TUTORIAL 5 SOLUTIONS

1, Prove that $\sqrt{3}$ is an irrational number.

Assume that $\sqrt{3}$ is a rational number. Hence

$$\sqrt{3} = \frac{a}{b} \text{ with } a, b \in \mathbb{Z},\ b \neq 0 \text{ and } \gcd(a,b) = 1 \quad (*)$$

$$a = \sqrt{3}\, b \Rightarrow a^2 = 3b^2 \quad (1)$$

Hence a^2 is divisible by 3 $\Rightarrow$ a is divisible by 3:

$$a = 3a_1 \text{ for some } a_1 \in \mathbb{Z} \quad (2)$$

Sub (2) into (1):

$$3b^2 = a^2 = (3a_1)^2 = 9a_1^2 \Rightarrow b^2 = 3a_1^2$$

Hence b^2 is divisible by 3 $\Rightarrow$ b is divisible by 3:

$$b = 3b_1 \text{ for some } b_1 \in \mathbb{Z} \quad (3)$$

(2) & (3) give a contradiction to $(*)$.

$\therefore$ $\sqrt{3}$ is an irrational number.

2) Prove that for any $n \geq 1$

b) $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \dots + n(n+1)(n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$ $(*)$

<u>Basis step</u>: It's clear that $\circledast$ is true for $n=1$: $1 \cdot 2 \cdot 3 = \dfrac{1 \cdot 2 \cdot 3 \cdot 4}{4}$

<u>Inductive step</u>: Assume that $\circledast$ true for $n=k$ with $k \geq 1$:

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \dots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4} \quad (1)$$

We need to prove $1 \cdot 2 \cdot 3 + \dots + (k+1)(k+2)(k+3) = \dfrac{(k+1)(k+2)(k+3)(k+4)}{4}$

Add $(k+1)(k+2)(k+3)$ to both sides of (1):

$$\begin{aligned} 1 \cdot 2 \cdot 3 + \dots + (k+1)(k+2)(k+3) &= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \\ &= (k+1)(k+2)(k+3)\left(\frac{k}{4} + 1\right) \\ &= \frac{(k+1)(k+2)(k+3)(k+4)}{4}, \end{aligned}$$

proving $(*)$ for $n = k+1$.

$\therefore 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \dots + n(n+1)(n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$ for any $n \geq 1$

c) $\dfrac{1}{2} \cdot \dfrac{3}{4} \cdots \dfrac{2n-1}{2n} \leq \dfrac{1}{\sqrt{3n+1}}$ $(*)$

<u>Basis step</u>: It's clear that $\circledast$ is true for $n=1$: $\dfrac{1}{2} \leq \dfrac{1}{\sqrt{4}}$.

<u>Inductive step</u>: Assume $\circledast$ is true for some $n = k \geq 1$:

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2k-1}{2k} \leq \frac{1}{\sqrt{3k+1}} \quad (1)$$

We need to prove $\dfrac{1}{2} \cdot \dfrac{3}{4} \cdots \dfrac{2k-1}{2k} \cdot \dfrac{2k+1}{2k+2} \leq \dfrac{1}{\sqrt{3k+4}}$

Multiply both sides of (1) by $\dfrac{2k+1}{2k+2}$:

$$\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2k-1}{2k}\cdot\frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{3k+1}}\cdot\frac{2k+1}{2k+2}$$

To prove (*) for $n=k+1$, it remains to show that

$$\frac{1}{\sqrt{3k+1}}\cdot\frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{3k+4}} \iff (2k+1)\sqrt{3k+4} \leq (2k+2)\sqrt{3k+1}$$

$$\iff (2k+1)^2(3k+4) \leq (2k+2)^2(3k+1)$$

$$\iff (4k^2+4k+1)(3k+4) \leq (4k^2+8k+4)(3k+1)$$

$$\iff \cancel{12k^3} + \cancel{12k^2} + 3k + \cancel{16k^2} + 16k + 4$$

$$\leq \cancel{12k^3} + \cancel{24k^2} + 12k + \cancel{4k^2} + 8k + 4$$

$$\iff 19k \leq 20k$$

$$\iff k \geq 0,$$

which is automatically true.

$$\therefore \frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}} \quad \text{for any } n \geq 1.$$

