Chapter 12

Potential Energy

12.1 Conservative force

Potential energy is only defined for conservative force in which a particle moving under the force influence has constant mechanical energy.

Examples of conservative force:

1) Spring
2) Gravitational force
3) Coulomb force

Example of non-conservative force - friction.

Question: Any rigorous definition for conservative force?

Definition:

A conservative force is a force such that if a particle moves under the influence of this force, the work done by the force on displaying the particle from an arbitrary point A to another arbitrary point B would be the same along any arbitrarily chosen path.

Note that:

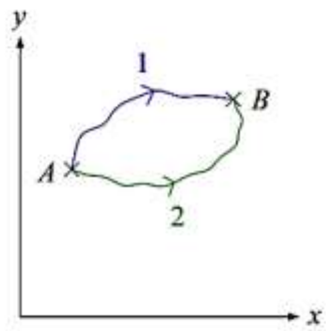
1) $\vec{F}(\vec{r})$ is conserved if and only if there exists a scalar function $\phi(\vec{r})$ such that

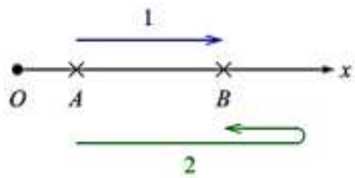
$$\nabla\phi(\vec{r}) = \vec{F}(\vec{r})$$

2)
$$\vec{\nabla} \times \vec{F} = 0$$

Work done of closed path (i. e.starting and ending at the same point) is zero.

Proof:





$$\int_{\text{path1}} \vec{F} \cdot d\vec{r} = \int_{\text{path2}} \vec{F} \cdot d\vec{r}$$

$\therefore$ Travelling from point A to B, then back to A, the work done is:

$$W_{A \to B \to A} = W_{A \to B} + W_{B \to A} = \int_{\text{path1}} \vec{F} \cdot d\vec{r} + \left(-\int_{\text{path2}} \vec{F} \cdot d\vec{r} \right) = 0$$

12.2 Potential Energy

Consider a particle moves in the influence of a conservative force, which is position dependent, i. e. $F(x)$. Now the particle displaces from x_i to x_f, potential difference ΔU is defined:

$$\Delta U = U_f - U_i = -W$$

where W is the work done by the force during the displacement x_i to x_f.

Or $$\boxed{\Delta U = U(x_f) - U(x_i) = -\int_{x_i}^{x_f} F(x)dx}$$

If for a particle reference point x_0, the potential energy is defined as zero, i. e. $U(x_0) \stackrel{\text{def}}{=} 0$.

$$\boxed{U(x) = -\int_{x_0}^{x} F(x)dx}$$

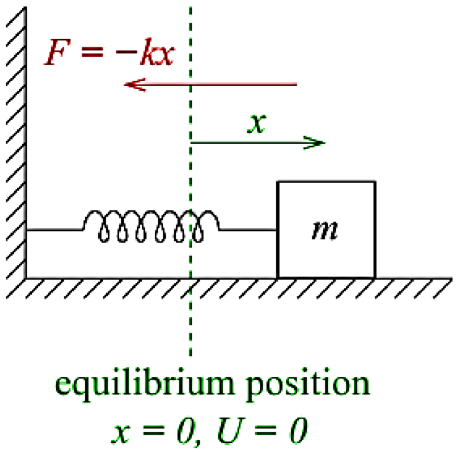
In particular,

$$U(x) - U(0) = -\int_{0}^{x} F(x)dx$$

$$\therefore \quad \frac{d}{dx}\left[U(x) - U(0)\right] = -\frac{d}{dx}\int_{0}^{x} F(x)dx$$

$$\Rightarrow \quad \boxed{\frac{dU}{dx} = -F(x)}$$

Spring



$$F = -kx$$

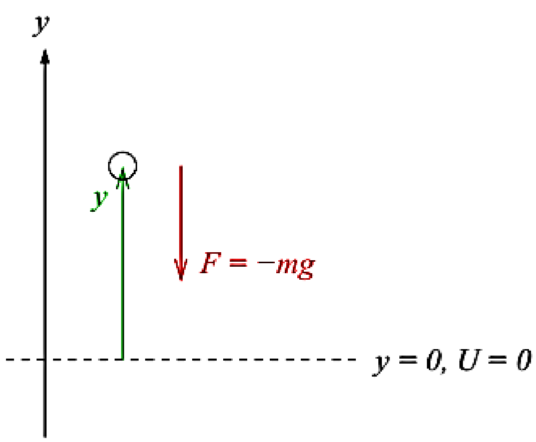
Take the equilibrium position to be $x = 0$ so that $U(0) = 0$.

