Important Continuous Distributions

Lecture-03

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Recall from lesson-01

- For a function to be continuous probability distributions, two conditions must be satisfied. $(i) f(x) \geq 0, \forall\, x \epsilon D\ and\ (ii)$ $\int_{x=l}^{x=u} f(x)dx = 1$ (Area under the curve is 1.0)
- Recall how to calculated the following for a pdf.

- Mean: $\mathrm{E(x)} = \int_{x=l}^{u} x.f(x)dx$

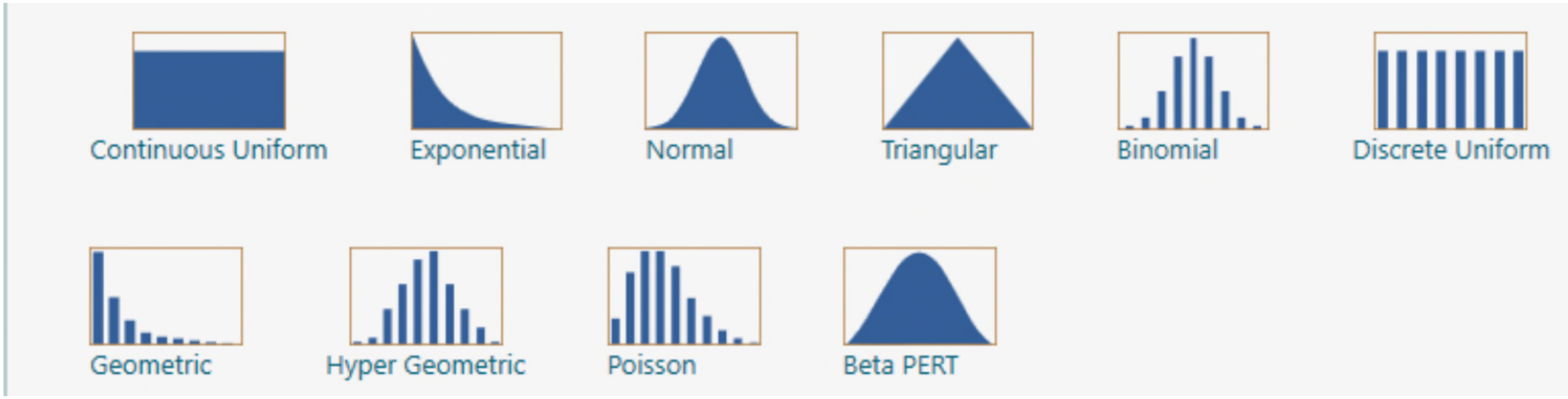
- Variance: $E(x^2) - [E(x)]^2$ where $E(x^2) = \int_{x=l}^{u} x^2.f(x)dx$

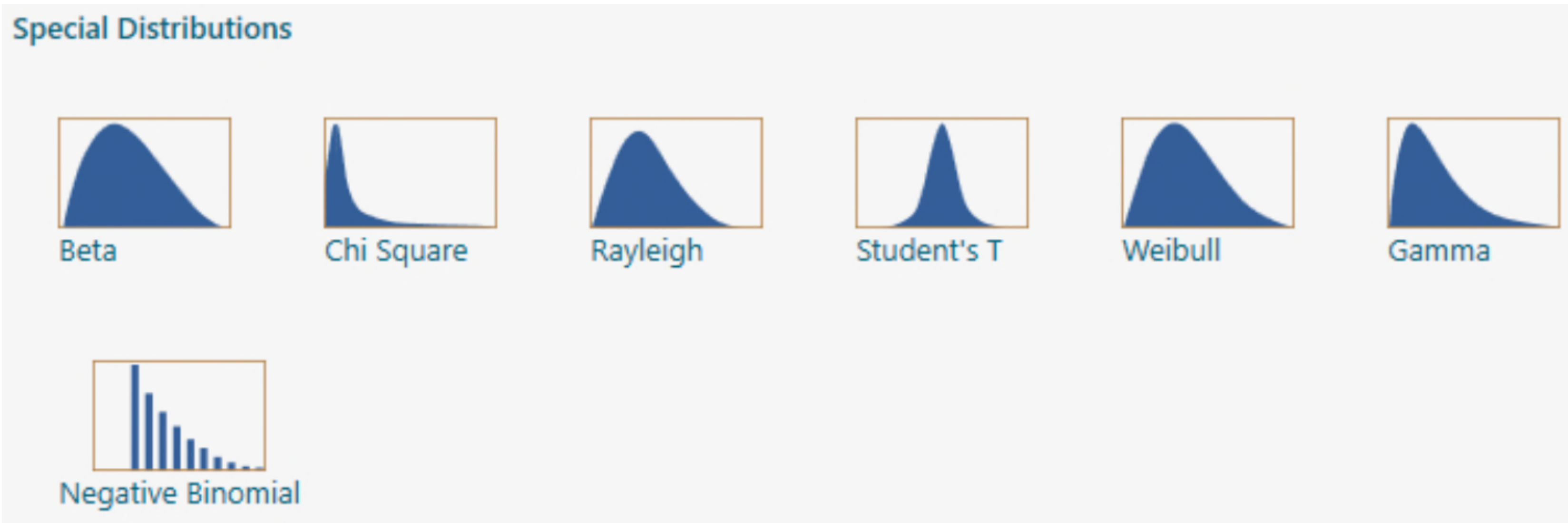
- Median: $\int_{x=l}^{M} f(x)dx = \frac{1}{2}$

