Distributions and Good Coding Practice

February 13, 2024

1 Applied Data Science 1

1.0.1 Module Leader: Dr. William Cooper

1.1 Distributions

Expanding on our basic statistics, let's think here about distributions.

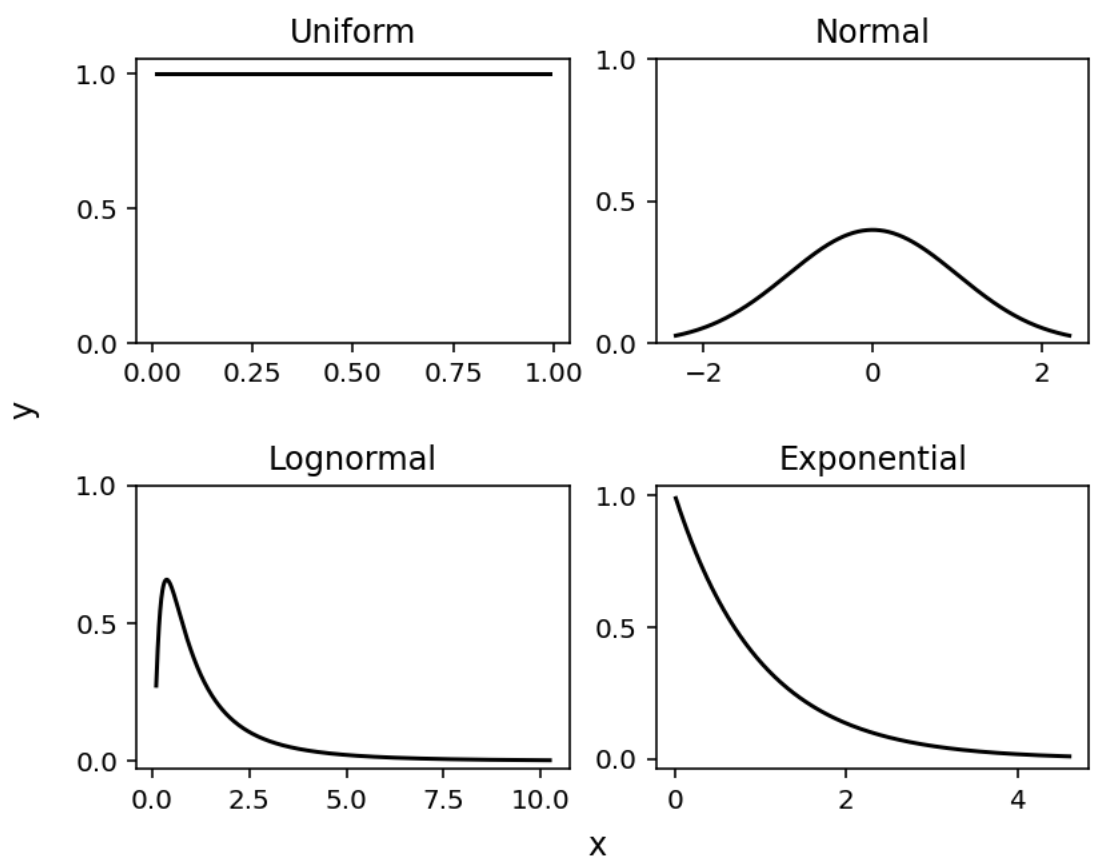
[2]:
```python
import matplotlib.pyplot as plt
import numpy as np
import pandas as pd
import scipy.stats as ss
```

[7]:
```python
x_uniform = np.linspace(ss.uniform.ppf(0.01), ss.uniform.ppf(0.99), 1000)
x_normal = np.linspace(ss.norm.ppf(0.01), ss.norm.ppf(0.99), 1000)
x_lognormal = np.linspace(ss.lognorm.ppf(0.01, 1), ss.lognorm.ppf(0.99, 1),␣
 ↪1000)
x_exp = np.linspace(ss.expon.ppf(0.01), ss.expon.ppf(0.99), 1000)

y_uniform = ss.uniform.pdf(x_uniform)
y_normal = ss.norm.pdf(x_normal)
y_lognormal = ss.lognorm.pdf(x_lognormal, 1)
y_exp = ss.expon.pdf(x_exp)

x_distributions = (x_uniform, x_normal, x_lognormal, x_exp)
y_distributions = (y_uniform, y_normal, y_lognormal, y_exp)
distributions = ('Uniform', 'Normal', 'Lognormal', 'Exponential')
```

