Lecture_13

Angular momentum

In physics, momentum is a fundamental concept. Momentum is determined by multiplying the velocity of a particle by its mass. Its significance lies in that when a force acts on a particle, its momentum changes. Conversely, in the absence of force, the momentum remains constant. When dealing with multiple particles, the total momentum equals the sum of the momenta of all individual particles. It is crucial to note that momentum is a vector quantity.

Today's lecture focuses on angular momentum, which is equally fundamental. This lecture will explore how angular momentum shares properties with regular momentum and possesses additional characteristics. This distinguishes it from a general definition of angular momentum.

Angular momentum of a single particle:

Angular momentum describes the rotational motion of an object. It is a vector quantity defined as the cross product of the object's position vector and its linear momentum vector, for a chosen origin, as shown in figure 13.1.

Mathematically,

$$\vec{L} = \vec{r} \times \vec{p}$$

There are different ways to write angular momentum.

$$L = r p \sin\theta$$
$$L = (r\sin\theta)\, p = r_{\perp} p$$
$$L = r(p\sin\theta) = r\, p_{\perp}$$

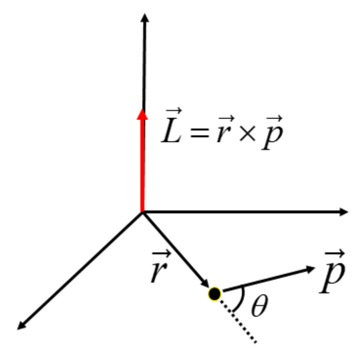


Figure13.1: Relation between angular and linear momentum.

Angular momentum of projectile:

A projectile is launched, following the trajectory of a parabola, originating from a specified point. Our objective is to determine its angular momentum for point O, as shown in figure 13.2.

It is crucial to clearly define point O, as the calculation of angular momentum depends on the reference point from which the projectile is thrown.

Let's find angular momentum ***L*** about origin O after some time t:

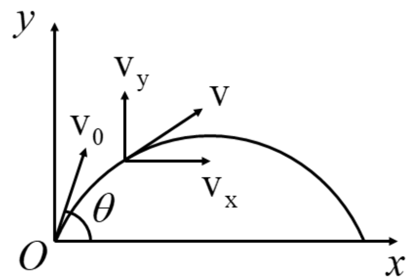


Figure13.2: Velocity components of a projectile.

$$x = \left(v_0 \cos\theta\right) t \qquad \text{eq(1)}$$

$$y = \left(v_0 \sin\theta\right) t - \frac{1}{2} g t^2 \qquad \text{eq(2)}$$

$$v_x = v_0 \cos\theta \qquad \text{eq(3)}$$

$$v_y = v_0 \sin\theta - gt \qquad \text{eq(4)}$$

$$\vec{L} = \vec{r} \times \vec{p} = \left(x\hat{i} + y\,\hat{j}\right) \times \left(v_x \hat{i} + v_y\,\hat{j}\right) m$$

$$\vec{L} = m\left(xv_x\,\hat{i}\times\hat{i} + xv_y\,\hat{i}\times\hat{j} + yv_x\,\hat{j}\times\hat{i} + yv_y\,\hat{j}\times\hat{j}\right)$$

$$\vec{L} = m\left(xv_x\,(0) + xv_y\,(\hat{k}) + yv_x\,(-\hat{k}) + yv_y\,(0)\right) \qquad \because \hat{i}\times\hat{i} = \hat{k}\times\hat{k} = \hat{j}\times\hat{j} = 0$$

$$\vec{L} = m\left(x\,v_y - y\,v_x\right)\hat{k} \qquad \because\ \hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{i} = -\hat{k}$$

putting values from eq(1-4), we get

$$\vec{L} = m\left(\left(tv_0\cos\theta\right)(v_0\sin\theta - gt) - \{(tv_0\sin\theta - \frac{1}{2}gt^2)v_0\cos\theta\}\right)\hat{k}$$

$$\vec{L} = m\left((tv_0\cos\theta v_0\sin\theta - gt^2 v_0\cos\theta) - \{tv_0\sin\theta v_0\cos\theta - \frac{1}{2}gt^2\, v_0\cos\theta\}\right)\hat{k}$$

$$\vec{L} = m\left(tv_0\cos\theta v_0\sin\theta - gt^2 v_0\cos\theta - tv_0\sin\theta v_0\cos\theta + \frac{1}{2}gt^2\, v_0\cos\theta\right)\hat{k}$$

$$\vec{L} = m\left(\frac{1}{2}gt^2 v_0\cos\theta - gt^2 v_0\cos\theta\right)\hat{k}$$

$$\vec{L} = mgt^2 v_0\cos\theta\left(\frac{1}{2} - 1\right)\hat{k}$$

$$\vec{L} = mgt^2 v_0\cos\theta\left(-\frac{1}{2}\right)\hat{k}$$

$$\vec{L} = -\frac{m}{2}gt^2 v_0\cos\theta\,\hat{k}$$

Here we are seeing that the angular momentum is proportional to time. It changes with t-square and is increasing in the negative k-direction because it has a minus angular momentum.

