Second Midterm Review Sheet

Ec240a – Second Half, Fall 2021

In preparing for the exam review your lecture notes, assigned course readings, problem sets and prepare answers to the questions on the review sheet(s). The exam is open book, however you are not allowed to consult with classmates or other people (you may freely uses books, notes etc.).

[1] You observe a simple random sample of size N from the population

$$Y_0 \sim N\left(\mu, \sigma^2\right)$$

as well as a second, independent, simple random sample, also of size N, from the population

$$Y_1 \sim N\left(\mu, 4\sigma^2\right).$$

The value of σ^2 is known. Consider the family of estimates of μ

$$\hat{\mu}\left(c_0, c_1\right) = c_0 \bar{Y}_0 + c_1 \bar{Y}_1,$$

where $\bar{Y}_0 = \frac{1}{N}\sum_{i=1}^{N} Y_{0i}$ and $\bar{Y}_1 = \frac{1}{N}\sum_{i=1}^{N} Y_{1i}$.

[a] Show that mean squared error equals

$$\mathbb{E}\left[\left(\hat{\mu}\left(c_0, c_1\right) - \mu\right)^2\right] = \frac{c_0^2 \sigma^2}{N} + \frac{c_1^2 4\sigma^2}{N} + \left(1 - c_0 - c_1\right)^2 \mu^2. \tag{1}$$

[b] Derive the oracle estimator (within the family) which minimizes (1).

[c] Show that

$$\hat{R}\left(c_0, c_1\right) = \frac{c_0^2 \sigma^2}{N} + \frac{c_1^2 4\sigma^2}{N} + \left(1 - c_0 - c_1\right)^2 \frac{1}{2}\left\{\bar{Y}_0^2 + \bar{Y}_1^2 - \frac{5\sigma^2}{N}\right\} \tag{2}$$

is an unbiased estimate of of (2). Can you propose another unbiased risk estimate? Why would you prefer one unbiased risk estimate over another?

[d] Describe in *words* how one might use (2) to construct an implementable estimator of μ.

[2] Let $m(Z) = \mathbb{E}[X|Z]$ and consider the linear regression

$$\mathbb{E}^*\left[Y|X, m(Z), A\right] = \alpha_0 + \beta_0 X + \gamma_0 m(Z) + A.$$

[a] Show that

$$\mathbb{E}^*\left[m(Z)|X\right] = \delta_0 + \xi_0 X$$

with

$$\begin{aligned} \delta_0 &= (1 - \xi_0)\,\mathbb{E}[X] \\ \xi_0 &= \frac{\mathbb{V}\left(\mathbb{E}[X|Z]\right)}{\mathbb{E}\left[\mathbb{V}(X|Z)\right] + \mathbb{V}\left(\mathbb{E}[X|Z]\right)}. \end{aligned}$$

[b] Assume the population under consideration is working age adults who grew up in the San Francisco Bay Area. Let Y denote a adult log income, let X denote the log income of one's parents as a child and let

Z be a vector of dummy variables denoting an individual's neighborhood of residence as a child. Provide an interpretation of ξ_0 as a measure of residential stratification by income.

[c] Establish the notation $\rho = \text{corr}(A, X)$, $\mu_A = \mathbb{E}[A]$, $\mu_X = \mathbb{E}[X]$, $\sigma_A^2 = \mathbb{V}(A)$ and $\sigma_X^2 = \mathbb{V}(X)$. Show that

$$\mathbb{E}^*[Y|X] = \alpha_0 + \gamma_0(1-\xi_0)\mu_X + \left(\mu_A - \rho\frac{\sigma_A}{\sigma_X}\mu_X\right) + \left\{\beta_0 + \gamma_0\xi_0 + \rho\frac{\sigma_A}{\sigma_X}\right\}X.$$

[d] Your research assistant computes an estimate of $\mathbb{E}^*[Y|X]$ using random sample from San Francisco. She computes a separate estimate using a random sample from New York City. Assume that there is more residential stratification by income in New York than in San Francisco. How would you expect the intercept and slope coefficients to differ across the two regression fits?

