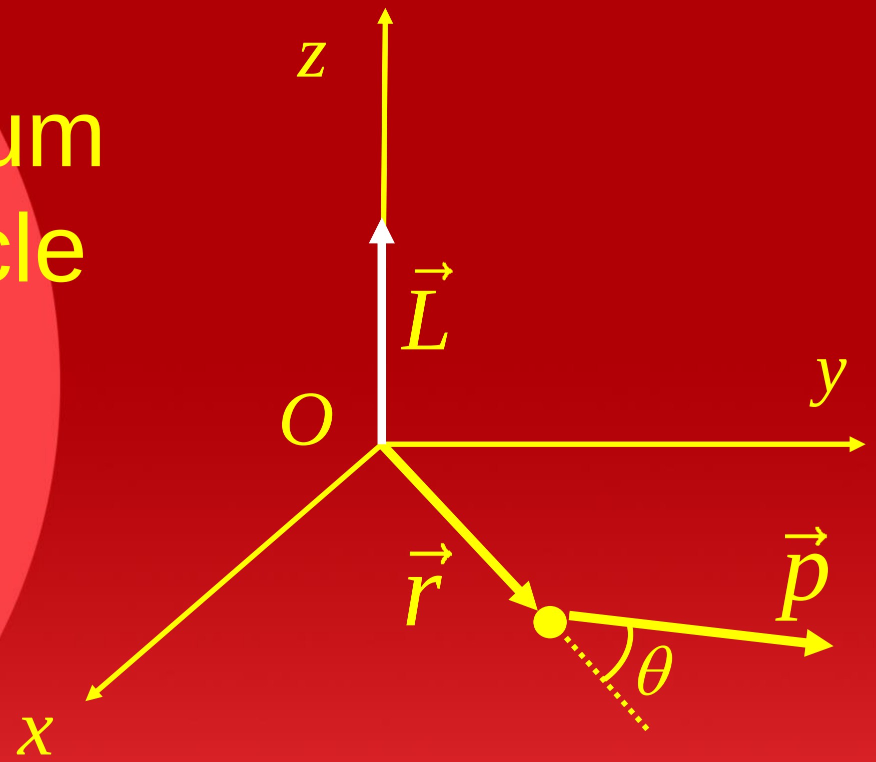
Some concepts are indispensable in physics. Momentum, defined as mv, is one: It is constant unless a force acts. No matter how complicated a system, momentum is conserved in collisions. A similar concept is useful in rotations: angular momentum. It is the subject of today’s lecture.

Physics

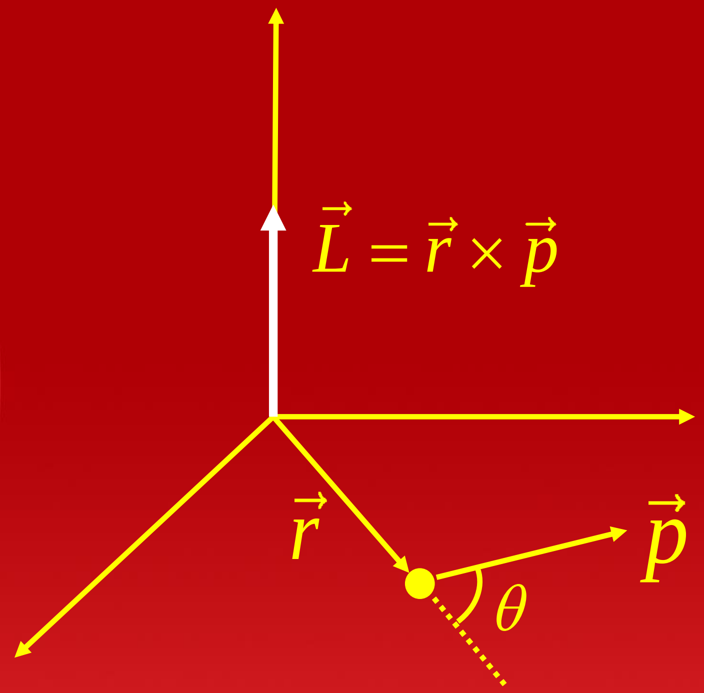
Angular Momentum

Angular momentum of a single particle



$$\vec{L} = \vec{r} \times \vec{p}$$

Angular Momentum



$$L = r\,p\sin\theta$$

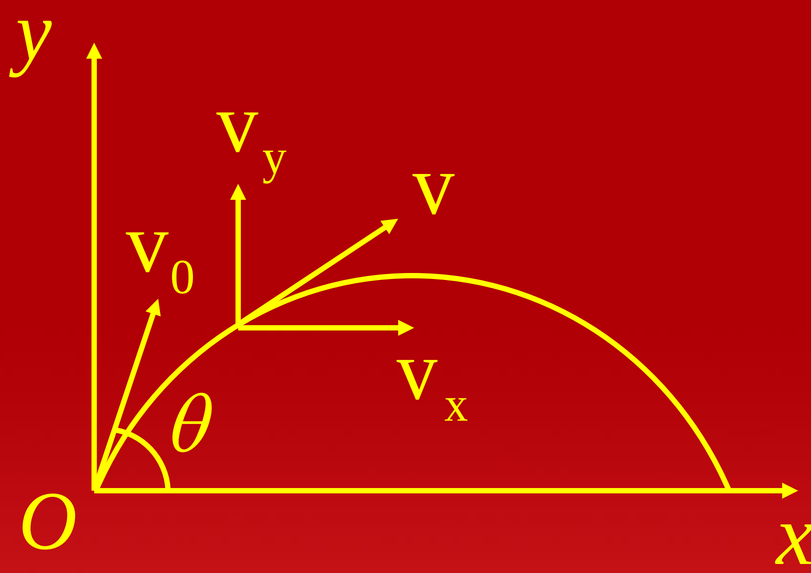
$$L = \left(r\sin\theta\right)p = r_{\perp}\,p$$

$$L = r\left(p\sin\theta\right) = r\,p_{\perp}$$

Just different ways of writing L !!

Angular momentum of projectile

Find L about O after time t:



$$x = \left(v_0 \cos\theta\right)t$$

$$y = \left(v_0 \sin\theta\right)t - \frac{1}{2}gt^2$$

$$v_x = v_0 \cos\theta$$

$$v_y = v_0 \sin\theta - gt$$

$$\vec{L} = \vec{r} \times \vec{p} = \left(x\hat{i} + y\,\hat{j} \right) \times \left(\mathrm{v}_x \hat{i} + \mathrm{v}_y\,\hat{j} \right) m$$

$$= m\left(x\,\mathrm{v}_x - y\,\mathrm{v}_y \right) \mathrm{v}_x\,\hat{k}$$

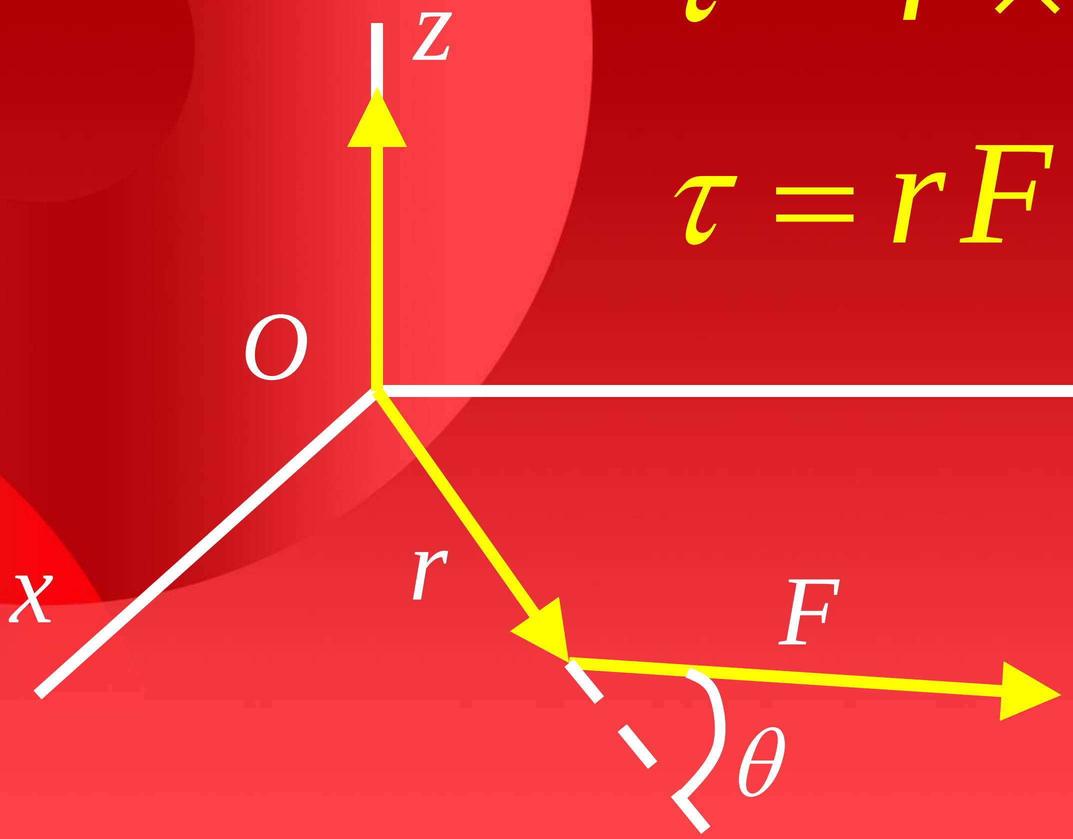
$$= m\left(\frac{1}{2} g t^2 \mathrm{v}_0 \cos\theta - g t^2 \mathrm{v}_0 \cos\theta \right) \hat{k}$$

$$= -\frac{m}{2} g t^2 \mathrm{v}_0 \cos\theta\,\hat{k}$$

Torque (reminder)

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\tau = rF\sin\theta$$



Relation between torque and angular momentum

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{L} + \Delta\vec{L} = \left(\vec{r} + \Delta\vec{r}\right) \times \left(\vec{p} + \Delta\vec{p}\right)$$

$$\vec{L} + \Delta\vec{L} = \vec{r} \times \vec{p} + \vec{r} \times \Delta\vec{p} + \Delta\vec{r} \times \vec{p} + \Delta\vec{r} \times \Delta\vec{p}$$

$$\Delta\vec{L} = \vec{r} \times \Delta\vec{p} + \Delta\vec{r} \times \vec{p}$$

$$\frac{\Delta \vec{L}}{\Delta t} = \frac{\vec{r} \times \Delta \vec{p} + \Delta \vec{r} \times \vec{p}}{\Delta t} = \vec{r} \times \frac{\Delta \vec{p}}{\Delta t} + \frac{\Delta \vec{r}}{\Delta t} \times \vec{p}$$

Take limit as $\Delta t \to 0$:

$$\because \quad \lim_{\Delta t \to 0} \frac{\Delta \vec{L}}{\Delta t} = \frac{d\vec{L}}{dt}$$

$$\therefore \quad \frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$$

But $\dfrac{d\vec{r}}{dt}$ is $\vec{v}$ and $\vec{p} = m\vec{v}$!

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = m\left(\vec{v} \times \vec{v}\right) = 0$$

and we are left with only

$$\boxed{\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}}$$

Now use Newton's second law:

$$\vec{F} = \frac{d\,\vec{p}}{dt}$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$$

$$\boxed{\therefore \quad \frac{d\vec{L}}{dt} = \vec{\tau}}$$

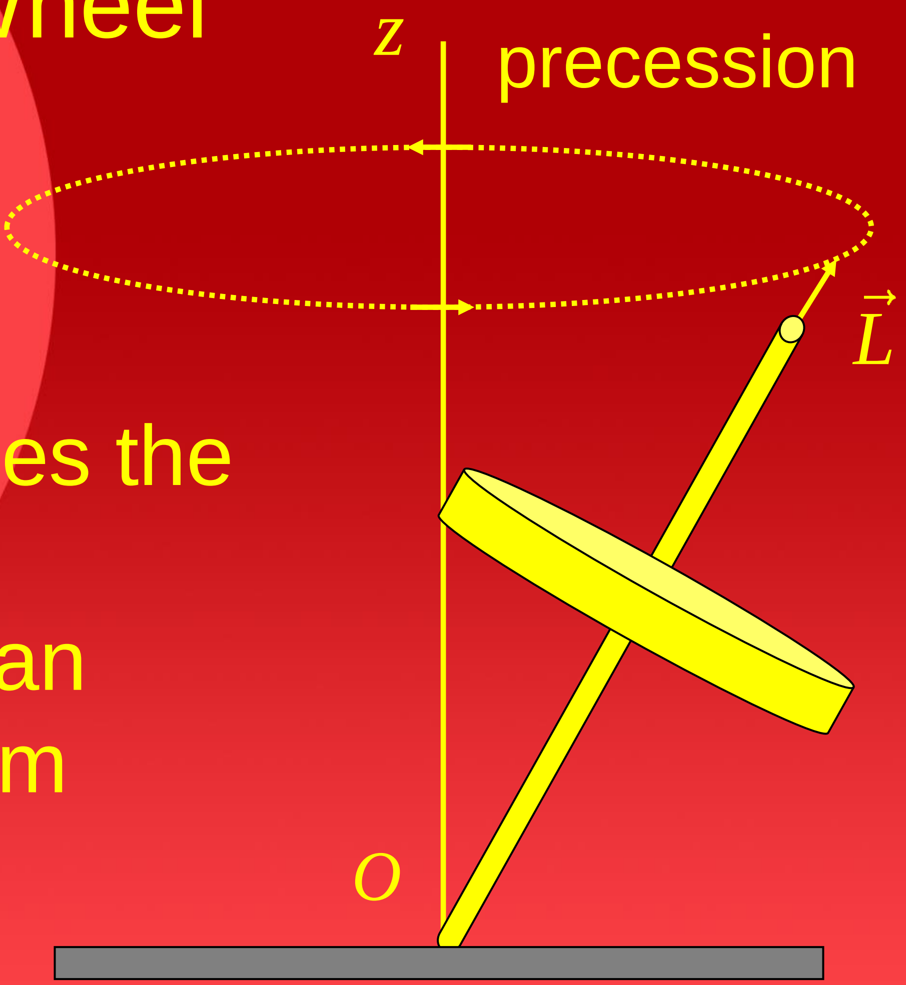
$$\boxed{\frac{d\vec{L}}{dt} = \sum \vec{\tau}}$$

