MAT 140

Linear Algebra and Affine Geometry

Dr Ton Boerkoel

DigiPen Institute of Technology

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Contents

Introduction

Part 1

An Exploration of 2 and 3 Dimensional Euclidean Space

Part II

Matrices, Systems of Equations and Row reduction

Part III

Linear Transformations

Part IV

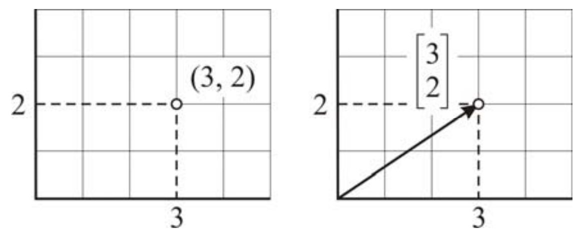
Affine Transformations

16. Linear Transformations.

We are going to focus next on transformations of two and three dimensional space. A lot of the transformations we are interested in, like reflections, projections, rotations and translations, can be very nicely described using vectors and matrices. Hence we are going to change our notation entirely in terms of vectors.

We used to distinguish between points and vectors:

$$(3, 2) \text{ referred to point, and } \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ referred to a vector.}$$



But there is no need to use different notations. The vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is pointing exactly at the point $(3, 2)$. The only reason we insisted on a different notation was to make you aware of the fact that these are entirely different objects. But as far as notation goes we could have used the same notation for both. From now on we will only use *vector* notation for *both* of them:

When we talk about the *point* $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ we are referring to the endpoint of the vector, i.e. the point $(3, 2)$.

When we talk about the *vector* $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ we are referring to the actual vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, starting at the origin $(0, 0)$ and ending at the point $(3, 2)$.

Hence from now on we will no longer use the following equations of a line

$$l: \quad (x, y) = (3, 2) + t\begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

and a plane

$$\alpha: \quad (x, y, z) = (5, 1, 2) + t\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Instead we will use the vector notations

$$l: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + t\begin{bmatrix} 4 \\ -1 \end{bmatrix} \quad \text{and} \quad \alpha: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

There is actually an advantage to this notation. The equations of the line

$$l: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + t\begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

in $\mathbb{R}^2$ can be transformed into normal form by simply dotting the entire vector equation with the normal vector $\vec{n} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, i.e. on both sides of the equation

$$\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} + t\begin{bmatrix} 4 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \Rightarrow \quad x + 4y = 11$$

Similarly, dotting the equation of the plane $\alpha: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ with the normal $\vec{n} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ would give is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ which

immediately simplifies to the normal equation of the plane:

$$x - y - z = 2$$

In general the normal equation for a *line* $(ax + by = c)$ in $\mathbb{R}^2$ can now be written as

$$\vec{n} \cdot \vec{x} = c \quad \text{or} \quad \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0 \quad \text{or} \quad \vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$$

And the *same* equations can be used for a *plane* in $\mathbb{R}^3$:

$$\vec{n} \cdot \vec{x} = c \quad \text{or} \quad \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0 \quad \text{or} \quad \vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$$

except that in last case $\vec{n}, \vec{x}, \vec{x}_0 \in \mathbb{R}^3$.

Example 1: **(a)** The line $x + 4y = 11$ in $\mathbb{R}^2$, can be written as $\begin{bmatrix} 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 11$

$$\text{or} \quad \begin{bmatrix} 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 \\ 4 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) = 0 .$$

(b) The plane $x-y-z=2$ in $\mathbb{R}^3$ can be written as: $\begin{bmatrix}1\\-1\\-1\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=2$

or $\begin{bmatrix}1\\-1\\-1\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\-1\\-1\end{bmatrix}\cdot\begin{bmatrix}5\\1\\2\end{bmatrix}$ *or* $\begin{bmatrix}1\\-1\\-1\end{bmatrix}\cdot\left(\begin{bmatrix}x\\y\\z\end{bmatrix}-\begin{bmatrix}5\\1\\2\end{bmatrix}\right)=0$

Note that we get the same parametric equations from either the old notation or the new vector notation:

$$\left.\begin{array}{l} l:\ (x,y)=(3,2)+t\begin{bmatrix}4\\-1\end{bmatrix} \\ l:\ \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}+t\begin{bmatrix}4\\-1\end{bmatrix}\end{array}\right\}\Rightarrow\begin{cases}x=3+4t\\ y=2-t\end{cases}$$

or for example for a plane

$$\left.\begin{array}{l} \alpha:\ (x,y,z)=(5,1,2)+t\begin{bmatrix}3\\2\\1\end{bmatrix}+s\begin{bmatrix}1\\1\\0\end{bmatrix} \\ \alpha:\ \begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}5\\1\\2\end{bmatrix}+t\begin{bmatrix}3\\2\\1\end{bmatrix}+s\begin{bmatrix}1\\1\\0\end{bmatrix}\end{array}\right\}\Rightarrow\begin{cases}x=5+3t+s\\ y=1+2t+s\\ z=2+t\end{cases}$$

When we talk about the line l given by the vector equation $l:\ \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}+t\begin{bmatrix}4\\-1\end{bmatrix}$ we of course refer to the points on the line, which are found in this vector equation as the *endpoints* of the vectors, not to the actual vectors. For example when $t=3$ the vector equation really give us the vector $\begin{bmatrix}3\\2\end{bmatrix}+3\begin{bmatrix}4\\-1\end{bmatrix}=\begin{bmatrix}15\\-1\end{bmatrix}$. The vector is not actually *on* the line but its endpoint *is*! The vector in this case it is pointing at the point $(15,-1)$ which is on the line.

Similarly when we refer to the plane $\alpha:\ \begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}5\\1\\2\end{bmatrix}+t\begin{bmatrix}3\\2\\1\end{bmatrix}+s\begin{bmatrix}1\\1\\0\end{bmatrix}$ we are referring to the *endpoints* of the vectors that this equation gives us. Those endpoints are the points of the plane.

Now we are going to use matrices to describe **transformations** between the spaces $\mathbb{R}^n$.
For example take the function $T:\ \mathbb{R}^2 \to \mathbb{R}^3$ defined by a matrix multiplication as follows:

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Hence e.g. $T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$. We say that the **image** of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$, or that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is **mapped to** $\begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$. The image of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is $T\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 11 \end{bmatrix}$.

We will see that a lot of the transformations we are interested in can be defined using a matrix multiplication: $T(\vec{x}) = A\vec{x}$. These turn out to be linear transformations.

Linear Transformations

Definition: Let V and W be two real vector spaces. A map (function) $T:\ V \to W$ is called a **linear transformation** if

(a) $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$ for all $\vec{x}, \vec{y} \in V$

(b) $T(t\cdot\vec{x}) = t\cdot T(\vec{x})$ for all $\vec{x} \in V$ and $t \in \mathbb{R}$

Example 2: $T:\ \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a-b \\ 2a+b \\ 3a-4b \end{bmatrix}$ is a linear transformation. The easiest way to show this is by rewriting T as a matrix multiplication:

$$T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

Then if we take two vectors $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\vec{y} = \begin{bmatrix} A \\ B \end{bmatrix} \in \mathbb{R}^2$ and use properties of matrices we get

$$T(\vec{x}+\vec{y}) = T\left(\begin{bmatrix} a \\ b \end{bmatrix}+\begin{bmatrix} A \\ B \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & -4 \end{bmatrix}\left(\begin{bmatrix} a \\ b \end{bmatrix}+\begin{bmatrix} A \\ B \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}+\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = T(\vec{x})+T(\vec{y})$$

And $$T(t\cdot\vec{x}) = T\left(t\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & -4 \end{bmatrix}\left(t\begin{bmatrix} a \\ b \end{bmatrix}\right) = t\cdot\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = t\cdot T(\vec{x})$$

Theorem 16.1: If a map $T:\ \mathbb{R}^n \to \mathbb{R}^m$ is defined by a matrix multiplication

$$T(\vec{x}) = A\vec{x}$$

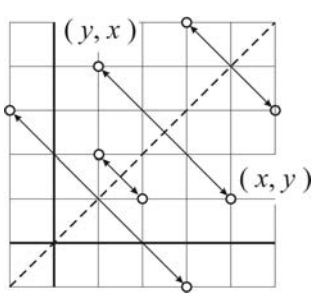
where A is a $m\times n-$matrix with real entries, then T is a linear transformation.

Proof: **(a)** $T(\vec{x}+\vec{y}) = A(\vec{x}+\vec{y}) = A\vec{x}+A\vec{y} = T(\vec{x})+T(\vec{y})$ for all $\vec{x},\vec{y}\in V$

(b) $T(t\cdot\vec{x}) = A(t\cdot\vec{x}) = t\cdot A\vec{x} = t\cdot T(\vec{x})$ for all $\vec{x}\in V$ and $t\in\mathbb{R}$

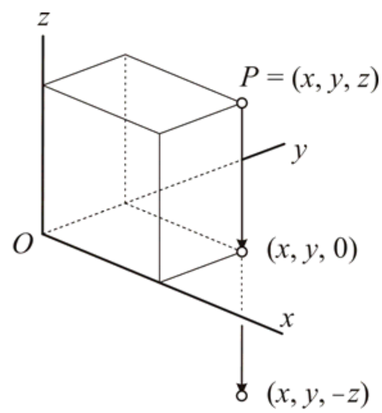
A lot of the transformations we will be looking at are of the form $T(\vec{x}) = A\vec{x}$.

Example 3: Let the map $T:\ \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(\vec{x}) = A\vec{x}$ with $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, i.e. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Note that $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$. Hence this transformation maps the point (x, y) to (y, x); i.e. it flips coordinates. Therefore this map is the reflection in the line $y = x$.



Example 4: Let $T:\ \mathbb{R}^3 \to \mathbb{R}^3$ be the map that reflects points in the *xy*-plane, i.e. the point $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is mapped to $\begin{bmatrix} x \\ y \\ -z \end{bmatrix}$. This can be done with the following matrix multiplication:

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$



Example 5: Let $T:\ \mathbb{R}^3 \to \mathbb{R}^3$ be the map defined by $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, i.e. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$. This is an orthogonal projection onto the *xy*-plane.

Example 6: Let the map $T:\ \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

Let's try to figure out what this map does by answering the following questions:

(a) Are there any points that are fixed under this transformation: i.e. $T(\vec{x}) = \vec{x}$?

(b) What points are mapped to zero?

(c) What is the range of T, i.e. the set of all images of T?

(d) What are all the points that get mapped to the point $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$?

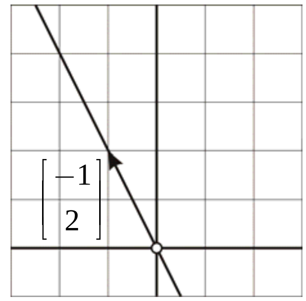
Solutions: **(a)** $T(\vec{x}) = \vec{x}$

$$\Rightarrow \frac{1}{5}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5x \\ 5y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x - 2y \\ -2x + 4y \end{bmatrix} = \begin{bmatrix} 5x \\ 5y \end{bmatrix} \quad \Rightarrow \begin{bmatrix} -4x - 2y \\ -2x - y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note that the equations $\begin{cases} -4x-2y = 0 \\ -2x-y = 0 \end{cases}$ both are multiples of $2x+y=0$, hence we can conclude that all the points that are fixed satisfy $2x+y=0$,



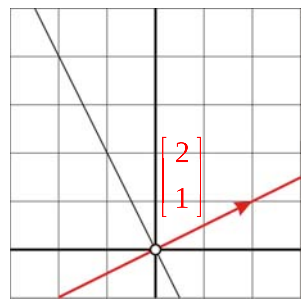
i.e. all fixed points are given by $\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} -1 \\ 2 \end{bmatrix}$:

The entire line $y=-2x$ is fixed, mapped to itself.

(b) To find what gets mapped to zero we need to solve: $T(\vec{x}) = \vec{0}$

$$\frac{1}{5}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



Note that the equations $\begin{cases} x-2y = 0 \\ -2x+4y = 0 \end{cases}$ are multiples of $x-2y=0$, hence all points $\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are mapped to the origin $(y=\frac{1}{2}x)$.

Note this line is perpendicular to the line we found in **(a)**.

(c) When we rewrite
$$T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\left(x\begin{bmatrix} 1 \\ -2 \end{bmatrix} + y\begin{bmatrix} -2 \\ 4 \end{bmatrix}\right) = \frac{1}{5}\left(x\begin{bmatrix} 1 \\ -2 \end{bmatrix} - 2y\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \frac{x-2y}{5}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

we see that the image $T(\vec{x})$ is a multiple of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. These were all the fixed points too.

Apparently *all* points are mapped onto this line.

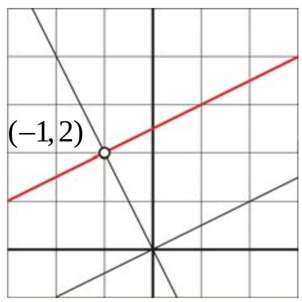
Note that we can also easily read off the answer to **(b)** from $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{x-2y}{5}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

(d) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow \frac{1}{5}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x - 2y = -5 \\ -2x + 4y = 10 \end{cases}$$

Note that the equations $\begin{cases} x - 2y = -5 \\ -2x + 4y = 10 \end{cases}$ are multiples of $x - 2y = -5$. Hence all points on the line $x - 2y = -5$ get mapped to the point $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

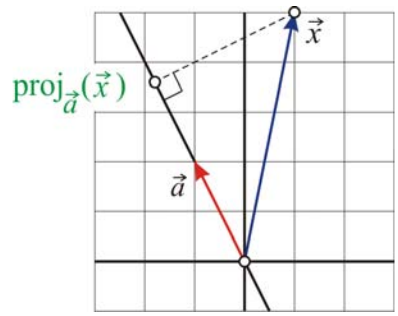


By now it may have become clear that this map seems to be the orthogonal projection onto the line $2x + y = 0$.

[Recall that $2x + y = 0$ was the fixed line **(a)**, and all points of $\mathbb{R}^2$ were mapped onto this line **(c)**; further more we just found in **(b)** and **(e)** that the two lines $x - 2y = 0$ and $x - 2y = -5$, each perpendicular to the fixed line, both get mapped to just *one* point: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ resp.]

Now let's prove that indeed T is the orthogonal projection onto the line $2x + y = 0$. After all we know how to project vectors onto a line through $\vec{a} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$:

$$\text{proj}_{\vec{a}}(\vec{x}) = \text{proj}_{\left[\begin{smallmatrix}-1\\2\end{smallmatrix}\right]}\begin{bmatrix}x\\y\end{bmatrix} = \frac{\begin{bmatrix}-1\\2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}}{\begin{bmatrix}-1\\2\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}}\begin{bmatrix}-1\\2\end{bmatrix}$$
$$= \frac{-x+2y}{5}\begin{bmatrix}-1\\2\end{bmatrix}$$
$$= \frac{1}{5}\begin{bmatrix}x-2y\\-2x+4y\end{bmatrix}$$
$$= \frac{1}{5}\begin{bmatrix}1 & -2\\-2 & 4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

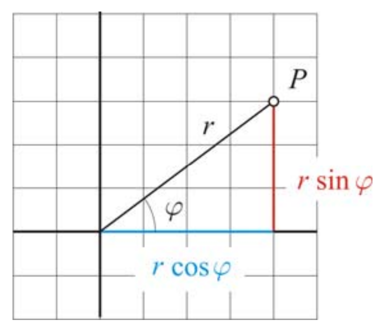


And indeed $T\begin{bmatrix}x\\y\end{bmatrix} = \frac{1}{5}\begin{bmatrix}1 & -2\\-2 & 4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$ was exactly the transformation we were given.

Example 7: Let the map $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation around the origin over the angle θ [in degrees or radians]. We'll show this is a linear map by finding a matrix A such that $T(\vec{x}) = A\vec{x}$.

Recall that the point P with polar coordinates r and φ has the following rectangular coordinates:

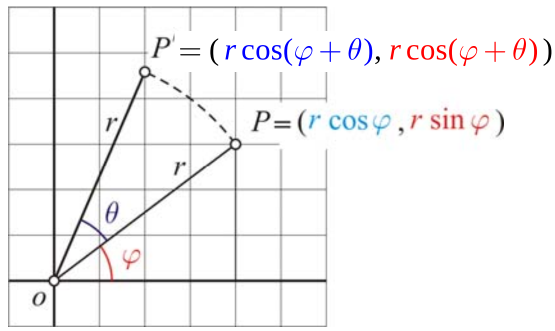
$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$



Also recall the following trig identities:

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

Let $P = \begin{bmatrix}x\\y\end{bmatrix}$ be a point in $\mathbb{R}^2$ and $P' = \begin{bmatrix}x'\\y'\end{bmatrix} = T\begin{bmatrix}x\\y\end{bmatrix}$ the image of $\begin{bmatrix}x\\y\end{bmatrix}$ after rotation around the origin over θ. We can now compute the coordinates of the point $\begin{bmatrix}x'\\y'\end{bmatrix}$:



$$\begin{cases} x' = r\cos(\varphi+\theta) \\ y' = r\sin(\varphi+\theta) \end{cases} \Rightarrow \begin{cases} x' = r\cos\varphi\cos\theta - r\sin\varphi\sin\theta \\ y' = r\sin\varphi\cos\theta + r\cos\varphi\sin\theta \end{cases}$$

$$\Rightarrow \begin{cases} x' = x\cos\theta - y\sin\theta \\ y' = y\cos\theta + x\sin\theta \end{cases}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ y\cos\theta + x\sin\theta \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Hence we have found that $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ is the rotation around the origin over the angle θ, and since it is written as $T(\vec{x}) = A\vec{x}$ it is a linear transformation.

Here are some examples of rotations and their matrices:

$$\text{If } \theta = 30^\circ \text{ then } T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$\text{If } \theta = 90^\circ \text{ then } T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$\text{If } \theta = 45^\circ \text{ then } T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Linear Transformation are determined by the images of basis vectors

One of the great properties of linear transformations is that they are completely determined by what happens to a basis.

Example 8: Let $T:\ \mathbb{R}^3 \to \mathbb{R}^2$ be linear, and suppose we know that

$$T\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix},\quad T\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}1\\4\end{bmatrix},\quad T\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}5\\6\end{bmatrix}$$

i.e. we know the **images** of the basis vectors $\vec{e}_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{e}_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{e}_1=\begin{bmatrix}0\\0\\1\end{bmatrix}$,

then

$$\begin{aligned}
T\begin{bmatrix}x\\y\\z\end{bmatrix} &= T\left(\begin{bmatrix}x\\0\\0\end{bmatrix}+\begin{bmatrix}0\\y\\0\end{bmatrix}+\begin{bmatrix}0\\0\\z\end{bmatrix}\right)\\
&= T\left(x\begin{bmatrix}1\\0\\0\end{bmatrix}+y\begin{bmatrix}0\\1\\0\end{bmatrix}+z\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = T\left(x\begin{bmatrix}1\\0\\0\end{bmatrix}\right)+T\left(y\begin{bmatrix}0\\1\\0\end{bmatrix}\right)+T\left(z\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\\
&= x\,T\begin{bmatrix}1\\0\\0\end{bmatrix}+y\,T\begin{bmatrix}0\\1\\0\end{bmatrix}+z\,T\begin{bmatrix}0\\0\\1\end{bmatrix}\\
&= x\begin{bmatrix}3\\2\end{bmatrix}+y\begin{bmatrix}1\\4\end{bmatrix}+z\begin{bmatrix}5\\6\end{bmatrix}\\
&= \begin{bmatrix}3&1&5\\2&4&6\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}
\end{aligned}$$

Note that the columns of the matrix $A=\begin{bmatrix}3&1&5\\2&4&6\end{bmatrix}$ are precisely the **images** of the standard basis vectors $\vec{e}_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{e}_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec{e}_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$, i.e

$$A=\begin{bmatrix}3&1&5\\2&4&6\end{bmatrix}=\begin{bmatrix}\uparrow&\uparrow&\uparrow\\T(\vec{e}_1)&T(\vec{e}_1)&T(\vec{e}_1)\\\downarrow&\downarrow&\downarrow\end{bmatrix}.$$

This makes perfect sense since:

$$T(\vec{e}_1) = A\vec{e}_1 = \begin{bmatrix} 3 & 1 & 5 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \qquad T(\vec{e}_2) = A\vec{e}_2 = \begin{bmatrix} 3 & 1 & 5 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix},$$

and

$$T(\vec{e}_3) = A\vec{e}_3 = \begin{bmatrix} 3 & 1 & 5 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

This is true in general:

Theorem 16.2: Let $T:\ \mathbb{R}^n \to \mathbb{R}^m$ be defined by a matrix multiplication

$$T(\vec{x}) = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}\vec{x}$$

where $A = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$ is the $m \times n$ matrix with columns $\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n$ then

$$T(\vec{e}_1) = \vec{a}_1,\ \ T(\vec{e}_2) = \vec{a}_2,\ \ \cdots\ ,\ \ T(\vec{e}_n) = \vec{a}_n$$

where $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, $\cdots$, $\vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ are the standard basis vectors of $\mathbb{R}^n$.

Proof: Obvious. $\square$

Of course it is easy to find the matrix A when we know what happens to the *standard* basis, but it is actually not too hard to find the matrix when we know the images of *any other* basis:

Example 10: Let $T:\ \mathbb{R}^3 \to \mathbb{R}^2$ be linear, and suppose we know that

$$T\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad T\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad T\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

Then, if we let $T(\vec{x}) = A\vec{x}$ we have

$$A\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix},\quad A\begin{bmatrix}1\\2\\1\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix},\quad A\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}4\\4\end{bmatrix}$$

Combining those gives us $A\begin{bmatrix}1&1&1\\1&2&1\\1&1&0\end{bmatrix}=\begin{bmatrix}1&3&4\\2&1&4\end{bmatrix}$

so that $A = \begin{bmatrix}1&3&4\\2&1&4\end{bmatrix}\begin{bmatrix}1&1&1\\1&2&1\\1&1&0\end{bmatrix}^{-1} = \begin{bmatrix}2&2&-3\\5&-1&-2\end{bmatrix}$

Let's check that the columns of the matrix A are indeed $T(\vec{e}_1)$, $T(\vec{e}_2)$ and $T(\vec{e}_3)$:

$$T\begin{bmatrix}1\\0\\0\end{bmatrix}=T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}-\begin{bmatrix}1\\2\\1\end{bmatrix}+\begin{bmatrix}1\\1\\0\end{bmatrix}\right)=T\begin{bmatrix}1\\1\\1\end{bmatrix}-T\begin{bmatrix}1\\2\\1\end{bmatrix}+T\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}-\begin{bmatrix}3\\1\end{bmatrix}+\begin{bmatrix}4\\4\end{bmatrix}=\begin{bmatrix}2\\5\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\\0\end{bmatrix}=T\left(\begin{bmatrix}1\\2\\1\end{bmatrix}-\begin{bmatrix}1\\1\\1\end{bmatrix}\right)=T\begin{bmatrix}1\\2\\1\end{bmatrix}-T\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix}-\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$$

$$T\begin{bmatrix}0\\0\\1\end{bmatrix}=T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}-\begin{bmatrix}1\\1\\0\end{bmatrix}\right)=T\begin{bmatrix}1\\1\\1\end{bmatrix}-T\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}-\begin{bmatrix}4\\4\end{bmatrix}=\begin{bmatrix}-3\\-2\end{bmatrix}$$

Of course one might ask how did we get that, e.g. $\begin{bmatrix}1\\0\\0\end{bmatrix}=1\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}-1\cdot\begin{bmatrix}1\\2\\1\end{bmatrix}+1\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}$? Well that follows immediately (after some thinking) from the fact that

$$\begin{bmatrix}1&1&1\\1&2&1\\1&1&0\end{bmatrix}^{-1}=\begin{bmatrix}1&-1&1\\-1&1&0\\1&0&-1\end{bmatrix}\text{ and thus }\begin{bmatrix}1&1&1\\1&2&1\\1&1&0\end{bmatrix}\begin{bmatrix}1&-1&1\\-1&1&0\\1&0&-1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

In the next linear algebra class we will discuss in more detail how we can change coordinates systems, i.e. how to go from describing a vector space with respect to one basis to describing it with respect to another basis. And also how the matrix of a transformation changes when we are working with different bases.

