# Lecture 4: Logics (continued)

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# Arguments

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  - The final proposition is called **conclusion**.

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  - All but the final proposition are called **premises**.
  - The final proposition is called **conclusion**.
- An argument is **valid** if

  whenever the premises are all true, the conclusion is true.

# Example 1

2 premises

The following is a valid argument

"If you have access to the network, you can change your grade."
"You have access to the network."
∴ "You can change your grade."

conclusion

# Argument forms

- The **argument form** of an argument is obtained by expressing it by propositional variables $(p, q, r, \dots)$ and logical connectives $(\neg, \wedge, \vee, \dots)$.

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- An **argument form** is **valid** if

  whenever the premises are all true, the conclusion is true
- The notation

  $$\therefore = \text{“therefore”}$$

  is used to write the conclusion of an argument (or an argument form).

# Example of argument form

Consider the argument

" If you have access to the network, you can change your grade."

"You have access to the network."

$\therefore$ " You can change your grade."

# Example of argument form

Consider the argument

$p$ $q$

" If you have access to the network, you can change your grade."

"You have access to the network."

$\therefore$ " You can change your grade."

- $p$ : You have access to the network.
  $q$ : You can change your grade.

- The argument form is

$$p \to q$$
$$p$$
$$\therefore q$$

# Argument forms

- Now we verify that the argument is valid. Valid argument means

  "if all premises are true, the conclusion is true"

$$p \to q,$$
$$p,$$
$$\therefore q.$$

| p | q | $p \to q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

# Argument forms

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| p | q | $p \to q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

- The last row shows that

  if both premises are true, the conclusion is true.

# Test validity of an argument form

We call a row in the truth table of an argument form **critical row** if all premises are true

1. Identify premises and conclusion.

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We call a row in the truth table of an argument form **critical row** if all premises are true

1. Identify premises and conclusion.
2. Construct the truth table showing all possible values of the premises and the conclusion.
3. Check the conclusion in each **critical row**
   - If the conclusion is false in any of these rows, the argument form is invalid.
   - If the conclusion is true at all these rows, the argument is valid.

# Example 2

Show that the following argument form is invalid.

$$p \rightarrow q \vee \neg r$$
$$q \rightarrow p \wedge r$$
$$\therefore p \rightarrow r.$$

The third critical row shows the argument is invalid.

| $p$ | $q$ | $r$ | $\neg r$ | $q \vee \neg r$ | $p \to q \vee \neg r$ | $p \wedge r$ | $q \to p \wedge r$ | $p \to r$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 |
| 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 |

# Exercise 1

Show that the following argument form is valid.

$$p \vee (q \vee r)$$
$$\neg r$$
$$\therefore p \vee q.$$

The conclusion is true in all 3 critical rows.
∴ The argument is valid.

| p | q | r | q∨r | p∨(q∨r) | ¬r | p∨q |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 1 | 1 | 1 | 0 | 0 |
| 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 0 | 1 |

# Rules of inference

- A **rule of inference** is a valid argument form
- For example, in the ~~last~~ ~~lecture~~, we learnt that the argument form example

$$p \to q$$
$$p$$
$$\therefore q.$$

is a rule of inference

# Modus ponens and modus tollens

- **Modus ponens** is the following valid argument form

$$p \to q$$
$$p$$
$$\therefore q$$

- **Modus tollens** is the following valid argument form

$$p \to q$$
$$\neg q$$
$$\therefore \neg p.$$

# Example 3: Validity of modus ponens and modus tollens

Prove that modus tollens is a valid argument form.

$$p \to q$$
$$\neg q$$
$$\therefore \neg p$$

Sol 1. Truth table

| $p$ | $q$ | $\neg q$ | $p \to q$ | $\neg p$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 |

Sol 2. From last lecture

$p \to q \equiv \neg q \to \neg p$

The argument form is a modus ponens

$\neg q \to \neg p$

$\neg q$

$\therefore \neg p$

# Example 4

Write conclusions for the following arguments

(a) If my CGPA at the end of 4 years is below 2.00, then I cannot graduate in 4 years.

$p$ $q$

My CGPA is 1.49 at the end of 4 years. $p$

Conclusion:

I cannot graduate in 4 years.

(b) If my CGPA at the end of 4 years is below 3.00, then I cannot graduate with honours in 4 years.

I graduated with honours in 4 years.

Conclusion:

My CGPA at the end of 4 years is 3.00 or above.

