Control Systems - 7^{th} Semester - Week 4

Step Response of Systems

Dr. Salman Ahmed

DCSE

Topics of last week

In last weeks, we studied about obtaining state-space models from differential equations

Then we studied about converting state-space model to transfer functions using formula

We also studied on converting transfer function to state-space model using canonical forms

Model - Recalling concepts

A model is representation or abstraction of reality/system.

Who invent model? We, human beings, invent model based on our knowledge.

This means the more knowledge a person has, the better he/she can write a model.

What is mathematical model? A set of equations (linear or differential) that describes the relationship between input and output of a system.

Types of Model

There are three types of mathematical models

- Black Box
- Grey Box
- White Box

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Introduction to Transient Analysis

Sometimes we cannot write white box models for the systems

Either we do NOT know what is inside the system or either the system is too complex to verify the components

Perhaps sometimes the components are not easily identifiable and their configuration or layout is not readable

So another way to obtain model of a system is to apply a *test* input signal and obtain the output signal

Standard Input Signals

Though there are many possible combinations of input signals, the following are famous or popular input signals

- Impulse Signal
- Step Signal
- Ramp Signal
- Parabolic Signal

Step Signal

The step signal is used to imitate the sudden change of a signal

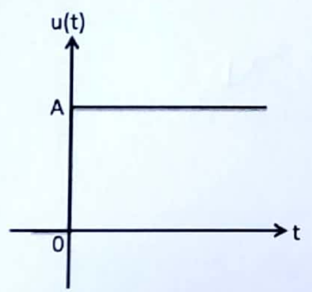

Figure: Step Signal

$$u(t) = \begin{cases} A & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

If $A = 1$, the step signal is called unit step signal

Step Response

Impulse response is the best response because transfer function is defined as impulse response of a system

However, in practical life generating impulse signal is not easy

We use unit step signal as test input signal and then analyze the output of the system

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First Order System

A first order system has a single pole (irrespective of number of zeros).

Many systems are of first order. Examples include

- velocity of a car on road
- control of angular velocity of rotating systems
- an RLC circuit with only one capacitor and no inductor
- an RLC circuit with only one inductor and no capacitor
- fluid flow in a pipeline
- level control in a tank
- pressure control in a gas cylinder

First Order System - Examples

Examples of first order system from computing system domain:

- speed control of dc motor in hard disk system
- time taken by queries in database management system, e.g.
  `SELECT * from STUDENT`
  `where ATTENDANCE_PERCENTAGE > 75;`
- time taken to read a temperature sensor interfaced with an embedded system
- energy consumed by an IoT device
- control of collision detection in networks e.g. CSMA/CA
- time taken by PS4 or Xbox one device to boot

Step Response of First Order System

A general first order system without zeros can be written as follows:

$$G(s) = \frac{b}{s+a}$$

Let $C(s)$ be the output of a system having transfer function $G(s)$ (expressed above). If the input to $G(s)$ is a unit step, then the output can be expressed as follows:

$$\text{Output Signal} = \text{Input Signal} \times \text{Transfer function}$$

We can further write the following:

$$C(s) = \text{Unit step signal} \times G(s)$$

$$C(s) = \frac{1}{s} \times \frac{b}{s+a} = \frac{b}{s(s+a)}$$

Step Response of First Order System

The term a is an important term. The inverse of a is called time constant i.e.

$$\tau = \frac{1}{a}$$

where τ is called time-constant of first order systems. For example compute τ of the following system:

$$G(s) = \frac{3}{s+2}$$

Here $\tau = \frac{1}{2} = 0.5$ and gain K is computed as $\frac{3}{2} = 1.5$

The value of gain K indicates the final steady-state value of the step response

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Step Response of First Order System

In order to compute transfer function from a plot, we need to define a few more terminologies

Rise Time: T_r, time taken to reach 90% or 0.9 of final value from 10% or 0.1. Mathematically:

$$T_r = \frac{2.2}{a}$$

Settling Time: T_s, time taken to stay within 2% of its final value (or reach 98% of final value). Mathematically:

$$T_s = \frac{4}{a}$$

Step Response of First Order System

Can you compute the transfer function?

