

ELEC 3224 — Frequency Domain Control Design

Part I

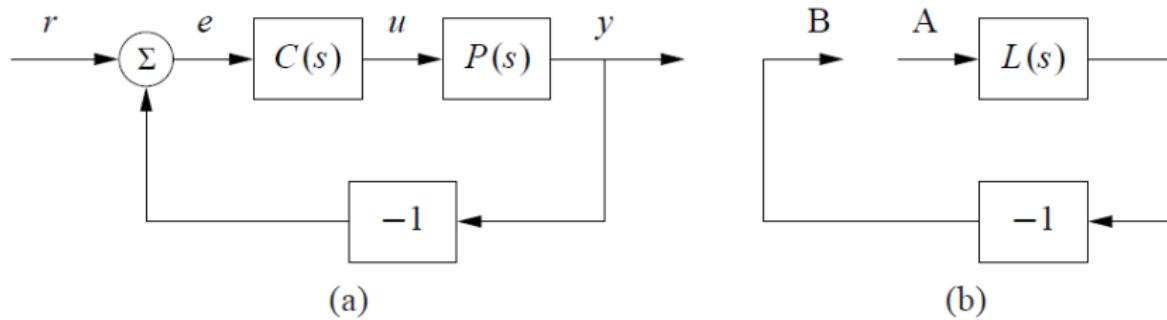
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Introduction

- ▶ The starting point is transfer-function descriptions of both the plant (system, process) to be controlled and the controller.
- ▶ In this set of notes $P(s)$ is the transfer-function of the plant and $C(s)$ is the transfer-function of the controller (often the dependence on s will be suppressed for ease of notation).
- ▶ The most common arrangement is unit negative feedback control.
- ▶ **Assumption:** The plant can be described by **linear time-invariant dynamics** and the model is **sufficiently accurate**.
- ▶ What happens if this is not the case will be considered later.

Introduction — A General Schematic



Introduction

- ▶ The assumptions guarantee that both $P(s)$ and $C(s)$ are rational and hence described by the ratio of two polynomials

$$P(s) = \frac{n_p(s)}{d_p(s)}, \quad C(s) = \frac{n_c(s)}{d_c(s)}$$

- ▶ It is assumed that there are no common factors in $n_p(s), d_p(s)$ and also $n_c(s), d_c(s)$ — often termed **coprimeness**. It is also assumed that there is no cancellations in forming the product $L = PC$.
- ▶ For physically realisable systems, the degree of the numerator in P (respectively C) is not greater than that of the denominator of P (respectively C).

Introduction

- ▶ The **poles** of P (respectively C) are the roots of

$$d_p(s) = 0 \quad (= \text{respectively } d_c(s) = 0)$$

- ▶ The **zeros** of P (respectively C) are the roots of

$$n_p(s) = 0 \quad (= \text{respectively } n_c(s) = 0)$$

- ▶ **Closed-loop transfer-function**

$$G_{yr}(s) = \frac{PC}{1 + PC} = \frac{n_p(s)n_c(s)}{d_p(s)d_c(s) + n_p(s)n_c(s)}$$

Stability

- ▶ The closed-loop characteristic polynomial for the control scheme considered is

$$\lambda(s) = d_p(s)d_c(s) + n_p(s)n_c(s)$$

- ▶ The closed loop system is **stable** if and only if a **bounded-input** produces a **bounded-output**.
- ▶ This property holds if and only if **all roots of $\lambda(s) = 0$ have strictly negative real parts.**
- ▶ 'Difficult' to design with this equation, i.e., for stability, transient response and steady state error.
- ▶ **Nyquist** investigated conditions under which oscillations can occur in the feedback control loop.

Stability

- ▶ This is explained in terms of the loop transfer-function

$$L(s) = P(s)C(s)$$

- ▶ This is the transfer-function obtained by breaking the feedback loop (see (b) in the above figure).
- ▶ $L(s)$ is the transfer-function from A to B in (b) of the above figure multiplied by -1 to account for the **the unity negative feedback loop**.
- ▶ When does a periodic oscillation arise in this loop?
- ▶ Inject a sinusoid of frequency ω_0 at A. Then in the steady state the signal at B will also be a sinusoid of the same frequency.

