Lecture 4: Angles and distances

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Cross product

- Computation

$$[\vec{u}\ \vec{v}] = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix} \Rightarrow \vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

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- Geometric properties
  1. $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$
  2. $||\vec{u} \times \vec{v}|| = ||\vec{u}|| ||\vec{v}|| \sin\theta =$ area of parallelogram formed by $\vec{u}, \vec{v}$

Cross product

- Computation

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- Geometric properties
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- Algebraic properties
  1. $\vec{u} \times \vec{v} = \vec{0} \Leftrightarrow \vec{u}$ and $\vec{v}$ are parallel
  2. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u}) \Rightarrow$ same length and opposite directions.

Intersections

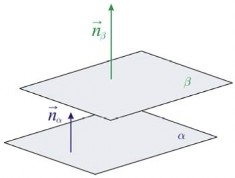
We will study intersections of

1. Two planes
2. A line and a plane
3. Two lines

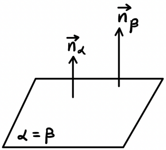
Intersection of 2 planes

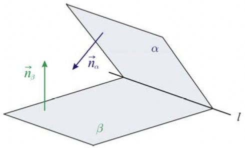
To find the intersection of α, β, we solve for common points on both α, β

- No solution $\Rightarrow \alpha \parallel \beta$.



- Infinitely many solutions $\Rightarrow$ *same plane* or *intersection* $=$ *a line*

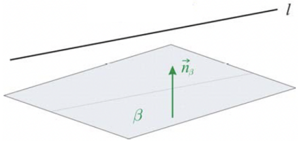




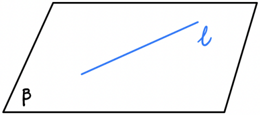
Intersection of line and plane by solving equations

To find intersection of l and β, we solve for common points on l and β.

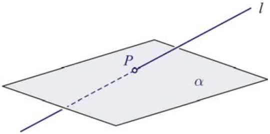
1. No solution $\Rightarrow$ l is parallel to β



2. Infinitely many solutions $\Rightarrow$ l lines on β



3. 1 solution $\Rightarrow$ l and β intersect at a point



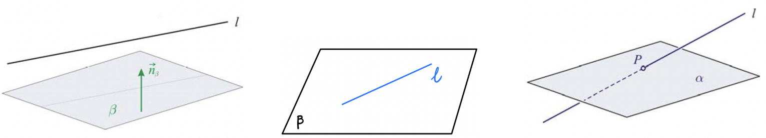
Example 1

Find the intersection of the line l and the plane β in following cases.

$$\beta : 3x - 2y - z = -4 \text{ and } l : (x, y, z) = (2, 1, 4) + t \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$$

Question

Assume l contains point P and has direction $\vec{d}$. Assume β has normal $\vec{n}_\beta$. When is $l \parallel \beta$? When is l on β? When does l intersect β?



Summary

Let l be a line through a point P and having direction $\vec{d}$.
Let β be a plane with normal vector $\vec{n}$. Then

- l is a line on $\beta \Leftrightarrow P$ is on β and $\vec{d} \cdot \vec{n} = 0$
- l is parallel to $\beta \Leftrightarrow P$ is not on β and $\vec{d} \cdot \vec{n} = 0$
- l intersects β at a unique point $\Leftrightarrow \vec{d} \cdot \vec{n} \neq 0$

Example 2

Find the intersection of

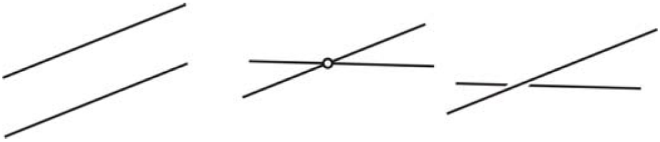
$$\beta : 3x - 2y - z = -4 \text{ and } l : (x, y, z) = (1, 2, 3) + t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Lines in $\mathbb{R}^3$

- In $\mathbb{R}^2$, two *different lines* either "parallel" or "intersect at a point".

