Exercise 6.6

3. $\frac{1}{2} t e^{-3t}$

Let, $g(t) = \frac{1}{2} t e^{-3t}$

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Let, $f(t) = \frac{1}{2} e^{-3t}$

Taking Laplace transform.

$$F(s) = \frac{1}{2} \cdot \frac{1}{s+3}$$

Taking derivative on b.s. w.r.t. s.

$$F'(s) = \frac{1}{2} \cdot \frac{-1}{(s+3)^2}$$

Multiply by -1.

$$-F'(s) = \frac{1}{2} \cdot \frac{1}{(s+3)^2}$$

Hence;

$$\mathcal{L}\left\{\frac{1}{2} t e^{-3t}\right\} = \frac{1}{2(s+3)^2}$$

5 $t \cos \omega t$

Let, $f(t) = t \cos \omega t$

Let, $g(t) = \cos \omega t$

By Taking laplace transform

$$F(s) = \frac{s}{s^2+w^2}$$

By taking derivative w.r.t. s.

$$F'(s) = \frac{(s^2+w^2)(1) - s(2s)}{(s^2+w^2)^2} = \frac{s^2+w^2-2s^2}{(s^2+w^2)^2}$$

$$F'(s) = \frac{w^2-s^2}{(s^2+w^2)^2}$$

Multiply -1 on b.s.

$$-F'(s) = \frac{s^2-w^2}{(s^2+w^2)^2}$$

Hence, $\mathcal{L}\{t\cos wt\} = \frac{s^2-w^2}{(s^2+w^2)^2}$

7 $t^2 \cosh 2t$

Let, $g(t) = t^2 \cosh 2t$

Let $f(t) = \cosh 2t$

By taking L.T.

$$F(s) = \frac{s}{s^2-4}$$

Diff. w.r.t. s.

$$F'(s) = \frac{(s^2-4)(1) - s(2s)}{(s^2-4)^2} = \frac{s^2-4-2s^2}{(s^2-4)^2}$$

$$F'(s) = \frac{-4-s^2}{(s^2-4)^2} = \frac{-(s^2+4)}{(s^2-4)^2} =$$

$$+F''(s) = \frac{(s^2-4)^2(-2s) + (s^2+4)\,2(s^2-4)(2s)}{(s^2-4)^4}$$

$$F''(s) = \frac{-2s(s^2-4)^2 + 4s(s^2+4)(s^2-4)}{\quad}$$

$$= \frac{\cancel{(s^2-4)}\left[-2s(s^2-4) + 4s(s^2+4)\right]}{(s^2-4)^{\cancel{4}\,3}}$$

$$F''(s) = \frac{-2s(s^2-4) + 4s(s^2+4)}{(s^2-4)^3}$$

Hence,

$$\mathcal{L}\{t^2\cosh 2t\} = \frac{-2s(s^2-4) + 4s(s^2+4)}{(s^2-4)^3}$$

9 $\frac{1}{2}t^2 \sin \pi t$

Let, $g(t) = \frac{1}{2}t^2 \sin \pi t$

Let, $f(t) = \frac{1}{2}\sin \pi t$

Taking L.T.

$$F(s) = \frac{1}{2} \cdot \frac{\pi}{s^2+\pi^2}$$

$$F'(s) = \frac{1}{2} \cdot \frac{(s^2+\pi^2)(0) - \pi(2s)}{(s^2+\pi^2)^2}$$

$$F'(s) = \frac{1}{2} \frac{-2\pi s}{(s^2+\pi^2)^2} = \frac{-\pi s}{(s^2+\pi^2)^2}$$

$$\cancel{F''(s) = \frac{(s^2+\pi^2)(-\pi) - (-\pi s)(2s)}{(s^2+\pi^2)^4}}$$

$$\cancel{F''(s) = \frac{-\pi(s^2+\pi^2) + 2\pi s^2}{}}$$

$$= \cancel{\frac{\pi(s^2 - \pi^2 - 2s^2)}{(s^2+\pi^2)^4}} = \cancel{\frac{\pi(-\pi^2)}{}}\ \pi^2$$

$$F''(s) = -\pi\left[\frac{(s^2+\pi^2)^2(1) - s(2)(s^2+\pi^2)(2s)}{(s^2+\pi^2)^4}\right]$$

$$= -\pi \left[\frac{(s^2+\pi^2)^2 - 4s^2(s^2+\pi^2)}{(s^2+\pi^2)^4} \right]$$

$$= \frac{-\pi(s^2+\pi^2)\left[s^2+\pi^2-4s^2\right]}{(s^2+\pi^2)^4} = \frac{-\pi(s^2+\pi^2-4s^2)}{//}$$

$$= \frac{-\pi(-3s^2+\pi^2)}{(s^2+\pi^2)^4} = \frac{\pi(3s^2-\pi^2)}{(s^2+\pi^2)^4}$$

