A FIRST COURSE IN GRAPH THEORY

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Graph Theory

- Simple graph
- Non-Simple graph
  - → Multigraph
  - → Pseudograph
- Subgraph
  - → Spanning Subgraph
  - → Induced subgraph

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* **Adjacent vertices** are those which are joined by an edge
* Otherwise, they are non-adjacent
* Simply, we say that two vertices are adjacent if there is a common edge ~~vertex~~ between them.
* Similarly, two edges are adjacent if there is the common vertex between them.

- If two vertices u and v are joined by an edge e, Then both u and v are incident with e and vice-versa.
- **Loop:** An edge joining a vertex v to itself. In this case, the vertex v is said to be self adjacent.
- **Parallel edges:** If two vertices are joined by two or more than two finite number of edges, then that edges are called parallel edges. Two or more than two edges joining a vertex to itself are also parallel, which are called parallel loops

  * parallel loops $\longrightarrow$ Parallel edges $\not\longrightarrow$ parallel ~~edges~~ loops.
- **Simple graph:** A graph which is free from loops as well as parallel edge. Otherwise, the graph is non-simple

* A non-simple is of two types:

Multigraph

A graph which has only parallel edges (they also can be parallel loops), and no simple loop

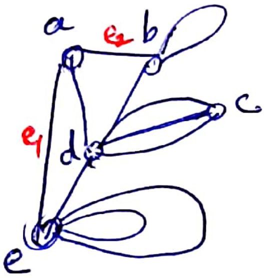
Pseudograph

A graph which has only loops with at least one simple loop

or

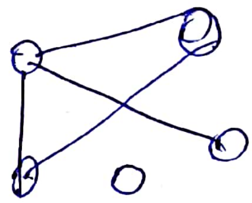
A graph which has both parallel edges and loops with at least one simple loop.

G_1:



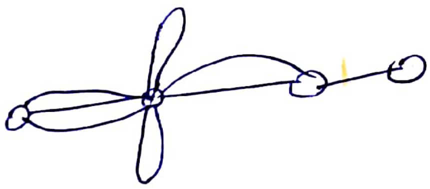
(Non-Simple)

G_2:



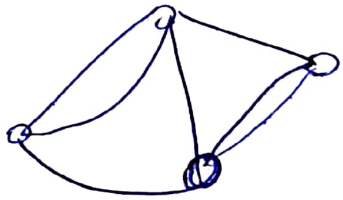
(Simple)

G_3:



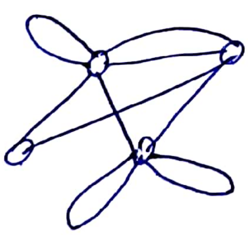
(Multigraph)

G_4:

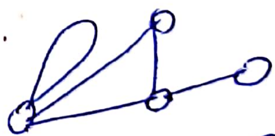


(Multigraph)

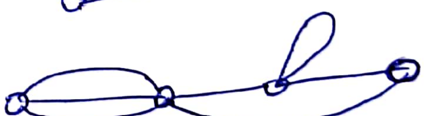
G_5:



← pseudographs → G_6:



G_7:



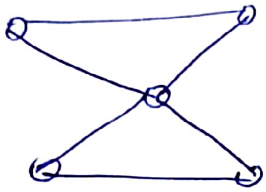
Remark Let G be a graph of order n and size $S(G)$. Then

(1) The maximum ~~ord~~ size of a simple graph G is $\binom{n}{2} = \frac{n!}{(n-2)!\,2!}$

(2) If $S(G) \leq \binom{n}{2}$, then G can be simple, ~~as well~~ but if $S(G) > \binom{n}{2}$, then G cannot be simple.

Subgraph: A graph H is said to be a subgraph of a graph G iff $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

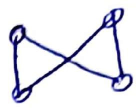
G:



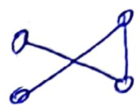
H_1:

H_2:

H_3:



H_4:

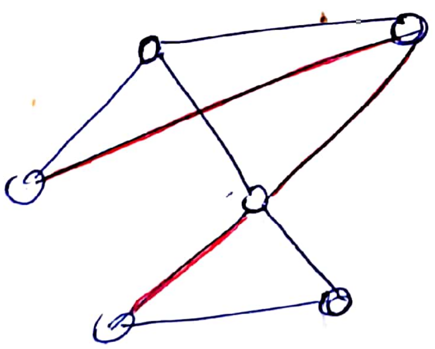


* From a given graph G, subgraphs canbe obtained in two ways:.

- **Spainning subgraph:** obtained by deleting one or more edges, denoted by $G-e$.
    - * $G-3e$ means we should delete any three edge.
    - * $G-\{e_1, e_2\}$ means we should delete the said edges only not of our own choice.

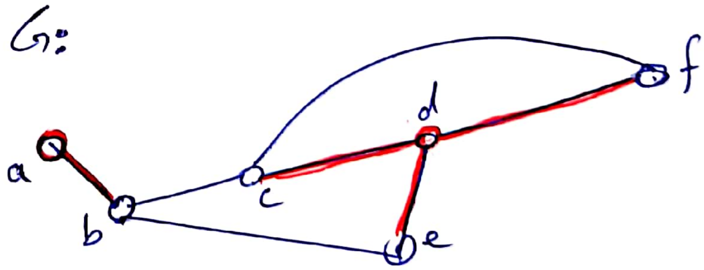
- **Induced subgraph:** obtained by deleting one or more vertices, denoted by $G-V$.
    - * $G-3V$ means we should delete any three vertices.
    - * $G-2e-3V$ means we should delete any two edges firstly and then any three vertices.

G:



$G-3e$ is a subgraph without red edges

G:



$G-\{a, d\}$ is a subgraph without red portion.

Assignment

* Write the order and size of each graph (shown) provided in the lecture.

* Make all possible different simple graphs of order 4.

* Draw two non-simple graphs: one is multi and one is pseudo.

* Can we draw a simple graph of order 5 and size 12? If no, then draw that non-simple graph.

* Draw a simple graph size 7.

Graph Theory

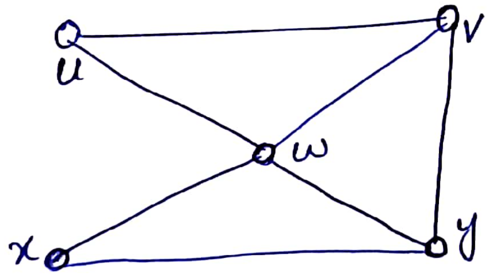
Connected and disconnected graphs

Let G be a graph. Then In G, between two vertices, starting from u and ending at v, we have

- **Walk**: is an alternating sequence of vertices and edges.
- **Trail**: is a walk without repeating any edge.
- **Path**: is a walk without repeating any vertex.
- **Circuit**: is a closed trail (start and end are same).
- **Cycle**: is a closed path.
- **Length**: is the number of edges in a walk.
- **Geodesic**: is a shortest path between two vertices.
- **Detoure**: is a longest path between two vertices.
- **K-cycle**: is a cycle of length K.
- **Odd (even) cycle**: is a cycle of odd (even) length.
- **Triangle and square**: is a 3-cycle and 4-cycle, respectively.

* The girth of a graph G is the length of a shortest cycle in G, denoted by $\omega(G)$.

G:



- $x-y-w-v-y-w-x$ is a walk of length 6. (also closed)
- $u-w-y-x-w-v$ is a trail of length 5.
- $w-y-v-u$ is a path of length 4.
- $u-w-y-x-w-v-u$ is a circuit of length 6.
- $u-v-y-x-w-u$ is a cycle of length 5. (5-cycle) $\rightarrow$ odd cycle.
- $u-w-x$ is a geodesic between u and x.
- $u-w-v-y-x$ is a detoure between u and x.
- A 3-cycle $w-v-y-w$ is a triangle in G.
- A 4-cycle $x-w-v-y-x$ is a square in G.

The shortest cycle in G is a 3-cycle, so $\omega(G)=3$.

* In a graph G, two vertices are connected is there is a path between them

adjacent vertices → connected vertices ↛ adjacent vertices.

* If every two vertices of a graph G are connected, then G is a connected graph. Otherwise, G is disconnected.

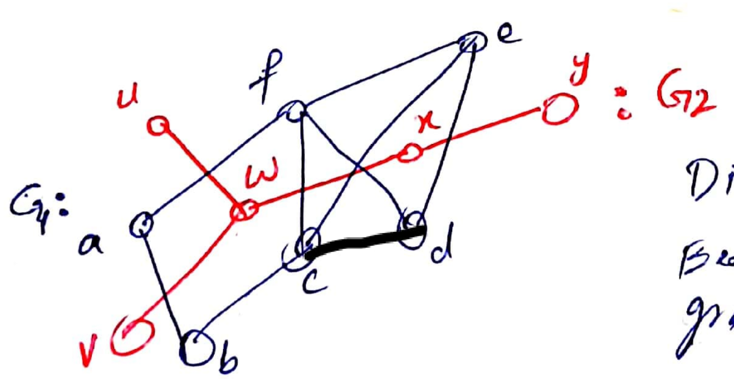
* A disconnected graph has connected parts as subgraphs each of which is called a component. The number of components in a graph G is denoted by K(G).

* A graph G is connected ⟺ K(G) = 1.

* The distance between two vertices u and v in a graph G is the length of a geodesic between u and v, denoted by $d_G(u,v)$ or simply $d(u,v)$.

* If there is no path between u and v (i.e., u and are disconnected), then $d(u,v) = \infty$.

* The diameter of a connected graph G is the maximum distance between two vertices in G, denoted by diam(G).



Disconnected graph with K(G) = 2.
Each component is a connected graph

$d(a,d) = 2$, $d(a,y) = \infty$

$d(f,c) = 1$, $d(f,w) = \infty$

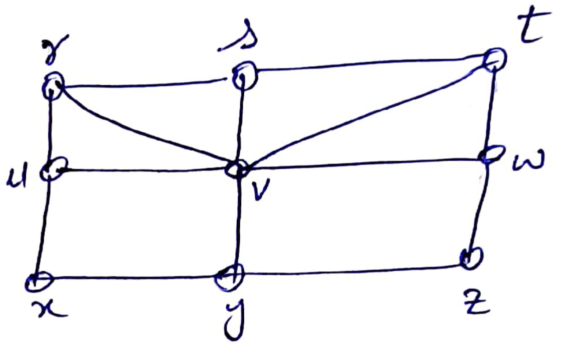
$diam(G_1) = 2$
$diam(G_2) = 3$

Assignment

* Make all possible different connected simple graphs of order 4.

