Chapter 6

Distributed Circuit Coefficients and Physical Design

6.1. Introduction.

Chapters 2 through 5 have developed transmission line analysis as an investigation of the propagation of voltage and current waves on uniform transmission lines defined by uniformly distributed electric circuit coefficients. Chapter 7 and subsequent chapters continue the same analysis. The present chapter, dealing with the derivation of expressions relating the distributed circuit coefficients of a uniform line to its dimensions and materials, involves topics and methods that are not directly part of this mainstream of transmission line theory.

Books on introductory circuit analysis, using the circuit element concepts of lumped resistance, inductance and capacitance, almost invariably omit any reference to the physical nature or construction of the units embodying these circuit properties. It is assumed that information is available in other sources on how to calculate the equivalent circuit of a specific real object, or how to design an assemblage of metals and dielectrics and ferromagnetic substances that will provide a circuit element meeting a desired specification. Since resistance, inductance and capacitance are in effect shorthand notations for relations between currents, charges and electromagnetic fields in bounded physical structures, the creation of formulas for the circuit representation of such structures is undertaken, in various degrees, by textbooks on electricity and magnetism or electromagnetic theory.

It is equally true that many books on electromagnetic theory develop equations for some or all of the distributed circuit coefficients for at least the simpler configurations of uniform transmission lines, and this could be used as a justification for omitting all such information from a transmission line textbook. There is, however, a fundamental difference of intent between the study of elementary circuit analysis and the study of transmission line engineering. The former seeks to convey a working knowledge of a few abstract relations between currents, voltages and circuit elements, with no thought of the specific situations in which these may occur. The latter, on the other hand, has an inherent concern to maintain a contact with physical reality, and to illustrate its theoretical analyses in terms of actual lines used for the transmission of signals and power. To achieve this purpose, it is essential that a transmission line textbook present a full discussion of the ways in which the distributed circuit coefficients of a line are dependent on its geometry and materials. Most of the standard textbooks on the subject have accepted this obligation.

The solution of boundary value problems in electromagnetic theory has been the business of mathematical physicists for a century. So far as the calculation of resistance, inductance and capacitance for either lumped or distributed devices is concerned, there now exists a voluminous amount of data on a wide variety of structures, but the mathematical form of the results is reasonably elementary only for constant unidirectional currents or voltages, and for structures with the simplest of geometries, such as concentric spheres, concentric circular cylinders, or infinite parallel planes. When the frequency is other than zero, even for these idealized geometries, calculations of resistance and reactance in some frequency ranges require uncommon mathematical functions for their expression, with the consequence that approximate formulas and graphical representations are widely used. Slightly less simple symmetries, such as parallel or concentric square or rectangular conductors, pose very difficult mathematical problems. Computer methods are needed to obtain adequate solutions, and the results are given in tables or charts.

It is an exceedingly fortunate fact that the transmission line constructions which are found to be experimentally optimal, in the sense of making most effective use of materials by providing minimum attenuation or maximum power handling capacity at the lowest cost and in the least space, involve only the simplest possible geometries. These are the lines whose cross sections are illustrated in Fig. 2-2, page 9. No important advantages, on either technical or economic grounds, have ever been claimed for lines with more unusual cross sections.

6.2. Distributed resistance and internal inductance of solid circular conductors.

By far the most widely used transmission line conductors are solid homogeneous wires of circular cross section. They are used as the center conductors of coaxial lines, the conductors of parallel wire or shielded pair or multi-conductor lines, and as the single conductor of image lines. Next in importance are tubular conductors of circular periphery, which are used in all of the above applications, and also as the outer conductor of coaxial lines and as the shield of shielded pair lines. The analysis that follows shows that an exact solution in functional form can be found for the distributed resistance and distributed internal inductance of homogeneous isotropic circular conductors, both solid and tubular, for all frequencies at which such conductors are used in transmission lines. Such exact solutions are not possible for the "stripline" constructions shown in Fig. 2-2, nor for any other practical transmission line cross section involving finite widths of plane surfaces.

If a wire of circular cross section has radius a meters, and is made of homogeneous isotropic material of conductivity σ mhos/m, its resistance per unit length at zero frequency (i.e. its d-c distributed resistance) is given by

$$R_{\text{d-c}} = \frac{1}{\sigma\pi a^2}\,\text{ohms/m} \tag{6.1}$$

For the same isolated wire, considered indefinitely long, the inductance per unit length derived as the total magnetic flux linking unit length of the wire when the wire carries unit current is found to be infinite. It does not follow from this that the distributed circuit coefficient L for a transmission line is infinite. The conductors of a transmission line are not infinitely long isolated wires, and one of them always prescribes a finite limit for the integral determining the magnetic flux linking the other.

The distributed inductance of any conductor, whether isolated or not, consists of two parts. One part is caused by flux-current linkages inside the conductor itself, the other by linkages of the total conductor current with flux external to the conductor. These will be designated as L_i and L_x, respectively, in units of henries/m. For a solid circular conductor of radius a carrying current I of zero frequency, an expression for the internal inductance $L_{i\text{d-c}}$ is found as follows, referring to Fig. 6-1 below.

The current in an infinitesimal tube of radius r_1 and thickness dr_1 in the conductor's cross section is $(I/\pi a^2)(2\pi r_1\,dr_1)$. Referring to the definition that inductance is the flux linking a "circuit" per unit current in the circuit, this tube of radius r_1 and thickness dr_1 evidently constitutes a fraction $(2\pi r_1\,dr_1)/(\pi a^2)$ of the conductor as a "circuit".

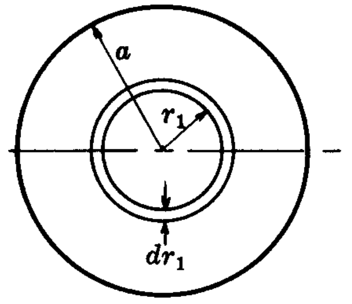

Fig. 6-1. Fractional "circuit" within a solid circular conductor carrying a d-c current.

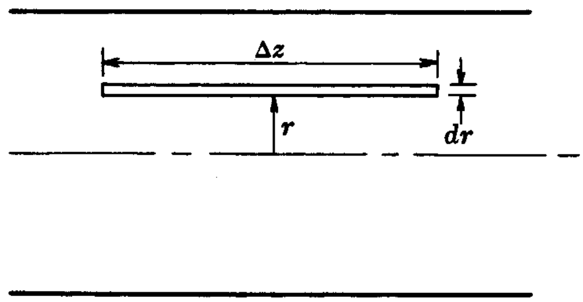

Fig. 6-2. Longitudinal cross section of solid circular conductor in diametral plane.

This fractional circuit is linked by all the magnetic flux inside the conductor between the radii r_1 and a (the flux lines being circles concentric with the conductor). At any radius r in this interval the magnetic flux density $B(r)$ is given by

$$B(r) \quad = \quad \frac{\mu I}{2\pi r}\left(\frac{\pi r^2}{\pi a^2}\right) \quad \text{teslas} \tag{6.2}$$

where μ is the mks permeability of the conductor material. Thus the contribution to the distributed internal inductance $L_{i\,\text{d-c}}$ of the fractional circuit consisting of the tube of thickness dr_1 at radius r_1 is

$$dL_{i\,\text{d-c}} \quad = \quad \frac{1}{I}\,\frac{2\pi r_1\,dr_1}{\pi a^2}\int_{r_1}^{a} dr \int_0^1 dz\,\frac{\mu I}{2\pi r}\,\frac{r^2}{a^2} \quad \text{henries/m} \tag{6.3}$$

where z is the coordinate in the direction of the length of the line, and r_1 is a constant during the integration with respect to r. The result is

$$dL_{i\,\text{d-c}} \quad = \quad \mu r_1\,dr_1\,(a^2 - r_1^2) \text{ henries/m} \tag{6.4}$$

The total internal inductance at zero frequency is obtained by integrating this with respect to r_1 from 0 to a, with the result

$$L_{i\,\text{d-c}} \quad = \quad \frac{\mu}{8\pi} \text{ henries/m} \tag{6.5}$$

Thus the internal inductance of a solid circular conductor, when the current is uniformly distributed over the conductor's cross section, is independent of the radius of the conductor. Calculations later in this chapter show that the internal inductance of the conductors of a transmission line may constitute 10% or even more of the line's total distributed inductance at low frequencies. At much higher frequencies the distributed internal inductance of a circular conductor becomes very small compared to its d-c value, and the distributed reactance of its internal inductance asymptotically approaches being identically equal to the frequency-dependent high frequency distributed resistance of the conductor.

Inspection of equation (*6.4*) indicates that for tubes of constant annular cross-sectional area, i.e. $2\pi r_1\,dr_1 =$ constant, the distributed internal inductance is greatest for small values of r_1 and approaches zero as r_1 approaches a. This means that at a given frequency the distributed internal reactance of a small circular area at the center of a circular conductor is much greater than the distributed reactance of the same area of conductor near the periphery. If an a-c voltage exists between the ends of a section of the conductor, less current will flow in the high reactance region at the center of the conductor than in an equal area of cross section at a greater radius. The current distribution will then no longer be one of constant density as was the case at zero frequency. The effect becomes more pronounced the higher the frequency, until at sufficiently high frequencies the current flows only in a very thin *skin* at the conductor's surface. Known as *skin effect*, this phenomenon causes the resistance of conductors of any shape or material to increase markedly and continuously with frequency for frequencies above some minimum value that depends on the conductor's size, permeability and conductivity, while at the same time the internal inductance decreases continuously.

A quantitative analysis for skin effect in a homogeneous isotropic circular conductor is obtained by applying Faraday's law to a rectangular path in a radial plane of the conductor as illustrated in Fig. 6-2 above. The radius of the conductor is a. The rectangle has length Δz in the coordinate direction z parallel to the length of the conductor, and infinitesimal width dr in the radial direction. It is located at distance r from the center of the conductor.

A postulate of this analysis is that an external source is causing current to flow in the conductor in the z direction, and that the resulting current density J_z at any point in the conductor's cross section is in general a function of r, but for symmetry reasons is not a function of angular position around the center of the conductor. The purpose of the analysis is to find the manner in which $J_z(r)$ varies with r, and from this result to find the effective resistance and internal inductance of the conductor per unit length, as a function of frequency and conductor material.

At any radius r in the conductor there will be an electric field $E_z(r)$ associated with the total current density $J_z(r)$ according to the time-harmonic electromagnetic relation

$$J_z \;=\; \sigma E_z + j\omega\epsilon E_z \tag{6.6}$$

where σ is the conductivity of the conductor and ϵ its permittivity. For the metals used in transmission line conductors, information about the value of the permittivity ϵ is nebulous, but there is no reason to believe that it differs appreciably from the value for free space, $\epsilon_0 = 8.85 \times 10^{-12}$ farads/m. Since the conductivity of the metals is 10^7 mhos/m or higher, it is readily seen that at all conceivable transmission line frequencies equation (6.6) becomes

$$J_z(r) \;=\; \sigma E_z(r) \tag{6.7}$$

which means that $J_z(r)$ is entirely conduction current density.

Voltage will be induced in the rectangle of Fig. 6-2 because of the time-changing magnetic flux through it. This flux is produced by all the conductor current inside radius r. Faraday's law $\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{\partial}{\partial t}\int_S \mathbf{B}\cdot d\mathbf{S}$ becomes

$$\left\{E_z(r) + \frac{\partial E_z(r)}{\partial r}\right\}\Delta z \;-\; E_z(r)\,\Delta z \;=\; \frac{j\omega\, dr\, \Delta z}{2\pi r}\int_0^r \mu\, J_z(r')\, 2\pi r'\, dr' \tag{6.8}$$

where r' is a dummy radial variable of integration, not to be confused with the coordinate r giving the location of the rectangle.

Substituting $E_z(r) = J_z(r)/\sigma$, multiplying both sides by $r/(dr\,\Delta z)$, differentiating both sides with respect to r, and finally dividing all terms by r,

$$\frac{\partial^2 J_z(r)}{\partial r^2} + \frac{1}{r}\frac{\partial J_z(r)}{\partial r} - j\omega\mu\sigma\, J_z(r) \;=\; 0 \tag{6.9}$$

The differentiation on the right merely removes the integration and substitutes the upper limit for the variable.

Equation (6.9) is a modified form of Bessel's equation of order zero. The order is zero because the term $-\nu^2 J_z(r)/r^2$ in a Bessel equation of order ν has coefficient zero. The equation is a "modified" Bessel equation because the coefficient of the term in $J_z(r)$ is a negative imaginary number rather than a positive real number. Formally, the solution of *(6.9)* can be written

$$J_z(r) \quad = \quad A_1\, J_0(\sqrt{-j\omega\mu\sigma}\, r) \;+\; A_2\, Y_0(\sqrt{-j\omega\mu\sigma}\, r) \tag{6.10}$$

where the symbols J_0 and Y_0 stand for Bessel functions of the first and second kinds respectively, of order zero. (Other symbols are frequently used for Y_0.) For real variables these functions are evaluated from infinite series and are readily available in mathematical tables. Such tables are not applicable, however, when the coefficient of r in the variable is the square root of a negative imaginary number, since in this case the series expansion will contain both real and imaginary terms. Equation *(6.9)* is of sufficient importance that separate names have been given to the functions which are respectively the real and imaginary parts of the Bessel functions of the first and second kinds of order zero, for the specific form of complex variable occurring in that equation. One of several equivalent sets of definitions for these special functions is

$$\mathrm{ber}\,(x) \;=\; \text{real part of } J_0(\sqrt{-j}\, x) \tag{6.11}$$

$$\mathrm{bei}\,(x) \;=\; \text{imaginary part of } J_0(\sqrt{-j}\, x) \tag{6.12}$$

$$\mathrm{ker}\,(x) \;=\; \text{real part of } Y_0(\sqrt{-j}\, x) \tag{6.13}$$

$$\mathrm{kei}\,(x) \;=\; \text{imaginary part of } Y_0(\sqrt{-j}\, x) \tag{6.14}$$

where x is real.

For reasons explained later, it is desirable to write the variable in *(6.10)* in the form $\sqrt{-j}\,\sqrt{2}\, r/\delta$, where

$$\delta \;=\; \sqrt{\frac{2}{\omega\mu\sigma}} \tag{6.15}$$

Using *(6.11)* to *(6.15)*, the general solution to *(6.10)* is

$$J_z(r) \;=\; A_1\left(\mathrm{ber}\,\frac{\sqrt{2}\, r}{\delta} + j\, \mathrm{bei}\,\frac{\sqrt{2}\, r}{\delta}\right) \;+\; A_2\left(\mathrm{ker}\,\frac{\sqrt{2}\, r}{\delta} + j\, \mathrm{kei}\,\frac{\sqrt{2}\, r}{\delta}\right) \tag{6.16}$$

Because ker (x) is infinite at $x = 0$, A_2 must equal zero when equation *(6.16)* is applied to a solid circular conductor, since the location $r = 0$ is a line within the conductor. For a tubular conductor the location $r = 0$ is not within the conductor's cross section and A_2 is in general not zero. For the solid circular conductor, then

$$J_z(r) \;=\; A_1\left(\mathrm{ber}\,\frac{\sqrt{2}\, r}{\delta} + j\, \mathrm{bei}\,\frac{\sqrt{2}\, r}{\delta}\right) \tag{6.17}$$

Although the primary purpose of this analysis is to find expressions for the distributed resistance and internal inductance of a homogeneous solid circular conductor, as a function of the conductor material and the frequency, it is of incidental interest to note the distribution of current density over the conductor's cross section, which is given directly by equation *(6.17)*.

Convenient tables of ber (x) and bei (x) are available in H. B. Dwight's *Tables of Integrals and Other Mathematical Data*. Fig. 6-3 shows graphs of the magnitude and phase of $J_z(r)$ plotted from equation *(6.17)* using these tables, with $A_1 = 1$. Since ber$\,(0) = 1$ and bei$\,(0) = 0$, these graphs all show the magnitude and phase angle of the current density at any value of r/δ relative to the magnitude and zero reference phase angle respectively at the center of the conductor.

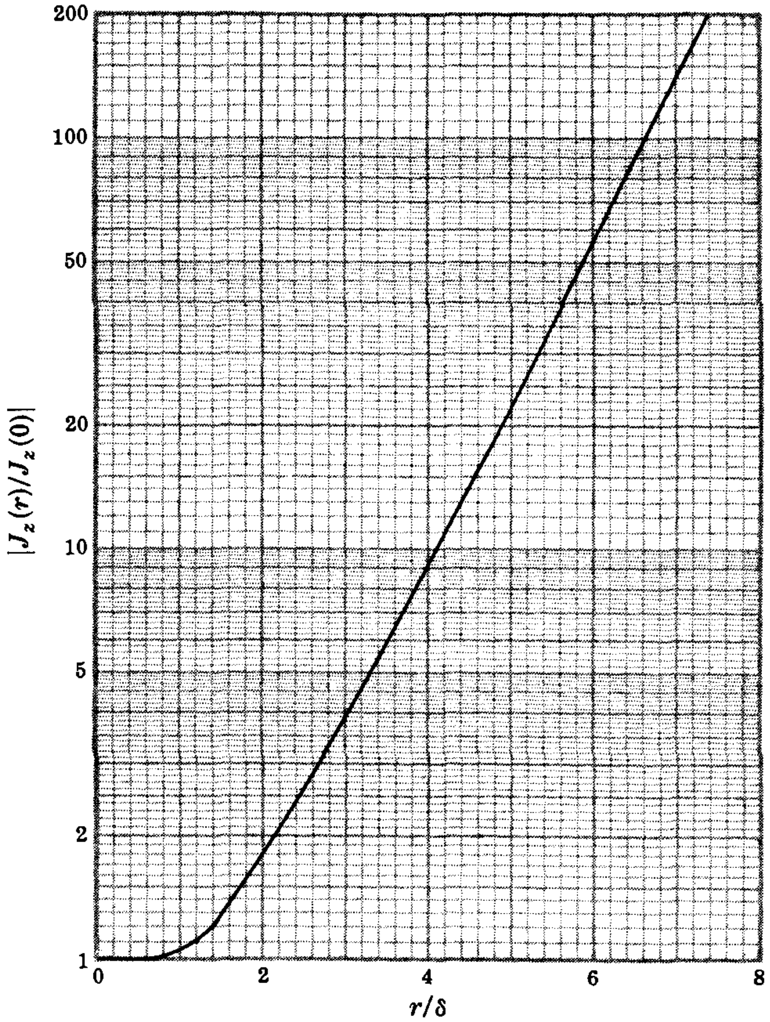

Fig. 6-3(a). Ratio of the magnitude of the current density $|J_z(r)|$ at any radius r, inside a solid circular conductor, to the magnitude of the current density $|J_z(0)|$ at the center of the conductor, as a function of r in skin depths.

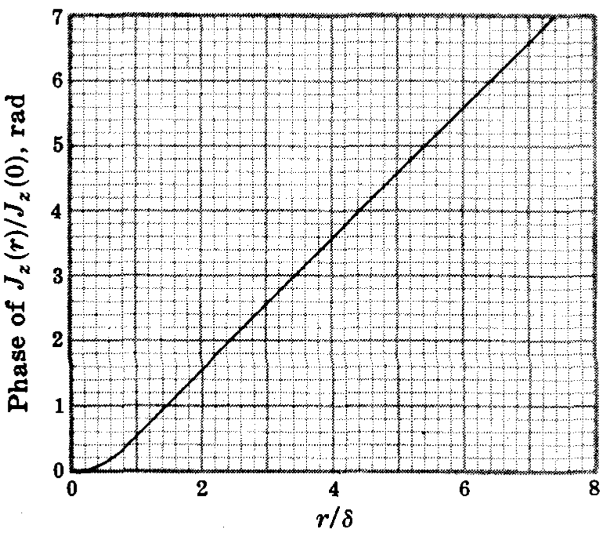

Fig. 6-3(b). Phase of the current density $J_z(r)$ at any radius r, inside a solid circular conductor, relative to the phase of the current density $J_z(0)$ at the center of the conductor, as a function of r in skin depths.

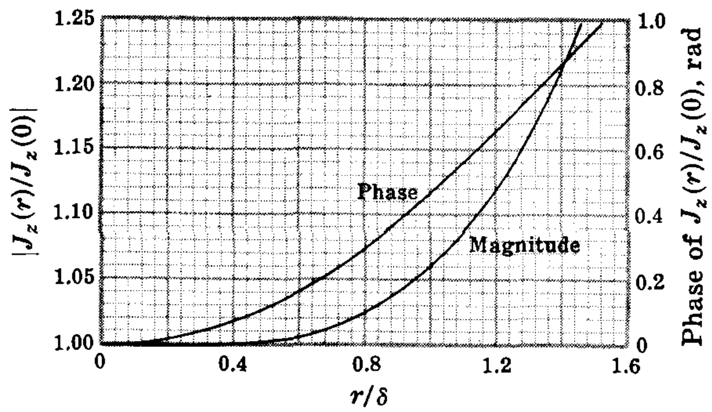

Fig. 6-3(c). An expanded linear presentation of the portions of Fig. 6-3(a) and (b) at low values of r/δ.

According to electromagnetic theory the currents and fields inside a solid circular conductor are to be regarded as having penetrated into the conductor from the interconductor fields of the transmission line at the conductor's surface. It is therefore quantitatively more significant to consider the current density at any radius r relative to the current density at the periphery of the conductor, in magnitude and phase. The current density at the surface is

$$J_z(a) \quad = \quad A_1\left(\operatorname{ber}\frac{\sqrt{2}\,a}{\delta} + j\operatorname{bei}\frac{\sqrt{2}\,a}{\delta}\right) \tag{6.18}$$

where a is the radius of the conductor. Substituting for A_1 from (*6.18*) into (*6.17*),

$$\frac{J_z(r)}{J_z(a)} \quad = \quad \frac{\operatorname{ber}\sqrt{2}\,r/\delta \; + \; j\operatorname{bei}\sqrt{2}\,r/\delta}{\operatorname{ber}\sqrt{2}\,a/\delta \; + \; j\operatorname{bei}\sqrt{2}\,a/\delta} \tag{6.19}$$

which is the desired relation. The same result can be obtained graphically, for a specific conductor at a specific frequency, by evaluating a/δ for the conductor and marking a vertical line at that value of r/δ on one of the graphs of Fig. 6-3. If then all the current density magnitude values for smaller values of r/δ are divided by the current density magnitudes at $r/\delta = a/\delta$, and the phase angle at $r/\delta = a/\delta$ is subtracted from the phase angle at all smaller values of r/δ, the graph of the resulting magnitudes and phase angles versus $(r/\delta)/(a/\delta)$ or r/a will show the magnitude and phase angle of the current density at any radius r, relative to the magnitude and zero reference phase angle of the current density at the conductor's surface. Fig. 6-4 is such a graph for $a/\delta = 4$.

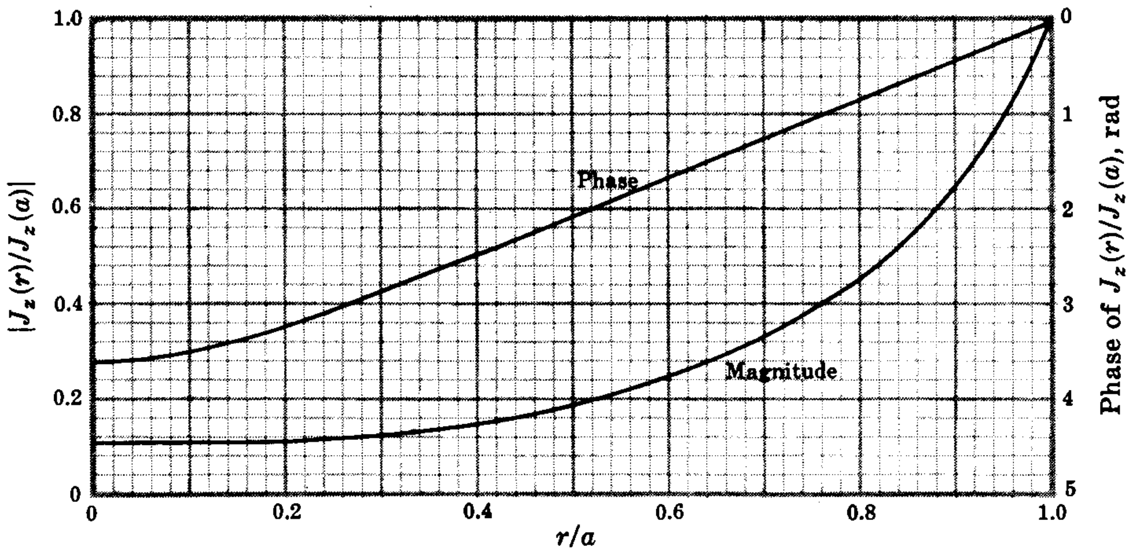

Fig. 6-4. Phase and magnitude relations for the current density at radius r in a solid circular conductor of radius a relative to the current density at the surface, at a frequency for which $a/\delta = 4$.

It is evident from Fig. 6-3 that for values of a/δ less than 0.5, the current distribution in a solid circular conductor is not perceptibly different from that for d-c, when $a/\delta = 0$. At $a/\delta = 1$, however, the change is quite noticeable, and when $a/\delta = 5$ there is a very marked concentration of current close to the conductor's surface. As a numerical example, a/δ has the value 0.5 for a copper conductor 2 millimeters in diameter at a frequency of 1090 hertz, the value 1 at 4360 hertz, and the value 5 at 109,000 hertz. The suggestion from Fig. 6-3 that at high enough values of a/δ a solid circular conductor might be replaced by a thin-walled circular tube with negligible change in current distribution, and consequently in a-c resistance, is perfectly correct. Surprisingly, it turns out that for a wall thickness of about 1.6δ the distributed resistance of a tubular conductor at high values of a/δ is actually a few percent less than the distributed resistance of a solid conductor of the same metal and outer diameter at the same frequency.

The distributed resistance R and distributed internal inductance L_i of a solid circular conductor at angular frequency ω rad/sec can be combined in the concept of the distributed internal impedance Z_i of the conductors,

$$Z_i = R_i + j\omega L_i \quad \text{ohms/m} \tag{6.20}$$

where it is not necessary to use a symbol R_i since the distributed internal resistance of the conductor is its total distributed resistance. In terms of electrical variables, the distributed internal impedance of the conductor is the ratio of the longitudinal potential difference over unit length of the conductor at the conductor's surface to the total current in the conductor. The longitudinal potential difference per unit length is identically the longitudinal electric field at the conductor's surface, which from equation (6.7) is $E_z(a) = J_z(a)/\sigma$. Then if the total current in the conductor is I_z,

$$Z_i = R + j\omega L_i = \frac{J_z(a)}{\sigma I_z} \tag{6.21}$$

To relate the total current I_z to quantities that have already appeared in the analysis, it is necessary to refer to another electromagnetic relation, the Maxwell equation for time-harmonic fields: curl $\mathbf{E} = -j\omega\mu\mathbf{H}$. From the postulated symmetry of the problem, the only field components present are E_z and H_ϕ, and these quantities are functions only of the coordinate r. The Maxwell equation in cylindrical coordinates then reduces to the single relationship

$$\frac{\partial E_z}{\partial r} = j\omega\mu\, H_\phi(r) \tag{6.22}$$

From Ampere's law (i.e. the integral form of the second Maxwell curl equation), the total current in the conductor will be equal to the line integral of H_ϕ around the conductor's periphery,

$$I_z = 2\pi a\, H_\phi(a) \tag{6.23}$$

Combining equations (6.7), (6.17) and (6.22),

$$H_\phi(r) = \frac{1}{j\omega\mu\sigma}\frac{\partial J_z(r)}{\partial r} = \frac{\sqrt{2}/\delta}{j\omega\mu\sigma} A_1\left(\text{ber}'\frac{\sqrt{2}\,r}{\delta} + j\,\text{bei}'\frac{\sqrt{2}\,r}{\delta}\right) \tag{6.24}$$

where $\text{ber}'(x) = d\,\text{ber}(x)/dx$ and $\text{bei}'(x) = d\,\text{bei}(x)/dx$.

