Dynamic Programming and Applications

Deterministic Dynamic Programming in Continuous Time

Lectures 3 – 4

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Part 1: Differential Equations

1. Continuous time limit

- Consider the two key difference equations:

$$K_{t+1} = I_t + (1-\delta)K_t$$

  and

$$\frac{1}{C_t} = \beta R_t \frac{1}{C_{t+1}}$$

- On the board: (i) generalized discrete time step Δ and (ii) continuous time limit

2. Ordinary differential equations

- Consider the "discrete-time" equation

$$X_{t+\Delta t} - X_t = G(X_t, t, \Delta t)$$

- *Continuous-time limit*: consider the limit as $\Delta t \to 0$

$$\dot{X}_t \equiv \frac{dX}{dt} \equiv \lim_{\Delta t \to 0} \frac{X_{t+\Delta t} - X_t}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} G(X_t, t, \Delta t) \equiv g(X_t, t)$$

- $\dot{X}_t = g(X_t)$ is *autonomous* and dropping subscripts: $\dot{X} = g(X)$

- This is a *first-order (ordinary) differential equation*, second-order equations are:

$$\frac{d^2 X_t}{dt^2} = g\left(\frac{dX_t}{dt}, X_t, t\right)$$

- We often consider ODEs in the *time dimension* but ODEs can be defined on any state space (e.g., space dimensions)

3. Boundary conditions

- Boundary conditions are critical for characterizing differential equations
- Consider an ODE on the time interval $t \in [0, 1]$. We call $[0, 1]$ the *state space*. $(0, 1)$ is the *interior of the state space* and $\{0, 1\}$ is the *boundary*
- The way to think about it: differential equations are defined on the interior of the state space but not on the boundary
- To characterize the function that satisfies the ODE on the interior on the *full* state space, we need a set of boundary conditions to also characterize the behavior on the boundary
- Heuristically: we need as many boundary conditions as the order of the differential equation

- Similar to discrete-time difference equations: forward equations have initial conditions, backward equations have terminal conditions

- For ODEs, you will often see the terminology:
  - *Initial value problems* specify a differential equation for X_t with some *initial condition* X_0
  - *Terminal value problems* instead specify X_T

- More broadly: We need sufficient information to characterize the function of interest along the boundary

- Types of boundary conditions: Dirichlet ($X_0 = c$), von-Neumann ($\frac{dX_0}{dt} = c$), reflecting boundaries, ...

- Boundary conditions are very important and can be very subtle (especially for PDEs)

4. Linear first-order ODEs

- Consider the equation:
$$\dot{X}(t) = a(t)X(t) + b(t) \tag{1}$$

- If $b(t) = 0$, (1) is a *homogeneous* equation, if $a(t) = a$ and $b(t) = b$ we say (1) has *constant coefficients*

- Start with $\dot{X}(t) = aX(t)$, divide by $X(t)$ and integrate with respect to t
$$\begin{aligned} \int \frac{\dot{X}(t)}{X(t)} dt &= \int a dt \\ \log X(t) + c_0 &= at + c_1 \\ X(t) &= Ce^{at} \end{aligned}$$
where $C = e^{c_1 - c_0}$

- Pin down constant C by using the boundary condition (we need 1)

- Consider time-varying coefficient with $\dot{X}(t) = a(t)X(t)$ with initial condition $X(0) = \bar{x}$
- Dividing by $X(t)$, integrating, and exponentiating yields

$$X(t) = Ce^{\int_0^t a(s)ds}$$

- Constant of integration again pinned down by boundary condition: $C = \bar{x}$
- Finally, for $\dot{X}(t) = aX(t) + b$, we find

$$X(t) = -\frac{b}{a} + Ce^{at}$$

  after using change of variables $Y(t) = X(t) + \frac{b}{a}$

- Many results for systems of linear differential equations: $\dot{\boldsymbol{X}}(t) = \boldsymbol{A}\boldsymbol{X}(t)$

5. Examples of differential equations in macro

Capital accumulation:

$$\dot{K}_t = I_t - \delta K_t$$

- We can always map back and forth between DT and CT
- In discrete time with *unit* time steps, $K_{t+1} = I_t + (1-\delta)K_t$
- With arbitrary Δ time step, $K_{t+\Delta} = K_t + \Delta(I_t - \delta K_t)$
- Continuous-time limit:

$$\begin{aligned}
K_{t+\Delta} &= K_t + \Delta(I_t - \delta K_t) \\
\frac{K_{t+\Delta} - K_t}{\Delta} &= I_t - \delta K_t \\
\dot{K}_t &= I_t - \delta K_t
\end{aligned}$$

