Fundamental Theorem of Calculus

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AY 23/24 Trimester 1

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General Information

- Mode of teaching: online lectures, on-campus tutorials
- Instructors:
  - Dr. Ronald Koh (Course Coordinator)
  - Dr. Lin Qinjie
  - Dr. Tay Bee Yen
- Class schedule:
  - Lectures: Mondays 0900 - 1130 HRS, **ONLINE**
  - Tutorials by sections (A) - (D), timings are 1400 - 1630:
    - (A) Fridays at ??, Instructor: Dr. Ronald Koh
    - (B) Tuesdays at Edison (SR2E), Instructor: Dr. Tay Bee Yen
    - (C) Thursdays at LT4B, Instructor: Dr. Lin Qinjie
    - (D) Fridays at ??, Instructor: Dr. Lin Qinjie
  - Tutorials by CSD2200 sections (A) and (B), timings also 1400 - 1630:
    - Fridays at ??, Instructor: Dr. Ronald Koh
    - Thursdays at LT4B, Instructor: Dr. Lin Qinjie
- Consultation: By appointment (Teams is preferred over email)

Course Assessment Tasks

Assessment Task	Weight	Tentative dates
Homeworks (5)	10%	Weeks 2, 3, 5, 9, 11
Quizzes (3)	60%	Weeks 4, 8, 12
Final Exam (1)	30%	Week 14

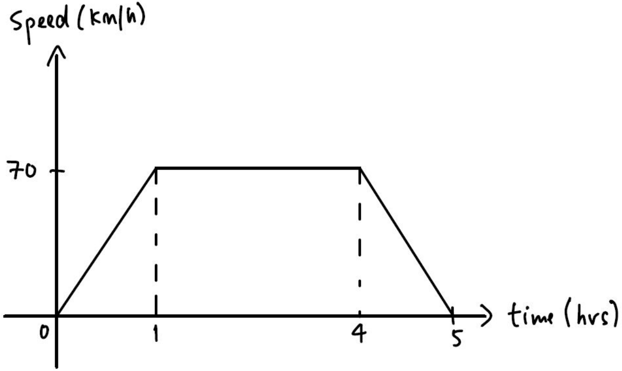
Like in CSD1251, a list of trigonometric formulae will be provided in quizzes and the final exam whenever required.

What's changed from CSD1251/1250?

- Tutorial sessions cover the material from the previous week's lecture (ignoring the recess week, Week 8 will be Quiz 2). This gives you more time to revise and consolidate the information learnt in class. Week 1 tutorial will focus on getting you back up to speed; recapping key concepts from CSD1251/1250.

Exercise 1

Below is a speed-time graph of a car travelling on a road.

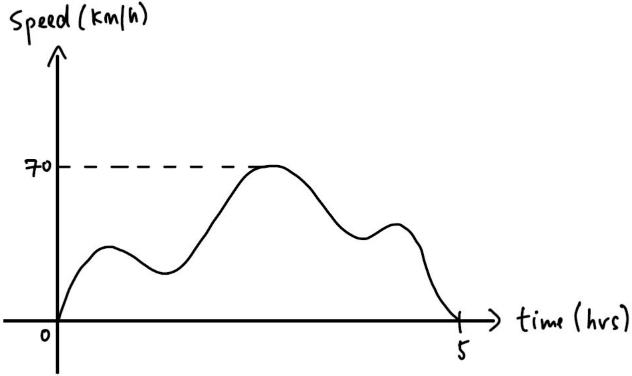


Find the **total distance** covered by the car from $t = 0$ to $t = 5$ hours.

Exercise 1

What about a generic speed function?

