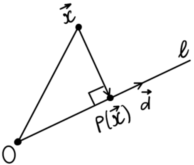
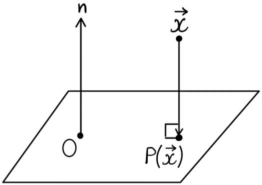
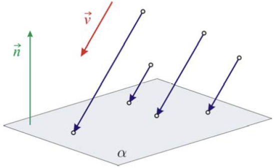
Week 11: Scaling, Shear, Rotation in 3D

Table of contents

Projections in $\mathbb{R}^3$





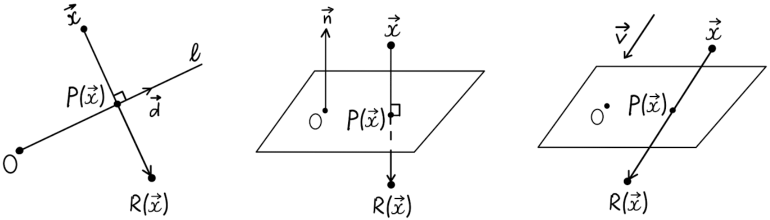


$$M = \frac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T \qquad M = I_3 - \frac{1}{||\vec{n}||^2||}\vec{n}\vec{n}^T \qquad M = I_3 - \frac{1}{\vec{v}\cdot\vec{n}}\vec{v}\vec{n}^T$$

Image of a plane under projection

Question 1: Let T be the orthogonal projection onto plane α. What are the possibilities for the image of a plane β under T?

Reflections in $\mathbb{R}^3$



$$M = \frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_3 \qquad M = I_3 - \frac{2}{||\vec{n}||^2}\vec{n}\vec{n}^T \qquad M = I_3 - \frac{2\vec{v}\vec{n}^T}{\vec{v}\cdot\vec{n}}$$

Image of a plane under reflection

Question 2: Let T be the orthogonal reflection through plane α. What are the possibilities for the image of a plane β under T?

Example 1

Let $\alpha : 3x + 2y - z = 0$ and let $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

(a) Compute the matrix M of the reflection through α in the direction $\vec{v}$.

(b) Find the image of the plane $\beta : 4x - 9y - 6z = 7$

Scaling in $\mathbb{R}^3$

- The scaling $S : \mathbb{R}^3 \to \mathbb{R}^3$ that scales all $x, y, z-$coordinates by factors a, b, c is

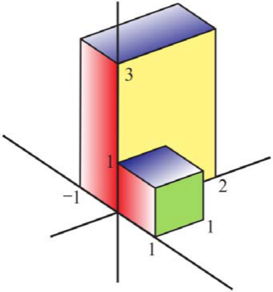
$$S \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax \\ by \\ cz \end{pmatrix}$$

- The matrix of S is

$$M = M_{a,b,c} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

Example 2

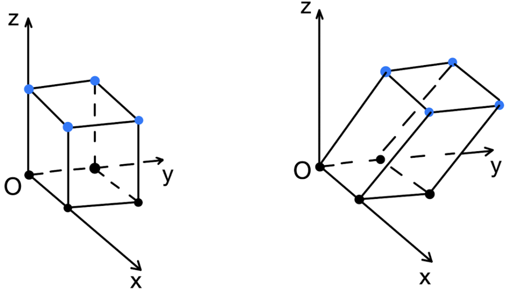
(a) Verify that the map $S\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ 2y \\ 3z \end{pmatrix}$ maps the unit cube to the rectangular cuboid.



(b) Compute the volume of the rectangular cuboid.

Shear

- The shear in $\mathbb{R}^3$ maps a cube to a parallelepiped

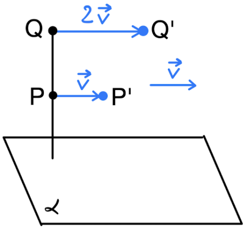


Shear

- The shear $S : \mathbb{R}^3 \to \mathbb{R}^3$ w.r.t. a plane $\alpha : \vec{n} \cdot \vec{x} = 0$ in the direction of **shearing vector** $\vec{v}$ ($\vec{v} \parallel \alpha$) is defined by

$$S(\vec{x}_0) = \vec{x}_0 + \frac{\vec{n} \cdot \vec{x}_0}{||\vec{n}||} \vec{v}$$

