P-set 3

P-1(a)

B_t is Browinian motion. Then, suppose $t \geq s$ then $B_t - B_s \sim N(0, t-s)$.

Then, $B_t = B_s + B_t - B_s$

is sum of two normal random variable. Also, we can write

$$E[B_t B_s] = E[B_s^2] + E[(B_t - B_s) B_s]$$

Now, $B_t - B_s$ is independent of B_s as $B_s = B_s - B_0$, So,

$$E[(B_t - B_s) B_s] = E[B_t - B_s] E[B_s]$$

$$= 0$$

$\Rightarrow$

$$E[B_t B_s] = s = E[B_s^2].$$

Also, $E[B_t] = E[B_s] = 0$

$$Cov(B_t, B_s) = s \quad \text{if} \quad t \geq s$$

$$\Rightarrow \quad Cov(B_t, B_s) = \min\{t, s\}$$

(b)

$X_t = B_t^2$ Since $E[B_t] = 0$,

$$E[X_t] = E[B_t^2] = Var(B_t) = t$$

Since B_t, B_s are jointly normal. We can use Isserlis' Theorem to write,

$$E[B_t^2 B_s^2] = E[B_t B_t B_s B_s]$$
$$= E[B_t^2] E[B_s^2] + E[B_t B_s] E[B_t B_s] + E[B_t B_s] E[B_t B_s]$$

$$E[B_t^2 B_s^2] = t^2 s^2 + 2 \min\{t, s\}^2$$

so,

$$Cov(B_t^2, B_s^2) = t^2 s^2 + 2 \min\{t, s\}^2 - ts$$

(C)

$$X_t = B_{t+s} - B_s$$

$$Y_t = \frac{1}{\sqrt{\lambda}} B_{\lambda t}$$

X_t: $X_0 = B_s - B_s = 0$

Since, $B_{t+s} \sim N(0, t+s)$, $B_s \sim N(0, s)$

$\Rightarrow X_t = B_{t+s} - B_s \sim N(0, t)$ – stationary normal

$$\Rightarrow X_{t_1} - X_{t_2} = B_{t_1+s} - B_s - B_{t_2+s} + B_s$$

$$= B_{t_1+s} - B_{t_2+s}$$

$$\Rightarrow X_{t_3} - X_{t_4} = B_{t_3+s} - B_{t_4+s}$$

Since, B_t is Brownian motion,

$B_{t_1+s} - B_{t_2+s}$ is independent of

$B_{t_4+s} - B_{t_3+s}$ for $0 \le t_1 < t_2 < t_3 < t_4$.

So,
$$X_{t_2} - X_{t_1} \perp X_{t_4} - X_{t_3}.$$

Continuity: For a given outcome ω, $B_{t+s}(\omega)$ is continuous & $B_s(\omega)$ is fixed number. Hence, difference of two functions is continuous.

$$Y_t = \frac{1}{\sqrt{h}} B_{ht}$$

$$Y_0 = \frac{1}{\sqrt{h}} B_0 = 0$$

Since, $B_{ht} \sim N(0, ht)$

$$\Rightarrow \frac{1}{\sqrt{h}} B_{ht} \sim N(0, t) = Y_t$$

Hence stationary normal.

Independent Increments :-

$$Y_{t_2} - Y_{t_1} = \frac{1}{\sqrt{\lambda}}\left(B_{\lambda t_2} - B_{\lambda t_1}\right)$$

$$Y_{t_4} - Y_{t_3} = \frac{1}{\sqrt{\lambda}}\left(B_{\lambda t_4} - B_{\lambda t_3}\right)$$

for $0 \le t_1 < t_2 < t_3 < t_4$,

$0 \le \lambda t_1 < \lambda t_2 < \lambda t_3 < \lambda t_4$, $\lambda > 0$

so, $B_{\lambda t_2} - B_{\lambda t_1} \perp B_{\lambda t_4} - B_{\lambda t_3}$.

this constant scalar will also be independent. so,

$$Y_{t_4} - Y_{t_3} \perp Y_{t_2} - Y_{t_1}$$

Continuous :- For any $\omega \in \Omega$,

$B_{\lambda t}(\omega)$ is continuous as function of $h t$. Then its scaling with $\frac{1}{\sqrt{h}}$ will also be continuous.

