Theorem 6.3.2 If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a constant, then

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c), \qquad s > a + c. \tag{7}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$e^{ct}f(t) = \mathcal{L}^{-1}\{F(s-c)\}. \tag{8}$$

According to Theorem 6.3.2, multiplication of $f(t)$ by e^{ct} results in a translation of the transform $F(s)$ a distance c in the positive s direction, and conversely. To prove this theorem, we evaluate $\mathcal{L}\{e^{ct}f(t)\}$. Thus

$$\mathcal{L}\{e^{ct}f(t)\} = \int_0^\infty e^{-st}e^{ct}f(t)\,dt = \int_0^\infty e^{-(s-c)t}f(t)\,dt$$
$$= F(s-c),$$

which is Eq. (7). The restriction $s > a + c$ follows from the observation that, according to hypothesis (ii) of Theorem 6.1.2, $|f(t)| \leq Ke^{at}$; hence $|e^{ct}f(t)| \leq Ke^{(a+c)t}$. Equation (8) is obtained by taking the inverse transform of Eq. (7), and the proof is complete.

The principal application of Theorem 6.3.2 is in the evaluation of certain inverse transforms, as illustrated by Example 5.

EXAMPLE 5

Find the inverse transform of

$$G(s) = \frac{1}{s^2 - 4s + 5}.$$

By completing the square in the denominator, we can write

$$G(s) = \frac{1}{(s-2)^2 + 1} = F(s-2),$$

where $F(s) = (s^2+1)^{-1}$. Since $\mathcal{L}^{-1}\{F(s)\} = \sin t$, it follows from Theorem 6.3.2 that

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{2t}\sin t.$$

The results of this section are often useful in solving differential equations, particularly those that have discontinuous forcing functions. The next section is devoted to examples illustrating this fact.

PROBLEMS

In each of Problems 1 through 6 sketch the graph of the given function on the interval $t \geq 0$.

1. $g(t) = u_1(t) + 2u_3(t) - 6u_4(t)$
2. $g(t) = (t-3)u_2(t) - (t-2)u_3(t)$
3. $g(t) = f(t-\pi)u_\pi(t)$, where $f(t) = t^2$
4. $g(t) = f(t-3)u_3(t)$, where $f(t) = \sin t$
5. $g(t) = f(t-1)u_2(t)$, where $f(t) = 2t$
6. $g(t) = (t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t)$

In each of Problems 7 through 12:
(a) Sketch the graph of the given function.
(b) Express $f(t)$ in terms of the unit step function $u_c(t)$.

7. $f(t) = \begin{cases} 0, & 0 \leq t < 3, \\ -2, & 3 \leq t < 5, \\ 2, & 5 \leq t < 7, \\ 1, & t \geq 7. \end{cases}$

8. $f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ -1, & 1 \leq t < 2, \\ 1, & 2 \leq t < 3, \\ -1, & 3 \leq t < 4, \\ 0, & t \geq 4. \end{cases}$

9. $f(t) = \begin{cases} 1, & 0 \leq t < 2, \\ e^{-(t-2)}, & t \geq 2. \end{cases}$

10. $f(t) = \begin{cases} t^2, & 0 \leq t < 2, \\ 1, & t \geq 2. \end{cases}$

11. $f(t) = \begin{cases} t, & 0 \leq t < 1, \\ t-1, & 1 \leq t < 2, \\ t-2, & 2 \leq t < 3, \\ 0, & t \geq 3. \end{cases}$

12. $f(t) = \begin{cases} t, & 0 \leq t < 2, \\ 2, & 2 \leq t < 5, \\ 7-t, & 5 \leq t < 7, \\ 0, & t \geq 7. \end{cases}$

In each of Problems 13 through 18 find the Laplace transform of the given function.

13. $f(t) = \begin{cases} 0, & t < 2 \\ (t-2)^2, & t \geq 2 \end{cases}$

14. $f(t) = \begin{cases} 0, & t < 1 \\ t^2 - 2t + 2, & t \geq 1 \end{cases}$

15. $f(t) = \begin{cases} 0, & t < \pi \\ t - \pi, & \pi \leq t < 2\pi \\ 0, & t \geq 2\pi \end{cases}$

16. $f(t) = u_1(t) + 2u_3(t) - 6u_4(t)$

17. $f(t) = (t-3)u_2(t) - (t-2)u_3(t)$

18. $f(t) = t - u_1(t)(t-1), \quad t \geq 0$

In each of Problems 19 through 24 find the inverse Laplace transform of the given function.