* Draw a simple connected graph of order 6 and size 5.

* Consider the graph G:



Find

(a) a x – y walk of length 6.

(b) a v – w trail that is not a path of length 6

(c) a z – r path of length 5

(d) a ciruit of length 10

(e) a 8 – cycle.

(f) a r – z geodesic

(g) a u – x detoure

(i) diam (G).

(h) the girth of G.

* Draw a connected graph G containing three vertices u, v and w such that $d(u,v) = d(u-w) = d(v,w) = diam(G) = 3$.

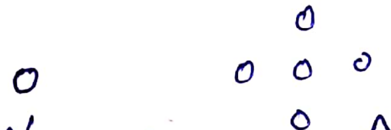
* Draw all different simple connected graphs in which distance between every two non-adjacent vertices is same and is 2.

Graph Theory

Common classes of graphs (All these concepts we will study in simple graphs)

Null graph: A graph having no edge, denoted by N_n for $n \geq 1$.

* N_1 is the trivial graph

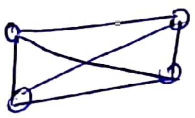


N_1 N_5

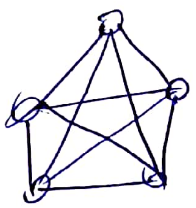
* A Null graph is disconnected with n components.

Complete graph: A graph in which every two vertices are adjacent, denoted by K_n for $n \geq 1$.

* $K_1 = N_1$: 



K_4



K_5

Path graph: A graph which just a single path, denoted by P_n; $n \geq 2$.

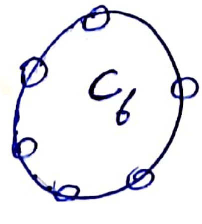
: P_3

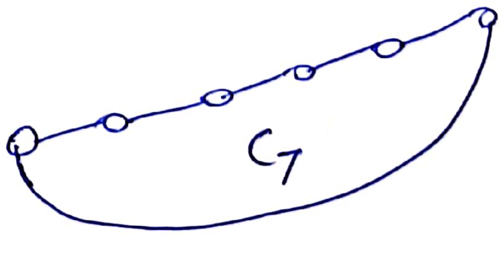
: P_6

Cycle graph: A closed path graph, denoted by C_n; $n \geq 3$.

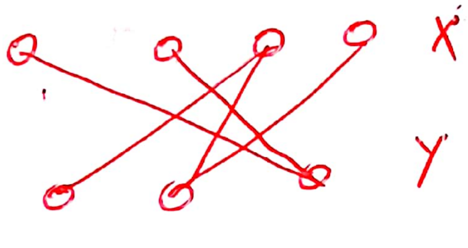
* C_1 :  non-simple, C_2 :  Multigraph

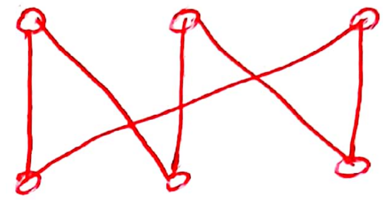
So for simplicity, $n \geq 3$.





Bipartite graph: A graph G is a bipartite graph whose edge set can be divided into two parts X and Y in such a way that each edge of G has one end in X and one end in Y.





C_6 is a bipartite.

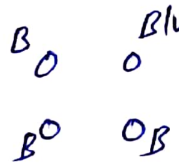
* Method to check a graph for bipartiteness.

Proper coloring of a graph: A coloring of vertices of a graph G is said to be proper if no two adjacent vertices have the same color.

* The minimum number of colors used for proper coloring of G is called the chromatic number of G, denoted by $\chi(G)$.

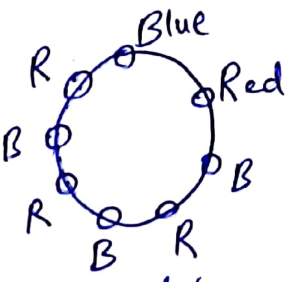
* If $\chi(G) = K \geq 1$, then G is a K-partite graph, or a multipartite graph.

N_4:



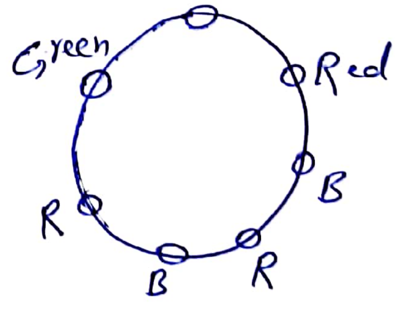
is 1-partite.

C_8:



is bipartite

C_7:

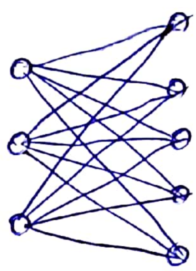


3-partite or tri-partite so it is not bipartite.

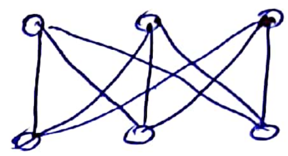
Complete bipartite graph: A bipartite graph with parts X and Y is a complete bipartite graph if each vertex of X is adjacent with every vertex in Y, denoted by $K_{m,n}$ for $m, n \geq 1$.

* Similarly, we have complete t-partite graph $K_{n_1, n_2, \dots, n_t}$ for $n_1, n_2, \dots, n_t \geq 1$.

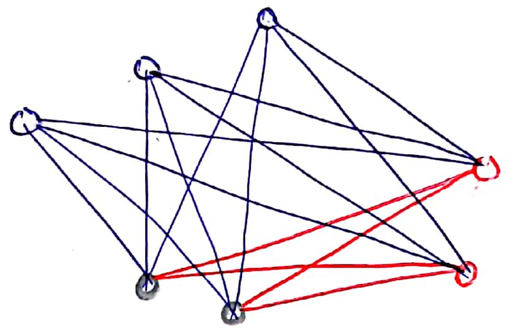
$K_{3,5}$:



$K_{3,3}$:

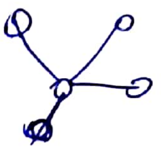


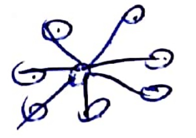
$K_{3,2,2}$



Star graph: A complet bipartite graph $K_{1,n}$ ($n \geq 1$) or $K_{1,n-1}$ ($n \geq 2$) is a star graph, also denoted by ~~$S_{1,n}$ or $S_{1,n-1}$~~ S_{n+1} or S_n

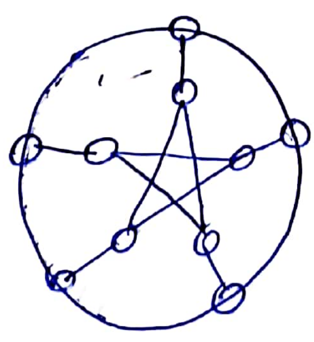
$K_{1,4} = S_5$





: $K_{1,7} = S_8$

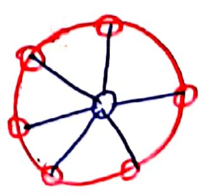
Petersen graph:



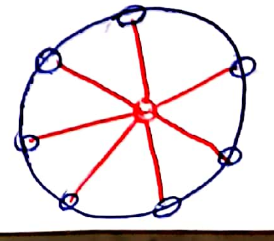
Wheel graph: It can be obtained by joining one vertex with the vertices of a cycle graph.

$W_n = K_1 + C_{n-1}$ or $W_{1,n} = K_1 + C_n$.

W_7: $K_1 + C_6$



$W_{1,7} = K_1 + C_7$



Assignment

* Determine that which graphs of these families are the same?
  For example $N_1 = K_1$, $K_2 = P_2$ etc.
* Find the order and size of each graph in general.
* The Petersen graph and each wheel are how much partite graphs?
* Find the diameter of each of these graphs in general.
* Suppose that the vertices of a graph G are integers, and two integers a and b are adjacent if a+b is odd. Then determin to which will-known class of graphs is G belonging?

Topic 5: Degree, regularity and degree sequence

Let G be a graph, v be any vertex of G and $e = xy$ be any edge of G. Then

* The degree of v is the number incidencies of edges with v. It is denoted by $d(v)$ or d_v.
* The degree of an edge $e = xy$ is $d(e) = d(x) + d(y) - 2$.

+ A vertex of degree zero is an isolated vertex.
+ A vertex of degree one is a leaf.
+ A vertex of even (odd) degree is known as even (odd) vertex
+ If all the vertices of G have the same degree r, then G is called r-regular or simply regular graph.
+ A 3-regular graph is called a cubic graph.
+ A graph is edge regular if all the edges have the same degree.

* The first theorem of graph theory (Hand shake Lemma):

In a graph G, the sum of the degrees of all the vertices is twice the number of edges.

