Physics-PHY101-Lecture

14 Equilibrium of rigid bodies:

Countless examples of stillness and motionlessness are found in our environment, for example, a ball or a chair lying on the floor or a ship sailing at a steady speed or the car is moving at a slow speed, these are some examples which are related to the ground but if you look from the sky then you will also see all the solar system is at a state of peace i.e. it is in equilibrium, so this lecture is about equilibrium of rigid/solid body.

During the last few lectures, we had talked to you in detail about momentum and angular momentum and before that we had also discussed centre of mass and moment of inertia**.** A body which is composed of many smaller objects, no matter how complex the interactions between them may be, but there is a point which is called the centre of mass and as this centre of mass moves, it carries momentum, depends on how much force acts on it.

"The rate of change of this momentum is equal to the net external force acting on the body".

Therefore, if no external force acts on this body, then this is the state of equilibrium. This is a necessary condition for equilibrium, as if the total external force has to vanish. It certainly does not mean that all the forces vanish. If forces are in different directions and they cancel each other out, only then you will say that the net external force is zero.

For equilibrium, it is not enough that we say that the net external force should be zero.

Conditions for Equilibrium:

A rigid body is in mechanical equilibrium if both the linear momentum $\vec{P}$ and angular momentum $\vec{L}$ have a constant value.

$$\text{i.e.,} \quad \frac{d\vec{P}}{dt} = 0 \quad \text{and} \quad \frac{d\vec{L}}{dt} = 0$$

$$\vec{P} = 0 \quad \text{and} \quad \vec{L} = 0 \quad \Rightarrow \quad \text{static equilibrium}$$

Examples of Equilibrium:

Here are some practical examples of equilibrium.

1) *How does a lever work?*

Solution:

Torques balance about an axis through the fulcrum, as shown in the figure 14.1:

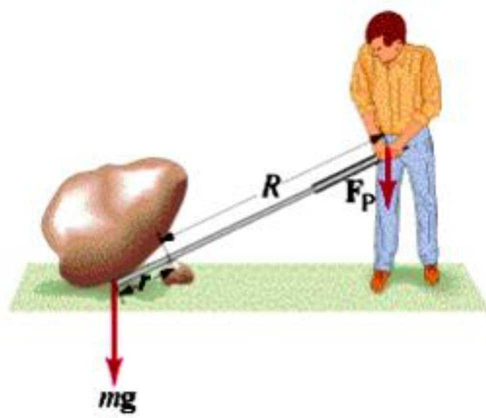


Figure 14.1: Torques by applied force and by force of gravity balance about an axis through the fulcrum.

Tourqe produced by person = Tourqe produced by stone

$$\vec{R} \times \vec{F_P} = \vec{r} \times \vec{F_g} \qquad \because F_g = mg$$

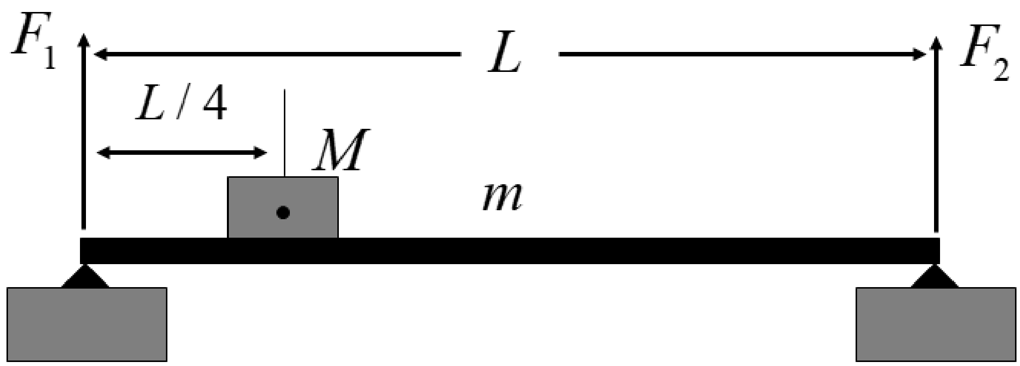
$$F_P \cdot R = mg \cdot r \qquad \because \theta = 90 \; so \; \sin\theta = 1$$

Solve for the applied force (F_P):

$$F_P = mg\frac{r}{R}$$

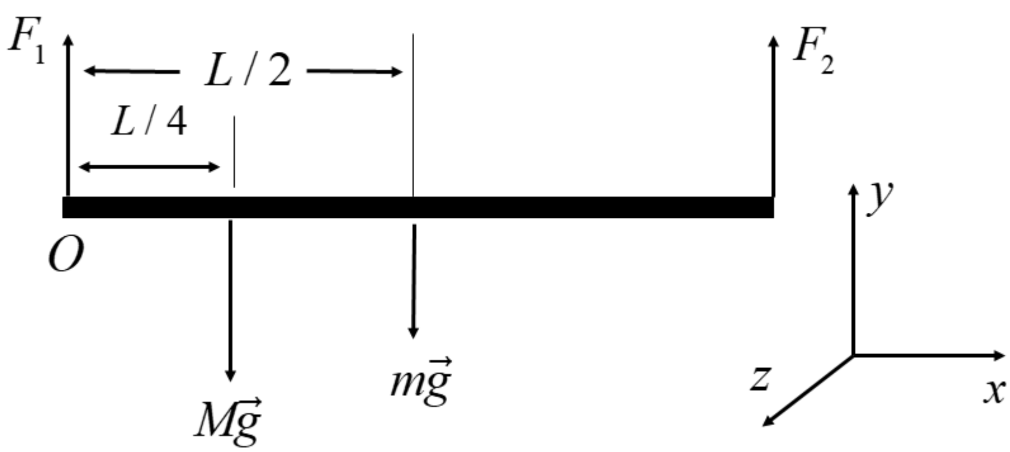
F_p (force applied by person) can be rather small depending on r and R. In the given figure R is bigger than r so a small F_p is required to lift the stone.

2) Consider a uniform rod of mass m (lies at center of mass) and length L. A mass M is placed on it as shown in figure. Find forces F_1 and F_2?



Solution:

For equilibrium, the sum of all forces and sum of all torque must be equal to zero.



