Discrete Structures

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Text book

Discrete Mathematics and Its Application, 7^{th} Edition

Kenneth H. Rosen

References

Discrete Mathematics and Its Application, 7h Edition

By Kenneth H. Rose

Probability & Statistics for Engineers & Scientists, Ninth edition, Ronald E. Walpole, Raymond H. Myer

Probability Demystified, Allan G. Bluman
https://en.wikipedia.org/wiki/Law_of_large_numbers

These slides contain material from the above resources.

Discrete Probability

- In computer science, **probability theory** plays an important role in the study of the **complexity of algorithms**.

- In particular, ideas and techniques from probability theory are used to determine the **average-case complexity of algorithms**.

Discrete Probability Cont.

- **Probabilistic algorithms** can be used to solve many problems that cannot be easily or practically solved by **deterministic algorithms.**

- In a **probabilistic algorithm**, instead of always following the same steps when given the same input, as a **deterministic algorithm does**, the algorithm makes **one or more random choices**, which may lead to different output.

Discrete Probability

- In combinatorics, probability theory can even be used to show that objects **with certain properties exist**

- Probability theory can help us answer questions that **involve uncertainty**, such as determining whether we should **reject an incoming mail message as spam** based on the words that appear in the message.

Basic concepts [1]

- **Probability** can be defined as the mathematics of chance.

- **Statisticians** use the word **experiment** to describe any process that **generates a set of data**.

OR

- A **probability experiment** is a chance process that leads to well defined outcomes or results. **For example**, tossing a coin can be considered a probability experiment since there are two well-defined outcomes—heads and tails.

Basic concepts [2]

- In probability theory, an **experiment or trial** is any procedure that can be **infinitely repeated** and has a **well-defined** set of possible **outcomes**, known as the **sample space**.

- An **outcome** of a probability experiment is the result of a single trial of a probability experiment.

Basic concepts [3]

- The set of all possible outcomes of a statistical experiment is called the **sample space** and is represented by the symbol ***S***.

OR

- The set of all outcomes of a probability experiment is called a **sample space**. Some sample spaces for various probability experiments are shown here.

Experiment	Sample space
Toss one coin	H, T
Roll a die	1, 2, 3, 4, 5, 6
Toss two coins	HH, HT, TH, TT

Basic concepts [4]

- Each outcome in a sample space is called an **element** or a **member** of the sample space, or simply a **sample point.**

- Each outcome of a probability experiment occurs at **random**.

- Each outcome of the experiment is **equally likely unless otherwise stated**.

Basic concepts [5]

- An **event** then usually consists of one or more outcomes of the sample space.

OR

- An **event** is a subset of a sample space.

- An event with one outcome is called a **simple event**.

- An event consists of two or more outcomes, it is called a **compound event**.

Example

A single die is rolled. List the outcomes in each event:

a. Getting an odd number

b. Getting a number greater than four

c. Getting less than one

Example cont.

Solution:

S = {1, 2, 3, 4, 5, 6}

a. Let **A** be the event contains the outcomes 1, 3, and 5.

A = {1, 3, 5}, n(A) = 3

b. Let **B** be the event contains the outcomes 5, and 6.

B = {5, 6}, n(B) = 2

c. Let **C** be the event that contains a number less than one

C = {}

Basic concepts [6]

Classical Probability:

The formula for determining the probability of an event **E** is

$$P(E) = \frac{n(E)}{n(S)}$$

OR

$$P(E) = \frac{\textbf{Number of outcomes contained in the event E}}{\textbf{Total number of outcomes in the sample space}}$$

Example:

Two coins are tossed; find the probability that both coins land heads up.

Solution:

S = {HH, HT, TH, and TT}

n(S) = 4

Let **A** be the event of getting a both heads

A = {HH}

n(A) = 1

P (A) = ¼ = **0.25 (or 25 %)**

Example:

A die is tossed; find the probability of each event:

a. Getting a two

b. Getting an even number

c. Getting a number less than 5

Example cont.

