202A: Dynamic Programming and Applications

Homework #5 Solutions

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The theoretical part of this Homework is very short. In Problem 1, you will set up the model of Aiyagari (1994) in continuous time and define its competitive equilibrium. In the numerical part of the Homework, you will then write code to solve this model numerically.

Problem 1: Aiyagari (1994)

Time is continuous, $t \in [0, \infty)$, and there is no aggregate uncertainty. We allow for one-time, unanticipated ("MIT") shocks to TFP (agents have perfect foresight with respect to this aggregate shock). Concretely, we denote by A_t the TFP of the economy at date t, and $\{A_t\}$ is an exogenously given but deterministic sequence.

Households. The economy is populated by a continuum of measure one of households. Consider a household $i \in [0, 1]$. Household i's preferences are encoded in her lifetime utility

$$V_{i,0} = \max_{\{c_{i,t}\}_{t \geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_{i,t}) dt.$$

The economy features a single asset—capital. Households are the owners of capital and rent it to firms. Denote the stock of capital owned by household i at date t by $k_{i,t}$. The household faces the budget constraints

$$c_{i,t} + i_{i,t} = z_{i,t} w_t + r_t^k k_{i,t}$$

$$\dot{k}_{i,t} = i_{i,t} - \delta k_{i,t}.$$

The first line of the budget constraint says that household expenditures—consumption and investment—must equal her income. Income comprises labor income and income from renting capital to firms at rate r_t^k. We assume that households supply one unit of labor inelastically, earning wage rate w_t. But households

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also face idiosyncratic earnings risk: $z_{i,t}$ denotes the household's idiosyncratic labor productivity, and the wage w_t is paid on efficiency units of work, rather than hours of work. We assume that $z_{i,t}$ follows a two-state Markov chain, with individual labor productivity taking on values in $\in \{z^L, z^H\}$. You may interpret z^L as unemployment and z^H as employment. Households transition between these dates at transition rates λ (symmetric for simplicity).

The second line summarizes the household's capital accumulation technology. Putting these two equations together, the household's stock of capital evolves simply as

$$\dot{k}_{i,t} = r_t k_{i,t} + z_{i,t} w_t - c_{i,t},$$

where we denote by r_t the effective real rate of return on owning capital. Finally, we assume that households also face a short-sale constraint on capital

$$k_{i,t} \geq 0.$$

(a) Write down household i's problem in sequence form.

The problem of household i in sequence form is:

$$\max_{\{c_{i,t}\}_{t\geq 0}} V_{i,0} = E_0 \int_0^\infty e^{-\rho t} u(c_{i,t})\, dt$$

$$\text{s.t.} \quad \dot{k}_{i,t} = (r_t^k - \delta) k_{i,t} + z_{i,t} w_t - c_{i,t} = r_t k_{i,t} + z_{i,t} w_t - c_{i,t} \tag{1'}$$

$z_{i,t}$ follows a two-state Markov chain with $z_{i,t} \in \{z_L, z_H\}$ and transition rates λ

$k_{i,t} \geq 0$

taking as given the initial capital and labor productivity $(k_{i,0}, z_{i,0})$. A solution to the household's problem is a stochastic process $\{c_{i,t}, k_{i,t}\}_{t\geq 0}$.

(b) Next, we characterize a recursive representation for the household problem. In terms of which state variables will we be able to derive a recursive representation? Write down the HJB for this problem. Is the HJB stationary—why or why not? Why are we now dropping the i subscript when writing down this recursive representation of the household problem?

The household's value function $V_t(k, z)$ satisfies:

$$\rho V_t(k,z) = \max_c \left\{ u(c) + \partial_k V_t(k,z) \left[r_t k + z w_t - c \right] + \lambda \left[V_t(k,z') - V_t(k,z) \right] + \partial_t V_t(k,z) \right\} \tag{1a}$$

where z' is the alternate state of $z \in \{z_L, z_H\}$.

The HJB is non-stationary because $V_t(k,z)$ depends on r_t and w_t, which vary over t due to the unanticipated TFP shock. We drop the i subscript because the value function $V_t(k,z)$ represents the problem for any household, parameterized by its state variables (k,z). In other words, since this model assumes no permanent or ex-ante heterogeneity, but only ex-post heterogeneity, households can be identified by their state variables (k,z).

(c) Where in the HJB does the short-sale constraint show up? Write down the explicit condition for this.

The short-sale constraint $k_{i,t} \geq 0$ appears as a state constraint boundary condition in the HJB:

$$\partial_k V_t(0, z) \geq u'(zw_t) \tag{1b}$$

(d) Take the first-order condition.