Lines in $\mathbb{R}^3$

- In $\mathbb{R}^2$, two *different lines* either "parallel" or "intersect at a point".
- The situation in $\mathbb{R}^3$ is different .

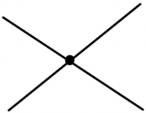


parallel intersecting crossing

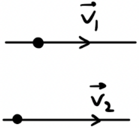
Intersection of 2 lines in $\mathbb{R}^3$

Assume l_1, l_2 have directions $\vec{v}_1, \vec{v}_1$.

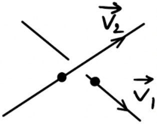
Solving for common points on l_1 and l_2, we have 3 cases



1 solution $\Rightarrow$ intersecting



No solution & $\vec{v}_1 \parallel \vec{v}_2$
$\Rightarrow$ parallel



No solution & $\vec{v}_1 \nparallel \vec{v}_2$
$\Rightarrow$ skew

Example 7

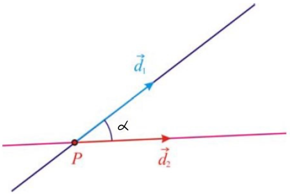
Find the relative position (parallel, intersecting, skew) and the intersection of

$$k : \begin{cases} x = 1 + t \\ y = 2 + 2t \\ z = 3 - t \end{cases} \quad \text{and} \quad m : \begin{cases} x = 7 + r \\ y = 13 + 4r \\ z = -3 - 3r \end{cases}$$

Example 7

Angle between intersecting lines

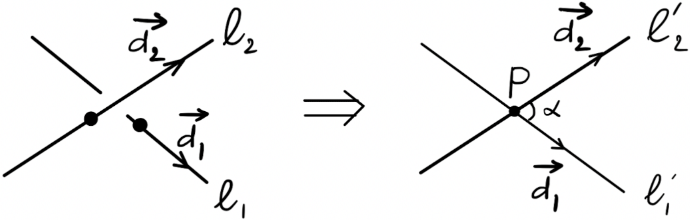
- Let l_1 and l_2 be two lines which intersect at P.
- The angle α between l_1 and l_2 is the smallest angle at P



- *Remark.*

$$0^\circ \leq \alpha \leq 90^\circ \text{ and } \alpha = 0^\circ \text{ if } l_1 \parallel l_2$$

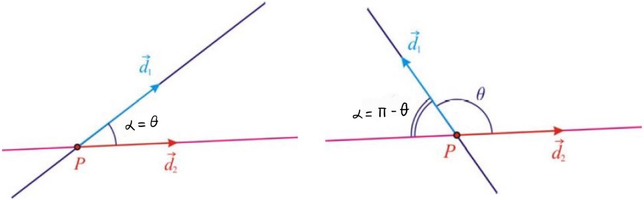
Angle between skew lines



1. Pick a point P and draw lines l_1', l_2' parallel to l_1, l_2
2. The angle between l_1, l_2 is equal to the angle between l_1', l_2'

Angle b.w. lines vs angle b.w. direction vectors

- $\vec{d_1}, \vec{d_2}$ = direction vectors of l_1, l_2
- Put $\theta = \angle(\vec{d_1}, \vec{d_2})$ and $\alpha = \angle(l_1, l_2)$



- The angle α between l_1, l_2 is

$$\alpha = \min(\theta, 180^\circ - \theta) = \begin{cases} \theta \text{ if } 0 \leq \theta \leq 90^\circ \\ 180^\circ - \theta \text{ if } \theta \leq 90^\circ \end{cases}$$

Observation

- Observation

$$\alpha = \theta \quad \Rightarrow \quad \cos\alpha = \cos\theta = \frac{\vec{d}_1 \cdot \vec{d}_2}{||\vec{d}_1||||\vec{d}_2||}$$

$$\alpha = 180^o - \theta \quad \Rightarrow \quad \cos\alpha = -\cos\theta = -\frac{\vec{d}_1 \cdot \vec{d}_2}{||\vec{d}_1||||\vec{d}_2||}$$

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$$\alpha = 180^o - \theta \quad \Rightarrow \quad \cos\alpha = -\cos\theta = -\frac{\vec{d_1}\cdot\vec{d_2}}{||\vec{d_1}||||\vec{d_2}||}$$

