L

SECOND-ORDER LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

SECOND-ORDER LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

A *second-order linear differential equation* is one of the form

$$\frac{d^2 y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x) \tag{1}$$

or, in alternative notation,

$$y'' + p(x)y' + q(x)y = r(x)$$

If $r(x)$ is identically 0, then (1) reduces to

$$\frac{d^2 y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

which is called the second-order linear **homogeneous** differential equation.

In order to discuss the solutions to a second-order linear homogeneous differential equation, it will be useful to introduce some terminology. Two functions f and g are said to be **linearly dependent** if one is a *constant* multiple of the other. If neither is a constant multiple of the other, then they are called **linearly independent**. Thus,

$$f(x) = \sin x \quad \text{and} \quad g(x) = 3 \sin x$$

are linearly dependent, but

$$f(x) = x \quad \text{and} \quad g(x) = x^2$$

are linearly independent. The following theorem is central to the study of second-order linear homogeneous differential equations.

> **L.1** **THEOREM** *Consider the homogeneous equation*
>
> $$\frac{d^2 y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \tag{2}$$
>
> *where the functions $p(x)$ and $q(x)$ are continuous on a common open interval. Then there exist linearly independent solutions $y_1(x)$ and $y_2(x)$ to (2) on that interval. Furthermore, given any such pair of linearly independent solutions $y_1(x)$ and $y_2(x)$, a general solution of (2) on the interval is given by*
>
> $$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{3}$$
>
> *That is, every solution of (2) on the interval can be obtained from (3) by choosing appropriate values of the constants c_1 and c_2; conversely, (3) is a solution of (2) for all choices of c_1 and c_2.*

We will restrict our attention to second-order linear homogeneous equations of the form

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0 \tag{4}$$

where p and q are *constants*. Since the constant functions $p(x) = p$ and $q(x) = q$ are continuous on $(-\infty, +\infty)$, it follows from Theorem L.1 that to determine a general solution to (4) we need only find two linearly independent solutions $y_1(x)$ and $y_2(x)$ on $(-\infty, +\infty)$. The general solution will then be given by $y(x) = c_1 y_1(x) + c_2 y_2(x)$, where c_1 and c_2 are arbitrary constants.

We will start by looking for solutions to (4) of the form $y = e^{mx}$. This is motivated by the fact that the first and second derivatives of this function are multiples of y, suggesting that a solution of (4) might result by choosing m appropriately. To find such an m, we substitute

$$y = e^{mx}, \quad \frac{dy}{dx} = me^{mx}, \quad \frac{d^2y}{dx^2} = m^2 e^{mx} \tag{5}$$

into (4) to obtain

$$(m^2 + pm + q)e^{mx} = 0 \tag{6}$$

which is satisfied if and only if

$$m^2 + pm + q = 0 \tag{7}$$

since $e^{mx} \neq 0$ for every x.

Equation (7), which is called the **auxiliary equation** for (4), can be obtained from (4) by replacing d^2y/dx^2 by m^2, dy/dx by m $(= m^1)$, and y by 1 $(= m^0)$. The solutions, m_1 and m_2, of the auxiliary equation can be obtained by factoring or by the quadratic formula. These solutions are

> Recall that the complex solutions of a polynomial equation, and in particular of a quadratic equation, occur as conjugate pairs $a + bi$ and $a - bi$, provided the coefficients in the equation are real.

$$m_1 = \frac{-p + \sqrt{p^2 - 4q}}{2}, \quad m_2 = \frac{-p - \sqrt{p^2 - 4q}}{2} \tag{8}$$

Depending on whether $p^2 - 4q$ is positive, zero, or negative, these roots will be distinct and real, equal and real, or complex conjugates. We will consider each of these cases separately.

