ELEC 3224 — State-Space based Control

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Introduction

- Frequency domain design does not extend to nonlinear dynamics (except locally based on linearising the nonlinear dynamics or for certain types of actuator and sensor nonlinearities such as saturation and dead-zones where harmonic linearisation or describing functions may be applicable.
- This set of notes introduces **state-space model based analysis,** which applies to both linear and nonlinear dynamics.
- For linear dynamics, there is a direct linear between the state-space and transfer-function descriptions of the system dynamics.
- Next some background is given — this will lead on to the properties of controllability and observability, minimality, state feedback control, observers and optimal control.

Introduction

- To follow from now on — see next figure — just look at the chessboard.
- The state contains all the relevant information about the future behavior of the system.
- The state is the memory of the system.
- Independence of past and future given the state.

Introduction

Introduction

- In the rest of this module, two points of view adopted on state and state representations.
- 1) The state is given as a "natural" set of attributes splitting past and future of the system.
- 2) The state is constructed from the equations/transfer function of the system, in order to exploit the properties and insight offered by state-space equations.
- Contradictory points of view, in some sense, but each with its merits.

Definitions

- **State** — minimum amount of information about a system at a time, say t_0, that is necessary to enable its **future behaviour** at a time, say t_1, to be determined.
- **State Variables:** a subset of n **linearly independent** system variables that represent the entire state of the system at any given time. Together with the inputs the provide a **complete description** of the system behaviour.
- **The choice of state is not unique.**
- **State-space:** n-dimensional Euclidean space where the system state can be represented.

Examples

- **A state-space model (or representation):** a **mathematical model of a physical system as a set of input, output and state variables related by 1st order ODES (Ordinary Differential Equations).**
- **Example 1:**
$$\dot{x} + ax = u$$
- **Example 2:**
$$m\ddot{x} + b\dot{x} + cx = u$$
- This ODE is a model for a linear spring mass damper system, where the underdamped case arises when $b = 2\zeta\omega_n$ and $c = \omega_n^2$, $0 < \zeta < 1$.

Examples

- The time response of Examples 1 and 2 to a unit step command can be found as an analytic formula.
- These formulas provide the basis of classical control.
- If a system is governed by a higher order ODE then finding an analytic solution will not be possible.
- Hence the state-space model represents the system equivalently as a set of coupled first order ODES.
- For example 2 introduce the so-called **state variables** as

$$x_1 = x,\ \dot{x}_1 = \dot{x} = x_2$$

- Then the defining equation can written as the following **set of coupled ODES.**

Examples

- $$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{c}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u\end{aligned}$$
- Introduce the state vector
$$X = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \quad (1)$$
- Then the system can be written in matrix form as follows.

Examples

- $$\begin{aligned} \dot{X} &= AX + Bu \\ y &= CX + Du \end{aligned}$$

  where
  $$A = \begin{bmatrix} 0 & 1 \\ -\frac{c}{m} & -\frac{b}{m} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

- $$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

Examples

- Consider the RCL circuit in the next figure.
- Kirchoff's voltage law

$$v_C(t) = L\frac{di_L(t)}{dt} + Ri_L(t)$$

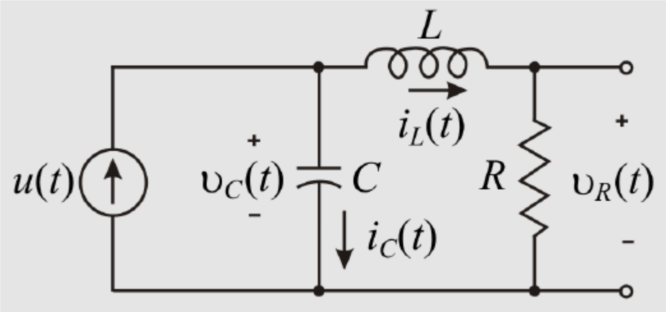
- Kirchoff's current law

$$i_C(t) = C\frac{dv_C(t)}{dt} = u(t) - i_L(t)$$

- Ohm's law (output)

$$v_R(t) = Ri_L(t)$$

Examples



Examples

- Chose the state variables as x_1 — the capacitor voltage $v_C(t)$ and x_2 — the inductor current $i_L(t)$.
- The previous equations can now be written as

$$\begin{aligned} \dot{x}_1 &= -\frac{1}{C}x_2 + \frac{1}{C}u \\ \dot{x}_2 &= \frac{1}{L}x_1 - \frac{R}{L}x_2 \\ y(t) &= v_r = Rx_2 \end{aligned}$$

