Lecture 6: Revision

Table of contents

Reminders on Midterm Exam

- Time and locations : Thursday 2-4pm, LT4A and LT4B

A/B & 230-254

C,D & 255-278

- Scope: Weeks 1-5
- Exam format:

12 questions

  1. Part A: MCQs + fill-in-blank questions
  2. Part B written questions

3

- Things to bring in
  1. One A4-size cheat sheet
  2. One calculator
- Wifi devices, notes, books, etc. are **not allowed**

Parallelism and orthogonality

Consider $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

- $\vec{a} \parallel \vec{b} \Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$ or $\vec{a} = c\vec{b}$
- $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$ $\Leftrightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0$

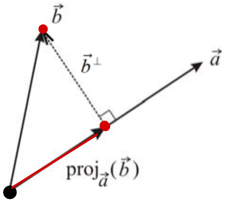
Parallelism and orthogonality

Consider $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

- $\vec{a} \parallel \vec{b} \Leftrightarrow \dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$ or $\vec{a} = c\vec{b}$
- $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$
- Orthogonal projection

$\vec{a} \neq \vec{0}$

$$\mathrm{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}\vec{a}$$



Vector operations

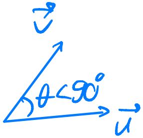
Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \theta = \angle(\vec{u}, \vec{v}) \in [0°, 180°]$

Vector operations

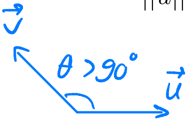
Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \theta = \angle(\vec{u}, \vec{v}) \in [0°, 180°]$

- Dot product: $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$

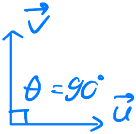
$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||||\vec{v}||}$$



$\vec{u} \cdot \vec{v} > 0$



$\vec{u} \cdot \vec{v} < 0$



$\vec{u} \cdot \vec{v} = 0$

Vector operations

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \theta = \angle(\vec{u}, \vec{v}) \in [0°, 180°]$

$\vec{u} \times \vec{v}$ is a normal vector to any plane having $\vec{u}, \vec{v}$ as direction vectors.

- Cross product: $[\vec{u}\ \vec{v}] = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ \cancel{u_3} & \cancel{v_3} \end{bmatrix} \Rightarrow \vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$
  1. $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$
  2. $||\vec{u} \times \vec{v}||$ = area of parallelogram formed by $\vec{u}, \vec{v}$

Lines in $\mathbb{R}^2$

1. The line through $P_0 = (x_0, y_0)$ with direction $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ has vector equation and parametric equation

$$(x, y) = (x_0, y_0) + t \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad \begin{cases} x = x_0 + at \\ y = y_0 + bt \end{cases}$$

$$(x, y) = P_0 + t\,\vec{v}$$

Lines in $\mathbb{R}^2$

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2. The line through $P_0 = (x_0, y_0)$ with normal $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ has equation

$$a(x - x_0) + b(y - y_0) = 0$$

Put $c = ax_0 + by_0$. The general equation and normal equation are

$$ax + by - c = 0 \text{ and } ax + by = c.$$

Lines in $\mathbb{R}^3$

- The line through $P_0 = (x_0, y_0, z_0)$ with direction vector $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is

$(x, y, z) = P_0 + t\vec{v}$

$$(x, y, z) = (x_0, y_0, z_0) + t \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Leftrightarrow \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

Lines in $\mathbb{R}^3$

- The line through $P_0 = (x_0, y_0, z_0)$ with direction vector $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is

$$(x, y, z) = (x_0, y_0, z_0) + t \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Leftrightarrow \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

- The line passing through 2 points P, Q has vector equation

$$(x, y, z) = P + t\overrightarrow{PQ}$$

Planes in $\mathbb{R}^3$

- The plane through $P(x_0, y_0, z_0)$ with normal $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Planes in $\mathbb{R}^3$

- The plane through $P(x_0, y_0, z_0)$ with normal $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

- The plane through $P(x_0, y_0, z_0)$ with direction vectors $\vec{u}, \vec{v}$ has vector equation and parametric equation

$$(x, y, z) = P + s\vec{u} + t\vec{v} \quad \text{and} \quad \begin{cases} x = x_0 + su_1 + tv_1 \\ y = y_0 + su_2 + tv_2 \\ z = z_0 + su_3 + tv_3 \end{cases}$$

s, t are parameters.

