Fundamentals of Differentiation Part 1

Dr. Ronald Koh
ronald.koh@digipen.edu (Teams preferred over email)

AY 22/23 Trimester 2

Table of contents

Slope/gradient of a straight line

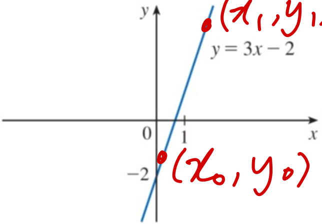
Let f be a linear function, where the graph of f is a straight line

$$f(x) = mx + c,$$

$y = mx + c$

where m is the slope/gradient and c is the y-intercept.

Example: The graph of $f(x) = 3x - 2$ can be found below.



(x_1, y_1)

$m = \dfrac{y_1 - y_0}{x_1 - x_0}$

(x_0, y_0)

Recall: How do we find $m = 3$ here?

Slope/gradient of a straight line

The constant m for a linear function may be found by picking out *any* two points $(x_0, y_0), (x_1, y_1)$ and computing the following quantity:

$$m = \frac{y_1 - y_0}{x_1 - x_0}.$$

For the example above, we can pick two "easy" points $(x_0, y_0) = (0, -2)$ and $(x_1, y_1) = (1, 1)$ and so

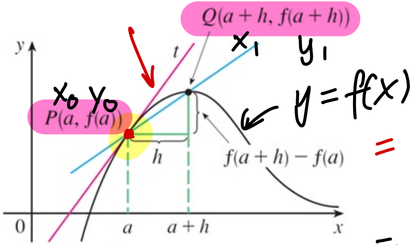
$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{1 - (-2)}{1 - 0} = 3.$$

The reason that this calculation works for *any* two points on the straight line is that the gradient of the line at any point is constant.

Question: How do we find the slope/gradient of a generic function $y = f(x)$ at a point a?

Visualization

We can make use of what we know about the gradient of a line.



gradient of blue line

$$= \frac{f(a+h) - f(a)}{a+h-a} = \frac{f(a+h) - f(a)}{h}$$

1. Suppose the line in black is the graph of the function $y = f(x)$.

   Important: The gradient of the magenta line is the gradient of the function f at $x = a$. The point $(a, f(a))$ on both of these graphs is labelled P.

   Thus, we want to find the gradient of this magenta line.

Visualization explanation

2. Consider a point Q on the graph of $y = f(x)$ that is a "small step of size h" away (move from a to $a+h$): $(a+h, f(a+h))$.
3. Connect the two points P and Q together to form a straight line in blue. We know how to find the gradient of this line:

$$\frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}.$$

4. As h becomes smaller, the blue line will get closer to the magenta line.
5. As such, as $h \to 0$, we see that the gradient of the blue line tends to the gradient of the magenta line.
6. Therefore, the gradient of the function $y = f(x)$ at a is

$$\boxed{(\star)\ \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.}$$

$f'(a)$

$$f \xrightarrow{\text{differentiate}} f' \xrightarrow{\text{differentiate}} f''$$

$$f'''$$

$$f^{(4)}$$

$$f^{(5)}$$

Definition of the derivative

Definition

The *derivative* of a function $y = f(x)$ at a point a, denoted by $f'(a)$, is the limit