The net torque acting on a particle is equal to the time rate of change of its angular momentum.

$$\boxed{\frac{d\vec{L}}{dt} = \sum \vec{\tau}}$$

The Spinning Wheel

The torque changes the direction but NOT the magnitude of an angular momentum



$$\vec{\tau} = \vec{r} \times \vec{F}$$

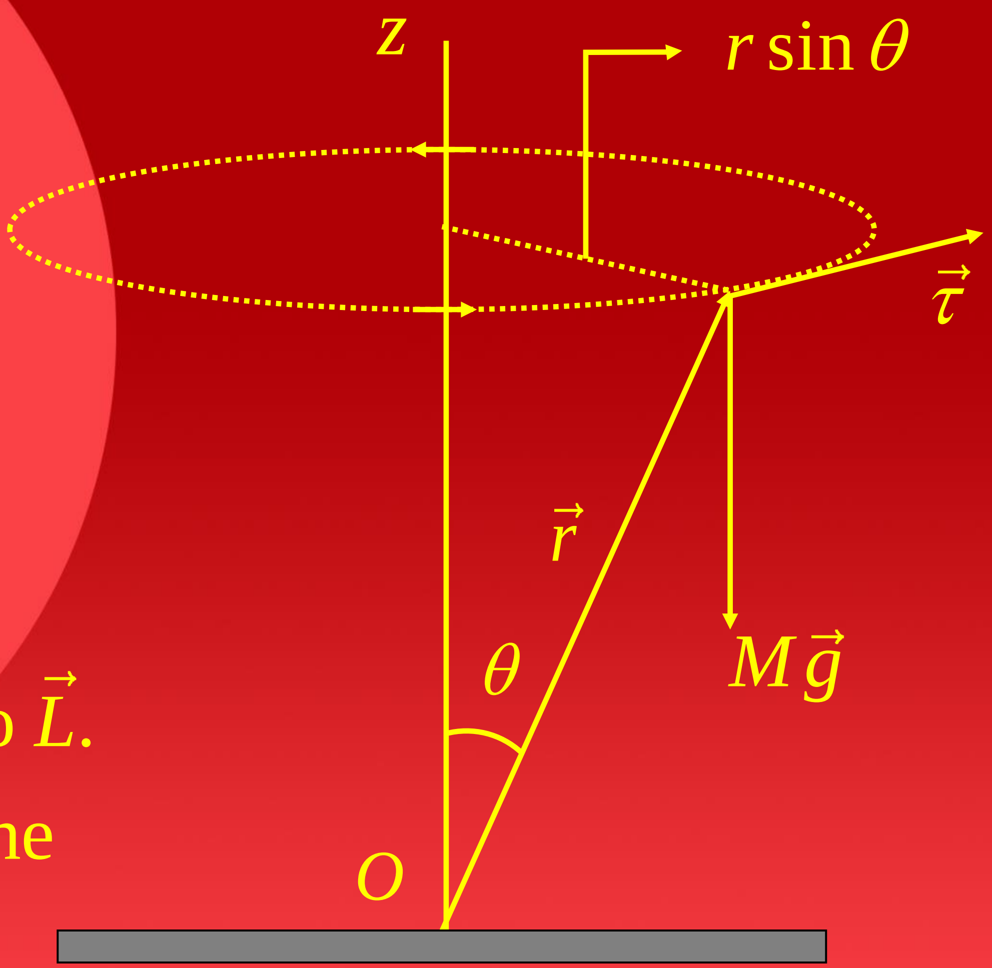
where $\vec{F} = m\vec{g}$

$$\therefore \ \tau = Mgr\sin\theta$$

$\vec{\tau}$ is perpendicular to $\vec{L}$.

$\therefore$ it cannot change the

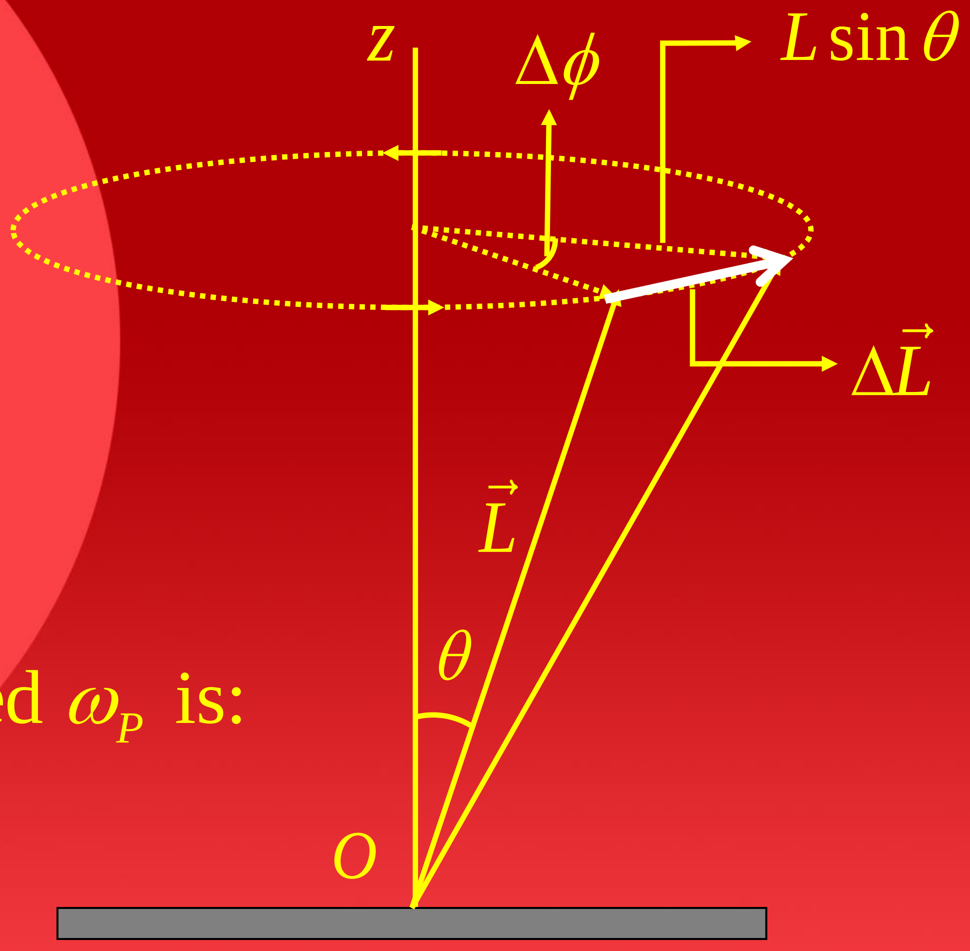
magnitude of $\vec{L}$!!



$$\Delta\vec{L} = \vec{\tau}\ \Delta t$$

$$\Delta\phi = \frac{\Delta L}{L\sin\theta}$$

$$= \frac{\tau\ \Delta t}{L\sin\theta}$$



Precession speed ω_P is:

$$\omega_P = \frac{\Delta\phi}{\Delta t}$$

$$= \frac{\tau}{L\sin\theta} = \frac{Mgr\sin\theta}{L\sin\theta} = \frac{Mgr}{L}$$

