Eigenvalues and Eigenvectors

The eigen value problem is one of the most important problem in linear algebra.

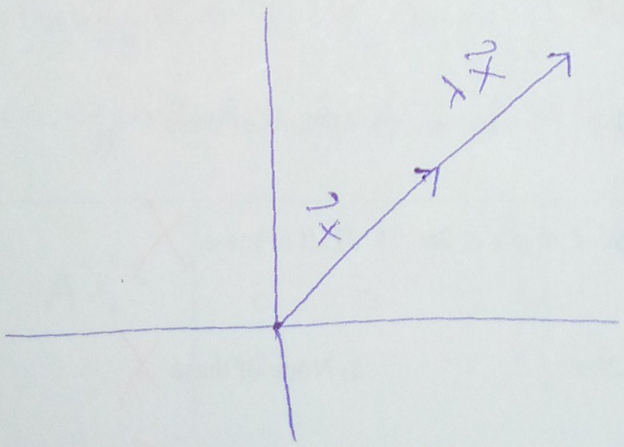
If A is an $n \times n$ matrix, do non zero vectors $\vec{x}$ in R^n exist such that $A\vec{x}$ is a scalar multiple of $\vec{x}$?

The scalar, denoted by the Greek letter λ, is called an eigenvalue of the matrix A and the non-zero vector $\vec{x}$ is called an eigenvector of A corresponding to λ.

<u>Definition</u> Let A be an $n \times n$ matrix. The scalar λ is called an eigenvalue of A if there is a non-zero vector $\vec{x}$ s.t

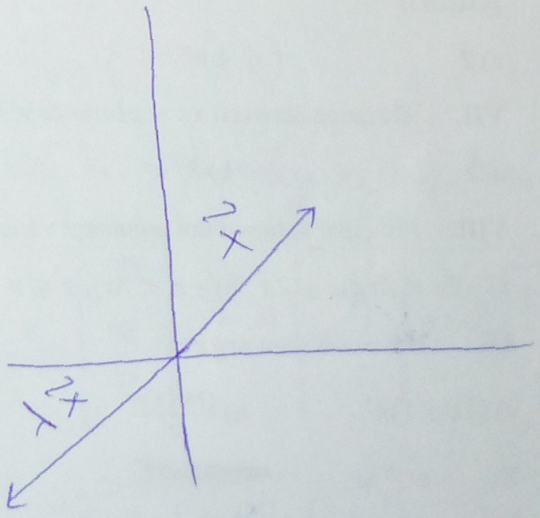
$$A\vec{x} = \lambda\vec{x}.$$

The vector $\vec{x}$ is called eigenvector of A corresponding to λ.



$A\vec{x} = \lambda\vec{x}$

$\lambda > 0$



$A\vec{x} = \lambda\vec{x}$

$\lambda < 0$

Example 1 For the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

a verify that $\vec{X}_1 = (1, 0)$ is an eigenvector of A corresponding to eigenvalue $\lambda_1 = 2$ and that $A\vec{X}_1 = \lambda_1 \vec{X}_1$

b. $\vec{X}_2 = (0, 1)$ is an eigenvector of A corresponding to eigenvalue $\lambda_2 = -1$.

$$A\vec{X}_1 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_1 \vec{X}_1$$

$$A\vec{X}_2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lambda_2 \vec{X}_2$$

Example 2 For the matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

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verify that $\vec{X}_1 = (-3, -1, 1)$ and $\vec{X}_2 = (1, 0, 0)$ are eigenvectors of A and find their corresponding eigenvalues.

$$A\vec{X}_1 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \lambda_1 \vec{X}_1$$

$\lambda_1 = 0$

$$A\vec{X}_2 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \lambda_2 \vec{X}_2$$

$\lambda_2 = 1$

<u>Eigenspace</u> The set of all eigenvectors of a given value λ, together with zero vector, is a subspace of R^n. This subspace is called the eigenspace of λ.

<u>Theorem</u> If A is an $n \times n$ matrix with eigenvalue λ, then the set of all eigenvectors of λ, together with zero vector

$$\{\vec{0}\} \cup \{\vec{x} : \vec{x} \text{ is an eigenvector of } \lambda\},$$

is a subspace of R^n. This subspace is called the eigenspace of λ.

<u>Computing Eigenvalues and Eigenvectors</u>

To find the eigenvalues and eigenvectors of an $n \times n$ matrix A, let I be the $n \times n$ identity matrix.

$$A\vec{x} = \lambda\vec{x} \Rightarrow A\vec{x} = \lambda I\vec{x}$$

$$\Rightarrow (\lambda I - A)\vec{x} = \vec{0}.$$

Support System of Equations because:
AX = 0 (matrix*vector = vector)

This homogeneous system of equations has non-zero solutions if and only if the coefficient matrix $(\lambda I - A)$ is not invertible i.e, if and only if the determinant of $(\lambda I - A)$ is zero.

Theorem Let A be an $n \times n$ matrix.