- In any case $\cos\alpha \geq 0$ (note that $0^\circ \leq \alpha \leq 90^\circ$)

$$\cos\alpha = \frac{|\vec{d_1}\cdot\vec{d_2}|}{||\vec{d_1}||||\vec{d_2}||}$$

Formula for angle between 2 lines

Theorem 1

Assume l_1, l_2 have direction vectors $\vec{d_1}, \vec{d_2}$. Put $\theta = \angle(\vec{d_1}, \vec{d_2})$. Then

(a) The angle α between l_1 and l_2 is

$$\alpha = \min\left(\theta, 180^o - \theta\right)$$

(b) α can be computed by

$$\cos a = \frac{|\vec{d_1} \cdot \vec{d_2}|}{||\vec{d_1}||||\vec{d_2}||}$$

Example 3

Find the angle between any two of the following lines

(a) In $\mathbb{R}^2$:

$$l_1 : 2x + y + 1 = 0, \quad l_2 : (x, y) = (1, 0) + t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

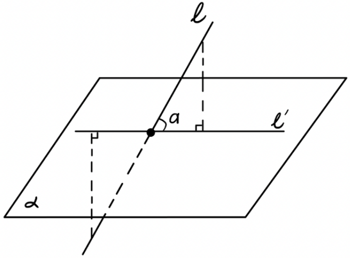
Example 3

(b) In $\mathbb{R}^3$:

$$l_1 : (x, y, z) = (1, 0, 1) + t \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \; l_2 : \begin{cases} x = 1 \\ y = 2 + 5t \\ z = 3 + 6t \end{cases}$$

Angle between a line and a plane

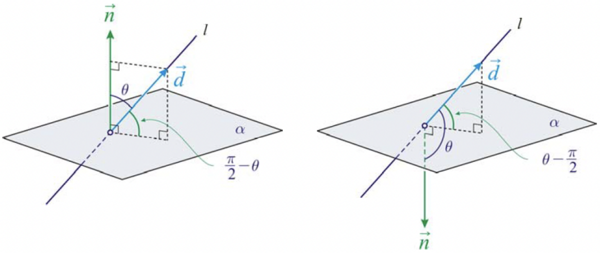
- Let l be a line and let α be a plane.
- The angle a between l and α is the angle between l and *its orthogonal projection* l' onto α.



- *Remark*

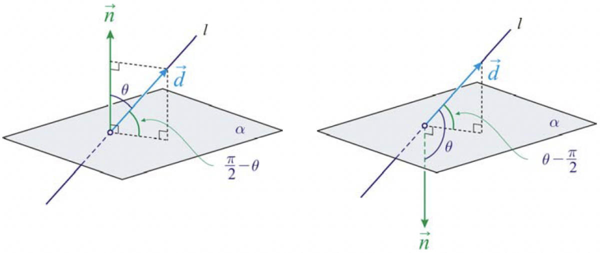
$$0^\circ \leq a \leq 90^\circ \text{ and } a = 0^\circ \text{ if } l \parallel \alpha$$

Angle between a line and a plane



- Put $\vec{d}$ = direction vector of l, $\vec{n}$ = normal vector of α, $\theta = \angle(\vec{d}, \vec{n})$

Angle between a line and a plane



- Put $\vec{d}$ = direction vector of l, $\vec{n}$ = normal vector of α, $\theta = \angle(\vec{d}, \vec{n})$
- The angle between l and α is $a = \begin{cases} 90^\circ - \theta \text{ if } \theta \leq 90^0 \\ \theta - 90^\circ \text{ if } \theta > 90^\circ \end{cases}$. So

$$a = \left|\theta - 90^0\right|$$

Angle between a line and a plane

Theorem 2

Let l be a line with direction $\vec{d}$ and let α be a plane with normal $\vec{n}$. Put $\theta = \angle(\vec{n}, \vec{d})$. The angle $a \in [0^0, 90^0]$ between l and α is

$$a = |\theta - 90^o|$$

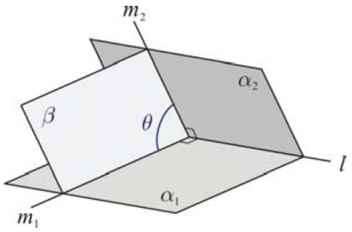
Example 4

Find the angle a between l and α in the following case

$$l : \begin{cases} x = 1 + 2t \\ y = 3 - 5t \\ z = 2 + 10t \end{cases} , \quad \alpha : (x, y, z) = (1, 2, -1) + s \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ -5 \\ -1 \end{bmatrix}$$

