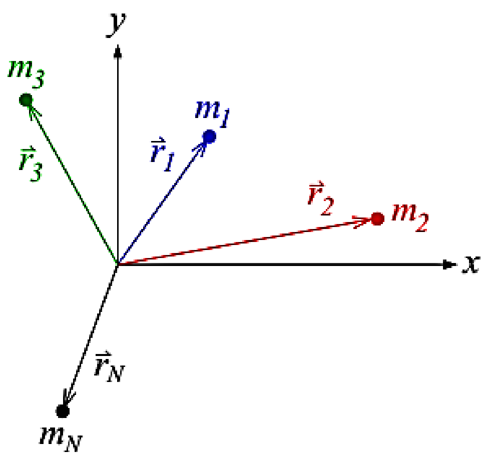
Chapter 7

Systems of Particles

In previous chapters, we have deal with problems of point mass or particle. Now we turn our focus to system containing many particles, e.g. rigid body.

7.1 Center of mass



Consider a system containing N particles $m_1, m_2, \ldots, m_N$.

$\vec{r}_i(t)$: position of m_i at time t

$\vec{v}_i(t)$: velocity of m_i at time t

$\vec{a}_i(t)$: acceleration of m_i at time t

Definition

$$\text{center of mass} \quad \vec{r}_{\mathrm{CM}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_N\vec{r}_N}{m_1 + m_2 + \ldots + m_N} = \frac{\sum_i m_i\vec{r}_i}{M}$$

where $M = \sum_i m_i$.

$$\therefore \quad \vec{v}_{\mathrm{CM}} = \frac{d\vec{r}_{\mathrm{CM}}}{dt} = \frac{1}{M}(m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots + m_N\vec{v}_N)$$

$$\vec{a}_{\mathrm{CM}} = \frac{d\vec{v}_{\mathrm{CM}}}{dt} = \frac{1}{M}(m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots + m_N\vec{a}_N)$$

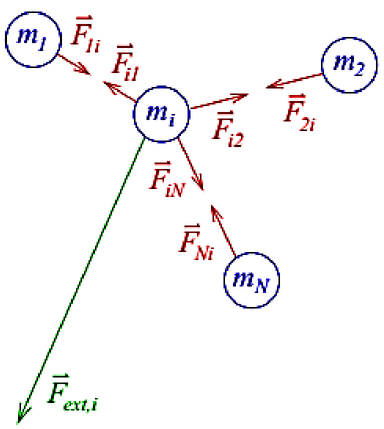
Hence,

$$M\vec{a}_{\rm CM} = m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots + m_N\vec{a}_N = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_N \tag{7.1}$$

$\vec{F}_i$ is indeed the total force experienced by mass m_i.

$$\vec{F}_i = \vec{F}_{{\rm int},i} + \vec{F}_{{\rm ext},i}$$

where $\vec{F}_{{\rm int},i}$ — total internal force acting on m_i originated from other particles $m_{j\neq i}$,
$\vec{F}_{{\rm ext},i}$ — total external force acting on m_i.



$$\vec{F}_{{\rm int},i} = \vec{F}_{i1} + \vec{F}_{i2} + \ldots + \vec{F}_{iN} \quad (\text{no } \vec{F}_{ii})$$

$\therefore$ From (7.1),

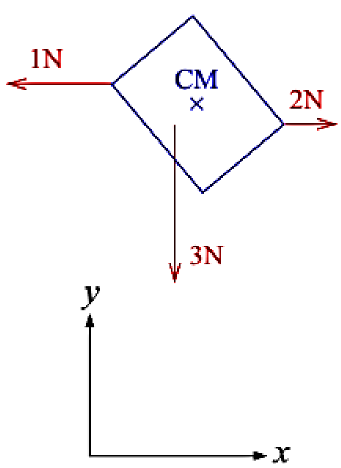
$$
\begin{aligned}
& M\vec{a}_{\rm CM} \\
= & \quad \vec{F}_{{\rm ext},1} + (\quad\quad \vec{F}_{12} + \vec{F}_{13} + \ldots + \vec{F}_{1,N-1} + \vec{F}_{1N}) \\
& + \vec{F}_{{\rm ext},2} + (\vec{F}_{21} \quad\quad + \vec{F}_{23} + \ldots + \vec{F}_{2,N-1} + \vec{F}_{2N}) \\
& + \vec{F}_{{\rm ext},3} + (\vec{F}_{31} + \vec{F}_{32} \quad\quad + \ldots + \vec{F}_{2,N-1} + \vec{F}_{2N}) \\
& + \quad \ldots \\
& + \vec{F}_{{\rm ext},N} + (\vec{F}_{N1} + \vec{F}_{N2} + \vec{F}_{N3} + \ldots + \vec{F}_{N,N-1} \quad\quad)
\end{aligned}
$$

