Discrete Structures

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Text book

Discrete Mathematics and Its Application, 7th Edition

Kenneth H. Rosen

References

Discrete Mathematics and Its Application, 7h Edition

By Kenneth H. Rose

Probability & Statistics for Engineers & Scientists, Ninth edition,

Ronald E. Walpole, Raymond H. Myer

These slides contain material from the above resources.

“Yesterday is history, tomorrow is a mystery, but today is a gift. That’s why we call it the present.”

—Attributed to A. A. Milne, Bill Keane, and Oogway, the wise turtle in Kung Fu Panda

Conditional Probability

A **conditional probability** is the probability of an event occurring, given that another event has already occurred. The conditional probability of event *B* occurring, given that event *A* has occurred, is denoted by **P(B|A)** and is read as "probability of *B*, given *A*."

Example: Two cards are selected in sequence from a standard deck of 52 playing cards. Find the probability that the second card is a queen, given that the first card is a king. (Assume that the king is not replaced.)

Solution

Because the first card is a king and is not replaced, the remaining deck has 51 cards, 4 of which are queens. So,

P(B|A) = $\frac{4}{51}$ = 0.078.

Independent and Dependent Events [1]

Two events A and B are **independent** if and only if **$P(B|A) = P(B)$ or $P(A|B) = P(A)$,** assuming the existences of the conditional probabilities. Otherwise, A and B are **dependent**.

OR

Two events, A and B, are said to be **independent** if the fact that **event A** occurs does not affect the probability that **event B** occurs.

Independent and Dependent Events [2]

Two events are **independent** when the occurrence of one of the events does not affect the probability of the occurrence of the other event. Two events A and B are independent when

P(B|A) = P(B) or P(A|B) = P(A)

Events that are **not independent** are **dependent.**

Independent and Dependent Events [3]

To determine whether *A* and *B* are independent, first calculate **P(B)** , the probability of **event *B***. Then calculate **P(B|A)**, the probability of ***B*, given *A***. If the values **are equal**, then the events are **independent**. If ***P*(B) $\neq$ P(B|A),** then *A* and *B* are dependent events.

Independent and Dependent Events [4]

Example 1: If a coin is tossed and then a die is rolled, the **outcome of the coin in no way affects** or changes the probability of the outcome of the die.

Example 2: Selecting a card from a deck, replacing it, and then selecting a second card from a deck. The outcome of the first card, as long as it is **replaced**, has no effect on the probability of the outcome of the second card.

Classifying Events as Independent or Dependent

Example Determine whether the events are independent or dependent.

1. Selecting a king (A) from a standard deck of 52 playing cards, not replacing it, and then selecting a queen (B) from the deck

2. Tossing a coin and getting a head (A), and then rolling a six-sided die and obtaining a 6 (B)

3. Driving over 85 miles per hour (A), and then getting in a car accident (B)

Solution

1. $\mathbf{P(B|A) = \frac{4}{51}}$ and $\boldsymbol{P(B) = \frac{4}{51}}$. The occurrence of *A* changes the probability of the occurrence of B, so the events are dependent.

2. $\mathbf{P(B|A) = \frac{1}{6}}$ and **$\mathbf{P(B) = \frac{1}{6}}$**. The occurrence of *A* does not change the probability of the occurrence of *B*, so the events are independent.

3. Driving over 85 miles per hour increases the chances of getting in an accident, so these events are dependent.

First Multiplication Rule [1]

- Before explaining the first multiplication rule, consider the Example of tossing two coins. The sample space is **HH, HT, TH, TT**. From classical probability theory, it can be determined that the probability of getting two heads is $1/4$.

- However, there is another way to determine the probability of getting two heads. In this case, the probability of getting a head on the first toss is $1/2$, and the probability of getting a head on the second toss is also $1/2$.

First Multiplication Rule [2]

- So the probability of getting two heads can be determined by multiplying $^{1}/_{2} \times {}^{1}/_{2} = {}^{1}/_{4}$.

Multiplication Rule I [1]

Multiplication Rule I: For two **independent** events A and B,

$P(A \text{ and } B) = P(A) \times P(B)$.

In other words, when two independent events occur in sequence, the probability that both events will occur can be found by multiplying the probabilities of each individual event.

The word **“and”** is the key word and means that both events occur in sequence and to multiply.

Multiplication Rule I [2]

Example: A coin is tossed and a die is rolled. Find the probability of getting a **tail on the coin** and a **5 on the die**.

Solution:

Let A be the event of getting a tail on the coin

P(A) = $^{1}/_{2}$ = **0.5 (or 50%)**

Let B be the event of getting a 5 on the die

P(B) = $^{1}/_{6}$ = **0.1667 (or 16.67 %)**

Since A and B are **independent** events, so

P(A and B) = P(A) × P(B)

= $^{1}/_{2} \times ^{1}/_{6} = ^{1}/_{12}$

= **0.0833 (or 8.33 %)**

Multiplication Rule II [1]

- When two sequential events are **dependent**, a slight variation of the multiplication rule is used to find the probability of both events occurring.

