# Lecture#15 PHY101

## Oscillations

Oscillation and vibration are fundamental physical phenomena. Oscillation is a back-and-forth movement, like stirring a pan or playing cards. It's all around us, seen in a swinging pendulum or a plucked guitar string. Even a floating object on water demonstrates oscillatory motion. This repetitive pattern continues unless acted upon by outside forces. For example, a spring with added weight will oscillate continuously. While some oscillations are visible, others need special tools like an oscilloscope to detect. Interestingly, the quartz crystal in a watch uses oscillations for accurate timekeeping.

We're exploring the idea of oscillation.

The **time period** is the amount of time it takes to reach a certain point again, often referred to as "T". It's the time it takes to complete one full cycle. The time period is measured in seconds.

Now, let's talk about **frequency** - it's closely related to the time period. Frequency is how often a cycle happens in one second. It's the inverse of the time period and is measured in hertz (Hz).

Finally, there's **amplitude** - this is the extent or size of the oscillation. Amplitude reflects how much the movement varies. A larger amplitude means a greater range of motion.

Let's understand these ideas using **a simple pendulum**. Picture a weight hanging from a string. When you move it sideways and it swings back, that's its time period - the time it takes to complete one full swing. Now, how many times does it swing back and forth in a second? That's the frequency. Imagine it takes 3 seconds to complete a full swing; that means the frequency is one-third swing per second. Finally, consider the amplitude - how far the pendulum swings. Notice how changing the amplitude changes the motion. In an ideal situation without air resistance, the pendulum would keep swinging forever. But air resistance gradually drains its energy over time, causing it to eventually stop.

Now, let's e-xplore the concept of **equilibrium position** - the point where a body naturally stays. Any force- acting on it near this position causes it to oscillate. Consider a pendulum: when displaced, gravity pulls it towards equilibrium, accelerating its motion. As it moves away, its velocity decreases. Notice, the pivotal force always directs towards equilibrium, governing the body's movement dynamics.

In the pendulum example, we observed that the force is directly proportional to the height it moves. When this force aligns horizontally, it's called a **linear restoring force**. However, if it's proportional to $x^2$, it pulls in the opposite direction. Whether the force is linear depends on not over stretching the spring. A mass attached to a spring as shown in Fig 15.1 will cause the system to oscillate, as the mass experiences the force of gravity. Let's see this in action by adding a mass to the spring.

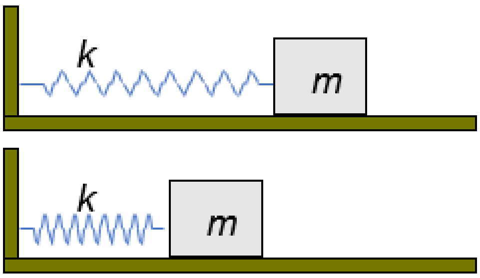


**Fig 15. 1:** Mass spring system

Now, observe that the system is at a balanced position when still. When released, it starts moving up and down in a repeating cycle. We call this the equilibrium point, which we'll mark as $x = 0$. As we change the mass attached to the spring, we notice the frequency of the oscillations changing. A heavier mass requires more force to stretch the spring, affecting the balance between gravity and spring force. This causes the oscillation frequency to decrease. This demonstrates how adjusting the mass can alter the oscillation frequency, an important concept to understand in oscillatory systems.

**What describes oscillations?**

- *period* T = time for completing one cycle
- *frequency* f = 1 / T
- *amplitude* A = maximum displacement from *equilibrium* position

**Restoring force**

- Suppose a force is always directed towards a central *equilibrium* position
- Force always acts to return the object to its equilibrium position
- The object will oscillate around the equilibrium position
- This “back-and-forth” motion around an equilibrium position is called: Periodic motion.

