Week 10

shape/curvedness of graph

SDT

Concavity and the Second Derivative Test

Alternative to FDT

Dr. Ronald Koh
ronald.koh@digipen.edu (Teams preferred over email)

AY 22/23 Trimester 2

Table of contents

Second derivative
curvature of graph

Example 1

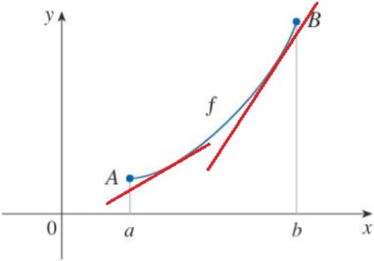
Below are graphs of two different functions f and g, both are **increasing** on $[a, b]$.

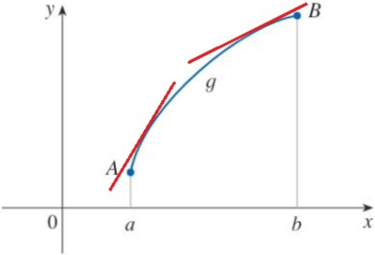
gradient of f is increasing

Notice that both "curve" in different directions. How can we distinguish between these two types of behaviours?

Example 1: Observation

We can observe the **gradient** of these two functions.





Notice that for the graph of f, the gradient f' is **increasing**, while for g, the gradient g' is **decreasing**.

Using the I/D Test on f' and g', we can say that $f''(x) > 0$ and $g''(x) < 0$ on an interval $[a, b]$.

I/D Test

$f''(x) > 0$ on I $\iff$ f' is increasing

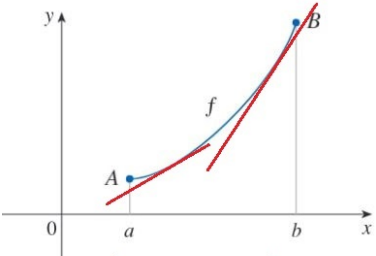
$f''(x) < 0$ on I $\iff$ f' is decreasing

Concavity of graphs definition

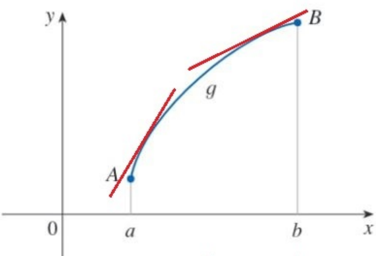
Concavity: The curved-ness (similar to a circle) of a graph.

Definition

1. If a graph of a function f lies **above** all its tangents on an interval I, then f is called **concave upward (CU)** on I.
2. If a graph of a function f lies **below** all its tangents on an interval I, then f is called **concave downward (CD)** on I.



f concave upwards



g concave downwards

Concavity Test

Similar to the I/D Test, we also have a test to check the intervals of CU and CD for a function.

Theorem (Concavity Test)

1. If $f''(x) > 0$ on an interval I, then the graph of f is concave upwards (CU) on I.
2. If $f''(x) < 0$ on an interval I, then the graph of f is concave downwards (CD) on I.

Example 2

similar to finding intervals of increasing or decreasing

Let $f(x) = x^3 - 3x^2 - 9x + 4$.
Find the intervals on which f is CU or CD.

$f'(x) = $ $3x^2 - 6x - 9$.

$f''(x) = $ $6x - 6$.

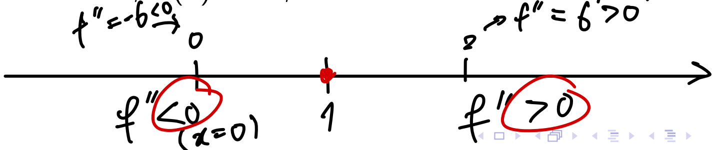
$f''(c) = 0$

inflection point.

$f''(x) = 0 \iff$ $6x - 6 = 0$ $\iff$ $x = 1$.

Therefore, for $x <$ 1, $f'(x)$ $\leq$ 0, thus f is CD on $(-\infty, 1)$.

