

2nd Coursework ELEC2220:

Question 1: Root Locus plots, design of forward path controllers and closed loop system response

A highly oscillatory plant, $P(s)$ is to be controlled using a forward path controller $K(s)$ and a unity feedback loop. An approximate model of the plant is given by

$$P(s) = \frac{1}{s^2 + 4}$$

It is proposed to use an ideal PD controller to ensure that the overall system does not have an oscillatory impulse response. The controller is defined as

$$K(s) = 4k + ks$$

where k is a tuneable scalar gain.

- (a) Sketch the root locus and determine the minimum allowable value of k to ensure that the closed loop impulse response is not oscillatory. (12 Marks)

- (b) It is decided not to use the ideal PD controller and instead the implemented controller is defined as

$$K(s) = \frac{120k+30ks}{50+s}$$

Sketch the new root locus and comment on system response as k is increased from zero to plus infinity. (8 Marks)

- (c) In practice it is found that there is a maximum value of k for which the closed loop system is stable. Further tests reveal that the plant has an additional open loop pole at $s = -100$. Sketch the root locus diagram and explain why there is a finite maximum value of k . (5 Marks)

Question 2: Nyquist stability criterion, gain and phase margin, stability and the effect of time delays

An industrial process has the open loop transfer function

$$GH(s) = \frac{k}{s(s+2)(s+8)}$$

- (a) Using the Nyquist stability criterion, plot the locus of the closed-loop system and clearly indicate the gain and phase margins. (8 Marks)
- (b) Determine the maximum allowable gain, k , and frequency, ω (rad s⁻¹), for the system to remain stable. (7 Marks)

- (c) Sketch the locus of the closed loop system if the open loop transfer function also includes a pure time delay of 10 ms.

(5 Marks)

- (d) Show that the phase of the time delayed closed loop system is 180° when

$$\frac{\pi}{2} - 0.01\omega = \tan^{-1}\left(\frac{10\omega}{16-\omega^2}\right)$$

Given

$$\tan^{-1} x \pm \tan^{-1} y = \tan^{-1}\left(\frac{x \pm y}{1 \mp xy}\right)$$

(5 Marks)

Question 1: Root Locus plots, design of forward path controllers and closed loop system response

A highly oscillatory plant, $P(s)$ is to be controlled using a forward path controller $K(s)$ and a unity feedback loop. An approximate model of the plant is given by

$$P(s) = \frac{1}{s^2 + 4}$$

It is proposed to use an ideal PD controller to ensure that the overall system does not have an oscillatory impulse response. The controller is defined as

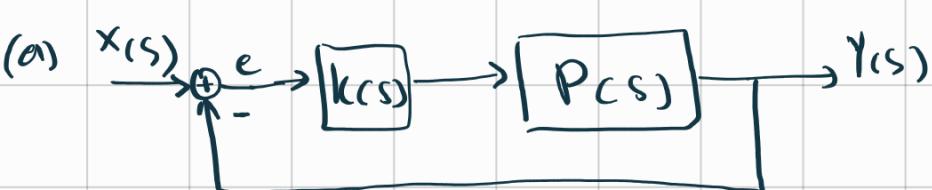
$$K(s) = 4k + ks$$

where k is a tuneable scalar gain.

- (a) Sketch the root locus and determine the minimum allowable value of k to ensure that the closed loop impulse response is not oscillatory.

