Discrete Strutrues

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Text book

Discrete Mathematics and Its Application, 7th Edition

Kenneth H. Rosen

References

Discrete Mathematics and Its Application, 7th Edition
by Kenneth H. Rose

Discrete Mathematics with Applications
by Thomas Koshy

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by Cinda Heeren, Siebel Center

http://raider.mountunion.edu/ma/MA125/Fall2011/Chapter7/IntroToFunctions.html

Functions

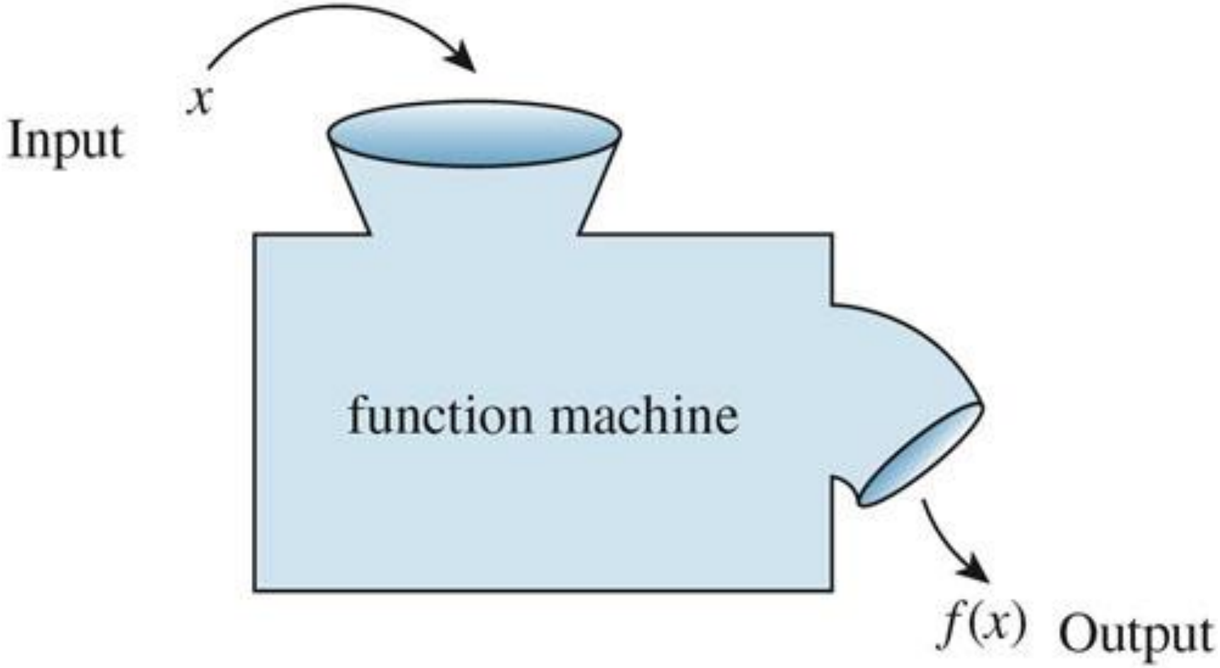
- The concept of a function is extremely important in mathematics and computer science.

- For example, in **discrete mathematics functions** are used in the definition of such **discrete structures as sequences** and **strings**.

- Functions are also used to represent how long it takes a **computer to solve problems of a given size**.

- Many computer programs and subroutines are designed to calculate **values of functions**.

Function Machines



Function Machines

One way to think of a **function is as a machine**. You drop a **domain element** into the input hopper, and it produces a **codomain** element from the output chute.

There is a rule or **formula hiding inside** the machine.

We do have to specify what the **domain** is for the rule, so we **don't drop things into the machine that might "break"** it. (It may not know how to handle certain inputs.)

