0.1 Section A: Sigma Notation

Recall,

$$\boxed{\sum_{i=1}^{n} f(i) = f(1) + f(2) + \cdots + f(n)}$$

1. Expand and simplify the following:

1.1. $\sum_{i=1}^{10} (4i - 1)$

1.2. $\sum_{r=1}^{5} r^2$

1.3. $\sum_{j=4}^{7} j\,(j + 1)$

1.4. $\sum_{k=1}^{10} 4$

1.5. $\sum_{i=1}^{6} \frac{i}{i+1}$

1.6. $\sum_{t=3}^{7}(2t - t^2)$

1.7. $\sum_{q=1}^{n} b$, where b is a constant

1.8. $\sum_{r=6}^{12}\left(\frac{r}{2} + 4\right)$

1.9. $\sum_{i=3}^{17} 10$

2. Write the following in sigma notation:

2.1. $1 + 2 + 3 + 4 + 5 + 6$

2.2. $1 + 4 + 9 + 16 + 25 + 36 + 49$

2.3. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{10}$

2.4. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \cdots + \frac{20}{21}$

2.5. $2x + 4x + 6x + \cdots$ to n terms

2.6. $p^2 + 3p^4 + 5p^6 + \cdots$ to n terms

2.7. $1.2 + 2.3 + 3.4 + \cdots$ to n terms

0.2 Section B: Arithmetic Sequences / Progressions

$$\boxed{T_n = a + (n-1)\,d}$$

1. Determine the 11^{th} term of the sequence: $4, 7, 10, \cdots$

2. Determine which term of the arithmetic progression: $25, 14, 3, \cdots$ is -52

3. Determine the second, third and fourth terms of the arithmetic sequence in

which the 1^{st} term is 10 and the 6^{th} term is 85

4. Determine the first four terms of an arithemtic sequence and the 10^{th} term in which the 3^{rd} term is -4 and the 7^{th} term is -20

5. The first three terms of an arithmetic sequence are: $3p-4, 4p-3$ and $7p-6$. Determine:

5.1. the value of p,

5.2. the first three terms of the sequence,

5.3. the value of the 16^{th} term

6. Determine the 10^{th} and 21^{st} terms of the following arithmetic sequences:

6.1. $4+7x, 5+9x, 6+11x, \cdots$

6.2. $a=-5$ and $T_4=4$

7. Determine the arithemtic progression for which the 4^{th} term is -13 and the 7^{th} term is -25

8. Determine the first three terms and the 12^{th} term of the arithmetic sequence in which $T_4=\log x$ and $T_9=\log x^8$

9. For the arithmetic sequence: $-5, -4\frac{1}{4}, -3\frac{1}{2}, \cdots$

9.1. which term is equal to -13?

9.2. calculate the 13^{th} term

10. $p-4, 8p+3$ and $10p-5$ are the $4^{\text{th}}, 6^{\text{th}}$ and 8^{th} terms of an arithmetic sequence respectively. Find:

10.1. the 1^{st} term of the sequence,

10.2. the common difference,

10.3. which term of the sequence will have a value of -70

11. In an arithmetic sequence, the 17^{th} term is 9 times the first term and the 9^{th} term is 6 less than 3 times the 3^{rd} term. Find the first three terms of the sequence

0.3 Section C: Arithmetic Series

$$\boxed{S_n = \frac{n}{2}\left[2a + (n-1)\,d\right] = \frac{n}{2}\,(a+L)}$$

1. Find the sum of the first 12 terms of the arithemtic series: $2 + 5 + 8 + \cdots$

2. Find the sum of the arithmetic series which extends from 0.5 to 2.4

3. How many terms are there in the following series and what is the sum of the series: $2 + 5 + 8 + \cdots + 62$?

4. What is the sum of the first 1000 natural numbers?

5. How many terms are there in the following series and what is the sum of the series: $\frac{1}{2} + 1 + 1\frac{1}{2} + \cdots + 21\frac{1}{2}$?

6. Find the sum of the following arithemtic series: $-7 - 5 - 3 - \cdots - 25$

7. How many terms of the arithmetic sequence: $34, 40, 46, \cdots$ will sum to 1 530?

8. The first three terms of a finite arithmetic sequence is $-20, -15$ and -10. If the sum of the series is -35, how many terms are there in the series?

9. The sum of the first 5 terms of an arithmetic series is 40. The last term is 14.

Determine the first term and the common difference.

10. The sum of the first 10 terms of an arithmetic series is 60. The 2^{nd} and 6^{th} terms add up to 9. Determine the first three terms.

11. The sum of the first 8 terms of an arithmetic series is -4.5. The 2^{nd} and 8^{th} terms add up to -1. What is the 9^{th} term?

