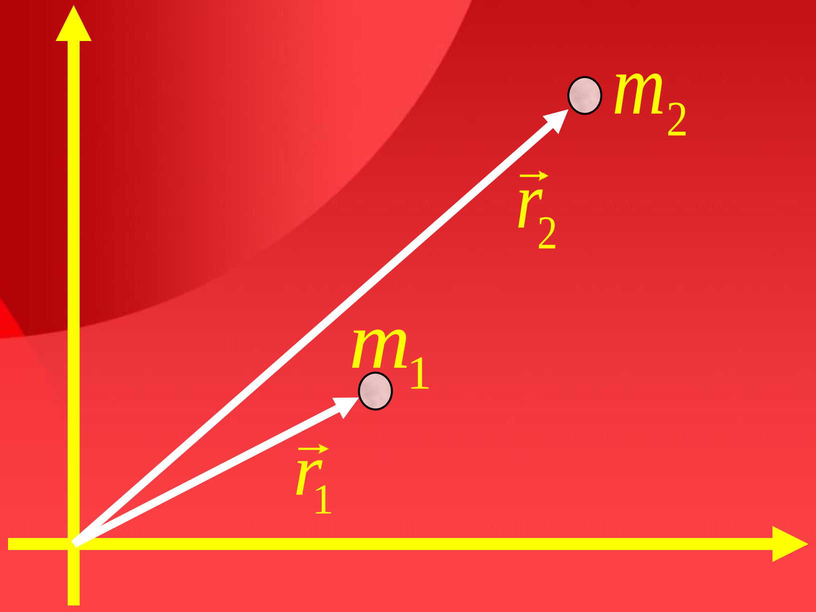
Physics

Of Many Particles

For two masses the centre of mass is:

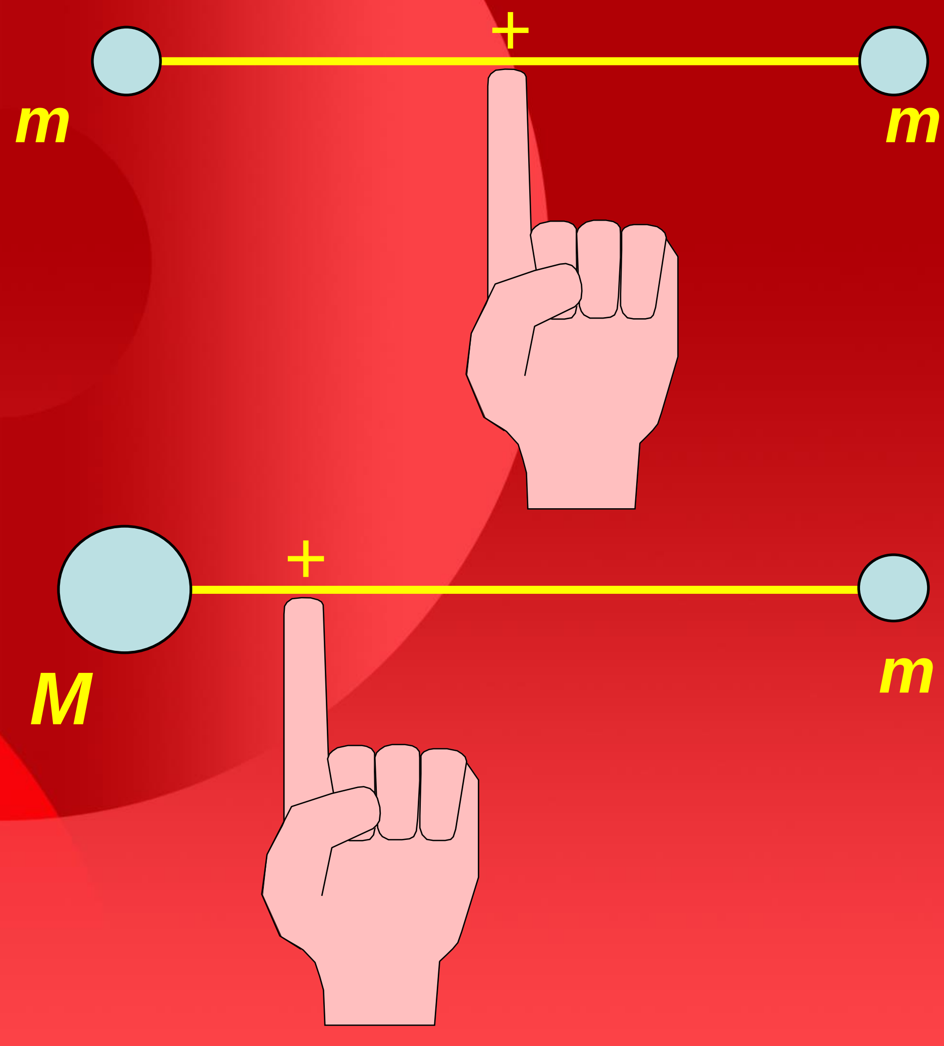
$$\vec{r}_{cm} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

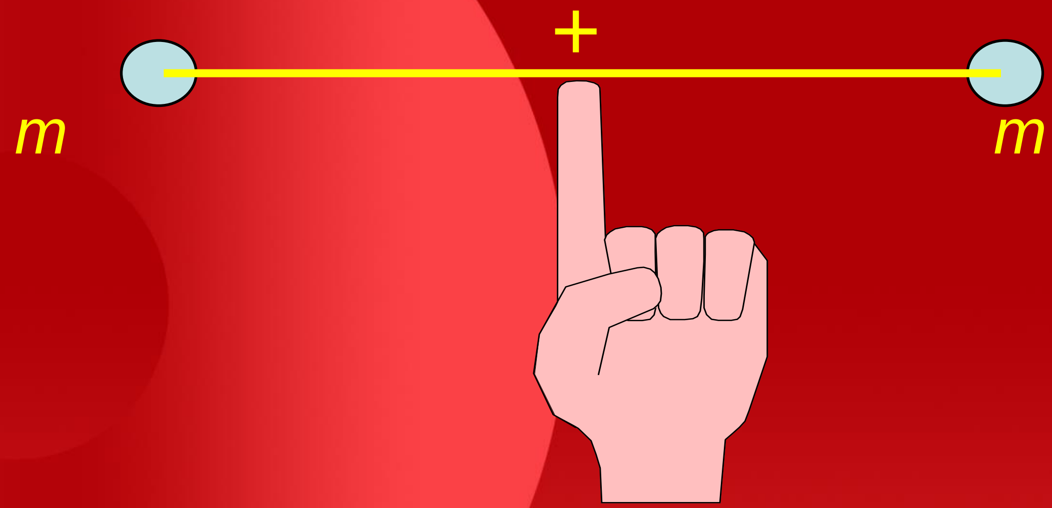


$$\vec{r}_{cm} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

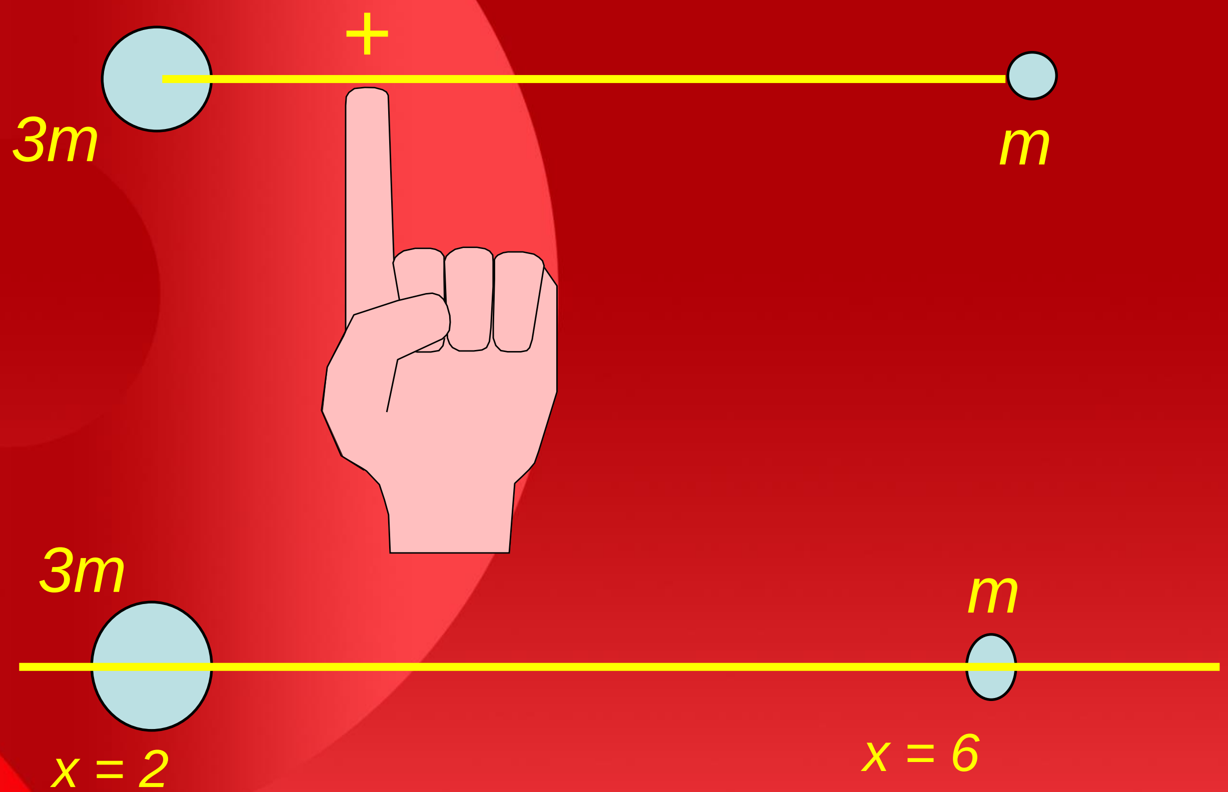
$$\Rightarrow \qquad x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$