(d)

$$dX_t = \mu X_t dt + \sigma X_t dB_t \quad , X_0 \text{ given}$$

We are given that our process X_t has above property. Let us define

$$Y_t = \log(X_t)$$

as another stochastic process. Then, Ito's Lemma gives

$$df(X_t) = f'(X_t)\, dX_t + \frac{1}{2} f''(X_t)(dX_t)^2$$

$$= \frac{dX_t}{X_t} - \frac{1}{2}\frac{1}{X_t^2}(dX_t)^2$$

We can write,

$$(dX_t)^2 = \mu^2 X_t^2 dt^2 + \sigma^2 X_t^2 dB_t^2 + 2\mu\sigma X_t^2 dB_t dt$$

Now when we take limit,

$dt \to 0$, first term goes to 0,

$dB_t dt \to 0$ & $dB_t^2 \to dt$,

$$(dX_t)^2 = \sigma^2 X_t^2 dt$$

$$\Rightarrow d\ln(X_t) = \frac{dX_t}{X_t} - \frac{1}{2}\frac{1}{X_t^2}\sigma^2 X_t^2 dt$$

$$= \frac{\mu X_t dt}{X_t} + \frac{\sigma X_t dB_t}{X_t} - \frac{1}{2}\sigma^2 dt$$

$$= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dB_t$$

Integrating we get

$$\ln(X_t) - \ln(X_0) = \mu t - \frac{1}{2}\sigma^2 t + \sigma \int_0^t dB_s ds$$

$\ln(X_t) = \ln(X_0) + \mu t - \frac{1}{2}\sigma^2 t + \sigma B_t$

when integral of Brownian motion follows from Ito's lemma.

$\Rightarrow X_t = X_0 e^{\mu t - \frac{1}{2}\sigma^2 t + \sigma B_t}$

Since $B_t \sim N(0, t)$

We use properties of Normal distribution to get,

$$E[X_t] = X_0 e^{\mu t - \frac{1}{2}\sigma^2 t} E\left[e^{\sigma B_t}\right]$$

$$= X_0 e^{\mu t - \frac{1}{2}\sigma^2 t} e^{\frac{1}{2}\sigma^2 t}$$

$$= X_0 e^{\mu t}$$

(e)

The OU process has stochastic differential equation

$$dX_t = -\mu X_t dt + \sigma dB_t$$

let us guess solution

$$Y_t = f(t, X_t) = X_t e^{\mu t}$$

Then by Ito's lemma,

$$dY_t = f_t dt + f_x dX + f_{xx}(dx)^2$$

$$= X_t e^{\mu t} \cdot \mu + e^{\mu t}(-\mu X_t dt + \sigma dB_t)$$

$$+ 0$$

$$dY_t = \mu X_t e^{\mu t} dt - \mu X_t e^{\mu t} dt + e^{\mu t} \sigma dB_t$$

$$dY_t = \sigma e^{\mu t} dB_t$$

$$\Rightarrow \quad Y_t = \sigma \int_0^t e^{\mu s}\, dB_s + X_0$$

$$X_t e^{\mu t} = \sigma \int_0^t e^{\mu s}\, dB_s + X_0$$

$$X_t = e^{-\mu t} X_0 + \sigma \int_0^t e^{-\mu(t-s)}\, dB_s$$

(f)

$$E[X_t] = \int_0^t E[B_s]\, ds = 0$$

$$E[Y_t] = \int_0^t E[B_s^2]\, ds$$

but $E[B_s^2] = Var(B_s) = s$

$$E[Y_t] = \int_0^t s\, ds = \frac{s^2}{2}\Big|_0^t = \frac{t^2}{2}$$

$$Var(X_t) = E[X_t^2] - E[X_t]^2$$

$$= E[X_t^2]$$

But by Isometry formula for Ito's Calculus,

$$\left(\int_0^t B_s ds\right)^2 = \int_0^t B_s^2 ds$$

so,

$$E[X_t^2] = E\left[\int_0^t B_s ds \int_0^t B_u du\right]$$

$$= \int_0^t E[B_s B_u] ds du$$

$$= \int_0^t \min\{s, u\} ds du$$

Problem - 2

The holding time is poisson random variable. So, pdf is

$$f(t) = \lambda e^{-\lambda t} \qquad t \geq 0$$

but the Expected value of this is actually time of transition which is $1/\lambda$.

