Physics

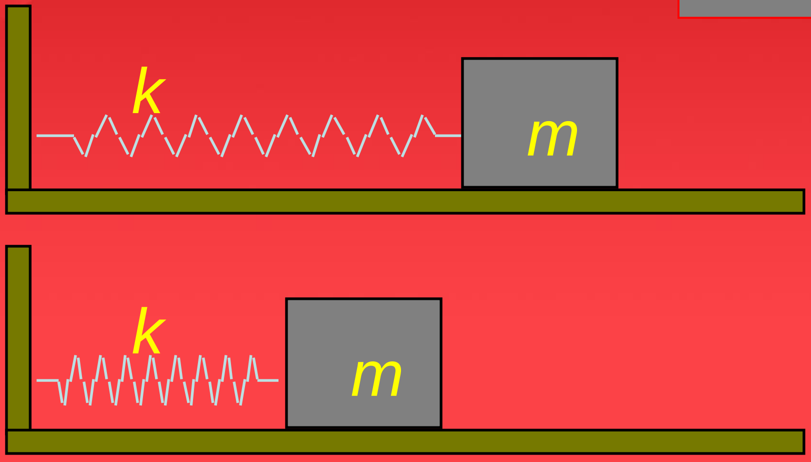
Oscillations

Periodic Motion

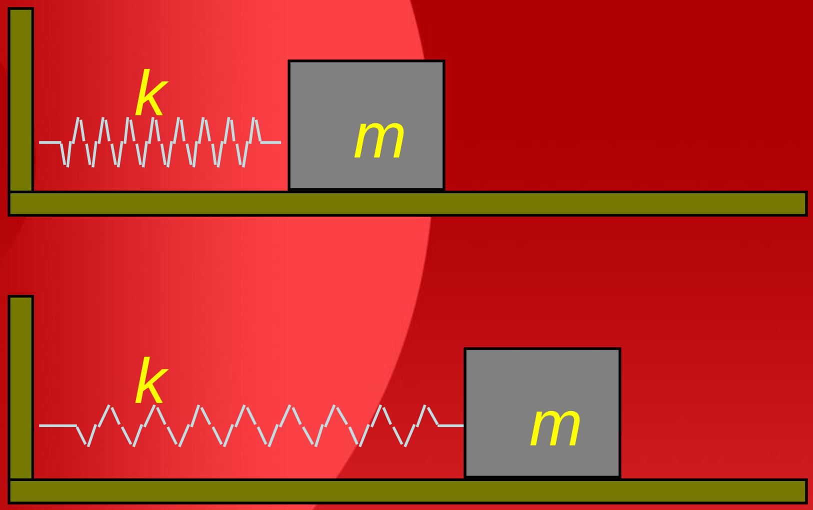
Oscillations are everywhere!

- pendulum
- sitar and guitar strings
- boats bobbing at anchor
- quartz crystal in a watch
- masses on springs

motion that repeats itself



What describes oscillations ?



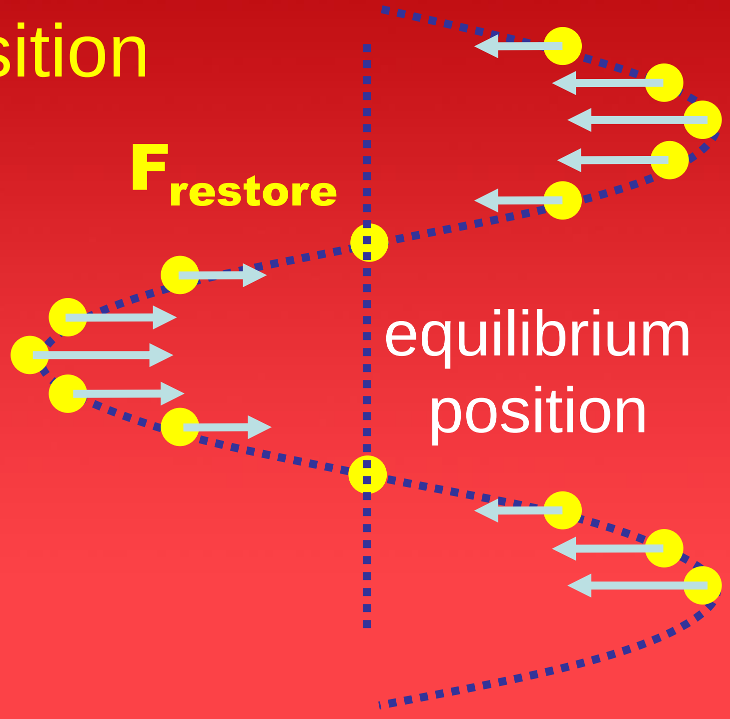
- *period* **T** = time for completing one cycle
- *frequency* **f** = 1 / T
- *amplitude* **A** = maximum displacement from *equilibrium* position

- Suppose a force is always directed towards a central *equilibrium* position

restoring force

- force always acts to return the object to its *equilibrium* position

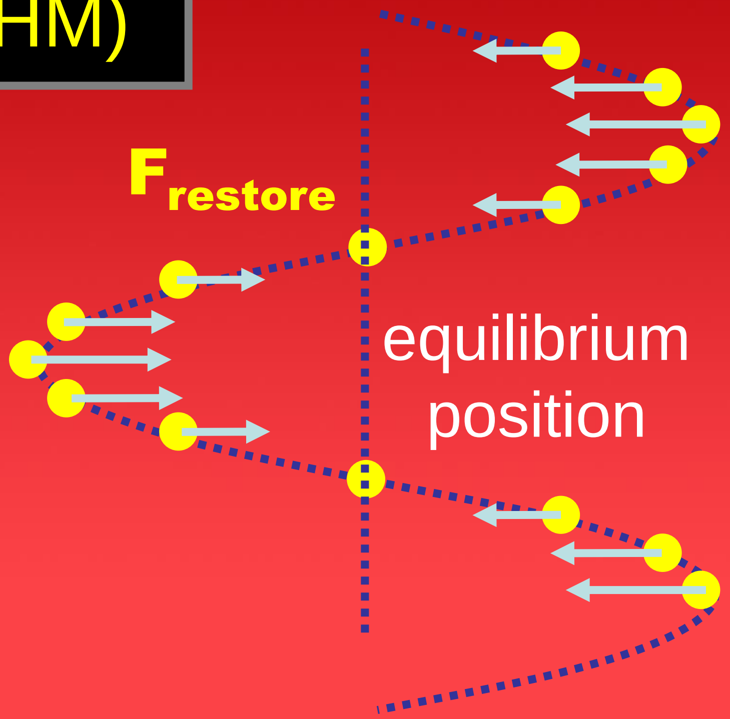
- the object will *oscillate* around the *equilibrium* position



- This "back-and-forth" motion around an *equilibrium* position is called: **periodic motion**

restoring force

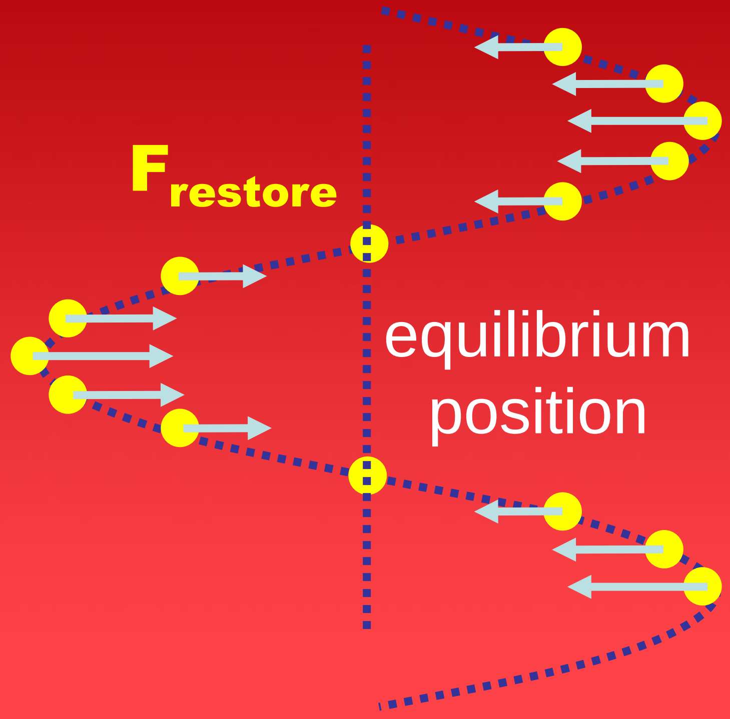
Simple Harmonic Motion (SHM)



- For there to be periodic motion, there must be:
  - an equilibrium position
  - a restoring force
  - energy transformation (kinetic $\Leftrightarrow$ potential)
- The restoring force depends on the displacement Δx

$$F_{\text{restore}} = -k\,\Delta x$$

“back” towards equilibrium point



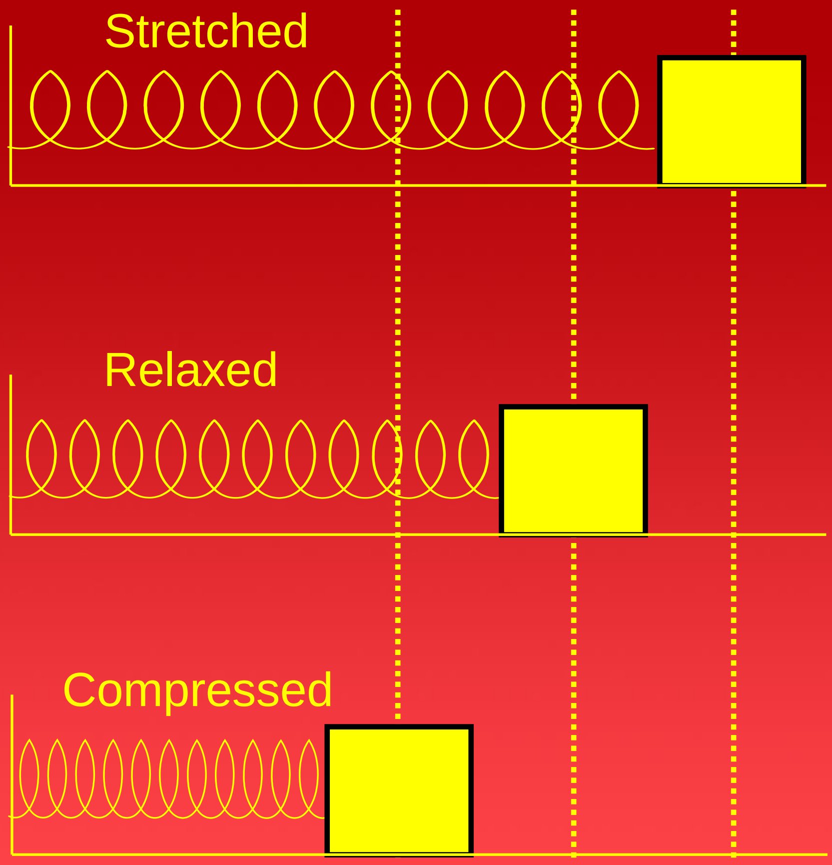
The simple harmonic oscillator

$$F(x) = -kx$$

where k is the force constant and measures stiffness of the spring

$$U(x) = \frac{1}{2}kx^2$$

(stored energy)



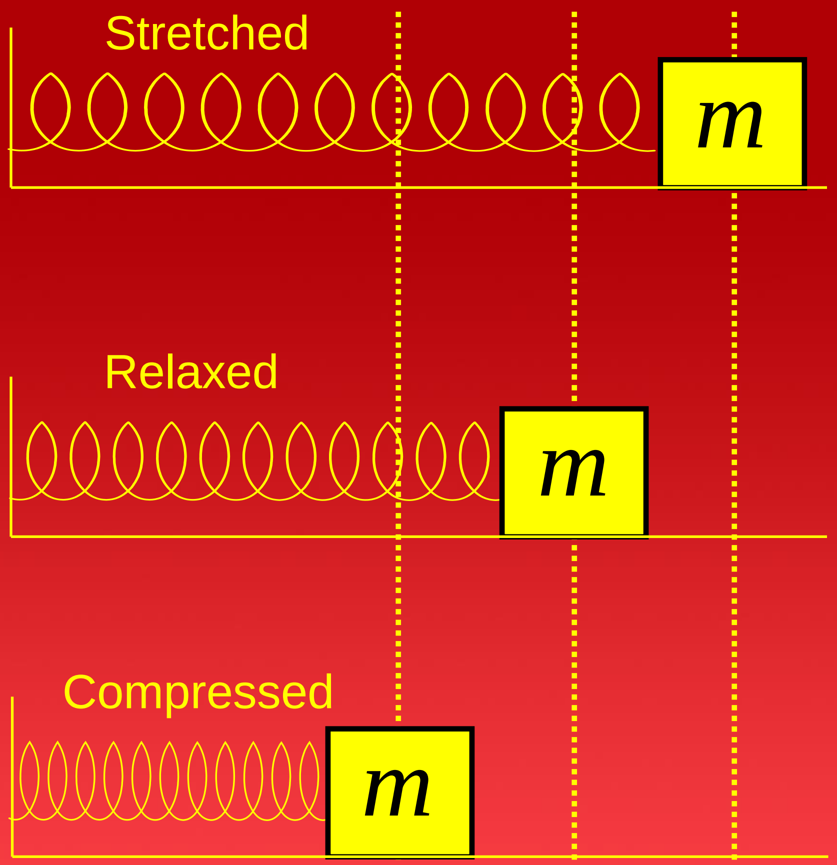
$$F(x) = -kx$$

$$ma = F \Rightarrow$$

$$m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

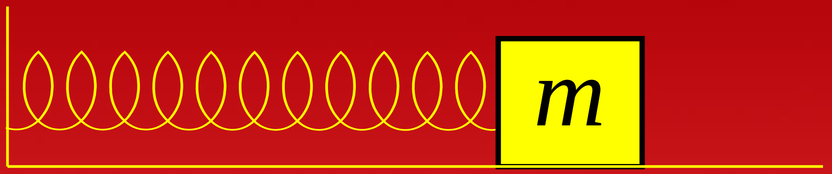
This is called equation of motion of the simple harmonic oscillator



