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Chapter 5
Statistical Models in Simulation

Banks, Carson, Nelson & Nicol

Discrete-Event System Simulation

Purpose & Overview

- The world the model-builder sees is probabilistic rather than deterministic.
  - Some statistical model might well describe the variations.

- An appropriate model can be developed by sampling the phenomenon of interest:
  - Select a known distribution through educated guesses
  - Make estimate of the parameter(s)
  - Test for goodness of fit

- In this chapter:
  - Review several important probability distributions
  - Present some typical application of these models

Review of Terminology and Concepts

- In this section, we will review the following concepts:
  - Discrete random variables
  - Continuous random variables
  - Cumulative distribution function
  - Expectation

Discrete Random Variables

- X is a discrete random variable if the number of possible values of X is finite, or countably infinite.
- Example: Consider jobs arriving at a job shop.
  - Let X be the number of jobs arriving each week at a job shop.
  - R_x = possible values of X (range space of X) = $\{0,1,2,\dots\}$
  - $p(x_i)$ = probability the random variable is $x_i = P(X = x_i)$
  - $p(x_i)$, $i = 1,2, \dots$ must satisfy:

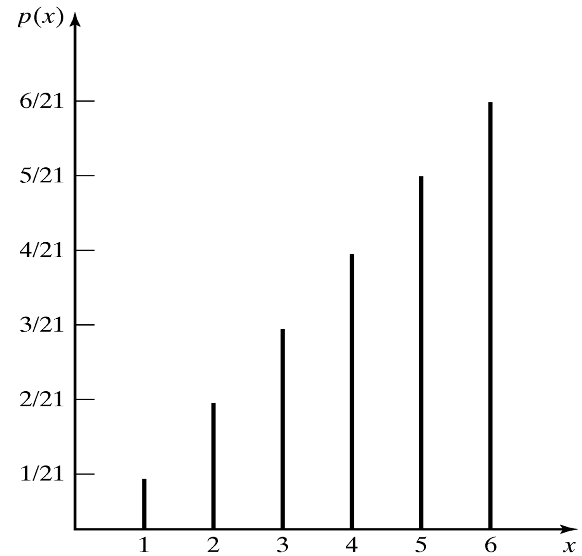
    1. $p(x_i) \geq 0, \text{ for all } i$
    2. $\sum_{i=1}^{\infty} p(x_i) = 1$

  - The collection of pairs $[x_i, p(x_i)]$, $i = 1,2,\dots$, is called the probability distribution of X, and $p(x_i)$ is called the probability mass function (pmf) of X.

Discrete Random Variables

- Example: Assume the die is loaded so that the probability that a given face lands up is proportional to the number of spots showing.

x_i	1	2	3	4	5	6
$P(x_i)$	1/21	2/21	3/21	4/21	5/21	6/21



  - $p(x_i)$, $i = 1,2, \ldots$ must satisfy:

    1. $p(x_i) \geq 0$, for all i
    2. $\sum_{i=1}^{\infty} p(x_i) = 1$

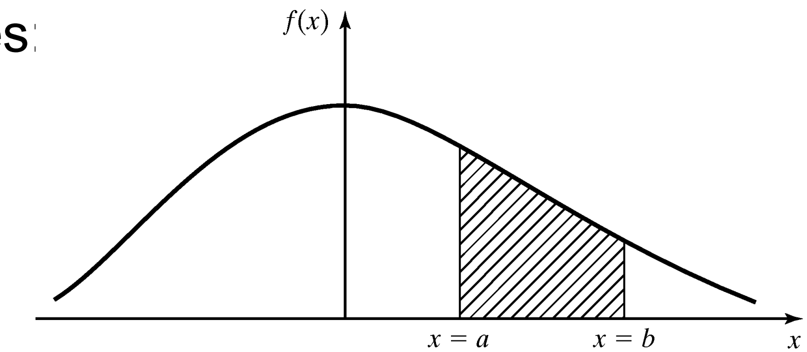
Continuous Random Variables [Probability Review]

- X is a continuous random variable if its range space R_x is an interval or a collection of intervals.
- The probability that X lies in the interval $[a,b]$ is given by:

$$P(a \le X \le b) = \int_a^b f(x)dx$$

- $f(x)$, denoted as the pdf of X, satisfies:

1. $f(x) \ge 0$, for all x in R_X
2. $\int_{R_X} f(x)dx = 1$
3. $f(x) = 0$, if x is not in R_X



- Properties

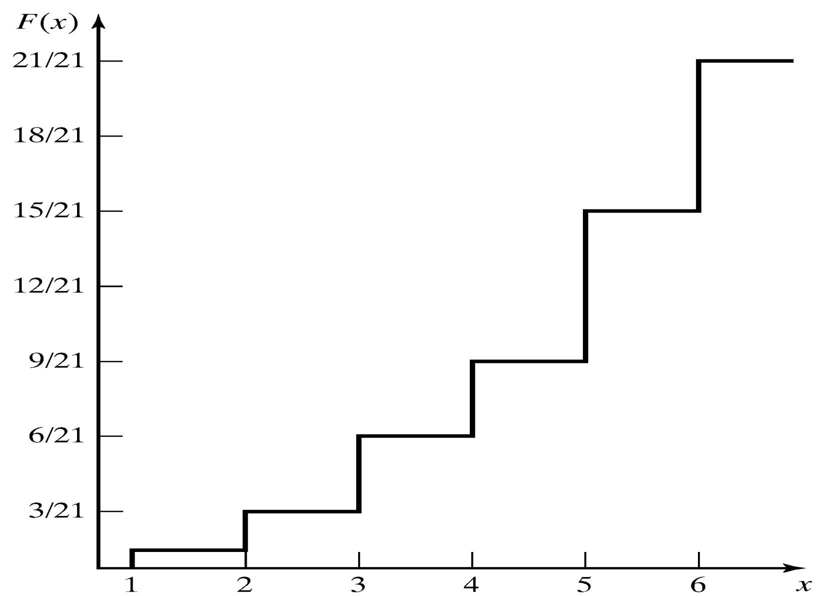
1. $P(X = x_0) = 0$, because $\int_{x_0}^{x_0} f(x)dx = 0$
2. $P(a \le X \le b) = P(a \prec X \le b) = P(a \le X \prec b) = P(a \prec X \prec b)$

Continuous Random Variables

- Example: The die-tossing experiment described in last example has a cdf given as follows:

x	(-∞,1)	[1,2)	[2,3)	[3,4)	[4,5)	[5,6)	[6,∞)
F(x)	0	1/21	3/21	6/21	10/21	15/21	21/21

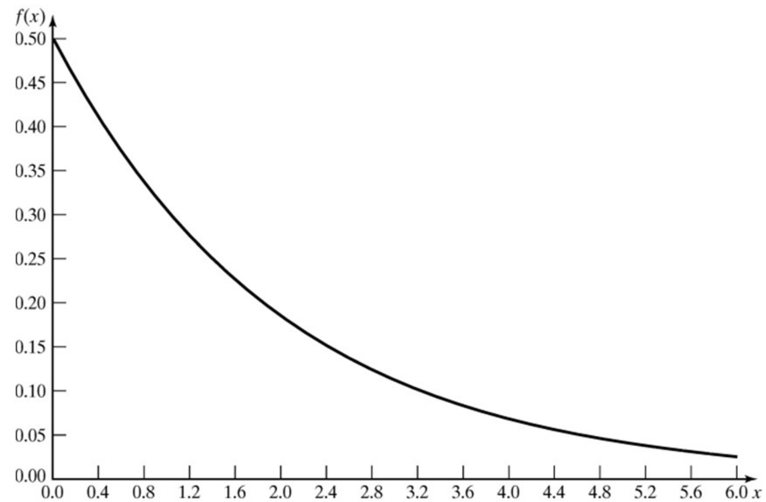
  - [a,b) = {a ≤ x ≤ b}



Continuous Random Variables

- Example: Life of an inspection device is given by *X*, a continuous random variable with pdf:

$$f(x) = \begin{cases} \frac{1}{2} e^{-x/2}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



  - *X* has an exponential distribution with mean 2 years
  - Probability that the device's life is between 2 and 3 years is:

$$P(2 \leq x \leq 3) = \frac{1}{2} \int_2^3 e^{-x/2} dx = 0.14$$

Continuous Distributions

- Continuous random variables can be used to describe random phenomena in which the variable can take on any value in some interval.
- In this section, the distributions studied are:
  - Uniform
  - Exponential
  - Normal
  - Weibull
  - Lognormal

Uniform Distribution [Continuous Dist'n]

- A random variable *X* is uniformly distributed on the interval (*a,b*), *U*(*a,b*), if its pdf and cdf are:

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases} \qquad F(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x < b \\ 1, & x \ge b \end{cases}$$

- Properties
  - $P(x_1 < X < x_2)$ is proportional to the length of the interval $[F(x_2) - F(x_1) = (x_2-x_1)/(b-a)]$
  - $E(X) = (a+b)/2$ $\qquad$ $V(X) = (b-a)^2/12$
- U(0,1) provides the means to generate random numbers, from which random variates can be generated.

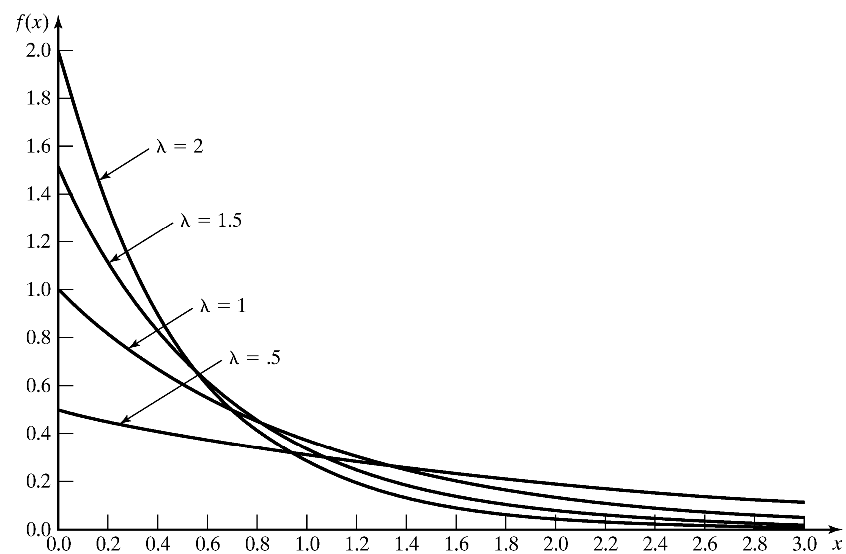
Exponential Distribution [Continuous Dist'n]

- A random variable X is exponentially distributed with parameter $\lambda > 0$ if its pdf and cdf are:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

$$F(x) = \begin{cases} 0, & x < 0 \\ \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$

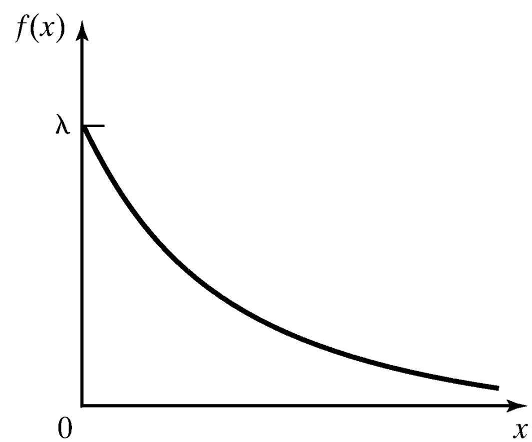
  - $E(X) = 1/\lambda$ $V(X) = 1/\lambda^2$
  - Used to model interarrival times when arrivals are completely random, and to model service times that are highly variable
  - For several different exponential pdf's (see figure), the value of intercept on the vertical axis is λ, and all pdf's eventually intersect.

