Discrete Structures

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Text book

Discrete Mathematics and Its Application, 7th Edition

Kenneth H. Rosen

References

Chapter 9

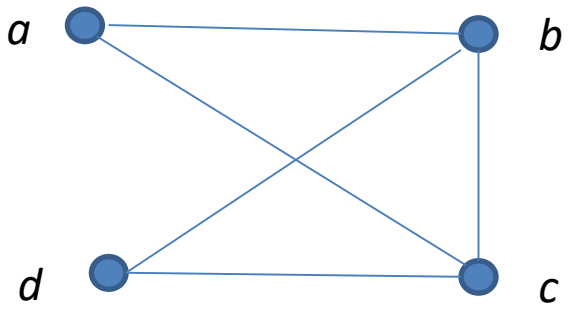
Discrete Mathematics and Its Application, 7th Edition

by Kenneth H. Rose

These slides contain material from the above resource.

Graph

A **graph G = (V ,E)** consists of V, a **nonempty** set of vertices (or nodes) and E, a set of edges. Each **edge** has either **one or two** vertices associated with it, called its **endpoints**. An edge is said to connect its endpoints.



Example: This is a graph with four vertices and five edges.

- The set of vertices *V* of a graph *G* may be **infinite**.
- A graph with an **infinite vertex** set or an infinite number of edges is called an **infinite graph**
- A graph with a **finite vertex set** and a **finite edge set** is called a **finite graph**.
- We will usually consider only finite graphs.

Types of Undirected Graphs:

- **Undirected graphs:** Graphs whose **edges undirected** are called undirected graphs.
- **Simple graph:** A graph in which **each edge** connects **two different vertices** and where **no two edges** connect the same pair of vertices is called a **simple graph**.
- **Multigraphs:** Graphs that may have **multiple edges** connecting the same vertices are called **multigraphs.**
- **Pseudographs**: Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices or a vertex to itself, are sometimes called **pseudographs**.

Directed graph

A **directed graph (or digraph) (V, E)** consists of a **nonempty set of vertices V** and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.

Types of Directed Graphs:

- **Simple directed graph:** When a directed graph **has no loops** and has no multiple directed edges, it is called a **simple directed graph**.

- **Directed multigraphs**: Directed graphs that may have **multiple** directed edges from a vertex to a second (possibly the same) vertex are **directed multigraphs.**

- **Mixed graph:** A graph with both **directed** and **undirected edges** is called a **mixed graph**.

Undirected vs. Directed Graphs

Type	Edges	Multiple Edges?	Loops?
1. Simple graph	undirected	no	no
2. Multigraph	undirected	yes	no
3. Pseudograph	undirected	yes	yes
4. Simple Directed graph	directed	no	no
5. Directed multigraph	directed	yes	yes
6. Mixed graph	directed and undirected	yes	yes

TABLE 1 Graph Terminology.

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

Example of Simple Graph

Now suppose that a network is made up of **data centers** and **communication links** between computers. We can represent the location of **each data center** by **a point** and **each communications link** by a **line segment**, as shown in Figure 1.