- or with state vector X as in (1)

Examples

- $$\begin{aligned} \dot{X} &= \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} X + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u(t) \\ y &= \begin{bmatrix} 0 & R \end{bmatrix} X \end{aligned}$$
- What happens if the state vectors are chosen as the capacitor voltage $v_C(t)$ and the voltage drop across in the inductor $v_L(t)$?
- **Fact:** The choice of **state variables and hence the state vector is not unique.**
- **Question: What is the link to a transfer-function description?**
- **What is the minimum number of state variables needed for a given problem?**

Solving the State Equation

- Consider the scalar differential equation

$$\dot{x} - ax = bu, \;\; x(0) = x_0$$

for input u and output x.
- Then for a given u x can be computed using an **integrating factor.**
- Multiply both sides of this differential equation by e^{-at} — the **integrating factor.** Hence
- $$\frac{d(e^{-at}x(t))}{dt} = e^{-at}u(t)$$
- Integrating from 0 to t gives

Solving the State Equation

- $$x(t) = e^{at}x_0 + b\int_0^t e^{a(t-\tau)}u(\tau)d\tau$$
- Particular case: RC circuit with $u(t) = V$ – dc voltage applied at $t = 0$.
- Output – voltage across the capacitor, with zero initial voltage.
- $$x(t) = V(1 - e^{-\frac{t}{RC}})$$
- Physically consistent — in the steady state $(t \to \infty)$ the capacitor is an open-circuit!

Solving the State Equation

- **Exponential function:**

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

- This can be generalized to a **square matrix.**
- **Matrix Exponential**

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$$

Solving the State Equation

- Use this to solve the state equation

$$\dot{X} = AX + Bu, \;\; X(0) = x_0$$

- resulting in

$$X(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

- Substitution in the output equation

$$y(t) = CX(t) + Du(t)$$

gives a solution of the state-space model.

Solving the State Equation

- How to compute the matrix exponential?
- Most basic method — compute the infinite series that defines e^{At}.
- An example — the double integrator (Newtons law)

$$\ddot{x}(t) = u(t)$$

- State-space model

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Solving the State Equation

- In this case

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

-

$$e^{AT} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

- This is an **exact** formula (no numerical error!)
- Does this last result generalize?
- **Definition:** A **nilpotent matrix** is a square matrix for which

$$A^k = 0$$

for some integer k – known as **the index.**

Solving the State Equation

- **In the general case** there are **many methods.**
- **An 'interesting' paper.**
- C. Moler and C. van Loan: Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later, SIAM Review, 45(1), pp. 3-49, 2003.
- Fortunately it is often not necessary to do this computation!!

Link to the Transfer-Function

- **Fact:** the choice of the state vector **is not unique.**
- Consider the state-space model

$$\begin{aligned} \dot{X} &= AX + Bu \\ y &= CX + Du \end{aligned} \qquad (2)$$

- **A state transformation for this system is defined by**

$$X = TZ$$

where T is **a nonsingular matrix.**

- $$\dot{X} = T\dot{Z}$$

Link to the Transfer-Function

- Substituting in (2) gives

$$
\begin{aligned}
\dot{Z} &= T^{-1}ATZ + T^{-1}Bu \\
y &= CTZ + Du
\end{aligned} \tag{3}
$$

- **Fact:** (2) and (3) are **equivalent descriptions of the dynamics.**
- **Fact:** The choice of T has a major role in the solution of many state-space based control problems.

