Week 9: Scaling, Rotation, Shear in 2D

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Linear transformation

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation** if it
  1. preserves addition
  $$T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$$
  2. and preserves scalar multiplication
  $$T(c\vec{x}) = cT(\vec{x})$$

inputs

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear $\Leftrightarrow$ each component in $T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is a linear combination of $x_1, \ldots, x_n$.

$\Updownarrow$

an expression of the form $a_1x_1 + \cdots + a_nx_n$

Matrix representation of linear transformation

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear $\Leftrightarrow$ there exists an $m \times n$ matrix M:

$$T(\vec{x}) = M\vec{x}$$

M is called the **matrix representation** of T.

Matrix representation of linear transformation

- There are 2 ways to determine M

1. $T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix} \Rightarrow M = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$
2. If $\vec{e}_1, \ldots, \vec{e}_n$ are *standard unit vectors* of $\mathbb{R}^n$, then

$$M = [T(\vec{e}_1) \ \cdots \ T(\vec{e}_n)]$$

Exercise 1

Determine whether T is linear. Find its matrix if it is linear.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + y \\ 2x + y \end{pmatrix}$

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - \sqrt{y} \\ 2x + y + 1 \end{pmatrix}$

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y - x \\ 2x + y \end{pmatrix}$

$$M = \begin{array}{cc} \multicolumn{2}{c}{\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}} \\ x & y \end{array}$$

Exercise 2

In this exercise, we learn that dot product and cross product can be explained as linear transformations!

1×3 matrix

(a) Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and let $T_{\vec{u}} : \mathbb{R}^3 \to \mathbb{R}^1$ be defined by $T_{\vec{u}}(\vec{x}) = \vec{u} \cdot \vec{x}$.

Show that $T_{\vec{u}}$ is a linear transformation. Write out its matrix.

$$T_{\vec{u}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x + 2y + z$$

$\Rightarrow T_{\vec{u}}$ is a linear transformation with matrix

$$M = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

matrix size = 3×3

(b) Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and let $C_{\vec{u}} \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $C_{\vec{u}}(\vec{x}) = \vec{u} \times \vec{x}$.

Show that $C_{\vec{u}}$ is a linear transformation. Write out its matrix.

$$C_{\vec{u}}(\vec{x}) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2z - y \\ x - z \\ y - 2x \end{pmatrix}$$

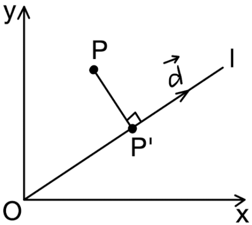
$\Rightarrow C_{\vec{u}}$ is linear with matrix

$$M = \begin{array}{ccc} x & y & z \\ \hline \end{array}$$
$$M = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}$$

Projections in $\mathbb{R}^2$

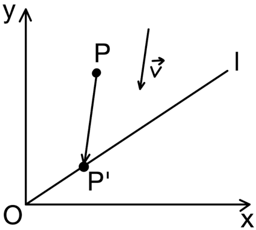
I_n = an $n \times n$ identity matrix , $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Orthogonal projection



$$M = \frac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T$$

Skew projection

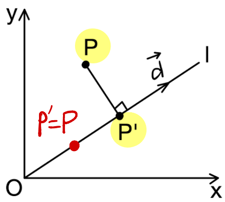


$$M = I_2 - \frac{\vec{v}\vec{n}^T}{\vec{v} \cdot \vec{n}}$$

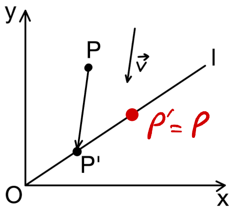
Question 1

A point $\vec{x}$ is called fixed by a map $T \Leftrightarrow T(\vec{x}) = \vec{x}$.
Which points are fixed by orthogonal projection and skew projection?