Angle between 2 planes

- Let α_1, α_2 be two planes which intersect at the line l. Let β be any plane which is *perpendicular* to l.



Angle between 2 planes

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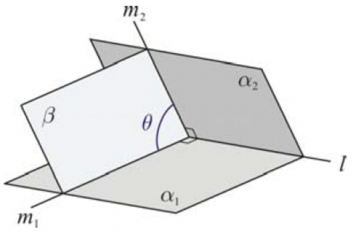
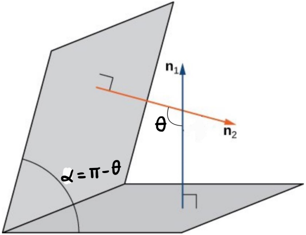


- $m_1, m_2 =$ intersections between β and α_1, α_1.
  The angle θ between α_1 and α_2 is the angle between m_1 and m_2.

Angle between 2 planes

- $\vec{n}_1, \vec{n}_2 =$ normal vectors of α_1, α_2. Put $\theta = \angle(\vec{n}_1, \vec{n}_2)$

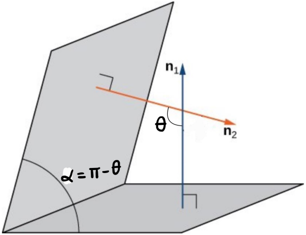


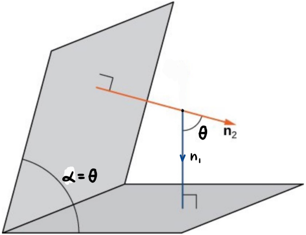
- The angle between α_1, α_2 is

$$a = \min\left(\theta, 180^0 - \theta\right)$$

Angle between 2 planes

- $\vec{n}_1, \vec{n}_2 =$ normal vectors of α_1, α_2. Put $\theta = \angle(\vec{n}_1, \vec{n}_2)$





- The angle between α_1, α_2 is

$$a = \min\left(\theta, 180^0 - \theta\right)$$

- a can be computed by $\cos a = \dfrac{|\vec{n}_1 \cdot \vec{n}_2|}{||\vec{n}_1||||\vec{n}_2||}$

Angle between 2 planes

Theorem 3

Let α_1, α_2 be planes with normal vectors $\vec{n}_1, \vec{n}_2$. Put $\theta = \angle(\vec{n}_1, \vec{n}_2)$. Then

(a) The angle between α_1 and α_2 is

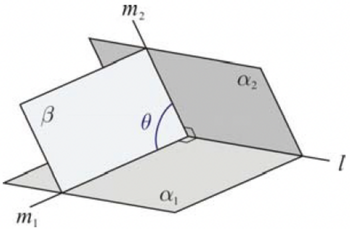
$$a = \min\left(\theta, 180^o - \theta\right)$$

(b) a can be computed by

$$\cos a = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{||\vec{n}_1||||\vec{n}_2||}$$

Question

Can we take *any two lines* l_1, l_2 from α_1, α_2 and define the angle between α_1, α_2 to be the angle between l_1 and l_2?



Example 5

Find the angle between 2 planes

$$\alpha_1 : x + 3y - z = 1 \text{ and } \alpha_2 : (x, y, z) = (1, 0, -1) + s \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

Distance

- The **distance** between any two geometrical objects (points, lines, planes) is the *shortest direct path* which connect one object to the other.
- Question: What if the two objects have a common point?