Link to the Transfer-Function

- The transfer-function of a system is defined for zero initial conditions.
- Taking the Laplace transform of (2) gives
- $$\begin{aligned}(sI - A)X(s) &= Bu(s) \\ y(s) &= CX(s) + Du(s)\end{aligned} \tag{4}$$
- or
  $$y(s) = (C(sI - A)^{-1}B + D)u(s) = G(s)u(s)$$
- where I denotes the identity matrix and $G(s)$ is the **transfer-function.**

Link to the Transfer-Function

- Taking the Laplace transform of (3) gives

$$\begin{aligned}(sI - T^{-1}AT)Z(s) &= T^{-1}Bu(s)\\ y(s) &= CTZ(s) + Du(s)\end{aligned}$$

- or

$$y(s) = (CT(sI - T^{-1}AT)^{-1}T^{-1}B + D)u(s) = \tilde{G}(s)$$

- **Fact:**

$$G(s) = \tilde{G}(s)$$

Link to the Transfer-Function

- To prove this last fact, consider two square matrices E and F of the same dimensions and both nonsingular. Then

$$(EF)^{-1} = F^{-1}E^{-1}$$

- Applied to $\tilde{G}(s)$ results in $G(s)$.
- **Fact:** For physical systems $D = 0$ in the state-space model, i.e.,no **direct feedthrough from input to output.**
- **Standing assumption from this point onways.**

Link to the Transfer-Function

- As one possible choice for the state transformation matrix consider the case when this matrix is chosen as an eigenvector matrix of A, i.e., considering for simplicity the case when A has real and distinct eigenvalues λ_i, $1 \leq i \leq n$,

$$T^{-1}AT = \Lambda = \mathrm{diag}\{\lambda_i\}_{1 \leq i \leq n}$$

- and it can be shown that

$$e^{At} = Te^{\Lambda t}T^{-1}$$

Stability

- Consider again the solution of the state equation, i.e.,

$$X(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{5}$$

- Suppose that the control input u is bounded (finite energy). Then $X(t)$ is also bounded if and only if **all eigenvalues of the state matrix A have strictly negative real parts.**
- This fact holds in **the more general case, i.e., all eigenvalues of A must have strictly negative real parts.**

Link to the Transfer-Function

- In the transfer-function case, write

$$G(s) = \frac{p(s)}{q(s)}$$

- $$q(s) = 0$$

  defines the system poles and each pole must have **a strictly negative real part** for stability.

$$G(s) = \frac{C\,\mathrm{adj}(sI - A)B}{\det(sI - A)}$$

Link to the Transfer-Function

- **Conjecture:** the system poles are the eigenvalues of the state matrix A?
- An example

$$\begin{aligned} \dot{X} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} X \end{aligned} \tag{6}$$

- In this case, the eigenvalues of A are 0 (twice) as $\det(sI - A) = s^2$.
- The transfer-function is

$$G(s) = \frac{1}{s}$$

Link to the Transfer-Function

- Hence the above **conjecture is not true for all cases.**
- Is it true in some cases?
- Yes — go back to (6) and replace the B matrix, i.e., $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- In this case $G(s) = \frac{1}{s^2}$.
- In general?
- **The answer requires the properties of controllability and observability.**

Controllability and Observability

- **Controllability characterises the link between the control input and the ability to 'steer' the state vector. Hence the focus is on the state equation.**

$$\dot{X} = AX + Bu, \;\; X \in \mathbb{R}^n \tag{7}$$

- Let $X(t_0)$ and $X(t_1)$ be two arbitrary state vectors and consider the problem of steering from $X(t_0)$ to $X(t_1)$ in finite time by bounded control action (actuators cannot generate infinite energy!!).
- **The system is said to be controllable if their exists a bounded control signal that will steer the system from any $X(t_0)$ to any $X(t_1)$ in finite time.**

Controllability and Observability

- This is a well studied problem and numerous equivalent tests for this property have been developed.
- **The system (7) is controllable if and only if**

$$\mathrm{rank}\left[\begin{array}{cccc} B & AB & A^2B & \ldots A^{n-1}B \end{array}\right] = n$$

- **In the special case of a single input single-output system controllability holds if and only if**

$$\det(\left[\begin{array}{cccc} B & AB & A^2B & \ldots A^{n-1}B \end{array}\right]) \neq 0$$

Controllability and Observability

- Earlier we considered the example when $n = 2$ and

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- This case is **not controllable** since

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0$$

- Consider the case when A is as before but $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- In this case

$$\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \neq 0$$

- In this case the **controllability property** holds.

