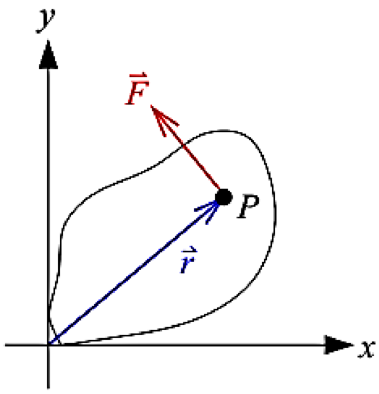
Chapter 9

Rotational Dynamics

9.1 Torque

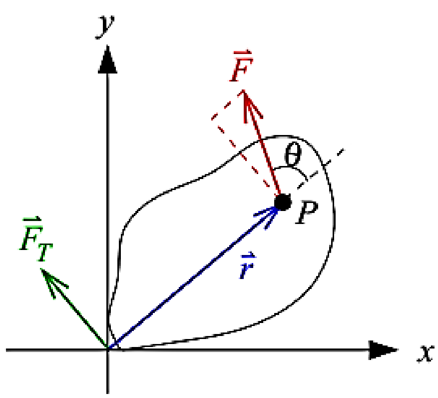
<u>Definition</u>



A force $\vec{F}$ was acting on a body at a point P. The torque about the point O is defined as:

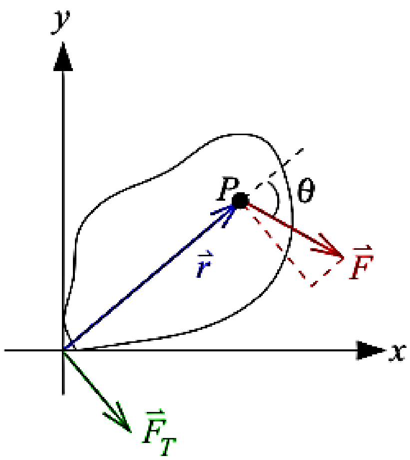
$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

where $\vec{r} = \vec{OP}$.



$$|\vec{\tau}| = |\vec{r}||\vec{F}| \sin\theta$$

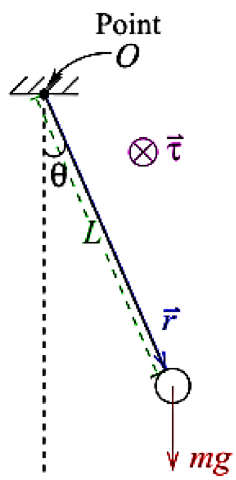
pointing out of paper



$$\tau = rF\sin\theta$$

pointing into the paper

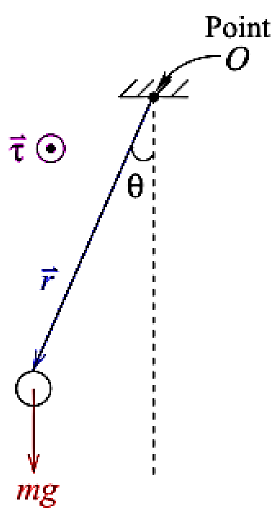
Example:



$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\therefore\ \tau = Lmg\sin\theta$$

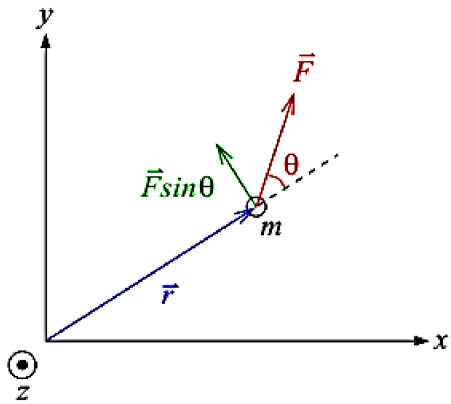
with direction pointing inward



$$\tau = Lmg\sin\theta$$

with direction pointing outward

9.2 Rotational Inertia and Newton's 2nd Law in Rotation



Single particle

- A particle is connected to the z-axis by a massless rod of length r.

From previous results (in the last chapter),

$$a_T = \alpha r$$

$$\therefore \; F_T = ma_T = m\alpha_z r$$

But $$F_T = F\sin\theta$$

$$\therefore \; F\sin\theta = m\alpha_z r$$

As $\tau_z = Fr\sin\theta$,

$$\tau_z = m\alpha_z r^2$$

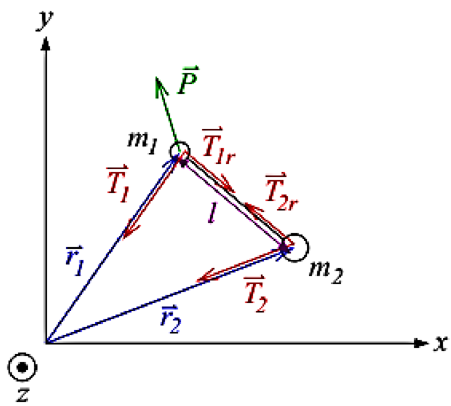
N. B. Subscript z is added to specify the axis of rotation.

Define $I = mr^2$ for single particle so that

$$\tau_z = I\alpha_z$$

- Newton 2nd law for rotation.

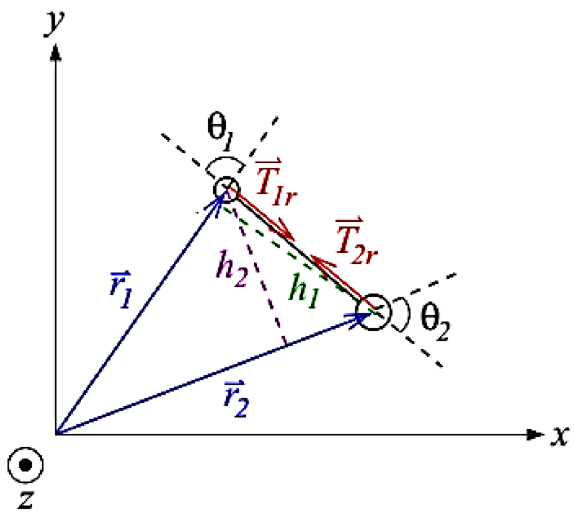
More than one particle



- Two masses m_1 and m_2 are linked by massless rods to the z-axis. m_1 and m_2 are linked to each other by a similar rod.
- Rotation axis: z-axis.
- $\vec{P}$ is an external force.