Solution:

S = {1, 2, 3, 4, 5, 6}

n(S) = 6

$$\textbf{P(E)} = \frac{\textbf{Number of outcomes contained in the event E}}{\textbf{Total number of outcomes in the sample space}}$$

a. Let A be the event of getting a "two"

A = {2}

n(A) = 1

P (A) = $^{1}/_{6}$ = 0.1667 (or 16.67%)

Example cont.

b. a. Let **B** be the event of getting a **“even number”**

A = {2, 4, 6}

n(A) = 3

P (B) = $^{3}/_{6}$ = $^{1}/_{2}$ **= 0.5 (or 50%)**

c. a. Let **C** be the event of getting a **“less than 5”**

C = {1, 2, 3, 4}

n(C) = 4

P (C) = $^{4}/_{6}$ = $^{2}/_{3}$ **= 0.6666 (or 66.67%)**

Basic concepts [7]

Rule 1: The probability of any event will always be a number from **zero to one**. Probabilities cannot be **negative** nor can they **be greater than one**.

Rule 2: When an event cannot occur, the probability will be **zero.**

Example: A die is rolled; find the probability of getting a 7.

Basic concepts [8]

Rule 3: When an event is certain to occur, the probability is **1**.

Example: A die is rolled; find the probability of getting a number less than 7.

Rule 4: The sum of the probabilities of all of the outcomes in the **sample space** is 1.

Example: $P(H) = {}^{1}/_{2}$, $P(T) = {}^{1}/_{2}$, $P(H) + P(T) = 1$.

Basic concepts [9]

Complement : The **complement** of an event A with respect to S is the subset of all elements of S that are not in A. We denote the complement of A by the symbol $\boldsymbol{A'}$ ***or*** $\boldsymbol{\overline{A}}$ ***or*** $\boldsymbol{A^c}$

Rule 5: The probability that an event will not occur is equal to 1 minus the probability that the event will occur.

Example: P(H) = $^1/_2$, P(T) = 1- P(H) = $^1/_2$

Basic concepts

The **probability** of an event A is the sum of the weights of all **sample points** in A.

Therefore,

I. $0 \leq P(A) \leq 1$

II. $P(\varphi) = 0$

III. $P(S) = 1.$

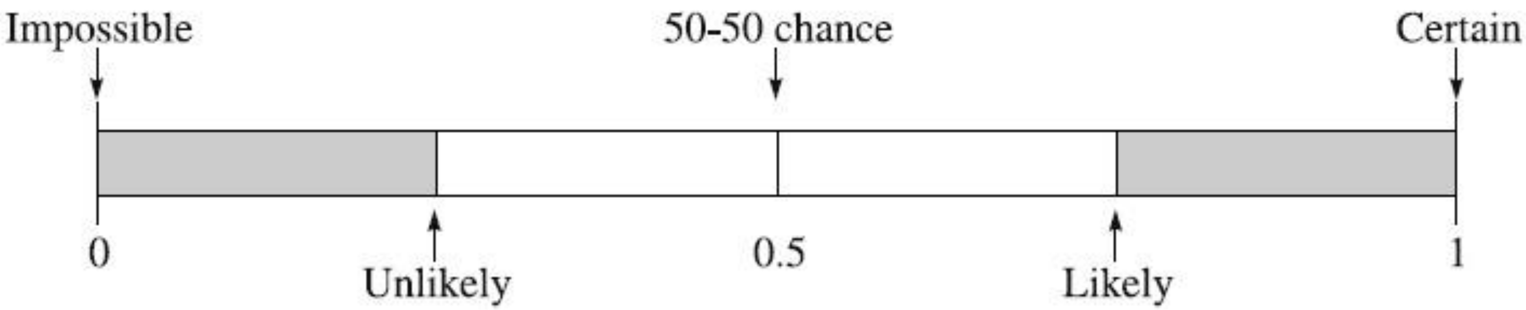
Basic concepts

- When the probability of an event is close to **zero**, the occurrence of the event is relatively **unlikely**. For example, if the chances that you will win a certain lottery are **0.00l** or one in one thousand, you probably won't win, unless of course, you are very **"lucky."**

- When the probability of an event is **0.5** or $^{1}/_{2}$, there is a **50–50 chance** that the event will happen—the same.

Basic concepts

When the probability of an event is close to one, the event is almost sure to occur. For example, if the chance of it snowing tomorrow is **90%,** more than **likely**, you'll see some snow.



