Discrete Mathematics for Computer Science

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Reference Book: Discrete Mathematics and its applications BY Kenneth H. Rosen – 8th edition

1.5 Rules of Inference

- ***An argument***: is a list of statements called **premises** (or **assumptions** or **hypotheses**) followed by a statement called the **conclusion.**
- Some forms of argument ("valid") never lead from correct statements to an incorrect conclusion. Some other forms of argument ("fallacies") can lead from true statements to an incorrect conclusion.
- ***A logical argument*** consists of a list of (possibly compound) propositions called premises/hypotheses and a single proposition called the conclusion.
- ***Logical rules of inference***: methods that depend on logic alone for deriving a new statement from a set of other statements. (Templates for constructing valid arguments.)

Valid Arguments (I)

- Example: A logical argument

  If I dance all night, then I get tired.

  I danced all night.

  Therefore I got tired.

- Logical representation of underlying variables:

  p: I dance all night. q: I get tired.

- Logical analysis of argument:

$$\begin{array}{ll} p \rightarrow q & \text{premise 1} \\ \underline{p} & \underline{\text{premise 2}} \\ \therefore q & \text{conclusion} \end{array}$$

Valid Arguments (II)

- A form of logical argument is ***valid*** if whenever every premise is true, the conclusion is also true. A form of argument that is not valid is called a ***fallacy***.

Inference Rules: General Form

- An *Inference Rule* is
  - A pattern establishing that if we know that a set of *premise* statements of certain forms are all true, then we can validly deduce that a certain related *conclusion* statement is true.

$$\begin{array}{l} \textit{premise 1} \\ \textit{premise 2} \\ \ldots \\ \hline \therefore \textit{ conclusion} \end{array}$$

"$\therefore$" means "therefore"

Inference Rules & Implication

- Each valid logical inference rule corresponds to an implication that is a tautology.

$$\begin{array}{l} \quad \textit{premise 1} \\ \quad \textit{premise 2} \\ \quad \ldots \\ \hline \therefore \textit{ conclusion} \end{array}$$

Inference rule

- Corresponding tautology:

$$((\textit{premise 1}) \wedge (\textit{premise 2}) \wedge \square\,) \rightarrow \text{conclusion}$$

Modus Ponens

- $$\begin{array}{l} \quad p \\ \quad p \to q \\ \hline \therefore\ q \end{array}$$ Rule of ***Modus ponens*** (a.k.a. *law of detachment*)

  "the mode of affirming"

- $(p \land (p \to q)) \to q$ is a tautology

p	q	$p \to q$	$p \land (p \to q)$	$(p \land (p \to q)) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

- Notice that the first row is the only one where premises are all true

Modus Ponens: Example

$$\text{If} \left\{ \begin{array}{ll} p \to q & : \text{“If it snows today then we will go skiing”} \\ p & : \text{“It is snowing today”} \end{array} \right\} \text{ assumed TRUE}$$

$$\text{Then } \therefore q \quad : \text{“We will go skiing”} \quad \text{is TRUE}$$

$$\text{If} \left\{ \begin{array}{ll} p \to q & : \text{“If } n \text{ is divisible by 3 then } n^2 \text{ is divisible by 3”} \\ p & : \text{“}n \text{ is divisible by 3”} \end{array} \right\} \text{ assumed TRUE}$$

$$\text{Then } \therefore q \quad : \text{“}n^2 \text{ is divisible by 3”} \quad \text{is TRUE}$$

Modus Tollens

- $$\begin{array}{l} \neg q \\ \underline{p \to q} \\ \therefore \neg p \end{array}$$ Rule of ***Modus tollens***

"the mode of denying"

- $(\neg q \wedge (p \to q)) \to \neg p$ is a tautology

- Example

If $\left\{\begin{array}{l} p \to q : \text{“If this jewel is really a diamond then it will scratch glass”} \\ \neg q : \text{“The jewel doesn't scratch glass”} \end{array}\right\}$ assumed TRUE