Note that not all linear transformations are necessarily at first presented using a single matrix. For example the following transformations are all linear but described in different ways:

(1) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} + z\begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}$ **(2)** $T(\vec{x}) = \vec{v} \times \vec{x}$

(3) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (x-2y)\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + (x+y+z)\begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$ **(4)** $T(\vec{x}) = \dfrac{\vec{v}\cdot\vec{x}}{\|\vec{v}\|^2}\,\vec{v}$

(5) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x+2y-z \\ -x+y-3z \\ 2x-4y+z \end{bmatrix}$ **(6)** $T(\vec{x}) = \left(I - \dfrac{\vec{n}\cdot\vec{n}^{\mathsf{T}}}{\|\vec{n}\|^2}\right)\vec{x}$

(7) $T(\vec{x}) = \big(1-\cos(\theta)\big)\dfrac{\vec{v}\,\vec{v}^{\mathsf{T}}}{\|\vec{v}\|^2}\vec{x} + \cos(\theta)\,\vec{x} + \dfrac{\sin(\theta)}{\|\vec{v}\|}(\vec{v}\times\vec{x})$

… and yet all of these can be reformulated using just a matrix multiplication. For example

(1) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ -1 & 2 & 3 \\ 2 & -4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ **(3)** $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

(3) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 & 1 & 3 \\ 1 & 4 & 2 \\ -2 & -8 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ **(4)** $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dfrac{1}{v_1^2+v_2^2+v_3^2}\begin{bmatrix} v_1^2 & v_1v_2 & v_1v_3 \\ v_1v_2 & v_2^2 & v_2v_3 \\ v_1v_3 & v_2v_3 & v_3^2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(5) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ -1 & 1 & -3 \\ 2 & -4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, here $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

(6) etc.

(7) We will encounter this matrix in chapter 28 on 3D rotations.

17. Compositions of Linear Transformations

Recall the definition of a linear transformation (over $\mathbb{R}$)

Definition: Let V and W be two real vector spaces. A map (function) $T: V \to W$ is called a **linear transformation** if

(a) $T(\vec{x}+\vec{y}) = T(\vec{x})+T(\vec{y})$ for all $\vec{x}, \vec{y} \in V$

(b) $T(t \cdot \vec{x}) = t \cdot T(\vec{x})$ for all $\vec{x} \in V$ and $t \in \mathbb{R}$

Let's take for example, two linear transformations $T: \mathbb{R}^2 \to \mathbb{R}^3$ and $U: \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}10\\-9\\7\end{bmatrix} \qquad T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-6\\5\\1\end{bmatrix}$$

and

$$U\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix} \qquad U\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}8\\4\end{bmatrix} \qquad U\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}12\\11\end{bmatrix}$$

This information completely determines both the transformations T and U, as well as their composition $U \circ T$:

$$T\begin{bmatrix}x\\y\end{bmatrix} = T\left(x\begin{bmatrix}1\\0\end{bmatrix}+y\begin{bmatrix}0\\1\end{bmatrix}\right) = x\,T\begin{bmatrix}1\\0\end{bmatrix} + y\,T\begin{bmatrix}0\\1\end{bmatrix} = x\begin{bmatrix}10\\-9\\7\end{bmatrix} + y\begin{bmatrix}-6\\5\\1\end{bmatrix} = \begin{bmatrix}10x-6y\\-9x+5y\\7x+y\end{bmatrix}$$

and

$$U\begin{bmatrix}x\\y\\z\end{bmatrix} = T\left(x\begin{bmatrix}1\\0\\0\end{bmatrix}+y\begin{bmatrix}0\\1\\0\end{bmatrix}+z\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = x\,T\begin{bmatrix}1\\0\\0\end{bmatrix}+y\,T\begin{bmatrix}0\\1\\0\end{bmatrix}+z\,T\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= x\begin{bmatrix}2\\3\end{bmatrix} + y\begin{bmatrix}8\\4\end{bmatrix} + z\begin{bmatrix}12\\11\end{bmatrix} = \begin{bmatrix}2x+8y+12z\\3x+4y+11z\end{bmatrix}$$

i.e.

$$T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}10x-6y\\-9x+5y\\7x+y\end{bmatrix} \quad \text{and} \quad U\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2x+8y+12z\\3x+4y+11z\end{bmatrix}$$

In particular

$$U\begin{bmatrix}10\\-9\\7\end{bmatrix}=\begin{bmatrix}2\cdot 10+8\cdot(-9)+12\cdot 7\\3\cdot 10+4\cdot(-9)+11\cdot 7\end{bmatrix}=\begin{bmatrix}32\\71\end{bmatrix}$$

and

$$U\begin{bmatrix}-6\\5\\1\end{bmatrix}=\begin{bmatrix}2\cdot(-6)+8\cdot 5+12\cdot 1\\3\cdot(-6)+4\cdot 5+11\cdot 1\end{bmatrix}=\begin{bmatrix}40\\13\end{bmatrix}$$

So that

$$(U\circ T)\begin{bmatrix}x\\y\end{bmatrix}=U\left(T\begin{bmatrix}x\\y\end{bmatrix}\right)=U\begin{bmatrix}10x-6y\\-9x+5y\\7x+y\end{bmatrix}=x\,U\begin{bmatrix}10\\-9\\7\end{bmatrix}+y\,U\begin{bmatrix}-6\\5\\1\end{bmatrix}$$
$$=x\begin{bmatrix}32\\71\end{bmatrix}+y\begin{bmatrix}40\\13\end{bmatrix}=\begin{bmatrix}32x+40y\\71x+13y\end{bmatrix}$$

Hence

$$T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}10x-6y\\-9x+5y\\7x+y\end{bmatrix},\quad U\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2x+8y+12z\\3x+4y+11z\end{bmatrix}\quad\text{and}\quad(U\circ T)\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}32x+40y\\71x+13y\end{bmatrix}$$

If we now recall the ***weird*** definition of matrix multiplication, then we see that

$$T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}10&-6\\-9&5\\7&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix},\quad U\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2&8&12\\3&4&11\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}\quad\text{and}\quad(U\circ T)\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}32&40\\71&13\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

and amazingly these three matrices are related by this weird matrix multiplication

$$\begin{bmatrix}2&8&12\\3&4&11\end{bmatrix}\begin{bmatrix}10&-6\\-9&5\\7&1\end{bmatrix}=\begin{bmatrix}32&40\\71&13\end{bmatrix}$$

We see that the matrix of the **composition** is just the **product** of the matrices of the separate transformations!

This is so in general, and this is the main reason why we defined matrix multiplication the weird way we did: so that this would be true.

We can prove this fact in general, but in this class that would in my opinion just obscure this reality. It would involve some algebra and manipulation of double summations ($\sum\sum$). This is better left for the next Linear Algebra class. In this class we will use this observation whenever we need to compose two or more transformations.

To summarize:

Theorem 17.1: If $T:\ \mathbb{R}^m \to \mathbb{R}^n$ and $U:\ \mathbb{R}^n \to \mathbb{R}^p$ are two linear transformations, and the matrices $M_T \in M_{n\times m}(\mathbb{R})$ and $M_U \in M_{p\times n}(\mathbb{R})$ are such that

$$T(\vec{x}) = M_T\,\vec{x} \quad\text{and}\quad U(\vec{x}) = M_U\,\vec{x}$$

then $U \circ T:\ \mathbb{R}^m \to \mathbb{R}^p$ is defined by $(U\circ T)(\vec{x}) = M_{U\circ T}\,\vec{x}$ with

$$\boxed{M_{U\circ T} = M_U M_T}$$

so that

$$(U\circ T)(\vec{x}) = U\big(T(\vec{x})\big) = M_U(M_T\,\vec{x}) = M_U M_T\,\vec{x} = M_{U\circ T}\,\vec{x}$$

Proof: See the next Linear Algebra class, or any linear algebra book.

Note: This fact gives us an easy proof of $A(BC) = (AB)C$

Just let $T(\vec{x}) = C\,\vec{x}$, $S(\vec{x}) = B\,\vec{x}$ and $R(\vec{x}) = A\,\vec{x}$ then

$$\big((R\circ S)\circ T\big)(\vec{x}) = (R\circ S)\big(T(\vec{x})\big) = (AB)(C\,\vec{x}) = (AB)C\,\vec{x}$$

and

$$\big(R\circ(S\circ T)\big)(\vec{x}) = R\big((S\circ T)(\vec{x})\big) = A\big((BC)\,\vec{x}\big) = A(BC)\vec{x}$$

But since composition of functions is associative

$$\big((R\circ S)\circ T\big)(\vec{x}) = \big(R\circ(S\circ T)\big)(\vec{x})$$

then

$$(AB)C\,\vec{x} = A(BC)\vec{x}$$

Since this is true for all $\vec{x}$ we find that

$$(AB)C = A(BC)$$

Example: Let $\mathbb{R}^3 \xrightarrow{T} \mathbb{R}^4 \xrightarrow{S} \mathbb{R}^2 \xrightarrow{R} \mathbb{R}^2$ be three linear transformations defined by

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad S\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \quad \text{and} \quad R\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix}$$

Then

$$\left((R\circ S)\circ T\right)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (R\circ S)\left(T\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = R\left(S\left(T\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\right)$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\right)$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -1 & -2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

So for example

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \xrightarrow{T} \begin{bmatrix} -2 \\ 5 \\ 2 \\ 4 \end{bmatrix} \xrightarrow{S} \begin{bmatrix} 4 \\ -3 \end{bmatrix} \xrightarrow{R} \begin{bmatrix} -2 \\ 7 \end{bmatrix}$$

and indeed: $\begin{bmatrix} 6 & -1 & -2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$

Also note that

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{T} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{S} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \xrightarrow{R} \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{T} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{S} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \xrightarrow{R} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{T} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{S} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \xrightarrow{R} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

18. Basic Linear Transformation Matrices in $\mathbb{R}^2$ and $\mathbb{R}^3$

In the following sections we will discuss the matrices of some of the most important 2D and 3D linear transformations. Since there is a lot of material covered with many proofs and examples, let's start with a brief overview:

2D and 3D matrices

1. The matrix of a **projection** onto the line $\vec{x} = t\vec{v}$ **:** $\quad M = \boxed{\dfrac{1}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}}}$

2. The matrix of a **projection** onto $\vec{n}\cdot\vec{x} = 0$: $\quad M = \boxed{I - \dfrac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}}$

3. The matrix of a **skew projection** onto $\vec{n}\cdot\vec{x} = 0$ in the direction of the vector $\vec{v}$ $\quad M = \boxed{I - \dfrac{1}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}}$

4. The matrix of a **reflection** in the line $\vec{x} = t\vec{v}$ **:** $\quad M = \boxed{\dfrac{2}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}} - I}$

5. The matrix of a **reflection** in $\vec{n}\cdot\vec{x} = 0$: $\quad M = \boxed{I - \dfrac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}}$

6. The matrix of a **skew reflection** in $\vec{n}\cdot\vec{x} = 0$ in the direction of the vector $\vec{v}$ $\quad M = \boxed{I - \dfrac{2}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}}$

7. The matrix of a **shear** with respect to $\vec{n}\cdot\vec{x} = 0$ and $\vec{v}$: $\quad M = \boxed{I + \dfrac{1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}}}$

8. The matrix of a 2D and a 3D **scaling**: $\quad M = \boxed{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}\,,\ \boxed{\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}}$

9. The matrix of a 2D **rotation** around the origin over θ: $\quad M = \boxed{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$

10. The matrix of a 3D **rotation** around the vector $\vec{v}$ over θ:

$$\boxed{M = (1-\cos\theta)\frac{\vec{v}\,\vec{v}^{\mathsf{T}}}{\|\vec{v}\|^2} + \cos\theta\, I + \frac{\sin\theta}{\|\vec{v}\|}C_{\vec{v}}}$$

A useful Lemma

For some of the matrices we'll be using our basic projection operation $\operatorname{proj}_{\vec{a}}(\vec{x}) = \dfrac{\vec{a}\cdot\vec{x}}{\vec{a}\cdot\vec{a}}\cdot\vec{a}$.

This expression is in terms of dot products. To get everything in terms of matrices we will need to re-write this expression a bit. The following lemma allows us to do just that:

Lemma: Let $\vec{a},\ \vec{x},\ \vec{b} \in \mathbb{R}^2$ or $\mathbb{R}^3$ (or $\mathbb{R}^n$) then $\boxed{(\vec{a}\cdot\vec{x})\cdot\vec{b} = \vec{b}\ \vec{a}^{\mathsf{T}}\ \vec{x}}$

Observe here the different types of 'multiplications' : $(\vec{a}\cdot\vec{x})\cdot\vec{b} = \vec{b}\ \vec{a}^{\mathsf{T}}\ \vec{x}$

dot product

scalar multiplication

matrix multiplications

Proof: We'll first give the following straight forward 2D and 3D proof:

(a) Let $\vec{a} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} A \\ B \end{bmatrix}$ then

$$(\vec{a}\cdot\vec{x})\cdot\vec{b} = \left(\begin{bmatrix} a \\ b \end{bmatrix}\cdot\begin{bmatrix} x \\ y \end{bmatrix}\right)\cdot\begin{bmatrix} A \\ B \end{bmatrix} = (ax+by)\cdot\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} aAx+bAy \\ aBx+bBy \end{bmatrix}$$

$$= \begin{bmatrix} aA & bA \\ aB & bB \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}\begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \vec{b}\cdot\vec{a}^{\mathsf{T}}\cdot\vec{x}$$

(b) Let $\vec{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ then

$$(\vec{a}\cdot\vec{x})\cdot\vec{b} = \left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\cdot\begin{bmatrix} A \\ B \\ C \end{bmatrix} = (ax+by+cz)\cdot\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} aAx+bAy+cAz \\ aBx+bBy+cBz \\ aCx+bCy+cCz \end{bmatrix}$$

$$= \begin{bmatrix} aA & bA & cA \\ aB & bB & cB \\ aC & bC & cC \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{b}\cdot\vec{a}^{\mathsf{T}}\cdot\vec{x}$$

Of course this proof works for $\vec{a},\ \vec{x},\ \vec{b} \in \mathbb{R}^n$, but it can be proven more elegantly.

Alternative Proof: First, two observations

(1) $$t\cdot\vec{a} = t\cdot\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1\cdot t \\ a_2\cdot t \\ \vdots \\ a_n\cdot t \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}[t] = \vec{a}\,[t]$$

a **scalar** multiplication (for $t\cdot\vec{a}$) a **matrix** multiplication (for $\vec{a}\,[t]$)

(2) $$\vec{a}^{\mathsf{T}}\vec{x} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} a_1x_1 + a_2x_2 + \cdots + a_nx_n \end{bmatrix} = \begin{bmatrix} \vec{a}\cdot\vec{x} \end{bmatrix}$$

a **matrix** multiplication (for $\vec{a}^{\mathsf{T}}\vec{x}$) a $1\times1-$**matrix** with a **dot** product as its only entry (for $[\vec{a}\cdot\vec{x}]$)

Combining these two observations gives us a simple and general proof of our lemma:

$$(\vec{a}\cdot\vec{x})\cdot\vec{b} = \vec{b}\,[\vec{a}\cdot\vec{x}] = \vec{b}\;\vec{a}^{\mathsf{T}}\vec{x}$$

Voila!

We will be using this shift from using a dot product to matrix multiplications a lot, so it is good to memorize this fact!

In particular we'll see the following cases appear in our transformation matrices:

$$\boxed{\begin{aligned} (\vec{v}\cdot\vec{x})\cdot\vec{v} &= \vec{v}\,\vec{v}^{\mathsf{T}}\vec{x} \\ (\vec{n}\cdot\vec{x})\cdot\vec{n} &= \vec{n}\,\vec{n}^{\mathsf{T}}\vec{x} \\ (\vec{n}\cdot\vec{x})\cdot\vec{v} &= \vec{v}\,\vec{n}^{\mathsf{T}}\vec{x} \end{aligned}}$$

Example 1: If $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and $\vec{n} = \begin{bmatrix} 5 \\ 3 \\ -4 \end{bmatrix}$ then

$$(\vec{n}\cdot\vec{x})\cdot\vec{v} = \vec{v}\,\vec{n}^{\mathsf{T}}\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}\begin{bmatrix} 5 & 3 & -4 \end{bmatrix}\vec{x} = \begin{bmatrix} 5 & 3 & -4 \\ -10 & -6 & 8 \\ 15 & 9 & -12 \end{bmatrix}\vec{x}$$

Example 2: We can re-write our basic projection now in terms of a matrix multiplication as follows:

$$\text{proj}_{\vec{v}}(\vec{x}) = \frac{\vec{v}\cdot\vec{x}}{\vec{v}\cdot\vec{v}}\cdot\vec{v} = \frac{1}{\vec{v}\cdot\vec{v}}(\vec{v}\cdot\vec{x})\cdot\vec{v} = \frac{1}{\vec{v}\cdot\vec{v}}\cdot\vec{v}\,\vec{v}^{\mathsf{T}}\vec{x} = \frac{1}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}}\vec{x}$$

For example, if $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then

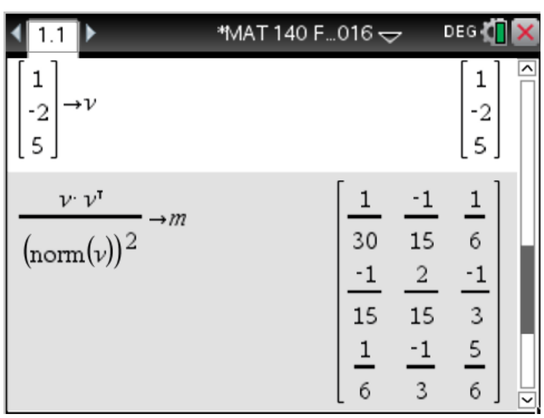
$$\text{proj}_{\vec{v}}(\vec{x}) = \frac{1}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}}\vec{x} = \frac{1}{30}\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & -2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 1 & -2 & 5 \\ -2 & 4 & -10 \\ 5 & -10 & 25 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

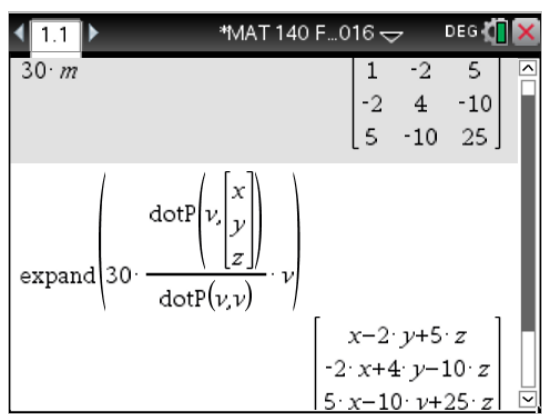
Of course we get the same result using the dot product formulation of our projection:

$$\text{proj}_{\vec{v}}(\vec{x}) = \frac{\vec{v}\cdot\vec{x}}{\vec{v}\cdot\vec{v}}\cdot\vec{v} = \frac{1}{30}(x-2y+5z)\cdot\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} x-2y+5z \\ -2x+4y-10z \\ 5x-10y+25z \end{bmatrix}$$

$$= \frac{1}{30}\begin{bmatrix} 1 & -2 & 5 \\ -2 & 4 & -10 \\ 5 & -10 & 25 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The TI-Nspire will do this for us as follows:

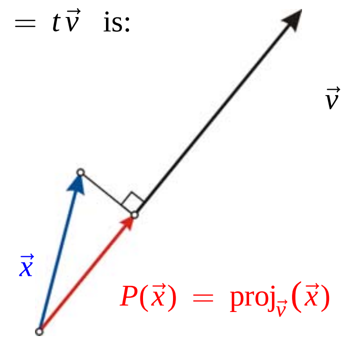




19. Projection Matrices in $\mathbb{R}^2$

Theorem 19.1: The matrix of a **projection** onto the line $\vec{x} = t\vec{v}$ is:

$$M = \boxed{\frac{1}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}}}$$

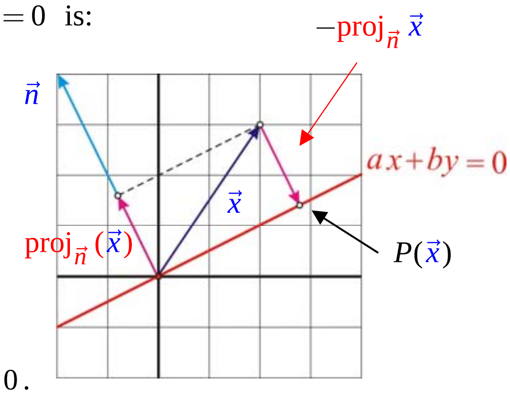


Furthermore:

- $M = M^{\mathsf{T}}$
- $M^2 = M$
- $\text{Trace}(M) = 1$
- M is not invertible: $\det(M) = 0$.
- The only fixed points of this transformation are the points on the line.

Theorem 19.2: The matrix of a **projection** onto $\vec{n}\cdot\vec{x} = 0$ is:

$$M = \boxed{I - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}}$$

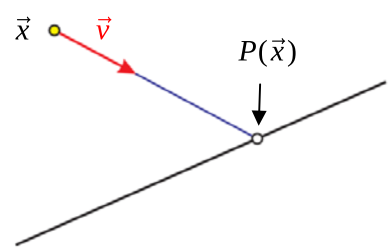


Furthermore:

- $M = M^{\mathsf{T}}$
- $M^2 = M$
- $\text{Trace}(M) = 1$
- M is not invertible: $\det(M) = 0$.
- The only fixed points of this transformation are the points on the line.

Theorem 19.3: The matrix of a **skew projection** onto $\vec{n}\cdot\vec{x} = 0$ in the direction of the vector $\vec{v}$ is

$$M = \boxed{I - \frac{1}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}}$$



Furthermore:

- $M \neq M^{\mathsf{T}}$
- $M^2 = M$
- $\text{Trace}(M) = 1$
- M is not invertible: $\det(M) = 0$.
- The only fixed points of this transformation are the points on the line.

Orthogonal Projections

In this section we will discuss the matrices of three important 2D projections.

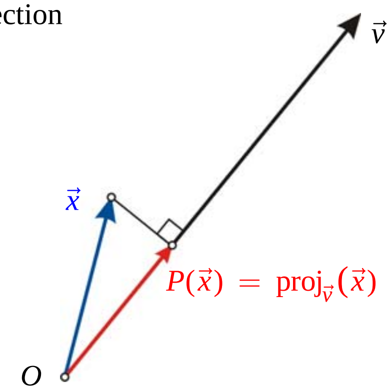
Projection onto a line through the origin

When we say projection onto a line we mean "orthogonal" projection. We basically dealt with this kind of projection before when we introduced projection onto a vector. The following method works in any dimension: $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4\cdots$ etc.

