DCSE 7th Sem (CS)

Control Systems - 7^{th} Semester - Week 2

Mathematical Modeling of Systems

Dr. Salman Ahmed

Model

A model is representation or abstraction of reality/system.

Who invent model? We, human beings, invent model based on our knowledge.

This means the more knowledge a person has, the better he/she can write a model.

What is mathematical model? A set of equations (linear or differential) that describes the relationship between input and output of a system.

Types of Model and System

In mathematics, we broadly classify systems into 2 types, namely stochastic (random, probabilistic, uncertain) and deterministic (fixed relation between input and output)

To write model for a deterministic system, there are three techniques

- Black Box
- Grey Box
- White Box

Black Box Model

It is used when only input and output data are available

The internal dynamics are either too complex or totally unknown (sometimes for cyber security purposes, we do not want to label/show the hardware)

Input → **BLACK BOX** → **Output**

Figure: Black Box Model of a System

It is very hard to analyze or conclude something based on I/O data without having knowledge about the system

Grey Box Model

It is used when input and output data is known (known means labeled), plus some information (information means knowledge) about internal dynamics of the system are known

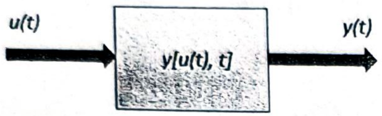

Figure: Grey Box Model of a System

In complex systems, we use grey box modeling to identify or estimate the system model

White Box Model

It is used when the input, output and internal dynamics of the system are known

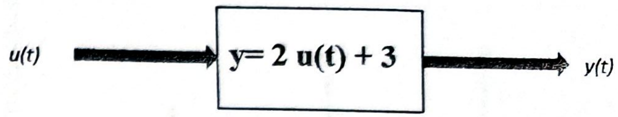

Figure: White Box Model of a System

White box models are very easy to predict any future values

Obtaining white box models requires us to know exact mathematical formulas and equations

Summary of Model Types

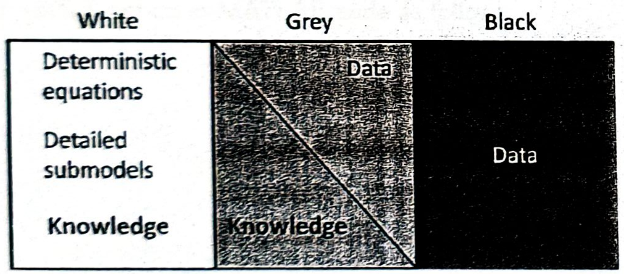

Figure: Techniques for obtaining models of a system

Equation Writing

In mathematics, we can write static equations as follows:

$$y = 2u + 3$$

If the equations are a function of time (means time-varying), then we equations as follows:

$$y(t) = 2u(t) + 3$$

Remember: If one variable is changing with time (like $u(t)$), then the whole equation changes with time. Mathematically, the following equation is incorrect

$$y = 2u(t) + 3$$

Equation Example

For example: if $u(t)$ is given as follows:

Time	Value
1	1
2	3
3	5
4	8

Table: Example of $u(t)$

MATLAB code for plotting the above signal $u(t)$

```
clear;

clc;

u=[1 3 5 8];

stem(u)
```


Equation Plot

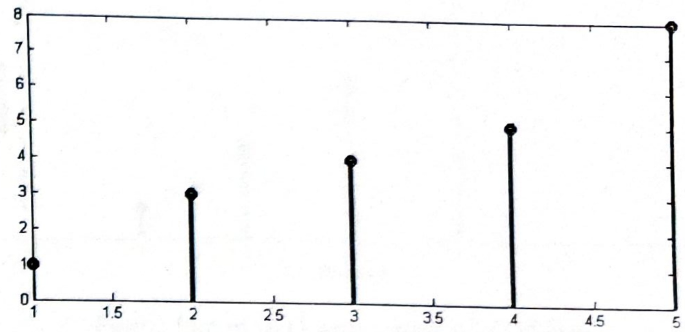

Figure: Plot of $u(t)$

Equation Plot

Let us put `axis` function in MATLAB code as follows:

```
clear;

clc;

u=[1 3 5 8];

stem(u)

axis([0 6 0 9])
```


