

# ELEC 3224 — Guidance Laws

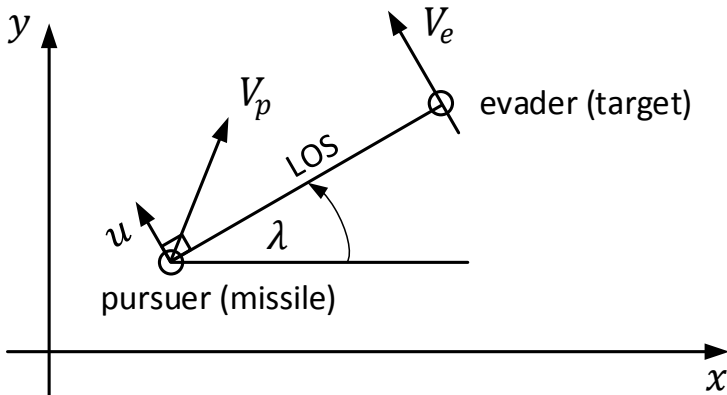
Professor Eric Rogers

School of Electronics and Computer Science  
University of Southampton  
etar@ecs.soton.ac.uk  
Office: Building 1, Room 2037

# Proportional Navigation (PN)

- ▶ In a PN guidance law the acceleration command is **perpendicular** to the **instantaneous line-of-sight (LOS)** and **proportional to the LOS rate** and **closing velocity**.
- ▶ The next figure shows the 2D geometry.
- ▶ LOS rate  $\dot{\lambda}$  is measured by the seeker (pursuer) and the closing velocity is 'guestimated'.
- ▶ How does it work?

## 2D Geometry



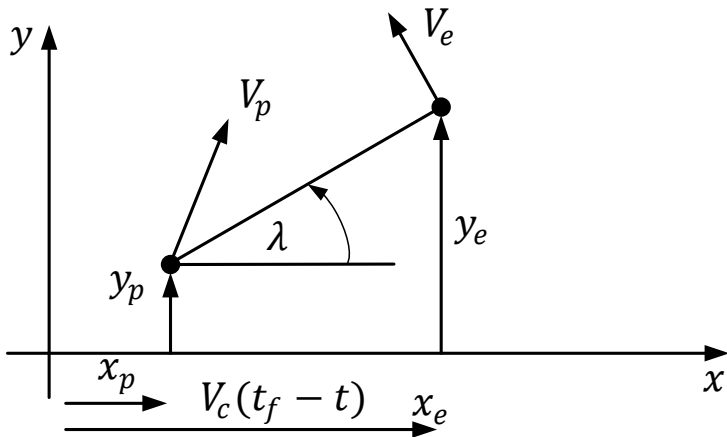
# Proportional Navigation (PN)

- ▶ If  $V_e$  and  $V_p$  are constant and evader and pursuer are non-maneuvering,  $\dot{\lambda} = 0$  and a collision course is ensured.
- ▶ In the next figure,  $V_c$  is the closing velocity in the x-direction,  $V_c(t_f - t)$  is the range to go.
- ▶ Also

$$y = y_e - y_p$$

- ▶  $t_f - t$  is the **time to intercept**.

# Linearised Geometry



# Proportional Navigation (PN)

- ▶ **Control law:** lateral acceleration demand (LATAX) is equal to

$$K_N V_c \dot{\lambda}$$

$K_N$  is a constant and is often termed the PN constant.



$$y = y_e - y_p$$

at  $t = t_f$  is the **terminal miss distance**.

# Proportional Navigation (PN)



$$\dot{y} = \dot{y}_e - \dot{y}_p$$

$$\ddot{y} = \ddot{y}_e - \ddot{y}_p$$

- ▶ The next step is to include the pursuer and evader dynamics.
- ▶ Simple form of pursuer dynamics is a first order lag, i.e., a transfer-function of the form

$$\frac{1}{1 + Ts}$$

(a first order lag). If  $t = 0$  then this is an instantaneous response.

- ▶ Equation of motion

$$\ddot{y} = \ddot{y}_e - \ddot{y}_p = \ddot{y}_e - K_N V_c \dot{\lambda}$$

# Proportional Navigation (PN)

- ▶ Particular case — non manoeuvring target, i.e.,  $\ddot{y}_e = 0$ .
- ▶ In this case,

$$\ddot{y} + K_N V_c \dot{\lambda} = 0$$

- ▶ integrating from 0 to  $t$  now gives

$$\dot{y}(t) - \dot{y}(0) + K_N V_c (\lambda(t) - \lambda(0)) = 0$$

- ▶ or

$$\dot{y}(t) + K_N V_c \lambda(t) = \dot{y}(0) + K_N V_c \lambda(0)$$

# Proportional Navigation (PN)

- ▶ **Fact: the right-hand side of this last differential equation is a constant (denoted by  $C$ ).**
- ▶ Need the solution of this last differential equation at  $t = t_f$ .
- ▶ It is possible to use

$$\lambda(t) \approx \frac{y(t)}{V_c(t_f - t)}$$

- ▶ Hence we need to solve

$$\dot{y} + \frac{K_N y}{t_f - t} = C$$

# Proportional Navigation (PN)

- ▶ **An Aside:** Integrating factor solution of an ordinary differential equation.