Putting $r = a$ in equation (6.24) and combining with (6.23),

$$I_z = \frac{2\pi a\, A_1\sqrt{2}/\delta}{j\omega\mu\sigma}\left(\text{ber}'\frac{\sqrt{2}\,a}{\delta} + j\,\text{bei}'\frac{\sqrt{2}\,a}{\delta}\right) \tag{6.25}$$

Substituting for A_1 from (6.18),

$$I_z = \frac{2\pi a\sqrt{2}/\delta}{j\omega\mu\sigma} J_z(a)\left(\frac{\text{ber}'\sqrt{2}\,a/\delta + j\,\text{bei}'\sqrt{2}\,a/\delta}{\text{ber}\sqrt{2}\,a/\delta + j\,\text{bei}\sqrt{2}\,a/\delta}\right) \tag{6.26}$$

Finally, substituting this value of I_z in (6.21) gives

$$R + j\omega L_i = \frac{jR_s}{\sqrt{2}\,\pi a}\left(\frac{\text{ber}\sqrt{2}\,a/\delta + j\,\text{bei}\sqrt{2}\,a/\delta}{\text{ber}'\sqrt{2}\,a/\delta + j\,\text{bei}'\sqrt{2}\,a/\delta}\right) \tag{6.27}$$

Here the symbol R_s has been adopted for $\dfrac{1}{\sigma\delta} = \sqrt{\dfrac{\omega\mu}{2\sigma}}$.

R_s is a surface resistivity in ohms per square, sometimes called the skin resistivity or high frequency surface resistivity, of the material defined by μ and σ at angular frequency ω. Physically it is the d-c resistance between opposite edges of a square sheet of the metal having thickness equal to the skin depth δ, since it is experimentally confirmed for the usual non-ferromagnetic conductor materials that the d-c value of σ continues to hold at all frequencies used on transmission lines.

Separate expressions for R and ωL_i are easily obtained from equation (*6.27*), and with the help of (*6.1*) and (*6.5*), equations giving the ratios of the distributed resistance and distributed internal inductance at any frequency to the same quantities at zero frequency can be formulated. (See Problem 6.36.)

For practical purposes, calculations of data from (*6.27*) or from the equations derived in Problem 6.36 can be divided into five sections, according to the value of a/δ:

(1) When a/δ is less than about 0.5, the distributed a-c resistance of a conductor increases over its d-c value given in (*6.1*) by less than $\frac{1}{2}\%$, and the distributed internal inductance decreases by less than $\frac{1}{4}\%$ from the d-c value given in (*6.5*).

(2) For all values of a/δ less than about 1.5, better than $\frac{1}{2}\%$ accuracy is obtained from the approximate formulas

$$R/R_{\text{d-c}} = 1 + (a/\delta)^4/48 \tag{6.28}$$

$$L_i/L_{i\,\text{d-c}} = 1 - (a/\delta)^4/96 \tag{6.29}$$

(3) When a/δ is greater than about 100, $\frac{1}{2}\%$ accuracy is given by the very simple formulas

$$R = R_s/(2\pi a) \tag{6.30}$$

where $R_s = 1/(\sigma\delta) = \sqrt{\omega\mu/(2\sigma)}$ as defined in equation (*6.27*), and

$$\omega L_i = R \quad \text{or} \quad L_i = R/\omega \tag{6.31}$$

Equation (*6.30*) states that for very high values of a/δ, which occur at frequencies of tens or hundreds of megahertz for typical conductor diameters, the distributed a-c resistance of a solid circular conductor is equal to the d-c resistance per unit length of a plane strip of the conductor material having thickness δ and a width equal to the periphery of the circular conductor. This is an expression of the skin effect theorem derived in Section 6.3. Alternative interpretations of (*6.30*) are that the distributed a-c resistance of a solid circular conductor at sufficiently high frequencies is equal to the d-c resistance per unit length of a surface skin of the conductor of thickness δ, or is equal to the resistance per unit length of a strip of indefinitely thin sheet resistance material having surface resistivity R_s ohms/square, wrapped around a nonconducting cylinder of radius equal to that of the conductor.

Equation (*6.31*) states that under the same conditions, the distributed internal reactance of the solid circular conductor is equal to its distributed a-c resistance.

It is obvious that (*6.30*) and (*6.31*) also apply to tubular circular conductors whose wall thickness is great enough to contain essentially the whole of the current distribution. For high values of a/δ a wall thickness greater than about 3δ ensures $\frac{1}{2}\%$ accuracy in using the equations. (See Section 6.3 and Fig. 6-7.)

(4) Equation (*6.30*) has better than $\frac{1}{2}\%$ accuracy for all values of a/δ greater than 4 if the effective circumference of the peripheral skin of the conductor is calculated more appropriately from a radius $(a - \frac{1}{2}\delta)$ instead of a. Then

$$R = \frac{1}{2\pi\sigma(a - \frac{1}{2}\delta)\delta} = \frac{R_s}{2\pi(a - \frac{1}{2}\delta)} \tag{6.32}$$

and

$$\frac{R}{R_{\text{d-c}}} = \frac{\frac{1}{2}a^2}{(a - \frac{1}{2}\delta)\delta} = \frac{(a/\delta)^2}{2(a/\delta) - 1} \tag{6.33}$$

For the same range of a/δ the inductance ratio $L_i/L_{i\,\text{d-c}}$ is given with equal accuracy by an empirical formula

$$\frac{L_i}{L_{i\,\text{d-c}}} = \frac{1}{R/R_{\text{d-c}}} + \frac{1}{(R/R_{\text{d-c}})^3} \tag{6.34}$$

(5) For values of a/δ between 1.5 and 4, there is no standard alternative to using data calculated directly from equation (*6.27*). Tables and graphs of such data are available in many sources. Table 6.1 shows the variation of $R/R_{\text{d-c}}$ and $L_i/L_{i\,\text{d-c}}$ for small intervals of a/δ in the range 0 to 4. Linear interpolation in the intervals is accurate enough for engineering purposes.

Table 6.1

a/δ	$R/R_{\text{d-c}}$	$L_i/L_{i\,\text{d-c}}$	a/δ	$R/R_{\text{d-c}}$	$L_i/L_{i\,\text{d-c}}$
0	1.000	1.000	2.3	1.404	0.805
0.5	1.001	1.000	2.4	1.454	0.783
0.7	1.005	0.998	2.5	1.505	0.760
0.8	1.009	0.996	2.6	1.557	0.737
0.9	1.014	0.993	2.7	1.610	0.715
1.0	1.021	0.989	2.8	1.663	0.693
1.1	1.030	0.984	2.9	1.716	0.672
1.2	1.042	0.978	3.0	1.769	0.652
1.3	1.057	0.971	3.1	1.821	0.632
1.4	1.075	0.962	3.2	1.873	0.613
1.5	1.097	0.951	3.3	1.924	0.595
1.6	1.122	0.938	3.4	1.974	0.578
1.7	1.152	0.924	3.5	2.024	0.562
1.8	1.187	0.908	3.6	2.074	0.547
1.9	1.225	0.890	3.7	2.124	0.533
2.0	1.266	0.870	3.8	2.174	0.520
2.1	1.309	0.849	3.9	2.224	0.507
2.2	1.355	0.827	4.0	2.274	0.495

Example 6.1.

Determine the distributed resistance and distributed internal reactance in ohms/m of a 19 gauge copper wire at frequencies of 0, 60, 10^3, 10^4, 10^5, 10^6, 10^8 and 10^{10} hertz.

From wire tables the radius of a 19 gauge wire is 0.4558×10^{-3} m, and $R_{\text{d-c}}$ is given as 26.42 ohms/km at 20°C. These figures are consistent with the copper having a conductivity of 5.80×10^7 mhos/m, which is officially defined as "100% conductivity" for copper at 20°C, and is the value invariably used for "room temperature" resistance calculations on copper conductors, unless some other specific value is stated.

For this value of conductivity the skin depth in copper as given by equation (*6.15*) is $\delta = 0.0661/\sqrt{f}$ m, where f is the frequency in hertz. Thus for the 19 gauge wire of this problem, the ratio a/δ at the frequencies listed has the sequence of values 0, 0.0533, 0.218, 0.689, 2.18, 6.89, 68.9 and 689. The first three values fall in the first category of calculations listed above. These are followed by one in the second category, one in the fifth, two in the fourth, and the final one in the third category.

The reference values for R and L_i are $R_{\text{d-c}} = 0.0264$ ohms/m and $L_{i\,\text{d-c}} = \mu_0/8\pi = 5.00 \times 10^{-8}$ henries/m. These are then also the values of R and L_i for the 19 gauge copper wire at frequencies of 0, 60 and 10^3 hertz, where $a/\delta < 0.5$.

At 10 kilohertz, with $a/\delta = 0.689$, equations (*6.28*) and (*6.29*) should be used. The results are $R/R_{\text{d-c}} = 1.005$ and $L_i/L_{i\,\text{d-c}} = 0.998$, which could also have been taken from Table 6.1 at $a/\delta = 0.7$.

At 100 kilohertz, Table 6.1 is used, to find $R/R_{\text{d-c}} = 1.346$ and $L_i/L_{i\,\text{d-c}} = 0.831$.

For frequencies of 1 megahertz and higher, (*6.33*) and (*6.34*) are used. These automatically reduce to the simpler forms of (*6.30*) and (*6.31*) when a/δ is large enough.

A tabulation of all the results for 19 gauge copper wire at frequencies from 0 to 10^{10} hertz is given in Table 6.2 below.

Table 6.2

Frequency hertz	a/δ	$R/R_{d\text{-}c}$	R ohms/m	$L_i/L_{i\,d\text{-}c}$	L_i henries/m	ωL_i ohms/m	$\omega L_i/R$
0	0	1.000	0.0264	1.000	5.00×10^{-8}	0	0
60	0.0533	1.000	0.0264	1.000	5.00×10^{-8}	1.88×10^{-5}	7.1×10^{-4}
10^3	0.218	1.000	0.0264	1.000	5.00×10^{-8}	3.14×10^{-4}	0.0119
10^4	0.689	1.005	0.0265	0.998	4.99×10^{-8}	3.13×10^{-3}	0.118
10^5	2.18	1.346	0.0355	0.831	4.16×10^{-8}	0.0261	0.736
10^6	6.89	3.71	0.0980	0.289	1.45×10^{-8}	0.0909	0.930
10^8	68.9	34.7	0.914	0.0288	1.44×10^{-7}	0.905	0.991
10^{10}	689	344	9.09	0.00291	1.45×10^{-10}	9.09	1.000

It is quite clear from Tables 6.1 and 6.2 that a/δ increasing above unity marks the beginning of rapid increases with frequency for the ratios $R/R_{d\text{-}c}$ and $L_i/L_{i\,d\text{-}c}$. Also, from Table 6.2 it can be seen that when a/δ approaches 100, the distributed resistance R begins to increase quite precisely in proportion to the square root of the frequency, while the distributed internal inductance L_i begins to vary inversely as the square root of the frequency.

The ratio a/δ varies directly with conductor radius a, and with the square root of the frequency, for solid circular conductors of the same material. Thus the same values of $R/R_{d\text{-}c}$ and $L_i/L_{i\,d\text{-}c}$ that apply to a conductor of radius a at frequency f, will hold for a conductor of the same material with radius $10a$ at a frequency $f/100$, or with radius $a/10$ at a frequency $100f$. Pursuing these figures in both directions, a 40 gauge copper wire shows no perceptible change in resistance from its d-c value for frequencies up to 1 megahertz, while a solid copper conductor 2″ in diameter shows about 10% increase of resistance from skin effect at a frequency of 60 hertz.

The ratio a/δ varies directly as the square root of the permeability and the square root of the conductivity of the conductor material. Hence conductors of any non-magnetic material except silver will have smaller values of a/δ than copper wires of the same diameter at the same frequency, and the changes in their values of distributed resistance R and distributed internal inductance L_i from the d-c values will be less than for the copper conductors. Wires of iron, nickel, or other ferromagnetic material may have values of a/δ, and hence of $R/R_{d\text{-}c}$, either larger or smaller than for copper wires of the same diameter at the same frequency, depending on whether or not their relative permeability at the frequency exceeds the ratio of the conductivity of copper to the conductivity of the ferromagnetic material. Iron wires may have quite large values of relative permeability at frequencies up to the low megahertz range, in which case the ratio of the distributed resistance of iron wires to the distributed resistance of copper wires of the same diameter may become much higher than would be determined by the ratio of their conductivities alone.

Table 6.3 below lists the conductivities at 20°C, and the temperature coefficient of the conductivity at that temperature, for some of the metals most commonly used as transmission line conductors, or as resistor materials or plating materials in high frequency applications.

From Example 6.1 it has been seen that the skin depth δ for 100% conductivity copper at frequency f hertz and 20°C is given by $\delta = 0.0661/\sqrt{f}$ m. Fig. 6-3 shows that for a/δ greater than about 5 or 10, most of the a-c current in a solid circular conductor flows in a peripheral skin of the conductor about two or three skin depths in thickness. At a frequency of 1 megahertz in copper this thickness is approximately 0.1 millimeters, and it is smaller

Table 6.3. Conductivities and temperature coefficients of common metals at 20°C.

Metal	Conductivity mhos/m	Temperature Coefficient /°C (all negative)
Aluminum	3.54×10^7	0.0039
Brass (somewhat variable)	1.4×10^7	0.002
Copper (annealed)	5.80×10^7	0.00393
Copper (hard drawn)	5.65×10^7	0.00382
Constantan	2.04×10^6	0.000008
Gold (pure)	4.10×10^7	0.0034
Iron* (pure)	1.00×10^7	0.0050
Lead	4.54×10^6	0.0039
Mercury	1.04×10^6	0.00089
Nickel*	1.28×10^7	0.0006
Silver	6.15×10^7	0.0038
Tin	8.67×10^6	0.0042
Zinc	1.76×10^7	0.0037

*The permeability of iron and nickel is very much dependent on processing techniques, and must be determined experimentally for any specific conductors.

at higher frequencies. This suggests the possibility of using plated conductors in high frequency applications, having a thin skin of more expensive high conductivity metal on a core of inexpensive material whose conductivity does not affect the situation. The technique is extensively used. For many years it was thought that silver was necessarily the best plating material, since silver has the highest conductivity of all metals. However, careful measurements have shown that the corrosion products on a silver surface in ordinary atmospheres have intermediate conductivity, while those on a copper surface have very low conductivity. The result is that high frequency currents in a copper conductor flow almost entirely in the copper, below the surface corrosion layers, and the conductor's effective conductivity is that of the copper. For a silver conductor, on the other hand, an appreciable fraction of the current flows in the corrosion material of intermediate conductivity (the corrosion products are generally oxides and sulfides) and the effective conductivity of the conductor as a whole may be substantially less than that of silver. If a silver surface is protected against corrosion, including oxidation, by an extremely thin layer of plated or evaporated gold or by a low-loss dielectric coating, a silver plated conductor will have the lowest possible distributed resistance.

It is theoretically true, when $a/\delta \gg 1$, that a plating thickness of 1.2δ or greater, on any base material whether conducting or not, will ensure a distributed conductor resistance equal to or less than that of a solid circular conductor of the plating material, if there is adequate protection from corrosion and from surface roughness effects.

The problem of surface roughness exists whether a conductor is solid or plated, and is an obvious consequence of the small values of δ at high frequencies. In the commercial fabrication of circular metal rods or wire, microscopic surface imperfections appear, in the form of pits, cracks, grooves, fissures, etc. At frequencies in the hundreds or thousands of megahertz, the skin depth in the material may be no greater than the dimensions of these imperfections, with the result that the current flow path along the conductor's surface is not a straight line equal to the conductor's length, but is a meandering path that may be much longer than the conductor itself. The high frequency resistance will increase by a corresponding factor over the theoretical value. At microwave frequencies the realization of distributed conductor resistances close to theoretical values for a given conductor material may require special surface polishing techniques in addition to protection against corrosion.

6.3. Distributed resistance and internal inductance of thick plane conductors.

Plane conductors of finite width occur in several of the transmission lines illustrated in Fig. 2-2, page 9, and it would be useful to be able to calculate the distributed resistance and internal inductance of those conductors. However, a reasonably simple analysis of the high frequency current distribution and distributed internal impedance of plane conductors is possible only if the conductors are postulated to extend indefinitely in both directions in the conductor plane. The general case of the distributed internal impedance of plane parallel conductors of finite width, or such special cases as the distributed internal impedance of finite width plane conductors within a rectangular, elliptical or circular shield or outer conductor, can be solved only by approximation and computer methods, and there are no universally accepted presentations of results for these cases, in the form of equations, tables, or graphs. In practice, experimental measurements are often likely to provide information on the distributed resistance and inductance of uniform transmission lines involving plane conductors as quickly and accurately as attempts to solve the analytical problem.

The analysis of the idealized unbounded-plane case is worth inspection in spite of these limitations, because for high values of a/δ it offers an easier solution to the practical problem of the thin-walled circular tube than can be obtained from the equations of Section 6.2, and because it provides a basis for approximate calculations in various other cases.

The longitudinal currents postulated in the distributed circuit analysis of transmission lines will flow only if there is a longitudinal electric field at the conductor's surface. An electric field component normal to the surface will not generate such currents, nor will a component parallel to the surface but transverse to the direction of propagation. In an investigation of the power losses associated with high frequency current flow in a plane metal surface, it is therefore most appropriate to consider a constant amplitude plane transverse electromagnetic harmonic wave incident normally on such a metal surface and partly reflected from it. The field components of the incident and reflected waves will combine at the surface to produce a total tangential electric field component which is continuous across the boundary, and an accompanying tangential magnetic field component, perpendicular to the electric field component and also continuous across the boundary. Let these field components, as phasors, have directions and rms magnitudes given by E_{z0} and H_{x0} at the metal surface, and by $E_z(y)$ and $H_x(y)$ at any distance y measured into the metal normally from the surface.

The unbounded plane conductor can be thought of as one of two such conductors constituting a parallel plane transmission line, with voltage and current waves propagating in the longitudinal z direction on the line. The normal to the conductor surfaces is in the y direction, and the conductor surfaces extend indefinitely in the transverse x direction. The distributed resistance to be determined is the resistance of the conductors per unit length in the z direction, per unit width in the x direction. Initially the conductors are assumed to be indefinitely thick in the y direction. The distance between the conductors is not relevant to the problem, and attention is directed to only one of the conductors, whose surface is in the coordinate plane at $y = 0$, the direction of increasing y being into the metal. The coordinate relations are shown in Fig. 6-5, below.

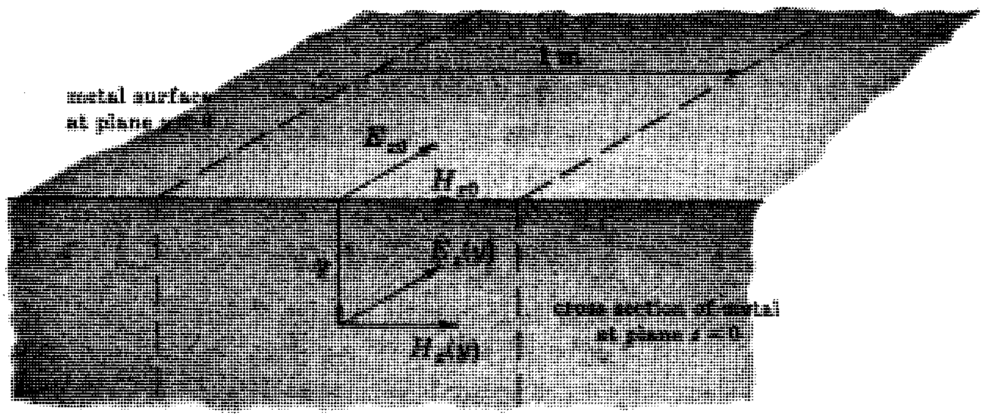

Fig. 6-5. Coordinate relations for investigating a-c current flow in a thick metal sheet.

Maxwell's equations for these time-harmonic fields are

$$\text{curl}\ \mathbf{E} \;=\; -j\omega\mu\mathbf{H} \tag{6.35}$$

$$\text{curl}\ \mathbf{H} \;=\; (\sigma + j\omega\epsilon)\mathbf{E} \tag{6.36}$$

It has been seen in connection with equation (*6.6*) that the displacement current density $\omega\epsilon E$ in (*6.36*) is negligible compared to the conduction current density σE for all ordinary metals at all frequencies up to 10^{12} hertz or higher. (This would not be true if the conductors were made, for example, of silicon.)

Dropping the term $j\omega\epsilon$, the wave equation is obtained from (*6.35*) and (*6.36*) by taking the curl of either and substituting into it from the other. This leads to

$$\nabla^2\mathbf{E} \;=\; j\omega\mu\sigma\mathbf{E} \tag{6.37}$$

and the same equation in **H**, where it is to be understood that (*6.37*) is actually three equations, one for each separate rectangular coordinate component of **E**. Since it has been postulated that the only component of **E** is E_z, equation (*6.37*) becomes

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \;=\; j\omega\mu\sigma E_z \tag{6.38}$$

Since it has also been postulated that the wave planes of the plane electromagnetic wave being considered extend indefinitely in the x and z directions, the derivatives $\partial/\partial x$ and $\partial/\partial z$ must both be zero. Finally, then, (*6.38*) takes the elementary form

$$\frac{\partial^2 E_z}{\partial y^2} \;=\; j\omega\mu\sigma E_z \tag{6.39}$$

whose solution is

$$E_z(y) \;=\; E_{z0}\, e^{\pm\gamma y} \tag{6.40}$$

where $\gamma = \sqrt{j\omega\mu\sigma} = (1+j)/\delta$ and $\delta = \sqrt{2/\omega\mu\sigma}$ is the same skin depth defined by equation (*6.15*). It follows that

$$E_z(y) \;=\; E_{z0}\, e^{-y/\delta}\, e^{-jy/\delta} \tag{6.41}$$

for a wave traveling into the metal in the positive y direction.

The meaning of equation (*6.41*) is that the electric field of the wave suffers an attenuation of 1 neper and a phase delay of 1 radian in traveling distance δ through the metal. (The propagation of voltage and current waves on a transmission line would be analogous if the line had the properties $R \ll \omega L$ and $G \gg \omega C$. The corresponding value of δ would be $\sqrt{2/(\omega LG)}$.)

At any point in the metal the current density produced by the electric field of equation (*6.41*) flows in the z direction and is given by

$$J_z(y) = \sigma E_z(y) \tag{6.42}$$

The power loss per unit volume in the metal at any point is $|J_z(y)|^2/\sigma = \sigma|E_z(y)|^2$, and the power loss per unit area of metal surface is

$$\begin{aligned} P_L &= \int_0^1 dx \int_0^\infty dy \int_0^1 dz \,\{\sigma|E_z(y)|^2\} = \int_0^\infty \sigma|E_z(y)|^2\,dy \\ &= \int_0^\infty \sigma|E_{z0}|^2 e^{-2y/\delta}\,dy = \sigma|E_{z0}|^2\,\delta/2 \end{aligned} \tag{6.43}$$

where $|E_z(y)|$ is obtained from equation (*6.41*).

From (*6.41*), the field E_z and consequently the current density J_z fall to negligible fractions of the surface values at a depth of less than 10δ, which at high frequencies can be a very small distance. The limit of infinity on the integrals with respect to y in (*6.43*) could be replaced by this distance.

A more tangible electrical variable in this skin effect situation than the tangential electric field at the surface of the metal is the total current in the conductor per unit width of surface. This is contained within a thickness of less than 10δ at the surface of the metal. It must be identified notationally as a *surface-current density* (mks units, amperes/meter), physically distinct from the current density J_z at a point (mks units, amperes/square meter). If J_{sz} is this *surface* current density,

$$J_{sz} = \int_0^1 dx \int_0^\infty dy\, J_z(y) = \int_0^\infty \sigma E_{z0}\, e^{-(1+j)y/\delta}\,dy = \sigma E_{z0}\,\delta/(1+j) \tag{6.44}$$

The phase relation here is meaningful, establishing that the total surface-current density J_{sz} lags the surface electric field E_{z0} by 45°. (Note, however, that the *current density at a point at the surface* J_{z0} must be in phase with E_{z0} according to equation (*6.42*). The distinction between the total surface-current density and the current density at a point at the surface is a vital one.)

In the case of the solid circular conductor, a distributed internal conductor impedance was defined in equation (*6.21*) by the ratio of the tangential electric field at the surface to the total current in the conductor. The corresponding concept for plane conductors is a longitudinally distributed internal impedance for unit width of conductor. Designating this as $Z_s = R_s + jX_s$, equation (*6.44*) shows that

$$Z_s = R_s + jX_s = (1+j)/\sigma\delta = R_s(1+j) \tag{6.45}$$

Thus the real and imaginary parts of this distributed internal impedance, for unit width of a plane conductor having thickness not less than about 10 skin depths, are both equal to the quantity R_s previously defined in (*6.27*), commonly known as the surface resistivity of the material and numerically equal to the d-c resistance between opposite edges of any square sheet of the material of thickness δ. The units of R_s and hence of X_s and Z_s are ohms, or ohms/square.

Substituting for $|E_{z0}|$ from (*6.44*) into (*6.43*),

$$P_L = |J_{sz}|^2/\sigma\delta = |J_{sz}|^2 R_s \tag{6.46}$$

According to (*6.46*) the power loss per unit area associated with the a-c surface-current density J_{sz} in amperes/(meter width of surface), for which the point current density J_z diminishes exponentially with depth into the metal, is the same as the power loss per unit area that would occur if a current of the same total rms value (whether a-c or d-c) flowed with constant point current density in a skin of the conductor of thickness δ. This result is known as the *skin effect theorem*. Its application to solid circular conductors when a/δ is of the order of 100 or greater has already been seen in equation (*6.30*).

The agreement of equations (*6.30*) and (*6.31*) with (*6.45*) for very large values of a/δ is a recognition of the fact that when a circular conductor carries an a-c current, if the radius of curvature of the conductor's surface is very large compared to the skin depth, then the surface can be regarded as a plane, and the skin effect calculations for plane surfaces are applicable.

The use of the name "skin depth" and a superficial misinterpretation of the skin effect theorem sometimes lead to the erroneous impression that high frequency currents are physically totally contained within a skin of thickness δ, at the surface of a plane conductor or of a curved conductor whose radius of curvature is very large compared with δ. The actual distribution of high frequency current in such cases, as given by (*6.41*) and (*6.42*) is of course that the magnitude of the current density at a point diminishes exponentially with distance into the metal, while the phase retards linearly with distance. At any point within the metal, the amplitude reduction of the current density in nepers and the phase lag in radians, relative to the surface values, are numerically equal.

