Physics-PHY101-Lecture 8

Conservation of Energy

Potential energy:

Potential energy is, as the word suggests, the energy "locked up" up somewhere and which can do work. It can be defined as, "**energy possessed by a body due to its position**".

It has the ability or capacity to do work like other forms of energies. Potential energy can be converted into kinetic energy ($1/2\ mv^2$) and vice versa.

Formula: Potential energy can be calculated using the following formula

$$P.E = m.g.h$$

Where,

- m is the mass in kilograms
- g is the acceleration due to gravity ($10\ m/s^2$)
- h is the height in meters

SI unit:

$$P.E = m.g.h = kg.ms^{-2}.m = kgm^2s^{-2} = \text{Joule (J)}$$

Types of potential energy:

The types of potential energy are,

1. Elastic potential energy
2. Gravitational potential energy

Elastic potential energy:

When an object is compressed or stretched, the energy stored in object is called elastic potential energy. The more the object stretch or compress, the more elastic potential energy is.

Examples:

- A twisted rubber band that powers a toy plane
- An archer's stretched bow
- A bent diver's board just before a diver dive in

Formula: Elastic potential energy can be calculated using the following formula,

$$U = \frac{1}{2}kx^2$$

Where,

- U is the elastic potential energy
- k is the spring force constant
- x is the string stretch length in m

Problem 1:

Suppose you pull on a spring and stretch it by an amount away from its normal (equilibrium) position. How much energy is stored in the spring?

Solution:

Obviously, the spring gets harder and harder to pull as it becomes longer. When it is extended by length x and you pull it a further distance dx, the small amount of work done is dW = Fdx = kxdx. Adding up all the small bits of work gives the total work:

$$W = \int_0^x F dx = \int_0^x kx dx = \frac{1}{2}kx^2$$

Gravitational potential energy:

The gravitational potential energy of an object is defined as "the energy possessed by an object that raise to a certain height against gravity. As the object is raised against the force of gravity, some amount of work (W) is done on it.

Work done on the object = force × displacement

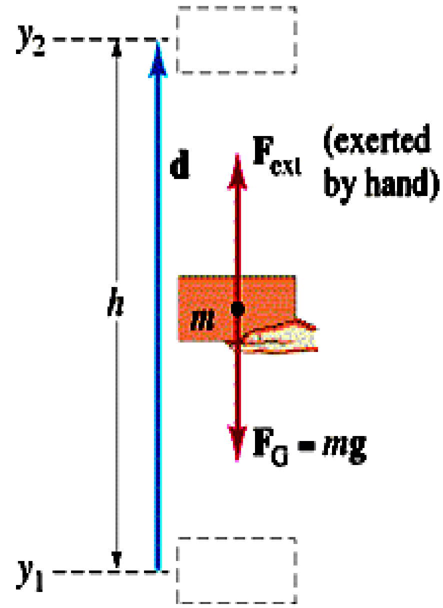
W = F.h

W = mg.h

(a = g = gravitational acceleration)

$$W = P.E = m.g.h$$

Example: If you lift a stone of mass from the ground up a distance, you have to do work against gravity. By conservation of energy, the work done by you was transformed into gravitational potential energy whose values is exactly equal to mgx. Where is the energy stored?



Answer: it is stored neither in the mass nor in the earth. It is stored in the gravitational field of the combined system of stone + earth.

1. How much work does it take to lift a mass m to height h, as shown in figure?

Answer:

$$W_{ext} = F_{ext}\, d$$
$$W_{ext} = mg\, h$$
$$W_{ext} = m\, g\, (y_2 - y_1)$$

You did work on the object. Therefore, its energy increased.

Key points:

- PE is measured with respect to some reference level.
- Only changes in PE actually have physical meaning.
- Changes in PE do not depend on path.
- Energy is a shared property.
- Work is force x distance.
- Energy is the capacity to do work.
- Power is the "rate of doing work"
- $\text{Power} = \dfrac{\text{Work done}}{\text{Time taken}}$

Problem 2:

Your heart is working as a pump and the volume of blood lifted daily is 8000 liters and the blood is lifted up to a height of 1.5 m. calculate its potential energy and power.

