Applications of Linear Systems

1) Network Analysis

The concept of network appears in a variety of applications. A network is a set of branches through which something flows. For example, the branches might be

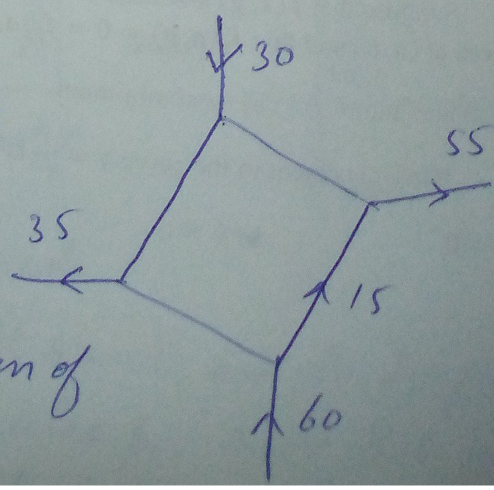
→ electrical wires through which electricity flows.
→ pipes through which water or oil flows.
→ traffic lanes through which vehicular traffic flows.
→ ecomonic linkages through which money flows.

In most networks, the branches meet at points, called nodes or junctions, where the flow divides. For example

→ in an electrical network, nodes occur where three or more wires join.
→ in a traffic network, junctions occur at street intersections.
→ in a fininical network they occur at banking centers where incoming money is distributed to individuals or other institutions.

In a network model, we assume that the total flow into a junction is equal to the total flow out of the junction.

Example 1 The figure shows a network with four nodes in which the flow rate and direction of flow in certain branches are known. Find the flow rates and direction of flow in the remaining branches.

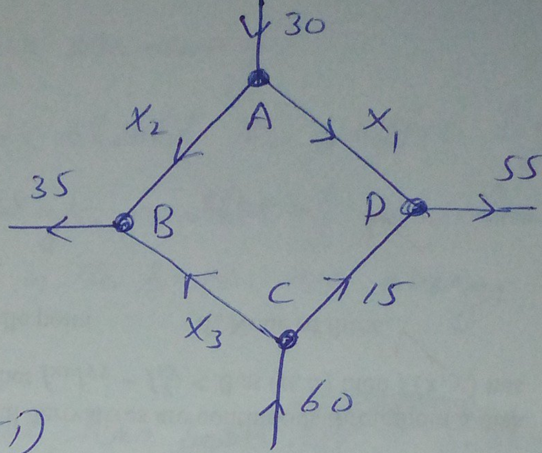


A $\quad 30 = x_1 + x_2$

B $\quad x_2 + x_3 = 35$

C $\quad 60 = x_3 + 15$

D $\quad x_1 + 15 = 55$



$$x_1 + x_2 = 30 \longrightarrow (1)$$
$$x_2 + x_3 = 35 \longrightarrow (2)$$
$$x_3 + 15 = 60 \longrightarrow (3)$$
$$x_1 + 15 = 55 \longrightarrow (4)$$

$(3) \Rightarrow \boxed{x_3 = 45} \qquad (4) \Rightarrow \boxed{x_1 = 40}$

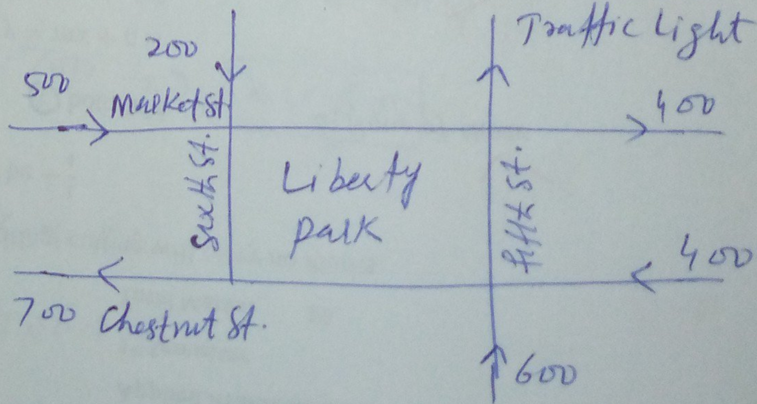
$(2) \Rightarrow x_2 + 45 = 35 \Rightarrow \boxed{x_2 = -10}$

$(1) \Rightarrow x_1 + x_2 = 30 \Rightarrow 40 + (-10) = 30$

$30 = 30$ (Satisfied).