Also, for $x >$ 1, $f'(x)$ $>$ 0, thus f is CU on $(1, \infty)$.



Exercise 1

Let $f(x) = x^4 - 2x^2 + 3$.
Find the intervals on which f is CU or CD.

$f'(x) = 4x^3 - 4x$

$f''(x) = 12x^2 - 4 = 0$

$\iff 4(3x^2 - 1) = 0$

$\iff 12\left(x^2 - \frac{1}{3}\right) = 0 \iff x = \pm\frac{1}{\sqrt{3}}$

$\left(x + \frac{1}{\sqrt{3}}\right)\left(x - \frac{1}{\sqrt{3}}\right)$

-1 inflection 0 inflection 1

$-\frac{1}{\sqrt{3}}$ $\frac{1}{\sqrt{3}}$

$f'' > 0$ $f'' < 0$ $f'' > 0$.

$\therefore f$ is CU on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$.

f is CD on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

Exercise 2

Let $f(x) = \dfrac{x}{x^2+1}$. Find the intervals on which f is CU or CD.

$x(x^2+1)^{-1}$

$$f'(x) = (x^2+1)^{-1} + x(-1)(x^2+1)^{-2} \cdot 2x.$$

$$= \frac{x^2+1 - 2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

$$= (1-x^2)(x^2+1)^{-2}$$

$$f''(x) = -2x(x^2+1)^{-2} + (1-x^2)(-2)(x^2+1)^{-3} \cdot 2x$$

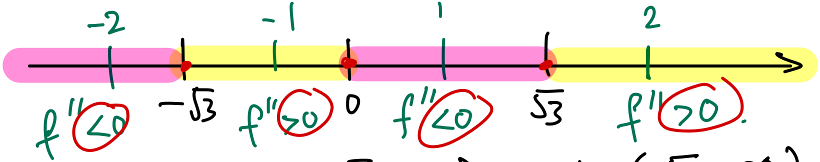
$$= \frac{-2x(x^2+1) - 4x(1-x^2)}{(x^2+1)^3}$$

Exercise 2

$$= \frac{2x\left(-x^2-1-2(1-x^2)\right)}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3} = 0.$$

$$\underset{\substack{\uparrow \\ x=0}}{2x}\,(\underset{\substack{\uparrow \\ x=\pm\sqrt{3}}}{x^2-3}) = 0 \iff \underline{x=0,\ x=-\sqrt{3},\ x=\sqrt{3}.}$$

inflection points.



f is CU on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$,

f is CD on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$.

Inflection points

local extrema / local extreme points

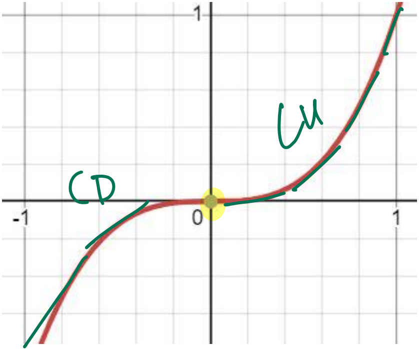
Local maxima and minima occur where f' changes sign.
Where f'' changes sign, i.e. where f changes from CU to CD or vice versa, we call them *inflection points*.

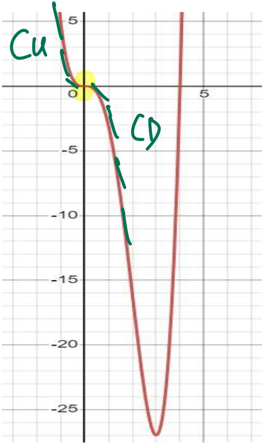
Definition

A point c on a curve $y = f(x)$ is called an **inflection point** if

1. f is continuous at c, and
2. f changes from CU to CD or CD to CU at c.
   Alternatively, f'' changes sign from positive to negative or negative to positive (in view of the Concavity Test).

