Lecture 5: Matrices and determinants

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Midterm exam

C + D + half of E

- Date and Time: Thursday Oct 6, 2-4pm
- Location: LT4A and LT4B (check announcement on Moodle or Ms Teams) → A + B + half of group E
- Format: MCQs + written questions
- Scope: Week 1-5 lectures and tutorials
- What is allowed to bring in?
  1. One A4-size cheat sheet
  2. One calculator

Matrices

- A **matrix** is a rectangular array of numbers.
  The numbers in the array are called **entries** of the matrix.
- The **size** of a matrix is described in terms of

$$(\#\text{ rows}) \times (\#\text{ columns})$$

- $\mathbf{A} = (\mathbf{a_{ij}})_{\mathbf{m\times n}}$ is the following $m \times n$ matrix

the $(i,j)^{th}$ entry is a_{ij}

i^{th} row j^{th} column

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Remark: We also use $(A)_{ij}$ to denote the (i,j)th entry of A.

Example 1

Determine the dimensions of the following matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}_{3\times 2}, \quad \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}_{1\times 4},$$

$$\begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & 0.1 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}_{2\times 1}, \quad \begin{bmatrix} 4 \end{bmatrix}_{1\times 1}$$

Rows, columns, square matrices

- A **row matrix** (or a **row**) = matrix with only one row.

  A **column matrix** (or a **column**) = matrix with only one column.

$$a = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}_{1 \times n}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$$

Rows, columns, square matrices

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  A **column matrix** (or a **column**) = matrix with only one column.

$$a = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

- A **square matrix of order n** has exactly n rows and n columns.

$$A = (a_{ij})_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$ $n \times n$

Diagonal entries

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

- The **diagonal entries** of A are $a_{11}, a_{22}, \ldots, a_{nn}$.
- The **non-diagonal entries** of A are a_{ij} with $i \neq j$.

Diagonal matrices

- A **diagonal matrix** if has all non-diagonal entries equal 0.

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Diagonal matrices

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- Example 2: Which of the following matrices are diagonal matrices?

$$\underset{\checkmark}{\begin{bmatrix} 4 \end{bmatrix}}, \quad \underset{\checkmark}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \quad \underset{\times}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}}, \quad \underset{\checkmark}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix}}$$

Identity matrix and zero matrix

- The **identity matrix of order n**, denoted by $\mathbf{I_n}$, is

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}$$

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- The **zero matrix** $\mathbf{0_{m\times n}}$ has all entries equal to 0.
  Sometimes, we simply write 0 for the zero matrix if there is no danger of confusion.

$$0_{\mathrm{m}\times\mathrm{n}} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{\mathrm{m}\times\mathrm{n}}$$

Equal matrices

- Two matrices A and B are **equal** if and only if they have the **same size** and the **same corresponding entries**

Equal matrices

- Two matrices A and B are **equal** if and only if they have

  the **same size** and the **same corresponding entries**
- $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{p \times q}$ are equal if and only if
  1. $m = p, n = q$ and
  2. $a_{ij} = b_{ij}$ for all i, j

Example 3

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix}_{2\times 2}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}_{2\times 2}, \quad C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 5 & 0 \end{bmatrix}_{2\times 3}$$

What can be the value of x such that $A = B$? $B = C$? $A = C$?

$A = B \iff x = 5$

Since C has different size than A and B, we always have $C \neq A$ and $C \neq B$.

Algebraic operations on matrices

We will discuss three algebraic operations on matrices

1. Matrix addition
2. Scalar multiplication
3. Matrix multiplication

Matrix addition and subtraction

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- The difference $A - B$ is defined by

$$(A-B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij}$$

Example 4

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}_{3\times 4}, \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix}_{3\times 4}, \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}_{2\times 2}$$

Find $A+B, A-B, A+C, A-C, B+C, B-C$.

$$A+B = \begin{pmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{pmatrix}, \quad A-B = \begin{pmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{pmatrix}$$

$A+C$, $A-C$, $B+C$, $B-C$ are all undefined.

