"HEAT EQUATION. CRANK-NICOLSON METHOD"

Question : The one − dimension heat equation ut = c ^ 2 uxx is a parabolic equation that governs, for instance, the heat flow in a bar where u (x, t) is the temperature at a point x and time t. Solve the corresponding difference equation with c ^ 2 = 1 on the interval 0 ≤ x ≤ 1 (the bar extending from x = 0 to x = 1 along the x − axis) subject to initial temperature u (x, 0) = sin (pix) by the Crank − Nicolson method with x − step h = 0.2 and time step k = 0.04 doing 5 time steps.

```
In[1]:= ClearAll["Global`*"]

In[2]:= n = 4; r = 1; h = 0.2;

In[3]:= A = Table[Switch[j - k, 0 , 4 , 1, -1, -1, -1, _, 0], {j, n}, {k, n}];

In[4]:= MatrixForm[A]
```

Out[4]//MatrixForm=

$$\begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}$$

```
In[5]:= Do[u[k] = N[Sin[Pi k h]], {k, 0, n + 1}]

In[6]:= Table[u[k], {k, 0, n + 1}]
```

Out[6]= $\{0., 0.587785, 0.951057, 0.951057, 0.587785, 1.22465 \times 10^{-16}\}$

```
In[7]:= T0 = Table[{0.2 k, u[k]}, {k, 0, n + 1}]
```

Out[7]= $\{\{0., 0.\}, \{0.2, 0.587785\}, \{0.4, 0.951057\}, \{0.6, 0.951057\}, \{0.8, 0.587785\}, \{1., 1.22465 \times 10^{-16}\}\}$

```
In[8]:= Table[Temp[k], {k, 1, n + 1}];

In[9]:= M = 5;

In[10]:= Do[
    b = Table[0, {i, 1, n}];
    Do[b[[k]] = u[k - 1] + u[k + 1], {k, 1, n}];
    v = N[LinearSolve[A, b]];
        Print[v]; Temp[j] = v;
     Do[u[k] = v[[k]], {k, 1, n}],
     {j, 1, M}
     ];
```

```
{0.399274, 0.646039, 0.646039, 0.399274}

{0.271221, 0.438844, 0.438844, 0.271221}

{0.184236, 0.2981, 0.2981, 0.184236}

{0.125149, 0.202495, 0.202495, 0.125149}

{0.0850118, 0.137552, 0.137552, 0.0850118}
```

```
In[11]:= v = Table[Table[Temp[k][[j]], {j, 1, M - 1}], {k, 1, n + 1}]

Out[11]= {{0.399274, 0.646039, 0.646039, 0.399274},
  {0.271221, 0.438844, 0.438844, 0.271221}, {0.184236, 0.2981, 0.2981, 0.184236},
  {0.125149, 0.202495, 0.202495, 0.125149}, {0.0850118, 0.137552, 0.137552, 0.0850118}}

In[12]:= v[[3]]

Out[12]= {0.184236, 0.2981, 0.2981, 0.184236}

In[13]:= v[[3, 2]]

Out[13]= 0.2981

In[14]:= w = Table[Join[{0}, v[[j]], {0}], {j, 1, M}]

Out[14]= {{0, 0.399274, 0.646039, 0.646039, 0.399274, 0},
  {0, 0.271221, 0.438844, 0.438844, 0.271221, 0},
  {0, 0.184236, 0.2981, 0.2981, 0.184236, 0}, {0, 0.125149, 0.202495, 0.202495, 0.125149, 0},
  {0, 0.0850118, 0.137552, 0.137552, 0.0850118, 0}}

In[15]:= w[[1]]

Out[15]= {0, 0.399274, 0.646039, 0.646039, 0.399274, 0}

In[16]:= T = Table[Table[{0.2 i, w[[p]][[i + 1]]}, {i, 0, n + 1}], {p, 1, M}]

Out[16]= {{{0., 0}, {0.2, 0.399274}, {0.4, 0.646039}, {0.6, 0.646039}, {0.8, 0.399274}, {1., 0}},
  {{0., 0}, {0.2, 0.271221}, {0.4, 0.438844}, {0.6, 0.438844}, {0.8, 0.271221}, {1., 0}},
  {{0., 0}, {0.2, 0.184236}, {0.4, 0.2981}, {0.6, 0.2981}, {0.8, 0.184236}, {1., 0}},
  {{0., 0}, {0.2, 0.125149}, {0.4, 0.202495}, {0.6, 0.202495}, {0.8, 0.125149}, {1., 0}},
  {{0., 0}, {0.2, 0.0850118}, {0.4, 0.137552}, {0.6, 0.137552}, {0.8, 0.0850118}, {1., 0}}}

In[17]:= T[[1]]

Out[17]= {{0., 0}, {0.2, 0.399274}, {0.4, 0.646039}, {0.6, 0.646039}, {0.8, 0.399274}, {1., 0}}

In[18]:= x = Table[x, {x, 0, 1, 0.2}]

Out[18]= {0., 0.2, 0.4, 0.6, 0.8, 1.}

In[19]:= U = Table[Sin[Pi k h], {k, 0, n + 1}]
```