[3] Let Y equal tons of banana's harvested in a given season for a randomly sampled Honduran banana planation. Output is produced using labor and land according to $Y = AL^{\alpha_0}D^{1-\alpha_0}$, where L is the number of employed workers and D is the size of the plantation in acres and we assume that $0 < \alpha_0 < 1$. The price of a unit of output is P, while that of a unit of labor is W. These prices may vary across plantations (e.g., due to transportation costs, labor market segmentation etc.). We will treat D as a fixed factor; A captures sources of plantation-level differences in farm productivity due to unobserved differences in, for example, soil quality and managerial capacity. Plantation owners choose the level of employed labor to maximize profits. The observed values of L are therefore solutions to the optimization problem:

$$L = \arg\max_l P \cdot Al^{\alpha_0}D^{1-\alpha_0} - W \cdot l.$$

[a] Show that the amount of employed labor is given by

$$L = \left\{\alpha_0 \frac{P}{W} A\right\}^{\frac{1}{1-\alpha_0}} D. \tag{3}$$

[b] Let $a_0 = \frac{1}{1-\alpha_0}\ln\alpha_0 + \frac{1}{1-\alpha_0}\mathbb{E}[\ln A]$, $b_0 = \frac{1}{1-\alpha_0}$, and $V = \frac{1}{1-\alpha_0}\{\ln A - \mathbb{E}[\ln A]\}$. Show that the log of the labor-land ratio is given by

$$\ln\left(\frac{L}{D}\right) = a_0 + b_0\ln\left(\frac{P}{W}\right) + V \tag{4}$$

and that, letting $c_0 = \mathbb{E}[\ln A]$ and $U = \ln A - \mathbb{E}[\ln A]$, the log of planation yield (output per unit of land) is given by

$$\ln\left(\frac{Y}{D}\right) = c_0 + \alpha_0\ln\left(\frac{L}{D}\right) + U. \tag{5}$$

[c] Briefly discuss the content and plausibility of the restriction

$$\mathbb{E}[\ln A|\ln(P/W)] = \mathbb{E}[\ln A]. \tag{6}$$

[d] Using (4), (5) and (6) show that the coefficient on $\ln(L/D)$ in $\mathbb{E}^*[\ln(Y/D)|\ln(L/D)]$ equals

$$\alpha_0 + (1-\alpha_0)\frac{\mathbb{V}(\ln A)}{\mathbb{V}(\ln A) + \mathbb{V}(\ln(P/W))}.$$

Provide some economic intuition for this result.

[e] Using (4), (5) and (6) show that the coefficient on $\ln(L/D)$ in $\mathbb{E}^*[\ln(Y/D)|\ln(L/D), V]$ equals α_0.

Provide some economic intuition for this result.

[f] Assume that all plantations face the same output price (P) and labor cost (W). What value does the coefficient on $\ln(L/D)$ in $\mathbb{E}^*[\ln(Y/D)|\ln(L/D)]$ equal now? Why?

[4] Let Y denote log-earnings and X years of completed schooling for a cohort of workers. Assume a random sample of size N is available from this population. Let $D_x = 1$ if $X = x$ and zero otherwise. Assume that $X \in \{0, \ldots, 16\}$ with positive probability attached to each support point.

[a] Let

$$\mathbb{E}^*[Y|D_1, \ldots, D_L] = \alpha_0 + \sum_{l=1}^{16} \gamma_{0l} D_l.$$

What is the relationship between this linear predictor and $\mathbb{E}[Y|X = x]$?

[b] Assume that $\Pr(X = 6) = 0$. Is the linear predictor defined in part [a] still well-defined? Why or why not?

[c] You hypothesize that $\mathbb{E}[Y|X = x]$ is linear in x. Consider the linear predictor in part [a] and let $\beta = (\alpha, \gamma_1, \ldots, \gamma_{16})'$. Show how your hypothesis may be equivalently expressed as set of linear restrictions of the form $C\beta_0 = c$. Provide explicit expressions for C and c. Describe how you would construct a test statistic for your hypothesis. What is the asymptotic sampling distribution of your statistic under the null? Assume that your have a consistent estimate $\hat{\Lambda}$ of the asymptotic variance-covariance matrix of $\sqrt{N}\left(\hat{\beta} - \beta\right)$, with $\hat{\beta}$ the least squares estimate.