[16]:
```python
def plot_distributions():
    """
    Plots the four distributions, uniform, normal, lognormal and exponential
    """
    fig, axs = plt.subplots(2, 2, dpi=144)
    axs = axs.flatten()

    for i, dist in enumerate(distributions):
        axs[i].plot(x_distributions[i], y_distributions[i], 'k-')
```

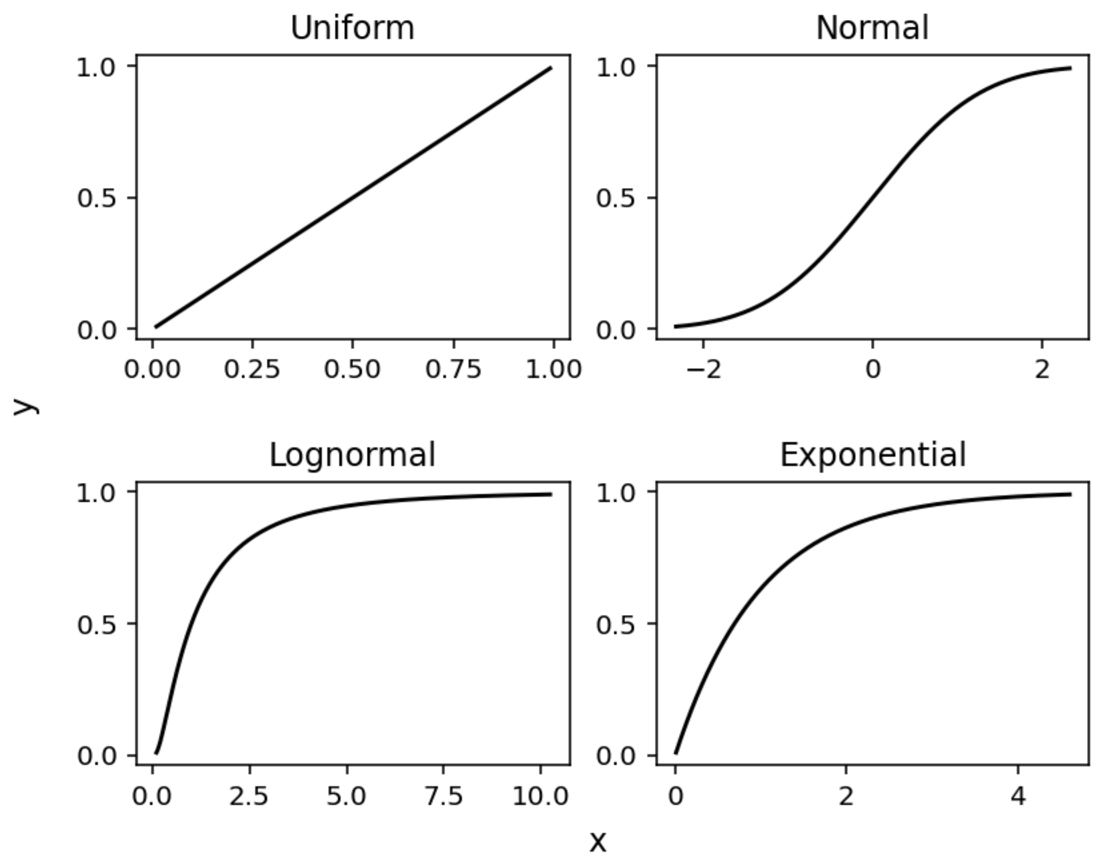
```
        axs[i].set_title(dist)
        axs[i].set_yticks(np.arange(0, 1.5, 0.5))

    fig.supxlabel('x')
    fig.supylabel('y')
    fig.subplots_adjust(hspace=0.5)
    plt.show()
    return
```

[17]:
```
plot_distributions()
```



These are the probability density functions, but what about the cumulative distributions?

[18]:
```
y_uniform = ss.uniform.cdf(x_uniform)
y_normal = ss.norm.cdf(x_normal)
y_lognormal = ss.lognorm.cdf(x_lognormal, 1)
y_exp = ss.expon.cdf(x_exp)

y_distributions = (y_uniform, y_normal, y_lognormal, y_exp)
plot_distributions()
```



Focusing just on the normal distribution, this can be modified by the four statistical moments. Let's look at the first moment:

$$\mu_1 = \int_{-\infty}^{\infty} x^1 F(x)\, dx = \frac{1}{T}\sum_{i=1}^{T} x_i^1$$

The sum is just the well known expression for the average $\overline{x}$. Carrying out the integration over the normal distribution:

$$N(x; a, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\frac{(x-a)^2}{\sigma^2}\right)$$

results in the peak (average) value a.

