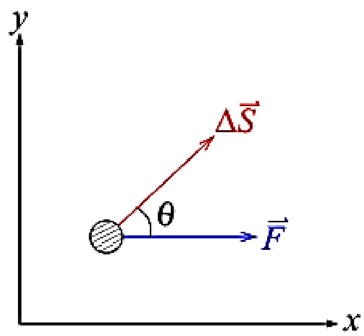
# Chapter 11

# Work and Kinetic Energy

## 11.1 Work done by a constant force




Consider a point mass $m$ in a time interval of $\Delta t$ is experiencing a constant force $\vec{F}$. During this time interval, the displacement of $m$ is $\Delta\vec{S}$.

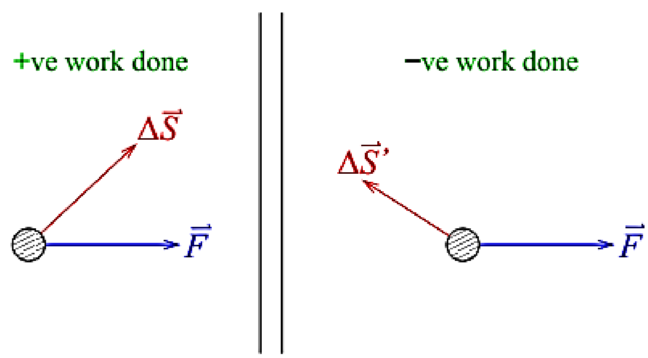
Work done by the force on the mass:

$$W \underbrace{=}_{\text{def}} \vec{F}\cdot\Delta\vec{S} = F\Delta S\cos\theta\ .$$

Work can be either positive or negative.

Power is defined by:

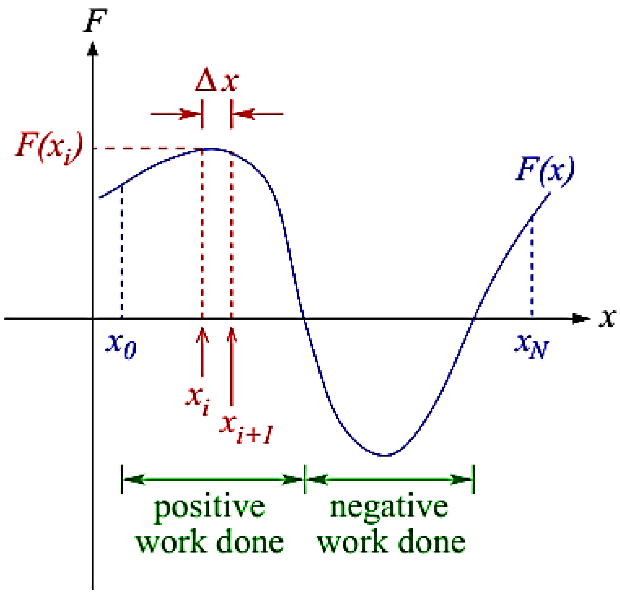
$$\boxed{P = \frac{dW}{dt}}$$



## 11.2 Work done by a variable force

### 11.2.1 One dimensional case

Suppose there is a position dependent force $F(x)$.




- Divide the whole displacement from $x_0$ to $x_N$ into $N$ partitions with separation $\Delta x$.
- Consider the $i$-th partition, $x_i \to x_{i+1}$ and in this very small interval, $F$ is approximately constant at $F(x_i)$.

$\therefore$ Work done in this time interval:

$$\Delta W(x_i) = F(x_i)\Delta x$$

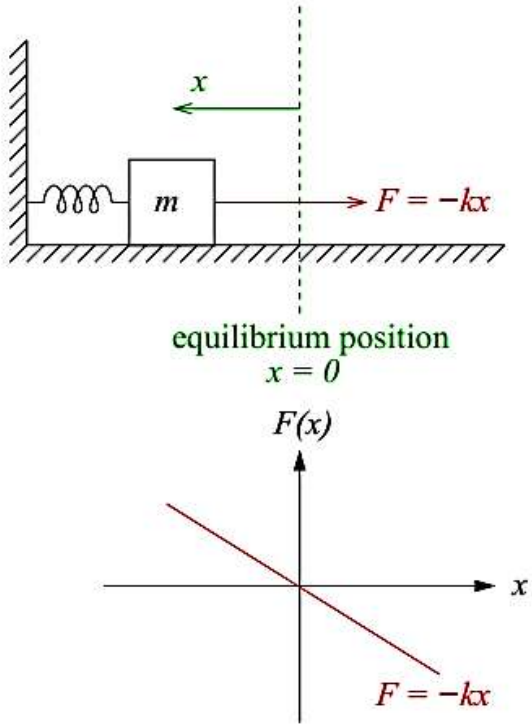
or $$\boxed{F(x) = \frac{dW}{dx}}$$

Total work done for the displacement from $x_0$ to $x_N$:

$$\boxed{W_{x_0 \to x_N} = \sum_i \Delta W_i = \sum_i F(x_i)\Delta x = \int_{x_0}^{x_N} F(x)dx}$$

or it is equal to the total area of the figure with positive area for positive $F(x)$ and negative area for negative $F(x)$.

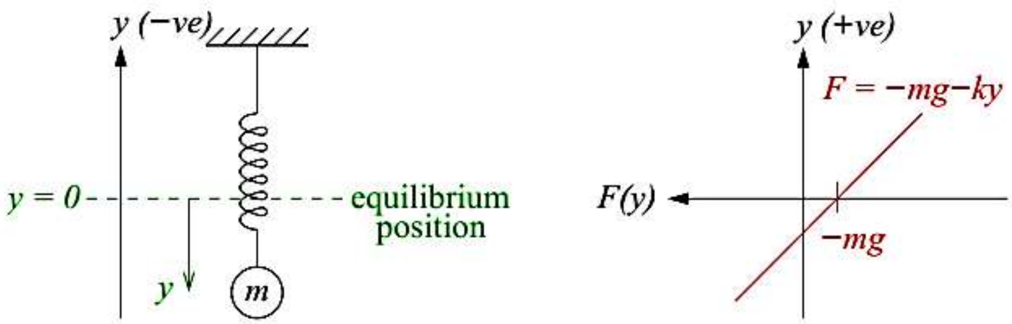
Example:




Spring force

$$
\begin{aligned}
W_{A\to B} &= \int_{x_A}^{x_B} F(x)dx \\
&= \int_{x_A}^{x_B} -kx dx \\
&= -\frac{1}{2}k(x_B^2 - x_A^2)
\end{aligned}
$$

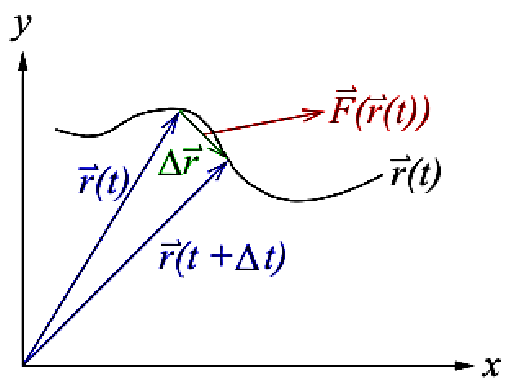
Example:




Consider the restoring force $F$,

$$
\begin{aligned}
F &= -mg - ky \\
W_{A\to B} &= \int_{y_A}^{y_B} F(y)dy \\
&= \int_{y_A}^{y_B} (-mg - ky)dy \\
&= -mg(y_B - y_A) - \frac{1}{2}k(y_B^2 - y_A^2)
\end{aligned}
$$

### 11.2.2 Two dimensional case




Trajectory of a particle is given by:

$$\vec{r}(t) = f_x(t)\hat{i} + f_y(t)\hat{j}$$

Force at any point $\vec{r}$ is given by:

$$\vec{F}(t) = F_x(\vec{r})\hat{i} + F_y(\vec{r})\hat{j}$$

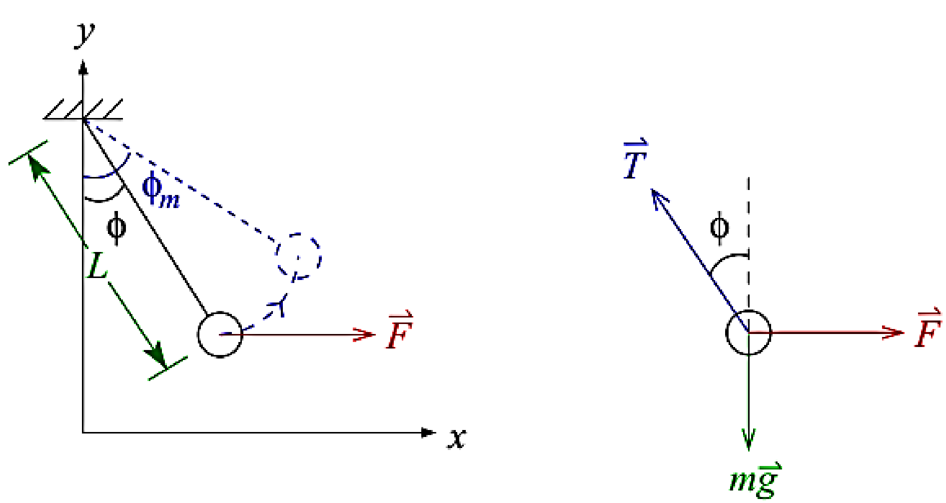
Consider a particle moving from $\vec{r}(t)$ to $\vec{r}(t+\Delta t)$ during the time interval $\Delta t$.
If $\Delta t \to 0$, force experienced by particle in this time interval is constant and $\approx \vec{F}(\vec{r}(t))$.
Work done in this small time interval with displacement $\Delta\vec{r}$:

$$\Delta W = \vec{F}(\vec{r}(t)) \cdot \Delta\vec{r}$$

(Only the tangential force component contributes!!)

$$\therefore \quad \boxed{W = \int \vec{F} \cdot d\vec{r}}$$

Example:

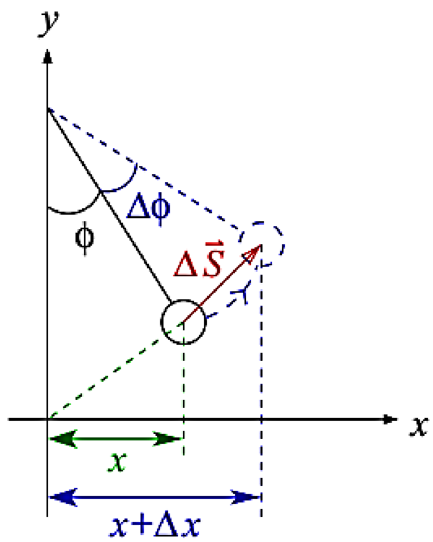



A mass $m$ is hanged by a string with length $L$ initially. A force $F$ which is always horizontal is applied to lift the mass up to an angle $\phi$. During the process, the mass moves with constant speed so small that the centripetal force can be neglected. Find the work done by the force $F$.

Answer:

If centripetal force approaches zero, $a_x = 0$ and $a_y = 0$.

$$F - T\sin\phi = 0 \quad \text{and} \quad T\cos\phi - mg = 0$$
$$\Rightarrow \quad F = mg\tan\phi$$




Consider the displacement $\Delta S$ from $\phi \to \phi + \Delta\phi$:

$$\begin{aligned} \Delta W &= \vec{F}\cdot\Delta\vec{S} = \vec{F}_{\text{tang}}\cdot\Delta\vec{S} \\ \text{but also:} &= F\Delta x \end{aligned}$$

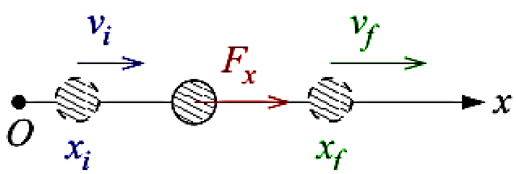
But

$$x = L\sin\phi \quad \Rightarrow \quad dx = L\cos\phi\, d\phi$$
$$\therefore \quad \Delta W = mg\tan\phi\; L\cos\phi\; d\phi = mgL\sin\phi\; d\phi$$

Hence,

$$W = \int_0^{\phi_m} mgL\sin\phi\; d\phi = mgL(1 - \cos\phi_m)$$

## 11.3 Work-energy theorem




Consider a particle $m$ displaces from $x_i$ to $x_f$. During this displacement, the $x$-component of the net force acting on $m$ is $F_x$.

$$\therefore \quad F_x = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = mv\frac{dv}{dx} \tag{11.1}$$

Work done on the mass by the force:

$$\begin{aligned} W &= \int_{x_i}^{x_f} F_x dx \\ &= \int_{x_i}^{x_f} mv\frac{dv}{dx}dx \quad \text{(using eq. (11.1))} \\ &= \int_{v_i}^{v_f} mv dv \\ \therefore \quad W &= \frac{1}{2}m(v_f^2 - v_i^2) \end{aligned}$$

