TUTORIAL 4 Solutions

1) (a) The following argument form is invalid (converse error)

$$p \to q$$
$$q$$
$$\therefore p$$

The 1st critical row shows that the argument is invalid.

conclusion p	premise 2 q	premise 1 $p \to q$	
0	0	1	
0	1	1	critical row 1
1	0	0	
1	1	1	critical row 2

(b) The following argument form is invalid (inverse error)

$$p \to q$$
$$\neg p$$
$$\therefore \neg q$$

The 1st critical row shows that the argument is invalid.

p	q	premise 1 $\neg p$	premise 1 $p \to q$	conclusion $\neg q$
0	0	1	1	1
0	1	1	1	0
1	0	0	0	1
1	1	0	1	0

c) $p \vee q$

$p \to r$

$q \to r$

$\therefore r$

Since the conclusion is true in all critical rows, the argument is valid.

p	q	r	$p \vee q$	$p \to r$	$q \to r$
0	0	0	0	1	1
0	0	1	0	1	1
0	1	0	1	1	0
0	1	1	1	1	1
1	0	0	1	0	1
1	0	1	1	1	1
1	1	0	1	0	0
1	1	1	1	1	1

2) Prove that the following argument form is valid

(a) $\neg p \to r \wedge \neg s$

(b) $t \to s$

(c) $u \to \neg p$

(d) $\neg w$

(e) $u \vee w$

given to be true

(f) $\therefore \neg t$

prove that this is true

Step 1:

$u \to \neg p$ (a)

$\neg p \to r \wedge \neg s$ (c)

$\therefore u \to r \wedge \neg s$ (transitivity)

Step 2:

$u \vee w$ (e)

$\neg w$ (d)

$\therefore u$ (elimination)

Step 3:

$u \to r \wedge \neg s$ (step 1)

u (step 2)

$\therefore r \wedge \neg s$ (modus ponens)

$\therefore \neg s$ (specialization)

Step 4:

$t \to s$ (b)

$\neg s$ (step 3)

$\therefore \neg t$ (modus tollens)

3/ Consider following predicates on domain $\mathbb{Z}$

$$\begin{aligned} O(x) &: \quad x \text{ is odd,} \\ P(x) &: \quad x \text{ is prime,} \\ S(x) &: \quad x \text{ is a perfect square.} \end{aligned}$$

(a) There exists an integer which is a prime and not an odd number.

$$\exists x \, (P(x) \wedge \neg O(x))$$

(b) If an integer is a prime, then it is not a perfect square.

$$\forall x \, (P(x) \rightarrow \neg S(x))$$

(c) There are integers which are both odd numbers and perfect squares.

$$\exists x \, (O(x) \wedge S(x))$$

4) (a) In the lectures, we studied that

$$\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$$

Give a counterexample to show that this relation hold for the operator $\lor$, that is,

$$\forall x(P(x) \lor Q(x)) \not\equiv \forall x P(x) \lor \forall x Q(x).$$

Domain $\mathbb{Z}$, $P(x): x \geq 0$, $Q(x): x < 0$

$\forall x (P(x) \lor Q(x))$ is always true $\Rightarrow \forall x (P(x) \lor Q(x)) = 1$

$\forall x P(x)$ is false $\Rightarrow \forall x P(x) = 0$

$\forall x Q(x)$ is false $\Rightarrow \forall x Q(x) = 0$

$\Big\}\ \forall x P(x) \lor \forall x Q(x) = 0$

$\therefore\ \forall x (P(x) \lor Q(x)) \not\equiv \forall x P(x) \lor \forall x Q(x)$

c) Give a counter example to show that

$$\exists x (P(x) \land Q(x)) \not\equiv \exists x P(x) \land \exists x Q(x)$$

Domain $\mathbb{Z}$, $P(x): x \geq 0$, $Q(x): x < 0$

$\exists x (P(x) \land Q(x))$ is false $\rightarrow \exists x (P(x) \land Q(x)) = 0$

$\exists x P(x)$ is true $\Rightarrow \exists x P(x) = 1$

$\exists x Q(x)$ is true $\Rightarrow \exists Q(x) = 1$

$\Big\}\ \exists x P(x) \land \exists x Q(x) = 1$

$\therefore\ \exists x (P(x) \land Q(x)) \not\equiv \exists x P(x) \land \exists x Q(x).$

b) Prove $\exists x\,(P(x) \vee Q(x)) \equiv \exists x\, P(x) \vee \exists x\, Q(x)$

