System of Linear Equations

One of the most frequently recurring practical problems in many fields of study such as mathematics, physics and engineering is that of solving a system of linear equations.

Linear Equations A linear equation in the variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b \longrightarrow (1)$$

where b and the coefficients $a_1, a_2, \ldots a_n$ are real or complex numbers, usually known in advance.

Examples

$$4x_1 - 5x_2 + 2 = x_1 \qquad \text{(linear)}$$

$$x_2 = 2(\sqrt{6} - x_1) + x_3 \qquad \text{(linear)}$$

$$4x_1 - 5x_2 = x_1 x_2 \qquad \text{(non linear)}$$

$$x_2 = 2\sqrt{x_1} - 6 \qquad \text{(non-linear)}$$

$$x_1 + 3x_2^2 = 4 \qquad \text{(non-linear)}$$

In the special case where $b = 0$, (1) has the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

which is called a homogeneous linear equation in the variables $x_1, x_2, \ldots, x_n$.

$$2x_1 + 3x_2 = 0$$

$$-6x_1 + 6x_2 + x_3 = 0$$

A Linear System (A System of Linear Equations)

A system of linear equations is a collection of one or more linear equations involving the same variables, say $x_1, x_2, --- x_n$. More generally, a system of m linear equations in n unknowns, $x_1, x_2, --, x_n$ is a set of m linear equations in n unknowns. A linear system can conveniently be written as

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \quad \longrightarrow ② \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad & \quad \vdots \\
a_{m1} + a_{m2} + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

Solution of Linear System A solution of the system ② is a list $(s_1, s_2, --, s_n)$ of numbers which has the property that each equation in system ② is satisfied when $x_1 = s_1$, $x_2 = s_2$, $--$ $x_n = s_n$ are substituted.

A system of linear equations has

1) no solution
2) exactly one solution
3) infinitely many solutions.

A system of linear equations is said to be consistent if it has either one solution or infinitely many solutions. A system is inconsistent if it has no solution.

<u>Homogeneous System</u> If $b_1 = b_2 = \cdots = b_m = 0$, then the system ② is called a homogeneous system.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad\quad \vdots \qquad\qquad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned} \longrightarrow ③$$

<u>Trivial Solution:</u> The homogeneous system ③ always has the solution $x_1 = x_2 = \cdots = x_n = 0$, it is called the trivial solution.

<u>Nontrivial Solution:</u> A solution to a homogeneous system ③ in which not all of $x_1, x_2, \ldots, x_n$ are zero is called a nontrivial solution.

<u>Equivalent Systems</u> Consider a system of r linear equations in n unknowns

$$\begin{aligned} c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n &= d_1 \\ c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n &= d_2 \\ \vdots \qquad\quad \vdots \qquad\qquad & \vdots \\ c_{r1}x_1 + c_{r2}x_2 + \cdots + c_{rn}x_n &= d_r \end{aligned} \longrightarrow ④$$

The systems ② and ④ are equivalent if they both have exactly the same solutions.

Example The linear system

$$x_1 - 3x_2 = -7$$
$$2x_1 + x_2 = 7 \longrightarrow (5)$$

has only the solution $x_1 = 2,\ x_2 = 3$.

The linear system

$$8x_1 - 3x_2 = 7$$
$$3x_1 - 2x_2 = 0 \longrightarrow (6)$$
$$10x_1 - 2x_2 = 14$$

has only solution $x_1 = 2,\ x_2 = 3$. Thus the systems (5) and (6) are equivalent.

Linear Systems with Two Unknowns

Consider a linear system of two equations in the unknowns x_1 and x_2.

$$a_1 x_1 + a_2 x_2 = c_1$$
$$b_1 x_1 + b_2 x_2 = c_2 \longrightarrow (7)$$

To find the solution to linear system (7), we shall use a technique called method of elimination.

