Tutorial 5

Q1. $y = \underbrace{e^{2x}}_{f(x)}\underbrace{[A\cos(3x) + B\sin(3x)]}_{g(x)}$.

(a) $y' = f'(x)g(x) + f(x)g'(x)$.

$= (2x)' e^{2x}[A\cos(3x) + B\sin(3x)]$

$+ e^{2x}\{[A\cos(3x)]' + [B\sin(3x)]'\}$

$= 2e^{2x}[A\cos(3x) + B\sin(3x)] + e^{2x}(A \cdot -\sin(3x)\cdot 3 + B\cos(3x)\cdot 3)$

$= 2e^{2x}[A\cos(3x) + B\sin(3x)] + e^{2x}(-3A\sin(3x) + 3B\cos(3x))$

$= e^{2x}[(2A+3B)\cos(3x) + (2B-3A)\sin(3x)] \leftarrow$

$y'' = \{\underbrace{e^{2x}}_{g(x)}\underbrace{[(2A+3B)\cos(3x) + (2B-3A)\sin(3x)]}_{f(x)}\}'$

$= \underbrace{2e^{2x}}_{g'(x)}\underbrace{[(2A+3B)\cos(3x) + (2B-3A)\sin(3x)]}_{f(x)}$

$+ \underbrace{e^{2x}}_{g(x)}\underbrace{[(2A+3B)\cos(3x)' + (2B-3A)\sin(3x)']}_{f'(x)}$

$= e^{2x}[(4A+6B)\cos(3x) + (4B-6A)\sin(3x)]$

$+ e^{2x}[-3(2A+3B)\sin(3x) + 3(2B-3A)\cos(3x)]$

$$= e^{2x}\left[(4A+6B+6B-9A)\cos(3x) + (4B-6A-6A-9B)\sin(3x)\right]$$

$$= \boxed{e^{2x}\left[(12B-5A)\cos(3x) + (-12A-5B)\sin(3x)\right]}$$

(b) $y'' - 4y' + 13y = 0$

$$\text{LHS} = e^{2x}\left[(12B-5A)\cos(3x) + (-12A-5B)\sin(3x)\right]$$
$$- 4\left[e^{2x}(2A+3B)\cos(3x) + (2B-3A)\sin(3x)\right] + 13e^{2x}\left[A\cos(3x) + B\sin(3x)\right]$$

$$= e^{2x}\left[\left[(\cancel{12B}-\cancel{5A}) - \cancel{8A} - \cancel{12B} + \cancel{13A}\right]\cos(3x) + (-\cancel{12A}-\cancel{5B} - \cancel{8B} + \cancel{12A} + \cancel{13B})\sin(3x)\right]$$

$$= e^{2x}\left[[0] + 0\right] = 0 = \text{RHS} \checkmark$$

Q2: $f(x) = \sin x$

(a)
$$\begin{cases} f'(x) = \cos x \\ f''(x) = -\sin x \\ f'''(x) = -\cos x \\ f^{(4)}(x) = \sin x \end{cases}$$

$f^{(5)}(x) = \cos x$

$f^{(6)}(x) = -\sin x$

$f^{(7)}(x) = -\cos x$

$f^{(8)}(x) = \sin x$

$$f^{(n)} = \begin{cases} \sin x & \text{if } n \bmod 4 = 0 \\ \cos x & \text{if } n \bmod 4 = 1 \\ -\sin x & \text{if } n \bmod 4 = 2 \\ -\cos x & \text{if } n \bmod 4 = 3 \end{cases}$$

(b) $f^{(6311)}(x) = -\cos x$. because $6311 \bmod 4 = 3$.

Q3.

(a) $f(x) = 2x^3 - 15x^2 + 24x - 5 \leftarrow$

$f'(x) = 6x^2 - 30x + 24 = 6(x^2 - 5x + 4) = \boxed{6(x-4)(x-1) = 0}$

critical points $x=1$, and $x=4$.

