

Discrete Mathematics for Computer Science

Department of Computer Science

Lecturer: Nazeef Ul Haq

Reference Book: Discrete Mathematics and its applications
BY Kenneth H. Rosen – 8th edition



Honor Code Of Class

- RESPECT YOURSELF!
- Maintain silence
- Use of mobile phones are not allowed
- Cheating/Plagiarism case will be dealt strictly
- Avoid cross talking
- Avoid copy paste submission of work



Course Introduction

- What we will cover in this course?
- Proofs and logics,
- Hashing function, Pseudorandom numbers
- Check Digits – UPCs, ISBNs, Airline ticket number
- Cryptography
- Mathematical Induction, Counting Techniques, Relations, Graphs and Trees
- Reference Book: Discrete Mathematics and its applications BY Kenneth H. Rosen – 8th edition



Grading Policy

- Mid Exam 30%
 - Final Exam (Complete Syllabus) 40%
 - Quizzes (3 to 4) 10%
 - Assignments (2 to 3) 10%
 - Surprise Quiz/Class Participation 10%
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- Minimum 75% attendance is MUST.



Lecture 1

Course Overview

Chapter 1. The Foundations

1.1 Propositional Logic

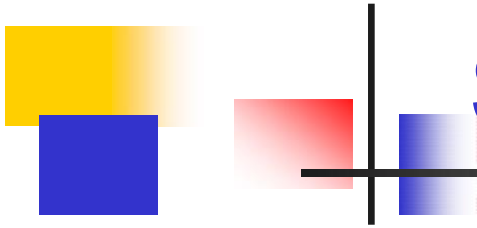


What is Mathematics, really?

- It's *not* just about numbers!
- Mathematics is *much* more than that:

Mathematics is, most generally, the study of
any and all *absolutely certain* truths about
any and all *perfectly well-defined* concepts.

- These concepts can be *about* numbers, symbols, objects, images, sounds, *anything*!
- It is a way to interpret the world around you.



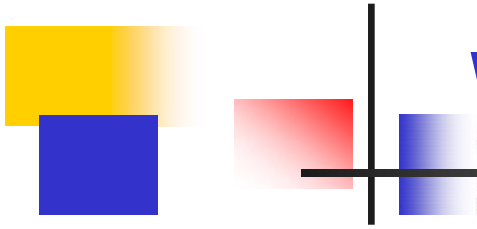
So, what's *this* class about?

What are “discrete structures” anyway?

- “**Discrete**” - Composed of distinct, separable parts. (Opposite of *continuous*.)

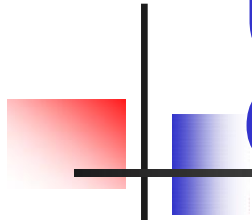
discrete:continuous :: digital:analog

- “**Discrete Mathematics**” - concerns processes that consist of a sequence of individual steps.



Why Study Discrete Math?

- The basis of all of digital information processing is: Discrete manipulations of discrete structures represented in memory.
- It's the basic language and conceptual foundation for all of computer science.
- Discrete math concepts are also widely used throughout math, science, engineering, economics, biology, *etc.*, ...
- A generally useful tool for rational thought!



Uses for Discrete Math in Computer Science

- Advanced algorithms & data structures
- Programming language compilers & interpreters
- Computer networks
- Operating systems
- Computer architecture
- Database management systems
- Cryptography
- Error correction codes
- Graphics & animation algorithms, game engines, *etc....*
- *i.e., the whole field!*



1.1 Propositional Logic

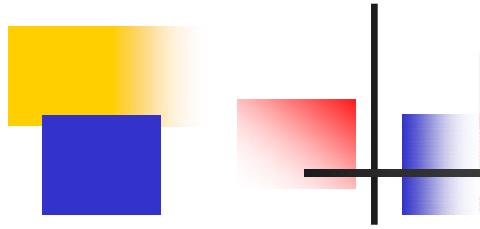
- Logic -- Logic is the study of the principles and methods that distinguishes between a valid and an invalid argument.
 - Focuses on the relationship among statements, not on the content of any particular statement.
 - Gives precise meaning to mathematical statements.
- ***Propositional Logic*** is the logic that deals with statements (propositions) and compound statements built from simpler statements using so-called *Boolean connectives*.
- Some applications in computer science:
 - Design of digital electronic circuits.
 - Expressing conditions in programs.
 - Queries to databases & search engines.