Total force on m_1: $\sum \vec{F}_1 = \vec{P} + \vec{T}_1 + \vec{T}_{1r}$

Total force on m_2: $\sum \vec{F}_2 = \vec{T}_2 + \vec{T}_{2r}$



Total torque on the system about z-axis:

$$
\begin{aligned}
\tau_z &= (\vec{r}_1 \times \sum \vec{F}_1) + (\vec{r}_2 \times \sum \vec{F}_2) \\
&= (\vec{r}_1 \times \vec{P}) + \underbrace{(\vec{r}_1 \times \vec{T}_1)}_{\to 0} + (\vec{r}_1 \times \vec{T}_{1r}) \\
&\quad + \underbrace{(\vec{r}_2 \times \vec{T}_2)}_{\to 0} + (\vec{r}_2 \times \vec{T}_{2r}) \\
&\quad (\because \ \vec{r}_1 // \vec{T}_1 \text{ and } \vec{r}_2 // \vec{T}_2) \\
&= (\vec{r}_1 \times \vec{P}) + (\vec{r}_1 \times \vec{T}_{1r}) + (\vec{r}_2 \times \vec{T}_{2r})
\end{aligned}
$$

Notice that:

$$
\begin{aligned}
|\vec{r}_1 \times \vec{T}_{1r}| &= r_1 T_{1r} \sin\theta_1 \\
|\vec{r}_2 \times \vec{T}_{2r}| &= r_2 T_{2r} \sin\theta_2
\end{aligned}
\tag{9.1}
$$

$(\vec{r}_1 \times \vec{T}_{1r})$ and $(\vec{r}_2 \times \vec{T}_{2r})$ are in opposite direction.

Let the distance between m_1 and m_2 be ℓ.

$$\therefore h_1 = \ell \sin\theta_1 \quad \& \quad h_2 = \ell \sin\theta_2$$

But area of triangle Om_1m_2 is equal to:

$$
\begin{aligned}
\tfrac{1}{2} h_1 r_1 &= \tfrac{1}{2} h_2 r_2 \\
\Rightarrow \ \ell \sin\theta_1 r_1 &= \ell \sin\theta_2 r_2 \\
\Rightarrow \ r_1 \sin\theta_1 &= r_2 \sin\theta_2
\end{aligned}
\tag{9.2}
$$

Since action-reaction forces are equal in magnitude

$$T_{1r} = T_{2r} \tag{9.3}$$

Substitute eq. (9.2) and (9.3) into eq. (9.1), we obtain:

$$|\vec{r}_1 \times \vec{T}_{1r}| = |\vec{r}_2 \times \vec{T}_{2r}|$$

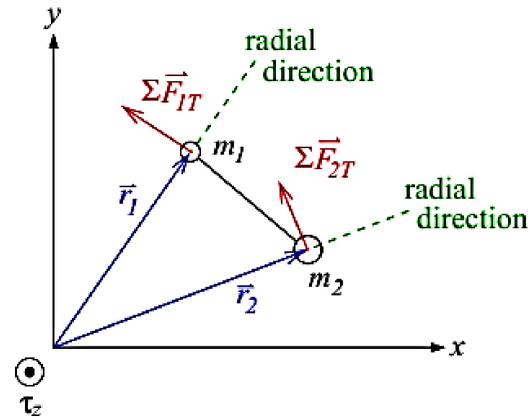
$\therefore$ Total torque:

$$\tau_z = \vec{r}_1 \times \vec{P}$$

which is only dependent on external force. Torque created by internal force are cancelled out.

Or
$$\tau_z = \sum_i \tau_{\text{ext},i}.$$

Consider the total force $\sum \vec{F}_i$ on mass m_i with $i = 1$ or 2.



- $\sum \vec{F}_i$ can be decomposed into two components, namely the tangential and the radial.
- $\tau_z \;=\; (\vec{r}_1 \times \sum \vec{F}_1) + (\vec{r}_2 \times \sum \vec{F}_2)$
  $\;=\; r_1 \underbrace{(\sum F_{1T})}_{m_1 a_{1T}} + r_2 \underbrace{(\sum F_{2T})}_{m_2 a_{2T}}$

$\because$ Radial components has no contribution to torque, i. e.

$$\begin{aligned} \tau_z &= r_1(m_1 \underbrace{a_{1T}}_{\alpha_z r_1}) + r_2(m_2 \underbrace{a_{2T}}_{\alpha_z r_2}) \\ &= (m_1 r_1^2 + m_2 r_2^2)\alpha_z \end{aligned}$$

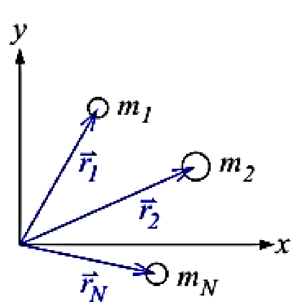
where a_{iT} are tangential acceleration of m_i and α_z is the angular acceleration of the system about the z axis.

Or

$$\tau_z = I\alpha_z$$

where $I = m_1 r_1^2 + m_2 r_2^2$.

Or in general for a N-particle rigid body:



Masses $m_1, m_2, \ldots, m_N$ located at $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$.