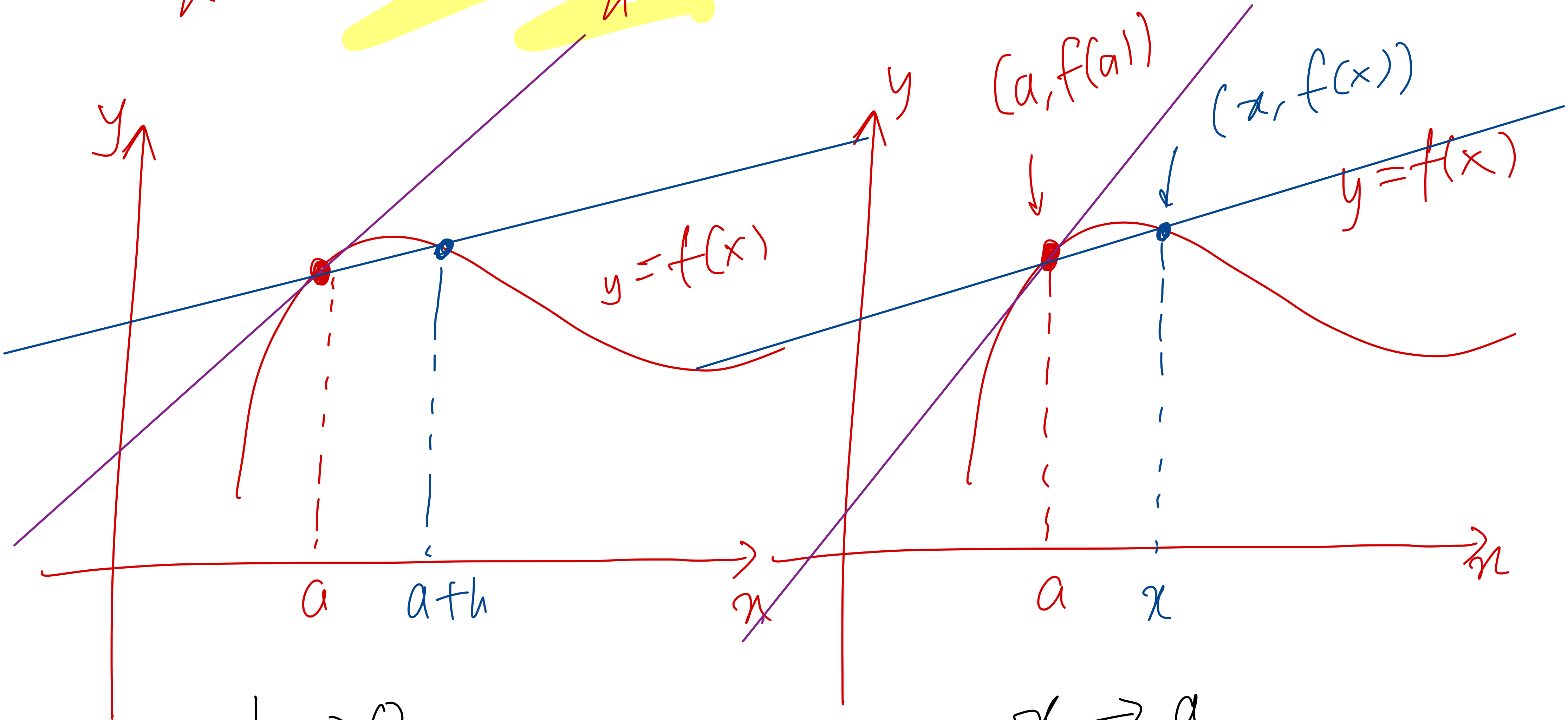
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \qquad (1)$$

if it exists. If this limit exists, we say that f is differentiable at the point a. Otherwise, f is not differentiable at $x = a$.

Alternatively, the above limit can also be interpreted as the following limit (let $h = x - a$, as $h \to 0$, $x \to a$)

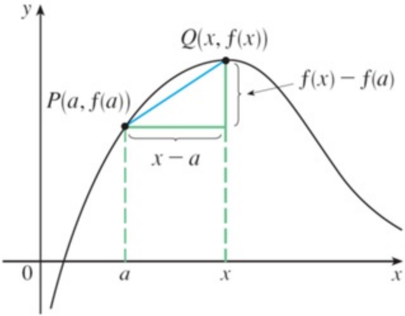
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \qquad (2)$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



$h \to 0$ $\qquad$ $x \to a$

Second interpretation of the derivative



The explanation follows similarly, but without using h. We just choose a point x that is near a.

