Physics

Motion (continued)

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots$$

dimensions must match!

$$Dim[c_0] = L$$

$$Dim[c_1] = L/T$$

$$Dim[c_2] = L/T^2$$

$$Dim[c_3] = L/T^3$$

Define the derivative of x(t) with respect to t:

$$\frac{dx}{dt} \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

How small should Δt be?

$$x(t) = t$$

$$\Delta x = x(t + \Delta t) - x(t)$$

$$= (t + \Delta t) - t = \Delta t$$

$$\Rightarrow \frac{dx}{dt} \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = 1$$

$$x(t) = t^2$$

$$\Delta x = \left(t + \Delta t\right)^2 - t^2$$

$$= t^2 + \left(\Delta t\right)^2 + 2t\Delta t - t^2$$

$$\frac{\Delta x}{\Delta t} = \Delta t + 2t$$

$$\Rightarrow \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = 2t$$

$$x(t) = t^3$$

$$\Delta x = (t + \Delta t)^3 - t^3$$

$$= t^3 + 3t^2 \Delta t + 3t \Delta t^2 + \Delta t^3 - t^3$$

$$\frac{\Delta x}{\Delta t} = \left(\Delta t\right)^2 + 3t^2 + 3t\Delta t$$

$$\Rightarrow \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = 3t^2$$

If $x(t) = t^n$

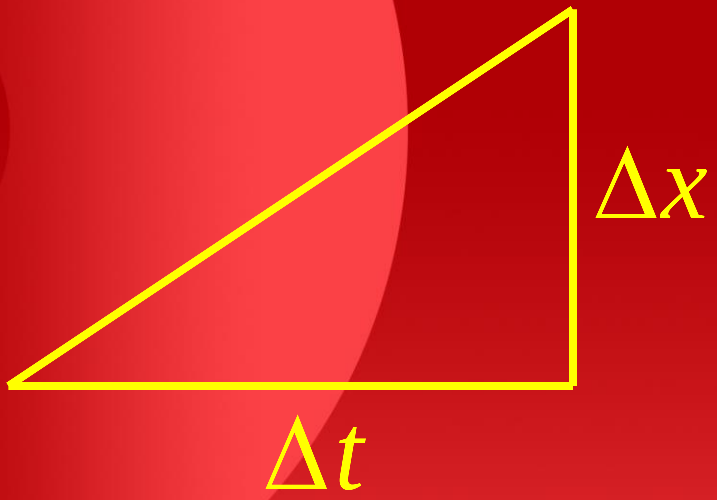
then:

$$\frac{dx}{dt} \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = n t^{n-1}$$

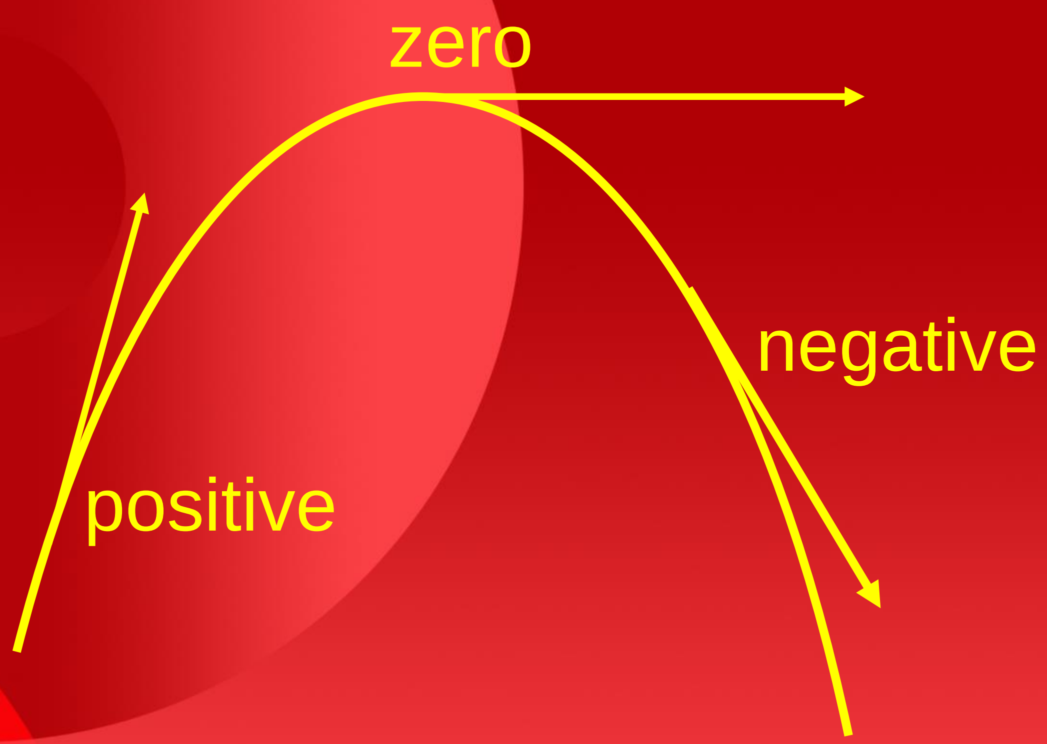
What is the derivative of a constant?

ZERO!

