Physics

Oscillations

Simple pendulum

$$F = -mg\sin\theta$$

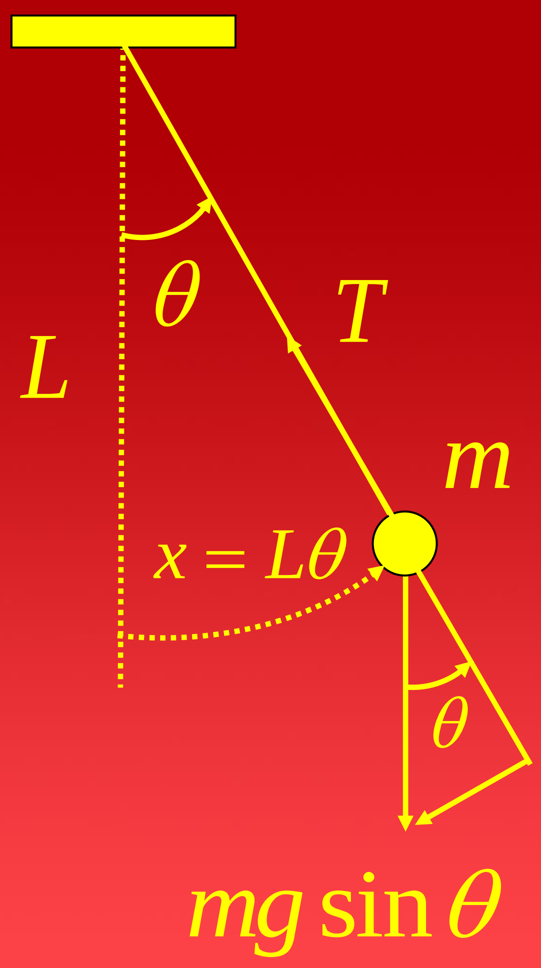
For small value of θ

$$\sin\theta \approx \theta$$

$$x = L\theta$$

$$F = -mg\theta = -mg\frac{x}{L}$$

$$= -\left(\frac{mg}{L}\right)x$$

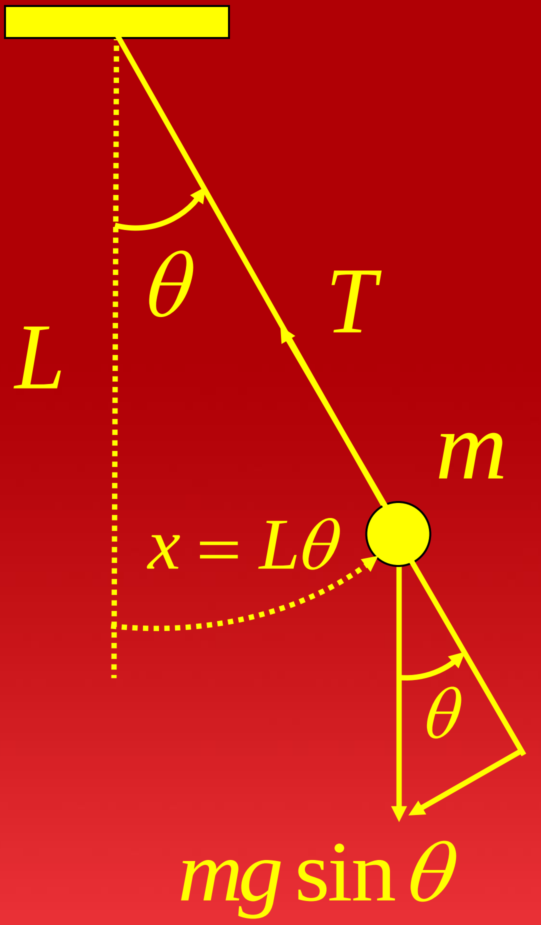


$$F = m\frac{d^2x}{dt^2} = -\left(\frac{mg}{L}\right)x$$

$$\frac{d^2x}{dt^2} = -\left(\frac{g}{L}\right)x$$

Solution: $x = x_m \cos \omega t$

$$\omega = \sqrt{\frac{g}{L}}$$



The physical pendulum

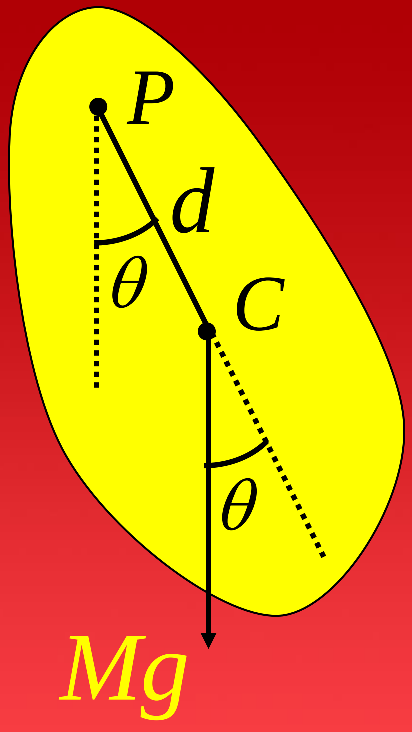
$$\tau = -Mgd\sin\theta$$

For small θ, $\sin\theta \approx \theta$

$$\therefore \tau = -Mgd\theta$$

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2} = -Mgd\theta$$

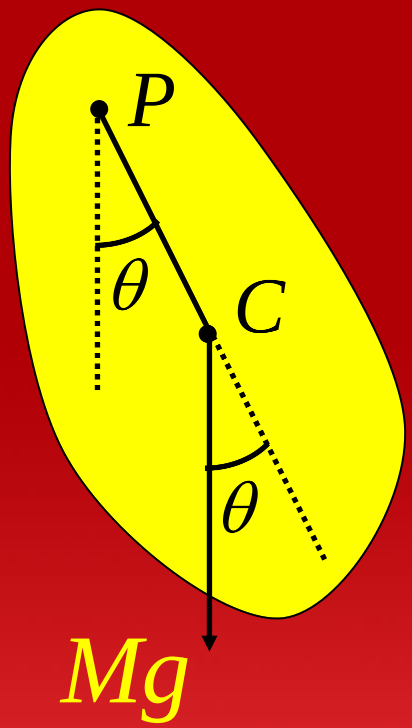
$$\frac{d^2\theta}{dt^2} = -\left(\frac{Mgd}{I}\right)\theta$$



Its time period will be

$$\omega = \sqrt{\frac{Mgd}{I}}$$

$$\text{or } I = \frac{Mgd}{\omega^2}$$



The quantities on the right are all measurable.
Using this formula we can determine
the rotational inertia of any body
about an axis of rotation other than
through the center of mass.

