CSD2301 Lecture
3. Kinematics in 2D and 3D

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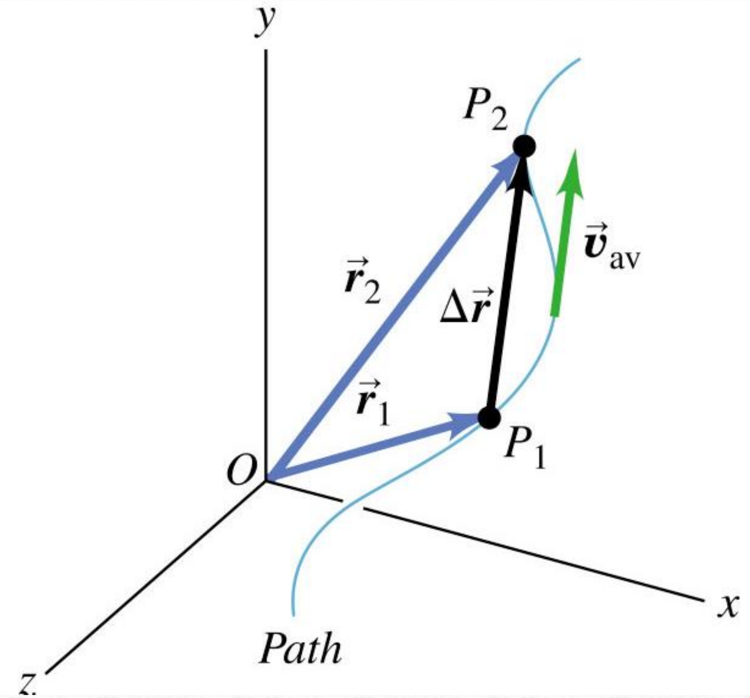
Outline

- Projectile motion
- Circular motion
- Relative velocity

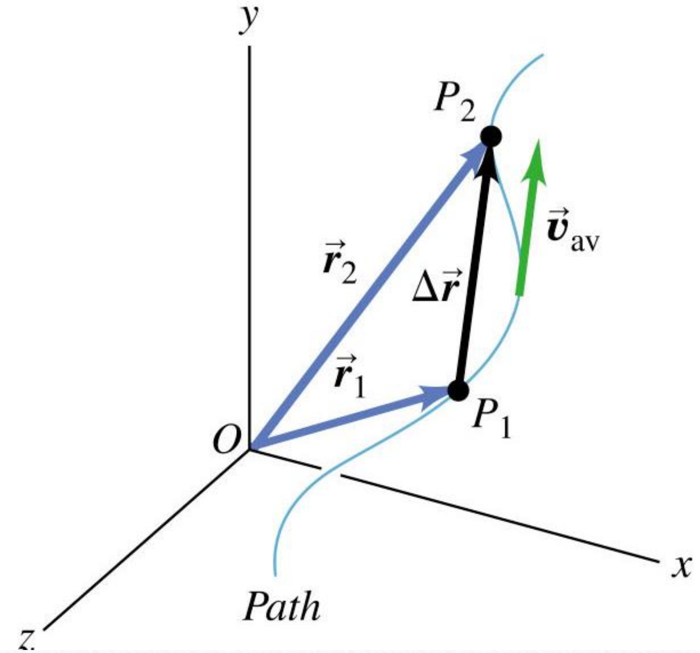
Displacement in 2D and 3D

- Similar to 1D motion, the kinematic equations for 2D & 3D motion can be derived by using the **vector properties of displacement, velocity and acceleration**
- Position vectors r, is the vector from origin of coordinate system 0 to P.
- Displacement is change of position during time interval: $\boxed{\Delta\vec{r} = \vec{r}_2 - \vec{r}_1}$



Velocity

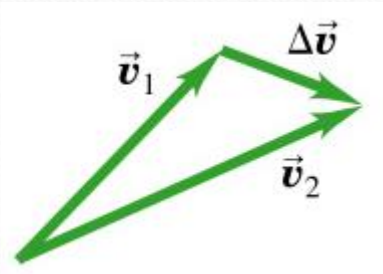
- Average velocity: $$\vec{v}_{av} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{\Delta \vec{r}}{\Delta t}$$
- Instantaneous velocity: $$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$
- Direction of **v** is a **tangent** to the path at that point in the direction of motion
  - A tangent is a straight line which touches the circle/path at only 1 point.
- Magnitude of **v** is the **speed**.

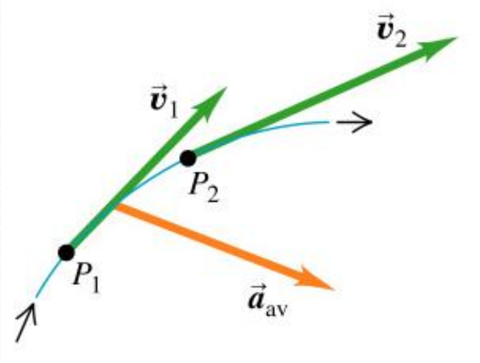


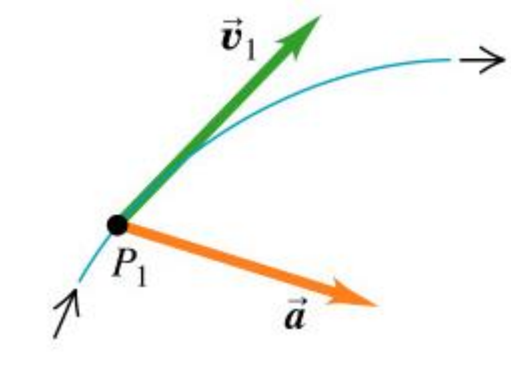
Acceleration

- Average acceleration: $\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta \vec{v}}{\Delta t}$

- Instantaneous acceleration: $\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$







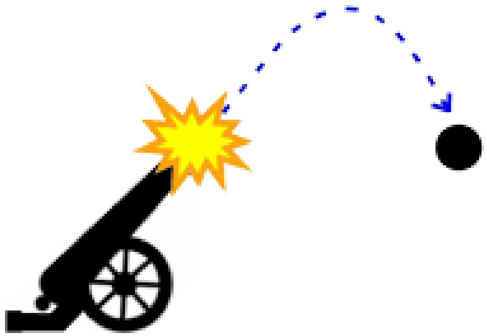
2D motion with constant acceleration

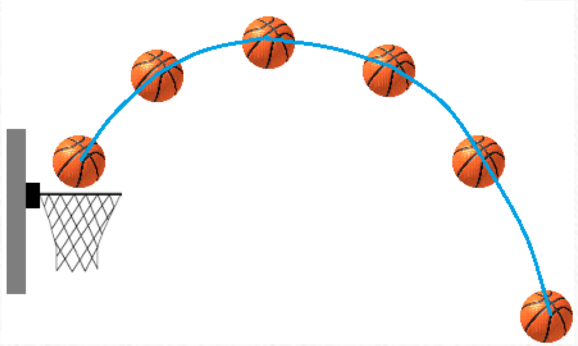
$$\vec{v} = \vec{v}_0 + \vec{a}t \quad \Leftrightarrow \quad \begin{cases} v_x = v_{x0} + a_x t \\ v_y = v_{y0} + a_y t \end{cases}$$