Question 1

Let A be an $m \times n$ matrix. What is $A + 0_{m\times n}$? What is $A - A$?

$$A + 0_{m\times n} = A$$

$$A - A = 0_{m\times n}$$

Scalar multiplication

- Let $A = (a_{ij})_{m\times n}$ and let c be any scalar
- The product cA, called **scalar multiple** of A, is defined by

$$\underbrace{(cA)_{ij}}_{\text{the } (i,j)^{th} \text{ entry of } cA} = ca_{ij},$$

that is

$$c\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

Example 5

Find $2A,\ -3B,\ \dfrac{1}{3}C$

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix},\quad B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix},\quad C = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$$

$$2A = \begin{pmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{pmatrix},\quad (-3)B = \begin{pmatrix} 0 & -6 & -21 \\ 3 & -9 & 15 \end{pmatrix}$$

$$\frac{1}{3}C = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{pmatrix}$$

Properties of matrix addition and scalar multiplication

Theorem 1

Matrix addition and scalar multiplication have properties similar to normal addition and multiplication of real numbers.

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(a) Matrix addition

- Associativity: $(M+N)+P = M+(N+P)$
- Commutativity: $M+N = N+M$
- Zero matrix: $M+0 = 0+M = M$

(b) Scalar multiplication: $r(sM) = (rs)M$ and $1M = M$

(c) Distributive rules

$$(r+s)M = rM + sM \text{ and } r(M+N) = rM + rN$$

Matrix multiplication

- $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times q}$ can be multiplied in the order AB only if

$$\# \text{ columns of } A = \# \text{ rows of } B, \text{ that is, } n = p$$

- The resulting matrix AB is an $m \times q$ matrix:

$$\text{“}(m \times \cancel{n}) \cdot (\cancel{n} \times q) = m \times q\text{”}$$

Matrix multiplication

- $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times q} \Rightarrow AB$ is an $m \times q$ matrix
- Its (i,j)th entry is

$$
\begin{aligned}
(AB)_{ij} &= (i\text{th row of } A) \times (j\text{th column of } B) \\
&= \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} \\
&= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}
\end{aligned}
$$

The symbol $\sum$

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- Examples

$$\sum_{i=1}^{10} a_i = a_1 + a_2 + \cdots + a_{10},$$

$$\sum_{x=1}^{\infty} f(x) = f(1) + f(2) + \cdots + f(x) + \cdots$$

$$\sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}.$$

Summary on matrix multiplication

- If $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{n\times q}$, then AB has size $m \times q$.
- Its (i, j)th entry is

$$(AB)_{ij} = (i\text{th row of } A) \times (j\text{th column of B}) = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Example 6

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3} \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix}_{3\times 2} = \begin{pmatrix} 1\cdot 1+2\cdot 2+3\cdot(-1) & 1\cdot 1+2\cdot 3+3\cdot(-2) \\ 4\cdot 1+5\cdot 2+6\cdot(-1) & 4\cdot 1+5\cdot 3+6\cdot(-2) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix}_{2\times 2}$$

Example 7

Let $A = (a_{ij})_{2\times 3}$ and $B = (b_{ij})_{3\times 4}$.

(a) What is the size of AB?

(b) Find the general formula for $(AB)_{13}$ and $(AB)_{24}$.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{pmatrix}$$

(a) AB has size $(2\times \cancel{3})\cdot(\cancel{3}\times 4) = 2\times 4$

(b) $$AB = \begin{pmatrix} (AB)_{11} & (AB)_{12} & (AB)_{13} & (AB)_{14} \\ (AB)_{21} & (AB)_{22} & (AB)_{23} & (AB)_{24} \end{pmatrix}$$

$(AB)_{13} = (1^{st}$ row of $A) \times (3^{rd}$ column of $B)$

$$= (a_{11} \quad a_{12} \quad a_{13}) \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33}$$

$(AB)_{24} = \ldots$

Question 2

Does matrix product behave like the usual multiplication of numbers?
Guess the answer for the following

- Is there a special matrix $\mathbf{1}$ so that $A\mathbf{1} = A$ for any matrix A?

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- Is matrix multiplication commutative, say $AB = BA$ for any A, B?

Maybe no!

