"HEAT EQUATION. CRANK–NICOLSON METHOD"

Question : The one – dimension heat equation ut = c^2 uxx is a parabolic equation that governs, for instance, the heat flow in a bar where u (x, t) is the temperature at a point x and time t. Solve the corresponding difference equation with c^2 = 1 on the interval 0 ≤ x ≤ 1 (the bar extending from x = 0 to x = 1 along the x – axis) subject to initial temperature u (x, 0) = sin (pix) by the Crank – Nicolson method with x – step h = 0.2 and time step k = 0.04 doing 5 time steps.

In[1]:=
```
ClearAll["Global`*"]
```

In[2]:=
```
n = 4; r = 1; h = 0.2;
```

In[3]:=
```
A = Table[Switch[j - k, 0, 4, 1, -1, -1, -1, _, 0], {j, n}, {k, n}];
```

In[4]:=
```
MatrixForm[A]
```

Out[4]//MatrixForm=

$$\begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}$$

In[5]:=
```
Do[u[k] = N[Sin[Pi k h]], {k, 0, n + 1}]
```

In[6]:=
```
Table[u[k], {k, 0, n + 1}]
```

Out[6]=
```
{0., 0.587785, 0.951057, 0.951057, 0.587785, 1.22465*10^-16}
```

In[7]:=
```
T0 = Table[{0.2 k, u[k]}, {k, 0, n + 1}]
```

Out[7]=
```
{{0., 0.}, {0.2, 0.587785}, {0.4, 0.951057},
 {0.6, 0.951057}, {0.8, 0.587785}, {1., 1.22465*10^-16}}
```

In[8]:=
```
Table[Temp[k], {k, 1, n + 1}];
```

In[9]:=
```
M = 5;
```

In[10]:=
```
Do[
    b = Table[0, {i, 1, n}];
    Do[b[[k]] = u[k - 1] + u[k + 1], {k, 1, n}];
    v = N[LinearSolve[A, b]];
        Print[v]; Temp[j] = v;
    Do[u[k] = v[[k]], {k, 1, n}],
    {j, 1, M}
    ];
```

```
{0.399274, 0.646039, 0.646039, 0.399274}

{0.271221, 0.438844, 0.438844, 0.271221}

{0.184236, 0.2981, 0.2981, 0.184236}

{0.125149, 0.202495, 0.202495, 0.125149}

{0.0850118, 0.137552, 0.137552, 0.0850118}
```

In[11]:=
```
v = Table[Table[Temp[k][[j]], {j, 1, M - 1}], {k, 1, n + 1}]
```

Out[11]=
```
{{0.399274, 0.646039, 0.646039, 0.399274},
 {0.271221, 0.438844, 0.438844, 0.271221}, {0.184236, 0.2981, 0.2981, 0.184236},
 {0.125149, 0.202495, 0.202495, 0.125149}, {0.0850118, 0.137552, 0.137552, 0.0850118}}
```

In[12]:=
```
v[[3]]
```

Out[12]=
```
{0.184236, 0.2981, 0.2981, 0.184236}
```

In[13]:=
```
v[[3, 2]]
```

Out[13]=
```
0.2981
```

In[14]:=
```
w = Table[Join[{0}, v[[j]], {0}], {j, 1, M}]
```

Out[14]=
```
{{0, 0.399274, 0.646039, 0.646039, 0.399274, 0},
 {0, 0.271221, 0.438844, 0.438844, 0.271221, 0},
 {0, 0.184236, 0.2981, 0.2981, 0.184236, 0}, {0, 0.125149, 0.202495, 0.202495, 0.125149, 0},
 {0, 0.0850118, 0.137552, 0.137552, 0.0850118, 0}}
```

In[15]:=
```
w[[1]]
```

Out[15]=
```
{0, 0.399274, 0.646039, 0.646039, 0.399274, 0}
```

In[16]:=
```
T = Table[Table[{0.2 i, w[[p]][[i + 1]]}, {i, 0, n + 1}], {p, 1, M}]
```