+
m
m
+
M
m



$$x_{cm} = \frac{mx_1 + mx_2}{m + m} = \frac{2m + 6m}{2m} = 4$$

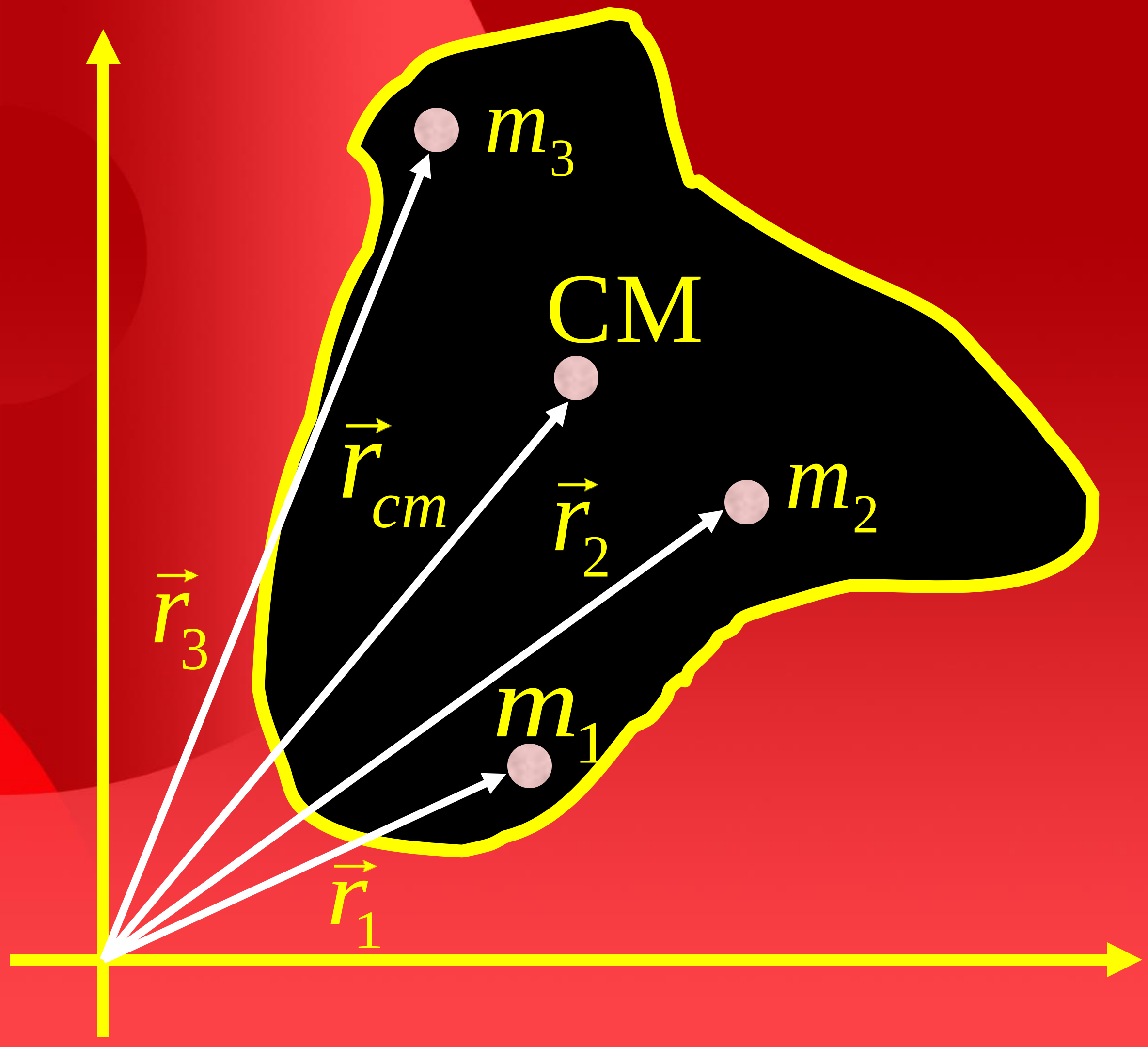


$$x_{cm} = \frac{(3m)x_1 + mx_2}{3m + m} = \frac{2(3m) + 6m}{4m} = 3$$

For N masses the centre of mass is:

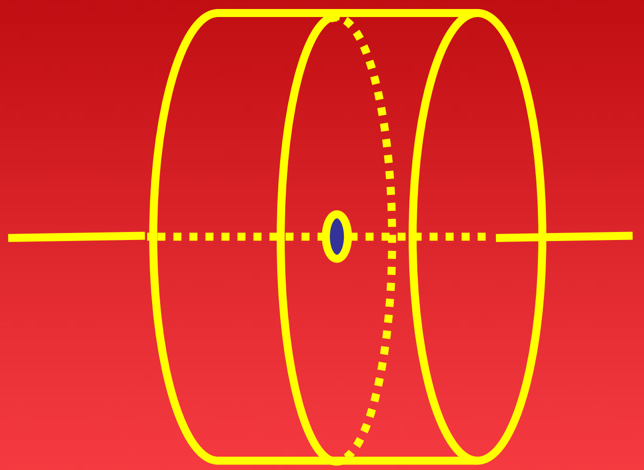
$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N}{m_1 + m_2 + \cdots + m_N}$$

$$= \frac{1}{M}\left(\sum m_n \vec{r}_n\right)$$



m_3
CM
$\vec{r}_{cm}$
m_2
$\vec{r}_2$
$\vec{r}_3$
m_1
$\vec{r}_1$

- For symmetrical objects, CM is easy to guess.



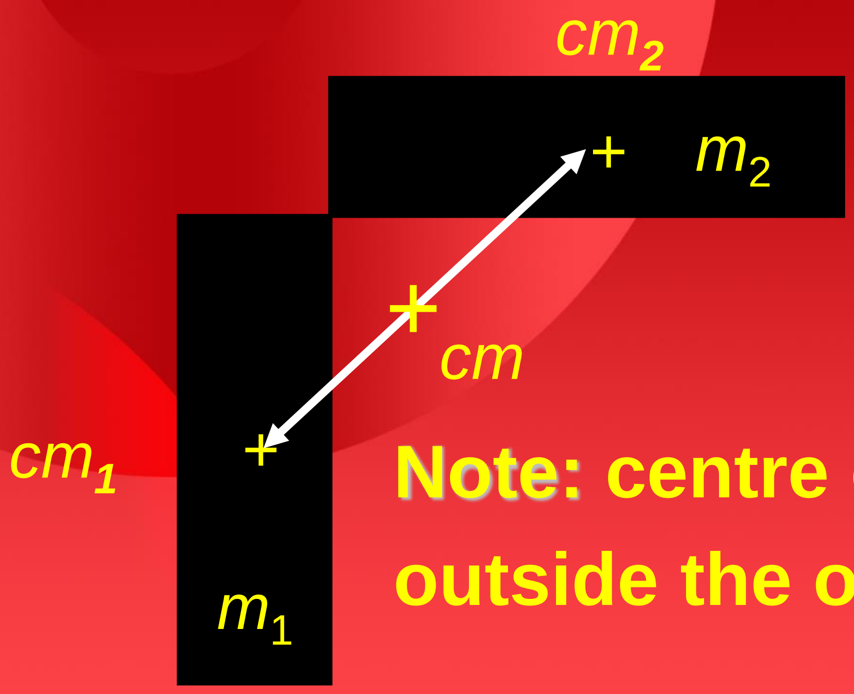
$$\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt} = \frac{1}{M}\left(\sum m_n \vec{v}_n\right)$$

$$\vec{a}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{1}{M}\left(\sum m_n \vec{a}_n\right)$$

$$M\vec{a}_{cm} = \sum \vec{F}_n = \sum\left(\vec{F}_{ext} + \vec{F}_{int}\right)$$

$$\boxed{\sum \vec{F}_{ext} = M\vec{a}_{cm}}$$

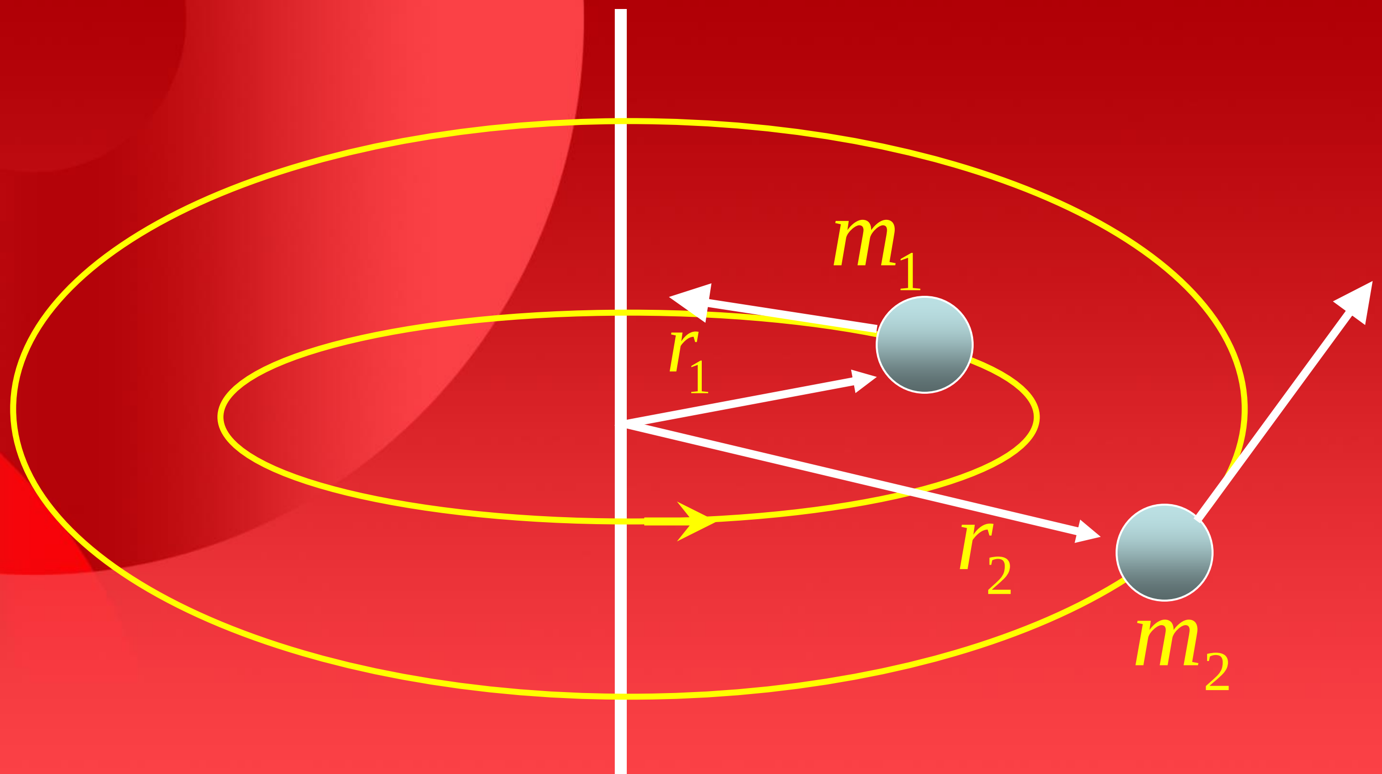
The centre of mass for a combination of objects is the average center of mass location of the objects !!



Note: centre of mass can lie outside the object !!

