PHY101-Lecture#11

Rotational Kinematics

When an object undergoes rotation or revolution, it means that the distance from a specific point, known as the center, remains constant. This signifies that during rotation, regardless of the object's position, this distance remains unchanged. We refer to this fixed distance as the radius. As the radius vector changes its position, an associated angle ϕ (phi), also changes accordingly. Every rotation entails an angle, ϕ, which defines the rotational position. Unlike Cartesian coordinates (X and Y) used to pinpoint a location, in rotational motion, we simplify this to a single value, ϕ. Furthermore, during rotation, a fixed distance 's' known as arc length is maintained. This arc length corresponds to the radius vector, and as ϕ increases, so does the arc length, denoted by 's'. It's noted that 's' is directly proportional to ϕ, indicating that as ϕ increases, the arc length also increases.

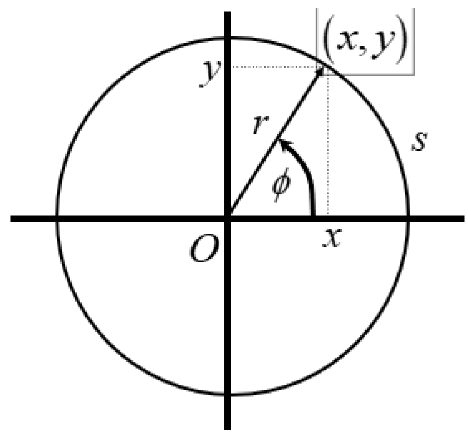


Fig. 11. 1: For a circle of radius r, one radian is the angle subtended by an arc length equal to r.

arc length = radius $\times$ angular displacement

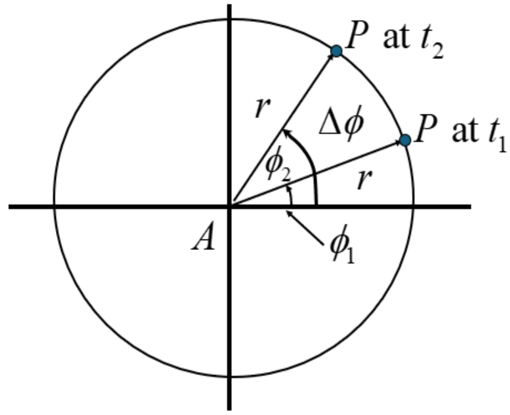
$$s = r\phi$$

$$\begin{aligned} \text{one revolution} &= 2\pi \text{ radians} \\ &= 360 \text{ degrees} \\ 1 \text{ radian} &= 57.3^0 \\ 1 \text{ radian} &= 0.159 \text{ revolution} \end{aligned}$$

$$s = 2\pi r = \text{total circumference}$$

Angular Speed

Consider an object at this moment, it occupies a certain position, let's denote the fixed distance from the center as 'A'. Initially positioned at ϕ_1, it then rotates to another position, which we'll refer to as ϕ_2. Consequently, it maintains a fixed angular difference of $\phi_2 - \phi$. Observe its depiction: 'r' has a complex value. Initially at ϕ_1 and later at ϕ_2. Suppose it's at ϕ_1 at time t_1 and at ϕ_2 at time t_2. From this, we derive the concept of angular speed.



$$\bar{\omega} = \frac{\phi_2 - \phi_1}{t_2 - t_1} = \frac{\Delta\phi}{\Delta t}$$

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\phi}{\Delta t}$$

$$\omega = \frac{d\phi}{dt}$$

Fig. 11. 2: A point on a rotating circular path for t_1 and t_2 displaced through the angle $\Delta\phi = \phi_2 - \phi_1$.

as $\Delta\phi$ decreases, we assume that $\Delta\phi$ and Δt are very small. This difference occurs over a brief period, causing Δt to decrease. Consequently, ω (Omega), our angular speed, approaches the value of $\frac{d\phi}{dt}$. We denote this angular speed as *'ω'*.

Consider the example of a clock. You'll notice that the second hand moves rapidly, followed by the minute hand, and then the hour hand. Now, let's explore their respective angular speeds. To calculate angular speed, we must remember that each hand covers a fixed angle during every revolution. For the second hand, this angle is traversed within 60 seconds. Therefore, its angular speed is 2π divided by 60 seconds. For the minute hand, this distance is covered within one hour, which means 60 minutes times 60 seconds equals 3600 seconds. For the hour hand, spanning 12 hours, you'd divide this time by 12, then calculate its respective values.