Controllability and Observability

- Another example

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{3}{10} & 0 & 0 \\ 0 & \frac{7}{10} & \frac{1}{10} \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

- This example is not controllable since

$$\det\left(\begin{bmatrix} -1 & \frac{3}{2} & -\frac{3}{20} \\ 3 & -\frac{3}{10} & \frac{9}{20} \\ 0 & \frac{21}{10} & 0 \end{bmatrix}\right) = 0$$

Controllability and Observability

- Immediate since multiplying the first column by $\frac{3}{20}$ gives the third column (linear dependence).
- Are x_1, x_2 and x_3 uncontrollable or just one of them or 2 of them? (Back to this later.)
- Next the case when a state transform based on an eigenvalue-eigenvector decomposition of the state matrix is used. Again, for simplicity, the case when the state matrix has real distinct eigenvalues is considered. The other cases follow as generalisations.

Controllability and Observability

- Return to the case when

$$T^{-1}AT = \Lambda = \mathrm{diag}\{\lambda_i\}_{1\leq i\leq n}$$

- Then under the state transformation $Z = TX$, the state equation becomes

$$\dot{Z} = \Lambda Z + T^{-1}BU$$

- write

$$Z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Controllability and Observability

- The state variable in Z satisfy

$$\dot{z}_i = \lambda_i z_i + b_i u, \;\; 1 \leq i \leq n$$

- Hence **they are decoupled from each other.**
- Suppose that $b_i = 0$ for some i.
- Then the corresponding z_i **is independent of the control signal.**
- It is therefore **uncontrollable.**
- **Fact: Controllability if and only if** $b_i \neq 0, \; 1 \leq i \leq n$.

Controllability and Observability

- Suppose that $z_i(0) \neq 0$.
- Then
$$z_i(t) = e^{\lambda_i(t)} z_i(0)$$
- If $\lambda_i > 0$ **then this state vector entry is unstable and control action cannot stabilize it.**
- **Fact: These two tests for controllability are equivalent.**
- **Fact: This is also true for all other conditions for this property.**

Controllability and Observability

- **Controllability examines the links between the input and the state vector.**
- **The output y is a linear combination of the state vector entries.** $(y = Cx)$.
- **Observability examines the links between the state vector and the output.**
- **The requirement is that each state vector entry must make a contribution to y.**
- **If this is not the case and at least one entry in the state vector is uncontrollable and unstable then it makes no contribution to y.**

Controllability and Observability

- **Observability** — determine $x(0)$ given $y(t)$ and $u(t)$.
- Why only $x(0)$?
- If $x(0) \neq 0$ then

$$x(t) = e^{At}x(0)$$

  This is a well studied problem and numerous equivalent tests for this property have been developed.
- **The system**

$$\dot{X} = AX + Bu \quad (8)$$

$$y = CX \quad (9)$$

is observable if and only if

Controllability and Observability

- $$\text{rank}\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$
- **In the special case of a single input single-output system observability holds if and only if**

$$\det\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) \neq 0$$

Controllability and Observability

- **Fact:** the **rank** of a matrix is equal to that of its **transpose.**
- Observability in the single-input single-output case if and only if

$$\det\left(\left[\begin{array}{ccccc} C^T & A^T C^T & (A^2)^T C^T & \ldots & (A^{n-1})^T C^T \end{array}\right]\right) \neq 0$$

- Return to the case of the state transformation $X = TZ$ (where T is the eigenvector of A which has real distinct eigenvalues λ_i).
- Write

$$CT = \left[\begin{array}{ccc} c_1 & \ldots & c_n \end{array}\right]$$

Controllability and Observability

- $$y(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} z_i(0)$$
- Hence if $c_i = 0$ then **the corresponding state vector entry makes no contribution** to $y(t)$.
- An **unobservable system could be unstable but this feature is not present in the output measurement.**
- Hence the system is **observable if and only if** $c_i \neq 0, \ 1 \leq i \leq n$.