- Kth order raw moment: $\mathrm{E}\left(x^k\right) = \int_{x=l}^{u} x^k.f(x)dx$

Common and Special Distributions

- There are hundreds of probability distributions in various fields. Such as engineering, genetics, social sciences, linguistics, business and economics etc.





Uniform distribution

- Continuous probability distribution.

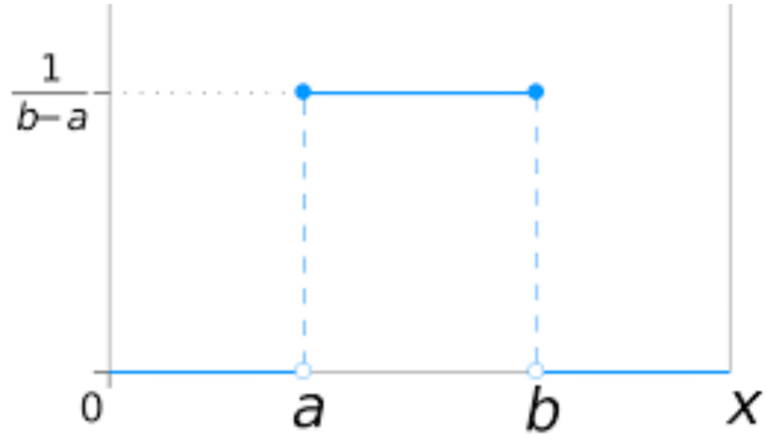
$$f(x)=\begin{cases}\frac{1}{b-a} & \text{for } a\le x\le b,\\ 0 & \text{for } x<a \text{ or } x>b\end{cases}$$

- a is the location parameters (starting point) and b-a is the scale parameter (spread).
- If a=0 and b=1, then f(x) =1 (Standard Uniform distribution)

$$f(x)=1;\ \ for\ 0\le x\le 1$$

- You can test the effect of parameters a and b using:

https://www.geogebra.org/m/jmfsagfz



- Symmetric distribution
- Mean $=\frac{a+b}{2}$
- Variance $=\frac{(b-a)^2}{12}$

$$F(x)=\begin{cases}0 & \text{for } x<a\\ \frac{x-a}{b-a} & \text{for } a\le x\le b\\ 1 & \text{for } x>b\end{cases}$$

Uniform distribution

- $Mean = E(x) = \int_{x=a}^{b} x.f(x) = \int_{x=a}^{b} x.\frac{1}{b-a}$

$$= \frac{1}{b-a}.\left|\frac{x^2}{2}\right|_a^b = \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{a+b}{2}$$

- $E(x^2) = \int_{x=a}^{b} x^2.f(x) = \int_{x=a}^{b} x^2.\frac{1}{b-a}$

$$= \frac{1}{b-a}.\left|\frac{x^3}{3}\right|_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{(b^2+ab+a^2)}{3}$$

- $V(x) = E(x^2) - [E(x)]^2 = \frac{(b-a)^2}{12}$ **(D.I.Y)**

Uniform distribution

- **K^th order raw moment: Another approach to derive mean, variance.**
- $E\left(x^k\right) = \int_{x=a}^{b} x^k . f(x) = \int_{x=a}^{b} x^k . \frac{1}{b-a}$

$$= \frac{1}{b-a} . \left| \frac{x^{k+1}}{k+1} \right|_a^b = \frac{b^{k+1} - a^{k+1}}{(k+1)(b-a)}$$

