Physics-PHY101-Lecture 16

OSCILLATIONS: II

Oscillations are a fundamental concept in physics that describes the repetitive motion of a system around a central equilibrium point. In this chapter, we will explore various types of oscillatory motion and their properties. We begin with the simple pendulum, which is a classic example of a harmonic oscillator. Next, we will discuss the physical pendulum, which is a more complex system than the simple pendulum. We will also discuss the composition of two harmonic oscillators with the same period, which leads to interesting phenomena like beats and resonance. Additionally, we will look into damped oscillators, which describe systems where the amplitude of oscillations decreases over time due to damping forces. We will analyze the behavior of damped oscillators and derive their equations of motion. Finally, we will examine forced and free oscillations. **Forced oscillations** occur when an external force is applied to a system, while **free oscillations** occur without any external force. By the end of this chapter, you will have a deep understanding of the fundamental concepts of oscillatory motion and be able to analyze and describe a wide range of oscillatory systems.

Simple Pendulum:

A simple pendulum is an ideal body consisting of a body of mass m and suspended by a light in-extensible string. The pendulum oscillates periodically under the action of gravity when it is pulled to one side of its equilibrium point and then released.

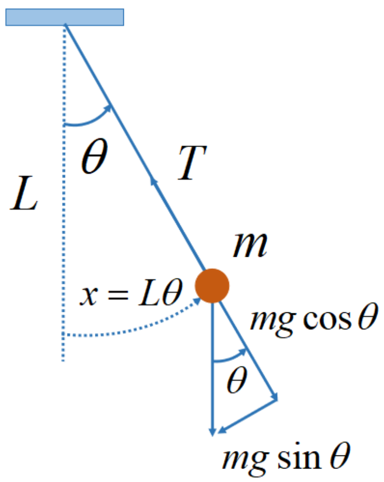


Figure 16. 1: The simple pendulum. A pendulum of mass m with a length L and making an angle with the vertical. We choose the x axis to be in the tangential direction and the y axis to be in the radial direction at this particular time.

The forces acting on mass are the weight mg and the tension T in the string. The weight mg is resolved into a radial component of magnitude mg cos θ and a tangential component of magnitude mg sin θ. The centripetal acceleration required to keep the particle travelling in a circular path is provided by the radial components of the forces. The restoring force on m that pushes it towards equilibrium is represented by the tangential component. The restoring force is thus

$$F = -mg\sin\theta \qquad (1)$$

The negative sign shows that F is opposite to the direction of increasing L= x / θ.

It should be noted that the restoring force is proportional to sin θ rather θ. As a result, the motion is not simply harmonic. However, for small angles, the sine function can be approximated using the small-angle approximation:

$$\sin\theta \approx \theta$$

Thus

$$x = L\theta \rightarrow \theta = \frac{x}{L}$$

Putting the value in eq. (1) we get

$$F = -mg\frac{x}{L}$$

where $a = -g\frac{x}{L}$ (2)

$$F = -\left(\frac{mg}{L}\right)x$$

By Newton's second law , F = ma where $a = \frac{d^2x}{dt^2}$. Hence

$$F = m\frac{d^2x}{dt^2}$$

Also, $F = -\left(\frac{mg}{L}\right)x$

On comparing, we get,

$$\frac{d^2x}{dt^2} = -\left(\frac{g}{L}\right)x$$

$$\frac{d^2x}{dt^2} + \left(\frac{g}{L}\right)x = 0$$

The general solution to this differential equation is of the form:

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

Where A and B are constants, and ω is the angular frequency, related to g and L by the equation ω = √(L/g) . For the specific, where the pendulum is released from its maximum displacement (x_m), the initial conditions are $x(0) = x_m$ and $\frac{dx}{dt}(0) = 0$. Solving for A and B based on these conditions yields:

$$x(0) = Acos(0) + Bsin(0) \rightarrow x_m = A$$

So, A = x_m and B = 0. Therefore, the solution for the given initial condition is:

$$x = x_m \cos \omega t$$

In SHM acceleration is given by

$$a = -\omega^2 x \qquad (3)$$

After comparing eq. 2 and 3, we will get the expression of angular frequency defined as,

$$\omega^2 = \frac{g}{L}$$

$$\omega = \sqrt{\frac{g}{L}}$$

The period of the oscillation of physical pendulum is given by,

$$T = \frac{2\pi}{\omega}.$$

Substituting ω into the expression, we get

$$T = 2\pi\sqrt{\frac{m}{mg/L}} = 2\pi\sqrt{\frac{L}{g}}$$

Physical Pendulum

A physical pendulum is a rigid body suspended from a pivot point that oscillates back and forth in a vertical plane under the influence of gravity. Unlike a simple pendulum, which consist of a mass suspended from weightless string, a physical pendulum has a non-negligible mass and size. The motion of a physical pendulum can be described by using the principle of rotational motion and the conservation of energy.

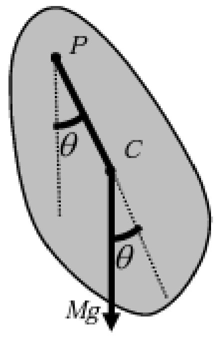


Figure 16. 2: Physical pendulum consisting of a rigid body of mass M.

Let us consider a physical pendulum consisting of a rigid body of mass M and length L, suspended from a pivot point P.

The pivot point P is fixed, and the pendulum is free to oscillate about it. Let the pendulum make a small angular displacement θ from its equilibrium position. The torque acting on the pendulum on the pendulum due to gravity is given by,

$$\tau = -Mgdsin\,\theta \qquad (4)$$

Where g is the acceleration due to gravity and L is the distance between the pivot point P and the center of mass of the pendulum. The negative sign indicates that the torque acts in the opposite direction to the displacement.

By Newton's second law for rotational motion, the torque is defined as,

$$\tau = I\,\alpha \qquad (5)$$

where I is the moment of inertia of the pendulum and α is the angular acceleration, $\alpha = \frac{d^2\theta}{dt^2}$.

Using the small angle approximation $sin\,\theta \approx \theta,$ and the definition of angular acceleration, we can write the above equation as,

$$I\frac{d^2\theta}{dt^2} = -Mgdsin\,\theta,$$

Dividing both sides by I and rearranging, we get

$$\frac{d^2\theta}{dt^2} + \left(\frac{Mgd}{I}\right)\,\theta = 0. \qquad (6)$$

The solution to this differential equation is a sinusoidal function of the form,

$$\theta(t) = Acos(\omega t + \varphi),$$

where A is the amplitude of oscillation, ω is the angular frequency, and φ is the phase angle. To find ω, take the second derivative of $\theta(t)$.

$$\theta''(t) = -A\omega^2 \cos\,(\omega t + \varphi)$$

Substituting this into the differential equation, we get

$$-A\omega^2\cos(\omega t + \varphi) + \left(\frac{Mgd}{I}\right)\,Acos(\omega t + \varphi) = 0$$

Dividing both sides by $Acos(\omega t + \varphi)$ and after simplification we get

$$\omega^2 = \left(\frac{Mgd}{I}\right)$$

After substituting this into the expression for the period, we get

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{Mgd}}$$

Simple Harmonic Motion and Uniform Circular Motion

Simple harmonic motion (SHM) and uniform circular motion (UCM) are both types of periodic motion, but they differ in their nature and characteristics. Although SHM and UCM have different characteristics, they are related in some ways.

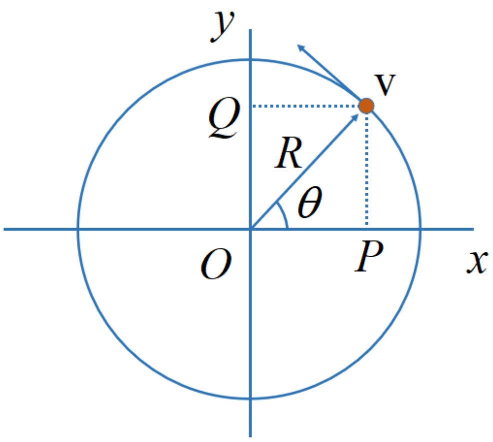


Figure 16. 3: Particle P executing uniform circular motion.

