SEMESTER 1 EXAMINATION 2020 - 2021

GUIDANCE, NAVIGATION AND CONTROL

DURATION MINS (Hours)

This paper contains 1 questions

Answer **all parts of the** question

An outline marking scheme is shown in brackets to the right of each question.

This examination contributes 100% of the marks for the module

University approved calculators MAY be used.

A foreign language dictionary is permitted ONLY IF it is a paper version of a direct Word to Word translation dictionary AND it contains no notes, additions or annotations.

7 page examination paper.

Answers

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Question 1.

(a) A linear time-invariant system is described by the differential equation

$$\ddot{x} + \omega_0^2 x = u$$

Develop a state-space model for this system with state matrix

$$A = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix}$$

Hence show that the state transition matrix in this case is

$$e^{At} = \begin{bmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{bmatrix}$$

Hint: Consider the derivative of the matrix exponential.

[15 marks]

(b) A coupled mechanical system is described by the equations of motion

$$\begin{aligned} \ddot{x} &= -2ex - f\dot{x} + ey \\ \ddot{y} &= ex - 2ey - f\dot{y} + eu \end{aligned}$$

where u denotes the input and e and f are real constants. Use the state vector

$$X = \begin{bmatrix} x & y & \dot{x} & \dot{y} \end{bmatrix}^T$$

to rewrite these equations of motion as

$$\dot{X} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2e & e & -f & 0 \\ e & -2e & 0 & -f \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 0 \\ e \end{bmatrix} u$$

Use of the state transformation

$$Z = TX,\; T = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Answers

results in the state matrix

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -e & -f & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3e & -f \end{bmatrix}$$

Show that stability of this system requires that all roots of the equations

$$s^2 + es + f = 0$$

and

$$s^2 + fs + 3e = 0$$

have strictly negative real parts.
Hint: The determinant of a $2n \times 2n$ block diagonal matrix $H = \mathrm{diag}\{H_1, H_2\}$ where H_1 and H_2 are $n \times n$ matrices is $\det H = \det(H_1) \times \det(H_2)$. [20 marks]

(c) A system is described by the state-space model

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & -2 \\ 1 & -0.4 \end{bmatrix} x + \begin{bmatrix} 1 \\ -2 \end{bmatrix} u \\ y &= \begin{bmatrix} 3 & 4 \end{bmatrix} x \end{aligned}$$

Show that this system is controllable and observable and that its transfer-function is

$$G(s) = \frac{-5s + 9.2}{s^2 + 1.4s + 2.4}$$

Apply the state transformation

$$z = Tx$$

$$T = \begin{bmatrix} B & AB \end{bmatrix}$$

to this system and then design a stabilizing state feedback control law to place the closed-loop system poles at $-1, -3$.

[20 marks]

TURN OVER

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Answers

(d) Consider the system described by the state-space model

$$\dot{x} \;=\; \begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

with cost function

$$J = \frac{1}{2}\int_0^\infty (x^T Q x + u^T R u) dt$$

where $R = 1$ and the following two possibilities of Q are to be considered

$$Q = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}, \quad \text{or} \quad Q = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$$

Determine which of these choices can be used and then give the algebraic Riccati equation for this case. Hence compute the optimal input. [20 marks]

(e) Figure 1 shows a schematic of one form of pursuit guidance, where all variables have their normal meanings. Develop the equation for the bank angle command, show all working and explain all assumptions made.

[5 marks]

Answers

Indicative Solution for Question 1.

(a) [15 marks]

Chose $\dot{q}_1 = \omega_0 q_2$, $\dot{q}_2 = -\omega_0 \dot{q}_1$.

$$\frac{d}{dt}e^{At} = \begin{bmatrix} -\omega_0 \sin\omega_0 t & \omega_0 \cos\omega_0 t \\ -\omega_0 \cos\omega_0 t & -\omega_0 \cos\omega_0 t \end{bmatrix}$$

or

$$\frac{d}{dt}e^{At} = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix} \begin{bmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{bmatrix} = Ae^{At}$$

(b) [20 marks]

The state-space model follows immediately on writing the defining second order differential equations as a set of coupled first order differential equations.

The poles of the system are the eigenvalues of the state matrix from either model. In the second model the state matrix is block diagonal. Let $\rho(s)$ denote the characteristic polynomial of the state matrix. Then

$$\rho(s) = \rho_1(s)\rho_2(s)$$

where $\rho_1(s)$ is the characteristic polynomial of the first diagonal block entry and $\rho_2(s)$ that for the second. Both of these block entries are in companion form and hence

$$\begin{aligned} \rho_1(s) &= s^2 + es + f \\ \rho_2(s) &= s^2 + fs + 3e \end{aligned}$$

and all roots of these equations must have strictly negative parts for stability.