Controllability and Observability

- **Fact: These two tests for observability are equivalent.**
- **Fact: This is also true for all other conditions for this property.**
- Example

$$A = \begin{bmatrix} -1 & 0 & \beta \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$$

- β — a real scalar.

Controllability and Observability

- In this case
$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{bmatrix}$$
- The **determinant of this matrix is zero and hence this system is unobservable for all** β.

Minimality

- Now we return to the link between the transfer-function and state-space model descriptions of the dynamics.
- Continuing with the assumption that the state matrix A has real and distinct eigenvalues.
- Then with Z as the state vector

$$G(s) = \sum_{i=1}^{n} \frac{c_i b_i}{s - \lambda_i}$$

- If $b_i = 0$ then the corresponding term $\frac{1}{s-\lambda_i}$ is **deleted from the transfer-function** — this is termed an **input decoupling zero** arising from an example that is **not controllable.**

Minimality

- If $c_j = 0$ then the corresponding term $\frac{1}{s-\lambda_j}$ is **deleted from the transfer-function** — this is termed an **output decoupling zero** arising from an example that is **not observable.**
- If $b_h = 0$ and $c_h = 0$ then the corresponding term $\frac{1}{s-\lambda_h}$ is **deleted from the transfer-function** — this is termed an **input-output decoupling zero** arising from an example that is **not controllable and not observable.**
- The **system poles** are the **eigenvalues of the state matrix if and only if the system is controllable and observable.**

Minimality

- Return to the case when

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- and add an output equation with

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- It has already been established that this case **is uncontrollable.**
- This example is observable since

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \neq 0.$$

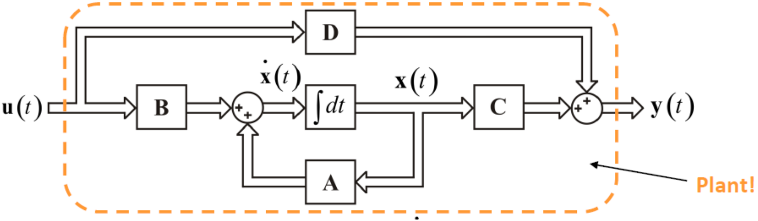
- This system has an **input decoupling zero** at $s = 0$.

Minimality

- In the case when $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ the system is **controllable and observable and there are no input and no output decoupling zeros.**
- **A system is minimal if and only if it is controllable and observable.**
- **Realisation:** having obtained a $G(s)$, one way to obtain a**minimal realization is to construct a controllable and observable state-space model.**

Internal/External Viewpoint

- The state can be viewed as the **internal dynamics**



State Feedback Control

- **A state feedback control law has the form**

$$u = FX + Gr$$

where r is the reference signal (e.g., a unit step applied at $t = 0$.)

- Controlled system state-space model

$$\begin{aligned} \dot{X} &= (A + BF)X + BGr \\ y &= CX \end{aligned} \qquad (10)$$

- The system is **stable** if and only if

$$\det(sI - A - BF) \neq 0$$

for all s such that $Re(s) \geq 0$.

- **or all eigenvalues of** $A + BF$ **must have strictly negative real parts.**

State Feedback Control Design

- **Problem?** — how to find F?
- This if often termed **the pole placement problem.**
- (Wonham 1967) **The pole placement problem has a solution if and only if the uncontrolled system is controllable, i.e., the pair $\{A, B\}$ is controllable.**
- How to compute F having established when it exists?
- Two general types of solution — **direct and indirect.**

Direct Pole Placement

- Let $\gamma_i,\ 1 \leq i \leq n$, are the desired locations (with real parts less than zero) of the poles for the controlled system, i.e.,

$$\begin{aligned} \rho_c(s) &= \prod_{i=1}^{n}(s-\gamma_i) \\ &= s^n + c_1 s^{n-1} + \ldots + c_{n-1}s + c_n \end{aligned} \tag{11}$$

