Exercise 6.5

17 $\dfrac{5.5}{(s+1.5)(s-4)}$

Let $F(s) = \dfrac{5.5}{s+1.5}$ & $G(s) = \dfrac{1}{s-4}$

$$\mathcal{L}\{F(s)\} = 5.5\,\mathcal{L}\left\{\frac{1}{s+1.5}\right\};$$

$$f(t) = 5.5\,e^{-1.5t} \quad ; \quad g(t) = e^{4t}$$

کسی ایک کو τ اور دوسرے کو $t-\tau$ کہہ دینا ہے۔

$$f(\tau) = 5.5\,e^{-1.5\tau} \quad ; \quad g(t-\tau) = e^{4(t-\tau)}$$

$$\because h(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

$$= \int_0^t 5.5\,e^{-1.5\tau}\,e^{4(t-\tau)}\,d\tau$$

$$= 5.5\int_0^t e^{-1.5\tau}\cdot e^{4t-4\tau}\,d\tau$$

$$= 5.5\int_0^t e^{-1.5\tau-4\tau}\cdot e^{4t}\,d\tau$$

$$= 5.5\int_0^t e^{4t}\cdot e^{-5.5\tau}\,d\tau$$

$$= 5.5\,e^{4t}\int_0^t e^{-5.5\tau}\,d\tau$$

$$= \cancel{5.5}\, e^{4t} \left| \frac{e^{-5.5T}}{-\cancel{5.5}} \right|_0^t$$

$$= -e^{4t} \left[e^{-5.5t} - e^{0} \right]$$

$$= -e^{4t-5.5t} + e^{+4t+0}$$

$$= -e^{-1.5t} + e^{+4t}$$

$$h(t) = e^{4t} - e^{-1.5t}$$

19 $$\frac{2\pi s}{(s^2+\pi^2)^2} = \frac{2\pi}{s^2+\pi^2} \cdot \frac{s}{s^2+\pi^2}$$

Let, $F(s) = \frac{2\pi}{s^2+\pi^2}$; $G(s) = \frac{s}{s^2+\pi^2}$

$f(t) = 2\sin\pi t$; $g(t) = \cos\pi t$

$f(t-T) = 2\sin\pi(t-T)$; $g(T) = \cos\pi T$

$$h(t) = \int_0^t f(t-T)\, g(T)\, dT$$

$$= \int_0^t 2\sin\pi(t-T) \cos\pi T \, dT$$

$$= 2\int_0^t \sin\pi(t-\tau)\cos\pi\tau \, d\tau$$

$$\cancel{= 2\int_0^t (\sin\pi t\cos\tau - \cos t\sin\pi\tau)(\cos\pi\tau)\, d\tau}$$

$$\cancel{\because \sin(A+B) = \sin A\cos B + \cos A\sin B:}$$

$$\cancel{= 2\int_0^t (\sin t\cos\tau \cdot \cos\pi\tau - \cos t\sin\pi\tau\cos\pi\tau)\, d\tau}$$

$$= 2\int_0^t (\sin\pi t\cos\pi\tau - \cos\pi t\sin\pi\tau)(\cos\pi\tau)\, d\tau$$

$$= 2\int_0^t (\sin\pi t\cos^2\pi\tau - \cos\pi t\sin\pi\tau\cos\pi\tau)\, d\tau$$

$$= 2\sin\pi t\int_0^t \frac{1+\cos 2\pi\tau}{2}\, d\tau - 2\int_0^t (\cos\pi t \cdot \sin\pi\tau \cdot \cos\pi\tau)\, d\tau$$

$$= \sin\pi t\int_0^t (1+\cos 2\pi\tau)\, d\tau - \frac{2\cos\pi t}{\pi}\int_0^t \sin\pi\tau(\pi)\cos\pi\tau$$

$$= \sin\pi t \cdot \left|\tau + \frac{\sin 2\pi\tau}{2\pi}\right|_0^t - \frac{2\cos\pi t}{\pi}\left|\frac{\sin^2\pi\tau}{2}\right|_0^t$$

$$= \sin\pi t\left(t + \cancel{\sin 2\pi t}\, 0 - 0\right) - \frac{2\cos\pi t}{\pi}\cdot\frac{1}{2}(0-0)$$

$$= t\sin\pi t - 0 = t\sin\pi t$$

$h(t) = t \sin \omega t$.

