1.2 MATRICES

Ex

If we examine the method of elimination described in Section 1.1, we make the following observation. Only the numbers in front of the unknowns $x_1, x_2, \ldots, x_n$ are being changed as we perform the steps in the method of elimination. Thus we might think of looking for a way of writing a linear system without having to carry along the unknowns. In this section we define an object, a matrix, that enables us to do this—that is, to write linear systems in a compact form that makes it easier to automate the elimination method on a computer in order to obtain a fast and efficient procedure for finding solutions. The use of a matrix is not, however, merely that of a convenient notation. We now develop operations on matrices (plural of matrix) and will work with matrices according to the rules they obey; this will enable us to solve systems of linear equations and solve other computational problems in a fast and efficient manner. Of course, as any good definition should do, the notion of a matrix provides not only a new way of looking at old problems, but also gives rise to a great many new questions, some of which we study in this book.

DEFINITION An $m \times n$ **matrix** A is a rectangular array of mn real (or complex) numbers arranged in m horizontal **rows** and n vertical **columns**:

$$A = \begin{array}{c} \begin{array}{ccccccc} & & & & a_{ij} & & \end{array} \\ \left[\begin{array}{ccccccc} a_{11} & a_{12} & \cdots & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mj} & \cdots & a_{mn} \end{array}\right] \\ \begin{array}{c} a_{mj} \\ j\text{th column} \end{array} \end{array} \begin{array}{l} \\ \\ \\ \leftarrow i\text{th row} \\ \\ \\ \end{array} \tag{1}$$

The ith **row** of A is

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \qquad (1 \le i \le m);$$

the jth **column** of A is

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \qquad (1 \le j \le n).$$

We shall say that A is m by n (written as $m \times n$). If $m = n$, we say that A is a **square matrix of order** n and that the numbers $a_{11}, a_{22}, \ldots, a_{nn}$ form the **main diagonal** of A. We refer to the number a_{ij}, which is in the ith row and jth column of A, as the i,j**th element** of A, or the (i, j) **entry** of A, and we often write (1) as

$$A = \begin{bmatrix} a_{ij} \end{bmatrix}.$$

For the sake of simplicity, we restrict our attention in this book, except for Appendix A, to matrices all of whose entries are real numbers. However, matrices with complex entries are studied and are important in applications.

EXAMPLE 1 Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 \\ 2 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} 3 \end{bmatrix}, \quad F = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix}.$$

Then A is a 2×3 matrix with $a_{12} = 2$, $a_{13} = 3$, $a_{22} = 0$, and $a_{23} = 1$; B is a 2×2 matrix with $b_{11} = 1$, $b_{12} = 4$, $b_{21} = 2$, and $b_{22} = -3$; C is a 3×1 matrix with $c_{11} = 1$, $c_{21} = -1$, and $c_{31} = 2$; D is a 3×3 matrix; E is a 1×1 matrix; and F is a 1×3 matrix. In D, the elements $d_{11} = 1$, $d_{22} = 0$, and $d_{33} = 2$ form the main diagonal. ■

For convenience, we focus much of our attention in the illustrative examples and exercises in Chapters 1–7 on matrices and expressions containing only real numbers. Complex numbers will make a brief appearance in Chapters 8 and 9. An introduction to complex numbers, their properties, and examples and exercises showing how complex numbers are used in linear algebra may be found in Appendix A.

A $1 \times n$ or an $n \times 1$ matrix is also called an ***n*-vector** and will be denoted by lowercase boldface letters. When n is understood, we refer to n-vectors merely as **vectors**. In Chapter 4 we discuss vectors at length.

EXAMPLE 2 $\mathbf{u} = \begin{bmatrix} 1 & 2 & -1 & 0 \end{bmatrix}$ is a 4-vector and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ is a 3-vector. ■

The n-vector all of whose entries are zero is denoted by $\mathbf{0}$.

Observe that if A is an $n \times n$ matrix, then the rows of A are $1 \times n$ matrices and the columns of A are $n \times 1$ matrices. The set of all n-vectors with real entries is denoted by R^n. Similarly, the set of all n-vectors with complex entries is denoted by C^n. As we have already pointed out, in the first seven chapters of this book we will work almost entirely with vectors in R^n.