Moment of inertia is defined:

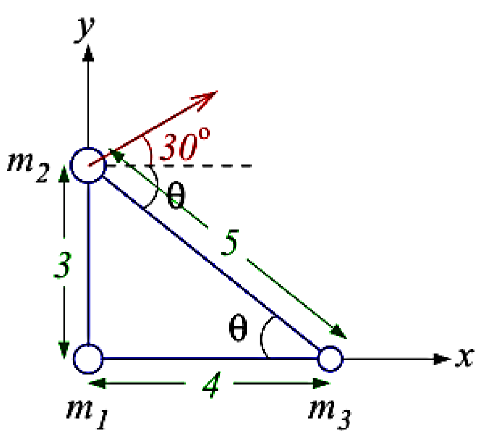
$$\boxed{I = \sum_i m_i r_i^2}$$

The total torque acting on this system:

$$\boxed{\tau_z = \sum_i \vec{r}_i \times \vec{F}_i = I\alpha_z}$$

N. B. Rigid body means the particles within the body have fixed relative position with respect to each others, i. e. as if linked by massless rods.

Example



$m_1 = 2.3$ kg
$m_2 = 3.2$ kg
$m_3 = 1.5$ kg

(a) Find moment of inertia of the system about axis perpendicular to xy plane and passing m_1, m_2 and m_3 respectively.

(b) If a 4.5 N force is applied to m_2 as shown and the system is free to rotate about the axis perpendicular to the xy plane and passing through m_3. What is the angular acceleration?

Answer:

(a) By definition,

$$\begin{aligned} I_1 &= \sum_i m_i r_i^2 = (2.3\ \text{kg})(0\ \text{m})^2 + (3.2\ \text{kg})(3\ \text{m})^2 + (1.5\ \text{kg})(4\ \text{m})^2 \\ &= 53\ \text{kg}\,\text{m}^2 \end{aligned}$$

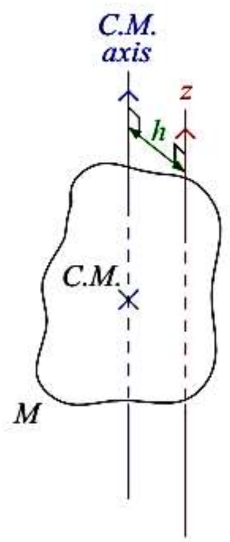
Similarly,

$$I_2 = (2.3\ \text{kg})(3\ \text{m})^2 + (3.2\ \text{kg})(0\ \text{m})^2 + (1.5\ \text{kg})(5\ \text{m})^2 = 58\ \text{kg}\,\text{m}^2$$
$$I_3 = (2.3\ \text{kg})(4\ \text{m})^2 + (3.2\ \text{kg})(5\ \text{m})^2 + (1.5\ \text{kg})(0\ \text{m})^2 = 117\ \text{kg}\,\text{m}^2$$

(b)
$$\theta = \sin^{-1}(3/5) \;\Rightarrow\; \theta = 37°$$
$$\therefore\ \tau_z = 4.5\ \mathrm{N} \times 5\ \mathrm{m} \times \sin(30° + 37°) = 20.7\ \mathrm{N\,m}$$

But
$$\tau_z = I\alpha_z$$
$$\therefore\ \alpha_z = \tau_z/I_3 = 0.18\ \mathrm{rad\,s^{-2}} \quad \text{in clockwise direction}$$

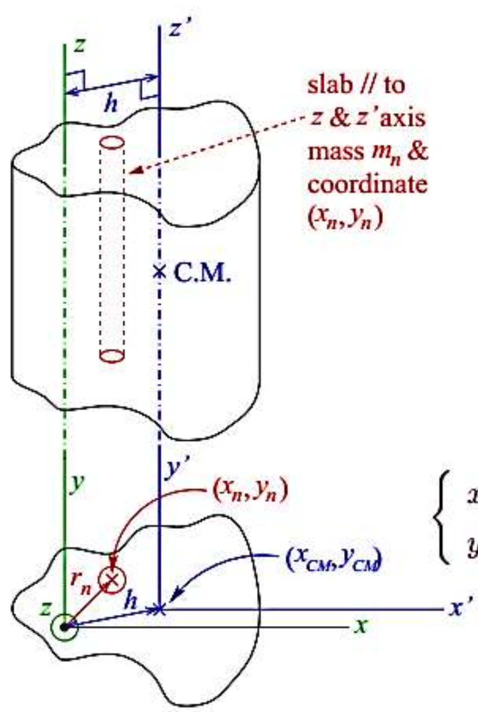
8.3 Parallel axis theorem



$$I_z = I_{\mathrm{CM}} + Mh^2$$

I_z = Moment of inertia rotating about z-axis,
I_{CM} = Moment of inertia rotating about the axis passing through C. M.,
z-axis is parallel to the C. M. axis and h is the distance between the two parallel axes.

Proof



For the I_z about the z-axis:

$$I_z = \sum_i m_i r_i^2 = \sum_i m_i(x_i^2 + y_i^2)$$

Let $(x_{\mathrm{CM}}, y_{\mathrm{CM}})$ be the x, y coordinates of the C. M. measured from the x, y coordinate system.

$$\begin{cases} x_i = x_i' + x_{\mathrm{CM}} \\ y_i = y_i' + y_{\mathrm{CM}} \end{cases}$$

$$
\begin{aligned}
\therefore\ I_z &= \sum_i m_i[(x'_i + x_{\rm CM})^2 + (y'_i + y_{\rm CM})^2] \\
&= \sum_i m_i(x'^2_i + 2x'_i x_{\rm CM} + x^2_{\rm CM} + y'^2_i + 2y'_i y_{\rm CM} + y^2_{\rm CM}) \\
&= \underbrace{\sum_i m_i(x'^2_i + y'^2_i)}_{=I_{CM}} + 2x_{\rm CM}\underbrace{\sum_i m_i x'_i}_{=Mx'_{CM}=0} + 2y_{\rm CM}\underbrace{\sum_i m_i y'_i}_{=My'_{CM}=0} + \underbrace{(x^2_{\rm CM} + y^2_{\rm CM})}_{=h^2}\underbrace{\sum_i m_i}_{=M} \\
&= I_{\rm CM} + Mh^2
\end{aligned}
$$

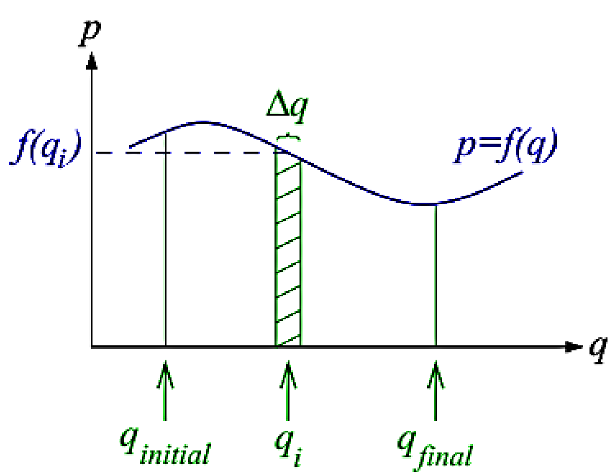
N. B. The axis passing through the CM is the axis that has the smallest moment of inertia as compared to other parallel axis.