21 $\dfrac{\omega}{s^2(s^2+\omega^2)} = \dfrac{1}{s^2} \cdot \dfrac{\omega}{s^2+\omega^2}$

Let, $F(s) = \dfrac{1}{s^2}$

$G(s) = \dfrac{\omega}{s^2+\omega^2}$

$f(t) = t$

$g(t) = \sin \omega t$

(Now,)

$f(t-\tau) = t-\tau$

$g(\tau) = \sin \omega \tau$

$\therefore \quad h(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$

$= \int_0^t (t-\tau) \sin \omega \tau \, d\tau$

$t-\tau$	$+$	$\sin \omega \tau$
-1	$-$	$-\dfrac{\cos \omega \tau}{\omega}$
0		$\dfrac{\sin \omega \tau}{\omega^2}$

$$= \left| (t-\tau)(-)\frac{\cos\omega\tau}{\omega} + (-1)(-)\left(\frac{\sin\omega\tau}{\omega^2}\right) \right|_0^t$$

$$= \frac{1}{\omega}\left| (-)(t-\tau)\cos\omega\tau \right|_0^t + \frac{1}{\omega^2}\left| \sin\omega\tau \right|_0^t$$

$$= -\frac{1}{\omega}\left[0 - t\right] + \frac{1}{\omega^2}\left[\sin\omega t - 0\right]$$

$$= \frac{t}{\omega} + \frac{1}{\omega^2}\sin\omega t$$

$$h(t) = \frac{\omega t - \sin\omega t}{\omega^2}$$

23 $\dfrac{40.5}{s(s^2-9)} = \dfrac{1}{s} \cdot \dfrac{40.5}{s^2-9}$

$f(t) = 1$

$g(t) = 13.5 \sinh 3t$

$g(\tau) = 13.5 \sinh 3\tau$

$$h(t) = \int_0^t (1 * 13.5 \sinh 3\tau)\, d\tau$$

$$= 13.5 \left| \frac{\cosh 3\tau}{3} \right|_0^t +$$

$h(t) = 4.5\,(\cosh 3t - 1)$

25. $\dfrac{18s}{(s^2+36)^2} = \dfrac{18}{s^2+36} \cdot \dfrac{s}{s^2+36}$

$F(s) = \dfrac{18}{s^2+36}$ & $G(s) = \dfrac{s}{s^2+36}$

$f(t) = 3\sin 6t$ & $g(t) = \cos 6t$

$f(t-\tau) = 3\sin 6(t-\tau)$ & $g(\tau) = \cos 6\tau$

$$h(t) = \int_0^t \cos 6\tau \cdot 3\sin 6(t-\tau)\, d\tau$$

$$= \int_0^t \left(\cos 6\tau \cdot 3\left[\sin 6t \cos 6\tau - \cos 6t \sin \tau\right]\right) d\tau$$

$$= \int_0^t 3\cos 6\tau\,(\sin 6t \cos 6\tau - \cos 6t \sin 6\tau)\, d\tau$$

$$= \int_0^t 3\cos^2 6\tau \sin 6t\, d\tau - \int_0^t 3\cos 6\tau \cos 6t \sin 6\tau\, d\tau$$

$$= 3\sin 6t \int_0^t \frac{1+\cos 6\tau}{2}\, d\tau - 3\cos 6t \int_0^t \sin 6\tau \cdot \cos 6\tau\, d\tau$$

$$= \frac{3}{2}\sin 6t \int_0^t (1+\cos 6\tau)\,d\tau - \frac{3\cos 6t}{6}\int_0^t \sin 6\tau \cdot \cos 6\tau \cdot (6)\,d\tau$$

$$= \frac{3}{2}\sin 6t \cdot \left| t + \frac{\sin 6\tau}{6} \right|_0^t - \frac{1\cos 6t}{2} \left| \frac{\sin^2 6\tau}{2} \right|_0^t$$

$$= \frac{3}{2}\sin 6t \left[t + \frac{\sin 6t}{6} - 0 \right] - \frac{\cos 6t}{2} \cdot \frac{1}{2}\left(\sin^2 6t - 0 \right)$$

$$= \frac{3}{2}\sin 6t + \frac{3}{12}\sin^2 6t - \frac{\cos 6t \sin^2 6t}{4}$$

8) $y(t) + 4\int_0^t y(\tau)(t-\tau)\,d\tau = 2t$

Writing in terms of convolution.

$$\cancel{y(t) + 4\int_0^t y(t)\,t\,dt}$$

$$y(t) + 4\ y(t) * t = 2t$$

Taking L.T

$$\mathcal{L}\{y(t)\} + 4\mathcal{L}\{y(t) * t\} = 2\mathcal{L}\{t\}$$

$$Y + 4\mathcal{L}\{y(t)\} * \mathcal{L}\{t\} = 2\frac{1}{s^2}$$

$$Y + 4Y \cdot \frac{1}{s^2} = \frac{2}{s^2}$$

$$Y\left(1 + \frac{4}{s^2}\right) = \frac{2}{s^2}$$

$$Y\left(\frac{s^2+4}{\cancel{s^2}}\right) = \frac{2}{\cancel{s^2}}$$

$$Y = \frac{2}{s^2+4}$$

taking I.L.T

$$y(t) = \sin 2t$$

9. $y(t) - \int_0^t y(\tau)\, \cancel{\sin(t-\tau)}\, d\tau = 1$

Writing in terms of convolution

$y(t) - y(t) * 1 = 1$

Taking L.T.