- Derive the mean and variance using $E\left(x^k\right)$. **(D.I.Y)**

The median of uniform distribution:

$$\int_{x=a}^{x=M} \frac{1}{b-a} = 0.5 \Rightarrow \frac{M-a}{b-a} = 0.5 \Rightarrow$$

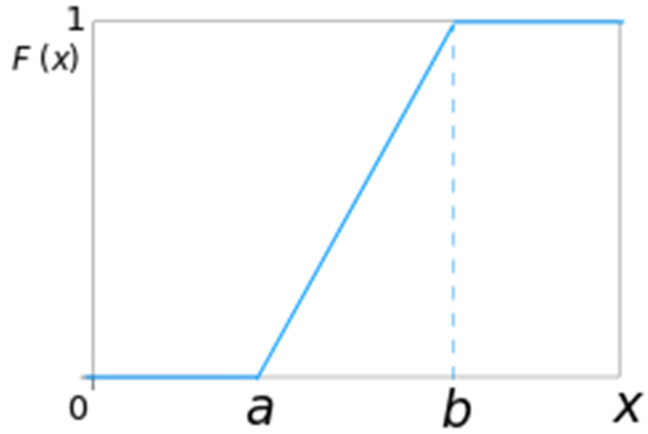
$$M = a + 0.5(b-a)$$

CDF of Uniform distribution

- $F(x) = P(X \leq x) = \int_{x=a}^{x} f(x)\, dx = \int_{a}^{x} \frac{1}{b-a}$

$$= \frac{1}{b-a} . |x|_a^x$$

$$F(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \leq x \leq b \\ 1 & \text{for } x > b \end{cases}$$



- Random number generator: Let $\frac{x-a}{b-a} = u \Rightarrow x = a + (b-a)u$ is the random number generator for Uniform distribution where u~U(0,1)

Question-1:

Q1. Consider the Uniform distribution:

$$f(x) = \frac{1}{2}\ ; -1 \le x \le 1$$

Calculate the following: (i). $P\left(\frac{-1}{2} \le x \le \frac{1}{2}\right)$ (ii) $P\left(x \ge \frac{1}{2}\right)$, (iii) mean and variance (iv) the median of f(x). (v). $P(-10 \ \le x \le 10)$.

Solution:

(i). $P\left(\frac{-1}{2} \le x \le \frac{1}{2}\right) = \int_{-1/2}^{1/2} \left(\frac{1}{2}\right) dx = \frac{1}{2}\left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right) = \frac{1}{2}.$

(ii). $P\left(x \ge \frac{1}{2}\right) = \int_{1/2}^{\infty} \left(\frac{1}{2}\right) dx = \int_{1/2}^{1} \frac{1}{2}\ dx = \frac{1}{2}\left(1 - \left(\frac{1}{2}\right)\right) = \frac{1}{4}.$

(iii). Mean = $\frac{a+b}{2} = \frac{1-(-1)}{2} = 1; Variace = \frac{(b-a)^2}{12} = \frac{1}{3}$

(iv) Median = $M = a + 0.5(b-a)$

(v). $P(-10 \ \le x \le 10) = P(x \le -1) + P(-1 \ \le x \le 1) + P(x \ge 1)$

Questions:

Q2: The random variable X is Uniformly probability distributed with mean= 5 and variance = 3. Find $P(|x - 4| < 1.5)$. Find the mean and variance. (Note: use calculus to solve absolute function).

Exponential Probability Distribution

Applications:

- Service time modelling (Queuing theory). e.g. traffic control.
- Reliability theory (e.g. electronic components)
- Time to event problems (e.g. Half life of a radioactive element)

Exponential distribution

- Recall: Poisson distribution counts the number of occurrences of some event in given fixed interval of time.
- An exponential distribution is reciprocal to Poisson distribution and measures the time between two occurrences.
  - e.g. waiting time between two buses.
  - e.g. time between two pandemic waves in a location.
  - Time between two faults by a machine.
  - The amount of time until the customer finishes browsing and actually purchases something in your store (success).
- The general formula for exponential distribution is:

$$f(x) = \frac{1}{\beta} e^{-(x-\mu)/\beta} \qquad x \geq \mu; \beta > 0$$

Here μ is the location parameter and β is the scale parameter (the scale parameter is often referred to as λ which equals 1/β). Standard exponential distribution: If $\mu = 0$ and $\beta = 1$.

