SEMESTER 2 EXAMINATION 2021 - 2022

GUIDANCE, NAVIGATION AND CONTROL

DURATION MINS (Hours)

This paper contains 5 questions

Answer **three** questions

An outline marking scheme is shown in brackets to the right of each question.

This examination contributes 100% of the marks for the module

University approved calculators MAY be used.

A foreign language dictionary is permitted ONLY IF it is a paper version of a direct Word to Word translation dictionary AND it contains no notes, additions or annotations.

17 page examination paper.

Question 1.

(a) The errors in one axis of an inertial navigation system are described by

$$\begin{aligned}\Delta\dot{P} &= \Delta V \\ \Delta\dot{V} &= -g\Delta\Theta + B \\ \Delta\dot{\Theta} &= \frac{1}{R}\Delta V + W\end{aligned}$$

where ΔP is the position error, ΔV is the velocity error, $\Delta\Theta$ is the tilt error, W is the gyro drift error, B is the accelerator bias error, R is the radius of the earth, and g is the acceleration due to gravity. Use the state variables $\Delta P = x_1$, $\Delta V = x_2$ and $\Delta\Theta = x_3$ to obtain a state-space model of the dynamics of this system. Determine the equation that governs the stability of this system. [13 marks]

(b) Consider the system described by

$$\dot{x}(t) = \begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

which is to be controlled by minimising the cost function

$$J = \frac{1}{2}\int_0^\infty (x^T(t)Qx(t) + u^T(t)Ru(t))dt$$

where $R = 1$ and

$$Q = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$$

Compute the optimal input in this case. You may make use of the following equation where all symbols have their normal meanings

$$A^TP + PA + Q - PBR^{-1}B^TP = 0$$

and assume that P is a diagonal matrix. [20 marks]

Indicative Solution for Question 1.

(a) [13 marks] Given the state variables, the system dynamics are described by

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -gx_3 + B\\ \dot{x}_3 &= \frac{1}{R}x_2 + W\end{aligned}$$

Hence

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -g \\ 0 & \frac{1}{R} & 0 \end{bmatrix} X + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} B \\ W \end{bmatrix}$$

The modes of the system are the eigenvalues of the state matrix, and hence

$$\rho(s) = s(s^2 + \frac{g}{R}) = s^3 + \frac{g}{R}s$$

(b) [20 marks]

$$A^T P + PA + Q - PBR^{-1}B^T P = 0$$

is the algebraic Riccati matrix equation and P **is required to be diagonal, i.e.,** $P = \mathrm{diag}\{p_1, p_2\}$.
With this choice the ARE is

$$\begin{bmatrix} 4 - 6p_1 & 2 - 3p_1 \\ 2 - 3p_1 & 6 - \frac{p_2^2}{2} \end{bmatrix} = 0$$

This leads to the following solutions

$$P_a = \mathrm{diag}\{\frac{2}{3}, -2\sqrt{3}\}, \quad P_b = \mathrm{diag}\{\frac{2}{3}, 2\sqrt{3}\}$$

P_a **is not positive definite where as** P_b **is and is the valid choice.**

Stabilizing state feedback control law is

$$u = -R^{-1}B^T P_b x$$

TURN OVER

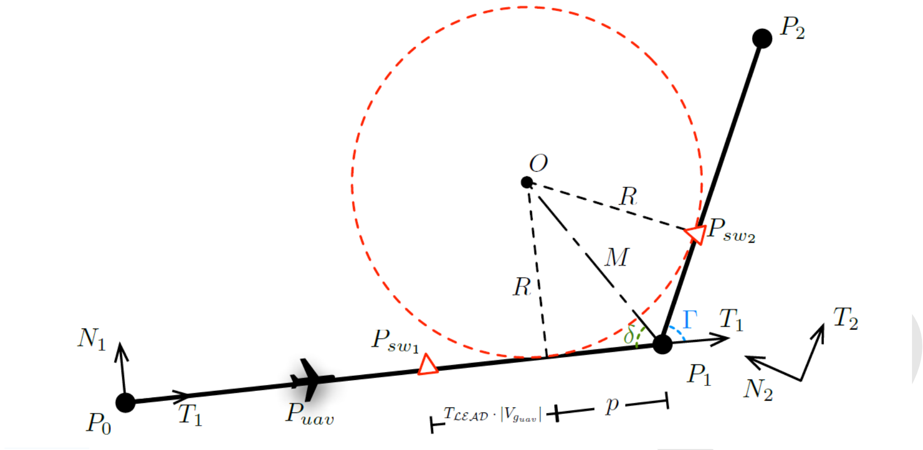


Figure Q2

Question 2.

