Chapter 8
Statistical Intervals for a Single Sample

Part 1: Confidence intervals (CI) for population mean μ

Section 8.1:
CI for μ when σ^2 known & drawing from normal distribution

Section 8.1.2:
Sample size calculation for estimating μ with specified error, σ^2 known

Section 8.2:
CI for μ when σ^2 unknown & drawing from normal distribution

Confidence Intervals

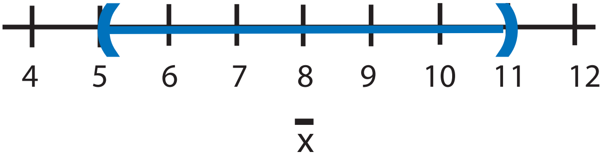
- We end the last chapter with a phrase:
  "Moving beyond point estimates"
- Point estimates are a good start, but we should also give the client some idea of the confidence in our estimate.
- More data gives more information. We will have more confidence in an estimate for μ from an $n = 50$ sample, than an estimate from an $n = 3$ sample.
- The confidence in an estimate is related to the size (or width) of such an interval.

Confidence Interval for μ - Normal parent population, known σ^2

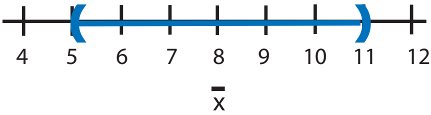
- We use the observed $\bar{x}$ as the point estimate for μ.
- We provide a two-sided CI for μ as a 'window' or interval for which we are fairly confident the unknown population mean μ lies.
- $\bar{x}$ will be at the center of our two-sided CIs

$$[\bar{x} - cushion, \bar{x} + cushion]$$

- For example, suppose $\bar{x} = 8$ and our cushion is 3



Confidence Interval for μ - Normal parent population, known σ^2



- We want to have high confidence that our interval contains μ.
- How do we choose this $\pm$'cushion' so that we have high confidence that it contains μ? Or the length of our interval?
- We use the behavior (or probability distribution) of $\bar{X}$...

$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) \qquad \text{for any sample size } n$$

to form our CI in such a way that we can say something very powerful, like...

"We are 95% confident that the true mean μ falls in this interval."

Confidence Interval for μ - Normal parent population, known σ^2

Right now, we are estimating μ and we say that we know σ^2.
Perhaps not terribly realistic, but we will loosen this up later...

- Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from a normal distribution with $X_i \sim N(\mu, \sigma^2)$ for all i, then

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

- Using this probability distribution, we have

$$P(-z_{0.025} \leq Z \leq z_{0.025}) = 0.95$$
$$P(-z_{0.025} \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq z_{0.025}) = 0.95$$

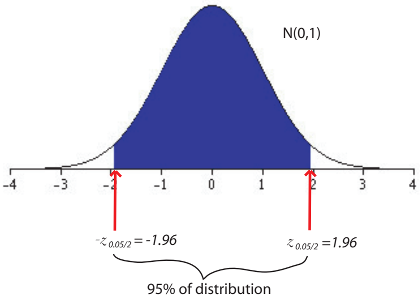
where $z_{0.025}$ is the 97.5th percentile of the standard normal (next slide).

Confidence Interval for μ - Normal parent population, known σ^2

$$P(-z_{0.025} \leq \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \leq z_{0.025}) = 0.95$$

- Manipulating what's inside the parentheses give us the Upper and Lower end-points for our 95% CI for μ...

$$P(\bar{X} - z_{0.025}\frac{\sigma}{\sqrt{n}} \leq \quad \mu \quad \leq \bar{X} + z_{0.025}\frac{\sigma}{\sqrt{n}}) = 0.95$$



NOTATION: $z_{0.025}$ is the z-value such that 97.5% of the distribution is below and 2.5% is above it (an upper tail z-value).

Confidence Interval for μ - Normal parent population, known σ^2

- We can state the lower and upper end-points of the 95% CI for μ from a random sample of size n drawn from a normally distributed population with variance σ^2 and sample mean of $\bar{x}$ as:

$$\text{Lower end-point (L)} = \bar{x} - z_{0.025}\frac{\sigma}{\sqrt{n}}$$

$$\text{Upper end-point (U)} = \bar{x} + z_{0.025}\frac{\sigma}{\sqrt{n}}$$

NOTE: $\bar{x}$ lies in the center of the 2-sided confidence interval.

