Recap

- Hashing
  - Hash function
  - Hash table
- Collision
  - Collision resolution
  - Linear probing

Considerations

- If the table is sparsely populated, searching is fast since we'd expect to perform one or two probes.
- If the table is nearly full, we will be spending most of our time resolving collisions. What is the worst case?
  - Probing for an open slot handles collisions, but won't help if we run out of slots.
- Collision tend to form groups of items
  - We call these groups **clusters**.
- Clusters tend to grow quickly. (Snowball effect)

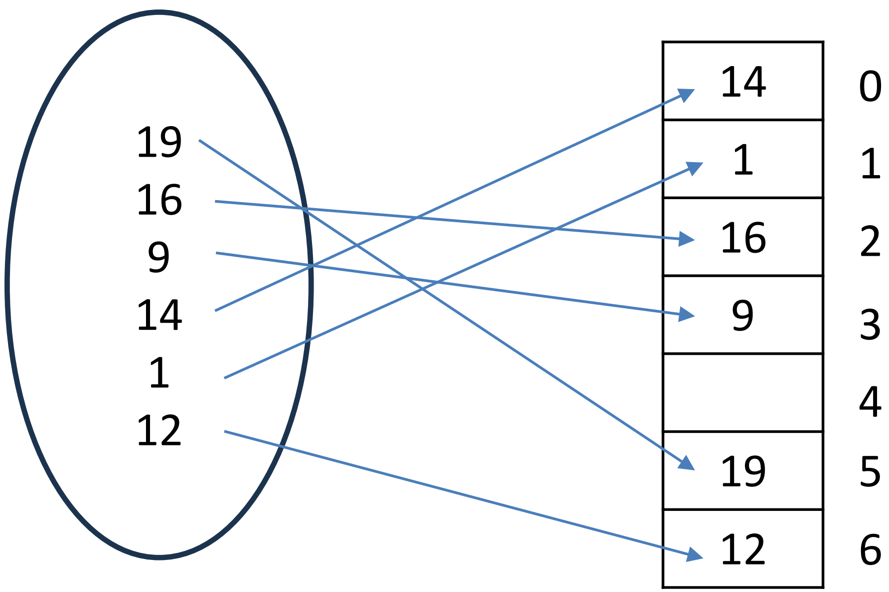
Load Factor

- Load Factor = (Items in table)/(Size of the hash table)
- The current value of load factor affects the performance significantly
  - Let's define a hit as finding an item.
  - Let's define a miss as discovering that an item doesn't exist.

Simple Example

- Hash function: $H(k)=k\%7$

Number of probes = 1 + 1 + 2 + 1 + 1 + 2 = 8



Load Factor = 6/7 = 0.86

Knuth's Formulas

- Show how probing is directly related to the load factor x, for a non-full table.
- Average number of probes for a hit: $\dfrac{1+\dfrac{1}{1-x}}{2}$
- Average number of probes for a miss: $\dfrac{1+\dfrac{1}{(1-x)^2}}{2}$

Knuth's Formulas

Load Factor (%)	Probe hits	Probe misses
5	1.03	1.05
10	1.09	1.12
20	1.13	1.28
30	1.21	1.52
40	1.33	1.89
50	1.50	2.50
60	1.75	3.62
70	2.17	6.06
80	3.00	13.00
90	5.50	50.50
95	10.5	200.50

Other Probing Methods

- The fundamental problem with linear probing is that all of the probes trace the same sequence.

- Quadratic probing: 1, 4, 9, 16, 25, etc.
- Pseudo-random probing: Probe by a random value
  - Must use key as the **seed** to ensure repeatability
- Double hashing: Use another hash function to determine the probe sequence.
  - Hash function: P(K), primary hash gives starting point (index)
  - Probe function: S(K), second hash gives the stride (offset for subsequent probes)

Other Probing Methods

- Quadratic probing: Probe 1, 4, 9, 16, 25, etc.
- Index from the hash function:

$$y, y+1, y+2^2, y+3^2 \ldots \ldots.$$

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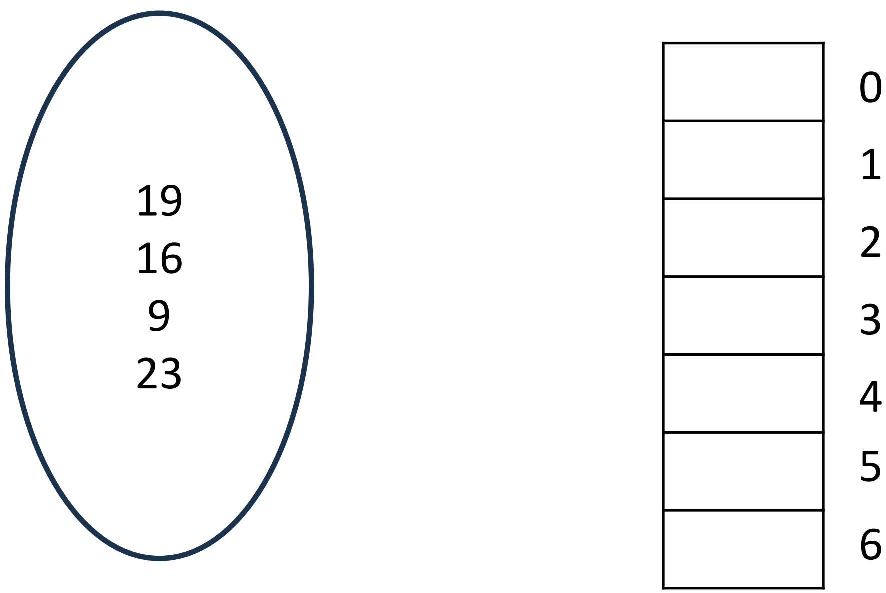