EXAMPLE 3 (Tabular Display of Data) The following matrix gives the airline distances between the indicated cities (in statute miles).

$$\begin{array}{l} \\ \text{London} \\ \text{Madrid} \\ \text{New York} \\ \text{Tokyo} \end{array}
\begin{array}{c} \begin{array}{cccc} \text{London} & \text{Madrid} & \text{New York} & \text{Tokyo} \end{array} \\ \begin{bmatrix} 0 & 785 & 3469 & 5959 \\ 785 & 0 & 3593 & 6706 \\ 3469 & 3593 & 0 & 6757 \\ 5959 & 6706 & 6757 & 0 \end{bmatrix} \end{array}$$

■

EXAMPLE 4 (Production) Suppose that a manufacturer has four plants each of which makes three products. If we let a_{ij} denote the number of units of product i made by plant j in one week, then the 4×3 matrix

$$\begin{array}{l} \\ \text{Plant 1} \\ \text{Plant 2} \\ \text{Plant 3} \\ \text{Plant 4} \end{array}
\begin{array}{c} \begin{array}{ccc} \text{Product 1} & \text{Product 2} & \text{Product 3} \end{array} \\ \begin{bmatrix} 560 & 340 & 280 \\ 360 & 450 & 270 \\ 380 & 420 & 210 \\ 0 & 80 & 380 \end{bmatrix} \end{array}$$

gives the manufacturer's production for the week. For example, plant 2 makes 270 units of product 3 in one week. ■

EXAMPLE 5

The wind chill table that follows shows how a combination of air temperature and wind speed makes a body feel colder than the actual temperature. For example, when the temperature is 10°F and the wind is 15 miles per hour, this causes a body heat loss equal to that when the temperature is −18°F with no wind.

mph	°F 15	10	5	0	−5	−10
5	12	7	0	−5	−10	−15
10	−3	−9	−15	−22	−27	−34
15	−11	−18	−25	−31	−38	−45
20	−17	−24	−31	−39	−46	−53

This table can be represented as the matrix

$$A = \begin{bmatrix} 5 & 12 & 7 & 0 & -5 & -10 & -15 \\ 10 & -3 & -9 & -15 & -22 & -27 & -34 \\ 15 & -11 & -18 & -25 & -31 & -38 & -45 \\ 20 & -17 & -24 & -31 & -39 & -46 & -53 \end{bmatrix}.$$

■

EXAMPLE 6

With the linear system considered in Example 5 in Section 1.1,

$$\begin{aligned} x + 2y &= 10 \\ 2x - 2y &= -4 \\ 3x + 5y &= 26, \end{aligned}$$

we can associate the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \\ 3 & 5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 10 \\ -4 \\ 26 \end{bmatrix}.$$

In Section 1.3, we shall call A the coefficient matrix of the linear system. ■

DEFINITION

A square matrix $A = \left[a_{ij}\right]$ for which every term off the main diagonal is zero, that is, $a_{ij} = 0$ for $i \neq j$, is called a **diagonal matrix**.

EXAMPLE 7

$$G = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

are diagonal matrices. ■

DEFINITION A diagonal matrix $A = [a_{ij}]$, for which all terms on the main diagonal are equal, that is, $a_{ij} = c$ for $i = j$ and $a_{ij} = 0$ for $i \neq j$, is called a **scalar matrix.**

EXAMPLE 8 The following are scalar matrices:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}.$$

■

The search engines available for information searches and retrieval on the Internet use matrices to keep track of the locations of information, the type of information at a location, keywords that appear in the information, and even the way Web sites link to one another. A large measure of the effectiveness of the search engine Google© is the manner in which matrices are used to determine which sites are referenced by other sites. That is, instead of directly keeping track of the information content of an actual Web page or of an individual search topic, Google's matrix structure focuses on finding Web pages that match the search topic and then presents a list of such pages in the order of their "importance."

Suppose that there are n accessible Web pages during a certain month. A simple way to view a matrix that is part of Google's scheme is to imagine an $n \times n$ matrix A, called the "connectivity matrix," that initially contains all zeros. To build the connections proceed as follows. When it is detected that Web site j links to Web site i, set entry a_{ij} equal to one. Since n is quite large, about 3 billion as of December 2002, most entries of the connectivity matrix A are zero. (Such a matrix is called sparse.) If row i of A contains many ones, then there are many sites linking to site i. Sites that are linked to by many other sites are considered more "important" (or to have a higher rank) by the software driving the Google search engine. Such sites would appear near the top of a list returned by a Google search on topics related to the information on site i. Since Google updates its connectivity matrix about every month, n increases over time and new links and sites are adjoined to the connectivity matrix.

