

Lecture No.9**Set identities****SET IDENTITIES:**

Let A, B, C be subsets of a universal set U.

1. Idempotent Laws
 - a. $A \cup A = A$
 - b. $A \cap A = A$
2. Commutative Laws
 - a. $A \cup B = B \cup A$
 - b. $A \cap B = B \cap A$
3. Associative Laws
 - a. $A \cup (B \cap C) = (A \cup B) \cap C$
 - b. $A \cap (B \cup C) = (A \cap B) \cup C$
4. Distributive Laws
 - a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
 - b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. Identity Laws
 - a. $A \cup \emptyset = A$
 - b. $A \cap \emptyset = \emptyset$
 - c. $A \cup U = U$
 - d. $A \cap U = A$
6. Complement Laws
 - a. $A \cup A^c = U$
 - b. $A \cap A^c = \emptyset$
 - c. $U^c = \emptyset$
 - d. $\emptyset^c = U$
8. Double Complement Law
$$(A^c)^c = A$$
9. DeMorgan's Laws
 - a. $(A \cup B)^c = A^c \cap B^c$
 - b. $(A \cap B)^c = A^c \cup B^c$
10. Alternative Representation for Set Difference
$$A - B = A \cap B^c$$
11. Subset Laws
 - a. $A \cup B \subseteq C$ iff $A \subseteq C$ and $B \subseteq C$
 - b. $C \subseteq A \cap B$ iff $C \subseteq A$ and $C \subseteq B$
12. Absorption Laws
 - a. $A \cup (A \cap B) = A$
 - b. $A \cap (A \cup B) = A$

EXERCISE:

1. $A \subseteq A \cup B$
2. $A - B \subseteq A$
3. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$
4. $A \subseteq B$ if, and only if, $B^c \subseteq A^c$

1. Prove that $A \subseteq A \cup B$

SOLUTION

Here in order to prove the identity you should remember the definition of Subset of a set. We will take the arbitrary element of a set then show that, that element is the member of the other then the first set is the subset of the other. So

Let x be an arbitrary element of A, that is $x \in A$.

$$\Rightarrow x \in A \text{ or } x \in B$$

$$\Rightarrow x \in A \cup B$$

But x is an arbitrary element of A .

$$\therefore A \subseteq A \cup B \quad (\text{proved})$$

1. Prove that $A - B \subseteq A$

SOLUTION

Let $x \in A - B$

$$\Rightarrow x \in A \text{ and } x \notin B \quad (\text{by definition of } A - B)$$

$$\Rightarrow x \in A \quad (\text{in particular})$$

But x is an arbitrary element of $A - B$

$$\therefore A - B \subseteq A \quad (\text{proved})$$

1. Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

SOLUTION

Suppose that $A \subseteq B$ and $B \subseteq C$

Consider $x \in A$

$$\Rightarrow x \in B \quad (\text{as } A \subseteq B)$$

$$\Rightarrow x \in C \quad (\text{as } B \subseteq C)$$

But x is an arbitrary element of A

$$\therefore A \subseteq C \quad (\text{proved})$$

1. Prove that $A \subseteq B$ iff $B^c \subseteq A^c$

SOLUTION:

Suppose $A \subseteq B$ {To prove $B^c \subseteq A^c$ }

Let $x \in B^c$

$$\Rightarrow x \notin B \quad (\text{by definition of } B^c)$$

$$\Rightarrow x \notin A$$

$$\Rightarrow x \in A^c \quad (\text{by definition of } A^c)$$

Now we know that implication and its contrapositivity are logically equivalent and the contrapositive statement of if $x \in A$ then $x \in B$ is: if $x \notin B$ then $x \notin A$ which is the definition of the $A \subseteq B$. Thus if we show for any two sets A and B , if $x \notin B$ then $x \notin A$ it means that

$$A \subseteq B. \text{ Hence}$$

But x is an arbitrary element of B^c

$$\therefore B^c \subseteq A^c$$

Conversely,

Suppose $B^c \subseteq A^c$ {To prove $A \subseteq B$ }

Let $x \in A$

$$\Rightarrow x \notin A^c \quad (\text{by definition of } A^c)$$

$$\Rightarrow x \notin B^c \quad (\because B^c \subseteq A^c)$$

$$\Rightarrow x \in B \quad (\text{by definition of } B^c)$$

But x is an arbitrary element of A .

$$\therefore A \subseteq B \quad (\text{proved})$$

EXERCISE:

Let A and B be subsets of a universal set U .

