| Physics | PHY101_Lecture#03

Kinematics-II

Student Learning Outcomes

After listening the lecture, students will be able to,

- Understanding Position Functions
- Analyzing Constants and Dimensions
- Exploring Derivatives and Velocity
- Motion in Two Dimensions
- Vector Operations and Applications

Here are some questions for the students,

1. Is it possible for a car to have acceleration while at rest?
2. Are velocity and acceleration always in the same direction? If a car is accelerating, does it acceleration with constant acceleration?

To comprehend these questions, one must understand the term 'motion.' In this lecture, we will delve into the concept of motion and explore some functions that elucidate this term.

Function

The function represents a machine where an input is inserted, producing an output. The emphasis is specifically on the position function, denoted as X, which is a function of time, t.

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \ldots..$$

Where $c_0, c_{1,} c_{2,} c_3$ are constants and remain fixed. 'Constant' refers to values that do not change over time. It is essential to focus initially on the dimensions on both the left and right side of the equation, ensuring their equality.

Upon observing the dimension on the left side, it is identified as length, even if t is not explicitly stated. This implies that the function describes the movement of a particle in a straight line, indicating the distance covered over time, t. So, this is a function that says the particle is moving in a straight line and the distance it covers in time t. This is what we called $x(t)$.

The dimensions must match,

$$Dim[c_0] = L$$

$$Dim[c_1] = L/T$$
$$Dim[c_2] = L/T^2$$
$$Dim[c_3] = L/T^3$$

And obviously, when the value of t is zero, then x is C_0. So, the dimension of C_0 is equal to L. Now, focus on the dimension of C_1. The dimension of C_1 is equal to L/T. Similarly, the dimension of C_2 is L/T^2, and the dimension of C_3 is L/T^3. In short, if the values of C_0, C_1, C_2, etc., are unknown, then at any point, the value of X(t) can be calculated. If dimensions of constants, and function $x(t)$ are known, the dimensions of "t" can easily be determined.

Derivatives

Derivative is the invention of Newton. In differential calculus we take the differences and then dividing them by the difference the ratio of both becomes finite. Let's try to understand the concept of derivatives, which represents the change. This change might be in position, velocity, acceleration etc. The change in position "x" w.r.t time is written in form of derivative is,

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

Never make the mistake of cancelling the "d" in numerator and denominator; this is not possible and doesn't hold any meaning.

The meaning of Δt is clear in this definition. On one value of "t", you take a value of x, and then on the second value of "t", you take another value of x. Then, take the difference, which we call Δx, and divide by Δt. Here we emphasize that Δt should be very small. You might ask how small? 0.1 is not enough? no. Then 0.01 is not enough either. Even if you take 0.0001, that is still not enough. But if we make it small enough so that Δt approaches 0."

We will discuss a little detail about it now,

$$x(t) = t$$
$$\Delta x = x(t + \Delta t) - x(t)$$
$$= (t + \Delta t) - t = \Delta t$$
$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = 1$$

From this it is clear that the function whose value is 1 is called a linear function and the derivative of linear function is constant.

Let's calculate the derivative of "t^2",

$$x(t) = t^2$$

$$\Delta x = x(t + \Delta t) - x(t)$$

$$\Delta x = (t + \Delta t)^2 - t^2$$
$$= t^2 + (\Delta t)^2 + 2t\Delta t - t^2$$
$$\frac{\Delta x}{\Delta t} = \Delta t + 2t$$
$$\Rightarrow \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = 2t$$

This case is different from the previous case. Now dx/dt the derivative of x of with respect to "t" is not constant, but it depends on function. And we know that dx/dt is known as speed or velocity. As in this case dx/dt is not constant but proportional to t.

Similarly,

$$x(t) = t^3$$
$$\Delta x = (t + \Delta t)^3 - t^3$$
$$= t^3 + 3t^2\Delta t + 3t\Delta t^2 + \Delta t^3 - t^3$$
$$\frac{\Delta x}{\Delta t} = (\Delta t)^2 + 3t^2 + 3t\Delta t$$
$$\Rightarrow \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = 3t^2$$

Now generalize it for t^4, t^5…. etc.

