CSD1240 Homework 3 Solutions

Problem 1. Find the area of the triangles with given vertices A, B, C.
(a) $A(2, 6, 1),\ B(1, 1, 1),\ C(-1, 2, 3)$.
(b) $A(2, 0), B(3, 5), C(-1, -2)$.

Solution. (a) We first compute $\overrightarrow{AB} \times \overrightarrow{AC}$ by putting them into columns of a matrix

$$[\overrightarrow{AB}\ \overrightarrow{AC}] = \begin{bmatrix} -1 & -3 \\ -5 & -4 \\ 0 & 2 \end{bmatrix} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} -10 \\ 2 \\ -11 \end{bmatrix}$$

We obtain

$$\text{Area } \triangle ABC = \frac{1}{2}||\overrightarrow{AB} \times \overrightarrow{AC}|| = \frac{1}{2}\sqrt{(-10)^2 + 2^2 + (-11)^2} = 7.5$$

(b) To use the formula Area $\triangle ABC = \frac{1}{2}||\overrightarrow{AB} \times \overrightarrow{AC}||$, we put A, B, C into xyz-space by assigning z-coordinate=0: $A = (2, 0, 0), B = (3, 5, 0), C = (-1, -2, 0)$.

$$[\overrightarrow{AB}\ \overrightarrow{AC}] = \begin{bmatrix} 1 & -3 \\ 5 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} 0 \\ 0 \\ 13 \end{bmatrix}$$

We obtain

$$\text{Area } \triangle ABC = \frac{1}{2}||\overrightarrow{AB} \times \overrightarrow{AC}|| = \frac{1}{2}\sqrt{0^2 + 0^2 + 13^2} = 6.5$$

□

Problem 2. Find both the *parametric equation* and the *general equation* of the plane β containing three points P, Q, R in the following cases. Further, let $A = (1, 2, 3)$. find the point B on β which is at the closest distance to A.

(a) $P(3, -1, 4),\ Q(6, 0, 2),\ R(5, 1, 1)$.
(b) $P(2, 1, 3),\ Q(1, 3, 4),\ R(-2, -1, -5)$

Solution. (a) The plane β has direction vectors $\overrightarrow{PQ} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$ and $\overrightarrow{PR} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$. So it has parametric equation

$$(x,y,z) = (6,0,2) + s\begin{bmatrix}3\\1\\-2\end{bmatrix} + t\begin{bmatrix}2\\2\\-3\end{bmatrix} \Rightarrow \begin{cases} x = 6+3s+2t \\ y = s+2t \\ z = 2-2s-3t \end{cases}$$

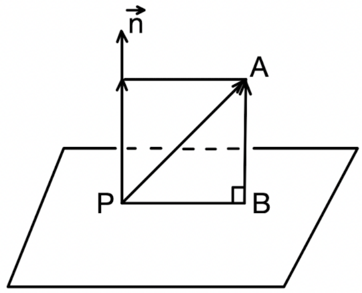
To find general equation of β, we compute its normal vector $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$

$$[\overrightarrow{PQ}\ \overrightarrow{PR}] = \begin{bmatrix}3 & 2\\1 & 2\\-2 & -3\end{bmatrix} \Rightarrow \vec{n} = \begin{bmatrix}1\\5\\4\end{bmatrix}$$

Since β passes through Q and has normal vector $\vec{n}$, it has general equation

$$1(x-6) + 5(y-0) + 4(z-2) = 0 \Leftrightarrow x + 5y + 4z - 14 = 0.$$

Next we find the orthogonal projection B of A onto β. Note that $\vec{n} = \begin{bmatrix}1\\5\\4\end{bmatrix}$, $\overrightarrow{PA} = \begin{bmatrix}-2\\3\\-1\end{bmatrix}$ and $\overrightarrow{BA} = \text{proj}_{\vec{n}}(\overrightarrow{PA})$.



We have

$$A - B = \overrightarrow{BA} = \text{proj}_{\vec{n}}(\overrightarrow{PA}) = \frac{\overrightarrow{PA}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n} = \frac{9}{42}\begin{bmatrix}1\\5\\4\end{bmatrix} = \begin{bmatrix}3/14\\15/14\\6/7\end{bmatrix}$$

We obtain

$$B = A - \overrightarrow{BA} = (1,2,3) - \begin{bmatrix}3/14\\15/14\\6/7\end{bmatrix} = \left(\frac{11}{14}, \frac{13}{14}, \frac{15}{7}\right).$$

(b) Similar to (a). $\square$

Problem 3. Find the intersection of the lines l_1 and l_2 (in $\mathbb{R}^2$) in following cases.

