Revision

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AY 23/24 Trimester 1

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Overview of methods

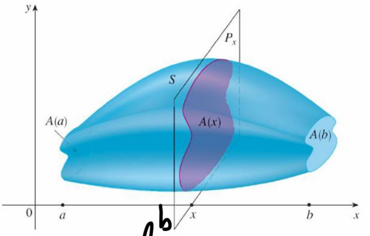
disk

washer

revolve around

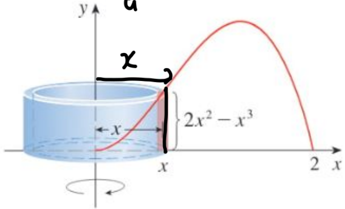
x, y-axis

- Cross-sectional method



- Cylindrical shells method

$$\int_a^b 2\pi x\, f(x)\, dx$$



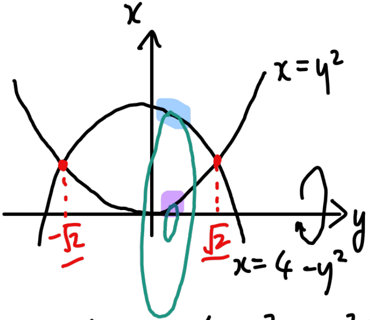
How to start a volumes of revolution question

visualize disk/washer/ cylindrical shells

1. Sketch the region.
2. **Determine the axis of revolution.** x
3. Roughly visualize the 3D solid (don't need to be 100% accurate, just a rough idea is enough).
4. **Determine the method you want to use**; this will determine your **variable of integration**:
   - Cross-sectional method: **same variable** as axis of revolution. $x \to A(x)$
   - Cylindrical shells method: **different variable** from axis of revolution. $y \to$ Height, radius in y
5. For cross-sectional method: determine the cross-sectional area $A(x)$ or $A(y)$, depending on the variable in Step 4.
6. For cylindrical shells method: determine the height and radius, depending on the variable in Step 4.

Example 1

Let R be the region enclosed by the curves $x = y^2$ and $x = 4 - y^2$. Find the volume of the solid obtained by rotating R about the y-axis.



$$y^2 = 4 - y^2$$

$$\iff 4 - 2y^2 = 0 \iff y = \pm\sqrt{2}$$

Cross-sectional method

$\to$ Variable in y

$r_{out} = 4 - y^2 \quad r_{in} = y^2$

$$A(y) = \pi\left(r_{out}^2 - r_{in}^2\right) = \pi\left((4-y^2)^2 - (y^2)^2\right)$$

$$= \pi\left(4 - y^2 + y^2\right)\left(4 - y^2 - y^2\right) \qquad -\sqrt{2} \le y \le \sqrt{2}$$

$$= 4\pi(4 - 2y^2) = 8\pi(2 - y^2)$$

Example 1

$$\text{Volume} = \int_{-\sqrt{2}}^{\sqrt{2}} A(y)\,dy = \int_{-\sqrt{2}}^{\sqrt{2}} 8\pi(2-y^2)\,dy$$

$$= 2\cdot 8\pi \int_0^{\sqrt{2}} 2-y^2\,dy \quad (\text{integrand is even})$$

not necessary

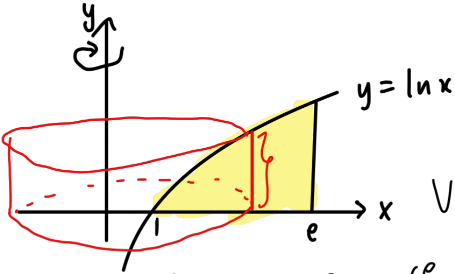
$$= 16\pi\left[2y-\frac{y^3}{3}\right]_0^{\sqrt{2}} = 16\pi\left[2\sqrt{2}-\frac{2\sqrt{2}}{3}\right]$$

$$= \frac{64\pi\sqrt{2}}{3}.$$

Example 2

x-axis

Let R be the region enclosed by the curves $y = \ln x$, $y = 0$ and $x = e$. Find the volume of the solid obtained by rotating R about the y-axis.