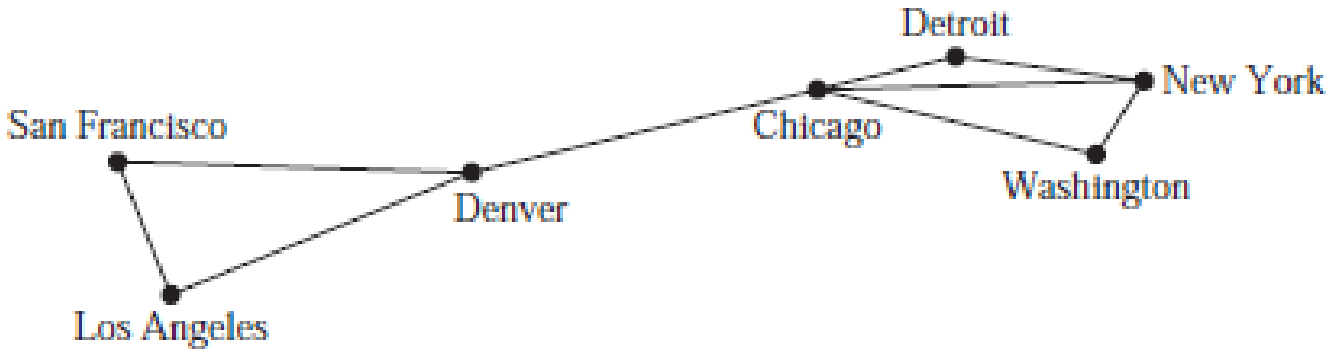


FIGURE 1 A Computer Network.

Example of Multigraph

A computer network may contain **multiple links** between data centers, as shown in Figure 2.

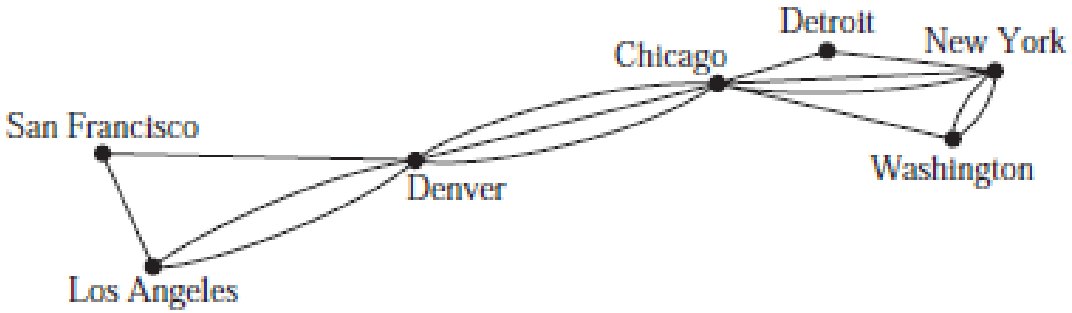


FIGURE 2 **A Computer Network with Multiple Links between Data Centers.**

Example of Pseudograph

Sometimes a **communications link** connects a data center with itself, perhaps a **feedback loop** for **diagnostic purposes**. Such a network is illustrated in Figure 3.

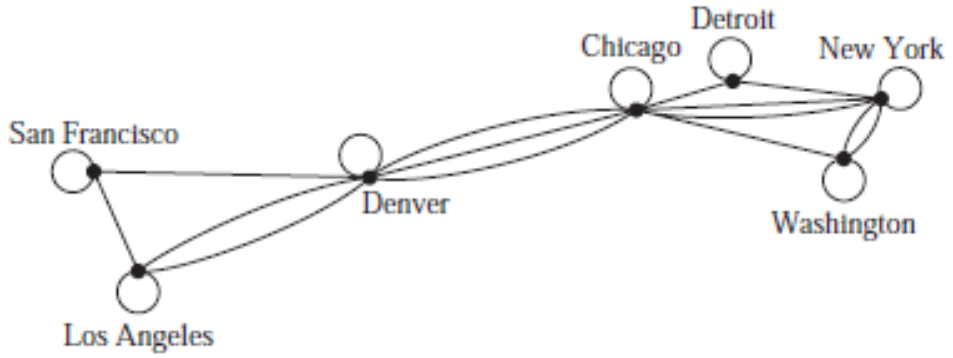


FIGURE 3 **A Computer Network with Diagnostic Links.**

Example of Simple Directed Graph

In a computer network, some links may operate in only one direction (such links are called single duplex lines). This may be the case if there is a large amount of traffic sent to some data centers, with little or no traffic going in the opposite direction. Such a network is shown in Figure 4.