The 2nd moment is

$$\mu_2 = \int_{-\infty}^{\infty} x^2 F(x)\, dx = \frac{1}{T}\sum_{i=1}^{T} x_i^2$$

However, it is often preferred to look at the *centralised* 2nd and higher moments, because they often carry more meaningful information.

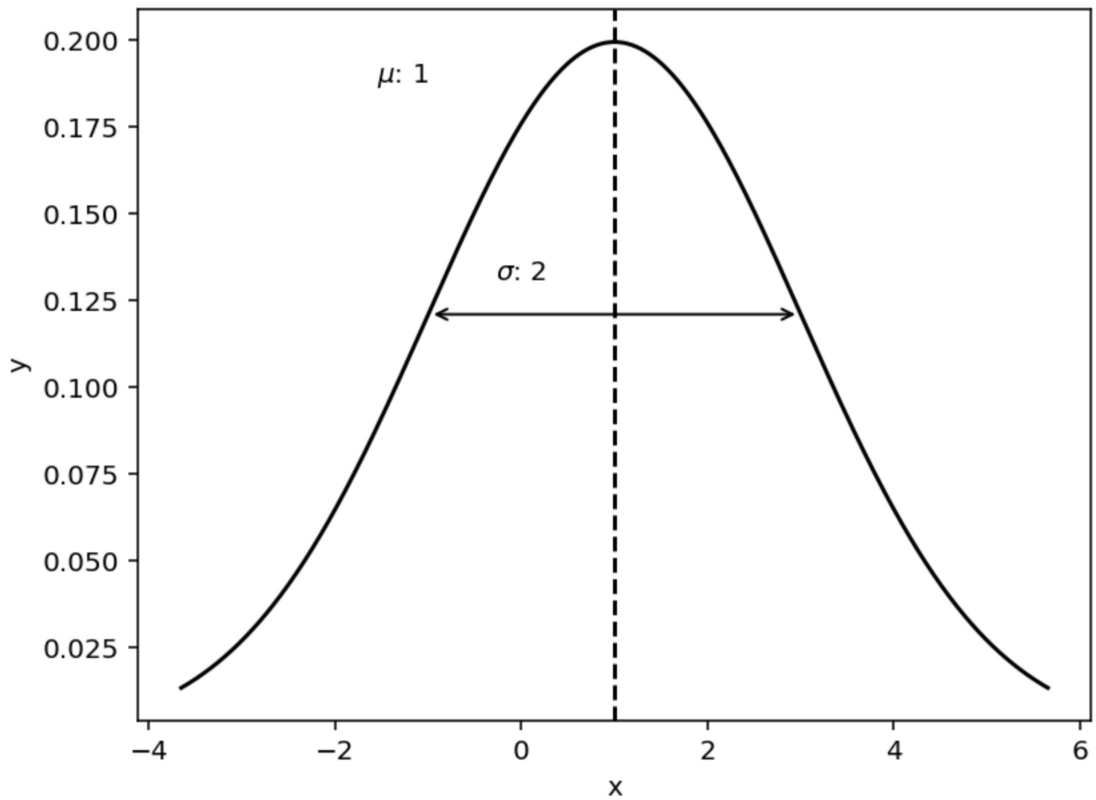
$$\mu_2^{(c)} = \int_{-\infty}^{\infty} (x - \overline{x})^2 F(x)\, dx = \frac{1}{T} \sum_{i=1}^{T} (x_i - \overline{x})^2$$

Carrying out this integration for the normal distribution gives the variance σ^2.

[21]:
```
norm = ss.norm(loc=1, scale=2)
x_normal = np.linspace(norm.ppf(0.01), norm.ppf(0.99), 1000)
y_normal = norm.pdf(x_normal)
mean = norm.mean()
std = norm.std()
```

[37]:
```
def plot_normal():
    """
    Plots the normal distribution
    """
    plt.figure(dpi=144)

    plt.plot(x_normal, y_normal, 'k-')

    plt.axvline(x=mean, color='k', linestyle='--', label='Mean')
    plt.text(mean-std, plt.ylim()[1]*0.9, f'$\mu$: {mean:0g}',␣
 ↪horizontalalignment='right')

    plt.annotate('', xy=(mean-std, norm.pdf(mean-std)), xytext=(mean+std, norm.
 ↪pdf(mean+std)),
                 arrowprops=dict(arrowstyle='<->', lw=1))
    plt.text(mean-0.5*std, norm.pdf(mean-std)+0.01, f'$\sigma$: {std:0g}',␣
 ↪horizontalalignment='center')

    plt.xlabel('x')
    plt.ylabel('y')
    plt.show()
    return
```

[38]:
```
plot_normal()
```



The 3rd moment is

$$\mu_3 = \int_{-\infty}^{\infty} x^3 F(x)\, dx = \frac{1}{T} \sum_{i=1}^{T} x_i^3$$

It is usual to look at the *centralised and normalised* 3rd and higher moments. That gives a number, which allows us to interpret the relevant property of the distributions, independent of average and variance.

$$\mu_3^{(\mathrm{n})} = \int_{-\infty}^{\infty} \left(\frac{x - \overline{x}}{\sigma} \right)^3 F(x)\, dx = \frac{1}{T} \sum_{i=1}^{T} \left(\frac{x_i - \overline{x}}{\sigma} \right)^3$$

The result of this integration for the normal distribution is 0.

