CSD1241 Tutorial 7

Remarks. The solution should only be used as guidance for your study. There is no guarantee on errors and typos. Would appreciate if you let me know the errors.

Problem 1. Let $a, b, c \in \mathbb{R}$ be constants and let $\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ be a vector. Define the cross-product map $T_{\vec{u}} : \mathbb{R}^3 \to \mathbb{R}^3$ as follows

$$T_{\vec{u}}(\vec{x}) = \vec{u} \times \vec{x}$$

(a) Is T linear? Justify your answer.
(b) If T is linear, write out its matrix.

Solution. For $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, we have

$$T_{\vec{u}}(\vec{x}) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} bz - cy \\ cx - az \\ ay - bx \end{pmatrix}.$$

Since each component in $T_{\vec{u}}(\vec{x})$ is a linear combination of x, y, z, the map $T_{\vec{u}}$ is linear. Further, its matrix is

$$M = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}.$$

□

Problem 2. Let $a, b > 0$. In this exercise, we learn that the scaling $S : \mathbb{R}^2 \to \mathbb{R}^2$

$$S \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix}$$

scales the area of a region in $\mathbb{R}^2$ by the factor ab.
(a) What is the matrix representation of S?
(b) Find the image $A'B'C'$ of the triangle ABC with

$$A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \; C = \begin{pmatrix} 7 \\ 10 \end{pmatrix}$$

(c) Compare the areas of the triangles $\triangle ABC$ and $\triangle A'B'C'$.

Solution. (a) The matrix of S is $M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$.

(b) The points A', B', C' are

$$\begin{aligned} A' &= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ B' &= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} a \\ 4b \end{bmatrix} \\ C' &= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 7 \\ 10 \end{bmatrix} = \begin{bmatrix} 7a \\ 10b \end{bmatrix} \end{aligned}$$

(c) The areas of ABC and $A'B'C'$ are

$$\begin{aligned} \text{Area}(ABC) &= \frac{1}{2} \left| \overrightarrow{AB}\ \overrightarrow{AC} \right| = \frac{1}{2} \begin{vmatrix} 1 & 7 \\ 4 & 10 \end{vmatrix} = 9 \\ \text{Area}(A'B'C') &= \frac{1}{2} \left| \overrightarrow{A'B'}\ \overrightarrow{A'C'} \right| = \frac{1}{2} \begin{vmatrix} a & 7a \\ 4b & 10b \end{vmatrix} = 9ab \end{aligned}$$

Therefore, the area of $\triangle A'B'C'$ differs from the area of $\triangle ABC$ by the factor ab. $\square$

Problem 3. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the counter-clockwise rotation around O over the angle $\theta = 120^0$.
(a) Find the matrix representation of R.
(b) Find the images of the points $\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$.
(c) Find the image of the line $m : x - \sqrt{3}y = 0$ under T (find general equation).
(d) Find the image of the line $n : y = 2$ under T (find general equation).

Solution. (a) The matrix of R is

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 120^0 & -\sin 120^0 \\ \sin 120^0 & \cos 120^0 \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

(b) The images of $\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$ are

$$\begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

(c) The line $m : x - \sqrt{3}y = 0$ has vector equation $\vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$. The image of m is

$$m' : \vec{x} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix},$$

which is a line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and having normal vector $\begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$. The normal equation to m' is

$$-x + \sqrt{3}y = 0.$$

(d) The line $n : y = 2$ $(0x + 1y = 2)$ has vector equation $\vec{x} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. The image of n is

$$m' : \vec{x} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \left(\begin{pmatrix} 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} -\sqrt{3} \\ -1 \end{pmatrix} + \frac{t}{2} \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix},$$

which is a line through $\begin{pmatrix} -\sqrt{3} \\ -1 \end{pmatrix}$ and having normal vector $\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$. The normal equation to m' is

$$\sqrt{3}(x + \sqrt{3}) + 1(y + 1) = 0 \Leftrightarrow \sqrt{3}x + y = -4$$

$\square$

Problem 4. In this problem, we will learn that a **shear** not only transforms a square into a parallelogram, it also preserves the area of the square.

Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the shear with respect to $l : 2x - 5y = 0$ in the direction of $\vec{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

(a) Find the matrix representation of S.

(b) Find the image $A'B'C'D'$ (under S) of the unit square $ABCD$ with

$$A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, D = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(c) Verify that $A'B'C'D'$ is a parallelogram, that is, $\vec{A'B'} = \vec{D'C'}$. Further, verify that the area of $A'B'C'D'$ is equal to 1 (the same as the area of $ABCD$).

