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Week 12 Q2 (c), (d)

Q3 (f), (g)

Week 13 Q2, Q3, Q4

Q2 (c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{0.3}}$

$\sum a_n$ absolutely conv. $\Leftrightarrow$ $\sum |a_n|$ converges
$\sum a_n$ conditionally conv. $\Leftrightarrow$ $\sum |a_n|$ diverges but $\sum a_n$ converges

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^{0.3}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{0.3}} \text{ divergent}$$

(p-series, $p = 0.3 \leq 1$)

Is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{0.3}}$ convergent? This is an alternating series.

Let $b_n = \frac{1}{n^{0.3}}$
- $b_n > 0$ clear
- b_n decreasing $\boxed{b_{n+1} \leq b_n \text{ for all } n}$
- $b_n \to 0$ as $n \to \infty$ clear

$\lim_{n\to\infty} b_n = 0$.

$\frac{1}{n} > 0$

$\Rightarrow \left(\frac{1}{n}\right)^{0.3} > 0$

2. b_n decreasing. Consider $f(x) = \frac{1}{x^{0.3}} = x^{-0.3}$ $n \geq 1$.

$$f'(x) = \underbrace{-0.3}_{<0}\underbrace{x^{-0.7}}_{>0} \leq 0$$

$\therefore$ b_n must be decreasing because f is decreasing.

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{0.3}}$ is convergent by Alternating Series Test.

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{0.3}}$ is conditionally convergent.

(d) $\sum_{n=1}^{\infty} (-1)^n \underbrace{\frac{2n^2+n}{n^2+1}}_{b_n}$ $\quad b_n \to 2 \neq 0$. as $n \to \infty$

Consider $a_n = (-1)^n \dfrac{2n^2+n}{n^2+1}$

$$a_{2n} = \frac{2(2n)^2+2n}{(2n)^2+1} \to 2 \qquad a_{2n+1} = -\frac{2(2n+1)^2+(2n+1)}{(2n+1)^2+1} \to -2$$

details omitted   details omitted

$\therefore$ Even and odd subsequences converge to different limits, hence by the Subsequence Test, a_n is divergent.

$\therefore$ By the Divergence Test, $\sum a_n = \sum (-1)^n \dfrac{2n^2+n}{n^2+1}$ is divergent

Radius of convergence

(f) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $(= \sin x)$ Ratio / Root Test

$x \neq 0$ ($x=0 \to$ series is convergent).

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{|x|^2}{(2n+2)(2n+3)} = 0 < 1 \text{ no matter what } x \text{ is.}$$

$\therefore$ By the Ratio Test, $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ is absolutely convergent for all $x \in \mathbb{R}$.

$\therefore R = \infty$. $R = (-\infty, \infty)$

(g) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{2^{2n+2} ((n+1)!)^2} \cdot \frac{2^{2n} (n!)^2}{(-1)^n x^{2n}} \right| \qquad \frac{(n!)^2}{((n+1)!)^2}$$

$$= \lim_{n\to\infty} \frac{|x|^2}{4(n+1)^2} = 0 \qquad = \frac{1}{(n+1)^2}$$

$\therefore$ For all $x \in \mathbb{R}$, $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ is absolutely convergent.

$\therefore R = \infty$.

Week 13 Taylor series of f centered at a

$$\sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(a)}{n!}}_{c_n} (x-a)^n$$

Q4 Taylor series for f centered at $a=2$

$$f^{(n)}(2) = \frac{(2n)!}{9^n n!}$$

$\rightarrow$ Taylor series for f centered at $a=2$

$$\sum_{n=0}^{\infty} \frac{(2n)!}{9^n (n!)^2} (x-2)^n$$

$x \neq 2$ ($x=2$ series is convergent)

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(2n+2)!\,(x-2)^{n+1}}{9^{n+1}((n+1)!)^2} \cdot \frac{9^n (n!)^2}{(2n)!\,(x-2)^n} \right|$$

$$= \lim_{n\to\infty} |x-2| \cdot \frac{(2n+2)!}{(2n)!} \cdot \frac{1}{9} \cdot \frac{(n!)^2}{((n+1)!)^2}$$

$$= \lim_{n\to\infty} |x-2| \cdot (2n+1)(2n+2) \cdot \frac{1}{9} \cdot \frac{1}{(n+1)^2}$$

$$= \frac{|x-2|}{9} \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)(n+1)} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \frac{|x-2|}{9} \lim_{n\to\infty} \frac{(2+\frac{1}{n})(2+\frac{2}{n})}{(1+\frac{1}{n})(1+\frac{1}{n})}$$

$$= \frac{|x-2|}{9} \cdot 4 < 1$$

$$\Leftrightarrow |x-2| < \frac{9}{4}$$

$\therefore R = \frac{9}{4}$.