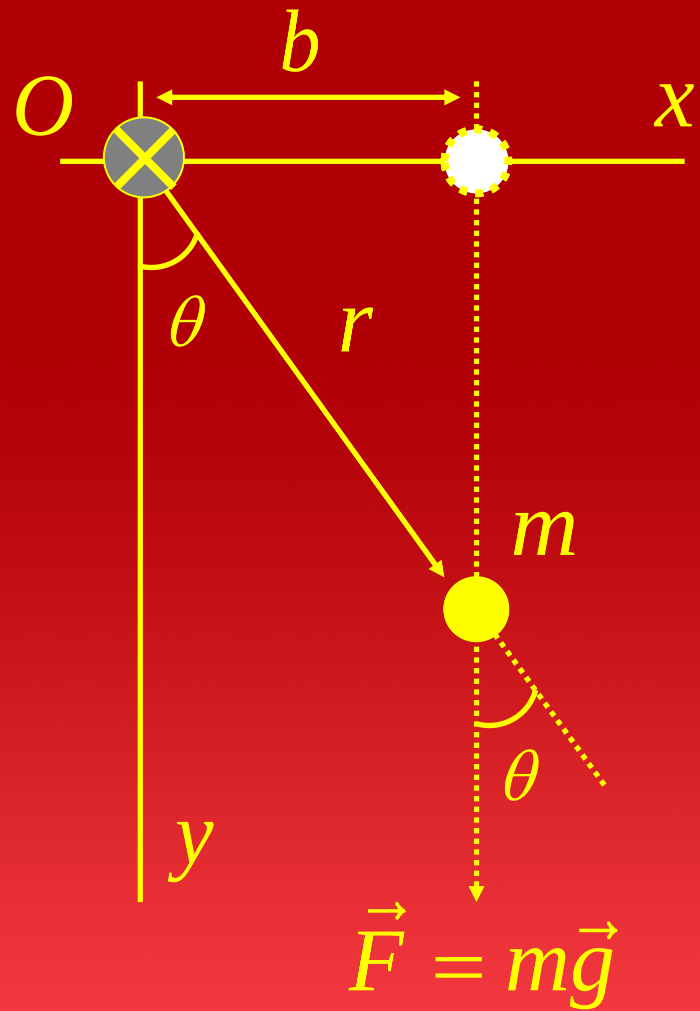
Example

$$\tau = rF\sin\theta$$

where $r\sin\theta = b$

and $F = mg$

$\therefore \quad \tau = mgb$



Right hand rule shows

that $\vec{\tau}$ is directed inwards. $\otimes\ \vec{\tau}$

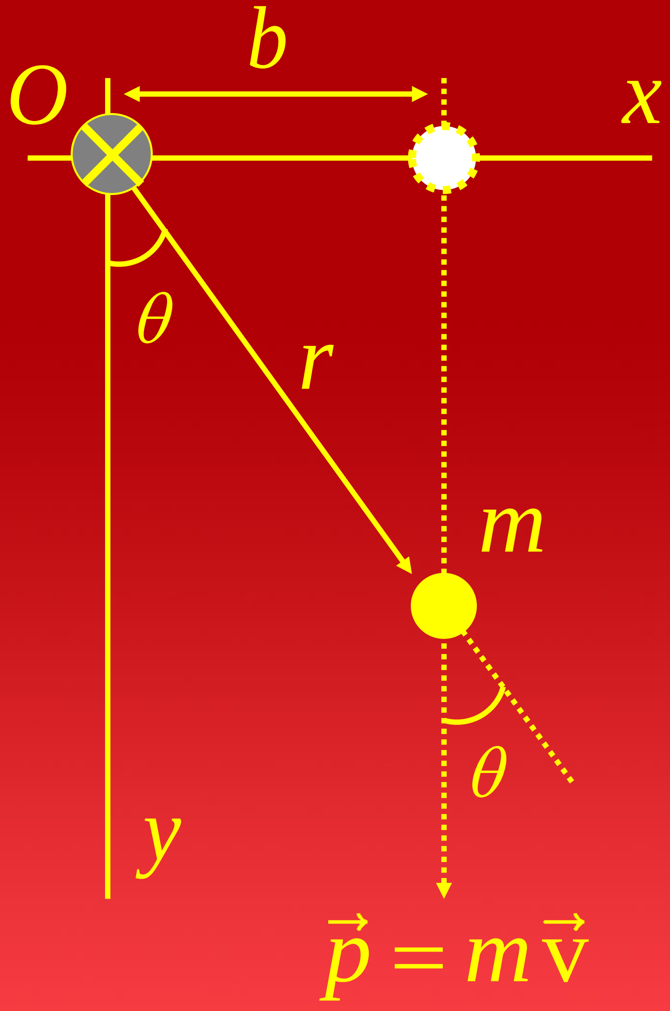
$$L = rp\sin\theta$$

where $r\sin\theta = b$

and $p = mv = mgt$

$\therefore \; L = mgbt$

$\vec{L}$ is directed inwards

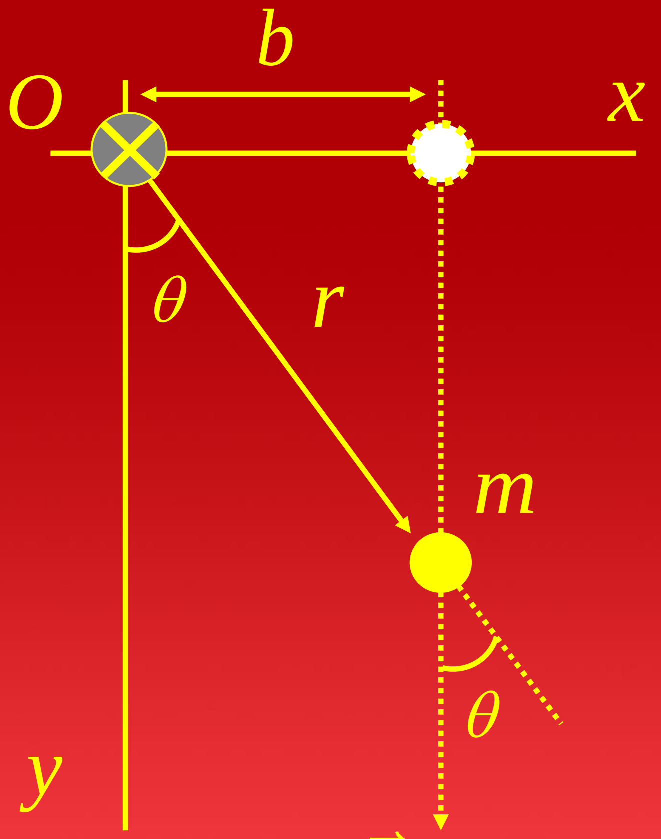


$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(mgtb\right)\hat{k}$$

$$= mgb\,\hat{k}$$

$$\tau = mgb\,\hat{k}$$

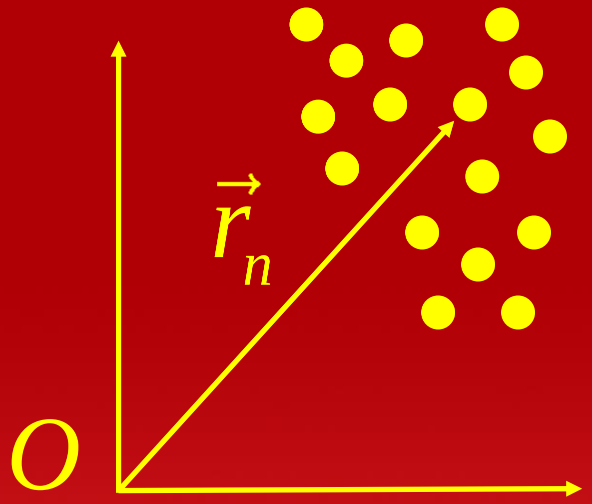
both are equal !



$$\vec{F} = m\vec{g}$$

$$\vec{p} = m\vec{\mathrm{v}}$$

Angular momentum for a system of particles



$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \cdots + \vec{L}_N = \sum_{n=1}^{N} \vec{L}_n$$

$$\frac{d\vec{L}}{dt} = \frac{d\vec{L}_1}{dt} + \frac{d\vec{L}_2}{dt} + \cdots + \frac{d\vec{L}_N}{dt} = \sum_{n=1}^{N} \frac{d\vec{L}_n}{dt}$$

$$\text{Since } \frac{d\vec{L}_n}{dt} = \vec{\tau}_n$$

$$\frac{d\vec{L}}{dt} = \sum_{n=1}^{N} \vec{\tau}_n$$

Thus the time rate of change of the total angular momentum of a system of particles equals the net torque acting on the system.

There are two sources of the torque acting on the system

1) The torque exerted on the particles of the system by internal forces between the particles

2) The torque exerted on the particles of the system by external forces

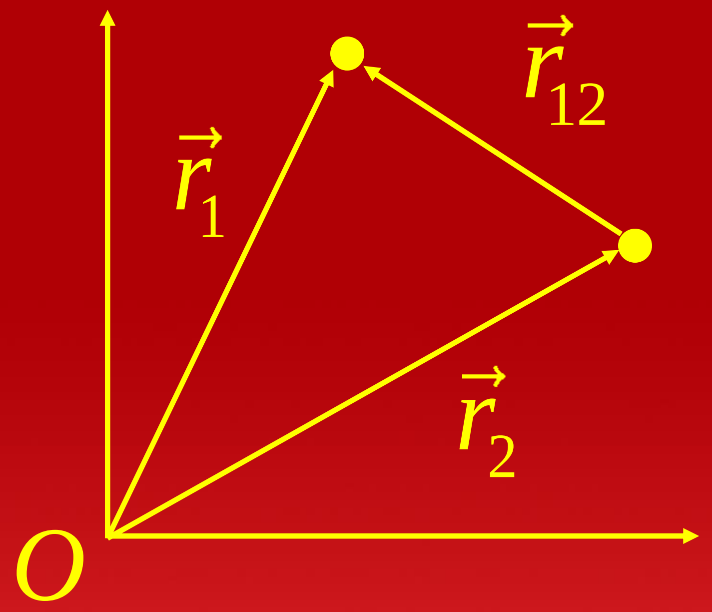
$$\sum \vec{\tau} = \sum \vec{\tau}_{\text{int}} + \sum \vec{\tau}_{ext}$$

If the forces between two particles not only are equal and opposite but are also directed along the line joining the two particles, then the total internal torque is zero.

$$\sum \vec{\tau}_{\text{int}} = 0$$

$$\sum \vec{\tau}_{\text{int}} = \vec{\tau}_1 + \vec{\tau}_2$$

$$= \vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21}$$



but

$$\vec{F}_{12} = -\vec{F}_{21} = F\hat{r}_{12}$$

$$\therefore \sum \vec{\tau}_{\text{int}} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12} = \vec{r}_{12} \times (F\hat{r}_{12})$$

$$= F(\vec{r}_{12} \times \hat{r}_{12}) = 0$$

Hence

$$\sum \vec{\tau} = \sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$

The net external torque acting on a system of particles is equal to the time rate of change of the of the total angular momentum of the system

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt} \quad \Leftrightarrow \quad \sum \vec{F}_{ext} = \frac{d\vec{L}}{dt}$$

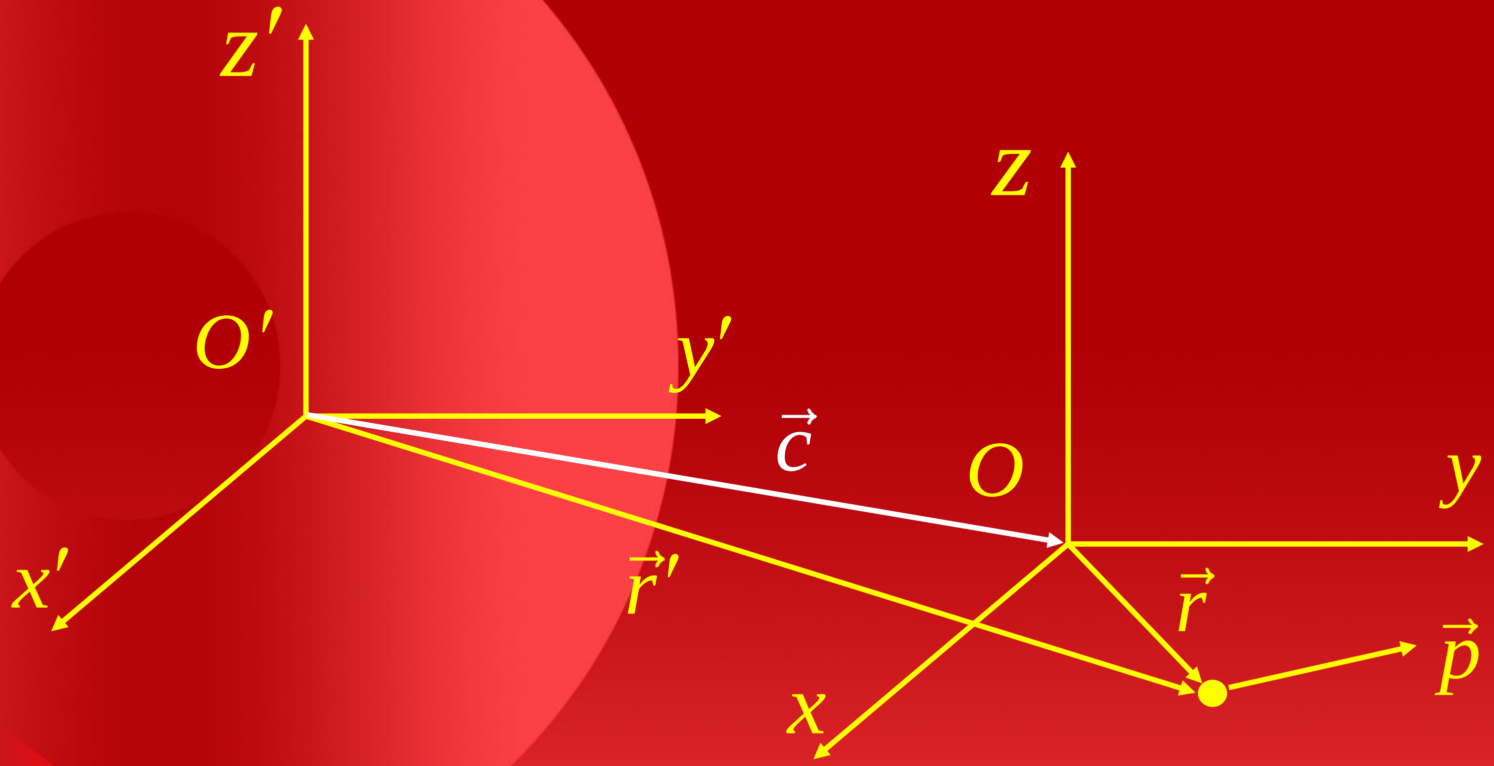
Conservation of Angular Momentum

If no net external torque acts on the system, then the angular momentum of the system does not change with the time

$$\frac{d\vec{L}}{dt} = 0 \quad \Rightarrow \quad \vec{L} = \text{a constant}$$

$$\vec{F} = \frac{d\,\vec{p}}{dt} \Leftrightarrow \vec{\tau} = \frac{d\,\vec{L}}{dt}$$

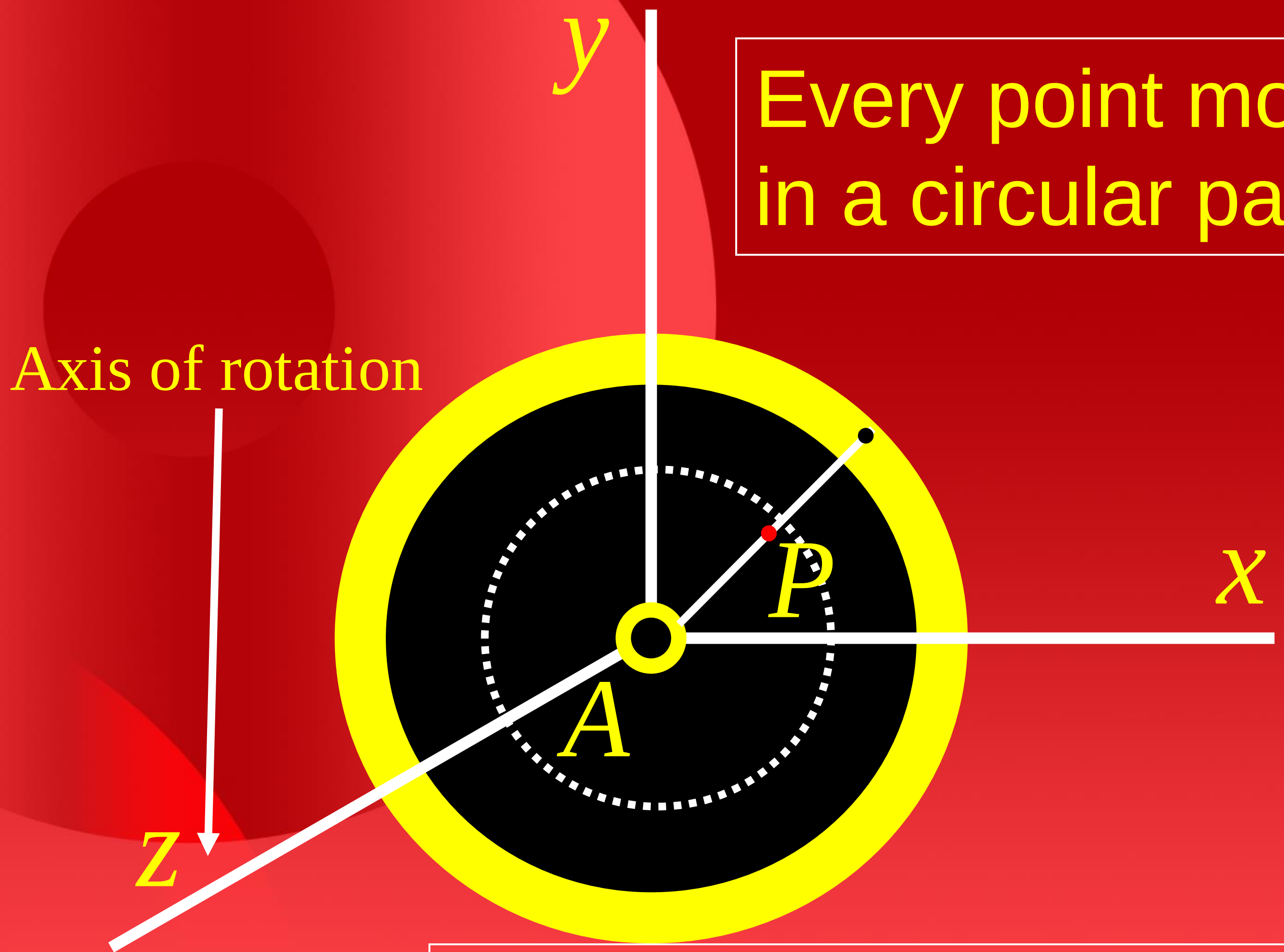
$$\vec{p} = m\vec{\mathrm{v}} \Leftrightarrow \vec{\mathrm{L}} = \ ??$$