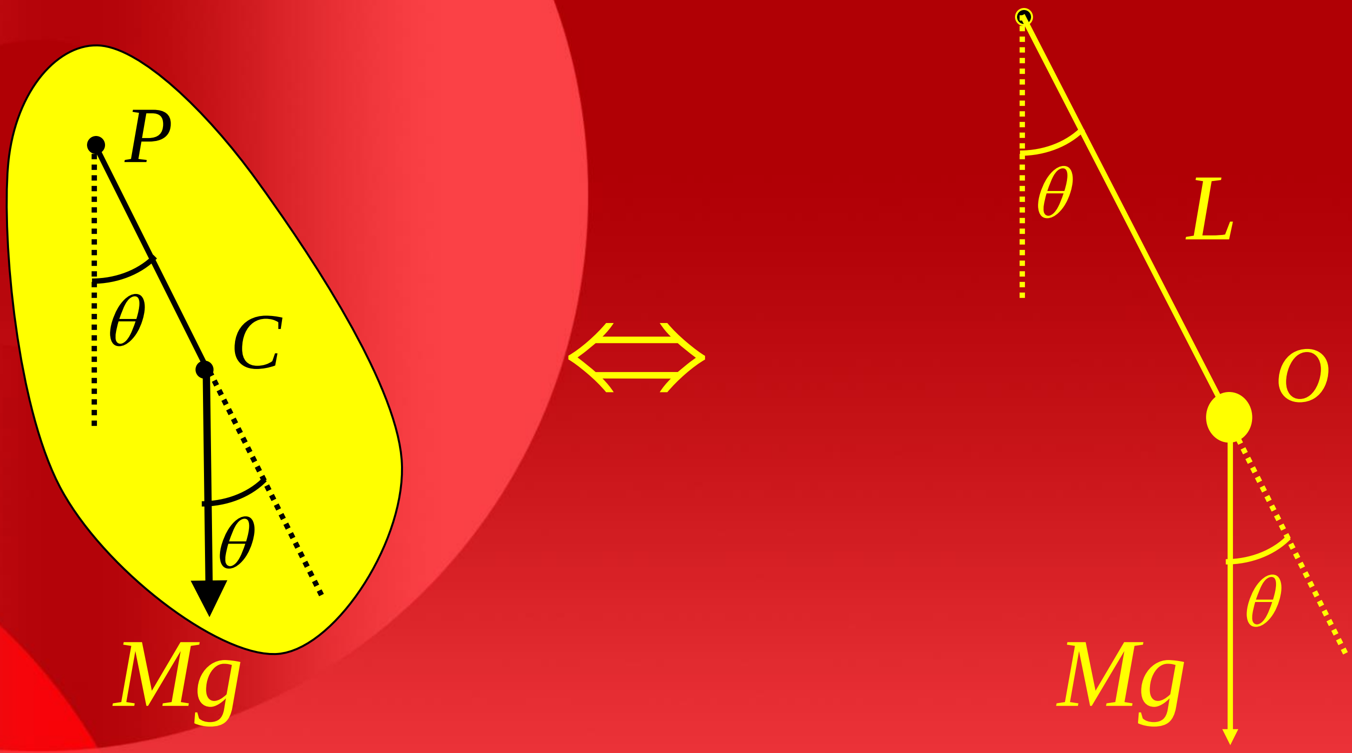
The physical pendulum includes the simple pendulum as a special case

$$d = L \text{ and } I = ML^2$$

$$T = 2\pi\sqrt{\frac{I}{Mgd}} = 2\pi\sqrt{\frac{ML^2}{MgL}}$$

$$= 2\pi\sqrt{\frac{L}{g}}$$

Center of gyration



$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{I}{Mgd}} \Rightarrow L = \frac{I}{Md}$$

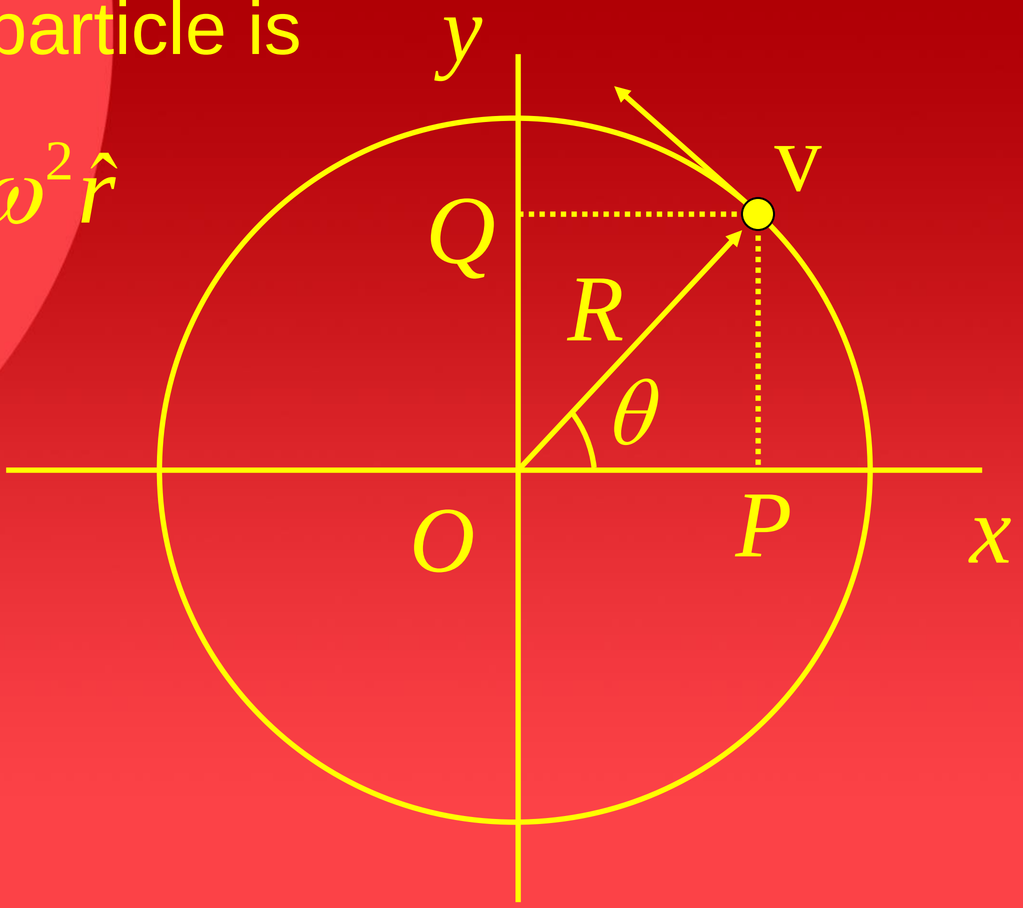
Simple harmonic motion and uniform circular motion

Acceleration of the particle is

$$\vec{a} = -\frac{\mathrm{v}^2}{R}\hat{r} = -R\omega^2\hat{r}$$

acceleration along x direction is:

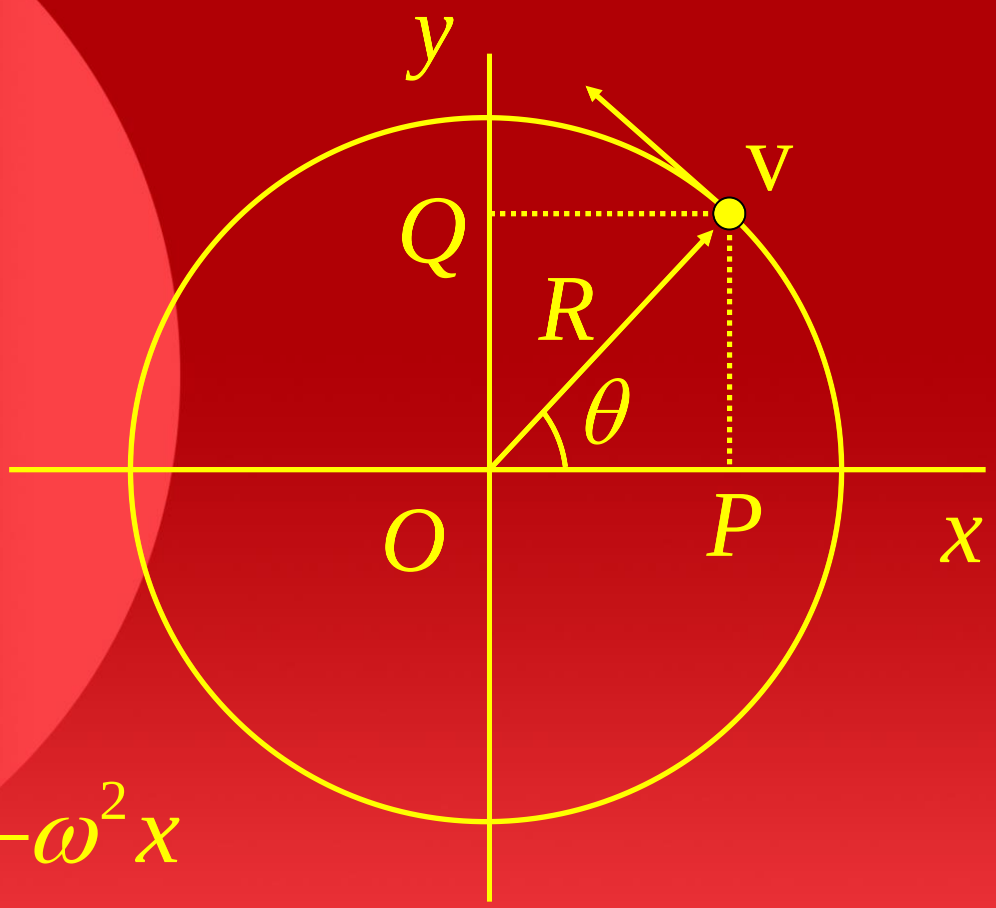
$$a_x = -R\omega^2 \cos\theta$$



but

$$x = R\cos\theta$$

$$\therefore a_x = \frac{d^2x}{dt^2} = -\omega^2 x$$



Thus point P executes simple harmonic motion

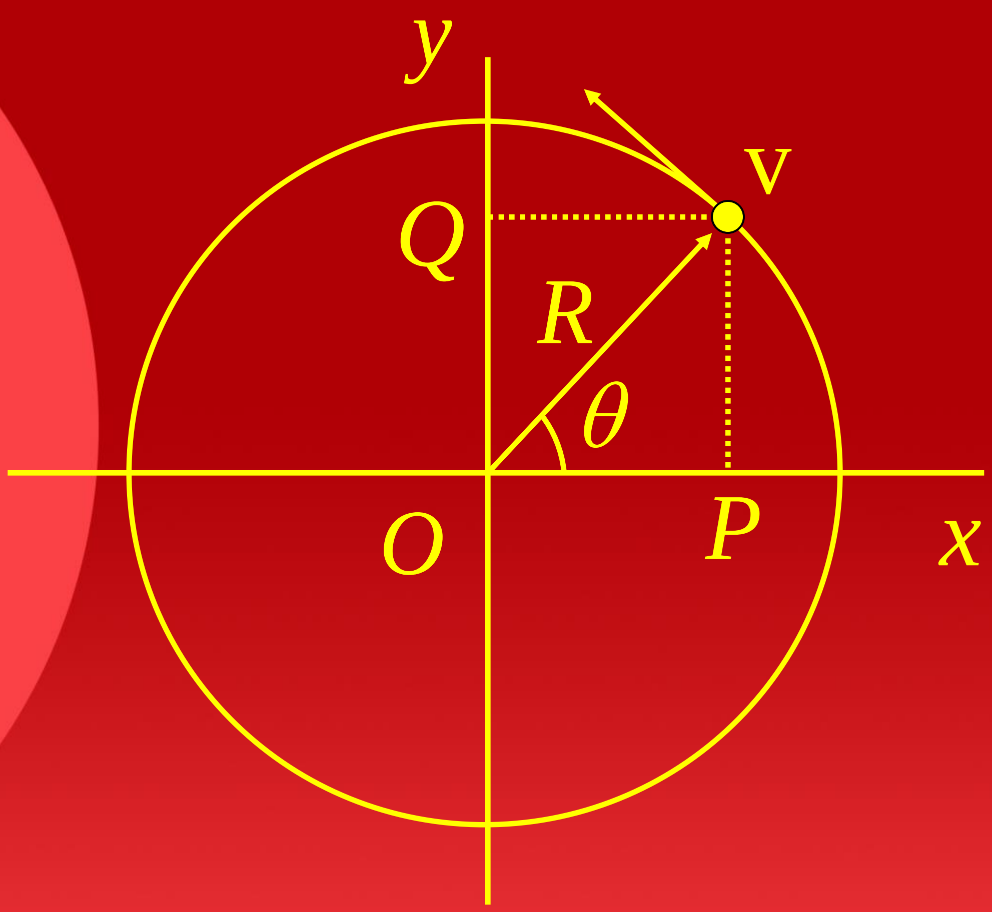
Acceleration of the point Q is:

$$a_y = -R\omega^2 \sin\theta$$

but

$$y = R\sin\theta$$

$$\therefore a_y = \frac{d^2 y}{dt^2} = -\omega^2 y$$



Q also executes simple harmonic motion

Composition of two simple harmonic motion of the same period along the same line

$$x_1 = A_1 \sin \omega t \text{ and } x_2 = A_2 \sin\left(\omega t + \phi\right)$$

The resultant displacement

$$x = x_1 + x_2$$

$$= A_1 \sin \omega t + A_2 \sin\left(\omega t + \phi\right)$$

$$= A_1 \sin \omega t + A_2 \sin \omega t \cos \phi + A_2 \sin \phi \cos \omega t$$

$$= \sin \omega t \left(A_1 + A_2 \cos \phi\right) + \cos \omega t \left(A_2 \sin \phi\right)$$

Let $A_1 + A_2 \cos\phi = R\cos\theta$

and $A_2 \sin\phi = R\sin\theta$

We get $x = R\sin(\omega t + \theta)$

Thus the resultant motion is also simple harmonic motion along the same line and has the same time period. Its amplitude is

$$R = \sqrt{A_1^2 + A_2^2 + A_1 A_2 \cos\phi}$$

$$\tan\theta = \frac{A_2 \sin\phi}{A_1 + A_2 \cos\phi}$$

Special cases:

If $\phi = 0$ then

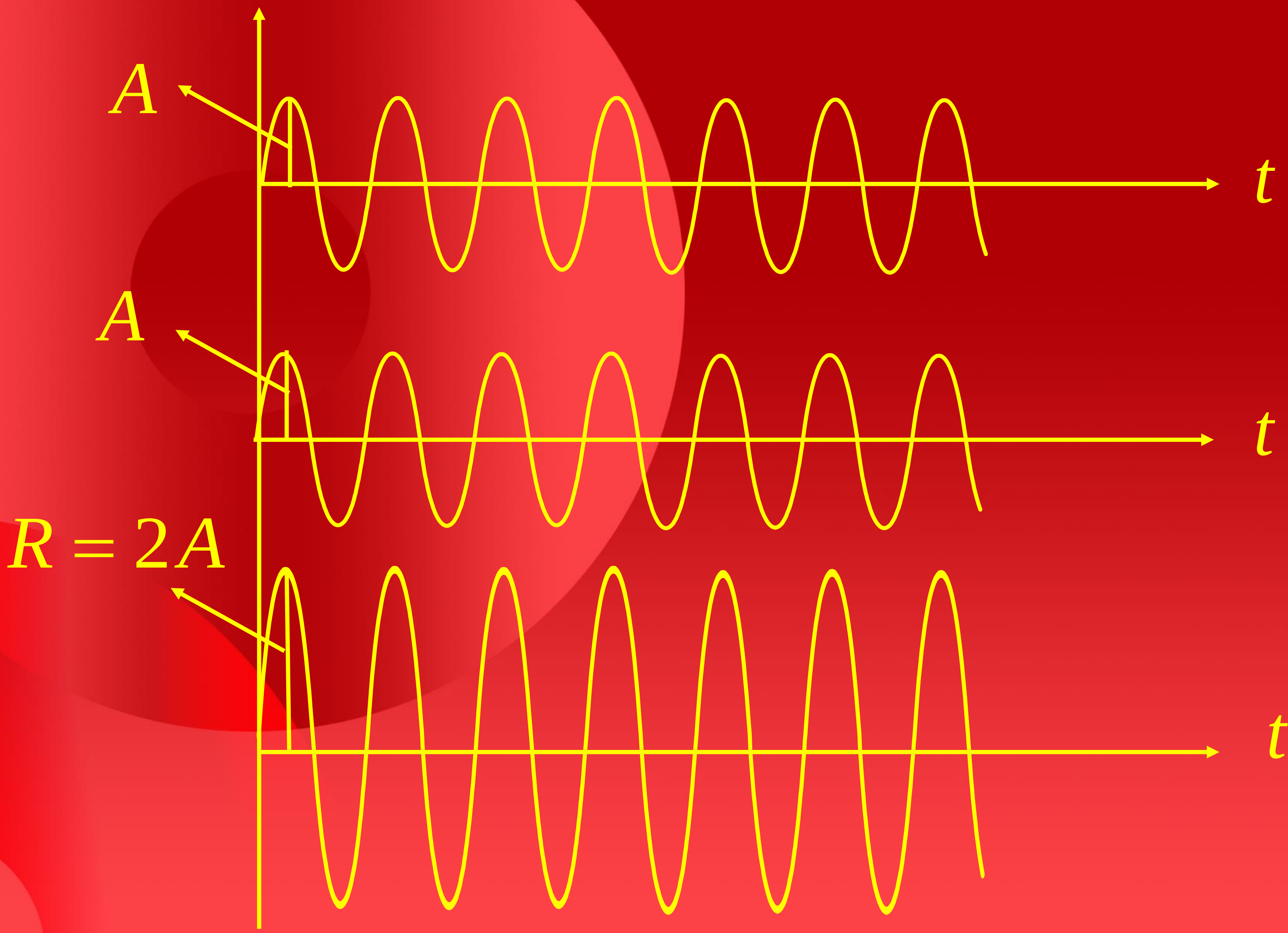
$$R = \sqrt{A_1^2 + A_2^2 + A_1 A_2} = \sqrt{\left(A_1 + A_2\right)^2} = A_1 + A_2$$

and

$$\tan\theta = 0 \Rightarrow \theta = 0$$

We get $x = \left(A_1 + A_2\right)\sin\omega t$

This is constructive interference



A
t
A
t
$R = 2A$
t

If $\phi = \pi$ then

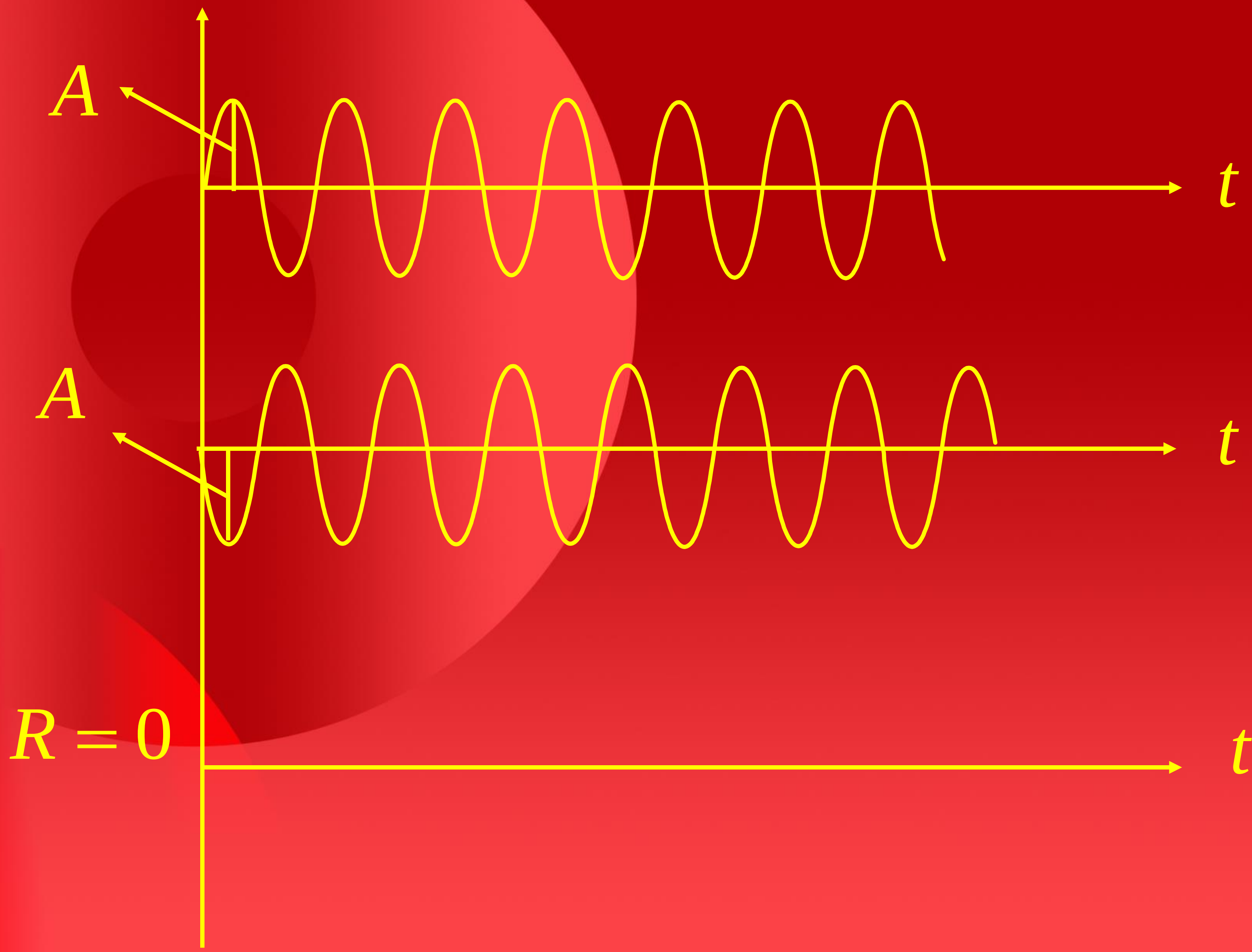
$$R = \sqrt{A_1^2 + A_2^2 - A_1 A_2} = \sqrt{\left(A_1 - A_2\right)^2} = A_1 - A_2$$

and

$$\tan\theta = 0 \Rightarrow \theta = 0$$

We get $x = \left(A_1 - A_2\right)\sin\omega t$

This is destructive interference


A
t
A
t
R = 0
t

Composition of two simple harmonic motions
Of the same period at right angles to each other

$$x = A\sin\omega t \text{ and } y = B\sin\left(\omega t + \phi\right)$$

$$\sin\omega t = \frac{x}{A} = \text{and } \cos\omega t = \sqrt{1 - x^2 / A^2}$$

$$\frac{y}{B} = \sin\omega t\cos\phi + \sin\phi\cos\omega t$$

$$= \frac{x}{A}\cos\phi + \sin\phi\sqrt{1 - x^2 / A^2}$$

squaring and rearranging

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}-2\frac{xy}{AB}\cos\phi=\sin^2\phi$$

This is the equation of an ellipse.