**Periodic motion**

- Simple Harmonic Motion (SHM)
- The restoring force depends on the displacement $\Delta x$

$$F_{\text{restore}} = -k\,\Delta x$$

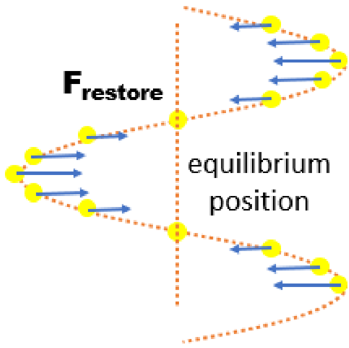


**Fig 15. 2:** Restoring force of a body at its equilibrium position.

**Derivation of Equation of Motion for a Mass-Spring System**

**1. Hooke's Law**

Hooke's Law describes the behaviour of an ideal spring:

$$F(x) = -kx$$

where:

- $F(x)$ is the force exerted by the spring
- $k$ is the spring constant (a measure of the stiffness of the spring)
- $x$ is the displacement from the equilibrium position.

Hooke's Law tells us that the force exerted by the spring is directly proportional to the displacement from its equilibrium position, with the negative sign indicating that the force is opposite to the direction of displacement.

**2. Stored Energy in the Spring**

When a spring is stretched or compressed, it stores potential energy. The amount of energy stored in the spring is given by:

$$U(x) = \frac{1}{2}kx^2$$

where:

- $U(x)$ is the potential energy stored in the spring
- $k$ is the spring constant, and
- $x$ is the displacement from the equilibrium position.

This equation shows that the potential energy stored in the spring is directly proportional to the square of the displacement from its equilibrium position.

### 3. Equating Force and Acceleration

According to Newton's Second Law, the force acting on an object is equal to the product of its mass and acceleration:

$$F = ma$$

Substituting $F = -kx$ (from Hooke's Law), we have:

$$ma = -kx$$

This equation relates the acceleration *(a)* of the mass to the force *(F)* exerted by the spring.

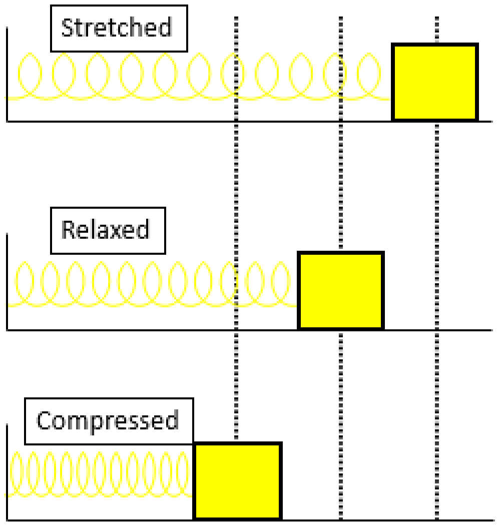


**Fig 15. 3:** A spring at equilibrium, neither compressed nor stretched. A block of mass m on a frictionless surface is pushed against the spring. If x is the compression in the spring, the potential energy stored in the spring is ½ $kx^2$. When the block is released, this energy is transferred to the block in the form of kinetic energy.

### 4. Differential Equation of Motion

By expressing acceleration as the second derivative of displacement with respect to time, we obtain:

$$a = \frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} = -kx$$

This equation represents the differential equation of motion for the mass-spring system, where:

- $m$ is the mass of the object,
- $\frac{d^2x}{dt^2}$ is the acceleration of the mass, and
- $-kx$ is the restoring force exerted by the spring.

### 5. Solving the Differential Equation

Dividing both sides by $m$, we get:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

This is a second-order linear homogeneous ordinary differential equation with constant coefficients. Its solution provides the equation of motion for the mass-spring system, describing how the displacement ($x$) of the mass varies with time during simple harmonic motion.

**<u>Simple Harmonic Motion</u>**

**1. Equation of Motion for Simple Harmonic Oscillator:**

We start with the general equation of motion for a simple harmonic oscillator:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

This equation represents the second-order linear ordinary differential equation governing the motion of the oscillator.