Simple calculations show that the current density falls to somewhat less than 1% of its surface magnitude at a depth of 5δ, and consequently to less than 0.01% of its surface magnitude at a depth of 10δ. Thus conductors carrying time-harmonic currents of any frequency need never be more than 5δ to 10δ thick for that frequency. Additional metal would serve no electrical purpose. For reasons to be explained, a plane conductor or a circular metal tube conductor of fixed outside radius $a \gg \delta$ actually have lower a-c resistance when the metal thickness is about 1.6δ than for either smaller or larger thicknesses, and their distributed resistance per unit width of surface remains within $\frac{1}{2}$% of R_s for all thicknesses greater than 3δ.

Closer inspection of equations (*6.41*) and (*6.42*) reveals an interesting phenomenon associated with a-c current flow in plane conductors. If the plane conductor is imagined to consist of a large number of uniform plane layers of equal thickness, the thickness Δy of each layer being conveniently about 0.3δ, then the longitudinal phasor current ΔJ_{sz} in unit width of any layer (this is an increment of the total surface-current density J_{sz}) would be given by $\Delta J_{sz}(y) = \Delta J_{sz}(0)\, e^{-(1+j)y/\delta}$ for a layer at depth y below the surface, where $\Delta J_{sz}(0)$ is the phasor current in unit width of the layer at the surface. Relative to the phasor current in the surface layer, the phasor currents in layers farther into the metal are smaller in magnitude and retarded in phase. Fig. 6-6 is a phasor diagram showing the cumulative addition of the phasor currents in consecutive layers, for layers totalling 3δ in thickness.

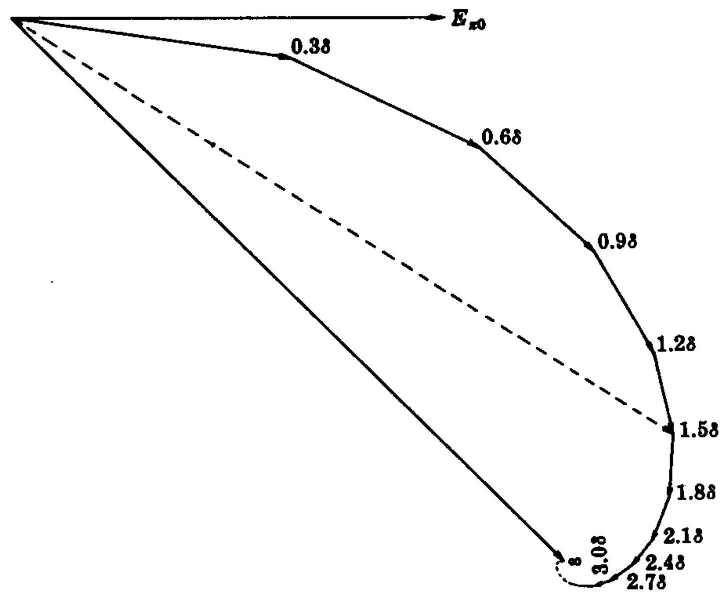

Fig. 6-6. Summation of surface-current density phasors in consecutive layers of a thick metal sheet carrying a-c current. The layers are 0.3 skin depths thick in the direction normal to the metal's plane surface, and 1 m wide parallel to the surface. The current flow is parallel to the surface. The largest surface-current density phasor at the top of the diagram is for the layer at the surface of the metal. The reference phasor is E_{z0}, the tangential electric field at the surface of the metal. The dashed line suggests that if the current beyond 1.5 skin depths into the metal could be eliminated, the total surface-current density would be the same as in an indefinitely thick sheet, but the losses would be reduced.

The value of y for each layer has been measured to the center of the layer. The value of $|\Delta J_{sz}(0)|$ in the first layer is an arbitrary scale factor. Since the center of the first layer is at $y = 0.15\delta$, its phase angle is -0.15 rad relative to the zero reference phasor given by the tangential electric field at the surface.

Fig. 6-6 could also be obtained as the envelope of the successive phasors given by the integral used in obtaining equation (*6.44*), with the upper limit varied through a sequence of values ranging from 0 to 3δ in suitable steps of 0.3δ.

The procedure of equation (*6.43*) for obtaining the power loss per unit area in an indefinitely thick plane conductor for a-c currents is equivalent to evaluating $|\Delta J_{sz}(y)|^2/(\sigma\,\Delta y)$ over an infinite set of layers, each having thickness Δy, on letting Δy become infinitesimally small. The result would be only slightly different for $\Delta y = 0.3$ skin depths. It is obvious from the convoluted form of the cumulative-phasor curve in Fig. 6-6 that the summation or integral of (*6.43*) must be quite a bit greater, for the same total conductor current magnitude, than if the incremental current phasors in successive layers all had the same phase.

These facts suggest qualitatively that some reduction of distributed surface resistance for plane conductors (and circular tube conductors with large a/δ) might be achieved by making the metal thin enough to eliminate the "backward current" portion of the phasor diagram of Fig. 6-6, without reducing the total current magnitude. This would make the length of the cumulative-phasor curve more nearly equal to the chord representing the total conductor current phasor.

The dashed total current phasor in Fig. 6-6 illustrates this hypothesis for a conductor 1.5 skin depths thick. The total current phasor in this case has almost exactly the same magnitude as for the indefinitely thick metal, but the curve constructed by adding the current phasors in successive layers is much shorter than before. The implication is that for the same total current the losses should be less in metal of thickness 1.5δ than for greater thicknesses. Although the model from which this conclusion has been drawn is somewhat oversimplified, because the postulated electromagnetic waves in the metal are actually reflected from the second surface of the thin sheet, and the combination of reflected waves with original waves gives a current phasor pattern differing appreciably from that for 1.5δ in Fig. 6-6, the reduction of a-c resistance in thin metal surfaces is nevertheless real.

Quantitative proof is given in Problem 7.10, page 147, where the analysis is made of a transmission line analog. A graph of the variation with thickness of the distributed resistance of a plane conductor, relative to the distributed resistance of an indefinitely thick conductor, is shown in Fig. 6-7 below. It appears that the resistance can be reduced by a maximum of about 8% for thicknesses of 1.5δ to 1.6δ, and that 5% reduction is obtained from about 1.3δ to 2.0δ. Since the abscissa of the curve is metal thickness in skin depths, it can be converted to a linear scale of thickness in meters at constant frequency, or to a scale proportional to the square root of the frequency for a given metal of constant thickness.

The quantitative conclusions drawn from Fig. 6-7 are directly applicable to tubular circular conductors for which the outer radius is very large compared to the skin depth.

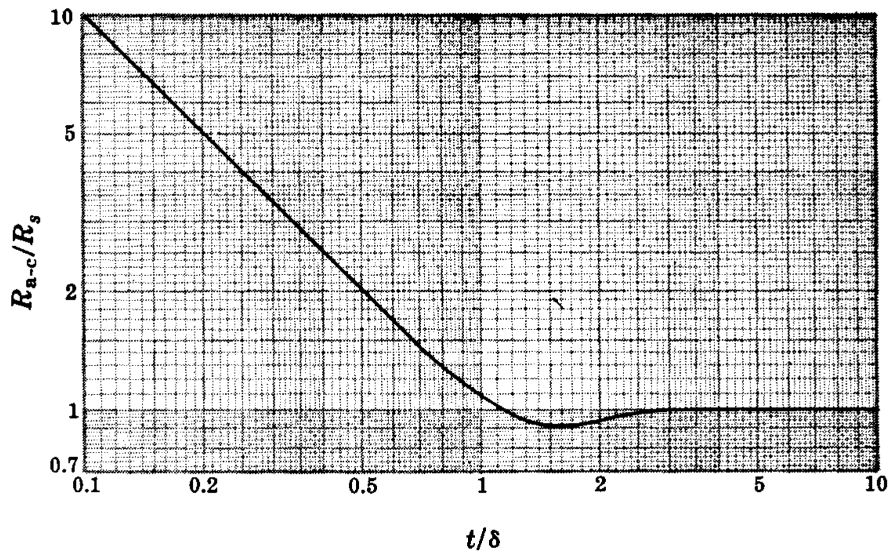

Fig. 6-7. Ratio of the distributed a-c resistance per unit width for an unbounded plane conductor of thickness t/δ skin depths to the distributed a-c resistance per unit width of an indefinitely thick conductor of the same metal. For t/δ less than about 0.5, the distributed a-c resistance is equal to the distributed d-c resistance. For t/δ greater than about 3, the distributed a-c resistance is equal to R_s.

6.4. Distributed resistance of tubular circular conductors.

Circular metal tubes are the optimum outer conductors for a coaxial transmission line, and they may be used for the center conductor of such a line also, or for the two identical conductors of a parallel wire line, in cases where solid conductors would be excessively heavy or would involve unduly inefficient use of metal.

Unfortunately, rigorous analysis of the distributed resistance and distributed internal inductance of circular tubular conductors, for all ranges of the ratio of wall thickness to outside diameter and of the ratio of outside diameter to skin depth, is a considerably more complicated and tedious procedure than the corresponding analysis for solid circular conductors given in Section 6.2. Equations (*6.9*) and (*6.10*) are still the basic equations, but boundary conditions must be met at two boundaries instead of one, and the coefficient A_2 in (*6.10*) is not zero.

The results of a complete analysis of the distributed internal *inductance* of tubular circular conductors do not appear to be available in convenient form. Discussion of the relative magnitudes of the distributed *internal* inductance and the distributed *external* inductance for various transmission line designs is continued in Section 6.8, which also includes a review of the procedures for estimating the value of the former.

The distributed a-c *resistance* of an isolated circular tubular conductor for a wide range of the variables was computed several decades ago by H. B. Dwight, with results shown graphically in Fig. 6-8 below. The top linear horizontal scale in $\sqrt{f/R_{d\text{-}c}}$, where f is the frequency in hertz and $R_{d\text{-}c}$ the distributed d-c resistance in ohms/meter length, applies to both solid and tubular circular conductors made of nonmagnetic material. For solid circular nonmagnetic conductors $\sqrt{f/R_{d\text{-}c}}$ is easily shown to be directly proportional to the variable a/δ already used in discussing the distributed a-c resistance of such conductors. The constant of proportionality is such that $\sqrt{f/R_{d\text{-}c}} = 892$ corresponds to $a/\delta = 1$. In calculating this, it is found to be given by $1/\sqrt{\mu_0}$ where μ_0 ($= 4\pi \times 10^{-7}$ henries/m) is the mks value of the permeability of nonmagnetic materials. Since $t/a = 1$ corresponds to a solid conductor, the curve in Fig. 6-8 for $t/a = 1$ is a plot of the data of Table 6.1.

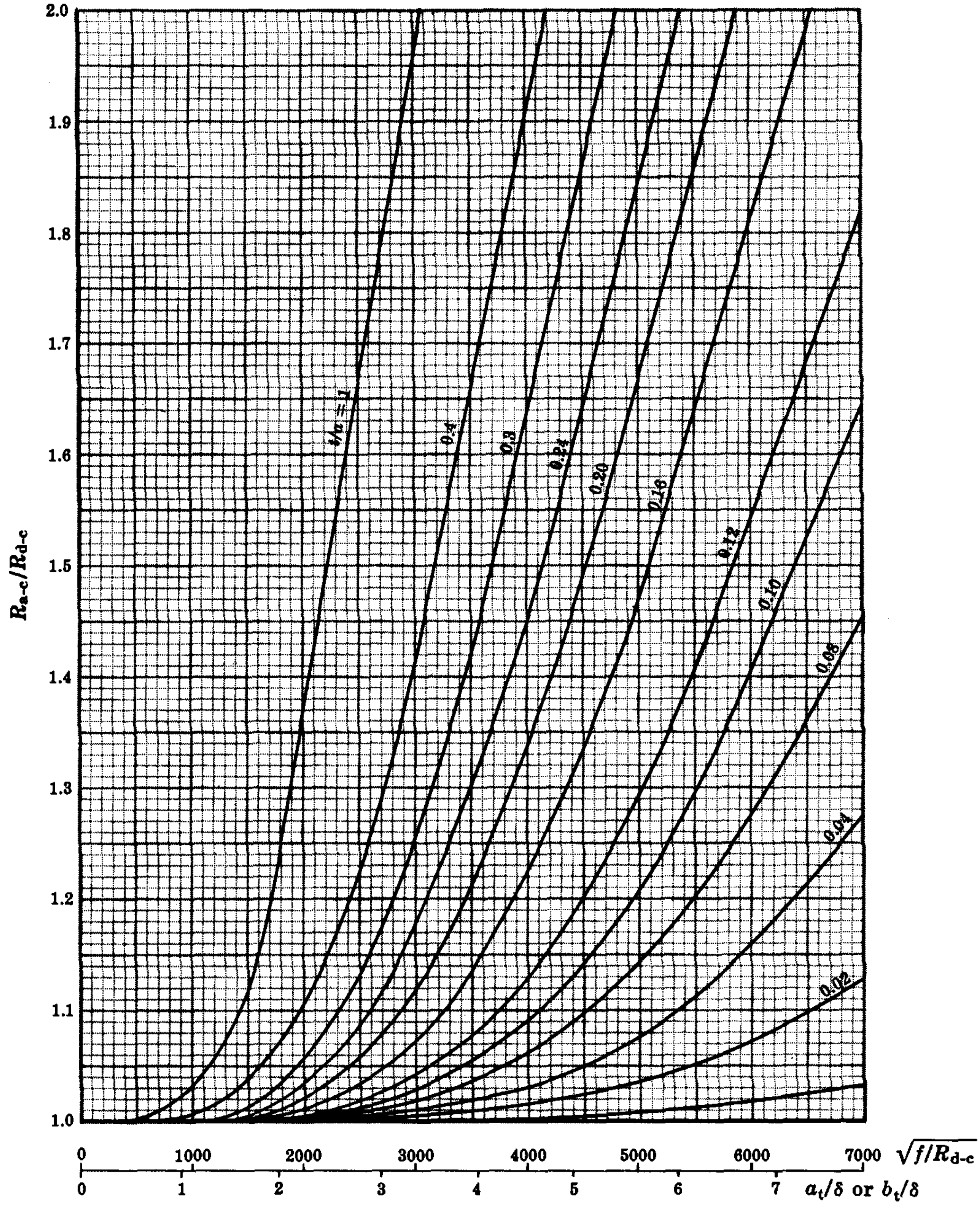

Fig. 6-8. Ratio of the distributed a-c resistance to the distributed d-c resistance of a circular tubular conductor of outside radius a, inside radius b, and wall thickness t as a function of dimension parameters in skin depths, for several values of the ratio of wall thickness to outside radius. The variables a_t and b_t are given by equations (*6.47*) and (*6.48*) respectively. The horizontal scale $\sqrt{f/R_{d\text{-}c}}$, for which f is in hertz and $R_{d\text{-}c}$ is in ohms/m, applies only to nonmagnetic conductor materials. The scales a_t and b_t apply to both magnetic and nonmagnetic materials. The ratio of the two scales at any point is 892. (After H. B. Dwight.)

For tubular conductors the scale conversion in Fig. 6-8 is a less direct one. The quantity corresponding to a/δ for the solid circular conductor is $\sqrt{2at-t^2}/\delta$ where a is the *outside* radius of the conductor and t its wall thickness. It reduces to a/δ when $t = a$. To permit further reference, this quantity is given the symbol a_t/δ. Then

$$a_t/\delta \;=\; \sqrt{2at-t^2}/\delta \tag{6.47}$$

When reference to the inside radius b of a tubular conductor is more convenient, the variable b_t/δ is defined by

$$b_t/\delta \;=\; \sqrt{2bt+t^2}/\delta \tag{6.48}$$

For a given tube a_t and b_t have identical values. The bottom horizontal scale of Fig. 6-8 is the same for a/δ, a_t/δ and b_t/δ.

The coordinate quantity $\sqrt{f/R_{\text{d-c}}}$ in Fig. 6-8 is unaffected by whether the conductor is magnetic or not. The question therefore arises as to how the graph should be used for conductors made of magnetic metals. The fact that the curve in Fig. 6-8 for solid conductors ($t/a = 1$) agrees with the $R/R_{\text{d-c}}$ figures from Table 6.1 where the permeability of the metal is accounted for in the skin depth δ, indicates that the abscissa scale of a_t/δ and b_t/δ (or a/δ for solid conductors) is applicable to either magnetic or nonmagnetic conductors. For conductors made of magnetic materials the scale $\sqrt{f/R_{\text{d-c}}}$ must not be used. It is clear from Dwight's writings that Fig. 6-8 was in fact constructed from expressions that derive from equations (*6.10*) or (*6.16*), in which the variable has the form $\sqrt{2}\,r/\delta$.

Since there are no electromagnetic fields in the interior space of the tubular conductor for which Fig. 6-8 was computed, and since the external fields and those within the metal must be the same at all angular positions around such a conductor, the only transmission line conductors to which Fig. 6-8 is directly applicable are the cases of a circular metal tube used as either the center conductor of a coaxial line or as a conductor of a parallel wire line whose two conductors are widely separated. However, if the ratio of a tube's wall thickness t to its external radius a does not exceed about 20%, the distributed resistance of a circular tube used as the *outer* conductor of a coaxial line can also be taken from Fig. 6-8 with adequate accuracy.

It is useful to classify the procedures for calculating the distributed a-c resistance of circular tubular conductors into the same five categories that were established in Section 6.2 for solid circular conductors.

(1) At the lowest frequencies, the quantitative relation for solid circular conductors stated in Section 6.2, that $R_{\text{a-c}}/R_{\text{d-c}} < 1.005$ for a/δ from 0 to 0.5, applies directly to tubular conductors if the modified variable $(\sqrt{t/a})(a/\delta)$ is substituted for a/δ. This statement is derived empirically from Fig. 6-8 and is a consequence of the shapes and interrelations of the curves of that figure. It has no connection with the variables a_t/δ or b_t/δ derived from the horizontal scale conversions.

(2) Using the modified variable $(\sqrt{t/a})(a/\delta)$ in place of a/δ, equation (*6.28*) gives the ratio of the distributed a-c resistance to the distributed d-c resistance for circular tubular conductors within $\frac{1}{2}\%$ for values of the modified variable up to 1.5. Equation (*6.28*) then covers all of the portion of Fig. 6-8 for which $R_{\text{a-c}}/R_{\text{d-c}}$ lies between 1.0 and 1.1.

(3) For values of the *unmodified* variable a/δ greater than 100, the distributed a-c resistance of circular tubular conductors is calculated from the same plane surface approximation used for solid circular conductors. Equation (*6.30*) will give results accurate to $\frac{1}{2}\%$, when a is the *radius of the surface carrying the surface current*, provided the tube's wall is "thick" (i.e. t/δ greater than about 3), a condition that will almost invariably be satisfied for such large values of a/δ. For tubular conductors used in transmission lines the current carrying surface is the outside surface except when the tube is the

outer conductor of a coaxial line, in which case it is the inside surface. In the unlikely event that $t/\delta < 3$ when $a/\delta > 100$, the distributed a-c resistance value given by (*6.30*) for the appropriate surface must be multiplied by a correction factor from Fig. 6-7 for conductors of "finite" thickness.

(4) For values of the unmodified variable a/δ down to 4, equations (*6.32*) and (*6.33*) derived for solid circular conductors also apply to tubular circular conductors, subject to the two conditions stated in (3), and with the further modification that if the tube is the *outer* conductor of a coaxial line, the sign in the denominator of each equation must be changed to positive.

As in (3), if $t/\delta < 3$, the multiplying factor from Fig. 6-7 must be used. With a/δ as low as 4, if t/a is very small then t/δ may be considerably less than unity, in which case the distributed a-c resistance of the conductor becomes equal to its d-c resistance, as can in fact be seen directly in Fig. 6-8. What has happened in this case is that conditions have fallen into a category already covered in (1) or (2) for the modified variable $(\sqrt{t/a})(a/\delta)$.

(5) For combinations of a/δ and t/a or t/δ not covered by any of the four categories above, Fig. 6-8 is used directly, with the horizontal scale in a_t or b_t, after one of those quantities has been calculated from (*6.47*) or (*6.48*).

A final precaution must be repeated, that for the unusual combination of a/δ fairly small and t/a fairly large (i.e. a very thick walled tube at a low to intermediate frequency) the value given by Fig. 6-8 for $R_{a\text{-}c}/R_{d\text{-}c}$ may be as much as several percent in error for a tubular conductor used as the *outer* conductor of a coaxial line. The inaccuracy is partially discounted if the distributed resistance of the outer conductor of the coaxial line is much less than that of the inner conductor, as is often true.

Example 6.2.

A coaxial transmission line has a solid copper center conductor 0.100″ in diameter. The outer conductor is a tube of outside diameter 0.375″ and wall thickness 0.0100″. The line is used at carrier frequencies between 60 kilohertz and 10 megahertz. Determine the distributed resistance of each of the conductors at the highest and lowest frequencies used on the line.

Using a for the outer radius of the inner conductor and b for the inner radius of the outer conductor, the quantities a/δ, b/δ, t/δ, a_t/δ, and t/b must be evaluated first, to identify the category of each of the four distributed resistance calculations.

The skin depth δ is given by equation (*6.15*) which for 100% conductivity copper ($\sigma = 5.80 \times 10^7$ mhos/m) becomes $\delta = 0.0661/\sqrt{f}$ m when f is in hertz. From this, $\delta = 2.70 \times 10^{-4}$ m at 60 kilohertz, and $\delta = 2.09 \times 10^{-5}$ m at 10 megahertz. The required values of the five ratios at the two frequencies are:

	60 kilohertz	10 megahertz
For the inner conductor, a/δ ($a = 1.27 \times 10^{-3}$ m)	4.71	61
For the outer conductor, b/δ ($b = 4.51 \times 10^{-3}$ m)	16.7	216
t/δ ($t = 2.54 \times 10^{-4}$ m)	0.94	12.2
a_t/δ (equation (*6.47*))	5.7	73.5
t/b	0.056	0.056

Two of the distributed resistance calculations fall clearly into category (4), one clearly into (3), and one marginally ($a/\delta = 61$) into (3). For both calculations in category (4) the values of a/δ or a_t/δ are low enough for the results to be checked by the procedure of category (5), i.e. by Fig. 6-8. For one of the calculations in category (4) the value of t/δ is small enough to require a correction for insufficient wall thickness. Evaluating the modified variable $(\sqrt{t/a})(a/\delta)$ does not change the classification of any of the calculations.

The distributed resistance calculations are as follows.

Inner conductor at 60 kilohertz, category (4):

$$R_{\text{a-c}} = \frac{1}{\sigma 2\pi(a - \frac{1}{2}\delta)\delta} = \frac{1}{2\pi(5.80 \times 10^7)(1.27 \times 10^{-3} - 1.35 \times 10^{-4})(2.70 \times 10^{-4})} = 0.00894 \text{ ohms/m}$$

In this calculation the length units for a, b and δ must be the same as for σ, in meters.

This result can be checked on Fig. 6-8 by a linear extrapolation of the curve for $t/a = 1$ to the ordinate $a/\delta = 4.71$. The result is $R_{\text{a-c}}/R_{\text{d-c}} = 2.63$ and since $R_{\text{d-c}}$ from equation (*6.1*) is found to be 0.00340 ohms/m, $R_{\text{a-c}}$ is given as 0.00894 ohms/m.

Inner conductor at 10 megahertz, category (4):

$$R_{\text{a-c}} = \frac{1}{2\pi(5.80 \times 10^7)(1.27 \times 10^{-3} - 1.04 \times 10^{-5})(2.09 \times 10^{-5})} = 0.104 \text{ ohms/m}$$

If this case is calculated as category (5) the result is 0.103 ohms/m, the difference being very small for such a high value of a/δ.

Outer conductor at 60 kilohertz, category (4):

$$R_{\text{a-c}} = \frac{1}{2\pi(5.80 \times 10^7)(4.51 \times 10^{-3} + 1.35 \times 10^{-4})(2.70 \times 10^{-4})} = 0.00219 \text{ ohms/m}$$

The change of sign in the denominator term, relative to equation (*6.32*) as used in the two previous calculations, results from this being the *outer* conductor of a coaxial line. The current carrying surface is the inside surface of the tube.

Since t/δ is only 0.94 for this tube at this frequency, there is a correction factor from Fig. 6-7 of 1.14 and the distributed resistance of the outer tube at 60 kilohertz is $0.00219 \times 1.14 = 0.00250$ ohms/m.

The result can be checked directly from Fig. 6-8. For $a_t/\delta = 5.7$, $R_{\text{a-c}}/R_{\text{d-c}}$ is found to be between 1.07 and 1.08 for a tube whose ratio of wall thickness to radius is 0.056. From a modification of equation (*6.1*), $R_{\text{d-c}} = \dfrac{1}{\sigma 2\pi(b + \frac{1}{2}t)t} = 0.00233$ ohms/m, and hence $R_{\text{a-c}} \doteq 0.00233 \times 1.07 \doteq 0.00250$ ohms/m, within about $\frac{1}{2}\%$.

Outer conductor at 10 megahertz, category (5):

$$R_{\text{a-c}} = \frac{1}{\sigma 2\pi b\delta} = \frac{1}{2\pi(5.80 \times 10^7)(4.51 \times 10^{-3})(2.09 \times 10^{-5})} = 0.0292 \text{ ohms/m}$$

6.5. Distributed circuit coefficients of coaxial lines.

(a) Distributed resistance.

Methods for calculating distributed resistances have been fully covered in Sections 6.2, 6.3 and 6.4, including Example 6.2. For coaxial lines used at frequencies of tens to hundreds of megahertz and higher, a specific expression obtained from equation (*6.30*) is worth noting:

$$R_{\text{hf}}(\text{coax}) = \frac{R_s}{2\pi b}\left(1 + \frac{b}{a}\right) \tag{6.49}$$

where $R_{\text{hf}}(\text{coax})$ is the *total* distributed resistance of the coaxial line at very high frequencies. Here a is the outside radius of the inner conductor, b the inside radius of the outer conductor, and $R_s = 1/(\sigma\delta) = \sqrt{\omega\mu/2\sigma}$ is the surface resistivity of the conductor material defined in connection with equation (*6.27*) and assumed the same for both conductors. This expression requires that the wall thickness of both conductors be greater than 3δ, and will give better than $\frac{1}{2}\%$ accuracy when $a/\delta > 100$, which ensures that $b/\delta \gg 100$. If the two conductors are of different materials, equation (*6.30*) must be applied separately to each. For values of $a/\delta < 100$ the distributed resistance must be calculated by one of the other methods described in the preceding sections.

(b) Distributed capacitance.

Derivation of an expression for the distributed capacitance of a coaxial line is the simplest of all the distributed circuit coefficient derivations, and no case has fewer complicating factors in its practical calculations.

Fig. 6-9 shows the central cross section, in a plane containing the axis, for a portion of coaxial line included between two transverse planes separated by distance Δl. The facing surfaces of the two conductors have radii a and b respectively, as prescribed in (a) above. It is postulated that there is free charge $+\Delta Q$ coulombs uniformly distributed over the length Δl of the outside surface of the inner conductor of the line and free charge $-\Delta Q$ coulombs similarly uniformly distributed over the length Δl of the inside surface of the outer conductor. It is also postulated that the electric charge distribution on the two conductors is continuous on either side of the length Δl so that the electric flux lines throughout Δl are everywhere radial. From symmetry the electric flux density at any point in the interconductor space over the length Δl is independent of angular position around the axis of the conductors.