Hence,

$$\mathcal{L}\left\{\frac{1}{2}t^2\sin\pi t\right\} = \frac{\pi(3s^2-\pi^2)}{(s^2+\pi^2)^4}$$

11 $4t\cos\frac{1}{2}\pi t$

Let $f(t) = 4t\cos\frac{1}{2}\pi t$

Let, $g(t) = 4\cos(\frac{1}{2}\pi t)$

Taking L.T.

$$F(s) = 4 \cdot \frac{s}{s^2+(\frac{1}{2}\pi)^2} = \cancel{4} \frac{s}{s^2+(\frac{\pi}{2})^2}$$

$$F'(s) = \frac{2s}{\left(s^2 + \left(\frac{\pi}{2}\right)^2\right)^2}$$

$$F'(s) = 4\,\frac{\left(s^2 + \left(\frac{\pi}{2}\right)^2\right)(1) - s(2s)}{\left(s^2 + \left(\frac{\pi}{2}\right)^2\right)^2}$$

$$F'(s) = 4\,\frac{s^2 + \frac{\pi^2}{4} - 2s^2}{\left(s^2 + \left(\frac{\pi}{2}\right)^2\right)^2} = 4\,\frac{-s^2 + \frac{\pi^2}{4}}{\left(s^2 + \frac{\pi^2}{4}\right)^2}$$

$$F'(s) = 4\,\frac{\frac{-4s^2 + \pi^2}{4}}{\left(s^2 + \frac{\pi^2}{4}\right)^2} = \frac{-4s^2 + \pi^2}{\left(s^2 + \frac{\pi^2}{4}\right)^2}$$

Multiply by -1.

$$-F'(s) = \frac{-\pi^2 + 4s^2}{\left(s^2 + \frac{\pi^2}{4}\right)^2}$$

Hence

$$\mathcal{L}\left\{4t\cos\left(\tfrac{1}{2}\pi\right)t\right\} = \frac{4s^2 - \pi^2}{\left(s^2 + \frac{\pi^2}{4}\right)^2}$$

14 $\dfrac{s}{(s^2+16)^2}$

Let $F(s) = \dfrac{s}{(s^2+16)^2} = \dfrac{1}{2}\dfrac{2s}{(s^2+16)^2}$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \frac{1}{2}\int_s^\infty \frac{2s}{(s^2+16)^2}\,ds = \frac{1}{2}\int_s^\infty (s^2+16)^{-2}\,2s\,ds$$

$$= \frac{1}{2}\left|\frac{(s^2+16)^{-1}}{-1}\right|_s^\infty$$

$$= -\frac{1}{2}\left|\frac{1}{s^2+16}\right|_s^\infty$$

$$= -\frac{1}{2}\left[\frac{1}{\infty} - \frac{1}{s^2+16}\right] = \frac{1}{2}\left[\frac{1}{s^2+16}\right]$$

By taking I.L.T.

$$\mathcal{L}^{-1}\left\{\frac{f(t)}{t}\right\} \quad \frac{1}{2}\cdot\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2+16}\right\}$$

$$\frac{f(t)}{t} = \frac{1}{8}\sin 4t$$

$$f(t) = \frac{1}{8} + \sin 4t$$

15 $\dfrac{s}{(s^2-9)^2}$

Let, $F(s) = \dfrac{s}{(s^2-9)^2}$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \frac{s}{(s^2-9)^2}\,ds$$