(c) If my CGPA at the end of 4 years is below 2.00, then I cannot graduate in 4 years.

My CGPA at the end of four years is 3.11.

Conclusion:

I can graduate in 4 years.

# Basic rules of inference

| Name | Argument form | Example |
|---|---|---|
| Generalization | $p$<br>$\therefore p \vee q$ | $x = 3$<br>$\therefore x = 3$ or $x = -3$ |
| Specialization | $p \wedge q$<br>$\therefore p$ | $y > 0$ and $y$ is an integer<br>$\therefore y > 0$ |
| Elimination | $p \vee q$<br>$\neg q$<br>$\therefore p$ | $x - 3 = 0$ or $x + 2 = 0$<br>$x \neq -2$<br>$\therefore x - 3 = 0$ |

# Basic rules of inference

| | | |
|---|---|---|
| Transitivity | $p \to q$<br>$q \to r$<br>$\therefore p \to r$ | If $x > a$, then $x > b$<br>If $x > b$, then $x > c$<br>$\therefore$ if $x > a$, then $x > c$ |
| Division into cases | $p \vee q$<br>$p \to r$<br>$q \to r$<br>$\therefore r$ | $x$ is positive or $x$ is negative<br>If $x$ is positive, then $x^2 > 0$<br>If $x$ is negative, then $x^2 > 0$<br>$\therefore x^2 > 0$ |
| Contradiction | $\neg p \to \mathbf{F}$<br>$\therefore p$ | If everyone sleeps before 12am, then there is no Covid-19<br>$\therefore$ Not everyone sleeps before 12am |

# Example 5

Show that the premises

- It is not sunny this afternoon $\neg p$ and it is colder than yesterday $q$,
- We will go swimming $r$ only if it is sunny, $p$
- If we do not go swimming $\neg r$, then we will take a canoe trip, $s$
- If we take a canoe trip $s$, then we will be home by sunset, $t$

lead to the conclusion

- We will be home by sunset $t$

# Example 5 solution

(1) $\neg p \wedge q$
(2) $r \to p$
(3) $\neg r \to s$
(4) $s \to t$

$\therefore t$

Step 1:

$\neg r \to s$ (by (3))
$s \to t$ (by (4))
$\therefore \neg r \to t$ (transitivity)

Step 2:

$\neg p \wedge q$
$\therefore \neg p$ (specialization)

Step 3:

$r \to p$ (by (2))
$\neg p$ (step 2)
$\therefore \neg r$ (modus tollens)

Step 4:

$\neg r \to t$ (step 1)
$\neg r$ (step 3)
$\therefore t$

## Exercise 2

Show that the premises

- If you send me an e-mail message $p$, then I will finish writing the $q$ program,
- If you do not send me an e-mail message $\neg p$, then I will go to $r$ sleep early,
- If I go to sleep early $r$, then I will wake up feeling refreshed $s$

lead to the conclusion

- If I do not finish writing the program $\neg q$, then I will wake up $s$ feeling refreshed.

# Exercise 2 solution

(1) $p \to q$

(2) $\neg p \to r$

(3) $r \to s$

$\therefore \neg q \to s$

Step 1: $p \to q$

$\therefore \neg q \to \neg p$ (contrapositive)

Step 2: $\neg q \to \neg p$ (step 1)

$\neg p \to r$ (by (2))

$r \to s$ (by (3))

$\therefore \neg q \to s$ (transitivity)

# Predicates and domains

- A **predicate** is a sentence that
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  - becomes a proposition when specific values are substituted for the variables.

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- A **predicate** is a sentence that
  - contains a **finite number of variables** and
  - becomes a proposition when specific values are substituted for the variables.
- The **domain** of a predicate is the set of all values that can be substituted in the variables.

# Examples

- "$P(n) : n$ is a prime, domain: natural numbers" is a predicate defined on the domain of natural numbers.

$$P(1) \text{ is false, } P(2) \text{ is true.}$$

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$$P(1) \text{ is false, } P(2) \text{ is true.}$$

- "$P(x) : x^2 > x$, domain: $\mathbb{R}$" is a predicate defined on the domain of real numbers.

$$P(2) \text{ is true, } P(0.5) \text{ is false.}$$

# Truth sets

Let $P(x)$ be a predicate on domain $D$.

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- The **truth set** of $P(x)$ is the set of all $x \in D$ which makes $P(x)$ true.
- Example: The predicate "$P(x) : x > 0$ on the domain of integers $\mathbb{Z}$" has truth set $\mathbb{Z}^+$.