Torque (reminder):

As we know torque is given as,

$$\vec{\tau} = \vec{r} \times \vec{F}$$
$$\tau = rF\sin\theta$$

Relation between torque and angular momentum

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{L} + \Delta\vec{L} = \left(\vec{r} + \Delta\vec{r}\right) \times \left(\vec{p} + \Delta\vec{p}\right)$$

$$\vec{L} + \Delta\vec{L} = \vec{r} \times \vec{p} + \vec{r} \times \Delta\vec{p} + \Delta\vec{r} \times \vec{p} + \Delta\vec{r} \times \Delta\vec{p}$$
$$\vec{L} + \Delta\vec{L} = \vec{L} + \vec{r} \times \Delta\vec{p} + \Delta\vec{r} \times \vec{p}$$
$$\because \Delta\vec{r} \times \Delta\vec{p} = \text{ignored, very small term}$$

$$\Rightarrow \Delta\vec{L} = \vec{r} \times \Delta\vec{p} + \Delta\vec{r} \times \vec{p}$$

$$\frac{\Delta\vec{L}}{\Delta t} = \frac{\vec{r} \times \Delta\vec{p} + \Delta\vec{r} \times \vec{p}}{\Delta t} = \frac{\vec{r} \times \Delta\vec{p}}{\Delta t} + \frac{\Delta\vec{r} \times \vec{p}}{\Delta t}$$
$$= \vec{r} \times \frac{\Delta\vec{p}}{\Delta t} + \frac{\Delta\vec{r}}{\Delta t} \times \vec{p}$$

Take limit as $\Delta t \to 0$:

$$\because \quad \lim_{\Delta t \to 0} \frac{\Delta\vec{L}}{\Delta t} = \frac{d\vec{L}}{dt}$$

$$\Rightarrow \quad \frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$$

But $\dfrac{d\vec{r}}{dt}$ is $\vec{\mathrm{v}}$ and $\vec{p} = m\vec{\mathrm{v}}$!

$$\because \quad \frac{d\vec{r}}{dt} \times \vec{p} = \vec{\mathrm{v}} \times m\vec{\mathrm{v}} = m\left(\vec{\mathrm{v}} \times \vec{\mathrm{v}}\right) = 0$$

And we are left with only,

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$$

Now use Newton's second law:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$$

$$\therefore \quad \frac{d\vec{L}}{dt} = \vec{\tau}$$

$$\boxed{\frac{d\vec{L}}{dt} = \sum \vec{\tau}}$$

The net torque acting on a particle is equal to the time rate of change of its angular momentum.

The Spinning Top:

As we observed from experiment, the torque changes the direction but not the magnitude of the angular momentum and the motion is called precession, as shown in figure 13.3.

Let's do it mathematically, starting from torque.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

where $\vec{F} = m\vec{g}$, so

$$\tau = Mgr\sin\theta \quad \text{eq(1)}$$

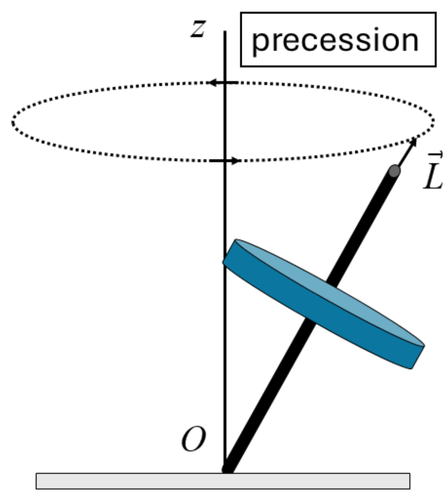


Figure 13.3: The precession about the z-axis.

$\vec{\tau}$ is perpendicular to $\vec{L}$ as shown in figure 13.4

$\therefore$ it cannot change the magnitude of $\vec{L}$!!

we know form figure 13.5

$$\Delta\vec{L} = \vec{\tau}\ \Delta t$$

using $\theta = \frac{s}{r}$, we get

$$\Delta\phi = \frac{\Delta L}{L\sin\theta}$$

or,

$$\Delta\phi = \frac{\tau\ \Delta t}{L\sin\theta} \quad \text{eq(2)}$$

Precession speed ω_P is:

$$\omega_P = \frac{\Delta\phi}{\Delta t}$$

using $\Delta\phi$ from eq(2)

$$= \frac{\tau\ \Delta t}{L\sin\theta}\frac{1}{\Delta t}$$

$$= \frac{\tau}{L\sin\theta}$$

using τ from eq(1)

$$= \frac{Mgr\sin\theta}{L\sin\theta}$$

$$\omega_P = \frac{Mgr}{L}$$

Here, precession is proportional to 1/L. So, as angular momentum decreses with time the precession increase.

