The MathPhys Aficionado

0.1 Section A: Limits

Find the limits of the following functions:

1. $\lim_{x \to 2} \frac{x^2-4}{x-2}$

2. $\lim_{x \to 3/2} \frac{4x^2-9}{2x+3}$

3. $\lim_{x \to 1} \frac{x^2+x-2}{x-1}$

4. $\lim_{x \to -2} \frac{x^2+3x+2}{x+2}$

5. $\lim_{x \to 2} \frac{x^2+x-6}{x-2}$

6. $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$

7. $\lim_{x \to 1} \left(\frac{x^2-3x+2}{x^2+x-2}\right)^2$

8. $\lim_{x \to 3} \sqrt{\frac{x^2-2x-3}{x-3}}$

9. $\lim_{x \to 1} \frac{\frac{1}{x}-1}{x-1}$

0.2 Section B: Differentiation Via First Principles

Find the derivative of the following functions using the definition of the derivative, also known as first principles. Recall for a function $f(x)$, the deriavtive is given by,

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

1. $f(x) = x^2$
2. $g(x) = -12x$
3. $h(x) = \frac{1}{2}x$
4. $f(x) = \frac{3x^2}{4}$
5. $f(x) = 1 - 3x$
6. $g(x) = ax^2 + bx + c$, where a, b and c are constants.
7. $h(x) = \frac{1}{x}$
8. $g(x) = \frac{1}{x^2}$
9. $f(x) = \frac{1}{x^3}$

0.3 Section C: Differentiation - Using The Rules and Higher Order Derivatives

Recall, for the function $f(x)$, the notation for derivatives are:

$$\boxed{f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = D_x[f(x)] = f_x}$$

The notation for higher order derivatives are as follows:

$$\underline{\text{First derivative:}}\;\; f'(x) = \frac{\mathrm{d}f}{dx}$$

$$\underline{\text{Second derivative:}}\;\; f''(x) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$$

$$\underline{\text{Third derivative:}}\;\; f^{(3)}(x) = \frac{\mathrm{d}^3 f}{\mathrm{d}x^3}$$

$$\vdots$$

$$\underline{k^{\text{th}}\text{ deriavtive:}}\;\; f^{(k)}(x) = \frac{\mathrm{d}^k f}{\mathrm{d}x^k}$$

Recall, the rules for differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}x}(k) = 0, \quad \text{the derivative of a constant is zero}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[kf(x)] = k \times \frac{\mathrm{d}f}{\mathrm{d}x}, \quad \text{where } k \text{ is a constant}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) + g(x)] = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) - g(x)] = \frac{\mathrm{d}f}{\mathrm{d}x} - \frac{\mathrm{d}g}{\mathrm{d}x}$$

Find the derivatives of the following functions:

1. $f(x) = x^{100} - 3x^{50} + 4 - 2x^{-50} + 3x^{-100}$
2. $g(t) = \frac{t^3}{3} + \frac{3}{t^3}$
3. $h(x) = 3\sqrt{t} - 4t^{-\pi} + \frac{6}{t^{1/3}} + \frac{t^3-2t^2-t+2}{t+1}$
4. $f(x) = \frac{x+1}{\sqrt{x}}$
5. $f(x) = (x+1)^2(x-2)$
6. $M(z) = \frac{1}{3}z^6 - \frac{3}{4}z^8 + \frac{1}{2}z^{-2} - 4 \cdot 3^2$
7. Find the second and third derivative of $g(x) = 3x^{3/2} - 3\sqrt{x} + \frac{6}{\sqrt{x}}$
8. Find the second and third derivative of $h(\theta) = \theta^{\sqrt{2}} + \theta - \frac{1}{\theta^{\sqrt{2}}}$
9. $N(q) = \sqrt{q} + 3q^3 - 4q^{-2.5} + 4 + \left(q^{1/4} + q^{-1/4}\right)^2 + 3q^{\pi} - 4q^{\sqrt{2}} + q^{0.01}$
10. Find the second derivative of $g(y) = (y^2 - 1)(3y + 4) + \frac{2}{y^3} + y^0$

0.4 Section D: Minima and Maxima of Functions and Other Applications

Recall that the **gradient of a tangent** of a function $f(x)$ at a point (x_1, y_1) is given by

$$\boxed{m_T = f'(x_1) = \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_1}}$$

