Fundamentals of Differentiation Part 2

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Recap

1. Defn of the derivative of f at a point a

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

2. Defn of the derivative of f (or the derivative function of f)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{y\to x} \frac{f(y) - f(x)}{y - x}$$

3. Differentiation rules: constant, power rule, trigo, expo, and log.
4. Algebraic differentiation rules: constant multiple, addition, difference, product, and quotient.

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Differentiating composite functions

The differentiation rules we have learnt in the last lecture cover most of the functions, with the exception of composite functions, for example

$$f(x) = \ln(\cos x), \quad g(x) = \sqrt{1 - x^2}.$$

How do we differentiate such functions? Using the *Chain Rule*.

Chain Rule

Theorem

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $f \circ g$ is differentiable at x and the derivative of $f \circ g$, $(f \circ g)'$ is given by

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

An alternative form of the chain rule: if $y = f(u)$ and $u = g(x)$ are differentiable, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

TLDR: Differentiate outer function f, sub in inner function g, then multiply by the derivative of the inner function.

Example 1

We differentiate $f(x) = \ln(\cos x)$. Set $g(x) = \ln x$ (outer function) and $h(x) = \cos x$ (inner function). Note that $f = g \circ h$. Then $g'(x) = \dfrac{1}{x}$ and $h'(x) = -\sin x$. Therefore, by the Chain Rule,

$$\begin{aligned} f'(x) = (g \circ h)'(x) &= g'(h(x)) \cdot h'(x) \\ &= \frac{1}{h(x)} \cdot (-\sin x) \\ &= \frac{1}{\cos x} \cdot (-\sin x) \\ &= -\frac{\sin x}{\cos x} \\ &= -\tan x. \end{aligned}$$

Example 2

We differentiate $f(x) = \sqrt{1 - x^2}$. Set $g(x) = \sqrt{x}$ and $h(x) = 1 - x^2$. Then $g'(x) = \dfrac{1}{2\sqrt{x}}$ and $h'(x) = -2x$. By the Chain Rule,

$$\begin{aligned} f'(x) = (g \circ h)'(x) &= g'(h(x)) \cdot h'(x) \\ &= \frac{1}{2\sqrt{h(x)}} \cdot (-2x) \\ &= \frac{1}{2\sqrt{1 - x^2}} \cdot (-2x) \\ &= -\frac{-2x}{2\sqrt{1 - x^2}} \\ &= -\frac{x}{\sqrt{1 - x^2}}. \end{aligned}$$

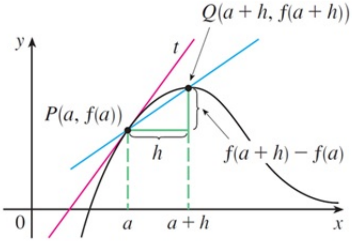
Exercise 1

Differentiate the following functions.

1. $\sin^2(x)$
2. $\sin(x^2)$
3. $(x^2 + 1)^6$
4. $(\star)$ $e^{\sin(x^2)}$
5. $(\star)$ $x^2 \ln(\tan x)$

Exercise 1

Tangent line to f



The magenta line here is called the *tangent line to the function f at the point* $(a, f(a))$. It has several properties:

1. It has the **same gradient** as the function f at the point $(a, f(a))$, i.e. its gradient is $f'(a)$.
2. It intersects the graph of $y = f(x)$ at only at the point $(a, f(a))$.

Tangent line equation

Using the information above, we can find the equation of the tangent line to f at $(a, f(a))$. Let

$$y = mx + c$$

be the equation of this tangent line, where m and c are unknown constants.

Since the gradient of this line is $f'(a)$, $m = f'(a)$, we have

$$y = f'(a)x + c.$$

This line contains the point $(a, f(a))$, so

$$f(a) = f'(a)a + c \implies c = f(a) - f'(a)a.$$

Therefore the equation of the tangent line to f at $(a, f(a))$ is

$$y = f'(a)x + f(a) - f'(a)a = f'(a)(x - a) + f(a).$$

Tangent line equation

Theorem

The equation of *tangent line to f at $(a, f(a))$* is

$$y = f'(a)(x - a) + f(a).$$

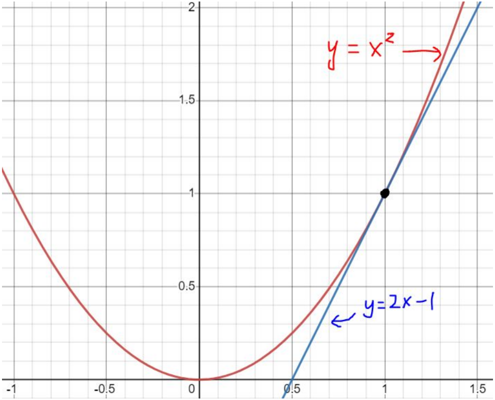
Example 3

We find the tangent line to $f(x) = x^2$ at the point $(1, 1)$.

