Physics-PHY101-Lecture 12

PHYSICS OF MANY PARTICLES

Introduction to center of mass:

Everybody is made up of particles and every particle has a direction with every other particle, that it either pulls it towards itself or pushes it. There are some quantities about which if we have the information then we can say something about the system as a whole which means they include the center of mass. So where the center of the mass will be, there will be a kind of concentration of mass.

Let's say there is an elephant. An ant moves on top of the elephant. It will not make much difference if the ant moves here or there. The mass of the elephant is so much that the center of mass will be the center of elephant. We would like to define the **center of mass** as,

A system's or object's center of mass is the location where the mass is considered to be concentrated. Simply, it's the average position of the mass distribution.

Mathematically for two masses the center of mass r_{cm} is:

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

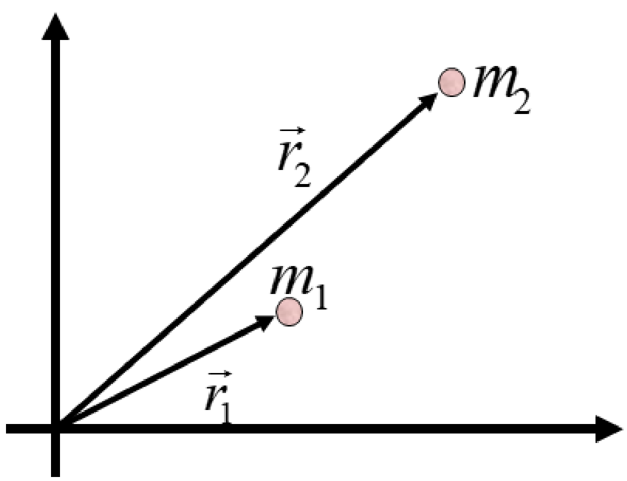


Figure 12.1. Distribution of masses ‘m_1 and m_2’ with their position vectors ‘r_1 and r_2’.

The center of mass equation can be written in two or three dimensions, depending on the given situation in question. In the case of two dimensions, its coordinates will be x and y; in the case of three dimensions, there will be x, y, and z. Equation of center of mass for two dimension is:

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$

The center of mass will lie between two masses if their masses are equal ($m_1 = m_2 = m$), and closer to the heavier mass if one mass is significantly greater than the other. If M is heavier than m, the center of mass is near mass M as can be seen from figure 12.2.

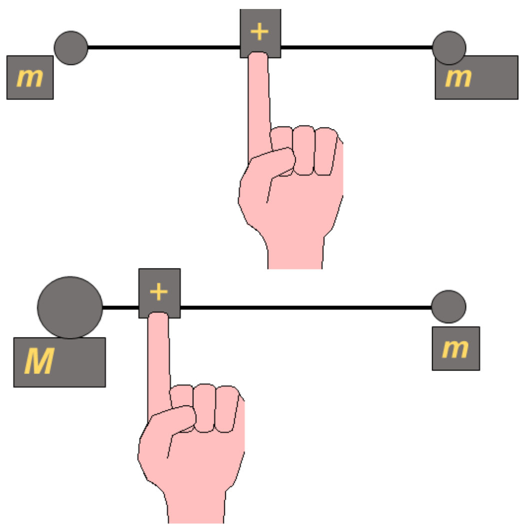


Figure 12.2. Center of mass alignment for unequal masses. Equal masses ($m_1 = m_2 = m$) result in a center of mass equidistant between them. For unequal masses (M > m), the center of mass is closer to the heavier mass.

Let's take another example. Consider two bodies of equal masses, one is placed at x = 2 m and other is placed at x = 6 m as shown in figure 12.3, then center of mass will be ,

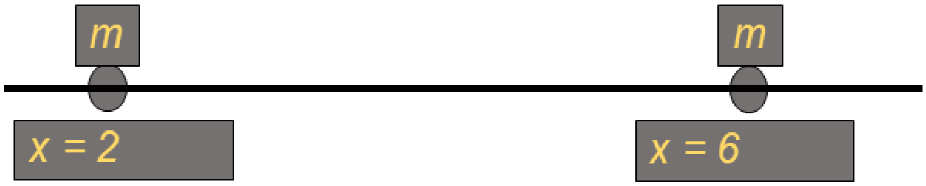


Figure 12.3. Two bodies of equal masses are positioned at x = 2 m and x = 6 m. The center of mass location is determined by their equilibrium.

$$x_{cm} = \frac{mx_1 + mx_2}{m + m}$$

$$x_{cm} = \frac{m(2\ meter) + m(6\ meters)}{2m}$$

$$x_{cm} = \frac{8m}{2m} = 4m\ \text{(4 meters)}$$

Now consider two bodies of unequal masses. One is a lighter 'mass m' placed at x = 6 m and other is a heavier 'mass 3m' placed at x = 2 m as shown in figure 12.4.