(12 Marks)

long derivatives
are calculated on
different pages to
reduce clutter.



highest number of poles
or zeros are 2 (from poles)
so number of line = 2.

$$OLTF = \frac{Y(s)}{X(s)} = \frac{k(s)P(s)}{1} = \frac{k(4+s)}{(s^2+4)}, \therefore \begin{aligned} \text{OL zeros} &: s = -4 \\ \text{OL poles} &: s = 2j, -2j \end{aligned}$$

Rank 1 as numerator highest order is 1
and denominator highest order is 2, thus $2-1=1$.

$$\text{Angle of asymptote} = \phi_A = \frac{(2q+1) \cdot 180^\circ}{n-m} \therefore Q_A = 180^\circ$$

$q: 0, 1, 2, \dots, (n-m-1)$

where n = number of pole/zero (whichever higher) = 2

m = number of pole or zero (whichever lower) = 1

$$\text{Centroid, } \alpha = \frac{\sum \text{real part of OL poles} - \sum \text{real part of OL zeros}}{n-m} = \frac{0 - -4}{1} = 4$$

angle of departure from a complex pole, $\theta_{p_j} = 180^\circ + \sum \text{angles from all zeros to } p_j - \sum \text{angles from all other poles to } p_j$

$$\therefore @ 2j, \theta_{p_{j1}} = 180^\circ + \tan^{-1}\left(\frac{2}{4}\right) - 90^\circ = 116.565^\circ$$

$$\therefore @ -2j, \theta_{p_{j2}} = 180^\circ + \tan\left(-\frac{2}{4}\right) - 270^\circ = 243.434$$

Finding break-in point : ① Rewrite in form CLTF : $1 + G(s) H(s) = 0$

$$\therefore 1 + \frac{k(s+4)}{(s^2+4)} = 0 \quad \text{② let } k \text{ be subject.}$$

$$\therefore k = \frac{-(s^2+4)}{s+4} \quad \text{③ differentiate, then equate to 0.}$$

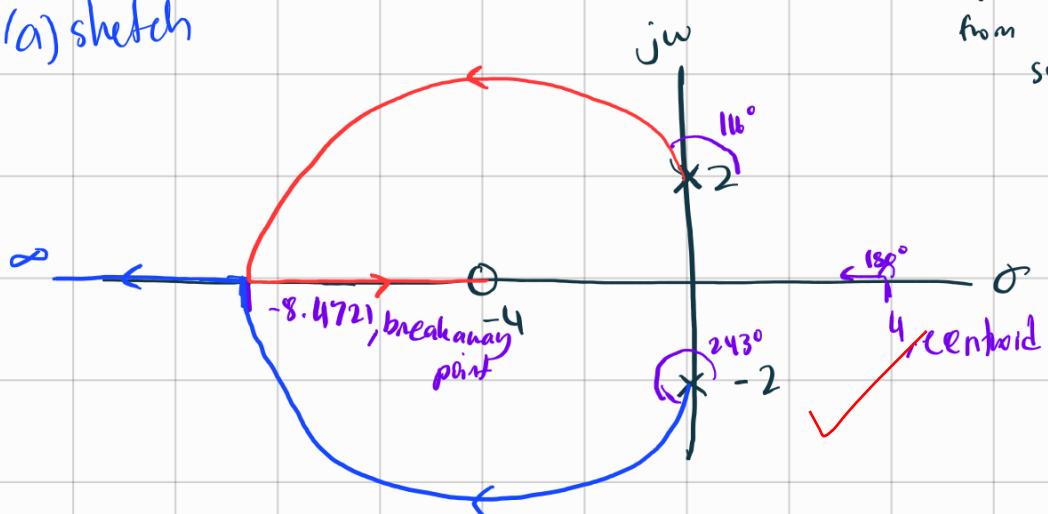
$$\frac{dk}{ds} = -\frac{(s^2+8s-4)}{(s+4)^2} = 0 \quad \text{④ solve and pick suitable } s \text{ as solution}$$

$$\therefore s \approx -8.4721 \quad \text{or} \quad s \approx 0.47214$$

as poles loci move to the left from $(0, 2) \leftarrow (0, -2)$, the solution is $s \approx -8.4721$

∴ break-in point $= -8.4721$

(a) sketch



To find minimum allowable value of k so that closed loop impulse response is not oscillatory, we can find the gain at breakaway point. as the loci before that had damping < 1 (underdamped ratio, 3 oscillatory in this case)

$$\therefore s = -8.4721$$

from CLTF char eqn:

$$k = \frac{-(s^2+4)}{s+4} = \frac{-[(-8.4721)^2+4]}{(-8.4721+4)} = 16.944$$

✓

12
12

$$\therefore k > 16.944$$

(b)

It is decided not to use the ideal PD controller and instead the implemented controller is defined as

$$K(s) = \frac{120k+30ks}{50+s}$$

Sketch the new root locus and comment on system response as k is increased from zero to plus infinity.