Cylindrical shells method

Variable in $\underline{x}$

Radius $= x$, Height $= \ln x$

$$V = \int_1^e 2\pi x \ln x \, dx$$

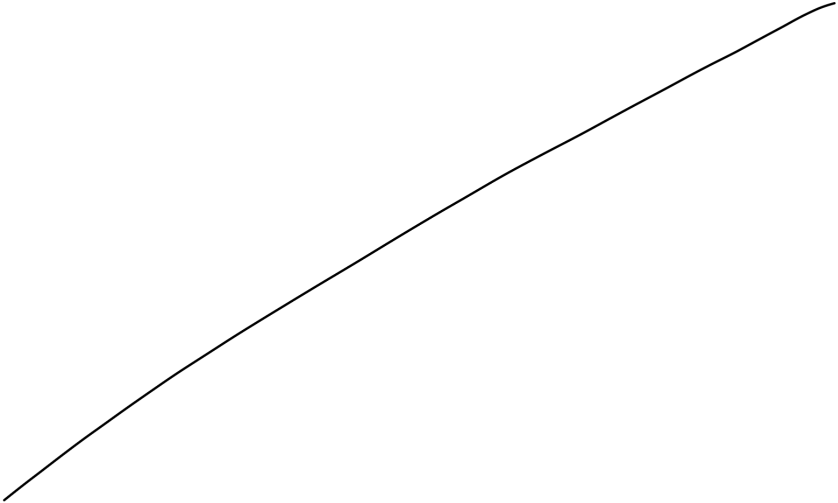
$u = \ln x$, $dv = x$

$du = \frac{1}{x}$, $v = \frac{x^2}{2}$

$$= 2\pi \left(\left[\frac{x^2}{2} \ln x \right]_1^e - \frac{1}{2}\int_1^e x \, dx \right) \quad \text{by parts}$$

$$= 2\pi \left(\frac{e^2}{2} - \frac{1}{4}\left[x^2\right]_1^e \right) = 2\pi \left(\frac{e^2}{2} - \frac{e^2 - 1}{4} \right) = \frac{(e^2+1)\pi}{2}$$

Example 2



Recap of sequences

- A **sequence** is an infinite list of numbers written in order:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

indices

- We are interested in finding the **limit** of a sequence

$$\lim_{n\to\infty} a_n.$$

- The limit of a sequence is either a number L or does not exist (oscillating or $\pm\infty$).
- If the limit of a sequence is a number L, $\lim_{n\to\infty} a_n$ is said to **exist** and $\lim_{n\to\infty} a_n = L$, otherwise $\lim_{n\to\infty} a_n$ **does not exist**.

DNE

Sequence limit evaluation techniques

$n^2 < n^4$

- **Limit Laws**: add/subtract/product/quotient (limit of denominator must be non-zero)/power/continuous function.
- a_{2n} even and odd a_{2n+1} **Subsequence Test**: Showing a sequence is **divergent** by finding two subsequences that converge to **two different limits**.
- **Rational/Power functions** in n: dividing by highest power, e.g.

$$\lim_{n\to\infty} \frac{2n^3+3}{3n^3+2n^2+4n} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{2+\frac{3}{n^3}}{3+\frac{2}{n}+\frac{4}{n^2}} = \frac{2}{3}.$$

- **Squeeze Theorem**, e.g.

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n} \implies \lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

$$0 \le \frac{n^2}{n^4} = \frac{\cancel{n}\cdot\cancel{n}\cdot 1\cdots 1}{\cancel{n}\cdot\cancel{n}\cdot n\cdots n} \le \frac{1}{n} \to 0.$$

- **L'Hôpital's Rule**: a is a number, or $\pm\infty$.

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \text{ or } \pm\infty \implies \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Example 3

① $\lim_{n\to\infty} \frac{3n^5 - n^4}{8n^5+2n^2+3} \frac{\frac{1}{n^5}}{\frac{1}{n^5}} = \lim_{n\to\infty} \frac{3-\frac{1}{n}}{8+\frac{2}{n^3}+\frac{3}{n^5}} = \frac{3}{8}.$

Evaluate the following limits.

1. $\lim_{n\to\infty} \frac{3n^5 - n^4}{8n^5 + 2n^2 + 3}$
2. $\lim_{n\to\infty} (-1)^n \frac{n+1}{n}$
3. $\lim_{n\to\infty} n \ln\left(1 + \frac{1}{n}\right)$

② $\lim_{n\to\infty} \underbrace{(-1)^n \frac{n+1}{n}}_{\text{divergent}}$ $\rightarrow$ $\lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$

Let $a_n = (-1)^n \frac{n+1}{n}$

$a_{2n} = (-1)^{2n} \frac{2n+1}{2n} = \frac{2n+1}{2n} \to 1$

$a_{2n+1} = (-1)^{2n+1} \frac{2n+2}{2n+1} = -\frac{2n+2}{2n+1} \to -1$ as $n\to\infty$

$\therefore$ a_n has two subsequences that tend to different limits. Hence a_n is divergent by the subsequence test.

Example 3

Wrong example ✗ ✓

$$\lim_{n\to\infty}(-1)^n\frac{n+1}{n} = \lim_{n\to\infty}(-1)^n \cdot \lim_{n\to\infty}\frac{n+1}{n}$$

divergent

not Convergent

Cannot split like this!

$$\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$$

both are convergent.