Functions Defined by Formulas

1. **f : $\mathbb{R} \rightarrow \mathbb{R}$ defined by f(x) = x^2**

This is a well-defined function, since the rule inside the function machine can **handle all possible values of x** from the domain real numbers

2. **f : $\mathbb{R} \rightarrow \mathbb{R}$ defined by f(x) = $\sqrt{x}$**

This is not a **well-defined function** (that is, it is not a function!), since the rule inside the function machine can not handle all possible values of x from the domain real numbers

What if we changed the domain to $\mathbb{R}^+$?

Functions

- **Definition 1** Let A and B be nonempty sets. A *function* ***f*** from ***A*** to ***B*** is an assignment of exactly **one element** of ***B*** to **each element of *A***. We write ***f (a) = b*** if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

- **Remark:** Functions are sometimes also called **mappings** or **transformations.**

Functions

- In many instances **we assign to each element of a set a particular element of a second set** (which may be the same as the first).
- For example, suppose that each student in a **discrete mathematics class** is assigned a letter grade from the set **{A, B, C, D, F}.** And suppose that the grades are A for Adams, C for Chou, B for Good friend, A for Rodriguez, and F for Stevens.

Functions

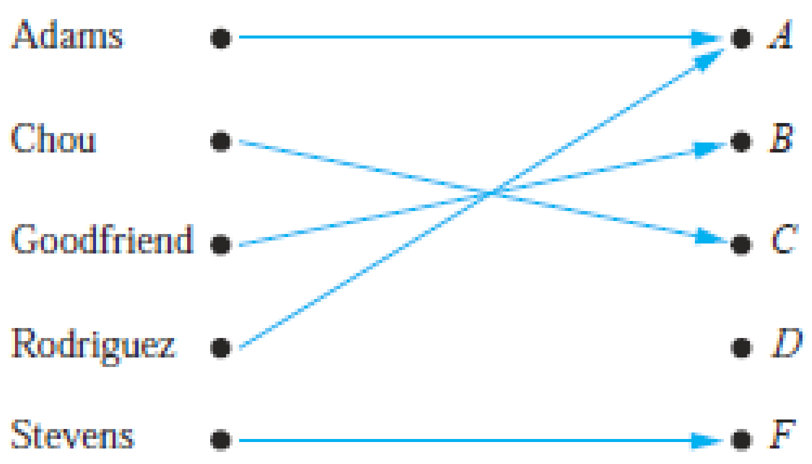


FIGURE 1 Assignment of Grades in a Discrete Mathematics Class.

Functions

- Functions are specified in many different ways. Sometimes we explicitly state the assignments, as in Figure 1.

- Often we give a **formula**, such as $f(x) = x + 1$, to define a function.

- Other times we use a **computer program** to specify a **function**.

Functions

- A function $f : A \to B$ can also be defined in terms of a **relation from *A* to *B***.

- Recall from that a **relation from *A* to *B*** is just a subset of ***A* × *B***.

- A relation from A to B that contains one, and **only one, ordered pair *(a, b)*** for **every element $a \in A$**, defines a function f from A to B.

- This function is defined by the assignment $f(a) = b$, where (a, b) is the **unique ordered pair** in the relation that has a as its first element

Functions

- **Definition 2** If f is a function from A to B, we say that A is the **domain** of f and B is the **codomain** of f. If f (a) = b, we say that **b** is the **image** of **a** and **a** is a **preimage** of b. The **range**, of f is **the set of all images of elements** of A. Also, if f is a function from A to B, we say that f maps A to B.

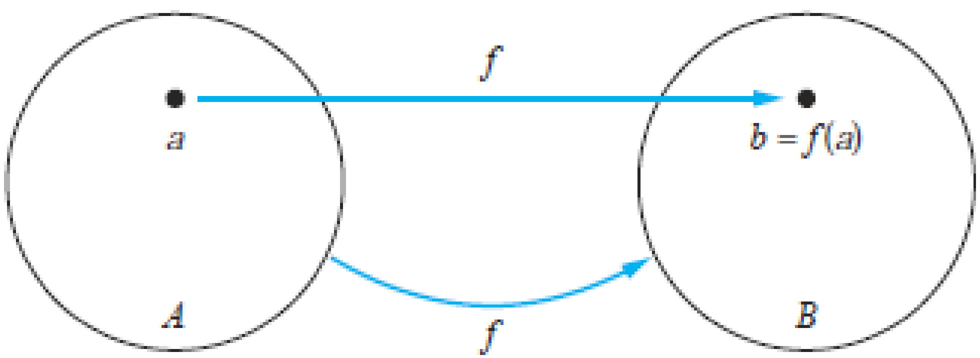


FIGURE 2 The Function f Maps A to B.