(8 Marks)

OLTF : $K(s)P(s) = \left(\frac{120k + 30ks}{50+s} \right) \left(\frac{1}{s^2+4} \right) = k \frac{30s+120}{(s^2+4)(s+50)}$

\therefore OL zeros : $30s = -120, s = -4$ $\phi_{A0} = 90^\circ$

\therefore OL poles : $s = -50, -2j, 2j$ $\phi_{A1} = 270^\circ$

rank 2 as $3 - 1 = 2$. $\alpha = \frac{(-50) - (-4)}{2} = -23$

$\phi_A = \frac{(2q+1) \cdot 180^\circ}{n-m+2}, q = 0, 1, \dots, (n-m-1)$ Number of lines = 3

@ $2j, \theta_{p1} = 114.3^\circ$ @ $-2j, \theta_{p1} = -114.3^\circ = 245.7^\circ$

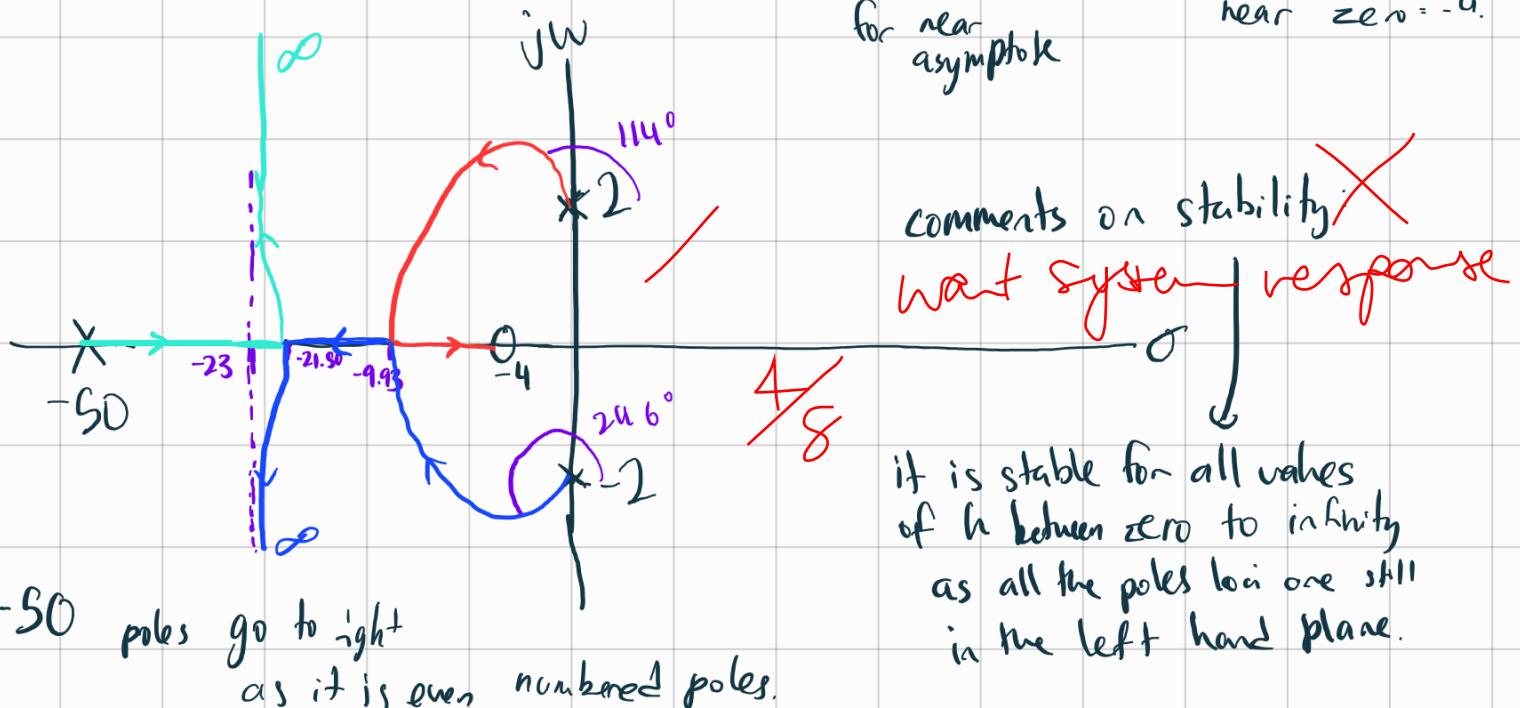
break-in point: ① $1 + \frac{k(30s+120)}{(s^2+4)(s+50)} = 0$