[5] Let $X \in \{0, 1, 2\}$ and $Y \in \{0, 1, 2\}$. The probability of the event $X = x$ and $Y = y$ for all possible combinations of x and y is given in the following table:

X\Y	0	1	2
0	$\frac{15}{100}$	$\frac{10}{100}$	$\frac{5}{100}$
1	$\frac{10}{100}$	$\frac{20}{100}$	$\frac{10}{100}$
2	$\frac{5}{100}$	$\frac{10}{100}$	$\frac{15}{100}$

[a] Calculate $\mathbb{E}[Y]$ and $\mathbb{E}[Y|X = 1]$. Are X and Y independent?

[b] Calculate $\mathbb{E}[X]$, $\mathbb{E}\left[X^2\right]$ and hence $\mathbb{V}(X)$.

[c] Calculate $\mathbb{C}(X, Y)$ and also the coefficient on X in $\mathbb{E}^*[Y|X]$.

[d] Calculate the intercept of $\mathbb{E}^*[Y|X]$.

[e] Repeat [a] to [d] above for the following joint distribution

X\Y	0	1	2
0	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
1	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
2	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

[6] Let Y be a scalar random variable, X a K vector of covariates (which includes a constant), and W a vector of additional covariates (which excludes a constant). Consider the long (linear) regression

$$\mathbb{E}^*[Y|W, X] = X'\beta_0 + W'\gamma_0. \tag{7}$$

Next define the short and auxiliary regressions

$$\mathbb{E}^*[Y|X] = X'b_0 \tag{8}$$
$$\mathbb{E}^*[W|X] = \Pi_0 X. \tag{9}$$

[a] Let $V = W - \mathbb{E}^*[W|X]$ be the projection error associated with the auxiliary regression. Show that

$$\begin{aligned}\mathbb{E}^*[Y|V,X] &= \mathbb{E}^*[Y|X] + \mathbb{E}^*[Y|1,V] - \mathbb{E}[Y] \\ &= \mathbb{E}^*[Y|X] + \mathbb{E}^*[Y|V]\end{aligned}$$

where $\mathbb{E}^*[Y|1,V]$ denotes the linear regression of Y onto a constant and V, while $\mathbb{E}^*[Y|V]$ denotes the corresponding regression without a constant (Hint: Observe that $\mathbb{C}(X,V)=0$).

[b] Next show that $\mathbb{E}^*[Y|V,X] = \mathbb{E}^*[Y|W,X]$ and hence that the coefficient on V in $\mathbb{E}^*[Y|V,X]$ coincides with that on W in $\mathbb{E}^*[Y|W,X]$.

[c] Let $U = Y - \mathbb{E}^*[Y|X]$ be the projection error associated with the short regression. Derive the coefficient on V in the linear regression of U onto V (excluding a constant).

[d] Discuss the possible practical value of the results shown in [b] and [c] above.

[7] This question is about the Rambly Shambly Hex Bolt Corporation. Let Y_t be the number of hex bolts produced by Rambly Shambly Hex in year t, M_t tons of steel used in production, K_t total factory capital stock, and L_t total person-hours worked. We assume that

$$Y_t = A_t M_t^{\alpha} K_t^{\beta} L_t^{\gamma}.$$

[a] Rambly Shambly Hex is owned by an eccentric billionaire who chooses M_t, K_t and L_t each year randomly using an eternally unchanging roulette-wheel-like-device (i.e., inputs are chosen independently of each other and independently of A_t). Further assume that the distribution of A_t is i.i.d. over time. Show that under this input choice mechanism that

$$\mathbb{E}^*[\ln Y_t|\ln M_t, \ln K_t, \ln L_t] = \lambda + \alpha \ln M_t + \beta \ln K_t + \gamma \ln L_t$$

with $\lambda = \mathbb{E}[\ln A_t]$. Is this same result likely to hold if Rambly Shambly instead chose input levels to maximize profits? Why or why not?