Out[19]= $\{0., 0.587785, 0.951057, 0.951057, 0.587785, 1.22465\times10^{-16}\}$

```
In[20]:= W = Transpose[w]

Out[20]= {{0, 0, 0, 0, 0}, {0.399274, 0.271221, 0.184236, 0.125149, 0.0850118},
  {0.646039, 0.438844, 0.2981, 0.202495, 0.137552},
  {0.646039, 0.438844, 0.2981, 0.202495, 0.137552},
  {0.399274, 0.271221, 0.184236, 0.125149, 0.0850118}, {0, 0, 0, 0, 0}}
```

In[21]:=
```
table =
 Table[{x[[l]], U[[l]], W[[l, 1]], W[[l, 2]], W[[l, 3]], W[[l, 4]], W[[l, 5]]}, {l, 1, 6}]
```

Out[21]=
```
{{0., 0., 0, 0, 0, 0, 0}, {0.2, 0.587785, 0.399274, 0.271221, 0.184236, 0.125149, 0.0850118},
 {0.4, 0.951057, 0.646039, 0.438844, 0.2981, 0.202495, 0.137552},
 {0.6, 0.951057, 0.646039, 0.438844, 0.2981, 0.202495, 0.137552},
 {0.8, 0.587785, 0.399274, 0.271221, 0.184236, 0.125149, 0.0850118},
 {1., 1.22465 × 10^-16, 0, 0, 0, 0, 0}}
```

In[22]:=
```
table1 = Prepend[table, {"x", "u(x,t),t=0", "u(x,t),t=0.04",
   "u(x,t),t=0.08", "u(x,t),t=0.12", "u(x,t),t=0.16", "u(x,t),t=0.20"}]
```

Out[22]=
```
{{x, u(x,t),t=0, u(x,t),t=0.04, u(x,t),t=0.08,
  u(x,t),t=0.12, u(x,t),t=0.16, u(x,t),t=0.20}, {0., 0., 0, 0, 0, 0, 0},
 {0.2, 0.587785, 0.399274, 0.271221, 0.184236, 0.125149, 0.0850118},
 {0.4, 0.951057, 0.646039, 0.438844, 0.2981, 0.202495, 0.137552},
 {0.6, 0.951057, 0.646039, 0.438844, 0.2981, 0.202495, 0.137552},
 {0.8, 0.587785, 0.399274, 0.271221, 0.184236, 0.125149, 0.0850118},
 {1., 1.22465 × 10^-16, 0, 0, 0, 0, 0}}
```

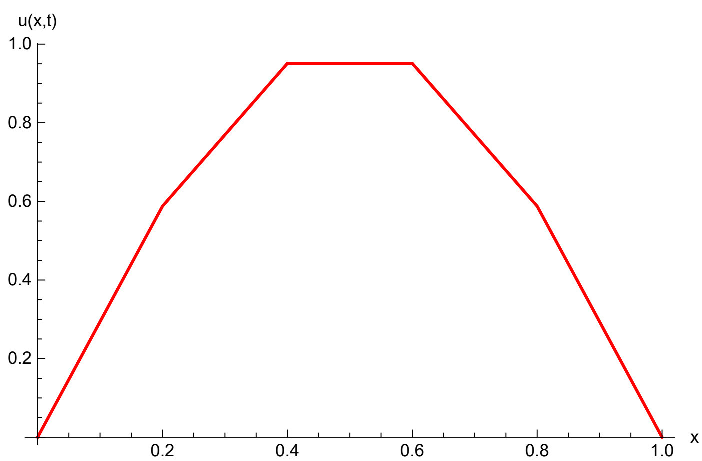
In[23]:=
```
Grid[table1, Frame → All, Spacings → {1, 1}]
```

Out[23]=

x	u(x,t),t=0	u(x,t),t=0.04	u(x,t),t=0.08	u(x,t),t=0.12	u(x,t),t=0.16	u(x,t),t=0.20
0.	0.	0	0	0	0	0
0.2	0.587785	0.399274	0.271221	0.184236	0.125149	0.0850118
0.4	0.951057	0.646039	0.438844	0.2981	0.202495	0.137552
0.6	0.951057	0.646039	0.438844	0.2981	0.202495	0.137552
0.8	0.587785	0.399274	0.271221	0.184236	0.125149	0.0850118
1.	1.22465×10^{-16}	0	0	0	0	0

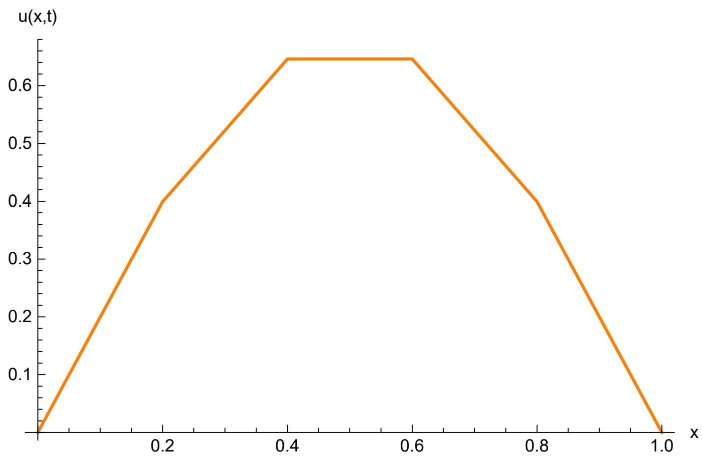
In[24]:=
```
P0 = ListLinePlot[T0, PlotStyle → Red, AxesLabel → {"x", "u(x,t)"}]
```