$\therefore k = \frac{-(s^2+4)(s+50)}{(30s+120)} \quad$ ② $\frac{dk}{ds} = -\frac{(s^3 + 31s^2 + 200s - 92)}{1s(s+4)^2} = 0$

$\therefore s^3 + 31s^2 + 200s - 92 = 0, s \approx -21.50, -9.93, 0.43$

most suitable $s = -21.50$ and $s = -9.93$.

↑
for near asymptote
↓
near zero = -4.



(c)

In practice it is found that there is a maximum value of k for which the closed loop system is stable. Further tests reveal that the plant has an additional open loop pole at $s = -100$. Sketch the root locus diagram and explain why there is a finite maximum value of k .

(5 Marks)

$$OLTF: h(s)Pcs = k \frac{(30s + 120)}{(s^2 + 4)(s + 50)(s + 100)}$$

OL zeros : $s = -4$

$$\gamma_A = \frac{(2q+1) \cdot 180^\circ}{n-m+3}, \quad q = 0, 1, 2, \dots, (n-m-1)$$

$$\text{OL poles : } s = -100, -80, -2j, 2j \quad \phi_{\infty} = 60^\circ$$

$$\angle AOD = 60^\circ$$

rank 3 as $4-1=3$

$$\theta_A = 180^\circ$$

$$\alpha = \frac{(-100 - 50) - (-4)}{3} = -48.67$$

