password: dull.

Question 1: Consider the function

$$f(x) = \begin{cases} ce^{-2x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \rightarrow f(x) = 2e^{-2x}$$

$X \sim Exp(2)$.

Solution (a):

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1.$$

$$\int_0^{+\infty} ce^{-2x}\,dx = 1$$

$$\lim_{t\to\infty} \int_0^t ce^{-2x}\,dx = 1$$

$$\lim_{t\to\infty} \left(\frac{c}{-2} e^{-2x}\right)\Big|_0^t = 1$$

$$\lim_{t\to\infty} \frac{c}{-2} e^{-2t} - \left(\frac{c}{-2}\cdot 1\right) = 1$$

$$\frac{c}{2} - \lim_{t\to\infty} \frac{2c}{e^{2t}} = 1 \quad e^{2t} \to +\infty$$

$$\frac{c}{2} = 1 \Rightarrow c = 2.$$

(b) CDF:

$$f(x) = \begin{cases} 2e^{-2x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

For $x<0$, it is clear that.

$$F(x) = \int_{-\infty}^{x} f(u)\,du = \int_{-\infty}^{x} 0\,du = 0.$$

For $x \geq 0$ We have

$$F(x) = \int_{-\infty}^{x} f(u)\,du = \int_{0}^{x} 2e^{-2u}\,du.$$

$$= -e^{-2u}\Big|_{0}^{x} = 1 - e^{-2x}.$$

Conclusion:

$$F(x) = \begin{cases} 1 - e^{-2x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

(c) Compute: $P(1 \leq X \leq 10)$

$$P(1 \leq X \leq 10) = F(10) - F(1)$$

$$= (1 - e^{-20}) - (1 - e^{-2})$$

$$= e^{-2} - e^{-20}.$$

Question 2: $X \sim N(\overset{\mu}{165}, \underset{\sigma^2}{400})$

(a). $\underbrace{Z = \dfrac{X - 165}{20}} \sim N(0, 1)$

We have.

$$P(X > 190) = P\Big(\overbrace{\frac{X-165}{20}}^{Z} > \frac{190-165}{20}\Big)$$

$$= P(Z > 1.25)$$

$$= 1 - P(Z \leq 1.25)$$

$$= 1 - 0.89435 = 0.10565 = 10.565\%.$$

(b) Let h be the minimum height such that the probability of meeting a person with height $> h$ is 0.05.

$$P(X > h) = 0.05$$

$$P\left(X > \frac{h-165}{20}\right) = 0.05$$

$$P\left(X \le \frac{h-165}{20}\right) = 0.95$$

$$\Phi\left(\frac{h-165}{20}\right) = 0.95$$

By the CDF table for $N(0,1)$

$$\frac{h-165}{20} = 1.65$$

$$h-165 = 33.$$

$$h = 165+\boxed{33}.$$

Therefore, the person should be at least 33 cm taller than the average height of 165cm.

<u>Question 3</u>:

$$X \sim N(165, 400).$$

$$Z = \frac{X-165}{20} \sim N(0,1).$$

let $\Phi(z)$ be the CDF of z.

(a) $P(X \geq 175) = P\left(Z > \frac{175-165}{20}\right)$

$$= P(Z > 0.5).$$

$= 1 - P(Z \le 0.5)$

$= 1 - \Phi(0.5) = 1 - 0.69146 \approx 30.854\%$.

(b) $P(X \ge m) = 0.5$ is equivalent to $P(Z > \frac{m-165}{20}) = 0.5$

Hence $\underbrace{\Phi(\frac{m-165}{20})}_{=0} = 0.5$ $\quad m = 165$.

Therefore, the median of the population is the same as μ., the median and the mean for any normal distribution are the same.

(c). The upper 5% point of the population is the number c such that $P(X > c) = 0.05$.

Hence. $\Phi(\frac{c-165}{20}) = 1 - 0.05 = 0.95$ → Look up the table.

which implies $\frac{c-165}{20} = 1.65$ $\quad c = 198\,cm$

the upper 30% point of the population is the number d such that $P(X > d) = 0.3$.

Hence $\Phi(\frac{d-165}{20}) = 1 - 0.3 = 0.7$ → look up the table.

$\frac{d-165}{20} \approx 0.53$ $\quad$ So $d = 175.6\,cm$.

Question 4:

$$f(x) = \begin{cases} \frac{1+\alpha x}{2} & \text{if } -1 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) (i) $f(x) \geq 0$ for all x
(ii) $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

① the condition $f(x) \geq 0$ for all x is equivalent to

$$\alpha x \geq -1 \quad \text{for all } x \in [-1, 1].$$

If $x=0$, it is clearly true.

If $0 < x \leq 1$, then $\alpha \geq -\frac{1}{x}$ for any $0 < x \leq 1$.

So, $\alpha \geq \max_{x \in (0,1]} -\frac{1}{x} = -1$ ②

If $-1 \leq x < 0$, then $\alpha \leq -\frac{1}{x}$ for any $-1 \leq x < 0$.