Then $\therefore \neg p$: "The jewel is not a diamond" is TRUE

More Inference Rules

- $$\frac{p}{\therefore\ p \vee q}$$

  Rule of ***Addition***

  Tautology: $p \rightarrow (p \vee q)$

- $$\frac{p \wedge q}{\therefore\ p}$$

  Rule of ***Simplification***

  Tautology: $(p \wedge q) \rightarrow p$

- $$\frac{\begin{array}{c} p \\ q \end{array}}{\therefore\ p \wedge q}$$

  Rule of ***Conjunction***

  Tautology: $[(p) \wedge (q)] \rightarrow p \wedge q$

Examples

- State which rule of inference is the basis of the following arguments:
  - It is below freezing now. Therefore, it is either below freezing or raining now.
  - It is below freezing and raining now. Therefore, it is below freezing now.
- p: It is below freezing now.
  q: It is raining now.
  - $p \to (p \lor q)$ (rule of addition)
  - $(p \land q) \to p$ (rule of simplification)

Hypothetical Syllogism

- $$\boxed{\begin{array}{l} \quad p \to q \\ \quad q \to r \\ \hline \therefore p \to r \end{array}}$$

  Rule of ***Hypothetical syllogism***

  Tautology:

  $$[(p \to q) \wedge (q \to r)] \to (p \to r)$$

- Example: State the rule of inference used in the argument:

  "If $\underbrace{\text{it rains today}}_{p}$, then $\underbrace{\text{we will not have a barbecue today}}_{q}$. If $\underbrace{\text{we do not have a barbecue today}}_{q}$, then $\underbrace{\text{we will have a barbecue tomorrow}}_{r}$. Therefore, if $\underbrace{\text{it rains today}}_{p}$, then $\underbrace{\text{we will have a barbecue tomorrow}}_{r}$."

Disjunctive Syllogism

- $$\begin{array}{l} \quad p \lor q \\ \quad \neg p \\ \hline \therefore q \end{array}$$

Rule of ***Disjunctive syllogism***

Tautology: $[(p \lor q) \land (\neg p)] \rightarrow q$

- Example
  - Ed's wallet is in his back pocket or it is on his desk. $(p \lor q)$

    $\underbrace{\text{Ed's wallet is in his back pocket}}_{p}$ or $\underbrace{\text{it is on his desk}}_{q}$
  - Ed's wallet is not in his back pocket. $(\neg p)$
  - Therefore, Ed's wallet is on his desk. (q)

Resolution

- $$\boxed{\begin{array}{r} p \vee q \\ \neg p \vee r \\ \hline \therefore\; q \vee r \end{array}}$$

  Rule of ***Resolution***

  Tautology:

  $$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$$

- When $q = r$:

  $$[(p \vee q) \wedge (\neg p \vee q)] \rightarrow q$$

- When $r = \mathbf{F}$:

  $[(p \vee q) \wedge (\neg p)] \rightarrow q$ (Disjunctive syllogism)

Resolution: Example

$$\begin{array}{r} p \vee q \\ \neg p \vee r \\ \hline \therefore\ q \vee r \end{array}$$

- Example: Use resolution to show that the hypotheses "$\underbrace{\textit{Jasmine is skiing}}_{r}$ *or* $\underbrace{\textit{it is not snowing}}_{\neg p}$" and "$\underbrace{\textit{It is snowing}}_{p}$ *or* $\underbrace{\textit{Bart is playing hockey}}_{q}$" imply that "$\underbrace{\textit{Jasmine is skiing}}_{r}$ *or* $\underbrace{\textit{Bart is playing hockey}}_{q}$"

$$(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$$

Rules of Inference-- Summary

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$	Resolution

Formal Proofs

- A formal proof of a conclusion *C*, given premises $p_1, p_2, \ldots, p_n$ consists of a sequence of *steps*, each of which applies some inference rule to premises or previously-proven statements to yield a new true statement (the *conclusion*).
- A proof demonstrates that *if* the premises are true, *then* the conclusion is true.