Example 1

$a = 1$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Let $f(x) = x^3$ and $a = 1$.
We check if f is differentiable at a using the definition.

$$\begin{aligned} f'(1) &= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x \to 1} \frac{x^3 - 1}{x - 1} \\ &= \lim_{x \to 1} \frac{(x-1)(x^2 + x + 1)}{x - 1} \\ &= \lim_{x \to 1} (x^2 + x + 1) = 3. \end{aligned}$$

$f(x) = x^3$

$f'(x) = 3x^2$

$f'(1) = 3.$

So, f has a derivative at $a = 1$ and $f'(1) = 3$.

gradient

Exercise 1

Compute $f'(a)$, using the definition of the derivative, whichever interpretation you prefer, for the following functions and points.

1. $f(x) = x^2$, $a = 2$
2. $f(x) = \sqrt{x}$, $a = 4$

$$f'(2) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(1) $f(x) = x^2$, $a = 2$

$f'(x) = 2x$

$$= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{\overset{a}{(h+2)}^2 - \overset{b}{2}^2}{h} = \lim_{h \to 0} \frac{((h+2)+2)\overbrace{((h+2)-2)}^{=\cancel{h}}}{\cancel{h}}$$

$$= \lim_{h \to 0} (h+4) = 4$$

Exercise 1

$f(x) = \sqrt{x}$, $a = 4$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{4+h} - 2)}{h} \cdot \frac{(\sqrt{4+h} + 2)}{(\sqrt{4+h} + 2)}$$

$$= \lim_{h \to 0} \frac{4+h-4}{\cancel{h}(\sqrt{4+h} + 2)} = \lim_{h \to 0} \frac{1}{\sqrt{4+h} + 2} = \frac{1}{4}$$

The derivative function

Previous: a constant ; calculate $f'(a)$

a constant
x variable.

We have learnt how to find the derivative of a function f at a point a. We now let this point vary by replacing a by a variable x.

$$\longrightarrow \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \qquad (1)$$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

or alternatively,

$$\longrightarrow \quad f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}. \qquad (2)$$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

f' is called the *derivative function*, or simply, the *derivative* of f.
We also say that we *differentiate* f to get f'.

$$f \xrightarrow{\text{differentiate}} f'$$

Example 2

Let $f(x) = x^2$, we use the definition of the derivative to find $f'(x)$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(\overset{a}{x+h})^2 - \overset{b}{x}^2}{h} \\
&= \lim_{h\to 0} \frac{((\overset{a}{x+h}) - \overset{b}{x})((\overset{a}{x+h}) + \overset{b}{x})}{h} \\
&= \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} (2x+h) = 2x.
\end{aligned}$$

Therefore, $f'(x) = 2x$. Looks familiar?

Exercise 2

$f(x+h) = \sqrt{x+h}$

Use the definition of the derivative to find the derivative of $f(x) = \sqrt{x}$.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \quad h$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$(x+h)^2 = (x+h)(x+h)$$

$$\left(\sqrt{x+1} - \sqrt{3}\right)^2$$

Continuity of functions with derivatives

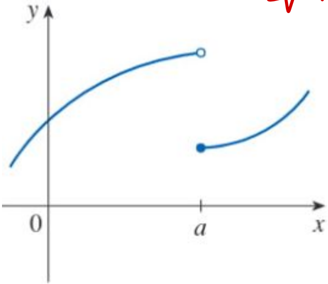
Theorem

If a function f is differentiable at a point a, then it is also continuous at a.

This tells you that functions with graphs that "break" at certain points cannot be differentiable at those points. An example of such a function:

TLDR

If you can differentiate f, then f is continuous.



If f is not continuous at a, then f is not differentiable at a.

Differentiation operator

There are also other ways of writing $f'(x)$, they all refer to $f'(x)$.

1. Using the *differentiation operator* $\dfrac{d}{dx}$:

$$f'(x) = \frac{d}{dx}f(x).$$

2. Let $y = f(x)$, then

$$f'(x) = \frac{dy}{dx}.$$

3. Let $y = f(x)$, then

$$f'(x) = y'.$$

We will use the first two interchangeably, and occasionally, the third.

Derivative of a constant function

Theorem

For any constant $c \in \mathbb{R}$,

$$\frac{d}{dx}(c) = 0.$$

Proof.

Let $c \in \mathbb{R}$, and set $f(x) = c$. Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0.$$

$\square$

Power Rule

Theorem

For any $n \in \mathbb{R}$,

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

We defer the proof for a later tutorial problem (Week 4).