$$\phi_{A2} = 300^\circ$$

$$@2_j, \theta_{p_j} = 113.1^\circ$$

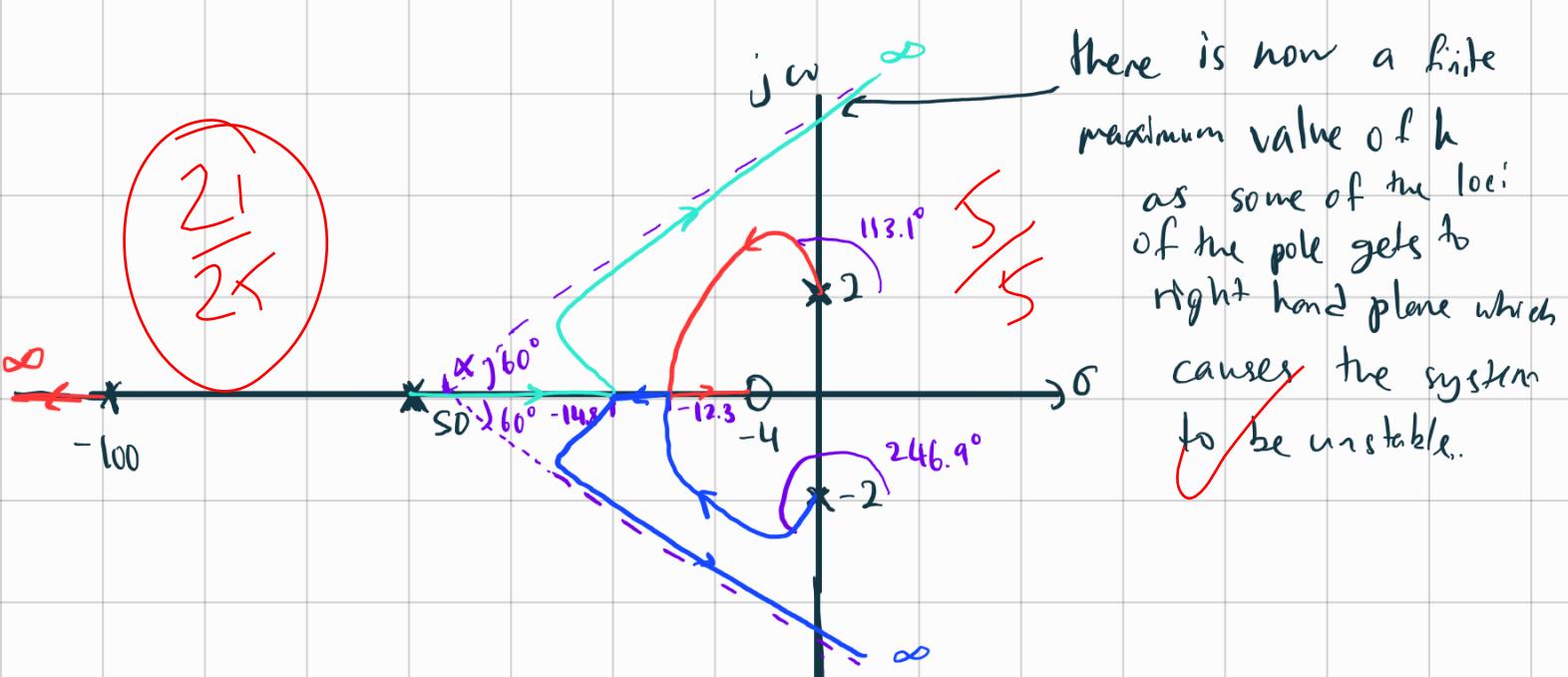
$$@-2j, \theta_{pj} = -173.1^\circ = 246.9^\circ$$

$$\text{breakaway point: } \textcircled{1} \text{ CLTF: } 1 + k \frac{(30, +120)}{(s^2 + 4)(s + 30)(s + 100)}$$

$$(2) \quad h = \frac{-(s^2+4)(s+s_0)(s+100)}{(30s+120)} \quad (3) \quad \frac{dh}{ds} = \frac{3s^4 + 316s^3 + 6804s^2 + 40032s - 17600}{30(s+4)^2} = 0$$

$$\textcircled{4} \text{ solve } s \approx -78.7, -14.8, -12.3, 0.41.$$

most suitable's as break-away and $s = -14.8$ & $s = -12.3$.
break-in point respectively.



Question 2: Nyquist stability criterion, gain and phase margin, stability and the effect of time delays

An industrial process has the open loop transfer function

$$GH(s) = \frac{k}{s(s+2)(s+8)}$$

- (a) Using the Nyquist stability criterion, plot the locus of the closed-loop system and clearly indicate the gain and phase margins.