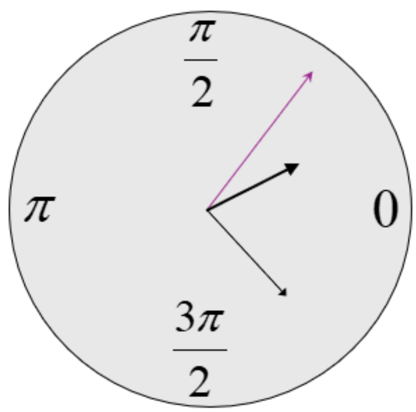


Fig. 11. 3: Three different angular speeds in a wall clock are connected to the radian angles.

$$\omega = \frac{2\pi}{T}$$

$$\omega_{\text{second}} = \frac{2\pi}{60} = 0.105\ rad\ /\ s$$

$$\omega_{\text{minute}} = \frac{2\pi}{60\times 60} = 1.75\times 10^{-3}\ rad\ /\ s$$

$$\omega_{\text{hour}} = \frac{2\pi}{60\times 60\times 12} = 1.45\times 10^{-4}\ rad\ /\ s$$

Problem 11.1: Our sun is 2.3 x 10^4 light years away from the center of our Milky Way galaxy. It moves in a circle around this center at 250 km/s.

(a) How long does it take the sun to make one revolution about the galactic center?

(b) How many revolutions has the sun completed since it was formed about 4.5 x 10^9 years ago?

Solution:

$$\text{a) 1 Light Year} = 9.46\times 10^{15}\, m$$

$$\text{v} = R\omega = R\frac{\theta}{t} = R\frac{2\pi}{T}$$

$$\therefore \text{ for one revolution } T = \frac{2\pi R}{\text{v}}$$

$$T = 5.5\times 10^{15}\, s = 1.74\times 10^{8} \text{ years}$$

$$b)\ \frac{4.5\times 10^{9}}{1.74\times 10^{8}} = 26 \text{ revolutions}$$

Angular acceleration

Angular Acceleration is defined as the time rate of change of angular velocity. It is usually expressed in radians per second per second. Thus,

$$a = \frac{d\omega}{dt}$$

The angular acceleration is also known as rotational acceleration. It is a quantitative expression of the change in angular velocity per unit time.

The average angular acceleration ($\bar{\alpha}$) over a specific time interval. Here, ω represents angular velocity, and t represents time. ω_1 and ω_2 are the initial and final angular velocities respectively, and t_1 and t_2 are the initial and final times respectively.

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\frac{d\phi}{dt} = \frac{d^2\phi}{dt^2}$$

Here, ϕ represents angular displacement. This equation shows that angular acceleration (α) can also be expressed as the second derivative of angular displacement with respect to time. It represents how rapidly the angular velocity is changing, which in turn indicates how quickly an object is rotating.

When the angle changes, the distance also changes accordingly. This implies that the length of the arc also undergoes alteration. As I rotate my hand in a certain direction, the angle increases, and there's a corresponding change in the distance, referred to as *'s'*, which equals *'r'* multiplied by *' ϕ '*, the angle through which it has moved. This *' ϕ '* varies with time, so *'s'* equals *'r'* multiplied by ' ϕ '. Now, let's differentiate it with respect to time. Since *'r'* is constant, it becomes *'r'* times the derivative of *' ϕ '* with respect to *'t'*, denoted as *d/dt* of *'r'* times ' ϕ ', which is *' ω '*. The formula for the speed at which this object is moving is *'r'* times *' ω '*. If *' ω '* is constant, then differentiating it again is straightforward.

$$s = r\phi$$

$$\frac{ds}{dt} = r\frac{d\phi}{dt}$$

$$\mathrm{v} = r\omega$$

$$\frac{d\mathrm{v}}{dt} = r\frac{d\omega}{dt}$$

$$\mathrm{a_T} = r\alpha$$

the acceleration 'a_T' in the tangent direction equals 'r' times 'α'. The tangent direction refers to the direction tangent to the motion path. I'll illustrate this with a diagram shortly in Fig.11.4. There exists a close relationship between linear motion and angular motion, despite the differences: one involves straight-line movement while the other involves rotation with constant distance. However, their mathematical principles differ, although to a similar extent. Let's revisit the formulas we derived earlier for linear motion.