Orthogonal projection



Skew projection

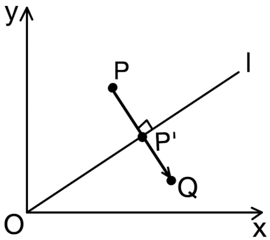


P is fixed $\Leftrightarrow P' = P$
$\Leftrightarrow P$ is on l

All fixed points are on l

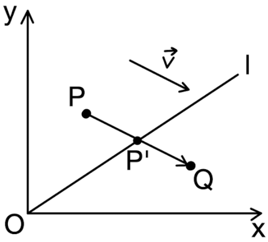
Reflections in $\mathbb{R}^2$

Orthogonal reflection



$$M = \frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_2$$

Skew reflection

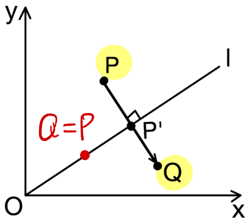


$$M = I_2 - \frac{2}{\vec{v}\cdot\vec{n}}\vec{v}\vec{n}^T$$

Question 2

Which points are fixed by reflections?

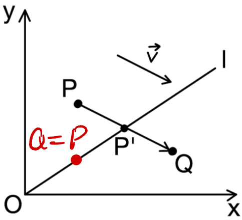
Orthogonal reflection



P is fixed $\Leftrightarrow$ $Q = P$

$\Leftrightarrow$ P is on ℓ

Skew reflection



All fixed points are on ℓ

Scaling

- A **scaling** (centered at the origin) is a map $S : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

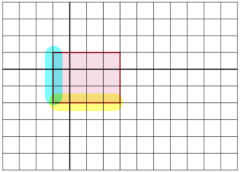
$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix}$$

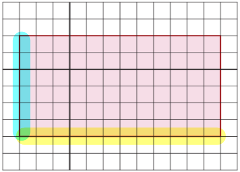
  for some constants $a, b \in \mathbb{R}$.

- All x-coordinates are scaled by a, all y-coordinates are scaled by b.

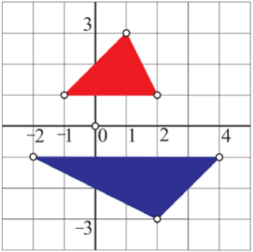
Example

- The small rectangle is scaled into large rectangle





- The red triangle is scaled into blue triangle



Scaling matrix

Theorem 1

The representation matrix of the scaling $S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix}$ is

$$M = M_{a,b} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + 0y \\ 0x + by \end{pmatrix} \Rightarrow M = M_{a,b} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

Example 1

Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the scaling which scale all x-coordinates by 2 and scale all y-coordinates by -1.

(a) What is the matrix representation M of S?

$$M = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

(b) What are the images of the points $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

Image of $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

Image of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} 2 \times 2 \\ (-1) \times 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

Example 1

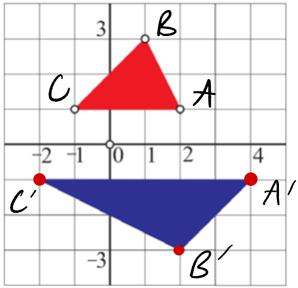
(c) Check that S maps the red triangle into the blue triangle below.

$$M = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

$A\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ has image $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} = A'$

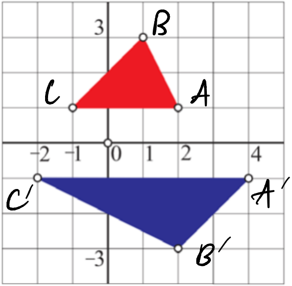
$B\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ —— $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} = B'$

$C\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ —— $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} = C'$



(d) Can you compare the areas of these 2 triangles?

Guess: blue triangle has area 2 times larger than red triangle.



$$\text{Area } ABC = \frac{1}{2}\left|\det\left(\vec{AB}\ \vec{AC}\right)\right|$$

$$= \frac{1}{2}\left|\det\begin{pmatrix} -1 & -3 \\ 2 & 0 \end{pmatrix}\right| = \frac{1}{2}\times 6 = 3$$

$$\text{Area } A'B'C' = \frac{1}{2}\left|\det\left(\vec{A'B'}\ \vec{A'C'}\right)\right|$$

$$= \frac{1}{2}\left|\det\begin{pmatrix} -2 & -6 \\ -2 & 0 \end{pmatrix}\right| = 6$$

$$A = \begin{pmatrix} 2 \\ 1 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 3 \end{pmatrix},\ C = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$A' = \begin{pmatrix} 4 \\ -1 \end{pmatrix},\ B' = \begin{pmatrix} 2 \\ -3 \end{pmatrix},\ C' = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

