First Derivative Test

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Table of contents

Applications of differentiation in optimization

We will start to unravel some of the important applications of differentiation. One of these types of applications are in **optimization** problems, which is the key focal point of this course.

In optimization problems, we are trying to find the most optimal/best way of doing something. Examples of such problems include:

- At what speed should you drive your car in order to minimize fuel consumption?
- What is the minimum amount you need to spend to get the maximum amount of cashback for a credit card?
- What is the minimum amount of material needed to manufacture a cylindrical can which can contain a fixed volume of liquid?

Global extreme values and points

c extreme point (domain)
f(c) extreme value (range)

Definition

Let c be a number in the domain D of a function f. Then we say that the number $f(c)$ is the

f(c)

- **global maximum value** of f on D if $f(c) \geq f(x)$ for all points $x \in D$. (c here is called a **global maximum point** of f)
- **global minimum value** of f on D if $f(c) \leq f(x)$ for all points $x \in D$. (c here is called a **global minimum point** of f)

E.g: The highest point on Earth is Mount Everest (global maximum: 8,848.86 m), and the lowest point on Earth is the Challenger Deep (global minimum: $-10,929$ m).

Local extreme values and points

$f(c)$ largest among $f(x)$

Definition

where x near c

The number $f(c)$ is a

d

- **local maximum value** of f if $f(c) \geq f(x)$ when x is near c.
  (c here is called a **local maximum point** of f)
- **local minimum value** of f if $f(c) \leq f(x)$ when x is near c.
  (c here is called a **local maximum point** of f)

Note: x is near c means on some open interval containing the point c. eg. $x \in (c-\frac{1}{2}, c+\frac{1}{2})$

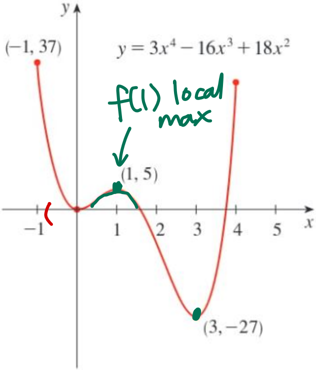
A function may have many *peaks* (local maxima) and *troughs* (local minima) but only one highest value (global maximum) and one lowest value (global minimum).

Example 1

The graph of the function $f(x) = 3x^4 - 16x^3 + 18x^2$ with domain $[-1, 4]$ is:

$f(0) \leq f(x)$

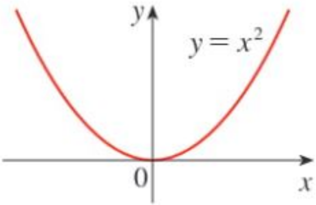
x near 0



$f(-1) = 37$ is the global maximum, where $f(3) = -27$ is the global minimum, and it is also a local minimum. $f(0) = 0$ is a local minimum, and $f(1) = 5$ is a local maximum.

Example 2

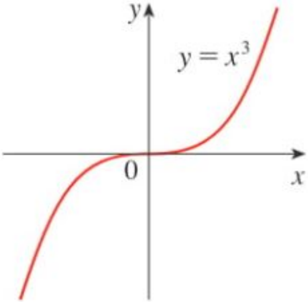
The graph of the function $f(x) = x^2$ with domain $\mathbb{R}$ is:



$f(0) = 0$ is both a local and the global minimum, but f has no local maxima nor global maximum.

Example 3

The graph of the function $f(x) = x^3$ with domain $\mathbb{R}$ is:



By observation, the function has no global extreme points nor local extreme points.

Extreme Value Theorem and Critical points

The previous examples beg the question: under what circumstances do functions have global extreme points?

Theorem (Extreme Value Theorem (EVT))

A **continuous** function f with domain $[a, b]$ attains its global maximum and minimum value.

closed and bounded interval

Recall that *critical points* are important in optimization:

Definition

$(\star)$ A **critical point** of a function f is a point c where either

1. $f'(c) = 0$ (i.e. c is a *stationary point*), or
2. f is not differentiable at c.

1/D Test very impt!

Fermat's Theorem

Theorem (Fermat's)

If f has a local maximum or local minimum at c, then c is a critical point of f.

Fermat's Theorem does not tell us how to find local extreme points, but it definitely tells us what type of points to check first (critical points).

To assist us with checking which critical points are local maxima or local minima, we have the **First Derivative Test**.

First Derivative Test

↳ to determine which critical points are local max / local min

The first derivative test tells us which critical points are local max, local min, or neither.