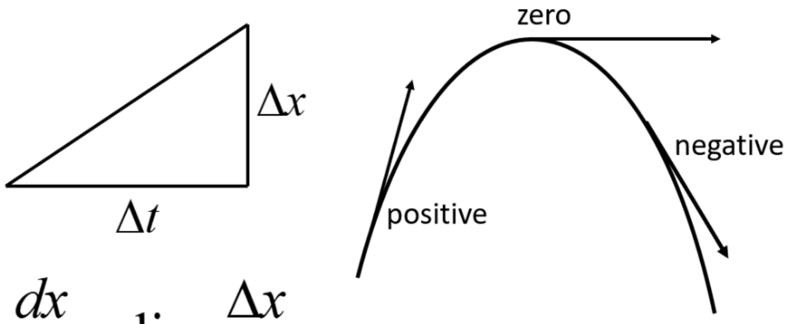
consider, the function with power "n", where n=integer

$$x(t) = t^n$$
$$then:$$
$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = nt^{n-1}$$

Then value of dx/dt is nt^{n-1}. Look for previous cases when n = 1 value of dx/dt is 1. And for n = 2 value of dx/dt is 2t. And for n = 0 the value of dx/dt should be 0. The derivative of a constant is always zero.

Geometrical interpretation of derivative

When value of t is increased to Δt, the value of x increases as well as Δx. As if dx/dt is like a gradient and in gradients, when you go a bit horizontal, you also go a bit vertical, resulting in a slope.



$$\frac{dx}{dt} \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

Figure 3.1. present the concept of derivatives.

In Figure 3.1. Three arrows represent the change w.r.t time. The arrow pointing upward indicates a positive gradient, meaning its derivative (dx/dt) is positive or the rate of change with respect to time (t) is positive. As we progress to the right, the value of t increases, and the arrow's direction changes. It transitions from an upward orientation to a horizontal one and eventually moves downward. Consequently, the sign of the derivative changes from positive to zero and then from zero to negative. This shift in sign signifies the geometrical significance of the derivative.

The second eq. of motion is,

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

Now let’s see how the derivative formulas we have drawn apply to this.

Now differentiate x with respect to t

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

$$\frac{dx}{dt} = 0 + v_0 + \frac{1}{2} a(2t)$$

$$\Rightarrow v = \frac{dx}{dt} = v_0 + at$$

$$\frac{dv}{dt} = 0 + a = a$$

x_0, v_0 is the initial displacement and velocity, respectively.

Now answer the question asked by a student about how the car will accelerate when it is parked. A car at rest can be accelerating very fast

$$v = at$$
$$\frac{dv}{dt} = a \neq 0$$

The car is not moving at t = 0, then it starts to move with some acceleration. So, from here it is clear that speed is a separate thing and acceleration is separate thing. It is possible that the object is moving in a positive direction with a constant velocity, while its acceleration is directed in the negative direction. Both speed and acceleration are always considered relative to a specific origin.

Now take the example of a stone. Stone always falls downward. A stone can be at rest yet accelerating.

$$v = -gt$$

$$\frac{dv}{dt} = -g \neq 0$$

The value of "g" does not remain constant at 9.8 m/s^2. If an object goes upward, it will decrease, and as they go further up, it will decrease even more. If an object goes out into space, the value of "g" will be zero. However, it's important to note that the units of "g" are in meters per second squared. Since we live on Earth, it would be beneficial to remember this value as 9.8 m/s^2.

A useful notation:

$$\frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$$
$$= \frac{d^2x}{dt^2}$$

When we take second derivative of x then we write it as: $\frac{d^2x}{dt^2}$

Remember that speed is $\frac{dx}{dt}$ and acceleration is $\frac{d^2x}{dt^2}$.

Motion in 2-dimension

Let's discuss some characteristics of vectors. Each vector has a specific direction and a certain length. Now, there are special vectors known as unit vectors. Their characteristic is that their length is equal to one, and they only indicate directions. For example, one unit vector may point in a certain direction, while another may be perpendicular to it. A unit vector is a vector with a magnitude of 1 (no units), and it is obtained by dividing a vector by its length or magnitude.

$$\hat{A} = \bar{A}/A$$

Example of unit vector are î and $\hat{j}$ in 2 dimensions. The vector i and j are perpendicular to each other, and their magnitude is 1. The resultant vector is written as,

$$\vec{A} = A_x\hat{\imath} + A_Y\hat{\jmath}$$

Resolution of vectors into its components

The vector component along the x direction is called x-component and the component along the y direction is called y-component.

$$\vec{A} = A_x\hat{\imath} + A_Y\hat{\jmath}$$

In this way, we can resolve a vector into its components as shown in Figure 3.2. Resolution of vectors means breaking down a vector into its components. We can do this in 3D too, except that it requires three vectors, î, $\hat{\jmath}$ and $\hat{k}$.