- When we define a function we specify its **domain**, its **codomain**, and the mapping of elements of the **domain to elements in the codomain**.

- Two functions are **equal** when they have the **same domain**, have **the same codomain**, and map each element of their common domain to the same element in their common codomain.

Note that if we change either **the domain or the codomain** of a function, then we obtain a **different function**. If we change the mapping of elements, then we also obtain a different function.

Domain, codomain, and range of a function

Example What are the domain, codomain, and range of the function that assigns grades to students described in Figure 1?

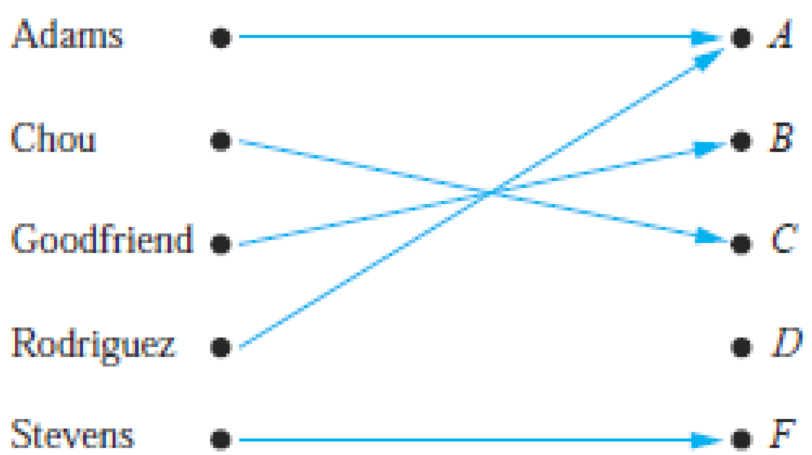


FIGURE 1 Assignment of Grades in a Discrete Mathematics Class.

Solution: Let **G** be the function that assigns a grade to a student in our discrete mathematics class.

Note that G(Adams) = A

domain of G = {Adams, Chou, Goodfriend, Rodriguez, Stevens}

codomain = {A, B, C, D, F}.

range of G = {A, B, C, F}, because each grade **except D** is assigned to some student.

Domain, codomain, and range of a function

ExampleLet R be the relation with ordered pairs (Abdul, 22), (Brenda, 24), (Carla, 21), (Desire, 22), (Eddie, 24), and (Felicia, 22). Here each pair consists of a graduate student and this student's age. Specify a function determined by this relation.

Solution:

f (x) is the age of x, where x is a student.

If f is a **function** specified by R, then

f (Abdul) = 22, **f (Brenda) = 24, f (Carla) = 21, f (Desire) = 22, f (Eddie) = 24**, and **f (Felicia) = 22**.

domain of F = {Abdul, Brenda, Carla, Desire, Eddie, Felicia}

codomain of F = contain all possible ages of students

Or

⇒ **codomain** of F = the set of positive integers less than 100

range of F = { 21, 22, 24}

Domain, codomain, and range of a function

Example: Let f be the function that assigns the last two bits of a bit string of length 2 or greater to that string.

For example,

$f(11010) = 10$.

Solution

domain of f = the set of all bit strings of length 2 or greater

codomain of f = {00, 01, 10, 11}.

range of f = {00, 01, 10, 11}.

Domain, codomain, and range of a function

Example: Let f: **Z** → **Z** assign the square of an integer to this integer. Then, $f(x) = x^2$. Find domain, codomain, and range of f.

Solution

domain of f = set of all integers

codomain of f = set of integers

range of f = the set of all integers that are perfect squares

or

range of f = {0, 1, 4, 9, . . . }.