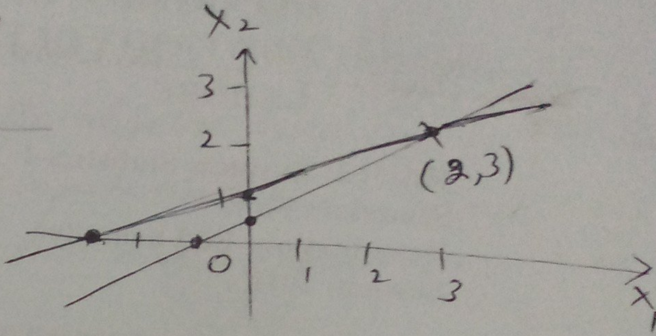
Example (1) Solve the linear system

$$x_1 - 2x_2 = -1 \longrightarrow (8)$$
$$-x_1 + 3x_2 = 3 \longrightarrow (9)$$

Adding both equations 8 and 9

$$\cancel{x_1} - 2x_2 = -1$$
$$\cancel{-x_1} + 3x_2 = 3$$

$$\boxed{x_2 = 2}$$



Substituting $x_2 = 2$ in 8

$$x_1 - 2(2) = -1$$
$$x_1 - 4 = -1 \Rightarrow \boxed{x_1 = 3}$$

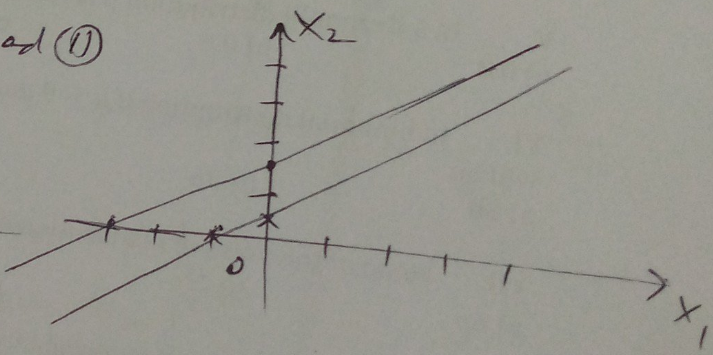
Example 2 Solve the linear system

$$x_1 - 2x_2 = -1 \longrightarrow 10$$
$$-x_1 + 2x_2 = 3 \longrightarrow 11$$

Adding both equations 10 and 11

$$\cancel{x_1} - \cancel{2x_2} = -1$$
$$\cancel{-x_1} + \cancel{2x_2} = 4$$

$$0 = 5$$



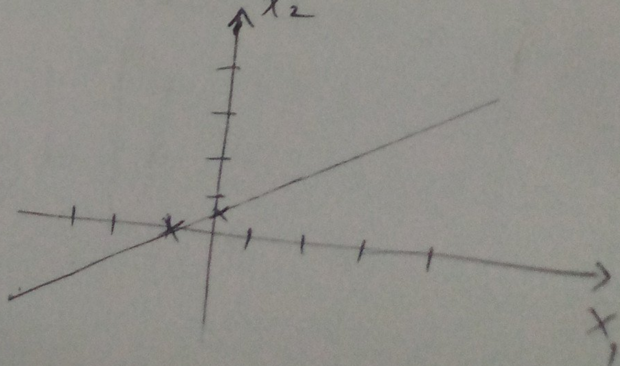
Example 3 Solve the linear system

$$x_1 - 2x_2 = -1 \longrightarrow 12$$
$$-x_1 + 2x_2 = 1 \longrightarrow 13$$

Adding both equations 12 & 13

$$\cancel{x_1} - \cancel{2x_2} = \cancel{-1}$$
$$\cancel{-x_1} + \cancel{2x_2} = 1$$

$$0 = 0$$



Linear Equations with Three or More Unknowns

As the number of equations and unknowns in a linear system increases, so does the complexity of the algebra involving in finding the solutions. The required computations can be made more managable by simplifying notation and standardizing procedures.

The basic strategy is to replace one system with an equivalent system that is easier to solve.

We can abbreviated the system by writing only the rectangular array of numbers

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{12}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \qquad \vdots \qquad \qquad & \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

This is called the augmented matrix for the system. For example, the augmented matrix for the system of equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned} \quad \text{is} \quad \begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}.$$

The basic method for solving a linear system is perform appropriate algebraic operations on the system that do not altee the solution set and that produce a succession of increasingly simpler system. There are three basic operations used to simplify a linear system.

