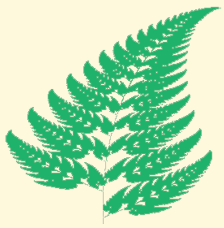
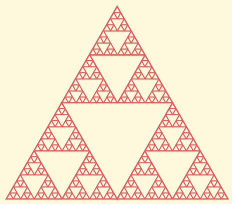
Iteration and Dynamical Systems

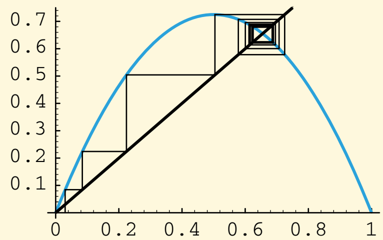
What do the four figures below have in common? The answer is that all of them are of interest in contemporary research and all involve a mathematical process called ***iteration****. In this module we will introduce this concept and touch on some of the fascinating ideas to which it leads.*



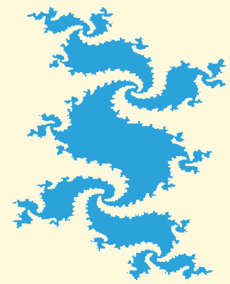
Barnsley's fern



The Sierpinski triangle



A cobweb diagram



A Julia set

Iterative Processes

Recall that in the notation $y = f(x)$, the variable x is called an ***input*** of the function f, and the variable y is called the corresponding ***output***. Suppose that we start with some input, say $x = c$, and each time we compute an output we feed it back into f as an input. This generates the following sequence of numbers:

$$f(c), \quad f(f(c)), \quad f(f(f(c))), \quad f(f(f(f(c)))), \ldots$$

This is called an ***iterated function sequence*** for f (from the Latin word *iteratus*, meaning "repeated"). The number c is called the ***seed value*** for the sequence, the terms in the sequence are called ***iterates***, and each time f is applied we say that we have performed an ***iteration***. Iterated function sequences arise in a wide variety of physical processes that are collectively called ***dynamical systems***.

Exercise 1 Let $f(x) = x^2$.

(a) Calculate the first 10 iterates in the iterated function sequence for f, starting with seed values of $c = 0.5$, 1, and 2. In each case make a conjecture about the ***long-term behavior*** of the iterates, that is, the behavior of the iterates as more and more iterations are performed.

(b) Try your own seed values, and make a conjecture about the effect of a seed value on the long-term behavior of the iterates.

Recursion Formulas

The proliferation of parentheses in an iterated function sequence can become confusing, so for simplicity let us introduce the following notation for the successive iterates

$$y_0 = c, \quad y_1 = f(c), \quad y_2 = f(f(c)), \quad y_3 = f(f(f(c))), \quad y_4 = f(f(f(f(c)))), \ldots$$

or expressed more simply,

$$y_0 = c, \quad y_1 = f(y_0), \quad y_2 = f(y_1), \quad y_3 = f(y_2), \quad y_4 = f(y_3), \ldots$$

Thus, successive terms in the sequence are related by the formulas

$$y_0 = c, \quad y_{n+1} = f(y_n) \qquad (n = 0, 1, 2, 3, \ldots)$$

These two formulas, taken together, comprise what is called a *recursion formula* for the iterated function sequence. In general, a ***recursion formula*** is any formula or set of formulas that provides a method for generating the terms of a sequence from the preceding terms and a seed value. For example, the recursion formula for the iterated function sequence of $f(x) = x^2$ with seed value c is

$$y_0 = c, \quad y_{n+1} = y_n^2$$

As shown in Exercise 25 of Section 4.7*, the recursion formula

$$y_0 = 1, \quad y_{n+1} = \frac{1}{2}\left(y_n + \frac{p}{y_n}\right) \tag{1}$$

produces an iterated function sequence whose iterates can be used to approximate $\sqrt{p}$ to any degree of accuracy.

Exercise 2 Use (1) to approximate $\sqrt{5}$ by generating successive iterates on a calculator until you encounter two successive iterates that are the same. Compare this approximation of $\sqrt{5}$ to that produced directly by your calculator.

Exercise 3

(a) Find iterates y_1 up to y_6 of the sequence that is generated by the recursion formula

$$y_0 = 1, \quad y_{n+1} = \tfrac{1}{2}y_n$$

(b) By examining the terms generated in part (a), find a formula that expresses y_n as a function of n.

Exercise 4 Suppose that you deposit $1000 in a bank at 5% interest per year and allow it to accumulate value without making withdrawals.

(a) If y_n denotes the value of the account at the end of the nth year, how could you find the value of y_{n+1} if you knew the value of y_n?

(b) Starting with $y_0 = 1000$ (dollars), use the result in part (a) to calculate y_1, y_2, y_3, y_4, and y_5.

(c) Find a recursion formula for the sequence of yearly account values assuming that $y_0 = 1000$.

(d) Find a formula that expresses y_n as a function of n, and use that formula to calculate the value of the account at the end of the 15th year.

Exploring Iterated Function Sequences

Iterated function sequences for a function f can be explored in various ways. Here are three possibilities:

- Choose a specific seed value, and investigate the long-term behavior of the iterates (as in Exercise 1).
- Let the seed value be a variable x (in which case the iterates become functions of x), and investigate what happens to the graphs of the iterates as more and more iterations are performed.
- Choose a specific iterate, say the 10th, and investigate how the value of this iterate varies with different seed values.

Exercise 5 Let $f(x) = \sqrt{x}$.

(a) Find formulas for the first five iterates in the iterated function sequence for f, taking the seed value to be x.