How to get vector equation from normal equation?

Example:

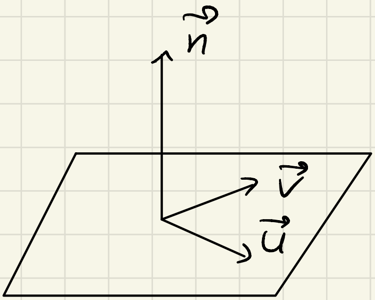
Write a vector equation to the plane

$x + y + z = 0$ — through $(0,0,0)$

normal $\vec{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Choose $\vec{u} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$$(x,y,z) = (0,0,0) + s\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$



Assume $\vec{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Then

$$\vec{u} \cdot \vec{n} = 0 \iff a + b + c = 0$$

Angles

- $\theta =$ angle between $\vec{u}$ and $\vec{v} \Rightarrow \theta \in [0^0, 180^0]$ and $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{||\vec{u}||||\vec{v}||}$
- The angle a between 2 lines, or between a line and a plane, or between 2 planes is always between 0^0 and 90^0: $0^0 \leq a \leq 90^0$

Angles

- The angle a between 2 lines, or between a line and a plane, or between 2 planes is always between 0^0 and 90^0: $0^0 \leq a \leq 90^0$
    1. Angle between lines l_1 : direction $\vec{d_1}$ and l_2 : direction $\vec{d_2}$

$$a = \min(\theta, 180^0 - \theta) \text{ and } \cos a = \frac{|\vec{d_1} \cdot \vec{d_2}|}{||\vec{d_1}||||\vec{d_2}||}$$

with $\theta = \angle(\vec{d}_1, \vec{d}_2)$

Angles

- The angle a between 2 lines, or between a line and a plane, or between 2 planes is always between 0^0 and 90^0: $0^0 \leq a \leq 90^0$

2. Angle between line l : direction $\vec{d}$ and plane α : normal $\vec{n}$

   $a = |\theta - 90^0|$ with $\theta =$ angle b.w. $\vec{d}$ and $\vec{n}$

Angles

- The angle a between 2 lines, or between a line and a plane, or between 2 planes is always between 0^0 and 90^0: $0^0 \leq a \leq 90^0$

3. Angle between planes α_1 : normal $\vec{n}_1$ and plane α_2 : normal $\vec{n}_2$

$$a = \min(\theta, 180^0 - \theta) \text{ and } \cos a = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{||\vec{n}_1|| ||\vec{n}_2||}$$

Point-line distances

(general equation)

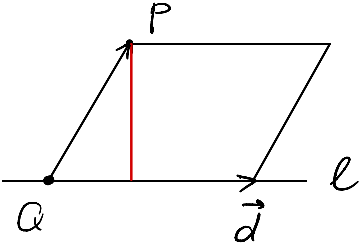
- In $\mathbb{R}^2$:

  Point $P = (x_0, y_0)$ and line $l : ax + by + c = 0.$

$$d(P, l) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

Point-line distances

Area $= \|\vec{QP} \times \vec{d}\|$
$=$ base $\times$ height



- In $\mathbb{R}^3$:

  Point P and line $l : (x, y, z) = Q + t\vec{d}$.