Sum of all forces must be equal to zero

$$\sum F_y = F_1 + F_2 - Mg - mg = 0$$

$$\Rightarrow F_1 + F_2 - Mg - mg = 0$$

$$\Rightarrow F_1 + F_2 = Mg + mg \ \ or \ \ F_1 = \ Mg + mg - F_2 \ ....(eq \ \ 14.1)$$

Sum of all torques must be equal to zero

$$\sum \tau_y = \left(F_1\right)\left(0\right) + \left(F_2\right)\left(L\right) - \left(Mg\right)\left(L/4\right) - \left(mg\right)\left(L/2\right) = 0$$

$$\Rightarrow F_2 L - Mg\frac{L}{4} \ - mg\frac{L}{2} = 0$$

$$or \ F_2 L = Mg\frac{L}{4} \ - mg\frac{L}{2} \ or \ L(Mg\frac{1}{4} \ + mg\frac{1}{2})$$

$$F_2 L = L(Mg\frac{1}{4} \ + mg\frac{1}{2})$$

$$Cancelling \ L \ on \ both \ sides$$

$$F_2 = (Mg\frac{1}{4} \ + mg\frac{1}{2})$$

$$F_2 = \frac{Mg + 2mg}{4} = \frac{(M + 2m)g}{4}$$

$$F_2 = \frac{(M + 2m)g}{4}$$

$$Now \ put \ this \ valuse \ of \ F_2 \ in \ above \ eq 14.1,$$

$$we \ get \ \ F_1 = \frac{(3M + 2m)g}{4}$$

For a body in equilibrium, the choice of origin for calculating torques is unimportant. Torque must vanish everywhere irrespective of position of O.

Proof:

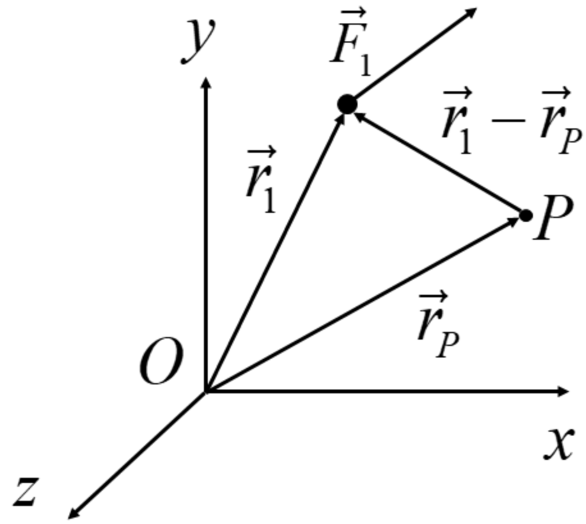


Figure 14.2: Torque about a general point P is being described.

Total torque is the sum of torques

$$\vec{\tau}_O = \vec{\tau}_1 + \vec{\tau}_2 + \cdots + \vec{\tau}_N$$
$$= \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \cdots\cdots + \vec{r}_N \times \vec{F}_N$$

Torque about point P as shown in figure 14.2, written as

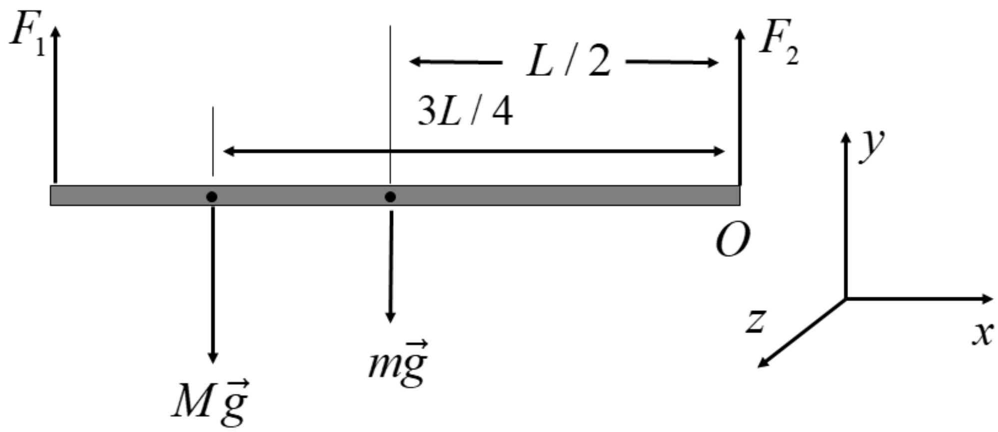
$$\vec{\tau}_P = (\vec{r}_1 - \vec{r}_P) \times \vec{F}_1 + (\vec{r}_2 - \vec{r}_P) \times \vec{F}_1 + \cdots\cdots + (\vec{r}_N - \vec{r}_P) \times \vec{F}_N$$
$$= \left[\vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \cdots + \vec{r}_N \times \vec{F}_N\right] - \left[\vec{r}_P \times \vec{F}_1 + \vec{r}_P \times \vec{F}_2 + \cdots + \vec{r}_P \times \vec{F}_N\right]$$
$$= \vec{\tau}_O - \left[\vec{r}_P \times \left(\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N\right)\right]$$
$$= \vec{\tau}_O - \left[\vec{r}_P \times \left(\sum \vec{F}_{ext}\right)\right]$$

but $\sum \vec{F}_{ext} = 0,$ for a body in translational equilibrium

$$\therefore \quad \vec{\tau}_P = \vec{\tau}_O$$

Hence the torque about any two points has the same value when the body is in translational equilibrium.

Now, origin O is at the right side (as shown in figure) lets recalculate forces to see the effect of different origin.