The fundamental technique used by Google© to rank sites uses linear algebra concepts that are somewhat beyond the scope of this course. Further information can be found in the following sources.

1. Berry, Michael W., and Murray Browne. *Understanding Search Engines—Mathematical Modeling and Text Retrieval.* Philadelphia: Siam, 1999.
2. www.google.com/technology/index.html
3. Moler, Cleve. "The World's Largest Matrix Computation: Google's Page Rank Is an Eigenvector of a Matrix of Order 2.7 Billion," MATLAB *News and Notes*, October 2002, pp. 12–13.

Whenever a new object is introduced in mathematics, we must define when two such objects are equal. For example, in the set of all rational numbers, the numbers $\frac{2}{3}$ and $\frac{4}{6}$ are called equal although they are not represented in the same manner. What we have in mind is the definition that $\frac{a}{b}$ equals $\frac{c}{d}$ when $ad = bc$. Accordingly, we now have the following definition.

Two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be **equal** if $a_{ij} = b_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$, that is, if corresponding elements are equal.

EXAMPLE 9 The matrices

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 4 \\ 0 & -4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & w \\ 2 & x & 4 \\ y & -4 & z \end{bmatrix}$$

are equal if $w = -1$, $x = -3$, $y = 0$, and $z = 5$. ■

We shall now define a number of operations that will produce new matrices out of given matrices. These operations are useful in the applications of matrices.

MATRIX ADDITION

DEFINITION If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices, then the **sum** of A and B is the $m \times n$ matrix $C = [c_{ij}]$, defined by

$$c_{ij} = a_{ij} + b_{ij} \quad (1 \le i \le m, 1 \le j \le n).$$

That is, C is obtained by adding corresponding elements of A and B.

EXAMPLE 10 Let

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 2 & -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 & -4 \\ 1 & 3 & 1 \end{bmatrix}.$$

Then

$$A + B = \begin{bmatrix} 1+0 & -2+2 & 4+(-4) \\ 2+1 & -1+3 & 3+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 4 \end{bmatrix}.$$ ■

It should be noted that the sum of the matrices A and B is defined only when A and B have the same number of rows and the same number of columns, that is, only when A and B are of the same size.

> We shall now establish the convention that when $A + B$ is formed, both A and B are of the same size.

Thus far, addition of matrices has only been defined for two matrices. Our work with matrices will call for adding more than two matrices. Theorem 1.1 in the next section shows that addition of matrices satisfies the associative property: $A + (B + C) = (A + B) + C$. Additional properties of matrix addition are considered in Section 1.4 and are similar to those satisfied by the real numbers.

EXAMPLE 11 **(Production)** A manufacturer of a certain product makes three models, A, B, and C. Each model is partially made in factory F_1 in Taiwan and then finished in factory F_2 in the United States. The total cost of each product consists of the manufacturing cost and the shipping cost. Then the costs at each factory (in dollars) can be described by the 3×2 matrices F_1 and F_2:

$$F_1 = \begin{array}{c} \begin{array}{cc} \text{Manufacturing cost} & \text{Shipping cost} \end{array} \\ \begin{bmatrix} 32 & 40 \\ 50 & 80 \\ 70 & 20 \end{bmatrix} \end{array} \begin{array}{l} \text{Model A} \\ \text{Model B} \\ \text{Model C} \end{array}$$

$$F_2 = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cc} \text{Manufacturing cost} & \text{Shipping cost} \\ \left[\begin{array}{c} 40 \\ 50 \\ 130 \end{array}\right. & \left.\begin{array}{c} 60 \\ 50 \\ 20 \end{array}\right] \end{array}\begin{array}{l} \\ \text{Model A} \\ \text{Model B} \\ \text{Model C} \end{array}$$

The matrix $F_1 + F_2$ gives the total manufacturing and shipping costs for each product. Thus the total manufacturing and shipping costs of a model C product are \$200 and \$40, respectively. ■

SCALAR MULTIPLICATION

DEFINITION

If $A = [a_{ij}]$ is an $m \times n$ matrix and r is a real number, then the **scalar multiple** of A by r, rA, is the $m \times n$ matrix $B = [b_{ij}]$, where

$$b_{ij} = ra_{ij} \qquad (1 \le i \le m, 1 \le j \le n).$$

That is, B is obtained by multiplying each element of A by r.

