application of integration

Volumes of Revolution

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AY 23/24 Trimester 1

Quiz 2 — Week 1-6 material, 1 hr

in tutorials — 6+2 format

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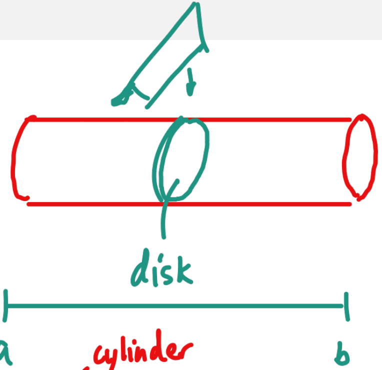
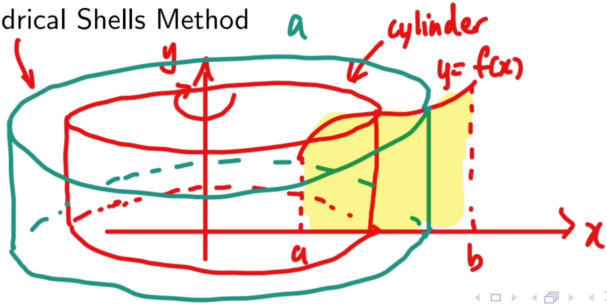




Vertical plane slicing

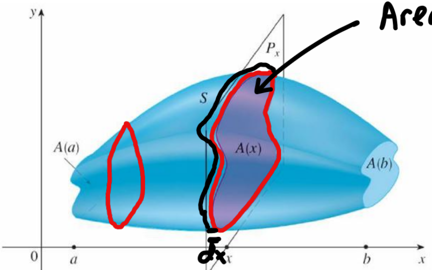
variable in x, revolve around x-axis

slice perpendicular to the x-axis

1. We slice the solid S vertically using a plane.
2. We calculate, **for every** $x \in [a, b]$, the **cross-sectional area function** $A(x)$.
3. Summing all these cross-sections means integrating $A(x)$ from $x = a$ to $x = b$, which gives us the volume of the solid, i.e.

$$\text{Volume of solid } S = \int_a^b \underbrace{A(x)}_{\text{area}} \underbrace{dx}_{\text{thickness}} .$$

Area of cross-section



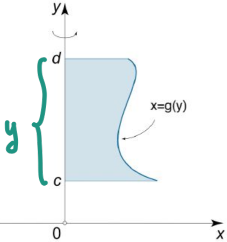
Horizontal plane slicing

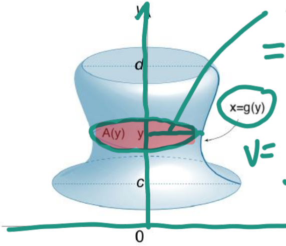
→ Variable in y, revolve around y-axis

→ perpendicular to y-axis

1. We slice the solid S horizontally using a plane.
2. We calculate, **for every** $y \in [c, d]$, the **cross-sectional area function** $A(y)$.
3. Summing all these cross-sections means integrating $A(y)$ from $y = c$ to $y = d$, which gives us the volume of the solid, i.e.

$$\text{Volume of solid } S = \int_c^d \underbrace{A(y)}_{\text{area}} \underbrace{dy}_{\text{thickness}} .$$





$A(y) = \pi(g(y))^2$

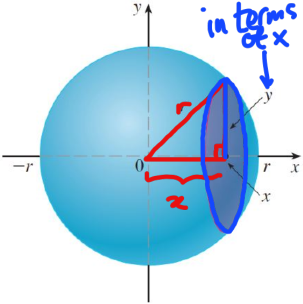
$V = \int_c^d \pi(g(y))^2 \, dy$

Example 1

$$x^2+y^2=r^2 \Rightarrow y=\sqrt{r^2-x^2} \quad y>0$$

Show that the volume V of a sphere of radius $r>0$ is $\dfrac{4}{3}\pi r^3$.