$$f(x) = e^{-x} \qquad \text{for } x \geq 0$$

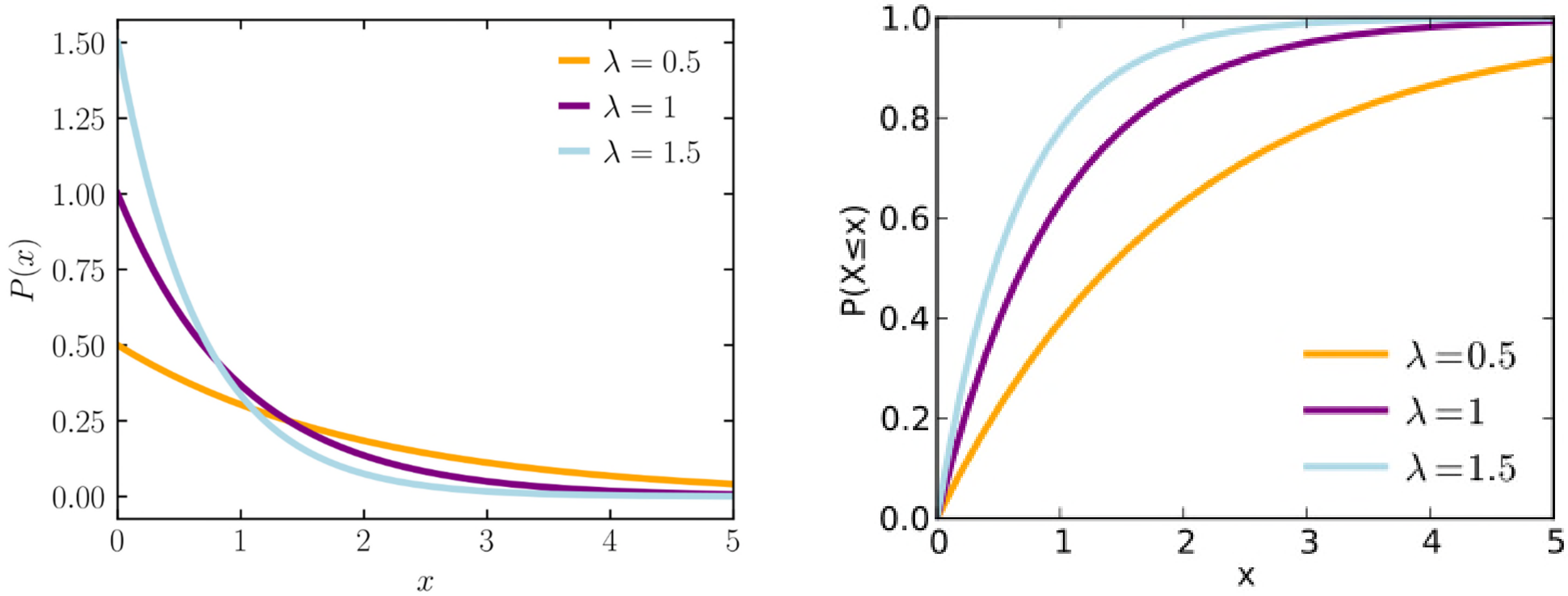
Exponential distribution

- Exponential distribution pdf :

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0, \\ 0 & x < 0. \end{cases}$$

- The cumulative density function (CDF) is:

$$F(x;\lambda) = \begin{cases} 1 - e^{-\lambda x} & x \ge 0, \\ 0 & x < 0. \end{cases}$$



Negative Exponential distribution

- Other form of exponential pdf (negative exponential distribution):

$$f(x;\beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & x \ge 0, \\ 0 & x < 0. \end{cases} \qquad F(x;\beta) = \begin{cases} 1 - e^{-x/\beta} & x \ge 0, \\ 0 & x < 0. \end{cases}$$

- For large values of parameter λ, the distribution decays faster.