If A and B are $m \times n$ matrices, we write $A + (-1)B$ as $A - B$ and call this the **difference** of A and B.

EXAMPLE 12 Let

$$A = \begin{bmatrix} 2 & 3 & -5 \\ 4 & 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 5 & -2 \end{bmatrix}.$$

Then

$$A - B = \begin{bmatrix} 2-2 & 3+1 & -5-3 \\ 4-3 & 2-5 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & -8 \\ 1 & -3 & 3 \end{bmatrix}.$$ ■

EXAMPLE 13 Let $\mathbf{p} = [18.95 \quad 14.75 \quad 8.60]$ be a 3-vector that represents the current prices of three items at a store. Suppose that the store announces a sale so that the price of each item is reduced by 20%.

(a) Determine a 3-vector that gives the price changes for the three items.

(b) Determine a 3-vector that gives the new prices of the items.

Solution (a) Since each item is reduced by 20%, the 3-vector

$$\begin{aligned} 0.20\mathbf{p} &= [(0.20)18.95 \quad (0.20)14.75 \quad (0.20)8.60] \\ &= [3.79 \quad 2.95 \quad 1.72] \end{aligned}$$

gives the price reductions for the three items.

(b) The new prices of the items are given by the expression

$$\begin{aligned} \mathbf{p} - 0.20\mathbf{p} &= [18.95 \quad 14.75 \quad 8.60] - [3.79 \quad 2.95 \quad 1.72] \\ &= [15.16 \quad 11.80 \quad 6.88]. \end{aligned}$$

Observe that this expression can also be written as

$$\mathbf{p} - 0.20\mathbf{p} = 0.80\mathbf{p}.$$ ■

Def:

If $A_1, A_2, \ldots, A_k$ are $m \times n$ matrices and $c_1, c_2, \ldots, c_k$ are real numbers, then an expression of the form

$$c_1A_1 + c_2A_2 + \cdots + c_kA_k \tag{2}$$

is called a **linear combination** of $A_1, A_2, \ldots, A_k$, and $c_1, c_2, \ldots, c_k$ are called **coefficients**.

EXAMPLE 14 (a) If

$$A_1 = \begin{bmatrix} 0 & -3 & 5 \\ 2 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 5 & 2 & 3 \\ 6 & 2 & 3 \\ -1 & -2 & 3 \end{bmatrix},$$

then $C = 3A_1 - \frac{1}{2}A_2$ is a linear combination of A_1 and A_2. Using scalar multiplication and matrix addition, we can compute C:

$$C = 3\begin{bmatrix} 0 & -3 & 5 \\ 2 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 5 & 2 & 3 \\ 6 & 2 & 3 \\ -1 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{2} & -10 & \frac{27}{2} \\ 3 & 8 & \frac{21}{2} \\ \frac{7}{2} & -5 & -\frac{21}{2} \end{bmatrix}.$$

(b) $2\begin{bmatrix} 3 & -2 \end{bmatrix} - 3\begin{bmatrix} 5 & 0 \end{bmatrix} + 4\begin{bmatrix} -2 & 5 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 3 & -2 \end{bmatrix}$, $\begin{bmatrix} 5 & 0 \end{bmatrix}$, and $\begin{bmatrix} -2 & 5 \end{bmatrix}$. It can be computed (verify) as $\begin{bmatrix} -17 & 16 \end{bmatrix}$.

(c) $-0.5\begin{bmatrix} 1 \\ -4 \\ -6 \end{bmatrix} + 0.4\begin{bmatrix} 0.1 \\ -4 \\ 0.2 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ -4 \\ -6 \end{bmatrix}$ and $\begin{bmatrix} 0.1 \\ -4 \\ 0.2 \end{bmatrix}$.

It can be computed (verify) as $\begin{bmatrix} -0.46 \\ 0.4 \\ 3.08 \end{bmatrix}$. ■

THE TRANSPOSE OF A MATRIX

DEFINITION If $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is an $m \times n$ matrix, then the $n \times m$ matrix $A^T = \begin{bmatrix} a_{ij}^T \end{bmatrix}$, where

$$a_{ij}^T = a_{ji} \qquad (1 \le i \le n,\ 1 \le j \le m)$$

is called the **transpose** of A. Thus, the entries in each row of A^T are the entries in the corresponding column of A.