Inflection point examples





Left: $y = x^3$
Right: $y = x^4 - 4x^3$

DESMOS

Both graphs have inflection points at $x = 0$.

Finding inflection points

f is a local min / local max at c then $f'(c)=0$.

Like Fermat's Theorem in narrowing the amount of points we need to check (critical points) to find local maxima and minima, we also have something similar for inflection points.

Theorem

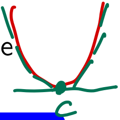
If f is twice differentiable at c and has an inflection point at c, then $f''(c) = 0$.

Exercise 3: Find the inflection points for the functions in Example 2, Exercises 1 and 2.

T

$C=1$

Second Derivative Test



As a result of the Concavity Test, we get the Second Derivative Test.

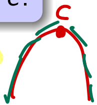
c is a critical point of f.

Theorem (Second Derivative Test)

Suppose f'' is continuous near c.

CU

1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.



The Second Derivative Test serves as an alternative to the First Derivative Test, but has some noticeable drawbacks; when $f''(c) = 0$, the Second Derivative Test is inconclusive. There could be a local maximum there, a local minimum there, or neither (See Examples 4, 5 and 6). **If this happens, we need to fall back to the First Derivative Test.**

what about $f''(c) = 0$?

Example 3

Let $f(x) = \dfrac{x}{x^2+1}$. → Ex 2.

→ $x=-2$ $f'<0$
→ $x=0$ $f'>0$

We know that $f'(x) = \dfrac{1-x^2}{(x^2+1)^2}$, so the critical points are $x = \pm 1$.

We have previously shown that $x = -1$ is a local minimum point, and $x = 1$ is a local maximum point. We can verify this using the Second Derivative Test. → can only use $\underbrace{c=-1,\ c=1}_{\text{critical points.}}$

$$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}$$

$f''(-1)$

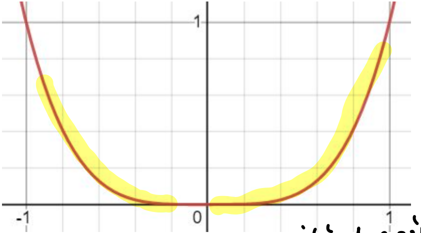
$f''(1)$

$f''(-1)$ $>$ 0, so $x = -1$ is a local Minimum point.

$f''(1)$ $<$ 0, so $x = 1$ is a local Maximum point.

Example 4

Local minimum at c where $f'(c) = 0$ and $f''(0) = 0$:
$f(x) = x^4$ and $c = 0$.



SDT inconclusive

critical point

Here, $f'(x) = 4x^3 = 0 \iff c = 0 \iff$ $f'(c)=0$.
Also, $f''(x) = 12x^2 = 0 \iff c = 0 \iff$ $f''(c)=0$.

If we use the First Derivative Test for sign changes of f' (or observe the graph directly), $c = 0$ is a local minimum point.

Example 5

Local maximum at c where $f'(c) = 0$ and $f''(0) = 0$:
$f(x) = -x^4$ and $c = 0$.

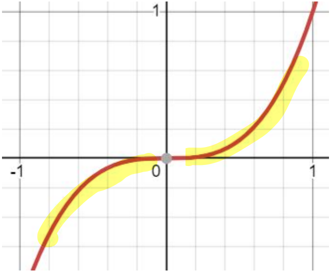
SDT inconclusive

Here, $f'(x) = -4x^3 = 0 \iff c = 0 \iff$ $f'(c)=0$.
Also, $f''(x) = -12x^2 = 0 \iff c = 0 \iff$ $f''(c)=0$.

If we use the First Derivative Test for sign changes of f' (or observe the graph directly), $c = 0$ is a local maximum point.

Example 6

Neither local max/ min at c where $f'(c) = 0$ and $f''(0) = 0$:
$f(x) = x^3$ and $c = 0$.



SDT
inconclusive

Here, $f'(x) = 3x^2 = 0 \iff c = 0 \iff$ $f'(c) = 0$.
Also, $f''(x) = 6x = 0 \iff c = 0 \iff$ $f''(c) = 0$.