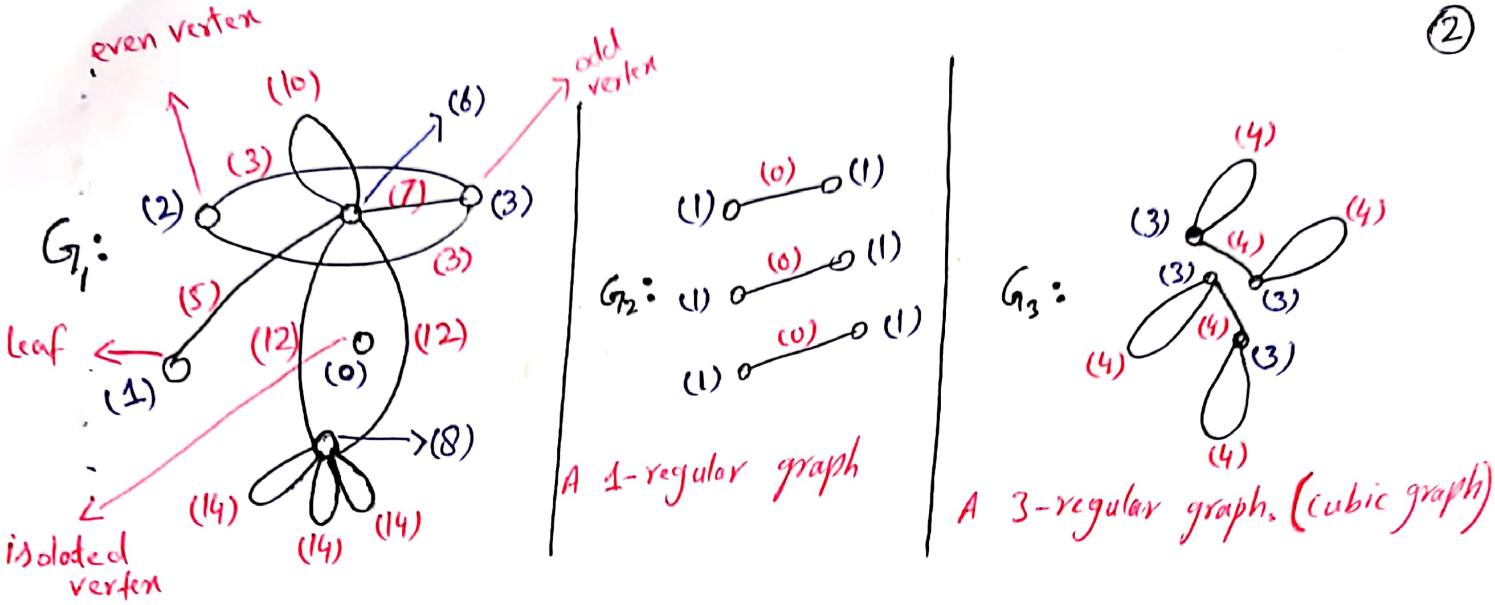
That is, mathematically if $v_1, v_2, \ldots, v_n$ are the vertices in G and $S(G)$ is the size of G (the number of edges). Then

$$\sum_{i=1}^{n} d(v_i) = 2 \times S(G). \quad \text{(even number)} \qquad \text{—— (1)}$$

* The number of odd vertices in a graph is even.

(This is the consequence of hand shake lemma)

Reason: (Because, if the number odd vertices in G is odd, then the sum of all the degrees would be odd, which contradicts the hand shake lemma.)



* <u>Degree sequence of a graph</u> Let G be a graph. Then the arrangement of the degrees of all the vertices of G in non-increasing or non-decreasing order is called the degree sequence of G.

For example, The degree sequence of G_1 is 8, 6, 3, 2, 1, 0

The degree sequence of G_2 is 1, 1, 1, 1, 1, 1

The degree sequence of G_3 is 3, 3, 3, 3.

* A degree sequence is trivial if all of terms are zero. That is, 0, 0, 0, 0, ---- 0.

<u>Graph construction from a given degree sequence.</u>

Remember the following steps:

<u>Step-I</u>: check by using the hand shake lemma that the given sequence is a degree sequence. Sum all the terms of the given sequence, and then sum should be even. Otherwise, hand shake lemma is not verifived, and write that, "this is not a degree sequence and we cannot draw a graph from this sequence".

<u>Step-II</u>: After varification of hand shake lemma, draw the vertices. The number of terms in the sequence is the number of vertices in the graph.

Step-III: Now, we draw the edges betwee the already drawn vertices. Since each term in the sequence is a degree, so it provides the information of the number edge incidencies with each vertex. Complete each term information on every vertex, then we get the graph.

Explaining the steps

consider three sequences:

(Hand Shake Lemma)

S_1: 7, 5, 4, 4, 1, 1, 0 — $7+5+4+4+1+1+0=22$ even, HSL verified.

S_2: 5, 4, 3, 3, 2, 0, 0 — $5+4+3+3+2+0+0=17$ odd, HSL is not verified

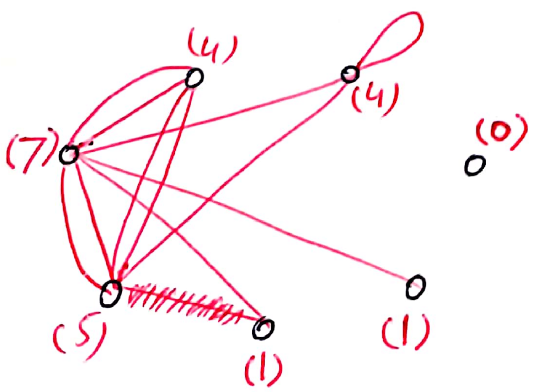
S_3: 3, 3, 2, 2, 1, 1 — $3+3+2+2+1+1=12$ even, HSL verified.

Here, S_1 and S_3 are the degree sequences, but S_2 is not. So, we draw graphs from S_1 and S_3, but cannot draw from S_2.

Remember: using HSL, we get the size of G, (the number of edges), by adding all the terms and then by deviding with 2, see the formula (1) on the first page.

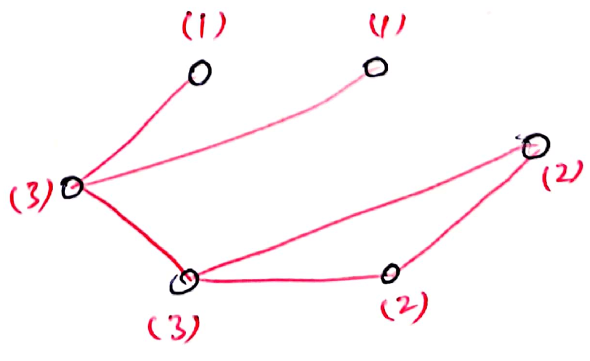
S_1: 7, 5, 4, 4, 1, 1, 0

we have seven verticas, and 11 edges



S_3: 3, 3, 2, 2, 1, 1

We have six verticas six edges.



From the previouse example, note that we construct a simple graph from the sequence S_3, and non-simple graph from S_1. Therefore, some time we can construct both simple and non-simple graphs from a given degree sequence, and some time we can construct only non-simple graph from a given degree sequence

Now, what is the identification of a sequence from whose we can construct a simple graph?

The answer is that we can construct a simple graph from a sequence which is graphical, and if a sequence is not graphical, then we can construct only non-simple graph from it.

Graphical sequence: A given degree sequence is said to be graphical if it can be reduced to the trivial sequence

Procedure to reduce (to check) a sequence for graphical and not graphical.

step-I * If there is a term in the sequence which is equal or greater to the total number of terms, then the sequence is not graphical.

For example, the sequence S_1: 7, 5, 4, 4, 1, 1, 0

step-II If the given sequence is not satisfy step-I, then arrange the terms in non-increasing order (if already not in that form).

step-III Skip the first term, which is say d_1, and subtract 1 from each next d_1 terms.

step-IV Re-arrange again in non-increasing order, and do step-III.

continue this procedure untill we get any -ve term or we reach the trivial sequence.

* **Decesion**: If we successfull reduced to the trivial sequence, then the given sequence is graphical, otherwise in any situation the sequence will not graphical.

Example

consider the sequence

$\cancel{3}, \overline{3, 3}, 1$

3−1, 3−1, 1−1

$\cancel{2}, \overline{2, 0}$

1, −1 -ve term so stop

The given sequence is non-graphical.

Consider the sequence

$\cancel{4}, \overline{3, 2, 2, 2}, 1, 1, 1, 0$

$\cancel{2}, \overline{1, 1}, 1, 1, 1, 1, 0$

0, 0, 1, 1, 1, 1, 0

$\cancel{1}, \overline{1}, 1, 1, 0, 0, 0$ (re-arrange)

0, 1, 1, 0, 0, 0

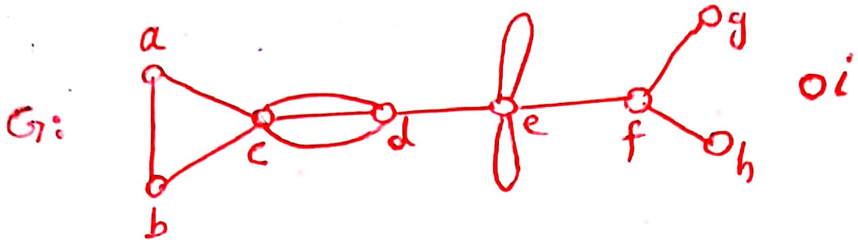
$\cancel{1}, \overline{1}, 0, 0, 0, 0$ (re-arrange)

0, 0, 0, 0, 0 trivial sequence

Hence the given sequence is graphical.

Topic: Neighborhood and Twin class

* Let G be a graph. Two vertices in G are said to be **neighbors** of eachother iff they are adjacent.



c, a and b are neighbors, but a is not neighbor of d, e, f, g, h, i

* For $v \in V(G)$, The **neighborhood** of v is:

open: $N(v) = \{u \in V(G) \mid u \text{ is adjacent with } v\}$
(The collection of neighbors of v)

closed: $N[v] = N(v) \cup \{v\}$

$N(c) = \{a, b, d\}$ $\qquad$ $N[c] = \{a, b, c, d\}$

* A graph G may has three types of vertices according to neighborhoods.

1. A vertex v such that
$N(v) = \{\cdot\}$ empty set
$N[v] = \{v\}$ singleton set
(This type of vertex should be an isolated vertex which is not adjacent with any vertex)
The vertex i in G.
$N(i) = \{\ \}$
$N[i] = \{i\}$

2. A vertex v for which $N(v)$ does not exist. only we can write its closed nbhd. $N[v]$
(A vertex having loop(s) is of this type)
The vertex e in G
$N[e] = \{e, d, f\}$

3. A vertex v having both non-empty neighborhoods.
(other than isolated and loop-free vertex is of this type)
All the vertices in G except e and i

* Two vertices u and v are Twins:

True: If $N[u] = N[v]$

In G, the vertices a and b are True Twins, because $N[a] = \{a, b, c\} = N[b]$

False: If $N(u) = N(v)$

In G, the vertices g and h are False twins, because $N(g) = \{f\} = N(h)$

* If vertices are True twins, then they would not be false twins, and vice-versa.

