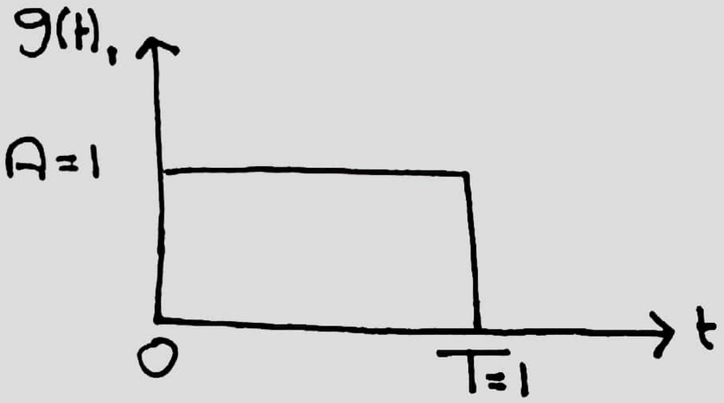
Part II)

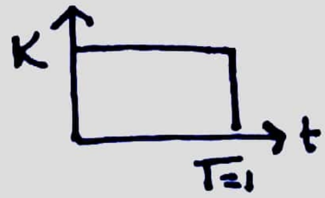


- $C(t) = 1$
- $w(t_0) \sim N(\mu = 0, \sigma = \frac{N}{2})$
- $g(t)_0 = -g(t)_1$ ← "1 was sent" ; ↖ "0 was sent"

$$y(T) = \begin{cases} g_0(T)_1 + n(T) & \text{"1 was sent"} \\ g_0(T)_0 + n(T) & \text{"0 was sent"} \end{cases}$$

→ Once we've found the distribution of $y(T)$ we can derive the Probability of error.

a) Consider a matched filter of unit energy

$h(t) = Kg(T-t) =$  $= Kg(t)_1$

- Has energy $K^2 \times 1$ and hence $K = 1$ (Satisfies unit energy)

$$\Rightarrow g_0(T)_1 = \int_{-\infty}^{\infty} g(\tau)_1 h(t-\tau)\Big|_{t=T} d\tau$$

$$= \int_{-\infty}^{\infty} g(\tau)_1 h(T-\tau) d\tau$$

• $h(t) = g(T-t) \rightarrow g(t) = h(T-t)$

$$g_o(T)_1 = \int_{-\infty}^{\infty} (g(\tau))^2 \, d\tau = 1$$

• $g(t)_1$'s energy

$$\Rightarrow g_o(T)_0 = \int_{-\infty}^{\infty} g(\tau)_0 \, h(T-\tau) \, d\tau$$

$$= \int_{-\infty}^{\infty} -g(\tau)_1 \cdot g(\tau)_1 \, d\tau = -1$$

$$\Rightarrow n(T) = h(t) * w(t) \big|_T$$

$$= \int_{-\infty}^{\infty} w(\tau) h(T-\tau) \, d\tau$$

$$= \int_{-\infty}^{\infty} w(\tau) \cdot g(\tau)_1 \, d\tau$$

$$= \int_{0}^{1} w(\tau) \, d\tau$$

• $n(T_p)$ is a continuous sum of random Gaussian variables and hence is a gaussian random variable itself.

can swap $\int$ E

$$\mu_{n(T_p)} = E[n(T)] = E\left[\int_0^1 w(\tau) \, d\tau\right]$$

$$= \int_0^1 \overset{0}{E[w(\tau)]} \, d\tau = 0$$

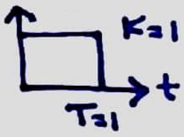
$$\sigma^2_{n(T)} = Var[n(T)]$$

$$= E[n^2(T)] - (\mu_{n(T)})^2$$

$$= E[n^2(T)] = R_n(0)$$

$$= \int_{-\infty}^{\infty} S_n(f)\,df = \int_{-\infty}^{\infty} S_w(f)\,|H(f)|^2 df$$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df$$