$\vec{L}$ depends on the choice of the origin:

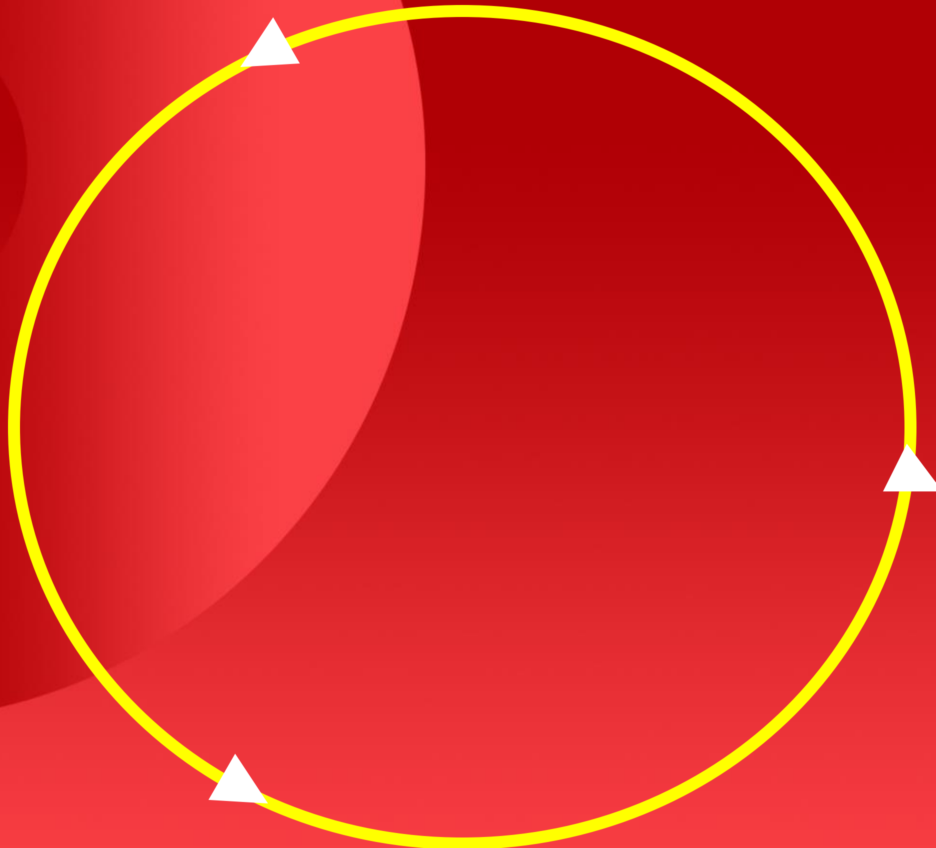
$$\vec{L}' = \vec{r}' \times \vec{p} = \left(\vec{c} + \vec{r}\right) \times \vec{p} = \vec{c} \times \vec{p} + \vec{L}$$

