Physics-PHY101-Lecture 9

MOMENTUM

Momentum is the "quantity of motion" possessed by a body. More precisely, it is defined as,

"The product of mass and velocity of a body".

$$p = mv$$

Dimensions of momentum:

The dimensions of momentum are MLT^{-1}.

Unit of momentum:

The units of momentum are kg-m/s.

Momentum is a vector quantity and has both magnitude and direction.

Newton's Second Law and Momentum:

Newton's Second Law can be expressed in terms of momentum. It is defined as,

"The rate of change of momentum of a body is equal to the resultant force acting on the body and is in the direction of that force".

$$m\vec{\mathbf{a}} = \vec{\mathbf{F}} \text{ (old form)}$$

$$\frac{d\vec{\mathbf{p}}}{dt} = \vec{\mathbf{F}} \quad \text{(new form of Newton's Law)}$$

They are the same:

$$\frac{d\vec{\mathbf{p}}}{dt} = \frac{d(m\vec{\mathbf{v}})}{dt} = m\frac{d\vec{\mathbf{v}}}{dt} = m\vec{\mathbf{a}} = \vec{\mathbf{F}}$$

Newton's 2nd law for several particles:

When there are many particles, then total momentum is,

$$
\begin{aligned}
\vec{\mathbf{P}} &= \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 + \cdots \vec{\mathbf{p}}_N \\
\frac{d}{dt}\vec{\mathbf{P}} &= \frac{d}{dt}\vec{\mathbf{p}}_1 + \frac{d}{dt}\vec{\mathbf{p}}_2 + \cdots \frac{d}{dt}\vec{\mathbf{p}}_N \\
\frac{d}{dt}\vec{\mathbf{P}} &= \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots \vec{\mathbf{F}}_N \\
\frac{d}{dt}\vec{\mathbf{P}} &= \sum_{i=1}^{i=N} \vec{\mathbf{F}}_i = \text{total external force}
\end{aligned}
$$

This shows that when there are several particles, the rate at which the total momentum changes is equal to the total force.

Conservation of Linear Momentum:

If the sum of the total external forces vanishes, then the total momentum is conserved.

$$
\begin{aligned}
&\sum \vec{\mathbf{F}}_{ext} = 0 \text{ Then,} \\
&\frac{d\vec{\mathbf{P}}}{dt} = 0 \\
&\vec{\mathbf{P}} = \text{constant}
\end{aligned}
$$

Momentum is conserved for an isolated system. This is quite independent of what sort of forces act between the bodies i.e., electric force, gravitational force, etc. - or how complicated these are. We shall see why this is so important from the following examples.

Problem 1:

Two balls, which can only move along a straight line, collide with each other as shown in figure 9.1. The initial momentum is $p_i = m_1u_1 + m_2u_2$ and the final momentum is $p_f = m_1v_1 + m_2v_2$. Obviously one ball exerts a force on the other when they collide, so their momentum changes. But, from the fact that there is no external force acting on the balls,

$$p_i = p_f$$

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

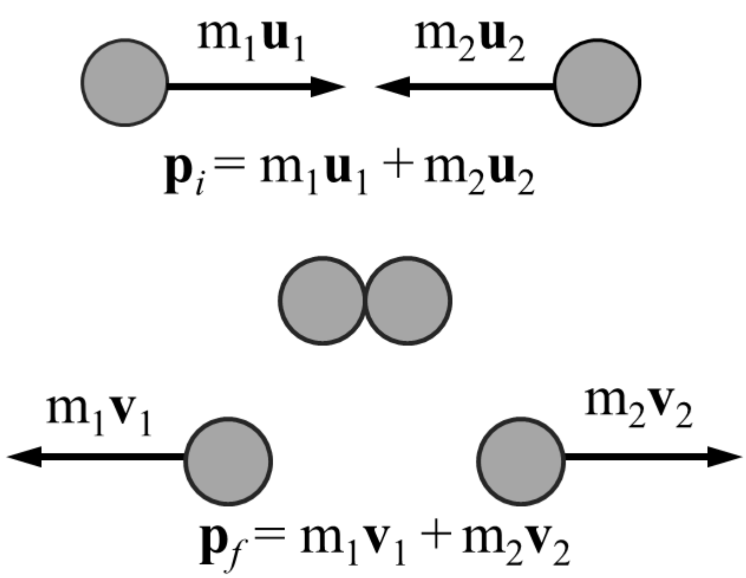


Figure 9.1. Momentum Transfer in a one-dimensional collision between the two balls along a Straight Path.

