PID controllers

- Hence **the steady state error in response to a unit step reference signal is zero.**
- If **zero steady state error in response to a unit step reference** is required then PC **must have pole at the origin or else this pole must be introduced by the controller**.

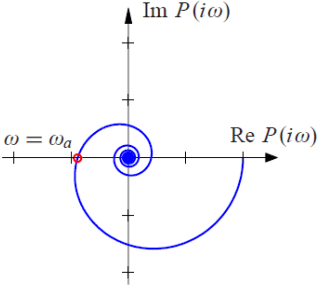
PID controllers — Auto-Tuning

- In the 1940's Ziegler and Nichols developed two methods for PID tuning based on simple characteristics of the process dynamics in time and frequency domains.
- The unit step response — left-hand plot in the next figure is based on applying a unit step and recording the response. This is supposed to be characterised by the parameters α and τ.
- These are the intercepts of the steepest tangent of the step response with the co-ordinate axes.
- τ is an approximation of the time delay of the system and $\frac{a}{\tau}$ is the steepest slope of the step response.
- Frequency response method — proportional control (critical value k_c), increased to oscillation with period T_c occurs.

PID controllers — Auto-Tuning



(a) Step response method



(b) Frequency response method

PID controllers — Auto-Tuning

. Ziegler–Nichols tuning rules. (a) The step response methods give the parameters in terms of the intercept a and the apparent time delay τ. (b) The frequency response method gives controller parameters in terms of *critical gain* k_c and *critical period* T_c.

Type	k_p	T_i	T_d
P	1/a		
PI	0.9/a	3τ	
PID	1.2/a	2τ	0.5τ

(a) Step response method

Type	k_p	T_i	T_d
P	$0.5k_c$		
PI	$0.4k_c$	$0.8T_c$	
PID	$0.6k_c$	$0.5T_c$	$0.125T_c$

(b) Frequency response method

- $$u = k_p\left(e + \frac{1}{T_i}\int_0^t e(\tau)d\tau + T_d\frac{de}{dt}\right)$$

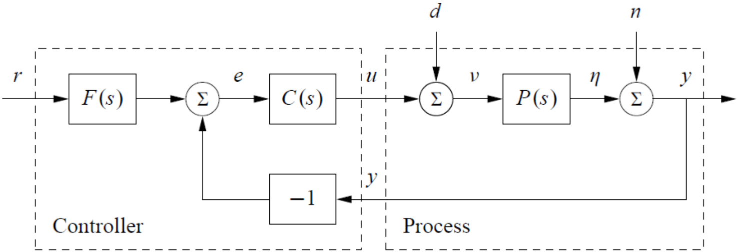
PID controllers — Auto-Tuning

- In the previous case, the **key message is that controller design and hence the closed-loop system can be undertaken by 'shaping'** PC **— the open-loop transfer-function.**
- This is much more easier than working with

$$G_{yr} = \frac{PC}{1 + PC}$$

- This analysis ignores the effects of **load disturbances and measurement noise.**
- Also it assumes that all **objectives can be met with a single controller.**
- Next figure — more general case.

General Control Architecture



General Control Architecture

- This is a **two-degree of freedom** scheme with feedforward controller $F(s)$ and feedback controller $C(s)$.
- d is the **load disturbance** and n is the **measurement noise.**
- r is the reference signal.
- η is the true system output.
- y is the **measured system** output.
- This is a 3 input (u, d and n) 1 output – y — system.

General Control Architecture Relationships

$$\begin{bmatrix} y \\ \\ n \\ v \\ u \\ e \end{bmatrix} = \begin{bmatrix} \frac{PCF}{1+PC} & \frac{P}{1+PC} & \frac{1}{1+PC} \\ \frac{PCF}{1+PC} & \frac{P}{1+PC} & -\frac{PC}{1+PC} \\ \frac{CF}{1+PC} & \frac{1}{1+PC} & -\frac{C}{1+PC} \\ \frac{CF}{1+PC} & -\frac{PC}{1+PC} & -\frac{C}{1+PC} \\ \frac{F}{1+PC} & -\frac{P}{1+PC} & -\frac{1}{1+PC} \end{bmatrix} \begin{bmatrix} r \\ d \\ n \end{bmatrix}$$

General Control Architecture Relationships

- The transfer-function between the reference signal r and output η (not an explicit signal in the system)

$$\epsilon = r - \eta = \left(1 - \frac{PCF}{1+PC}\right) r - \frac{P}{1+PC} d + \frac{PC}{1+PC} n$$

- Many of the transfer-functions are the same and many relations are given by the following 6.

