Q-1(a)

If your current draw is same as discounted future expected draw, then you are indiffent. Also, if current draw is strictly better than next period's draw, you stop. You only redraw if your expected value from next draw is higher than current draw. This justfies structure of max operator.

Q-1(b)

This summer I was searching for place to rent. Clearly, as I searched more I could find a better, cheaper place.

But eventually I had to stop. Based on my belief about market & distribution of house on the market, I made choice after searching for one and half month. This was my optimal stopping problem.

(C)

Take f & g two bounded continuous functions over $[0,1]$.

Suppose $f(x) \geq g(x)$, then

$$(Bf)(x) = \max\{x, \beta E f(x')\}$$

$$(Bg)(x) = \max\{x, \beta E g(x')\}$$

Since, $E\,s(x') \leq E(f(x'))$

$$(Bf)(x) \geq (Bs)(x) \qquad \forall x$$

B is monotone.

$$[B(f+a)](x) = \max\{x, \beta E(f+a)(x')\}$$

$$= \max\{x, \beta E(f(x')) + \beta a\}$$

obviously,

$$\max\{x, \beta E(f(x'))\} \leq \max\{x, \beta E(f(x')) + \beta a\}$$

for $\beta \geq 0$, $a \geq 0$. Hence, discount factor β exists. So, this Bellman operator satisfies Blackwell sufficient conditions and hence is contraction map.

(d)

Contraction mapping property implies $B^n w$ converges to unique fixed point of Bellman operator which will be our value function.

(e)

$$(Bw)(x) = \max\{x, \beta E w(x)\}$$

$$E w(x) = \int_0^1 1\, dx = 1$$

$$(Bw)(x) = \max\{x, \beta\}$$

$$(B^2 w)(x) = \max\left\{ x, \beta E \max\{x, \beta\} \right\}$$

$$E \max\{x, \beta\} = \int_0^{\beta} \beta \, dx + \int_{\beta}^{1} x \, dx$$

$$= \beta^2 + \left[\frac{1}{2} - \frac{\beta^2}{2} \right]$$

$$= \frac{\beta^2}{2} + \frac{1}{2}$$

$$(B^2 w)(x) = \max\left\{ x, \beta \left(\frac{\beta^2 + 1}{2} \right) \right\}$$

let us refer to fixed terms inside fixed operator c_n. So,

$c_1 = \beta$ & $(Bw)(x) = \max\{ x, \beta \}$

$c_2 = \beta\left(\frac{\beta^2}{2} + \frac{1}{2} \right)$

$$(B^3 w)(x) = \max\{x, \beta E \max\{x, c_2\}\}$$

$$E\max\{x, c_2\} = \int_0^{c_2} c_2 \, dx + \int_{c_2}^{1} x' \, dx'$$

$$= \frac{1}{2} + \frac{c_2^2}{2}$$

$$(B^3 w)(x) = \max\left\{x, \beta\left(\frac{1}{2} + \frac{c_2^2}{2}\right)\right\}$$

Define is $\quad c_3 = \beta\left(\frac{1}{2} + \frac{c_2^2}{2}\right)$

then we will have

$$(B^n w)(x) = \max\{x, c_n\}$$

where

$C_0 = 1, \quad C_1 = \beta, \quad C_2 = \beta\left(\frac{1+\beta^2}{2}\right)$

$C_3 = \beta\left(\frac{1+C_2^2}{2}\right)$

therefore,

$$C_n = \beta\left(\frac{1+C_{n-1}^2}{2}\right), \quad C_0 = 1$$

$$\Rightarrow (B^n w)(x) = \max\left\{x, C_n\right\}$$

Now we need to show that seq C_n is convergent.

