CH.NO.2 ①

Numerical NEWTON RAPHSON METHOD.

Statement.

If x_n is an approximation to an exact root 'α' of an equation $f(x)=0$, then a better approximation is in general is given by x_{n+1} where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Proof.

Let $f(x)=0$ be a given function. Let x_0 and x_1 be the points in the interval in which at least one real root of the equation lies.

As $x_1 = x_0 + h$ ——— (i)

where h is +ive and very very small.

$\therefore$ we have

$$f(x_1) = f(x_0+h) = 0$$

Applying Taylor's theorm.

$$f(x_1) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots = 0$$

Neglecting higher powers of h.

$$f(x_1) = f(x_0) + hf'(x_0) = 0$$

$$\Rightarrow f(x_0) + hf'(x_0) = 0$$

$$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)} \text{ ——— (ii)}$$

putting Eq (ii) in Eq (i)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \text{ ——— (a)}$$

Now suppose that x_2 be the root of $f(x)=0$

$$\therefore f(x_2) = 0$$

$$\Rightarrow f(x_1+h) = 0$$

Applying Taylor's theorm.

$$f(x_1) + hf'(x_1) + \frac{h^2}{2!} f''(x_1) + \cdots = 0$$

Neglecting higher powers of h.

$$f(x_1) + hf'(x_1) = 0$$

$$\Rightarrow h = -\frac{f(x_1)}{f'(x_1)}$$

$$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad \text{——— (b)}$$

Similarly

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \quad \text{——— (c)}$$

In general

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{——— (d)}$$

(d) is Newton Raphson method.

Questions related to Newton Raphson method.

Q 1

Using Newton Raphson method find to 4D the root near 0.5 of the Equation.

$$e^{-x} - \sin x = 0$$

Sol.

$$f(x) = e^{-x} - \sin x = 0$$

$$f'(x) = -(e^{-x} + \cos x) = 0$$

$$x_0 = 0.5$$

Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{——— (a)}$$

Put $n=0$ in Eq (a)

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

As $f(x_0) = 0.12711$

and $f'(x_0) = -1.48411$

$$x_1 = 0.5 + \frac{0.12711}{1.48411}$$

$$x_1 = 0.58565$$

Put $n=1$ in Eq (a)

$$\therefore\ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

As $f(x_1) = 0.00400$

and $f'(x_1) = -1.39010$

$$\therefore\ x_2 = 0.58585 + \frac{0.00400}{1.39010}$$

$$x_2 = 0.58853$$

Put $n=2$ in Eq (a)

$$\therefore\ x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

As $f(x_2) = 0.00006381$

$\Rightarrow$ 0.58853 is the root of given Eq.

Q-2.

Find the root of $x^3 - 3x - 3 = 0$ to four decimal places using Newton Raphson method, that lies near $2 = x$.

Sol.

$$f(x) = x^3 - 3x - 3 = 0$$

$$f'(x) = 3x^2 - 3 = 0$$

$$x_0 = 2$$

Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{——— (a)}$$

put $n=0$ in Eq (a)

$$\therefore\ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

As $f(x_0) = -1$

and $f'(x_0) = 9$

$$\therefore\ x_1 = 2 + \frac{1}{9}$$

$$= 2.11111$$

put $n=1$ in Eq (a)

$$\therefore\ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

As $f(x_1) = 0.07543$

and $f'(x_1) = 10.37036$

$$\therefore\ x_2 = 2.11111 - \frac{0.07543}{10.37036}$$

$$= 2.10384$$

put $n=2$ in Eq (a)

$\therefore \quad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

As $\quad f(x_2) = 0.00036$

and $\quad f'(x_2) = 10.27843$

$\therefore \quad x_3 = 2.10384 - \frac{0.00036}{10.27843}$

$x_3 = 2.10380$

put $n = 3$ in Eq (a)

$\Rightarrow \quad x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$

As $\quad f(x_3) = -0.00003$

Hence 2.10380 is the root of given Eq.

Q-3.

Using Newton Raphson method find to 2D the root b/w 0 and 1 of the equation $e^x - 3x = 0$.

Sol.