Out[16]=
```
{{{0., 0}, {0.2, 0.399274}, {0.4, 0.646039}, {0.6, 0.646039}, {0.8, 0.399274}, {1., 0}},
 {{0., 0}, {0.2, 0.271221}, {0.4, 0.438844}, {0.6, 0.438844}, {0.8, 0.271221}, {1., 0}},
 {{0., 0}, {0.2, 0.184236}, {0.4, 0.2981}, {0.6, 0.2981}, {0.8, 0.184236}, {1., 0}},
 {{0., 0}, {0.2, 0.125149}, {0.4, 0.202495}, {0.6, 0.202495}, {0.8, 0.125149}, {1., 0}},
 {{0., 0}, {0.2, 0.0850118}, {0.4, 0.137552}, {0.6, 0.137552}, {0.8, 0.0850118}, {1., 0}}}
```

In[17]:=
```
T[[1]]
```

Out[17]=
```
{{0., 0}, {0.2, 0.399274}, {0.4, 0.646039}, {0.6, 0.646039}, {0.8, 0.399274}, {1., 0}}
```

In[18]:=
```
x = Table[x, {x, 0, 1, 0.2}]
```

Out[18]=
```
{0., 0.2, 0.4, 0.6, 0.8, 1.}
```

In[19]:=
```
U = Table[Sin[Pi k h], {k, 0, n + 1}]
```

Out[19]=
$$\left\{0., 0.587785, 0.951057, 0.951057, 0.587785, 1.22465 \times 10^{-16}\right\}$$

In[20]:=
```
W = Transpose[w]
```

Out[20]=
```
{{0, 0, 0, 0, 0}, {0.399274, 0.271221, 0.184236, 0.125149, 0.0850118},
 {0.646039, 0.438844, 0.2981, 0.202495, 0.137552},
 {0.646039, 0.438844, 0.2981, 0.202495, 0.137552},
 {0.399274, 0.271221, 0.184236, 0.125149, 0.0850118}, {0, 0, 0, 0, 0}}
```

In[21]:=
```
table =
 Table[{x[[l]], U[[l]], W[[l, 1]], W[[l, 2]], W[[l, 3]], W[[l, 4]], W[[l, 5]]}, {l, 1, 6}]
```

Out[21]=
```
{{0., 0., 0, 0, 0, 0, 0}, {0.2, 0.587785, 0.399274, 0.271221, 0.184236, 0.125149, 0.0850118},
 {0.4, 0.951057, 0.646039, 0.438844, 0.2981, 0.202495, 0.137552},
 {0.6, 0.951057, 0.646039, 0.438844, 0.2981, 0.202495, 0.137552},
 {0.8, 0.587785, 0.399274, 0.271221, 0.184236, 0.125149, 0.0850118},
 {1., 1.22465 × 10^-16, 0, 0, 0, 0, 0}}
```

In[22]:=
```
table1 = Prepend[table, {"x", "u(x,t),t=0", "u(x,t),t=0.04",
    "u(x,t),t=0.08", "u(x,t),t=0.12", "u(x,t),t=0.16", "u(x,t),t=0.20"}]
```

Out[22]=
```
{{x, u(x,t),t=0, u(x,t),t=0.04, u(x,t),t=0.08,
  u(x,t),t=0.12, u(x,t),t=0.16, u(x,t),t=0.20}, {0., 0., 0, 0, 0, 0, 0},
 {0.2, 0.587785, 0.399274, 0.271221, 0.184236, 0.125149, 0.0850118},
 {0.4, 0.951057, 0.646039, 0.438844, 0.2981, 0.202495, 0.137552},
 {0.6, 0.951057, 0.646039, 0.438844, 0.2981, 0.202495, 0.137552},
 {0.8, 0.587785, 0.399274, 0.271221, 0.184236, 0.125149, 0.0850118},
 {1., 1.22465 × 10^-16, 0, 0, 0, 0, 0}}
```