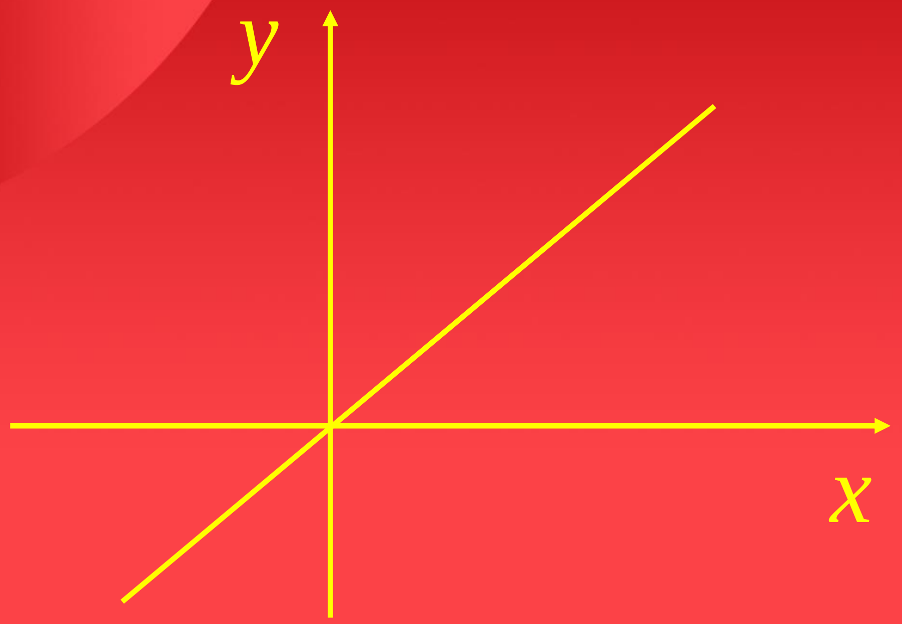
Special cases:

If $\phi=0$ then

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}-2\frac{xy}{AB}=0\Rightarrow\left(\frac{x}{A}-\frac{y}{B}\right)^2=0$$

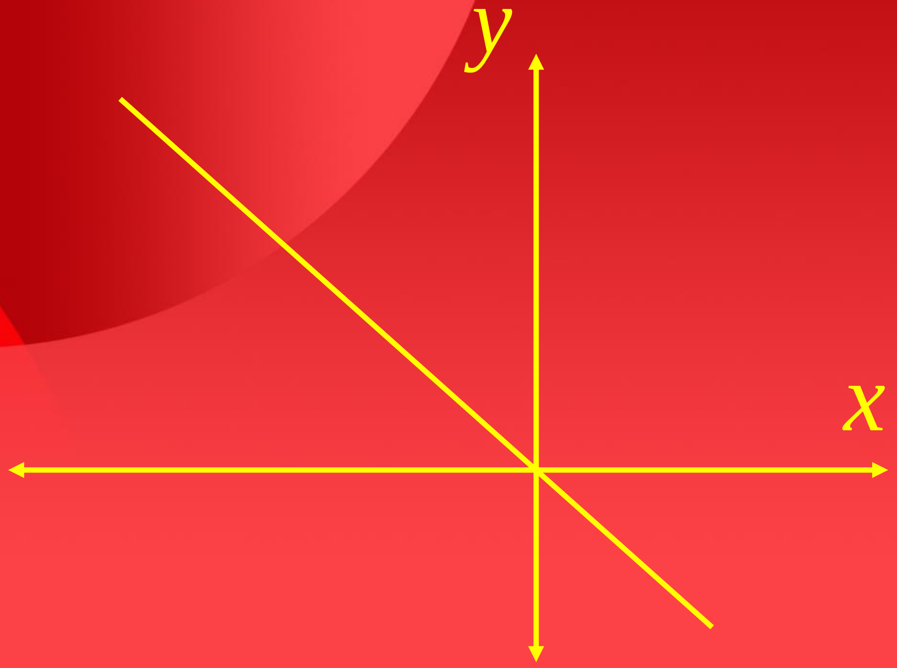
$$\frac{x}{A} - \frac{y}{B} = 0 \text{ or } y = \left(\frac{B}{A}\right)x$$

This is the equation of a straight line. Thus the resultant motion is a S.H.M. along a straight line passing through the origin.



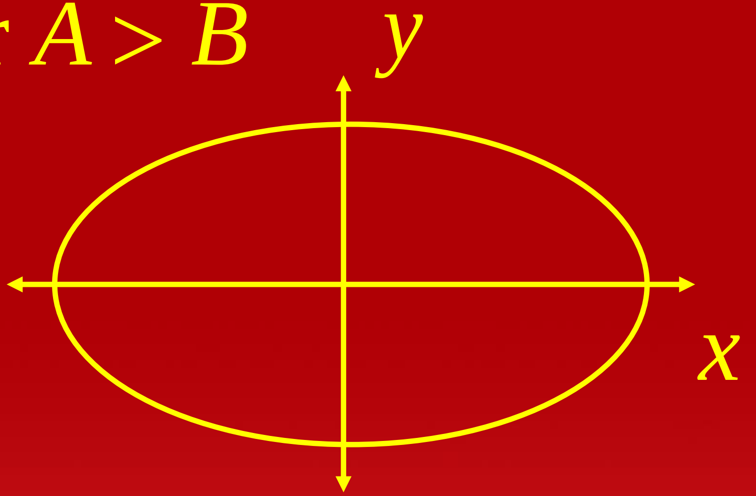
If $\phi = \pi$ we get

$$\frac{x}{A} + \frac{y}{B} = 0 \text{ or } y = -\left(\frac{B}{A}\right)x$$



If $\phi = \dfrac{\pi}{2}$ then

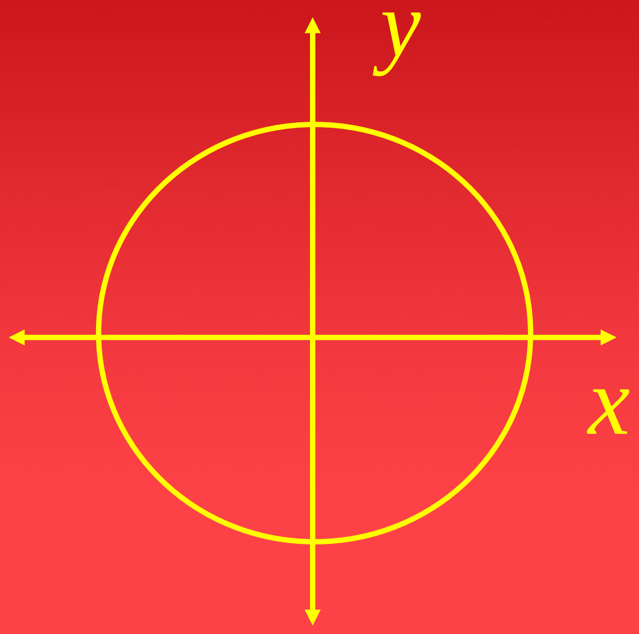
For $A > B$



$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

which is an ellipse

If $A = B$ then $x^2 + y^2 = A^2$



which is an circle

Lissajous Figures

$$x = A\sin\omega_x t \text{ and } y = B\sin\left(\omega_y t + \phi\right)$$

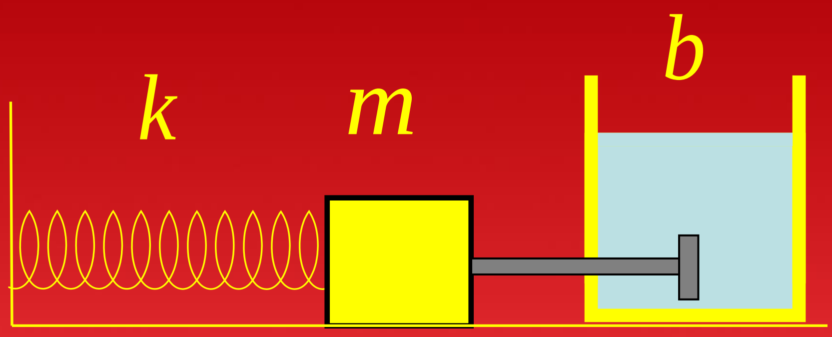
If two oscillations of different frequencies at right angles are combined, the resulting motion is more complicated. It is not even periodic unless the two frequencies are in the ratio of integers. This resulting curve are called Lissajous figures.

