Lecture 3: Logics

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Propositions

- A **proposition** (more precisely **simple proposition**), or a **statement**, is a *declarative sentence* that is **either true or false**, but not both.
- Small letters $p, q, r, \dots$ are used to denote propositions. These letters are called **propositional variables**.
- 1 (or T) $=$ **truth value** of a *true statement*,
  0 (or F) $=$ truth value of a *false statement*.

Example 1

Which of the following are propositions? Give their truth values.

(a) $2+4=7$.

(b) Julius Caesar was a president of the United States.

(c) What time is it?

Example 1

(d) Be quiet.

(e) $2 + 2 = 4$.

(f) $x + y = z$.

(g) $x + 2 = 5$.

(h) $x + 2 = 5$ when $x = 3$.

Negation

- The **negation** of proposition p, denoted by $\neg p$ (read "not p"), is the proposition "it is not the case that p".
- $\neg p$ is true when p is false, and $\neg p$ is false when p is true.

$$\neg p = \begin{cases} 0 \text{ if } p = 1, \\ 1 \text{ if } p = 0. \end{cases}$$

Example 2

Let p be the proposition "$\sqrt{2}$ is an irrational number". Which of the following is $\neg p$?

(a) $\sqrt{3}$ is irrational.

(b) Every number is irrational except $\sqrt{2}$.

(c) $\sqrt{2}$ is not an irrational number.

(d) $\sqrt{2}$ is a rational number.

Example 3

Let p be the proposition "there are six stones on the book cover". Which of the following is $\neg p$?

(a) There are seven birds on the book cover.

(b) There are no stones on the book cover.

(c) There are six stones on the table.

Conjunction, disjunction

- The **conjunction** of p and q, denoted by $p \wedge q$, is the proposition "**p and q**".

$$p \wedge q \text{ is } \begin{cases} \text{true when both p and q are true,} \\ \text{false otherwise.} \end{cases}$$

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- The **disjunction** of p and q, denoted $p \vee q$, is "p or q".

$$p \vee q \text{ is } \begin{cases} \text{true if either p or q is true,} \\ \text{false otherwise.} \end{cases}$$

Exclusive-or

- The **exclusive-or** (read "ex-or") of p and q, denoted by $p \oplus q$, is the proposition "*exactly one of p or q*".

$$p \oplus q \text{ is } \begin{cases} \text{true if exactly one of p or q is true,} \\ \text{false otherwise.} \end{cases}$$

Exclusive-or

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- Another way to remember $p \oplus q$

$$p \oplus q = (p + q) \mod 2$$

In particular,

$$\begin{aligned} 0 \oplus 0 &= 1 \oplus 1 = 0, \\ 0 \oplus 1 &= 1 \oplus 0 = 1 \end{aligned}$$

Operations and names

Notation	Name/Read	Example
$\neg$	negation/not	$\neg p$
$\wedge$	conjunction/and	$p \wedge q$
$\vee$	disjunction/or	$p \vee q$
$\oplus$	exclusive or/ex-or	$p \oplus q$

Compound propositions

- The statements which are formed by *connectives* $\neg, \vee, \wedge, \oplus$ and propositional variables $p, q, \ldots$ are called **compound propositions**:

$$\neg p,\ \ p \vee q,\ \ p \wedge q,\ \ p \oplus q$$

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- The propositions which make up a compound proposition are called **propositional variables**. For examples,
  - p is a propositional variable of $\neg p$.
  - p and q are propositional variables of $p \vee q, p \wedge q, p \oplus q$.

Example 4

Construct $p \wedge q$, $p \vee q$ and $p \oplus q$ and find their truth values.