Equation Plot

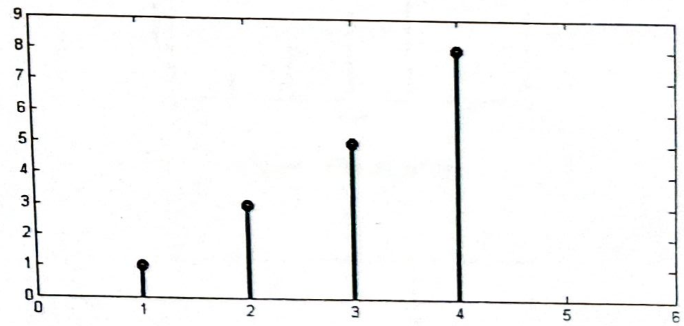

Figure: Plot of $u(t)$ with extended axis

Still missing something: the labels for x-axis and y-axis

Equation Plot

Let us put `xlabel` and `ylabel` function in MATLAB code as follows:

```
clear;
clc;
u=[1 3 5 8];
stem(u)
axis([0 6 0 9])
xlabel('Time (sec)') ; ylabel('Amplitude')
```


Equation Plot

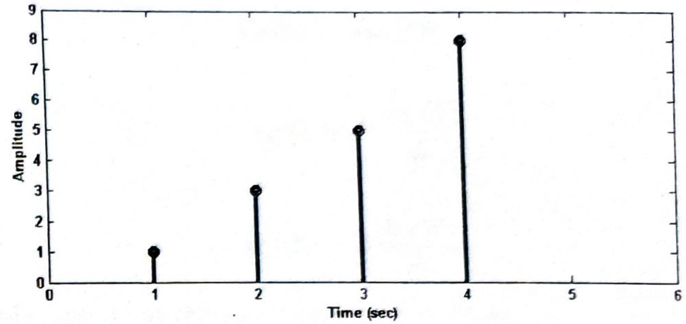

Figure: Plot of $u(t)$ with correct labels of axes

Now, we have the following equation:

$$y = 2u(t) + 3$$

Equation Plot

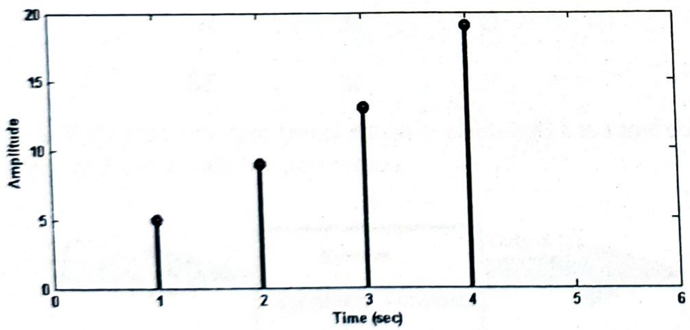

Figure: Plot of $y(t) = 2u(t) + 3$

Equation Plots

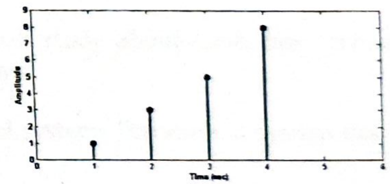

Figure: Plot of $u(t)$

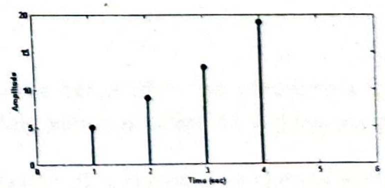

Figure: Plot of $y(t) = 2u(t) + 3$

Formulas Recap

Formulas used in electrical systems are as follows:

$$V_R = I_R R$$

$$i_c = C\frac{dv_c}{dt}$$

$$v_L = L\frac{di_L}{dt}$$

Formulas used in mechanical systems are as follows:

$$F = f_v\frac{dp}{dt}$$

$$F = Kp$$

$$F = Ma = M\frac{d^2p}{dt^2}$$

Formulas Recap

If the variables are time-varying, then the formulas are as follows:

$$V_R(t) = I_R(t)R$$

$$i_c(t) = C\frac{dv_c(t)}{dt}$$

$$v_L(t) = L\frac{di_L(t)}{dt}$$

Formulas used in mechanical systems are as follows:

$$F(t) = f_v\frac{dp(t)}{dt}$$

$$F(t) = Kp(t)$$

$$F(t) = Ma(t) = M\frac{d^2p(t)}{dt^2}$$

Formulas Recap

If you notice, the following six (6) terms remain constant:

$$R \quad , \quad L \quad , \quad C$$

$$M \quad , \quad K \quad , \quad f_v$$

So, we can state that a system (mechanical or electrical) has some constant parameters and some variable parameters