Stability

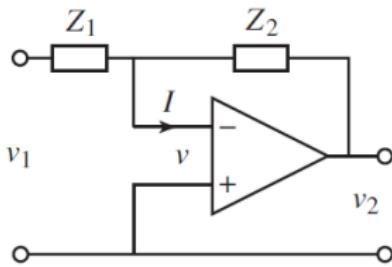
- ▶ The signals at A and B are identical if

$$L(j\omega) = -1$$

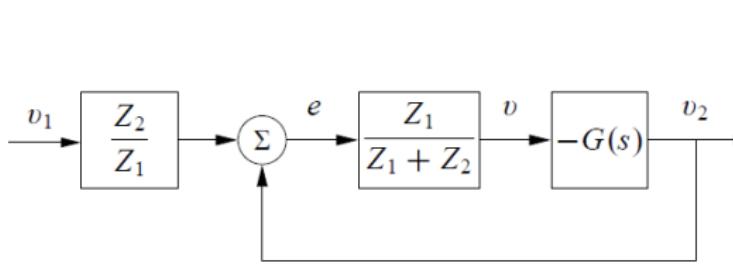
$$(j = \sqrt{-1})$$

- ▶ Under this condition an oscillation will be maintained.
- ▶ What happens in a more general case?
- ▶ All to do with poles in the right-half plane or not.

Op-Amp Example



(a) Amplifier circuit



(b) Block diagram

- ▶ Z_1 and Z_2 the transfer-functions of the feedback elements from voltage to current — feedback because v_2 is related to v through the transfer-function $-G$ (op-amp dynamics) and v is related to v_2 through the transfer-function $\frac{Z_1}{Z_1+Z_2}$.

Op-Amp Example cont'd

- ▶ In this case

$$L = \frac{GZ_1}{Z_1 + Z_2}$$

- ▶ **Assumption:** $I = 0$ and hence current through Z_1 and Z_2 is the same, therefore

$$\frac{v_1 - v}{Z_1} = \frac{v - v_2}{Z_2}$$

- ▶ Hence

$$v = \frac{Z_2}{Z_1} \frac{L}{G} v_1 - Lv$$

Op-Amp Example cont'd

- ▶ $v_2 = -Gv$ and hence (the system transfer-function)

$$G_{v_2 v_1} = -\frac{Z_2}{Z_1} \frac{L}{1 + L}$$

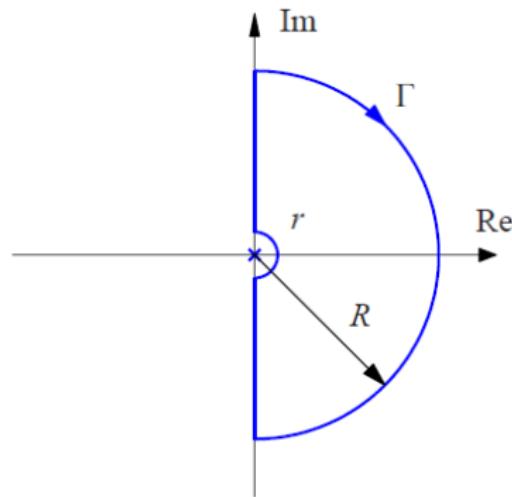
- ▶ Oscillation in the op-amp when

$$L(j\omega) = \frac{Z_1(j\omega))G(j\omega)}{Z_1(j\omega) + Z_2(j\omega)} = -1$$

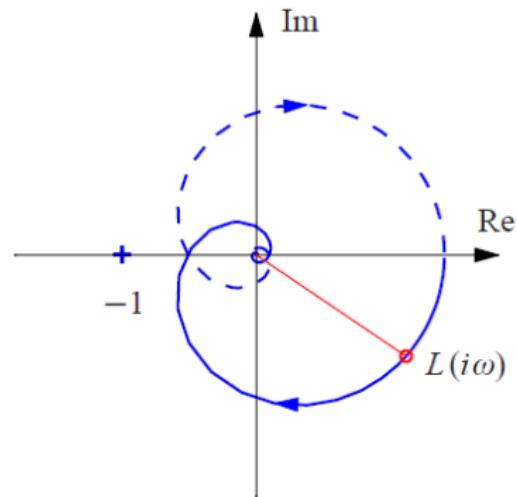
Nyquist Stability and Design

- ▶ Major plus point — study the stability of the closed-loop system in terms of $L(j\omega)$.
- ▶ Advantage – see how $C(j\omega)$ should be chosen to shape $L(j\omega)$
- ▶ For example, if **gain** of the controller is changed (a constant term) the loop transfer-function is scaled accordingly.
- ▶ To stabilize an unstable system the gain should be reduced to avoid the -1 point.
- ▶ This leads to **loop shaping design**.