(a) $p : 5 > 9$, $q : 9 > 7$

(b) $p :$ Today is Wednesday, $q :$ It is raining

Order of operations

- If a compound proposition consists of $\neg, \vee$ and $\wedge$, the operation $\neg$ is always performed first. For example,

$$\neg p \vee q = (\neg p) \vee q$$

- When $\oplus$ is used, we use brackets to clearly indicate its order of precedence. For example,

$$(p \oplus q) \vee r, \;\; p \wedge (q \oplus r)$$

- When $\vee$ and $\wedge$ are used, we use brackets to indicate the order. For example

$$(p \wedge q) \vee r, \;\; p \wedge (q \vee r)$$

Example 5

Find the truth value of $\neg(p \wedge q) \vee r$

p : Today is Wednesday.

$q : 2 + 1 = 3.$

r : There is no Covid-19 infection in Singapore.

Truth table

- The **truth table** of a compound proposition is the table of all possible truth values for that proposition.
- We can make a single table for many propositions that share the same propositional variables: $\neg p, p \wedge q, p \vee q, p \oplus q$

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$
0	0	1	0	0	0
0	1	1	0	1	1
1	0	0	0	1	1
1	1	0	1	1	0

  - The first 2 columns are the propositional variables p and q.
  - Starting from the 2nd row, the table contains all possible combinations of p and q $(4$ in total).

Exercise 1

Construct the truth table (one single table) for

$$(p \oplus q) \oplus r, (p \oplus q) \wedge r.$$

p	q	r	$p \oplus q$	$(p \oplus q) \oplus r$	$(p \oplus q) \wedge r$
0	0	0			
0	0	1			
0	1	0			
0	1	1			
1	0	0			
1	0	1			
1	1	0			
1	1	1			

Tautology, contradiction, contingency

- A **tautology** is a compound proposition that is always true.
- A **contradiction** is a compound proposition that is always false.
- A **contingency** is a compound proposition that is neither a tautology nor a contradiction.
- Notations

$$\mathbf{T} = \text{tautology}, \ \mathbf{F} = \text{contradiction}.$$

Example 6

- Tautology

  "Today is Sunday or today is not Sunday"

- Contradiction

  "Today is Sunday and today is not Sunday"

- Contingency

  "I am going to get an A for CSD2258"

Exercise 2

Let p, q be propositions. Using truth table, show that

(a) Using truth table, show that $p \vee \neg p$ is a tautology and $p \wedge \neg p$ is a contradiction.

Exercise 2

(b) Construct the truth table for $(p \wedge q) \vee (\neg p \vee \neg q)$. Determine if this proposition is a tautology, a contradiction or a contingency.

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \wedge q$	$(p \wedge q) \vee (\neg p \vee \neg q)$
0	0					
0	1					
1	0					
1	1					

Logical equivalence

- Two compound propositions are **equivalent** if they have exactly the same truth values *under all possibilities*.
- If p and q are equivalent, we write

$$p \equiv q$$

- If p and q are not equivalent, we write

$$p \not\equiv q$$

Logical equivalence rules

Theorem 1

Let p, q, r be simple propositions. **T**=tautology, **F**=contradiction. The following logical equivalences hold.

(a) Identity laws

$$p \wedge \mathbf{T} \equiv p$$

$$p \vee \mathbf{F} \equiv p$$

(b) Negation laws

$$p \vee \neg p \equiv \mathbf{T}$$

$$p \wedge \neg p \equiv \mathbf{F}$$

Theorem 1 continued

(c) Domination laws

$$\begin{aligned} p \vee \mathbf{T} &\equiv \mathbf{T} \\ p \wedge \mathbf{F} &\equiv \mathbf{F} \end{aligned}$$

(d) Commutative laws

$$\begin{aligned} p \wedge q &\equiv q \wedge p \\ p \vee q &\equiv q \vee p \end{aligned}$$

(e) Associative laws

$$\begin{aligned} (p \wedge q) \wedge r &\equiv p \wedge (q \wedge r) \\ (p \vee q) \vee r &\equiv p \vee (q \vee r). \end{aligned}$$

Theorem 1 continued

(f) Distributive laws

$$
\begin{aligned}
p \wedge (q \vee r) &\equiv (p \wedge q) \vee (p \wedge r) \\
p \vee (q \wedge r) &\equiv (p \vee q) \wedge (p \vee r)
\end{aligned}
$$