Notice $\vec{F}_{ij} = -\vec{F}_{ji}$ and thus

$$\boxed{M\vec{a}_{\rm CM} = \vec{F}_{{\rm ext},1} + \vec{F}_{{\rm ext},2} + \ldots + \vec{F}_{{\rm ext},N} = \sum_i \vec{F}_{{\rm ext},i}}$$

i. e. to say N-particle system with external forces acting on individual particles and internal forces between each of the particle behaves as if a single point mass at the position $\vec{r}_{\rm CM}$ experiencing a force of $\sum_i \vec{F}_{{\rm ext},i}$.

<u>Example:</u>



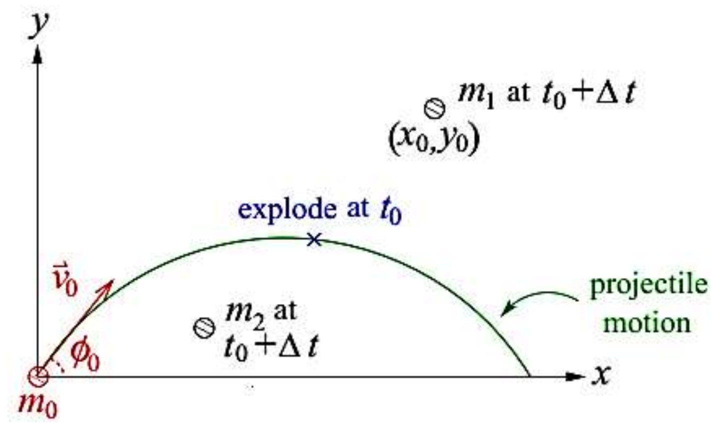
Uniform laminar, mass = 1 kg

$$1 \times \vec{a}_{\rm CM} = -1\hat{\rm i} + 2\hat{\rm i} - 3\hat{\rm j} = \hat{\rm i} - 3\hat{\rm j}$$

$$\therefore \quad \vec{a}_{{\rm CM},x} = 1 \ {\rm ms}^{-1}, \quad \vec{a}_{{\rm CM},y} = -3 \ {\rm ms}^{-1}$$

(Only motion of center of mass is known, but not the rotation.)

Example:



Particle m_0 explodes at time t_0 into two masses of m_1 and m_2. Δt after the explosion, m_1 was found at (x_0, y_0). Find the position of m_2 at $t_0 + \Delta t$.

Consider m_1 and m_2 as a system.
Before the explosion, the CM of the system is just the position of the mass m_0.

$$\therefore \quad \frac{d^2\vec{r}_{\rm CM}}{dt^2} = -g\hat{\rm j} = \vec{a}_{\rm CM}$$

$$\vec{v}_{\rm CM} = (v_0\cos\phi_0)\hat{\rm i} + (v_0\sin\phi_0 - gt)\hat{\rm j}$$

$$\vec{r}_{\rm CM} = (v_0\cos\phi_0)t\hat{\rm i} + [(v_0\sin\phi_0)t - \frac{1}{2}gt^2]\hat{\rm j}$$

At the explosion, m_0 splitted into m_1 and m_2. Therefore, m_1 and m_2 have their own positions $\vec{r}_1(t)$ and $\vec{r}_2(t)$. But as the explosion only involves internal forces, the CM's position will follow the original parabola.
Now, $t = t_0 + \Delta t$

