Sequence Fundamentals

Dr. Ronald Koh
ronald.koh@digipen.edu (Teams preferred over email)

AY 23/24 Trimester 1

Table of contents

What is a sequence?

- A **sequence** is a list of numbers written in order:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

- a_n is the nth term of the sequence, and n is the **index** of a_n; this is akin to indexing lists in coding.
- We denote the entire sequence as $\{a_n\}_{n=1}^{\infty}$, $\{a_n\}$, or sometimes a_n.
- Two ways of writing an entire sequence:
  - **General formula/Closed form**: e.g.

$$a_n = \frac{\sin(n^2)}{n}.$$

  - **Recursive relation**: e.g. the Fibonacci sequence:

$$a_n = a_{n-1} + a_{n-2}.$$

Convergence of sequences (informal)

- In a sequence, we have an infinite list of numbers.
- It is thus natural to ask: "Where does this lead to?" or "What number does this list approach as the index n gets **very large**?".
- We refer to this as the **limit** L of a sequence $\{a_n\}$. We say that the sequence $\{a_n\}$ has limit L if the terms are 'close' to L when n gets (arbitrarily) large. It is written as

$$\lim_{n\to\infty} a_n = L.$$

- If such a number L exists, we say that $\{a_n\}$ is **convergent** (or **converges**).
- Otherwise, we say that $\{a_n\}$ is **divergent** (or **diverges**), or the limit of $\{a_n\}$ **does not exist**.

Convergence of sequences (formal, optional)

- Formal definition of $\lim_{n\to\infty} a_n = L$:
  For every $\varepsilon > 0$, there is a positive integer N such that
  $$|a_n - L| < \varepsilon \quad \text{whenever} \quad n > N.$$
- We understand $|a_n - L|$ as the distance from the nth term of the sequence a_n to a number L.
- A layman's way of saying this formal definition is:
  $\{a_n\}$ converges to L if and only if

  for every $\varepsilon > 0$, we go down the list $\{a_n\}$ long enough, **eventually**, after a certain index N, all terms of the sequence with index greater than N would be within ε distance from L.

Example 1 (optional)

Let's try to get an understanding of the formal definition of a sequence limit by considering the sequence $a_n = \dfrac{1}{n}$. Computing the first few terms, we get

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \dots$$

At first glance, the sequence looks to be approaching 0. For an example, let's consider $\varepsilon = 1$. For this ε, we can choose $N = 1$, then any term after the index 2 will be within distance of 1 from 0:

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \dots$$

Example 1 (optional)

For $\varepsilon = \dfrac{1}{2500}$, we can choose $N = 2500$, then any term after index 2500 will be within distance of $\dfrac{1}{2500}$ from 0:

$$\frac{1}{2501}, \frac{1}{2502}, \frac{1}{2503}, \frac{1}{2504} \dots$$

A suitable index N can actually be chosen for any choice of $\varepsilon > 0$. Thus

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Subsequences

- A **subsequence** of $\{a_n\}_{n=1}^{\infty}$ is a sequence of the form

$$a_{n_1}, a_{n_2}, a_{n_3}, \ldots$$

  where $1 \leq n_1 < n_2 < n_3 < \cdots$
- Some notable subsequences include:
  - $\{a_1, a_3, a_5, \ldots\}$ is called the **odd subsequence** of $\{a_n\}_{n=1}^{\infty}$.
  - $\{a_2, a_4, a_6, \ldots\}$ is called the **even subsequence** of $\{a_n\}_{n=1}^{\infty}$.
- A subsequence **preserves the order** of its original sequence;

  e.g. for a sequence $\{a_n\}_{n=1}^{\infty}$,
  - $\{a_1, a_5, a_9, \ldots\}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$.
  - $\{a_2, a_1, a_3, a_6, a_4, a_5, \ldots\}$ is **NOT** a subsequence of $\{a_n\}_{n=1}^{\infty}$.

Subsequence Test

Theorem

If a sequence $\{a_n\}$ converges to L, then **every** subsequence of $\{a_n\}$ converges to L.

As a consequence, if there are two subsequences of $\{a_n\}$ that do not converge to the same limit, then $\{a_n\}$ is divergent.

Note: This test is usually used to show that a particular sequence is divergent.

Example 2

Use the Subsequence Test to show that the sequence

$$a_n = (-1)^n$$

is divergent.

Exercise 1

Use the Subsequence Test to show that the sequence

$$a_n = \sin\left(\frac{n\pi}{2}\right)$$

is divergent.