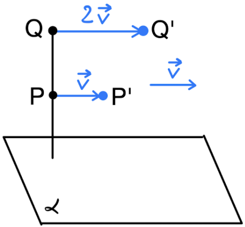
Comments



- S moves $\vec{x}_0$ in the direction of $\vec{v}$ by the factor $\dfrac{\vec{n}\cdot\vec{x}_0}{||\vec{n}||}$

$$S(\vec{x}_0) = \vec{x}_0 + \frac{\vec{n}\cdot\vec{x}_0}{||\vec{n}||}\vec{v}$$

Comments



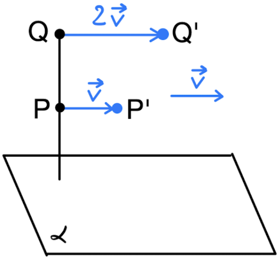
- S moves $\vec{x}_0$ in the direction of $\vec{v}$ by the factor $\dfrac{\vec{n}\cdot\vec{x}_0}{||\vec{n}||}$

$$S(\vec{x}_0) = \vec{x}_0 + \frac{\vec{n}\cdot\vec{x}_0}{||\vec{n}||}\vec{v}$$

- The factor $\dfrac{\vec{n}\cdot\vec{x}_0}{||\vec{n}||}$ has *magnitude* equal to $d(\vec{x}_0,\alpha)$: $d(\vec{x}_0,\alpha) = \dfrac{|\vec{n}\cdot\vec{x}_0|}{||\vec{n}||}$

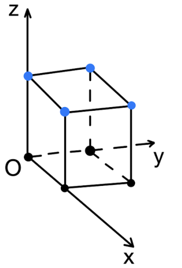
Question

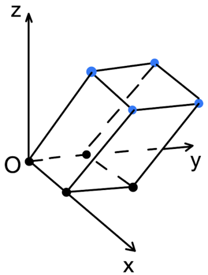
Consider the shear S with respect to the plane α and the shearing vector $\vec{v}$. What points are fixed by S?



Example 3

Verify that the shear w.r.t. the xy-plane in the direction of the shearing vector $\vec{j}$ maps the unit cube to the parallelepiped.





Matrix of shear in $\mathbb{R}^3$

Theorem 1

The shear with respect to the plane $\alpha : \vec{n} \cdot \vec{x} = 0$ in the direction of the vector $\vec{v}$ ($\vec{v} \perp \vec{n}$) has matrix representation

$$M = M_{\vec{n},\vec{v}} = I_3 + \frac{1}{||\vec{n}||}\vec{v}\vec{n}^T$$

Proof

- Let $\vec{x}_0$ be any point.
- The image of $\vec{x}_0$ is

$$
\begin{aligned}
S(\vec{x}_0) &= \vec{x}_0 + \frac{\vec{n}\cdot\vec{x}_0}{||\vec{n}||}\vec{v} = \vec{x}_0 + \frac{1}{||\vec{n}||}(\vec{n}\cdot\vec{x}_0)\vec{v} \\
&= \vec{x}_0 + \frac{1}{||\vec{n}||}\vec{v}\vec{n}^T\vec{x}_0 = \left(I_2 + \frac{\vec{v}\vec{n}^T}{||\vec{n}||}\right)\vec{x}_0
\end{aligned}
$$

Example 4

Consider $\alpha : 3x - 4y - 12 = 0$ and $\vec{v} = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$.

(a) Compute the matrix of the shear wrt to α and shearing vector $\vec{v}$.

(b) What are the images of $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$?

(c) Show that the image of $m : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$ is itself.

(d) What is the image of $n : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$?

(e) What is the image of $\beta : x + y - z = 1$?

Example 5

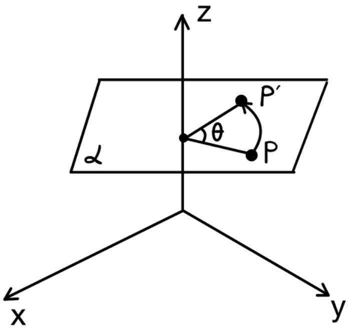
Consider the same shear as in the previous example. Find the images $\vec{i'}, \vec{j'}, \vec{k'}$ of the unit cube formed by $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

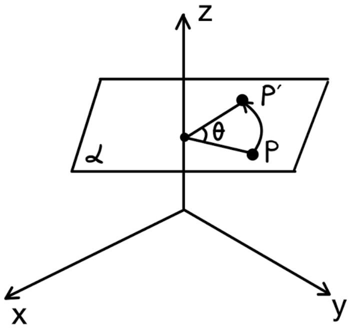
Example 5

(b) The unit cube is mapped to a parallelepiped formed by $\vec{i}', \vec{j}', \vec{k}'$.
What is the volume of this parallelepiped?