- Hence the requirement is that

$$\det\left(sI - A - BF\right) = \rho_c(s)$$

Direct Pole Placement

- An $n \times n$ matrix, say H, has the **companion** structure if

$$H = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & 0 & 1 \\ -h_n & -h_{n-1} & \dots & -h_2 & -h_1 \end{bmatrix}$$

- Also

$$\det(sI - H) = s^n + h_1 s^{n-1} + \ldots + h_{n-1} s + h_n$$

Direct Pole Placement

- **Controllable Canonical Form:** Suppose that the system is controllable. Then there exists a state transformation $X = TZ$ such that the state equation has the form

$$\dot{Z} = A_c Z + B_c u$$

- where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & \dots & -a_2 & -a_1 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix}$$

Direct Pole Placement

- where $b \neq 0$.
- Also

$$\det(sI - A) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$$

- Now chose the state feedback matrix as

$$F = \frac{1}{b}\left[\begin{array}{cccc} a_n - c_n & a_{n-1} - c_{n-1} & \ldots & a_1 - c_1 \end{array}\right] \tag{12}$$

- With Z as the state vector, the state matrix is

Direct Pole Placement

- $$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & 0 & 1 \\ -c_n & -c_{n-1} & \dots & -c_2 & -c_1 \end{bmatrix}$$
- Hence a stabilizing state feedback control law with Z as the state vector is
$$u = FZ$$
where F is given by (12).

Direct Pole Placement

- **For implementation it is necessary to revert to the state vector** X and hence **the final form of the stabilising state feedback control law is**

$$u = FT^{-1}X$$

- **Example 1**

$$\dot{Z} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -10 \end{bmatrix} Z + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Direct Pole Placement

- The task is to design a state feedback law that places the closed-loop poles at

$$s = -1 \pm j, \;\; -5$$

- In this case

$$\rho_c(s) = (s+1+j)(s+1-j)(s+5) = s^3 + 7s^2 + 12s + 10$$

- Hence

$$F = \begin{bmatrix} -8 & -7 & 3 \end{bmatrix}$$

Direct Pole Placement

- Example 2
- $$G(s) = \frac{26s + 13}{s^3 + 13s^2 + 33s + 13}$$
- In controllable companion form the state dynamics are

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -13 & -33 & -13 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Direct Pole Placement

- Design a state feedback control law to place the closed-loop poles at $s = -1, -2, -3$.
- In this case

$$\rho_c(s) = (s+1)(s+2)(s+3) = s^3 + 6s^2 + 11s + 6$$

- Hence

$$F = \left[\begin{array}{ccc} -7 & -22 & -7 \end{array} \right]$$

Indirect Pole Placement

- There are a very large number of direct pole placement design algorithms.
- These require the characteristic polynomial of the controlled system.
- Forming this polynomial is feasible for relatively low order systems.
- These locations have to chosen to meet, as closely as possible, the performance specifications for the controlled dynamics.
- An alternative approach is to use **indirect pole placement.**

Indirect Pole Placement

- Basic premise: solve an **auxiliary problem that gives a stabilising state feedback control law.**
- In **direct pole placement, the closed-loop poles are specified but in indirect pole placement the guarantee is that the poles of the controlled system are stable.**
- There are many approaches to indirect pole placement but here attention is restricted to **Linear Quadratic Optimal Control in the form of Linear Quadratic Regulator (LQR) control.**
- The basis is the specification of a **cost function, which is then minimised.**

Indirect Pole Placement

- **Basic matrix property:** The eigenvalues of a symmetric matrix are real.
- **A quadratic form has the structure X^TQX where $X \in \mathbb{R}^n$ and Q is a symmetric matrix.**
- A quadratic form is **positive definite** if $X^TQX > 0$ for all $X \neq 0$.
- **One condition for $X^TQX > 0$ is that all eigenvalues of Q are positive.**

Indirect Pole Placement

- **A linear quadratic cost function** for (the multiple-input multiple-output case)

$$\begin{aligned} \dot{X} &= AX + Bu \\ y &= CX \end{aligned} \tag{13}$$

- is

$$J = \frac{1}{2}\int_0^t \left(X^T(\tau)QX(\tau) + u^T(\tau)Ru(\tau)\right) d\tau$$

where the cost function matrices Q and R, also referred to as **weighting matrices** and are **required to be symmetric positive-definite** (denoted by $\succ 0$).