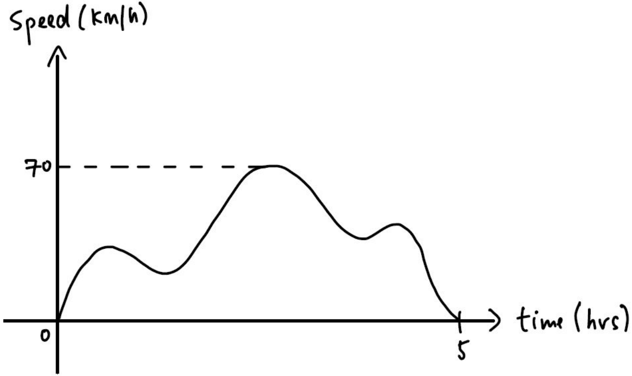
Realistically, a speed-time graph of a car travelling on a road is more like the graph below, with multiple accelerations and decelerations.



What is the total distance covered by the car?

What about a generic speed function?

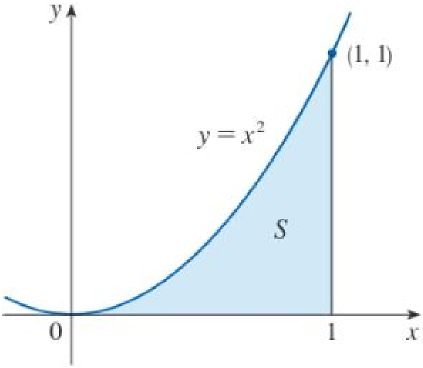
Realistically, a speed-time graph of a car travelling on a road is more like the graph below, with multiple accelerations and decelerations.



What is the total distance covered by the car? Area under this curve.

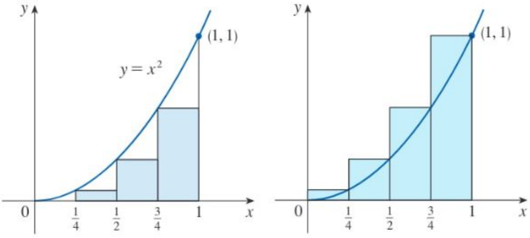
Using rectangles as approximation

Finding the area under the graph of a generic function is not straightforward; we use rectangles to approximate this area.
Let's approximate the area under $f(x) = x^2$ on $[0, 1]$ as an example.



We will also use $n = 4$ rectangles to approximate the area under this graph. Divide $[0, 1]$ into $n = 4$ subintervals with equal length, each has length $\frac{1}{4}$. Each rectangle has a base length of $\frac{1}{4}$.

Since we already have the base length of each rectangle, we need the height. There are many ways to do this, but commonly, we use *left* and *right endpoints*:

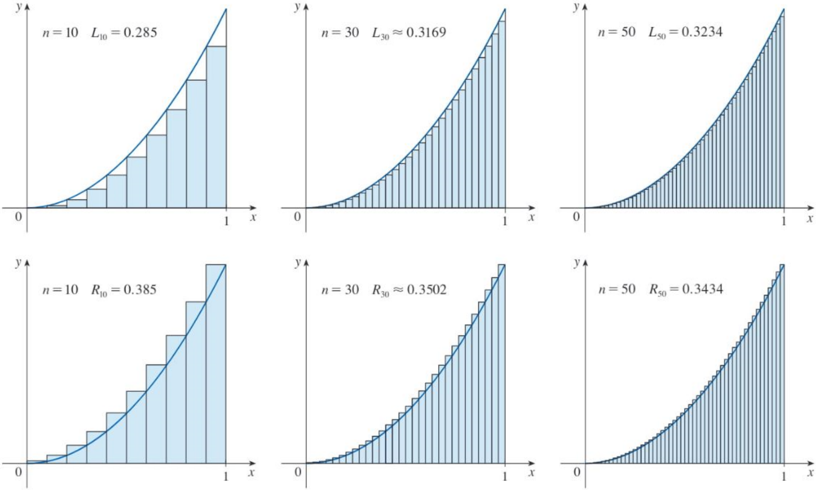


Exercise 2

Compute L_4 and R_4, the total area of the 4 rectangles using left (L) and right (R) endpoints respectively.