(8 Marks)

OLTF : $GH(s) = \frac{k}{s(s+2)(s+8)}$

let $G(j\omega) = GH(j\omega)$

\therefore let $s = j\omega$

$$G(j\omega) = \frac{k}{j\omega(j\omega+2)(j\omega+8)}$$

$$|G(j\omega)| = \frac{k}{\sqrt{\omega^2(\omega^2+2^2)(\omega^2+8^2)}} = \frac{k}{\omega\sqrt{(\omega^2+4)(\omega^2+64)}}$$

$$\angle G(j\omega) = \left[-\frac{\pi}{2} - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{8}\right) \right]$$

$$\lim_{\omega \rightarrow 0^+} |G(j\omega)| = |\infty| \quad \angle \left[-\frac{\pi}{2} - 0 - 0 \right] = |\infty| \left[-\frac{\pi}{2} \right] = -90^\circ = 270^\circ$$

$$\lim_{\omega \rightarrow \infty} |G(j\omega)| = |0| \quad \angle \left[-\frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} \right] = |0| \left[-\frac{3\pi}{2} \right] = -270^\circ = 90^\circ$$

$$\omega \rightarrow \infty$$

Phase Cross over Frequency

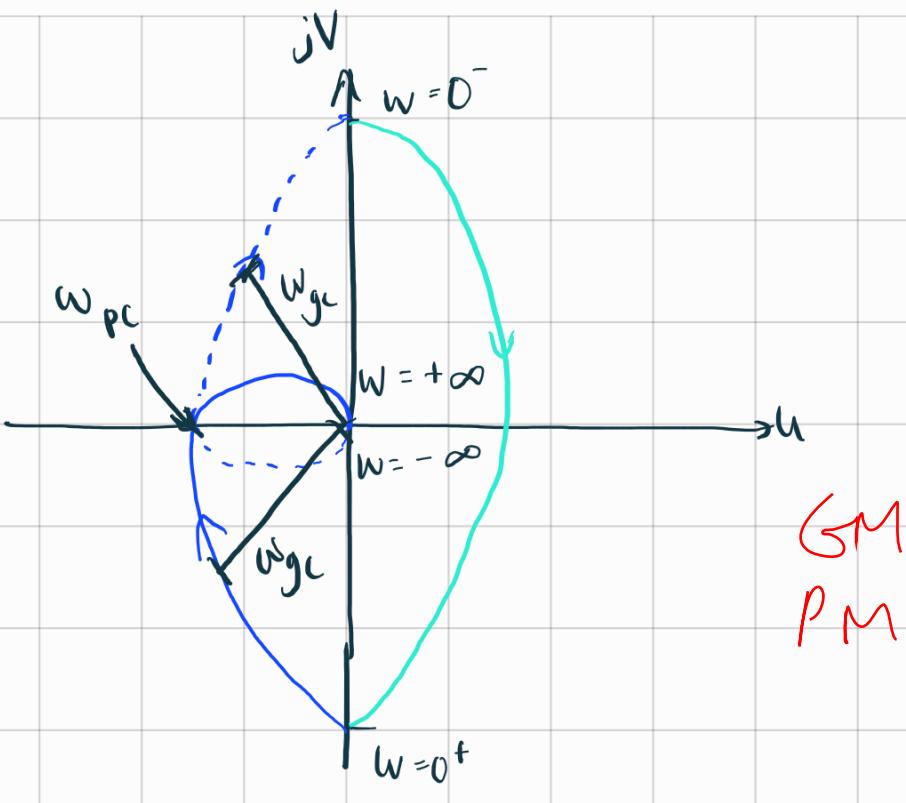
The frequency at which the Nyquist plot intersects the negative real axis (phase angle is 180°) is known as the **phase cross over frequency**. It is denoted by ω_{pc} .

$$M = 1 = 0 \text{ dB}$$



Gain Cross over Frequency

The frequency at which the Nyquist plot is having the magnitude of one is known as the **gain cross over frequency**. It is denoted by ω_{gc} .



