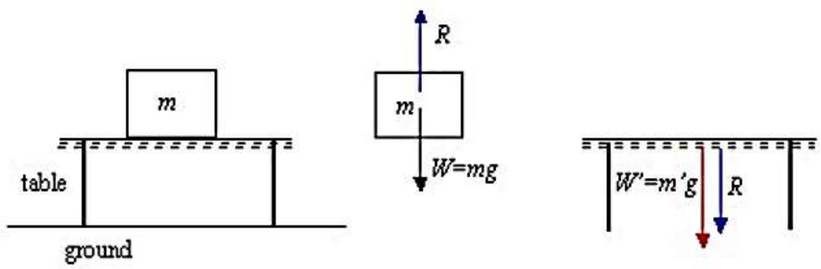
Chapter 5

Application of Newton's Law

5.1 Examples with tension and normal force

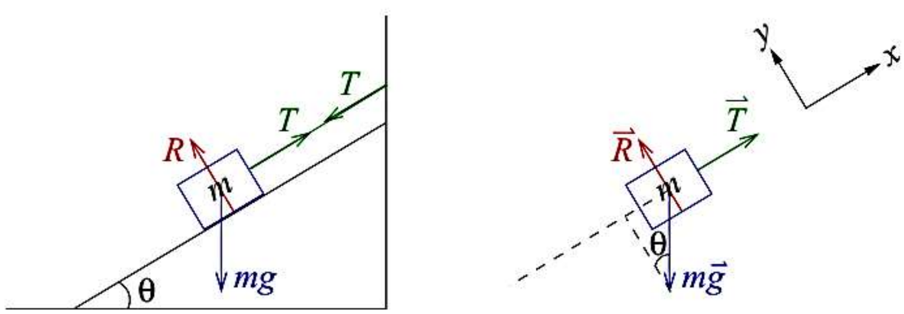
Example 1



Mass m is in equilibrium on the table: $R - W = 0$, where $W = mg$. Hence, we know the reaction force from the ground on the mass: $R = W = mg$. And, the action force from m on the table: $R = mg$ according to the Newton's three law of motion. Note that m' and W' are the mass and the weight of the table respectively.

Example 2

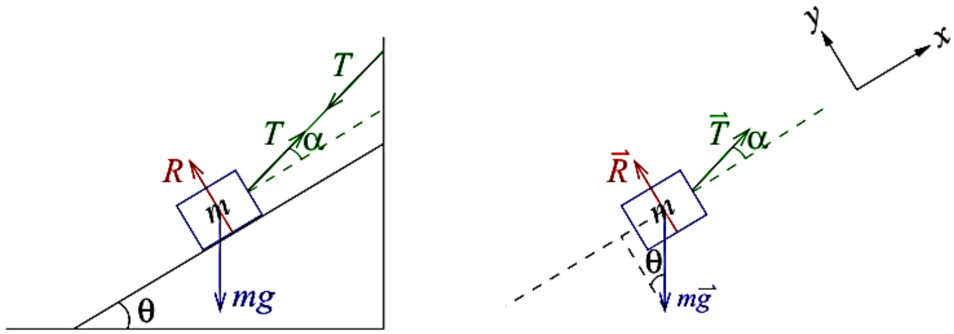
Mass hanged by a string on a frictionless inclined plane



$$\vec{R} + \vec{T} + m\vec{g} = 0$$

$$x\text{-direction:} \quad T - mg\sin\theta = 0 \quad \Rightarrow \quad T = mg\sin\theta$$
$$y\text{-direction:} \quad R - mg\cos\theta = 0 \quad \Rightarrow \quad R = mg\cos\theta$$

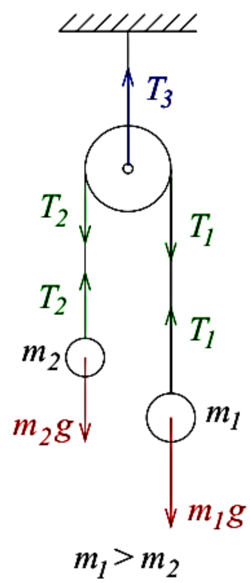
Example 3



$$x\text{-direction:} \quad T\cos\alpha - mg\sin\theta = 0 \quad \Rightarrow \quad T = mg\sin\theta\sec\alpha$$
$$y\text{-direction:} \quad R + T\sin\alpha - mg\cos\theta = 0 \quad \Rightarrow \quad R = mg\cos\theta - mg\sin\theta\tan\alpha$$

