Basis for a Vector Space

A set of vectors $S = \{v_1, v_2, \ldots v_r\}$ in a vector space V is a basis for V when the following conditions are true.

1) S spans V 2) S is linearly independent.

Example 1 Show that $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis for R^3.

1) S spans V. Let $(u_1, u_2, u_3) \in R^3$.

$$(u_1, u_2, u_3) = u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1)$$

k1+0k2+0k3 = u1
0k1+1k2+0k3 = u2
0k1+0k2+1k3 = u3

2) S is linearly independent

$$k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) = (0,0,0)$$

$$(k_1, k_2, k_3) = (0,0,0)$$

$$\Rightarrow k_1 = 0, \quad k_2 = 0, \quad k_3 = 0$$

S is a basis for R^3.

The basis $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ is the standard basis for R^3. The vectors

$$\hat{e}_1 = (1, 0, 0, \ldots, 0)$$
$$\hat{e}_2 = (0, 1, 0, \ldots, 0)$$
$$\vdots$$
$$\hat{e}_n = (0, 0, 0, \ldots, 1)$$

form the standard basis for R^n.

<u>Example 2</u> Show that the set $S = \{(1,1), (1,-1)\}$ is a basis for R^2.

1) <u>S spans V</u> Let $(u_1, u_2) \in R^2$.

$$(u_1, u_2) = k_1(1,1) + k_2(1,-1)$$
$$= (k_1 + k_2, k_1 - k_2)$$

$$k_1 + k_2 = u_1 \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$k_1 - k_2 = u_2$$

$\det A = -2$. The system has a unique solution.

2) <u>S is linearly independent</u>

$$k_1(1,1) + k_2(1,-1) = (0,0)$$

$$k_1 + k_2 = 0$$
$$k_1 - k_2 = 0 \quad \Rightarrow k_1 = 0, \; k_2 = 0$$

S is linearly independent. Therefore S is a basis.

<u>Example 3</u> $S = \{(1,2,3), (0,1,2), (-2,0,1)\}$

S spans R^3 and is linearly independent. So S is a basis for R^3.

Example (4) Show that the vector space P_3 has the basis.

$$S = \{1, x, x^2, x^3\}$$

1) S spans V

$$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

2) S is linearly independent

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 0$$

$$a_0 = 0,\ a_1 = 0,\ a_2 = 0,\ a_3 = 0.$$

S is linearly independent. Therefore S is a basis for P_3.

$S = \{1, x, x^2, \ldots x^n\}$ is a basis for the vector space P_n of polynomials of degree n or less.

Example (5) The set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

for a basis for the vector space $M_{2,2}$ of 2×2 matrices.

1) S spans V Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2,2}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

2) <u>S is linearly independent</u>

$$k_1\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + k_2\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + k_3\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} + k_4\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}k_1 & k_2\\ k_3 & k_4\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$k_1 = 0, \quad k_2 = 0, \quad k_3 = 0, \quad k_4 = 0.$$

The set S is linearly independent. Therefore from the basis for $M_{2,2}$.

<u>Example 8</u> Show that the vectors $\vec{v_1} = (1,2,1)$, $\vec{v_2} = (2,9,0)$, $\vec{v_3} = (3,3,4)$ form basis for R^3.

s spans v = ham nai vectors ko let karna hi k sai multiply karna hi aur then determenent find karna hi. Agar tou determinent non zero ata hi tou phir inverse possible ho ga. aur then s spans v true ho ga

s is linealy independent = ham homogenious equations consider kar laitay hin then solve kar kai k ki values find kartay hin. Agar tou k = 0 ho ga tab s is linealy independent ho ga

1) <u>S spans V</u> Let $(u_1, u_2, u_3) \in R^3$.

$$(u_1, u_2, u_3) = k_1(1,2,1) + k_2(2,9,0) + k_3(3,3,4)$$

$$= (k_1 + 2k_2 + 3k_3,\ 2k_1 + 9k_2,\ 3k_3,\ k_1 + 4k_3)$$

$$\begin{aligned} k_1 + 2k_2 + 3k_3 &= u_1\\ 2k_1 + 9k_2 + 3k_3 &= u_2\\ k_1 \quad\quad + 4k_3 &= u_3 \end{aligned} \qquad A = \begin{bmatrix}1 & 2 & 3\\ 2 & 9 & 3\\ 1 & 0 & 4\end{bmatrix}$$

$\det A = -1$. The system has unique solution.