Geometrical interpretation of derivative



$$\frac{dx}{dt} \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$



Geometrical interpretation of derivative

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

$$\frac{dx}{dt} = 0 + v_0 + \frac{1}{2} a(2t)$$

$$\Rightarrow v \equiv \frac{dx}{dt} = v_0 + at$$

$$\frac{dv}{dt} = 0 + a = a$$

A car at rest can be accelerating very fast!

$$v = at$$

$$\text{but } \frac{dv}{dt} = a \neq 0$$

A stone can be at rest yet be accelerating!

$$v = -gt$$

$$\frac{dv}{dt} = -g \neq 0$$

$$g \approx 9.81 \text{metres/sec}^2$$

A useful notation:

$$\frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$$

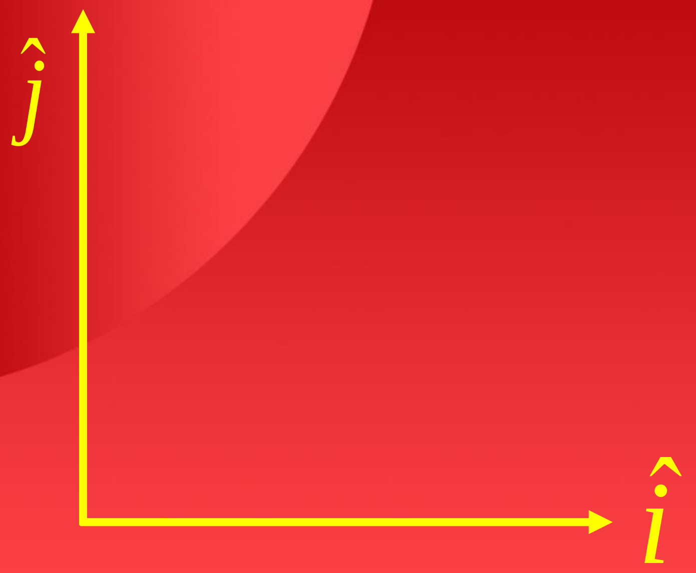
$$= \frac{d^2x}{dt^2}$$

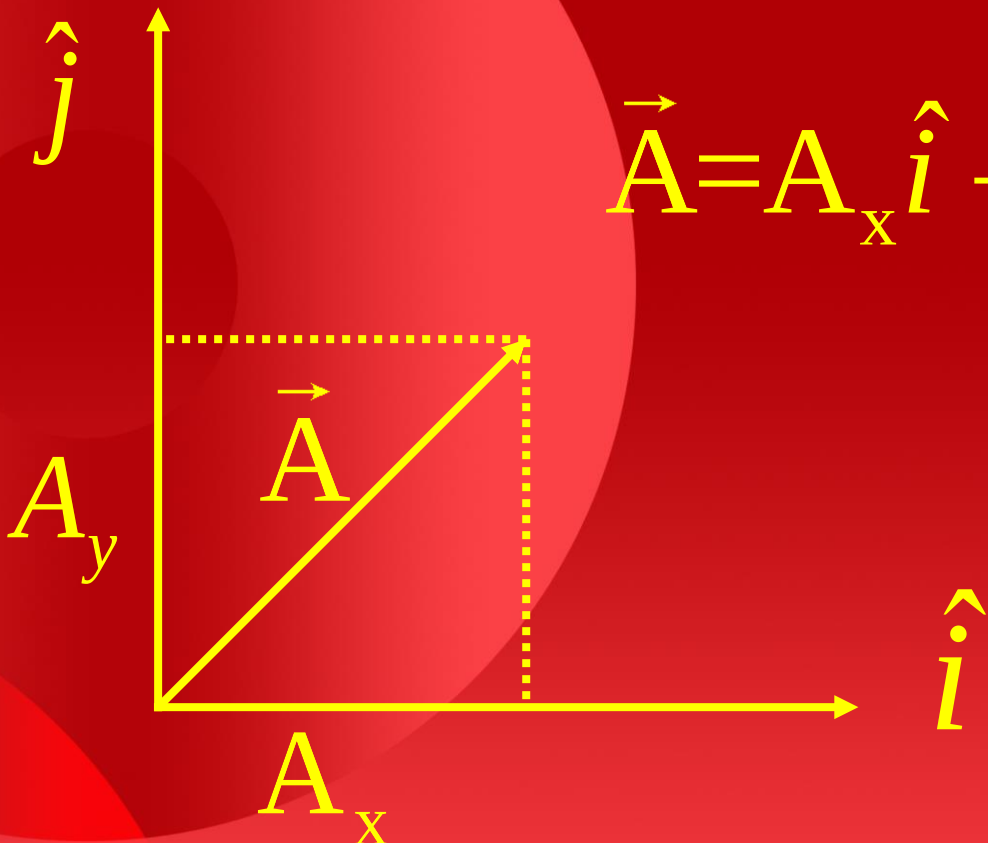
A unit vector is a vector that has magnitude 1 (no units).

A unit vector is obtained by dividing a vector by its length.

$$\hat{A} = \frac{\vec{A}}{A}$$

Examples of unit vectors are $\hat{i}, \hat{j}$ in 2-dimensional space.





