Let's work through a simple example to calculate **eigenvalues** and **eigenvectors** manually using a small matrix.

Example:

Given a 2x2 matrix:

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

We will calculate its **eigenvalues** and **eigenvectors** step by step.

Step 1: Find the Eigenvalues

Eigenvalues λ are found by solving the **characteristic equation**:

$$\det(A - \lambda I) = 0$$

Where:

- I is the identity matrix.
- λ is the eigenvalue.

1.1 Subtract λ from the diagonal elements of A:

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix}$$

1.2 Find the determinant of this matrix:

$$\det(A - \lambda I) = (4-\lambda)(3-\lambda) - (2 \cdot 1)$$

Expand this equation:

$$(4-\lambda)(3-\lambda) = 12 - 4\lambda - 3\lambda + \lambda^2 = \lambda^2 - 7\lambda + 12$$

Now subtract the product of 2 and 1:

$$\lambda^2 - 7\lambda + 12 - 2 = \lambda^2 - 7\lambda + 10$$

1.3 Solve the quadratic equation:

$$\lambda^2 - 7\lambda + 10 = 0$$

Using the quadratic formula:

$$\lambda = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(10)}}{2(1)}$$

$$\lambda = \frac{7 \pm \sqrt{49-40}}{2} = \frac{7 \pm \sqrt{9}}{2} = \frac{7 \pm 3}{2}$$

So, the two eigenvalues are:

$$\lambda_1 = \frac{7+3}{2} = 5, \quad \lambda_2 = \frac{7-3}{2} = 2$$

Step 2: Find the Eigenvectors

Now we find the eigenvectors corresponding to each eigenvalue.

2.1 Eigenvector for $\lambda_1 = 5$:

We solve the equation:

$$(A - 5I)v = 0$$

$$A - 5I = \begin{bmatrix} 4-5 & 1 \\ 2 & 3-5 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$$

We solve:

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the first row, we get:

$$-1x_1 + 1x_2 = 0 \quad \Rightarrow \quad x_1 = x_2$$

Thus, the eigenvector corresponding to $\lambda_1 = 5$ is:

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Any scalar multiple of this vector is also valid, so v_1 could be $c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where c is any constant.

2.2 Eigenvector for $\lambda_2 = 2$:

Now solve for $\lambda_2 = 2$:

$$(A - 2I)v = 0$$

$$A - 2I = \begin{bmatrix} 4-2 & 1 \\ 2 & 3-2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

We solve:

We solve:

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the first row, we get:

$$2x_1 + x_2 = 0 \quad \Rightarrow \quad x_2 = -2x_1$$

Thus, the eigenvector corresponding to $\lambda_2 = 2$ is:

$$v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Again, any scalar multiple of this vector is also valid.

Summary:

- Eigenvalues: $\lambda_1 = 5$, $\lambda_2 = 2$
- Eigenvector for $\lambda_1 = 5$: $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- Eigenvector for $\lambda_2 = 2$: $v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

This simple example illustrates how to manually calculate eigenvalues and eigenvectors for a 2x2 matrix.