[b] Further show that under the completely random input choice scheme described above that:

$$\mathbb{E}^*[\ln Y_t|\ln M_t, \ln K_t, \ln L_t] = \mathbb{E}^*[\ln Y_t|\ln M_t] + \mathbb{E}^*[\ln Y_t|\ln K_t] + \mathbb{E}^*[\ln Y_t|\ln L_t] - 2\mathbb{E}[\ln Y_t].$$

[c] Let $X_t = (\ln M_t, \ln K_t, \ln L_t)'$ and $\sigma^2 = \mathbb{V}(\ln A_t)$. Argue that under the completely random input choice scheme:

$$\sqrt{T}\left(\hat{\theta} - \theta_0\right) \xrightarrow{D} N(0, \Lambda),$$

for $\theta = (\alpha, \beta, \gamma)'$, $\Lambda = \sigma^2 \mathbb{V}(X_t)^{-1}$ and $\hat{\theta}$ estimated by the OLS fit of $\ln Y_t$ onto a constant and X_t for $t = 1, \ldots, T$. What are the values of the off-diagonal elements of $\mathbb{V}(X_t)$?

[d] For $T = 3,859$, an OLS fit gives

$$\hat{\theta} = \begin{pmatrix} 0.32 \\ 0.36 \\ 0.42 \end{pmatrix}, \quad \hat{\Lambda} = \begin{pmatrix} 5 & 1/200 & 2/1000 \\ 1/200 & 3 & 3/1000 \\ 2/1000 & 3/1000 & \frac{198}{100} \end{pmatrix}.$$

Construct a Wald Statistic (carefully explaining each step in the construction) for the null hypothesis of constant returns to scale (i.e., $H_0 : \alpha + \beta + \gamma = 1$). What is the appropriate reference distribution and critical value for a two-sided test with size $\alpha = 0.05$? Do you reject the null?

[8] Available is a random sample of $i = 1, \ldots, N$ farmers. We have measures of output, Y_t, and (labor) input, X_t, for each farmer (on a per hectare basis) for each of $t = 1, \ldots, T$ years. The sample is

$$\left\{ (Y_{i1}, \ldots, Y_{iT}, X_{i1}, \ldots, X_{iT})' \right\}_{i=1}^{N}.$$

We assume that, for $Y_t = \ln O_t$ and $X_t = \ln L_t$, output per hectare, O_t, equals

$$O_t = L_t^{\beta} Q^{\gamma} \exp(U_t)$$

with L_t labor, Q soil quality – which is unobserved by the econometrician – and U_t a stochastic, and also unobserved, input outside of the farmer's control (e.g., rainfall).

Our behavioral assumption is that the farmer chooses L_t to maximize expected profits. She knows period t output and input prices, respectively P_t and W_t, as well as her soil quality, Q.

$$L_t = \arg\max_{l} \mathbb{E}\left[P_t l^{\beta} Q^{\gamma} \exp(U_t) - W_t l \,\middle|\, P_t, W_t, Q \right]$$

She does not know rainfall, assumed non-forecastable by anything in her information set, with marginal distribution

$$U_t \sim N\left(0, \sigma_U^2\right).$$

You may also assume that (P_t, W_t) varies independently of all other variables in the model and is also independently and identically distributed across farms and over time.

[a] Show that the farmer's t labor input is

$$X_t = \mu + \frac{1}{1-\beta} A + V_t$$

with $\mu = \frac{1}{1-\beta}\left(\ln \beta + \frac{\sigma_U^2}{2}\right)$, $A = \gamma \ln Q$ and $V_t = \frac{1}{1-\beta} \ln\left(\frac{P_t}{W_t}\right)$. How does rainfall risk affect the farmer's chosen labor input level? Soil quality?

[b] Show that

$$\mathbb{E}\left[Y_t \mid X_t, A \right] = \beta X_t + A.$$

Why is conditioning on land quality alone sufficient to identify β? What maintained assumption is important for this result?

