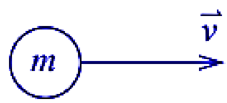
Chapter 6

Impulse and Momentum

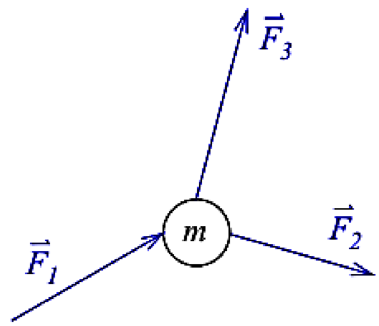
6.1 Definition of linear momentum



$$\vec{p} = m\vec{v}$$

<u>Newton's 2nd law stated in momentum:</u>

The rate of change of the momentum of a body is equal to the total external force acting on the body.



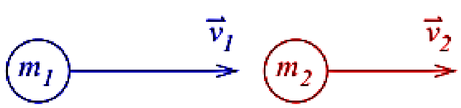
$$\text{i. e. } \sum_i \vec{F}_i = \frac{d\vec{p}}{dt}$$

If the mass of the body is unchanged,

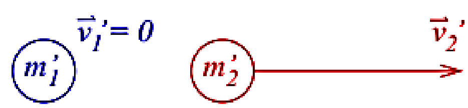
$$\sum_i \vec{F}_i = m\frac{d\vec{v}}{dt} = m\vec{a}$$

6.2 Collision between bodies

In a collision between particles, momentum and mass may transfer. For example,



Before collision

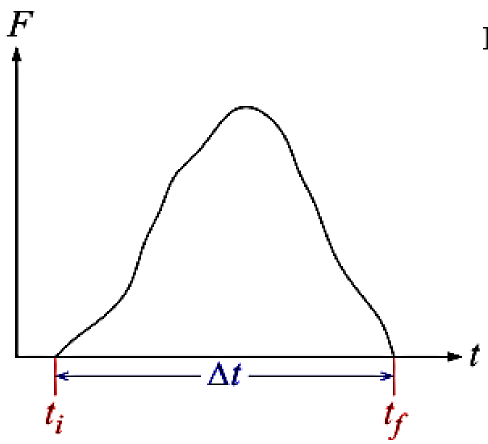


After collision

$$m_1\vec{v}_1 + m_2\vec{v}_2 = m_1'\vec{v'}_1 + m_2'\vec{v'}_2$$

6.3 Impulse and momentum

Consider two bodies in collision and pay attention to one of these two bodies. It experiences a force (e. g. EM force) during the collision.



For the small time interval dt,

$$d\vec{p} = \vec{F}dt$$

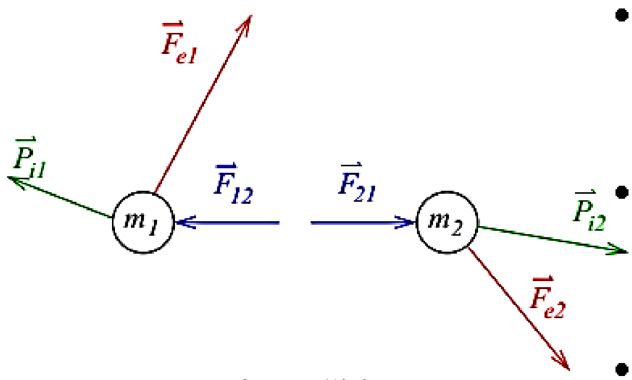
$$\Rightarrow \int_{\vec{p}(t_i)}^{\vec{p}(t_f)} d\vec{p} = \int_{t_i}^{t_f} \vec{F}dt$$

$$\Rightarrow \underbrace{\vec{p}(t_f) - \vec{p}(t_i)}_{\text{change in momentum after the collision}} = \int_{t_i}^{t_f} \vec{F}dt \stackrel{\text{def}}{=} \underbrace{\vec{J}}_{\text{impulse}}$$

This equation can also be applied generally on a body experiencing a force $\vec{F}(t)$ for a time interval from t_i to t_f.

6.4 Conservation of linear momentum

Rather than looking into the experience of one of the body in the collision process, we look into the total momentum of the two bodies before and after the collision.



Before collision

- Consider two particles having initial momentum $\vec{p}_{i1}$ and $\vec{p}_{i2}$ before the collision.
- During the collision process, the impulse force acting on m_1 from m_2 is $\vec{F}_{12}$ and that on m_2 from m_1 is $\vec{F}_{21}$.
- They are also experiencing external forces of $\vec{F}_{e1}$ and $\vec{F}_{e2}$.

