Pset 5 Analytical

Mohammad Bashir

Q-2(a)

The problem in sequence form is

$$V_{i,0} = \max_{\{c_{i,t}\}_{t\geq 0}} E_0 \int_0^\infty e^{-\rho t} u(c_{i,t})\,dt$$

s.t

$$\dot{k}_{i,t} = r_t k_{i,t} + z_{i,t} w_t - c_{i,t}$$

$z_{i,t} \{z^L, z^H\}$ with poisson intensities λ

$$k_{i,t} \geq 0$$

$k_{i,0}, y_{i,0}$ as given

Q-1(b)

Our state variables are capital k, employment status z, and calender time t which captures r_t, w_t.

Beause of this, our value function will depend on time t.
We no long care about 'i' because for the sake of model, a household is determined by its capital stock and employment status. Once we know these two variables, household identity does not matter anymore.

The HJB will be,

$$\rho V_t(a, z) = \max_{c} \{ u(c) + EdV \}$$

To get EdV, we need generator which is

$$EdV = \frac{\partial V_t(a,z)}{\partial t} + \frac{\partial V}{\partial k}\dot{k} + \lambda\left[V_t(a,\bar{z}) - V_t(a,z)\right]$$

or

$$\rho V_t(a,z) = \max_c \left\{ u(c) + (r_t k + z w_t - c)\frac{\partial V}{\partial k} + \lambda\left[V_t(k,\bar{z}) - V_t(k,z)\right] + \frac{\partial V}{\partial t} \right\}$$

(c)

The shortsale constraint shows up in boundry of value function and hence,

$$\frac{\partial V_t(0,z)}{\partial k} \geq u'(w_t z)$$

(d)

FOC:

$$u'(c(a,z)) = \frac{\partial V(a,z)}{\partial R}$$

Aggregation:-

$$\partial_t g_t(a, z_j) = -\partial_a\left[(r_t R - w_t z_j - c_t(a, z_j)\right] g_t(a, z_j) - \lambda g_t(a, z_j) + \lambda g_t(a, z_{-j})$$

(e)

The Kolmogor Forward Equation givs evolution of density function $g_t(R, z)$.

So, define capital accumulation function $I_t(k, y)$. Then,

$$\partial_t s_t(k, z_j) = -\partial_a \left[I_t(k, z_j)\, g_t(k, z_j) \right]$$
$$- \lambda \left[g_t(k, z_j) \right] + h \left[g_t(k, z_{-j}) \right]$$

(f)

$$\max_{K_t, L_t} \quad A_t K_t^{\alpha} L_t^{1-\alpha} - w_t L_t - r_t^{k} K_t$$

$$\Rightarrow \quad \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} = w_t$$

$$\alpha A_t \left(\frac{K_t}{L_t} \right)^{\alpha-1} = w_t$$

Similarly,

$$(1-\alpha) A_t \left(\frac{K_t}{L_t} \right)^{\alpha} = r_t^{k}$$

The profits will be zero when there are constant returns to scale i.e $0<\alpha<1$.

(g)

We have market for capital, labor, output good and a technology for transfer of capital from current period to next period at rate r_t. The return to holding capital is $r_t^K - \delta$ as capital depreciates at rate δ and you get a return of r_t^n on it.

Hence,

Labor Market:- $L_t = \int z\, g_t(k,z)\, dk\, dz$

$$= \sum_j z_j g_t(k, z_j)\, dk$$

Capital Market:- $K_t = \int k\, g_t(k,z)\, dk\, dz$

$$K_t = \sum_j k\, g_t(k, z_j)\, dz$$

Goods Market:-

$$Y_t = C_t + I_t$$

when C_t is aggregate consumption

$$C_t = \int c_t(k,z)\, dk\, dz$$

and

$$I_t = \int I_t(k,z)\,dk\,dz$$

(h)

Taking as given initial distribution $g_0(k,z)$ and exogenous path of TFP $\{A_t\}$, a competitive equilibrium comprises functions

$$\{V_t(k,z),\ c_t(k,z),\ g_t(k,z)\}$$

and aggregates

$\{ Y_t, K_t, L_t, r_t, w_t, r_t^k \}$

such that households optimize,
② firms optimize
and ③ markets clear and
that joint density $g_t(k,z)$ evolves
according to household expectation/ choices.

(1)

FOC:

$$u'(c_t(k,z)) = V_K(k,z)$$

Law of motion of capital:

$$\dot{k}_t(k,z) = r_t k + z w_t - c_t(k,z)$$

Borrowing constrint:-

$k \geq 0$ or

Kormogov Forward Equation:-

$$\frac{\partial g_t(k,z_j)}{\partial t} = -\frac{\partial}{\partial k}\left[\left[r_t k + z w_t - c_t(k,z)\right] g_t(k,z)\right] - \lambda g_t(k,z_j) + \lambda g_t(k,z_{-j})$$

Firm optimization:-

$$w_t = (1-\alpha) A_t \left(\frac{K_t}{L_t}\right)^{\alpha}$$

$$r_t^k = \alpha A_t \left(K_t/L_t\right)^{1-\alpha}$$

and $r_t = r^k_t - \delta$

Market Clearing Condition :-

$$K_t = \int \sum_{z \in \{z_L, z_H\}} k \, g_t(k, z) \, dk$$

$$L_t = \int \sum_{z \in \{z_L, z_H\}} z \, g_t(k, z) \, dk$$

$$Y_t = C_t + I_t$$

$$C_t = \int \sum_{z \in \{z_L, z_H\}} c_t(k, z) \, dk$$

$$I_t = \dot{K}_t - \delta K_t$$

However, we will drop gods

market clearing by Walras' Law.

(i)

Stationary Competitive Equilibrium :-

At stationary competitive equilibrium,

$$\rho V(R, z_i) = u(c(R, z_i)) + \dot{R}(R, z_i) \frac{\partial V(R, z_i)}{\partial R} + \lambda \left(V(a, z_{-i}) - V(a, z_i) \right)$$

$$u'(c(R,z)) = \frac{\partial V}{\partial R}(R,z) \quad \text{—— FOC}$$

$$\frac{\partial V}{\partial R}(0, z_i) \geq u'(c(0, z_i))$$

$$0 = -\frac{\partial}{\partial R}\left([rR + zw - c(R,z)]\, g(R,z) \right) - \lambda\, g(R,z) + \lambda g(R, z')$$

i.e the distribution does not change

$$w = (1-\alpha) A \left(\frac{K}{L}\right)^{\alpha}$$

$$r^R = \alpha A \left(\frac{K}{L}\right)^{1-\alpha}$$

$$r = r^R - \delta$$