■ DISTINCT REAL ROOTS

If m_1 and m_2 are distinct real roots, then (4) has the two solutions

$$y_1 = e^{m_1 x}, \quad y_2 = e^{m_2 x}$$

Neither of the functions $e^{m_1 x}$ and $e^{m_2 x}$ is a constant multiple of the other (Exercise 29), so the general solution of (4) in this case is

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} \tag{9}$$

▶ **Example 1**   Find the general solution of $y'' - y' - 6y = 0$.

Solution.   The auxiliary equation is

$$m^2 - m - 6 = 0 \quad \text{or equivalently} \quad (m + 2)(m - 3) = 0$$

so its roots are $m = -2$, $m = 3$. Thus, from (9) the general solution of the differential equation is

$$y = c_1 e^{-2x} + c_2 e^{3x}$$

where c_1 and c_2 are arbitrary constants. ◀

■ **EQUAL REAL ROOTS**

If m_1 and m_2 are equal real roots, say $m_1 = m_2 (= m)$, then the auxiliary equation yields only one solution of (4):

$$y_1(x) = e^{mx}$$

We will now show that

$$y_2(x) = xe^{mx} \tag{10}$$

is a second linearly independent solution. To see that this is so, note that $p^2 - 4q = 0$ in (8) since the roots are equal. Thus,

$$m = m_1 = m_2 = -p/2$$

and (10) becomes

$$y_2(x) = xe^{(-p/2)x}$$

Differentiating yields

$$y_2'(x) = \left(1 - \frac{p}{2}x\right)e^{(-p/2)x} \quad \text{and} \quad y_2''(x) = \left(\frac{p^2}{4}x - p\right)e^{-(p/2)x}$$

so

$$y_2''(x) + py_2'(x) + qy_2(x) = \left[\left(\frac{p^2}{4}x - p\right) + p\left(1 - \frac{p}{2}x\right) + qx\right]e^{(-p/2)x}$$

$$= \left[-\frac{p^2}{4} + q\right]xe^{(-p/2)x} \tag{11}$$

But $p^2 - 4q = 0$ implies that $(-p^2/4) + q = 0$, so (11) becomes

$$y_2''(x) + py_2'(x) + qy_2(x) = 0$$

which tells us that $y_2(x)$ is a solution of (4). It can be shown that

$$y_1(x) = e^{mx} \quad \text{and} \quad y_2(x) = xe^{mx}$$

are linearly independent (Exercise 29), so the general solution of (4) in this case is

$$y = c_1e^{mx} + c_2xe^{mx} \tag{12}$$

▶ **Example 2**   Find the general solution of $y'' - 8y' + 16y = 0$.

Solution.   The auxiliary equation is

$$m^2 - 8m + 16 = 0 \quad \text{or equivalently} \quad (m - 4)^2 = 0$$

so $m = 4$ is the only root. Thus, from (12) the general solution of the differential equation is

$$y = c_1e^{4x} + c_2xe^{4x} \quad ◀$$

■ **COMPLEX ROOTS**

If the auxiliary equation has complex roots $m_1 = a + bi$ and $m_2 = a - bi$, then both $y_1(x) = e^{ax}\cos bx$ and $y_2(x) = e^{ax}\sin bx$ are linearly independent solutions of (4) and

$$y = e^{ax}(c_1\cos bx + c_2\sin bx) \tag{13}$$

is the general solution. The proof is discussed in the exercises (Exercise 30).

▶ **Example 3**   Find the general solution of $y'' + y' + y = 0$.

Solution.   The auxiliary equation $m^2 + m + 1 = 0$ has roots

$$m_1 = \frac{-1 + \sqrt{1-4}}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$m_2 = \frac{-1 - \sqrt{1-4}}{2} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Thus, from (13) with $a = -1/2$ and $b = \sqrt{3}/2$, the general solution of the differential equation is

$$y = e^{-x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) \quad \blacktriangleleft$$

■ INITIAL-VALUE PROBLEMS

When a physical problem leads to a second-order differential equation, there are usually two conditions in the problem that determine specific values for the two arbitrary constants in the general solution of the equation. Conditions that specify the value of the solution $y(x)$ and its derivative $y'(x)$ at $x = x_0$ are called **initial conditions**. A second-order differential equation with initial conditions is called a **second-order initial-value problem**.