Formal Proof Example

- Suppose we have the following premises:
  "It is not sunny and it is cold."
  "We will swim only if it is sunny."
  "If we do not swim, then we will canoe."
  "If we canoe, then we will be home by sunset."

- Given these premises, prove the conclusion **"We will be home by sunset"** using inference rules.

Proof Example *cont.*

- Step 1: Identify the propositions (Let us adopt the following abbreviations)
  - *sunny* = **"It is sunny"**; *cold* = **"It is cold"**; *swim* = **"We will swim"**; *canoe* = **"We will canoe"**; *sunset* = **"We will be home by sunset"**.
- Step 2: Identify the argument. (Build the argument form)

$$\neg sunny \land cold$$
$$swim \to sunny$$
$$\neg swim \to canoe$$
$$canoe \to sunset$$
$$\overline{\therefore\ sunset}$$

It is not sunny and it is cold.

We will swim only if it is sunny.

If we do not swim, then we will canoe.

If we canoe, then we will be home by sunset.

We will be home by sunset.

Proof Example *cont.*

- Step 3: Verify the reasoning using the rules of inference

$$\begin{array}{l} \neg sunny \wedge cold \\ swim \rightarrow sunny \\ \neg swim \rightarrow canoe \\ canoe \rightarrow sunset \\ \hline \therefore\ sunset \end{array}$$

Step	Proved by
1. $\neg sunny \wedge cold$	Premise #1.
2. $\neg sunny$	Simplification of 1.
3. $swim \rightarrow sunny$	Premise #2.
4. $\neg swim$	Modus tollens on 2 and 3.
5. $\neg swim \rightarrow canoe$	Premise #3.
6. $canoe$	Modus ponens on 4 and 5.
7. $canoe \rightarrow sunset$	Premise #4.
8. $sunset$	Modus ponens on 6 and 7.

Common Fallacies

- A ***fallacy*** is an inference rule or other proof method that is not logically valid.
  - A fallacy may yield a false conclusion!
- *Fallacy of affirming the conclusion*:
  - "$p \rightarrow q$ is true, and q is true, so p must be true." (No, because **F** $\rightarrow$ **T** is true.)
- Example
  - If David Cameron (DC) is president of the US, then he is at least 40 years old. ($p \rightarrow q$)
  - DC is at least 40 years old. (q)
  - Therefore, DC is president of the US. (p)

Common Fallacies (*cont'd*)

- *Fallacy of denying the hypothesis*:
  - "$p \to q$ is true, and p is false, so q must be false." (No, again because $\mathbf{F} \to \mathbf{T}$ is true.)
- Example
  - If a person does arithmetic well then his/her checkbook will balance. ($p \to q$)
  - I cannot do arithmetic well. ($\neg p$)
  - Therefore my checkbook does not balance. ($\neg q$)

Inference Rules for Quantifiers

- $$\frac{\forall x\, P(x)}{\therefore P(c)}$$ **Universal instantiation**
  (substitute *any* specific member c in the domain)

- $$\frac{P(c)}{\therefore \forall x\, P(x)}$$ (for an arbitrary element c of the domain)
  Universal generalization

- $$\frac{\exists x\, P(x)}{\therefore P(c)}$$ **Existential instantiation**
  (substitute an element c for which $P(c)$ is true)

- $$\frac{P(c)}{\therefore \exists x\, P(x)}$$ (for some element c in the domain)
  Existential generalization

Example

- Every man has two legs. John Smith is a man.
  Therefore, John Smith has two legs.
- Proof
  - Define the predicates:
    - $M(x)$: x is a man
    - $L(x)$: x has two legs
    - J: John Smith, a member of the universe
  - The argument becomes

    1. $\forall x\,[M(x) \rightarrow L(x)]$

    2. $M(J)$

    $\therefore\ L(J)$

Example *cont.*

$\forall x\,(M(x) \longrightarrow L(x))$
$M(J)$
$\therefore L(J)$

- The proof is

1. $\forall x\,[M(x) \rightarrow L(x)]$ — *Premise 1*
2. $M(J) \rightarrow L(J)$ — *U. I. from (1)*
3. $M(J)$ — *Premise 2*
4. $L(J)$ — *Modus Ponens from (2) and (3)*

- Note: Using the rules of inference requires lots of practice.
  - Try example problems in the textbook.