Op-Amp Example



(a) Nyquist D contour



(b) Nyquist plot

Nyquist Stability and Design

- ▶ The Nyquist contour Γ (a), in the previous figure, encloses the right-half plane with s small semi-circle around any poles of $L(s)$ and an arc at infinity, represented by $R \rightarrow \infty$.
- ▶ The Nyquist plot (or diagram), (b) in the previous figure, is the image of $L(s)$ as s traverses Γ clockwise. The solid line is for $\omega > 0$ and the dashed line is for $\omega < 0$. The latter is the mirror image of the former.
- ▶ The **gain** at frequency ω is $|L(j\omega)|$ and the **phase** is $\phi = \angle L(j\omega)$.
- ▶ In this figure $L(s) = \frac{1.4e^{-s}}{(s+1)^2}$.

Nyquist Stability and Design

- ▶ If $L(s)$ goes to zero as s gets larger (the usual cases as physical systems often have more poles than zeros), the contour at 'infinity' maps to the origin.
- ▶ The condition for oscillation means that the Nyquist plot of the loop transfer-function must go through the point $L = -1$ — known as the **critical point**.
- ▶ Let ω_c be the frequency at which $\angle L(j\omega) = 180^\circ$, corresponding to where the Nyquist plot **crosses the negative real axis**.
- ▶ **Stability** provided $|L(j\omega_c)| < 1$.

Nyquist Stability Criterion

Let $L(s)$ be the loop transfer function and assume that L has no poles in the closed right half-plane ($\text{Re}s > 0$ except for single poles on the imaginary axis).

Then the closed-loop system is stable if and only if the closed contour given by

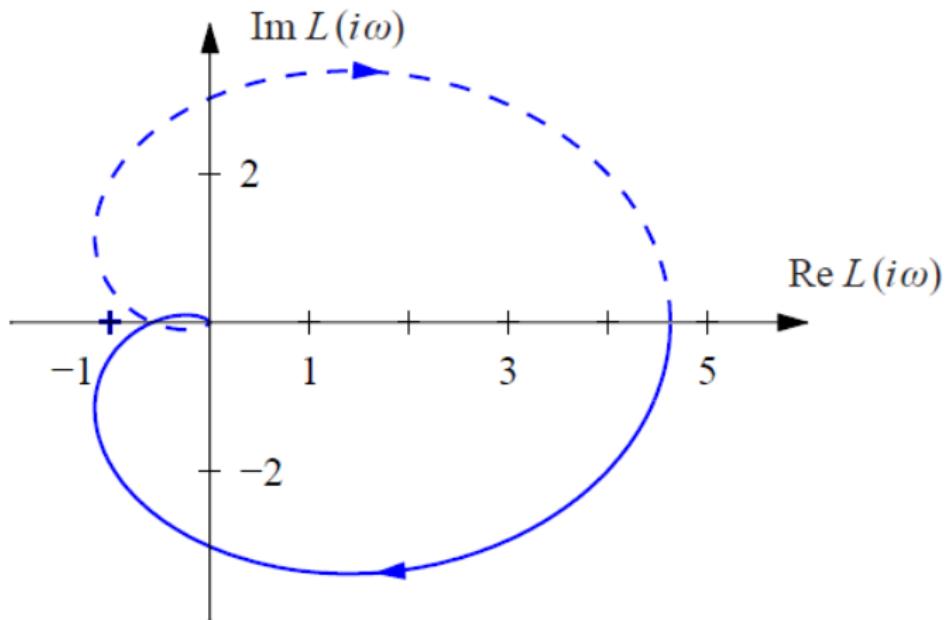
$$\Omega = \{L : -\infty < \omega < \infty\}$$

has **no encirclements** of the critical point $s = -1$.

An Example

- ▶
$$L(s) = \frac{1}{(s + a)^3}$$
- ▶
$$L(j\omega) = \frac{a^3 - 3a\omega^2}{(\omega^2 + a^2)^3} + j \frac{\omega^3 - 3a^2\omega}{(\omega^2 + a^2)^3}$$
- ▶ Nyquist plot shown in the next figure.
- ▶ Scaling by $k > 0$ (a proportional controller) can lead to instability (why?)