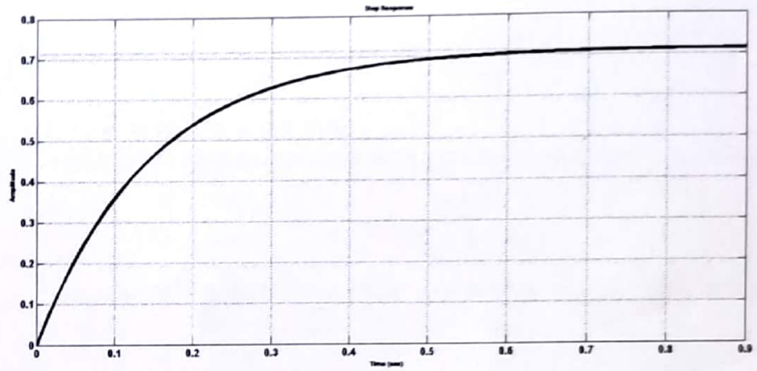

Figure: Step Response of a transfer function

Step Response of First Order System

Time constant: Time to reach 63% of final value. Compute the transfer function from the previous plot.

Final value = steady-state value = gain $K = 0.72$
63% of final value is $0.63 \times 0.72 = 0.4464$
Time taken to reach 0.45 value is 0.15 seconds
The final transfer function is

$$G(s) = \frac{\frac{0.72}{0.15}}{s + \frac{1}{0.15}}$$

OR

Pole is inverse of time constant which comes out to be $\frac{1}{0.15} = 6.67$

Another way of writing the transfer function is

$$G(s) = \frac{4.802}{s + 6.67}$$

Step Response of First Order System

The previous step-response was obtained for the following actual transfer function:

$$G(s) = \frac{5}{s+7}$$

MATLAB code for obtaining step response

```
num = [5] ;

den = [1 7] ;

step(num,den)
```

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Step Response of First Order System

Effects of decreasing time constant

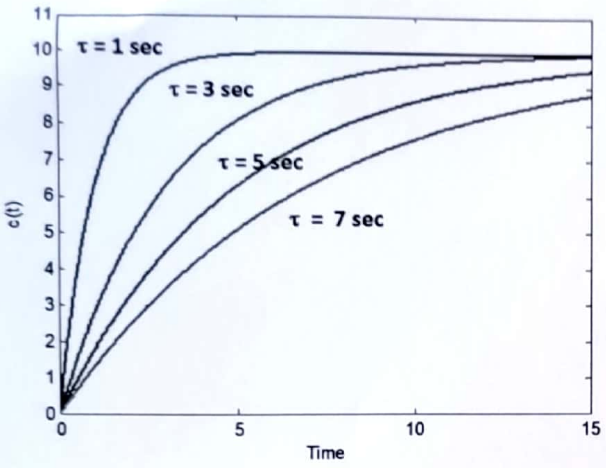

Figure: Effects of decreasing time constants of first order transfer function

Step Response of First Order System

Effects of increasing gains (remember its K not the term b)

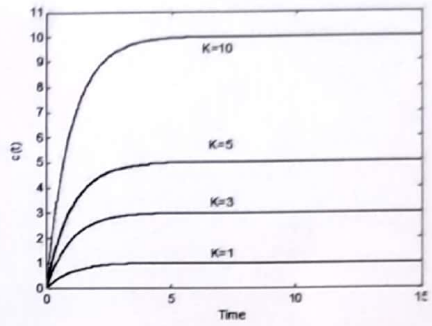

Figure: Effects of increasing gains of first order transfer function

Step Response of First Order System

Question: NADRA manages the registration database of Pakistani citizens. Previously, till **2001** people had **11** digit NIC numbers. Each citizen of Pakistan is issued a **13** digit CNIC number. The first **5** digits in a CNIC are based on a citizen locality, the next **7** numbers are random, and the final last digit is gender based (even for females, and odd for males). The current database entries in NADRA are estimated as **90** million. If a person details are required and his/her CNIC is entered in the NADRA database, its take **1** min to search **98%** of the records in a NADRA database. Assuming the query search process to be a first-order system, find the time constant.

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Linear Systems and Control - Week 5

Step Response of Second Order Systems

Dr. Salman Ahmed

Model - Recalling concepts

A model is representation or abstraction of reality/system.