Empirical Probability [1]

Probabilities can be computed for situations that do not use sample spaces. In such cases, frequency distributions are used and the probability is called **empirical probability**.

Rank	Frequency
Freshmen	4
Sophomores	6
Juniors	8
Seniors	7
TOTAL	25

Empirical Probability [2]

$$\text{P(E)} = \frac{\text{Frequency of E}}{\text{Sum of the frequencies}}$$

$\text{P(E)} = {}^{1}/_{4}$

Empirical probability is sometimes called **relative frequency probability**.

Law of large numbers

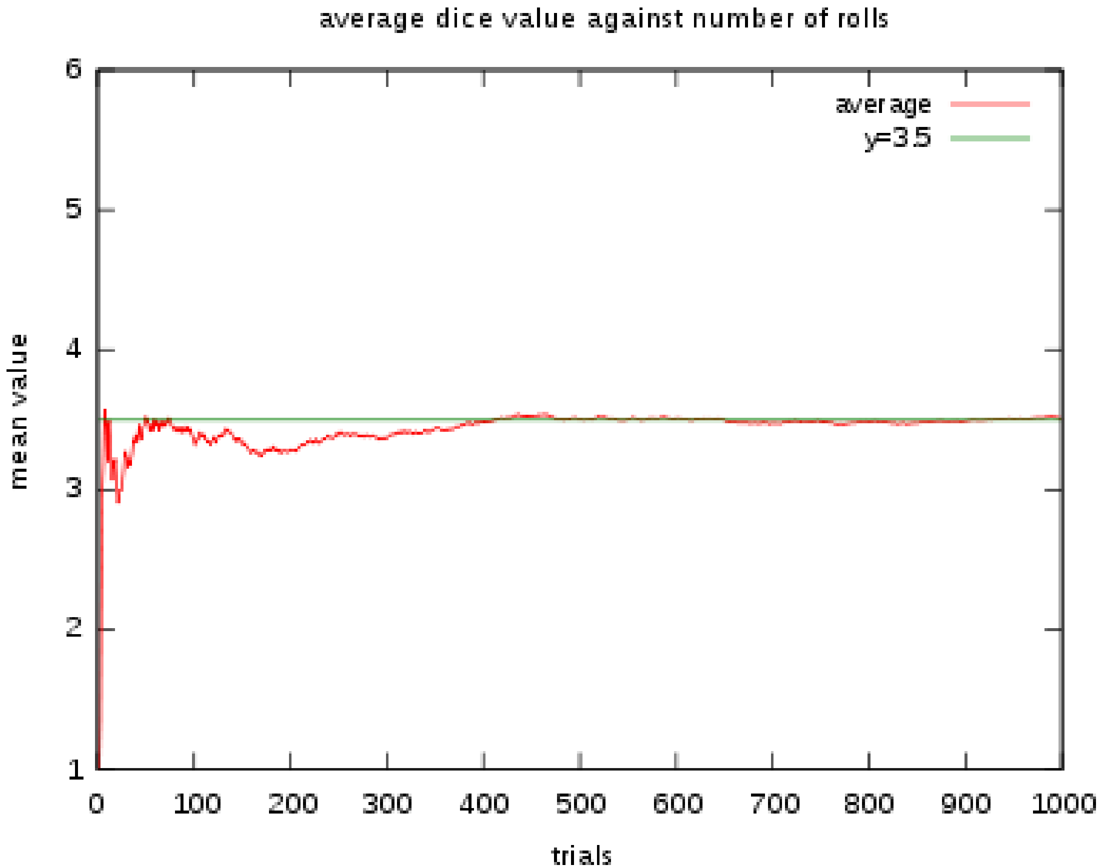
- In probability theory, the **law of large numbers (LLN)** is a theorem that describes the **result** of performing the **same experiment a large number of times**.

- According to the law, the **average** of the results obtained from a **large number of trials** should be close to the **expected value**, and will tend to become **closer** as more trials are performed.

Law of large numbers

- The LLN is important because it **"guarantees" stable long-term** results for the averages of some random events.

- **For example**, while a casino may lose money in a single spin of the **roulette** wheel, its earnings will tend towards a predictable percentage over a large number of spins.



An illustration of the law of large numbers using a particular run of rolls of a single die. As the number of rolls in this run increases, the **average** of the values of all the results approaches **3.5**.