Zero distance

In any of the following cases, the distance is 0

- Distance from a point to a line containing it
- Distance from a point to a plane containing it
- Distance between two intersecting lines
- Distance between two intersection planes
- Distance between a line and a plane intersecting it

Distance in $\mathbb{R}^2$

In $\mathbb{R}^2$, there are 3 distances all of which we knew how to compute.

1. The distance between $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is
$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
2. The distance from $P = (x_0, y_0)$ to $l : ax + by + c = 0$ is
$$d(P, l) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$
3. The distance between 2 different lines l_1 and l_2
   - l_1 and l_2 intersect $\Rightarrow d(l_1, l_2) = 0$
   - l_1 and l_2 are parallel
$$d(l_1, l_2) = d(P, l_2) \text{ with } P = \text{any point on } l_1.$$

Geometric objects and distances in $\mathbb{R}^3$

We would like to measure the distances between

Points, Lines, Planes

Point-point and point-plane distances

- Points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ is

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- Point $P = (x_0, y_0, z_0)$ and plane $\alpha ax + by + cz + d = 0$

$$d(P, \alpha) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Distance between 2 planes

Consider 2 planes α and β. There are 2 cases

- α and β intersect

$$d(\alpha, \beta) = 0$$

- α and β are parallel

$$d(\alpha, \beta) = d(P, \beta) \text{ with } P = \text{any point on } \alpha.$$

When α and β are parallel?

- Assume

$$\alpha : \text{through } P \text{ and normal vector } \vec{n}_\alpha$$

$$\beta : \text{through } Q \text{ and normal vector } \vec{n}_\beta$$

When α and β are parallel?

- Assume

$$\alpha : \text{through } P \text{ and normal vector } \vec{n}_\alpha$$

$$\beta : \text{through } Q \text{ and normal vector } \vec{n}_\beta$$

- α and β are parallel if and only if
  1. $\vec{n}_\alpha \parallel \vec{n}_\beta$ and
  2. P is not on β

Example 6

Find the distance between two planes α and β in following cases.

(a) $\alpha : 3x - 2y - z = -4$ and $\beta : 3x + 2y + z = 4$

Example 6

(b) $\alpha : (x, y, z) = (1, 2, 3) + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$ and $\beta : x - z = 1.$

Summary on point-point, point-plane and plane-plane distances

- The distance between $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ is

$$d(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- The distance from $P_0 = (x_0, y_0, z_0)$ to $\mathcal{P} : ax + by + cz + d = 0$ is

$$d(P_0, \mathcal{P}) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

- Consider 2 planes α and β
  1. α and β intersect $\Rightarrow d(\alpha, \beta) = 0$
  2. α and β are parallel

$$d(\alpha, \beta) = d(P, \beta) \text{ for any point } P \text{ on } \alpha$$

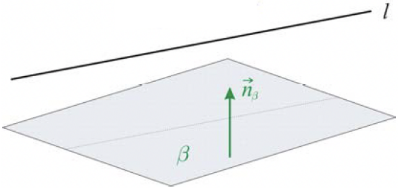
Distance between a line and a plane

Consider the line l and the plane α

1. l and α intersect:
$$d(l,\alpha) = 0$$
2. l parallel to α:
$$d(l,\alpha) = d(P,\alpha) \text{ for any point } P \text{ on } l.$$

Question

When are l and α parallel?



Distance between a line and a plane

Theorem 4

Let $l : (x, y, z) = (x_0, y_0, z_0) + t\vec{d}$ be a line and let $\alpha : ax + by + cz + d = 0$ be a plane. Let $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a normal vector to α.

Distance between a line and a plane

Theorem 4

Let $l : (x, y, z) = (x_0, y_0, z_0) + t\vec{d}$ be a line and let $\alpha : ax + by + cz + d = 0$ be a plane. Let $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a normal vector to α.

(a) If $\vec{n} \cdot \vec{d} = 0$ (that is $l \parallel \alpha$), then

$$d(l, \alpha) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Distance between a line and a plane

Theorem 4

Let $l : (x, y, z) = (x_0, y_0, z_0) + t\vec{d}$ be a line and let $\alpha : ax + by + cz + d = 0$ be a plane. Let $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a normal vector to α.

