UNIT-III

UNIT-III: Combinatorics and Recurrence Relations:
Basis of Counting, Permutations, Permutations with Repetitions, Circular and Restricted Permutations, Combinations, Restricted Combinations, Binomial and Multinomial Coefficients and Theorems.

Recurrence Relations:
Generating Functions, Function of Sequences, Partial Fractions, Calculating Coefficient of Generating Functions, Recurrence Relations, Formulation as Recurrence Relations, Solving Recurrence Relations by Substitution and Generating Functions, Method of Characteristic

BASIS OF COUNTING: Counting is a fundamental concept in combinatorics and forms the basis for permutations and combinations. Two primary principles are used:
Addition Principle (Sum Rule): If there are n_1 ways of doing one thing and n_2 ways of doing another thing and these two things cannot happen simultaneously, then there are $n_1 + n_2$ ways of doing either.

Multiplication Principle (Product Rule): If there are n_1 ways of doing one thing and n_2 ways of doing another thing after the first has been done, then there are $n_1 * n_2$ ways of doing both.

PERMUTATIONS refer to the different ways of arranging a set of objects in a specific order. In permutations, the order of the objects is important, meaning that rearranging the objects leads to a different permutation.

TYPES OF PERMUTATIONS

1. **PERMUTATIONS OF DISTINCT OBJECTS (WITHOUT REPETITION)**

Definition: When all the objects are distinct and we want to arrange them in a specific order, the total number of permutations is the factorial of the number of objects.

Formula: For **n** distinct objects, the number of permutations is given by: $\mathbf{n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1}$

Explanation: Here, **n!** represents the total number of ways to arrange **n** distinct objects in a sequence. It assumes that all **n** objects are being used and that no object is repeated.

Example 1: The number of ways to arrange 3 distinct letters (A, B, C) is **3!=6**, which are: ABC, ACB, BAC, BCA, CAB, CBA.

Example 2: Let us determine the number of distinguishable permutations of the letters ELEMENT.

Solution: Suppose we make all the letters different by labelling the letters as follows. E1LE2ME3NT
Now, all the letters are different from each other. In this case, there are (7-1)! = 6! different circular permutations possible.

2. **PERMUTATIONS OF A SUBSET OF OBJECTS (WITHOUT REPETITION)**

Definition: This formula is used when you want to arrange only **r** objects out of a total of **n** distinct objects.

Formula: $\dfrac{\boldsymbol{n}!}{(\boldsymbol{n-r})!}$

Explanation: Here, $\frac{n!}{(n-r)!}$ represents the number of ways to choose and arrange **r** objects from **n** distinct objects, without repetition and with order mattering. The formula works by first calculating **n!** (the number of ways to arrange all **n** objects) and then dividing by **(n−r)!** which accounts for the fact that only **r** objects are being arranged, so the remaining **(n−r)** objects are not considered.

Example: If you have 5 distinct letters (A, B, C, D, E) and you want to find the number of ways to arrange 3 of these letters, the number of permutations is $\frac{5!}{(5-3)!} = \frac{5!}{(2)!} = \frac{120}{2} = 60$ Some of the permutations include ABC, ACB, BAC, etc.

Example: Calculate the following:
(i) ^nP_r when n = 12, r = 5
(ii) $^9P_4$
Solution:
(i) ^nP_r when n = 12, r = 5
$^nP_r = {}^{12}P_5 = 12!/(12 - 5)!$
$= 12!/7!$
$= (12 \times 11 \times 10 \times 9 \times 8 \times 7!)/7!$
$= 12 \times 11 \times 10 \times 9 \times 8$
$= 95040$
(ii) $^9P_4$
$^9P_4 = 9!/(9 - 4)! = 9!/5!$
$= (9 \times 8 \times 7 \times 6 \times 5!)/5!$
$= 3024$

3. **PERMUTATIONS OF DISTINCT OBJECTS (WITH REPETITION ALLOWED)**

Definition: When the objects are distinct, but repetition is allowed, the total number of permutations is determined by the number of positions and the number of objects available for each position.

Formula: If there are n distinct objects and r positions to fill, the number of permutations is given by: n^r

Example: If you have 3 distinct letters (A, B, C) and you want to form 2-letter words with repetition allowed, the number of permutations is **3^2=9**, which are: **AA, AB, AC, BA, BB, BC, CA, CB, CC.**

Example: A person has to choose three-digits from the set of following seven numbers to make a three-digit number. {1, 2, 3, 4, 5, 6, 7} How many different arrangements of the digits are possible?