Hash function: H(k)=k%7

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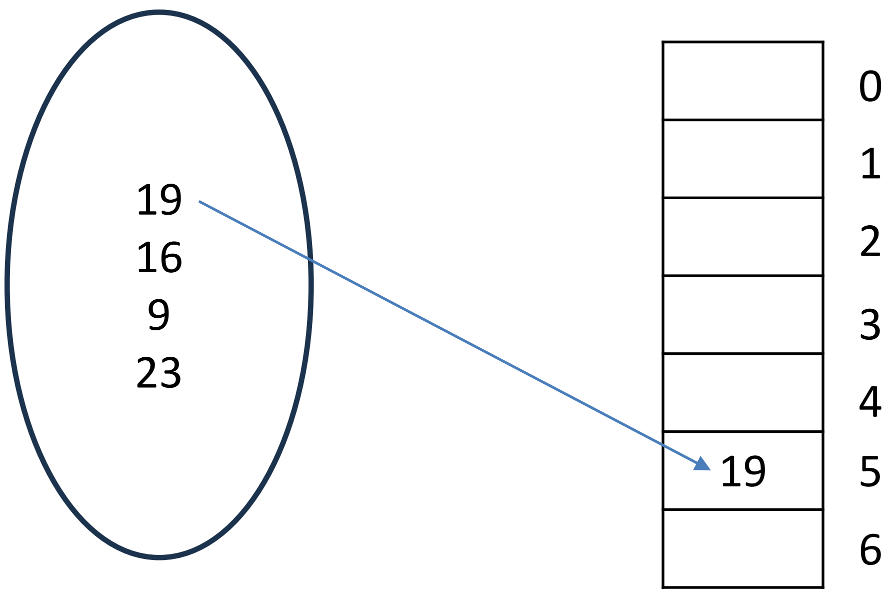

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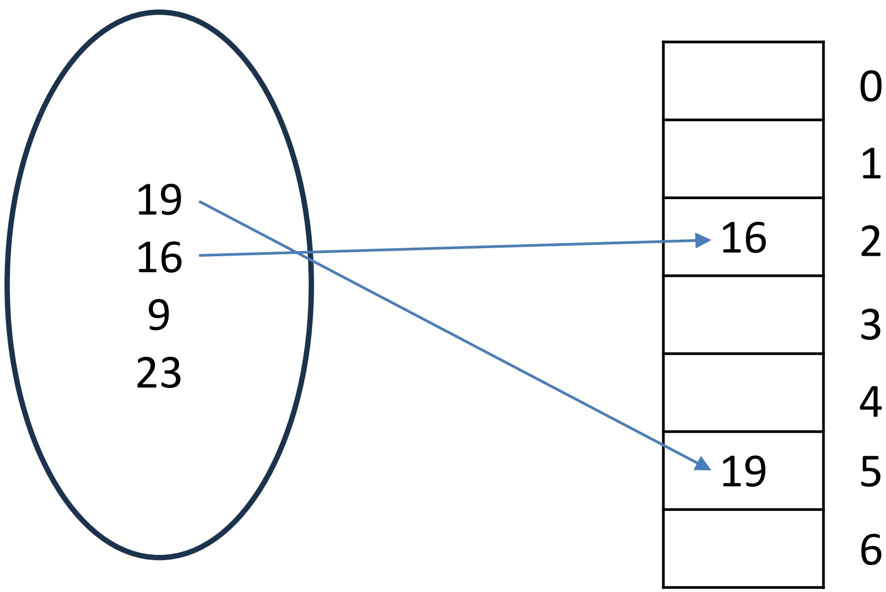

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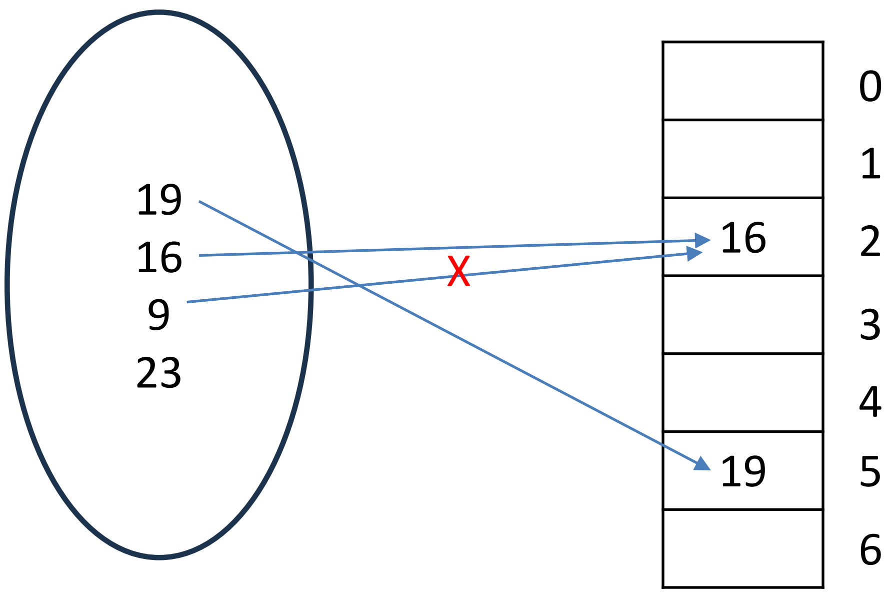

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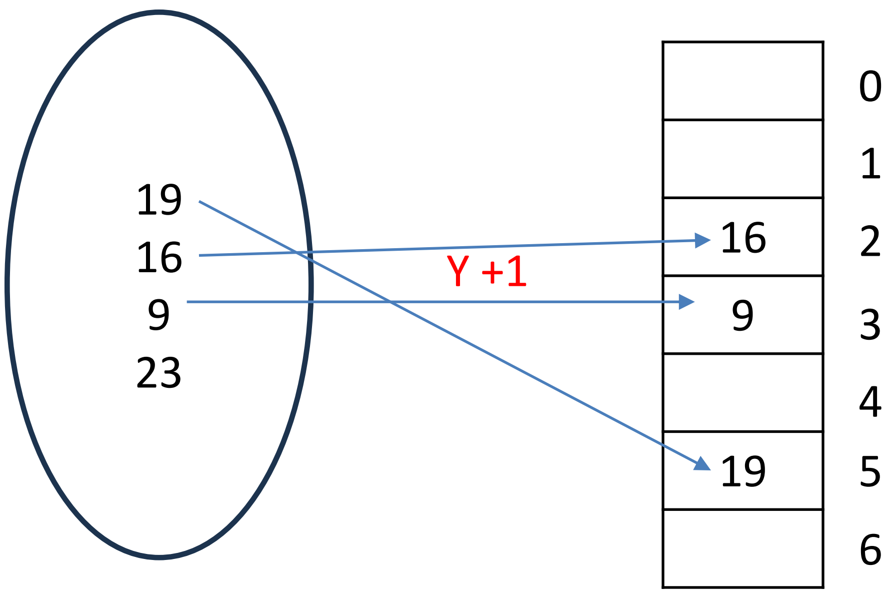

