ELEC 3224 — Kalman Filtering Examples

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Example 1

- Consider an unstable first order system

$$\begin{aligned} \dot{x} &= x + u + w_1 \\ y &= x + w_2 \end{aligned}$$

- The uncorrelated noise signals w_1 and w_2 are white noise signals with intensities V_1 and V_2.
- In this case the Riccati equation is

$$2P + V_1 - \frac{P^2}{V_2} = 0$$

Example 1

- This last equation has the positive solution

$$P = V_2 + V_2\sqrt{1 + \frac{V_1}{V_2}}$$

- **In this case, the Kalman filter gain is only a function of the ratio $\beta = \frac{V_1}{V_2}$**
- The Kalman filter gain is

$$H = \frac{P}{V_2} = 1 + \sqrt{1+\beta}$$

- The Kalman filter error dynamics are
-

$$\begin{aligned} \dot{\tilde{x}} &= (A - HC)\tilde{x} + w_1 - Kw_2 \\ &= -\sqrt{1+\beta}\tilde{x} + w_1 - (1 + \sqrt{1+\beta})w_2 \end{aligned}$$

Example 1

- The pole of the Kalman filter pole is at $-\sqrt{1+\beta}$. Hence as β increases, the pole moves to the left on the real axis. Hence the Kalman filter very much 'trusts' the measurements. Conversely, if $\beta \to 0$, the pole of the Kalman filter approaches -1, i.e., the Kalman filter pole is as fast as the system pole. In this case the filter 'trusts' the system model much more than the measurements.

Example 2

- For a Kalman filter design, the Riccati equation

$$AP + PA^T + V_1 - PC^T V_2^{-1} CP = 0$$

with

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

results in the following three simultaneous equations to be solved to result is a **symmetric positive-definite matrix.**

$$\begin{aligned} -p_1^2 + 2p_2 + 3 &= 0 \\ p_1 + p_3 - p_1 p_2 &= 0 \\ -p_2^2 + 2p_2 + 3 &= 0 \end{aligned}$$

Example 2

- **There are many equivalent conditions** for a symmetric matrix to be positive definite.
- One condition is that all eigenvalues, which are real, are positive.
- Another is that **all principal minors** are positive.
- This last condition for the 2×2 case is $p_1 > 0$ and $(p_1 p_3 - p_2^2) > 0$ or $p_3 > 0$ and $(p_1 p_3 - p_2^2) > 0$.
- From the last equation, the candidate values for p_2 are $p_2 = -1$ or $p_2 = 3$. item Now consider the 1st equation for these possible values of p_2 and the requirement that $p_1 > 0$. Hence the only feasible choices are $p_1 = 3$ or $p_1 = 1$.
- Now consider the second equation, where $p_3 > 0$ is required. Consequently $p_3 = 6$ is the required value.

Example 2

- Hence

$$P = \begin{bmatrix} 3 & 3 \\ 3 & 6 \end{bmatrix}$$

- The state-space models in this case are

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- Hence

$$H = PC^T = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Example 2

- The Kalman filter is

$$\dot{\hat{x}} = (A - KC)\hat{x} + Bu + Ky$$

- or

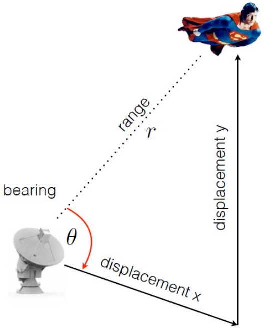
$$\begin{aligned} \dot{\hat{x}}_1 &= -3\hat{x}_1 + \hat{x}_2 + u + 3y \\ \dot{\hat{x}}_2 &= -2\hat{x}_1 + 3y \end{aligned}$$

Example 3 — Extended Kalman Filtering

- The derivation of the Kalman filter is based on a **model for linear dynamics, or, more accurately, a linear Gaussian model.**
- **Fact:** In many applications, including in aerospace and hence guidance and control, the dynamics are **nonlinear.**
- One option in such cases is to use an **extended Kalman filter (EKF).**
- The EKF does not assume linear Gaussian models, but does assume Gaussian noise. Also it uses local linear approximations of the model dynamics to retain the efficiency of the Kalman filter setting.

Example 3 — Extended Kalman Filtering

- One problem of interest – 2D example of range bearing.



Example 3 — Extended Kalman Filtering

- State vector — position velocity

$$x = \begin{bmatrix} x & \dot{x} & y & \dot{y} \end{bmatrix}^T$$

- Constant velocity model (discrete time) — state matrix (see the next section)

$$A = \begin{bmatrix} 0 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- T is the sampling time.

Example 3 — Extended Kalman Filtering

- Assuming additive (white) noise, **the motion model is linear.**
- Measurement – **range bearing** – is

$$z = \begin{bmatrix} r \\ \Theta \end{bmatrix} = \begin{bmatrix} \sqrt{x^2 + y^2} \\ \tan^{-1} \frac{y}{x} \end{bmatrix}$$

- Assuming additive (white) noise, the **measurement is nonlinear.**
- **Solution** – linearise the measurement equation and then apply the Kalman filter — this is the **extended Kalman filter.**

Example 3 — Extended Kalman Filtering

- **Fact:** the extended Kalman filter can be ‘problematic’ to apply. Reasons include:
- Linearisation by Taylor series expansion can result in a poor approximation of nonlinear functions.
- The success of the linearisation depends on a limited level of uncertainty and the strength of the nonlinearity.
- Requires the computation of partial derivatives — often not easy.
- Drifts when the linearisation is a poor approximation of the dynamics.
- Cannot be applied with multi-modal (multi-hypothesis) distributions.