As $\vec{F}_{12}$ and $\vec{F}_{21}$ are action-reaction pair.

$$\vec{F}_{12} = -\vec{F}_{21}$$

Before the collision, total momentum of the system:

$$\vec{\mathcal{P}}_i = \vec{p}_{i1} + \vec{p}_{i2}$$

In the collision time interval Δt,

$$\Delta\vec{p}_1 = (\vec{F}_{12} + \vec{F}_{e1})\Delta t \quad \text{and} \quad \Delta\vec{p}_2 = (\vec{F}_{21} + \vec{F}_{e2})\Delta t$$

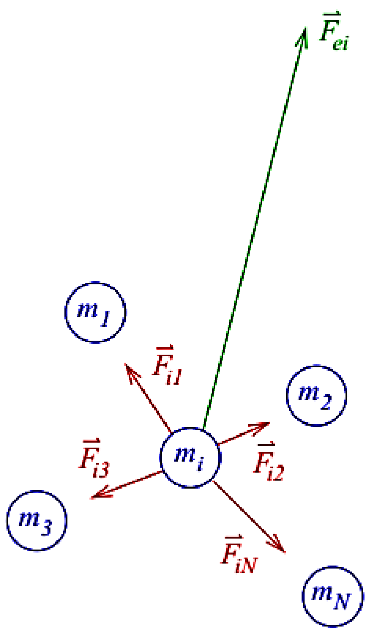
where $\Delta\vec{p}_j$ is the change in momentum of mass m_j during the collision.
$\therefore$ After the collision, final momentum of m_1 and m_2:

$$\vec{p}_{f1} = \vec{p}_{i1} + \Delta\vec{p}_1 = \vec{p}_{i1} + (\vec{F}_{12} + \vec{F}_{e1})\Delta t$$
$$\vec{p}_{f2} = \vec{p}_{i2} + \Delta\vec{p}_2 = \vec{p}_{i2} + (\vec{F}_{21} + \vec{F}_{e2})\Delta t$$

$\therefore$ After the collision, total momentum of the system:

$$\vec{\mathcal{P}}_f = \vec{p}_{f1} + \vec{p}_{f2} = \vec{p}_{i1} + \vec{p}_{i2} + \underbrace{(\vec{F}_{e1} + \vec{F}_{e2})}_{=\vec{F}_{e,\text{tot}}}\Delta t + \underbrace{(\vec{F}_{12} + \vec{F}_{21})}_{=0}\Delta t$$

Hence, $$\vec{\mathcal{P}}_f - \vec{\mathcal{P}}_i = \Delta\vec{\mathcal{P}} = \vec{F}_{e,\text{tot}}\Delta t \;\Rightarrow\; \frac{\Delta\vec{\mathcal{P}}}{\Delta t} = \vec{F}_{e,\text{tot}}$$



To generalize, consider a N particles system consisting of masses $m_1, m_2, \ldots, m_N$.
Define: $\vec{\mathcal{P}} = \sum_i m_i\vec{v}_i$.
For each particle m_i, it experiences an external force $\vec{F}_{ei}$ and other forces arising from its neighbours m_j, i. e. $\vec{F}_{ij}$.

$$\therefore \frac{d\vec{\mathcal{P}}}{dt} = \sum_i \vec{F}_{e,i} = \vec{F}_{e,\text{tot}}$$

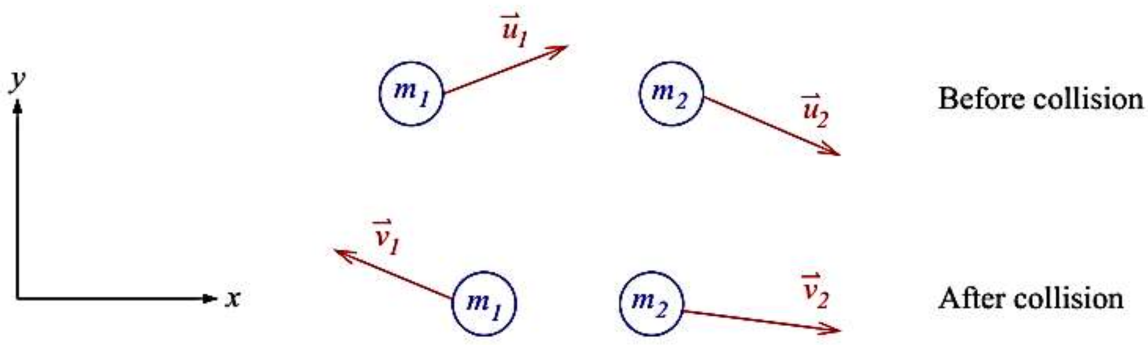
Only the external forces, but NOT the internal forces, play roles on changing the total momentum of the system.