$C_0 = 1, \quad C_2 = \beta, \quad C_3 = \frac{\beta}{2}(1+\beta^2)$

We know $C_2 \leq \beta$. Suppose

$C_{n-1} \leq \beta$

$$c_{n-1}^2 \le \beta^2 \le 1$$

$$1 + c_{n-1}^2 \le 1$$

$$\frac{1 + c_{n-1}^2}{2} \le 1/2$$

$$\left(\frac{1 + c_{n-1}^2}{2}\right)\beta \le \beta/2 \le \beta$$

$$\Rightarrow \quad c_n \le \beta \qquad \forall n$$

So seqn c_n is bounded in $\mathbb{R}$ and by completeness in $\mathbb{R}$, it has limit.

Suppose $c_n \longrightarrow c^*$

$$c_n = \frac{\beta}{2}\left(1 + c_{n-1}^2\right)$$

$$c^* = \frac{\beta}{2}\left(1 + c^{*2}\right)$$

$$2c_* = \beta + \beta c^{*2}$$

$$\beta c^{*2} - 2c^* + \beta = 0$$

$$c^* = \frac{1}{\beta}\left(1 \pm \sqrt{1-\beta^2}\right)$$

Since $c_n > 0$ $\forall n$, $c^* > 0$ So

$$c^* = \frac{1}{\beta}\left(1 - \sqrt{1-\beta^2}\right)$$

Using $\rho = -\log(\beta) \Rightarrow \beta = e^{-\rho}$

$$\boxed{c^* = e^{\rho}\left(1 - \left(1 - e^{-2\rho}\right)^{1/2}\right)}$$

Hence,

$$(B^n w)(x) \longrightarrow \max\{x, c^*\} = v(x)$$

$$V(x) = \begin{cases} x^* & \text{if } x \le x^* \\ x & \text{if } x \ge x^* \end{cases}$$

where $x^* = e^{\rho}\left(1 - \left(1 - e^{-2\rho}\right)^{1/2}\right)$

Q-2

Once you get draw of cost, you can choose to work on project and pay cost c, then you expect pay-off in terms of cost is

$$\underbrace{(1-p)\,c}_{\text{complete}} + \underbrace{p\,(c + l + EV(c'))}_{c,\ l\ \text{cost}\ \text{r}\ \text{new draw}\ c'}$$

If you don't work on it, you just get $EV(c')$ but pay late fee l.

$$V(c) = \min\{(1-p)\,c + p(c + l + EV(c')),\ l + EV(c')\}$$

Its minimum base problem is set in terms of cost,

$$V(c) = \min\{c + p[l + EV(c')], l + E(V(c'))\}$$

To match it with our usual specification,

$$\min\{c + p(l + EV(c')), l + E[V(c')]\} = V(c)$$

$$= \min\{c, (1-p)(l + EV(c'))\}$$

Since there is late fee and you are incurring cost, it is ok to not assume discounting. We can say $(1-p)$ is discounting as you pay l with prob $1-p$.

Assuming fixed point exists,

$$V(c) = \min\left\{ c + p(l + EV(c')),\ l + EV(c') \right\}$$

At optimum,

$$c + p(l + EV(c')) = l + EV(c')$$

Now,

$$EV(c') = \int_0^{c^*} \left(c + pl + pEV(c')\right) dc + \int_{c^*}^{1} \left(l + EV(c')\right) dc$$

$$= pl + \frac{c^{*2}}{2} + pEV(c') + l + EV(c') - (l + EV(c'))c^*$$

$$0 = (p+1)l + \frac{c^{*2}}{2} + (p - c^*)EV(c') - lc^*$$

(c)

$$V(c) = \min\left\{ c \; , \beta\left(l + V(c')\right)\right\}$$

$$c^* = \beta\left\{l + \int_0^1 V(c')\,dc\right\}$$

$$= \beta l + \beta \int_0^{c^*} c + \beta \int_{c^*}^1 c^* dc$$

$$= \beta\left\{ l + \frac{c^{*2}}{2} + c^* - c^{*2} \right\}$$

$$c^* = \beta\left\{ l + c^* - \frac{c^{*2}}{2} \right\}$$

$$\frac{\beta}{2} c^{*2} + (1-\beta) c^* - \beta l = 0$$

$$c^* = \frac{(\beta-1) \pm \sqrt{(1-\beta)^2 - 4\left(\frac{\beta}{2}\right)(-\beta e)}}{2\frac{\beta}{2}}$$