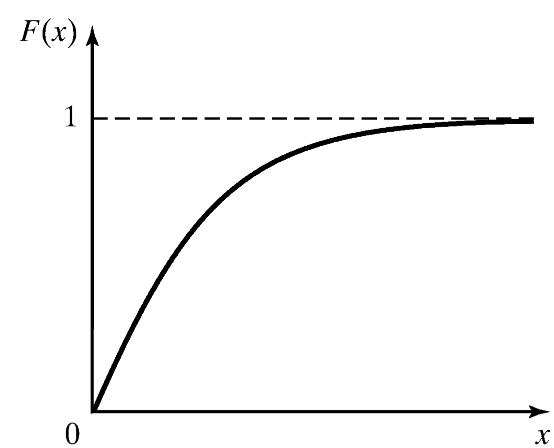


Exponential Distribution [Probability Review]

- Model times between events
  - Times between arrivals
  - Times between failures
  - Times to repair
  - Service Times
- A random variable X is said to be exponentially distributed with parameter if its PDF is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$





Exponential Distribution [Continuous Dist'n]

- Memoryless property
  - For all s and t greater or equal to 0:

    $$P(X > s+t \mid X > s) = P(X > t)$$

  - Example: A lamp ~ exp(λ = 1/3 per hour), hence, on average, 1 failure per 3 hours.
    - The probability that the lamp lasts longer than its mean life is:
      $P(X > 3) = 1-(1-e^{-3/3}) = e^{-1} = 0.368$
    - The probability that the lamp lasts between 2 to 3 hours is:
      $P(2 <= X <= 3) = F(3) – F(2) = 0.145$
    - The probability that it lasts for another hour given it is operating for 2.5 hours:
      $P(X > 3.5 \mid X > 2.5) = P(X > 1) = e^{-1/3} = 0.717$

Gamma Distribution [Probability Review]

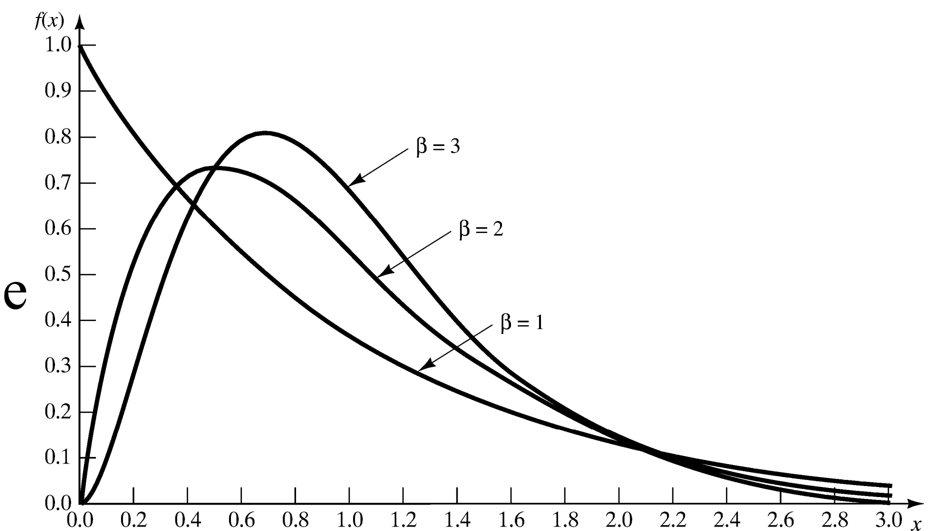
- A function used in defining the gamma distribution is the gamma function, which is defined for all $\beta > 0$ as

$$\Gamma(\beta) = \int_0^{\infty} x^{\beta-1} e^{-x} dx$$

- A random variable X is gamma distributed with parameters β and θ if its PDF is given by

$$f(x) = \begin{cases} \dfrac{\beta\theta}{\Gamma(\beta)} (\beta\theta x)^{\beta\text{-}1} \mathrm{e}^{\text{-}\beta\theta x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$E(X) = \frac{1}{\theta} \qquad V(X) = \frac{1}{\beta\theta^2}$$

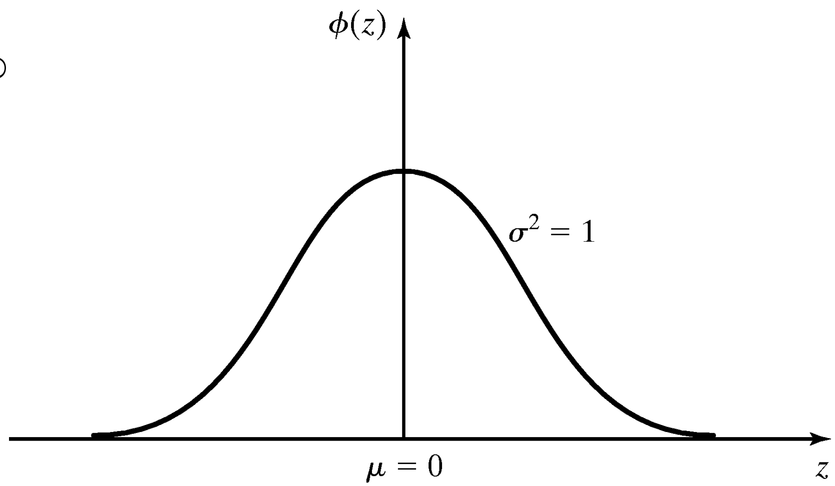


Normal Distribution

[Continuous Dist'n]

- A random variable X is normally distributed has the pdf:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], \quad -\infty$$



  - Mean: $-\infty < \mu < \infty$
  - Variance: $\sigma^2 > 0$
  - Denoted as $X \sim N(\mu, \sigma^2)$
- Special properties:
  - $\lim_{x\to-\infty} f(x) = 0, \text{ and } \lim_{x\to\infty} f(x) = 0$.
  - $f(\mu-x)=f(\mu+x)$; the pdf is symmetric about μ.
  - The maximum value of the pdf occurs at $x = \mu$; the mean and mode are equal.

Normal Distribution

[Continuous Dist'n]

- Evaluating the distribution:
  - Use numerical methods (no closed form)
  - Independent of μ and σ, using the standard normal distribution:

    Z ~ N(0,1)
  - Transformation of variables: let Z = (X - μ) / σ,

$$F(x) = P(X \le x) = P\left(Z \le \frac{x-\mu}{\sigma}\right)$$

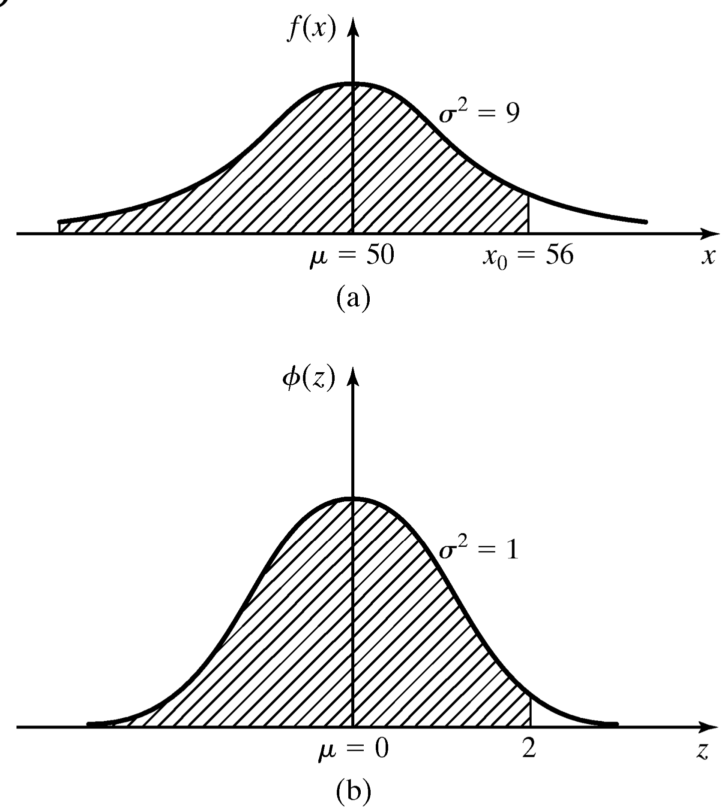
$$= \int_{-\infty}^{(x-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \int_{-\infty}^{(x-\mu)/\sigma} \phi(z) dz = \Phi\left(\tfrac{x-\mu}{\sigma}\right) \quad \text{, where } \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

Normal Distribution [Probability Review]