Hash function: H(k)=k%7

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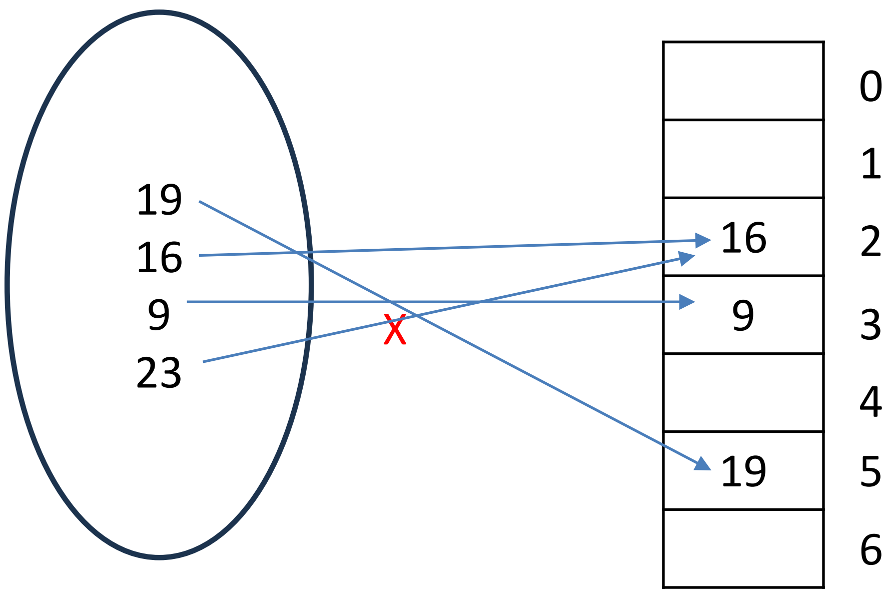

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Other Probing Methods

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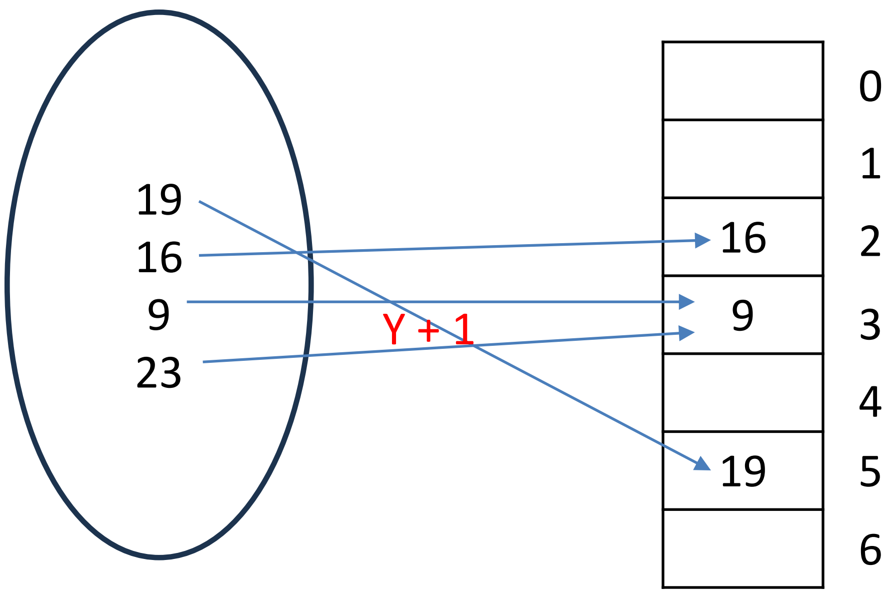

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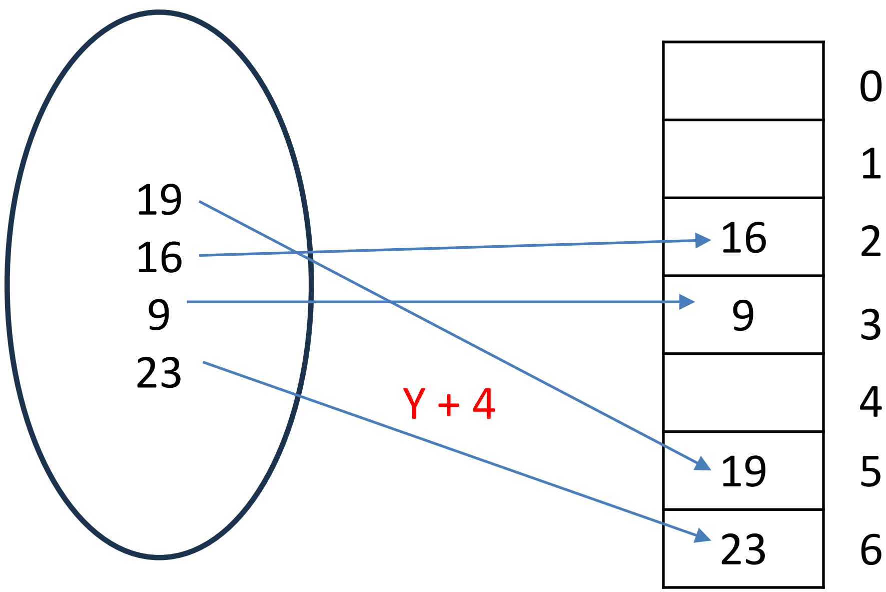