An Example



Bode Plots

- ▶ 2 plots — gain in dBs against ω , where the latter is plotted on a logarithmic scale.
- ▶ $20 \log_{10} |L(j\omega)|$ – Bode Gain.
- ▶ Bode phase plot — $\phi(\omega)$ versus ω on a logarithmic scale.
- ▶ There also the Bode straight line approximations.
- ▶ Example

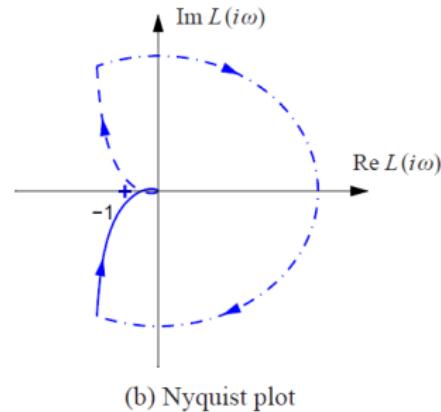
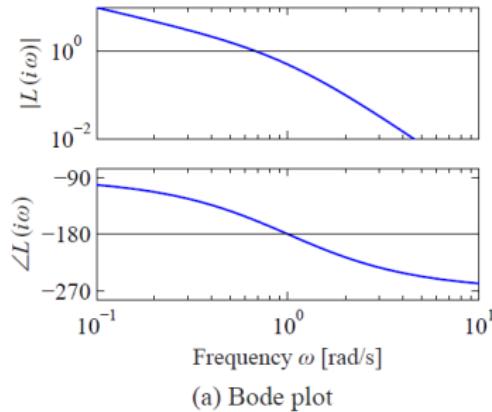
$$L(s) = \frac{k}{s(s+1)^2}$$

- ▶ $k = 1$.

Bode Plots

- ▶ This system has no zeros, a pole at $s = 0$ and a double pole at $s = -1$, whereas the previous example has all three poles at $s = -1$.
- ▶ For low frequencies the Bode gain plot has slope -1 or -20 dB/decade.
- ▶ At $s = 1$ the slope changes to -3 or -60 dB/decade.
- ▶ For low frequencies $L(s) \approx \frac{k}{s}$.
- ▶ The phase plot starts at -90° , for low frequencies it is -120° and for high frequencies it is -270° .

Bode and Nyquist Plots for the Last Example



Stability of the Last Example



$$L(j) = -\frac{k}{2}$$

- ▶ i.e. the Nyquist response is **negative real** when $\omega = 1$.
- ▶ Hence the locus crosses the negative real axis when $\omega = 1$.

$$|L(j)| = \frac{k}{2}$$

- ▶ System is unstable for all $k > 2$.

General Nyquist Stability Criterion

Consider a closed loop system with the loop transfer function $L(s)$ that has P poles in the region enclosed by the Nyquist contour. Let N be the net number of clockwise encirclements of -1 by $L(s)$ when s encircles the Nyquist contour Γ in the clockwise direction. The closed-loop system then has

$$Z = N + P$$

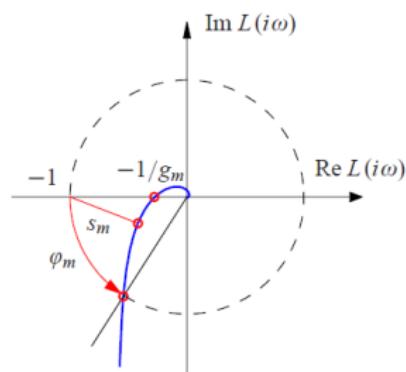
poles in the right half-plane.

- ▶ For a **stable closed-loop system** $Z = 0$ and hence

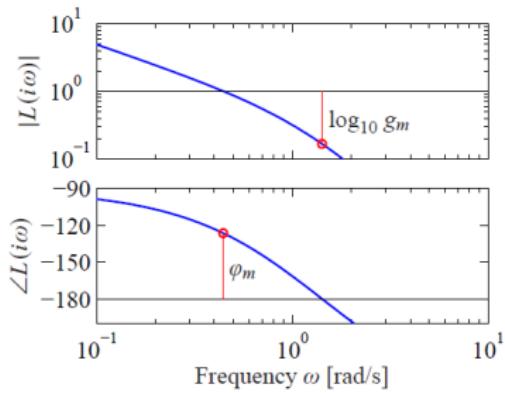
$$N = -P$$

i.e., N anti-clockwise encirclements of -1 .

