Discrete Structures

Syed Faisal Bukhari, PhD

Associate Professor

Department of Data Science (DDS), Faculty of Computing and Information Technology (FCIT), University of the Punjab (PU)

Text book

Discrete Mathematics and Its Application, 7^{th} Edition

Kenneth H. Rosen

References

Chapter 9

Discrete Mathematics and Its Application, 7th Edition

by Kenneth H. Rose

These slides contain material from the above resource.

Representing Relations

- Representing Relations using **Matrices**
- Representing Relations using **Digraphs**

Representing Relations Using Matrices

A **relation** between **finite sets** can be represented using a **zero–one matrix**. Suppose that R is a relation from $A = \{a_1, a_2, . . . , a_m\}$ to $B = \{b_1, b_2, . . . , b_n\}$. (Here the elements of the sets A and B have been listed in a particular, but arbitrary, order. Furthermore, when $A = B$ we use the same ordering for A and B.) The relation R can be represented by the matrix $\mathbf{M}_R = [m_{ij}]$,

$$\text{Where } M_{ij} = \begin{cases} 1 = \text{if } (a_i, b_j) \in R, \\ 0 = \text{if} (a_i, b_j) \notin R. \end{cases}$$

In other words, the **zero–one matrix** representing R has a **1** as its (i, j) entry when a_i is related to b_j, and a **0** in this position if a_i is not related to b_j. (Such a representation depends on the orderings used for A and B.)

Examples of Representing Relations Using Matrices

Example: Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

Solution:

A **= {1, 2, 3}** and ***B*** **= {1, 2}.**

$\boldsymbol{A \times B = \{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2)\}}$

Because ***R*** **= {(2, 1), (3, 1),(3, 2)},** the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Examples of Representing Relations Using Matrices (*cont.*)

Example: Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that:

$R = \{(a_1, b_2), (a_2, b_1),(a_2, b_3), (a_2, b_4),(a_3, b_1), \{(a_3, b_3), (a_3, b_5)\}.$

Matrices of Relations on Sets

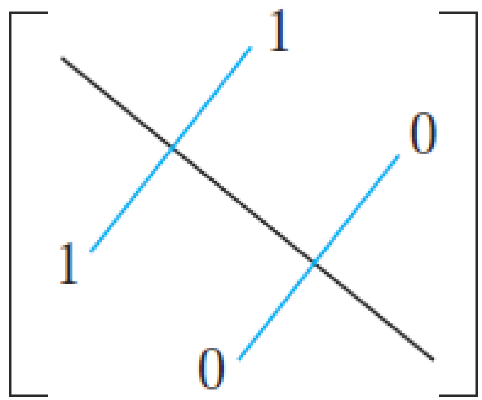
The matrix of a relation on a set, which is **a square matrix**, can be used to determine whether the **relation** has certain properties.

- If ***R*** is a **reflexive relation**, all the elements on the **main diagonal** of M_R are equal to **1**.

$$\begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}$$

Note that the elements off the main diagonal can be either **0 or 1**.

- ***R*** is a **symmetric relation**, if and only if $\boldsymbol{m_{ij} = 1}$ whenever $\boldsymbol{m_{ji} = 1}$



In terms of the entries of M_R, *R* is **symmetric** if and only if $\boldsymbol{m_{ji} = 1}$ whenever $\boldsymbol{m_{ij} = 1}$. This also means $\boldsymbol{m_{ji} = 0}$ whenever $\boldsymbol{m_{ij} = 0}$. Consequently, ***R* is symmetric** if and only if $\boldsymbol{m_{ij} = m_{ji}}$, for all pairs of integers *i* and *j* with $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

R is symmetric if and only if

$\mathbf{M}_R = (\mathbf{M}_R)^t$

that is, if $\mathbf{M}_R$ is a symmetric matrix.

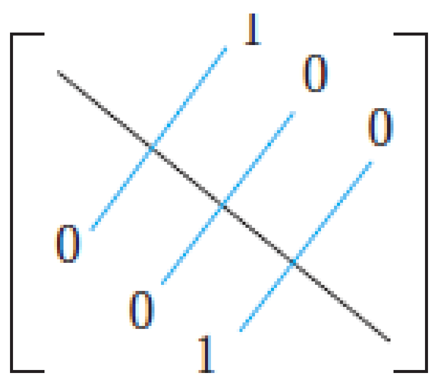
Consequently, the matrix of an antisymmetric relation has the property that if $m_{ij} = 1$ with $i = j$, then $m_{ji} = 0$. Or, in other words, either $m_{ij} = 0$ or $m_{ji} = 0$ when $i = j$.

The relation R is **antisymmetric** if and only if ***(a, b) ∈ R*** and ***(b, a) ∈ R*** imply that $a = b$.

Consequently, the matrix of an **antisymmetric relation** has the property that **if $m_{ij} = 1$** with $i \neq j$, **then** $m_{ji} = 0$.