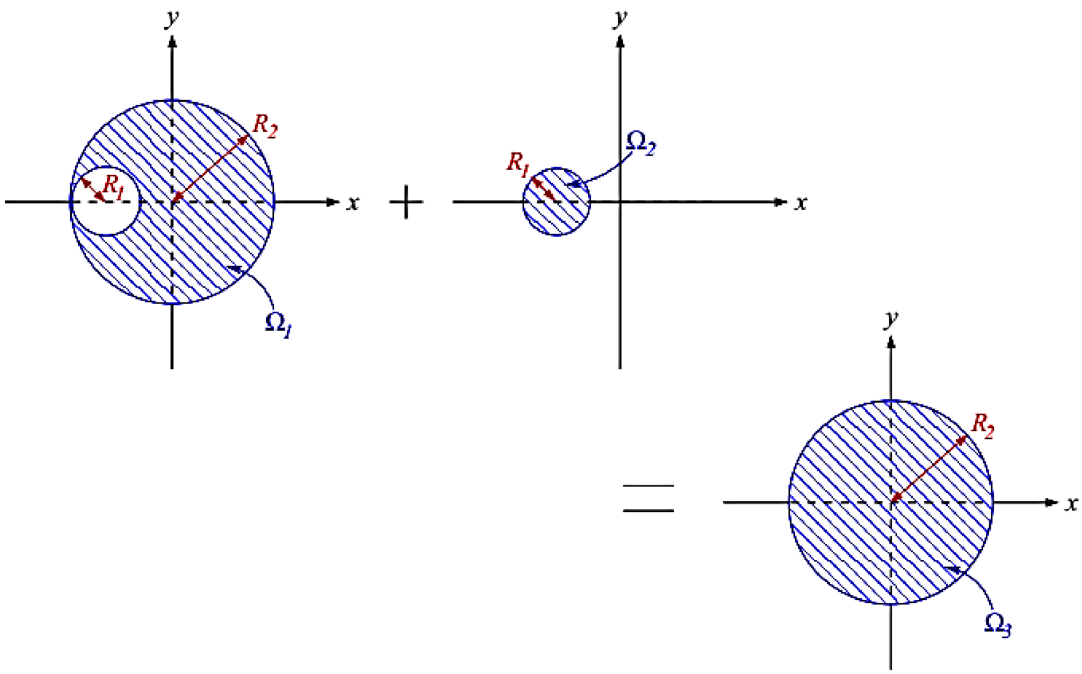
$$\vec{r}_1(t_0+\Delta t) = x_0\hat{\rm i} + y_0\hat{\rm j} \quad \text{and} \quad \vec{r}_{\rm CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_0}$$

$$\begin{aligned}\Rightarrow \vec{r}_2(t_0+\Delta t) &= \frac{m_0}{m_2}\vec{r}_{\rm CM} - \frac{m_1}{m_2}\vec{r}_1 \\ &= \left(\frac{m_0}{m_2}v_0\cos\phi_0 t - \frac{m_1}{m_2}x_0\right)\hat{\rm i} + \left[\frac{m_0}{m_2}\left(v_0\sin\phi_0 t - \frac{1}{2}gt^2\right) - \frac{m_1}{m_2}y_0\right]\hat{\rm j}\end{aligned}$$

7.2 Center of mass of some rigid bodies

Example:

An uniform circular laminar of radius R_2 has a hole in it. The hole is in the form of circle and it has a radius R_1. The circumferences of the hole and the laminar meet tangentially.



