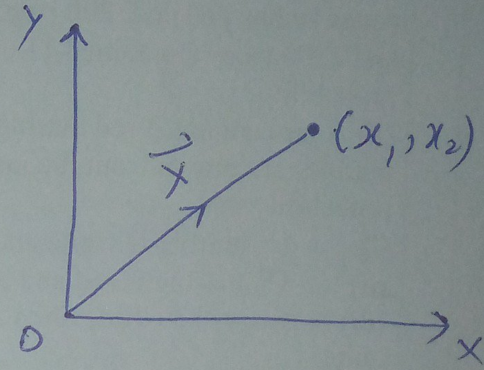
ctor Spaces

Vectors in R^n

In physics and engineering, a vector is characterized by two quantities (length and direction) and is represented by a directed line segment.

Vectors in the plane R^2

A vector in the plane is represented by a directed line segment with its initial point at the origin and its terminal point at (x_1, x_2) as shown in the figure.

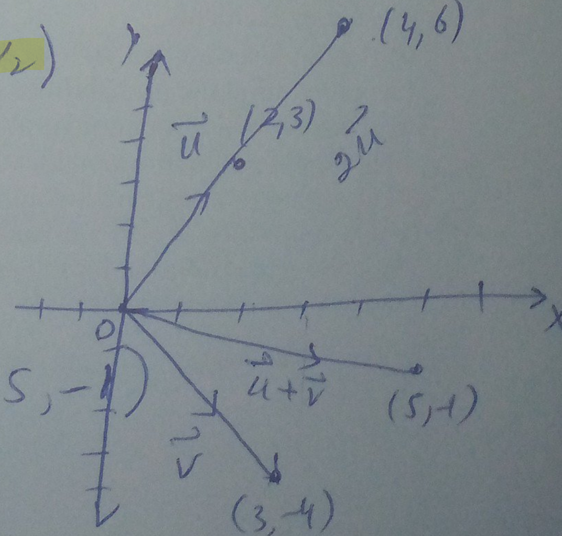


The set of all points in the plane is denoted by R^2; it is called 2-space.

Example Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$. Then

1) $\vec{u} + \vec{v} = (u_1 + u_2, v_1 + v_2)$

2) $Ku = (Ku_1, Ku_2)$



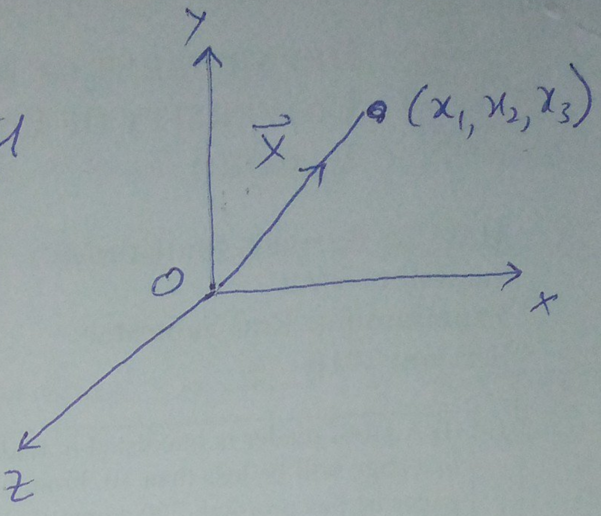
$\vec{u} = (2, 3)$, $\vec{v} = (3, -4)$

$\vec{u} + \vec{v} = (2+3, 3-4) = (5, -1)$

$(4, 6)$

$3\vec{u} =$

Vectors in Space R^3



A vector in the space is represented by a directed line segment with its initial point at the origin and its terminal point at (x_1, x_2, x_3).

The set of all points in the space is denoted by R^3; it is called 3-space.

<u>Example</u> Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$

1) $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

2) $k\vec{u} = (ku_1, ku_2, ku_3)$

$\vec{u} = (2, 3, -1)$, $\vec{v} = (3, -4, 2)$

$\vec{u} + \vec{v} = (2+3, 3+(-4), -1+2) = (5, -1, 1)$

$2\vec{u} = (4, 6, -2)$

<u>Vectors in R^n</u> The discussion of vectors in the plane can be extended to a discussion of vectors in n-space.
An ordered n-tuple represents a vector in n-space and is denoted by R^n. A general ordered n-tuple has the form

$(x_1, x_2, --, x_n)$.