Example 4

Non-stretchable string, frictionless & weightless pulley



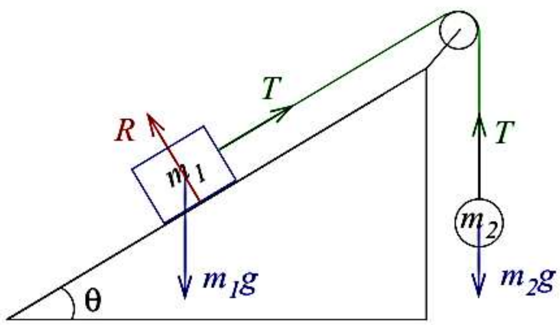
Note that $T_3 = T_1 + T_2$, where
$T_1 = T_2 = T$. If $T_1 \neq T_2$, there is a non-zero torque τ acting on the pulley and the pulley will has angular acceleration. As the mass of the pulley is zero and thus its moment of inertia I is also zero. Then $\tau = I\ddot{\theta} \neq 0$ implies $\ddot{\theta} \to \infty$ which is impossible.
Analysing the force on the two masses, we find that

$$m_1 g - T = m_1 a \quad \text{and} \quad T - m_2 g = m_2 a$$

Then a and T can be found by solving these two equations.

Example 5

Frictionless inclined plane, frictionless & weightless pulley



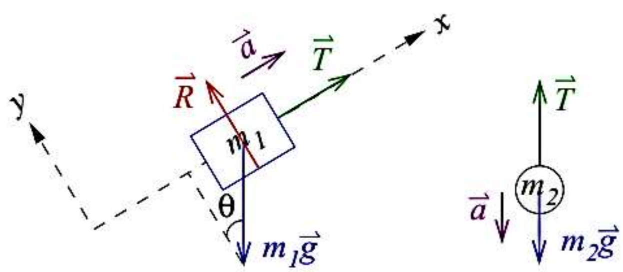
For the block m_1:

x-direction: $T - m_1 g \sin\theta = m_1 a$

$\Rightarrow \quad T = m_1(a + g\sin\theta)$

y-direction: $R - m_1 g\cos\theta = 0$

$\Rightarrow \quad R = m_1 g\cos\theta$



For the block m_2:

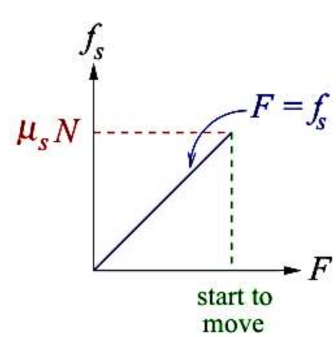
$$m_2 g - T = m_2 a \quad \Rightarrow \quad T = m_2(g - a)$$

Combining the two expressions of T, we obtain:

$$m_2(g-a) = m_1(a + g\sin\theta) \quad \Rightarrow \quad a = \frac{(m_2 - m_1\sin\theta)g}{m_1 + m_2}$$

5.2 Frictional force

5.2.1 Static friction



If a mass m is placed on a table with friction. We have experience that we must have large enough force to have it moved.

Before the mass is moved,

$$F = f_s$$

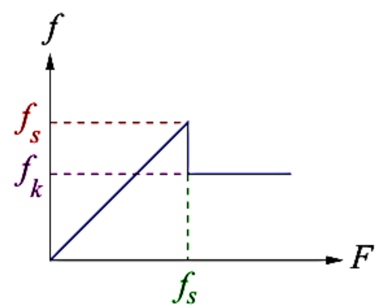
and they have opposite direction. Thus the net force on the mass is zero.
There exists a maximum static friction force $f_{s,\max}$ such that

$$f_s \le f_{s,\max} = \mu_s N$$

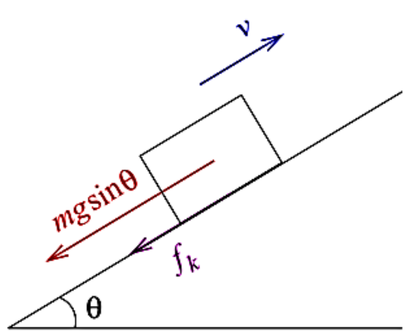
where μ_s is the coefficient of static friction.