Rotation of Rigid Bodies

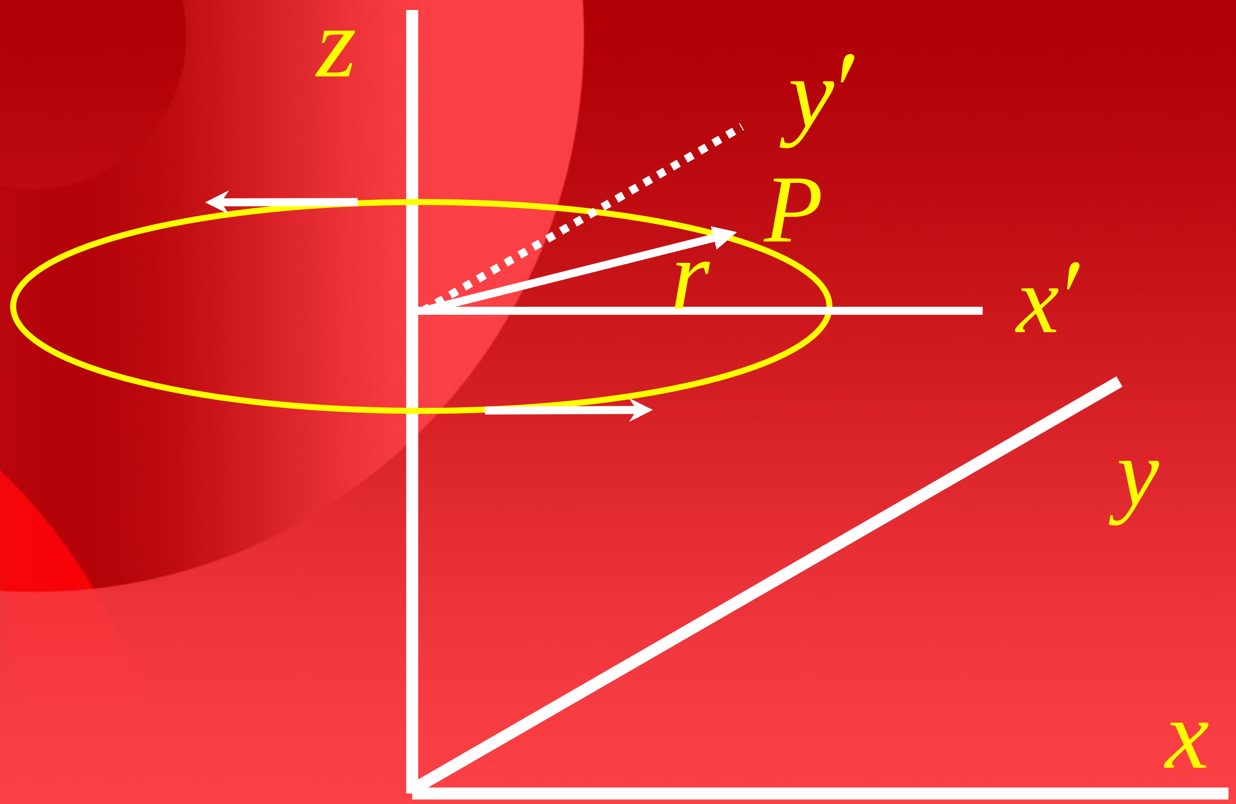


Every point moves in a circular path

Reference line AP moves through the same angle

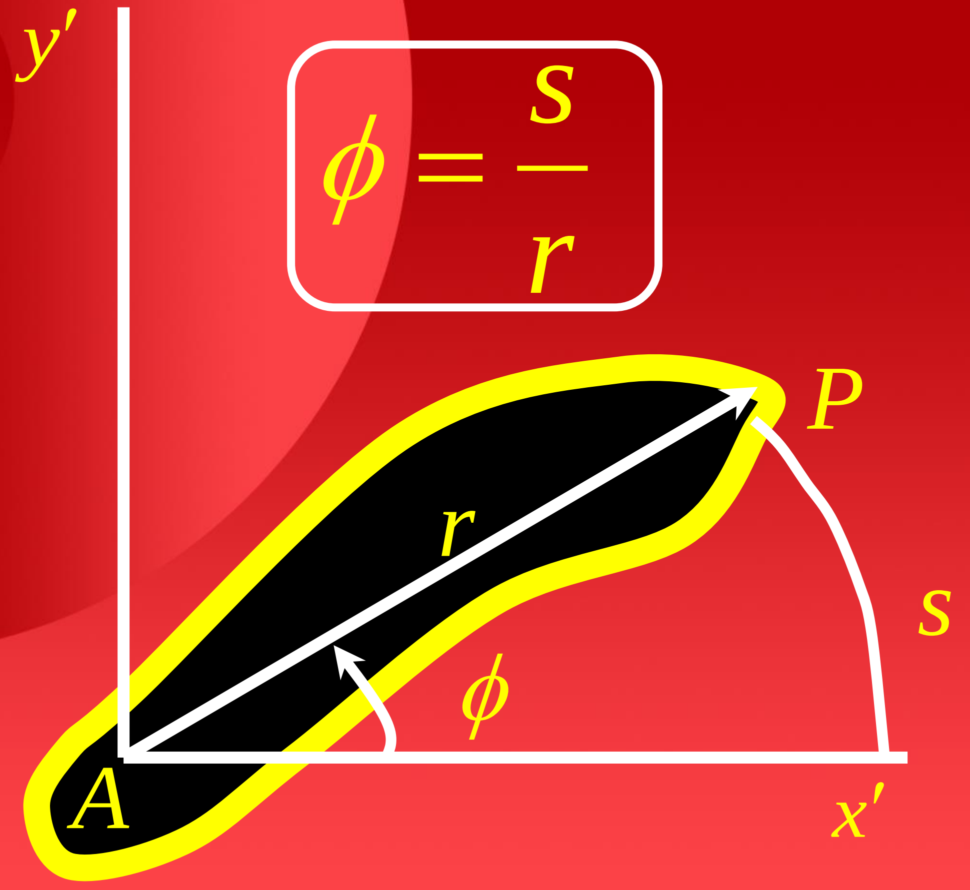


- Kinematics of a rigid body can be described by the motion of point P



- Cross-sectional Slice

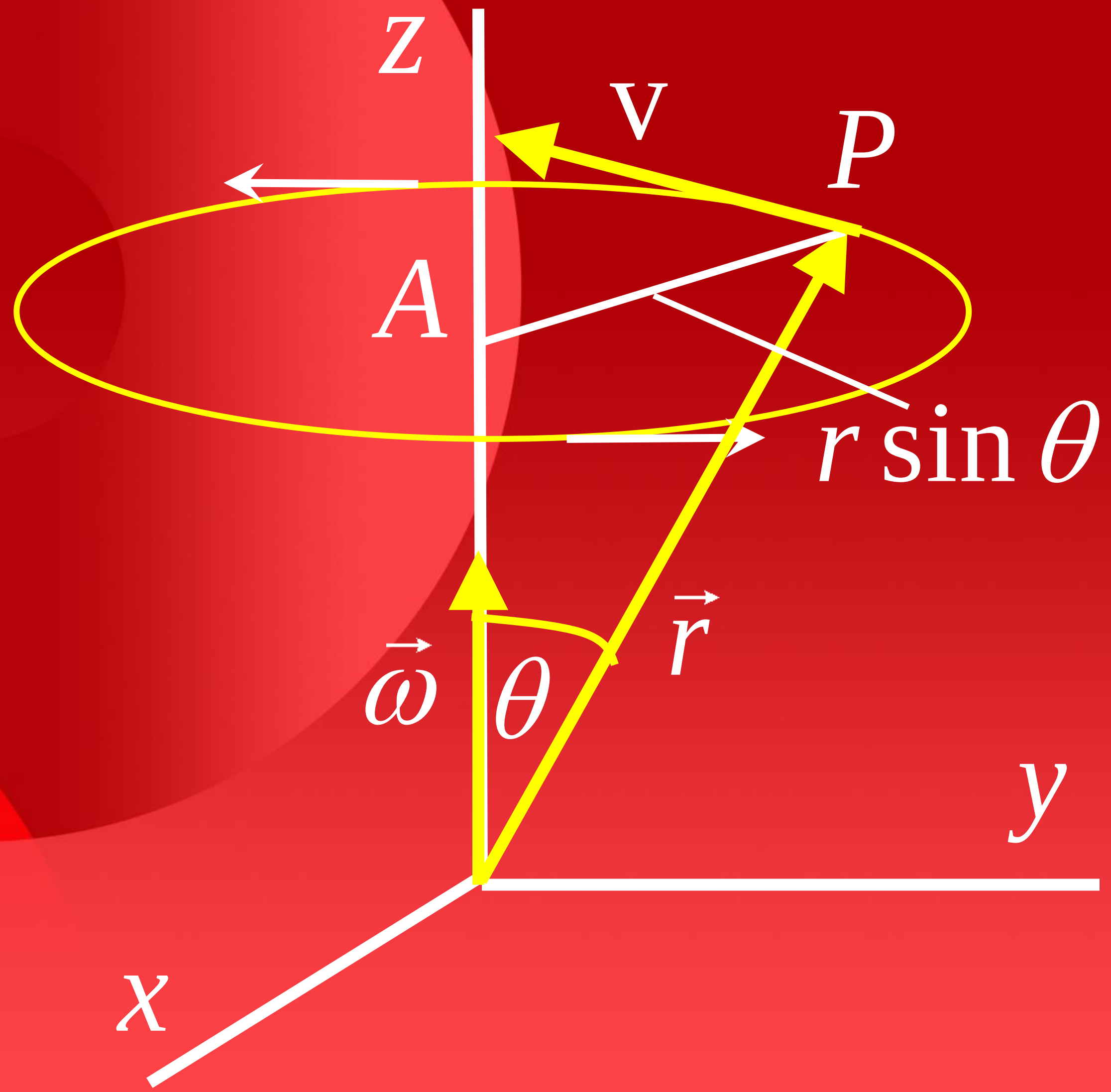
$$\boxed{\phi = \frac{s}{r}}$$



Linear and angular velocity

$$\vec{\mathrm{v}} = \vec{\omega} \times \vec{r}$$

$$\mathrm{v} = \omega r \sin\theta$$



z
V
P
A
$r \sin\theta$
$\vec{r}$
$\vec{\omega}$
θ
y
x

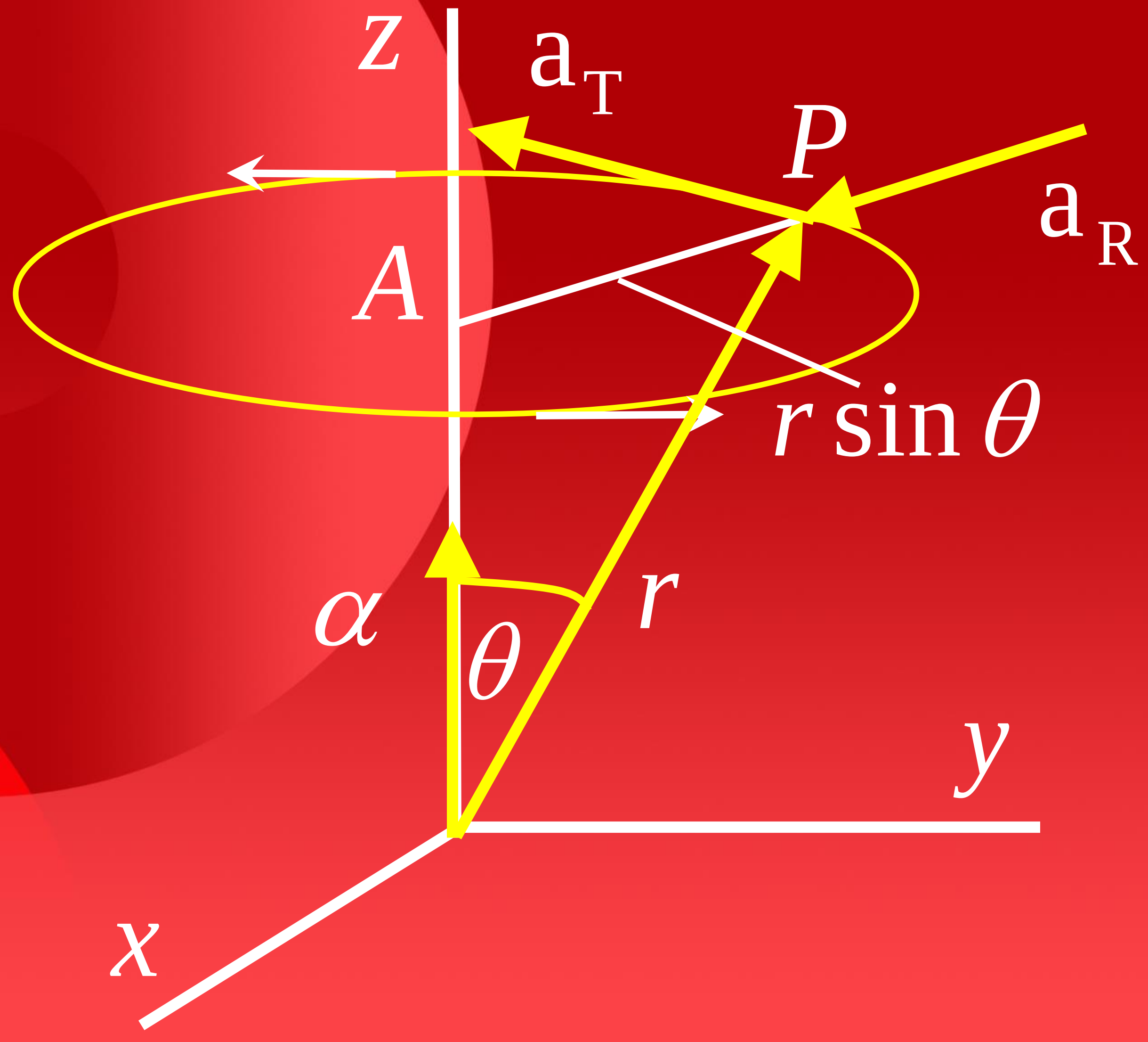
Linear and angular acceleration

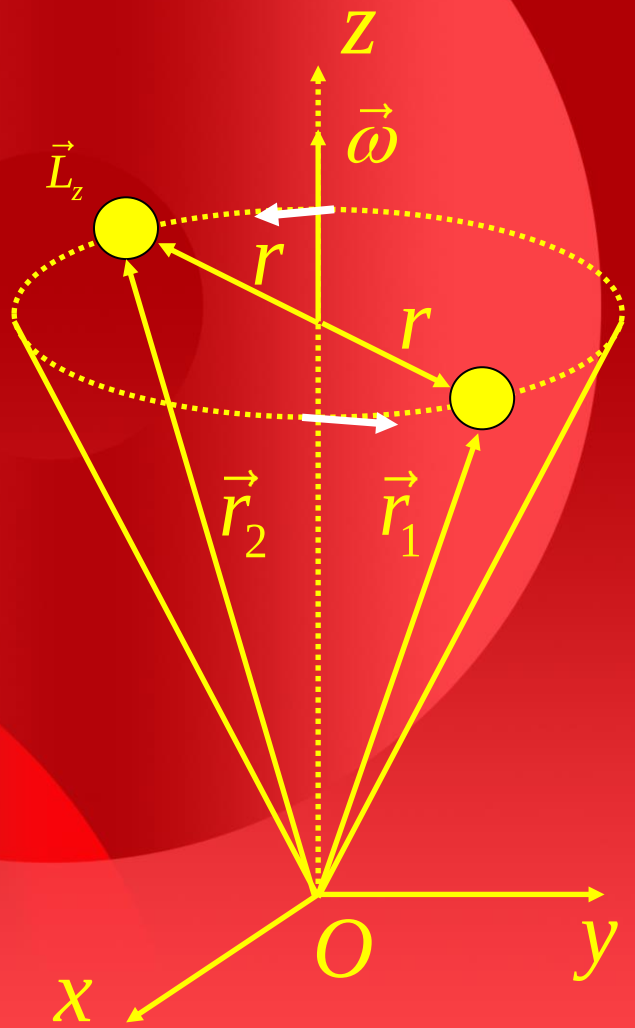
$$\vec{\mathrm{a}} = \frac{d\vec{\mathrm{v}}}{dt} = \frac{d}{dt}\left(\vec{\omega}\times\vec{r}\right)$$

$$= \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\frac{d\vec{r}}{dt}$$

$$= \vec{\alpha}\times\vec{r} + \vec{\omega}\times\vec{\mathrm{v}}$$

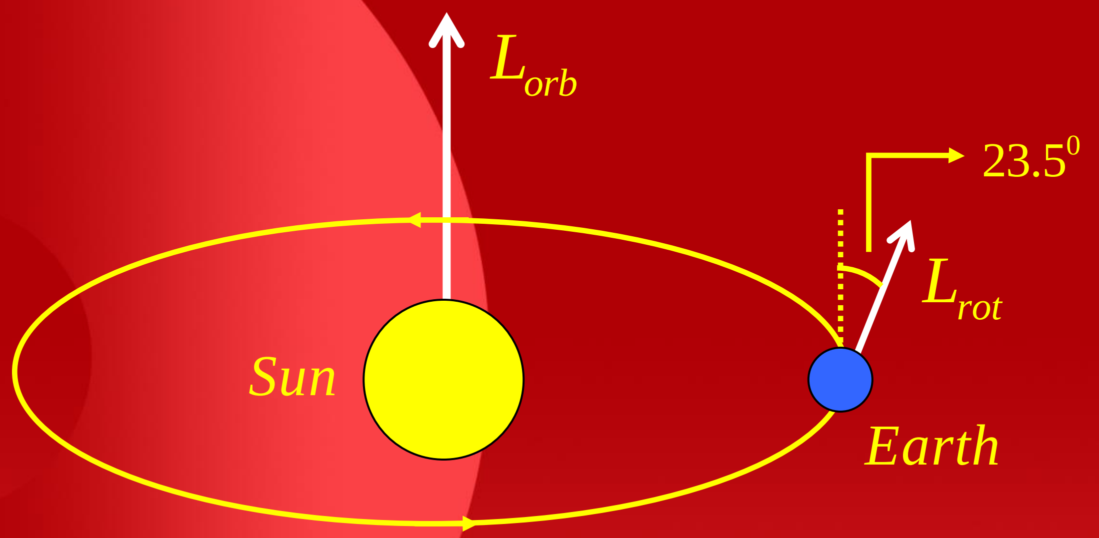
$$\vec{\mathrm{a}} = \vec{\mathrm{a}}_{\mathrm{T}} + \vec{\mathrm{a}}_{\mathrm{R}}$$


z
a T
P
a R
A
r sin θ
r
α
θ
y
x

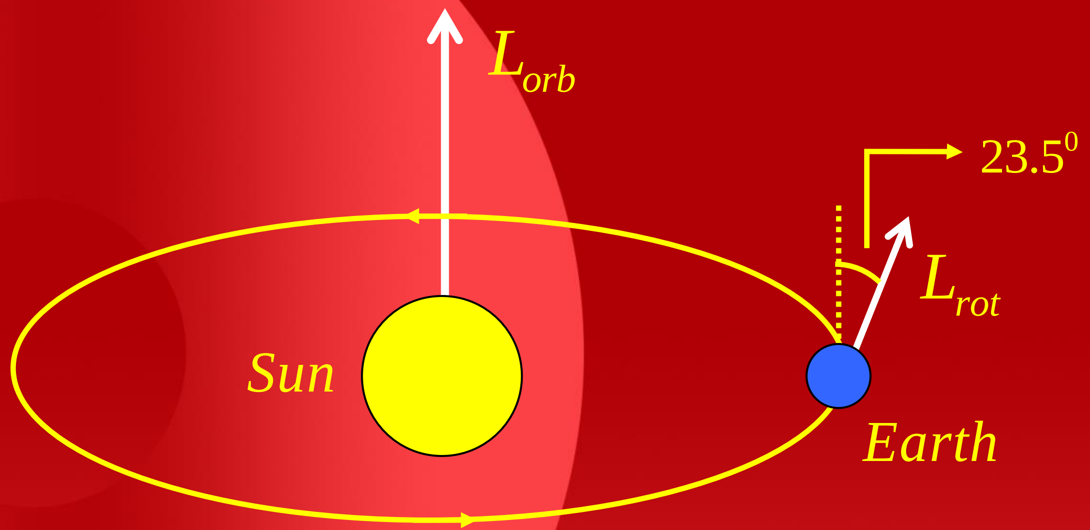


$$\vec{L} = \left(2mr^2\right)\vec{\omega}$$

$$= I\vec{\omega}$$



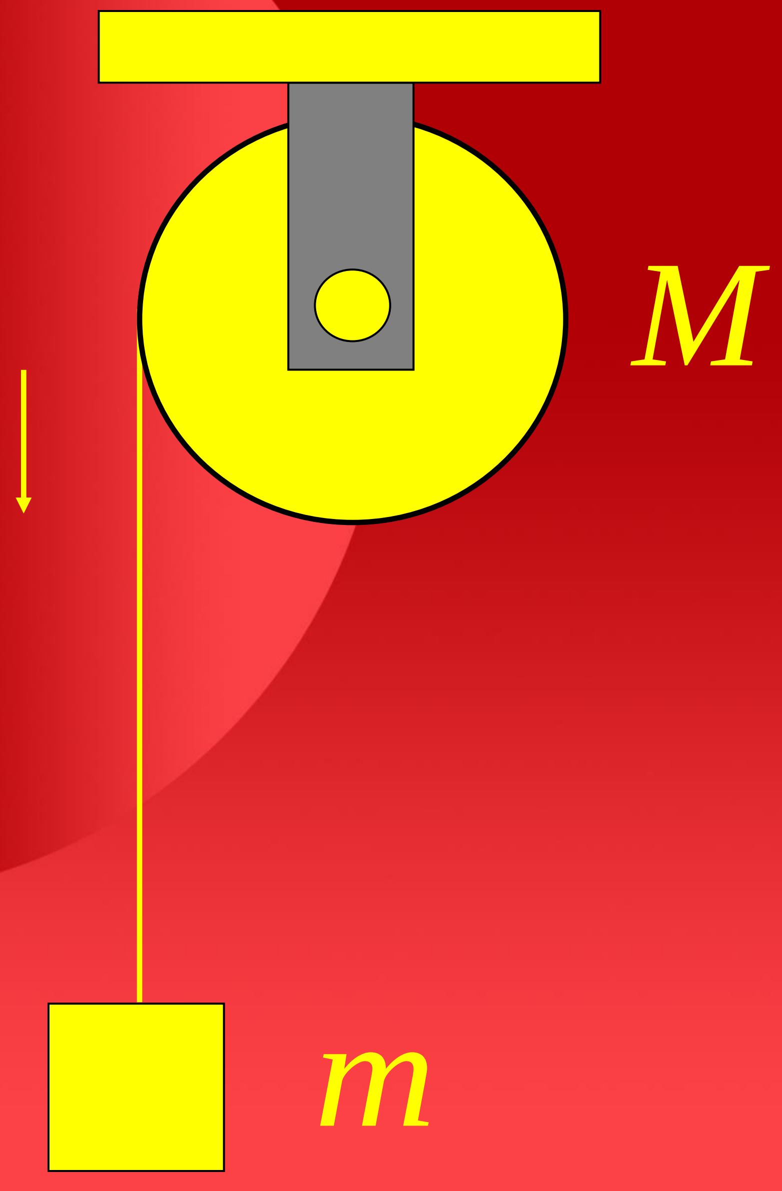
Which is greater?
(a) The angular momentum of the Earth due to rotation on its axis.
(b) The angular momentum of the Earth due to its orbital motion around the sun.