* If a vertex is not twin with any other vertex, then it is said to be self-twin.

* Relation: Two vertices u and v are twin equivalent (written as $u \overset{T}{\sim} v$) iff either $N(u) = N(v)$ or $N[u] = N[v]$.

In G, $a \overset{T}{\sim} b$ and $g \overset{T}{\sim} h$, where all other vertices are self twin equivalent.

* The relation of twin equivalent is an equivalence relation.

Reflexive: $v \overset{T}{\sim} v$, because $N(v) = N(v)$

Symmetric: $v \overset{T}{\sim} u \Rightarrow N(v) = N(u)$
$\Rightarrow N(u) = N(v)$
$\Rightarrow u \overset{T}{\sim} v$

Transitive: $v \overset{T}{\sim} u$ and $u \overset{T}{\sim} w$
$\Downarrow$ $\Downarrow$
$N(v) = N(u)$ and $N(u) = N(w)$
$\Rightarrow N(v) = N(w)$
$\Rightarrow v \overset{T}{\sim} w$.

* This equivalence relation defines a partition of $V(G)$, called twin partition, whose each part is called a twin class.

Mathematically, a twin class of a vertex v is

$$T_v = \{u \in V(G) \mid u \overset{T}{\sim} v\}.$$

Properties

(1) $v \in T_v$ is a class representative.

(2) Each vertex belongs to exactly one twin class, because for $u \neq v$ either $T_u = T_v$ or $T_u \cap T_v = \phi$ (empty set).

(3) A twin class T_v is also the twin class for each of the vertex in T_v.

Consider the graph G on page 1.

Neighborhoods of each vertex are:

$N(a) = \{b, c\}$ $\quad$ $N[a] = \{a, b, c\} = N[b]$

$N(b) = \{a, c\}$ $\quad$ $N[c] = \{a, b, c, d\}$

$N(c) = \{a, b, d\}$ $\quad$ $N[d] = \{c, d, e\}$

$N(d) = \{c, e\}$ $\quad$ $N[e] = \{d, e, f\}$

$N(f) = \{e, g, h\}$ $\quad$ $N[f] = \{e, f, g, h\}$

$N(g) = \{f\} = N(h)$ $\quad$ $N[g] = \{f, g\}$

$N[h] = \{f, h\}$

$N(i) = \{\ \}$ $\quad$ $N[i] = \{i\}$.

Twin partition is:

$T_a = \{a, b\} = T_b$, $T_c = \{c\}$, $T_d = \{d\}$, $T_e = \{e\}$, $T_f = \{f\}$,

$T_g = \{g, h\} = T_h$, $T_i = \{i\}$.

Practice question: Find the twin partition, generally of graphs studied in the common classes of graphs.

Topic: Matrices on Graphs.

Let G be a graph with vertices $v_1, v_2, \cdots, v_n$ and edges $e_1, e_2 \cdots e_m$. Then

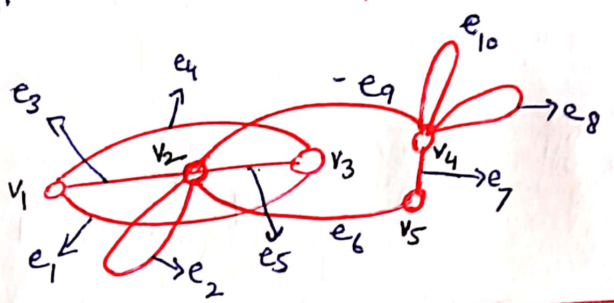
The adjacency matrix of G, $A(G)$ has order $n \times n$, and entries can be obtained as follows:

$$A(G) = [a_{ij}]_{n\times n} = \begin{cases} 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent} \\ \text{The number of edges between } v_i \text{ and } v_j, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \end{cases}$$

the incidence matrix of G, $I(G)$, has order $n \times m$, and ijth entry can be found as follows:

$$I(G) = [I_{ij}]_{n\times m} = \begin{cases} 0, & \text{if the vertex } v_i \text{ is not incident with the edge } e_j \\ 1, & \text{if } v_i \text{ is incident with } e_j. \end{cases}$$

G:



$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Remember that

- The order of $A(G)$ is order of G X order of G
- The order of $I(G)$ is order of G X size of G.

$$I(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Properties of $A(G)$:

(1) It should be a symmetric matrix with non-negative integer entries, and a diagonal entry must be even.

(2) The sum of each row is the degree of corresponding vertex. So the degree sequence and the size of G can be found.

(3) Each non-zero diagonal entry indicates a loop, and each non-diagonal non-binary entry indicates parallel edges.

(4) If a row in entirely zero, then the corresponding vertex is an isolated vertex.

(5) The entries of $A(G)$ are the binary digits if and only if G in a simple graph.

Remember: Binary digits are 0 and 1

Properties of $I(G)$:

(1) Each entry should be a binary digit. Each column sum should be either 1 or 2.

(2) An edge whose column sum is 1 is a loop, and that column in called a loop column. Two identical columns indicate parallel edge in G.

(3) The sum of each row, by taking the loop column entry of that row twice time, in the degree of corresponding vertex. So the degree sequence of G can be found.

(4) If a row is entirely zero, then the corresponding vertex is an isolated vertex.

(5) Each column sum is 2 and no two columns are identical if and only if G is a simple graph.

$$M_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 4 & 6 & 1 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 3 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 4 & 2 & 0 & 2 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 2 & 4 & 1 & 0 \\ 4 & 0 & -1 & 3 \\ 1 & -1 & 6 & 2 \\ 0 & 3 & 2 & 0 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 4 & 1 \\ 0 & 4 & 1 & 1 & 1 \\ \frac{1}{2} & 1 & 0 & 2 & 2/3 \\ 4 & 1 & 2 & 0 & 0 \\ 1 & 1 & 2/3 & 0 & 2 \end{bmatrix} \qquad M_4 = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 4 & 3 & 1 \\ 0 & 3 & 2 & 3 \\ 2 & 1 & 2 & 6 \end{bmatrix}$$

$$M_5 = \begin{array}{c} \\ \left[\begin{matrix} 0 & 0 & 1 & 2 & 3 \\ 0 & 4 & 2 & 0 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 2 & 0 & 2 & 0 & 1 \\ 3 & 1 & 2 & 1 & 0 \end{matrix}\right] \end{array} \begin{array}{c} \text{Degrees} \\ 6 \\ 7 \\ 9 \\ 5 \\ 7 \end{array} \qquad M_6 = \begin{array}{c} \\ \left[\begin{matrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{matrix}\right] \end{array} \begin{array}{c} \text{Degrees} \\ 4 \\ 3 \\ 4 \\ 0 \\ 3 \\ 4 \end{array}$$

Matrix	Adjacency matrix	not adjacency matrix
M1	Not	Yes, because one diagonal entry is odd
M2	Not	Yes, because it has negative entries.
M3	Not	Yes, because it has ~~radical~~ fractional entries.
M4	Not	Yes, because it is not symmetric
M5	Yes	Not
M6	Yes	Not

M_5 for non-simple graph and M_6 is for simple graph.

$$I_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix},$$

identical columns

(4) degree

$$I_2 = \begin{array}{c} \left[\begin{matrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & \overset{\times 2}{1} \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \overset{\times 2}{1} & 0 & 0 & 0 \end{matrix}\right] \\ \begin{matrix} 2 & 2 & 2 & 1 & 2 & 2 & 1 \end{matrix} \end{array} \begin{matrix} 3 \\ 2 \\ 5 \\ 2 \\ 2 \\ \end{matrix}$$

Loop columns

column sums

$$I_3 = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix},$$

degree

$$I_4 = \begin{array}{c} \left[\begin{matrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{matrix}\right] \\ \begin{matrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{matrix} \end{array} \begin{matrix} 4 \\ 3 \\ 4 \\ 0 \\ 4 \\ 3 \\ \end{matrix}$$

Column sums →

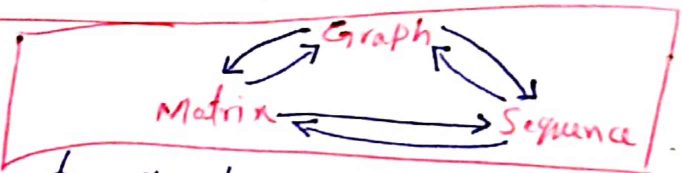
I_1 is not incidence matrix because it has one column sum zero.

I_2 is an incidence matrix for a non-simple graph.

I_3 is not incidence matrix because it has one column sum greater than 2.

I_4 is an incidence matrix for simple graph

Transformations



① Transformation of a graph into the degree sequence

② Transformation of a degree sequence into a graph

(Both these transformations has been studied in the topic of degree and degree sequence)

③ Transformation of a graph into matrix

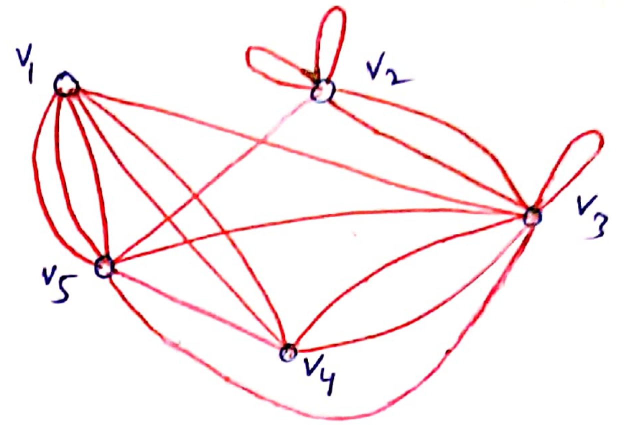
(studied in this lecture)

④ Transformation of a ~~graph~~ matrix into graph.

(i) From $A(G)$ to G.

