Problem Set 4

Q-1

$$u(c) = \frac{c^{1-1/\sigma} - 1}{1-1/\sigma}$$

First not that at $\sigma = 1$, we get $0/0$ which is tractble indeterminate form, so, Take deivative wrt, σ

$$\frac{\frac{\partial}{\partial\sigma}\left(c^{1-1/\sigma} - 1\right)}{\frac{\partial}{\partial\sigma}\left(1-1/\sigma\right)} = \frac{c^{1-1/\sigma} \cdot \ln(c) \left[+1/\sigma^2\right]}{1/\sigma^2}$$

Now taking $\sigma \longrightarrow 1$, $\ln(c)$.

(b)

$$u'(c) = c^{-1/\sigma}$$

$$u''(c) = -\frac{1}{\sigma} c^{-\frac{1}{\sigma}-1}$$

$$u'''(c) = \frac{1}{\sigma}\left(\frac{1}{\sigma}+1\right) c^{-\frac{1}{\sigma}-2}$$

$$-\frac{u'''(c)}{u''(c)}c = \left(\frac{1}{\sigma}+1\right)\frac{c}{c} = \frac{1}{\sigma}+1$$

This measure strongness of pre-cautionary savings and decress with σ.

2 (a)

We can write sequence problem

$$\max_{c_1, c_2} \frac{c_1^{1-1/\sigma}}{1-1/\sigma} + \beta E \frac{c_2^{1-1/\sigma}}{1-1/\sigma}$$

$$\text{s.t } c_1 + \frac{c_2}{1+r} = y_0 + \frac{y_1}{1+r}$$

$$\mathcal{L} = \frac{c_1^{1-1/\sigma}}{1-1/\sigma} + \beta E \frac{c_2^{1-1/\sigma}}{1-1/\sigma} + \lambda\left[y_0 + \frac{y_1}{1+r} - c_1 - \frac{c_2}{1+r}\right]$$

$$\frac{\partial \mathcal{L}}{\partial c_1} = c_1^{-1/\sigma} - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = \beta E c_2^{-1/\sigma} - \frac{\lambda}{1+r} = 0$$

$$\Rightarrow (1+r)\beta E c_2^{-1/\sigma} = \lambda = c_1^{-1/\sigma}$$

$$\Rightarrow \quad c_1^{-1/\sigma} = (1+r)\beta E c_2^{-1/\sigma}$$

$$\Rightarrow \quad u'(c_1) = (1+r)\beta E[u'(c_2)]$$

which is or usul Eulor equation.

If $c_1 > 0$ and $\sigma \longrightarrow \infty$, then LHS

gous to 1, then it has to be that

$$1 = (1+r)\beta.$$

as if c_2 has to stay positive, the

this must be the case.

(c)

Since it has to be true that

$$c_0^{-1/\sigma} = (1+r)\beta E[c_1^{-1/\sigma}]$$

and also

$$c_0 + \frac{c_1}{1+r} = y_0 + \frac{y_1}{1+r}$$

$$c_1 = (1+r)(y_0 - c_0) + y_1$$

For this budget constrait, whatever saving, $c_1 = (1+r)(y_0 - c_0) + y_1^L$

or $c_1 = (1+r)(y_0 - c_0) + y_1^H$

depending on realization of y_1.

PART-3

$$C_0 = \frac{1}{1+\beta^{\sigma}(1+r)^{\sigma-1}} y_0$$

a) $$\frac{\partial c_0}{\partial(1+r)} = \frac{-y_0}{\left[1+\beta^{\sigma}(1+r)^{\sigma-1}\right]^2} \times \beta^{\sigma}(\sigma-1)(1+r)^{\sigma-2}$$

If $\sigma > 1$, then a decrease in r leads to increase in C_0. Decrease in r means consumption tomorrow is expensive and hence you consume more today.

when $\sigma > 1$, individual is very elastic and substitution effect dominates. However, if $\sigma < 1$, the income effect dominates and individual saves even more to have more income tomorrow. Both effects cancel at $\sigma = 1$