Problem 2:

A bomb at rest explodes into two fragments as shown in figure 9.2. Before the explosion the total momentum was zero. So obviously it is zero after the explosion as well.

$$\mathbf{P}_i = \mathbf{0} \quad \text{and}$$
$$\mathbf{P}_f = \mathbf{0} \quad \text{but } \mathbf{P}_f = m_1 \mathrm{v}_1 + m_2 \mathrm{v}_2$$
$$\therefore \; m_1 \mathrm{v}_1 = -m_2 \mathrm{v}_2$$

During the time when the explosion happens, the forces acting upon the pieces are very complicated and change rapidly with time. But when all is said and done, there are two pieces flying away with a total zero final momentum. In other words, the fragments fly apart with equal momentum but in opposite directions. The center-of-mass stays at rest. So, knowing the velocity of one fragment permits knowing the velocity of the other fragment.

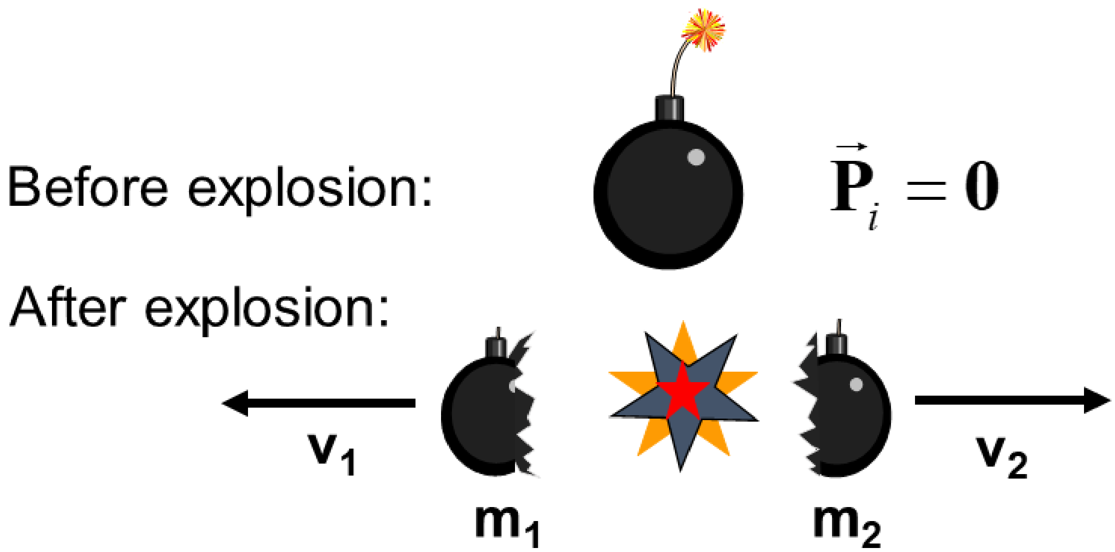


Figure 9.2. The physics of explosions as a bomb at rest divides into two fragments, shows momentum conservation.

<u>Problem 3:</u>

A bomb of mass 10 kg, initially at rest, explodes into two pieces of masses 4 kg and 6 kg. If the speed of the 4 kg piece is 12 m/s, find the speed of the 6 kg piece.

<u>Solution:</u>

Mass of bomb=$M = 10kg$

After explosion,

$m_1 = 4kg$

$V_1 = 12m/s$

$m_2 = 6kg$

$V_2 = ?$

Using formula,

$m_1V_1 = m_2V_2$

$$V_2 = \frac{m_1V_1}{m_2} = \frac{4\times12}{6} = 8m/s$$

$V_2 = 8m/s$

Problem 4:

A rocket conserves momentum through the principle of action and reaction, as described by Newton's third law of motion: "For every action, there is an equal and opposite reaction." In the context of a rocket, this law explains how momentum is conserved during the propulsion process as shown in figure 9.3.