Matrix of rotation

Theorem 2

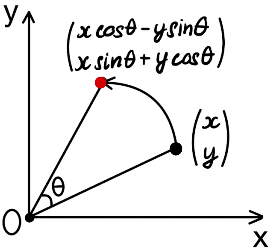
The counter-clockwise rotation $R : \mathbb{R}^2 \to \mathbb{R}^2$ around the origin over the angle θ has matrix representation

$$M = M(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

counter-clockwise

clockwise

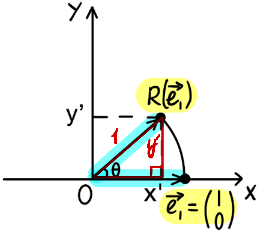
$$R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

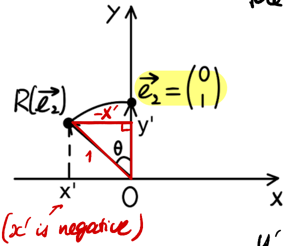


Proof (sketch)

The matrix is $M = [R(\vec{e}_1)\ R(\vec{e}_2)]$

R: rotation around O over the angle θ





$R(\vec{e}_1) = \begin{pmatrix} x' \\ y' \end{pmatrix}$ $\begin{cases} \cos\theta = \frac{x'}{1} = x' \\ \sin\theta = \frac{y'}{1} = y' \end{cases}$

$R(\vec{e}_2) = \begin{pmatrix} x' \\ y' \end{pmatrix}$ $\begin{cases} \cos\theta = \frac{y'}{1} = y' \\ \sin\theta = \frac{-x'}{1} = -x' \end{cases}$

$R(\vec{e}_1) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ $\quad$ $R(\vec{e}_2) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Example 2

(a) What is the counter-clockwise rotation (around O) matrix over 90^0?

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \overset{\theta = 90^\circ}{=\!=} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(b) What is the counter-clockwise rotation (around O) matrix over 135^0?

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \overset{\theta = 135^\circ}{=\!=} \begin{pmatrix} -\sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

Example 2

(c) Find the images of $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ by 135° rotation about O.

$$\frac{\sqrt{2}}{2}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} & -5\sqrt{2}/2 \\ 3\sqrt{2} & -\sqrt{2}/2 \end{pmatrix}$$

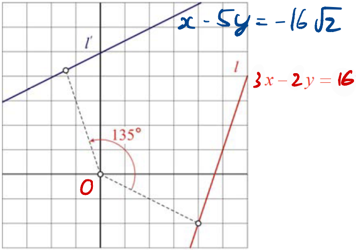
(d) Find the image of the line $3x - 2y = 16$ by 135° rotation about O.

The line ℓ: $3x - 2y = 16$ has vector equation

$$\ell : \vec{x} = \vec{x}_0 + t\vec{d} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

The image of ℓ is

$$\ell' : \vec{x} = M\left(\begin{pmatrix} 4 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right)$$



$$\ell': \vec{x} = M\begin{pmatrix} 4 \\ -2 \end{pmatrix} + tM\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ 3\sqrt{2} \end{pmatrix} + t\begin{pmatrix} -5\sqrt{2}/2 \\ -\sqrt{2}/2 \end{pmatrix}$$

$$\ell': \vec{x} = \begin{pmatrix} -\sqrt{2} \\ 3\sqrt{2} \end{pmatrix} + \frac{-t\sqrt{2}}{2}\begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

contain $\begin{pmatrix} -\sqrt{2} \\ 3\sqrt{2} \end{pmatrix}$

direction $\begin{pmatrix} 5 \\ 1 \end{pmatrix} \Rightarrow$ normal $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$

$$\ell': \quad 1(x+\sqrt{2}) - 5(y - 3\sqrt{2}) = 0$$

$$x - 5y = -16\sqrt{2}$$

Example 3

(a) What is matrix M of the rotation by 60^0?

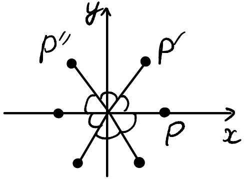
$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \overset{\theta=60^\circ}{=\!=} \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

(b) Computation the matrix of rotation by $120^0, 180^0, 360^0$ by computing M^2, M^3, M^6. Check that $M^6 = I_2$.

$$M^2 = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$M^3 = M^2 M = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M^6 = M^3 M^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

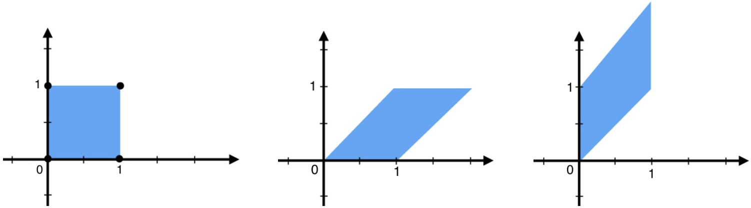


$P' = MP$,

$P'' = MP' = M^2 P$

Shear

- A **shear** is a map which transforms a square into a parallelogram.