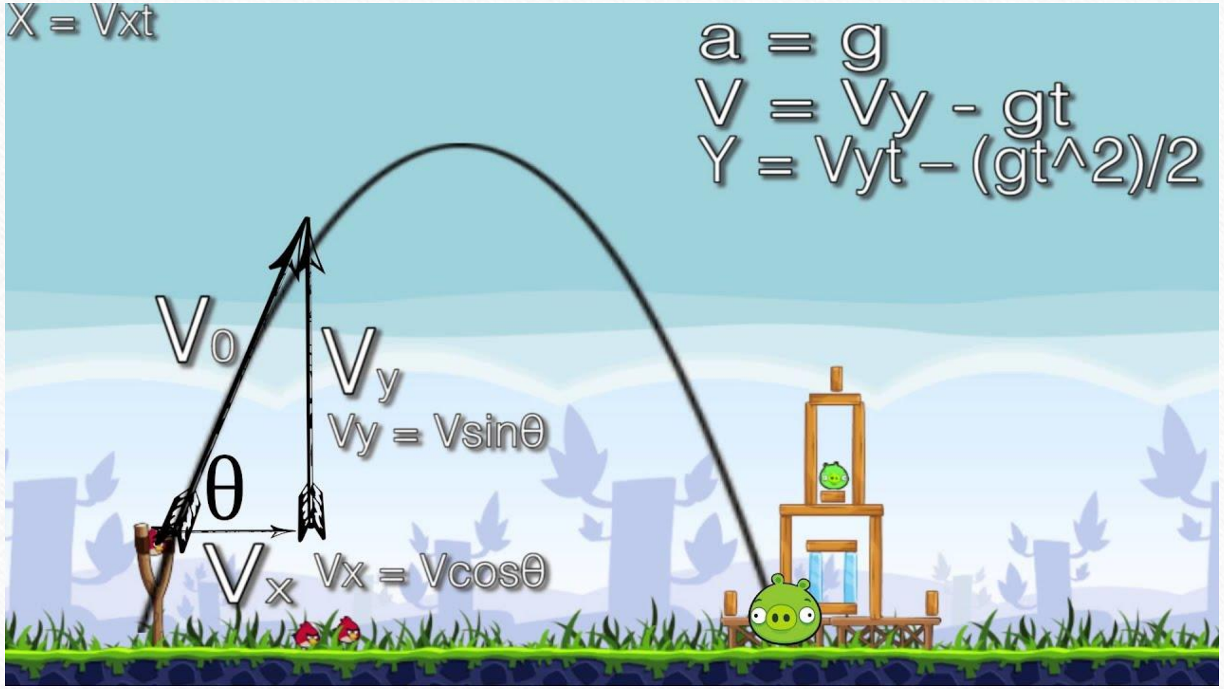
$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 \quad \Leftrightarrow \quad \begin{cases} x = x_0 + v_{x0}t + \frac{1}{2}a_x t^2 \\ y = y_0 + v_{y0}t + \frac{1}{2}a_y t^2 \end{cases}$$

2D motion with constant acceleration is equivalent to 2 independent motions in the x and y directions with constant accelerations a_x and a_y.

Projectile Motion

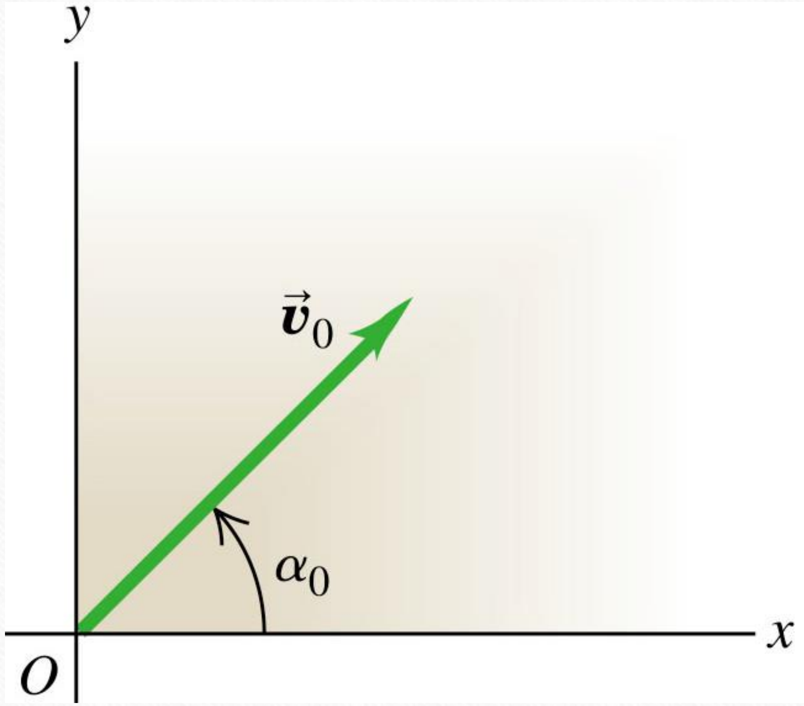




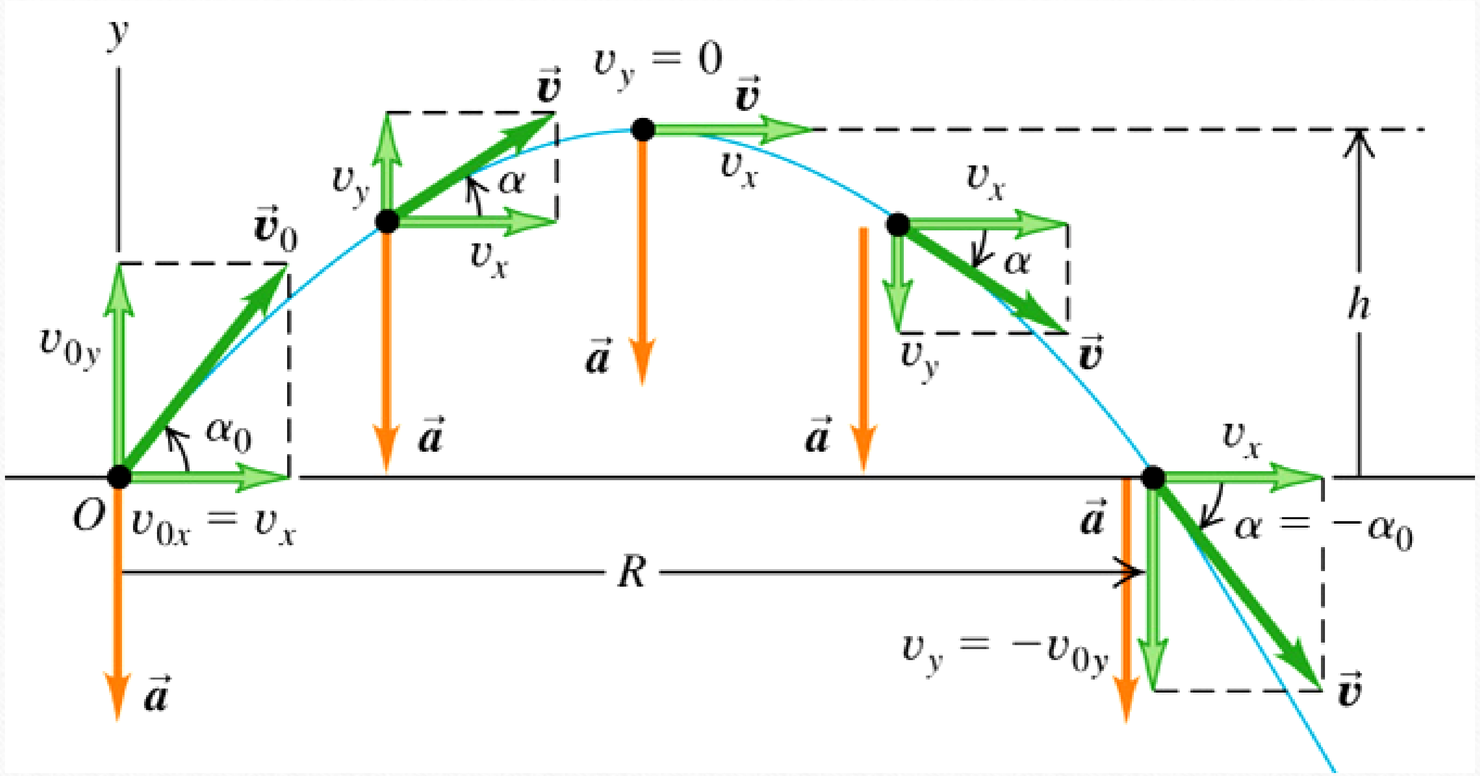


Projectile Motion

- Assumptions:
  - (1) free-fall acceleration
  - (2) neglect air resistance
- Usually choose the y direction as positive upward.
- $a_x = 0$; $a_y = -g$ (a constant)
- Take $x_0 = y_0 = 0$ at $t = 0$
- Initial velocity $\mathbf{v_0}$ makes an angle a_0 with the horizontal



