examples of equilibrium: ball on floor, ladder resting against a wall, a car moving at steady speed or an airplane in steady flight, the complete solar system. Of course nothing is perfect, but we can improve approximations to any degree of accuracy.

How do we define equilibrium?

Physics

Equilibrium of rigid bodies

The translational motion of the center of mass of a rigid body is governed by

$$\vec{F} = \frac{d\,\vec{p}}{dt}$$

where $\vec{F} = \sum \vec{F}_{ext}$ is net external force.

In equilibrium: $\dfrac{d\,\vec{p}}{dt} = 0 \;\Rightarrow\; \vec{F} = 0$

(Must hold for all components !!).

The rotational motion of a rigid body is governed by:

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

where $\vec{\tau} = \sum \vec{\tau}_{ext}$ is net external torque.

In equilibrium $\frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{\tau} = 0.$

Conditions for Equilibrium

A rigid body is in mechanical equilibrium if both the linear momentum $\vec{P}$ and angular momentum $\vec{L}$ have a constant value.

i.e., $\dfrac{d\vec{P}}{dt} = 0$ and $\dfrac{d\vec{L}}{dt} = 0$

$\vec{P} = 0$ and $\vec{L} = 0 \Rightarrow$ static equilibrium

How does a lever work ?

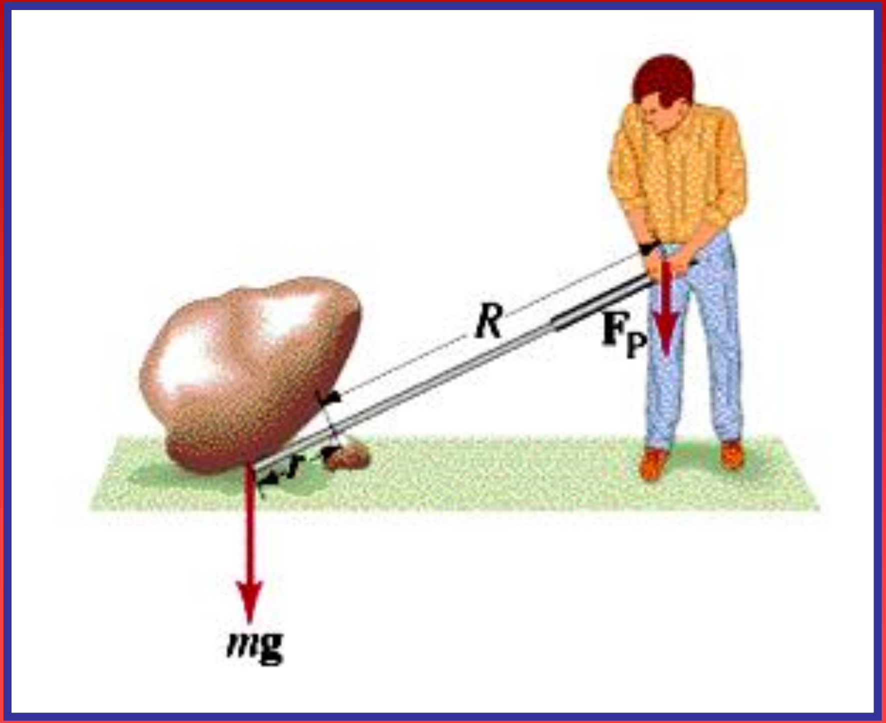
Torques balance about an axis through the fulcrum:

$$F_P \cdot R = mg \cdot r$$

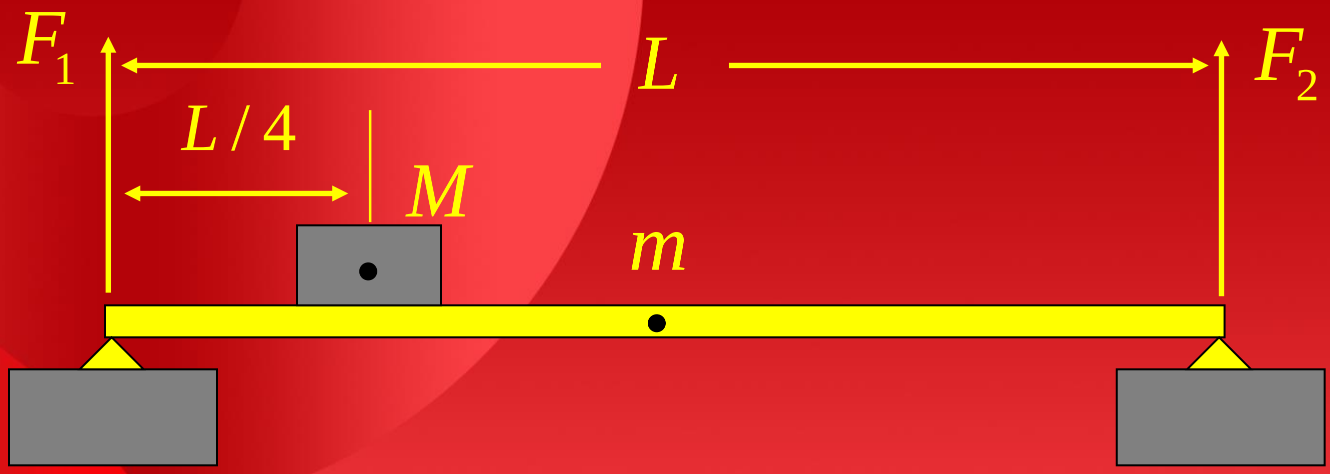
Solve for the applied force:

$$F_P = mg\,\frac{r}{R}$$

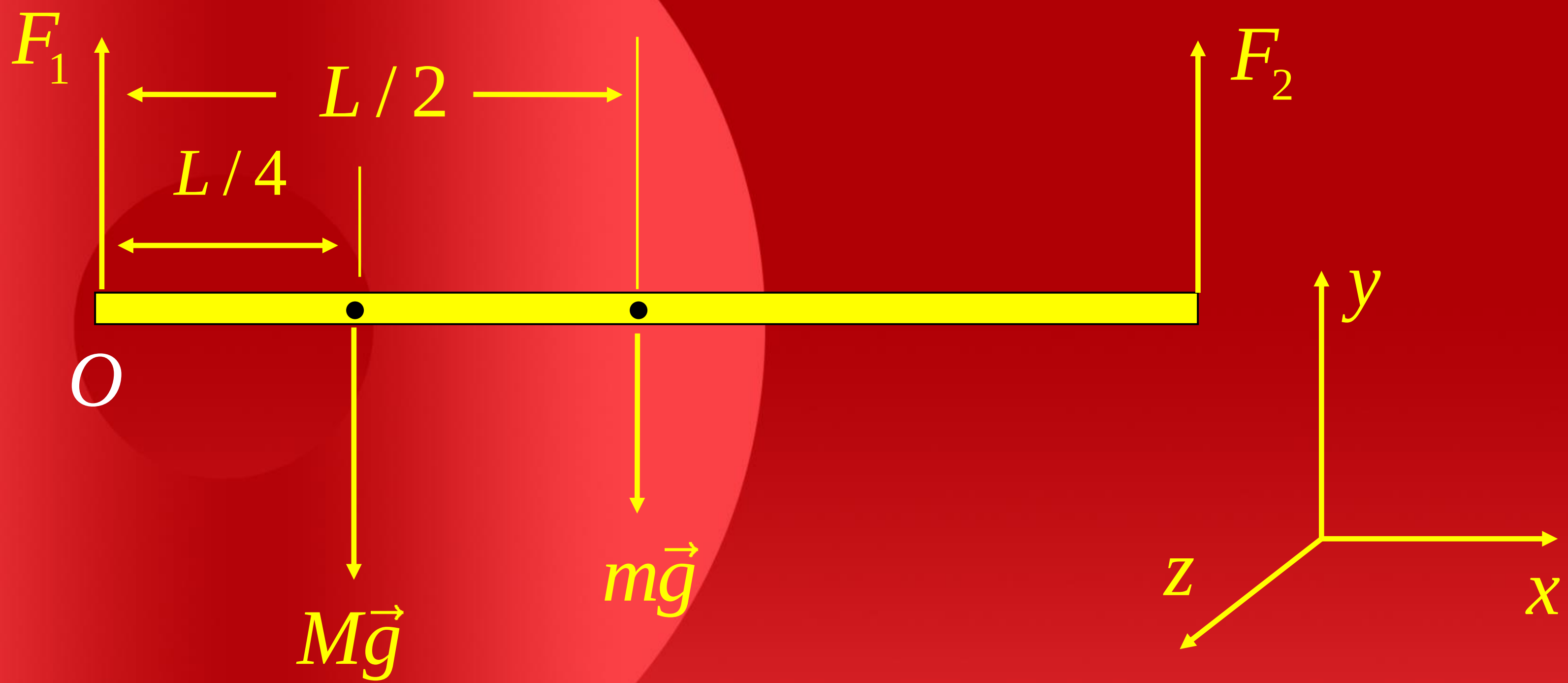
F_p can be rather small depending on *r* and *R*



Example of equilibrium



Find forces F_1 and F_2



$$\sum F_y = F_1 + F_2 - Mg - mg = 0$$

$$\sum \tau_y = (F_1)(0) + (F_2)(L) - (Mg)(L/4) - (mg)(L/2) = 0$$

$$\therefore \quad F_2 = \frac{(M+2m)g}{4}$$

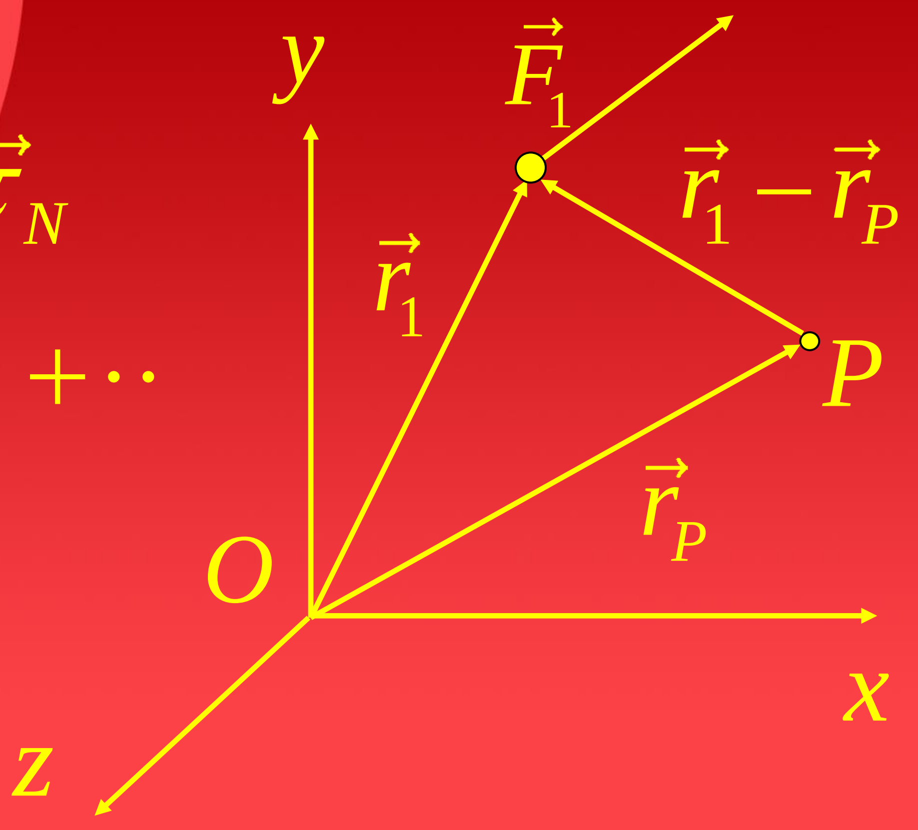
$$F_1 + F_2 = (M+m)g$$

$$F_1 = \frac{(3M+2m)g}{4}$$

For a body in equilibrium, the choice of origin for calculating torques is unimportant.