1) An eigenvalue of A is a scalar λ such that

$$\det(\lambda I - A) = 0$$

This is called the characteristic equation of A

1.determinant find kar kai lemda find kar lena hi. matlab eigon values find kar leni hin
2.lemda ki values ko equation mai put kartay jana hi.aur eigon vectors miltay jain gain.

2) The eigenvectors of A corresponding to λ are the non zero solutions of $(\lambda I - A)\vec{x} = \vec{0}$.

Example 3 Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

$$\underbrace{(\lambda I - A)}_{\det(\lambda I - A) = 0}\vec{x} = \vec{0}$$

The characteristic polynomial of A is

$$\det(\lambda I - A) = 0$$

Simple way diagonal pai (lemda - value) likh dena hi aur baqi sab entries kai sign change kar denay hin.

$$\begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} = 0$$

$$(\lambda - 2)(\lambda + 5) - (-12) = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda + 1)(\lambda + 2) = 0$$

$$\lambda_1 = -1 \qquad \lambda_2 = -2$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be eigenvector of the matrix A corresponding to eigenvalue λ.

$$(\lambda I - A)\vec{x} = \vec{0}$$

$$\begin{pmatrix} \lambda-2 & 12 \\ -1 & \lambda+5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For $\lambda_1 = -1$ $\begin{pmatrix} -3 & 12 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The augmented matrix is

$$\begin{pmatrix} -3 & 12 & 0 \\ -1 & 4 & 0 \end{pmatrix}$$

The echelon form is

$$\begin{pmatrix} 1 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 - 4x_2 = 0, \quad x_2 = t$

$x_1 = 4t$

$\vec{x} = (4t, t) = t(4,1) = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \; t \neq 0.$

For $\lambda_2 = -2$ $\begin{pmatrix} -4 & 12 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The augmented matrix is

$$\begin{pmatrix} -4 & 12 & 0 \\ -1 & 3 & 0 \end{pmatrix}$$

The echelon form is

$$\begin{pmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 - 3x_2 = 0 \qquad x_2 = t$

$x_1 = 3t$

$\vec{x} = (3t, t) = t(3,1) = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \; t \neq 0$

<u>Example (4)</u> Find the eigenvalues of

Shukar hi eigenvalues find karnay ka bola hi. eigon vectors ni. Shukar hi

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

The Characteristic polynomial of A is

$\det(\lambda I - A) = 0$

$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{vmatrix} = 0$$

$\lambda[\lambda(\lambda-8)+17] + 1(0-4) + 0 = 0$

$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0 \longrightarrow (1)$

para okha kam hi. koi ni.
aik root find karna hi hit and trial method sai lagay raho.

The possible integer solutions of (1) are the divisors of -4

$\pm 1, \pm 2, \pm 3, \pm 4$

$\lambda = 4$ is a solution $\quad 4^3 - 8(4)^2 + 17(4) - 4 = 0$

$$(\lambda-4)(\lambda^2-4\lambda+1)=0$$

4	1	-8	17	-4
		4	-16	4
	1	-4	1	0

$$\lambda^2-4\lambda+1=0$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\lambda = \frac{4 \pm \sqrt{16-4(1)(1)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

$$\lambda_1 = 4, \quad \lambda_2 = 2+\sqrt{3}, \quad \lambda_3 = 2-\sqrt{3}$$

<u>Example 5</u> Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The characteristic polynomial is

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda-2 & -1 & 0 \\ 0 & \lambda-2 & 0 \\ 0 & 0 & \lambda-2 \end{vmatrix} = 0$$

$$(\lambda-2)^3 = 0$$

$$\lambda = 2$$

is the eigen value of the matrix A.

$$(\lambda I - A)\vec{X} = \vec{0}$$

$$\begin{pmatrix} \lambda-2 & -1 & 0 \\ 0 & \lambda-2 & 0 \\ 0 & 0 & \lambda-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda = 2$ $\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

iss ko echelon mai kar lena bacho
row 1 ko -1 sai divide kar kai
aik tou ye echelon bi jan ni
chortay.

$x_1 = \text{free} = s$

$x_2 = 0$

$x_3 = \text{free} = t$

$$\vec{X} = (s, 0, t) = (s, 0, 0) + (0, 0, t)$$

$$= s(1, 0, 0) + t(0, 0, 1)$$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors

of A corresponding to eigen value $\lambda = 2$.

Eigenvalues of an upper Triangular Matrix

The eigenvalues of the upper triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

are $\lambda_1 = a_{11}$, $\lambda_2 = a_{22}$, $\lambda_3 = a_{33}$, $\lambda_4 = a_{44}$.

<u>Theorem</u> If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, diagonal), then the eigenvalues of A are the entries on the main diagonal of A.

<u>Example ①</u> Find the eigenvalues of the lower triangular matrix.

$$A = \begin{bmatrix} 1/2 & 0 & 0 \\ -1 & 2/3 & 0 \\ 5 & -8 & -1/4 \end{bmatrix}$$

$\lambda_1 = 1/2$, $\lambda_2 = 2/3$, $\lambda_3 = -1/4$.

