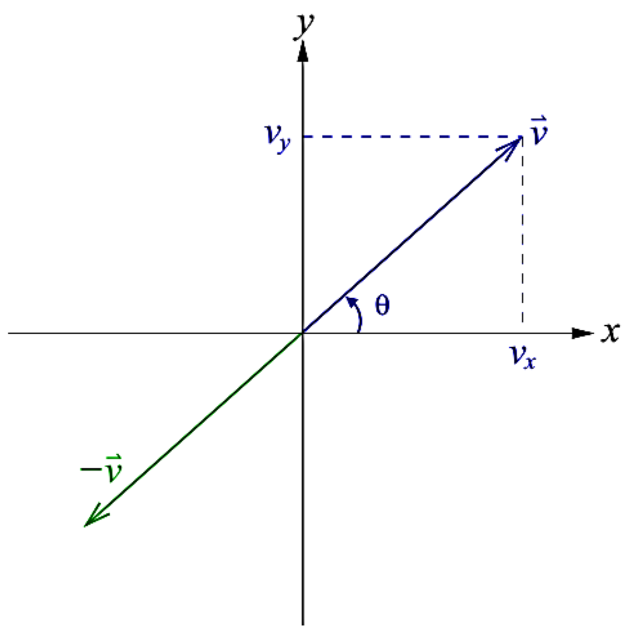
Chapter 2

Motion in One Dimension

2.1 Vector

a) Definition

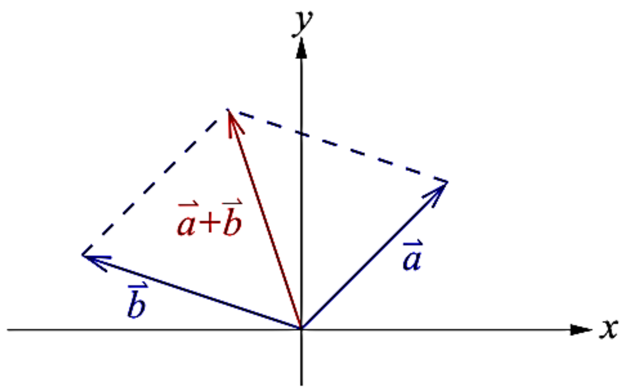
A vector has both magnitude and direction. It is represented by an arrow.



v_x is called the x-component of vector $\vec{v}$ and v_y is called the y-component of vector $\vec{v}$. The magnitude of the vector $\vec{v}$ is denoted by $|\vec{v}|$ or v where

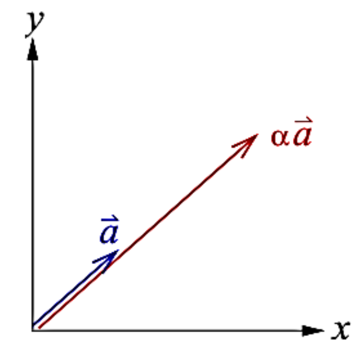
$$|\vec{v}| = v = \sqrt{v_x^2 + v_y^2}$$

b) Addition of vector



c) Multiplication of vector by a scalar

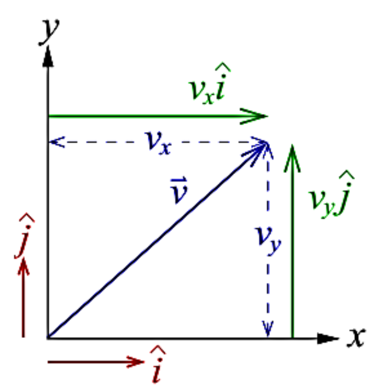
$\alpha\vec{a}$ has the same direction as $\vec{a}$ but has a magnitude equal to α times the magnitude of $\vec{a}$, i. e. $|\alpha\vec{a}| = |\alpha||\vec{a}|$.



d) Component form of a vector

$\hat{\mathrm{i}}$ and $\hat{\mathrm{j}}$ are called the unit vector of x and y directions which have magnitude of unity, i. e. $|\hat{\mathrm{i}}| = |\hat{\mathrm{j}}| = 1$.
Thus, $\vec{v} = v_x\hat{\mathrm{i}} + v_y\hat{\mathrm{j}}$.
With component form,

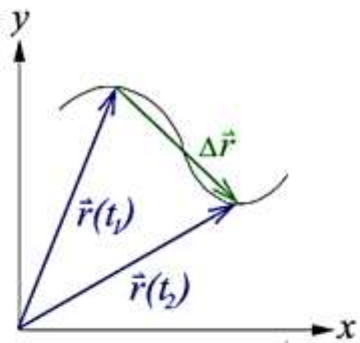
$$\vec{u} = u_x\hat{\mathrm{i}} + u_y\hat{\mathrm{j}}, \quad \vec{v} = v_x\hat{\mathrm{i}} + v_y\hat{\mathrm{j}}$$
$$\vec{u} + \vec{v} = (u_x + v_x)\hat{\mathrm{i}} + (u_y + v_y)\hat{\mathrm{j}}$$
$$\alpha\vec{u} = \alpha u_x\hat{\mathrm{i}} + \alpha u_y\hat{\mathrm{j}}$$