Rotational Energy of Rigid Bodies

Consider a <u>rigid</u> body rotating about a fixed axis:



Total kinetic energy :

$$K = \frac{1}{2} m_1 \mathrm{v}_1^2 + \frac{1}{2} m_2 \mathrm{v}_2^2 + \frac{1}{2} m_3 \mathrm{v}_3^2 + \cdots$$

$$= \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \frac{1}{2} m_3 r_3^2 \omega^2 + \cdots$$

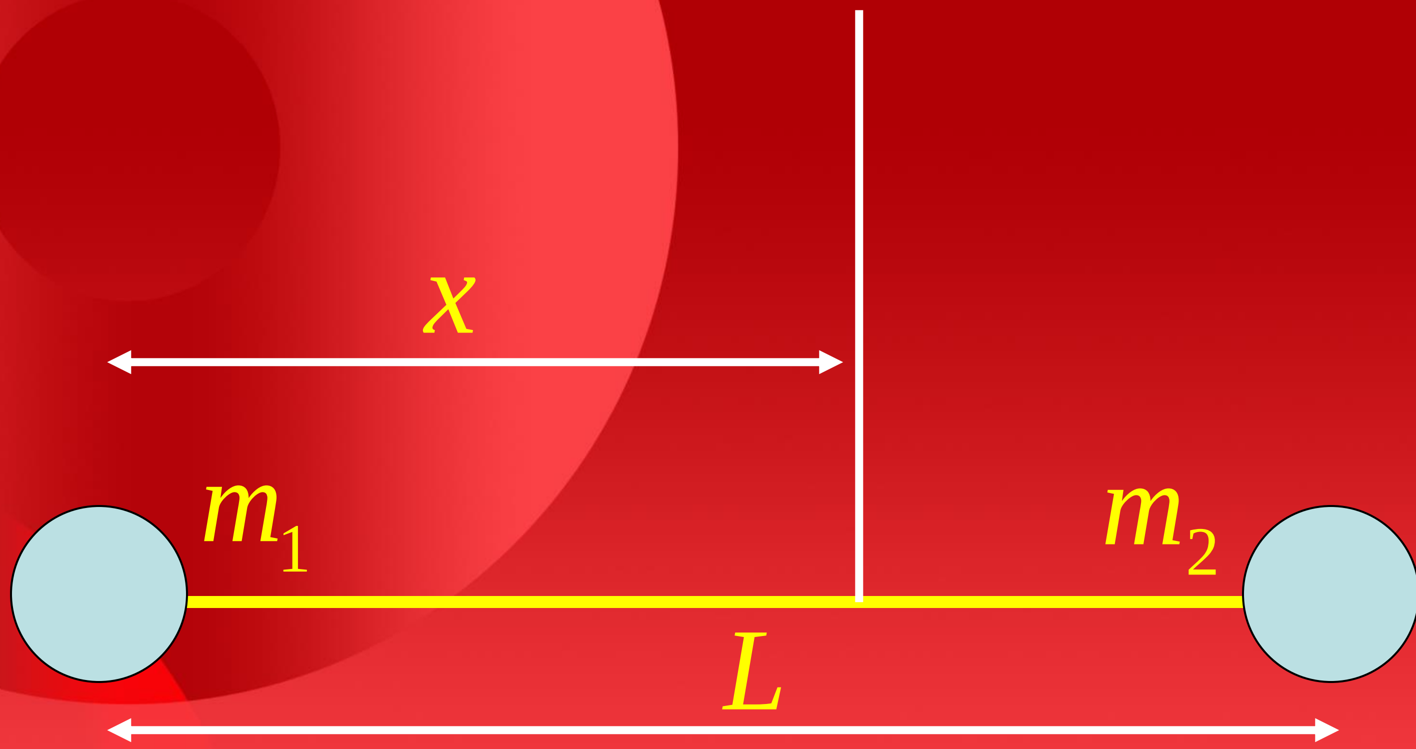
$$K = \frac{1}{2} \left(\sum m_i r_i^2 \right) \omega^2$$

❖Rotational Inertia

$$K = \tfrac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

$$\Rightarrow K = \tfrac{1}{2} I\omega^2, \text{ where } I \equiv \sum m_i r_i^2$$

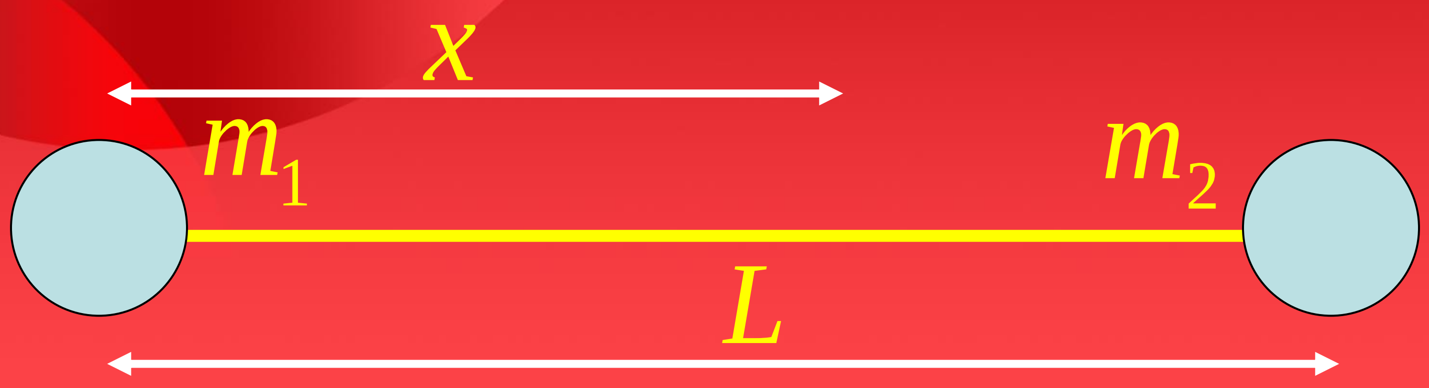
$$\text{Compare with } K = \frac{1}{2} M \mathrm{v}^2 \text{ !!}$$

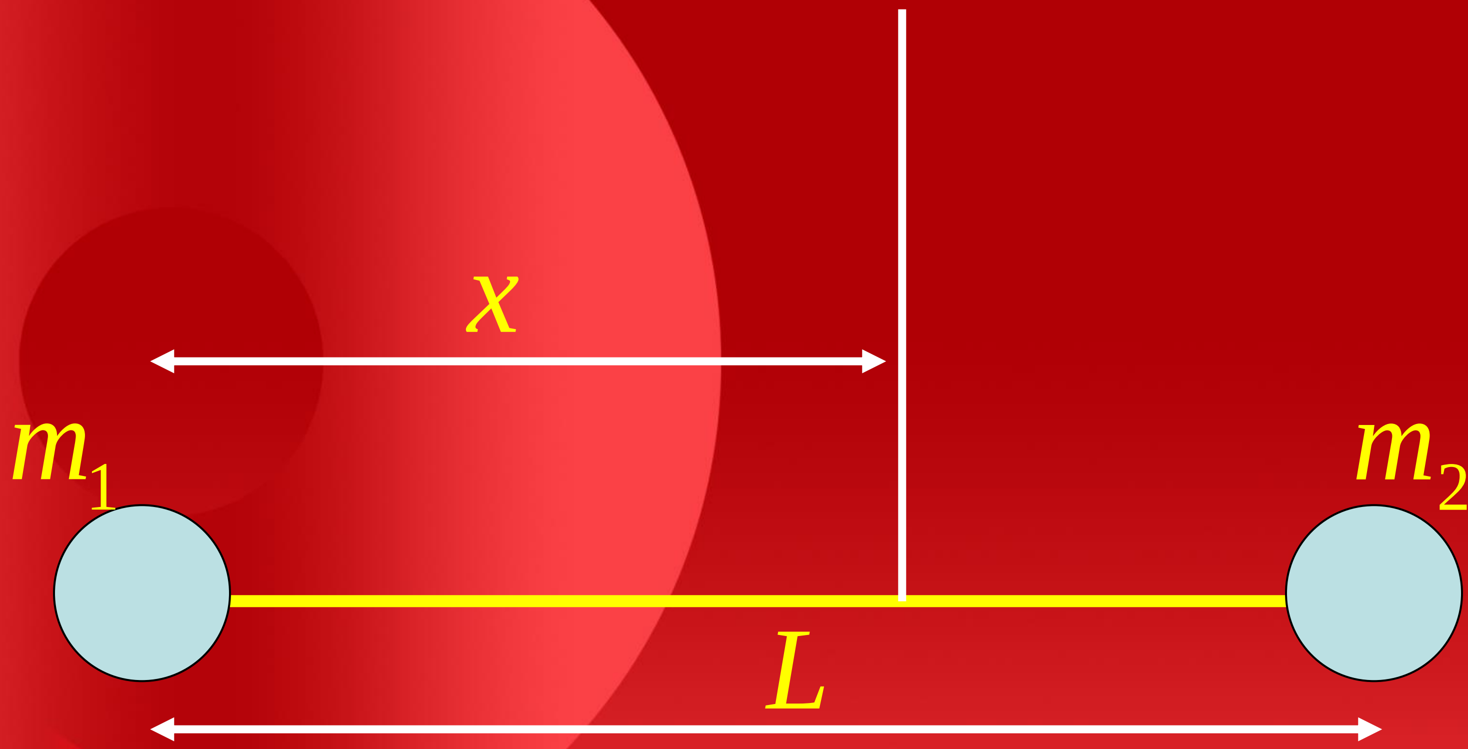


x
m1
m2
L

Two particles m_1 and m_2 are connected by a light rigid rod of length L.

Neglecting the mass of the rod, find the rotational inertia I of this system about an axis perpendicular to the rod and at a distance x from m_1.