▶ **Example 4**   Solve the initial-value problem

$$y'' - y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

Solution.   We must first solve the differential equation. The auxiliary equation

$$m^2 - 1 = 0$$

has distinct real roots $m_1 = 1$, $m_2 = -1$, so from (9) the general solution is

$$y(x) = c_1 e^x + c_2 e^{-x} \tag{14}$$

and the derivative of this solution is

$$y'(x) = c_1 e^x - c_2 e^{-x} \tag{15}$$

Substituting $x = 0$ in (14) and (15) and using the initial conditions $y(0) = 1$ and $y'(0) = 0$ yields the system of equations
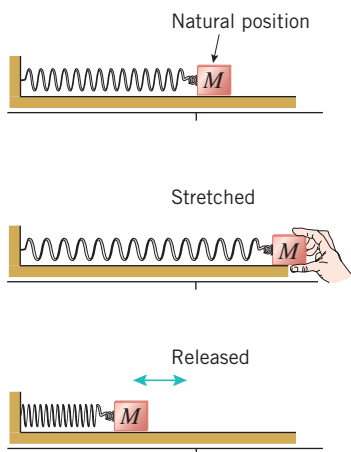
$$c_1 + c_2 = 1$$
$$c_1 - c_2 = 0$$

Solving this system yields $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2}$, so from (14) the solution of the initial-value problem is

$$y(x) = \tfrac{1}{2}e^x + \tfrac{1}{2}e^{-x} = \cosh x \quad \blacktriangleleft$$
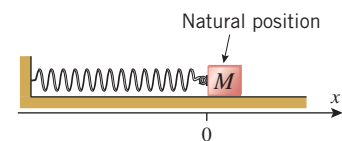
The following summary is included as a ready reference for the solution of second-order homogeneous linear differential equations with constant coefficients.

<div align="center">

Summary

</div>

EQUATION: $y'' + py' + qy = 0$
AUXILIARY EQUATION: $m^2 + pm + q = 0$

CASE	GENERAL SOLUTION
Distinct real roots m_1, m_2 of the auxiliary equation	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
Equal real roots $m_1 = m_2$ $(= m)$ of the auxiliary equation	$y = c_1 e^{mx} + c_2 x e^{mx}$
Complex roots $m_1 = a + bi$, $m_2 = a - bi$ of the auxiliary equation	$y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$

■ VIBRATIONS OF SPRINGS

Natural position

Stretched

Released

▲ **Figure L.1**

Natural position

▲ **Figure L.2**

Confirm that the functions in family (19) are solutions of (18).

In Section 8.1 we developed a mathematical model that describes the vibratory motion of a block of mass M attached to the end of a horizontal spring that is set into motion by pulling the spring beyond its natural position and releasing it at time $t = 0$ (Figure L.1).[*] If we introduce a horizontal x-axis whose origin is at the natural length of the spring and whose positive direction is to the right (Figure L.2), and if we let $x = x(t)$ be the coordinate of the point of attachment of the block to the spring, then $x(t)$ satisfies the second-order differential equation

$$M\frac{d^2 x}{dx^2} = -kx \tag{16}$$

If at time $t = 0$ the mass is released from rest at position $x_0 = x(0)$, then $x(t)$ can be found by solving the initial-value problem

$$M\frac{d^2 x}{dx^2} = -kx, \quad x(0) = x_0, \quad x'(0) = 0 \tag{17}$$

To solve this initial-value problem, we first rewrite (16) as the homogeneous equation

$$x''(t) + \left(\frac{k}{M}\right) x(t) = 0 \tag{18}$$

The auxiliary equation for (18) is

$$m^2 + \frac{k}{M} = 0$$

which has imaginary roots $m_1 = \sqrt{k/M}\,i$, $m_2 = -\sqrt{k/M}\,i$ (since k and M are positive). It follows that the general solution of (18) is

$$x(t) = c_1 \cos\left(\sqrt{\frac{k}{M}}\,t\right) + c_2 \sin\left(\sqrt{\frac{k}{M}}\,t\right) \tag{19}$$

