Sequence-Space Jacobians and Fake-News Algorithm

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Abstract

This note presents a sequence-space representation of the canonical, one-asset HANK model with wage rigidity. It then develops an implementation Auclert et al. (2021)'s fake-news algorithm that differs slightly from theirs.

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1 The Canonical One-Asset HANK Model

In this section, I briefly describe the canonical one-asset HANK model with wage rigidity. Time is continuous, $t \in [0, T]$, with $T \leq \infty$. There is no aggregate uncertainty and I focus on one-time, unanticipated shocks. The model sketched here follows closely that of Dávila and Schaab (2022).

1.1 Environment

Households. There is a unit mass of households whose lifetime utility is

$$V_0 = \max \mathbb{E}_0 \int_0^\infty e^{-\rho dt} U(c_t, n_t) dt, \tag{1}$$

where $U(\cdot)$ denotes instantaneous utility flow from consumption c_t and hours worked n_t. ρ is a common discount factor.

Households can trade a bond subject to the budget constraint

$$\dot{a}_t = r_t a_t + z_t w_t n_t + \tau_t(z_t) - c_t, \tag{2}$$

where a_t denotes the household's stock of bond holdings. Households take as given the relevant prices, the real interest rate r_t and the real wage rate w_t. Households receive a lump-sum rebate $\tau_t(z_t)$ which will be 0 in equilibrium. Household earnings risk is encoded in z_t. For now, we assume that z_t follows a two-state Markov chain. Finally, households face the borrowing constraint $a_t \geq \underline{a}$.

We denote by $g_t(a, z)$ the cross-sectional household distribution or, formally, the joint density over bond holdings and earnings at time t.

Labor markets. We adopt the textbook model of nominal wage stickiness. Labor unions intermediate labor supply decisions and ration labor across households. That is, $n_t = N_t$, where N_t is aggregate labor. Nominal wages are sticky, and unions pay a quadratic Rotemberg (1982) adjustment cost to change wages, $-\frac{\delta}{2}(\pi_t^w)^2$, where π_t^w denotes wage inflation and the parameter $\delta \geq 0$ modulates the degree of wage rigidity.

The union's dynamic wage-setting problem gives rise to a non-linear New Keynesian wage Phillips curve, given by

$$\dot{\pi}_t^w = \rho \pi_t^w + \frac{\epsilon}{\delta} \iint n_t \left(\frac{\epsilon - 1}{\epsilon} (1 + \tau^L) w_t z u'(c_t) - v'(n_t) \right) g_t(a, z) \, da \, dz, \tag{3}$$

where ϵ is the elasticity of substitution across unions. We allow for an employment subsidy τ^L to potentially offset unions' desired markup.

Production. A representative firm produces the final consumption good using labor,

$$Y_t = A_t N_t, \tag{4}$$

where total factor productivity (TFP) A_t is potentially time-varying and a source of exogenous productivity shocks. Under perfect competition and flexible prices, profits from production are zero and the marginal cost of labor is equal to its marginal product, with

$$w_t = A_t, \tag{5}$$

so the real wage w_t is equal to the marginal rate of transformation (MRT) A_t.

Government. The fiscal authority pays for the employment subsidy with a lump-sum tax. Assuming proportionality to earnings potential z_t, households receive a net fiscal rebate of exactly 0, i.e., $\tau_t(z_t) = 0$.

Monetary policy follows a Taylor rule, given by

$$i_t = r_{ss} + \lambda_\pi \pi_t + \lambda_Y \frac{Y_t - Y_{ss}}{Y_{ss}} + \theta_t, \tag{6}$$

where θ_t is a monetary policy shock. A Fisher relation holds, with

$$r_t = i_t - \pi_t, \tag{7}$$

where π_t is consumer price inflation. Finally, we can relate price inflation to wage inflation by differentiating equation (5), which yields

$$\pi_t = \pi_t^w - \frac{\dot{A}_t}{A_t}. \tag{8}$$

1.2 Equilibrium

Definition 1 (Competitive Equilibrium)**.** *Given an initial cross-sectional distribution $g_0(a,z)$ and predetermined sequences of shocks $\{A_t, \xi_t\}$, an equilibrium is defined as paths for prices $\{\pi_t^w, \pi_t, w_t, r_t\}$, aggregates $\{Y_t, N_t, C_t, B_t\}$, individual allocation rules $\{c_t(a,z)\}$, and for the joint distribution over household bond holdings and idiosyncratic labor productivities $\{g_t(a,z)\}$, such that households optimize, unions optimize, labor is rationed, firms optimize, and markets for goods and bonds clear, that is,*

$$Y_t = C_t = \iint c_t(a,z)\, g_t(a,z)\, da\, dz \tag{9}$$

$$0 = B_t = \iint a\, g_t(a,z)\, da\, dz. \tag{10}$$

We can summarize a minimal representation of equilibrium conditions in terms of two blocks, a micro block and a macro block. In particular, equilibrium comprises the 10 functions

$$\Big\{c_t(a,z), V_t(a,z), g_t(a,z), Y_t, N_t, r_t, i_t, w_t, \pi_t, \pi_t^w\Big\},$$

which solve the following equations.