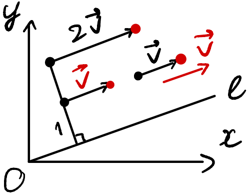
- Distance from a point $\vec{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ to a line $\ell : \vec{n} \cdot \vec{x} = 0$

$(ax + by = 0)$

$\vec{n} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$d(\vec{x}_0, \ell) = \frac{|ax_0 + by_0|}{\sqrt{a^2+b^2}} = \frac{|\vec{n} \cdot \vec{x}_0|}{\|\vec{n}\|}$$

Shear



(through O)

- The **shear** with respect to the line $l : \vec{n} \cdot \vec{x} = 0$ in the direction of the **shearing vector** $\vec{v}$ ($\vec{v} \parallel l$) is a map $S : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$S(\vec{x}_0) = \vec{x}_0 + \frac{\vec{n} \cdot \vec{x}_0}{||\vec{n}||}\vec{v}$$

scalar

magnitude = distance from $\vec{x}_0$ to l

Remark

$$S(\vec{x}_0) = \vec{x}_0 + \frac{\vec{n} \cdot \vec{x}_0}{||\vec{n}||}\vec{v}$$

- $\vec{v}$ must be parallel to l for the **shear S to be defined**. So

$$\vec{v} \cdot \vec{n} = 0 \quad \text{and } \vec{v} \parallel \vec{d}$$

Remark

$$S(\vec{x}_0) = \vec{x}_0 + \frac{\vec{n} \cdot \vec{x}_0}{||\vec{n}||}\vec{v}$$

- l has equation $\vec{n} \cdot \vec{x} = 0$. The distance from the point $\vec{x}_0$ to l is

$$d(\vec{x}_0, l) = \frac{|\vec{n} \cdot \vec{x}_0|}{||\vec{n}||}$$

Remark

$$S(\vec{x}_0) = \vec{x}_0 + \frac{\vec{n} \cdot \vec{x}_0}{||\vec{n}||}\vec{v}$$

- l has equation $\vec{n} \cdot \vec{x} = 0$. The distance from the point $\vec{x}_0$ to l is

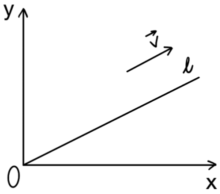
$$d(\vec{x}_0, l) = \frac{|\vec{n} \cdot \vec{x}_0|}{||\vec{n}||}$$

- The shear S shifts $\vec{x}_0$

  in the direction $\vec{v}$ by the factor $\pm d(\vec{x}_0, l)$.

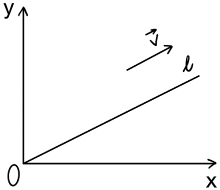
Example

- A shear is defined based on a line l through O and a vector $\vec{v} \parallel l$.

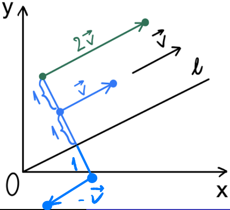


Example

- A shear is defined based on a line l through O and a vector $\vec{v} \parallel l$.



- Points at distance 1 are shifted by $\vec{v}$. Points at distance 2 are shifted by $2\vec{v}$.



Example 4

Consider the shear S w.r.t. the **x-axis** in the direction of $\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

$\ell : y = 0$

(a) Check that

$$S\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + y_0 \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

w.r.t : with respect to

that is, S shifts any point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ parallel to the x-axis by $y_0\vec{v}$.