9.4 Rotational inertia of solid bodies

$$I = \sum_i m_i r_i^2$$

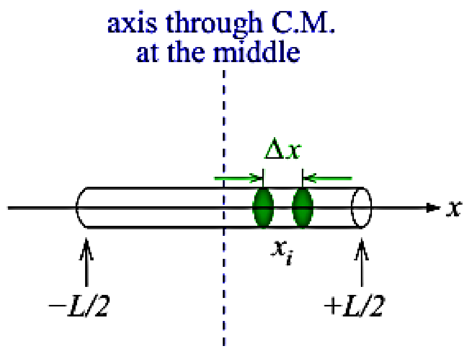
Integral form: $$I = \int r^2 dm$$

Recall: What is definite integral?



$$\int_{q_{\rm initial}}^{q_{\rm final}} f(q)\,dq = \sum_i f(q_i)\Delta q$$

Example:



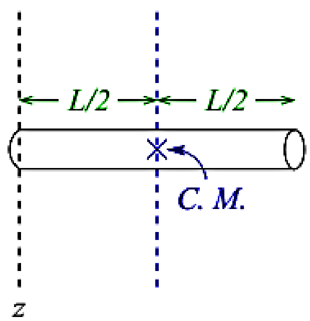
Uniform rod with mass M and length L.
Partition the whole rod into segments with length Δx.
Consider the segment at x_i:

$$\Delta m_i = \lambda \Delta x, \quad \lambda = \text{density (i. e. kg m}^{-1}\text{)}$$

$$\therefore \ I = \sum_i \Delta m_i x_i^2 = \int_{-L/2}^{+L/2} x^2 dm = \int_{-L/2}^{+L/2} x^2 \lambda dx$$

But $\lambda = M/L$.

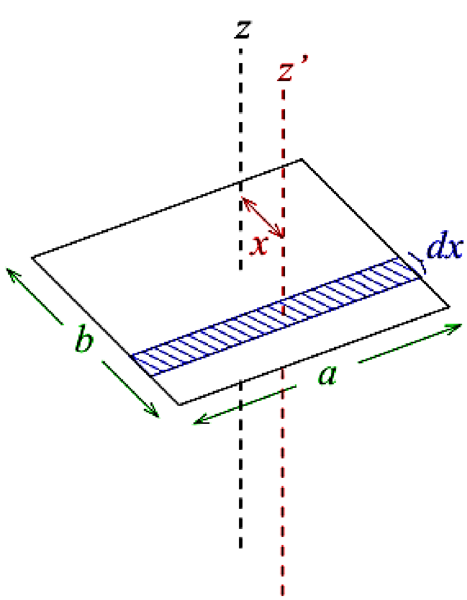
$$\therefore \ I = \frac{1}{3}[x^3]_{-L/2}^{+L/2} \times \left(\frac{M}{L}\right) = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{1}{12}ML^2$$



By parallel axis theorem,

$$\begin{aligned} I_z &= I_{\text{CM}} + Mh^2 \\ &= \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 \\ &= \frac{1}{3}ML^2 \end{aligned}$$

Example:



An uniform rectangular plate rotating about an axis through the center.
Partition the plate into strips with width dx.
Mass at each of these strip:

$$dm = (adx)\sigma, \quad \sigma = \text{density}$$

Moment of inertia at the strip about z':

$$dI_{\text{CM}} = \frac{1}{12}dma^2 = \frac{1}{12}\sigma a^3 dx$$

Moment of inertia of the strip about z:

$$dI = dI_{\text{CM}} + x^2 dm = \frac{1}{12}\sigma a^3 dx + a\sigma x^2 dx$$

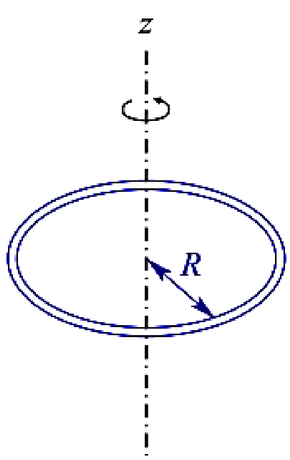
But $\sigma = M/(ab)$.

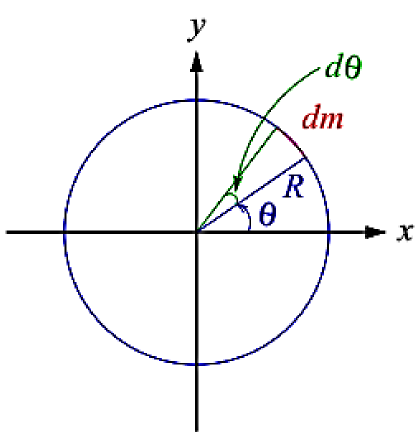
$$\therefore\ dI = \frac{1}{12}\frac{M}{ab}a^3dx + a\frac{M}{ab}x^2dx = \left(\frac{1}{12}\frac{M}{b}a^2 + \frac{M}{b}x^2\right)dx$$

$\therefore$ Total moment of inertia about z-axis:

$$\begin{aligned} I_z &= \int_{-b/2}^{b/2} dI = \int_{-b/2}^{b/2} \frac{M}{b}\left(\frac{1}{12}a^2 + x^2\right)dx \\ &= \frac{M}{b}\frac{1}{12}a^2(b) + \frac{M}{b}\frac{1}{3}[x^3]_{-b/2}^{b/2} \\ &= \frac{M}{12}a^2 + \frac{M}{12}b^2 \\ &= \frac{M}{12}(a^2+b^2) \end{aligned}$$

<u>Example:</u>





An uniform circular ring rotating about the circle center.