$$L_{rot} = I\omega = \left(\frac{2}{5}MR_E^2\right)\omega$$

$$L_{orb} = R_{orb}p = R_{orb}Mv = R_{orb}M\left(R_{orb}\omega\right) = MR_{orb}^2\omega$$

$$L_{orb}\big/L_{rot} = \frac{5}{2}\left(\frac{R_{orb}}{R_E}\right)^2 \gg 1 \Rightarrow L_{orb} \gg L_{rot}$$



M
m

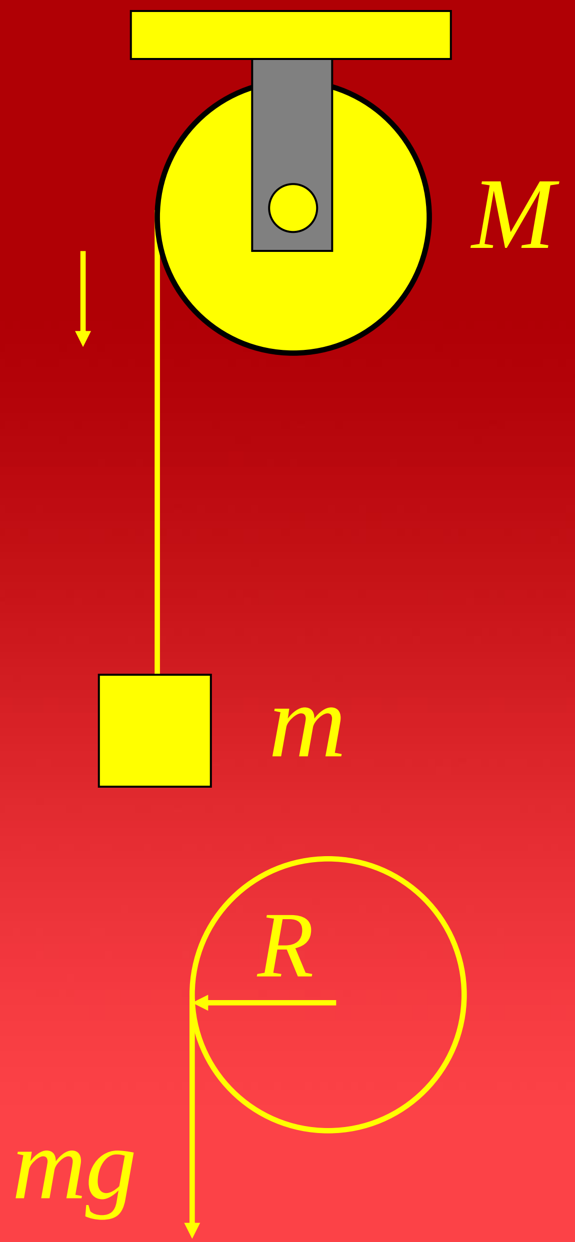
$$L = I\omega + mvR$$

$$\tau = \frac{dL}{dt}$$

$$(mg)R = \frac{d}{dt}(I\omega + mvR)$$

$$= I\left(\frac{d\omega}{dt}\right) + mR\left(\frac{dv}{dt}\right)$$

$$= I\alpha + mRa$$

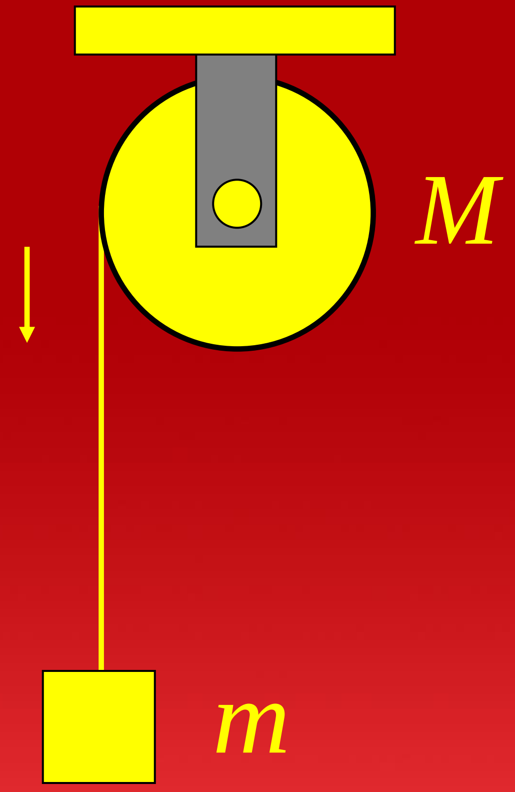


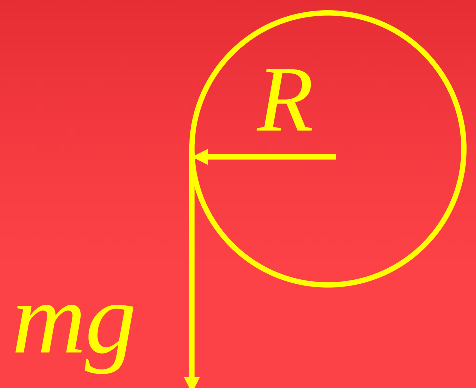
$$a = \alpha R$$

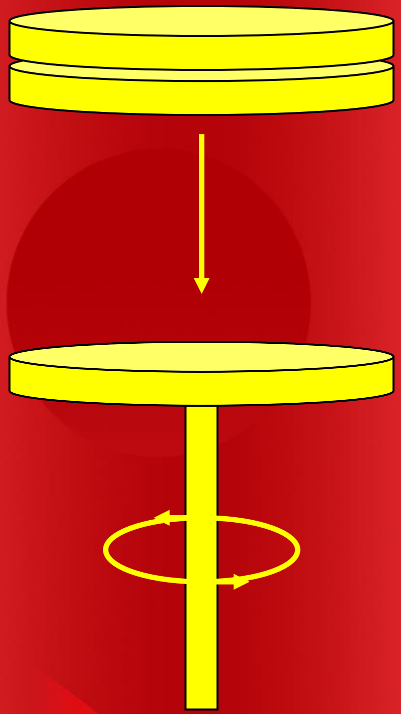
$$I = \frac{1}{2} MR^2$$

$$mgR = \left(\frac{1}{2} MR^2 \right) (a / R) + mRa$$

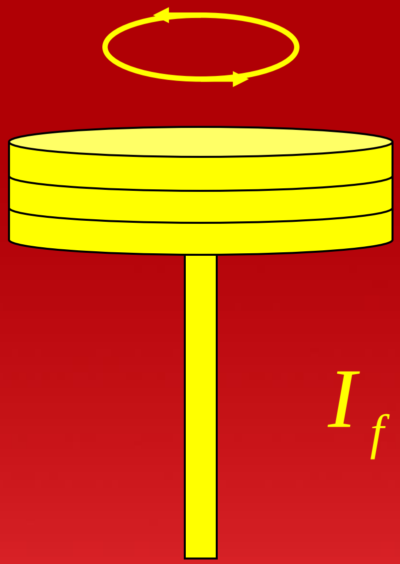
$$a = \frac{2mg}{M + 2m}$$







$$I_i = \frac{MR^2}{2}$$

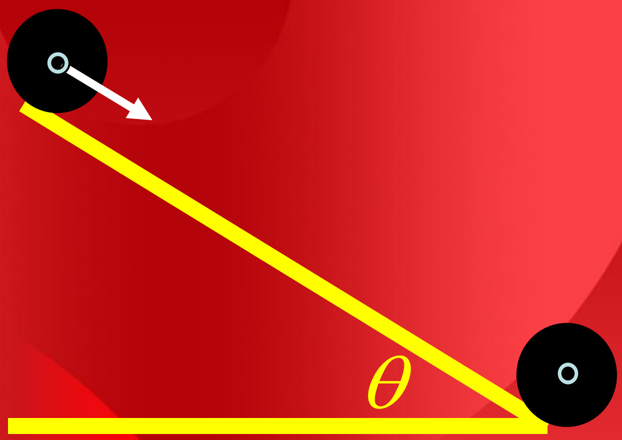


$$I_f = \frac{3MR^2}{2}$$

$$I_i \omega_i = I_f \omega_f \;\Rightarrow\; \omega_f = \omega_i \left(\frac{I_i}{I_f} \right), \text{ for one disc } I = \frac{1}{2} MR^2$$

$$\omega_f = \omega_i \left(\frac{MR^2}{2} \times \frac{2}{3MR^2} \right) = \frac{1}{3} \omega_i$$

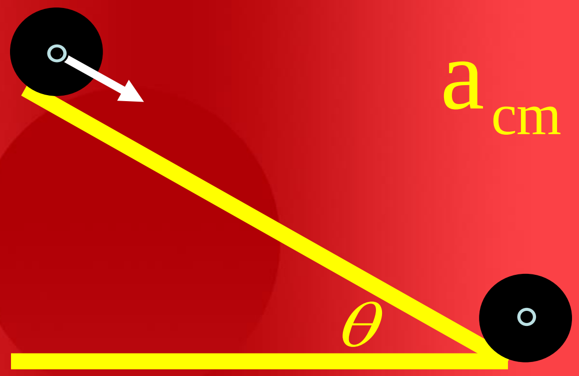
Rolling a sphere, cylinder and hoop:



$$Mg\sin\theta - f = Ma_{cm}$$

$$f = I_{cm}\frac{\alpha}{R} = I_{cm}\frac{a_{cm}}{R^2}$$

$$a_{cm} = \frac{g\sin\theta}{1 + I_{cm}/MR^2}$$



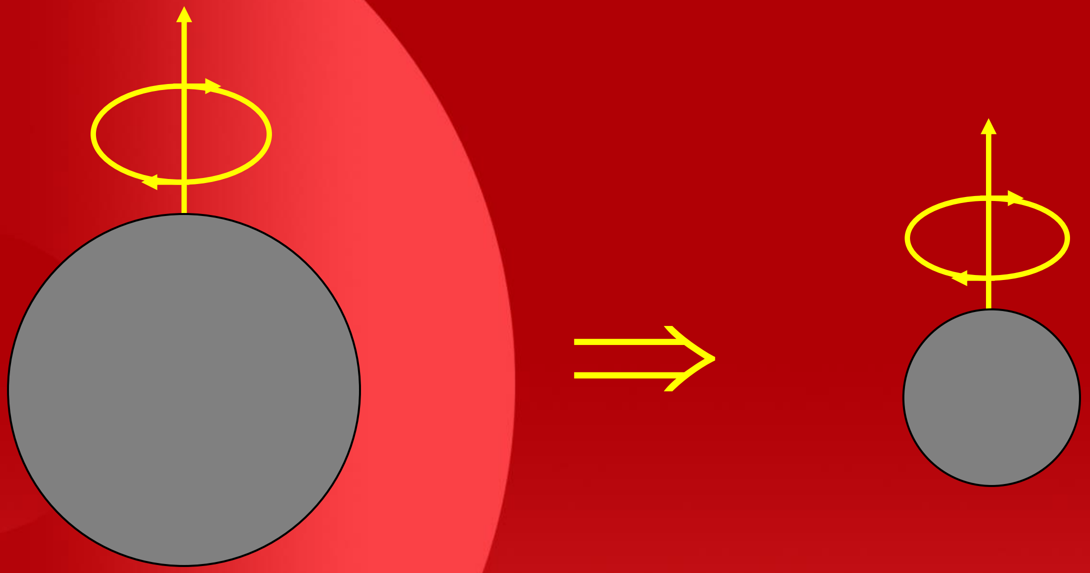
$$a_{cm}\left(\text{sphere}\right)=\frac{5}{7}g\sin\theta$$

$$a_{cm}\left(\text{cylinder}\right)=\frac{2}{3}g\sin\theta$$

$$a_{cm}\left(\text{hoop}\right)=\frac{1}{2}g\sin\theta$$

Sphere will reach first !!