Law of Large Numbers

Questions:

What happens if we toss the coin **100 times** ? Will we get **50** heads?

What will happen if we toss a coin **1000 times**? Will we get exactly **500** heads?

Law of Large Numbers

- **Solution:** Probably not.

- However, as the number of **tosses increases**, the ratio of the number of heads to the total number of tosses will get closer to $^{1}/_{2}$.

- This phenomenon is known as the **law of large numbers**.

Subjective Probability

A third type of probability is called **subjective probability**. Subjective probability is based upon an **educated guess**, **estimate**, **opinion**, or **inexact information**.

Sample Spaces

There are two **specific devices** that will be used to find sample spaces for probability experiments. They are **tree diagrams** and **tables**.

A **tree diagram** consists of **branches** corresponding to the outcomes of two or more probability experiments that are done in sequence.

Sample Spaces

- In order to construct a tree diagram, use branches corresponding to the outcomes of the **first experiment**. These branches will emanate from a single point.

- Then from each branch of the first experiment draw branches that represent the outcomes of the **second experiment**.

- You can continue the process for further experiments of the sequence if necessary.

Tree Diagram [1]

Example: A **coin** is tossed and a **die** is rolled. Draw a tree diagram and find the sample space.

Solution:

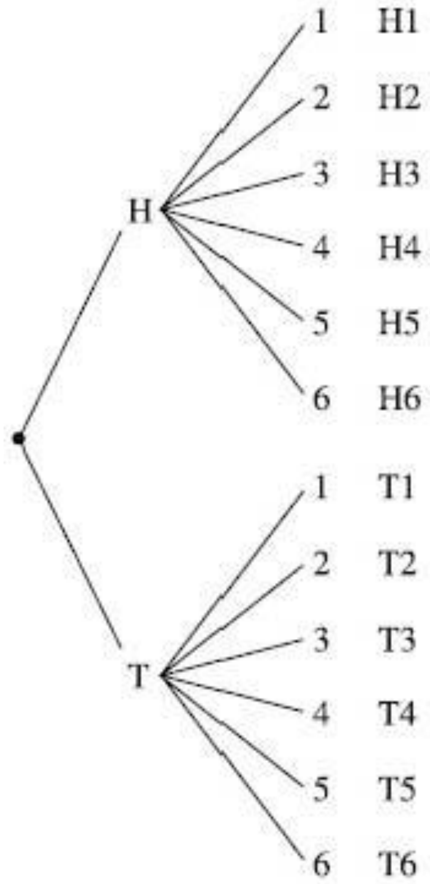
Since there are two outcomes (heads and tails for the coin), draw two branches from a single point and label one **H** for head and the other one **T** for tail.

From each one of these outcomes, draw and label six branches representing the outcomes **1, 2, 3, 4, 5,** and **6** for the die.

Trace through each branch to find the outcomes of the experiment.

Tree Diagram [2]

Example: A coin is tossed and a die is rolled. Draw a tree diagram and find the sample space.



Tree Diagram [3]

Example: A coin is tossed and a die is rolled. Find the probability of getting

a. A head on the coin and a 3 on the die.

b. A head on the coin.

c. A 4 on the die.

Solution:

a. $P(H3) = {}^{1}/_{12}$ = **0.0833 (or 8.33%)**

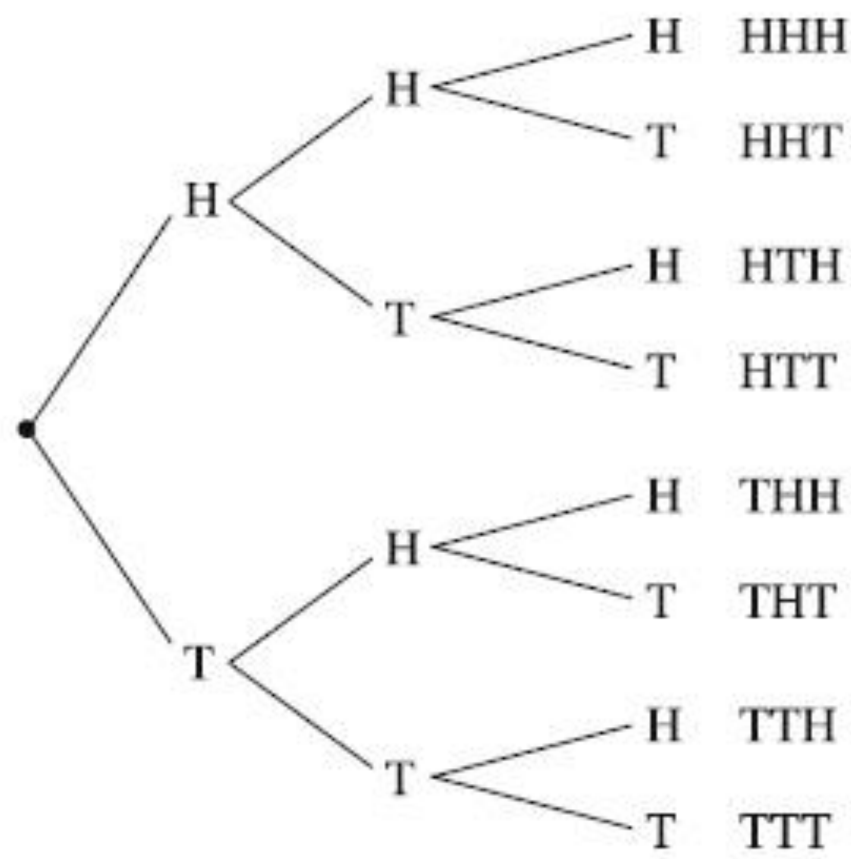
b. $P(\text{head on the coin}) = {}^{6}/_{12} = {}^{1}/_{2}$ = **0.5 (or 50%)**

c. $P(\text{4 on the die}) = {}^{2}/_{12} = {}^{1}/_{6}$ = **0.1667 (16.67%)**

Tree Diagram [4]

Example: Three coins are tossed. Draw a tree diagram and find the sample space.

Solution



Tree Diagram [5]

Example: Three coins are tossed. Find the probability of getting

a. Two heads and a tail in any order.

b. Three heads.

c. No heads.

d. **At least** two tails.

e. **At most** two tails.

Solution

a. P(2 heads and a tail) = 3/8 = **0.375 (or 37.5 %)**

b. P(HHH) = ${}^{1}/_{8}$ = **0.125 (or 12.5 %)**

c. P(TTT) = ${}^{1}/_{8}$ = **0.125 (or 12.5 %)**

d. P(at least two tails) = ${}^{4}/_{8}$ = ${}^{1}/_{2}$ = **0.5 (or 50 %)**

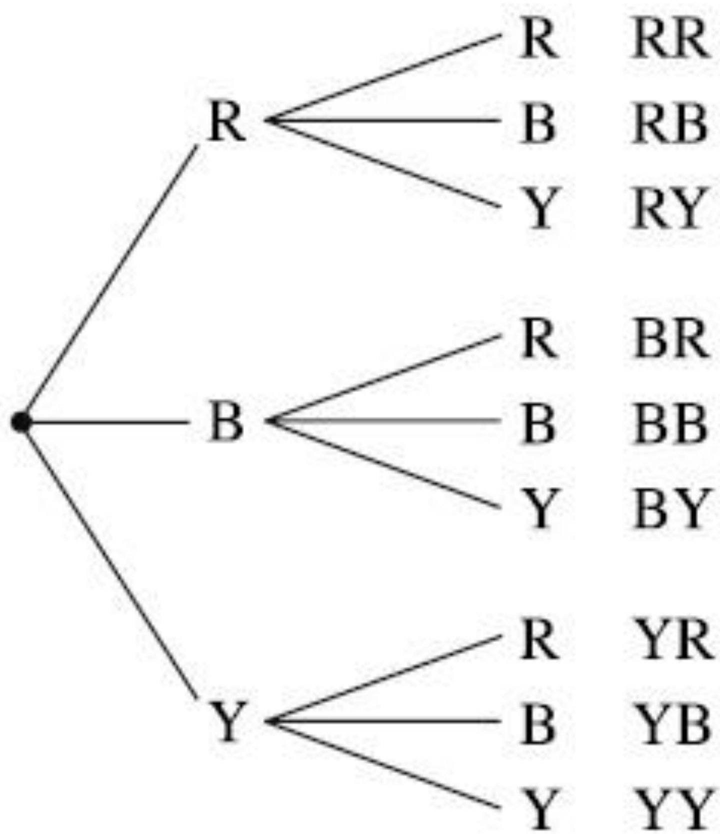
e. P(at most two tails) = ${}^{7}/_{8}$ = **0.875 (or 87.5 %)**