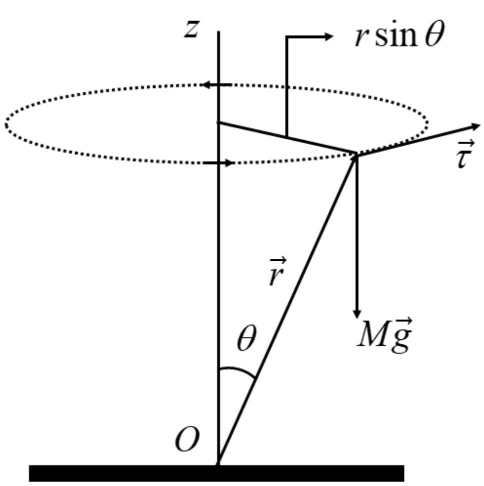


Figure 13.4: Torque, weight, and component of radius is being drawn.

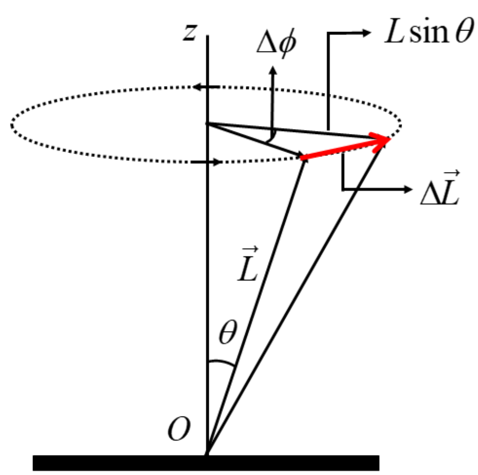


Figure 13.5: The incremental change in angular momentum and angular displacement is presented.

Example:

A mass m is released from a distance b along x-axis. A Position vector is r from origin O, as shown in the figure 13.6. Calculate torque and angular momentum.

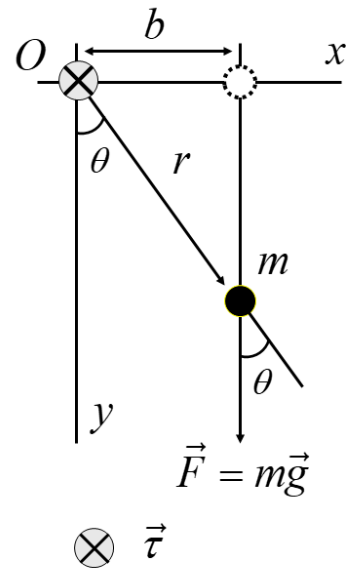


Figure 13.6: Direction of torque is into the paper.

Solution:

Let's calculate torque,

$$\vec{\tau} = \vec{r} \times \vec{F}$$
$$\tau = rF\sin\theta$$
putting $r\sin\theta = b$ and $F = mg$ from figure 13.6
we get
$$\tau = mgb$$

Right hand rule shows that $\vec{\tau}$ is directed inwards, as shown in figure 13.6.

Let's calculate angular momentum,

$$\vec{L} = \vec{r} \times \vec{p}$$
$$L = rp\sin\theta$$
putting $r\sin\theta = b$ and $p = m\mathrm{v} = mgt$
form figure 13.7, we get
$$L = mgbt$$

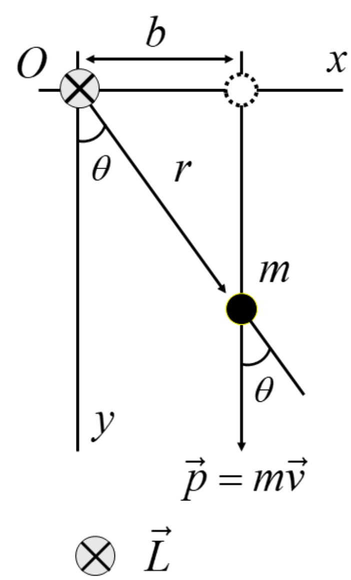


Figure 13.7: The direction of angular momentum is into the paper.

$\vec{L}$ is directed inwards, as shown in figure 13.7.

We know angular momentum and torque are connected, so let's calculate torque from angular momentum and check our expression.

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(mgtb)\hat{k}$$
$$= mgb\,\hat{k}$$
$$\tau = mgb\,\hat{k}$$

As we expected, it is the same.

Angular momentum for a system of particles:

Suppose we have large number of particles in a system, as shown in figure 13.8, to calculate its total angular momentum we do vector addition of angular momentum for individual particles.