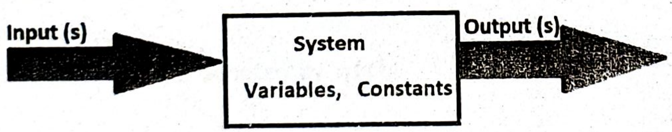

Figure: Graphical sketch of a system

System Modeling

In this week, we will study about white-box techniques used to model deterministic systems

To recap, in control systems literature, a system has

1. input
2. output
3. variables
4. constants

In order to obtain white-box models, we introduce a famous (very popular) technique of modeling which is called as state-space model

In state-space modeling of a system, we classify system elements (or parameters) as either constants or variables

System Modeling

To summarize again, we are studying state-space models of deterministic systems

Inside a system, we have either constant or variable parameters

State-space variables: Those variables which completely describe the behavior of a system

State-space variables are abbreviated as *ss* variables (or sometimes state variables)

State-space Model

State-space variables are used to obtain mathematical model a system

In a system, we can have one or two or many state-space variables

If there is one (1) state-space variable, then we denote it by x

In case of more than one (1) state-space variables, then we stack them in a vector and denote it by x

State-space Model

In control systems literature, we use the symbol $u(t)$ to denote input and $y(t)$ to denote output.

Before showing you mathematics, I summarize again the main points:

1. input denoted by $u(t)$
2. output denoted by $y(t)$
3. variables
4. constants
5. state-space variables denoted by $x(t)$

If any parameter is just constant (not a function of time t), then we do not write the term (t)

State-space Model

The standard state-space model (or template) is expressed as follows:

$$\frac{dx}{dt} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

where

- $x(t)$ denotes the vector having state-space variables
- $\frac{dx}{dt}$ denotes the derivative of state-space variables
- $u(t)$ denotes input and
- $y(t)$ denotes the output

State-space Model

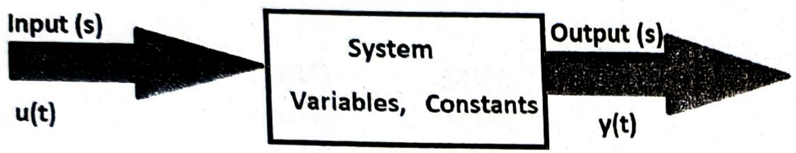

Figure: Graphical sketch of a system

State-space Model

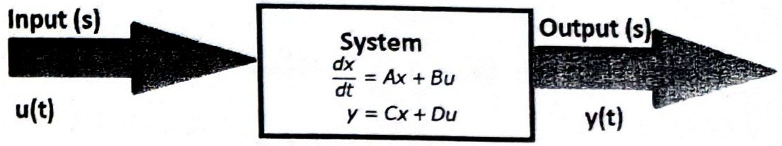

Figure: Graphical sketch of a system

State-space Model

Now, we will study examples to understand state-space model

In electric circuits: every circuit must have some input and some output

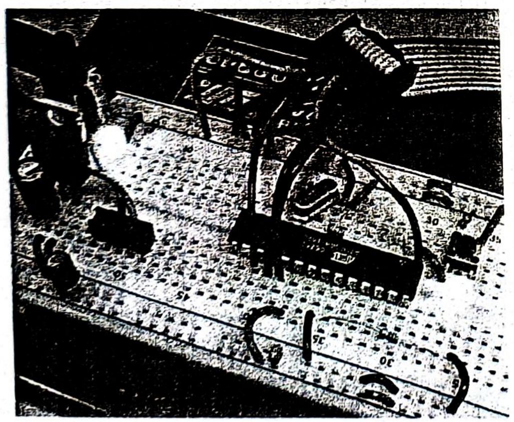

Figure: Circuit representation on bread-board

Electrical Circuit Model

Obtain the state-space model of the following circuit. Choose current across the resistor as output variable.

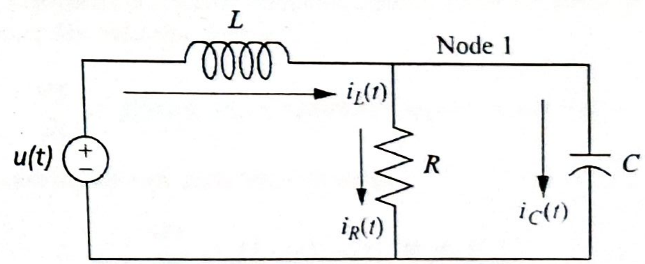

Figure: RLC example used to obtain state-space model

Solution: Step 1. Identify input, output, variables and constants in this circuit.