The **equation of the tangent line** is given by

$$\boxed{y - y_1 = m_T\,(x - x_1)}$$

The **gradient of a normal line** of a function $f(x)$ at a point (x_1, y_1) is given by

$$\boxed{m_N = -\frac{1}{m_T} = -\frac{1}{f'(x_1)}}$$

The **equation of the normal line** is given by

$$\boxed{y - y_1 = m_N\,(x - x_1)}$$

1. Find the equation of the tangent and normal lines of the following curves at the indicated point:

1.1. $y = -x^2 + 2x + 3$ at $x = 2$

1.2. $y = 2x^2 + 3x - 2$ at $x = -2$

1.3. $xy = 8$ at $x = 1$

1.4. $y = -\frac{6}{x}$ at $x = -2$

1.5. $y = \sqrt{x} - 1$ at $x = 1$

1.6. $y = \frac{4}{x} + 1$ at $x = 2$

1.7. $y = (3x - 2)^2$ at $x = 6$

0.5 Section E: Curve Sketching

In order to graph / skecth a function, perform the following steps:

1. Find the $y-$intercept of the graph: Set $x = 0$

2. Find the $x-$intercept/s of the graph: Set $y = 0$. **Note:** It is possible that the graph may have one $x-$intercept or no $x-$intercepts altogether!

3. Find the stationary/turning point/s on the graph: Set $f'(x) = 0$

4. Classify the turning point/s

   - If $f''(\text{point}) > 0$ this means that a local minimum occurs at this point

   - If $f''(\text{point}) < 0$ this means that a local maximum occurs at this point

5. Find the inflection point/s, i.e. the point/s where the graph changes shape

6. Find the regions where the graphs is increasing and / or decreasing. You may use the numberline method of the second derivative method

Sketch the following functions:

1. $y = x^3 - 4x^2 + 4x$

2. $y = -(x+1)^3 + 8$

3. $y = 2x^2 + x - 6$

4. $y = (x-2)^2 (x-5)$

5. $y = x^3 + 3x^2 + 3x + 2$

6. $y = -9x^3 + 45x^2 - 72x + 36$

7. $y = 2x^3 + 9x^2 + 12x - 1$

0.6 Section F: Optimization Word Problems

1. The length of a rectangle is x mm and its width is $(200 - x)$ mm. Find the dimensions of the rectangle of maximum area.

2. The sum of two positive numbers is 12. Find the two numbers when their product is a maximum.

3. The perimeter of a rectangle is 600 mm. Find the dimensions of the rectangle that has maximum area.

4. The edges of a rectangular prism (box) are x $mm \times x$ $mm \times (240 - 2x)$ mm respectively.

4.1. Show that the volume of the prism is $V = 240x^2 - 2x^3$.

4.2. Find the value of x which gives maximum volume and hence find the maximum volume of the prism.

5. The edges of a rectangular box are $x\ mm \times 2x\ mm \times (180-3x)\ mm$ respectively.

5.1. Find an expression for the volume of the box.

5.2. Find the values of x that gives maximum volume.

6. The radius of a solid metal cylinder is $x\ mm$ and its height is $(300-x)\ mm$.

6.1. Show that the volume of the cylinder is given by $V = 300\pi x^2 - \pi x^3$.

6.2. Find the value of x which results in maximu volume.

7. A cylindrical rod is made with a hemispherical end. The radius of the cylindrical section is $x\ mm$ while the total length of the rod is $(600-x)\ mm$.

7.1. Show that the volume of the rod is $V = 600\pi x^2 - \frac{4}{3}\pi x^3$.

7.2. Find the value of x at which the maximum volume occurs.

8. The manner in which the temperature $T(°C)$ at the center of a smelting pot in a blast furnace increases with time t (min) is given by

$$T(t) = t^2\,(45-t) \times 10^{-1} + 15\ \ (°C/\text{min})$$

8.1. Determine the rate of increase of temperature at

8.1.1. $t = 5$ min

8.1.2. $t = 20$ min

8.1.3. $t = 600\ s$

8.2. Find the maximum temperature.

9. The mass of bateria culture $P \ (mg)$ varies with time according to the equation

$$P(t) = 500 + 200t + 15t^2 \ \ (mg/\text{min})$$

Find how fast the mass of the culture is growing after 5 minutes.

10. The volume of water in a tank is governed by the equation

$$V(t) = 5 + 10t - t^2 \ \left(m^3/\text{min}\right)$$

10.1. Find the rate at which the volume is increasing at $t = 2$ min.

10.2. At what time does the volume start decreasing?

0.7 Section G: Exponents and Surds

Recall the rules for exponents and surds:

$$a^0 = 1 \quad \text{any number to the power zero is always one}$$

$$a^1 = a \quad \text{any number to the power one is always itself}$$

$$a^{-1} = \frac{1}{a} \quad \text{and} \quad a^{-m} = \frac{1}{a^m}$$

$$a^m \cdot a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{m \times n}$$

$$\sqrt[n]{a^m} = a^{m/n}$$

$$\sqrt[n]{\sqrt[m]{a^k}} = \sqrt[m \times n]{a^k} = a^{k/mn}$$

$$(ab)^m = a^m \cdot b^m \quad \text{and} \quad (a^m b^n)^k = a^{m \times k} b^{n \times k}$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \qquad \text{and} \quad \left(\frac{a^m}{b^n}\right)^k = \frac{a^{m \times k}}{b^{n \times k}}$$