Projectile Motion



$$v_{0x} = v_0 \cos \alpha_0$$

$$v_{0y} = v_0 \sin \alpha_0$$

Projectile Motion

- Velocity: $v_x = v_{0x} = v_0 \cos\alpha_0 = \text{const}$

  $v_y = v_{0y} - gt = v_0 \sin\alpha_0 - gt$

  From: $\boxed{v = v_0 + at}$

- Displacement: $x = v_{0x}t = v_0 \cos\alpha_0 t$

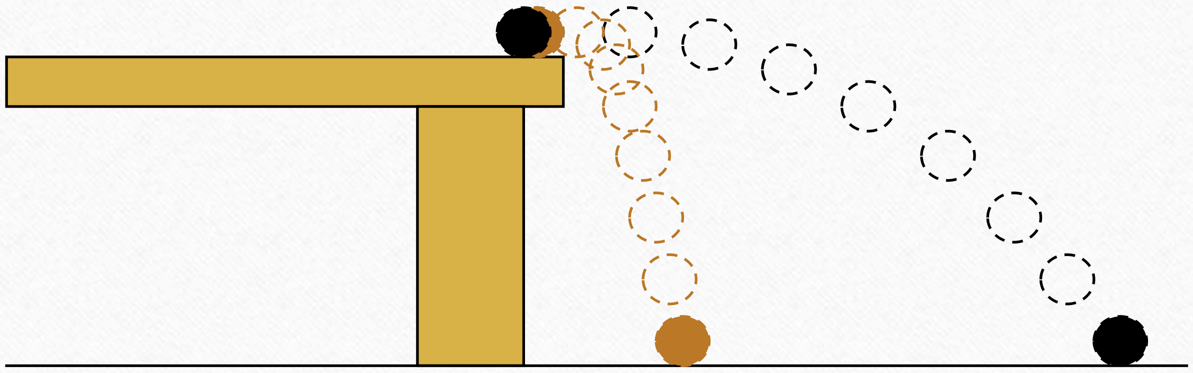
  $$y = v_{0y}t - \frac{1}{2}gt^2 = v_0 \sin\alpha_0 t - \frac{1}{2}gt^2$$

  From: $\boxed{x = x_0 + v_0 t + \frac{1}{2}at^2}$

- Projectile motion is the superposition of 2 motions:
  - (1) constant horizontal velocity,
  - (2) vertical free fall

Concept Question

Place two tennis balls at the edge of a tabletop. Sharply snap one ball horizontally off the table with one hand while gently tapping the second ball off with your other hand. Which ball will fall to the ground first? Explain your answer.



Trajectory

Eliminating *t* from these equations

$$x = v_0 \cos \alpha_0 t$$
$$y = v_0 \sin \alpha_0 t - \frac{1}{2} g t^2$$

We obtain the trajectory y = f(x):

$$\boxed{y = (\tan \alpha_0) x - \left(\frac{g}{2 v_0^2 \cos^2 \alpha_0} \right) x^2}$$

which is a parabola of the form: $y = ax - bx^2$

Maximum Height

At the peak of its trajectory, in the vertical direction: $v_y = 0$ ⇨ $v_0 \sin\alpha_0 - gt = 0$

Time t_1 to reach the peak: $t_1 = \dfrac{v_0 \sin\alpha_0}{g}$

Substituting into $y = v_0 \sin\alpha_0 t - \dfrac{1}{2} g t^2$

We get maximum height: $\boxed{h = y_{\max} = \dfrac{v_0^2 \sin^2\alpha_0}{2g}}$

Horizontal Range

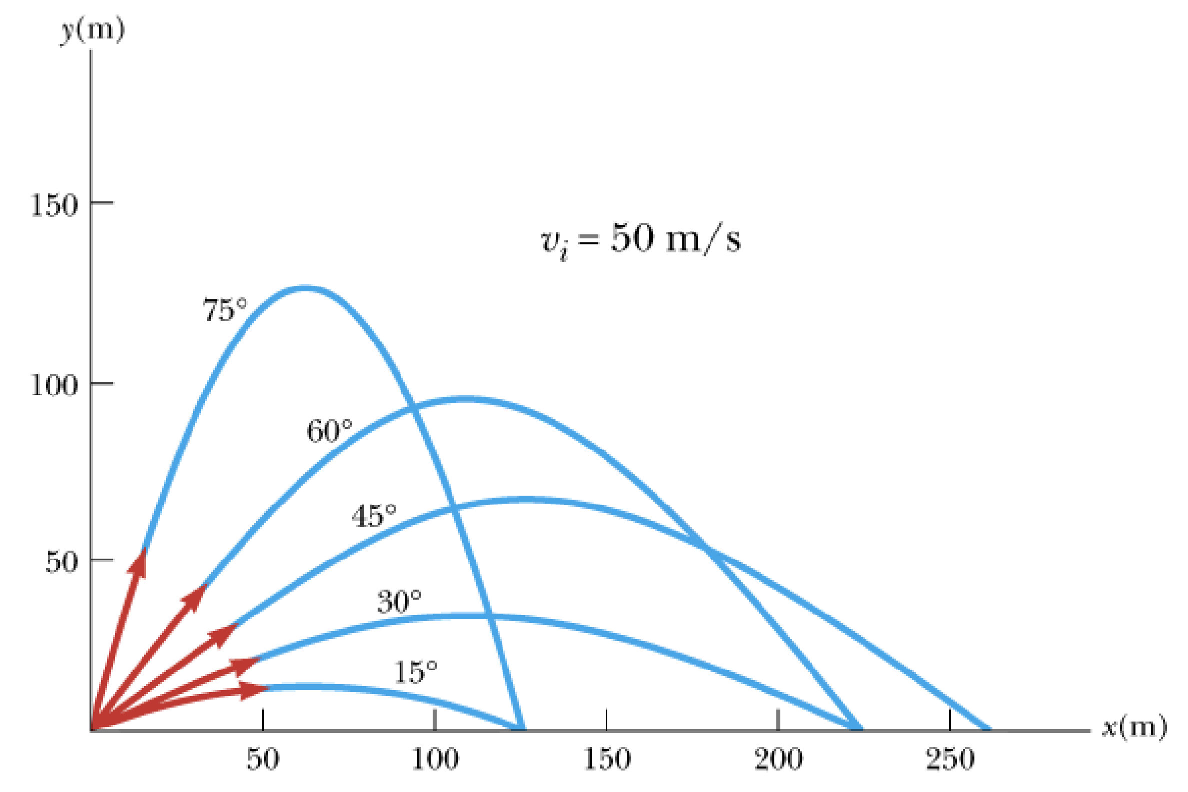
When the projectile reaches the ground:

$$y = (\tan\alpha_0)x - \left(\frac{g}{2v_0^2\cos^2\alpha_0}\right)x^2 = 0$$

$$\tan\alpha_0 = \left(\frac{g}{2v_0^2\cos^2\alpha_0}\right)x \Rightarrow \boxed{R = x = \frac{v_0^2\sin(2\alpha_0)}{g}}$$

R is a maximum when $\boxed{\alpha_0 = 45^\circ}$

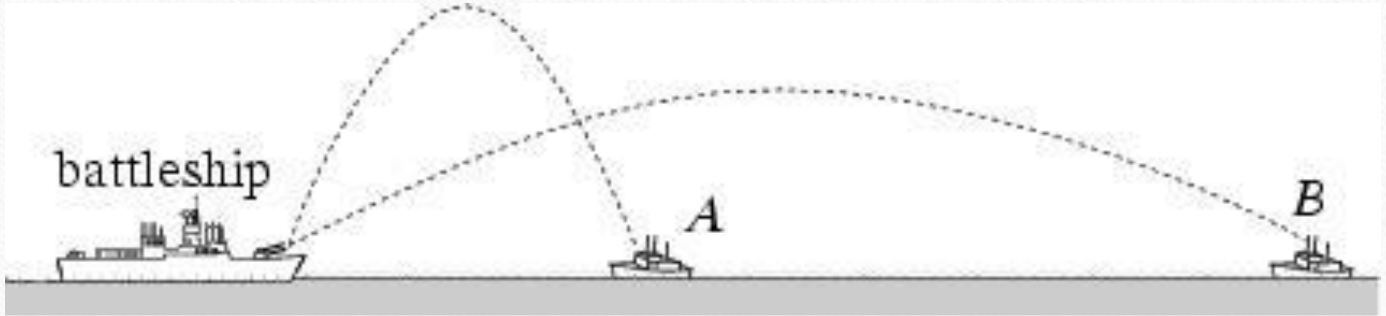
Horizontal Range



$$\boxed{R = \frac{v_0^2 \sin(2\alpha_0)}{g}}$$

Concept Question

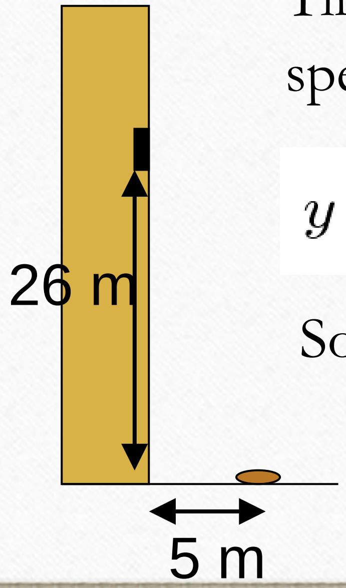
A battleship simultaneously fires two shells at enemy ships. If the shells follow the parabolic trajectories shown, which ship gets hit first?



A. Ship A
B. Ship B
C. Both ships got hit at the same time
D. The shells collide in midair and no ship was hit.
E. Not enough information to tell.