- For Example, when a card is selected from an ordinary deck of **52 cards** the probability of getting a specific card is $^{1}/_{52}$, but the probability of getting a specific card on the second draw is $^{1}/_{51}$ since 51 cards remain.

Example: Two cards are selected from a deck and the first card is **not replaced**. Find the probability of getting **two kings**.

Solution

P(two kings) = $^{4}/_{52}$ x $^{3}/_{51}$

= $^{12}/_{2652}$

= $^{1}/_{221}$

= **0.0045 (or 0.45 %)**

Multiplication Rule II [4]:

When two events are **dependent**, the probability of both events occurring is **$P(A \text{ and } B) = P(A) \times P(B|A)$**

Example: A box contains **24 toasters**, **3** of which are **defective**. If **two toasters** are selected and tested, find the probability that **both are defective** (assume toasters are not replaced).

Solution

Let $\mathbf{D_1}$ be the event that **first toaster** is **defective**.

Let $\mathbf{D_2}$ be the event that **second toaster** is **defective**.

$P(D_1 \text{ and } D_2) = P(D_1) \times P(D_2 | D_1)$

$= {}^{3}/_{24} \times {}^{2}/_{23}$

$= {}^{1}/_{8} \times {}^{2}/_{23}$

$= {}^{1}/_{92}$

= **0.0109 (or 1.0870 %)**

Example: An industrial psychologist administered a personality inventory test for passive-aggressive traits to 150 employees. Each individual was given a score from 1 to 5, where 1 is extremely passive and 5 is extremely aggressive. A score of 3 indicated neither trait. The results are shown at the left. Construct a probability distribution for the random variable *x*. Find mean and variance of x.

Frequency Distribution

Score, *x*	frequency, f
1	24
2	33
3	42
4	30
5	21

Probability Distribution

x	P(x)	xP(x)	x^2P(x)
1	$\frac{24}{150} = 0.1600$	0.1600	0.1600
2	$\frac{33}{150} = 0.2200$	0.4400	0.8800
3	$\frac{42}{150} = 0.2800$	0.8400	2.5200
4	$\frac{30}{150} = 0.2000$	0.8000	3.2000
5	$\frac{21}{150} = 0.1400$	0.7000	3.5000
	$\sum_{i=1}^{5} P_i = 1.0000$	$\sum \mathrm{x}\mathrm{P}(\mathrm{x}) = 2.9400$	$\sum \mathrm{x}^2\mathrm{P}(\mathrm{x}) = 10.2600$

Note:

1. $0 \le P(x) \le 1$

2. $\sum P(x) = 1$

$$\mu = E(x) = \sum xP(x) = 2.9400$$

$$E(x^2) = \sum x^2 P(x) = 10.2600$$

$$\sigma^2 = E(x^2) - [E(x)]^2 = 10.2600 - (2.9400)^2$$

$$\sigma^2 = 1.6164$$

Concept of a Random Variable [1]

- **Statistics** is concerned with making **inferences** about **populations** and **population characteristics**. Experiments are conducted with results that are subject to chance.

- The testing of a number of electronic components is an example of a **statistical experiment**, a term that is used to describe any process by which several chance **observations are generated**.

- It is often important to allocate a **numerical description** to the outcome.

Concept of a Random Variable [2]

- **For example**, the sample space giving a detailed description of each possible outcome when **three electronic components** are tested may be written

S = {NNN, NND, NDN, DNN, NDD, DND, DDN, DDD}

where **N** denotes **nondefective** and **D** denotes **defective**.

Concept of a Random Variable [3]

- One is naturally concerned with the number of defectives that occur. Thus, each point in the sample space will be assigned a numerical value of **0, 1, 2, or 3**.

These values are, of course, **random quantities** determined by the outcome of the experiment

Random variables [1]

Variables whose values are **due to chance** are called random variables.

OR

When an experiment is performed and it **produced different results under the same condition** is called random variable (r.v). It is usually denoted by capital letter **X**.

OR

A **random variable** is a function that associates **a real number** with **each element in the sample space**.

Random variables [2]

OR

A **random variable** is a variable (typically represented by x) that has **a single numerical value**, determined by chance, for each outcome of a procedure.

Random variables [3]

- We shall use a capital letter, say **X**, to denote a random variable and its **corresponding small letter**, **x** in this case, for one of its values.
- In the electronic component testing illustration above, we notice that the random variable **X** assumes the value **2** for all elements in the subset

E = {DDN, DND, NDD} of the **sample space S**.