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Other Probing Methods

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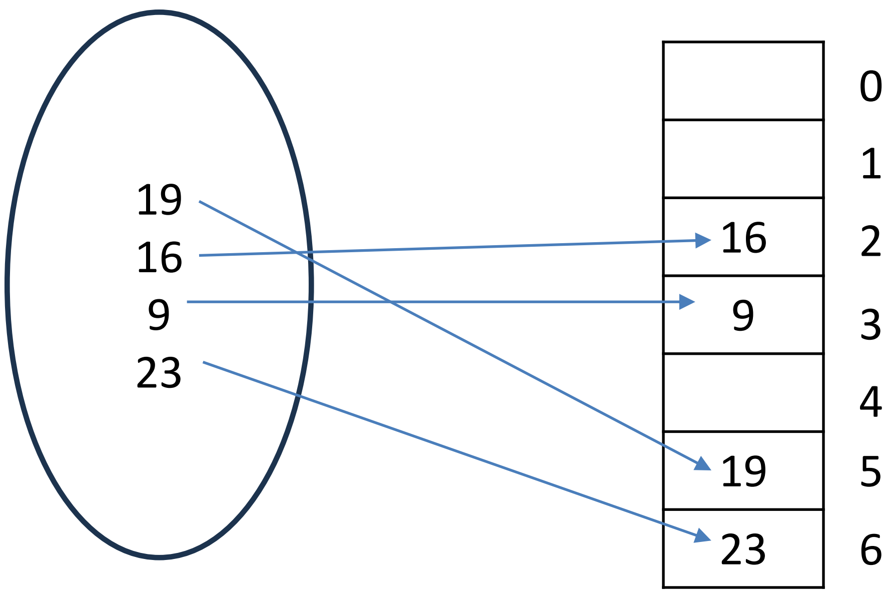


Number of probes = 1 + 1 + 2 + 3 = 7

Hash function: H(k)=k%7

Other Probing Methods

- Pseudo-random probing: Probe by a random value
  - Must use key as the **seed** to ensure repeatability

Other Probing Methods

- Double hashing: Use another hash function to determine the probe sequence.
  - Hash function: P(K), primary hash gives starting point (index)
  - Probe function: S(K), second hash gives the stride (offset for subsequent probes)

Double Hashing

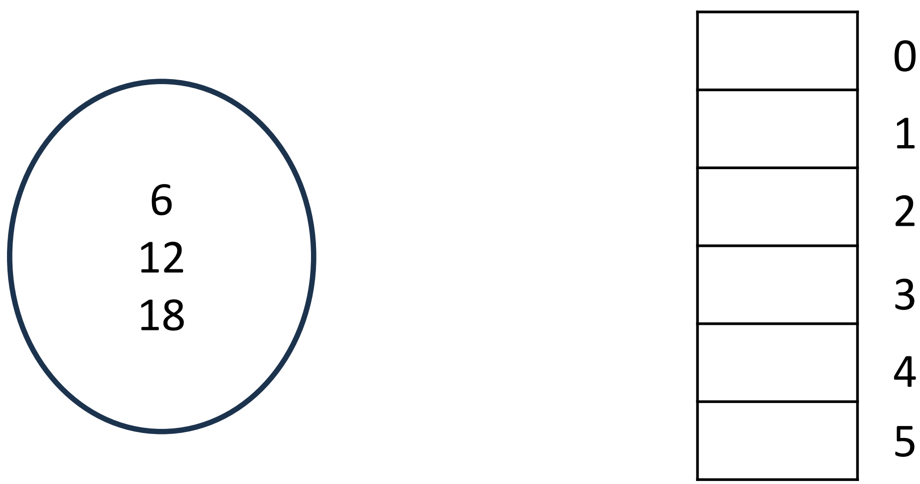
- $\mathrm{P(k)}$ is the primary Hash function and is computed once for searches.

- $\mathrm{S(k)}$ is the Secondary Hash and is computed once only if there was a collision with $\mathrm{P(k)}$.

- First probe is just for the primary hash: $\mathrm{P(k)}$
- Second probe: $\mathrm{P(k) + S(k)}$
- Third probe: $\mathrm{P(k) + 2S(k)}$
- Fourth probe: $\mathrm{P(k) + 3S(k)}$, etc.