$$c^* = \frac{(\beta-1) \pm \sqrt{(1-\beta)^2 + 2\beta^2 e}}{\beta}$$

$$c^* = \frac{1}{\beta}\left[\beta - 1 + \sqrt{(1-\beta)^2 + 2\beta^2 e}\right]$$

becu $c \geq 0$.

(d)

Since you are weighting different periods now at weight β, p does matter as it determines how likely you are to succeed in given period.

(e)

$P(\text{undertake}) = \Pr(c \leq c^*) = c^*$

since c is uniform.

$P_{\text{complete}|\text{undertake}} = 1-p$

$P_{\text{success}} = P_{\text{under}} \times P_{\text{complete}|\text{success}} = c^*(1-p)$

This probability is constant each period as all draw are independent.

Then its geometric distribution with prob of success $c^*(1-p)$.

Then mean is $\dfrac{1}{c^*(1-p)}$.

Q2-3 (a)

The flow budget constraint is,

$$\frac{d}{dt}(P_t a_t) = i_t (P_t a_t) - P_t c_t + P_t y_t$$

$$P_t \dot{a}_t + a_t \dot{P}_t = i_t P_t a_t - P_t c_t + P_t y_t$$

$$\dot{a}_t = i_t a_t - a_t \frac{\dot{P}_t}{P_t} - c_t + y_t$$

$$\dot{a}_t = (i_t - \bar{\pi}_t) a_t - c_t + y_t$$

$$\dot{a}_t - (i_t - \bar{\pi}_t) a_t = y_t - c_t$$

(b)

We can solve this differential equation to get lifetime budget constraint.

Define real interest rate $r(t) = i_t - \bar{\pi}_t$

$$\dot{a}_t - r(t)a_t = y_t - c_t$$

$$e^{-\int_0^t r(s)ds}\dot{a}_t - e^{-\int_0^t r(s)ds} r(t)a_t = (y_t - c_t)\, e^{-\int_0^t r(s)ds}$$

$$\frac{d}{dt}\left(e^{-\int_0^t r(s)ds} a_t\right) = e^{-\int_0^t r(s)ds}(y_t - c_t)$$

Taking lifetime integral,

$$\int_0^\infty \frac{d}{dt}\left(e^{-\int_0^t r(s)ds} a_t\right)dt = \int_0^\infty e^{-\int_0^t r(s)ds}(y_t - c_t)\,dt$$

$$\lim_{t\to\infty} e^{-\int_0^t r(s)ds} a_t - a_0 = \int_0^\infty e^{-\int_0^t r(s)ds} y_t - \int_0^\infty e^{-\int_0^t r(s)ds} c_t$$

It says present value of terminal wealth minus current wealth plus present value of total income is equal to present value of lifetime consumption. Usually we assume in models that $\lim_{t \to \infty} e^{-\int_0^t r(s)ds} a_t = 0$ as you don't end up with final wealth if you optimize.

$$a_0 + \int_0^\infty e^{-\int_0^t r(s)ds} y_t = \int_0^\infty e^{-\int_0^t r(s)ds} c_t$$

Although I think the other part in a_t should only drop out when we optimize. Budget constrit itself is correct as solution to DE.

