Question 1

(a) $f(x) = 12 + 4x - x^2 \quad [0,5]$

$f'(x) = 4 - 2x = 0 \quad x=2 \quad$ critical $\underline{x=2}$.

$f(0) = 12$.

$f(2) = 12 + 2\times4 - 2^2 = 12 + 8 - 4 = 16$. global max.

$f(5) = 12 + 20 - 25 = 12 - 5 = 7$ global min.

(b) $f(x) = x^3 - 6x^2 + 5 \quad [-3,5]$

$f'(x) = 3x^2 - 12x = 0 \Rightarrow x(3x-12) = 0 \Rightarrow x = 0, 4$.

$f(-3) = (-3)^3 - 6(9) + 5 = -76$ .. smallest global min.

$f(0) = 5$ .... largest global max

$f(4) = 4^3 - 6(4)^2 + 5 = -27$

$f(5) = 5^3 - 6(5^2) + 5 = -20$.

(c) $f(x) = x + \frac{1}{x} \quad [0.2, 4]$

$f'(x) = 1 - \frac{1}{x^2} = 0 \quad x = \pm 1$. $\quad x = -1 \notin [0.2, 4]$.

$f(0.2) = 0.2 + 5 = 5.2$ .... largest global max

$f(1) = 1 + 1 = 2$ ..... smallest global min.

$f(4) = 4 + \frac{1}{4} = 4.25$

(d) $f(x) = \frac{x}{x^2 - x + 1} \quad [0,3]$

$$f'(x) = \frac{(x^2-x+1) - x(2x-1)}{(x^2-x+1)^2} = \frac{x^2-x+1-2x^2+x}{(x^2-x+1)^2} = \frac{-x^2+1}{(x^2-x+1)^2} = 0$$

$-x^2 + 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$f(0) = 0$ .... smallest. global min.

$f(1) = 1$ .... largest global max

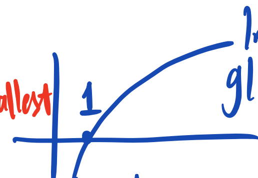
$f(3) = \frac{3}{9-3+1} = \frac{3}{7}$

(e). $f(x) = \ln(x^2+x+1)$, $[-1,1]$

$f'(x) = \frac{1}{x^2+x+1}\cdot(2x+1) = 0.$ $\Rightarrow$ $2x+1=0 \Rightarrow x = -\frac{1}{2}.$

$f(-1) = \ln(1-1+1) = \ln(1) = 0$

$f(-\frac{1}{2}) = \ln(\frac{1}{4}-\frac{1}{2}+1) = \ln(\frac{3}{4}) < 0$ ... smallest



global min.

$f(1) = \ln(3) > 0$ ...... largest   global max

(f) $f(x) = \frac{\ln x}{x^2}$ $\boxed{[\frac{1}{2}, 4]}$

$\boxed{e^{\ln x}} = e^{\frac{1}{2}}.$
$x = e^{\frac{1}{2}}.$

$f'(x) = \frac{\frac{1}{x}\cdot x^2 - \ln x \cdot 2x}{x^4} = \frac{x - 2x\ln x}{x^4} = 0.$

$x = 2x\ln x \Rightarrow 1 = 2\ln x \Rightarrow \ln x = \frac{1}{2} \Rightarrow x = e^{\frac{1}{2}} = 1.6487$

$f(\frac{1}{2}) = \frac{\ln(\frac{1}{2})}{\frac{1}{4}} = -2.77$ .... smallest   global min

$f(e^{\frac{1}{2}}) = \frac{\frac{1}{2}}{e} = 0.1839$ ... largest   global max

$f(4) = \frac{\ln 4}{16} = 0.0866$

Question 2

$C(t) = 0.135t\,e^{-2.802t}.$ $[0,3]$

$C'(t) = 0.135e^{-2.802t} + (0.135t)(-2.802)e^{-2.802t}.$

$= \underbrace{e^{-2.802t}}_{\neq 0}(0.135 - 0.3727t) = 0$

$t = \frac{0.135}{0.3727} \approx 0.3622.$

$C(0) = 0$

$C(0.3622) = (0.135)(0.3622)e^{-2.802\cdot 0.3622} \approx 0.0177$

$C(3) = 0.0000905$

Question 3.

$$C(t) = 8(e^{-0.4t} - e^{-0.6t}) \quad [0, 12]$$

$$C'(t) = 8(-0.4e^{-0.4t} + 0.6e^{-0.6t}) = 0.$$

$$0.4e^{-0.4t} = 0.6e^{-0.6t}$$

$$\frac{e^{-0.4t}}{e^{-0.6t}} = \frac{0.6}{0.4}$$

$$e^{-0.4t+0.6t} = \frac{3}{2}.$$

$$\ln e^{0.2t} = \ln\frac{3}{2}. \Rightarrow 0.2t = \ln\frac{3}{2}$$

$$t = 5\ln\frac{3}{2}. \approx 2.0273.$$

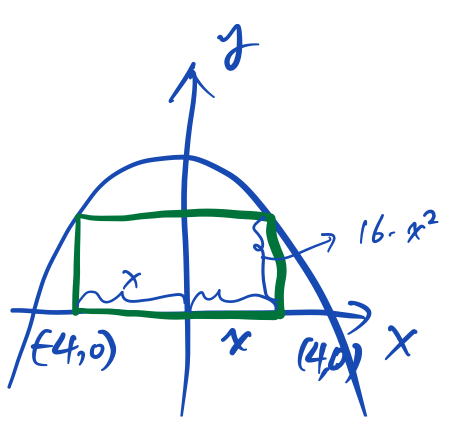
$$C(0) = 8(e^0 - e^0) = 0$$

$$C(2.0273) = 8(e^{-0.4\cdot 2.0273} - e^{-0.6\cdot 2.0273}) = 1.18519 \text{ largest}$$