Out[24]=



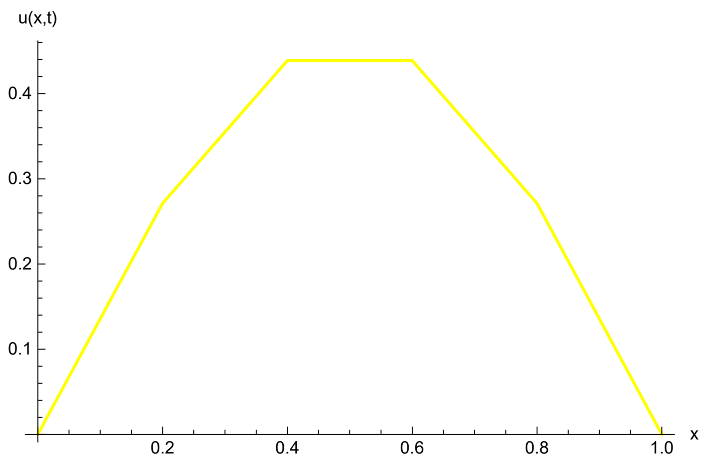
In[25]:=
```
P1 = ListLinePlot[T[[1]], PlotStyle → Orange, AxesLabel → {"x", "u(x,t)"}]
```

Out[25]=



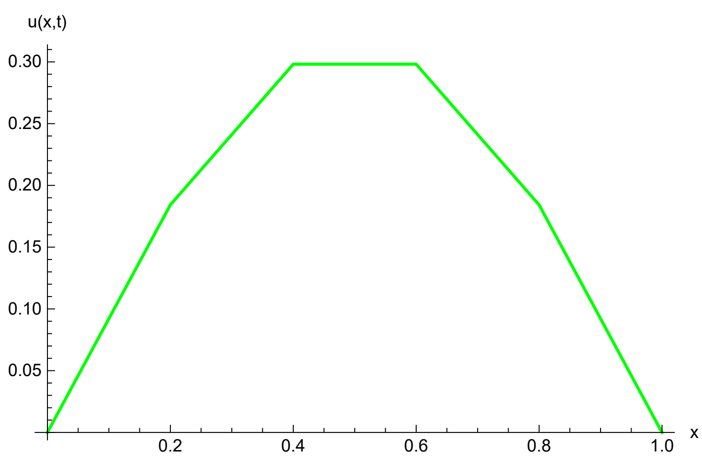
In[26]:=
```
P2 = ListLinePlot[T[[2]], PlotStyle → Yellow, AxesLabel → {"x", "u(x,t)"}]
```

Out[26]=



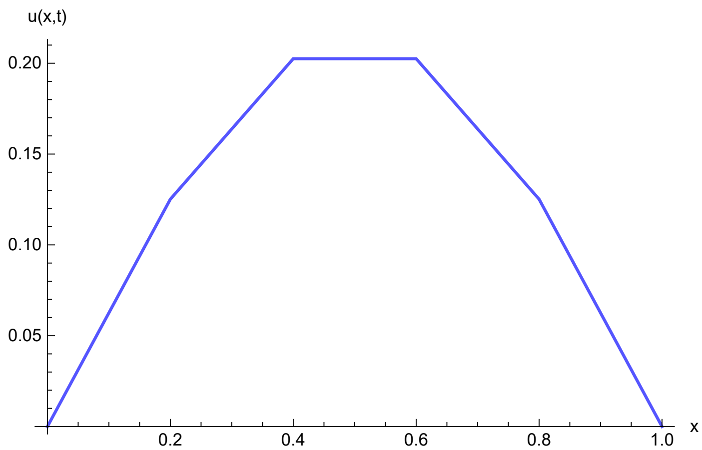
In[27]:=
```
P3 = ListLinePlot[T[[3]], PlotStyle → Green, AxesLabel → {"x", "u(x,t)"}]
```

Out[27]=



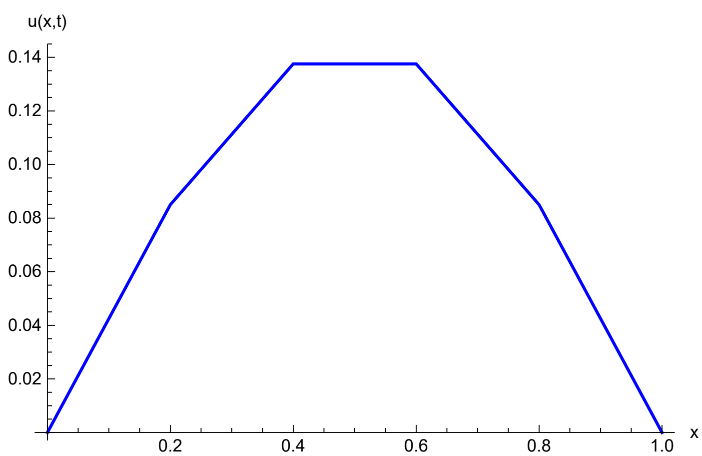
In[28]:= `P4 = ListLinePlot[T[[4]], PlotStyle → Lighter[Blue], AxesLabel → {"x", "u(x,t)"}]`