(a) Projection when the line is given in vector form

Let $l:\ \vec{x} = t\vec{v}$ where $\vec{v}$ is a direction vector of the line.
In this case projection onto the line is the same as projection onto the vector $\vec{v}$ which we discussed in chapter 3:

$$P(\vec{x}) = \mathrm{proj}_{\vec{v}}(\vec{x}) = \frac{\vec{v}\cdot\vec{x}}{\vec{v}\cdot\vec{v}}\cdot\vec{v} = \frac{1}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}}\vec{x}$$



Hence the projection matrix is $\frac{1}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}}$ or $\frac{\vec{v}\,\vec{v}^{\mathsf{T}}}{\|\vec{v}\|^2}$

Example 1. **2D**: Let $l:\ \vec{x} = t\begin{bmatrix}1\\-2\end{bmatrix}$ then $\vec{v} = \begin{bmatrix}1\\-2\end{bmatrix}$ and the projection onto this line

is $$P(\vec{x}) = \frac{1}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}}\vec{x} = \frac{1}{5}\begin{bmatrix}1\\-2\end{bmatrix}\begin{bmatrix}1 & -2\end{bmatrix}\vec{x} = \frac{1}{5}\begin{bmatrix}1 & -2\\-2 & 4\end{bmatrix}\vec{x}$$

In a higher dimensions this works as well:

Example 2. **3D**: Let $l:\ \vec{x} = t\begin{bmatrix}1\\-2\\5\end{bmatrix}$ then $\vec{v} = \begin{bmatrix}1\\-2\\5\end{bmatrix}$ and the projection onto this line

is $$P(\vec{x}) = \frac{1}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}}\vec{x} = \frac{1}{30}\begin{bmatrix}1\\-2\\5\end{bmatrix}\begin{bmatrix}1 & -2 & 5\end{bmatrix}\vec{x} = \frac{1}{30}\begin{bmatrix}1 & -2 & 5\\-2 & 4 & -10\\5 & -10 & 25\end{bmatrix}\vec{x}$$

The above derivation proves the first part of the following theorem:

Theorem 19.1: Let the line l be given in vector form $\vec{x} = t\vec{v}$, then the projection onto this line is given by

$$P(\vec{x}) = \frac{1}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}}\vec{x} \quad \text{or} \quad P(\vec{x}) = \frac{\vec{v}\,\vec{v}^{\mathsf{T}}}{\vec{v}\cdot\vec{v}}\vec{x}$$

P is a linear transformation with matrix $\boxed{M = \frac{1}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}}}$

i.e. if $\vec{v} = \begin{bmatrix} A \\ B \end{bmatrix}$ then $M = \frac{1}{A^2+B^2}\begin{bmatrix} A^2 & AB \\ AB & B^2 \end{bmatrix}$

or if $\vec{v} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ then $M = \frac{1}{A^2+B^2+C^2}\begin{bmatrix} A^2 & AB & AC \\ AB & B^2 & BC \\ AC & BC & C^2 \end{bmatrix}$

Furthermore:

- $M = M^{\mathsf{T}}$
- $M^2 = M$
- $\text{Trace}(M) = 1$
- M is not invertible: $\det(M) = 0$
- The only fixed points of this transformation are the points on the line.

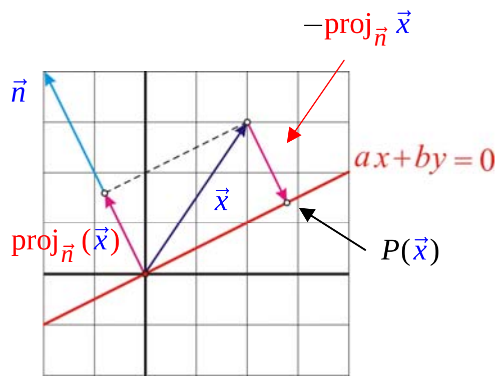
Only the ‘Furthermore’ points still need to be proven. Proofs of these you can find at the end of this chapter.

(b) Projection when the line is given in normal form

Let $l:\ ax+by = 0$ be a line through the origin, with normal $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$, i.e. $\vec{n}\cdot\vec{x} = 0$.

The projection $P:\ \mathbb{R}^2 \to \mathbb{R}^2$ onto this line can be computed as follows

$$\begin{aligned} P(\vec{x}) &= \vec{x} - \operatorname{proj}_{\vec{n}}(\vec{x}) \\ &= \vec{x} - \frac{\vec{n}\cdot\vec{x}}{\vec{n}\cdot\vec{n}}\,\vec{n} \\ &= \vec{x} - \frac{1}{\|\vec{n}\|^2}(\vec{n}\cdot\vec{x})\,\vec{n} \\ &= \vec{x} - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\vec{x} \\ &= I_2\,\vec{x} - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\vec{x} \\ &= \left(I_2 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right)\vec{x} \end{aligned}$$



Hence the projection matrix is: $I_2 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}$.

The explicit expression for this matrix in terms of a and b is

$$\begin{aligned} I_2 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} &= \frac{1}{a^2+b^2}\begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix} - \frac{1}{a^2+b^2}\begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix} \\ &= \frac{1}{a^2+b^2}\left(\begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix} - \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}\right) \\ &= \frac{1}{a^2+b^2}\begin{bmatrix} b^2 & -ab \\ -ab & a^2 \end{bmatrix} \end{aligned}$$

We have proven the first part of the following theorem:

Theorem 19.2: Let the line l be given in normal form $\vec{n}\cdot\vec{x} = 0$ [i.e. $ax+by = 0$], then the projection onto this line is given by

$$P(\vec{x}) = \left(I_2 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} \right)\vec{x}$$

P is a linear transformation with matrix $\boxed{M = I_2 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}}$

i.e. if $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ then $M = \dfrac{1}{a^2+a^2}\begin{bmatrix} b^2 & -ab \\ -ab & a^2 \end{bmatrix}$

Furthermore:

- $M = M^{\mathsf{T}}$
- $M^2 = M$
- $\mathrm{Trace}(M) = 1$
- M is not invertible: $\det(M) = 0$.
- The only fixed points of this transformation are the points on the line.

Note: Theorem 19.2 is specific to $\mathbb{R}^2$. $\mathbb{R}^2$ is a special space. In $\mathbb{R}^2$ a line has both a vector equation **and** the normal equation. In higher dimensions a line can**not** be written in normal form! In higher dimensions a line *has* to be written in vector form $l: \ \vec{x} = t\vec{v}$, in which case we can use Theorem 19.1 which holds for any line in 2D, 3D, 4D $\cdots$ etc. On the other hand the equation $\vec{n}\cdot\vec{x} = 0$ represents a line *only* in $\mathbb{R}^2$. In $\mathbb{R}^3$ it represents a plane. In $\mathbb{R}^n$ it would represent a $n-1$ dimensional hyper-plane. Projections on (hyper) planes we will discuss in a later chapter.

Only the 'Furthermore' points of the theorem still need to be proven. Although some of them are trivial, we will give the proofs at the end of this chapter. Right now we'll proceed with some examples:

Example 3: Let $l: \ x-2y = 0$ be a line in $\mathbb{R}^2$, then the matrix of the projection P onto the line l is

$$I_2 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 1 \\ -2 \end{bmatrix}\begin{bmatrix} 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

or

$$\frac{1}{a^2+b^2}\begin{bmatrix} b^2 & -ab \\ -ab & a^2 \end{bmatrix} = \frac{1}{1^2+(-2)^2}\begin{bmatrix} (-2)^2 & -1\cdot(-2) \\ -1\cdot(-2) & 1^2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

Hence

- The image of $\vec{x} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$ is $P(\vec{x}) = \frac{1}{5}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

- $\vec{x} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$ is mapped to $P(\vec{x}) = \frac{1}{5}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -5 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

- The image of the line $m: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} + t\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is

$$P\begin{bmatrix} x \\ y \end{bmatrix} = P\left(\begin{bmatrix} 5 \\ 5 \end{bmatrix} + t\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = P\begin{bmatrix} 5 \\ 5 \end{bmatrix} + tP\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} + t\begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Hence if we call the images $\begin{bmatrix} x' \\ y' \end{bmatrix} = P\begin{bmatrix} x \\ y \end{bmatrix}$ we get

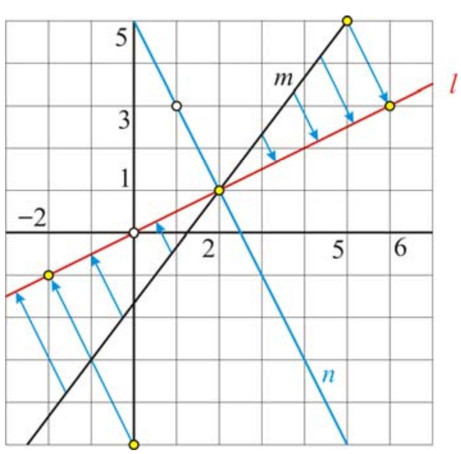
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} + t\begin{bmatrix} 4 \\ 2 \end{bmatrix} = (3+2t)\begin{bmatrix} 2 \\ 1 \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

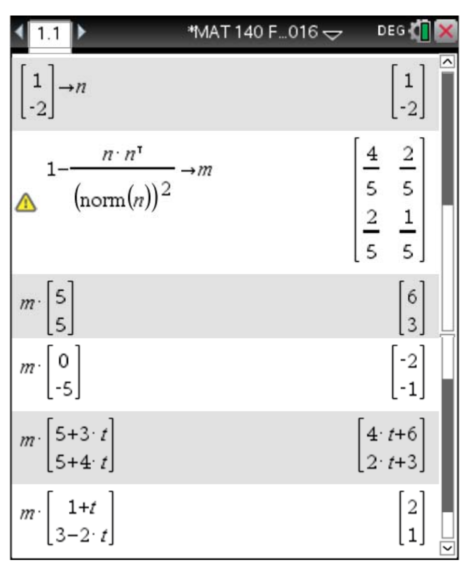
so that x', y' satisfy the equation $x' - 2y' = 0$, i.e. all the images (x', y') lie on the line l. In fact the image of the entire line m is l.

- The image of e.g. the line $n: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is

$$P\begin{bmatrix} x \\ y \end{bmatrix} = P\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = P\begin{bmatrix} 1 \\ 3 \end{bmatrix} + tP\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

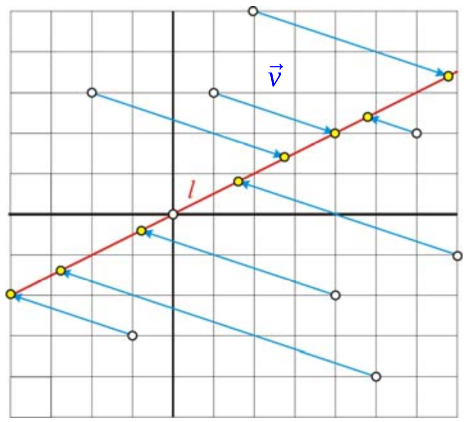
Hence the entire line n is mapped to the point $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, after all n is perpendicular to the line l onto which we project.





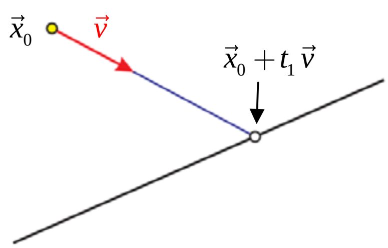
Skew Projections

Suppose we want to project onto a line in a given direction, not necessarily orthogonal to the line:



Let $l: \ \vec{n}\cdot\vec{x} = 0$ be a line with normal $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} A \\ B \end{bmatrix}$ a vector, not parallel to the line l, i.e. $\vec{n}\cdot\vec{v} \neq 0$. We want to project $\vec{x}_0$ onto l in the direction of $\vec{v}$, hence for some t_1 we have that $\vec{x}_0 + t_1\vec{v}$ is on the line

To compute the t_1 value that puts us on the line we solve the equation



$$\vec{n}\cdot(\vec{x}_0 + t_1\vec{v}) = 0$$

i.e.

$$t_1 = -\frac{\vec{n}\cdot\vec{x}_0}{\vec{n}\cdot\vec{v}}$$

Hence

$$\begin{aligned} T(\vec{x}_0) &= \vec{x}_0 + t_1 v \\ &= \vec{x}_0 - \frac{\vec{n}\cdot\vec{x}_0}{\vec{n}\cdot\vec{v}}\vec{v} \\ &= \vec{x}_0 - \frac{1}{\vec{n}\cdot\vec{v}}(\vec{n}\cdot\vec{x}_0)\,\vec{v} = \vec{x}_0 - \frac{1}{\vec{n}\cdot\vec{v}}\vec{v}\,\vec{n}^{\mathsf{T}}\vec{x}_0 = \left(I - \frac{1}{\vec{n}\cdot\vec{v}}\vec{v}\,\vec{n}^{\mathsf{T}}\right)\vec{x}_0 \end{aligned}$$

So that the matrix of the skew projection is given by $I - \dfrac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}$.

An explicit matrix with $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} A \\ B \end{bmatrix}$ then becomes

$$I - \frac{1}{aA+bB}\begin{bmatrix} A \\ B \end{bmatrix}\begin{bmatrix} a & b \end{bmatrix} = I - \frac{1}{aA+bB}\begin{bmatrix} aA & bA \\ aB & bB \end{bmatrix} = \frac{1}{aA+bB}\begin{bmatrix} bB & -bA \\ -aB & aA \end{bmatrix}$$

We have proven the first part of the following theorem.

Theorem 19.3: The skew projection $P: \mathbb{R}^2 \to \mathbb{R}^2$ onto the line $l: \vec{n}\cdot\vec{x} = 0$, with normal $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$, in the direction of the vector $\vec{v} = \begin{bmatrix} A \\ B \end{bmatrix}$ has as matrix

$$\boxed{M = I - \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}} \qquad \text{i.e.} \qquad M = \frac{1}{aA+bB}\begin{bmatrix} bB & -bA \\ -aB & aA \end{bmatrix}$$

Furthermore:
- $M \neq M^{\mathsf{T}}$ (unless it is an orthogonal projection: $\vec{v} = \vec{n}$)
- $M^2 = M$
- $\text{Trace}(M) = 1$
- M is not invertible: $\det(M) = 0$.
- The only fixed points of this transformation are the points on the line.

Again only the ‘Furthermore’ points still need to be proven, even though some of them are trivial. We will give the proofs at the end of this chapter.

Note: If $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} = \vec{n}$ i.e. the normal of the line $l: ax + by = 0$, we get the usual *orthogonal* projection matrix: $I - \dfrac{\vec{n}\,\vec{n}^{\mathsf{T}}}{\vec{n}\cdot\vec{n}}$ i.e. $\dfrac{1}{aa+bb}\begin{bmatrix} bb & -ba \\ -ab & aa \end{bmatrix} = \dfrac{1}{a^2+b^2}\begin{bmatrix} b^2 & -ab \\ -ab & a^2 \end{bmatrix}$

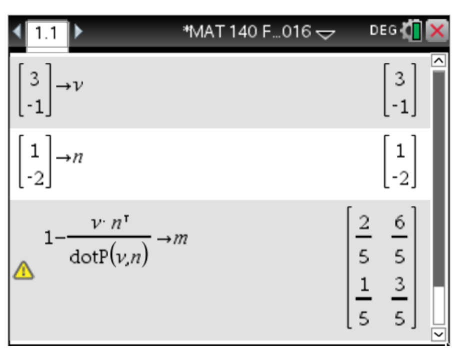
We’ll proceed with an example:

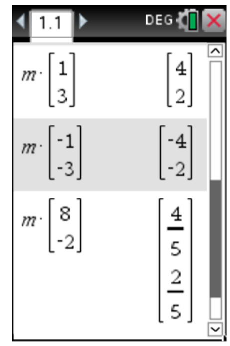
Example 4: Let $l: x - 2y = 0$ and $\vec{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$,

then $I - \dfrac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}} = I - \dfrac{1}{5}\begin{bmatrix} 3 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -2 \end{bmatrix} = \dfrac{1}{5}\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$

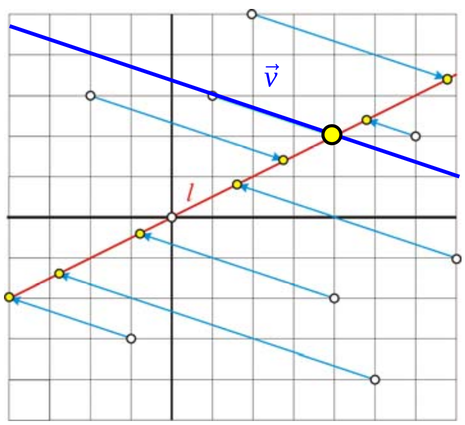
or

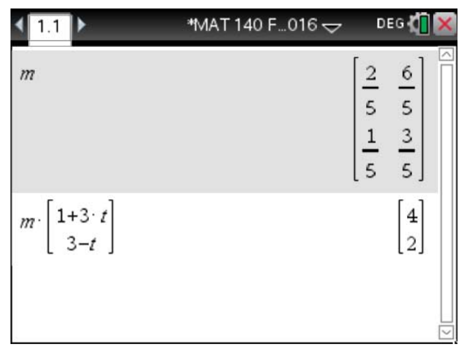
$$\frac{1}{aA+bB}\begin{bmatrix} bB & -bA \\ -aB & aA \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$$





- The image of $\begin{bmatrix}1\\3\end{bmatrix}$ is $P\begin{bmatrix}1\\3\end{bmatrix} = \frac{1}{5}\begin{bmatrix}2 & 6\\1 & 3\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$
- The image of $\begin{bmatrix}-1\\-3\end{bmatrix}$ is $P\begin{bmatrix}-1\\-3\end{bmatrix} = \frac{1}{5}\begin{bmatrix}2 & 6\\1 & 3\end{bmatrix}\begin{bmatrix}-1\\-3\end{bmatrix} = \begin{bmatrix}-4\\-2\end{bmatrix}$
- The image of $\begin{bmatrix}8\\-2\end{bmatrix}$ is $P\begin{bmatrix}8\\-1\end{bmatrix} = \frac{1}{5}\begin{bmatrix}2 & 6\\1 & 3\end{bmatrix}\begin{bmatrix}8\\-1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$
- The line $l:\ x+3y=10$, i.e. $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix} + t\begin{bmatrix}3\\-1\end{bmatrix}$ is parallel to the skew direction of this projection, hence gets mapped to one point: $P\begin{bmatrix}1+3t\\3-t\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$



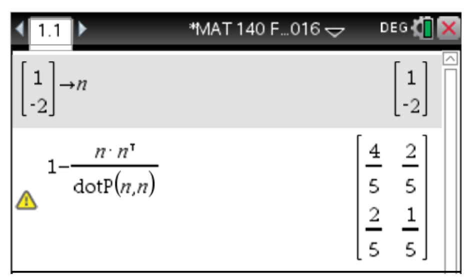


Example 5: Let $l:\ x-2y=0$ and $\vec{v} = \begin{bmatrix}1\\-2\end{bmatrix}$,

then $I - \dfrac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}} = I - \dfrac{1}{5}\begin{bmatrix}1\\-2\end{bmatrix}\begin{bmatrix}1 & -2\end{bmatrix} = \dfrac{1}{5}\begin{bmatrix}4 & 2\\2 & 1\end{bmatrix}$

or

$$\frac{1}{aA+bB}\begin{bmatrix}bB & -bA\\-aB & aA\end{bmatrix} = \frac{1}{5}\begin{bmatrix}4 & 2\\2 & 1\end{bmatrix}$$



Note that this is the orthogonal projection onto the line l, and note that this matrix *is* symmetric!

Proofs of the "Furthermore" points

Proof of the 'Furthermore' points of Theorem 19.1:

- $M^\intercal = M$ is obvious from the matrices $\frac{1}{A^2+B^2}\begin{bmatrix} A^2 & AB \\ AB & B^2 \end{bmatrix}$ and $\frac{1}{A^2+B^2+C^2}\begin{bmatrix} A^2 & AB & AC \\ AB & B^2 & BC \\ AC & BC & C^2 \end{bmatrix}$. But it is even true in higher dimensional spaces which we'll demonstrate next by a different proof (not using coordinates):
$$M^\intercal = \left(\frac{1}{\|\vec{v}\|^2}\cdot \vec{v}\,\vec{v}^\intercal\right)^\intercal = \frac{1}{\|\vec{v}\|^2}\cdot\left(\vec{v}\,\vec{v}^\intercal\right)^\intercal = \frac{1}{\|\vec{v}\|^2}\cdot(\vec{v}^\intercal)^\intercal\,\vec{v}^\intercal = \frac{1}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^\intercal = M$$
- $$M^2 = \left(\frac{1}{\|\vec{v}\|^2}\cdot \vec{v}\,\vec{v}^\intercal\right)\cdot\left(\frac{1}{\|\vec{v}\|^2}\cdot \vec{v}\,\vec{v}^\intercal\right)$$
$$= \frac{1}{\|\vec{v}\|^4}\cdot\vec{v}\,\vec{v}^\intercal\,\vec{v}\,\vec{v}^\intercal = \frac{1}{\|\vec{v}\|^4}\cdot\left(\vec{v}\,\vec{v}^\intercal\,\vec{v}\right)\vec{v}^\intercal = \frac{1}{\|\vec{v}\|^4}\cdot\left(\left[\,\vec{v}\cdot\vec{v}\,\right]\cdot\vec{v}\right)\vec{v}^\intercal$$
$$= \frac{\vec{v}\cdot\vec{v}}{\|\vec{v}\|^4}\cdot\vec{v}\,\vec{v}^\intercal = \frac{1}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^\intercal = M$$

- $\text{Trace}(M) = 1$.

  Obvious for the matrices $\frac{1}{A^2+B^2}\begin{bmatrix} A^2 & AB \\ AB & B^2 \end{bmatrix}$ and $\frac{1}{A^2+B^2+C^2}\begin{bmatrix} A^2 & AB & AC \\ AB & B^2 & BC \\ AC & BC & C^2 \end{bmatrix}$.

  But it is even true in higher dimensional spaces which we prove as follows:
$$\text{Trace}(M) = \frac{1}{\|\vec{v}\|^2}\cdot\text{Trace}(\vec{v}\cdot\vec{v}^\intercal) = \frac{\vec{v}\cdot\vec{v}}{\|\vec{v}\|^2} = 1.$$

- $\det(M) = 0$:

  There are infinitely many vectors $\vec{x}\perp\vec{v}$ and they are **all** mapped to $\vec{0}$, as can be seen from $\frac{1}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^\intercal\vec{x} = \frac{1}{\|\vec{v}\|^2}(\vec{v}\cdot\vec{x})\cdot\vec{v} = \frac{1}{\|\vec{v}\|^2}(0)\cdot\vec{v} = \vec{0}$.

  So this map cannot be invertible, and hence $\det(M) = 0$

- Fixed points:

  Note that any vector can be written as $\vec{x} = \text{proj}_{\vec{v}}(\vec{x}) + \vec{x}^{\perp} = P(\vec{x}) + \vec{x}^{\perp}$ where $\vec{x}^{\perp}$ is the orthogonal complement of $\vec{x}$ with respect to $\vec{v}$ so that we know that $P\left(\vec{x}^{\perp}\right) = \vec{0}$. To find fixed points we need to solve

$$\begin{aligned} P(\vec{x}) = \vec{x} \quad &\Rightarrow \quad P\left(P(\vec{x}) + \vec{x}^{\perp}\right) = P(\vec{x}) + \vec{x}^{\perp} \\ &\Rightarrow \quad P\left(P(\vec{x})\right) + P(\vec{x}^{\perp}) = P(\vec{x}) + \vec{x}^{\perp} \\ &\Rightarrow \quad P(\vec{x}) + \vec{0} = P(\vec{x}) + \vec{x}^{\perp} \\ &\Rightarrow \quad \vec{0} = \vec{x}^{\perp} \end{aligned}$$

  Hence the only points that are fixed (i.e. for which $P(\vec{x}) = \vec{x}$) are those points with $\vec{x}^{\perp} = \vec{0}$ i.e. all those points whose perpendicular component to $\text{proj}_{\vec{v}}(\vec{x})$ is zero. Hence all points on the line.

Proof of the 'Furthermore' points of Theorem 19.2:

- $M = M^{\mathsf{T}}$ and $\text{Trace}(M) = 1$ are obvious from $M = \dfrac{1}{a^2+a^2}\begin{bmatrix} b^2 & -ab \\ -ab & a^2 \end{bmatrix}$.