$$d(P, l) = \frac{||\overrightarrow{QP} \times \vec{d}||}{||\vec{d}||}$$

Point-plane, plane-plane, line-plane distances

- Point $P_0 = (x_0, y_0, z_0)$ and plane α : $ax + by + cz + d = 0$

$$d(P_0, \alpha) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

general equation

Point-plane, plane-plane, line-plane distances

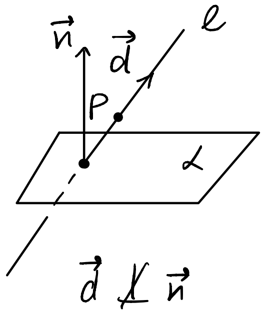
- Planes α and β with normal $\vec{n}_\alpha$ and $\vec{n}_\beta$
  1. $\vec{n}_\alpha \nparallel \vec{n}_\beta \Rightarrow \alpha$ and β intersect

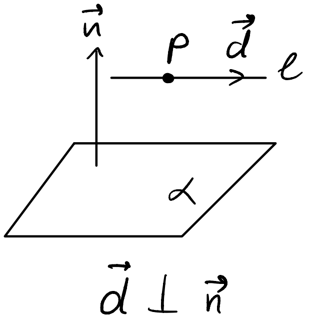
     $$d(\alpha, \beta) = 0$$

  2. $\vec{n}_\alpha \parallel \vec{n}_\beta \Rightarrow \alpha \parallel \beta$

     $d(\alpha, \beta) = d(P, \beta)$ for any point P on α

Point-plane, plane-plane, line-plane distances





- Line $l : (x, y, z) = P + t\vec{d}$ and plane α with normal vector $\vec{n}$.
  1. $\vec{n} \cdot \vec{d} \neq 0 \Rightarrow l$ and α intersect $\Rightarrow d(l, \alpha) = 0$
  2. $\vec{n} \cdot \vec{d} = 0 \Rightarrow l \parallel \alpha \Rightarrow d(l, \alpha) = d(P, \alpha)$

Line-line distance

Line $l_1 : (x, y, z) = Q_1 + t\vec{d_1}$ and $l_2 : Q_2 + t\vec{d_2}$

- $l_1 \parallel l_2$ $(\vec{d_1} \parallel \vec{d_2})$

$$d(l_1, l_2) = d(Q_1, l_2) = \frac{||\overrightarrow{Q_2Q_1} \times \vec{d_2}||}{||\vec{d_2}||}, \quad \text{or}$$

$$d(l_1, l_2) = d(Q_2, l_1) = \frac{||\overrightarrow{Q_1Q_2} \times \vec{d_1}||}{||\vec{d_1}||}$$

Line-line distance

Line $l_1 : (x, y, z) = Q_1 + t\vec{d}_1$ and $l_2 : Q_2 + t\vec{d}_2$

- l_1 and l_2 are skew or intersecting ($\vec{d}_1 \nparallel \vec{d}_2$)

$$d(l_1, l_2) = ||\,\mathrm{proj}_{\vec{d}_1 \times \vec{d}_2}(\overrightarrow{Q_1Q_2})||$$

$\vec{d} = \vec{d}_1 \times \vec{d}_2$

$\mathrm{proj}_{\vec{d}}(\overrightarrow{Q_1Q_2})$

Matrix multiplication and determinant

- $A = (a_{ij})_{m\times n}$ means that A has size $m \times n$ in which the (i,j)th entry of A is a_{ij}.
- A is called a squared matrix if and only if $m = n$.
- If $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times p}$, then AB has size $m \times p$.

$$(AB)_{ij} = (i\text{th row of } A) \times (j\text{th column of } B)$$

- For a 2×2 or a 3×3 matrix A

$$\det(A) = \text{main diagonal - anti diagonal}$$

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13}) - (a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{23}a_{32}a_{11})$$

Area of parallelogram and volume of parallelepiped

- Area of the parallelogram spanned by $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is

(vectors in $\mathbb{R}^2$)

$$\text{Area} = \left| \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \right|$$

- Volume of parallelepiped spanned by $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

$$\left| \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \right|$$

Exercise 1

Consider two vectors $\vec{u} = \begin{bmatrix} 2 \\ -4 \\ c \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ c \\ -1 \end{bmatrix}$. Find c such that

(a) $\vec{u}$ is parallel to $\vec{v}$.

