No tutorials next week

Fundamentals of Differentiation Part 3

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AY 22/23 Trimester 2

Week 1-5 material tested for midterms.

Recap

$\sin(x^2) \longrightarrow \cos(x^2) \cdot 2x.$

1. Chain Rule

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

2. Implicit and explicit equations, how to differentiate implicitly
3. Explicit equations: Tangent line equation to f at $(a, f(a))$

$$y = f'(a)(x - a) + f(a)$$

Implicit equations: Tangent line equation to graph at (x_0, y_0)

$$y = \underbrace{\frac{dy}{dx}(x_0, y_0)}_{\substack{\text{derivative of } y \\ \text{at } (x_0, y_0)}}(x - x_0) + y_0$$

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Higher order derivatives

explicit equations $y = f(x)$

If a function $y = f(x)$ is differentiable, then we have a function $f'(x)$, the derivative of $y = f(x)$. $f^{(1)}(x)$

We also have the derivative of $f'(x)$, which is a function $f''(x)$, called the *second derivative* of $f(x)$, is the derivative of the first derivative

$$f''(x) = (f')'(x).$$ $f^{(2)}(x)$

The second derivative can also be written as

$$\underbrace{\frac{d}{dx}}_{\substack{\text{derivative}\\ \text{of}}} \underbrace{\left(\frac{dy}{dx}\right)}_{\substack{\text{first}\\ \text{derivative}}} = \underbrace{\frac{d^2y}{dx^2}}_{\substack{\text{second}\\ \text{derivative}}}$$

Exercise 1

Find the second derivative for each of the following functions.

1. $f(x) = 3x^2 + 2x + 1$ $\rightarrow f'(x) = 6x+2,\ f''(x) = 6$
2. $f(x) = \sin(2x)$ $\rightarrow f'(x) = \cos(2x)\cdot 2 = 2\cos(2x)$

   $f''(x) = -2\sin(2x)\cdot 2$

   $= -4\sin(2x)$
3. $f(x) = e^{5x}$
4. $f(x) = \ln(x^2 + 1)$

$f'(x) = 5e^{5x} \rightarrow f''(x) = 25e^{5x}$

$f'(x) = \frac{1}{x^2+1}\cdot 2x = \frac{2x}{x^2+1} = 2x(x^2+1)^{-1}$

$f''(x) = 2(x^2+1)^{-1} + 2x(-1)(x^2+1)^{-2}\cdot 2x$

$= \frac{2}{x^2+1} - \frac{4x^2}{(x^2+1)^2} = \frac{2(x^2+1) - 4x^2}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2}$

$$\frac{2-2x^2}{(1+x^2)^2} = \frac{-2(x^2-1)}{(1+x^2)^2}$$

Third derivative onwards

The *third derivative* $f'''(x)$ of $y = f(x)$ is the derivative of the second derivative

$$f'''(x) = (f'')'(x).$$

$f^{(3)}(x)$

The *fourth derivative* and onwards are abbreviated slightly differently (for pretty obvious reasons)

$$f^{(4)}(x) = (f''')'(x).$$

In general, the *n-th derivative* $f^{(n)}(x)$ is obtained by differentiating the $(n-1)$-th derivative:

$$f^{(n)}(x) = (f^{(n-1)})'(x).$$

Exercise 2

Find the third derivative for each of the functions. (These were the functions in Exercise 1)

1. $f(x) = 3x^2 + 2x + 1$ → ezpz 0
2. $f(x) = \sin(2x)$ → $f''(x) = -4\sin(2x)$,
   $f'''(x) = -4\cos(2x) \cdot 2$
   $= -8\cos(2x)$.
3. $f(x) = e^{5x}$
4. $f(x) = \ln(x^2 + 1)$

$f'''(x) = 5^3 e^{5x}$.

$f''(x) = \dfrac{2-2x^2}{(x^2+1)^2}$

Exercise 2

$$f''(x) = \frac{2-2x^2}{(x^2+1)^2} = (2-2x^2)(x^2+1)^{-2}$$

$$f'''(x) = -4x(x^2+1)^{-2} + (2-2x^2)(-2)(x^2+1)^{-3} \cdot 2x$$

$$= \frac{-4x}{(x^2+1)^2} + \frac{(-4x)(2-2x^2)}{(x^2+1)^3}$$

$$= \frac{-4x(x^2+1) + 8x^3 - 8x}{(x^2+1)^3}$$

$$= \frac{4x^3 - 12x}{(x^2+1)^3}$$

Definitions of increasing and decreasing functions

Definition

Let A be any subset of the domain of a function f. For $x_1, x_2 \in A$,

- f is *increasing* on A if

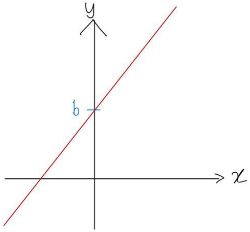
$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2,$$

- f is *decreasing* on A if

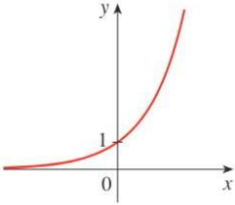
$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2.$$

Important: For a function to be increasing/decreasing, $f(x_1) \leq f(x_2)/f(x_1) \geq f(x_2)$ must hold for all pairs of $x_1 < x_2$!