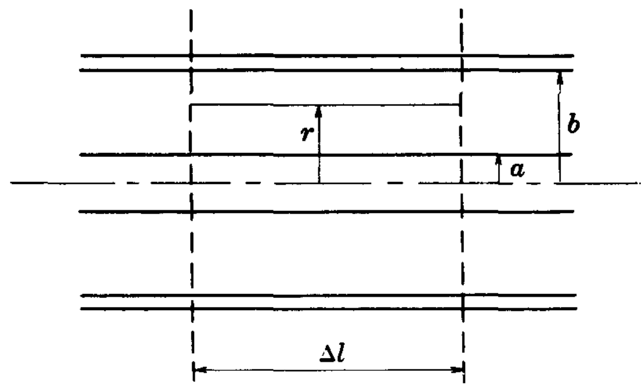

Fig. 6-9. Cross section of a coaxial transmission line in a longitudinal diametral plane with notation for determining distributed capacitance.

Crossing the length Δl of any cylinder of radius r concentric with the line's conductors, where $a<r<b$, the total electric flux is ΔQ coulombs, and the radial electric flux density $D_r(r)$ at any point on the cylinder is

$$D_r(r) \quad = \quad \frac{\Delta Q}{\Delta l}\frac{1}{2\pi r} \quad = \quad \frac{\rho_l}{2\pi r} \quad \text{coulombs/m}^2 \tag{6.50}$$

where ρ_l is the uniformly distributed longitudinal charge density in coulombs/m over Δl.

If the space between the conductors is filled with a homogeneous lossless isotropic medium of (real) permittivity ϵ' farads/m, the electric field $E_r(r)$ at radius r is

$$E_r(r) \quad = \quad \frac{D_r(r)}{\epsilon'} \quad = \quad \frac{\rho_l}{2\pi\epsilon' r} \quad \text{volts/m} \tag{6.51}$$

Postulate 4 of transmission line analysis, as given in Chapter 2, states that the potential difference between the two conductors of a line at any cross section has a unique value given by the line integral of the electric field along any path between the conductor surfaces, which are equipotentials. Since in the present derivation Δl can always be taken small enough to ensure that there is negligible longitudinal potential difference along the length Δl of the conductors, it follows that the potential difference between the two conductors at any cross section within Δl is given by $V_a - V_b = \int_a^b -E_r(r)\,dr$, or

$$V_b \;-\; V_a \quad = \quad \int_a^b \frac{\rho_l\,dr}{2\pi\epsilon' r} \quad = \quad \frac{\rho_l}{2\pi\epsilon'}\log_e\frac{b}{a} \quad \text{volts} \tag{6.52}$$

By definition the capacitance between any two conductors or conductor elements is the ratio of the magnitude of either of the equal and opposite charges on them to the potential difference associated with the charges. Then if ΔC is the capacitance between the conductors of a coaxial line for length Δl, $\Delta C = \Delta Q/(V_b - V_a)$, and the *distributed* capacitance of the line is $C = \Delta C/\Delta l = (\Delta Q/\Delta l)/(V_b - V_a)$. Using equation (*6.52*) and the relation $\Delta Q/\Delta l = \rho_l$,

$$C = \frac{2\pi\epsilon'}{\log_e b/a} \quad \text{farads/m} \tag{6.53}$$

If k'_e is the real dielectric constant of the lossless material filling the interconductor space, $k'_e = \epsilon'/\epsilon_0$, where $\epsilon_0 = 8.85 \times 10^{-12}$ farads/m is the permittivity of free space. Equation (*6.53*) can be written

$$C = \frac{55.6\,k'_e}{\log_e b/a} \quad \text{micromicrofarads/m} \tag{6.54}$$

For most practical coaxial transmission lines the ratio b/a lies within a fairly narrow range from about 2 to less than 10, and the logarithm is then confined to an even smaller range from about 0.7 to 2. Except in unusual cases k'_e is the low dielectric constant of a low-loss low-density material, or is a small average value for a line partly filled with dielectric beads or discs. The distributed capacitance of a coaxial transmission line therefore usually lies within the range of about 25 to 200 micromicrofarads/m, and values between 50 and 100 micromicrofarads/m are most common.

Distributed capacitance calculations differ from those for distributed resistance and distributed inductance in that the effects of fields and currents within the metal of the conductors are completely negligible, for all conceivable physical conditions. Equation (*6.54*) is therefore a highly accurate and complete expression for the case to which it applies, of smooth uniform conductors with isotropic homogeneous dielectric filling the interconductor space. The complications introduced by a multi-strand center conductor or a braided outer conductor are not amenable to simple analytical treatment, and the distributed capacitance of such lines is usually determined by experimental measurement.

The dielectric constant k'_e of the insulating materials most widely used in coaxial transmission lines, such as teflon, polyethylene and polystyrene plastics, and steatite ceramics, varies by less than $\frac{1}{2}$% over the frequency range from 60 hertz to 10 gigahertz or more, so that the distributed capacitance is generally a true "constant" of the line. Other less common insulating materials, including bakelite, glasses, acrylic plastics, rubbers, etc., show several percent variation of k'_e over the same frequency range.

(c) *Distributed conductance.*

In very rare instances the interconductor space of a coaxial transmission line is filled with material which conducts electricity by the actual flow of charged carriers, either electrons or ions. Damp earth, electrolytic solutions, and plasters or ceramics containing dispersed carbon would be examples of such materials, whose current carrying properties would be described by a true value of conductivity having the same significance as the conductivity of a metal. The distributed conductance of a transmission line filled with such a conducting dielectric is found, by appropriate application of equation (*6.1*), as the conductance between the inner and outer cylindrical surfaces for the piece of material filling unit length of the line (see Problem 6.31). Depending on the material, such a charge-flow conductance value may remain constant from zero frequency up to a few hertz, or to a few kilohertz, or even into the gigahertz frequency range.

Apart from these exceptional cases, which would exist only for special purposes and not as lines intended for efficient transmission of signals or power, the zero frequency distributed conductance of coaxial transmission lines insulated with plastics or ceramics is normally so small as to be difficult to measure, and the distributed conductance at any operating frequency is not caused by the flow of free charges but is a measure of internal dielectric losses in the insulating material resulting from repeated reversals of the dielectric polarization by the a-c electric field. Since the loss is thus on a per cycle basis, it tends to be directly proportional to frequency over wide ranges of frequency. It is then desirable to have an expression for the distributed conductance of a transmission line that contains the frequency explicitly, the remaining terms in the expression being more or less independent of frequency. This goal is achieved by the standard convention of designating the permittivity or dielectric constant of a lossy (but nonconducting for d-c) dielectric by a complex number. Thus for such a material

$$\epsilon \equiv \epsilon' - j\epsilon'', \qquad k_e \equiv k_e' - jk_e'' = \epsilon'/\epsilon_0 - j\epsilon''/\epsilon_0 \tag{6.55}$$

The distributed admittance of the distributed capacitance of a uniform transmission line is given by $G + jB$, where G is one of the line's four distributed circuit coefficients and $B = \omega C$ is the distributed susceptance of the line's distributed capacitance C. If in equation (*6.53*) for the distributed capacitance of a coaxial line the permittivity ϵ' or the dielectric constant k_e' is replaced by a complex number form from (*6.55*), then the quantity calculated as the distributed susceptance B will have both real and imaginary parts. The negative imaginary part of ϵ' contributes a positive real part to the distributed admittance, which is the distributed conductance of the line. Thus $G + jB = j\omega C = j\omega 2\pi(\epsilon' - j\epsilon'')/(\log_e b/a) = \omega\epsilon'' 2\pi/(\log_e b/a) + j\omega\epsilon' 2\pi/(\log_e b/a)$. From the real parts of this relation,

$$G = \omega\frac{\epsilon''}{\epsilon'}\frac{2\pi\epsilon'}{\log_e b/a} = \omega C \tan\delta \tag{6.56}$$

where $C = 2\pi\epsilon'/(\log_e b/a)$ as in equation (*6.53*), and $\tan\delta = \epsilon''/\epsilon'$.

The quantity $\tan\delta$ used to designate the lossiness of a dielectric in a-c electric fields is called the loss factor or the tangent of the loss angle for the material. *It cannot be too strongly emphasized that this use of the Greek letter δ is in no way connected with the previous use of the letter δ for the skin depth in a conductor carrying a-c currents.* It is an unfortunate duplication that is firmly entrenched as the established notation in each case.

Equation (*6.56*) shows that the distributed conductance of a coaxial transmission line is directly proportional to frequency, if the loss factor $\tan\delta$ and the line's distributed capacitance C are independent of frequency. The constancy of C has been discussed in (*b*) above. For the most commonly used low loss insulating materials mentioned there, the loss factor $\tan\delta$ lies between 10^{-3} and 10^{-5}. Tabulated values often indicate only that the loss factor of polyethylene, for example, is less than 0.0002 for frequencies from hertz to gigahertz. This usually means that the value was below the sensitivity of the measuring equipment. Except at gigahertz frequencies, the value of G resulting from such low values of $\tan\delta$ is likely to be too small to affect the attenuation factor α as determined by the methods of Chapter 5. There is evidence that the precise value of $\tan\delta$ may vary considerably among different samples of polymer plastic dielectrics, depending on impurities and thermal history. Values of $\tan\delta$ listed in standard handbooks for low loss materials should usually be considered as approximate typical values only.

Example 6.3.

If the interconductor space of the coaxial transmission line of Example 6.2 is filled with teflon dielectric having constant $k_e' = 2.10$ and constant $\tan\delta = 0.00015$ over the frequency range 10 megahertz to 10 gigahertz, determine the distributed capacitance C and the distributed conductance G of the line at frequencies of 10^7, 10^8, 10^9 and 10^{10} hertz.

The distributed capacitance C has the same value at all the frequencies listed, given by equation (*6.53*) as

$$C = 2\pi(2.10 \times 8.85 \times 10^{-12})/(\log_e 0.1775/0.050) = 92.2 \text{ micromicrofarads/m}$$

The distributed conductance G is then directly proportional to frequency, according to (*6.56*), and at 10^7 hertz has the value

$$G = 2\pi \times 10^7(92.2 \times 10^{-12}) \times 0.00015 = 0.87 \text{ micromhos/m}$$

The values of G at the other frequencies are respectively 8.7, 87 and 870 micromhos/m.

(d) Distributed inductance.

Of the three distributed circuit coefficients so far considered for a coaxial line, the distributed resistance R is a quantity entirely "internal" to the conductors, being determined completely by the conductor materials and dimensions and the frequency. It is in no way dependent on the properties of the uniform medium filling the interconductor space. The distributed capacitance C and the distributed conductance G, on the other hand, are "external" quantities, being totally independent of the material of the conductors or of the transverse extension of the conductors on either side of the interconductor space. They are functions only of the nature and dimensions of the material filling the interconductor space, and of the frequency.

The fourth distributed circuit coefficient, distributed inductance, is the only one of the four such coefficients for a transmission line that in some cases has to be determined as the sum of both "external" and "internal" components. It is true that at frequencies above some minimum frequency that is typically in the megahertz region the distributed reactance of the distributed internal inductance of circular or plane conductors becomes equal to the distributed resistance of the conductors, as shown by equations (*6.31*) and (*6.45*), and since $\omega L/R \gg 1$ always under such conditions, it follows that the distributed external inductance must be very much greater than the distributed internal inductance for coaxial lines at such frequencies. However, at low frequencies the distributed internal inductance of a solid circular conductor as given by equation (*6.5*) can be a substantial fraction of the total distributed inductance, for a coaxial line.

The expression for the distributed external inductance L_x to be derived for a coaxial line is independent of frequency if the magnetic properties of the material in the interconductor space are not functions of frequency. It is the total distributed inductance of the coaxial line at frequencies high enough to make $a/\delta \gg 100$. At lower frequencies it must be added to a distributed internal inductance term L_i determined by methods reviewed in Section 6.8.

Referring to Fig. 6-10, a uniform coaxial transmission line has circular conductors, the outside radius of the inner conductor being a and the inside radius of the outer conductor being b, as in Fig. 6-9. Distributed external inductance is a measure of the linkage of magnetic flux in the interconductor space with the center conductor of the line as a distributed "circuit".

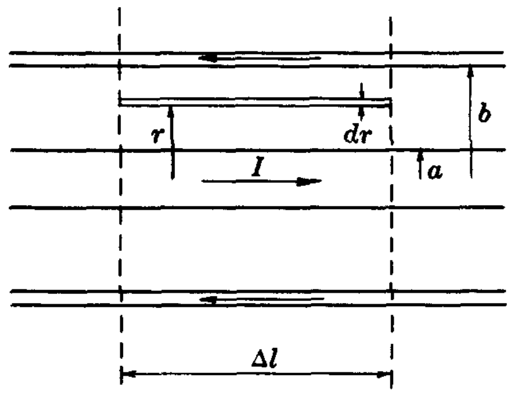

Fig. 6-10. Cross section of a coaxial transmission line in a longitudinal diametral plane with notation for determining distributed external inductance.

Considering a section of the transmission line of length Δl between transverse planes, a total current I flows longitudinally in one direction in the inner conductor, and an equal current flows in the opposite direction in the outer conductor. From the symmetry of the line the currents are uniformly distributed with respect to angular position around the periphery of the conductors, and the magnetic flux lines produced in the interconductor space by the current in the center conductor are circles concentric with the conductor. Δl is assumed short enough that no quantities in the problem vary in the longitudinal direction.

The only component of magnetic flux density is B_ϕ and its value $B_\phi(r)$ at any point in the interconductor space distant r from the central axis is

$$B_\phi(r) \quad = \quad \mu_m' I/2\pi r \ \text{teslas} \tag{6.57}$$

where μ_m' is the (real) permeability of *the medium in the interconductor space.*

The magnetic flux passing through the small rectangle of length Δl and radial width dr at coordinate dr in the radial plane is $d\psi = B_\phi(r)\,\Delta l\, dr$. Then the total flux in the interconductor space linking length Δl of the center conductor is

$$\Delta\psi \quad = \quad \Delta l \int_a^b (\mu_m' I)/(2\pi r)\, dr \quad = \quad (1/2\pi)\,\Delta l\, \mu_m' I \log_e b/a$$

The distributed external inductance of the line, L_x, defined as the amount of external flux-circuit linkage per unit length per unit current, is then

$$L_x \quad = \quad \frac{1}{I}\frac{1}{\Delta l}\Delta\psi \quad = \quad \frac{\mu_m'}{2\pi}\log_e\frac{b}{a} \ \text{henries/m} \tag{6.58}$$

Except for peculiar situations, such as filling the interconductor space of a coaxial line with a ferrite material, the permeability μ_m' always has the value for free space, $\mu_0 = 4\pi \times 10^{-7}$ henries/m, which is an exactly defined value in the mks system of units.

The comments in (c) above about the limited variation of b/a and of $\log_e(b/a)$ for practical coaxial lines are directly applicable here again, and mean that the distributed external inductance L_x of coaxial lines usually lies between about 0.1 and 10 microhenries/m. (In contrast to the small range of values of L_x and C for practical coaxial lines, the distributed resistance R varies by a factor of 10^5 among common types of lines.)

Example 6.4.

For the transmission line of Examples 6.2 and 6.3 determine the distributed external inductance. Then at a frequency of 10 megahertz find the characteristic impedance, phase velocity and attenuation factor of the line.

The distributed external inductance L_x at all frequencies is given by equation (*6.58*) with $\mu_m' = \mu_0$ for all nonmagnetic media in the interconductor space. Thus $L_x = 2 \times 10^{-7} \log_e 0.1775/0.050 = 0.253$ microhenries/m.

From Example 6.2, the values of a/δ and b/δ for this coaxial line at 10 megahertz are 61 and 216 respectively, which according to Table 6.2 ensures that the distributed internal inductance of the center conductor of the line is less than about 3% of its zero frequency value of 0.05 microhenries/m. It is therefore somewhat under 1% of the distributed external inductance and can be neglected. At the value $b/\delta = 216$ for the tubular outer conductor, equation (*6.31*) applies and $L_i = R/\omega = 0.00046$ microhenries/m, where R has the value 0.0292 ohms/m given in Example 6.2. This also is negligible in comparison with L_x. Hence for this coaxial line at a frequency of 10 megahertz, $L = L_x + L_i = L_x = 0.253$ microhenries/m. A check shows that at 10 megahertz $\omega L/R \doteq 120$, where R is the total distributed resistance of the line, and $\omega C/G = 1/(\tan\delta) \doteq 6000$. Hence the high frequency approximate equations of Chapter 5 are to be used in determining Z_0, v_p and α. The results are

$$Z_0 \quad = \quad \sqrt{L/C} \quad = \quad \sqrt{(2.53 \times 10^{-7})/(92.2 \times 10^{-12})} \quad = \quad 52.4 \text{ ohms}$$

$$v_p \quad = \quad 1/\sqrt{LC} \quad = \quad 2.07 \times 10^8 \text{ m/sec}$$

$$\alpha \quad = \quad R/2Z_0 + GZ_0/2 \quad = \quad 0.00127 + 0.000023 \quad = \quad 0.00129 \text{ nepers/m}$$

It is to be noted that dielectric losses contribute about 2% of the attenuation factor at this frequency.

(e) Summary of high frequency relations for the coaxial line.

If the conditions $a/\delta > 100$, $b/\delta > 100$, $\omega L/R > 10$ and $\omega C/G > 10$ *all* hold for a coaxial line, which is likely to be the case for most lines at frequencies above some value between 1 and 100 megahertz depending on the line dimensions and materials, the following simplified expressions are easily derived

$$Z_0 \quad = \quad \sqrt{\frac{\mu_m'/2\pi}{2\pi e'}}\log_e\frac{b}{a} \quad = \quad \frac{1}{2\pi\sqrt{k_e'}}\sqrt{\frac{\mu_0}{\epsilon_0}}\log_e\frac{b}{a}$$
$$= \quad \frac{60}{\sqrt{k_e'}}\log_e\frac{b}{a} \quad = \quad \frac{138}{\sqrt{k_e'}}\log_{10}\frac{b}{a} \ \text{ohms} \tag{6.59}$$

$$v_p \;=\; \frac{1}{\sqrt{\mu_m'\epsilon'}} \;=\; \frac{1}{\sqrt{k_e'}}\frac{1}{\sqrt{\mu_0\epsilon_0}} \;=\; \frac{3.00\times 10^8}{\sqrt{k_e'}} \text{ m/sec} \tag{6.60}$$

Equations (*6.59*) and (*6.60*) both make the assumption, usually valid, that the material in the interconductor space of the coaxial line has the magnetic properties of free space. If this is not the case, Z_0 must be multiplied by, and v_p must be divided by, the square root of the relative permeability of the medium.

6.6. Distributed circuit coefficients of transmission lines with parallel circular conductors.

Measured by the number of miles in practical operation, parallel wire transmission lines must constitute a large majority of transmission line installations. Telephone pole lines and cable pairs and television antenna lead-in lines are among the most common examples.

The derivation of expressions for the distributed circuit coefficients of parallel wire lines follows the same basic procedures used in the derivations for coaxial lines, but the departure from cylindrical symmetry adds a complication. The effects of this distortion of symmetry are known collectively as "proximity effect". In the analyses for distributed capacitance, conductance and external inductance of parallel wire lines, proximity effect can be allowed for without too much difficulty, but the complete analyses of distributed resistance and internal inductance for solid and circular tubular conductors are mathematically too cumbersome to be presented here. Tabulated data and approximate expressions are given in (*a*) below and in Section 6.8.

(*a*) Distributed resistance.

For parallel wire transmission lines with identical solid circular conductors, the amount by which proximity effect increases the resistance depends on the material and radius of the conductors, and the frequency, combined in the variable a/δ as previously, and on the "proximity" of the conductors, expressed by the ratio of the separation s of their centers to the diameter $2a$ of each of them. The most complete analysis available is that of A. H. M. Arnold.

In a d-c circuit ($a/\delta = 0$) there is no proximity effect even when the conductors are virtually in contact at their adjacent surfaces ($s/2a \doteq 1$). At any finite value of the conductor separation there is a minimum value of a/δ at which proximity effect increases the distributed resistance perceptibly. With the conductors almost touching, for example, the distributed resistance will increase about $\frac{1}{2}\%$ from proximity effect when a/δ is approximately 0.5; but if the conductor axes are 8 diameters apart, the same increase in distributed resistance will not occur until a/δ is increased to about 2. As $s/2a$ increases from 8 to just over 10, the minimum value of a/δ at which proximity effect increases the distributed resistance by $\frac{1}{2}\%$ rises from 2 to infinity; and for axial separations greater than 10 or 12 conductor diameters, proximity effect is negligible at all frequencies. This statement applies to all the distributed circuit coefficients, and to lines with either solid or tubular circular conductors. In some cases the effects become negligible at smaller separations.

At high frequencies for which $a/\delta > 100$, proximity effect increases the distributed resistance of solid circular conductors by a factor $P_{\rm hf}$ given by the simple formula

$$P_{\rm hf} \;=\; \frac{1}{\sqrt{1 - 1/(s/2a)^2}} \tag{6.61}$$

While the implication of infinite distributed resistance for conductors approaching contact is not to be taken literally, predicted values greater than 2 for $P_{\rm hf}$ when the adjacent surfaces of the line's conductors are only 1% of a diameter apart ($s/2a = 1.01$) have been confirmed experimentally.

Equation (*6.61*) applies also to circular tubular conductors for the stated conditions of $a/\delta > 100$, if the tube wall is thick, i.e. $t/\delta > 3$.

Example 6.5.

If the axes of the copper conductors in the 19 gauge cable pair transmission line of Example 5.4, page 52, are separated on the average by 2.0 conductor diameters, determine the lowest frequency at which proximity effect will increase the distributed resistance of the line by about $\frac{1}{2}\%$, using data for isolated 19 gauge conductors from Table 6.2, page 80. At a frequency of 3×10^8 hertz, find the distributed resistance of the line.

Interpolating among the approximate figures in the second paragraph of (*a*) above, it appears that for $s/2a = 2.0$ there should be a $\frac{1}{2}\%$ increase in distributed resistance for $a/\delta = 0.7$ approximately. From Table 6.2 this ratio occurs for 19 gauge copper wires at a frequency close to 10,000 hertz.

At a frequency of 3×10^8 hertz for the same wires, $a/\delta \doteq 100$. The distributed a-c resistance at that frequency from equation (*6.30*) is 1.57 ohms/m, for each conductor of the transmission line. The high frequency proximity effect factor from equation (*6.61*) is $P_{\text{hf}} = 1/\sqrt{1-(1/2)^2} = 1.15$. Hence the distributed resistance of the parallel wire line at 3×10^8 hertz is $1.57 \times 2 \times 1.15 = 3.61$ ohms/m. A small additional factor caused by the increase in effective length due to twist of the wire pair has been neglected.

After equation (*6.61*), the next simplest approximate equation for the proximity effect factor in parallel wire transmission lines having solid circular conductors applies to lines for which $s/2a > 2$, for all values of a/δ. Designating the factor as P_s under these conditions,

$$P_s = \frac{1}{\sqrt{1 - f_1(\sqrt{2}\,a/\delta)/(s/2a)^2}} \tag{6.62}$$

where $f_1(\sqrt{2}\,a/\delta)$ is tabulated in Table 6.4 as given by Arnold. The actual function f_1 derives from a Bessel equation similar to equation (*6.27*).

Table 6.4

$\sqrt{2}\,a_t/\delta$	f_1	f_4	$\sqrt{2}\,a_t/\delta$	f_1	f_4	$\sqrt{2}\,a_t/\delta$	f_1	f_4
0.2	0.000	0.000	2.3	0.436	0.201	4.8	0.731	0.240
0.3	0.000	0.000	2.4	0.470	0.205	5.0	0.739	0.236
0.4	0.001	0.001	2.5	0.502	0.208	5.5	0.760	0.223
0.5	0.002	0.002	2.6	0.530	0.210	6.0	0.778	0.209
0.6	0.004	0.004	2.7	0.556	0.213	6.5	0.795	0.196
0.7	0.007	0.007	2.8	0.578	0.215	7.0	0.809	0.185
0.8	0.013	0.012	2.9	0.598	0.218	7.5	0.821	0.174
0.9	0.020	0.019	3.0	0.614	0.221	8.0	0.832	0.165
1.0	0.030	0.029	3.1	0.629	0.224	9	0.849	0.148
1.1	0.044	0.040	3.2	0.641	0.227	10	0.864	0.135
1.2	0.061	0.054	3.3	0.652	0.230	11	0.876	0.123
1.3	0.081	0.070	3.4	0.661	0.233	12	0.886	0.113
1.4	0.106	0.088	3.5	0.668	0.235	14	0.902	0.098
1.5	0.135	0.106	3.6	0.675	0.238	16	0.914	0.086
1.6	0.167	0.123	3.7	0.681	0.240	18	0.923	0.077
1.7	0.203	0.140	3.8	0.687	0.242	20	0.931	0.069
1.8	0.240	0.156	3.9	0.692	0.244	25	0.944	0.056
1.9	0.280	0.169	4.0	0.696	0.245	30	0.953	0.047
2.0	0.320	0.180	4.2	0.705	0.246	35	0.960	0.040
2.1	0.360	0.189	4.4	0.714	0.245	40	0.965	0.035
2.2	0.399	0.196	4.6	0.722	0.243	50	0.972	0.028
						Above 50	$1 - \delta/a_t$	δ/a_t

Finally, an equation stated by Arnold to give the value of the proximity effect factor for all circular solid and tubular conductors at all separations and all frequencies, with an accuracy of better than $\frac{1}{2}\%$ in almost all cases, is

$$P = \frac{1}{\sqrt{1 - A_1/(s/2a)^2 + [A_2/(s/2a)^4]/[1 - A_3/(s/2a)^2]}} \tag{6.63}$$

where

$$A_1 = f_1(\sqrt{2}\,a_t/\delta) + \{1 - (a_t/a)^2 - (a_t/a)[1 - (a_t/a)]\,f_7(\sqrt{2}\,a_t/\delta)\}\,f_4(\sqrt{2}\,a_t/\delta)$$
$$A_2 = f_2(\sqrt{2}\,a_t/\delta) + [1 - (a_t/a)^2]\,f_5(\sqrt{2}\,a_t/\delta)$$
$$A_3 = f_3(\sqrt{2}\,a_t/\delta) + [1 - (a_t/a)^2]\,f_6(\sqrt{2}\,a_t/\delta)$$

In these equations, $a_t = \sqrt{2at - t^2}$ for a circular tubular conductor of outside radius a and wall thickness t, as in equation (*6.47*). For a solid circular conductor $t = a$ and $a_t = a$.

The functions f_1, f_2, f_3, f_4, f_5 and f_6 are tabulated in Tables 6.4, 6.5 and 6.6 for all values of the variable $\sqrt{2}\,a_t/\delta$ (which is $\sqrt{2}\,a/\delta$ for solid conductors). The function f_7 is an empirical one given by $f_7 = (\sqrt{2}\,a_t/\delta)^3/[400 + (\sqrt{2}\,a_t/\delta)^3]$.