3, $\{F_n\}_{n=1}^{\infty}$ is defined by

$$F_1 = F_2 = 1,$$

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3$$

(a) Prove that

$$F_1 + F_3 + \dots + F_{2n-1} = F_{2n} \text{ for any } n \geq 1 \quad (*)$$

<u>Basis step</u>: It's clear that $(*)$ is true for $n = 1$: $F_1 = F_2$

<u>Inductive step</u>: Assume that for some $k \geq 1$

$$F_1 + F_3 + \dots + F_{2k-1} = F_{2k} \quad (1)$$

We need to prove $(*)$ for $n = k+1$: $F_1 + \dots + F_{2k-1} + F_{2k+1} = F_{2k+2}$

Add F_{2k+1} to both sides of (1):

$$F_1 + \dots + F_{2k-1} + F_{2k+1} = F_{2k} + F_{2k+1}$$

$$= F_{2k+2} \text{ (by def of Fibonacci sequence),}$$

proving $(*)$ for $n = k+1$.

$\therefore$ $F_1 + F_3 + \dots + F_{2n-1} = F_{2n}$ for any $n \geq 1$.

4, $\{a_n\}_{n=1}^{\infty}$ is defined by

$$a_1 = 5,\ a_2 = 13,\ a_n = 5a_{n-1} - 6a_{n-2} \text{ for } n \geq 3$$

Prove that

$$a_n = 2^n + 3^n \text{ for any } n \geq 1. \ (*)$$

<u>Basis step</u>: It's clear that (*) is true for $n=1$ and $n=2$

$$a_1 = 2^1 + 3^1 \quad \text{and} \quad a_2 = 2^2 + 3^2$$

<u>Inductive step</u>: Assume (*) is true $n = 1, \ldots, k$ with $k \geq 2$:

$$a_n = 2^n + 3^n \text{ for } n \in \{1, \ldots, k\} \qquad (1)$$

We need to prove (*) for $n = k+1$: $a_{k+1} = 2^{k+1} + 3^{k+1}$.

Since $k+1 \geq 3$, we have

this relation only applies for $k+1 \geq 3$

$$a_{k+1} = 5a_k - 6a_{k-1}$$

$$= 5(2^k + 3^k) - 6(2^{k-1} + 3^{k-1}) \quad ((1) \text{ for } n=k \ \& \ n=k-1)$$

$$= (5 \cdot 2^k - 6 \cdot 2^{k-1}) + (5 \cdot 3^k - 6 \cdot 3^{k-1})$$

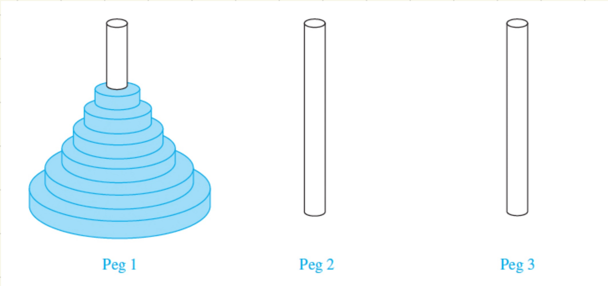
$$= 2^{k-1}(5 \cdot 2 - 6) + 3^{k-1}(5 \cdot 3 - 6)$$

$$= 2^{k+1} + 3^{k+1},$$

proving (*) for $n = k+1$.

$\therefore \quad a_n = 2^n + 3^n$ for any $n \geq 1$.

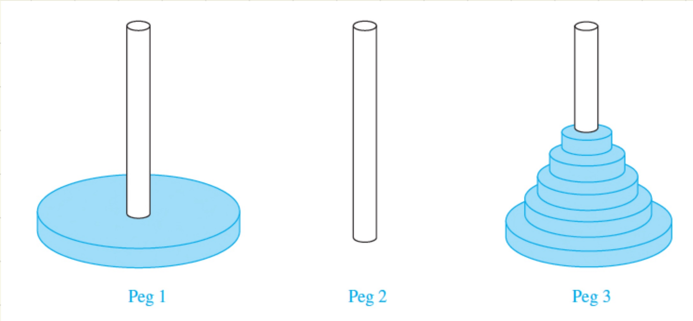
5)



H_n = minimum number of moves to move n disks from Peg 1 to Peg 2

Rule: Larger disks are always below smaller disks

One way to move is



① Move n-1 disks from Peg 1 to Peg 3 → H_{n-1} moves

② Move the bottom disk to Peg 2 → 1 move

③ Move n-1 disks from Peg 3 to Peg 2 → H_{n-1} moves

In this way, # moves $= 2H_{n-1} + 1$. Hence

$$H_n \le 2H_{n-1} + 1 \quad (1)$$

We show that the equality happens by proving $H_n \ge 2H_{n-1} + 1$.

