Discrete Structures

Syed Faisal Bukhari, PhD
Associate Professor
Department of Data Science (DDS), Faculty of Computing and Information Technology (FCIT), University of the Punjab (PU)

Text book

Discrete Mathematics and Its Application, 7th Edition

Kenneth H. Rosen

References

Chapter 5

1. Discrete Mathematics and Its Application, 7h Edition

By Kenneth H. Rose

2. Discrete Mathematics with Applications

By Thomas Koshy

These slides contain material from the above resources.

The Growth of Combinations of Functions

Theorem Suppose that $f_1(x)$ is $O(g_1(x))$ and that $f_2(x)$ is $O(g_2(x))$. Then $(f_1 + f_2)(x)$ is $O(\max(|g_1(x)|, |g_2(x)|))$.

Corollary Suppose that $f_1(x)$ and $f_2(x)$ are both $O(g(x))$. Then $(f_1 + f_2)(x)$ is $O(g(x))$.

Theorem Suppose that $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$. Then $(f_1f_2)(x)$ is $O(g_1(x)g_2(x))$.

Example Give a big-O estimate for $f(n) = 3n \log(n!) + (n^2 + 3) \log n$, where n is a positive integer.

Solution:

$f(n) = 3n \log(n!) + (n^2 + 3) \log n$

$\log(n!) = O(n \log n)$

$3n = O(n)$

Suppose that $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$. Then $(f_1f_2)(x)$ is $O(g_1(x)g_2(x))$.

$\Rightarrow 3n \log(n!)$ is $O(n^2 \log n)$.

$(n^2 + 3) \log n$?

Because $(n^2 + 3) < 2n^2$ when $n > 2$, it follows that

$n^2 + 3$ is **$O(n^2)$.**

$\Rightarrow$ **$(n^2 + 3) \log n$** is **$O(n^2 \log n)$.**

Theorem Suppose that $f_1(x)$ is $O(g_1(x))$ and that $f_2(x)$ is $O(g_2(x))$. Then $(f_1 + f_2)(x)$ is $O(\max(|g_1(x)|, |g_2(x)|))$.

Corollary Suppose that $f_1(x)$ and $f_2(x)$ are both $O(g(x))$. Then $(f_1 + f_2)(x)$ is $O(g(x))$.

$\because f(n) = 3n \log(n!) + (n^2 + 3) \log n$

$\Rightarrow$ $3n \log(n!)$ is $O(n^2 \log n)$.

$\Rightarrow$ $(n^2 + 3) \log n$ is $O(n^2 \log n)$.

Corollary Suppose that $f_1(x)$ and $f_2(x)$ are both $O(g(x))$. Then $(f_1 + f_2)(x)$ is $O(g(x))$.

$\because (f_1 + f_2)(x) = O(n^2 \log n)$

Example: Let $f(n) = 6n^2 + 5n + 7\lg n!$ Estimate the growth of f(n)

Solution:

$f(n) = 6n^2 + 5n + 7\lg n!$

$6n^2 = O(n^2)$

$5n = O(n)$

$\because$ **$(f_1 + f_2)(x)$** is **$O(\max(|g_1(x)|, |g_2(x)|))$**

$6n^2 + 5n = O(n^2)$

$7 = O(1)$

$\lg n! = O(n\lg n)$

$\because$ **$(f_1f_2)(x)$ is $O(g_1(x)g_2(x))$**

$7\lg n! = O(1.n\lg n) = O(n\lg n)$

$\because$ **$(f_1 + f_2)(x)$** is **$O(\max(|g_1(x)|, |g_2(x)|))$**

$6n^2 + 5n + 7\lg n!$ is $O(\max(n^2, n\lg n)) = O(n^2)$

big-Omega (Ω)

Let **f** and **g** be functions from the set of integers or the set of real numbers to the set of real numbers. We say that f(x) is **Ω(g(x))** if there are positive constants C and k such that

|f (x)| ≥ C|g(x)| whenever x > k. [This is read as "f (x) is big-Omega of g(x)."]

Note: There is a strong connection between big-O and big-Omega notation. In particular, **f(x) is Ω (g(x))** if and only **if g(x) is O(f(x))**

Example The function $\mathbf{f(x) = 8x^3 + 5x^2 + 7}$ is $\Omega(g(x))$, where $g(x)$ is the function $g(x) = x^3$.

Solution

$f(x) = 8x^3 + 5x^2 + 7$

$|f(x)| \geq C|g(x)|$ whenever $x > k$.