$\mathcal{L}\{y(t)\} - \mathcal{L}\{y(t)\} = \mathcal{L}\{1\}$

$Y - Y =$

$\mathcal{L}\{y(t)\} - \mathcal{L}\{y(t) * 1\} = \mathcal{L}\{1\}$

$Y - \mathcal{L}\{y(t)\}\mathcal{L}\{1\} = \frac{1}{s}$

$Y - Y \cdot \frac{1}{s} = \frac{1}{s}$

$Y\left(1 - \frac{1}{s}\right) = \frac{1}{s}$

$Y\left(\frac{s-1}{\cancel{s}}\right) = \frac{1}{\cancel{s}}$

$Y = \frac{1}{s-1}$

taking I.L.T.

$y(t) = e^{t}$

$$y(t) - \int_0^t y(T)\sin 2(t-T)\,dT = \sin 2t$$

(10) $\cancel{y(t) + \int_0^t (t-T)\,y(t)} = 1$

Writing in terms of convolution.

$$y(t) - y(t) * \sin 2t = \sin 2t$$

$$\mathcal{L}\{y(t)\} - \mathcal{L}\{y(t)\}\,\mathcal{L}\{\sin 2t\} = \mathcal{L}\{\sin 2t\}$$

$$Y - Y \cdot \frac{2}{s^2+4} = \frac{2}{s^2+4}$$

$$Y\left(1 - \frac{2}{s^2+4}\right) = \frac{2}{s^2+4}$$

$$Y\left(\frac{s^2+4-2}{\cancel{(s^2+4)}}\right) = \frac{2}{\cancel{s^2+4}}$$

$$Y = \frac{1}{\sqrt{2}} \cdot \frac{2\sqrt{2}}{s^2+(\sqrt{2})^2}$$

Taking $\mathcal{L}^{-1}$

$$y(t) = \frac{2}{\sqrt{2}}\sin\sqrt{2}\,t = \sqrt{2}\sin\sqrt{2}\,t$$

II $y(t) + \int_0^t (t-\tau)\, y(\tau)\, d\tau = 1$

Writing in terms of convolution.

$y(t) + t * y(t) = 1$

taking L.T.

$\mathcal{L}\{y(t)\} + \mathcal{L}\{y(t) * t\} = \mathcal{L}\{1\}$

$Y + Y \cdot \frac{1}{s^2} = \frac{1}{s}$

$Y\left(1 + \frac{1}{s^2}\right) = \frac{1}{s}$

$Y\left(\frac{s^2+1}{s^{\cancel{2}}}\right) = \frac{1}{\cancel{s}}$

$Y = \frac{s}{s^2+1}$

taking g. L.T

$y(t) = \cancel{\sin t}\ \cos t$

$$y(t) + \int_0^t y(\tau)\cosh(t-\tau)\,d\tau = t + e^t$$

12 $\cancel{y(t) + 2e^t \int_0^t y(\tau)\, e^{-\tau}\, d\tau = t}$

Writing in terms of convolution

$$y(t) + y(t) * \cosh(t) = t + e^t$$

taking L.T.

$$\mathcal{L}\{y(t)\} + \mathcal{L}\{y(t)\}\mathcal{L}\{\cosh(t)\} = \mathcal{L}\{t\} + \mathcal{L}\{e^t\}$$

$$Y + Y \cdot \frac{s}{s^2-1} = \frac{1}{s^2} + \frac{1}{s-1}$$

$$Y\left(1 + \frac{s}{s^2-1}\right) = \frac{s-1+s^2}{s^2(s-1)}$$

$$Y\left(\cancel{\frac{s^2-1+s}{s^2-1}}\right) = \cancel{\frac{(s^2+s-1)}{s^2(s-1)}}$$

$$Y\left(\frac{1}{s+1}\right) = \frac{1}{s^2}$$

$$Y = \frac{s+1}{s^2}$$

$$Y = \frac{s}{s^2} + \frac{1}{s^2} = \frac{1}{s} + \frac{1}{s^2}$$

taking g.L.T

$$y(t) = 1 + t$$