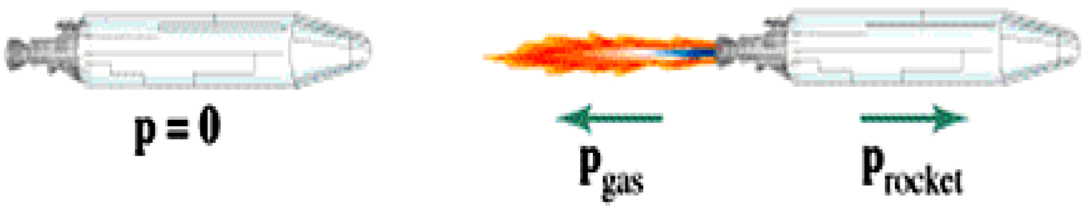


Figure 9.3. Momentum Conservation in the Propulsion Process.

When a rocket engine expels exhaust gases at high velocity in one direction, it generates a force in the opposite direction according to Newton's third law. This force is what propels the rocket forward. The expelled gases have mass and velocity, contributing to a significant momentum in the opposite direction. Since momentum is conserved in an isolated system, the rocket gains an equal and opposite momentum. This results in the rocket moving forward. Mathematically, this can be expressed using the equation for momentum:

$$p_i = p_f$$
$$p_{\text{gas}} + p_{\text{rocket}} = 0$$
$$p_{\text{gas}} = -p_{\text{rocket}}$$

Since the initial momentum is small or zero, the final momentum of the rocket is primarily determined by the momentum of the expelled exhaust gases. The larger the momentum of the gases, the larger the momentum gained by the rocket.

Problem 5:

When a gun is fired, there are two main components involved: the gun itself and the bullet (projectile) being expelled from the gun as shown in figure 9.4. The gun and the bullet are initially at rest, so their initial momentum is zero. When the gun is fired, the bullet accelerates out of gun and gains momentum in one direction (forward), the gun and the shooter gain an equal and opposite momentum in the opposite direction (backward).