Gain and Phase Margins



(a) Nyquist plot



(b) Bode plot

Gain and Phase Margins

- ▶ The **gain margin** is the **smallest amount by which the gain of $L(s)$ can be increased without the closed-loop system becoming unstable**. Let ω_{pc} be the frequency where $L(j\omega)$ is negative real — termed the **phase crossover frequency**. Then the gain margin is

$$g_m = \frac{1}{|L(j\omega_{pc})|}$$

- ▶ The **phase margin**, is the amount of phase lag required to reach the stability limit.

Gain and Phase Margins

- ▶ Consider again the case when

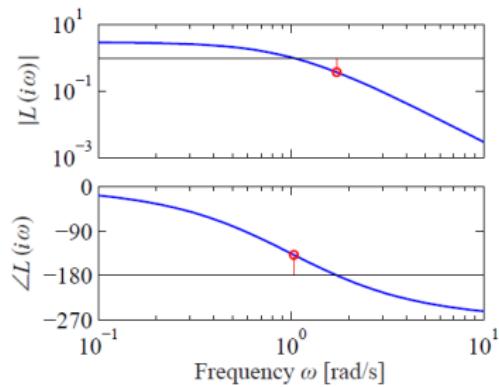
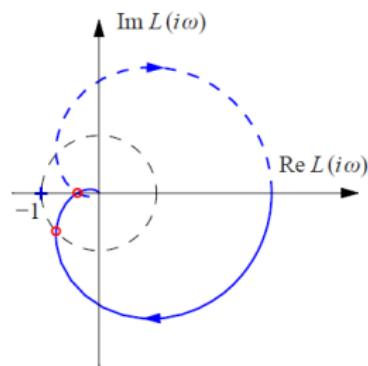
$$L(s) = \frac{3}{(s+1)^3}$$

- ▶ In this case

$$g_m = 2.67, \phi_m = 41.7^\circ, s_m = 0.464$$

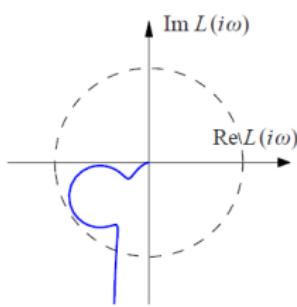
- ▶ Gain and phase margins are classical and long standing robustness margins and have been extensively used in design.
- ▶ s_m is the **stability margin** – shortest distance to the critical -1 point (very relevant to disturbance attenuation (covered later)).

Gain and Phase Margins

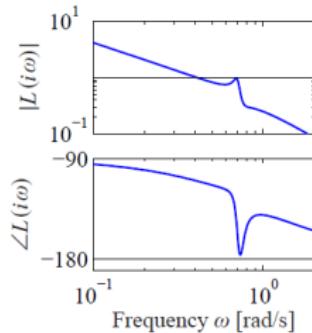


- ▶ Margins for the last example.

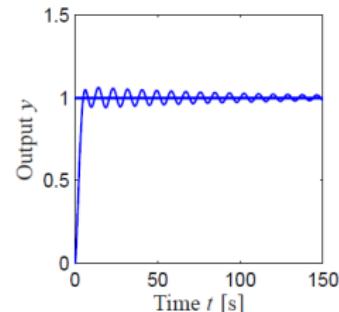
Gain and Phase Margins



(a)



(b)



(c)



$$L(s) = \frac{0.38(s^2 + 0.1s + 0.55)}{s(s + 1)(s^2 + 0.06s + 0.5)}$$

Gain and Phase Margins

- ▶ In this case $g_m = 266$, phase margin is 70° .
- ▶ The Nyquist curve is ‘close’ to the critical -1 point.
- ▶ $s_m = 0.27$ — very low.
- ▶ This system has 2 resonant modes – with damping ratios 0.81 and $0.014 \Rightarrow$ highly oscillatory step response ((c) in the figure above).
- ▶ Target values to achieve — phase margin $30^\circ - 60^\circ$, gain margin $2 - 5$ $s_m = 0.5 - -0.8$.

Generalised Concepts of Gain and Phase

- ▶ **System Gain:** Consider the scalar static linear system $y = Ax$, where A is a matrix with complex entries and not necessarily square.
- ▶ The **Euclidean norm** of the input vector with elements u_i :

$$\|u\| = \sqrt{\sum |u_i|^2}$$

- ▶ The norm of the output is

$$\|y\|^2 = u^* A^* A u$$

- ▶ * denotes the complex conjugate transpose.