Out[28]=



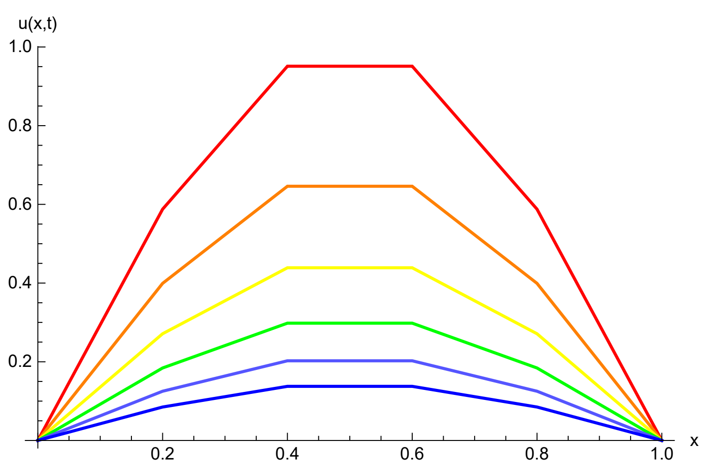
In[29]:= `P5 = ListLinePlot[T[[5]], PlotStyle → Blue, AxesLabel → {"x", "u(x,t)"}]`

Out[29]=



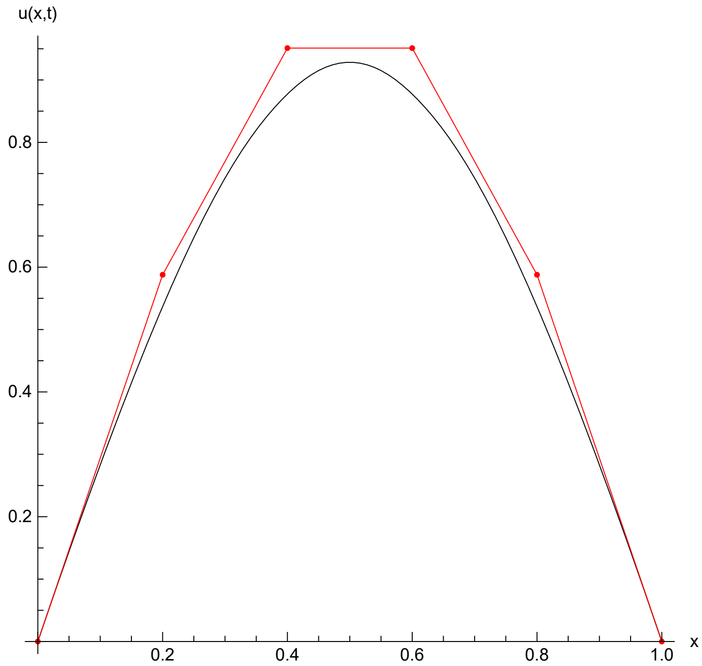
In[30]:= `Show[P0, P1, P2, P3, P4, P5, PlotRange → Automatic]`

Out[30]=



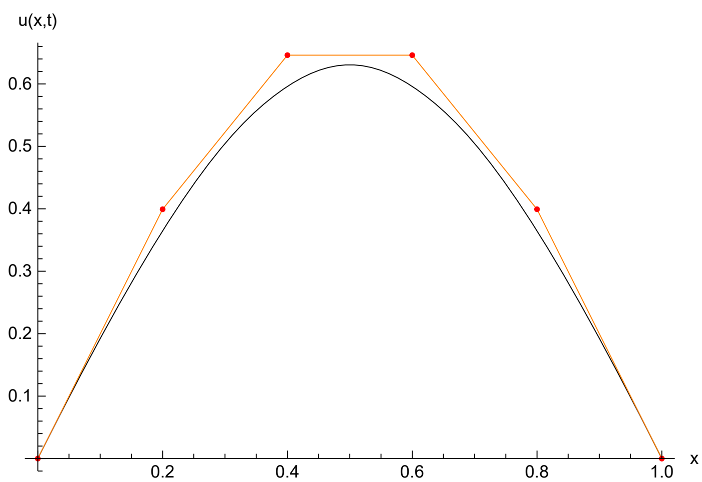
In[31]:=
```
S0 = Graphics[{BSplineCurve[T0], Red, Line[T0], Red, Point[T0]},
  Axes → True, AxesLabel → {"x", "u(x,t)"}]
```

Out[31]=



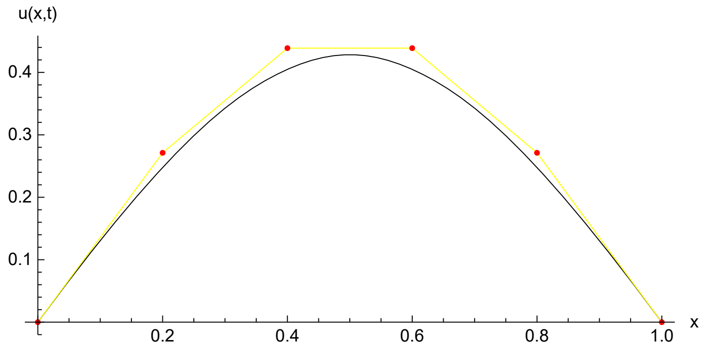
In[32]:=
```
S1 = Graphics[{BSplineCurve[T[[1]]], Orange, Line[T[[1]]], Red, Point[T[[1]]]},
  Axes -> True, AxesLabel → {"x", "u(x,t)"}]
```