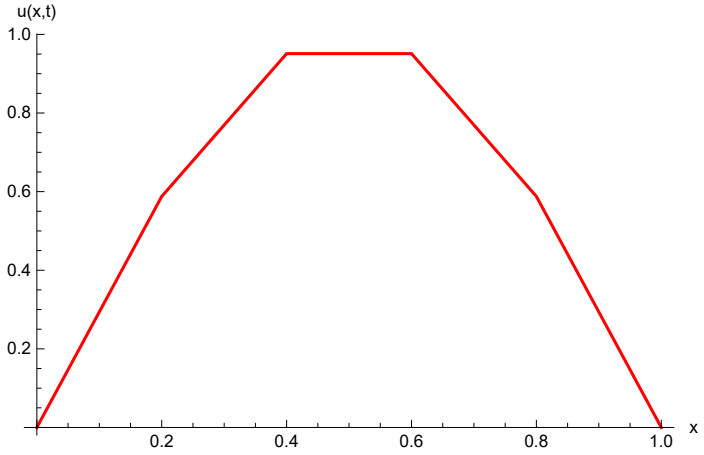
In[23]:=
```
Grid[table1, Frame → All, Spacings → {1, 1}]
```

Out[23]=

x	u(x,t),t=0	u(x,t),t=0.04	u(x,t),t=0.08	u(x,t),t=0.12	u(x,t),t=0.16	u(x,t),t=0.20
0.	0.	0	0	0	0	0
0.2	0.587785	0.399274	0.271221	0.184236	0.125149	0.0850118
0.4	0.951057	0.646039	0.438844	0.2981	0.202495	0.137552
0.6	0.951057	0.646039	0.438844	0.2981	0.202495	0.137552
0.8	0.587785	0.399274	0.271221	0.184236	0.125149	0.0850118
1.	1.22465 × 10^-16	0	0	0	0	0

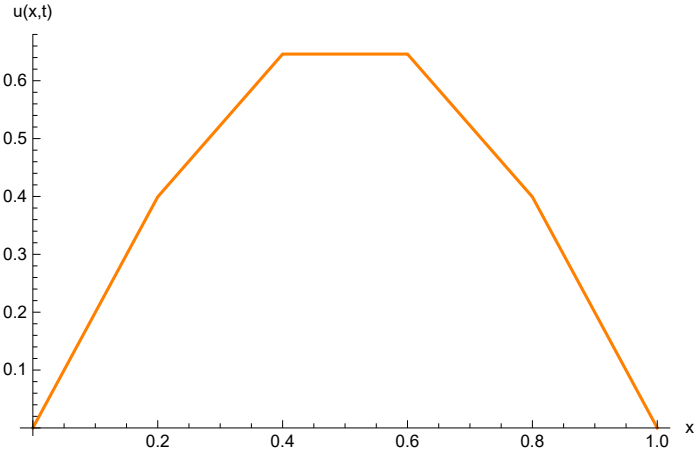
In[24]:=
```
P0 = ListLinePlot[T0, PlotStyle → Red, AxesLabel → {"x", "u(x,t)"}]
```

Out[24]=



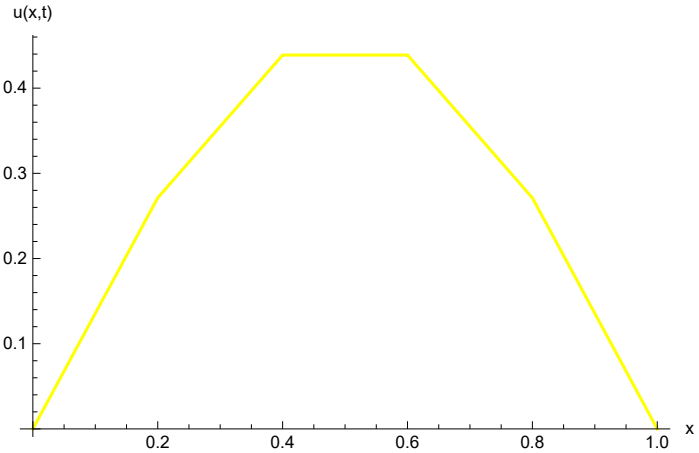
In[25]:=
```
P1 = ListLinePlot[T[[1]], PlotStyle → Orange, AxesLabel → {"x", "u(x,t)"}]
```