$\Rightarrow \boldsymbol{At\ most\ two\ tails\ mean\ no\ three\ tails}$

Tree Diagram [6]

Example: A box contains a **red ball (R),** a **blue ball (B),** and a **yellow ball (Y**). **Two balls** are selected at random in succession. Draw a **tree diagram** and find the sample space if the first ball is **replaced** before the **second ball** is selected.

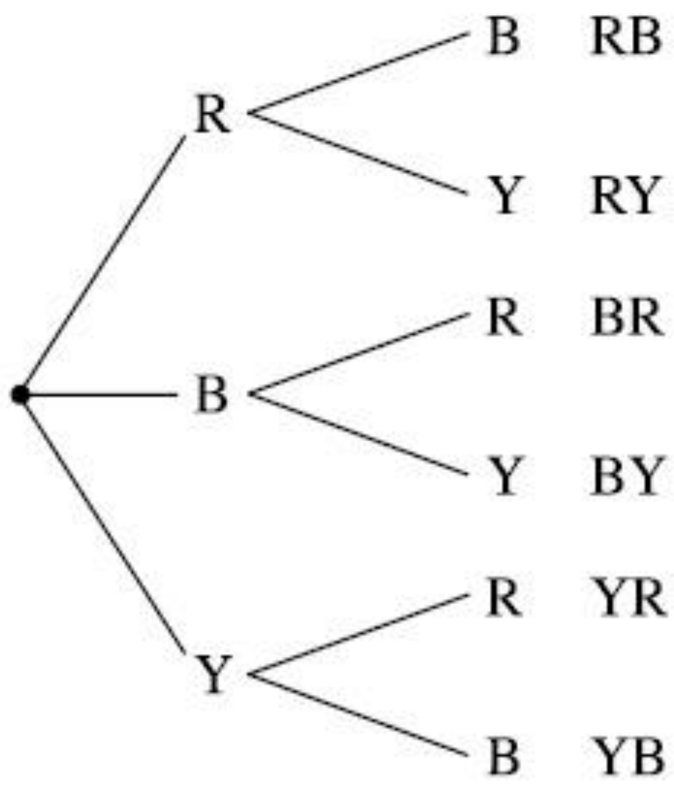
Solution



Tree Diagram [7]

Example: A box contains a **red ball (R)**, a **blue ball (B)**, and a **yellow ball (Y)**. Two balls are selected at random in **succession**. Draw a **tree diagram** and find the sample space if the first ball is **not replaced** before the second ball is selected.