Simple Harmonic Motion

Let us solve the equation of motion of the simple harmonic oscillator

$$\frac{d^2x}{dt^2}+\frac{k}{m}x=0$$



$$\frac{d^2x}{dt^2}=-\omega^2x$$

$$\omega^2=\frac{k}{m}$$

How to calculate $\dfrac{\mathrm{d}}{\mathrm{dt}}\cos\omega t$?

$$x(t) = \cos\omega t$$

$$x(t+\Delta t) = \cos\omega(t+\Delta t)$$

$$x(t+\Delta t) - x(t) = \cos\omega(t+\Delta t) - \cos\omega t$$

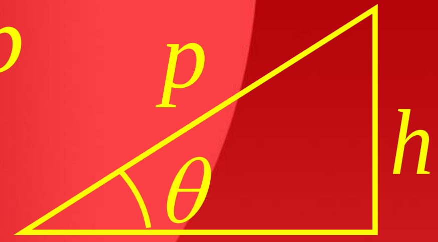
$$= -\sin\omega\Delta t\sin(\omega t + \omega\Delta t/2)$$

$$\approx -\omega\Delta t\sin\omega t$$

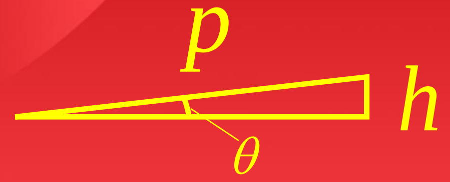
$$\boxed{\therefore \frac{\mathrm{d}}{\mathrm{dt}}\cos\omega t = -\omega\sin\omega t}$$

Why is $\sin\theta \approx \theta$ for small θ?

$$\sin\theta = \frac{h}{p}$$



$$\theta \to 0$$



$$h \to p\theta$$

$$\sin\theta = \frac{h}{p} \to \theta$$

Remember two important results:

$$\frac{d}{dt}\left(\sin\omega t\right) = \omega\cos\omega t$$

$$\frac{d}{dt}\left(\cos\omega t\right) = -\omega\sin\omega t$$

What happens if you differentiate twice?

- $$\frac{d^2}{dt^2}\left(\sin\omega t\right)=\omega\frac{d}{dt}\cos\omega t$$
  $$=-\omega^2\sin\omega t$$
- $$\frac{d^2}{dt^2}\left(\cos\omega t\right)=-\omega\frac{d}{dt}\sin\omega t$$
  $$=-\omega^2\cos\omega t$$

Any function of the form:

$$x = a\cos\omega t + b\sin\omega t$$

is a solution of $\dfrac{d^2x}{dt^2} = -\omega^2 x$

But what do ω, a, b represent ?

Physical significance of constant ω

$$x = x_m \cos\omega\left(t + \frac{2\pi}{\omega}\right)$$

$$= x_m \cos\left(\omega t + 2\pi\right)$$

$$= x_m \cos\omega t$$

That is, the function merely repeats itself after a time $2\pi / \omega$

So $2\pi/\omega$ is the period of the motion T

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

The frequency ν of the oscillator is the number of complete vibrations per unit time:

$$\boxed{\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}$$

Hence,

$$\omega = 2\pi\nu = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

ω is called the angular frequency

$$\dim[\omega] = T^{-1}$$

Unit of ω is radian/second

$$x(t) = a\cos\omega t + b\sin\omega t$$

x(0)=a

$$\frac{\mathrm{d}}{\mathrm{dt}}x(t) = -\omega a\sin\omega t + \omega b\cos\omega t$$

$$= \omega b \text{ (at t=0)}$$

The solution can also be written as:

$$x(t) = x_m\cos\left(\omega t + \phi\right)$$

Physical significance of constant x_m

$$x = x_m \cos(\omega t + \phi)$$

$$\Rightarrow \quad -x_m \leq x \leq +x_m$$

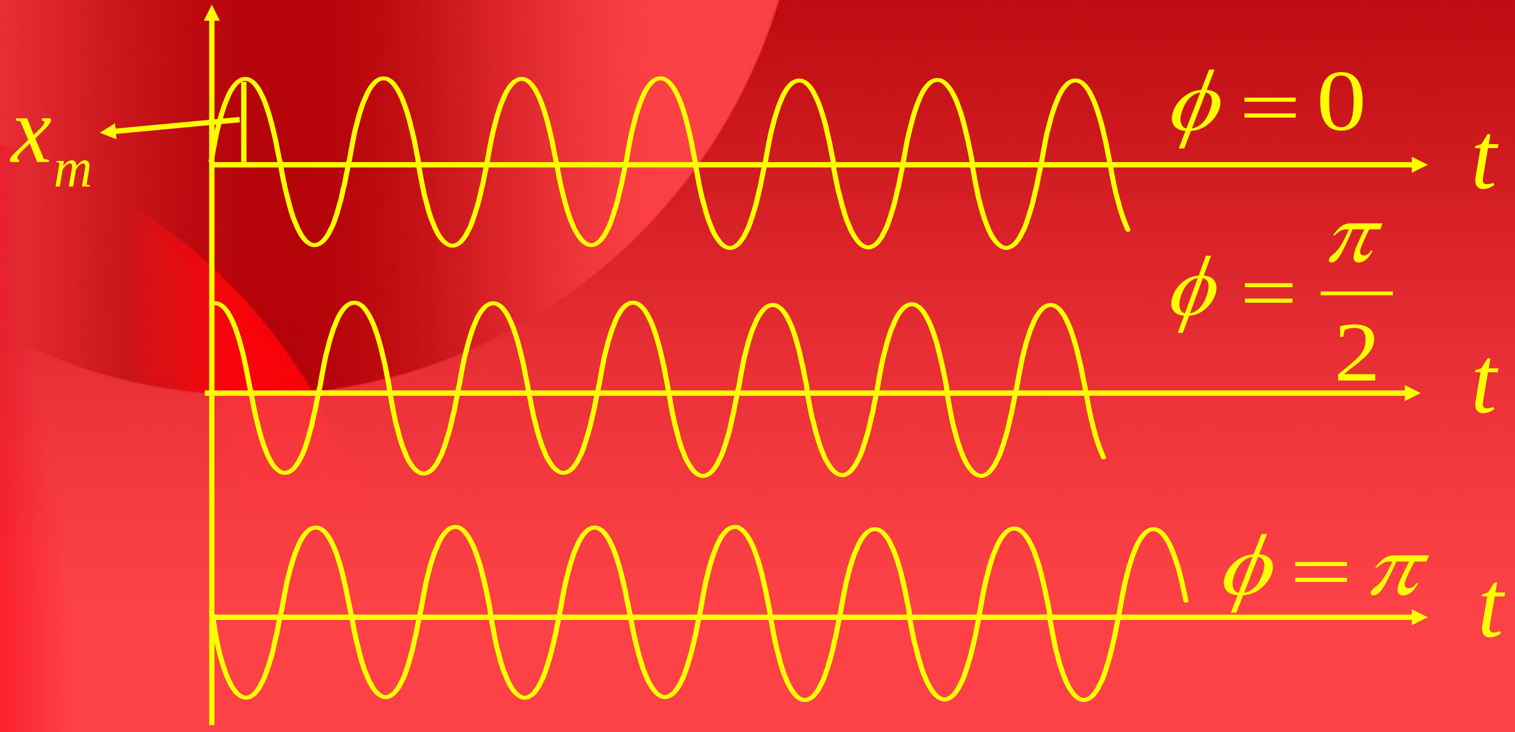
x_m is called the amplitude of the motion

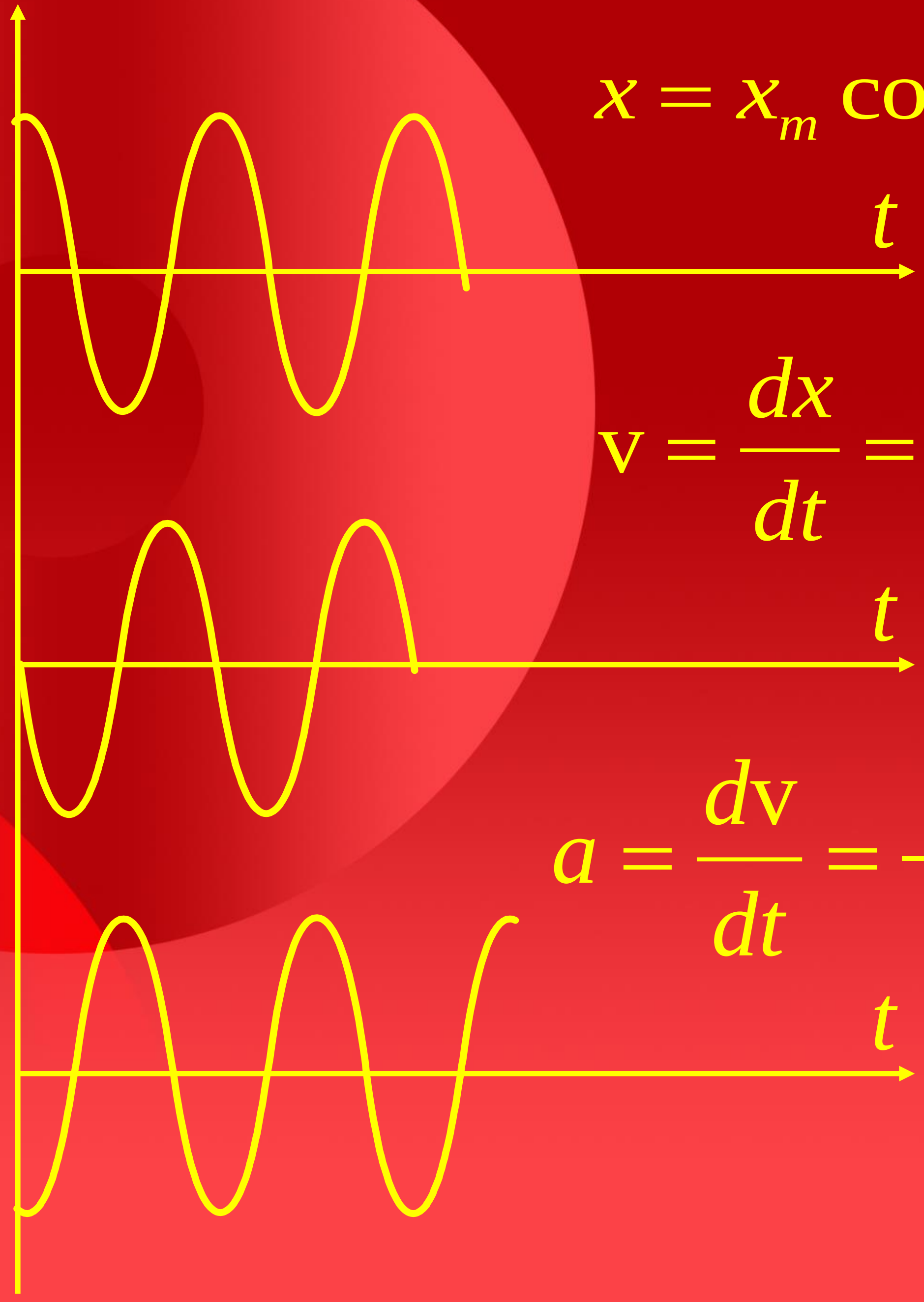
The frequency of the simple harmonic motion is independent of the amplitude of the motion

Phase

$$x = x_m \sin(\omega t + \phi)$$

The quantity $\theta = \omega t + \phi$ is called the phase of the motion. The constant ϕ is called the phase constant.