(b)

The generator matrix Q for this is,

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}$$

Suppose in the long run, process spends fraction of time $\bar{\pi}_1$ in Y_1 & $\bar{\pi}_2$ in Y_2.

Then, $\bar{\pi}_1 + \bar{\pi}_2 = 1$ and also rates between two states has to be equal

i.e. $\bar{\pi}_1 \lambda_1 = \bar{\pi}_2 \lambda_2$

$\Rightarrow$ Together with $\bar{\pi}_1 + \bar{\pi}_2 = 1$

this gives $\bar{\pi}_1 = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$ & $\bar{\pi}_2 = \dfrac{\lambda_2}{\lambda_1 + \lambda_2}$

Problem - 3 (a)

$$dX_t = -\alpha X_t dt + \sigma dB_t$$

$$f(t, X_t) = e^{\alpha t} X_t$$

$$df = f_t dt + f_x dX_t + \frac{1}{2} f_{xx}$$

$$= \alpha e^{\alpha t} X_t dt + e^{\alpha t} dX_t + 0\, dX_t^2$$

$$df = \alpha e^{\alpha t} X_t dt + e^{\alpha t} dX_t$$

$$= \alpha e^{\alpha t} X_t dt - \alpha e^{\alpha t} X_t dt + e^{\alpha t} \sigma dB_t$$

$$\boxed{df = \sigma e^{\alpha t} dB_t}$$

(b)

$$dV_t = V_t\,dt + V_K\,dK_t - \frac{1}{2} V_{KK}\,dK_t^2$$
$$= 0 + V_K\left[(\iota-\ell)K_t\,dt + \sigma K_t\,dB_t\right]$$
$$- \frac{1}{2} V_{KK}\,dK_t^2$$

$$dV_t = V_K(\ell-\delta)K_t\,dt + \sigma K_t\,dB_t + \frac{1}{2}K_t^2\,dt$$

$$dV_t = \left[V_K(\ell-\delta)K_t + \frac{1}{2}V_{KK}K_t^2\right]dt + V_K\sigma K_t\,dB_t$$

(c)

$$df = f_t\,dt + f_B\,dB_t + \frac{1}{2}f_{BB}\,dB^2$$
$$= \left(f_t + \frac{1}{2}f_{BB}B_t^2\right)dt + \sigma(B_t,t)\,f_B\,dB_t$$

$$\Rightarrow \quad f_t + \frac{1}{2}f_{BB} = 0, \quad \sigma(B_t,t)f_B = f$$

For sim f,

$$f_t = e^{a-\frac{1}{2}t}\left(-1/2\right)$$

$$f_B = e^{B-\frac{1}{2}t}, \quad f_{BB} = e^{\beta - 1/2 t}$$

$$-\frac{1}{2}e^{B-1/2 t} + \frac{1}{2}e^{B-\frac{1}{2}t} = 0$$

Also, $f_B = f$.

Problem-4 (a)

$$da_t = (r a_t - y_t - c_t)\,dt + \sigma dB_t$$

$$dy_t = \theta(\bar{y} - y_t)\,dt + \nu W_t$$

$$\Rightarrow AV = (r a_t + y_t - c_t) V_a + \theta(\bar{y} - y_t) V_y + \frac{1}{2}(\sigma^2 V_{aa} + \nu^2 V_{yy})\,dt$$

(b)

The generator of markov process is

$$A = \begin{pmatrix} -h & h \\ h & -h \end{pmatrix}$$

then

$$AV(k, A) = V_k(i - \delta) k + h(V(k, A_{-t}) - V(k, A_t))$$

P-5 (a)

$$dy_t = \theta(\bar{y} - y_t)\,dt + \sigma\,dB_t$$

In sequle form,

$$V(a_t, y_t) = \max_c \left\{ u(c)\Delta t + \frac{1}{1+\rho\Delta} E_t V(a_{t+\Delta}, y_{t+\Delta}) \right\}$$