(g) Double negation

$$\neg(\neg p) \equiv p$$

De Morgan's laws

Theorem 2

Let p, q be propositions. Then

$$\begin{aligned} \neg(p \vee q) &\equiv \neg p \wedge \neg q \\ \neg(p \wedge q) &\equiv \neg p \vee \neg q. \end{aligned}$$

Proof: $\neg(p \vee q) \equiv \neg p \wedge \neg q$

We prove by truth table.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

Example 7

Use De Morgan's law to write negations for the following.

(a) I am dreaming or today is Sunday.

(b) $-3 \leq x \leq 5$.

Exercise 3

Is $\neg(p \vee q) \equiv \neg p \vee \neg q$? Justify your answer.

Exercise 4

Prove that

$$\neg(\neg p \wedge q) \wedge (p \vee q) \equiv p.$$

Exercise 5

Prove that the following is a tautology

$$(p \wedge q) \vee (\neg p \vee \neg q)$$

Conditional statement

The **conditional statement** $p \to q$ is the proposition

"if p, then q" or "p implies q" or "p only if q"

- p is called **hypothesis** (or **premise**), q is called **conclusion**.

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- $p \to q$ is false only when "p is true and q is false".

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Conditional statement

The **conditional statement** $p \to q$ is the proposition

"if p, then q" or "p implies q" or "p only if q"

- p is called **hypothesis** (or **premise**), q is called **conclusion**.
- $p \to q$ is false only when "p is true and q is false".
  - True hypothesis $\to$ true conclusion.
  - False hypothesis $\to$ conclusion can be anything: **true by default** (or **vacuously true**). For example, "If I am superman, you are spiderman".

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Example 8

(a) Using truth table, prove that $p \to q \equiv \neg p \vee q$.

(b) Using (a), rewrite the following sentence in if-then form.
"Either you get to work on time or you are fire".

Exercise 6: Negation of a conditional statement

(a) Prove that $\neg(p \to q) \equiv p \wedge \neg q$.

(b) Using (a), write negations for the following statements

Caution. Negation of an if-then statement *does not start with if*.

"If you attend all lectures, then you will get an A".

"If you are caught littering, then you will be fined".

Converse, inverse and contrapositive

- The **converse** of $p \to q$ is $q \to p$.
- The **inverse** of $p \to q$ is $\neg p \to \neg q$.
- The **contrapositive** of $p \to q$ is $\neg q \to \neg p$.

Example 9

Write the converse, inverse and contrapositive of

"If today is Friday, then I have a test today".

Exercise 7

Using a truth table, prove the following

(a) $p \to q \not\equiv q \to p$.

(b) $p \to q \not\equiv \neg p \to \neg q$.

(c) $p \to q \equiv \neg q \to \neg p$.

Conditional statement vs converse, inverse, contrapositive

- Neither the converse nor the inverse of a conditional statement is equivalent to the statement itself.

$$p \to q \quad \not\equiv \quad q \to p$$
$$p \to q \quad \not\equiv \quad \neg p \to \neg q$$

Conditional statement vs converse, inverse, contrapositive

- Neither the converse nor the inverse of a conditional statement is equivalent to the statement itself.

$$p \to q \quad \not\equiv \quad q \to p$$
$$p \to q \quad \not\equiv \quad \neg p \to \neg q$$

- The contrapositive of a conditional statement is equivalent to the statement itself.

$$p \to q \equiv \neg q \to \neg p.$$

Biconditional statement

- The **biconditional** statement $p \leftrightarrow q$ is the proposition

  "p if and only if q"

- It is true if both p and q have the same truth value, and it is false otherwise.

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

Example 10

Using truth table to show that $p \leftrightarrow q$ is logically equivalent to the conjunction of $p \rightarrow q$ and $q \rightarrow p$, that is,

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p).$$