2024 Fall CSD3240/3241/SEM4108 - Probability and Statistics Homework 9

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1. Let X be a geometric random variable with parameter p. Find the maximum likelihood estimator of p based on a random sample of size n.

   A) $\hat{p} = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$
   B) $\hat{p} = n\,(X_1 + X_2 + \cdots + X_n)$
   C) $\hat{p} = \dfrac{n}{X_1 + X_2 + \cdots + X_n}$
   D) $\hat{p} = X_1 + X_2 + \cdots + X_n - n$

2. Let X be random variable with probability density function

   $$f(x) = \theta x^{\theta-1}\,, \qquad 0 < \theta < \infty\,, \quad 0 < x < 1$$

   A random sample of n is taken, find the maximum likelihood estimator for θ.

   A) $\hat{\theta} = -nX_1X_2\cdots X_n$
   B) $\hat{\theta} = -\dfrac{n}{X_1X_2\cdots X_n}$
   C) $\hat{\theta} = \dfrac{1 \pm \sqrt{1 - 4nX_1X_2\cdots X_n}}{2X_1X_2\cdots X_n}$
   D) $\hat{\theta} = -\dfrac{n}{\ln(X_1X_2\cdots X_n)}$

3. Let X be random variable with probability density function

   $$f(x) = \frac{1}{\theta}x^{(1-\theta)/\theta}\,, \qquad 0 < x < 1\,, \quad 0 < \theta < \infty$$

   If the random sample of size n are taken, find the maximum likelihood estimator for θ.

   A) $\hat{\theta} = -\dfrac{1}{n}\ln(X_1)\ln(X_2)\cdots\ln(X_n)$
   B) $\hat{\theta} = -\dfrac{1}{n}\left[\ln(X_1) + \ln(X_2) + \cdots + \ln(X_n)\right]$
   C) $\hat{\theta} = -\dfrac{X_1X_2\cdots X_n}{n}$
   D) $\hat{\theta} = -\dfrac{1}{n}\left[\ln(X_1 + X_2 + \cdots + X_n)\right]$

4. A random variable x has probability density function

   $$f(x) = \frac{1}{2\lambda^3}x^2e^{-x/\lambda}\,, \qquad 0 < x < \infty\,, \quad 0 < \lambda < \infty$$

   If the random sample of size n are selected, find the maximum likelihood estimator for λ.

   A) $\hat{\lambda} = \dfrac{X_1 + X_2 + \cdots + X_n}{3n}$
   B) $\hat{\lambda} = \dfrac{\ln(X_1 + X_2 + \cdots + X_n)}{3n}$
   C) $\hat{\lambda} = \dfrac{\ln(X_1) + \ln(X_2) + \cdots + \ln(X_n)}{3n}$
   D) $\hat{\lambda} = 3n\ln(X_1 + X_2 + \cdots + X_n)$

5. The Rayleigh distribution has probability density function

   $$f(x) = \frac{x}{\theta}e^{-x^2/2\theta}\,, \qquad x > 0, \quad 0 < \theta < \infty$$

   i) Find the maximum likelihood estimator of θ.

   A) $\hat{\theta} = \dfrac{2\left(X_1^2 + X_2^2 + \cdots + X_n^2\right)}{n}$
   B) $\hat{\theta} = \dfrac{X_1^2 + X_2^2 + \cdots + X_n^2}{3n}$
   C) $\hat{\theta} = \dfrac{X_1^2 + X_2^2 + \cdots + X_n^2}{2n}$
   D) $\hat{\theta} = \dfrac{X_1^2 + X_2^2 + \cdots + X_n^2}{n}$

   ii) If the maximum likelihood estimator of $\hat{\theta}$ is an unbiased estimator, find $E(X^2)$.

   A) $E(X^2) = 2\theta$ B) $E(X^2) = \frac{1}{2}\theta$ C) $E(X^2) = \theta$ D) $E(X^2) = 3\theta$

6. Let X denote the proportion of allotted time that a randomly selected student spends working on a certain aptitude test. Suppose the probability density function of X is

$$f(x) = \begin{cases} (\theta+1)x^{\theta}\,, & 0 \le x \le 1 \\ 0\,, & \text{otherwise} \end{cases}$$

where $-1 < \theta$. A random sample of ten students yields data

$$x_1 = 0.92\,,\quad x_2 = 0.79\,,\quad x_3 = 0.90\,,\quad x_4 = 0.65\,,\quad x_5 = 0.86\,,\quad x_6 = 0.47\,,\quad x_7 = 0.73\,,$$
$$x_8 = 0.97\,,\quad x_9 = 0.94\,,\quad x_10 = 0.77$$

i) Find the maximum likelihood estimator of θ,

A) $\hat{\theta} = \dfrac{n}{\sum_{i=1}^{n} X_i} - 1$

B) $\hat{\theta} = -\dfrac{n}{\ln(X)} + 1$

C) $\hat{\theta} = -\dfrac{n}{\Pi_{i=1}^{n} \ln(X_i)} - 1$

D) $\hat{\theta} = -\dfrac{n}{\sum_{i=1}^{n} \ln(X_i)} - 1$

ii) Compute the maximum likelihood estimate for the given data.

A) 3.00 B) 3.12 C) 2.96 D) 2.54

7. For the standard normal distribution, $X \sim N(0,1)$, what is the range of z score for 95% confidence level?

A) $-1.645 < z < 1.645$ B) $-1.96 < z < 1.96$ C) $-2.11 < z < 2.11$ D) $-2.576 < z < 2.576$

8. In a random sample of 3534 participants attending the examination of the Offspring Heart Study it shows a point estimate for the mean of the population having systolic blood pressure is $\hat{X} = 127.3$ and standard deviation $\sigma = 19.0$. Find the 95% confidence interval.
(Hint: For the large sample n, the lower and higher interval limit can be calculated as $\hat{X} \pm \frac{z\sigma}{\sqrt{n}}$)

A) $p_l = 117.7\,, p_h = 120.7$

B) $p_l = 119.7\,, p_h = 122.9$

C) $p_l = 122.7\,, p_h = 125.8$

D) $p_l = 126.7\,, p_h = 127.9$