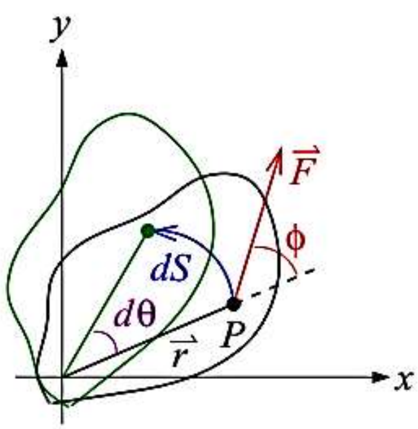
Define kinetic energy $K = \frac{1}{2}mv^2$.

$$W = K_f - K_i = \Delta K$$

If W is positive, $v_f > v_i$ and $\Delta K > 0$.
If W is negative, $v_f < v_i$ and $\Delta K < 0$.

N. B. In inertia frames having relative motion, the absolute value of kinetic energy are not the same, but the theorem $W = \Delta K$ holds in all inertia frames.

## 11.4 Work done and kinetic energy in rotational motion




Consider a rigid body moving through an angle $d\theta$ about the rotational $z$-axis with a force acting on point $P$.
Work done by the force:

$$dW = (F\sin\phi)dS = F\sin\phi\, r d\theta = \tau_z\, d\theta$$

where $\tau_z$ is the $z$-component of the torque about $O$.

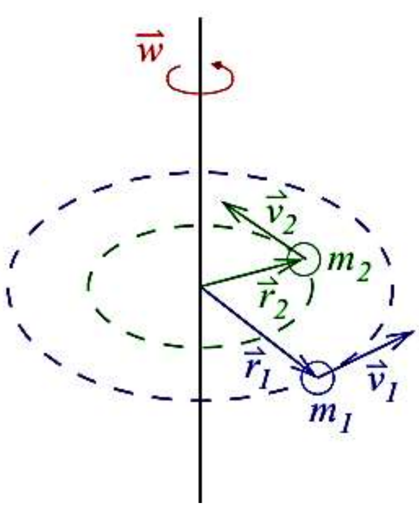
$\therefore$ If the rigid body is to displace from $\theta_i$ to $\theta_f$,

$$\boxed{W = \int_{\theta_i}^{\theta_f} \tau_z d\theta}$$

If the torque is constant,

$$\boxed{W = \tau_z \theta}$$

Power: $$P = \frac{dW}{dt} = \tau_z \frac{d\theta}{dt} = \tau_z \omega.$$




Consider each point of the rigid body, say $m_1, m_2, \ldots, m_N$.
Kinetic energy of particle $i$ is given by:

$$K_i = \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i(r_i\omega)^2 = \frac{1}{2}m_i r_i^2 \omega^2$$

$\therefore$ Total rotational kinetic energy of the rigid body:

$$\begin{aligned} K &= \sum_i K_i = \sum_i \frac{1}{2}m_i r_i^2 \omega^2 \\ &= \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 \end{aligned}$$

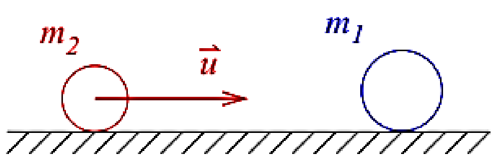
$$\therefore \quad \boxed{K = \frac{1}{2} I \omega^2}$$

$I$ is the moment of inertia of the rigid body about the rotational axis.

## 11.5 Kinetic energy in collision

For elastic collision, $\Delta K = 0$, i. e. $K_f = K_i$.
For inelastic collision, $\Delta K < 0$, i. e. $K_f < K_i$.
For complete inelastic collision, the two colliding objects stick together after collision.

Example:
Consider an elastic collision as shown in the figure below.




