ELEC 3224 — Robust Control

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Introduction

- The designs so far have assumed that **an exact and linear model of the dynamics to be controlled is available.**
- If the model used for design is **'not accurate'** then controller design may perform **'poorly'** in implementation.
- We have already seen the use of gain and phase margins and the stability margin and these are basic counters against poor/unmodeled dynamics.
- In the more general case **a robust control design is needed.**
- This general area has been the subject of intense research effort since the mid 1970's (at least).

Introduction

- The result of this effort has been a number of ‘general’ design methods and supporting software.
- Some of these designs have led to at least **laboratory-based experimental validation.**
- **What are the sources of this general problem?**
- One cause is **parametric uncertainty** where the parameters describing the system are unknown.
- **Example:** Variation of the mass of a vehicle — **changes with the number of passengers and the weight of the contents.**

Introduction

- When **linearizing a nonlinear system** the parameters of the linearised model **also depend on the operating conditions.**
- **Example:** A spring-mass-damper system. If the spring obeys **Hooke's law** (force is proportional to displacement) and the damper is linear the for input u and output x and assuming unit mass (for simplicity), a model is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = u$$

- where $0 < \zeta < 1$ is the **damping ratio** and ω_n is the **undamped natural frequency.**

Introduction

- Or in transfer-function terms

$$X(s) = G(s)U(s), \;\; G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- In application, **either the spring or the damper may deviate from the linear operating condition.**
- For example, the spring no longer obeys Hookes law and instead 'goes brittle' — the so-called hardened spring!
- As another example, the damping element may no longer provide a restoring force that is proportional to $\dot{x}$.

Introduction

- In the former case a polynomial representation is one model for the nonlinear spring, resulting in the system description

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x + k_1 x^n = u$$

- where $k_1 > 0$ is a constant.
- This is a **nonlinear model** and for **small values** of x $k_1 x^n$ can be neglected.
- For large x this is not realistic.
- Even if $k_1 = 0$ then the values of ζ and/or ω_n may not be accurately known. **This is parametric uncertainty.**

Introduction

- Unmodeled dynamics can be accounted for by developing a more complex model.
- Such models are commonly used for controller development, but substantial effort is required to develop them.
- An alternative is to investigate if the closed-loop system is sensitive to **generic forms of unmodeled dynamics.**
- The basic idea is to describe the unmodeled dynamics by including a transfer-function in the system description whose frequency response is bounded but otherwise unspecified.

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- Basic idea is to explore if additional linear dynamics may cause difficulties.
- A simple way is to assume that the transfer-function of the process is

$$P(s) + \Delta$$

- where $P(s)$ is the **nominal simplified transfer-function** and Δ represents the **unmodeled dynamics in terms of additive uncertainty.**
- Different representations of uncertainty are shown in the next figure.

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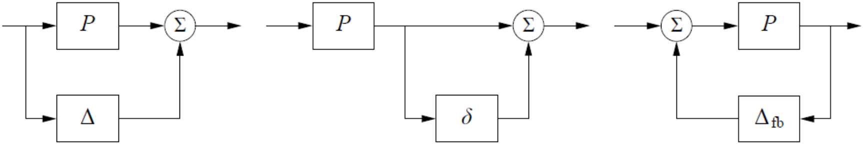

Figure 1

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- In this figure P in each case is the **nominal model.**
- Left-hand case – **additive uncertainty** Δ.
- Middle case — **multiplicative uncertainty** $\delta = \frac{\Delta}{P}$.
- Right-hand case — **feedback uncertainty** Δ_{fb}.
- Δ, δ and Δ_{fb}, respectively, represent **unmodeled dynamics.**

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- **Stability in the presence of uncertainty.**
- The Nyquist criterion provides a powerful and elegant way to study the effects of uncertainty for linear systems.
- A simple criterion is that the Nyquist curve is 'sufficiently far' from the critical point -1.
- The shortest distance from the Nyquist curve to the critical point is $s_m = \frac{1}{M_s}$, where M_s is the maximum of the sensitivity function and s_m is the stability margin from earlier.
- The maximum sensitivity M_s or the stability margin s_m is therefore a good robustness measure — see next figure.