$\because f(x) = 8x^3 + 5x^2 + 7 \geq 8x^3$ for all positive real numbers x.

This is equivalent to saying that $g(x) = x^3$ is $O(8x^3 + 5x^2 + 7)$, which can be established directly by turning the inequality around.

big-Theta(Θ)

- Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that f(x) is **Θ(g(x)) if f(x) is O(g(x)) and f(x) is Ω(g(x)).** When f(x) is Θ(g(x)) we say that "f is big-Theta of g(x)'', that f(x) is of order g(x), and that f (x) and g(x) are of the same order.

- When f (x) is Θ(g(x)), it is also the case that g(x) is Θ(f (x)). Also note that f(x) is Θ(g(x)) if and only if f (x) is O(g(x)) and g(x) is O(f (x)).

- **f (x) is** Θ**(g(x))** if and only if there are **real numbers C_1 and** C_2 and a positive real number k such that **$C_1|g(x)| \leq |f(x)| \leq C_2|g(x)|$** whenever $x > k$. The existence of the constants C_1, C_2, and k tells us that f (x) is Ω(g(x)) and that f (x) is O(g(x)), respectively

Example Show that $3x^2 + 8x \log x$ is $\Theta(x^2)$.

Solution:

$C_1|g(x)| \leq |f(x)| \leq C_2|g(x)|$ whenever $x > k$

For O:

$|f(x)| \leq C_2|g(x)|$ whenever $x > k$.

$0 \leq 3x^2 + 8x \log x \leq 3x^2 + 8x^2 \leq 11x^2$ whenever **$x > 1$**

$3x^2 + 8x \log x \leq \mathbf{11}x^2$

$3x^2 + 8x \log x \leq Cg(x)$ whenever **$x > 1$**

$C_2 = 11, k = 1$

Note: C_2 and k are constants, whereas k is a positive real number and C is a real number

For Ω:

$|f(x)| \geq C_1|g(x)|$ whenever $x > k$.

$3x^2 + 8x \log x \geq \mathbf{3}x^2$ whenever $x > 1$.

$3x^2 + 8x \log x \geq C_2|g(x)|$

$C_1 = 3, k = 1$

Note: C_1 and k are constants, whereas k is a positive real number and C is a real number

$3x^2 + 8x \log x$ is $\Theta(x^2)$.

Theorem Let $\mathbf{f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}$, where $a_0, a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$. Then ***f(x)*** is of order $\mathbf{x^n}$.

Example The polynomials

$3x^8 + 10x^7 + 221x^2 + 1444$,

$x^{19} - 18x^4 - 10{,}112$,

and

$-x^{99} + 40{,}001x^{98} + 100{,}003x$

are of orders x^8, x^{19}, and x^{99}, respectively.

Complexity of Algorithms

Questions such as these involve the **computational complexity** of the algorithm.

- An analysis of the **time** required to solve a problem of a particular size involves the **time complexity** of the algorithm.

- An analysis of the **computer memory** required involves the **space complexity** of the algorithm.

TABLE 1 Commonly Used Terminology for the Complexity of Algorithms.

Complexity	*Terminology*
$\Theta(1)$	Constant complexity
$\Theta(\log n)$	Logarithmic complexity
$\Theta(n)$	Linear complexity
$\Theta(n \log n)$	Linearithmic complexity
$\Theta(n^b)$	Polynomial complexity
$\Theta(b^n)$, where $b > 1$	Exponential complexity
$\Theta(n!)$	Factorial complexity

Induction and Recursion

Many **mathematical statements** assert that a property is true for all **positive integers**.

Examples of such statements are that for every positive integer n:

- $n! \leq n^n$
- $n^3 - n$ is divisible by 3
- A set with n elements has 2^n subsets
- The sum of the first n positive integers is $n(n + 1)/2$.

Proofs using mathematical induction have two parts.

First, they show that the statement holds for the **positive integer 1** .

Second, they show that if the statement holds for a **positive integer** then it must also hold for the **next larger integer.**

Mathematical induction is based on the **rule of inference** that tells us that if **$P(1)$** and **$\forall k(P(k) \rightarrow P(k + 1))$** are true for the domain of positive integers, then **$\forall P(n)$** is true.