1) Multiply a row by a non-zero constant (KR_i)
2) Interchange two rows (R_{ij})
3) Add a constant time one row to another $(KR_i + R_j)$

These are called elementary row operations on a matrix.

Example (4) Solve the system

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 8$$
$$5x_1 \quad - 5x_3 = 10.$$

The augmented matrix is

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \quad -5R_1 + R_3$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix} \quad \frac{1}{2}R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{bmatrix} \quad -10R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{bmatrix} \quad \frac{1}{30} R_3$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$x_1 - 2x_2 + x_3 = 0$$

$$x_2 - 4x_3 = 4$$

$$\boxed{x_3 = -1}$$

$$x_2 - 4(-1) = 4 \Rightarrow \boxed{x_2 = 0}$$

$$x_1 - 2(0) + (-1) = 0$$

$$\boxed{x_1 = 1}$$

Example ⑤ Solve the following system

$$x_2 - 4x_3 = 8$$
$$2x_1 - 3x_2 + 2x_3 = 1$$
$$4x_1 - 8x_2 + 12x_3 = 1$$

The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix} R_{12}$$

$$\sim \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{bmatrix} -2R_1 + R_3$$

$$\sim \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{bmatrix} 2R_2 + R_3$$

$$\sim \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

$$2x_1 - 3x_2 + 2x_3 = 1$$
$$x_2 - 4x_3 = 8$$
$$0 = 15$$

The system has no solution.

Example 6 Find the solution to the linear system

$$x_2 - x_3 = 3$$
$$x_1 \quad + 2x_3 = 2$$
$$-3x_2 + 3x_3 = -9$$

The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -1 & 3 \\ 1 & 0 & 2 & 2 \\ 0 & -3 & 3 & -9 \end{bmatrix} R_{12}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & -3 & 3 & -9 \end{bmatrix} 3R_2 + R_3$$

$$\begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 2x_3 = 2 \Rightarrow x_1 = 2 - 2x_3$$
$$x_2 - x_3 = 3 \Rightarrow x_2 = 3 - x_3$$

x_3 is free

one particular solution is

$\boxed{x_3 = 0}$ $\boxed{x_1 = 2}$ $\boxed{x_2 = 3}$

Another particular solution is

$\boxed{x_3 = 1}$ $\boxed{x_1 = 0}$ $\boxed{x_2 = 2}$

Example (7) Given the augmented matrices, find the solution of the linear systems.

(a) $\begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -1 & 0 & 2 & 4 \\ 0 & 0 & 1 & -3 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & -1 & 7 \\ 0 & 0 & 1 & 2 & 3 & 7 \\ 0 & 0 & 0 & 1 & 2 & 9 \end{bmatrix}$

(a) $x_1 + 2x_2 + 3x_3 = 9$ $\Rightarrow \boxed{x_2 = -1}$

$x_2 + x_3 = 2$ $\Rightarrow \boxed{x_1 = 2}$

$\boxed{x_3 = 3}$

(b) $x_1 + 2x_2 + 3x_3 = 6$ $\Rightarrow \boxed{x_2 = -2}$

$x_2 + 2x_3 = 4$ $\boxed{x_1 = 1}$

$\boxed{x_3 = 3}$

(c) $x_1 - x_2 + x_4 = 4$ x_2 is free.

$x_3 - 3x_4 = 7$ x_4 is free.

$\Rightarrow x_1 = 4 + x_2 - x_4$ $\boxed{x_2 = 0}$ $\boxed{x_4 = 1}$

$x_3 = 7 + 3x_4$ $\boxed{x_1 = 3}$ $\boxed{x_3 = 10}$

(d) $x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 6$

$x_2 + 2x_3 + 3x_4 - x_5 = 7$ x_5 is free.

$x_3 + 2x_4 + 3x_5 = 7$

$x_4 + 2x_5 = 9$

Row Echelon (Echelon) Form

A rectangular $m \times n$ matrix is in echelon form if it has the following properties.

1) All zero rows, if there are any, appear at the bottom of the matrix.
2) The first nonzero entry from the left of a nonzero row is 1. This entry is called a leading one. (leading 1)
3) In any two successive rows, the leading 1 in the lower row occurs further to the right than the leading 1 in the higher row.
4) All entries in a column below leading 1 are zeros.

