Discrete Structures

Syed Faisal Bukhari, PhD

Associate Professor

Department of Data Science (DDS), Faculty of Computing and Information Technology (FCIT), University of the Punjab (PU)

Text book

Discrete Mathematics and Its Application, 7th Edition

Kenneth H. Rosen

References

Chapter 3

Discrete Mathematics and Its Application, 7th Edition

by Kenneth H. Rose

These slides contain material from the above resource.

Binary Relations [1]

Definition 1 Let A and B be sets. A *binary relation from A to B* is a subset of $\boldsymbol{A \times B}$.

OR

A *binary relation* R from a set A to a set B is a subset $\boldsymbol{R \subseteq A \times B}$.

- In other words, a **binary relation** from A to B is a set R of **ordered pairs** where the **first element of each ordered pair** comes from $\boldsymbol{A}$ and the **second element** comes **from** $\boldsymbol{B}$.

- We use the notation $a\ R\ b$ to denote that $\boldsymbol{(a, b) \in R}$ and $a \not{R}\ b$ to denote that $\boldsymbol{(a, b) \notin R}$. Moreover, when (a, b) belongs to R, $\boldsymbol{a}$ is said to be **related to $\boldsymbol{b}$** by R.

Binary Relations [2]

- **Example** Let ***A*** be the set of **students in your school**, and let ***B*** be the **set of courses**. Let ***R*** be the relation that consists of those pairs *(a, b)*, where ***a*** is **a student** enrolled in **course *b***.

- For instance, if Jason Good friend and Deborah Sherman are enrolled in CS518, the pairs **(Jason Good friend, CS518)** and **(Deborah Sherman, CS518)** belong to ***R***.

- However, if Deborah Sherman is not enrolled in **CS510**, then the pair **(Deborah Sherman, CS510)** is **not in *R***.

Binary Relations [3]

- **Note** that if a student is not currently enrolled in any courses there will be **no pairs in *R*** that have this student as the **first element**.

- Similarly, if a course is not currently being offered there will be **no pairs in *R*** that have this course as their **second element**.

Example Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.

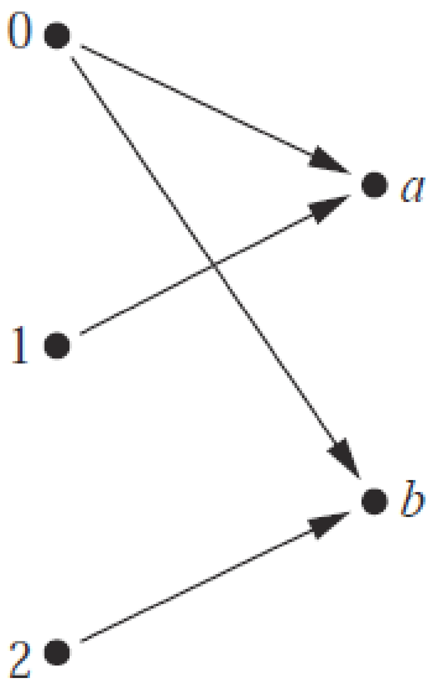
- This means, for instance, that **$0\ R\ a$**, but that $1 \not R\, b$. Relations can be represented graphically, as shown in Figure 1, using **arrows to represent ordered pairs**.

- Another way to represent this relation is to **use a table**, which is also done in Figure 1.

Let ***A* = {0, 1, 2}** and ***B* = {*a*, *b*}**

{(0, *a*), (0, *b*), (1, *a*) , (2, *b*)} is a relation from *A* to *B*.

We can represent relations from a set *A* to a set *B* graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Figure 1 Displaying the Ordered Pairs in the Relation *R* from the previous Example.

Function vs. Relation

- A **relation** can be used to express a **one-to-many** relationship between the elements of the sets *A* and *B* where an element of ***A*** may be related to more than **one element of *B***.

- A **function** represents a relation where exactly one **element of *B*** is related to **each element of *A***.

Note: Relations are more general than **functions**. A **function** is a **relation** where **exactly one element of *B*** is related to **each element of *A*.**

Relations on a Set

Definition A *relation on a* ***set A*** is a relation from *A* to *A*.