2.2 Position, velocity and acceleration vectors

Position vector

- vector is usually used to describe the position of a particle at time t.



$$\vec{r}(t) = r_x(t)\hat{\mathrm{i}} + r_y(t)\hat{\mathrm{j}}$$
$$\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_2)$$

↑
displacement
vector

Average velocity in time period $t_1 \to t_2$:

$$\vec{v}_{\text{ave}} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1}$$

If the time interval Δt is infinitesimal small, i. e. $\Delta t \to 0$, the instantaneous velocity at time t_1:

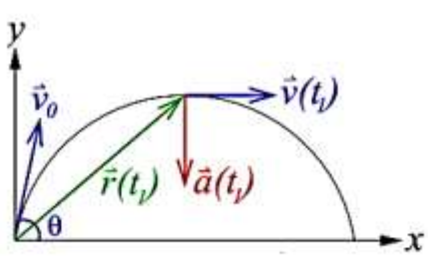
$$\vec{v}_{\text{inst}} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}(t)}{dt} \text{ (tangential to curve } \vec{r}(t))$$

Likewise, the average acceleration in time period $t_1 \to t_2$:

$$\vec{a}_{\text{ave}} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}(t_2) - \vec{v}(t_1)}{t_2 - t_1}$$
$$\vec{a}_{\text{inst}} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}(t)}{dt} = \frac{d^2\vec{r}(t)}{dt^2}$$

If we talk about velocity or acceleration, we are usually talking about instantaneous velocity and acceleration.

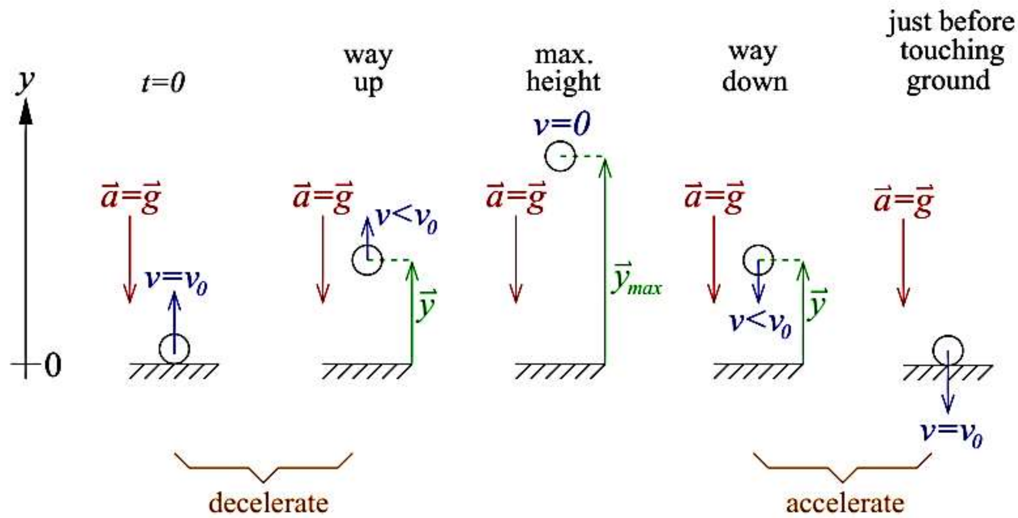
E. g. Projectile motion



$$\vec{v}_0 = v_0 \cos\theta\hat{\mathrm{i}} + v_0 \sin\theta\hat{\mathrm{j}}$$
$$\vec{r}(t) = (v_0 \cos\theta)t\hat{\mathrm{i}} + [(v_0 \sin\theta)t - \frac{1}{2}gt^2]\hat{\mathrm{j}}$$
$$\vec{v}(t) = (v_0 \cos\theta)\hat{\mathrm{i}} + [(v_0 \sin\theta) - gt]\hat{\mathrm{j}}$$
$$\vec{a}(t) = -g\hat{\mathrm{j}}$$

2.3 One dimensional motion

An example: Throwing a stone towards the sky vertically with a speed of v_0.



Take upward as positive y direction.

$$\therefore \quad a = \frac{dv}{dt} = \frac{d^2y}{dt^2} = -g$$
$$v = \frac{dy}{dt} = -\int g dt = -gt + A, \quad A = \text{constant}$$

To determine A, substitute the initial condition at $t = 0$,

$$\frac{dy}{dt}|_{t=0} = v(t=0) = A = +v_0$$
$$\therefore \quad v = \frac{dy}{dt} = v_0 - gt \tag{2.1}$$
$$y = \int (v_0 - gt)dt = v_o t - \frac{1}{2}gt^2 + B, \quad B = \text{constant}$$