Note: A function is called **real-valued** if its codomain is the set of real numbers, and it is called **integer-valued** if its codomain is the set of integers.

Domain, codomain, and range of a function

Example The domain and codomain of functions are often specified in programming languages. For instance, the **Java** statement

```
int floor(float real){. . .}
```

and the **C++** function statement

```
int function (float x){. . .}
```

both tell us that the **domain** of the **floor function** is the set of **real numbers** (represented by floating point numbers) and its **codomain** is **the set of integers**.

One-to-One and Onto Functions

Some functions never assign **the same value** to **two different domain elements**. These functions are said to be **one-to-one**.

.

One-to-One and Onto Functions

Definition 5 function f is said to be **one-to-one**, or an **injunction**, if and only if **f (a) = f (b)** implies that **a = b** for all a and b in the domain of f. A function is said to be **injective** if it is **one-to-one**

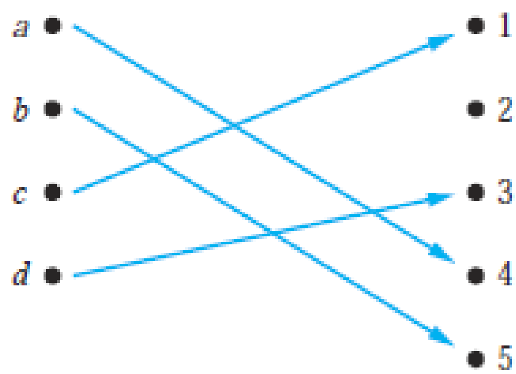


FIGURE 3 A One-to-One Function.

Note: A function f is one-to-one if and **only if $f(a) \neq f(b)$ whenever $a \neq b$**. This way of expressing that f is one-to-one is obtained by taking the contrapositive of the implication in the definition.

Example Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.

Solution

The function f is one-to-one because f takes on different values at the four elements of its domain

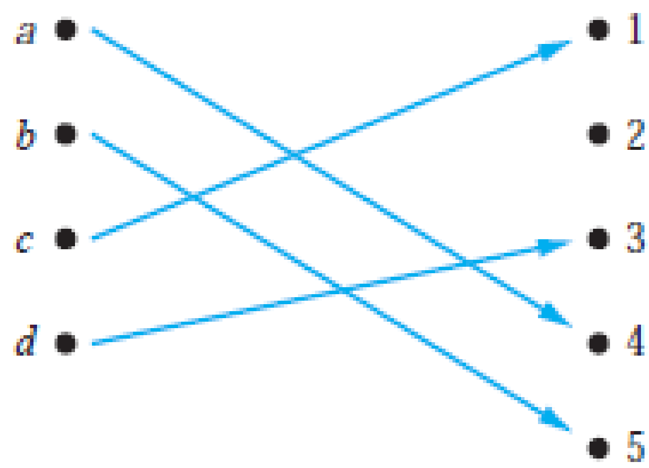


FIGURE 3 A One-to-One Function.

Addition and multiplication of functions

Two real-valued functions or **two integer valued functions** with the **same domain** can be added, as well as multiplied.

Definition 3 Let f_1 and f_2 be functions from A to **R**. Then $f_1 + f_2$ and f_1f_2 are also functions from A to **R** defined for all $x \in A$ by

1. $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
2. $(f_1f_2)(x) = f_1(x)f_2(x)$.

Example Let f_1 and f_2 be functions from **R** to **R** such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and f_1f_2?

Solution: From the definition of the sum and product of functions, it follows that

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$$

and

$$(f_1f_2)(x) = x^2(x - x^2) = x^3 - x^4$$

When f is a function from A to B, the image of a subset of A can also be defined.

Image

Example Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$, and $f(e) = 1$. What is the **image** of the **subset** $S = \{b, c, d\}$

Solution:

The **image** of the **subset** $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$.