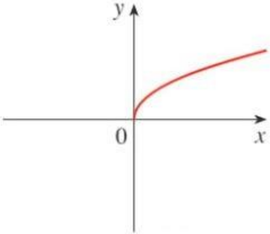
Examples of increasing functions



$y = mx + b \ (m > 0)$



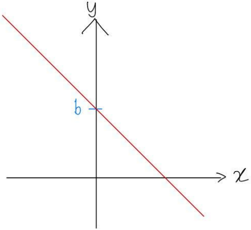
$y = a^x \ (a > 0)$



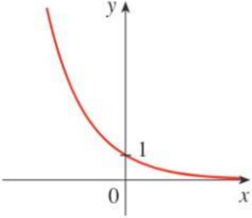
$y = \sqrt{x}$

What do you observe about the gradient of the functions here?

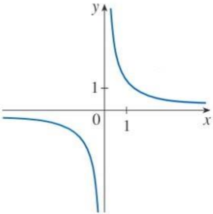
Examples of decreasing functions



$y = mx + b \ (m < 0)$



$y = a^x \ (0 < a < 1)$



$y = \frac{1}{x}$

What do you observe about the gradient of the functions here?

Increasing/Decreasing Test

By observing the sign (positive/negative) of the gradient of a differentiable function f, we can tell if f is increasing or decreasing.

Theorem (Increasing/Decreasing Test or I/D Test)

Let I be an interval which is a subset of the domain of f. If f is differentiable on I, then

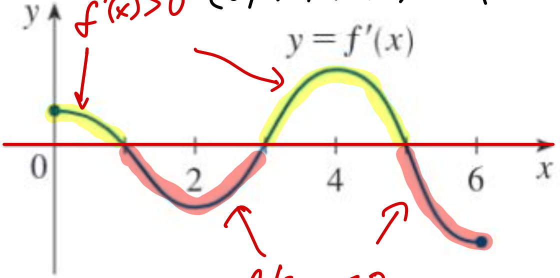
- f is increasing on I if and only if $f'(x) > 0$ on $\cancel{A}$ I,
- f is decreasing on I if and only if $f'(x) < 0$ on $\cancel{A}$ I

Example 1

The graph of the *derivative* f' of a function f is shown below. Determine the intervals for which f is increasing or decreasing.

$(0,1),(3,5) \rightarrow f$ increasing

$f'(x)>0$



$f'(x)<0$.

$(1,3),(5,6) \rightarrow f$ decreasing.

Critical points of f

Definition

(⋆) A *critical point* of a function f is a point c where either

1. $f'(c) = 0$, or
2. f is not differentiable at c.

Critical points play an important role in the identification of intervals where a function is increasing or decreasing, and they also play a big role in optimization (later in the course).

Exercise 3

Let $f(x) = \dfrac{x}{x^2+1}$. $= x(x^2+1)^{-1}$

1. What is the domain of f? $\rightarrow \mathbb{R}$.
2. Find the intervals for which f is increasing or decreasing.

① Find critical points of f.

$$f'(x) = (x^2+1)^{-1} + x(-1)(x^2+1)^{-2} \cdot 2x$$

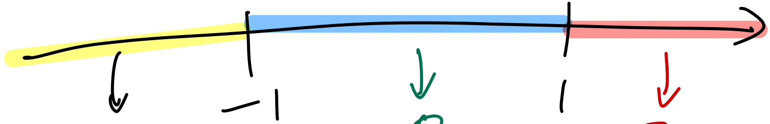
$$= \frac{1}{(x^2+1)} + \frac{-2x^2}{(x^2+1)^2}$$

$$= \frac{(x^2+1) - 2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

Exercise 3

(1) $1 - x^2 = 0 \Rightarrow x = -1, \quad x = 1$

(2)



$-2 \qquad\qquad 0 \qquad\qquad 2$

$f'(-2) < 0 \qquad f'(0) > 0 \qquad f'(2) < 0$

(3) f is decreasing $(-\infty, -1), (1, \infty)$

f is increasing $(-1, 1)$