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The following are famous or popular input signals

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Step Response of First Order System

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Many systems are of first order for example

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- an RLC circuit with only one inductor and no capacitor
- fluid flow in a pipeline
- level control in a tank
- pressure control in a gas cylinder

Step Response of First Order System

A first order system without zeros can be written as follows:

$$G(s) = \frac{b}{s+a}$$

The inverse of a is called time constant i.e.

$$\tau = \frac{1}{a}$$

The gain K is also called as dc-gain or steady-state gain of a system.

$$K = \frac{b}{a}$$

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Step Response of First Order System

Rise Time: T_r, time taken to reach 90% or 0.9 of final value from 10% or 0.1. Mathematically:

$$T_r = \frac{2.2}{a}$$

Settling Time: T_s, time taken to stay within 2% of its final value (or reach 98% of final value). Mathematically:

$$T_s = \frac{4}{a}$$

First Order Systems Summary

In first order system, we only have 2 parameters: dc gain and time-constant

Varying these two parameters only change the speed or amplitude of step response

Which parameter changes the speed of first order transfer function?

Which parameter changes the amplitude of first order transfer function?

Poles Location of Second Order System

A second order system has 2 poles. So, the following possibilities can occur:

- Both poles are real and equal
- Both poles are real and unequal
- Both poles are complex conjugate

Wait, one more possibility is also there

- Both poles are complex conjugate with real part equal to zero

General Second Order System

A general second order system can be written as follows:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ω_n is called the natural frequency of second order system and ζ is called damping ratio

ω_n is pronounced as omega n

ζ is pronounced as zeta

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General Second Order System

Analyze this second order transfer function and determine ω_n and ζ

$$G(s) = \frac{4}{s^2 + 2s + 4}$$

$\zeta = 0.5$ and $\omega_n = 2$

General Poles of Second Order System

You can apply quadratic formula and compute the poles of transfer function:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The poles of the transfer function are

$$-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

Response Types of Second Order System

Now, we can have the following four possibilities:

- Overdamped response: The system has two real poles which are unequal - in this case $\zeta > 1$
- Critically damped response: The system has two real poles which are equal - in this case $\zeta = 1$
- Underdamped response: The system has two complex conjugate poles with some real part - in this case $0 < \zeta < 1$
- Undamped response: The system has two imaginary poles with zero real part - in this case $\zeta = 0$

Poles Location of Over Damped Second Order System

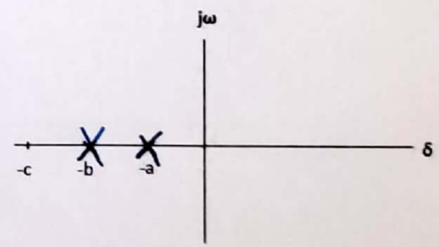

Figure: Over Damped System

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Poles Location of Critically Damped Second Order System

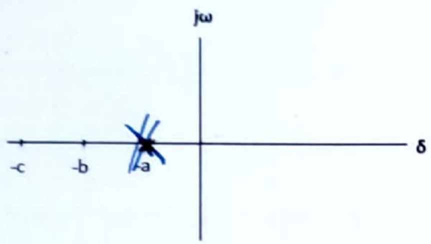

Figure: Critically Damped System

Poles Location of Under Damped Second Order System

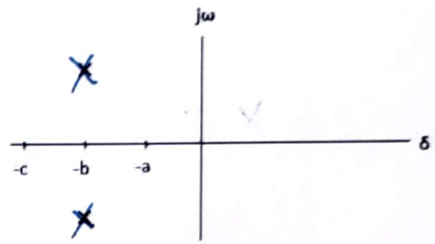

Figure: Under Damped System

Poles Location of Un Damped Second Order System

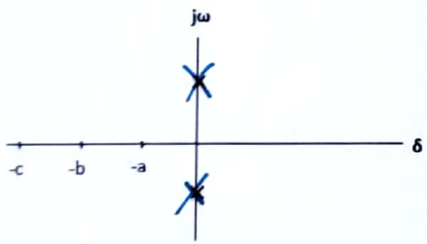

Figure: Un Damped System

Step responses of undamped second order systems

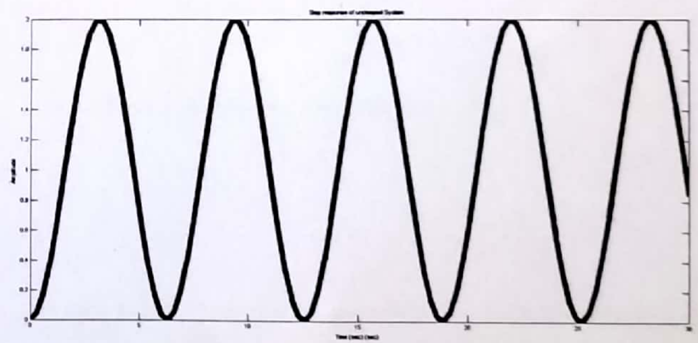

Figure: Step response of undamped system

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Step responses of under damped second order systems