(a) State the purpose of waypoint switching for UAVs. What is the distinctive feature of the L_2^+ implementation?

[6 marks]

(b) Figure Q2 shows the waypoint switching geometry for an UAV, where all terms have their usual meanings.

The radius of the circle in this figure is

$$R = \frac{(U_c + |\text{wind}|)^2}{a_{\max}}$$

Define all the variables in this formula. The actual radius of curvature of the vehicle can change. Explain why and give a possible solution.

[8 marks]

(c) What is the meaning of the point P_{sw1} in Figure Q2? Show also that

$$p = \frac{R}{\tan \delta}$$

[11 marks]

(d) Why in flight tests can commencing the turn just at the switch point cause a problem? Explain the use of a lead time as a counter to this problem and develop the formula for computing the resulting switch point. Also, what happens in this case if the vehicle is already beyond the next segment switching point? [8 marks]

Indicative Solution for Question 2.

(a) [6 marks] The guidance strategy is to move from one segment to the next of the flight mission by connecting the two segments with a circular arc. An L_2^+ implementation uses early waypoint switching to give priority to path-following over precision in the waypoint.

(b) [8 marks]

U_c **— commanded airspeed**
wind **— the wind speed**
$a_{\mathbf{max}}$ **— maximum acceleration of the UAV**

In the presence of wind the actual radius of curvature of the vehicle over the ground changes as the track angle changes. To avoid this problem in defining switching points, the maximum possible ground speed should be used in the numerator, which creates circle with radius larger than any curve over the ground.

(c) [11 marks]

The first tangent point, $P_{\mathbf{sw1}}$, determines the location where the UAV will switch to start tracking the next waypoint segment.

$$\Gamma = \arccos T_1^T T_2$$
$$\delta = \frac{\Gamma - \pi}{2}$$

Also

$$p = M \cos \delta, \, : M = \frac{R}{\sin \delta}$$

Hence

$$p = \frac{R}{\tan \delta}$$

(d) [8 marks]

Initiating the turn just at the switch point given by the last equation is not adequate due to the lag time of the roll dynamics of the UAV. To counter this a lead time, say $\tau_{\mathbf{lead}}$ can be introduced to initiate the turn. Multiplying $\tau_{\mathbf{lead}}$ by the groundspeed, gives the extra distance from the waypoint to the switch point. The new switch point distance is

$$p = \tau_{\mathbf{lead}} |V_g| + \frac{R}{\tan \delta}$$

and

$$P_{\mathbf{sw1}} = P_1 - pT_1$$

During the transition from missions, the L_2 vector intercepts the circular arc, not the straight line segments. Also, depending on the exact geometry of the waypoints, and the aircraft speed, it may be that just after switching legs, the aircraft is already beyond the next segment switching point. In this case, the logic immediately switches again to the next leg.

Question 3.

(a) The equations of motion of a space vehicle in the earth's gravitational field are

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\frac{ga^2}{r^2} \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= 0 \end{aligned}$$

where all terms have their usual meanings. Also show that the second equation of motion in this case can be written as

$$r^2\dot{\theta} = r_0 V \cos\gamma$$

and again define the terms in this equation. Give also the conditions under which this equation is valid. [13 marks]

(b) What is the reasoning for using complementary filtering in aerospace applications? Support your answer by explaining how this approach can be applied to **any three of** i) roll angle estimation, ii) pitch angle estimation, iii) altitude estimation, and iv) altitude rate estimation. [20 marks]

TURN OVER

Indicative Solution for Question 3.

(a) [13 marks]

The vehicle is treated as a point mass m and there is no thrust applied. Also g de notes the acceleration at the earth's surface. At $t = 0,$ the radial distance $r(t) = r_0$ and the tangential velocity of the vehicle is $V \cos\gamma,$ where V and γ are constants.