Out[25]=



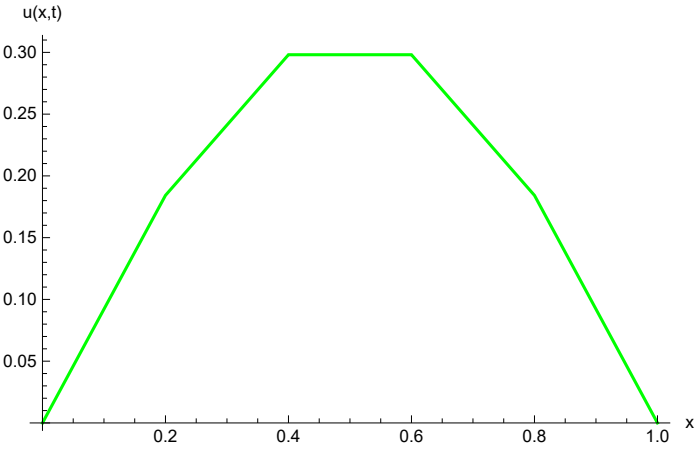
In[26]:=
```
P2 = ListLinePlot[T[[2]], PlotStyle → Yellow, AxesLabel → {"x", "u(x,t)"}]
```

Out[26]=



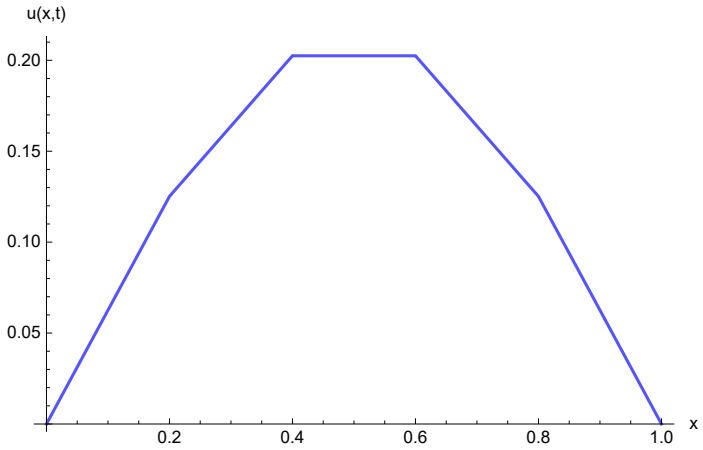
In[27]:=
```
P3 = ListLinePlot[T[[3]], PlotStyle → Green, AxesLabel → {"x", "u(x,t)"}]
```

Out[27]=



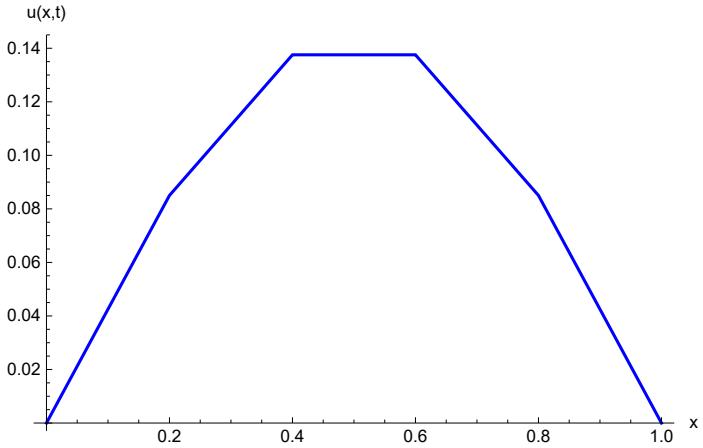
In[28]:=
```
P4 = ListLinePlot[T[[4]], PlotStyle → Lighter[Blue], AxesLabel → {"x", "u(x,t)"}]
```