(c)

$$c_1 = (1+r)[y_0 - c_0] = \left(1 - \frac{1}{1+\beta^\sigma (1+r)^{\sigma-1}}\right) y_0 \,(1+r)$$

$$c_1 = \frac{\beta^\sigma (1+r)^\sigma}{1+\beta^\sigma (1+r)^{\sigma-1}} y_0$$

(d)

$$\frac{\partial c_1}{\partial (1+r)} = \frac{\sigma \beta^\sigma (1+r)^{\sigma-1}}{[1+\beta^\sigma (1+r)^{\sigma-1}]} y_0 - \frac{\beta^\sigma (1+r)^\sigma (\sigma-1)\beta^\sigma (1+r)^{\sigma-2}}{[1+\beta^\sigma (1+r)^{\sigma-1}]^2}$$

Simplification would give sign this path.

$$\frac{\partial c_1}{\partial (1+r)} = \frac{\sigma \beta^\sigma (1+r)^{\sigma-1} [}{[1+\beta^\sigma (1+r)^{\sigma-1}]^2} y_0$$

In t_0 you are forward looking. When r changes you do adjust c_0 but either income or substitution effect dominates it does not depend on σ.
When $\sigma > 1$ & substitution effect dominates & you save less when r goes down and consume less c_1. However, even if when $\sigma < 1$ and income effect dominates which is you decrease c_0 when $r \downarrow$ and save more but still not to fully offset reduction in c_1 as r also goes down & you end up less in period 2.

Q-2

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$w_{t+1} = R(w_t - c_t)$$

a) We are not given initial condition of W & we have to assume $W_0 > 0$. We are assuming you start with wealth W_0 and consume out of it. I think only slightly non-sensible thing is that R is fixed. It is eat the pie because you have W_0 & you use R to shift it across states. you consume pie over time.

(b)

You start with some wealth w and end up with $V(R(w-c))$ & consume c. So,

$$V(w) = \sup_{c \in [0,w]} \{ u(c) + \beta V(R(w-c)) \}$$

The wealth process is completely deterministic & depends only on starting state w. Also, R is fixed across time. So, need to know calendar time but just initial wealth.

(c)

Let u & f be two functions s.t

$u(c) \geq f(c)$

then

$(Bu)(c) \geq (Bf)(c)$

because $\beta V(R(W-c))$ is same & other part is u or f

$u(c) \geq f(c)$

$u(c) + \beta V(R(W-c)) \geq f(c) + \beta V(R(W-c))$

$\sup\{u(c) + \beta V(R(W-c))\} \geq \sup\{f(c) + \beta V(R(W-c))\}$

$(Bu)(c) \geq (Bf)(c)$

For any v,

$$B(f+a)(c) = \sup\{u(c) + \beta f(R(v-c)) + \beta a\}$$

So it satisfies second condition as well.

Boundedness is necessary bese otherwise supremum may not exist and Bellman operator would not be defined.

Blackwell says we can start with an arbitrary operator V/guess and once start iterating eventually we will reach fixed point which will be our value function.

(d)

if $\gamma = 1$, $u(c) = \ln(c)$

$$\phi + \psi \ln(W) = \sup_{c \in [0,W]} \left\{ \ln(c) + \beta \{ \phi + \psi \ln [R(W-c)] \} \right\}$$

Supremum is resolved when we use optimal choice of c & here,

$$\phi + \frac{1}{1-\beta} \ln W = \ln(1-\beta) + \ln(W) + \beta \left\{ \phi + \frac{1}{1-\beta} [\ln(\beta R) + \ln W] \right\}$$

$$\Rightarrow \phi = \frac{\ln(1-\beta) + \frac{\beta}{1-\beta} \ln(\beta R)}{1-\beta}$$

When $\gamma \neq 1$,

$$\psi \frac{w^{1-\gamma}}{1-\gamma} = \sup_{c \in [0, w]} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta \psi \frac{R(w-c)^{1-\gamma}}{1-\gamma} \right\}$$

Again using optimal choice of c,

$$\frac{\psi w^{1-\gamma}}{1-\gamma} = \frac{\left(\psi^{-1/\gamma} w\right)^{1-\gamma}}{1-\gamma} + \beta \psi \frac{\left[R(w - \psi^{-1/\gamma} w)\right]^{1-\gamma}}{1-\gamma}$$

$$\Rightarrow \psi = \left(1 - \beta^{1/\gamma} R^{\frac{1}{\gamma}-1}\right)^{-\gamma}$$

(c)

when $\gamma = 1$, income & substitution effects always cancel out and therefore the price (R) does not matter.