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- Is matrix multiplication associative, say $(AB)C = A(BC)$?
- Is matrix multiplication is distributive, say

$$(A + B)C = AC + BC, \text{ and } A(B + C) = AB + AC?$$

YES

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- Is matrix multiplication commutative, say $AB = BA$ for any A, B?
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- Is matrix multiplication is distributive, say

$$(A + B)C = AC + BC, \text{ and } A(B + C) = AB + AC?$$

- Does $AB = 0$ implies either $A = 0$ or $B = 0$? No!

While most questions have answer yes, there are cases where matrix multiplication is different from the usual multiplication of numbers.

Exercise 1. Find an example of 2 matrices $A, B \in M_{2\times 2}(\mathbb{R})$ such that

$$AB \neq BA.$$

In other words, matrix multiplication is not commutative.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \neq AB$$

Exercise 2

(a) Find an example of 2 nonzero matrices $A, B \in M_{2\times 2}(\mathbb{R})$ such that

$$AB = 0.$$

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}}_{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Exercise 2

(b) Find an example of 3 nonzero matrices $A, B, C \in M_{2\times 2}(\mathbb{R})$ such that

$$AB = AC, \text{ but } B \neq C.$$

$$AB = AC \Leftrightarrow AB - AC = 0 \Leftrightarrow A(B-C) = 0$$

Choose $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $\underbrace{B - C = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}}$

choose $C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$,

$B = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

Summary of exercises 1,2

Matrix multiplication is **not commutative** and **doesn't allow cancellation law**. In general, we have

Summary of exercises 1,2

Matrix multiplication is **not commutative** and **doesn't allow cancellation law**. In general, we have

- $AB \neq BA$
- $AB = 0 \nRightarrow A = 0$ or $B = 0$
- $AB = AC \nRightarrow B = C$

Properties of matrix multiplication

Theorem 2

Let A, B, C be three matrices so that whenever we write the product of any two of them, it is well-defined. The following hold.

(i) Identity matrix I: $AI = A$.

(ii) Zero matrix 0: $A0 = 0$.

(iii) Associativity: $(AB)C = A(BC)$.

(iv) Distributivity:

$$A(B + C) = AB + AC \text{ and } (A + B)C = AC + BC.$$

Transpose of a matrix

- Let A be an $m \times n$ matrix.
- The **transpose** of A, denoted by A^T, is an $n \times m$ matrix that is obtained by interchanging the rows and columns of A.

$$(A^T)_{ij} = (A)_{ji}$$

Transpose of a matrix

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$$(A^T)_{ij} = (A)_{ji}$$

- Here is the general formula

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n} \Rightarrow A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}_{n \times m}$$

Example 8

Given the following matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}_{3\times 4}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \quad C = [1 \quad 3 \quad 5], \quad D = [4]$$

(a) Find A^T, B^T, C^T, D^T.

$$A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{pmatrix}_{4\times 3}, \quad B^T = \begin{pmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{pmatrix}, \quad C^T = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \quad D^T = [4]$$

Example 8

(b) Find $(A^T)^T, (B^T)^T, (C^T)^T, (D^T)^T$.

$$(A^T)^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} = A$$

$$(B^T)^T = \ldots = B$$

$$(C^T)^T = \ldots = C$$

$$(D^T)^T = \ldots = D$$

Determinant as a function on matrices

- **Determinant** is a function "$\mathbf{det}$" which takes input as square matrices and gives outputs as real numbers.

$$\det : \{\text{square matrices}\} \to \mathbb{R}.$$

(input) (output)

Determinant as a function on matrices

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- In this course, we focus on 2×2 and 3×3 matrices.

Determinant of 1×1 and 2×2 matrices

- For 1×1 matrices, determinant is a "trivial function".

$$\det([c]) = c.$$

- For 2×2 matrices

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (\text{main diagonal}) - (\text{anti diagonal}) = ad - bc.$$

Determinant of 3×3 matrices

- For $A = (a_{ij})_{3\times 3}$, the **main diagonals** of A includes the *usual main diagonal* $a_{11} - a_{22} - a_{33}$ and *another two* which are parallel to it.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \text{ main diagonals } a_{11}a_{22}a_{33},\ a_{12}a_{23}a_{31},\ a_{21}a_{32}a_{13}$$