Example

In a detective story, a body is found 5.0 m from the base of a building and beneath an open window 26 m above. Would you have guessed the death to be accidental? Why?



Time taken for fall, assuming zero initial vertical speed is given by

$$y = -\frac{1}{2}gt^2 \Rightarrow -26 = -\frac{1}{2}(9.80)t^2 \Rightarrow t = 2.3 \text{ s}$$

So his initial horizontal speed must be

$$v_{x0} = \frac{5.0 \text{ m}}{2.3 \text{ s}} = 2.2 \text{ m/s} = 7.8 \text{ km/h}$$

Example

In a detective story, a body is found 5.0 m from the base of a building and beneath an open window 26 m above. Would you have guessed the death to be accidental? Why?

Could it be accidental?

- If it was accidental, we expect the horizontal velocity to be close to zero, and he will be found much closer to the base of building.
- Normal walking speed = 4 km/h. His horizontal velocity was double that a normal person moves.
- He jumped off or someone threw him down.

Example

A physics book slides off a horizontal tabletop with a speed of 1.10 m/s. It strikes the floor in 0.350 s. Ignore air resistance.

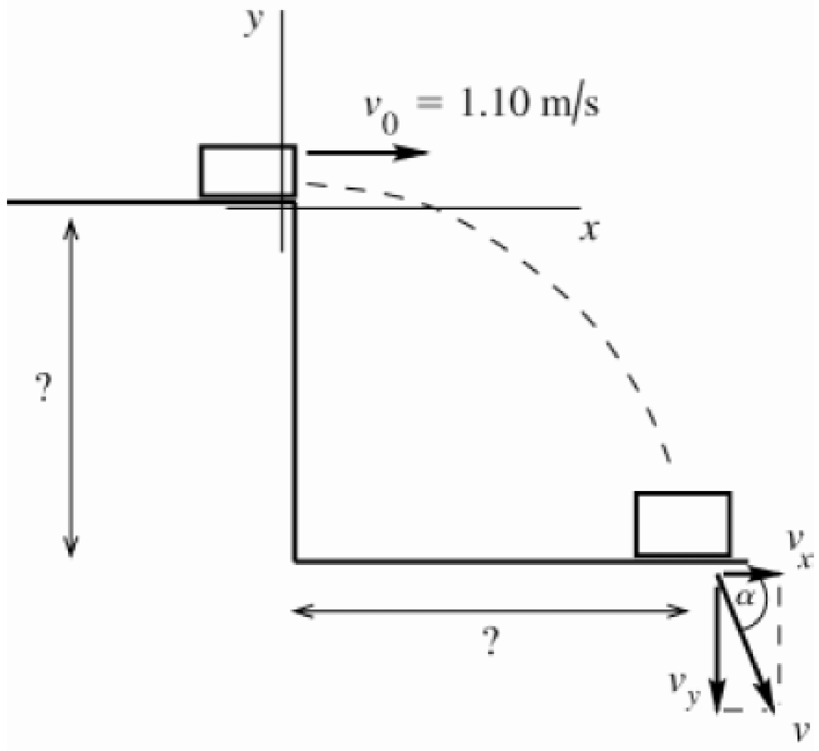
a) Find the height of the tabletop above the floor

b) Find the horizontal distance from the edge of the table to the point where the book strikes the floor

c) Find the horizontal and vertical components of the book's velocity, and the magnitude and direction of its velocity, just before the book reaches the floor.

Example

Solution:



<u>x-component</u>:

$a_x = 0, \quad v_{0x} = 1.10$ m/s,

$t = 0.350$ s

<u>y-component</u>:

$a_y = -9.80 \text{ m/s}^2,$

$v_{0y} = 0,$

$t = 0.350$ s

Example

a) Find the height of the tabletop above the floor

(a) $y - y_0 = ?$

$$y - y_0 = v_{0y}t + \tfrac{1}{2}a_y t^2 = 0 + \tfrac{1}{2}(-9.80\ \text{m/s}^2)(0.350\ \text{s})^2 = -0.600\ \text{m}.$$

The table top is 0.600 m above the floor.

Example

b) Find the horizontal distance from the edge of the table to the point where the book strikes the floor

(b) $x - x_0 = ?$

$$x - x_0 = v_{0x}t + \tfrac{1}{2}a_x t^2 = (1.10\ \text{m/s})(0.350\ \text{s}) + 0 = 0.358\ \text{m}.$$

Example

c) Find the horizontal and vertical components of the book's velocity, and the magnitude and direction of its velocity, just before the book reaches the floor.

(c) $v_x = v_{0x} + a_x t = 1.10\ \text{m/s}$ (The x-component of the velocity is constant, since $a_x = 0$.)

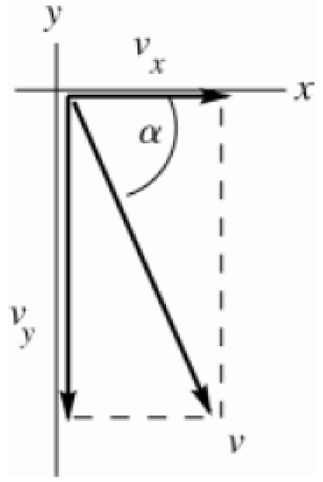
$$v_y = v_{0y} + a_y t = 0 + (-9.80\ \text{m/s}^2)(0.350\ \text{s}) = -3.43\ \text{m/s}$$

$$v = \sqrt{v_x^2 + v_y^2} = 3.60\ \text{m/s}$$

$$\tan\alpha = \frac{v_y}{v_x} = \frac{-3.43\ \text{m/s}}{1.10\ \text{m/s}} = -3.118$$

$$\alpha = -72.2°$$

Direction of $\vec{v}$ is 72.2° below the horizontal



Example

A military helicopter flying horizontally at a speed of 60.0 m/s accidentally drops a bomb at an elevation of 300 m. You can ignore air resistance.

a) How much time is required for the bomb to reach the earth?

b) How far does it travel horizontally while falling?

c) Find the horizontal and vertical components of its velocity just before it strikes the earth.

Example

a) How much time is required for the bomb to reach the earth?

SET UP: The initial velocity of the bomb is the same as that of the helicopter. Take $+y$ downward, so $a_x = 0$, $a_y = +9.80\ \text{m/s}^2$, $v_{0x} = 60.0\ \text{m/s}$ and $v_{0y} = 0$.

EXECUTE: **(a)** $y - y_0 = v_{0y}t + \frac{1}{2}a_y t^2$ with $y - y_0 = 300\ \text{m}$ gives $t = \sqrt{\dfrac{2(y-y_0)}{a_y}} = \sqrt{\dfrac{2(300\ \text{m})}{9.80\ \text{m/s}^2}} = 7.82\ \text{s}$.

Example

b) How far does it travel horizontally while falling?

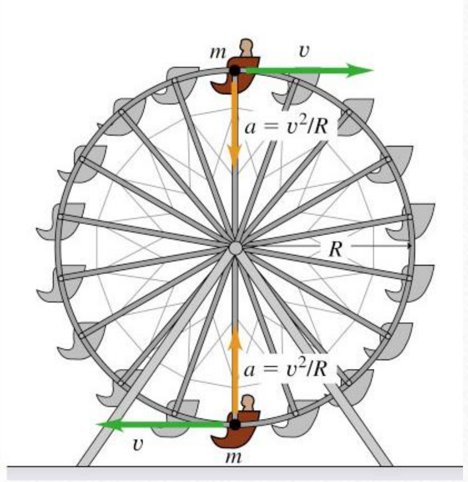
(b) The bomb travels a horizontal distance $x - x_0 = v_{0x}t + \frac{1}{2}a_x t^2 = (60.0 \text{ m/s})(7.82 \text{ s}) = 470 \text{ m}$.