Mathematical Induction

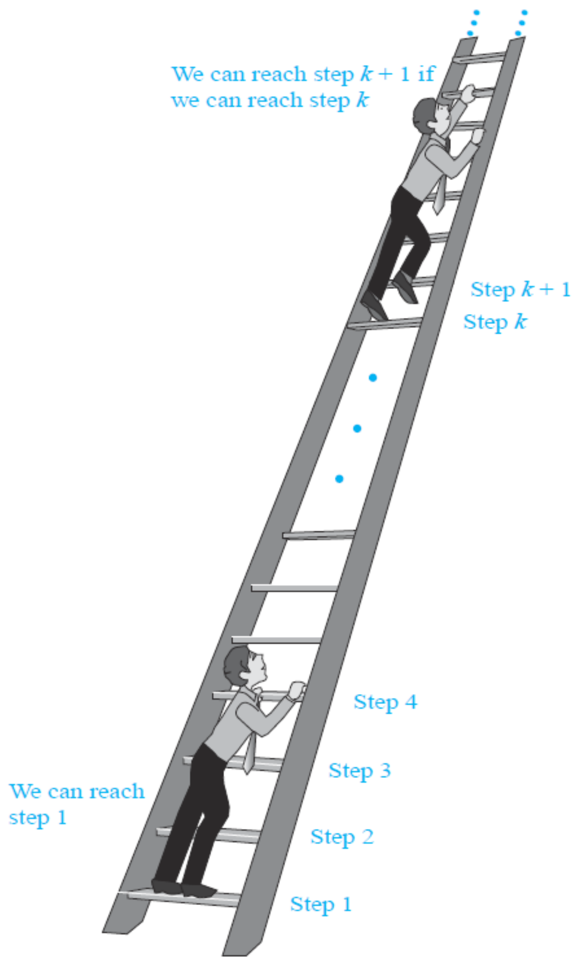


FIGURE 1 **Climbing an Infinite Ladder.**

Mathematical Induction

Introduction: Suppose that we have an **infinite ladder**, as shown in **Figure 1** , and we want to know whether we can reach every step on this ladder.

We know two things:

1 . We can reach the **first rung** of the ladder.

2. If we can reach a particular **rung of the ladder**, then we can reach **the next rung**.

Mathematical Induction

- Can we conclude that we can reach every rung? By (1), we know that we can reach the first rung of the ladder.

- Moreover, because we can reach the first rung, by (2), we can also reach the second rung; it is the next rung after the first rung. Applying (2) again, because we can reach the second rung, we can also reach the third rung.

- Continuing in this way, we can show that we can reach the fourth rung, the fifth rung, and so on.

Mathematical Induction

- For example, after 100 uses of (2), we know that we can reach the 101^{st} rung. But can we conclude that we are able to reach every rung of this infinite ladder?

- The **answer is yes**, something we can verify using an important proof technique called **mathematical induction**. That is, we can show that **P(n)** is true for **every positive integer** n, where P(n) is the statement that we can reach the nth rung of the ladder.

Principle of Mathematical Induction

To prove that $P(n)$ is true for all positive integers n , where $P(n)$ is a propositional function, we complete two steps:

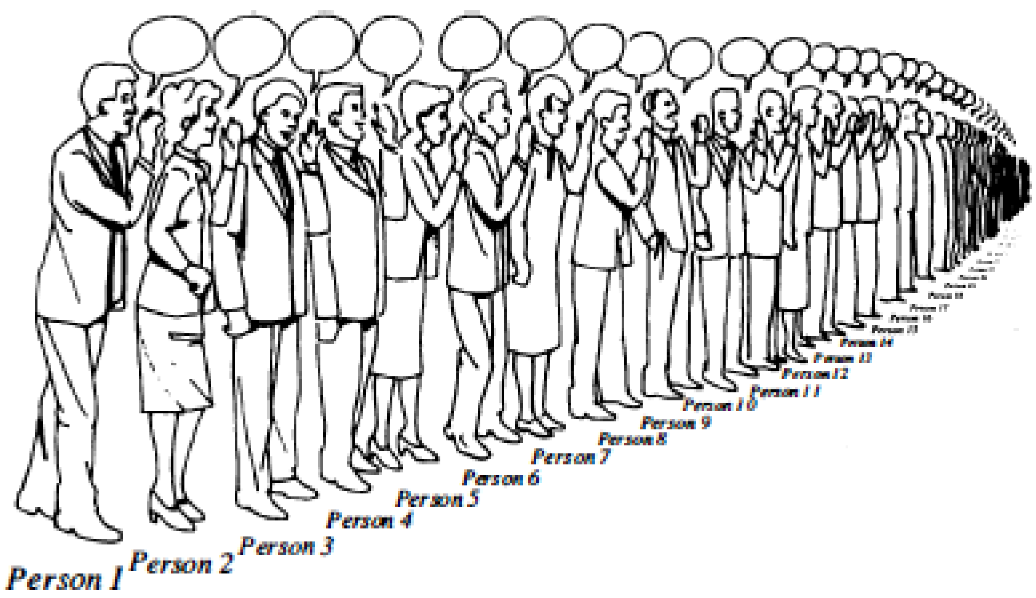
Basis Step: We verify that $P(1)$ is true.