(x_1, x_2, x_3) ordered triple

(x_1, x_2, x_3, x_4) ordered quadruple

R^1 = 1-space = Set of all real numbers
R^2 = 2-space = Set of all ordered pairs of real numbers
R^3 = 3-space = Set of all ordered triples of real numbers.
⋮
R^n = n-space = Set of all ordered n-tuples of real numbers.

Example Let $\vec{u} = (u_1, u_2, ---, u_n)$
$\vec{v} = (v_1, v_2, ---, v_n)$

be vectors in R^n and let k by any real number.

1) $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, ---, u_n + v_n)$
2) $k\vec{u} = (ku_1, ku_2, ----, ku_n)$

$\vec{u} = (2, 3, 4, 5)$ $\vec{v} = (4, -1, 2, 6)$

$\vec{u} + \vec{v} = (2+4, 3+(-1), 4+2, 5+6)$
$= (6, 2, 6, 11)$

$2\vec{u} = (4, 6, 8, 10)$

Notations for vectors

$\vec{v} = (v_1, v_2, v_3 ---, v_n)$ Comma-delimited

$\vec{v} = [v_1 \; v_2 \; --- \; v_n]$ Row-matrix

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ column-matrix.

Linear Combinations of Vectors

If $\vec{w}$ is a vector in R^n, then $\vec{w}$ is said to be a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_r$ in R^n if it can be expressed in the form

$$\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r$$

where $k_1, k_2, \ldots k_r$ are scalars and called coefficients of linear combinations.

Example 1 Let $\vec{w} = (-1,-2,-2)$, $\vec{v}_1 = (0,1,4)$, $\vec{v}_2 = (-1,1,2)$, $\vec{v}_3 = (3,1,2)$ in R^3. Find scalars k_1, k_2, k_3 such that

$$\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3$$

$$(-1,-2,-2) = k_1(0,1,4) + k_2(-1,1,2) + k_3(3,1,2)$$

K1 ka phela element aur phir k2 ka aur k3 ka bi

$$= (-k_2 + 3k_3,\ k_1 + k_2 + k_3,\ 4k_1 + 2k_2 + 2k_3)$$

$$-1 = -k_2 + 3k_3 \Rightarrow -k_2 + 3k_3 = -1$$
$$-2 = k_1 + k_2 + k_3 \Rightarrow k_1 + k_2 + k_3 = -2$$
$$-2 = 4k_1 + 2k_2 + 2k_3 \Rightarrow 4k_1 + 2k_2 + 2k_3 = -2$$

$$\begin{bmatrix} 0 & -1 & 3 & -1 \\ 1 & 1 & 1 & -2 \\ 4 & 2 & 2 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$\boxed{k_1 = 1}$, $\boxed{k_2 = -2}$, $\boxed{k_3 = -1}$

$$\vec{w} = \vec{v}_1 - 2\vec{v}_2 - \vec{v}_3$$

different hin tou multipy sai kar raha hi

Example 2 Consider the vectors $\vec{u} = (1,2,-1)$, $\vec{v} = (6,4,2)$ in R^3. Show that $\vec{w} = (9,2,7)$ is a linear combination of $\vec{u}$ and $\vec{v}$, and that $\vec{w}' = (4,-1,8)$ is not a linear combination of $\vec{u}$ and $\vec{v}$.