Solution:

Volume of blood lifted daily = 8000 litres

Average height lifted = 1.5 m

$\therefore$ Density of blood $\approx$ 1 kg/litre

$P.E = mgh$ $\qquad$ $\because$ density = mass/volume $\Rightarrow$ mass = density * volume

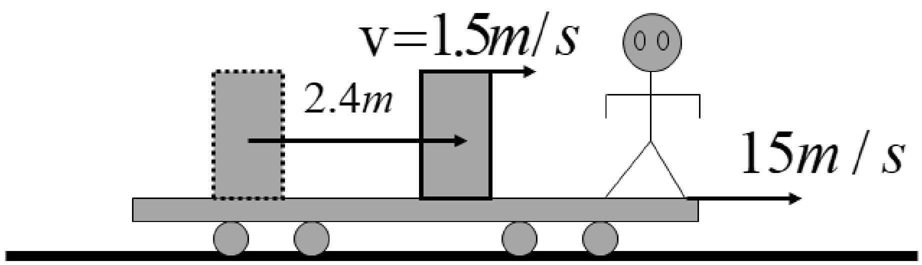
$Work\ done = P.E$ = 8000 x 1 x 9.8 x 1.5

$\approx$ 120,000 J in 24 hours

$$\therefore \quad \text{Power} = \frac{Work\ done}{time} = \frac{120000}{24 \text{ x } 60 \text{ x } 60} \approx 1.4 \text{ W}$$

Problem 3:

A box of mass 12kg is pushed with a constant force so that its speed goes from zero to 1.5m/sec (as measured by the person at rest on the cart) and it covers a distance of 2.4m. Assume there is no friction.



Solution:

$$\text{mass of box} = 12\ kg$$

$$\Delta K = K_f - K_i = \frac{1}{2}(12kg)(1.5m/s)^2 - 0 = 13.5J$$

$$a = \frac{v_f^2 - v_i^2}{2(x_f - x_i)} = \frac{(1.5m/s)^2 - 0}{2(2 \cdot 4m)} = 0.469m/s^2$$

This acceleration results from a constant net force given by:

$$F = ma = (12kg)(0.469\,m/s^2) = 5.63N$$

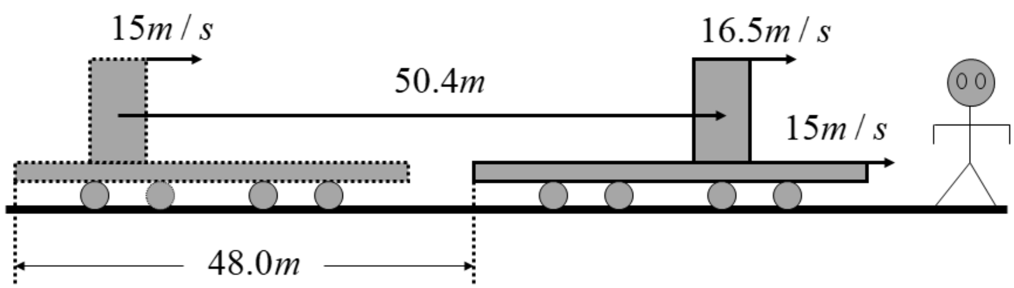
Work done on the crate is:

$$W = F\Delta x = (5.63N)(2.4m) = 13.5J$$

(same as $\Delta K = 13.5J$!)

Problem 4:

Suppose there is somebody standing on the ground, and that the trolley moves at 15 m/sec relative to the ground.



Solution:

Let us repeat the same calculation:

$$\Delta K' = K'_f - K'_i = \frac{1}{2}m\text{v}_\text{f}^2 - \frac{1}{2}m\text{v}_\text{i}^2$$
$$= \frac{1}{2}(12kg)(16.5m/s)^2 - \frac{1}{2}(12kg)(15.0m/s)^2$$
$$= 284J$$

(This is not equal to $\Delta K = 13.5J$)

This example clearly shows that work and energy have different values in different frames. Now, if this person calculate the force to be 5.63 N, then calculate work done by the person.

$$\because\ a' = a\ \therefore\ F' = F = 5.63N$$

$$t = \frac{\text{v}_f - \text{v}_i}{a} = \frac{1.5m/s}{0.469m/s^2} = 3.2s$$
and train moves (15m/s)(3.2s) $= 48m$

$$\text{Total displacement} = \Delta x'$$
$$= 48m + 2.4m = 50.4m$$

The ground-based observer also concludes that the work is:

$$W' = F'\Delta x' = (5.63N)(50.4m) = 284J$$

Conclusion:

- If you calculate work or kinetic energy in different frame of reference, the results would be different in both frames.