<u>Finding Basis for Eigenspaces.</u> Let A be an $n \times n$ matrix. The eigenspace of A corresponding to the eigenvalue λ is the solution space of the homogeneous system

$$(\lambda I - A)\vec{x} = \vec{0}.$$

<u>Example 7</u> Find basis for the eigenspace of

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}.$$

The Characteristic equation is

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{vmatrix} = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$\lambda = 1$ is a root $\quad 1^3 - 5(1)^2 + 8(1) - 4 = 0$

1	1	-5	8	-4
		1	-4	4
	1	-4	4	0

$$(\lambda - 1)(\lambda^2 - 4\lambda + 4) = 0$$

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = 2.$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigenvector of A corresponding to eigenvalue λ.

$$(\lambda I - A)\vec{x} = \vec{0}$$

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_1 = 1$ $\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solving this system, $x_1 = -2s$, $x_2 = s$, $x_3 = s$

$$\vec{x} = \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$B_1 = \{(-2, 1, 1)\}$ is a basis for eigenspace corresponding to $\lambda = 1$

For $\lambda_2 = 2$ $\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solving this system yields

$$x_1 = -s, \quad x_2 = t, \quad x_3 = s.$$

$$\vec{x} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}$$

$$= s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$B_2 = \{(-1, 0, 1), (0, 1, 0)\}.$$

The vectors are linearly independent. These vectors form a basis for the eigenspace corresponding to $\lambda = 2$.

<u>Example 8</u> Find the eigenvalues of

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

and find a basis for each of the corresponding eigenspace.

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda-1 & 0 & 0 & 0 \\ 0 & \lambda-1 & -5 & 10 \\ -1 & 0 & \lambda-2 & 0 \\ -1 & 0 & 0 & \lambda-3 \end{vmatrix} = 0$$

$$(\lambda-1)^2(\lambda-2)(\lambda-3)=0$$

$\lambda_1=1, \quad \lambda_2=2, \quad \lambda_3=3$

$$(\lambda I - A)\vec{x}=0.$$

$$\begin{bmatrix} \lambda-1 & 0 & 0 & 0 \\ 0 & \lambda-1 & -5 & 10 \\ -1 & 0 & \lambda-2 & 0 \\ -1 & 0 & 0 & \lambda-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_1=1$ $\quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1=-2t, \quad x_2=s, \quad x_3=2t, \quad x_4=t$

$$\vec{x} = \begin{bmatrix} -2t \\ s \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2t \\ 0 \\ 2t \\ t \end{bmatrix}$$

$$= s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$B_1 = \{(0,1,0,0), (-2,0,2,1)\}$ is a basis for the eigenspace corresponding to $\lambda_1 = 1$.

For $\lambda_2 = 2$ $\quad B_2 = \{(0,5,1,0)\}$ (verify)

For $\lambda_3 = 3$ $\quad B_3 = \{(0,-5,0,1)\}$. (verify)

Eigenvalues of powers of a Matrix

If λ is an eigenvalue of matrix A and $\vec{x}$ is a corresponding eigenvector, then

$$A^2\vec{x} = A(A\vec{x}) = A(\lambda\vec{x}) = \lambda(A\vec{x}) = \lambda(\lambda\vec{x})$$

$$A^2\vec{x} = \lambda^2\vec{x}.$$

This shows that λ^2 is an eigenvalue of A^2 and $\vec{x}$ is a corresponding eigenvectors.

<u>Theorem</u> If K is a positive integer, λ is an eigenvalue of a matrix A, and $\vec{x}$ is a corresponding eigenvector, then λ^K is an eigenvalue of A^K and $\vec{x}$ is a corresponding eigenvector.

<u>Example(9)</u> Given the matrix

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Find eigenvalues and eigenvectors of A^7.

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda-1 & -3 & 0 \\ -3 & \lambda-1 & 0 \\ 0 & 0 & \lambda+2 \end{vmatrix} = 0$$

$$(\lambda-1)\left[(\lambda-1)(\lambda+2)\right] + 3\left[3(\lambda+2)\right] - 0 = 0$$

$$(\lambda+2)(\lambda-1)^2 - 9(\lambda+2) = 0$$

$$(\lambda+2)\left[(\lambda-1)^2 - 9\right] = 0$$

$\lambda_1 = -2$ $\quad$ $\lambda_2 = 4$ are eigenvalues of A

$(-2)^7$, $(4)^7$ are eigenvalues of A^7

$= -128$ $\quad$ $= 16384$

For $\lambda_1 = -2$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are

eigenvectors of A^T (verify).

For $\lambda_2 = 4$ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector of A^T (verify)

if lemda = 0
inverse of given vector will not possible.

Eigenvalues and Invertibility

A square matrix A is invertible iff $\lambda = 0$ is not an eigenvalue of A.

Example 10 The matrix $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

is invertible since it has $\lambda_1 = -2$, $\lambda_2 = 4$, neither of which is zero. (check $\det A \neq 0$).

$\boxed{\text{Exercise 5.1 Q1 – 29 odd}}$