Proof:

$$\vec{\tau}_O = \vec{\tau}_1 + \vec{\tau}_2 + \cdots + \vec{\tau}_N$$
$$= \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \cdots$$
$$\cdots + \vec{r}_N \times \vec{F}_N$$

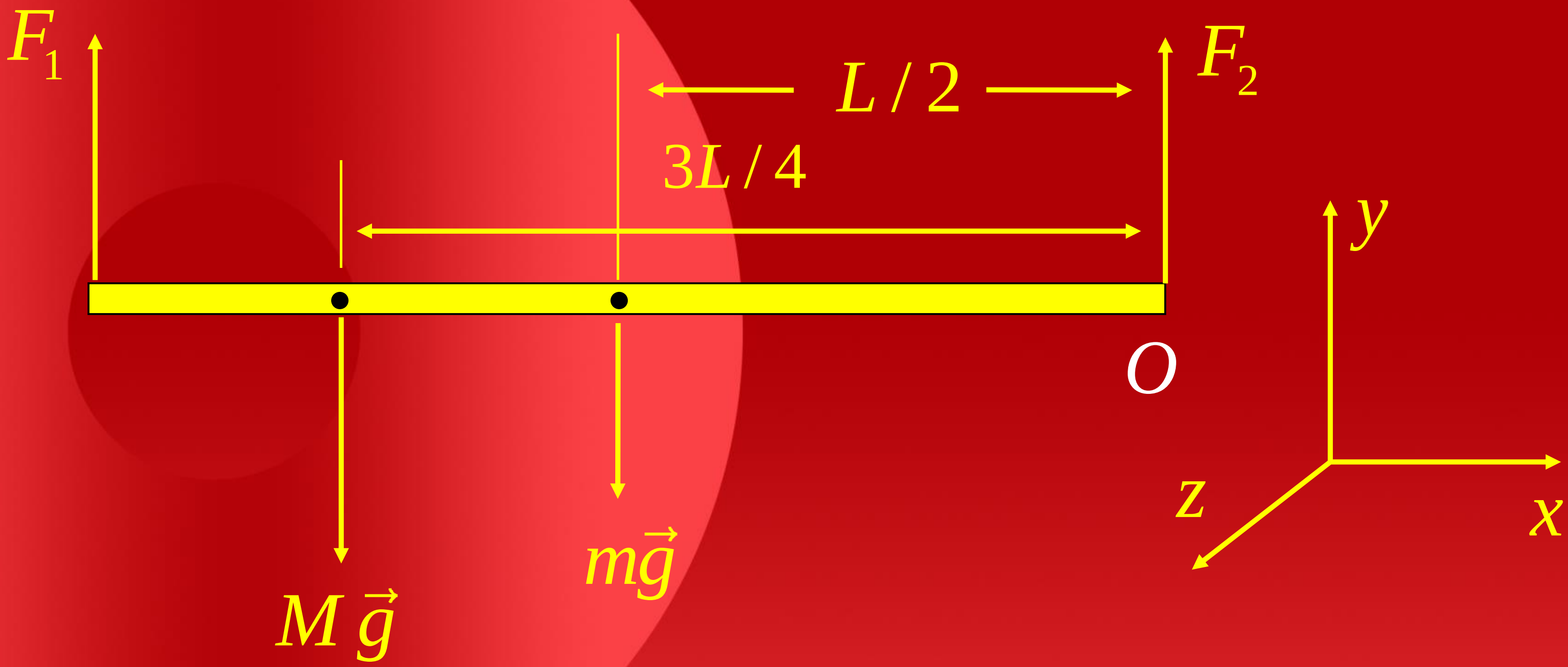


$$
\begin{aligned}
\vec{\tau}_P &= \left(\vec{r}_1 - \vec{r}_P\right)\times\vec{F}_1 + \left(\vec{r}_2 - \vec{r}_P\right)\times\vec{F}_1 + \cdots \\
&\quad \cdots + \left(\vec{r}_N - \vec{r}_P\right)\times\vec{F}_N \\
&= \left[\vec{r}_1\times\vec{F}_1 + \vec{r}_2\times\vec{F}_2 + \cdots + \vec{r}_N\times\vec{F}_N\right] \\
&\quad - \left[\vec{r}_P\times\vec{F}_1 + \vec{r}_P\times\vec{F}_2 + \cdots + \vec{r}_P\times\vec{F}_N\right] \\
&= \vec{\tau}_O - \left[\vec{r}_P\times\left(\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N\right)\right] \\
&= \vec{\tau}_O - \left[\vec{r}_P\times\left(\sum\vec{F}_{ext}\right)\right]
\end{aligned}
$$

but $\sum \vec{F}_{ext} = 0,$ for a body in translational equilibrium

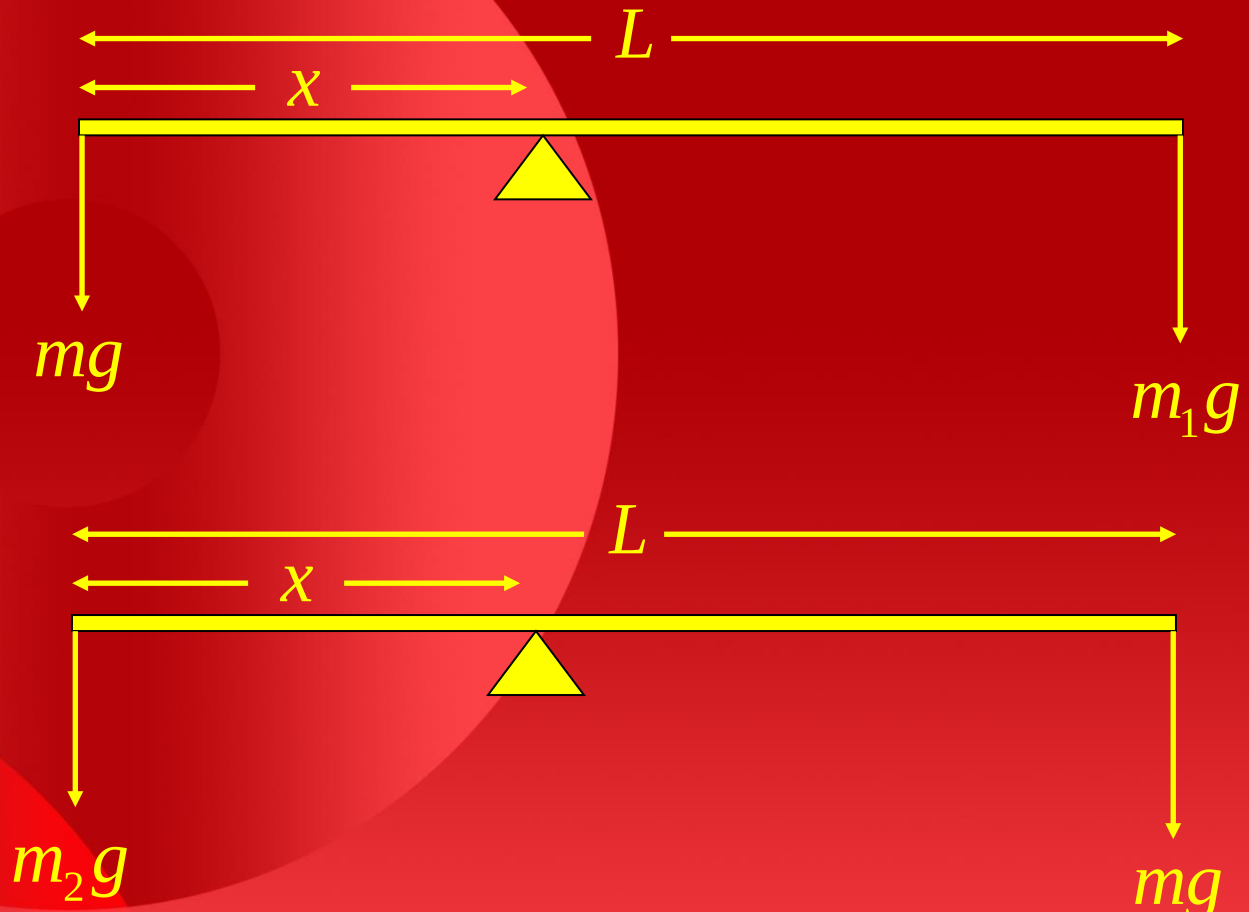
$$\therefore \quad \vec{\tau}_P = \vec{\tau}_O$$

Hence the torque about any two points has the same value when the body is in translational equilibrium



$$\sum \tau_y = -(F_1)(L) + (F_2)(0) + (Mg)(3L/4) + (mg)(L/2) = 0$$

$$F_1 = \left(\frac{3M + 2m}{4}\right)\mathrm{g} \;\text{ and }\; F_2 = \frac{(M + 2m)g}{4}$$