After collision,

$$m_1 v_1 + m_2 v_2 = m_2 u \tag{11.2}$$

$$\tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_2 u^2 \tag{11.3}$$

From (11.2),

$$v_1 = \frac{m_2 u - m_2 v_2}{m_1} \tag{11.4}$$

Substitute (11.4) into (11.3), we get:

$$m_2 v_2^2 + m_1 \left( \frac{m_2 u - m_2 v_2}{m_1} \right)^2 = m_2 u^2$$

$$\Rightarrow \quad m_1 m_2 v_2^2 + m_2^2 u^2 + m_2^2 v_2^2 - 2 m_2^2 u v_2 = m_1 m_2 u^2$$

$$\Rightarrow \quad (m_1 m_2 + m_2^2) v_2^2 - 2 m_2^2 u v_2 + (m_2^2 u^2 - m_1 m_2 u^2) = 0$$

Solve for $v_2$ and then $v_1$, we find

$$v_1 = \left( \frac{2 m_2}{m_1 + m_2} \right) u \quad \text{and} \quad v_2 = \left( \frac{m_2 - m_1}{m_1 + m_2} \right) u \ .$$

## 11.6 Conservative force

Potential energy is only defined for conservative force in which a particle moving under the force influence has constant mechanical energy.

Examples of conservative force:

1) Spring
2) Gravitational force
3) Coulomb force

Example of non-conservative force - friction.

Question: Any rigorous definition for conservative force?

Definition:

A conservative force is a force such that if a particle moves under the influence of this force, the work done by the force on displaying the particle from an arbitrary point $A$ to another arbitrary point $B$ would be the same along any arbitrarily chosen path.

Note that:

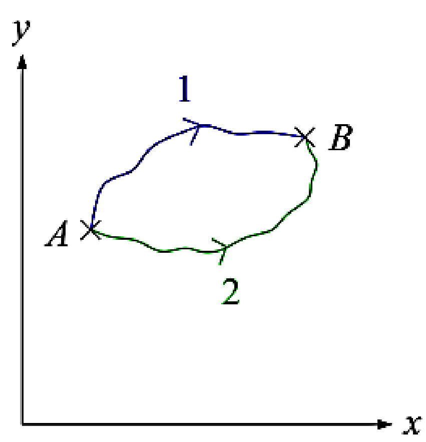
1) $\vec{F}(\vec{r})$ is conserved if and only if there exists a scalar function $\phi(\vec{r})$ such that

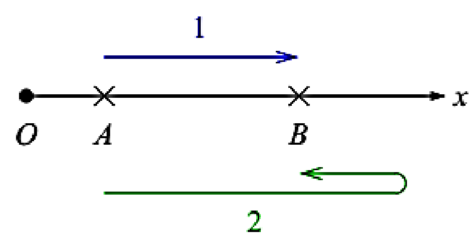
$$\nabla\phi(\vec{r}) = \vec{F}(\vec{r})$$

2) $$\vec{\nabla} \times \vec{F} = 0$$

Work done of closed path (i. e.starting and ending at the same point) is zero.

Proof:



$$\int_{\text{path1}} \vec{F} \cdot d\vec{r} = \int_{\text{path2}} \vec{F} \cdot d\vec{r}$$

$\therefore$ Travelling from point $A$ to $B$, then back to $A$, the work done is:

$$W_{A\to B\to A} = W_{A\to B} + W_{B\to A} = \int_{\text{path1}} \vec{F} \cdot d\vec{r} + \left( - \int_{\text{path2}} \vec{F} \cdot d\vec{r} \right) = 0$$

## 11.7 Potential Energy

Consider a particle moves in the influence of a conservative force, which is position dependent, i. e. $F(x)$. Now the particle displaces from $x_i$ to $x_f$, potential difference $\Delta U$ is defined:

$$\Delta U = U_f - U_i = -W$$

where $W$ is the work done by the force during the displacement $x_i$ to $x_f$.

Or $$\boxed{\Delta U = U(x_f) - U(x_i) = -\int_{x_i}^{x_f} F(x)dx}$$

If for a particle reference point $x_0$, the potential energy is defined as zero, i. e. $U(x_0) \stackrel{\text{def}}{=} 0$.

$$\boxed{U(x) = -\int_{x_0}^{x} F(x)dx}$$

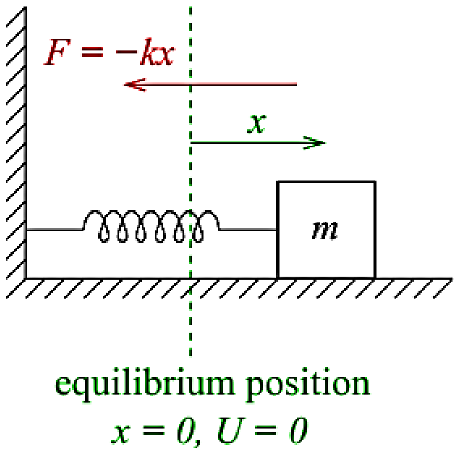
In particular,

$$U(x) - U(0) = -\int_0^x F(x)dx$$

$$\therefore \quad \frac{d}{dx}\left[U(x) - U(0)\right] = -\frac{d}{dx}\int_0^x F(x)dx$$

$$\Rightarrow \quad \boxed{\frac{dU}{dx} = -F(x)}$$

Spring




$$F = -kx$$

Take the equilibrium position to be $x = 0$ so that $U(0) = 0$.