Radius of the disk (in terms of x) $=\sqrt{r^2-x^2}$



Area of cross-section $A(x)$ $=\pi\left(\sqrt{r^2-x^2}\right)^2$

$$=\pi(r^2-x^2)$$

$$V=\int_{-r}^{r}A(x)\,dx=\pi\int_{-r}^{r}\underbrace{r^2-x^2}_{\text{even}}\,dx$$

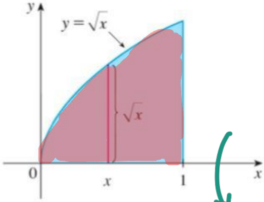
$$=2\pi\int_0^r r^2-x^2\,dx=2\pi\left[r^2x-\frac{x^3}{3}\right]_0^r=2\pi\left[r^3-\frac{r^3}{3}\right]$$

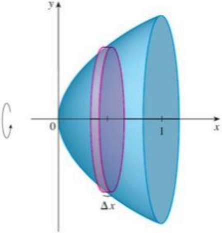
$$=\frac{4\pi r^3}{3}$$

Solids by revolving a curve about an axis

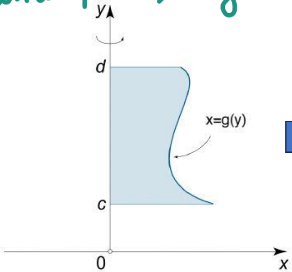
Vertical plane slicing

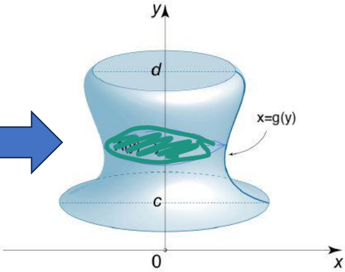
donut





Horizontal plane slicing





Cross-sectional area

TLDR $\rightarrow \pi R^2$

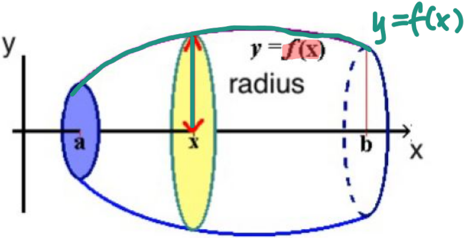
To find these volumes of revolution, the key step is to find the cross-sectional area (either $A(x)$ or $A(y)$ depending on vertical or horizontal planar slicing). We focus on two cases:

- Cross-section is a **disk**. 
- Cross-section is a **washer**. 

Area of big disk
− Area of small disk

Cross-section is a disk

sphere example



- We use vertical slicing (since function $y = f(x)$ is in variable x).
- Cross-section is a disk with radius $f(x)$. Thus the cross-sectional area function is

$$A(x) = \pi(f(x))^2.$$

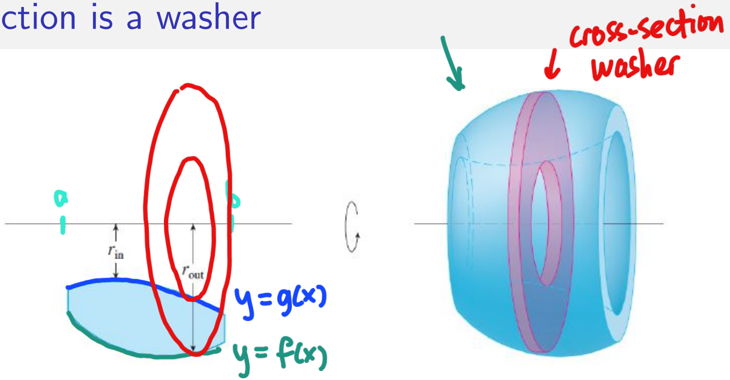
$A(y) = \pi(g(y))^2$

horizontal slicing

- By summing all cross-sectional areas from $x = a$ to $x = b$, we obtain the volume V of the solid. Thus

$$V = \int_a^b A(x)\,dx = \int_a^b \pi(f(x))^2\,dx.$$

Cross-section is a washer



- The cross-sectional area is the difference of the area of two disks:

$$A(x) = \underbrace{\pi r_{out}^2}_{\text{bigger disk}} - \overbrace{\pi r_{in}^2}^{\text{smaller disk}}.$$