$x_2 = -10$ indicates that direction assigned to that flow is incorrect, i.e, The flow in that branch is into node A.

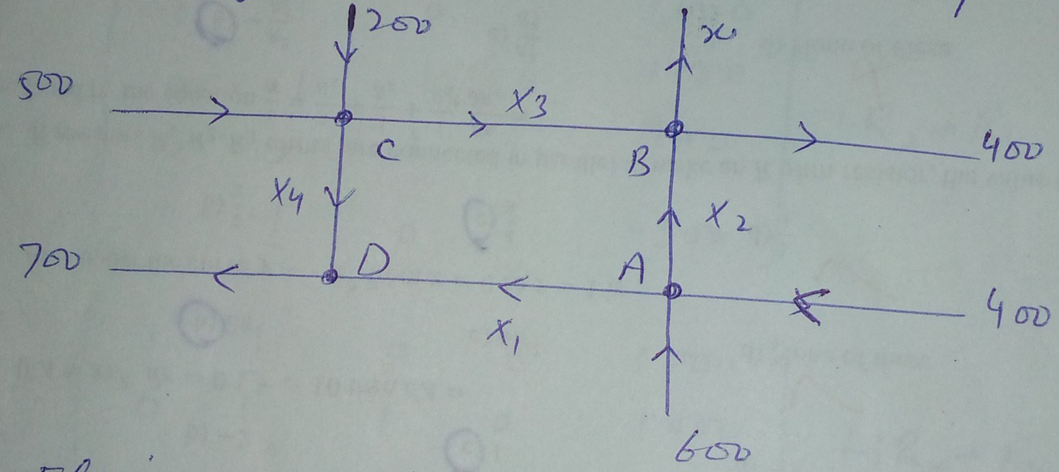
Example (2)



The network in the figure shows a purposed plan for the traffic around a new park. The plan calls for a Computerized traffic light at north exit on fifth street and the diagram indicates the average number of vehicles per hour that are expected to flow in and flow out of the streets that border the complex. All streets are one-way.

(a) How many vehicles per hour should the traffic light let through to ensure that average number of vehicles per hour flowing into the complex is the same as average no. of vehicles flowing out.

(b) Assuming that the traffic light has been set to balance the total flow in and out of the complex, what can you say about the average number of vehicles per hour that will flow along the streets that border the complex.



(a) Flow in: $500 + 400 + 600 + 200 = 1700$

Flow out: $x + 700 + 400 = x + 1100$

$x + 1100 = 1700 \Rightarrow \boxed{x = 600}$

(b) <u>A</u> $400 + 600 = x_1 + x_2$

<u>B</u> $x_2 + x_3 = 400 + x$

<u>C</u> $500 + 200 = x_3 + x_4$

<u>D</u> $x_1 + x_4 = 700$

$$x_1 + x_2 = 1000$$
$$x_2 + x_3 = 1000$$
$$x_3 + x_4 = 700$$
$$x_1 + x_4 = 700$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1000 \\ 0 & 1 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \\ 1 & 0 & 0 & 1 & 700 \end{bmatrix} \quad -1R_1 + R_4$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 & 1000 \\ 0 & 1 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \\ 0 & -1 & 0 & 1 & -300 \end{bmatrix} \quad 1 \cdot R_2 + R_4$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 & 1000 \\ 0 & 1 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \\ 0 & 0 & 1 & 1 & 700 \end{bmatrix} \quad -1R_3 + R_4$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 & 1000 \\ 0 & 1 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + x_2 = 1000$

$x_2 + x_3 = 1000$ $\qquad$ $x_4 = \text{free}$.

