Ordinal Collapsing Function for the Proof-Theoretic Treatment of the least Karanwan Cardinal

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In this paper, we define an ordinal collapsing function for the subsystem of ZF (Zermelo-Frankael set theory) known as KP, augmented by the axiom schema asserting the existence of a Karanwan cardinal

I. Preliminaries

Roughly speaking, the theory of KP arises from ZF by completely omitting the powerset axiom and restricting the axiom of induction to set-bounded formulae.

Definition I.1 An ~~reg~~ uncountable cardinal κ is called a Karanwan cardinal if κ cannot be defined with $<\kappa$ many symbols using the infinitary language ~~$L_{\omega,\omega}$~~ $L_{\kappa,\omega}$, using the symbols $\forall, \exists, \in, \neg, \cup, \cap$, parenthesis, in addition to ordinal, cardinal and set variables.

Definition I.2 The infinitary logic $L_{\alpha,\beta}$ for regular α and $\beta = 0$ or $\omega \leq \beta \leq \alpha$ use the same set of symbols as a finitary logic and may use the same rules of formulation as formulae of a ~~in~~ finitary logic, in addition to:

- Given a set of formulae $A = \{A_\gamma \mid \gamma < \delta < \alpha\}$, then $(A_0 \vee A_1 \vee \ldots)$ and $(A_0 \wedge A_1 \wedge \ldots)$ are formulae. (sequence have length δ)
- Given a set of variables $V = \{V_\gamma \mid \gamma < \delta < \beta\}$ and a formula A_0 then $\forall A_0 : \forall A_1 \cdots (A_0)$ and $\exists V_0 : \exists V_1 \cdots (A_0)$ are formulae (sequence has length δ)

As with the case of finitary formulae, a formulae with no bounded quantifiers in an infinitary language is call a sentence.

So the infinitary language $L_{\kappa,\omega}$ allows for logical concatenation of up to but not including κ many formulae and Levy formulae of ~~up to~~ arbitrarily finitely many alternations of quantifiers.

Throughout this paper, we use the symbol ~~Kr(α)~~ Kr(α) to denote the $(1+\alpha)^{th}$ Karauwan cardinal

Collorary I.3 ~~Continuum Hypothesis~~ Generalized Continuum Hypothesis

The powerset of a set with cardinality $\aleph_\alpha$ is $\aleph_{\alpha+1}$, for any ordinal α. Or more generally speaking, there are no sets with cardinality strictly between $\aleph_\alpha$ and $\aleph_{\alpha+1}$.

Lemma I.4 (Follows from Col. I.3). The existence of n-ply iteration of the powerset operation is ~~the~~ equipotent with the existence of n many regular ~~cardinals~~ (uncountable) cardinals.

For the rest of this thesis, we denote ω as the smallest transfinite ordinal appended with its usual fundamental sequence ($= \omega_0$, the FS is $\omega[n] = n$), equivalently also as the set of natural numbers, and $P(S)$ for a set S denotes the powerset of S. We would also assume GCH.

Collorary I.5 Lower Bound for Karauwan Cardinals (Hypothesis)

A reasonable lower bound for $Kr(0)$ would be the 1st ordinal that is Π_2-reflecting onto the set of regular cardinals below it. This is because that the jump from computable ordinals to ω_1^{CK} is analogous to the jump from the initial segment of the hierarchy of uncountable regular cardinals to ~~$\sup\{$cardinal β that can be defined with α symbols $\mid \alpha<\beta\}$~~
$\sup\{$cardinals α that can be defined with β symbols $\mid \beta<\alpha\}$.

Assuming collorary I.5, then it follows that an ordinal analysis for the proof-theoretic ordinal of KP + there exists a Karauwan cardinal would then be superficially similar to a proof theoretic ordinal of a theory which permits unboundably but computably many iterations of powerset axiom. The outline of such a theory is given in [1].