For example, the projection of UCM onto any diameter of the circle is SHM, with the amplitude of the SHM equal to the radius of the circle and the period of the SHM equal to the period of the UCM. Similarly, the motion of a particle in SHM can be considered as the projection of UCM onto a straight line. Furthermore, the equation for SHM can be derived from the equation for UCM by considering the projection of UCM onto a diameter of the circle.

Figure shows a particle P in uniform circular motion where **v** is the linear velocity and Ris radius of circle.

At a time t, the vector R which locates point P relative to the origin O, makes an angle θ with the x axis and the x component of vector R is,

$$x = Rcos\theta$$

Differentiating above equation two times w.r.t time we will get

$$\frac{dx}{dt} = R(sin\theta)\frac{d\theta}{dt}$$

$$\frac{d^2x}{dt^2} = -Rcos\theta\frac{d^2\theta}{dt^2}$$

$$\frac{d^2x}{dt^2} = -x\frac{d^2\theta}{dt^2}$$

As we knew that the left side of above equation is the definition of acceleration and right side is the definition of angular displacement so above equation simplified as,

$$a = -\omega^2 x$$

Which is the expression of the body executing simple harmonic motion. Therefore, we can conclude that uniform circular motion can be considered a form of simple harmonic motion. When referring to "circular motion," the acceleration connected to this concept is referred to as centripetal acceleration, which is defined as:

$$\vec{a} = -\frac{v^2}{R}\hat{r} = -R\omega^2\hat{r}$$

where $v = R\omega$. The acceleration along x direction is

$$a_x = -R\omega^2 \cos\theta$$

By substituting $R\cos\theta = x$ (i.e. $\cos\theta = \frac{x}{R}$) in above equation we will get

$$a_x = -\omega^2 x$$

which shows that point P executes simple harmonic motion.

Similarly, the y component of acceleration is

$$a = -R\omega^2 \sin\theta = -\omega^2 y \qquad \because\ y = R\sin\theta.$$

Composition of Two Simple Harmonic Oscillator along Same Line

When two simple harmonic motions with the same period are added together, the resulting motion is also a simple harmonic motion with the same period. The amplitude and phase of the resulting motion depend on the amplitudes and phases of the two individual motions.

Let's consider two simple harmonic motions with the same period T,

$$x_1(t) = A_1 \sin\omega t,$$

$$x_2(t) = A_2 \sin(\omega t + \varphi),$$

where A_1 and A_2 are the amplitudes and φ is any fixed angle and ω is the angular frequency. The sum of two motions is,

$$x(t) = x_1(t) + x_2(t),$$

$$x(t) = A_1 \sin\omega t + A_2 \sin(\omega t + \varphi).$$

To simplify this expression, we can use the identity

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Using this identity we get,

$$x(t) = A_1 \sin\omega t + A_2 \sin\omega t \cos\varphi + A_2 \cos\omega t \sin\varphi,$$

$$x(t) = \sin\omega t(A_1 + A_2\cos\varphi) + \cos\omega t(A_2 \sin\varphi).$$

Let $A_1 + A_2\cos\varphi = R\cos\varphi$ and $A_2 \sin\varphi = R\sin\theta$. After using some simple trigonometry we can write x in the form, $x = \sin(\omega t + \theta)$. Thus, the resultant motion is also simple harmonic motion along the same line and has the same time period. Its **amplitude is R** which is defined as:

$$R = \sqrt{\left(R_x^2 + R_y^2\right)}$$
$$R = \sqrt{(A_1 + A_2 \cos\phi)^2 + (A_2 \sin\phi)^2}$$
$$R = \sqrt{(A_1^2 + A_2^2 \cos^2\phi + 2A_1A_2\cos\phi + A_2^2 \sin^2\phi)}$$
$$R = \sqrt{(A_1^2 + A_2^2(\cos^2\phi + \sin^2\phi) + 2A_1A_2\cos\phi)}$$
$$R = \sqrt{(A_1^2 + A_2^2(1) + 2A_1A_2\cos\phi)}$$
$$R = \sqrt{(A_1^2 + A_2^2 + 2A_1A_2\cos\phi)}$$