$$\begin{aligned} TF &= \frac{PCF}{1+PC}, \; T = \frac{PC}{1+PC}, \; PS = \frac{P}{1+PC} \\ CFS &= \frac{CF}{1+PC} \; CS = \frac{C}{1+PC}, \; S = \frac{1}{1+PC} \end{aligned}$$

General Control Architecture Relationships

- Special case of $F = 1$ is termed a system with (pure) feedback and the system **is completely specified by the following four transfer-functions:**
- **Sensitivity Function**

$$S = \frac{1}{1 + PC}$$

- **Load Sensitivity Function**

$$PS = \frac{P}{1 + PC}$$

General Control Architecture Relationships

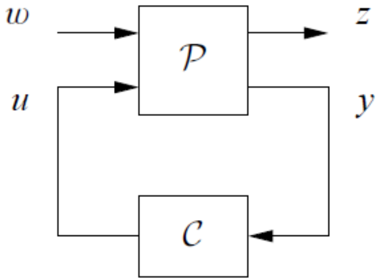
- **Complementary Sensitivity Function**

$$T = \frac{PC}{1 + PC}$$

- **Noise Sensitivity Function**

$$CS = \frac{C}{1 + PC}$$

General Control Architecture



General Control Architecture Relationships

- Previous figure — a more general representation of a feedback system.
- u – the control signal (or signals in the MIMO case) that can be manipulated.
- w – other signal (or signals in the MIMO case) that influences the dynamics.
- y — measured variable (or variables in the MIMO case).
- z — other signals of interest.

General Control Architecture Relationships

- **Example**

$$P(s) = \frac{1}{s-a}, \quad C(s) = k\frac{s-a}{s}$$

- In this case

$$L = \frac{k}{s}$$

$$T = \frac{k}{s+k}, \quad PS = \frac{s}{(s+a)(s+k)}$$

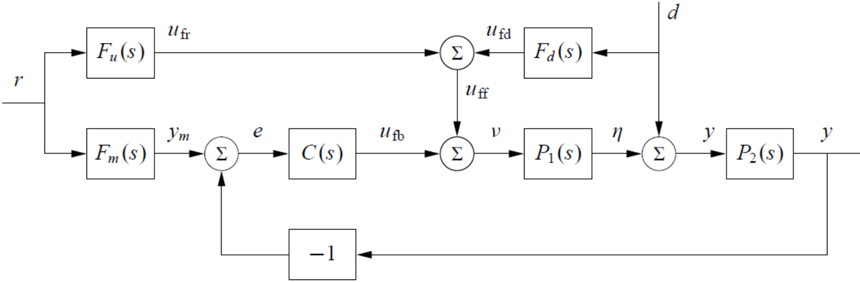
$$CS = k\frac{(s-a)}{s+k}, \quad S = \frac{s}{s+k}$$

- **The factor** $s-a$ is cancelled when computing the loop transfer-function. Major difficulty when $a>0$ – **unstable pole-zero cancellation.**

General Control Architecture Relationships

- The transfer-function PS relating load disturbances to the process output is unstable and hence a small disturbance d can lead to an unbounded output.
- Generalisation to MIMO systems is immediate.

Feedforward Control



Feedforward Control

- Previous figure — 2 Degrees Of Freedom (DOF) system — feedforward controllers: $F_m(s)$ sets the desired output value, ii) $F_u(s)$ generates the feedforward command u_{fr} and iii) $F_d(s)$ attempts to cancel disturbances.
- Feedforward is a simple and powerful technique that complements feedback.
- It can be used both to improve the response to reference signals and to reduce the effect of measurable disturbances.
- Feedforward compensation admits perfect elimination of disturbances, but it is much more sensitive to process variations than feedback compensation.

Feedforward Control

- 2 DOF schemes have the advantage that the **response to reference signals can be designed independently of the design for disturbance attenuation and robustness.**
- Set $d = 0$, i.e., no load disturbance and use F_m to denote the ideal response to the reference signal. Then the feedforward compensator is characterised by the transfer-functions F_u and F_m.
- When the reference is changed, the transfer-function F_u generates the signal u_{fr}, which is chosen to give the desired output when applied as input to the process.

Feedforward Control

- Under ideal conditions the result is that $y = y_m$, the error signal is zero and then there will be no feedback action.
- If there are disturbances or modeling errors, the feedback control action will attempt to reduce the error to zero.
- $$G_{yr} = \frac{P(CF_m + F_u)}{1 + PC} = F_m + \frac{PF_u - F_m}{1 + PC}, \;\; P = P_2 P_1$$
- F_m — desired transfer-function.
- How to make the second term on the right-hand side 'small'?