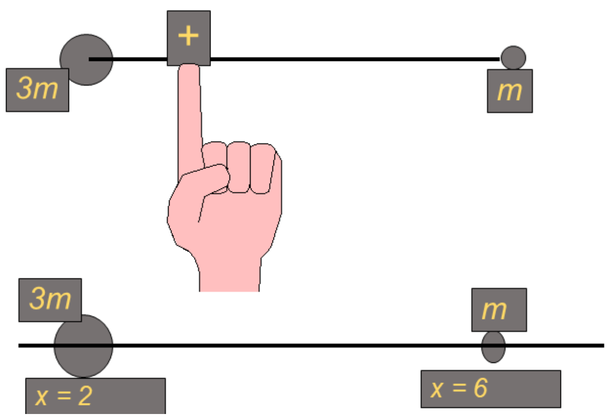


Figure 12.4. A lighter mass 'm' is positioned at x = 6 m and a heavier mass '3m' at x = 2 m. The center of mass is influenced by the masses and their respective positions.

$$x_{cm} = \frac{(3m)x_1 + mx_2}{3m + m} = \frac{(3m)2 + 6m}{4m} = \frac{12m}{4m} = 3\ meters$$

For N masses, the center of mass is:

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N}{m_1 + m_{2+} + \cdots + m_N}$$

$$\vec{r}_{cm} = \frac{1}{M}\left(\sum m_n\vec{r}_n\right)$$

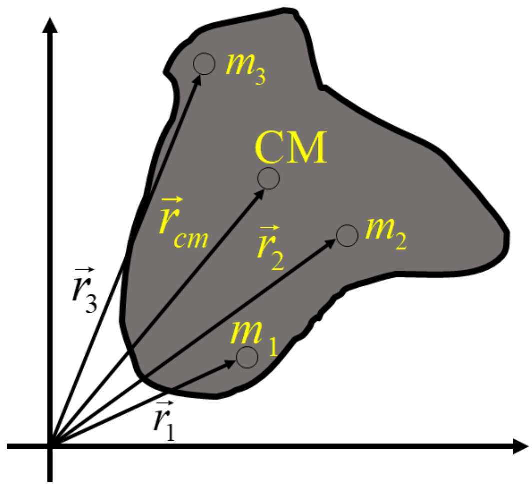
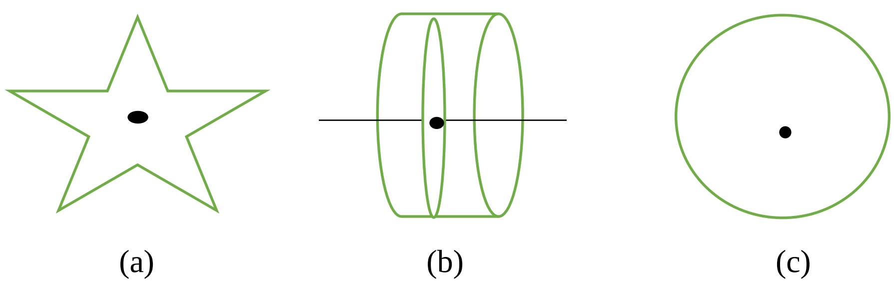


Figure 12.5: There is a point CM which we call center of mass, from the origin O, vector r be the position vector.

For symmetrical object, the center of mass is easy to guess as can be seen from the figures below.



In figure (a), the center of mass is in the center of star and in (b) the center of mass is in the center of cylinder if the material of cylinder will be uniform and in figure (c) if it's a uniform sphere then its center of mass is also in the center of sphere.

In order to understand that why the center of mass is useful concept, we have to apply newton's law. The velocity and acceleration of the center of mass can be calculated by the definition of the center of mass and differentiating it w.r.t time.

$$\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt} = \frac{1}{M}\left(\sum m_n\vec{v}_n\right) \quad \therefore \vec{r}_{cm} = \frac{1}{M}\left(\sum m_n\vec{r}_n\right)$$

$$\vec{a}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{1}{M}\left(\sum m_n \vec{a}_n\right)$$

It is known as the acceleration of a single point, and we know that a particle possesses acceleration only when a force act on it. Here, the force can act for two reasons.

- One reason is that the particle is in external field. For example, the particle is inside the gravity and the gravity is pulling it down or up or in some direction.
- The second reason is that the other particles exert force and push or pull the particle in different directions.

The two types of forces that act on every particle are divided into external and internal forces. Therefore, the acceleration of each particle is due to the internal and external forces, but the internal force gets cancelled (because action and reactions are equal and opposite) and only external force remains.

$$M\vec{a}_{cm} = \sum \vec{F}_n = \sum\left(\vec{F}_{ext} + \vec{F}_{\text{int}}\right)$$

$$\sum \vec{F}_{ext} = M\vec{a}_{cm}$$

In conclusion, the center of mass moves in such a way to concentrate the whole mass on it.