(b) $\vec{u}$ is perpendicular to $\vec{v}$. Further, compute the area of the parallelogram formed by $\vec{u}, \vec{v}$.

(a) $\vec{u} \parallel \vec{v} \iff \frac{2}{1} = \frac{-4}{c} = \frac{c}{-1}$

$$\iff \begin{cases} -\frac{4}{c} = 2 \Rightarrow c = -2 \\ \frac{c}{-1} = 2 \Rightarrow c = -2 \end{cases} \iff c = -2$$

(b) $\vec{u} \cdot \vec{v} = 0 \iff 2 - 5c = 0 \iff c = \frac{2}{5}$.

Exercise 1

Consider two vectors $\vec{u} = \begin{bmatrix} 2 \\ -4 \\ c \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ c \\ -1 \end{bmatrix}$. Find c such that

(a) $\vec{u}$ is parallel to $\vec{v}$.

(b) $\vec{u}$ is perpendicular to $\vec{v}$. Further, compute the area of the parallelogram formed by $\vec{u}, \vec{v}$.

(b) For $c = \frac{2}{5}$,

$$\vec{u} \times \vec{v} = \begin{pmatrix} 2 \\ -4 \\ 2/5 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2/5 \\ -1 \end{pmatrix} = \begin{pmatrix} 96/25 \\ 12/5 \\ 24/5 \end{pmatrix}$$

$$\text{Area} = \|\vec{u} \times \vec{v}\| \simeq 6.6.$$

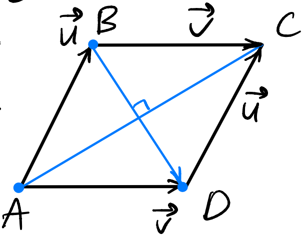
Exercise 2

The parallelogram formed by 2 vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$ of equal length is called a **rhombus**. Prove that the diagonals of this rhombus are perpendicular.

The 2 diagonal vectors are

$$\vec{u} + \vec{BD} = \vec{v} \Rightarrow \vec{BD} = \vec{v} - \vec{u}$$

$$\vec{u} + \vec{v} = \vec{AC} \Rightarrow \vec{AC} = \vec{v} + \vec{u}$$



Hence

$$\begin{aligned} \vec{BD} \cdot \vec{AC} &= (\vec{v} - \vec{u}) \cdot (\vec{v} + \vec{u}) \\ &= \vec{v} \cdot (\vec{v} + \vec{u}) - \vec{u} \cdot (\vec{v} + \vec{u}) \\ &= \vec{v} \cdot \vec{v} + \cancel{\vec{v} \cdot \vec{u}} - \cancel{\vec{u} \cdot \vec{v}} - \vec{u} \cdot \vec{u} = \|\vec{v}\|^2 - \|\vec{u}\|^2 = 0 \end{aligned}$$

$\vec{BD} \perp \vec{AC}$.

Exercise 3

Consider 2 planes $\alpha : x - y + 2z = 1$ and $\beta : 2x - y = 0$.

(a) Find the line l which is the intersection of α and β.

(b) Find the plane γ containing the point $(1, 3, 5)$ and perpendicular to l.