Another example

- Correct or incorrect: "At least one of the 20 students in the class is intelligent. John is a student of this class. Therefore, John is intelligent."
- First: Separate premises from conclusion
  - *Premises*:
    1. At least one of the 20 students in the class is intelligent.
    2. John is a student of this class.
  - *Conclusion*: John is intelligent.

Answer

- Next, translate the example in logic notation.
  - <u>*Premise 1*</u>: At least one of the 20 students in the class is intelligent.

    Let the domain = all people

    $C(x)$ = "x is in the class"

    $I(x)$ = "x is intelligent"

    Then *Premise 1* says: $\boldsymbol{\exists x(C(x) \land I(x))}$

  - <u>*Premise 2*</u>: John is a student of this class.

    Then *Premise 2* says: **C(John)**

  - And the *Conclusion* says: ***I*****(John)**

$$\begin{array}{l} \exists x\,(C(x) \land I(x)) \\ C(John) \\ \hline \therefore\ I(John) \end{array}$$

Answer (*cont'd*)

$$\begin{array}{l} \exists x\,(C(x) \wedge I(x)) \\ \underline{C(John)\qquad\qquad} \\ \therefore\; I(John) \end{array}$$

- No, the argument is invalid; we can disprove it with a counter-example, as follows:
- Consider a case where there is only one intelligent student A in the class, and A $\neq$ John.
  - Then by existential instantiation of the premise **$\exists x\,(C(x) \wedge I(x))$, C(A) $\wedge$ I(A)** is true,
  - But the conclusion ***I*(John)** is false, since A is the only intelligent student in the class, and John $\neq$ A.
- Therefore, the premises *do not* imply the conclusion.

More Proof Examples

- Is this argument correct or incorrect?
  - "All TAs compose easy quizzes.
    Mike is a TA.
    Therefore, Mike composes easy quizzes."
- First, separate the premises from conclusion:
  - *Premise 1*: All TAs compose easy quizzes.
  - *Premise 2*: Mike is a TA.
  - *Conclusion*: Mike composes easy quizzes.

Answer

- Next, re-render the example in logic notation.
  - *Premise 1*: All TAs compose easy quizzes.
    - Let the domain = all people
    - Let $T(x)$ = "x is a TA"
    - Let $E(x)$ = "x composes easy quizzes"
    - Then *Premise 1* says: $\forall x(T(x) \rightarrow E(x))$
  - *Premise 2*: Mike is a TA.
    - Let M = Mike
    - Then *Premise 2* says: T(M)
  - And the *Conclusion* says: E(M)

$$\begin{array}{l} \forall x\,(T(x) \longrightarrow E(x)) \\ T(M) \\ \hline \therefore\ E(M) \end{array}$$

The Proof in Gory Detail

- The argument is correct, because it can be reduced to a sequence of applications of valid inference rules, as follows:

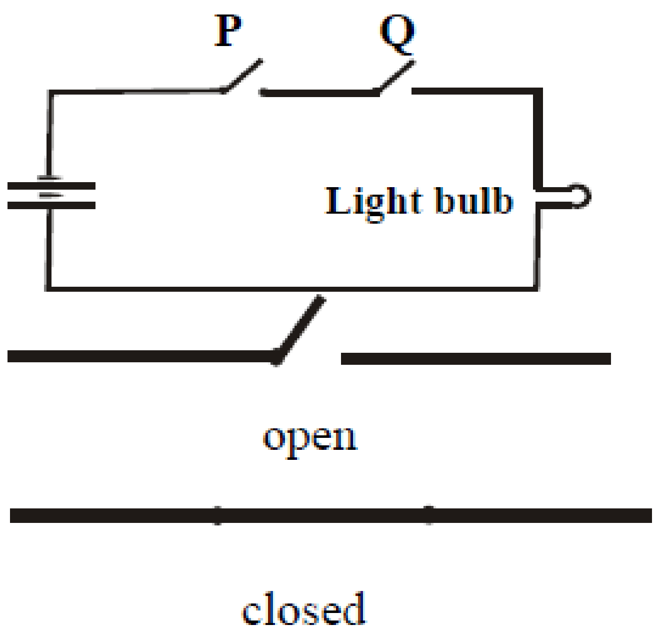
$$\begin{array}{l} \forall x\,(T(x) \longrightarrow E(x)) \\ \underline{T(M)\qquad\qquad\qquad} \\ \therefore\ E(M) \end{array}$$

- <u>Statement</u> <u>How obtained</u>

1. $\forall x(T(x) \rightarrow E(x))$ **(Premise #1)**
2. $T(\mathrm{M}) \rightarrow E(\mathrm{M})$ **(Universal Instantiation)**
3. $T(\mathrm{M})$ **(Premise #2)**
4. $E(\mathrm{M})$ **(*Modus Ponens* from #2 and #3)**

Applications of Logic

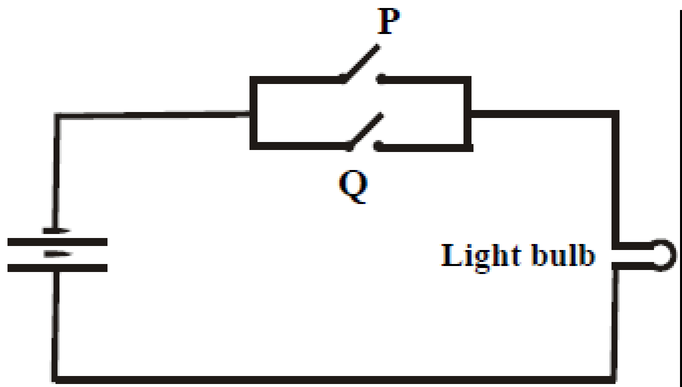
SWITCHES IN SERIES



Switches		Light Bulb
P	**Q**	**State**
Closed	Closed	On
Closed	Open	Off
Open	Closed	Off
Open	Open	Off

Applications of Logic

SWITCHES IN PARALLEL:



Switches		Light Bulb
P	**Q**	**State**
Closed	Closed	On
Closed	Open	On
Open	Closed	On
Open	Open	Off

SWITCHES IN SERIES:

Switches		Light Bulb
P	**Q**	**State**
Closed	Closed	On
Closed	Open	Off
Open	Closed	Off
Open	Open	Off