(c) [20 marks]

Controllability

$$\det(\begin{bmatrix} B & AB \end{bmatrix}) \neq 0$$

$$\det(\begin{bmatrix} 1 & 3 \\ -2 & 1.8 \end{bmatrix}) \neq 0$$

and hence the system is controllable.

Observability

$$\det(\begin{bmatrix} C \\ CA \end{bmatrix}) \neq 0$$

$$\det(\begin{bmatrix} 3 & 4 \\ -1 & -7.6 \end{bmatrix}) \neq 0$$

The matrix T is, see above, invertible and it is a necessary and sufficient condition for controllability. This state transformation puts the state-equation into controllable canonical form, i.e.

$$\dot{z} = \begin{bmatrix} 0 & 1 \\ -2.4 & -1.4 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

TURN OVER

Answers

Desired closed-loop polynomial

$$\rho_c(s) = s^2 + 4s + 3$$

Control law using z as the state vector is

$$u = \begin{bmatrix} -0.6 & -2.6 \end{bmatrix} z$$

$$x = T^{-1}z$$

and hence with x as the state vector

$$u = \begin{bmatrix} -0.6 & -2.6 \end{bmatrix} T^{-1}x$$

(d) [20 marks]

The first choice for Q is a nonsingular symmetric matrix and the second is not symmetric (and is singular). Hence only the first choice can be used.

$$A^T P + PA + Q - PBR^{-1}B^T P = 0$$

is the algebraic Riccati matrix equation and P is required to be diagonal, i.e., $P = \mathrm{diag}\{p_1, p_2\}$. With this choice the ARE is

$$\begin{bmatrix} 4 - 6p_1 & 2 - 3p_1 \\ 2 - 3p_1 & 6 - \frac{p_2^2}{2} \end{bmatrix} = 0$$

This leads to the following solutions

$$P_a = \mathrm{diag}\{\frac{2}{3}, -2\sqrt{3}\}, \quad P_b = \mathrm{diag}\{\frac{2}{3}, 2\sqrt{3}\}$$

P_a is not positive definite where as P_b is and is the valid choice.

Stabilizing state feedback control law is

$$u = -R^{-1}B^T P_b x$$

(e) [5 marks] Setting the commanded acceleration proportional to $\sin\eta$ as in the figure below is one of number of possible guidance laws for UAVs. The basic guidance problem is to determine an aim point and steer the velocity vector towards it.

In this figure V_g is the UAV groundspeed and C is a circular arc of radius $R,$ which originates at the UAV and intercepts the desired path and L_1 is a constant lookahead distance from the UAV to the path in the desired direction of travel.

By basic trigonometry

$$\frac{|L_1|}{2} = R\sin\eta \tag{1}$$

The centripetal acceleration $a_c,$ is

$$a_c = \frac{|V_g|^2}{R} \tag{2}$$

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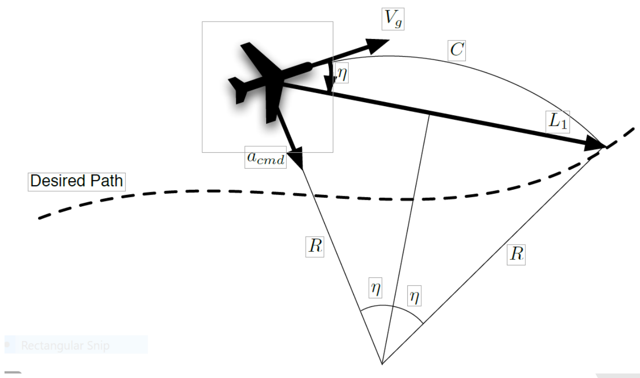

Hence the UAV must command a lateral acceleration a_c. Solving (1) for R and substituting in (2) gives the commanded acceleration

$$a_{cmd} = 2\frac{|V_g|^2}{|L_1|}\sin\eta \tag{3}$$

To implement this control law it is required to select the lookahead distance $|L_1|$ and determine $\sin\eta$ – the sine of the angle from the velocity vector to L_1. Moreover, η is also referred to the line of sight angle. Choosing $|L_1|$ is analogous to selecting a feedback gain in control — larger L_1 corresponds to smaller gains.

Also

$$\sin\eta = \frac{V_g \times L_1}{|V_g||L_1|} \tag{4}$$

$\times$ **— vector product. For the UAV to actually track the desired trajectory, the lateral acceleration command must be converted to an appropriate bank angle command using the steady-state turn equation**

$$\phi = \tan^{-1}\frac{a_{cmd}}{g}$$

Answers

END OF PAPER

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