Taylor and Maclaurin Series
Some Quiz 2 Problems

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Power Series

- A power series centered at a is the series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

- A power series centered at a is always convergent at $x = a$

$$x = a \implies \sum_{n=0}^{\infty} c_n(x-a)^n = c_0.$$

- The radius of convergence of $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ is a number R such that

$$\sum_{n=0}^{\infty} c_n(x-a)^n \text{ converges if } |x-a| < R \text{ and diverges if } |x-a| > R.$$

Power series $\longrightarrow$ function

- We can find R using either the Ratio/Root Test.
- Suppose this power series has a radius of convergence R.
- We can then define a function f as

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad |x-a| < R. \quad (1)$$

- In this case, the function was defined using a power series, and we call equation (1) as the **power series representation** of f.
- **Question**: What about the converse, i.e. given a function f, can we find a power series representation of f?
- We can find an answer using **Taylor/Maclaurin series**.

Observation about the coefficients

Let's suppose that a function f has a power series representation

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

We have

- $f(a) = c_0$.
- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \implies c_1 = f'(a)$.
- $f''(x) = 2c_2 + 6c_3(x-a) + 12c_4(x-a)^2 + \cdots \implies c_2 = \dfrac{f''(a)}{2}$.
- $f'''(x) = 6c_3 + 24c_4(x-a) + \cdots \implies c_3 = \dfrac{f'''(a)}{6}$.
- We can continue the rest for $c_4, c_5, \ldots$

Question: What's the general formula for c_n?

General formula for c_n

- We have

$$c_0 = f(a),\ c_1 = f'(a),\ c_2 = \frac{f''(a)}{2},\ c_3 = \frac{f'''(a)}{6}, \dots$$

- What is the general formula for c_n?

Function $\longrightarrow$ power series

Theorem

If f has a power series representation centered at $x = a$, i.e. if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \; |x-a| < R,$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Taylor series and Maclaurin series

- Let f be an **infinitely differentiable** function on an open interval centered at a: $(a-R, a+R)$ for some $R>0$.
- The **Taylor series of** f **at** a (or about a, or centered at a) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

- The Taylor series centered at $a=0$ is called the **Maclaurin series** of f.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

Remark on Taylor/Maclaurin series

- Given a function f defined on $\mathbb{R}$, we do not always have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

- The reason is that the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

may not be convergent for all x.

Example 1

Find the Maclaurin series for $f(x) = e^x$ and find its radius of convergence.

Example 1

Exercise 1

Find the Taylor series for $f(x) = e^x$ centered at $a = 3$ and find its radius of convergence.

Exercise 1

Exercise 2

Find the Maclaurin series for $f(x) = \sin x$ and find its radius of convergence.

Exercise 2

Question 2

For the integral $\int_0^2 f(x)\,dx$, the approximations T_1 and M_1 give the values 5 and 4 respectively. Determine S_2.

Question 3

Let f be defined on $[-1, 3]$. Suppose $-8 \leq f^{(4)}(x) \leq 1$ for all $x \in [-1, 3]$. What is the error bound for S_4, as an approximation for the integral

$$\int_{-1}^{3} f(x)\, dx?$$

Note: About 90 - 95% of students got this question wrong.

Question 3

Question 4/5

Find α, where

$$\int_0^2 \frac{5x+15}{x^3+x^2+4x+4}\,dx = \alpha + \ln\left(\frac{9}{2}\right).$$

Question 4/5

Question 4/5

Question 7

Evaluate $\int \frac{1}{x^8 - x}\,dx$. *Hint:* Recommended to use substitution first.

Question 7