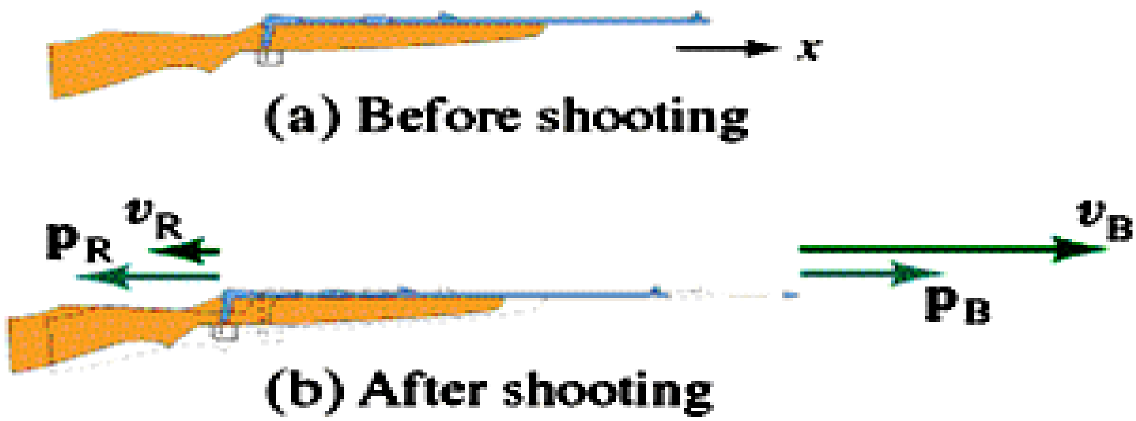


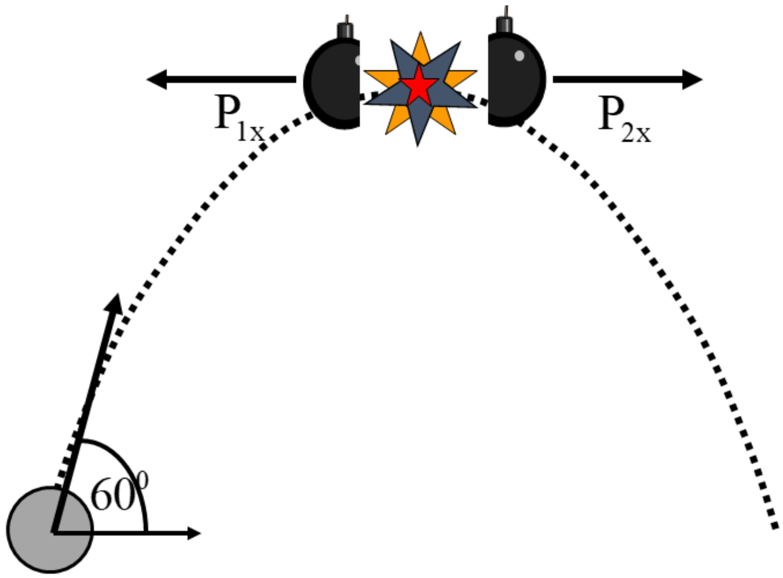
Figure 9.4. The dynamics of gun firing, where bullet acceleration and recoil impart equal and opposite momentum to maintain conservation.

$$p_i = p_f$$
$$p_{\text{rifle}} + p_{\text{bullet}} = 0$$
$$p_{\text{rifle}} = -p_{\text{bullet}}$$

This conservation of momentum ensures that the total momentum before and after the gunshot remains zero, as no external forces are acting on the system.

<u>Problem 6:</u>

A shell is fired from a cannon with a speed of 10 m/s at an angle 60^0 with the horizontal. At the highest point in its path, it explodes into two pieces of equal masses. One of the pieces retraces its path to the cannon. Find the velocity of the other piece immediately after the explosion.



Solution:

Before explosion:

$V_{ix} = V_i \cos\theta = 10\cos 60 = 5\,m/s$

$P_x = M.V_x = 5\,\text{M kg}\,m/s$

After explosion:

$$MV_{ix} = -\frac{M}{2}V_{1x} + \frac{M}{2}V_2$$

$$\therefore V_1 = V_{ix}, \text{ and } V_2 = V_{2x}$$

$$V_{ix} = -\frac{1}{2}V_{ix} + \frac{1}{2}V_{2x}$$

$$2V_{ix} = -V_{ix} + V_{2x}$$

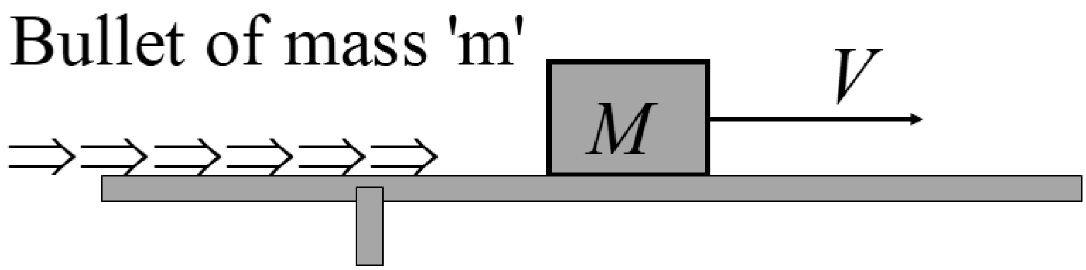
$$2V_{ix} + V_{ix} = V_{2x}$$

$$V_{2x} = 3V_{ix}$$

$$V_{2x} = 3\times 5 = 15\,m/s$$

Problem 7:

A stream of bullets, each of mass m, is fired horizontally with a speed v into a large wooden block of mass M that is initially at rest on a horizontal table. If the block is free to slide without friction, what speed will it acquire after it has absorbed N bullets?