- That is, each possible value of **X represents an event** that is a **subset of the sample space** for the given experiment

Random variables [1]

Example: Two balls are drawn in succession **without replacement** from an urn containing **4 red balls** and **3 black balls**. The possible outcomes and the **values y** of the random variable Y, where ***Y* is the number of red balls**, are

Sample Space	y
RR	2
RB	1
BR	1
BB	0

Discrete Sample Space vs. Continuous Sample Space.

- If a sample space contains a **finite number of possibilities or an unending sequence** with as many elements as there are whole numbers, it is called a **discrete sample space**.

- If a sample space contains an **infinite number of possibilities** equal to the number of points on a line segment, it is called a **continuous sample space**.

Discrete Random Variable

- A random variable defined over **the discrete sample space** is called **discrete random variable.**

OR

- A random variable is called a **discrete random variable** if its set of possible outcomes is **countable**.

OR

- **Discrete random variable** has either **a finite number of values** or **a countable number of values**, where **"countable"** refers to the fact that there might be **infinitely many values**, but they can be associated with a **counting process**.

Devices Used to Count and Measure Discrete

Discrete Random Variable: Count of the number of movie patrons.

Examples: Discrete

- **Examples:**
- Let x the **number of eggs** that a hen lays in a day. This is a *discrete* random variable
- The **count of the number of statistics** students present in class on a given day is a whole number and is therefore a discrete random variable

Probability Distribution

A **probability distribution** consists of the values of a random variable and their corresponding probabilities.

A **discrete variable** has a countable number of values (countable means values of zero, one, two, three, etc.). For example, **when four coins are tossed**, the outcomes for the number of heads obtained are zero, one, two, three, and four. **When a single die is rolled**, the outcomes are one, two, three, four, five, and six. These are examples of discrete variables.

Discrete Probability Distributions [1]

Example: Construct a discrete probability distribution for the number of heads when three coins are tossed.

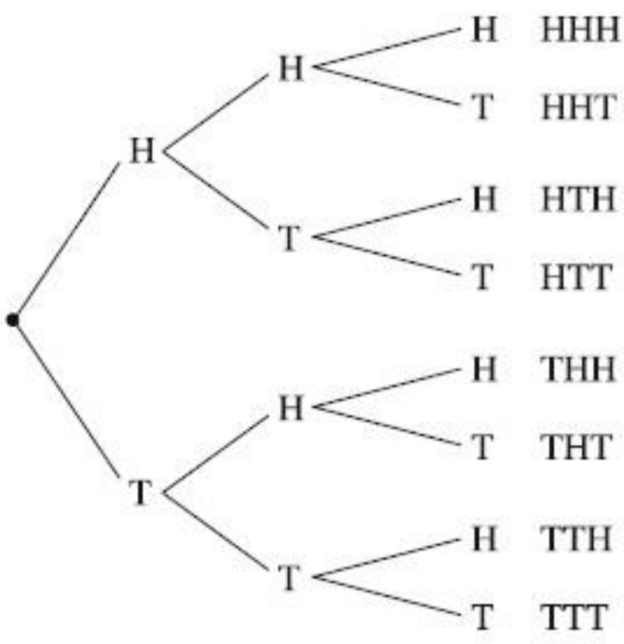
Solution: Recall that the sample space for tossing three coins is TTT, TTH, THT, HTT, HHT, HTH, THH, and HHH. The outcomes can be arranged according to the number of heads, as shown.

0 heads TTT

1 heads TTH

2 heads THH

3 heads HHH



Probability Distribution	
Value, x	Probability, P(x)
0	$\frac{1}{8} = 0.1250$
1	$\frac{3}{8} = 0.3750$
2	$\frac{3}{8} = 0.3750$
3	$\frac{1}{8} = 0.1250$
	$\sum P(x) =$ $\frac{8}{8} = 1$

Discrete Probability Distributions [2]

- The **probability distribution** of a **discrete random variable X** is a list or table of the distinct numerical values of X and the probabilities associated with those values.
- The probability distribution is usually given in tabular form or in the form of an equation. For example, the discrete probability distribution for the previous problem is

x	1	2	3	4	5	6	
P(x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	**$\sum P(x)$ =** $\frac{6}{6}$ =1

Properties Of Discrete Probability Distribution

- Every probability distribution must satisfy the following two properties:

- Probability ranges from 0 to 1 i.e.,

  $$0 \leq P(x) \leq 1$$

- Sum of probabilities is one i.e.,

  $$\sum P(x) = 1$$

Discrete Probability Distributions

Some of the important type of discrete probability distributions are:

1. **Binomial Probability Distribution**

2. **Hypergeometric Distribution**

Binomial Distribution [1]

A **binomial distribution** is obtained from a probability experiment called a binomial experiment. The experiment must satisfy these conditions:

1. Each trial can have only **two outcomes** or outcomes that can be **reduced to two outcomes**. The outcomes are usually considered as a success or a failure.
2. There is a **fixed number** of trials.
3. The outcomes of each trial are **independent** of each other.
4. The probability of a **success** must remain the **same** for each trial.