[c] Show that, for $\sigma_A^2 = \mathbb{V}(A)$, and $\sigma_V^2 = \mathbb{V}(V_t)$ for $t = 1, \ldots, T$, that

$$\begin{aligned}
\mathbb{C}(A, X_t) &= \frac{1}{1-\beta}\sigma_A^2 \\
\mathbb{V}(X_t) &= \left(\frac{1}{1-\beta}\right)^2 \sigma_A^2 + \sigma_V^2 \\
\mathbb{E}[X_t] &= \mu + \frac{1}{1-\beta}\mathbb{E}[A] + \mathbb{E}[V_t].
\end{aligned}$$

[d] Next use your results in part [c] to show that

$$\mathbb{E}^*[A|\, X_t] = \eta_0 + \eta_1 X_t$$

where

$$\begin{aligned}
\eta_1 &= (1-\beta)\left[1 + (1-\beta)^2 \frac{\sigma_V^2}{\sigma_A^2}\right]^{-1} \\
\eta_0 &= E[A] - \eta_1 \left[\mu + \frac{1}{1-\beta}\mathbb{E}[A] + \mathbb{E}[V_t]\right].
\end{aligned}$$

[e] Using your results from parts [b] and [d] solve for the coefficient, say b, on X_t in $\mathbb{E}^*[Y_t|\, X_t]$. Does $b = \beta$? Discuss economic conditions under which $b \approx \beta$ as well as conditions where $b >> \beta$.

[f] Let $X = (X_1, \ldots, X_T)'$. Solve for $\mathbb{V}(X)$, $\mathbb{C}(A, X)$ and hence the coefficients $\delta = (\delta_1, \ldots, \delta_T)'$ on $X_1, \ldots, X_T$ in the linear regression

$$\mathbb{E}^*[A|\, X] = \lambda + X'\delta.$$

[g] Let $Y = (Y_1, \ldots, Y_T)'$ and $W = (\mathbf{1}, X')'$ Solve for the multivariate linear predictor

$$\mathbb{E}^*[Y|\, X] = \Pi W.$$

Specifically, express the elements of the $T \times (1+T)$ matrix Π in terms of β, λ and δ.

[h] Let $\theta = (\beta, \lambda, \delta')'$. Let $\pi = \mathrm{vec}(\Pi')$ be the $T(T+1) \times 1$ vector constructed by vertically stacking the columns of Π. Let G be a $T(T+1) \times (1+1+T)$ matrix such that

$$\pi = G\theta.$$

Derive the form of G.

[i] Consider the estimator

$$\hat{\pi} = \left[\frac{1}{N}\sum_{i=1}^{N}(I_T \otimes W_i)(I_T \otimes W_i)'\right]^{-1} \times \left[\frac{1}{N}\sum_{i=1}^{N}(I_T \otimes W_i)\, Y_i\right].$$

Further define $U = Y - (I_T \otimes W_i)'\,\pi$ and

$$\Gamma = I_T \otimes \mathbb{E}[WW']$$

and

$$\Omega = \mathbb{E}\left[(I_T \otimes W_i) UU' (I_T \otimes W_i)'\right].$$

Argue that $\hat{\pi} \xrightarrow{p} \pi$ and furthermore also argue that $\sqrt{N}(\hat{\pi} - \pi) \xrightarrow{D} \mathcal{N}(0, \Lambda)$ for $\Lambda = \Gamma^{-1}\Omega\Gamma^{-1}$.

[j] $\boxed{\text{Optional}}$ Construct a minimum distance estimate of θ and derive its asymptotic sampling distribution.

[9] Consider the following joint probability density function for x and y

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{11}\left(x^2 + y\right), & (x,y) \in [0,2] \times [0,1] \\ 0 & (x,y) \notin [0,2] \times [0,1] \end{cases}$$

[a] Show that $f_{X,Y}(x,y)$ is a valid probability density function.

[b] Compute the conditional density $f_{Y|X}(y|\, x)$.

[c] Compute the conditional expectation function $\mathbb{E}[Y|\, X = x]$.

[d] Compute the linear predictor $\mathbb{E}^*[Y|\, X = x]$.