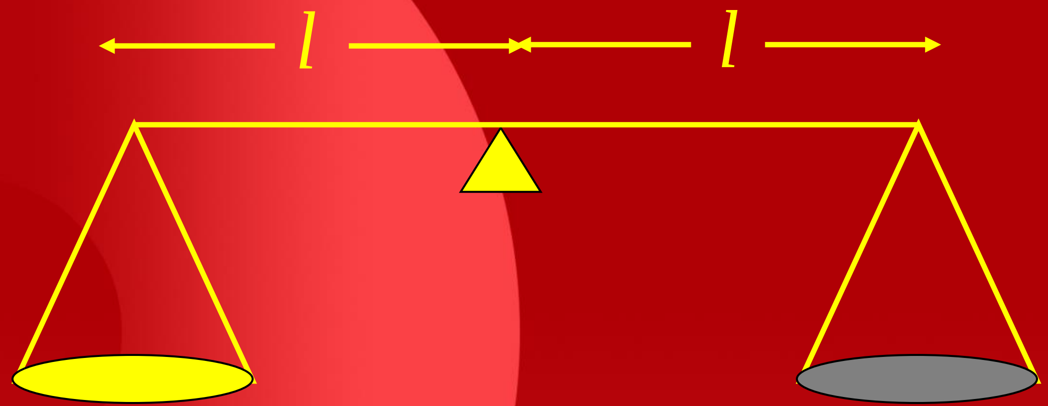
$$m = ?$$

Taking the torques about the knife edge in the two cases, we have

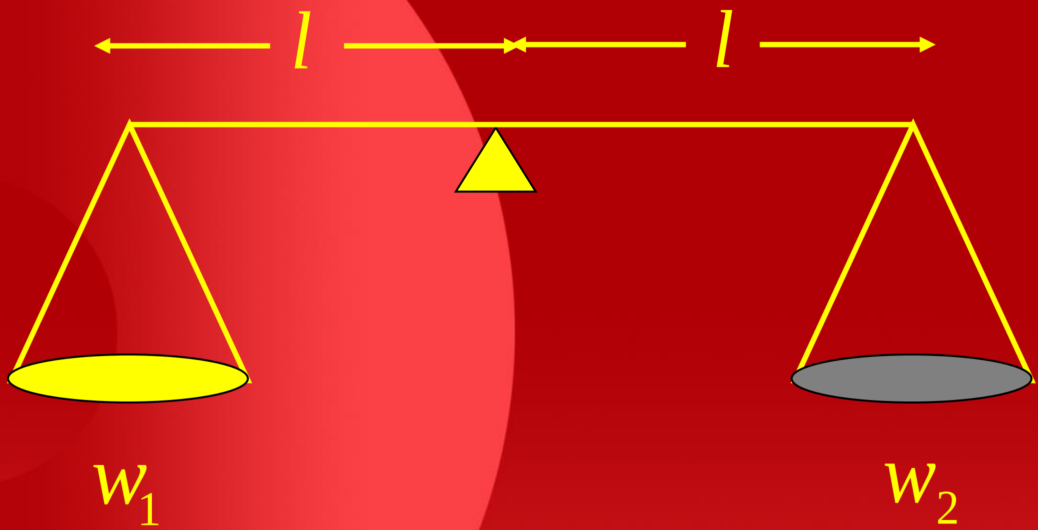
$$mgx = m_1 g\left(L - x\right)$$

$$m_2 g x = mg\left(L - x\right)$$

$$\Rightarrow \frac{m}{m_2} = \frac{m_1}{m} \text{ or } m = \sqrt{m_1 m_2}$$



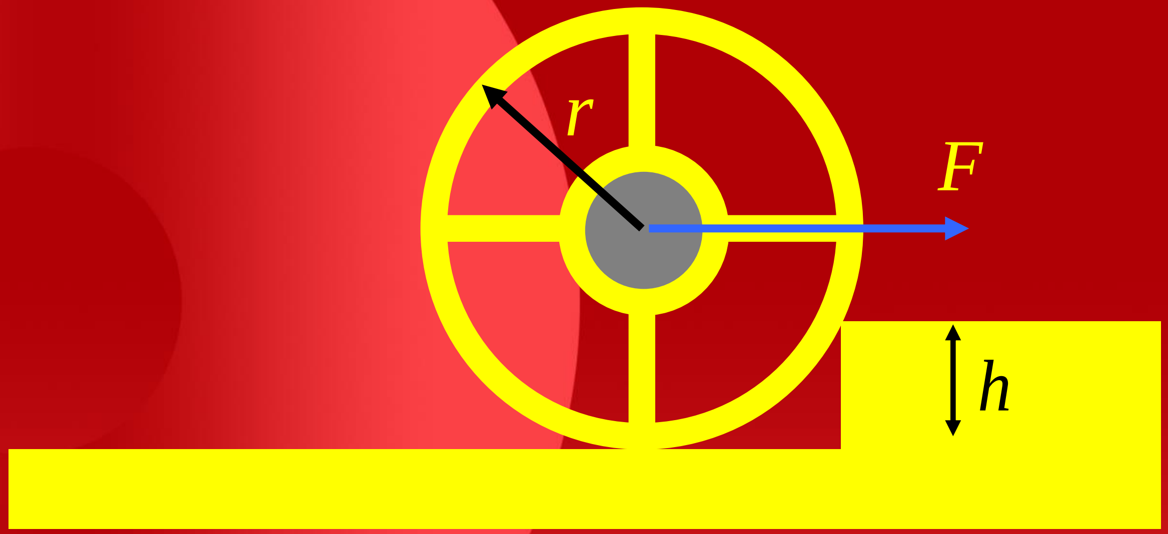
A false balance has equal arms. An object weighs x when placed in one pan and y when placed in the other pan. What is the true weight ?



Let the weights of the pans be w_1 and w_2 and the true weight be w. Then:

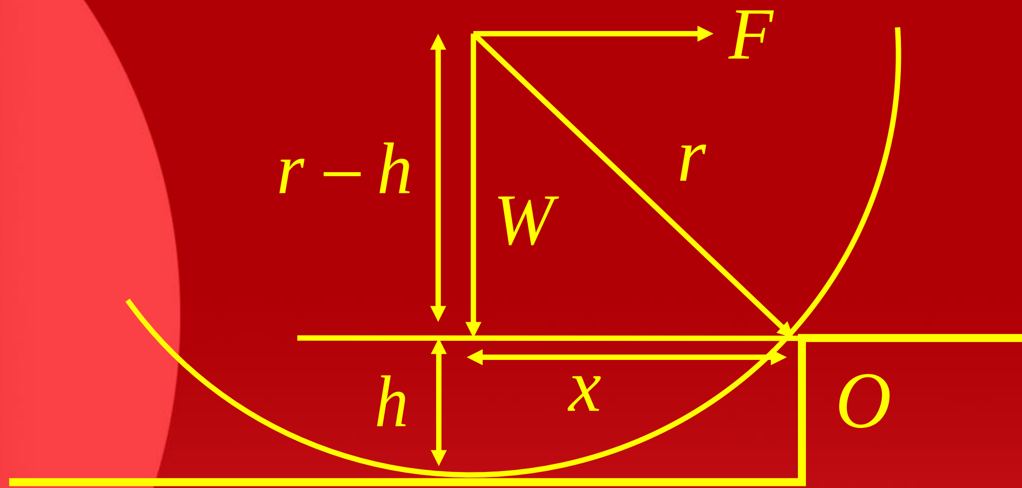
$$(w+w_1)l=(x+w_2)l$$
$$(w+w_2)l=(y+w_1)l$$

$$\Rightarrow\ w=\frac{(x+y)}{2}$$



What minimum force F applied horizontally at the axle of the wheel is necessary to raise the wheel over an obstacle of height h ?

The normal force vanishes as wheel leaves the ground.



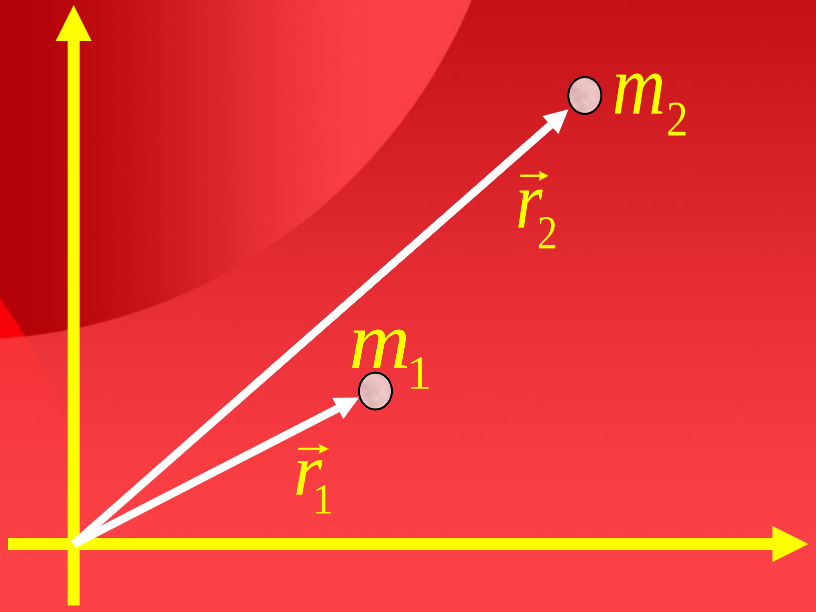
Take torques about O: $\quad Wx = F(r-h)$.

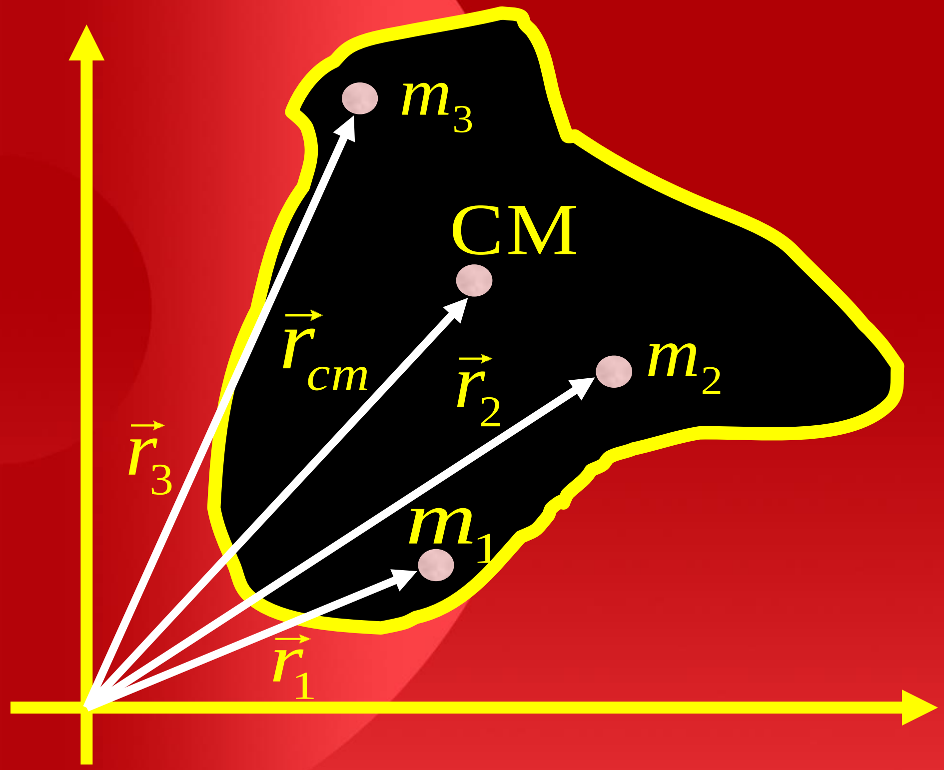
Use: $x^2 + (r-h)^2 = r^2$

$$\Rightarrow\ F = W\frac{\sqrt{2rh-h^2}}{r-h}.$$

Review: For two masses the CM is:

$$\vec{r}_{cm} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$



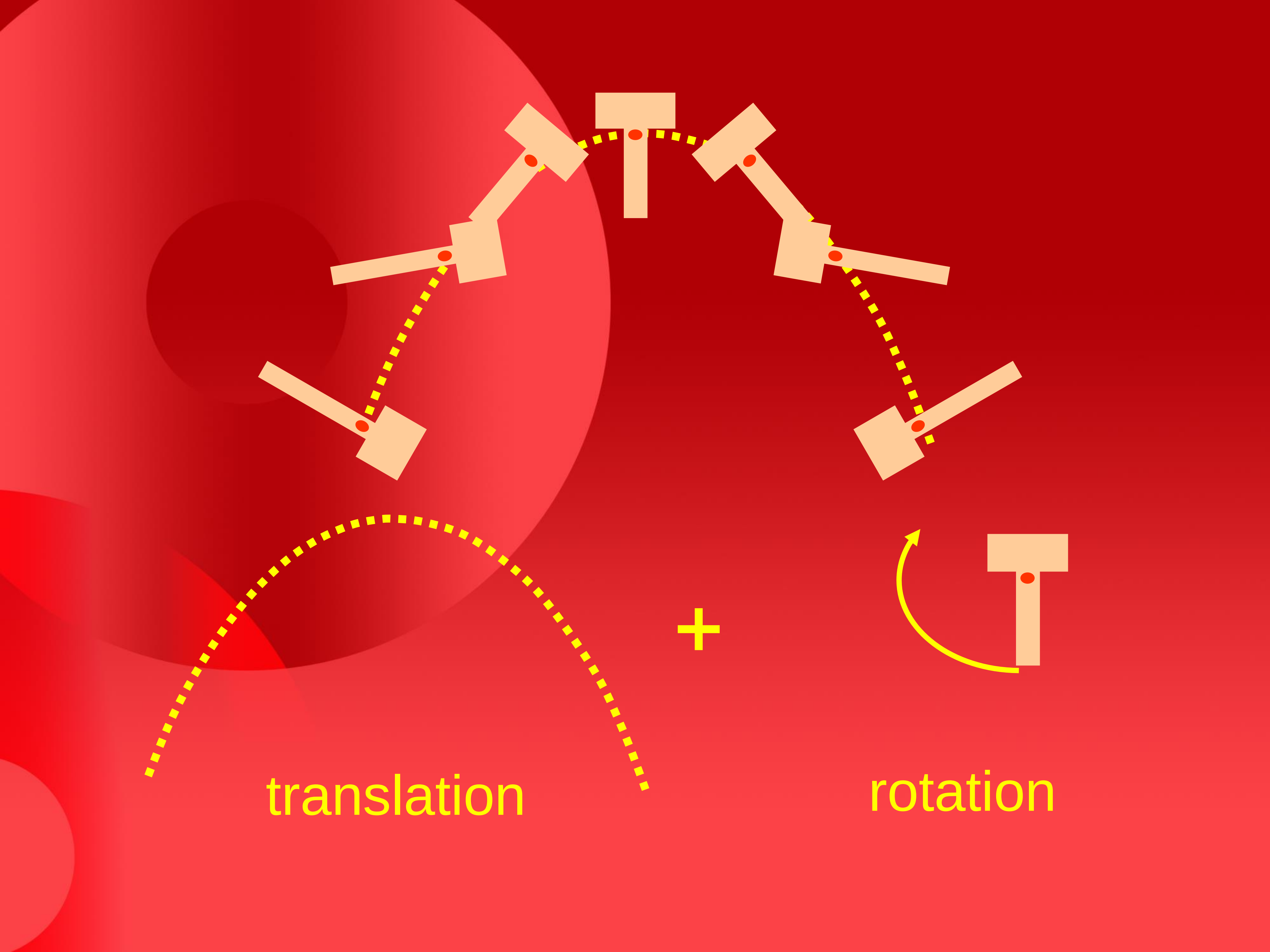


$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N}{m_1 + m_2 + \cdots + m_N}$$