Theorem (First Derivative Test)

Let f be a continuous function where $f(c)$ **is defined**. Also, let c be a critical point of f.

- If f' changes from postive to negative at c, then f has a local maximum at c.
- If f' changes from negative to positive at c, then f has a local minimum at c.
- If f' is positive to both the left and right of c, or negative to both the left and right of c, then f has neither a local maximum nor a local minimum at c.

Example 4

Let $f(x) = \dfrac{x}{x^2+1}$. $= x(x^2+1)^{-1}$ $\qquad x = \pm 1$

Step 1: We find local extreme values of f by first checking the critical points of f:

$$f'(x) = (x^2+1)^{-1} + x(-1)(x^2+1)^{-2} \cdot 2x$$

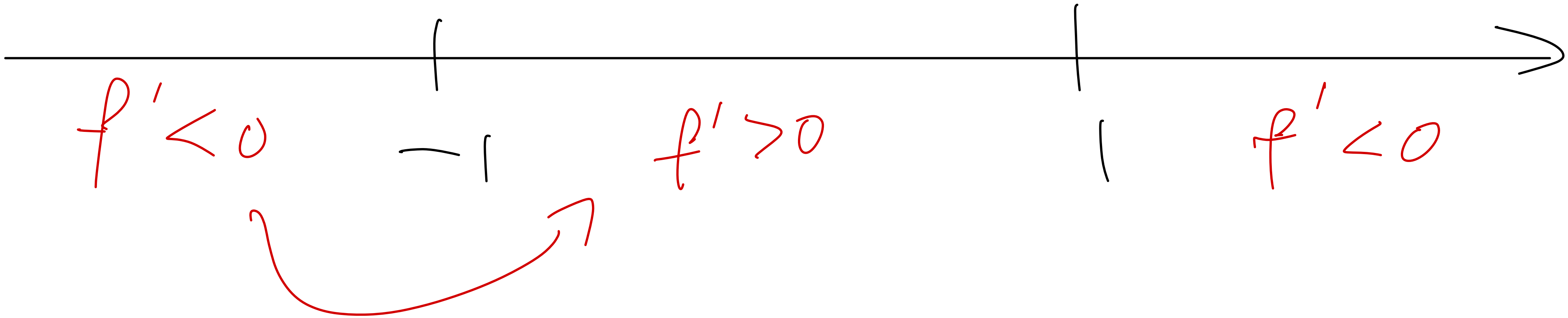
$$= \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = 0$$

Step 2: We check for sign changes of f' (First Derivative Test).

We see that for the critical point $x = -1$, f' changes from negative to positive, so $f(-1)$ is a local minimum value.

We see that for the critical point $x = 1$, f' changes from positive to negative, so $f(1)$ is a local maximum value.

$$f'(x) = \frac{1 - x^2}{(x^2+1)^2}$$



Exercise 1

Let $f(x) = 2x^3 - 15x^2 + 24x - 5$.

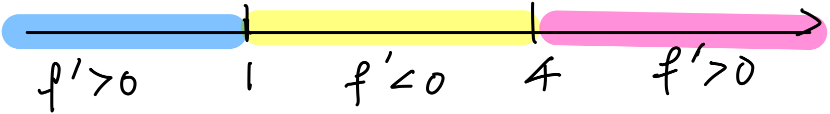
Find all local extreme points and values of f.

$$\begin{array}{cc|c} x & -4 & -4x \\ x & -1 & -x \\ \hline x^2 & +4 & -5x \end{array}$$

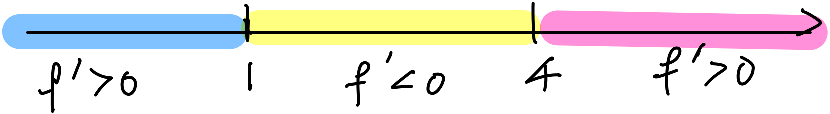
$$f'(x) = 6x^2 - 30x + 24 = 6(\underbrace{x^2 - 5x + 4})$$

$$f'(x) = 0 \Leftrightarrow 6(x-1)(x-4) = 0.$$

$\therefore$ Critical points are $x = 1$, $x = 4$



Exercise 1



For the critical pt $x = 1$, f' sign changes from positive to negative, so $x = 1$ is a local maximum point. $f(1) = 6$ is a local maximum value.

For the critical point $x = 4$, f' sign changes from negative to positive, so $x = 4$ is a local minimum point. $f(4) = -21$ is a local minimum value.