(a) If $\vec{n} \cdot \vec{d} = 0$ (that is $l \parallel \alpha$), then

$$d(l, \alpha) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

(b) If $\vec{n} \cdot \vec{d} \neq 0$ (that is $l \nparallel \alpha$), then

$$d(l, \alpha) = 0.$$

Example 7

Find the distance between the line l and the plane α in the following cases

(a) $l : (x, y, z) = (2, 1, 4) + t \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$ and $\alpha : 3x - 2y - z = -4$.

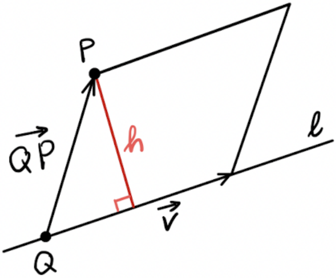
(b) $l : (x, y, z) = (1, 2, 3) + t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\alpha : 3x - 2y - z = -4.$

Remaining distances

1. Distance between a point to a line
2. Distance between lines

Distance from a point to a line

- Point P, line l : through Q and direction $\vec{v}$



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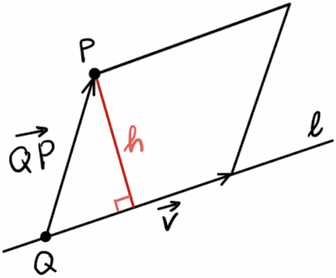


- $d(P,l) =$ height h of parallelogram

$$d(P,l) = \frac{\text{Area of parallelogram}}{||\vec{v}||} = \frac{||\overrightarrow{QP} \times \vec{v}||}{||\vec{v}||}$$

Distance from a point to a line in $\mathbb{R}^3$

Theorem 5

The distance from the point P to the line $l : (x, y) = Q + t\vec{v}$ is

$$d(P, l) = \frac{||\overrightarrow{QP} \times \vec{v}||}{||\vec{v}||}$$

Remark. To use this formula, we need

1. A point Q on l and
2. A direction vector $\vec{v}$ of l

Example 8

Find the distance from $P = (1, 1)$ to the line $l : \begin{cases} x = 1 + 5t \\ y = -6 + 7t \\ z = 2 + 2t \end{cases}$

Distance between 2 lines in $\mathbb{R}^3$

l_1 : through Q_1 and direction $\vec{d}_1$, l_2 : through Q_2 and direction $\vec{d}_2$

1. l_1 and l_2 intersect $\Rightarrow d(l_1, l_2) = 0$

Distance between 2 lines in $\mathbb{R}^3$

l_1 : through Q_1 and direction $\vec{d}_1$, l_2 : through Q_2 and direction $\vec{d}_2$

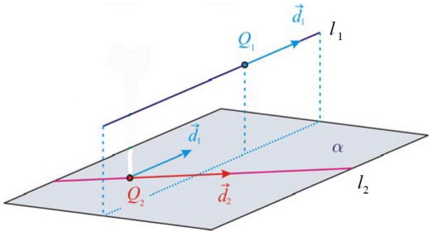
1. l_1 and l_2 intersect $\Rightarrow d(l_1, l_2) = 0$
2. $l_1 \parallel l_2 \Rightarrow d(l_1, l_2) = d(P, l_2)$ for any point P on l_1

Distance between skew lines

l_1 : through Q_1 and direction $\vec{d_1}$,

l_2 : through Q_2 and direction $\vec{d_2}$

2. l_1 and l_2 are skew



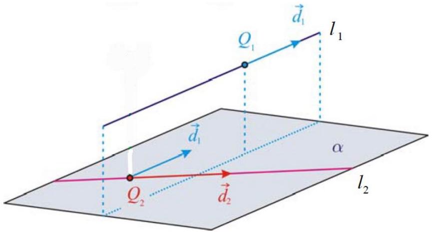
- $\alpha =$ plane containing l_2 and parallel to l_1

$$\alpha : (x, y, z) = Q_2 + s\vec{d_1} + t\vec{d_2}$$

Distance between skew lines

l_1 : through Q_1 and direction $\vec{d_1}$,
l_2 : through Q_2 and direction $\vec{d_2}$

