Trigonometric Integrals Part 2
Method of Partial Fractions Part 1

Dr. Ronald Koh
ronald.koh@digipen.edu (Teams preferred over email)

AY 23/24 Trimester 1

Table of contents

Integration by parts, sine/cosine integrals

- We have learnt how to integrate the product of functions; i.e. **integration by parts**:

$$\int u\,dv = uv - \int v\,du.$$

- We have learned how to choose u using the **LIATE prioritization tool**, based on the difficulty of integration.
- We have also learned how to integrate $\sin^m x \cos^n x$:
  - <u>m is odd</u>: take out one copy of $\sin x$, convert rest to $\cos x$ using $\sin^2 x = 1 - \cos^2 x$, sub $u = \cos x$.
  - <u>n is odd</u>: take out one copy of $\cos x$, convert rest to $\sin x$ using $\cos^2 x = 1 - \sin^2 x$, sub $u = \sin x$.
  - <u>Both m and n even</u>: use double angle formulae:

$$\sin^2 x = \frac{1-\cos(2x)}{2} \quad \text{and} \quad \cos^2 x = \frac{1+\cos(2x)}{2}.$$

Example 1

Evaluate $\displaystyle\int \sec^4 x\, dx$.

Example 2

Evaluate $\int \tan^3 x \sec x \, dx$.

Example 3

Evaluate $\int \sec x \, dx$.

Method for integrating powers of tangent/secant (1)

Method for integrating $\int \tan^m x \sec^n x \, dx$**:**

- If n is **even**, then $n = 2k$ for some integer k. Then

$$\begin{aligned}\int \tan^m x \sec^n x \, dx &= \int \tan^m x \sec^{2k-2} x \sec^2 x \, dx \\ &= \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \, dx \\ &= \int \tan^m x (\tan^2 x + 1)^{k-1} \sec^2 x \, dx.\end{aligned}$$

Then apply substitution $u = \tan x$. See Example 1.

Method for integrating powers of tangent/secant (2)

- If m is **odd** and $n \geq 1$, then $m = 2k + 1$ for some integer k. Then

$$\begin{aligned}
\int \tan^m x \sec^n x \, dx &= \int \tan^{m-1} x \sec^{n-1} x \sec x \tan x \, dx \\
&= \int \tan^{2k} x \sec^{n-1} x \sec x \tan x \, dx \\
&= \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \, dx \\
&= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx.
\end{aligned}$$

Then apply substitution $u = \sec x$. See Example 2.

Method for integrating powers of tangent/secant (3)

There are other cases, e.g.

$$\int \tan^2 x \sec^3 x \, dx,$$

where things are not so "black and white". They usually can be done by converting all the $\tan^2 x$ to $\sec^2 x$ (since m is even), then applying integration by parts on integrals of $\sec^n x$. **See Exercise 1 Q3** for a base example. We will cover more of these cases in Week 4 Tutorial.

Exercise 1

Evaluate the following integrals.

1. $\displaystyle\int \tan^2 x \sec^4 x \, dx$
2. $\displaystyle\int \tan^3 x \sec^3 x \, dx$
3. $\displaystyle\int \sec^3 x \, dx$

Exercise 1

What are rational functions?

For the remaining of this lecture, we focus on the integration of **rational functions**. Rational functions are functions of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. A polynomial of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

with $a_n \neq 0$ has degree n, denoted by $\deg(P) = n$.
If $\deg(P) < \deg(Q)$, we say that the rational function f is **proper**.
If $\deg(P) \geq \deg(Q)$, we say that the rational function f is **improper**.

Integration of rational functions

If the integrand is an **improper** rational function, then **long division** is required to convert it from improper to proper before integration; we need to find functions S and R such that

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where $\dfrac{R(x)}{Q(x)}$ is a proper rational function. Then

$$\int f(x)\,dx = \int \underbrace{\frac{P(x)}{Q(x)}}_{\text{improper}}\,dx = \int S(x) + \underbrace{\frac{R(x)}{Q(x)}}_{\text{proper}}\,dx.$$

Exercise 2

Figure out which of these rational functions are proper or improper. For those that are improper, use long division to convert it to $S(x) + \dfrac{R(x)}{Q(x)}$ where $\dfrac{R(x)}{Q(x)}$ is proper.

1. $\dfrac{x^3 + x}{x - 1}$
2. $\dfrac{x + 5}{x^2 + x - 2}$
3. $\dfrac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}$

Exercise 2

Partial fraction decomposition

From this point on, we will assume that you have converted the improper rational function to a proper one. The whole idea of partial fraction decomposition is

1. Factorizing $Q(x)$ into linear and irreducible quadratic factors. E.g. if $Q(x) = x^4 - 16$, then

$$Q(x) = (x^2 - 4)(x^2 + 4) = (x + 2)(x - 2)(x^2 + 4).$$

2. Writing $\dfrac{R(x)}{Q(x)}$ as a decomposition into different fractions, each fraction tagged to a linear/irreducible quadratic factor of Q. E.g. for the example $Q(x) = x^4 - 16$,

$$\frac{R(x)}{Q(x)} = \frac{A}{x+2} + \frac{B}{x-2} + \frac{Cx + D}{x^2 + 4}.$$

Partial fraction decomposition

Depending on the factors of Q, we have different partial fraction decompositions. There are **four** different cases:

1. Q is a product of distinct, non-repeating linear factors.
   E.g. $Q(x) = (x-2)(x+2)$.
2. Q contains repeated linear factors.
   E.g. $Q(x) = (x-2)^3(2x+2)(x-3)^2$.
3. Q contains a non-repeated irreducible quadratic factor.
   E.g. $Q(x) = (x-2)^3(6x-3)(x^2+9)$.
4. Q contains repeated irreducible quadratic factors.
   E.g. $Q(x) = (x-1)^2(2x^2+1)^2$.

In this course, we only cover cases (1), (2) and (3).

Method for distinct, non-repeating linear factors

The first case is when $Q(x)$ factors only into distinct, non-repeating factors $(a_1x + b_1)$, $(a_2x + b_2)$,..., $(a_nx + b_n)$. Then there exist constants $A_1, A_2, \ldots, A_n$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{(a_1x + b_1)} + \frac{A_2}{(a_2x + b_2)} + \cdots + \frac{A_n}{(a_nx + b_n)}.$$

We can find these constants A_i by multiplying $Q(x)$ to both sides of the equation, substituting the roots corresponding to these linear factors to solve for A_i.

Example 4

Let a be a constant. Evaluate $\displaystyle\int \frac{1}{x^2 - a^2}\,dx$.

Example 4

Method for Repeated Linear Factors

When $Q(x)$ has a repeated linear factor, i.e. $(ax+b)^m$ where m is the **multiplicity** of the factor, then there must be m repeating terms in the partial fraction decomposition of $\frac{R(x)}{Q(x)}$; there exists constants $B_1, B_2, \ldots, B_m$ such that

$$\frac{B_1}{(ax+b)} + \frac{B_2}{(ax+b)^2} + \cdots + \frac{B_m}{(ax+b)^m}.$$

Example 5

Evaluate $\displaystyle\int \frac{x^2+2x}{x^3-x^2-x-1}\,dx.$

Example 5

Exercise 3

Evaluate the following integrals.

1. $$\int \frac{x+5}{x^2+x-2}\,dx$$
2. $$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$$

Exercise 3

Exercise 4

Evaluate the following integrals.

1. $$\int \frac{1}{x^3 + x^2}\, dx$$
2. $$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx$$

Exercise 4

Exercise 4