And its direction is,

$$\tan\theta = \frac{A_2 \sin\varphi}{A_1 + A_2\cos\varphi}.$$

__Special Cases__

1. Constructive Interference

If $\boldsymbol{\phi = 0}$, the waves are in phase, resulting in a rise in the total amplitude to a certain degree from the combination of wave amplitudes.

$$\mathrm{R} = \sqrt{{A_1}^2 + {A_2}^2 + 2A_1A_2} = \sqrt{(A_1 + A_2)^2} = (A_1 + A_2),$$

And $\tan\theta = 0$,

which means that if displacements of two harmonic oscillators is equal then the resultant displacement is sum of two individual displacements.

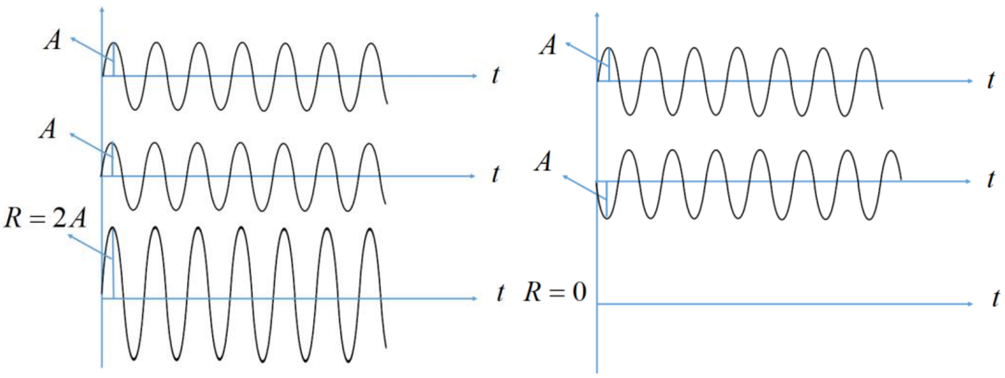


Figure 16. 4: Figure illustrates both constructive and destructive interference: the left side shows waves synchronizing in phase, resulting in reinforcement (constructive), while the right side displays waves being out of phase, causing them to nullify each other (destructive).

2. Destructive Interference

If $\boldsymbol{\phi} = \boldsymbol{\pi}$, the waves are out of phase, their combined amplitudes result in a reduction of the overall amplitude to a certain extent resultant is given by

$$\mathrm{R} = \sqrt{{A_1}^2 + {A_2}^2 - 2A_1A_2} = \sqrt{(A_1 - A_2)^2} = (A_1 - A_2),$$

and $\tan\theta = 0$,

and this is the case of destructive interference. The resultant will be zero, if the amplitude of both waves are equal.

Composition of Two Simple Harmonic Oscillator at Right angles to each other

Until this point, we have only discussed the one-dimensional simple harmonic oscillator, which implies motion along a single direction. However, a particle can also move in two or three dimensions. Therefore, let's now turn our attention to the two-dimensional simple harmonic oscillator.