What happens when we use more rectangles?

Notice we get closer and closer to the area under the graph!



Generic Riemann Sums

What happens in a general context, i.e. $y = f(x)$ on $[a, b]$? We follow these steps:

1. Divide $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. Let $a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b$ where

$$x_i = x_0 + i\Delta x.$$

The n subintervals are

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$$

Generic Riemann Sums

2. Within each of these subintervals, choose a **sample point** x_i^* (previously, our sample points were either left or right endpoints). For $i = \{1, \ldots, n\}$,
   - Left endpoints $x_i^* = x_{i-1}$,
   - Right endpoints $x_i^* = x_i$.
3. Construct the *Riemann sum*:

$$S_n = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots f(x_n^*)\Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

   n here represents the number of rectangles used.
4. When n gets larger and larger, we get the **definite integral of** f **from** a **to** b:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Definition

The **definite integral of** f **from** a **to** b is the following expression

$$\int_a^b f(x)\,dx.$$

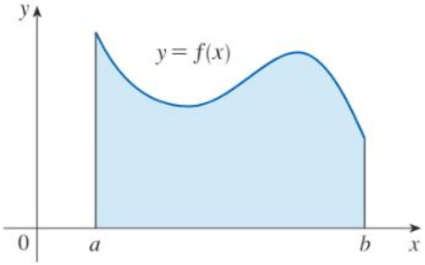
- The symbol $\displaystyle\int$ is an elongated S, because the definite integral is the limit of *sums*.
- The function $f(x)$ here is called the **integrand**.
- a and b are the **lower** and **upper limits** of integration.
- dx refers to the **variable of integration**, in this case, x.

Meaning of the definite integral

(1) When $f(x) \geq 0$ on $[a, b]$, i.e. the graph of f is above the x-axis for $x \in [a, b]$, then

$$\int_a^b f(x)\,dx$$

measures the area under the graph of f from a to b.



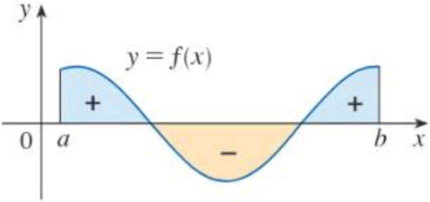
Meaning of the definite integral

(2) If f takes on both negative and positive values on $[a, b]$, then

$$\int_a^b f(x)\,dx$$

measures the **net area** $A_1 - A_2$ where

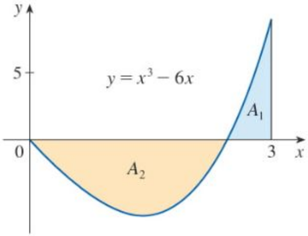
- A_1 is the total area of the region **above** the x-axis and **below the graph** of f, and
- A_2 is the total area of the region **below** the x-axis and **above the graph** of f.



Example 2

The graph of a function $y = x^3 - 6x$ on the interval $[0, 3]$ can be found below. Given that $A_1 = 2.25$ and $A_2 = 9$, evaluate

$$\int_0^3 x^3 - 6x \, dx.$$



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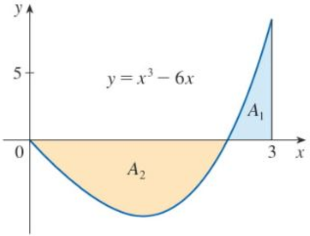
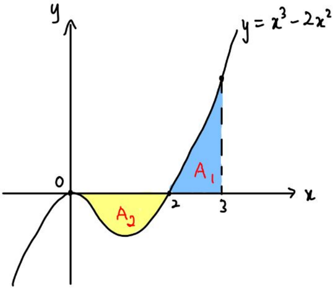