$$\frac{\omega_x}{\omega_y} = \text{integers} \Rightarrow \text{periodic motion}$$

Damped harmonic motion

$$\text{Damping force} = -b\frac{dx}{dt} \qquad \text{where } b > 0$$

From Newton's second law



$$\sum \vec{F} = m\vec{a}$$

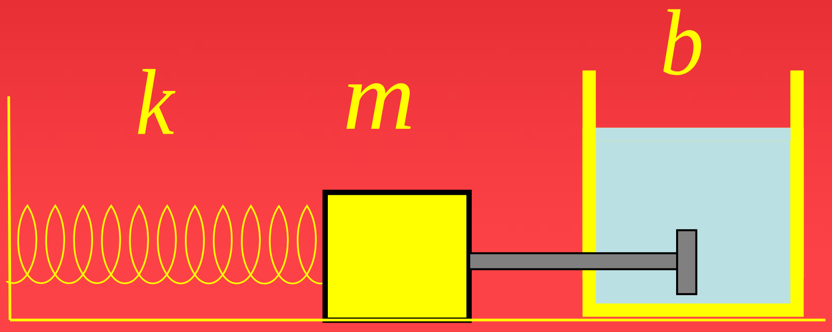
$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

$$\Rightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Its solution for $\dfrac{k}{m} \geq \left(\dfrac{b}{2m}\right)^2$ is

$$x = x_m e^{-bt/2m} \cos(\omega' t + \phi)$$

where $$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$



Forced oscillation and resonance

$$m\frac{d^2x}{dt^2}+kx=F_0\cos\omega t$$

Solution: $$x=\frac{F_0}{m\left(\omega_0^2-\omega^2\right)}\cos\omega t$$

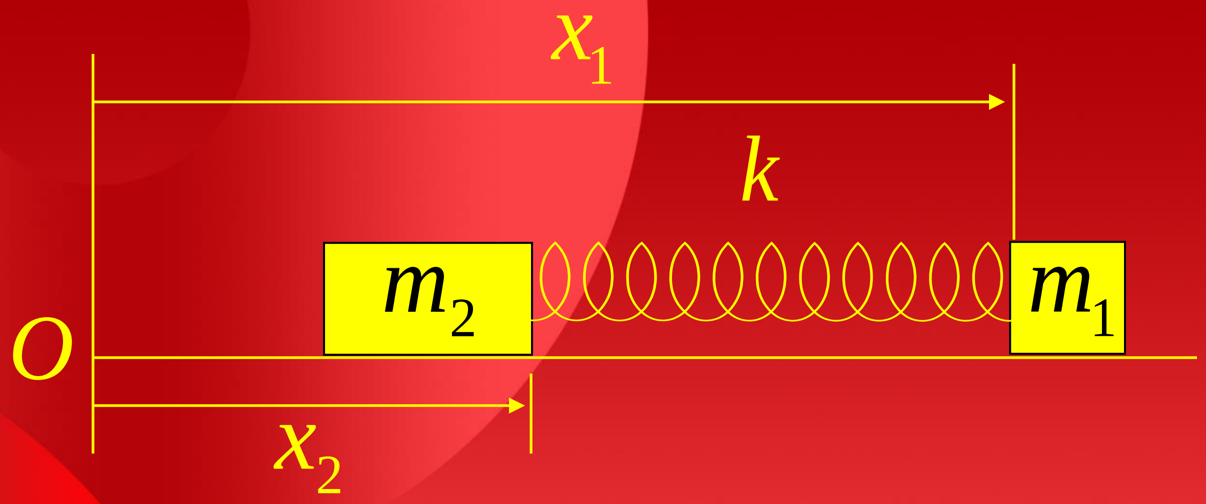
Check:

$$\mathrm{LHS}=\frac{mF(-\omega^2)+kF}{m\left(\omega_0^2-\omega^2\right)}\cos\omega t=\mathrm{RHS}$$

Here ω_0 is the natural frequency of the system and is given by $\omega_0 = \sqrt{\frac{k}{m}}$

There is a characteristic value of the driving frequency ω at which the amplitude of oscillation is a maximum. This condition is called resonance. For negligible damping resonance occurs at $\omega = \omega_0$

Two body oscillations



$$m_1 \frac{d^2 x_1}{dt^2} = -kx \qquad m_2 \frac{d^2 x_2}{dt^2} = +kx$$

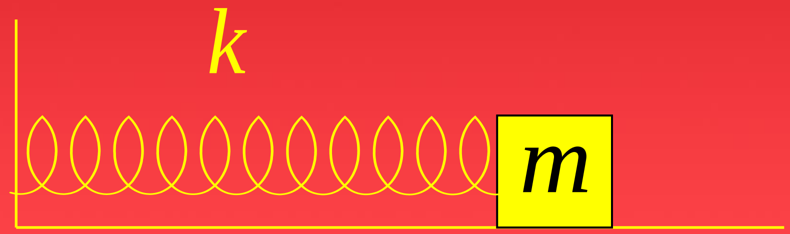
$$m_1 m_2 \frac{d^2 x_1}{dt^2} - m_1 m_2 \frac{d^2 x_2}{dt^2} = -m_2 kx - m_1 kx$$

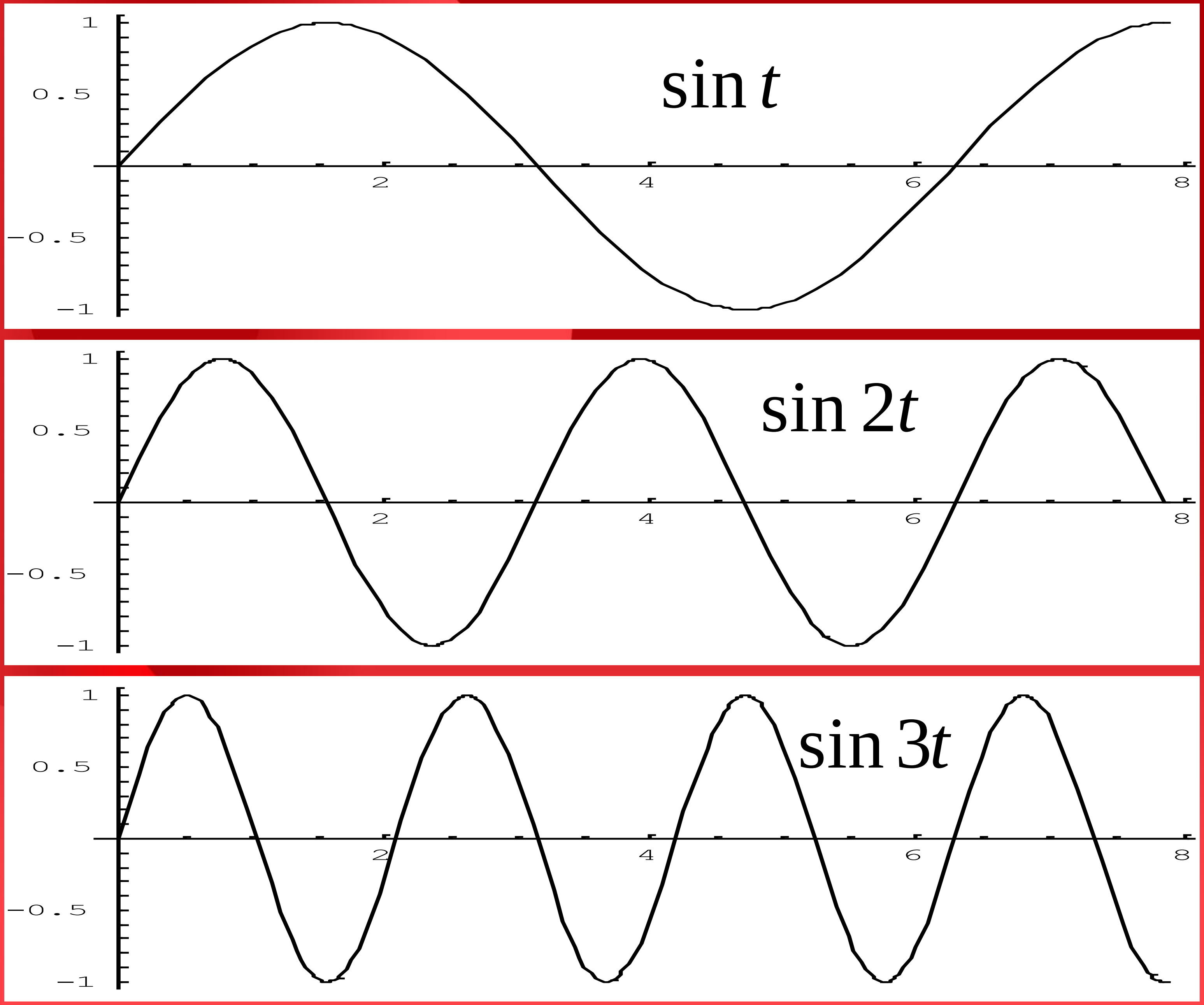
$$\left(\frac{m_1 m_2}{m_1 + m_2} \right) \frac{d^2}{dt^2} \left(x_1 - x_2 \right) = -kx$$