Table 6.5

$\sqrt{2}\,a_t/\delta$	f_2	f_5	$\sqrt{2}\,a_t/\delta$	f_2	f_5
0.5	0.000	0.000	3.6	0.053	−0.026
1.0	−0.001	−0.002	3.8	0.046	−0.021
1.2	−0.001	−0.003	4.0	0.039	−0.017
1.4	−0.000	−0.005	4.2	0.033	−0.014
1.6	+0.003	−0.006	4.4	0.027	−0.011
1.8	0.011	−0.007	4.6	0.023	−0.009
2.0	0.022	−0.010	4.8	0.020	−0.008
2.2	0.037	−0.015	5.0	0.018	−0.007
2.4	0.051	−0.022	5.5	0.014	−0.006
2.6	0.062	−0.028	6.0	0.012	−0.006
2.8	0.068	−0.033	7	0.009	−0.006
3.0	0.069	−0.034	8	0.007	−0.005
3.2	0.066	−0.033	10	0.005	−0.004
3.4	0.060	−0.030	Above 10	$\frac{1}{4}(\delta/a_t)^2$	$2(\delta/a_t)^4 - \frac{1}{4}(\delta/a_t)^2$

Table 6.6

$\sqrt{2}\,a_t/\delta$	f_3	f_6	$\sqrt{2}\,a_t/\delta$	f_3	f_6
0.0	0.09	0.03	4.0	0.41	0.33
0.2	0.09	0.03	4.2	0.46	0.30
0.4	0.09	0.03	4.4	0.51	0.27
0.6	0.08	0.02	4.6	0.56	0.24
0.8	0.08	0.02	4.8	0.60	0.21
1.0	0.06	0.00	5.0	0.64	0.19
1.2	0.02	*0.00 (−0.05)	5.2	0.67	0.17
1.4	*0.00 (−1.2)	*0.00 (0.91)	5.4	0.69	0.16
1.6	*0.02 (0.17)	*0.00 (−3.2)	5.6	0.70	0.15
1.8	*0.05 (0.11)	*0.00 (−0.09)	5.8	0.71	0.15
2.0	*0.07 (0.09)	0.44	6	0.72	0.14
2.2	0.08	0.44	7	0.76	0.13
2.4	0.08	0.44	8	0.79	0.11
2.6	0.10	0.44	9	0.83	0.08
2.8	0.12	0.44	10	0.85	0.07
3.0	0.15	0.44	12	0.87	0.05
3.2	0.19	0.43	14	0.89	0.03
3.4	0.24	0.42	16	0.90	0.02
3.6	0.29	0.39	18	0.93	0.00
3.8	0.35	0.36	20	0.93	*0.00
			Above 20	$1 - \delta/a_t$	*0.00

Values marked with an asterisk are artificial values which should be used in equations (*6.62*) and (*6.63*). When $\sqrt{2}\,a_t/\delta > 20$, the true value of f_6 is $\delta/a_t - 5/64$.

Arnold notes that the peculiar behavior of equation (*6.63*) when $A_3/(s/2a)^2 = 1$ has no physical meaning and the arithmetical anomaly should be avoided by using the starred figures in Table 6.6 instead of the proper values of the functions f_3 and f_6 in a certain range of $\sqrt{2}\,a/\delta$. This is a very minor complication that will seldom be encountered.

(b) Distributed capacitance.

The derivation of an expression for the distributed capacitance of a parallel wire transmission line, including proximity effect, is a somewhat indirect procedure. Consider two indefinitely long parallel lines lying in the xz coordinate plane at locations $x = +d$ and $x = -d$, as shown in cross section in Fig. 6-11. On each line, electric charge is uniformly distributed longitudinally, the linear charge density being $+\rho_l$ coulombs/m for the line at $x = +d$, and $-\rho_l$ coulombs/m for the line at $x = -d$.

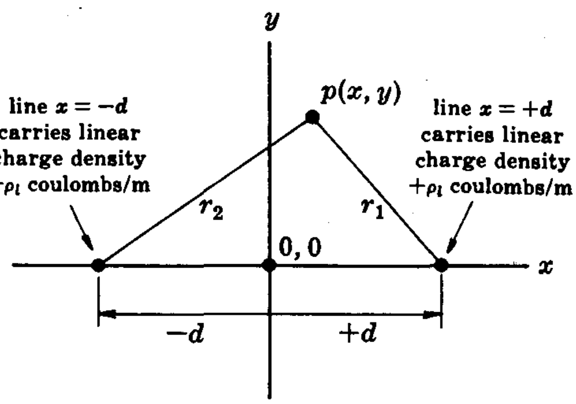

Fig. 6-11. Coordinates in a plane transverse to two infinitely long parallel lines of charge, having equal and opposite linear densities of charge.

The electric potential difference between any two points in the field of an indefinitely long uniformly distributed line of charge is a function of the linear charge density on the line, the permittivity of the surrounding medium, and the radial distances of the two points from the line. At any point $p(x, y)$ in an xy plane transverse to the charge lines of Fig. 6-11, the potential relative to a zero reference potential on the axis $x = y = 0$ is $V_p^+ = +(\rho_l/2\pi\epsilon')(\log_e d/r_1)$ from the field of the positively charged line, and $V_p^- = -(\rho_l/2\pi\epsilon')(\log_e d/r_2)$ from the field of the negatively charged line, r_1 and r_2 being respectively the distances from the point p to the positively and negatively charged lines. The total potential difference between the point p and the axis $x = y = 0$ is then

$$V_p \quad = \quad \frac{\rho_l}{2\pi\epsilon'} \log_e \frac{r_2}{r_1} \quad \text{volts} \tag{6.64}$$

An equipotential line in the transverse plane will be described by the relation

$$r_2/r_1 \quad = \quad \text{constant } K \quad = \quad e^{2\pi\epsilon' V_p/\rho_l} \tag{6.65}$$

From the coordinates of the point p, $r_1 = \sqrt{(d-x)^2 + y^2}$ and $r_2 = \sqrt{(d+x)^2 + y^2}$. Combining these expressions through $r_2/r_1 = K$, the equation of an equipotential line becomes

$$x^2 \; - \; 2xd\left(\frac{1+K^2}{1-K^2}\right) \; + \; y^2 \quad = \quad -d^2 \tag{6.66}$$

where the actual potential at any specific equipotential line (relative to zero potential at $x = y = 0$) is determined by the value of K, using (*6.65*).

Adding $d^2\left(\dfrac{1+K^2}{1-K^2}\right)^2$ to both sides of (*6.66*) completes a square with the first two terms, resulting in a more comprehensible equation for an equipotential line

$$\left[x - d\left(\frac{1+K^2}{1-K^2}\right)\right]^2 \; + \; y^2 \quad = \quad \left(\frac{2Kd}{1-K^2}\right)^2 \tag{6.67}$$

Equation (*6.67*) is the equation of a family of circles, with K as a parameter and d as a scale factor. For any potential (i.e. value of K) an equipotential line is a circle of radius $2Kd/(1-K^2)$ whose center has x coordinate $d(1+K^2)/(1-K^2)$ and y coordinate zero. Fig. 6-12 shows a few equipotential circles as given by equation (*6.67*) for parallel line charges.

The derivation of an expression for the capacitance of a parallel wire transmission line from equation (*6.67*) is accomplished by making use of the physical fact that at any cross section the circumferences of a transmission line's conductors are equipotential lines in the electric field. Hence the field pattern of any specific parallel wire transmission line will be found by fitting the cross section pattern of the outside surface of the two conductors to a pair of equipotential circles in Fig. 6-12, making whatever scale changes are required.

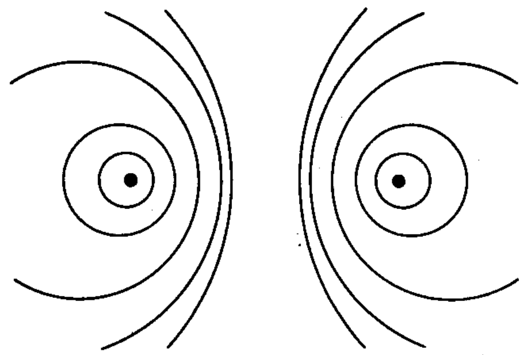

Fig. 6-12. Equipotential lines in a plane transverse to two infinitely long parallel lines of charge, having equal and opposite linear densities of charge.

For any specified potential difference between the conductors (balanced relative to the center point between them) the equivalent linear charge density ρ_l can be determined from the equations, and the distributed capacitance is the ratio of the distributed charge to the potential difference.

It is important to note that for conductors of finite radius, the separation $2d$ of the equivalent lines of charge producing the field is not the same as the separation s of the axes of the conductors. The difference between these two quantities represents the embodiment of proximity effect in the calculation.

If a parallel wire transmission line has circular conductors of radius a, the axes of the two conductors being separated by distance s, then from equation (*6.67*)

$$a \;=\; 2Kd/(1-K^2) \tag{6.68}$$

$$s/2 \;=\; d(1+K^2)/(1-K^2) \tag{6.69}$$

Eliminating d from these equations and solving for $s/2a$,

$$s/2a \;=\; \tfrac{1}{2}(K+1/K) \tag{6.70}$$

From (*6.65*), $K = e^{2\pi\epsilon' V_p/\rho_l}$. If there is a potential difference V between the conductors of a parallel wire line, balanced with respect to the central axis, then one conductor is at potential $+V/2$ and the other at potential $-V/2$. In (*6.65*) this corresponds to $V_p = \pm V/2$. For $V_p = +V/2$, $K = e^{+\pi\epsilon' V/\rho_l}$, and substituting this into (*6.70*),

$$s/2a \;=\; \tfrac{1}{2}(e^{+\pi\epsilon' V/\rho_l} + e^{-\pi\epsilon' V/\rho_l}) \;=\; \cosh \pi\epsilon' V/\rho_l \tag{6.71}$$

Since ρ_l/V is the distributed capacitance C of the line, being the ratio of the magnitude of one of the equal and opposite linear charge densities on the conductors to the potential difference between them, it follows from (*6.71*) that

$$C \;=\; \frac{\pi\epsilon'}{\cosh^{-1} s/2a} \quad \text{farads/m} \tag{6.72}$$

or

$$C \;=\; \frac{27.8\,k_e'}{\cosh^{-1} s/2a} \quad \text{micromicrofarads/m}$$

It is an identity that $\cosh^{-1} x = \log_e (x + \sqrt{x^2 - 1})$. For $x \gg 1$ this becomes $\cosh^{-1} x \doteq \log_e 2x$. The approximation involves an error of less than $\frac{1}{2}\%$ for $x > 5$. Applying this result to (*6.72*),

$$C \doteq \frac{\pi\epsilon'}{\log_e s/a} \text{ farads/m} \quad \text{if } s/a > 10 \tag{6.73}$$

Equation (*6.73*) is the expression for the distributed capacity of a parallel wire transmission line that would be arrived at by elementary methods ignoring proximity effect, for $s/a \gg 1$. Referring back to comments in (*a*) above, it appears that in the range of $s/2a$ from 5 to 10, proximity effect modifies the distributed resistance of a parallel wire transmission line appreciably, but has no effect on the distributed capacitance.

Comparison of numerical values for the distributed capacitance of parallel wire transmission lines from (*6.73*) with values for the distributed capacitance of coaxial lines from (*6.53*) shows that for lines of average design the former are usually less than 50% of the latter.

(c) *Distributed conductance.*

The distributed conductance of any transmission line depends on the loss factor or conductivity of the medium surrounding the conductors, and on the geometry of the line, in exactly the same manner that the distributed capacitance depends on the permittivity of the medium and the line's geometry. This means that equation (*6.56*), presented in the discussion of coaxial lines, is in fact directly applicable to uniform transmission lines having conductors of any shape or arrangement, provided all of the electric field pattern of the line lies within the medium described by the dielectric constant k_e' and loss factor $\tan \delta$. The bounded geometry of a coaxial line, or of any other line with a self-shielding configuration of conductors, assures that this requirement is satisfied if the medium fills the interconductor space inside the outer conductor. For a coaxial line whose interconductor space is only partly filled with insulating beads or discs, separate determinations of G and C can be made for the air-dielectric and material-dielectric fractions of the line, and from these reliable average values for the line as a whole can be calculated.

For a parallel wire transmission line, on the other hand, the requirement that the electric field surrounding the conductors be entirely contained within a surrounding dielectric medium would literally demand that the medium be of indefinitely great extent, and even a more practical goal such as 99% containment would necessitate that the medium around and between the conductors have a thickness considerably greater than the distance between the conductor axes.

The presence of solid dielectric material in the interconductor electric field of any transmission line increases the distributed circuit coefficients C and G, relative to air dielectric, without affecting R or L. Both of the changes increase the attenuation factor α. This can be seen directly, for high frequencies, from equations (*5.9*) and (*5.11*); and a consideration of phase angle relations in equation (*5.1*) shows that it is true at all useful frequencies. For the efficient transmission of signals and power, there is therefore always a premium on minimizing the amount of solid dielectric material in the interconductor electric fields of a transmission line.

The many commercial types of small diameter coaxial transmission lines whose interconductor space is filled with plastic dielectric have been designed primarily to achieve a high degree of flexibility, combined with adequate electrical uniformity and mechanical stability. This objective is attained at an increase in cost and an increase in attenuation factor over a line with identical conductors but with an interconductor space only partially filled with dielectric.

To minimize the cost and the undesirable effects of dielectric material, practical parallel wire transmission lines use the smallest possible amount of insulating material for periodic supports or spacers, or in the form of a thin web maintaining the spacing between the conductors of a flexible line. Unfortunately for analytical purposes, expressions for the distributed capacitance and conductance of such lines cannot be derived in any generalized form.

Frequently an estimate of relative volumes of air and solid dielectric can suggest that the solid insulating material in the structure of a particular line occupies too little space to produce a significant value of distributed conductance or to have any measurable effect on the distributed capacitance. For lines incorporating a substantial amount of solid dielectric, such as the standard "Twinline" used for connecting television receivers to antennas, the values of C and G must be determined by direct measurement at any operating frequency.

Example 6.6.

The conductors of the 19 gauge cable pair transmission line of Example 6.5 are insulated by a machine which embeds them in liquid paper pulp. With the separation between conductor axes maintained at 2.0 conductor diameters, the pulp is dried and forms a solid dielectric that extends far enough in all directions around the conductors to protect the line electrically and mechanically from dozens of similar parallel wire transmission lines in the same cable. The average values of distributed capacitance and distributed conductance for the 19 gauge cable pair line are respectively $C = 0.062$ microfarads/mile and $G = 1.0$ micromhos/mile at a frequency of 1 kilohertz. Assuming the electric field of the line is completely contained within the homogeneous paper insulation, and that the value of distributed conductance is entirely due to dielectric loss rather than charge-flow conductivity, determine the average value of the dielectric constant k_e' and the loss factor $\tan\delta$ for the insulating material.

It is convenient to do the calculation in metric units, using (*6.72*). The distributed capacitance of the line is $(0.062 \times 10^{-6})/1609 = 38.5$ micromicrofarads/m. From the dimension data, $s/2a = 2.0$ and $\cosh^{-1} s/2a = 1.32$. Then from the second equation of (*6.72*), $27.8\, k_e'/1.32 = 38.5$, or $k_e' = 1.83$, a reasonable value for the dielectric constant of a rather porous material.

In using (*6.56*) to determine $\tan\delta$, any length units may be used for C and G provided they are the same for both. Using the values per mile, and a frequency of 1 kilohertz,

$$1.0 \times 10^{-6} = (2\pi \times 1000)(0.062 \times 10^{-6})(\tan\delta) \quad \text{or} \quad \tan\delta = 0.0026$$

This is typical of the values of $\tan\delta$ for many organic materials, such as paper, wood, and bakelite and acetate plastics, but is higher by a factor of 10 or more than the values for the best low loss plastics.

(d) Distributed inductance.

All the introductory remarks in Section 6.4(*d*) about the relative magnitudes of the distributed internal inductance and the distributed external inductance of coaxial lines also apply to parallel wire lines. Although the distributed external inductance of parallel wire lines tends to be considerably higher than that of coaxial lines, for the same geometrical reasons that make the distributed capacitance tend to be lower, the distributed internal inductance also tends to be somewhat higher because the parallel wire line has two identical conductors, usually solid, while one of the coaxial line's conductors is a thin walled tube. The distributed internal inductance of the two cases at various frequencies is discussed in Section 6.8. The net result is that the ratio of the distributed internal inductance to the distributed external inductance is typically only a little smaller for parallel wire transmission lines than for coaxial lines.

The derivation of an expression for the distributed external inductance L_x of a parallel wire transmission line, taking proximity effect into account, follows a pattern identical to that of the derivation of equation (*6.72*) for the distributed capacitance. Magnetic quantities take the place of electric quantities, and the conductor circumferences are identified as lines of constant magnetic vector potential. The result is

$$L_x = (\mu_m'/\pi)(\cosh^{-1} s/2a) \text{ henries/m} \quad \text{or} \quad L_x = 0.4 \cosh^{-1} s/2a \text{ microhenries/m} \qquad (6.74)$$

where μ_m' is the real part of the mks permeability of the medium surrounding the conductors. For all insulating materials used in practical transmission lines, $\mu_m' = \mu_0 = 4\pi \times 10^{-7}$ henries/m.

Example 6.7.

To complete the calculations on the 19 gauge cable pair transmission line of Examples 6.5 and 6.6, determine its distributed external inductance and total distributed inductance at a frequency of 1 kilohertz, and compare the results with the stated distributed inductance at that frequency (Example 5.4, page 52) of 1.00 millihenries/mile.

For $s/2a = 2.0$, $\cosh^{-1}(s/2a) = 1.32$, as in Example 6.5. From equation (*6.74*) the distributed external inductance of the line is then $0.4 \times 1.32 = 0.53$ microhenries/m $= 0.53 \times 1609 = 0.85$ millihenries/mile. At the low frequency of 1 kilohertz, Table 6.2 shows that the distributed internal inductance of a 19 gauge isolated wire is the same as its d-c value of 0.050 microhenries/m = 0.080 millihenries/mile.

At the frequency of 1 kilohertz, a/δ from Table 6.2 is 0.22, and from Fig. 6-8 or the discussion in Section 6.6(a) the distributed resistance of the conductors is not increased by proximity effect under these conditions. It can therefore be assumed that the distributed internal inductance is also not affected.

The total distributed internal inductance for the two wires of the line is then 0.160 millihenries/mile, and the total distributed inductance of the line is $0.85 + 0.16 = 1.01$ millihenries/mile. The difference between this calculated value and the stated average value of 1.00 millihenries/mile is covered by the slightly nominal nature of the latter figure.

(e) *Summary of high frequency relations for the parallel wire transmission line.*

If the conditions $a/\delta > 100$, $\omega L/R > 10$ and $\omega C/G > 10$ *all* hold for a parallel wire transmission line, which is usually the case at frequencies above some value between 1 and 100 megahertz depending on the line dimensions and materials, the following simplified expressions can be obtained from equations (*6.72*) and (*6.74*),

$$Z_0 = \sqrt{\frac{\mu_m'/\pi}{\pi\epsilon'}} \cosh^{-1}(s/2a) = \frac{1}{\pi\sqrt{k_e'}}\sqrt{\frac{\mu_0}{\epsilon_0}} \cosh^{-1} s/2a = \frac{120}{\sqrt{k_e'}} \cosh^{-1} s/2a$$
$$= \frac{120}{\sqrt{k_e'}} \log_e\left(\frac{s}{2a} + \sqrt{\left(\frac{s}{2a}\right)^2 - 1}\right) = \frac{276}{\sqrt{k_e'}} \log_{10}\left(\frac{s}{2a} + \sqrt{\left(\frac{s}{2a}\right)^2 - 1}\right) \quad \text{ohms} \tag{6.75}$$

and if $s/2a > 10$, then within $\frac{1}{2}\%$,

$$Z_0 \doteq \frac{120}{\sqrt{k_e'}} \log_e s/a \doteq \frac{276}{\sqrt{k_e'}} \log_{10} s/a \quad \text{ohms} \tag{6.76}$$

For all values of $s/2a$,

$$v_p = \frac{1}{\sqrt{\mu_m'\epsilon'}} = \frac{1}{\sqrt{k_e'}}\frac{1}{\sqrt{\mu_0\epsilon_0}} = \frac{3.00 \times 10^8}{\sqrt{k_e'}} \quad \text{m/sec} \tag{6.77}$$

Equations (*6.60*) and (*6.77*), for the high frequency phase velocity on coaxial lines and parallel wire lines respectively, are identical. They carry no information about the geometry of the lines or the metal of their conductors. The numerical result is the value of the phase velocity for plane transverse electromagnetic waves in an unbounded medium having the properties of the insulating material filling or surrounding the line conductors. This follows from the fact, discussed in Chapter 2, Section 2.3, that the electric power traveling along a transmission line is a plane transverse electromagnetic (TEM) wave guided by the conductors. At high enough frequencies the propagation of the waves in the interconductor medium is not affected by the bounding conductors. At lower frequencies equation (*5.6*), page 49, shows that either distributed resistance or distributed conductance separately will reduce the phase velocity below the value for an unbounded medium, but if both are present there is some mutual cancellation of effects according to the term $\{(R/2\omega L) - (G/2\omega C)\}^2$.

As was the case for equations (*6.59*) and (*6.60*), the interconductor medium is assumed in (*6.75*), (*6.76*) and (*6.77*) to have the magnetic properties of free space. Otherwise Z_0 must be multiplied by and v_p divided by the square root of the relative permeability of the medium. If the relative permeability is complex and has a sufficiently large phase angle, an additional contribution to the distributed resistance of the line will be created. This situation is likely to occur only in a research context. Its analysis is easily developed by analogy with the method used in Section 6.5(*c*) to handle complex dielectric permittivity. (See Problem 6.32.)

Example 6.8.

Three sizes of circular copper tubing, having outside diameters respectively of 1.00″, 1.50″ and 4.00″ and all with wall thickness 0.100″, are available to make a transmission line to be used at 100 megahertz. The maximum transverse dimensions of the line must not exceed 4.00″. Assuming that dielectric supports will have negligible effect on either the distributed capacitance or the losses of any proposed line, what arrangement of the conductors, as either a coaxial line, or a parallel wire line with identical conductors, will result in the lowest attenuation factor? (Make a guess before proceeding.)

There are five possible designs of transmission line that use the available conductors and meet the conditions stated. Two are coaxial with the 4″ tube as outer conductor, one is coaxial with the 1.5″ tube as outer conductor, and two are parallel wire lines with the 1″ and 1.5″ tubes respectively as conductors. The coaxial line with the 1.5″ tube as outer conductor will obviously have higher attenuation than the coaxial lines with a 4″ tube as outer conductor, since its distributed resistance is higher and its characteristic impedance is lower than for either of them. No calculations need therefore be made for it. There is no self-evident basis for rejecting any of the other four possibilities. (Some design criteria for "optimum" transmission lines for various purposes and subject to various specifications are developed in Section 6.9.)

Since δ for copper at 100 megahertz is 6.61×10^{-6} m, it is obvious that $a/\delta \gg 100$ for all of the conductors, and the resistance calculations can be made by the simplest high frequency formulas (*6.30*) or (*6.49*), with the simplest correction for proximity effect, equation (*6.61*), used where necessary. The attenuation calculations for each of the four lines are then as follows:

(1) *Coaxial line with 4″ outer conductor and 1″ inner conductor.*

From (*6.59*), characteristic impedance $= 138 \log_{10} 1.90/0.50 = 80.1$ ohms.

From (*6.49*), distributed resistance $= \dfrac{0.00261}{2\pi(1.90 \times 0.0254)}\left(1 + \dfrac{1.90}{0.50}\right) = 0.0413$ ohms/m. (Here use has been made of $R_s = 1/\sigma\delta = 2.61 \times 10^{-7}\sqrt{f}$ ohms/square for copper when f is in hertz, and of the conversion factor $1'' = 0.0254$ m.)

From (*5.9*), attenuation factor $= 0.0413/(2 \times 80.1) = 2.57 \times 10^{-4}$ nepers/m.

(2) *Coaxial line with 4″ outer conductor and 1.5″ inner conductor.*

From (*6.59*), characteristic impedance $= 138 \log_{10} 1.90/0.75 = 57.9$ ohms.

From (*6.49*), distributed resistance $= \dfrac{0.00261}{2\pi(1.90 \times 0.0254)}\left(1 + \dfrac{1.90}{0.75}\right) = 0.0304$ ohms/m.

From (*5.9*), attenuation factor $= 0.0304/(2 \times 57.9) = 2.62 \times 10^{-4}$ nepers/m.

(3) *Parallel wire line with 1″ conductors.*

Minimum attenuation for this line will occur with the conductors as far apart as possible, since this will maximize Z_0 in equation (*5.9*) and will at the same time minimize R by minimizing proximity effect. Therefore $s = 4.00'' - 2a = 3.00''$.

From (*6.75*), characteristic impedance $= 276 \log_{10}\left(3.00/1.00 + \sqrt{(3.00/1.00)^2 - 1}\right) = 211.6$ ohms.

From (*6.30*), distributed resistance of conductors if isolated $= 2\dfrac{0.00261}{2\pi(0.050 \times 0.0254)} = 0.0656$ ohms/m.

From (*6.61*), proximity effect factor $= 1/\sqrt{1 - 1/(3.00/1.00)^2} = 1.060$.

Distributed resistance of conductors including proximity effect = 0.0695 ohms/m.

From (*5.9*), attenuation factor $= 0.0695/(2 \times 211.6) = 1.64 \times 10^{-4}$ nepers/m.

(4) *Parallel wire line with 1.5″ conductors,* $s = 2.50''$.

From (*6.75*), characteristic impedance $= 276 \log_{10}\left(2.50/1.50 + \sqrt{(2.50/1.50)^2 - 1}\right) = 131.9$ ohms.

From (*6.30*), distributed resistance of conductors if isolated $= 2\dfrac{0.00261}{2\pi(0.75 \times 0.0254)} = 0.0437$ ohms/m.

From (*6.61*), proximity effect factor $= 1/\sqrt{1 - 1/(2.50/1.50)^2} = 1.118.$

Distributed resistance of conductors including proximity effect = 0.0489 ohms/m.

From (*5.9*), attenuation factor $= 0.0489/(2 \times 131.9) = 1.86 \times 10^{-4}$ nepers/m.

Worth noting from these results are the fact that the two parallel wire lines have substantially lower attenuation factors than either of the coaxial lines, in spite of having higher distributed resistance values, and the fact that for each type of line the 1″ conductor gives lower attenuation than the 1.5″ conductor. Both facts are of course due to the relative values of the characteristic impedances involved. It does not follow that conductors of 0.75″ or 0.50″ outside diameter will give still lower attenuation in either type of line. For each case, subject to the fixed maximum lateral dimension, there is an optimum value of conductor diameter for the parallel wire line or inner conductor diameter for the coaxial line that provides minimum attenuation. With smaller conductors the distributed resistance rises more rapidly than the characteristic impedance, and the attenuation factor increases. (See Section 6.9 and Problem 6.15.)

6.7. Distributed circuit coefficients of transmission lines with parallel plane conductors.

Elementary methods can derive exact expressions for the distributed circuit coefficients of transmission lines with parallel plane conductors only in the idealized case of conductors which are portions of infinite parallel planes. The infinite plane specification eliminates the effects of the curved electric and magnetic field lines that occur at the edges of finite plane conductors, effects which are not easy to analyze mathematically. The equations for the idealized case apply with useful accuracy to lines with conductors of finite width if the conductor width is sufficiently large compared with the separation of the conductors.