By the time we move the bottom disk, we already moved the top n-1 disks → need $\ge H_{n-1}$ moves.

Now move the bottom disk → 1 move.

Move the $n-1$ disks to peg 2 → need $\geq H_{n-1}$ moves

All together, we need $\geq 2H_{n-1}+1$ moves, that is

$$H_n \geq 2H_{n-1}+1 \quad (2)$$

By (1) & (2), we obtain $H_n = 2H_{n-1}+1$.

b) $H_1 = 1$, $H_2 = 3$, $H_3 = 7$, $H_4 = 15$, $H_5 = 31$

Guess: $H_n = 2^n - 1$

We prove for any $n \geq 1$

$$H_n = 2^n - 1 \quad (*)$$

Basis step: It's clear that (*) is true for $n = 1$: $H_1 = 1$.

Inductive step: Assume that (*) is true for $n = k$:

$$H_k = 2^k - 1 \quad (1)$$

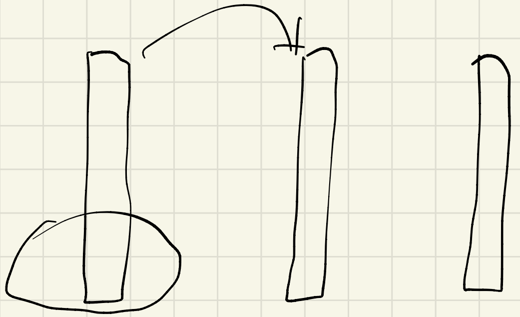
We need to prove (*) for $n = k+1$: $H_{k+1} = 2^{k+1} - 1$. We have

$$H_{k+1} = 2H_k + 1 = 2(2^k - 1) + 1 = 2^{k+1} - 1,$$

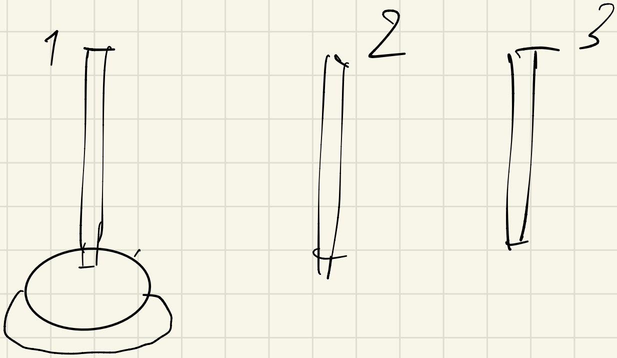
proving (*) for $n = k+1$.

$\therefore H_n = 2^n - 1$ for any $n \geq 1$.

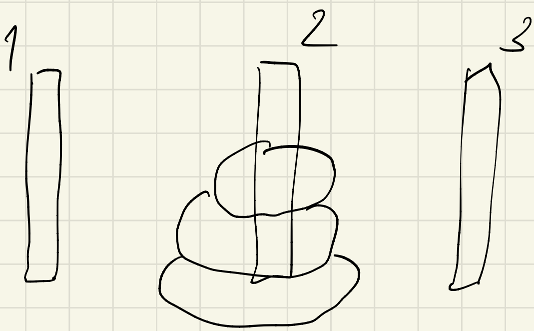
c) The time for the monks to move is $H_{64} = 2^{64} - 1$ seconds, which is about 584 billion years.



$\Rightarrow \quad H_1 = 1$



$\Rightarrow \quad H_2 = 3$



① Move top 2 disks to peg 3

→ 3 steps

② Move peg 1 → 2

→ 1 step

③ Move back 2 disks to peg 2

→ 3 steps