- 95% CI for μ when σ^2 known and drawing from a normally distributed population:

$$\boxed{\bar{x} - z_{0.025}\frac{\sigma}{\sqrt{n}} \leq \quad \mu \quad \leq \bar{x} + z_{0.025}\frac{\sigma}{\sqrt{n}}}$$

Or...

$$\boxed{\bar{x} - 1.96\frac{\sigma}{\sqrt{n}} \leq \quad \mu \quad \leq \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}}$$

Confidence Interval for μ - Normal parent population, known σ^2

Example (Fill weights of boxes)

The sample mean for the fill weights of 100 boxes is $\bar{x} = 12.050$. The population variance of the fill weights is known to be $(0.100)^2$. Find a **95% confidence interval** for the population mean μ fill weight of the boxes.

ANS:

$$L = \bar{x} - z_{0.025} \cdot \frac{\sigma}{\sqrt{n}} = 12.050 - 1.96 \cdot \frac{0.100}{\sqrt{100}} = 12.030.$$

$$U = \bar{x} + z_{0.025} \cdot \frac{\sigma}{\sqrt{n}} = 12.050 + 1.96 \cdot \frac{0.100}{\sqrt{100}} = 12.070.$$

The 95% confidence interval for μ is $[12.030,\ 12.070]$.

We are 95% confident that the true parameter value lies in this interval.

NOTE: Because σ^2 was very small and n was fairly large, we have a very narrow confidence interval for μ (which is good).

Confidence Interval for μ - Normal parent population, known σ^2

CI for any choice of confidence level, or 100$(1\text{-}\alpha)$% confidence

- The confidence level of choice is stated as $100(1-\alpha)\%$.

  For a 95% confidence interval, $\alpha = 0.05$.
  For an 80% confidence interval, $\alpha = 0.20$.

- We can re-write the earlier Z probability as

$$P(-z_{\alpha/2} \leq \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}) = 1-\alpha$$

  and this leads to the $100(1-\alpha)\%$ confidence interval for μ

$$P(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \quad \mu \quad \leq \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) = 1-\alpha$$

- In a two-sided confidence interval, the α amount is split between the two tails, thus we see $\alpha/2$ or specifically, $z_{\alpha/2}$ in the formula.

Confidence Interval for μ - Normal parent population, known σ^2

- **100(1-α)% Confidence interval on the mean, variance known**
  If $\bar{x}$ is the sample mean of a random sample of size n from a normal population with known variance σ^2, a $100(1-\alpha)\%$ confidence interval for μ is given by

$$\boxed{\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \quad \mu \quad \leq \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}}$$

  where $z_{\alpha/2}$ represents the z-value from the standard normal distribution with $\alpha/2$ in the upper tail (e.g. if $\alpha = .05$, $z_{\alpha/2} = z_{.025} = 1.96$).

- Commonly used z scores

Conf. Level	α	$\alpha/2$	$z_{\alpha/2}$
90%	0.10	0.05	1.645
95%	0.05	0.025	1.96
99%	0.01	0.005	2.576

Confidence Interval for μ - Normal parent population, known σ^2

Example (Fill weights of boxes)

The sample mean for the fill weights of 100 boxes is $\bar{x} = 12.050$. The population variance of the fill weights is known to be $(0.100)^2$. Find a **80% confidence interval** for the population mean μ fill weight of the boxes.

ANS:

$$L = \bar{x} - z_{0.10} \cdot \frac{\sigma}{\sqrt{n}} = 12.050 - 1.28 \cdot \frac{0.100}{\sqrt{100}} = 12.037.$$

$$U = \bar{x} + z_{0.10} \cdot \frac{\sigma}{\sqrt{n}} = 12.050 + 1.28 \cdot \frac{0.100}{\sqrt{100}} = 12.063.$$

The 80% confidence interval for μ is $[12.037,\ 12.063]$.

We are 80% confident that the true parameter value lies in this interval.

NOTE: Because σ^2 was very small and n was fairly large, we have a very narrow confidence interval for μ (which is good).

- Compare the 80% and 95% confidence intervals:

  The 80% confidence interval for μ is $[12.037,\ 12.063]$
  (The width of this interval is 0.026)

  The 95% confidence interval for μ is $[12.030,\ 12.070]$
  (The width of this interval is 0.040)

  The 95% CI is wider... i.e. All else being held constant, if you want to be more confident you capture μ, you'll have to make your *net* bigger.