- Ω_1 is symmetric about x-axis, CM must be on x-axis.
- Integral form of CM:

$$x_{\text{CM}} = \sum_i \frac{m_i x_i}{M} = \frac{\int x dm}{M}$$

$$y_{\text{CM}} = \sum_i \frac{m_i y_i}{M} = \frac{\int y dm}{M}$$

- Using the integral form, we find

$$x_{\text{CM},\Omega_3} = \frac{\int_{\Omega_3} x dm}{M_{\Omega_3}} = 0$$

$$x_{\text{CM},\Omega_2} = \frac{\int_{\Omega_2} x dm}{M_{\Omega_2}} = R_1 - R_2$$

But
$$\int_{\Omega_3} x dm = \int_{\Omega_1} x dm + \int_{\Omega_2} x dm$$

$$\therefore \quad x_{\text{CM},\Omega_1} = \frac{\int_{\Omega_1} x dm}{M_{\Omega_3} - M_{\Omega_2}} = \frac{0 - M_{\Omega_2}(R_1 - R_2)}{M_{\Omega_3} - M_{\Omega_2}}$$

Let ρ be the density of the shapes. Then

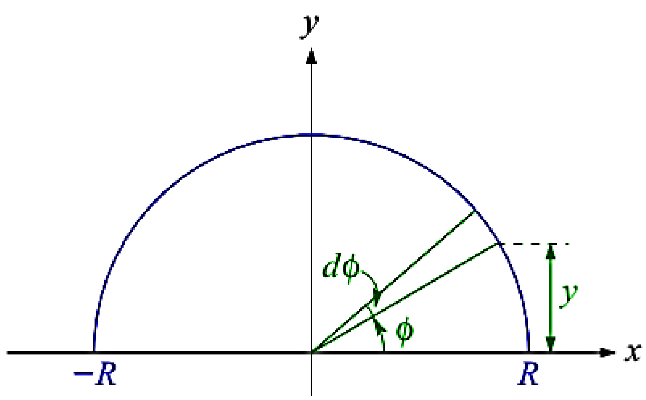
$$M_{\Omega_2} = \pi R_1^2 \rho \quad \text{and} \quad M_{\Omega_3} = \pi R_2^2 \rho$$

Hence

$$x_{\mathrm{CM},\Omega_1} = \frac{-\pi R_1^2 \rho (R_1 - R_2)}{M_{\Omega_3} - M_{\Omega_2}} = \frac{R_1^2}{R_1 + R_2}$$

Example:

A wire is bent into a semi-circle



- Symmetric about y-axis and the CM must be on the y-axis.
- Consider the segment as shown in the figure:

$$y = R \sin\phi$$

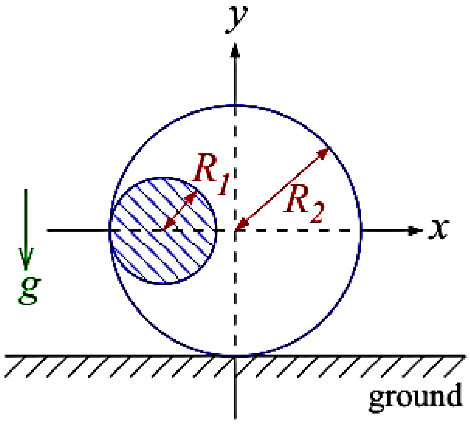
Mass of the segment:

$$dm = \rho R d\phi$$

where ρ is the denity of the wire.

$$\therefore \quad y_{\mathrm{CM}} = \frac{\int y dm}{M} = \frac{\int_0^\pi \rho R^2 \sin\phi d\phi}{\pi R \rho} = \frac{\int_0^\pi R \sin\phi d\phi}{\pi} = \frac{2R}{\pi}$$

Example:



A solid ball with radius R_1 is placed inside a hollow sphere with radius R_2, as shown in the figure. The ball is then released both the ball and the sphere roll back and forth. What is the final equilibrium position? The masses of the ball and the sphere are both m.

Consider motion in x-direction, as external force is zero, the x component of the CM does not change.