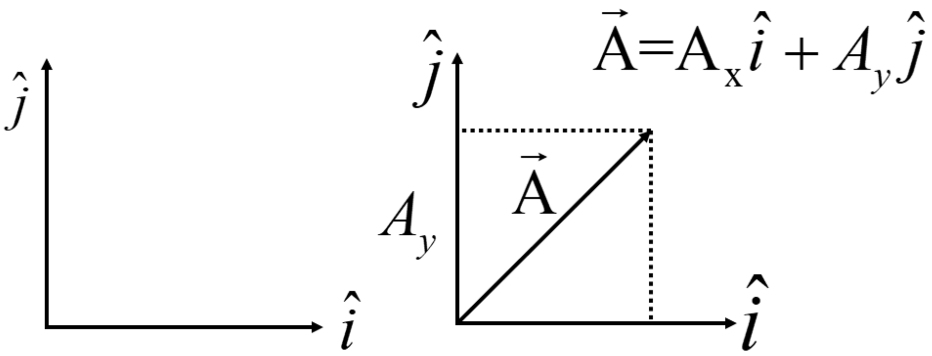


Figure 3.2. Illustrate the concept of addition of vectors.

Velocity in 2 dimensions

We have already discussed velocity in 1 dimension in detail. Velocity in 2 dimensions come from differentiating a displacement vector in 2 dimensions.

$$\vec{r} = x(t)\hat{\imath} + y(t)\hat{\jmath}$$

$$\vec{v} = \frac{\overrightarrow{dr}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath}$$

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$$

Acceleration in 2 dimensions

Acceleration is the rate of change of velocity. When the velocity in 2-dimension is differentiated with respect to time, the acceleration is

$$\vec{a} = \frac{d\vec{v}}{dt}$$
$$= \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath}$$
$$= a_x\hat{\imath} + a_y\hat{\jmath}$$

Addition of vectors

Two vectors **A** and **B** can be added by head to tail rule. Mathematically,

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$$
$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$$
$$\vec{R} = \vec{A} + \vec{B}$$
$$= \left(A_x\hat{\imath} + A_y\hat{\jmath}\right) + \left(B_x\hat{\imath} + B_y\hat{\jmath}\right)$$
$$= (A_x + B_x)\hat{\imath} + \left(A_y + B_y\right)\hat{\jmath}$$
$$= R_x\hat{\imath} + R_y\hat{\jmath}$$

Example

$$\vec{A} = 6\hat{\imath} + 5\hat{\jmath}$$
$$\vec{B} = 8\hat{\imath} + 7\hat{\jmath}$$

What is the magnitude of of $2\vec{A} - \vec{B}$?

Letting

$$\vec{R} = 2\vec{A} - \vec{B}$$

$$= 2\ (\ 6\hat{\imath} + 5\hat{\jmath}) - (\ 8\hat{\imath} + 7\hat{\jmath})$$

$$= (\ 12 - 8\ )\hat{\imath} + (\ 10 - 7\ )\ \hat{\jmath}$$

$$\vec{R}\ = 4\hat{\imath} + 3\hat{\jmath}$$

The magnitude is,

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{4^2 + 3^2} = 5$$

Consider two vectors $\vec{A}$ and $\vec{B}$ making an angle θ with each other as shown in Figure 3.3,

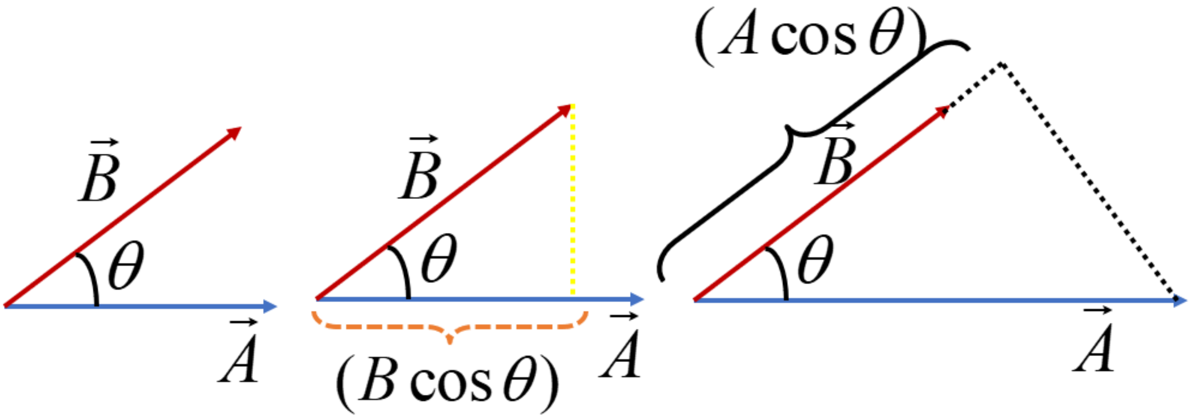


Figure 3.3. Illustrates the schematic of dot product of two vectors.