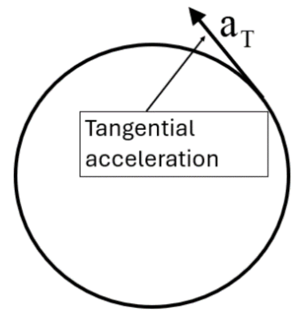


Fig. 11. 4: Tangential acceleration is tangent of a circle.

Relationship between linear and angular variables:

Translational Motion	**Rotational Motion**
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2} a t^2$	$\phi = \phi_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\phi - \phi_0)$

Problem 11.2: A point on the rim of a 0.75 m diameter grinding wheel changes speed from 12 m/s to 25 m/s in 6.2 s. What is the angular acceleration during this interval?

Solution:

$$a = \frac{v_f - v_i}{t} = 2.1 \text{ m/s}^2$$
$$\because \quad a = r\alpha$$
$$r = d/2 = 0.75/2 = 0.375$$
$$\alpha = \frac{a}{r} = \frac{2.1}{0.375} = 5.6 \text{ rad/s}^2$$

Problem 11.3: The angular speed of a car engine is increased from 1170 rev/min to 2880 rev/min in 12.6 s.

(a) Find the average angular acceleration in rev/ min^2.

(b) How many revolutions does the engine make during this time?

Solution:

$$\omega_i = 1170 \text{ rev/min}$$
$$\omega_f = 2880\ rev/\min$$
$$t = 12.6\ s = 12.6/60 = 0.21\min$$
$$\alpha = \frac{\omega_f - \omega_i}{t} = \frac{(2880 - 1170) \text{ rev/min}}{0.21\min} = 8140\,\text{rev/min}^2$$
$$\phi = \omega_i t + \frac{1}{2}\alpha t^2 = 425\,\text{rev}$$

Problem 11.4: A diver makes 2.5 complete revolutions on the way from a 10 m platform to the water below shown in Fig.11.5. Assuming zero initial vertical velocity, calculate the average angular velocity.

$$h = v_i t + \frac{1}{2}gt^2$$
$$\therefore v_i = 0$$
$$h = \frac{1}{2}gt^2$$
$$h = 10\ m\ ,\ \text{g} = 10 \text{ m/s}^2$$
$$t = \sqrt{\frac{2h}{g}} = 1.43\ s$$
$$\therefore \omega = \frac{\phi}{t} = \frac{2\pi(\frac{rad}{rev})n}{t}$$
$$n = 2.5\ (rev)$$
$$\omega = \frac{2\pi(\frac{rad}{rev})\times 2.5\ (rev)}{1.43\ s} = 11\,\text{rad/s}$$

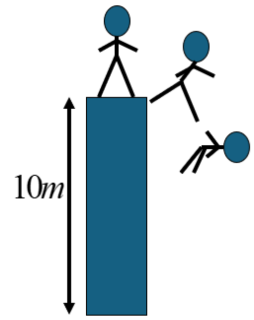


Fig. 11. 5: Path of a diver from a 10 m Platform.

Rotation is a common phenomenon in our surroundings, not limited to just the wheels of a car. If you observe any machinery, whether it's a car, motorcycle, or in a lathe shop, you'll notice the interaction between small and large wheels. Understanding the concepts of angular speed and angular acceleration allows us to comprehend many aspects of our surroundings.

Consider a bicycle as an example. In a bicycle, there are two gears: a large one driving a smaller one. As the large gear rotates once, the smaller gear rotates several times. We can calculate this rotation ratio using their respective radii. If the radius of the larger gear is represented as 'r_c' and that of the smaller gear as 'r_a' as shown in Fig.11.6, then the smaller gear will rotate ' r_c / r_a ' times for every rotation of the larger gear. If there's a threefold difference between their radii, the smaller gear will rotate three times for every rotation of the larger gear. Now, let's focus on the chain mechanism. The chain's purpose is to transmit force effectively. Despite its motion, the chain maintains a constant speed throughout its length. This uniformity ensures that as one part of the chain moves forward, the other part moves backward with the same speed, ensuring effective force transmission. When acceleration occurs, it is also transmitted backward through the chain. Now, let's work through a problem related to this concept.

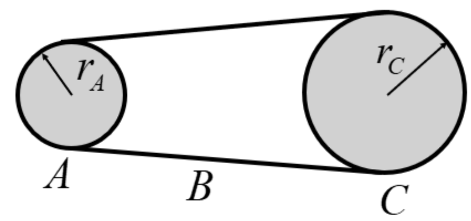


Fig. 11. 6: Angular speed between gears of a bicycle.