The third moment is called *skewness*. It measures the asymmetry of distribution functions. The normal distribution is symmetric and has a skewness of 0.

- *Skewness = 0*. The distribution is symmetric.
- *Skewness < 0*. Negative skew. Stronger left tail. The distribution is said to be *left-tailed* or *left skewed*. The average is larger than the median.
- *Skewness > 0*. Positive skew. Stronger right tail. The distribution is said to be *right-tailed* or *right skewed*. The average is smaller than the median.

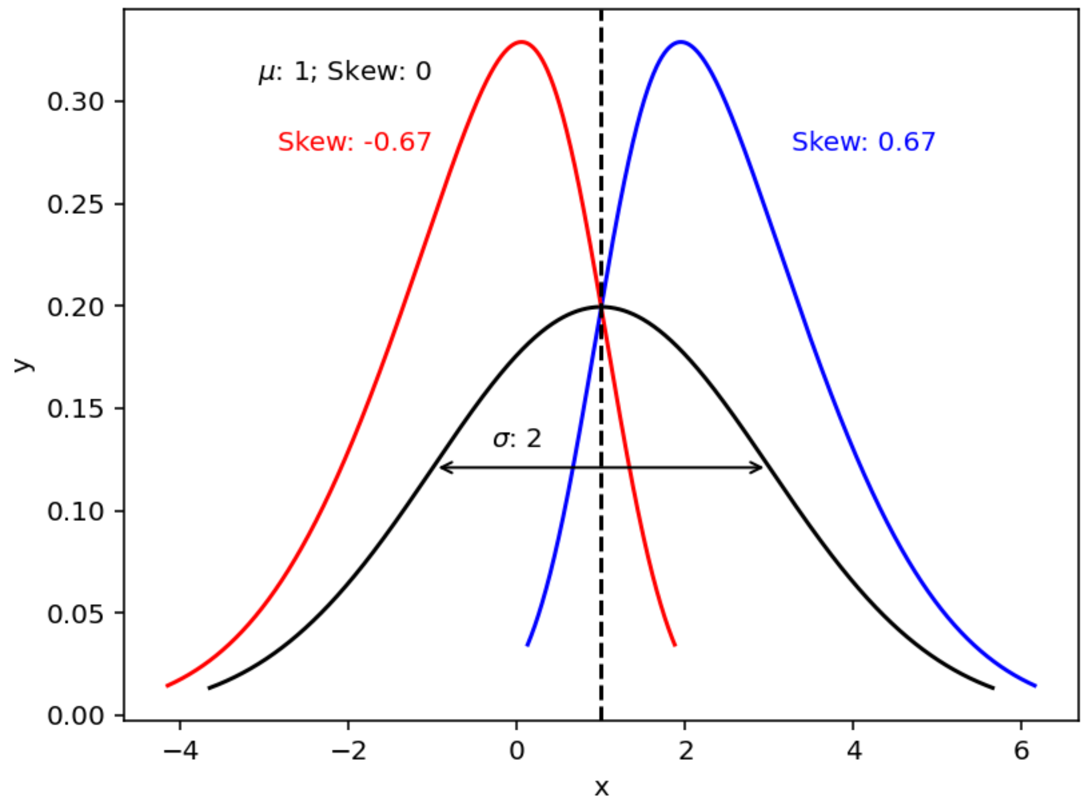
Both distributions have identical average and variance.

The skewness of the normal distribution is 0.