Indirect Pole Placement

- The LQR problem for systems described by (13) is to **minimize J with respect to U over the infinite time interval, i.e.,**
- $$J_\infty := \min_U \frac{1}{2}\int_0^t \left(X^T(\tau)QX(\tau) + u^T(\tau)Ru(\tau)\right) d\tau$$
- The solution to this problem is

$$u = -R^{-1}B^TPX = -FX \qquad (14)$$

Indirect Pole Placement

- where P is the solution of the associated **algebraic Riccati equation**

$$PA + A^T P + Q - PBR^{-1}B^T P = 0 \qquad (15)$$

- **Fact:** $P \succ 0$.
- **Fact:** The control law (14) when applied to (13) results in **stable dynamics.**
- **This design method requires the selection** of Q and R.
- The simplest case is to select Q and R as **diagonal.**

Indirect Pole Placement

- A special case of the choice of diagonal matrices is

$$Q = qI, \;\; R = rI$$

where $q > 0$ and $r > 0$ are real scalars.

- In this last case the cost function integrand is

$$q(x_1^2 + x_2^2 + \ldots + x_{n-1}^2 + x_n^2) + r(u_1^2 + u_2^2 + \ldots + u_{m-1}^2 + x_m^2)$$

where m denotes the number of inputs.

Indirect Pole Placement

- Consider the case when

$$Q = \mathrm{diag}\{q_i\}_{1 \le i \le n}, \quad R = \mathrm{diag}\{r_i\}_{1 \le i \le m}$$

 the integrand of the cost function is

$$\frac{1}{2}\left(\sum_{i=1}^{n} q_i x_i^2 + \sum_{i=1}^{m} r_i u_i^2\right)$$

 i.e., a **weighted sum of the squares of the states plus a weighted sum of the squares of the control inputs.**
- **It is important to remember that optimal here means optimal with respect to the cost function.**

Indirect Pole Placement

- If the output is to regulated then replace Q by $C^T Q' C$.
- The solution here is for the infinite time horizon and results in the need to solve an algebraic Riccati equation. A finite time interval version is also possible but requires the solution of the differential Riccati equation backwards in time.
- Optimal control is one of the most studied problems in control systems and has seen many applications.
- Matlab routines for computing the solution exist.
- The solution algorithm is a ‘one shot’ computation, if **constrained optimal control** is considered then iterative solution algorithms are needed.
- Later optimal control in the presence of noise will be covered.

Indirect Pole Placement

- As an analytic example consider the system

$$\dot{X} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

- $Q' = I_2$, $R = 1$.
- $$Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
- Set (this matrix has to be symmetric)

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

Indirect Pole Placement

- $P \succ 0$ is required and extensive algebraic manipulations give that this property is determined by the following **three simultaneous equations.**

$$\begin{aligned} 2p_{11} - p_{11}^2 + 4p_{12} &= 0 \\ \\ p_{12} + 2p_{22} - p_{11}p_{12} &= 0 \\ 1 - p_{12}^2 &= 0 \end{aligned}$$

- The last equation gives $p_{12} = \pm 1$.
- If $p_{12} = -1$ the **first equation will have complex roots — not appropriate.**
- Hence select $p_{12} = 1$ and then the first equation gives

$$p_{11} = 1 \pm \sqrt{5}.$$

Indirect Pole Placement

- p_{11} must be positive (principal minors test for a positive-definite matrix).
- Hence
$$p_{11} = 1 + \sqrt{5} = 3.236$$
- Also
$$p_{22} = \frac{\sqrt{5}}{2} = 1.118$$
- The state feedback control law is
$$u = -R^{-1}B^{T}Px = -\left[\begin{array}{cc} 3.236 & 1 \end{array}\right] X$$
- The closed-loop poles are at
$$-\frac{\sqrt{5}}{2} \pm j\frac{\sqrt{3}}{2}$$