Resolving max operator gives the first-order condition (FOC) for c, which defines consumption policy function:

$$u'(c_t(k, z)) = \partial_k V_t(k, z) \tag{1c}$$

for all t, k, and z.

Aggregation. We denote by $g_t(k, z)$ the joint density of households over capital and individual labor productivities at date t.

(e) Write down the Kolmogorov forward equation for $g_t(k, z)$.

The joint density $g_t(k, z)$ solves the Kolmogorov forward (KF) equation:

$$\frac{\partial g_t(k, z)}{\partial t} = -\frac{\partial}{\partial k} \left[g_t(k, z) \left(r_t k + z w_t - c \right) \right] + \lambda \left[g_t(k, z') - g_t(k, z) \right] \tag{3}$$

where z' is the alternate state of $z \in \{z_L, z_H\}$, and $c = c_t(k, z)$ is the optimal consumption.

Firms. There is a representative firm that operates the production technology

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha},$$

where the deterministic sequence $\{A_t\}$ denotes TFP. The firm maximizes profit $Y_t - w_t L_t - r_t^k K_t$. We use capital letters here to signify that K_t and L_t denote aggregate capital and labor.

(f) Write down the firm problem and take first-order conditions. This should yield two conditions for factor prices w_t and r_t^k. Why are the implied profits zero?

The firm's problem:

$$\begin{aligned} &\max_{K_t, L_t} Y_t - r_t^k K_t - w_t L_t \\ &\text{s.t.} \quad Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} \end{aligned} \tag{2}$$

First-order conditions:

$$w_t = (1-\alpha) A_t \left(\frac{K_t}{L_t} \right)^{\alpha} \tag{2a}$$

$$r_t^k = \alpha A_t \left(\frac{L_t}{K_t} \right)^{1-\alpha} \tag{2b}$$

Profits are zero due to constant returns to scale and perfect competition.

Markets and equilibrium.

(g) Which markets have to clear in equilibrium? Write down each market clearing condition.

(i) Goods market:

$$Y_t = C_t + I_t \quad \text{where } C_t \equiv \sum_z \int_0^\infty c_t(k,z) g_t(k,z)\, dk$$
$$I_t \equiv \sum_z \int_0^\infty (\dot{k}_t(k,z) + \delta k)\, dk \tag{4}$$

(ii) Labor market:

$$L_t^D = L_t^S \quad \text{where } L_t^S \equiv \sum_z \int_0^\infty (z \cdot 1) g_t(k,z)\, dk \tag{5}$$

(iii) Capital market:

$$K_t^D = K_t^S \quad \text{where } K_t^S \equiv \sum_z \int_0^\infty k g_t(k,z)\, dk \tag{6}$$

(h) Define (recursive) competitive equilibrium.

The (recursive) competitive equilibrium is defined as follows:

(a) Taking as given:

(i) an initial joint density $g_0(k,z)$

(ii) an exogenous path of TFP, $\{A_t\}_{t\geq 0}$

(b) a competitive equilibrium consists of the following **functions**:

$$\{V_t(k,z), c_t(k,z), g_t(k,z)\}_{t\geq 0} \quad \text{and} \quad \{Y_t, L_t, K_t, r_t, r_t^k, w_t\}_{t\geq 0}$$

(c) satisfying the following conditions:

(i) households optimize: (1a), (1b), (1c)

(ii) firms optimize: (2)

(iii) markets clear: (4), (5), (6)

(iv) the joint density evolves consistently with household behavior: (3)

(i) Explicitly write down the system of equations that competitive equilibrium characterizes. Convince yourself that you have account for $\{c_t(k,z), V_t(k,z), g_t(k,z), Y_t, L_t, K_t\}_{t\geq 0}$ and $\{r_t, r_t^k, w_t\}_{t\geq 0}$.

(i) **Household optimization:** The value function $V_t(k,z)$ and policy function $c_t(k,z)$ solve the time-dependent HJB equation:

$$\rho V_t(k,z) = \max_c \left\{ u(c) + \partial_k V_t(k,z) \left[r_t k + z w_t - c \right] + \lambda \left[V_t(k,z') - V_t(k,z) \right] + \partial_t V_t(k,z) \right\} \tag{1a}$$