$$\vec{F} = M\,\vec{a}_{cm}$$

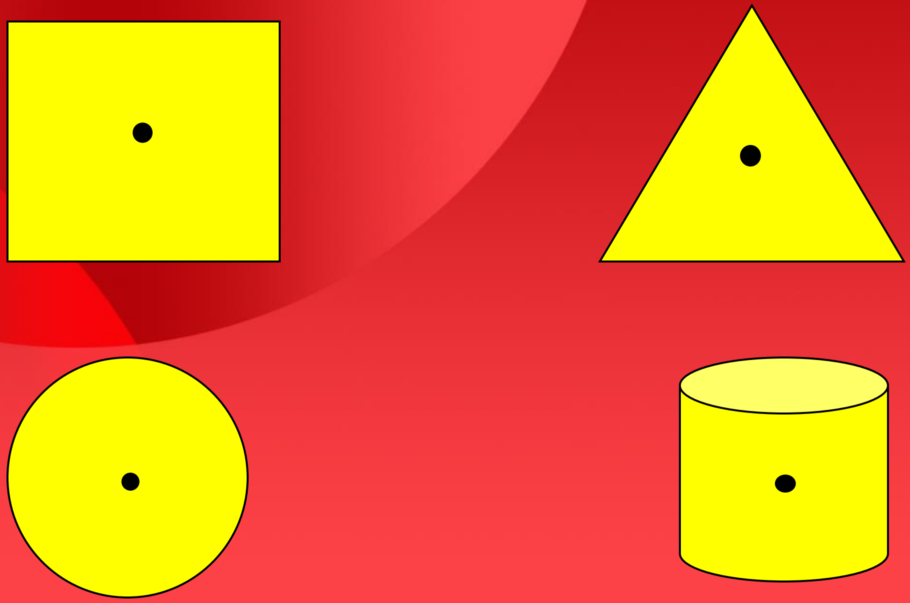
where,

$$\vec{F} = \sum \vec{F}_{ext}$$


+
translation
rotation

Centre of gravity

The centre of gravity is the average location of the weight of an object.



Suppose the gravitational acceleration $\vec{g}$ has the same value at all points of a body.

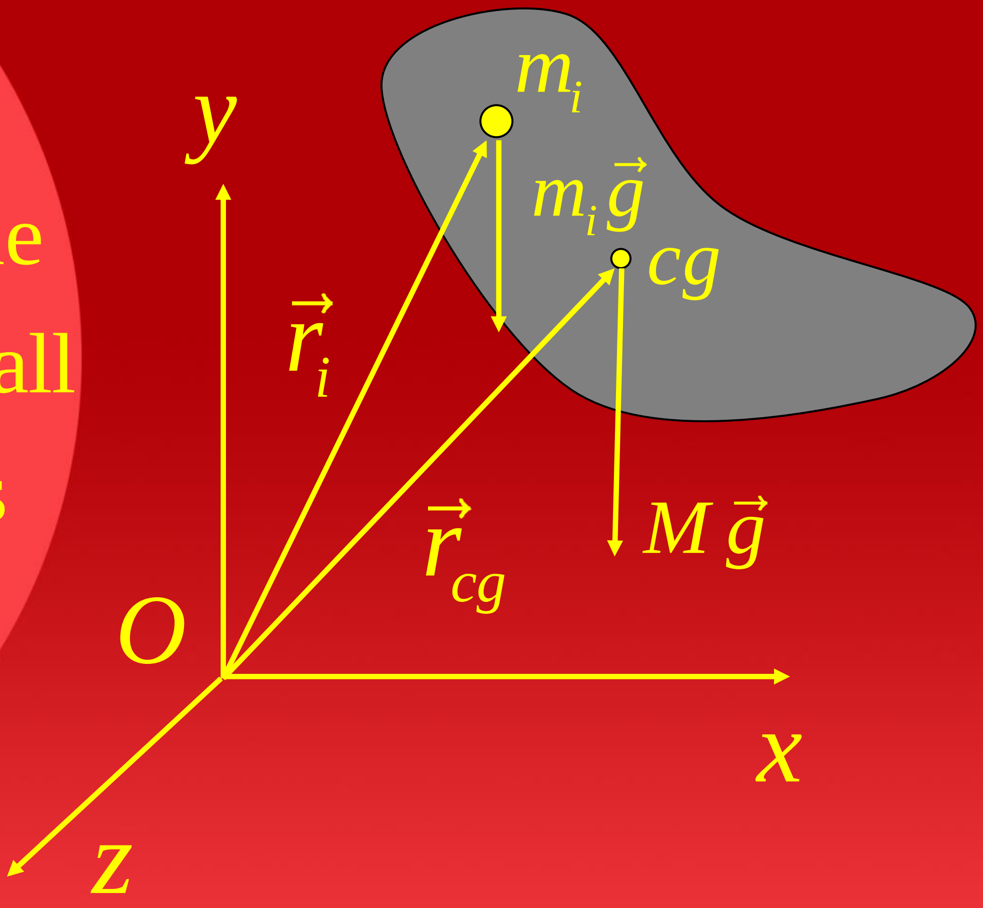
Then:

1) The weight is equal to $M\,\vec{g}$, and

2) The center of gravity coincides with the centre of mass

The net force on the whole = sum over all individual particles

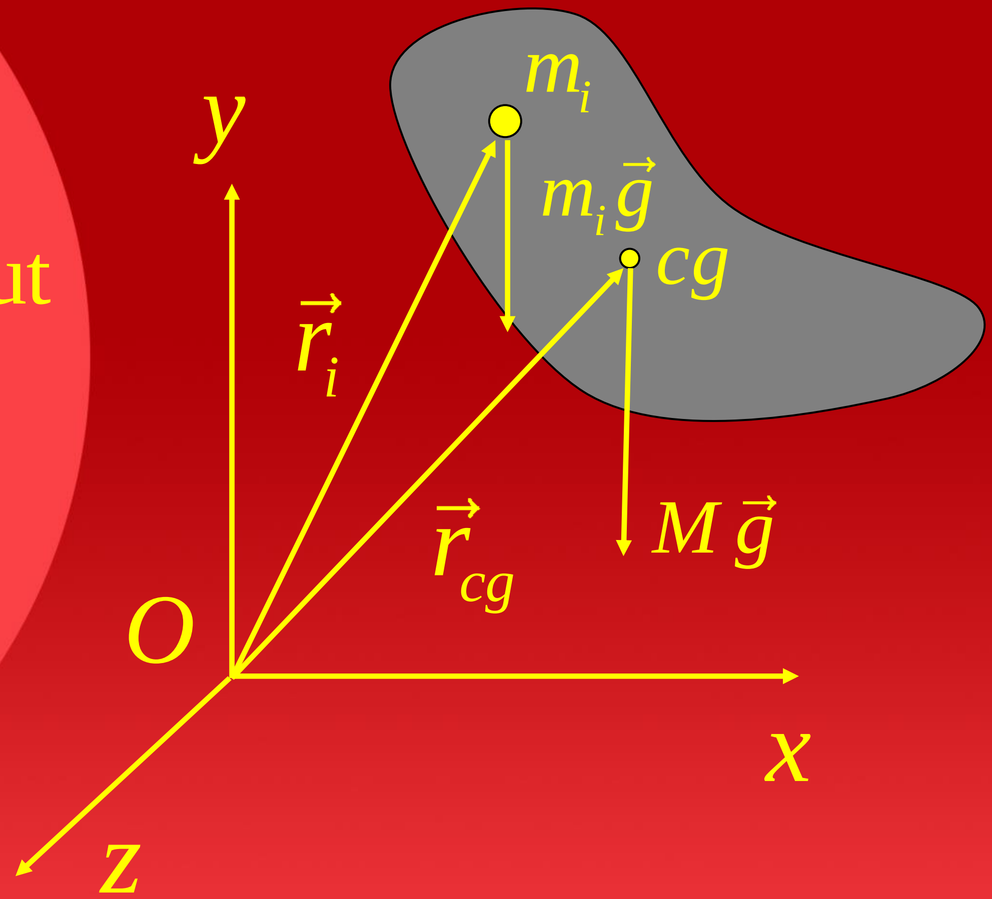
$$\sum \vec{F} = \sum m_i \vec{g}$$



Since $\vec{g}$ has the same value for every particle of the body $\therefore \sum \vec{F} = \vec{g} \sum m_i = M\vec{g}$

The net torque about the origin O:

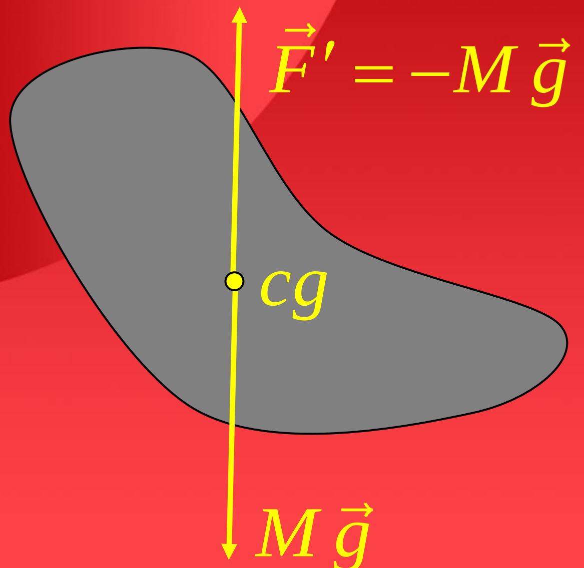
$$\sum \vec{\tau} = \sum \left(\vec{r}_i \times m_i \vec{g} \right)$$
$$= \sum \left(m_i \vec{r}_i \times \vec{g} \right)$$