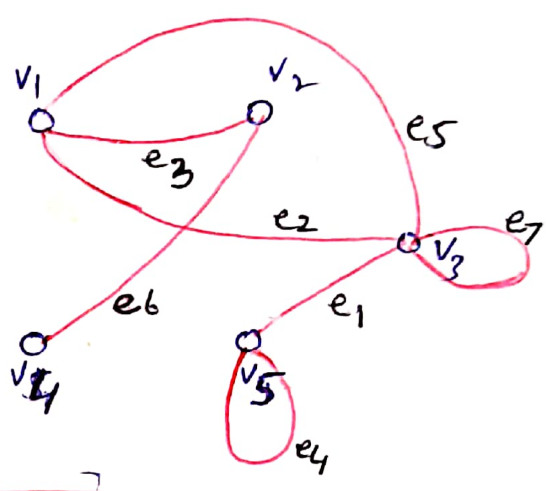
Let us consider the adjacency matrix M_5 given at page-3.

$$\begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 0 & 1 & 2 & 3 \\ v_2 & 0 & 4 & 2 & 0 & 1 \\ v_3 & 1 & 2 & 2 & 2 & 2 \\ v_4 & 2 & 0 & 2 & 0 & 1 \\ v_5 & 3 & 1 & 2 & 1 & 0 \end{array} \longrightarrow$$



(ii) From $I(G)$ to G : Let us consider the incidency matrix ~~given at page 4~~.

$$\begin{array}{c|ccccccc|c} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & \text{degree} \\ \hline v_1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 3 \\ v_2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ v_3 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 5 \\ v_4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ v_5 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 3 \end{array} \longrightarrow$$



(5) Transformation of Matrix into degree sequence

(i) From the adjacency matrices M_5 and M_2 given at Page-3,

$M_5 \rightarrow (9, 7, 7, 6, 5)$

$M_6 \rightarrow (0, 3, 3, 4, 4, 4)$

(ii) From the incidence matrices I_2 and I_4 given at Page-4

$I_2 \rightarrow (2, 2, 2, 3, 5)$

$I_4 \rightarrow (4, 4, 4, 3, 3, 0)$

(6) Transformation of a degree sequence into an adjacency matrix.

Let the degree sequence

$(0, 1, 1, 3, 4, 6, 9)$ order of the matrix will 7×7

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 & 4 \end{bmatrix} \begin{matrix} 0 \\ 1 \\ 1 \\ 3 \\ 4 \\ 6 \\ 9 \end{matrix}$$

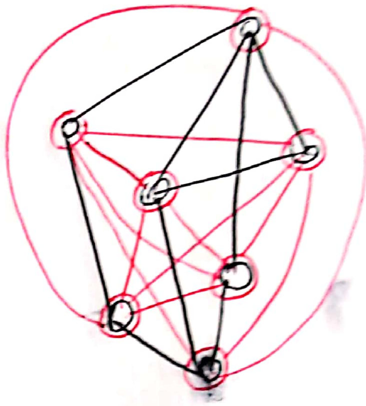
Topic: Graph operations (8)

Here we discuss five graph operations:

(1) The compliment of a graph

Let G be a simple graph. Then the compliment of G is denoted by $\bar{G}$, and is defined as follows:

- The vertex set of $\bar{G}$ is same as the vertex set of G.
- Two vertices in $\bar{G}$ will be adjacent iff they are non-adjacent in G.



* If $\bar{G} = G$, then G is a ~~self-centered~~ self complimentary graph.

* If $d(v) = k$ in G, then $d(v) = |G| - k - 1$ in $\bar{G}$

* $|G| = |\bar{G}|$ and $s(\bar{G}) = \binom{|G|}{2} - s(G)$

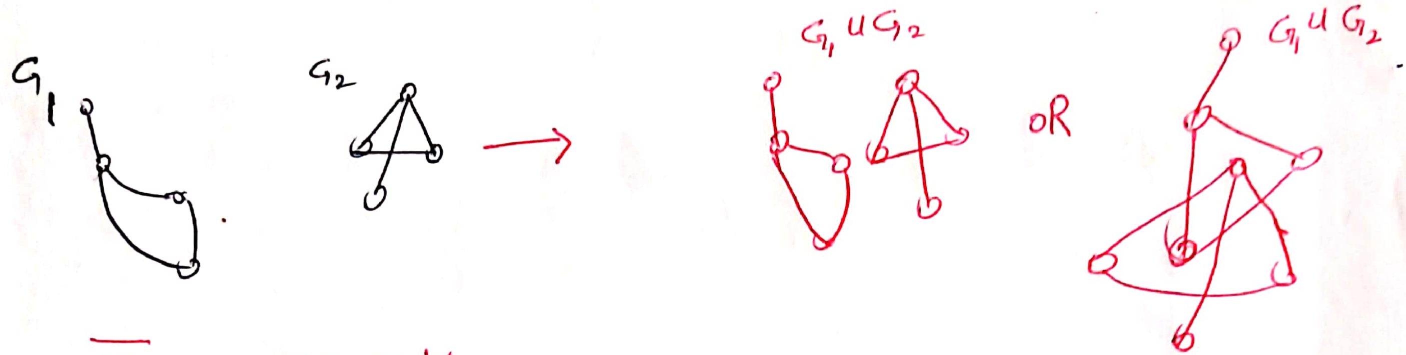
* $\bar{K}_n = N_n$ and $\bar{N}_n = K_n$ for all $n \geq 2$.

* $\bar{\bar{G}} = G$ as usual.

* If v is a leaf in a graph G of order n, then in $\bar{G}$, $d(v) = n - 2$.

(2) Union of graphs (non-labelled graphs)

If G_1 and G_2 are two graphs, then $G_1 \cup G_2$ has vertex set $V(G_1) \cup V(G_2)$ and edge set of $G_1 \cup G_2$ is $E(G_1) \cup E(G_2)$



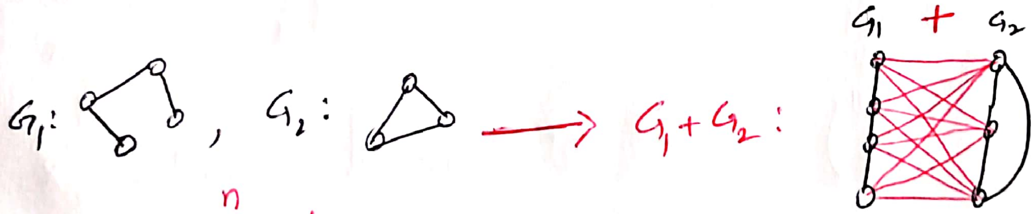
* $\overline{K_{1,n}} = K_1 \cup K_n$
* $\overline{K_{m,n}} = \cancel{K_m \cup N_m}\ K_m \cup K_n$
* $N_n = \bigcup_{i=1}^{n} N_1$

* $|G_1 \cup G_2| = |G_1| + |G_2|$
and $S(G_1 \cup G_2) = S(G_1) + S(G_2)$
* $G_1 \cup G_2 = G_2 \cup G_1$

(3) Sum (Join) of graphs

Let G_1 and G_2 be two graphs. Then The sum (join) of these graphs is denoted by $G_1 + G_2$ ($G_1 \vee G_2$), Can can be obtained as follows:

* Take the union of G_1 and G_2
* Join each vertex of G_1 with every vertex of G_2 by one edge.



* $K_n = \sum_{i=1}^{n} N_1 = N_1 + N_1 + \cdots + N_1$
* $K_{m,n} = N_m + N_n$
* $W_{1,n} = K_1 + C_n$
* $|G_1 + G_2| = |G_1| + |G_2|$

* Let $v \in V(G_1 + G_2)$, then

$$d(v) = \begin{cases} d(v) \text{ in } G_1 + |G_2|, & \text{if } v \in V(G_1), \\ d(v) \text{ in } G_2 + |G_1|, & \text{if } v \in V(G_2). \end{cases}$$

* $S(G_1 + G_2) = S(G_1) + S(G_2) + |G_1||G_2|$
* $G_1 + G_2 = G_2 + G_1$

(4) <u>Product of graphs</u>: Let G_1 and G_2 be two graphs. Then The product (cartesian) of G_1 and G_2 is denoted by $G_1 \times G_2$ or $G_1 \square G_2$, and is obtained as follows:

* $V(G_1 \times G_2) = V(G_1) \times V(G_2) = \{(x, y) ; x \in V(G_1), y \in V(G_2)\}$.

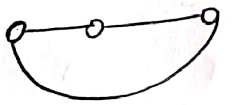
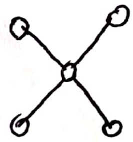
* Two pairs (x_1, y_1) and (x_2, y_2) will be adjacent iff one of the followings hold:

$x_1 = x_2$ and y_1 is adjacent with y_2 in G_2

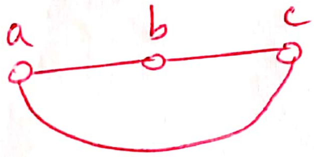
OR

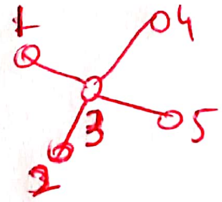
$y_1 = y_2$ and x_1 is adjacent with x_2 in G_1

* Easy steps to find the product of two graphs:

Let two graphs G_1:  and G_2: 

<u>step-I</u>: Label the vertices of both graphs if already not labelled

G_1: 

G_2: 

<u>step-II</u>: Take $V(G_1) \times V(G_2) = V(G_1 \times G_2)$

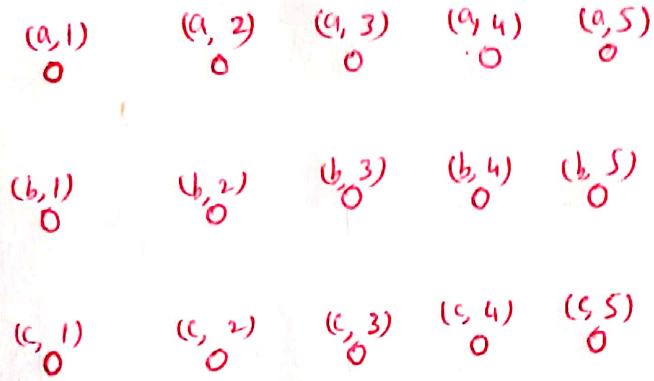
$V(G_1) \times V(G_2) = \{(a,1), (a,2), (a,3), (a,4), (a,5), (b,1), (b,2), (b,3), (b,4), (b,5), (c,1), (c,2), (c,3), (c,4), (c,5)\}$

<u>step-III</u>: Arrange all the pairs row-wise ~~or~~ or column-wise.