(a) Any point on l satisfies

$$\begin{cases} x - y + 2z = 1 \Rightarrow 2z = 1 - x + y \Rightarrow z = \dfrac{1+x}{2} \\ 2x - y = 0 \Rightarrow y = 2x \end{cases}$$

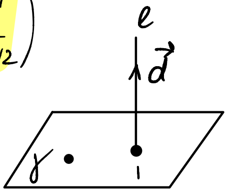
Hence

$$(x, y, z) = \left(x, 2x, \frac{1+x}{2}\right) = \left(0, 0, \frac{1}{2}\right) + x \begin{pmatrix} 1 \\ 2 \\ 1/2 \end{pmatrix}$$

(b) A normal vector for γ is $\vec{n}_\gamma = \vec{d} = \begin{pmatrix} 1 \\ 2 \\ 1/2 \end{pmatrix}$.

$$1(x-1) + 2(y-3) + \frac{1}{2}(z-5) = 0$$

$$2x + 4y + z - 19 = 0$$

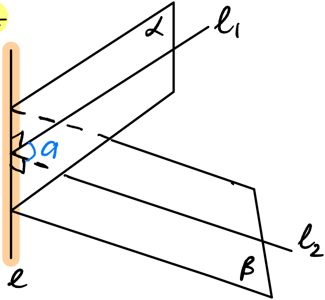


Exercise 3

(c) Find the intersection l_1 of γ and α and the intersection l_2 of γ and β.

(d) Find the angle a between l_1 and l_2. Verify that $a = \angle(\alpha, \beta)$.

(c) l_1 satisfies $\begin{cases} x - y + 2z = 1 \Rightarrow x = 1 + y - 2z \\ 2x + 4y + z = 19 \end{cases}$



$2(1+y-2z) + 4y + z = 19$

$6y - 3z = 17 \Rightarrow z = \frac{6y-17}{3} = 2y - \frac{17}{3}$

Hence $x = 1 + y - 2\left(2y - \frac{17}{3}\right) = -3y + \frac{40}{3}$

$l_1: (x, y, z) = \left(\frac{40}{3} - 3y, y, -\frac{17}{3} + 2y\right) = \left(\frac{40}{3}, 0, -\frac{17}{3}\right) + y \underbrace{\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}}_{\vec{d}_1}$

Exercise 3

(c) Find the intersection l_1 of γ and α and the intersection l_2 of γ and β.

(d) Find the angle a between l_1 and l_2. Verify that $a = \angle(\alpha, \beta)$.

The line l_2 has direction

$$\vec{d}_2 = \vec{n}_\beta \times \vec{n}_\gamma = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 10 \end{pmatrix}$$

$$\vec{d}_1 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \quad \vec{n}_\alpha = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \vec{n}_\beta = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

(d) The angle b/w l_1 & l_2 is

$$\cos a = \frac{|\vec{d}_1 \cdot \vec{d}_2|}{\|\vec{d}_1\| \, \|\vec{d}_2\|} = \frac{21}{\sqrt{14}\sqrt{105}} \Rightarrow a \cong 57^\circ.$$

Further

$$\angle(\alpha, \beta) = \cos^{-1}\left(\frac{|\vec{n}_\alpha \cdot \vec{n}_\beta|}{\|\vec{n}_\alpha\| \, \|\vec{n}_\beta\|}\right) = \cos^{-1}\left(\frac{3}{\sqrt{6}\sqrt{5}}\right) \cong 57^\circ.$$

Exercise 4

Given 3 points $A = (1, 2, 0), B = (0, 3, 1), C = (-1, 0, 1)$ and the line

$$l : (x, y, z) = (3, 5, -1) + t \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}.$$

(a) Let m be the line containing A, B. Find the intersection, the angle and the distance between l and m.

m has equation $(x,y,z) = A + s\vec{AB} = (1,2,0) + s\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

First, we solve for the intersection. Let $P=(x,y,z)$ be a common point of l, m

$$\begin{cases} x = 3-t \\ y = 5+t \\ z = -1-t \end{cases} \text{ and } \begin{cases} x = 1-s \\ y = 2+s \\ z = s \end{cases} \Rightarrow \begin{cases} 3-t = 1-s \Rightarrow t = s+2 \\ 5+t = 2+s \Rightarrow 5+(s+2) = s+2 \Rightarrow 5=0, \\ -1-t = s \end{cases}$$

contradiction.

The lines l & m do not intersect.