Micro block. At the micro level, we have the Hamilton-Jacobi-Bellman (HJB) equation for household value $V_t(a,z)$, the household's consumption-savings optimality condition, and the Kolmogorov forward (KF) equation for the evolution of the distribution. These are given by

$$\rho V_t(a,z) = U(c_t, N_t) + \partial_t V_t(a,z) + \mathcal{A}_t V_t(a,z) \tag{11}$$

$$u'(c_t(a,z)) = \partial_a V_t(a,z) \tag{12}$$

$$\partial_t g_t(a,z) = \mathcal{A}_t^* g_t(a,z), \tag{13}$$

where the generator satisfies, using the short-hand $s_t(a,z) = r_t a + z w_t N_t - c_t(a,z)$,

$$\mathcal{A}_t f_t(a,z) = s_t(a,z)\partial_a f_t(a,z) + \mathcal{A}^z f_t(a,z),$$

where $\mathcal{A}^z$ is an operator (transition matrix) that characterizes the exogenous earnings process z_t. Similarly, the adjoint is characterized by

$$\mathcal{A}_t^* g_t(a,z) = -\partial_a \Big[s_t(a,z) g_t(a,z)\Big] - \mathcal{A}^{z,*} g_t(a,z),$$

where $\mathcal{A}^{z,*}$ is the adjoint of $\mathcal{A}^z$.

Macro block. The macro block comprises the equations (3) through (10), which we restate here for convenience

$$
\begin{aligned}
Y_t &= A_t N_t \\
\dot{\pi}_t^w &= \rho \pi_t^w + \frac{\epsilon}{\delta} \iint N_t \left(\frac{\epsilon - 1}{\epsilon} (1 + \tau^L) w_t z u'(c_t(a,z)) - v'(N_t) \right) g_t(a,z) \, da \, dz \\
r_t &= i_t - \pi_t \\
w_t &= A_t \\
i_t &= r_{ss} + \lambda_\pi \pi_t + \lambda_Y \frac{Y_t - Y_{ss}}{Y_{ss}} + \theta_t \\
\pi_t &= \pi_t^w - \frac{\dot{A}_t}{A_t} \\
0 &= \iint a \, g_t(a,z) \, da \, dz.
\end{aligned}
$$

We have dropped the goods market clearing condition (9) by Walras' law.

2 Transition Dynamics in Sequence Space

2.1 Sequence-Space Representation of Equilibrium

It will be convenient, especially for numerical implementation, to develop a *discretized* representation of the equilibrium conditions of our model. To that end, we now restate a result from Dávila and Schaab (2022), which in turn builds on Achdou et al. (2022) and Schaab and Zhang (2022).

Discretization. We first discretize the equations that characterize competitive equilibrium in both time and space. We use a finite-difference discretization scheme building on Achdou et al. (2022). In particular, we discretize the time dimension over a finite horizon, $t \in [0, T]$, where T can be arbitrarily large, using N discrete time steps, which we denote by $n \in \{1, \ldots, N\}$. With a step size $dt = \frac{T}{N-1}$, we have $t_n = dt(n-1)$. We similarly discretize the idiosyncratic state space over (a, z) using J grid points. Using bold-faced notation, we denote the discretized consumption policy function of the household at time t_n as the $J \times 1$ vector $\boldsymbol{c}_n$, where the ith element corresponds to $c_{t_n}(a_i, z_i)$.

We can now state an intermediate result that presents the discretized representation of our key equilibrium conditions.

Lemma 2. *A consistent finite-difference discretization of the equilibrium conditions is as follows. For the HJB equation, we have*

$$\rho \boldsymbol{V}_n = \frac{\boldsymbol{V}_{n+1} - \boldsymbol{V}_n}{dt} + u(\boldsymbol{c}_n) - v(N_n) - \frac{\delta}{2}(\pi_n^w)^2 + \boldsymbol{s}_n \cdot \frac{\boldsymbol{D}_a}{da}\boldsymbol{V}_n + \boldsymbol{A}^z \boldsymbol{V}_n$$

For the consumption first-order condition of the household, we simply have

$$u'(\boldsymbol{c}_{n,[2:J]}) = \left(\frac{\boldsymbol{D}_a}{da}\boldsymbol{V}_{n+1}\right)_{[2:J]}$$

For the Kolmogorov forward equation, we have

$$\frac{\boldsymbol{g}_{n+1} - \boldsymbol{g}_n}{dt} = (\boldsymbol{A}^z)'\boldsymbol{g}_n + \frac{\boldsymbol{D}_a'}{da}\left[\boldsymbol{s}_n \cdot \boldsymbol{g}_n\right]$$

Finally, for the resource constraint we simply have

$$A_n N_n = \boldsymbol{c}_n' \boldsymbol{g}_n d\boldsymbol{x}$$

and for the Phillips curve

$$\frac{\pi_{n+1}^w - \pi_n^w}{dt} = \rho \pi_n^w + \frac{\epsilon}{\delta}\left[\frac{\epsilon - 1}{\epsilon}(1 + \tau^L) A_n (\boldsymbol{z} \cdot u'(\boldsymbol{c}_n))' \boldsymbol{g}_n d\boldsymbol{x} - v'(N_n)\right] N_n$$

and we have already used $c_{n,1} = i_n a_1 - \pi_n^w a_1 + \frac{A_{n+1}-A_n}{dt A_n} a_1 + z_1 A_n N_n$.