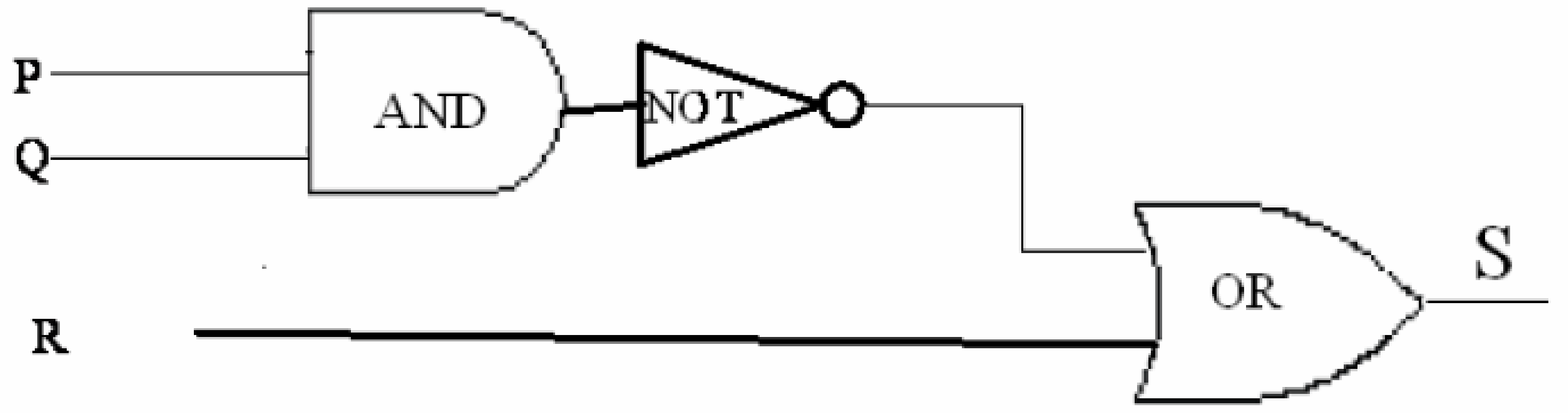
$\Leftrightarrow$

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

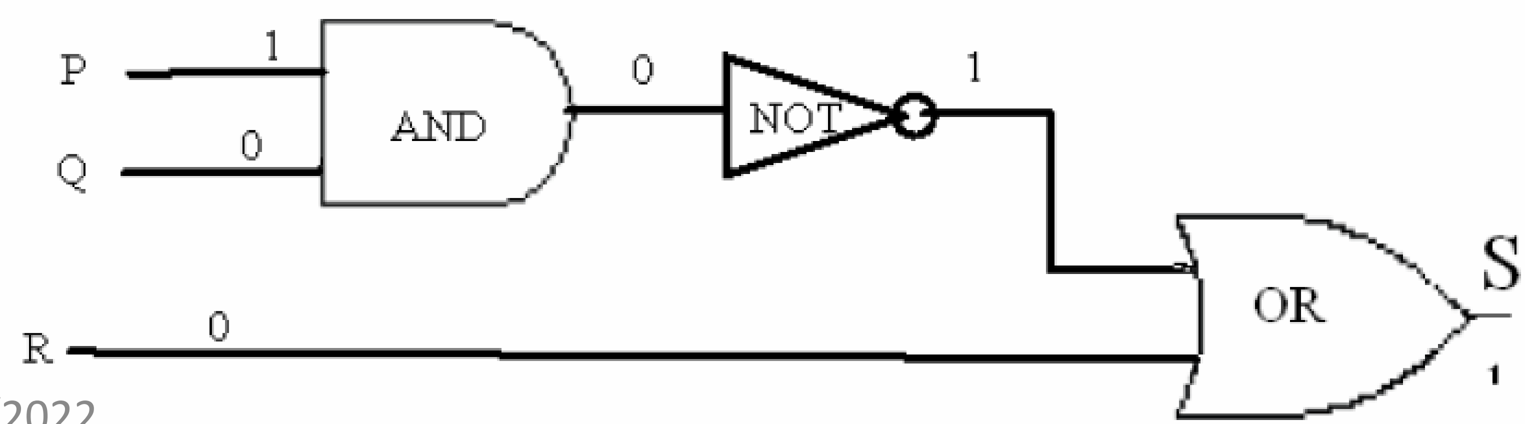
Applications of Logic

DETERMINING OUTPUT FOR A GIVEN INPUT:

Indicate the output of the circuit below when the input signals are P = 1, Q = 0 and R = 0