Out[32]=



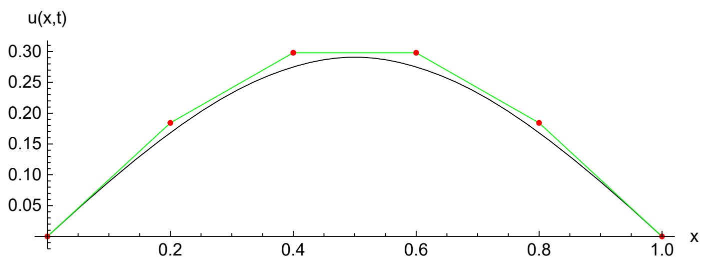
In[33]:=
```
S2 = Graphics[{BSplineCurve[T[[2]]], Yellow, Line[T[[2]]], Red, Point[T[[2]]]},
  Axes -> True, AxesLabel → {"x", "u(x,t)"}]
```

Out[33]=



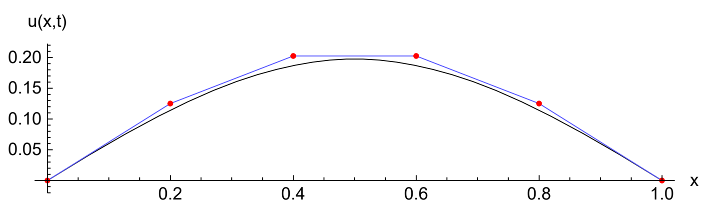
In[34]:=
```
S3 = Graphics[{BSplineCurve[T[[3]]], Green, Line[T[[3]]], Red, Point[T[[3]]]},
  Axes -> True, AxesLabel → {"x", "u(x,t)"}]
```

Out[34]=



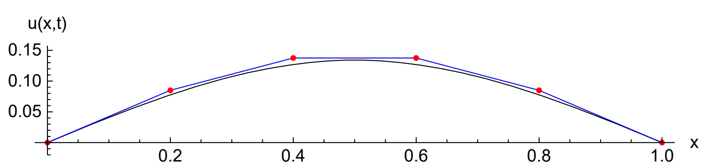
In[35]:=
```
S4 = Graphics[{BSplineCurve[T[[4]]], Lighter[Blue], Line[T[[4]]], Red, Point[T[[4]]]},
  Axes -> True, AxesLabel → {"x", "u(x,t)"}]
```

Out[35]=



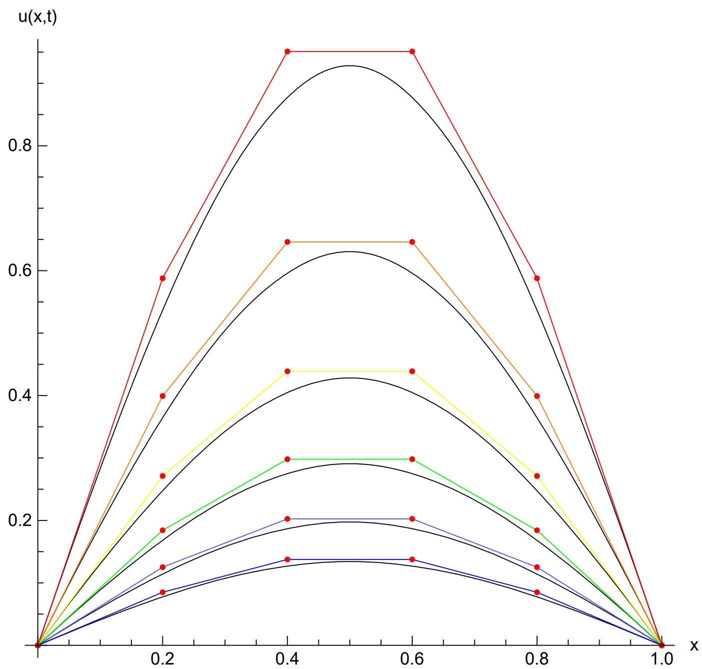
In[36]:=
```
S5 = Graphics[{BSplineCurve[T[[5]]], Blue, Line[T[[5]]], Red, Point[T[[5]]]},
  Axes -> True, AxesLabel → {"x", "u(x,t)"}]
```

Out[36]=



In[37]:=
```
Show[S0, S1, S2, S3, S4, S5]
```

Out[37]=



Another Question : **Solve the corresponding difference equation with c^2 = 1 on the interval 0 ≤ x ≤ 1 subject to the initial temperature u (x, 0) = sin (pix) by Crank – Nicolson method with x – steps h = 0.25 and time t – step k = 0.5 doing 5 time steps.**

In[38]:=
```
ClearAll["Global`*"]
```

In[39]:=
```
n = 3; r = 8; h = 0.25;
```

In[40]:=
```
A = Table[Switch[j - k, 0, 18, 1, -8, -1, -8, _, 0], {j, n}, {k, n}];
```

In[41]:=
```
MatrixForm[A]
```

Out[41]//MatrixForm=

$$\begin{pmatrix} 18 & -8 & 0 \\ -8 & 18 & -8 \\ 0 & -8 & 18 \end{pmatrix}$$

In[42]:=
```
Do[u[k] = N[Sin[Pi k h]], {k, 0, n + 1}]
```

In[43]:=
```
Table[u[k], {k, 0, n + 1}]
```

Out[43]= $\{0., 0.707107, 1., 0.707107, 1.22465 \times 10^{-16}\}$

In[44]:=
```
T0 = Table[{0.25 k, u[k]}, {k, 0, n + 1}]
```

Out[44]= $\{\{0., 0.\}, \{0.25, 0.707107\}, \{0.5, 1.\}, \{0.75, 0.707107\}, \{1., 1.22465 \times 10^{-16}\}\}$