A center of mass is a point but it’s not necessarily for a mass to be actually there. It's a common misconception that there's actually something. It is only appearing that all the masses concentrated there but when Newton's Law is applied, this imaginary point moves just like a mass that is concentrated at one point.

The center of mass for a combination of objects is the average center of mass location of objects. The center of mass can be outside the body; it does not have to be inside for all the structures as can be seen from figure 12.6.

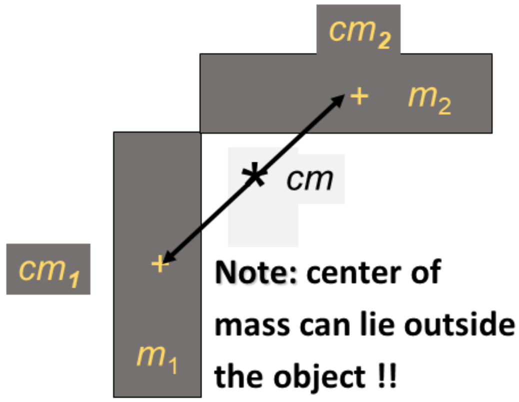


Figure 12.6. The center of mass represents the average location, and it can be located outside the physical boundaries of the objects.

Rotational Energy of Rigid Bodies:

Where there is movement there will be energy called kinetic energy. Motion is not only in one direction it can also be rotational motion. Consider a rigid body rotating about a fixed axis as shown in figure 12.7.

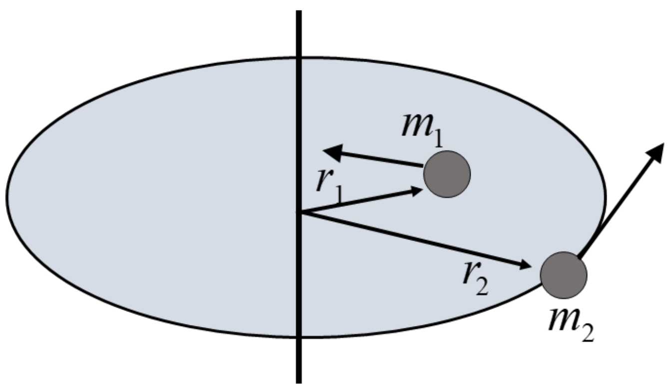


Figure 12.7. Rigid body rotation about a fixed axis.

Now let's know what a rigid body is. A body which has no elasticity, in which all its particles move together, such as wheel. If you calculate the total kinetic energy of a rigid body, then it is the sum of all the kinetic energies of each body.

Total kinetic energy is:

$$K = \frac{1}{2} m_1 v_1{}^2 + \frac{1}{2} m_2 v_2{}^2 + \frac{1}{2} m_3 v_3{}^2 + \cdots$$
$$\because v = r\omega$$

For a rigid body or an inelastic body, all the particles are moving together, which means that each particle has the same angular velocity ‘ω’ as the other particles, when we add all the K.E we get,

$$K = \frac{1}{2} m_1 r_1{}^2 \omega^2 + \frac{1}{2} m_2 r_2{}^2 \omega^2 + \frac{1}{2} m_3 r_3{}^2 \omega^2 + \cdots$$

$$K = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

Where $\sum m_i r_i^2$ is called as ‘moment of inertia I’.

Rotational inertia:

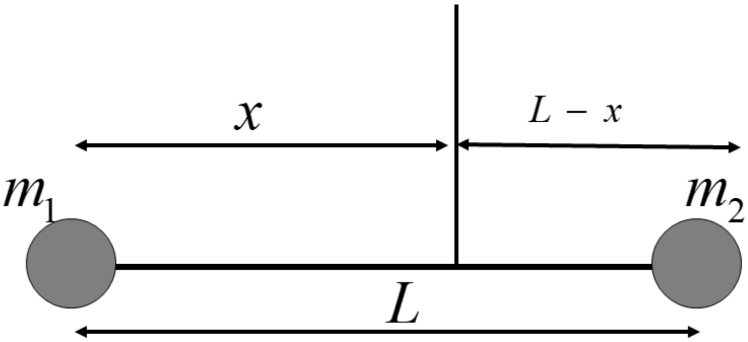
$$K = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$
$$K = \frac{1}{2} I \omega^2 \qquad \therefore I = \sum m_i r_i^2$$

This implies that we will take the mass of each body and use the square of the distance to calculate each body's moment of inertia. Then, we will add up all of the body's moments of inertia to get the total moment of inertia. When a body moves on straight line than its KE is:

$$K = \frac{1}{2} M v^2 \qquad \therefore \text{ v is the linear velocity.}$$

<u>Problem:</u>

Two particles m_1 and m_2 are connected by a light rigid rod of length L. neglect the mass of rod, find the rotational inertia I of this system about an axis perpendicular to the rod and at a distance x from m_1.