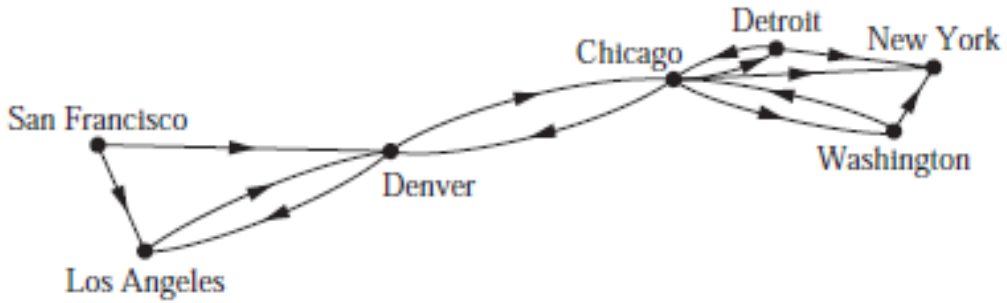


FIGURE 4 **A Communications Network with One-Way Communications Links.**

Example of directed Multigraph

In some computer networks, multiple communication links between two data centers may be present, as illustrated in Figure 5.

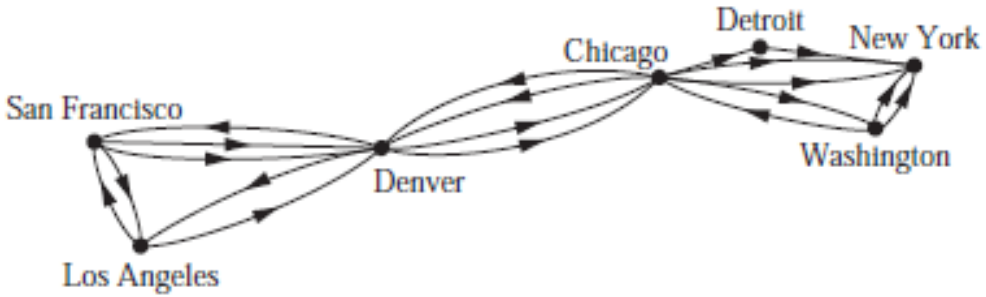


FIGURE 5 A Computer Network with Multiple One-Way Links.

Graph Models

Social Networks: Graphs are extensively used to model social structures based on different kinds of relationships between **people or groups of people**.

These social structures, and the graphs that represent them, are known as **social networks**.

- **Individuals** or **organizations** are represented by **vertices**.
- **Relationships** between **individuals** or **organizations** are represented by **edges**.

Types of Social Networks

- Acquaintanceship and Friendship
- Influence Graphs
- Collaboration Graphs

Acquaintanceship

We can use a **simple graph** to represent whether two people know each other, that is, whether they are acquainted or whether they are friends (either in the real world in the virtual world via a social networking site such as **Facebook**).

- **Each person** in a particular group of people is represented by a **vertex**.
- An **undirected edge** is used to connect **two people** when these people know each other, when we are concerned only with acquaintanceship, or whether they are friends.
- **No multiple edges** and usually **no loops** are used.

Example of Acquaintanceship

The acquaintanceship graph of all people in the world has more than **six billion vertices** and probably **more than one trillion edges**!