Let's consider two simple harmonic motions with different amplitudes i.e.

$$\mathrm{x} = \mathrm{A}\sin\omega\mathrm{t} \quad \Longrightarrow \quad \frac{\mathrm{x}}{A} = \sin\omega\mathrm{t} \tag{7}$$

and

$$\mathrm{y} = \mathrm{B}\sin(\omega\mathrm{t} + \phi) \;\Longrightarrow\; \frac{\mathrm{y}}{\mathrm{B}} = \sin(\omega\mathrm{t} + \phi) \tag{8}$$

Using eq. 7 in the trigonometric relation $\cos\theta = \sqrt{1 - sin^2\theta}$, we can write

$$\cos\omega\mathrm{t} = \sqrt{1 - \frac{\mathrm{x}}{\mathrm{A}^2}^2}. \tag{9}$$

Similarly, by using another trigonometric identity $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ in eq. 8 we can write

$$\frac{\mathrm{y}}{\mathrm{B}} = \sin\omega\mathrm{t}\cos\phi + \cos\omega\mathrm{t}\sin\phi \tag{10}$$

Substituting the values of $\cos\omega t$ and $\sin\omega t$ from eq. 9 and from eq. 7 into eq. 10 we get

$$\frac{\mathrm{y}}{\mathrm{B}} = \frac{\mathrm{x}}{\mathrm{A}}\cos\phi + \sqrt{1 - \frac{\mathrm{x}}{\mathrm{A}^2}^2}\sin\phi$$

$$\frac{\mathrm{y}}{\mathrm{B}} - \frac{\mathrm{x}}{\mathrm{A}}\cos\phi = \sqrt{1 - \frac{\mathrm{x}}{\mathrm{A}^2}^2}\sin\phi$$

Squaring both sides $\quad \therefore (a - b)^2 = a^2 + b^2 - 2ab$

$$\left(\frac{\mathrm{y}}{\mathrm{B}} - \frac{\mathrm{x}}{\mathrm{A}}\cos\phi\right)^2 = \left(\sqrt{1 - \frac{\mathrm{x}}{\mathrm{A}^2}^2}\sin\phi\right)^2$$

$$\frac{y^2}{B^2}+\frac{x^2}{A^2}\cos^2\phi-2\frac{y}{B}\frac{x}{A}\cos\phi=\left(1-\frac{x^2}{A^2}\right)\sin^2\phi$$

$$\therefore \cos^2\phi = 1-\sin^2\phi$$

$$\frac{y^2}{B^2}+\frac{x^2}{A^2}(1-\sin^2\phi)-2\frac{y}{B}\frac{x}{A}\cos\phi=\left(1-\frac{x^2}{A^2}\right)\sin^2\phi$$

$$\frac{y^2}{B^2}+\frac{x^2}{A^2}-\frac{x^2}{A^2}\sin^2\phi-2\frac{y}{B}\frac{x}{A}\cos\phi=\left(1-\frac{x^2}{A^2}\right)\sin^2\phi$$

$$\frac{y^2}{B^2}+\frac{x^2}{A^2}-2\frac{y}{B}\frac{x}{A}\cos\phi=\left(1-\frac{x^2}{A^2}\right)\sin^2\phi+\frac{x^2}{A^2}\sin^2\phi$$

$$\frac{y^2}{B^2}+\frac{x^2}{A^2}-2\frac{y}{B}\frac{x}{A}\cos\phi=\left(1-\frac{x^2}{A^2}+\frac{x^2}{A^2}\right)\sin^2\phi$$

Thus, combined displacement of both harmonic motions is given by the relation

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}-2\frac{x}{A}\frac{y}{B}\cos\phi=\sin^2\phi \tag{11}$$

<u>Special Cases</u>

<u>Case I</u>

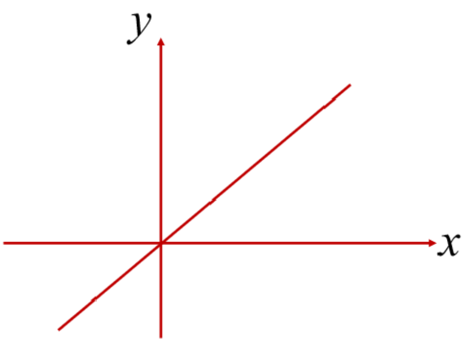
If $\phi = 0$, eq (11) becomes $\therefore Sin\ 0^o = 0,\ Cos\ 0^o = 1$