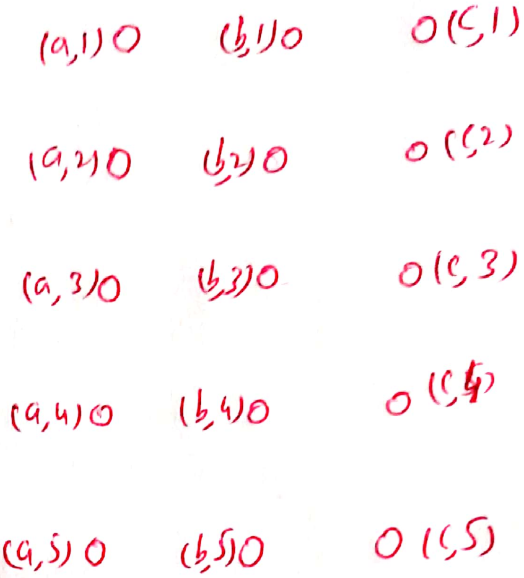
Either make rows correspond to each vertex of G_1

OR

make columns correspond to each vertex of G_2.

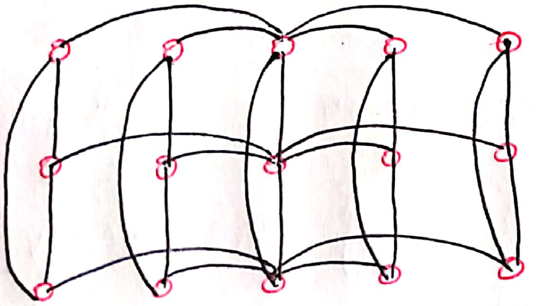


OR

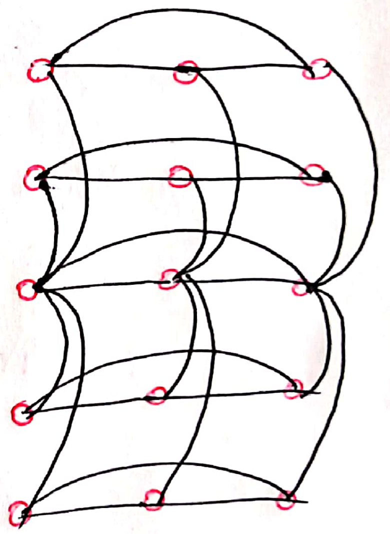


Step-IV

- If rows are correspond to the vertices of G_1, then draw G_1 in each column on the vertices in the column and draw G_2 in each row on the vertices in that row.
- If rows are correspond to the vertices of G_2, then row-wise draw G_1 and column-wise draw G_2.



$G_1 \times G_2$



* There is no need to label the vertices in $G_1 \times G_2$ if you understand the steps.

* Must remember that, in $G_1 \times G_2$, we get $|G_1|$ times G_2 and $|G_2|$ times G_1. Accordingly, we have

$$|G_1 \times G_2| = |G_1||G_2|$$

and $$S(G_1 \times G_2) = |G_1|\, S(G_2) + |G_2|\, S(G_1).$$

* $G_1 \times G_2 \cong G_2 \times G_1$

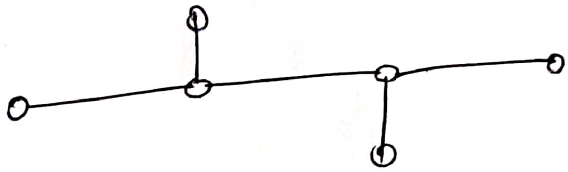
(5) <u>corona (Product) of graphs</u>: Let G_1 and G_2 be any two graphs. Then the corona of G_1 and G_2 is denoted by $G_1 \circ G_2$, and can be obtained as follows:

<u>step-I</u> Make G_1 one time ~~and G_2~~

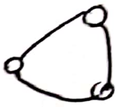
<u>step-II</u> Make G_2 $|G_1|$-times for each vertex of G_1

<u>step-III</u> Join each vertex of G_1 to the every vertex of its corresponding copy of G_2.

G_1 :

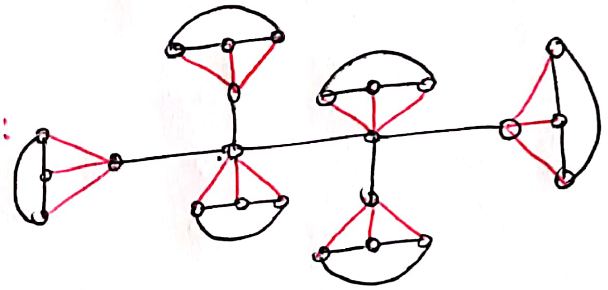


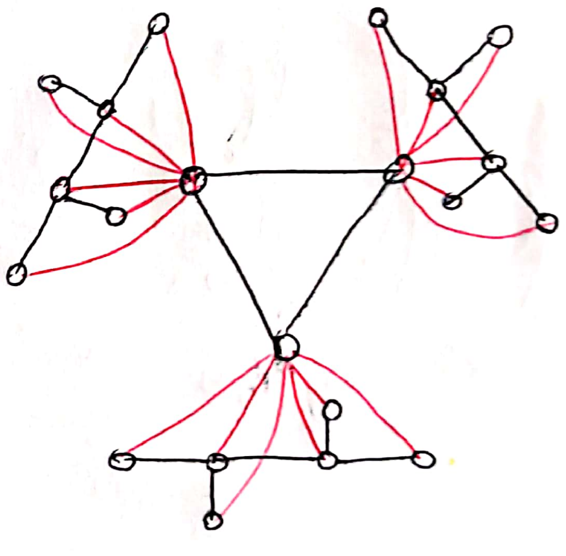
and G_2 :



$G_2 \circ G_1$:

$G_1 \circ G_2$:





* $G_1 \circ G_2 = G_2 \circ G_1$ iff $G_1 = G_2$
* $|G_1 \circ G_2| = |G_1| + |G_1||G_2|$
* $S(G_1 \circ G_2) = S(G_1) + |G_1| S(G_2) + |G_1||G_2|$
* Let $v \in V(G_1 \circ G_2)$, then

$$d(v) = \begin{cases} d(v) \text{ in } G_1 + |G_2|, & \text{if } v \in V(G_1), \\ d(v) \text{ in } G_2 + 1, & \text{if } v \in V(G_2). \end{cases}$$

Topic: Graph duality and Line graph

⑨

- The crossing number of a graph (simple) G is the minimum number of crossings of edges in the graph G, denoted by $cr(G)$.

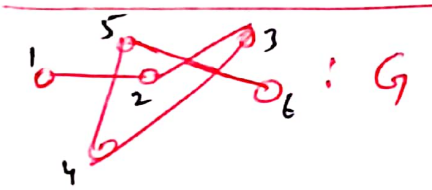
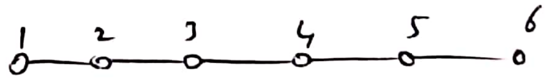
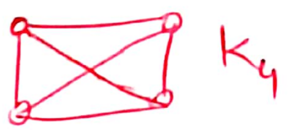
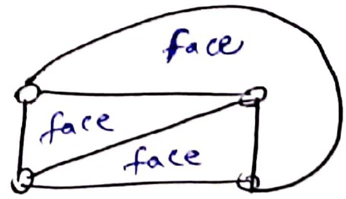
- A graph G is planar if $cr(G) = 0$. That is, there is no edge crossing in G.

- In a planar graph G, a region surrounded by edges but not crossed by any edge is called a face.

- Faces in a graph are of two types

unbounded face: The only one face which is the outside region of the graph

bounded faces: May or may not exist. These are the closed inside regions.

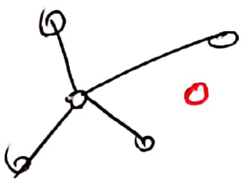
Graph	Nature
 : G It has no bounded face	Planar, because it can be reshaped to remove the crossings of edges  $cr(G)=0$
 K_4	Planar  $cr(K_4)=0$ 3 bounded faces, one unbounded face
C_n a cycle graph, $cr=0$	Planar with one bound face and one unbounded face.
K_n :	Planar only for $1 \le n \le 4$. $cr(K_5) = 1$.

- The **dual** of a planar graph G is denoted by $dual(G)$, and can be obtained with the following steps:

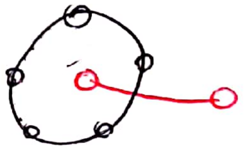
  Step-I Make one vertex on each face of G.

  Step-II Join two new vertices by an edge iff their corresponding faces share a common edge.

- The **bounded dual** of G, is the dual of G obtained by using only bounded faces of G, denoted by $dual^*(G)$.

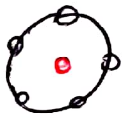
G_1: 

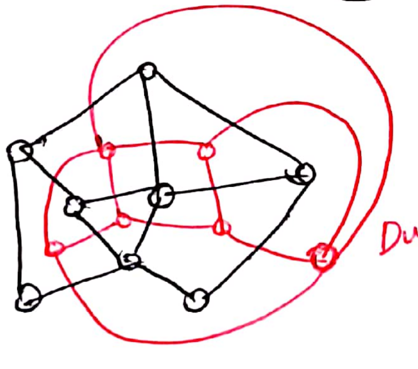
The dual of G_1 is the trivial graph, but bounded dual does not exist.

G_2: 

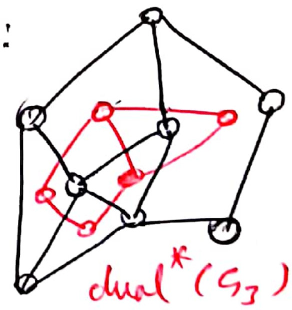
$dual(G_2) = P_2$

$dual^*(G_2) = K_1$ the trivial graph



G_3: 

Dual (G_3)

G_3: 

$dual^*(G_3)$

Remarks: ① If $dual(G)$ is trivial, then $dual^*(G)$ does not exist.

② $dual(G)$ is also a planar graph.

③ $dual(W_{1,n}) = W_{1,n}$ and $dual^*(W_{1,n}) = C_n$.

