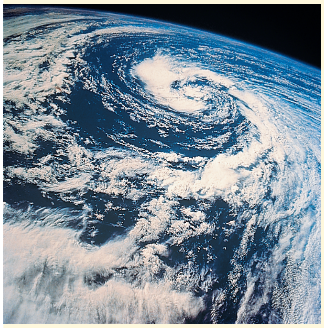

Hurricane Modeling

Each year population centers throughout the world are ravaged by hurricanes, and it is the mission of the National Hurricane Center to minimize the damage and loss of life by issuing warnings and forecasts of hurricanes developing in the Caribbean, Atlantic, Gulf of Mexico, and Eastern Pacific regions. Your assignment as a trainee at the Center is to construct a simple mathematical model of a hurricane using basic principles of fluid flow and properties of vector fields.

Modeling Assumptions

You have been notified of a developing hurricane in the Bahamas (designated hurricane *Isaac*) and have been asked to construct a model of its velocity field. Because hurricanes are complicated three-dimensional fluid flows, you will have to make many simplifying assumptions about the structure of a hurricane and the properties of the fluid flow. Accordingly, you decide to model the moisture in Isaac as an ***ideal fluid***, meaning that it is ***incompressible*** and its ***viscosity*** can be ignored. An incompressible fluid is one in which the density of the fluid is the same at all points and cannot be altered by compressive forces. Experience has shown that water can be regarded as incompressible but water vapor cannot. However, incompressibility is a reasonable assumption for a basic hurricane model because a hurricane is not restricted to a closed container that would produce compressive forces.

All fluids have a certain amount of viscosity, which is a resistance to flow—oil and molasses have a high viscosity, whereas water has almost none at subsonic speeds. Thus, it is reasonable to ignore viscosity in a basic model. Next, you decide to assume that the flow is in a ***steady state***, meaning that the velocity of the fluid at any point does not vary with time. This is reasonable over very short time periods for hurricanes that move and change slowly. Finally, although hurricanes are three-dimensional flows, you decide to model a two-dimensional horizontal cross section, so you make the simplifying assumption that the fluid in the cross section flows horizontally.

The photograph of Isaac shown at the beginning of this module reveals a typical pattern of a Caribbean hurricane—a counterclockwise swirl of fluid around the ***eye*** through which the fluid exits the flow in the form of rain. The lower pressure in the eye causes an inward-rushing air mass, and circular winds around the eye contribute to the swirling effect.

Your first objective is to find an explicit formula for Isaac's velocity field $\mathbf{F}(x, y)$, so you begin by introducing a rectangular coordinate system with its origin at the eye and its y-axis pointing north. Moreover, based on the hurricane picture and your knowledge of meteorological theory, you decide to build up the velocity field for Isaac from the velocity fields of simpler flows—a counterclockwise "vortex flow" $\mathbf{F}_1(x, y)$ in which fluid flows counterclockwise in concentric circles around the eye and a "sink flow" $\mathbf{F}_2(x, y)$ in which the fluid flows in straight lines toward a sink at the eye. Once you find explicit formulas for $\mathbf{F}_1(x, y)$ and $\mathbf{F}_2(x, y)$, your plan is to use the ***superposition principle*** from fluid dynamics to express the velocity field for Isaac as $\mathbf{F}(x, y) = \mathbf{F}_1(x, y) + \mathbf{F}_2(x, y)$.

Modeling a Vortex Flow

A ***counterclockwise vortex flow*** of an ideal fluid around the origin has four defining characteristics (Figure 1a):

- The velocity vector at a point (x, y) is tangent to the circle that is centered at the origin and passes through the point (x, y).

- The direction of the velocity vector at a point (x, y) indicates a counterclockwise motion.
- The speed of the fluid is constant on circles centered at the origin.
- The speed of the fluid along a circle is inversely proportional to the radius of the circle (and hence the speed approaches $+\infty$ as the radius of the circle approaches 0).

In fluid dynamics, the ***strength*** k of a vortex flow is defined to be 2π times the speed of the fluid along the unit circle. If the strength of a vortex flow is known, then the speed of the fluid along any other circle can be found from the fact that speed is inversely proportional to the radius of the circle. Thus, your first objective is to find a formula for a vortex flow $\mathbf{F}_1(x, y)$ with a specified strength k.

Exercise 1 Show that

$$\mathbf{F}_1(x, y) = -\frac{k}{2\pi(x^2 + y^2)}(y\mathbf{i} - x\mathbf{j})$$

is a model for the velocity field of a counterclockwise vortex flow around the origin of strength k by confirming that

(a) $\mathbf{F}_1(x, y)$ has the four properties required of a counterclockwise vortex flow around the origin;

(b) k is 2π times the speed of the fluid along the unit circle.