$\vec{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\vec{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$$S(\vec{x}_0) = \vec{x}_0 + \frac{\vec{n} \cdot \vec{x}_0}{\|\vec{n}\|}\vec{v}$$

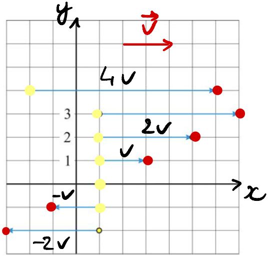
$\vec{v} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$$S(x_0) = \vec{x}_0 + \frac{y_0}{1}\vec{v} \iff S\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + y_0 \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Example 4

(b) Check that in the figure below, all "yellow" points are shifted to "red" points. Could you explain why the points above x-axis are shifted to the right and the points below x-axis are shifted to the left?

$S(\vec{x}_0) = \vec{x}_0 + y_0 \vec{v}$, that is, the point $\vec{x}_0$ is shifted by the $y_0 \vec{v}$



Question

For what points $\vec{x}_0$ do we have $S(\vec{x}_0) = \vec{x}_0$, that is, $\vec{x}_0$ is fixed by the shear?

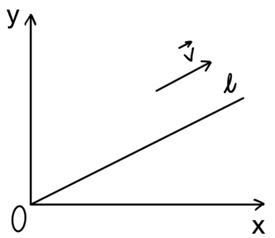
$$S(\vec{x}_0) = \vec{x}_0 + \frac{\vec{n}\cdot\vec{x}_0}{\|\vec{n}\|}\vec{v}$$

$$S(\vec{x}_0) = \vec{x}_0 \iff \frac{\vec{n}\cdot\vec{x}_0}{\|\vec{n}\|} = 0$$

In this case

$$d(\vec{x}_0, \ell) = \frac{|\vec{n}\cdot\vec{x}_0|}{\|\vec{n}\|} = 0$$

$\therefore$ $\vec{x}_0$ is on ℓ $\Rightarrow$ all fixed points are on ℓ.



$\ell: \vec{n}\cdot\vec{x} = 0$

Matrix of shear

Theorem 3

The shear w.r.t. $l : \vec{n} \cdot \vec{x} = 0$ in the direction of the shearing vector $\vec{v}$ for which $\vec{n} \cdot \vec{v} = 0$ has matrix

$$M = M_{\vec{n},\vec{v}} = I_2 + \frac{1}{||\vec{n}||}\vec{v}\vec{n}^T$$

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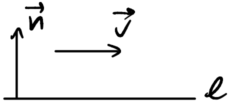
Question. Why is there condition $\vec{n} \cdot \vec{v} = 0$?

Ans. To guarantee $\vec{v} \parallel l$



Proof

Let $\vec{x}_0$ be any point

$$
\begin{aligned}
S(\vec{x}_0) &= \vec{x}_0 + \frac{\vec{n} \cdot \vec{x}_0}{||\vec{n}||}\vec{v} \\
&= \vec{x}_0 + \frac{1}{||\vec{n}||}(\vec{n} \cdot \vec{x}_0)\vec{v} \\
&= \vec{x}_0 + \frac{1}{||\vec{n}||}\vec{v}\vec{n}^T\vec{x}_0 \\
&= \left(I_2 + \frac{\vec{v}\vec{n}^T}{||\vec{n}||}\right)\vec{x}_0
\end{aligned}
$$

Example 5

Let $l : 3x + 4y = 0$ and $\vec{v} = \begin{bmatrix} 8 \\ -6 \end{bmatrix}$. $\quad \vec{n} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

(a) What is $M_{\vec{n},\vec{v}}$?

$$M = I_2 + \frac{1}{\|\vec{n}\|} \vec{v}\,\vec{n}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 8 \\ -6 \end{pmatrix}\begin{pmatrix} 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 24 & 32 \\ -18 & -24 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 24 & 32 \\ -18 & -24 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 29 & 32 \\ -18 & -19 \end{pmatrix}$$

(b) What are the images of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$?

The images are

$$\frac{1}{5}\begin{pmatrix} 29 & 32 \\ -18 & -19 \end{pmatrix}\begin{pmatrix} -1 & 4 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 7 & 55.2 \\ -4 & -33.4 \end{pmatrix}$$

Example 5

(c) What is the image of the line $n : 3x - y = 5$?