Write the position (in range and bearing format) as

$$P = re^{j\theta}$$

Hence the velocity is

$$V = \dot{r}e^{j\theta} + j\dot{\theta}e^{j\theta}$$

Differentiating again gives the radial acceleration as

$$a_r = \ddot{r} - r\dot{\theta}^2$$

Equation of motion

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{CMm}{r^2}$$

(C is a constant and M is the mass of the earth). At $r = a,$

$$mg = \frac{CmM}{a^2}$$

and hence $CM = ga^2.$ Hence the radial equation is

$$\ddot{r} - r\dot{\theta}^2 = -\frac{ga^2}{r^2}$$

Considering the second equation and multiplying through by r gives

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0$$

or

$$\frac{d}{dr}(r^2\dot{\theta}) = 0$$

Hence (by integrating)

$$r^2\dot{\theta} = C$$

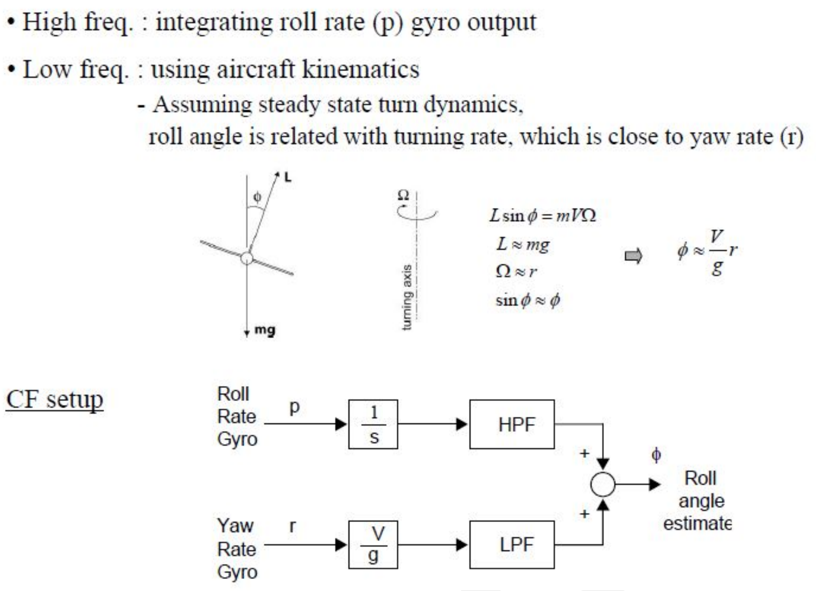
and

$$C = r_0 V \cos\gamma$$

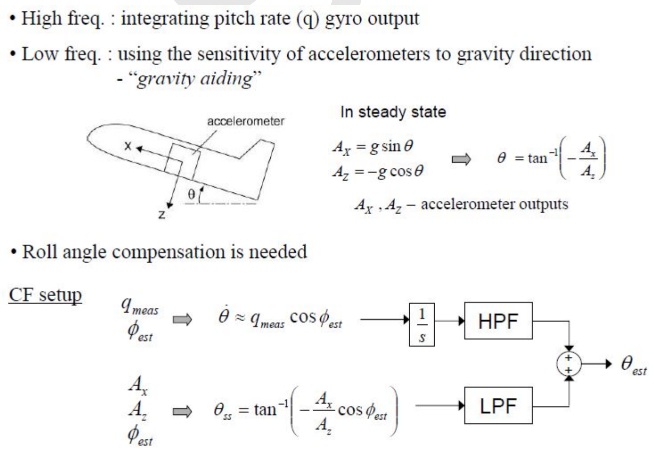
and the formula is established.

(b) [20 marks]

Often there are cases where two different measurement sources are possible for estimating one variable. Moreover, the noise properties of the two measurements are such that one source gives good information only in the low frequency region and the other only in the high frequency region. This is the general application area for complementary filters. Roll angle estimation



Roll angle estimation.

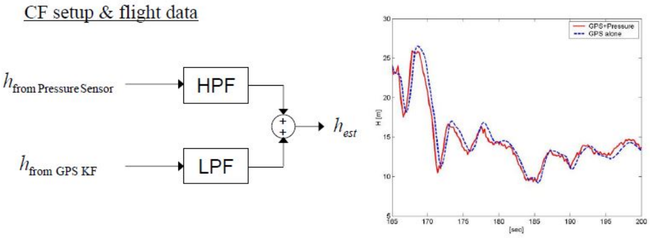


Pitch angle estimation.

TURN OVER

- Motivation : GPS receiver gives altitude output, but it has ~0.4 seconds of delay. In order of overcome this, pressure sensor was added.
- Low freq. : from GPS receiver
- High freq. : from pressure sensor

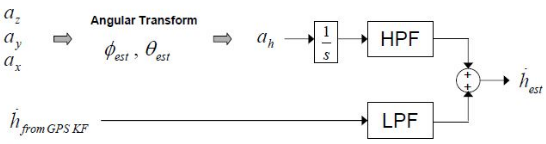
CF setup & flight data



Altitude estimation.