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \cdots + \vec{L}_N = \sum_{n=1}^{N} \vec{L}_n$$

$$\frac{d\vec{L}}{dt} = \frac{d\vec{L}_1}{dt} + \frac{d\vec{L}_2}{dt} + \cdots + \frac{d\vec{L}_N}{dt} = \sum_{n=1}^{N} \frac{d\vec{L}_n}{dt}$$

Since $\dfrac{d\vec{L}_n}{dt} = \vec{\tau}_n$

$$\frac{d\vec{L}}{dt} = \sum_{n=1}^{N} \vec{\tau}_n$$

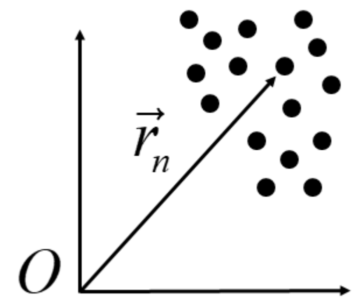


Figure 13.8: A general position vector for a system of particle.

Thus, the time rate of change of the total angular momentum of a system of particles equals the net torque acting on the system.

There are two sources of the torque acting on the system:

1) The torque exerted on the particles of the system by internal forces between the particles.

2) The torque exerted on the particles of the system by external forces.

$$\sum \vec{\tau} = \sum \vec{\tau}_{\text{int}} + \sum \vec{\tau}_{ext}$$

If the forces between two particles, as shown in figure 13.8 not only are equal and opposite but are also directed along the line joining the two particles, then the total internal torque is zero.

$$\sum \vec{\tau}_{\text{int}} = 0$$

$$\sum \vec{\tau}_{\text{int}} = \vec{\tau}_1 + \vec{\tau}_2$$

$$= \vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21}$$

but

$$\vec{F}_{12} = -\vec{F}_{21} = F\hat{r}_{12}$$

$$\therefore \sum \vec{\tau}_{\text{int}} = \left(\vec{r}_1 - \vec{r}_2\right) \times \vec{F}_{12} = \vec{r}_{12} \times \left(F\hat{r}_{12}\right)$$
$$= F\left(\vec{r}_{12} \times \hat{r}_{12}\right) = 0$$

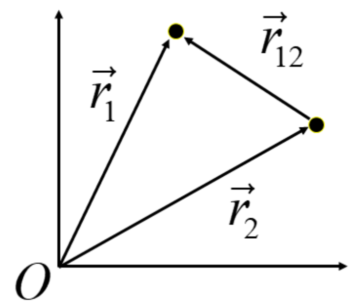


Figure 13.9: Two particles at position vector r_1 and r_2 respectively.

Hence

$$\sum \vec{\tau} = \sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$

The net external torque acting on a system of particles is equal to the time rate of change of the total angular momentum of the system.

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$

$$\sum \vec{F}_{ext} = \frac{d\vec{L}}{dt}$$

Conservation of Angular Momentum:

If no net external torque acts on the system, then the angular momentum of the system does not change with time.

$$\frac{d\vec{L}}{dt} = 0 \quad \Rightarrow \quad \vec{L} = \text{a constant}$$

Mathematical formulism of linear motion and angular motion shows some resemblance. As clear from eq(1) now go for angular momentum L since it is origin dependent while linear momentum p is not.

$$\vec{F} = \frac{d\vec{p}}{dt} \Leftrightarrow \vec{\tau} = \frac{d\vec{L}}{dt} \quad \text{eq(1)}$$

$$\vec{p} = m\vec{\mathrm{v}} \Leftrightarrow \vec{\mathrm{L}} = \;??$$

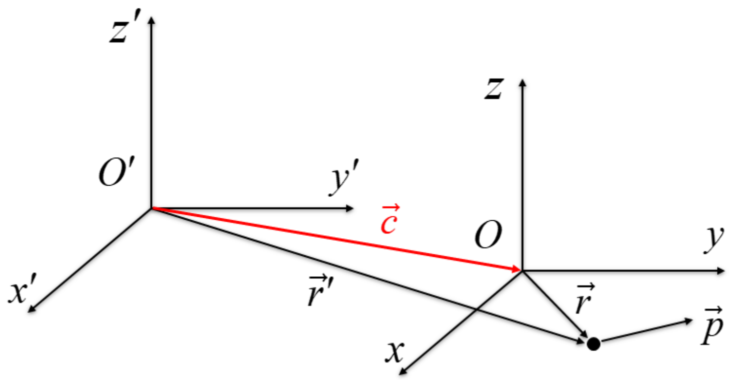


Figure 13.10: Angular momentum for two different origins is different.

$\vec{L}$ depends on the choice of the origin, from figure 13.10 we get

$\vec{L}' = \vec{r}' \times \vec{p} = (\vec{c} + \vec{r}) \times \vec{p} = \vec{c} \times \vec{p} + \vec{L}$

Rotation of Rigid Bodies:

A rigid body is an object whose shape doesn't deform under the influence of external forces. In other words, the distances between points on the object remain constant. Consider a rigid body that rotates about z-axis, as shown in figure 13.11. As reference line AP rotates through an angle all the points on it move with the same angular speed.