Exercise 4

Given 3 points $A = (1, 2, 0), B = (0, 3, 1), C = (-1, 0, 1)$ and the line

$$l : (x, y, z) = (3, 5, -1) + t \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}.$$

$\vec{AP} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \Rightarrow \frac{\vec{AP}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\vec{d} = \frac{-10}{8}\vec{d} = \begin{pmatrix} 5/2 \\ 5/2 \\ 0 \end{pmatrix}$

(a) Let m be the line containing A, B. Find the intersection, the angle and the distance between l and m.

Note that m goes through $A = (1,2,0)$ & has direction $\vec{d}_m = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

l ———— $P = (3,5,-1)$ ———— $\vec{d}_l = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

The angle b.w. m & l is $a = \cos^{-1}\left(\frac{|\vec{d}_m \cdot \vec{d}_l|}{\|\vec{d}_m\|\,\|\vec{d}_l\|}\right) \approx 70.53°$.

Since $\frac{-1}{-1} = \frac{1}{1} \neq \frac{1}{-1}$, $\vec{d}_m \not\parallel \vec{d}_l \Rightarrow m \not\parallel l$. Put

$\vec{d} = \vec{d}_m \times \vec{d}_l = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix} \Rightarrow d(l,m) = \|\mathrm{proj}_{\vec{d}}(AP)\| = \left\|\frac{\vec{AP}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\vec{d}\right\| = \frac{5\sqrt{2}}{2}$

Exercise 4

(b) Let α be the plane through A, B, C. Find the intersection and the angle between l and α.

α contains $A = (1,2,0)$ and has normal

$$\vec{n}_\alpha = \vec{AB} \times \vec{AC} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$$

So α has equation

$$(x-1) - 3(y-2) + 4z = 0 \Leftrightarrow x - 3y + 4z + 5 = 0$$

Let $Q = (x, y, z)$ be a common point of l & α. We have

$$\begin{cases} x = 3 - t \\ y = 5 + t \\ z = -1 - t \end{cases} \quad \text{and} \quad x - 3y + 4z + 5 = 0$$

$$(3-t) - 3(5+t) + 4(-1-t) + 5 = 0$$

$$-8t - 11 = 0 \Rightarrow t = -\frac{11}{8}$$

$\therefore Q = \left(\frac{35}{8}, \frac{29}{8}, \frac{3}{8}\right)$.

Exercise 5

Consider 2 lines $l_1 : \begin{cases} x = -1 + s \\ y = 10 + 2s \\ z = 10 \end{cases}$ and $l_2 : \begin{cases} x = t \\ y = 4 + 2t \\ z = 5 - 3t \end{cases}$.

(a) Find the distance $d(l_1, l_2)$ between l_1 and l_2.

(b) Find the equation of the plane β containing l_2 and parallel to l_1.

(a) l_1 contains $Q_1 = (-1, 10, 10)$ & has direction $\vec{d_1} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

l_2 contains $Q_2 = (0, 4, 5)$ & ________ $\vec{d_2} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

Since $\frac{1}{1} = \frac{2}{2} \neq \frac{0}{-3}$, $\vec{d_1} \nparallel \vec{d_2} \Rightarrow l_1 \nparallel l_2$.

Put $\vec{d} = \vec{d_1} \times \vec{d_2} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \\ 0 \end{pmatrix}$. Hence

Exercise 5

Consider 2 lines $l_1 : \begin{cases} x = -1 + s \\ y = 10 + 2s \\ z = 10 \end{cases}$ and $l_2 : \begin{cases} x = t \\ y = 4 + 2t \\ z = 5 - 3t \end{cases}$.

(a) Find the distance $d(l_1, l_2)$ between l_1 and l_2.

(b) Find the equation of the plane β containing l_2 and parallel to l_1.