# Example 6

Consider predicates

$P(x) : |x| < 4$ and $Q(x) : x^2 = 8$ both defined on the domain of integers.

$\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$

Find truth sets of $P(x)$ and $Q(x)$.

$|x| < 4 \Leftrightarrow x \in \{-3, -2, -1, 0, 1, 2, 3\}$

$\therefore$ The truth set for $P(x)$ is $\{-3, -2, -1, 0, 1, 2, 3\}$.

$x^2 = 8 \Leftrightarrow x = 2\sqrt{2}$ or $x = -2\sqrt{2}$, none of which is an integer

$\therefore$ The truth set for $Q(x)$ is $\phi$.

# Universal quantifier

Let $P(x)$ be a predicate on domain $D$.

- The **universal quantifier** $\forall$ is the notation for **for all**.
- A **universal statement** is a statement of the form

$$\forall\ x \in D\ P(x), \text{ or } \forall\ x\ P(x) \tag{1}$$

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$$\forall\ x \in D\ P(x), \text{ or } \forall\ x\ P(x) \tag{1}$$

- (1) is true if $P(x)$ is true for all $x \in D$.
- (1) is false if there exists $x \in D$ for which $P(x)$ is false.
  The value $x$ for which $P(x)$ is false is called a **counterexample**.

# Example 7

(a) Let $D = \{2, 3, 4\}$. Show that the following statement is true

$$\forall\, x \in D, x > \frac{1}{x}.$$

$x = 2$ : $2 > \frac{1}{2}$ is true

$x = 3$ : $3 > \frac{1}{3}$ is true $\quad \therefore\ \forall x \in D, x > \frac{1}{x}$ is true.

$x = 4$ : $4 > \frac{1}{4}$ is true

(b) Is the statement "$\forall\, x \in \mathbb{Z} - \{0\}, x \geq \frac{1}{x}$" true?

False.

Counterexample: $x = -2 \rightarrow -2 \geq -\frac{1}{2}$ is false.

# Existential quantifier

Let $P(x)$ be a predicate on domain $D$.

- The **existential quantifier** $\exists$ is the notation for **there exists**.
- An **existential statement** is a statement of the form

$$\exists \, \mathbf{x} \in \mathbf{D} \; \mathbf{P}(\mathbf{x}), \text{ or } \exists \, \mathbf{x} \; \mathbf{P}(\mathbf{x}). \tag{2}$$

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- (2) is true if $P(x)$ is true for at least one $x \in D$.
- (2) is false if $P(x)$ is false for all $x \in D$.

# Example 8

Which of the following statements are true? Justify your answer.

(a) $\exists\, x \in \mathbb{R}$ such that $x^4 < x^2$.

True. For example with $x = 0.1$

$0.1^4 < 0.1^2$ is true.

(b) Let $D = \{3, 4, 5\}$. Then

$$\exists\, x \in D \text{ such that } x^4 < x^2.$$

False: $x = 3$: $3^4 < 3^2$ is false

$x = 4$: $4^4 < 4^2$ is false

$x = 5$: $5^4 < 5^2$ is false

# Universal quantifier vs existential quantifier

Predicate $P(x)$, domain $D$.

- $\forall$ = for all, $\exists$ = there exists.
- Universal statement:

$$\forall\ x \in D\ P(x), \text{ or } \forall\ x\ P(x)$$

# Universal quantifier vs existential quantifier

Predicate $P(x)$, domain $D$.

- $\forall$ = for all, $\exists$ = there exists.
- Universal statement:

$$\forall\ x \in D\ P(x), \text{ or } \forall\ x\ P(x)$$

- Existential statement:

$$\exists\ \mathbf{x} \in \mathbf{D}\ \mathbf{P}(\mathbf{x}), \text{ or } \exists\ \mathbf{x}\ \mathbf{P}(\mathbf{x}).$$

# Precedence of quantifiers

- $\forall$ and $\exists$ have higher precedence than $\neg, \wedge, \vee, \oplus$.
- For example,

$$\begin{aligned} \forall\ x\ P(x) \vee Q(x) &= (\forall x P(x)) \vee Q(x) \\ \exists\ x\ P(x) \wedge Q(x) &= (\exists x P(x)) \wedge Q(x). \end{aligned}$$

# Precedence of quantifiers and logical operators

| Operators | Precedence |
|---|---|
| $\forall$<br>$\exists$ | 1 |
| $\neg$ | 2 |
| $\wedge$<br>$\vee$ | 3 |
| $\rightarrow$<br>$\leftrightarrow$ | 4 |

# Quantified statements, logical equivalence

- A **quantified statement** is a statement which involves predicates $P(x), Q(x), \ldots$ and quantifiers $\forall, \exists$.