$$x = x_m \cos \omega t$$

$$\mathrm{v} = \frac{dx}{dt} = -\omega x_m \sin \omega t$$

$$a = \frac{d\mathrm{v}}{dt} = -\omega^2 x_m \cos \omega t$$

Energy of simple harmonic motion

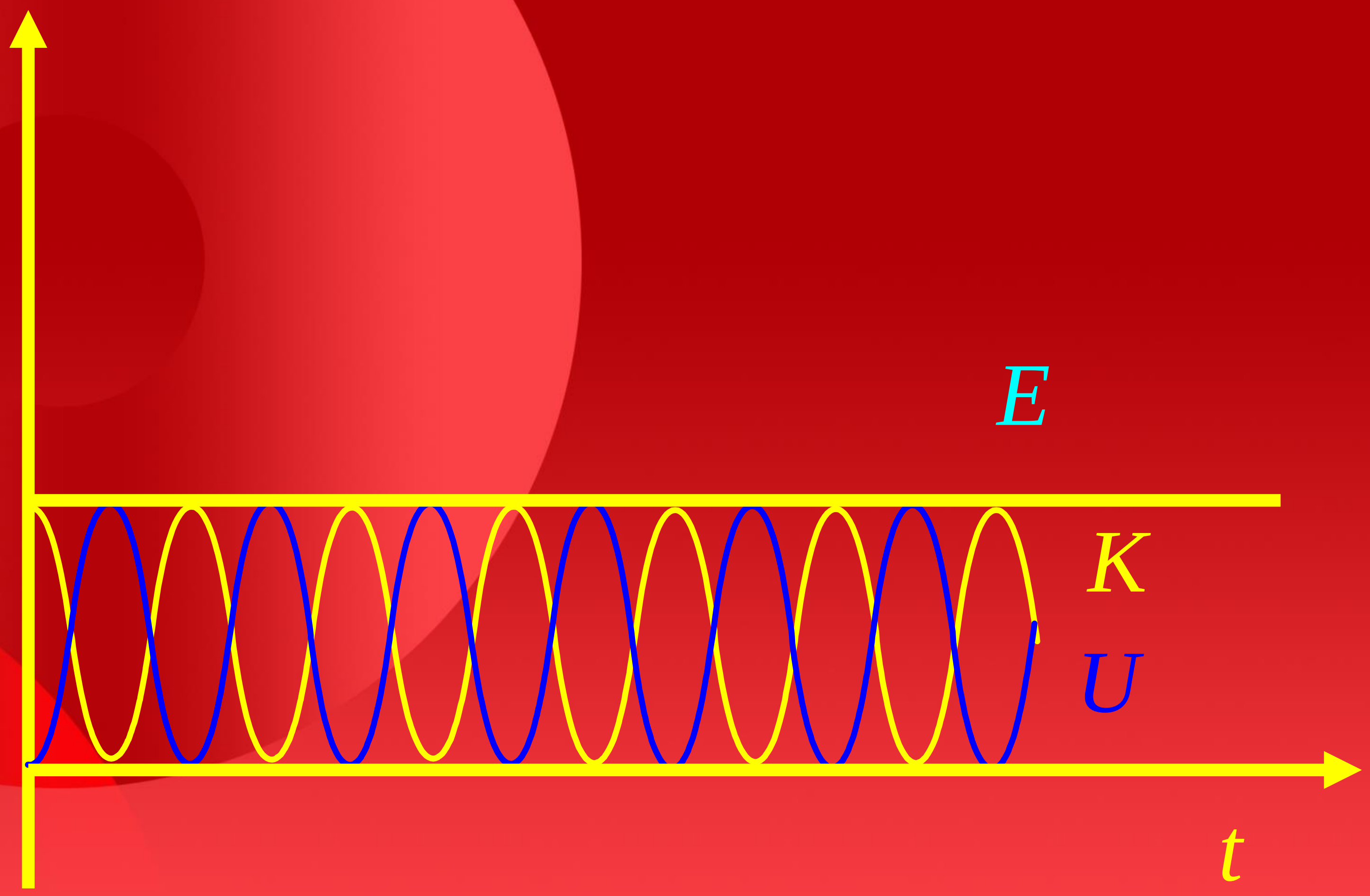
$$x = x_m \cos \omega t$$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2 \cos^2 \omega t$$

$$K = \frac{1}{2}m\mathrm{v}^2 = \frac{1}{2}m\omega^2 x_m^2 \sin^2 \omega t$$

$$= \frac{1}{2}kx_m^2 \sin^2 \omega t$$

$$
\begin{aligned}
E &= K + U \\
&= \frac{1}{2} k x_m^2 \cos^2 \omega t + \frac{1}{2} k x_m^2 \sin^2 \omega t \\
&= \frac{1}{2} k x_m^2 \left(\cos^2 \omega t + \sin^2 \omega t \right) \\
&= \frac{1}{2} k x_m^2
\end{aligned}
$$



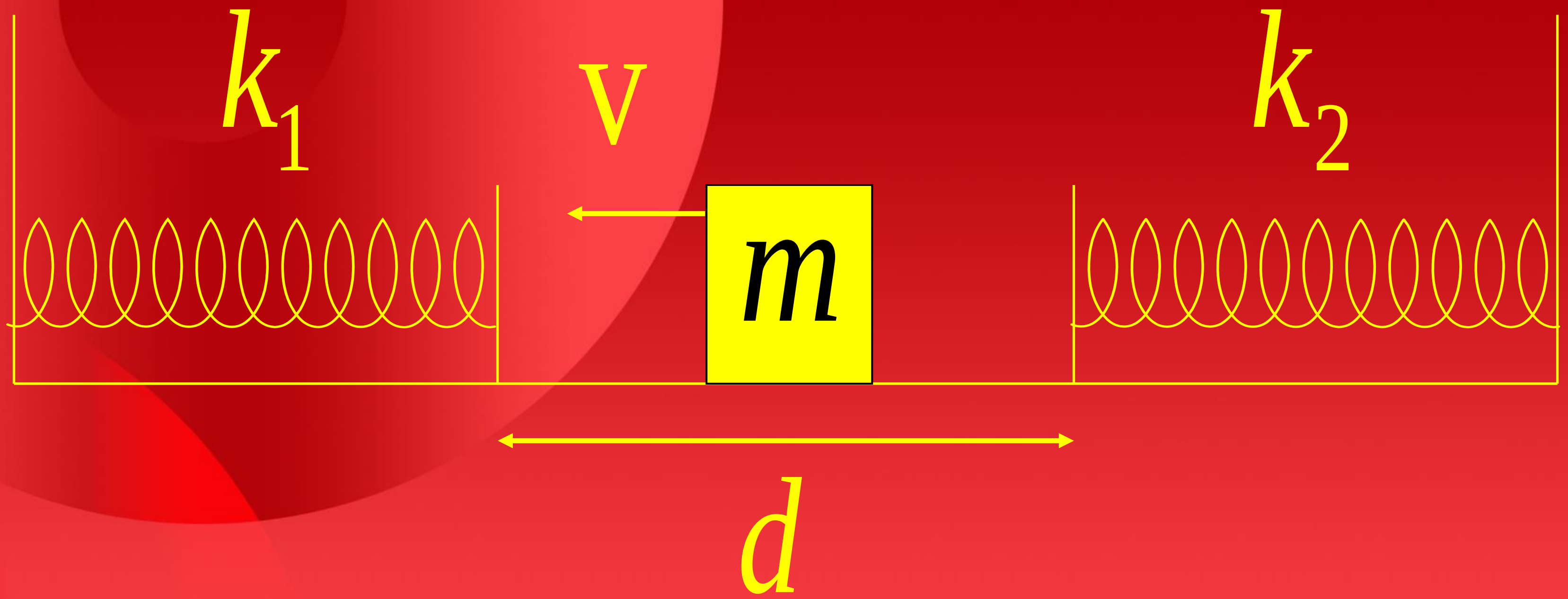
E
K
U
t

$$E = \frac{1}{2} m\mathrm{v}^2 + \frac{1}{2} kx^2 = \frac{1}{2} kx_m^2$$

$$\mathrm{v} = \frac{dx}{dt} = \pm\sqrt{\frac{k}{m}\left(x_m^2 - x^2\right)}$$

speed is maximum at x=0

speed is zero at $x = \pm x_m$



k1
V
k2
m
d

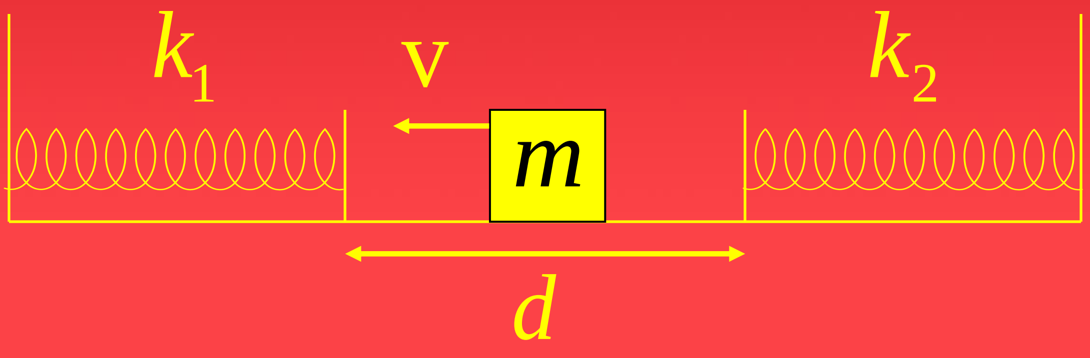
$$T = t_1 + t_2 + t_3$$

t_1 = period of oscillation of spring k_1

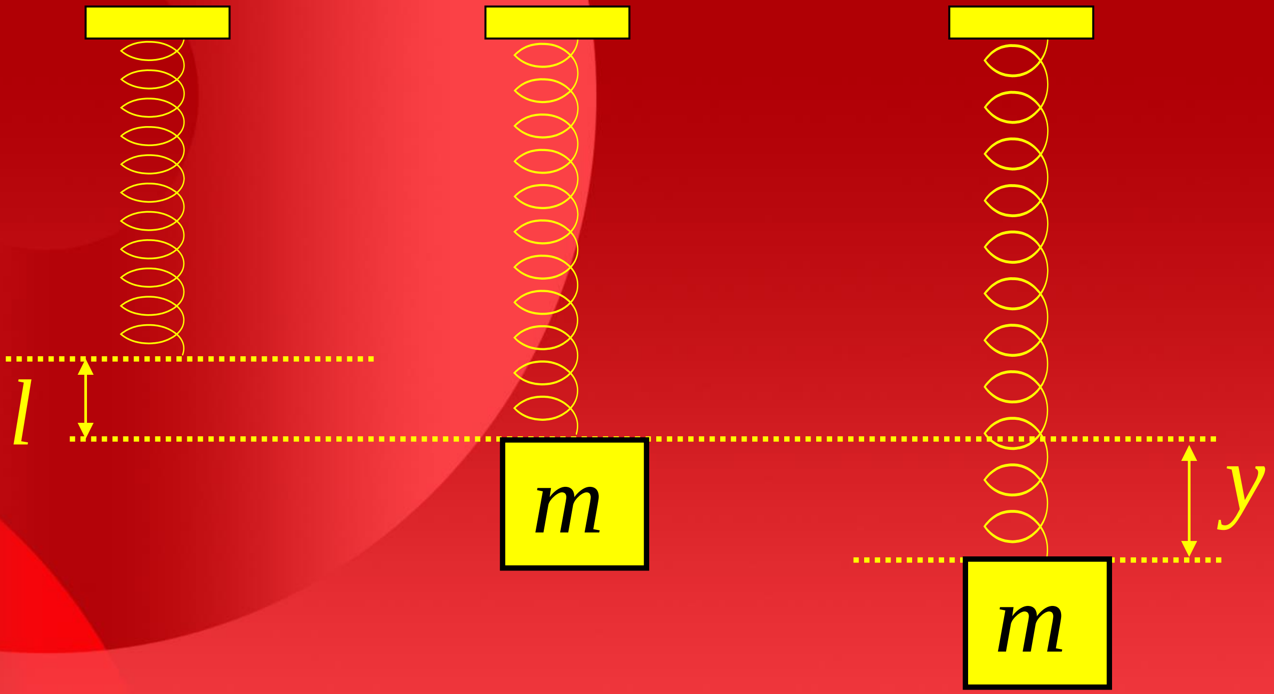
t_2 = period of oscillation of spring k_2

t_3 = time to cover the distance d

$$T = 2\pi\sqrt{\frac{m}{k_1}} + 2\pi\sqrt{\frac{m}{k_2}} + \frac{2d}{\mathrm{v}}$$



Vertical oscillations



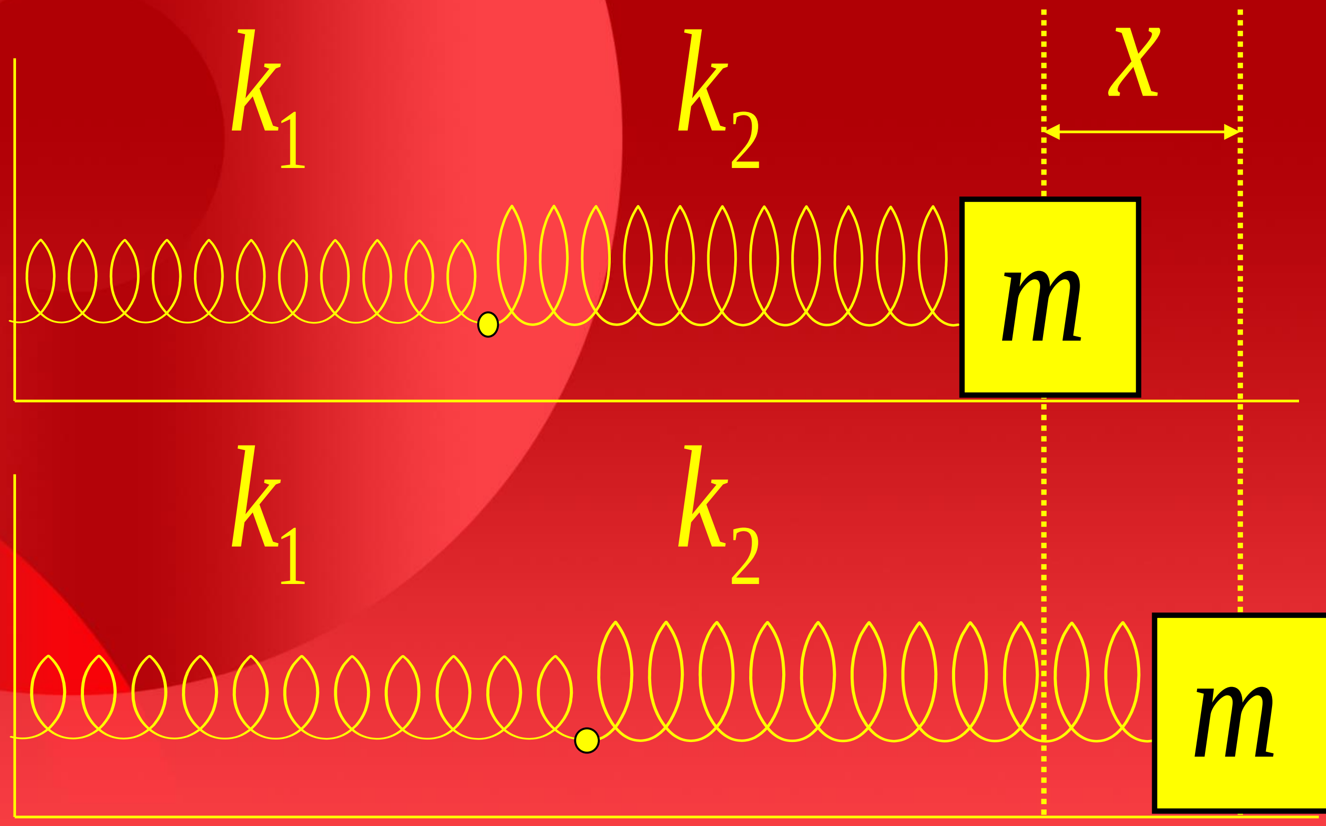
$$kl = mg \quad \Rightarrow k = \frac{mg}{l}$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{mg / l}{m}}$$