Feedforward Control

- Method one: make $PF_u - F_m$ 'small'.
- Method two: make $1 + PC$ 'large'.
- Perfect feedforward compensation

$$F_u = \frac{F_m}{P}$$

- This involves P^{-1}!!

Feedforward Control

- Feedback and feedforward have different properties.
- Feedforward action is obtained by **matching two transfer functions, requiring precise knowledge of the process dynamics, while feedback attempts to make the error small by dividing it by a large quantity.**
- For a controller with **integral action,** ($\frac{1}{s}$ and hence an open-loop pole at the origin) the loop gain is large for low frequencies and it is thus sufficient to make sure that the condition for ideal feedforward holds at higher frequencies. This is easier than trying to satisfy the last condition for all frequencies.

Feedforward Control

- Consider the reduction of the load disturbance d by feedforward control.
- Assume that the disturbance signal is measured and that the disturbance enters the process dynamics in a known way, represented by P_1 and P_2.
- The effect of the disturbance can be reduced by feeding the measured signal through a dynamic system with the transfer-function F_d.
- Set $r = 0$ for analysis.

Feedforward Control

- By block diagram algebra

$$G_{yd} = \frac{P_2(1 + F_d P_1)}{1 + PC}$$

- Choices now are to make $1 + F_d P_1$ small (feedforward) or make $1 + PC$ large (feedback).
- **Perfect compensation** with

$$F_d = P^{-1}$$

- but needs the plant inverse transfer-function!

Feedforward Control

- As in the case of reference tracking, disturbance attenuation can be accomplished by combining feedback and feedforward control.
- Since low-frequency disturbances can be eliminated by feedback, feedforward only can be used for high frequency disturbances and the transfer-function F_d in the last equation can then be computed using an approximation of P_1 for high frequencies.
- To obtain a transfer-function that can be implemented without difficulties requires that i) the feedforward is stable and does not require differentiation.
- Hence there may be constraints on possible choices of the desired response F_m and approximations are needed if the process has zeros in the right half-plane or time delays (see below).

Performance Specifications

- Time and frequency domain specifications are possible.
- Time domain specifications are often given in terms of robustness to process variations and disturbances.
- Performance specifications: step response command. Steady-state error to a unit step command, rise time, setting time (based on a dominant 2nd order pole model).
- Robustness specifications — later.

Performance Specifications

- Common features of frequency responses are resonant peak, peak frequency, gain crossover frequency and bandwidth.
- A **resonant peak** is a maximum of the gain, and **the peak frequency** is the corresponding frequency.
- The **gain crossover frequency** is the frequency where the open loop gain is equal one.
- The **bandwidth** is defined as the frequency range where the closed loop gain is $\frac{1}{\sqrt{2}}$ of the low-frequency gain (low-pass), mid-frequency gain (band-pass) or high-frequency gain (high-pass).
- Time response in the initial stages corresponds to high-frequency.

Performance Specifications

- **Response to reference signals:**
- $$G_{yr} = \frac{PCF}{1+PC}, \quad G_{ur} = \frac{CF}{1+PC}$$
- Control over the input is important in many applications.
- Too large an input signal could result in actuator problems, e.g., saturation.

Performance Specifications

- **Response to Load Disturbances and Measurement Noise:**
- A simple criterion for disturbance attenuation is to compare the output of the closed- loop system with the output of the corresponding open-loop system, obtained by setting $C = 0$.
- If the disturbances for the open and closed-loop systems are identical, the output of the closed-loop system is then obtained simply by passing the open-loop output through a system with the transfer function S.
- The sensitivity function shows how the variations in the output are influenced by feedback.

Performance Specifications

- Disturbances with frequencies such that $S|(j\,\omega)| < 1$ are **attenuated**, but disturbances with frequencies such that $S|(j\,\omega)| > 1$ are **amplified** by feedback.
- The maximum sensitivity M_s, which occurs at the frequency ω_{ms}, is hence a measure of the **largest amplification of the disturbances.**
- The maximum magnitude of $\frac{1}{1+L}$ is also the minimum of $|1+L|$, which is precisely the stability margin s_m.

Performance Specifications

- Hence

$$M_s = \frac{1}{s_m}$$

- the maximum sensitivity is also robustness measure.
- If the sensitivity function is known, the potential improvements by feedback can be evaluated simply by recording a typical output and filtering it through the sensitivity function.
- A plot of the gain curve of the sensitivity function is a good way to make an assessment of the disturbance attenuation.
- The sensitivity function

$$S = \frac{1}{1 + PC}$$

depends only on the loop transfer-function.