This is subject to the borrowing constraint:

$$\partial_k V_t(0,z) \geq u'(z w_t) \tag{1b}$$

and the first-order condition:

$$u'(c_t(k,z)) = \partial_k V_t(k,z) \tag{1c}$$

(ii) **Income-wealth distribution:** The joint density $g_t(k,z)$ solves the KF equation:

$$\frac{\partial g_t(k,z)}{\partial t} = -\frac{\partial}{\partial k}\left[g_t(k,z)\left(r_t k + z w_t - c\right)\right] + \lambda\left[g_t(k,z') - g_t(k,z)\right] \tag{3}$$

(iii) **Market clearing conditions**[1]**:**

$$\text{Labor market:} \quad L_t^D = L_t^S \equiv \sum_z \int_0^\infty (z \cdot 1) g_t(k,z)\, dk \tag{5}$$

$$\text{Capital market:} \quad K_t^D = K_t^S \equiv \sum_z \int_0^\infty k g_t(k,z)\, dk \tag{6}$$

(iv) **Production technology:** Output Y_t satisfies the production function:

$$Y_t = A_t K_t^\alpha L_t^{1-\alpha} \tag{2'}$$

(v) **Prices:** Prices w_t, r_t^k, and r_t satisfy:

$$w_t = (1-\alpha) A_t \left(\frac{K_t}{L_t}\right)^\alpha \tag{2a}$$

$$r_t^k = \alpha A_t \left(\frac{L_t}{K_t}\right)^{1-\alpha} \tag{2b}$$

$$r_t^k - \delta = r_t$$

(j) Define stationary competitive equilibrium. Dropping time subscripts to denote allocation and prices in steady state, make sure you have accounted for $\{c(k,z), V(k,z), g(k,z), Y, L, K\}$ and $\{r, r^k, w\}$.

The stationary competitive equilibrium is a special case of the recursive competitive equilibrium defined in (h), where all variables become time-invariant. Specifically, the system of equations in (i) still characterizes the equilibrium, but all time derivatives (e.g., $\partial_t V_t(k,z)$, $\frac{\partial g_t(k,z)}{\partial t}$) drop to zero, as we are now in steady state.

The (recursive) competitive equilibrium is defined as follows:

(a) Taking as given:

  (i) a constant TFP, A

(b) a stationary competitive equilibrium consists of the following functions:

$$\{V(k,z), c(k,z), g(k,z)\} \quad \text{and} \quad \{Y, L, K, r, r^k, w\}$$

(c) satisfying the following conditions:

[1] Goods market clearing condition is dropped by Walras' law.

(i) households optimize: (1a^s), (1b^s), (1c^s)
(ii) firms optimize: (2a^s), (2b^s)
(iii) markets clear: (4^s), (5^s), (6^s)
(iv) the joint density evolves consistently with household behavior: (3^s)

(i) **Household optimization:** The value function $V(k,z)$ and policy function $c(k,z)$ solve the stationary HJB equation:

$$\rho V(k,z) = \max_c \left\{ u(c) + \partial_k V(k,z)\left[rk + zw - c\right] + \lambda \left[V(k,z') - V(k,z)\right] \right\} \tag{1a^s}$$

This is subject to the borrowing constraint:

$$\partial_k V(0,z) \geq u'(zw) \tag{1b^s}$$

and the first-order condition:

$$u'(c(k,z)) = \partial_k V(k,z) \tag{1c^s}$$

(ii) **Income-wealth distribution:** The joint density $g(k,z)$ solves the KF equation:

$$0 = \frac{\partial g(k,z)}{\partial t} = -\frac{\partial}{\partial k}\left[g(k,z)\left(rk + zw - c\right)\right] + \lambda \left[g(k,z') - g(k,z)\right] \tag{3^s}$$

(iii) **Market clearing conditions**[2]**:**

$$\text{Labor market:} \quad L^D = L^S \equiv \sum_z \int_0^\infty (z \cdot 1) g(k,z)\, dk \tag{5^s}$$

$$\text{Capital market:} \quad K^D = K^S \equiv \sum_z \int_0^\infty k g(k,z)\, dk \tag{6^s}$$

(iv) **Production technology:** Output Y satisfies the production function:

$$Y = AK^\alpha L^{1-\alpha} \tag{2'^{s}}$$

(v) **Prices:** Prices w, r^k, and r satisfy:

$$w = (1-\alpha)A\left(\frac{K}{L}\right)^\alpha \tag{2a^s}$$

$$r^k = \alpha A\left(\frac{L}{K}\right)^{1-\alpha} \tag{2b^s}$$

$$r^k - \delta = r$$

[2] The goods market clearing condition is expressed as follows but is dropped by Walras' law.:

$$Y = C + I \quad \text{where } C \equiv \sum_z \int_0^\infty c(k,z) g(k,z)\, dk \tag{4^s}$$