[e] Consider a joint distribution with a conditional that coincides with the one derived in part [b] above, but where the marginal distribution of X is uniform on $[0,2]$. Explain, qualitatively, how this change would affect your answers in parts [c] and [d] above?

[10] Consider two large populations of households and firms, both of which may costlessly migrate across $i = 1, \ldots, N$ different localities. Conditional on choosing to reside in location i, households $t = 1, \ldots, M_i$, choose the amount of unimproved land, l, and composite commodity, x, to consume in order to maximize utility:

$$\max_{l,x} Q_i l^{\gamma} x^{1-\gamma} \ s.t. \ H_{it}W_i \leq R_i l + x, \tag{10}$$

where Q_i, W_i and R_i respectively denote the "quality of life", wage rate per efficiency unit of labor, and rent per unit of unimproved land, in locality i. The price of the composite commodity is fixed on world markets and normalized to one. A household's endowment of efficiency units of labor is given by H_{it} so that their total budget conditional on residence in locality i is $H_{it}W_i$. Let L_{it} and X_{it} denote the household's utility-maximizing land and composite commodity consumption.

Equilibrium requires that households are indifferent between locations, or that the indirect utility associated with one efficiency unit of labor is constant across communities. Under (10) this condition takes the form

$$V(W,R;Q) = \psi \frac{QW}{R^{\gamma}} = \bar{u}, \ \psi = \gamma^{\gamma}(1-\gamma)^{1-\gamma}, \tag{11}$$

where $\bar{u}$ it the common, nationwide, utility level (available to an owner of an efficiency unit of labor).

Firm $f = 1, \ldots, L_i$ in locality i produces X_{if} units of the composite commodity using a constant returns to scale production technology requiring three inputs: (i) efficiency units of labor, n ; (ii) units of improved land, l , and (iii) capital, k. Labor and land prices in a locality are determined in equilibrium, while the cost of capital is constant and equal to σ. The firm's problem is to minimize costs:

$$\min_{n,l,k} W_i n + R_i l + \sigma k, \ s.t. \ A_i n^{\alpha} l^{\gamma} k^{1-\alpha-\beta} = X_{if}, \tag{12}$$

where A_i is locality-specific total factor productivity. Free entry ensures that profits are zero in equilibrium. The zero profit condition requires that unit costs equal the normalized price of the composite good:

$$C(W,R,A) = \xi \frac{W^\alpha R^\beta}{A} = 1,\ \xi = \left(\frac{1}{\alpha}\right)^\alpha \left(\frac{1}{\beta}\right)^\beta \left(\frac{1}{1-\alpha-\beta}\right)^{1-\alpha-\beta} \sigma^{1-\alpha-\beta} \tag{13}$$

[a] Use (11) and (13) to show that the equilibrium wage and rent in locality i is given by

$$\begin{aligned}
\ln W_i &= \kappa_W + \frac{\gamma}{\alpha\gamma+\beta}\ln A_i - \frac{\beta}{\alpha\gamma+\beta}\ln Q_i \\
\ln R_i &= \kappa_R + \frac{1}{\alpha\gamma+\beta}\ln A_i + \frac{\alpha}{\alpha\gamma+\beta}\ln Q_i,
\end{aligned} \tag{14}$$

with

$$\begin{aligned}
\kappa_W &= \frac{1}{\alpha\gamma+\beta}\left(\beta\ln\bar{u} - \beta\ln\psi - \gamma\ln\xi\right) \\
\kappa_R &= \frac{1}{\alpha\gamma+\beta}\left(\alpha\ln\bar{u} - \alpha\ln\psi - \ln\xi\right).
\end{aligned}$$

HINT: You do not need to verify the form of the intercepts in (14).

[b] Assume that A_i and Q_i vary independently. Let $\phi = \frac{\gamma^2\mathbb{V}(\ln A_i)}{\gamma^2\mathbb{V}(\ln A_i)+\beta^2\mathbb{V}(\ln Q_i)}$. Show that the coefficient on $\ln W_i$ in the (mean squared error minimizing) linear predictor of $\ln R_i$ onto a constant and $\ln W_i$ equals

$$b = \frac{1}{\gamma}\phi - \frac{\alpha}{\beta}(1-\phi).$$

Interpret this expression. Under what conditions is b positive? Negative? Why?