$$\therefore\ U(x) - U(0) = -\int_0^x F(x)dx$$
$$\Rightarrow\ U(x) = -\int_0^x (-kx)dx$$
$$\Rightarrow\ U(x) = \frac{1}{2}kx^2$$

Thus

$$\frac{dU}{dx} = \frac{1}{2}k(2x) = kx = -F$$

Force of gravity



Take $U(0) = 0$.

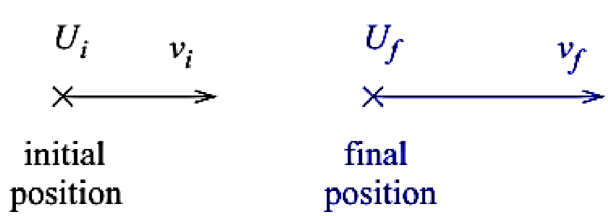
$$\therefore\ U(y) - U(0) = -\int_0^y F(y)dy$$
$$\Rightarrow\ U(y) = -\int_0^x (-mg)dy$$
$$\Rightarrow\ U(y) = mgy$$

Thus

$$\frac{dU}{dy} = mg = -F$$

12.3 Conservation of Mechanical Energy

$$\Delta U = U_f - U_i = -W \tag{11.5}$$



But $W = \int_{x_i}^{x_f} F(x)dx$ is the work done by the force in the journey from $x_i \to x_f$.
From previous chapter,

$$W = \int_{x_i}^{x_f} F(x)dx = \frac{1}{2}m(v_f^2 - v_i^2) = K_f - K_i = \Delta K \qquad (11.6)$$

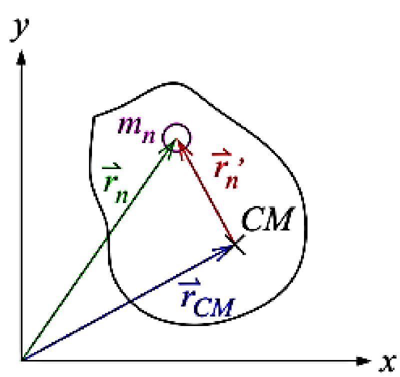
Substitute (11.6) into (11.5), we have

$$U_f - U_i = K_i - K_f$$
$$\Rightarrow \quad U_i + K_i = U_f + K_f$$
$$\Rightarrow \quad \Delta U = -\Delta K$$

In an isolating system whose only conservative force exists, mechanical energy of a particle conserves.

Revisiting combined rotational and translational motion

In previous sections, we have considered cases of pure translational (i. e. movement of C.M. of rigid body) or pure rotational (about a fixed axis) motion. Now we turn into case such that both the CM is moving and the rigid body is rotating.



Consider a rigid body consisted of particles $m_1, m_2, \ldots, m_N$.
Total K.E. of the body:

$$K = \frac{1}{2}\sum_i m_i v_i^2 \qquad (11.7)$$

Note that

$$\vec{r}_i = \vec{r}_{\rm CM} + \vec{r}_i' \quad \Rightarrow \quad \vec{v}_i = \vec{v}_{\rm CM} + \vec{v}_i'$$

where $\vec{v}_i$ = velocity of mass i with respect to the Earth's frame,
$\vec{v}_{\rm CM}$ = velocity of the body's center of mass with respect to the Earth's frame,
$\vec{v}_i'$ = velocity of mass i with respect to the body's center of mass.

