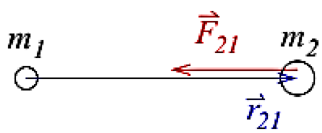
Chapter 14

Gravitation

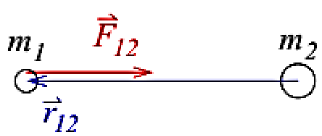
14.1 Newton's Law of Universal Gravitation



$$\vec{F}_{21} = -\frac{Gm_1m_2}{r_{21}^2}\hat{r}_{21}$$

$\vec{r}_{21}$: position of m_2 relative to m_1

$\vec{F}_{21}$: force experienced by m_2 due to m_1



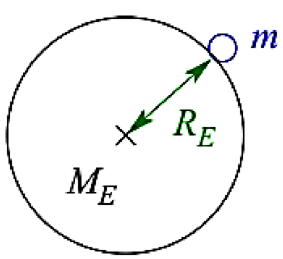
$$\vec{F}_{12} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12}$$

$\vec{r}_{12}$: position of m_1 relative to m_2

$\vec{F}_{12}$: force experienced by m_1 due to m_2

14.2 Gravitation near the Earth's surface

If we assume the Earth to be a stationary uniform sphere,



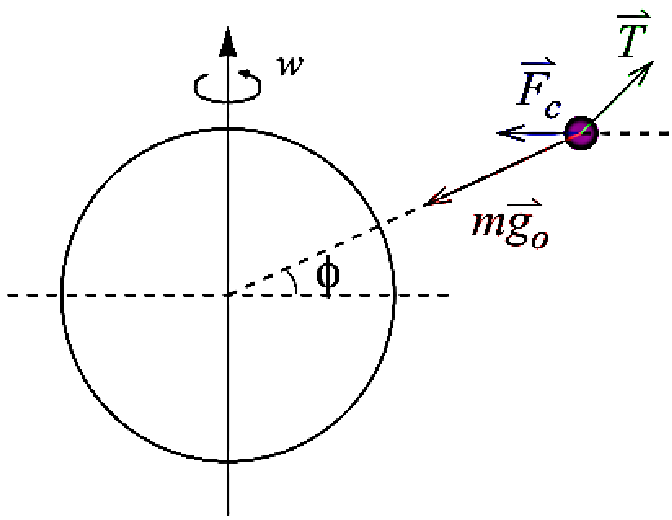
M_E = Mass of the Earth

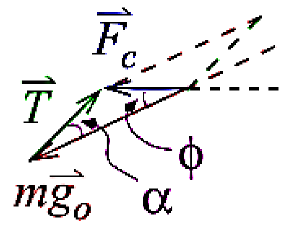
R_E = Radius of the Earth

Gravitation pull on the mass m:

$$F = \frac{GM_Em}{R_E^2}, \quad \text{i. e. } g_0 = \frac{GM_E}{R_E^2}$$

14.3 Effect of Earth's Rotation





Consider a mass m hanged by a tsring and the string is deviated from the true vertical line.

$$\vec{F}_c = m \underbrace{\vec{g}_0}_{\substack{\text{true}\\ \text{vertical}\\ \text{direction}}} + \vec{T}$$

Cosine Law:

$$T^2 = F_c^2 + (mg_0)^2 - 2F_c mg_0 \cos\phi$$

Note that $T = mg_{\text{eff}}$ where $g_{\text{eff}} =$ effective measured gravity.

For the case at the equator, $\phi = 0$.

$$T^2 = F_c^2 + (mg_0)^2 - 2F_c mg_0$$

$$|T| = mg_{\text{eff}} = mg_0 - m\omega^2 R_E \quad \Rightarrow \quad g_{\text{eff}}(\phi = 0) = g_0 - \omega^2 R_E$$

Let α be the angle between the string and the real vertical axis.
Using Sine Law and let R be the distance between the particle and the z-axis.

$$\frac{\sin\alpha}{F_c} = \frac{\sin(\pi - \alpha - \phi)}{mg_0}$$

$$\Rightarrow \quad \frac{g_0 \sin\alpha}{\omega^2 R} = \sin(\alpha + \phi)$$

$$\Rightarrow \quad \frac{g_0}{\omega^2 R} = \cos\phi + \cot\alpha \sin\phi$$
$$\Rightarrow \quad \frac{g_0}{\omega^2 R \sin\phi} = \cot\phi + \cot\alpha$$
$$\Rightarrow \quad \cot\alpha = \frac{g_0 - \omega^2 R\cos\phi}{\omega^2 R \sin\phi}$$