**2. Rearrangement:**

We can rearrange the equation by moving the term involving $x$ to one side:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

This equation shows that the acceleration of the oscillator ($\frac{d^2x}{dt^2}$) is directly proportional to its displacement ($x$) from the equilibrium position, with a negative sign indicating that the acceleration is opposite to the displacement.

**3. Introduction of Angular Frequency:**

We introduce the concept of angular frequency ($\omega$) for the oscillator:

$$\omega^2 = \frac{k}{m}$$

The angular frequency ($\omega$) represents the rate at which the oscillator oscillates back and forth. It is related to the stiffness of the oscillator (determined by $k$) and its mass (determined by $m$). Squaring both sides simplifies the equation.

How to calculate $\frac{\mathrm{d}}{\mathrm{dt}}\cos\omega t$ ?

$x(t) = \cos\omega t$

$x(t+\Delta t) = \cos\omega(t+\Delta t)$

Differece between $x(t+\Delta t)$ and x(t)

Apply Trigonometric Identity (Angle Addition Formula for Cosine):

$$\cos(A+B) - \cos(A) = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Applying this formula with $A = \omega t + \omega\Delta t$ and $B = -\omega t$

$$\cos(\omega t + \omega\Delta t) - \cos(\omega t) = -2\sin\left(\frac{\omega t + \omega\Delta t - (-\omega t)}{2}\right)\sin\left(\frac{\omega t + \omega\Delta t - \omega t}{2}\right)$$

Simplifying the expression inside the sine functions:

$$\sin\left(\frac{\omega t+\omega\Delta t+\omega t}{2}\right)\sin\left(\frac{\omega\Delta t}{2}\right)=\sin(\omega t+\omega\Delta t/2)\sin(\omega\Delta t/2)$$

Approximation:

Since $\Delta t$ is very small, $\omega\Delta t$ is slso very small.

For small angles, $\sin(\theta)\approx\theta$

So, $\sin(\omega\Delta t/2)\approx\omega\Delta t/2$

Therefore, $\sin(\omega t+\omega\Delta t/2)\sin(\omega\Delta t/2)$ is approximated as $(\omega\Delta t/2)^2$,

and the negative sign from the trigonometric identity is retained.

Result:

Substituting the approximation into the equation:

$$x(t+\Delta t)-x(t)\approx-\sin\omega\Delta t\cdot\sin(\omega t+\omega\Delta t/2)\approx-\omega\Delta t\cdot\sin\omega t$$

$$\therefore\frac{\mathrm{d}}{\mathrm{dt}}\cos\omega t=-\omega\sin\omega t$$

Why is $\sin\theta\approx\theta$ for small $\theta$?

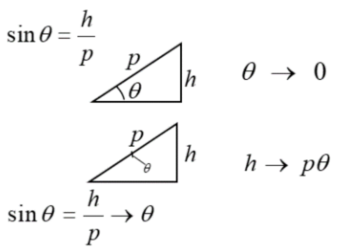


**Fig 15. 4:** For small angles, $\sin\theta$ is approximately equal to $\theta$ due to the linear behaviour near the origin."

Remember two important results:

$$\frac{d}{dt}(\sin\omega t)=\omega\cos\omega t$$

$$\frac{d}{dt}(\cos\omega t)=-\omega\sin\omega t$$

What happens if you differentiate twice?

$$\frac{d^2}{dt^2}(\sin\omega t)=\omega\frac{d}{dt}\cos\omega t$$
$$=-\ \omega^2\sin\omega t$$

$$\frac{d^2}{dt^2}(\cos\omega t)=-\omega\frac{d}{dt}\sin\omega t$$
$$=-\ \omega^2\cos\omega t$$

Any function of the form:

$x=a\cos\omega t+b\sin\omega t$

is a solution of $\dfrac{d^2x}{dt^2}=-\omega^2x$

Physical significance of constant $\omega$

$$x=x_m\cos\omega\left(t+\frac{2\pi}{\omega}\right)$$

$$x=x_m\cos(\omega t+2\pi)$$

$$x=x_m\cos\omega t$$

That is, the function merely repeats itself after a time $2\pi / \omega$

So $2\pi / \omega$ is the period of the motion $T$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