$x_3 + x_4 = 700$

$x_4 = t$

$x_3 = 700 - x_4 = 700 - t$

$x_2 = 1000 - x_3 = 1000 - (700 - t) = 300 + t$

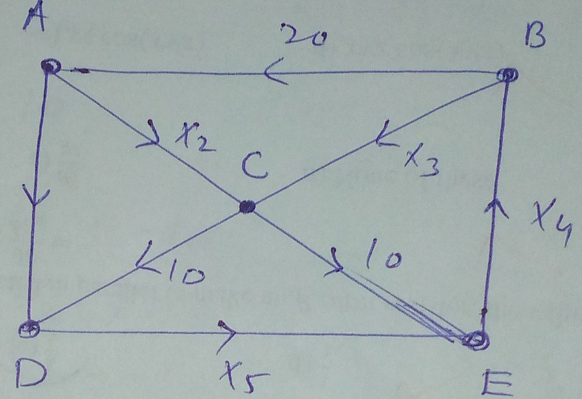
$x_1 = 1000 - x_2 = 1000 - (300 + t) = 700 - t$

$0 \le t \le 700 \quad 0 \le x_1 \le 700, \quad 300 \le x_2 \le 1000$

$0 \le x_4 \le 700.$

Example (3)

Set up a system of linear equations to represent the network show in the figure. Then solve the system.



$\underline{A}$ $\quad 20 = x_1 + x_2$

$\underline{B}$ $\quad x_4 = 20 + x_3$

$\underline{C}$ $\quad x_3 + x_2 = 10 + 10$

$\underline{D}$ $\quad x_1 + 10 = x_5$

$\underline{E}$ $\quad x_5 + 10 = x_4$

$x_1 + x_2 = 20$

$-x_3 + x_4 = 20$

$x_2 + x_3 = 20$

$x_1 - x_5 = -10$

$-x_4 + x_5 = -10$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 0 & 0 & -1 & 1 & -10 \end{bmatrix}$$

Gauss-Jordan Elimination produces the matrix.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 - x_5 = -10$

$x_2 + x_5 = 30$ $\quad\quad x_5 = \text{free}$

$x_3 - x_5 = -10$

$x_4 - x_5 = 10$

$x_1 = -10 + x_5 = t - 10$

$x_2 = 30 - x_5 = 30 - t$

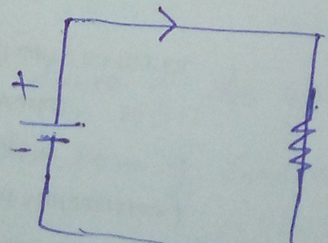
$x_3 = -10 + x_5 = t - 10$

$x_4 = 10 + x_5 = t + 10$

$x_5 = t.$

Electrical Circuits Electrical network is another type of network where analysis is commonly applied. An electrical circuit consists of batteries and resistors.

A battery is a source of electric energy and a resistor (such as light bulb) is an element that dissipates electrical energy.



→ Electrical current, which is flow of electrons through wires, behaves like flow of water through pipes.

→ A battery acts like a pump that creates electrical pressure to increase the flow rate of electrons

→ A resistor acts like a restriction in a pipe that reduces the flow rate of electrons.

Electric potential The technical term for electrical pressure is electric potential. It is measured in volts (V).

Resistance The degree to which a resistor reduces the electrical potential is called its resistance. It is measured in ohms (Ω).

Current The rate of Flow of electrons in a wire is called current. It is measured in amperes (A).

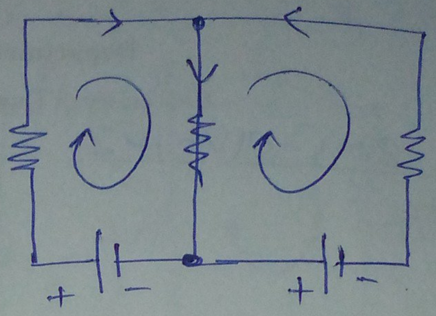
Ohm's Law If a current of I amperes passes through a resistor with resistance of R ohms, then there is a resulting drop of E volts in electrical potential that is the product of the current I and resistance R.

$$E = IR$$

Kirchhoff's current Law The sum of the currents flowing into any node is equal to the sum of currents flowing out.

Kirchhoff's Voltage Law If one transversal of any closed loop, the sum of the voltage rise equal to sum of voltage drops.

In circuits with multiple loops and batteries there is no way to tell in advance which way the currents are flowing. The procedure is to assign arbitrary directions to the current flows in the branches and let the mathematical computations determine whether the assignments are correct.