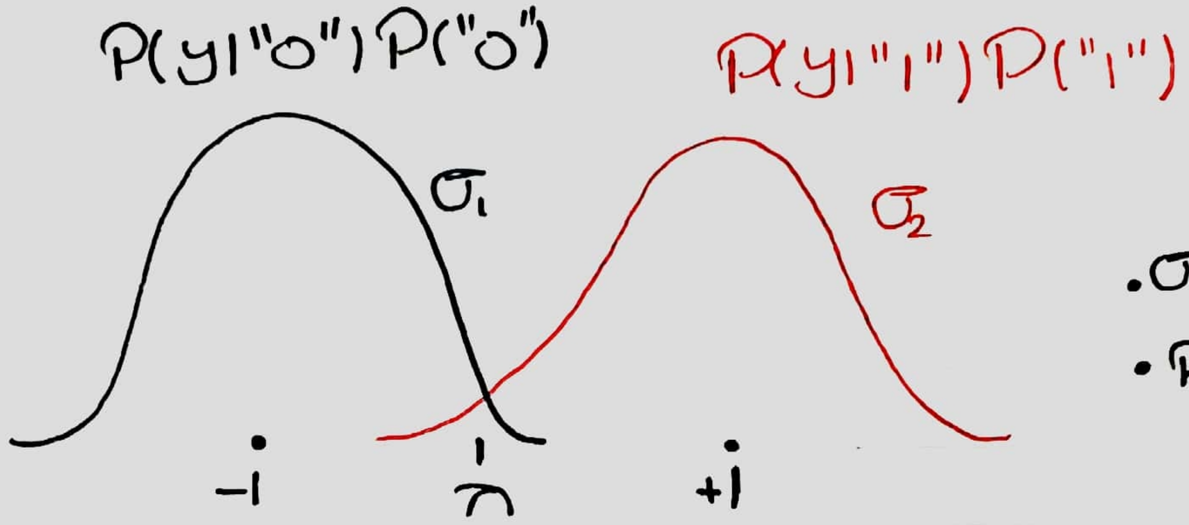
$$= \frac{N_0}{2} \int_{-\infty}^{\infty} |h(t)|^2 dt \qquad (h(t) =$$  $)$

$$= \frac{N_0}{2}$$

$$y(T) = \begin{cases} 1 + n(T) & \text{"1 was sent"} \\ -1 + n(T) & \text{"0 was sent"} \end{cases}$$

$$y(T) \mid \text{"1"} \sim N\left(\mu = 1, \sigma = \sqrt{\frac{N_0}{2}}\right)$$

$$y(T) \mid \text{"0"} \sim N\left(\mu = -1, \sigma = \sqrt{\frac{N_0}{2}}\right)$$



- $\sigma_1 = \sigma_2 = \sqrt{\frac{N_0}{2}}$
- $P(``0") = P(``1") = \frac{1}{2}$

By Symmetry (or by the derivation in the appendix)

$$\lambda_{opt} = \frac{-1+1}{2} = 0$$

Hence,

$$P(error) = P(y > \lambda | ``0")P(``0") + P(y < \lambda | ``1")P(``1")$$

$$= \frac{1}{2}\left(P(y > \lambda | ``0") + P(y > \lambda | ``0")\right)$$

$$= P(y > \lambda | ``0") \qquad \text{(by Symmetry again)}$$

$$y | ``0" \sim N\left(\mu = -1, \sigma = \sqrt{\frac{N_0}{2}}\right)$$

Let $Z = \frac{y - \mu}{\sigma} \rightarrow Z | ``0" \sim N(\mu = 0, \sigma = 1)$

$$= P\left(Z > \frac{\lambda + 1}{\sqrt{\frac{N_0}{2}}} \Big| ``0"\right)$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2}\, dt$$

Thus, $P\left(Z > \frac{\eta+1}{\sqrt{N./2}}\right) = Q\left(\frac{\eta+1}{\sqrt{N./2}}\right)$

$$= Q\left(\sqrt{\frac{2}{N.}}\right) = Q\left(\frac{1}{\sigma}\right)$$

b)

$$y(T) = \begin{cases} +1 + w(T) & \text{"1 Sent"} \\ -1 + w(T) & \text{"0 Sent"} \end{cases}$$

$\because g(t)_1 =$  $\therefore g(T)_1 = 1$

$\because g(t)_0 = -g(t)_1 \therefore g(T)_0 = -1$

• $g(t) = g_0(t)$ as

$g(t) = g(t) * \delta(t)$

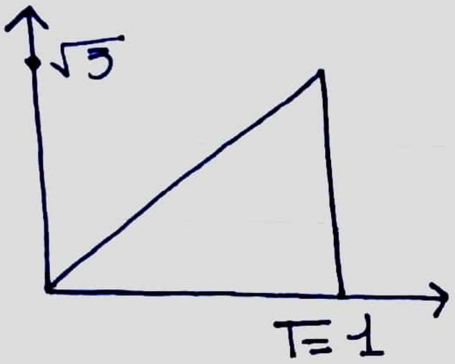
$$w(T) \sim N\left(\mu = 0, \sigma = \sqrt{\frac{N_0}{2}}\right)$$