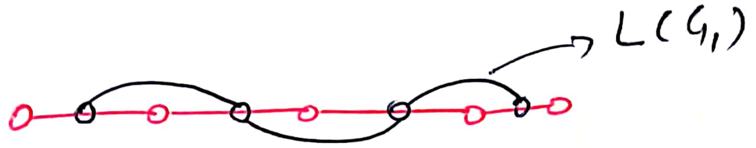
④ The Euler's formula: If n is the number of vertices, e is the number of ~~fa~~ edges and f is the number of faces (bounded + unbounded) in a planar graph, then $n+f-e=2$.

The Line graph of a simple graph is denoted by $L(G)$, and can be obtained by using the following steps:

<u>Step-I</u> Make one ~~edge~~ vertex on each edge of G.

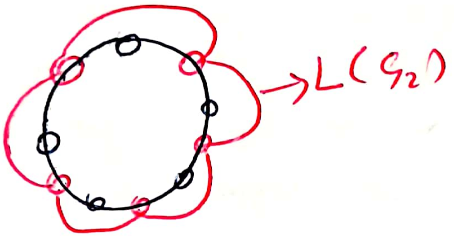
<u>Step-II</u> Join two new vertices by an edge iff their corresponding edges share a common vertex.

G_1 :



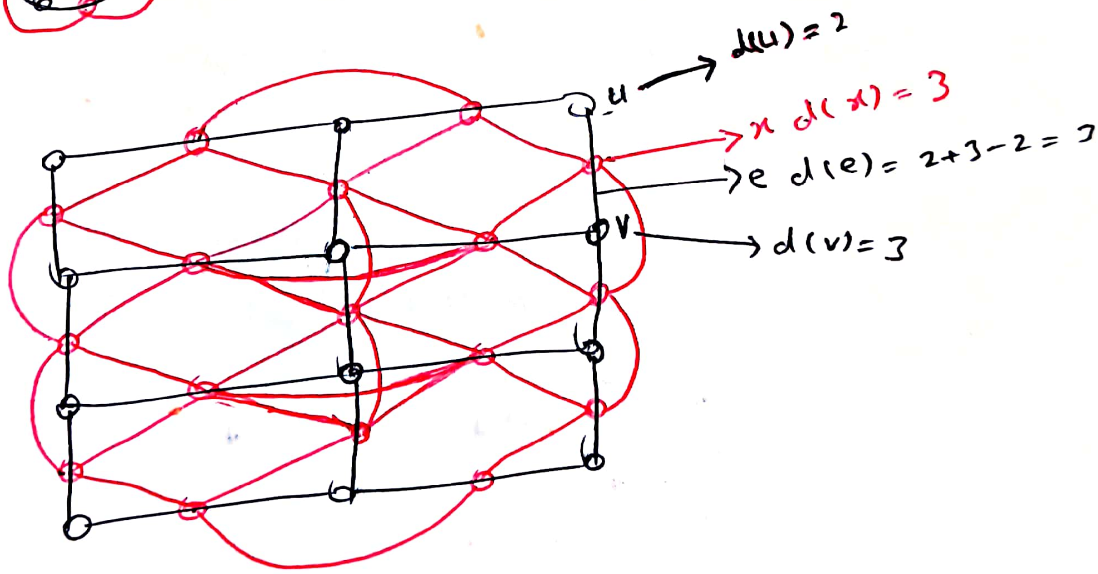
In fact, $L(P_n) = P_{n-1}$

G_2 :



In fact, $L(C_n) = C_n$

G_3 :



<u>Check</u>: If $e = uv$ be an edge in G and x is the vertex on e in $L(G)$, Then

$$d(x) = d(e) = d(u) + d(v) - 2.$$

For example, in G_3 let $e = uv$ and x is the vertex on e, then

$$d(x) = d(e) = 3.$$

This formula must be verified for each new vertex in $L(G)$.

Topic: Connectivity numbers, [redacted]

* A graph is connected if there exists a path between every two vertices in the graph.

– A connected graph can be disconnected in two ways:

- **By deleting edge(s):**

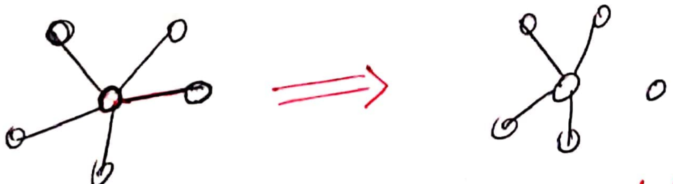
* A single edge, whose deletion from a connected graph G disconnects the graph G, is known as a bridge in G.

* A set of ~~minimum~~ ~~se~~ edges whose removal from G disconnects G is called a separating set for G.

* The cardinality of a smallest separating set for G is called the edge connectivity number of G, denoted by $\lambda(G)$.

* $\lambda(G) = 1 \iff G$ has a bridge.

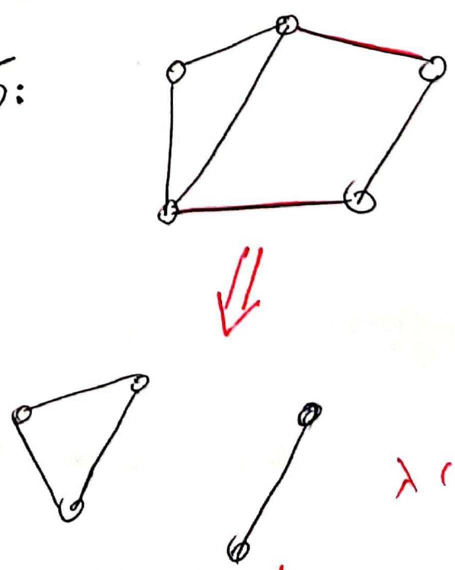
G:



Disconnected

every edge is a bridge
$\lambda(G) = 1$

G:



$\lambda(G) = 2$

Disconnected

- **By deleting vertex (vertices):**

* A single vertex, whose deletion from a connected graph G disconnects the graph G, is known as cut vertex in G.

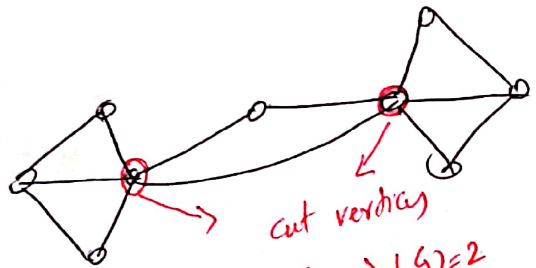
* A set of vertices whose removal from G disconnects G is called a cut-set for G.

* The cardinality of a smallest cut-set for G is called the connectivity number of G, denoted by $K(G)$.

* $K(G)=1 \Leftrightarrow$ G has a cut vertex.

* $K(G) \le \lambda(G)$.

G:



$K(G)=1$, $\lambda(G)=2$

* $K(P_n) = \lambda(P_n) = 1$

* $K(C_n) = \lambda(C_n) = 2$

* $K(K_n) = \lambda(K_n) = n-1$.

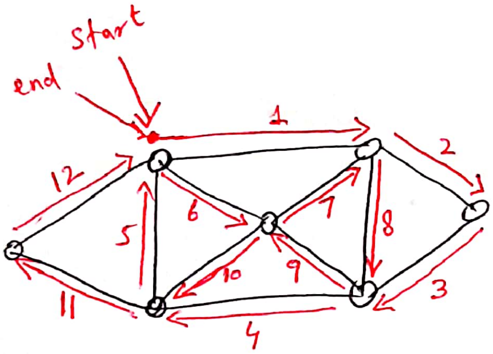
* If the connectivity number of a connected graph G is K, then G cannot be disconnect by deleting any number of vertices less than K.

Eulerian graph: Let G be a connected simple graph.

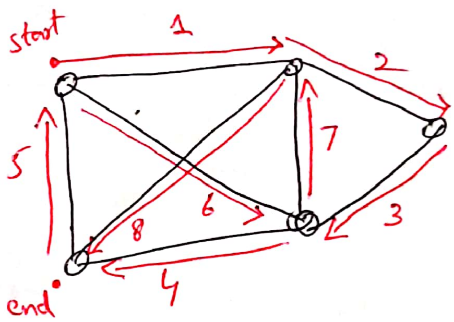
* A trail in G passing through each edge of G, and is closed is called an **Eulerian trail**. ~~and~~
* If such a trail is not closed then that trail is a **semi-Eulerian trail**.
* G is **Eulerian** if it has an Eulerian trail.
* G is **semi-Eulerian** if it has a semi-Eulerian trail.
* Otherwise, G is **non-Eulerian**.

Hamiltonian graph: Let G be a connected simple graph.

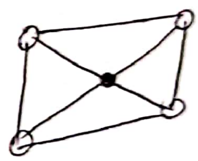
* A cycle in G passing through each vertex of G is called a **Hamiltonian cycle**.
* A path in G passing through each vertex of G is called a **Hamiltonian path**.
* G is **Hamiltonian** if it has a Hamiltonian cycle.
* G is **semi-Hamiltonian** if it has a Hamiltonian path.
* Otherwise, G is **non-Hamiltonian**.



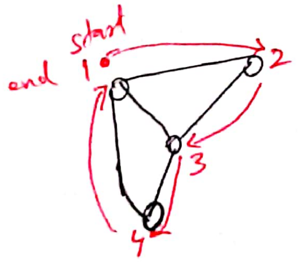
Eulerian graph



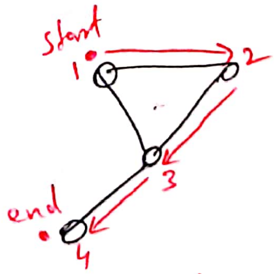
Semi-Eulerian graph



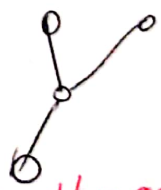
Non-Eulerian graph.



Hamiltonian graph



Semi-Hamiltonian graph



Non-Hamiltonian graph.

Remarks ① A connected graph G is an Eulerian graph $\Leftrightarrow$ it has no vertex of odd degree.

② A connected graph G is semi-Eulerian $\Leftrightarrow$ it has exactly two vertices of odd degree.