- Example: Suppose that *X ~ N (50, 9).*

$$F(56) = \Phi\left(\frac{56-50}{3}\right) = \Phi(2) = 0.9772$$



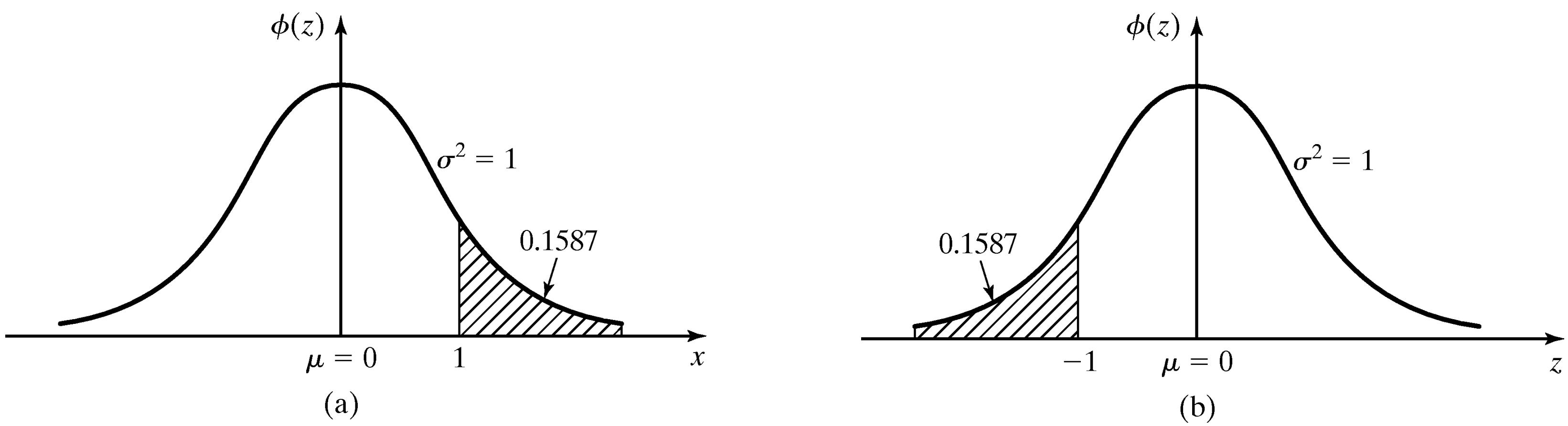
Normal Distribution

[Continuous Dist'n]

- Example: The time required to load an oceangoing vessel, *X*, is distributed as *N(12,4)*
  - The probability that the vessel is loaded in less than 10 hours:

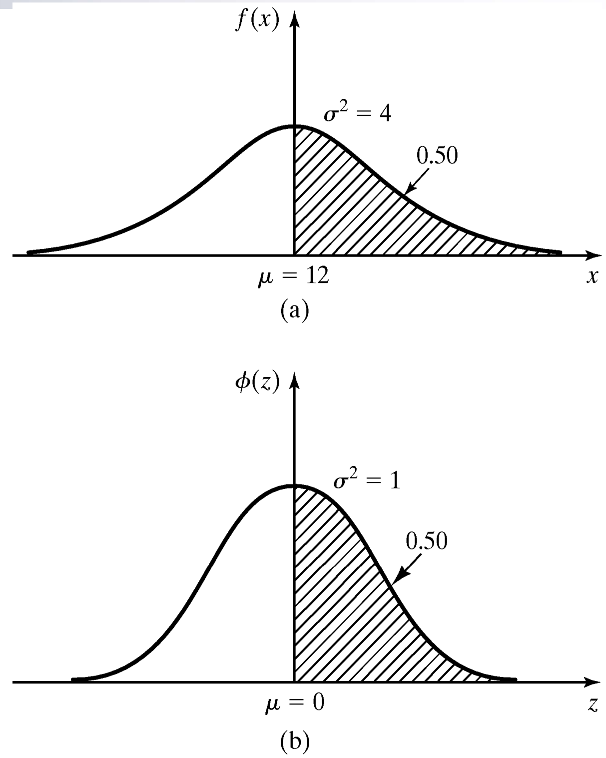
$$F(10) = \Phi\left(\frac{10-12}{2}\right) = \Phi(-1) = 0.1587$$

- Using the symmetry property, $\Phi(1)$ is the complement of Φ (-1)



Normal Distribution [Probability Review]

- Example: The time in hours required to load a ship, X, is distributed as $N(12, 4)$. The probability that 12 or more hours will be required to load the ship is:



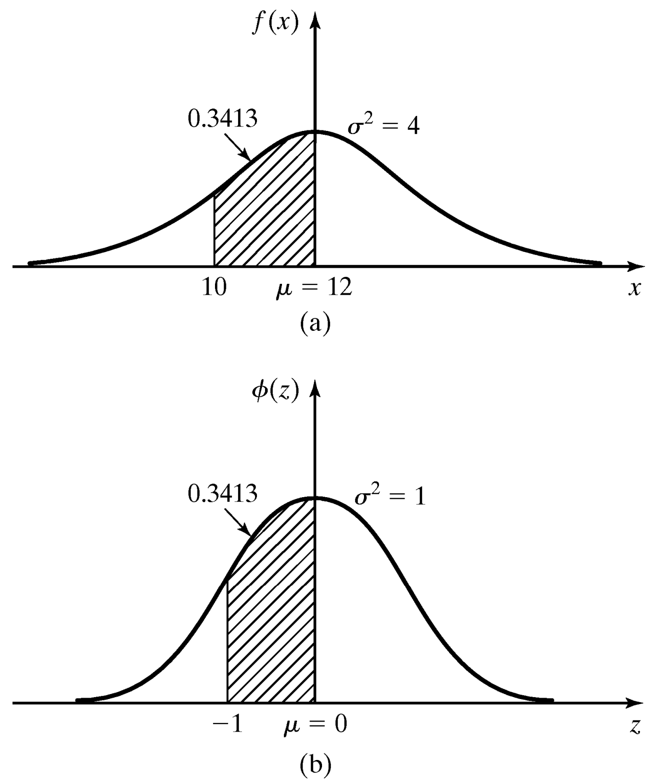
$$P(X > 12) = 1 - F(12) = 1 - 0.50 = 0.50$$

(The shaded portions in both figures)

Normal Distribution [Probability Review]

- Example (cont.):

  The probability that between 10 and 12 hours will be required to load a ship is given by



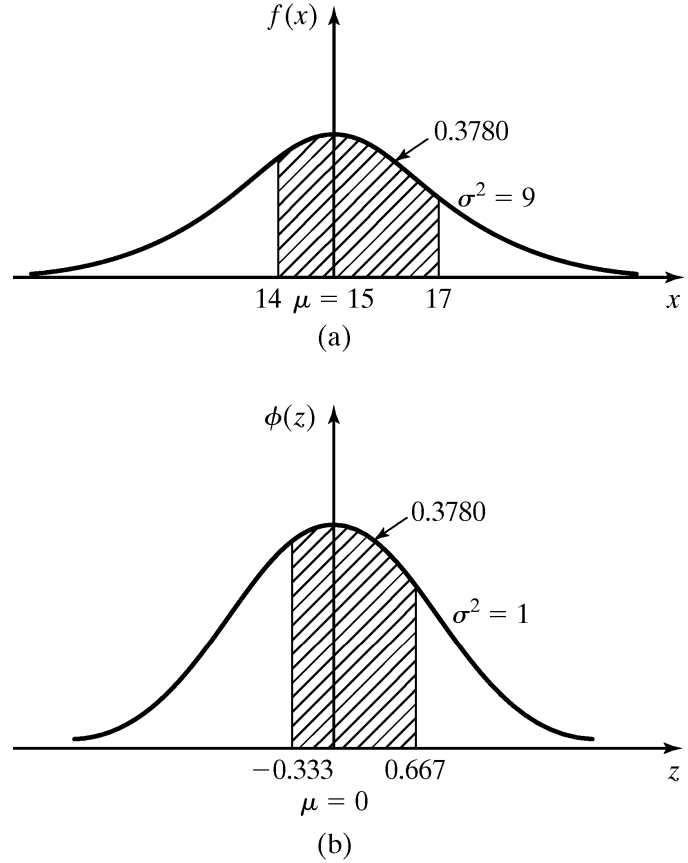
$$P(10 \le X \le 12) = F(12) - F(10) = 0.5000 - 0.1587 = 0.3413$$

The area is shown in shaded portions of the figure

Normal Distribution [Probability Review]

- Example: The time to pass through a queue is *N(15, 9)*. The probability that an arriving customer waits between 14 and 17 minutes is:

Type equation here.



$P(14 \le X \le 17) = F(17) - F(14) =$

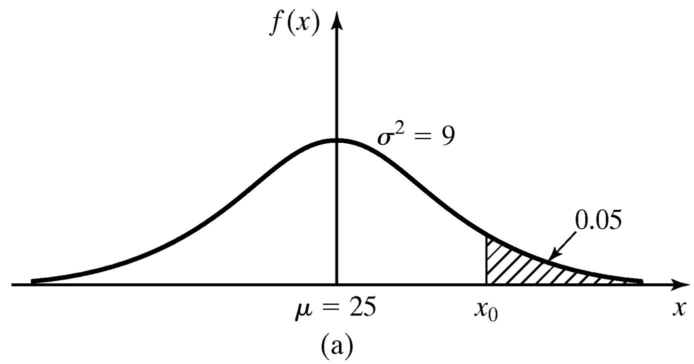
$$\Phi\left(\frac{17-15}{3}\right) - \Phi\left(\frac{14-15}{3}\right) = \Phi(0.667) - \Phi(-0.333) = 0.7476 - 0.3696 = 0.3780$$

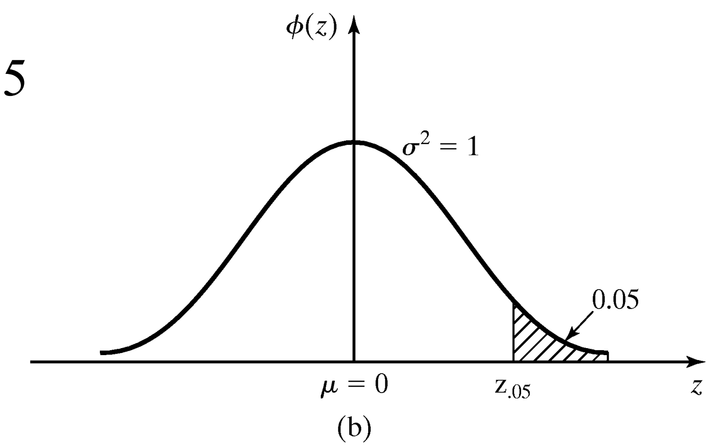
$\varphi(-.333) = 1 - \varphi(0.333) = 1 - 0.6304 = 0.3696$

Normal Distribution [Probability Review]

- Example: Lead-time demand, *X*, for an item is *N(25, 9)*.