(a) $l_1 : \begin{cases} x = -3 + t \\ y = 1 - t \end{cases}$ and $l_2 : (x, y) = (7, 0) + s \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(b) $l_1 : x + 4y = 13$ and l_2 : go through $(4, 0)$ and $(5, -1)$.

(c) $l_1 : y - 1 = -(x + 3)$ and l_2 : go through $(4, 0)$ and perpendicular to $x + 4y = 13$.

Solution. Let $P = (x, y)$ be a common point of l_1 and l_2.

(a) Since P lies on both l_1 and l_2, we have

$$\begin{cases} x = -3 + t \\ y = 1 - t \end{cases} \quad \text{and} \quad \begin{cases} x = 7 + 2s \\ y = s \end{cases} \quad \Rightarrow \begin{cases} -3 + t = 7 + 2s \\ 1 - t = s \end{cases} \quad \Rightarrow s = -3, t = 4$$

Therefore, $P = (x, y) = (1, -3)$ is the only common point of l_1 and l_2, that is, l_1 and l_2 intersect at the point $(1, -3)$.

(b) Put $A = (4, 0)$ and $B = (5, -1)$. The line l_2 goes through $A = (4, 0)$ and has direction vector $\overrightarrow{AB} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So it has parametric equation

$$(x, y) = (4, 0) + t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Leftrightarrow \begin{cases} x = 4 + t \\ y = -t \end{cases}$$

Assume that $P = (x, y)$ is a common point of l_1 and l_2.

- P is on $l_2 \Rightarrow x = 4 + t,\ y = -t$ for some t
- P is on $l_1 \Rightarrow x + 4y = 13 \Rightarrow (4 + t) + 4(-t) = 13 \Rightarrow t = -3$

We obtain $P = (4 + t, -t) = (1, 3)$, that is, l_1 and l_2 intersect at the point $(1, 3)$.

(c) The line l_1 has general equation

$$x + y + 2 = 0$$

The line l_2 goes through $(4, 0)$ and has a normal vector $\vec{n}$ equal to the direction vector of $x + 4y - 13 = 0$, that is, $\vec{n} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$. So l_2 has general equation

$$-4(x - 4) + 1(y - 0) = 0 \Leftrightarrow -4x + y + 16 = 0.$$

Since $P = (x, y)$ is a common point of l_1 and l_2, its coordinates satisfy both equations of l_1 and l_2. We have

$$\begin{cases} x + y + 2 = 0 \\ -4x + y + 16 = 0 \end{cases} \quad \Rightarrow x = \frac{14}{5}, y = -\frac{24}{5}.$$

Therefore, l_1 and l_2 intersect at the point $\left(\frac{14}{5}, -\frac{24}{5}\right)$. □

Problem 4. Find the relative position (intersecting, parallel, skew) and the intersections between any two of the three lines k, l, m

$$k : \begin{cases} x = 1 + 2r \\ y = -1 + 2r \\ z = 2 + 3r \end{cases} \quad l : \begin{cases} x = -1 + 2t \\ y = t \\ z = 2 + 2t \end{cases} \quad m : (x, y, z) = (2, 0, 1) + s \begin{bmatrix} 1 \\ 1 \\ 3/2 \end{bmatrix}$$

Solution. To determine the relative position of 2 lines and find their intersection, we start by solving the equations of both lines. Note the following possibilities.

1. Exactly one solution $\Rightarrow$ 2 lines intersect at one point
2. Infinitely many solution $\Rightarrow$ 2 lines are the same
3. No solution
   - Parallel direction vectors $\Rightarrow$ parallel lines
   - Non-parallel direction vectors $\Rightarrow$ skew lines

(a) Let $P = (x, y, z)$ be a common point of k and l. We have

$$\begin{cases} x = 1 + 2r \\ y = -1 + 2r \\ z = 2 + 3r \end{cases} \text{and} \begin{cases} x = -1 + 2t \\ y = t \\ z = 2 + 2t \end{cases},$$

which implies

$$\begin{cases} 1 + 2r = -1 + 2t \\ -1 + 2r = t \\ 2 + 3r = 2 + 2t \end{cases} \Leftrightarrow \begin{cases} r = t - 1 \\ -1 + 2(t - 1) = t \\ 2 + 3(t - 1) = 2 + 2t \end{cases} \Leftrightarrow \begin{cases} r = t - 1 \\ t = 3 \\ t = 3 \end{cases} \Leftrightarrow t = 3, r = 2$$

Therefore, k and l intersect at $P = (x, y, z) = (1 + 2r, -1 + 2r, 2 + 3r) = (5, 3, 8)$.