II. Ordinal Collapsing Function

Contrary to popular beliefs, an ordinal collapsing function is not meant to be an ordinal notation which defines fundamental sequences for large computable ordinals. Instead, it defines ordinals by set theoretic conditions which restrict their size. Hence, an ordinal collapsing function is not necessarily a well-defined ordinal notation.

For a proof-theoretic treatment of KP + "there exists a Karauwan cardinal", it is sufficient to define the $\psi_{Kr(0)}$ collapsing function as an ennumeration of ordinals that can be defined by iterating the powerset axiom. However, in order to make it an ordinal notation, we need to provide it with more fine structures, so it's possible to express every ordinal below the proof-theoretic ordinal of the said theory.

Hence, we would be using multiple sub-functions to define our ordinal collapsing function. The approach we use would be as follows:

Let Card(S) to denote the cardinality of a set S

Let P(S) to denote the powerset of a set S. i.e, the set consisting of all subsets of set S.

Define the class of sets $C(\alpha)$ inductively as follows:

$$\begin{cases} C(0) = N \text{ or the set of natural numbers} \\ C(\alpha+1) = P(C(\alpha)) \\ C(\beta) = \cancel{\forall_{C(\alpha)}} \bigcup_{\alpha<\beta} C(\alpha) \end{cases}$$

Define the class of Levy formulas $D(n)$ (for finite n) inductively as follows:

$$\begin{cases} D(0) = \text{formulas with no unbounded } \cancel{\text{qu}} \text{ quantifiers} \\ D(\alpha+1) = \text{formulas of the form} \\ \forall\alpha_1 \forall\alpha_2 \forall\alpha_3 \cdots \forall\alpha_n \exists\beta_1 \exists\beta_2 \exists\beta_3 \cdots D(\alpha) \end{cases}$$

Define iterations of admissibility and reflections ~~as follows:~~ $E(\alpha)$ as follows ~~(up to transfinite α)~~ (α is a reflecting instance)

~~$E(0)$ = admissible ordinals (= Π_2-reflecting ordinals)~~

~~$E(\alpha+1)$ = ordinals Π_2-reflecting onto $E(\alpha)$.~~

~~$E(\beta)$ = enum $\{y \mid \forall \alpha \in \beta\ (y \in E(\alpha))\}$ for limit ordinal β~~

~~And, let~~ Let α be the strings as follows:

If $\alpha = (0,0,0,\ldots,0)$ with n entries, then $E(\alpha)$ are Π_n-reflecting ordinal.

If $\alpha = (X,0,0,\ldots,0)$ with n total entries, then $E(\alpha)$ are Π_n-reflecting ordinals onto class X.

If $\alpha = (0,0,0,\ldots,X,\ldots,0,0,0)$ with m entries ~~before X and n entries after X, then~~ in total and X being the n^{th} entry counting from the right, then $E(\alpha)$ is the class of Π_m-reflecting ordinals that are Π_n-reflecting onto class X.

Define $\psi^{\alpha,\beta}_{Kr(0)}(y)(\delta)$ as the ennumeration function of $\{\delta \mid L_\delta \models KP + Card(C(\alpha))$ exists $\wedge$ δ can be defined using β $D(\beta)$ $\wedge$ δ have $E(y)$ properties over class of ordinals $\psi^{\alpha,\beta-1}_{Kr(0)}(y)(\delta)\}$.

III Analysis

If this ordinal collapsing function is well-defined, then the associated ordinal notation would have a limit of the proof-theoretic ordinal of KP + "there exists a Karauwan cardinal"

$\psi^{0,1}_{Kr(0)}(0,0)(0)$ corresponds to the 1^{st} ~~uncountable reg~~ admissible ordinal.

$\psi^{0,1}_{Kr(0)}(0,0)(1)$ corresponds to the 2^{nd} admissible ordinal (ω_2^{CK})

$\psi^{0,1}_{Kr(0)}((0,0))(0)$ corresponds to the 1^{st} limit of admissible ordinals (ω_ω^{CK})