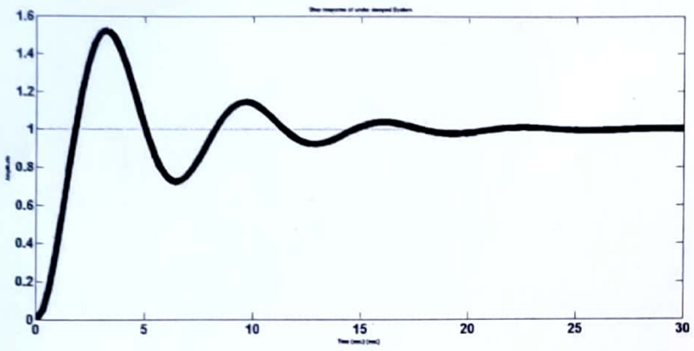

Figure: Step response of under damped system

Step responses of Critically Damped second order systems

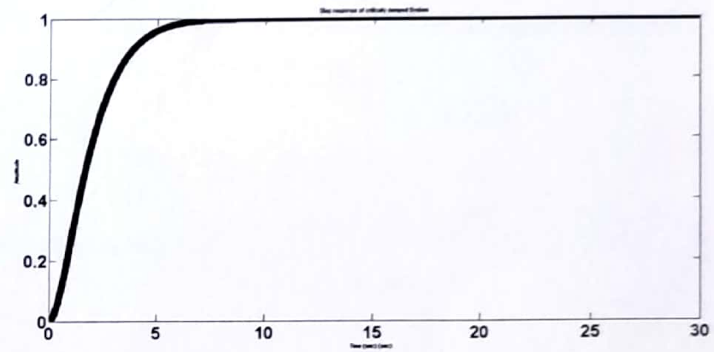

Figure: Step response of Critically damped system

Step responses of over damped second order systems

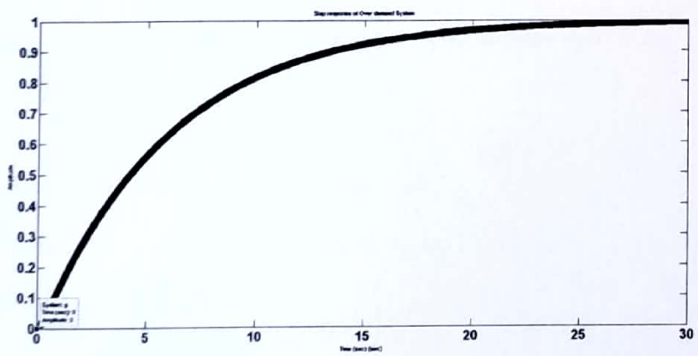

Figure: Step response of over damped system

Step response Analysis

The under damped system has interesting graph which needs more analysis

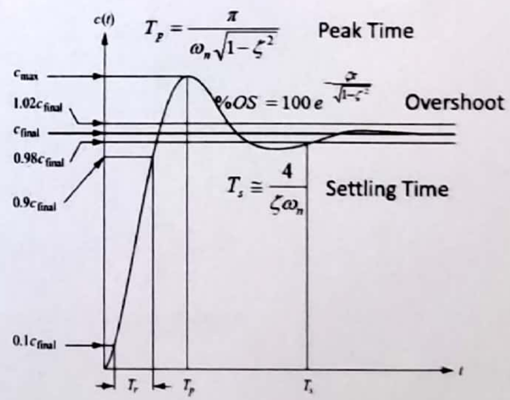

Figure: Step response of under damped system

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Second Order System Analysis

You should know (for examination purposes):

- The four types of step responses of second order systems
- Being able to identify from graph, the type of response

Analysis of under damped systems involve many formulae, and we will use MATLAB software to analyze and compute them

Second Order System Analysis - Example

Compute ζ and ω_n for the following transfer function

$$G(s) = \frac{36}{s^2 + 4.2s + 36}$$

Let us compare it with general form of second order systems

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Second Order System Analysis - Example of Mechanical System

Obtain transfer function of this system (assume $m = 3$, $k = 2$ and $b = 8$)

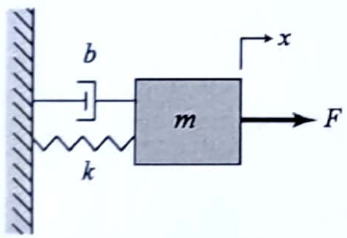

Figure: Obtain transfer function for this system

Second Order System Analysis - Example of Mechanical System

The transfer function is as follows:

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

$$\frac{X(s)}{F(s)} = \frac{1}{3s^2 + 8s + 2}$$

Now what we do, what is ω_n and what is ζ? Let us compare it with more general form.