$$\therefore\ U(x) - U(0) = -\int_0^x F(x)dx$$

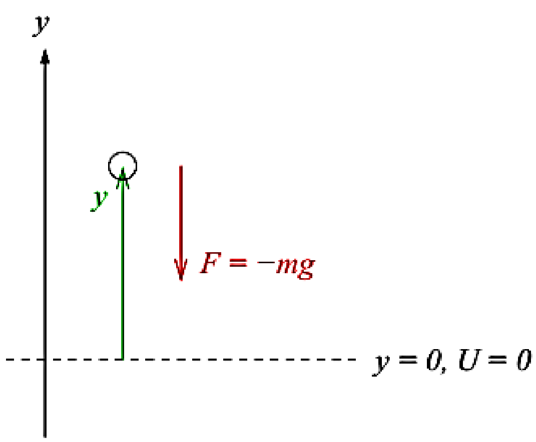
$$\Rightarrow\ U(x) = -\int_0^x (-kx)dx$$

$$\Rightarrow\ U(x) = \frac{1}{2}kx^2$$

Thus

$$\frac{dU}{dx} = \frac{1}{2}k(2x) = kx = -F$$

Force of gravity




Take $U(0) = 0$.

$$\therefore\ U(y) - U(0) = -\int_0^y F(y)dy$$

$$\Rightarrow\ U(y) = -\int_0^x (-mg)dy$$

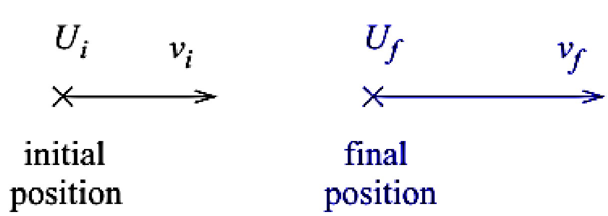
$$\Rightarrow\ U(y) = mgy$$

Thus

$$\frac{dU}{dy} = mg = -F$$

## 11.8 Conservation of Mechanical Energy

$$\Delta U = U_f - U_i = -W \tag{11.5}$$



But $W = \int_{x_i}^{x_f} F(x)dx$ is the work done by the force in the journey from $x_i \to x_f$.
From previous chapter,

$$W = \int_{x_i}^{x_f} F(x)dx = \frac{1}{2}m(v_f^2 - v_i^2) = K_f - K_i = \Delta K \tag{11.6}$$

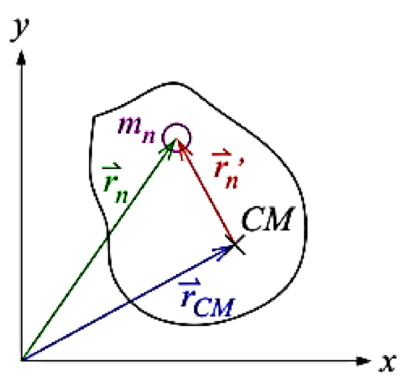
Substitute (11.6) into (11.5), we have

$$U_f - U_i = K_i - K_f$$
$$\Rightarrow \quad U_i + K_i = U_f + K_f$$
$$\Rightarrow \quad \Delta U = -\Delta K$$

In an isolating system whose only conservative force exists, mechanical energy of a particle conserves.

Revisiting combined rotational and translational motion

In previous sections, we have considered cases of pure translational (i. e. movement of C.M. of rigid body) or pure rotational (about a fixed axis) motion. Now we turn into case such that both the CM is moving and the rigid body is rotating.




Consider a rigid body consisted of particles $m_1, m_2, \ldots, m_N$.
Total K.E. of the body:

$$K = \frac{1}{2}\sum_i m_i v_i^2 \tag{11.7}$$

Note that

$$\vec{r}_i = \vec{r}_{\mathrm{CM}} + \vec{r}_i' \quad \Rightarrow \quad \vec{v}_i = \vec{v}_{\mathrm{CM}} + \vec{v}_i'$$

where $\vec{v}_i$ = velocity of mass $i$ with respect to the Earth's frame,
$\vec{v}_{\mathrm{CM}}$ = velocity of the body's center of mass with respect to the Earth's frame,
$\vec{v}_i'$ = velocity of mass $i$ with respect to the body's center of mass.