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- The **anti diagonals** of A includes the *usual anti diagonal* $a_{13} - a_{22} - a_{31}$ and *another two* which are parallel to it.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \text{ anti diagonals } a_{13}a_{22}a_{31},\ a_{12}a_{21}a_{33},\ a_{23}a_{32}a_{11}$$

Determinant of 3×3 matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\begin{aligned} \det(A) &= \text{(main diagonals)} - \text{(anti diagonals)} \\ &= (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13}) - \\ &\quad (a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{23}a_{32}a_{11}) \end{aligned}$$

Example 9

Find the determinants of $A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 7 & 0 \\ -5 & -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & -7 \\ 5 & 3 & 1 \end{bmatrix}$.

$\det(A) = (\text{main diagonals}) - (\text{anti diagonals})$

$= \left(1\cdot 7\cdot 1 + (-1)\cdot 0\cdot(-5) + 2\cdot(-3)\cdot 5\right) - \left(5\cdot 7\cdot(-5) + (-1)\cdot 2\cdot 1 + 0(-3)\cdot 1\right)$

$= (-23) - (-177) = 154$

$\det(B) = \left(1\cdot 0\cdot 1 + 1\cdot(-7)\cdot 5 + 1\cdot 3\cdot 3\right) - \left(3\cdot 0\cdot 5 + 1\cdot 1\cdot 1 + 3\cdot(-7)\cdot 1\right)$

$= (-26) - (-20) = -6$

Determinant by row/column expansion

- Consider the 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Determinant by row/column expansion

- Consider the 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
- Determinant of A by **row expansion along the 1st row**:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) +$$
$$+ a_{13}(a_{21}a_{32} - a_{23}a_{31})$$

Determinant by row/column expansion

- Consider the 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

- Determinant of A by **column expansion along the 1st column**:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - \ldots$$

Example 10

Compute $\det \begin{pmatrix} 10 & 0 & 5 \\ 1 & 2 & 1 \\ 5 & -1 & 3 \end{pmatrix} = 10 \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} - 0 \begin{vmatrix} \dots \end{vmatrix} + 5 \begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix}$

$= 10 \cdot 7 + 5 \cdot (-11)$

$= 15.$

Summary on determinants of 2×2 and 3×3 matrices

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ or $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$\det(A) =$ (main diagonal) - (anti-diagonal)

- $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$
- $\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13}) - (a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{23}a_{32}a_{11})$

Triangular matrices

- $A = (a_{ij})_{n\times n}$ is an **upper triangular matrix** if all entries below the main diagonal are equal to 0

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{pmatrix}$$

entries below main diagonal = 0

all other entries can be 0 or nonzero

Triangular matrices

- $A = (a_{ij})_{n\times n}$ is an **upper triangular matrix** if all entries below the main diagonal are equal to 0

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{pmatrix}$$

- A is a **lower triangular matrix** if all entries above the main diagonal are equal to 0

$$A = \begin{pmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

Triangular matrices

- $A = (a_{ij})_{n\times n}$ is an **upper triangular matrix** if all entries below the main diagonal are equal to 0

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{pmatrix}$$

- A is a **lower triangular matrix** if all entries above the main diagonal are equal to 0

$$A = \begin{pmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

- A is a **triangular matrix** if it is either an upper triangular matrix or a lower triangular matrix.

2×2 triangular matrices

Find $\det \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$ and $\det \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}$

$$\det \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = a_{11} a_{22}$$

$$\det \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} = a_{11} a_{22}$$

3×3 triangular matrices

Find $\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$ and $\det \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ 0 & a_{33} \end{vmatrix} - 0 \left| \cdots \right| + 0 \left| \cdots \right|$$

$$= a_{11} a_{22} a_{33}$$

$$\det \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} \begin{vmatrix} a_{22} & 0 \\ a_{32} & a_{33} \end{vmatrix} - 0 \left| \cdots \right| + 0 \left| \cdots \right| = a_{11} a_{22} a_{33}$$

Determinant of triangular matrices

Theorem 3

If $A = (a_{ij})_{n \times n}$ is a triangular matrix (upper triangular or lower triangular), say