5.2.2 Kinetic friction

As a mass m is moving on a frictional surface, there exists a frictional force, for which its direction is against the motion.
Its magnitude is always: $f_k = \mu_k N$.
N. B.: $\mu_k < \mu_s$ for the same surface.



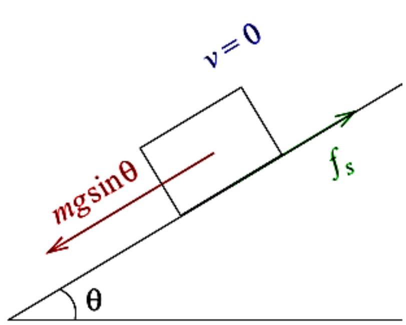
Examples:



$$-\mu_k mg\cos\theta - mg\sin\theta = ma$$

$$\Rightarrow \quad a = -\mu_k g\cos\theta - g\sin\theta$$

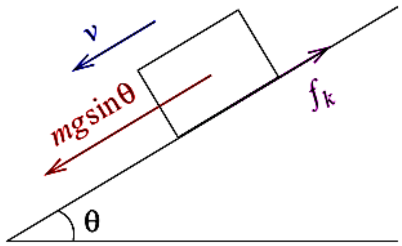
i. e. downwards with $\mu_k g\cos\theta + g\sin\theta$



As the mass come to rest, its subsequent motion depends on the coefficient of static friction μ_s:

a) if $mg\sin\theta < f_{s,\max} = \mu_s mg\cos\theta$ or $\mu_s > \tan\theta$, then it will continue stay at rest.

b) if $mg\sin\theta > f_{s,\max}$ or $\mu_s < \tan\theta$, it will slide down again.



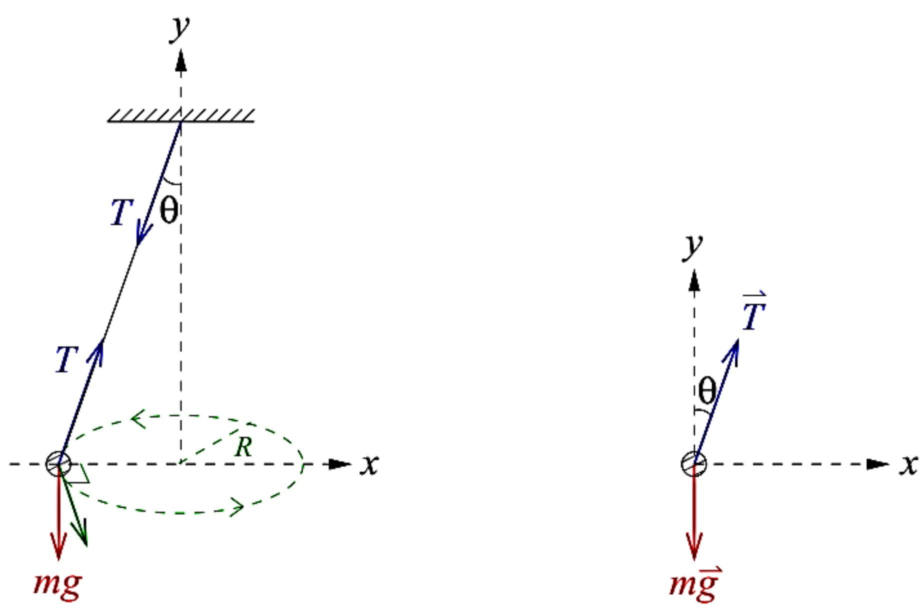
If $mg\sin\theta > f_{s,\max}$ and on its way down,

$$\mu_k mg\cos\theta - mg\sin\theta = ma$$

$$\Rightarrow \quad a = \mu_k g\cos\theta - g\sin\theta < 0, \text{accelerate.}$$

5.3 Some examples concerning uniform circular motion

a) Conical pendulum



y-direction:

$$T\cos\theta - mg = ma_y = 0 \qquad (\because \text{ no acceleration in } y\text{-direction})$$

$$\Rightarrow \qquad T = \frac{mg}{\cos\theta}$$

x-direction:

$$T\sin\theta = ma_x \;\Rightarrow\; a_x = T\frac{\sin\theta}{m} = \frac{mg}{\cos\theta}\frac{\sin\theta}{m} = g\tan\theta$$

N. B. a_x is pointing towards the circle center and is also perpendicular to $\vec{v}$. It is thus the centripetal acceleration for the uniform circular motion.