- ▶ Consider

$$\frac{dy}{dx} + p(x)y = q(x)$$

- ▶ Then

$$\frac{d}{dx} \left[ y e^{\int p(x) dx} \right] = q(x) e^{\int p(x) dx}$$

- ▶ Hence

$$y(x) e^{\int p(x) dx} = \int q(x) e^{\int p(x) dx} dx + \text{const}$$

# Proportional Navigation (PN)

- ▶ Integrating factor in the case considered is

$$e^{\int \frac{K_N dt}{(t_f - t)}} = e^{-K_N \ln(t_f - t)} = \frac{1}{(t_f - t)^{K_N}}$$

- ▶ Hence

$$\frac{d}{dt} \left[ \frac{y(t)}{(t_f - t)^{K_N}} \right] = \frac{C}{(t_f - t)^{K_N}}$$

# Proportional Navigation (PN)

- ▶ Moreover

$$\frac{y(t)}{(t_f - t)^{K_N}} = \frac{C}{(K_N - 1)(t_f - t)^{K_N - 1}} + C_2$$

- ▶ Hence

$$y(t) = \frac{C}{K_N - 1}(t_f - t) + C_2(t_f - t)^{K_N}$$

- ▶ Hence **when**  $K_N > 1$ ,

$$y(t_f) = 0$$

# Proportional Navigation (PN)

- ▶ Vehicle lateral acceleration

$$\ddot{y} = \ddot{y}_e - \ddot{y}_p$$

- ▶ Hence

$$\ddot{y}_p = -\ddot{y}$$

- ▶ and

$$|\ddot{y}_p| = C_2 K_N (K_N - 1) (t_f - t)^{K_N - 2}$$

- ▶ Next, consider  $|\ddot{y}_p(t)|$  against  $t$ .

# Proportional Navigation (PN)

- ▶ **Case 1:**  $K_N = 2$ .

- ▶ In this case

$$|\ddot{y}_p(t)| = 2C_2, \text{ for all } t$$

- ▶ **Case 2:**  $K_N = 3$ .

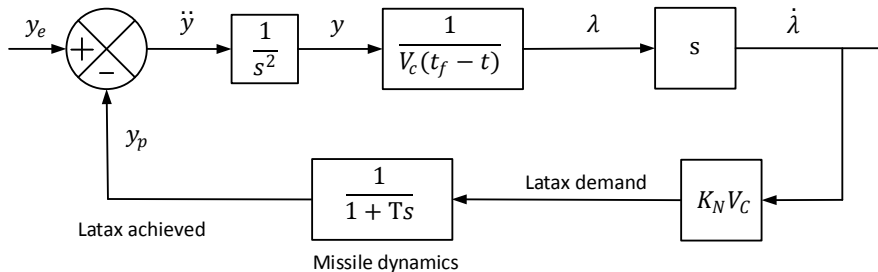
- ▶ In this case

$$|\ddot{y}_p(t)| = 6C_2(t_f - t)$$

decays linearly to zero at  $t = t_f$ .

- ▶ **Case 3:**  $K_N > 3$  – nonlinear relationship —  $K_N = 3$  is a good compromise!

# Block Diagram Interpretation



# Proportional Navigation (PN) – State-Space Model

- ▶ Consider the case when  $T = 0$ .
- ▶ State variables

$$\begin{aligned}\frac{d}{dt}y &= \dot{y} \\ \frac{d}{dt}\dot{y} &= \ddot{y}_e - K_N V_c \frac{d}{dt} \left( \frac{y}{V_c(t_f - t)} \right)\end{aligned}$$

- ▶ State dynamics

$$\dot{x} = A(t)x + bu$$



$$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u = \ddot{y}_e$$

# Proportional Navigation (PN) – State-Space Model



$$A(t) = \begin{bmatrix} 0 & 1 \\ -\frac{K_N}{(t_f - t)^2} & -\frac{K_N}{(t_f - t)} \end{bmatrix}$$

- ▶ This is a **time-varying state-space model**.
- ▶ **Consequence:** The block diagram (Laplace transform) analysis can only be applied by 'freezing'  $t$ .

# Proportional Navigation (PN) – State-Space Model

- ▶ Interpretation in terms of **Zero Effort Miss (ZEM)**.
- ▶ Consider the predicted miss distance assuming the pursuer does not manoeuvre.
- ▶ Then

$$\ddot{y} = \ddot{y}_e - \ddot{y}_p = 0 \quad (1)$$

- ▶ Suppose also that at some time  $t_0$ , say

$$y = y(t_0) \text{ \& } \dot{y} = \dot{y}(t_0)$$

- ▶ Then

$$\lambda(t) = \frac{y}{V_c(t_f - t)} \quad (2)$$

$$\dot{\lambda}(t) = \frac{\dot{y}}{V_c(t_f - t)} + \frac{y}{V_c(t_f - t)^2} \quad (3)$$