In addtion, Kirchhoff's voltage law requires a direction of travel for each closed loop. The choice is arbitrary but we always take This direction to be clockwise.

we make the following conventions:

Voltage at Resister:
Voltage drops if Current through Resister and direction of loop is same
Voltage Rises if Current through Resister and direction of loop is opposite

1) A voltage drop occurs at a resistor if the direction assigned to the current through the resistor is the same as direction assigned to the loop.

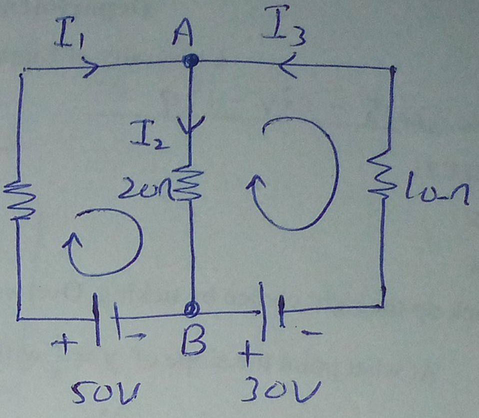
A voltage rise occurs at a resistor if the direction assigned to the current through the resistor is the opposite to that assigned to the loop.

2) A voltage drop occurs at a battery if the direction assigned to the loop is from (+) to (−) through the battery.

A voltage rise occurs at a battery if the direction assigned to the loop is from (−) to (+) through the battery.

Voltage at Battery:
V drop if loop direction + to -
V Rise if loop direction - to +

Example (4) Determine the currents I_1, I_2 and I_3 in the circuit shown in figure.



A $\quad I_1 + I_3 = I_2$

B $\quad I_2 = I_1 + I_3$

Dono same eqs. hin tou aik tou lai lia just

$I_1 + I_3 - I_2 = 0$

$I_1 - I_2 + I_3 = 0 \longrightarrow (1)$

East or West Rules are Best

	Voltage Rises	Voltage Drops
Left	50	$5I_1 + 20I_2$
Right	$30 + 20I_2 + 10I_3$	0

$50 = 5I_1 + 20I_2$.

$30 + 20I_2 + 10I_3 = 0$

$5I_1 + 20I_2 = 50 \longrightarrow (2)$

$20I_2 + 10I_3 = -30 \longrightarrow (3)$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 5 & 20 & 0 & 50 \\ 0 & 20 & 10 & -30 \end{bmatrix} \quad -5R_1 + R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 25 & -5 & 50 \\ 0 & 20 & 10 & -30 \end{bmatrix} \quad \frac{1}{25}R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1/5 & 2 \\ 0 & 20 & 10 & -30 \end{bmatrix} \quad -20R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1/5 & 2 \\ 0 & 0 & 14 & -70 \end{bmatrix} \quad \frac{1}{14} R_3$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1/5 & 2 \\ 0 & 0 & 1 & -5 \end{bmatrix}$$

$\boxed{I_3 = -5}$

$I_2 - \frac{1}{5} I_3 = 2$

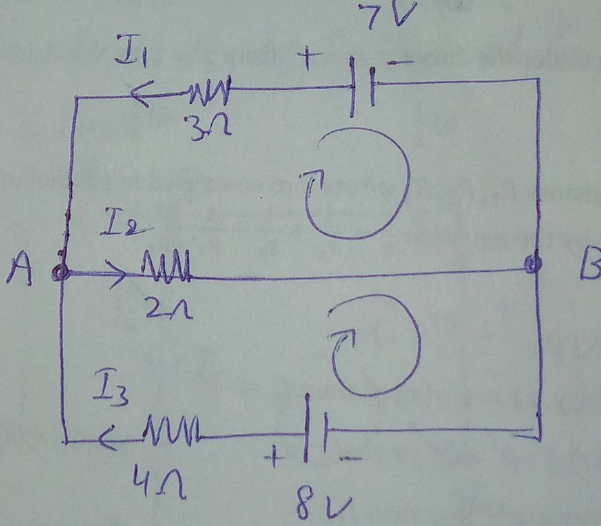
$I_2 - \frac{1}{5}(-5) = 2 \Rightarrow \boxed{I_2 = 1}$

$I_1 - I_2 + I_3 = 0$

$I_1 - 1 + (-5) = 0 \Rightarrow \boxed{I_1 = 6}$

<u>Example (5)</u>

Determine the currents I_1, I_2 and I_3 for the electrical network shown in the figure.