Definition function f is said to be **one-to-one**, or an **injunction**, if and only if f (a) = f (b) implies that a = b for all a and b in the domain of f. A function is said to be *injective* if it is one-to-one

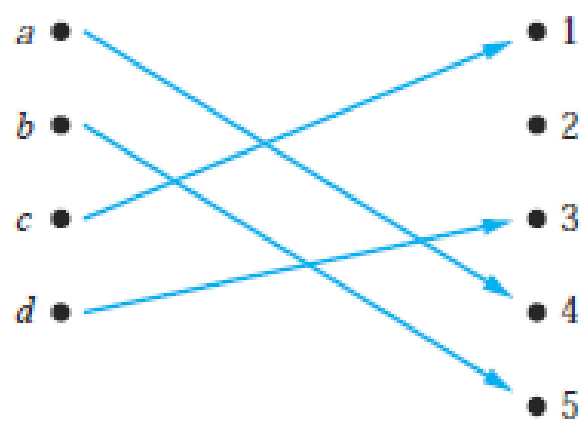


FIGURE 3 A One-to-One Function.

Determine whether the function *f* from {*a, b, c, d*} to {1, 2, 3, 4, 5} with ***f (a)* = 4, *f (b)* = 5, *f (c)* = 1**, and ***f (d)* = 3** is **one-to-one**.

Solution: The function f is one-to-one because f takes on different values at the four elements of its domain. This is illustrated in Figure 3.

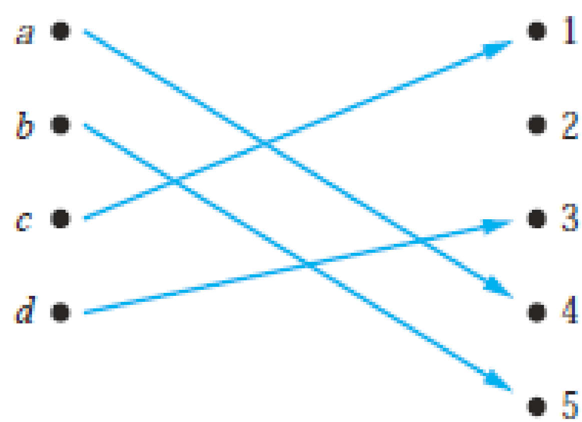


FIGURE 3 A One-to-One Function.

Example Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is **one-to-one**.

Solution:

The function $f(x) = x^2$ is **not one-to-one because**, for instance, $f(1) = f(-1) = 1$, but $1 = -1$.

Note that the function $f(x) = x^2$ with its domain restricted to $\mathbf{Z}^+$ is one-to-one.

Remark: We can express that f is one-to-one using quantifiers as

$$\forall a \forall b (f(a) = f(b) \rightarrow a = b)$$

or $\forall a \forall b (a \neq b \rightarrow f(a) \neq f(b))$,

where the universe of discourse is the domain of the function.

Example Determine whether the function $f(x) = x + 1$ from the set of real numbers to itself is one-to-one.

The function $f(x) = x + 1$ is a one-to-one function. To demonstrate this, note that

$x + 1 \neq y + 1$ when $x \neq y$.

Recall:

$\forall a \forall b(f(a) = f(b) \rightarrow a = b)$

or $\forall a \forall b(a \neq b \rightarrow f(a) \neq f(b))$

Increasing and Decreasing Functions

Definition A function f whose domain and codomain are subsets of the set of **real numbers** is called

1. **increasing if f (x) ≤ f (y),**
2. **strictly increasing** if f(x) < f(y), whenever x < y and x and y are in the domain of f.
3. **decreasing if f (x) ≥ f (y)**
4. **strictly decreasing** if f(x) > f(y), whenever x < y and x and y are in the domain of f.

Increasing and Decreasing Functions

- A function f is **increasing** if $\forall x \forall y(x < y \rightarrow f(x) \leq f(y))$
- **Strictly increasing** if $\forall x \forall y(x < y \rightarrow f(x) < f(y))$
- **Decreasing** if $\forall x \forall y(x < y \rightarrow f(x) \geq f(y))$
- **Strictly decreasing** if $\forall x \forall y(x < y \rightarrow f(x) > f(y))$, where the universe of discourse is the domain of f.