The frequency $\nu$ of the oscillator is the number of complete vibrations per unit time:

$$\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$\omega = 2\pi\nu = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

$\omega$ is called the angular frequency

$$\dim[\omega] = T^{-1}$$

Unit of $\omega$ is radian/second

$$x(t) = a\cos\omega t + b\sin\omega t$$

$$\mathrm{x(0)=a}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}x(t) = -\omega a\sin\omega t + \omega b\cos\omega t$$

$$= \omega b \text{ (at t=0)}$$

This solution can also be written as:

$$x(t) = x_m\cos(\omega t + \phi)$$

Physical significance of constant $x_m$

$$x = x_m\cos(\omega t + \phi)$$

$$\Rightarrow \ -x_m \le x \le +x_m$$

$x_m$ is called the amplitude of the motion

The frequency of the simple harmonic motion is independent of the amplitude of the motion

**Phase**

We're exploring the concept of "phi" (a Greek letter) in sinusoidal waves. Phi represents a phase shift, which changes where the wave begins on the vertical axis. When we assign differe-nt constant values to phi, we see distinct changes in the waveform. For example, setting phi to

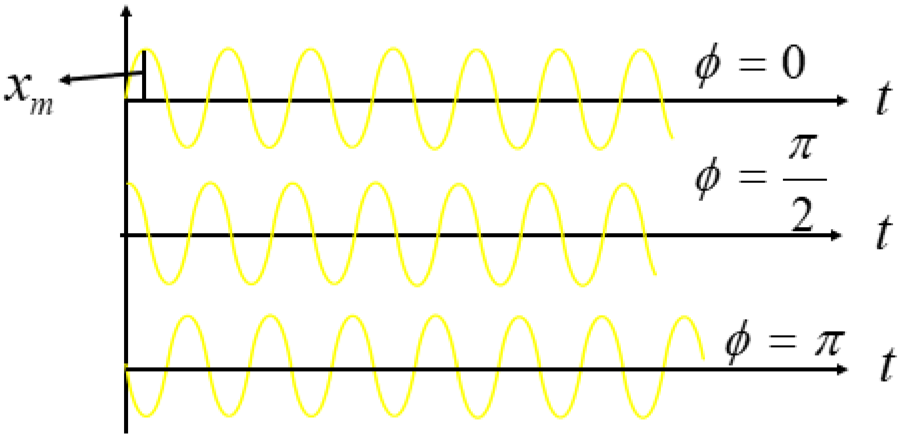


**Fig 15. 5**: Impact of Phase Shift (Phi) on Sinusoidal Waves: Different constant values of phi shift waveforms along the vertical axis."

zero gives us a standard sine wave with a frequency denoted by omega as shown in Fig.15.5. Increasing phi shifts the wave along the axis, moving its starting point from a peak to a trough. We can visually see this shift through plots. Additionally, differentiating the wave reveals its acceleration, which is linked to omega and the displacement amplitude. By exploring phi, we grasp its significant role in shaping sinusoidal waves and its importance in wave analysis.

$$x = x_m \sin(\omega t + \phi)$$

The quantity $\theta = \omega t + \phi$ is called the phase of the motion. The constant $\phi$ is called the phase constant.

## Energy of simple harmonic motion

### 1. Expressing Displacement ($x$):

In simple harmonic motion, the object oscillates back and forth around a central point as shown in Fig 15.6. We express this oscillation as $x = x_m \cos(\omega t)$

Where:

$x$ is the displacement from the central point at time $t$.

$x_m$ is the amplitude, representing the maximum displacement from the central point.

$\omega$ is the angular frequency, determining the rate of oscillation.