- Potential energy and kinetic energy depend on the frame you choose to measure it in. If you are running with a ball, it has zero kinetic energy with respect to you. But someone who is standing will see that it has kinetic energy.
- In the absence of friction, the total energy of the system is conserved.
- Total mechanical energy is:

  $E_{mech} = KE + PE$
- If no friction then E_{mech} is conserved:

  $\Delta(E_{mech}) = \Delta(KE) + \Delta(PE) = 0$
- $E_{mech} = KE + PE$ is constant

Problem 5:

A ball is thrown upwards at speed v_o. How high will it go before it stops?

Solution:

The loss of potential energy is equal to the gain of potential energy,

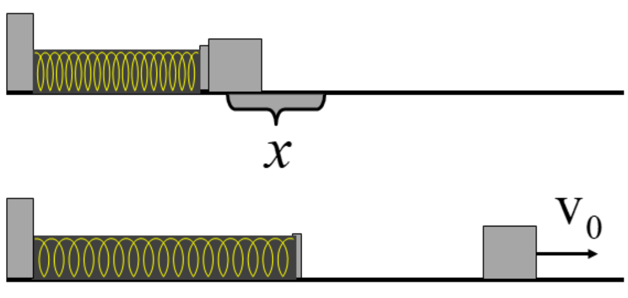
$$K.E = P.E$$

$$\frac{1}{2}mv_0^2 = mgh$$

$$h = \frac{v_0^2}{2g}$$

Problem 6:

A mass is attached to a spring, this spring is compressed, and elastic potential energy is stored in it. The extension created in this string is x. When mass m is released from the spring, it moves with velocity V_o. Here, we want to calculate that velocity.



Solution:

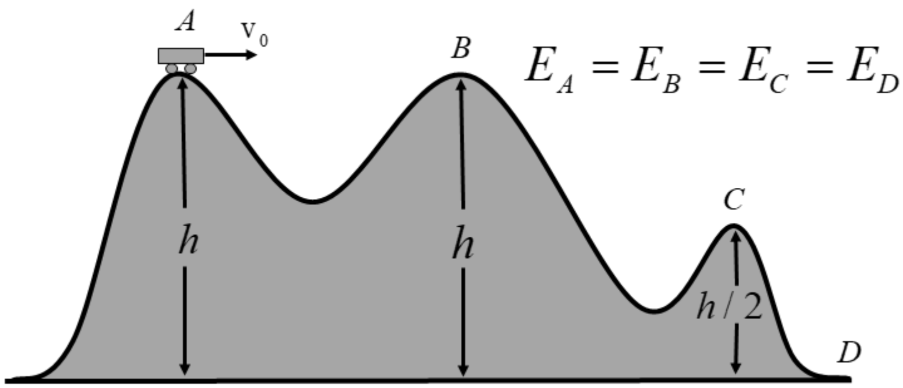
As there is no friction, the kinetic energy of the spring becomes equal to the elastic potential energy.

$$\frac{1}{2}m\mathrm{v}_0^2 = \frac{1}{2}kx^2$$

$$\mathrm{v}_0 = \sqrt{\frac{k}{m}}x$$

Problem 7:

Now look at the smooth, frictionless motion of a car over the hills as shown below:



What will be the speed of the car,

a) at point B,

b) at point C,

c) at point D?

Solution: As, in the absence of friction, energy at every point would be the same.

$$E_A = E_B = E_C = E_D$$

According to the law of conservation of energy, From A to B;

$$\frac{1}{2}m\mathrm{v}_A{}^2 + mgh = \frac{1}{2}m\mathrm{v}_B{}^2 + mgh$$

$$\mathrm{v}_A = \mathrm{v}_B = \mathrm{v}_0 \rightarrow 1$$

From B to C;

$$\frac{1}{2}m\mathrm{v}_B{}^2 + mgh = \frac{1}{2}m\mathrm{v}_C{}^2 + mg\frac{h}{2}$$

$$\frac{1}{2}m\mathrm{v}_B{}^2 + mgh = \frac{1}{2}\left(m\mathrm{v}_C{}^2 + mgh\right)$$