  Compute the value for lead-time that will be exceeded only *5%* of time.





$$P(X > x_0) = P(Z > \frac{x_0 - 25}{3}) = 1 - \Phi(\frac{x_0 - 25}{3}) = 0.05$$

$$\Phi(1.645) = 0.95.$$

$$\Phi\left(\frac{x_0 - 25}{3}\right) = 0.95$$

$$\frac{x_0 - 25}{3} = 1.645$$

$$x_0 = 29.935$$

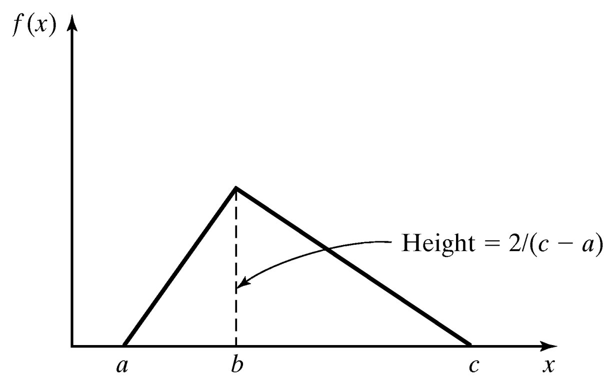
Triangular Distribution [Probability Review]

- A random variable X has a triangular distribution if its PDF is given by

$$f(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(c-a)}, & a \le x \le b \\[2ex] \dfrac{2(c-x)}{(c-b)(c-a)}, & b < x \le c \\[2ex] 0, & \text{elsewhere} \end{cases}$$

where $a \le b \le c$.

$$E(X) = \frac{a+b+c}{3}$$



Triangular Distribution

- A burger franchise planning a new outlet in Auckland wants to determine the probability the new outlet will have weekly sales of less than $2000. If the weekly sales are less than this the outlet is unlikely to cover its costs.
- So, they wish to calculate $\Pr(X < 2000)$ They use a triangular distribution to model the future weekly sales with a minimum value of *a*=$1000, and maximum value of *b*=$6000 and a peak value of *c*=$3000.

Triangular Distribution

Area corresponding to $\Pr(X < 2000)$

The area of a triangle is $area = \frac{1}{2} base \times height$. In this case $base = 2000 - 1000 = 1000$

the height is given by $f(2000)$ using the probability density function formula. Since 2000 is between *a*=1000 and *c*=3000 we have:

$$height = f(2000) = \frac{2(2000 - a)}{(b - a)(c - a)} = \frac{2 \times (2000 - 1000)}{(6000 - 1000) \times (3000 - 1000)} = 0.0002$$

So $area = \frac{1}{2} \times 1000 \times 0.0002 = 0.1$ so $\Pr(X < 2000) = 0.1$ (or 10%)

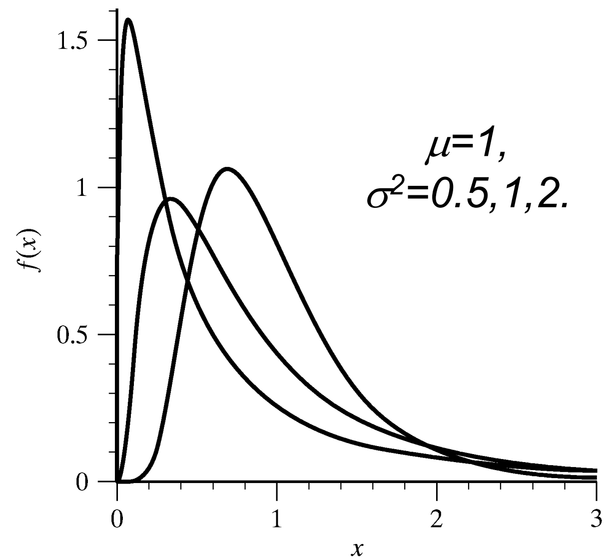
Lognormal Distribution [Continuous Dist'n]

- A random variable X has a lognormal distribution if its pdf has the form:

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\sigma x} \exp\left[-\dfrac{(\ln x - \mu)^2}{2\sigma^2} \right], & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

  - Mean $E(X) = e^{\mu+\sigma^2/2}$
  - Variance $V(X) = e^{2\mu+\sigma^2/2}(e^{\sigma^2} - 1)$

median $= e^{\mu}$ mode $= e^{\mu-\sigma^2}$



- Relationship with normal distribution
  - When $Y \sim N(\mu, \sigma^2)$, then $X = e^Y \sim$ lognormal(μ, σ^2)
  - Parameters μ and σ^2 are not the mean and variance of the lognormal

Lognormal Distribution

[Continuous Dist'n]

- If the mean and variance of the lognormal are known to be μ_L and σ_L^2, respectively,
- then the parameters μ and σ^2 are given by

$$\mu = \ln\left(\frac{\mu_L^2}{\sqrt{\mu_L^2 + \sigma_L^2}}\right)$$

$$\sigma^2 = \ln\left(\frac{\mu_L^2 + \sigma_L^2}{\mu_L^2}\right)$$

Lognormal Distribution [Continuous Dist'n]

- The rate of return on a volatile investment is modeled as having a lognormal distribution with mean 20% and standard deviation 5%. Compute the parameters for the lognormal distribution.

$$\mu = \ln\left(\frac{20^2}{\sqrt{20^2 + 5^2}}\right) \doteq 2.9654$$

$$\sigma^2 = \ln\left(\frac{20^2 + 5^2}{20^2}\right) \doteq 0.06$$

Beta Distribution

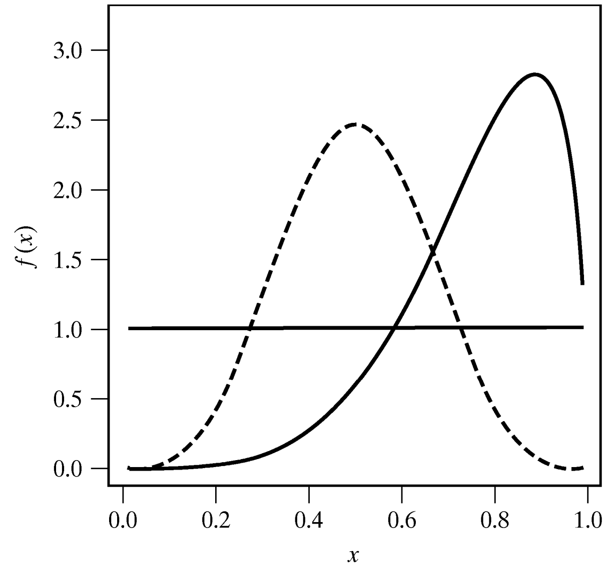
[Probability Review]

- A random variable *X* is beta-distributed with parameters $\beta_1 > 0$ and $\beta_2 > 0$ if its PDF is given by

$$f(x) = \begin{cases} \dfrac{x^{\beta_1 - 1}(1 - x)^{\beta_2 - 1}}{B(\beta_1, \beta_2)}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

where

$$B(\beta_1, \beta_2) = {\Gamma(\beta_1)\Gamma(\beta_2)} \big/ {\Gamma(\beta_1 + \beta_2)}$$



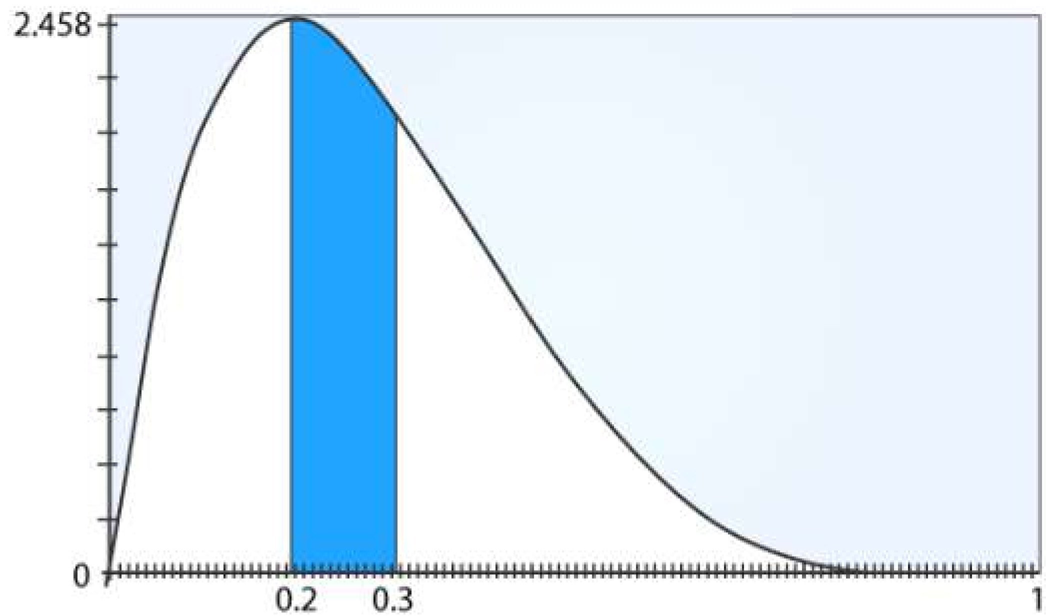
Beta Distribution

Problem: Suppose, if in a basket there are balls which are defective with a Beta distribution of α=2 and β=5. Compute the probability of defective balls in the basket from 20% to 30%.

$$P(x) = x^{\alpha-1}(1-x)^{\beta-1}/B(\alpha,\beta)$$

$$P(0.2 \leq x \leq 0.3) = \sum_{0.2}^{0.3} x^{2-1}(1-x)^{5-1}/B(2,5)$$

$=0.235185$



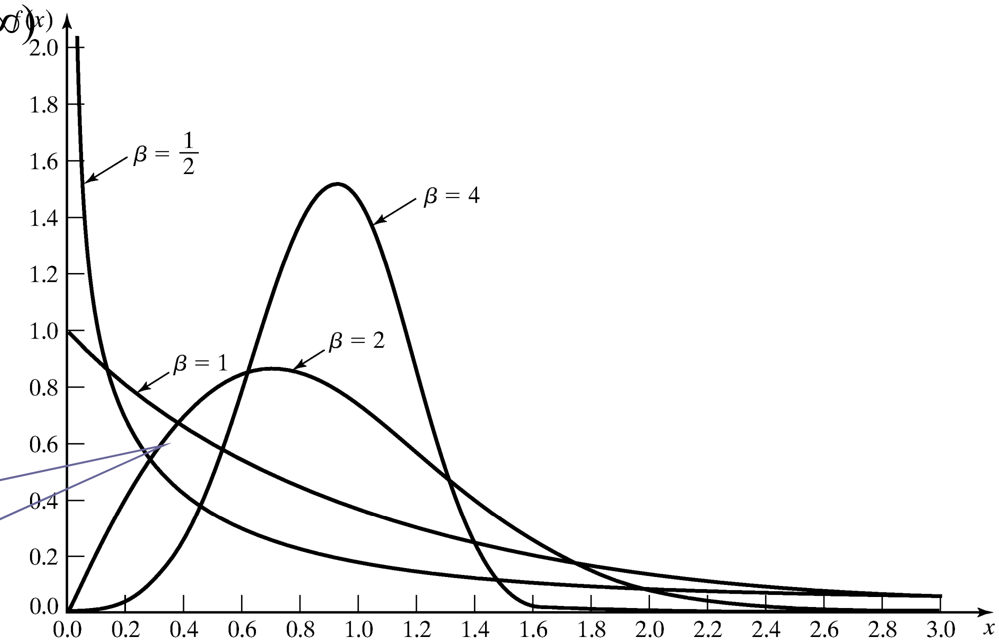
Weibull Distribution

[Continuous Dist'n]