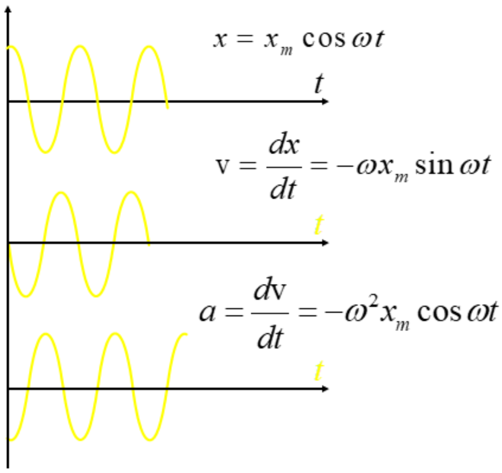


**Fig 15. 6:** Representation of Displacement in Simple Harmonic Motion: Explaining the components of displacement (x) in the oscillatory motion.

### 2. Calculating Potential Energy ($U$):

The potential energy of the oscillator, $U$, arises from the restoring force exerted by the spring or restoring element. We use Hooke's Law to calculate it:

$$U = \frac{1}{2} k x^2$$

Substituting $x = x_m \cos(\omega t)$, we find $U = \frac{1}{2}kx_m^2 \cos^2(\omega t)$

This equation shows how the potential energy of the oscillator varies with its displacement from the central point.

### 3. Calculating Kinetic Energy (*K*):

The kinetic energy of the oscillator, *K*, arises from its motion. We calculate it using the formula:

$$K = \frac{1}{2}mv^2$$

Since velocity $v = \frac{dx}{dt}$ , and $x = x_m \cos(\omega t)$, we find $v = -x_m \omega \sin(\omega t)$

Substituting this into the equation for kinetic energy, we find

$$K = \frac{1}{2}m\omega^2 x_m^2 \sin^2(\omega t)$$

This equation illustrates how the kinetic energy of the oscillator varies with its velocity and displacement.

### 4. Calculating Total Mechanical Energy (*E*):

The total mechanical energy (*E*) of the oscillator is the sum of its potential and kinetic energies:

$$E=K+U$$

Substituting the expressions for *K* and *U*, we find

$$E = \frac{1}{2}kx_m^2 \cos^2(\omega t) + \frac{1}{2}kx_m^2 \sin^2(\omega t)$$

Since $\cos^2(\omega t) + \sin^2(\omega t) = 1$, we find $E = \frac{1}{2}kx_m^2$

This equation shows that the total mechanical energy of the oscillator remains constant throughout its motion, illustrating the principle of energy conservation as shown in Fig 15.7.

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2$$

$$\mathrm{v} = \frac{dx}{dt} = \pm\sqrt{\frac{k}{m}\left(x_m^2 - x^2\right)}$$

speed is maximum at x=0

speed is zero at $x = \pm x_m$

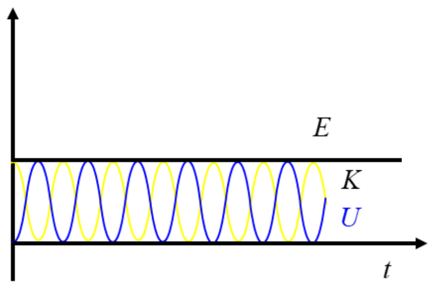


**Fig 15. 7:** Total mechanical energy of the oscillator remains constant.

**Problem: 15.1:** Two spring systems, illustrated in Fig 15.8, require determination of their respective time period.

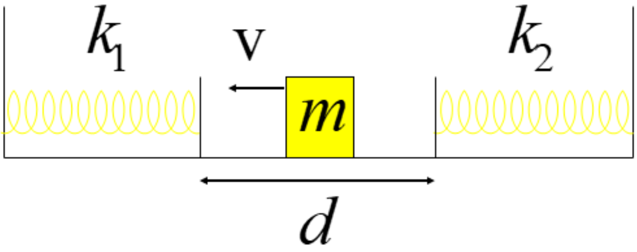


**Fig 15. 8:** Illustration of Two Spring System.

**Solution:**

$T = t_1 + t_2 + t_3$

$t_1 =$ period of oscillation of spring $k_1$

$t_2 =$ period of oscillation of spring $k_2$

$t_3 =$ time to cover the distance $d$

$$T = 2\pi\sqrt{\frac{m}{k_1}} + 2\pi\sqrt{\frac{m}{k_2}} + \frac{2d}{\text{v}}$$

**<u>Springs Coupled in Series</u>**

Consider a system where two springs, each with spring constants $k_1$ and $k_2$, are connected end to end, such that they act in series.