Inductive Step: We show that the conditional statement **$P(k) \rightarrow P(k+1)$** is true for all positive integers k.

Expressed as a **rule of inference**, this proof technique can be stated as

$(P(1) \wedge \forall k\,(P(k) \rightarrow P(k+1))) \rightarrow \forall n\, P(n)$, when the domain is the set of positive integers.

WAYS TO REMEMBER HOW MATHEMATICAL INDUCTION WORKS



WAYS TO REMEMBER HOW MATHEMATICAL INDUCTION WORKS

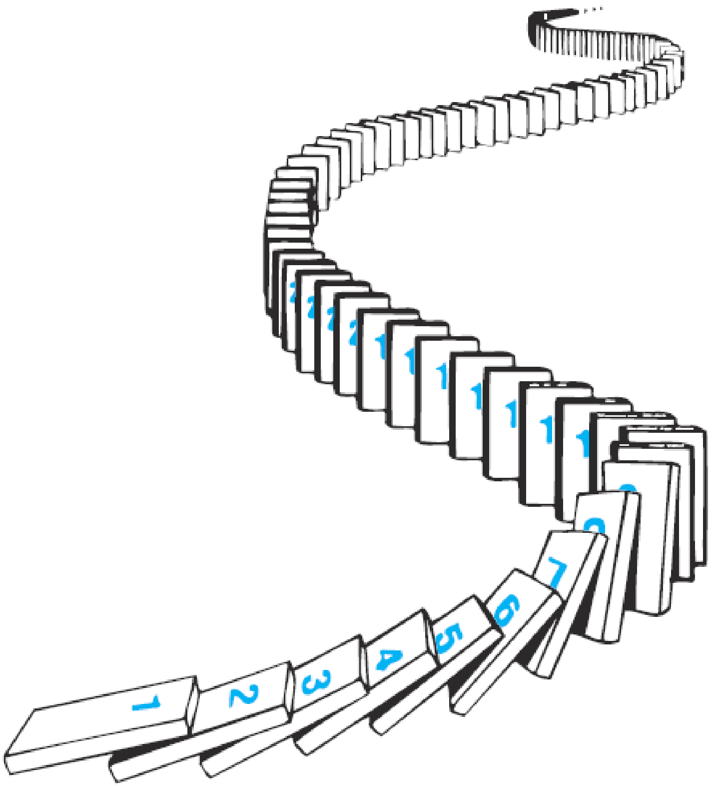


FIGURE 2 Illustrating How Mathematical Induction Works Using Dominoes.

Mathematical induction: Example 1

Example Show that if n is a positive integer, then

$$1 + 2 + \cdot \cdot \cdot + n = \frac{n(n+1)}{2}.$$

Solution

$1 + 2 + \cdot \cdot \cdot + n = \frac{n(n+1)}{2}.$

Let $P(n) = \frac{n(n+1)}{2}$

Basis Step: P(1) is true

$\because 1 = \frac{1(1+1)}{2} = 1$

Inductive Step: Let it will be true n = k

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

Under this assumption, it must be shown that **P(k + 1)** is true

Adding k+1 on both sides

$$1 + 2 + \cdots + k + k + 1 = \frac{k(k+1)}{2} + k + 1$$

$$1 + 2 + \cdots + k + k + 1 = \frac{k(k+1) + 2(k+1)}{2}$$

$$1 + 2 + \cdots + k + k + 1 = \frac{(k+1)(k+2)}{2}$$

$$1+2 + \cdots + k + \overline{k+1} = \frac{(\overline{k + 1})(\overline{k + 1}+1)}{2}$$

It true for n = k+1

We have completed the **basis step** and the **inductive step**, so by mathematical induction we know that **P(n)** is true for all positive integers n.

That is, we have proven that

$1 + 2 + . . . + n = n(n + 1)/2$ **for all positive integers n.**

Suggested Readings

5.1 Mathematical Induction