0.4 Section D: Geometric Sequences / Progressions

$$\boxed{T_n = ar^{n-1}}$$

1. Determine the 5^{th} and the 8^{th} terms of the geometric sequence: $9, 3, 1, \cdots$

2. Determine the first three terms of the geometric progression with the 2^{nd} term being -4 and the 5^{th} term being $\frac{4}{125}$.

3. The geometric sequence: $1, \frac{3}{2}, \frac{9}{4}, \cdots$ has a term equal to $\frac{243}{32}$. What is the number of the term?

4. Determine the first three terms of the geometric sequence in which the 8^{th} term is 270 and the constant ratio is 3.

5. Determine the 50^{th} term of the geometric sequence in which $T_1 = 2\left(x^2 + y^2\right)$ and $T_4 = 16\left(x^2 + y^2\right)^4$.

6. Which term of the geometric progression: $0.2, 2.2, 24.2, \cdots$ is 2 928.2?

7. Find term 2 and term 6 of the geometric progression in which $T_1 = -2 \times 10^{-1}$ and $T_7 = -1.28 \times 10^{-5}$.

8. $x-4, x+2, 3x+1$ are the $4^{\text{th}}, 5^{\text{th}}$ and 6^{th} term respectively of a geometric sequence. Find two possible values for

8.1. the common ratio,

8.2. the first term.

9. Which term of the geometric progression: $8, 6, 4\frac{1}{2}, \cdots$ will be the first to be less than $\frac{1}{100}$?

10. Which term of the geometric sequence: $6, 5, \frac{25}{6}, \cdots$ will be less than $\frac{1}{1\,000}$?

0.5 Section E: Geometric Series

$$\boxed{S_n = \frac{a\left(r^n-1\right)}{r-1} = \frac{a\left(1-r^n\right)}{1-r}}$$

$$\boxed{S_\infty = \frac{a}{1-r}, \text{ for } -1<r<1}$$

1. Find the sum of the following geometric series:

1.1. $2+6+18+54+162+486$

1.2. $27-9+3-1+\frac{1}{3}$

1.3. $\frac{1}{3}-1+3-\cdots$ to 5 terms.

2. The sum of the first n terms of the series: $\frac{3}{4}+\frac{3}{2}+3+\cdots$ is $23\frac{1}{4}$. Determine n.

3. The 1^{st} term of a geometric series is 3, the last term is 48 and the sum of the series is 93.

3.1. What is the constant ratio?

3.2. How many terms are there in the series?

4. What is the lowest value of $m\,(m \in \mathbb{N})$, for which $\sum_{r=1}^{m} \frac{4^{r-1}}{16} > 5$?

5. What is the greatest value of $m\,(m \in \mathbb{N})$, for which $\sum_{k=1}^{m} 7 \times 3^{k-1} < 10^6$?

6. The constant ratio of a geomertic series is $-5\frac{1}{2}$. The sum of the 1st four terms is $17\frac{2}{5}$. Calculate the 1st term.

7. The 1st term of a geometric sequence is 27, the last term is 8 and the sum of the series is 65. What is the constant ratio and how many terms are there in the series?

8. The 1st term of a geometric series is 1 and the 4th term is -125. Find the sum of the series to 20 terms.

9. Find S_∞ of the series: $\frac{3}{10} + \frac{3}{100} + \frac{3}{1\,000} + \cdots$

10. Find S_∞ for the following geometric series:

10.1. $7 + \frac{7}{5} + \frac{7}{25} + \cdots$

10.2. $5x + \frac{5x}{4} + \frac{5x}{16} + \cdots$

10.3. $18 - 3 + \frac{1}{2} - \frac{1}{12} + \cdots$

11. The sum to infinity of a geometric series is 15. The constant ratio is $\frac{2}{3}$. Determine the first 5 terms of the series.

12. For a geometric series, $S_\infty = 5$ and $r = -0.11$. Find the first term.

13. Consider the geometric series: $1 + \frac{1}{2} + \frac{1}{4} + \cdots$

13.1. Calculate S_∞ for the series.

13.2. Find the least value of n such that $S_\infty - S_n$ is less than:

13.2.1. 10^{-3},

13.2.2. 10^{-6}.

14. For which value of x will the following geometric series converge: $1 + \frac{1}{2x-1} + \frac{1}{(2x-1)^2} + \frac{1}{(2x-1)^3} + \cdots$?