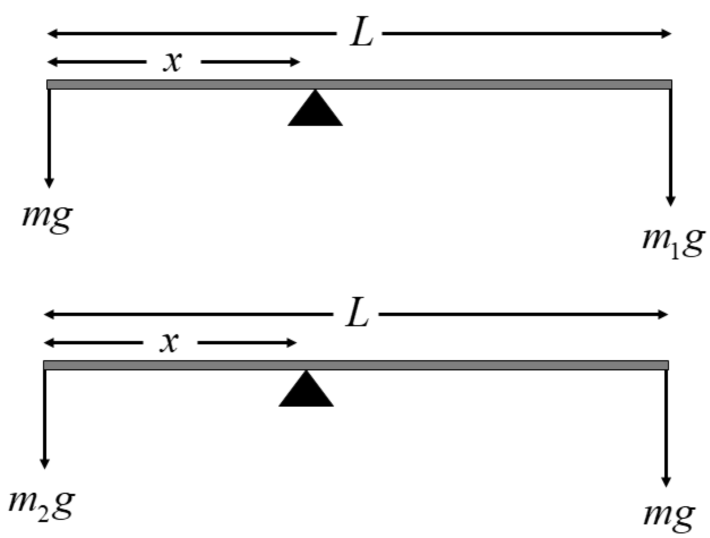


$$\sum F_y = F_1 + F_2 - Mg - mg = 0$$
$$\Rightarrow F_1 + F_2 - Mg - mg = 0$$
$$\Rightarrow F_1 + F_2 = Mg + mg \;\; or \;\; F_2 = \;\; Mg + mg - F_1 \;\; (eq\, ii)$$
$$\sum \tau_y = -(F_1)(L) + (F_2)(0) + (M\,g)(3L/4) + (mg)(L/2) = 0$$
$$\Rightarrow -F_1 L + Mg\frac{3L}{4} \;\; + mg\frac{L}{2} = 0$$
$$or\; F_1 L = Mg\frac{3L}{4} \;\; + mg\frac{L}{2} \;\; or\; L(Mg\frac{3}{4} \;\; + mg\frac{1}{2})$$
$$F_1 L = L(Mg\frac{3}{4} \;\; + mg\frac{1}{2})$$
Cancelling L on both sides
$$F_1 = (Mg\frac{3}{4} \;\; + mg\frac{1}{2})$$
$$F_1 = \frac{3Mg + 2\,mg}{4} = \frac{(3M + 2\,m)g}{4}$$
$$F_1 = \frac{(3M + 2\,m)g}{4} \text{ Now put this valuse of } F_1 \text{ in eq ii above}$$
$$\text{we get } \;\; F_2 = \frac{(M + 2\,m)g}{4}$$

Conclusion: As we expected, change of origin does not affect the forces.

3) A massless rod of length L is placed on a fulcrum at a distance x from the left side. An unknown mass m is placed first on the left side and then on the right side of the rod (as shown in figure). If rod is balanced (by m_1 and m_2) in both situations, then calculate the value of m?

Solution:



$$m = ?$$

Taking the torques about the knife edge in the two cases (as shown in figure), we have,

$$mgx = m_1 g(L-x)$$
$$m_2 gx = mg(L-x)$$

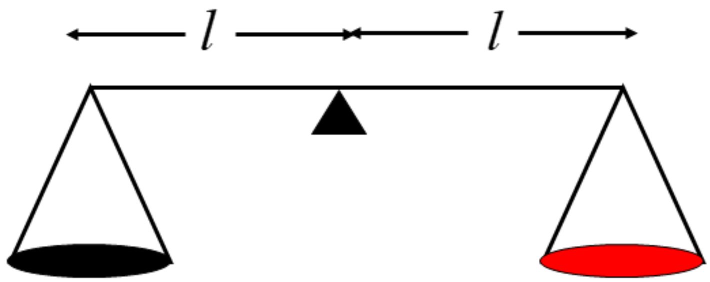
by dividing these equations we get

$$\frac{mgx}{m_2 gx} = \frac{m_1 g(L-x)}{mg(L-x)}$$

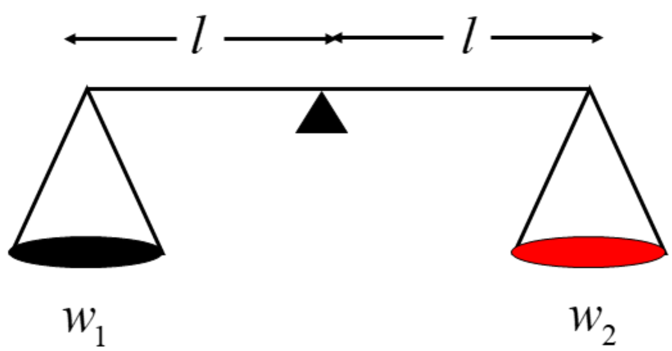
$$\Rightarrow \frac{m}{m_2} = \frac{m_1}{m} \text{ or } m = \sqrt{m_1 m_2}$$

So, we can find out the unknown mass with the help of two known masses using this methodology.

4) A false balance (weight of both pans are not equal) has equal arms. An object weights x when placed in one pan and y when placed in the other pan. What is the true weight of the object?



Solution:



Let the weights of the pans be w_1 and w_2 and the true weight be w.

Taking torques about the knife edge gives:

$$(w+w_1)l=(x+w_2)l$$
$$\Rightarrow(w+w_1)=(x+w_2) \quad \text{eq(1)}$$

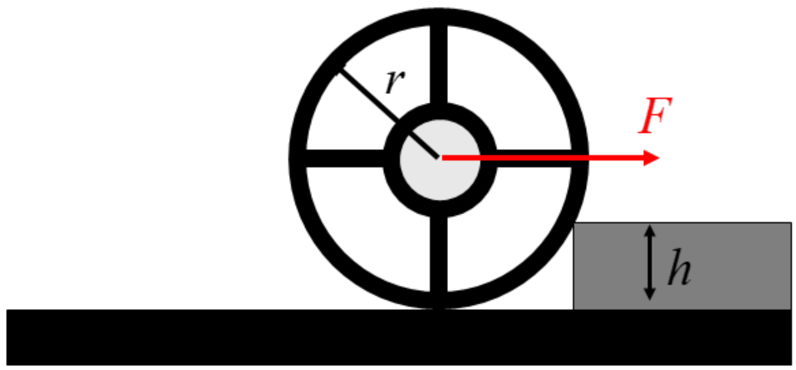
$$(w+w_2)l=(y+w_1)l$$
$$\Rightarrow(w+w_2)=(y+w_1) \quad \text{eq(2)}$$

Adding eq(1) and eq(2)

$$(w+w_2)+(w+w_1)=(x+w_2)+(y+w_1)$$
$$2w+(w_1+w_2)=x+y+(w_1+w_2)$$
$$2w=x+y$$
$$\Rightarrow w=\frac{x+y}{2}$$

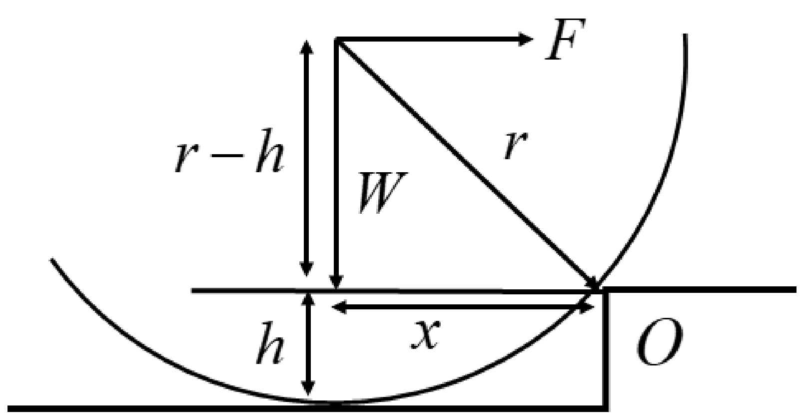
Conclusion: The actual weight is the average of both weights.

