Week 10 Lecture I: Sums and Products of Random Variables

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INTRODUCTION

Sums and Products

Goal

Given X and Y, study the distributions, expectations and variances of $X + Y$ and XY.

- Let X take values in $x_1, \ldots, x_n$ with probabilities $p_1, \ldots, p_n$
- Y in $y_1, \ldots, y_m$ with probabilities $q_1, \ldots, q_m$.
- Let $Z = X + Y$.

For a given $x_i + y_j$, how can we compute

$$P(X + Y = x_i + y_j) \text{ or } P(XY = x_i y_j)?$$

There are some difficulties, namely if there are multiple pairs of x_i and y_j that have the same sum or product.

Example 1

Let X and Y be Bernoulli r.v.s with joint distribution

Joint Prob.	$Y = 0$	$Y = 1$
$X = 0$	0.4	0.2
$X = 1$	0.35	0.05

- What is $P(X + Y = 0)$? This must be when both X and Y are 0, i.e.
$$P(X + Y = 0) = P(X = 0, Y = 0) = 0.4.$$
- What is $P(X + Y = 1)$? Either $X = 0$ and $Y = 1$ or vice versa. Since both events are mutually exclusive,
$$\begin{aligned} P(X + Y = 1) &= P(X = 0, Y = 1) + P(X = 1, Y = 0) \\ &= 0.2 + 0.35 = 0.55. \end{aligned}$$

Example 1 Cont'd

Joint Prob.	$Y=0$	$Y=1$
$X=0$	0.4	0.2
$X=1$	0.35	0.05

- What is $P(XY=0)$? Either $X=0$ or $Y=0$ or both, so

$$\begin{aligned} P(XY=0) &= P(X=0, Y=0) + P(X=0, Y=1) \\ &\quad + P(X=1, Y=0) \\ &= 0.95. \end{aligned}$$

- What is $E(2X)$? $2X$ takes values in $\{0,2\}$ with $P(2X=2)=0.4$. Then

$$E(2X) = 0(0.6) + 2(0.4) = 0.8 = 2E(X).$$

Joint Prob.	$Y=0$	$Y=1$	P_X
$X=0$	0.4	0.2	
$X=1$	0.35	0.05	
P_Y			

- What is $E(X+Y)$? Let $Z = X + Y$ then

$Z = X + Y$	0	1	2
Prob.	0.4	0.55	0.05

  Thus

  $$E(X+Y) = E(Z) = 0(0.4) + 1(0.55) + 2(0.05) = 0.65.$$

  So

  $$E(X+Y) = 0.65 = E(X) + E(Y).$$

  Looking a bit closer,

  $$\begin{aligned} E(X+Y) &= 0.55 + 0.05 + 0.05 \\ &= {\color{red}0.35 + 0.05} + {\color{blue}0.2 + 0.05} = {\color{red}E(X)} + {\color{blue}E(Y)} \end{aligned}$$

ADDITION

Addition

We can compute $P(X+Y=x_i+y_j)$ and then $E(X+Y)$ as follows:

- Let $Z=X+Y$ and $\{z_1,\ldots,z_l\}$ be the sample space for Z.
- Each $z_k = x_i + y_j$ for some (maybe more than one pair of) (i,j).
- Then

$$P(Z=z_k) = \sum_{\text{All } (i,j): x_i+y_j=z_k} P(X=x_i, Y=y_j).$$

- We can write

$$E(X+Y) = E(Z) = \sum_k z_k \Pr(Z=z_k).$$

Then...

Expectation of $E(X+Y)$

$$\begin{aligned}
\sum_k z_k P(Z=z_k) &= \sum_k z_k \sum_{(i,j):x_i+y_j=z_k} P(X=x_i, Y=y_j) \\
&= \sum_k \sum_{(i,j):x_i+y_j=z_k} z_k P(X=x_i, Y=y_j) \\
&= \sum_{\text{all } (i,j)} (x_i+y_j) P(X=x_i, Y=y_j) \\
&= \sum_{\text{all } (i,j)} (x_i P(X=x_i, Y=y_j) + y_j P(X=x_i, Y=y_j)) \\
&= \sum_{\text{all } (i,j)} x_i P(X=x_i, Y=y_j) + \sum_{\text{all } (i,j)} y_j P(X=x_i, Y=y_j) \\
&= \sum_i x_i P(X=x_i) + \sum_j y_j P(Y=y_j)
\end{aligned}$$

Theorem

Theorem

For any r.v.s X and Y:

1. $E(X+Y) = E(X) + E(Y)$ *Proved in the previous slides*
2. *For any a, $E(aX) = aE(X)$.*
3. *For any r.v.s $X_1, \ldots, X_N$ an constants $a_1, \ldots, a_N$,*

$$E(a_1X_1 + \cdots + a_NX_N) = a_1E(X_1) + \cdots + a_NE(X_N).$$

Proof.