Hence, Clearly

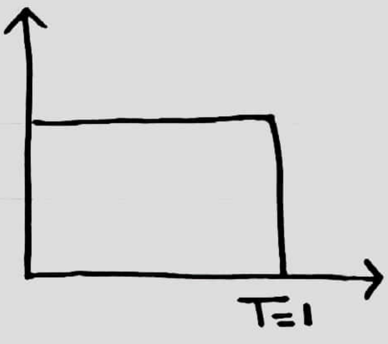
$$P(e) = Q\left(\sqrt{\frac{2}{N.}}\right) = Q\left(\frac{1}{\sigma}\right)$$

c)

• $h(t) =$



• $g(t)_1 =$

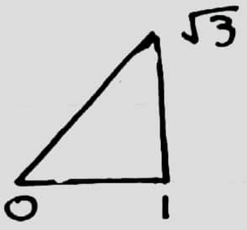
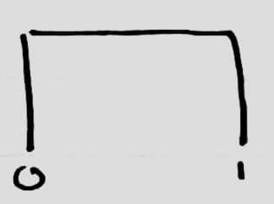


• $g(t)_0 = -g(t)_1$

$$y(T) = \begin{cases} g_0(T)_1 + n(T) \\ g_0(T)_0 + n(T) \end{cases}$$

• $g_0(T)_1 = g(t)_1 * h(t) \big|_{t=T}$

$$= \int_{-\infty}^{\infty} h(\tau)\, g(T-\tau)\, d\tau$$

$= \int_0^1 ($  $\times$  $)\, d\tau$

$$= \frac{\sqrt{3}}{2}$$

• $g_0(T)_0 = -\frac{\sqrt{3}}{2}$ (by homogeity of convolution)

As have been demonstrated before

$$n(T) = h(t) * w(t)\big|_T$$

Gaussian random

Deterministic Gaussian random

$$= \int_{-\infty}^{\infty} w(\tau)\, h(T-\tau)\, d\tau$$

Thus, $\mu_{n(T)} = E[n(T)] = E\left[\int_{-\infty}^{\infty} w(\tau) h(T-\tau) d\tau\right]$

$$= \int_{-\infty}^{\infty} \underbrace{h(T-\tau)}_{\text{deterministic}} \underbrace{E[w(\tau)]}_{0}\, d\tau$$

$$= 0$$

$$\sigma^2_{n(T)} = E[n^2(T)] - \mu^2_{n(T)}$$

$$= \int_{-\infty}^{\infty} S_n(f)\, df$$

$$= \int_{-\infty}^{\infty} S_w(f) |H(f)|^2 df$$

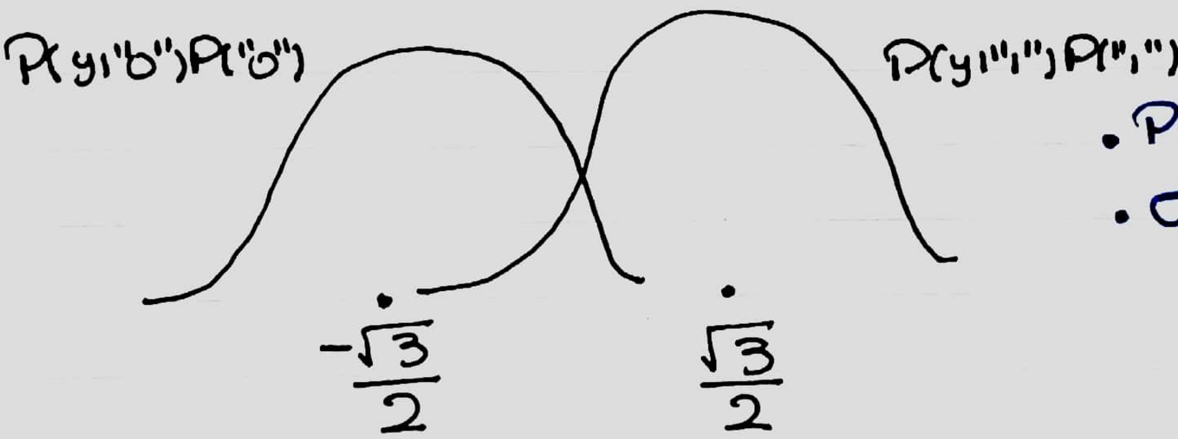
$$= \frac{N_0}{2} \int_0^1 |h(t)|^2 dt$$