- The volume V of the solid is

$$V = \int_a^b A(x)\,dx.$$

Example 2

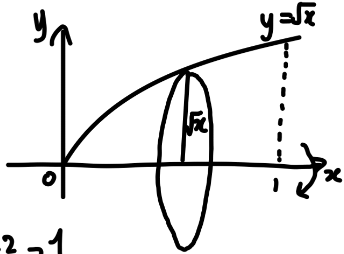
Find the volume of the solid obtained by rotating about the x-axis, the region under the curve $y = \sqrt{x}$ from $x = 0$ to $x = 1$.

$$A(x) = \pi(\sqrt{x})^2 = \pi x$$

$$\text{Volume of solid} = \int_0^1 A(x)\,dx$$

$$= \int_0^1 \pi x\,dx = \pi\left[\frac{x^2}{2}\right]_0^1$$

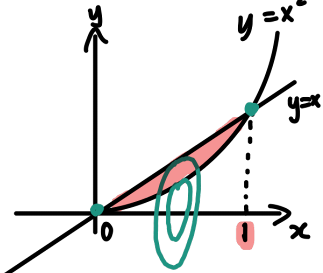
$$= \frac{\pi}{2}.$$



Example 2

Exercise 1 Washer

The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.



$r_{out} = x \quad , \quad r_{in} = x^2$

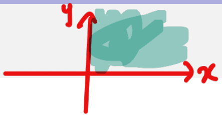
$$A(x) = \pi r_{out}^2 - \pi r_{in}^2$$
$$= \pi x^2 - \pi x^4$$

$$\text{Volume of solid} = \pi \int_0^1 x^2 - x^4 \, dx$$
$$= \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1$$
$$= \pi \left[\frac{1}{3} - \frac{1}{5} \right] = \frac{2\pi}{15}$$

$$x^2 = x$$
$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0 \text{ or } x = 1$$

Exercise 1

Exercise 2



Cross-sectional method variable in y

Find the volume of the solid generated by rotating the region bounded by $y = x^{\frac{1}{3}}$ and $y = \frac{x}{4}$ that lies in the first quadrant, about the y-axis.

$x = y^3$ $\quad$ $x = 4y$

$y = x^3$ $\quad$ $r_{out} = 4y, \quad r_{in} = y^3$

$$V = \int_0^2 \pi(4y)^2 - \pi(y^3)^2 \, dy$$

$$= \pi \int_0^2 16y^2 - y^6 \, dy$$

$$= \pi \left[\frac{16y^3}{3} - \frac{y^7}{7} \right]_0^2 = \pi \left[\frac{16 \cdot 8}{3} - \frac{2^7}{7} \right]$$

$\frac{512\pi}{21}$

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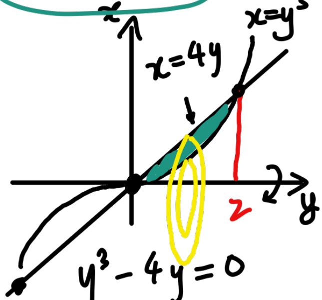


$y^3 - 4y = 0$

$y(y^2 - 4) = 0$

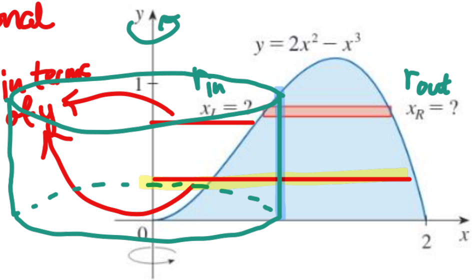
$y = 0, \; y = \pm 2$

Exercise 2

Cross-sectional method can be difficult

- Some volumes of revolution can be difficult to handle with the cross-sectional method.
- Let's suppose we wish to find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$ (see figure below).
- If we use the cross-sectional method, we would need to find the area of the washer, and as we can see, it is not easy to find r_{out} and r_{in}: x in terms of y.