To determine the constants c_1 and c_2 in (19) we will take as our initial conditions the position and velocity at time $t = 0$. Specifically, we will ask you to show in Exercise 36 that if the position of the block at time $t = 0$ is x_0, and if the initial velocity of the block is zero (i.e., it is *released* from rest), then

$$x(t) = x_0 \cos\left(\sqrt{\frac{k}{M}}\,t\right) \tag{20}$$

---

[*]We use "M" to denote mass to avoid a conflict with the use of the letter "m" in the auxiliary equation.
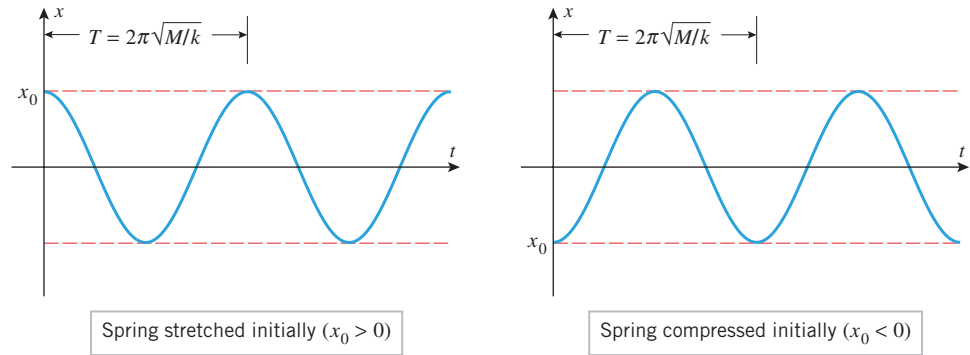
This formula describes a periodic vibration with an amplitude of $|x_0|$, a period T given by

$$T = \frac{2\pi}{\sqrt{k/M}} = 2\pi\sqrt{M/k} \tag{21}$$

and a frequency f given by

$$f = \frac{1}{T} = \frac{\sqrt{k/M}}{2\pi} \tag{22}$$

(Figure L.3).

▲ **Figure L.3**

▶ **Example 5**   Suppose that the block in Figure L.2 has a mass of 20 g and that the spring constant is 0.98 N/m. Suppose also that the block is stretched 0.5 m beyond its equilibrium position and released at time $t = 0$.

(a) Find the position function $x(t)$ of the block.

(b) Find the amplitude, period, and frequency of the vibration.

Solution (a).   The appropriate formula is (20). Since the mass M is 0.02 kg,

$$\frac{k}{M} = \frac{0.98}{0.02} = 49 \text{ s}^{-2} \tag{23}$$

and substituting this in (20) yields

$$x(t) = x_0 \cos 7t$$

where x_0 is the coordinate of the block at time $t = 0$. However, we are given that the block is initially 0.5 m *beyond* the equilibrium position, so $x_0 = 0.5$ and hence the position function of the block is $x(t) = 0.5 \cos 7t$.