- Suppose $\{I_t\}_{t\geq 0}$ exogenously given
- Solving this *inhomogeneous equation*, we use *integrating factor*:

$$\dot{K}_t + \delta K_t = I_t$$

$$e^{\int_0^t \delta ds}\dot{K}_t + e^{\int_0^t \delta ds}\delta K_t = e^{\int_0^t \delta ds} I_t$$

- Notice that $\int_0^t \delta ds = \delta \int_0^t ds = \delta [s]_0^t = \delta(t-0) = \delta t$, so

$$e^{\delta t}\dot{K}_t + e^{\delta t}\delta K_t = e^{\delta t} I_t$$

- We have $e^{\delta t}\dot{K}_t + e^{\delta t}\delta K_t = \frac{d}{dt}(K_t e^{\delta t})$, integrating:

$$K_t e^{\delta t} = \tilde{C} + \int_0^t e^{\delta s} I_s ds$$

$$K_t = C + \int_0^t e^{-\delta(t-s)} I_s ds$$

- Integrating constant solves initial condition: $C = K_0$

Wealth dynamics (*very important equation in this course*):

$$\dot{a}_t = r_t a_t + y_t - c_t$$

- r_t is the real rate of return on wealth, y_t is income, and c_t is consumption
- Structure of the equation similar to capital accumulation equation

Consumption Euler equation:

$$\frac{\dot{C}_t}{C_t} = r_t - \rho$$

- The Euler equation typically takes the form of a *backward equation* and comes with a terminal condition (C_T) or transversality condition ($\lim_{T\to\infty} C_T$)
- Stationary point only if $r_t = \rho$
- Suppose we are at $r_t = r = \rho$ and a shock is realized. $r_0 > r$ what happens? $r_0 < r$ what happens?

6. Example: Solow growth model

- As before, $Y_t = C_t + I_t$ and

$$\dot{K}_t = Y_t - C_t - \delta K_t$$

- Representative firms operates neoclassical production function

$$Y_t = F(K_t, L_t, A_t)$$

- Normalize labor to $L_t = 1$ and hold TFP constant $A_t = A$

- We again assume constant savings rate: $Y_t - C_t = I_t = sY_t$

- Assume Cobb-Douglas $Y_t = AK_t^\alpha$ so equilibrium allocation

$$\dot{K}_t = sAK_t^\alpha - \delta K_t$$

- Steady state is given by

$$K_{ss} = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}$$

- Key equilibrium condition in $\dot{K}_t$ is *non-linear* — how to proceed?

- Let $X_t = K_t^{1-\alpha}$, then

$$\begin{aligned} \dot{X}_t &= (1-\alpha)K_t^{-\alpha}\dot{K}_t \\ &= (1-\alpha)K_t^{-\alpha}(sAK_t^{\alpha} - \delta K_t) \\ &= (1-\alpha)sA - (1-\alpha)K_t^{1-\alpha}\delta \\ &= (1-\alpha)sA - (1-\alpha)\delta X_t \end{aligned}$$

- Solution with initial condition X_0 (work this out):

$$X_t = X_{ss} + e^{-(1-\alpha)\delta t}\Big[X_0 - X_{ss}\Big], \quad \text{where } X_{ss} = \frac{sA}{\delta}$$

- Transition dynamics (rate of convergence) governed by $-(1-\alpha)\delta$

7. What are partial differential equations?

- Partial differential equations (PDEs) generalize ODEs to higher-dimensional state spaces
- PDEs are at the heart of (i) continuous-time **dynamic programming** and (ii) heterogeneous-agent models in macro
- PDEs have long been a core tool in physics, applied math, ...
  $\implies$ increasingly used in economics

- Consider a function $u(x_1, x_2, \ldots, x_n)$ where $x_1, \ldots, x_n$ are coordinates in $\mathbb{R}^n$
- Partial derivatives of $u(\cdot)$

$$\frac{\partial u}{\partial x_i} \equiv \partial_{x_i} u \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x_i \partial x_j} = \partial_{x_i x_j} u$$

- A PDE is an equation in u and its partial derivatives — fully generally:

$$0 = G(u, \partial_{x_1} u, \ldots, \partial_{x_n} u, \partial_{x_1 x_1} u, \ldots)$$