Solution:

$$I = I_1 + I_2$$

$$I = m_1 x^2 + m_2 \left(L - x\right)^2$$

For what x, is I the largest?

$$\frac{dI}{dx} = \frac{d}{dx}\left(m_1 x^2 + m_2 \left(L - x\right)^2\right)$$

$$\frac{dI}{dx} = m_1\left(2x\right) + m_2 2\left(L - x\right)\left(-1\right)$$

For a maxima, $\frac{dI}{dx} = 0$

$$0 = 2m_1 x - 2m_2\left(L - x\right)$$

$$0 = 2m_1 x - 2m_2 L + 2m_2 x$$

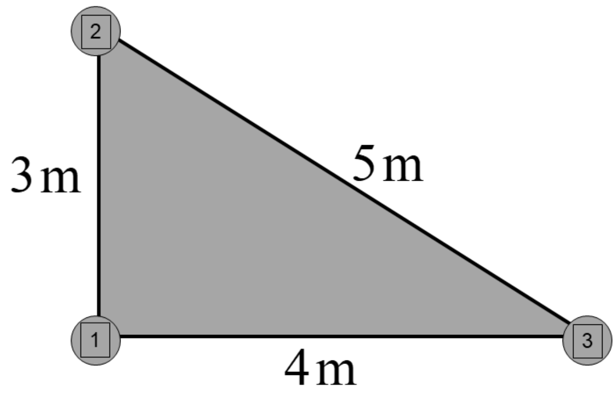
$$\left(2m_1 + 2m_2\right)x = 2m_2 L$$

$$x = \frac{2m_2 L}{2(m_1 + m_2)} = \frac{m_2 L}{(m_1 + m_2)}$$

Conclusion: At this distance, the system of particle possess the highest value of moment of inertia. If $m_1 = m_2$ then x = L/2 for maxima.

Problem:

Three particles of masses m_1 (2.3 kg), m_2 (3.2 kg) and m_3 (1.5 kg) are at the vertices of a triangle.



Part-I: Find the rotational inertia about axes perpendicular to the xy plane and passing through each of the particles.

Part-II: What is the moment of inertia about the center of mass?

Solution:

Moment of inertia about each axis of rotation =?

When passing through particle 1:

$$I = I_1 + I_2 + I_3 = mr_1^2 + mr_2^2 + mr_3^2$$

$$I = 2.3kg(0) + 3.2kg(3)^2 + 1.5kg(4)^2 = 52.8kg$$

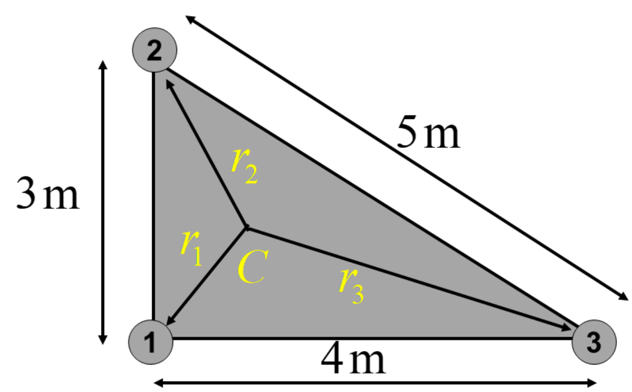
When passing through particle 2:

$$I = 2.3kg(3)^2 + 3.2kg(0) + 1.5kg(5)^2 = 58.2kg$$

When passing through particle 3:

$$I = 2.3kg(4)^2 + 3.2kg(5)^2 + 1.5kg(0) = 116.8kg$$

Part-II: What is the moment of inertia about the center of mass?



Lets find the center of mass first.

Since, masses are in xy-plane so,

$$\overrightarrow{r_{cm}} = x_{cm}\hat{i} + y_{cm}\,j$$

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} = \frac{m_1(0) + m_2(0) + m_3(4)}{2.3 + 3.2 + 1.5} = \frac{6}{7} = 0.86m$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} = \frac{m_1(0) + m_2(3) + m_3(0)}{2.3 + 3.2 + 1.5} = \frac{9.6}{7} = 1.37m$$

Using pythagoras theorem,

$$r_1^2 = x_{cm}^2 + y_{cm}^2 = (0.86)^2 + (1.37)^2 = 2.62\text{m}^2$$

$$r_2^2 = x_{cm}^2 + (y_2 - y_{cm})^2 = (0.86)^2 + (3 - 1.37)^2 = 3.40\,\text{m}^2$$

$$r_3^2 = (x_3 - x_{cm})^2 + y_{cm}^2 = (4 - 0.857)^2 + (1.37)^2 = 11.74\text{m}^2$$

$$I_{cm} = \sum m_i r_i^2 = m_1r_1^2 + m_2r_2^2 + m_3r_3^2$$

$$I_{cm} = 2.3 * 2.62 + 3.2 * 3.40 + 1.5 * 11.74$$

$$I_{cm} = 34.5\,\text{kg}\,\text{m}^2$$

This problem is for three masses, you may solve it for infinite masses. We use integration if we have infinite masses. If we divide a body into small parts, if mass of a small parts is dm. Take the distance r from the center and then take square of distance and then multiply with dm and then sum up over all the parts of the body that make up. This is called integration, and the total moment of inertia is the integration or the integral of the entire body of r square times dm.