- central force motion



If the radius the earth, assumed to be a perfect sphere, suddenly shrinks to half its present value, the mass of the Earth remaining unchanged, what will be the duration of one day?

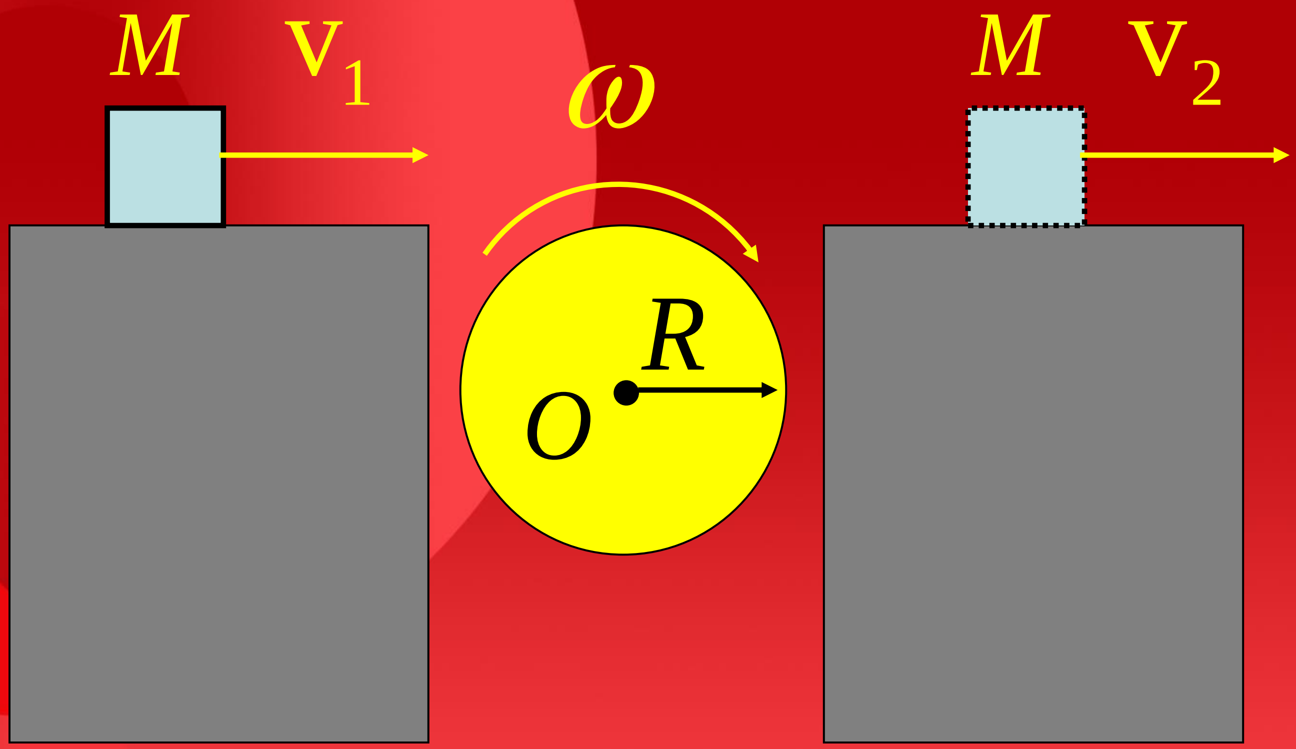
Applying the law of conservation of angular momentum

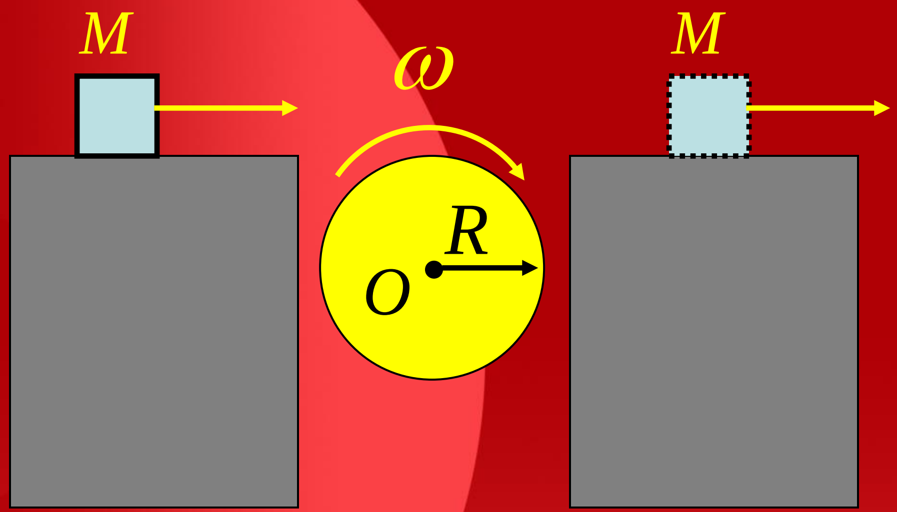
$$I_i\omega_i = I_f\omega_f$$

$$\left(\frac{2}{5}MR_i^2\right)\left(\frac{2\pi}{T_i}\right) = \left(\frac{2}{5}MR_f^2\right)\left(\frac{2\pi}{T_f}\right)$$

$$\frac{R_i^2}{T_i} = \frac{R_f^2}{T_f} \quad or \quad T_f = \left(\frac{R_f}{R_i}\right)^2 T_i = \left(\frac{1}{2}\right)^2 (24\text{hours})$$

$$\therefore T_f = \left(\frac{1}{4}\right)(24\text{hours}) = 6 \text{ hours}$$


M
v_1
ω
R
O
M
v_2



Decrease in angular momentum

of the block about O $= M(\mathrm{v}_1 - \mathrm{v}_2)R$

If cylinder acquires an angular velocity ω

as a result of collision, then its angular

momentum $= I\omega$

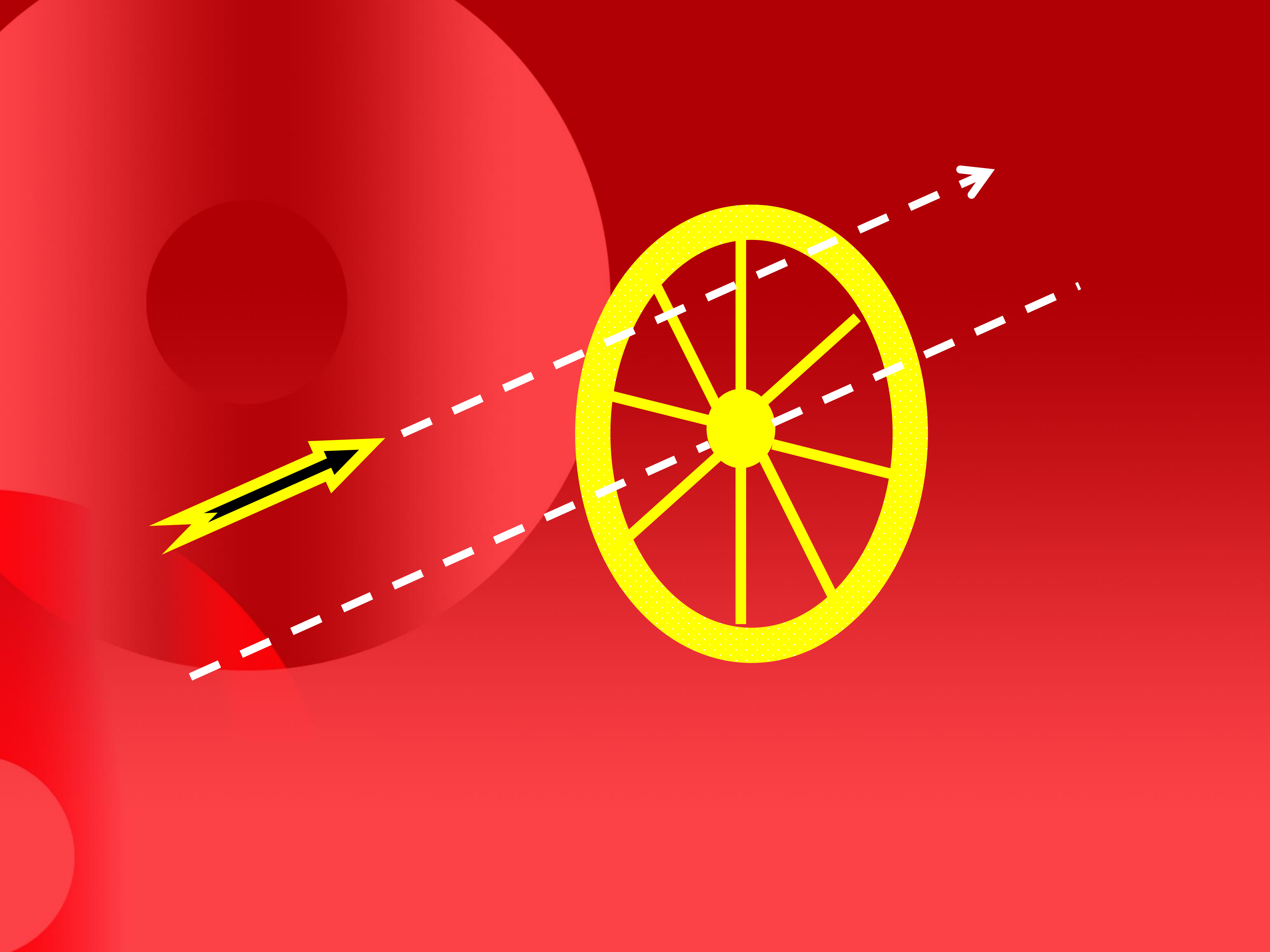
Use angular momentum conservation:

$$I\omega = M(\mathrm{v}_1 - \mathrm{v}_2)R$$

The linear velocity of the cylinder at its periphery is identical with the final velocity of the block:

$$\Rightarrow \omega = \frac{\mathrm{v}_2}{R}. \qquad \text{Hence } I\frac{\mathrm{v}_2}{R} = M(\mathrm{v}_1 - \mathrm{v}_2)R$$

$$\boxed{\mathrm{v}_2 = \frac{\mathrm{v}_1}{(1 + I/MR^2)}}$$



A wheel has eight spokes and radius of 30 cm. It is mounted on a fixed axle and is spinning at 2.5 rev/s. You want to shoot a 24-cm arrow parallel to this axle and through the wheel without hitting any of the spokes. Assume that the arrow and the spokes are very thin. (a) What minimum speed must the arrow have? (b) Does it matter where between the axle and the rim of the wheel you aim? If so, where is the best location?

$$\text{minimum speed} = \frac{\text{length of the arrow}}{\text{time to pass one spoke}}$$

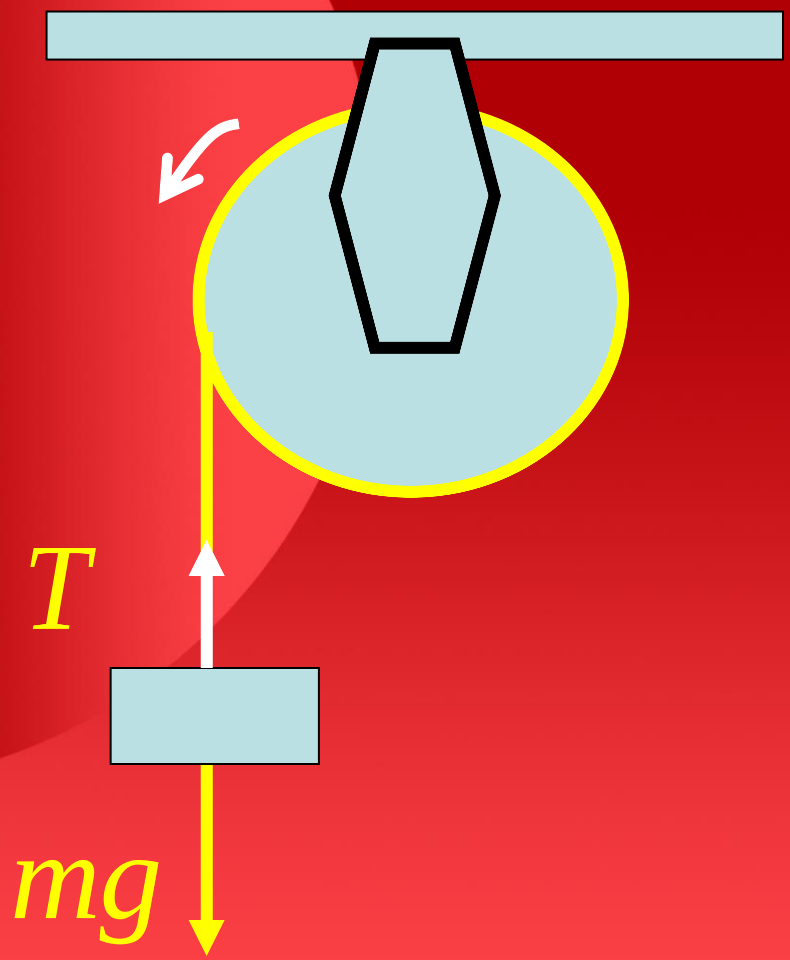
$$s = \text{distance traveled by one spoke} = \frac{2\pi r}{8}$$

$$\text{time to pass one spoke} = \frac{\text{distance traveled by one spoke}}{\text{speed of spoke}}$$