Moreover, if $\vec{F}_{e,\text{tot}} = 0$, then

$$\frac{d\vec{\mathcal{P}}}{dt} = 0$$

i. e. if the total external force acting on a system of particles is zero, the total momentum of the system conserves.

6.5 Two particles collision in two dimensional space



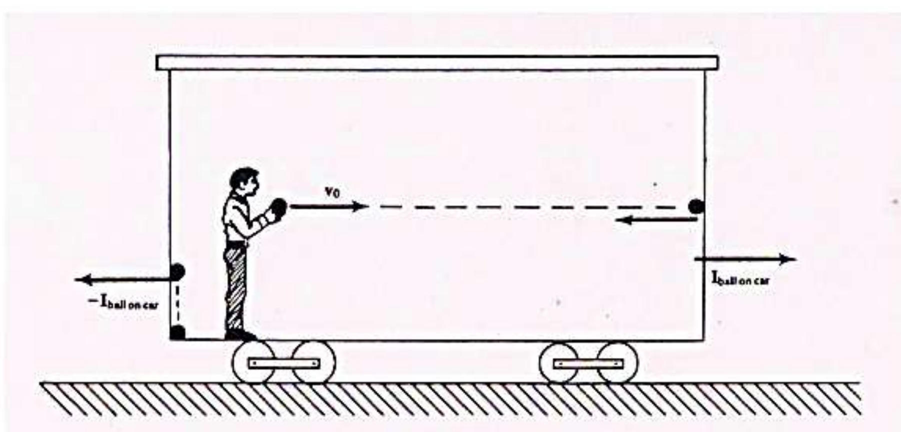
$$m_1 u_{1x} + m_2 u_{2x} = m_1 v_{1x} + m_2 v_{2x}$$

$$m_1 u_{1y} + m_2 u_{2y} = m_1 v_{1y} + m_2 v_{2y}$$

Example

Suppose that a boy stands at one end of a boxcar sitting on a railroad track. Let the mass of the boy and the boxcar be M. He throws a ball of mass m with velocity $\vec{v}_0$ (relative to the ground) toward the other end, where it collides elastically with the wall and travels back down the length (L) of the car, striking the other side inelastically and come to rest. If there is no friction in the wheels of the boxcar, describe the motion of the boxcar.

Solution



All forces are internal, as shown in the figure. Therefore, if $\vec{V}$ and $\vec{v}$ are the velocities of the boxcar plus boy and the ball, conservation of momentum gives

$$M\vec{V} + m\vec{v} = 0 \;\Rightarrow\; \vec{V} = -\frac{m}{M}\vec{v} \tag{6.1}$$

at all times.

Before the first collision, $\vec{v} = \vec{v}_0$ and so

$$\vec{V} = -\frac{m}{M}\vec{v}_0 \quad \text{for} \quad 0 < t < \frac{L}{(1+m/M)v_0}$$

where we have on the right the time for the ball to reach the boxcar wall, traveling at speed

$$v_0 + V = v_0 + \frac{m}{M}v_0$$

with respect to the floor of the boxcar.

The effect of the first collision will be simply to reverse both velocity vectors (eq. (6.1) and elasticity). Thus,

$$\vec{V} = +\frac{m}{M}\vec{v}_0 \quad \text{for} \quad \frac{L}{(1+m/M)v_0} < t < \frac{2L}{(1+m/M)v_0}$$

Finally, after the second collision, the ball and boxcar have common velocity. Hence,

$$\vec{V} = 0 \quad \text{for} \quad t > \frac{2L}{(1+m/M)v_0}$$

Notice that a nonzero common velocity will move the center of mass of the system, which is not allowed as there is no external force acting on the system.

It is seen that the boxcar first moves to the left a distance

$$\left(\frac{m}{M}v_0\right)\frac{L}{(1+m/M)v_0} = \left(\frac{m}{m+M}\right)L$$

and then moves an equal distance to the right, coming to rest at its starting point. This result — that there is no net displacement of the boxcar if the ball return to its initial position within the boxcar — holds whether or not the first collision is elastic (because the center of mass of the system must remain at rest).