We can also write it as:

$$\vec{A}.\vec{B} = (A)(B\cos\theta)$$

= (length of A) × (projection of B on A)

Scalar product of unit vectors

The dot product of vector with itself is always 1, and dot product of two mutually perpendicular vectors is always zero.

$$\hat{i}.\hat{i} = \hat{j}.\hat{j} = (1)(1)\cos(0) = 1$$
$$\hat{i}.\hat{j} = (1)(1)\cos(90^\circ) = 0$$

We can take dot product easily between any two vectors.

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$$
$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$$
$$\vec{A}.\vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath}).(B_x\hat{\imath} + B_y\hat{\jmath})$$
$$= A_xB_x\hat{\imath}.\hat{\imath} + A_xB_y\hat{\imath}.\hat{\jmath} + A_tB_x\hat{\jmath}.\hat{\imath} + A_yB_y\hat{\jmath}.\hat{\jmath}$$
$$= A_xB_x + A_yB_y$$

Now you can do all these things in the same way in 3 dimensions. There would be only one more unit vector needed which is $\hat{k}$.

Generalization in 3 dimensions

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

$$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Application:

Projectile Motion:

Consider the example of a ball being thrown. When the ball is thrown, it follows a trajectory until it reaches its destination. The ball's velocity is composed of two distinct components: one in the y-direction and another in the x-direction. Notably, the x and y components of velocity are independent of each other. Acceleration is a factor affecting the ball's motion. Upon release, the ball descends in the negative y-direction, experiencing acceleration in the y-direction. However, there is no acceleration acting in the x-direction. For the sake of clarity, let's denote the acceleration of the ball in the y-direction as "a_y."

- Acceleration along y-axis is $a_y = -g$
- Velocity along x is constant
- Acceleration along x-axis is $a_x = 0$

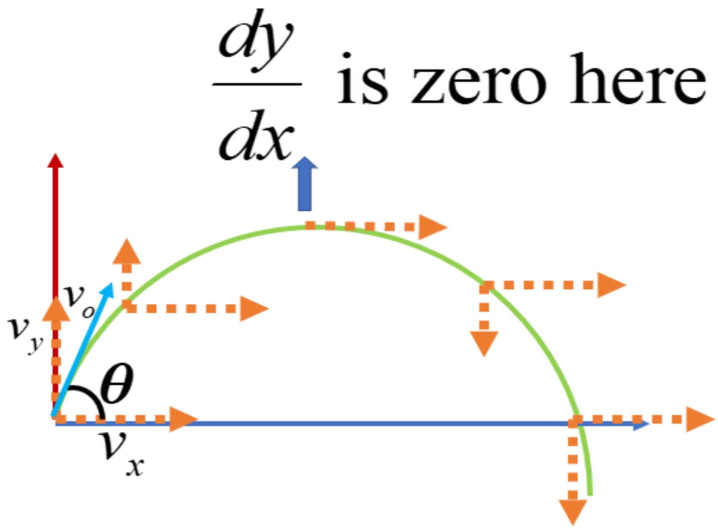


Figure 3.4. Shows the projectile motion of an object at various instants.

$$V_{0x} = V_0 \cos\theta$$
$$V_{0y} = V_0 \sin\theta$$

As presented in Figure 3.4 the velocity V of the ball is represented by two components: horizontal V_x and vertical V_y. As the ball traverses its trajectory, its velocity undergoes changes. Notably, the value of V_x remains constant because the acceleration a_x in the horizontal x direction is equal to zero, indicating no acceleration along the x-direction. In contrast, the value of V_y changes as the ball moves. At its highest point in the trajectory, V_y becomes zero. As discussed earlier, the gradient dy/dx is also zero at this point. As x increases beyond this point, dy/dx continues to be zero. Subsequently, the ball descends, causing its vertical component V_y to become negative. V_y eventually reaches zero again at a lower point in the trajectory, and further descent results in negative values for V_y. A more detailed examination using mathematical formulas will provide further insight into this phenomenon.

Along x-axis

As the variable "x" undergoes change, it is important to note that when "x" is constant, it equals the product of velocity (v) and time (t). The initial velocity "v" is the same as the velocity with which the ball was initially thrown. When the ball is thrown, the component along the horizontal direction remains constant when time is equal to zero. At this point, the value of "x" is zero, and this choice of coordinates is convenient, as it aligns with the starting point of the ball. While alternative coordinate choices are valid, setting "x" to zero at "t" not equal to zero proves to be a practical choice.