[c] Assume that the $\gamma = 1/3$, $\alpha = 3/5$, and $\beta = 1/5$. A macro economist, with considerable central banking experience, asserts that the "quality-of-life" differences across cities are "overblown". An econometrician, with almost no "real world" experience, asserts that "firms are equally unproductive in all places". An urban economist claims that "actually both are equally important".

Using a random sample of US cities you compute an estimate of b equal to -3. Who is correct? The macro-economist, econometrician or urban economist?

[d] It turns out that you made a Python coding error and that the correct estimate of b is zero. Who do you believe to be correct now?

[e] Use (14) to solve for $\ln A_i$ and $\ln Q_i$ (again you may ignore constants). Use the parameter values given in (c) to simplify your expressions.

[f] You observe mean January temperature for each city in your sample. Can you compute consistent estimates of the semi-elasticity of total factor productivity and quality of life with respect to temperature? Describe your procedure.

[11] Let $X \sim \text{Uniform}[-1,1]$ and assume that

$$Y = -\frac{2}{3} + X^2 + V,\ \ V|\,X \sim \mathcal{N}\left(0,\sigma^2\right) \tag{15}$$

[a] Calculate $\mathbb{E}[Y|\,X]$

[b] Calculate $\mathbb{E}\left[X^2\right]$ and $\mathbb{V}(X)$

[c] Calculate $\mathbb{E}[Y]$

[d] Calculate $\mathbb{C}(X,Y)$ and also the coefficient on X in $\mathbb{E}^*[Y|X]$.

[e] Let $U = Y - \mathbb{E}^*[Y|X]$. Show that $\mathbb{C}(U,X) = 0$. Give an intuitive explanation for this result.

[f] Find a function $g(X)$ such that that $\mathbb{C}(U, g(X)) = \mathbb{V}\left(X^2\right)$. Give an intuitive explanation for your answer.

[g] Describe how your answers in (d) to (f) would change if (15) held but now $X \sim \text{Uniform}[0,2]$. You may find it helpful to sketch a figure.

[12] Let $A \in \{a_l, a_h\}$ denote an individual's unobserved 'entrepreneurial acumen', and X be a binary indicator taking a value of one if an individual completed an undergraduate degree and zero otherwise. Let Y equal annual earnings. The following table gives the conditional mean of Y for each of the four possible 'entrepreneurial acumen' and schooling combinations

	$A = a_h$	$A = a_l$
$X = 1$	\$45,000	\$35,000
$X = 0$	\$50,000	\$15,000

.

Assume that $m(x,a) = \mathbb{E}[Y|X = x, A = a]$ is a structural function in the following sense: in subpopulations homogenous in 'entrepreneurial acumen', $m(x,a)$, traces out how average earnings would change with external manipulations in college completion behavior. The population frequency of each of the four schooling and 'entrepreneurial acumen' combinations is

	$A = a_h$	$A = a_l$
$X = 1$	0.20	0.10
$X = 0$	0.05	0.65

.

[a] While on the elevator in Evans Hall you heard a grumpy individual (possibly a professor) claim "the best students should just start a tech firm in their parents' garages, we can train the rest to become corporate lawyers". Comment with reference to the population described above.

[b] Calculate the average annual earnings level in this economy, $\mathbb{E}[Y]$, and the averages conditional on college completion, $\mathbb{E}[Y|X = 1]$, and not, $\mathbb{E}[Y|X = 0]$.

[c] Calculate average earnings in a counterfactual world where $\Pr(X = 1|A = a_l) = 1$ and $\Pr(X = 1|A = a_h) = 0$.

[d] What is the expected earnings gain associated with college completion for a random draw from the population?

[e] Let $W = 1$ if an individual operated a lemonade stand at some point during childhood and zero otherwise. Assume that (i) $0 < \Pr(X = 1|W = w) < 1$ for $w \in \{0,1\}$ and (ii) that X is conditionally independent of A given W. Show that for $q(x,w) = \mathbb{E}[Y|X = x, W = w]$ we have $\mathbb{E}[q(x,W)] = \mathbb{E}[m(x,A)]$. How would your answer change if $\Pr(X = 1|W = 1)$ were equal to one?