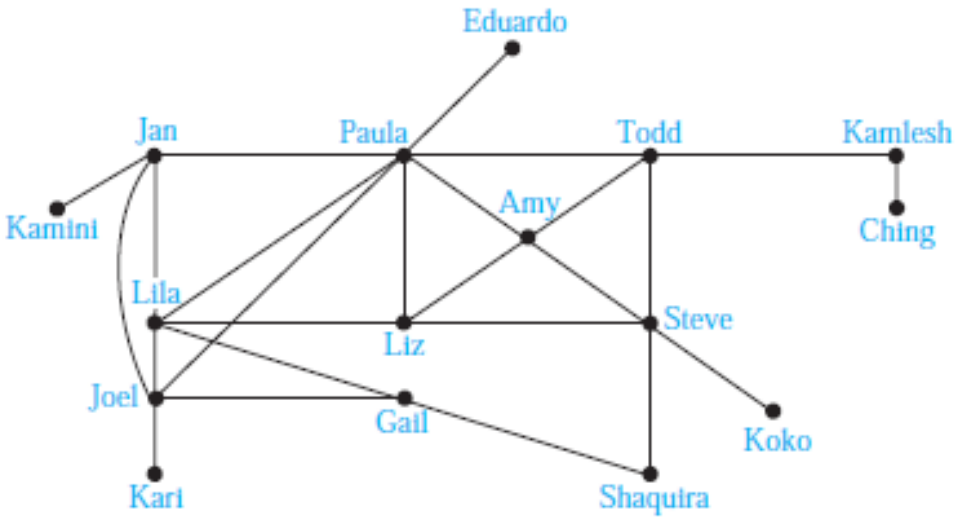


FIGURE 6 **An Acquaintanceship Graph.**

Influence Graphs

In studies of group behavior it is observed that **certain people can influence** the thinking of others. A **directed graph** called an **influence graph** can be used to model this behavior.

- **Each person** of the group is represented **by a vertex**.
- There is a directed edge from **vertex *a* to vertex *b*** when the person represented by **vertex *a* can influence** the person represented by **vertex *b***.
- This graph does **not contain loops** and it does **not contain multiple directed edges**.

Example of Influence Graph

An example of an influence graph for members of a group is shown in **Figure 7.**

In the group modeled by this influence graph, **Deborah cannot** be influenced, but she can **influence Brian**, **Fred, and Linda**. Also, Yvonne and Brian can influence each other.

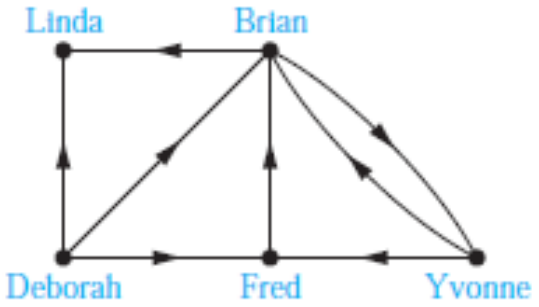


FIGURE 7 An Influence Graph.

Collaboration Graphs

Collaboration graph is used to model social networks where **two people** are related by working together in a particular way.

- **Collaboration graphs** are **simple graphs**, as edges in these graphs are undirected and there are no multiple edges or loops.
- **Vertices** in these graphs represent **people**; two people are connected by **an undirected edge** when the people have collaborated.
- There are **no loops nor multiple edges** in these graphs.

Collaboration Graphs

The **Hollywood graph** is a **collaborator graph** that represents **actors by vertices** and connects **two actors with an edge** if they have **worked together** on a **movie or television show**. The Hollywood graph is a huge graph with more than 1.5 million vertices (as of early 2011).

In an **academic collaboration graph**, vertices represent people (perhaps restricted to members of a certain academic community), and edges link two people if they **have jointly published a paper**. The collaboration graph for people who have published research papers in mathematics was found in 2004 to have more than 400,000 vertices and 675,000 edges, and these numbers have grown considerably since then.

COMMUNICATION NETWORKS We can model different communications networks using **vertices** to represent **devices** and **edges** to represent the particular type of **communications links** of interest.

Call Graphs

- Graphs can be used to model **telephone calls** made in a network, such as a long distance telephone network.
- In particular, a **directed multigraph** can be used to **model calls** where each **telephone number** is represented by **a vertex** and **each telephone call** is represented by a **directed edge**.
- The **edge** representing **a call** starts at the **telephone number** from which the **call was made** and **ends** at the telephone number to which **the call was made**.
- We need **directed edges** because the direction in which the call is made matters.
- We need **multiple directed edges** because we want to represent each call made from a particular telephone number to a second number.