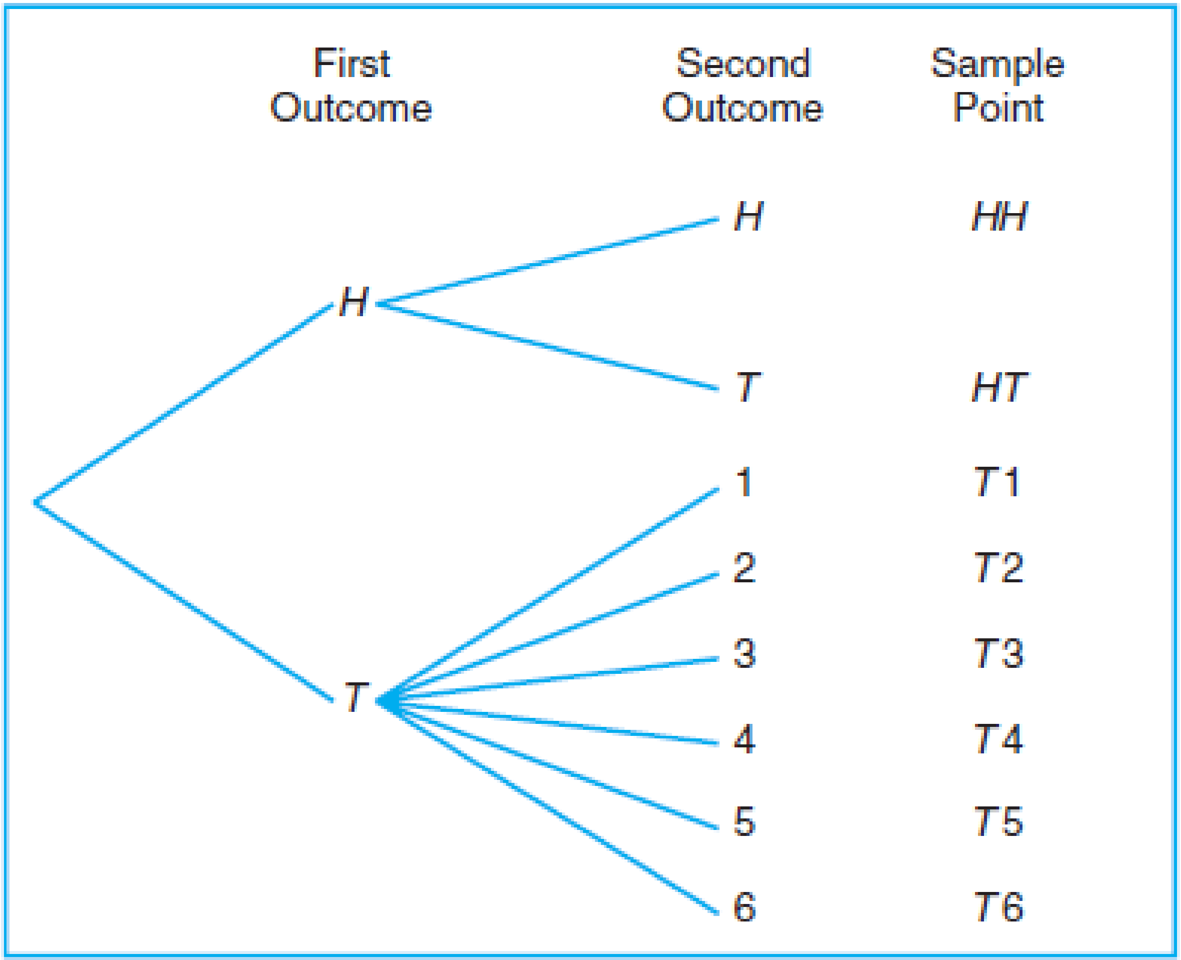
Solutions



Tree Diagram [9]

Example An experiment consists of **flipping a coin** and then flipping it a **second time** if a **head occurs**. If a **tail** occurs on the **first flip**, then a **die is tossed** once. To list the elements of the sample space providing the most information, construct the **tree diagram**