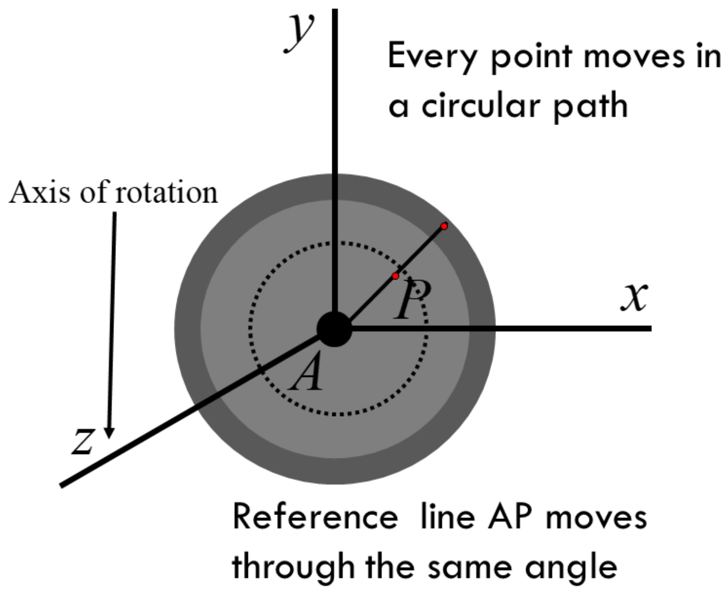


Figure 13.11: Rigid body rotates about z-axis.

Kinematics of a rigid body can be described by the motion of point P. There are two ways to understand one is by observing the motion of the point P as shown in figure 13.12 and the other is by observing the motion of cross-sectional slice of the rigid body, as shown in figure 13.13.

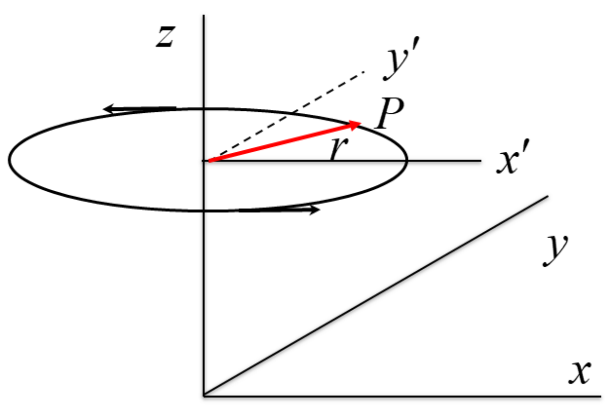


Figure 13.12: Motion of point P with translated coordinates.

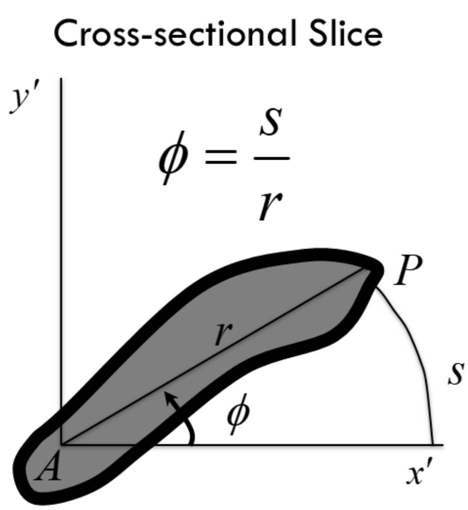


Figure 13.13: Motion of a cross-section of rigid body.

Linear and angular velocity:

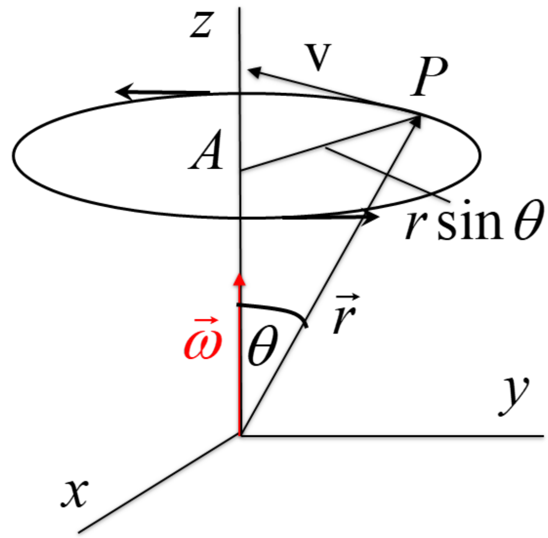


Figure 13.14: Linear and angular velocities of point P.