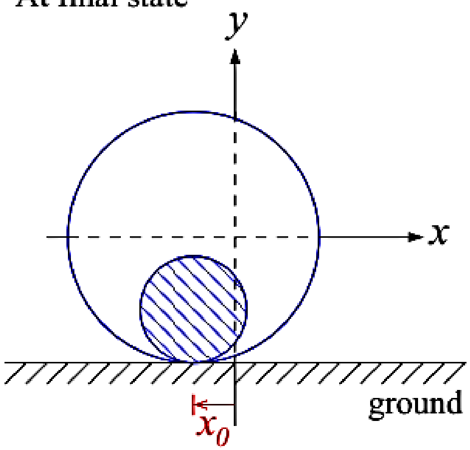
Before release,

$$
\begin{aligned}
x_{\text{CM}} &= \frac{m \times 0 + m \times (R_1 - R_2)}{2m} \\
&= \frac{R_1 - R_2}{2}
\end{aligned}
$$

After reaching equilibrium,

$$
\begin{aligned}
x_{\text{CM}} &= \frac{m \times x_0 + m \times x_0}{2m} \\
&= x_0 \quad = \quad \frac{R_1 - R_2}{2}
\end{aligned}
$$

At final state



7.3 Momentum of system of particles

Consider a system of N particles $m_1, m_2, \ldots, m_N$ having position vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$.

Total momentum of the system

$$\vec{P} = \sum_i m_i \vec{v}_i = \sum_i \vec{p}_i \tag{7.2}$$

But since

$$\vec{r}_{\text{CM}} = \frac{\sum_i m_i \vec{r}_i}{M}.$$

$$\Rightarrow \; M \frac{d\vec{r}_{\text{CM}}}{dt} = \sum_i m_i \frac{d\vec{r}_i}{dt}$$

$$\Rightarrow \; M \vec{v}_{\text{CM}} = \sum_i \vec{p}_i \tag{7.3}$$

Combining (7.2) and (7.3):

$$\boxed{\vec{\mathcal{P}} = M\vec{v}_{\mathrm{CM}}} \quad \text{and} \quad \boxed{\frac{d\vec{\mathcal{P}}}{dt} = M\vec{a}_{\mathrm{CM}}}$$

i. e. to find the total momentum, other than adding all $\vec{p}_i$, we can also get it by finding $M\vec{v}_{\mathrm{CM}}$. Or the system of N particles behaves as if it is a point mass having mass M, velocity $\vec{v}_{\mathrm{CM}}$ and acceleration $\vec{a}_{\mathrm{CM}}$.

Moreover, from last chapter or the recall at the beginning of this chapter:

$$\frac{d\vec{\mathcal{P}}}{dt} = \sum_i \vec{F}_{\mathrm{ext},i} = \vec{F}_{\mathrm{ext,tot}}$$

Therefore,

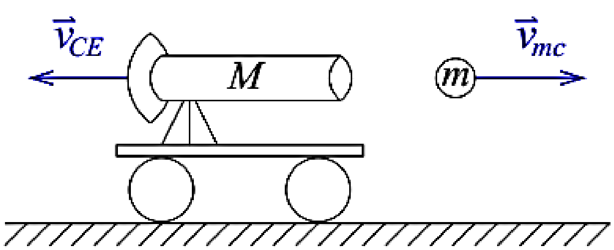
$$\boxed{M\vec{a}_{\mathrm{CM}} = \sum_i \vec{F}_{\mathrm{ext},i}}$$

If total external force is zero, then

$$\frac{d\vec{\mathcal{P}}}{dt} = 0$$

— Conservation of linear momentum for system of particles!

Example:



A canon on a frictionless ground fires a cannon ball. The canon ball is fired with speed of v_{mc} relative to the canon.

In x-direction, there is no external force.

$\therefore$ Momentum conserved which implies

$$0 = Mv_{\mathrm{cE}} + mv_{\mathrm{mE}}$$

But

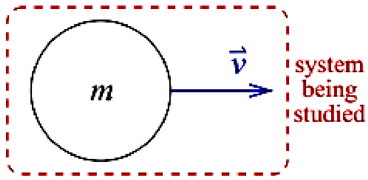
$$v_{\mathrm{mE}} = v_{\mathrm{mc}} + v_{\mathrm{cE}}$$

$$\Rightarrow \; 0 = Mv_{\mathrm{cE}} + mv_{\mathrm{mc}} + mv_{\mathrm{cE}}$$

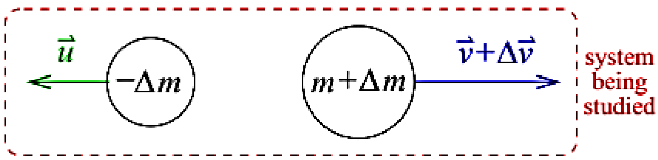
$$\Rightarrow \; v_{\mathrm{cE}} = \frac{-mv_{\mathrm{mc}}}{m+M}$$

7.4 System of variable mass

Time t



Time $t + \Delta t$



- At $t + \Delta t$, a small mass was ejected out from the original mass.
- $\vec{u}$ is the velocity of the smalll mass relative to the Earth. However, it should be noted that the small mass velocity is usually given in relative to the original mass.