$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

Decomposition of a vector into components

Velocity in 2 dimensions

$$\vec{r} = x(t)\hat{i} + y(t)\hat{j}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

$$= v_x\hat{i} + v_y\hat{j}$$

Acceleration in 2-d

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j}$$

$$= a_x\hat{i} + a_y\hat{j}$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j}, \quad \vec{B} = B_x\hat{i} + B_y\hat{j}$$

$$\vec{R} = \vec{A} + \vec{B}$$

$$= (A_x\hat{i} + A_y\hat{j}) + (B_x\hat{i} + B_y\hat{j})$$

$$= (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$$

$$= R_x\hat{i} + R_y\hat{j}$$

Example:

$$\vec{A} = (6\hat{i} + 5\hat{j})$$

$$\vec{B} = (8\hat{i} + 7\hat{j})$$

What is the magnitude of $2\vec{A} - \vec{B}$?

Letting $\vec{R} = 2\vec{A} - \vec{B}$, we have

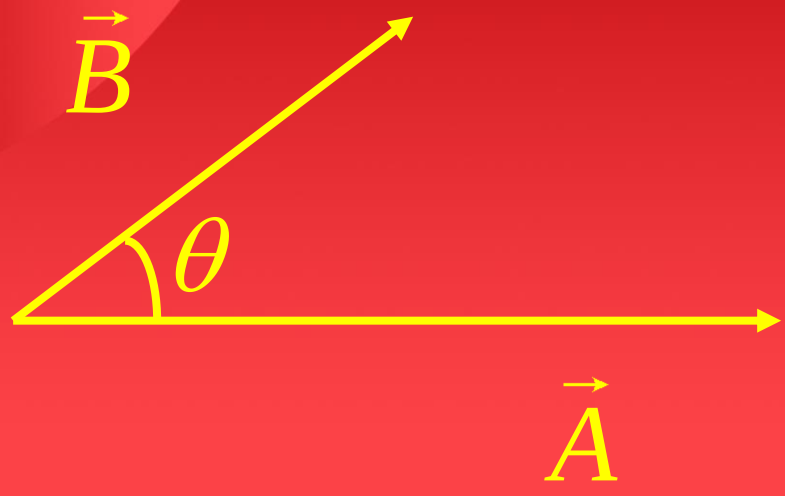
$$\vec{R} = 2(6\hat{i} + 5\hat{j}) - (8\hat{i} + 7\hat{j})$$

$$= (12 - 8)\hat{i} + (10 - 7)\hat{j}$$

$$= (4\hat{i} + 3\hat{j})$$

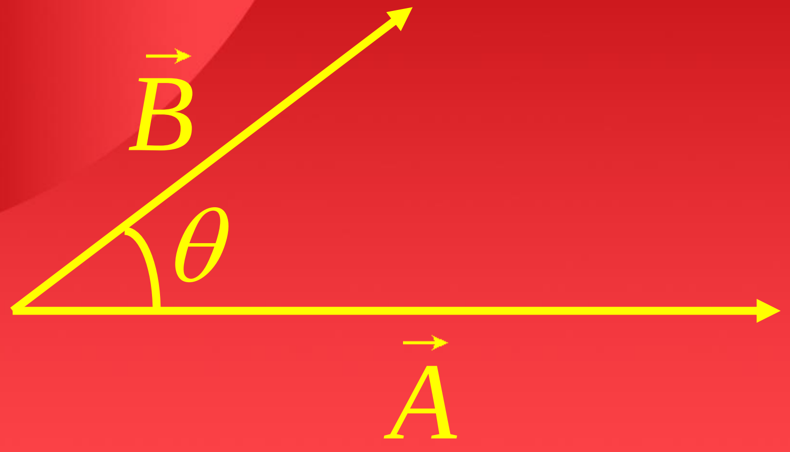
$$R = \sqrt{R_x{}^2 + R_y{}^2} = \sqrt{4^2 + 3^2} = 5$$

Consider two vectors $\vec{A}$ and $\vec{B}$ making an angle θ with each other.



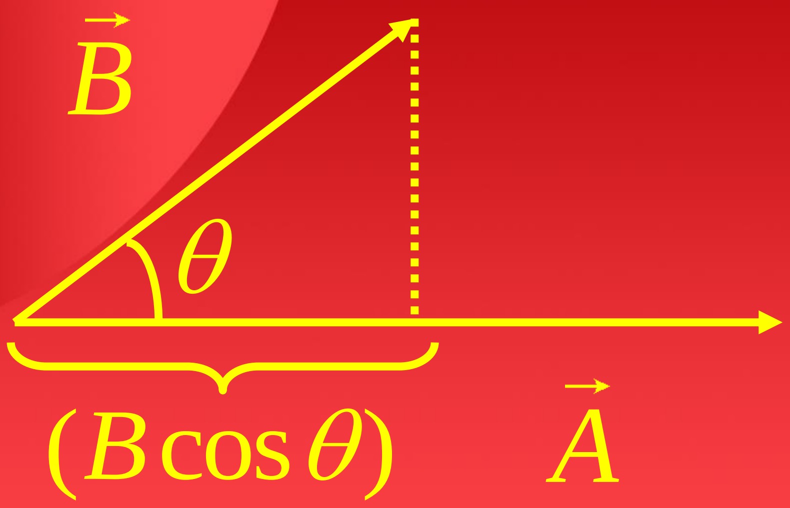
The scalar product of $\vec{A}$ and $\vec{B}$ is defined as:

$$\vec{A}\cdot\vec{B} = AB\cos\theta, \quad 0 < \theta < \pi$$



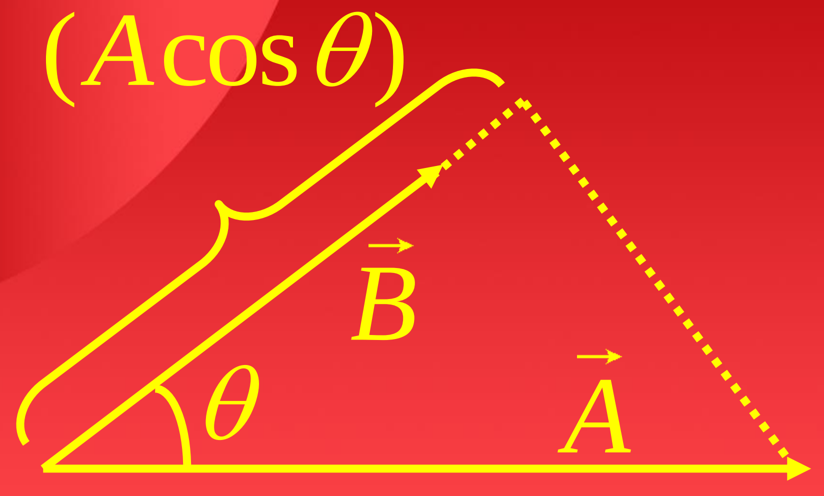
$$\vec{A}\cdot\vec{B} = (A)(B\cos\theta)$$

$$= (\text{length of } \vec{A}) \times (\text{projection of } \vec{B} \text{ on } \vec{A})$$



$$\vec{A}\cdot\vec{B} = (B)(A\cos\theta)$$

$$= (\text{length of } \vec{B}) \times (\text{projection of } \vec{A} \text{ on } \vec{B})$$



Scalar products of $\hat{i}$, $\hat{j}$ are

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = (1)(1)\cos(0) = 1$$

$$\hat{i}\cdot\hat{j} = (1)(1)\cos(90^0) = 0$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j}, \quad \vec{B} = B_x\hat{i} + B_y\hat{j}$$

$$\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j})\cdot(B_x\hat{i} + B_y\hat{j})$$