Consider the segment at angle θ.

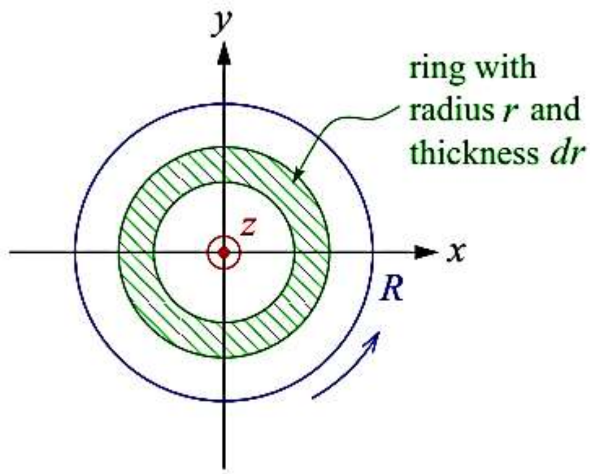
$$dm = (Rd\theta)\lambda, \quad \lambda = \text{linear density } (\text{kg m})^{-1}$$

Hence, its moment of inertia is given by

$$\begin{aligned} I &= \int r^2 dm = \int_0^{2\pi} R^2 R\lambda d\theta \\ &= \int_0^{2\pi} R^3 \underbrace{\left(\frac{M}{2\pi R}\right)}_{=\lambda} d\theta \\ &= \frac{MR^2}{2\pi}2\pi \\ &= MR^2 \end{aligned}$$

<u>Example:</u>

An uniform circular disk rotating about the circle center.



Consider the ring with radius r and thickness dr.

$$
\begin{aligned}
dm &= \underbrace{\left(\frac{M}{\pi R^2}\right)}_{\text{surface density (kg s}^{-1}\text{)}} [\pi(r+dr)^2 - \pi r^2] \\
&= \frac{M}{\pi R^2}[\underbrace{\pi dr^2}_{\to 0} + 2\pi r dr] \\
&\approx \frac{M}{\pi R^2} 2\pi r dr \\
&= \frac{2Mr}{R^2} dr
\end{aligned}
$$

Moment of inertia of the ring rotating about the center:

$$dI = (dm)r^2 = \frac{2Mr^3}{R^2} dr$$

$\therefore$ Total moment of inertia of the disk about the center:

$$I = \sum dI = \int_0^R \frac{2Mr^3}{R^2} dr = \frac{2M}{R^2}\frac{R^4}{4} = \frac{1}{2}MR^2$$

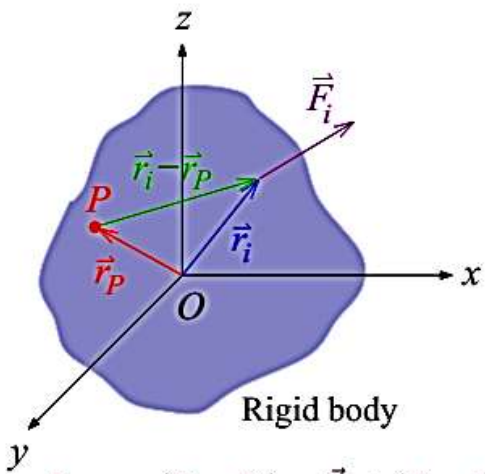
9.5 Equilibrium of rigid body

For a rigid body to stay at equilibrium, the following conditions must be satisfied:

1) $\sum \vec{F}_{\text{ext}} = 0$

2) $\sum \vec{\tau}_{\text{ext}} = 0$ about choice of reference point

Question: If we know $\sum \vec{F}_{\text{ext}} = 0$ and $\sum \vec{\tau} = 0$ about one particular point, say O, can we conclude the total torque about any other choice of point?



Refer to point O,

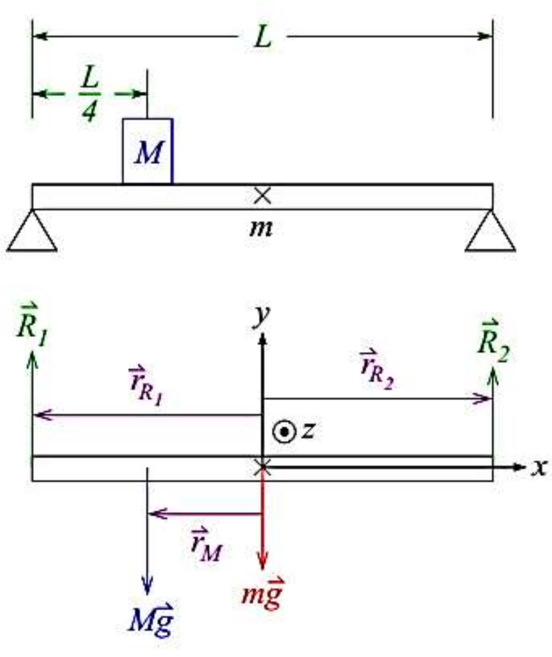
$$\begin{aligned}\vec{\tau}_o &= \vec{\tau}_1 + \vec{\tau}_2 + \ldots \vec{\tau}_N \\ &= (\vec{r}_1 \times \vec{F}_1) + (\vec{r}_2 \times \vec{F}_2) + \ldots + (\vec{r}_N \times \vec{F}_N) \\ &= 0\end{aligned}$$

Now, we consider the torque about another point P,

$$\begin{aligned}\vec{\tau}_P &= (\vec{r}_1 - \vec{r}_P) \times \vec{F}_1 + (\vec{r}_2 - \vec{r}_P) \times \vec{F}_2 + \ldots + (\vec{r}_N - \vec{r}_P) \times \vec{F}_N \\ &= \underbrace{[(\vec{r}_1 \times \vec{F}_1) + (\vec{r}_2 \times \vec{F}_2) + \ldots + (\vec{r}_N \times \vec{F}_N)]}_{\text{Given condition} = 0} - \underbrace{[\vec{r}_P \times (\vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_N)]}_{\sum_i \vec{F}_i = 0} \\ &= 0\end{aligned}$$

i. e. if $\sum_i \vec{F}_i = 0$ (i. e. translational equilibrium established) and $\sum \vec{\tau}_{\text{ext}} = 0$ about a given point, then $\sum \vec{\tau}_{\text{ext}} = 0$ about any point and thus, equilibrium must be established.

