1 Predicte Logic

- Propositional logic, cannot adequately express the meaning of all statements in mathematics and in natural language.
- For example, suppose that we know that "Every computer connected to the university network is functioning properly."
- No rules of propositional logic allow us to conclude the truth of the statement "MATH3 is functioning properly" where MATH3 is one of the computers connected to the university network.
- We will introduce a more powerful type of logic called **predicate logic**.

1.1 Predicates

- The statement **"x is greater than 3"** has **two parts**.
- The first part, the variable x, is the **subject** of the statement.
- The second part the **predicate**, is greater than 3 refers to a **property** that the subject of the statement can have.
- We can denote the statement "x is greater than 3" by $P(x)$, where P denotes the predicate "is greater than 3" and x is the variable.
- The statement $P(x)$ is also said to be the value of the **propositional function** P at x.
- Once a value has been assigned to the variable x, the statement $P(x)$ **becomes a proposition** and has a truth value.

Example

- Let $P(x)$ denote the statement $x > 3$. What is the truth values of $P(4)$ and $P(2)$?

1.2 Quantifiers

- When the variables in a **propositional function** are assigned values, the resulting statement becomes a **proposition** with a certain **truth value**.
- However, there is another important way, called **quantification**, to create a **proposition** from a **propositional function**.
- Quantification expresses the **extent** to which a predicate is true over a range of elements.
- In English, the words all, some, many, none, and few are used in quantifications.
- We will focus on two types of quantification here:
  - **Universal quantification**, which tells us that a predicate is true for every element under consideration.
  - **Existential quantification**, which tells us that there is one or more element under consideration for which the predicate is true.
- The area of logic that deals with predicates and quantifiers is called the **predicate calculus**.

1.2.1 DEFINITION 1

- The universal quantification of $P(x)$ is the statement $P(x)$ for all values of x in the domain.
- The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$.
- Here $\forall$ is called the **universal quantifier**.
- We read $\forall x P(x)$ as "for all $xP(x)$ or "for every $xP(x)$".
- An element for which $P(x)$ is false is called a **counterexample** of $\forall x P(x)$

Example

- Let $P(x)$ be the statement $x + 1 > x$. What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

Note

- When all the elements in the domain can be listed – say $x_1, x_2, .., x_n$ it follows that the universal quantification $\forall x P(x)$ is the same as the **conjunction** $P(x_1) \wedge P(x_2) \wedge .. \wedge P(x_n)$
- Because this conjunction is true if and only if $P(x_1), P(x_2), .., P(x_n)$ are all true.

Example

- Let $Q(x)$ be the statement $x < 2$. What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

1.2.2 DEFINITION 2

- The existential quantification of $P(x)$ is the proposition
- There exists an element x in the domain such that $P(x)$
- We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$.
- Here $\exists$ is called the **existential quantifier**

Notes

- When all elements in the domain can be listed say, $x_1, x_2, ..., x_n$ the existential quantification $\exists x P(x)$ is the same as the disjunction $P(x_1) \vee P(x_2) \vee ... \vee P(x_n)$ because this disjunction is true if and only if at least one of $P(x_1), P(x_2), ..., P(x_n)$

Example

- Let $P(x)$ denote the statement $x > 3$. What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

1.2.3 The Uniqueness Quantifier

- Denoted by $\exists!$ or $\exists_1$.
- The notation $\exists! x P(x)$ or $\exists_1 x P(x)$ states "There exists a unique x such that $P(x)$ is true".
- Other phrases for uniqueness quantification include there is exactly one and there is one and only one"
- For instance $\exists! x(x - 1 = 0)$
- Generally, it is best to stick with existential and universal quantifiers so that rules of inference for these quantifiers can be used.

Example

- In the statement, $\exists x(x + y = 1)$, the variable x is bound by the existential quantification $\exists x$, but the variable y is free becuase it is not bound by a quantifier and no value is assigned to this variable.
- In the statement, $\exists x(P(x) \wedge Q(x)) \vee \forall x R(x)$, all variables are bound.
- The scope of the first quantifier, $\exists x$, is the expression $P(x) \wedge Q(x)$ because $\exists x$ is applied only to $P(x) \wedge Q(x)$, and not the rest of the statement.
- Similarly, the scope of the second quantifier, $\forall x$, is the expression $R(x)$.
- That is, the existential quantifier binds the variable x in $P(x) \wedge Q(x)$ and the universal quantifier $\forall x$ binds the variable x in $R(x)$.

2 Nested Quantifiers

- Nested quantifiers, where one quantifier is within the scope of another.
- such as, $\forall x \exists y(x + y = 0)$
- Note that **everything** within the scope of a quantifier can be thought of as a propositional function
- For example, $\forall x \exists y(x + y = 0)$ is the same thing as $\forall x Q(x)$, where $Q(x)$ is $\exists y P(x, y)$, where $P(x, y)$ is $x + y = 0$

2.1 Understanding Statements Involving Nested Quantifiers

Example

- Assume that the domain for the variables x and y consists of all real numbers.
- The statement $\forall x \forall y(x + y = y + x)$ says that $x + y = y + x$ for all real numbers x and y.
- This is the **commutative law** for addition of real numbers.