$f(x) = e^x - 3x = 0$

$f'(x) = e^x - 3 = 0$

formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (a)$$

put $n=0$ in Eq (a)

$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

As $\quad f(x_0) = 1$

$f'(x_0) = -2$

$\Rightarrow \quad x_1 = 0 + \frac{1}{2}$

$x_1 = 0.5000$

Put $n = 1$ in Eq (a)

$\therefore \quad x_2 = x_1 - \frac{f(x_1)}{f(x_1)}$

As $\quad f(x_1) = 0.14872$

and $f'(x_1) = -1.35128$

$\Rightarrow \quad x_2 = 0.5 + \frac{0.14872}{1.35128}$

$x_2 = 0.61006$

Put $n = 2$ in Eq (a)

$x_3 = x_2 \ldots$

As $f(x_2) \ldots$

and $f'(x_2) \ldots$

$\Rightarrow \ldots x_3 \ldots$

put n…

$\therefore$

As

Hence

Q-4.

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Sol.

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

As $f(x_2) = 0.01036$

and $f'(x_2) = -1.15946$

$$\Rightarrow x_3 = 0.61006 + \frac{0.01036}{1.15946}$$

$$= 0.67900 \simeq 0.62$$

put $n = 3$ in Eq (a)

$$\therefore x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

As $f(x_3) = 0.00007$

Hence 0.62 is the root of given Eq.

Q_4.

Using Newton Raphson method find to 4D the root b/w 0.4 and 0.6 of the equation $\sin x = 5x - 2$.

Sol.

-0.0205

-4.1224

$$f(x) = \sin x - 5x + 2 = 0 \quad 0.3894$$

$$f'(x) = \cos x - 5 = 0 \quad 0.1121$$

formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{———— (a)}$$

put $n = 0$ in Eq (a)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

As $f(x_0) = 0.38942$

$f'(x_0) = -4.07894$

$$\Rightarrow x_1 = 0.4 + \frac{0.38942}{4.07894}$$

$$x_1 = 0.49547$$

put $n = 1$ in Eq (a)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

As $f(x_1) = -0.00190$

$f'(x_1) = -4.12025$

$$x_2 = 0.49547 - \frac{0.00190}{4.12025}$$

$$x_2 = 0.49501$$

put $n = 2$ in Eq a

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

As $f(x_2) = -0.00001$

Hence 0.49501 is the root of given E

Q-5. Using Newton Raphson method fi to 4D the root b/w 0 and 1 of the equa tion $e^{-x} - \sin(\frac{\pi x}{2})$.

Sol.

$$f(x) = e^{-x} - \sin(\frac{\pi x}{2})$$

$$f'(x) = -\{e^{-x} + \frac{\pi}{2}\cos(\frac{\pi x}{2})\}$$

formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{——— (a)}$$

put $n=0$ in Eq (a) ———

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

As $f(x_0) = 1$

and $f'(x_0) = -2.57080$

$$\Rightarrow x_1 = 0 + \frac{1}{2.57080}$$

$$x_1 = 0.38898$$

put $n=1$ in Eq (a)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

As $f(x_1) = 0.10405$

and $f'(x_1) = -2.59553$

$$\Rightarrow x_2 = 0.38898 + \frac{0.10405}{2.59553}$$

$$x_2 = 0.42907$$

put $n=2$ in Eq (a)

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

As $f(x_2) = 0.02701$

and $f'(x_2) = -2.66149$

$$\Rightarrow x_3 = 0.42907 + \frac{0.02701}{2.66149}$$

$$x_3 = 0.43922$$

put $n=3$ in Eq (a)

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

As $f(x_3) = 0.00806$

$f'(x_3) = -2.68117$

$\Rightarrow$ $x_4 = 0$

$x_4 =$

put $n=4$

x_5

As $f(x$

$f'(x$

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$\Rightarrow \quad x_4 = 0.43922 + \frac{0.00806}{2.68117}$

$x_4 = 0.44223$

Put $n = 4$ in Eq (a)

$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)}$

As $f(x_4) = 0.00248$

$f'(x_4) = -2.68718$

$\Rightarrow \quad x_5 = 0.44223 + \frac{0.00248}{2.68718}$

$x_5 = 0.44315$

Put $n=5$ in Eq (a)