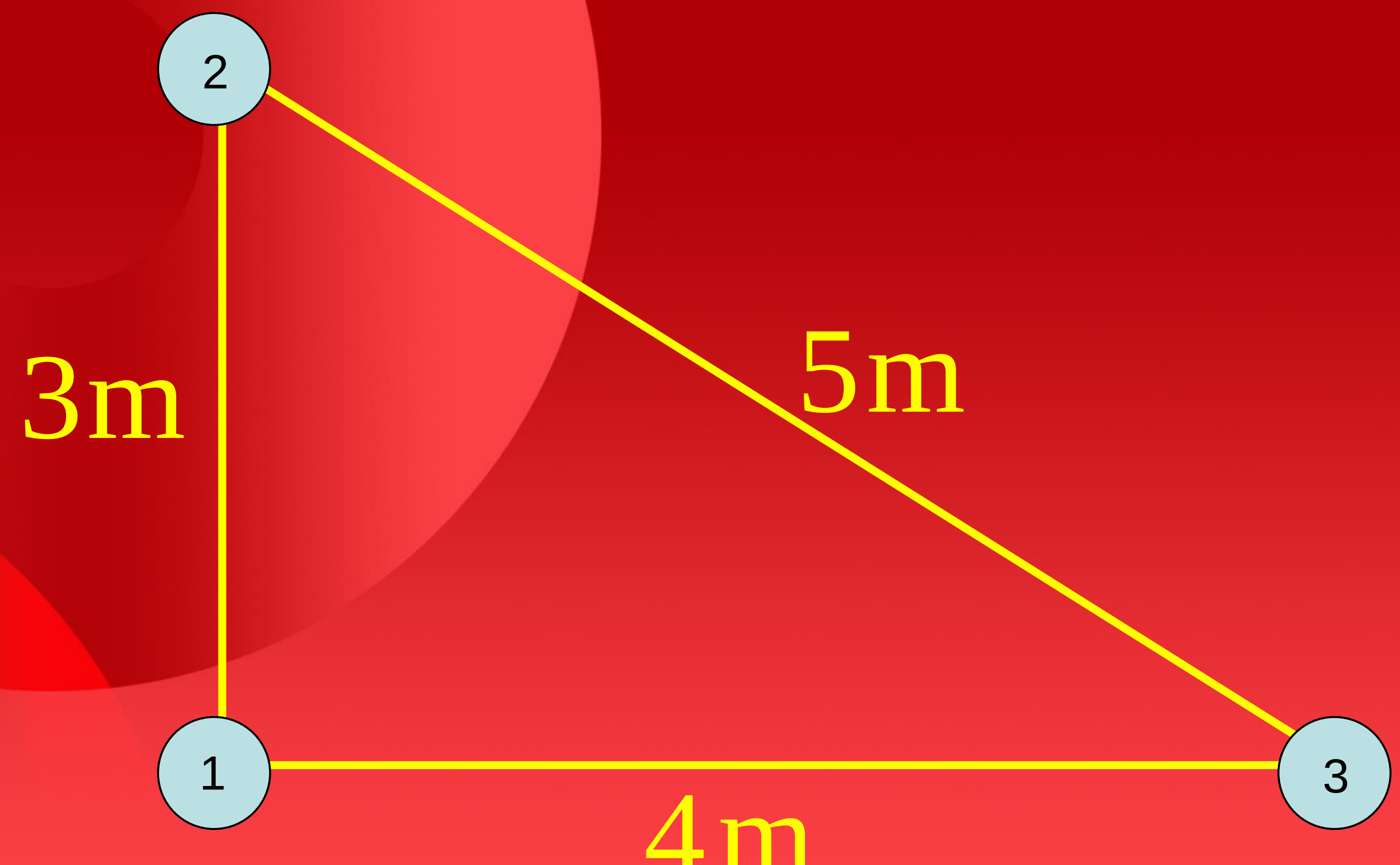
$$I = m_1 x^2 + m_2 \left(L - x\right)^2$$

For what x is I the largest?

$$I = m_1 x^2 + m_2 \left(L - x\right)^2$$

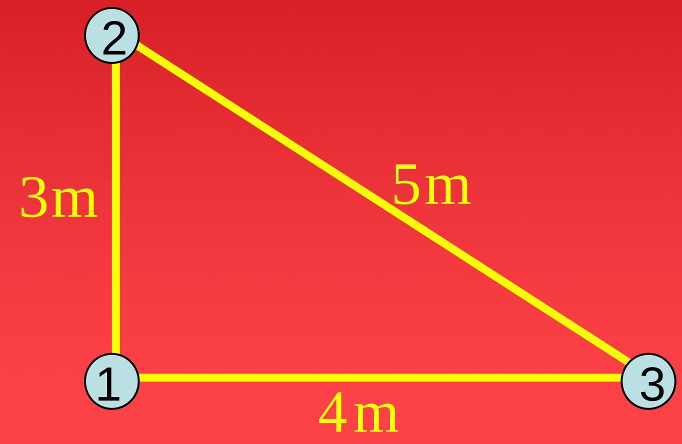
$$\frac{dI}{dx} = 2m_1 x - 2m_2 \left(L - x\right) = 0$$

$$x = \frac{m_2 L}{m_1 + m_2}$$


2
3m
5m
1
4m
3

3 particles of masses m_1 (2.3 kg), m_2 (3.2 kg) and m_3 (1.5 kg) are at the vertices of this triangle.

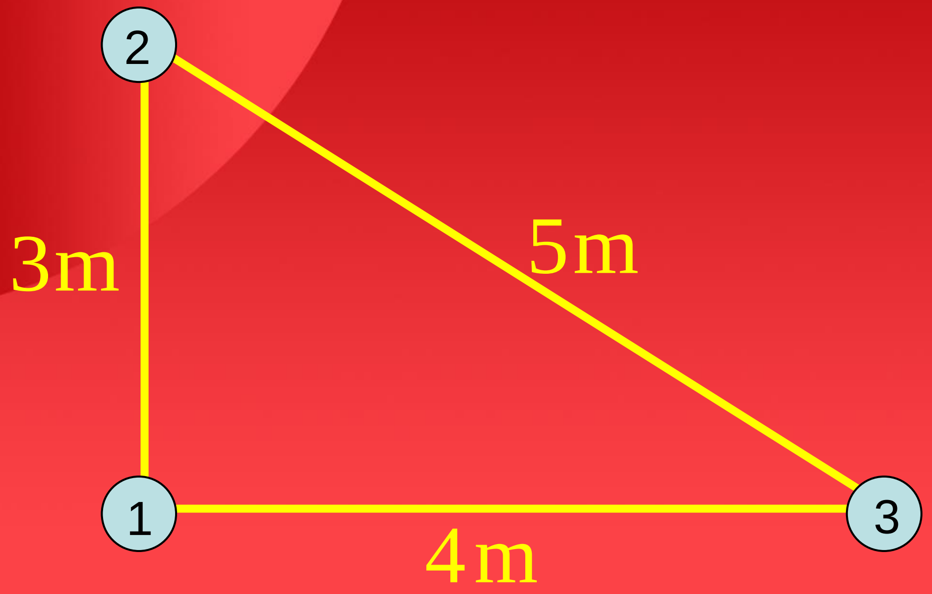
Find the rotational inertia about axes perpendicular to the xy plane and passing through each of the particles.

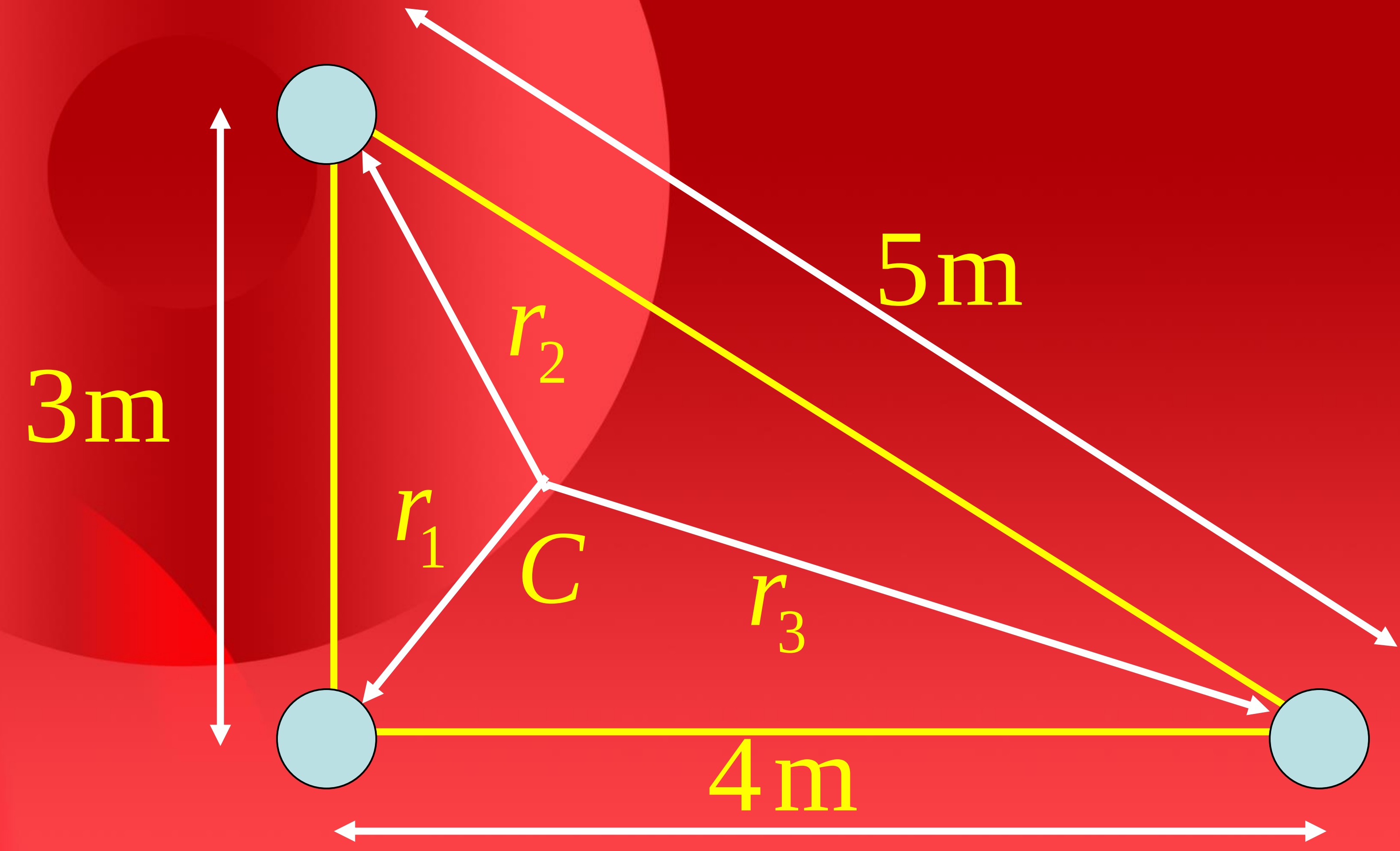


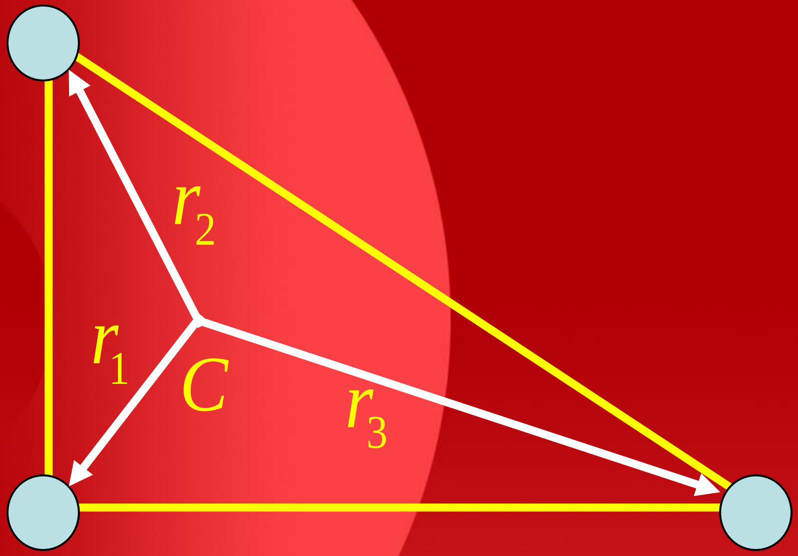
$$I_{m_1} = \sum m_i r_i^2 = 52.8\,\mathrm{kg\,m^2}$$