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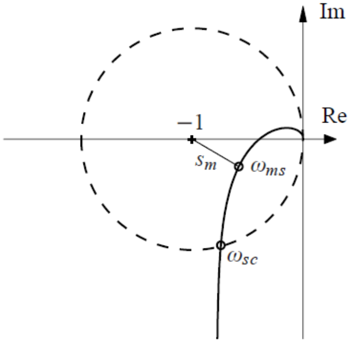
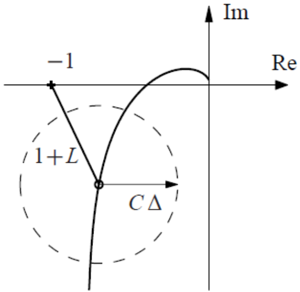


(a) Nyquist plot



(b) Additive uncertainty

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- To obtain explicit conditions for permissible process uncertainties, consider a stable feedback system with system P and controller C.
- If P is replaced by $P + \Delta$, the loop transfer-function L changes from PC to $PC + C\Delta$ — see the right-hand plot in the last figure.
- If a bound on the size of Δ exists, represented by the dashed circle in the figure, then the system remains stable provided the variations never overlap the -1 point, since this leaves the number of encirclements of -1 unchanged.

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- Some additional assumptions are required for the analysis to hold.
- Most importantly, the perturbation Δ be **stable** and hence no extra right half-plane poles are introduced that would require additional encirclements in the Nyquist criterion.
- An analytical bound on the allowable disturbances can now be developed.
- The distance from the critical point -1 to L is $|1 + L|$.

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- The perturbed Nyquist curve will not reach the critical point -1 point provided

$$|C\Delta| < |1 + L|$$

or

$$|\Delta| < |\frac{1 + PC}{C}|$$

- or

$$|\delta| = |\frac{\Delta}{P}| < \frac{1}{|T|}$$

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- This **condition must be valid for all points on the Nyquist curve, i.e., pointwise for all frequencies.**
- The condition for **robust stability** can therefore be written as

$$|\delta(j\omega)| = |\frac{\Delta(j\omega)}{P(j\omega)}| < \frac{1}{|T(j\omega)|}, \ \text{for all} \ \omega \geq 0 \qquad (1)$$

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- This **last condition is conservative** because it follows from the last figure that the critical perturbation is in the direction toward the critical point -1.
- **Larger perturbations can be permitted in the other directions.**
- This last condition enables **reasoning about the effects of uncertainty without exact knowledge of the perturbations.**
- It allows verification of stability for **any uncertainty** Δ that satisfies a given bound.

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- From an analysis perspective, this gives us a measure of the robustness for a given design. Conversely, if we require robustness of a given level, we can attempt to choose our controller C such that the desired level of robustness is available (by asking that T is small) in the appropriate frequency bands.

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- The mathematical models used to design control systems are often simplified, and the properties of a system may change during operation.
- This equation **implies that the closed-loop system will at least be stable for substantial variations in the system dynamics.**
- This equation also shows **that the variations can be 'large' for frequencies where T is 'small' and 'smaller' variations are allowed for frequencies where T is 'large'.**

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- A **conservative estimate of permissible system variations that will not cause instability** is

$$|\delta(j\omega)| = |\frac{\Delta(j\omega)}{P(j\omega)}| < \frac{1}{M_t} \tag{2}$$

- where M_t is the largest value of the complementary sensitivity function

$$M_t = \sup_{\omega} |T(j\omega)| = ||\frac{PC}{1+PC}||_\infty$$

- This condition is **conservative** because it follows from Figure 1 that the critical perturbation is in the direction toward the critical point -1. Larger perturbations can be permitted in the other directions.

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- Equation (2) enables reasoning about uncertainty **without exact knowledge** of the process perturbations.
- Stability can be verified for any uncertaintyΔ that satisfies the given bound.
- For analysis perspective, gives a measure of the robustness for a given design.
- Conversely, if robustness of a given level is required, the problem is to choose the controller C such that the desired level of robustness is available (making (if possible) T small) in the appropriate frequency bands.