- **Binomial probability distributions** allow us to deal with **circumstances** in which the outcomes belong to ***two* relevant categories**, such as **acceptable defective** or **survived died.** Other requirements are given in the following definition.

A Binomial Probability Distribution

- A **binomial probability distribution** results from a procedure that meets all the following requirements:

1. The procedure has a ***fixed number of trials***.

2. The trials must be ***independent.*** (The outcome of any individual trial doesn't affect the probabilities in the other trials.)

3. Each trial must have all outcomes **classified into *two categories*** (commonly referred to as *success* and *failure*).

4. The probability of a **success remains the same** in all trials.

Notation for Binomial Probability Distributions

- **S and F (success and failure)** denote the two possible categories of all outcomes; ***p*** and ***q*** will denote the **probabilities of S and F**, respectively.

- **Note:** Be sure that x and p both refer to the *same* category being called a success.

Notation for Binomial Probability Distributions

$P(S) = p$	(p probability of a success)
$P(F) = 1 – p = q$	(q probability of a failure)
n	denotes the fixed number of trials.
x	denotes a specific number of successes in *n* trials, so *x* can be any whole number between 0 and *n*, inclusive.
p	denotes the probability of *success* in *one* of the *n* trials.
q	denotes the probability of *failure* in *one* of the *n* trials.
P(x)	denotes the probability of getting exactly *x* successes among the *n* trials.

Binomial Distribution [2]

Example: Explain why the probability experiment of tossing three coins is a binomial experiment.

Solution:

1. There are only **two outcomes** for each trial, head and tail. Depending on the situation, either heads or tails can be defined as a success and the other as a failure.
2. There is a **fixed number** of trials. In this case, there are three trials since three coins are tossed or one coin is tossed three times.
3. The outcomes are **independent** since tossing one coin does not effect the outcome of the other two tosses.
4. The probability of a **success** (say heads) is $\frac{1}{2}$ and it **does not change**. Hence the experiment meets the conditions of a binomial experiment.

Binomial Distribution [3]

The binomial probability formula is used to compute **probabilities for binomial random** variables. The binomial probability formula is given as:

$$b(x; n, p) = c_x^n p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

where $c_x^n = \dfrac{n!}{x!(n-x)!}$

OR

$$b(x; n, p) = \binom{n}{x} p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

where **n** = the total number of trials

x = the number of successes (0, 1, 2, 3, . . . , n)

p = the probability of a success

q = the probability of a failure

p + q = 1

- **Example** Five fair coins are flipped. If the outcomes are assumed independent, find the probability mass function of the number of heads obtained.

Solution

Here n = 5 **(Total number of coins)**

$p = \frac{1}{2}$ **(Probability of head)**

$q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$ **(Probability of tail)**

Let X denotes number of heads

$$\mathbf{b(x; n, p) = \binom{n}{x} p^x q^{n-x}}, \quad \mathbf{x = 0, 1, 2, \ldots, n}$$

$$\therefore b\left(x; 5, \frac{1}{2}\right) = \binom{5}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}, \quad x = 0, 1, 2, \ldots, 5$$

$$P(X = 0) = \binom{5}{0}\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{5} = \frac{1}{32}$$

$$P(X = 1) = \binom{5}{1}\left(\frac{1}{2}\right)^{1}\left(\frac{1}{2}\right)^{4} = \frac{5}{32}$$

$$P(X = 2) = \binom{5}{2}\left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right)^{3} = \frac{10}{32}$$

$$P(X = 3) = \binom{5}{3}\left(\frac{1}{2}\right)^{3}\left(\frac{1}{2}\right)^{2} = \frac{10}{32}$$

$$P(X = 4) = \binom{5}{4}\left(\frac{1}{2}\right)^{4}\left(\frac{1}{2}\right)^{1} = \frac{5}{32}$$

$$P(X = 5) = \binom{5}{5}\left(\frac{1}{2}\right)^{5}\left(\frac{1}{2}\right)^{0} = \frac{1}{32}$$

Probability Distribution

x	P(X = x)	
0	$= \binom{5}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5 = \frac{1}{32}$	
1	$= \binom{5}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4 = \frac{5}{32}$	
2	$= \binom{5}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = \frac{10}{32}$	
3	$= \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = \frac{10}{32}$	
4	$= \binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1 = \frac{5}{32}$	
5	$= \binom{5}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0 = \frac{1}{32}$	
	$\sum_{i=0}^{5} P_i = 1$	