$$I_{m_2} = \sum m_i r_i^2 = 58.2\,\mathrm{kg\,m^2}$$

$$I_{m_3} = \sum m_i r_i^2 = 116.8\,\mathrm{kg\,m^2}$$

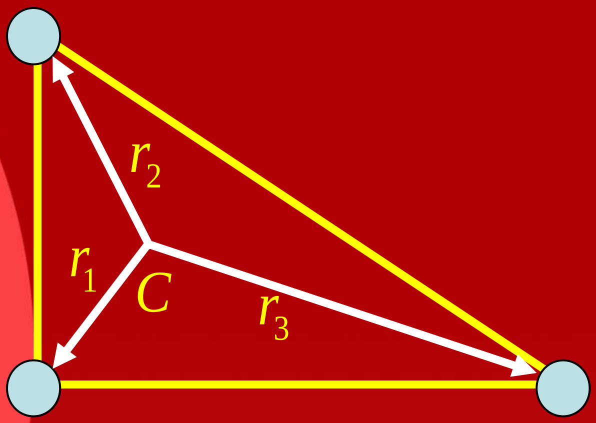



5m
3m
4m
r_1
r_2
r_3
C



$$x_{cm} = \frac{\sum m_i x_i}{\sum m_i} = 0.86\,m$$

$$y_{cm} = \frac{\sum m_i y_i}{\sum m_i} = 1.37\,m$$



$$r_1^2 = x_{cm}^2 + y_{cm}^2 = 2.62\text{m}^2$$

$$r_2^2 = x_{cm}^2 + \left(y_2 - y_{cm}\right)^2 = 3.40\,\text{m}^2$$

$$r_3^2 = \left(x_3 - x_{cm}\right)^2 + y_{cm}^2 = 11.74\text{m}^2$$

$$I_{cm} = \sum m_i r_i^2 = 34.5\,\text{kg}\,\text{m}^2$$

For solid bodies

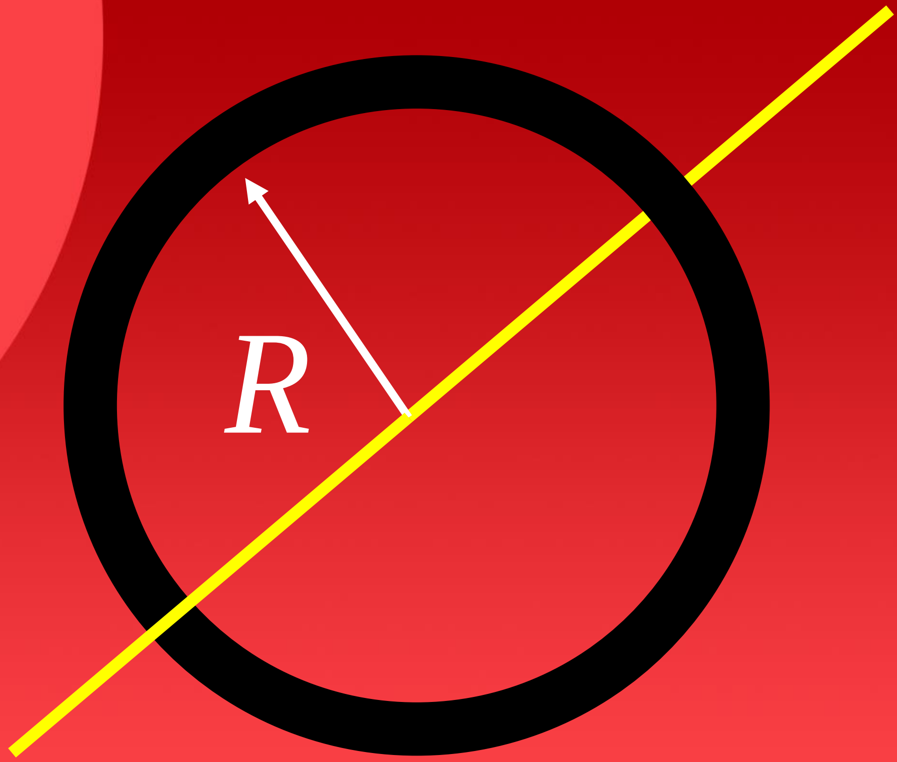
$$I = \int r^2 dm$$

Hoop about cylinder axis

$$I = \int r^2 dm$$

$$= R^2 \int dm$$

$$= MR^2$$



Solid plate about cylinder axis

$$I = \int r^2 dm \quad (dm = 2\pi r dr \rho_0)$$