Definition of a *Proposition*

Definition: A *proposition* (denoted p, q, r, \dots) is simply:

- a *statement* (i.e., a declarative sentence)
 - with some definite meaning,
(not vague or ambiguous)
- having a *truth value* that's either *true* (**T**) or *false* (**F**)
 - it is **never** both, neither, or somewhere “in between!”
 - However, you might not *know* the actual truth value,
 - and, the truth value might *depend* on the situation or context.
- Later, we will study *probability theory*, in which we assign *degrees of certainty* (“between” **T** and **F**) to propositions.
 - But for now: think True/False only! (or in terms of **1** and **0**)



Examples of Propositions

- It is raining. (In a given situation)
- Beijing is the capital of China. (T)
- $2 + 2 = 5$. (F)
- $1 + 2 = 3$. (T)
- A fact-based declaration is a proposition, even if no one knows whether it is true
 - 11213 is prime.
 - There exists an odd perfect number.



Proposition

- Rule -- If the sentence is preceded by other sentences that make the pronoun or variable reference clear, then the sentence is a statement.

Example:

$x = 1$ and $x > 2$

$x > 2$ is a statement with truth-value FALSE.

COMPOUND STATEMENT:

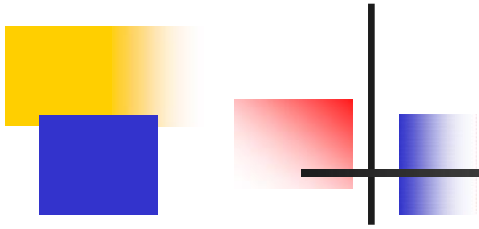
Simple statements could be used to build a compound statement.

EXAMPLES:

LOGICAL CONNECTIVES

1. “ $3 + 2 = 5$ ” **and** “Lahore is a city in Pakistan”
2. “The grass is green” or “It is hot today”
3. “Discrete Mathematics is **not** difficult to me”

AND, OR, NOT are called LOGICAL CONNECTIVES.



Examples of Non-Propositions

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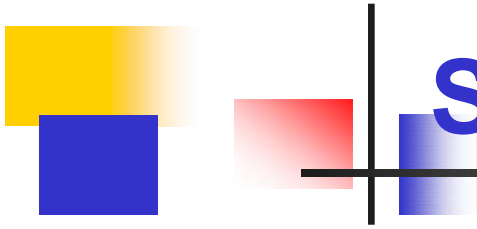
The following are **NOT** propositions:

- Who's there? (interrogative, question)
- Just do it! (imperative, command)
- La la la la la. (meaningless interjection)
- Yeah, I sorta dunno, whatever... (vague)
- $1 + 2$ (expression with a non-true/false value)
- $x + 2 = 5$ (declaration about semantic tokens of non-constant value)



Truth Tables

- An *operator* or *connective* combines one or more *operand* expressions into a larger expression. (e.g., “+” in numeric expressions.)
- **Unary** operators take *one* operand (e.g., -3);
Binary operators take *two* operands (e.g. 3×4).
- **Propositional** or **Boolean operators** operate on propositions (or their truth values) instead of on numbers.
- The **Boolean domain** is the set $\{T, F\}$. Either of its elements is called a **Boolean value**.
An n -tuple (p_1, \dots, p_n) of Boolean values is called a **Boolean n -tuple**.
- An n -operand truth table is a table that assigns a Boolean value to the set of all Boolean n -tuples.