$$\text{let } m = \left(\frac{m_1 m_2}{m_1 + m_2} \right)$$

$$\frac{d}{dt}\left(x_1 - x_2\right) = \frac{d}{dt}\left(x + L\right) = \frac{dx}{dt}$$

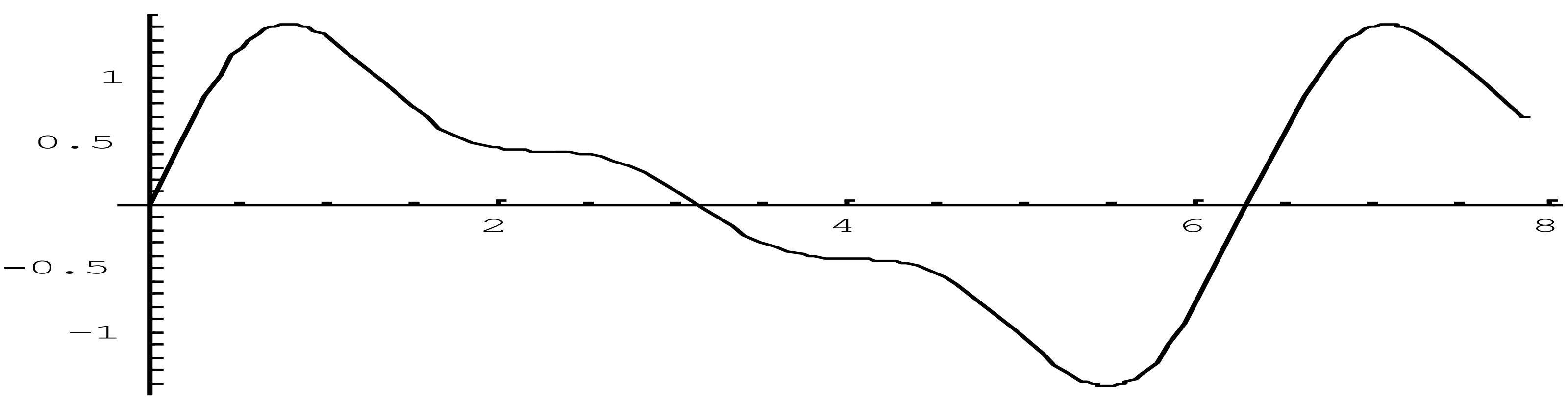
$$\frac{d^2 x}{dt^2} + \frac{k}{m}x = 0$$



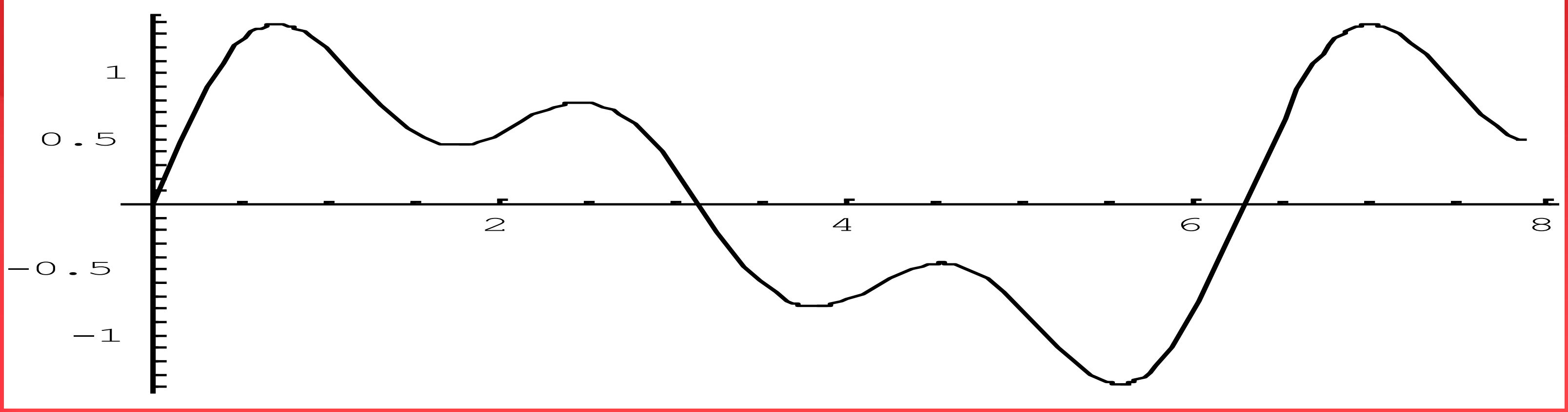


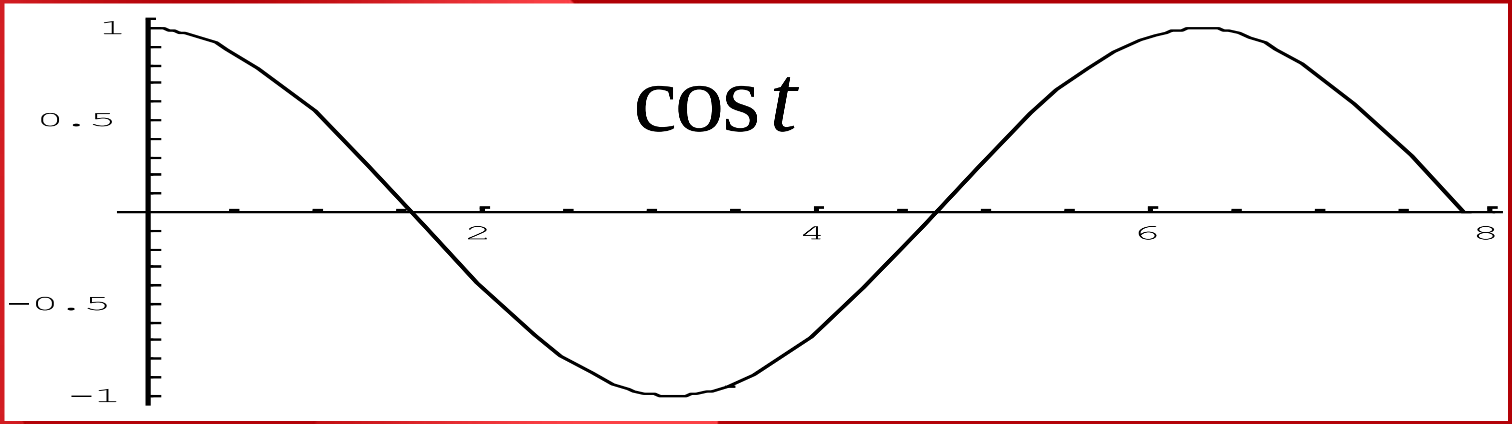
1
0.5
−0.5
−1
2
4
6
8
sin t
sin $2t$
sin $3t$