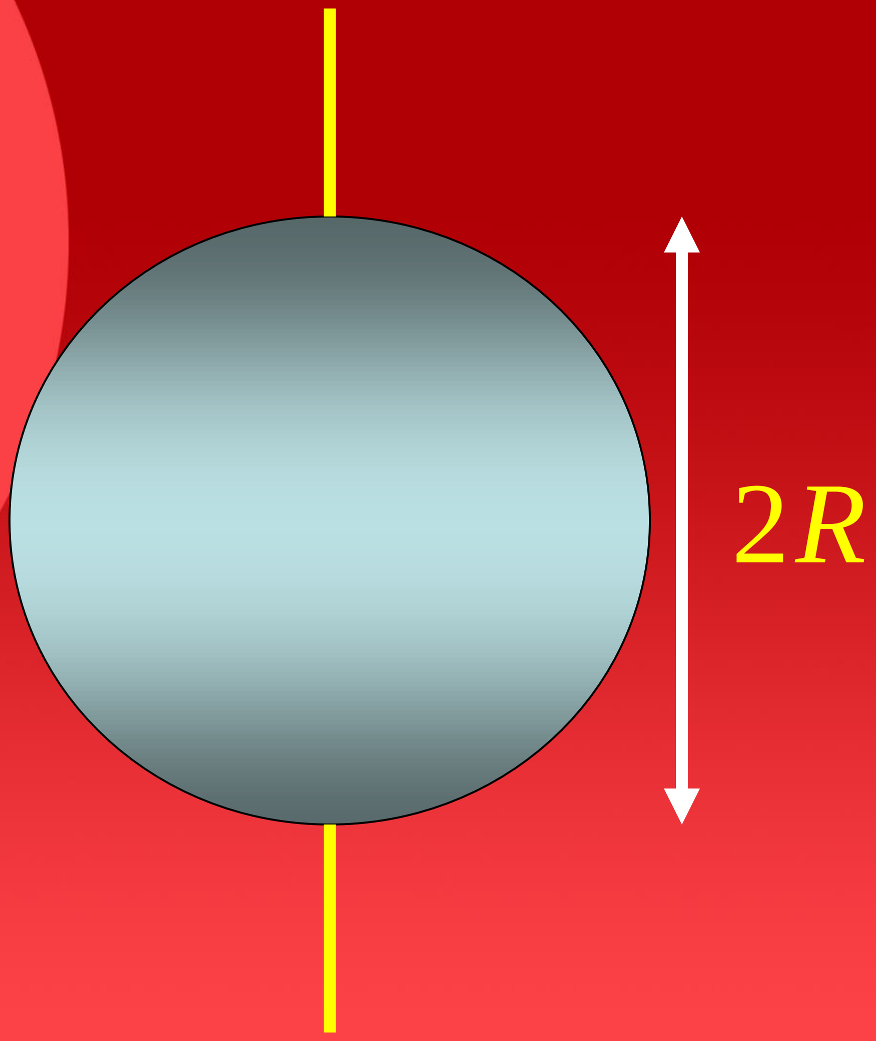
$$= \int_0^R 2\pi r^3 dr \rho_0$$

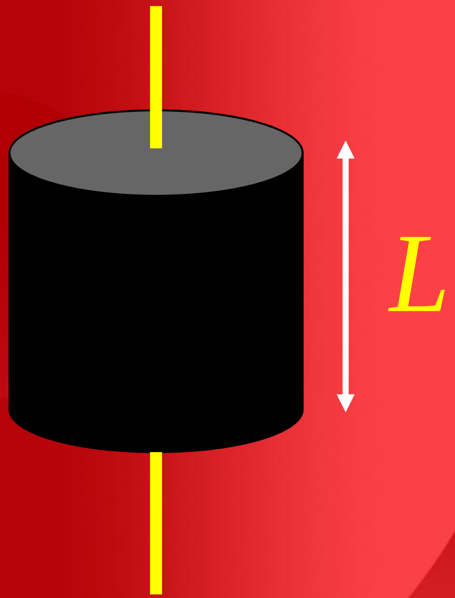
$$= \frac{1}{2}(\pi R^2 \rho_0) R^2$$

$$= \frac{1}{2} M R^2$$

Solid sphere about diameter

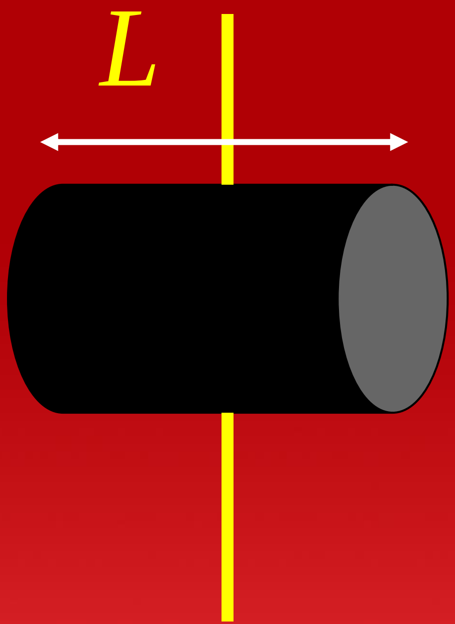
$$I = \frac{2}{5}MR^2$$





Solid cylinder or disk about cylinder axis

$$I = \frac{1}{2} MR^2$$



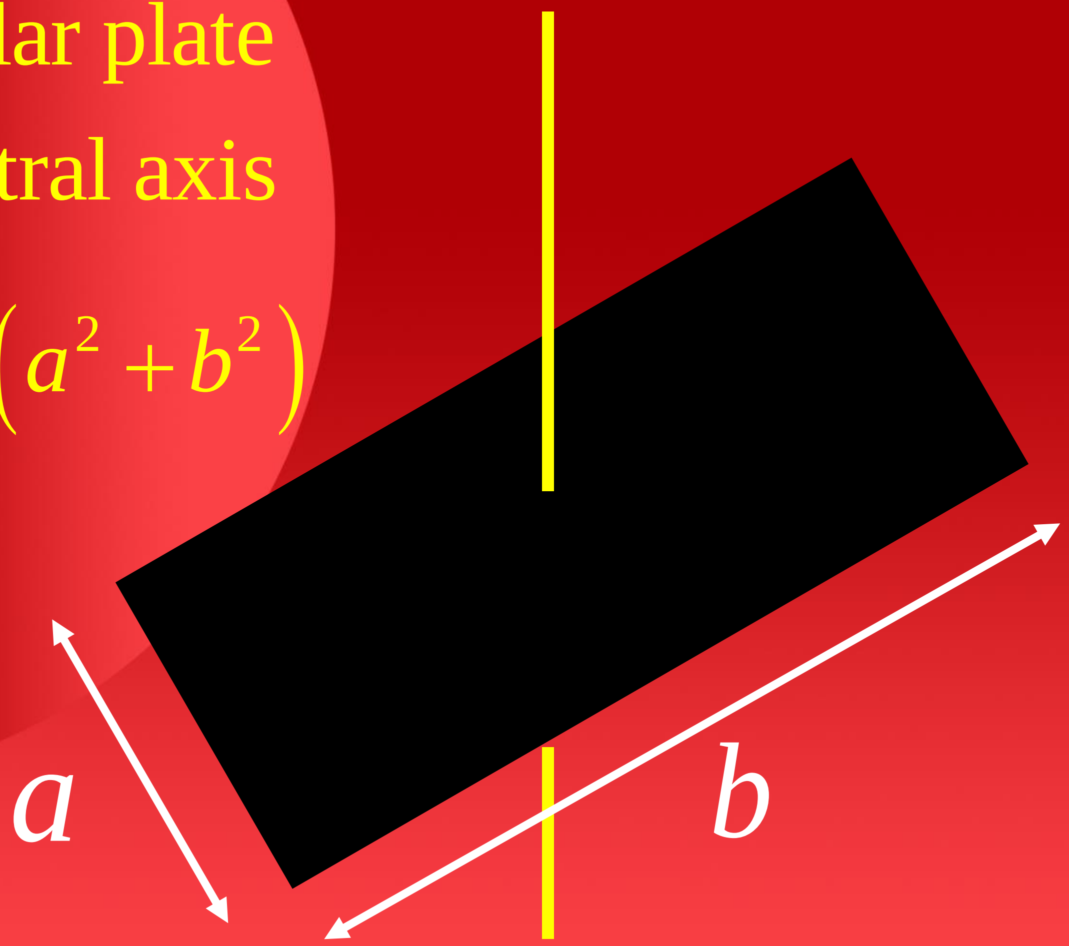
Solid cylinder or disk about central diameter

$$I = \frac{1}{4} MR^2 + \frac{1}{12} ML^2$$

Rectangular plate

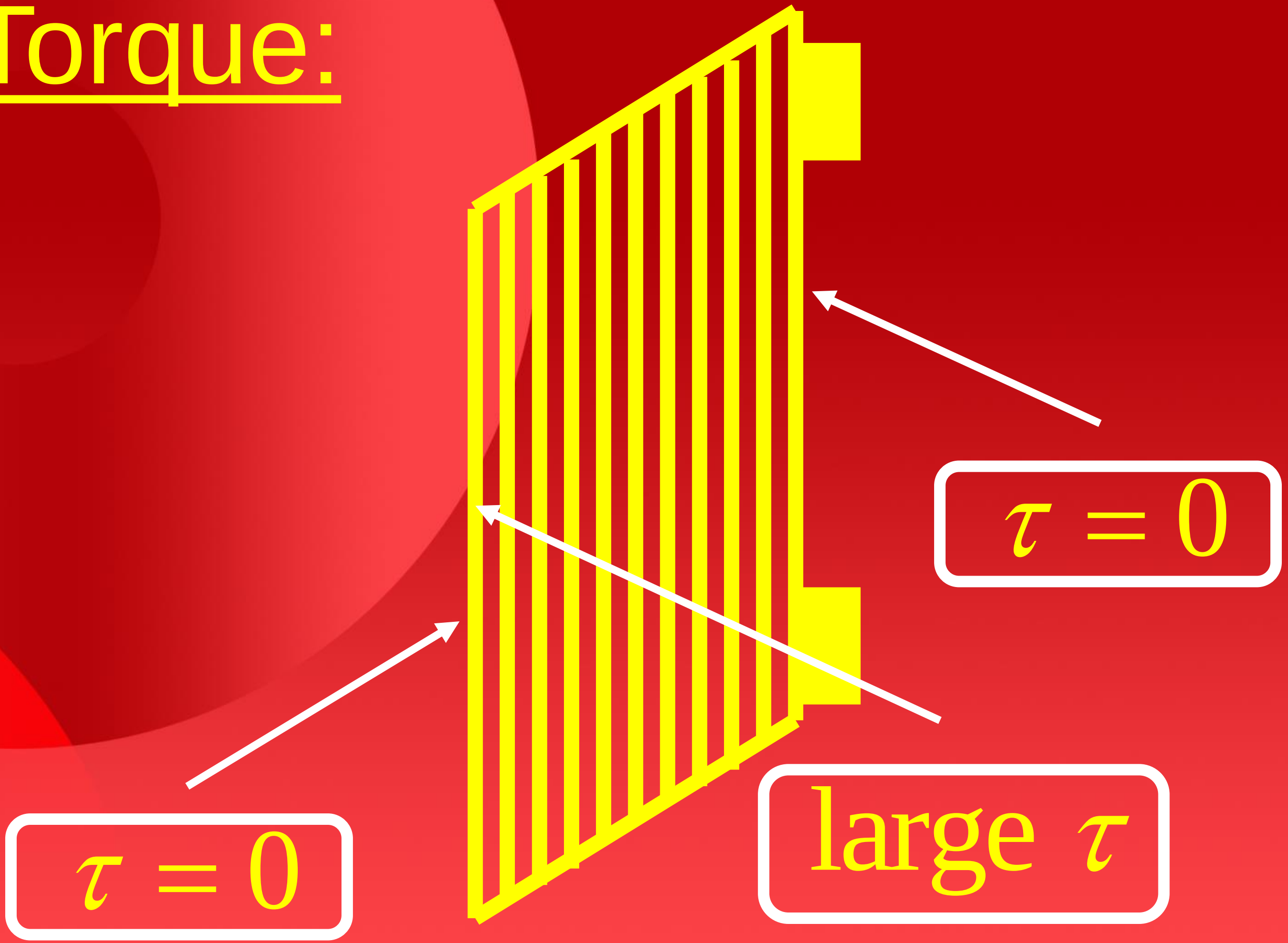
about central axis

$$I = \frac{1}{12} M \left(a^2 + b^2\right)$$



Rotational Dynamics of Rigid Bodies

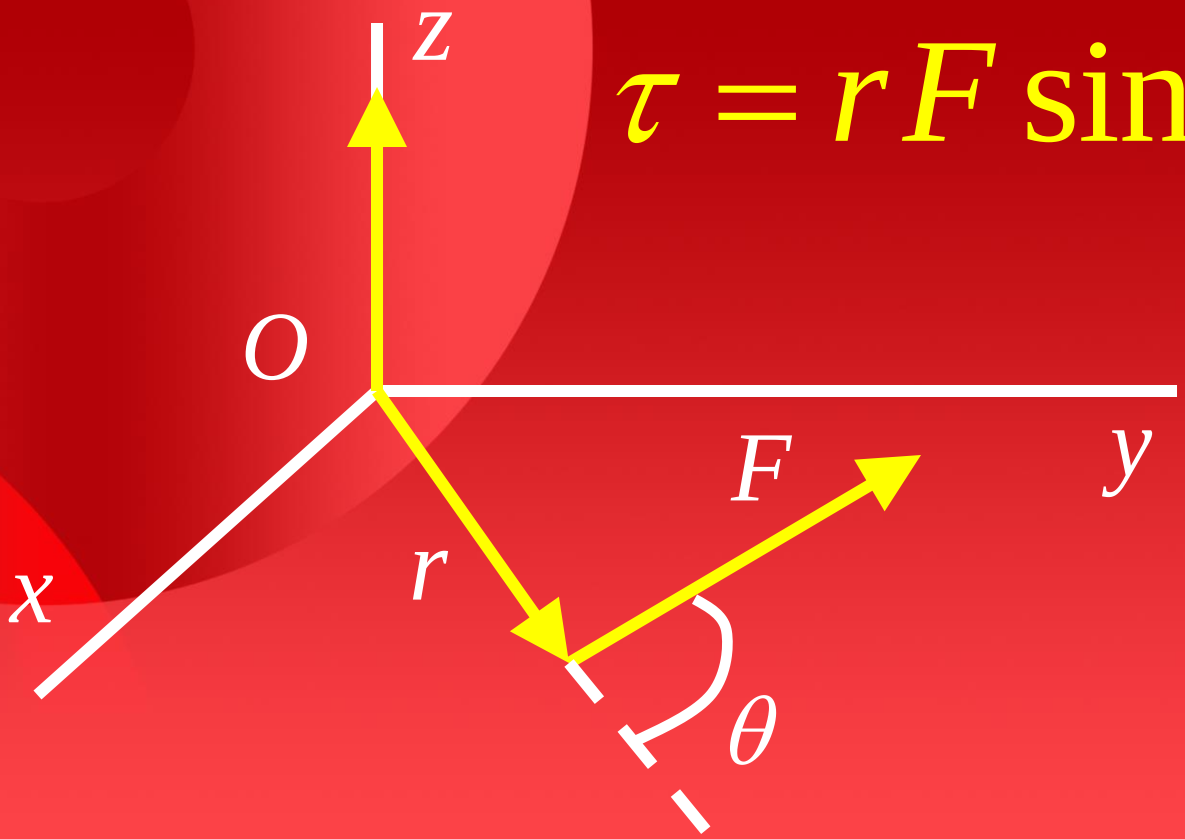
Torque:



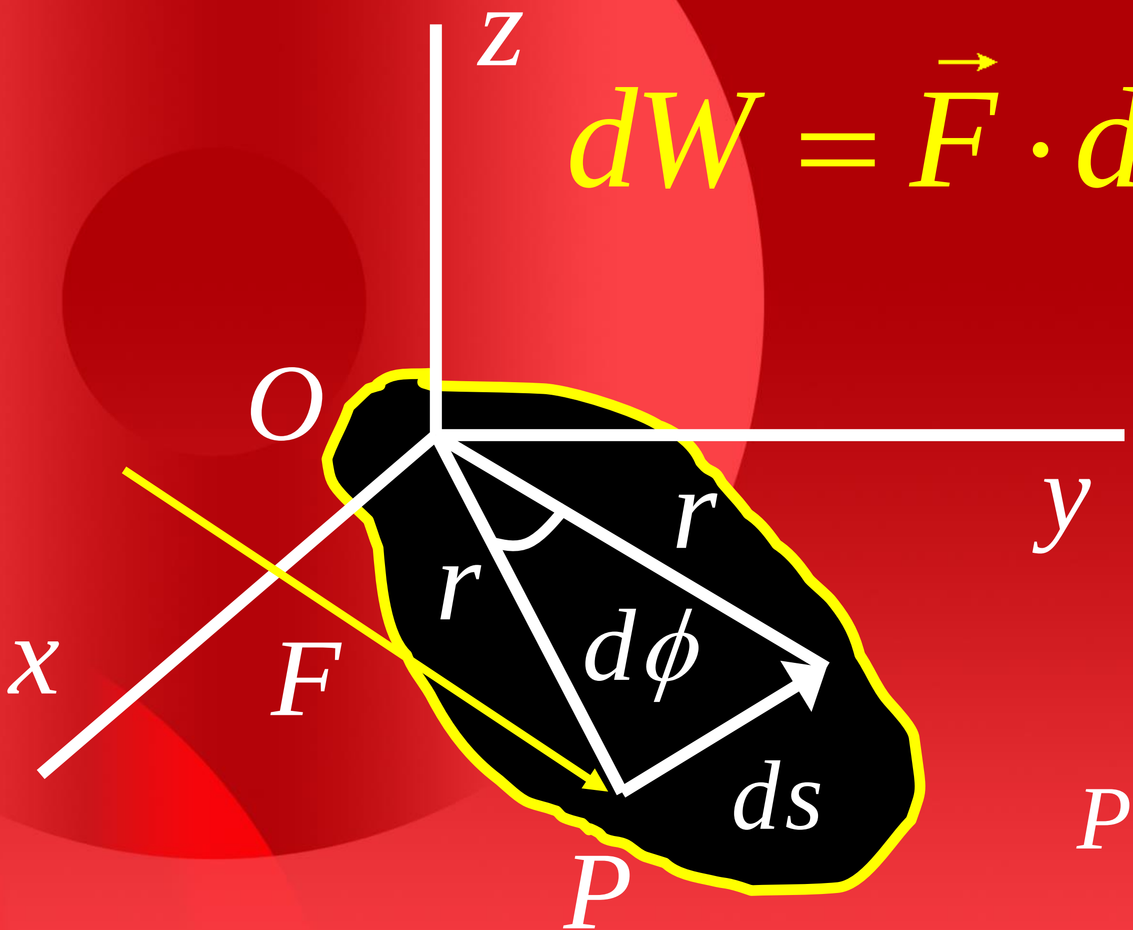
Torque:

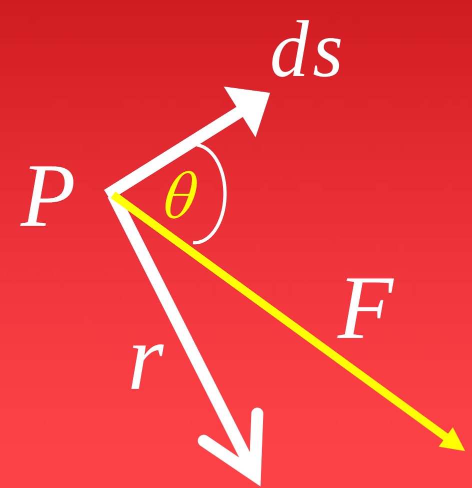
$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\tau = rF\sin\theta$$



$$dW = \vec{F} \cdot d\vec{s}$$





$$dW = \vec{F} \cdot d\vec{s} = F\cos\theta\, ds$$

$$= \left(F\cos\theta\right)\left(r\,d\phi\right)$$

$$dW = \tau\, d\phi$$

$$dW_{net} = \left(F_1\cos\theta_1\right)r_1 d\phi + \left(F_2\cos\theta_2\right)r_2 d\phi +$$

$$\cdots + \left(F_n\cos\theta_n\right)r_n d\phi$$

$$dW_{net} = \left(\tau_1 + \tau_2 + \cdots + \tau_n\right)d\phi$$

$$dW_{net} = \left(\sum \tau_{ext}\right) d\phi = \left(\sum \tau_{ext}\right) \omega dt$$

$$dK = d\left(\frac{1}{2} I \omega^2\right) = I\omega d\omega = (I\alpha)\omega dt$$

$$dW_{net} = dK \Rightarrow \sum \tau_{ext} = I\alpha$$

Like Newton's second law !!!

Translational

$$x, M$$

$$\mathrm{v} = \frac{dx}{dt}$$

$$\mathrm{a} = \frac{d\mathrm{v}}{dt}$$

Rotational

$$\phi, I$$

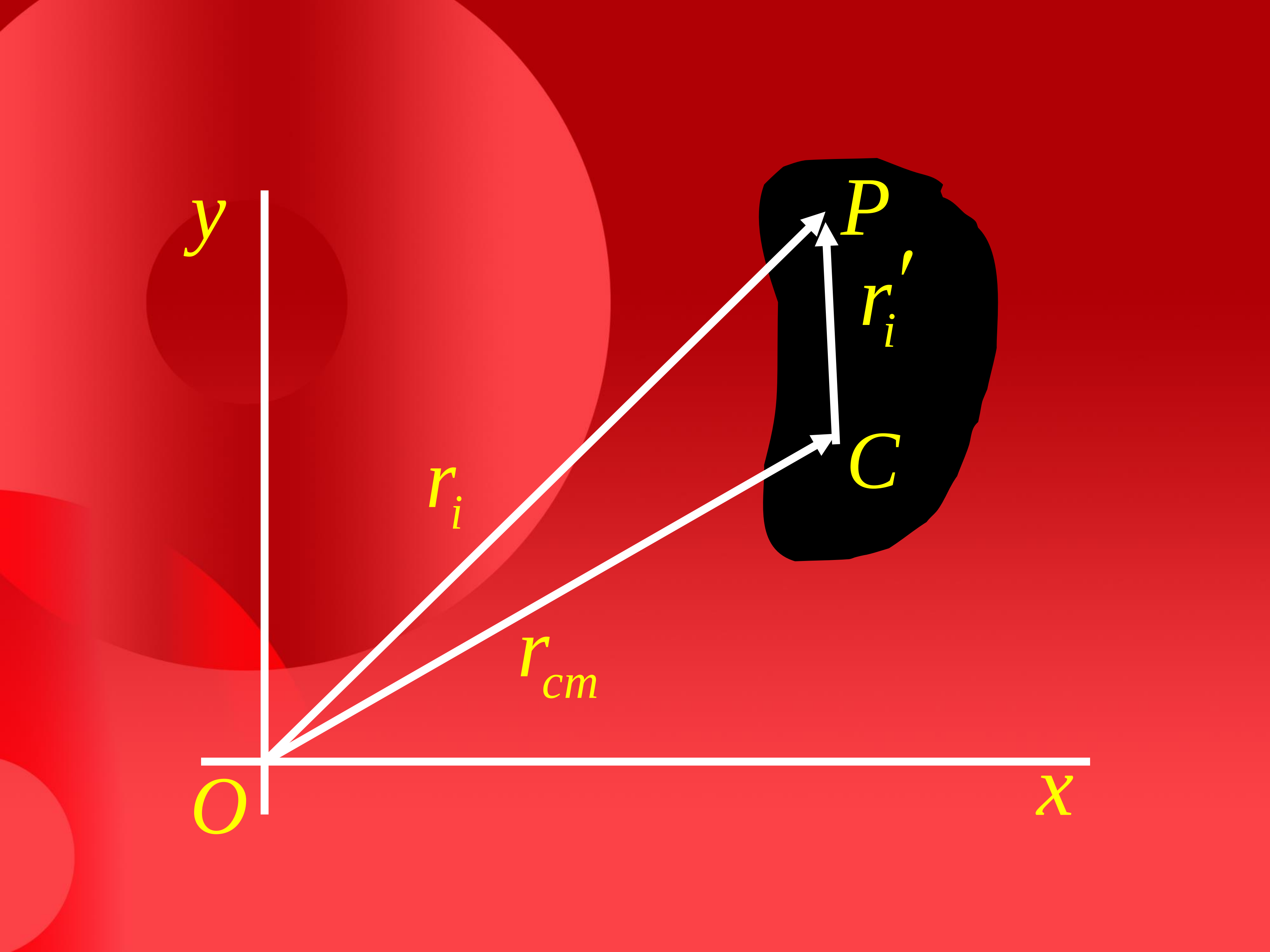
$$\omega = \frac{d\phi}{dt}$$

$$\alpha = \frac{d\omega}{dt}$$

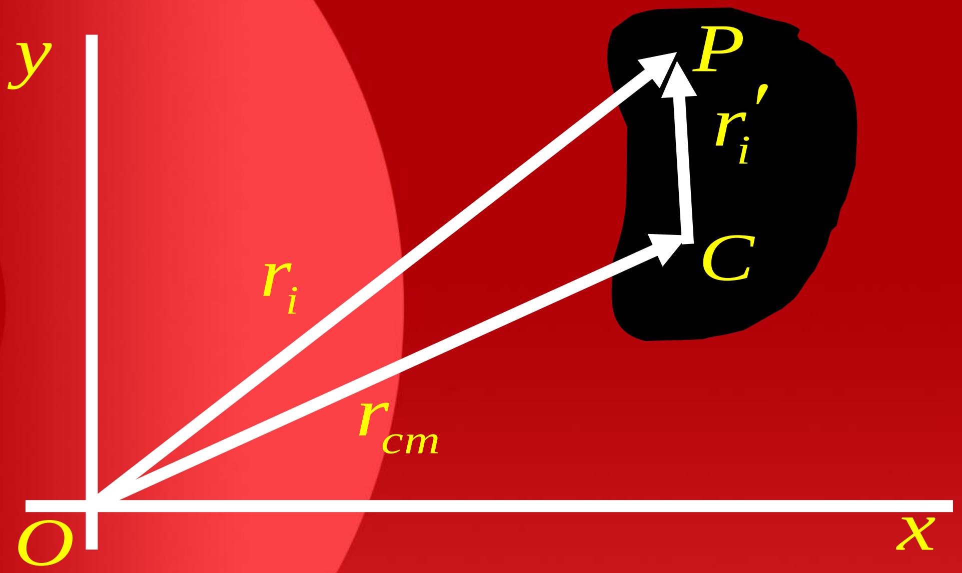
Translational	Rotational

Translational	Rotational
$F = M\mathrm{a}$	$\tau = I\alpha$
$W = \int F dx$	$W = \int \tau d\phi$
$K = \frac{1}{2} M\mathrm{v}^2$	$K = \frac{1}{2} I\omega^2$

Combined Rotational and Translational Motion



y
P
r_i'
r_i
C
r_{cm}
O
x



$$K = \frac{1}{2} M\, \mathrm{v}_{\mathrm{cm}}^2 + \frac{1}{2} I_{cm}\,\omega^2$$

WHY ??