5) What minimum force F applied horizontally at the axle of the wheel (of the radius r) is necessary to raise the wheel over an obstacle of height h, as shown in the figure?



Solution:

The normal force vanishes as the wheel leaves the ground. The remaining two forces (applied F and weight w) participate in torque.



Take torques about O (the point of contact):

$$Wx = F\left(r - h\right)$$

$$x = \frac{F\left(r - h\right)}{W} \qquad \text{eq(1)}$$

Using Pythagoras Theorem on right-angle triangle:

$$x^2 + \left(r - h\right)^2 = r^2$$

$$x^2 = r^2 - \left(r - h\right)^2$$

$$x^2 = r^2 - \left(r^2 + h^2 - 2rh\right)$$

$$x^2 = r^2 - r^2 - h^2 + 2rh$$

$$x^2 = 2rh - h^2$$

using equation (1)

$$\left\{\frac{F\left(r - h\right)}{W}\right\}^2 = 2rh - h^2$$

$$\Rightarrow \frac{F\left(r - h\right)}{W} = \sqrt{2rh - h^2}$$

$$\Rightarrow\ F = W\,\frac{\sqrt{2rh - h^2}}{r - h}.$$

So, this F will raise the wheel over an obstacle of height h.

Review: Center of Mass (CM)

The center of mass (CM) is a theoretical point in a system of particles where, on average, the mass of the system is concentrated.

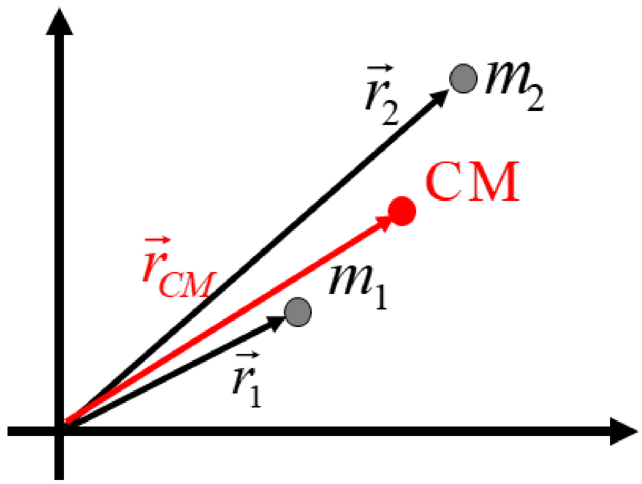


Figure 14.3: CM and r_{CM} for system of two masses.

Mathematical expression of figure 14.3 is:

$$\vec{r}_{cm} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

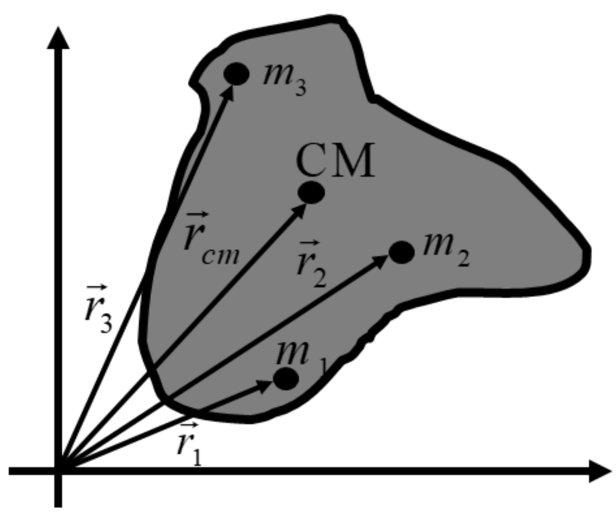


Figure 14.4: CM and r_{CM} for system of N masses is presented.

For N number of masses, as shown in figure 14.4 it can be written as

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N}{m_1 + m_2 + \cdots + m_N}$$

We also know

$$\vec{F} = M\,\vec{a}_{cm}$$

where,

$$\vec{F} = \sum \vec{F}_{ext}$$

The specialty of the center of mass is that if we throw a body of any shape, then the trajectory of CM, the curve on which that body will move will be a parabola, its shape may be complicated, but its one point, which is the center of mass, moves on a parabola. The parabolic path (dotted line) of the center of mass (red dot) on the trajectory of the object is shown in figure 14.5.

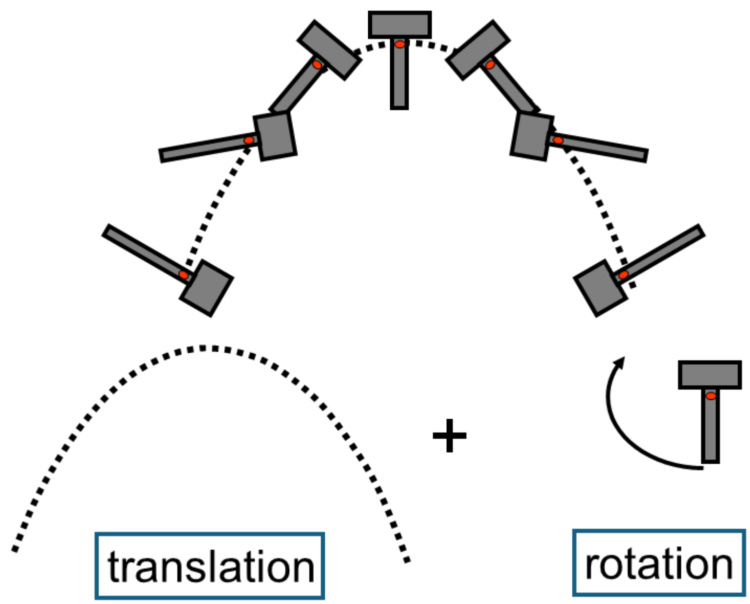


Figure 14.5: An object of arbitrary shape has translational and rotational motion while CM (red dot) has a parabolic path.

Centre of gravity:

The center of gravity is the average location of the weight of an object. As it is shown for different shapes in figure 14.6.

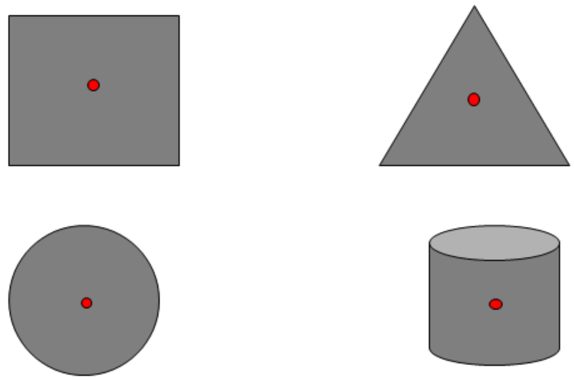


Figure 14.6: Center of gravity (as a red dot) is shown for different shapes.