In[45]:=
```
Table[Temp[k], {k, 1, n + 1}];
```

In[46]:=
```
M = 4;
```

In[47]:=
```
Do[
    b = Table[0, {i, 1, n}];
    Do[b[[k]] = u[k - 1] + u[k + 1], {k, 1, n}];
    v = N[LinearSolve[A, b]];
        Print[v]; Temp[j] = v;
     Do[u[k] = v[[k]], {k, 1, n}],
     {j, 1, M}
     ];
```

```
{0.14956, 0.211509, 0.14956}

{0.0316333, 0.0447362, 0.0316333}

{0.00669074, 0.00946213, 0.00669074}

{0.00141515, 0.00200133, 0.00141515}
```

In[48]:=
```
v = Table[Table[Temp[k][[j]], {j, 1, M - 1}], {k, 1, n + 1}]
```

Out[48]=
```
{{0.14956, 0.211509, 0.14956}, {0.0316333, 0.0447362, 0.0316333},
 {0.00669074, 0.00946213, 0.00669074}, {0.00141515, 0.00200133, 0.00141515}}
```

In[49]:=
```
v[[1]]
```

Out[49]=
```
{0.14956, 0.211509, 0.14956}
```

In[50]:=
```
v[[1, 3]]
```

Out[50]=
```
0.14956
```

In[51]:=
```
w = Table[Join[{0}, v[[j]], {0}], {j, 1, M}]
```

Out[51]=
```
{{0, 0.14956, 0.211509, 0.14956, 0}, {0, 0.0316333, 0.0447362, 0.0316333, 0},
 {0, 0.00669074, 0.00946213, 0.00669074, 0}, {0, 0.00141515, 0.00200133, 0.00141515, 0}}
```

In[52]:=
```
w[[3]]
```

Out[52]=
```
{0, 0.00669074, 0.00946213, 0.00669074, 0}
```

In[53]:=
```
T = Table[Table[{0.25 i, w[[p]][[i + 1]]}, {i, 0, n + 1}], {p, 1, M}]
```

Out[53]=
```
{{{0., 0}, {0.25, 0.14956}, {0.5, 0.211509}, {0.75, 0.14956}, {1., 0}},
 {{0., 0}, {0.25, 0.0316333}, {0.5, 0.0447362}, {0.75, 0.0316333}, {1., 0}},
 {{0., 0}, {0.25, 0.00669074}, {0.5, 0.00946213}, {0.75, 0.00669074}, {1., 0}},
 {{0., 0}, {0.25, 0.00141515}, {0.5, 0.00200133}, {0.75, 0.00141515}, {1., 0}}}
```

In[54]:=
```
T[[2]]
```

Out[54]=
```
{{0., 0}, {0.25, 0.0316333}, {0.5, 0.0447362}, {0.75, 0.0316333}, {1., 0}}
```

In[55]:=
```
x = Table[x, {x, 0, 1, 0.25}]
```

Out[55]=
```
{0., 0.25, 0.5, 0.75, 1.}
```

In[56]:=
```
U = Table[Sin[Pi k h], {k, 0, n + 1}]
```

Out[56]= $\{0., 0.707107, 1., 0.707107, 1.22465 \times 10^{-16}\}$

In[57]:=
```
W = Transpose[w]
```

Out[57]=
```
{{0, 0, 0, 0}, {0.14956, 0.0316333, 0.00669074, 0.00141515},
 {0.211509, 0.0447362, 0.00946213, 0.00200133},
 {0.14956, 0.0316333, 0.00669074, 0.00141515}, {0, 0, 0, 0}}
```

In[58]:=
```
table = Table[{x[[l]], U[[l]], W[[l, 1]], W[[l, 2]], W[[l, 3]], W[[l, 4]]}, {l, 1, 5}]
```

Out[58]=
```
{{0., 0., 0, 0, 0, 0}, {0.25, 0.707107, 0.14956, 0.0316333, 0.00669074, 0.00141515},
 {0.5, 1., 0.211509, 0.0447362, 0.00946213, 0.00200133},
 {0.75, 0.707107, 0.14956, 0.0316333, 0.00669074, 0.00141515},
 {1., 1.22465 × 10^-16, 0, 0, 0, 0}}
```

In[59]:=
```
table1 = Prepend[table,
  {"x", "u(x,t),t=0", "u(x,t),t=0.5", "u(x,t),t=1.0", "u(x,t),t=1.5", "u(x,t),t=2.0"}]
```

Out[59]=
```
{{x, u(x,t),t=0, u(x,t),t=0.5, u(x,t),t=1.0, u(x,t),t=1.5, u(x,t),t=2.0},
 {0., 0., 0, 0, 0, 0}, {0.25, 0.707107, 0.14956, 0.0316333, 0.00669074, 0.00141515},
 {0.5, 1., 0.211509, 0.0447362, 0.00946213, 0.00200133},
 {0.75, 0.707107, 0.14956, 0.0316333, 0.00669074, 0.00141515},
 {1., 1.22465 × 10^-16, 0, 0, 0, 0}}
```