15. Evaluate: $\frac{401+403+405+\cdots+499}{1+3+5+\cdots+99}$.

16. For which value of a will the series: $2\left(3a-1\right) + 2\left(3a-1\right)^2 + \cdots$ converge?

17. The first term of a geometric series is $\sqrt{3}$, the second term is $\sqrt{3}-1$.

17.1. Write down, in surd form, the 12^{th} term.

17.2. Calculate the sum to infinity of the series.

0.6 Section F: Quadratic Sequences and Series

0.6.1 Shortcut Method: Not Taught in Schools

Consider the quadratic sequence: $q_1, q_2, q_3, q_4, \cdots$ We note that this sequence has a constant second difference! The formula to generate the n^{th} term of this sequence is

$$T_n = an^2 + bn + c,$$

$$a = \frac{1}{2}\left(q_1 - 2q_2 + q_3\right),$$

$$b = \frac{1}{2}\left(-5q_1 + 8q_2 - 3q_3\right),$$

$$c = 3q_1 - 3q_2 + q_3.$$

0.6.2 Quadratic Series - Never Ever Tested, and Probably Will Not Ever!

Consider the quadratic series: $S_n = q_1 + q_2 + q_3 + q_4 + \cdots + q_n$. The sum of this series is given by

$$S_n = an^3 + bn^2 + cn + d,$$

$$a = \frac{1}{6}\left(q_2 - 2q_3 + q_4\right),$$

$$b = \frac{1}{2}\left(-3q_2 + 5q_3 - 2q_4\right),$$

$$c = \frac{1}{6}\left(26q_2 - 31q_3 + 11q_4\right),$$

$$d = q_1 - 3q_2 + 3q_3 - q_4.$$

1. Write down the n^{th} terms of the following quadratic sequences:

1.1. $5, 12, 23, 38, \cdots$

1.2. $3, 6, 10, 15, 21, \cdots$

1.3. $2, 10, 26, 50, 82, \cdots$

1.4. $31, 30, 27, 22, 15, \cdots$

1.5. $16, 27, 42, 61, \cdots$

2. Only do this question if you have time and don't have anything else to do!

2.1. Find the sum: $1^2 + 2^2 + 3^2 + 4^2 + \cdots + 1000^2$.

2.2. Find the sum: $16 + 27 + 42 + 61 + \cdots + 16\,296$.

2.3. Find the sum: $31 + 30 + 27 + 22 + 15 + \cdots + (-22\,170)$.

0.7 Section G: Word Problems Involving Sequences and Series

1. In the 6 weeks prior to the New York Marathon, Janice's training schedule demands that she run a total of 800 km. If she starts off by running 100 km in the first week, by what regular amount must she increase her distance each week in order to log the 800 km planned in the schedule?

2. Pierre can jump over a rope 90 times in a minutes when he starts training. Each week he increases his recod by 5 jumps. Marie starts with 60 jumps per minute and increases her record by 10 jumps per minute each week.

2.1. After how many weeks will their number of jumps per minute be the same?

2.2. How many jumps per minute will they each do in the competition which is in 6 weeks' time?

3. A ladder has 50 rungs. The bottom rung is 1 m long. Each rung is 12.5 mm shorter than the rung beneath it. What is the total length of steel required to make the 50 rungs?

4. In an infinite sequence of circle, the radius of the 1^{st} circle is 90 cm and the radius of each successive cirlce is two-thirds that of its predecessor. Find the sum of the areas of the circles.

0.8 The R1 000 Question

Whichever student can solve this problem first, I promise to give them R1 000.

Find the sum:

$$S = 12 + 1000 + 13\ 866 + 93\ 186 + 415\ 564 + 1\ 422\ 792 + 4\ 049\ 970 + 10\ 058\ 746$$

$$+ 22\ 495\ 836 + 46\ 296\ 984 + \cdots + 326\ 698\ 001\ 922$$

0.9 Section H: Logarithms

Recall the laws of logarithms,

$$\log_a(a) = 1$$

$$\log_a(1) = 0$$

$$\log_{10}(x) = \log(x)$$

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

$$\log_a(x^n) = n \times \log_a(x)$$

$$a^{\log_a(x)} = x$$

$$n = n \times 1 = n\log_a(a)$$

$$\log_a x = \frac{\log x}{\log a} \quad \text{and} \quad \log_a x = \frac{\log_b x}{\log_b a}$$

1. Simplify the following:

1.1. $3\log xy - 2\log x - 2\log y$

1.2. $2\log a + 3\log b + 2\log 10 - 5\log c$

1.3. $5\log p - 3\log q - 2\log r$

1.4 $\log_2 32 - \log_2 8$

1.5. $\frac{\log 32}{\log 8}$

1.6. $\log_2 \frac{5}{27} - 2\log_2 \frac{4}{3} + \log_2 \left(1\frac{1}{5}\right)$