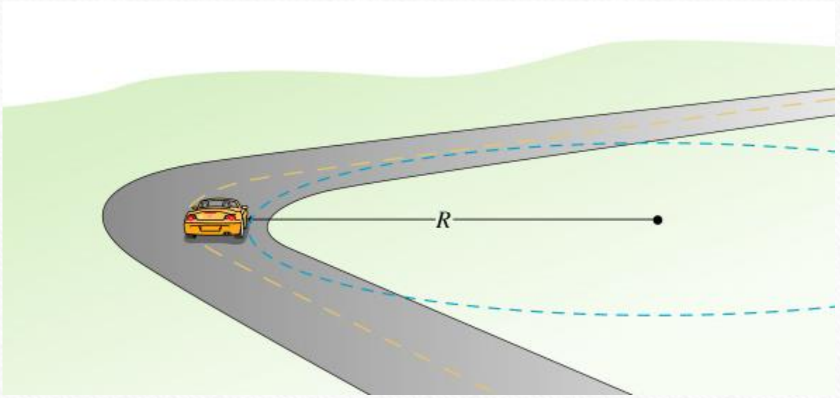
Example

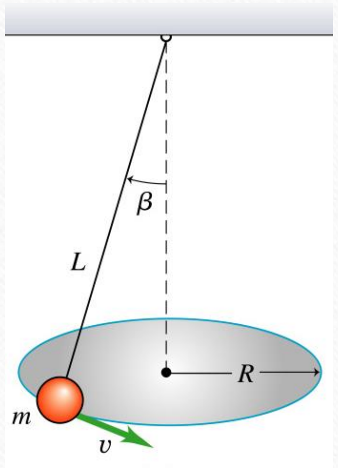
c) Find the horizontal and vertical components of its velocity just before it strikes the earth.

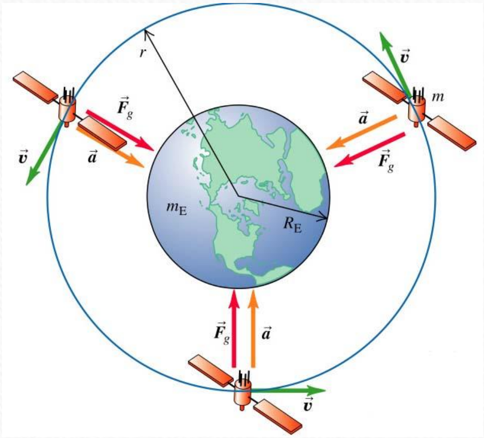
(c) $v_x = v_{0x} = 60.0 \text{ m/s}$. $v_y = v_{0y} + a_y t = (9.80 \text{ m/s}^2)(7.82 \text{ s}) = 76.6 \text{ m/s}$.

Circular Motion







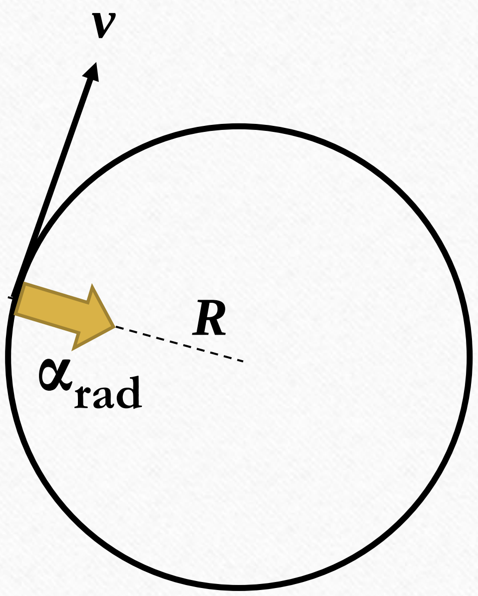


Circular Motion

- Acceleration occurs when there is a change in magnitude and/or direction of velocity.
- Recall: $\vec{a} = \dfrac{\vec{v}_f - \vec{v}_i}{t_f - t_i} = \dfrac{\Delta\vec{v}}{\Delta t}$
- Uniform circular motion occurs when there is a **change of direction of velocity only**, i.e. the object moves at constant speed.

Centripetal Acceleration

- Acceleration experienced while in circular motion
- Perpendicular to the tangential velocity



$$\boxed{|\vec{a}| = a_{\mathrm{rad}} = \frac{v^2}{R}}$$

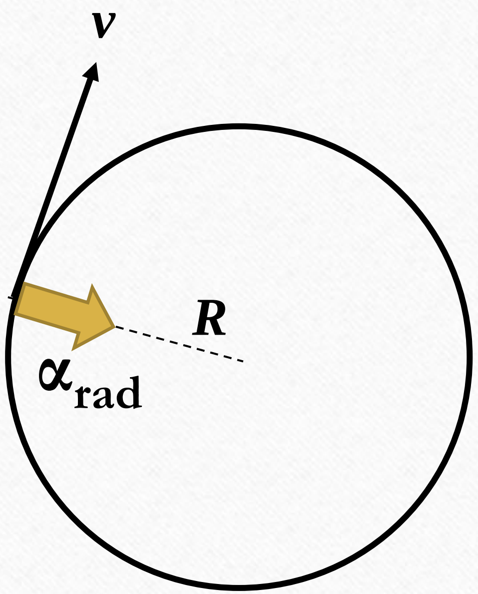
Centripetal Acceleration

- Time for one revolution, T is given by:

$$T = \frac{2\pi R}{v} \quad \text{or} \quad v = \frac{2\pi R}{T}$$

- Centripetal acceleration can therefore be written as:

$$a_{rad} = \frac{v^2}{R} = \frac{4\pi^2 R}{T^2}$$



Example

A model of a helicopter rotor has four blades, each 3.40 m long from the central shaft to the blade tip. The model is rotated at 550 revolutions per minute (rev/min). (a) What is the speed of the blade tip, in m/s? (b) What is the radial acceleration of the blade tip expressed as a multiple of the acceleration of gravity, g?

Example

Ans:

IDENTIFY: Each blade tip moves in a circle of radius $R = 3.40\text{ m}$ and therefore has radial acceleration $a_{\text{rad}} = v^2 / R$.

SET UP: $550\text{ rev/min} = 9.17\text{ rev/s}$, corresponding to a period of $T = \dfrac{1}{9.17\text{ rev/s}} = 0.109\text{ s}$.

EXECUTE: **(a)** $v = \dfrac{2\pi R}{T} = 196\text{ m/s}$.

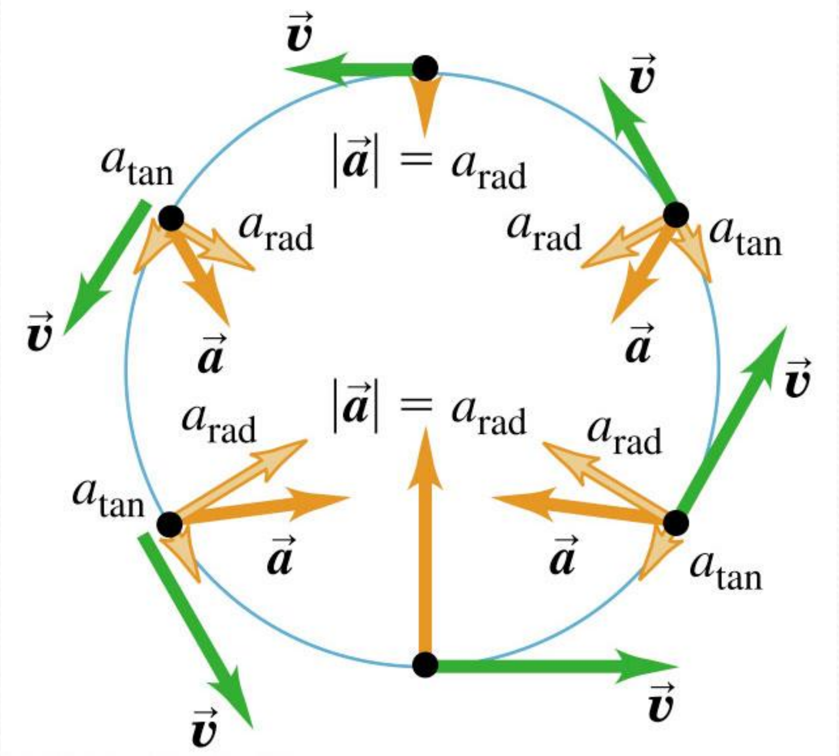
(b) $a_{\text{rad}} = \dfrac{v^2}{R} = 1.13\times 10^4\text{ m/s}^2 = 1.15\times 10^3 g$.

EVALUATE: $a_{\text{rad}} = \dfrac{4\pi^2 R}{T^2}$ gives the same results for a_{rad} as in part (b).

Non-uniform Circular Motion

If the speed is **not** a constant, there will also be a **tangential acceleration**, equals to the rate of change of speed.