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}-2\frac{x}{A}\frac{y}{B}=0$$

It is a special condition in which the ellipse degenerates (i.e. width of one axis becomes zero).

$$\left(\frac{x}{A}-\frac{y}{B}\right)^2=0$$

$$\frac{x}{A}-\frac{y}{B}=0$$



Rearranging the above equation

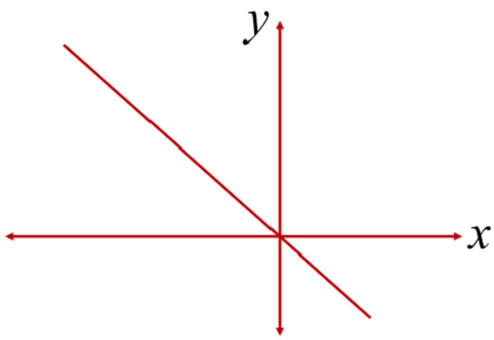
$$y=\left(\frac{B}{A}\right)x$$

This is the equation of a straight line (y = mx + c) with slope $c=\frac{B}{A}$. Thus, the resultant motion is a S. H. M. along a straight line passing through the origin.

<u>Case II</u>

If $\phi = \pi$, then eq. 11 becomes

$\therefore Sin\ 180^o = 0,\ Cos\ 180^o = -1$



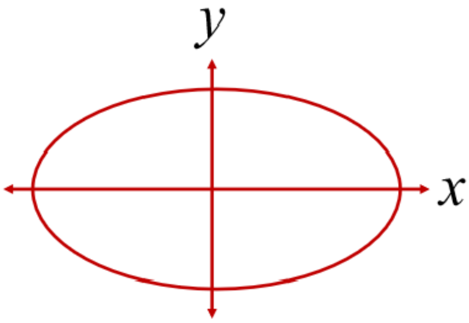
$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + 2\frac{x}{A}\frac{y}{B} = 0$$

$$\left(\frac{x}{A} + \frac{y}{B}\right)^2 = 0$$

$$\frac{x}{A} + \frac{y}{B} = 0$$

$$y = -\left(\frac{B}{A}\right)x$$

This is also the equation of straight line with negative slope c $= -\frac{B}{A}$.



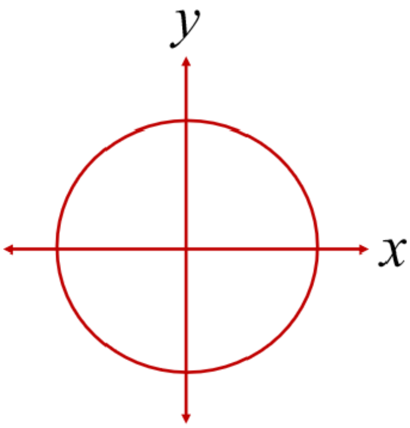
Case III

If $\phi = \frac{\pi}{2}$ (for A > B), we get

$\therefore Sin\ 90^o = 1, Cos\ 90^o = 0$

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1,$$

which is the equation of an ellipse.



Case IV

If $\phi = \frac{\pi}{2}$ (for A = B), we get

$\therefore Sin\ 90^o = 1, Cos\ 90^o = 0$

$$x^2 + y^2 = A^2$$

which is a circle with radius A.

Lissajous Figures

If two oscillations of different frequencies at right angles are combined, the resulting motion is more complicated. It is not even periodic unless the two frequencies are in the ratio of integers. This resulting curve is called Lissajous figures.

$$\frac{\omega_x}{\omega_y} = \text{integers} \Rightarrow \text{periodic motion}$$

Such a curve can be represented by the following pair of equations.

$$x = A\sin\omega_x t$$
$$y = B\sin\left(\omega_y t + \phi\right)$$

Damped Harmonic Motion

Damped harmonic motion refers to the motion of a system that is subjected to both a restoring force and a damping force. The restoring force tends to return the system to its equilibrium position, while the damping force tends to reduce the amplitude of the oscillation over time. This reduction of amplitude is due to the damping force. There are many cases of damping force including friction, air resistance, and internal forces.