Solution: A three-digit number can have 2 or three identical numbers. Similarly, in a number, the order of digits is important.

It is given that the person can select 3 digits from the set of 7 numbers. Hence, n = 7 and r = 3. Substitute these values in the formula below to get the number of possible arrangements.

$n^r = 7^3 = 343$

4. **PERMUTATIONS OF NON-DISTINCT OBJECTS (MULTISETS)**

Definition: When some of the objects in a set are identical, we must adjust the permutation count to avoid overcounting the identical arrangements.

Formula: For a set of **n** objects where there are $\mathbf{n_1}$ identical objects of one type, $\mathbf{n_2}$ identical objects of another type and so on, the number of distinct permutations is given by:

$$\frac{\boldsymbol{n}!}{\boldsymbol{n_1}!\,\boldsymbol{X}\boldsymbol{n_2}!\,\boldsymbol{X}\ldots\boldsymbol{n_k}!}$$

Example: Consider the word "BANANA". Here, n=6 (total letters), $\mathbf{n_1}$ =3 (A's), $\mathbf{n_2}$ =2 (N's), $\mathbf{n_3}$ =1 (B). The number of distinct permutations is:

$$\frac{6!}{3!X2!X1!} = \frac{720}{12} = \mathbf{60}$$

1. Find the number of permutations of the letters of the word MICROSOFT.

A. The word MICROSOFT consists of 9 letters, in which the letter 'O' is repeated two times. Therefore, the number of permutations of the letters of the word MICROSOFT = $\frac{9!}{2!1!1!1!1!1!1!1!}$
= 181440.

2. John owns six-coloured pairs of shoes (two red, two blue and two black). He wants to put all these pairs of shoes on the shoe rack. How many different arrangements of shoes are possible?

A. The total number of pair of shoes = n = 6
Number of red shoes p = 2
Number of blue shoes q = 2
Number of black shoes= r = 2
This is an example of permutation with repetition because the elements are repeated and their order is important.
Put the above values in the formula below to get the number of permutations:
$\frac{6!}{2!2!2!}$ = 7208 = 90
Hence, shoes can be arranged on the shoe rack in 90 ways.

3. Consider arranging the letters of the word "MATH". We want to find the number of distinct permutations possible.

A. Given all the letters in the word "MATH" are distinct, we can directly apply the permutation formula without considering repetition.
Using the formula for permutations without repetition:
P = $\frac{n!}{n_1!Xn_2!X....Xn_{k!}}$
In this case, since all the letters are distinct, we have:
P = $\frac{4!}{1!1!1!1!}$
P = $\frac{24}{1*1*1*1}$
P = 24
Therefore, there are 24 distinct permutations of the letters in the word "MATH".

5. **CIRCULAR PERMUTATIONS**

Definition: Circular permutations consider arrangements around a circle where the order is significant, but rotations of the same arrangement are considered identical.

Formula: For n objects arranged in a circle, the number of distinct permutations is:

(n−1)!

Example: For 4 people sitting around a circular table, the number of distinct seating arrangements is $(4-1)! = 3! = 6$.

Example: Let us determine the number of distinguishable permutations of the letters ELEMENT.
A. Suppose we make all the letters different by labelling the letters as follows.
E1LE2ME3NT
Now, all the letters are different from each other. In this case, there are (7-1)! = 6! different circular permutations possible.

6. **PERMUTATIONS WITH RESTRICTIONS**

Definition: Sometimes, there are specific constraints on the arrangement, such as certain objects must or must not be next to each other. These problems require customized approaches based on the given restrictions.

Example: If you need to arrange 4 people (A, B, C, D) such that A must always be next to B, you can treat A and B as a single unit, reducing the problem to arranging 3 "units" (AB, C, D). Then, the number of arrangements would be $3!\times 2!=12$, where 2! accounts for the internal arrangement of A and B.

COMBINATIONS refer to the selection of items from a larger set, where the order of selection does not matter. Unlike permutations, where the arrangement or order of items is important, combinations are simply about which items are selected, regardless of their arrangement.