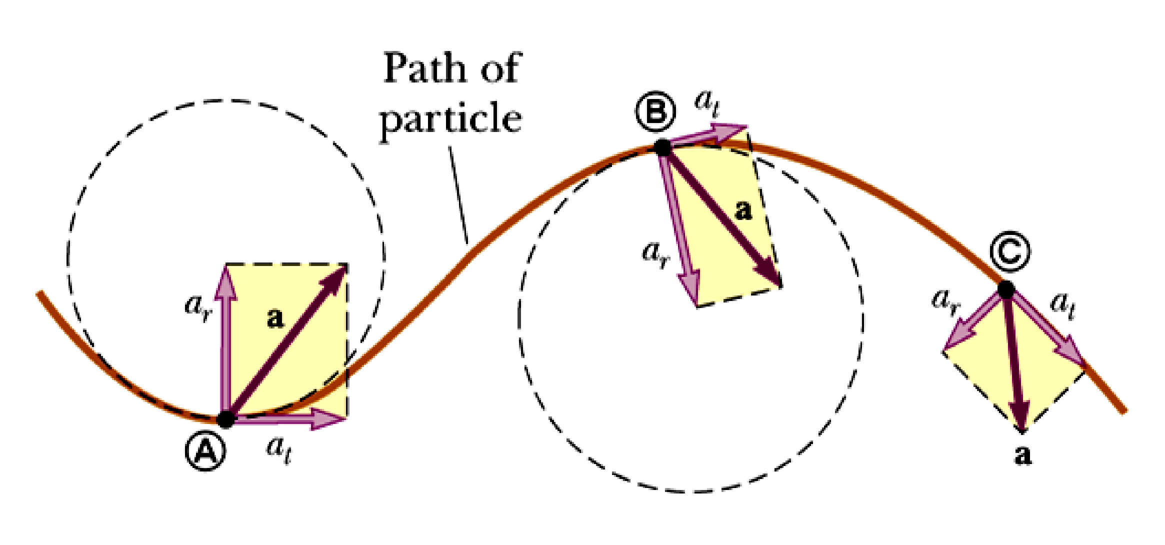
$$\boxed{|\vec{a}_t| = a_t = \frac{d|\vec{v}|}{dt}}$$



Tangential & Radial Acceleration

In general, for any motion, where there is a change in the direction and magnitude of velocity, the acceleration is the vector **sum** of the tangential and radial accelerations:

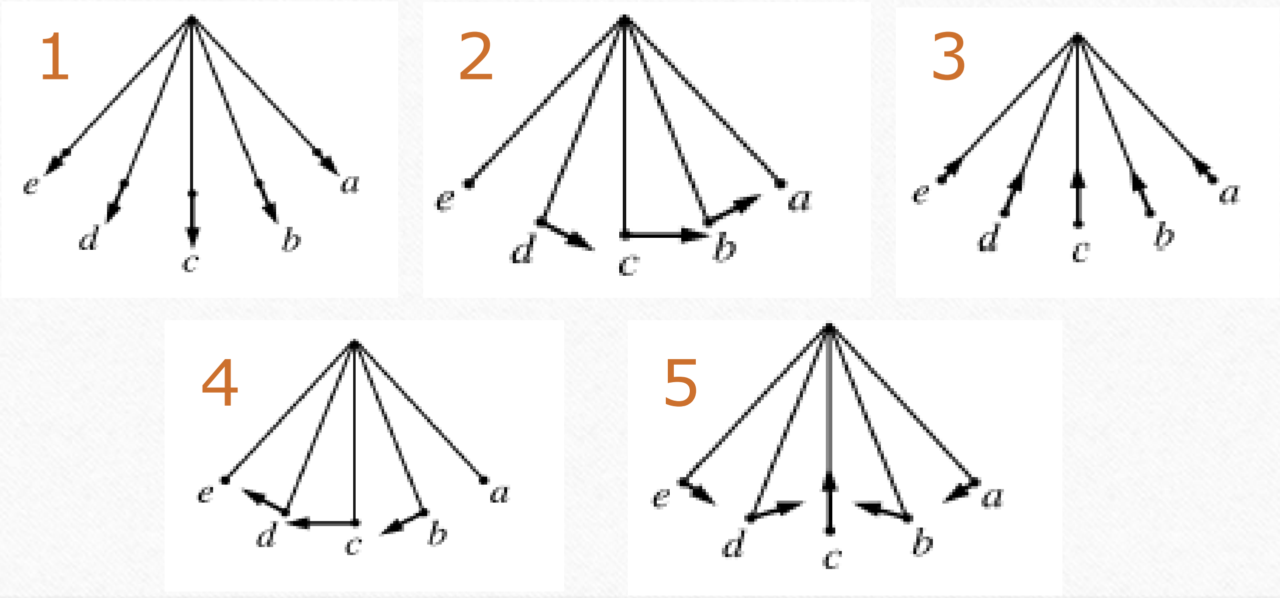
$$\boxed{\vec{a} = \vec{a}_r + \vec{a}_t}$$



where $|\vec{a}_r| = a_r = \dfrac{v^2}{r}$ & $|\vec{a}_t| = a_t = \dfrac{d|\vec{v}|}{dt}$

Concept Question

Which of the following best illustrates the acceleration of a pendulum bob at points a through e ?



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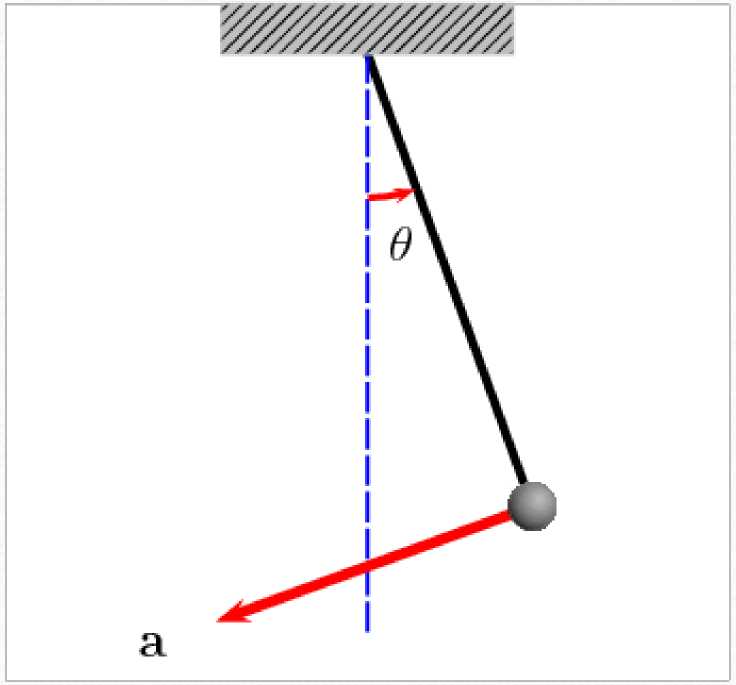
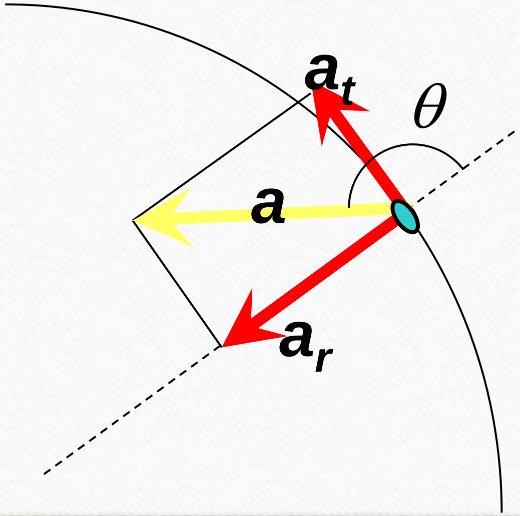


Example

A car whose speed is increasing at a rate of 0.600 m/s^2 travels along a circular road of radius 20.0 m. When the instantaneous speed of the car is 4.00 m/s, find (a) the tangential acceleration component, (b) the centripetal acceleration component, and (c) the magnitude and direction of the total acceleration.



(a) $a_t = 0.600 \text{ m/s}^2$

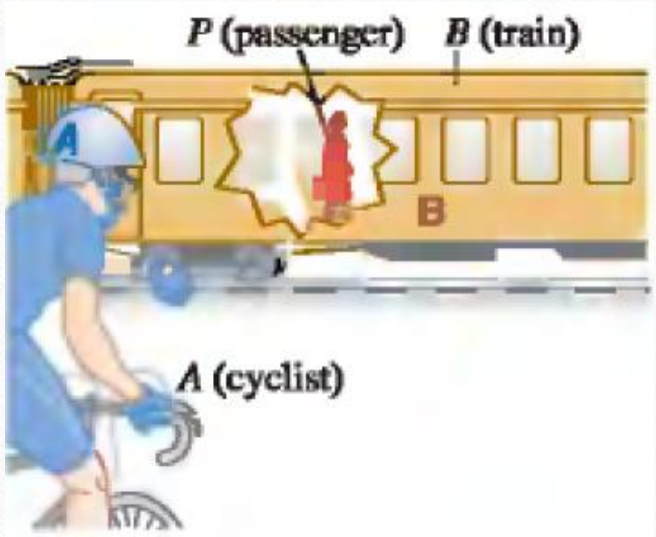
(b) $a_r = \frac{v^2}{r} = \frac{4.00^2}{20.0} = 0.80 \text{ m/s}^2$

(c) $a = \sqrt{a_t^2 + a_r^2} = 1.00 \text{ m/s}^2$

$$\theta = \tan^{-1}\left(-\frac{a_t}{a_r}\right) = 143^\circ$$

Relative Velocity

- **Relative velocity** is the velocity of an object A with respect to another object B.
- Every object can be a **frame of reference** - with a coordinate system and a time scale.
- Suppose we have 3 objects shown in the image.
  - A is the cyclist's frame of reference
  - B is the frame of reference of the moving train.
- For notation: the position of P relative to frame B is given by $x_{P/B}$.