Terminologies

Input: $u(t)$

Output: $i_R(t)$

Variables: (Total 6)

$$\begin{matrix} v_R(t) & i_R(t) \\ v_C(t) & i_C(t) \\ v_L(t) & i_L(t) \end{matrix}$$

Constants: (Total 3)

$$R \quad L \quad C$$

Next question: Identify state-space variables. Let us write formulas for all 6 variables

Electrical Circuit Formulas

$$v_R(t) = i_R(t)R$$

$$i_R(t) = \frac{v_R(t)}{R}$$

$$i_C(t) = C\frac{dv_c}{dt}$$

$$v_C(t) = \frac{1}{C}\int_0^t i_C dt$$

$$v_L(t) = L\frac{di_L}{dt}$$

$$i_L(t) = \frac{1}{L}\int_0^t v_L dt$$

Now, which variables derivatives are used in formulas:

$$v_C(t) \qquad \text{and} \qquad i_L(t)$$

So, these 2 variables are state-space variables.

Electrical Circuit Formulas

Our state-space variable x will be a vector in this case, and expressed as follows:

$$x(t) = \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}$$

$$\frac{dx}{dt} = \begin{bmatrix} \frac{dv_C}{dt} \\ \frac{di_L}{dt} \end{bmatrix}$$

Alternatively, some one may write the following also (which is perfectly correct):

$$x(t) = \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix}$$

$$\frac{dx}{dt} = \begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv_C}{dt} \end{bmatrix}$$

Standard state-space template equations

Now the standard state-space template demands that we write (or obtain) equations in the following formats:

$$\frac{dx}{dt} = f(\text{state-space variables, inputs, constants})$$

In other words, we can state the following:

$$\frac{dx}{dt} = f\Big(x(t), u(t), R, L, C\Big)$$

First state-space equation

Let us write the equations in our example and then we compare it with standard format

$$\frac{dv_C}{dt} = \frac{1}{C} i_C(t)$$
$$\frac{di_L}{dt} = \frac{1}{L} v_L(t)$$

Let us analyze the first equation and each term in the first equation (I have ignored writing (t) and switched to abusive notation)

$$\frac{dv_C}{dt} = \frac{1}{C} i_C$$

When analyzing each term on right side of the equation, we can say

- C is constant and $\frac{1}{C}$ is also constant term
- i_C is not a constant, not an input, nor an output term

i_C is a problematic term and let us eliminate it

First state-space equation

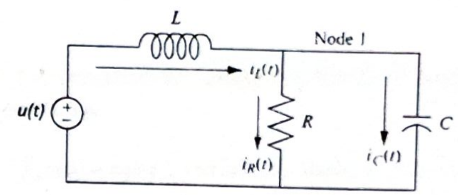

Figure: RLC example used to obtain state-space model

We can write the following:

$$i_L = i_R + i_C$$
$$i_C = i_L - i_R$$

Let us analyze again (by substituting it in the equation)

$$\frac{dv_C}{dt} = \frac{1}{C} i_C$$
$$\frac{dv_C}{dt} = \frac{1}{C}[i_L - i_R]$$

First state-space equation

Let us analyze each term in this equation:

$$\frac{dv_C}{dt} = \frac{1}{C}[i_L - i_R]$$

It seems i_R is problematic term, as $\frac{1}{C}$ is constant and i_L is state-space variable. Let us further solve i_R. We can write the following:

$$i_R = \frac{v_R}{R} = \frac{v_C}{R}$$

We obtain the first state-space equation as follows:

$$\frac{dv_C}{dt} = \frac{1}{C}\{i_L - \frac{v_C}{R}\}$$

$$\frac{dv_C}{dt} = \frac{i_L}{C} - \frac{v_C}{RC}$$

Second state-space equation

The second ss variable was i_L. Let us obtain the second state-space equation.

$$\frac{di_L}{dt} = f(\text{state-space variables, inputs, constants})$$

$$\frac{di_L}{dt} = f(i_L, v_C, \text{ inputs, R, L, C})$$

We have the following equation:

$$\frac{di_L}{dt} = \frac{1}{L} v_L$$

Let us analyze each term, v_L is problematic term and needs to be eliminated.