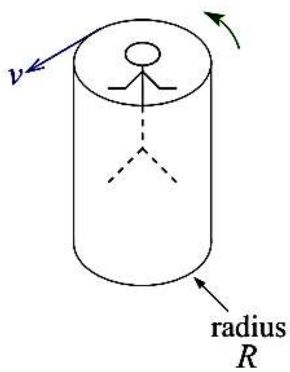
$$\therefore\; a_x = g\tan\theta = \frac{v^2}{R} \;\Rightarrow\; v = \sqrt{gR\tan\theta}$$

Period T is given by:

$$T = \frac{2\pi R}{v} = 2\pi\sqrt{\frac{R\cot\theta}{g}}$$

b) The rotor

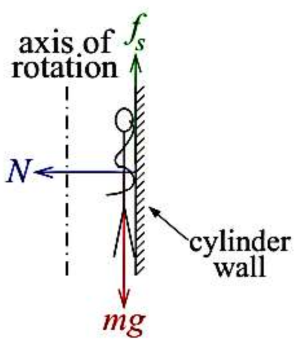
A hollow cylinder is rotating about its axis with uniform circular motion. A person with his hands back on the cylinder wall.



For the person to stick on the wall without falling down,

$$mg < f_{s,\max} = \mu_s N \tag{5.1}$$

The person is also under uniform circular motion and the normal reaction from the wall provides the required centrpetal acceleration.



$$\therefore\ N = \frac{mv^2}{R}$$

Thus, from (5.1),

$$mg < \mu_s \frac{mv^2}{R} \quad \text{or} \quad v > \sqrt{\frac{gR}{\mu_s}} \quad \text{i. e. } v_{\text{min}} = \sqrt{\frac{gR}{\mu_s}}$$

If the rotor spins too slow, the person will fall down.

c) A bicycle moving around a curve on a level road

The bicycle is moving with constant speed v.

$$N = mg$$

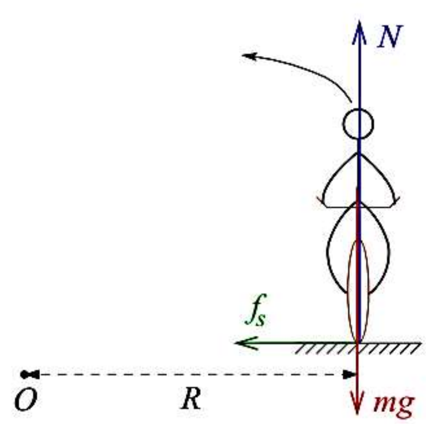
and the friction provides the required centrpetal force.

$$\therefore\ f_s = \frac{mv^2}{R}$$

For the bicycle not slipping,

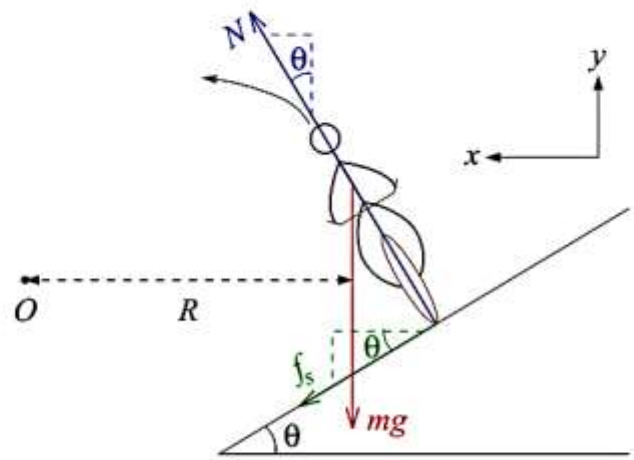
$$f_s = \frac{mv^2}{R} < f_{s,\max} = \mu_s N = \mu_s mg$$

$$\Rightarrow \quad v^2 < \mu_s Rg \quad \text{or} \quad v < \sqrt{\mu_s Rg}$$



i. e. if the bicycle runs too fast, it will slip.

d) A bicycle moving around a curve on a banked road



y-direction:

$$N\cos\theta - f_s \sin\theta - mg = 0 \;\Rightarrow\; N = \frac{mg + f_s \sin\theta}{\cos\theta} \tag{5.2}$$

x-direction:

$$N\sin\theta + f_s\cos\theta = ma_x = \frac{mv^2}{R} \tag{5.3}$$

N. B.: Even if there is no friction between the bicycle and the road, the bicycle can also turn around the curve without slipping, i. e.