$$\vec{w} = k_1 \vec{u} + k_2 \vec{v}$$

$$(9,2,7) = k_1(1,2,-1) + k_2(6,4,2)$$

$$= (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

$$k_1 + 6k_2 = 9$$
$$2k_1 + 4k_2 = 2$$
$$-k_1 + 2k_2 = 7$$

$$\begin{bmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{bmatrix} \begin{matrix} -2R_1 + R_2 \\ 1R_1 + R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{bmatrix} -\frac{1}{8} R_2 \sim \begin{bmatrix} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 8 & 16 \end{bmatrix} -8R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} -6R_2 + R_1 \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{k_1 = -3}$, $\boxed{k_2 = 2}$

$$\vec{w} = -3\vec{u} + 2\vec{v}$$

$$\vec{w}' = k_1 \vec{u} + k_2 \vec{v}$$

$$(4, -1, 8) = (k_1 + 6k_2,\ 2k_1 + 4k_2,\ -k_1 + 2k_2)$$

$k_1 + 6k_2 = 4$

$2k_1 + 4k_2 = -1$

$-k_1 + 2k_2 = 8$

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{bmatrix} \begin{matrix} -2R_1 + R_2 \\ 1R_1 + R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 8 & 12 \end{bmatrix} \quad -\frac{1}{8}R_2$$

$$\sim \begin{bmatrix} 1 & 6 & 4 \\ 0 & 1 & 9/8 \\ 0 & 8 & 12 \end{bmatrix} \quad -8R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & 6 & 4 \\ 0 & 1 & 9/8 \\ 0 & 0 & 3 \end{bmatrix}$$

$k_1 + 6k_2 = 4$

$\boxed{k_2 = 9/8}$

$k_1 = 4 - 6(9/8) = -11/4$

$2k_1 + 4k_2 = -1 \Rightarrow \cancel{2}\left(\frac{-11}{\cancel{4}_2}\right) + \cancel{4}\left(\frac{9}{\cancel{8}_2}\right) = -1$

$-\frac{11}{2} + \frac{9}{2} = -1 \Rightarrow \boxed{-1 = -1}$ satisfied

$-k_1 + 2k_2 = 8$

$-\left(-\frac{11}{4}\right) + 2\left(\frac{9}{8}\right) = 8$

$\frac{11}{4} + \frac{9}{4} = 8$

$\frac{20}{4} = 8 \Rightarrow 5 = 8$ contradiction.

Example 3 write $\vec{w} = (1,1,1)$ as a linear combination of $\vec{v}_1 = (1,2,3)$, $\vec{v}_2 = (0,1,2)$, $\vec{v}_3 = (-1,0,1)$.

$$\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3$$

$$(1,1,1) = k_1(1,2,3) + k_2(0,1,2) + k_3(-1,0,1)$$

$$= (k_1 + 0k_2 - k_3,\ 2k_1 + k_2 + 0k_3,\ 3k_1 + 2k_2 + k_3)$$

$$k_1 - k_3 = 1$$
$$2k_1 + k_2 = 1$$
$$3k_1 + 2k_2 + k_3 = 1$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & +1 \\ 3 & 2 & 1 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$k_1 - k_3 = 1 \qquad k_3 = t$$
$$k_2 + 2k_3 = -1$$

$$k_2 = -1 - 2t$$
$$k_1 = 1 + t$$

For $t = 1$ $\quad k_1 = 2,\ k_2 = -3,\ k_3 = 1$

$$\vec{w} = 2\vec{v}_1 - 3\vec{v}_2 + \vec{v}_3$$

Example (4) write $\vec{w} = (1, -2, 2)$ as a linear combination of $\vec{v}_1 = (1, 2, 3)$, $\vec{v}_2 = (0, 1, 2)$, $\vec{v}_3 = (-1, 0, 1)$.

$$\vec{w} = k_1 \vec{v}_1 + k_2 \vec{v}_2 + k_3 \vec{v}_3$$

$$(1, -2, 2) = k_1 (1, 2, 3) + k_2 (0, 1, 2) + k_3 (-1, 0, 1)$$

$$= (k_1 - k_3 , 2k_1 + k_2 , 3k_1 + 2k_2 + k_3)$$

$$k_1 - k_3 = 1$$
$$2k_1 + k_2 = -2$$
$$3k_1 + 2k_2 + k_3 = 2$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{bmatrix} \begin{matrix} -2R_1 + R_2 \\ -3R_1 + R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 2 & 4 & -1 \end{bmatrix} \quad -2R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{bmatrix} \quad \frac{1}{7} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$k_1 - k_3 = 1$$
$$k_2 + 2k_3 = -4$$
$$\boxed{0 = 1}$$