1.7. $\frac{\log 16 - \log 4}{\log 16 + \log 4}$

1.8. $\frac{\log_a 16 - \log_b 4}{\log_a 4 - \log_b 2}$

1.9. $\frac{\log_x 243}{\log_x \frac{1}{27}}$

1.10. $\log_3 27 - \log_3 3$

1.11. $\frac{1}{2}\log 2 + 2\log \frac{1}{2}$

1.12. $\log_2 16 - 3\log_3 \frac{1}{3} + \log_{25} 5$

1.13. $\log_4 0.5 - \log_2 0.25 - \log 0.01$

1.14. $\log_a a^2 - \log_2 \frac{1}{64} - \log_x x^6 - \log_5 25$

2. Solve the following log equations:

2.1. $3\log x - 9 = 0$

2.2. $3\log x - 2\log x = 10$

2.3. $\log_2 (x-2) + \log_2 (x-3) = 1$

2.4. $\log (x+2)^2 = 2$

2.5. $2\log \left(\frac{1}{x}\right) + 1 = 0$

2.6. $\log x + \log (x-1) = \log 12$

2.7. $\log (2x+1) - \log (x-1) = 1$

2.8. $\log (x-4) + \log (x-3) = 2\log x$

2.9. $2^{x+1} - 2^{x-1} = \log_3 81 + \log_5 25$

2.10. $3^x = 2$

2.11. $3^{2x} - 5 \cdot 3^x + 6 = 0$

3. Sketch the following log graphs, its inverses and state the domain and range:

3.1. $y = \log_2 x$

3.2. $y = \log_2 (2x - 3)$

3.3. $y = \log_2 (7x - 5)$

3.4. $y = \log_3 (4 - x)$

3.5. $y = \log (x^2 + x - 6)$

3.6. $y = \log_2 (15 - x^2 - 2x)$

Consider the inequality: $\log_a [f(x)] > \log_a [g(x)]$ or $\log_a [f(x)] < \log_a [g(x)]$ or $\log_a [f(x)] \geq \log_a [g(x)]$ or $\log_a [f(x)] \leq \log_a [g(x)]$

1. If $a > 1$: The solution is given in the region where $f(x) > g(x)$ or $f(x) < g(x)$ or $f(x) \geq g(x)$ or $f(x) \leq g(x)$ respectively. Additionally, you need to make $f(x) > 0$ and $g(x) > 0$.

2. If $a < 1$: The solution is then reversed: To illustrate, consider $\log_{1/3} (3x - 1) \geq \log_{1/3} (x + 2) \implies 3x - 1 \leq x + 2$. Additionally, you need to make $f(x) > 0$ and $g(x) > 0$.

3. When log inequalities have different bases, you have to use the change of base formula.

4. Solve the following log inequalities: (These problems get sneakily tested, so it's good to know how to do them)

4.1. $\log_3 x < 3$

4.2. $\log_4 x \geq 5$

4.3. $\log_{16} x > \frac{3}{2}$

4.4. $\log_2 (2x+3) > \log_2 (x-2)$

4.5. $\log_5 (x+2) > \log_5 (5x+1)$

4.6. $\log_2 (2x+3) > \log_2 3x$

4.7. For all positive integers: $n \geq 2$, find $\log (n-1) + \log (n+1) > 2 \log n$

4.8. $\log_2 [\log_3 (4x+1)] > \log_2 [\log_3 (2x+3)]$

4.9. Given that $a \in \mathbb{R}$, solve: $1 + \log_5 (x^2+1) \geq \log_5 (ax^2+4x+a)$

4.10. $\log_{1/2} 3x > \log_{1/2} (2x+3)$

4.11. $\log_7 (x+5) > \log_5 (x+5)$

4.12. $\log_{0.3} (x-1) < \log_{0.09} (x-1)$

4.13. $\log_2 x + (\log_2 x)^2 > 6$. **Hint:** Use an appropriate substitution!

4.14. $\log_{1/2} (x+2) < -2 < \log_{1/4} (2x)$

0.10 Section I: Fibonnaci Sequences and Series

0.10.1 Fibonnaci Seuquence

Not really tested formally, however, occasionally examiners like to sneakily test it! Just be aware of what it is and how to generate each term; for $n \geq 2$, the term that follows is the sum of the previous two terms.

$$\boxed{T_n = F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]}$$

The number $\frac{1+\sqrt{5}}{2}$ is called the *golden ratio* and is denoted by the letter ϕ, i.e.

$$\boxed{\phi = \varphi = \frac{1+\sqrt{5}}{2}}$$

The Fibonnaci sequence goes as follows: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \cdots$ Alternatively, the Fibonnaci sequence is defined as:

$$T_1 = 1,$$

$$T_2 = 1,$$

$$T_{n+1} = T_n + T_{n-1} \text{ for } n \geq 2.$$

0.10.2 Fibonnaci Series

Consider the series $1+1+2+3+5+8+13+\cdots$ The sum of the general Fibonnaci series is given by the rather unusual formula

$$\boxed{S_n = \frac{2^{-(n+1)}}{5}\left[\left(5+3\sqrt{5}\right)\left(1+\sqrt{5}\right)^n + \left(5-3\sqrt{5}\right)\left(1-\sqrt{5}\right)^n - 5\times 2^{n+1}\right]}$$