Optimization II

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The FDT-GEV

We finish off this chapter with the last theorem of the course.

Theorem (First Derivative Test for Global Extreme Values)

Suppose that c is a critical point of a continuous function f defined on an interval I.

1. If $f'(x) > 0$ for **all** $x < c$ and $f'(x) < 0$ for **all** $x > c$, then $f(c)$ is the global maximum value of f on I.
2. If $f'(x) < 0$ for **all** $x < c$ and $f'(x) > 0$ for **all** $x > c$, then $f(c)$ is the global minimum value of f on I.

Note that this interval I can be **unbounded**, e.g. $(2, \infty)$, $(-\infty, -1]$, $(-\infty, \infty)$, unlike in the ICBM, where the interval has to be bounded; $[a, b]$.

FDT versus FDT-GEV

Key differences between the FDT-GEV compared to the FDT:

- FDT shows local max/min, unlike FDT-GEV that shows global max/min.
- FDT only concerns with the function for x that is **near** c, while FDT-GEV looks at the function for all values of x (before and after c) in I.

Visualization

Example 1

A cylindrical can is to be made to hold 1000 cm^3 of oil. Find the dimensions of the can that will minimize the cost of metal to manufacture.

The volume of the can is

$$V = \underline{\hspace{3cm}},$$

where r and h is the radius and the height (in cm) of the can respectively. The function that we are aiming to minimize is the ________________ of the can, which is

$$A = \underline{\hspace{7cm}}.$$

We need to express A in terms of one variable, thus we use the volume equation to express h in terms of r:

$$h = \underline{\hspace{3cm}}..$$

Example 1

Then A can be written in terms of r only:

$$A(r) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

The domain of this function is __________.

We find the critical points of this function by first differentiating A with respect to r:

$$A'(r) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

Critical points are __________ (ignored), and __________.

Observe from the expression of A' that $A'(r)$ _____ when $r <$ _____, and $A'(r)$ _____ when $r >$ _____. Therefore $r =$ _____ is a global ____________ point.

Example 1

We need the dimensions r and h, we already know r, so we can compute h because we have a relation between h and r:

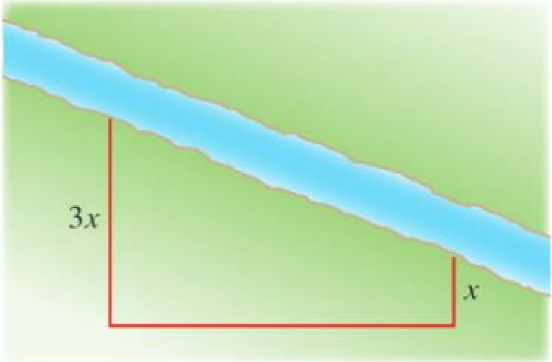
Hence, the dimensions of the cylindrical can that minimizes the cost of metal to manufacture are

$$\text{radius} = r = \_\_\_\_\_\_ \text{ cm and height} = h = \_\_\_\_\_\_\_\_\_\_\_\_ \text{ cm.}$$

Observation: In this case, the height is twice the size of the radius.

Example 2

A farmer has 400 m of fencing of negligible thickness for enclosing a trapezoidal field along a river as shown below. One of the parallel sides is three times longer than the other. No fencing is needed along the river. Find the largest area the farmer can enclose.



Example 2

Let x be the shorter end on one of the widths. Let y be the length of the field. Then the area A of the trapezoidal field in terms of x and y is

$$A = \underline{\hspace{6cm}}.$$

Since the farmer has 400 m of fencing, we have another relation between x and y:

$$4x + y = 400.$$

We want A to be in terms of x only, so $y = 400 - 4x$, and thus

$$A(x) = \underline{\hspace{9cm}}.$$

Differentiating A with respect to x gives

$$A'(x) = \underline{\hspace{9cm}}.$$

Example 2

Thus the only critical point of A is $x =$ _____.

We also notice that $A'(x)$_______________ for $x <$ _____ and $A'(x)$_______________ for $x >$ _____. So by the FDT-GEV, $x =$ _____ is a global $x =$ _______________. Thus $y =$ __________, and the largest area the farmer can enclose is

$$A(\_\_\_) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

Note: This example can also be done using the ICBM. You only need to realize that the interval for x is $[0, 100]$.

Exercise 1

Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

Exercise 1