From (11.7), we obtain

$$K = \frac{1}{2}\sum_i m_i(\vec{v}_{\mathrm{CM}} + \vec{v}_i') \cdot (\vec{v}_{\mathrm{CM}} + \vec{v}_i') = \frac{1}{2}\sum_i m_i(v_{\mathrm{CM}}^2 + 2\vec{v}_{\mathrm{CM}} \cdot \vec{v}_i' + v_i'^2)$$

But consider the second term:

$$\sum_i \vec{v}_{\mathrm{CM}} \cdot (m_i\vec{v}_i') = \sum_i \vec{v}_{\mathrm{CM}} \cdot (m_i\vec{v}_i - m_i\vec{v}_{\mathrm{CM}}) = \vec{v}_{\mathrm{CM}} \cdot \sum_i m_i\vec{v}_i - Mv_{\mathrm{CM}}^2$$

As $\vec{v}_{\rm CM} = (\sum_i m_i \vec{v}_i)/M$,

$$\therefore \sum_i \vec{v}_{\rm CM} \cdot (m_i \vec{v}_i') = \vec{v}_{\rm CM} \cdot M\vec{v}_{\rm CM} - M v_{\rm CM}^2 = 0$$

And the third term:

$$\frac{1}{2}\sum_i m_i {v_i'}^2 = \frac{1}{2}\sum_i m_i (r_i'\omega)^2 = \frac{1}{2} I\omega^2$$

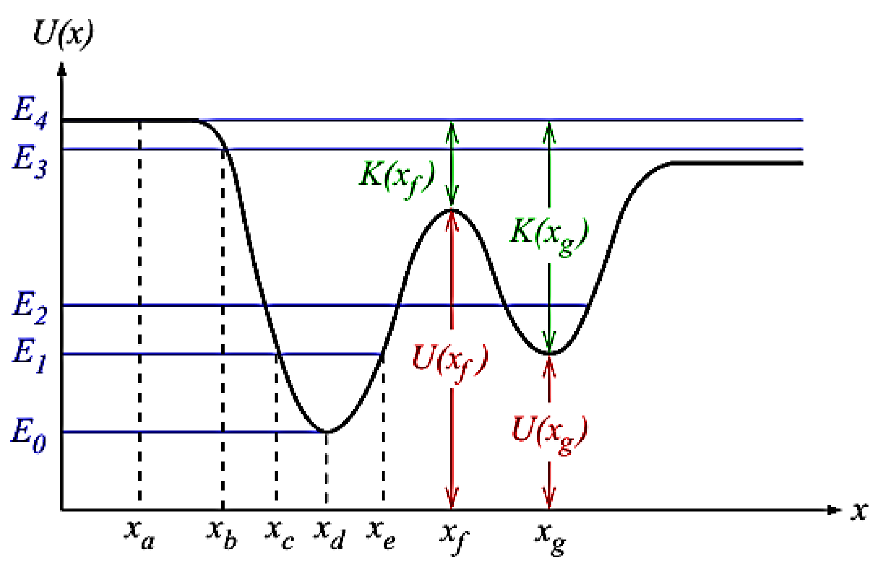
where $\omega$ is the angular velocity about an axis passing through the center of mass.

$$\therefore \boxed{K = \frac{1}{2}Mv_{\rm CM}^2 + \frac{1}{2}I\omega^2}$$

1st term: Translational term of the C.M. as if there is no rotation.

2nd term: Rotational term with rotation about the axis passing through the C.M. as if the rotational axis does not move.

## 11.9 One dimensional conservative system




- Particle experienced a conservative force field with potential energy $U(x)$.
- $$\boxed{F(x) = -\frac{dU}{dx}}$$

$\therefore$ At $x = x_a, x_d, x_f, x_g$, $F = 0$.

$x = x_d, x_g$: stable equilibrium - slightly displaced particle experiences a restoring force

$x = x_f$: unstable equilibrium - displaced particle experiences a force in the same direction as displacement

$x = x_a$: neutral equilibrium - displaced particle experiences no force

- $U(x) + \frac{1}{2}mv^2 = E$, where $E$ is the conserved total energy.