OR

In other words, a relation on a **set** ***A*** is a subset of ***A* × *A***.

Example Let A be the set {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a divides b}?

Solution:

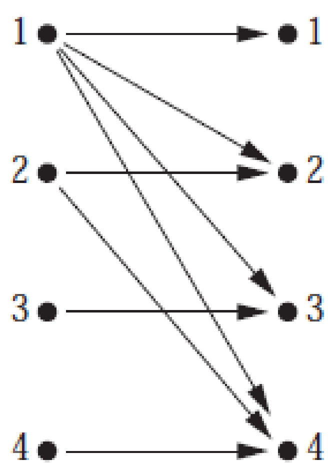
A = {1, 2, 3, 4}

$$A \times A =$$
$$\{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),$$
$$(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\ \}$$

R = {(a, b) | a divides b}?

Because **(a, b) is in R** if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that

R = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)}.



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Binary Relations on a Set (*cont.*)

Example: Consider these relations on the set of integers:

$R_1 = \{(a, b) \mid a \leq b\}$,

$R_2 = \{(a, b) \mid a > b\}$,

$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a, b) \mid a = b\}$,

$R_5 = \{(a, b) \mid a = b + 1\}$,

$R_6 = \{(a, b) \mid a + b \leq 3\}$.

Note that these relations are on an infinite set and each of these relations is an infinite set.

Which of these relations contain each of the pairs

$(1, 1)$, $(1, 2)$, $(2, 1)$, $(1, -1)$, and $(2, 2)$?

Solution: Checking the conditions that define each relation, we see that the pair

- **(1,1)** is in R_1, R_3, R_4 , and R_6
- **(1, 2)** is **in** R_1 and R_6
- **(2, 1)** is in R_2, R_5, and R_6
- **(1, -1)** is in R_2, R_3, and R_6
- **(2, 2)** is in R_1, R_3, and R_4

Binary Relation on a Set (*cont.*)

Question: How many relations are there on a set A?

Solution: A relation on a set A is a subset of $A \times A$. Because $A \times A$ has $\boldsymbol{n^2}$ elements when A has n elements, and a set with m elements has 2^m subsets, there are $\boldsymbol{2^{n^2}}$ subsets of $A \times A$. Thus, there are 2^{n^2} relations on a set with n elements.

Example: There are how many relations on the set **{a, b, c}**

Solution: There are $2^{3^2} = 2^9 = 512$ relations on the set **{*a, b, c*}.**

Properties of Relations

Definition A relation R on a set A is called ***reflexive*** if ***(a, a)*** $\in R$ for **every element** $a \in A$.

Remark: Using quantifiers we see that the **relation *R*** on the set A is reflexive if $\forall a((a, a) \in R)$, where the universe of discourse is the set of all elements in A.

Example: The following relations on the integers are reflexive:

$R_1 = \{(a, b) \mid a \leq b\}$,

$R_3 = \{(a, b) \mid a = b$ or $a = -b\}$,

$R_4 = \{(a, b) \mid a = b\}$.

The following relations are not reflexive:

$R_2 = \{(a, b) \mid a > b\}$ (note that $3 \ngtr 3$), for e.g., (3, 3)

$R_5 = \{(a, b) \mid a = b + 1\}$ (note that $3 \neq 3 + 1$), for e.g., (3, 3)

$R_6 = \{(a, b) \mid a + b \leq 3\}$ (note that $4 + 4 \nleq 3$) for e.g., (4, 4)

Example Consider the following relations **on {1, 2, 3, 4}:**

R_1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)},

R_2 = {(1, 1), (1, 2), (2, 1)},

R_3 = **{(1, 1),** (1, 2), (1, 4), (2, 1), **(2, 2), (3, 3),** (4, 1), **(4, 4)**},

R_4 = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)},

R_5 = {**(1, 1)**, (1, 2), (1, 3), (1, 4), **(2, 2),** (2, 3), (2, 4), **(3, 3),** (3, 4), **(4, 4)**},

R_6 = {(3, 4)}.

Which of these relations are **reflexive**?