TYPES OF COMBINATIONS

1. **COMBINATIONS WITHOUT REPETITION (SIMPLE COMBINATIONS)**

Definition: This is the most basic type of combination, where each item can be selected only once, and the order of selection does not matter.

Formula: If you have **n** distinct objects and you want to choose **r** of them, the number of combinations is given by:

$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$ This is often read as "n choose r."

Example: If you have 5 fruits (Apple, Banana, Cherry, Date, and Elderberry) and you want to choose 3, the number of ways to do this is $C(5, 3) = \binom{5}{3} = \frac{5!}{3!(5-3)!} = 10$.

The combinations are: ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE.

2. **COMBINATIONS WITH REPETITION (MULTICHOOSE)**

Definition: This type of combination allows for the same item to be selected more than once, meaning repetition is allowed.

Formula: If you have **n** distinct objects and you want to choose **r** of them, allowing for repetition, the number of combinations is given by:

$C_{rep}(n\ \ r) = (\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}$

Example: If you have 3 types of candies (A, B, C) and you want to pick 4 candies (allowing for repeats), the number of combinations is $C_{rep}(3\ \ 4) = (\binom{3+4-1}{4} = \frac{(3+4-1)!}{4!(3-1)!} = 15$. Some of the combinations include AAAB, AABC, BBBB, etc.

3. **COMBINATIONS OF NON-DISTINCT OBJECTS (MULTISETS)**

Definition: This is similar to combinations with repetition, but it focuses on selecting items from a set that may include identical objects.

Formula: This situation is typically handled using the same formula as combinations with repetition, depending on the exact problem structure.

Example: Suppose you have a set of letters {A, A, B, B, C}, and you want to choose 3 letters. The approach would depend on how many of each type are chosen. For instance, choosing {A, B, C} would be one combination, and {A, A, B} would be another.

4. **COMBINATIONS IN A MULTISET WITH SPECIFIC CONSTRAINTS**

Definition: Sometimes, combinations involve specific rules or constraints, such as requiring a certain number of items from different categories.

Example: Suppose you have 10 balls, 3 red, 4 blue, and 3 green, and you need to choose 5 balls with at least one from each color. The problem can be approached by first ensuring the constraint is met (e.g., choosing one ball from each color) and then selecting the remaining items.

5. **CIRCULAR COMBINATIONS**

Definition: Circular combinations consider selections made in a circular or cyclical manner, where the arrangement still does not matter, but the cyclic nature might impose specific conditions.

Example: Consider seating 5 people around a circular table where the order does not matter. However, in combinations, this is not as common because the focus is on selection, not arrangement.

RESTRICTED COMBINATIONS:

Restricted combinations involve selecting objects under certain conditions.

Example: Suppose you must choose **r** objects from **n**, but one specific object cannot be selected. The number of such restricted combinations is: C(n−1, r)

These concepts form the backbone of combinatorics and are widely applied in various fields, including probability, statistics, and computer science.

BINOMIAL AND MULTINOMIAL COEFFICIENTS AND THEOREMS:

Binomial Coefficients and Theorems:

Definition: The binomial coefficient $\binom{n}{r}$ represents the number of ways to choose r elements from a set of n elements without regard to the order of selection. A Binomial Theorem describe the algebraic expansion of powers of a binomial with two variables. It is defined as:

Let n ∈ N, (x,y) ∈ R then

$$\mathbf{(x+y)^n = \sum_{r=0}^{n} \; n_{c_r}\, x^{n-r} y^r \quad \text{where} \quad n_{c_r} = \frac{n!}{(n-r)!r!}}$$

Here n_{c_0}, n_{c_1}, n_{c_2}...... n_{c_n} are called binomial coefficients and also represented by $C_0, C_1......C_n$.

Multinomial Coefficients and Theorems:

Definition: The multinomial coefficient $\binom{n}{n_1 * n_2 * \dots * n_k}$ represents the number of ways to divide n items into m groups of sizes $n_1 * n_2 * \dots * n_k$ where $n_1 + n_2 + \cdots + n_k$=n. multinomial theorem in algebra, a generalization of the binomial theorem to mor than two variables. It is defined as:

$$\boldsymbol{(x_1 + x_2 + \cdots + x_k)^n = \sum_{n_1 * n_2 * \dots * n_k} \frac{n!}{n_1! * n_2! * \dots * n_k!} (x_1)^{n_1} (x_2)^{n_2} \dots\dots (x_k)^{n_k}}$$