Reduced Row Echelon Form

5) If a column contains leading 1, then all other entries in that column are zero.

Examples Echelon Form

$$A = \begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reduced Echelon Form

$$A = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The following matrices are not reduced echelon form.

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & -2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Elimination Methods to Solve Linear Systems

1) Gauss Elimination
2) Gauss-Jordan Elimination

1) Gauss Elimination

Row echelon form

(a) Reduce the augmented matrix to row echelon form
(b) use backward substitution to find the solution.

2) Gauss-Jordan Elimination

Reduced echelon form

(a) Reduce the agumented matrix to reduced row echelon form
(b) Direct solution is obtained.

Example ⑧ Given the agumented matrix is reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

$x_1 = 3, \quad x_2 = -1, \quad x_3 = 0, \quad x_4 = 5$

Example ⑨

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Agar sab ka answer 0 a jai tou koi solution ni hota

The system has no solution.

Example 10

(a) $\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(a) $x_1 + 3x_3 = -1$ x_3 is free

$x_2 - 4x_3 = 2$

$x_1 = -1 - 3x_3 = -1 - 3t$

$x_2 = 2 + 4x_3 = 2 + 4t$

$x_3 = t$

For $t = 0$, $x_1 = -1$, $x_2 = 2$, $x_3 = 0$

For $t = 1$, $x_1 = -4$, $x_2 = 6$, $x_3 = 1$

etc.

(b) $x_1 - 5x_2 + x_3 = 4$ x_2, x_3 free

$x_1 = 4 + 5x_2 - x_3$

$x_1 = 4 + 5s - t$

$x_2 = s$

$x_3 = t$

$x_2 = 0$, $x_3 = 1$ $x_1 = 3$

$x_2 = 1$, $x_3 = 1$ $x_1 = 8$

etc.

<u>General Solution</u> If a linear system has infinitely many solutions, then a set of parametric equations from which all solutions can be obtained by assigning numerical values to the parameters is called a general solution of the system.

<u>Example 11</u> Solve the system by Gauss-Jordan elimination.

$$-2x_3 + 7x_5 = 12$$
$$2x_1 + 4x_2 - 10x_3 + 6x_4 + 12x_5 = 28$$
$$2x_1 + 4x_2 - 5x_3 + 6x_4 - 5x_5 = -1$$

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} R_{12}$$

$$\sim \begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \tfrac{1}{2}R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} -2R_1 + R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} -\tfrac{1}{2}R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} -5R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix} \quad 2R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad +\frac{7}{2}R_3+R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad 5R_2+R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 3 & 6 & 19 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad -6R_3+R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$x_1 + 2x_2 + 3x_4 = 7$

$x_3 = 1$ $\qquad$ x_2, x_4 free

$x_5 = 2$

$x_1 = 7 - 2x_2 - 3x_4$

$x_1 = 7 - 2s - 3t$

$x_2 = s$

$x_3 = 1$

$x_4 = t$

$x_5 = 2$

Solution to Homogeneous Linear Systems

Example 2 Use Gauss-Jordan elimination to solve homogeneous linear system.

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$
$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0$$
$$5x_3 + 10x_4 + 15x_6 = 0$$
$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 3x_2 + 4x_4 + 2x_5 = 0$$
$$x_3 + 2x_4 = 0$$
$$x_6 = 0$$

x_2, x_4, x_5 free

$$x_1 = -3x_2 - 4x_4 - 2x_5$$
$$x_3 = -2x_4$$
$$x_6 = 0$$

$x_2 = r$

$x_4 = s$

$x_5 = t$

$$x_1 = -3r - 4s - 2t$$
$$x_2 = r$$
$$x_3 = -2s$$
$$x_4 = s$$
$$x_5 = t$$
$$x_6 = 0$$

Free Variables for Homogeneous System

If a homogeneous linear system has n unknowns, and if the reduced row echelon form of its agumented matrix has r non zero rows, then the system has $n-r$ free variables.

Theorem A homogeneous linear system with more unknowns that equations has infinitely many solutions

$$\boxed{\begin{array}{l} \text{Exercise 1.1 Q1–17} \\ \text{Exercise 1.2 Q1–43} \end{array}}$$