- A random variable X has a Weibull distribution if its pdf has the form:

$$f(x) = \begin{cases} \dfrac{\beta}{\alpha}\left(\dfrac{x-\nu}{\alpha}\right)^{\beta-1} \exp\left[-\left(\dfrac{x-\nu}{\alpha}\right)^{\beta}\right], & x \geq \nu \\ 0, & \text{otherwise} \end{cases}$$

- 3 parameters:
  - Location parameter: υ, $(-\infty < \nu < \infty)$
  - Scale parameter: β, $(\beta > 0)$
  - Shape parameter. α, (> 0)
- Example: $\upsilon = 0$ and $\alpha = 1$:



When $\beta = 1$,
$X \sim exp(\lambda = 1/\alpha)$

Weibull Distribution

[Continuous Dist'n]

- The time it takes for an aircraft to land and clear the runway at a major international airport has a Weibull distribution with v = 1.34 minutes, β = 0.5, and α= 0.04 minute. Find the probability that an incoming airplane will take more than 1.5 minutes to land and clear the runway.

$$P(X \le 1.5) = F(1.5)$$

$$= 1 - \exp\left[-\left(\frac{1.5 - 1.34}{0.04}\right)^{0.5}\right]$$

$$= 1 - e^{-2} = 1 - 0.135 = 0.865$$

- Therefore, the probability that an aircraft will require more than 1.5 minutes to land and clear the runway is 0.135.

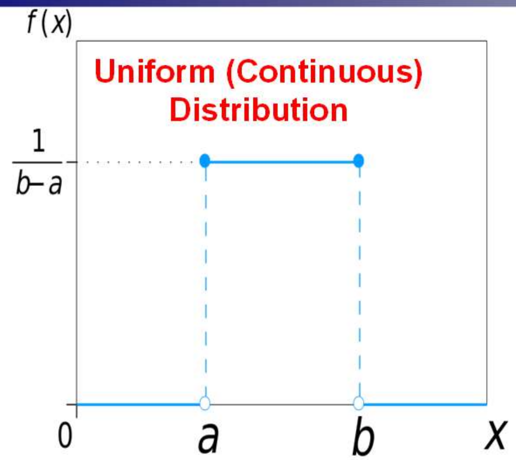
Continuous Distributions

- For events that are highly variable (interarrival times) or instantaneous occurrences (failure of a light bulb) → exponential distribution
- Sum of independent exponential distributions → Gamma distribution. Extremely flexible, used to model non-negative variables
- The sum of *k* independent random variables → normal distribution
- The product of *k* independent random variables → lognormal distribution
- Bounded random variables → beta distribution

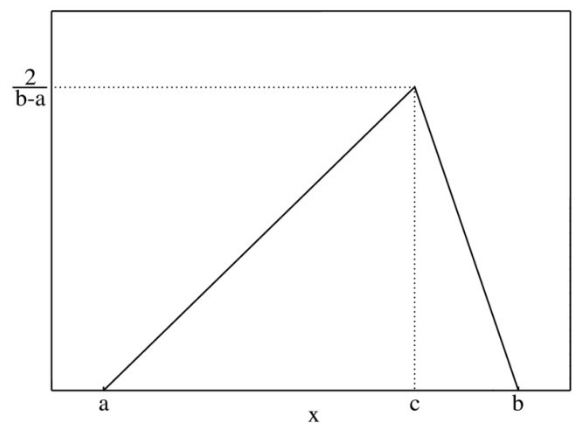
Continuous Distributions

- Weibull distribution can be thought of as a stretched exponential distribution. That's why it has a longer tail.
- Uniform distribution → complete uncertainty
- Triangular distribution → maximum and minimum are known
- When no theoretical distribution seems appropriate → Empirical data