An example of Call Graph

A small telephone call graph is displayed in Figure 8(a), representing seven telephone numbers. This graph shows, for instance, that three calls have been made from **732-555-1234** to **732-555-9876** and two in the other direction, but no calls have been made from **732-555-4444** to any of the other six numbers except **732-555-0011**.

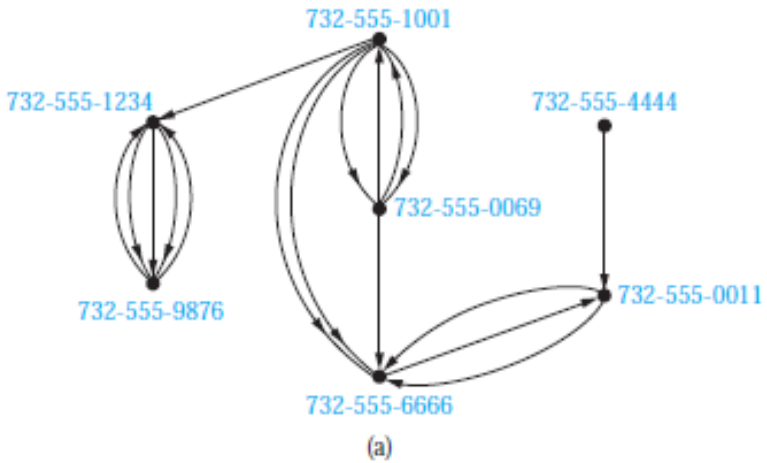
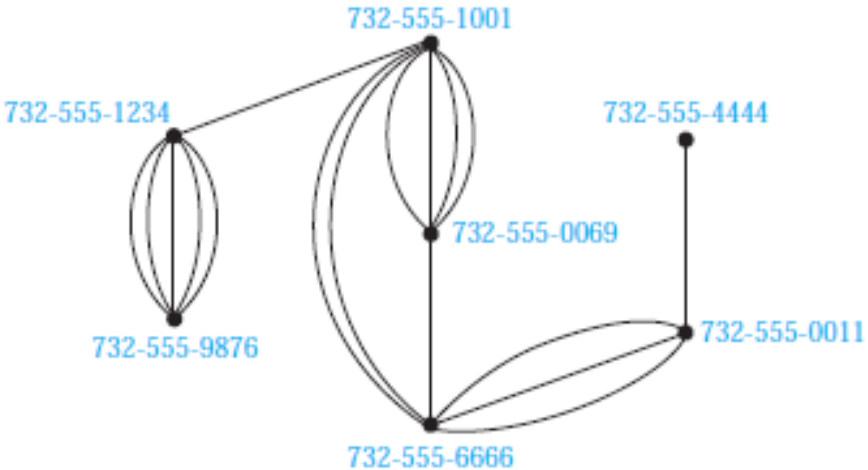


FIGURE 8 A Call Graph.

An example of Call Graphs

When we care only whether there has been a **call connecting two telephone numbers**, we use an undirected graph with an edge connecting telephone numbers when there has been a call between these numbers. This version of the call graph is displayed in Figure 8(b).



INFORMATION NETWORKS

Graphs can be used to model various networks that link particular types of information. Here, we will describe how to model the World Wide Web using a graph.

The Web Graph

The Web Graph The World Wide Web can be modeled as a **directed graph** where **each Web page is represented by a vertex** and where an **edge starts at the Web page *a*** and **ends at the Web page *b* if there is a link on *a* pointing to *b***. Because new Web pages are created and others removed somewhere on the Web almost every second, the Web graph changes on an almost continual basis.

Citation Graphs

Citation Graphs can be used to represent **citations** in different types of documents, including **academic papers, patents, and legal opinions**. In such graphs, each **document** is represented by **a vertex**, and there is an edge from one document to a second document if the **first document cites** the **second** in its citation list

Software Design Applications

Software Design Applications Graph models are useful tools in the design of software.