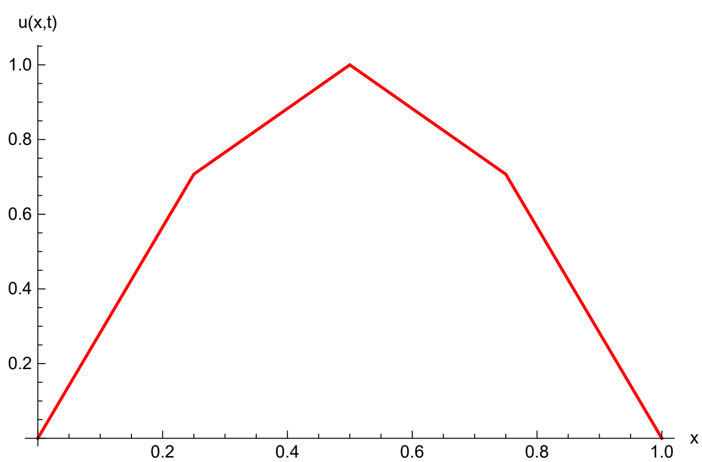
In[60]:=
```
Grid[table1, Frame → All, Spacings → {1, 1}]
```

Out[60]=

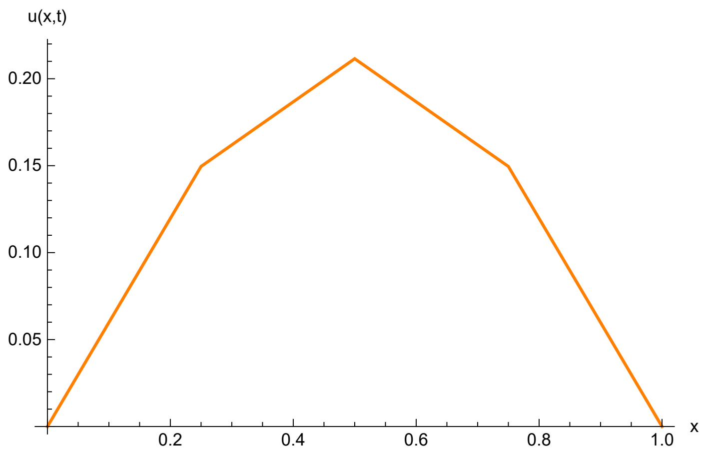
x	u(x,t),t=0	u(x,t),t=0.5	u(x,t),t=1.0	u(x,t),t=1.5	u(x,t),t=2.0
0.	0.	0	0	0	0
0.25	0.707107	0.14956	0.0316333	0.00669074	0.00141515
0.5	1.	0.211509	0.0447362	0.00946213	0.00200133
0.75	0.707107	0.14956	0.0316333	0.00669074	0.00141515
1.	1.22465×10^{-16}	0	0	0	0

In[61]:=
```
P0 = ListLinePlot[T0, PlotStyle → Red, AxesLabel → {"x", "u(x,t)"}]
```

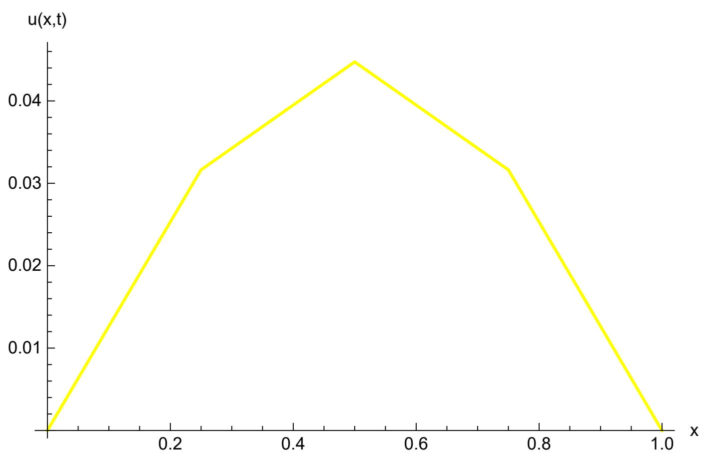
Out[61]=



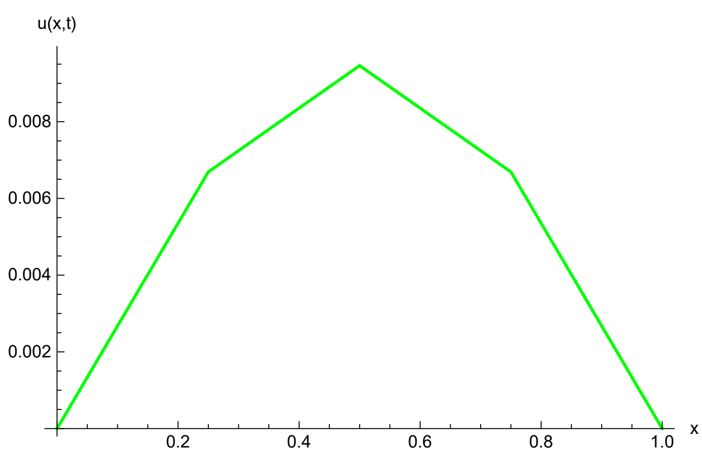
In[62]:=
```
P1 = ListLinePlot[T[[1]], PlotStyle → Orange, AxesLabel → {"x", "u(x,t)"}]
```

Out[62]=


In[63]:=
```
P2 = ListLinePlot[T[[2]], PlotStyle → Yellow, AxesLabel → {"x", "u(x,t)"}]
```