Solution:

Linear momentum is conserved:

$$P_f = P_i$$

$$(M + Nm)V = N(m\text{v})$$

$$V = \frac{Nm}{(M + Nm)}\text{v}$$

Problem 7:

A cannon with mass M equal to 1300 kg fires a 72 kg ball in horizontal direction with a muzzle speed v of 55 m/s. The cannon is mounted so that it can recoil freely.

a) What is the velocity V of the recoiling cannon with respect to the earth?

b) What is the initial velocity V_E of the ball with respect to the earth?

Solution:

a) Linear momentum is conserved:

$$P_f = P_i$$

$$MV + m(\text{v} + V) = 0$$

$$V = -\frac{m\text{v}}{m + M} = -\frac{(72kg)(55m/s)}{1300kg + 72kg}$$

$$V = -2.9\ m/s$$

b) $V_E = \text{v} + V$

$$V_E = 55m/s + (-2.9\ m/s)$$

$$V_E = 52m/s$$

Problem 8:

Consider a body at point A which is at height h from the ground. The body has no friction, when it is at point A, but when it comes down straight to earth it has friction which renders the body to move again to the curve as shown in fig. For that case, we want to know,

a) Is the momentum conserved?
b) Where does the particle finally come to rest?

Solution:

Suppose the total distance moved on the flat part before it comes to rest is ‘x’,

$$mgh = \frac{1}{2}mv^2$$
$$\therefore \quad \frac{1}{2}mv^2 = f.x$$
$$\frac{1}{2}mv^2 = \mu Nx = \mu mgx$$
$$mgh = \mu mgx$$
$$x = \frac{h}{\mu}$$

Impulse and Momentum:

When you hit your thumb with a hammer it hurts, doesn't it? Why? Because a large amount of momentum has been destroyed in a short amount of time. If you wrap your thumb with foam, it will hurt less. To understand this better, remember that force is the rate of change of momentum,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

$$d\mathbf{p} = \mathbf{F}dt = I$$

Impulse:

It is defined as, "force acting over time to change the momentum of an object".

If the force changes with time between the limits, then one should define impulse as,

$$\mathrm{I} \equiv \int_{t_1}^{t_2} \mathrm{F}dt$$

$$\text{Since } \int_{t_1}^{t_2} \mathrm{F}dt = \int_{p_i}^{p_f} dp$$

$$\mathrm{I} = \mathbf{p}_f - \mathbf{p}_i$$

In words, the change of momentum equals the impulse, which is equal to the area under the curve of force versus time. Even if you wrap your thumb in foam, the impulse is the same. But the force is definitely not.

Sometimes we only know the force numerically (i.e., there is no expression like F=something). But we still know what the integral means: it is the area under the curve of force versus time. The curve here is that of a hammer striking a table. Before the hammer strikes, the force is zero, reaches a peak, and goes back to zero.

Graphical representation:

1. Let us draw a graph between force F(t) and time (Δt) as shown in figure 9.5. From this graph, one can see the area under the curve gives impulse and the value of force is maximum at t_2 but minimum at t_1 and t_3 (also, F(t) is same at t_1 and t_3). Below t_1 and above t_3, F(t) begins to zero.