Parallel plane transmission lines with conductors many times wider than their separation have no particular electrical virtues, their attenuation being greater than for parallel wire or coaxial transmission lines of comparable maximum transverse dimensions, or containing comparable amounts of metal, but profitable use can sometimes be made of their space-saving geometry and the fact that they have a higher degree of self shielding than lines with parallel circular conductors. The stripline constructions of Fig. 2-2, page 9, are commercial types of line that take advantage of these properties.

Even though the expressions for the distributed circuit coefficients of the idealized parallel plane transmission line are seldom accurately applicable to practical situations, they are worth noting because their simple form is so easily remembered, and they can often serve as a basis for a useful estimate of coefficient values for a line having some other design.

(a) Distributed resistance.

The distributed resistance per unit width of single infinite plane conductors, in which the currents are excited by fields from one side only, has been fully dealt with in Section 6.3. For a two-conductor line with plane parallel conductors of width w and thickness t, assuming the idealized fields of infinite planes, the distributed resistance at any value of the ratio t/δ will be $2/w$ times the values given in Section 6.3 for conductors of finite thickness. At low frequencies the result is equal to the distributed d-c resistance of the conductors, as usual. At frequencies sufficiently high that the conductors are at least three skin depths thick, the result is

$$R = 2R_s/w \text{ ohms/m} \tag{6.78}$$

For intermediate frequencies the distributed resistance is the value given by equation (*6.78*) multiplied by a factor from Fig. 6-7, page 87.

(b) Distributed capacitance.

If two infinite parallel plane conductors carry equal and opposite uniform surface charge densities of magnitude ρ_s coulombs/m², the electric flux field is entirely confined to the space between them, is normal to the conductor surfaces, and has constant density $D = \rho_s$ coulombs/m². The electric field E, also constant everywhere and normal to the surfaces, is given by $E = D/\epsilon' = \rho_s/\epsilon'$ volts/m, where ϵ' is the real part of the permittivity of the medium between the conductors. The potential difference V between the conductors is then $V = Ed = \rho_s d/\epsilon'$ volts. The capacitance per unit area, being the ratio of the magnitude of one of the surface charges per unit area to the potential difference between the conductors, is ϵ'/d farads/m². The distributed capacitance of the parallel plane conductor is therefore given approximately, when $w/d \gg 1$, by

$$C = \epsilon' w/d \text{ farads/m} = 8.85\, k_e' w/d \text{ micromicrofarads/m} \tag{6.79}$$

(c) Distributed conductance.

The distributed conductance of all transmission lines is given by equation (*6.56*), in terms of the distributed capacitance of the line and the loss factor of the interconductor medium, assuming that the distributed conductance is due entirely to molecular dielectric loss mechanisms. For the idealized parallel plane transmission line neglecting edge effects this takes the specific form

$$G = \omega\, 8.85\, k_e' w/d \tan\delta \text{ micromicromhos/m} \tag{6.80}$$

For this expression to be applicable, the interconductor medium whose loss factor is $\tan\delta$ must contain all of the electric field surrounding the line conductors. The condition is fulfilled for parallel plane conductors if the medium fills the space between them.

(d) Distributed inductance.

Practical applications of parallel plane transmission lines are based primarily on the geometrical fact that the total volume occupied by the line can be made small and one dimension can be made very small, with no reduction in the surface area of the conductors or increase in the high frequency distributed resistance R. At the same time the lines retain a high degree of self-shielding.

If a transmission line has to be designed so that its cross section is contained within a rectangular area of which one edge is much longer than the other, a line with parallel plane conductors will have a lower attenuation factor and much better power handling capacity than either a coaxial line or a parallel wire line with circular conductors when the aperture ratio of the rectangle exceeds some minimum value, and will have much smaller external fields than the parallel wire line.

It is shown in Section 6.8 that the distributed internal inductance of tubular and plane conductors is proportional to the thickness t at low frequencies for which $\delta > t$, and to the skin depth δ at high frequencies for which $t > \delta$. The expression derived in this section for the distributed external inductance of a line with parallel plane conductors shows that it is proportional to the separation d between the facing surfaces of the conductors. Since d might be comparable to or even smaller than t for a parallel plane line, it is possible at low frequencies for the distributed internal inductance of such a line to constitute the greater part of the total distributed inductance, a result not possible for any reasonable design of coaxial or parallel wire line with circular conductors.

In calculating the distributed inductance at low and intermediate frequencies for lines with parallel plane conductors, therefore, particular attention must be paid to the relative importance of the distributed internal inductance. If $d/\delta > 100$ the distributed internal inductance is always negligible, for all values of conductor thickness t.

A single isolated infinite plane conductor parallel to the xz plane and carrying an instantaneous surface current in the z direction of density J_{sz} amperes/(meter width of plane in the x direction) has a constant magnetic field of value $H_x = J_{sz}/2$ throughout the whole of space on one side of the plane, and a constant field of equal magnitude and opposite sign on the other side of the plane. When two such conductors are parallel to one another and carry currents of equal density in opposite directions, the fields between the conductors are additive and those outside the conductors cancel. Thus the magnetic flux density between the conductors of a parallel plane transmission line, neglecting edge effects caused by finite conductor width, is constant at the value $B_x = \pm\mu_m' J_{sz}$ teslas, where μ_m' is the real part of the mks permeability of the medium between the conductors. The sign is determined by the sign of the z directed current in one of the conductors. The magnitude of the total magnetic flux $|\psi|$ between the conductors per unit length of conductor in the z direction is $|\psi| = |B_x|\,d = \mu_m'\,d\,J_{sz}$. The conclusion that all of this flux links all of the current in the conductors as a "circuit" requires the hypothesis that the flux lines after extending indefinitely laterally in the space between the conductors are self-closing by returning in the unbounded space outside the conductors, where their vanishingly small density constitutes zero field.

If the actual conductors in the parallel plane transmission line have finite width w, but the currents and fields are those of infinite planes, the current magnitude in each line conductor will be $I = J_{sz}w$ amperes. The distributed external inductance of the line, defined as the total flux external to the conductors linking the circuit per unit current, becomes

$$L_x = \frac{|\psi|}{I} = \mu_m' \frac{d}{w} \text{ henries/m} \qquad (6.81)$$

For a nonmagnetic medium in the interconductor space, $\mu_m' = \mu_0 = 4\pi \times 10^{-7}$ henries/m and $L_x = 1.256d/w$ microhenries/m.

(e) Summary of high frequency relations for the parallel plane transmission line.

Subject to the conditions that $d/\delta > 100$, $w/d \gg 1$, $\omega L/R > 10$ and $\omega C/G > 10$, the following simplified expressions can be obtained from equations (*6.79*) and (*6.81*):

$$Z_0 = \sqrt{\frac{\mu_m' d/w}{\epsilon' w/d}} = \frac{1}{\sqrt{k_e'}}\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{d}{w} = \frac{377}{\sqrt{k_e'}}\frac{d}{w} \text{ ohms} \qquad (6.82)$$

$$v_p = \frac{1}{\sqrt{(\mu_m' d/w)(\epsilon' w/d)}} = \frac{1}{\sqrt{k_e'}}\frac{1}{\sqrt{\mu_0\epsilon_0}} = \frac{3.00 \times 10^8}{\sqrt{k_e'}} \text{ m/sec} \qquad (6.83)$$

In these equations it is again assumed that the interconductor medium is nonmagnetic.

Example 6.9.

A parallel plane transmission line used at 10 megahertz has copper conductors 1.00″ wide and 0.050″ *thick, spaced 0.100″. The interconductor space is filled with material of dielectric* constant 2.25 and loss factor 0.00025. Neglecting edge effects, determine the characteristic impedance, the attenuation factor and the phase velocity at the frequency of operation. Compare the results with the values for a coaxial line whose total conductor periphery is equal to the combined width of the two plane conductors (i.e. $2\pi a + 2\pi b = 2w$) and for which the ratio $b/a = 3.5$, the interconductor space being filled with the same medium. (The reason for choosing $b/a = 3.5$ is given in Section 6.9.)

The skin depth in copper at 10 megahertz is 2.09×10^{-5} m so that $d/\delta > 100$ and $t/\delta > 100$. A check shows that $\omega L/R > 10$ and $\omega C/G > 10$ are both satisfied.

The characteristic impedance of the parallel conductor line, from the high frequency equation (*6.82*) is 25.1 ohms.

The distributed resistance from (*6.78*) is $R = 2 \times 0.000825/0.0254 = 0.0650$ ohms/m.

The distributed capacitance from (*6.79*) is 199 micromicrofarads/m, and the distributed conductance from (*6.80*) is 3.13×10^{-6} mhos/m.

The attenuation factor is therefore

$$\alpha \;=\; 0.0650/(2\times 25.1) + \tfrac{1}{2}(3.13\times 10^{-6})\times 25.1 \;=\; 0.00129 + 0.000039 \;=\; 1.33\times 10^{-3} \text{ nepers/m}$$

The phase velocity from (*6.88*) is

$$v_p \;=\; 3.00\times 10^8/1.5 \;=\; 2.00\times 10^8 \text{ m/sec}$$

For the coaxial line meeting the stated conditions, the relations $a+b = w/\pi$ and $b/a = 3.5$ give $a = 0.0707'' = 1.79\times 10^{-3}$ m, and $b = 3.5a = 0.247'' = 6.28\times 10^{-3}$ m.

The characteristic impedance from equation (*6.59*) is 50.1 ohms.

The distributed capacitance from (*6.54*) is 99.7 micromicrofarads/m, and the distributed conductance from (*6.56*) is then 1.56×10^{-6} mhos/m.

Finally the attenuation factor of the coaxial line is

$$\alpha \;=\; 0.094/(2\times 50.1) + \tfrac{1}{2}(1.56\times 10^{-6})\times 50.1 \;=\; 0.000937 + 0.000039 \;=\; 9.8\times 10^{-4} \text{ nepers/m}$$

The coaxial line in the above example has an attenuation factor about 25% smaller than that of the parallel plane line, with the same area of metal sheet of the same thickness in unit length of each. The calculation for the parallel plane line neglected edge effects, and allowance for these would increase the advantage of the coaxial line by a few percent. The equality of the contribution to the attenuation factor from the distributed conductance in the two cases is an identity (see Problem 6.29).

If the same amount of metal sheet were made into two circular conductors for a parallel wire transmission line with the maximum transverse dimension equal to that of the parallel plane transmission line, the conductors being fully surrounded by the same medium, the attenuation factor would be much less than for either the coaxial or the parallel plane lines, but the external fields of the parallel wires line would be appreciable at greater distances from the conductors than for the parallel plane line.

Reducing the separation of the parallel plane conductors by 50% would reduce the characteristic impedance, as given by the idealized equations, by the same amount. The distributed resistance would not change, and the contribution of the distributed resistance to the attenuation factor would therefore be doubled. The contribution from the distributed conductance would remain constant, so that the total attenuation factor would increase by somewhat less than 100%.

If the separation between the parallel plane conductors is steadily increased, the entire field pattern begins to change drastically, becoming similar to that of a parallel wire transmission line for values of d/w much greater than unity. The attenuation factor decreases sharply, but the idealized theory ceases to be usefully accurate when d/w exceeds 0.1.

6.8. Distributed internal inductance for plane and tubular circular conductors of finite thickness.

In Sections 6.2 and 6.3, full derivations were presented of expressions for the distributed resistance of solid circular conductors, and plane conductors of unlimited area and thickness, respectively, as a function of frequency. An inherent feature of those derivations was that the resulting expressions, equations (*6.27*) and (*6.45*), also gave values for the distributed internal inductance of the two cases.

The much more elaborate mathematical investigations that resulted in Fig. 6-8, page 88, for the distributed resistance of circular tubular conductors at low and intermediate values of the parameter a_t/δ, and in equation (*6.63*) for the influence of proximity effect on the distributed resistance of solid and tubular circular conductors under all conditions, would also have provided information about distributed internal inductance. Unfortunately, the main practical application which the authors of those two major studies had in mind was power transmission at frequencies of 50 and 60 hertz, where ωL_i is generally too small to merit attention. Their published works contain the very complicated equations from which L_i can be determined, with or without proximity effect, but the extensive numerical computations required were made only for the real parts of the expressions, yielding $R_{a\text{-}c}/R_{d\text{-}c}$.

It is the purpose of this section to discuss the conditions under which the distributed internal inductance of a transmission line's conductors is a significant part of its total distributed inductance, and to review the methods and data available for calculating distributed internal inductance for the types of transmission lines dealt with in Sections 6.5, 6.6 and 6.7. For some situations exact values are readily determined, for others adequate approximations can be made, and for still others a judicious guess may be the best available solution.

A simple and useful criterion for estimating whether or not the distributed internal inductance can be ignored in any particular transmission line application, can be derived from the flux-circuit linkage definition of inductance, combined with the fundamental electromagnetic fact that across any boundary between two materials the tangential component of magnetic flux density B is continuous.

The tangential B-field at the surface of a conductor being continuous means that its value just inside the metal is equal to its value just outside the metal in the adjacent interconductor space. Inside the metal the B-field diminishes to zero, either exponentially (as in a plane conductor several skin depths thick), or linearly (as in an isolated solid circular conductor at zero frequency) or according to some other law. The diminishing flux densities farther from the surface into the metal also link a decreasing fraction of the conductor current as a "circuit". If the tangential B-field at the surface of a conductor in a transmission line is designated B_s, and most of the flux of this field is contained within a metal thickness l_i from the surface, then as a rough average estimate, about half the flux within the metal will link about half the circuit, per unit length, and a resulting approximate expression for the distributed internal inductance is $\frac{1}{4}B_s l_i/i_C$, where i_C is the instantaneous total current in the conductor, causing the field B_s at the surface. (It is an encouraging coincidence that this procedure happens to produce exactly the correct answer for the distributed internal inductance of a solid circular conductor at zero frequency. For that case $B_s = \mu i_C/(2\pi a)$, $l_i = a$, and the resulting distributed internal inductance is given as $\mu_0/8\pi$ henries/m, the value found in equation (*6.5*) by formal analysis.)

In the interconductor space, the value of B does not fall to zero, and all of its flux links the whole circuit. If the distance between facing conductor surfaces of a transmission line is l_x, the amount of flux-circuit linkage in the interconductor space will be fairly represented in most cases by the product $B_s l_x$. It may be greater or less than this, depending on the cross section of the line, but usually by only a small factor less than 2. The distributed external inductance of the line will therefore be given approximately by $B_s l_x/i_C$.

The ratio of distributed external inductance to distributed internal inductance is finally $2l_x/l_i$, where an additional factor of 2 has been introduced because a line has two conductors.

The distance l_x is an obvious quantity, independent of frequency, for any transmission line. The distance l_i, on the other hand, varies widely with frequency. At frequencies low enough for the current density to be nearly constant over a conductor's cross section, l_i is the radius of a solid circular conductor, or the thickness of tubular or plane conductors. At frequencies high enough to make a/δ or t/δ greater than 3 or 4, l_i can be equated with sufficient accuracy to the skin depth δ. In the small range of intermediate frequencies between these two regions, l_i can be taken conservatively as whichever value is greater.

Example 6.10.

For the 19 gauge cable pair line of Examples 6.5 to 6.7, the ratio $s/2a = 2.0$, and $a = 4.56 \times 10^{-4}$ m. Estimate what percent of the total distributed inductance of the line is distributed internal inductance, at frequencies of 10^4, 10^6 and 10^8 hertz. Make the same estimate at the same frequencies for a $3\frac{1}{8}''$ standard rigid transmission line whose inner conductor has an O.D. of 3.34×10^{-2} m with wall thickness 2.14×10^{-3} m, and whose outer conductor has I.D. of 7.80×10^{-2} m and wall thickness 2.49×10^{-3} m.

At each separate frequency the skin depth in the copper conductor metal is the same for the two lines, with values 6.61×10^{-4} m at 10^4 hertz, 6.61×10^{-5} m at 10^6 hertz, and 6.61×10^{-6} m at 10^8 hertz.

For the 19 gauge cable pair the distance between facing conductor surfaces is $2a = 9.12 \times 10^{-4}$ m. At 10^4 hertz, l_i must be taken as a since $a/\delta < 1$. Therefore $2l_x/l_i \doteq 4$ and the distributed internal inductance must not be disregarded. At 10^6 hertz, $a/\delta \doteq 10$ and l_i can be taken as δ. Then $2l_x/l_i \doteq 30$. In this case the distributed internal inductance is estimated as being a few percent of the total, and if not considered negligible would at least not need to be calculated very accurately.

At 10^8 hertz the distributed internal inductance will clearly be negligible.

The distance between facing surfaces of the $3\frac{1}{8}''$ standard rigid coaxial line is 2.23×10^{-2} m. At 10^4 hertz the wall thicknesses are somewhat over 3 skin depths and $l_i = \delta$ can be used, giving $2l_x/l_i \doteq 35$. The situation is the same as for the 19 gauge cable pair at 10^6 hertz. At 10^6 hertz and 10^8 hertz distributed internal inductance is negligible for this large diameter coaxial line.

Distributed resistance values for all the types of conductors considered earlier in this chapter are always given, in the low frequency range from zero to some upper limit, by a factor multiplying the distributed d-c resistance for the entire cross section of the conductor (Table 6.1 and Fig. 6-8, for example). At frequencies diminishing from very high values down to some lower limit, calculations make use of the skin effect theorem and distributed resistance values are given by a factor multiplying the distributed d-c resistance of a peripheral skin of the conductor of thickness δ, this skin depth δ being itself a function of frequency (Fig. 6-7, for example). Use of the theorem requires that metal thicknesses and conductor radii be not less than 3 or 4 skin depths. At frequencies between the upper and lower limiting frequencies for these two categories of calculations, tables, graphs, or elaborate formulas are required.

The same classifications apply to calculations of distributed internal inductance of circular conductors. Expressions for the d-c distributed internal inductance are therefore needed for each type of conductor investigated. The only such expression developed so far is equation (*6.5*) for the solid circular conductor.

The d-c distributed internal inductance of a circular tubular conductor depends somewhat, in the case of coaxial lines, on whether the tube is the inner or outer conductor of the line. The difference in the two values, for the same tube, is large in the case of thick walled tubes ($t/a = 0.3$, for example), but only a few percent for thin walled tubes ($t/a = 0.05$) and disappears for extremely thin walled tubes ($t/a = 0.005$). The value for a tube used as the inner conductor of a coaxial line also applies to the same tube used in a parallel wire line or an image line, with correction by a proximity effect factor if necessary.

Expressions for the d-c distributed internal inductance of a circular tubular conductor are developed through the same mathematical procedures employed to obtain equation (*6.5*) for a solid circular conductor. The result for an isolated tube, which applies to the inner conductor of a coaxial line and the other equivalent cases, is

$$L_{i\,\text{d-c}}\ \text{(tube)} \quad = \quad \frac{\mu}{8\pi} \cdot \frac{1 - 4(a_i/a)^2 + 3(a_i/a)^4 + 4(a_i/a)^4 \log_e (a/a_i)}{[1 - (a_i/a)^2]^2} \quad \text{henries/m} \qquad (6.84)$$

where a is the external radius of the tube and a_i its internal radius. Equation (*6.84*) reduces to (*6.5*) when $a_i = 0$.

Making the substitution $t = a - a_i$, and expanding the logarithm as a power series in (t/a), the first few terms of (*6.84*) are

$$L_{i\,\text{d-c}}\ \text{(tube)} \quad \doteq \quad \frac{\mu}{8\pi}\left[\frac{4}{3}\left(\frac{t}{a}\right) - \frac{2}{15}\left(\frac{t}{a}\right)^3 - \frac{1}{10}\left(\frac{t}{a}\right)^4\right] \quad \text{henries/m} \qquad (6.85)$$

The accuracy of this expression is better than ½% for $t/a < 0.5$, and for small values of t/a it is much more convenient to use than equation (*6.84*).

When a circular tubular conductor is the outside conductor of a coaxial line, the magnetic flux density in the metal of the outer conductor at any point is caused by the whole of the current in the inner conductor and a portion of the current in the outer conductor. The flux in any increment of radius in the outer conductor also links all of the current in the inner conductor and a portion of the current in the outer conductor. The total situation can be expressed by an integral similar to that used in obtaining (*6.84*), with the result

$$L_{i\,\text{d-c}}\ (\text{coax. outer}) \;=\; \frac{\mu}{8\pi}\cdot\frac{4\log_e(a/a_i) - 3 + 4(a_i/a)^2 - (a_i/a)^4}{[1-(a_i/a)^2]} \quad \text{henries/m} \qquad (6.86)$$

which expanded in powers of t/a becomes

$$L_{i\,\text{d-c}}\ (\text{coax. outer}) \;\doteq\; \frac{\mu}{8\pi}\left[\frac{4}{3}\left(\frac{t}{a}\right) + \frac{4}{3}\left(\frac{t}{a}\right)^2 + \frac{6}{5}\left(\frac{t}{a}\right)^3 + \frac{31}{30}\left(\frac{t}{a}\right)^4\right] \quad \text{henries/m} \qquad (6.87)$$

Equation (*6.87*) converges less rapidly than (*6.85*) but is accurate to better than ½% for t/a as large as 0.25.

Equations (*6.84*) and (*6.86*) are plotted in Fig. 6-13 for a wider range of wall thicknesses than would ever be encountered in practical transmission lines. Inspection of the numerical values shows that for the same outside diameter a, the ratio of the d-c distributed internal inductance of a circular tube to that of a solid circular conductor is roughly equal to the ratio of their metal cross sections.

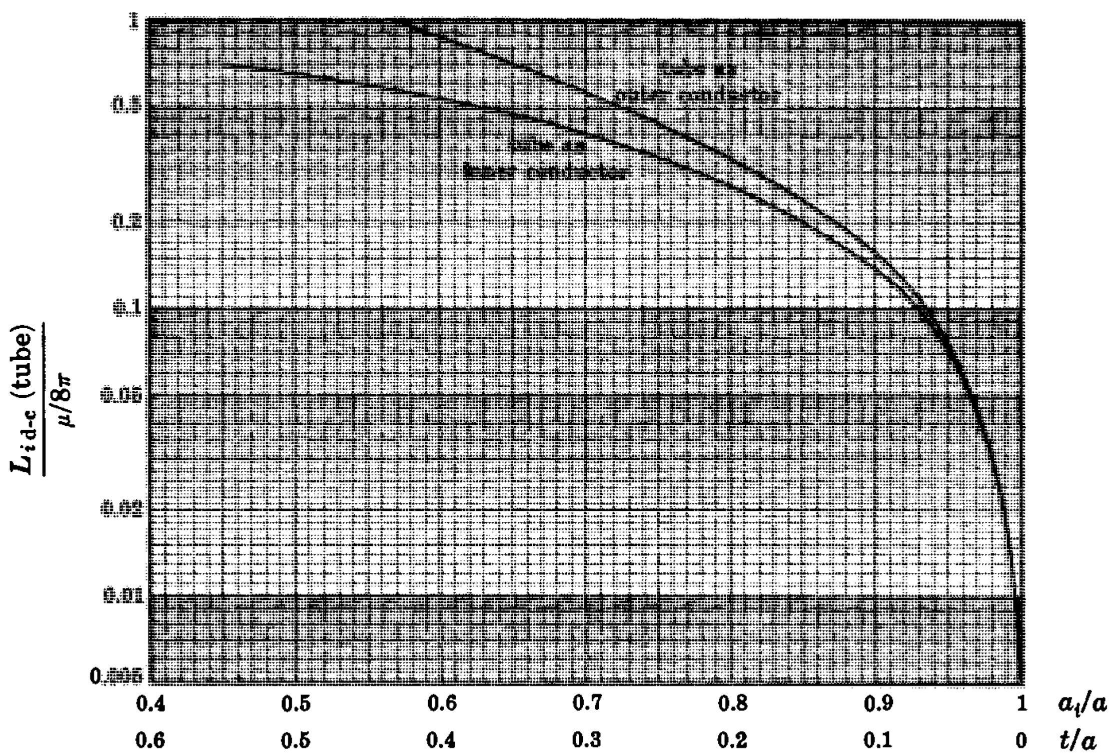

Fig. 6-13. Ratio of the d-c distributed internal inductance of a circular metal tubular conductor of inside radius a_i, outside radius a, and wall thickness t, to the distributed d-c internal inductance of a solid circular conductor of the same metal and same outside radius.

An expression for the d-c distributed internal inductance per unit width of infinite plane conductors of finite thickness is obtained by extending the analysis of Section 6.7(*d*). It derives from the fact that if the longitudinal d-c current in the plane conductors has density

J_{sz} as a surface current, and flows in opposite directions in the two conductors, the tangential B field at the facing conductor surfaces is μJ_{sz}. Inside the conductors the field diminishes linearly from this value to zero at the outside surfaces. The resulting value for the distributed internal inductance per unit width of plane is

$$L_{i\,\text{d-c}} \text{ (plane, per unit width)} \quad = \quad \tfrac{1}{3}\mu t \quad \text{henries/m} \qquad (6.88)$$

where t is the thickness of each conductor. For a parallel plane transmission line with identical conductors of width w, but with t/w small enough that the field pattern in the interconductor space is essentially that for infinite planes, equation (6.88) gives

$$L_{i\,\text{d-c}} \text{ (plane transmission line)} \quad = \quad \tfrac{2}{3}\mu t/w \quad \text{henries/m} \qquad (6.89)$$

Applied to a tubular conductor of radius a, with $t \ll a$, the conductor would approximate a plane of width $2\pi a$, with the result $L_{i\,\text{d-c}}$ (tube) $\doteq (\mu/8\pi)(4/3)(t/a)$ henries/m, in agreement with the first terms in equations (6.85) and (6.87).

In Section 6.3 a transmission line analog was used (Problem 7.10) to investigate the ratio $R_{\text{a-c}}/R_s$ for unit width of plane conductors of thickness t, over the critical range of t/δ from about 0.5 to 3, the results being presented in Fig. 6-7. The same analog provides information about the ratio $\omega L_i/R_s$ for the plane conductors, as shown in Fig. 6-14. It is known from equation (6.45) that for large enough values of t/δ, $R_{\text{a-c}} = \omega L_i = R_s$ for unit width of infinite plane conductors, and it is known that for low enough frequencies ($t/\delta < 0.5$), $\omega L_i = \omega L_{i\,\text{d-c}}$. Fig. 6-14 covers the transition between these two limits. For small values of t/a it applies also to tubular conductors.