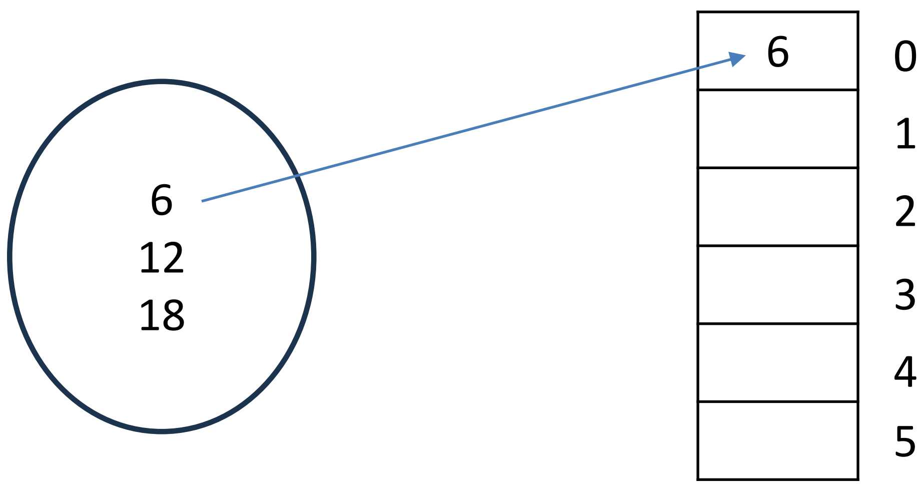
Double Hashing

$$h(x) = x \% 6 \qquad g(x) = (x\%5) + 1$$



Double Hashing

$h(x) = x \% 6$ $g(x) = (x\%5) + 1$



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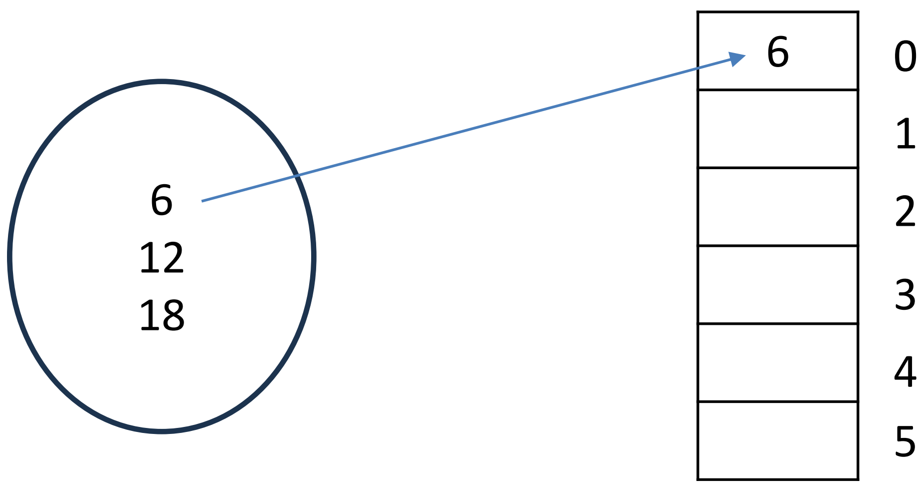
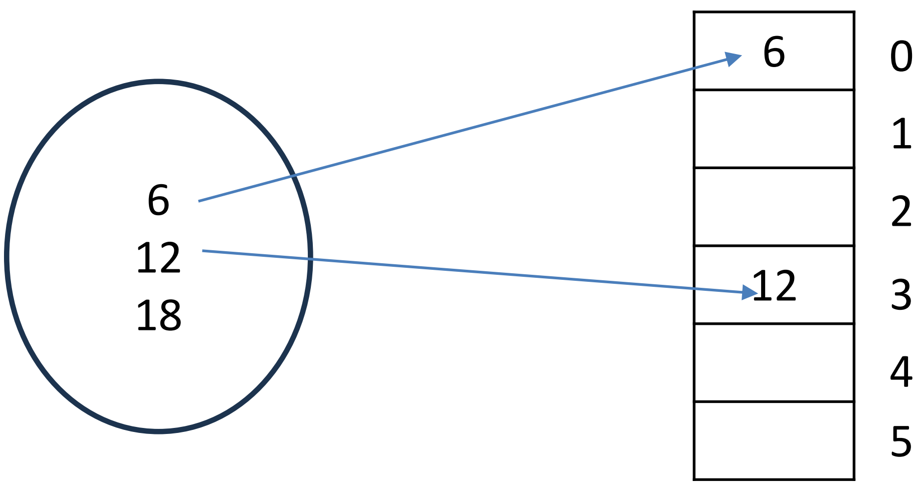


For 12
12%6 + 12%5 + 1
= 0 + 3 = 3

Double Hashing

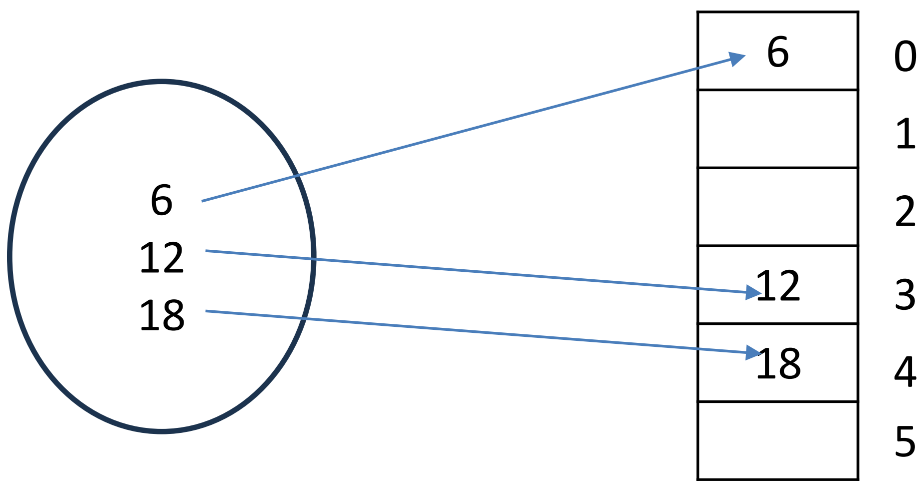
$h(x) = x \% 6$ $\qquad$ $g(x)=(x\%5)+1$