Onto Function

Definition A function f from A to B is called ***onto***, or a *surjection,* if and only if for **every element $b \in B$** there is an **element $a \in A$** with $f(a) = b$. A function f is called *surjective* if it is onto.

Example Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

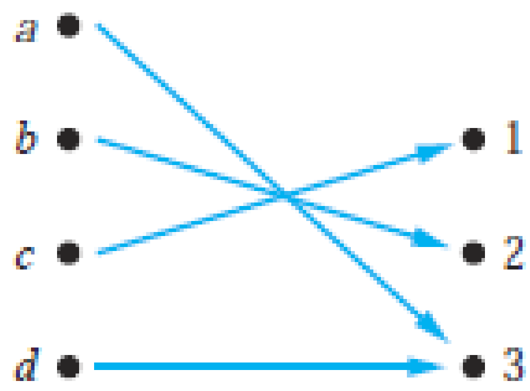


FIGURE 4 An Onto Function.

Because all **three elements of the codomain** are images of elements in the domain, we see that f is onto

Example of different types of correspondences

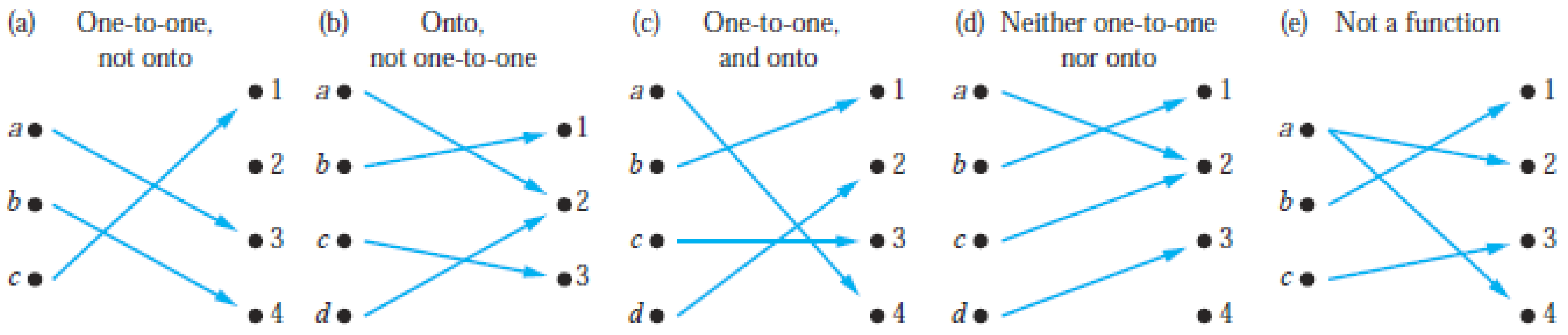


FIGURE 5 Examples of Different Types of Correspondences.

Inverse function

Definition Let f be a **one-to-one correspondence** from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that f (a) = b. The inverse function of f is denoted by f^{-1}

Hence, f

f^{-1} (a) = b when

f (a) = b.

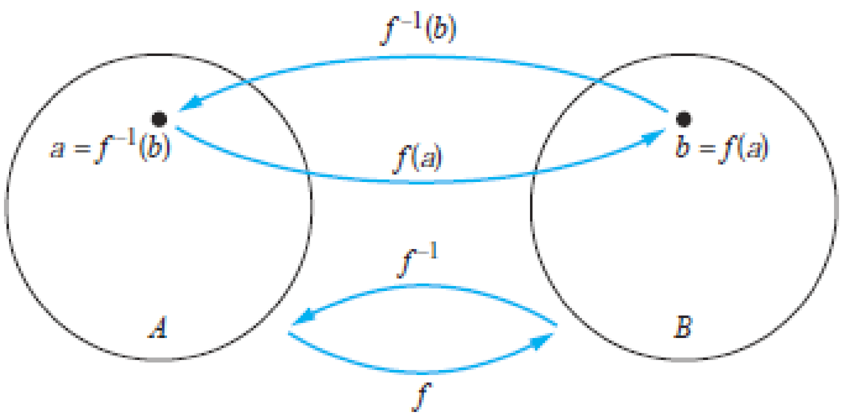


FIGURE 6 The Function f^{-1} Is the Inverse of Function f.