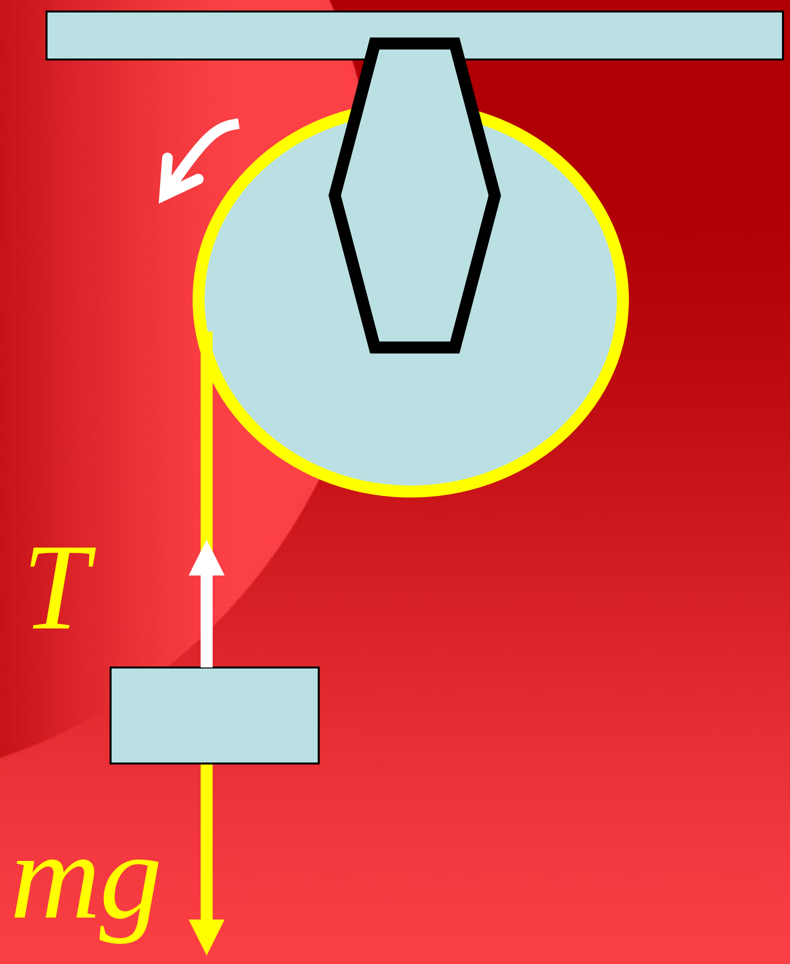
$$\text{time to pass one spoke} = \frac{s}{v} = \frac{2\pi r}{8r\omega}$$

$$\text{So minimum speed} = \frac{\ell \times 8r\omega}{2\pi r} = 4.8\,\text{m/s}$$

Does not matter where we aim!!


T
mg

A disk of mass M = 2.5kg and radius R=20cm is mounted on a fixed horizontal axle. A block of mass m = 1.2kg hangs from a light cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the tension in the cord, and the angular acceleration of the disk.


T
mg

$$\sum F = mg - T = m\text{a}$$

$$\sum \tau = TR = \frac{1}{2} MR^2 \left(\frac{\text{a}}{R} \right)$$

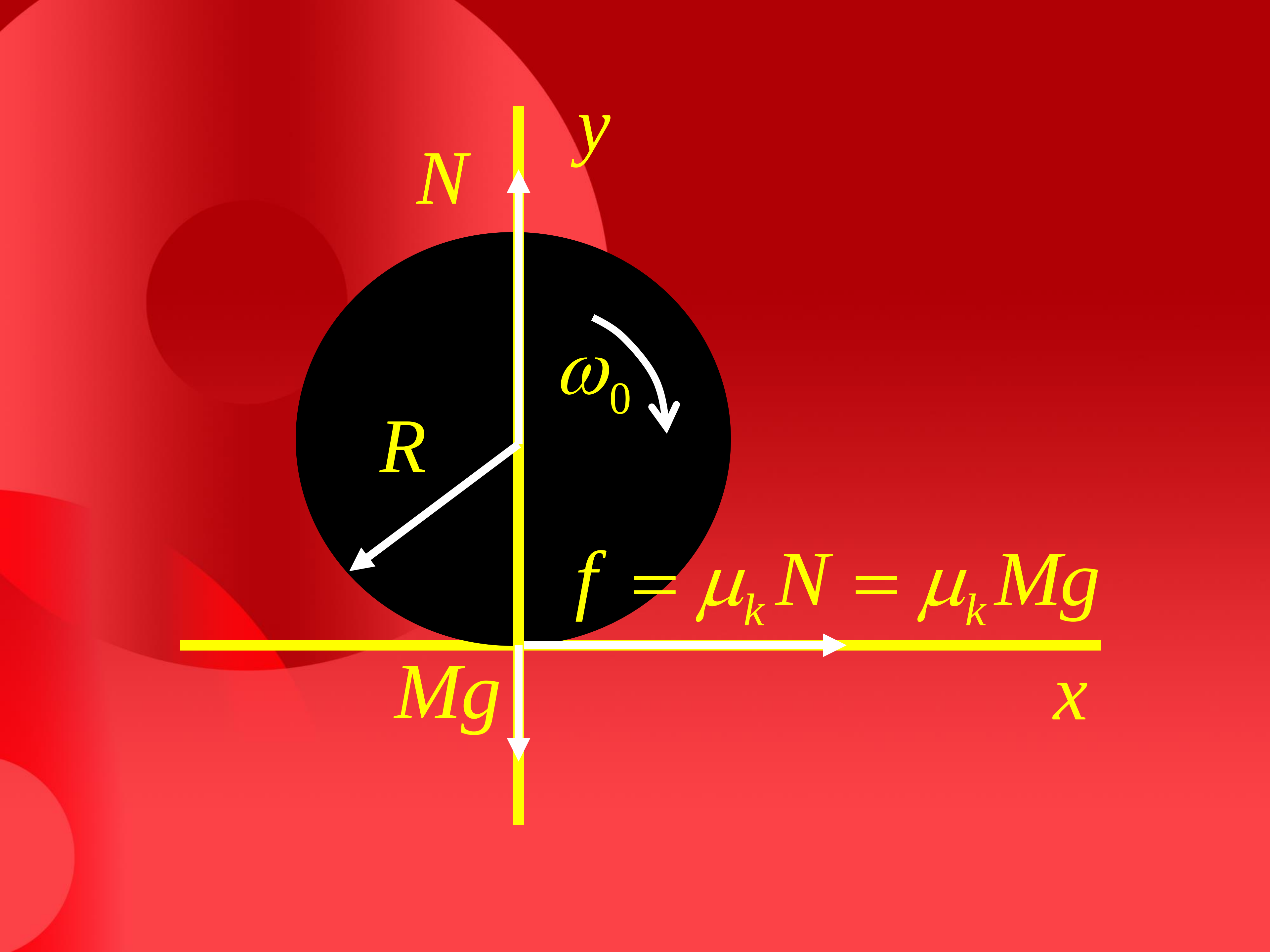
$$T = \frac{1}{2} M\text{a}$$

$$\text{a} = g \frac{2m}{M + 2m} = 4.8\,\text{m/s}^2$$

$$T = mg \frac{M}{M + 2m} = 6.0\,N$$

$$\alpha = \frac{\text{a}}{R} = 3.8\,\text{rev/s}^2$$

A uniform solid cylinder of radius R = 12cm and mass M = 3.2kg is given an initial clockwise angular velocity ω_0 of 15rev/s and then lowered on to a flat horizontal surface. The coefficient of kinetic friction between the surface and the cylinder is $\mu_k = 0.21$. Initially, the cylinder slips as it moves along the surface, but after a time t pure rolling without slipping begins. (a) What is the velocity v_{cm}? (b) What is the value of t?



y
N
ω_0
R
$f = \mu_k N = \mu_k Mg$
Mg
x

$$\sum F_x = Ma_{\mathrm{cm}} = M\left(\frac{\mathrm{v}_{\mathrm{cm}} - 0}{t-0}\right)$$

$$\mu_k Mg = M\,\frac{\mathrm{v}_{\mathrm{cm}}}{t}$$

$$\sum \tau = I_{cm}\alpha = I_{cm}\left(\frac{\omega_f - \omega_i}{t-0}\right)$$

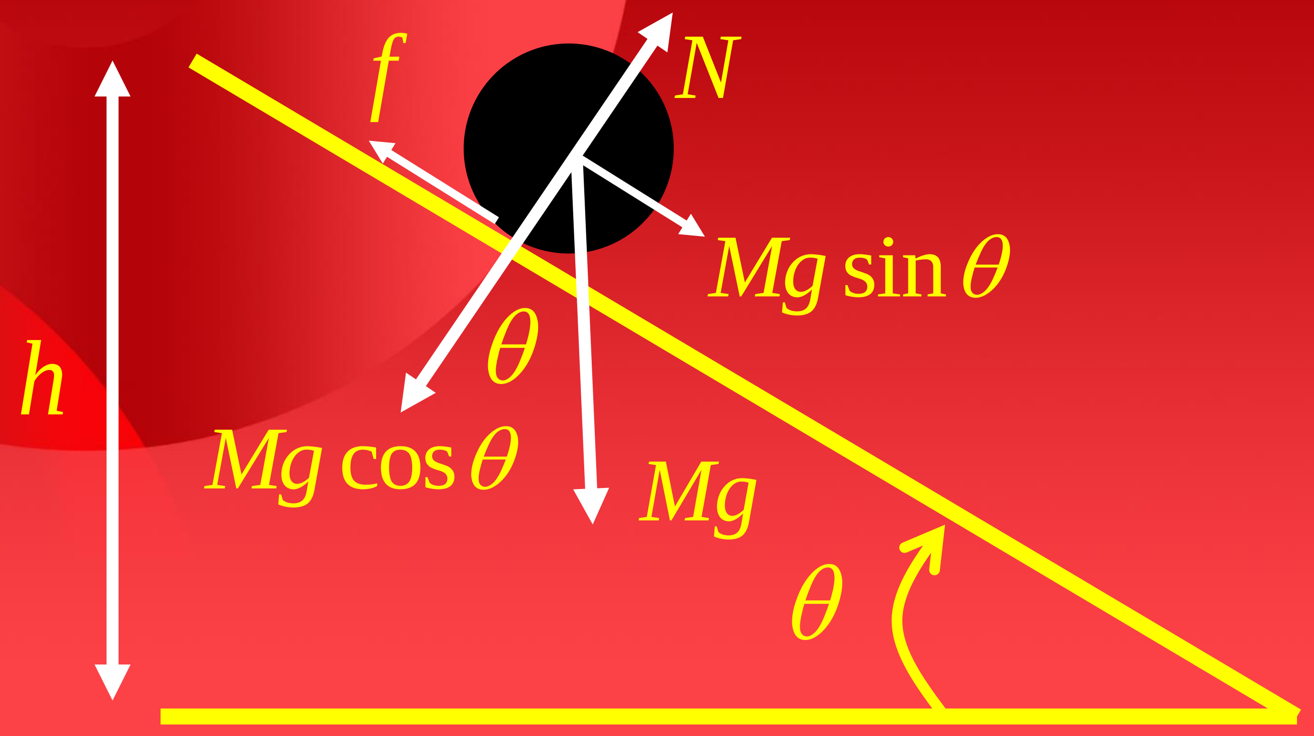
$$\mu_k MgR = \left(\frac{1}{2}MR\right)^2\left(\frac{-\mathrm{v}_{\mathrm{cm}}/R - \left(-\omega_0\right)}{t}\right)$$

$$\mathrm{v}_{\mathrm{cm}} = \frac{1}{3}\omega_0 R = 3.8 m/s$$

$$t = \frac{\omega_0 R}{3\mu_k g} = 1.8 s$$

A solid cylinder of mass M and radius R rolls without slipping down an inclined plane of length L and height h. Find the speed of its center of mass when the cylinder reaches the bottom.

Solve same problem using forces and torques:



$$N - Mg\cos\theta = 0$$

$$Mg\sin\theta - f = M\mathrm{a}_{\mathrm{cm}}$$

$$fR = I_{cm}\alpha$$

$$f = \frac{I_{cm}\alpha}{R}$$

$$\mathrm{v}_{\mathrm{cm}} = R\omega$$

$$\mathrm{a}_{\mathrm{cm}} = R\alpha$$

$$f = \left(\frac{1}{2}MR^2\right)\left(\frac{\mathrm{a}_{\mathrm{cm}}}{R^2}\right) = \frac{1}{2}M\mathrm{a}_{\mathrm{cm}}$$

$$\mathrm{a}_{\mathrm{cm}} = \frac{2}{3}g\sin\theta$$

$$\mathrm{v}^2 = \mathrm{v}_0^2 + 2\mathrm{a}x$$

$$\mathrm{v}_{\mathrm{cm}}^2 = 2\mathrm{a}_{\mathrm{cm}}L$$

$$\mathrm{v}_{\mathrm{cm}}^2 = \frac{4}{3}g\left(\frac{h}{L}\right)L = \frac{4}{3}gh$$

$$\mathrm{v}_{\mathrm{cm}} = \sqrt{\frac{4}{3}gh}$$