EXAMPLE 15 Let

$$A = \begin{bmatrix} 4 & -2 & 3 \\ 0 & 5 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 2 & -4 \\ 3 & -1 & 2 \\ 0 & 4 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & 4 \\ -3 & 2 \\ 2 & -3 \end{bmatrix},$$

$$D = \begin{bmatrix} 3 & -5 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

Then

$$A^T = \begin{bmatrix} 4 & 0 \\ -2 & 5 \\ 3 & -2 \end{bmatrix}, \quad B^T = \begin{bmatrix} 6 & 3 & 0 \\ 2 & -1 & 4 \\ -4 & 2 & 3 \end{bmatrix},$$

$$C^T = \begin{bmatrix} 5 & -3 & 2 \\ 4 & 2 & -3 \end{bmatrix}, \quad D^T = \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}, \quad \text{and} \quad E^T = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}.$$

BIT MATRICES (OPTIONAL)

The majority of our work in linear algebra will use matrices and vectors whose entries are real or complex numbers. Hence computations, like linear combinations, are determined using matrix properties and standard arithmetic base 10. However, the continued expansion of computer technology has brought to the forefront the use of binary (base 2) representation of information. In most computer applications like video games, FAX communications, ATM money transfers, satellite communications, DVD videos, or the generation of music CDs, the underlying mathematics is invisible and completely transparent to the viewer or user. Binary coded data is so prevalent and plays such a central role that we will briefly discuss certain features of it in appropriate sections of this book. We begin with an overview of binary addition and multiplication and then introduce a special class of binary matrices that play a prominent role in information and communication theory.

Binary representation of information uses only two symbols 0 and 1. Information is coded in terms of 0 and 1 in a string of **bits.*** For example, the decimal number 5 is represented as the binary string 101, which is interpreted in terms of base 2 as follows:

$$5 = 1(2^2) + 0(2^1) + 1(2^0).$$

The coefficients of the powers of 2 determine the string of bits, 101, which provide the binary representation of 5.

Just as there is arithmetic base 10 when dealing with the real and complex numbers, there is arithmetic using base 2; that is, binary arithmetic. Table 1.1 shows the structure of binary addition and Table 1.2 the structure of binary multiplication.

Table 1.1

+	0	1
0	0	1
1	1	0

Table 1.2

×	0	1
0	0	0
1	0	1

The properties of binary arithmetic for combining representations of real numbers given in binary form is often studied in beginning computer science courses or finite mathematics courses. We will not digress to review such topics at this time. However, our focus will be on a particular type of matrix and vector that contain entries that are single binary digits. This class of matrices and vectors are important in the study of information theory and the mathematical field of *error-correcting codes* (also called *coding theory*).

*A bit is a binary digit; that is, either a 0 or 1.

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DEFINITION An $m \times n$ **bit matrix**[†] is a matrix all of whose entries are (single) bits. That is, each entry is either 0 or 1.

A bit **n-vector (or vector)** is a $1 \times n$ or $n \times 1$ matrix all of whose entries are bits. ∎

EXAMPLE 16 $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is a 3×3 bit matrix. ∎

EXAMPLE 17 $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is a bit 5-vector and $\mathbf{u} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$ is a bit 4-vector. ∎

The definitions of matrix addition and scalar multiplication apply to bit matrices provided we use binary (or base 2) arithmetic for all computations and use the only possible scalars 0 and 1.

EXAMPLE 18 Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. Using the definition of matrix addition and Table 1.1, we have

$$A + B = \begin{bmatrix} 1+1 & 0+1 \\ 1+0 & 1+1 \\ 0+1 & 1+0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$ ∎

Linear combinations of bit matrices or bit n-vectors are quite easy to compute using the fact that the only scalars are 0 and 1 together with Tables 1.1 and 1.2.

EXAMPLE 19 Let $c_1 = 1, c_2 = 0, c_3 = 1, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then

$$\begin{aligned} c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 &= 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} (1+0)+1 \\ (0+0)+1 \end{bmatrix} \\ &= \begin{bmatrix} 1+1 \\ 0+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$ ∎

Q-12

From Table 1.1 we have $0 + 0 = 0$ and $1 + 1 = 0$. Thus the additive inverse of 0 is 0 (as usual) and the additive inverse of 1 is 1. Hence to compute the difference of bit matrices A and B we proceed as follows:

$$A - B = A + (\text{inverse of } 1)\, B = A + 1B = A + B.$$

We see that the difference of bit matrices contributes nothing new to the algebraic relationships among bit matrices.

[†]A bit matrix is also called a Boolean matrix.