**1. Total Displacement ( $y$ )**

When a force (*F*) is applied to the system, it causes a total displacement ( $y$ ) in the system, which can be expressed as the sum of the displacements ( $y_1$ and $y_2$ ) caused by each spring as shown in Fig 15.9:

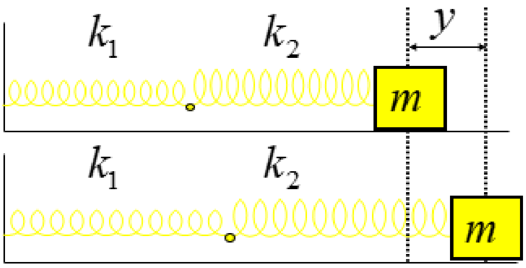


**Fig 15. 9**: Illustration of Springs Coupled in Series.

$$y = y_1 + y_2$$

**2. Hooke's Law for Each Spring**

According to Hooke's law, the force exerted by each spring (*F*) is proportional to its displacement ( $y_1$ and $y_2$ ). Therefore:

$$F = -k_1 y_1 = -k_2 y_2$$

where $k_1$ and $k_2$ are the spring constants of the respective springs.

**3. Displacement of Each Spring**

Solving for the displacements $y_1$ and $y_2$ in terms of the force *F*:

$$y_1 = -\frac{F}{k_1}\ , y_2 = -\frac{F}{k_2}$$

**4. Total Displacement and Effective Spring Constant**

Substituting the expressions for $y_1$ and $y_2$ into the total displacement equation, we get:

$$y = -F\left(\frac{1}{k_1} + \frac{1}{k_2}\right) = -F\left(\frac{k_1 + k_2}{k_1 k_2}\right)$$

This expression shows the total displacement $y$ in terms of the force $F$ and the inverse of the sum of the reciprocals of the spring constants $k_1$ and $k_2$.

### 5. Effective Spring Constant ($k_{eff}$)

The effective spring constant ($k_{eff}$) of the coupled springs can be obtained by equating the force $F$ to $-k_{eff}\, y$, which yields:

$$F = -\left(\frac{k_1 k_2}{k_1 + k_2}\right) y \;,\; k_{eff} = \left(\frac{k_1 k_2}{k_1 + k_2}\right)$$

This equation provides a way to calculate the effective spring constant of the series-coupled springs based on their individual spring constants.

### 6. Period of Oscillation ($T$)

Finally, the period of oscillation ($T$) of the system can be determined using the effective spring constant ($k_{eff}$) and the mass ($m$) connected to the springs:

$$\frac{1}{k_{eff}} = \frac{1}{k_1} + \frac{1}{k_2} \;,\; T = 2\pi\sqrt{\frac{m}{k_{eff}}}$$

This equation relates the period of oscillation to the effective spring constant and the mass of the system, providing insight into the dynamics of the coupled springs in series.

**<u>Springs in parallel</u>**

### 1. Introduction to Displacement ($x$):

We begin by defining $x$ as the displacement of the mass from its equilibrium position.

### 2. Force Exerted by Spring 1 ($F_1$)

Spring 1 exerts a force ($F_1$) on the mass, given by Hooke's Law:

$F_1 = -k_1 x$

Here,

$k_1$ represents the spring constant of spring 1.