Q-3 (a)

Households lifetime wealth is derived by integrating wealth differential Equation,

$$\dot{a}_t - r_t a_t = y_t - c_t$$

Natural borrowing constraint says that any point, net present value of income

$$\int_t^{\infty} e^{-r(s-t)} y_s \, ds$$

and you can only borrow against it i.e

$$a_t \geq - \int_t^{\infty} e^{-r(s-t)} y_s \, ds$$

This is also equivalent to non-Ponzi scheme when you apply transversality condition.

So you cannot be so low in debt that even if you consume zero after time t you cannot pay it back.

(b)

We can solve ODE for wealth to get

$$\dot{a}_t - r_t a_t = y_t - c_t$$

$$a_t e^{\int_0^t r_s ds} \Big|_0^t = \int_0^t (y_s - c_s) \, ds$$

$$\underbrace{a_t e^{-\int_0^t r_s ds}}_{\to 0 \text{ in the limit } t \to \infty} - a_0 = \int_0^t e^{-\int_0^t r_s ds} (y_s - c_s) ds$$

$\Rightarrow$

$$a_0 + \int_0^\infty e^{-\int_0^t r_s ds} y_s ds = \int_0^\infty e^{-\int_0^t r_s ds} c_s ds$$

So, initial lifetime wealth is

$$W(a_0, \{r_t\}, \{y_t\}) = a_0 + \int_0^\infty e^{-\int_0^t r_s ds} y_t dt$$

$$dW = da_0 + \int_0^\infty e^{-\int_0^t r_s ds} dy_t dt + \int_0^\infty e^{-\int_0^t r_s ds} y_t dr_t dt$$

(d)

The calender time captures r_t & y_t which are exogenous time dependent processes.

HJB:-

$$\rho V_t(a) = \frac{\partial V_t(a)}{\partial t} + \max_{c} \left\{ u(c) + \frac{\partial V_t(a)}{\partial a} \left[ra + y - c \right] \right\}$$

(e)

FOC: $$u'(c) - \frac{\partial V_t(a)}{\partial a} = 0$$

$$\Rightarrow \quad u'(c) = \frac{\partial V_t(a)}{\partial a}$$

Envelop:-

$$\rho \frac{\partial V_t(a)}{\partial a} = \frac{\partial^2 V_t(a)}{\partial t \partial a} + \frac{\partial^2 V_t(a)}{\partial a^2} \dot{a} + r \frac{\partial V_t(a)}{\partial a}$$

$$(\rho - r)\frac{\partial V_t(a)}{\partial a} = \frac{\partial^2 V_t(a)}{\partial t \partial a} + \frac{\partial^2 V_t(a)}{\partial a^2}\dot{a} \quad \text{— } (a)$$

Taking time derivative of FOC,

$$u''(c)\frac{dc}{dt} = \frac{\partial^2 V_t(a)}{\partial^2 a^2}\dot{a} + \frac{\partial^2 V_t(a)}{\partial a \partial t}$$

But this is RHS of Envelop (a)

$$(\rho - r)\, u'(c) = u''(c)\frac{dc}{dt}$$

$$\boxed{(\rho - r_t)\, dt = \frac{u''(c)}{u'(c)}\, dc}$$

(f)

We know growth rate of marginal utility is $\rho - r_t$, so we can write marginal utility as

$$u'(c_t) = C_0 \, e^{\int_0^t (\rho - r_k) dk}$$

$$u'(c_s) = c_0 \, e^{\int_0^s (\rho - r_k) dk}$$

Since $t > s$, we can write

$$u'(c_t) = \underbrace{c_0 \, e^{\int_0^s (\rho - r_k) dk}} \cdot e^{\int_s^t (\rho - r_k) dk}$$

$$u'(c_t) = u'(c_s) \, e^{\int_s^t (\rho - r_k) dk}$$

$$u'(c_t) = u'(c_s) \cdot e^{\rho(t-s)} \, e^{\int_s^t r_k dk}$$

OR $$u'(c_s) = u'(c_t)\, e^{-\rho(t-s)}\, e^{-\int_s^t r_k dk}$$

$$u'(c_s) = u'(c_t)\, e^{-\rho(t-s)}\, R_{s,t}$$

(5)