$$= A_xB_x\hat{i}\cdot\hat{i} + A_xB_y\hat{i}\cdot\hat{j}$$

$$+ A_yB_x\hat{j}\cdot\hat{i} + A_yB_y\hat{j}\cdot\hat{j}$$

$$= A_xB_x + A_yB_y$$

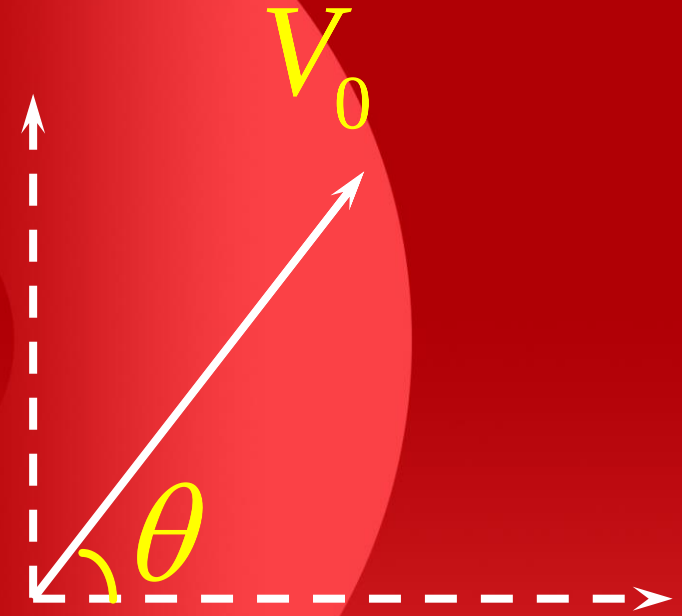
Generalization to three dimensions

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

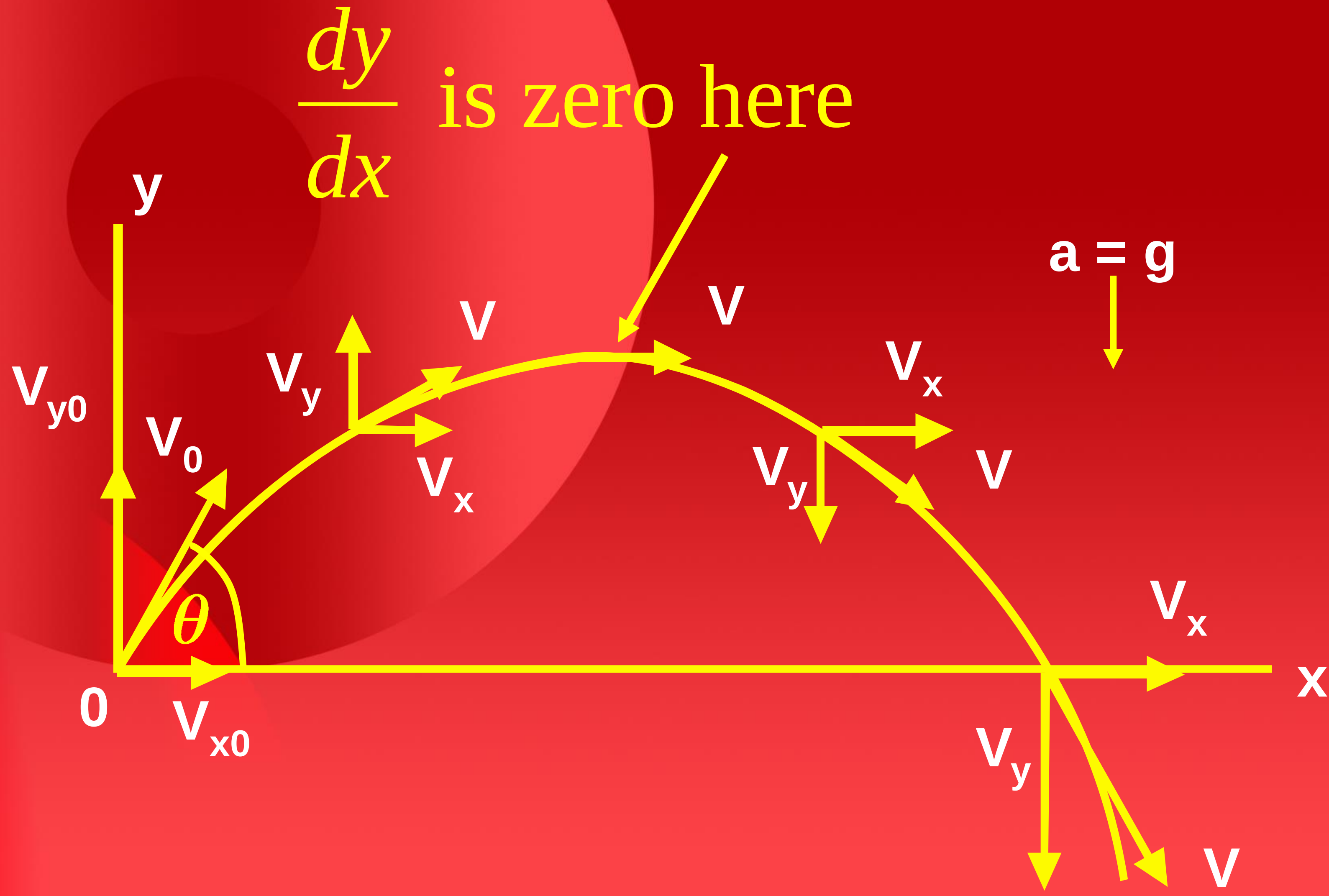
$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

- Acceleration along y is $a_y = -g$
- Acceleration along x is $a_x = 0$
- Velocity along x is *constant*



$$V_{0x} = V_0 \cos\theta$$

$$V_{0y} = V_0 \sin\theta$$



$\frac{dy}{dx}$ is zero here
a = g
y
x
0
V_{y0}
V_0
θ
V_{x0}
V_y
V
V_x
V
V_x
V_y
V
V_x
V_y
V