$$\sin t + 0.5\sin 2t + 0.3\sin 3t$$

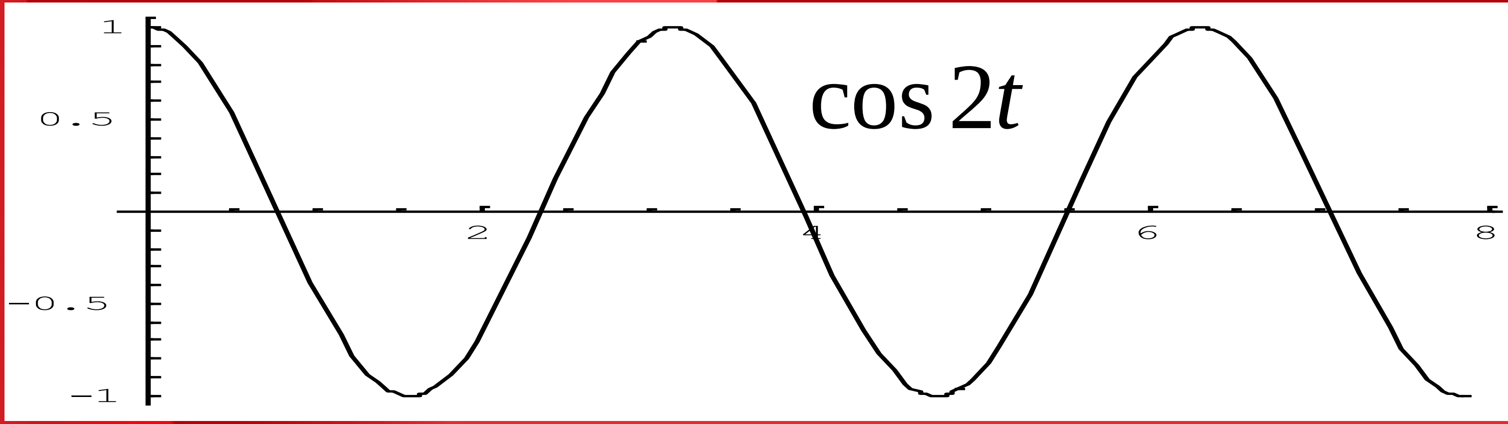


$$\sin t + 0.3\sin 2t + 0.5\sin 3t$$

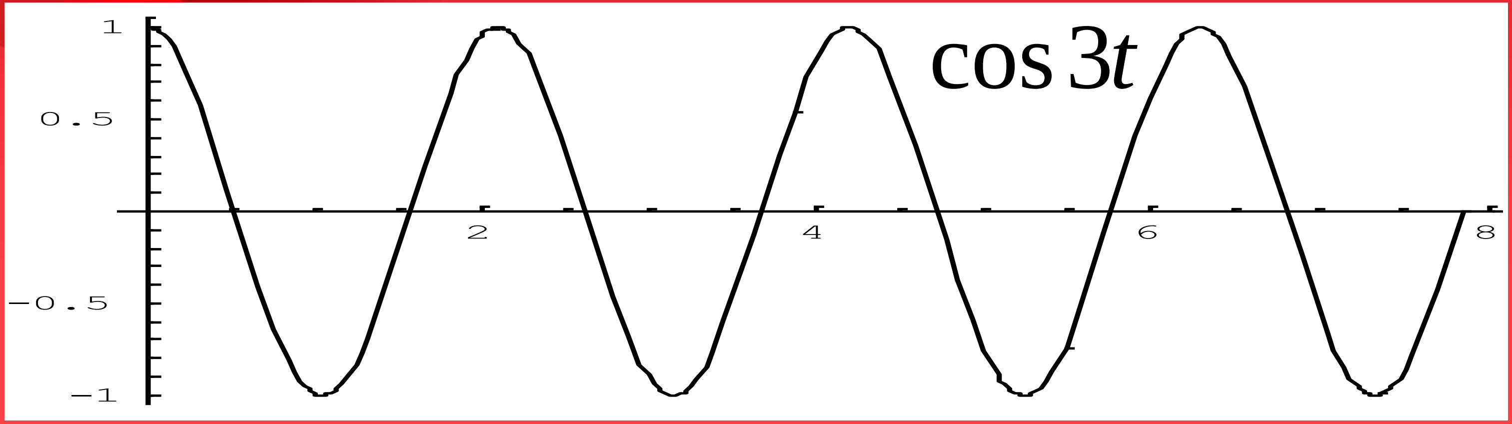




cos t
1
0.5
-0.5
-1
2
4
6
8

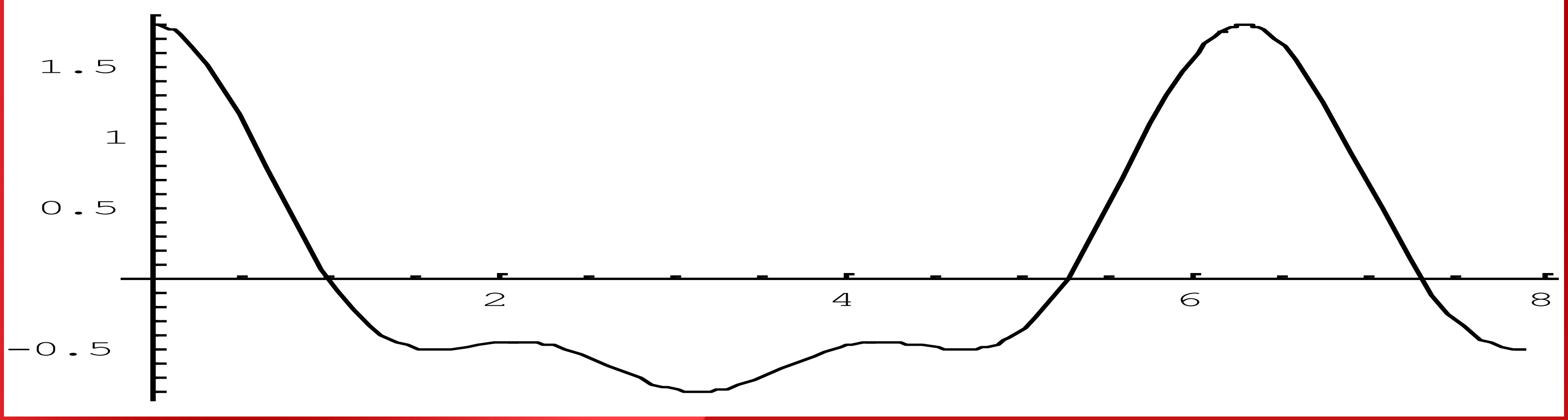


cos $2t$
1
0.5
-0.5
-1
2
4
6
8



cos $3t$
1
0.5
-0.5
-1
2
4
6
8

$$\cos t + 0.5\cos 2t + 0.3\cos 3t$$



$$\cos t + 0.3\cos 2t + 0.5\cos 3t$$