So in CRRA, MU is $c^{-\gamma}$

$$c_s^{-\gamma} = c_t^{-\gamma}\, e^{-\rho(t-s)}\, R_{s,t}$$

Now, suppose $s=0$;

$$c_0^{-\gamma}\, e^{\rho(t-s)}\, \frac{1}{R_{0,t}} = c_t^{-\gamma}$$

$$c_t = c_0 \left[e^{-\rho t}\, R_{0,t} \right]^{1/\gamma}$$

(h)

Now lifetime consumption is,

$$\int_0^\infty e^{-\int_0^t r_s ds} c_t dt = \int_0^\infty e^{-\int_0^t r_s ds} c_0 e^{-\rho/\gamma t} \cdot R_{0,t}^{1/\gamma} dt$$

$$= c_0 \int_0^\infty e^{-\rho/\gamma t} \times R_{0,t}^{\frac{1}{\gamma}-1} dt$$

$$= c_0 \int_0^\infty e^{-\rho/\gamma dt} R_{0,t}^{\frac{1-\gamma}{\gamma}} dt$$

The first term $-\rho/\gamma$ gives PV of consumption in utility terms while $R_{0,t}^{\frac{1-\gamma}{\gamma}}$ takes effect of interest (kind of income effect)

y_s impacts c_t only through c_0. So,

$$\frac{\partial c_t}{\partial y_s} = \frac{\partial c_0}{\partial y_s}\left[e^{-\rho t} R_{0,t}\right]^{1/r} = \frac{\partial c_0}{\partial y_s}\left[e^{\frac{r-\rho}{r}t}\right]$$

c_0 is given by

$$c_0 = w\left[\int_0^\infty e^{\frac{\rho - \lambda(1-r)}{r}t}\,dt\right]$$

$$\frac{\partial c_0}{\partial y_s} = \frac{\partial w}{\partial y_s}\left[\int_0^\infty e^{\frac{\rho - \lambda(1-r)}{r}t}\,dt\right]$$

But y_s adds following present discounted value into wealth,

$$\frac{\partial w}{\partial y_s} = e^{-\lambda s}$$

$$\frac{\partial c_0}{\partial y_s} = e^{-rs} \int_0^{\infty} e^{\frac{\rho - r(1-\gamma)}{\gamma} t} \, dt$$

$$\frac{\partial c_t}{\partial y_s} = e^{\frac{r-\rho}{\gamma} t - rs} \int_0^{\infty} e^{\frac{\rho - r(1-\gamma)}{\gamma} t} \, dt$$

assuming that $\frac{\rho - r(1-\gamma)}{\gamma} < 0$ so that integral converges, we get

$$\frac{\partial c_t}{\partial y_s} = e^{\frac{r-\rho}{\gamma} t - rs} \cdot \frac{\gamma}{\rho - r(1-\gamma)}$$

Q-4

$$Af = \frac{\partial f}{\partial a}[r a_t - c_t] + \frac{1}{2}\frac{\partial^2 f}{\partial^2 a}\sigma^2 a_t^2$$

(b)

Since r is fixed, after controlling for a, we don't need any other time dependent information. Here,

$$\rho V(a) = \max_c \left\{ u(c) + \frac{dv}{dt} \right\}$$

using generator & diffusion process of a,

$$\frac{dv}{dt} = V'(a)[r a_t - c_t] + \frac{1}{2}\sigma^2 a_t^2 \, V''(a)$$

So,

$$\rho V(a) = \max_c \left\{ u(c) + V'(a)[ra - c] + \frac{1}{2}\sigma^2 a^2 V''(a) \right\}$$

Since we have only one state variable, this is 2nd order partial differential equation.