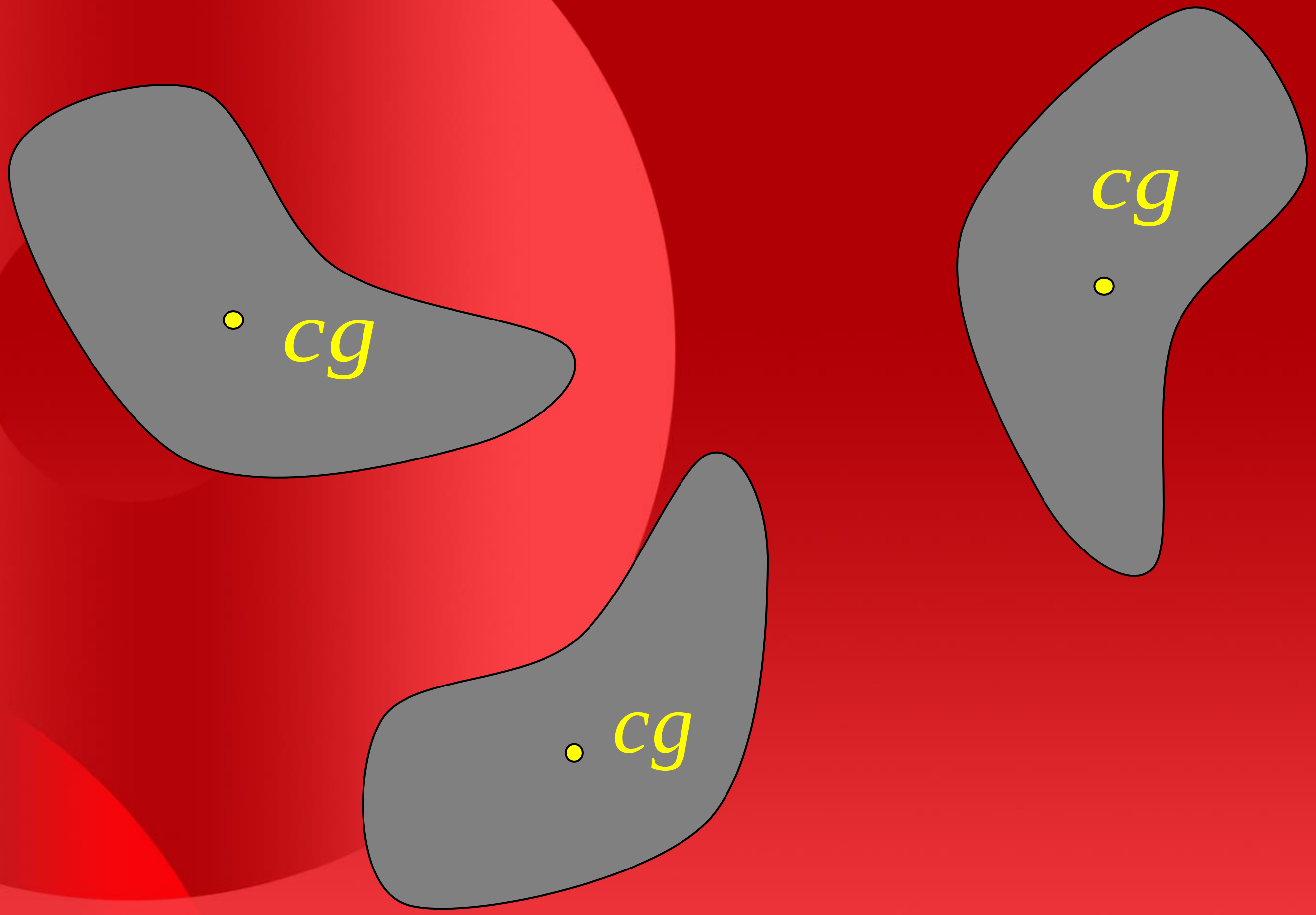


$$\therefore \sum \vec{\tau} = M \vec{r}_{cm} \times \vec{g} = \vec{r}_{cm} \times M \vec{g}$$

The torque due to gravity about the centre of mass of a body is zero !!

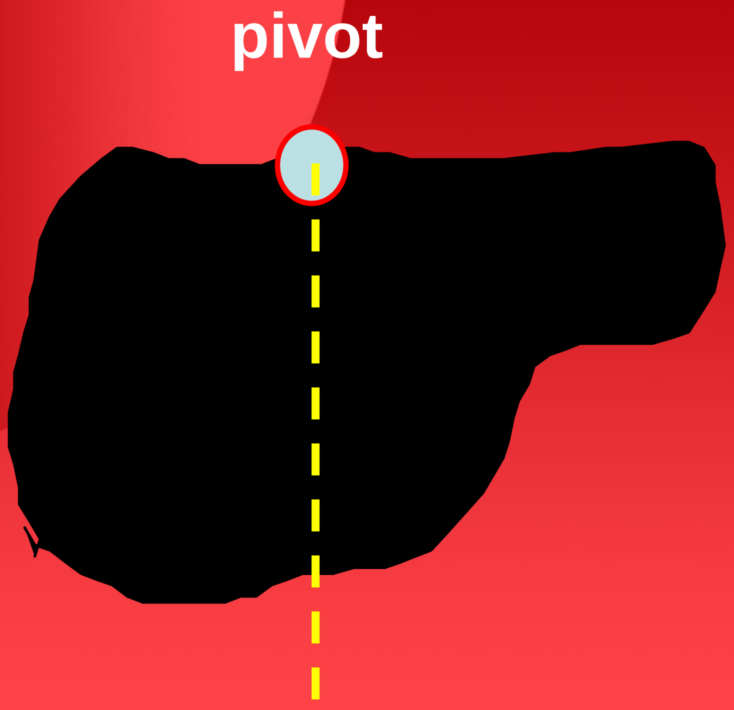
Under what conditions will a body in the Earth's gravity be in equilibrium?

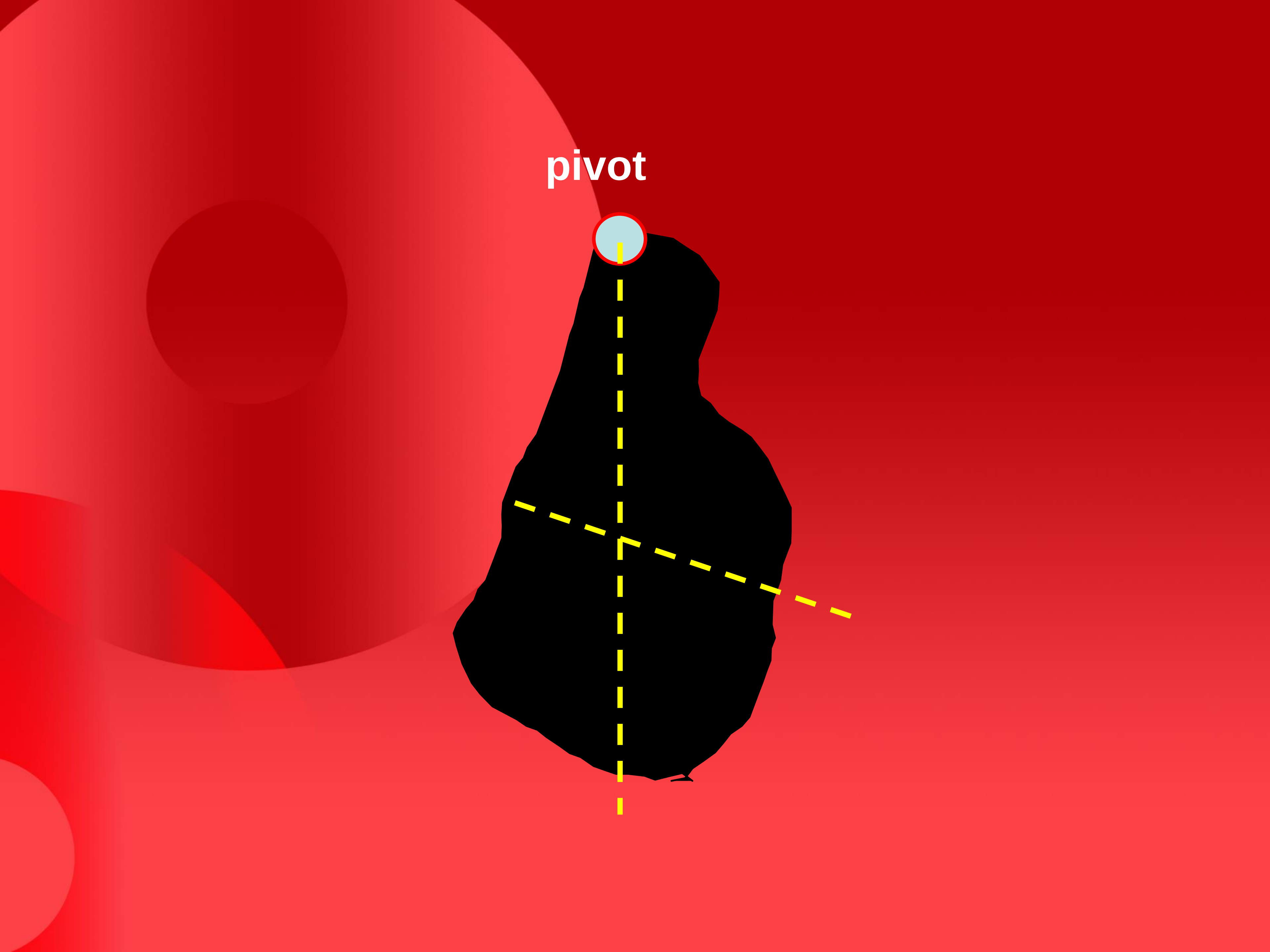


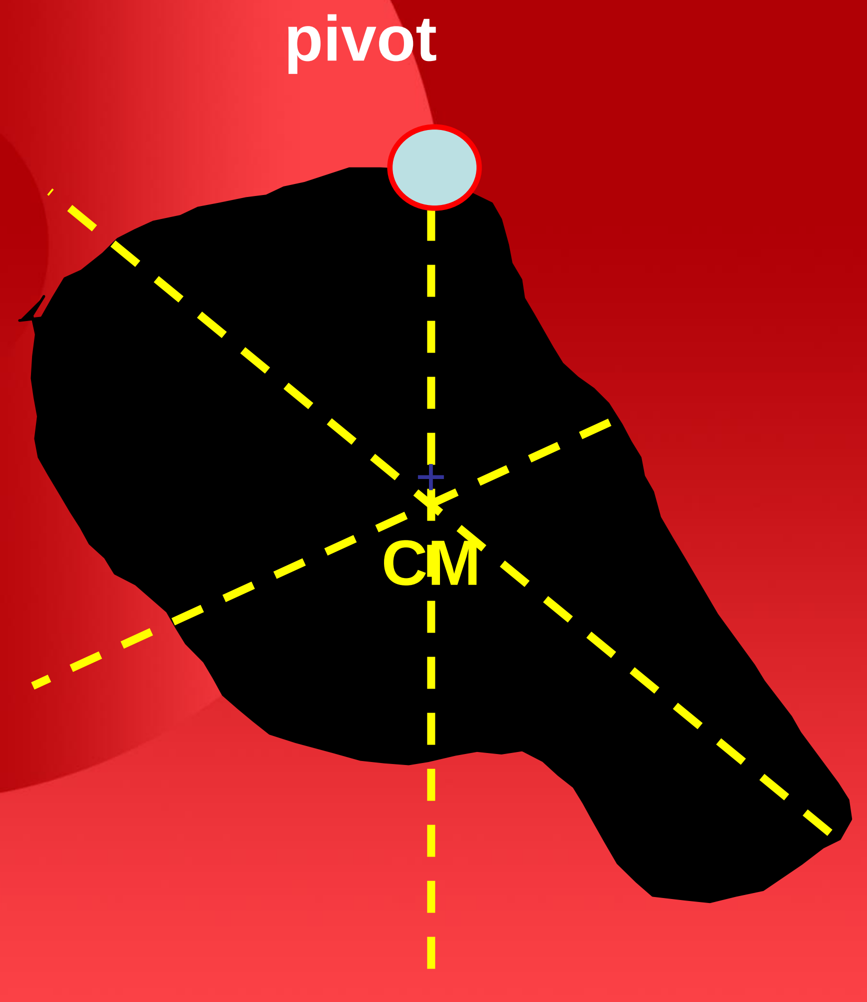


The object will be in equilibrium no matter what its orientation.

How do we find the center of mass of an object with irregular shape?