Let us consider a simple model of damped oscillator in which the block slides on a friction-less surface.

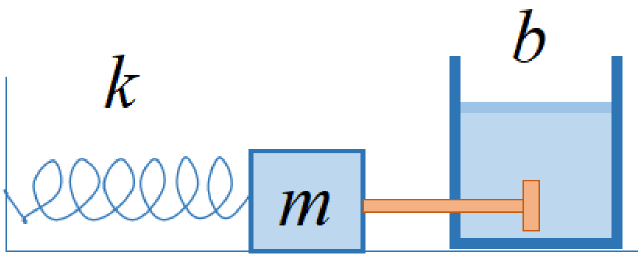


Figure 16. 5: A representation of damped harmonic oscillator.

The damping force is represented in terms of a (mass-less) vane that moves in a viscous fluid. This damping force due to fluid is $-bv_x$, where b is a positive constant called the damping constant that depends on the properties of the fluid and the size and shape of the object. With $\sum F_x = -kx - bv_x$, Newtons second law gives,

$$-kx - bv_x = ma_x$$

with $v_x = \frac{dx}{dt}$ and $a_x = \frac{d^2x}{dt^2}$ the above equation becomes,

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0. \qquad (12)$$

The solution of above equation is,

$$x(t) = x_m e^{\frac{-bt}{2m}} \cos(\omega' t + \varphi) \qquad (13)$$

Use the above equation to calculate $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$. Then by putting the values of x, $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ in eq. 12 to calculate the value of ω as

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)}.$$

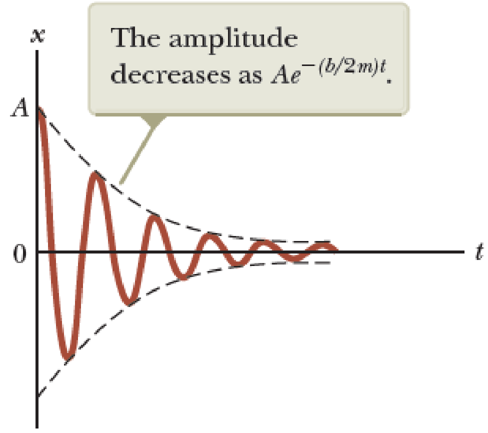


Figure 16. 6: Graph of displacement versus time showing decrease in amplitude with time for a damped oscillator.

The above solution is same as we obtain in the case of simple and physical pendulum except the extra term $e^{\frac{-bt}{2m}}$ which is due to damping force. This solution assumes that the damping constant is small so that the quantity under the square root in eq. 16.40 cannot be negative.

Figure shows the position as a function of time for an object oscillating in the presence of a retarding force. When the retarding force is small, the oscillatory character of the motion is preserved but the amplitude decreases exponentially in time, with the result that the motion ultimately becomes undetectable. Any system that behaves in this way is known as a **damped oscillator.** The dashed black lines in Figure, which define the *envelope* of the oscillatory curve, represent the exponential factor in Equation 13. This envelope shows that the amplitude decays exponentially with time.

When the magnitude of the retarding force is small such that $\frac{b}{2m} < \frac{k}{m}$, the system is said to be under-damped**.**

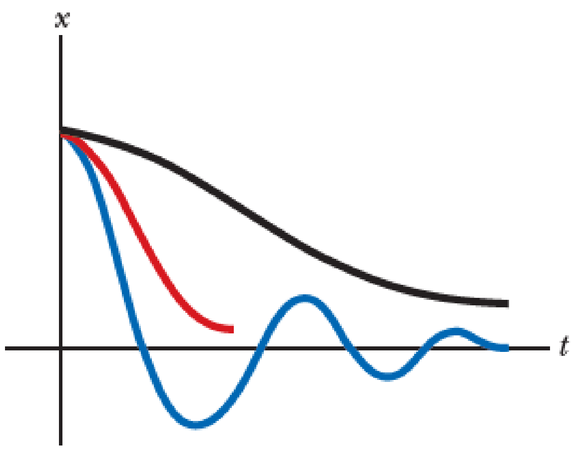


Figure 16. 7: Displacement versus time graphs of an underdamped oscillator (blue line), a critically damped oscillator (red), and an overdamped oscillator (black).