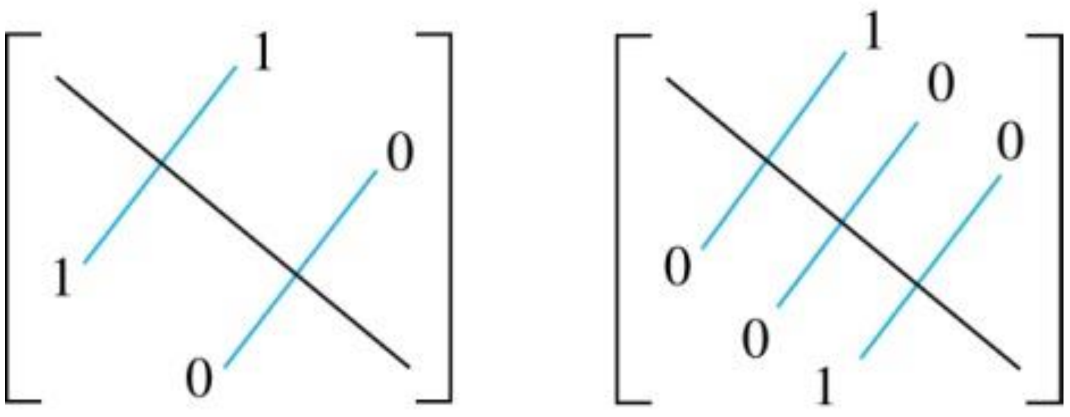
OR

R is an **antisymmetric relation**, if and only if **$m_{ij} = 0$ or $m_{ji} = 0$** when **$i \neq j$.**



Consequently, the matrix of an **antisymmetric relation** has the property that if ***m_{ij} = 1*** with ***$i \neq j$***, then ***m_{ji} = 0***. Or, in other words, either $m_{ij} = 0$ or $m_{ji} = 0$ when ***$i \neq j$*** .

Consequently, the matrix of an **antisymmetric relation** has the property that if $\boldsymbol{m_{ij} = 1}$ with $\boldsymbol{i \neq j}$, then $\boldsymbol{m_{ji} = 0}$. Or, in other words, either $\boldsymbol{m_{ij} = 0}$ **or** $\boldsymbol{m_{ji} = 0}$ when $\boldsymbol{i \neq j}$.



(a) Symmetric (b) Antisymmetric

Example of a Relation on a Set

Example 3: Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R **reflexive, symmetric, and/or antisymmetric**?

Solution:

Reflexive: Because all the diagonal elements are equal to 1, R is reflexive.

Symmetric: Because M_R is symmetric, R is symmetric $m_{1,2}$ and $m_{2,1}$ are 1

Antisymmetric: It is not antisymmetric because $m_{1,2}$ and $m_{2,1}$ are 1.

If $m_{1,2} = 1$ then $m_{2,1} = 0$ in order to make it antisymmetric

Transitive

Find the **non-zero entries** in M_R^2. If M_R=1 then M_R^2 = 1 then the relation is transitive. If M_R=1 then M_R^2 = 0 then the relation is not transitive. If (a, b) ∈ M_R^2, then (a, b) ∈ M_R

Note: Look for the non-zero entries in M_R^2 (say 1 or 2) then M_R must contain at least 1 in the corresponding position.

Example: Consider the following relation on {a, b, c}

$R = \{(a, a),(a, b),(b, c),(a, c)\}$. Is R is transitive?

Given

$A = \{a, b, c\}$

$R = \{(a, a),(a, b),(b, c),(a, c)\}.$

$$M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_R^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_R^2 = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example: Consider the following relation on {1, 2, 3, 4}

R = **{(2, 1), (3, 1), (3, 2),** (4, 1), (4, 2), (4, 3)}. Is R is transitive?

$R =$ **{(2, 1), (3, 1), (3, 2),** (4, 1), (4, 2), (4, 3)}

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \mathbf{1} & 1 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 1 & 0 \end{bmatrix}$$

$$M_R^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$M_R^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ \mathbf{2} & \mathbf{1} & 0 & 0 \end{bmatrix}$$

Union and Intersection

The matrix representing the union of these relations has a 1 in the positions where either $\mathbf{M}_{R1}$ or $\mathbf{M}_{R2}$ has a_1. The matrix representing the intersection of these relations has a 1 in the positions where both $\mathbf{M}_{R1}$ and $\mathbf{M}_{R2}$ have a 1. Thus, the matrices representing the union and intersection of these relations are

$\mathbf{M}_{R1 \cup R2} = \mathbf{M}_{R1} \vee \mathbf{M}_{R2}$ and $\mathbf{M}_{R1 \cap R2} = \mathbf{M}_{R1} \wedge \mathbf{M}_{R2}$

Example Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$\mathrm{M}_{R1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \; and \; \mathrm{M}_{R2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $\mathbf{M}_{R1\cup R2}$ and $\mathbf{M}_{R1\cap R2}$?