Suppose the gravitational acceleration g has the same value at all points of a body. Then

- The weight is equal to mg
- And the center of gravity "coincides" with the center of mass.

Lets take look on equilibrium conditions, here

The net force on the whole = sum over all individual particles

$$\sum \vec{F} = \sum m_i \vec{g}$$

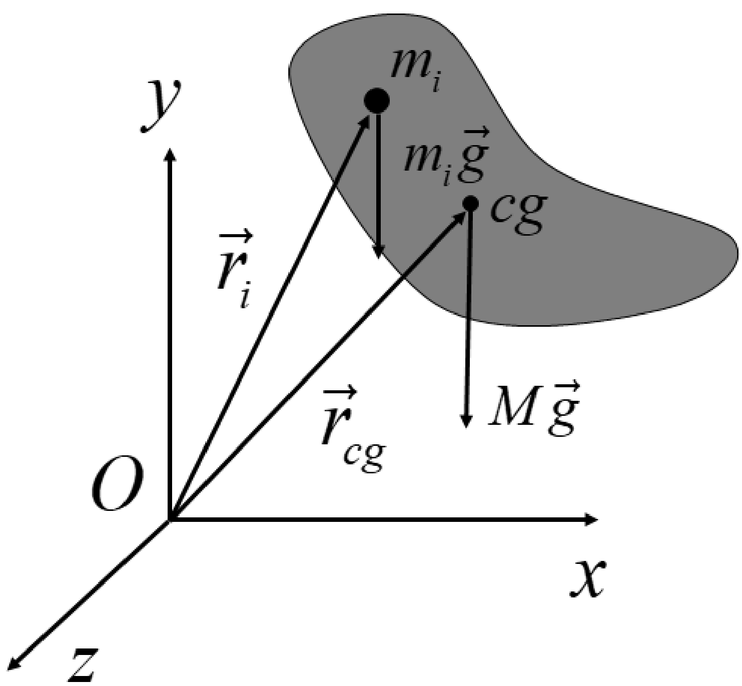


Figure 14.7: Position vector r_{cg} and cg is center of gravity for an arbitrary shape is depicted.

Since $\vec{g}$ has the same value for every particle of the body

$$\therefore \sum \vec{F} = \vec{g} \sum m_i = M\vec{g}$$

The net torque about the origin $O,$ as shown in figure 14.7 :

$$\sum \vec{\tau} = \sum (\vec{r}_i \times m_i \vec{g})$$

$$= \sum (m_i \vec{r}_i \times \vec{g}) \qquad \because \vec{r}_{CM} = \frac{m_i \vec{r}_i}{M}$$

$$\therefore \sum \vec{\tau} = M\vec{r}_{cm} \times \vec{g} = \vec{r}_{cm} \times M\vec{g}$$

Figure 14.8 shows the torque due to gravity about the centre of mass of a body is zero, and object is balanced at CM/CG.

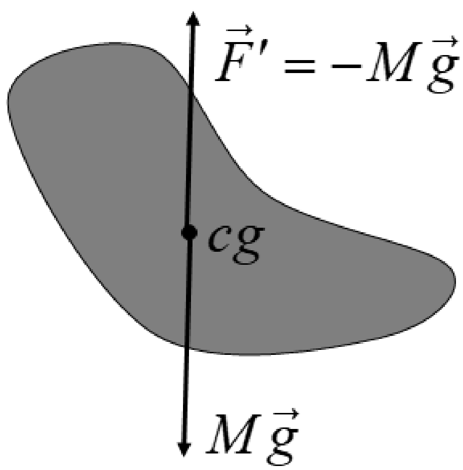


Figure 14.8: The object is balanced at CM/CG.

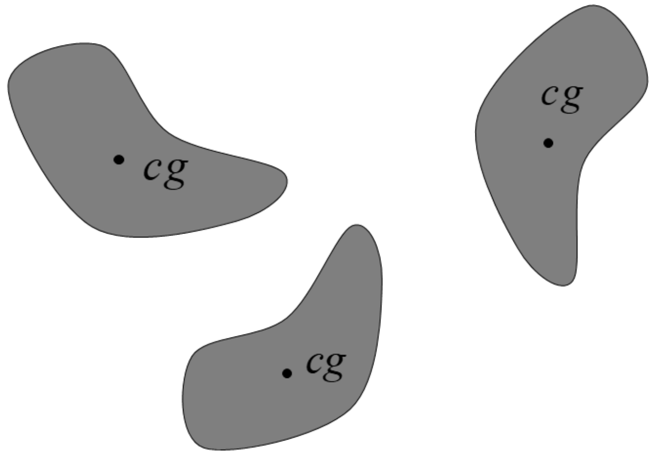


Figure 14.9: Center of gravity for different shapes is shown.

The object will be in equilibrium no matter what its orientation.

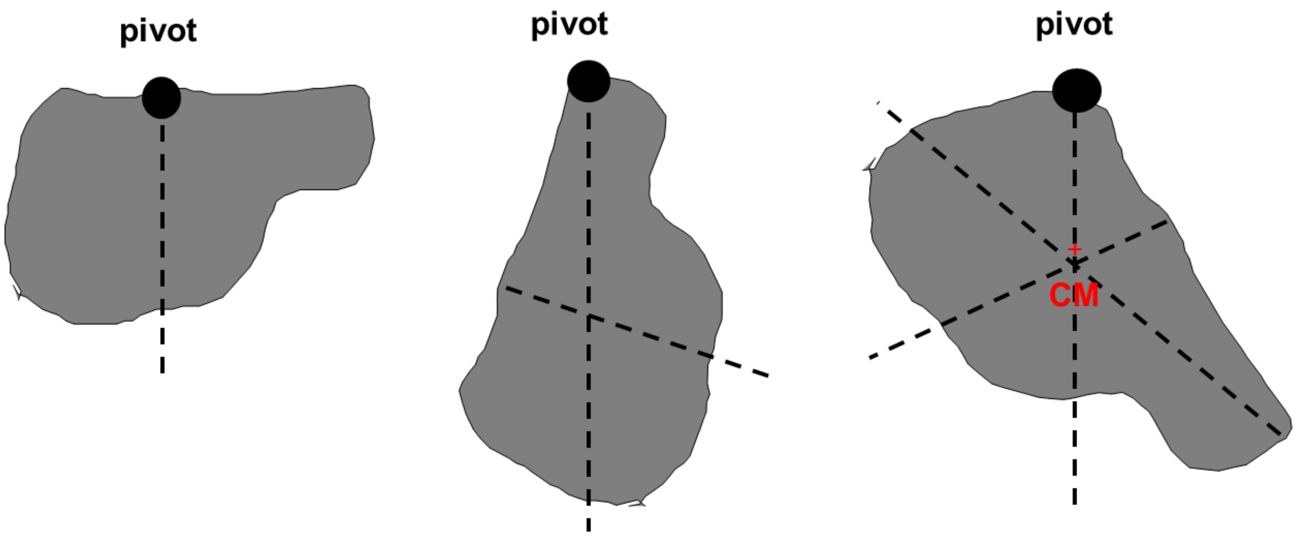


Figure 14.10: The arbitrary shaped object is being pivoted at different edges to find out CM.