14.4 Gravitational force due to an uniform spherical shell

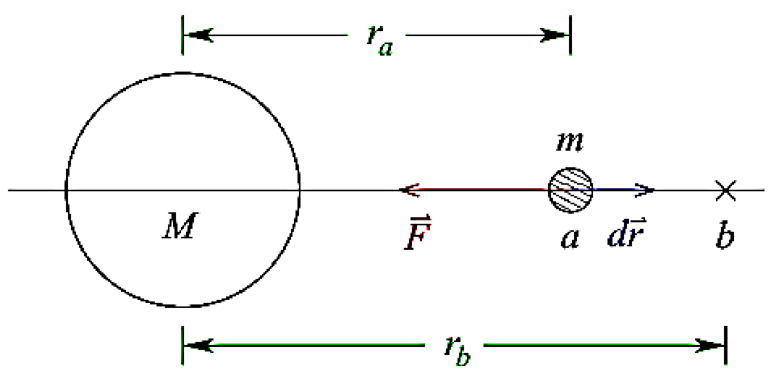
Theorem 1:

An uniform spherical shell attracts an external particle as if all the mass of the shell was concentrated at the center.

Theorem 2:

An uniform spherical shell exerts no force on a particle located inside the sphere.

14.5 Gravitational potential energy



Recall: $\Delta U = U_f - U_i = -W_{if}$.

Consider a mass m displaces from a to b.

$$W_{ab} = \int_a^b \vec{F}\cdot d\vec{r} = -\int_{r_a}^{r_b} \frac{GMm}{r^2} dr$$
$$= -GMm\left[-\frac{1}{r}\right]_{r_a}^{r_b}$$

$$= +GMm\left[\frac{1}{r_b}-\frac{1}{r_a}\right]$$

If $r_b > r_a$, $W_{ab} < 0$; if $r_b < r_a$, $W_{ab} > 0$.
Notice that W_{ab} is the work done by the gravitational force in bringing m from a to b.

$\therefore$ In bringing the mass m from a to b,

$$\Delta U = U_b - U_a = -W_{ab} = -GMm\left[\frac{1}{r_b}-\frac{1}{r_a}\right]$$

Now, we take $r_a = r$, $U_a = U(r)$ and $r_b \to \infty$, where $U(\infty) = 0$.

$$U(\infty) - U(r) = -GMm\left(-\frac{1}{r}\right)$$

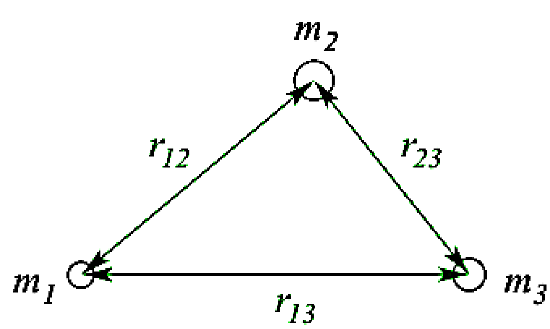
$$\therefore \quad U(r) = -\frac{GMm}{r} = W_{r\infty}$$

<u>Escape speed</u>

A particle having initial speed of v_0 is fixed on the surface of the Earth. For the particle to escape from the Earth's gravitational force field, it is energetic possible for it to travel to infinity.

$$\therefore \ \frac{1}{2}mv_{0,\min}^2 + \left(-\frac{GMm}{R_E}\right) = 0$$

14.6 Potential energy of many particle system

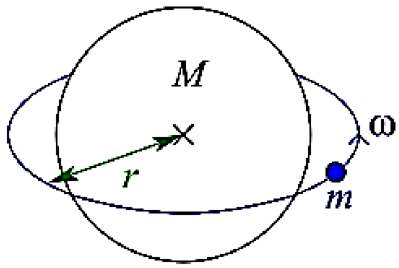


$$U = -\left(\frac{Gm_1m_2}{r_{12}}+\frac{Gm_1m_3}{r_{13}}+\frac{Gm_2m_3}{r_{23}}\right)$$

Energy required to take these three particles to separated infinity:

$$E = -U$$

14.7 Energy consideration of satellite motion



Consider a satellite orbiting a planet.

$$U = -\frac{GMm}{r}$$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(\omega r)^2 = \frac{1}{2}m\omega^2 r^2$$

If the gravitational force provides the centripetal force,

$$\frac{GMm}{r^2} = m\omega^2 r \quad \Rightarrow \quad \frac{GM}{r} = \omega^2 r^2$$

$$\therefore \quad K = \frac{1}{2}\frac{GMm}{r}$$

$$\therefore \quad E = K + U = \frac{1}{2}\frac{GMm}{r} - \frac{GMm}{r} = -\frac{GMm}{2r}$$