CSD2259 Tutorial 5

Problem 1. Prove that $\sqrt{3}$ is an irrational number.

Problem 2. Using mathematical induction, prove that for any integer $n \geq 1$, we have

(a) $1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \dfrac{n(n+1)(n+2)}{3}$.

(b)$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n \cdot (n+1) \cdot (n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$.

(c) $\dfrac{1}{2} \cdot \dfrac{3}{4} \cdots \dfrac{2n-1}{2n} \leq \dfrac{1}{\sqrt{3n+1}}$.

Problem 3. The Fibonacci sequence $\{F_n\}_{n=1}^{\infty}$ is defined by

$$\begin{aligned} F_1 &= F_2 = 1, \\ F_n &= F_{n-1} + F_{n-2} \text{ for any } n \geq 3. \end{aligned}$$

(a) Prove that $F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$ for any $n \geq 1$.

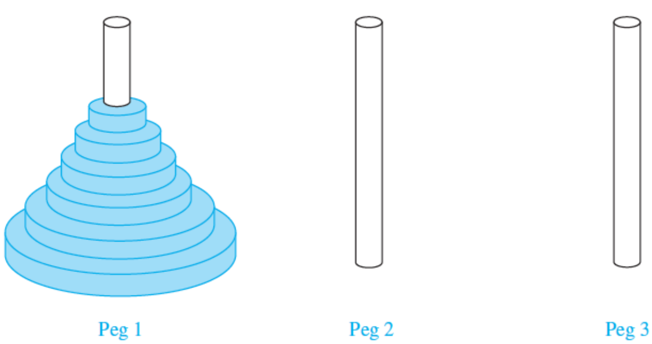
(b) Prove that $F_n = \dfrac{1}{\sqrt{5}} \left(\left(\dfrac{1+\sqrt{5}}{2} \right)^n - \left(\dfrac{1-\sqrt{5}}{2} \right)^n \right)$ for any $n \geq 1$.

Problem 4. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence defined by

$$a_1 = 5,\ a_2 = 13 \text{ and } a_{n+1} = 5a_n - 6a_{n-1} \text{ for any } n \geq 2$$

(a) Compute a_1, a_2, a_3, a_4, a_5. Observe that $a_n = 2^n + 3^n$ for all these values of n.
(b) Prove that $a_n = 2^n + 3^n$ for any $n \geq 1$.

Problem 5. (Tower of Hanoi)
A popular puzzle of the late 29th century, called the Tower of Hanoi, consists of three pegs mounted on a board together with disks of different sizes. Initially these disks are placed on the first peg in order of size, with the largest on the bottom.



Rule: Move disks one at a time from one peg to another such that a disk is never placed on top of a smaller disk.
Goal: Move all the disks to the second peg.
Let H_n denote the minimum number of moves needed to solve the Tower of Hanoi puzzle with n disks.
(a) Prove that $H_n = 2H_{n-1} + 1$.
(b) Guess a formula for H_n (in terms of n). Further, prove your guess using mathematical induction.
(c) A myth created to accompany the puzzle tells of a tower in Hanoi where monks are transferring 64 gold disks from one peg to another, according to the rules of the puzzle. The myth says that the world will end when they finish the puzzle. How long after the monks started will the world end if the monks take one second to move a disk?

Hints and Instructions

1. Prove by contradiction. See example from the lectures.

2. Try it.

3. Inductive step: Assume that for some $k \geq 1$ we have

$$F_1 + F_3 + \cdots + F_{2k-1} = F_{2k} \tag{1}$$

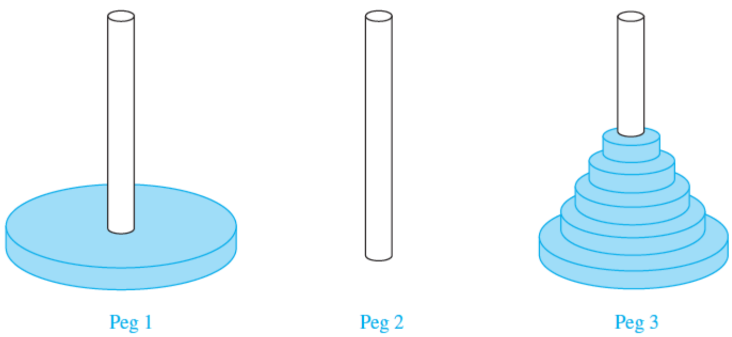
We need to prove

$$F_1 + F_3 + \cdots + F_{2k+1} = F_{2k+2} \tag{2}$$

Try to add F_{2k+1} to both sides of (1) and see if you can get (2).

4a. Use mathematical induction. 4b. Use induction

5. To move all n disks to peg 3, you can



(1) Move the top $n-1$ disks of peg 1 to peg 3 $\rightarrow H_{n-1}$ moves

(2) Transfer the largest disk of peg 1 to peg 2 $\rightarrow$ 1 move

(3) Transfer the $n-1$ disks from peg 3 to peg 2 $\rightarrow H_{n-1}$ moves