Key Terms

Matrix
Rows
Columns
Size of a matrix
Square matrix
Main diagonal of a matrix
Element (or entry) of a matrix
ijth element
(i, j) entry
n-vector (or vector)
Diagonal matrix
Scalar matrix
0, the zero vector
R^n, the set of all n-vectors
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Equal matrices
Matrix addition
Scalar multiplication
Scalar multiple of a matrix
Difference of matrices
Linear combination of matrices
Transpose of a matrix
Bit
Bit (or Boolean) matrix
Upper triangular matrix
Lower triangular matrix

1.2 Exercises

1. Let

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 6 & -5 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 7 & 3 & 2 \\ -4 & 3 & 5 \\ 6 & 1 & -1 \end{bmatrix}.$$

(a) What is a_{12}, a_{22}, a_{23}?

(b) What is b_{11}, b_{31}?

(c) What is c_{13}, c_{31}, c_{33}?

2. If

$$\begin{bmatrix} a+b & c+d \\ c-d & a-b \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 10 & 2 \end{bmatrix},$$

find a, b, c, and d.

H.W

3. If

$$\begin{bmatrix} a+2b & 2a-b \\ 2c+d & c-2d \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 4 & -3 \end{bmatrix},$$

find a, b, c, and d.

In Exercises 4 through 7, let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix},$$

$$E = \begin{bmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} -4 & 5 \\ 2 & 3 \end{bmatrix},$$

and $$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

4. If possible, compute the indicated linear combination:

(a) $C + E$ and $E + C$ (b) $A + B$

(c) $D - F$ (d) $-3C + 5O$

(e) $2C - 3E$ (f) $2B + F$

5. If possible, compute the indicated linear combination:

(a) $3D + 2F$

(b) $3(2A)$ and $6A$

(c) $3A + 2A$ and $5A$

(d) $2(D + F)$ and $2D + 2F$

(e) $(2 + 3)D$ and $2D + 3D$

(f) $3(B + D)$

6. If possible, compute:

(a) A^T and $(A^T)^T$

(b) $(C + E)^T$ and $C^T + E^T$

(c) $(2D + 3F)^T$

(d) $D - D^T$

(e) $2A^T + B$

(f) $(3D - 2F)^T$

7. If possible, compute:

(a) $(2A)^T$ (b) $(A - B)^T$

(c) $(3B^T - 2A)^T$

(d) $(3A^T - 5B^T)^T$

(e) $(-A)^T$ and $-(A^T)$

(f) $(C + E + F^T)^T$

8. Is the matrix $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ a linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$? Justify your answer.

H.W

9. Is the matrix $\begin{bmatrix} 4 & 1 \\ 0 & -3 \end{bmatrix}$ a linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$? Justify your answer.

10. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & -2 & 3 \\ 5 & 2 & 4 \end{bmatrix} \quad \text{and} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If λ is a real number, compute $\lambda I_3 - A$.

H.W

Exercises 11 through 15 involve bit matrices. P-17

11. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Compute each of the following.

(a) $A + B$ (b) $B + C$ (c) $A + B + C$
(d) $A + C^T$ (e) $B - C$

H.W

12. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, and $D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Compute each of the following.

(a) $A + B$ (b) $C + D$ (c) $A + B + (C + D)^T$
(d) $C - B$ (e) $A - B + C - D$

13. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(a) Find B so that $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(b) Find C so that $A + C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

H.W

14. Let $\mathbf{u} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$. Find the bit 4-vector $\mathbf{v}$ so that $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$.

15. Let $\mathbf{u} = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$. Find the bit 4-vector $\mathbf{v}$ so that $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$.

Theoretical Exercises

T.1. Show that the sum and difference of two diagonal matrices is a diagonal matrix.

T.2. Show that the sum and difference of two scalar matrices is a scalar matrix.

T.3. Let

$$A = \begin{bmatrix} a & b & c \\ c & d & e \\ e & e & f \end{bmatrix}.$$

(a) Compute $A - A^T$.

(b) Compute $A + A^T$.

(c) Compute $(A + A^T)^T$.

T.4. Let O be the $n \times n$ matrix all of whose entries are zero. Show that if k is a real number and A is an $n \times n$ matrix such that $kA = O$, then $k = 0$ or $A = O$.

T.5. A matrix $A = [a_{ij}]$ is called **upper triangular** if $a_{ij} = 0$ for $i > j$. It is called **lower triangular** if $a_{ij} = 0$ for $i < j$.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

Upper triangular matrix
(The elements below the main diagonal are zero.)