Prove that $A - B = A \cap B^c$.

SOLUTIONLet $x \in A - B$ $\Rightarrow x \in A$ and $x \notin B$ (definition of set difference) $\Rightarrow x \in A$ and $x \in B^c$ (definition of complement) $\Rightarrow x \in A \cap B^c$ (definition of intersection)But x is an arbitrary element of $A - B$ so we can write

$$\therefore A - B \subseteq A \cap B^c \dots\dots\dots(1)$$

Conversely,let $y \in A \cap B^c$ $\Rightarrow y \in A$ and $y \in B^c$ (definition of intersection) $\Rightarrow y \in A$ and $y \notin B$ (definition of complement) $\Rightarrow y \in A - B$ (definition of set difference)But y is an arbitrary element of $A \cap B^c$

$$\therefore A \cap B^c \subseteq A - B \dots\dots\dots(2)$$

From (1) and (2) it follows that

$$A - B = A \cap B^c \quad (\text{as required})$$

EXERCISE:Prove the DeMorgan's Law: $(A \cup B)^c = A^c \cap B^c$ **PROOF**Let $x \in (A \cup B)^c$ $\Rightarrow x \notin A \cup B$ (definition of complement) $x \notin A$ and $x \notin B$ (DeMorgan's Law of Logic) $\Rightarrow x \in A^c$ and $x \in B^c$ (definition of complement) $\Rightarrow x \in A^c \cap B^c$ (definition of intersection)But x is an **arbitrary** element of $(A \cup B)^c$ so we have proved that

$$\therefore (A \cup B)^c \subseteq A^c \cap B^c \dots\dots\dots(1)$$

Converselylet $y \in A^c \cap B^c$ $\Rightarrow y \in A^c$ and $y \in B^c$ (definition of intersection) $\Rightarrow y \notin A$ and $y \notin B$ (definition of complement) $\Rightarrow y \notin A \cup B$ (DeMorgan's Law of Logic) $\Rightarrow y \in (A \cup B)^c$ (definition of complement)But y is an arbitrary element of $A^c \cap B^c$

$$\therefore A^c \cap B^c \subseteq (A \cup B)^c \dots\dots\dots(2)$$

From (1) and (2) we have

$$(A \cup B)^c = A^c \cap B^c$$

Which is the Demorgan's Law.

EXERCISE:**Prove the associative law: $A \cap (B \cap C) = (A \cap B) \cap C$** **PROOF:**Consider $x \in A \cap (B \cap C)$ $\Rightarrow x \in A$ and $x \in B \cap C$ (definition of intersection) $\Rightarrow x \in A$ and $x \in B$ and $x \in C$ (definition of intersection) $\Rightarrow x \in A \cap B$ and $x \in C$ (definition of intersection) $\Rightarrow x \in (A \cap B) \cap C$ (definition of intersection)

But x is an arbitrary element of $A \cap (B \cap C)$

$$\therefore A \cap (B \cap C) \subseteq (A \cap B) \cap C \dots (1)$$

Conversely

let $y \in (A \cap B) \cap C$

$$\Rightarrow y \in A \cap B \text{ and } y \in C \quad (\text{definition of intersection})$$

$$\Rightarrow y \in A \text{ and } y \in B \text{ and } y \in C \quad (\text{definition of intersection})$$

$$\Rightarrow y \in A \text{ and } y \in B \cap C \quad (\text{definition of intersection})$$

$$\Rightarrow y \in A \cap (B \cap C) \quad (\text{definition of intersection})$$

But y is an arbitrary element of $(A \cap B) \cap C$

$$\therefore (A \cap B) \cap C \subseteq A \cap (B \cap C) \dots (2)$$

From (1) & (2), we conclude that

$$A \cap (B \cap C) = (A \cap B) \cap C \quad (\text{proved})$$

EXERCISE:

Prove the distributive law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

PROOF:

Let $x \in A \cup (B \cap C)$

$$\Rightarrow x \in A \text{ or } x \in B \cap C \quad (\text{definition of union})$$

Now since we have $x \in A$ or $x \in B \cap C$ it means that either x is in A or in $A \cap B$

it is in the $A \cup (B \cap C)$ so in order to show that

$A \cup (B \cap C)$ is the subset of $(A \cup B) \cap (A \cup C)$ we will consider both the cases when x is in A or x is in $B \cap C$ hence we will consider the two cases.

CASE I:

(when $x \in A$)

$$\Rightarrow x \in A \cup B \text{ and } x \in A \cup C \quad (\text{definition of union})$$

Hence,

$$x \in (A \cup B) \cap (A \cup C) \quad (\text{definition of intersection})$$

CASE II:

(when $x \in B \cap C$)

We have $x \in B$ and $x \in C$ (definition of intersection)

Now $x \in B \Rightarrow x \in A \cup B$ (definition of union)

and $x \in C \Rightarrow x \in A \cup C$ (definition of union)

Thus $x \in A \cup B$ and $x \in A \cup C$

$$\Rightarrow x \in (A \cup B) \cap (A \cup C)$$

In both of the cases $x \in (A \cup B) \cap (A \cup C)$

Accordingly,

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C) \dots (1)$$

Conversely,

Suppose $x \in (A \cup B) \cap (A \cup C)$

$$\Rightarrow x \in (A \cup B) \text{ and } x \in (A \cup C) \quad (\text{definition of intersection})$$

Consider the two cases $x \in A$ and $x \notin A$

CASE I: (when $x \in A$)

We have $x \in A \cup (B \cap C)$ (definition of union)

CASE II: (when $x \notin A$)

Since $x \in A \cup B$ and $x \notin A$, therefore $x \in B$

Also, since $x \in A \cup C$ and $x \notin A$, therefore $x \in C$. Thus $x \in B$ and $x \in C$
That is, $x \in B \cap C$

$$\Rightarrow x \in A \cup (B \cap C) \quad (\text{definition of union})$$

Hence in both cases

$$x \in A \cup (B \cap C)$$

$$\therefore (A \cup B) \cap C \subseteq A \cup (B \cap C) \dots\dots\dots(2)$$

By (1) and (2), it follows that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad (\text{proved})$$

EXERCISE:

For any sets A and B if $A \subseteq B$ then

$$(a) \quad A \cap B = A \qquad (b) \quad A \cup B = B$$

SOLUTION:

$$(a) \quad \text{Let } x \in A \cap B$$

$$\Rightarrow x \in A \text{ and } x \in B$$

$$\Rightarrow x \in A \quad (\text{in particular})$$

$$\text{Hence } A \cap B \subseteq A \dots\dots\dots(1)$$

Conversely,

$$\text{let } x \in A.$$

$$\text{Then } x \in B \quad (\text{since } A \subseteq B)$$

$$\text{Now } x \in A \text{ and } x \in B, \text{ therefore } x \in A \cap B$$

$$\text{Hence, } A \subseteq A \cap B \dots\dots\dots(2)$$

From (1) and (2) it follows that

$$A = A \cap B \quad (\text{proved})$$

(b) Prove that $A \cup B = B$ when $A \subseteq B$

SOLUTION:

Suppose that $A \subseteq B$. Consider $x \in A \cup B$.