Simplify the following:

1. $\frac{x^{1/2} \times \sqrt[4]{y^3} \times (xy)^{1/4}}{(x^3)^{1/4}}$
2. $\frac{2^{2n-1} \cdot 4^{n+1} \cdot 2}{16^n}$
3. $32^{3/5} \times \left(\frac{3}{2}\right)^2 \div \sqrt{\frac{81}{16}}$

4. $\frac{1}{(x+y)^{-1}} - \left(x^{1/2} - y^{1/2}\right)^2$

5. $\frac{45^{-n+1} \times 5^{n-1} \times 81^{-1}}{4^{n+2} \times 36^{-n-1}}$

6. $\frac{2^n - 2^{n-1}}{3 \cdot 2^n - 4 \cdot 2^{n-2}}$

7. $\frac{2^{2x}}{2^{-x} \cdot (8^{-1})^{x+2} \cdot 4^{3x}}$

8. $\frac{(3^y)^{1-y}}{6^{-1} \cdot 3^{-y-1}} \div \frac{\left(3^{1-y}\right)^{y+1}}{2^{-1} \cdot 9^{-y-1}}$

9. $\sqrt{\sqrt{0.0016}}$

10. $\frac{2 \cdot 3^x + 3^{x-2}}{5 \cdot 3^{x+1} - 7 \cdot 3^{x-1}}$

11. $\sqrt{\sqrt{2} \times \sqrt{3}}$

12. Rationalise the denominator: $\frac{1}{5 - 2\sqrt{2}}$

13. Rationalise the denominator: $\frac{1 - \sqrt{8}}{\left(5 + \sqrt{2}\right)\left(2 - \sqrt{2}\right)}$

14. Rationalise the denominator: $\frac{2}{1 + \sqrt{2} + \sqrt{3}}$

15. $\frac{2^{2n} - 2^{n+2} + 4}{2^n - 2}$. **Hint:** Use an appropriate substitution!

16. Solve the following equations:

16.1. $27^x \times 9^{x-2} = 1$

16.2. $125^x = \frac{1}{25}$

16.3. $2^{x+2} + 2^{x-2} + 2^x = 42$

16.4. $2^{2x+2} - 5 \cdot 2^x = -1$

16.5. $3^x - 3^{1-x} - 2 = 0$

16.6. $\frac{1}{2^x} + 5 \cdot 2^{-x} + 2^{1-x} = 8$

16.7. $3^x + 3^{x+1} + \frac{3^x}{8} = 1\frac{3}{8}$

16.8. Solve for x and y:

$$8^x = 4^{2+y}$$
$$9 \cdot 3^y = 27^x$$

16.9. Solve for x and y:

$$4^{2x} = 8^{3x-5}$$
$$2^x - 2^{x-1} = 4$$

0.8 Section H: Kinematics (The Calculus of Motion)

Recall the relationship between speed, distance and time,

$$v = \frac{\mathrm{d}s}{\mathrm{d}t}$$
$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$$

1. A radar tells an operator that a motorbike is moving according to the equation:

$$s(t) = \frac{1}{15}t^3 - 2t^2 + 20t \ \ (m/s)$$

1.1. Find how fast the bike was travelling when initially detected.

1.2. How long did it take the bike to stop?

1.3. Find the distance covered while being tracked.

1.4. Find the acceleration of the bike at $t = 5$ sec.

2. The motion of a particle is described by

$$s(t) = 5t^2 - 2t \ \ (km/h)$$

2.1. Find an expression for the velocity of the particle.

2.2. Find the velocity at $t = 2\ h$.

2.3. Find the acceleration of the particle.

3. A stone is thrown vertically upwards and its height at any point in time is given by

$$s(t) = 20t - 5t^2 \ \ (m/s)$$

3.1. Find at what height the stone is after $1\ s, 2\ s$ and $3\ s$.

3.2. Find the velocity of the stone at $1\ s, 2\ s$ and $3\ s$.

3.3. What is the maximum height reached by the stone and what is the speed at maximum height?

0.9 Section I: Inflection Points, Concave Up and Concave Down

Recall for the function $f(x)$ the following:

Inflection (point where graph changes shape): Set $f''(x) = 0$ and solve for x

Concave up: Set $f''(x) > 0$ and solve the inequality for x

Concave down: Set $f''(x) < 0$ and solve the inequality for x

Find the infelction points and the regions where the following functions are concave up and concave down:

1. $f(x) = x^3 - 12x^2 + 36x$
2. $f(x) = x^3 - 6x^2 + 9x + 1$
3. $f(x) = x^2 + 4x - 2$
4. $f(x) = x^3 + 10x^2 + 25x - 25$
5. $g(x) = -2x^3 + 15x^2 - 36x + 7$