Questions for ELEC3224

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1 Range bearing model for extended Kalman filter

In the extended Kalman filter, the state transition and observation models don't need to be linear functions of the state but may instead be differentiable functions.

$$x_k = f\left(x_{k-1}\right) + w_{k-1}$$
$$z_k = h\left(x_k\right) + v_k$$

Here $\mathbf{w}_k$ and $\mathbf{v}_k$ are the process and observation noises which are both assumed to be zero mean multivariate Gaussian noises with covariance $\mathbf{Q}_k$ and $\mathbf{R}_k$ respectively.

The function f is used to compute the predicted state from the previous estimate. However, f cannot be applied to the covariance directly. Instead a matrix of partial derivatives (the Jacobian) is computed.

The Jacobian is evaluated with current predicted states. The Jacobian matrix is used in the Kalman filter equations. This process linearizes the non-linear function around the current estimate.

The state transition and observation matrices are defined to be the following Jacobians

$$F_k = \left.\frac{\partial f}{\partial x}\right|_{\hat{x}_{k-1|k-1}}$$
$$H_k = \left.\frac{\partial h}{\partial x}\right|_{\hat{x}_{k|k-1}}$$

$$\hat{x}_{k|k-1} = f\left(\hat{x}_{k-1|k-1}, u_{k-1}\right)$$
$$P_{k|k-1} = F_k P_{k-1|k-1} F_k^T + Q_{k-1}$$

$$\tilde{y}_k = z_k - h\left(\hat{x}_{k|k-1}\right)$$
$$S_k = H_k P_{k|k-1} H_k^T + R_k$$
$$K_k = P_{k|k-1} H_k^T S_k^{-1}$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \tilde{y}_k$$
$$P_{k|k} = \left(I - K_k H_k\right) P_{k|k-1}$$

where the state transition and observation matrices are defined to be the following Jacobians

$$F_k = \left.\frac{\partial f}{\partial x}\right|_{\hat{x}_{k-1|k-1}}$$
$$H_k = \left.\frac{\partial h}{\partial x}\right|_{\hat{x}_{k|k-1}}$$

Suppose that the observation is a range and bearing of the target with zero mean Gaussian noise. Determine the observation matrix, i.e. consider

$$z = \begin{bmatrix} r \\ \theta \end{bmatrix}$$

and

$$H = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial \dot{x}} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial \dot{y}} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial \dot{x}} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial \dot{y}} \end{bmatrix}.$$

2 Gaussian sum filter

Let us consider a filtering problems under the assumptions that the dynamics and observation characteristics of the object is linear, with Gaussian noise, so that

$$f_{k|k-1}(x \mid y) = \mathcal{N}\left(x; F_{k-1}y, Q_{k-1}\right)$$
$$g_k(z \mid x) = \mathcal{N}\left(z; H_k x, R_k\right),$$

where $\mathcal{N}(z; m, P)$ denotes a multi-variate Gaussian density with mean m and covariance P, i.e.

$$\mathcal{N}(z; m, P) = |2\pi P|^{-1/2} \exp\left(-\frac{1}{2}(x-m)^T P^{-1}(x-m)\right),$$

F_{k-1} is a state transition matrix, Q_{k-1} is the process noise covariance, H_k is the matrix that projects the state onto the observation space, and R_k is the observation covariance noise matrix.

The following two identities can be assumed to be true:

Identity 1: Given F, Q, m, P of suitable dimensions, and Q and P are positive definite, then

$$\int \mathcal{N}(x; Fy, Q)\mathcal{N}(y; m, P)dy = \mathcal{N}\left(x; Fm; Q + FPF^T\right)$$

Identity 2: Given H, R, m, P of suitable dimensions, and R and P are positive definite, then

$$\mathcal{N}(z; Hx, R)\mathcal{N}(x; m, P) = q(z)\mathcal{N}(x; \bar{m}, \bar{P}),$$

where

$$
\begin{aligned}
q(z) &= \mathcal{N}\left(z; HM, R + HPH^T\right) \\
\bar{m} &= m + K(z - Hm) \\
\bar{P} &= (I - KH)P \\
K &= PH^T \left(HPH^T + R\right)^{-1}.
\end{aligned}
$$

1. Suppose that the posterior density at time $k-1$ is a Gaussian mixture of the form

$$
p_{k-1}(x) = \sum_{i=1}^{N} w_{k-1}^{(i)} \mathcal{N}\left(x; m_{k-1}^{(i)}, P_{k-1}^{(i)}\right),
$$

where the weights are positive and sum to one.

Then, under linear Gaussian assumptions, show that the the predicted density for time k is also a Gaussian mixture using the Chapman-Kolmogorov equation

$$
p_{k|k-1}\left(x_k \mid z_{1:k-1}\right) = \int f_{k|k-1}\left(x_k \mid x\right) p_{k-1}\left(x \mid z_{1:k-1}\right) dx
$$

2. Now suppose that the predicted density for time k is a Gaussian mixture as above. Using Bayes' rule, given below, show that the updated density is a Gaussian mixture and give its form.

$$
p_k\left(x_k \mid z_{1:k}\right) = \frac{g_k\left(z_k \mid x_k\right) p_{k|k-1}\left(x_k \mid z_{1:k-1}\right)}{\int g_k\left(z_k \mid x\right) p_{k|k-1}\left(x \mid z_{1:k-1}\right) dx}
$$

3 Particle filter

Let us consider a filtering problems under the assumptions that the dynamics and observation characteristics of the object is non-linear, with Gaussian noise, so that

$$
\begin{aligned}
f_{k|k-1}(x \mid y) &= \mathcal{N}\left(x; f(y), Q_{k-1}\right) \\
g_k(z \mid x) &= \mathcal{N}\left(z; h(x), R_k\right),
\end{aligned}
$$

where $\mathcal{N}(z; m, P)$ denotes a multi-variate Gaussian density with mean m and covariance P, i.e.

$$
\mathcal{N}(z; m, P) = |2\pi P|^{-1/2} \exp\left(-\frac{1}{2}(x-m)^T P^{-1}(x-m)\right),
$$

f_k is a nonlinear state transition function, Q_{k-1} is the process noise covariance, h_k is the nonlinear that projects the state onto the observation space, and R_k is the observation covariance noise matrix.

1. Suppose that the posterior density at time $k-1$ is a Dirac δ mixture of the form

$$
p_{k-1}(x) = \sum_{i=1}^{N} w_{k-1}^{(i)} \delta(x - x_{k-1}^{(i)}),
$$

where the weights are positive and sum to one.

Then, under the assumptions above, determine the predicted density with the Chapman-Kolmogorov equation, i.e.

$$p_{k|k-1}\left(x_k \mid z_{1:k-1}\right) = \int f_{k|k-1}\left(x_k \mid x\right) p_{k-1}\left(x \mid z_{1:k-1}\right) dx.$$

2. Now suppose that the predicted density for time k is a Dirac δ mixture. Using Bayes' rule, given below, determine the updated density.

$$p_k\left(x_k \mid z_{1:k}\right) = \frac{g_k\left(z_k \mid x_k\right) p_{k|k-1}\left(x_k \mid z_{1:k-1}\right)}{\int g_k\left(z_k \mid x\right) p_{k|k-1}\left(x \mid z_{1:k-1}\right) dx}$$