Second state-space equation

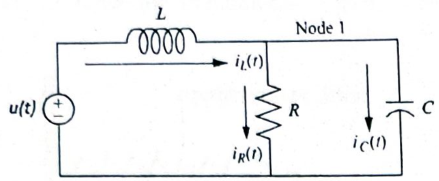

Figure: RLC example used to obtain state-space model

We can write the following:

$$u(t) = v_L + v_C$$
$$v_L = u(t) - v_C$$
$$\frac{di_L}{dt} = \frac{1}{L}v_L = \frac{1}{L}[u(t) - v_C]$$

Output equation

Now, let us write the equation for output in standard form. The output is current across resistor i_R

$$i_R = f(\text{state-space variables, inputs, constants})$$
$$i_R = \frac{v_R}{R} = \frac{v_C}{R}$$

As output is denoted by $y(t)$, so we can further write the following:

$$y = \frac{v_R}{R} = \frac{v_C}{R}$$

State-space Model

The state-space model is as follows:

$$\frac{dv_C}{dt} = \frac{i_L}{C} - \frac{v_C}{RC} \quad (1)$$
$$\frac{di_L}{dt} = \frac{1}{L}[u(t) - v_C] \quad (2)$$
$$i_R = \frac{v_C}{R} \quad (3)$$

Now, let us convert it to matrix form.

State-space Model

$$\begin{bmatrix} \frac{dv_c}{dt} \\ \frac{di_L}{dt} \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} u(t)$$
$$y = \begin{bmatrix} \cdot & \cdot \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + [\cdot]\, u(t)$$

Substituting values from equations, we obtain the following:

$$\begin{bmatrix} \frac{dv_c}{dt} \\ \frac{di_L}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t)$$
$$y = \begin{bmatrix} \frac{1}{R} & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + [0]\, u(t)$$

Example of Mechanical System 1

Write equation of the following mechanical system (assuming frictionless surface).

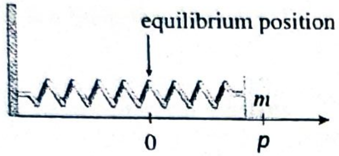

Figure: Mechanical System Example 1

Force on spring + Force on mass = 0

$$kp + m\frac{d^2p}{dt^2} = 0$$

Example of Mechanical System 2

Write equation of the following mechanical system (assuming frictionless surface).

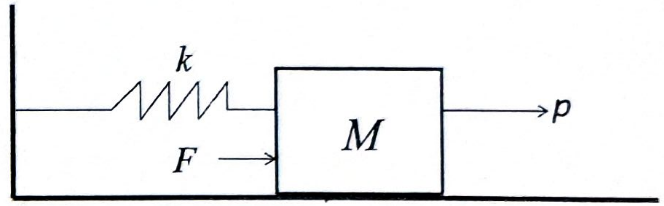

Figure: Mechanical System Example 2

$$kp + M\frac{d^2p}{dt^2} = F$$

Example of Mechanical System 3

Write equation of the automobile shock absorber system shown below

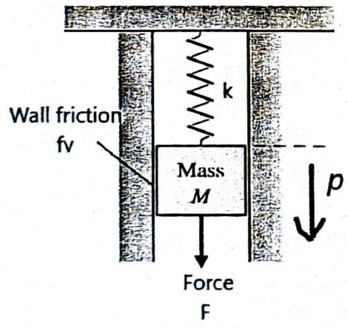

Figure: Mechanical System Example 3

Force on spring + Force on mass + Wall Friction = Force, F

$$kp + M\frac{d^2p}{dt^2} + f_v\frac{dp}{dt} = F$$

Example of Mechanical System 4

Write equation of the system shown below (here f_v is represented by b)

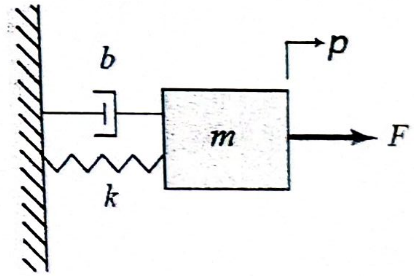

Figure: Mechanical System Example 4

Control Systems - 7^{th} Semester - Week 3

State-space Modeling of Systems

Dr. Salman Ahmed

Model of Systems

Last week, we studied about obtaining models for electrical systems

Last week, we also studied how to obtain differential equations for mechanical systems

This week we will study how to obtain state-space model from differential equations

Let us recall our lecture of last week in which we studied Example 2 of mechanical systems

Example 2 of Mechanical System

If you recall, this was from last week. Obtain state-space model (assuming frictionless surface)

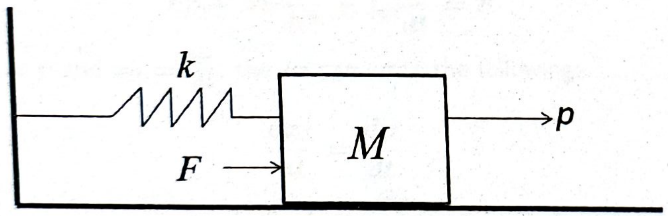

Figure: Mechanical System Example 2

$$kp + M\frac{d^2p}{dt^2} = F$$

To obtain state-space model, we need to know the input and output of the system.