Confidence Interval for μ - Normal parent population, known σ^2

- Looking at the form of the confidence interval:

$$\underbrace{\bar{x}}_{\substack{\text{Sample}\\ \text{mean}}} \quad \pm \quad \underbrace{z_{\alpha/2}}_{\substack{\text{multiplier}\\ \text{based on}\\ \text{confidence}\\ \text{level}}} \quad \cdot \quad \underbrace{\frac{\sigma}{\sqrt{n}}}_{\substack{\text{value}\\ \text{based on}\\ \sigma \text{ and}\\ \text{sample size}}}$$

$\uparrow$
changes
for different
% CI

$\uparrow$
standard
error of the
sample mean

Confidence Interval for μ - Normal parent population, known σ^2

- More narrow CIs are desirable.
- How can this be achieved?

$$\underbrace{\bar{x}}_{\text{Sample mean}} \pm \underbrace{z_{\alpha/2}}_{\text{multiplier based on confidence level}} \cdot \underbrace{\frac{\sigma}{\sqrt{n}}}_{\text{value based on } \sigma \text{ and sample size}}$$

- Increase your sample size (Good idea if possible)
- Decrease σ? Not an option, it's fixed by original distribution
- Decrease your confidence level? (Not a great idea. You reduce the CI width, but you're less likely to capture μ)

Confidence Interval Interpretation

- Once the confidence interval is formed (based on observed $\bar{x}$), it either does or does not contain the fixed unknown value μ

- For example, the 95% CI for box fill weights was: $[12.030,\ 12.070]$

  and the true population mean either is or isn't in this interval.

- The confidence interval level arises based on the randomness of the interval. BEFORE we collect the data, the CI is a random interval and it could take on many different values due to the *randomness* of $\bar{X}$.

Confidence Interval Interpretation

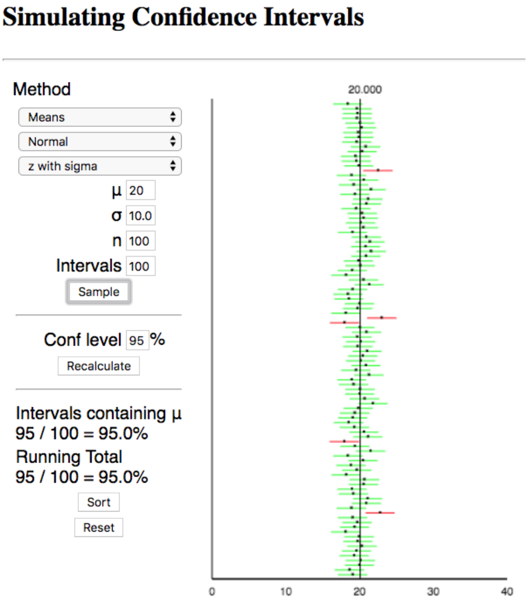
- For a 95% CI, we are 95% confident that the true μ lies in the interval. This statement of confidence reflects the following...

  If we repeated this process 100 times (i.e. collect a sample, compute $\bar{x}$, compute the CI), 95 out of 100 times we will capture the true μ on average, in the long run.

  The confidence relates to the method used to calculate the CI. We don't know if our CI captured μ or not (μ is unknown), but using the same method, 95 out of 100 times I'll get it (on average).

- See confidence interval applet website: *http://www.rossmanchance.com/applets/ConfSim.html*

Confidence Interval Interpretation & Simulation



Sample Size Calculation for μ

- The length of the CI is a measure of precision of estimation.
- Precision is related to sample size n.
  Higher precision coincides with a larger sample size
  (all else being held constant).
- What sample size should you choose? (when you CAN choose)

  Let E be the error in estimating μ, distance of observed $\bar{x}$ from target.

  $E = |\bar{x} - \mu|$

  Other books may state this error E as the **'Margin of Error'** .
- Choose a sample size that gives you a pre-specified level of precision.

Sample Size Calculation for μ

- Choose n to provide a certain bound on the error E with confidence $100(1-\alpha)$.

$$\bar{x} \pm \underbrace{z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}}_{\substack{\uparrow \\ \text{CI half-width or } E}}$$

- Pre-specified error: $E = z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \quad \Rightarrow \quad n = \left(\frac{z_{\alpha/2} \cdot \sigma}{E}\right)^2$

- Sample size for estimating μ with 100(1-α)% confidence and error E:

$$\boxed{n = \left(\frac{z_{\alpha/2} \cdot \sigma}{E}\right)^2}$$

Sample Size Calculation for μ

Example (The fill weight example)

In the fill weight example, how many boxes must be sampled to obtain a 99% confidence interval of full width 0.024 oz.? (i.e. $E = 0.012$)

ANS: $\sigma = 0.100$ from before, and we want 99% CI,
so $\alpha = 0.01$ and $z_{0.005} = 2.576.$

Error E is set at 0.012 (half-width of CI).

$$n = \left(\frac{z_{\alpha/2} \cdot \sigma}{E}\right)^2 = \left(\frac{z_{0.01/2} \cdot 0.100}{0.012}\right)^2 = \left(\frac{2.576 \times 0.1}{0.012}\right)^2 = 460.8$$

We can't sample a fraction of a box, so we **round-up** to ensure our confidence level is at least 99%, thus the required **sample size** is n=461.