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Second Order System Analysis - Example of Mechanical System

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Now, we have the following:

$$\frac{X(s)}{F(s)} = \frac{1}{3s^2 + 8s + 2}$$

Let us first eliminate the term 3 from this transfer function.

$$\frac{X(s)}{F(s)} = \frac{1/3}{3/3s^2 + 8/3s + 2/3}$$

$$\frac{X(s)}{F(s)} = \frac{1/3}{s^2 + 8/3s + 2/3}$$

Second Order System Analysis - Example of Mechanical System

$$\frac{X(s)}{F(s)} = \frac{1/3}{s^2 + 8/3s + 2/3}$$

Comparing with the standard form, we obtain $\omega_n^2 = 2/3$, which gives us $\omega = 0.8165$. Next, we compute the dc-gain of the transfer function which can be obtained from the numerator part as follows:

$$K\omega_n^2 = \frac{1}{3}$$

$$K \times \frac{2}{3} = \frac{1}{3}$$

which gives us $K = 1/2 = 0.5$. So in the step-response, the steady-state amplitude would be 0.5.

Second Order System Analysis - Example of Mechanical System

Let us determine ζ now, which can be computed as follows:

$$(2)(\zeta)(\omega_n) = \frac{8}{3}$$

$$(2)(\zeta)(0.8165) = \frac{8}{3}$$

which gives us $\zeta = 1.6330$. Now based on ζ, what would be the type of step response (underdamped or overdamped or undamped or critically damped)

Response Types of Second Order System

Now, we can have the following four possibilities:

- Overdamped response: The system has two real poles which are unequal - in this case $\zeta > 1$
- Critically damped response: The system has two real poles which are equal - in this case $\zeta = 1$
- Underdamped response: The system has two complex conjugate poles with some real part - in this case $0 < \zeta < 1$
- Undamped response: The system has two imaginary poles with zero real part - in this case $\zeta = 0$

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Second Order System Analysis - Example of Mechanical System

So, for the mechanical system, the response type will be over damped and the poles would be real and unequal. Let us use MATLAB to verify the same.

$$-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

Step Response of Second Order System - MATLAB Analysis

MATLAB code for obtaining step response

```
num = [1] ;

den = [3 8 2] ;

step(num,den)
```


Step Response of Second Order System - MATLAB Analysis

MATLAB code for analyzing step response

```
zeta=1.6330;

omegan=0.8165;

-zeta*omegan + (omegan*sqrt(zeta*zeta-1))

-zeta*omegan - (omegan*sqrt(zeta*zeta-1))
```


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Linear Systems and Control - Week 6

Block Reduction of Complex Systems

Dr. Salman Ahmed

Contents that we have covered till now

We studied the following topics till now:

- Converting state-space to transfer function using formula
- Converting transfer functions to state-space models using canonical forms
- Obtaining transfer functions and state-space models for RLC circuits
- Obtaining transfer functions and state-space models for mechanical system
- Analyzing step responses of first order systems (time constant and dc-gain)
- Analyzing step responses of second order systems (underdamped, undamped, over damped, critically damped)

Contents that we will cover before mid term exam

We will study the following topics before mid term exam

- Block reduction of complex systems (today lecture)
- Stability of systems in time-domain and transfer function domain

Block reduction algebra

First we analyze a simple transfer function block.

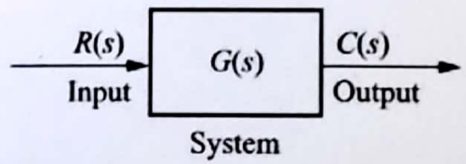

Figure: Transfer function block

The input signal is denoted by $R(s)$ and output signal by $C(s)$. We can write the following:

$$C(s) = G(s)R(s)$$

Sometimes, we skip the term (s) and write the following abusive notation:

$$C = GR$$

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Block reduction algebra - Summer