# Proportional Navigation (PN) – State-Space Model

- ▶ From (1),

$$\dot{y}(t) - \dot{y}(t_0) = 0$$

- ▶ Hence

$$y(t) - y(t_0) - \dot{y}(t_0)(t - t_0) = 0$$

- ▶ and therefore the ZEM is

$$y(t_f) = y(t_0) + \dot{y}(t_0)(t_f - t_0) \quad (4)$$

## Proportional Navigation (PN) – State-Space Model

- ▶ From (3) and (4)

$$\dot{\lambda}(t_0) = \frac{y(t_f)}{V_c(t - t_0)^2}$$

- ▶ Control demand at  $t = t_0$

$$K_N V_c \dot{\lambda}(t_0) = \frac{K_N y(t_f)}{(t_f - t_0)^2} = \frac{K_N ZEM}{(t - t_0)^2} \quad (5)$$

- ▶ The significance of this last equation is that this principle it can be used (intuitively) for more complex dynamics, i.e., compute the ZEM, estimate the time-to-go, and then the **control action demand is proportional to the square of the time-to-go.**

# Proportional Navigation (PN) – Pursuer Dynamics with $T \neq 0$

- ▶ Start from

$$\ddot{y} = \ddot{y}_e - \ddot{y}_p$$

- ▶ and

$$(1 + Ts)\ddot{y}_p = u = K_N V_c \dot{\lambda}$$

- ▶ State variables

$$\begin{aligned}\frac{dy}{dt} &= \dot{y} \\ \frac{d}{dt}\dot{y} &= \ddot{y}_e - \ddot{y}_p\end{aligned}$$

# Proportional Navigation (PN) – Pursuer Dynamics with $T \neq 0$

- ▶ Start from

$$\frac{d}{dt}\ddot{y}_p = -\frac{1}{T}\ddot{y}_p + \frac{K_N V_c}{T} \left[ \frac{\dot{y}}{V_c(t_f - t)} + \frac{y}{V_c(t_f - t)^2} \right]$$

# Proportional Navigation (PN) – Pursuer Dynamics with $T \neq 0$

- ▶ In state-space model form

$$\frac{d}{dt} \begin{bmatrix} y \\ \dot{y} \\ \ddot{y}_p \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ \frac{K_N}{T(t_f-t)^2} & \frac{K_N}{T(t_f-t)} & -\frac{1}{T} \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \ddot{y}_p \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \ddot{y}_e$$

- ▶ or

$$\dot{x} = A(t)x + ba_e$$

# Proportional Navigation (PN) – Pursuer Dynamics with $T \neq 0$

- ▶ Again a time-varying state equation.
- ▶ Formal solution

$$x(t) = \Phi(t, 0)x(0) + \int_0^t \Phi(t, t_1)ba_e(t_1)dt_1$$

- ▶  $\Phi$  is the **transition matrix** and in the **time-invariant case is the matrix exponential**.
- ▶ The interest is often in  $y(t_f)$ .

$$y(t_f) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t_f) = c^T x(t_f)$$

- ▶ Consider now (with no loss of generality) the case when  $x(0) = 0$ , and  $a_e \neq 0$ .

# Proportional Navigation (PN) – Pursuer Dynamics with $T \neq 0$

- ▶ Then

$$y(t) = \int_0^t c^T \Phi(t, t_1) b a_e(t_1) dt_1$$

- ▶ The right-hand side is the convolution of the system impulse response and the input.
- ▶ **Problem:** It would be **necessary to recompute the impulse response for each  $t_1$ .**

# Proportional Navigation (PN) – Pursuer Dynamics with $T \neq 0$

- ▶ This problem can be avoided by using the **adjoint** system.
- ▶ Let  $x(0) = 0$ , then if  $x(t_f)$  is found,  $y(t_f)$  follows.
- ▶ Now

$$x(t_f) = \int_0^{t_f} \Phi(t_f, t_1) bu(t_1) dt_1$$

- ▶ Now introduce the new system (the adjoint)

$$\dot{p} = -A^T p$$

# Proportional Navigation (PN) – Pursuer Dynamics with $T \neq 0$

- ▶ Now introduce

$$v = x^T p$$

- ▶ Then

$$\dot{v} = \dot{x}^T p + x^T \dot{p}$$

- ▶ and hence

$$\dot{v} = (Ax + bu)^T p + x^T (-A^T p)$$

- ▶ Then

$$\dot{v} = b^T p u$$

## Proportional Navigation (PN) – Pursuer Dynamics with $T \neq 0$

- ▶ This last equation is a **scalar**.
- ▶ integrating this equation gives

$$v(t) - v(0) = \int_0^t b^T p(t_1) u(t_1) dt_1$$

and  $v(0) = 0$ .

- ▶ Let

$$p(t_f) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$$

- ▶ Then

$$x_1(t_f) = \int_0^{t_f} b^T p(t_1) u(t_1) dt_1$$

# Proportional Navigation (PN) – Pursuer Dynamics with $T \neq 0$

- ▶ to obtain the miss distance, solve

$$\dot{p} = -A^T p$$

**backward (or in reverse) time once.**

- ▶ This is more efficient than the **forward time solution**.
- ▶ **Aside:** The adjoint system plays a critical role in optimal control problems!!