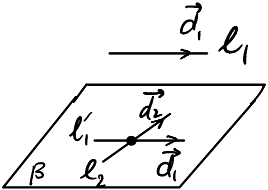
(a) $d(l_1, l_2) = \|\text{proj}_{\vec d}(\overrightarrow{Q_1Q_2})\|$

$\overrightarrow{Q_1Q_2} = \begin{pmatrix}1\\-6\\-5\end{pmatrix}$

$\vec d = \begin{pmatrix}-6\\3\\0\end{pmatrix}$

$= \left\| \frac{\overrightarrow{Q_1Q_2}\cdot \vec d}{\vec d \cdot \vec d}\vec d \right\| = \left\| \frac{-24}{45}\vec d \right\|$

$= \frac{24}{45}\sqrt{(-6)^2+3^2} = \frac{24\sqrt{45}}{45} = \frac{8\sqrt5}{5}$.



(b) β passes through $Q_2 = (0,4,5)$ & has normal $\vec n_\beta = \vec d_1 \times \vec d_2 = \begin{pmatrix}-6\\3\\0\end{pmatrix}$

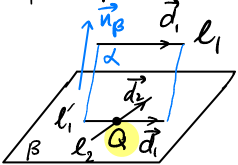
$-6x + 3(y-4) = 0 \Leftrightarrow -2x + y - 4 = 0$

(c) Find the point Q on l_2 which at the closet distance to l_1.

$Q = l_1' \cap l_2$, where l_1' = projection of l_1 onto β

let α = plane containing l_1 & l_1'. Then

Q = intercept of α & l_2.



For α: 1 direction vector is $\vec{d}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$\alpha \perp \beta \Rightarrow$ another direction vector is $\vec{n}_\beta = \begin{pmatrix} -6 \\ 3 \\ 0 \end{pmatrix} \parallel \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

Hence $\vec{n}_\alpha = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$. α contains $Q_1 = (-1, 10, 10)$:

$$5(z-10) = 0 \Leftrightarrow z = 10.$$

Q has coordinates satisfying $\begin{cases} x = t & \Rightarrow x = -5/3 \\ y = 4+2t & \Rightarrow y = 2/3 \\ z = 5-3t = 10 & \Rightarrow t = -\frac{5}{3} \end{cases}$

$\therefore Q = \left(\frac{-5}{3}, \frac{2}{3}, 10\right)$

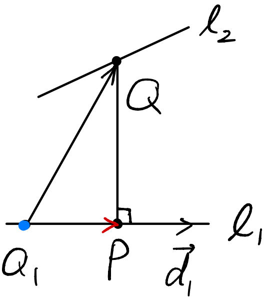
(d) Find the point P on l_1 which at the closet distance to l_2.

P: projection of Q onto l_1

$$\overrightarrow{Q_1P} = \text{proj}_{\vec{d_1}} \overrightarrow{Q_1Q} = \frac{\overrightarrow{Q_1Q} \cdot \vec{d_1}}{\vec{d_1} \cdot \vec{d_1}} \vec{d_1}$$

$$P - Q_1 = \frac{\overrightarrow{Q_1Q} \cdot \vec{d_1}}{\vec{d_1} \cdot \vec{d_1}} \vec{d_1}$$

$$P = \vec{Q_1} + \frac{\overrightarrow{Q_1Q} \cdot \vec{d_1}}{\vec{d_1} \cdot \vec{d_1}} \vec{d_1} = \ldots$$



Exercise 6

(a) Given 3 points A, B, C in $\mathbb{R}^2$. How to determine whether they lie on a line?

Method 1: Write equation of the line l through 2 points A, B
check if C is on l.

Method 2: Check whether $\vec{AB} \parallel \vec{AC}$

Examples:

(1) $A = (0,1)$, $B = (1,2)$, $C = (10,11)$

$\vec{AB} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \parallel \vec{AC} = \begin{pmatrix} 10 \\ 10 \end{pmatrix} \Rightarrow A, B, C$ are on the same line

(2) $A = (0,1)$, $B = (1,2)$, $C = (10,10)$

$\vec{AB} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \nparallel \vec{AC} = \begin{pmatrix} 10 \\ 9 \end{pmatrix} \Rightarrow A, B, C$ are not on the same line.