Performance Specifications

- Hence its properties can also be visualized graphically using the Nyquist plot of the loop transfer-function.
- See the next figure.
- The sensitivity is less than 1 for **all points outside a circle with radius** 1 **and center at** -1.
- Disturbances with frequencies in this range are attenuated by the feedback.

Performance Specifications

- The transfer function G_{yd} from load disturbance d to the output y is

$$G_{yd} = \frac{P}{1+PC} = PS = \frac{T}{C}$$

- Since load disturbances typically have low frequencies, it is natural to focus on the behavior of the transfer function at low frequencies. For a system with $P(0) \neq 0$, and a controller with integral action, the controller gain goes to infinity for small frequencies and we have the following approximation for small s:

$$G_{yd} = \frac{T}{C} \approx \frac{1}{C} \approx \frac{s}{k_i}$$

  k_i integral gain.
- Also $S \to 1$ for 'large' s, $G_{yd} \approx P$ for high frequencies.

Performance Specifications

- Measurement noise, which typically has high frequencies, generates rapid variations in the control variable that are detrimental because they cause wear in many actuators and can even saturate an actuator.
- variations in the control signal due to measurement noise must be kep at reasonable levels – a typical requirement is that the variations are only a fraction of the span of the control signal.
- The variations can be influenced by filtering and by proper design of the high-frequency
- The effects of measurement noise are captured by the transfer function from the measurement noise to the control signal.

Performance Specifications

- $$-G_{un} = \frac{C}{1+PC} = CS = \frac{T}{P}$$
- The complementary sensitivity function T is close to 1 for low frequencies ($\omega < \omega_{gc}$, and G_{un} can be approximated by $-\frac{1}{P}$.
- The sensitivity function is close to 1 for high frequencies ($\omega > \omega_{gc}$ and G_{un} can be approximated by $-C$.

Performance Specifications

- An example

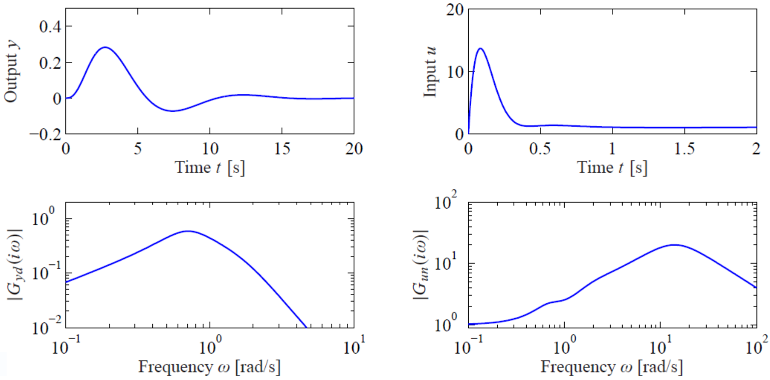
$$P = \frac{1}{(s+1)^3}$$

- Controller: PID with $k_p = 0.6, k_1 = 0.5$ and $k_d = 2$.
- Controller is augmented by a second-order noise filter with $T_f = 0.1$, resulting in

$$C = \frac{k_d s^2 + k_p s + k_1}{s(\frac{s^2 T_f}{2} + T_f s + 1)}$$

- Next figure, system responses.

Performance Specifications



(a) Output load response (b) Input noise response

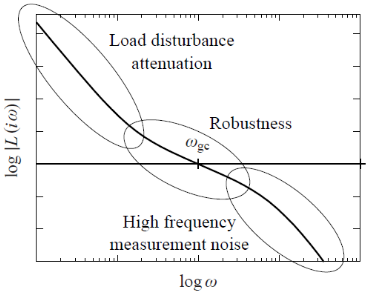
Performance Specifications

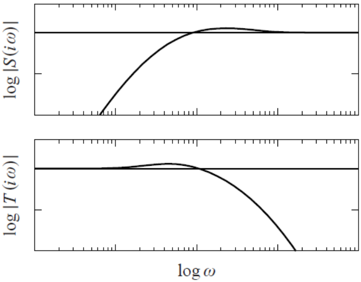
- Top-left plot: response of the output to a step in the load disturbance has a peak of 0.28 at time $t = 2.73s$.
- The frequency response, bottom left, shows that the gain has a maximum of 0.58 at $\omega = 0.7$ rad/sec.
- The high-frequency roll-off of $G_{un}(j\,\omega)$ is due to filtering, without it the gain curve, bottom-right, would continue to rise after 20 rad/sec.
- The step response has a peak of 13 at $t = 0.08$ secs. The frequency response has its peak 20 at $\omega = 14$ rad/sec. This peak occurs far above the peak of the response to load disturbances and far above the gain crossover frequency $\omega_{gc} = 0.78$ rad/secs.