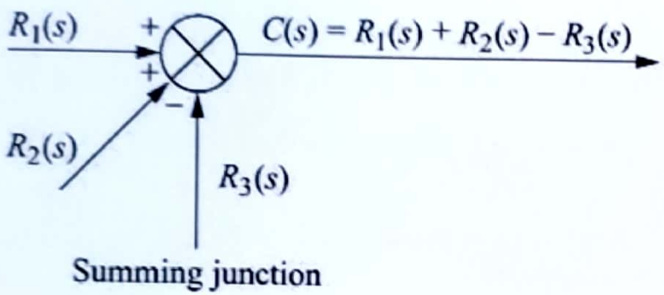

Figure: Summing Junction Symbol

Block reduction algebra - Pick off point

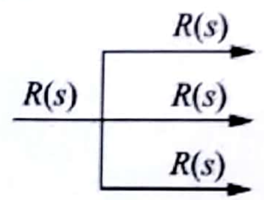

Figure: Pick Off point

Types of known interconnections

There are 3 types of known interconnections which are

- Series interconnection
- Parallel interconnection
- Feedback interconnection

Let us study each interconnection first, and then we study a few operations on transfer functions

Series Interconnection

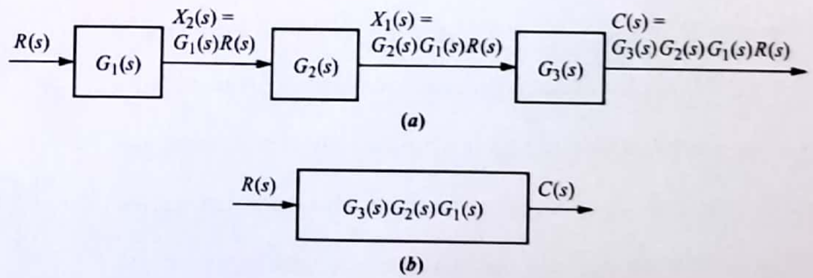

Figure: Series Interconnection of transfer functions

We can write the equivalent transfer function as $G_e = G_3G_2G_1$

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Parallel Interconnection

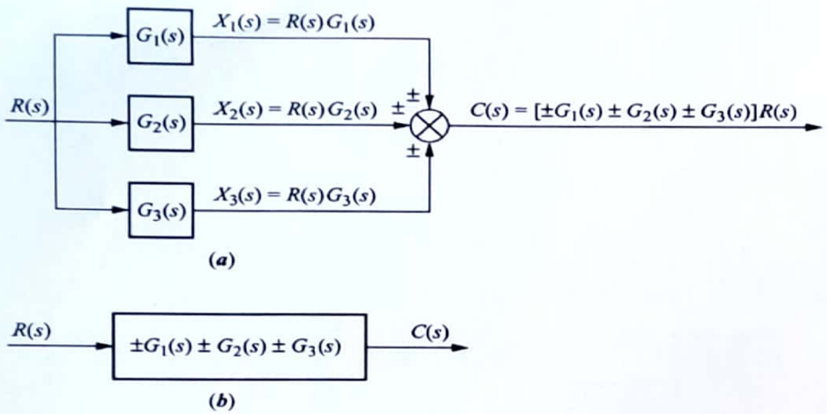


Figure: Parallel Interconnection of transfer functions

We can write $G_e = \pm G_3 \pm G_2 \pm G_1$

Few Important Points

Series interconnection involves product of transfer functions.

In parallel interconnection, be careful to identify the transfer functions correctly.

Two blocks are in parallel if they have same input signal and the output goes towards same summing junction.

Parallel interconnection involves sum or different of transfer functions.

Feedback Interconnection

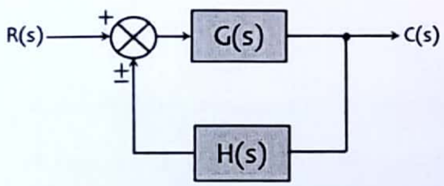
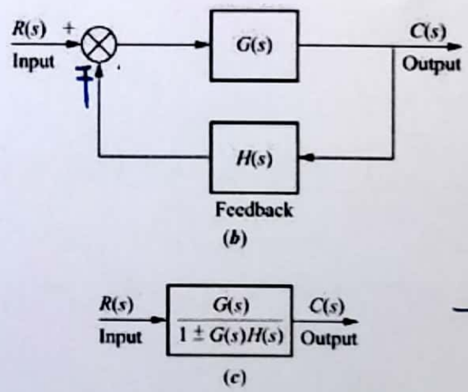