1. Find out the coefficients of x^9y^3 in the expansion of $(x+2y)^{12}$

Solution: Binomial theorem $(x+y)^n = \sum_{r=0}^{n} n_{c_r} x^{n-r} y^r$

According to the binomial theorem **x=x, y=2y, n=12.**

$$(x+y)^{12} = \sum_{r=0}^{12} 12_{c_r} x^{12-r} 2y^r$$

$$= \sum_{r=0}^{12} 12_{c_r} 2^r x^{12-r} y^r \text{-----------}1$$

We have to find out x^9y^3

$x^9 = x^{12-r}$

$9 = 12-r$

$r = 12-9$

r =3

$y^3 = y^r$

r = 3

'**r**' value substituted in equation 1

$= 12_{c_3} 2^3 x^{12-3} y^3$

$= \frac{12*11*10}{1*2*3} 2^3 x^9 y^3$

$= 1760\, x^9 y^3$

∴ the coefficients of x^9y^3 in the expansion $(x+2y)^{12}$ is 1760.

2. Find out the coefficients of x^5y^2 in the expansion of $(2x-3y)^7$?

3. Compute the following $\binom{7}{2,\ 3,\ 2}$

Solution: multinomial theorem $\binom{n}{n_1 * n_2 * \ldots * n_k}$ where $n_1+n_2+\ldots+n_k=n$

By applying the multinomial theorem $\frac{n!}{n_1!*n_2!*\ldots*n_k!}$ where $n_1=2$, $n_2=3$, $n_3=2$ and $n_1+n_2+n_3=n$

$\frac{7!}{2!*3!*2!}$ **= 210.**

4. Compute the following $\binom{8}{4,\ 2,\ 2,\ 0}$

5. Compute the following $\binom{12}{5,\ 3,\ 2,\ 2}$

6. Compute the following $\binom{4}{1,\ 1,\ 2}$

7. Determine the coefficients of xyz^2 in the expansion of $(2x-y-z)^4$?

Solution: By applying the multinomial theorem

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{n_1*n_2*\ldots*n_k} \frac{n!}{n_1!*n_2!*\ldots*n_k!} (x_1)^{n_1}(x_2)^{n_2}\ldots\ldots(x_k)^{n_k} \text{ where } n_1 + n_2 + \ldots + n_k = n$$

The given expression is $(2x-y-z)^4$ where n=4, x_1 =2x, x_2=-y, x_3=-2

$= \frac{4!}{n_1!*n_2!*n_3!}\ (2x)^{n_1}(-y)^{n_2}(-z)^{n_3}$-----------------**1**

We have to find out the coefficients xyz^2 where $n_1=1$, $n_2=1$, $n_3=2$

Substituting n_1, n_2, n_3 values in equation 1

$= \frac{4!}{1!1!2!}\ (2x)^1(-y)^1(-z)^2$

$= \frac{4*3*2*1}{2*1}(2)^1\ xyz^2$

= -24 xyz^2

∴ the coefficients of xyz^2 in the expansion $(2x-y-z)^4$ is -24.

8. Determine the coefficients of $a^2b^3c^2d^5$ in the expansion of $(a+2b-3c+2d+5)^4$?
9. Find the coefficients of $x^3y^3z^2$ in the expansion of $(2x-3y+5z)^8$?
10. Find the coefficients of $x^{11}y^4z^2$ in the expansion of $(2x^3-3xy^2+z^2)^6$?

GENERATING FUNCTIONS AND FUNCTION OF SEQUENCES:

Consider a sequence of real numbers a_0, a_1, a_2….. let us denote the sequence by $a_r = a_0, a_1, a_2$….. where r = 0,1,2….

Suppose there exists a function f(x) where expansion in a series of powers of x is

$f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \ldots$..

f(x) can also written as **f(x) = $\sum_{r=0}^{n} a_n\ x^r$** here f(X) is called as a generating function for the sequence of a_0, a_1, a_2…..