Limit Laws

Like in functional limits in Calculus I, we also have limit laws for sequences. Let $\{a_n\}$ and $\{b_n\}$ be **convergent** sequences. Then

(a) $\lim\limits_{n\to\infty} ca_n = c \lim\limits_{n\to\infty} a_n$.

(b) $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$.

(c) $\lim\limits_{n\to\infty} (a_n b_n) = \left(\lim\limits_{n\to\infty} a_n\right) \cdot \left(\lim\limits_{n\to\infty} b_n\right)$.

(d) $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ if $\lim\limits_{n\to\infty} b_n \neq 0$.

(e) $\lim\limits_{n\to\infty} a_n^p = \left(\lim\limits_{n\to\infty} a_n\right)^p$ if $p > 0$ and $a_n > 0$ for all n.

(f) $\lim\limits_{n\to\infty} f(a_n) = f\left(\lim\limits_{n\to\infty} a_n\right)$ if f is continuous at $\lim\limits_{n\to\infty} a_n$.

Exercise 2

Evaluate the following limits.

1. $\lim_{n\to\infty} \frac{1}{n^2}$
2. $\lim_{n\to\infty} 3 - \frac{1}{n}$
3. $\lim_{n\to\infty} \cos\left(\frac{4}{n}\right)$

Sequences defined by a function

Theorem

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for all n, then

$$\lim_{n\to\infty} a_n = L.$$

Visualization Example:

Sequences that diverge to ∞

- We have seen divergent sequences that have an oscillating behavior.
- There is also another type of divergent sequence:

$$\lim_{n\to\infty} a_n = \infty \ \text{ or } \ -\infty.$$

- Examples of such sequences include n^p (for $p > 0$), $\ln n$, $\ln\left(\frac{1}{n}\right)$, 3^n, etc.
- This can be easily observed using the theorem in the previous slide.

Limit Evaluation Technique 1: Squeeze Theorem

Theorem

If there are three sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$ that obey

1. $a_n \leq b_n \leq c_n$ for all $n > N$, where N is a fixed integer,
2. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

then

$$\lim_{n\to\infty} b_n = L.$$

TLDR: If b_n is (eventually) sandwiched/squeezed in between two other sequences a_n and c_n, and a_n, c_n both converge to the same limit L, then b_n also converges to L.

Example 3 (Geometric Sequence)

Let r be a fixed number. The sequence $\{a_n\}$ defined by

$$a_n = r^n$$

is called the **geometric sequence with rate** r. We show that

1. $\{a_n\}$ is convergent for $-1 < r \leq 1$ with

$$r^n = \begin{cases} 0 & \text{if } -1 < r < 1, \\ 1 & \text{if } r = 1. \end{cases}$$

2. $\{a_n\}$ is divergent for $r <= -1$ and $r > 1$.

We do this by cases. We have already shown in Example 2 that when $r = -1$, r^n is divergent.

When $r = 1$, $r^n = 1^n = 1$, so r^n converges to 1.

Also, when $r = 0$, $r^n = 0$, so r^n converges to 0.

Example 3 (Geometric Sequence: Case 1)

The remaining cases are

1. $-1 < r < 1$, $r \neq 0$
2. $r > 1$
3. $r < -1$

For case 1, we first consider $0 < r < 1$. Consider the graph $f(x) = r^x$:

We have $\lim_{x \to \infty} r^x = 0$. As $a_n = r^n = f(n)$, by the theorem on slide 14 (sequences defined by a function),

$$\lim_{n \to \infty} r^n = 0.$$

Example 3 (Geometric Sequence: Case 1)

For $-1 < r < 0$, we first note that $r^n \leq |r^n| = ||r|^n|$, so

$$-|r|^n \leq r^n \leq |r|^n.$$

Since $-1 < r < 0$, we have $0 < |r| < 1$, so by the last slide,

$$\lim_{n\to\infty} -|r|^n = \lim_{n\to\infty} |r|^n = 0.$$

Thus by the Squeeze Theorem,

$$\lim_{n\to\infty} r^n = 0.$$

Case 1 is thus proved.

Example 3 (Geometric Sequence: Case 2)

For $r > 1$, we consider, again, the graph $f(x) = r^x$.

Clearly, $\lim_{x\to\infty} f(x) = \infty$, thus by the Theorem on slide 14, $\lim_{n\to\infty} r^n = \infty$, and r^n is divergent.

Example 3 (Geometric Sequence: Case 3)

When $r < -1$, consider the even subsequence of r^n:

$$|r|^2, |r|^4, |r|^6, \ldots$$

Since $r < -1$, we have $|r| > 1$.