At time t, total momentum:

$$\vec{\mathcal{P}}(t) = m\vec{v}$$

At time $t + \Delta t$, total momentum:

$$\begin{aligned}
\vec{\mathcal{P}}(t+\Delta t) &= (m+\Delta m)(\vec{v}+\Delta\vec{v}) + (-\Delta m)\vec{u} \\
&= m\vec{v} + m\Delta\vec{v} + \Delta m\vec{v} + \Delta m\Delta\vec{v} - \Delta m\vec{u} \\
&= m\vec{v} + m\Delta\vec{v} + \Delta m(\vec{v}-\vec{u}) + \Delta m\Delta\vec{v}
\end{aligned}$$

Hence, we find

$$\Delta\vec{\mathcal{P}} = \vec{\mathcal{P}}(t+\Delta t) - \vec{\mathcal{P}}(t) = m\Delta\vec{v} + \Delta m(\vec{v}-\vec{u}) + \Delta m\Delta\vec{v}$$

$$\Rightarrow\ \vec{F}_{\text{ext}} = \lim_{\Delta t\to 0}\frac{\Delta\vec{\mathcal{P}}}{\Delta t} = m\frac{d\vec{v}}{dt} + \frac{dm}{dt}(\vec{v}-\vec{u})$$

$$\text{or}\quad \boxed{\vec{F}_{\text{ext}} = m\frac{d\vec{v}}{dt} - \frac{dm}{dt}\,\vec{v}_{\text{rel}}}$$

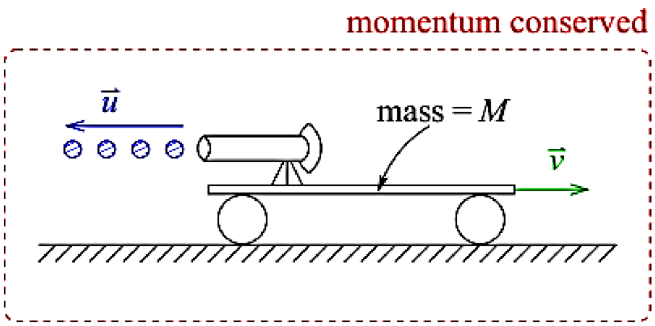
where $\vec{v}_{\text{rel}} = \vec{u} - \vec{v}$ is the velocity of the small particle relative to the original mass.

Notes:

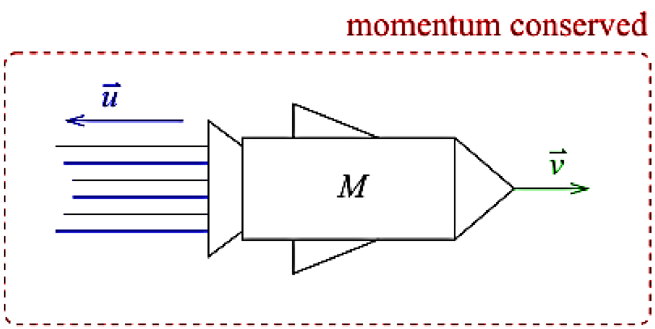
- $$\lim_{\Delta t\to 0}\left(\frac{\Delta m\Delta\vec{v}}{\Delta t}\right) = 0 \text{ since } \Delta m, \Delta\vec{v} \to 0 \text{ as } \Delta t \to 0.$$
- $$\because\ \vec{v} = \vec{v}_{m\text{E}}, \qquad \vec{u} = \vec{v}_{\Delta m,\text{E}}$$

$$\therefore\ \vec{v} - \vec{u} = \vec{v}_{m\text{E}} + \vec{v}_{\text{E},\Delta m} = \vec{v}_{m,\Delta m} = -\vec{v}_{\Delta m,m} = -\vec{v}_{\text{rel}}$$

Example:



A train on a frictionless rail with a machine gun firing at a rate of n bullets per second. Mass of bullet is m.