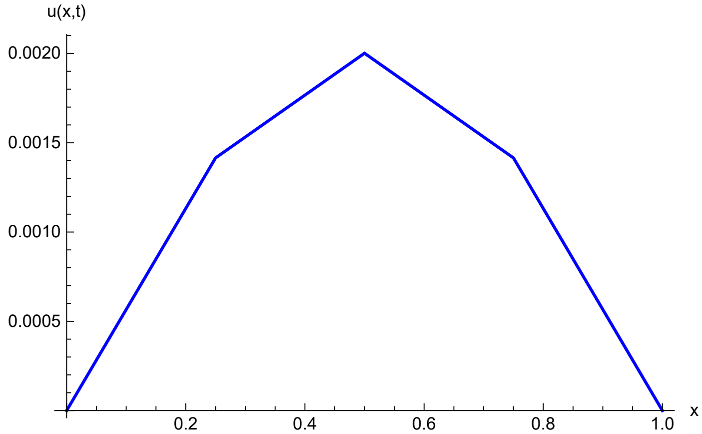
Out[63]=


In[64]:=
```
P3 = ListLinePlot[T[[3]], PlotStyle → Green, AxesLabel → {"x", "u(x,t)"}]
```

Out[64]=


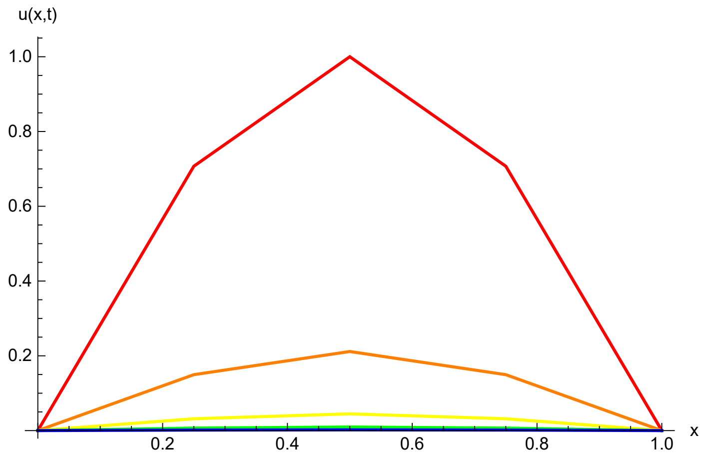
In[65]:=
```
P4 = ListLinePlot[T[[4]], PlotStyle → Blue, AxesLabel → {"x", "u(x,t)"}]
```

Out[65]=



In[66]:=
```
Show[P0, P1, P2, P3, P4, PlotRange → All]
```

Out[66]=



In[67]:=
```
S0 = Graphics[{BSplineCurve[T0], Red, Line[T0], Red, Point[T0]},
   Axes -> True, AxesLabel → {"x", "u(x,t)"}];
```

In[68]:=
```
S1 = Graphics[{BSplineCurve[T[[1]]], Orange, Line[T[[1]]], Red, Point[T[[1]]]},
   Axes -> True, AxesLabel → {"x", "u(x,t)"}];
```

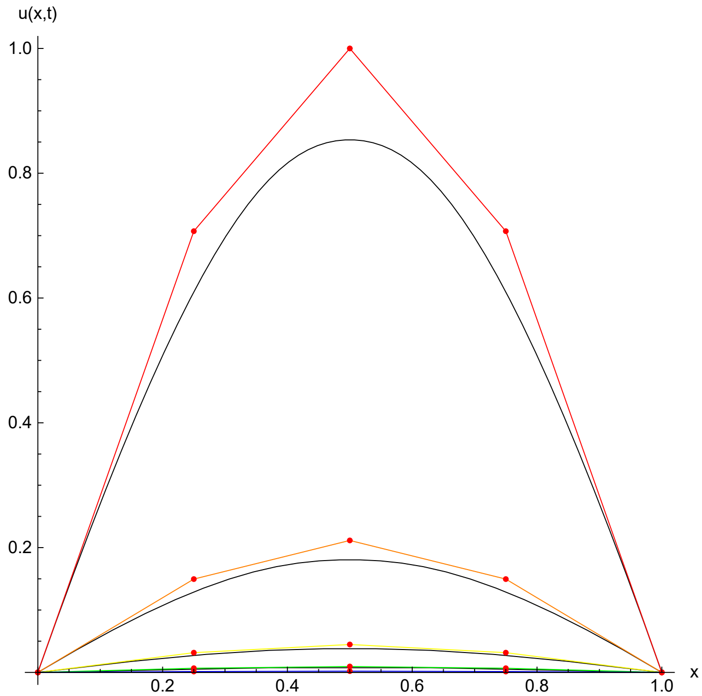
In[69]:=
```
S2 = Graphics[{BSplineCurve[T[[2]]], Yellow, Line[T[[2]]], Red, Point[T[[2]]]},
   Axes -> True, AxesLabel → {"x", "u(x,t)"}];
```

In[70]:=
```
S3 = Graphics[{BSplineCurve[T[[3]]], Green, Line[T[[3]]], Red, Point[T[[3]]]},
   Axes -> True, AxesLabel → {"x", "u(x,t)"}];
```

In[71]:=
```
S4 = Graphics[{BSplineCurve[T[[4]]], Blue, Line[T[[4]]], Red, Point[T[[4]]]},
   Axes -> True, AxesLabel → {"x", "u(x,t)"}];
```

In[72]:=
```
Show[S0, S1, S2, S3, S4]
```

Out[72]=



In[73]:=
```
ClearAll["Global`*"]
```