x direction

$$V_x = V_{0x}$$

$$x = x_0 + V_{0x}t$$

$$a_x = 0$$

y direction

$$a_y = -g$$

$$V_y = V_{0y} - gt$$

$$y = y_0 + V_{0y}t - \frac{1}{2}gt^2$$

horizontal motion

Constant Velocity

$x = x_o + v_{ox}t$

$v_x = v_{ox}$, $a_x = 0$

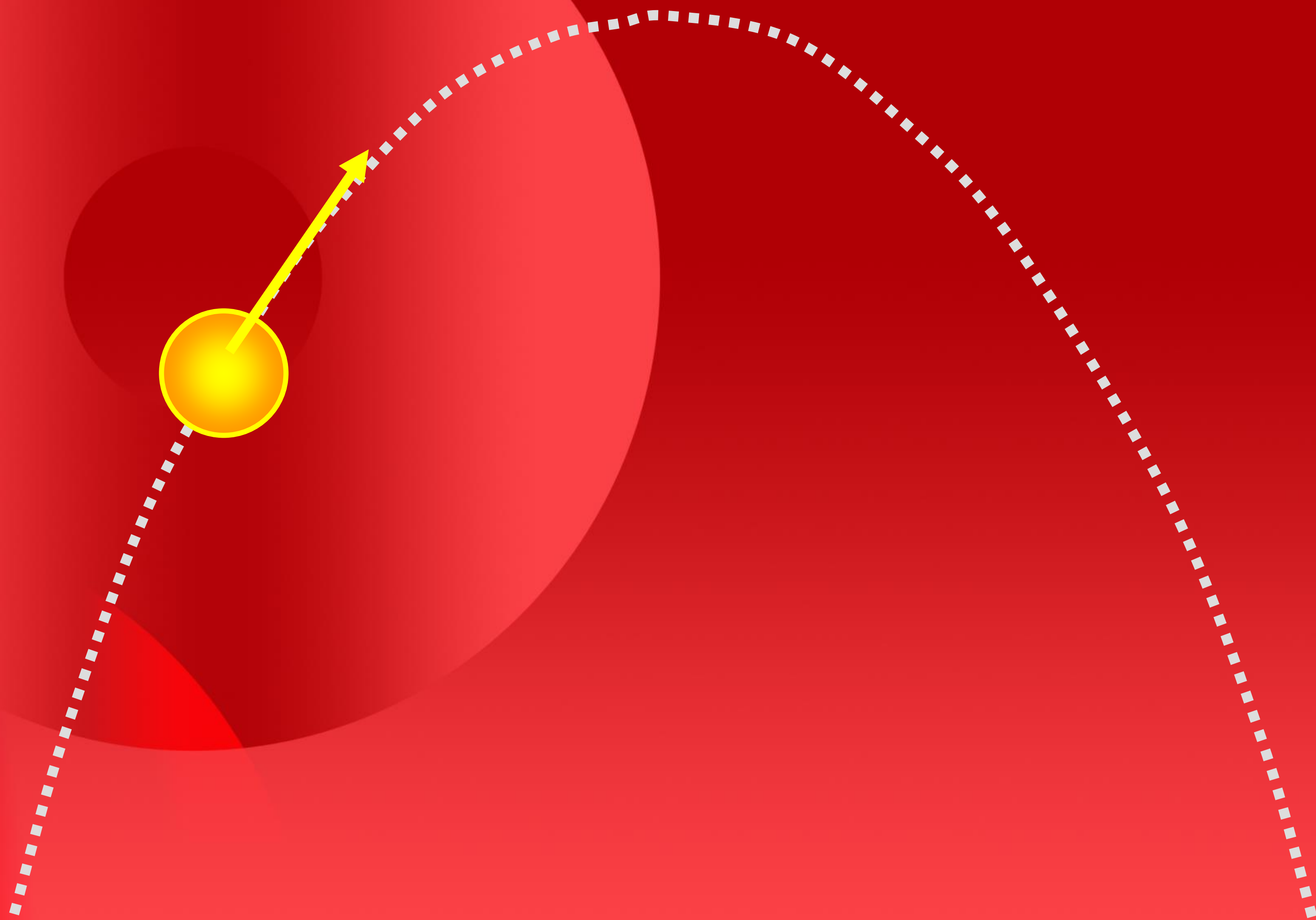
vertical **motion**

Free Fall

$$y = y_o + v_{oy}t + \tfrac{1}{2}a_yt^2$$

$$v_y = v_{oy} + a_yt$$

$$a_y = -g$$



Is the vertical acceleration constant?

YES! It is always $-g$ in free fall.

Is the horizontal acceleration constant?

YES! It is zero.

Is the vertical component of velocity constant?

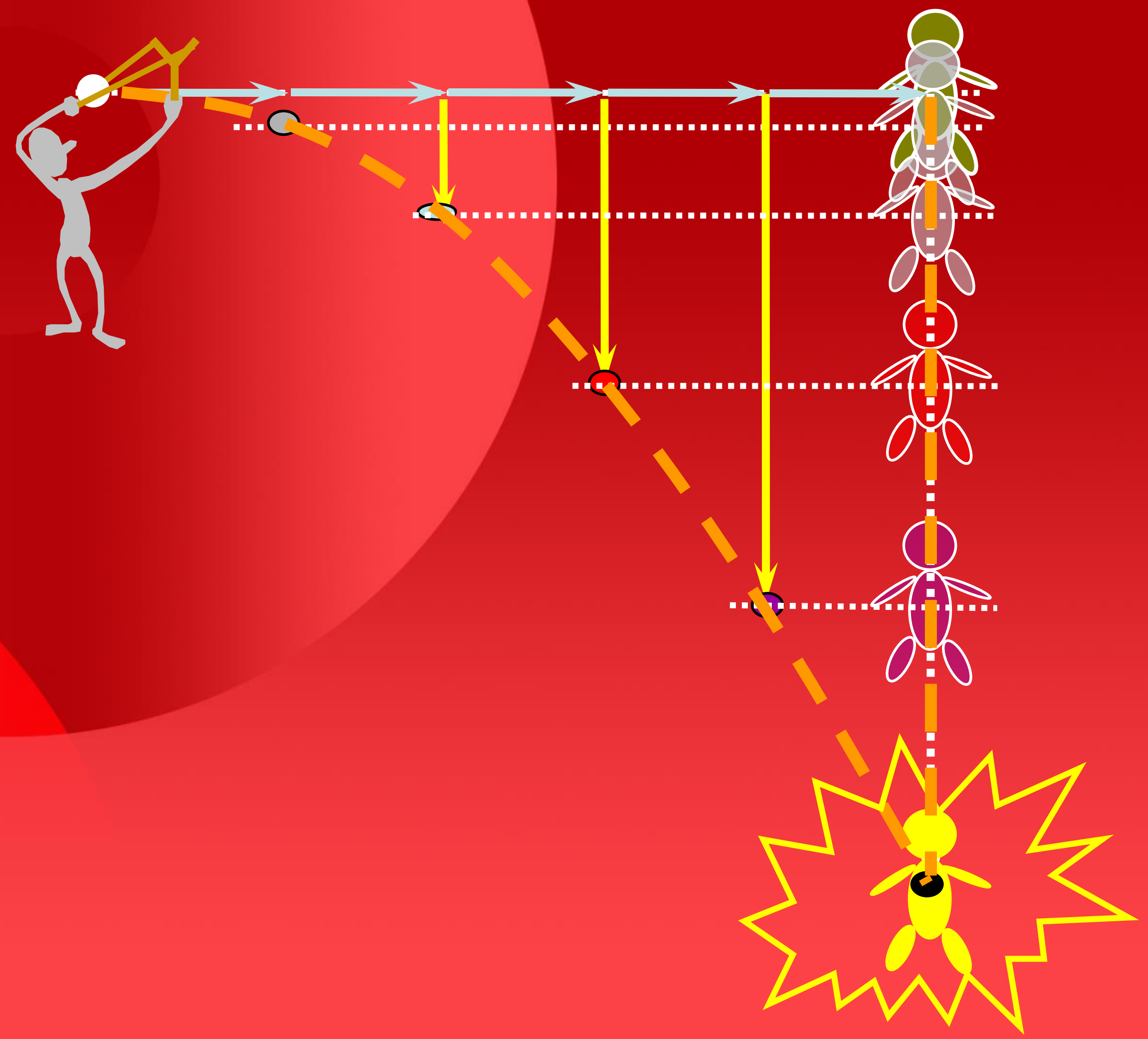
NO! Ball thrown straight up does not have constant velocity.