For ii. note that $\Pr(aX = ax_i) = \Pr(X = x_i)$, thus

$$E(aX) = \sum_i ax_i \Pr(aX = ax_i) = a \sum_i x_i \Pr(X = x_i) = aE(X).$$

iii. follows from i. and ii. $\square$

Example 2

Find $E(2X + 3Y)$, where

	$Y = 1$	$Y = 2$	$Y = 3$	P_X
$X = 0$	0.05	0.04	0.01	
$X = 1$	0.10	0.08	0.02	
$X = 2$	0.35	0.28	0.07	
P_Y				

Then

$$E(X) = 0(0.1) + 1(0.2) + 2(0.7) = 1.6,$$

and

$$E(Y) = 1(0.5) + 2(0.4) + 3(0.1) = 1.6.$$

We know

$$E(2X + 3Y) = 2E(X) + 3E(Y) = 2(1.6) + 3(1.6) = 8.$$

$E(2X+3Y)$

Let $Z = 2X + 3Y$. Sample space is all possible $2x + 3y$ where $x = 0, 1, 2$ and $y = 1, 2, 3$:

X	0	0	0	1	1	1	2	2	2
Y	1	2	3	1	2	3	1	2	3
Z	3	6	9	5	8	11	7	10	13
$P(Z)$	0.05	0.04	0.01	0.10	0.08	0.02	0.35	0.28	0.07

We can compute

$$\begin{aligned} E(2X+3Y) &= 3(0.05) + 6(0.04) + 9(0.01) + 5(0.1) + 8(0.08) \\ &\quad + 11(0.02) + 7(0.35) + 10(0.28) + 13(0.07) \\ &= 8 \end{aligned}$$

Which method is better?

MULTIPLICATION

Example 3: $E(XY)$ for X, Y independent

Consider

	$Y=0$	$Y=1$	P_X
$X=0$	0.36	0.24	
$X=1$	0.24	0.16	
P_Y			

- $E(X) = 0.4$ and $E(Y) = 0.4$.
- Since X and Y are independent (Why?),

$$E(XY) = E(X)E(Y) = 0.16.$$

- The r.v. $Z = XY$ takes values in $\{0, 1\}$,

$$P(XY = 0) = 0.84, \quad P(XY = 1) = 0.16.$$

- Thus

$$E(XY) = 0(0.84) + 1(0.16) = 0.16 = (0.4)^2 = E(X)E(Y).$$

Example 4: $E(XY)$ for X, Y dependent

Consider

	$Y=0$	$Y=1$	P_X
$X=0$	0.48	0.12	0.6
$X=1$	0.12	0.28	0.4
P_Y	0.6	0.4	

- X and Y are dependent:
  $P(X=1, Y=1) = 0.28 \neq P(X=1)P(Y=1) = 0.16.$
- $E(X) = 0.4$ and $E(Y) = 0.4$, but

  $$P(XY=0) = 0.48+0.12+0.12 = 0.72, \quad P(XY=1) = 0.28.$$

- Then

  $$E(XY) = 0(0.72) + 1(0.28) = 0.28 \neq E(X)E(Y) = 0.16.$$

Variance

- Recall
$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y),$$
- If X and Y are independent then $\mathrm{Cov}(X, Y) = 0$. (This was the result of a previous theorem.)
- For any r.v. X, its variance is
$$\mathrm{var}(X) = \sigma^2 = E(X^2) - (E(X))^2 = E(X^2) - (\overline{X})^2.$$

Question

What is $\mathrm{var}(X + Y)$?

Theorem: var$(X+Y)$

Theorem

If X and Y are any r.v.s then

$$var(X+Y) = var(X) + var(Y) + 2Cov(X,Y)$$
$$\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y + 2\rho_{X,Y}.$$

Thus if X and Y are independent then

$$var(X+Y) = var(X) + var(Y)$$
$$\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y.$$

Proof of Theorem

Proof.

$$
\begin{aligned}
\mathrm{var}(X+Y) &= E\left[(X+Y-\overline{X+Y})^2\right] \\
&= E\left[(X+Y-\overline{X}-\overline{Y})^2\right] \\
&= E\left[(X-\overline{X}+Y-\overline{Y})^2\right] \\
&= E\left[(X-\overline{X})^2+2(X-\overline{X})(Y-\overline{Y})+(Y-\overline{Y})^2\right] \\
&= E(X-\overline{X})^2+E(Y-\overline{Y})^2+2E[(X-\overline{X})(Y-\overline{Y})] \\
&= \mathrm{var}(X)+\mathrm{var}(Y)+2\mathrm{Cov}(X,Y).
\end{aligned}
$$

Further, if X and Y are independent, then $\mathrm{Cov}(X, Y) = 0$. $\square$

General Version

For $Y = X_1 + X_2 + X_3 + \cdots + X_n$, we obtain a more general version of the above equation. We write

$$\mathrm{var}(Y) = \mathrm{var}\left(\Sigma_{i=1}^{n} X_i\right) = \Sigma_{i=1}^{n}\,\mathrm{var}(X_i) + 2\Sigma_{i<j}\mathrm{Cov}(X_i, X_j)$$