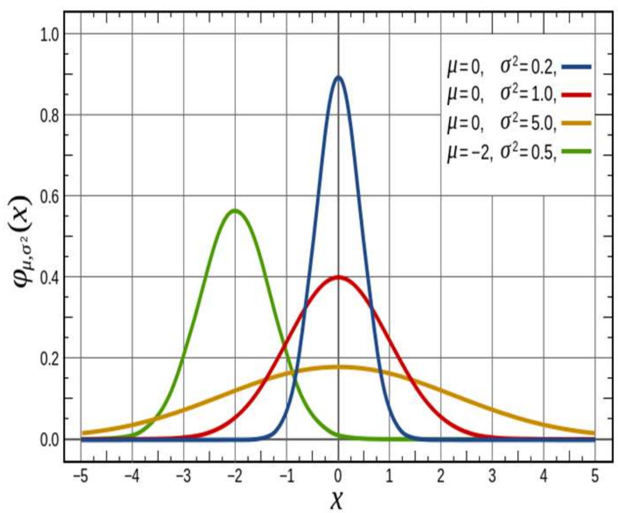
Continuous Probability Distributions



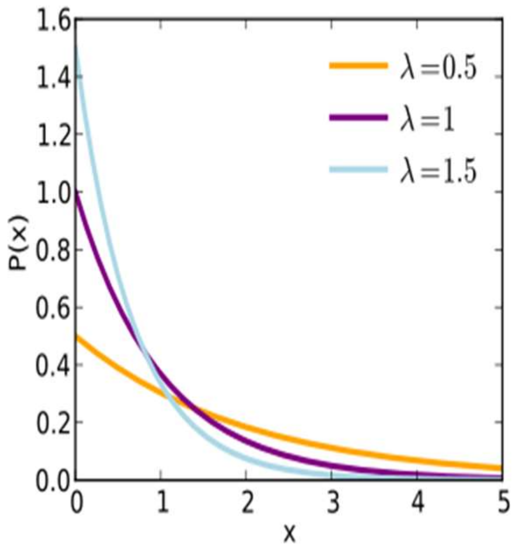
Triangular Distribution



Normal Distribution

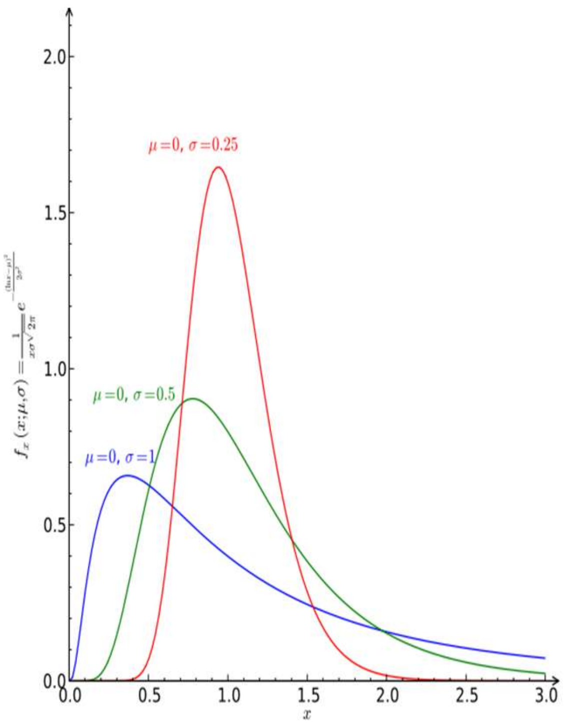


Exponential Distribution

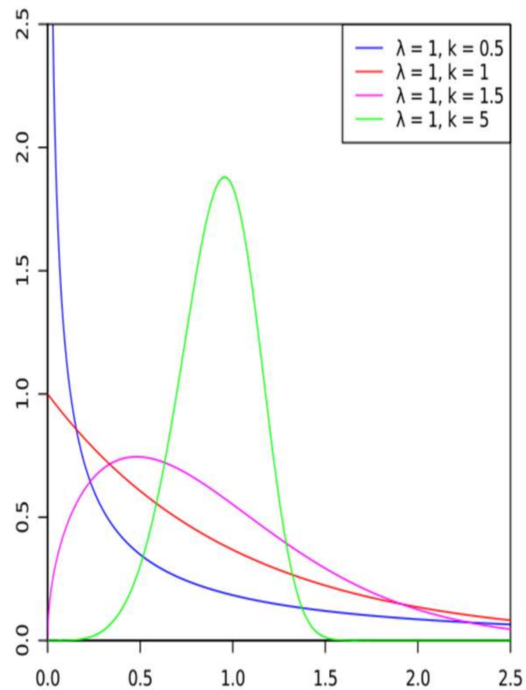


Continuous Probability Distributions

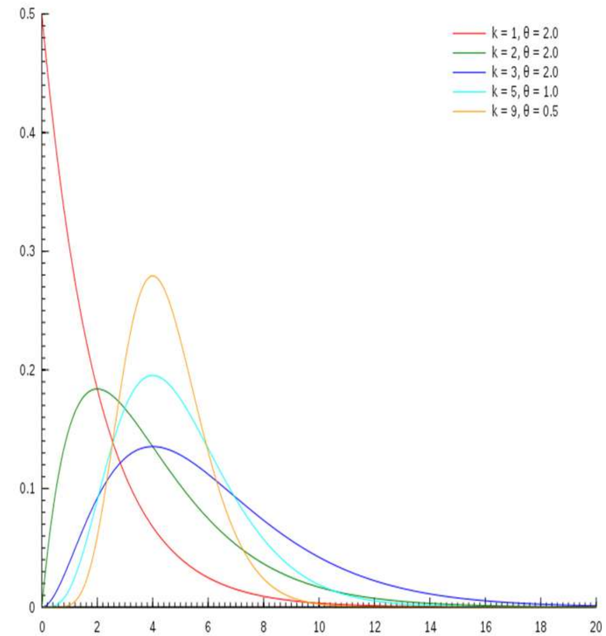
Lognormal Distribution



Weibull Distribution



Gamma Distribution

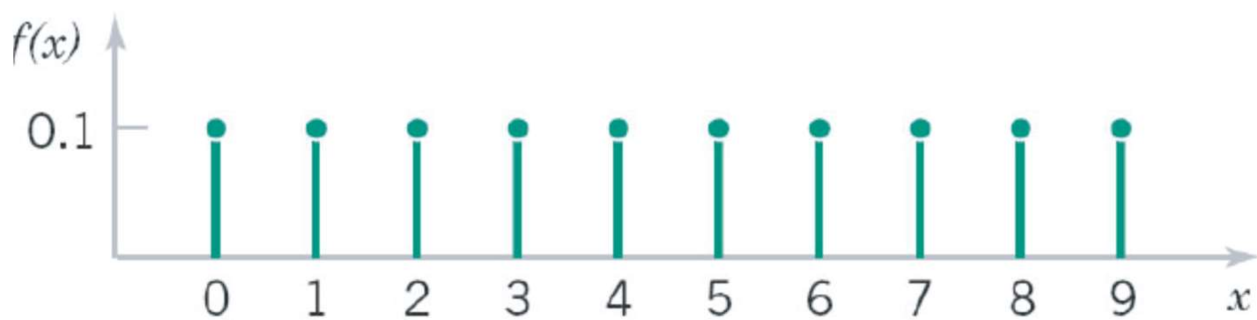


Discrete Distributions

- Discrete random variables are used to describe random phenomena in which only integer values can occur.
- In this section, we will learn about:
  - Bernoulli trials and Bernoulli distribution
  - Binomial distribution
  - Geometric and negative binomial distribution
  - Poisson distribution

Uniform Distribution

- A random variable X has a discrete uniform distribution if each of the n values in its range, say $x_1, x_2, \ldots\ldots\ldots\ x_n$, has equal probability. Then, $f(x_i) = 1/n$
- Let X represent a random variable taking on the possible values of {0; 1; 2; 3; 4; 5; 6; 7; 8; 9}, and each possible value has equal probability.
- This is a discrete uniform distribution and the probability for each of the 10 possible value is $P(X = x_i) = f(x_i) = 1/10 = 0.10$



Bernoulli Trials and Bernoulli Distribution

- Bernoulli Trials:
  - Consider an experiment consisting of n trials, each can be a success or a failure.
    - Let $X_j = 1$ if the jth experiment is a success
    - and $X_j = 0$ if the jth experiment is a failure
  - The Bernoulli distribution (one trial):

$$p_j(x_j) = p(x_j) = \begin{cases} p, & x_j = 1, j = 1,2,\ldots,n \\ 1 - p = q, & x_j = 0, j = 1,2,\ldots,n \\ 0, & \text{otherwise} \end{cases}$$

  - where $E(X_j) = p$ and $V(X_j) = p(1-p) = pq$
- Bernoulli process:
  - The n Bernoulli trials where trails are independent:

$$p(x_1, x_2, \ldots, x_n) = p_1(x_1)\, p_2(x_2) \ldots p_n(x_n)$$

Binomial Distribution

[Discrete Dist’n]

- The number of successes in *n* Bernoulli trials, *X*, has a binomial distribution.

$$p(x) = \begin{cases} \binom{n}{x} \underbrace{p^x q^{n-x}}, & x = 0,1,2,\ldots,n \\ 0, & \text{otherwise} \end{cases}$$

The number of outcomes having the required number of successes and failures

Probability that there are x successes and (n-x) failures

  - The mean, $E(x) = p + p + \ldots + p = n{*}p$
  - The variance, $V(X) = pq + pq + \ldots + pq = n{*}pq$

Binomial Distribution

- Suppose 40% of a very large population of registered voters favor candidate Obama. A random sample of n = 5 voters will be selected, and X, the number favoring Obama out of 5, is to be observed.
- $p = P(\text{success}) = 0.40$
- $1 - p = P(\text{failure}) = 0.60$

Binomial Distribution

- What is the probability of getting no one who favors Obama, i.e. what is P(X = 0)?
- P(X = 0) = (0.6)(0.6)(0.6)(0.6)(0.6)

  No No No No No

  $= (0.6)^5$

  $= 0.07776$

What is the probability of getting 1 person who favors Obama?

P(X = 1) $= \binom{5}{1} (0.4)^1 (0.6)^4 = 0.25920$

Binomial Distribution

- What is the probability of getting 2 person who favors Obama? P(X = 2) =?

$$\binom{5}{2} = \frac{5!}{2!\,3!} = 10 \begin{cases} \text{Y Y N N N} & (0.4)(0.4)(0.6)(0.6)(0.6) \\ \text{Y N Y N N} & (0.4)(0.6)(0.4)(0.6)(0.6) \\ \vdots & \vdots \\ \text{N N N Y Y} & (0.6)(0.6)(0.6)(0.4)(0.4) \end{cases}$$

10 configurations

$$P(X = 2) = \binom{5}{2}(0.4)^2(0.6)^3 = 10 \cdot (0.4)^2(0.6^3) = 0.34560$$

Geometric Distribution

[Discrete Dist'n]

- Geometric distribution
  - The number of Bernoulli trials, *X*, to achieve the 1st success:

$$p(x) = \begin{cases} q^{x-1}p, & x = 0,1,2,\ldots,n \\ 0, & \text{otherwise} \end{cases}$$

  - *E(x) = 1/p, and V(X) = q/p²*

- Let X be a geometric random variable with p = 0.25. What is the probability that X = 4 (i.e. that the first success occurs on the 4th trial)?
- ANS: For X to be equal to 4, we must have had 3 failures, and then a success.

Negative Binomial Distribution

- Negative binomial distribution
  - The number of Bernoulli trials, X, until the *k*th success
  - If Y is a negative binomial distribution with parameters p and k, then:

$$p(x) = \begin{cases} \begin{pmatrix} y-1 \\ k-1 \end{pmatrix} q^{y-k} p^k, & y = k, k+1, k+2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

  - *E(Y) = k/p, and V(X) = kq/p*^{2}

Poisson Distribution

- Definition: *N(t)* is a counting function that represents the number of events occurred in *[0,t].*
- A counting process {*N(t), t>=0*} is a Poisson process with mean rate λ if:
  - Arrivals occur one at a time
  - {*N(t), t>=0*} has stationary increments
  - {*N(t), t>=0*} has independent increments
- Properties

$$P[N(t)=n]=\frac{e^{-\lambda t}(\lambda t)^n}{n!}, \qquad \text{for } t \geq 0 \text{ and } n = 0,1,2,\ldots$$

  - Equal mean and variance: $E[N(t)] = V[N(t)] = \lambda t$
  - Stationary increment: The number of arrivals in time *s* to *t* is also Poisson-distributed with mean $\lambda(t-s)$

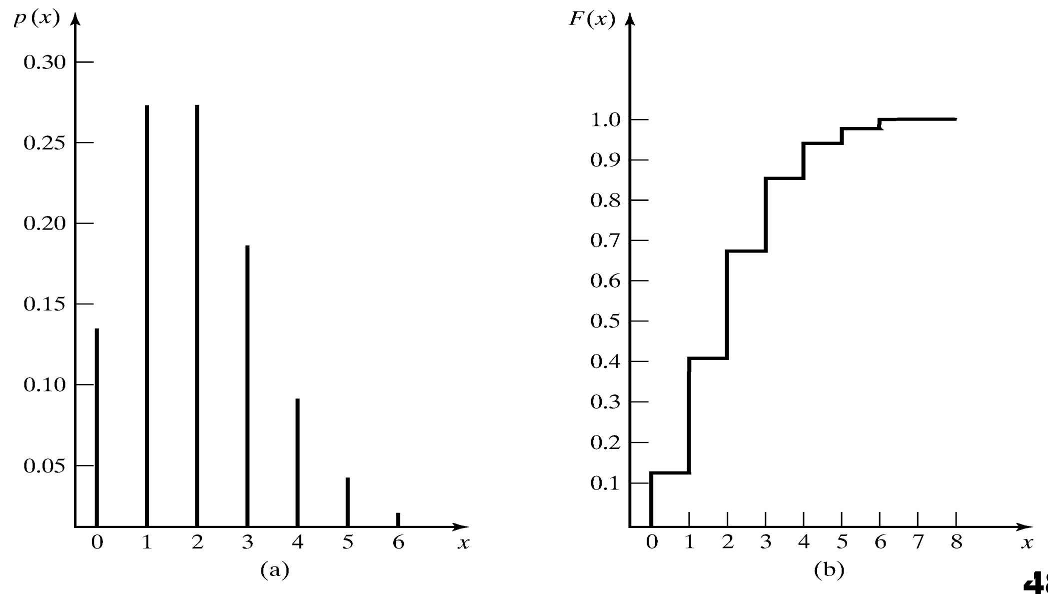
Poisson Distribution

- Poisson distribution describes many random processes quite well and is mathematically quite simple.
  - where $\alpha > 0$, pdf and cdf are:

$$p(x) = \begin{cases} \dfrac{e^{-\alpha}\alpha^{x}}{x!}, & x = 0,1,\ldots \\ 0, & \text{otherwise} \end{cases}$$

$$F(x) = \sum_{i=0}^{x} \frac{e^{-\alpha}\alpha^{i}}{i!}$$

  - $E(X) = \alpha = V(X)$



Poisson Distribution

[Discrete Dist'n]

If the random variable X follows a Poisson distribution with mean 3.4, find $P(X=6)$.

This can be written more quickly as: if $X \sim Po(3.4)$ find $P(X=6)$.

Now $$P(X=6) = \frac{e^{-\lambda}\lambda^{6}}{6!}$$

$$= \frac{e^{-3.4}(3.4)^{6}}{6!}$$

= 0.072

Poisson Distribution

[Discrete Dist'n]

- Example: A computer repair person is "beeped" each time there is a call for service. The number of beeps per hour ~ Poisson(α = 2 per hour).
  - The probability of three beeps in the next hour:

    $p(3) = e^{-2}2^3/3! = 0.18$

    also, $p(3) = F(3) – F(2) = 0.857-0.677=0.18$

  - The probability of two or more beeps in a 1-hour period:

    $p(2 \text{ or more}) = 1 – p(0) – p(1)$

    $= 1 – F(1)$

    $= 0.594$

Poisson Distribution

The number of industrial injuries per working week in a particular factory is known to follow a Poisson distribution with mean 0.5.