- **Module Dependency Graphs**
- **Precedence Graphs and Concurrent Processing**

Module Dependency Graph

A **module dependency graph** provides a useful tool for understanding how different modules of a program interact. In a program dependency graph, each module is represented by a vertex. There is a **directed edge** from a module to a second module if the **second module** depends on the **first.**

Example of Module Dependency Graph

An example of a program dependency graph for a web browser is shown in Figure 9

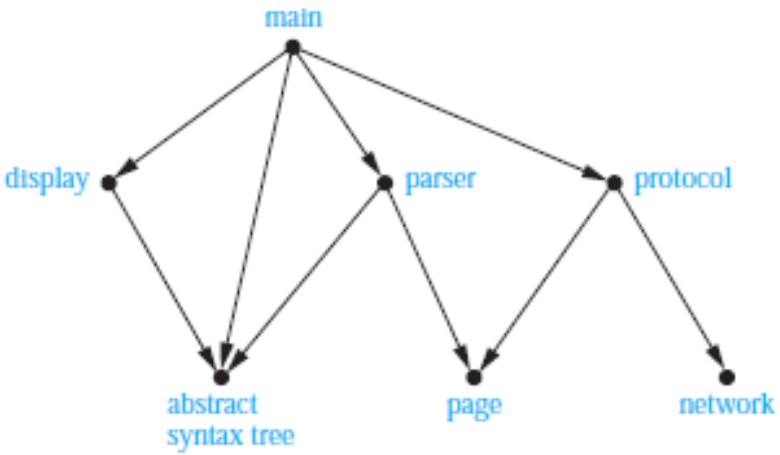


FIGURE 9 A Module Dependency Graph.

Precedence Graphs

Computer programs can be executed more rapidly by executing certain statements concurrently. It is important not to execute a statement that requires results of statements not yet executed. The dependence of statements on previous statements can be represented by a directed graph.

Each statement is represented by **a vertex**, and there is **an edge** from **one statement to a second statement** if **the second statement cannot be executed before the first statement**. This resulting graph is called a **precedence graph**

Precedence Graph

A computer program and its graph are displayed in Figure 10. For instance, the graph shows that statement **S_5 cannot be executed** before statements **S_1, S_2, and S_4** are executed.

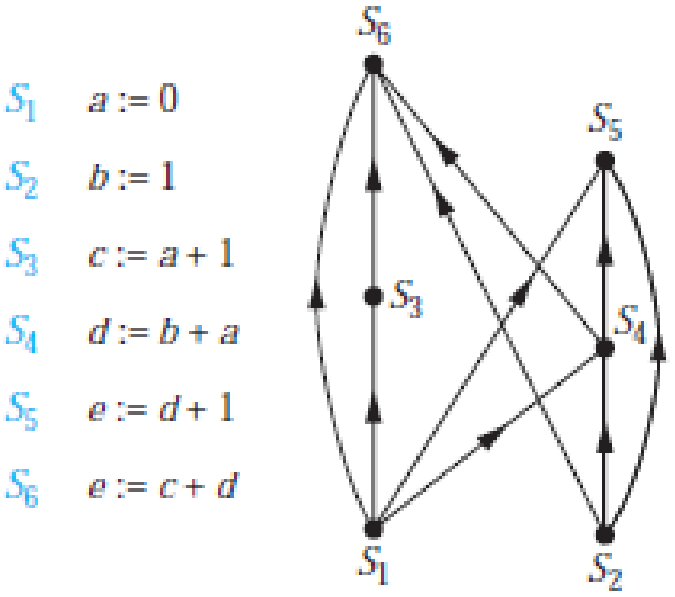


FIGURE 10 A Precedence Graph.

Suggested Readings

10.1 Graphs and Graph Models.