2. l_1 and l_2 are skew



- α = plane containing l_2 and parallel to l_1

$$\alpha : (x, y, z) = Q_2 + s\vec{d_1} + t\vec{d_2}$$

- $d(l_1, l_2) = d(l_1, \alpha) = d(Q_1, \alpha)$

Summary on distance between lines

l_1 : through Q_1 and direction $\vec{d_1}$,

l_2 : through Q_2 and direction $\vec{d_2}$

- Case 1: $\vec{d_1} \parallel \vec{d_2} \Rightarrow l_1 \parallel l_2$. So

$$d(l_1, l_2) = d(Q_1, l_2) = \frac{||\overrightarrow{Q_2Q_1} \times \vec{d_2}||}{||\vec{d_2}||}$$

Summary on distance between lines

l_1 : through Q_1 and direction $\vec{d_1}$,
l_2 : through Q_2 and direction $\vec{d_2}$

- Case 2: $\vec{d_1} \nparallel \vec{d_2} \Rightarrow l_1$ and l_2 are not parallel
  1. Find the plane α through Q_2 and parallel to l_2

     α goes through Q_2 and has normal $\vec{n} = \vec{d_1} \times \vec{d_2}$
  2. $d(l_1, l_2) = d(Q_1, \alpha)$

Example 9

Find the distance between

$$l_1 : \begin{cases} x = 2 - 4t \\ y = 3 + 4t \\ z = 3 \end{cases} \quad \text{and} \quad l_2 : (x, y, z) = (1, 1, 2) + t \begin{bmatrix} 1 \\ -1 \\ -3/4 \end{bmatrix}$$

Distance between 2 lines in $\mathbb{R}^3$ (another way)

Theorem 6

Let $l_1 : (x, y, z) = Q_1 + t\vec{d_1}$ and $l_2 : (x, y, z) = Q_2 + t\vec{d_2}$ be 2 lines.

(a) If l_1 and l_2 are parallel ($\vec{d_1} \parallel \vec{d_2}$), then

$$d(l_1, l_2) = \frac{||\overrightarrow{Q_2Q_1} \times \vec{d_1}||}{||\vec{d_1}||}$$

(b) If l_1 and l_2 are not parallel (intersect, or skew), then

$$d(l_1, l_2) = ||\operatorname{proj}_{\vec{d_1} \times \vec{d_2}}(\overrightarrow{Q_1Q_2})||$$

Example 10

Using Theorem 6, find the distance between

$$l_1 : \begin{cases} x = 2 - 4t \\ y = 3 + 4t \\ z = 3 \end{cases} \quad \text{and} \quad l_2 : (x, y, z) = (1, 1, 2) + t \begin{bmatrix} 1 \\ -1 \\ -3/4 \end{bmatrix}$$

Summary on distance between 2 lines

Assume $l_1 : (x, y, z) = Q_1 + t\vec{d_1}$ and $l_2 : (x, y, z) = Q_2 + t\vec{d_2}$

- Method 1
  1. $\vec{d_1} \parallel \vec{d_2} \Rightarrow d(l_1, l_2) = d(Q_1, l_2)$
  2. $\vec{d_1} \nparallel \vec{d_2}$

     $\alpha =$ plane containing l_2 & parallel to l_1 $\Rightarrow d(l_1, l_2) = d(Q_1, \alpha)$

Summary on distance between 2 lines

Assume $l_1 : (x, y, z) = Q_1 + t\vec{d_1}$ and $l_2 : (x, y, z) = Q_2 + t\vec{d_2}$

- Method 2 (use Theorem 6)

$$d(l_1, l_2) = \begin{cases} \dfrac{||\overrightarrow{Q_2Q_1} \times \vec{d_1}||}{||\vec{d_1}||} & \text{if } \vec{d_1} \parallel \vec{d_2} \\ ||\operatorname{proj}_{\vec{d_1}\times\vec{d_2}}(\overrightarrow{Q_1Q_2})|| & \text{if } \vec{d_1} \nparallel \vec{d_2} \end{cases}$$