Find the probability that

(a) in a particular week there will be:

(i) less than 2 accidents,

(ii) more than 2 accidents;

(b) in a three week period there will be no accidents.

Let A be 'the number of accidents in one week', so $A \sim P_0(0.5)$.

Poisson Distribution

$$P(A<2) = P(A=0)+P(A=1)$$

$$= e^{-0.5} + \frac{e^{-0.5}\times 0.5}{1!}$$

$$= \frac{3}{2}e^{-0.5}$$

$$\simeq 0.9098.$$

(ii) $\quad P(A>2) = 1-P(A\leq 2)$

Poisson Distribution [Discrete Dist'n]

$$1 - [P(A=0) + P(A=1) + P(A=2)]$$

$$= 1 - \left[e^{-0.5} + e^{-0.5} 0.5 + \frac{e^{-0.5}(0.5)^2}{2!} \right]$$

$$= 1 - e^{-0.5}(1 + 0.5 + 0.125)$$

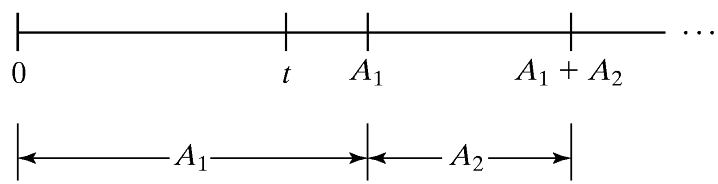
$$= 1 - 1.625\, e^{-0.5}$$

$$\approx 0.0144.$$

(b) $P\,(0 \text{ in } 3 \text{ weeks}) = \left(e^{-0.5}\right)^3 \approx 0.223.$

Interarrival Times [Poisson Dist'n]

- The experiment results in outcomes that can be classified as successes or failures.
- Consider the interarrival times of a Possion process ($A_1, A_2, \ldots$), where A_i is the elapsed time between arrival i and arrival $i+1$



  - The 1st arrival occurs after time t iff there are no arrivals in the interval [0,t], hence:

$$P\{A_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$$

$$P\{A_1 <= t\} = 1 - e^{-\lambda t} \qquad \text{[cdf of exp}(\lambda)]$$

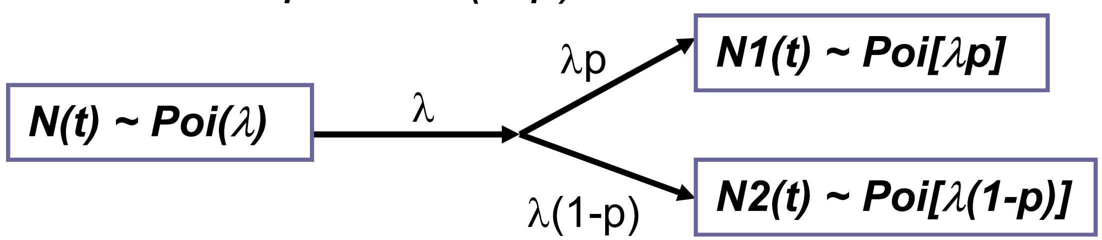
  - Interarrival times, $A_1, A_2, \ldots$, are exponentially distributed and independent with mean $1/\lambda$

Arrival counts ~ $Poi(\lambda)$ ⇔ Interarrival time ~ $Exp(1/\lambda)$

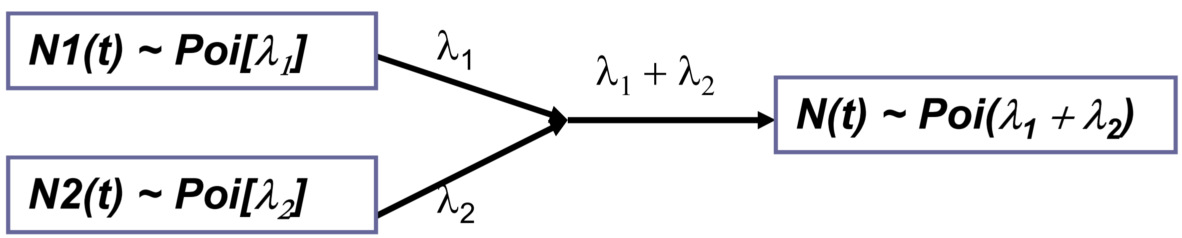
Stationary & Independent | Memoryless

Splitting and Pooling

- Splitting:
  - Suppose each event of a Poisson process can be classified as Type I, with probability *p* and Type II, with probability *1-p.*
  - *N(t)* = *N1(t)* + *N2(t)*, where *N1(t)* and *N2(t)* are both Poisson processes with rates λp and $\lambda(1-p)$



- Pooling:
  - Suppose two Poisson processes are pooled together
  - *N1(t)* + *N2(t)* = *N(t)*, where *N(t)* is a Poisson processes with rates $\lambda_1 + \lambda_2$



Nonstationary Poisson Process (NSPP)

- Poisson Process without the stationary increments, characterized by $\lambda(t)$, the arrival rate at time t.
- The expected number of arrivals by time t, $\Lambda(t)$:

$$\Lambda(t) = \int_0^t \lambda(s)ds$$

- Relating stationary Poisson process $n(t)$ *with rate* $\lambda=1$ and NSPP $N(t)$ with rate $\lambda(t)$:
  - Let arrival times of a stationary process with rate $\lambda = 1$ be $t_1, t_2, \ldots,$ and arrival times of a NSPP with rate $\lambda(t)$ be $T_1, T_2, \ldots$, we know:

$$t_i = \Lambda(T_i)$$
$$T_i = \Lambda^{-1}(t_i)$$

Nonstationary Poisson Process (NSPP)

[Poisson Dist'n]

- Example: Suppose arrivals to a Post Office have rates 2 per minute from 8 am until 12 pm, and then 0.5 per minute until 4 pm.
- Let t = 0 correspond to 8 am, NSPP *N(t)* has rate function:

$$\lambda(t) = \begin{cases} 2, & 0 \le t < 4 \\ 0.5, & 4 \le t < 8 \end{cases}$$

Expected number of arrivals by time t:

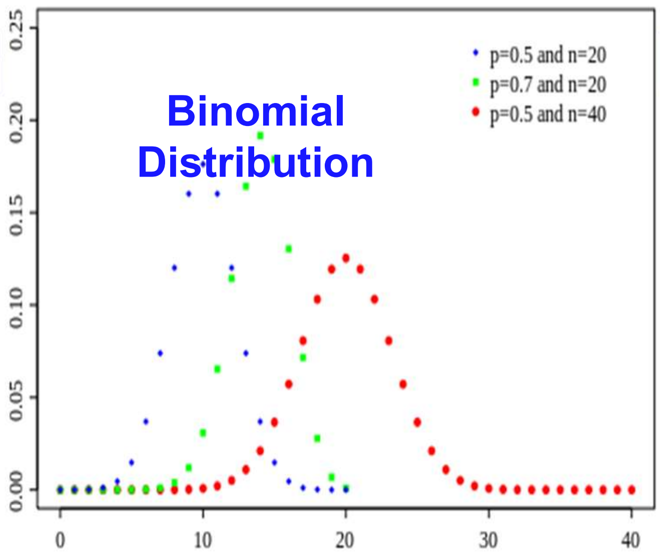
$$\Lambda(t) = \begin{cases} 2t, & 0 \le t < 4 \\ \int_0^4 2ds + \int_4^t 0.5ds = \frac{t}{2} + 6, & 4 \le t < 8 \end{cases}$$

- Hence, the probability distribution of the number of arrivals between 11 am and 2 pm.

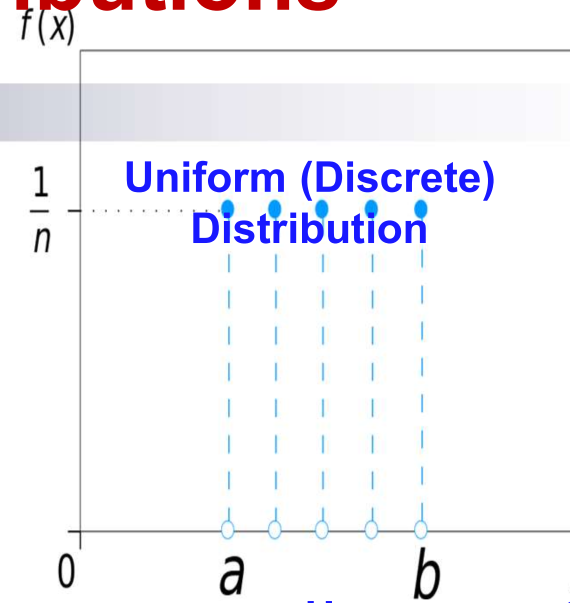
$$\begin{aligned} P[N(6) - N(3) = k] &= P[N(\Lambda(6)) - N(\Lambda(3)) = k] \\ &= P[N(9) - N(6) = k] \\ &= e^{(9-6)}(9-6)^k/k! \quad = e^3(3)^k/k! \end{aligned}$$