Solution: The matrices of these relations are

$\mathbf{M}_{R1 \cup R2} = \mathbf{M}_{R1} \vee \mathbf{M}_{R2}$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\mathbf{M}_{R1 \cap R2} = \mathbf{M}_{R1} \wedge \mathbf{M}_{R2}$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \wedge \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Representing Relations Using Digraphs

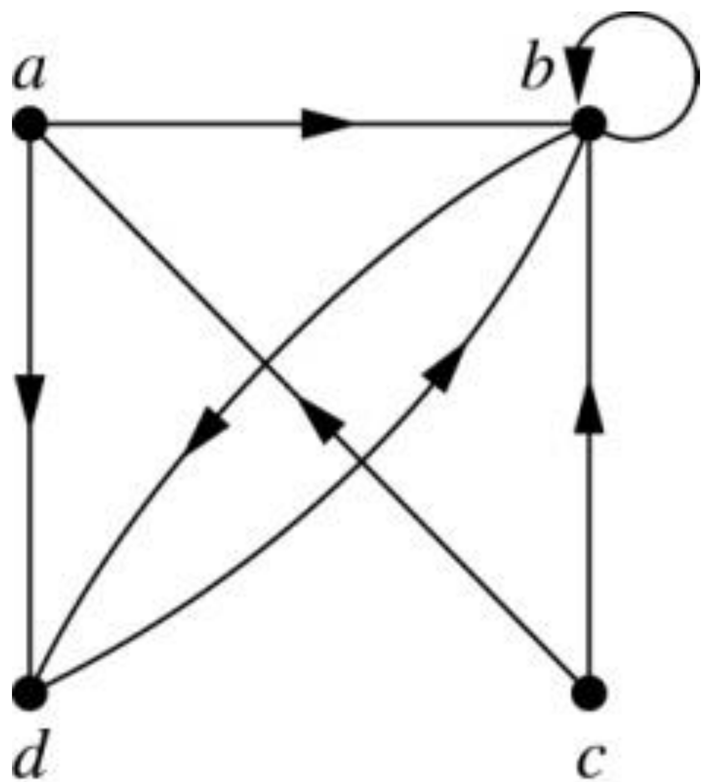
Definition: A **directed graph**, or **digraph**, consists of a set V of **vertices (or nodes)** together with a set E of ordered pairs of elements of V called **edges (or arcs).** The vertex **a** is called the **initial vertex** of the edge (a, b), and the **vertex b** is called the **terminal vertex** of this edge.

- An edge of the form *(a, a)* is represented using an arc from the vertex *a* back to itself. Such an edge is called a **loop**.

Example: A drawing of the directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), **(b, b),** (b, d), (c, a), (c, b), and (d, b) is shown in the next slide.

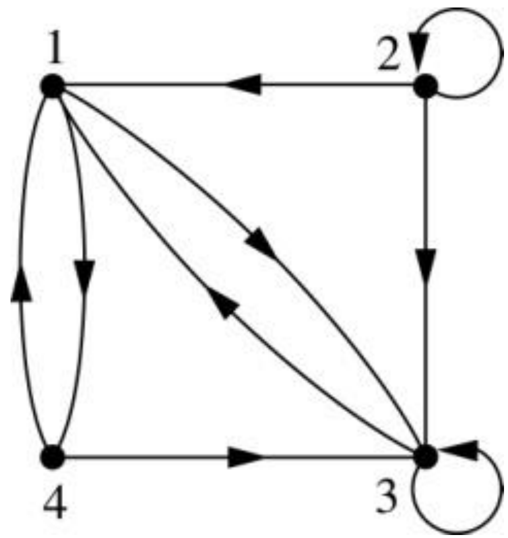
Solution

directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), **(b, b),** (b, d), (c, a), (c, b), and (d, b)

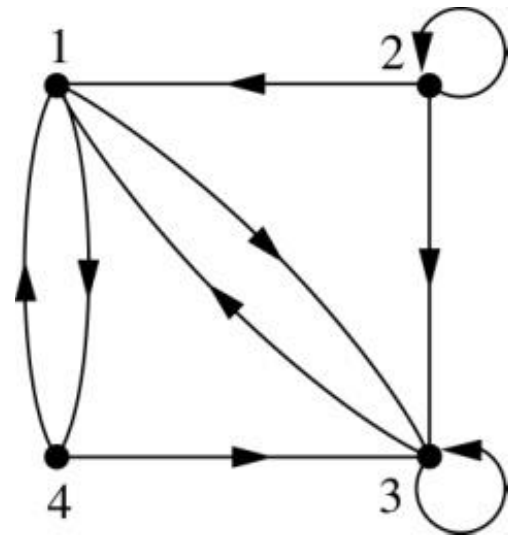


Examples of Digraphs Representing Relations

Example: What are the ordered pairs in the relation represented by this directed graph?



Solution:



The ordered pairs in the relation are

(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), and (4, 3)

Example The directed graph of the relation $R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$ on the set $\{1, 2, 3, 4\}$

Solution: The directed graph of the relation R = **{(1, 1),** (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)} on the set {1, 2, 3, 4}