Examples:

1. $f(x) = x^2 \implies f'(x) = 2x.$
2. $f(x) = x^4 \implies f'(x) = 4x^3.$
3. $f(x) = \sqrt{x} = x^{\frac{1}{2}} \implies f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$
4. $f(x) = (\sqrt[4]{x})^3 = x^{\frac{3}{4}} \implies f'(x) = \frac{3}{4}x^{-\frac{1}{4}} = \frac{3}{4\sqrt[4]{x}}.$

$f''(x) = 12x^2$

$x^{-1} = \frac{1}{x}$

$x^{-\frac{1}{2}} = \frac{1}{x^{\frac{1}{2}}}$

$= \frac{1}{\sqrt{x}}$

Secant function

Definition

The *secant function* $\sec x$ is the reciprocal of the cosine function

$$\sec x = \frac{1}{\cos x}.$$

Its domain is the set of real numbers excluding the values x where $\cos x = 0$, i.e. $\mathbb{R} \setminus \{\frac{\pi}{2} + n\pi\}$.

Note: The secant function has the same domain as the tangent function $\tan x$.

Derivatives of trigonometric functions

Theorem

The following are derivatives of some of the common trigonometric functions.

$\frac{d}{d☺} \sec ☺$

$= \sec ☺ \tan ☺$

$$(1) \quad \frac{d}{dx}(\sin x) = \cos x$$

$$(2) \quad \frac{d}{dx}(\cos x) = -\sin x$$

$$(3) \quad \frac{d}{dx}(\tan x) = \sec^2 x$$

$= (\sec x)^2$

$\neq \sec(x^2)$ ✗

$$(4) \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$

Derivatives of exponential and log functions

Theorem

The following are derivatives of exponential and logarithmic functions. Let $a > 0$ be a constant.

(1) $\dfrac{d}{dx}(e^x) =$ [redacted] e^x

(2) $$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

(3) $$\frac{d}{dx}(a^x) = a^x \ln a$$

Algebraic properties of derivatives (part 1)

Using our knowledge of derivatives of basic functions, we can use algebraic operations (addition/subtraction/multiplication/division) to obtain derivatives of these combinations of basic functions.

Theorem

Let c be a fixed constant and f, g be functions. We differentiate constant multiples and sums/subtractions of f and g in the following manner.

$$\frac{d}{dx}(2x) = 2\left(\frac{d}{dx}x\right)$$
$$= 2 \cdot 1 = 2.$$

$$(1) \quad \frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$$

$$(2) \quad \frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

$$\frac{d}{dx}(\sin x + e^x) = \frac{d}{dx}\sin x + \frac{d}{dx}e^x$$
$$= \cos x + e^x$$

Exercise 3

For each of these functions, find their derivatives.

1. $f(x) = 2\sin x + 3\ln x$
2. $g(t) = 5^t - 10\tan t$
3. $h(\theta) = \dfrac{1}{10}\theta^5 + \sec\theta$
4. $p(x) = \dfrac{2}{5}x^3 + \dfrac{7}{4}x^2 + 3$

① $f'(x) = 2\dfrac{d}{dx}\sin x + 3\dfrac{d}{dx}\ln x$

$= 2\cos x + \dfrac{3}{x}$

② $g'(t) = \dfrac{d}{dt}5^t - 10\dfrac{d}{dt}\tan t$

$= 5^t\ln 5 - 10\sec^2 t.$

③ $h'(\theta)$

$= \dfrac{1}{10}\dfrac{d}{d\theta}\theta^5 + \dfrac{d}{d\theta}\sec\theta$

$= \dfrac{1}{10}\cdot 5\theta^4 + \sec\theta\tan\theta$

$= \dfrac{1}{2}\theta^4 + \sec\theta\tan\theta$

Exercise 4

(4) $p(x) = \frac{2}{5}x^3 + \frac{7}{4}x^2 + 3$

$$p'(x) = \frac{2}{5}\underbrace{\frac{d}{dx}x^3}_{3x^2} + \frac{7}{4}\underbrace{\frac{d}{dx}x^2}_{2x} + \frac{d}{dx}3.$$

$$= \frac{6}{5}x^2 + \frac{7}{2}x$$

Algebraic properties of derivatives (part 2)