We have $f(1) = 1$. Note that $f'(x) = 2x$, therefore $f'(1) = 2$. Putting this together, the equation of the tangent line to $f(x) = x^2$ at $(1, 1)$ is

$$\begin{aligned} y &= f'(1)(x-1) + f(1) \\ &= 2(x-1) + 1 \\ &= 2x - 1. \end{aligned}$$

Graph of Example 3



The black point is the point $(1, 1)$. One can observe that the line $y = 2x - 1$ is tangent to the graph of $f(x) = x^2$ at $a = 1$.

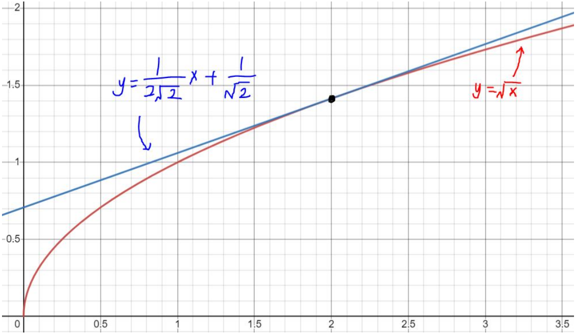
Example 4

We find the tangent line to $f(x) = \sqrt{x}$ at the point $(2, \sqrt{2})$.

We have $f(2) = \sqrt{2}$. Note that $f'(x) = \dfrac{1}{2\sqrt{x}}$, hence $f'(2) = \dfrac{1}{2\sqrt{2}}$. We put this together to get the equation of the tangent line to $f(x) = \sqrt{x}$ at $(2, \sqrt{2})$:

$$\begin{aligned} y &= f'(2)(x-2) + f(2) \\ &= \frac{1}{2\sqrt{2}}(x-2) + \sqrt{2} \\ &= \frac{1}{2\sqrt{2}}x - \frac{1}{\sqrt{2}} + \sqrt{2} \\ &= \frac{1}{2\sqrt{2}}x + \frac{1}{\sqrt{2}}. \end{aligned}$$

Graph of Example 4



Exercise 2

Find the tangent line for each of the following functions at the given points.

1. $f(x) = \dfrac{x+2}{x-3}$, at $(2, -4)$
2. $f(x) = \sqrt{1-3x}$, at $(-1, 2)$

Exercise 2

Explicit and implicit functions

Most of the functions so far which we have seen are written in an *explicit* form, where a variable y is expressed explicitly in terms of another variable x, called *explicit functions*, for example,

$$y = x^2 + 1, \quad \text{or} \quad y = \sqrt{1 - 3x}$$

or in general, $y = f(x)$. On the other hand, there are functions y in terms of x which are defined *implicitly*, called *implicit functions*, for example

$$x^2 + y^2 = 1, \quad \text{or} \quad \sin(y^2 + x) + \cos(x^2 + y) = 0.$$

Explicit and implicit functions

It is not always possible to feasibly find an explicit formula for y given an implicit definition.

1. Let $x^2 + y^2 = 1$, where y is defined implicitly here. We can make y the subject of this equation, which yields two *explicit* functions:

$$y = \sqrt{1 - x^2}, \quad \text{or} \quad y = -\sqrt{1 - x^2}.$$

2. On the other hand, let $\sin(y^2 + x) + \cos(x^2 + y) = 0$, where y is also defined implicitly here. It is clearly not obvious nor is feasible to try to make y the subject of this equation.

Implicit differentiation

Problem: The derivatives which we have found so far are for explicit functions.

Question: Can we still find $\dfrac{dy}{dx}$ for implicit functions?

The answer to this question is yes. We can differentiate an implicit function y, provided y is a differentiable function.

($\star$) In this course, with respect to implicit differentiation, it is always assumed that y is a differentiable function.

Example 5

Consider $x^2 + y^2 = 1$. We differentiate both sides of the equation, taking into account that y is a function of x:

$$\begin{aligned} \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(1) \\ \implies \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= 0 \\ \implies 2x + \frac{d}{dx}(y^2) &= 0. \end{aligned}$$

Now, since y is a function of x, y^2 is therefore a composite function of x, thus the Chain Rule applies (then make $\dfrac{dy}{dx}$ the subject of the equation):

$$2x + 2y \cdot \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}.$$

Exercise 3

For the following equations, find $\dfrac{dy}{dx}$.

1. $x^3 + y^3 = 6xy$
2. $2x^2 + xy - y^2 = 2$
3. $e^x \sin(y) = x + y$

Exercise 3

Exercise 4

Find the equation of the tangent line to the graph of $x^2 + 3y^2 = 16$ at the point $(2, 2)$.