For 18
18%6 + 18%5 + 1
= 0 + 4 = 4

Double Hashing

$h(x) = x \% 6$ $\qquad$ $g(x)=(x\%5)+1$



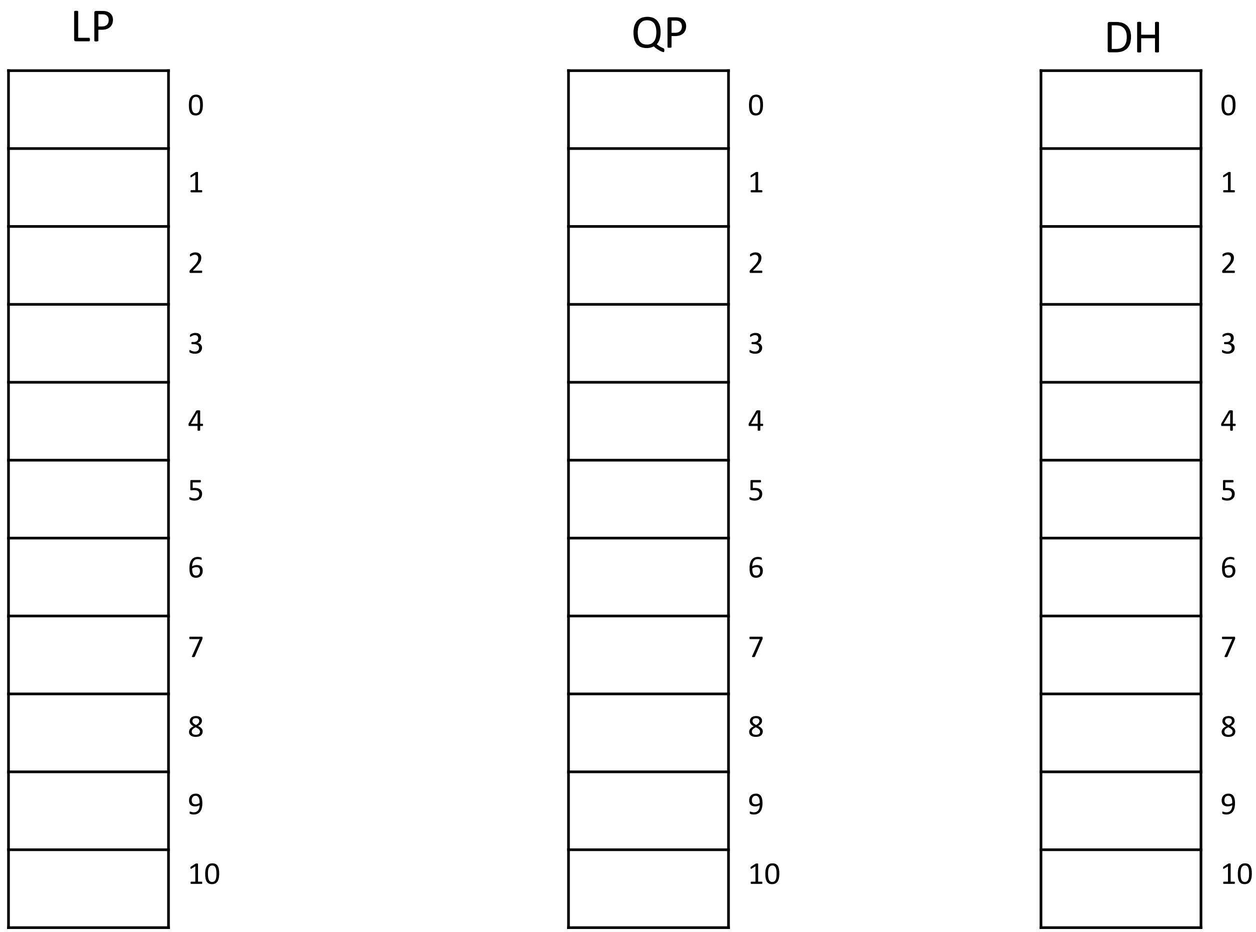
For 18
18%6 + 18%5 + 1
= 0 + 4 = 4

Examples

- Insert the following keys into a hash table of size 11, using $\mathrm{P(k)=k\%11}$
  - Linear probing
  - Quadratic probing
  - Double hashing $\mathrm{S(k)=k\%7+1}$
- 11,22,33,12,13,25,18

Example

11,22,33,12,13,25,18



$h(x) = x \ \% \ 11$

$g(x) = (x\%7)+1$

Example

11,22,33,12,13,25,18

LP

1	11	0
		1
		2
		3
		4
		5
		6
		7
		8
		9
		10

QP

1	11	0
		1
		2
		3
		4
		5
		6
		7
		8
		9
		10

DH

1	11	0
		1
		2
		3
		4
		5
		6
		7
		8
		9
		10

$h(x) = x \% 11$
$g(x) = (x\%7)+1$

Example

11,22,33,12,13,25,18

LP

1	11	0
2	22	1
		2
		3
		4
		5
		6
		7
		8
		9
		10

QP

1	11	0
2	22	1
		2
		3
		4
		5
		6
		7
		8
		9
		10

DH

1	11	0
		1
2	22	2
		3
		4
		5
		6
		7
		8
		9
		10

For 22

22%11 + 22%7 + 1

= 0 + 2 = 2

$h(x) = x \% 11$

$g(x)=(x\%7)+1$

Example

11,22,33,12,13,25,18

LP

1	11	0
2	22	1
3	33	2
		3
		4
		5
		6
		7
		8
		9
		10

QP

1	11	0
2	22	1
		2
		3
3	33	4
		5
		6
		7
		8
		9
		10

For 33
33%11 + 4
= 4

DH

1	11	0
		1
2	22	2
		3
		4
		5
2	33	6
		7
		8
		9
		10

For 33
33%11 + 33%7 + 1
= 0 + 6 = 6

$h(x) = x \% 11$
$g(x)=(x\%7)+1$

Example

11,22,33,12,13,25,18

LP

1	11	0
2	22	1
3	33	2
3	12	3
		4
		5
		6
		7
		8
		9
		10

QP

1	11	0
2	22	1
2	12	2
		3
3	33	4
		5
		6
		7
		8
		9
		10

DH

1	11	0
1	12	1
2	22	2
		3
		4
		5
2	33	6
		7
		8
		9
		10

$h(x) = x \ \% \ 11$

$g(x)=(x\%7)+1$

Example

11,22,33,12,13,25,18

LP

1	11	0
2	22	1
3	33	2
3	12	3
3	13	4
		5
		6
		7
		8
		9
		10

QP

1	11	0
2	22	1
2	12	2
2	13	3
3	33	4
		5
		6
		7
		8
		9
		10

DH

1	11	0
1	12	1
2	22	2
		3
		4
		5
2	33	6
		7
		8
2	13	9
		10

For 13

13%11 + 13%7 + 1

= 3 + 6 = 9

$h(x) = x \ \% \ 11$

$g(x)=(x\%7)+1$

Example

11,22,33,12,13,25,18

Index	LP probes	LP	QP probes	QP	DH probes	DH
0	1	11	1	11	1	11
1	2	22	2	22	1	12
2	3	33	2	12	2	22
3	3	12	2	13	1	25
4	3	13	3	33		
5	3	25				
6					2	33
7			3	25		
8						
9					2	13
10						

For 25

25%11 + 4

= 3 + 4 = 7

$h(x) = x \% 11$

$g(x)=(x\%7)+1$

Example

11,22,33,12,13,25,18

	LP		QP		DH	
1	11	0	1	11	1	11
2	22	1	2	22	1	12
3	33	2	2	12	2	22
3	12	3	2	13	1	25
3	13	4	3	33		
3	25	5				
		6			2	33
1	18	7	3	25	1	18
		8	2	18		
		9			2	13
		10				