Solution (b).   The amplitude of the vibration is

$$\text{amplitude} = |x_0| = |0.5| = 0.5 \text{ m}$$

and from (21), (22), and (23) the period and frequency are

$$\text{period} = T = 2\pi\sqrt{\frac{M}{k}} = 2\pi\sqrt{\frac{1}{49}} = \frac{2\pi}{7} \text{ s}, \quad \text{frequency} = f = \frac{1}{T} = \frac{7}{2\pi} \text{ Hz} \ ◀$$

1. Verify that the following are solutions of the differential equation $y'' + y' - 2y = 0$ by substituting these functions into the equation.
   (a) e^{-2x} and e^x
   (b) $c_1 e^{-2x} + c_2 e^x$ $(c_1, c_2$ constants)

2. Verify that the following are solutions of the differential equation $y'' + 4y' + 4y = 0$ by substituting these functions into the equation.
   (a) e^{-2x} and xe^{-2x}
   (b) $c_1 e^{-2x} + c_2 xe^{-2x}$ $(c_1, c_2$ constants)

3–16 Find the general solution of the differential equation. ∎

3. $y'' + 3y' - 4y = 0$
4. $y'' + 5y' + 6y = 0$

5. $y'' - 2y' + y = 0$
6. $y'' - 6y' + 9y = 0$

7. $y'' + y = 0$
8. $y'' + 5y = 0$

9. $\dfrac{d^2 y}{dx^2} - \dfrac{dy}{dx} = 0$
10. $\dfrac{d^2 y}{dx^2} + 3\dfrac{dy}{dx} = 0$

11. $\dfrac{d^2 y}{dt^2} - 4\dfrac{dy}{dt} + 4y = 0$
12. $\dfrac{d^2 y}{dt^2} - 10\dfrac{dy}{dt} + 25y = 0$

13. $\dfrac{d^2 y}{dx^2} + 4\dfrac{dy}{dx} + 13y = 0$
14. $\dfrac{d^2 y}{dx^2} - 6\dfrac{dy}{dx} + 25y = 0$

15. $8y'' - 2y' - y = 0$
16. $9y'' - 6y' + y = 0$

17–22 Solve the initial-value problem. ∎

17. $y'' + 2y' - 3y = 0,$ $y(0) = 1,$ $y'(0) = 9$

18. $y'' - 6y' - 7y = 0,$ $y(0) = 5,$ $y'(0) = 3$

19. $y'' + 6y' + 9y = 0,$ $y(0) = 2,$ $y'(0) = -5$

20. $y'' + 4y' + y = 0,$ $y(0) = 5,$ $y'(0) = 4$

21. $y'' + 4y' + 5y = 0,$ $y(0) = -3,$ $y'(0) = 0$

22. $y'' - 6y' + 13y = 0,$ $y(0) = -2,$ $y'(0) = 0$

23. In each part, find a second-order linear homogeneous differential equation with constant coefficients that has the given functions as solutions.
   (a) $y_1 = e^{5x},$ $y_2 = e^{-2x}$ (b) $y_1 = e^{4x},$ $y_2 = xe^{4x}$
   (c) $y_1 = e^{-x}\cos 4x,$ $y_2 = e^{-x}\sin 4x$

24. Show that if e^x and e^{-x} are solutions of a second-order linear homogeneous differential equation, then so are $\cosh x$ and $\sinh x$.

25. Find all values of k for which the differential equation $y'' + ky' + ky = 0$ has a general solution of the given form.
   (a) $y = c_1 e^{ax} + c_2 e^{bx}$ (b) $y = c_1 e^{ax} + c_2 xe^{ax}$
   (c) $y = c_1 e^{ax}\cos bx + c_2 e^{ax}\sin bx$

26. The equation
$$x^2 \frac{d^2 y}{dx^2} + px\frac{dy}{dx} + qy = 0 \quad (x > 0)$$
where p and q are constants, is called *Euler's equidimensional equation*. Show that the substitution $x = e^z$ trans-

forms this equation into the equation
$$\frac{d^2 y}{dz^2} + (p-1)\frac{dy}{dz} + qy = 0$$

27. Use the result in Exercise 26 to find the general solution of
   (a) $x^2\dfrac{d^2 y}{dx^2} + 3x\dfrac{dy}{dx} + 2y = 0$ $(x > 0)$
   (b) $x^2\dfrac{d^2 y}{dx^2} - x\dfrac{dy}{dx} - 2y = 0$ $(x > 0)$.