$$= \frac{1}{2}\int_s^\infty (s^2-9)^{-2}\,2s\,ds$$

$$= \frac{1}{2}\left|\frac{(s^2-9)^{-1}}{-1}\right|_s^\infty$$

$$= -\frac{1}{2}\left|\frac{1}{s^2-9}\right|_s^\infty = -\frac{1}{2}\left[\frac{1}{\infty} - \frac{1}{s^2-9}\right]$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2-9}\right] = \frac{1}{2(s^2-9)}$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \frac{1}{2} \cdot \frac{1}{3} \frac{3}{s^2-9}$$

$$\frac{f(t)}{t} = \frac{1}{6} \mathcal{L}^{-1}\left\{\frac{3}{s^2-(3)^2}\right\}$$

$$\frac{f(t)}{t} = \frac{1}{6} \sinh 3t$$

$$f(t) = \frac{t \sinh 3t}{6}$$

1b) $$\frac{2s+6}{(s^2+6s+10)^2}$$

$$F(s) = \frac{2s+6}{(s^2+6s+10)^2}$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \frac{2s+6}{(s^2+6s+10)^2} = \int_s^\infty (s^2+6s+10)^{-2} (2s+6)\, ds$$

$$= \left[\frac{(s^2+6s+10)^{-1}}{-1}\right]_s^\infty$$

$$= -\left[\frac{1}{s^2+6s+10}\right]_s^{\infty}$$

$$= -\left[\frac{1}{\infty} - \frac{1}{s^2+6s+10}\right] = \frac{1}{s^2+6s+10}$$

$$= \frac{1}{s^2+6s+10+9-9} = \frac{1}{s^2+6s+9+1}$$

$$= \frac{1}{(s+3)^2+1} = e^{-3t}\sin t$$

$$\frac{f(t)}{t} = e^{-3t}\sin t$$

$$f(t) = t e^{-3t}\sin t$$

X 17) $F(s) = \ln\frac{s}{s-1} = \ln(s) - \ln(s-1)$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} \ln(s) - \int_s^{\infty} \ln(s-1)$$

$$= \left|\frac{1}{s}\right|_s^{\infty} - \left|\frac{1}{s-1}\right|_s^{\infty}$$

$$= \left[\frac{1}{\infty} - \frac{1}{s}\right] - \left[\frac{1}{\infty} - \frac{1}{s-1}\right]$$

$$= -\frac{1}{s} + \frac{1}{s-1} = \frac{1}{s-1} - \frac{1}{s}$$

$$\frac{f(t)}{t} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$= \overset{e^t}{\cancel{\sinh t}} - 1$$

18 $\text{arc cot } \frac{s}{\pi} = \cot^{-1}\left(\frac{s}{\pi}\right)$

$$F(s) = \cot^{-1}\left(\frac{s}{\pi}\right) \qquad \because \frac{d}{du}\cot^{-1}\left(\frac{s}{\pi}\right) = \frac{-1}{1+u^2}$$

$$F'(s) = \frac{-1}{1+\left(\frac{s}{\pi}\right)^2} = \frac{-1}{1+\frac{s^2}{\pi^2}} = \frac{-1}{\frac{\pi^2+s^2}{\pi^2}}$$

$$F'(s) = \frac{-\pi^2}{\pi^2+s^2}$$

$$\because \; t f(t) = -1\, F'(s)$$

$$-t f(t) = F'(s)$$

$$\mathcal{L}^{-1} F'(s) = -1\, \mathcal{L}\, \frac{\pi}{s^2+\pi^2}$$

$$-t f(t) = -1 \cdot \sin \pi t$$

$$f(t) = \frac{\sin \pi t}{t}$$

$$= -\frac{2s}{s^2+1} + \frac{2}{s-1}$$

$$\mathcal{L}^{-1}\left\{\frac{f(t)}{t}\right\} = \cancel{-2\mathcal{L}^{-1}\frac{s}{s^2+1}} + 2\mathcal{L}^{-1}\frac{1}{s-1}$$

$$= -2\cos t + 2e^{t}$$

20 $\ln\dfrac{s+a}{s+b}$

$$F(s) = \ln(s+a) - \ln(s+b)$$

$$F'(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

$$-t\,f(t) = \mathcal{L}^{-1}\frac{1}{s+a} - \mathcal{L}^{-1}\frac{1}{s+b}$$

$$-t\,f(t) = e^{-at} - e^{-bt}$$

$$f(t) = \frac{e^{-bt} - e^{-at}}{t}$$