- $M^2 = M$

  Note that
$$\begin{aligned} M^2 &= \left(I_2 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right)^2 \\ &= \left(I_2 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right)\left(I_2 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right) \\ &= I_2^2 - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} + \frac{1}{\|\vec{n}\|^4}\vec{n}\,\vec{n}^{\mathsf{T}}\,\vec{n}\,\vec{n}^{\mathsf{T}} \\ &= I_2 - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} + \frac{1}{\|\vec{n}\|^4}(\vec{n}\cdot\vec{n})\vec{n}\,\vec{n}^{\mathsf{T}} \\ &= I_2 - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} + \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} \\ &= I_2 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} = M \end{aligned}$$

- That M is not invertible is clear from the fact that $M\vec{n} = \vec{0}$ which follows from

$$M\vec{n} = \left(I_2 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right)\vec{n} = \vec{n} - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\vec{n} = \vec{n} - \frac{1}{\|\vec{n}\|^2}(\vec{n}\cdot\vec{n})\vec{n} = \vec{n} - \vec{n} = \vec{0}$$

- Fixed points: $M\vec{x} = \vec{x} \Rightarrow \left(I_2 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right)\vec{x} = \vec{x}$

$$\Rightarrow \vec{x} - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\vec{x} = \vec{x}$$

$$\Rightarrow \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\vec{x} = \vec{0}$$

$$\Rightarrow \vec{n}\,\vec{n}^{\mathsf{T}}\vec{x} = \vec{0}$$

$$\Rightarrow (\vec{n}\cdot\vec{x})\,\vec{n} = \vec{0}$$

$$\Rightarrow \vec{n}\cdot\vec{x} = 0 \qquad \Rightarrow \quad \text{all points on the line } l$$

Proof of the 'Furthermore' points of Theorem 19.3:

- $M \neq M^{\mathsf{T}}$ in general, is clear from the matrix $\frac{1}{aA+bB}\begin{bmatrix} bB & -bA \\ -aB & aA \end{bmatrix}$

- $$M^2 = \left(I - \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}\right)\left(I - \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}\right) = I - 2\frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}} + \frac{1}{(\vec{v}\cdot\vec{n})^2}\vec{v}\,\vec{n}^{\mathsf{T}}\vec{v}\,\vec{n}^{\mathsf{T}}$$

$$= I - 2\frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}} + \frac{1}{(\vec{v}\cdot\vec{n})^2}(\vec{n}\cdot\vec{v})\vec{v}\,\vec{n}^{\mathsf{T}} = I - 2\frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}} + \frac{1}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}} = I - \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}} = M$$

- M is not invertible since clearly $M\vec{v} = \vec{0}$ and $M\vec{0} = \vec{0}$

$$\left(I - \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}\right)\vec{v} = \vec{v} - \frac{1}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}\vec{v} = \vec{v} - \frac{1}{\vec{v}\cdot\vec{n}}(\vec{n}\cdot\vec{v})\vec{v} = \vec{0}$$

- Finally, fixed points: $M\vec{x} = \vec{x} \Rightarrow \left(I - \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}\right)\vec{x} = \vec{x}$

$$\Rightarrow \vec{x} - \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}\vec{x} = \vec{x}$$

$$\Rightarrow \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}\vec{x} = \vec{0}$$

$$\Rightarrow (\vec{n}\cdot\vec{x})\vec{v} = \vec{0}$$

$$\Rightarrow \vec{n}\cdot\vec{x} = 0 \qquad \text{i.e. all points on the line.}$$

20. Reflection matrices in $\mathbb{R}^2$

In this sections we will discuss the matrices of 2D line reflections. Here is the condensed version of the main theorems

Theorem 20.1: The matrix of a **reflection** in the line $\vec{x} = t\vec{v}$: $\quad M = \boxed{\dfrac{2}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}} - I}$

Theorem 20.2: The matrix of a **reflection** in $\vec{n}\cdot\vec{x} = 0$: $\quad M = \boxed{I - \dfrac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}}$

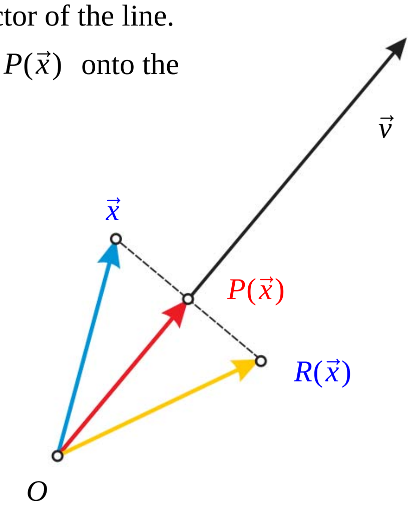
Theorem 20.3: The matrix of a **skew-reflection** in $\vec{n}\cdot\vec{x} = 0$ in the direction $\vec{v}$: $\quad M = \boxed{I - \dfrac{2}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}}$

Reflections

With only minor modifications we can produce the reflection matrices using the methods we used for projections in the previous section.

(a) The line is given in vector form (The method presented here also works in higher dimensional space)

Let $l:\ \vec{x} = t\vec{v}$ where $\vec{v}$ is a direction vector of the line.
Notice the relationship between the *projection* $P(\vec{x})$ onto the line and the *reflection* $R(\vec{x})$ in the line



$$\frac{\vec{x} + R(\vec{x})}{2} = P(\vec{x})$$

Hence $\quad R(\vec{x}) = 2P(\vec{x}) - \vec{x}$

So that

$$R(\vec{x}) = \frac{2}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}}\vec{x} - \vec{x}$$

Hence the reflection matrix is $\dfrac{2}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}} - I$

or if $\vec{v} = \begin{bmatrix} A \\ B \end{bmatrix}$ then $M = \dfrac{1}{A^2 + B^2}\begin{bmatrix} A^2 - B^2 & 2AB \\ 2AB & B^2 - A^2 \end{bmatrix}$

Example 1: **2D**: Let $l: \ \vec{x} = t\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ then $\vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and the reflection in this line

is $$R(\vec{x}) = \left(\frac{2}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}} - I\right)\vec{x} = \left(\frac{2}{5}\begin{bmatrix} 1 \\ -2 \end{bmatrix}\begin{bmatrix} 1 & -2 \end{bmatrix} - I\right)\vec{x} = \frac{1}{5}\begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix}\vec{x}$$

This approach also works in higher dimensions:

Example 2: **3D**: Let $l: \ \vec{x} = t\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ then $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ and the reflection in this line

is $$R(\vec{x}) = \left(\frac{2}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}} - I\right)\vec{x} = \left(\frac{2}{5}\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & -2 & 5 \end{bmatrix} - I\right)\vec{x} = \frac{1}{5}\begin{bmatrix} -3 & -4 & 2 \\ -4 & 3 & -4 \\ 2 & -4 & 9 \end{bmatrix}\vec{x}$$

Theorem 20.1: Let the line l be given in vector form $\vec{x} = t\vec{v}$, then the reflection in this line is given by

$$R(\vec{x}) = \left(\frac{2}{\|\vec{v}\|^2}\,\vec{v}\,\vec{v}^{\mathsf{T}} - I\right)\vec{x} \quad \text{or} \quad R(\vec{x}) = \left(\frac{2\,\vec{v}\,\vec{v}^{\mathsf{T}}}{\|\vec{v}\|^2} - I\right)\vec{x}$$

R is a linear transformation with matrix $\boxed{M = \frac{2}{\|\vec{v}\|^2}\,\vec{v}\,\vec{v}^{\mathsf{T}} - I}$

If $\vec{v} = \begin{bmatrix} A \\ B \end{bmatrix}$ then $M = \frac{1}{A^2+B^2}\begin{bmatrix} A^2-B^2 & 2AB \\ 2AB & B^2-A^2 \end{bmatrix}$

or

if $\vec{v} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ then $M = \frac{1}{A^2+B^2+C^2}\begin{bmatrix} A^2-B^2-C^2 & 2AB & 2AC \\ 2AB & -A^2+B^2-C^2 & 2BC \\ 2AC & 2BC & -A^2-B^2+C^2 \end{bmatrix}$

Furthermore

- $M = M^{\mathsf{T}}$
- $M^2 = I$ i.e. $M^{-1} = M$
- $\text{Trace}(M_2) = 0$ in the 2D case
- $\text{Trace}(M_3) = -1$ in the 3D case [In $\mathbb{R}^n$: $\text{Trace}(M_n) = 2-n$]
- $\det(M_2) = -1$ in the 2D case
- $\det(M_3) = +1$ in the 3D case [In $\mathbb{R}^n$: $\det(M_n) = (-1)^{n-1}$]
- The only fixed points of this transformation are the points on the line.

Proof: Only the 'furthermore' points remain to be proven.

- $$M^{\mathsf T} = \left(\frac{2}{\|\vec v\|^2}\cdot\vec v\,\vec v^{\mathsf T} - I\right)^{\mathsf T} = \frac{2}{\|\vec v\|^2}\cdot(\vec v^{\mathsf T})^{\mathsf T}\,\vec v^{\mathsf T} - I^{\mathsf T} = \frac{2}{\|\vec v\|^2}\cdot\vec v\,\vec v^{\mathsf T} - I = M$$

- $$\begin{aligned} M^2 &= \left(\frac{2}{\|\vec v\|^2}\cdot\vec v\,\vec v^{\mathsf T} - I\right)\cdot\left(\frac{2}{\|\vec v\|^2}\cdot\vec v\,\vec v^{\mathsf T} - I\right) \\ &= \frac{4}{\|\vec v\|^4}\cdot\vec v\,\vec v^{\mathsf T}\,\vec v\,\vec v^{\mathsf T} - \frac{4}{\|\vec v\|^2}\cdot\vec v\,\vec v^{\mathsf T} + I^2 \\ &= \frac{4}{\|\vec v\|^4}\cdot\left([\,\vec v\cdot\vec v\,]\cdot\vec v\right)\vec v^{\mathsf T} - \frac{4}{\|\vec v\|^2}\cdot\vec v\,\vec v^{\mathsf T} + I = I \end{aligned}$$

  And since $M^2 = I$ we also can conclude that $M^{-1} = M$

- $\mathrm{Trace}(M) = 2 - n$ ($\mathbb{R}^2, \mathbb{R}^3$ clear from the matrices. The general case: an exercise!)

- $\det(M_n) = (-1)^{n-1}$ (A direct calculation suffices for the 2D and 3D matrices. The general case requires more work, but is easy using eigenvalues.)

- Fixed points: $\left(\frac{2}{\|\vec v\|^2}\vec v\,\vec v^{\mathsf T} - I\right)\vec x = \vec x \;\Rightarrow\; \frac{1}{\|\vec v\|^2}\vec v\,\vec v^{\mathsf T}\vec x = \vec x \;\Rightarrow\; P(\vec x) = \vec x$

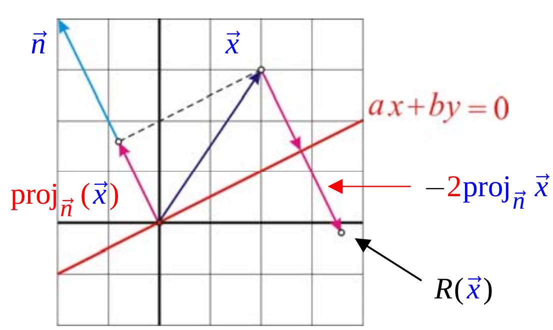
  i.e. all those points on the line (as we proved for the projection P).

(b) The line is given in normal form (2D only! There is **no** normal form of lines in higher dimensions)

Let $l:\ ax + by = 0$ be a line through the origin, with normal $\vec n = \begin{bmatrix} a \\ b \end{bmatrix}$.

The reflection $R:\ \mathbb{R}^2 \to \mathbb{R}^2$ in this line can be computed as follows

$$\begin{aligned} R(\vec x) &= \vec x - 2\mathrm{proj}_{\vec n}(\vec x) \\ &= \vec x - 2\frac{\vec n\cdot\vec x}{\vec n\cdot\vec n}\vec n \\ &= \vec x - \frac{2}{\|\vec n\|^2}(\vec n\cdot\vec x)\,\vec n \\ &= \vec x - \frac{2}{\|\vec n\|^2}\vec n\,\vec n^{\mathsf T}\vec x \\ &= \left(I_2 - \frac{2}{\|\vec n\|^2}\vec n\,\vec n^{\mathsf T}\right)\vec x \end{aligned}$$



Hence the reflection matrix is $I_2 - \frac{2}{\|\vec n\|^2}\vec n\,\vec n^{\mathsf T}$. An explicit form of this matrix would be

$$M = \frac{1}{a^2+b^2}\begin{bmatrix} -a^2+b^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix}$$ which can be easily computeded as follows

$$\begin{aligned} I_2 - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^\mathsf{T} &= \frac{1}{a^2+b^2}\begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix} - \frac{1}{a^2+b^2}\begin{bmatrix} 2a^2 & 2ab \\ 2ab & 2b^2 \end{bmatrix} \\ &= \frac{1}{a^2+b^2}\left(\begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix} - \begin{bmatrix} 2a^2 & 2ab \\ 2ab & 2b^2 \end{bmatrix}\right) \\ &= \frac{1}{a^2+b^2}\begin{bmatrix} -a^2+b^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix} \end{aligned}$$

Theorem 20.2: Let $l:\ \vec{n}\cdot\vec{x} = 0$ be a line in $\mathbb{R}^2$, i.e. $l:\ ax+by = 0$ when $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$,

then the matrix of the reflection in the line l is given by

$$M = \boxed{I_2 - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^\mathsf{T}} \qquad \text{i.e.} \qquad M = \frac{1}{a^2+b^2}\begin{bmatrix} -a^2+b^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix}$$

Furthermore:

- $M = M^\mathsf{T}$
- $M^2 = I_2$ hence $M^{-1} = M$ (i.e. M is invertible.)
- $\text{Trace}(M) = 0$
- $\det(M) = -1$
- The only fixed points of this transformation are the points on the line.

Proof: Since $M = \frac{1}{a^2+b^2}\begin{bmatrix} -a^2+b^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix}$

- $M = M^\mathsf{T}$ and $\text{Trace}(M) = 0$ are pretty obvious
- $M^2 = M$ follows by direct computation. [This also implies that $M^{-1} = M$.]
- $\det(M) = -1$ follows by direct computation.
- Fixed points: $M\,\vec{x} = \vec{x} \Rightarrow \left(I_2 - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^\mathsf{T}\right)\vec{x} = \vec{x}$

$$\begin{aligned} &\Rightarrow \vec{x} - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^\mathsf{T}\vec{x} = \vec{x} \\ &\Rightarrow \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^\mathsf{T}\vec{x} = \vec{0} \\ &\Rightarrow \vec{n}\,\vec{n}^\mathsf{T}\vec{x} = \vec{0} \\ &\Rightarrow (\vec{n}\cdot\vec{x})\,\vec{n} = \vec{0} \\ &\Rightarrow \vec{n}\cdot\vec{x} = 0 \qquad \text{i.e. all points on the line } l \end{aligned}$$

Example 3: Let $l:\ x-2y = 0$ be a line in $\mathbb{R}^2$, hence $\vec{n} = \begin{bmatrix}1\\-2\end{bmatrix}$, then the matrix of the reflection R in the line l is

$$I_2 - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} = \begin{bmatrix}1&0\\0&1\end{bmatrix} - \frac{2}{5}\begin{bmatrix}1\\-2\end{bmatrix}\begin{bmatrix}1&-2\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix} - \frac{2}{5}\begin{bmatrix}1&-2\\-2&4\end{bmatrix} = \frac{1}{5}\begin{bmatrix}3&4\\4&-3\end{bmatrix}$$

or

$$\frac{1}{a^2+b^2}\begin{bmatrix}-a^2+b^2 & -2ab\\ -2ab & a^2-b^2\end{bmatrix} = \frac{1}{1^2+(-2)^2}\begin{bmatrix}-1^2+(-2)^2 & -2\cdot1\cdot(-2)\\ -2\cdot1\cdot(-2) & 1^2-(-2)^2\end{bmatrix} = \frac{1}{5}\begin{bmatrix}3&4\\4&-3\end{bmatrix}$$

- The image of $\begin{bmatrix}5\\5\end{bmatrix}$ and $\begin{bmatrix}0\\-5\end{bmatrix}$ are $\frac{1}{5}\begin{bmatrix}4&2\\2&1\end{bmatrix}\begin{bmatrix}5\\5\end{bmatrix} = \begin{bmatrix}7\\1\end{bmatrix}$ and $\frac{1}{5}\begin{bmatrix}4&2\\2&1\end{bmatrix}\begin{bmatrix}0\\-5\end{bmatrix} = \begin{bmatrix}-4\\3\end{bmatrix}$

- The image of $m:\ \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}5\\5\end{bmatrix} + t\begin{bmatrix}3\\4\end{bmatrix}$

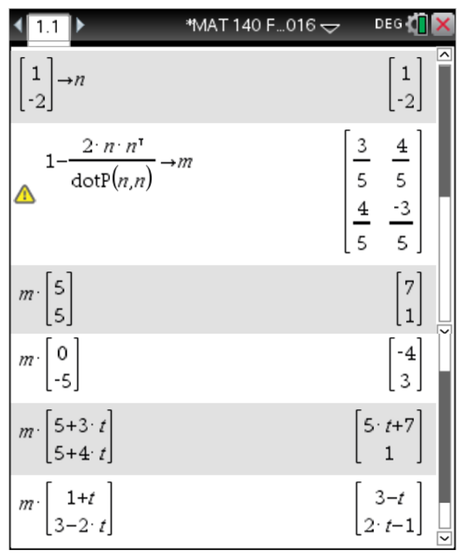
  is

  $$R\begin{bmatrix}x\\y\end{bmatrix} = R\left(\begin{bmatrix}5+3t\\5+4t\end{bmatrix}\right) = \begin{bmatrix}7\\1\end{bmatrix} + t\begin{bmatrix}5\\0\end{bmatrix}$$

  so that

  $$\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}7\\1\end{bmatrix} + t\begin{bmatrix}5\\0\end{bmatrix} = \begin{bmatrix}7+5t\\1\end{bmatrix}$$

  i.e. the images $(x',\ y')$ are on the horizontal line $y=1$.

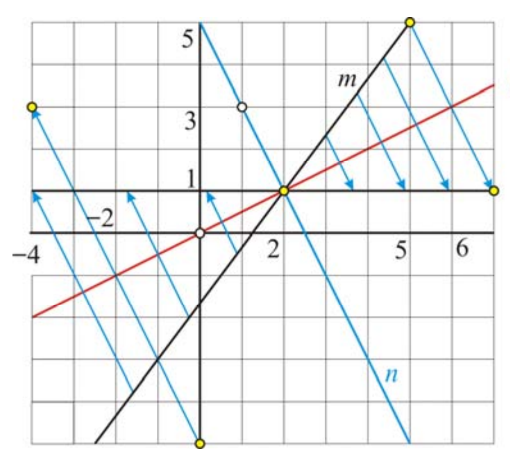


- The image of the line $n:\ \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix} + t\begin{bmatrix}1\\-2\end{bmatrix}$ (i.e. $2x+y=5$) is

  $$R\begin{bmatrix}x\\y\end{bmatrix} = R\left(\begin{bmatrix}1\\3\end{bmatrix} + t\begin{bmatrix}1\\-2\end{bmatrix}\right) = \begin{bmatrix}3\\-1\end{bmatrix} + t\begin{bmatrix}-1\\2\end{bmatrix}$$

  Closer inspection reveals this is the same line $2x+y=5$. This makes sense once we realize that n is perpendicular to the line l in which we reflect.

  (See picture)

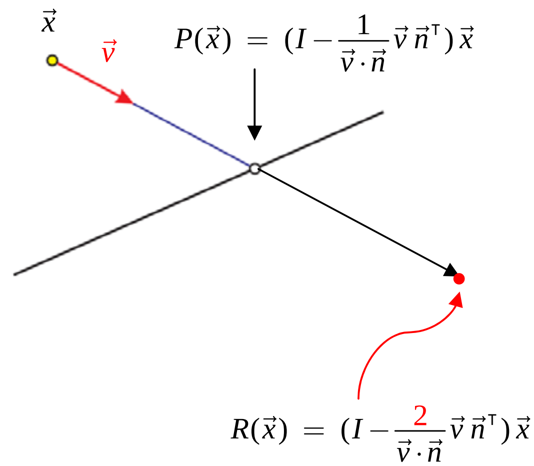


Skew Reflections

We can also create a skew-reflection, which we can derive from a skew-projection, in pretty much the same way as we derived a reflection from a projection:

Theorem 20.3: The matrix of a **skew reflection** onto $\vec{n}\cdot\vec{x} = 0$ in the direction of the vector $\vec{v}$ is

$$M = \boxed{I - \frac{2}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}}$$



If $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} A \\ B \end{bmatrix}$ then

$$M = \frac{1}{aA+bB}\begin{bmatrix} -aA+bB & -2bA \\ -2aB & aA-bB \end{bmatrix}$$

Furthermore:

- $M \neq M^{\mathsf{T}}$ (unless $\vec{v} = \vec{n}$
- $M^2 = I$ so that $M^{-1} = M$
- $\text{Trace}(M) = 0$
- $\det(M) = -1$.
- The only fixed points of this transformation are the points on the line.

Proof:

Again $\dfrac{\vec{x} + R(\vec{x})}{2} = P(\vec{x})$ so that

$$R(\vec{x}) = 2P(\vec{x}) - \vec{x} = 2\left(I - \frac{1}{\vec{v}\cdot\vec{n}}\vec{v}^{\mathsf{T}}\vec{n}\right)\vec{x} - \vec{x} = \left(I - \frac{2}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}\right)\vec{x}$$

Computing $M = I - \dfrac{2}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}$ with $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} A \\ B \end{bmatrix}$ we get, after some algebra,

$$M = \frac{1}{aA+bB}\begin{bmatrix} -aA+bB & -2bA \\ -2aB & aA-bB \end{bmatrix}$$

It is then clear that, in general, $M \neq M^{\mathsf{T}}$, $\text{Trace}(M) = 0$ and after simple computations that $M^2 = I$ and $\det(M) = -1$. That the only fixed points are the points on the line we leave as an exercise (or compare with the skew-reflection in a plane in chapter 25).

21. Scalings in $\mathbb{R}^2$

In this section we will discuss the matrices of 2D scalings. The condensed version of the main theorem is:

Theorem 21.1: The matrix of a 2D **scaling** is : $M = \boxed{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}$

Scalings

Definition: A scaling (centered at the origin) is defined by $S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a\,x \\ b\,y \end{bmatrix}$.

i.e. all x-coordinates are scaled by a constant a, and all y-coordinates are scaled by a constant b.

The following theorem should be obvious, since $\begin{bmatrix} a\,x \\ b\,y \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$:

Theorem 21.1: The matrix of the scaling $S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a\,x \\ b\,y \end{bmatrix}$ is $M = M_{a,b} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

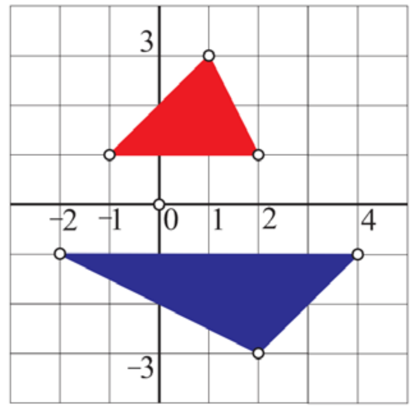
Furthermore:

- M is invertible *iff* both $a \neq 0$ **and** $b \neq 0$, in which case:
$$M^{-1} = M_{a,b}^{-1} = \begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix} = M_{1/a,1/b}$$
- $M^2 = M_{a,b}^2 = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} = M_{a^2,b^2}$ [In general $M^m = M_{a,b}^m = M_{a^m,b^m}$]
- $M = M^{\mathsf{T}}$
- $\text{Trace}(M) = a + b$ and $\det(M) = a\,b$
- If $a \neq 1$ and $b \neq 1$ then the only fixed point of S is $(0, 0)$.

  If $a = 1$ and $b \neq 1$, i.e. $M = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$ then *only* the points on the x-axis are fixed.

  If $b = 1$ and $a \neq 1$, i.e. $M = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ then *only* the points on the y-axis are fixed.
- If both $a = b = 1$ then $M = I_2$ and *all* points are fixed.

Example 1: Let $S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ -y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ then S maps the red triangle in the picture to the blue triangle.