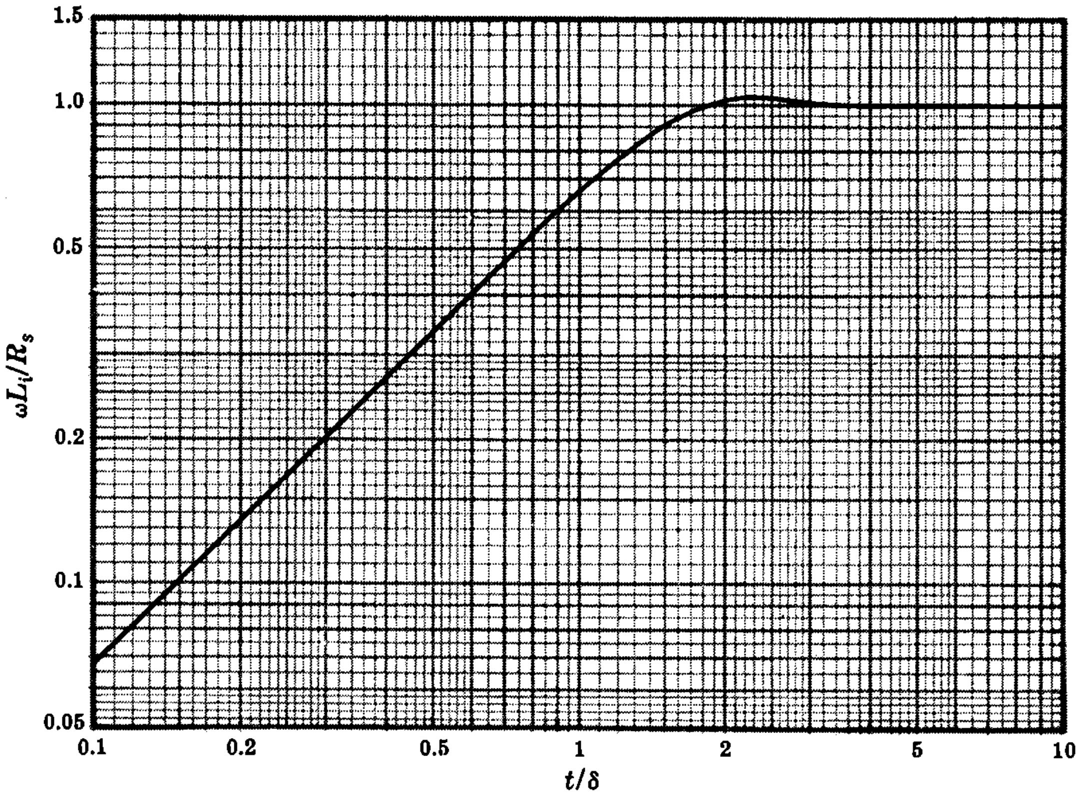

Fig. 6-14. Ratio of the distributed internal inductance per unit width of an unbounded plane conductor of thickness t/δ skin depths to the limiting high frequency value R_s/ω for a conductor of indefinite thickness. For values of t/δ less than about 0.5, the value of L_i is equal to the distributed d-c internal inductance of the metal sheet.

For $t/\delta < 0.5$ in Fig. 6-14 inspection shows that $\omega L_i/R_s = \tfrac{2}{3}t/\delta$ with high accuracy. From this $L_i = 2R_s t/(3\omega\delta) = 2t/(3\omega\sigma\delta^2) = 2\omega\mu\sigma t/6\omega\sigma = \mu t/3$ henries/m for unit width of the plane. From equation (6.88) this is equal to $L_{i\,\text{d-c}}$.

Analogous to equation (*6.29*) for solid circular conductors, a power series expression can be used to express the variation of $L_i/L_{i\,\text{d-c}}$ for t/δ as high as 1.5. This is

$$\frac{L_i}{L_{i\,\text{d-c}}}\ (\text{plane}) \quad = \quad 1 - 0.020(t/\delta)^4 \tag{6.90}$$

This equation is also adequately accurate for tubular conductors having the reasonable values of t/a likely to be encountered in practical transmission line conductors.

A similar expression can be found for $R/R_{\text{d-c}}$ for plane and tubular conductors,

$$\frac{R}{R_{\text{d-c}}}\ (\text{plane}) \quad = \quad 1 + 0.075(t/\delta)^4 \tag{6.91}$$

The question to which no universal answer seems to have been published is that of the influence of proximity effect on distributed internal inductance. Considering the evidence from equations (*6.28*) and (*6.29*) for solid circular conductors and from equations (*6.90*) and (*6.91*) for plane and tubular conductors, it can be concluded that proximity effect like skin effect will reduce the distributed internal inductance of a conductor and that the percent reduction will be considerably less in any situation than the percent by which proximity effect increases the distributed resistance of the same conductor.

Fortunately, the combination of circumstances that would require accurate information about the proximity effect factor for distributed internal inductance occurs rather rarely in transmission line practice. The most unfavorable situation would be a parallel wire line with solid circular conductors, the facing surfaces of the conductors being separated by only a few percent of a conductor radius, operating at a frequency to make a/δ have a value near 2. These conditions make the distributed internal inductance comparable in magnitude to the distributed external inductance, with a proximity effect factor that might be as small as 0.8 or 0.85. There is no recognized basis for making an accurate analysis of the total distributed inductance of a line for such a case.

Appropriate changes in any of the factors specified can improve the situation. Increasing the conductor separation, changing to tubular conductors, or increasing the frequency *will all reduce the relative value of the distributed internal inductance, and the first change* will reduce the proximity effect factor. When the facing conductor surfaces are at least a conductor diameter apart ($s/2a = 2$), the distributed internal inductance will be less than 20% of the total distributed inductance, and the proximity effect factor will be not less than 0.87 according to equation (*6.62*) and Table 6.4. Proximity effect can then not modify the total distributed inductance value by more than about 2%, and the factor need be known only very roughly.

All of the above facts combine to justify the general conclusion that the influence of proximity effect on the total distributed inductance of a parallel wire transmission line will always be less than, and except in inconceivably extreme situations much less than, its influence on the line's distributed resistance. It is therefore suggested as a working hypothesis that unless the proximity effect *factor for distributed resistance is found from* equations (*6.61*), (*6.62*), or (*6.63*) to be at least 1.05, it can be assumed that proximity effect will not change the value of the line's total distributed inductance. If the proximity effect factor for distributed resistance lies between 1.05 and 1.25 it may be a fair guess that the distributed *internal* inductance value should be divided by the square root of that factor. For higher values of the distributed resistance factor no suggestions are offered for accurate determination of the total distributed inductance, and experimental measurements may be the best procedure.

Example 6.11.

Complete the analysis of the 19 gauge cable pair transmission line of Examples 6.5, 6.6 and 6.7, by finding the distributed internal inductance at frequencies of 10^2, 10^4, 10^6 and 10^8 hertz.

The information about the line needed for purposes of the calculation is $s/2a = 2.0$, $a = 4.56 \times 10^{-4}$ m, and from Table 6.2, page 80, $a/\delta = 0.0689, 0.689, 6.89$ and 68.9 at the four frequencies listed.

At 10^2 hertz, with $a/\delta = 0.0689$, $L_i = L_{i\,\text{d-c}} = \mu_0/8\pi$ henries/m for an isolated 19 gauge conductor, from Tables 6.1 or 6.2 and equation (*6.5*). From equation (*6.62*) and Table 6.4, page 98, the proximity effect factor for resistance for solid conductors at $a/\delta = 0.0689$ is 1.000. Hence the total distributed internal inductance of the line is (for two conductors) 0.100 microhenries/m. The distributed external inductance of this line was determined in Example 6.6 as 0.53 microhenries/m. The distributed internal inductance is therefore about 16% of the total.

At 10^4 hertz, with $a/\delta = 0.689$, the situation has changed by only about 0.3% from the value of distributed internal inductance at 10^2 hertz.

At 10^6 hertz, with $a/\delta = 6.89$, Table 6.2 shows that the distributed internal inductance for the isolated conductors has dropped to 29% of its d-c value. Equation (*6.62*) shows a proximity effect factor of 1.12 for resistance for these solid conductors, close to the limiting high frequency value from equation (*6.61*) of 1.15. The procedure suggested above is that the distributed internal inductance in this case be estimated as being reduced by a factor $1/\sqrt{1.12} \doteq 0.95$. The total distributed internal inductance of the line is then $2(\mu_0/8\pi) \times 0.29 \times 0.95 = 0.0274$ microhenries/m, which should be accurate within better than 2%. Since this is only about 5% of the total distributed inductance of the line, it need not be known with better than 10% accuracy, and the inclusion of the proximity effect factor has no effect on the final total value. This illustrates the typical phenomenon that, as proximity effect becomes greater, the fraction of the total distributed inductance affected by it becomes less. Only for extremely closely spaced conductors will a critical calculation problem arise, in a small range of frequencies.

At the frequency of 10^8 hertz, $L_i/L_{i\,\text{d-c}}$ is about 3%, and this percentage of $2\mu_0/8\pi$ has dropped to the negligible amount of about $\frac{1}{2}$% of the total distributed inductance of the line.

6.9. Optimum geometries for coaxial lines.

In designing a coaxial transmission line, the diameter of the outer conductor is often determined by considerations of space or cost. The size of the center conductor does not change the space occupied by the line and its cost is a minor part of the total. It can then be adjusted to achieve various desirable electrical properties. It can be seen, for example, from equation (*6.49*) for the high frequency distributed resistance of a coaxial line, and from equation (*6.59*) for its high frequency characteristic impedance, that the high frequency attenuation factor of such a line as given by equation (*5.9*), page 49, becomes indefinitely large when a is approximately equal to b $(Z_0 \doteq 0)$ and when a becomes vanishingly small (R increases to large values more rapidly than the logarithmic term in Z_0.) There is therefore an optimum intermediate value of a, when b is fixed, for which the line has a minimum high frequency attenuation factor. (Note, however, that if a is fixed and b allowed to vary, the attenuation factor will diminish continuously for indefinite increase of b, and there is no optimum value for b other than infinity.)

Making the indicated substitutions for R from (*6.49*) and for Z_0 from (*6.59*) into (*5.9*), the high frequency attenuation factor of a coaxial line having $G = 0$ is

$$\alpha_{\text{hf}}\,(\text{coax.}) \;=\; (R_s/2\pi b)[1+(b/a)]/[120 \log_e (b/a)] \text{ nepers/m} \qquad (6.92)$$

Differentiating with respect to b/a and equating to zero leads to $\log_e(b/a) = 1 + (a/b)$. This is a transcendental equation which must be solved graphically or from tables. The result is $b/a = 3.592$, and the corresponding characteristic impedance for an air dielectric line from equation (*6.59*) is 76.64 ohms. The minimum in the attenuation factor as a function of b/a is a broad one, showing less than $\frac{1}{2}$% variation from $b/a = 3.2$ to $b/a = 4.1$, and only 5% increase at $b/a = 2.6$ and $b/a = 5.2$.

If a transmission line is to handle a maximum amount of power for a fixed value of b, the design must be optimized to avoid breakdown rather than to reduce the attenuation factor. The principal types of failure are dielectric breakdown due to excessive electric field in the interconductor space, and thermal breakdown due to excessive temperature rise of the center conductor.

From the geometry of a coaxial line, maximum electric field will always occur at the surface of the inner conductor. Substituting into equation (*6.51*) an expression for the distributed longitudinal charge density ρ_l obtained from (*6.52*), the maximum electric field at $r = a$ is found in terms of the potential difference $V_b - V_a$ between the conductors to be $E_{\max} = (V_b - V_a)/[a \log_e(b/a)]$. If the voltage on the line varies harmonically with time and has rms value V volts, the peak value of $V_b - V_a$ will be $\sqrt{2}\,V$, and the power transmitted by the line is V^2/Z_0 since Z_0 is real at high frequencies. The final expression for the maximum electric field in terms of the power level and the line dimensions is

$$E_{\max} \;=\; \sqrt{120P}/[a\sqrt{\log_e(b/a)}\,] \text{ volts/m}$$

where P is the power level in watts. The differentiation is much easier if the expression is inverted. Taking $d(1/E_{\max})/da = 0$ gives directly $\log_e(b/a) = \frac{1}{2}$, $b/a = 1.649$, and $Z_0 = 30$ ohms.

Optimum design for protection against thermal breakdown through overheating of the center conductor requires assumptions about the processes of heat transfer between the conductors and from the outer conductor to its surroundings, both of which are very sensitive to the condition of the conductor surfaces. On the simple but rather inaccurate assumption that the outer conductor temperature remains close to the line's ambient temperature, regardless of the temperature of the inner conductor, it is easily shown that minimum power dissipation occurs in the inner conductor, for constant transmitted power and constant dimension b, with $b/a = 1$ and $Z_0 = 60$ ohms if the dielectric is air.

If a section of coaxial line is used as a capacitor, or as a delay line or wave-shaping network, the desired objective may be that it should withstand the highest possible value of applied voltage, for a fixed value of outside diameter. From the equation for $E_{\max}$ in terms of $V_b - V_a$ given above, this requires maximizing $a \log_e(b/a)$ with b constant. The result is $b/a = 1$ and $Z_0 = 60$ ohms.

Large rigid conductor coaxial lines for high power use are generally designed with $Z_0 = 50$ ohms, which appears to be a compromise value for optimization against breakdown. The attenuation factor is 10% higher than for a line of the same outside diameter having $Z_0 = 76.6$ ohms. Low power dielectric filled flexible coaxial lines are available in several values of Z_0, the most widely used having characteristic impedances near 50 ohms or near 75 ohms. The latter are effectively optimum design for minimum attenuation factor at constant outside diameter. The former are close to optimum design for minimum center conductor heating, for constant diameter and temperature of the outside conductor. When the interconductor space of a coaxial line is filled with solid dielectric, the assumption becomes reasonably correct that the temperature of the outer conductor is not affected by the heating of the center conductor. Dielectric, whether lossy or lossless, does not affect the optimum design for minimum attenuation factor.

Solved Problems

6.1. An indefinitely long straight solid circular conductor has radius a, carries a d-c current of I amperes, and is made of metal whose mks permeability is μ henries/m. The surrounding medium has zero conductivity and permeability μ_m henries/m. Describe the variation of the magnetic flux density B in the conductor and the surrounding medium, as a function of the radial distance r from the central axis of the conductor.

The H field within and surrounding the wire is determined by the current distribution alone and is independent of the magnetic properties of the materials. From the symmetry of the problem the H field and the B field have only H_ϕ and B_ϕ components in the cylindrical coordinate system whose z axis coincides with the axis of the conductor. Applying Ampere's law to any transverse circular path of radius r inside the conductor and concentric with it gives $H_\phi = Ir/(2\pi a^2)$ amperes/m, since the current enclosed by the path is the fraction r^2/a^2 of the total conductor current I. Outside the conductor all paths enclose the total current I, and $H_\phi = I/(2\pi r)$ amperes/m. The B field is everywhere equal to the H field multiplied by the mks permeability. Thus for $r < a$, $B_\phi = \mu I r/2\pi a^2$ teslas (or webers/m²) and increases linearly with r from the center of the conductor to its periphery. For $r > a$, $B_\phi = \mu_m I/2\pi r$. There is a discontinuity in B_ϕ at $r = a$ unless $\mu_m = \mu$. In the medium outside the conductor, B_ϕ falls off inversely as the distance r from the conductor's axis. For a copper conductor in air or in a plastic dielectric, $\mu = \mu_m = \mu_0$, the mks permeability of free space; but for an iron conductor in air or for a copper conductor embedded in ferrite, the two permeabilities will have different values and B_ϕ will either decrease or increase discontinuously at the conductor surface.

6.2. (*a*) A current of 5 amperes d-c flows in a 16 gauge copper wire having 100% conductivity. Determine the current density as a function of position inside the wire.

(*b*) At a frequency which makes $\sqrt{2}\,a/\delta = 10$, a current of rms value 5 amperes flows in the same wire. What is the ratio of the rms magnitude of the current density at the surface of the wire to the value found in (*a*)?

(*c*) What is the frequency in (*b*)?

(*a*) The radius of 16 gauge wire is 6.455×10^{-4} m. The d-c current density is constant over the cross section of the wire at the value $J_{z\,\text{d-c}} = I/(\pi a^2) = 3.82 \times 10^6$ amperes/m².

(*b*) The surface current density in the a-c case is given in terms of the total conductor current by equation (*6.26*), page 77, where $J_z(a)$ will be an rms current density if I_z is an rms current. From tables, ber (10) = 138.84, bei (10) = 56.37, ber′ (10) = 51.20 and bei′ (10) = 135.31. Substituting $I_z = \pi a^2 J_{z\,\text{d-c}}$, and noting that $2/(\omega\mu\sigma) = \delta^2$, equation (*6.26*) gives

$$|J_z(a)/J_{z\,\text{d-c}}| \;=\; \tfrac{1}{2}(\sqrt{2}\,a/\delta)\,\sqrt{\{[\text{ber}\,(10)]^2 + [\text{bei}\,(10)]^2\}/\{[\text{ber}'\,(10)]^2 + [\text{bei}'\,(10)]^2\}} \;=\; 5.19$$

(*c*) When $\sqrt{2}\,a/\delta = 10$, $\delta = 9.13 \times 10^{-5}$ m for 16 gauge wire. From $\delta = 0.0661/\sqrt{f}$ for 100% conductivity copper when f is in hertz, the frequency is 523 kilohertz.

6.3. If a solid circular conductor of radius a carries an a-c current of sufficiently high frequency, the reactance of the distributed internal inductance ωL_i is equal to the distributed internal resistance R according to equation (*6.31*), page 78. This requires $a/\delta = 100$. Show that under these conditions $L_i/L_{i\,\text{d-c}} = 1/(R/R_{\text{d-c}})$, a relation that is confirmed in Table 6.2, page 80, for large values of a/δ.

It is known from equation (*6.1*), page 71, that $R_{\text{d-c}} = 1/(\sigma\pi a^2)$ ohms/m, and from (*6.5*) that $L_{i\,\text{d-c}} = \mu 8\pi$ henries/m. Since under the conditions stated $L_i = R/\omega$, the problem is to demonstrate the identity $(R/\omega)/(\mu/8\pi) = (1/\sigma\pi a^2)/R$, or $R^2 = (\omega\mu)/(8\pi^2\sigma a^2)$. Using $\omega\mu\sigma/2 = 1/\delta^2$, this becomes $R^2 = 1/(4\pi^2 a^2 \sigma\delta)$, which is in agreement with equation (*6.30*), using $R_s = 1/(\sigma\delta)$. Hence the identity is established.

6.4. An iron wire of diameter 0.128″ used in a telephone circuit has relative permeability 150 at frequency 1000 hertz. Determine the distributed resistance and distributed internal inductance of the wire at that frequency at 20°C.

The radius a of the wire in metric units is 1.63×10^{-3} m. The conductivity of iron at 20°C from Table 6.3 is 1.00×10^7 mhos/m. The skin depth δ in iron at 1000 hertz must be calculated directly from equation (*6.15*), page 74, and is 4.12×10^{-4} m. The value of a/δ is then 3.96. From Table 6.1, $R/R_{\text{d-c}}$ for this value of a/δ is about 2.25, and $L_i/L_{i\,\text{d-c}}$ is about 0.499. The value of $R_{\text{d-c}}$ for the iron wire from equation (*6.1*) is 0.0120 ohms/m. Hence the distributed resistance R for the wire at 1000 hertz is 2.70 ohms/m. Since $L_{i\,\text{d-c}}$ from (*6.5*) is 7.50 microhenries/m, the distributed internal inductance of the wire at 1000 hertz is 3.74 microhenries/m.

6.5. For an a-c surface current produced in a plane conductor of indefinite thickness by a uniform tangential a-c electric field, determine the percent of the total power loss in the conductor, per unit area of surface, that occurs in distances from the surface of 0.5, 1, 1.5, 1.6, 2, 3, 4 and 5 skin depths.

The result for any distance is obtained from equation (*6.43*) on changing the upper limit of the integral from infinity to the desired distance y in skin depths. The percent is then found to be given by $100(1 - e^{-2y/\delta})$. Numerical values are

Distance from surface in skin depths	0.5	1	1.5	1.6	2	3	4	5
Percent of total power loss, from surface to that distance	63.2	86.5	95.0	95.9	98.2	99.75	99.97	99.995

Although the magnitude of the current density diminishes as $e^{-y/\delta}$, so that at a distance of 5δ into the metal from the surface the current density is still nearly 1% of the surface value, the losses vary as the *square* of the current density at any distance and it can be seen that more than 99% of the losses occur in about 2.5 skin depths.

It is mentioned in Section 6.3, referring to the results of a transmission line analog in Problem 7.10, that if an unbounded plane conductor of thickness about 1.6δ is substituted for an unbounded plane conductor of indefinite thickness (i.e. any thickness greater than a few δ) the power loss per unit area of surface in the total metal thickness is reduced about 8%. Fig. 6-6, page 85, shows that the magnitude of the total surface current density for a given tangential electric field is the same, within about 1%, in a surface layer of thickness 1.6δ on a thick plane conductor as in an indefinitely thick surface layer (compare also the results of Problem 6.6 below). The data table of this problem, however, suggests that "removing" the current pattern beyond a thickness of 1.6δ would reduce the losses by only 5%. The extra 3% is a consequence of the reflection of the electromagnetic waves from the second surface of the metal, mentioned in Section 6.3. The resulting modified current distribution in the metal is slightly more uniform and hence produces less loss for the same total current than the exponentially decaying current distribution from which Fig. 6-6 was constructed. An alternative explanation is that the presence of the reflected wave modifies the distributed internal admittance of the conductor, making the real part a little smaller than the value given by the reciprocal of equation (*6.45*).

6.6. A tangential electric field of rms phasor value E_{z0} exists over the surface of a plane conductor. The plane surface of the conductor extends indefinitely in the x and z directions and the conductor is indefinitely thick in the y direction perpendicular to the surface. Using E_{z0} as a real reference phasor, determine an equation for the total phasor surface-current density per unit width of surface in the x direction, contained between the surface plane $y=0$, and a parallel plane at distance y into the metal from the surface. Make calculations from the equation, a graph of which would give the curve of Fig. 6-6, page 85, for the case $\Delta y = 0$.

The answer is obtained in the same manner as equation (*6.44*), page 84, but with the upper limit of the integral changed to y instead of infinity. Thus

$$J_{sz}\,(0 \text{ to } y) \quad = \quad \int_0^1 dx \int_0^y dy\,[\sigma E_{z0}(1+j0)\,e^{-(1+j)y/\delta}]$$

Carrying out the integration, and using the relation $R_s = 1/\sigma\delta$, this becomes

$$J_{sz}\,(0 \text{ to } y) \quad = \quad \{E_{z0}/[R_s(1+j)]\}(1 - e^{-(1+j)y/\delta})$$

Rationalizing the first term and using $e^{-jx} = \cos x - j \sin x$,

$$J_{sz}\,(0 \text{ to } y) \quad = \quad (E_{z0}/2R_s)\{1 - e^{-y/\delta}(\cos y/\delta - \sin y/\delta) - j[1 - e^{-y/\delta}(\cos y/\delta + \sin y/\delta)]\}$$

Several points on the required curve are shown in the following table.

y/δ	$J_{sz}\,(0 \text{ to } y)/(E_{z0}/2R_s)$	Magnitude	Phase Angle
0.3	$0.512 - j0.075$	0.518	−8.3°
0.6	$0.856 - j0.250$	0.891	−16.3°
0.9	$1.064 - j0.428$	1.147	−21.9°
1.2	$1.171 - j0.611$	1.323	−27.6°
1.5	$1.206 - j0.762$	1.430	−32.3°
1.8	$1.198 - j0.877$	1.486	−36.2°
2.1	$1.167 - j0.957$	1.510	−39.4°
2.4	$1.128 - j1.006$	1.511	−41.8°
2.7	$1.088 - j1.032$	1.502	−43.5°
3.0	$1.057 - j1.042$	1.486	−44.6°
4	$0.999 - j1.025$	1.433	−45.7°
5	$0.992 - j1.004$	1.412	−45.3°
10	$1.000 - j1.000$	1.414	−45°

The result is constant at the last value for all thicknesses greater than 10δ, and is in agreement with the magnitude and phase relations given by equation *(6.45)*, page 84. When the above points are plotted on the graph of Fig. 6-6, the locations of the first ten points agree very precisely with the locations of the points determined by summing surface-current incremental phasors for layers 0.3δ thick.

6.7. For the same plane conductor and tangential phasor value of electric field described in Problem 6.6, determine the length of the arc of the curve of Fig. 6-6 from the origin to each of the points tabulated in Problem 6.6, relative to the length of the chord from the origin to each point, and relative to the length of the chord from the origin to the point at infinite y.

The magnitude values in Problem 6.6 are the chord lengths from the origin to any point on the curve of Fig. 6-6, expressed in units of $E_{z0}/2R_s$, the chord length to the point for infinite y being 1.414 in these units. The length of the arc of the curve from the origin to any coordinate y in the conductor is given by the same integral used in Problem 6.6, with the phase information removed. The result has no particular physical meaning, but the ratio of the arc length from the origin to the chord length over the same portion of the curve gives some impression of the relative inefficiency of a "thick" conductor compared to a conductor of optimum thickness. In the same units used in Problem 6.6, the arc length from the origin to coordinate y is

$$\int_0^1 dx \int_0^y dy\, |\sigma E_{z0}(1+j0)|\, |e^{-(1+j)y/\delta}| \quad = \quad (E_{z0}/R_s)(1-e^{-y/\delta})$$

The results are tabulated below, with the chord lengths repeated from Problem 6.6 for comparison.

y/δ	(arc length)/(length of chord for infinite y)	(chord length)/(length of chord for infinite y)	(arc length)/(chord length)
0.3	0.366	0.366	1.00
0.6	0.637	0.630	1.01
0.9	0.838	0.812	1.03
1.2	0.987	0.938	1.05
1.5	1.097	1.012	1.08
1.8	1.179	1.050	1.12
2.1	1.240	1.068	1.16
2.4	1.285	1.068	1.20
2.7	1.318	1.062	1.24
3	1.343	1.050	1.28
4	1.388	1.014	1.37
5	1.405	0.998	1.41
10	1.414	1.000	1.414

6.8. The copper inner conductor of a coaxial transmission line is a circular tube of outside diameter 0.250″ and wall thickness 0.015″. Determine its distributed resistance at frequencies of 10, 10^3, 10^5, 10^7 and 10^9 hertz. Compare the range of frequencies over which its distributed resistance remains within ½% of its d-c distributed resistance with the corresponding frequency range for a solid circular copper conductor of the same outside diameter.

The calculation requires significant quantities which have the following values at the different frequencies:

	10 hertz	10^3 hertz	10^5 hertz	10^7 hertz	10^9 hertz
δ, m	2.09×10^{-2}	2.09×10^{-3}	2.09×10^{-4}	2.09×10^{-5}	2.05×10^{-6}
a/δ	0.152	1.52	15.2	152	1520
$a_t = 1.51\times10^{-3}$ m					
a_t/δ	0.0723	0.723	7.23	72.3	723
t/δ	0.0182	0.182	1.82	18.2	182
R_s, ohms	8.25×10^{-7}	8.25×10^{-6}	8.25×10^{-5}	8.25×10^{-4}	8.25×10^{-3}
$R_{d\text{-}c} = 2.42\times10^{-3}$ ohms/m					

At 10 hertz, with $a/\delta = 0.152$, $R_{a\text{-}c} = R_{d\text{-}c} = 2.42 \times 10^{-3}$ ohms/m.

At 10^3 hertz, with $a/\delta = 1.52$, $R_{a\text{-}c}/R_{d\text{-}c}$ would be almost 1.10 for a solid conductor, but on consulting Fig. 6-8 for $a_t/\delta = 0.723$ for a tubular conductor with $t/a = 0.12$, it is clear that $R_{a\text{-}c}/R_{d\text{-}c}$ is less than 1.005, and the distributed resistance is again 2.42×10^{-3} ohms/m.