# Quantified statements, logical equivalence

- A **quantified statement** is a statement which involves predicates $P(x), Q(x), \ldots$ and quantifiers $\forall, \exists$.
- 2 quantified statements are **logically equivalent** if they have the same truth values under all situations.
- We use $\equiv$ and $\not\equiv$ to denote **equivalent** and **non-equivalent**.

# Universal quantifier and conjunction

**Lemma 1**

Let $P(x)$ and $Q(x)$ be predicates defined on the same domain $D$. Then

(a) $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$.

(b) $\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$

**Proof.** Optional. See textbook.

$\forall x\,(P(x) \wedge Q(x))$: $P(x)$ and $Q(x)$ are both true for all $x$

$\forall x\, P(x) \wedge \forall x\, Q(x)$: $P(x)$ is true for all $x$ and $Q(x)$ is true for all $x$.

## De Morgan's laws for quantifiers

**Lemma 2**

Let $P(x)$ be a predicate defined on domain $D$. Then

$$\begin{aligned} \neg\exists x P(x) &\equiv \forall x \neg P(x) \\ \neg\forall x P(x) &\equiv \exists x \neg P(x) \end{aligned}$$

**Proof**. Optional. See textbook.

The law says that

$\forall$ and $\exists$ are negation of each other

# Example 9

Write negations of the following statements.

- There is an honest politician.

  All politicians are not honest.

- All Americans eat cheeseburgers.

  There exists an American who doesn't eat cheese burgers.

- No students sleep before 12am.

  There exists a student who sleeps before 12 am.

- $\exists\, x \in \mathbb{Z}^+$ such that $x^2 + 1$ is a square.

  $\forall\, x \in \mathbb{Z}^+$, $x^2 + 1$ is not a square. ✓

## Truth table for predicates?

- Let $P(x)$ be a predicate defined on the domain $D$.
    - For each $x \in D$, $P(x)$ is a proposition.

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  - The truth table for $P(x)$ would have $|D|$ propositions $\Rightarrow 2^{|D|}$ rows $\Rightarrow$ often too large (even infinite if $|D|$ is infinite, say $D = \mathbb{Z}$ or $D = \mathbb{R}$).

# Truth table for predicates?

- Let $P(x)$ be a predicate defined on the domain $D$.
  - For each $x \in D$, $P(x)$ is a proposition.
  - The truth table for $P(x)$ would have $|D|$ propositions $\Rightarrow 2^{|D|}$ rows $\Rightarrow$ often too large (even infinite if $|D|$ is infinite, say $D = \mathbb{Z}$ or $D = \mathbb{R}$).
- To prove the equivalence between quantified statements, we often don't use truth table ($D$ can be infinite).
  We use logical equivalence rules and Lemma 1+2.

## Exercise 3

Let $P(x), Q(x)$ be defined on the same domain $D$. Show that

$$\neg\forall x(P(x) \to Q(x)) \equiv \exists x(P(x) \land \neg Q(x)).$$

**Hint**: $P(x) \to Q(x) \equiv \neg P(x) \lor Q(x)$.

$\neg \forall x\, R(x) \equiv \exists x \neg R(x)$

$$\neg \forall x \left(P(x) \to Q(x)\right) \equiv \neg \forall x \left(\neg P(x) \lor Q(x)\right)$$

$$\equiv \exists x\, \neg\left(\neg P(x) \lor Q(x)\right) \quad \text{(De Morgan)}$$

$$\equiv \exists x \left(P(x) \land \neg Q(x)\right) \quad \text{(De Morgan)}$$

## Example 10 (Lewis Carroll)

Consider the following argument.

"All lions are fierce."

"Some lions do not drink coffee."

$\therefore$ "Some fierce creatures do not drink coffee."

Let $P(x), Q(x), R(x)$ be "x is a lion", "x is fierce", "x drinks coffee".

Assuming that the *domain consists of all creatures*, express the argument in its argument form using quantifiers and $P(x), Q(x)$, and $R(x)$.

$D$ = all creatures

$\forall x \; P(x) \rightarrow Q(x)$

$\exists x \; P(x) \wedge \neg R(x)$

$\therefore \exists x \; Q(x) \wedge \neg R(x)$