- The *order* of the PDE, is the order of the highest partial derivative
- Examples from physics
  - Heat equation: $\partial_t u = \partial_{xx} u$ (second-order, linear, homogeneous)
  - Wave equation: $\partial_{tt} u = \partial_{xx} u$ (second-order, linear, homogeneous)
  - Transport equation: $\partial_t u = \partial_x u$ (first-order, linear, homogeneous)
- Income distribution “solves heat equation”, wealth dynamics “solve transport equations”, dynamic programming often transport + heat

Part 2: Optimization with Deterministic Dynamics

1. Neoclassical growth model in continuous time

- The lifetime value of the representative household is

$$v(k_0) = \max_{\{c_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} u(c_t) dt$$

subject to

$$\dot{k}_t = F(k_t) - \delta k_t - c_t$$
$$k_0 \text{ given },$$

where $\dot{x}_t = \frac{d}{dt} x_t$, ρ is the discount rate, c_t is the rate of consumption, $u(\cdot)$ is instantaneous utility flow, and $\dot{k}_t$ is the rate of (net) capital accumulation

- No uncertainty for now

- This is the **sequence problem** in continuous time

2. Calculus of variations

- Resources:
  - LeVeque: Finite Difference Methods for Ordinary and Partial Differential Equations
  - Kamien and Schwartz: Dynamic Optimization
  - Gelfand and Fomin: Calculus of Variations

- This dynamic optimization problem is associated with the Lagrangian

$$L = \int_0^\infty e^{-\rho t} \left[u(c_t) + \mu_t \left(F(k_t) - \delta k_t - c_t - \dot{k}_t \right) \right] dt$$

- μ_t is the Lagrange multiplier on the capital accumulation ODE

- What do we do with $\dot{k}_t$??

- Integrate by parts:

$$\int_0^\infty e^{-\rho t}\mu_t \dot{k}_t dt = e^{-\rho t}\mu_t k_t\Big|_0^\infty - \int_0^\infty \frac{d}{dt}\left(e^{-\rho t}\mu_t\right)k_t dt$$
$$= -\mu_0 k_0 + \int_0^\infty e^{-\rho t}\rho\mu_t k_t dt - \int_0^\infty e^{-\rho t}\dot{\mu}_t k_t dt$$

- Plugging into Lagrangian:

$$L = \int_0^\infty e^{-\rho t}\left[u(c_t) + \mu_t\left(F(k_t) - \delta k_t - c_t\right) - \rho\mu_t k_t + \dot{\mu}_t k_t\right]dt + \mu_0 k_0$$

- What have we accomplished?

- Notice $\mu_0 k_0$, this is crucial. What's intuition?

$$L = \int_0^\infty e^{-\rho t}\Big[u(c_t) + \mu_t\Big(F(k_t) - \delta k_t - c_t\Big) - \rho\mu_t k_t + \dot{\mu}_t k_t\Big]dt + \mu_0 k_0$$

- The planner optimizes over paths $\{c_t\}$ and $\{k_t\}$
- At an optimum, there cannot be *any* small perturbation in these paths that the planner finds preferable
- Let $\{c_t\}$ and $\{k_t\}$ be *candidate* optimal paths. Consider $\hat{c}_t = c_t + \alpha h_t^c$ and $\hat{k}_t = k_t + \alpha h_t^k$ for arbitrary functions h_t^c and h_t^k

$$L(\alpha) = \int_0^\infty e^{-\rho t}\Big[u(c_t + \alpha h_t^c) + \mu_t\Big(F(k_t + \alpha h_t^k) - \delta k_t - \delta\alpha h_t^k - c_t - \alpha h_t^c\Big) \\ - \rho\mu_t(k_t + \alpha h_t^k) + \dot{\mu}_t(k_t + \alpha h_t^k)\Big]dt + \mu_0(k_0 + \alpha h_0^k)$$

- What about *boundary conditions*? At $t = 0$, capital stock is fixed (k_0 given) while consumption is free. So must have: $h_0^k = 0$ while h_0^c is free

Necessary condition for optimality: $\frac{d}{d\alpha}L(0) = 0$

$$L(\alpha) = \int_0^\infty e^{-\rho t}\Big[u(c_t + \alpha h_t^c) + \mu_t\Big(F(k_t + \alpha h_t^k) - \delta k_t - \delta \alpha h_t^k - c_t - \alpha h_t^c\Big)$$
$$- \rho \mu_t (k_t + \alpha h_t^k) + \dot{\mu}_t (k_t + \alpha h_t^k)\Big] dt + \mu_0 (k_0 + \alpha h_0^k)$$