$$K=\sum\frac{1}{2}m_i\mathrm{v}_i^{\,2}=\sum\frac{1}{2}m_i\vec{\mathrm{v}}_i.\vec{\mathrm{v}}_i$$

$$=\sum\frac{1}{2}m_i\left(\vec{\mathrm{v}}_{\mathrm{cm}}+\vec{\mathrm{v}}_i'\right).\left(\vec{\mathrm{v}}_{\mathrm{cm}}+\vec{\mathrm{v}}_i'\right)$$

$$=\sum\frac{1}{2}m_i\left(\mathrm{v}_{\mathrm{cm}}^2+2\vec{\mathrm{v}}_{\mathrm{cm}}.\vec{\mathrm{v}}_{\mathrm{i}}'+\mathrm{v}_{\mathrm{i}}'^{2}\right)$$

$$\sum m_i\vec{\mathrm{v}}_{\mathrm{cm}}\cdot\vec{\mathrm{v}}_{\mathrm{i}}'=\vec{\mathrm{v}}_{\mathrm{cm}}\cdot\sum m_i\vec{\mathrm{v}}_i'$$

$$\sum\vec{p}_i'=\sum m_i\vec{\mathrm{v}}_i'=M\,\vec{\mathrm{v}}_{\mathrm{cm}}'$$

$$\vec{\mathrm{v}}_{\mathrm{cm}}'=0,\ \text{in centre-of-mass frame}$$

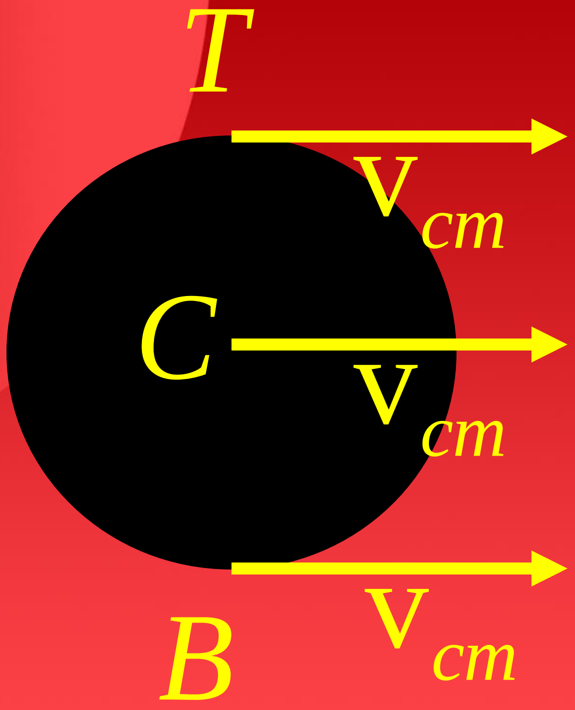
$$K = \sum \frac{1}{2} m_i \mathrm{v}_{\mathrm{cm}}^2 + \sum \frac{1}{2} m_i \mathrm{v}_{\mathrm{i}}'^2$$

$$= \frac{1}{2} M \mathrm{v}_{\mathrm{cm}}^2 + \sum \frac{1}{2} m_i r_{\mathrm{i}}'^2 \omega^2$$

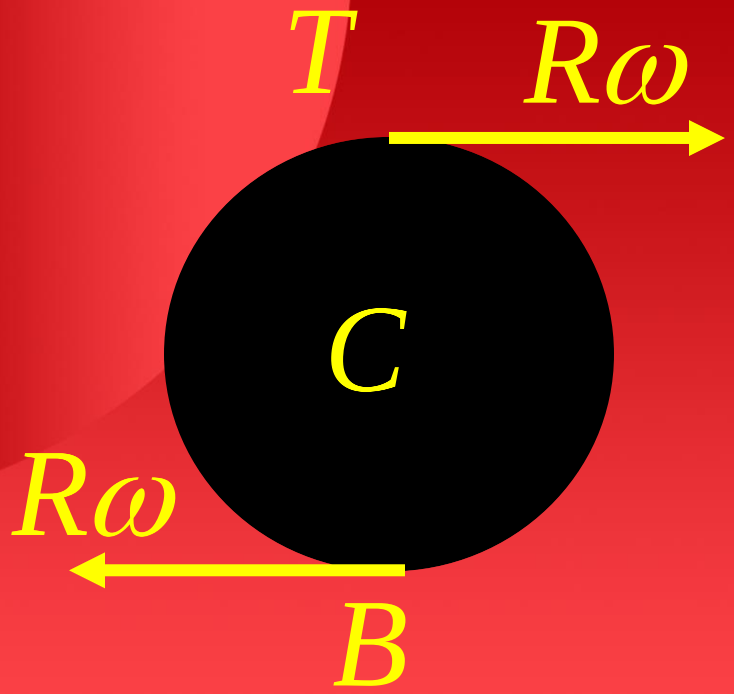
$$\boxed{K = \frac{1}{2} M \mathrm{v}_{\mathrm{cm}}^2 + \frac{1}{2} I_{cm} \omega^2}$$

Rolling without slipping

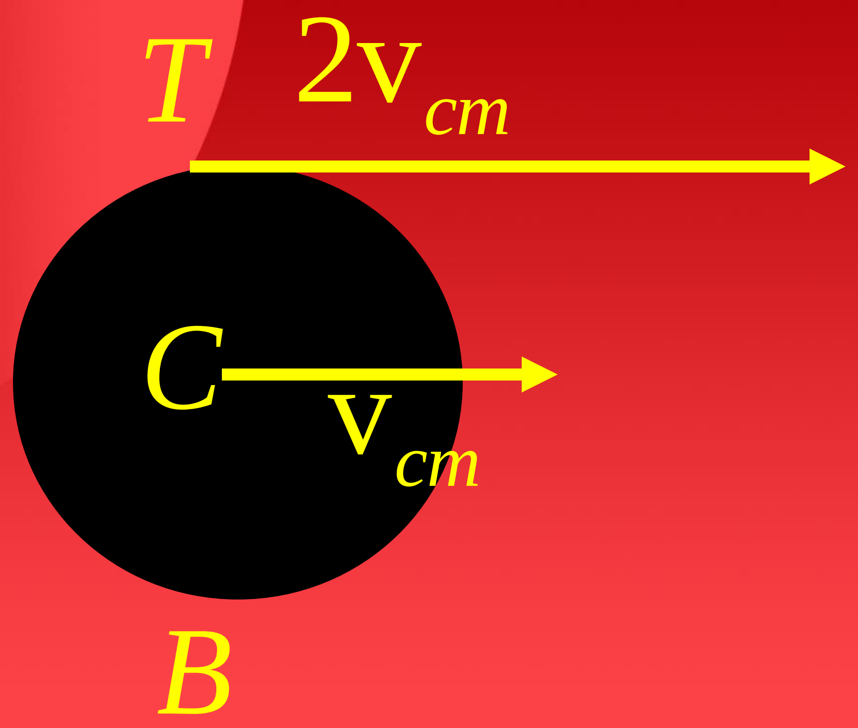
Translational motion



Rotational motion



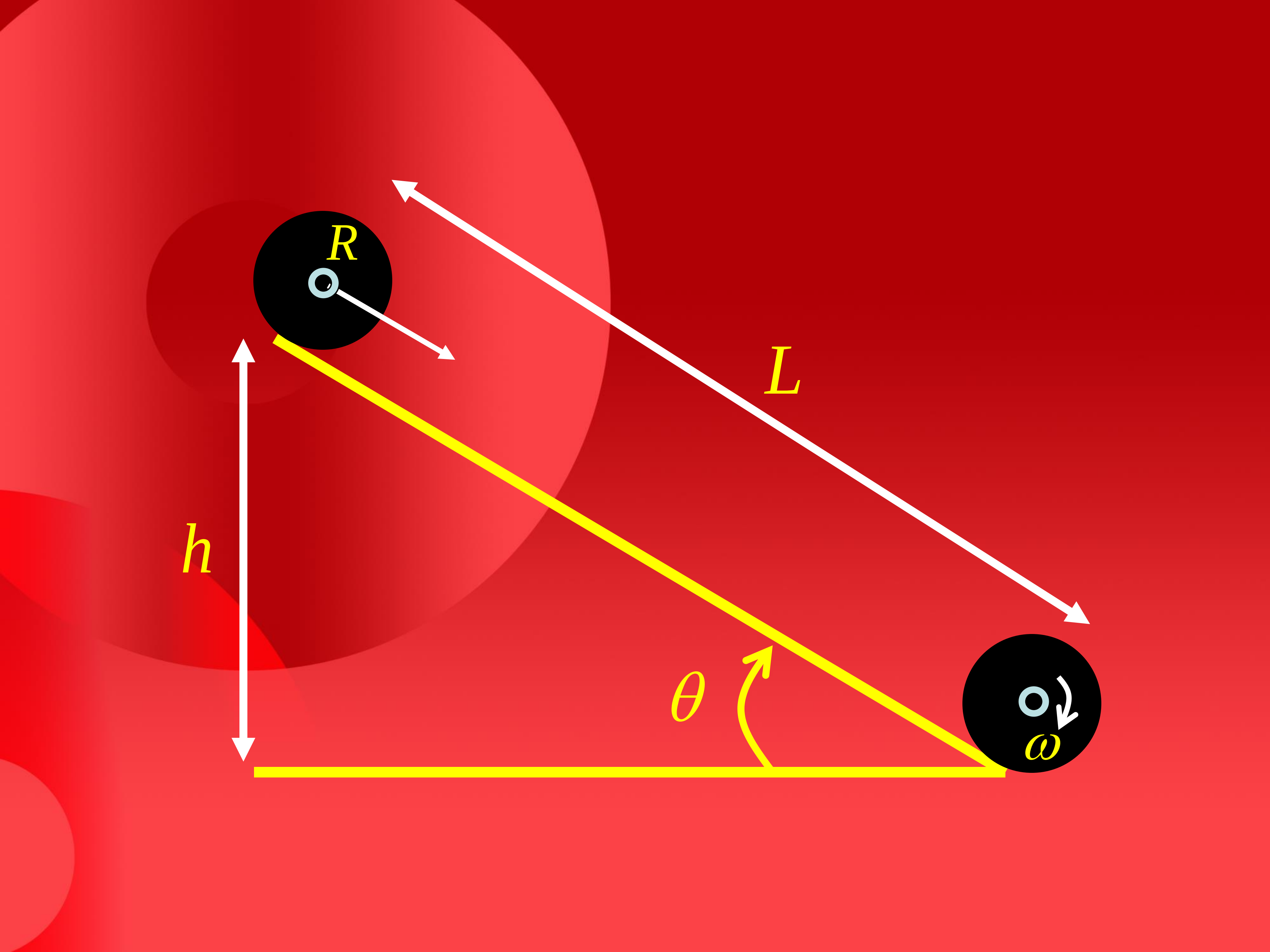
Translational + Rotational



$$\mathrm{v}_{\mathrm{cm}} = R\omega$$

$$K = \frac{1}{2} M \mathrm{v}_{\mathrm{cm}}^2 + \frac{1}{2} I_{cm} \left(\frac{\mathrm{v}_{\mathrm{cm}}^2}{R^2} \right)$$

$$\boxed{K = \frac{1}{2} M R^2 \omega^2 + \frac{1}{2} I_{cm} \omega^2}$$



R
L
h
θ
ω

$$K = \frac{1}{2} M \mathrm{v}_{\mathrm{cm}}^{2} + \frac{1}{2} I_{cm} \omega^{2}$$

$$Mgh = \frac{1}{2} M \mathrm{v}_{\mathrm{cm}}^{2} + \frac{1}{2}\left(\frac{1}{2} MR^{2}\right)\left(\frac{\mathrm{v}_{\mathrm{cm}}^{2}}{R}\right)^{2}$$

$$\mathrm{v}_{\mathrm{cm}} = \sqrt{\frac{4}{3} gh}$$