Let's look at linear and angular velocity of point P. Their vector representation along with components of position vector **r** is shown in figure 13.14.

$$\vec{v} = \vec{\omega} \times \vec{r} \quad \text{eq(1)}$$
$$v = \omega r \sin\theta$$

Linear and angular acceleration:

As we know acceleration is rate change of velocity so using eq(1) we can get,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\vec{\omega} \times \vec{r}\right)$$
$$= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$
$$= \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v} \quad \because \vec{\alpha} = \frac{d\vec{\omega}}{dt} \text{ and } \vec{v} = \frac{d\vec{r}}{dt}$$
$$\vec{a} = \vec{a}_T + \vec{a}_R$$

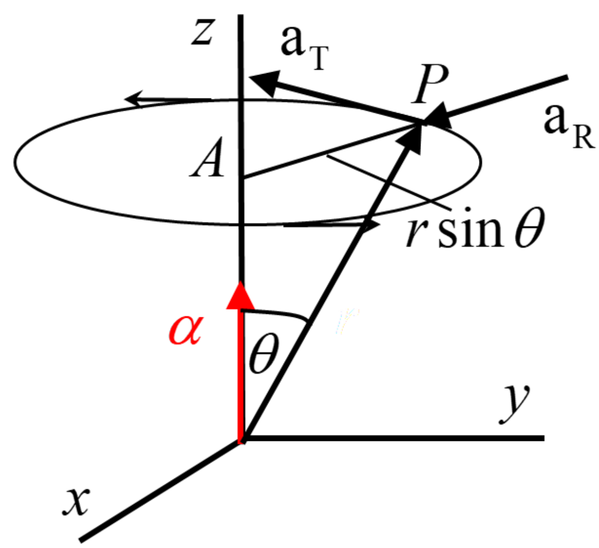


Figure 13.15: linear and angular acceleration of point P.

Where $\vec{a}_T$ and $\vec{a}_R$ are tangential and radial (towards the point A) components of acceleration, respectively, as shown in the figure 13.15.

Now let's discuss a special case in which there are two equal masses. In this way, the center of mass will be in between the two. Axis of rotation passes through the middle, as shown in figure 13.16. Here, the direction of angular momentum is along the z-axis same as of angular momentum.

$$\vec{L} = \left(2mr^2\right)\vec{\omega}$$

$$\vec{L} = I\vec{\omega}$$

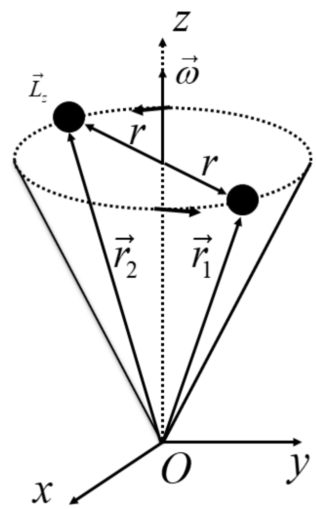


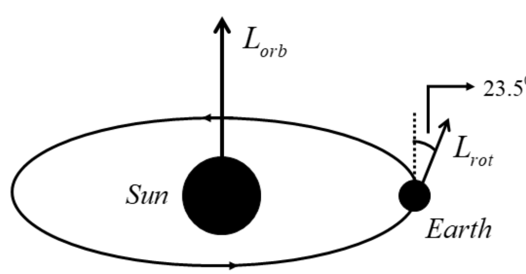
Figure 13.16: Two equal masses rotating about the z-axis.

Problem # 1:

Which is greater?
(a) The angular momentum of the Earth due to rotation on its axis.
(b) The angular momentum of the Earth due to its orbital motion around the sun.

Solution:

L_{rot} is the angular momentum of earth due to rotational motion about its own axis and L_{orb} is the angular momentum of earth due to the orbital motion about the sun. R_E is the radius of the earth and R_{orb} is the distance between the sun and the earth (clearly $R_{orb} \gg R_E$).



$$L_{rot} = I\omega = \left(\frac{2}{5}MR_E^2\right)\omega \quad \text{eq(1)} \quad \because I = \frac{2}{5}MR_E^2 \text{ for solid sphere}$$

$$L_{orb} = R_{orb}\,p = R_{orb}M\text{v} = R_{orb}M\left(R_{orb}\omega\right) = MR_{orb}^2\omega \quad \text{eq(2)}$$

By dividing eq(2) by eq(1), we get

$$\frac{L_{orb}}{L_{rot}} = \frac{MR_{orb}^2\omega}{\left(\frac{2}{5}MR_E^2\right)\omega}$$

$$L_{orb}/L_{rot} = \frac{5}{2}\left(\frac{R_{orb}}{R_E}\right)^2$$

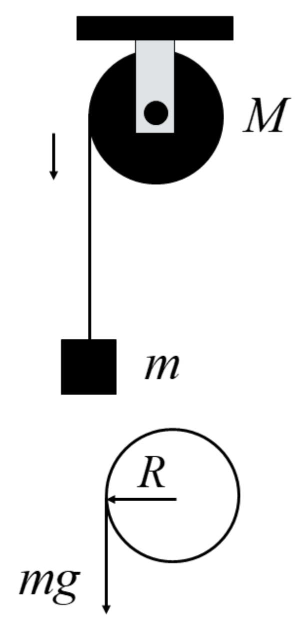
we know, $R_{orb} \gg R_E$ or $\frac{R_{orb}}{R_E} \gg 1$ so,

$$L_{orb}/L_{rot} = \frac{5}{2}\left(\frac{R_{orb}}{R_E}\right)^2 \gg 1 \Rightarrow L_{orb} \gg L_{rot}$$

The angular momentum of the earth due to its orbital motion aroud the sun is much greater than the angular momentum of the earth due to rotation about its axis because of the fact that the distance between the sun and the earth is much larger than the radius of the earth.

