Linear Dependence and Linear Independence

A set of vectors $S = \{V_1, V_2, --V_r\}$ in a vector space V is linearly independent when the vector equation

Linearly independent:

$$K_1 \vec{V_1} + K_2 \vec{V_2} + --- + K_r \vec{V_r} = \vec{0}$$

has only the trivial solution $K_1 = 0, K_2 = 0, --K_r = 0$.

If there is a non-trivial solution, then S is linearly dependent.

Example① Determine whether the set of vectors in R^3 is linearly independent or linearly dependent.

$$S = \{(1,2,3), (0,1,2), (-2,0,1)\}$$

$$K_1 \vec{V_1} + K_2 \vec{V_2} + K_3 \vec{V_3} = \vec{0}$$

Linearly independent houn gai agar tou solution 0 a jai sab ka

$$K_1(1,2,3) + K_2(0,1,2) + K_3(-2,0,1) = (0,0,0)$$

$$(K_1 - 2K_3, 2K_1 + K_2, 3K_1 + 2K_2 + K_3) = (0,0,0)$$

$$\begin{aligned} K_1 - 2K_3 &= 0 \\ 2K_1 + K_2 &= 0 \\ 3K_1 + 2K_2 + K_3 &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

The reduced row echelon matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{aligned} K_1 &= 0 \\ K_2 &= 0 \\ K_3 &= 0 \end{aligned}$$

The vectors in S are linearly independent.

Example (2) Determine the vectors $\vec{v}_1 = (1,-2,3)$, $\vec{v}_2 = (5,6,-1)$ and $\vec{v}_3 = (3,2,1)$ are linearly independent or linearly dependent in R^3.

$$k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{0}$$

$$k_1(1,-2,3) + k_2(5,6,-1) + k_3(3,2,1) = (0,0,0)$$

$$(k_1 + 5k_2 + 3k_3,\ -2k_1 + 6k_2 + 2k_3,\ 3k_1 - k_2 + k_3) = (0,0,0)$$

$$k_1 + 5k_2 + 3k_3 = 0$$

$$-2k_1 + 6k_2 + 2k_3 = 0$$

$$3k_1 - k_2 + k_3 = 0$$

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & +\frac{1}{2} & 0 \\ 0 & 1 & +\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$k_1 + \frac{1}{2}k_3 = 0$$

$$k_2 + \frac{1}{2}k_3 = 0$$

$k_3 =$ free

$$k_1 = -\frac{1}{2}t$$

$$k_2 = -\frac{1}{2}t$$

$$k_3 = t$$

The system has nontrivial solution. The vectors are linearly dependent.

non trivial solution = linearly dependent
trivial solution = linearly independent

Example ③ Determine whether the set of vectors in P_2 is linearly independent or linearly dependent.

$$\vec{p_1} = 1+x-2x^2, \quad \vec{p_2} = 2+5x-x^2, \quad \vec{p_3} = x+x^2$$

$$k_1\vec{v_1} + k_2\vec{v_2} + k_3\vec{v_3} = \vec{0}$$

$$k_1(1+x-2x^2) + k_2(2+5x-x^2) + k_3(x+x^2) = 0$$

$$(k_1+2k_2+0k_3) + (k_1+5k_2+k_3)x + (-2k_1-k_2+k_3) = 0$$

$$k_1+2k_2+0k_3 = 0$$

$$k_1+5k_2+k_3 = 0$$

$$-2k_1-k_2+k_3 = 0$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$k_1 + 2k_2 = 0 \qquad k_2 + \tfrac{1}{3}k_3 = 0 \qquad k_3 = t$$

The system has non trivial solution. The vectors $\vec{P_1}, \vec{P_2}, \vec{P_3}$ are linearly dependent.

non trivial = linearly dependent

Theorem A set S with two or more vectors is

1) Linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.
2) Linearly independent if and only if no vectors in S is expressible as a linear combination of the other vectors in S.

Example (4) In example (2), the vectors $\vec{v}_1 = (1,-2,3)$, $\vec{v}_2 = (5,6,-1)$, $\vec{v}_3 = (3,2,1)$ are linearly dependent.

$$k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{0}$$

$$k_1 = -\tfrac{1}{2}t, \quad k_2 = -\tfrac{1}{2}t, \quad k_3 = t$$

For $t = 1$, $k_1 = -\frac{1}{2}$, $k_2 = -\frac{1}{2}$, $k_3 = 1$

$$-\tfrac{1}{2}\vec{v}_1 - \tfrac{1}{2}\vec{v}_2 + \vec{v}_3 = 0$$

$$\vec{v}_3 = \tfrac{1}{2}\vec{v}_1 + \tfrac{1}{2}\vec{v}_2$$

Theorem Two vectors $\vec{u}$ and $\vec{v}$ in a vector space V are linearly dependent if and only if one is a scalar multiple of the other.