Feedback Control Design by Loop Shaping

- First consider loop transfer-function ($L(s)$ shapes to give good controlled loop performance and good stability margins.
- Good robustness margins require good stability margins (gain and phase margins).
- This last requirement imposes requirements on $L(s)$ around the cross-over frequencies ω_{pc} and ω_{gc}.
- The next figure illustrates these points.

Performance Specifications





Feedback Control Design by Loop Shaping

- The gain of L at low frequencies must be large in order to have good tracking of command signals and good attenuation of low frequency disturbances.
- Since $S = \frac{1}{1+L}$ it follows that for frequencies where $|L| > 101$ disturbances will be attenuated by a factor of 100 and the tracking error is less than 1%.
- It is therefore desirable to have a **large crossover frequency and a steep (negative) slope of the gain curve.**
- The gain at low frequencies can be increased by a controller with integral action, which is also termed **lag compensation.**

Feedback Control Design by Loop Shaping

- To avoid injecting too much measurement noise into the system, the loop transfer-function should have low gain at high frequencies — **known as high-frequency roll-off.**
- The choice of gain crossover frequency is a compromise among attenuation of load disturbances, injection of measurement noise and robustness.
- **Bode's** relations impose limitations on the shape of the loop transfer-function.

Bode's Relations

- The Bode plots suggest a relation between the gain curve and the phase curve.
- For the differentiator ($P(s) = s$) the slope is $+1$ and the phase is $\frac{\pi}{2}$.
-
- For the integrator ($P(s) = \frac{1}{s}$) the slope is -1 and the phase is $-\frac{\pi}{2}$.
- For the first order system $P(s) = s + a$ the gain curve has slope 0 for small frequencies and $\frac{\pi}{2}$ for high frequencies. The phase is 0 for low frequencies and $\frac{\pi}{2}$ for high frequencies.
- General case?
- **Assumption:** $P(s)$ has **no poles or zeros in the right-half of the complex plane.**

Bode's Relations

- Bode established that **the phase is uniquely given by the shape of the gain curve and vice versa** under the above assumption.

$$
\begin{aligned}
\arg P(j\omega_0) &= \frac{\pi}{2}\int_0^{\infty} f(\omega)\frac{d\log|G(j\omega)|}{d\log\omega}\,d\log\omega \\
&\approx \frac{\pi}{2}\frac{d\log|G(j\omega)|}{d\log\omega}
\end{aligned}
$$

- where f is the weighting kernel

$$
F(\omega) = \frac{2}{\pi^2}\log\left|\frac{\omega+\omega_0}{\omega-\omega_o}\right|
$$

Bode's Relations

- The phase curve is thus a weighted average of the derivative of the gain curve. If the gain curve has constant slope n, the phase curve has constant value $\frac{n\pi}{2}$.
- This result holds applies to **minimum phase** systems.
- **Non-minimum phase systems** are 'difficult' to control.
- **Poles are an intrinsic property** of systems.
- **Zeros depend on how the system output and input are coupled.**

Feedback Control Design by Loop Shaping

- Bode's relations **impose restrictions on the shape of the loop transfer-function.**
- The slope of the gain curve at gain crossover cannot be too steep.
- If the gain curve has a constant slope n_{gc} and the phase margin, ϕ_m then

$$n_{gc} = -2 + \frac{2\phi_m}{\pi}$$

Loop Shaping in Action

- Start with the Bode plot of the system transfer-function $P(s)$.
- Then change the plot by adding poles and zeros.
- The only limitation for minimum phase systems is that large phase leads and high controller gains may be required to obtain closed loop systems with a fast response.
- Simple compensator

$$C(s) = k\frac{s+a}{s+b}$$

- **Lead compensator** if $a < b$.
- **Lag compensator** if $a > b$.

Loop Shaping in Action

- **A PI controller is a special case of a lag compensator** with $b = 0$.
- An Ideal PD controller is a special case of a lead compensator with $a = 0$.
- **Lag compensation**, which increases the gain at low frequencies, is **typically used to improve tracking performance and disturbance attenuation at low frequencies.**
- Compensators that are tailored to specific disturbances can be also designed.
- **Lead compensation** is **typically used to improve the phase margin.**