CASE I (when $x \in A$)

$$\text{Since } A \subseteq B, x \in A \Rightarrow x \in B$$

CASE II (when $x \notin A$)

$$\text{Since } x \in A \cup B, \text{ we have } x \in B$$

Thus $x \in B$ in both the cases, and we have

$$A \cup B \subseteq B \dots\dots\dots(1)$$

Conversely

$$\text{let } x \in B. \text{ Then clearly, } x \in A \cup B$$

$$\text{Hence } B \subseteq A \cup B \dots\dots\dots(2)$$

Combining (1) and (2), we deduce that

$$A \cup B = B \quad (\text{proved})$$

USING SET IDENTITIES:

For all subsets A and B of a universal set U, prove that

$$(A - B) \cup (A \cap B) = A$$

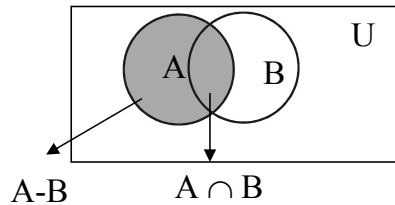
PROOF:

$$\text{LHS} = (A - B) \cup (A \cap B)$$

$$= (A \cap B^c) \cup (A \cap B) \quad (\text{Alternative representation for set difference})$$

$$\begin{aligned} &= A \cap (B^c \cup B) && \text{Distributive Law} \\ &= A \cap U && \text{Complement Law} \\ &= A && \text{Identity Law} \\ &= \text{RHS} && \text{(proved)} \end{aligned}$$

The result can also be seen by Venn diagram.

**EXERCISE:**

For any two sets A and B prove that $A - (A - B) = A \cap B$

SOLUTION

$$\begin{aligned} \text{LHS} &= A - (A - B) \\ &= A - (A \cap B^c) && \text{Alternative representation for set difference} \\ &= A \cap (A \cap B^c)^c && \text{Alternative representation for set difference} \\ &= A \cap (A^c \cup (B^c)^c) && \text{DeMorgan's Law} \\ &= A \cap (A^c \cup B) && \text{Double Complement Law} \\ &= (A \cap A^c) \cup (A \cap B) && \text{Distributive Law} \\ &= \emptyset \cup (A \cap B) && \text{Complement Law} \\ &= A \cap B && \text{Identity Law} \\ &= \text{RHS} && \text{(proved)} \end{aligned}$$

EXERCISE:

For all set A, B, and C prove that $(A - B) - C = (A - C) - B$

SOLUTION

$$\begin{aligned} \text{LHS} &= (A - B) - C \\ &= (A \cap B^c) - C && \text{Alternative representation of set difference} \\ &= (A \cap B^c) \cap C^c && \text{Alternative representation of set difference} \\ &= A \cap (B^c \cap C^c) && \text{Associative Law} \\ &= A \cap (C^c \cap B^c) && \text{Commutative Law} \\ &= (A \cap C^c) \cap B^c && \text{Associative Law} \\ &= (A - C) \cap B^c && \text{Alternative representation of set difference} \\ &= (A - C) - B && \text{Alternative representation of set difference} \\ &= \text{RHS} && \text{(proved)} \end{aligned}$$

EXERCISE:

Simplify $(B^c \cup (B^c - A))^c$

SOLUTION

$$\begin{aligned} (B^c \cup (B^c - A))^c &= (B^c \cup (B^c \cap A^c))^c \\ &\text{Alternative representation for set difference} \\ &= (B^c)^c \cap (B^c \cap A^c)^c && \text{DeMorgan's Law} \\ &= B \cap ((B^c)^c \cup (A^c)^c) && \text{DeMorgan's Law} \\ &= B \cap (B \cup A) && \text{Double Complement Law} \end{aligned}$$

$$= B$$

Absorption Law

is the simplified form of the given expression.

PROVING SET IDENTITIES BY MEMBERSHIP TABLE:

Prove the following using Membership Table:

- (i) $A - (A - B) = A \cap B$
- (ii) $(A \cap B)^c = A^c \cup B^c$
- (iii) $A - B = A \cap B^c$
 $A - (A - B) = A \cap B$

A	B	A-B	A-(A-B)	A∩B
1	1	0	1	1
1	0	1	0	0
0	1	0	0	0
0	0	0	0	0

$$(A \cap B)^c = A^c \cup B^c$$

A	B	A ∩ B	(A ∩ B) ^c	A ^c	B ^c	A ^c ∪ B ^c
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

SOLUTION (iii):

A	B	A - B	B ^c	A ∩ B ^c
1	1	0	0	0
1	0	1	1	1
0	1	0	0	0
0	0	0	1	0