Example (5) (a) The vectors $\vec{v}_1 = (1,2,0)$ and $\vec{v}_2 = (-2,2,1)$ are linearly independent because $\vec{v}_1$ and $\vec{v}_2$ are not scalar multiple of each other.

(b) The vectors $\vec{v}_1 = (4,-4,-2)$ and $\vec{v}_2 = (-2,2,1)$ are linearly dependent because $\vec{v}_1 = -2\vec{v}_2$.

Theorem Let $S = \{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_r\}$ be a set of vectors in R^n. If $r > n$, then S is linearly dependent.

Linear Independence of Functions

Let $\vec{f_1} = \sin^2 x$, $\vec{f_2} = \cos^2 x$, $\vec{f_3} = 5$.

Then $5\vec{f_1} + 5\vec{f_2} - \vec{f_3} = 5\sin^2 x + 5\cos^2 x - 5$

$$= 5(\sin^2 x + \cos^2 x) - 5$$

$$= 5(1) - 5 = \vec{0}$$

The vectors $\vec{f_1}, \vec{f_2}, \vec{f_3}$ are linearly dependent.

Wronskian If $\vec{f_1} = f_1(x)$, $\vec{f_2} = f_2(x)$, --- $\vec{f_n} = f_n(x)$ are functions that are $n-1$ times differentiable on the interval $(-\infty, \infty)$, then the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the Wronskian of functions $f_1(x), f_2(x)$ ---

Theorem If the functions $\vec{f_1}, \vec{f_2}, \ldots \vec{f_n}$ have $n-1$ derivatives on the interval $(-\infty, \infty)$, the functions linearly independent when the Wronskian of th is not identically zero on $(-\infty, \infty)$.

Linear Independence of Functions

Let $\vec{f}_1 = \sin^2 x$, $\vec{f}_2 = \cos^2 x$, $\vec{f}_3 = 5$.

Then $5\vec{f}_1 + 5\vec{f}_2 - \vec{f}_3 = 5\sin^2 x + 5\cos^2 x - 5$

$$= 5(\sin^2 x + \cos^2 x) - 5$$

$$= 5(1) - 5 = \vec{0}$$

The vectors $\vec{f}_1, \vec{f}_2, \vec{f}_3$ are linearly dependent.

Wronskian If $\vec{f}_1 = f_1(x)$, $\vec{f}_2 = f_2(x)$, --- $\vec{f}_n = f_n(x)$ are functions that are $n-1$ times differentiable on the interval $(-\infty, \infty)$, then the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the Wronskian of functions $f_1(x), f_2(x)$ -- $f_n(x)$.

Theorem If the functions $\vec{f}_1, \vec{f}_2$, -- $\vec{f}_n$ have $n-1$ continuous derivatives on the interval $(-\infty, \infty)$, the functions are linearly independent when the Wronskian of these functions is not identically zero on $(-\infty, \infty)$.

Example 6 Use the Wronskian to show that $\vec{f_1} = x$ and $\vec{f_2} = \sin x$ are linearly independent.

2 function diay gai hin tou ham n - 1 tak derivative find karin gai

$$W(x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x$$

$W(x) = x\cos x - \sin x$ is not identically zero on the interval $(-\infty, \infty)$. The functions are linearly independent.

Example 7 Use the Wronskian to show that the functions $\vec{f_1} = 1$, $\vec{f_2} = e^x$, $\vec{f_3} = e^{2x}$ are linearly independent.

$$W(x) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix}$$

$$= 1 \begin{vmatrix} e^x & 2e^{2x} \\ e^x & 4e^{2x} \end{vmatrix} = 4e^{3x} - 2e^{3x}$$

$$= 2e^{3x}$$

The Wronskian $W(x) = 2e^{3x}$ is not identically zero on $(-\infty, \infty)$. Therefore the functions $\vec{f_1}, \vec{f_2}, \vec{f_3}$ are linearly independent.

Example (8) $\vec{f_1} = x^2 \quad \vec{f_2} = x+3 \quad \vec{f_3} = 4x^2+2x+6$

$$W(x) = \begin{vmatrix} x^2 & x+3 & 4x^2+2x+6 \\ 2x & 1 & 8x+2 \\ 2 & 0 & 8 \end{vmatrix}$$

$$= x^2(8-0) - (x+3)(\cancel{16x} - \cancel{16x} - 4) + (4x^2+2x+6)(0-2)$$

$$= 8x^2 + 4(x+3) - 2(4x^2+2x+6)$$

$$= \cancel{8x^2} + \cancel{4x} + \cancel{12} - \cancel{8x^2} - \cancel{4x} - \cancel{12}$$

wronskan mai ulta hisab kitab hi

$$= 0$$

The functions are linerly dependent.

$$\boxed{\begin{array}{l} \text{Ex } 4.3 \\ \text{Q1} - 28 \end{array}}$$