n has vector equation $\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

The image of n is

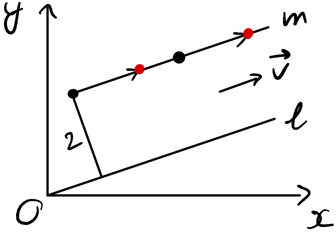
$$\vec{n}' : \vec{x} = \frac{1}{5}\begin{pmatrix} 29 & 32 \\ -18 & -19 \end{pmatrix}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right)$$

$$\vec{n}' : \vec{x} = \begin{pmatrix} 18 \\ -11 \end{pmatrix} + t\begin{pmatrix} 25 \\ -15 \end{pmatrix}$$

Example 5

(d) Show that the image of $m : 3x + 4y = 5$ is itself (note that $m \parallel l$).

m has vector equation $\vec{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} -4 \\ 3 \end{pmatrix}$



The image of m is

$$m' : \vec{x} = \frac{1}{5}\begin{pmatrix} 29 & 32 \\ -18 & -19 \end{pmatrix}\left(\begin{pmatrix} 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} -4 \\ 3 \end{pmatrix}\right)$$

$$m' : \vec{x} = \begin{pmatrix} 11 \\ -7 \end{pmatrix} + t\begin{pmatrix} -4 \\ 3 \end{pmatrix} \begin{cases} \text{through } \begin{pmatrix} 11 \\ -7 \end{pmatrix} \\ \text{direction } \begin{pmatrix} -4 \\ 3 \end{pmatrix} \Rightarrow \vec{n}' = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \end{cases}$$

$$m' : 3(x-11) + 4(y+7) = 0$$

$$3x + 4y = 5$$

$$\Rightarrow m' = m.$$

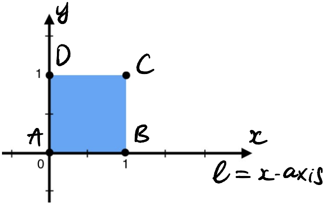
Exercise 2 (Horizontal shear)

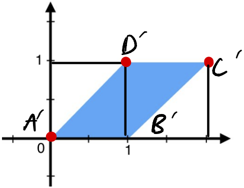
(a) Consider the shear w.r.t. $l : y = 0$ (the x-axis) in direction $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Show that the image of the unit square is the parallelogram as below.

What is the area of the resulting parallelogram?

$S(\vec{x}_0) = \vec{x}_0 + c\,\vec{i}$ with $c = d(\vec{x}_0, \ell)$ = y-coordinate





Because A, B are on the line ℓ, they are fixed by the shear

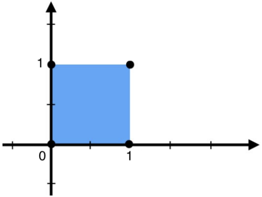
$A' = A, \quad B' = B$

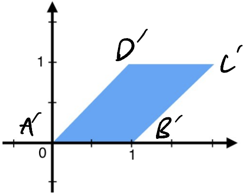
$D = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is mapped to $D' = D + 1\vec{i} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is mapped to $C' = C + 1\vec{i} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Exercise 2 (Vertical shear)

(b) Consider the shear w.r.t. y-axis in direction $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Show that the image of the unit square is the parallelogram as below. What is the area of the resulting parallelogram?





$$\text{Area of } A'B'C'D' = \left|\det\left(\vec{A'B'}\ \vec{A'D'}\right)\right| = \left|\det\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right| = 1$$

$$= \text{Area of } ABCD$$

Exercise 2

(c) Consider the shear w.r.t. $l : x - y = 0$ in direction $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Sketch the image of the unit square in part a and compute its area.

Exercise!

Composition of linear transformation

- Let $\mathbb{R}^m \xrightarrow{S} \mathbb{R}^n \xrightarrow{T} \mathbb{R}^k$ be a sequence of linear transformations. The composition $T \circ S : \mathbb{R}^m \to \mathbb{R}^k$ is defined by

$$T \circ S(\vec{x}) = T(S(\vec{x}))$$

- We will see that $T \circ S$ is another linear transformation. Further, if M_T, M_S are matrices of T, S, the matrix of $T \circ S$ is

$$M_{T \circ S} = M_T M_S.$$

Exercise 3

Let P be the projection onto $l : \sqrt{3}x - y = 0$ and let R be the reflection through $m : x - \sqrt{3}y = 0$.

(a) Describe $P \circ R$, that is, find a formula for $P \circ R\left(\vec{x}\right)$.

(b) Find the matrices of $M_P, M_R, M_{P\circ R}$ of $P, R, P \circ R$.

(c) Find the points which are fixed by $P \circ R$.