PARTIAL FRACTIONS (or) PROPERTIES OF GENERATING FUNCTION:

1) **$(1+x)^1 = (1+x)$**
2) **$(1-x)^1 = (1-x)$**
3) **$(1+x)^{-1} = 1-x+x^2-x^3+x^4-x^5+\ldots$.**
4) **$(1-x)^{-1} = 1+x+x^2+x^3+x^4+x^5+\ldots$.**
5) **$(1+x)^{-2} = 1-2x+3x^2-4x^3+5x^4-\ldots$.**
6) **$(1-x)^{-2} = 1+2x+3x^2+4x^3+\ldots$.**
7) **$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\ \frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$**

CALCULATING COEFFICIENT OF GENERATING FUNCTIONS:

1. Find out the sequence generated by the generating function$(1+x)^n$

Solution: Given generating function: $(1+x)^n$

We already know that **$(x+y)^n = \sum_{r=0}^{n}\ \ n_{c_r}\, x^{n-r}y^r$**

$(1+x)^n = n_{c_0}(1)^n x^0 + n_{c_1}(1)^{n-1} x^1 + n_{c_2}(1)^{n-2} x^2 + \ldots$

$= \frac{n!}{o!(n-0)!}(1).1 + \frac{n!}{1!(n-1)!}(1).x + \frac{n!}{2!(n-2)!}(1).x^2 + \frac{n!}{3!(n-3)!}(1).x^3 + \ldots$

$= \frac{n!}{1n!}.1 + \frac{n(n-1)!}{1(n-1)!}.x + \frac{n(n-2)!}{2(n-2)!}.x^2 + \frac{n(n-1)(n-2)(n-3)!}{6(n-3)!}.x^3 + \ldots$

$= 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)!}{3!}x^3+\ldots$

$= 1+n_{c_1}x+n_{c_2}x^2+n_{c_3}x^3+\ldots$

$= \sum_{r=0}^{n}\ \ n_{c_r}\, x^r$

$= \sum_{r=0}^{n}\frac{n!}{(n-r)!r!}\ x^r$

2. Find out the sequence generated by the following generating function $(3+x)^3$

Solution: Given generating function is $(3+x)^3$

We already know that $(1+x)^n = \sum_{r=0}^{n} n_{c_r} x^r$

$(3+x)^3 = (3(1+\frac{x}{3}))^3 = 3^3 (1+\frac{x}{3})^3$ where n=3, x=1, y=$\frac{1}{3}$

$= 3^3 [n_{c_0}(1)^n \left(\frac{x}{3}\right)^0 + n_{c_1}(1)^{n-1} \left(\frac{x}{3}\right)^1 + n_{c_2}(1)^{n-2} \left(\frac{x}{3}\right)^2 + n_{c_3}(1)^{n-3} \left(\frac{x}{3}\right)^3]$

$= 27[3_{c_0}(1)^3 \left(\frac{x}{3}\right)^0 + 3_{c_1}(1)^{3-1} \left(\frac{x}{3}\right)^1 + 3_{c_2}(1)^{3-2} \left(\frac{x}{3}\right)^2 + 3_{c_3}(1)^{3-3} \left(\frac{x}{3}\right)^3]$

$= 27[1.1.1+3.1.\frac{x}{3} +3.1.\frac{x^2}{9}+1.1.\frac{x^3}{27}]$

$= 27[1+x+\frac{x^2}{3} + \frac{x^3}{27}]$

$=27+27x+9x^2+x^3$

∴ the sequence generated by the generating function $(3+x)^3$ by 27,27,9,1.

3. Find out the sequence generated by the following generating function $3x^3+e^{2x}$?
4. Find out the sequence generated by the following generating function $(1+3x)^{\frac{-1}{2}}$?
5. Find the generating function for the following sequence 1,2,3,4…?

Solution: Given sequence is 1,2,3,4… ---------------1

We know that $f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \ldots..$ ---------------2

from equation 1 and 2

$a_0=1, a_1=2, a_2=3, a_3=4$ …

the above values are substituted in equation 2

$f(x) = 1 x^0 + 2 x^1 + 3 x^2 + 4 x^3 + \ldots..$

$f(x) = 1 + 2 x + 3 x^2 + 4 x^3 + \ldots..$

∴ the equivalent generating function is $f(x)=(1-x)^{-2}$.

6. Find the generating function for the following sequence 1,-2,3,-4…?
7. Find the generating function for the following sequence 0,1,2,3,4…?
8. Find the generating function for the following sequence 0,1,-2,3,-4…?

RECURRENCE RELATIONS:

A recurrence relation is an equation that recursively defines a sequence based on a rule that gives the next term in the sequence as a function of the previous terms when one or more initial terms are given.