Out[28]=



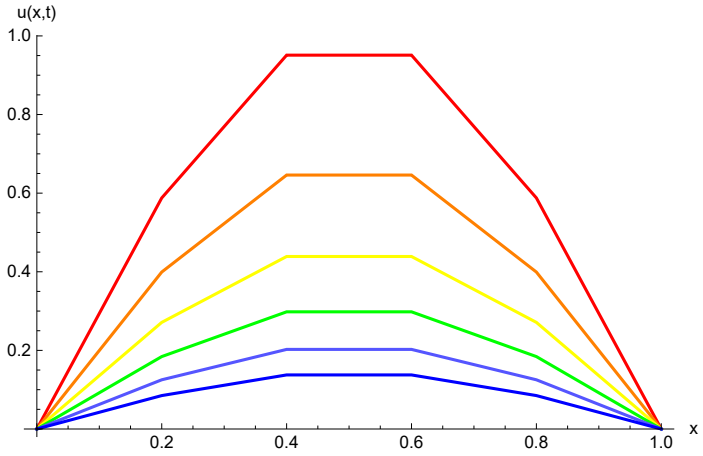
In[29]:=
```
P5 = ListLinePlot[T[[5]], PlotStyle → Blue, AxesLabel → {"x", "u(x,t)"}]
```

Out[29]=



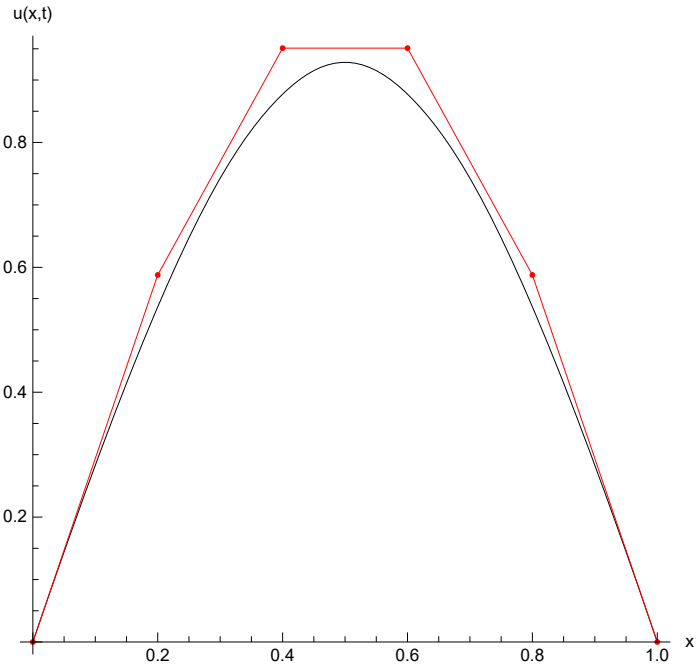
In[30]:=
```
Show[P0, P1, P2, P3, P4, P5, PlotRange → Automatic]
```

Out[30]=



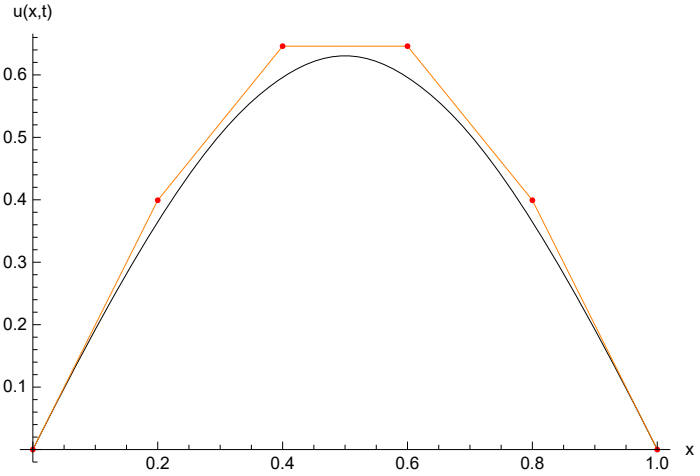
In[31]:=
```
S0 = Graphics[{BSplineCurve[T0], Red, Line[T0], Red, Point[T0]},
  Axes → True, AxesLabel → {"x", "u(x,t)"}]
```