<u>A</u> $I_1 + I_3 = I_2$

<u>B</u> $I_2 = I_1 + I_3$

$I_1 - I_2 + I_3 = 0 \longrightarrow (1)$

	Voltage Rises	Voltage Drops.
<u>Top</u>	$2I_2 + 3I_1$	7
<u>Bottom</u>	8.	$4I_3 + 2I_2$

$$I_1 - I_2 + I_3 = 0 \longrightarrow (1)$$

$$3I_1 + 2I_2 = 7 \longrightarrow (2)$$

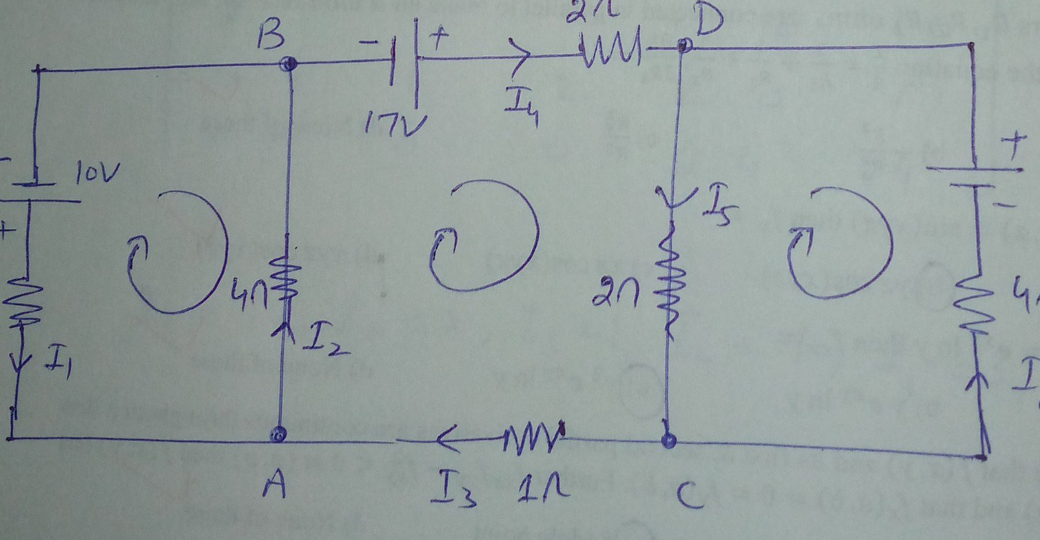
$$2I_2 + 4I_3 = 8 \longrightarrow (3)$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 3 & 2 & 0 & 7 \\ 0 & 2 & 4 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$I_1 = 1, \quad I_2 = 2, \quad I_3 = 1$$

Example (8) Determine the currents I_1, I_2, I_3, I_4, I_5 and I_6 for the electrical network shown below.



A $\quad I_1 + I_3 = I_2 \qquad I_1 - I_2 + I_3 = 0 \longrightarrow (1)$

B $\quad I_2 = I_1 + I_4 \qquad I_1 - I_2 + I_4 = 0 \longrightarrow (2)$

C $\quad I_5 = I_3 + I_6 \qquad I_3 - I_5 + I_6 = 0 \longrightarrow (3)$

D $\quad I_4 + I_6 = I_5 \qquad I_4 - I_5 + I_6 = 0 \longrightarrow (4)$

	Voltage Rises	Voltage Drops
Left	$2I_1 + 4I_2$	10
Middle	17	$4I_2 + I_3 + 2I_5$
Right	$2I_5 + 4I_6$	14

$$2I_1 + 4I_2 = 10 \longrightarrow (5)$$

$$4I_2 + I_3 + 2I_5 = 17 \longrightarrow (6)$$

$$2I_5 + 4I_6 = 14 \longrightarrow (7)$$

$$\begin{array}{cc}
 & \begin{array}{cccccc|c} I_1 & 2 & 3 & 4 & 5 & 6 & \end{array} \\
\begin{array}{c} \text{Nodes section} \\ \\ \\ \\ \text{Voltage Section} \\ \\ \\ \end{array} &
\left[\begin{array}{cccccc|c}
1 & -1 & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 1 & -1 & 1 & 0 \\
2 & 4 & 0 & 0 & 0 & 0 & 10 \\
0 & 4 & 1 & 2 & 2 & 0 & 17 \\
0 & 0 & 0 & 0 & 2 & 4 & 14
\end{array}\right]
\end{array}$$