28. Let $y(x)$ be a solution of $y'' + py' + qy = 0$. Prove: If p and q are positive constants, then $\lim_{x \to +\infty} y(x) = 0$.

29. Prove that the following functions are linearly independent.
   (a) $y_1 = e^{m_1 x},$ $y_2 = e^{m_2 x}$ $(m_1 \neq m_2)$
   (b) $y_1 = e^{mx},$ $y_2 = xe^{mx}$

30. Prove: If the auxiliary equation of
$$y'' + py' + qy = 0$$
has complex roots $a + bi$ and $a - bi$, then the general solution of this differential equation is
$$y(x) = e^{ax}(c_1 \cos bx + c_2 \sin bx)$$
[*Hint*: Using substitution, verify that $y_1 = e^{ax}\cos bx$ and $y_2 = e^{ax}\sin bx$ are solutions of the differential equation. Then prove that y_1 and y_2 are linearly independent.]

31. Suppose that the auxiliary equation of the equation $y'' + py' + qy = 0$ has distinct real roots μ and m.
   (a) Show that the function
$$g_\mu(x) = \frac{e^{\mu x} - e^{mx}}{\mu - m}$$
   is a solution of the differential equation.
   (b) Use L'Hôpital's rule to show that
$$\lim_{\mu \to m} g_\mu(x) = xe^{mx}$$
   [*Note*: Can you see how the result in part (b) makes it plausible that the function $y(x) = xe^{mx}$ is a solution of $y'' + py' + qy = 0$ when m is a repeated root of the auxiliary equation?]

32. Consider the problem of solving the differential equation
$$y'' + \lambda y = 0$$
subject to the conditions $y(0) = 0,$ $y(\pi) = 0$.
   (a) Show that if $\lambda \leq 0$, then $y = 0$ is the only solution.
   (b) Show that if $\lambda > 0$, then the solution is
$$y = c \sin \sqrt{\lambda} x$$
   where c is an arbitrary constant, if
$$\lambda = 1, 2^2, 3^2, 4^2, \ldots$$
   and the only solution is $y = 0$ otherwise.

33–35 These exercises involve vibrations of the block pictured in Figure L.1. Assume that the x-axis is as shown in Figure L.2 and that the simple harmonic model applies. ∎

33. Suppose that the block has a mass of 2 kg, the spring constant is $k = 0.5$ N/m, and the block is stretched 0.4 m beyond its equilibrium position and released at time $t = 0$.   *(cont.)*

(a) Find the position function $x(t)$ of the block.

(b) Find the period and frequency of the vibration.

(c) Sketch the graph of $x(t)$.

(d) At what time does the block first pass through the equilibrium position?

(e) At what time does the block first reach its maximum distance below the equilibrium position?

34. Suppose that the block has a weight of 64 lb, the spring constant is $k = 0.25$ lb/ft, and the block is stretched 1 ft beyond its equilibrium position and released at time $t = 0$.

(a) Find the position function $x(t)$ of the block.

(b) Find the period and frequency of the vibration.

(c) Sketch the graph of $x(t)$.

(d) At what time does the block first pass through the equilibrium position?

(e) At what time does the block first reach its maximum distance below the equilibrium position?

35. (a) For what values of x would you expect the block in Exercise 34 to have its maximum speed? Confirm your answer to this question mathematically.

(b) For what values of x would you expect the block to have its minimum speed? Confirm your answer to this question mathematically.

36. Use the initial position $x(0) = x_0$ and the initial velocity $v(0) = 0$ to find the constants c_1 and c_2 in (19).

37. A block attached to a horizontal spring is displaced from its equilibrium position and released, thereby causing it to vibrate with amplitude $|x_0|$ and period T.

(a) Show that the velocity of the block has maximum magnitude $2\pi|x_0|/T$ and that the maximum occurs when the block is at its equilibrium position.