Theorem

Let f and g be functions. We differentiate products and quotients of f and g in the following manner.

$$\text{(3)} \quad \frac{d}{dx}(f(x)g(x)) = \left(\frac{d}{dx}f(x)\right)g(x) + f(x)\left(\frac{d}{dx}g(x)\right)$$

$$\text{(4)} \quad \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\left(\frac{d}{dx}f(x)\right) - f(x)\left(\frac{d}{dx}g(x)\right)}{[g(x)]^2}$$

We refer to (3) as the *product rule* and (4) as the *quotient rule*.

$$\frac{d}{dx}(\sin x \cos x) = \left(\frac{d}{dx}\sin x\right)\cos x + \sin x\left(\frac{d}{dx}\cos x\right)$$

$$= \cos x \cos x + \sin x(-\sin x)$$

$$= \cos^2 x - \sin^2 x$$

$$= \cos(2x) \quad \left(\text{double angle formula}\right)$$

$$\frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$\tan x$

Exercise 5

For each of these functions, find their derivatives.

1. $f(x) = e^x \sin x + \cos x$
2. $g(\theta) = \sec\theta \tan\theta$
3. $h(\theta) = \dfrac{\sin\theta}{1 + \cos\theta}$
4. $q(x) = \dfrac{x}{x^2 + 1}$

(1) $f'(x)$

$$= \frac{d}{dx}(e^x \sin x) + \frac{d}{dx}\cos x$$

$$= \left(\frac{d}{dx}e^x\right)\cdot \sin x + e^x\left(\frac{d}{dx}\sin x\right) - \sin x$$

$$= e^x \sin x + e^x \cos x - \sin x.$$

② $g(\theta) = \sec\theta \tan\theta$

$$g'(\theta) = \left(\frac{d}{d\theta}\sec\theta\right)\tan\theta + \sec\theta\left(\frac{d}{d\theta}\tan\theta\right)$$

$$= \sec\theta\tan\theta \cdot \tan\theta + \sec\theta \cdot \sec^2\theta$$

$$= \sec\theta\tan^2\theta + \sec^3\theta.$$

$$= \sec\theta\left(\tan^2\theta + \sec^2\theta\right)$$

③ $h(\theta) = \dfrac{\sin\theta}{1+\cos\theta} \Rightarrow h'(\theta) = \dfrac{(1+\cos\theta)\cdot\left(\frac{d}{d\theta}\sin\theta\right) - \sin\theta\left(\frac{d}{d\theta}(1+\cos\theta)\right)}{(1+\cos\theta)^2}$

$$= \frac{(1+\cos\theta)\cos\theta - \sin\theta(-\sin\theta)}{(1+\cos\theta)^2}$$

$$= \frac{\cos\theta + \overbrace{\cos^2\theta + \sin^2\theta}^{=1}}{(1+\cos\theta)^2}$$

$$= \frac{1+\cos\theta}{(1+\cos\theta)^2} = \frac{1}{1+\cos\theta}.$$

$$g(x) = \frac{x}{x^2+1}$$

$$g'(x) = \frac{(x^2+1)\left(\frac{d}{dx}x\right) - x\left(\frac{d}{dx}(x^2+1)\right)}{(x^2+1)^2}$$

$$= \frac{(x^2+1) - x(2x)}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2} \le 0$$

$1-x^2 \le 0$

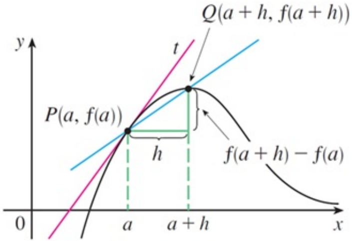
$(x^2+1)^2 > 0$

$$\boxed{x \ge 1}$$

$x^2 \ge 1$

$1 - x^2 \le 0$

Additional food for thought



Using the information we have learnt in the past two weeks and today's lecture, can you find the equation of the magenta line?