CSD1241 Tutorial 2 Answers Keys

Problem 1. Consider two points

$$P = (9, -1),\ Q = (5, -3)$$

(a) Find the general equation, the vector equation and the parametric equation of the line l passing through P and Q.
(b) Find the condition for a, b, c so that the line $l' : ax + by + c = 0$ is parallel to l.
(c) Find the condition on d, e, f so that $l'' : dx + ey + f = 0$ is perpendicular to l.
Hint. l has direction $\overrightarrow{PQ}$, l' has direction $\vec{v} = \begin{bmatrix} -b \\ a \end{bmatrix}$ and l'' has direction $\vec{w} = \begin{bmatrix} -e \\ d \end{bmatrix}$.

Solution. (a) Vector equation and parametric equation

$$(x, y) = P + t\vec{d} = (9, -1) + t \begin{bmatrix} -4 \\ -2 \end{bmatrix} \text{ and } \begin{cases} x = 9 - 4t \\ y = -1 - 2t \end{cases}.$$

General equation $x - 2y - 11 = 0$.
(b) $b = -2a$ and c can be any number.
(c) $d = 2e$ and f can be any number. □

Problem 2. Consider the point $P = (3, 2)$. In each of the following cases, find the distance from P to the given line l. Further, find the point Q on l which is at the shortest distance to P (Q is the orthogonal projection of P onto l).

(a) l has general equation

$$x - y - 3 = 0.$$

(b) l has vector equation

$$(x, y) = (1, -1) + t \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

(c) l has parametric equation

$$\begin{cases} x = 3t \\ y = 1 - 2t \end{cases}$$

(d) l passes through $A(0, 5)$ and $B(10, 1)$.

Solution. (a) $d(P, l) = \sqrt{2}$. There are two ways to find Q.

Solution 1. Assume $Q = (x, y)$. Then Q satisfies two conditions

1. $\overrightarrow{PQ} = \begin{bmatrix} x-3 \\ y-2 \end{bmatrix}$ is parallel to the normal vector $\vec{n} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ of l, which implies

$$\frac{x-3}{1} = \frac{y-2}{-1} \Leftrightarrow x + y - 5 = 0 \tag{1}$$

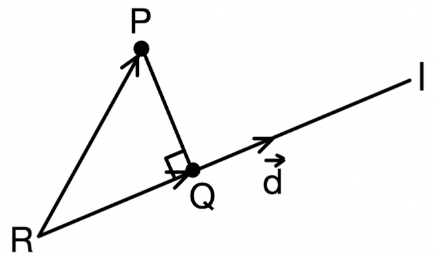
2. Q is on l, which implies

$$x - y - 3 = 0 \tag{2}$$

By (1) and (2), we obtain

$$Q = (x, y) = (4, 1).$$

Solution 2. Let $R = (3, 0)$ be a point on l and let $\vec{d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ be a direction vector of l.



Note that $\overrightarrow{RP} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Further, we observe that

$$\overrightarrow{RQ} = \text{proj}_{\vec{d}}(\overrightarrow{RP}) = \frac{\overrightarrow{RP} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \vec{d} = \vec{d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Since $\overrightarrow{RQ} = Q - R$, we obtain

$$Q = R + \overrightarrow{RQ} = (3, 0) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (4, 1).$$

(b) General equation of $l : x - 1 = 0$. $d(P, l) = 2$.
Since l is the vertical line $x = 1$, the orthogonal projection of P on l has

$$\text{x-coordinate} = 1 \text{ and y-coordinate} = \text{y-coordinate of P} = 2$$

Therefore, $Q = (1, 2)$.

(c) General equation of $l : 2x + 3y - 3 = 0$. $d(P, l) = \dfrac{9}{\sqrt{13}}$.
Applying one of the methods in a, you should get $Q = \left(\dfrac{21}{13}, -\dfrac{1}{13}\right)$.

(d) General equation of $l : 2x + 5y - 25 = 0$. $d(P, l) = \dfrac{9}{\sqrt{29}}$.

Applying one of the methods in a, you should get $Q = \left(\dfrac{105}{29}, \dfrac{103}{29}\right)$. $\square$

Problem 3. Find the normal equation (form $ax + by + cz = d$) of the plane β in the following cases

(a) β goes through $P = (1, -1, 2)$ and has normal vector $\vec{n} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

(b) β goes through $S = (1, 2, 3)$ and parallel to the plane $\alpha : 3x - 2y + z = 7$.
(c) β goes through $S = (1, 2, 3)$ and perpendicular to the line

$$l : (x, y, z) = (1, -1, 2) + t \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Solution. (a) $2x - y + z = 5$.
(b) $3x - 2y + z = 2$.
(c) $x + y - z = 0$. $\square$

Problem 4. Let α be the plane going through 3 points $P = (1, -1, 2), Q = (3, 1, 0), R = (2, 1, 1)$.
(a) Find the vector equation and the parametric equation of α.
(b) Find the general equation (form $ax + by + cz + d = 0$) of α.

Hint. Let $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a normal vector of α. Then $\vec{n} \cdot \overrightarrow{PQ} = 0$ and $\vec{n} \cdot \overrightarrow{PR} = 0$. You can find $\vec{n}$ from these 2 equations.
(c) Find the distance from the point $A = (1, 1, 1)$ to α.
(d) Find the point B on α which is at the closet distance to A (Hint. B = orthogonal projection of A onto α).

Solution. (a) Vector equation and parametric equation are

$$(x, y, z) = (1, -1, 2) + s \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{cases} x = 1 + 2s + t \\ y = -1 + 2s + 2t \\ z = 2 - 2s - t \end{cases}$$

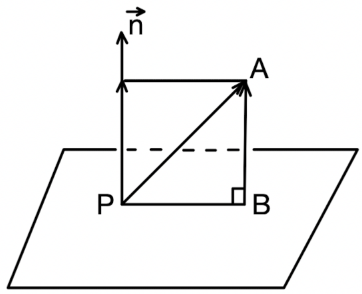
(b) Assume $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a normal vector of α. Then

$$\vec{n} \perp \overrightarrow{PQ} \quad \Rightarrow \quad \vec{n} \cdot \overrightarrow{PQ} = 0 \Rightarrow a + b - c = 0 \tag{3}$$
$$\vec{n} \perp \overrightarrow{PQ} \quad \Rightarrow \quad \vec{n} \cdot \overrightarrow{PQ} = 0 \Rightarrow a + 2b - c = 0 \tag{4}$$

Solving (3) and (4) gives b=0, a=c. So $\vec{n} = \begin{bmatrix} c \\ 0 \\ c \end{bmatrix}$. Choosing $c = 1$ gives $\vec{n} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. The general equation for α is $x + z - 3 = 0$.

(c) $d(A, \alpha) = \dfrac{1}{\sqrt{2}}$.

(d) Note that $\vec{n} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\overrightarrow{PA} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ and $\overrightarrow{BA} = \text{proj}_{\vec{n}}(\overrightarrow{PA})$.



We have

$$A - B = \overrightarrow{BA} = \text{proj}_{\vec{n}}(\overrightarrow{PA}) = \frac{\overrightarrow{PA} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n} = \frac{-1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ -1/2 \end{bmatrix}$$

We obtain

$$B = A - \overrightarrow{BA} = (1, 1, 1) - \begin{bmatrix} -1/2 \\ 0 \\ 01/2 \end{bmatrix} = \left(\frac{3}{2}, 1, \frac{3}{2} \right).$$

$\square$