$$N = \frac{mg}{\cos\theta} \qquad \& \qquad N\sin\theta = \frac{mv^2}{R}$$

$$\Rightarrow v^2 = \frac{R}{m}\frac{mg}{\cos\theta}\sin\theta = Rg\tan\theta$$

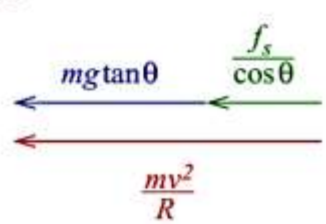
$$\Rightarrow v = \sqrt{Rg\tan\theta}$$

For a non-frictionless road and no slipping, (5.2) and (5.3) give

$$\frac{f_s}{\cos\theta} + mg\tan\theta = \frac{mv^2}{R}$$

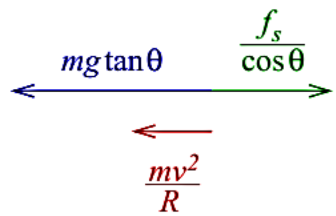
$$\Rightarrow f_s = \frac{mv^2}{R}\cos\theta - mg\sin\theta \tag{5.4}$$

Case I: $\frac{mv^2}{R}\cos\theta > mg\sin\theta$ or $v > \sqrt{Rg\tan\theta}$



$\frac{mv^2}{R}$ is large, both the friction term and the mg term contributes to the centripetal force.

Case II: $\dfrac{mv^2}{R}\cos\theta < mg\sin\theta$ or $v < \sqrt{Rg\tan\theta}$



The centripetal force required is smaller than the mg term and thus the frictional force has a direction against the mg term.

When will slipping occur?

Case I: $v > \sqrt{Rg\tan\theta}$ (Frictional force points downward)

v is too large such that the frictional force required to provide the centripetal force is larger than the maximum static friction $f_{s,\text{max}}$.
Or from (5.2):

$$N = \frac{mg + \mu_s N\sin\theta}{\cos\theta} \quad \Rightarrow \quad N = \frac{mg}{\cos\theta - \mu_s\sin\theta}$$

From (5.3):

$$N\sin\theta + \mu_s N\cos\theta = \frac{mv_{\text{max}}^2}{R}$$

$$\Rightarrow \left(\frac{mg}{\cos\theta - \mu_s\sin\theta}\right)(\sin\theta + \mu_s\cos\theta) = \frac{mv_{\text{max}}^2}{R}$$

$$\Rightarrow v_{\text{max}} = \sqrt{gR\left(\frac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\right)}$$

As $v > v_{\text{max}}$, the bicycle slips outward.

Case II: $v < \sqrt{Rg\tan\theta}$ (Frictional force points upward)

$\dfrac{f_s}{\cos\theta}$ is in opposite direction to that of $mg\tan\theta$, i. e.

$$|mg\tan\theta| - \left|\frac{f_s}{\cos\theta}\right| = \left|\frac{mv^2}{R}\right|$$

$\left|\dfrac{mv^2}{R}\right|$ may be so small such that

$$|mg\tan\theta| - \left|\frac{f_{s,\text{max}}}{\cos\theta}\right| = \left|\frac{mv_{\text{min}}^2}{R}\right|$$

Clearly, the bicycle slips inward when $v < v_{\min}$.

$$|mg\tan\theta| - \left|\frac{\mu_s N}{\cos\theta}\right| = \left|m\frac{v_{\min}^2}{R}\right|$$

But from (5.2):

$$N = \frac{mg - \mu_s N\sin\theta}{\cos\theta} \qquad \Rightarrow \; N = \frac{mg}{\cos\theta + \mu_s\sin\theta}$$

From (5.3):

$$N\sin\theta - \mu_s N\cos\theta = \frac{mv_{\min}^2}{R}$$

$$\Rightarrow \; \left(\frac{mg}{\cos\theta + \mu_s\sin\theta}\right)(\sin\theta - \mu_s\cos\theta) = m\frac{v_{\min}^2}{R}$$

$$\Rightarrow \; v_{\min} = \sqrt{gR\left(\frac{\sin\theta - \mu_s\cos\theta}{\cos\theta + \mu_s\sin\theta}\right)}$$

As $v < v_{\min}$, the bicycle slips inward.