Generalised Concepts of Gain and Phase

- ▶ The matrix A^*A is **symmetric** — **Fact:** a symmetric matrix only has **real** eigenvalues.
- ▶ Let $\lambda_{max}(A^*A)$ denote the largest eigenvalue of A^*A . Then

$$\|y\|^2 \leq \lambda_{max}(A^*A) \|u\|^2$$

- ▶ Then the **system gain is defined as the maximum ratio of the output to the input over all possible inputs, i.e,**

$$\gamma = \max_u \frac{\|y\|}{\|u\|} = \sqrt{\lambda_{max}(A^*A)}$$

Generalised Concepts of Gain and Phase

- ▶ For dynamic systems, it is necessary to consider signals.
- ▶ L_2 — **the space of square integrable functions** with norm

$$\|u\|_2 = \sqrt{\int_0^\infty |u|^2(\tau) d\tau}$$

- ▶ **The gain of a system mapping input $u_2 \in L_2$ to output $y \in L_2$ is**

$$\gamma = \sup_{u \in L_2} \frac{\|y\|}{\|u\|} \quad (1)$$

- ▶ sup — ‘more general’ version of max.

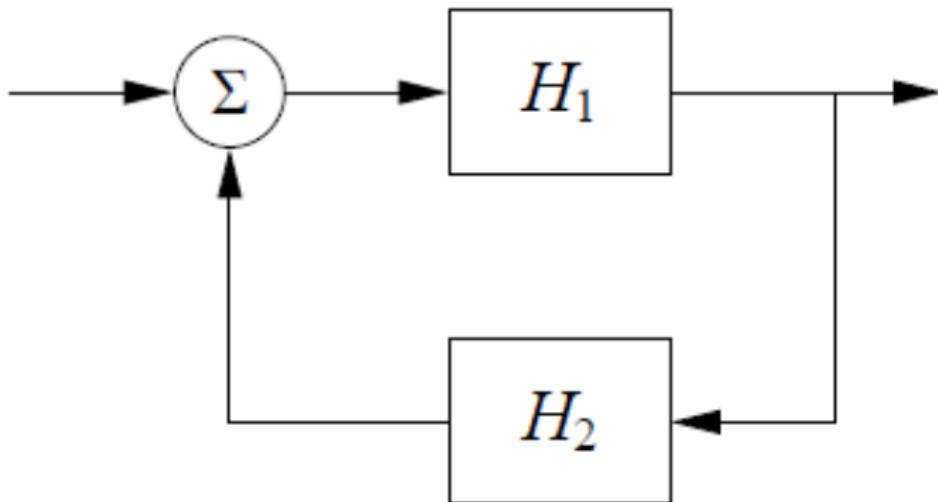
Generalised Concepts of Gain and Phase

- ▶ The norm (1) has some very useful properties for linear system.
- ▶ Consider a single-input single-output (SISO) stable linear system with transfer-function $G(s)$.
- ▶ The norm of the system in this case is

$$\gamma = \sup_{\omega} |G(j\omega)| = \|G\|_{\infty}$$

- ▶ **Physical meaning:** this norm is **the peak value of the frequency response.**
- ▶ Known as the H_{∞} norm of the transfer-function.
- ▶ Can be generalised to Multiple-Input Multiple-Output (MIMO) systems.

Small Gain Theorem



Small Gain Theorem

- ▶ For the closed-loop system in the previous figure, let H_1 and H_2 are stable transfer-functions with gains γ_1 and γ_2 (with appropriate signal spaces).
- ▶ The closed-loop system is stable if

$$\gamma_1 \gamma_2 < 1$$

- ▶ Also the gain of the closed-loop system is

$$\gamma = \frac{\gamma_1}{1 + \gamma_1 \gamma_2}$$

Small Gain Theorem

- ▶ For linear systems this result forces the Nyquist locus to **lie entirely within the unit circle in the complex plane for stability.**
- ▶ This is a **sufficient but not necessary condition** and there it is a **conservative condition**.
- ▶ This result can be extended to nonlinear systems.
- ▶ To generalise the phase, start with the inner product in L_2 defined as

$$\langle u, y \rangle = \int_0^{\infty} u(\tau) y(\tau) d\tau$$

Small Gain Theorem

- ▶ The **phase** φ between two signals u and y is

$$\langle u, y \rangle = ||u|| ||y|| \cos(\psi)$$

- ▶ Systems where the phase between inputs and outputs is 90° for all inputs are termed **passive** systems.
- ▶ By the Nyquist stability theorem, the closed-loop system is stable if the phase of the loop transfer-function (L) is between $-\phi$ and ϕ .
- ▶ An extension to nonlinear systems (passivity theory).