We find that the center of mass is at the “center” of the object.

<u>Difference Between Center of Mass and Center of Gravity:</u>

If g is not constant over the body, then the center of gravity and the center of mass do not coincide, as shown in figure 14.11.

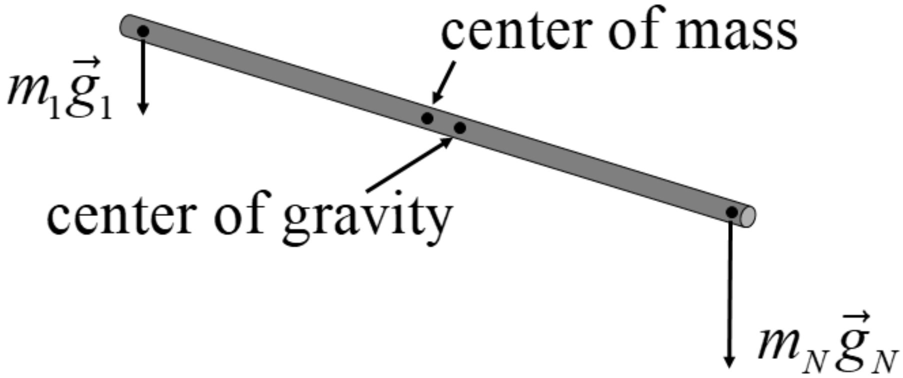


Figure 14.11: For different values of g over a body can led to different location of CM and CG of the body.

<u>Types of Equilibrium:</u>

- **Stable equilibrium:** Object returns to its original position if displaced slightly, as shown in figure 14.12.

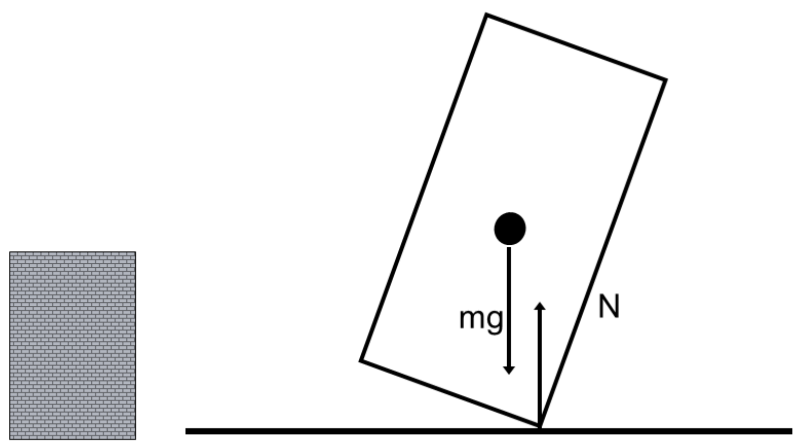


Figure 14.12: The object possesses stable equilibrium if slightly tilted.

- **Unstable equilibrium:** Object moves farther away from its original position if displaced slightly, as shown in figure 14.13.

Figure 14.13: The object possesses unstable equilibrium if slightly tilted.

- **Neutral equilibrium:** Object stays in its new position if displaced slightly, as shown in figure 14.14.

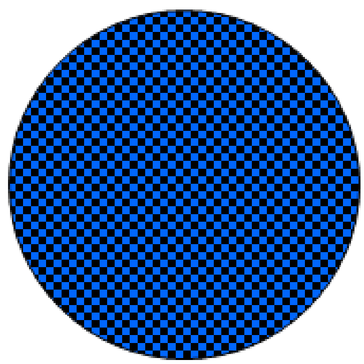


Figure 14.14: The object possesses neutral equilibrium if slightly tilted.

Figure 14.15: To achieve stable equilibrium while carrying a weight people tend to keep CG over their feet.

- We can observe that people try to keep the CG over their feet, to feel stable while carrying weight, as shown in figure 14.15.

6) A thin bar AB of negligible weight is pinned to a vertical wall at A and supported by a thin wire BC, a weight w can be moved along the bar.

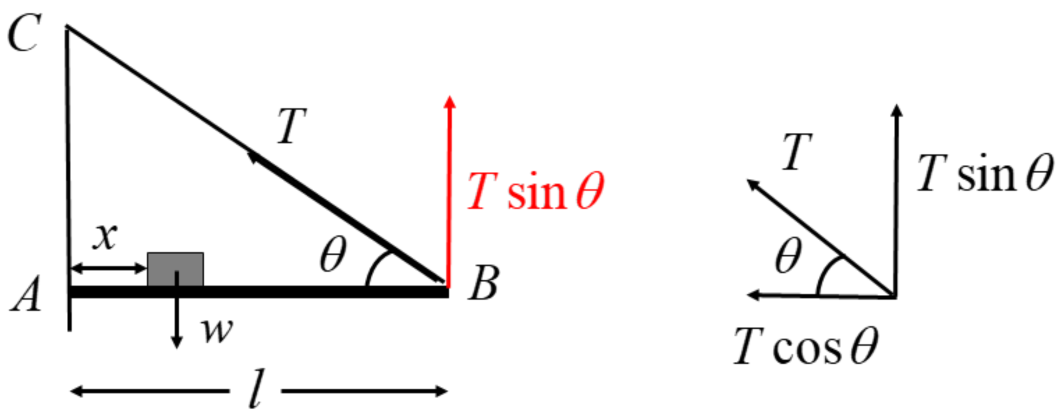
(a) Find T as a function of x. (b) Find the horizontal and vertical components of the force exerted on the bar by the pin at A.