Out[31]=



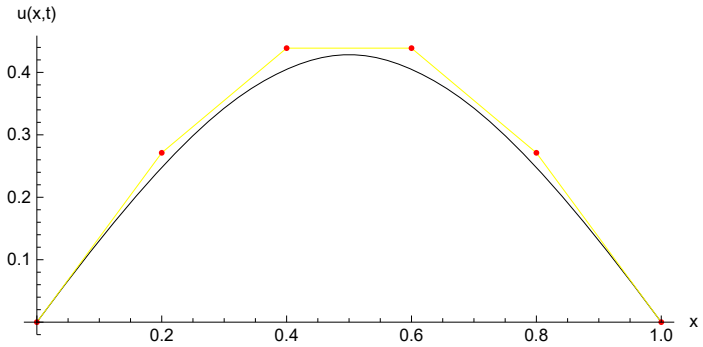
In[32]:=
```
S1 = Graphics[{BSplineCurve[T[[1]]], Orange, Line[T[[1]]], Red, Point[T[[1]]]},
  Axes -> True, AxesLabel → {"x", "u(x,t)"}]
```

Out[32]=



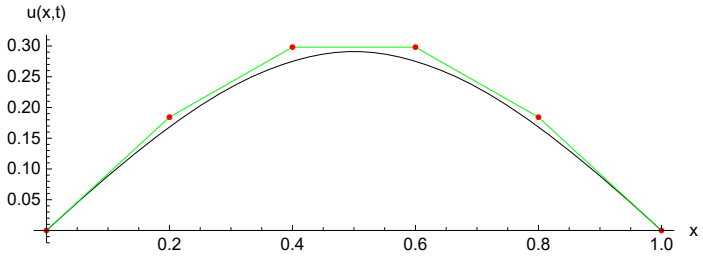
In[33]:=
```
S2 = Graphics[{BSplineCurve[T[[2]]], Yellow, Line[T[[2]]], Red, Point[T[[2]]]},
  Axes -> True, AxesLabel → {"x", "u(x,t)"}]
```

Out[33]=



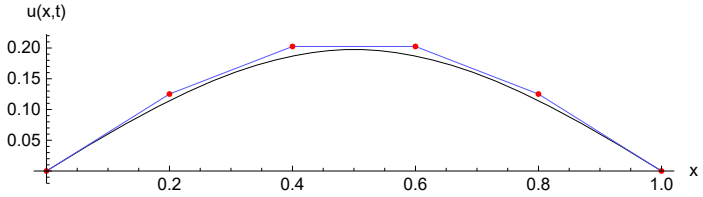
In[34]:=
```
S3 = Graphics[{BSplineCurve[T[[3]]], Green, Line[T[[3]]], Red, Point[T[[3]]]},
  Axes -> True, AxesLabel → {"x", "u(x,t)"}]
```

Out[34]=



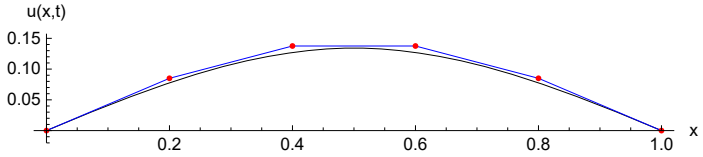
In[35]:=
```
S4 = Graphics[{BSplineCurve[T[[4]]], Lighter[Blue], Line[T[[4]]], Red, Point[T[[4]]]},
  Axes -> True, AxesLabel → {"x", "u(x,t)"}]
```

Out[35]=



In[36]:=
```
S5 = Graphics[{BSplineCurve[T[[5]]], Blue, Line[T[[5]]], Red, Point[T[[5]]]},
  Axes -> True, AxesLabel → {"x", "u(x,t)"}]
```

Out[36]=



In[37]:=
```
Show[S0, S1, S2, S3, S4, S5]
```

Out[37]=