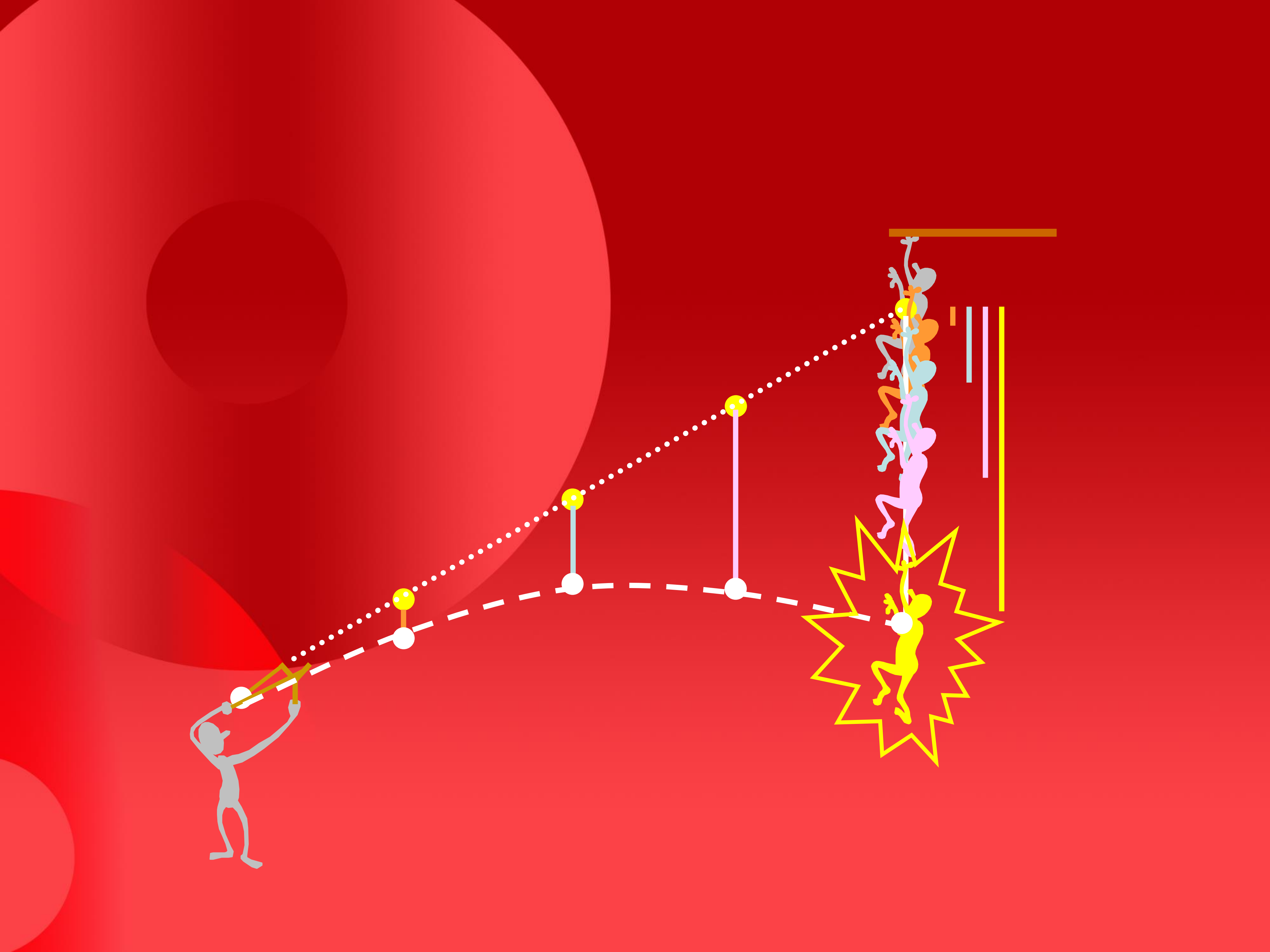
Is the horizontal component of velocity constant?

YES! There's no acceleration in the x direction.