(b) Graph the iterates in part (a) in the same coordinate system, and make a conjecture about the behavior of the graphs as more and more iterations are performed.

*This reference to Calculus, Early Transcendentals, 9th edition, corresponds to Exercise 23 in Section 3.7 of Calculus, 9th edition.

Continued Fractions and Fibonacci Sequences

If $f(x) = 1/x$, and the seed value is x, then the iterated function sequence for f flip-flops between x and $1/x$:

$$y_1 = \frac{1}{x}, \quad y_2 = \frac{1}{1/x} = x, \quad y_3 = \frac{1}{x}, \quad y_4 = \frac{1}{1/x} = x, \ldots$$

However, if $f(x) = 1/(x+1)$, then the iterated function sequence becomes a sequence of fractions that, if continued indefinitely, is an example of a ***continued fraction***:

$$\frac{1}{1+x}, \quad \frac{1}{1+\dfrac{1}{1+x}}, \quad \frac{1}{1+\dfrac{1}{1+\dfrac{1}{1+x}}}, \quad \frac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+x}}}}, \ldots$$

Exercise 6 Let $f(x) = 1/(x+1)$ and $c = 1$.

(a) Find *exact values* for the first 10 terms in the iterated function sequence for f; that is, express each term as a fraction p/q with no common factors in the numerator and denominator.

(b) Write down the numerators from part (a) in sequence, and see if you can discover how each term after the first two is related to its predecessors. The sequence of numerators is called a ***Fibonacci sequence*** [in honor of its medieval discoverer Leonardo ("Fibonacci") da Pisa]. Do some research on Fibonacci and his sequence, and write a paper on the subject.

(c) Use the pattern you discovered in part (b) to write down the exact values of the second 10 terms in the iterated function sequence.

(d) Find a recursion formula that will generate all the terms in the Fibonacci sequence after the first two.

(e) It can be proved that the terms in the iterated function sequence for f get closer and closer to one of the two solutions of the equation $q = 1/(1+q)$. Which solution is it? This solution is a number known as the ***golden ratio***. Do some research on the golden ratio, and write a paper on the subject.

Applications to Ecology

There are numerous models for predicting the growth and decline of populations (flowers, plants, people, animals, etc.). One way to model populations is to give a recursion formula that describes how the number of individuals in each generation relates to the number of individuals in the preceding generation. One of the simplest such models, called the ***exponential model***, assumes that the number of individuals in each generation is a fixed percentage of the number of individuals in the preceding generation. Thus, if there are c individuals initially and if the number of individuals in any generation is r times the number of individuals in the preceding generation, then the growth through successive generations is given by the recursion formula

$$y_0 = c, \quad y_{n+1} = ry_n \qquad (n = 0, 1, 2, 3, \ldots)$$

Exercise 7 Suppose that a population with an exponential growth model has c individuals initially.

(a) Express the iterates y_1, y_2, y_3, and y_4 in terms of c and r.

(b) Find a formula for y_{n+1} in terms of c and r.

(c) Describe the eventual fate of the population if $r = 1$, $r < 1$, and $r > 1$.

There is a more sophisticated model of population growth, called the ***logistic model***, that takes environmental constraints into account. In this model, it is assumed that there is some maximum population that can be supported by the environment, and the population is expressed as a fraction of the maximum. Thus, in each generation the population is represented as a number in the interval $0 \leq y_n \leq 1$. When y_n is near 0 the population has lots of room to grow, but when y_n is near 1 the population is close to the maximum and the environmental factors tend to inhibit further growth. Models of this type are given by recursion formulas of the form

$$y_0 = c, \quad y_{n+1} = ky_n(1 - y_n) \tag{2}$$

in which k is a positive constant that depends on the ecological conditions.

Figure 1 illustrates a graphical method for tracking the growth of a population described by (2). That figure, which is called a ***cobweb diagram***, shows graphs of the line $y = x$ and the curve $y = kx(1 - x)$.

Exercise 8 Explain why the values y_1, y_2, and y_3 are the populations for the first three generations of the logistic growth model given by (2).

Exercise 9 The cobweb diagram in Figure 2 tracks the growth of a population with a logistical growth model given by the recursion formula

$$y_0 = 0.1, \quad y_{n+1} = 2.9y_n(1 - y_n)$$

(a) Find the populations $y_1, y_2, \ldots, y_5$ of the first five generations.

(b) What happens to the population over the long term?

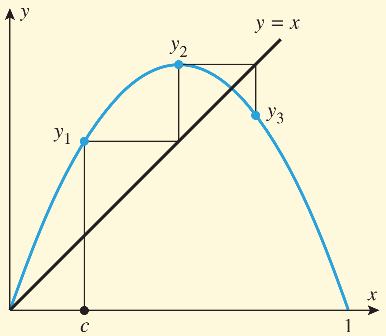


Figure 1

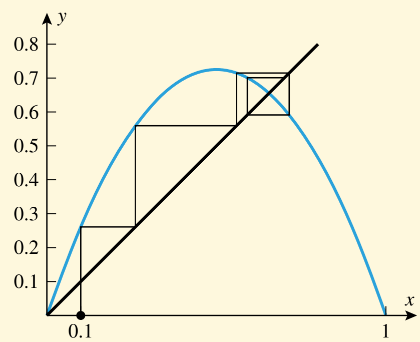


Figure 2

Chaos and Fractals

Observe that (2) is a recursion formula for the iterated function sequence of $f(x) = kx(1 - x)$. Iterated function sequences of this form are called ***iterated quadratic systems***. These are important not only in modeling populations but also in the study of ***chaos*** and ***fractals***—two important fields of contemporary research.

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