Input $= F$ and output $= p$

State-space Model of Example 2

$$kp + M\frac{d^2p}{dt^2} = F$$

Let $x_1 = p$ and $x_2 = \frac{dp}{dt}$, the we can write the following:

$$\frac{dx_1}{dt} = \frac{dp}{dt}$$
$$\frac{dx_2}{dt} = \frac{d^2p}{dt^2}$$

We can further write the following:

$$\frac{dx_1}{dt} = \frac{dp}{dt} = x_2$$
$$\frac{dx_2}{dt} = \frac{d^2p}{dt^2} = \frac{F - kp}{M} = \frac{F}{M} - \frac{kx_1}{M}$$

State-space Model of Example 2

$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} = \frac{F}{M} - \frac{kx_1}{M}$$
$$y = x_1$$

Let us convert these 3 equations into matrix form

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} F$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\, F$$

Example 3 of Mechanical System

Obtain state-space model of the automobile shock absorber system shown below

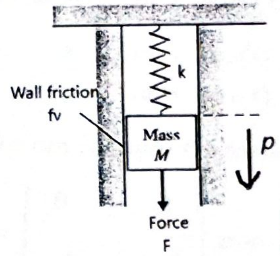

Figure: Mechanical System Example 3

Force on spring + Force on mass + Wall Friction = Force, F

$$kp + M\frac{d^2p}{dt^2} + f_v\frac{dp}{dt} = F$$

Let us write the input and output of this system.

Input $= F$ and output $= p$

State-space Model of Example 3

$$kp + M\frac{d^2p}{dt^2} + f_v\frac{dp}{dt} = F$$

Let $x_1 = p$ and $x_2 = \frac{dp}{dt}$, the we can write the following:

$$\frac{dx_1}{dt} = \frac{dp}{dt}$$
$$\frac{dx_2}{dt} = \frac{d^2p}{dt^2}$$

We can further write the following:

$$\frac{dx_1}{dt} = \frac{dp}{dt} = x_2$$
$$\frac{dx_2}{dt} = \frac{d^2p}{dt^2} = \frac{F}{M} - \frac{kp}{M} - \frac{f_v}{M}\frac{dp}{dt} = \frac{F}{M} - \frac{k}{M}x_1 - \frac{f_v}{M}x_2$$

State-space Model of Example 3

$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} = \frac{F}{M} - \frac{k}{M}x_1 - \frac{f_v}{M}x_2$$
$$y = z_1$$

Let us convert these 3 equations into matrix form

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{f_v}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} F$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\, F$$

State-space Model of Example 3

If we denote the derivative by the symbol $\dot{x} = \frac{dx}{dt}$, then let

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

The state-space that we obtained was as follows:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{f_v}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} F$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\, F$$

which can be written as follows:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{f_v}{M} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} F$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + [0]\, F$$

General template of *ss* model

The standard template for state-space model is as follows:

$$\dot{x} = Ax + Bu(t)$$
$$y = Cx + Du(t)$$

In the example case, we can conclude the following:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{f_v}{M} \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}}_{} F$$

$$A \quad x + Bu$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x + [0]\, F$$

$$C \quad x + Du$$

Converting *ss* to *tf*

The general form or template of *ss* model is as follows:

$$\dot{x} = Ax + Bu(t)$$
$$y = Cx + Du(t)$$

Let $G(s)$ denote the transfer function after converting to *tf* domain. The formula is:

$$G(s) = D + C[(sI - A)^{-1}B]$$

Example of conversion from *ss* to *tf*

Convert the following state-space model to transfer function

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 5 \\ 6 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} x$$

Let us first obtain $(sI - A)^{-1}$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$sI = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} s-1 & -2 \\ -3 & s-4 \end{bmatrix}$$

Example of conversion from *ss* to *tf*

$$sI - A = \begin{bmatrix} s-1 & -2 \\ -3 & s-4 \end{bmatrix}$$

Now let us find $(sI - A)^{-1}$

$$(sI - A)^{-1} = \frac{\text{adjoint}(sI - A)}{\det(sI - A)}$$

$$\text{adjoint}(sI - A) = \begin{bmatrix} s-4 & 2 \\ 3 & s-1 \end{bmatrix}$$

$$\begin{aligned} \det(sI - A) &= (s-1)(s-4) - (-2)(-3) \\ &= (s^2 - 5s + 4) - (6) \\ &= s^2 - 5s + 4 - 6 \\ &= s^2 - 5s - 2 \end{aligned}$$

$$(sI - A)^{-1} = \frac{\text{adjoint}(sI - A)}{\det(sI - A)} = \frac{1}{s^2 - 5s - 2} \begin{bmatrix} s-4 & 2 \\ 3 & s-1 \end{bmatrix}$$