Figure: Feedback Interconnection of transfer functions

We write the following:

$$G_e = \frac{G}{1 \mp GH}$$

Feedback Interconnection



$\frac{G}{1 \pm GH}$

Figure: Feedback Interconnection of transfer functions

We can write $G_e = \frac{G}{1 \pm GH}$

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Operations with transfer functions

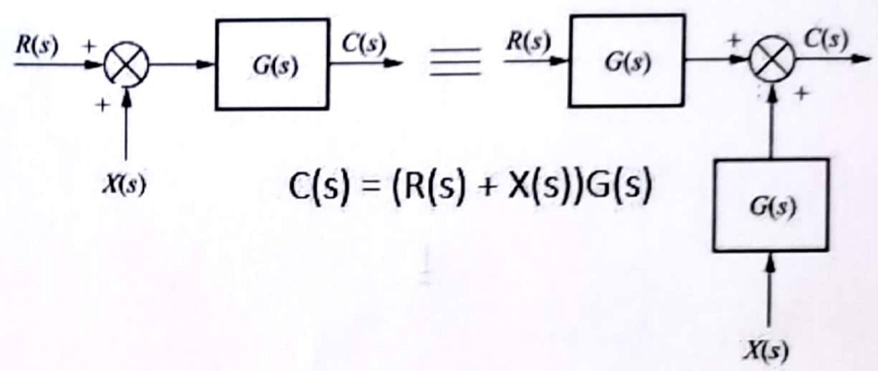
Sometimes in a complex system we can NOT easily identify series or parallel or feedback interconnections.

In those cases, we need to perform operations on transfer function block by either moving it before a point or summing junction or after a point/summing junction

Performing operations on transfer functions involve mathematics which we will study now.

There are 4 types of operations

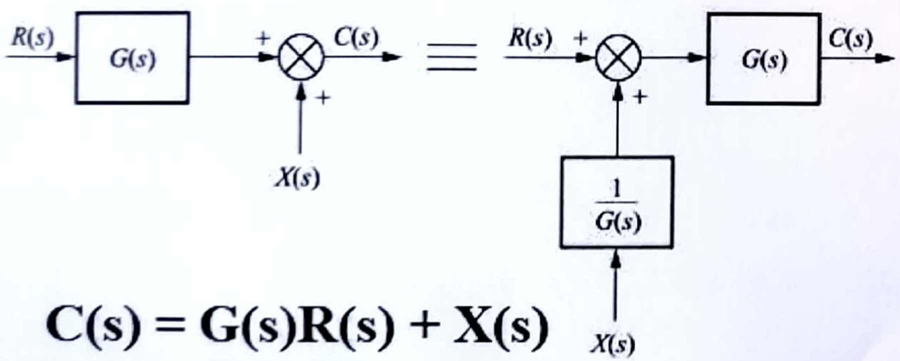
Moving summing junction after transfer function



$$C(s) = (R(s) + X(s))G(s)$$

Figure: Moving a summing junction after transfer function

Moving summing junction before transfer function



$$C(s) = G(s)R(s) + X(s)$$

Figure: Moving a summing junction before transfer function

Moving before pickoff point

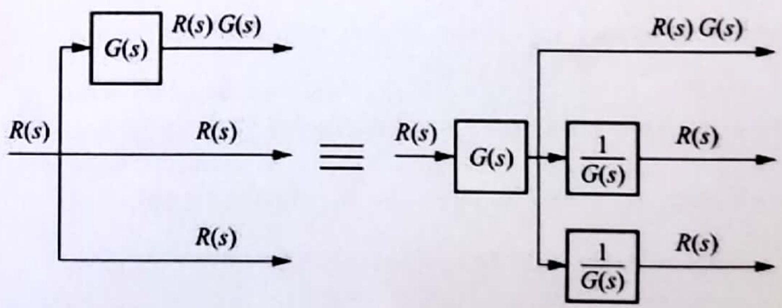


Figure: Moving before a pick-off point

Moving after pickoff point

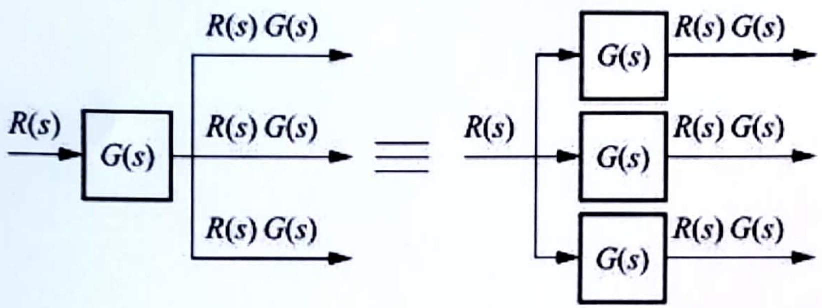


Figure: Moving after a pick-off point

Example 1 - Problem to solve

Can you obtain the transfer function, $\frac{C(s)}{R(s)}$?