From (11.7), we obtain

$$K = \frac{1}{2}\sum_i m_i(\vec{v}_{\rm CM} + \vec{v}_i')\cdot(\vec{v}_{\rm CM} + \vec{v}_i') = \frac{1}{2}\sum_i m_i(v_{\rm CM}^2 + 2\vec{v}_{\rm CM}\cdot\vec{v}_i' + {v_i'}^2)$$

But consider the second term:

$$\sum_i \vec{v}_{\rm CM}\cdot(m_i\vec{v}_i') = \sum_i \vec{v}_{\rm CM}\cdot(m_i\vec{v}_i - m_i\vec{v}_{\rm CM}) = \vec{v}_{\rm CM}\cdot\sum_i m_i\vec{v}_i - Mv_{\rm CM}^2$$

As $\vec{v}_{\rm CM} = (\sum_i m_i\vec{v}_i)/M$,

$$\therefore \sum_i \vec{v}_{\rm CM}\cdot(m_i\vec{v}_i') = \vec{v}_{\rm CM}\cdot M\vec{v}_{\rm CM} - Mv_{\rm CM}^2 = 0$$

And the third term:

$$\frac{1}{2}\sum_i m_i v_i'^2 = \frac{1}{2}\sum_i m_i(r_i'\omega)^2 = \frac{1}{2}I\omega^2$$

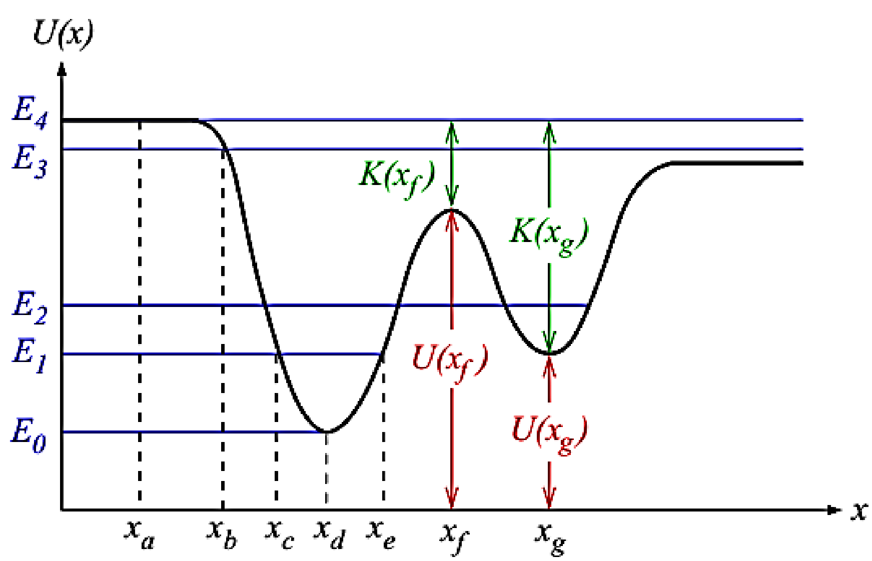
where ω is the angular velocity about an axis passing through the center of mass.

$$\therefore \boxed{K = \frac{1}{2}Mv_{\rm CM}^2 + \frac{1}{2}I\omega^2}$$

1st term: Translational term of the C.M. as if there is no rotation.

2nd term: Rotational term with rotation about the axis passing through the C.M. as if the rotational axis does not move.

12.4 One dimensional conservative system



- Particle experienced a conservative force field with potential energy $U(x)$.
- $$\boxed{F(x) = -\frac{dU}{dx}}$$

$\therefore$ At $x = x_a, x_d, x_f, x_g$, $F = 0$.

$x = x_d, x_g$: stable equilibrium - slightly displaced particle experiences a restoring force

$x = x_f$: unstable equilibrium - displaced particle experiences a force in the same direction as displacement

$x = x_a$: neutral equilibrium - displaced particle experiences no force

- $U(x) + \frac{1}{2}mv^2 = E$, where E is the conserved total energy.

  Example

  If $E = E_4$ as shown in the previous figure,

  $$E_4 = K(x_g) + U(x_g) \text{ at } x = x_g$$
  $$E_4 = K(x_f) + U(x_f) \text{ at } x = x_f$$

  If the energy of the particle E is different, it will have different behavior as follows:

  1) If $E = E_0$, particle stays stationary at $x = x_d$.
  2) If $E = E_1$, particle stays in the region $x = x_c \to x_e$.
  3) If $E = E_2$, particle may stay in the two valleys. However if it is in one of the valley, it does not have enough energy to go to the other valley.
  4) If $E = E_3$, particle can stay in the region $x > x_b$.
  5) If $E \geq E_4$, particle can be anywhere.