Ex: The Fibonacci sequence is defined by using the recurrence relation

$F_n = F_{n-1} + F_{n-2}$

FORMULATION AS RECURRENCE RELATIONS:

Formulating problems as recurrence relations involves breaking down a problem into smaller sub-problems and expressing the solution to the original problem in terms of solutions to those sub-problems. This approach is widely used in algorithm analysis, dynamic programming, combinatorics, and other fields where recursive structures naturally arise.

Steps to Formulate a Problem as a Recurrence Relation

1. **Identify the sub-problems**: Break the problem into smaller instances of the same problem.
2. **Define the recurrence relation**: Express the solution to the problem for an input size n in terms of solutions for smaller sizes, typically $n-1$, $n-2$, etc.
3. **Specify base cases**: Define the smallest possible input sizes for which the solution is known and does not depend on smaller sub-problems.

Examples of Formulating Recurrence Relations:

1. Fibonacci Numbers (Classic Example)

The Fibonacci sequence is defined as:

$F_n=F_{n-1}+F_{n-2}$ with base cases $F_0=0$ and $F_1=1$.

SOLVING RECURRENCE RELATIONS BY SUBSTITUTION AND GENERATING FUNCTIONS, METHOD OF CHARACTERISTIC:

Solving Recurrence Relations:

There are several methods for solving recurrence relations, depending on their type and complexity:

1. **Substitution (Iteration) Method:** Express the recurrence relation as a series of substitutions until a recognizable pattern emerges.

   - Example: $a_n=2a_{n-1}$, $a_0=3$

   1.find the recurrence relation for the given sequence 2,6,18,54,162….

   Solution: $a_1=2$

   $a_2=3a_1=3*2=6$

   $a_3=3a_2=3*6=18$

   $a_4=3a_3=3*18=54$

   $a_5=3a_4=3*54=162$….

   ∴ the equivalent recurrence relation is $a_n=3a_{n-1}$

   2. find the recurrence relation for the given sequence 20,17,14,11,8….

   3. find the recurrence relation for the given sequence 1,3,6,10,15,21….

   4.find the sequence generated by the recurrence relation $a_n=2a_{n-1}$ with $a_1=4$

   Solution: the given recurrence relation is $a_n=2a_{n-1}$ initial condition $a_1=4$

   $n=2 \equiv a_2=2a_{2-1} \equiv a_2=2a_1 \equiv a_2=2*4 \equiv 8$

   $n=3 \equiv a_3=2a_{3-1} \equiv a_3=2a_2 \equiv a_3=2*8 \equiv 16$

   $n= \equiv a_4=2a_{4-1} \equiv a_4=2a_3 \equiv a_4=2*16 \equiv 32$

   ∴ recurrence sequence is S=4,8,16,32……

   5. find the sequence generated by the recurrence relation $T_n=3T_{n-1}$ with $T_1=3$

2. **Characteristic Equation Method:** Used for linear homogeneous recurrence relations with constant coefficients. The relation is converted into a characteristic polynomial, which is solved to find the general form of the sequence.

   In this solved by any method based on the order. **First order of recurrence relation is $a_n = A(r)^n$**

   Second order of recurrence relation the general form is classified into 3 types

   1. **Distinct recurrence relation of real roots: $a_n = A(r_1)^n +B(r_2)^n$**
   2. **Repeated recurrence relation real roots: $a_n = A(r)^n +n.B(r)^n$**

Orders(first and second) followed by some instructions:

1. Arrange linear homogenous recurrence relation
2. Convert into root form(both orders)
3. Divided by smallest nth value(both orders)
4. Find out root value(both orders)
5. Substitute general forms(both orders)
6. Find out constants value with help of initial condition.

1. Solve the recurrence relation $a_{n+1} = 4a_n$ given $a_0=3$

Solution: given the equation $a_{n+1} = 4a_n$

Convert into root form $r^{n+1} = 4r^n$

n+1=n

n+1-n=0

∴ 1(first order)

Divided by the smallest power 'r^n' both

$$\frac{r^{n+1}}{r^n} = 4\frac{r^n}{r^n}$$

r=4 (Find out root value(both orders)

general form $\mathbf{a_n = A(r)^n}$

$a_n = A(4)^n$

with the help of initial condition, we have to find at constant value 'A'

$a_0 =3$, n=0

$a_0 = A(4)^0$

A=3

We have 'A' value in equation 1

$\therefore\ a_n = (3)(4)^n$.