[f] Maintaining the assumptions of part [e] above show that

$$\mathbb{E}\left[\frac{\mathbf{1}(X = x)Y}{\Pr(X = x|W)}\right] = \mathbb{E}[m(x,A)].$$

Provide an intuitive discussion of this result.

[g] Available is a random sample of size N from the population of high school graduates. For each unit we observe $Z = (W, X, Y)'$. Let

$$R_1 = \left(\mathbf{1}(X=0)\mathbf{1}(W=0), \mathbf{1}(X=0)\mathbf{1}(W=1), \mathbf{1}(X=1)\mathbf{1}(W=0), \mathbf{1}(X=1)\mathbf{1}(W=1)\right)'$$

where $\mathbf{1}(\bullet)$ denotes the indicator function and

$$\mathbf{S} = \begin{pmatrix} Y \\ W \\ X \\ WX \end{pmatrix}, \mathbf{R} = \begin{pmatrix} R_1' & \underline{0}_3' \\ \underline{0}_3\underline{0}_3' & I_3 \end{pmatrix}$$

with $\underline{0}_k$ a $k \times 1$ vector of zeros and I_k a $k \times k$ identity matrix. Establish the following notation: $\mu_{xw} = q(x, w)$, $\sigma_{xw}^2 = \mathbb{V}(Y|\, X = x, W = w)$, $p_x = \Pr(X = x)$, $q_w = \Pr(W = w)$, and $r_{xw} = \Pr(X = x, W = w)$. Assume (i) σ_{xw}^2 is finite and (ii) that $r_{xw} > 0$ for all four x and w combinations.

Consider the estimate

$$\hat{\beta} = \left[\frac{1}{N}\sum_{i=1}^{N}\mathbf{R}_i'\mathbf{R}_i\right]^{-1} \times \left[\frac{1}{N}\sum_{i=1}^{N}\mathbf{R}_i'\mathbf{S}_i\right].$$

[i] Show that $\hat{\beta} \to \beta_0$ and provide an expression for each of the seven elements of β_0 in terms of the notation established above (i.e. in terms of μ_{xw}, σ_{xw}^2 etc.). How would your analysis change if it were the case that $r_{11} = 0$?

[ii] Show that $\sqrt{N}\left(\hat{\beta} - \beta_0\right)$ converges in distribution to a normal random variable. Provide an explicit expression for the covariance matrix of this normal distribution in terms of the notation established above (i.e. in terms of μ_{xw}, σ_{xw}^2 etc.).

[iii] Using the elements of $\hat{\beta}$ construct estimates of $\mathbb{E}(q(1, W))$ and $\mathbb{E}(q(0, W))$. Establish the consistency of these estimates.

[iv] You are interested in the joint hypothesis that $\mu_{10} = \mu_{00}$ and $\mu_{11} = \mu_{01}$. Discuss the substance of this hypothesis in light of the empirical set-up developed in parts [a] to [f] above. Show that this hypothesis may be represented as a restriction of the form $C\beta_0 = c$ for some matrix C and column vector c.

[v] Your professor provides you will the following (consistent) estimates: $\hat{\mu}_{00} = 10,000$, $\hat{\mu}_{01} = 20,000$, $\hat{\mu}_{10} = 50,000$, $\hat{\mu}_{11} = 10,000$, $\hat{\sigma}_{00}^2 = 1,000$, $\hat{\sigma}_{01}^2 = 2,000$, $\hat{\sigma}_{10}^2 = 500$, $\hat{\sigma}_{11}^2 = 1,000$, $\hat{r}_{00} = 0.2$, $\hat{r}_{01} = 0.3$, and $\hat{r}_{10} = 0.25$. Construct a test statistic for the hypothesis described in part [iii] above and compare it to the appropriate critical value. For your reference the 0.95 quantiles of χ^2 random variables with parameters 1, 2 and 3 are, respectively, 3.84, 5.99 and 7.81.