Solution:

- The relations **R_3 and R_5 are reflexive** because they both contain all pairs of the form (a, a), namely, (1, 1), (2, 2), (3, 3), and (4, 4).

- The other relations are **not reflexive** because they do not contain all of **these ordered pairs**. In particular, R_1, R_2, R_4, and R_6 are not reflexive because **(3, 3)** is not in any of these relations.

Symmetric A relation R on a set A is called ***symmetric*** if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.

Antisymmetric A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called ***antisymmetric***.

Remark: Using quantifiers, we see that the relation R on the set A is symmetric if $\forall a \forall b((a, b) \in R \rightarrow (b, a) \in R)$.

Similarly, the relation R on the set A is antisymmetric if

$\forall a \forall b(((a, b) \in R \wedge (b, a) \in R) \rightarrow (a = b))$.

Example Consider the following relations on {1, 2, 3, 4}:

R_1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)},

R_2 = {(1, 1), **(1, 2), (2, 1)** },

R_3 = {(1, 1), **(1, 2), (1, 4), (2, 1),** (2, 2), (3, 3), **(4, 1),** (4, 4)},

R_4 = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)},

R_5 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)},

R_6 = {(3, 4)}.

Which of these relations are **symmetric** and ***antisymmetric***?

Solution:

Recall: A relation R on a set A is called **symmetric** if (b, a) ∈ R whenever (a, b) ∈ R, for all a, b ∈ A.

R_2 & R_3: symmetric each case (b, a) belongs to the relation whenever (a, b) does.

For R_2: only thing to check that both (1, 2) & (2, 1) belong to the relation

For R_3: it is necessary to check that both (1, 2) & (2, 1) belong to the relation.

None of the other relations is symmetric: find **a pair (a, b)** so that it is in the relation **but (b, a) is not**.

Recall: A relation *R* on a set *A* such that for all *a, b* ∈ *A*, if *(a, b)* ∈ *R* and *(b, a)* ∈ *R*, then *a* = *b* is called ***antisymmetric***.

R_4, R_5 and R_6: antisymmetric for each of these relations there is no pair of elements a and b with **$a \neq b$** such that both (a, b) and (b, a) belong to the relation.

None of the other relations is antisymmetric.: **find a pair (a, b)** with $a \neq b$ so that **(a, b)** and **(b, a)** are both in the relation.

Definition A relation R on a set A is called ***transitive*** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

Remark: Using quantifiers we see that the relation R on a set A is transitive if we have $\forall a \forall b \forall c(((a, b) \in R \wedge (b, c) \in R) \rightarrow (a, c) \in R)$.

Definition A relation R on a set A is called ***transitive*** if whenever ***$(a, b) \in R$*** and ***$(b, c) \in R$***, then ***$(a, c) \in R$***, for all $a, b, c \in A$.

Example Consider the following relations on {1, 2, 3, 4}:

R_1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)},

R_2 = {(1, 1), (1, 2), (2, 1)},

R_3 = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)},

R_4 = **{(2, 1), (3, 1), (3, 2),** (4, 1), (4, 2), (4, 3)},

R_5 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)},

R_6 = {(3, 4)}.

Which of these relations are ***transitive***?

Recall: A relation *R* on a set *A* is called ***transitive*** if whenever ***(a, b) ∈ R*** and ***(b, c) ∈ R***, then ***(a, c) ∈ R***, for all *a, b, c* ∈ *A*.

- **R_4 , R_5 & R_6 : transitive** verify that if (a, b) and (b, c) belong to this relation then (a, c) belongs also to the relation R_4 transitive since (3,2) and (2,1), (4,2) and (2,1), (4,3) and (3,1), and (4,3) and (3,2) are the only such sets of pairs, and **(3,1) , (4,1) and (4,2)** belong to R_4.
  Same reasoning for R_5 and R_6.
- **R_1 : not transitive** (3,4) and (4,1) belong to R_1, but **(3,1)** does not.
- **R_2 : not transitive** (2, 1) and (1, 2) belong to R_2, but **(2,2)** does not.
- **R_3 : not transitive** (4,1) and (1,2) belong to R_3, but **(4,2)** does not

Suggested Readings

9.1 Relations and Their Properties