For solid bodies:

$$I = \int r^2 dm$$

Hoop about cylinder axes:

$$I = \int r^2 dm \qquad \because \text{r = R= Fix}$$

$$I = R^2 \int dm$$

$$I = MR^2$$

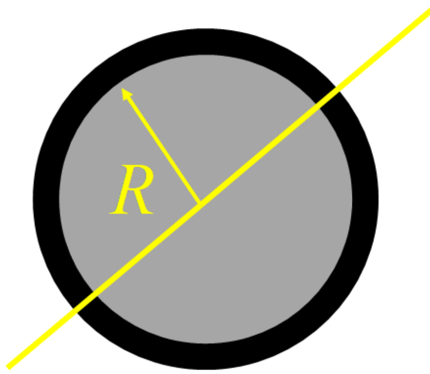


Figure 12.8. Hoop rotation about cylinder axis.

Solid plate about cylinder axes:

$$I = \int r^2 dm$$

$$\therefore \rho_O = \frac{M}{Area} = \frac{M}{\pi r^2}$$

$$if\, \rho_O = \frac{dm}{dA} = \frac{dm}{2\pi rdr}$$

$$dm = 2\pi rdr\rho_O$$

$$I = \int_0^R r^2 .2\pi rdr\rho_O$$

$$I = \int_0^R 2\pi r^3 dr\rho_O = 2\pi\rho_O \int_0^R r^3 dr$$

$$I = 2\pi\rho_O \left.\frac{r^4}{4}\right|_0^R$$

$$I = \frac{1}{2}\left(\pi R^4 \rho_O\right) = \frac{1}{2}\left(\pi R^4\right)\frac{M}{\pi R^2}$$

$$I = \frac{1}{2}MR^2$$

Solid sphere about diameter:

$$I = \frac{2}{5}MR^2$$

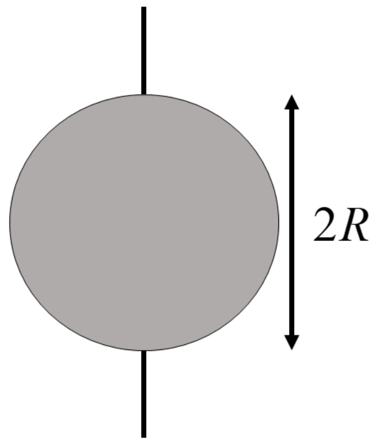


Figure 12.9. Solid sphere rotation about diameter.

For a hollow sphere, the mass is concentrated towards the outside, and you would get the result that a hollow sphere of mass m will have a greater moment of inertia than a sphere of mass m.

The moment of inertia measures how much energy is generated inside an object when you spin it. For example as shown in figure 12.10 (a) and (b).

(a) Solid cylinder or disk about cylinder axis

$$I = \frac{1}{2}MR^2$$

(b) Solid cylinder or disk about central diameter:

$$I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$$

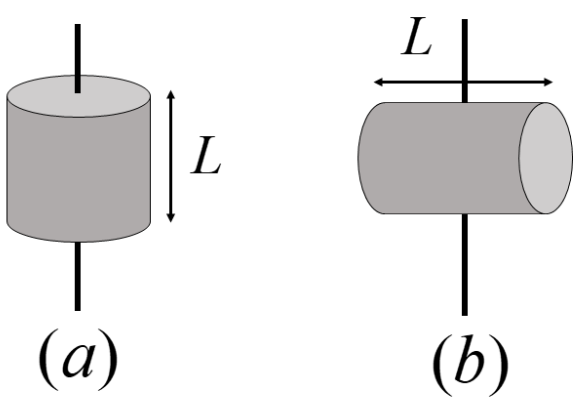


Figure 12.10. (a) solid cylinder or disk about cylinder axis, (b) solid cylinder or disk about central diameter .

Rectangular plate about central axis:

$$I = \frac{1}{12} M\left(a^2 + b^2\right)$$

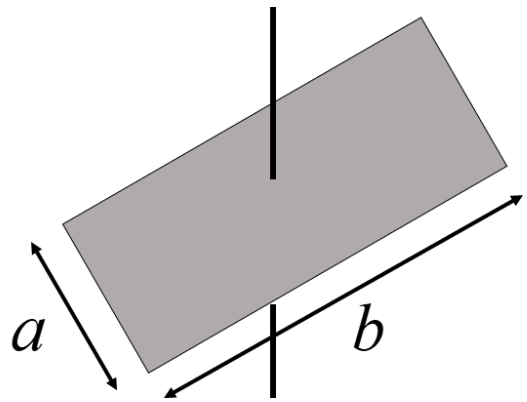


Figure 12.11. Rectangular plate about central axis.