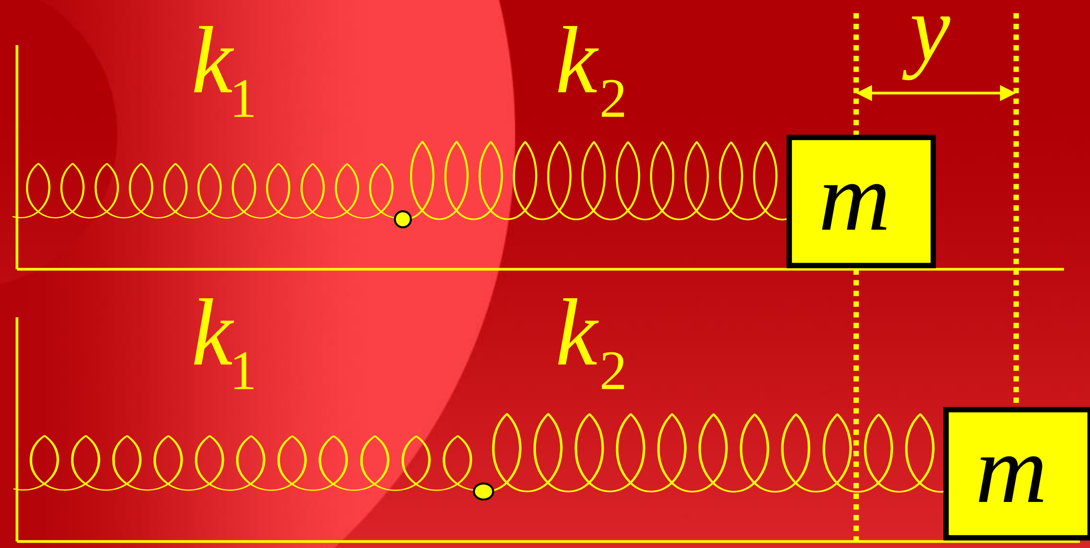
$$= \sqrt{\frac{g}{l}}$$

$$y = y_m \cos(\omega t + \phi)$$



k_1
k_2
x
m
k_1
k_2
m

Springs coupled in series



$$y = y_1 + y_2 \qquad F = -k_1 y_1 = -k_2 y_2$$

$$y_1 = -\frac{F}{k_1} \qquad y_2 = -\frac{F}{k_2}$$

$$F=-\left(\frac{k_1k_2}{k_1+k_2}\right)y$$

$$\therefore k_{eff}=\left(\frac{k_1k_2}{k_1+k_2}\right)=\left(\frac{k_1k_2}{k_1}+\frac{k_1k_2}{k_2}\right)=k_2+k_1$$

$$\frac{1}{k_{eff}}=\frac{1}{k_1}+\frac{1}{k_2}$$

$$F=-\left(\frac{k_1k_2}{k_1+k_2}\right)y$$

$$k_{eff}=\left(\frac{k_1k_2}{k_1+k_2}\right)$$

$$\boxed{\frac{1}{k_{eff}}=\frac{1}{k_1}+\frac{1}{k_2}}$$

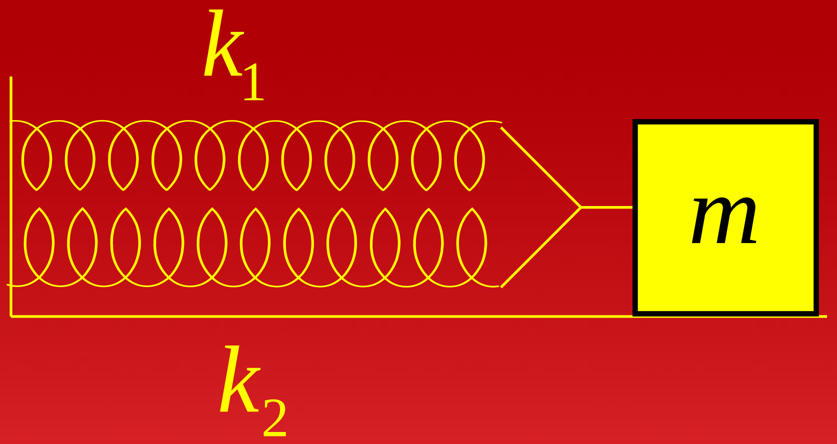
$$T=2\pi\sqrt{\frac{m}{k_{eff}}}$$

Springs in parallel

Let $x = d$isplacement of the mass

$$F_1 = -k_1 x$$

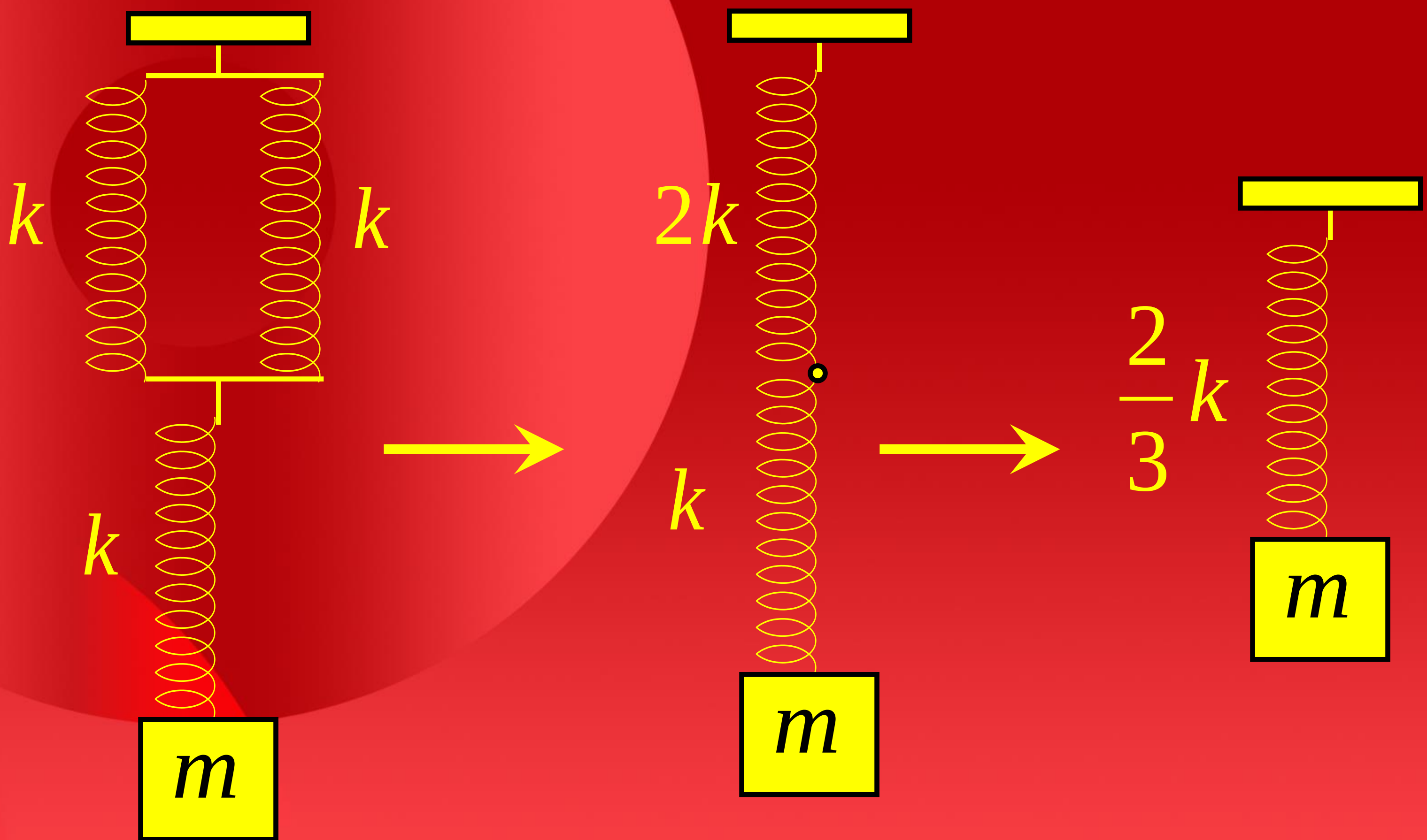
$$F_2 = -k_2 x$$

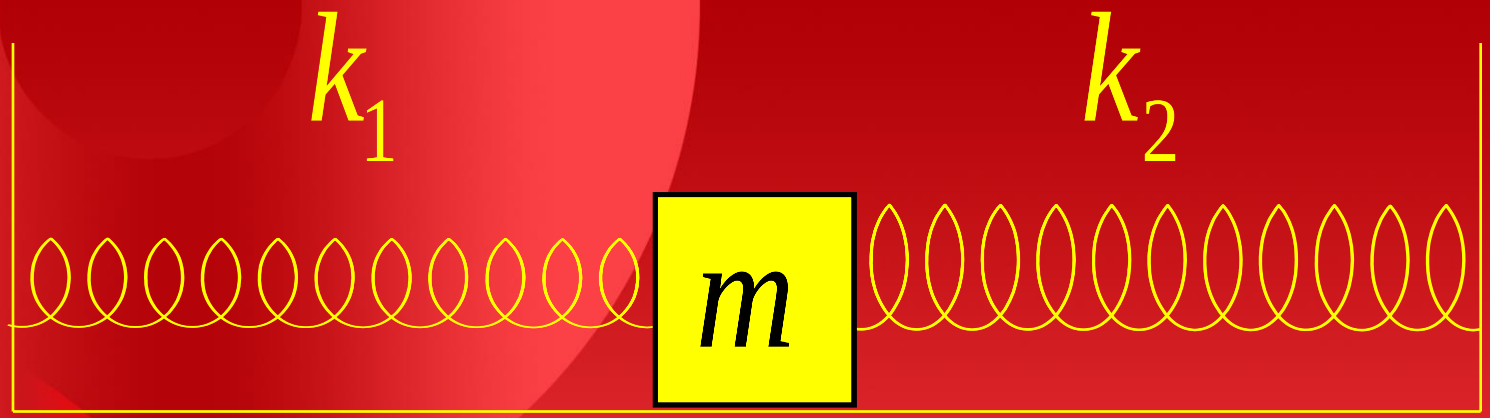


$$F = F_1 + F_2 = -\left(k_1 + k_2\right) x$$

$$k_{eff} = k_1 + k_2$$

$$\boxed{\omega = \sqrt{\frac{k_1 + k_2}{m}}}$$


k
k
k
m
2k
k
m
2
3
k
m


k_1
k_2
m

Mass connected between two springs

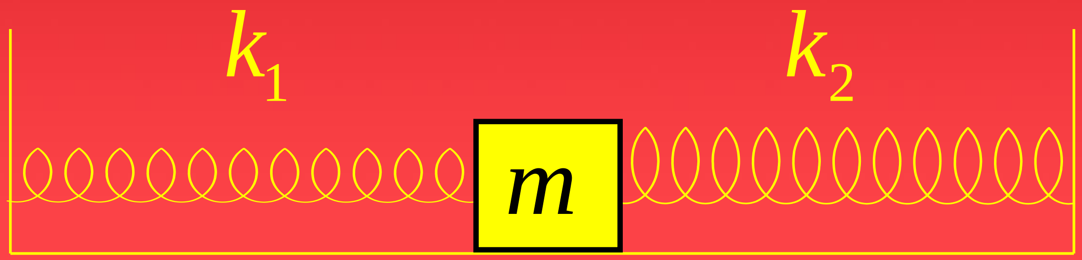
$$x = \text{displacement of body}$$

$$F_1 = -k_1 x \qquad F_2 = -k_2 x$$

$$\therefore F = F_1 + F_2 = -\left(k_1 + k_2\right) x$$

This shows that effective force constant is

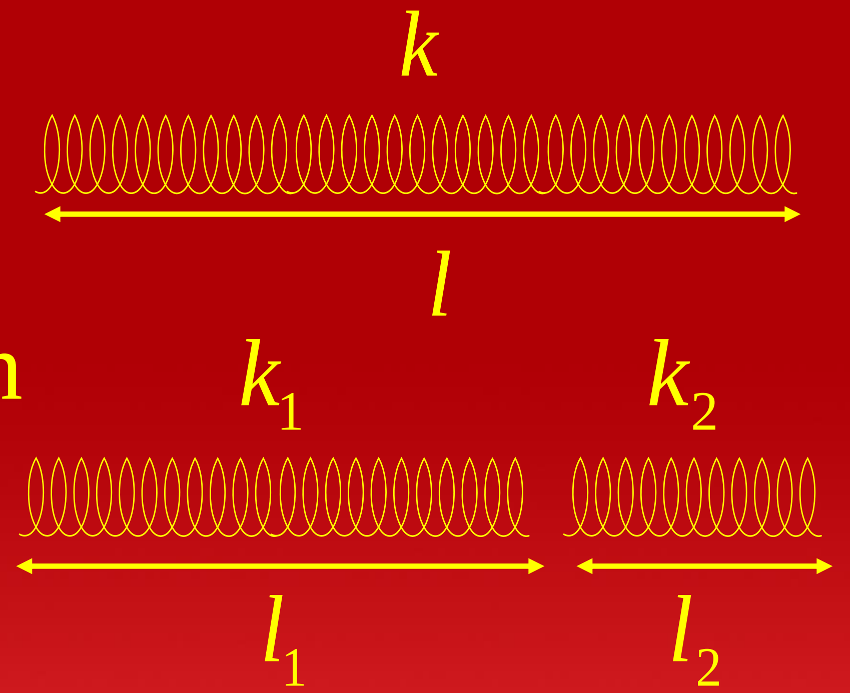
$$\boxed{k_{eff} = k_1 + k_2}$$



$$l_1 = 2l_2$$

If the whole spring undergoes an extension x by a force F, then

$$F = kx$$



The extension suffered by the parts l_1 and l_1 by the same force are $\frac{l_1}{l}x$ and $\frac{l_2}{l}x$ respectively. Therefore