Rotation about positive z-axis

- Assume we want to rotate a point P counter-clockwise about the positive z-axis over the angle θ.
- This means when we *look down* from above, the image P' of P is shifted by the angle θ.

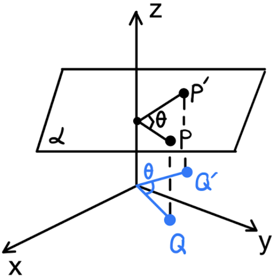




1. Draw the horizontal plane α through P
2. On α, rotate P counter-clockwise by the angle θ to get P'

Coordinates of P'

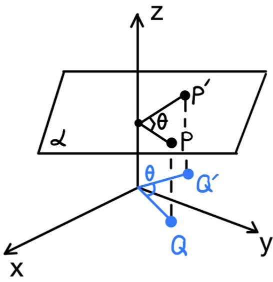
Assume $P = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \Rightarrow z$-coordinate of $P' = z_0$.



Coordinates of P'

Assume $P = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \Rightarrow z$-coordinate of $P' = z_0$.

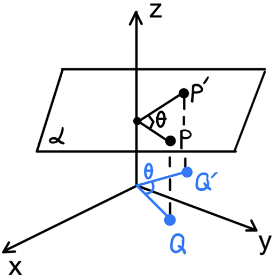
- Project P, P' onto xy-plane to get Q, Q'.



Coordinates of P'

Assume $P = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \Rightarrow z$-coordinate of $P' = z_0$.

- Project P, P' onto xy-plane to get Q, Q'.
- As points on xy-plane,
  $Q = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and $Q' =$ rotation of Q over θ
- $Q' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0\cos\theta - y_0\sin\theta \\ x_0\sin\theta + y_0\cos\theta \end{bmatrix}$



Rotation about positive z-axis

- In summary, P' has coordinates

$$P' = \begin{bmatrix} x_0 \cos\theta - y_0 \sin\theta \\ x_0 \sin\theta + y_0 \cos\theta \\ z_0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

- The matrix for counter-clockwise rotation about the positive z-axis over angle θ is

$$M_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotations about the axes

Theorem 2

The counter-clockwise rotation over the angle θ about

(a) x-axis has matrix $R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

(b) y-axis has matrix $R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

(c) z-axis has matrix $R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example 6

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation about the x-axis over 30^0.

(a) Find the matrix A of this transformation.

(b) Find the images of the points $\begin{bmatrix} 1 \\ 1 \\ \sqrt{3} \end{bmatrix}$ and $\begin{bmatrix} 11 \\ \sqrt{3} \\ 1 \end{bmatrix}$ under T.

(c) Find all points $\vec{x}$ that are fixed under T, that is, $T(\vec{x}) = \vec{x}$.

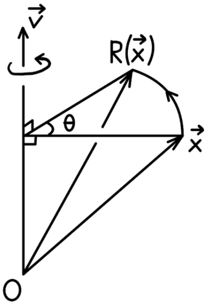
(d) Find the image of $\beta : x - 3y - z = 11$ under T.

(e) Find the image of the line $l : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{3} \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ \sqrt{3} \end{bmatrix}$ under T.

Rotation about any vector

The rotation in $\mathbb{R}^3$ can be defined around any nonzero vector $\vec{v}$. Assume we want to find the image of $\vec{x}$ when rotating about $\vec{v}$ over angle θ.

1. Draw a plane α through $\vec{x}$ and perpendicular to $\vec{v}$



2. On α, rotate $\vec{x}$ by angle θ to obtain its image $R(\vec{x})$.

Rotation about any vector

Theorem 3

The matrix of the counter clockwise rotation around the vector $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ over the angle θ is

$$M = (1 - \cos\theta)\frac{\vec{v}\vec{v}^T}{||\vec{v}||^2} + (\cos\theta)I_3 + \frac{\sin\theta}{||\vec{v}||}C_{\vec{v}},$$

where $C_{\vec{v}} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$ is the cross-product matrix induced by $\vec{v}$.

Example 7

Find the matrix of the rotation about the vector $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ over $\theta = 60^\circ$.