NOTE: Read sample size problems closely to determine if they are giving precision as a half-width of a CI which is E (the cushion up or down), or the full width of the CI which is $2E$.

One-sided Confidence Bounds for μ

Occasionally, you may be interested in finding a bound for μ on only one side.

- A $100(1-\alpha)\%$ upper-confidence bound for μ is

  $\mu \leq \bar{x} + z_\alpha \sigma/\sqrt{n}$ and this gives an interval $(-\infty,\ \bar{x} + z_\alpha \sigma/\sqrt{n})$.

  (an upper bound on μ)

- A $100(1-\alpha)\%$ lower-confidence bound for μ is

  $\bar{x} - z_\alpha \sigma/\sqrt{n} \leq \mu$ and this gives an interval $(\bar{x} - z_\alpha \sigma/\sqrt{n},\ \infty)$.

  (a lower bound on μ)

Confidence Interval for μ - Normal parent pop'n, unknown σ^2

- What if we don't know σ? Can I just plug-in my estimator for σ (or s) and again have the same 95% CI?

$$\hat{\sigma}^2 = s^2 = \frac{\sum(x_i - \bar{x})^2}{n-1}$$

This was a 95% CI for μ when σ was known

$$\bar{x} - z_{0.025}\frac{\sigma}{\sqrt{n}} \leq \quad \mu \quad \leq \bar{x} + z_{0.025}\frac{\sigma}{\sqrt{n}}$$

Is this a 95% CI for μ?

$$\bar{x} - z_{0.025}\frac{s}{\sqrt{n}} \leq \quad \mu \quad \leq \bar{x} + z_{0.025}\frac{s}{\sqrt{n}}$$

- HINT: This feels like cheating. If I don't know σ, I must have more uncertainty in trying to capture μ than when I do know σ.

  So, how do we incorporate this extra uncertainty (for not knowing σ)?

Confidence Interval for μ - Normal parent pop'n, unknown σ^2

- The answer comes from the t-distribution.

The 95% CI for μ when σ unknown

$$\bar{x} - t_{0.025,n-1}\frac{s}{\sqrt{n}} \leq \quad \mu \quad \leq \bar{x} + t_{0.025,n-1}\frac{s}{\sqrt{n}}$$

where...

- $t_{0.025,n-1}$ is the 97.5^{th} percentile of the t-distribution with $n-1$ degrees of freedom (next slide).
- s is the sample standard deviation $s = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n-1}}$
- n is the number of observations (the sample size)

Confidence Interval for μ - Normal parent pop'n, unknown σ^2

Connection to the Z-distribution...

- The Z random variable follows a $N(0,1)$ distribution

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

- A T random variable follows a t-distribution

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

where t_{n-1} is a t-distribution with $n-1$ degrees of freedom.

Confidence Interval for μ - Normal parent pop'n, unknown σ^2

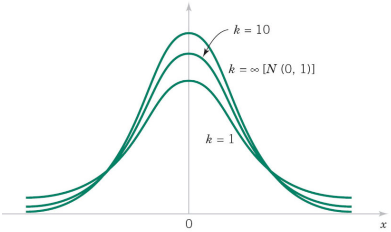
- What does the t-distribution look like?

  There is only one Z-distribution, but there are many t-distributions (distinguished by their degrees of freedom df as t_{df}). They look a lot like the $N(0,1)$, except they have heavier tails.

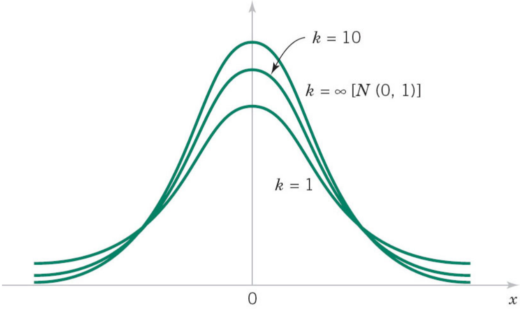
  For estimating a single parameter μ, the degrees of freedom is $n-1$.

  The heaviness of the tails depends on the degrees of freedom (the subscript on the t), so it depends on the sample size n.