If the X_i's are independent, then $\mathrm{Cov}(X_i, X_j) = 0$ for $i \neq j$. In this case, we can write

$$\mathrm{var}(Y) = \mathrm{var}\left(\Sigma_{i=1}^{n} X_i\right) = \Sigma_{i=1}^{n}\,\mathrm{var}(X_i)$$

Example 5: var$(X+Y)$

Recall the independent r.v.s X and Y:

	$Y=0$	$Y=1$	P_X
$X=0$	0.36	0.24	0.6
$X=1$	0.24	0.16	0.4
P_Y	0.6	0.4	

- $E(X)=0.4$ and $X=X^2$, so $E(X^2)=0.4$.
- Thus

$$\sigma_X^2 = E(X^2)-(E(X))^2 = 0.4-0.16 = 0.24,$$

and

$$\sigma_Y^2 = E(Y^2)-(E(Y))^2 = 0.4-0.16 = 0.24.$$

- Thus

$$\sigma_X^2+\sigma_Y^2 = 0.48.$$

Example 5 Cont'd

	$Y=0$	$Y=1$	P_X
$X=0$	0.36	0.24	0.6
$X=1$	0.24	0.16	0.4
P_Y	0.6	0.4	

We can compute the mean and variance of $X+Y$ by noting

$$E(X+Y)=E(X)+E(Y)=0.4+0.4=0.8.$$

and

$$E((X+Y)^2)=E(X^2)+2E(XY)+E(Y^2)=0.4+2(0.16)+0.4=1.12.$$

Thus

$$\sigma^2_{X+Y}=E\left[(X+Y)^2\right]-[E(X+Y)]^2=1.12-0.64=0.48=\sigma^2_X+\sigma^2_Y$$

Example 6: var$(X+Y)$ for dependent r.v.s

Consider again

	$Y=0$	$Y=1$
$X=0$	0.48	0.12
$X=1$	0.12	0.28

We have

$$E(X+Y)=E(X)+E(Y)=0.4+0.4=0.8,$$

and

$$E\left[(X+Y)^2\right]=E(X^2)+2E(XY)+E(Y^2)=0.4+2(0.28)+0.4=1.36.$$

Thus since $\sigma_X^2=E(X^2)-(E(X))^2=.4-.16=0.24$,

$$\sigma_{X+Y}^2=1.36-0.64=0.72\neq\sigma_X^2+\sigma_Y^2=0.48.$$

Theorem: var(aX)

Theorem

If X is an r.v. and a is constant, then

$$var(aX) = a^2\, var(X).$$

Proof.

$$\begin{aligned}
\mathrm{var}(aX) &= E(a^2X^2) - (E(aX))^2 \\
&= a^2E(X^2) - (aE(X))^2 \\
&= a^2\left[E(X^2) - (E(X))^2\right]. \\
&= a^2\,\mathrm{var}(X)
\end{aligned}$$

$\square$

Example 7: var($3X$)

Consider the r.v.s X and $3X$:

X	0	1
$3X$	0	3
Prob.	0.6	0.4

- $E(X) = 0.4$ and $\text{var}(X) = 0.24$
- We can compute

$$E(3X) = 3E(X) = 3(0.4) = 1.2,$$

- And,

$$E((3X)^2) = E(9X^2) = 9E(X^2) = 9(0.4) = 3.6.$$

- Thus

$$\text{var}(3X) = 3.6 - (1.2)^2 = 3.6 - 1.44 = 2.16 = 3^2(.24)$$

A special example and generalization

If $Y = -X$, then

$$E(Y) = E(-X) = -E(X),$$

and

$$\text{var}(Y) = \text{var}(-X) = (-1)^2\,\text{var}(X) = \text{var}(X).$$

The following theorem is a generalization of $\text{var}(X+Y)$:

Theorem

$$var(a_1X_1 + \cdots + a_nX_N) = a_1^2\, var(X_1) + \cdots + a_n^2\, var(X_n).$$

Example

N people sit around a round table, when $N > 5$. Each person tosses a coin. Anyone whose outcome is different from his/her two neighbors will receive a present. Let X be the number of people who receive present. Find $E(X)$.

Cont'd

Solution: Number the N people from 1 to N. Let X_i be the indicator random variable from the ith person. That is, $X_i = 1$ if the ith person receives a present and zero otherwise. Then

$$X = X_1 + X_2 + \cdots + X_N$$

Note that $P(X_i = 1) = 1/4$. This is the probability that the person to the right has a different outcome times the probability that the person to the left has a different outcome. In other words. If we define H_i and T_i be the events that the ith person's outcome is heads and tails respectively.

Cont'd

$$E(X_i) = P(X_i = 1) = P(H_{i-1}, T_i, H_{i+1}) + P(T_{i-1}, H_i, T_{i+1}) = 1/4$$

Thus we have

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_N) = \frac{N}{4}$$