[41]:
```
skewnorm_p = ss.skewnorm(3, loc=1, scale=2)
x_skewnormal_p = np.linspace(skewnorm_p.ppf(0.01), skewnorm_p.ppf(0.99), 1000)
y_skewnormal_p = skewnorm_p.pdf(x_skewnormal_p)

skewnorm_n = ss.skewnorm(-3, loc=1, scale=2)
x_skewnormal_n = np.linspace(skewnorm_n.ppf(0.01), skewnorm_n.ppf(0.99), 1000)
y_skewnormal_n = skewnorm_n.pdf(x_skewnormal_n)

skew = norm.stats(moments='s')
skew_p = skewnorm_p.stats(moments='s')
skew_n = skewnorm_n.stats(moments='s')
```

[50]:
```
def plot_skewed_normal():
    """
    Plots the skewed normal distribution
    """
    plt.figure(dpi=144)

    plt.plot(x_normal, y_normal, 'k-')
    plt.plot(x_skewnormal_p, y_skewnormal_p, 'b-')
    plt.plot(x_skewnormal_n, y_skewnormal_n, 'r-')

    plt.axvline(x=mean, color='k', linestyle='--', label='Mean')
    plt.text(mean-std, plt.ylim()[1]*0.9, f'$\mu$: {mean:0g}; Skew: {skew:0g}',
             horizontalalignment='right')

    plt.annotate('', xy=(mean-std, norm.pdf(mean-std)), xytext=(mean+std, norm.
 ↪pdf(mean+std)),
                 arrowprops=dict(arrowstyle='<->', lw=1))
    plt.text(mean-0.5*std, norm.pdf(mean-std)+0.01, f'$\sigma$: {std:0g}',␣
 ↪horizontalalignment='center')

    plt.text(mean+2*std, plt.ylim()[1]*0.8, f'Skew: {skew_p:.2f}',
             horizontalalignment='right', color='b')
    plt.text(mean-std, plt.ylim()[1]*0.8, f'Skew: {skew_n:.2f}',
             horizontalalignment='right', color='r')


    plt.xlabel('x')
    plt.ylabel('y')
    plt.show()
    return
```

[51]:
```
plot_skewed_normal()
```



The 4th moment is

$$\mu_4 = \int_{-\infty}^{\infty} x^4 F(x)\, dx = \frac{1}{T} \sum_{i=1}^{T} x_i^4$$

Again, usually the *centralised and normalised* 4th moment is preferred.

$$\mu_4^{(\mathrm{n})} = \int_{-\infty}^{\infty} \left(\frac{x - \overline{x}}{\sigma} \right)^4 F(x)\, dx = \frac{1}{T} \sum_{i=1}^{T} \left(\frac{x_i - \overline{x}}{\sigma} \right)^4$$

The result of this integration for the normal distribution is **3**.

The fourth moment is called *kurtosis*. This is called *Pearson's kurtosis*. It measures the strength of the tails.

One is usually interested in the question whether the tails are *stronger* or *weaker* than that of a normal distribution. Thus, an *excess kurtosis* is defined.

$$\mathrm{Kurt} = \mu_4^{\mathrm{n}} - 3$$

This is called *Fisher's kurtosis*.

Always check the documentation for which version of skewness and kurtosis are used. Sometimes they are implemented with modifications for low-number statistics.

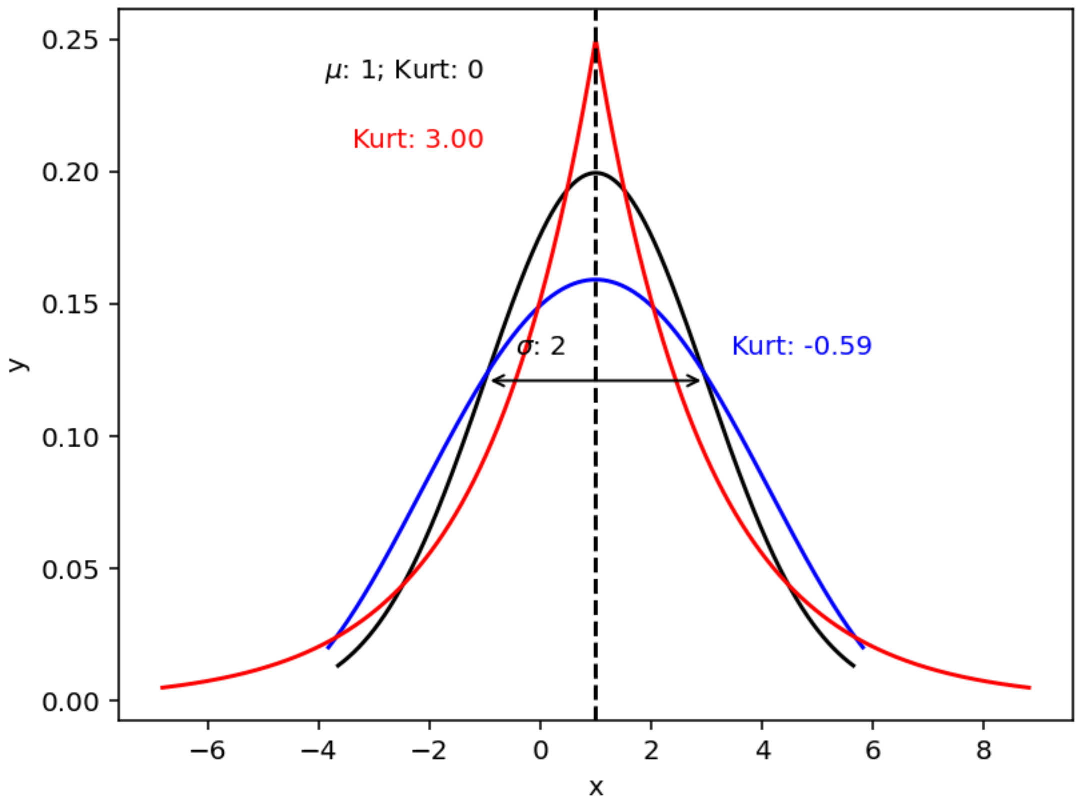
- *(Excess) kurtosis = 0.* The strengths of the tail is the same as for the normal distribution. The distribution is called *mesokurtic* (from Greek meso, meaning intermediate). In practical terms an empirical distribution with a skewness of 0 and an excess kurtosis of 0 is very likely normally distributed.
- *(Excess) kurtosis > 0.* The tails are stronger than for the normal distribution, the peak is steeper. These distributions are called *leptokurtic* (from Greek lepto, meaning narrow).
- *(Excess) kurtosis < 0.* The distribution has a broader, lower peak and thinner tails than the normal distribution. These distributions are called *platykurtic* (from Greek platy, meaning broad).