- If $U(x)$ is known, it is possible to work out the particle position.

  Example

  If at $t = 0$, $x(t = 0) = x_0$ and $v(t = 0) = 0$. Suppose $U(x) = \frac{1}{2}kx^2$.

  $$\therefore\ E = \frac{1}{2}k[x(0)]^2 + \frac{1}{2}m[v(0)]^2 = \frac{1}{2}kx_0^2 = \text{constant}$$

At time t,

$$U(x)+\frac{1}{2}mv^2=\frac{1}{2}kx_0^2$$

$$\Rightarrow \quad \frac{1}{2}kx^2+\frac{1}{2}mv^2=\frac{1}{2}kx_0^2, \qquad \text{where } x=x(t) \text{ and } v=v(t)$$

$$\Rightarrow \quad v=\frac{dx}{dt}=\pm\sqrt{\frac{k}{m}x_0^2-\frac{k}{m}x^2}$$

$$\Rightarrow \quad dt=\pm\sqrt{\frac{m}{k}}\frac{dx}{\sqrt{x_0^2-x^2}}$$

$$\Rightarrow \quad t=\pm\int_{x_0}^{x}\sqrt{\frac{m}{k}}\frac{dx}{\sqrt{x_0^2-x^2}}$$

To solve $\int \frac{dx}{\sqrt{x_0^2-x^2}}$,

let $x=x_0\sin\theta \quad \Rightarrow \quad dx=x_0\cos\theta d\theta$.

$$\therefore \int\frac{dx}{\sqrt{x_0^2-x^2}}=\int\frac{x_0\cos\theta d\theta}{\sqrt{x_0^2-x_0^2\sin^2\theta}}=\int\frac{\cos\theta}{\cos\theta}d\theta=\theta=\sin^{-1}\left(\frac{x}{x_0}\right)$$

$$\therefore t=\pm\sqrt{\frac{m}{k}}\left[\sin^{-1}\left(\frac{x}{x_0}\right)-\sin^{-1}\left(\frac{x_0}{x_0}\right)\right]=\pm\sqrt{\frac{m}{k}}\left[\sin^{-1}\left(\frac{x}{x_0}\right)-\frac{\pi}{2}\right]$$

If $t=+\sqrt{\frac{m}{k}}\left[\sin^{-1}\left(\frac{x}{x_0}\right)-\frac{\pi}{2}\right]$, then

$$\cos\left(\sqrt{\frac{k}{m}}t\right)=\cos\left[\sin^{-1}\left(\frac{x}{x_0}\right)-\frac{\pi}{2}\right]$$

$$\Rightarrow \quad \cos\left(\sqrt{\frac{k}{m}}t\right)=\cos\left[\sin^{-1}\left(\frac{x}{x_0}\right)\right]\cos\frac{\pi}{2}+\frac{x}{x_0}\sin\frac{\pi}{2}=\frac{x}{x_0}$$

$$\Rightarrow \quad x=x_0\cos\left(\sqrt{\frac{k}{m}}t\right)$$

If $t=-\sqrt{\frac{m}{k}}\left[\sin^{-1}\left(\frac{x}{x_0}\right)-\frac{\pi}{2}\right]$, then

$$\cos\left(\sqrt{\frac{k}{m}}t\right)=\cos\left[\frac{\pi}{2}-\sin^{-1}\left(\frac{x}{x_0}\right)\right]$$

$$\Rightarrow \quad \cos\left(\sqrt{\frac{k}{m}}t\right)=\cos\frac{\pi}{2}\cos\left[\sin^{-1}\left(\frac{x}{x_0}\right)\right]+\left(\sin\frac{\pi}{2}\right)\frac{x}{x_0}=\frac{x}{x_0}$$

$$\Rightarrow \quad x=x_0\cos\left(\sqrt{\frac{k}{m}}t\right)$$