2) <u>S is linearly independent</u>

$$k_1 + 2k_2 + 3k_3 = 0$$
$$2k_1 + 9k_2 + 3k_3 = 0$$
$$k_1 \quad + 4k_3 = 0$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 9 & 3 & 0 \\ 1 & 0 & 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$k_1 = 0, \quad k_2 = 0, \quad k_3 = 0.$

The vectors are linearly independent. Therefore the vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ form a basis for R^3.

<u>Theorem</u> uniqueness of Basis Representation

If $S = \{v_1, v_2, \ldots v_r\}$ is a basis for a vector space V, then every vector in V can be written as a linearly combination of vectors in S in one and only one way.

<u>Coordinates of a Vector Relative to Basis</u>

If $S = \{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is a basis for a vector space V and $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$ is the expression for a vector $\vec{v}$ in terms of S, then the scalars $c_1, c_2 \ldots c_n$ are called coordinates of $\vec{v}$ relative to the basis S.

$$(\vec{v})_S = (c_1, c_2, \ldots c_n).$$

is mai ham nai agar to R^3
v = k1v1 + k2v2 + k3v3
k1,k2,k3 ki values find kar kai dai deni hin
aur bus phir masla khatam

Example (7) (a) The vectors $\vec{v_1} = (1,2,1)$, $\vec{v_2} = (2,9,0)$, $\vec{v_3} = (3,3,4)$ form a basis for R^3. Find the coordinates of $\vec{v} = (5,-1,9)$ relative to the basis $S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$.

(b) Find the vector $\vec{v}$ in R^3 whose coordinate vector relative to S is $(\vec{v})_S = (-1,3,2)$.

(a) To find $(\vec{v})_S$, we express $\vec{v}$ as a linear combination of the vectors in S.

$$\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}$$

$$(5,-1,9) = c_1(1,2,1) + c_2(2,9,0) + c_3(3,3,4)$$

$$= (c_1 + 2c_2 + 3c_3,\ 2c_1 + 9c_2 + 3c_3,\ c_1 + 4c_3)$$

$$c_1 + 2c_2 + 3c_3 = 5$$
$$2c_1 + 9c_2 + 3c_3 = -1$$
$$c_1 + 4c_3 = 9$$

Solving this system, $c_1 = 1$, $c_2 = -1$, $c_3 = 2$.

$$(\vec{v})_S = (1,-1,2)$$

(b)

$$\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}$$

$$= (-1)(1,2,1) + 3(2,9,0) + 2(3,3,4)$$

$$= (11, 31, 7)$$

$\boxed{\text{Exercise 4.4 Q1-19}}$

Dimension of a Vector Space

see Lark

If a vector space V has a basis consisting of n vectors, then n is called the dimension of V, denoted by

$$\dim(V) = n.$$

If V consists of zero vector alone, the dimension of V is defined as zero.

1) $\dim(R^n) = n$ The standard basis has n vectors
2) $\dim(P_n) = n+1$ The standard basis has $n+1$ vectors
3) $\dim(M_{m,n}) = mn$ The standard basis has mn vectors.

Example 8 Determine the dimension of each subspace of R^3

(a) $H = \{(d, c-d, c) : c, d \in \mathbb{R}\}$

(b) $H = \{(2b, b, 0) : b \in \mathbb{R}\}$.

(a) $(d, c-d, c) = (0, c, c) + (d, -d, 0)$

$$= c(0,1,1) + d(1,-1,0)$$

H is spanned by the set $S = \{(0,1,1), (1,-1,0)\}$.

The set S is linearly independent (verify). So its a basis for H.

$$\dim(H) = 2.$$

(b) $(2b, b, 0) = b(2, 1, 0)$

H is spanned by the set $S = \{(2, 1, 0)\}$.

$$\dim(H) = 1$$

<u>Theorem</u> Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then S is a basis for V iff S spans V or S is linearly independent.

<u>Example 9</u> (a) By inspection, explain why $\vec{v_1} = (-3, 7)$ and $\vec{v_2} = (5, 5)$ form a basis for R^2.

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(b) By inspection, explain why $\vec{v_1} = (2, 0, -1)$, $\vec{v_2} = (4, 0, 7)$ and $\vec{v_3} = (-1, 1, 4)$ form a basis for R^3.

(a) Since neither vector is a multiple of the other, the two vectors form a linearly independent S in R^2. Hence they form a basis.

(b) The vectors $\vec{v_1}$ and $\vec{v_2}$ form a linearly independent set in xz-plane. The vector $\vec{v_3}$ is outside of xz-plane. So the set $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is linearly independent. Hence they form a basis for R^3.

<u>Theorem</u> Let S be a finite set of vectors in a dimensional vector space V.

(a) If S spans V but is not a basis for V, then S can be reduced to a basis by removing appropriate vectors from S.