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 0 \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

Lower triangular matrix
(The elements above the main diagonal are zero.)

(a) Show that the sum and difference of two upper triangular matrices is upper triangular.

(b) Show that the sum and difference of two lower triangular matrices is lower triangular.

(c) Show that if a matrix is both upper and lower triangular, then it is a diagonal matrix.

T.6. (a) Show that if A is an upper triangular matrix, then A^T is lower triangular.

(b) Show that if A is a lower triangular matrix, then A^T is upper triangular.

T.7. If A is an $n \times n$ matrix, what are the entries on the main diagonal of $A - A^T$? Justify your answer.

T.8. If $\mathbf{x}$ is an n-vector, show that $\mathbf{x} + \mathbf{0} = \mathbf{x}$.

Exercises T.9 through T.18 involve bit matrices.

T.9. Make a list of all possible bit 2-vectors. How many are there?

T.10. Make a list of all possible bit 3-vectors. How many are there?

T.11. Make a list of all possible bit 4-vectors. How many are there?

Ex: 1.2

Q1 Let $A=\begin{bmatrix} 2 & -3 & 5 \\ 6 & -5 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix}$ and $C=\begin{bmatrix} 7 & 3 & 2 \\ -4 & 3 & 5 \\ 6 & 1 & -1 \end{bmatrix}$.

a) $a_{12}=-3$, $a_{22}=-5$, $a_{23}=4$

b) $b_{11}=4$, $b_{31}=5$

c) $c_{13}=2$, $c_{31}=6$, $c_{33}=-1$

Q2 If $\begin{bmatrix} a+b & c+d \\ c-d & a-b \end{bmatrix}=\begin{bmatrix} 4 & 6 \\ 10 & 2 \end{bmatrix}$ find a, b, c & d.

Sol:

$a+b=4$ —— (i)
$c+d=6$ —— (ii)
$c-d=10$ —— (iii)
$a-b=2$ —— (iv)

(i)+(iv)

$a+\cancel{b}=4$
$a-\cancel{b}=2$

$2a=6 \Rightarrow \boxed{a=3}$

eq (i) $\Rightarrow 3+b=4 \Rightarrow b=4-3=1 \Rightarrow \boxed{b=1}$

(ii) + (iii)

$c+\cancel{d}=6$
$c-\cancel{d}=10$

$2c=16 \Rightarrow \boxed{c=8}$

eq (ii) $\Rightarrow 8+d=6 \Rightarrow d=6-8=-2 \Rightarrow \boxed{d=-2}$

Q3 is similarly to Q2.

In Exercises 4 through 7, let

$$A=\begin{bmatrix}1&2&3\\2&1&4\end{bmatrix},\quad B=\begin{bmatrix}1&0\\2&1\\3&2\end{bmatrix},\quad C=\begin{bmatrix}3&-1&3\\4&1&5\\2&1&3\end{bmatrix}$$

$$D=\begin{bmatrix}3&-2\\2&4\end{bmatrix},\quad E=\begin{bmatrix}2&-4&5\\0&1&4\\3&2&1\end{bmatrix},\quad F=\begin{bmatrix}-4&5\\2&3\end{bmatrix}$$

and $O=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$.

Q4 a) C+E & E+C

$$C+E=\begin{bmatrix}3&-1&3\\4&1&5\\2&1&3\end{bmatrix}+\begin{bmatrix}2&-4&5\\0&1&4\\3&2&1\end{bmatrix}=\begin{bmatrix}5&-5&8\\4&2&9\\5&3&4\end{bmatrix}$$

$$\& \ E+C=\begin{bmatrix}2&-4&5\\0&1&4\\3&2&1\end{bmatrix}+\begin{bmatrix}3&-1&3\\4&1&5\\2&1&3\end{bmatrix}=\begin{bmatrix}5&-5&8\\4&2&9\\5&3&4\end{bmatrix}.$$

commutative addition.

b) $A+B=\begin{bmatrix}1&2&3\\2&1&4\end{bmatrix}+\begin{bmatrix}1&0\\2&1\\3&2\end{bmatrix}$ addition is not

Possible because the order of matrices is not same.