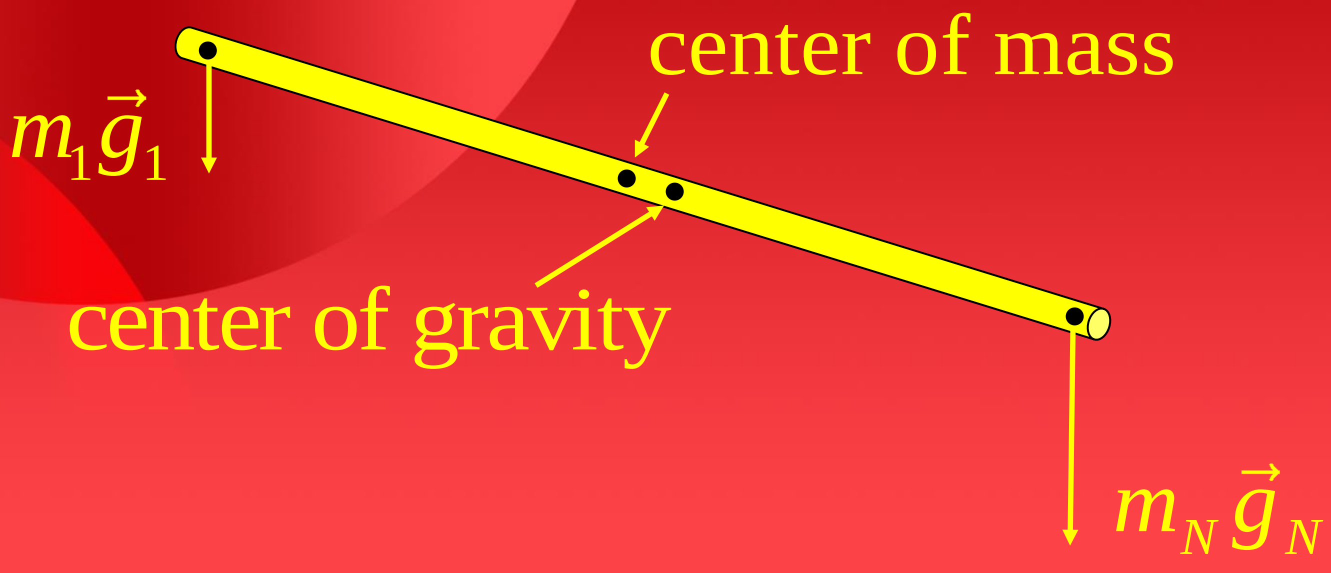

pivot


pivot
CM

We find that the center of mass is at the “centre” of the object.

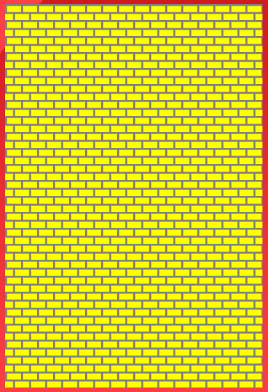
Difference between Center of Mass and Center of Gravity

If $\vec{g}$ is not constant over the body, then the center of gravity and center of mass do not coincide.



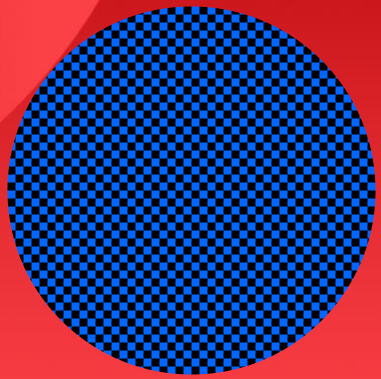
Types of Equilibrium

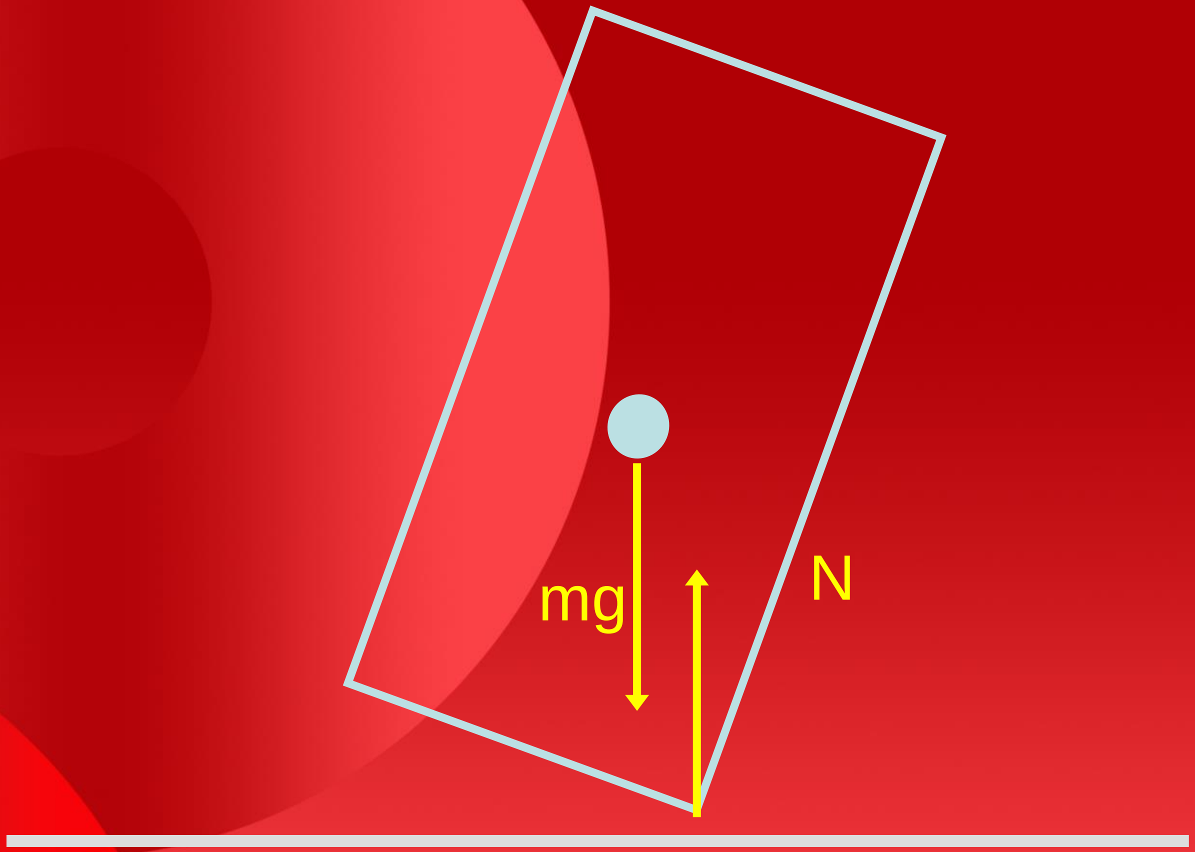
- Stable equilibrium: object returns to its original position if displaced slightly



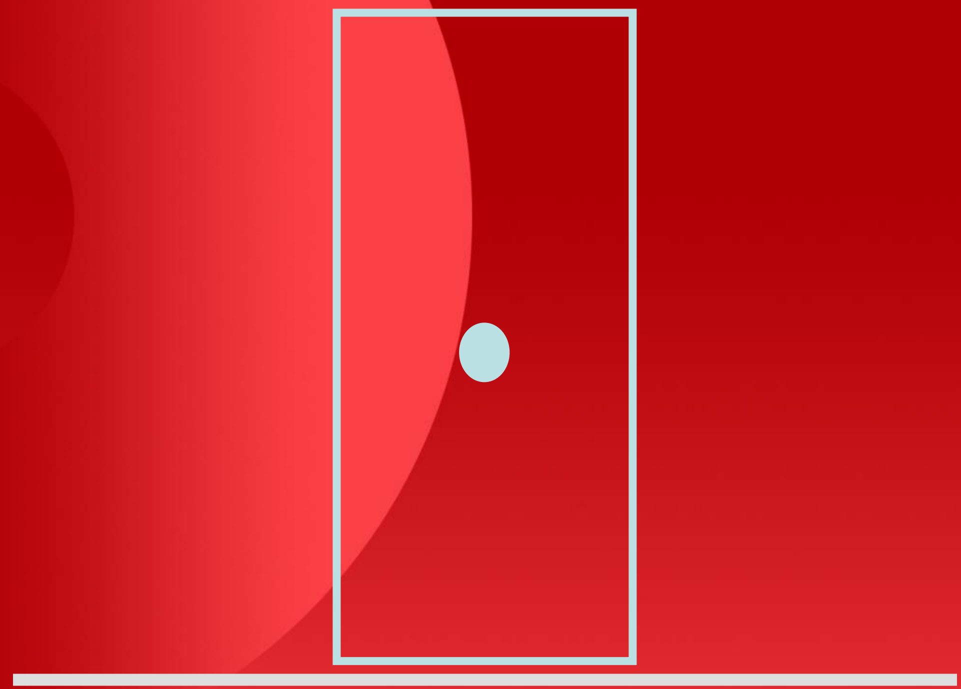
- Unstable equilibrium: object moves farther away from its original position if displaced slightly