So in conclusion, if an object rotates, its kinetic energy becomes equal to the ½ I ω^2, but the question arises here is that, why the object rotates? Of course, force helps to rotate the object, but it depends on where the force is applied. For example, we have wrenches of various sizes. Because of torque, when we use a small-length wrench, it is difficult to tighten the nut; however, when we use a large-length wrench, the nut is easily tightened. A wrench with a long length has higher torque than one with a little length.

Now we define **torque** mathematically,

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Where ‘r’ is the distance at which force acts. If we take its magnitude, we have

$$\tau = rF \sin\theta$$

And θ is the angle between r and F as shown in figure 12.12.

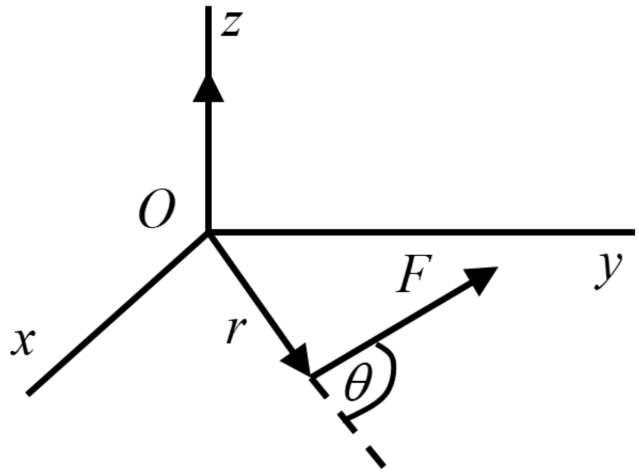


Figure 12.12. Illustration of torque, where 'r' is the distance at which force (F) acts, and θ is the angle between r and F.

Work can be done wherever a force occurs and when it acts and moves a distance S, the force into a distance is equal to work as shown in figure 12.13. So F.ds is the amount of work done.

$$dW = \vec{F}.\vec{d}s$$

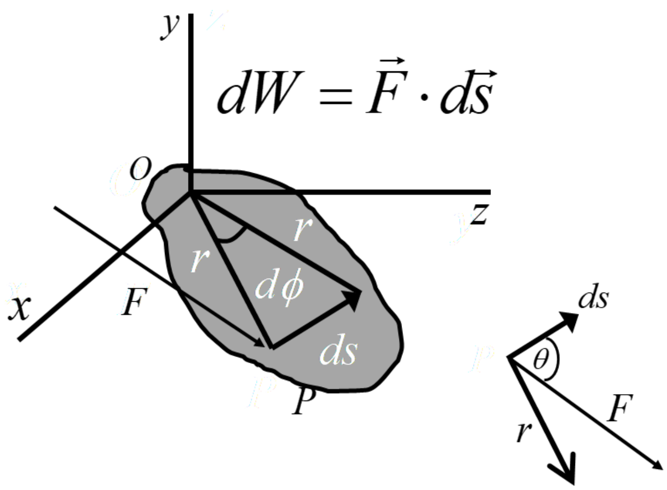


Figure 12.13. Work (W) is the product of force (F) and displacement (ds), where the force acting over a distance results in the performance of work.

So suppose there are several bodies and forces acting on them. Now the work done is,

$$
\begin{aligned}
dW &= \vec{F}.d\vec{s} = F\cos\theta ds \\
&= \left(F\cos\theta\right)\left(rd\phi\right) \\
&= \tau d\phi
\end{aligned}
$$

For all particles,

$$
\begin{aligned}
dW_{net} &= \left(F_1\cos\theta_1\right)r_1 d\phi + \left(F_2\cos\theta_2\right)r_2 d\phi + \cdots + \left(F_n\cos\theta_n\right)r_n d\phi \\
&= \left(\tau_1 + \tau_2 + \cdots + \tau_n\right)d\phi
\end{aligned}
$$

See, there's only one angle here, and that's because all the particles move together because it's a rigid body.

$$W_{net} = \left(\sum \tau_{ext}\right)d\phi = \left(\sum \tau_{ext}\right)\omega dt$$

When we differentiate the K.E with respect to ω,

$$
\begin{aligned}
&K = \left(\frac{1}{2}I\omega^2\right) \\
&\frac{dK}{d\omega} = \frac{d}{d\omega}\left(\frac{1}{2}I\omega^2\right) = \frac{1}{2}I(2\omega) = I\omega \\
&dK = I\omega d\omega \qquad \therefore \alpha = \frac{d\omega}{dt} \Rightarrow d\omega = \alpha dt \\
&dK = \left(I\alpha\right)\omega dt
\end{aligned}
$$

Remember that α is the angular acceleration.

$$dW_{net} = dK$$

Work has been done by you in rotating it so that work is converted into kinetic energy. Then the result is the total external torque and is equal to moment of inertia multiplied by angular acceleration,

$$\sum \tau_{ext} = I\alpha$$

This equation is analogous to the Newton’s law F = ma.