$h(x) = x \% 11$

$g(x) = (x\%7)+1$

Example

11,22,33,12,13,25,18

LP

1	11	0
2	22	1
3	33	2
3	12	3
3	13	4
3	25	5
		6
1	18	7
		8
		9
		10

#probes = 16

QP

1	11	0
2	22	1
2	12	2
2	13	3
3	33	4
		5
		6
3	25	7
2	18	8
		9
		10

#probes = 15

DH

1	11	0
1	12	1
2	22	2
1	25	3
		4
		5
2	33	6
1	18	7
		8
2	13	9
		10

#probes = 10

Performance of Double Hashing

- Average number of probes for a hit: $\frac{1}{x}\ln\left(\frac{1}{1-x}\right)$

- Average number of probes for a miss: $\frac{1}{1-x}$

- Recall the formulae in linear probing:

$$\frac{1+\frac{1}{1-x}}{2} \qquad \frac{1+\frac{1}{(1-x)^2}}{2}$$

Performance of Double Hashing

Load Factor (%)	Probe hits	Probe miss
5	1.03	1.05
10	1.05	1.11
20	1.12	1.25
30	1.19	1.43
40	1.28	1.67
50	1.39	2.00
60	1.53	2.50
70	1.72	3.33
80	2.01	5.00
90	2.56	10.00
95	3.15	20.00

Double Hashing

Probe hits	Probe misses
1.03	1.05
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2.17	6.06
3.00	13.00
5.50	50.50
10.5	200.50

Linear Probing

[Full Table]

Linear v.s. Double Probing

- If the table is sparse (and memory is available), linear probing is very fast, however
  - Performance can degrade rapidly once clusters start forming.
- Double hashing uses memory more efficiently (smaller table or more full), costs a little more to compute secondary hash.
- For sparse tables, linear probing and double hashing require about the same number of probes, but double hashing will take more time since it must compute a second hash.
- For nearly full tables, double hashing is better than linear probing due to the less likelihood of collisions.

Expanding the Hash Table

- The performance of the hash table algorithms depend on the load factor of the table.
- Tables must not get full (or near full) or performance degrades.
- If we cannot determine the amount of data we expect, we may need to grow it at runtime.
  - This essentially means creating a new table and re-inserting all of the items.
  - Expanding the table is costly, but is done infrequently.
  - The cost is amortized over the run time of the algorithm

Deletion From Hash Table

Deleting items: Linear Probing Hash Table

- Insert(SPINAL)
  - $h(S_{19}) = 5$
  - $h(P_{16}) = 2$
  - $h(I_9) = 2$
  - $h(N_{14}) = 0$
  - $h(A_1) = 1$
  - $h(L_{12}) = 5$

N	A	P	I		S	L
0	1	2	3	4	5	6

delete(S)

N	A	P	I		S	L
0	1	2	3	4	5	6

N	A	P	I			L
0	1	2	3	4	5	6

find(L) = NOT FOUND???

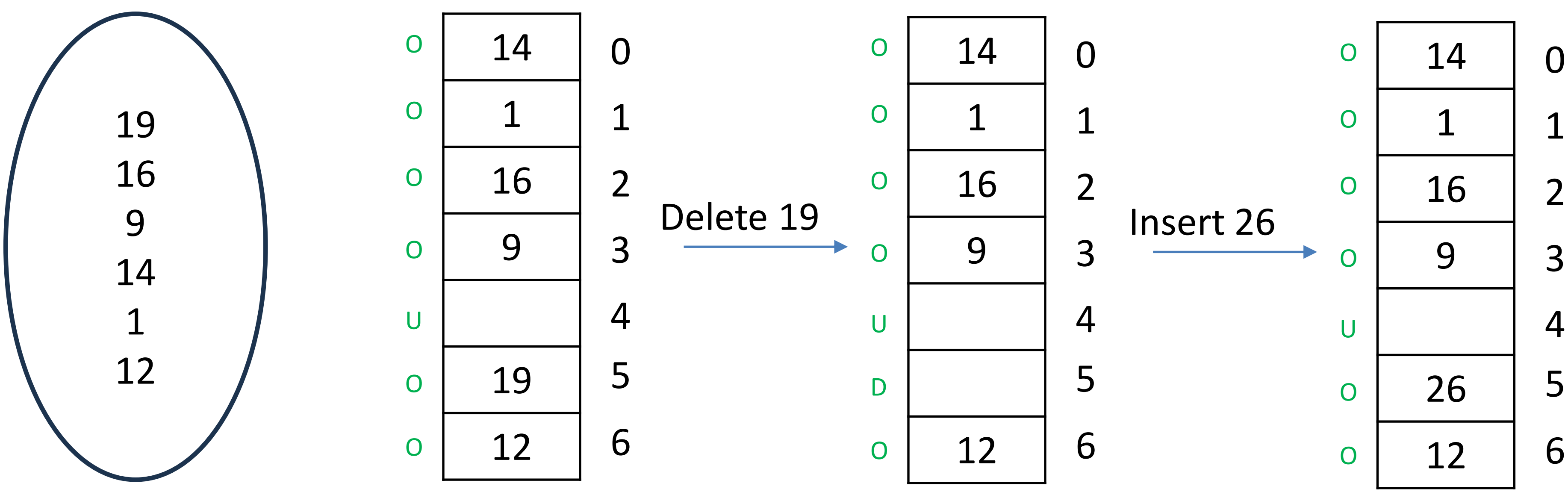
- <u>Deleting an item from a cluster presents a problem as the deleted item could be part of a linear probe sequence.</u>

Handling Deletions: Solution #1

- Marking slots as deleted (MARK)
- Each slot can be in one of three states:
  - Occupied
  - Unoccupied
  - Deleted.
- Search until we find the item or encounter the first unoccupied slot.
  - Insert at first *deleted* or *unoccupied* slot
  - Need to remember where first deleted slot is when we insert an item
- Load factor is decreased when a slot is marked as deleted.