Problem # 2:

A mass m is tied to a pulley of mass M with a massless rope, as shown. As mass m moves down, the pully starts rotating. What is angular acceleration?



Solution:

Total angular momentum of system consists of two sections, one is due to pully and other is due to mass m. Origin is at center of the pully so,

$$L = I\omega + m\text{v}R$$

We also know,

$$\tau = \frac{dL}{dt} \quad \because \tau = \left(mg\right)R$$

We get,

$$(mg)R = \frac{d}{dt}(I\omega + m\text{v}R)$$

$$(mg)R = I\left(\frac{d\omega}{dt}\right) + mR\left(\frac{d\text{v}}{dt}\right)$$

$$(mg)R = I\alpha + mRa \quad \because \alpha = \frac{d\omega}{dt} \text{ and } a = \frac{d\text{v}}{dt}$$

$a = \alpha R$ or $\frac{a}{R} = \alpha$ $\because$ relation between linear and angular accleration

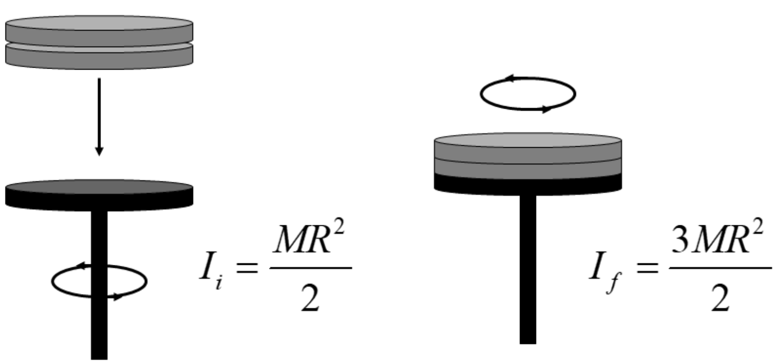
$I = \frac{1}{2}MR^2$ $\because$ moment of inertia of pully

$$\Rightarrow mgR = \left(\frac{1}{2}MR^2\right)(a/R) + mRa$$

$$a = \frac{2mg}{M + 2m}$$

Problem # 3:

A disc which comes rotating and gets attached to another disc below. This upper disc which is twice as heavy as the lower disc and twice as big, when it gets attached to the lower disc, then these three discs start rotating together. What is the angular velocity?



Solution:

Here, angular momentum is conserved because there is no external torque. So,

$$I_i\omega_i = I_f\omega_f$$
$$\Rightarrow\ \omega_f = \omega_i\left(\frac{I_i}{I_f}\right)$$

$\because$ For one disc $I_i = I = \frac{1}{2}MR^2$ and for three discs $I_f = I = \frac{3}{2}MR^2$

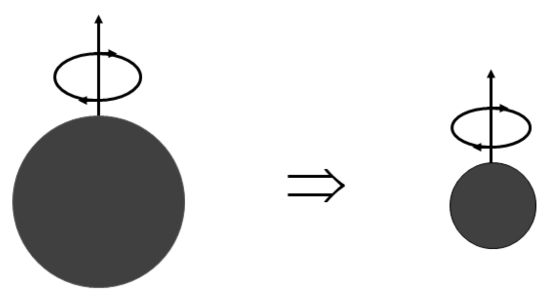
$$\omega_f = \omega_i\left(\frac{MR^2}{2}\times\frac{2}{3MR^2}\right)$$
$$\omega_f = \frac{1}{3}\omega_i$$

So, the final angular velocity is one third of the initial angular velocity.

Additional Problems:

Problem # 1:

If the radius of the earth suddenly shrinks to half its present value, while the mass of the Earth remains unchanged, what will be the duration of one day where earth assumed to be a perfect sphere?