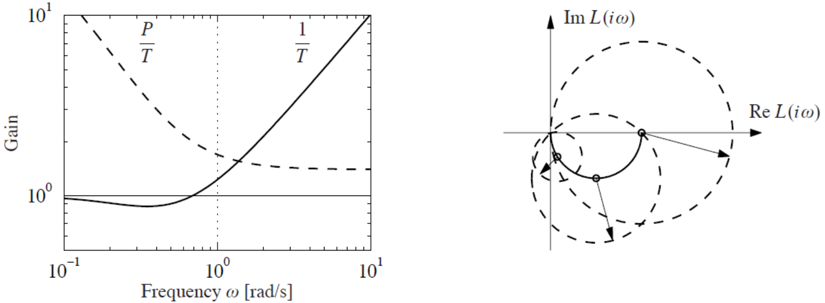
Robust Control — Example

- $$P(s) = \frac{1.38}{s + 0.0142}$$
- PI controller with $k_p = 0.2$ and $k_i = 0.18$.
- Next figure plots the allowable size of the uncertainty using the bound given by (2).
- At low frequencies $T(0) = 1$ and hence the perturbations can be as large ($|\delta| = |\frac{\Delta}{P}| < 1$).
- The complementary sensitivity has maximum value

$$M_t = 1.14$$

at $\omega_{mt} = 0.35$ rad/sec.

Robust Control — Example



Robust Control — Example

- Maximum allowable uncertainty $|\delta| < 0.87$ or $|\Delta| < 3.47$.
- At high frequencies $T \to 0$ and hence the relative error can be very large.
- As one case, at $\omega = 5$ rad/sec, $|T(j\omega)| = 0.195$ and hence the stability requirement is $|\delta| < 5.1$.
- The analysis clearly indicates that the system has good robustness and that the high-frequency properties of the system are not important for the design.
- The controller can tolerate large amounts of uncertainty and still maintain stability of the closed-loop.

Robust Control — Example

- The previous example is typical of many systems: moderately small uncertainties are required only around the gain crossover frequencies, but large uncertainties can be permitted at higher and lower frequencies.
- A consequence of this is that a simple model that describes the system dynamics well around the crossover frequency is often sufficient for design.
- Systems with many resonant peaks are an exception to this rule because the transfer-function for such systems may have large gains for higher frequencies — a example is given later.

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- The robustness condition (2) can be given **another interpretation by using the small gain theorem.**
- To apply the theorem, start with block diagrams of a closed-loop system with a perturbed systems and make a sequence of transformations of the block diagram that isolate the block representing the uncertainty — see the next figure.
- The result is a two block interconnection with loop transfer-function

$$L = \frac{PC}{1+PC}\frac{\Delta}{P} = T\delta$$

- The small gain theorem can be used to check robust stability for uncertainty in a variety of cases.

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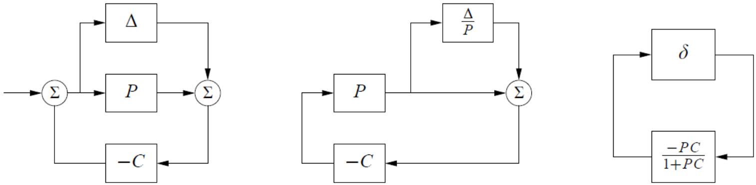

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- **Additive Uncertainty:** robust stability when

$$||CS\Delta||_\infty < 1$$

- **Multiplicative Uncertainty:** robust stability when

$$||T\delta||_\infty < 1$$

- **Feedback Uncertainty:** robust stability when

$$||PS\Delta_{fb}||_\infty < 1$$

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- **Disturbance attenuation — robust control case.**
- The sensitivity function S gives a rough characterization of the effect of feedback on disturbances.
- A more detailed characterization is given by the transfer-function from load disturbances to process output:

$$G_{yd} = \frac{P}{1 + PC} = PS$$

- Load disturbances typically have low frequencies, and it is therefore important that the transfer-function is small for low frequencies. For systems with constant low-frequency gain and a controller with integral action

$$G_{yd} \approx \frac{s}{k_i}$$

Robust Control

- The integral gain k_i is therefore a simple measure of the attenuation of load disturbances.
- To determine how the transfer-function G_{yd} is influenced by small variations in the system transfer-function, differentiating the last expression with respect to P gives

$$\frac{dG_{yd}}{dP} = \frac{1}{(1+PC)^2} = \frac{SP}{P(1+PC)} = S\frac{G_{yd}}{P}$$