$\Rightarrow$

$$\rho V(a_t, y_t) = \max_c \left\{ u(c) + E\frac{dV(a,y)}{dt} \right\}$$

We can define generator as

$$\mathcal{A} = V_a \dot{a} + \mu(t, y_t) V_y + \frac{1}{2}\sigma(t,x)^2 V_{yy}$$

$$\mathcal{A} = V_a \dot{a} + \theta(\bar{y} - y_t) V_y + \frac{1}{2}\sigma^2 V_{yy}$$

$$\rho V(a_t, y_t) = \max_c \Big\{ u(c) + V_a(ra_t + y_t - c_t) + \theta(\bar{y} - y_t) V_y + \frac{1}{2}\sigma^2 V_{yy} \Big\}$$

(b)

$$\rho V(a, y) = \max_c \Big\{ u(c) + V_a(ra + y - c) + \lambda^j \big(V(a, \bar{y}^j) - V(a, y^j)\big) \Big\}$$

(c)

$$\rho V(t, a, y) = \frac{\partial V}{\partial t} + \max_c \Big\{ u(c) + V_a(ra + y - c) + \lambda^j \big(V(a, \bar{y}^j) - V(a, y^j)\big) + \frac{\partial V}{\partial t} \Big\}$$

Not now we just get extra partial derivative wrt time. This just comes from the fact that t determines exogenous interest rate r_t.

P-6

$$dR = \frac{D}{Q}dt + \frac{dQ}{Q}$$

Assume diffusion is,

$$dR = \mu dt + \sigma dB$$

$$n = \theta n + (1-\theta)n = QR + Pb$$

$$\theta n = QR \Rightarrow R = \frac{\theta n}{Q}$$

$$(1-\theta)n = Pb \Rightarrow b = \frac{(1-\theta)n}{P}$$

We know for stock price,

$$dQ = QdR - Ddt$$

$$= Q\mu dt + Q\sigma dB - Ddt$$

For bond price, we have

$$dP = Pr\,dt$$

Since $n = QK + Pb$

$$dn = K\,dQ + Q\,dK + b\,dP + P\,db$$

$$= KQ\mu\,dt + Q\sigma\,dB - Q\,dK + b(Pr\,dt) + P\,db$$

$$dn = \theta n\mu\,dt + \theta n\sigma\,dB - \theta n\frac{D}{Q}\,dt + Q\,dK + (1-\theta)nr\,dt + P\,db$$

We can simply using budget constrit to get,

$$dn = \theta n\mu\,dt + \theta n\sigma\,dB - \theta n\frac{D}{Q}\,dt + \frac{\theta n}{Q}D\,dt - c\,dt + (1-\theta)\,nr\,dt$$

$$dn = (rn + \theta n(\mu - r) - c)dt + \theta n\sigma dB$$

(b)

$$\rho V(n) = \max_{c,\theta} \{u(c) + EV'(n')\}$$

By using generator,

$$EV'(n) = V'(n)[rn + \theta n(\mu - r) - c] + \frac{1}{2}(\theta n\sigma)^2 V''(n)$$

$$\rho V(n) = \max_{c,\theta} \Big\{ u(c) + V'(n)(rn + \theta n(\mu - r) - c) + \frac{1}{2}(\theta n\sigma)^2 V''(n) \Big\}$$

Having recursive representation in networth is useful as then we have to deal with single diffusion process and problem has only one state variable.

V is stationary as once you know net-worth n, you know whole information to compute value. So, time or any other exogenous variables do not enter into value function.

(c)

FOCs:

$$u'(c) = v'(n)$$

$$v'(n)\, n(\mu - r) + v''(n)(\theta n \sigma)(n\sigma) = 0$$

We can derive θ by

$$\theta\, v''(n)(n\sigma)^2 = -\, v'(n)\, n(\mu - r)$$

$$\theta = -\frac{v'(n)}{v''(n)} \frac{\mu - r}{\sigma^2 n} = \frac{r - \mu}{\sigma^2 n} \frac{v'(n)}{v''(n)}$$

(d)

Euler's Equation,