Solution. (a) Note that l has normal vector $\vec{n} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$. The matrix of S is

$$\begin{aligned} M &= I_2 + \frac{1}{||\vec{n}||}\vec{v}\vec{n}^T = I_2 + \frac{1}{\sqrt{29}}\begin{pmatrix} 5 \\ 2 \end{pmatrix}\begin{pmatrix} 2 & -5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\sqrt{29}}\begin{pmatrix} 10 & -25 \\ 4 & -10 \end{pmatrix} \\ &= \frac{1}{\sqrt{29}}\begin{pmatrix} \sqrt{29}+10 & -25 \\ 4 & \sqrt{29}-10 \end{pmatrix} \end{aligned}$$

(b) Note that A is on the line l, so it is fixed by the sheer, that is, $A' = A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The images of A, B, C, D are

$$\begin{aligned} B' &= \frac{1}{\sqrt{29}}\begin{pmatrix} \sqrt{29}+10 & -25 \\ 4 & \sqrt{29}-10 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} (\sqrt{29}+10)/\sqrt{29} \\ 4/\sqrt{29} \end{pmatrix} \\ C' &= \frac{1}{\sqrt{29}}\begin{pmatrix} \sqrt{29}+10 & -25 \\ 4 & \sqrt{29}-10 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} (\sqrt{29}-15)/\sqrt{29} \\ (\sqrt{29}-6)/\sqrt{29} \end{pmatrix} \\ D' &= \frac{1}{\sqrt{29}}\begin{pmatrix} \sqrt{29}+10 & -25 \\ 4 & \sqrt{29}-10 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -25/\sqrt{29} \\ (\sqrt{29}-10)/\sqrt{29} \end{pmatrix} \end{aligned}$$

(c) We have

$$\overrightarrow{D'C'} = C' - D' = \begin{pmatrix} (\sqrt{29}-15)/\sqrt{29} \\ (\sqrt{29}-6)/\sqrt{29} \end{pmatrix} - \begin{pmatrix} -25/\sqrt{29} \\ (\sqrt{29}-10)/\sqrt{29} \end{pmatrix} = \begin{pmatrix} (\sqrt{29}+10)/\sqrt{29} \\ 4/\sqrt{29} \end{pmatrix} = \overrightarrow{A'B'}$$

Thus, $A'B'C'D'$ is a parallelogram.

(d) The area of $A'B'C'D'$ is

$$\begin{aligned} \left|\det\begin{pmatrix} \overrightarrow{A'B'} & \overrightarrow{A'C'} \end{pmatrix}\right| &= \left|\det\begin{pmatrix} (\sqrt{29}+10)/\sqrt{29} & (\sqrt{29}-15)/\sqrt{29} \\ 4/\sqrt{29} & (\sqrt{29}-6)/\sqrt{29} \end{pmatrix}\right| \\ &= \left|\frac{(\sqrt{29}+10)(\sqrt{29}-6)}{29} - \frac{4(\sqrt{29}-15)}{29}\right| \\ &= \left|\frac{-31+4\sqrt{29}}{29} - \frac{4\sqrt{29}-60}{29}\right| = 1 \end{aligned}$$

$\square$

In the last problem, we study **composition of linear transformations**.
Let $\mathbb{R}^m \xrightarrow{S} \mathbb{R}^n \xrightarrow{T} \mathbb{R}^k$ be a sequence of linear transformations. The composition $T \circ S : \mathbb{R}^m \to \mathbb{R}^k$ is another linear transformation defined by

$$T \circ S(\vec{x}) = T(S(\vec{x}))$$

Further, if M_T, M_S are matrices of T, S, then the matrix of $T \circ S$ is

$$M_{T\circ S} = M_T M_S.$$

Problem 5. Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto the line $l : \sqrt{3}x - y = 0$ and let R be the reflection through the line $m : x - \sqrt{3}y = 0$.
(a) Find the matrices of $M_P, M_R, M_{P\circ R}, M_{R\circ P}$ of $P, R, P \circ R, R \circ P$.
(b) Describe $P \circ R$ and $R \circ P$, that is, find $P \circ R \begin{pmatrix} x \\ y \end{pmatrix}$ and $R \circ P \begin{pmatrix} x \\ y \end{pmatrix}$.
(c) Find the points which are fixed by $P \circ R$.

Solution. (a) The lines l and m have direction vectors $\vec{d_l} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$ and $\vec{d_m} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$. The matrices of P and R are

$$\begin{aligned} M_P &= \frac{1}{||\vec{d_l}||^2}\vec{d_l}\vec{d_l}^T = \frac{1}{4}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}\begin{pmatrix} 1 & \sqrt{3} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix} \\ M_R &= \frac{2}{||\vec{d_m}||^2}\vec{d_m}\vec{d_m}^T - I_2 = \frac{2}{4}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} - I_2 \\ &= \frac{1}{2}\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \end{aligned}$$

The matrices of $P \circ R$ and $R \circ P$ are

$$\begin{aligned} M_{P\circ R} &= M_P M_R = \frac{1}{8}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ \sqrt{3}/2 & 0 \end{pmatrix} \\ M_{R\circ P} &= M_R M_P = \frac{1}{8}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

(b) The composition $P \circ R$ is map $P \circ R : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$P \circ R \begin{pmatrix} x \\ y \end{pmatrix} = M_{P\circ R}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ \sqrt{3}/2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/2 \\ \sqrt{3}x/2 \end{pmatrix}$$

Similarly, the composition $R \circ P$ is map $R \circ P : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$R \circ P \begin{pmatrix} x \\ y \end{pmatrix} = M_{R\circ P}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (x + \sqrt{3}y)/2 \\ 0 \end{pmatrix}$$

(c) Assume that $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ is a fixed point of $P \circ R$. We have

$$P \circ R \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} x/2 \\ \sqrt{3}x/2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{cases} x/2 = x \\ \sqrt{3}x/2 = y \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

Therefore, the only fixed point of $P \circ R$ is the origin $\mathrm{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.