Topic: Distance in graphs

* Let G be a connected graph. For any two vertices u and v of G, then the length of a geodesic (shortest path) between u and v is called the distance, denoted by $d(u,v)$.

* $ecc(v) = \max\limits_{u \in V(G)} d(u,v)$ is called the eccentricity of v.

* $rad(G) =$ ~~$\min ecc(v)$~~ $\min\limits_{v \in V(G)} ecc(v)$ is called the radius of G.

* $diam(G) = \max\limits_{v \in V(G)} ecc(v)$ is called the diameter of G.

* A vertex v for which $ecc(v) = rad(G)$ is called a central vertex.

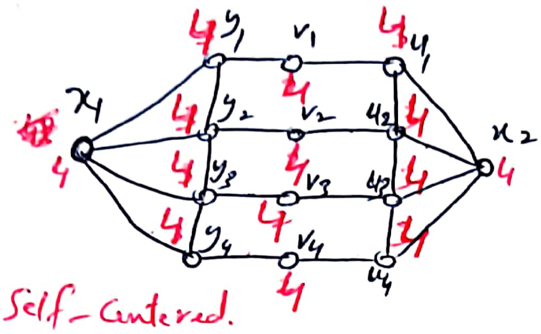
* A vertex v for which $ecc(v) = diam(G)$ is called a peripheral vertex

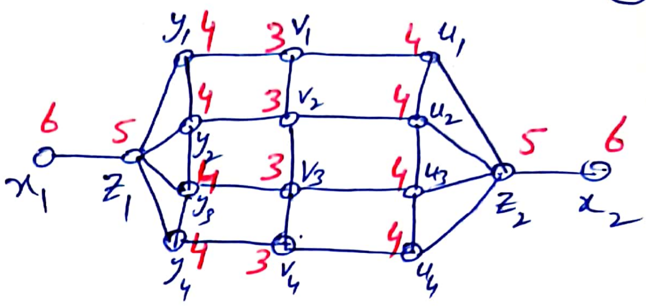
* The center of a graph G is induced by central vertices, denoted by $Cen(G)$.

* The Periphery of G is induced by peripheral vertices, denoted by $Per(G)$.

* If $rad(G) = diam(G)$, then G is self-centered graph.

G_1:





G_2

- G_1 is self-centered.
- $rad(G_2) = 3$, $diam(G_2) = 6$
- v_1, v_2, v_3, v_4 are central vertices, the center of G_2 is: $v_1 - v_2 - v_3 - v_4$, it is P_4.
- x_1 and x_2 are peripheral vertices, the periphery of G_2 is: x_1 x_2 , it is N_2.
- G_2 is not self centered

* every cycle graph C_n is self-centered.
* every complete graph is self-centered.

Generally, find the diameter and radius of

P_n, C_n, K_n, $K_{m,n}$, $K_{1,n}$, $W_{1,n}$, Petersen graph.

$P_m \times P_n$, $P_m + P_n$, $P_n \times C_m$, $C_m + C_n$, $C_n \circ N_1$,

$P_n \circ N_1$, $N_1 \circ P_n$, $N_1 \circ C_n$ $N_1 + (C_n \circ N_1)$.

Topic: Distance degree sequence

* let G be a connected graph. then for any vertex v of G, the sequence

$$dds(v) = (d_0(v), d_1(v), \dots d_{ecc(v)}(v))$$

is called the distance degree sequence of v, where $d_i(v)$ is the number of vertices lying at distance i from v, and is called the i^{th} distance degree of v.

Properties of $dds(v)$.

① $d_0(v) = 1$, $d_1(v) = d(v)$

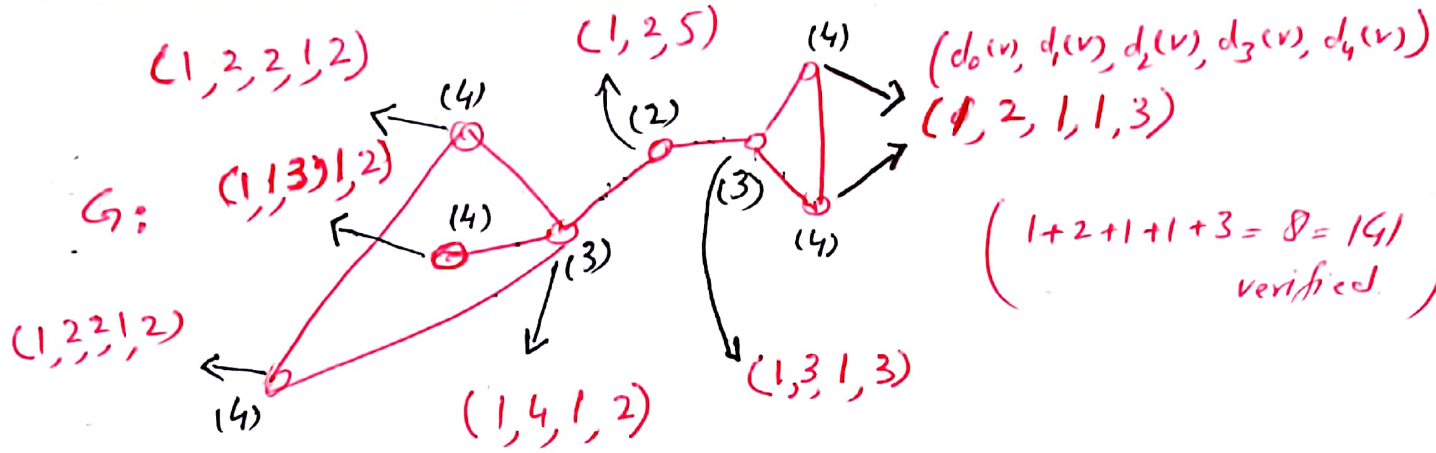
② The length of $dds(v)$ is $ecc(v)+1$.

③ $\sum_{i=0}^{ecc(v)} d_i(v) = |G|$ (the order of G).

* The collection of the $dds(v)$ for all v. is called the distance degree sequence of G, denoted by $dds(G)$.

* If $dds(v)$ is same for all v, then G is said to be distance degree regular. otherwise, G is distance degree injective.

Find the distance degree sequences of $C_5 \circ P_6$, $P_6 \circ C_5$, $K_m + N_n$, $K_1 + (N_n \cup K_m)$, $K_{m,n}$, $W_{1,n}$.



(1+2+1+1+3 = 8 = |G| verified)

$$dds(G) = \left((1,2,2,1,2)^2, (1,1,3,1,2), (1,4,1,2), (1,2,5), (1,3,1,3), (1,2,1,1,3)^2 \right)$$

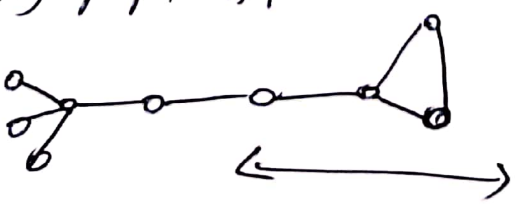
G is ~~not~~ distance degree injective

If we have the distance degree sequence of G as follows:

$$\left((1,2,1,1,3)^2 (1,1,3,1,1,2)^3, (1,3,1,1,3), (1,2,3,3), (1,2,4,2), (1,4,1,1,2) \right)$$

Then
$(1,2,1,1,3)^2$ provides that $|G| = 9$, two vertices have degree 2 and eccentricity $6-1=5$.
$(1,1,3,1,1,2)^3$ 〃 〃 〃 three 〃 〃 degree 1 〃 〃 $6-1=5$
$(1,3,1,1,3)$ 〃 〃 〃 〃, one vertex has degree 3 〃 〃 $5-1=4$
$(1,2,3,3)$ 〃 〃 〃 〃, 〃 〃 〃 〃 2 〃 〃 $4-1=3$
$(1,2,4,2)$ 〃 〃 〃 〃, 〃 〃 〃 〃 2 〃 〃 $4-1=3$
$(1,4,1,1,2)$ 〃 〃 〃 〃, 〃 〃 〃 〃 4 〃 〃 $5-1=4$

Thus $rad(G) = 3$, $diam(G) = 5$, G is not self centered.
G has two central vertices, 5 peripheral vertices
The degree sequence of G is 4, 3, 2, 2, 2, 2, 1, 1, 1.
The corresponding graph is:



it is simple

Topic: Tree and Spanning tree

Tree: A connected graph having no cycle.

Forest The union of trees

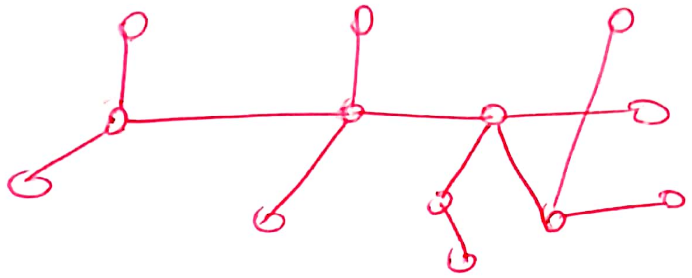
Properties of a Tree

Let T be a tree on n vertices.

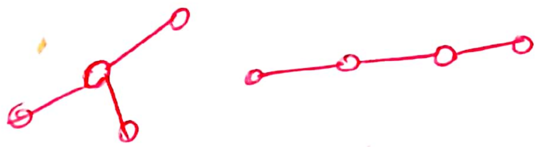
1. The number of edges in T is $n-1$.
2. Every edge in T is a bridge.
3. There is exactly one path between two vertices in T. So a geodesic is also a detour in T.
4. By adding one edge between any two vertices in T, we get exactly on. cycle

* **Spanning tree** let G be a connected graph. A tree spanned from G with following process is known as a. spanning tree:
Process: Find a cycle in G and make it non-cycle by removing any edge of that cycle. If there is no cycle remains, then the resultant graph is a spanning tree. Otherwise, find another cycle and make it non-cycle by removing its any edge. Continue this process untill all the cycles contained in G become non-cycles, then the final graph will be a spanning tree.

* Every path and Stare graph is a tree.



* There are only two trees on four vertices



* There are only three trees on five vertices

* Show that there only 6 different trees on 6 vertices, and only 11 different trees on 7 vertices

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