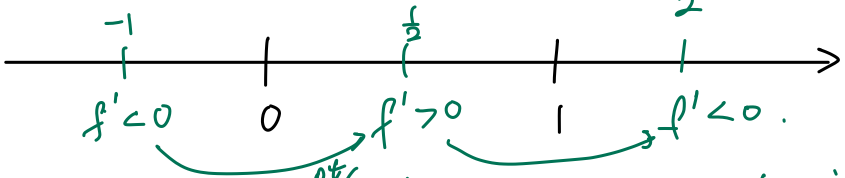
If we use the First Derivative Test for sign changes of f' (or observe the graph directly), $c = 0$ is neither local min or local max pt.

Exercise 3-1

Use the First Derivative Test to find the local extreme point(s) of $f(x) = 1 + 3x^2 - 2x^3$.

$$f'(x) = 6x - 6x^2 = \underset{>0}{6x}\underset{<0}{(1-x)} = 0 \rightarrow \begin{array}{l} x=0 \\ x=1 \end{array}$$

critical points



$\rightarrow x=0$ local min pt (f' change from negative to positive)

$\rightarrow x=1$ local max pt (f' change from positive to negative)

Exercise 3-2

Use the Second Derivative Test to find the local extreme point(s) of $f(x) = 1 + 3x^2 - 2x^3$.

$x=0$ $\quad 6(x-0)(1-x)$

$$f'(x) = 6x - 6x^2 = 6x(1-x) = 0$$

$x=0 \quad x=1$ critical pts

$$f''(x) = 6 - 12x$$

$f''(0) = 6 > 0 \xrightarrow{SDT} x=0$ is local min point

$f''(1) = 6 - 12 = -6 < 0 \xrightarrow{SDT} x=1$ is local max point

Very easy to apply SDT.

Exercise 4-1

Use the First Derivative Test to find the local extreme point(s) of
$f(x) = \frac{x^2}{x-1}$. $= x^2(x-1)^{-1} \leftarrow$ domain $\mathbb{R}\setminus\{1\}$

$$f'(x) = 2x(x-1)^{-1} + x^2(-1)(x-1)^{-2}$$

$$= \frac{2x(x-1) - x^2}{(x-1)^2} = \frac{x^2-2x}{(x-1)^2}$$

$x(x-2)$

Critical points $x=1$, $x=0$, $x=2$

~~$x=1$,~~ idc $f(1)$ not defined

local max pt

local min pt.

-1, 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, 3

$f'>0$ 0 $f'<0$ 1 $f'<0$ 2 $f'>0$

Exercise 4-2

Use the Second Derivative Test to find the local extreme point(s) of $f(x) = \dfrac{x^2}{x-1}$.

Critical points $x=0$, $x=2$

$$f'(x) = \frac{x^2-2x}{(x-1)^2} = (x^2-2x)(x-1)^{-2}$$

$$f''(x) = (2x-2)(x-1)^{-2} + (x^2-2x)(-2)(x-1)^{-3}\cdot 1$$

$$= \frac{(2x-2)(x-1) - 2(x^2-2x)}{(x-1)^3}$$

$$= \frac{2x^2 - 2x - 2x + 2 - 2x^2 + 4x}{(x-1)^3} = \frac{2}{(x-1)^3}$$

$f''(0) = \frac{2}{(-1)^3} = -2 < 0 \therefore x=0$ local max pt

$f''(2) = \frac{2}{1^3} = 2 > 0 \therefore x=2$ local min pt.

When to use FDT or SDT?

To find local extreme points of a function, we have either the First Derivative Test or the Second Derivative Test. Here are some tips:

- If the calculation of the second derivative is tedious/difficult, avoid SDT altogether and just stick to FDT.
- If the calculation of the second derivative is easy, it is usually more efficient to use SDT than FDT, but you also run the risk of running into inflection points (where SDT is inconclusive, from there you have to fall back to FDT).
- Experience (do more problems!) will help you determine which test to use quicker.