- Motivation : GPS receiver gives altitude rate, but it has ~0.4 seconds of delay. In order of overcome this, inertial sensor outputs were added.
- Low freq. : from GPS receiver
- High freq. : integrating acceleration estimate in altitude direction from inertial sensors

CF setup



note : $\begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} = \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix} - [\phi_{est}][\theta_{est}]\begin{Bmatrix} 0 \\ 0 \\ -g \end{Bmatrix}$ A_x , A_z – accelerometer outputs

$[\phi_{est}]$,$[\theta_{est}]$: angular transformation matrices

Altitude rate estimation.

Question 4.

(a) Figure Q4 shows a control scheme where all symbols have their normal meanings. Write down the equations that govern the dynamics of this system and give the formulas for i) the sensitivity function S, ii) the load sensitivity function PS, iii) the complementary sensitivity function T and iv) the noise sensitivity function CS. What is the significance of the relationship

$$S + T = 1$$

[10 marks]

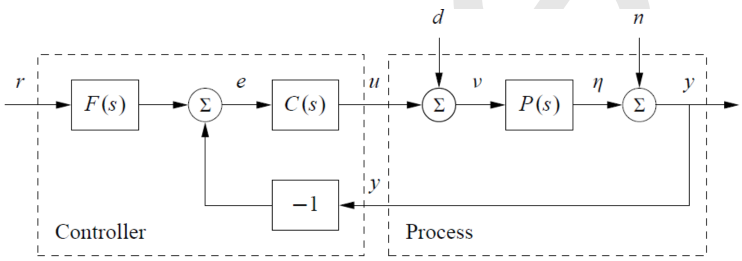


Figure Q4

(b) Consider the case of a system, for which

$$P(s) = \frac{1}{s-a}$$

where $a > 0$, is to be controlled by applying the scheme of the previous part of this question with $F(s) = 1$ and

$$C(s) = k\frac{s-a}{s}, \;\; k > 0$$

Compute the functions S, PS, T and CS in this case and hence explain why this scheme cannot be applied. What are the implications when $a < \epsilon$ in the cases when i) ϵ is a very small negative number and ii) ϵ is a very large negative number? [10 marks]

TURN OVER

(c) Consider a differential linear time-invariant system described by

$$P(s) = \frac{1}{s^2}$$

for which the performance specifications are less than 1% steady state error for step inputs and less than 10% tracking error up to 10 rad/sec. Give a sketch of the gain and phase responses of this system and explain why increasing the gain is not a feasible design. **Specify, but do not analyse,** the structure of a controller $C(s)$ that can be used to meet this specification. [13 marks]

Indicative Solution for Question 4.

(a) [10 marks] Governing equations

$$\begin{aligned} y(s) &= \eta(s) + n(s) \\ \eta(s) &= P(s)(u(s) + d(s)) \\ u(s) &= C(s)e(s) \\ e(s) &= F(s)r(s) - y(s) \end{aligned}$$

Sensitivity Function

$$S = \frac{1}{1 + PC}$$

Load Sensitivity Function

$$PS = \frac{P}{1 + PC}$$

Complementary Sensitivity Function

$$T = \frac{PC}{1 + PC}$$

Noise Sensitivity Function

$$CS = \frac{C}{1 + PC}$$

$S + T = 1$ **–** S **is sensitivity to noise and** T **is performance. Hence these two central design issues are coupled.**

(b) [10 marks] Routine algebra gives

$$\begin{aligned} T &= \frac{k}{s+k}, \;\; PS = \frac{s}{(s-b)(s+k)} \\ CS &= k\frac{s-a}{s+k}, \;\; S = \frac{s}{s+k} \end{aligned}$$