$I_1 = 1$, $I_2 = 2$, $I_3 = 1$, $I_4 = 1$, $I_5 = 3$, $I_6 = 2$

2) Balancing Chemical Equations

Chemical compounds are represented by chemical formulas that describe the atomic make up of their molecules. For example, water is composed of two hydrogen atoms and one oxygen atom, so its chemical formula is H_2O.

Chemical Equation when chemical compounds are combined under right conditions, the atoms in their molecules re-arrange to form new compounds. This is indicated by a chemical equation. For example

$$CH_4 + O_2 \longrightarrow CO_2 + H_2O$$

$$CH_4 + 2O_2 \longrightarrow CO_2 + 2H_2O$$

Example ① Balance the chemical equation.

$$CH_4 + O_2 \longrightarrow CO_2 + H_2O$$

Let x_1, x_2, x_3, x_4 be positive integers that balance the equation.

$$x_1(CH_4) + x_2(O_2) \longrightarrow x_3(CO_2) + x_4(H_2O)$$

	Left Side		Right Side
Carbon	x_1	$=$	x_3
Hydrogen	$4x_1$	$=$	$2x_4$
Oxygen	$2x_2$	$=$	$2x_3 + x_4$

$$x_1 - x_3 = 0$$
$$4x_1 - 2x_4 = 0$$
$$2x_2 - 2x_3 - x_4 = 0$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -2 & 0 \\ 0 & 2 & -3 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{bmatrix}$$

$$x_1 - \frac{1}{2}x_4 = 0 \qquad \Rightarrow \qquad x_1 = \frac{1}{2}x_4$$
$$x_2 - x_4 = 0 \qquad\qquad x_2 = x_4$$
$$x_3 - \frac{1}{2}x_4 = 0 \qquad\qquad x_3 = \frac{1}{2}x_4$$

Let $x_4 = t$

$$x_1 = \frac{1}{2}t, \quad x_2 = t, \quad x_3 = \frac{1}{2}t, \quad x_4 = t.$$

For $t = 2$ $\quad x_1 = 1, \quad x_2 = 2, \quad x_3 = 1, \quad x_4 = 2$

$$1(CH_4) + 2(O_2) \longrightarrow 1(CO_2) + 2(H_2O)$$

$$CH_4 + 2O_2 \longrightarrow CO_2 + 2H_2O.$$

Example (8) Balance the Chemical equation

$HCl + Na_3PO_4 \longrightarrow H_3PO_4 + NaCl$

Let x_1, x_2, x_3, x_4 be positive integers that can balance the equation.

$$x_1(HCl) + x_2(Na_3PO_4) \longrightarrow x_3(H_3PO_4) + x_4(NaCl)$$

Hydrogen	$x_1 = 3x_3$	$x_1 - 3x_3 = 0$
Chlorine	$x_1 = x_4$	$x_1 - x_4 = 0$
Sodium	$3x_2 = x_4$	$3x_2 - x_4 = 0$
phosphorous	$x_2 = x_3$	$x_2 - x_3 = 0$
Oxygen	$4x_2 = 4x_3$	$4x_2 - 4x_3 = 0$

$$\left[\begin{array}{ccccc} 1 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 4 & -4 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccccc} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - x_4 = 0$

$x_2 - \frac{1}{3}x_4 = 0$

$x_3 - \frac{1}{3}x_4 = 0$

$x_1 = \boxed{x_4 = t}$

$x_2 = \frac{1}{3}x_4 = \frac{1}{3}t$

$x_3 = \frac{1}{3}x_4 = \frac{1}{3}t$

$t = 3 \quad x_1 = 3, \ x_2 = 1, \ x_3 = 1, \ x_4 = 3$

3) Polynomial Interpolation

consider n points in the xy-plane

$$(x_1, y_1), (x_2, y_2) - - - (x_n, y_n).$$

that represent a collection of data, and we want to find a polynomial of degree $n-1$

$$p(x) = a_0 + a_1 x + a_2 x^2 + - - + a_{n-1} x^{n-1}$$

whose graph passes through points. This procedure is called polynomial interpolation (polynomial curve fitting).