Example of conversion from *ss* to *tf*

Next, we post-multiply with matrix B as follows:

$$(sI - A)^{-1} \times B = \frac{1}{s^2 - 5s - 2} \begin{bmatrix} s-4 & 2 \\ 3 & s-1 \end{bmatrix} \times \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$= \frac{1}{s^2 - 5s - 2} \begin{bmatrix} \left((s-4) \times 5\right) + \left(2 \times 6\right) \\ \left(3 \times 5\right) + \left((s-1) \times 6\right) \end{bmatrix}$$

$$= \frac{1}{s^2 - 5s - 2} \begin{bmatrix} 5s - 20 + 12 \\ 15 + 6s - 6 \end{bmatrix}$$

$$= \frac{1}{s^2 - 5s - 2} \begin{bmatrix} 5s - 8 \\ 6s + 9 \end{bmatrix}$$

Example of conversion from *ss* to *tf*

Now, let us pre-multiply with matrix C as follows:

$$\begin{aligned} C(sI - A)^{-1}B &= \frac{1}{s^2 - 5s - 2} \begin{bmatrix} 1 & 2 \end{bmatrix} \times \begin{bmatrix} 5s - 8 \\ 6s + 9 \end{bmatrix} \\ &= \frac{1}{s^2 - 5s - 2} \left[1 \times (5s - 8) + 2 \times (6s + 9)\right] \\ &= \frac{1}{s^2 - 5s - 2} \left[5s - 8 + 12s + 18\right] \\ &= \frac{1}{s^2 - 5s - 2} \left[17s + 10\right] \\ &= \frac{17s + 10}{s^2 - 5s - 2} \end{aligned}$$

Example of conversion from *ss* to *tf*

MATLAB code for conversion of *ss* to *tf*

```
A=[1 2; 3 4];

B=[5; 6];

C=[1 2];

D=[0];

[num,den] = ss2tf(A,B,C,D);

g=tf(num,den)
```

Conversion from tf to ss

Converting from tf to state-space is not a unique process

There are various techniques to convert form transfer function domain to state-space domain

We call each technique as canonical form. Let us study the first canonical form which is topic 3.5 in book

Conversion from tf to ss - Canonical Form 1

For a 2^{nd} order transfer function:

$$G(s) = \frac{b_1 s^1 + b_0}{s^2 + a_1 s + a_0}$$

We write the following state-space model (using Canonical Form 1):

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Conversion from tf to ss - Canonical Form 1

For a 3^{rd} order transfer function:

$$G(s) = \frac{b_2 s^2 + b_1 s^1 + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

We write the following state-space model (using Canonical Form 1):

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \qquad B = \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Conversion from tf to ss - Canonical Form 1

For a 4^{th} order transfer function:

$$G(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s^1 + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

We write the following state-space model (using Canonical Form 1):

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \qquad B = \begin{bmatrix} b_3 \\ b_2 \\ b_1 \\ b_0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

Conversion from *tf* to *ss* - Canonical Form 1

For n^{th} order transfer function:

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + ...b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_1 s + a_0}$$

We write the following state-space model (using Canonical Form 1):

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_{n-2} & -a_{n-1} \end{bmatrix} \qquad B = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

Conversion from *tf* to *ss* - Canonical Form 1

Example 3.4 Page 128: Convert the following transfer function to state-space domain

$$G(s) = \frac{24}{s^3 + 9s^2 + 26s + 24}$$

Solution: In this example, $a_0 = 24$, $a_1 = 26$, $a_2 = 9$ and $b_0 = 24$, we can obtain the following:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Conversion from *tf* to *ss* - Canonical Form 1

Example 3.5 Page 128: Convert the following transfer function to state-space domain

$$G(s) = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24}$$

Solution: Solution: In this example, $a_0 = 24$, $a_1 = 26$, $a_2 = 9$ and $b_0 = 2$, $b_1 = 7$ and $b_2 = 1$, we can obtain the following:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Conversion from *tf* to *ss* - Canonical Form 2

There is another canonical form which is as follows:

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + ...b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_2 s^2 + \ldots + a_1 s + a_0}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_{n-2} & -a_{n-1} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & b_2 & \ldots & b_{n-1} \end{bmatrix}$$

What is the difference between this canonical form and the previous one?