Solution:

Applying the law of conservation of angular momentum

$$I_i\omega_i = I_f\omega_f$$

$\because$ Moment of inertia of a solid sphere $I = \frac{2}{5}MR^2$

$\because\ \omega = \frac{\text{angular displacment}}{\text{time}} = \frac{2\pi}{T}$

$$\left(\frac{2}{5}MR_i^2\right)\left(\frac{2\pi}{T_i}\right) = \left(\frac{2}{5}MR_f^2\right)\left(\frac{2\pi}{T_f}\right)$$

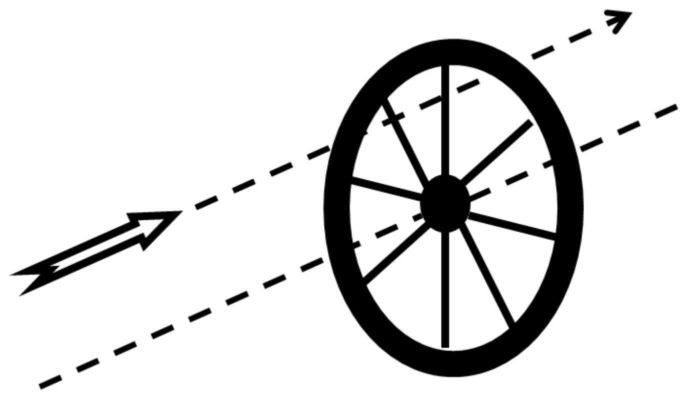
$$\frac{R_i^2}{T_i} = \frac{R_f^2}{T_f} \quad or \quad T_f = \left(\frac{R_f}{R_i}\right)^2 T_i = \left(\frac{1}{2}\right)^2 (24\text{hours})$$

$$T_f = \left(\frac{1}{4}\right)(24\text{hours}) = 6 \text{ hours}$$

So, duration of one day will be 6 hours.

Problem # 2:

A wheel has eight spokes and a radius of 30 cm. It is mounted on a fixed axle and is spinning at 2.5 rev/s. You want to shoot a 24 cm arrow parallel to this axle and through the wheel without hitting any of the spokes. Assume that the arrow and the spokes are very thin. (a) What minimum speed must the arrow have? (b) Does it matter where between the axle and the rim of the wheel you aim? If so, where is the best location?



Solution:

$$\text{minimum speed} = \frac{\text{length of the arrow}}{\text{time to pass one spoke}}$$

$$\text{s = distance traveled by one spoke} = \frac{2\pi \text{r}}{8}$$

$$\text{time to pass one spoke} = \frac{\text{distance traveled by one spoke}}{\text{speed of spoke}}$$

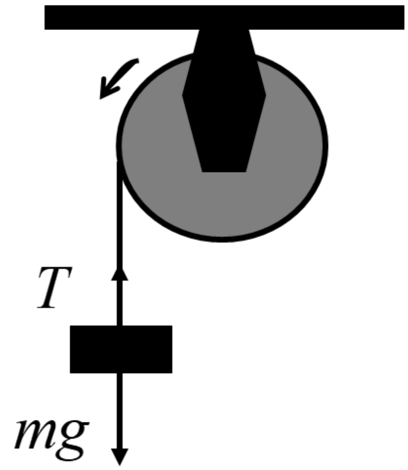
$$\text{time to pass one spoke} = \frac{\text{s}}{\text{v}} = \frac{2\pi \text{r}}{8\text{r}\omega}$$

$$\text{So minimum speed} = \frac{\ell \times 8\text{r}\omega}{2\pi \text{r}} = 4.8\,\text{m/s}$$

Does not matter where we aim!!

Problem # 3:

A disk of mass M = 2.5 kg and radius R = 20 cm is mounted on a fixed horizontal axle. A block of mass m = 1.2 kg hangs from a light cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the tension in the cord, and the angular acceleration of the disk.



Solution:

$$\sum F = mg - T = m\mathrm{a}$$

$$m\mathrm{a} = mg - T \ \ \mathrm{eq(1)}$$

$$\because \tau = \mathrm{r\ F\ also}\ \ \tau = \mathrm{I}\alpha$$

$$\sum \tau = TR \ \ \mathrm{eq(2)}$$

$$\sum \tau = \frac{1}{2} MR^2 \left(\frac{\mathrm{a}}{R}\right) \ \ \mathrm{eq(3)} \quad \because \mathrm{a} = R\alpha$$

from eq(2) and eq(3) we get

$$TR = \frac{1}{2} MR^2 \left(\frac{\mathrm{a}}{R}\right)$$

$$T = \frac{1}{2} M\mathrm{a} \ \ \mathrm{eq(4)}$$

substituting eq(4) into eq(1), we get

$$m\mathrm{a} = mg - \frac{1}{2} M\mathrm{a}$$

$$m\mathrm{a} + \frac{1}{2} M\mathrm{a} = mg$$

$$\mathrm{a}\,\frac{M + 2m}{2} = mg$$

$$\mathrm{a} = g\,\frac{2m}{M + 2m} \ \ \mathrm{eq(5)}$$

$$\mathrm{a} = 4.8\,\mathrm{m/s^2}$$

putting eq(5) into eq(4), we get

$$T = mg\,\frac{M}{M + 2m}$$

$$T = 6.0N$$

$$\alpha = \frac{\mathrm{a}}{R} = 3.8\,\mathrm{rev/s^2}$$