(b) If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

<u>Theorem</u> If H is a subspace of a finite dimensional vector space V, then

(a) H is finite dimensional

(b) $\dim(H) \leq \dim(V)$

(c) $H = V$ iff $\dim(H) = \dim(V)$.

<u>Example 10</u> Find the dimension of the subspace H of R^4 spanned by

$$S = \{(-1, 2, 5, 0), (3, 0, 1, -2), (-5, 4, 9, 2)\}$$

Although H is spanned by the set S, S is not a basis for H because S is linearly dependent set.

$$\vec{v}_3 = 2\vec{v}_1 - \vec{v}_2$$

we discard $\vec{v}_3$ and we have set

$$S_1 = \{(-1, 2, 5, 0), (3, 0, 1, -2)\}$$

than spans H. Moreover S_1 is linearly independent because neither vector is a scalar multiple of the other.

$$\dim(H) = 2$$

Example 11 Find the dimension of the subspace

$$H = \{(a-3b+6c,\ 5a+4d,\ b-2c-d,\ 5d) : a,b,c,d \in \mathbb{R}\}$$

$$(a-3b+6c,\ 5a+4d,\ b-2c-d,\ 5d)$$

$$= (a, 5a, 0, 0) + (-3b, 0, b, 0) + (6c, 0, -2c, 0) + (0, 4d, -d, 5d)$$

$$= a(1,5,0,0) + b(-3,0,1,0) + c(6,0,-2,0) + d(0,4,-1,5)$$

H is spanned by the set S given by

$$S = \{(1,5,0,0), (-3,0,1,0), (6,0,-2,0), (0,4,-1,5)\}$$

The vector $\vec{v}_3$ is a multiple of $\vec{v}_2$. We may discard $\vec{v}_3$ and still have a set

$$S_1 = \{(1,5,0,0), (-3,0,1,0), (0,4,-1,5)\}$$

that spans H. The set S_1 is linearly independent.

$$\dim(H) = 3.$$

Example 12 Find a basis and dimension of the solution space of the homogeneous system.

dimension ka matalb ye hi keh ham nai equations find karni hin jou hamin homogeneous system dia gia ho ga us sai.

$$2x_1 + 2x_2 - x_3 + x_5 = 0$$

$$-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0$$

$$x_1 + x_2 - 2x_3 - x_5 = 0$$

$$x_3 + x_4 + x_5 = 0$$

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

The reduced-echelon form is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + x_2 + x_5 = 0$

$x_3 + x_5 = 0$

$x_4 = 0$

$x_2 = \text{free} = s$

$x_5 = \text{free} = t$

variables ko aik sath likh dena hi

$x_1 = -s - t$

$x_2 = s$

$x_3 = -t$

$x_4 = 0$

$x_5 = t$

$(-s-t, s, -t, 0, t)$

s waly alg aur t walay alag

$= (-s, s, 0, 0, 0) + (-t, 0, -t, 0, t)$

s common lia hi

$= s(-1, 1, 0, 0, 0) + t(-1, 0, -1, 0, 1)$

The vectors $\vec{v_1} = (-1, 1, 0, 0, 0)$

$\vec{v_2} = (-1, 0, -1, 0, 1)$

spans solution space.

Since neither vector is a scalar multiple of other, they are linearly independent. Hence they form a basis for solution space.

dim = 2.

form the standard ...

Example 13 Find a basis and the dimension of the solution space of the homogenous system.

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$
$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0$$
$$5x_3 + 10x_4 + 15x_6 = 0$$
$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 5 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 3x_2 + 4x_4 = 0$$
$$x_3 + 2x_4 = 0$$
$$x_6 = 0$$

$x_2 = \text{free} = r$

$x_4 = \text{free} = s$

$x_5 = \text{free} = t$

$$x_1 = -3r - 4s - 2t$$
$$x_2 = r$$
$$x_3 = -2s$$
$$x_4 = s$$
$$x_5 = t$$
$$x_6 = 0$$

$(-3r+4s-2t,\ r,\ -2s,\ t,\ 0)$

$= (-3r, r, 0, 0, 0) + (-4s, 0, -2s, 0, 0)$

$+ (-2t, 0, 0, t, 0)$

$= r(-3, 1, 0, 0, 0) + s(-4, 0, -2, 0, 0)$

$+ t(-2, 0, 0, 1, 0)$

The vectors $\vec{v}_1 = (-3, 1, 0, 0, 0)$, $\vec{v}_2 = (-4, 0, -2, 0, 0)$

$\vec{v}_3 = (-2, 0, 0, 1, 0)$ span solution space. The vectors are linearly independent (verify).

dim $= 3$.

Exercise 4.5
Q1-22