Before continuing we should know where translational and rotational motions are similar.

‘x’ is displacement at the one side and Ø is angular displacement on the other side. As x increases with the time and we take the derivative dx/dt, likewise Ø increases with time and we take the derivative from there we get the angular speed.

Translational motion	**Angular motion**
x, M	ϕ, I
$v = \frac{dx}{dt}$	$\omega = \frac{d\phi}{dt}$
$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$

In term of dynamics, the moment of inertia has exactly the same role for rotational motion as that of mass in translational motion.

Translational	**Rotational motion**
$F = Ma$	$\tau = I\alpha$
$W = \int F dx$	$W = \int \tau d\phi$
$K = \frac{1}{2} M v^2$	$K = \frac{1}{2} I \omega^2$

Combined Rotational and Translational Motion:

In order to understand this type of motion, let’s take the example of a car. Take the wheel of a car, put a mark on the wheel, and put a mark on its rim. The car goes in a straight line, but at the same time, its wheel rotates. So, this is an example of combined rotational and translational motion.

Now we study it in detail, there is a body with two vectors on it. One vector is of center of mass and the second is at point p, point P is any random point on a body and a vector which goes from the center of mass to P is called r_i' as shown in figure 12.14.

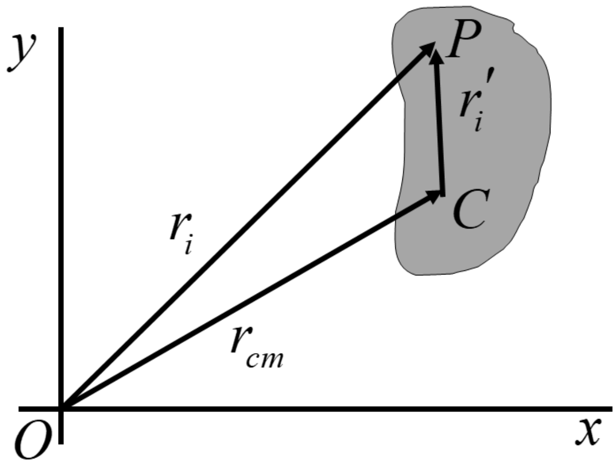


Figure 12.14. Body with center of mass vector and position vector at point P. The vector from the center of mass to a random point p on the body is defined as the position vector $\boldsymbol{r_i'}$.

If we take out the total kinetic energy of it, then it will have two parts, one part is the kinetic energy of the center of the mass and the other part is due to its rotation.

$$K = \frac{1}{2} M v_{cm}{}^2 + \frac{1}{2} I_{cm} \omega^2$$

Remember, we didn't take a regular shape here. The body is rolling along its path in addition to traveling in a straight line. We now aim to find out if it is indeed the case that this body possesses two different kinds of kinetic energy. Let's prove it.

First of all, we will extract the kinetic energies of all its different particles, then we divide the velocity of every particle into two parts. One is the part of center of mass and other is due to rotation.

$$K = \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i \vec{v}_i \,.\vec{v}_i$$
$$\sum \frac{1}{2} m_i \left(\vec{v}_{cm} + \vec{v}_i'\right).\left(\vec{v}_{cm} + \vec{v}'\right)$$
$$\sum \frac{1}{2} m_i \left(v^2{}_{cm} + 2\vec{v}_{cm}.\vec{v}_i' + v_i'^2\right)$$
$$\sum m_i \vec{v}_{cm}.\vec{v}_i' = \vec{v}_{cm}.\sum m_i \vec{v}_i'$$
$$\sum \vec{p}_i' = \sum m_i \vec{v}_i' = M\vec{v}_{cm}'$$
$$\vec{v}_{cm}' = 0$$

$\vec{v}_{cm}' = 0$ in the center of mass frame.

$$K = \sum \frac{1}{2} m_i v^2{}_{cm} + \sum \frac{1}{2} m_i v'^2$$
$$= \frac{1}{2} M v^2{}_{cm} + \sum \frac{1}{2} m_i r_i'^2 \omega^2$$
$$K = \frac{1}{2} M v^2{}_{cm} + \frac{1}{2} I_{cm} \omega^2$$

Here, we emphasized that it is a rigid body that moves along. However, we cannot apply this to a non-rigid body in such a way that, for example, rotating a bucket of water will cause the water inside to rotate at one speed while the water attached to the bucket rotates at a different speed, indicating that the bucket is not a rigid body.

Rolling without slipping:

Take a rigid body with a center of the mass speed (v_{cm}) moving toward the right side and this is just translational motion. And on the other hand it is just rotational motion that is rotating along the fix center and its speed is Rω on their edges as shown in figure 12.15.