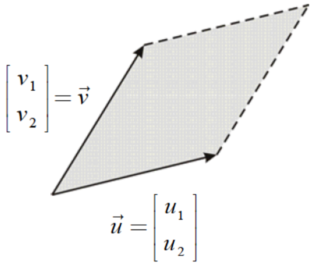
$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix},$$

then

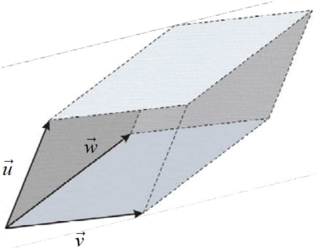
$$\det(A) = \text{product of diagonal entries} = a_{11}a_{22}\cdots a_{nn}$$

Parallelogram and parallelepiped

- A parallelogram can be formed by 2 vectors



- A parallelepiped can be formed by 3 vectors



Area of parallelogram and volume of parallelepiped

Theorem 4

(a) The area of the parallelogram spanned by $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is

(in R^2)

$$|\det[\vec{u}\ \vec{v}]| = \left| \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \right|$$

$|\det[\vec{u}\ \vec{v}\ \vec{w}]|$

$\parallel$

(b) The volume of the parallelepiped spanned by

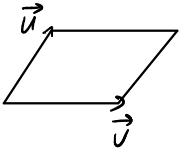
$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \text{ is } \left| \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \right|.$$

Example 11

(a) Find the area of the parallelogram spanned by $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

Sol 1.

Put $\vec{u}, \vec{v}$ into $\mathbb{R}^3$: $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \vec{v} = \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix}$



$$\vec{u} \times \vec{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -11 \end{pmatrix}$$

$$\text{Area} = \|\vec{u} \times \vec{v}\| = 11$$

Sol 2.

$$\text{Area} = \left|\det \begin{pmatrix} 1 & 5 \\ 2 & -1 \end{pmatrix}\right| = |-11| = 11$$

(b) Find the volume of the parallelepiped spanned by

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

$$\text{Volume} = \left| \det \begin{pmatrix} 1 & 5 & 2 \\ 2 & -1 & 5 \\ 5 & 0 & 7 \end{pmatrix} \right|$$

$$= \left| \left((-7) + 125 \right) - \left((-10) + 70 \right) \right|$$

$$= 58$$

Remark on Theorem 4

- The formula to compute area of parallelogram formed by $\vec{u}, \vec{v}$ is only applied for $\vec{u}, \vec{v} \in \mathbb{R}^2$:
$$\text{Area} = \left|\det[\vec{u}\ \vec{v}]\right|$$
- What happens if you use this formula for $\vec{u}, \vec{v} \in \mathbb{R}^3$? What formula do you need to use to compute area of parallelogram formed by 2 vectors in $\mathbb{R}^3$?

Example : $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\text{Area} = \left| \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}_{3 \times 2} \right|$ is undefined !

$\text{Area} = \| \vec{u} \times \vec{v} \|$

Remark on Theorem 4

Exercise 4

Consider 2 vectors $\vec{u} = \begin{bmatrix} c-3 \\ -3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ c+1 \end{bmatrix}$ in $\mathbb{R}^2$.

(a) Compute the area of the parallelogram formed by $\vec{u}, \vec{v}$ in terms of c?

(b) For what value of c does the parallelogram formed by $\vec{u}, \vec{v}$ has the smallest possible area?

(a) $\text{Area} = \left| \det \begin{pmatrix} c-3 & 3 \\ -3 & c+1 \end{pmatrix} \right| = |\underbrace{(c-3)(c+1)}_{(c^2-3c+c-3)} + 9|$

$\text{Area} = |c^2 - 2c + 6|$

(b) $\text{Area} = |(c^2 - 2c + 1) + 5| = |\underbrace{(c-1)^2}_{\geq 0} + 5|$

$= (c-1)^2 + 5 \geq 5.$

So minimum value of area is 5 at $c = 1$.

Challenge of the day!

Consider the volume formed by

$$\vec{u} = \begin{pmatrix} 1 \\ 1 \\ c \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 0 \\ c-1 \\ 2 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 0 \\ -10 \\ c+2 \end{pmatrix}$$

Find c s.t. the volume is smallest possible!