[55]:
```
platy = ss.cosine(loc=1, scale=2)
x_platy = np.linspace(platy.ppf(0.01), platy.ppf(0.99), 1000)
y_platy = platy.pdf(x_platy)

lepto = ss.laplace(loc=1, scale=2)
x_lepto = np.linspace(lepto.ppf(0.01), lepto.ppf(0.99), 1000)
y_lepto = lepto.pdf(x_lepto)

kurt = norm.stats(moments='k')
platy_kurt = platy.stats(moments='k')
lepto_kurt = lepto.stats(moments='k')
```

[57]:
```
def plot_excess_kurtosis_normal():
    """
    Plots the normal distribution against distributions with excess kurtosis
    """
    plt.figure(dpi=144)

    plt.plot(x_normal, y_normal, 'k-')
    plt.plot(x_platy, y_platy, 'b-')
    plt.plot(x_lepto, y_lepto, 'r-')

    plt.axvline(x=mean, color='k', linestyle='--', label='Mean')
    plt.text(mean-std, plt.ylim()[1]*0.9, f'$\mu$: {mean:0g}; Kurt: {kurt:0g}',
             horizontalalignment='right')

    plt.annotate('', xy=(mean-std, norm.pdf(mean-std)), xytext=(mean+std, norm.
 ↪pdf(mean+std)),
                 arrowprops=dict(arrowstyle='<->', lw=1))
    plt.text(mean-0.5*std, norm.pdf(mean-std)+0.01, f'$\sigma$: {std:0g}',␣
 ↪horizontalalignment='center')

    plt.text(mean+2.5*std, plt.ylim()[1]*0.5, f'Kurt: {platy_kurt:.2f}',
             horizontalalignment='right', color='b')
```

```
    plt.text(mean-std, plt.ylim()[1]*0.8, f'Kurt: {lepto_kurt:.2f}',
             horizontalalignment='right', color='r')


    plt.xlabel('x')
    plt.ylabel('y')
    plt.show()
    return
```

[58]:
```
plot_excess_kurtosis_normal()
```



2 Question 1

Load the *distributions* file, view some basic statistics, then, view them as histograms and print the 4 major statistical moments for each distribution (scipy.stats has inbuilt methods for skewness and kurtosis). Set the range and number of bins of the histograms so they are a fair comparison. What distributions do you think they are samplings of?

[ ]:
```
df_dists =
```

[ ]:
```python
def plot_file_distributions():
    """
    Plots the four distributions from the file
    """
    return
```

[ ]:
```python
plot_file_distributions()
```

[ ]:
```python
def print_stats(dist):
    """
    Prints the moments of each distribution
    """
    return
```

[ ]:
```python
for dist in df_dists.columns:
```

2.1 End Question 1

2.2 Errors and Probabilities

Science needs uncertainties estimates when making statements, how do we do this for unknown datasets when looking at the statistical moments?