Example:



As the bar is in equilibrium,

$$\sum_i \vec{F}_i = 0$$

$$\therefore \quad \vec{R}_1 + \vec{R}_2 + M\vec{g} + m\vec{g} = \vec{O}$$

$$\text{or} \quad \vec{R}_1 + \vec{R}_2 + (M+m)\vec{g} = \vec{O}$$

Take upward as positive:

$$R_1 + R_2 - (M+m)g = 0 \qquad (9.4)$$

Take moment about O,

$$\begin{aligned}\vec{\tau} &= \vec{r}_{R_1} \times \vec{R}_1 + \vec{r}_M \times (M\vec{g}) + \vec{r}_{R_2} \times \vec{R}_2 \\ \Rightarrow \tau &= -\frac{L}{2}R_1 + \frac{L}{4}Mg + \frac{L}{2}R_2 = 0\end{aligned}$$

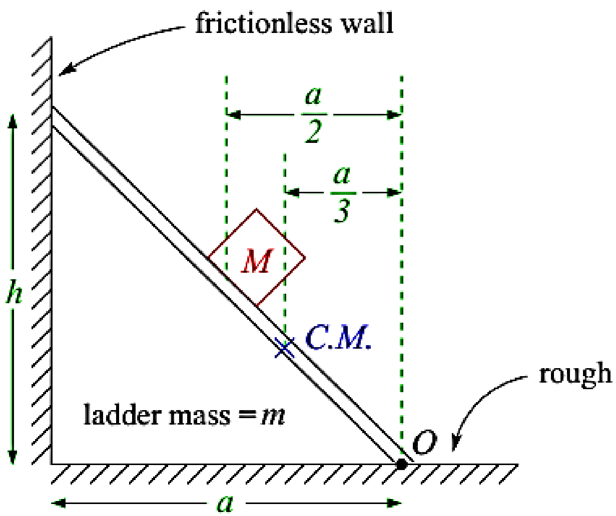
Therefore,

$$R_1 - R_2 - \frac{1}{2}Mg = 0 \tag{9.5}$$

(9.4) + (9.5):

$$2R_1 = (M+m)g + \frac{1}{2}Mg = mg + \frac{3}{2}Mg$$
$$R_1 = \frac{1}{2}mg + \frac{3}{4}Mg$$
$$R_2 = R_1 - \frac{1}{2}Mg = \frac{1}{2}mg + \frac{1}{4}Mg$$

Example:

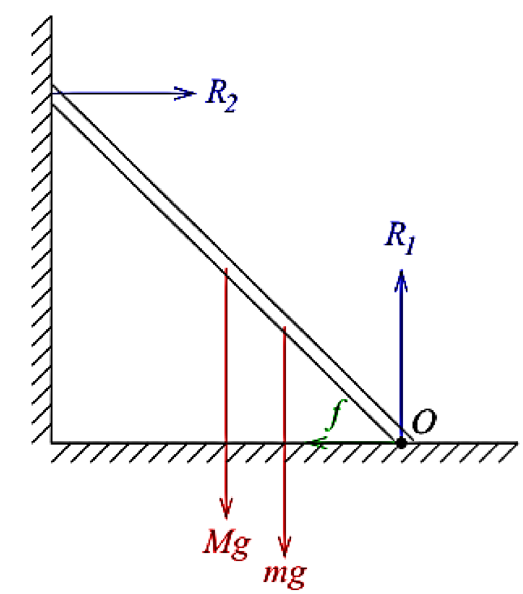


To obtain equilibrium, $\sum_i \vec{F}_i = 0$.

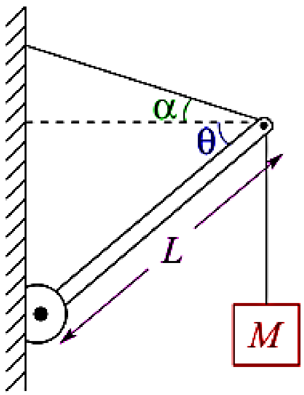
$$\Rightarrow \begin{cases} R_2 = f \\ R_1 = (m+M)g \end{cases}$$

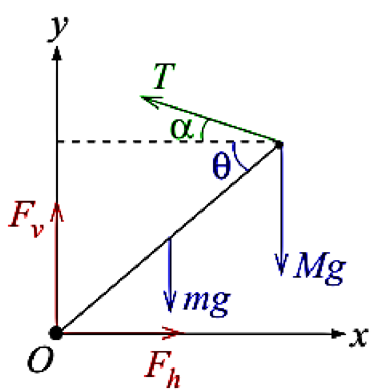
Finding torque about the axis through O and perpendicular to the paper.

$$Mg\tfrac{a}{2} + mg\tfrac{a}{3} = R_2 h$$
$$\Rightarrow \quad R_2 = f = \tfrac{a}{2h}Mg + \tfrac{a}{3h}mg$$



Example:





Consider a uniform beam of mass m.