Some Popular Boolean Operators

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<u>Formal Name</u>	<u>Nickname</u>	<u>Arity</u>	<u>Symbol</u>
Negation operator	NOT	Unary	\neg
Conjunction operator	AND	Binary	\wedge
Disjunction operator	OR	Binary	\vee
Exclusive-OR operator	XOR	Binary	\oplus
Implication operator	IMPLIES	Binary	\rightarrow
Biconditional operator	IFF	Binary	\leftrightarrow



The Negation Operator

- The unary ***negation operator*** “ \neg ” (*NOT*) transforms a proposition into its logical *negation*.
- *E.g.* If $p =$ “I have brown hair.”
then $\neg p =$ “It is not the case that I have brown hair” or “I do **not** have brown hair.”
- The *truth table* for NOT:

p	$\neg p$
T	F
F	T

Operand
column

Result
column



The Conjunction Operator

- The binary ***conjunction operator*** “ \wedge ” (*AND*) combines two propositions to form their logical *conjunction*.

- *E.g.* If p = “I will have salad for lunch.” and q = “I will have steak for dinner.”

then, $p \wedge q$ = “I will have salad for lunch **and** I will have steak for dinner.”



Conjunction Truth Table

Operand columns

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- Note that a conjunction $p_1 \wedge p_2 \wedge \dots \wedge p_n$ of n propositions will have 2^n rows in its truth table



The Disjunction Operator

- The binary ***disjunction operator*** “ \vee ” (*OR*) combines two propositions to form their logical *disjunction*.
- *E.g.* If p = “My car has a bad engine.” and q = “My car has a bad carburetor.”
then, $p \vee q$ = “My car has a bad engine, **or** my car has a bad carburetor.”

Meaning is like “and/or” in informal English.

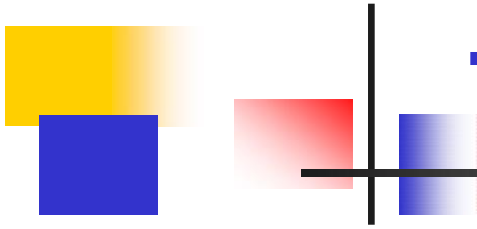


Disjunction Truth Table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Note difference
from AND

- Note that $p \vee q$ means that p is true, or q is true, **or both** are true!
- So, this operation is also called ***inclusive or***, because it **includes** the possibility that both p and q are true.



The Exclusive-Or Operator

- The binary ***exclusive-or operator*** “ \oplus ” (*XOR*) combines two propositions to form their logical “exclusive or”
- *E.g.* If $p =$ “I will earn an A in this course.” and $q =$ “I will drop this course.”, then
$$p \oplus q = \text{“I will **either** earn an A in this course, **or** I will drop it (**but not both!**)”}$$



Exclusive-Or Truth Table

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Note difference
from OR.

- Note that $p \oplus q$ means that p is true, or q is true, but **not both**!
- This operation is called ***exclusive or***, because it **excludes** the possibility that both p and q are true.



Natural Language is Ambiguous

- Note that the English “or” can be ambiguous regarding the “both” case!

- “Pat is a singer or Pat is a writer.” - ✓

- “Pat is a man or Pat is a woman.” - ⊕

p	q	p "or" q
T	T	?
T	F	T
F	T	T
F	F	F

- Need context to disambiguate the meaning!
- For this class, assume “or” means inclusive (✓).



The Implication Operator

- The conditional statement (aka ***implication***) $p \rightarrow q$ states that p implies q .
- */e.*, If p is true, then q is true; but if p is not true, then q could be either true or false.
- *E.g.*, let p = “You study hard.”
 q = “You will get a good grade.”
 $p \rightarrow q$ = “If you study hard, then you will get a good grade.” (else, it could go either way)
 - p : *hypothesis* or *antecedent* or *premise*
 - q : *conclusion* or *consequence*



Implication Truth Table

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The only
False case!

- $p \rightarrow q$ is **false** only when p is true but q is **not** true.
- $p \rightarrow q$ does **not** require that p or q are ever true!
- E.g. “ $(1=0) \rightarrow$ pigs can fly” is TRUE!