Exercise 2 Use a graphing utility that can generate vector fields to generate a vortex flow of strength 2π.

Modeling a Sink Flow

A ***uniform sink flow*** of an ideal fluid toward the origin has four defining characteristics (Figure 1*b*):

- The velocity vector at every point (x, y) is directed toward the origin.
- The speed of the fluid is the same at all points on a circle centered at the origin.
- The speed of the fluid at a point is inversely proportional to its distance from the origin (from which it follows that the speed approaches $+\infty$ as the distance from the origin approaches 0).
- There is a sink at the origin at which fluid leaves the flow.

As with a vortex flow, the ***strength*** q of a uniform sink flow is defined to be 2π times the speed of the fluid at points on the unit circle. If the strength of a sink flow is known, then the speed of the fluid at any point in the flow can be found using the fact that the speed is inversely proportional to the distance from the origin. Thus, your next objective is to find a formula for a uniform sink flow $\mathbf{F}_2(x, y)$ with a specified strength q.

Exercise 3 Show that

$$\mathbf{F}_2(x, y) = -\frac{q}{2\pi(x^2 + y^2)}(x\mathbf{i} + y\mathbf{j})$$

is a model for the velocity field of a uniform sink flow toward the origin of strength q by confirming the following facts:

(a) $\mathbf{F}_2(x, y)$ has the four properties required of a uniform sink flow toward the origin. [A reasonable physical argument to confirm the existence of the sink will suffice.]

(b) q is 2π times the speed of the fluid at points on the unit circle.

Exercise 4 Use a graphing utility that can generate vector fields to generate a uniform sink flow of strength 2π.

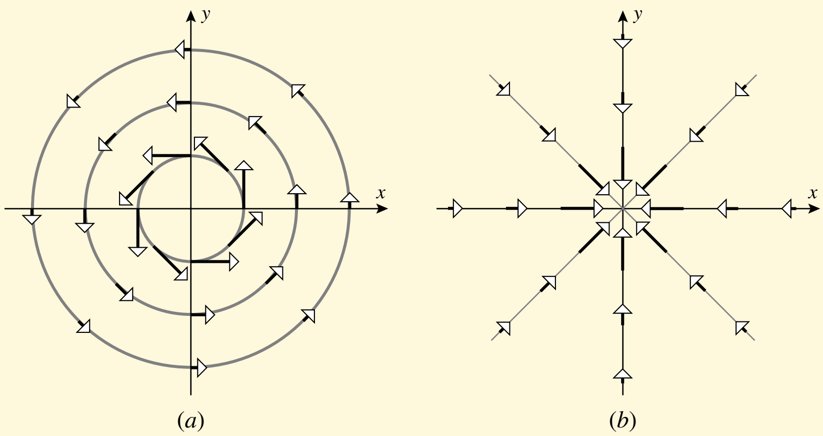

Figure 1

A Basic Hurricane Model

It now follows from Exercises 1 and 3 that the vector field $\mathbf{F}(x, y)$ for a hurricane model that combines a vortex flow around the origin of strength k and a uniform sink flow toward the origin of strength q is

$$\mathbf{F}(x, y) = -\frac{1}{2\pi(x^2 + y^2)}[(qx + ky)\mathbf{i} + (qy - kx)\mathbf{j}] \tag{1}$$

Exercise 5

(a) Figure 2 shows a vector field for a hurricane with vortex strength $k = 2\pi$ and sink strength $q = 2\pi$. Use a graphing utility that can generate vector fields to produce a reasonable facsimile of this figure.

(b) Make a conjecture about the effect of increasing k and keeping q fixed, and check your conjecture using a graphing utility.

(c) Make a conjecture about the effect of increasing q and keeping k fixed, and check your conjecture using a graphing utility.

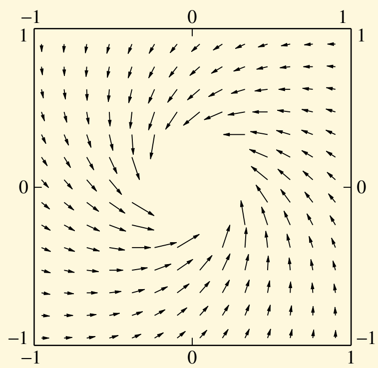

Figure 2

Modeling Hurricane Isaac

You are now ready to apply Formula (1) to obtain a model of the vector field $\mathbf{F}(x, y)$ of hurricane Isaac. You need some observational data to determine the constants k and q, so you call the Technical Support Branch of the Center for the latest information on hurricane Isaac. They report that 20 km from the eye the wind velocity has a component of 15 km/h toward the eye and a counterclockwise tangential component of 45 km/h.