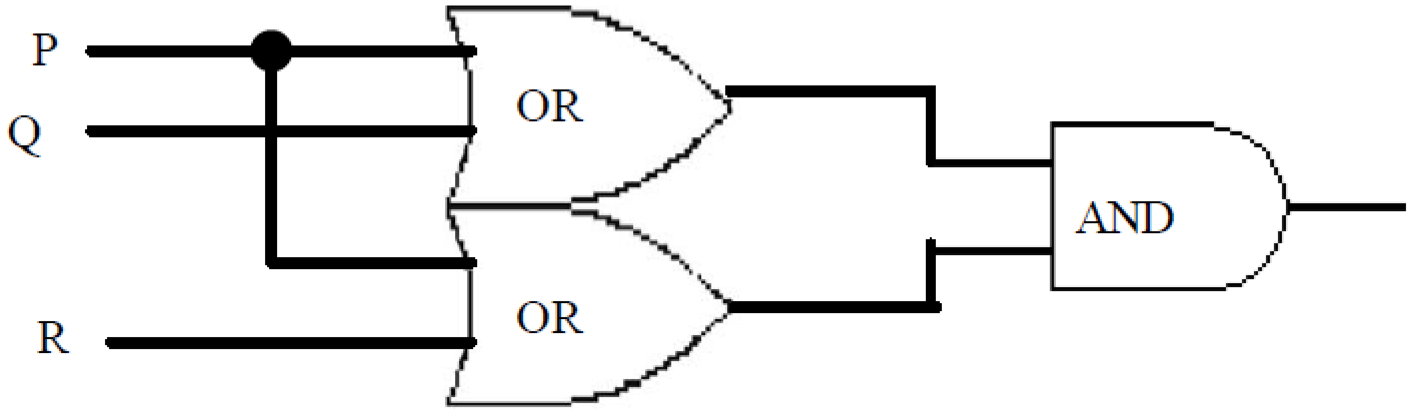


SOLUTION:



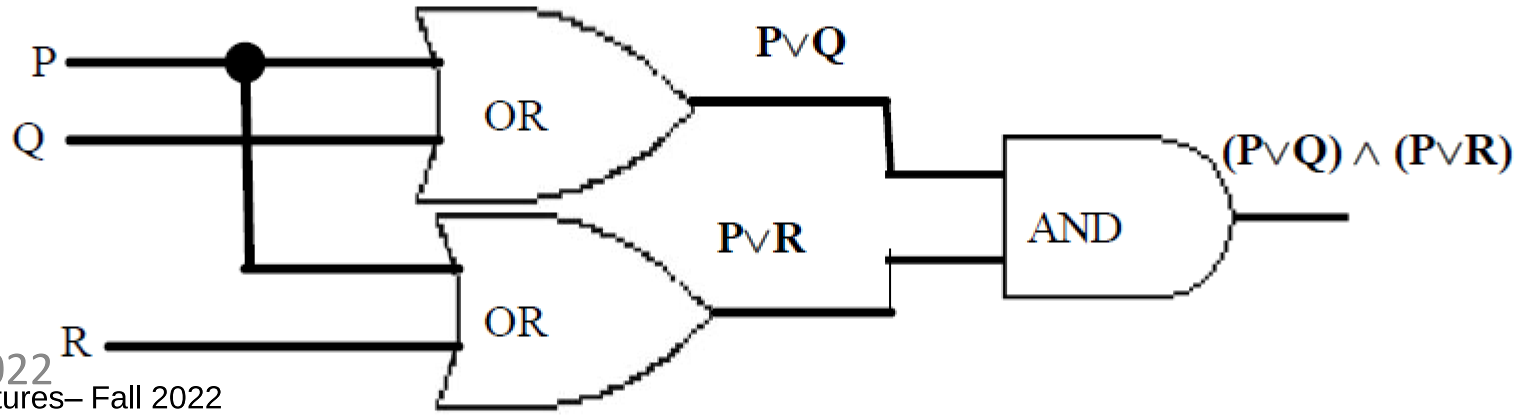
Applications of Logic

FINDING A BOOLEAN EXPRESSION FOR A CIRCUIT



SOLUTION:

Trace through the circuit from left to right, writing down the output of each logic gate.



Applications of Logic

- Construct circuit for the Boolean expression (P∧Q) ∨ ~R