Inverse function

- If a function f is not a **one-to-one correspondence**, we cannot define an inverse function of f . When f is not a one-to-one correspondence, either it is **not one-to-one** or it is **not onto.**

- Note: A function f is a **one-to-one correspondence** if it is one-to-one and onto

Example Let f be the function from {a, b, c} to {1, 2, 3} such that f (a) = 2, f (b) = 3, and f (c) = 1. Is f invertible, and if it is, what is its inverse?

Solution

The function f is invertible because it is a **one-to-one correspondence**. The inverse function f^{-1} reverses the correspondence given by f ,

$f^{-1}(1) = c$

$f^{-1}(2) = a$

$f^{-1}(3) = b$

Let $f : \mathbf{Z} \to \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

The function f has an inverse because it is a one-to-one correspondence, as follows from the previous examples.

$y = x + 1.$

$\Rightarrow x = y - 1$

$f^{-1}(y) = y - 1$

Example Let f be the function from **R** to **R** with $f(x) = x^2$. Is f invertible?

$f(-2) = f(2) = 4$,

f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible

Composition of the functions

Definition Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted for all a ∈ A by f ∘ g, is defined by (f ∘ g)(a) = f (g(a)).

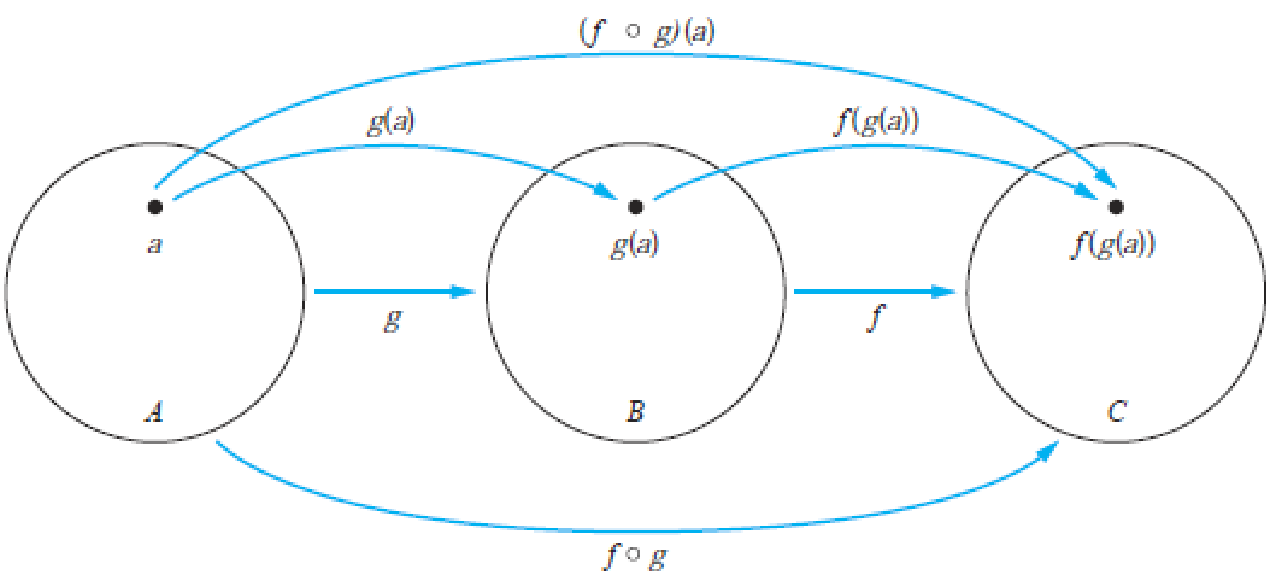


FIGURE 7 The Composition of the Functions f and g.