Solution

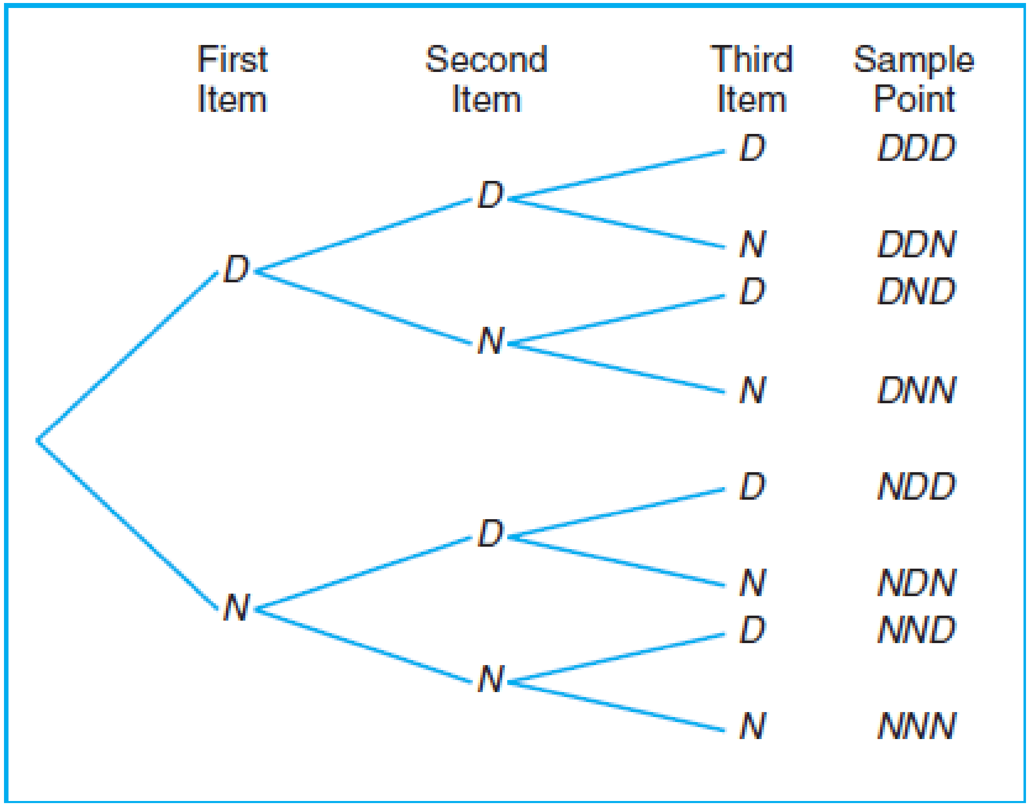


First Outcome
Second Outcome
Sample Point
H
T
H
HH
T
HT
1
T1
2
T2
3
T3
4
T4
5
T5
6
T6

Tree Diagram [10]

Example Suppose that **three items** are selected at random from a manufacturing process. Each item is inspected and classified **defective**, **D**, or **nondefective**, **N**. To list the elements of the sample space providing the most information, construct the **tree diagram.**

Solution



First Item
Second Item
Third Item
Sample Point
D
D
D
DDD
N
DDN
N
D
DND
N
DNN
N
D
D
NDD
N
NDN
N
D
NND
N
NNN

Tables [1]

Another way to find a sample space is to use a **table.**

Example: Find the sample space for selecting a card from a standard deck of 52 cards.

There are four suits—**hearts** and **diamonds**, which are **red**, and **spades** and **clubs**, which are **black**. Each suit consists of 13 cards—ace through king. Face cards are kings, queens, and jacks.

A standard deck of 52 cards

Heart	A ♥	2 ♥	3 ♥	4 ♥	5 ♥	6 ♥	7 ♥	8 ♥	9 ♥	10 ♥	J ♥	Q ♥	K ♥
Diamond	A ♦	2 ♦	3 ♦	4 ♦	5 ♦	6 ♦	7 ♦	8 ♦	9 ♦	10 ♦	J ♦	Q ♦	K ♦
Spade	A ♠	2 ♠	3 ♠	4 ♠	5 ♠	6 ♠	7 ♠	8 ♠	9 ♠	10 ♠	J ♠	Q ♠	K ♠
Club	A ♣	2 ♣	3 ♣	4 ♣	5 ♣	6 ♣	7 ♣	8 ♣	9 ♣	10 ♣	J ♣	Q ♣	K ♣

Tables [2]

Example: A single card is drawn at random from a standard deck of cards. Find the probability that it is

a. The 4 of diamonds.

b. A queen.

Solution:

a. P(The 4 of diamonds) = $^{1}/_{52}$ = **0.0192 (or 1.9231%)**

b. P(A queen) = $^{4}/_{52}$ = $^{1}/_{13}$ = **0.0769 (or 7.6923 %)**

Tables [3]

A table can be used for the sample space when two dice are rolled.

	Die 2					
Die 1	1	2	3	4	5	6
1	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

Tables [4]

Example: When two dice are rolled, find the probability of getting a sum of nine.

Solution:

Let A be the event of getting a "sum of 9"

P(A) = ${}^{4}/_{36}$ = ${}^{1}/_{9}$ = **0.1111 (or 11.11 %)**

Suggested Readings

Probability & Statistics for Engineers & Scientists, Ninth Edition, Ronald E. Walpole, Raymond H. Myer

2.1 Sample space

2.2 Events

2.3 Counting Sample Points

Discrete Mathematics and Its Application, 7th Edition Kenneth H. Rosen

7.1 An Introduction to Discrete Probability

7.2 Probability Theory