Work this out yourselves (many times, in many applications!)

$$\frac{d}{d\alpha}L(0) = \int_0^\infty e^{-\rho t}\Big[u'(c_t)h_t^c + \mu_t\Big(F'(k_t)h_t^k - \delta h_t^k - h_t^c\Big)$$
$$- \rho \mu_t h_t^k + \dot{\mu}_t h_t^k\Big] dt + \mu_0 h_0^k$$

where $h_0^k = 0$ because k_0 is fixed

Group terms:

$$\frac{d}{d\alpha}L(0) = \int_0^\infty e^{-\rho t}\left[\Big(u'(c_t) - \mu_t\Big)h_t^c + \Big(\mu_t\Big(F'(k_t) - \delta\Big) - \rho\mu_t + \dot{\mu}_t\Big)h_t^k\right]dt$$

Fundamental Theorem of the Calculus of Variations: Since h_t^c and h_t^k were arbitrary, we must have *pointwise*

$$\begin{aligned} 0 &= u'(c_t) - \mu_t \\ 0 &= \mu_t\Big(F'(k_t) - \delta\Big) - \rho\mu_t + \dot{\mu}_t \end{aligned}$$

Proposition. (Euler equation for marginal utility)

$$\frac{\dot{\mu}_t}{\mu_t} = \frac{\dot{u}_{c,t}}{u_{c,t}} = \rho - F'(k_t) + \delta = \rho - r_t$$

- We have now solved the neoclassical growth model in continuous time. Its solution is given by a system of two ODEs.

- Suppose $u(c) = \log(c)$ and $F(k) = k^\alpha$, then:

$$\frac{\dot{c}_t}{c_t} = \alpha k_t^{\alpha-1} - \delta - \rho$$
$$\dot{k}_t = k_t^\alpha - \delta k_t - c_t$$

  with k_0 given

- Derive the consumption Euler equation yourselves!

- What are the boundary conditions? (Always ask about BCs!)
  - Initial condition on capital: k_0 given
  - Terminal condition on consumption : $\lim_{T\to\infty} c_T = c_{\text{ss}}$

3. Optimal control theory

- Optimal control theory emerged from the calculus of variations

- Applies to dynamic optimization problems in continuous time that feature (ordinary) differential equations as constraints

- Again the neoclassical growth model:

$$v(k_0) = \max_{\{c_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} u(c_t) dt$$

  subject to

$$\dot{k}_t = F(k_t) - \delta k_t - c_t, \quad k_0 \text{ given}$$

- Three new terms:
  - **State variable**: k_t
  - **Control variable**: c_t
  - **Hamiltonian**: $H(c_t, k_t, \mu_t) = u(c_t) + \mu_t \left[F(k_t) - \delta k_t - c_t \right]$

- With Hamiltonian in hand, *copy-paste* formula that we can always use:
  - **Optimality condition**: $\frac{\partial}{\partial c} H = 0$
  - **Multiplier condition**: $\rho\mu_t - \dot{\mu}_t = \frac{\partial}{\partial k} H$
  - **State condition**: $\dot{k}_t = \frac{\partial}{\partial \mu} H$

- This gives us the same equations that we derived using calc of variations:

$$\begin{aligned} u'(c_t) &= \mu_t \\ \rho\mu_t - \dot{\mu}_t &= \mu_t(F'(k_t) - \delta) \\ \dot{k}_t &= F(k_t) - \delta k_t - c_t \end{aligned}$$

- We again get system of Euler equation and capital accumulation:

$$\begin{aligned} \dot{c}_t &= \frac{u'(c_t)}{u''(c_t)}\Big(\rho - F'(k_t) + \delta\Big) \\ \dot{k}_t &= F(k_t) - \delta k_t - c_t \end{aligned}$$

4. Simple example [*skip*]

- Credit: Kamien-Schwartz p. 129
- Simple problem: not much intuition, but illustrates mechanics

$$\max \int_0^1 (x+u)dt$$

subject to $\dot{x} = 1 - u^2$ and initial condition $x_0 = 1$

- Step 1: form Hamiltonian $H(t,x,u,\lambda) = x + u + \lambda(1-u^2)$
- Step 2: necessary conditions (note: no discounting here)

$$\begin{aligned} 0 &= H_u = 1 - 2\lambda u \\ -\dot{\lambda} &= H_x = 1 \end{aligned}$$

and terminal condition $\lambda_1 = 0$ (because u_1 is *free*)

