Concavity and the Second Derivative Test

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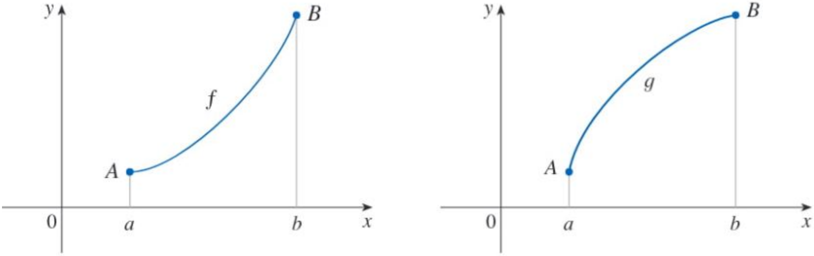
Example 1

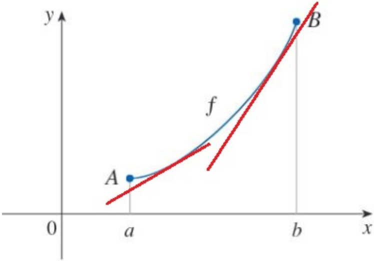
Below are graphs of two different functions f and g, both are **increasing** on $[a, b]$.

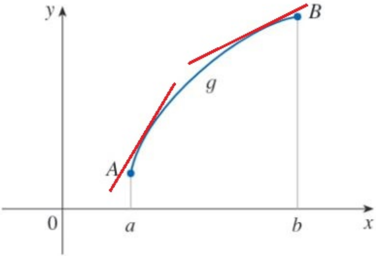


Notice that both "curve" in different directions. How can we distinguish between these two types of behaviours?

Example 1: Observation

We can observe the **gradient** of these two functions.





Notice that for the graph of f, the gradient f' is **increasing**, while for g, the gradient g' is **decreasing**.

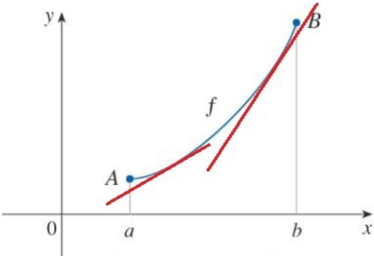
Using the I/D Test on f' and g', we can say that $f''(x) > 0$ and $g''(x) < 0$ on an interval $[a, b]$.

Concavity of graphs definition

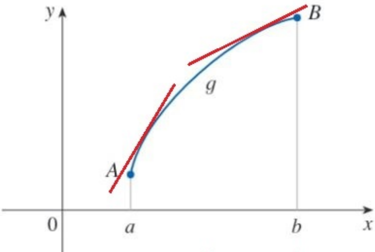
Concavity: The curved-ness (similar to a circle) of a graph.

Definition

1. If a graph of a function f lies **above** all its tangents on an interval I, then f is called **concave upward (CU)** on I.
2. If a graph of a function f lies **below** all its tangents on an interval I, then f is called **concave downward (CD)** on I.



f concave upwards



g concave downwards

Concavity Test

Similar to the I/D Test, we also have a test to check the intervals of CU and CD for a function.

Theorem (Concavity Test)

1. If $f''(x) > 0$ on an interval I, then the graph of f is concave upwards (CU) on I.
2. If $f''(x) < 0$ on an interval I, then the graph of f is concave downwards (CD) on I.

Example 2

Let $f(x) = x^3 - 3x^2 - 9x + 4$.
Find the intervals on which f is CU or CD.

$f'(x) = \rule{6cm}{0.4pt}.$

$f''(x) = \rule{4cm}{0.4pt}.$

$f''(x) = 0 \iff \rule{5cm}{0.4pt} \iff \rule{5cm}{0.4pt}.$

Therefore, for $x < \underline{\quad}$, $f'(x) \underline{\quad} 0$, thus f is $\underline{\qquad}$ on $\underline{\qquad\quad}$.

Also, for $x > \underline{\quad}$, $f'(x) \underline{\quad} 0$, thus f is $\underline{\qquad}$ on $\underline{\qquad\quad}$.

Exercise 1

Let $f(x) = x^4 - 2x^2 + 3$.
Find the intervals on which f is CU or CD.

Exercise 2

Let $f(x) = \dfrac{x}{x^2+1}$. Find the intervals on which f is CU or CD.

Exercise 2

Inflection points

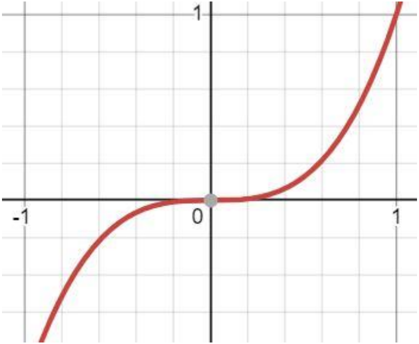
Local maxima and minima occur where f' changes sign.
Where f'' changes sign, i.e. where f changes from CU to CD or vice versa, we call them *inflection points*.