0 = 0
infinity many solutions

The system is inconsistent. We can not write $\vec{w}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

Example (5) Let $\vec{v}_1 = (1,3,2,1)$, $\vec{v}_2 = (2,-2,-5,4)$ $\vec{v}_3 = (2,-1,3,6)$. If $\vec{w} = (2,5,-4,0)$, write $\vec{w}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Let $\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3$

$$(2,5,-4,0) = k_1(1,3,2,1) + k_2(2,-2,-5,4) + k_3(2,-1,3,6)$$

$$(2,5,-4,0) = (k_1+2k_2+2k_3,\ 3k_1-2k_2-k_3,\ 2k_1-5k_2+3k_3,\ k_1+4k_2+6k_3)$$

$$k_1+2k_2+2k_3 = 2$$
$$3k_1-2k_2-k_3 = 5$$
$$2k_1-5k_2+3k_3 = -4$$
$$k_1+4k_2+6k_3 = 0$$

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 3 & -2 & -1 & 5 \\ 2 & -5 & 3 & -4 \\ 1 & 4 & 6 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$k_1 = 2$
$k_2 = 1$
$k_3 = -1$

$$\vec{w} = 2\vec{v}_1 + \vec{v}_2 - \vec{v}_3$$

Vector Spaces Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the listed axioms are satisfied for every $\vec{u}, \vec{v}, \vec{w}$ in V and every scalar (real numbers) K and m, then V is a vector space.

Addition

1) $\vec{u} + \vec{v}$ is in V (Closure under Addition)
2) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (Commutative property)
3) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ (Associative property)
4) V has a zero vector $\vec{0}$ such that for every $\vec{u}$ in V, (Additive Identity)

$$\vec{u} + \vec{0} = \vec{u}$$

5) For every $\vec{u}$ in V, there is a vector in V denoted by $-\vec{u}$ such that $\vec{u} + (-\vec{u}) = \vec{0}$ (Additive Inverse)

Scalar Multiplication

6) $K\vec{u}$ is in V (Closure under Scalar Multiplication)
7) $K(\vec{u} + \vec{v}) = K\vec{u} + K\vec{v}$ (Distributive property)
8) $(k+m)\vec{u} = K\vec{u} + m\vec{u}$ (Distributive property)
9) $K(m\vec{u}) = (Km)\vec{u}$ (Associative property)
10) $1(\vec{u}) = \vec{u}$ (Scalar Identity)

Steps to Show that a Set is a Vector Space

Step I Identify the set V of objects that will become vectors

Step II Verify axioms 1 and 6

Step III Confirm that axioms 2, 3, 4, 5, 7, 8, 9 and 10 holds.

Example ⑧ Let $V = \{(x, \frac{1}{2}x),\ x \in \mathbb{R}\}$ with standard operations. Is it a vector space?

1) For real numbers x and y

$$(x, \tfrac{1}{2}x) + (y, \tfrac{1}{2}y) = (x+y, \tfrac{1}{2}(x+y))$$

So V is closed under addition.

6) For a scalar K

$$K(x, \tfrac{1}{2}x) = (Kx, \tfrac{1}{2}(Kx))$$

So V is closed under scalar multiplication.

2) Clearly addition is commutative

3) Clearly addition is associative

4) The element $\vec{0} = (0,0)$ satisfy the property of the zero element.

5) We have $-(x, \frac{1}{2}x) = (-x, \frac{1}{2}(-x))$, so every element in V has additive inverse.

7) The distributive property $K(\vec{u}+\vec{v}) = K\vec{u} + K\vec{v}$ holds for $\vec{u}, \vec{v}$ in V.

8) The distributive property $(K+m)\vec{u} = K\vec{u} + m\vec{u}$ holds for $\vec{u}$ in V.

9) The associative property $K(m\vec{u}) = (Km)\vec{u}$ holds.

10) $1(\vec{u}) = \vec{u}$.

Example 7 Let $\vec{V} = \{(x, y) : x \geq 0,\ y \geq 0\}$ with standard operations. Is it a vector space?