Gain Margin

The gain margin GM is equal to the reciprocal of the magnitude of the Nyquist plot at the phase cross over frequency.

$$GM = \frac{1}{M_{pc}}$$

find ω_{pc} by finding arms intersection:

$$@ \omega_{pc} \angle G(j\omega) = -\frac{\pi}{2} - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{8}\right) = -\pi$$

try 2 and 4 for ω , $\omega = 4$.

$$\frac{1}{GM} = \frac{1}{4j(4j+2)(4j+8)} = \frac{-1}{160}, GM = -160$$

$$GM \text{ in dB} = -20 \log_{10} (-160) = -44.08 \text{ dB}$$

Phase Margin

The phase margin PM is equal to the sum of 180° and the phase angle at the gain cross over frequency.

$$PM = 180^\circ + \phi_{gc}$$

Where, ϕ_{gc} is the phase angle at the gain cross over frequency.

$$|G(j\omega)| = \sqrt{k} \frac{1}{\omega^2(\omega^2+4)(\omega^2+64)} = 1$$

$$\therefore \omega^6 + 68\omega^4 + 256\omega^2 - 1 = 0$$

$$\text{let } x = \omega^2.$$

$$x^3 + 68x^2 + 256x - 1 = 0$$

$$x = -64, x = 0.0039, x = -4.004$$

$$\omega^2 = -64, \omega^2 = 0.0039, \omega^2 = -4.004$$

$$\omega = \pm 8j, \underline{\omega = \pm 0.0062}, \omega = \pm 2j$$

$$\therefore @ \omega = 0.0062, \angle G(j\omega) = -\frac{\pi}{2} - \tan^{-1}\left(\frac{0.0062}{2}\right) - \tan^{-1}\left(\frac{0.062}{8}\right)$$

$$\phi_{gc} = -92.2^\circ$$

$$PM = -180^\circ - (-92.2^\circ) = -87.78^\circ$$

(b)

Determine the maximum allowable gain, k, and frequency, ω (rad s⁻¹), for the system to remain stable.

(7 Marks)

$$G_1(s) = \frac{k}{s(s+2)(s+8)}$$

OL zero : N/A

from gain margin:

max f, $\omega = 4$.

OL poles : $s = -8, -2, 0$

$$\text{rank } k = 3 - 0 = 3$$

$$G_{II}(j\omega) = \frac{k}{(j\omega)(j\omega+2)(j\omega+8)} = \frac{k}{-j\omega^3 - 10\omega^2 + 16j\omega}$$

\therefore system is stable as long as the contour didn't encircle point $(-1, 0)$ according to nyquist stability criterion.

real part

criterion.

$$\frac{k}{-j(4)^3 - 10(4)^2 + 16j(4)} = -1$$

$$\therefore k = -1 \times -160$$

imaginary part

$$k = 160.$$

is removed as k is real only.

✓✓✓

(c)

Sketch the locus of the closed loop system if the open loop transfer function also includes a pure time delay of 10 ms.

(5 Marks)