$$= \frac{N_0}{2} \int_0^1 (\sqrt{3}\, t)^2 dt = \frac{N_0}{2}$$

$$y(T) = \begin{cases} \frac{\sqrt{3}}{2} + n(T) & \text{"1"} \\ \\ -\frac{\sqrt{3}}{2} + n(T) & \text{"0"} \end{cases}$$

$$\rightarrow y \mid \text{"1"} \sim N\left(\mu = \frac{\sqrt{3}}{2}, \sigma = \sqrt{\frac{N_0}{2}}\right)$$

$$y \mid \text{"0"} \sim N\left(\mu = \frac{\sqrt{3}}{-2}, \sigma = \sqrt{\frac{N_0}{2}}\right)$$



- $P(\text{"0"}) = P(\text{"1"}) = \frac{1}{2}$
- $\sigma_1 = \sigma_2 = \sqrt{\frac{N_0}{2}}$

$\lambda_{opt} = 0$ by Symmetry

$$P(\text{error}) = P(y > \lambda \mid \text{"0"})$$

$$= P\left(Z > \frac{\lambda + \sqrt{3}/2}{\sqrt{N_0/2}}\right) \qquad \bullet\ Z = \frac{y-\mu}{\sigma}$$

$$= Q\left(\sqrt{\frac{3}{2N_0}}\right) = Q\left(\frac{\sqrt{3}}{2} \cdot \frac{1}{\sigma}\right)$$

Appendix

$$P(e) = Q\left(\frac{\lambda - \mu_1}{\sigma_1}\right) P(\text{"0"}) + Q\left(\frac{\mu_2 - \lambda}{\sigma_2}\right) P(\text{"1"})$$

$$\frac{\partial P(e)}{\partial \lambda} = 0 \longrightarrow P(\text{"0"})\, Q'\left(\frac{\lambda - \mu_1}{\sigma_1}\right) \cdot \frac{1}{\sigma_1} = -P(\text{"1"})\, Q'\left(\frac{\mu_2 - \lambda}{\sigma_2}\right) \cdot \frac{-1}{\sigma_2}$$

which by the fundamental theorem of calculus is equivalent to

$$P(\text{"0"}) P(y|\text{"0"}) = P(\text{"1"}) P(y|\text{"1"})$$

$$P(X|C_1)P(C_1) = P(X|C_2)P(C_2)$$

$$P(C_1).\frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(X-\mu_1)^2}{2\sigma_1^2}} = P(C_2).\frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(X-\mu_2)^2}{2\sigma_2^2}}$$

. Now to Simplify things (We'll generalize later) assume $\sigma_1 = \sigma_2$

$$P(C_1).e^{-\frac{(x-\mu_1)^2}{2\sigma^2}} = P(C_2).e^{-\frac{(x-\mu_2)^2}{2\sigma^2}}$$

take logarithm

$$\ln(P(C_1)) - \frac{(x-\mu_1)^2}{2\sigma^2} = \ln(P(C_2)) - \frac{(x-\mu_2)^2}{2\sigma^2}$$

$$\ln\left(\frac{P(C_1)}{P(C_2)}\right) = \frac{(x-\mu_1)^2}{2\sigma^2} - \frac{(x-\mu_2)^2}{2\sigma^2}$$

$$2\sigma^2 \ln\left(\frac{P(C_1)}{P(C_2)}\right) = (x^2 - 2\mu_1 x + \mu_1^2) - (x^2 - 2\mu_2 x + \mu_2^2)$$

$$= 2x(\mu_2 - \mu_1) + (\mu_1^2 - \mu_2^2)$$

Hence, decision boundary is at

$$x_d = \frac{2\sigma^2 \ln(P_{C_1}/P_{C_2}) - (\mu_1^2 - \mu_2^2)}{2(\mu_2 - \mu_1)}$$