2. Solve the recurrence relation $a_{n+1} = 2a_n$ given $a_0=1$?
3. Solve the recurrence relation $4a_n - 5a_{n+1} =0$ given $a_0=1$?
4. Solve the recurrence relation $3a_{n+1} - 4a_n =0$,$n\geq 0$ given $a_1=5$?
5. Using the generating function solve the recurrence relation $a_n-a_{n-1}-6a_{n-2}=0$ given $a_0=2$, $a_1=1$?

Solution: Given recurrence relation $a_n-a_{n-1}-6a_{n-2}=0$

Convert into root form $r^n-r^{n-1}-6r^{n-2}=0$

Divided by the smallest power 'r^{n-2}' both

[**order:** difference between the largest and smallest subscript is called order.]

Order: n-(n-2)

n-n+2

2 (second order)

$$\frac{r^n}{r^{n-2}} - \frac{r^{n-1}}{r^{n-2}} - 6\frac{r^{n-2}}{r^{n-2}} =0$$

$r^2-r-6=0$

$r^2-3r+2r-6=0$

$r(r-3)+2(r-3)=0$

$(r-3)(r+2)$

$\therefore\ r_1=3,\ r_2=-2$

General form for the 2nd order linear recurrence relation $a_n=A(r_1)^n+B(r_2)^n$

$a_n=A(3)^n+B(-2)^n$ -------------1

with the help of initial condition, we have to final at A & B values

for $a_0=2$, n=0

$a_0=A(3)^0+B(-2)^0$

$2=A+B$ ----------------2

for $a_1=1$, n=1

$a_1=A(3)^1+B(-2)^1$

$1=3A-2B$ -------------3

from equation 2 and equation 3, here equation 2 multiply with 2

$2A+2B=4$

$3A-2B=1$

$5A=5$

$A=\frac{5}{5}$

$A=1$ -----------------4

from equation 2 substitute A value

$A+B=2$

$B=2-1$

$B=1$ ------------------5

Substitute A & B values in general form $a_n=(1)(3)^n+(1)(-2)^n$

$\therefore\ a_n=(1)(3)^n+(1)(-2)^n$

7. Using the generating function solve the recurrence relation $a_n-2a_{n-1}-3a_{n-2}=0,\ n\geq 2$?
8. Using the generating function solve the recurrence relation $y_{n+2}-y_{n+1}-6y_n=0$ given $y_0=2,\ y_1=1$?
9. Using the generating function solve the recurrence relation $a_n=6a_{n-1}-9a_{n-2}$ given $a_0=1,\ a_1=1$?

   Solution: Given recurrence relation $a_n=6a_{n-1}-9a_{n-2}$

   Convert into root form $r^n-6r^{n-1}+9r^{n-2}=0$

   Divided by the smallest power 'r^{n-2}' both

   [**order:** difference between the largest and smallest subscript is called order.]

   Order: n-(n-2)

   n-n+2

   2 (second order)

$$\frac{r^n}{r^{n-2}}-6\frac{r^{n-1}}{r^{n-2}}+9\frac{r^{n-2}}{r^{n-2}}=0$$

$r^2-6+9=0$

$r^2-3r-3r+9=0$

$r(r-3)-3(r-3)=0$

$(r-3)(r-3)$

$\therefore\ r=3,\ r=3$

General form for the 2^{nd} order linear recurrence relation $a_n=A(r)^n+nB(r)^n$

$a_n=A(3)^n+nB(3)^n$ -------------1

with the help of initial condition, we have to final at A & B values

for $a_0=1$, $n=0$

$a_0=A(3)^0+nB(3)^0$

$1=A$ ----------------2

for $a_1=1$, $n=1$

$a_1=A(3)^1+1B(3)^1$

$1=3(1)+1.B(3)^1$

$1=3+3B$

$1-3=3B$

$B=\frac{-2}{3}$ -------------3

Substitute A & B values in general form $a_n=A(3)^n+nB(3)^n$

$\therefore\ a_n=(1)(3)^n+n(\frac{-2}{3})(3)^n$

10. Using the generating function solve the recurrence relation $a_n=4a_{n-1}-4a_{n-2}=0$, with $a_0=6$, $a_1=8$?