Series Fundamentals

Dr. Ronald Koh
ronald.koh@digipen.edu (Teams preferred over email)

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Series: an infinite sum of terms

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.

- A **series** is an infinite sum of all the terms of the sequence $\{a_n\}$
$$\sum_{n=1}^{\infty} a_n.$$
- Sometimes, the series can start summing from a later index, say $n = n_0$. This series is written as
$$\sum_{n=n_0}^{\infty} a_n.$$
- We can sum first N terms of $\{a_n\}_{n=1}^{\infty}$, this gives us a **sequence** $\{s_N\}_{N=1}^{\infty}$ **of partial sums**, i.e.
$$s_N = a_1 + a_2 + \cdots + a_N = \sum_{n=1}^{N} a_n.$$

Definition of an (infinite) series

- An infinite series is the **limit** of the sequence of partial sums

$$\sum_{n=1}^{\infty} a_n = S = \lim_{N\to\infty} s_N = \lim_{N\to\infty} (a_1 + a_2 + \cdots + a_N).$$

- If this limit exists, we say that the series $\sum_{n=1}^{\infty} a_n$ is **convergent**, and

$$S = \sum_{n=1}^{\infty} a_n.$$

Otherwise, $\sum_{n=1}^{\infty} a_n$ is **divergent**.

Example 1: Geometric series

The **geometric series** with **starting term** $a \neq 0$ and **common ratio** r is

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

Find the general formula for the sequence of partial sums s_N.

Example 1: Geometric series

Convergence of the Geometric series

Theorem (1)

The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ with starting term $a \neq 0$ and common ratio r is convergent if and only if $|r| < 1$. For $|r| < 1$, its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

Proof of Theorem (1)

Exercise 1

Determine if the following series are convergent. If they are, find their sum.

1. $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$
2. $\sum_{n=0}^{\infty} r^n, \ |r| < 1$

Exercise 1

Why is there a need for convergence/divergence tests?

- To evaluate the sum $S = \sum_{n=1}^{\infty} a_n$, we need to find the limit of partial sums:

$$S = \lim_{N\to\infty} s_N = \lim_{N\to\infty} a_1 + a_2 + \cdots + a_N.$$

- Finding this limit is no easy feat, e.g. for the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, the sequence of partial sums $\{s_N\}_{N=1}^{\infty}$ is

$$s_N = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{N^2}.$$

Why is there a need for convergence/divergence tests?

- The limit of partial sums is then

$$S = \lim_{N\to\infty} 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{N^2} = \frac{\pi^2}{6}.$$

- This is difficult to evaluate, and the difficulty is representative of most limits of partial sums, with some exceptions like geometric series.
- Instead of finding this limit S, we focus on showing if this limit exists or not (convergence or divergence).
- This is done by **convergence/divergence tests**.

Convergence of p-series

Theorem (p-series)

A p-series is a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for some fixed real number p.

This series is convergent if $p > 1$ and divergent if $p \leq 1$.

Some examples:

- $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the **harmonic series**. It is divergent.
- $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. $(p = 2)$
- $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent. $\left(p = \frac{1}{2}\right)$

Divergence Test

It turns out that one of the indicators of a divergent series is the **limit of the sequence** $\{a_n\}_{n=1}^{\infty}$ (NOT the limit of partial sums $\{s_N\}_{n=1}^{\infty}$).

Theorem (Divergence Test)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. If

$$\lim_{n\to\infty} a_n \neq 0 \,(\text{or does not exist}),$$

then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

Note: If $\displaystyle\lim_{n\to\infty} a_n = 0$, the test is **inconclusive**; there are examples of series $\displaystyle\sum_{n=1}^{\infty} a_n$ which are convergent/divergent. (Can you come up with examples?)

Example 2

Determine the convergence of $\sum_{n=1}^{\infty} \frac{n^2+1}{2n^2-1}$.

Exercise 2

Determine the convergence of the following series.

1. $$\sum_{n=1}^{\infty}(-1)^n$$
2. $$\sum_{n=1}^{\infty}\frac{n^3+n}{\sqrt{n^6+1}}$$

Series Laws

Like the limit laws for sequences, some of these laws extend to series, **except** for product, quotient and power laws.

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are **convergent**. Then

$$\text{(a)} \sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n.$$

$$\text{(b)} \sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n.$$

Exercise 3

Find the sum $\sum_{n=1}^{\infty}\left[\left(\frac{3}{4}\right)^n + 2^{2n}6^{1-n}\right]$.

Comparison Test

Theorem (Comparison Test)

Let n_0 be a fixed positive integer. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with $0 \leq a_n \leq b_n$ for $n \geq n_0$ (i.e. the terms of each series **eventually** obey this inequality; doesn't have to start from $n = 1$). Then

- If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
- If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is divergent.

The Comparison Test can be intuitively summed up in two sentences. If we have two non-negative series, with the larger of the two series being convergent, then the smaller series must also be convergent. Also, if the smaller series is divergent, then the larger series must also be divergent.

Example 3

Determine the convergence of $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n-1}$.

Exercise 4

Determine the convergence of the following series.

1. $$\sum_{n=2}^{\infty} \frac{n}{n^2 - 2}$$
2. $$\sum_{n=3}^{\infty} \frac{n}{n^3 - 8}$$

Exercise 4

Limit Comparison Test

Theorem (Limit Comparison Test, or the LCT)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with positive terms. If c is a number such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0,$$

then either both series converge, or both series diverge.

Note: The limit comparison test is NOT the comparison test.

Standard steps in the use of LCT

We usually start with a series $\sum_{n=1}^{\infty} a_n$, and we are asked to figure out if this series is convergent or divergent.

1. Find a series $\sum_{n=1}^{\infty} b_n$ which is *similar* to $\sum_{n=1}^{\infty} a_n$, and whose convergence/divergence is **known**.
2. Compute $\lim_{n\to\infty} \frac{a_n}{b_n}$.
3. If this limit exists, and is positive, we can apply the LCT.

Example 4

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2 + 2n}$.

Exercise 5

Determine the convergence of the following series.

1. $$\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^6 + n}}$$
2. $$\sum_{n=1}^{\infty} \frac{1}{2^n + 2}$$

Exercise 5