19. $F(s) = \dfrac{3!}{(s-2)^4}$

20. $F(s) = \dfrac{e^{-2s}}{s^2 + s - 2}$

21. $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2 - 2s + 2}$

22. $F(s) = \dfrac{2e^{-2s}}{s^2 - 4}$

23. $F(s) = \dfrac{(s-2)e^{-s}}{s^2 - 4s + 3}$

24. $F(s) = \dfrac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$

25. Suppose that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$.
(a) Show that if c is a positive constant, then

$$\mathcal{L}\{f(ct)\} = \frac{1}{c}F\left(\frac{s}{c}\right), \qquad s > ca.$$

(b) Show that if k is a positive constant, then

$$\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k}f\left(\frac{t}{k}\right).$$

(c) Show that if a and b are constants with $a > 0$, then

$$\mathcal{L}^{-1}\{F(as+b)\} = \frac{1}{a}e^{-bt/a}f\left(\frac{t}{a}\right).$$

31. $F(s) = n!/s^{n+1}$

32. $F(s) = n!/(s-a)^{n+1}$

33. $F(s) = 2b(s-a)/[(s-a)^2 + b^2]^2$

34. $F(s) = [(s-a)^2 - b^2]/[(s-a)^2 + b^2]^2$

36. (a) $Y' + s^2 Y = s$ (b) $s^2 Y'' + 2sY' - [s^2 + \alpha(\alpha+1)]Y = -1$

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7. (b) $f(t) = -2u_3(t) + 4u_5(t) - u_7(t)$

8. (b) $f(t) = 1 - 2u_1(t) + 2u_2(t) - 2u_3(t) + u_4(t)$

9. (b) $f(t) = 1 + u_2(t)[e^{-(t-2)} - 1]$

10. (b) $f(t) = t^2 + u_2(t)(1-t^2)$

11. (b) $f(t) = t - u_1(t) - u_2(t) - u_3(t)(t-2)$

12. (b) $f(t) = t + u_2(t)(2-t) + u_5(t)(5-t) - u_7(t)(7-t)$

13. $F(s) = 2e^{-s}/s^3$

14. $F(s) = e^{-s}(s^2+2)/s^3$

15. $F(s) = \dfrac{e^{-\pi s}}{s^2} - \dfrac{e^{-2\pi s}}{s^2}(1+\pi s)$

16. $F(s) = \dfrac{1}{s}(e^{-s} + 2e^{-3s} - 6e^{-4s})$

17. $F(s) = s^{-2}[(1-s)e^{-2s} - (1+s)e^{-3s}]$

18. $F(s) = (1-e^{-s})/s^2$

19. $f(t) = t^3 e^{2t}$

20. $f(t) = \frac{1}{3}u_2(t)[e^{t-2} - e^{-2(t-2)}]$

21. $f(t) = 2u_2(t)e^{t-2}\cos(t-2)$

22. $f(t) = u_2(t)\sinh 2(t-2)$

23. $f(t) = u_1(t)e^{2(t-1)}\cosh(t-1)$

24. $f(t) = u_1(t) + u_2(t) - u_3(t) - u_4(t)$

26. $f(t) = 2(2t)^n$

27. $f(t) = \frac{1}{2}e^{-t/2}\cos t$

28. $f(t) = \frac{1}{6}e^{t/3}(e^{2t/3} - 1)$

29. $f(t) = \frac{1}{2}e^{t/2}u_2(t/2)$

30. $F(s) = s^{-1}(1 - e^{-s}), \quad s > 0$

31. $F(s) = s^{-1}(1 - e^{-s} + e^{-2s} - e^{-3s}), \quad s > 0$

32. $F(s) = \dfrac{1}{s}[1 - e^{-s} + \cdots + e^{-2ns} - e^{-(2n+1)s}] = \dfrac{1 - e^{-(2n+2)s}}{s(1+e^{-s})}, \quad s > 0$

33. $F(s) = \dfrac{1}{s}\displaystyle\sum_{n=0}^{\infty}(-1)^n e^{-ns} = \dfrac{1/s}{1+e^{-s}}, \quad s > 0$

35. $\mathcal{L}\{f(t)\} = \dfrac{1/s}{1+e^{-s}}, \quad s > 0$

36. $\mathcal{L}\{f(t)\} = \dfrac{1-e^{-s}}{s(1+e^{-s})}, \quad s > 0$

37. $\mathcal{L}\{f(t)\} = \dfrac{1-(1+s)e^{-s}}{s^2(1-e^{-s})}, \quad s > 0$

38. $\mathcal{L}\{f(t)\} = \dfrac{1+e^{-\pi s}}{(1+s^2)(1-e^{-\pi s})}, \quad s > 0$

39. (a) $\mathcal{L}\{f(t)\} = s^{-1}(1-e^{-s}), \quad s > 0$

(b) $\mathcal{L}\{g(t)\} = s^{-2}(1-e^{-s}), \quad s > 0$

(c) $\mathcal{L}\{h(t)\} = s^{-2}(1-e^{-s})^2, \quad s > 0$

40. (b) $\mathcal{L}\{p(t)\} = \dfrac{1-e^{-s}}{s^2(1+e^{-s})}, \quad s > 0$

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