- Step 3: manipulate necessary conditions:

$$\lambda = 1 - t$$
$$u = \frac{1}{2\lambda}$$

and therefore: $u = \frac{1}{2}(1 - t)$

- Finally: solve for all paths (control, state, multiplier)

$$x_t = t - \frac{1}{4}(1 - t) + \frac{5}{4}$$
$$\lambda_t = 1 - t$$
$$u_t = \frac{1}{2}(1 - t)$$

5. Hamilton-Jacobi-Bellman equation

- Recall the neoclassical growth model in continuous time

$$v(k_0) = \max_{\{c_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} u(c_t) dt$$

subject to

$$\dot{k}_t = F(k_t) - \delta k_t - c_t$$
$$k_0 \text{ given },$$

where $\dot{x}_t = \frac{d}{dt} x_t$, ρ is the discount rate, c_t is the rate of consumption, $u(\cdot)$ is instantaneous utility flow, and $\dot{k}_t$ is the rate of (net) capital accumulation

- No uncertainty for now
- This is the infinite-horizon sequence problem, $t \in [0, \infty)$
- A function $v(\cdot)$ that solves this problem is a solution to the neoclassical growth model

- We will now work towards a recursive representation (good reference: Stokey textbook)
- The discrete-time Bellman equation would be

$$v(k_t) = \max_c \left\{ u(c)\Delta t + \frac{1}{1+\rho\Delta t} v(k_{t+\Delta}) \right\}$$

where $\beta = \frac{1}{1+\rho\Delta t}$

- Next: multiply by $1+\rho\Delta t$ and note that $(\Delta t)^2 \approx 0$

$$(1+\rho\Delta t)v(k_t) = \max_c \left\{ (1+\rho\Delta t)u(c)\Delta t + v(k_{t+\Delta}) \right\}$$

$$\rho\Delta t v(k_t) = \max_c \left\{ u(c)\Delta t + v(k_{t+\Delta}) - v(k_t) \right\}$$

$$\rho v(k_t) = \max_c \left\{ u(c) + \frac{v(k_{t+\Delta}) - v(k_t)}{\Delta t} \right\}$$

- Finally: take limit $\Delta t \to 0$ and drop t subscripts

$$\rho v(k) = \max_c \left\{ u(c) + dv \right\}$$

- We want to express dv in terms of $v'(\cdot)$ and dk
- Different ways to think about this: chain rule, Ito's lemma (though no uncertainty here), generator
- Recall generator of (stochastic) process dk_t: For any $f(\cdot)$

$$\mathcal{A}f(k_t) = \lim_{\Delta t \to 0} \mathbb{E}_t \frac{f(k_{t+\Delta t}) - f(k_t)}{\Delta t}$$

- For simple ODE (no uncertainty) $dk = (F(k) - \delta k - c)dt$, we have

$$\mathcal{A}f(k) = (F(k) - \delta k - c)f'(k)$$

- Therefore, we arrive at the **Hamilton-Jacobi-Bellman equation**:

$$\rho v(k) = \max_c \left\{ u(c) + \Big(F(k) - \delta k - c\Big) v'(k) \right\}$$

- Notice: We conjectured a stationary value function (what does this mean?)

6. First-order condition for consumption

- HJB still has “max” operator:

$$\rho v(k) = \max_c \left\{ u(c) + \Big(F(k) - \delta k - c \Big) v'(k) \right\}$$

- To get rid of this, we have to resolve optimal consumption choice
- First-order condition:

$$u'(c(k)) = v'(k)$$

- This defines the **consumption policy function**
- We can now plug back in, obtaining an ODE in $v'(k)$

$$\rho v(k) = u(c(k)) + \Big(F(k) - \delta k - c(k) \Big) v'(k)$$

- Why is this a “stationary” value function and ODE? What would a time-dependent ODE look like? When would we get one?