$$F = k_1\left(\frac{l_1}{l}x\right) = k_2\left(\frac{l_2}{l}x\right) = kx$$

$$k_1 l_1 = k_2 l_2 = kl \Rightarrow k_1 = k\frac{l}{l_1} \text{ and } k_2 = k\frac{l}{l_2}$$

$$l = l_1 + l_2 = 2l_2 + l_2 = 3l_2 = l_1\left(\frac{3}{2}\right)$$

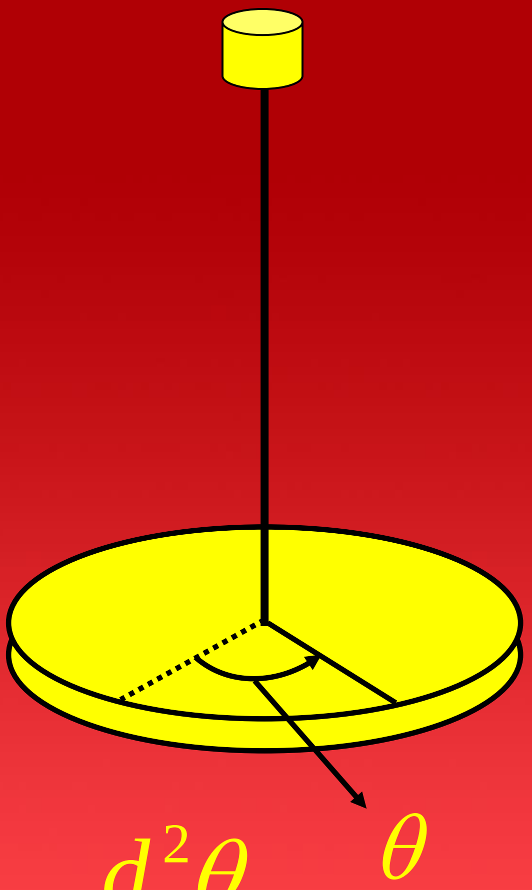
$$\frac{l}{l_1} = \frac{3}{2} \text{ and } \frac{l}{l_2} = 3$$

$$\therefore k_1 = k\left(\frac{3}{2}\right) \text{ and } k_2 = 3k$$

Torsional oscillator

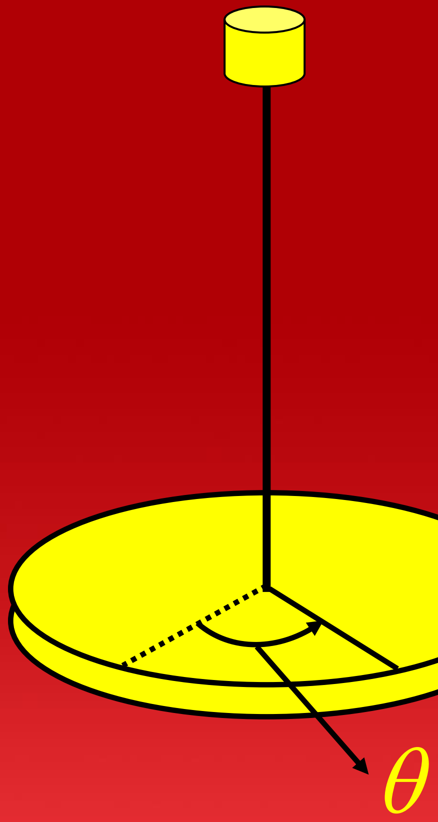
$$\tau = -\kappa\theta$$

Here τ is a constant that depends on the properties of wire and is called the torsional constant.



$$\tau = I\alpha = I\frac{d^2\theta}{dt^2} \quad \Rightarrow -\kappa\theta = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\left(\frac{\kappa}{I}\right)\theta$$



Solution: $\theta = \theta_m \cos \omega t$

$\theta = \theta_m \cos \omega t$ is the maximum angular displacement.

$$\omega = \sqrt{\frac{\kappa}{I}}$$

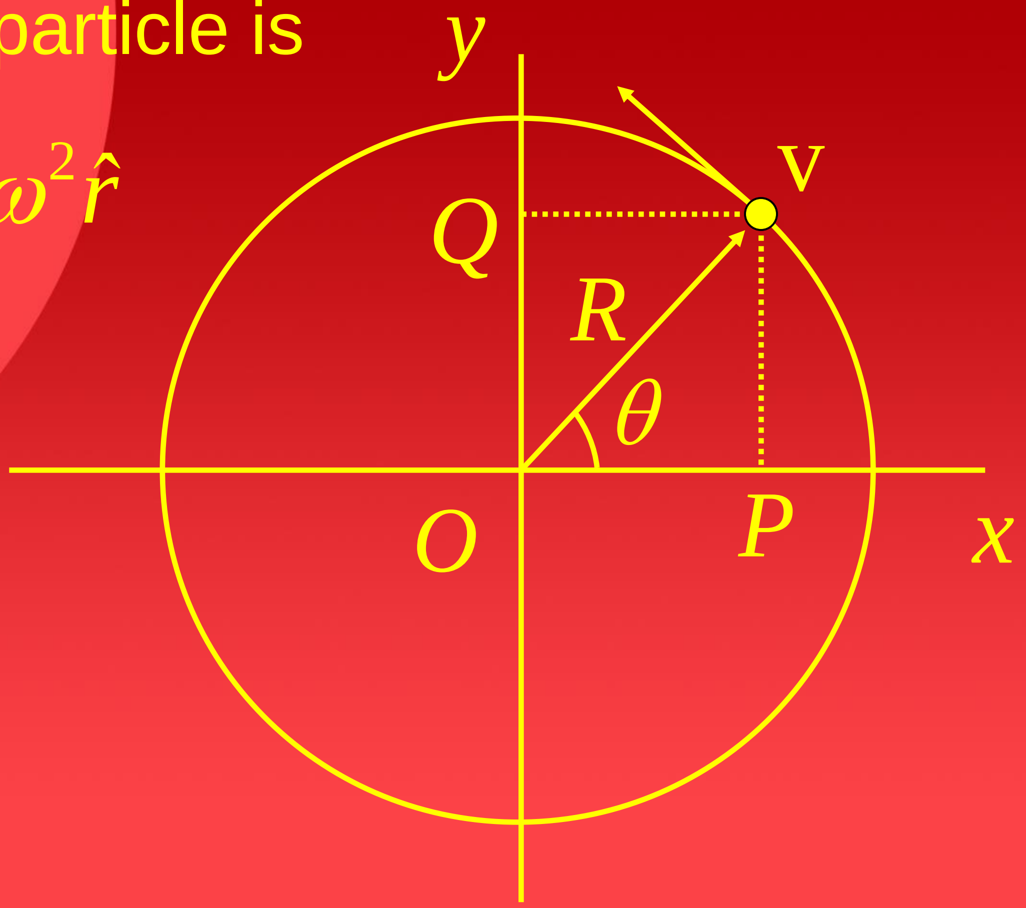
Simple harmonic motion and uniform circular motion

Acceleration of the particle is

$$\vec{a} = -\frac{\mathrm{v}^2}{R}\hat{r} = -R\omega^2\hat{r}$$

acceleration along x direction is:

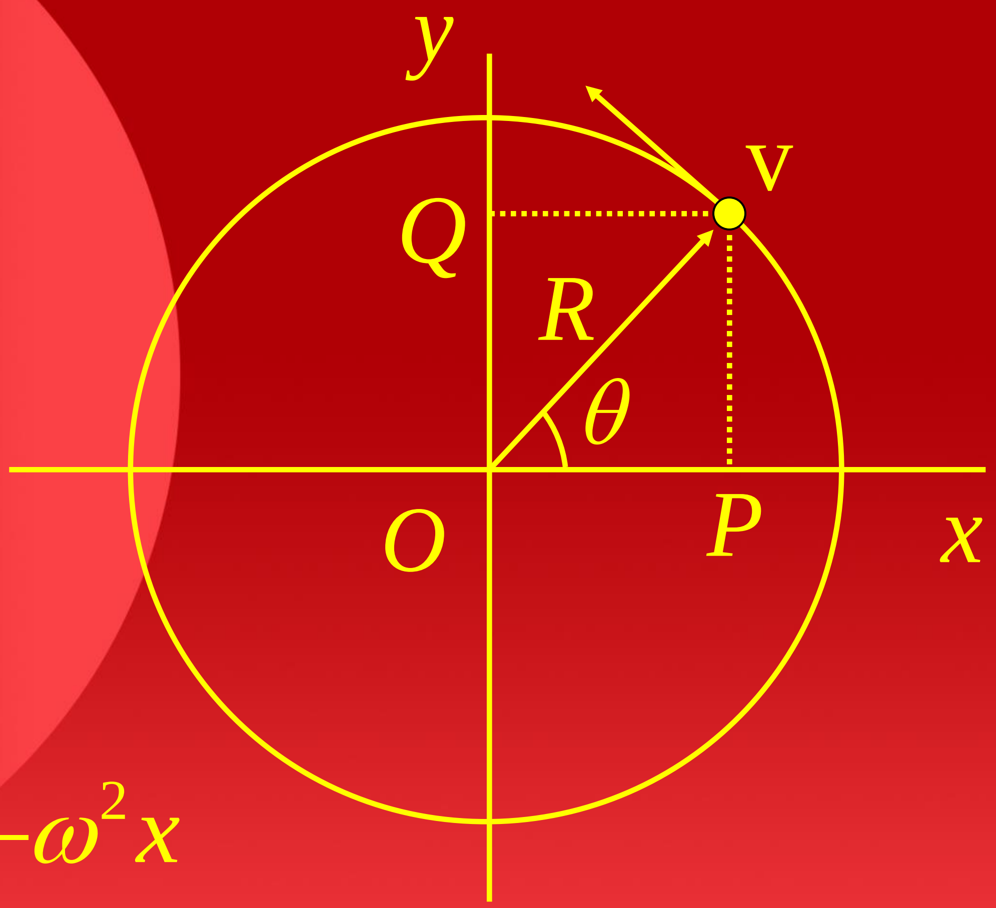
$$a_x = -R\omega^2\cos\theta$$



but

$$x = R\cos\theta$$

$$\therefore a_x = \frac{d^2x}{dt^2} = -\omega^2 x$$



Thus point P executes simple harmonic motion

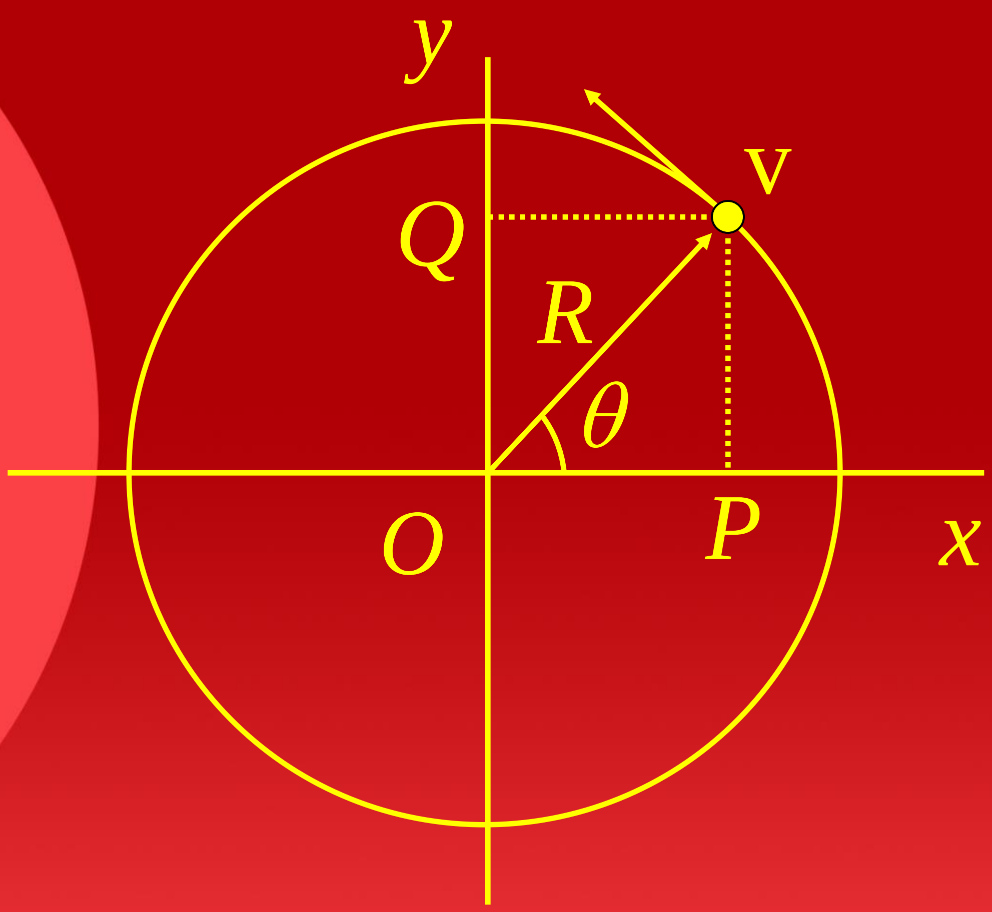
Acceleration of the point Q is:

$$a_y = -R\omega^2 \sin\theta$$

but

$$y = R\sin\theta$$

$$\therefore a_y = \frac{d^2 y}{dt^2} = -\omega^2 y$$



Q also executes simple harmonic motion

Composition of two simple harmonic motion of the same period along the same line

$$x_1 = A_1 \sin \omega t \text{ and } x_2 = A_2 \sin(\omega t + \phi)$$

The resultant displacement

$$x = x_1 + x_2$$

$$= A_1 \sin \omega t + A_2 \sin(\omega t + \phi)$$

$$= A_1 \sin \omega t + A_2 \sin \omega t \cos \phi + A_2 \sin \phi \cos \omega t$$

$$= \sin \omega t (A_1 + A_2 \cos \phi) + \cos \omega t (A_2 \sin \phi)$$

Let $A_1 + A_2 \cos\phi = R\cos\theta$

and $A_2 \sin\phi = R\sin\theta$

We get $x = R\sin(\omega t + \theta)$

Thus the resultant motion is also simple harmonic motion along the same line and has the same time period. Its amplitude is

$$R = \sqrt{A_1^2 + A_2^2 + A_1 A_2 \cos\phi}$$

$$\tan\theta = \frac{A_2 \sin\phi}{A_1 + A_2 \cos\phi}$$

Special cases:

If $\phi = 0$ then

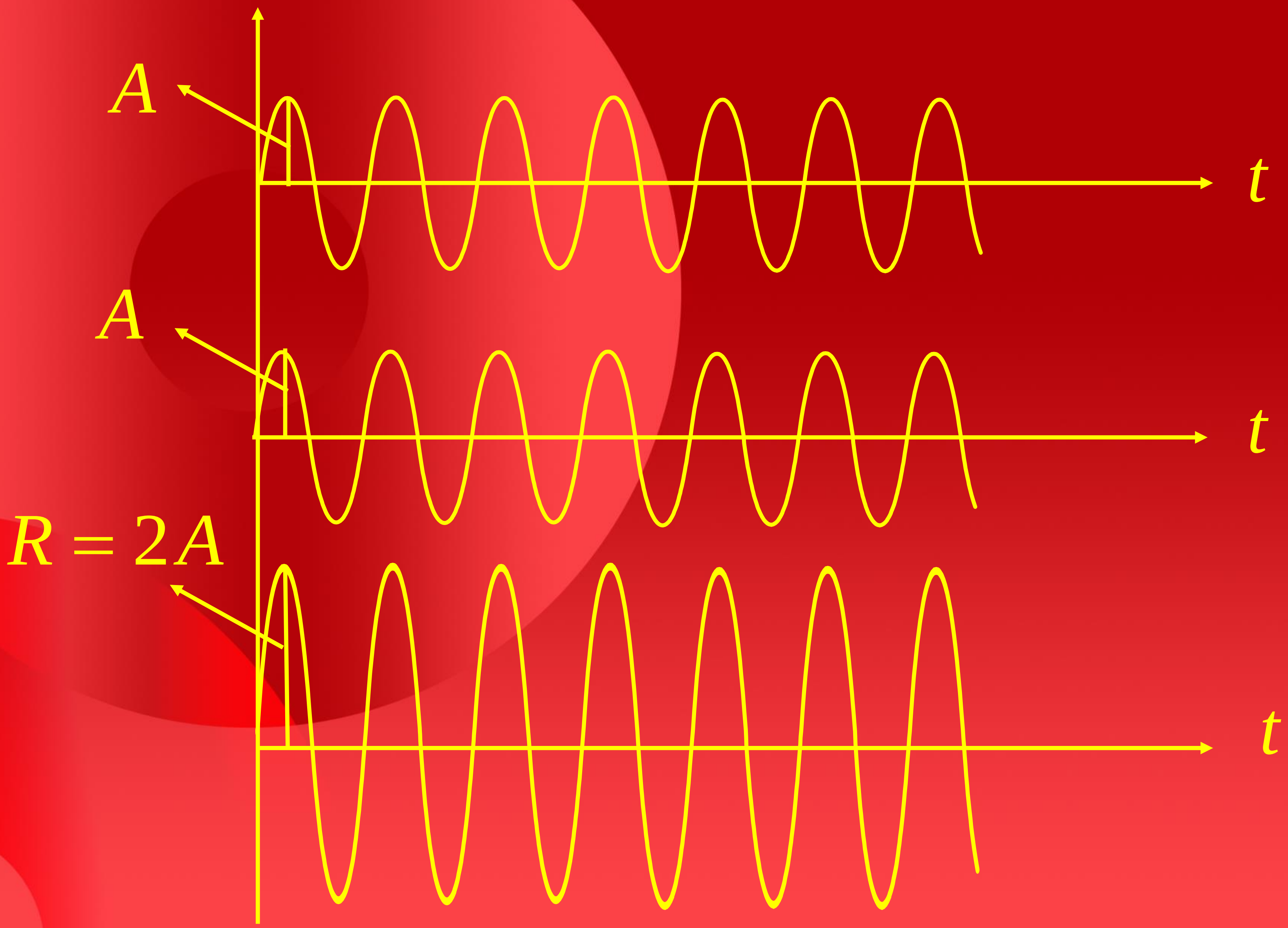
$$R = \sqrt{A_1^2 + A_2^2 + A_1 A_2} = \sqrt{\left(A_1 + A_2\right)^2} = A_1 + A_2$$

and

$$\tan\theta = 0 \Rightarrow \theta = 0$$

We get $x = \left(A_1 + A_2\right)\sin\omega t$

This is constructive interference



A
t
A
t
R = 2A
t

If $\phi = \pi$ then

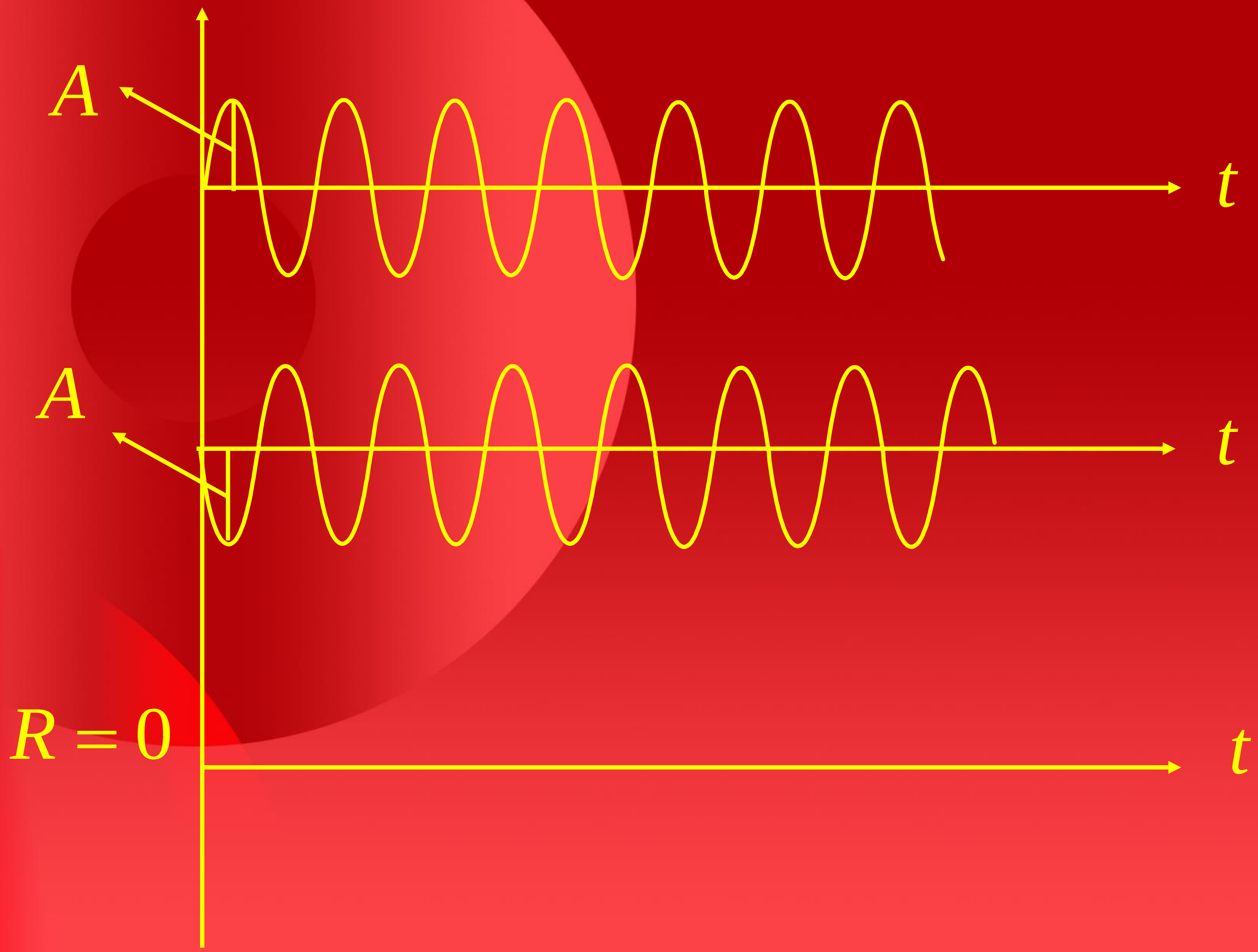
$$R = \sqrt{A_1^2 + A_2^2 - A_1 A_2} = \sqrt{\left(A_1 - A_2\right)^2} = A_1 - A_2$$