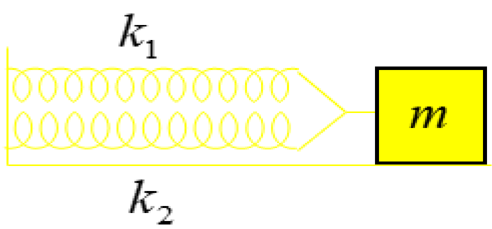


**Fig 15. 10:** Illustration of Springs in parallel.

### 3. Force Exerted by Spring 2 ($F_2$)

Similarly, spring 2 exerts a force ($F_2$) on the mass, also given by Hooke's Law:

$$F_2 = -k_2 x$$

$k_2$ is the spring constant of spring 2.

**4. Total Force ($F$):**

The total force ($F$) exerted on the mass is the sum of the forces exerted by both springs:

$$F = F_1 + F_2 = -(k_1 + k_2)x$$

This equation shows that the total force is the sum of the forces from each spring acting in the same direction.

**5. Effective Spring Constant ($k_{eff}$)**

We define an effective spring constant ($k_{eff}$), representing the combined stiffness of the two springs:

$$k_{eff} = k_1 + k_2$$

This effective spring constant accounts for the total restoring force acting on the mass due to both springs.

**6. Angular Frequency ($\omega$):**

The angular frequency ($\omega$) of the system is given by:

$$\omega = \sqrt{\frac{k_1 + k_2}{m}} = \sqrt{\frac{k_{\text{eff}}}{m}}$$

It represents the rate at which the mass oscillates back and forth under the influence of the combined springs.

**<u>Torsional Oscillator</u>**

Consider a system where a wire or a shaft is twisted, exerting a restoring torque ($\tau$) proportional to the angle of twist ($\theta$) as shown in Fig 15. The constant of proportionality ($\kappa$), known as the torsional constant, depends on the properties of the wire or shaft.

**1. Expression for Torque ($\tau$)**

The torque ($\tau$) acting on the system can be expressed as:

$$\tau = -\kappa\theta$$

Here, $\theta$ represents the angular displacement from the equilibrium position.

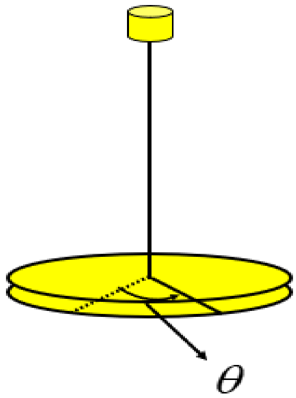


**Fig 15. 11:** Illustration of a Torsional System with Expression for Torque.

**2. Newton's Second Law for Rotation**

Applying Newton's second law for rotation, we equate the torque ($\tau$) to the moment of inertia ($I$) times the angular acceleration ($\alpha$):

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2}$$

This equation relates the torque to the angular acceleration, where $I$ is the moment of inertia of the system.

**3. Equating Torque and Angular Acceleration**

Equating the expressions for torque and angular acceleration, we get:

$$-\kappa\theta = I\frac{d^2\theta}{dt^2}$$

**4. Differential Equation for Angular Displacement**

Rearranging terms, we obtain the differential equation for angular displacement:

$$\frac{d^2\theta}{dt^2} = -\left(\frac{\kappa}{I}\right)\theta$$

This equation governs the motion of the torsional oscillator, relating the angular displacement ($\theta$) to its second derivative with respect to time.

**5. Solution for Angular Displacement**

Assuming harmonic motion, we propose a solution of the form:

$$\theta = \theta_m \cos\omega t$$

Here,

$\theta_m$ represents the maximum angular displacement, and $\omega$ is the angular frequency of oscillation.

**6. Angular Frequency ($\omega$)**

Substituting the proposed solution into the differential equation, we find the angular frequency ($\omega$) of the oscillation:

$$\omega = \sqrt{\frac{\kappa}{I}}$$

This equation gives the rate at which the angular displacement oscillates back and forth, determined by the torsional constant ($\kappa$) and the moment of inertia ($I$) of the system.