Answer: $\int_0^3 x^3 - 6x\,dx = A_1 - A_2 = 2.25 - 9 = -6.75.$

Exercise 3

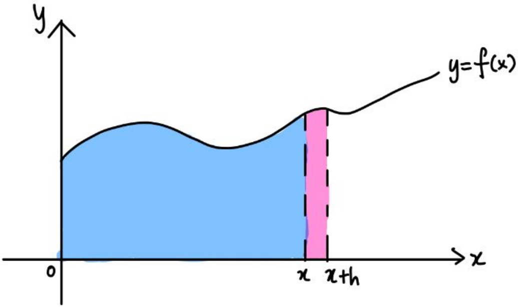
The graph of a function $y = x^3 - 2x^2$ on the interval $[0, 3]$ can be found below. Given that $A_1 = \frac{43}{12}$ and $A_2 = \frac{4}{3}$, evaluate

$$\int_0^3 x^3 - 2x^2 \, dx.$$



Thought experiment

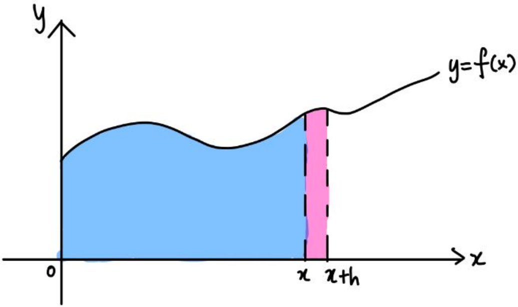
Let $A(x)$ be the area under the graph of $y = f(x)$ from 0 to x, shaded in blue. Then for a small $h > 0$, $A(x+h)$ is the area from 0 to $x+h$.



Thus, $A(x+h) - A(x)$ is the area shaded in pink. When h is very small, this area is approximately $f(x) \cdot h$. This implies that

$$A(x+h) - A(x) \approx f(x) \cdot h.$$

Thought experiment



We rearrange this to get

$$\frac{A(x+h) - A(x)}{h} \approx f(x).$$

Letting $h \to 0$, we get equality (details omitted)

$$f(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = A'(x).$$

Conclusions drawn from the thought experiment

The area under the graph of $y = f(x)$ from 0 to x is

$$A(x) = \int_0^x f(t)\,dt.$$

The previous slide tells us that differentiating both sides of this equation gives us

$$A'(x) = f(x)$$

i.e. after first integrating f, followed by differentiating, we get back f.

This means that differentiation and integration are inverse processes.

The Fundamental Theorem of Calculus

Definition

A function F is an **antiderivative** of another function f if $F'(x) = f(x)$.

Theorem (Fundamental Theorem of Calculus 1 and 2)

If f is continuous on $[a, b]$, then

1. the function g defined by
$$g(x) = \int_a^x f(t)\,dt \quad a \leq x \leq b$$
is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.
2. $$\int_a^b f(x)\,dx = F(b) - F(a)$$
where F is *any* antiderivative of f.

FTC 2 and indefinite integrals

FTC 2 tells us that we can evaluate any definite integral of f just by finding *one* antiderivative of f. We introduce the concept of antiderivatives using **indefinite integrals**.

We have an important fact about antiderivatives:

Lemma

If F and G are antiderivatives of a function f, then F and G differ by a constant, i.e.

$$F(x) - G(x) = C$$

for a fixed constant C.

Indefinite integrals

An **indefinite integral** is a definite integral without the limits a and b:

$$\int f(x)\,dx.$$

This indefinite integral is usually used to denote an antiderivative for f, i.e.

$$\int f(x)\,dx = F(x) \quad \text{means} \quad F'(x) = f(x).$$

Example

1. $$\int x^4\,dx = \frac{x^5}{5} + C \quad \text{because} \quad \frac{d}{dx}\left(\frac{x^5}{5} + C\right) = x^4.$$
2. $$\int \cos x\,dx = \sin x + C \quad \text{because} \quad \frac{d}{dx}(\sin x + C) = \cos x.$$

Table of antiderivatives/indefinite integrals (1)

Let c and k be constants.