Relative Velocity in 1D

- Displacement relationship is given by:

$$x_{P/A} = x_{P/B} + x_{B/A}$$

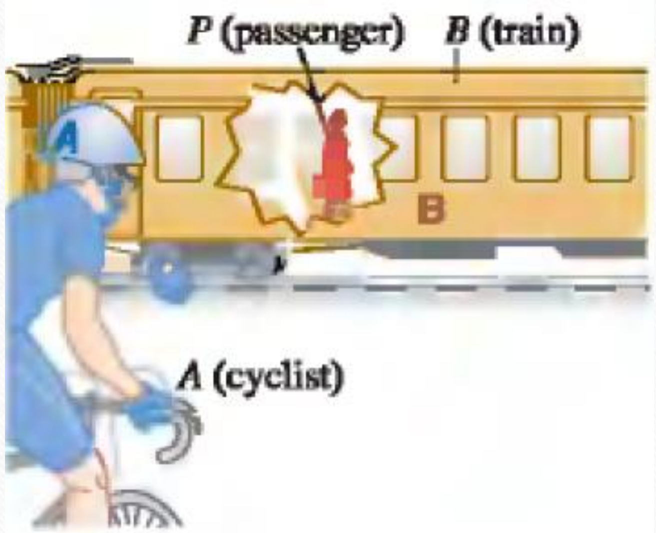
- Therefore, velocity relationship is given by:

$$\frac{dx_{P/A}}{dt} = \frac{dx_{P/B}}{dt} + \frac{dx_{B/A}}{dt}$$

$$v_{P/A\text{-}x} = v_{P/B\text{-}x} + v_{B/A\text{-}x}$$

means x-velocity of P relative to frame A

Imagine they are all in one line.



Relative Velocity in 1D

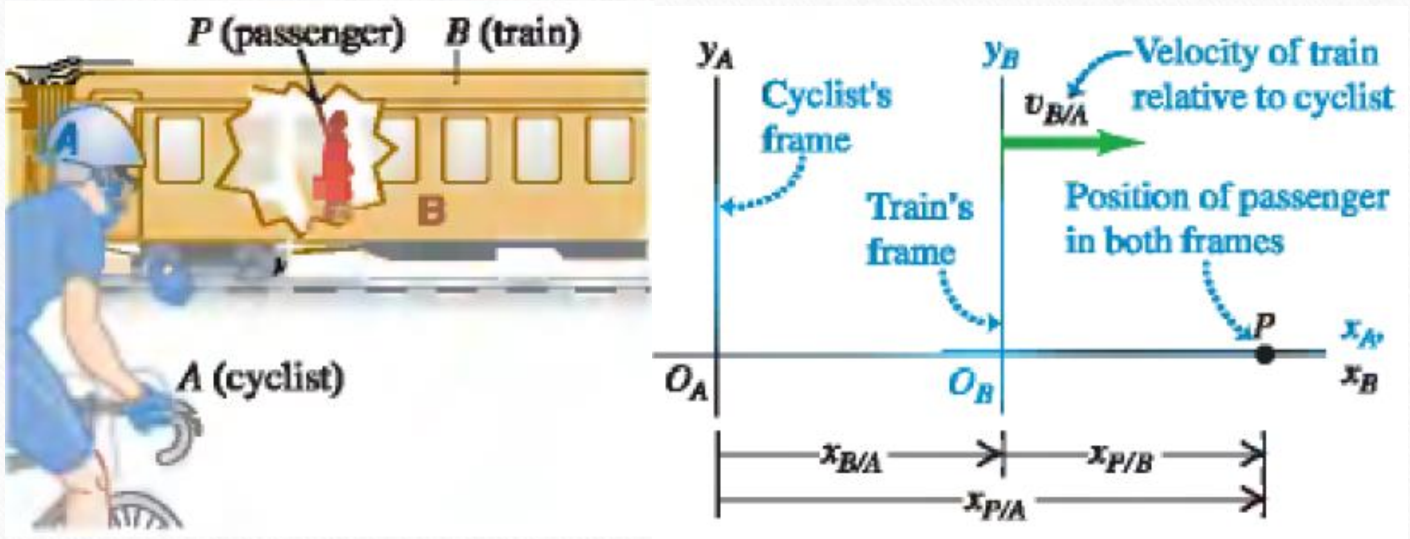
- If $v_{P/B\text{-}x} = +1.0\ \text{m/s}$

  $v_{B/A\text{-}x} = +3.0\ \text{m/s}$

- Then

$$v_{P/A\text{-}x} = +1.0\ \text{m/s} + 3.0\ \text{m/s}$$

$$= +4.0\ \text{m/s}$$



Imagine they are all in one line.

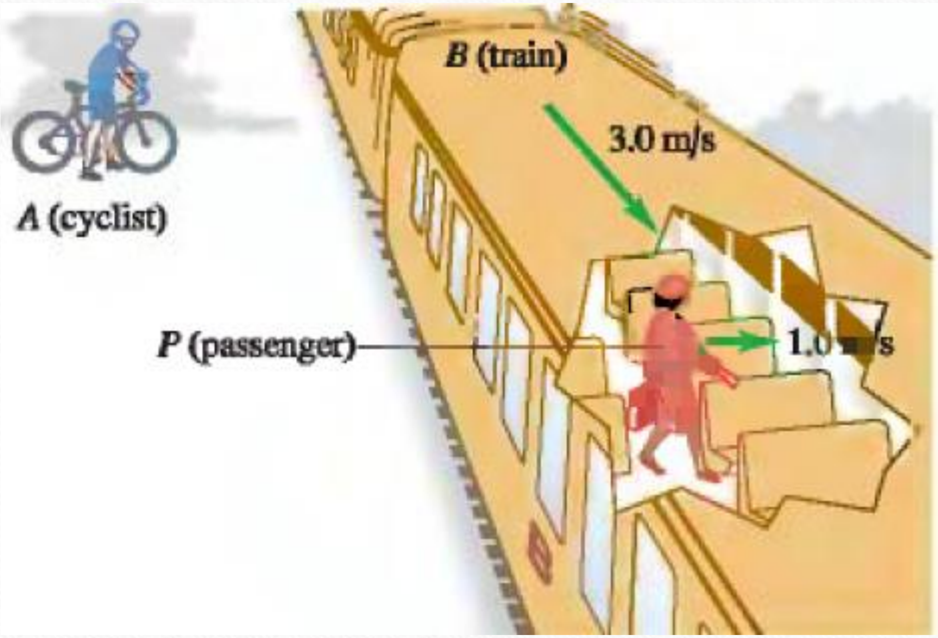
Relative Velocity in 2D or 3D

- Displacement relationship is given by:

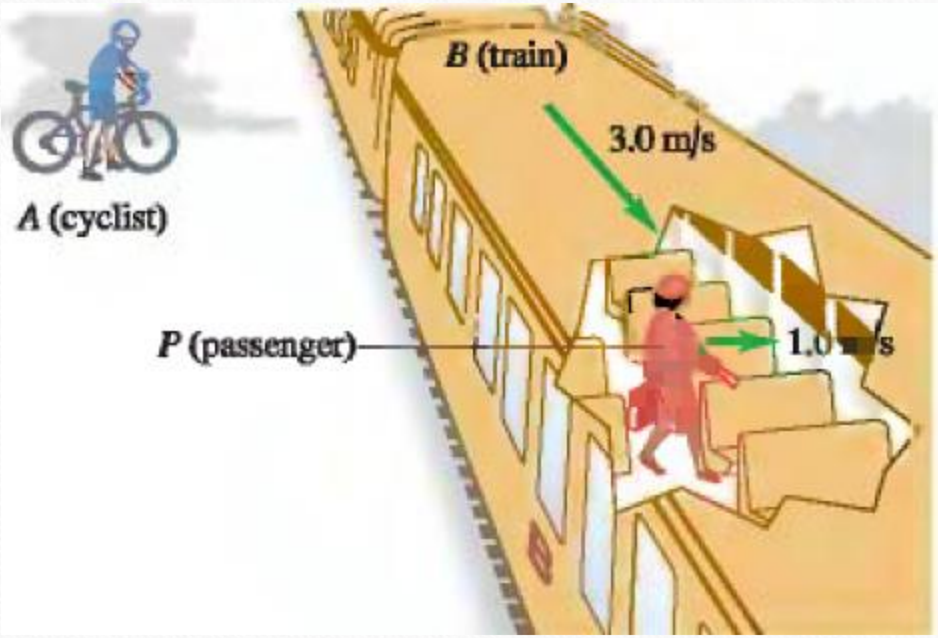
$$\vec{r}_{P/A} = \vec{r}_{P/B} + \vec{r}_{B/A}$$