- Try the visual effects of parameters and probabilities for modified exponential distribution https://www.geogebra.org/m/rGXMGrV4

K^{th} order mean (for mean and variance)

- $E(x^k) = \int_{x=0}^{\infty} x^k . \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} x^k . e^{-\lambda x}\ dx$

Change of variable (transformation)

$Let\ \ y = \lambda x \Rightarrow \frac{dy}{dx} = \lambda \Rightarrow \frac{dy}{\lambda} = dx$

$For\ x = 0 \Rightarrow y = 0\ and\ for x \to \infty \Rightarrow y \to \infty.$

$Also\ y = \lambda x \Rightarrow x = \frac{y}{\lambda}$

Therefore:

$$E(x^k) = \lambda \int_0^{\infty} \left(\frac{y}{\lambda}\right)^k . e^{-y}\ \frac{dy}{\lambda}\ \Rightarrow \frac{1}{\lambda^k} \int_0^{\infty} y^k . e^{-y}\ dy$$

K^th order mean (for mean and variance)

Using Gamma integration: $\int_0^\infty w^{k-1}.\, e^{-w}\, dw = (k-1)!$

Therefore:

$$E\left(x^k\right) = \frac{1}{\lambda^k}\int_0^\infty y^{(k+1)-1}.\, e^{-y} dy = \frac{(\mathrm{k}+1-1)!}{\lambda^k}$$

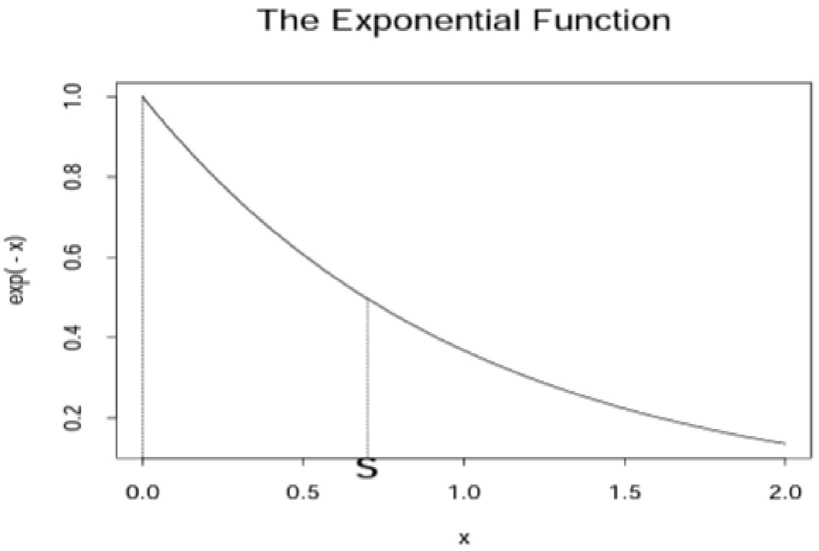
$$E\left(x^k\right) = \frac{\mathrm{k}!}{\lambda^k} \quad for\ k = 1,2,3,4$$

Question: Derive the mean and variance using $E\left(x^k\right)$

The Memoryless Property of Exponential pdf:

- The following plot illustrates a key property of the exponential distribution. The graph after the point s is an exact copy of the original function. The important consequence of this is that the distribution of X conditioned on {X > s} is again exponential.

$$
\begin{aligned}
P(Y > t | X > s) &= P(X > s + t | X > s) \\
&= \frac{P(X > s + t, X > s)}{P(X > s)} \\
&= \frac{P(X > s + t)}{P(X > s)} \\
&= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \\
&= e^{-\lambda t}.
\end{aligned}
$$



- You replaced a light bulb at your house, You won't worry about the total running life of the old plus new bulb but only care for the current new bulb.

Question-01:

1. The life of a certain type of device has an advertised failure rate of 0.01 per hour. The failure rate is constant and the exponential distribution is applies. What is the probability that atleast 200 hours will pass before a failure is observed?

Here $\beta = 1/\lambda$ = 0.01.

Thus, $P(x \geq 200) = 1 - \int_0^{200} \left(0.01.\, e^{-0.01x}\right) dx$

$= 1 - P(X \leq 200)$

$= 1 - F(200)$

$= 1 - \left(1 - e^{-0.01 \times 200}\right)$

$=$ 0.135335

Question-02:

1. Let X denotes the length in time of a long-distance telephone conversation. Assume that the density for X is given by

$$\boldsymbol{f(x) = \frac{1}{10}\, e^{-\frac{1}{10}x}, \qquad x \geq 0}$$

- Find the probability that a randomly selected call will last at most 7 minutes; at least 7 minutes, exactly 7 minutes.

  Hint: $P(x \leq 7); P(x \geq 7) = 1 - P(x \leq 7); and\ P(X = 7)$

- Would it be unusual for a call to last between 1 and 2 minutes? Explain, based on the probability of this occurring. (To find $P(1 < x \leq 2)$;

- Find mean and variance of f(x).

End of Lesson-03

Thank you