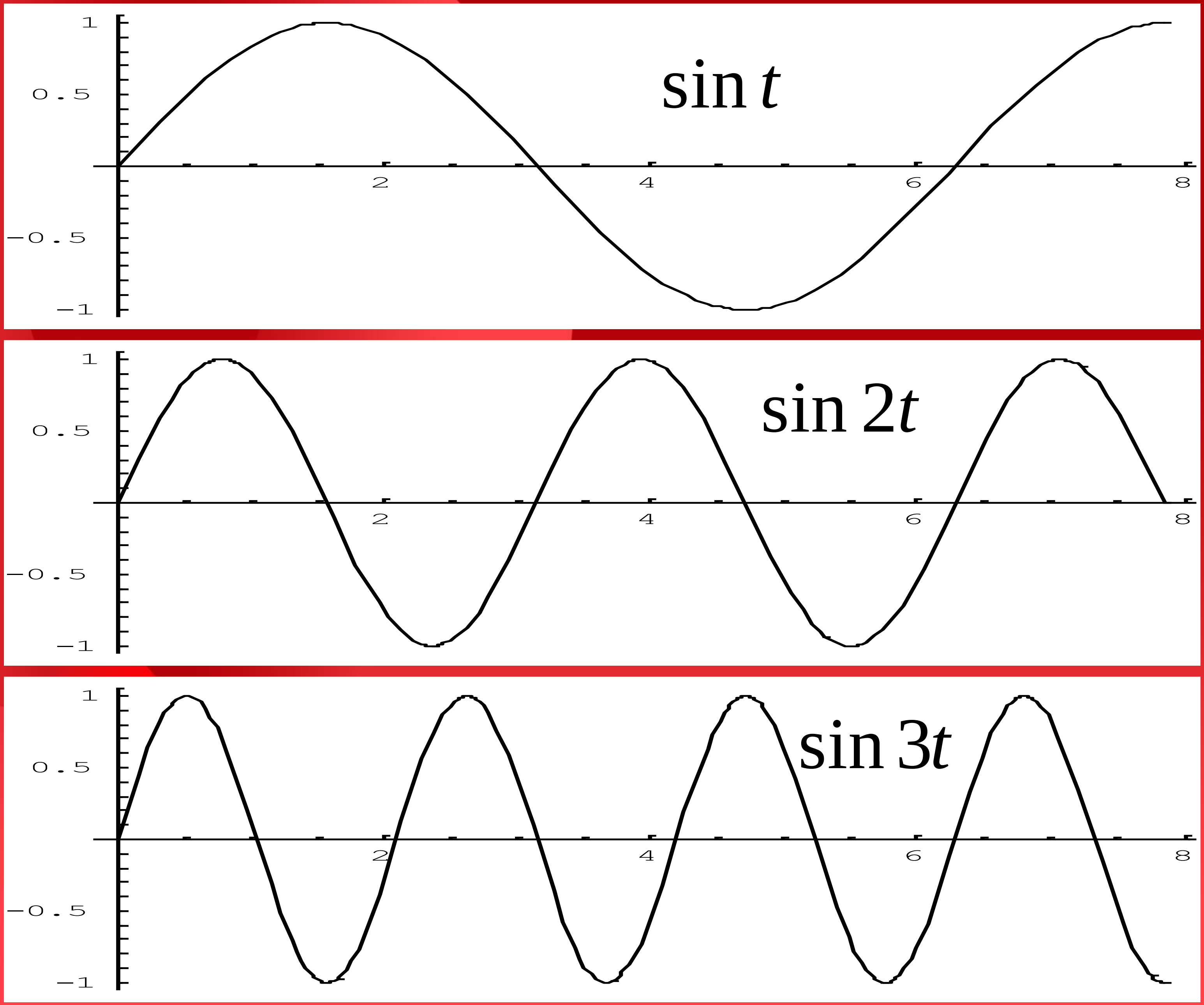
and

$$\tan\theta = 0 \Rightarrow \theta = 0$$

We get $x = \left(A_1 - A_2\right)\sin\omega t$

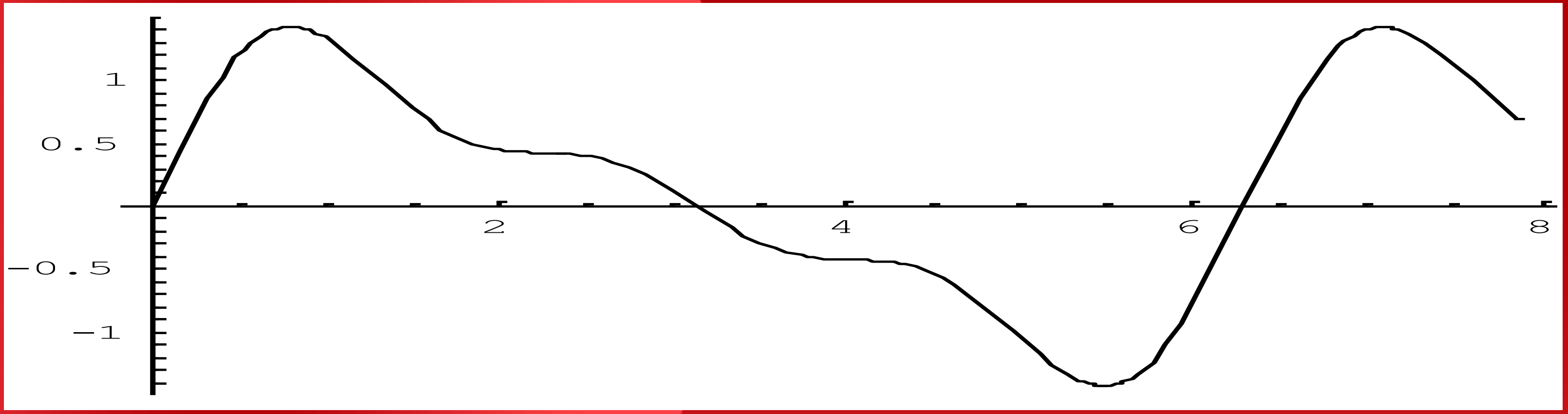
This is destructive interference


A
t
A
t
R = 0
t

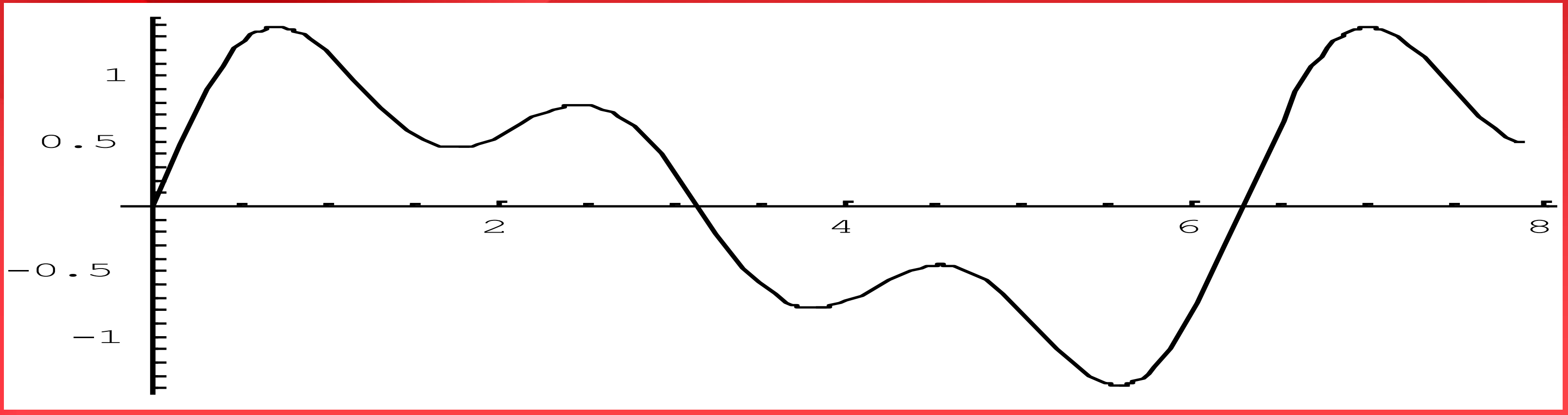


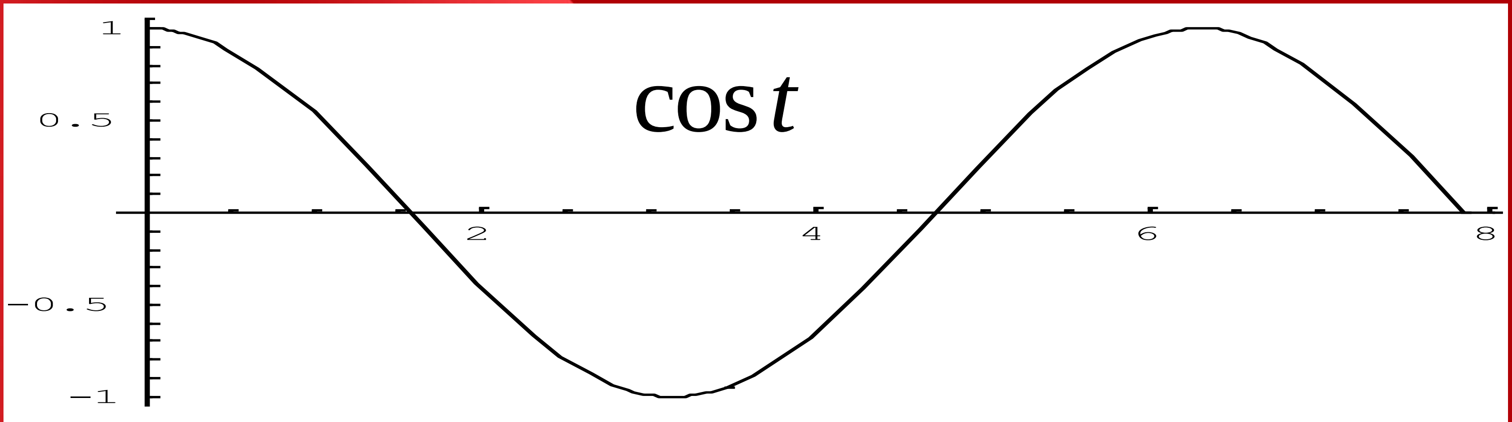
$\sin t$
$\sin 2t$
$\sin 3t$
1
0.5
−0.5
−1
2
4
6
8