Discrete Probability Distributions

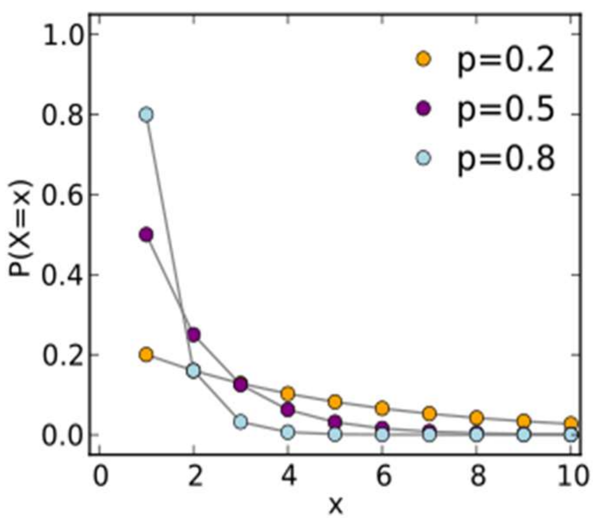
Binomial Distribution



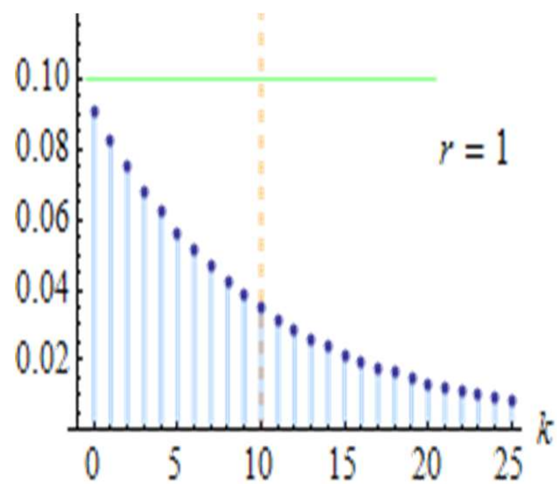
Uniform (Discrete) Distribution



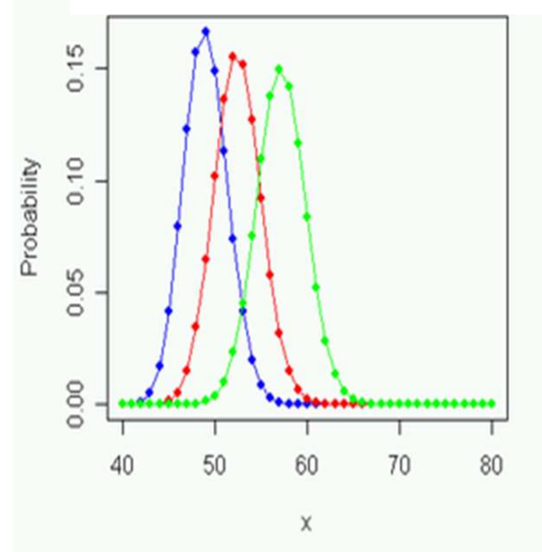
Geometric



Negative Binomial Distribution



Hypergeometric Distribution



Discrete Distributions

- If an experiment only has two possible outcomes → Binomial distribution (ex: packet successfully received or not)
- If we need to count the number of trials until the first success → geometric distribution
- If we need to count the number of trials until the *kth* success, $k = 1, 2, \ldots$ →negative binomial distribution.
- Negative binomial distribution can thought of as a sum of independent geometric distributions
- Ex: What is the probability that the third inspected product at a manufacturing plant is the second one accepted.
- If we need to count the number of event occurrences within a period of time → Poisson distribution (ex: number of calls in an hour)

Empirical Distributions

- A distribution whose parameters are the observed values in a sample of data.
  - May be used when it is impossible or unnecessary to establish that a random variable has any particular parametric distribution.
  - Advantage: no assumption beyond the observed values in the sample.
  - Disadvantage: sample might not cover the entire range of possible values.

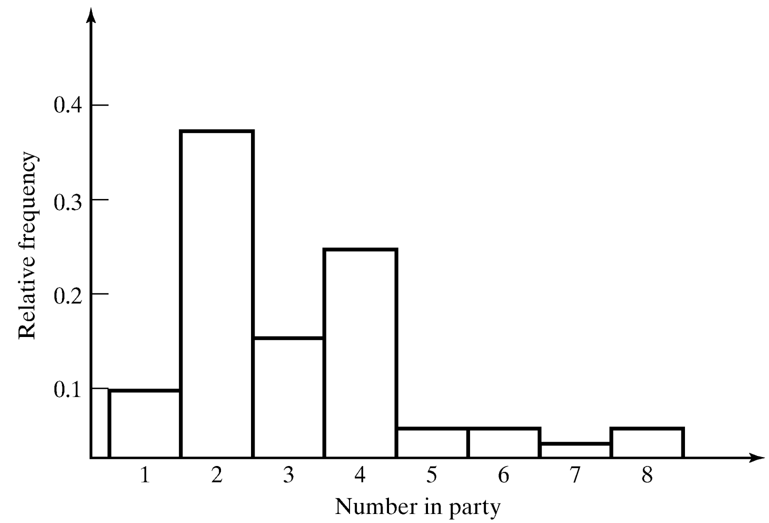
Empirical Distributions

- Example:
- Customers arrive at lunchtime in groups of from one to eight persons.
- The number of persons per party in the last 300 groups has been observed.
- The results are summarized in a table.
- The histogram of the data is also included.

Empirical Distributions (cont.)

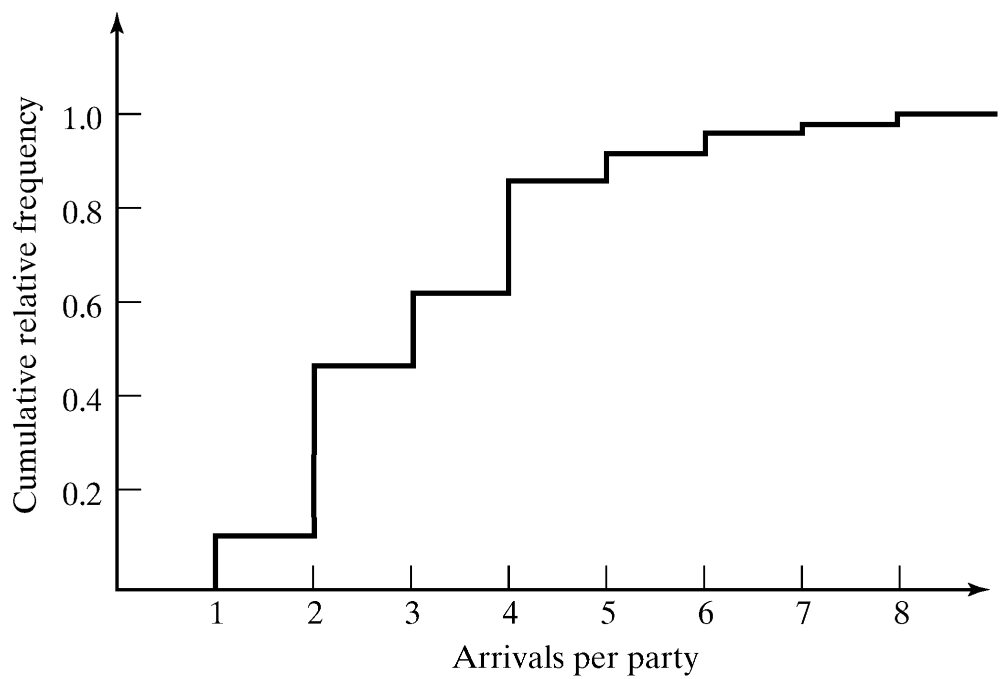
[Probability Review]

Arrivals per Party	Frequency	Relative Frequency	Cumulative Relative Frequency
1	30	0.10	0.10
2	110	0.37	0.47
3	45	0.15	0.62
4	71	0.24	0.86
5	12	0.04	0.90
6	13	0.04	0.94
7	7	0.02	0.96
8	12	0.04	1.00



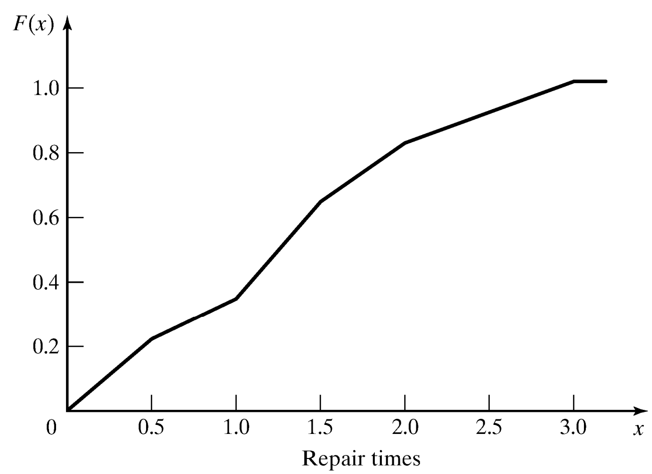
Empirical Distributions (cont.) [Probability Review]

- The CDF in the figure is called the empirical distribution of the given data.



Empirical Distributions

- Example:
- The time required to repair a system that has suffered a failure has been collected for the last 100 instances.
- The empirical CDF is shown in the figure



Empirical Distributions (cont.)

Intervals (Hours)	Frequency	Relative Frequency	Cumulative Frequency
0<x<0.5	21	0.21	0.21
0.5<x<1.0	12	0.12	0.33
1.0<x<1.5	29	0.29	0.62
1.5<x<2.0	19	0.19	0.81
2.0<x<2.5	8	0.08	0.89
2.5<x<3.0	11	0.11	1.00

Useful Statistical Models

- In this section, statistical models appropriate to some application areas are presented. The areas include:
  - Queueing systems
  - Inventory and supply-chain systems
  - Reliability and maintainability
  - Limited data

Queueing Systems

- In a queueing system, interarrival and service-time patterns can be probabilistic
- Sample statistical models for interarrival or service time distribution:
  - Exponential distribution: if service times are completely random
  - Normal distribution: fairly constant but with some random variability (either positive or negative)
  - Truncated normal distribution: similar to normal distribution but with restricted value.
  - Gamma and Weibull distribution: more general than exponential (involving location of the modes of pdf's and the shapes of tails.)

Inventory and supply chain [Useful Models]

- In realistic inventory and supply-chain systems, there are at least three random variables:
  - The number of units demanded per order or per time period
  - The time between demands
  - The lead time (to satisfy demands)
- Sample statistical models for lead time distribution:
  - Gamma
- Sample statistical models for demand distribution:
  - Poisson: simple and extensively tabulated.
  - Negative binomial distribution: longer tail than Poisson (more large demands).
  - Geometric: special case of negative binomial given at least one demand has occurred.

Reliability and maintainability [Useful Models]

- Time to failure (TTF)
  - Exponential: failures are random
  - Gamma: for standby redundancy where each component has an exponential TTF
  - Weibull: failure is due to the most serious of a large number of defects in a system of components
  - Normal: failures are due to wear
  - Lognormal distribution: time to failure for some types of components

Limited Data

- If the data obtained is limited or incomplete, there are usually three distributions that are used:

Uniform Distribution	Triangular Distribution	Beta Distribution
• Inter-arrival or service times are known to be random • No other information is known about the distribution	• When assumptions can be made about the maximum, minimum and modal values of the distribution	• Provides a variety of distribution forms on the unit interval • These distributions can be shifted to any other interval

Summary

- The world that the simulation analyst sees is probabilistic, not deterministic.
- In this chapter:
  - Reviewed several important probability distributions.
  - Showed applications of the probability distributions in a simulation context.
- Important task in simulation modeling is the collection and analysis of input data, e.g., hypothesize a distributional form for the input data. Reader should know:
  - Difference between discrete, continuous, and empirical distributions.
  - Poisson process and its properties.