- Hence

$$\frac{dG_{yd}}{G_{yd}} = S\frac{dP}{P}$$

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- The response to load disturbances is therefore insensitive to systems variations for frequencies where $|S(j\omega)|$ is small, i.e., for frequencies where load disturbances are important.
- A drawback with feedback is that the controller feeds measurement noise into the system. In addition to the load disturbance rejection, it is thus also important that the control actions generated by measurement noise are not too large.
- The transfer-function from measurement noise to output is

$$G_{un} = \frac{C}{1 + PC} = -\frac{T}{P}$$

Robust Control

- Measurement noise is high frequency.
- Hence G_{un} should not be too large for high frequencies.
- PC is typically small for high frequencies and hence for large s

$$G_{un} \approx C$$

- Hence it is important that $C(s)$ is small for large s — known as **high-frequency roll-off.**
- $$\frac{G_{un}}{dP} = \frac{d}{dP}\left(-\frac{C}{1+PC}\right) = \frac{C}{(1+PC)^2}C = T\frac{G_{un}}{P}$$

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- Hence

$$\frac{dG_{un}}{G_{un}} = T\frac{dP}{P}$$

- Since the complementary sensitivity function T is also small for high frequencies, the system uncertainty has little influence on the transfer-function G_{un} for frequencies where measurements are important.
- **Reference Signal Tracking**
- The transfer-function from reference to output is

$$G_{yr} = \frac{PCF}{1+PC} = TF$$

Robust Control

- To determine how variations in P affects system performance, differentiating with respect to P gives

$$\begin{aligned}\frac{dG_{yr}}{dP} &= \frac{CF}{1+PC} - \frac{PCFC}{(1+PC)^2}\\ &= \frac{CF}{(1+PC)^2} = S\frac{G_{yr}}{P}\end{aligned}$$

- Hence

$$\frac{dG_{yr}}{G_{yr}} = S\frac{dP}{P}$$

Robust Control

- Hence the relative error in the closed-loop transfer function is small when the sensitivity is small.
- This is one of the useful properties of feedback.
- Return to the Op-Amp example — see the next figure.
- This is an Operational amplifier with uncertain dynamics.
- The circuit on the left is modeled using the transfer-function $G(s)$ to capture its dynamic properties and it has a load at the output.
- The block diagram on the right shows the input/output relationships. The load is represented as a disturbance d applied at the output of $G(s)$.

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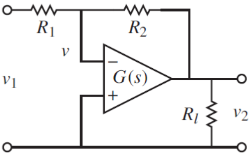

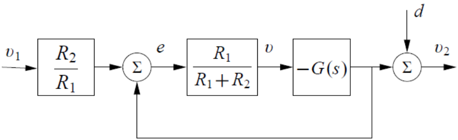

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- Set $v_2 = -G(s)v$

$$G_{v_1 v_2} = -\frac{R_2}{R_1}\frac{G(s)}{G(s) + \frac{R_2}{R_1} + 1}$$

- Hence if $G(s)$ is large over the desired frequency range then the closed-loop system is very close to the ideal response $\frac{R_2}{R_1}$.

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- Suppose that

$$G(s) = \frac{b}{s+a}$$

- Then

$$S = \frac{s+a}{s+a+ab}, \quad T = \frac{ab}{s+a+ab}$$

- In this case, the sensitivity function around the nominal values gives that the tracking response varies as a function of the system perturbations

$$\frac{dG_{yr}}{G_{yr}} = S\frac{dP}{P}$$

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- For low frequencies, where S is small, variations in the bandwidth a or the gain-bandwidth product b has relatively little effect on the performance of the amplifier under the assumption that b is sufficiently large.
- The disturbance d represents changes in the output voltage due to loading effects.
- The transfer-function $G_{yd} = S$ gives the response of the output to d.
- Also

$$\frac{dG_{yd}}{dP} = -\frac{C}{(1+PC)^2} = -\frac{T}{P}G_{yd}$$

Robust Control

- Hence

$$\frac{dG_{yd}}{G_{yd}} = -T\frac{dP}{P}$$

- Consequently, relative changes in the disturbance rejection are roughly the same as the system perturbations at low frequency when $T \approx 1$ and drop off at higher frequencies.
- However, G_{yd} itself is small at low frequency and so these variations in relative performance may not be an issue in many applications.