<u>Solution:</u>

Since the system is in rotational equilibrium, the net torque about A is zero.

$$\therefore\ w\cdot x-(T\sin\theta)l=0$$

$$\text{or}\quad T=\frac{w\cdot x}{l\sin\theta}\qquad \text{eq(1)}$$



Let F_H and F_V be the horizontal and verticalcomponents of the force exerted by the pin at A. Then since there is translational equilibrium we have

$$F_H = T\cos\theta$$

Using $eq(1)$

$$F_H = \frac{w \cdot x}{l\sin\theta}\cos\theta = \frac{wx\cot\theta}{l} \qquad \because \cot\theta = \frac{\cos\theta}{\sin\theta}$$

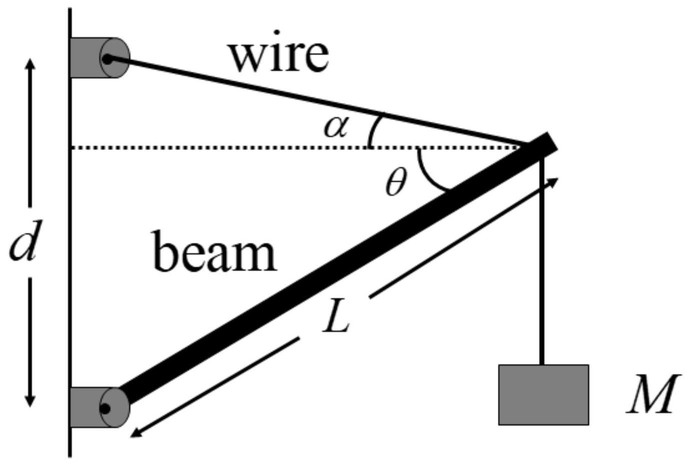
and

$$F_V = w - T\sin\theta$$

Using $eq(1)$

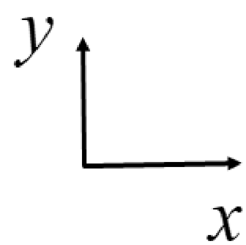
$$F_V = w - \frac{wx}{l\sin\theta}\sin\theta = w\left(1 - \frac{x}{l}\right)$$

7) A mass M is hanged over a pivoted beam of length L, making an angle α with the horizontal. Find the tension in the wire and the force exerted by the hinge on the beam.



Solution:

From the translational equilibrium



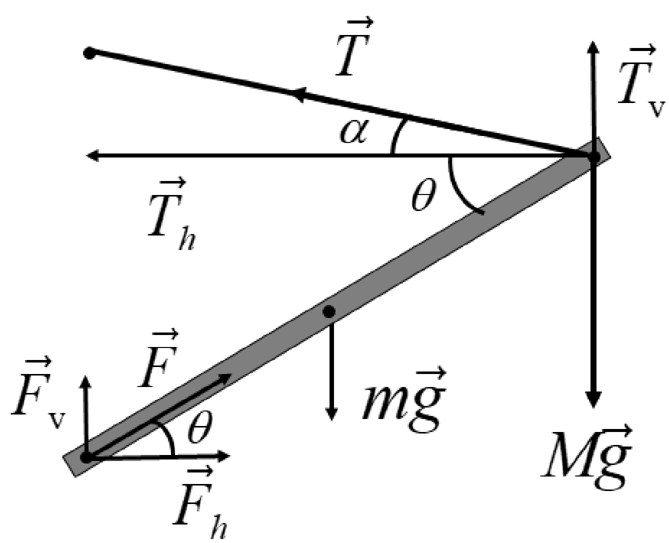
$$\sum F_x = F_h - T_h = 0$$

$$\sum F_y = F_\mathrm{v} + T_\mathrm{v} - mg - Mg = 0$$

$$T_\mathrm{v} = T_h \tan\alpha$$

$$\tan\alpha = \frac{\left(d - Lsin\theta\right)}{L\cos\theta}$$

Applying the rotational equilibrium around the upper end of the beam



$$\sum \tau_z = F_\mathrm{v}\left(L\cos\theta\right) + F_h\left(L\sin\theta\right)_h$$
$$+mg\left(\frac{L}{2}\cos\theta\right) = 0 \Rightarrow F_\mathrm{v} = F_h\tan\theta + \frac{mg}{2}$$

Solving these equations simultaneously, we get

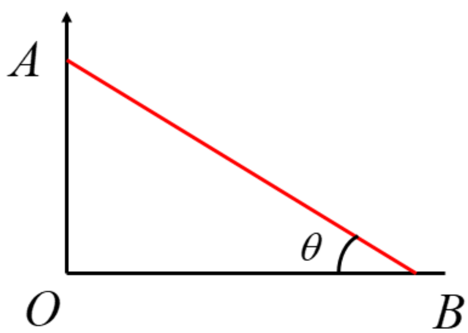
$$F_\mathrm{v} = \frac{\left(dm + L\left(m + 2M\right)\sin\theta\right)}{2d}g$$

$$F_h = \frac{L(2M+m)\cos\theta}{2d}g$$

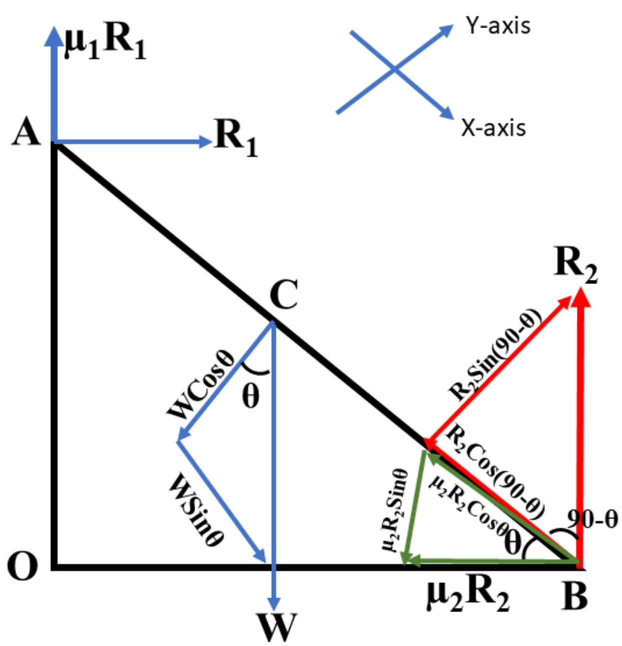
$$T_v = \frac{(m+2M)(d - L\cos\theta)}{2d}g$$

$$T_h = \frac{(m+2M)L\cos\theta}{2d}g$$

8) Show that the least angle θ at which the rod can lean to the horizontal without slipping is given by $\theta = \tan^{-1}\left(\frac{1\text{-}\mu_1\mu_2}{2\mu_2}\right)$



Solution:



W = weight of the rod

R_1 *and* R_2 = *normal force perpendicular to the surface.*

Acoording to the first condition of equilibrium,

$\sum F_x = 0, \ \sum F_y = 0$

Along x-axis:

$R_1 - \mu_2 R_2 = 0 \Rightarrow R_1 = \mu_2 R_2 \ \rightarrow$ eq(1)

Along y-axis:

$R_2 + \mu_1 R_1 - W = 0 \Rightarrow \ R_2 + \mu_1 R_1 = W \ \rightarrow$ eq(2)

substituting R_1 from eq(1) into eq(2), we get

$R_2 + \mu_1 \mu_2 R_2 = W$

$R_2 (1 + \mu_1 \mu_2) = W$

$\Rightarrow \quad R_2 = \dfrac{W}{(1 + \mu_1 \mu_2)}$ eq(3)

According to the second condition of equilibrium,

$\sum \tau = 0$

Since A is our center of rotation which means, there is no torque contribution from R_1 and $\mu_1 R_1$ (*i.e.*, τ = r*F where, r = 0)

Now Considering the rotational equilibrium about A,

$$R_2 \times OB = W \times OD + \mu_2 R_2 \times OA$$

$$R_2 \times AB \cos\theta = W \times \frac{AB\cos\theta}{2} + \mu_2 R_2 \times AB \sin\theta$$

$$R_2 \cos\theta = W \frac{\cos\theta}{2} + \mu_2 R_2 \sin\theta$$

$$R_2 \cos\theta - W \frac{\cos\theta}{2} = \mu_2 R_2 \sin\theta$$

or $\cos\theta \left(R_2 - \dfrac{W}{2} \right) = \mu_2 R_2 \sin\theta$

$$\Rightarrow \quad \tan\theta = \frac{R_2 - \frac{W}{2}}{\mu_2 R_2} \qquad \because \tan\theta = \frac{\sin\theta}{\cos\theta}$$

using the value of R_2 from eq(3), we get

$$\tan\theta = \frac{\dfrac{W}{(1+\mu_1\mu_2)} - \dfrac{W}{2}}{\mu_2 \dfrac{W}{(1+\mu_1\mu_2)}}$$

$$\tan\theta = \frac{\dfrac{2W - W(1+\mu_1\mu_2)}{2(1+\mu_1\mu_2)}}{\mu_2 \dfrac{W}{(1+\mu_1\mu_2)}}$$

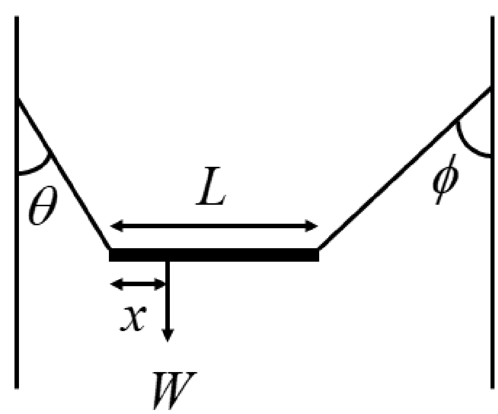
$$\tan\theta = \frac{2W - W(1+\mu_1\mu_2)}{2(1+\mu_1\mu_2)} \frac{(1+\mu_1\mu_2)}{W\mu_2}$$

$$\tan\theta = \frac{2W - W(1+\mu_1\mu_2)}{2W\mu_2}$$

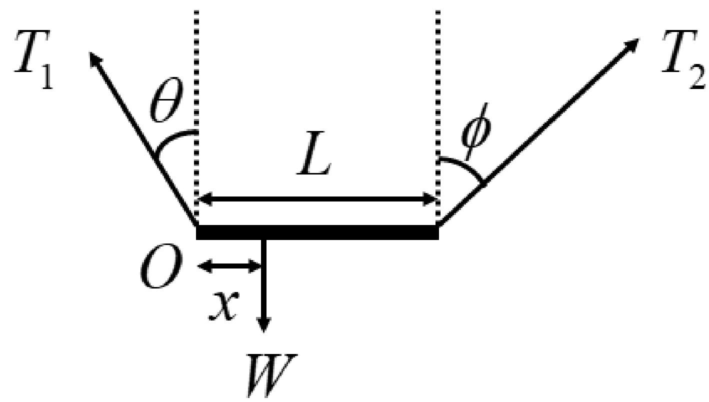
$$\tan\theta = \frac{W(2-1-\mu_1\mu_2)}{2W\mu_2}$$

$$\Rightarrow \quad \tan\theta = \frac{1\text{-}\mu_1\mu_2}{2\mu_2}$$

9) A non-uniform bar of weight W is suspended at rest in a horizontal position by two light cords. Find the distance x from the left-hand end to the center of gravity.



Solution:



Summition of horizontal components

$T_2 \sin\phi - T_1 \sin\theta = 0$

multiplying it with $\cos\theta$, we get

$T_2 \sin\phi\cos\theta - T_1 \sin\theta\cos\theta = 0$ eq(1)

Summition of vertical components

$T_2 \cos\phi + T_1 \cos\theta - W = 0$

multiplying it with $\sin\theta$, we get

$T_2 \cos\phi\sin\theta + T_1 \cos\theta\sin\theta = W\sin\theta$ eq(2)

adding equation 1 and 2

$T_2 \sin\phi\cos\theta - T_1 \sin\theta\cos\theta + T_2 \cos\phi\sin\theta + T_1 \cos\theta\sin\theta = W\sin\theta$

$T_2 \sin\phi\cos\theta + T_2 \cos\phi\sin\theta = W\sin\theta$

$T_2(\sin\phi\cos\theta + \cos\phi\sin\theta) = W\sin\theta$

$\because \sin(\theta+\phi) = \sin\phi\cos\theta + \cos\phi\sin\theta$

$\Rightarrow T_2 = \dfrac{W\sin\theta}{\sin(\theta+\phi)}$

Now for rotational equilibrium about O:

$$-Wx + (T_2\cos\phi)L = 0$$

$$\Rightarrow x = \frac{(T_2\cos\phi)L}{W} = \frac{W\sin\theta}{\sin(\theta+\phi)}\frac{\cos\phi L}{W}$$

$$x = \frac{L\sin\theta\cos\phi}{s\,\text{in}(\theta+\phi)}$$

Equilibrium means no net force and no net torque but, in the several physics books, it is called static equilibrium when a body is stationary, for example, a ladder is attached to a wall, then it is called static equilibrium or if a box is lying on a table, it is also static. However, if a plane is flying in the air and is moving at a constant speed if it is doing so, then it is called dynamic equilibrium. However, the difference between this and that is not in the sense that if you were to move along with that ship, then it would appear static to you, hence whether something is static or dynamic depends on your frame of reference so, I have not differentiated between these two. As you have seen during this lecture how many different types of systems, we can apply the concepts of equilibrium and how much we can learn about the subject of mechanics.