7. Envelope condition and Euler equation

- We now derive the Euler equation in continuous time
- We start with the **HJB envelope condition**. Differentiating in k:

$$\rho v'(k) = u'(c(k))c'(k) + \Big(F'(k) - \delta - c'(k)\Big)v'(k) + \Big(F(k) - \delta k - c(k)\Big)v''(k)$$
$$\rho v'(k) = \Big(\underbrace{F'(k) - \delta}_{\text{interest rate } r}\Big)v'(k) + \Big(F(k) - \delta k - c(k)\Big)v''(k)$$
$$(\rho - r)v'(k) = \Big(F(k) - \delta k - c(k)\Big)v''(k)$$

- Next, we characterize *process* $dv'(k)$. Using Ito's lemma (even though no uncertainty):

$$\begin{aligned} dv'(k) &= v''(k)dk \\ &= v''(k)(F(k) - \delta k - c(k))dt \\ &= (\rho - r)v'(k)dt. \end{aligned}$$

- Recall first-order condition $u'(c(k)) = v'(k)$.
- The **Euler equation for marginal utility** is given by

$$\frac{du'(c)}{u'(c)} = (\rho - r)dt.$$

- To go from marginal utility to consumption, we use CRRA utility: $u(c) = \frac{1}{1-\gamma}c^{1-\gamma}$. $u'(c) = c^{-\gamma}$ is a function of *process* c, so by Ito's lemma:

$$\begin{aligned} du'(c) &= -\gamma c^{-\gamma-1} dc \\ &= -\gamma u'(c) \frac{dc}{c} \end{aligned}$$

- Plugging in yields **Euler equation for consumption** in continuous time:

$$\frac{dc}{c} = \frac{r-\rho}{\gamma} dt$$

or (you'll often see this notation when no uncertainty): $\frac{\dot{c}}{c} = \frac{r-\rho}{\gamma}$

Connection between calculus of variations and HJB:

- What is the connection between costate / multiplier μ_t and marginal value of wealth $V'(k)$?

- What is the connection between multiplier equation and envelope condition?

9. Boundary conditions

- This is really important: everything we have done so far is only valid in the **interior of the state space**
- What's the state space of a model?
- For the neoclassical growth model without uncertainty, state space is $k \in [0, \infty)$, or

$$\mathcal{X} = \left\{ k \mid k \in [0, \bar{k}] \right\}$$

  where we impose an upper boundary $\bar{k}$
- This is like the domain of the function $v(k)$ that will be valid
- We say $\partial\mathcal{X} = \{0, \bar{k}\}$ is the **boundary** of the state space and $\mathcal{X} \backslash \partial\mathcal{X} = (0, \bar{k})$ is the **interior**
- As is the case **for all differential equations**, the HJB holds on the interior and we need **boundary conditions** to characterize $v(k)$ along the boundary

- What kind of differential equation is the HJB in this model?
- So how many boundary conditions do we need?
- In terms of the economics, what is the correct boundary condition? I.e., what is the correct economic behavior at the boundary $k \in \{0, \bar{k}\}$?
- Nice intuition: 2 boundary inequalities do same job as 1 boundary equality
- We want households to not leave the state space, so we impose that they do not dissave / borrow as $k \to 0$ and save as $k \to \bar{k}$
- This implies: (why?)

$$u'(c(0)) \geq v'(0)$$
$$u'(c(\bar{k})) \leq v'(\bar{k})$$

- If households ever hit the boundaries (in the neoclassical growth model, this doesn't really happen), then consumption behavior is no longer determined by the Euler equations but rather by the boundary conditions

Part 3: Applications

1. Labor: search and matching

- One of most important ideas in labor: frictional search and matching
  Diamond-Mortensen-Pissarides (DMP) model
- This is just a simple application of dynamic programming

Firms: Can post vacancies at cost (rate) c, vacancy filled at rate q. Value of a vacancy is given by HJB

$$rV = -c + q(J - V).$$

Assume firms post until $V = 0$. Once matched, workers produce revenue at rate p and cost wage w. Match separates at rate s. Value of job is given by HJB

$$rJ = p - w - sJ$$

Labor demand: In equilibrium, labor demand schedule given by

$$p - w = (r + s)\frac{c}{q}$$

Profit $p - w$ equalized with amortized cost of search / posting vacancies

Workers: When worker is unemployed, gets benefit b and can search at intensity λ, which costs $\psi(\lambda)$. When employed, gets wage w but separates at rate s. Let U be value of unemployment and E value of employment:

$$\begin{aligned} rU &= \max_{\lambda} \left\{ b - \psi(\lambda) + \lambda(E - U) \right\} \\ rE &= w + s(U - E) \end{aligned}$$