Exercise 6

(a) Find the strengths k and q of the vortex and sink for hurricane Isaac.

(b) Find the vector field $\mathbf{F}(x, y)$ for hurricane Isaac.

(c) Estimate the size of hurricane Isaac by finding a radius beyond which the wind speed is less than 5 km/h.

Streamlines for the Basic Hurricane Model

The paths followed by the fluid particles in a fluid flow are called the ***streamlines*** of the flow. Thus, the vectors $\mathbf{F}(x, y)$ in the velocity field of a fluid flow are tangent to the streamlines. If the streamlines can be represented as the level curves of some function $\psi(x, y)$, then the function ψ is called a ***stream function*** for the flow. Since $\nabla\psi$ is normal to the level curves $\psi(x, y) = c$, it follows that $\nabla\psi$ is normal to the streamlines; and this in turn implies that

$$\nabla\psi \cdot \mathbf{F} = 0 \tag{2}$$

Your plan is to use this equation to find the stream function and then the streamlines of the basic hurricane model.

Since the vortex and sink flows that produce the basic hurricane model have a central symmetry, intuition suggests that polar coordinates may lead to simpler equations for the streamlines than rectangular coordinates. Thus, you decide to express the velocity vector $\mathbf{F}$ at a point (r, θ) in terms of the orthogonal unit vectors

$$\mathbf{u}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} \quad \text{and} \quad \mathbf{u}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$$

The vector $\mathbf{u}_r$, called the ***radial unit vector***, points away from the origin, and the vector $\mathbf{u}_\theta$, called the ***transverse unit vector***, is obtained by rotating $\mathbf{u}_r$ counterclockwise 90° (Figure 3).

Exercise 7 Show that the vector field for the basic hurricane model given in (1) can be expressed in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$ as

$$\mathbf{F} = -\frac{1}{2\pi r}(q\mathbf{u}_r - k\mathbf{u}_\theta)$$

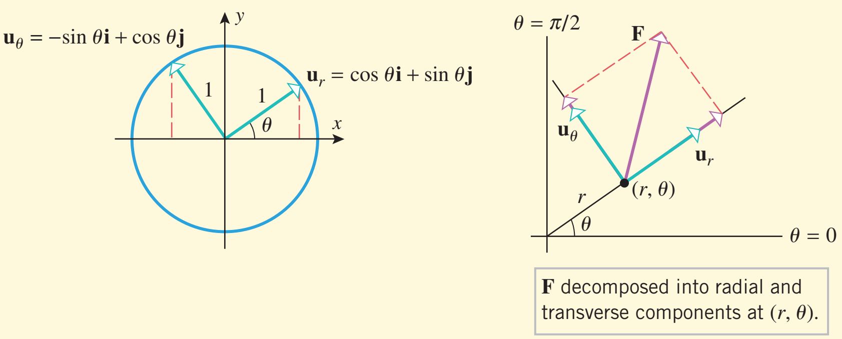

Figure 3

It follows from Exercise 79 of Section 13.6 that the gradient of the stream function can be expressed in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$ as

$$\nabla\psi = \frac{\partial\psi}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{u}_\theta$$

Exercise 8 Confirm that for the basic hurricane model the orthogonality condition in (2) is satisfied if

$$\frac{\partial\psi}{\partial r} = \frac{k}{r} \quad \text{and} \quad \frac{\partial\psi}{\partial\theta} = q$$

Exercise 9 By integrating the equations in Exercise 8, show that

$$\psi = k\ln r + q\theta$$

is a stream function for the basic hurricane model.

Exercise 10 Show that the streamlines for the basic hurricane model are logarithmic spirals of the form

$$r = Ke^{-q\theta/k} \quad (K > 0)$$

Exercise 11 Use a graphing utility to generate some typical streamlines for the basic hurricane model with vortex strength 2π and sink strength 2π.

Streamlines for Hurricane Isaac

Exercise 12 In Exercise 6 you found the strengths k and q of the vortex and sink for hurricane Isaac. Use that information to find a formula for the family of streamlines for Isaac; and then use a graphing utility to graph the streamline that passes through the point that is 20 km from the eye in the direction that is 45°NE from the eye.

Module by: *Josef S. Torok, Rochester Institute of Technology*
Howard Anton, Drexel University