As the value of b increases, the amplitude of the oscillations decreases more and more rapidly. When b reaches a critical value b_c such that $b_c/_{2m} = \frac{k}{m}$,the system does not oscillate and is said to be **critically damped.** In this case, the system, once released from rest at some non-equilibrium position, approaches but does not pass through the equilibrium position. The graph of position versus time for this case is the red curve in Figure.

If the medium is so viscous that the retarding force is large compared with the restoring force—that is, if $\frac{b}{2m} > \frac{k}{m}$ the system is over-damped**.** Again, the displaced system, when free to move, does not oscillate but rather simply returns to its equilibrium position. As the damping increases, the time interval required for the system to approach equilibrium also increases as indicated by the black curve in Figure.

Forced Oscillations and Resonance

An oscillating system undergoes **forced oscillations** whenever it is pushed periodically by an external force. The frequency at which a system would oscillate if neither a driving nor a damping force are present is known as the **natural frequency.**

Resonance is the phenomenon that occurs when the driving frequency ω is equal to the natural frequency ω_o and the forced oscillations reach their maximum displacement amplitude, when the damping is negligible.

$$\omega = \omega_o$$

Where
$$\omega_o = \sqrt{\frac{k}{m}}$$

The corresponding frequency is called the **resonant angular frequency.**

We take the driving force to be $F_o cos \boldsymbol{\omega} t$. Newtons second law gives,

$$\sum F_x = -kx + F_o \cos \omega t$$

$$-kx + F_o \cos \omega t = ma_x$$

with $v_x = \frac{dx}{dt}$ and $a_x = \frac{d^2x}{dt^2}$ the above equation becomes,

$$m\frac{d^2x}{dt^2} + kx = F_o \qquad (14)$$

The solution of above equation is,

$$x(t) = \frac{F_o}{m(\omega_o^2 - \omega^2)} \cos\ \omega' t \qquad (15)$$

Use eq. 15 to calculate $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$. Then by putting the values of x and $\frac{d^2x}{dt^2}$ in eq. 14 we can verify the solution of eq. 14 i.e.

$$\frac{dx(t)}{dt} = \frac{(-\omega)F_o}{m(\omega_o^2 - \omega^2)} sin\ \omega t \tag{16}$$

$$\frac{d^2x(t)}{dt^2} = -\frac{(-\omega^2)F_o}{m(\omega_o^2 - \omega^2)} cos\ \omega t \tag{17}$$

After substituting eq. 16 and eq. 17 in eq. 14 we get

$$m\left(\frac{(-\omega^2)F_o}{m(\omega_o^2 - \omega^2)} cos\ \omega t\right) + k\frac{F_o}{m(\omega_o^2 - \omega^2)} cos\ \omega t = F_o\, cos\, \omega t$$

$$\frac{(-\omega^2)mF_o + kF_o}{m(\omega_o^2 - \omega^2)} cos\ \omega t = F_o\, cos\, \omega t \tag{18}$$

Note that the amplitude becomes infinitely large when ω approaches ω_o. This occurs due to the absence of a damping term in the equation. However, when damping is present, the amplitude remains large as ω approaches ω_o but remains finite. All mechanical structures such as buildings, bridges, and airplanes have one or more natural frequencies of oscillation. If the structure is subject to a driving frequency that matches one of the natural frequencies, the resulting large amplitude of oscillation can have disastrous consequences. The collapse of roadways and bridges in earthquakes is a more serious outcome.