Labor supply schedule characterized by FOC for search intensity

$$\psi'(\lambda) = E - U$$

where E and U solve the coupled system of HJBs

2. Urban / trade / dynamic spatial: migration

- One of most important themes in urban, trade, international and dynamic spatial literatures: people move (migrate) in response to shocks
  For example: To what extent do households migrate in response to China trade shock or climate change?
- Turns out: state-of-the-art dynamic migration model (Caliendo-Dvorkin-Parro) is a simple application of our tools
- Here: also add dynamic consumption-savings problem

Households: There are N regions indexed by j. Consider a household i and denote her region $j_{i,t}$. Lifetime utility is

$$V_{i,0} = \max \mathbb{E} \int_0^\infty e^{-\rho t} u(c_{i,t}) dt$$

- Household inelastically supplies 1 unit of labor, earns wage $w_{j_{i,t}}$
- They have a checking account and face budget constraint

$$\dot{a}_{i,t} = r a_{i,t} + w_{j_{i,t}} - c_{i,t}$$

- Problem will be stationary because r and w_j are time-invariant

Migration: discrete-choice optimal stopping problem

- Households face fixed cost κ_{jk} to move from j to k
- **Key trick:** At rate μ, household draws opportunity and **extreme-value taste shock** ϵ_k with shape parameter θ for possible destinations k

Recursive representation:

$$\rho V(j,a) = \max_c \Big\{ u(c) \underbrace{+ (ra + w_j - c) V_a(j,a)}_{\text{consumption-savings}} \underbrace{+ \mu \Big(\mathbb{E}\Big[\max_k V(k,a) - \kappa_{jk} + \epsilon_k \Big] - V(j,a) \Big)}_{\text{migration}} \Big\}$$

3. Macro: sticky prices

- In the data, firms do not adjust prices instantly
- Price stickiness (nominal rigidities) is at the heart of the New Keynesian model
- We will derive the New Keynesian Phillips Curve as simple application of our tools
  $\Longrightarrow$ much easier to derive in continuous time!!
- Consider a continuum of firms indexed by j that compete monopolistically
- Firm j faces demand function

$$Y_{j,t} = \left(\frac{P_{j,t}}{P_t}\right)^{-\epsilon} Y_t$$

where Y_t and P_t are aggregate (industry) demand and price index

- Firms produce intermediate varieties with the linear production function

$$Y_{j,t} = A_t N_{j,t}$$

- A_t is aggregate productivity and $N_{j,t}$ firm j's labor demand
- Firm j sells at price $P_{j,t}$, profit = revenue net of operating expenses

$$\Pi_{j,t} = P_{j,t} Y_{j,t} - W_t N_{j,t}$$

- Firms maximize NPV of future profit streams, discounted at interest rate r

- Firms set prices optimally over time by choosing inflation $\dot{P}_{j,t} = P_{j,t}\pi_{j,t}$

- Firms pay quadratic adjustment cost $\frac{\delta}{2}\pi_{j,t}^2 P_t Y_t$ to adjust nominal price

- Firm problem:

$$\max_{\{\pi_{j,t}, N_{j,t}\}_{t\geq 0}} \int_0^\infty e^{-rds}\left(P_{j,t}Y_{j,t} - W_t N_{j,t} - \frac{\delta}{2}\pi_{j,t}^2 P_t\right)dt,$$

- Firms are small and take as given $\{W_t, Y_t, P_t\}_{t\geq 0}$ and initial condition $P_{j,0}$

- Any two firms j and j' with same initial price $P_{j,0} = P_{j',0}$ adopt identical inflation and production policies $\implies$ we get back to representative firm

- Hamiltonian (state: $P_{j,t}$, control: $\pi_{j,t}$, multiplier: $\eta_{j,t}$):

$$\mathcal{H}_t(P_{j,t},\pi_{j,t},\eta_{j,t}) = P_{j,t}^{1-\epsilon}P_t^{\epsilon}Y_t - \frac{W_t}{A_t}P_{j,t}^{-\epsilon}P_t^{\epsilon}Y_t - \frac{\delta}{2}\pi_{j,t}^2 P_t Y_t + \eta_{j,t}P_{j,t}\pi_{j,t}$$