Handling Deletions: Solution #1

19, 16, 9, 14, 1, 12 h(k) = k%7

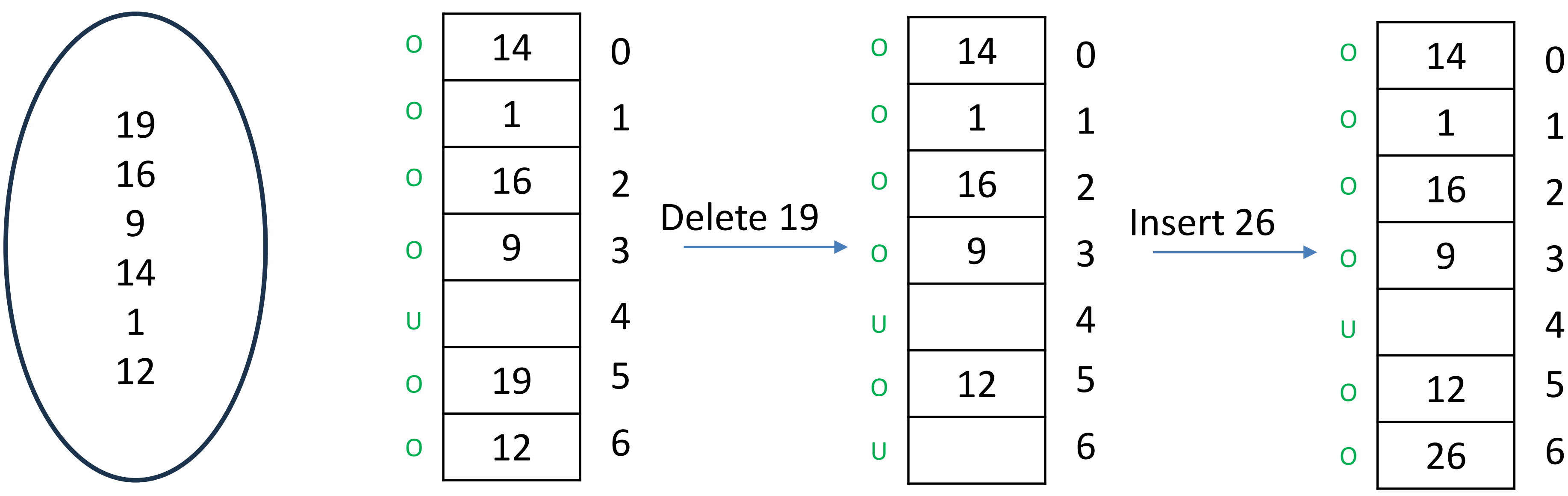


Handling Deletions: Solution #2

- Adjust the table (PACK) after a deletion.
- For each item after the deleted item that is in the cluster, mark its slot unoccupied and insert it back into the table.
- Works well for relatively sparse tables because the number of re-insertions is small.

Handling Deletions: Solution #2

19, 16, 9, 14, 1, 12 $h(k) = k\%7$



Collision Resolution by Chaining

Collision Resolution by Chaining

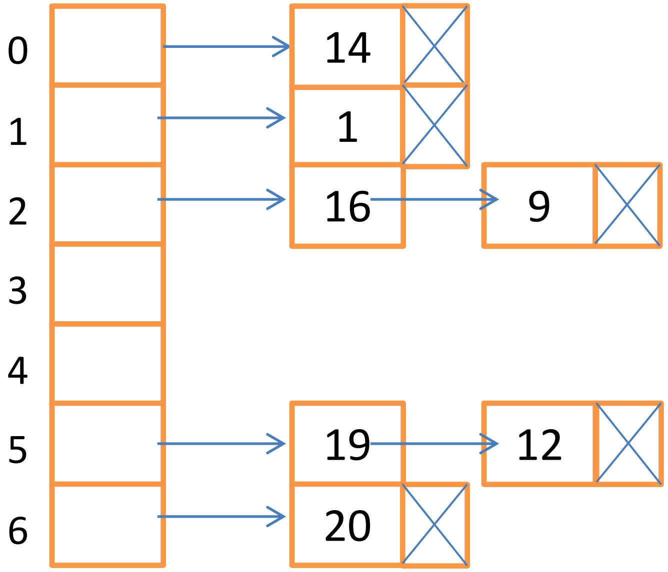
- With the open-addressing scheme, the data is stored in the hash table itself.

- In this scheme, the data is stored outside of the hash table.

- This method is called chaining (or separate chaining).
  - Instead of storing items in the hash table (in the slot indexed by the hashed key), we store them on a linked list.
  - The hash table simply contains pointers to the first item in each list.

Collision Resolution by Chaining

- Insert the following keys into the hash table: 19, 16, 9, 14, 1, 12, 20



Collision Resolution by Chaining

- Our data structure has been somewhat reduced to a singly linked list.

- Where do we insert into the list?
  - Front? Back? Middle?

- Should the list be sorted?

- Splay (caching) hash tables?

Considerations on Chaining

- We never run out of space (subject to the available memory).
- Implementing insert and delete is trivial compared to open-addressing above.
- Since we must ensure there are no duplicates, we must always look for an item before adding (inserting it).
  - Most time is spent searching through the linked lists.

Complexity of Chaining

- Recall the performance of linear probing:

$$\frac{1+\frac{1}{1-x}}{2} \qquad \frac{1+\frac{1}{(1-x)^2}}{2}$$

- With linear probing we minimize probing by keeping the hash table below 2/3 full.
  - An average of 2 probes for a successful search and 3 for an unsuccessful one (average cluster size of 3)

- Note that these are constants, not related to the number of elements in the table. $O(k)$

Complexity of Chaining

- There is no concept of "2/3" full.
- Load factor is still computed the same as before, but now it is likely to be greater than 1.
- Think of the load factor as being the average lengths of the lists

Advantages of Chaining

- Has the potential benefit that removing an item is trivial.
- Trivial to implement (linked list algorithms readily available).
- Node allocation can be expensive, but can be implemented efficiently with a memory manager(ObjectAllocator).
- Degrades gracefully as the average length of each lists grows. (No snowballing effect, i.e. clustering)
- Lists could be sorting using a BST or other data structure.

More Hashing

Summary

- There are two parts to hash-based algorithms that implementations must deal with:
  - Computing the hash function to produce an index from a key.
  - Dealing with the inevitable collisions
- Hash tables rely on the fact that the data is uniformly and randomly distributed
  - Since we cannot control the data that is provided from the user, we must ensure that it is randomly distributed by hashing it.
- Hashing algorithms are used in other areas as well (e.g. cryptography)

Interesting Links

- Hash function performance and distribution
- Performance of various hash functions
- Various hash-related information
- More from Bob Jenkins
- Fowler / Noll / Vo (FNV) Hash
- GNU perfect hash function generator
- C Minimal Perfect Hashing Library