L'H $\left(\frac{0}{0}\right)$ or $\left(\frac{\pm\infty}{\pm\infty}\right)$

convert to $\frac{1}{0}$

(3) $$\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \quad \left(\frac{0}{0}\right) \overset{L'H}{=} \lim_{n\to\infty}\frac{\frac{1}{1+\frac{1}{n}}\cdot\cancel{\left(-\frac{1}{n^2}\right)}}{\cancel{-\frac{1}{n^2}}}$$

$\infty \cdot 0$

indeterminate $\quad n = \frac{1}{\frac{1}{n}}$

$$= \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1.$$

Because $n\cdot\frac{1}{n} = 1$

$a\cdot b = 1$

$\Rightarrow a = \frac{1}{b}$ $b \neq 0$

Series

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.

- A **series** is an (infinite) sum of all the terms in $\{a_n\}_{n=1}^{\infty}$.
- It is defined as the **limit** of the **sequence of partial sums**:

Sequence of partial sums

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} s_N = \lim_{N\to\infty} (a_1 + a_2 + \cdots + a_N).$$

- If this limit exists, we say that the series $\sum_{n=1}^{\infty} a_n$ is **convergent**, and

Find this exact sum is hard

$$S = \sum_{n=1}^{\infty} a_n.$$

Otherwise, $\sum_{n=1}^{\infty} a_n$ is **divergent**.

Convergence/Divergence Tests for Series (1)

- Often, the sum S cannot be found easily (exception: geometric series), thus we only test for convergence/divergence.
- For **geometric series**, starting term a and common ratio r:

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1, \\ \text{divergent} & \text{if } |r| \geq 1. \end{cases}$$

- p-**series** (**IMPORTANT!**): p constant

it forms the base for CT/LCT

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is } \begin{cases} \text{convergent} & \text{if } p > 1, \\ \text{divergent} & \text{if } p \leq 1. \end{cases}$$

- First test to do everytime we test the convergence of a series - **Divergence Test**: ↳ if you don't know how to start

terms

$$\lim_{n\to\infty} a_n \neq 0 \implies \sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

← series

Convergence/Divergence Tests for Series (2)

a_n b_n

$\sum \frac{n^2+1}{n^3+2}$ $\frac{1}{n}$

- **Comparison Test**: If $0 \leq a_n \leq b_n$ for $n \geq n_0$,
  - $\sum b_n$ converges $\implies \sum a_n$ converges.
  - $\sum a_n$ diverges $\implies \sum b_n$ diverges.
- **Limit Comparison Test**: $a_n, b_n > 0$ with $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, c finite:

sums of rational func

$$\sum a_n \quad \text{and} \quad \sum b_n \text{ both converge or both diverge.}$$

$f'(x)$ or $b_{n+1} \leq b_n$ for all n

- **Alternating Series Test**: $b_n > 0$, decreasing, $\lim_{n\to\infty} b_n = 0$:

alternating factor

$$\sum (-1)^n b_n \text{ converges.}$$

- **Absolute Convergence Test**:

$$\sum |a_n| \text{ converges} \implies \sum a_n \text{ converges.}$$

Convergence/Divergence Tests for Series (3)

$|(-1)^n| = 1$

Let $\sum_{n=1}^{\infty} a_n$ be a series.

- **Ratio Test**: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$:
- **Root Test**: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$:

$$\sum_{n=1}^{\infty} a_n \text{ is } \begin{cases} \text{absolutely convergent} & \text{if } L < 1, \\ \text{divergent} & \text{if } L > 1, \\ \text{inconclusive} & \text{if } L = 1. \end{cases}$$

Try other tests

Tips for testing for convergence

$\sum a_n$

$a_n = \frac{2n^4 + n^2}{n^6 + 2n^2 + 2}$

- When unsure, use Divergence Test first.
- For series with rational function terms, use LCT with

$$b_n = \frac{\text{dominating term in numerator}}{\text{dominating term in denominator}}.$$

$b_n = \frac{n^4}{n^6} = \frac{1}{n^2}$

$a_n = (-1)^n b_n$

- For alternating series $\sum(-1)^n b_n$, if $\lim_{n\to\infty} b_n = b \neq 0$, use the Subsequence Test; even and odd subsequences of $a_n = (-1)^n b_n$, and Divergence Test to show $\sum(-1)^n b_n$ is divergent.
- If terms of series has mostly $n!$, heavily consider Ratio Test.
- If terms of series has mostly powers of n, heavily consider Root Test.
- If terms of series has both $n!$ and powers of n, priortize the Ratio Test over the Root Test.