Need to prove 2 things

1. If $\exists x\,(P(x) \vee Q(x))$ is true, then $\exists x\, P(x) \vee \exists x\, Q(x)$ is true.

$\exists x\,(P(x) \vee Q(x))$ is true

$\therefore$ $P(a) \vee Q(a)$ is true for some $a \in$ Domain

Subcase 1: $P(a)$ is true $\Rightarrow$ $\exists x\, P(x)$ is true

$\therefore$ $\exists x\, P(x) \vee \exists x\, Q(x)$ is true

Subcase 2: $Q(a)$ is true $\Rightarrow$ $\exists x\, Q(x)$ is true

$\therefore$ $\exists x\, P(x) \vee \exists x\, Q(x)$ is true

2. If $\exists x\, P(x) \vee \exists x\, Q(x)$ is true, then $\exists x\,(P(x) \vee Q(x))$ is true.

$\exists x\, P(x) \vee \exists x\, Q(x) = 1 \Rightarrow \left[\begin{array}{l} \exists x\, P(x) = 1 \\ \exists x\, Q(x) = 1 \end{array}\right.$

(true means equal 1)

Subcase 1: $\exists x\, P(x) = 1 \Rightarrow P(a) = 1$ for some $a \in$ Domain

$\therefore$ $P(a) \vee Q(a) = 1$ for some $a \in$ Domain

$\therefore$ $\exists x\,(P(x) \vee Q(x)) = 1$

Subcase 2: $\exists x\, Q(x) = 1 \Rightarrow Q(a) = 1$ for some $a \in$ Domain

$\therefore$ $P(a) \vee Q(a) = 1$ for some $a \in$ Domain

$\therefore$ $\exists x\,(P(x) \vee Q(x)) = 1$

5)

1. It is not raining or Yvette has her umbrella
2. Yvette does not have her umbrella or she does not get wet
3. It is raining or Yvette does not get wet

∴ Yvette does not get wet.

p: It is raining

q: Yvette has her umbrella

r: She gets wet

(1) $\neg p \vee q$

(2) $\neg q \vee \neg r$

(3) $p \vee \neg r$

$\therefore \neg r$

<u>Solution 1</u>

<u>Step 1</u>: $\neg r \vee \neg q$ (2)

$\neg r \vee p$ (3)

$\therefore (\neg r \vee p) \wedge (\neg r \vee \neg q)$

$\therefore \neg r \vee (p \wedge \neg q)$ (distributivity)

<u>Step 2</u>: $\neg p \vee q$ (1)

$\neg p \vee q \equiv \neg(p \wedge \neg q)$ (D'M)

$\therefore \neg(p \wedge \neg q)$

<u>Step 3</u>: $\neg r \vee (p \wedge \neg q)$ (Step 1)

$\neg(p \wedge \neg q)$ (Step 2)

$\therefore \neg r$ (elimination)

<u>Solution 2</u>: By (3)

$p \vee \neg r = 1 \Rightarrow$ $\neg r = 1$ or $p = 1$

<u>Case 1</u>: $\neg r = 1 \rightarrow$ done

<u>Case 2</u>: $p = 1$

$\underset{(0)}{\neg p} \vee q = 1 \Rightarrow q = 1$

$\underset{(0)}{\neg q} \vee \neg r = 1 \Rightarrow \neg r = 1$

$\therefore \neg r = 1$ in any case.

6) Domain = all English text

$P(x)$: x is a clear explanation

$Q(x)$: x is satisfactory

$R(x)$: x is an excuse

(a) All clear explanations are satisfactory.
(b) Some excuses are unsatisfactory.
(c) Some excuses are not clear explanations.
(d) Show that (c) follow from (a) and (b).

a) $\forall x\,(P(x) \to Q(x))$

b) $\exists x\,(R(x) \land \neg Q(x))$

c) $\exists x\,(R(x) \land \neg P(x))$

d) $\forall x\,(P(x) \to Q(x))$ (a)

$\exists x\,(R(x) \land \neg Q(x))$ (b)

$\therefore \exists x\,(R(x) \land \neg P(x))$

<u>Step 1</u>: $\exists x\,(R(x) \land \neg Q(x))$

$\therefore R(a) \land \neg Q(a)$ for some $a \in$ Domain

$\therefore R(a)$ (1)

$\therefore \neg Q(a)$

<u>Step 2</u>: $\forall x\,(P(x) \to Q(x))$

$\therefore P(a) \to Q(a)$

$\neg Q(a)$ (step 1)

$\therefore \neg P(a)$ (2)

<u>Step 3</u>: $R(a)$ (by (1))

$\neg P(a)$ (by (2))

$\therefore R(a) \land \neg P(a)$

$\therefore \exists x\,(R(x) \land \neg Q(x))$