To determine B, substitute $t = 0$ again,

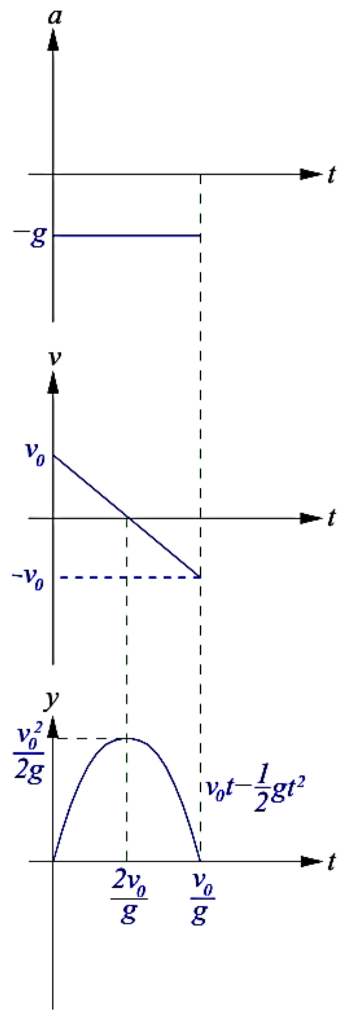
$$y(t=0) = B = 0$$
$$\therefore \quad y = v_0 t - \frac{1}{2}gt^2 \tag{2.2}$$

At maximum height, $\frac{dy}{dt}|_{t=t_{\max}} = 0$, i. e. $v_0 - gt_{\max} = 0 \Rightarrow t_{\max} = \frac{v_0}{g}$.

$$\therefore y_{\max} = y(t_{\max}) = \frac{v_0^2}{g} - \frac{1}{2}g\frac{v_0^2}{g^2} = \frac{1}{2}\frac{v_0^2}{g}$$

Time for touching the ground again, say t_0,

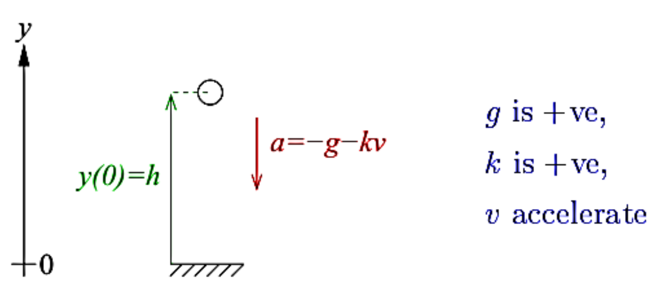
$$y = v_0 t_0 - \frac{1}{2}gt_0^2 = 0 \ \Rightarrow \ \underbrace{t_0 = 0}_{\text{initial at ground}} \quad \text{or} \ \ v_0 = \frac{1}{2}gt_0 \ (\text{i. e. } t_0 = \frac{2v_0}{g})$$



2.4 Another example: Non-constant acceleration

Dropping a stone at a height of h with the consideration of the air resistance.

Given: Air drag acceleration $= -kv$, $k =$ positive constant



$$a = -g - kv$$

$$\Rightarrow \quad \frac{dv}{dt} = -g - kv$$

$$\Rightarrow \quad \frac{dv}{-g-kv} = dt$$

$$\left(\because \frac{dy(t)}{dt} = f(y) \Rightarrow \frac{dy(t)}{f(y)} = dt\right)$$

$$\Rightarrow \int \frac{dv}{-g-kv} = \int dt + A, \qquad A = \text{constant}$$

$$\Rightarrow \quad -\frac{1}{k}\ln(-g-kv) = t + A$$

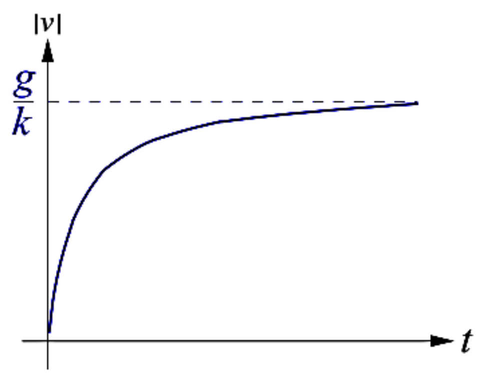
$$\Rightarrow \quad -g-kv = e^{-k(t+A)} = Be^{-kt}, \quad \text{where } B = e^{-kA}$$

$$\Rightarrow \quad v = \frac{1}{k}(-g - Be^{-kt})$$

$$\text{At } t=0,\ v(0)=0 \Rightarrow -g-B=0 \Rightarrow B=-g$$

$$\therefore \quad v = \frac{1}{k}(-g+ge^{-kt}) = -\frac{g}{k}(1-e^{-kt}),$$

$$\text{i. e. } \frac{dy}{dt} = -\frac{g}{k}(1-e^{-kt})$$



$$y = -\int \frac{g}{k}(1-e^{-kt})dt + C, \qquad C = \text{constant}$$

$$= -\frac{g}{k}t - \frac{g}{k^2}e^{-kt} + C$$

$$\text{At } t=0,\ y=h \Rightarrow h = -\frac{g}{k^2} + C \Rightarrow C = h + \frac{g}{k^2}$$

$$\therefore y = -\frac{g}{k}t - \frac{g}{k^2}e^{-kt} + h + \frac{g}{k^2}$$

$$= h - \frac{g}{k}t + \frac{g}{k^2}(1-e^{-kt})$$