  i.e. **a stable system.**

Observer based Pole Placement

- To implement **a state feedback control law** $u = Fx$ **requires that all entries in the state vector are available.**
- If this is not the case, the option is to use and estimated state vector, $\hat{x}$ of x by the design of a suitable observer.
- Consider the state-space model

$$\begin{aligned} \dot{X} &= AX + Bu \\ y &= CX \end{aligned} \tag{16}$$

- **Suppose that A is stable.**
- Then an **open-loop observer** is a simulation

$$\dot{\hat{X}} = A\hat{X} + Bu \tag{17}$$

Observer based Pole Placement

- By subtracting (16) from (16) gives the observer error dynamics
$$\dot{e} = Ae$$
- where
$$e = \hat{X} - X$$
- Since A is stable $\lim_{t \to \infty} e = 0$.
- **Problems with this design.**
- **i) Cannot be implemented if A is unstable or in the presence of distrubances.**
- ii) Does not use the output $y(t)$.

Observer based Pole Placement

- **Closed-Loop Observer** (invented by D Luenberger)

$$\dot{\hat{X}} = A\hat{X} + Bu - L(C\hat{X} - y)$$

- In this case, $C\hat{X} - y$ is **the output prediction error.** Also known as the **innovation.**
- L is the **observer gain.**
- Since $y = CX$, the error dynamics are described by

$$\dot{e} = Ae - LC(\hat{X} - X) = (A - LC)e$$

Observer based Pole Placement

- **Fact: If $A - LC$ is stable, i.e., all eigenvalues have strictly negative real parts**, then

$$\lim_{t \to \infty} e = 0$$

- The estimated state dynamics are governed by

$$\begin{bmatrix} \dot{\hat{X}} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A & -LC \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} \hat{X} \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

- (Note that the observer error e is not controllable from u.)

Observer Design

- How to chose L such that all eigenvalues of $A - LC$ have strictly negative real parts?
- **Fact:** For a square matrix, say F, $\det(F) = \det(F^T)$.
- Hence
$$\det(sI - F) = \det\left(sI - F^T\right)$$
i.e., F and F^T have the **same eigenvalues.**
- Hence **choosing** L **to assign the eigenvalues of** $A - LC$ **is the same as choosing** L **to assign the eigenvalues of**
$$(A - LC)^T = A^T - C^T L^T$$

Observer Design

- The design now is that for state feedback control of

$$\dot{X} = A^T X + C^T u$$

- using $u = -L^T x$.
- This problem has a **solution if and only if** (A^T, C^T) **is controllable or, equivalently,** (A, C) **is observable.**
- Hence the observer poles can be designed using any pole placement method.

Observer Design

- The state feedback control law based on the estimated state is

$$u - F\hat{X} + Gr$$

- Closed-loop dynamics

$$\begin{bmatrix} \dot{X} \\ \dot{\hat{X}} \end{bmatrix} = \begin{bmatrix} A & -BF \\ LC & A - BF - LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} r$$

Observer Design

- Applying the co-ordinate transform

$$e = \hat{X} - X$$

-

$$\begin{bmatrix} \dot{X} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BF & -BF \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} X \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$$

Observer Design

- **This last representation shows that poles of the controlled system is the union of the eigenvalues of $A - BF$ and those of $A - LC$.**
- **This is known as the separation principle.**
- **This means that the closed-loop poles and those of the observer can be designed independently.**
- **Clearly the observer poles should be as fast as possible — the estimation error will quickly and $\hat{X}$ converges rapidly to X.**

Observer Design

- **'Obvious strategy – place the observer poles far in the left-half plane.**
- **In applications this is not always the best strategy.**
- Why? **Measurement noise.**
- **Practical alternative — place the observer poles to be** $5 - 10$ **times faster than the system poles.**
- **If the noise is 'severe' stronger action is needed.**
- **This leads to the Kalman filter.**