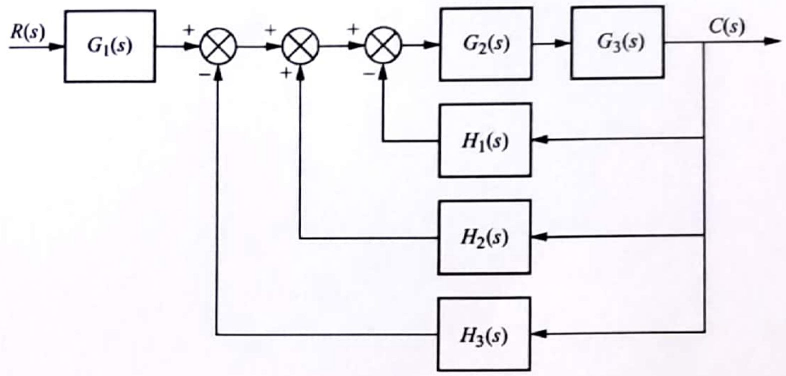


Figure: Example 1

Example 1 - Solution part a

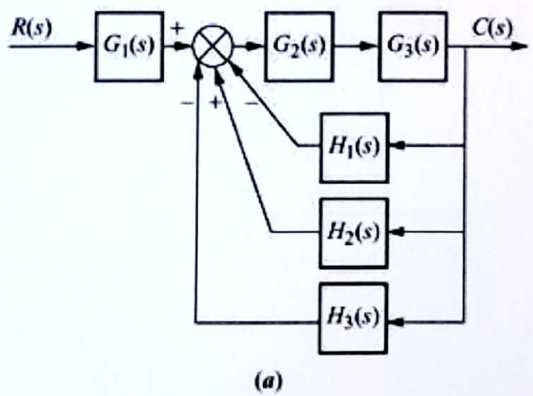


Figure: Example 1 - Solution part a

Example 1 - Solution part b

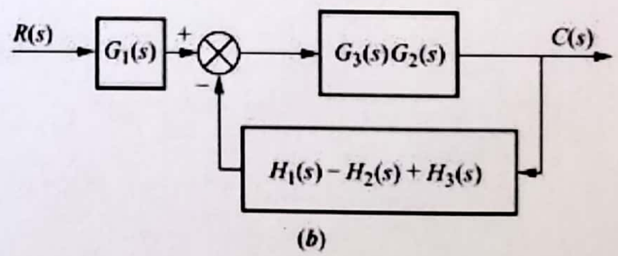


Figure: Example 1 - Solution part b

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Example 1 - Solution part c

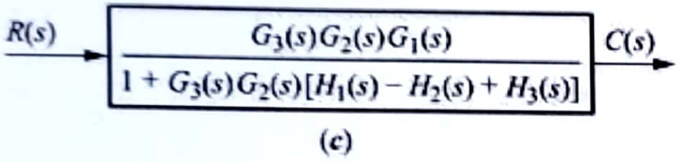

Figure: Example 1 - Solution part c

Example 2 - Problem to solve

Can you obtain the transfer function, $\frac{C(s)}{R(s)}$?

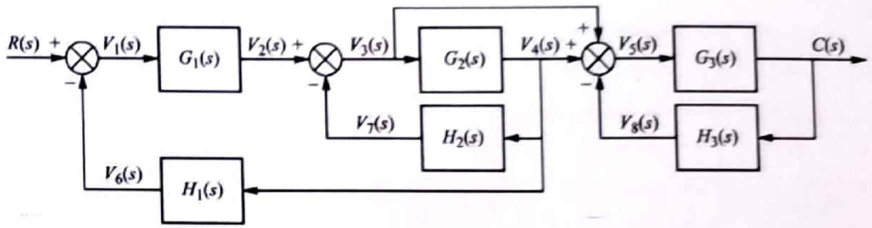

Figure: Example 2

Example 2 - Solution part a

Example 2 - Solution part b

Example 2 - Solution part c

Example 2 - Solution part d

Example 2 - Solution part e

Scanned with CamScanner