At 10^5 hertz, with $a/\delta = 15.2$, $R_{a\text{-}c}/R_{d\text{-}c}$ for a solid conductor can be found quickly from equation *(6.33)*, page 78, as 7.86. But $R_{d\text{-}c}$ for the tube is greater than that of a solid conductor of the same outside diameter by a factor of 4.43. Hence if *(6.33)* were directly applicable to the tube, the result would be $R_{a\text{-}c}/R_{d\text{-}c} = 7.86/4.43 = 1.77$. However, the tube wall at this frequency is only 1.84 skin depths thick, and there is a correction factor from Fig. 6-7 to be applied. The distributed a-c resistance for a tube of wall thickness 1.84 skin depths is in fact less than that for an indefinitely thick tube, to which equations *(6.32)* and *(6.33)* would apply, by a factor of about 0.93, indicating a corrected ratio for $R_{a\text{-}c}/R_{d\text{-}c}$ of $1.77 \times 0.93 = 1.65$. The graphical result from Fig. 6-8 for $a_t/\delta = 7.23$ and $t/\delta = 0.12$ is 1.66. Hence the distributed resistance of the conductor is 4.02×10^{-3} ohms/m.

At 10^7 hertz, with $a/\delta = 152$ and $t/\delta = 18.4$, the distributed resistance is given by equation *(6.30)* and is exactly the same as for a solid conductor of the same outside diameter. The result is 4.14×10^{-2} ohms/m, and $R_{a\text{-}c}/R_{d\text{-}c} = 17.1$.

At 10^9 hertz, equation *(6.30)* again applies, giving a distributed resistance value higher by precisely a factor of 10. The result is 0.414 ohms/m, and $R_{a\text{-}c}/R_{d\text{-}c} = 171$.

The highest frequency at which $R/R_{d\text{-}c}$ remains less than 1.005 for the solid conductor is taken as the frequency for which $a/\delta \doteq 0.5$. This is found to be about 104 hertz. For the tubular conductor it is not possible to calculate the corresponding limiting frequency directly and it must be found empirically from Fig. 6-8. With $t/a = 0.12$, it appears from Fig. 6-8 that $R_{a\text{-}c}/R_{d\text{-}c} < 1.005$ when $a_t/\delta < 2$, approximately. This occurs at a frequency of about 7700 hertz, with $\delta = 7.55 \times 10^{-4}$ m and $t/\delta = 0.50$. Although Fig. 6-7 cannot be considered applicable with high precision to this situation with a/δ as low as 4.4, it does show that for plane conductors the distributed a-c resistance is quite precisely equal to the d-c resistance when t/δ is as low as 0.5. Hence the figure of 7700 hertz should be reasonably accurate, indicating that the frequency range of constant distributed resistance for the tube is about 70 times as great as the range for a solid conductor of the same outside diameter.

6.9. A pair of circular coaxial metal tubes is used as an electrostatic capacitor. The outside diameter of the inner conductor is 2 cm and the inside diameter of the outer conductor is 15 cm. The tubes are 3.5 m long. The center conductor is supported by thin transverse dielectric discs, which occupy 5% of the interconductor volume and are made of material having dielectric constant 3.2.

(*a*) What is the total capacitance of the capacitor, neglecting anomalous "edge effects" at the ends?

(*b*) If breakdown occurs in the line when the electric field in the air dielectric portions exceeds 1.5×10^6 volts/m, what is the maximum voltage that can be applied to the capacitor?

(*c*) Find the breakdown voltage of the capacitor if the outside diameter of the inner conductor is increased to 4 cm without changing the outer conductor.

(*d*) Find the breakdown voltage of the capacitor if the outside diameter of the inner conductor is increased to 8 cm without changing the outer conductor.

(*a*) The capacitance must be calculated as the sum of two separate components, one for the air dielectric portion of the line, and the other for the portion filled with solid dielectric. Using equation *(6.54)*, page 93, the *distributed* capacitance of the air dielectric portion is $55.6/(\log_e 7.5) =$ 27.5 micromicrofarads/m. The air dielectric portion is 95% of the total length. Hence its capacitance is $3.5 \times 0.95 \times 27.5 = 91.5$ micromicrofarads. For the solid dielectric portion the capacitance is similarly $3.5 \times 0.05 \times 55.6 \times 3.2/(\log_e 7.5) = 15.5$ micromicrofarads. The total capacity of the line is then $91.5 + 15.5 = 107$ micromicrofarads.

(*b*) For any specific line geometry, equation *(6.52)* shows that the linearly distributed charges on coaxial conductors are directly proportional to the applied voltage. Equation *(6.51)* states that the electric field in the interconductor space is directly proportional to the linearly distributed charges, and is a maximum at the smallest value of r in the interconductor space, i.e. at the outer surface of the inner conductor.

When the breakdown field of 1.5×10^6 volts exists at the surface of the inner conductor in the air dielectric portions of the line, the magnitude of the distributed charge on each conductor is

$$2\pi(8.85 \times 10^{-12})(1 \times 10^{-2})(1.5 \times 10^6) = 8.33 \times 10^{-7} \text{ coulombs/m}$$

From equation *(6.52)* the voltage between the conductors will then be

$$8.33 \times 10^{-7} \log_e 7.5/(2\pi \times 8.85 \times 10^{-12}) = 30{,}300 \text{ volts}$$

(*c*) When the outside radius of the center conductor is 2 cm, the breakdown voltage is

$$(55.6 \times 10^{-12})(2 \times 10^{-2})(1.5 \times 10^6) \log_e 3.75/(55.6 \times 10^{-12}) = 39{,}700 \text{ volts}$$

(*d*) When the outside radius of the center conductor is 4 cm the breakdown voltage is

$$(4 \times 10^{-2})(1.5 \times 10^6) \log_e 1.875 = 37{,}600 \text{ volts}$$

The results of parts (*b*), (*c*) and (*d*) illustrate a phenomenon investigated analytically in Section 6.9, that for a fixed size of the outer conductor there is an optimum size of the inner conductor that results in minimum electric field in the interconductor space for a given applied voltage.

6.10. A coaxial transmission line has circular tubular copper conductors with walls 0.050″ thick, the inside diameter of the outer conductor being 1.25″, and the outside diameter of the inner conductor 0.375″. The center conductor is supported by a continuously spiraled dielectric webbing which fills 20% of the interconductor space. At a frequency of 200 megahertz the phase velocity on the line is measured to be 93%, and the attenuation factor to be 0.635 db/(100 ft). Determine the equivalent average dielectric constant and loss factor of the material in the interconductor space.

The frequency is high enough to ensure that $a/\delta \gg 100$ and $t/\delta \gg 1$ for both conductors. A rough check shows also that $\omega L/R \gg 1$ and $\omega C/G \gg 1$. It follows that the attenuation factor of the line is given by equation *(5.9)*, page 49, using equation *(6.59)* for the characteristic impedance Z_0, *(6.49)* for the distributed resistance R, and *(6.56)* for the distributed conductance G. Using *(6.60)* for the phase velocity, the real part of the average dielectric constant of the interconductor medium is found from

$$k_e' \text{ (average)} = [(3.00 \times 10^8)/(0.93 \times 3.00 \times 10^8)]^2 = 1.16$$

The portion α_R of the attenuation factor caused by conductor resistance is

$$[R_s(1 + b/a)/(2\pi b)]/[60 \log_e (b/a)/\sqrt{k_e'}] = 2.26 \times 10^{-3} \text{ nepers/m} = 0.598 \text{ db/(100 ft)}$$

The portion of the attenuation factor due to dielectric loss is then $\alpha_G = 0.635 - 0.598 = 0.037$ db/(100 ft) $= 1.40 \times 10^{-4}$ nepers/m. From *(5.9)* the distributed conductance that would produce this attenuation is $G = 2\alpha_G/Z_0 = 2.80 \times 10^{-4}/67.3 = 4.17 \times 10^{-6}$ mhos/m. Using equation *(6.56)*,

$$\tan \delta = G/\omega C = 4.17 \times 10^{-6}/[(2\pi \times 200 \times 10^6)(2\pi \times 1.16 \times 8.85 \times 10^{-12})/1.206] = 7.2 \times 10^{-5}$$

6.11. Show that the characteristic impedance Z_0 of a coaxial transmission line as given by equation *(6.59)*, page 96, is equal to the d-c resistance between two concentric circle metallic electrodes on a surface resistance sheet (such as a thin carbon film deposited on a plane of nonconducting material), the d-c surface resistivity of the sheet being $377/\sqrt{k_e'} \doteq 120\pi/\sqrt{k_e'}$ ohms/square, the outside radius of the inner circular electrode being a, and the inside radius of the outer circular electrode being b.

For a circular ring of radius r and radial width dr on the resistance sheet, the d-c resistance between the inner circumference and the outer circumference is $dR = \rho_s(dr)/2\pi r$ ohms, where ρ_s is the d-c surface resistivity of the sheet in ohms/square. The resistances of such rings filling the area between the contact electrodes at $r = a$ and $r = b$ are in series between the electrodes. Hence the total resistance between the electrodes is

$$R = \int_a^b dR = (\rho_s/2\pi) \log_e (b/a) \quad \text{ohms}$$

and if $R_s = 120\pi/\sqrt{k_e'}$ ohms/square, $R = (60/\sqrt{k_e'}) \log_e (b/a) = Z_0$ ohms.

This relation derives from the fact that the wave impedance or intrinsic impedance for plane transverse electromagnetic waves in an unbounded medium of dielectric constant k_e' and low loss factor is

$$Z_{\text{TEM}} = \sqrt{\mu/\epsilon} = (1/\sqrt{k_e'})\sqrt{\mu_0/\epsilon_0} = 377/\sqrt{k_e'} \text{ ohms}$$

6.12. Show that the thickness 1.5δ to 1.6δ of plane metal sheet conductor shown in Fig. 6-7, page 87, to have minimum distributed high frequency resistance is very close to $\frac{1}{4}$ wavelength thick for the waves propagating in the metal.

In the analysis of Section 6.3, all field quantities for the plane waves propagating in the metal vary in amplitude by a term $e^{-y/\delta}$ and in phase by a term $e^{-jy/\delta}$, when the wave is traveling in the direction of increasing y. According to the latter term, the phase will change by 2π radians in a distance Δy given by $\Delta y/\delta = 2\pi$. But in a harmonic space pattern, the distance in which the phase changes by 2π radians is defined as the wavelength λ of the pattern. Hence in the metal, $\lambda = 2\pi\delta \doteq 6.28\delta$, and $\frac{1}{4}$ wavelength would be 1.57δ, very close to the sheet thickness found to have minimum distributed a-c resistance.

6.13. At a high frequency ω rad/sec a coaxial transmission line has distributed external inductance L_x henries/m, distributed resistance R ohms/m, and negligible distributed internal inductance L_i when the interconductor space is air filled. Show that if the interconductor space could be filled with a magnetic medium having relative permeability k_m' and magnetic loss factor $\tan\delta_m$ such that $\sqrt{k_m'} - 1 > \omega L_x(\tan\delta_m)/R$, the attenuation factor of the line will be reduced if the medium has a dielectric constant of unity and no dielectric losses.

Since the attenuation factor of an air dielectric transmission line at high frequencies is given by $\alpha_R = R/(2\sqrt{L_x/C})$ nepers/m when there are no dielectric losses, the requirement on the medium is that the square root of the factor by which the distributed external inductance is increased on adding the medium must exceed the factor by which the distributed resistance is increased from magnetic losses. (See Problem 6.32.) The total distributed external inductance in the presence of the medium is $k_m'L_x$, and the total distributed resistance is $(R + \omega L_x \tan\delta_m)$. The properties of the medium must then satisfy $\sqrt{k_m'L_x/L_x} > (R + \omega L_x \tan\delta_m)/R$ which converts to the expression stated in the problem.

At low frequencies a useful amount of the desired result can be achieved by winding thin magnetic metal tape (such as Permalloy) around the center conductor of a coaxial line. At higher frequencies ferrite materials have high values of k_m' and low values of $\tan\delta_m$ but they also exhibit dielectric losses, which according to equation (*5.9*) are accentuated by the value of k_m'. Their use as an interconductor medium in coaxial lines can result in reduced attenuation for certain designs of line at low and intermediate frequencies.

6.14. A coaxial transmission line is to be designed to transmit 50 kilowatts of power at a frequency of 100 megahertz over a distance of 100 ft with at least 90% efficiency. What is the minimum diameter of a coaxial line with copper conductors and air dielectric that will meet the efficiency specification? Will the minimum diameter line handle the stated amount of power?

The attenuation of the line in decibels is $10 \log_{10}(50{,}000/45{,}000) = 0.457$ db $= 0.0526$ nepers. Since the line is 30.48 m long, the attenuation factor is $0.0526/30.48 = 0.00172$ nepers/m. For fixed outside diameter, a coaxial line of minimum attenuation has $b/a \doteq 3.6$ (see Section 6.9) and for an air dielectric line this corresponds to $Z_0 = 76.6$ ohms. Then $\alpha = [4.6R_s/(2\pi b)]/153.2$, and since $R_s = 2.61 \times 10^{-3}$ ohms for copper at 10^8 hertz, b to give the desired value of attenuation is found to be 0.725 cm.

The peak voltage on the line, assuming it to be terminated in its characteristic impedance will be $\sqrt{2}\sqrt{50{,}000 \times 76.6} = 2760$ volts. The interconductor distance being about 0.5 cm, the maximum electric field is of the order of 500,000 volts/m (a more precise value could be determined from equations (*6.51*) and (*6.52*)). This is a fairly high value of electric field, but would be tolerable in a well-maintained line. From the thermal point of view the line must dissipate 5 kilowatts in 100 feet, or 50 watts per foot of length. There is no simple basis for demonstrating that this would result in the temperature of the center conductor rising to more than 300°F, which may be considered excessive. For steady power transmission of 50 kilowatts at 100 megahertz, manufacturers recommend a copper transmission line with outside diameter about 3″.

6.15. Assuming that a parallel wire transmission line with circular tubular or solid conductors is operated at high enough frequencies that its characteristic impedance is given by equation (*6.75*), the resistance of its conductors if isolated is given by equation (*6.30*), and the proximity effect factor for resistance is given by (*6.61*), show that if the radius of the conductors is varied while the separation of their axes is held constant, there is an intermediate value of the conductor radius at which the line has a minimum attenuation factor. Find an approximate value for the ratio $s/2a$ (s being constant) that gives the minimum value for the attenuation factor.

Combining the conditions stated, the attenuation factor of the line is

$$\alpha = \frac{2R_s}{\pi s}\frac{s/2a}{\sqrt{1-(2a/s)^2}}\frac{1}{120\cosh^{-1}s/2a} \text{ nepers/m}$$

Letting $s/2a = x$, the problem stated in its simplest terms is to find the value of x that minimizes $x/[\sqrt{1-(1/x)^2}(\cosh^{-1}x)]$. Although this is not an impossible analytical task, an adequate solution is found much more quickly by making calculations for several values of x. The attenuation factor is found to be constant within about $\frac{1}{2}\%$ for values of $s/2a$ from 2.1 to almost 2.5, with a minimum close to 2.32.

6.16. Determine the attenuation factor at a frequency of 1 kilohertz for a parallel conductor transmission line whose copper conductors are tubes of outside diameter $\frac{3}{8}''$ and wall thickness $\frac{1}{16}''$, the separation between the adjacent surfaces of the conductors being $\frac{1}{8}''$. The line is assumed to have air dielectric.

Since $a/\delta \doteq 2.3$, equation (*6.61*) cannot be used for calculating the proximity effect factor for resistance, and since $s/2a = 1.333$, equation (*6.62*) is also not considered to be accurate. The full calculation of equation (*6.63*) must therefore be used.

The various quantities required in the calculation are $a_t = 3.55 \times 10^{-3}$, $a_t/a = 0.746$, $a_t/\delta = 1.70$, and $s/2a = 1.333$.

The next stage of the calculations gives $A_1 = 0.56$, $A_2 = 0.041$, $A_3 = 0.28$.

The relative effect of the various terms is then finally indicated by $P = 1/\sqrt{1 - 0.314 + 0.016} = 1.19$. It is evident that the error incurred in P by dropping A_2 and A_3 from the calculations would be about 1%. Equation (*6.62*) gives $P = 1.17$, a deviation of 2% from the value given by the more complete formula.

For $a_t/\delta = 1.70$ and $t/a = 0.333$, Fig. 6-8 gives $R_{\text{a-c}}/R_{\text{d-c}} \doteq 1.02$. $R_{\text{d-c}}$ is found by the usual formulas to be 8.70×10^{-4} ohms/m for the two conductors. Hence the value of the distributed line resistance $R = 1.19(1.02)(8.70 \times 10^{-4}) = 1.06 \times 10^{-3}$ ohms/m.

In Section 6.9 a rough criterion $L_x/L_i = 2l_x l_i$ is developed for the ratio of a line's distributed external inductance L_x to its distributed internal inductance L_i. In this expression l_x is the interconductor distance in which magnetic flux contributes to L_x, and l_i is the estimated thickness of the region inside the conductors in which magnetic flux contributes to L_i. In the present problem $t/\delta \doteq 0.8$, so l_i must be taken as equal to t. Since the facing surfaces of the conductors are separated by $2t$, $2l_x/l_i \doteq 4$, and L_i is a substantial part of the line's total distributed inductance L. The value of L_x from equation (*6.74*) is 0.318 microhenries/m. The value of $L_{i\,\text{d-c}}$ for the two tubular conductors from equation (*6.84*) is 0.044 microhenries/m. Equation (*6.90*) and the accompanying discussion in Section 6.8 suggest that this should be reduced about 1% because t/δ, being approximately 0.8, is a little higher than the maximum value at which L_i for tubular and plane sheet conductors can be assumed to remain within $\frac{1}{2}\%$ of $L_{i\,\text{d-c}}$. Section 6.8 emphasizes that there is no very accurate information available about the influence of proximity effect on distributed internal inductance, but that it is plausible to reduce L_i by the square root of the proximity effect factor for resistance, found above to be 1.19. The net result of all these corrections is to give a value 0.040 microhenries/m for L_i, which adds to L_x to give $L = 0.358$ microhenries/m, with a probable error not exceeding about 1%.

The distributed capacitance of the line from equation (*6.72*) is 35.0 micromicrofarads/m, and since it is obvious that $\omega L/R \gg 1$ and $\omega C/G \gg 1$, the characteristic impedance of the line is $\sqrt{L/C} = 101$ ohms, more than 5% higher than the value given by equation (*6.75*), which neglects distributed internal inductance.

Finally, the attenuation factor is $\alpha = R/2Z_0 = 5.25 \times 10^{-6}$ nepers/m.

Supplementary Problems

6.17. The distributed internal inductance and the distributed resistance of a solid circular conductor change by less than $\frac{1}{2}\%$ from the d-c values at all low frequencies for which $a/\delta < 0.5$, where a is the radius of the conductor, and δ the skin depth given by equation (*6.15*), page 74. Show that over this same range of frequencies $\omega L_i/R = \frac{1}{4}(a/\delta)^2$, within $\frac{1}{2}\%$.

6.18. (*a*) For isolated wires of each of the following AWG sizes (same as B & S sizes), determine the highest frequency at which the distributed resistance will remain within $\frac{1}{2}\%$ of the d-c value: 000, 0, 2, 4, 8, 12, 16, 20, 24, 28, 32, 36, and 40.

(*b*) Find an empirical equation from this data, relating $f_{1/2}(X)$ to $f_{1/2}(16)$, where $f_{1/2}(16)$ is the frequency determined in part (*a*) for 16 gauge wire, and $f_{1/2}(X)$ is the frequency for wire of any gauge X.

Ans. (*a*) 79.3, 126, 200, 318, 804, 2030, 5140, 13,000, 32,850, 83,100, 210,000, 530,000 and 1,337,000 hertz.

(*b*) The equation is $f_{1/2}(X) = 5140(1.261)^{(X-16)}$ hertz. For use in the equation the wire size 000 must be called size -2. The nature of the equation can be found from plotting $f_{1/2}(X)$ against X on various types of graph paper. The plot on semilog paper with X on the linear scale is a straight line.

6.19. From some source (e.g. Dwight's *Tables of Integrals and Other Mathematical Data*) find power series suitable for expressing ber x, bei x, ber$'$ x, and bei$'$ x, fcr small values of x. Then derive equations (*6.28*) and (*6.29*) from (*6.27*).

6.20. Show that for a solid circular conductor having $a/\delta \gg 100$, equation (*6.33*) is consistent with (*6.30*), and that under these conditions $R/R_{\text{d-c}} = \frac{1}{2}a/\delta$ and $L_i/L_{i\,\text{d-c}} = 1/(\frac{1}{2}a/\delta)$.

6.21. A solid circular aluminum conductor has diameter 1.50″. Determine the ratio of the current density at the center of the conductor to the current density at the surface, at frequencies of 60, 200 and 1000 hertz. Assume a temperature of 20°C.

Ans. 66% at 60 hertz, 22% at 200 hertz, 0.60% at 1000 hertz.

6.22. Extend Table 6.1, page 79, by calculating $R_{\text{a-c}}/R_{\text{d-c}}$ for a solid circular wire at the following additional values of a/δ: 5, 10, 20, 40, 60, 80, 100. *Ans.* 2.77, 5.26, 10.25, 20.25, 30.25, 40.25, 50.25.

6.23. From the preceding problem it appears that for $a/\delta > 20$, $R_{\text{a-c}}/R_{\text{d-c}} = \frac{1}{2}(a/\delta) + \frac{1}{4}$. Show that this is consistent with equation (*6.33*) when the division of the latter is carried out to two terms.

6.24. Determine the distributed internal impedance of a copper sheet at frequencies of 60, 10^3, 10^6 and 10^9 hertz, assuming copper of 100% conductivity at 20°C. Assume the metal to be many skin depths thick at each frequency.

Ans. $(2.02 + j2.02) \times 10^{-6}$ ohms/square at 60 hertz; $(8.23 + j8.23) \times 10^{-6}$ ohms/square at 10^3 hertz; $(2.61 + j2.61) \times 10^{-4}$ ohms/square at 10^6 hertz; $(8.23 + j8.23) \times 10^{-3}$ ohms/square at 10^9 hertz.

6.25. At a frequency of 10^6 hertz determine the distributed internal impedance of plane sheets of aluminum and lead, and of a plane sheet of iron having a relative permeability of 200 at that frequency. The temperature is 20°C in each case.

Ans. $(3.33 + j3.33) \times 10^{-4}$ ohms/square for aluminum; $(9.33 + j9.33) \times 10^{-4}$ ohms/square for lead; $(8.91 + j8.91) \times 10^{-3}$ ohms/square for iron.

6.26. What wall thickness should a circular tubular copper conductor of outside radius 0.100″ have if its distributed resistance is to remain within $\frac{1}{2}\%$ of the d-c value for all frequencies up to 10^8 hertz?

Ans. a/δ being very large, a tubular conductor with $t/\delta = 0.5$ or $t = 3.3 \times 10^{-6}$ m will achieve the desired result.

6.27. Show that for a circular metal tube of outside radius a, inside radius b, and made of *nonmagnetic* material, $\sqrt{f/R_{\text{d-c}}} = 892a_t/\delta = 892b_t/\delta$ where f is in hertz, $R_{\text{d-c}}$ is the distributed d-c resistance of the tube in ohms/m, and a_t and b_t are defined by equations (*6.47*) and (*6.48*) respectively.

6.28. A 12 gauge copper conductor carries a current of 5 amperes at a frequency of 10^7 hertz. What is the power loss per meter length of conductor, and how does it compare with the result for a d-c current of the same magnitude flowing in the same conductor?

Ans. At 10^7 hertz the power loss is 3.2 watts/m, about 25 times as great as the d-c power loss of 0.130 watts/m.

6.29. Show that if a medium of dielectric constant k_e' and loss factor $\tan \delta$ fills the interconductor space of any transmission line, the contribution of the distributed conductance to the attenuation factor at high frequencies is $\alpha_G = (\omega \tan \delta)/2v_p$ nepers/m.

6.30. If a coaxial transmission line with circular conductors has an outer conductor whose inside radius is 3.50″, what must be the radial distance between the facing conductor surfaces of the line to give the line a distributed capacitance of 1000 micromicrofarads/m, the interconductor medium being air?
Ans. 0.189 in.

6.31. Show that if the medium filling the interconductor space of a coaxial transmission line has a conductivity σ_m mhos/m, the distributed conductance of the line is given by $G = 2\pi\sigma_m/(\log_e b/a)$ mhos/m.

6.32. Show that if the interconductor space of any transmission line is filled with material having a complex permeability $\mu_m' - j\mu_m''$ in mks units, or relative permeability $(\mu_m' - \mu_m'')/\mu_0 = k_m' - jk_m''$, and $k_m''/km' = \tan \delta_m$, then the distributed external inductance of the line is increased by the factor k_m' over its value with a nonmagnetic medium in the space, and a contribution $R_m = \omega L_x \tan \delta_m$ ohms/m must be added to the usual distributed resistance of the conductors, where L_x is the distributed external inductance with the magnetic medium present.

6.33. For a coaxial line operated at 10^8 hertz, the outside diameter of the inner conductor is 0.500″ and the inside diameter of the outer conductor is 1.75″. Both conductors are several skin depths thick.

(*a*) What is the distributed resistance of the line if both conductors are copper?

(*b*) By what percent will the distributed resistance of the line be increased if the outer conductor is changed to aluminum, the center conductor being copper?

(*c*) By what percent will the distributed resistance of the line be increased if the inner conductor of the line is changed to aluminum, the outer conductor being copper?

(*d*) By what percent will the distributed resistance be increased if the outer conductor is changed to pure iron with a relative permeability of 50 at 10^8 hertz, the inner conductor being copper?

Ans. (*a*) 0.0842 ohms/m; (*b*) 4%; (*c*) 18%; (*d*) 113%. The result shows that changing the outer conductor of a coaxial transmission line from copper to lighter and less expensive aluminum has little effect on the distributed resistance and attenuation factor of the line. The magnetic permeability of iron makes it unsuitable for the same purpose.

6.34. Show that for copper $R_s = 2.61 \times 10^{-7}\sqrt{f}$ ohms/square, where f is in hertz.

6.35. Show that if the interconductor space of a coaxial transmission line is filled with a material of dielectric constant k_e' and loss factor $\tan \delta$, the ratio b/a of the radii of the facing conductor surfaces that will give minimum high frequency attenuation for a fixed size of outer conductor is 3.592, for all values of k_e' or $\tan \delta$.

6.36. From equation (*6.27*), page 77, determine explicit expressions for $R_{\text{a-c}}/R_{\text{d-c}}$ and $\omega L_i/R_{\text{d-c}}$ for a solid circular conductor of radius a defined by mks permeability μ and conductivity σ at angular frequency ω.

Ans.
$$\frac{R_{\text{a-c}}}{R_{\text{d-c}}} = \frac{x}{2}\cdot\frac{\operatorname{ber} x \operatorname{bei}' x - \operatorname{bei} x \operatorname{ber}' x}{\operatorname{ber}'^2 x + \operatorname{bei}'^2 x} \qquad \text{where } x = \sqrt{2}\,a/\delta$$
$$\frac{\omega L_i}{R_{\text{d-c}}} = \frac{x}{2}\cdot\frac{\operatorname{ber} x \operatorname{ber}' x + \operatorname{bei} x \operatorname{bei}' x}{\operatorname{ber}'^2 x + \operatorname{bei}'^2 x}$$