$$m\mathrm{v}_B{}^2 + mgh = \left(m\mathrm{v}_C{}^2 + mgh\right)$$

$$\mathrm{v}_B{}^2 + gh = \mathrm{v}_C{}^2$$

$$\mathrm{v}_\mathrm{C} = \sqrt{\mathrm{v}_0{}^2 + gh} \rightarrow 2$$

From C to D;

$$\frac{1}{2}m\mathrm{v}_C{}^2 + mg\frac{h}{2} = \frac{1}{2}m\mathrm{v}_D{}^2$$

$$\mathrm{v}_D{}^2 = \mathrm{v}_C{}^2 + gh \rightarrow 3$$

putting eq.2 in eq.3

$$\mathrm{v}_D{}^2 = \mathrm{v}_o{}^2 + gh + gh$$

$$\mathrm{v}_\mathrm{D} = \sqrt{\mathrm{v}_0{}^2 + 2gh} \rightarrow 4$$

Remember that potential energy has meaning only for a force that is conservative. A conservative force is that for which the work done in going from point A to point B is independent of the path chosen. Friction is an example of a non-conservative force, and a potential energy cannot be defined. For a conservative force,

$$F = -\frac{dV}{dx}$$

$$as, V = \frac{1}{2}kx^2$$

$$F = -\frac{d}{dx}\left[\frac{1}{2}kx^2\right] = -\frac{1}{2}k.\frac{d}{dx}\left[x^2\right]$$

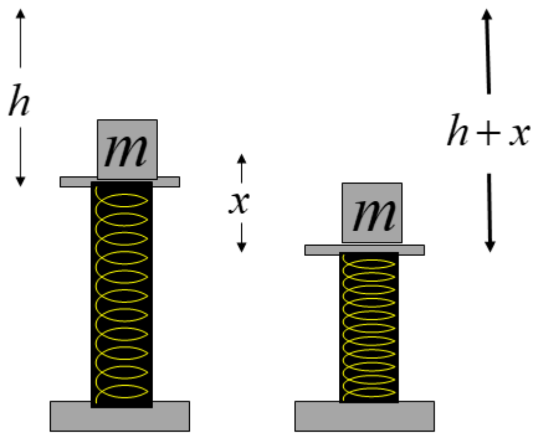
$$F = -\frac{1}{2}k.2x = -kx$$

<u>Problem 8:</u>

A mass m is taken on height h above spring. The mass compresses the spring upon falling on it as shown in the figure. The extension in the spring is x. We have to calculate that x.

Solution:

According to the conservation of energy, initial K.E becomes equal to the P.E stored in the spring,



$$\frac{1}{2}kx^2 = mg(x+h)$$

$$\frac{1}{2}kx^2 = mgx + mgh \Rightarrow \frac{1}{2}kx^2 - mgx - mgh = 0$$

$$a = \frac{1}{2}k, \quad \text{b}= -mg, \quad \text{c}= -mgh$$

On applying quadratic formula, we get,

$$\text{x}= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{x}= \frac{mg \pm \sqrt{(mg)^2 - 4\left(\frac{1}{2}k\right)(-mgh)}}{k} = \frac{mg \pm \sqrt{m^2g^2 + (2k)(mgh)}}{k}$$

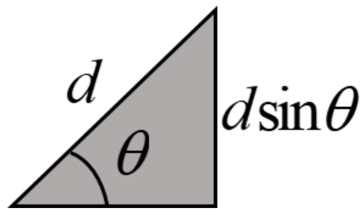
$$\text{x}= \frac{mg \pm mg\sqrt{1+\frac{2kh}{mg}}}{k}$$

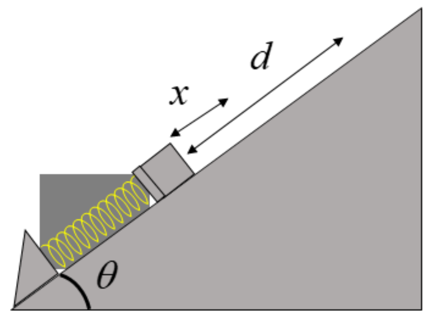
$$\Rightarrow x = \frac{mg}{k}(1 \pm \sqrt{1+2hk/mg})$$

Here, we find the two solutions for this problem. The two solutions prove the oscillation of the spring (up & down) when mass falls on it.