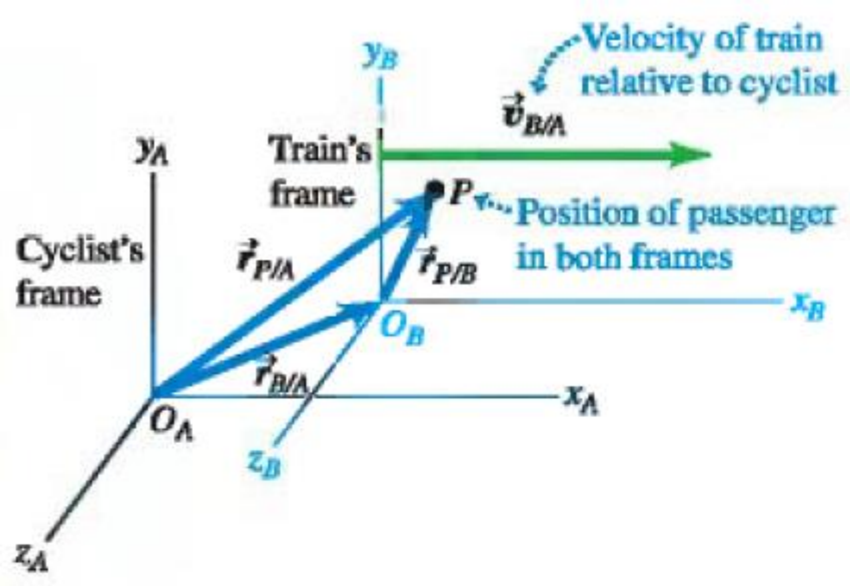
- Therefore, velocity relationship is given by:

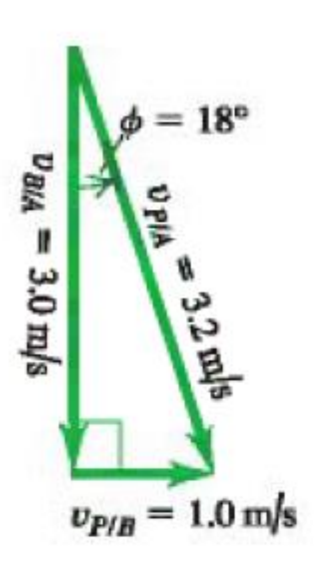
$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$$



Relative Velocity in 2D or 3D







Top view (as seen from above)

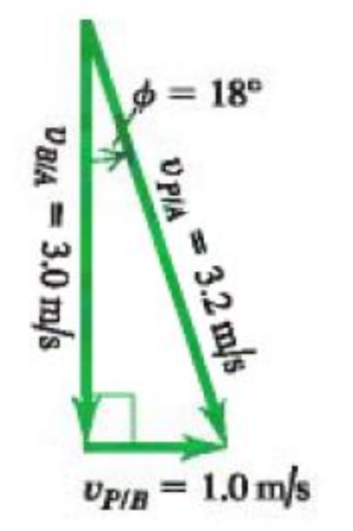
Relative Velocity in 2D or 3D

- To calculate P w.r.t A:

$$v_{P/A} = \sqrt{(3.0\ \text{m/s})^2 + (1.0\ \text{m/s})^2} = \sqrt{10\ \text{m}^2/\text{s}^2} = 3.2\ \text{m/s}$$

- The direction of P's velocity vector (relative to the ground/cyclist) makes an angle with the train's velocity vector:

$$\tan\phi = \frac{v_{P/B}}{v_{B/A}} = \frac{1.0\ \text{m/s}}{3.0\ \text{m/s}} \quad \text{and} \quad \phi = 18°$$



Example: Flying in a Crosswind

The compass of an airplane indicates that it is headed due north, and its airspeed indicator shows that it is moving through the air at 240 km/h. If there is a wind of 100 km/h from west to east, what is the velocity of the airplane relative to the earth?

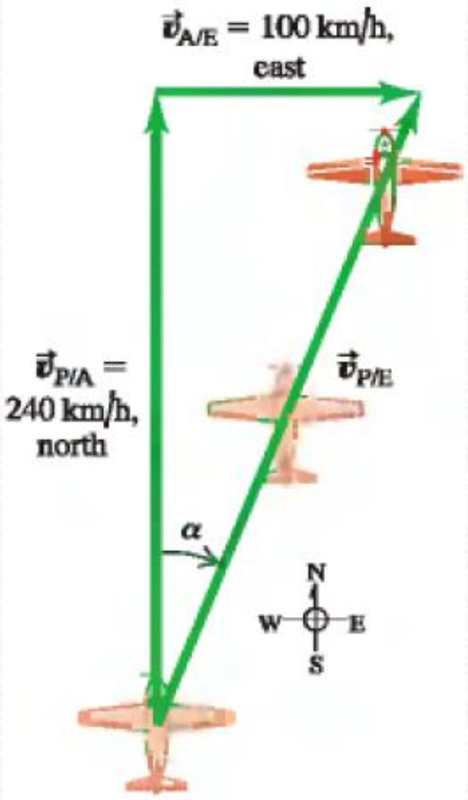
$\vec{v}_{P/A} = 240\ \text{km/h}$ due north

$\vec{v}_{A/E} = 100\ \text{km/h}$ due east

$$\vec{v}_{P/E} = \vec{v}_{P/A} + \vec{v}_{A/E}$$

$$v_{P/E} = \sqrt{(240\ \text{km/h})^2 + (100\ \text{km/h})^2} = 260\ \text{km/h}$$

$$\alpha = \arctan\left(\frac{100\ \text{km/h}}{240\ \text{km/h}}\right) = 23^\circ \text{ E of N}$$



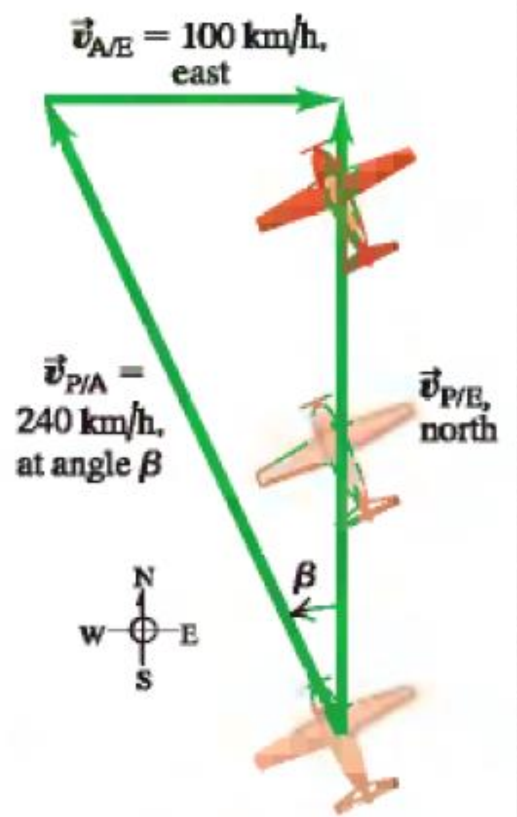
Example: Correcting for Crosswind

Continuing from previous example, in what direction should the pilot head to travel due north? What will be her velocity relative to the earth? Her plane's airspeed and the velocity of the wind are the same as in previous example.

$\vec{v}_{P/E}$	= magnitude unknown	due north
$\vec{v}_{P/A}$	= 240 km/h	direction unknown
$\vec{v}_{A/E}$	= 100 km/h	due east

$$v_{P/E} = \sqrt{(240\ \text{km/h})^2 - (100\ \text{km/h})^2} = 218\ \text{km/h}^2$$

$$\beta = \arcsin\left(\frac{100\ \text{km/h}}{240\ \text{km/h}}\right) = 25°$$



The End