$$I = F_{av}\Delta t$$

Area under the curve = impulse

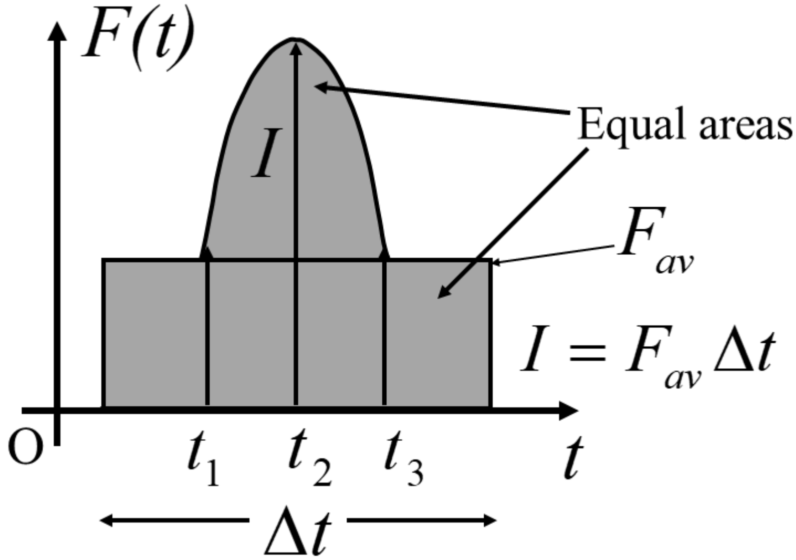


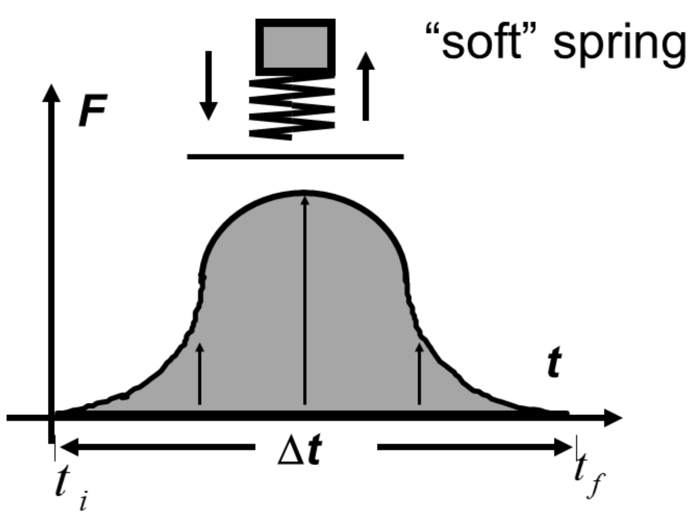
Figure 9.5. Graphical representation of F(t) and Δt, unraveling impulse as the area under the curve, with maximal force at t_2 and minimum force at t_1 and t_3.

2. Force may be of any type. Let us now consider a case where a soft spring is taken upon which when force is applied, it shows less resistance as being soft in nature. When a weight is taken over a soft spring it shows,

$$\Delta t \textit{ is big, } F \textit{ is small}$$

But the product of force and time gives us a constant value which is called impulse.

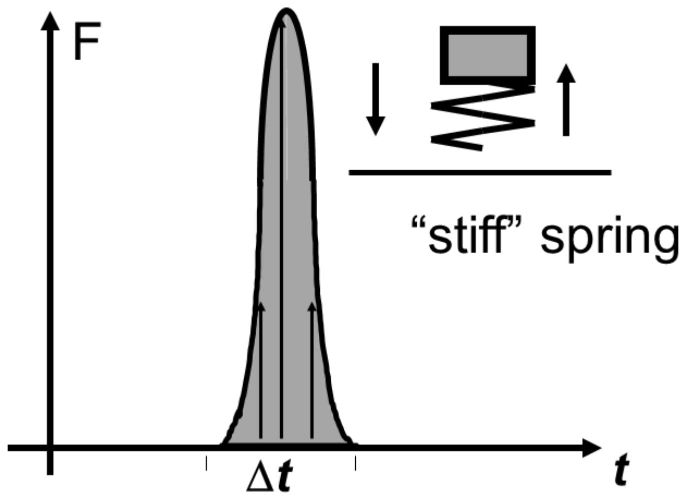
$$I = F\Delta t$$



3. Now consider a case of a stiff spring. When a small force is applied to that spring, it may create a significant resistance.

$$\Delta t \text{ is small, } F \text{ is big}$$

But the product of force and time again gives us a constant value which is called impulse.



Conclusion:

Momentum is a concept that is useful because of Newton's 2nd law. Physics is a quantitative field. It turns out that momentum conservation still holds even when we go beyond Newton's laws. But momentum is not just mv.

Practice Questions:

Q: Would you rather land with your legs bending or stiff?

Q: Why do cricket fielders move their hands backwards when catching a fast ball?

Q: Why do railway carriages have dampers at the front and back?