Time delay in terms of $j\omega$: $e^{-0.01s}$

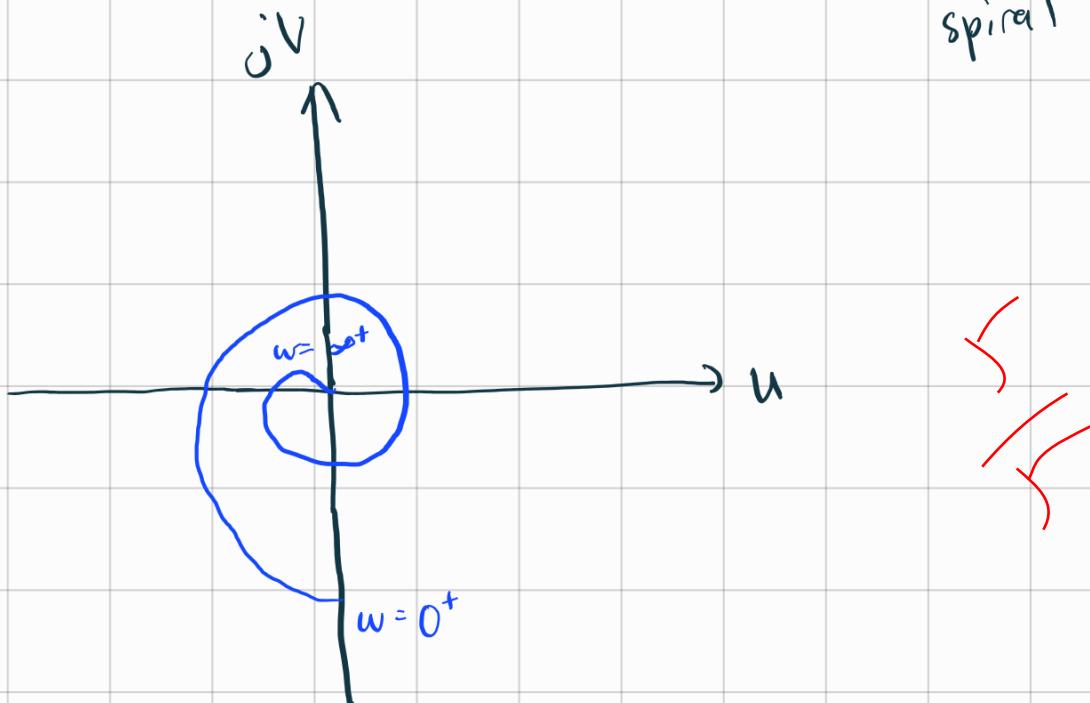
$$GH(s) = \frac{ke^{-0.01s}}{s(st^2)(s+8)}$$

$$GHI(j\omega) = \frac{ke^{-0.01j\omega}}{\omega \sqrt{(\omega^2+4)(\omega^2+64)}} \left[-\frac{\pi}{2} - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{8}\right) + 0.01\omega \right]$$

$$\lim_{\omega \rightarrow 0} = |0| \left[-\frac{\pi}{2} - 0 - 0 - 0 \right] = |0| \left[-\frac{\pi}{2} \right]$$

$$\lim_{\omega \rightarrow \infty} = |0| \left[-\frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} - \infty \right] = |0| \left[-\frac{3\pi}{2} - \infty \right]$$

↑ causes spiral



(d)

Show that the phase of the time delayed closed loop system is 180° when

$$\frac{\pi}{2} - 0.01\omega = \tan^{-1} \left(\frac{10\omega}{16-\omega^2} \right)$$

Given

$$\tan^{-1} x \pm \tan^{-1} y = \tan^{-1} \left(\frac{x \pm y}{1 \mp xy} \right) \quad (2)$$

(5 Marks)

$$\text{phase of } G(j\omega), \angle = \boxed{-\frac{\pi}{2} - \tan^{-1}\left(\frac{w}{2}\right) - \tan^{-1}\left(\frac{w}{8}\right) - 0.01w}$$

using (2) :

$$\therefore \angle = \left[-\frac{\pi}{2} - 0.01w - \left(\tan^{-1}\left(\frac{w}{2}\right) + \tan^{-1}\left(\frac{w}{8}\right) \right) \right]$$

$$= \left[-\frac{\pi}{2} - 0.01w - \tan^{-1} \left(\frac{\frac{w}{2} + \frac{w^2}{8}}{1 - \frac{w^2}{16}} \right) \right]$$

$$= \left| -\frac{\pi}{2} - 0.01w - \tan^{-1} \left(\frac{\frac{S_w}{8}}{\frac{16-w^2}{16}} \right) \right|$$

$$= \boxed{-\frac{\pi}{2} - 0.01w - \tan^{-1}\left(\frac{10w}{16-w^2}\right)}$$

using ①

$$\therefore L = \left| -\frac{\pi}{2} - 0.01w - \left(\frac{\pi}{2} - 0.01w \right) \right|$$

$$= \left\{ -\frac{\pi}{2} - \frac{\pi}{2} - 0.01w + 0.01w \right\}$$

$$= \underline{-\pi} \neq \underline{\pi} = \underline{180^\circ} \quad \times$$

1

23

44
—
50