(c)

Now we can write budget constrnt as

$$a_0 + \int_0^\infty e^{-\int_0^t r(s)ds} y_t \, dt = \int_0^\infty e^{-\int_0^t r(s)ds} c_t \, dt$$

$$\mathcal{L} = \int_0^\infty e^{-\rho t} u(c_t) dt + \lambda \left[\int_0^\infty e^{-\int_0^t r(s)ds} (y_t - c_t) dt + a_0 \right]$$

$$\frac{\partial \mathcal{L}}{\partial c_t} = e^{-\rho t} u'(c_t) - \lambda e^{-\int_0^t r(s)ds} = 0$$

$$\Rightarrow \quad u'(c_t) = e^{-\int_0^t r(s)ds} \quad e^{\rho t}$$

Taking log & then time derivative

$$\log(u'(c_t)) = -\int_0^t r(s)\,ds + \rho t$$

Derivative

$$\frac{1}{u'(c_t)} u''(c_t)\dot{c}_t = -r(t) + \rho = \rho - r(t)$$

$$\boxed{\frac{u''(c_t)}{u'(c_t)} \dot{c}_t = \rho - r(t)}$$

(d)

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$$

$$u'(c_t) = c_t^{-\gamma}$$

$$u''(c_t) = -\gamma c_t^{-(\gamma+1)}$$

$$\frac{-\gamma c_t^{-(\gamma+1)}}{c_t^{-\gamma}} \dot{c}_t = \rho - r(t)$$

$$-\gamma c_t^{-1} \dot{c}_t = \rho - r(t)$$

$$\boxed{\frac{\dot{c}_t}{c_t} = \frac{r(t) - \rho}{\gamma}}$$ — Euler Equation for CRRA

if $u(c_t) = \log(c_t)$

$$u'(c_t) = \frac{1}{c_t}$$

$$u''(c_t) = \frac{-1}{c_t^2}$$

$$\frac{-1}{c_t^2} \times c_t \cdot \dot{c}_t = \rho - r(t)$$

$$\boxed{\frac{\dot{c}_t}{c_t} = \rho - r(t)}$$

(e)

The optimal control problem is

$$\max_{\{c_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} u(c_t)\, dt$$

$$\text{s.t } \dot{a}_t - r(t) a_t = y_t - c_t$$

Control: c_t

State: $a_t, r(t), y_t$

Multiplier: μ_t

(f)

Hamiltonian,

$$H(c_t, y_t, a_t, r(t)) = u(c_t) + \mu_t [y_t - c_t + r(t) a_t]$$

(g)

$$0 = \frac{\partial H}{\partial c} = u'(c_t) - \mu_t = 0$$

$$\rho \mu_t - \dot{\mu}_t = \frac{\partial H}{\partial a_t} = \mu_t r(t) = 0$$

$$\dot{a}_t = y_t - c_t + r(t) a_t$$

$$\Rightarrow u'(c_t) = \mu_t$$

Taking time derivative,

$$u''(c_t)\dot{c}_t = \dot{\mu}_t = \rho\mu_t - \mu_t r(t)$$

$$u''(c_t)\dot{c}_t = \rho u'(c_t) - u'(c_t) r(t)$$

$$\boxed{\frac{u''(c_t)}{u'(c_t)} \dot{c}_t = \rho - r(t)}$$

This is same as Euler equation using simple Lagragian with lifetime budget constraint.

Q-4

$$V_0 = \max_{\{c_t\}_{t \geq 0}} \int_0^{\infty} e^{-\rho t} u(c_t)$$

$$\dot{a}_t = r_t a_t + y_t - c_t$$

a_t is only state variable that matters for optimal choice because r_t, y_t are exogenous and agent's choice does not change their path. Moreover, their path is perfectly known they are like constants.

The value function depends time because there are exogenous processes y_t, & r_t that are not state but time dependent process.