Let the reaction at the hinge has x, y components of F_v and F_h.

$$F_v - mg - Mg + T\sin\alpha = 0 \qquad (9.6)$$

$$F_h - T\cos\alpha = 0 \qquad (9.7)$$

Take the torque about the axis passing through O and perpendicular to the paper:

$$TL\sin(\alpha+\theta) - MgL\cos\theta - mg\tfrac{L}{2}\cos\theta = 0$$

$$\therefore \quad T = \frac{g(M+m/2)\cos\theta}{\sin(\alpha+\theta)}$$

From (9.6):

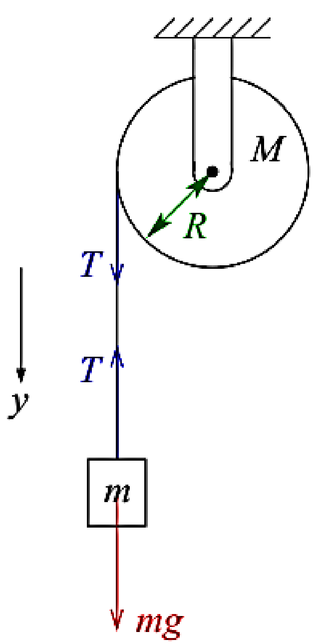
$$F_v = (m+M)g - \frac{g(M+m/2)\cos\theta\sin\alpha}{\sin(\alpha+\theta)}\ .$$

From (9.7):

$$F_h = \frac{g(M+m/2)\cos\theta\cos\alpha}{\sin(\alpha+\theta)}\ .$$

9.6 Non-equilibrium situation: pure rotation

Example:



Frictionless pulley with mass M and radius R.

$$mg - T = m\frac{d^2y}{dt^2}$$

$$\Rightarrow \quad T = mg - m\frac{d^2y}{dt^2}$$

Torque acting on pulley:

$$\tau = TR = I\alpha$$

where $I = \frac{1}{2}MR^2$ for disk rotating about its center.

$$\therefore \quad \left(mg - m\frac{d^2y}{dt^2}\right)R = \frac{1}{2}MR^2\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \quad 2mg - 2m\frac{d^2y}{dt^2} = MR\frac{d^2\theta}{dt^2} \qquad (9.8)$$

But if the rope run through the pulley without slipping:

$$\theta R = y \quad \Rightarrow \quad R\frac{d^2\theta}{dt^2} = \frac{d^2y}{dt^2}$$

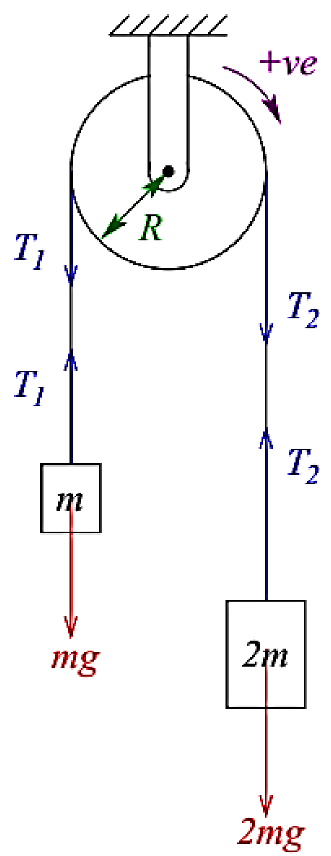
$\therefore$ (9.8) becomes:

$$2mg - 2mR\frac{d^2\theta}{dt^2} = MR\frac{d^2\theta}{dt^2}$$

Hence,

$$\alpha = \frac{d^2\theta}{dt^2} = \frac{2mg}{MR + 2mR}$$
$$a = \frac{d^2y}{dt^2} = \frac{2mg}{M + 2m}$$

Example:



Frictionless pulley with mass M and radius R.

$$T_1 - mg = ma \tag{9.9}$$
$$2mg - T_2 = 2ma \tag{9.10}$$

Total torque on pulley:

$$\tau = T_2R - T_1R = (\tfrac{1}{2}MR^2)\alpha \tag{9.11}$$
$$a = R\alpha \tag{9.12}$$

Put (9.9) and (9.10) into (9.11):

$$(2mg - 2ma)R - (ma + mg)R = \frac{1}{2}MR^2\alpha \tag{9.13}$$

Substitute (9.12) into (9.13), we obtain

$$2mg - 2m(R\alpha) - m(R\alpha) - mg = \frac{1}{2}MR\alpha$$
$$\Rightarrow \quad mg = \frac{1}{2}MR\alpha + 3mR\alpha$$
$$\Rightarrow \quad \alpha = \frac{mg}{MR/2 + 3mR} \quad \text{and} \quad a = R\alpha = \frac{mg}{M/2 + 3m}$$

9.7 Non-equilibrium situation: rotational and translational motion

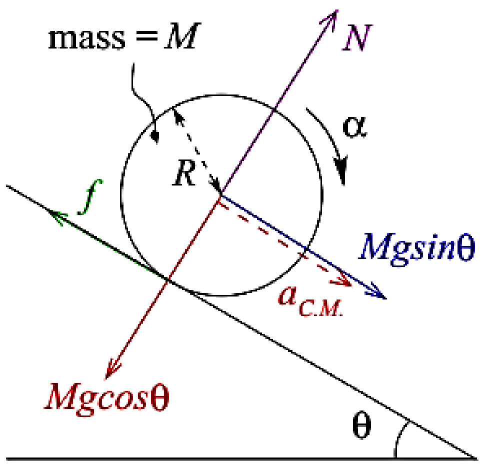
If $\sum_i \vec{F}_i \neq 0$ and $\sum_i \vec{\tau}_i \neq 0$ about any axis, the motion of the rigid body has both self-rotation and the motion of the C.M.

For the present course, we only focus on cases such that:

a) Axis of rotation passes through C.M.

b) Rotating axis always has the same direction in space.