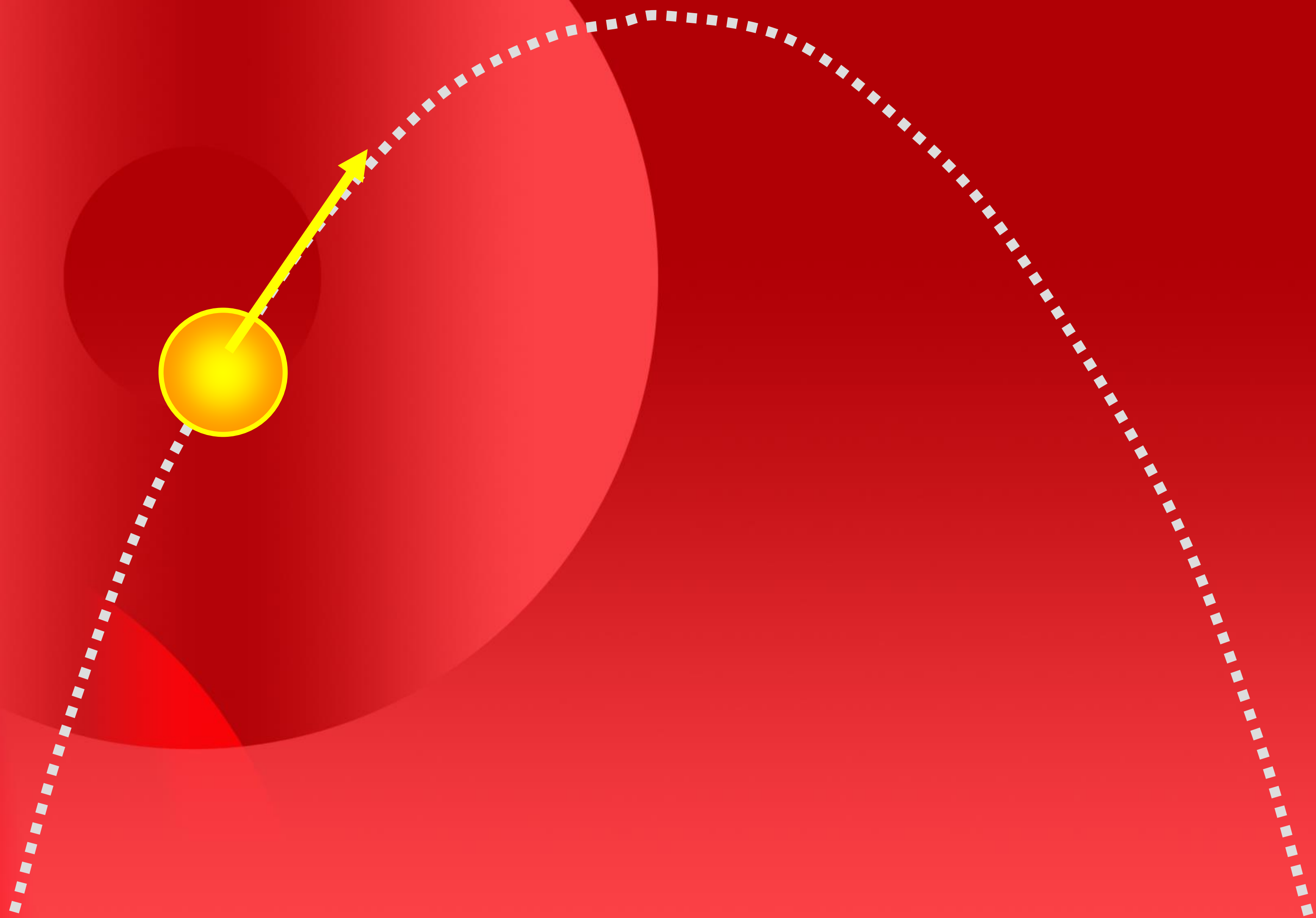
Is the speed constant?

NO! Vertical component of velocity is changing and horizontal is not, so speed must be changing.





Insert monkey phys_4_1



At $y_{max} = H$, $v_y = 0$

$$v_0 \sin\theta - gt = 0 \text{ and so } t = \frac{v_0 \sin\theta}{g}$$

$$y = (v_0 \sin\theta)t - \frac{1}{2}gt^2 \text{ becomes}$$

$$H = (v_0 \sin\theta)(\frac{v_0 \sin\theta}{g}) - \frac{1}{2}g(\frac{v_0 \sin\theta}{g})^2$$

$$H = \frac{(v_0 \sin\theta)^2}{2g}$$

$$x = (v_0 \cos\theta)t \Rightarrow t = \frac{x}{v_0 \cos\theta}$$

$$y = (v_0 \sin\theta)t - \frac{1}{2}gt^2$$

$$= (v_0 \sin\theta)\left(\frac{x}{v_0 \cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{v_0 \cos\theta}\right)^2$$

$$= x\tan\theta - x^2\left(\frac{g\sec^2\theta}{2{v_0}^2}\right)$$

$$y = x\left[\tan\theta - x\left(\frac{g}{2v_0^{\,2}\cos^2\theta}\right)\right] = 0$$

has two solutions for x !

x=0, AND $x = R = \dfrac{2v_0^{\,2}\sin\theta\cos\theta}{g}$

$$= \frac{v_0^{\,2}\sin 2\theta}{g}$$

Since $-1 \le \sin 2\theta \le 1$

therefore $(\sin 2\theta)_{\max} = 1$

$$\Rightarrow R_{\max} = \frac{v_0^{\ 2}}{g}(\sin 2\theta)_{\max} = \frac{v_0^{\ 2}}{g}$$

How long will the projectile take to arrive at R_{max}?

Recall: $R_{max} = \frac{v_0^{\ 2}}{g}$

$$T = \frac{R_{max}}{v_0 \sin 45} = \sqrt{2}\,\frac{v_0}{g}$$