(b)

Discreet time Bellman Equation is,

$$V_t(a_t) = \max_c \left\{ u(c_t)\Delta t + \frac{1}{1+\rho\Delta t} V_{t+\Delta}(a_{t+\Delta}) \right\}$$

$$(1+\rho\Delta t) V_t(a_t) = \max_c \left\{ u(c_t)\Delta t + \rho u(c_t)\Delta t^2 + V_{t+\Delta}(a_{t+\Delta}) \right\}$$

$$\rho\Delta t V_t(a_t) = \max_c \left\{ u(c_t)\Delta t + \rho u(c_t)\Delta t^2 + V_t(a_{t+\Delta}) - V_t(c_t) \right\}$$

$$\rho\Delta t V_t(a_t) = \max_{c_t} \left\{ u(c_t)\Delta t + \rho u(c_t)\Delta t^2 + V_{t+\Delta}(a_{t+\Delta}) - V_t(a_t) \right\}$$

Now we can take Taylor Series exponsion of $V_{t+\Delta t}(a_{t+\Delta t})$ around (a_t, t).

$$V(a_{t+\Delta t}, t+\Delta t) = V(a_t, t) + \frac{\partial V}{\partial t}\Delta t + \frac{\partial V}{\partial a_t}(a_{t+\Delta t} - a_t)$$

then

$$\rho V_t(a_t) = \max_{c_t} \left\{ u(c_t) + \rho u(c_t)\Delta t + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial a_t} \cdot \frac{(a_{t+\Delta t} - a_t)}{\Delta t} \right\}$$

Taking $\Delta t \to 0$

$$\rho V_t(a_t) = \max_{c_t} \left\{ u(c_t) + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial a_t} \dot{a}_t \right\}$$

$$= \frac{\partial V}{\partial t} + \max_{c_t} \left\{ u(c_t) + \frac{\partial V}{\partial a_t} \left(r_t a_t + y_t - c_t \right] \right\}$$

We can take out $\frac{\partial V}{\partial t}$ because it does not depend on our choice c_t but only on a_t.

(c)

$V_t(a)$ term appears becse our value function not only depend on initial state but actual time when you start optimizing as path y_t, r_t depends on that.

(d)

$$u'(c) - \frac{\partial V_t(a)}{\partial a} = 0$$

This says marginal utility of current consumption is equal to lifetime value of investing that extra unit.

A consumption policy function is $c_t(a_t)$ that takes current state at t time which gives $r_t r$ y_t and give optimal value of c_t in each time t.

Since u' is one-to-one function, we can define

$$c(a) = (u')^{-1}\left\{\frac{\partial V(a)}{\partial a}\right\}$$

which is in general non-linear PDE.

$$\rho V(a_t, t) = \frac{\partial V}{\partial t} + u\left((u')^{-1}\left(\frac{\partial V}{\partial a}\right)\right) + \frac{\partial V}{\partial a}\left[r_t a + y_t - (u')^{-1}\left\{\frac{\partial V}{\partial a}\right\}\right] \quad (*)$$

This is non-linear PDE as function $(u')^{-1}$ is non-linear.

(f)

$$\rho \frac{\partial V_t(a)}{\partial a} = \frac{\partial^2 V_t(a)}{\partial t \partial a} + \frac{\partial^2 V_t(a)}{\partial a^2}\left[r_t a + y_t - c\right] + \lambda_t \frac{\partial V_t(a)}{\partial a}$$

(g)

$$\frac{d V(t, a_t)}{dt} = \frac{\partial V(t, a_t)}{\partial t} + \frac{\partial V}{\partial a} \times \frac{\partial a}{\partial t}$$

(h)

$$\frac{d V(t, a_t)}{dt} = \frac{\partial V(t, a_t)}{\partial t} + \frac{\partial V}{\partial a} \times \frac{\partial a}{\partial t}$$

$$u'(c) = \frac{\partial V(t, a)}{\partial a}$$

$$\rho \frac{\partial V_t(a)}{\partial a} = \frac{\partial^2 V_t(a)}{\partial t \partial a} + \frac{\partial^2 V_t(a)}{\partial a^2}\left[r_t a + y_t - c\right] + \lambda_t \frac{\partial V_t(a)}{\partial a}$$