- Conditions for optimum:

$$\begin{aligned} i_t\eta_{j,t} - \dot{\eta}_{j,t} &= (1-\epsilon)P_{j,t}^{-\epsilon}P_t^{\epsilon}Y_t + \epsilon\frac{W_t}{A_t}P_{j,t}^{-\epsilon-1}P_t^{\epsilon}Y_t + \eta_{j,t}\pi_{j,t} \\ 0 &= -\delta\pi_{j,t}P_tY_t + \eta_{j,t}P_{j,t}, \end{aligned}$$

as well as the initial condition for the multiplier $\eta_{j,0} = 0$

- Now we can impose symmetric equilibrium: $P_{j,t} = P_t$ for all j

$$\begin{aligned} i_t\eta_t - \dot{\eta}_t &= (1-\epsilon)P_t^{-\epsilon}P_t^{\epsilon}Y_t + \epsilon\frac{W_t}{A_t}P_t^{-\epsilon-1}P_t^{\epsilon}Y_t + \eta_t\pi_t \\ 0 &= -\delta\pi_t P_t Y_t + \eta_t P_t \end{aligned}$$

- Or simply:

$$\begin{aligned} i_t\eta_t - \dot{\eta}_t &= (1-\epsilon)Y_t + \epsilon\frac{w_t}{A_t}Y_t + \eta_t\pi_t \\ \eta_t &= \delta\pi_t Y_t \end{aligned}$$

- Differentiating eq. 2 ($\dot{\eta}_t = \delta\dot{\pi}_t Y_t + \delta\pi_t\dot{Y}_t$) yields:

$$\dot{\pi}_t = \pi_t\left(i_t - \pi_t - \frac{\dot{Y}_t}{Y_t}\right) - \frac{\epsilon}{\delta}\left(\frac{w_t}{A_t} - \frac{\epsilon-1}{\epsilon}\right)$$

- From previous slide:

$$\dot{\pi}_t = \pi_t\left(i_t - \pi_t - \frac{\dot{Y}_t}{Y_t}\right) - \frac{\epsilon}{\delta}\left(\frac{w_t}{A_t} - \frac{\epsilon - 1}{\epsilon}\right)$$

- Last step: recall Euler equation of the representative household

$$\frac{\dot{C}_t}{C_t} = r_t - \rho$$

and use goods market clearing

$$Y_t = C_t$$

- **NKPC:**

$$\dot{\pi}_t = \rho\pi_t - \frac{\epsilon}{\delta}\left(\frac{w_t}{A_t} - \frac{\epsilon - 1}{\epsilon}\right)$$

4. IO: duopoly

- Consider continuous-time variant of Ericson-Pakes (1995) quality-ladder model

- Duopolistic competition: 2 firms $i \in \{A, B\}$ produce good with quality q_t^i and maximize NPV of profits: $\max \int_0^\infty e^{-rt} \pi_t^i dt$. They compete over investments ι_t^i:

$$\dot{q}^i = \iota_t^i - \delta q_t^i$$

- Profits π_t^i depend on both firms' product qualities $\rightarrow$ state variables for recursive representation are $\omega \equiv (\omega^A, \omega^B)$

- Best-response of firm A to firm B characterized by HJB

$$rV^A(\omega) = \pi^A(\omega) + \max_{\iota} \left\{ (\iota - \delta \omega^A) V^A_{\omega^A}(\omega) - \Phi(\iota) \right\} + (\iota^B - \delta \omega^B) V^A_{\omega^B}(\omega)$$

where $\Phi(\cdot)$ is cost of investment, and best-response takes ι^B as given

5. Public finance: tax competition

- Two countries, $i \in \{A, B\}$, setting corporate tax rates τ_t^i on firms operating / headquartered in country i

- Mass of multinational firms j, with μ_t denoting $\%$ in country A at time t

- Firms relocate activity / headquarters at rate θ towards low-tax country:
$$d\mu_t = \theta\mu_t(\tau_t^B - \tau_t^A)^\gamma dt$$

- Country A maximizes tax revenue: $\max \int_0^\infty e^{-\rho t}\tau_t^A \mu_t dt$. Countries compete over taxes $\{\tau_{it}\}$

- Dynamic Nash: country A sets τ_t^A as best response taking τ_t^B as given

- Recursive representation: the only state variable is μ_t
$$\rho V^A(\mu) = \max_{\tau^A} \left\{ \tau^A\mu + \theta\mu\Big(\tau^B(\mu) - \tau^A\Big)^\gamma \partial_\mu V^A(\mu) \right\}$$
Best response strategies: $0 = \mu + \gamma\theta\mu(\tau^B(\mu) - \tau^A)^{\gamma-1} V_\mu^A(\mu)$