  Differing t-distributions are shown below with $df = k$.



- For a large sample size n, $df = n - 1$ is very large, and the t_{n-1} looks just like the $N(0,1)$.
- So, $Z \sim N(0,1)$ is the limiting distribution for t_{n-1} as $n \to \infty$.



Confidence Interval for μ - Normal parent pop'n, unknown σ^2

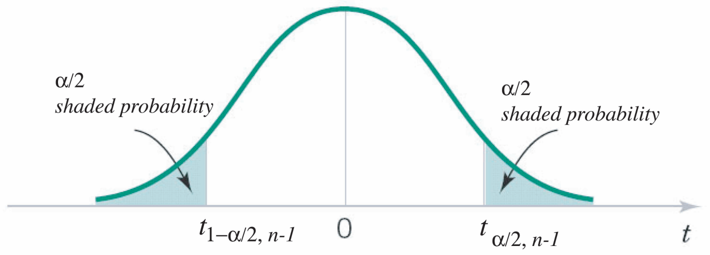
- **100(1-α)% Confidence interval for mean, variance unknown**
  If $\bar{x}$ is the sample mean and s is the sample standard deviation of a random sample of size n from a normal population, a $100(1-\alpha)$% confidence interval for μ is given by

$$\boxed{\bar{x} - t_{\alpha/2,df}\frac{s}{\sqrt{n}} \leq \quad \mu \quad \leq \bar{x} + t_{\alpha/2,df}\frac{s}{\sqrt{n}}}$$

- How do I get the $t_{\alpha/2,n-1}$ value? (next slide)

Confidence Interval for μ - Normal parent pop'n, unknown σ^2

- How do I get the $t_{\alpha/2,n-1}$ value? Similar to getting a z-value.



- A t-table can be found in your book Appendix A, Table V, page A-11.

- When $\alpha = 0.05$ (for 95% CI) and the sample size is $n = 10$,
  $t_{\alpha/2,n-1} = t_{0.025,9}$

- This is the t-value for a t_9 distribution with 2.5% above and 97.5% below. Looking at the table...

$$t_{0.025,9} = 2.262$$

Confidence Interval for μ - Normal parent pop'n, unknown σ^2

Example (CI for μ using t-distribution)

Suppose a sample of size $n = 10$ is taken from a normal population and $\bar{x} = 8.94$ and $s = 4.3$. Construct a 95% CI for the population mean.

Upper end-point:

$$\bar{x} + t_{\alpha/2,n-1} \cdot \tfrac{s}{\sqrt{n}} = \bar{x} + t_{0.025,9} \cdot \tfrac{s}{\sqrt{n}} = 8.94 + 2.262 \left(\tfrac{4.3}{\sqrt{10}}\right) = 10.02$$

Lower end-point:

$$\bar{x} - t_{\alpha/2,n-1} \cdot \tfrac{s}{\sqrt{n}} = 8.94 - 2.262 \left(\tfrac{4.3}{\sqrt{10}}\right) = 5.86$$

The 95% confidence interval for μ is $[5.86, 10.02]$.

We are 95% confident that the true mean μ is between 5.86 and 10.02.

Confidence Interval for μ - Normal parent pop'n, unknown σ^2

Normality assumption for these t-based confidence intervals:

- When σ^2 is unknown and we have a rather small sample, we need the parent population to be normally distributed (or nearly normal) to truly achieve our $100(1-\alpha)\%$ confidence level.
  - After we collect our data, we can check this assumption of normality by creating a *normal probability plot* (recall section 6.7).

- If the data are not normally distributed, we have to use a different approach. Something that doesn't depend on this normality assumption, such methods are called *nonparametric methods* (which we won't cover in in this class).

$100(1-\alpha)\%$ Confidence Interval for μ

RULE OF THUMB

- When σ is known, use $\boxed{\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \quad \mu \quad \leq \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}}$
- When σ is unknown, use $\boxed{\bar{x} - t_{\alpha/2,df}\frac{s}{\sqrt{n}} \leq \quad \mu \quad \leq \bar{x} + t_{\alpha/2,df}\frac{s}{\sqrt{n}}}$

When n is REALLY LARGE ($n > 60$) a 95% CI for μ can be

$$\bar{x} - z_{0.025}\frac{s}{\sqrt{n}} \leq \quad \mu \quad \leq \bar{x} + z_{0.025}\frac{s}{\sqrt{n}}$$

NOTE: At $n = 60$ the Z-table and t_{60}-table are very very similar.

But just use the rule of thumb,

which says s goes with t and σ goes with z.