Problem 11.5: Wheel A of radius r_A = 10.0 cm is coupled by a chain B to wheel C of radius r_C = 25.0 cm as shown in Fig.11.7. Wheel A increases its angular speed from rest at a uniform rate of 1.60 rad/s^2.

Determine the time for wheel C to reach a rotational speed of 100 rev/min.

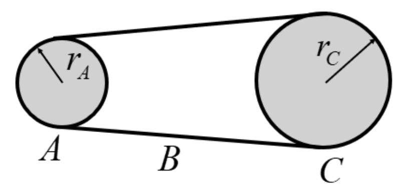


Fig. 11. 7: Angular speed between gears of a bicycle provides certain rotational speed for a specific time.

$Given:$

$r_A = 10.0\ cm$

$r_C = 25.0\ cm$

$Angular\ acceleration\ of\ wheel\ A,\ \alpha = 1.60\ rad/s^2$

$Desired\ rotational\ speed\ of\ wheel\ C,\ \omega_C = 100\ rev/\min$

$First,\ we\ need\ to\ convert\ the\ desired\ rotational\ speed\ of\ wheel\ C$

$from\ revolutions\ per\ \min ute\ to\ radians\ per\ \sec ond:$

$$Angular\ speed\ of\ wheel\ C = \omega_C = \frac{100 \times 2\pi}{60} = \frac{100 \times 2(3.14)}{60} = 10.5\ rad/s$$

$Now,\ we\ can\ use\ the\ given\ equations:$

$v_A = v_C \Rightarrow r_A\omega_A = r_C\omega_C$

1. $\omega_A = \dfrac{r_C\omega_C}{r_A}$

$$\omega_A = \frac{25.0\times10.5}{10.0} = 26.3\ rad/s$$

$$2.\ \alpha = \frac{\omega_A - 0}{t}$$

$$3.\ t = \frac{\omega_A}{\alpha} = \frac{26.3}{1.60} = 16.4\ s$$

Uniform circular motion

Consider an object rotating in a circle at a constant speed. For instance, imagine I've tied it to a thread and I'm rotating it. Now, the length of the thread remains constant, ensuring that the distance from the center, or the radius, remains unchanged, and the speed of the object also remains constant. However, does this imply that the speed remains constant without any acceleration? We need to delve deeper into this. It's essential to consider the direction of velocity. Initially, as mentioned before, the speed remains constant, denoted as v. However, there's a difference in direction: if v_1 is upward, then v_2 is slightly sideways, creating an angle between them. Moreover, v_2 is directed towards the center, giving rise to what we call centripetal acceleration. Now, let's determine its value. Understanding its direction is crucial; it's directed towards the center, thus termed centripetal acceleration.

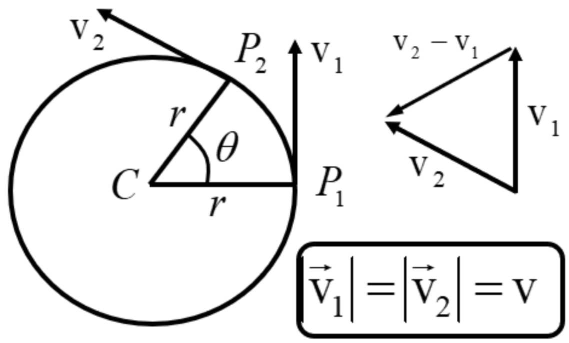


Fig. 11. 8: As object moves from P_1 to P_2 the direction of its velocity changes from v_1 to v_2. The resultant vector of velocity $v_2 - v_1$ is toward the center of the circle.

To find out the difference, we need to calculate v_2 - v_1. Consider this scenario: if the body is initially positioned at P_1 and after some time it moves to the point P_2, forming an angle θ (theta), Let's denote its length a_R, as the radius, and θ as the angle through which it has rotated. By drawing a triangle here, we can notice a minor discrepancy between the triangle and the actual path. This occurs when theta is very small. However, if I adjust it slightly, ensuring theta is small, there will be no discernible difference between the triangle and the actual path. Thus, Δr is approximately equal to $r\theta$.