Composition of the functions

In other words, $f \circ g$ is the function that assigns to the element a of A the element assigned by f to g(a). That is, to find $(f \circ g)(a)$ we first apply the function g to a to obtain g(a) and then we apply the function f to the result g(a) to obtain $(f \circ g)(a) = f(g(a))$.

Note that the composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f

Example Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of f and g, and what is the composition of g and f ?

Solution

Given

g be the function from the set {a, b, c} to itself

g(a) = b, g(b) = c, and **g(c) = a**

f be the function from the set {a, b, c} to the set {1, 2, 3}

f (a) = 3, f (b) = 2, and **f (c) = 1**

$f \circ g = (f \circ g)(a) = f(g(a)) = f(b) = 2$

$f \circ g = (f \circ g)(b) = f(g(b)) = f(c) = 1$

$f \circ g = (f \circ g)(c) = f(g(c)) = f(a) = 3$

$g \circ f = (g \circ f)(a) = g(f(a)) = g(3) = ?$

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Example Let f and g be the functions from the set of integers to the set of integers defined by **$f(x) = 2x + 3$** and **$g(x) = 3x + 2$**. What is the composition of f and g? What is the composition of g and f ?

Both the compositions $f \circ g$ and $g \circ f$ are defined.

$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$

Floor and ceiling functions

Definition The **floor function** assigns to the real number x the largest integer that is **less than or equal to x**. The value of the floor function at x is denoted by $\lfloor x \rfloor$. The **ceiling function** assigns to the real number x the smallest integer that is **greater than or equal to x**. The value of the ceiling function at x is denoted by $\lceil x \rceil$.

Floor and ceiling functions

TABLE 1 Useful Properties of the Floor and Ceiling Functions. (n is an integer, x is a real number)	
(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Example These are some values of the floor and ceiling functions:

$\left\lfloor \frac{1}{2} \right\rfloor$ = $\lfloor 0.5 \rfloor$ = 0 and $\left\lceil \frac{1}{2} \right\rceil$ = $\lceil 0.5 \rceil$ = 1

$\left\lfloor -\frac{1}{2} \right\rfloor$ = $\lfloor -0.5 \rfloor$ = -1 and $\left\lceil -\frac{1}{2} \right\rceil$ = $\lceil -0.5 \rceil$ = 0

$\lfloor 3.1 \rfloor$ = 3 and $\lceil 3.1 \rceil$ = 4

$\lfloor 7 \rfloor$ = 7 and $\lceil 7 \rceil$ = 7

Example: Data stored on a computer disk or transmitted over a data network are usually represented as a string of **bytes**. Each **byte** is made up of **8 bits**. How many **bytes** are required to encode **100 bits** of data?

Solution

$\left\lceil \frac{100}{8} \right\rceil = \lceil 12.5 \rceil = 13$ bytes

Reasons for not using floor function:

Since data is represented as a string of bytes, if we take **the floor function** of it then we might **lose some bits** while transmitted over a data network.

Example In **asynchronous transfer mode (ATM)** (a communications protocol used on backbone networks), data are organized into **cells of 53 bytes**. How many ATM cells can be transmitted in **1 minute** over a connection that transmits data at the rate of **500 kilobits per second**.

Solution

- In **1 minute**, this connection can transmit **500,000** $\times$ **60** = **30,000,000 bits**

- Each ATM cell is **53 bytes** long, which means that it is 53 $\times$ 8 = **424 bits** long.

- $\left\lfloor \frac{30,000,000}{424} \right\rfloor$ = 70,754 ATM cells can be transmitted in 1 minute over a **500 kilobit per second connection**.

Reasons for not using ceiling function:

- Data is organized into **cells of 53 bytes**. But connection transmits data at the rate of 500 kilobits per second (500kb per second bandwidth). It **means the maximum bandwidth** (of the channel) is **500 kb per second**.

- We cannot send more data than the bandwidth of the channel. That is the reason we take the ceiling function

Suggested Readings

2.3 Functions