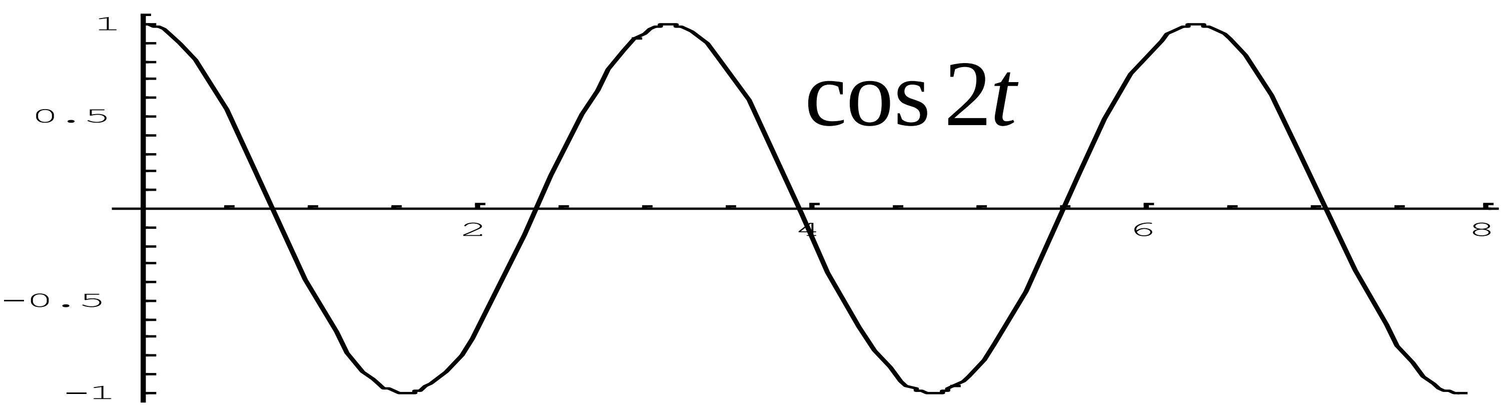
$$\sin t + 0.5\sin 2t + 0.3\sin 3t$$

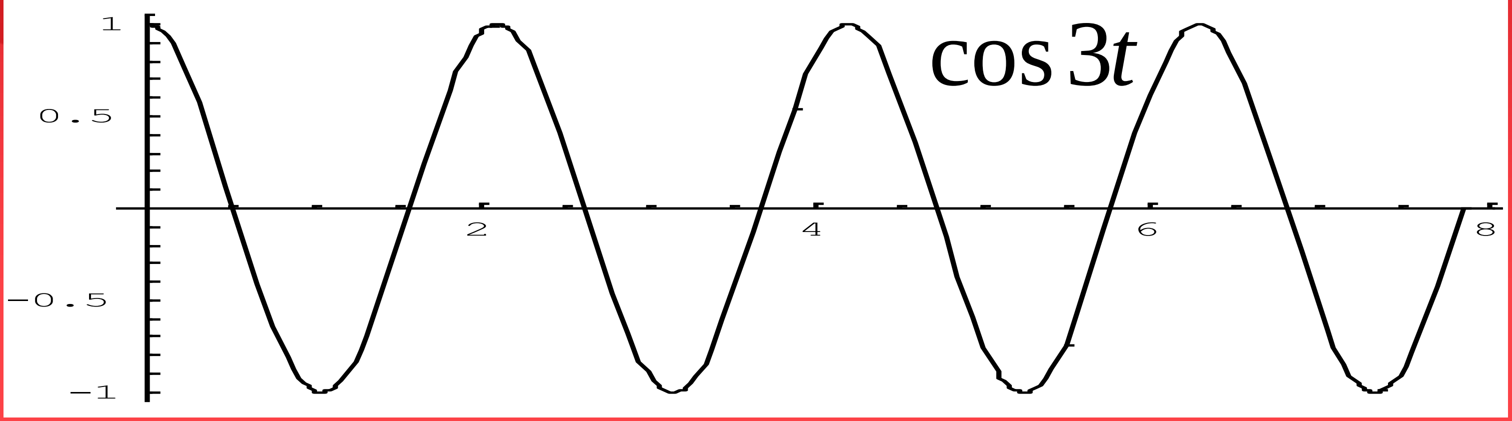


$$\sin t + 0.3\sin 2t + 0.5\sin 3t$$

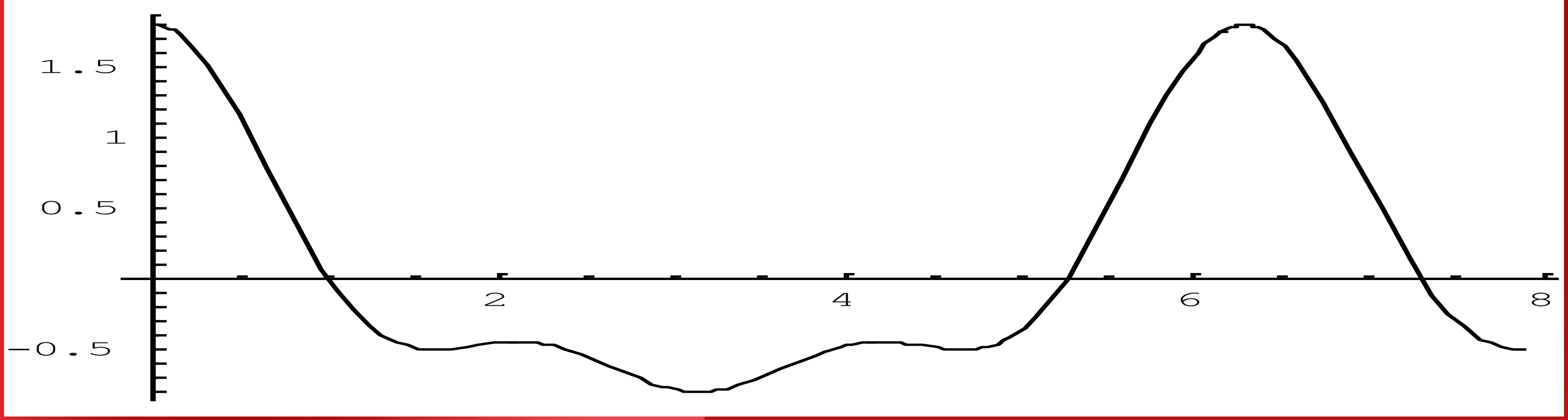



cos t
1
0.5
−0.5
−1
2
4
6
8

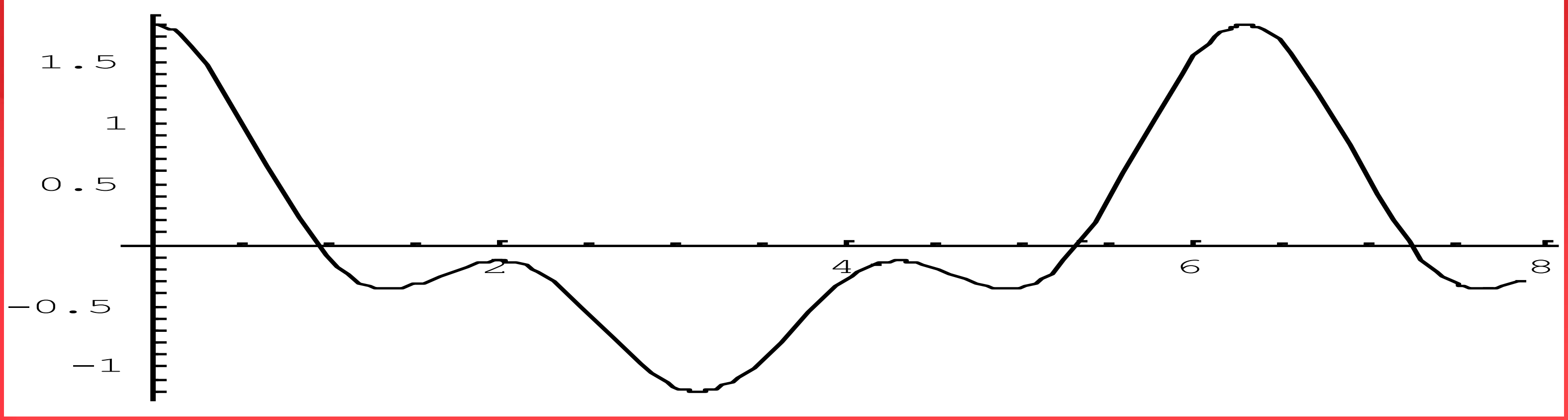

cos 2t
1
0.5
−0.5
−1
2
4
6
8


cos 3t
1
0.5
−0.5
−1
2
4
6
8

$$\cos t + 0.5\cos 2t + 0.3\cos 3t$$



$$\cos t + 0.3\cos 2t + 0.5\cos 3t$$



Composition of two simple harmonic motions
Of the same period at right angles to each other

$$x = A\sin\omega t \text{ and } y = B\sin\left(\omega t + \phi\right)$$

$$\sin\omega t = \frac{x}{A} = \text{and } \cos\omega t = \sqrt{1 - x^2 / A^2}$$

$$\frac{y}{B} = \sin\omega t\cos\phi + \sin\phi\cos\omega t$$

$$= \frac{x}{A}\cos\phi + \sin\phi\sqrt{1 - x^2 / A^2}$$

squaring and rearranging

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}-2\frac{xy}{AB}\cos\phi=\sin^2\phi$$

This is the equation of an ellipse.

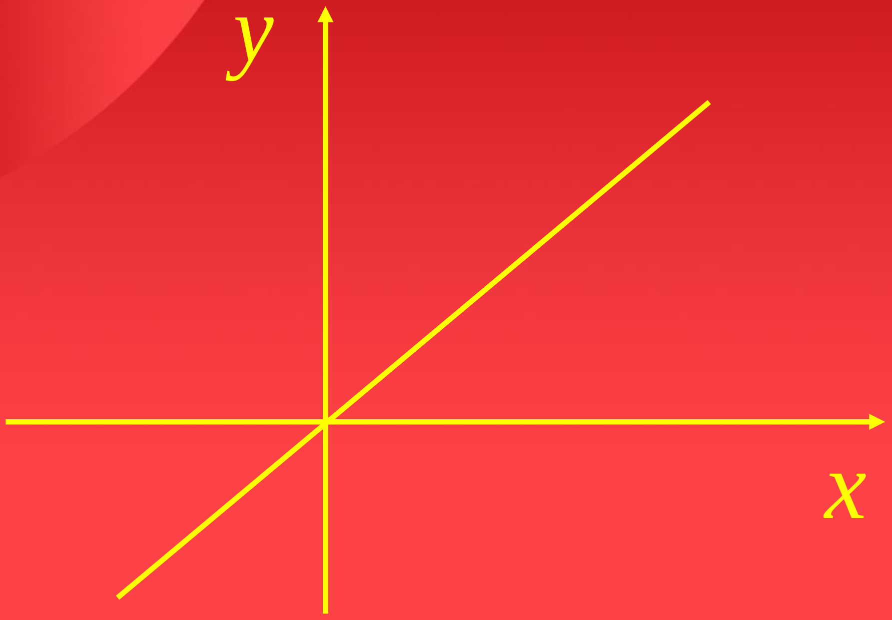
Special cases:

If $\phi=0$ then

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}-2\frac{xy}{AB}=0\Rightarrow\left(\frac{x}{A}-\frac{y}{B}\right)^2=0$$

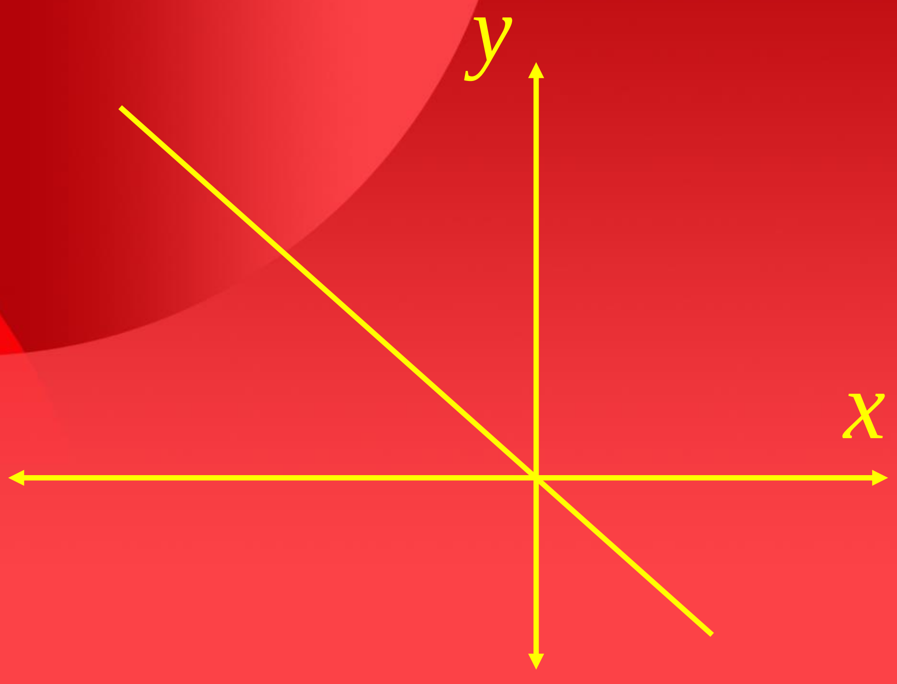
$$\frac{x}{A} - \frac{y}{B} = 0 \text{ or } y = \left(\frac{B}{A}\right)x$$

This is the equation of a straight line. Thus the resultant motion is a S.H.M. along a straight line passing through the origin.



If $\phi = \pi$ we get

$$\frac{x}{A} + \frac{y}{B} = 0 \text{ or } y = -\left(\frac{B}{A}\right)x$$

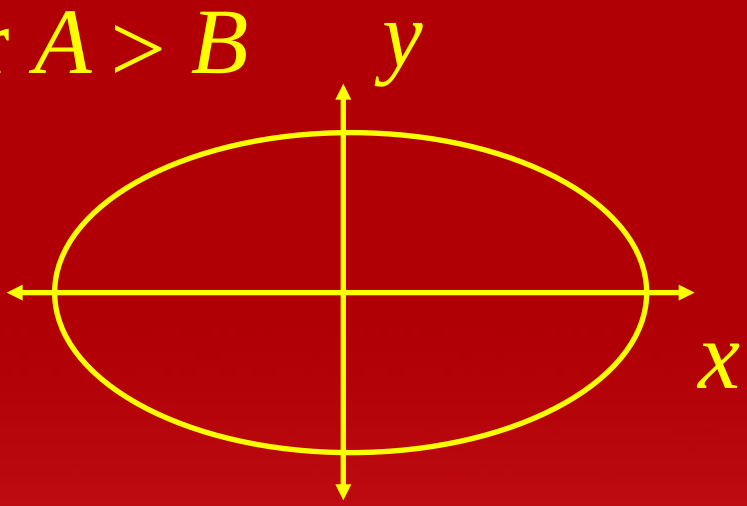


If $\phi = \dfrac{\pi}{2}$ then

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

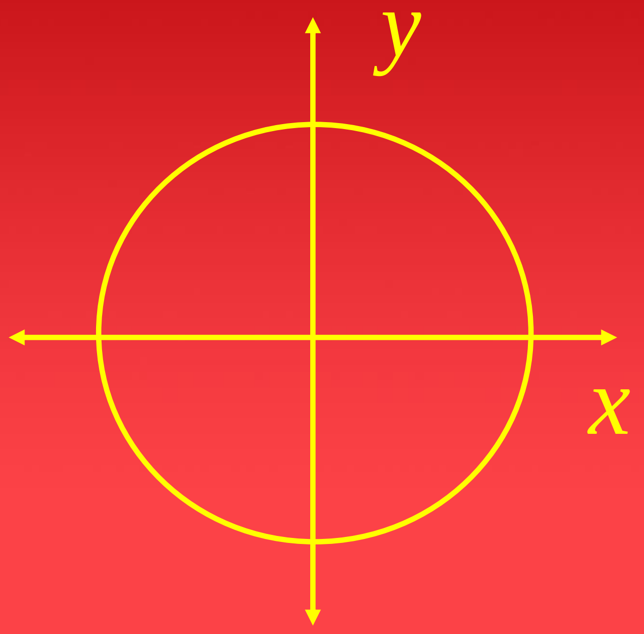
which is an ellipse

For $A > B$



If $A = B$ then $x^2 + y^2 = A^2$

which is an circle



Lissajous Figures

$$x = A\sin\omega_x t \text{ and } y = B\sin\left(\omega_y t + \phi\right)$$

If two oscillations of different frequencies at right angles are combined, the resulting motion is more complicated. It is not even periodic unless the two frequencies are in the ratio of integers. This resulting curve are called Lissagous Figures.

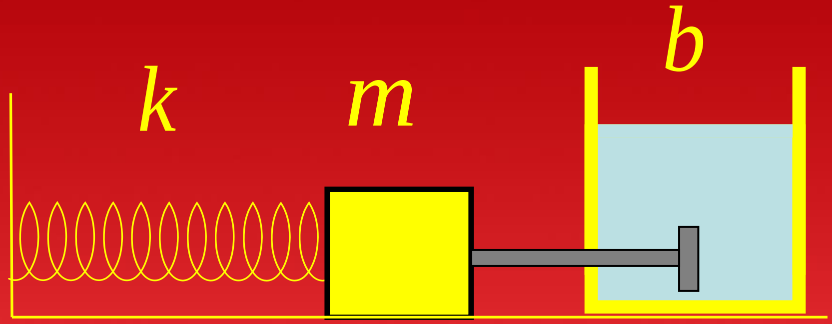
$$\frac{\omega_x}{\omega_y} = \text{integers} \Rightarrow \text{periodic motion}$$

Damped harmonic motion

$$\text{Damping force} = -b\frac{dx}{dt} \qquad \text{where } b > 0$$

From Newton's second law

$$\sum \vec{F} = m\vec{a}$$



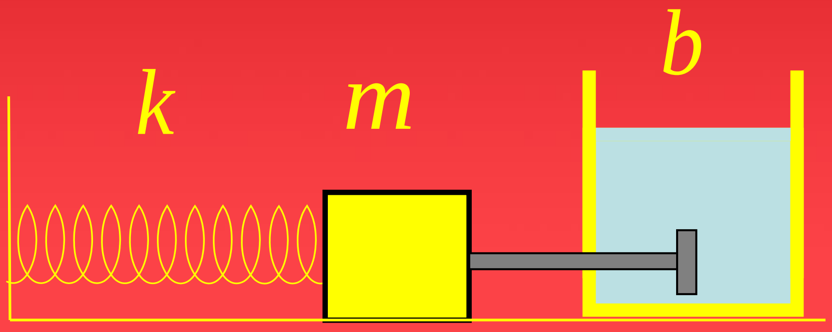
$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

$$\Rightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Its solution for $\frac{k}{m} \geq \left(\frac{b}{2m}\right)^2$ is

$$x = x_m e^{-bt/2m} \cos\left(\omega' t + \phi\right)$$

where $$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$



Forced oscillation and resonance

$$m\frac{d^2x}{dt^2}+kx=F_0\cos\omega t$$

Solution: $$x=\frac{F_0}{m\left(\omega_0^2-\omega^2\right)}\cos\omega t$$

Check:

$$\mathrm{LHS}=\frac{mF(-\omega^2)+kF}{m\left(\omega_0^2-\omega^2\right)}\cos\omega t=\mathrm{RHS}$$

Here ω_0 is the natural frequency of the system and is given by $\omega_0 = \sqrt{\dfrac{k}{m}}$

There is a characteristic value of the driving frequency ω at which the amplitude of oscillation is a maximum. This condition is called resonance. For negligible damping resonance occurs at $\omega = \omega_0$