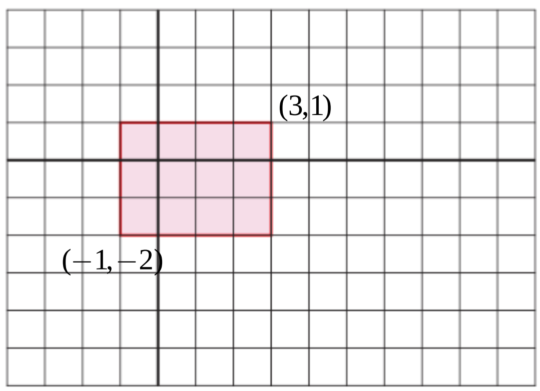
Notice that the -1 flipped everything about the x-axis (i.e. stretched by -1 in the y direction) and the 2 stretches everything in the x-direction.

S maps point $(1,3)$ to point $(2,-3)$,

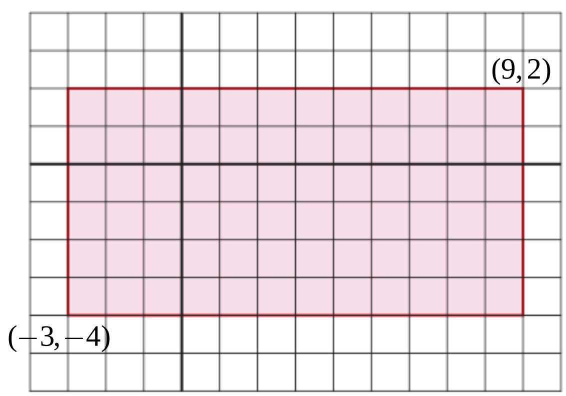
S maps point $(-1,1)$ to point $(-2,-1)$ and

S maps point $(2,1)$ to point $(4,-1)$

Example 2: $S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ transforms the rectangular red-shaded screen as shown



into the screen



22. Rotation Matrices in $\mathbb{R}^2$

In this section we will revisit the matrices of 2D rotations around the origin. The condensed version of the main theorem is:

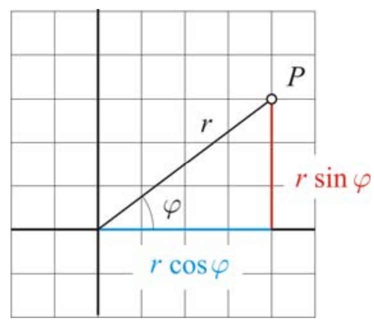
Theorem 22.1: The matrix of a 2D **rotation** around the origin over θ is given by:

$$M = \boxed{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$$

Rotations

Let's examine (as we did in chapter 16) the rotation $R:\ \mathbb{R}^2 \to \mathbb{R}^2$ around the origin over the angle θ (with the usual convention of 'positive' and 'negative' angles).

Recall that the point P with polar coordinates r and φ has the following rectangular coordinates:

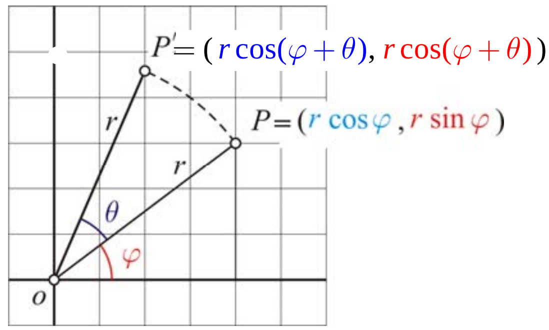


$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

Also recall the following trig identities:

$$\begin{aligned} \cos(A+B) &= \cos(A)\cos(B) - \sin(A)\sin(B) \\ \sin(A+B) &= \sin(A)\cos(B) + \cos(A)\sin(B) \end{aligned}$$

Let $P = \begin{bmatrix} x \\ y \end{bmatrix}$ be a point in $\mathbb{R}^2$ and $P' = \begin{bmatrix} x' \\ y' \end{bmatrix} = T\begin{bmatrix} x \\ y \end{bmatrix}$ the image of $\begin{bmatrix} x \\ y \end{bmatrix}$ after rotation around the origin over θ. We can now compute the coordinates of the point $\begin{bmatrix} x' \\ y' \end{bmatrix}$:



$$\begin{cases} x' = r\cos(\varphi+\theta) \\ y' = r\sin(\varphi+\theta) \end{cases} \Rightarrow \begin{cases} x' = r\cos\varphi\cos\theta - r\sin\varphi\sin\theta \\ y' = r\sin\varphi\cos\theta + r\cos\varphi\sin\theta \end{cases}$$

$$\Rightarrow \begin{cases} x' = x\cos\theta - y\sin\theta \\ y' = y\cos\theta + x\sin\theta \end{cases}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ y\cos\theta + x\sin\theta \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Hence we have found that $R\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, and we have proven the main part of the following theorem:

Theorem 22.1: The rotation $R: \mathbb{R}^2 \to \mathbb{R}^2$ around the origin over the angle θ has matrix

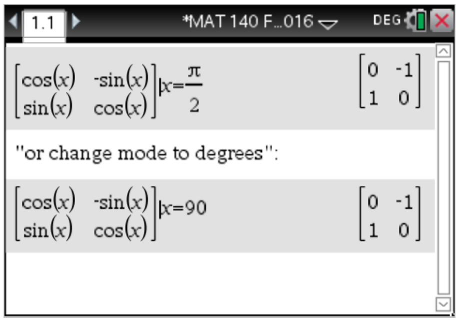
$$M = M(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Furthermore

- M is invertible: $M(\theta)^{-1} = M(-\theta)$ hence $M^{-1} = M^{\mathsf{T}}$.
- $M^2 = M(\theta)^2 = M(2\theta)$ [In general $M^m = M(\theta)^m = M(m\theta)$]
- $M \neq M^{\mathsf{T}}$ (unless $M = \pm I$)
- $\mathrm{Trace}(M) = \mathrm{Trace}(M(\theta)) = 2\cos(\theta)$
- $\det(M) = 1$
- The only fixed point is $(0, 0)$ [unless $\theta = k \cdot 360°$, then $M = I_2$ i.e. the identity matrix, in which case *all* points are fixed.]

Proof: The proofs of the 'Furthermore' points are pretty straightforward and left as an exercise.

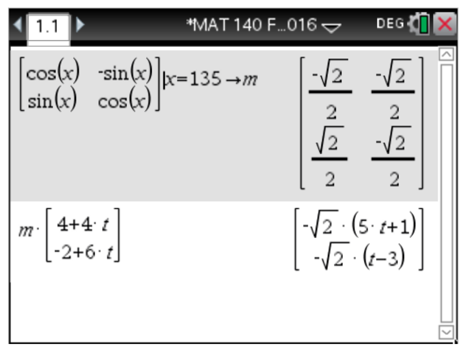
Example 1: The rotation over $90°$ has as matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$



[Check to see if your mode is set to radians or degrees, i.e. if you have to type in $\frac{\pi}{2}$ or 90]

Example 2:

The rotation over $135°$ has as matrix $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$.



a) What is the image of the line $l: \;\; 6x - 4y = 32$?

First let's write the line in vector form:

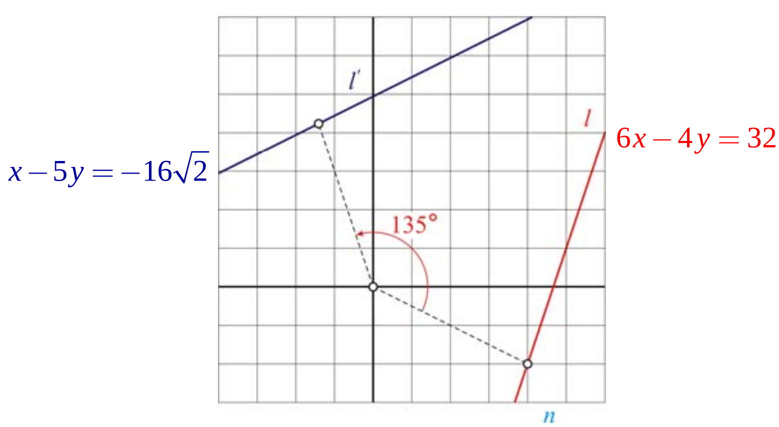
$$l: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} + t\begin{bmatrix} 4 \\ 6 \end{bmatrix} \quad \text{(check!)}$$

Hence $R\begin{bmatrix} x \\ y \end{bmatrix} = R\begin{bmatrix} 4+4t \\ -2+6t \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 3\sqrt{2} \end{bmatrix} + t\begin{bmatrix} -5\sqrt{2} \\ -\sqrt{2} \end{bmatrix}$

so that $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 3\sqrt{2} \end{bmatrix} + t\begin{bmatrix} -5\sqrt{2} \\ -\sqrt{2} \end{bmatrix}$,

i.e. $x' - 5y' = -16\sqrt{2}$, hence it is mapped to the line $\boxed{x - 5y = -16\sqrt{2}}$

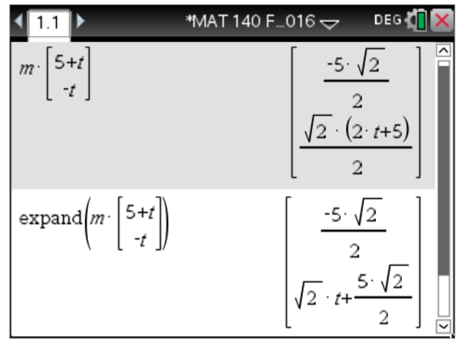
(see picture)



b) what is the image of the line $m:\ x+y=5$?

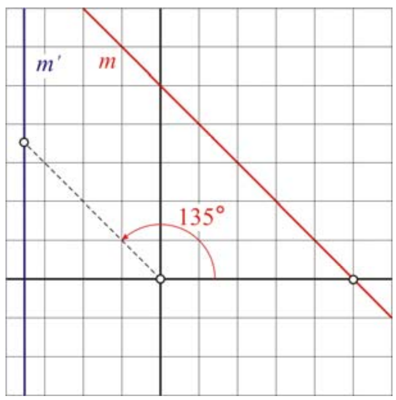
First let's write the line in vector form: $m: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Hence $R\begin{bmatrix} x \\ y \end{bmatrix} = R\left(\begin{bmatrix} 5+t \\ -t \end{bmatrix}\right) = \dfrac{1}{\sqrt{2}}\begin{bmatrix} -5 \\ 5 \end{bmatrix} + t\begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$



so that the images form the vertical line $x = -\dfrac{5}{\sqrt{2}}$.

(See picture)



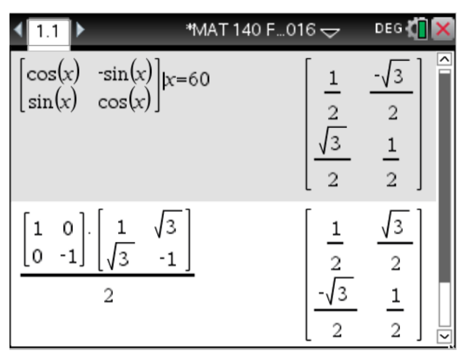
$x+y=5$

$x = -\dfrac{5}{\sqrt{2}}$

Example 3:

The rotation over $60°$ has as matrix $\dfrac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$.

Let's compare this transformation to the following **two reflections**:



the *reflection* in the *x*-axis has matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

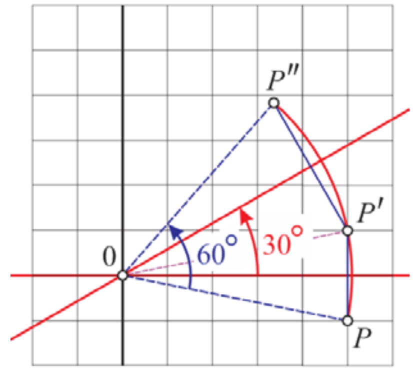
and the *reflection* in the line $x-\sqrt{3}y = 0$

has matrix $\dfrac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$.

So that their composition matrix is: $\dfrac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$

This shows that the composition of the reflection in the lines $y = 0$ followed by the reflection in the line $x - \sqrt{3}y = 0$ is a rotation around the origin over $60°$.

Note that the angle between the two lines is $30°$!



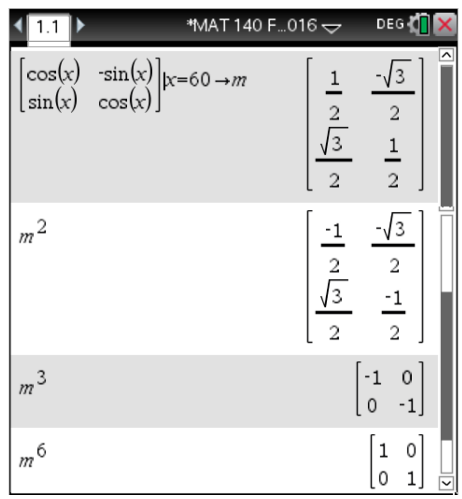
Example 4:

The rotation over $60°$ has as matrix $M = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$.

(a) If we perform this operation 2 times we get:

$$M^2 = \left(\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\right)^2 = \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$$

which is the rotation over $120°$.



(b) If we perform this operation 3 times we get

$$M^3 = \left(\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\right)^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

which is the rotation over $180°$ (which is the same as a *point reflection* in the origin)

(c) If we perform this operation 6 times we get

$$M^6 = \left(\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\right)^6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is the rotation over $360°$ (which is the same as the *identity transformation*: i.e. we're back where we started)

23. Shear Matrices in $\mathbb{R}^2$

In this section we will discuss the matrices of 2D shears with respect to lines. Most books only discuss horizontal or vertical shears. We will give a formula for general shears parallel to any line through the origin. The condensed version of the main theorem is:

Theorem 23.1: The matrix of a **shear** with respect to $\vec{n}\cdot\vec{x}=0$ and $\vec{v}$:

$$M = \boxed{I+\frac{1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}}}$$

Shears

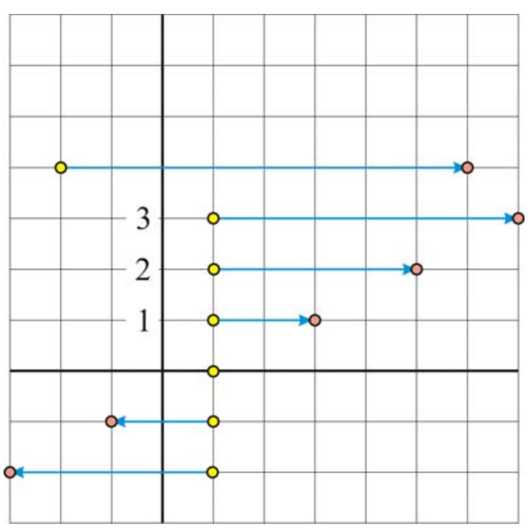
The matrices $\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\ b & 1\end{bmatrix}$ perform transformations that are called shears.

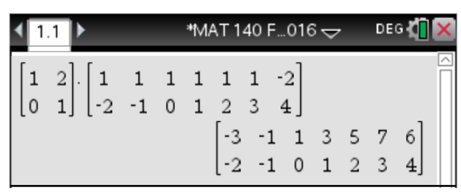
Example 1: $S:\ \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $S(\vec{x}) = \begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\vec{x}$ then

$$S\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}x\\ y\end{bmatrix}+y\begin{bmatrix}2\\ 0\end{bmatrix}$$

Hence the shear moves the point $\begin{bmatrix}x\\ y\end{bmatrix}$ over by $y\begin{bmatrix}2\\ 0\end{bmatrix}=\begin{bmatrix}2y\\ 0\end{bmatrix}$, i.e. a shift parallel to the x-axis.

How much we shift depends on the y-coordinate of the point. Let's look at some examples:





Note that the farther we are from the x-axis the more we shift. If we are 3 units above the x-axis we shift $3\begin{bmatrix}2\\0\end{bmatrix}$ over, if we are 2 units above the x-axis we shift $2\begin{bmatrix}2\\0\end{bmatrix}$ over, If we are 2 units *below* the x-axis (i.e. $y=-2$) we shift $-2\begin{bmatrix}2\\0\end{bmatrix}$ over. If we are on the x-axis we shift $0\begin{bmatrix}2\\0\end{bmatrix}$ over, i.e. we do *not* shift (the x-axis is fixed). We always shift parallel to the x-axis over a multiple of the vector $\begin{bmatrix}2\\0\end{bmatrix}$, where the multiple is determined by the y-coordinate, i.e. the signed distance from x-axis.

The Generalized Shear

We are now going to **generalize** this transformation to a shear parallel to any line through the origin:

Suppose we want to shear parallel to line $l:\ ax+by=0$ with shearing vector $\begin{bmatrix}A\\B\end{bmatrix}$. Of course we need the vector $\begin{bmatrix}A\\B\end{bmatrix}$ to be parallel to the line: $\begin{bmatrix}a\\b\end{bmatrix}\cdot\begin{bmatrix}A\\B\end{bmatrix}=0$ [i.e. $aA+bB=0$].

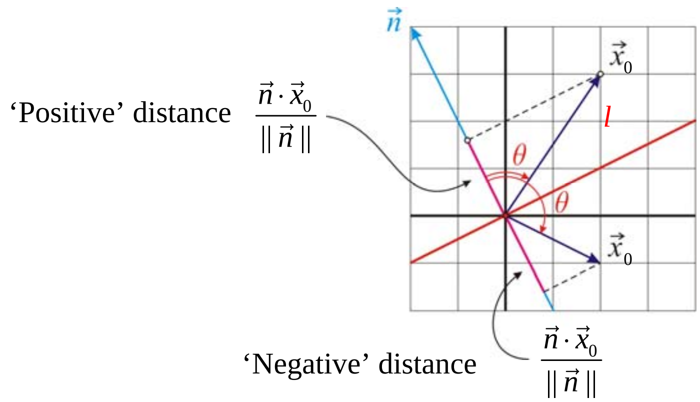
Recall that the shift depends on the distance from the line, and on the opposite side we shift in different directions. How can we accomplish that?

To find the distance of a point $\vec{x}_0 = \begin{bmatrix}x_0\\y_0\end{bmatrix}$ to the line we compute: $\dfrac{|ax_0+by_0-0|}{\sqrt{a^2+b^2}}$

i.e. this is computed by $\dfrac{|ax_0+by_0-0|}{\sqrt{a^2+b^2}} = \dfrac{|\vec{n}\cdot\vec{x}_0|}{\|\vec{n}\|}$ Notice the absolute values!

Recall that $\vec{n}\cdot\vec{x}_0 = \|\vec{n}\|\cdot\|\vec{x}_0\|\cdot\cos\theta$ which is **positive** if the angle $0\le\theta<90^\circ$ i.e. when $\vec{x}_0$ and $\vec{n}$ are on the same side of the line, and **negative** if the angle $90^\circ<\theta\le 180^\circ$ i.e. when $\vec{x}_0$ and $\vec{n}$ are on opposite sides of the line.

Hence $\dfrac{\vec{n}\cdot\vec{x}_0}{\|\vec{n}\|}$ gives us exactly the **signed distance** of point $\vec{x}_0$ to the line l : when $\vec{x}_0$ and $\vec{n}$ are on the **same** side of the line l then $\dfrac{\vec{n}\cdot\vec{x}_0}{\|\vec{n}\|}$ gives us exactly the **positive distance** of point $\vec{x}$ to the line l, and when $\vec{x}_0$ and $\vec{n}$ are on **opposite** sides of the line l then $\dfrac{\vec{n}\cdot\vec{x}_0}{\|\vec{n}\|}$ gives us exactly the **negative distance** of point $\vec{x}$ to the line l.



This **signed distance** $\frac{\vec{n}\cdot\vec{x}_0}{\|\vec{n}\|}$ is exactly what we need for our shear to work in the way we described before:

$$\begin{aligned} S(\vec{x}_0) &= \vec{x}_0 + \frac{\vec{n}\cdot\vec{x}_0}{\|\vec{n}\|}\vec{v} \\ &= \vec{x}_0 + \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\|\vec{n}\|}\vec{x}_0 \\ &= I_2\vec{x}_0 + \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\|\vec{n}\|}\vec{x}_0 \\ &= \left(I_2 + \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\|\vec{n}\|}\right)\vec{x}_0 \end{aligned}$$

Hence we find

$$M = I_2 + \frac{1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}}$$

or with $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} A \\ B \end{bmatrix}$

$$M = I_2 + \vec{v}\,\vec{n}^{\mathsf{T}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{\sqrt{a^2+b^2}}\begin{bmatrix} A \\ B \end{bmatrix}\begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} 1+\frac{aA}{\sqrt{a^2+b^2}} & \frac{bA}{\sqrt{a^2+b^2}} \\ \frac{aB}{\sqrt{a^2+b^2}} & 1+\frac{bB}{\sqrt{a^2+b^2}} \end{bmatrix}$$

We have proven the main part of the following theorem:

Theorem 23.1: The shear $S:\ \mathbb{R}^2 \to \mathbb{R}^2$ parallel to the line $l:\ \vec{n}\cdot\vec{x} = 0$, i.e.

$l:\ ax + by = 0$, where $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$, in the direction of the shearing vector

$\vec{v} = \begin{bmatrix} A \\ B \end{bmatrix}$ for which $\vec{n}\cdot\vec{v} = 0$, i.e. $\begin{bmatrix} a \\ b \end{bmatrix}\cdot\begin{bmatrix} A \\ B \end{bmatrix} = 0$ (or $aA + bB = 0$),

has matrix

$$M = M_{\vec{n},\vec{v}} = I_2 + \frac{1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}} \qquad \text{i.e.} \qquad M = M_{\vec{n},\vec{v}} = \begin{bmatrix} 1+\dfrac{aA}{\sqrt{a^2+b^2}} & \dfrac{bA}{\sqrt{a^2+b^2}} \\ \dfrac{aB}{\sqrt{a^2+b^2}} & 1+\dfrac{bB}{\sqrt{a^2+b^2}} \end{bmatrix}$$

Furthermore:

- M is invertible: $M^{-1} = M_{\vec{n},\vec{v}}^{-1} = I_2 + \dfrac{-1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}} = M_{\vec{n},-\vec{v}}$
- $M^2 = M_{\vec{n},\vec{v}}^2 = I_2 + \dfrac{2}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}} = M_{\vec{n},2\vec{v}}$ [In general $M^m = M_{\vec{n},\vec{v}}^m = M_{\vec{n},m\vec{v}}$]
- $M \neq M^{\mathsf{T}}$ (in general, unless $A = B = 0$ when $M = I$)
- $\text{Trace}(M) = 2$
- $\det(M) = 1$
- The only fixed points of S is are the points on the line l (unless $M = I$).

Proof: Most of these are simple to prove

e.g. $\text{Trace}(M) = 1+\dfrac{aA}{\sqrt{a^2+b^2}} + 1+\dfrac{bB}{\sqrt{a^2+b^2}} = 2+\dfrac{aA+bB}{\sqrt{a^2+b^2}} = 2$ since $aA + bB = 0$.

$$\begin{aligned}\det(M) &= \left(1+\frac{aA}{\sqrt{a^2+b^2}}\right)\left(1+\frac{bB}{\sqrt{a^2+b^2}}\right) - \frac{aB}{\sqrt{a^2+b^2}}\frac{bA}{\sqrt{a^2+b^2}} \\ &= 1+\frac{aA+bB}{\sqrt{a^2+b^2}} + \frac{aAbB}{a^2+b^2} - \frac{aAbB}{a^2+b^2} = 1 \qquad \text{since } aA+bB=0\end{aligned}$$

The rest is left as an exercise.