Ex 1.2

(c) $D-F = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -4 & 5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3+4 & -2-5 \\ 2-2 & 4-3 \end{bmatrix} = \begin{bmatrix} 7 & -7 \\ 0 & 1 \end{bmatrix}$

(d) $-3C+5O = -3\begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix} + 5\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} -9 & 3 & -9 \\ -12 & -3 & -15 \\ -6 & -3 & -9 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} -9 & 3 & -9 \\ -12 & -3 & -15 \\ -6 & -3 & -9 \end{bmatrix}$ Ans.

(f) $2B+F = 2\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} -4 & 5 \\ 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 \\ 4 & 2 \\ 6 & 4 \end{bmatrix} + \begin{bmatrix} -4 & 5 \\ 2 & 3 \end{bmatrix}$ addition is not Possible because the order of matrices is not same.

Q5 If Possible compute the indicated linear combination

(a) $3D+2F = 3\begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix} + 2\begin{bmatrix} -4 & 5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & -6 \\ 6 & 12 \end{bmatrix} + \begin{bmatrix} -8 & 10 \\ 4 & 6 \end{bmatrix}$

$= \begin{bmatrix} 1 & 4 \\ 10 & 18 \end{bmatrix}$.

(b) 3(2A) and 6A

$$3(2A) = 6A = 6\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 12 & 18 \\ 12 & 6 & 24 \end{bmatrix}.$$

(f) $3(B+D) = 3\left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}\right)$ addition is not

Possible because the order of matrices is not same.

Q6 If Possible, compute:

(a) A^T and $(A^t)^t$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}^t = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}.$$

$$d\ (A^t)^t = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}^t = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$$

So $A^t = (A^t)^t$.

Ex: 1.2

(b) $(C+E)^T$ and C^T+E^T.

$$C+E=\begin{bmatrix}3&-1&3\\4&1&5\\2&1&3\end{bmatrix}+\begin{bmatrix}2&-4&5\\0&1&4\\3&2&1\end{bmatrix}=\begin{bmatrix}5&-5&8\\4&2&9\\5&3&4\end{bmatrix}.$$

$$(C+E)^t=\begin{bmatrix}5&-5&8\\4&2&9\\5&3&4\end{bmatrix}^t=\begin{bmatrix}5&4&5\\-5&2&3\\8&9&4\end{bmatrix}.$$

$$C^t=\begin{bmatrix}3&4&2\\-1&1&1\\3&5&3\end{bmatrix} \text{ \& } E^t=\begin{bmatrix}2&0&3\\-4&1&2\\5&4&1\end{bmatrix}.$$

$$C^t+E^t=\begin{bmatrix}5&4&5\\-5&2&3\\8&9&4\end{bmatrix}.$$

So $(C+E)^t=C^t+E^t$ An..

Q7 is similarly to Q 6.

Q9 Is the matrix $\begin{bmatrix}4&1\\0&-3\end{bmatrix}$ a linear combination of the matrices $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ and $\begin{bmatrix}1&0\\0&0\end{bmatrix}$? Justify your answer.

Sol: $$\begin{bmatrix}4&1\\0&-3\end{bmatrix}=c_1\begin{bmatrix}1&0\\0&1\end{bmatrix}+c_2\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}c_1&0\\0&c_1\end{bmatrix}+\begin{bmatrix}c_2&0\\0&0\end{bmatrix}.$$

$$\begin{bmatrix}4&1\\0&-3\end{bmatrix}=\begin{bmatrix}c_1+c_2&0\\0&c_1\end{bmatrix}.$$

$c_1 + c_2 = 4$, $d\ c_1 = -3$ $\quad c_2 = 7$

So $\begin{bmatrix} 4 & 1 \\ 0 & -3 \end{bmatrix} \neq \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}$

So it's not a linear combination.

Q13 let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ bit matrices.

(a) Find B so that $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ bit.M.}$$

(b) Find C so that $A + C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - A$$

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Q15 Given that $U = [0\ 1\ 0\ 1]$ & $U + V = [1\ 1\ 1\ 1]$

let $V = [a, b, c, d]$

then $[0\ 1\ 0\ 1] + [a, b, c\ d] = [1\ 1\ 1\ 1]$

$[a \quad b+1 \quad c \quad d+1] = [1\ 1\ 1\ 1]$

$\boxed{a = 1}$ $\quad b + 1 = 1 \Rightarrow \boxed{b = 0}$ $\quad \boxed{c = 1}$ $\quad d + 1 = 1 \Rightarrow \boxed{d = 0}$

So $V = [a\ b\ c\ d] = [1\ 0\ 1\ 0]$.

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