1) For $x_1, x_2 \geq 0$ and $y_1, y_2 \geq 0$

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2)$$

So V is closed under addition.

6) For a scalar K

$$K(x, y) = (Kx, Ky)$$

The set V is not closed under multiplication when $K < 0$.

Therefore V is not a vector space.

Example 8 Let V be the set of all 2×2 matrices of the form

$$V = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

V is a vector space.

Example 9 The set of all polynomials of degree 4 (not a vector space)

The set of all polynomials of degree 4 or less (vector space).

Example (8) <u>Important vector spaces.</u>

1) R = Set of all real numbers
2) R^2 = Set of all ordered pairs
3) R^3 = Set of all ordered triples
4) R^n = Set of all n-tuples
5) $C(-\infty, \infty)$ = Set of all continuous functions defined on the real line
6) $C[a, b]$ = Set of all continuous functions defined on closed interval $[a, b]$ where $a \neq b$.
7) P = Set of all polynomials
8) P_n = Set of all polynomials of degree $\leq n$ (including zero polynomial)
9) $M_{m,n}$ = Set of all $m \times n$ matrices
10) $M_{n,n}$ = Set of all $n \times n$ Square matrices.

<u>Sets that are not Vector Spaces</u>

1) The set of integers is not a vector space.
Axiom 6 violated $\frac{1}{2}(3) = \frac{3}{2}$ not an integer

2) Set of polynomials of degree n. is not a vector space.
Axiom 1 violated
For $n = 2$ $\quad p(x) = x^2, \; q(x) = 1 + x - x^2$

$$p(x) + q(x) = \cancel{x^2} + 1 + x - \cancel{x^2} = 1 + x$$

The sum is a first degree polynomial.

$\boxed{\text{Exercise 4.1 Q1—27}}$

Subspaces of Vector Spaces

A nonempty subset H of a vector space V is called a subspace of V if H is a vector space under the operations of addition and scalar multiplication defined in V.

Example (11) (a) Let $H = \{(x, 0) : x \in \mathbb{R}\}$. Then $H \subseteq R^2$. ~~H~~ is a subspace of R^2.

(b) Let $H = \{(x, 0, z) : x, z \in \mathbb{R}\}$. H is a subspace of R^3.

Example (12) (a) The set H of invertible $n \times n$ matrices is not a subspace of M_{nn}.

Axiom 1 violated.

$$U = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \quad V = \begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}$$

(b) Let H be the set of all points on any given line $y = mx$ through the origin in the plane R^2. Then H is a subspace of R^2.

Theorem Test for a Subspace

If H is a nonempty subset of a vector space V, then H is a subspace of V if and only if the two conditions are satisfied.

1) If $\vec{u}$, $\vec{v}$ are in H, then $\vec{u} + \vec{v}$ in H.

2) If $\vec{u}$ is in H, then $k\vec{u}$ is in H for any scalar k.

Example (13) Determine whether each subset is a subspace of R^2.

(a) The set of points on the line $x+2y=0$

(b) The set of points on the line $x+2y=1$

(a) $x+2y=0 \Rightarrow x=-2y$ Let $y=t$

$x=-2t$

$(-2t, t)$

Let $\vec{u}=(-2t_1, t_1)$, $\vec{v}=(-2t_2, t_2)$

$\vec{u}+\vec{v}=(-2t_1+(-2t_2), t_1+t_2)$

$=(-2(t_1+t_2), t_1+t_2)$

$=(-2t_3, t_3)$ closed under addition.

$k\vec{u}=(-2(kt_1), kt_2)$

$=(-2t_3, t_3)$ closed under multiplication.

So this set is a subspace of R^2.

(b) $x=1-2y$, let $y=t$ $(1-2t, t)$

$x=1-2t$

$\vec{u}=(1-2t_1, t_1)$ $\vec{v}=(1-2t_2, t_2)$

$\vec{u}+\vec{v}=(1-2t_1+1-2t_2, t_1+t_2)$

$=(2-2t_1-2t_2, t_1+t_2)$

$=(2-2(t_1+t_2), t_1+t_2)$

$=(2-2t_3, t_3)$ not closed under addition