(b) Show that the acceleration of the block has maximum magnitude $4\pi^2|x_0|/T^2$ and that the maximum occurs when the block is at a top or bottom point of its motion.

38. Assume that the motion of a block of mass M is governed by the simple harmonic model (18). Define the *potential energy* of the block at time t to be $\frac{1}{2}k[x(t)]^2$, and define the *kinetic energy* of the block at time t to be $\frac{1}{2}M[x'(t)]^2$. Prove that the sum of the potential energy of the block and the kinetic energy of the block is constant.

39–43 The accompanying figure shows a mass–spring system in which an object of mass M is attached to a spring and linked to a piston that moves in a *dashpot* containing a viscous fluid. If there are no external forces acting on the system, then the object is said to have ***free motion*** and the motion of the object is completely determined by the displacement and velocity of the object at time $t = 0$, the stiffness of the spring as measured by the spring constant k, and the viscosity of the fluid in the dashpot as measured by a *damping constant* c. Mathematically, the displacement $x = x(t)$ of the object from its equilibrium position is the solution of an initial-value problem of the form

$$x'' + Ax' + Bx = 0, \quad x(0) = x_0, \quad x'(0) = v_0$$

where the coefficient A is determined by M and c and the coefficient B is determined by M and k. In our derivation of Equation (21) we considered only motion in which the coefficient A is zero and in which the object is released from rest, that is, $v_0 = 0$. In these exercises you are asked to consider initial-value problems for which both the coefficient A and the initial velocity v_0 are nonzero.

Dashpot
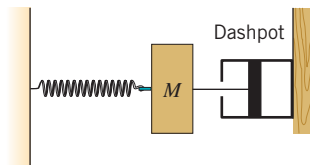
39. (a) Solve the initial-value problem $x'' + 2.4x' + 1.44x = 0$, $x(0) = 1$, $x'(0) = 2$ and graph $x = x(t)$ on the interval $[0, 5]$.

(b) Find the maximum distance to the right of the equilibrium position attained by the object.

(c) The graph of $x(t)$ suggests that the object does not pass through the equilibrium position. Show that this is so.

40. (a) Solve the initial-value problem $x'' + 5x' + 2x = 0$, $x(0) = 1/2$, $x'(0) = -4$ and graph $x = x(t)$ on the interval $[0, 5]$.

(b) Find the maximum distance to the left of the equilibrium position attained by the object.

(c) The graph of $x(t)$ suggests that the object passes through the equilibrium position exactly once. With what speed does the object pass through the equilibrium position?

41. (a) Solve the initial-value problem $x'' + x' + 5x = 0$, $x(0) = 1$, $x'(0) = -3.5$ and graph $x = x(t)$ on the interval $[0, 8]$.

(b) Find the maximum distance to the left of the equilibrium position attained by the object.

(c) Find the velocity of the object when it passes through the equilibrium position the first time.

(d) Find, by inspection, the acceleration of the object when it passes through the equilibrium position the first time. [*Hint:* Examine the differential equation and use the result in part (c).]

42. (a) Solve the initial-value problem $x'' + x' + 3x = 0$, $x(0) = -2$, $x'(0) = v_0$.

(b) Find the largest positive value of v_0 for which the object will move no further than 1 unit to the right of the equilibrium position. [*Hint:* Use a trial-and-error strategy. Estimate v_0 to the nearest hundredth.]

(c) Graph the solution of the initial-value problem on the interval $[0, 8]$ using the value of v_0 obtained in part (b).

43. (a) Solve the initial-value problem $x'' + 3.5x' + 3x = 0$, $x(0) = 1$, $x'(0) = v_0$.

(b) Use the result in part (a) to find the solutions for $v_0 = 2$, $v_0 = -1$, and $v_0 = -4$ and graph all three solutions on the interval $[0, 4]$ in the same coordinate system.

(c) Discuss the effect of the initial velocity on the motion of the object.