[97]:
```python
df_ftse = pd.read_csv('data/FTSE100_2022.csv', index_col='Date')
df_ftse.head()
```

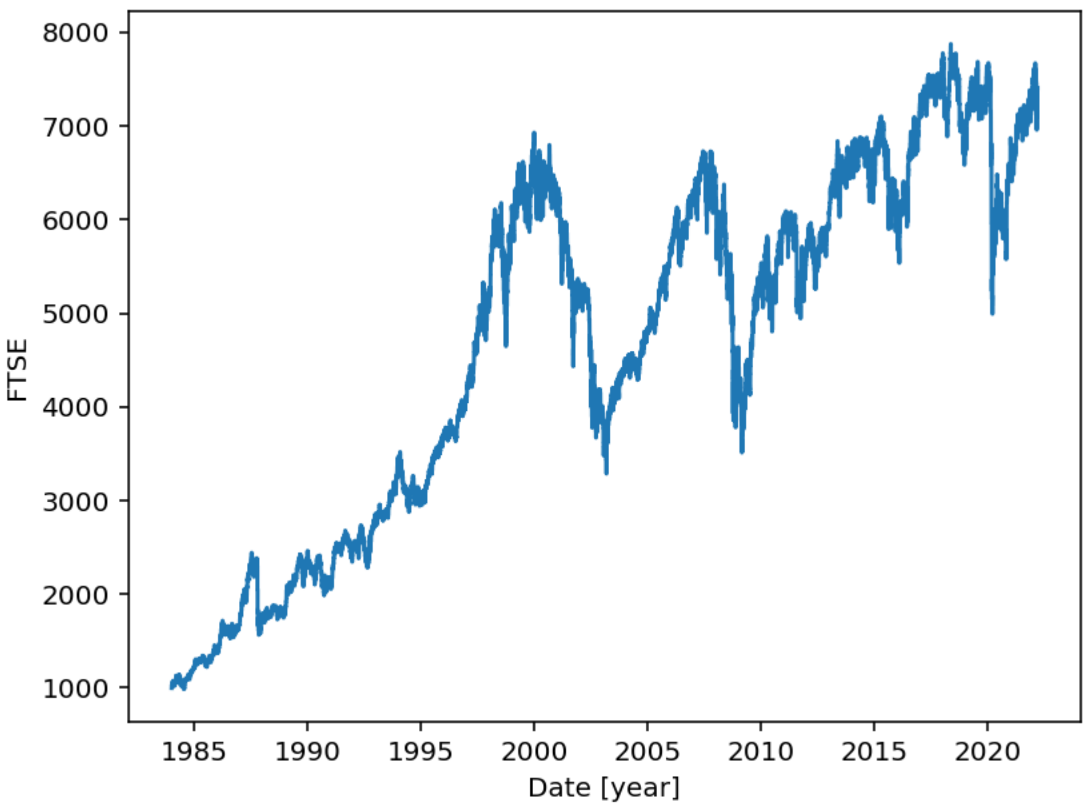
[97]:
```
            Close
Date
18/03/22  7404.73
17/03/22  7385.34
16/03/22  7291.68
15/03/22  7175.70
14/03/22  7193.47
```

[98]:
```python
df_ftse.index = pd.to_datetime(df_ftse.index, format="%d/%m/%y")
df_ftse.sort_index(inplace=True)
df_ftse.head()
```

[98]:
```
                  Close
Date
1984-01-03   997.500000
1984-01-04   998.599976
1984-01-05  1015.799988
1984-01-06  1029.000000
1984-01-09  1034.599976
```

[78]:
```python
def plot_ftse():
    """
    Plots the FTSE index
    """
    plt.figure(dpi=144)

    plt.step(df_ftse.index, df_ftse.Close)

    plt.xlabel('Date [year]')
    plt.ylabel('FTSE')
    plt.show()
    return
```

[79]:
```python
plot_ftse()
```



[99]:
```python
# Let's convert closing price into a return using a percentage change
df_ftse['Return'] = df_ftse['Close'].pct_change()
df_ftse.dropna(inplace=True)  # a shift introduces a NaN in the first N rows␣
 ↪shifted by
df_ftse.head()
```

[99]:
```
                 Close    Return
Date
1984-01-04   998.599976  0.001103
1984-01-05  1015.799988  0.017224
1984-01-06  1029.000000  0.012995
1984-01-09  1034.599976  0.005442
1984-01-10  1034.300049 -0.000290
```