Matrix B in canonical form 2 seems like transpose of matrix C in the (previous) canonical form 1 and vice versa.

Conversion from *tf* to *ss* - Canonical Form 2

Example 3.5 Page 128: Convert the following transfer function to state-space domain using canonical form 2

$$G(s) = \frac{s^2+7s+2}{s^3+9s^2+26s+24}$$

Solution:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix}$$

Controller Canonical Form - Canonical Form 3

There is another canonical form called controller canonical form which is as follows:

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + ...b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_2 s^2 + \ldots + a_1 s + a_0}$$

$$A = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} b_{n-1} & b_{n-2} & b_{n-3} & \cdots & b_0 \end{bmatrix}$$

Conversion from *tf* to *ss* - Controller Canonical Form

Example 3.5 Page 128: Convert the following transfer function to state-space domain using controller canonical form

$$G(s) = \frac{s^2+7s+2}{s^3+9s^2+26s+24}$$

Solution:

$$A = \begin{bmatrix} -9 & -26 & -24 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 7 & 2 \end{bmatrix}$$

Controller Canonical Form - Canonical Form 4

Another canonical form is observer canonical form, which is as follows:

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + ...b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_2 s^2 + \ldots + a_1 s + a_0}$$

$$A = \begin{bmatrix} -a_{n-1} & 1 & 0 & 0 & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

Conversion from *tf* to *ss* - Observer Canonical Form

Example 3.5 Page 128: Convert the following transfer function to state-space domain using observer canonical form

$$G(s) = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24}$$

Solution:

$$A = \begin{bmatrix} -9 & 1 & 0 \\ -26 & 0 & 1 \\ -24 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

MATLAB code for conversion from *tf* to *ss*

MATLAB code for conversion of *tf* to *ss*

```
num=[1 7 2] ;

den=[1 9 26 24];

[A,B,C,D]=tf2ss(num,den)

Aobsv=A'

Bobsv=B'

Cobsv=C'

Dobsv=D'
```


Common mistake by students using MATLAB code

Students think they are good programmers. They think they may used short variables. So, use the alphabet n for num and d for den.

```
n=[1 7 2] ;

d=[1 9 26 24];

[a,b,c,d]=tf2ss(n,d)
```

Problem in above code: state-space matrix d and transfer function denominator d have same alphabets.

Next week topics

- If we need a model of a system, we need to obtain differential equation and then state-space model.
- We can convert state-space to transfer function. Can we obtain transfer function from differential equation?
- We already know that stability in *tf* domain is determined by poles.
- How about stability in state-space domain
- How about stability if we donot know the model of a system

Assignment for you

A hard disk drive (HDD) is a data storage device. It is used almost in every computing device including laptops, desktop computers, video game consoles, digital video recorders, mobiles and tablets. A hard disk stores data and nowadays we have too much data to store. Therefore, we require hard disks which can store more data (more data per square inch), which means the storage density of data is high. A hard disk uses magnetic storage system along with electronic hardware to access the data as shown in Figure below .

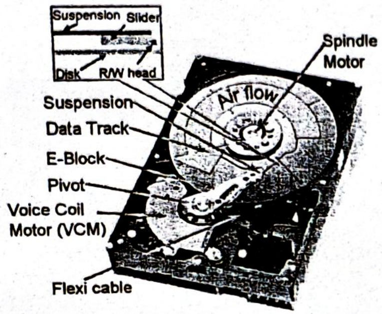

Figure: Hard disk drive schematic

Assignment for you

The electronic circuit of a hard disk consists of a dc motor. A dc motor has the following state-space:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\Theta} \\ \dot{i} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{-b}{J} & \frac{k}{J} \\ 0 & \frac{k}{L} & \frac{-R}{L} \end{bmatrix} \begin{bmatrix} \theta \\ \Theta \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \Theta \\ i \end{bmatrix}$$

The above state-space equation is taken from the website `http://ctms.engin.umich.edu/CTMS/index.php?example=MotorPosition&sectio ControlStateSpace`
Using the values of $J = 3.2$, $b = 3.5$, $k = 0.0274$, $R = 4$ and $L = 2.75$, perform the following 3 tasks:

- Convert the state-space model to transfer function
- Check the stability of the hard disk system (in state-space and transfer function)
- Can you compute the range of L such that the hard disk system would be stable