Examples of Implications

- “If this lecture ever ends, then the sun will rise tomorrow.” *True or False?* ($T \rightarrow T$)
- “If $1+1=6$, then Joe Biden is president.” *True or False?* ($F \rightarrow T$)
- “If the moon is made of green cheese, then I am richer than Bill Gates.” *True or False?* ($F \rightarrow F$)
- “If Tuesday is a day of the week, then I am a penguin.” *True or False?* ($T \rightarrow F$)



English Phrases Meaning $p \rightarrow q$

- “ p implies q ”
- “if p , then q ”
- “if p , q ”
- “when p , q ”
- “whenever p , q ”
- “ q if p ”
- “ q when p ”
- “ q whenever p ”
- “ p only if q ”
- “ p is sufficient for q ”
- “ q is necessary for p ”
- “ q follows from p ”
- “ q is implied by p ”

We will see some
equivalent logic
expressions later.



Converse, Inverse, Contrapositive

- Some terminology, for an implication $p \rightarrow q$:
- Its **converse** is: $q \rightarrow p$.
- Its **inverse** is: $\neg p \rightarrow \neg q$.
- Its **contrapositive**: $\neg q \rightarrow \neg p$.

p	q	$p \rightarrow q$	$q \rightarrow p$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

- One of these three has the *same meaning* (same truth table) as $p \rightarrow q$. Can you figure out which?

Contrapositive



Examples

- p : Today is Easter
 q : Tomorrow is Monday
- $p \rightarrow q$:
If today is Easter then tomorrow is Monday.
- **Converse:** $q \rightarrow p$
If tomorrow is Monday then today is Easter.
- **Inverse:** $\neg p \rightarrow \neg q$
If today is not Easter then tomorrow is not Monday.
- **Contrapositive:** $\neg q \rightarrow \neg p$
If tomorrow is not Monday then today is not Easter.



The Biconditional Operator

- The ***biconditional*** statement $p \leftrightarrow q$ states that p ***if and only if*** (*iff*) q .
- $p =$ “It is below freezing.”
 $q =$ “It is snowing.”
 $p \leftrightarrow q =$ “It is below freezing if and only if it is snowing.”

or

$=$ “That it is below freezing is
necessary and sufficient for it to be
snowing”

Biconditional Truth Table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Both same \Rightarrow true

- p is necessary and sufficient for q
- If p then q , and conversely
- p iff q

- $p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$. ✓
- $p \leftrightarrow q$ means that p and q have the **same** truth value.
- $p \leftrightarrow q$ does **not** imply that p and q are true.
- Note this truth table is the exact **opposite** of \oplus 's! Thus, $p \leftrightarrow q$ means $\neg(p \oplus q)$.



Boolean Operations Summary

- Conjunction: $p \wedge q$, (read p and q), “discrete math is a required course **and** I am a computer science major”.
- Disjunction: $p \vee q$, (read p or q), “discrete math is a required course **or** I am a computer science major”.
- Exclusive or: $p \oplus q$, “discrete math is a required course **or** I am a computer science major **but not both**”.
- Implication: $p \rightarrow q$, “**if** discrete math is a required course **then** I am a computer science major”.
- Biconditional: $p \leftrightarrow q$, “discrete math is a required course **if and only if** I am a computer science major”.

Boolean Operations Summary

- We have seen 1 unary operator and 5 binary operators. What are they? Their truth tables are below.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	F	T	T
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	F
F	F	T	F	F	F	T	T

- For an implication $p \rightarrow q$
- Its **converse** is: $q \rightarrow p$
- Its **inverse** is: $\neg p \rightarrow \neg q$
- Its **contrapositive**: $\neg q \rightarrow \neg p$