Example:



Consider a solid cylinder with mass M and radius R.

$$N = Mg\cos\theta$$

$$Mg\sin\theta - f = Ma_{\text{C.M.}} \qquad (9.14)$$

Total torque on the cylinder:

$$\tau = fR = I\alpha = \frac{1}{2}MR^2\alpha$$

$$\Rightarrow \quad f = \frac{1}{2}MR\alpha \qquad (9.15)$$

Put (9.15) into (9.14):

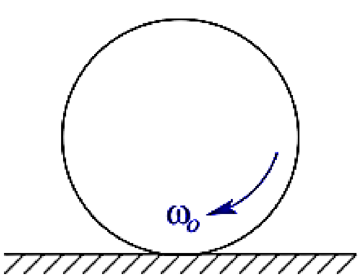
$$Mg\sin\theta - \frac{1}{2}MR\alpha = Ma_{\text{C.M.}}$$

But $R\alpha = a_{\text{C.M.}}$, thus

$$Mg\sin\theta - \frac{1}{2}Ma_{\text{C.M.}} = Ma_{\text{C.M.}}$$

$$\Rightarrow \quad a_{\text{C.M.}} = \frac{2}{3}g\sin\theta \quad \text{and} \quad \alpha = \frac{a_{\text{C.M}}}{R} = \frac{2g}{3R}\sin\theta$$

Example:



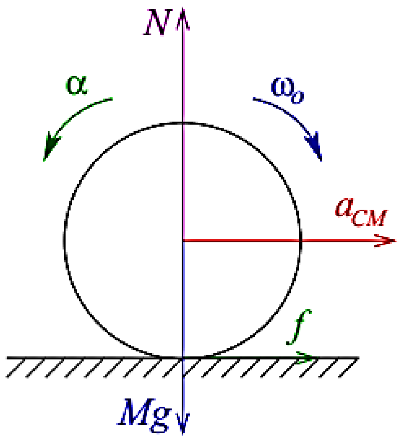
A uniform solid cylinder of radius R and mass M is given an initial velocity ω_0 and then lowered on a uniform horizontal surface. The coefficient of kinetic friction between the cylinder and the surface is μ_k. Initially the cylinder slips as it moves along the surface, but after a time t, pure rolling without slipping begins.

(a) What is the velocity v_{CM} at time t?

(b) What is the value of t?

Solutions:

(a) During the interval $0 \to t$:



$$f = \mu_k N = \mu_k Mg \tag{9.16}$$

f is constant within this period. We know that just at $t = 0$, velocity of C. M. is equal to zero.

$$\therefore \quad a_{\text{CM}} = \frac{v_{\text{CM}}}{t}$$

$$f = Ma_{\text{CM}} = M\frac{v_{\text{CM}}}{t} \tag{9.17}$$

(9.16) and (9.17) gives:

$$\mu_k Mg = M\frac{v_{\text{CM}}}{t} \quad \Rightarrow \quad \mu_k g = \frac{v_{\text{CM}}}{t} \tag{9.18}$$

Besides,

$$\begin{aligned} I\alpha &= fR \\ \Rightarrow \quad \frac{1}{2}MR^2\alpha &= fR \\ \Rightarrow \quad \alpha &= \frac{2f}{MR} \end{aligned} \tag{9.19}$$

Let ω_f be the angular velocity at t, where ω_f rotates in clockwise direction.

$$-\omega_f = -\omega_0 + \alpha t \Rightarrow \alpha = \frac{\omega_0 - \omega_f}{t}$$

At time t, no slipping occurs.

$$\therefore \quad v_{\text{CM}} = \omega_f R \quad \Rightarrow \quad \alpha = \frac{\omega_0 - v_{\text{CM}}/R}{t} \tag{9.20}$$

Put (9.19) into (9.20), we have:

$$\begin{aligned} \frac{2f}{MR} &= \frac{\omega_0 - v_{\text{CM}}/R}{t} \\ \Rightarrow \quad 2ft &= MR\omega_0 - Mv_{\text{CM}} \end{aligned} \tag{9.21}$$

Put (9.16) into (9.21), we have:

$$2t\mu_k Mg = MR\omega_0 - Mv_{\text{CM}}$$
$$\Rightarrow \quad 2t\mu_k g = R\omega_0 - v_{\text{CM}} \tag{9.22}$$

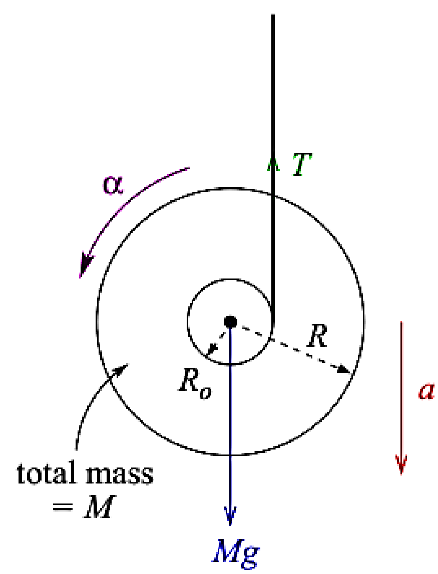
Put (9.18) into (9.22), we have:

$$2\mu_k g\left(\frac{v_{\text{CM}}}{\mu_k g}\right) = R\omega_0 - v_{\text{CM}}$$
$$\Rightarrow \quad v_{\text{CM}} = \frac{1}{3}\omega_0 R \tag{9.23}$$

(b) From (9.18),

$$t = \frac{v_{\text{CM}}}{\mu_k g} = \frac{\omega_0 R}{3\mu_k g}$$

<u>Example:</u>



Two solid cylinders are sticked together and string is winded on the cylinder with the smaller radius. Assume the thin slab is very light as compared to the large cylinder.

$$Mg - T = Ma$$
$$\tau = TR_0 = m(g-a)R_0 \tag{9.24}$$
$$\tau = I\alpha = \frac{1}{2}MR^2\alpha \tag{9.25}$$

(9.24) and (9.25) gives:

$$m(g-a)R_0 = \frac{1}{2}MR^2\alpha$$
$$\Rightarrow \quad \alpha = 2(g-a)\frac{R_0}{R^2} \tag{9.26}$$

For no slipping,

$$a = R_0\alpha \tag{9.27}$$

Substituting (9.27) into (9.26), we obtain:

$$\frac{a}{R_0} = 2(g-a)\frac{R_0}{R^2}$$
$$\Rightarrow \quad a = \frac{2gR_0^2}{R^2 + 2R_0^2}$$

Hence,

$$\alpha = \frac{a}{R_0} = \frac{2gR_0}{R^2 + 2R_0^2}$$