- Neutral equilibrium: object stays in its new position if displaced slightly

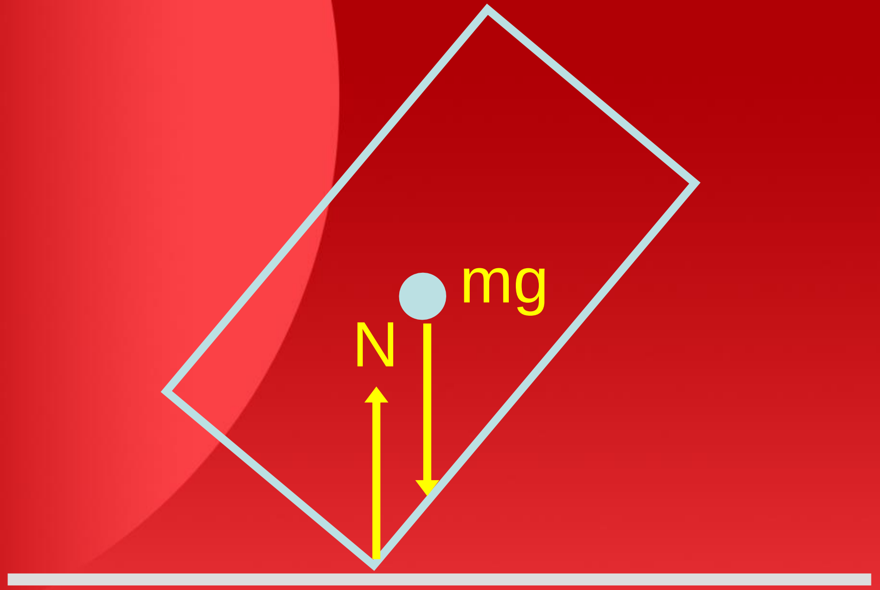




Restoring torque $\Rightarrow$ brick does not tip

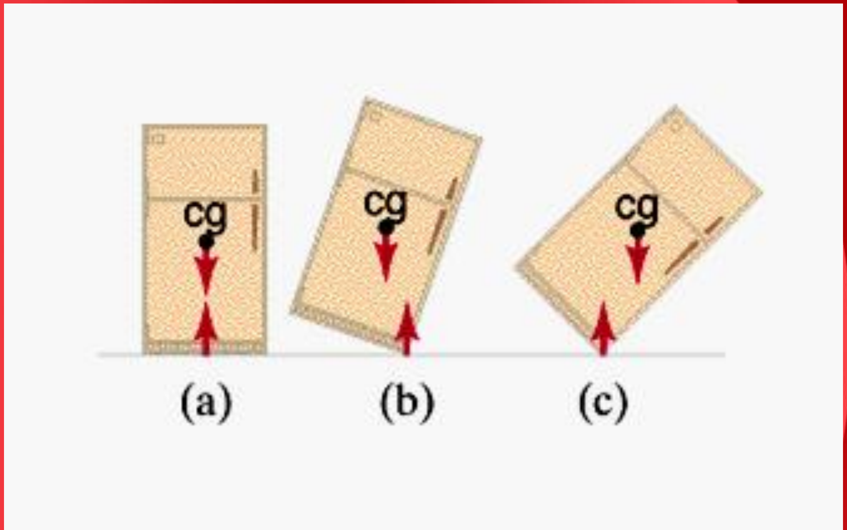


no torque



torque $\Rightarrow$ rotation

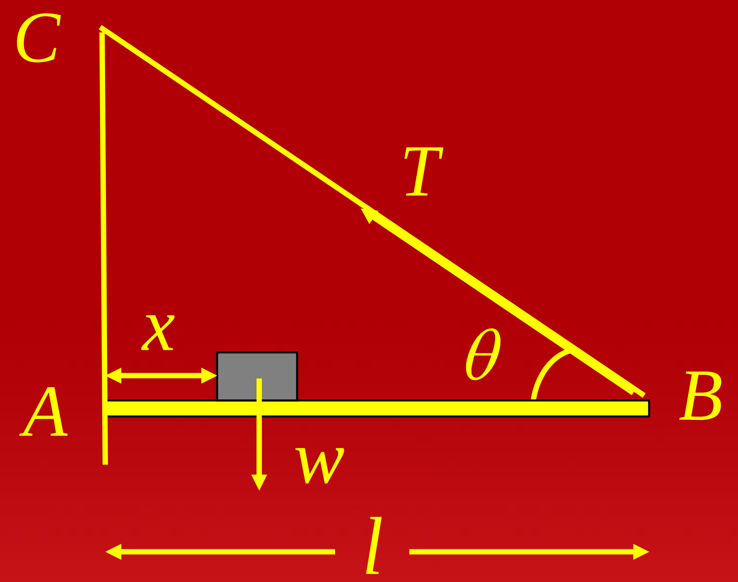
Brick tips !!



When CG of the refrigerator is no longer over the support point, it will tip over.

People try to keep the CG over their feet, in order to feel stable.

A thin bar AB of negligible weight is pinned to a vertical wall at A and supported by a thin wire BC. A weight *w* can be moved along the bar.

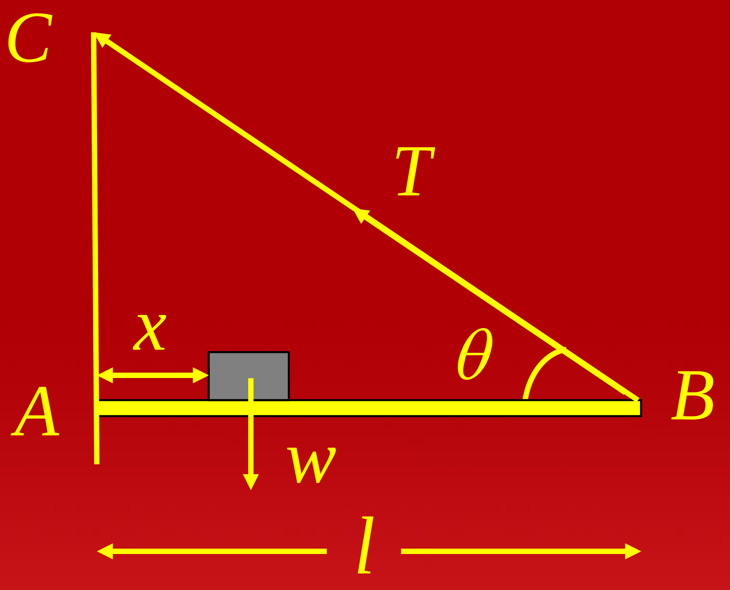


a) Find T as a function of x.

b) Find the horizontal and vertical components of the force exerted on the bar by the pin at A.

a)

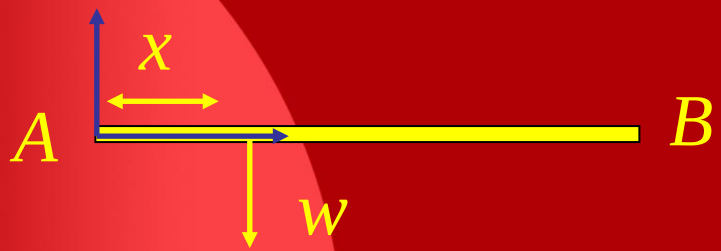
Since the system is in the rotational equilibrium, the net torque about A is zero.



$$\therefore \; w \cdot x - (T \sin\theta) l = 0$$

$$\text{or} \quad T = \frac{w \cdot x}{l \sin\theta}$$

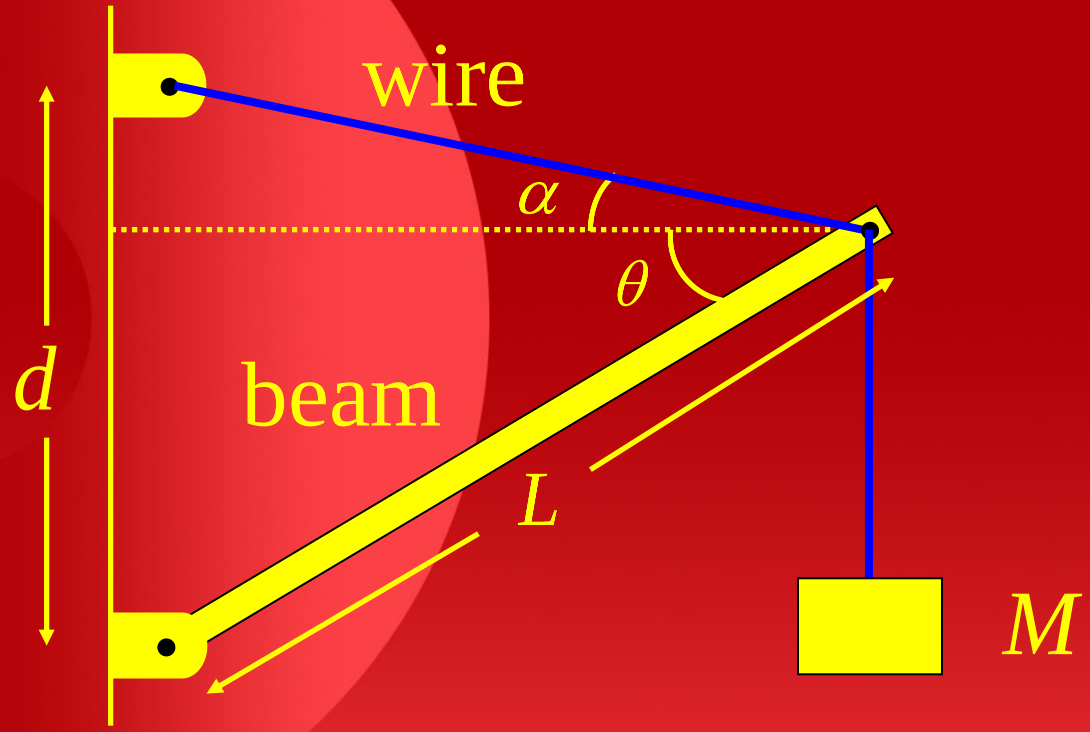
b)



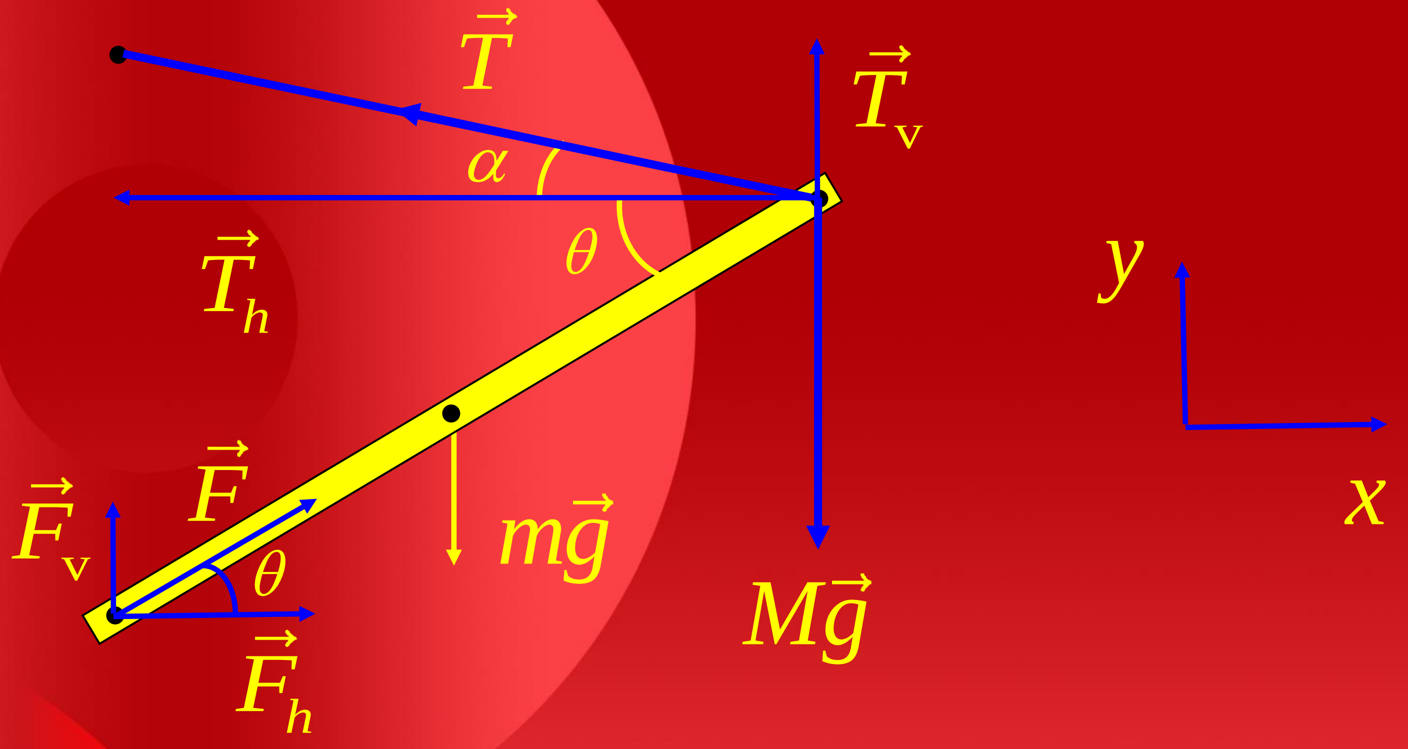
Let F_H and F_V be the horizontal and vertical components of the force exerted by the pin at A. Then since there is translational equilibrium we have