Taking time derivative of FOC,

$$u''(c)\,\dot{c} = \frac{\partial^2 V(t,a)}{\partial a \partial t} + \frac{\partial^2 V(t,a)}{\partial^2 a}\,\dot{a}$$

Now from HJB,

$$\frac{\partial V}{\partial t} = \rho V(t,a) - u(c) - \frac{\partial V}{\partial a}\,\dot{a}$$

$$\Rightarrow \frac{\partial V}{\partial t \partial a} = \rho \frac{\partial V}{\partial a} - \frac{\partial V^2}{\partial a^2}\,\dot{a} - \frac{\partial V}{\partial a}\,r_t$$

$$\Rightarrow u''(c)\,\dot{c} = \rho \frac{\partial V}{\partial a} - \frac{\partial V}{\partial a}\,r_t$$

$$\Rightarrow u''(c)\,\dot{c} = (\rho - r_t)\;u'(c)$$

$$\Rightarrow \quad \frac{u''(c)\dot{c}}{u'(c)} = \rho - r_t \qquad \text{Euler Equation}$$

(2-5)

~~At~~ any time t, you have extracted total oil of $\int_0^t c_s ds$, so remain oil at time t, $x_t = x_0 - \int_0^t c_s ds$

(b)

First we can find flow budget constraint,

$$\dot{x}_t = -c_t$$

State: x_t

Control: c_t

Multipliers: μ_t

then

$$H = u(c_t) + \lambda_t[-c_t]$$

(c)

$$\frac{\partial H}{\partial c} = 0 \quad = \quad u'(c) - \lambda = 0$$

$$\rho \lambda_t - \dot{\lambda}_t = \frac{\partial H}{\partial x} = 0$$

$$\rho \lambda_t = \dot{\lambda}_t$$

$$\dot{x}_t = -c_t$$

using $\quad \frac{\dot{\lambda}_t}{\lambda_t} = \rho$

$$\Rightarrow \quad \lambda_t = \lambda_0 e^{\rho t}$$

$$\Rightarrow \quad u'(c_t) = \lambda_0 e^{\rho t}$$

$$\Rightarrow \quad \frac{1}{c_t} = \mu_0 e^{\rho t}$$

$$\Rightarrow \quad c_t = \frac{1}{\mu_0} e^{-\rho t}$$

Now with budget constraint,

$$x_0 = \int_0^\infty \frac{1}{\mu_0} \cdot e^{-\rho t} \, dt$$

$$x_0 = \frac{1}{\mu_0} \times \left. \frac{e^{-\rho t}}{-\rho} \right|_0^\infty = \frac{1}{\mu_0} \times \frac{1}{\rho}$$

$$x_0 \rho = \frac{1}{\mu_0}$$

$$\Rightarrow \boxed{c_t = x_0 \rho e^{-\rho t}}$$

(d) – (e)

HJB equation is,

$$\rho V(x) = \max_{c \geq 0} \left\{ \ln(c) - \dot{x} \frac{\partial V}{\partial x} \right\}$$

$$= \max_{c \geq 0} \left\{ \ln(c) - c \frac{\partial V}{\partial x} \right\}$$

Now, $\frac{\partial V}{\partial x} = \frac{b}{x}$

$$\rho a + \rho b \ln(x_0) = \max_{c \geq 0} \left\{ \ln(c) - c \cdot \frac{b}{x} \right\}$$

RHS is maximized at

$$\frac{1}{c} = b/x$$

$$c = x/b$$

$$\rho a + \rho b \ln(x_0) = \ln(x/b) - \frac{x}{b} \times b/x$$

$$= \ln(x) - \ln(b) - 1$$

$$\Rightarrow \rho a = -\ln(b) - 1$$

$$\rho b = 1$$

$$\Rightarrow b = 1/\rho$$

$$a = \frac{-\ln(1/\rho) - 1}{\rho} = \frac{\ln(\rho) - 1}{\rho}$$