Problem 9:

Consider an inclined plane. A spring with attached mass is taken on it. The mass is stretched through a distance of d, which compresses the spring through x. We want to calculate the value of x.





Solution: According to the conservation of energy, K.E becomes equal to the P.E stored in the spring,

$$\frac{1}{2}kx^2 = mgd\sin\theta$$

$$\Rightarrow \quad d = \frac{kx^2}{2mg\sin\theta}$$

$$mgd\sin\theta = \frac{1}{2}m\mathrm{v}^2$$

$$\Rightarrow \quad \mathrm{v} = \sqrt{2gd\sin\theta}$$

To find d, we have,

$$mg(x+d)\sin\theta = \frac{1}{2}kx^2 \Rightarrow mgx\sin\theta + mgd\sin\theta = \frac{1}{2}kx^2$$

$$mgd\sin\theta = \frac{1}{2}kx^2 - mgx\sin\theta$$

$$d = \frac{kx^2}{2mg\sin\theta} - x$$

To find x, we have,

$$\frac{1}{2}kx^2 - mgx\sin\theta - mgd\sin\theta = 0$$

$$a = \frac{1}{2}k, \quad \mathrm{b} = -\mathrm{mg}\sin\theta, \quad \mathrm{c} = -mgd\sin\theta$$

on applying quadratic formula, we get,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{mg\sin\theta\pm\sqrt{(mg\sin\theta)^2-4\left(\frac{1}{2}k\right)(-mgd\sin\theta)}}{k}$$

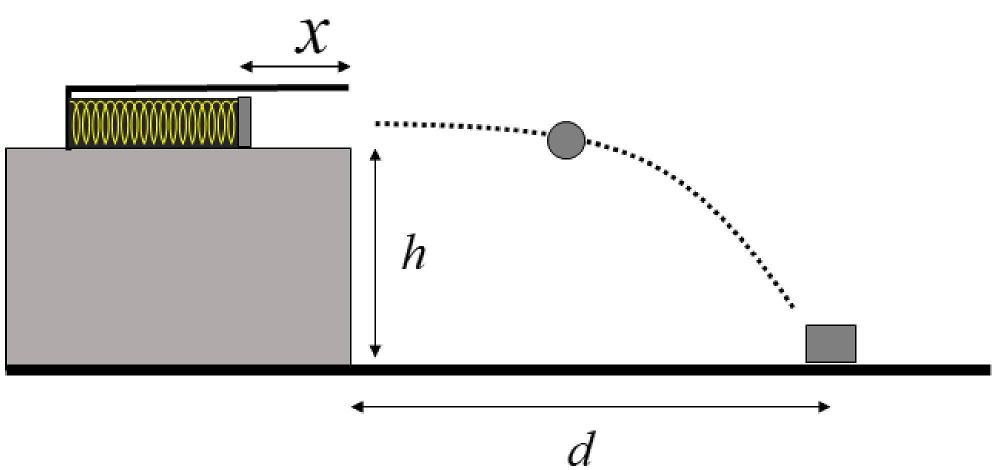
$$x=\frac{mg\sin\theta\pm\sqrt{(mg\sin\theta)^2+2k(mgd\sin\theta)}}{k}$$

$$x=\frac{mg\sin\theta\pm mg\sin\theta\sqrt{1+\left(\frac{2kd}{mg\sin\theta}\right)}}{k}$$

$$x=\frac{mg\sin\theta}{k}\left(1\pm\sqrt{1+\frac{2kd}{mg\sin\theta}}\right)$$

Problem 10:

A spring is taken at height h from ground level and a ball is attached to it. When the spring is compressed, the ball falls down to earth and make a parabolic path and touches the mass which is on ground at distance d. We want to know the value of x (extension of the spring) as shown in figure.