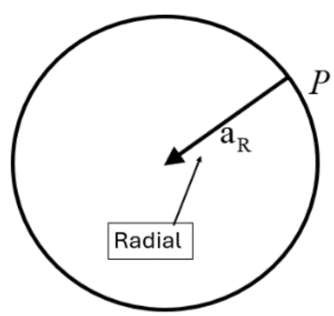


Fig. 11. 9: As the object moves along the cicular path the direction of its velocity changes so the object undergoes a radial acceleration.

$$\Delta r = \mathrm{v}\Delta t \approx r\theta$$

This same concept directly applies to velocity. Notice that another triangle is formed, where delta and velocity are both influenced by theta.

$$\Delta \mathrm{v} \approx \mathrm{v}\theta$$

Applying this concept again, we find that alpha, denoted by 'a', equals v^2 divided by 'r', representing the average acceleration. However, you'll learn that there's minimal disparity between average and instantaneous values, given negligible differences in distances and times. Consequently, as we let Δt approach zero, we obtain v^2/r, known as centripetal acceleration.

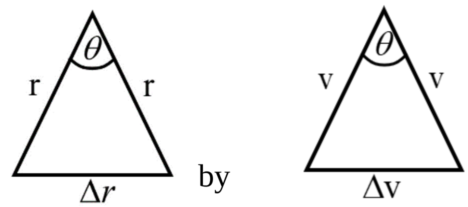


Fig. 11. 10: Triangular approximation for radius r and velocity v.

$$\overline{\mathrm{a}} = \frac{\Delta \mathrm{v}}{\Delta t} \approx \frac{\mathrm{v}\theta}{r\theta / \mathrm{v}} = \frac{\mathrm{v}^2}{r}$$

$$\mathrm{a} = \lim_{\Delta t \to 0} \frac{\Delta \mathrm{v}}{\Delta t} = \frac{\mathrm{v}^2}{r}$$

$$\vec{\mathrm{a}}_R = -\frac{\mathrm{v}^2}{r}\hat{r}$$

Negative sign shows the acceleration is readily inward.

If there's no other acceleration, meaning the object maintains a constant speed, then only radial acceleration exists. However, if you gradually increase its speed, acceleration occurs; likewise, when decelerating. This introduces another acceleration component: tangential acceleration as shown in Fig 11.11. The total acceleration becomes a vector with two components: one being centripetal and the other is tangential. The resultant acceleration of these two components equals the total acceleration ‘a’.

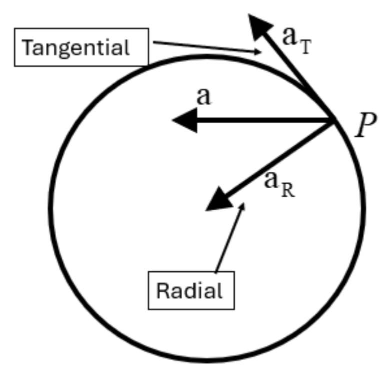


Fig. 11. 11: Total acceleration of the rotating object at a point P.

$$a_T = r\alpha \ , \ a_R = \frac{v^2}{r} = r\omega^2$$

$$a = \sqrt{a_T^{\ 2} + a_R^{\ 2}}$$

Problem 11.6: The Moon revolves about the Earth as shown in Fig.11.12, making a complete revolution in 27.3 days. Assume that the orbit is circular and has a radius of 238,000 miles. What is the magnitude of the acceleration of the Moon towards the Earth?

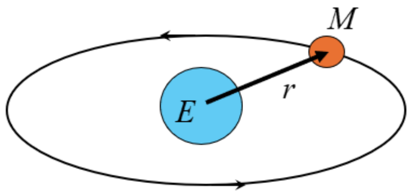
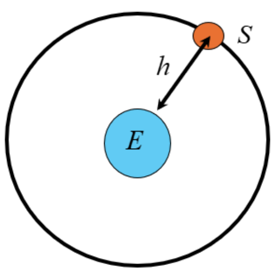


Fig. 11. 12: Illustration of the Moon and Earth during orbital motion.