Q2 $f(x) = \cos x$ $a = 0$

$f^{(0)}(x) = \cos x$ $f^{(4)}(x) = \cos x$

$f'(x) = -\sin x$ $f^{(5)}(x) = -\sin x$

$f''(x) = -\cos x$ $f^{(6)}(x) = -\cos x$

$f'''(x) = \sin x$ $f^{(7)}(x) = \sin x$

$f^{(0)}(0) = 1$ $f^{(4)}(0) = 1$ ...

$f'(0) = 0$ $f^{(5)}(0) = 0$ ...

$f''(0) = -1$ $f^{(6)}(0) = -1$ ...

$f'''(0) = 0$ $f^{(7)}(0) = 0$ ..

Maclaurin Series for $\cos x$ is

$$\frac{1}{0!} + 0 - \frac{1}{2!}x^2 + 0$$

$$+ \frac{1}{4!}x^4 + 0 - \frac{1}{6!}x^6 + 0 + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$R = \infty$.

Quiz 3 Radius of convergence of

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{\sqrt{n+1}} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt{n}}$$

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convergent?

(A) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$ (B) $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$ (C) $\sum_{n=1}^{\infty} \frac{n^3+n}{n^4+2n^2+23}$

(A) only (B) (B) & (C) only (C) (A) and (C) only

(d) (A) and (B) only (e) All of them

(B) is convergent $\quad b_n = \sin(\frac{1}{n}) > 0 \quad \because \frac{1}{n} \in (0, \frac{\pi}{2})$

$$f(x) = \sin(\frac{1}{x})$$
$$f'(x) = \cos(\frac{1}{x}) \cdot (-\frac{1}{x^2}) < 0$$

$$\lim_{n\to\infty} \sin(\frac{1}{n}) = \sin(\lim_{n\to\infty} \frac{1}{n}) = \sin 0 = 0.$$

Q6 Pick the correct statement.

(a) If $\sum a_n$ and $\sum b_n$ are convergent, so is $\sum a_n b_n$

(b) The alternating series $\sum (-1)^n \frac{1}{n^p}$ converges absolutely if $p > 1$.

(c) If $0 \le a_n \le b_n$ for $n \ge n_0$, if $\sum b_n$ divergent then $\sum a_n$ divergent.

(d) $\sum (-1)^n b_n$ $\quad b_n$ decreasing & positive, $\sum (-1)^n b_n$ convergent

(e) A geometric series converges if and only if the modulus of the common ratio is less than or equal to 1.

(a) wrong

$$a_n = b_n = (-1)^n \frac{1}{\sqrt{n}} \qquad \sum a_n b_n = \sum \frac{1}{n} \text{ diverges.}$$

$\sum a_n$, $\sum b_n$ convergent

$$\sum_{n=0}^{\infty} \frac{(3n)!}{23^n (n!)^3} (7x-2)^n \qquad x \neq \frac{2}{7}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(3n+3)!\,(7x-2)^{n+1}}{23^{n+1} ((n+1)!)^3} \cdot \frac{23^n (n!)^3}{(3n)!\,(7x-2)^n} \right|$$

$$= \lim_{n\to\infty} |7x-2| \frac{(3n+3)!}{(3n)!} \cdot \frac{1}{23} \cdot \frac{(n!)^3}{((n+1)!)^3}$$

$$= \frac{|7x-2|}{23} \lim_{n\to\infty} \frac{(3n+1)(3n+2)(3n+3)}{(n+1)(n+1)\ (n+1)} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}}$$

$$= \frac{|7x-2|}{23} \cdot 27 < 1$$

$$|7x-2| < \frac{23}{27} \iff |x - \frac{2}{7}| < \frac{23}{27 \times 7}$$