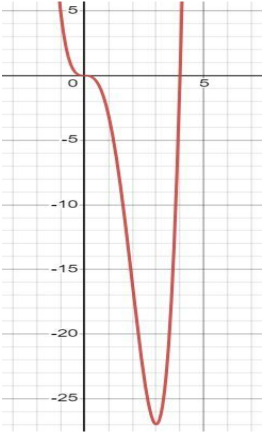
Definition

A point c on a curve $y = f(x)$ is called an **inflection point** if

1. f is continuous at c, and
2. f changes from CU to CD or CD to CU at c.
   Alternatively, f'' changes sign from positive to negative or negative to positive (in view of the Concavity Test).

Inflection point examples





Left: $y = x^3$
Right: $y = x^4 - 4x^3$

Both graphs have inflection points at $x = 0$.

Finding inflection points

Like Fermat's Theorem in narrowing the amount of points we need to check (critical points) to find local maxima and minima, we also have something similar for inflection points.

Theorem

If f is twice differentiable at c and has an inflection point at c, then $f''(c) = 0$.

Exercise 3: Find the inflection points for the functions in Example 2, Exercises 1 and 2.

Second Derivative Test

As a result of the Concavity Test, we get the Second Derivative Test.

Theorem (Second Derivative Test)

Suppose f'' is continuous near c.

1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

The Second Derivative Test serves as an alternative to the First Derivative Test, but has some noticeable drawbacks; when $f''(c) = 0$, the Second Derivative Test is inconclusive. There could be a local maximum there, a local minimum there, or neither (See Examples 4, 5 and 6). **If this happens, we need to fall back to the First Derivative Test.**

Example 3

Let $f(x) = \dfrac{x}{x^2+1}$.

We know that $f'(x) = \dfrac{1-x^2}{(x^2+1)^2}$, so the critical points are $x = \pm 1$.

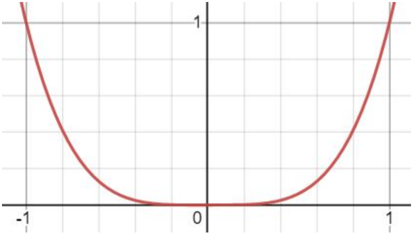
We have previously shown that $x = -1$ is a local ____________ point, and $x = 1$ is a local ____________ point. We can verify this using the Second Derivative Test.

$f''(x) =$

$f''(-1)$ ___ 0, so $x = -1$ is a local ____________ point.
$f''(1)$ ___ 0, so $x = 1$ is a local ____________ point.

Example 4

Local minimum at c where $f'(c) = 0$ and $f''(0) = 0$:
$f(x) = x^4$ and $c = 0$.



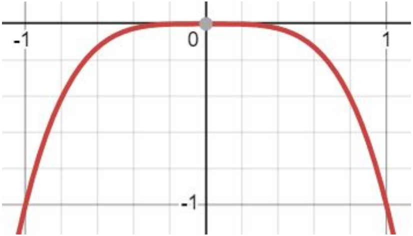
Here, $f'(x) = 4x^3 = 0 \iff c = 0 \iff$ ________________.

Also, $f''(x) = 12x^2 = 0 \iff c = 0 \iff$ ________________.

If we use the First Derivative Test for sign changes of f' (or observe the graph directly), $c = 0$ is a ____________________.

Example 5

Local maximum at c where $f'(c) = 0$ and $f''(0) = 0$:
$f(x) = -x^4$ and $c = 0$.

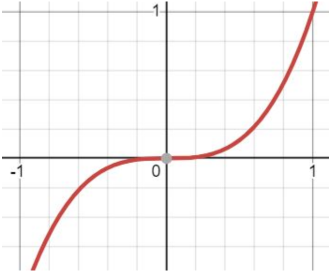


Here, $f'(x) = -4x^3 = 0 \iff c = 0 \iff$ ________________.
Also, $f''(x) = -12x^2 = 0 \iff c = 0 \iff$ ________________.

If we use the First Derivative Test for sign changes of f' (or observe the graph directly), $c = 0$ is a ________________.

Example 6

Neither local max/ min at c where $f'(c) = 0$ and $f''(0) = 0$:
$f(x) = x^3$ and $c = 0$.



Here, $f'(x) = 3x^2 = 0 \iff c = 0 \iff$ ________________.

Also, $f''(x) = 6x = 0 \iff c = 0 \iff$ ________________.

If we use the First Derivative Test for sign changes of f' (or observe the graph directly), $c = 0$ is ________________________________.

Exercise 3-1

Use the First Derivative Test to find the local extreme point(s) of $f(x) = 1 + 3x^2 - 2x^3$.

Exercise 3-2

Use the Second Derivative Test to find the local extreme point(s) of $f(x) = 1 + 3x^2 - 2x^3$.

Exercise 4-1

Use the First Derivative Test to find the local extreme point(s) of $f(x) = \dfrac{x^2}{x-1}$.

Exercise 4-2

Use the Second Derivative Test to find the local extreme point(s) of $f(x) = \dfrac{x^2}{x-1}$.

When to use FDT or SDT?

To find local extreme points of a function, we have either the First Derivative Test or the Second Derivative Test. Here are some tips:

- If the calculation of the second derivative is tedious/difficult, avoid SDT altogether and just stick to FDT.
- If the calculation of the second derivative is easy, it is usually more efficient to use SDT than FDT, but you also run the risk of running into inflection points (where SDT is inconclusive, from there you have to fall back to FDT).
- Experience (do more problems!) will help you determine which test to use quicker.