$\sqrt[n]{n!} \to ?$

Example 4

① Let $a_n = (-1)^n \frac{n+1}{n}$, we have shown in the earlier eg that $\lim_{n\to\infty} a_n$ D.N.E. $\therefore \sum (-1)^n \frac{n+1}{n}$ is divergent by the Divergence Test.

Determine the convergence of the following series.

1. $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n}$ $\quad \to 1 \neq 0$
2. $\sum_{n=1}^{\infty} \frac{3n^2}{n^4+1}$
3. $\sum_{n=1}^{\infty} \frac{n!2^n}{n^n}$

② Let $a_n = \frac{3n^2}{n^4+1}$ and $b_n = \frac{1}{n^2}$ $\quad a_n, b_n > 0$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{3n^2}{n^4+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{3n^4}{n^4+1} \cdot \frac{\frac{1}{n^4}}{\frac{1}{n^4}}$$

$$= \lim_{n\to\infty} \frac{3}{1+\frac{1}{n^4}} = 3 > 0$$

Since $\sum \frac{1}{n^2} < \infty$ (p-series, $p = 2 > 1$), by the LCT, $\sum \frac{3n^2}{n^4+1} < \infty$.

Convergent

Example 4 $\sum_{n=1}^{\infty} \frac{n!\,2^n}{n^n}$

Let $a_n = \frac{n!\,2^n}{n^n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!\,2^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)} \cdot 2 \cdot n^n}{(n+1)^n \cdot \cancel{(n+1)}} = \lim_{n\to\infty} 2 \cdot \left(\frac{n}{n+1}\right)^n$$

$e^{\ln x} = x$ method

$$= \lim_{n\to\infty} 2 \underbrace{\left(1 - \frac{1}{n+1}\right)^n}_{\to \frac{1}{e}} = \frac{2}{e} < 1$$

shown in Lecture 11 eg 4 Slide 21

$\therefore \sum_{n=1}^{\infty} \frac{n!\,2^n}{n^n}$ is absolutely convergent by the Ratio Test.

Power series

- A **power series** centered at a is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

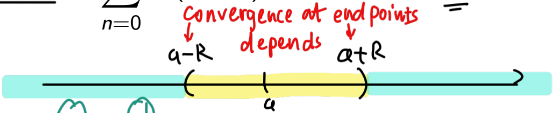
Taylor

Maclaurin

where x is a **variable** and c_n are constants called the **coefficients** of this power series.

$R \geqslant 0$

- The **radius of convergence** of $\sum_{n=0}^{\infty} c_n(x-a)^n$ is a **number** R such that



$$\sum_{n=0}^{\infty} c_n(x-a)^n \text{ converges if } |x-a| < R \text{ and diverges if } |x-a| > R.$$

Coefficient is 1

- R can be found using either the **Ratio Test** or the **Root Test**.

Example 5

$$a_n = \frac{(-1)^n x^{2n}}{n 9^n} \to \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\left| \frac{(-1)^n x^{2n}}{n 9^n} \right| \right)^{\frac{1}{n}}$$

Find the radius of convergence for the following power series.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n\,9^n}$ $\displaystyle = \lim_{n\to\infty} \left(\frac{|x|^{2n}}{n\ 9^n} \right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{|x|^2}{n^{\frac{1}{n}}\ 9}$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\displaystyle = \frac{|x|^2}{9} \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = \frac{|x|^2}{9} \frac{1}{\lim_{n\to\infty} n^{\frac{1}{n}}} = \frac{|x|^2}{9} < 1$

1

$\Rightarrow \quad |x|^2 < 9 \qquad \to \quad R = 9$ ✗

$\Rightarrow \quad |x| < 3 \quad \to \quad R = 3.$

Example 5 $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $x \neq 0$ $a_n = \frac{x^n}{n!}$

(2) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$

$= \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$ for all x

$\therefore \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent for all x

$\Rightarrow$ Radius of convergence is ∞

Taylor and Maclaurin series

- Let f be an **infinitely differentiable** function on an open interval centered at a: $(a-R, a+R)$ for some $R>0$.
- The **Taylor series of f at a** (or about a, or centered at a) is

coefficients

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

- The Taylor series centered at $a=0$ is called the **Maclaurin series** of f.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

Final exam details

- Thursday, 30 November 2023, 1400 - 1530 HRS.
- LT6C or LT6D depending on your surname in **alphabetical order**
  - LT6C: Surname/last name AARON SANUSI to LIONG
  - LT6D: Surname/last name LIU to ZHUO
- Material covered: Week 8 to Week 12; this may indirectly include material from the first half of the trimester.
- Format: 6/7 MCQ + 3 Open-ended
- Closed-book, no formula list will be provided.
- Graphic calculators or scientific calculators with definite integral calculation capabilities are **strictly not allowed**.