[101]:
```python
# Let's get uncertainties on our statistical moments
# Calculate 1 sigma bootstrapped confidence interval for average
data = (df_ftse["Return"].to_numpy(), )
sigma = ss.bootstrap(data, np.mean, confidence_level=0.682).standard_error

aver = np.mean(df_ftse["Return"])
print("Mean =  ", np.round(aver, 6), " +/-", np.round(sigma, 6),
      "   significance level ", np.round(np.abs(aver / sigma), 3))

# Calculate 1 sigma bootstrapped confidence interval for std. deviation
sigma = ss.bootstrap(data, np.std, confidence_level=0.682).standard_error

std = np.std(df_ftse["Return"])
print("Std. dev = ", np.round(std, 6), " +/-", np.round(sigma, 6),
      "   significance level", np.round(np.abs(std / sigma), 3))

# Calculate 1 sigma bootstrapped confidence interval for skew
sigma = ss.bootstrap(data, ss.skew, confidence_level=0.682).standard_error

skewness = ss.skew(df_ftse["Return"])
print("Skewness = ", np.round(skewness, 6), " +/-", np.round(sigma, 6),
      "   significance level ", np.round(np.abs(skewness / sigma), 3))

# Calculate 1 sigma bootstrapped confidence interval for kurtosis
sigma = ss.bootstrap(data, ss.kurtosis, confidence_level=0.682).standard_error

kurtosis = ss.kurtosis(df_ftse["Return"])
print("Kurtosis = ", np.round(kurtosis, 6), " +/-", np.round(sigma, 6),
      "   significance level ", np.round(np.abs(kurtosis / sigma), 3))
```

```
Mean =   0.00026  +/- 0.000109    significance level  2.393
Std. dev =  0.01083  +/- 0.000189    significance level 57.192
Skewness =  -0.372461  +/- 0.251674    significance level  1.48
Kurtosis =  9.804102  +/- 1.848724    significance level  5.303
```

Normal distribution

$$N(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

The probability of values falling in the -1σ and $+1\sigma$ range is

$$\int_{\mu-\sigma}^{\mu+\sigma} N(x)\,dx = 0.682$$

sigmas	probability	significance $\frac{x-\mu}{\sigma} > n$
1	0.682	0.318
2	0.954	0.046
3	0.997	0.003
5	0.9999994267	$5.73 \cdot 10^{-7}$

The chance of a 5σ deviation is 1 in 1,744,278. A 5σ deviation is highly significant. E.g. requirement by CERN to announce a new particle. Physical sciences in general consider 3σ results significant. One should at least require a significance of 5%.

3 Question 2

Amend the print_stats function from Question 1, to also include these bootstrapped errors. Run as before (beware this can take a while).

[ ]:
```
def print_stats(dist):
    """
    Prints the moments of each distribution
    """
    return
```

[ ]:
```
for dist in df_dists.columns:
```

3.1 End Question 2

3.2 Good Coding Practice

Read, and become comfortable with, PEP 8.

To summarise some of the most important features: - Spaces, not tabs, and 4 spaces per indent level (e.g. in functions) - Keep lines less than 80 characters long - Two empty lines between top-level functions/classes (only one between class methods) - Scripts should be in order: imports > functions/classes > main program - Imports should be grouped by their type (standard library/external library/local libraries) - Spaces around binary/assignment/Boolean/commas operators (unless it makes things harder to read, e.g. keywords) - Comments should come above code and well-explain what it does, logical blocks also good - Docstrings should be in all non-dunder functions/classes

4 Question 3

Fix up the following code:

```
[ ]: # doing all the imports HERE
import numpy as np
import matplotlib.pyplot as plt




def fsquare(x, l):
    # Calculates an approximation of a square wave using a Fourier series
    # Terms are added until a reasonable approximation is achieved.
    # x: x-values
    # l: length of the period

    f= nterm(x, l, 1) + nterm(x, l, 3) + nterm(x, l, 5) + nterm(x, l, 7) +␣
 ↪nterm(x, l, 9) + nterm(x, l, 11) + nterm(x, l, 13) + nterm(x, l, 15) +␣
 ↪nterm(x, l, 17)

    return f
def nterm(x, l, n):
    """ Calculates the nth term of the Fourier series. l is the period
        length
    """
    t = np.sin(2.0*n*np.pi*x /l ) / n

    return t


def square(x, l):
    """ Calculates an accurate square wave of period length l. """

    # calculate a sine wave
    f = np.sin(x)
    # use the sign function (+1 for positive numbers, -1 for negative numbers
    # to convert into a suqare wave
    f = np.sign(f)

    return f


l = 2.0*np.pi    # wavelength of the square wave
x=np.linspace(-2.0*np.pi,2.0*np.pi,10000)

plt.figure()
```

```python
# plot square wave and approximation
plt.plot(x, fsquare(x, l), label = "Fourier")
plt.plot(x, square(x, l), label = "square")

plt.xlim(-2.0*np.pi, 2.0*np.pi)
plt.xlabel("x")
plt.ylabel("f(x)")
plt.legend()

plt.show()
```