Example 2: Let $l:\ 3x+4y=0$ and $\vec{v} = \begin{bmatrix} 8 \\ -6 \end{bmatrix}$. Note that $\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ -6 \end{bmatrix} = 0$. Then

$$I_2 + \frac{1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}} = \begin{bmatrix} 1+\frac{24}{5} & \frac{32}{5} \\ -\frac{18}{5} & 1-\frac{24}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 29 & 32 \\ -18 & -19 \end{bmatrix}$$

- So the image of $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is $S\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 29 & 32 \\ -18 & -19 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$

- What is the image of the line $m:\ 3x+4y=5$? Note this is a line parallel to the plane with respect to which we are shearing; i.e. we should not expect to get another line … the line is moved to itself … not point wise though(!) … all points move but to other points on the same plane. Let's check this:

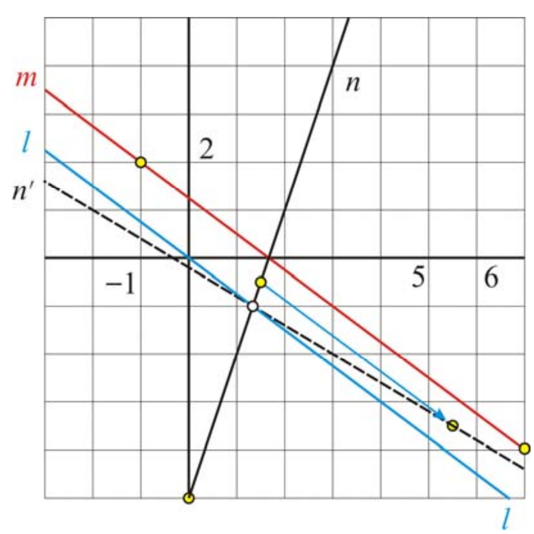
$$m:\ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

hence

$$m':\ S\begin{bmatrix} x \\ y \end{bmatrix} = S\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 4 \\ -3 \end{bmatrix}\right)$$

$$m':\ \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \end{bmatrix} + t\begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

which has a normal equation: $3x'+4y'=5$ which is the line m.
So indeed, the image of the line m is the line m itself.



- What is the image of the line $n: \; 3x - y = 5$? Rewrite n in vector form:

$$n: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.5 \\ -.5 \end{bmatrix} + t\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

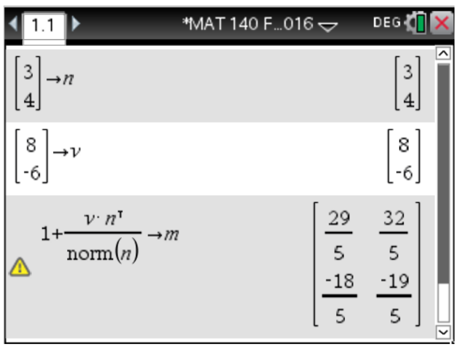
then

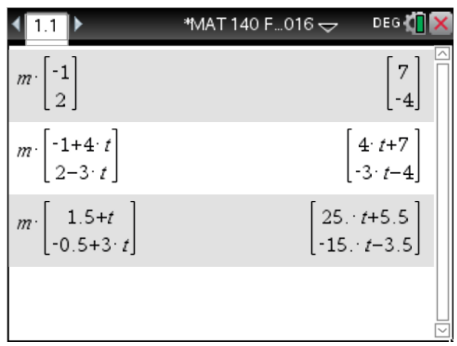
$$n': \; S\begin{bmatrix} x \\ y \end{bmatrix} = S\left(\begin{bmatrix} 1.5 \\ -.5 \end{bmatrix} + t\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$$

$$n': \; \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5.5 \\ -3.5 \end{bmatrix} + t\begin{bmatrix} 25 \\ -15 \end{bmatrix}$$

i.e. n' is the line $3x + 5y = -1$.

All these computations can be done with the TI-Nspire as follows:





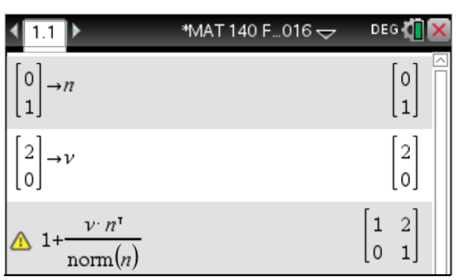
Example 3: Let $l: \; y = 0$ i.e. the x-axis ($\vec{n} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$), and $\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

Note that $\vec{n} \cdot \vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 0$.

Then

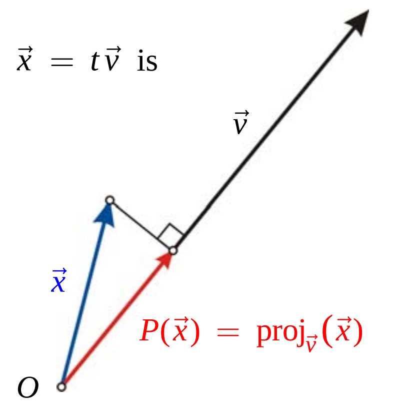
$$I_2 + \frac{1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{1}\begin{bmatrix} 2 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

This is exactly the shear we looked at in example 1.



Theorem 24.1: The matrix of the projection onto the line $\vec{x} = t\vec{v}$ is

$$\boxed{M = \frac{1}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}}}$$

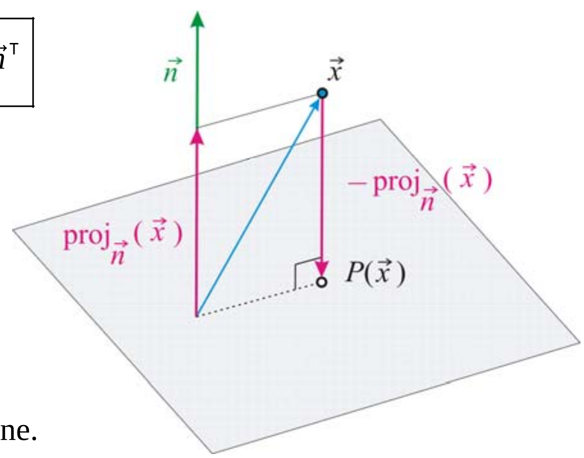


Furthermore:

- $M = M^{\mathsf{T}}$
- $M^2 = M$
- $\mathrm{Trace}(M) = 1$
- M is not invertible: $\det(M) = 0$.
- The only fixed points of this transformation are the points on the line.

Theorem 24.2: The matrix of the projection onto the plane $\vec{n} \cdot \vec{x} = 0$ is

$$M = \boxed{I - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}}$$

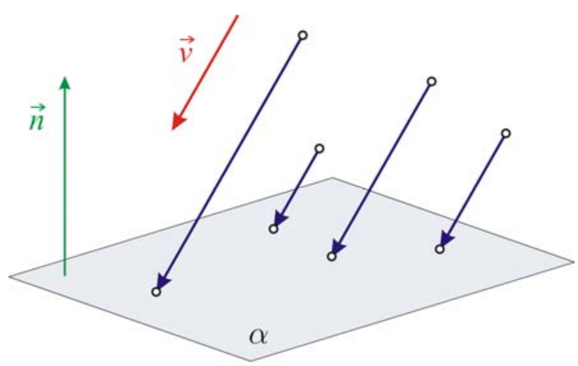


Furthermore:

- $M = M^{\mathsf{T}}$
- $M^2 = M$
- $\mathrm{Trace}(M) = 2$
- M is not invertible: $\det(M) = 0$.
- The only fixed points of this transformation are the points on the plane.

Theorem 24.3: The matrix of the skew projection onto the plane $\alpha: \vec{n} \cdot \vec{x} = 0$ in the direction of $\vec{v}$ is

$$M = \boxed{I - \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v} \cdot \vec{n}}}$$



Furthermore:

- $M \neq M^{\mathsf{T}}$
- $M^2 = M$
- $\mathrm{Trace}(M) = 2$
- M is not invertible: $\det(M) = 0$.
- The only fixed points of this transformation are the points on the plane.

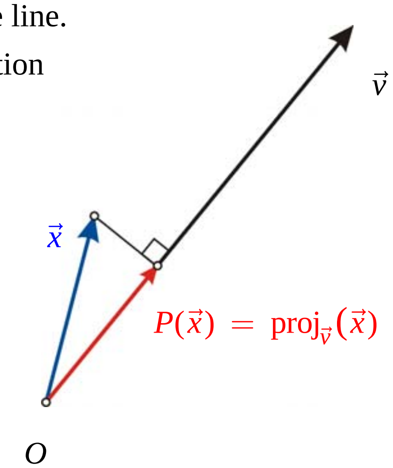
Orthogonal Projection onto a line through the origin

When we say projection onto a line we mean "orthogonal" projection. We basically dealt with this kind of projection before when we introduced projection onto a vector. Note that we already covered this transformation matrix in the section on the 2D projection on a line, since the method works in any dimension: $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4\cdots$ etc. In contrast to the 2D case where we have two distinct ways to represent a line, in higher dimensions a line has to be represented by a vector equation.

Let $l: \ \vec{x} = t\vec{v}$, where $\vec{v}$ is a direction vector of the line.
In this case projection onto the line is the same as projection onto $\vec{v}$:

$$P(\vec{x}) = \text{proj}_{\vec{v}}(\vec{x}) = \frac{\vec{v}\cdot\vec{x}}{\vec{v}\cdot\vec{v}}\cdot\vec{v} = \frac{1}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}}\vec{x}$$

Hence the projection matrix is $\dfrac{1}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}}$ or $\dfrac{\vec{v}\,\vec{v}^{\mathsf{T}}}{\|\vec{v}\|^2}$



Example 1: Let $l: \ \vec{x} = t\begin{bmatrix}1\\-2\\5\end{bmatrix}$ then $\vec{v} = \begin{bmatrix}1\\-2\\5\end{bmatrix}$ and the projection onto this line

is $P(\vec{x}) = \dfrac{1}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}}\vec{x} = \dfrac{1}{30}\begin{bmatrix}1\\-2\\5\end{bmatrix}\begin{bmatrix}1 & -2 & 5\end{bmatrix}\vec{x} = \dfrac{1}{30}\begin{bmatrix}1 & -2 & 5\\-2 & 4 & -10\\5 & -10 & 25\end{bmatrix}\vec{x}$

So the image of $\begin{bmatrix}4\\2\\12\end{bmatrix}$ is $\dfrac{1}{30}\begin{bmatrix}1 & -2 & 5\\-2 & 4 & -10\\5 & -10 & 25\end{bmatrix}\begin{bmatrix}4\\2\\12\end{bmatrix} = \begin{bmatrix}2\\-4\\10\end{bmatrix}$. Note that indeed $\begin{bmatrix}2\\-4\\10\end{bmatrix}$ is on the line and that $\begin{bmatrix}4\\2\\12\end{bmatrix} - \begin{bmatrix}2\\-4\\10\end{bmatrix} = \begin{bmatrix}2\\6\\2\end{bmatrix} \perp \begin{bmatrix}1\\-2\\5\end{bmatrix}$ so that this is indeed the *orthogonal* projection.

Theorem 24.1: Let the line l be given in vector form $\vec{x} = t\vec{v}$, then the projection onto this line is given by $P(\vec{x}) = \frac{1}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}}\vec{x}$ or $P(\vec{x}) = \frac{\vec{v}\,\vec{v}^{\mathsf{T}}}{\vec{v}\cdot\vec{v}}\,\vec{x}$

P is a linear transformation with matrix $\boxed{M = \frac{1}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}}}$

i.e. if $\vec{v} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ then $M = \frac{1}{A^2+B^2+C^2}\begin{bmatrix} A^2 & AB & AC \\ AB & B^2 & BC \\ AC & BC & C^2 \end{bmatrix}$

Furthermore:

- $M = M^{\mathsf{T}}$
- $M^2 = M$
- $\text{Trace}(M) = 1$
- M is not invertible
- The only fixed points of this transformation are the points on the line.

Proof: Only the 'furthermore' points remain to be proven. For this see the end of this section.

Example 2: Let $l: \vec{x} = t\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ be the same line in $\mathbb{R}^3$ as before, with projection

$$P(\vec{x}) = \frac{1}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}}\vec{x} = \frac{1}{30}\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & -2 & 5 \end{bmatrix}\vec{x} = \frac{1}{30}\begin{bmatrix} 1 & -2 & 5 \\ -2 & 4 & -10 \\ 5 & -10 & 25 \end{bmatrix}\vec{x}$$

For example: the image the line $l: \vec{x} = \begin{bmatrix} 4 \\ 2 \\ 12 \end{bmatrix} + t\begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$ is:

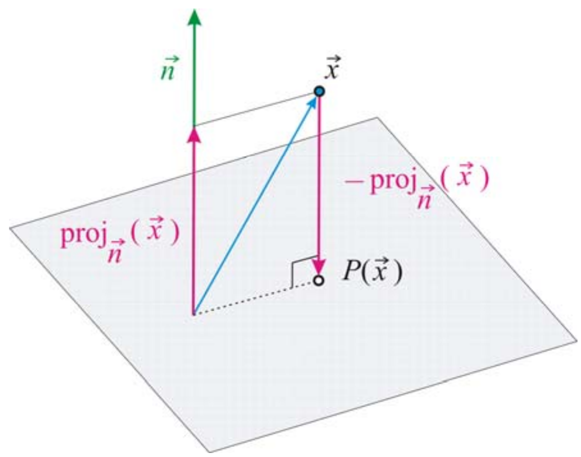
$$P\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P\left(\begin{bmatrix} 4 \\ 2 \\ 12 \end{bmatrix} + t\begin{bmatrix} 4 \\ -3 \\ -2 \end{bmatrix}\right) = \frac{1}{30}\begin{bmatrix} 1 & -2 & 5 \\ -2 & 4 & -10 \\ 5 & -10 & 25 \end{bmatrix}\begin{bmatrix} 4+4t \\ 2-3t \\ 12-2t \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 12 \end{bmatrix}$$

i.e. the entire line gets mapped to a point. [Note that $\begin{bmatrix} 4 \\ -3 \\ -2 \end{bmatrix} \perp \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$]

Projection onto a plane through the origin

Let $\alpha: \vec{n}\cdot\vec{x} = 0$ be a plane through the origin, i.e. if normal $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ then α has the normal equation $a\,x + b\,y + c\,z = 0$, then

$$\begin{aligned} P(\vec{x}) &= \vec{x} - \text{proj}_{\vec{n}}(\vec{x}) \\ &= \vec{x} - \frac{\vec{n}\cdot\vec{x}}{\vec{n}\cdot\vec{n}}\,\vec{n} \\ &= \vec{x} - \frac{1}{\|\vec{n}\|^2}(\vec{n}\cdot\vec{x})\,\vec{n} \\ &= \vec{x} - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\vec{x} \\ &= I_3\,\vec{x} - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\vec{x} \\ &= \left(I_3 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right)\vec{x} \end{aligned}$$



Hence the projection matrix is: $I_3 - \dfrac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}$ or $I_3 - \dfrac{\vec{n}\,\vec{n}^{\mathsf{T}}}{\|\vec{n}\|^2}$

The explicit expression for this matrix in terms of a, b and c is

$$I_3 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} = \frac{1}{a^2+b^2+c^2}\begin{bmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{bmatrix}$$

We have proven the main part of the following theorem:

Theorem 24.2: Let the plane α be given in normal form $\vec{n}\cdot\vec{x} = 0$, then the projection onto this plane is given by $P(\vec{x}) = \left(I_3 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right)\vec{x}$ i.e. its transformation matrix is

$$\boxed{M = I_3 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}}$$

i.e. if $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ then $M = \frac{1}{a^2+b^2+c^2}\begin{bmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{bmatrix}$

Furthermore:

- $M = M^{\mathsf{T}}$
- $M^2 = M$
- $\mathrm{Trace}(M) = 2$
- M is not invertible
- The only fixed points of this transformation are the points on the plane.

Proof: Only the 'furthermore' points remain to be proven. For this see the end of this section.

Example 3: Let $\alpha: \ 3x+2y-z = 0$ be a plane in $\mathbb{R}^3$, then the matrix of the projection P onto the plane α is

$$P(\vec{x}) = \left(I_3 - \frac{1}{14}\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}\begin{bmatrix} 3 & 2 & -1 \end{bmatrix}\right)\vec{x} = \frac{1}{14}\begin{bmatrix} 5 & -6 & 3 \\ -6 & 10 & 2 \\ 3 & 2 & 13 \end{bmatrix}\vec{x}$$

- $\mathrm{Trace}\left(\frac{1}{14}\begin{bmatrix} 5 & -6 & 3 \\ -6 & 10 & 2 \\ 3 & 2 & 13 \end{bmatrix}\right) = \frac{5+10+13}{14} = 2$

- $\text{Det}\left(\frac{1}{14}\begin{bmatrix} 5 & -6 & 3 \\ -6 & 10 & 2 \\ 3 & 2 & 13 \end{bmatrix}\right) = 0$

- The image of $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ is $P\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 5 & -6 & 3 \\ -6 & 10 & 2 \\ 3 & 2 & 13 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

- $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 6 \end{bmatrix}$ is mapped to $P\begin{bmatrix} 0 \\ -4 \\ 6 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 5 & -6 & 3 \\ -6 & 10 & 2 \\ 3 & 2 & 13 \end{bmatrix}\begin{bmatrix} 0 \\ -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$

- The image of the line $m: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -4 \\ 6 \end{bmatrix}$ is

$$P\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P\left(\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -4 \\ 6 \end{bmatrix}\right) = \frac{1}{14}\begin{bmatrix} 5 & -6 & 3 \\ -6 & 10 & 2 \\ 3 & 2 & 13 \end{bmatrix}\begin{bmatrix} 4 \\ 1-4t \\ 6t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$$

Call the images $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = P\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$ is another line. This line is of course *in* the plane! [To check this: $3(1+3t)+2(-1-2t)-(1+5t) = 0$ ✓].

- What is the image of the plane $\beta: \ 4x - 9y - 6z = 7$? First we rewrite the plane as

$$\beta: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -4 \\ 6 \end{bmatrix} + s\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$$P\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P\left(\begin{bmatrix} 4+3s \\ 1-4t+2s \\ 6t-s \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence the entire plane β is mapped to the line $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$ (in the plane),

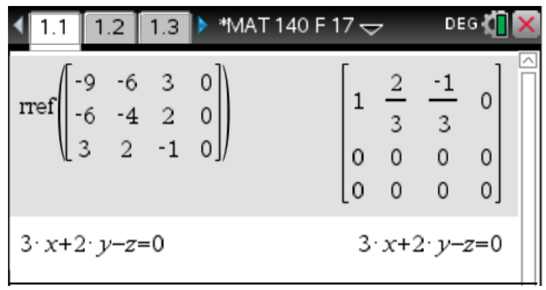
which makes sense since β is perpendicular to the plane α onto which we project.

- Let's compute the fixed points of this transformation: $T(\vec{x}) = \vec{x}$.

$$\frac{1}{14}\begin{bmatrix} 5 & -6 & 3 \\ -6 & 10 & 2 \\ 3 & 2 & 13 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{cases} 5x - 6y + 3z = 14x \\ -6x + 10y + 2z = 14y \\ 3x + 2y + 13z = 14z \end{cases} \Rightarrow \begin{cases} -9x - 6y + 3z = 0 \\ -6x - 4y + 2z = 0 \\ 3x + 2y - z = 0 \end{cases}$$

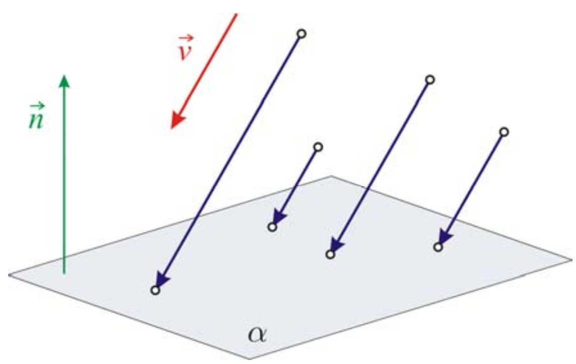
Note that we don't even have to rref here: these equations are all multiples of the same (plane) equation $3x + 2y - z = 0$.

But of course there is no harm in rref-ing:



Skew Projection onto a plane

Suppose we want to project onto a plane in a given direction, not necessarily orthogonal to the plane:



Let $\alpha: \vec{n} \cdot \vec{x} = 0$. Hence if $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ then α has the normal equation $ax + by + cz = 0$.

We pick a vector $\vec{v}$ not parallel to the plane (i.e. $\vec{n}\cdot\vec{v} \neq 0$. Hence if $\vec{v} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ then $aA + bB + cC \neq 0$)

If we want to project $\vec{x}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ onto α in the direction of $\vec{v}$ we could construct the line m through $\vec{x}_0$ in the direction of $\vec{v}$, and find the projection as the intersection of this line with the plane:

$$\left.\begin{aligned} m:\ \vec{x} &= \vec{x}_0 + t\,\vec{v} \\ \alpha:\ \vec{n}\cdot\vec{x} &= 0 \end{aligned}\right\} \Rightarrow \vec{n}\cdot(\vec{x}_0 + t\,\vec{v}) = 0$$

$$\Rightarrow \vec{n}\cdot\vec{x}_0 + t\,\vec{n}\cdot\vec{v} = 0$$

$$\Rightarrow t = -\frac{\vec{n}\cdot\vec{x}_0}{\vec{n}\cdot\vec{v}}$$

So that

$$\begin{aligned} P(\vec{x}_0) &= \vec{x}_0 - \frac{\vec{n}\cdot\vec{x}_0}{\vec{n}\cdot\vec{v}}\vec{v} \\ &= \vec{x}_0 - \frac{1}{\vec{n}\cdot\vec{v}}(\vec{n}\cdot\vec{x}_0)\vec{v} \\ &= \vec{x}_0 - \frac{1}{\vec{n}\cdot\vec{v}}\vec{v}\,\vec{n}^{\mathsf{T}}\vec{x}_0 = \left(I - \frac{1}{\vec{n}\cdot\vec{v}}\vec{v}\,\vec{n}^{\mathsf{T}}\right)\vec{x}_0 \end{aligned}$$

Hence the projection matrix is

$$M = I - \frac{1}{\vec{n}\cdot\vec{v}}\vec{v}\,\vec{n}^{\mathsf{T}} \quad \text{or} \quad I - \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}$$

In terms of $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ we find

$$M = \frac{1}{aA + bB + cC}\begin{bmatrix} bB + cC & -bA & -cA \\ -aB & aA + cC & -cB \\ -aC & -bC & aA + bB \end{bmatrix}$$

Note that if $\vec{v} = \vec{n}$ we get the usual *orthogonal* projection matrix $M = I - \dfrac{\vec{n}\,\vec{n}^{\mathsf{T}}}{\vec{n}\cdot\vec{n}}$

We have the following theorem

Theorem 24.3: The skew projection $P:\ \mathbb{R}^3 \to \mathbb{R}^3$ onto the plane $\alpha:\ \vec{n}\cdot\vec{x} = 0$ in the direction of $\vec{v}$ has as matrix

$$M = I - \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}$$

If $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ then $M = \dfrac{1}{aA+bB+cC}\begin{bmatrix} bB+cC & -bA & -cA \\ -aB & aA+cC & -cB \\ -aC & -bC & aA+bB \end{bmatrix}$

Furthermore:

- $M \neq M^{\mathsf{T}}$ (unless $\vec{v} \parallel \vec{n}$)
- $M^2 = M$
- $\text{Trace}(M) = 2$
- M is not invertible
- The only fixed points of this transformation are the points on the plane.

Proof: Only the 'furthermore' points remain to be proven. For this see the end of this section.