A rocket in space ejecting mass at rate of $|\frac{dM}{dt}|$.

where $\vec{u}$: velocity of ejecting mass or bullet relative to the Earth.

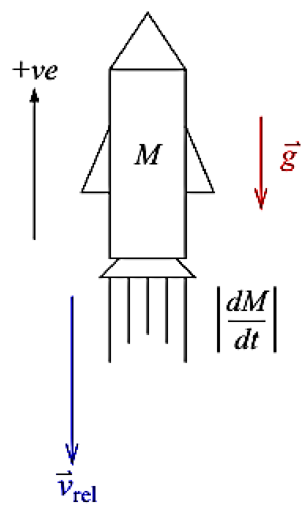
For both case, $\vec{F}_{\text{ext}} = 0$.

$$\therefore \quad M\frac{d\vec{v}}{dt} = \frac{dM}{dt}(\vec{u} - \vec{v})$$

$$= \begin{cases} (-mn)(\vec{u} - \vec{v}) = mn(\vec{v} - \vec{u}) & \text{for the train,} \\ \left(-\left|\frac{dM}{dt}\right|\right)(\vec{u} - \vec{v}) = \left|\frac{dM}{dt}\right|(\vec{v} - \vec{u}) & \text{for the rocket} \end{cases}$$

This implies both the train and the rocket will accelerate as if there were a force (called thrust). Indeed, it is not a real external force acting on the train or the rocket but only to maintain the conservation of momentum.

Example:



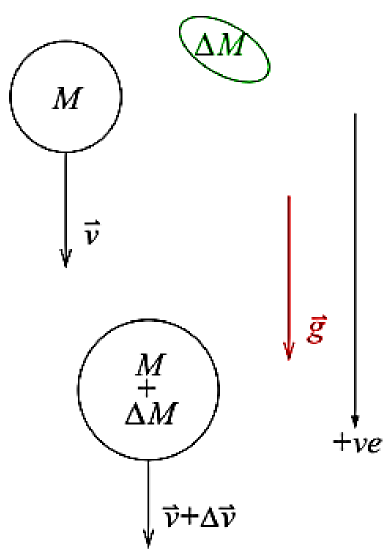
Rocket ascending on earth by ejecting mass at a rate of $|\frac{dM}{dt}|$ with velcoity $\vec{v}_{\rm rel}$ relative to the rocket.

$$\vec{F}_{\rm ext} = M\frac{d\vec{v}}{dt} - \frac{dM}{dt}\,\vec{v}_{\rm rel}$$

$$\Rightarrow \quad -M|g| = M\frac{dv}{dt} - \left(-\left|\frac{dM}{dt}\right|\right)(-|v_{\rm rel}|)$$

$$\Rightarrow \quad M\frac{dv}{dt} = \left|\frac{dM}{dt}\right||v_{\rm rel}| - M|g|$$

Example:



Raindrop falling and water vapour keeps on condensing on it with a rate of $|\frac{dM}{dt}|$.

$$\vec{F}_{\rm ext} = M\frac{d\vec{v}}{dt} - \frac{dM}{dt}\vec{v}_{\rm rel}$$

$$\vec{F}_{\rm ext} = M\frac{d\vec{v}}{dt} - \frac{dM}{dt}(\vec{v}_{\rm moisture} - \vec{u}_{\rm rain})$$

$$\Rightarrow \quad M|g| = M\frac{dv}{dt} - \left|\frac{dM}{dt}\right|(0 - v) \qquad \because \vec{u}_{\rm moisture} = 0$$

$$\Rightarrow \quad \frac{dv}{dt} + \frac{1}{M}\left|\frac{dM}{dt}\right| v = |g|$$