$$C(12) = 8(e^{-0.4\cdot 12} - e^{-0.6\cdot 12}) = 0.05986.$$

The max concentration of the antibiotic during the first 12 hours is 1.18519 mg/mL

Question 4:



$y = 16 - x^2$

$y = 0 \Rightarrow 16 - x^2 = 0$

$x = \pm 4.$

Assume one of the vertices on the base is $(x, 0)$.
The base of the rectangle is $2x$, and the height of the rectangle is $16 - x^2$. Then the area of the rectangle is $A = 2x(16 - x^2)$ $[0, 4]$

$$A = 32x - 2x^3$$

$$A' = 32 - 6x^2 = 0.$$

$$x^2 = \frac{32}{6} \qquad x = \pm\sqrt{\frac{32}{6}} = \pm\sqrt{\frac{16}{3}} = \pm\frac{4}{\sqrt{3}}.$$

$$x = \frac{4}{\sqrt{3}}. \qquad y = 16 - x^2 = 16 - \frac{16}{3} = \frac{32}{3}$$

$$\begin{cases} A\left(\frac{4}{\sqrt{3}}\right) = \frac{32}{3} \cdot \frac{4}{\sqrt{3}} \cdot 2 = \frac{256}{3\sqrt{3}} \to \text{largest} \\ A(0) = 0 \\ A(4) = 0 \end{cases}$$

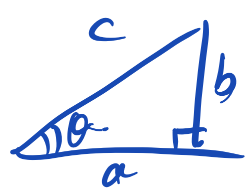
global max

Question 5

Let $u = \tan\theta$, $u > 0$. $\theta \in (0, \frac{\pi}{2})$ 1st
$\theta \in (\pi, \frac{3}{2}\pi)$.

(a)

$$\sin\theta = \frac{u}{\sqrt{u^2+1}}.$$



$$\text{RHS} = \frac{\tan\theta}{\sqrt{\tan^2\theta+1}} = \frac{\frac{b}{a}}{\frac{c}{a}}$$

$$= \frac{b}{a} \cdot \frac{a}{c} = \frac{b}{c} = \sin\theta$$

$$= \text{LHS}.$$

$$\tan\theta = \frac{b}{a}.$$

$$\tan^2\theta = \frac{b^2}{a^2}.$$

$$\tan^2\theta + 1 = \frac{b^2}{a^2} + 1$$

$= \frac{b^2+a^2}{a^2} = \frac{c^2}{a^2}$.

(b) Show that:

$$\frac{\mu}{\sqrt{\mu^2+1}} \le 1 \quad \text{and} \quad \frac{\mu}{\sqrt{\mu^2+1}} \le \mu.$$

$\sin\theta$ will never exceed "1"

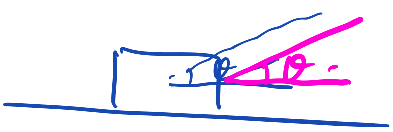
$$\frac{\mu}{\sqrt{\mu^2+1}} \le \frac{\mu}{1} \leftarrow$$

$$\sqrt{\mu^2+1} \ge 1 \leftarrow$$

$$\sqrt{\mu^2+1} \ge \sqrt{1}$$

$$\mu^2+1 \ge 1. \checkmark$$

Question 6



$$F(\theta) = \frac{\mu W}{\mu\sin\theta + \cos\theta}.$$

μ: coefficients of friction.

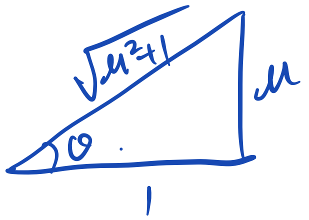
$$F'(\theta) = \frac{-\mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2} = 0.$$

$$\mu\cos\theta = \sin\theta \Rightarrow \mu = \tan\theta \Rightarrow \theta = \arctan\mu$$

$$F(0) = \frac{\mu W}{1} = \boxed{\mu W}$$

$$F(\tfrac{\pi}{2}) = \frac{\mu W}{\mu} = \boxed{W}$$

$$F(\arctan\mu) = \frac{\mu W}{\mu \sin(\arctan\mu) + \cos(\arctan\mu)}$$



$$= \frac{\mu W}{\mu \cdot \underbrace{\boxed{\frac{\mu}{\sqrt{\mu^2+1}}}}_{\le 1} + \frac{1}{\sqrt{\mu^2+1}}} = \frac{\mu W}{\frac{\mu^2+1}{\sqrt{\mu^2+1}}}$$

$$= \frac{\mu W}{\sqrt{\mu^2+1}} \le W \cdot \frac{\mu}{\sqrt{\mu^2+1}} \le W\mu .$$

F is minimized when $\tan\theta = \mu$.