Example 4: Let $\alpha:\ 3x+2y-z = 0$ and $\vec{v} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$, then

$$M = I - \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}} = I - \frac{1}{6}\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}\begin{bmatrix} 3 & 2 & -1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3 & -2 & 1 \\ -12 & -2 & 4 \\ -15 & -10 & 11 \end{bmatrix}$$

$$P(\vec{x}) = \frac{1}{6}\begin{bmatrix} 3 & -2 & 1 \\ -12 & -2 & 4 \\ -15 & -10 & 11 \end{bmatrix}\vec{x}$$

- $\text{Trace}\left(\dfrac{1}{6}\begin{bmatrix} 3 & -2 & 1 \\ -12 & -2 & 4 \\ -15 & -10 & 11 \end{bmatrix}\right) = \dfrac{3-2+11}{6} = 2$

- $\operatorname{Det}\left(\frac{1}{6}\begin{bmatrix} 3 & -2 & 1 \\ -12 & -2 & 4 \\ -15 & -10 & 11 \end{bmatrix}\right) = 0$

- The image of $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ is $P\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3 & -2 & 1 \\ -12 & -2 & 4 \\ -15 & -10 & 11 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ -5 \end{bmatrix}$

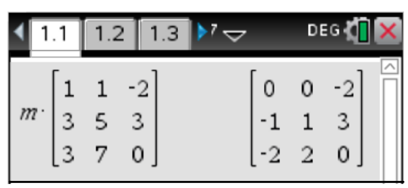
- The image of $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is $P\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3 & -2 & 1 \\ -12 & -2 & 4 \\ -15 & -10 & 11 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

- The line l : , i.e. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$ is parallel to the skew direction of this projection,

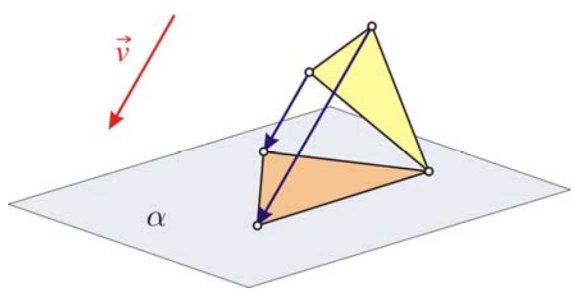
  hence gets mapped to **one** point: $P\begin{bmatrix} 2+t \\ 4t \\ 5t \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ -5 \end{bmatrix}$

- What is the image of the triangle with vertices $(1, 3, 3)$, $(1, 5, 7)$ and $(-2, 3, 0)$?
  Does the triangle and its image have the same area? [Note the first two points are on the same side of the plane as the normal (why?) And the last point is on the plane.]

$$P\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix},\quad P\begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},\quad P\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$$



Since lines are mapped to lines, the lines through the vertices are mapped to lines through the images of the vertices.



Let $\vec{u} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$

The area of the original triangle was: $\left\| \frac{1}{2}(\vec{u}-\vec{v})\times(\vec{w}-\vec{v}) \right\| = \sqrt{43}$

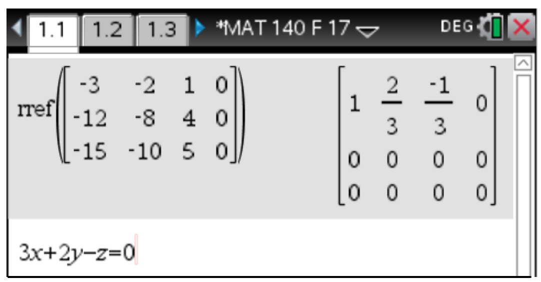
The area of the image triangle is $\left\| \frac{1}{2}(A\vec{u}-A\vec{v})\times(A\vec{w}-A\vec{v}) \right\| = \frac{\sqrt{14}}{3}$

- Let's compute the fixed points of this transformation: $T(\vec{x}) = \vec{x}$.

$$\frac{1}{6}\begin{bmatrix} 3 & -2 & 1 \\ -12 & -2 & 4 \\ -15 & -10 & 11 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad\Rightarrow\quad \begin{cases} 3x-2y+\ z = 6x \\ -12x-2y+4z = 6y \\ -15x-10y+11z = 6z \end{cases} \quad\Rightarrow\quad \begin{cases} -3x-2y+\ z = 0 \\ -12x-8y+4z = 0 \\ -15x-10y+5z = 0 \end{cases}$$

Note that we don't even have to rref here: these equations are all multiples of the same (plane) equation $3x+2y-z = 0$.

But of course there is no harm in rref-ing:



Proofs of the "Furthermore" points

Proof of the 'Furthermore' points of Theorem 24.1:

- $M^{\mathsf{T}} = M$ follows easily from $M = \frac{1}{A^2+B^2+C^2}\begin{bmatrix} A^2 & AB & AC \\ AB & B^2 & BC \\ AC & BC & C^2 \end{bmatrix}$ or we can follow the more general proof which works in all dimensions:

$$M^{\mathsf{T}} = \left(\frac{1}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}}\right)^{\mathsf{T}} = \frac{1}{\|\vec{v}\|^2}\cdot\left(\vec{v}\,\vec{v}^{\mathsf{T}}\right)^{\mathsf{T}} = \frac{1}{\|\vec{v}\|^2}\cdot(\vec{v}^{\mathsf{T}})^{\mathsf{T}}\,\vec{v}^{\mathsf{T}} = \frac{1}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}} = M$$

- $M^2 = M$ follows by direct computation from $M = \dfrac{1}{A^2+B^2+C^2}\begin{bmatrix} A^2 & AB & AC \\ AB & B^2 & BC \\ AC & BC & C^2 \end{bmatrix}$ or

  we can follow the more general proof which works in all dimensions:

$$\begin{aligned} M^2 &= \left(\frac{1}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}}\right)\cdot\left(\frac{1}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}}\right) \\ &= \frac{1}{\|\vec{v}\|^4}\cdot\vec{v}\,\vec{v}^{\mathsf{T}}\,\vec{v}\,\vec{v}^{\mathsf{T}} = \frac{1}{\|\vec{v}\|^4}\cdot\left(\vec{v}\,\vec{v}^{\mathsf{T}}\,\vec{v}\right)\vec{v}^{\mathsf{T}} = \frac{1}{\|\vec{v}\|^4}\cdot\left(\left[\,\vec{v}\cdot\vec{v}\,\right]\cdot\vec{v}\right)\vec{v}^{\mathsf{T}} \\ &= \frac{\vec{v}\cdot\vec{v}}{\|\vec{v}\|^4}\cdot\vec{v}\,\vec{v}^{\mathsf{T}} = \frac{1}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}} = M \end{aligned}$$

- $\text{Trace}(M) = 1$ follows immediately from $M = \dfrac{1}{A^2+B^2+C^2}\begin{bmatrix} A^2 & AB & AC \\ AB & B^2 & BC \\ AC & BC & C^2 \end{bmatrix}$ or

  we can follow the more general proof which works in all dimensions:

$$\text{Trace}(M) = \frac{1}{\|\vec{v}\|^2}\cdot\text{Trace}(\vec{v}\cdot\vec{v}^{\mathsf{T}}) = \frac{\vec{v}\cdot\vec{v}}{\|\vec{v}\|^2} = 1 \qquad (\text{ in 3D: } \ \frac{A^2+B^2+C^2}{A^2+B^2+C^2} = 1\ )$$

- $\det(M) = 0$. There are infinitely many vectors $\vec{x}\perp\vec{v}$ and they are **all** mapped to $\vec{0}$: $\dfrac{1}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}}\vec{x} = \dfrac{1}{\|\vec{v}\|^2}(\vec{v}\cdot\vec{x})\cdot\vec{v} = \dfrac{1}{\|\vec{v}\|^2}(0)\cdot\vec{v} = \vec{0}$. It therefore follows that this map cannot be invertible. Hence its determinant is zero.

- Fixed points:

  Note that any vector can be written as $\vec{x} = \text{proj}_{\vec{v}}(\vec{x}) + \vec{x}^{\perp} = P(\vec{x}) + \vec{x}^{\perp}$ where $\vec{x}^{\perp}$ is the orthogonal complement of $\vec{x}$ with respect to $\vec{v}$, hence we know that $P\left(\vec{x}^{\perp}\right) = \vec{0}$. Therefore to find fixed points we need to solve

$$\begin{aligned} P(\vec{x}) = \vec{x} \quad &\Rightarrow \quad P\left(P(\vec{x}) + \vec{x}^{\perp}\right) = P(\vec{x}) + \vec{x}^{\perp} \\ &\Rightarrow \quad P\left(P(\vec{x})\right) + P(\vec{x}^{\perp}) = P(\vec{x}) + \vec{x}^{\perp} \\ &\Rightarrow \quad P(\vec{x}) + \vec{0} = P(\vec{x}) + \vec{x}^{\perp} \\ &\Rightarrow \quad \vec{0} = \vec{x}^{\perp} \end{aligned}$$

  Hence the only points that are fixed (i.e. for which $P(\vec{x}) = \vec{x}$) are those points with $\vec{x}^{\perp} = \vec{0}$ i.e. all those points that have **zero** perpendicular component to $\text{proj}_{\vec{v}}(\vec{x})$. Hence all points on the line.

Proof of the 'Furthermore' points of Theorem 24.2:

- $M = M^{\mathsf{T}}$ and $\text{Trace}(M) = 2$ are obvious from the matrix

$$M = \frac{1}{a^2+b^2+c^2}\begin{bmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{bmatrix}$$

- $M^2 = M$:

$$\begin{aligned} M^2 &= \left(I_3 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right)^2 \\ &= \left(I_3 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right)\left(I_3 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right) \\ &= I_3^2 - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} + \frac{1}{\|\vec{n}\|^4}\vec{n}\,\vec{n}^{\mathsf{T}}\,\vec{n}\,\vec{n}^{\mathsf{T}} \\ &= I_3 - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} + \frac{1}{\|\vec{n}\|^4}(\vec{n}\cdot\vec{n})\vec{n}\,\vec{n}^{\mathsf{T}} \\ &= I_3 - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} + \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} \\ &= I_3 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} = M \end{aligned}$$

- That M is not invertible is clear from the fact that $M\,\vec{n} = \vec{0} = M\,\vec{0}$

  $$\left[\; M\,\vec{n} = \left(I_3 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right)\vec{n} = \vec{n} - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\vec{n} = \vec{n} - \frac{1}{\|\vec{n}\|^2}(\vec{n}\cdot\vec{n})\vec{n} = \vec{n} - \vec{n} = \vec{0} \;\right]$$

  Both $\vec{n}$ and $\vec{0}$ are mapped to the origin, hence the map is not one-to-one, and hence not invertible.

- Fixed points: $M\,\vec{x} = \vec{x} \;\Rightarrow\; \left(I_3 - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right)\vec{x} = \vec{x}$

  $$\Rightarrow\; \vec{x} - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\vec{x} = \vec{x}$$

  $$\Rightarrow\; \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\vec{x} = \vec{0}$$

  $$\Rightarrow\; \vec{n}\,\vec{n}^{\mathsf{T}}\vec{x} = \vec{0}$$

  $$\Rightarrow\; (\vec{n}\cdot\vec{x})\,\vec{n} = \vec{0}$$

  $$\Rightarrow\; \vec{n}\cdot\vec{x} = 0 \qquad \Rightarrow \text{ all points on the line } l$$

Proof of the 'Furthermore' points of Theorem 24.3:

- $M \neq M^{\mathsf{T}}$ and $\text{Trace}(M) = 2$ is clear from the matrix

  $$M = \frac{1}{aA + bB + cC}\begin{bmatrix} bB + cC & -bA & -cA \\ -aB & aA + cC & -cB \\ -aC & -bC & aA + bB \end{bmatrix}$$

  The matrix is only symmetric if $\vec{v}$ is a multiple of $\vec{n}$, i.e. when the "skew" projection is actually an "orthogonal" projection.

- $M^2 = M$: Of course direct computation would demonstrate this. Or we could argue as follows:

  $$M^2 = \left(I - \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}\right)\left(I - \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}\right) = I - 2\frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}} + \frac{1}{(\vec{v}\cdot\vec{n})^2}\vec{v}\,\vec{n}^{\mathsf{T}}\vec{v}\,\vec{n}^{\mathsf{T}}$$

  $$= I - 2\frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}} + \frac{1}{(\vec{v}\cdot\vec{n})^2}(\vec{n}\cdot\vec{v})\,\vec{v}\,\vec{n}^{\mathsf{T}} = I - 2\frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}} + \frac{1}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}} = I - \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}} = M$$

- M is not invertible since clearly $M\vec{0} = \vec{0}$ *and* $M\vec{v} = \vec{0}$ [which follows from $\left(I - \dfrac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}\right)\vec{v} = \vec{v} - \dfrac{1}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}\vec{v} = \vec{v} - \dfrac{1}{\vec{v}\cdot\vec{n}}(\vec{n}\cdot\vec{v})\vec{v} = \vec{0}$]. Since both $\vec{v}$ and $\vec{0}$ are mapped to the origin, the map is not one-to-one, and hence not invertible.

- Finally, fixed points: $M\vec{x} = \vec{x} \;\Rightarrow\; \left(I - \dfrac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}\right)\vec{x} = \vec{x}$

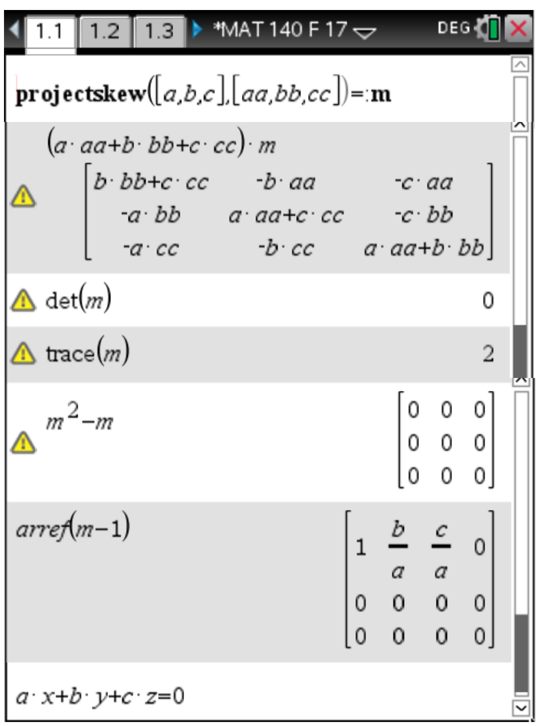
$$\Rightarrow\; \vec{x} - \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}\vec{x} = \vec{x}$$

$$\Rightarrow\; \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}\vec{x} = \vec{0}$$

$$\Rightarrow\; (\vec{n}\cdot\vec{x})\vec{v} = \vec{0}$$

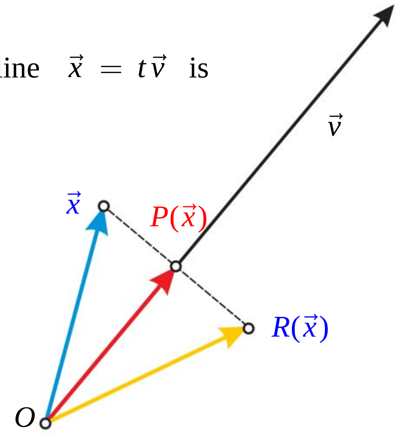
$$\Rightarrow\; \vec{n}\cdot\vec{x} = 0$$ i.e. all points on the plane.

Checks with the TI-Nspire ☺



Theorem 25.1: The matrix of the reflection in the line $\vec{x} = t\vec{v}$ is

$$M = \boxed{\frac{2}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}} - I}$$

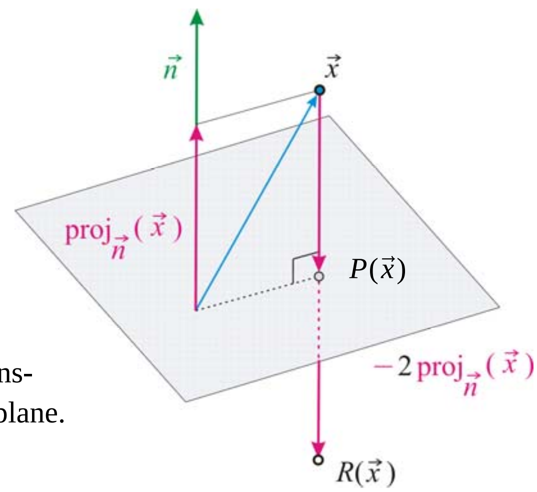


Furthermore:

- $M = M^{\mathsf{T}}$
- $M^2 = I\,,\ M^{-1} = M$
- $\mathrm{Trace}(M) = -1$
- $\det(M) = 1\,.$
- The only fixed points of this transformation are the points on the line.

Theorem 25.2: The matrix of the reflection in the plane $\alpha:\ \vec{n}\cdot\vec{x} = 0$ is

$$M = \boxed{I - \frac{2\,\vec{n}\,\vec{n}^{\mathsf{T}}}{\|\vec{n}\|^2}}$$

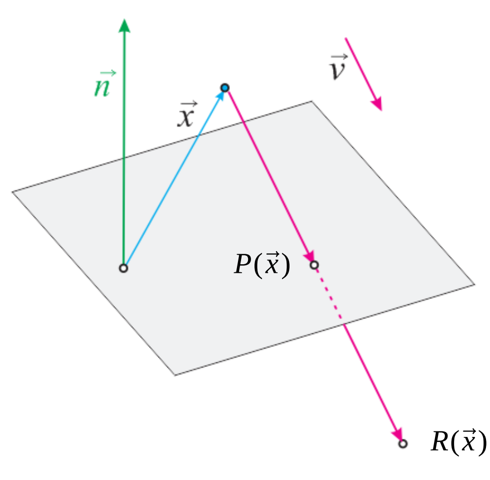


Furthermore:

- $M = M^{\mathsf{T}}$
- $M^2 = I\,,\ M^{-1} = M$
- $\mathrm{Trace}(M) = 1$
- $\det(M) = -1\,.$
- The only fixed points of this transformation are the points on the plane.

Theorem 25.3: The matrix of the skew-reflection in the plane $\alpha:\ \vec{n}\cdot\vec{x} = 0$ in the direction $\vec{v}$ ($\vec{n}\cdot\vec{v} \neq 0$) is

$$M = \boxed{I - \frac{2\,\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}}$$



Furthermore:

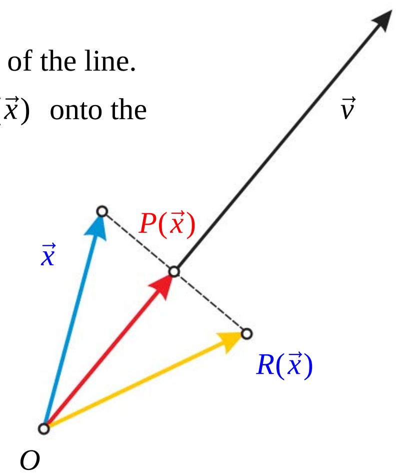
- $M \neq M^{\mathsf{T}}$ (unless $\vec{v} \parallel \vec{n}$)
- $M^2 = I\,,\ M^{-1} = M$
- $\mathrm{Trace}(M) = 1$
- $\det(M) = -1\,.$
- The only fixed points are the points on the plane.

Chapter 25 Reflection matrices in $\mathbb{R}^3$

With only minor modifications we can produce the reflection matrices using the methods we used for projections in the previous section.

Reflection in a line

Let $l: \ \vec{x} = t\vec{v}$ where $\vec{v}$ is a direction vector of the line. Notice the relationship between the projection $P(\vec{x})$ onto the line and the reflection $R(\vec{x})$ in the line



$$\frac{\vec{x}+R(\vec{x})}{2} = P(\vec{x})$$

Hence $\quad R(\vec{x}) = 2P(\vec{x}) - \vec{x}$

So that $\quad R(\vec{x}) = \frac{2}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}}\vec{x} - \vec{x}$

Hence the reflection matrix is $\quad \frac{2}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}} - I$

or

$$\text{if} \quad \vec{v} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} \quad \text{then} \quad M = \frac{1}{A^2+B^2+C^2}\begin{bmatrix} A^2-B^2-C^2 & 2AB & 2AC \\ 2AB & -A^2+B^2-C^2 & BC \\ 2AC & 2BC & -A^2-B^2+C^2 \end{bmatrix}$$

Theorem 25.1: Let the line l be given in vector form $\vec{x} = t\vec{v}$, then the reflection in this line is given by $R(\vec{x}) = \left(\frac{2}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}} - I\right)\vec{x}$

R is a linear transformation with matrix $\quad \boxed{M = \frac{2}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}} - I}$

$$\text{if} \quad \vec{v} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} \quad \text{then} \quad M = \frac{1}{A^2+B^2+C^2}\begin{bmatrix} A^2-B^2-C^2 & 2AB & 2AC \\ 2AB & -A^2+B^2-C^2 & 2BC \\ 2AC & 2BC & -A^2-B^2+C^2 \end{bmatrix}$$

Furthermore:

- $M = M^{\mathsf{T}}$
- $M^2 = I, \ M^{-1} = M$
- $\text{Trace}(M) = -1$
- $\det(M) = 1$
- The only fixed points of this transformation are the points on the line.

Proof: Only the 'furthermore' points remain to be proven. For this see the end of this section.

Example 1: Let $l: \ \vec{x} = t\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ then $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$. The reflection matrix is computed as

$$M = \frac{2}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}} - I = \frac{2}{30}\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & -2 & 5 \end{bmatrix} - I = \frac{1}{15}\begin{bmatrix} -14 & -2 & 5 \\ -2 & -11 & -10 \\ 5 & -10 & 10 \end{bmatrix}$$

so that the reflection in this line is given by $R(\vec{x}) = \frac{1}{15}\begin{bmatrix} -14 & -2 & 5 \\ -2 & -11 & -10 \\ 5 & -10 & 10 \end{bmatrix}\vec{x}$.

- The image of $\begin{bmatrix} 4 \\ 2 \\ 12 \end{bmatrix}$ is $R\begin{bmatrix} 4 \\ 2 \\ 12 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} -14 & -2 & 5 \\ -2 & -11 & -10 \\ 5 & -10 & 10 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \\ 8 \end{bmatrix}$. Notice that the midpoint $\frac{1}{2}\left(\begin{bmatrix} 4 \\ 2 \\ 12 \end{bmatrix} + \begin{bmatrix} 0 \\ -10 \\ 8 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -4 \\ 10 \end{bmatrix}$ is the projection onto the line (see example 1 of section 24)

- Let's use this transformation—just for fun—to calculate the distance of the point $P = (-7, -1, 13)$ to the line:

$$R\begin{bmatrix} -7 \\ -1 \\ 13 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} -14 & -2 & 5 \\ -2 & -11 & -10 \\ 5 & -10 & 10 \end{bmatrix}\begin{bmatrix} -7 \\ -1 \\ 13 \end{bmatrix} = \begin{bmatrix} 11 \\ -7 \\ 7 \end{bmatrix}$$

  So that $P' = (11, -7, 7)$.