$\begin{cases} 6(x-4)(x-1) > 0 & x > 4 \text{ or } x < 1 \quad \text{increasing} \\ & (4, +\infty) \quad (-\infty, 1) \\ 6(x-4)(x-1) < 0 & 1 < x < 4 \quad \text{decreasing.} \\ & (1, 4) \end{cases}$

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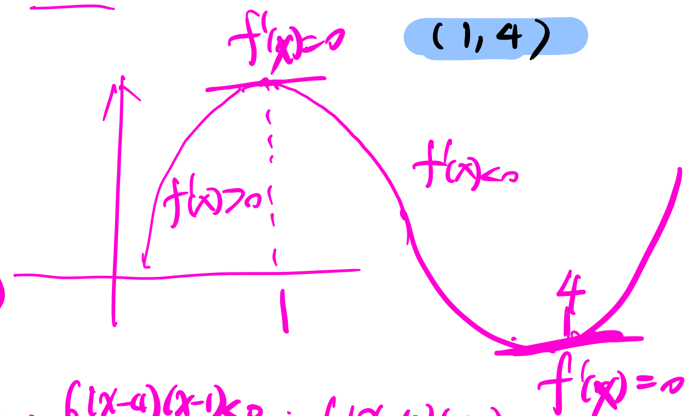
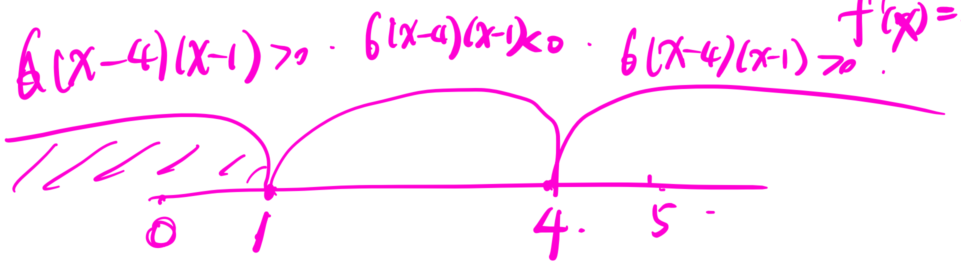


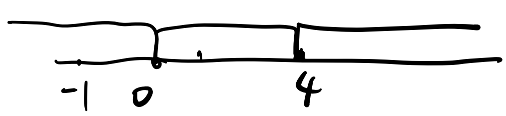


(b) $f(x) = \underbrace{x^4}_{f(x)} \underbrace{e^{-x}}_{g(x)}$

$f'(x) = f'(x)g(x) + f(x)g'(x)$

$= 4x^3 e^{-x} + x^4 \cdot (-e^{-x})$

$= x^3 \boxed{e^{-x}} (4 - x) = 0$ $\qquad x=0,\ x=4$



when $x=-1$ $\quad f'(-1) = (-1)^3 e^1 (4-(-1)) = -5e < 0$

$\boxed{f(x) \text{ decreases on } (-\infty, 0)}$.

when $x=1$ $\quad f'(1) = 1^3 (e^1) \cdot (4-1) = 3e > 0$.

$\boxed{f(x) \text{ increases on } (0, 4)}$.

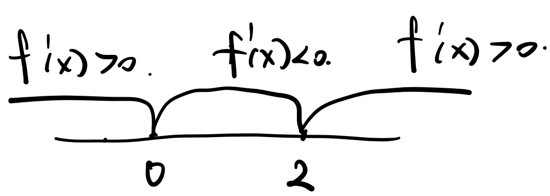
when $x=5$ $\quad f'(5) = 5^3 e^{-5} (-1) < 0$

$\boxed{f(x) \text{ decreases on } (4, +\infty)}$

(c) $f(x) = x + \frac{4}{x^2} = x + 4x^{-2}$.

$f'(x) = 1 - 8x^{-3} = 1 - \frac{8}{x^3} = 0 \Rightarrow 1 = \frac{8}{x^3} \Rightarrow \boxed{x=2}$

$x=0$ is also a critical point as f is not differentiable at $x=0$.



$f(x)$ decreases on $(0, 2)$

$f(x)$ increases on $(-\infty, 0)$ and $(2, +\infty)$