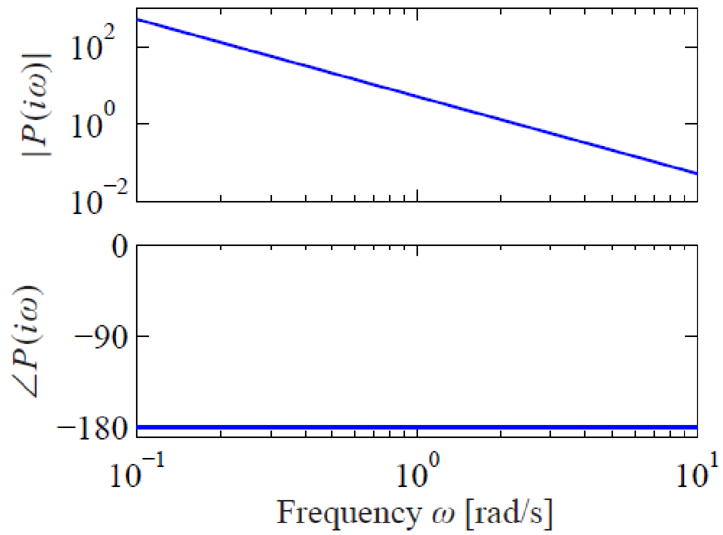
Unstable pole-zero cancellation when $b > 0$ **and** PS **is unstable. Hence the design is not feasible. In case i) this is a stable pole-zero cancellation but very close to the stability boundary — impinges on the transient dynamics and poor relative stability unless the cancellation is exact. Case ii) is feasible as the pole-zero cancellation is far away from the stability boundary.**

(c) [13 marks]

To achieve the specification, a gain of at least 10 **at** 10 **rad/s is needed, requiring the gain crossover frequency to be at a higher frequency. From the figure below, increasing the gain would give a very low phase margin. Hence the phase at the desired crossover frequency must be increased. This can be achieved by a phase lead controller**

$$C(s) = \frac{s+a}{s+b}, \;\; a < b$$

TURN OVER


$|P(i\omega)|$
10^2
10^0
10^{-2}
$\angle P(i\omega)$
0
−90
−180
10^{-1}
10^0
10^1
Frequency ω [rad/s]

Question 5.

(a) Explain the following for target tracking categories with respect to size, sensor resolution, and target-sensor distance: i) point target, ii) extended target, iii) unresolved targets, and iv) dim targets.

[10 marks]

(b) Calculate the transition matrix e^{AT} for the following cases

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Use the resulting matrices to give the target tracking state-space model for i) a nearly constant velocity and ii) a nearly constant acceleration model.

[14 marks]

(c) Explain the principles of the guidance laws for i) proportional navigation, ii) beam rider and iii) command-to-line-of-sight.

[9 marks]

TURN OVER

Indicative Solution for Question 5.

(a) [10 marks] Point target: A target that can result in at most a single measurement. This means its magnitude is comparable to sensor resolution.

Extended target: A target that can result in multiple measurements at a single scan. Sometimes, an extended target can also be treated as a point target by tracking its centroid or corners.

Unresolved targets: This term denotes a group of close targets that can collectively result in a single measurement in the sensor.

Dim target: This is a target which results in returns that cannot be separated from noise by simple thresholding. These can be tracked much better with track before detect (TBD) type approaches.

(b) [14 marks] These are both nilpotent matrices, where $A_1^2 = 0$ and $A_2^3 = 0$. Hence in the power series representation

$$e^{A_1 T} = I + A_1 T = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

and

$$e^{A_2 T} = I + A_2 T + \frac{1}{2} A_2^2 T^2 = \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}$$

Nearly constant velocity model

$$x_k = \begin{bmatrix} x_k \\ v_k^x \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{k-1} \\ v_{k-1}^x \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} a_k$$

where $a_k \sim \mathcal{N}(0, \sigma_a^2)$ is a white noise source.

Nearly constant acceleration model

$$x_k = \begin{bmatrix} x_k \\ v_k^x \\ a_k^x \end{bmatrix} = \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{k-1} \\ v_{k-1}^x \\ a_{k-1}^x \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} \\ T \\ 1 \end{bmatrix} \eta_k$$

where $\eta_k \sim \mathcal{N}(0, \sigma_\eta^2)$ is a white noise source.

(c) [9 marks]

PN —acceleration command is perpendicular to the instantaneous line-of-sight (LOS) and is proportional to the LOS rate and closing velocity

$$u = K_N v_c \dot{\lambda}$$

Beam rider – pursuit object should proceed along the beam, the beam tracks the target and the commands this object receives are a function of the angular deviation from the beam

$$u = k R_M (\theta_e - \theta_p)$$

CLOS modify beam rider law to include beam motion

$$u = KR_M(\theta_e - \theta_p) + f(\dot{\theta}_e, \ddot{\theta}_e)$$

END OF PAPER