Given Data:

$$Radius\ of\ the\ Moon's\ orbit: r = 3.81952\times10^{8}\,meters$$
$$Period\ of\ the\ Moon's\ orbit: T\ =\ 27.3\ days$$
$$Calculate\ Orbital\ Speed\ (v):$$
$$Convert\ the\ period\ of\ the\ Moon's\ orbit\ from\ days\ to\ seconds:$$
$$T = 27.3\times24\times60\times60\text{seconds}$$
$$T = 2.36\times10^{6}\,\text{seconds}$$
$$Use\ the\ formula\ for\ the\ circumference\ of\ a\ circle\ to\ find\ the\ orbital\ speed:$$
$$v = \frac{2\pi r}{T} = \frac{2\times3.14159\times3.81952\times10^{8}}{2.36\times10^{6}}\text{m/s}$$
$$v = \frac{23.95546\times10^{8}}{2.36\times10^{6}} = 1015.421\text{m/s}$$
$$Calculate\ Acceleration\ (\text{a}):$$
$$Use\ the\ centripetal\ acceleration\ formula:$$
$$a = \frac{v^2}{r}$$
$$Substitute\ the\ calculated\ values:$$
$$a = \frac{(1015.421\text{m/s})^2}{3.81952\times10^{8}}\text{m/s}^2 = \frac{1030885.961}{3.81952\times10^{8}}\text{m/s}^2$$
$$a \approx 2.719\times10^{-3}\,\text{m/s}^2$$

Problem 11.7: Calculate the speed of an Earth satellite as shown in Fig.11.13 that it is traveling at an altitude h of 210 km where g = 9.2 m/s^2. The radius R of the Earth is 6370 km.



Solution:

$$a = \frac{v^2}{r}$$
$$a = g \text{ and } r = R + h$$
$$g = \frac{v^2}{R+h}$$
$$v = \sqrt{(R+h)g} = 7780\, m/s$$

Fig. 11. 13: Artificial satellite 'S' orbiting around the Earth at a height h.

Vector Cross Products

Let's explore another method of defining the product of two vectors, known as the cross product. Imagine drawing lines on a piece of paper. First, we'll create vector A and then position the second vector, 'B', anywhere within the same plane on this paper. Between them, there exists an angle θ. Now, while the dot product involves the sine of theta multiplied by the magnitude of A times B, the cross product takes a different approach. It's perpendicular to both A and B as shown in Fig.11.14, extending in a direction we'll call the 'n' direction. This perpendicularity is similar to the tip of a pencil rooted on the paper. The magnitude of this vector, denoted as 'A cross B', equals the magnitude of A times B times sine theta, with this being its limit.

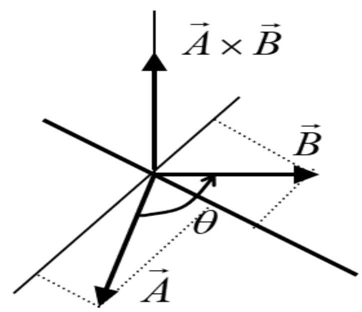


Fig. 11. 14: Vectors A and B and their cross-product A × B are perpendicular to each other.

$$\vec{A} \times \vec{B} = AB \sin\theta\, \hat{n}$$
$$\hat{n} \text{ is perpendicular to AB-plane}$$

Now, let's consider unit vectors in three dimensions, represented by $\hat{i}$, $\hat{j}$, and $\hat{k}$ cross products of these unit vectors are well defined. As the angle between any two different unit vectors is always 90 degrees, if we take the cross product of $\hat{i}$ and $\hat{j}$, it results in the third direction, $\hat{k}$, and similarly for other combinations.

$$\hat{i} \times \hat{j} = \hat{k}$$
$$\hat{k} \times \hat{i} = \hat{j}$$
$$\hat{j} \times \hat{k} = \hat{i}$$

Important properties

Several important properties characterize the cross product. For instance, if two vectors are parallel, their cross product is zero. When the angle between them is zero, the cross product of a vector with itself is also zero.

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$
$$\vec{A} \times \vec{A} = 0$$

There are other properties as well, such as the distributive property.

$$\left(\vec{A} + \vec{B}\right) \times \vec{C} = \left(\vec{A} \times \vec{B}\right) + \left(\vec{A} \times \vec{C}\right)$$

When expressed using unit vectors, A cross B can be represented as a determinant.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$
$$= (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$$

Summary

In this chapter, we explained the fundamental concepts of rotational kinematics, specifically focusing on angular speed, angular velocity, and angular acceleration. These concepts come into play when a body undergoes rotational motion, moving along a circular path. We questioned why objects tend to move in circular paths when, inherently, they tend to move in straight lines. Exploring these principles helps us understand the forces at play that compel objects to rotate. Overall, this chapter provided an overview of rotational kinematics, shedding light on the dynamics of circular motion.