To solve for n coefficients of $p(x)$, substitute each of n points into polynomial function and obtain n linear equations in n variables $a_0, a_1, - - a_{n-1}$.

$$\begin{aligned}
a_0 + a_1 x_1 + a_2 x_1^2 + - - + a_{n-1} x_1^{n-1} &= y_1 \\
a_0 + a_1 x_2 + a_2 x_2^2 + - - + a_{n-1} x_2^{n-1} &= y_2 \\
\vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_0 + a_1 x_n + a_2 x_n^2 + - - + a_{n-1} x_n^{n-1} &= y_n
\end{aligned}$$

The agumented matrix of the system is

$$\begin{bmatrix} 1 & x_1 & x_1^2 & - - - & x_1^{n-1} & y_1 \\ 1 & x_2 & x_2^2 & - - - & x_2^{n-1} & y_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & x_n & x_n^2 & - - - & x_n^{n-1} & y_n \end{bmatrix}$$

Example(9) Find a cubic polynomial whose graph passes through the points

$(1,3), \quad (2,-2), \quad (3,-5), \quad (4,0).$

(x1,y1) (x2,y2) (x3,y3) (x4,y4)

Since there are four points, we will use an interpolating polynomial of degree $n=3$. Polynomial max power = points(n) - 1

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

y

a. + a1x1 + a2x1^2 + a3x1^3 = y1
a. + a1x2 + a2x2^2 + a3x2^3 = y2
a. + a1x3 + a2x3^2 + a3x3^3 = y3

$$a_0 + a_1 + a_2 + a_3 = 3$$

$$a_0 + 2a_1 + 4a_2 + 8a_3 = -2$$

$$a_0 + 3a_1 + 9a_2 + 27a_3 = -5$$

$$a_0 + 4a_1 + 16a_2 + 64a_3 = 0$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & -2 \\ 1 & 3 & 9 & 27 & -5 \\ 1 & 4 & 16 & 64 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$a_0 = 4, \quad a_1 = 3, \quad a_2 = -5, \quad a_3 = 1$

$$p(x) = 4 + 3x - 5x^2 + x^3$$

Example 10 Find a polynomial that fits the points.

$(-2,3)$, $(-1,5)$, $(0,1)$, $(1,4)$, $(2,+10)$

(x1,y1) (x2,y2) (x3,y3) (x4,y4) (x5,y5)

Since five points are given, we will use an interpolating polynomial of degree $n=4$.

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4 = 3$$

$$a_0 - a_1 + a_2 - a_3 + a_4 = 5$$

$$a_0 = 1$$

$$a_0 + a_1 + a_2 + a_3 + a_4 = 4$$

$$a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 = 10$$

$$\boxed{a_0 = 1}$$

$$-2a_1 + 4a_2 - 8a_3 + 16a_4 = 2$$

$$-a_1 + a_2 - a_3 + a_4 = 4$$

$$a_1 + a_2 + a_3 + a_4 = 3$$

$$2a_1 + 4a_2 + 8a_3 + 16a_4 = 9$$

$$\begin{bmatrix} -2 & 4 & -8 & 16 & 2 \\ -1 & 1 & -1 & 1 & 4 \\ 1 & 1 & 1 & 1 & 3 \\ 2 & 4 & 8 & 16 & 9 \end{bmatrix}$$

$$\left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & -5/4 \\ 0 & 1 & 0 & 0 & 101/24 \\ 0 & 0 & 1 & 0 & 3/4 \\ 0 & 0 & 0 & 1 & -17/24 \end{array}\right]$$

$a_1 = -5/4, \quad a_2 = 101/24, \quad a_3 = 3/4, \quad a_4 = -17/24$

$$p(x) = 1 - \frac{5}{4}x + \frac{101}{24}x^2 + \frac{3}{4}x^3 - \frac{17}{24}x^4$$

$\boxed{\text{Exercise } 1.8 \quad Q1-18}$