Examining this concept in mathematical terms, the relationship is described through the following formulas.

X direction

$$V_x = V_{0x}$$
$$x = x_0 + V_{0x}t$$
$$a_x = 0$$

Y-direction

$$a_y = -g$$
$$V_y = V_{0y} - gt$$
$$y = y_0 + V_{0y}t - \frac{1}{2}gt^2$$

How much its velocity changes depend on the value of "t". In the equation, $V_y = V_{0y} - gt$, if "t" is equal to zero, then the values of V_y will be equal to the initial vertical velocity, but with the increase of "t", they are decreasing and finally when they reach their maximum, where V_y will be zero.

Consider the following scenario: an inquiry into the maximum height a ball can attain when thrown lightly versus when thrown with greater force. The analysis suggests that a faster throw results in a higher ascent. To quantify this, calculations are performed with the objective of determining the optimal height. Subsequently, attention is turned to calculating the maximum distance the ball can cover in the horizontal (x) direction—the maximum range. To address these questions, algebraic methods are employed to derive the necessary equations and relationships that govern the ball's trajectory.

$$V_{0x} = V_0 \cos\theta$$
$$V_{0y} = V_0 \sin\theta$$

$$V_y = V_{0y} - gt$$
$$v_y = 0$$
$$v_0 \sin\theta - gt = 0$$
$$t = \frac{v_0 \sin\theta}{g}$$
$$y = (v_0 \sin\theta)t - \frac{1}{2}gt^2$$
$$H = (v_0 \sin\theta)\left(\frac{v_0 \sin\theta}{g}\right) - \frac{1}{2}g\left(\frac{v_0 \sin\theta}{g}\right)^2$$
$$\boldsymbol{H = \frac{(v_0 \sin\theta)^2}{2g}}$$

In the quest to achieve the maximum height for a thrown ball, it is essential to set the angle 90, as the sine function attains its maximum value at 90 degrees. Thus, to propel the ball to its highest point, aligning it straight upward is optimal. On the other hand, if the objective is to maximize the horizontal distance covered by the ball, the strategy involves achieving the maximum range "R". Given that the speed of the ball remains constant in the horizontal direction, the distance covered "x" in time "t" is crucial. At two distinct points in time, "y" attains the value of zero: firstly, at the release point (initial point) and secondly, at the landing point (final point). Consequently, there are two unique values of "x" where "y" equals zero. The first is evident at "x = 0", and the second, denoted as x = R, represents the distance from the starting to the ending point. Notably, the choice of θ is pivotal for achieving the maximum range. To optimize the horizontal distance covered, it is recommended to throw the ball at an angle of θ = 45°.

$$x = (v_0 \cos\theta)t$$

$$t = \frac{x}{(v_0 \cos\theta)}$$

$$y = (v_0 \sin\theta)t - \frac{1}{2}gt^2$$

$$= (v_0 \sin\theta)\frac{x}{(v_0 \cos\theta)} - \frac{1}{2}g\left(\frac{x}{(v_0 \cos\theta)}\right)^2$$

$$= x \tan\theta - x^2\left(\frac{g \sec^2\theta}{2{v_0}^2}\right)$$

$$y = x\left[\tan\theta - x\left(\frac{g}{2{v_0}^2 \cos^2\theta}\right)\right] = 0$$

This equation has two solutions for x,

$$x = 0, \qquad and \qquad x = R\ (range)$$

$$\left[\tan\theta - R\left(\frac{g}{2{v_0}^2 \cos^2\theta}\right)\right] = 0$$

$$R\left(\frac{g}{2{v_0}^2 \cos^2\theta}\right) = \tan\theta$$

$$R = \frac{2{v_0}^2 \cos^2\theta}{g}.\tan\theta$$

$$R = \frac{2{v_0}^2 \cos^2\theta}{g}.\frac{\sin\theta}{\cos\theta}$$

$$R = \frac{2{v_0}^2 \sin\theta \cos\theta}{g}$$

$$\boldsymbol{R = \frac{{v_0}^2 \sin 2\theta}{g}}$$

Since $-1 \leq \sin 2\theta \leq 1$

therefore $(\sin 2\theta)_{\max} = 1$

$$\Rightarrow R_{\max} = \frac{{v_0}^2}{g}(\sin 2\theta)_{\max} = \frac{{v_0}^2}{g}$$