Translational motion **Rotational motion**

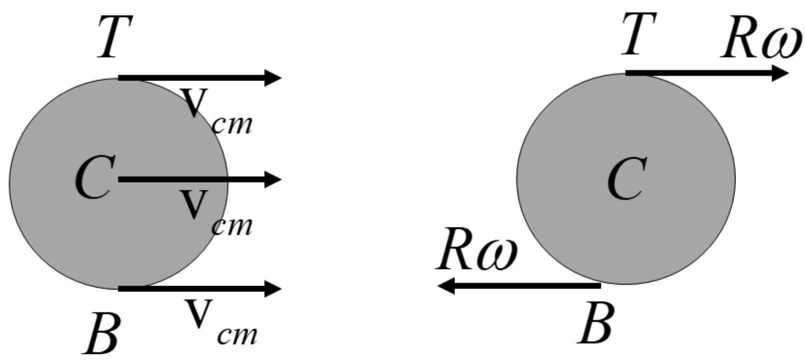


Figure 12.15. The body exhibits translational motion with center of mass speed (v_{cm}) to the right and rotational motion about a fixed center with speed Rω.

It is possible that it is rolling and rotating together, so it is a combination of translational plus rotational motion. Here point B is attached to the ground so it is at rest. It is moving at the speed of $V_{cm,}$ and the top point is moving at two times the velocity of center of mass as shown in figure 12.16.

Translational + Rotational motion

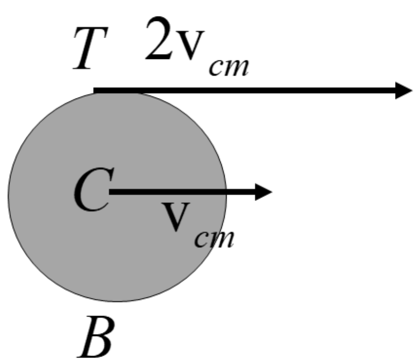


Figure 12.16. Rigid body with combined translational (vcm) and rotational (Rω) motion about a fixed axis.

If we find the total energy then we see the result of rolling without slipping is,

$$v_{cm} = R\omega$$

$$K = \frac{1}{2}Mv^2{}_{cm} + \frac{1}{2}I_{cm}\left(\frac{v^2{}_{cm}}{R^2}\right)$$

$$K = \frac{1}{2}MR^2\omega^2 + \frac{1}{2}I_{cm}\omega^2$$

We didn’t say that this is a sphere or a hoop or the other shape we only discuss it as a rigid body.

Example:

In the absence of friction, energy is conserved, meaning that the same amount of energy at the beginning remains at the end and since we know that energy is divided into two parts; kinetics energy of center of mass plus rotational motion so from this we can solve many types of questions.

Now we take the example of a body rolling down a slope as shown in figure 12.17. For example a hoop which has the moment of inertia ½ mR^2. Now we want to know how fast it will be when it leaves the slope.

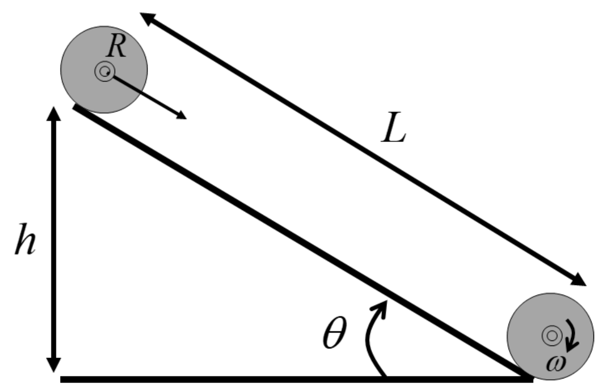


Figure 12.17. Rolling Motion of a Hoop on a Slope.

Solution:

Here we use the conversation of energy, in the start it has potential energy but later its loss the potential energy and only kinetic energy remains. When body leaves the slop its K.E is, At starting point, object possess gravitational P.E = Mgh

As it moves down, it possess rotational and translational K.E, which is:

$$K = \frac{1}{2}Mv^2{}_{cm} + \frac{1}{2}I_{cm}\omega^2$$

$$Mgh = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v_{cm}}{R}\right)^2 \qquad \therefore \mathrm{v}_{cm} = \mathrm{r}\omega$$

$$gh = \frac{v_{cm}^2}{2} + \frac{v_{cm}^2}{4} = \frac{3v_{cm}^2}{4}$$

$$v_{cm} = \sqrt{\frac{4}{3}gh}$$

Summary: Kinetic energy consists of two parts, one related to the center of mass and one related to its rotation. Center of mass and moment of inertia are two properties associated with a body and which define the body. What we've discussed is very important for mechanical engineers, people who design machines that have rotating objects inside, whether they're aircraft engineers or automobile engineers or others. We will go far with these concepts, which are essential in every other subject that we have learnt.