$x_6 = x_5 - \frac{f(x_5)}{f'(x_5)}$

As $f(x_5) = 0.00078$

$f'(x_5) = -2.68905$

$\Rightarrow \quad x_6 = 0.44315 + \frac{0.00078}{2.68905}$

$x_6 = 0.44344$

put $n=6$ in Eq (a)

$x_7 = x_6 - \frac{f(x_6)}{f'(x_6)}$

As $f(x_6) = 0.00025$

$f'(x_6) = -2.68965$

$\Rightarrow \quad x_7 = 0.44344 + \frac{0.00025}{2.68965}$

$x_7 = 0.44353$

put $n = 7$ in Eq (a)

$x_8 = x_7 - \frac{f(x_7)}{f'(x_7)}$

As $f(x_7) = 0.00008$

Hence 0.44353 in the root of given Eq.

Q-6.

Evaluate the five real roots of the following using Newton Raphson method.

(i) $x^4 + 0.5x^3 + 0.84x^2 - 1.66x - 1.32 = 0$

(ii) $x^4 - 3.2x^3 + 3.21x^2 - 12.6x + 13.23 = 0$

Sol.

(i) $f(x) = x^4 + 0.5x^3 + 0.84x^2 - 1.66x - 1.32 = 0$

$f'(x) = 4x^3 + 1.5x^2 + 1.68x - 1.66 = 0$

By Isolation of roots

$f(0) = -1.32$ i.e $-ive$

$f(1) = -0.64$ i.e $-ive$

$f(2) = 18.72$ i.e $+ive$

Hence the root of given Eq lies b/w the interval $[1, 2]$. i.e $x_0 = 1$

formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Put $n = 0$ in Eq (a)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

As $f(x_0) = -0.64$

$f'(x_0) = 5.52$

$$\Rightarrow x_1 = 1 + \frac{0.64}{5.52}$$

$x_1 = 1.1159$

Put $n = 1$ in Eq (a)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

As $f(x_1) = 0.1190$

$f'(x_1) = 7.6408$

$$\Rightarrow x_2 = 1.1159 - \frac{0.1190}{7.6408}$$

$x_2 = 1.1003$

Put $n = 2$ in Eq (a)

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

As $f(x_2) = 0.0022$

$f'(x_2) = 7.3329$

$$\Rightarrow x_3 = 1.1003 - \frac{0.0022}{7.3329}$$

$x_3 = 1.1000$

Put $n = 3$ in Eq (a)

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

As $f(x_3) = 0.0000$

Hence 1.1 is the root of given Eq.

(ii) $f(x) = x^4 - 3.2x^3 + 3.21x^2 - 12.6x + 13.23 = 0$

$f'(x) = 4x^3 - 9.6x^2 + 6.42x - 12.6 = 0$

By Isolation of roots.

$f(0) = 13.23$ i.e +ive

$f(1) = 1.64$ i.e +ive

$f(2) = -8.73$ i.e -ive

Hence the root of Equation lies b/w the interval $[1, 2]$ i.e $x_0 = 1$

formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{———} (a)$$

Put $n = 0$ in Eq (a)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

As $f(x_0) = 0.64$

and $f'(x_0) = -11.78$

$$\Rightarrow x_1 = 1 + \frac{0.64}{11.78}$$

$$x_1 = 1.1392$$

Put $n = 1^s$ in Eq (a)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

As $f(x_1) = -0.0048$

and $f'(x_1) = -11.8313$

$$\Rightarrow x_2 = 1.1392 - \frac{0.0048}{11.8313}$$

$$x_2 = 1.1388$$

put $n = 2$ in Eq (a)

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

As $f(x_2) = -0.00007$

Hence 1.1388 is the root of given Eq.

Q-7.

Determine the +ive real root of the following equations upto 4D using Newton Raphson method.

(i) $4 \sin x = e^x$ in the interval $(0, 0.5)$.

(ii) $e^x \sin x = \frac{1}{2}x$ in the interval $(0, \pi)$.

(iii) $2 \sinh x = \cosh x$ in the interval $(0, 1)$.