Cross-sectional method is infeasible to use



$y = 2x^2 - x^3$

Make x in terms of y

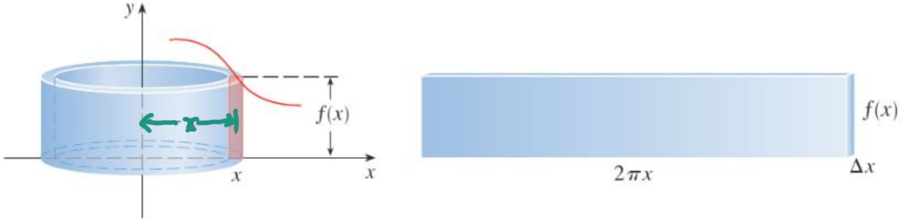
$y = x^2(2-x)$

Cylindrical Shells

different variable

- For each x, we rotate the height $f(x)$ around the y-axis, the resulting **cylindrical shell** has a radius x, with thickness dx.
- Each of these shells has a volume of $2\pi x f(x)\,dx$ (see figure below).
- Summing these shells from $x = a$ to $x = b$, we get the volume V of the solid is

$$V = \int_a^b \underbrace{2\pi x}_{\text{circumference}} \overbrace{f(x)}^{\text{height}} \overbrace{dx}^{\text{thickness}}.$$

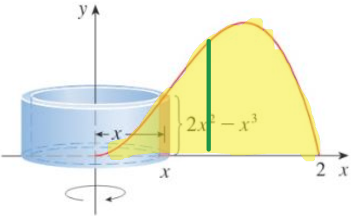


Example 3

Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = \underbrace{2x^2 - x^3}$ and $y = 0$.

Height $= 2x^2 - x^3$

radius $= x$



$$V = \int_0^2 2\pi x\,(2x^2 - x^3)\,dx$$

$$= \pi \int_0^2 4x^3 - 2x^4\,dx$$

$$= \pi \left[x^4 - \frac{2x^5}{5} \right]_0^2 = \pi \left[16 - \frac{64}{5} \right] = \frac{16\pi}{5}.$$

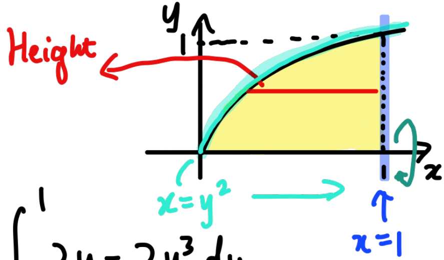
Example 4 Answer was $\frac{\pi}{2}$

$x = y^2$

Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis, the region under the curve $y = \sqrt{x}$ from $x = 0$ to $x = 1$.

Height $= 1 - y^2$

$x=$ $\quad$ $x=$



Radius $= y$

$$V = \int_0^1 2\pi y(1-y^2)\,dy = \pi \int_0^1 2y - 2y^3\,dy$$

$$= \pi\left[y^2 - \frac{y^4}{2}\right]_0^1 = \pi\left[1 - \frac{1}{2}\right]$$

$$= \frac{\pi}{2}$$

Example 4

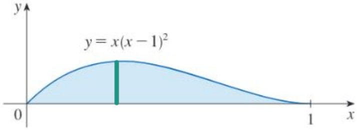
Exercise 3

Let S be the solid obtained by rotating the region shown in the figure below about the y-axis.

- Briefly explain why it might be awkward to find the volume V of S using the washer method. Difficult to make x in terms of y
- Find V using cylindrical shells.

Height $= x(x-1)^2$

Radius $= x$



$$V = \int_0^1 2\pi x \, x(x-1)^2 \, dx$$

$$= 2\pi \int_0^1 x^2(x-1)^2 \, dx = 2\pi \int_0^1 (x^2-x)^2 \, dx$$

Exercise 3

$$2\pi \int_0^1 (x^2-x)^2 \, dx$$

$$= 2\pi \int_0^1 x^4 - 2x^3 + x^2 \, dx$$

$$= 2\pi \left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3} \right]_0^1$$

$$= 2\pi \left[\frac{1}{5} - \frac{1}{2} + \frac{1}{3} \right] = \frac{\pi}{15}.$$