Thus by the previous slide, r^n diverges to ∞.

Since a subsequence of r^n diverges to ∞, it follows that r^n is divergent.

We have thus completed this proof. $\square$

Exercise 3

Evaluate the following limits.

1. $\lim_{n\to\infty} 3^n$
2. $\lim_{n\to\infty} (-0.5)^n$
3. $\lim_{n\to\infty} \frac{1}{2^n}$

Limit Evaluation Technique 2: Rational Functions

We start off with polynomials in n, say of degree 2, $n^2 - 2n + 3$. We know that when n is large ($n > 1$), $n^2 >> n >>$ constant. Thus the limit

$$\lim_{n \to \infty} n^2 - 2n + 3$$

is **fully dependent** on the 'highest power term' n^2. Since

$$\lim_{n \to \infty} n^2 = \infty,$$

it follows that

$$\lim_{n \to \infty} n^2 - 2n + 3 = \infty.$$

Limit Evaluation Technique 2: Rational Functions

We move on to limits of rational functions in n, for example

$$\lim_{n\to\infty} \frac{n^2 - 2n + 3}{2n^4 + 3n^3 + 2n^2}.$$

The numerator and denominator both tend to ∞. How do we handle these kind of limits? We **divide** both the numerator and denominator by the highest power of n in the fraction; here being n^4. We get

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^2 - 2n + 3}{2n^4 + 3n^3 + 2n^2} &= \lim_{n\to\infty} \frac{n^2 - 2n + 3}{2n^4 + 3n^3 + 2n^2} \cdot \frac{\frac{1}{n^4}}{\frac{1}{n^4}} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n^2} - \frac{2}{n^3} + \frac{3}{n^4}}{2 + \frac{3}{n} + \frac{2}{n^2}} \\
&= \frac{\lim\limits_{n\to\infty} \frac{1}{n^2} - \frac{2}{n^3} + \frac{3}{n^4}}{\lim\limits_{n\to\infty} 2 + \frac{3}{n} + \frac{2}{n^2}} = \frac{0}{1} = 0.
\end{aligned}$$

Example 4

Evaluate the following limits.

1. $\displaystyle\lim_{n\to\infty} \frac{n^3 - 2n^2 + 4n}{3n^3 + n^2 + 1}$
2. $\displaystyle\lim_{n\to\infty} \frac{-n^4 + n^2}{n^3 + n}$
3. $(*)\ \displaystyle\lim_{n\to\infty} \frac{3^n + 2^n}{4^n + 5^n}$

Example 4

Exercise 4

Evaluate the following limits.

1. $\lim_{n\to\infty} \frac{-2n^3 + 4n}{3n^3 + 3n^2}$
2. $\lim_{n\to\infty} \frac{n^3 + 2n^2}{n^2 - 2n}$
3. $(*) \lim_{n\to\infty} \frac{1 + 2^n}{6^n + 2^n}$

Exercise 4

Motivation for L'Hôpital's Rule: Indeterminate Cases

- Consider $\lim_{n\to\infty} \frac{a_n}{b_n}$.
- Under the limit laws, if $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ with $b \neq 0$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

- **Indeterminate cases**:

$$\frac{a_n}{b_n} \to \frac{\pm\infty}{\pm\infty} \text{ or } \frac{a_n}{b_n} \to \frac{0}{0}.$$

- We can use the Theorem in slide 14 (sequences defined by functions) along with **L'Hôpital's Rule** to solve these indeterminate cases.

L'Hôpital's Rule

Theorem (L'Hôpital's Rule)

Let a be any number, or $\pm\infty$. Assume f and g are differentiable functions with $g'(x) \neq 0$ on an open interval containing a. If

1. $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ exists, and
2. $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$ or $\pm\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Example 5

Evaluate the following limits.

1. $\lim_{n \to \infty} \frac{\ln n}{n}$
2. $(*)$ $\lim_{n \to \infty} n^{\frac{1}{n}}$

Example 5

Exercise 5

Evaluate the following limits.

1. $\lim_{n\to\infty} \dfrac{n^2+n}{e^n}$
2. $(*)\ \lim_{n\to\infty} \ln(n^2+2) - \ln(3n^2-1)$

Exercise 5

Food for thought

Should we use L'Hôpital's Rule to solve

$$\lim_{n\to\infty} \frac{n^{2023} + n^{2021}}{3n^{2023} - 3n^{2019}}?$$