$$F_H = T\cos\theta = \frac{wx\cot\theta}{l} \quad \text{and}$$

$$F_V = W - T\sin\theta = w - \frac{wx}{l} = w\left(1 - \frac{x}{l}\right)$$



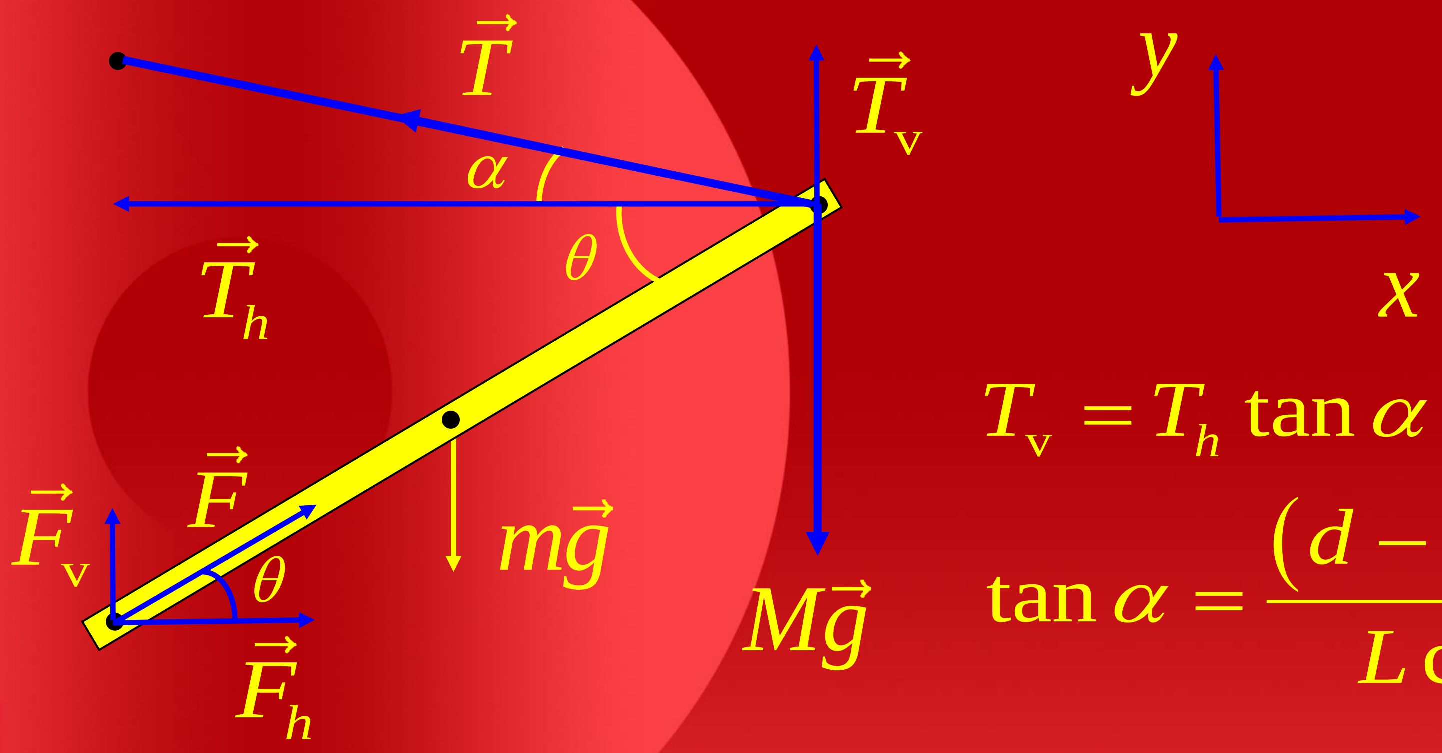
Find the tension in the wire and the force exerted by the hinge on the beam.



From the translational equilibrium

$$\sum F_x = F_h - T_h = 0$$

$$\sum F_y = F_v + T_v - mg - Mg = 0$$



$$T_v = T_h \tan\alpha$$

$$\tan\alpha = \frac{(d - Lsin\theta)}{L\cos\theta}$$

Applying the rotational equilibrium around the upper end of the beam

$$\sum \tau_z = F_v(L\cos\theta) + F_h(L\sin\theta)_h$$

$$+mg\left(\frac{L}{2}\cos\theta\right) = 0 \Rightarrow F_v = F_h\tan\theta + \frac{mg}{2}$$

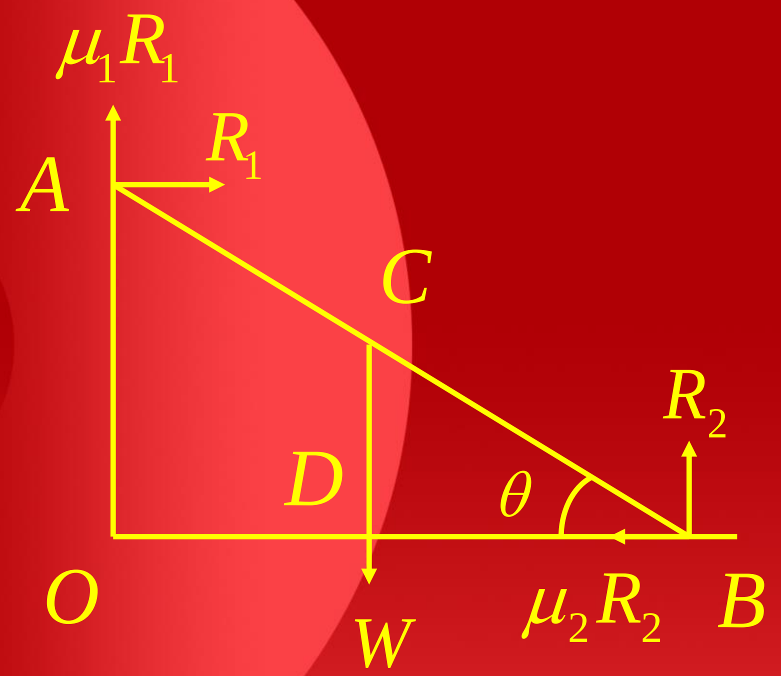
Solving these equations simultaneously, we get

$$F_{\mathrm{v}} = \frac{\left(dm + L\left(m + 2M\right)\sin\theta\right)}{2d}g$$

$$F_h = \frac{L\left(2M + m\right)\cos\theta}{2d}g$$

$$T_{\mathrm{v}} = \frac{\left(m + 2M\right)\left(d - L\cos\theta\right)}{2d}g$$

$$T_h = \frac{\left(m + 2M\right)L\cos\theta}{2d}g$$



Show that the least angle θ at which the rod can lean to the horizontal without slipping is given by

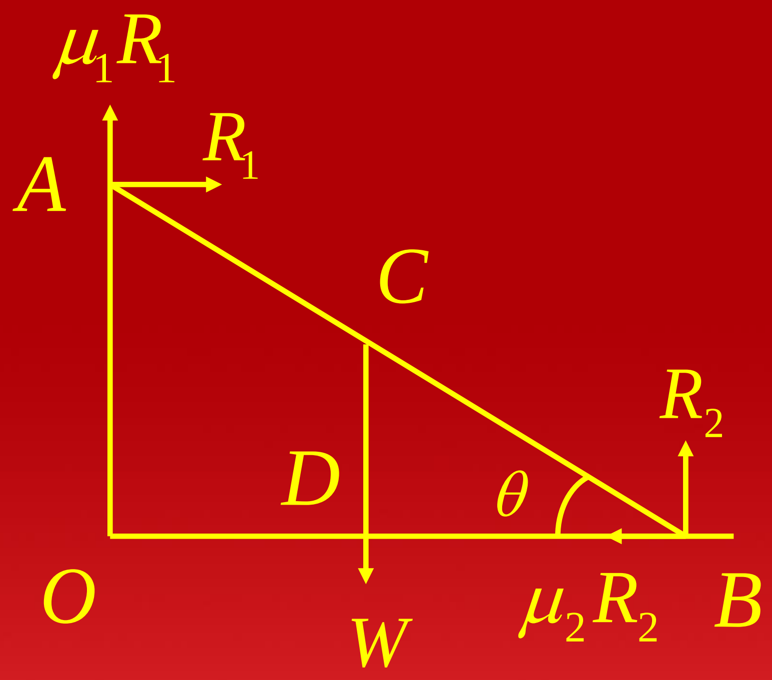
$$\theta = \tan^{-1}\left(\frac{1-\mu_1\mu_2}{2\mu_2}\right)$$

Considering the translational equilibrium of the rod,

$$R_1 = \mu_2 R_2$$

$$R_2 + \mu_1 R_1 = W$$

$$\Rightarrow \quad R_2 = \frac{W}{(1+\mu_1\mu_2)}$$



Now Considering the rotational equilibrium about A,

$$R_2 \times OB = W \times OD + \mu_2 R_2 \times OA$$

$$R_2 \times AB\cos\theta = W \times \frac{AB\cos\theta}{2} + \mu_2 R_2 \times AB\sin\theta$$

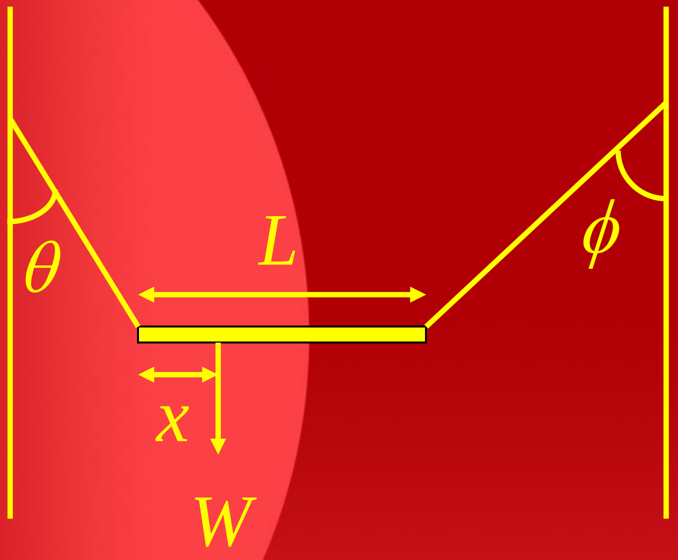
$$\text{or } \cos\theta\left(R_2 - \frac{W}{2}\right) = \mu_2 R_2 \sin\theta$$

$$\Rightarrow \quad \tan\theta = \frac{R_2 - \dfrac{W}{2}}{\mu_2 R_2}$$

$$\text{where } R_2 = \frac{W}{\left(1 + \mu_1\mu_2\right)}$$

using the value of R_2, we get

$$\Rightarrow \quad \tan\theta = \frac{1\text{-}\mu_1\mu_2}{2\mu_2}$$

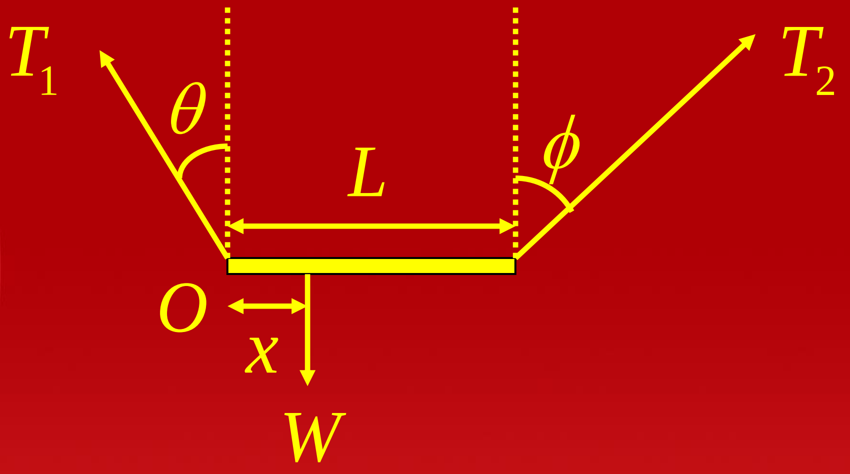


A non-uniform bar of weight W is suspended at rest in a horizontal position by two light cords. Find the distance x from the left-hand end to the center of gravity.

$$T_2 \sin\phi - T_1 \sin\theta = 0$$

$$T_2 \cos\phi + T_1 \cos\theta - W = 0$$

$$\Rightarrow T_2 = \frac{W}{\sin(\theta + \phi)}$$



Now for rotational equilibrium about O:

$$-Wx + (T_2 \cos\phi) L = 0$$

$$\Rightarrow x = \frac{(T_2 \cos\phi) L}{W} = \frac{L \cos\phi}{\sin(\theta + \phi)}$$