So, $\alpha \leq \min_{x \in [-1,0)} \frac{1}{-x} = 1.$ ③.

$$\boxed{-1 \leq \alpha \leq 1.}$$

② $\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-1}^{1} \frac{1+\alpha x}{2}\,dx$

$$= \left(\frac{x}{2} + \frac{\alpha x^2}{4}\right)\Big|_{-1}^{1}$$

$$= \frac{1}{2} + \frac{\alpha}{4} - \left(-\frac{1}{2} + \frac{\alpha}{4}\right) = 1.$$

The condition (ii) is automatically true for the function $f(x)$. Therefore, we obtain.

$$-1 \leq \alpha \leq 1.$$

1b) Find the CDF:

Case 1: $x < -1$. We have

$$F(x) = \int_{-\infty}^{x} f(y)\,dy = \int_{-\infty}^{x} 0\,dy = 0.$$

Case 2: $-1 \le x \le 1$ We have

$$F(x) = \int_{-\infty}^{x} f(y)\,dy = \int_{-1}^{x} \frac{1+\alpha y}{2}\,dy.$$

$$= \left(\frac{y}{2} + \frac{\alpha y^2}{4}\right)\Big|_{-1}^{x}$$

$$= \frac{x}{2} + \frac{\alpha x^2}{4} - \left(-\frac{1}{2} + \frac{\alpha}{4}\right)$$

$$= \frac{x}{2} + \frac{\alpha x^2}{4} + \frac{1}{2} - \frac{\alpha}{4}.$$

Case 3: $x > 1$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-1}^{1} \frac{1+\alpha y}{2}\,dy = 1.$$

$$F(x) = \begin{cases} 0 & \text{if } x < -1 \\ \underbrace{\frac{x}{2} + \frac{\alpha x^2}{4} + \frac{1}{2} - \frac{\alpha}{4}}_{} & \text{if } -1 \le x \le 1. \\ 1 & \text{if } x > 1. \end{cases}$$

(c). Let m be the median. We have $P(X > m) = 0.5$.

which is equivalent to $F(m) = 0.5$.

$$\frac{m}{2} + \frac{\alpha m^2}{4} + \frac{1}{2} - \frac{\alpha}{4} = 0.5 \quad \text{and } -1 \le m \le 1.$$

$$\alpha m^2 + 2m - \alpha + \cancel{1} = \cancel{1}$$

$$2m^2+2m-2=0 \qquad m\in[-1,1]$$

quadratic formula: $ax^2+bx+c=0$, $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$m=\frac{-2\pm\sqrt{4+4a^2}}{2a}=\frac{-2\pm\sqrt{4+4a^2}}{2a}.$$

$$=\frac{-2\pm2\sqrt{1+a^2}}{2}=-1\pm\sqrt{1+a^2}.$$

$$\boxed{m_1=-1+\sqrt{1+a^2}} \quad \in[-1,1].$$

$$m_2=-1-\sqrt{1+a^2} \quad \times \quad <-1.$$

Question 5: $\lambda=0.1$. $T\sim Exp(0.1)$.

PDF of T is $f(t)=0.1e^{-0.1t}$ $t>0$.

(a) $$P(T<10)=\int_{-\infty}^{10}f(t)\,dt.$$

$$=\int_0^{10}f(t)\,dt=\int_0^{10}0.1e^{-0.1t}\,dt$$

$$=-e^{-0.1t}\Big|_0^{10}=1-e^{-1}$$

(b) $$P(5<T<15)=\int_5^{15}0.1e^{-0.1t}\,dt=-e^{-0.1t}\Big|_5^{15}.$$

$$=e^{-0.5}-e^{-1.5}$$

(c). $P(T>t)=0.01$ $\quad t>0.$

$$\underline{0.01} = \int_t^{\infty} f(u)\,du = -e^{-0.1t}\Big|_t^{\infty} = \underline{e^{-0.1t}}.$$

$$\ln(0.01) = \ln\left(e^{-0.1t}\right) = -0.1t \ln e = -0.1t$$

$$t = \frac{\ln(0.01)}{-0.1} \approx 46.$$

Tutorial 5.

Question 1:

$$f(x) = \begin{cases} \frac{1+\alpha x}{2}. & \text{if } -1 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) $E(X)$ and $Var(X)$.

$$E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_{-1}^{1} x \cdot \frac{1+\alpha x}{2}\,dx.$$

$$= \left(\frac{x^2}{4} + \alpha \frac{x^3}{6}\right)\Big|_{-1}^{1} = \frac{1}{4} + \frac{\alpha}{6} - \left(\frac{1}{4} - \frac{\alpha}{6}\right) = \frac{\alpha}{3}.$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)\,dx = \int_{-1}^{1} x^2 \frac{1+\alpha x}{2}\,dx.$$

$$= \left(\frac{x^3}{6} + \alpha \frac{x^4}{8}\right)\Big|_{-1}^{1} = \frac{1}{6} + \frac{\alpha}{8} - \left(-\frac{1}{6} + \frac{\alpha}{8}\right) = \frac{1}{3}.$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{1}{3} - \left(\frac{\alpha}{3}\right)^2 = \frac{1}{3} - \frac{\alpha^2}{9}.$$