(b) Let $Q = (x, y, z)$ be a common point of k and m. We have

$$\begin{cases} x = 1 + 2r \\ y = -1 + 2r \\ z = 2 + 3r \end{cases} \quad \text{and} \quad \begin{cases} x = 2 + s \\ y = s \\ z = 1 + \dfrac{3}{2}s \end{cases},$$

which implies

$$\begin{cases} 1 + 2r = 2 + s \\ -1 + 2r = s \\ 2 + 3r = 1 + \dfrac{3}{2}s \end{cases} \Leftrightarrow \begin{cases} s = 2r - 1 \\ s = 2r - 1 \\ 2 + 3r = 1 + \dfrac{3}{2}(2r - 1) \end{cases} \Leftrightarrow \begin{cases} s = 2r - 1 \\ 2 = -\dfrac{1}{2} \end{cases}$$

The 2nd equation cannot happen. So k and l don't intersect. Further, the direction vectors $\vec{v}_k = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v}_l = \begin{bmatrix} 1 \\ 1 \\ 3/2 \end{bmatrix}$ of k and l are parallel ($\vec{v}_k = 2\vec{v}_l$). The lines k and l are parallel.

(c) Let $R = (x, y, z)$ be a common point of l and m. We have

$$\begin{cases} x = -1 + 2t \\ y = t \\ z = 2 + 2t \end{cases} \quad \text{and} \quad \begin{cases} x = 2 + s \\ y = s \\ z = 1 + \dfrac{3}{2}s \end{cases},$$

which implies

$$\begin{cases} -1 + 2t = 2 + s \\ t = s \\ 2 + 2t = 1 + \dfrac{3}{2}s \end{cases} \Leftrightarrow \begin{cases} t = s \\ -1 + 2s = 2 + s \\ 2 + 2s = 1 + \dfrac{3}{2}s \end{cases} \Leftrightarrow \begin{cases} t = s \\ s = 3 \\ s = -2 \end{cases}$$

The 2nd equation and the 3rd equation contradict each other. So l and m don't intersect. Further, the direction vectors $\vec{v}_l = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{v}_m = \begin{bmatrix} 1 \\ 1 \\ 3/2 \end{bmatrix}$ of l and m are not parallel. The lines l and m are skew. $\square$

Problem 5. Let l be the line going through $P = (2,3,1)$ and $Q = (5,-3,4)$. Let α be the plane going through $(0,2,-1)$ with direction vectors $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$.

(a) Find the intersection of l and α.

(b) Find the intersection of α and the plane β: through $(1,2,0)$ with normal $\vec{n} = \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix}$.

Solution. (a) The line l has equation

$$(x,y,z) = P + t\overrightarrow{PQ} = (2,3,1) + t\begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} \Leftrightarrow \begin{cases} x = 2+3t \\ y = 3-6t \\ z = 1+3t \end{cases}$$

The plane α has normal vector $\vec{n} = \vec{u} \times \vec{v}$

$$[\vec{u}\ \vec{v}] = \begin{bmatrix} 1 & -3 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \Rightarrow \vec{n} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

The general equation of β is

$$1(x-0) + (-1)(y-2) + 2(z-(-1)) = 0 \Leftrightarrow x - y + 2z + 4 = 0.$$

Let $P = (x,y,z)$ be a common point of l and α. We have

$$\begin{cases} x = 2+3t \\ y = 3-6t \\ z = 1+3t \end{cases} \quad \text{and} \quad x - y + 2z + 4 = 0,$$

which implies

$$(2+3t) - (3-6t) + 2(1+3t) + 4 = 0 \Leftrightarrow 15t + 5 = 0 \Leftrightarrow t = -\frac{1}{3}.$$

Therefore, l and α intersect at a unique point

$$P = (x,y,z) = (2+3t, 3-6t, 1+3t) = (1,5,0).$$

(b) The plane β has equation

$$1(x-1) - 2(y-2) + 6(z-0) = 0 \Leftrightarrow x - 2y + 6z + 3 = 0.$$

Let $P = (x, y, z)$ be a common point of α and β. We have

$$x - y + 2z + 4 = 0 \tag{1}$$

$$x - 2y + 6z + 3 = 0 \tag{2}$$

In solving (1) and (2), we express x and y in terms of z. Subtracting (1)-(2), we have

$$y - 4z + 1 = 0 \Rightarrow y = 4z - 1 \tag{3}$$

Substituting $y = 4z - 1$ into (1), we have

$$x = y - 2z - 4 = 2z - 5 \tag{4}$$

By (3) and (4), any common point $P = (x, y, z)$ of α and β need to have $x = 2z - 5$ and $y = 4z - 1$. So the intersection of α and β is

$$(x, y, z) = (2z - 5, 4z - 1, z) = (-5, -1, 0) + z \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix},$$

which is a line going through the point $(-5, -1, 0)$ with the direction vector $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$. $\square$