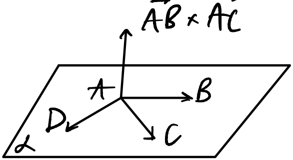
Exercise 6

(b) Given 4 points A, B, C, D in $\mathbb{R}^3$. How to determine whether they lie on a plane?

Method 1. Write the equation for plane α through A, B, C.
Check whether D is on α.

Method 2
α : $A, B, C \rightarrow$ normal $\vec{AB} \times \vec{AC}$
β : $A, B, D \rightarrow$ normal $\vec{AB} \times \vec{AD}$
check for parallelism!

Method 3: Check whether
$\vec{AD} \cdot (\vec{AB} \times \vec{AC}) = 0$

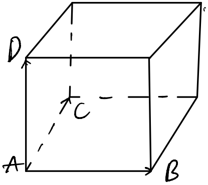


Exercise 6

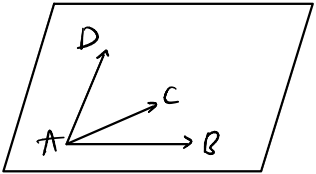
(b) Given 4 points A, B, C, D in $\mathbb{R}^3$. How to determine whether they line on a plane?

Method 4: If A, B, C, D lie on the same plane, the $\vec{AB}, \vec{AC}, \vec{AD}$ form a parallelepiped with volume $= 0$

$\Leftrightarrow \det(\vec{AB}\ \vec{AC}\ \vec{AD}) = 0$



Volume > 0



Volume $= 0$

Exercise 6

(b) Given 4 points A, B, C, D in $\mathbb{R}^3$. How to determine whether they line on a plane?

Examples: check for 2 cases

YES (a) $A = (0,0,0)$, $B = (1,2,3)$, $C = (-4,1,-3)$,
$D = (2,10,12)$

NO (b) $A = (0,0,0)$, $B = (1,2,3)$, $C = (-4,1,-3)$,
$D = (2,10,-1)$

Given 4 points A, B, C, D which form a parallelogram.

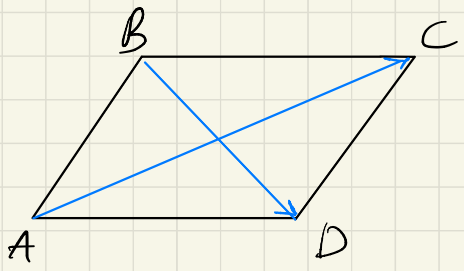
Which of the following vectors is a normal vector to the plane containing A, B, C, D?

$\vec{AB} \times \vec{BC}$ ✓

$\vec{AB} \times \vec{AD}$ ✓

$\vec{AC} \times \vec{BD}$ ✓ $(\vec{AC} \nparallel \vec{BD})$

$\vec{AB} \times \vec{CD}$ ✗



$\vec{AB} \parallel \vec{CD} \Rightarrow \vec{AB} \times \vec{CD} = \vec{0}$

HW3 Problem 6-8

- Let A be a **square matrix**.
  A matrix B is called the inverse of A, denoted $B = A^{-1}$, if $AB = I$.
- Our question: Compute A^2, A^4 with $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$.

$$A^2 = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A^4 = A^2 \cdot A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since $A^4 = I$, we have $A^{-1} = A^3$

Which matrix among A^3, A^7, A^{11} could be A^{-1}?

By the previous part, we already have

$$A^3 = A^{-1}$$

Further,

$$A^7 = A^3 A^4 = A^3 I = A^3 \Rightarrow A^7 = A^{-1}$$

$$A^{11} = A^7 A^4 = A^7 I = A^7 \Rightarrow A^{11} = A^{-1}$$