  Example

  If $E = E_4$ as shown in the previous figure,

$$E_4 = K(x_g) + U(x_g) \text{ at } x = x_g$$
$$E_4 = K(x_f) + U(x_f) \text{ at } x = x_f$$

  If the energy of the particle $E$ is different, it will have different behavior as follows:

  1) If $E = E_0$, particle stays stationary at $x = x_d$.
  2) If $E = E_1$, particle stays in the region $x = x_c \rightarrow x_e$.
  3) If $E = E_2$, particle may stay in the two valleys. However if it is in one of the valley, it does not have enough energy to go to the other valley.
  4) If $E = E_3$, particle can stay in the region $x > x_b$.
  5) If $E \geq E_4$, particle can be anywhere.

- If $U(x)$ is known, it is possible to work out the particle position.

  Example

  If at $t = 0$, $x(t = 0) = x_0$ and $v(t = 0) = 0$. Suppose $U(x) = \frac{1}{2}kx^2$.

$$\therefore\ E = \frac{1}{2}k[x(0)]^2 + \frac{1}{2}m[v(0)]^2 = \frac{1}{2}kx_0^2 = \text{constant}$$

At time $t$,

$$U(x) + \frac{1}{2}mv^2 = \frac{1}{2}kx_0^2$$

$$\Rightarrow \quad \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kx_0^2, \qquad \text{where } x = x(t) \text{ and } v = v(t)$$

$$\Rightarrow \quad v = \frac{dx}{dt} = \pm\sqrt{\frac{k}{m}x_0^2 - \frac{k}{m}x^2}$$

$$\Rightarrow \quad dt = \pm\sqrt{\frac{m}{k}}\frac{dx}{\sqrt{x_0^2 - x^2}}$$

$$\Rightarrow \quad t = \pm\int_{x_0}^{x}\sqrt{\frac{m}{k}}\frac{dx}{\sqrt{x_0^2 - x^2}}$$

To solve $\int \frac{dx}{\sqrt{x_0^2 - x^2}}$,

let $x = x_0 \sin\theta \quad \Rightarrow \quad dx = x_0 \cos\theta d\theta$.

$$\therefore \int \frac{dx}{\sqrt{x_0^2 - x^2}} = \int \frac{x_0 \cos\theta d\theta}{\sqrt{x_0^2 - x_0^2 \sin^2\theta}} = \int \frac{\cos\theta}{\cos\theta}d\theta = \theta = \sin^{-1}\left(\frac{x}{x_0}\right)$$

$$\therefore t = \pm\sqrt{\frac{m}{k}}\left[\sin^{-1}\left(\frac{x}{x_0}\right) - \sin^{-1}\left(\frac{x_0}{x_0}\right)\right] = \pm\sqrt{\frac{m}{k}}\left[\sin^{-1}\left(\frac{x}{x_0}\right) - \frac{\pi}{2}\right]$$

If $t = +\sqrt{\frac{m}{k}}\left[\sin^{-1}\left(\frac{x}{x_0}\right) - \frac{\pi}{2}\right]$, then

$$\cos\left(\sqrt{\frac{k}{m}}t\right) = \cos\left[\sin^{-1}\left(\frac{x}{x_0}\right) - \frac{\pi}{2}\right]$$

$$\Rightarrow \quad \cos\left(\sqrt{\frac{k}{m}}t\right) = \cos\left[\sin^{-1}\left(\frac{x}{x_0}\right)\right]\cos\frac{\pi}{2} + \frac{x}{x_0}\sin\frac{\pi}{2} = \frac{x}{x_0}$$

$$\Rightarrow \quad x = x_0 \cos\left(\sqrt{\frac{k}{m}}t\right)$$

If $t = -\sqrt{\frac{m}{k}}\left[\sin^{-1}\left(\frac{x}{x_0}\right) - \frac{\pi}{2}\right]$, then

$$\cos\left(\sqrt{\frac{k}{m}}t\right) = \cos\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{x}{x_0}\right)\right]$$

$$\Rightarrow \quad \cos\left(\sqrt{\frac{k}{m}}t\right) = \cos\frac{\pi}{2}\cos\left[\sin^{-1}\left(\frac{x}{x_0}\right)\right] + \left(\sin\frac{\pi}{2}\right)\frac{x}{x_0} = \frac{x}{x_0}$$

$$\Rightarrow \quad x = x_0 \cos\left(\sqrt{\frac{k}{m}}t\right)$$