(c)

$$c(a) = \rho a$$

Since utility is $\log(c)$, $V(a)$ will be log linear,

$$V(a) = A + B\log(a)$$

$$V'(a) = B/a$$

$$V''(a) = -\frac{B}{a^2}$$

$$\rho A + \rho B \log(a) = \log(\rho a) + B(r-\rho) + \frac{\sigma^2}{2}\left(-\frac{B}{a^2}\right) \times a^2$$

$$\Rightarrow \quad B = 1/\rho$$

$$\rho A = \log(\rho) + B(r-\rho) - \frac{\sigma^2}{2} B$$

$$\rho A = \log(\rho) + \frac{r-\rho}{\rho} - \frac{\sigma^2}{2\rho}$$

$$A = \frac{1}{\rho}\log(\rho) + \frac{r-\rho}{\rho^2} - \frac{\sigma^2}{2\rho^2}$$

$$\Rightarrow \boxed{V(a) = \frac{1}{\rho}\log(\rho a) + \frac{r-\rho}{\rho^2} - \frac{\sigma^2}{2\rho^2}}$$

Q-5 (a)

Since current marginal utility is non-random, household trades it off with an extra unit investment in any of j assets tomorrow. But for optimality, it must hold current MU for present discount expected marginal utility from investing in asset j.

For CRRA, $u'(c) = c^{-\gamma}$

So, $$\frac{u'(c_{t+1})}{u'(c_t)} = \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} = e^{\ln\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}}$$

$$= e^{-\gamma \ln(c_{t+1}/c_t)} = e^{-\gamma \Delta \ln c_{t+1}}$$

So,

$$1 = e^{-\rho} E\left[e^{\ln R^i_{t+1}} \cdot e^{-\gamma \Delta \ln c_{t+1}}\right]$$

$$1 = E\left[e^{r^i_{t+1} - \rho - \gamma \Delta \ln c_{t+1}}\right]$$

using, $R^i_{t+1} = e^{r^i_{t+1} + \sigma^i \varepsilon^i_{t+1} - \frac{1}{2}(\sigma^i)^2}$

our equation become,

$$1 = E\left[e^{\underbrace{r^i_{t+1} + \sigma^i \varepsilon^i_{t+1} - \frac{1}{2}(\sigma^i)^2 - \rho - \gamma \Delta \ln c_{t+1}}_{X_t}}\right]$$

Since this power (X_t) is sum of two normal, it is also normal

$$E[X_t] = r^j_{t+1} - \frac{1}{2}(\sigma^j)^2 - \rho - \gamma \mu_{c,t}$$

$$Var(X_t) = (\sigma^j)^2 + \gamma^2 \sigma^2_{c,t} - \sigma^j \gamma \, Cov(\varepsilon_{t+1}, \Delta \ln C_{t+1}) \, 2$$

$$= (\sigma^j)^2 + \gamma^2 \sigma^2_{c,t} - 2\sigma^j \gamma \, \rho_{j,c} \, \sigma_{c,t}$$

Since X_t is Normally distributed,

$$E[e^{X_t}] = e^{-\rho + r^j_{t+1} - \frac{1}{2}(\sigma^j)^2 - \gamma \mu_{c,t} + \frac{1}{2} Var_t(\sigma\varepsilon^j_{t+1} - \gamma \Delta \ln C_{t+1})}$$

$\log(1) = 0$ so,

$$0 = -\rho + r^j_{t+1} - \frac{1}{2}(\sigma_j)^2 - \gamma \mu_{c,t} + \frac{1}{2} Var_t(\sigma^j \varepsilon^j_{t+1} - \gamma \Delta \ln C_{t+1})$$

(d)

when $\sigma^j = 0$,

$$0 = -\rho + r^f_{t+1} - \gamma \mu_{c,t} + \frac{1}{2}\gamma^2 \sigma^2_{c,t}$$

$$r^f_{t+1} = \rho + \gamma \mu_{c,t} - \frac{1}{2}\gamma^2 \sigma^2_{c,t}$$

(e)

$$\pi^E_{t+1} = -\frac{1}{2}(\sigma^E)^2 + \frac{1}{2}(\sigma^E)^2 + \frac{1}{2}\left(2\,\mathrm{cov}(\sigma \varepsilon^E_{t+1}, \Delta \ln c_{t+1})\right) \times \gamma$$

$$\boxed{\pi^E_{t+1} = \gamma\, \sigma_{c,E}}$$

Q-6

$$\frac{dQ}{Q} = \mu_Q \, dt + \sigma_Q \, dB$$