Question 2 : $\underbrace{P(x) = \frac{\lambda^x}{x!} e^{-\lambda}}$, $x = 0, 1, \cdots$

(a) $\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$

As $p(x)$ is the PMF of X, we have $\sum P(x) = 1$.

$$\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} e^{-\lambda} = 1$$

$$\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$$

(b). $E(X) = \sum_{x=0}^{\infty} \frac{x\lambda^x}{x!} e^{-\lambda}$ when $x=0$ $\frac{0 \cdot \lambda^0}{0!} e^{-\lambda} = 0$,

$$= \sum_{x=1}^{\infty} \frac{x\lambda^x}{x!} e^{-\lambda} = \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} e^{-\lambda}$$

Put $y = x-1$. we obtain.

$$E(X) = \sum_{y=0}^{\infty} \frac{\lambda^{1+y}}{y!} e^{-\lambda}.$$

$$= \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} \cdot \boxed{\lambda e^{-\lambda}}$$

$$= \lambda e^{-\lambda} \boxed{\sum_{y=0}^{\infty} \frac{\lambda^y}{y!}} = e^{\lambda} \text{ from part (a)}$$

$$= \lambda e^{-\lambda} e^{\lambda} = \boxed{\lambda.}$$

The expectation for poisson distribution is λ.

Question 3.

$$f(x) = \begin{cases} \frac{6}{5}(x^2+x) & \text{if } 0 \le x \le 1 \\ 0 & \text{else.} \end{cases}$$

(a) $E(X) = \int_0^1 x\, f(x)\, dx.$

$= \int_0^1 x \cdot \frac{6}{5}(x^2+x)\, dx.$

$= \frac{6}{5}\int_0^1 (x^3+x^2)\, dx = \frac{6}{5}\left[\frac{x^4}{4}+\frac{x^3}{3}\right]\Big|_0^1.$

$= \frac{6}{5}\left[\frac{1}{4}+\frac{1}{3}\right] = \frac{6}{5}\cdot\frac{7}{12} = \frac{7}{10}.$

(b) $E(X^2) = \int_0^1 x^2 \frac{6}{5}(x^2+x)\, dx$

$= \frac{6}{5}\int_0^1 (x^4+x^3)\, dx = \frac{6}{5}\left[\frac{x^5}{5}+\frac{x^4}{4}\right]\Big|_0^1.$

$= \frac{6}{5}\left[\frac{1}{5}+\frac{1}{4}\right] = \frac{\cancel{6}^{3}}{5}\cdot\frac{9}{\cancel{20}_{10}} = \frac{27}{50}.$

$Var(X) = E(X^2) - [E(X)]^2 = \frac{27}{50} - \left(\frac{7}{10}\right)^2$

$= \frac{27}{50} - \frac{49}{100} = \frac{54-49}{100} = \frac{1}{20}.$

$\sigma_X = \sqrt{\frac{1}{20}}$

Question 4.

$$f(x) = \begin{cases} x + 0.5 & 0 \le x \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Find $E(X^n)$

$$E(X^n) = \int_0^1 x^n (x+0.5)\,dx.$$

$$= \int_0^1 \left(x^{n+1} + \frac{x^n}{2}\right) dx.$$

$$= \left[\frac{x^{n+2}}{n+2} + \frac{x^{n+1}}{2(n+1)}\right]\Bigg|_0^1$$

$$= \frac{1}{n+2} + \frac{1}{2n+2}$$

Question 5.

$$f(x) = \begin{cases} 4x^3 & 0 \le x \le 1 \\ 0 & \text{else}. \end{cases}$$

Find $P(X \le \frac{2}{3} \mid X > \frac{1}{3}) = \dfrac{P(\frac{1}{3} < X \le \frac{2}{3})}{P(X > \frac{1}{3})}$

$P(X > \frac{1}{3}) \to 1 - P(X \le \frac{1}{3})$

$$= \frac{\int_{\frac{1}{3}}^{\frac{2}{3}} (4x^3)\,dx}{1 - \int_0^{\frac{1}{3}} (4x^3)\,dx}$$

$$= \frac{x^4 \Big|_{\frac{1}{3}}^{\frac{2}{3}}}{1 - x^4 \Big|_0^{\frac{1}{3}}} = \boxed{\frac{(\frac{2}{3})^4 - (\frac{1}{3})^4}{1 - (\frac{1}{3})^4}} = \frac{\frac{2^4 - 1^4}{3^4}}{\frac{3^4 - 1}{3^4}}.$$

$$= \boxed{\frac{2^4 - 1}{3^4 - 1}} = \frac{15}{80} = \frac{3}{16}.$$

Question 6. Binomial distribution

$E(X) = 7$

$Var(X) = 6$.

$E(X) = np = 7$

$Var(X) = np(1-p) = 6$.

$1 - p = \frac{6}{7}$

$\boxed{p = \frac{1}{7}}$ → probability of success.

$n = 49$ ← # of independent Bernoulli trials.