Leveraging the boundary condition at the borrowing constraint, note that we have

$$s_n = \begin{pmatrix} 0 \\ i_n \boldsymbol{a}_{[2:J]} - \pi_n^w \boldsymbol{a}_{[2:J]} + \frac{A_{n+1}-A_n}{dt A_n} \boldsymbol{a}_{[2:J]} + \boldsymbol{z}_{[2:J]} A_n N_n - \boldsymbol{c}_{n,[2:J]} \end{pmatrix}.$$

Sequence-space representation of equilibrium. After discretizing our model, the resulting equations satisfy the general model representation of heterogeneous-agent economies presented in Auclert et al. (2021). To facilitate comparison, we follow their notation. In comparison to Appendix C in Dávila and Schaab (2022), we abstract from optimal policy and consider the path of optimal interest rates as one of the aggregate equilibrium objects.

We consider a general representation of a heterogeneous-agent problem as a mapping from time paths of aggregate inputs $(\boldsymbol{X}, \boldsymbol{Z})$ to time paths of aggregate outputs $\boldsymbol{Y}$. We use bold-faced notation here to indicate time paths, with $\boldsymbol{X} = \{X_n\}_{n=1}^N$. It will be useful to explicitly distinguish between the time paths for exogenous shock $\boldsymbol{Z}$ on the one hand and the time paths for other aggregate inputs $\boldsymbol{X}$ on the other hand. To simplify the exposition, we assume that there is only one aggregate input variable other than the shock, so that $X_n \in \mathbb{R}$. We generalize this below.

Denoting the discretized cross-sectional distribution by the $J \times 1$ vector $\boldsymbol{g}_n$, our main focus will be on outcome variables that take the form $Y_n = \boldsymbol{y}_n' \boldsymbol{g}_n$, where $\boldsymbol{y}_n$ is a $J \times 1$ vector that represents an individual outcome.[1] For example, aggregate consumption takes the form $C_n = \boldsymbol{c}_n' \boldsymbol{g}$. Given an initial distribution $\boldsymbol{g}_0$, aggregate outcomes $\boldsymbol{Y}$ then solve the system of equations

$$\boldsymbol{c}_n = \mathcal{C}(\boldsymbol{c}_{n+1}, X_n, Z_n) \tag{14}$$

$$\boldsymbol{g}_{n+1} = \Lambda(\boldsymbol{c}_{n+1}, X_n, Z_n) \boldsymbol{g}_n \tag{15}$$

which is exactly as in Auclert et al. (2021). To be clear, equation (14) does not refer to individual consumption but rather to any forward-looking individual decision variable. We use the notation $\boldsymbol{c}_n$ here because it is suggestive of consumption, which is the main such variable in the simple model.

Given individual outcomes $\boldsymbol{c}_n$ and the distribution $\boldsymbol{g}_n$, the remaining equilibrium conditions satisfy an *equilibrium map*

$$\mathcal{H}(\boldsymbol{X}, \boldsymbol{Z}) = 0,$$

which implicitly also takes as inputs individual outcomes and the distribution. Using the language of micro-macro block from above, equation (14) represents the HJB and consumption-savings optimality conditions, and (15) the KF equation. The equilibrium map then represents the macro

[1] We normalize the discretized distribution representation so that $\boldsymbol{g}_n$ sums to 1, i.e., $\boldsymbol{1}' \boldsymbol{g}_n = 1$, where $\boldsymbol{1}$ is a $J \times 1$ vector of 1s.

block.

A minimal representation for our baseline model takes the form

$$\boldsymbol{Z}_n = \{A_n, \xi_n\}$$

$$\boldsymbol{X}_n = \{N_t, M_t\},$$

where we define M_t as the aggregate labor wedge

$$M_t = \iint N_t \left(\frac{\epsilon - 1}{\epsilon} (1 + \tau^L) w_t z u'(c_t(a, z)) - v'(N_t) \right) g_t(a, z) \, da \, dz.$$

Taking as given the inputs of $\boldsymbol{X}$, we obtain Y_t from the production function, π_t^w and π_t from the Phillips curve, we set $w_t = A_t$, we directly obtain i_t from the Taylor rule, and r_t from the Fisher relation.

2.2 Sequence-Space Jacobians

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