Compound Propositions

- A **propositional variable** is a variable such as p, q, r (possibly subscripted, e.g. p_i) over the Boolean domain.
- An **atomic proposition** is either Boolean constant or a propositional variable: e.g. T, F, p
- A **compound proposition** is derived from atomic propositions by application of propositional operators: e.g. $\neg p, p \vee q, (p \vee \neg q) \rightarrow q$
- Precedence of logical operators: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
- Precedence also can be indicated by parentheses.
 - e.g. $\neg p \wedge q$ means $(\neg p) \wedge q$, not $\neg(p \wedge q)$



An Exercise

- Any compound proposition can be evaluated by a truth table
- $(p \vee \neg q) \rightarrow q$

p	q	$\neg q$	$p \vee \neg q$	$(p \vee \neg q) \rightarrow q$
T	T	F	T	T
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F



Translating English Sentence

- Let p = “It rained last night”,
 q = “The sprinklers came on last night,”
 r = “The lawn was wet this morning.”

Translate each of the following into English:

$\neg p$ = “It didn’t rain last night.”

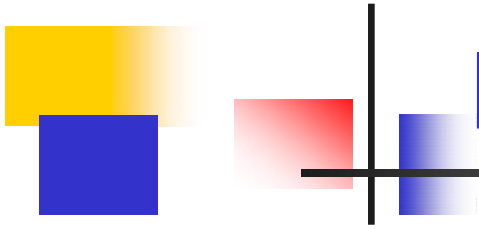
$r \wedge \neg p$ = “The lawn was wet this morning,
and it didn’t rain last night.”

$\neg r \vee p \vee q$ = “The lawn wasn’t wet this
morning, or it rained last night, or
the sprinklers came on last night.”



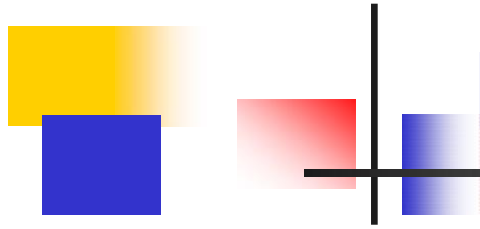
Another Example

- Find the converse of the following statement.
 - “Raining tomorrow is a sufficient condition for my not going to town.”
- **Step 1:** Assign propositional variables to component propositions.
 - p : It will rain tomorrow
 - q : I will not go to town
- **Step 2:** Symbolize the assertion: $p \rightarrow q$
- **Step 3:** Symbolize the converse: $q \rightarrow p$
- **Step 4:** Convert the symbols back into words.
 - “If I don’t go to town then it will rain tomorrow” or
 - “Raining tomorrow is a *necessary condition* for my not going to town.”



Logic and Bit Operations

- A **bit** is a **b**inary (base 2) dig**it**: 0 or 1.
- Bits may be used to represent truth values.
 - By convention:
0 represents “False”; 1 represents “True”.
- A **bit string of length n** is an ordered sequence of $n \geq 0$ bits.
- By convention, bit strings are (sometimes) written left to right:
 - e.g. the “first” bit of the bit string “1001101010” is 1.
 - What is the length of the above bit string?



Bitwise Operations

- Boolean operations can be extended to operate on bit strings as well as single bits.

- Example:

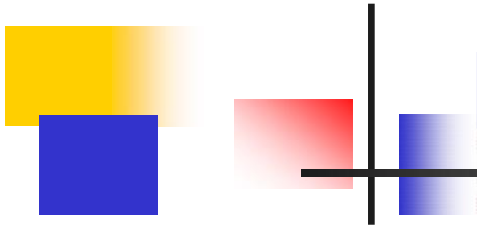
01 1011 0110

11 0001 1101

11 1011 1111 Bit-wise OR

01 0001 0100 Bit-wise AND

10 1010 1011 Bit-wise XOR



End of 1.1

You have learned about:

- Propositions: what they are
- Propositional logic operators'
 - symbolic notations, truth tables, English equivalents, logical meaning
- Atomic vs. compound propositions
- Bits, bit strings, and bit operations
- Next section:
 - Propositional equivalences
 - Equivalence laws
 - Proving propositional equivalences