- $\displaystyle\int cf(x)\,dx = c\int f(x)\,dx$
- $\displaystyle\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
- $\displaystyle\int k\,dx = kx + C$
- $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$
- $\displaystyle\int e^x\,dx = e^x + C$
- $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

Table of antiderivatives/indefinite integrals (2)

- $\displaystyle\int \sin x \, dx = -\cos x + C$
- $\displaystyle\int \cos x \, dx = \sin x + C$
- $\displaystyle\int \sec^2 x \, dx = \tan x + C$
- $\displaystyle\int \sec x \tan x \, dx = \sec x + C$
- $\displaystyle\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$
- $\displaystyle\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$

Note: These antiderivatives are only valid on an interval where the integrand is continuous!

Example 3 (Indefinite integrals)

Evaluate the following integrals.

1. $\int 3x^2 + 4x + 2\,dx$

Example 4 (Indefinite integrals)

Evaluate the following integrals.

2. $\displaystyle\int 6x^{-1} + 4\sin x - \frac{7}{1+x^2}\,dx$

Exercise 4

Evaluate the following integrals.

1. $\displaystyle\int 2x^4 + 3x^2 + 5x\,dx$
2. $\displaystyle\int \frac{3\tan x}{\sec x} - \frac{2}{\sqrt{1-x^2}}\,dx$

Exercise 4

How to evaluate definite integrals

The most basic rule for evaluation of definite integral of a function f is the FTC2; we find an antiderivative F (see table of integrals on slide 29 and 30, just set $C = 0$ because the C cancels each other below):

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Rules governing definite integrals (1)

The rules governing definite integrals include

1. $\displaystyle\int_a^a f(x)\,dx = 0$
2. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
3. $\displaystyle\int_a^b c\,dx = c(b-a)$, where c is any constant

Rules governing definite integrals (2)

4. $\displaystyle\int_a^b cf(x) = c\int_a^b f(x)\,dx$, where c is any constant
5. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
6. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for any c

Example 5 (Definite integrals)

Evaluate the following integrals.

1. $\displaystyle\int_1^3 3x^2 + 4x + 2\,dx$

Example 6 (Definite integrals)

Evaluate the following integrals.

2. $\displaystyle\int_0^1 x^2 + \frac{1}{\sqrt{1-x^2}}\, dx$

Exercise 5

Evaluate the following integrals.

1. $\displaystyle\int_{-1}^{1} 4x^3 + 2x \, dx$
2. $\displaystyle\int_{0}^{2} 2x^3 - 6x + \frac{3}{1+x^2} \, dx$

Exercise 5

Common mistake

The output of an indefinite integral vs the output of a definite integral:

- $\int f(x)\,dx$ outputs **a (family of) functions**.
- $\int_a^b f(x)\,dx$ outputs a **number**.

Summary

- We can use rectangles of equal base length $\Delta x = \dfrac{b-a}{n}$ and **sample points** (most commonly left/right endpoints x_i^* to approximate the **net area** under the graph. This approximation is known as a **Riemann Sum**:

$$S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

- Letting n (the number of rectangles) $\to \infty$, we get the net area under the graph:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Summary

- If $f(x) \geq 0$, the definite integral of f from a to b measures the area under the graph and above the x-axis.
- If f takes both positive and negative values on $[a, b]$, the definite integral of f from a to b measures the net area $A_1 - A_2$ where
  - A_1 is the total area of the region **above** the x-axis and **below the graph** of f, and
  - A_2 is the total area of the region **below** the x-axis and **above the graph** of f.

Summary

- By FTC1, integration and differentiation are inverse processes; to evaluate integrals of a function f, one must know an **antiderivative** of f.
- We can find antiderivatives of standard functions using the table of indefinite integrals.
- By FTC2, we can evaluate definite integrals of f from a to b using an antiderivative F of f:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$