Solution:

According to the conservation of energy, K.E becomes equal to the P.E stored in the spring,

$$\frac{1}{2}m\mathrm{v}_0^2 = \frac{1}{2}kx^2$$

$$\Rightarrow \quad x = \sqrt{\frac{m}{k}}\mathrm{v}_0$$

$$\mathrm{v}_0 = \mathrm{v}_{ix} = \sqrt{\frac{kx}{m}}$$

$$y-component;$$

$$S_y = \mathrm{v}_{iy}t + \frac{1}{2}a_y t^2$$

$$S_y = h, \quad a_y = 9.8$$

$$h = o - \frac{1}{2}gt^2 = \frac{1}{2}gt^2$$

$$\Rightarrow \quad t = \sqrt{\frac{2h}{g}}$$

$$x-component;$$

$$S_x = \mathrm{v}_{ix}t + \frac{1}{2}a_x t^2$$

$$d = \sqrt{\frac{kx}{m}}t + 0$$

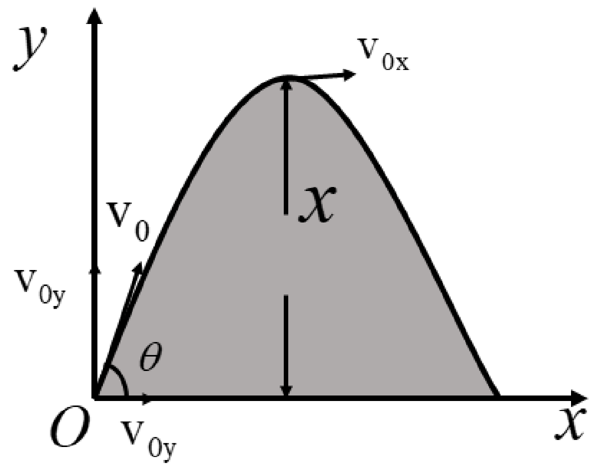
$$d = \sqrt{\frac{kx}{m}}.\sqrt{\frac{2h}{g}} = \sqrt{\frac{2kh}{mg}}x$$

$$\Rightarrow \quad x = d\sqrt{\frac{mg}{2hk}}$$

Problem 11:

A ball of mass m is falling vertically upward which performs a projectile motion under constant acceleration. We want to calculate the maximum height of the ball reaches during its motion.

Solution: According to the conservation of energy, initial K.E and P.E of the ball becomes equal to the final K.E and P.E,



$$K_0 + U_0 = K + U$$

$$\frac{1}{2}mv_{0x}^2 + \frac{1}{2}mv_{0y}^2 = \frac{1}{2}mv_{0x}^2 + mgh$$

$$\frac{1}{2}mv_{0y}^2 = mgh$$

$$\because v_{0y} = v_0 \sin\theta$$

$$v_{0y}^2 = 2gh$$

$$v_0{}^2 \sin^2\theta = 2gh$$

$$h = \frac{v_0^2 \sin^2\theta}{2g}$$

At different angles, the ball reaches to different heigh but it reaches its maximum height at 90 degree.

$\because \sin 90 = 1$

$$h_{\max} = \frac{v_0^2}{2g}$$

Conservative force:

Conservative force means, work does not depend on the path taken.

- Gravitational force
- Electrostatic force
- Elastic force

Note: Remember that potential energy has meaning only for a force that is conservative. A conservative force is that for which the work done in going from point A to point B is independent of the path chosen.

Non-Conservative force:

Non-Conservative force means work is dependent on the path taken.

For Example:

- Frictional force
- air resistance force
- velocity dependent forces

Potential energy cannot be defined for non-conservative forces.

Problem 12:

Suppose a particle has potential energy as V(x), and the particle followed path as shown in figure.

Solution:

If the particle moves Δx

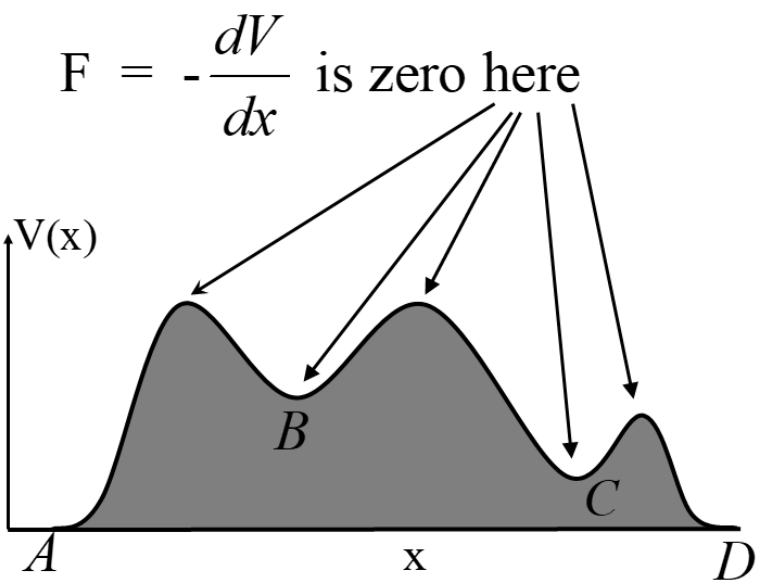
Then change in PE is ΔV

Where: $\Delta V = - F \, \Delta x$

$$\Rightarrow \qquad F = - \frac{\Delta V}{\Delta x}$$

Now let $\Delta x \rightarrow 0$

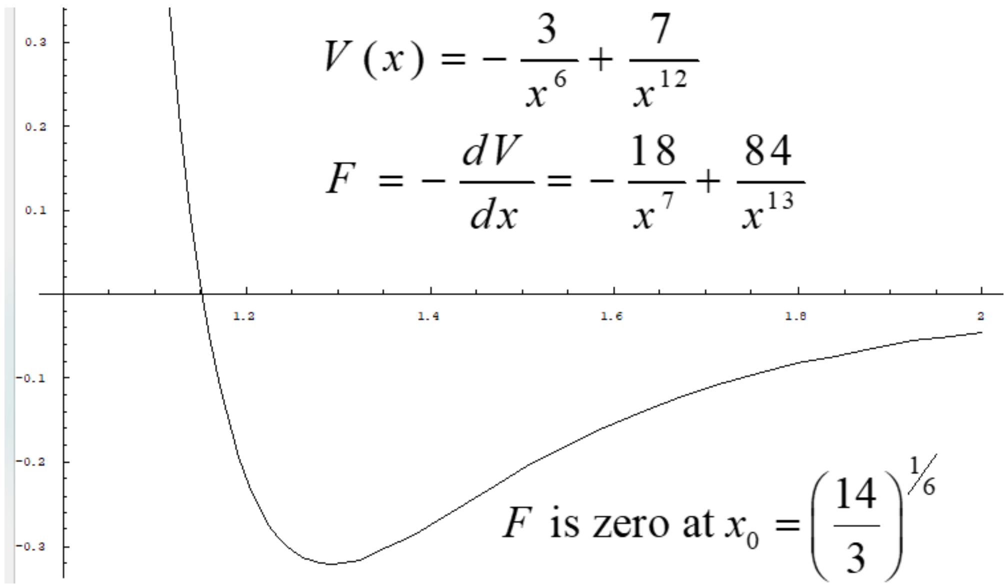
$$F = - \lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} = - \frac{dV}{dx}$$



The tangent shows that the derivative of potential is zero at the top point of the curve, where the tangent is flat to curve, force is also zero there.

Example:

Suppose we have a potential function as shown in the figure.



The potential function is given as,

$$V(x) = -\frac{3}{x^6} + \frac{7}{x^{12}}$$

$$As, \Rightarrow F = -\frac{dV}{dx} = -\frac{d}{dx}\left[-\frac{3}{x^6} + \frac{7}{x^{12}}\right],$$

Using power rule nx^{n-1}, and taking differential w.r.t "x",

$$F = 3\left(6.x^{-6-1}\right) + 7\left(12.x^{-12-1}\right) = \left[-\frac{18}{x^7} + \frac{84}{x^{13}}\right]$$

F is zero at x_0, *Hence,*

$$0 = \left[-\frac{18}{x_0^{\ 7}} + \frac{84}{x_0^{\ 13}}\right]$$

$$\frac{x_0^{\ 13}}{x_0^{\ 7}} = \frac{84}{18} \Rightarrow x_0^{\ 13-7} = \frac{28}{6}$$

$$x_0^{\ 6} = \frac{14}{3} \Rightarrow x_0 = \left(\frac{14}{3}\right)^{\frac{1}{6}}$$

Conclusion: We can conclude that potential can be defined only for the conservative forces. We cannot define potential for non-conservative forces.

Because non-conservative forces do not satisfy the path-independence property. The work done by non-conservative forces depends on the specific path taken, and as a result, it is not possible to define a potential energy associated with non-conservative forces in the same way. Examples of non-conservative forces include friction and air resistance.