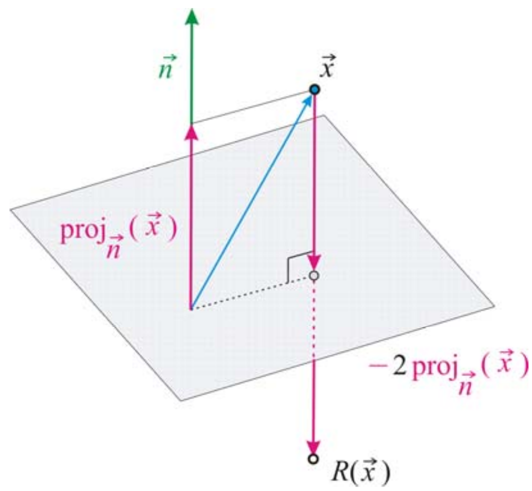
  Hence the distance of P to the line is **half** the distance between P and P':

$$\mathrm{Dist}(P, P') = \frac{1}{2}\sqrt{\left(11-(-7)\right)^2 + \left(-7-(-1)\right)^2 + (7-13)^2} = 3\sqrt{11}$$

Reflection in a plane

Let $\alpha:\ \vec{n}\cdot\vec{x} = 0$ be a plane through the origin. If normal $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ the equation of the plane is $\alpha:\ ax+by+cz = 0$. The reflection $R:\ \mathbb{R}^3 \to \mathbb{R}^3$ in this line can be computed as follows

$$\begin{aligned} R(\vec{x}) &= \vec{x} - 2\mathrm{proj}_{\vec{n}}(\vec{x}) \\ &= \vec{x} - 2\frac{\vec{n}\cdot\vec{x}}{\vec{n}\cdot\vec{n}}\vec{n} \\ &= \vec{x} - \frac{2}{\|\vec{n}\|^2}(\vec{n}\cdot\vec{x})\,\vec{n} \\ &= \vec{x} - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\vec{x} \\ &= \left(I_2 - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right)\vec{x} \end{aligned}$$



Hence the reflection matrix is $I_2 - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}$. An explicit form of this matrix would be

$$M = \frac{1}{a^2+b^2+c^2}\begin{bmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}$$

which can be easily derived as follows

$$\begin{aligned} I_2 - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} &= \\ &= \frac{1}{a^2+b^2+c^2}\begin{bmatrix} a^2+b^2+c^2 & 0 & 0 \\ 0 & a^2+b^2+c^2 & 0 \\ 0 & 0 & a^2+b^2+c^2 \end{bmatrix} - \frac{1}{a^2+b^2+c^2}\begin{bmatrix} 2a^2 & 2ab & 2ac \\ 2ab & 2b^2 & 2bc \\ 2ac & 2bc & 2c^2 \end{bmatrix} \\ &= \frac{1}{a^2+b^2}\left(\begin{bmatrix} a^2+b^2+c^2 & 0 & 0 \\ 0 & a^2+b^2+c^2 & 0 \\ 0 & 0 & a^2+b^2+c^2 \end{bmatrix} - \begin{bmatrix} 2a^2 & 2ab & 2ac \\ 2ab & 2b^2 & 2bc \\ 2ac & 2bc & 2c^2 \end{bmatrix}\right) \\ &= \frac{1}{a^2+b^2+c^2}\begin{bmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix} \end{aligned}$$

We have proven the main part of the following theorem:

Theorem 25.2: Let the plane $\alpha:\ \vec{n}\cdot\vec{x} = 0$, then the reflection in this plane is given by

$$R(\vec{x}) = \left(I - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right)\vec{x}$$

i.e. R is a linear transformation with matrix $M = \boxed{I - \dfrac{2\,\vec{n}\,\vec{n}^{\mathsf{T}}}{\|\vec{n}\|^2}}$

If $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ then $M = \dfrac{1}{a^2+b^2+c^2}\begin{bmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}$

Furthermore:
- $M = M^{\mathsf{T}}$
- $M^2 = I$, $M^{-1} = M$
- $\text{Trace}(M) = 1$
- $\det(M) = -1$.
- The only fixed points of this transformation are the points on the plane.

Proof: Only the ‘furthermore’ points remain to be proven. For those, see the end of this chapter.

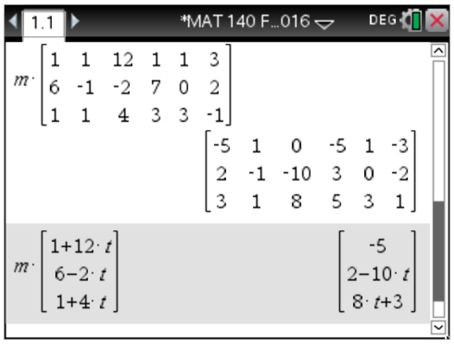
Example 2: Let $\alpha:\ 3x+2y-z = 0$ be a plane in $\mathbb{R}^3$, then the matrix of the reflection P in the plane α is

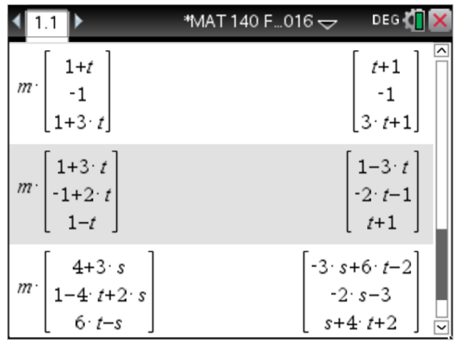
$$I - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} = I - \frac{2}{14}\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}\begin{bmatrix} 3 & 2 & -1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -2 & -6 & 3 \\ -6 & 3 & 2 \\ 3 & 2 & 6 \end{bmatrix}$$

- The image of $\begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix}$ is $R\begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -2 & -6 & 3 \\ -6 & 3 & 2 \\ 3 & 2 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ 3 \end{bmatrix}$
- The image of $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is $R\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -2 & -6 & 3 \\ -6 & 3 & 2 \\ 3 & 2 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$
- The image of the line $m:\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix} + t\begin{bmatrix} 12 \\ -2 \\ 4 \end{bmatrix}$ is

$$R\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \; = \frac{1}{7}\begin{bmatrix} -2 & -6 & 3 \\ -6 & 3 & 2 \\ 3 & 2 & 6 \end{bmatrix}\begin{bmatrix} 1+12t \\ 6-2t \\ 1+4t \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} 0 \\ -10 \\ 8 \end{bmatrix}$$

Call the images $R(\vec{x}) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$: we get $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} 0 \\ -10 \\ 8 \end{bmatrix}$ which is another line.

- 



  Here the TI-Nspire mapped some points (all at once), as well as two lines and a planes.

- The image of the line $n: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ is $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, the *same* line, which makes sense since the line is *in* the plane.

- The image of the line $k: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ is $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$, i.e. k is also mapped to *itself*, but not point wise. Note this line is perpendicular to the plane α.

- What is the image of the plane $\beta:\; 4x-9y-6z=7$? First we rewrite the plane as

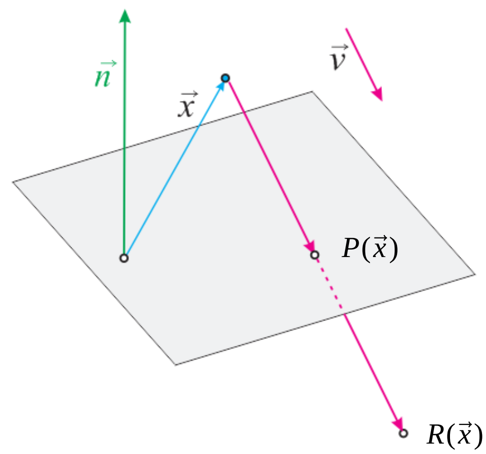
$$\beta: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -4 \\ 6 \end{bmatrix} + s\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$$R\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -2 & -6 & 3 \\ -6 & 3 & 2 \\ 3 & 2 & 6 \end{bmatrix}\begin{bmatrix} 4+3s \\ 1-4t+2s \\ 6t-s \end{bmatrix} \Rightarrow \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix} + t\begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix} + s\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

  Note this is again a plane, with normal equation $4x-9y-6z=7$. Hence the entire plane β is mapped to *itself* (though not point- wise, but as a plane), which makes sense since β is perpendicular to the plane α in which we reflect.

Skew-Reflection in a plane

Let $\alpha: \vec{n}\cdot\vec{x} = 0$ be a plane through the origin. If normal $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ the equation of the plane is $\alpha: ax+by+cz = 0$. Let also a direction $\vec{v} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ be given, with $\vec{v}\cdot\vec{n} \neq 0$, i.e. $aA+bB+cC \neq 0$. The skew-reflection $R: \mathbb{R}^3 \to \mathbb{R}^3$ in this line can be computed as follows



$$\frac{\vec{x}+R(\vec{x})}{2} = P(\vec{x})$$

Hence $\quad R(\vec{x}) = 2P(\vec{x}) - \vec{x}$

where $\quad P(\vec{x}) = \left(I - \dfrac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}\right)\vec{x}$, so that

$$R(\vec{x}) = 2\left(I - \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}\right)\vec{x} - \vec{x}$$

$$= \left(I - 2\frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}\right)\vec{x}$$

So that the matrix of this transformation is $M = I - \dfrac{2\,\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}$

In terms of $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ we find

$$M = I - 2\frac{1}{aA+bB+cC}\begin{bmatrix} A \\ B \\ C \end{bmatrix}\begin{bmatrix} a & b & c \end{bmatrix} = I - 2\frac{1}{aA+bB+cC}\begin{bmatrix} aA & bA & cA \\ aB & bB & cB \\ aC & bC & cC \end{bmatrix}$$

so that $$M = \frac{1}{aA+bB+cC}\begin{bmatrix} -aA+bB+cC & -2bA & -2cA \\ -2aB & aA-bB+cC & -2cB \\ -2aC & -2bC & aA+bB-cC \end{bmatrix}.$$

We have proven the main part of the following theorem:

Theorem 25.3: The matrix of the skew-reflection in the plane $\alpha:\ \vec{n}\cdot\vec{x} = 0$ in the direction $\vec{v}$ ($\vec{n}\cdot\vec{v}\neq 0$) is

$$M = \boxed{I - \frac{2\,\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}} \quad \text{and when } \vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

$$M = \frac{1}{aA+bB+cC}\begin{bmatrix} -aA+bB+cC & -2bA & -2cA \\ -2aB & aA-bB+cC & -2cB \\ -2aC & -2bC & aA+bB-cC \end{bmatrix}$$

Furthermore:

- $M \neq M^{\mathsf{T}}$ (unless $\vec{v}\ \|\ \vec{n}$)
- $M^2 = I\,,\quad M^{-1} = M$
- $\text{Trace}(M) = 1$
- $\det(M) = -1$.
- The only fixed points of this transformation are the points on the plane.

The proof of the 'furthermore' points of this theorem will be coming right up, with all the others.

Proofs of the "Furthermore" points

Proof of the 'Furthermore' points of Theorem 25.1:

- $M^{\mathsf{T}} = M$: This is immediate from the matrix

$$M = \frac{1}{A^2+B^2+C^2}\begin{bmatrix} A^2-B^2-C^2 & 2AB & 2AC \\ 2AB & -A^2+B^2-C^2 & 2BC \\ 2AC & 2BC & -A^2-B^2+C^2 \end{bmatrix}$$

  or as done in the section on the 2D reflection matrix, we can give the proof that works in general: $M^{\mathsf{T}} = \left(\frac{2}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}} - I\right)^{\mathsf{T}} = \frac{2}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}} - I = M$

- $M^2 = I$: $\quad M^2 = \left(\frac{2}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}} - I\right)\cdot\left(\frac{2}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}} - I\right)$

$$= \frac{4}{\|\vec{v}\|^4}\cdot\vec{v}\,\vec{v}^{\mathsf{T}}\,\vec{v}\,\vec{v}^{\mathsf{T}} - \frac{4}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}} + I^2$$

$$= \frac{4}{\|\vec{v}\|^4}\cdot\left(\left[\,\vec{v}\cdot\vec{v}\,\right]\cdot\vec{v}\right)\vec{v}^{\mathsf{T}} - \frac{4}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}} + I = I$$

- Trace$(M) = -1$ (Immediate from the matrix, or using eigenvalues)
- $M^2 = I$ implies $M^{-1} = M$.
- $\det(M) = 1$: Check by direct computation that $\det(M) = 1$ (or using eigenvalues.)
- Fixed points: $\left(\frac{2}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}} - I\right)\vec{x} = \vec{x} \;\Rightarrow\; \frac{1}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}}\vec{x} = \vec{x} \;\Rightarrow\; P(\vec{x}) = \vec{x}$

  i.e. all those points on the line (as we proved for the projection).

Proof of the 'Furthermore' points of Theorem 25.2:

- $M = M^{\mathsf{T}}$ follows immediately from the matrix

$$M = \frac{1}{a^2+b^2+c^2}\begin{bmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}$$

- Trace$(M) = 1$ also follows immediately from the above matrix

- $M^2 = I$ follows by direct computation using the matrix, or by banking on the 'geometry' of a reflection: if you do the same reflection twice you are back where you started. Or algebraically, in general, as follows

$$\begin{aligned} M^2 &= \left(I - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right)^2 \\ &= \left(I - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right)\left(I - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right) \\ &= I - \frac{4}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} + \frac{4}{\|\vec{n}\|^4}\vec{n}\,\vec{n}^{\mathsf{T}}\vec{n}\,\vec{n}^{\mathsf{T}} \\ &= I - \frac{4}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} + \frac{4}{\|\vec{n}\|^4}(\vec{n}\cdot\vec{n})\vec{n}\,\vec{n}^{\mathsf{T}} \\ &= I - \frac{4}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} + \frac{4}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} \\ &= I \end{aligned}$$

- $M^{-1} = M$: follows immediately from $M^2 = I$:
- $\det(M) = -1$: Check--by hand--that $\det(M) = -1$, (or using eigenvalues.)

- Fixed points: $M\vec{x} = \vec{x} \;\Rightarrow\; \left(I - \dfrac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right)\vec{x} = \vec{x}$

$$\Rightarrow\; \vec{x} - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\vec{x} = \vec{x}$$

$$\Rightarrow\; \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\vec{x} = \vec{0}$$

$$\Rightarrow\; \vec{n}\,\vec{n}^{\mathsf{T}}\vec{x} = \vec{0}$$

$$\Rightarrow\; (\vec{n}\cdot\vec{x})\,\vec{n} = \vec{0}$$

$$\Rightarrow\; \vec{n}\cdot\vec{x} = 0 \qquad \Rightarrow \quad \text{all points on the plane } \alpha$$

Proof of the 'Furthermore' points of Theorem 25.3:

- $M \neq M^{\mathsf{T}}$ follows immediately from the matrix (unless $\vec{v} \parallel \vec{n}$)

$$M = \frac{1}{aA+bB+cC}\begin{bmatrix} -aA+bB+cC & -2bA & -2cA \\ -2aB & aA-bB+cC & -2cB \\ -2aC & -2bC & aA+bB-cC \end{bmatrix}$$

- $\mathrm{Trace}(M) = 1$ also follows immediately from the above matrix.

- $M^2 = I$ could also be computed from the matrix, or in general as follows:

$$M^2 = \left(I - \frac{2\,\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}\right)\left(I - \frac{2\,\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}}\right) = I - \frac{4\,\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}} + \frac{4\,\vec{v}\,\vec{n}^{\mathsf{T}}\vec{v}\,\vec{n}^{\mathsf{T}}}{(\vec{v}\cdot\vec{n})^2}$$

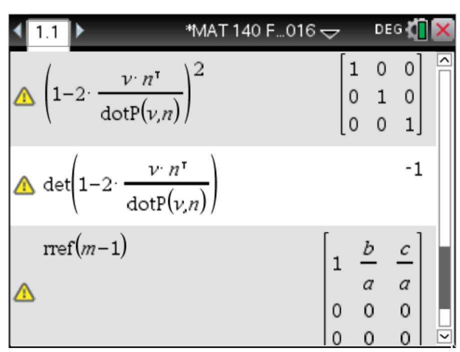
$$= I - \frac{4\,\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}} + \frac{4\left((\vec{n}\cdot\vec{v})\,\vec{v}\right)\vec{n}^{\mathsf{T}}}{(\vec{v}\cdot\vec{n})^2}$$

$$= I - \frac{4\,\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}} + \frac{4\,\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}} = I$$

Hence $M^{-1} = M$.

- Direct computation shows: $\det(M) = -1$

Of course the TI-Nspire has no problem with this.



- Even the fixed point we could compute with the TI-Nspire :

$$a\,x + b\,y + c\,z = 0$$

26. Scalings in $\mathbb{R}^3$

In this section we will discuss the matrices of 3D scalings.

Definition: A 3D scaling (centered at the origin) is defined by $S\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax \\ by \\ cz \end{bmatrix}$.

i.e. all x-coordinates are scaled by a constant a, all y-coordinates are scaled by a constant b, and all z-coordinates are scaled by a constant c.

The following theorem should be obvious, since $\begin{bmatrix} ax \\ by \\ cz \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$:

Theorem 26.1: The matrix of the scaling $S\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax \\ by \\ cz \end{bmatrix}$ is $M = M_{a,b,c} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

Furthermore:

- M is invertible *iff* $a \neq 0$ **and** $b \neq 0$ **and** $c \neq 0$, in which case:

  $$M^{-1} = M_{a,b,c}^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix} = M_{1/a,1/b,1/c}$$

- $M^2 = M_{a,b,c}^2 = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{bmatrix} = M_{a^2,b^2,c^2}$ [In general $M^m = M_{a,b,c}^m = M_{a^m,b^m,c^m}$]

- $M = M^{\mathsf{T}}$

- $\text{Trace}(M) = a + b + c$

- $\det(M) = abc$.

- When $a \neq 1$ and $b \neq 1$ and $c \neq 1$ then the origin O is the only fixed point.
  When $a = b = c = 1$ then $M = I$ and *all* points are fixed.
  The other combinations leave either one of the axis or one of the base planes fixed:

  $a = 1$ and $b \neq 1$ and $c \neq 1$ leaves the x-axis fixed.

  $a \neq 1$ and $b = 1$ and $c \neq 1$ leaves the y-axis fixed.

  $a \neq 1$ and $b \neq 1$ and $c = 1$ leaves the z-axis fixed.

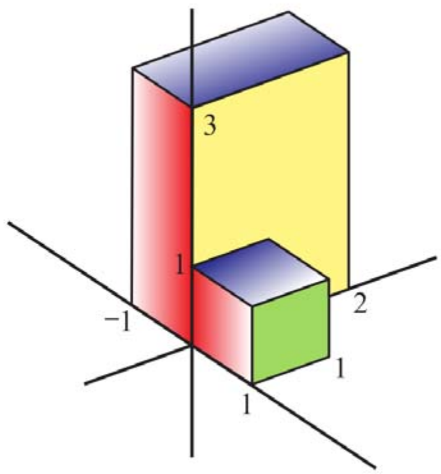
  $a \neq 1$ and $b = 1$ and $c = 1$ leaves the yz-plane [$x = 0$] fixed.

  $a = 1$ and $b \neq 1$ and $c = 1$ leaves the xz-plane [$y = 0$] fixed.

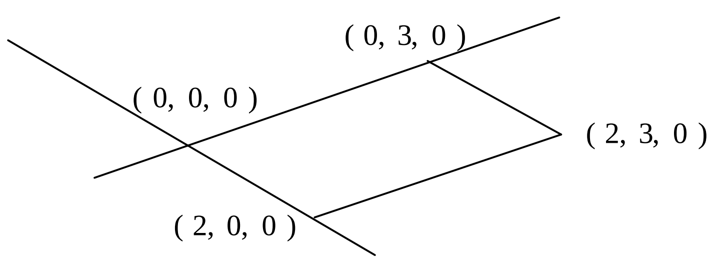
  $a = 1$ and $b = 1$ and $c \neq 1$ leaves the xy-plane [$z = 0$] fixed.

Proof: The proofs of these are straightforward, and are left to the reader.

Example 1: Let $S\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x \\ 2y \\ 3z \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then S maps the unit cube in the picture to the rectangular brick.



Example 2: Let $S\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then S stretches x coordinates by a factor of two and y coordinates by a factor of 3. The z coordinates are collapsed to zero. This map is not invertible (the determinant = 0). It has one fixed point: (0, 0, 0).
The cube (shown above) is mapped to a rectangle in the $x\,y$ – plane with vertices (0, 0, 0), (2, 0, 0), (2, 3, 0) and (0, 3, 0)



27. Shears

In this chapter we will prove the following theorem

Theorem 27.1: The matrix of the 3D shear with respect to the plane $\vec{n}\cdot\vec{x}=0$ in the direction of the vector $\vec{v}$ parallel to the plane is

$$M \;=\; \boxed{I+\frac{1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}}}$$

Furthermore

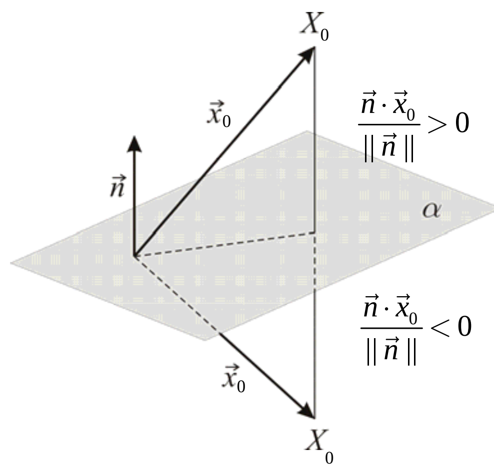
- $M^{-1} = M^{-1}_{\vec{n},\vec{v}} = I_2+\frac{-1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}} = M_{\vec{n},-\vec{v}} \quad\Rightarrow\quad M^{-1} = 2I - M$
- $\det(M) = 1$
- $M^2 = M^2_{\vec{n},\vec{v}} = I_2+\frac{2}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}} = M_{\vec{n},2\vec{v}}$ [In general $M^m = M^m_{\vec{n},\vec{v}} = M_{\vec{n},m\vec{v}}$]
- $M \neq M^{\mathsf{T}}$ (unless $\vec{v}=\vec{0}$ in which case $M=I$)
- $\operatorname{Trace}(M) = 3$
- The only fixed points of S is are the points on the plane l. (Unless $\vec{v} = \vec{0}$)

We will now develop a formula for general shears parallel to any plane through the origin.

Let α be the plane $\vec{n}\cdot\vec{x}=0$, i.e. when $\vec{n}=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ we have $a\,x+b\,y+c\,z=0$.

Recall that the distance from a point X_0 [with position vector $\vec{x}_0$] to the plane α : $\vec{n}\cdot\vec{x}=0$ is given by:

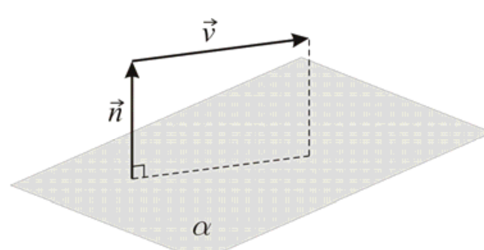
$$\text{dist}(X_0,\alpha) = \frac{|\vec{n}\cdot\vec{x}_0|}{\|\vec{n}\|}.$$



Also recall that $\vec{n}\cdot\vec{x}_0 = \|\vec{n}\|\cdot\|\vec{x}_0\|\cdot\cos\theta$ which is **positive** if the angle $0\le\theta<90°$ i.e. when $\vec{x}_0$ and $\vec{n}$ are on the same side of the plane, and **negative** if the angle $90°<\theta\le180°$ i.e. when $\vec{x}_0$ and $\vec{n}$ are on opposite sides of the plane.

Hence $\frac{\vec{n}\cdot\vec{x}_0}{\|\vec{n}\|}$ gives us exactly the **signed distance** of point $\vec{x}_0$ to the plane: when $\vec{x}_0$ and $\vec{n}$ are on the **same** side of the plane then $\frac{\vec{n}\cdot\vec{x}_0}{\|\vec{n}\|}$ gives us exactly the **positive distance** of point $\vec{x}$ to the plane, and when $\vec{x}_0$ and $\vec{n}$ are on **opposite** sides of the plane then $\frac{\vec{n}\cdot\vec{x}_0}{\|\vec{n}\|}$ gives us exactly the **negative distance** of point $\vec{x}$ to the plane.

Let $\vec{v}=\begin{bmatrix} A \\ B \\ C \end{bmatrix}$ be a vector parallel to α, hence perpendicular to $\vec{n}$: i.e. $\vec{n}\cdot\vec{v}=0$.



We'll define the shear $S:\ \mathbb{R}^3\to\mathbb{R}^3$ with respect to plane α and vector $\vec{v}$ as follows

$$S(\vec{x}) = \vec{x}+\frac{\vec{n}\cdot\vec{x}}{\|\vec{n}\|}\,\vec{v}$$

Hence

$$S(\vec{x}) \;=\; \vec{x} + \frac{1}{\|\vec{n}\|}\,\vec{v}\,\vec{n}^{\mathsf{T}}\vec{x} \;=\; \left(I + \frac{1}{\|\vec{n}\|}\,\vec{v}\,\vec{n}^{\mathsf{T}}\right)\vec{x}$$

If we let $\vec{v} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$, with $\vec{n} \perp \vec{v}$ then

$$M \;=\; \begin{bmatrix} 1+\dfrac{aA}{\sqrt{a^2+b^2+c^2}} & \dfrac{bA}{\sqrt{a^2+b^2+c^2}} & \dfrac{cA}{\sqrt{a^2+b^2+c^2}} \\ \dfrac{aB}{\sqrt{a^2+b^2+c^2}} & 1+\dfrac{bB}{\sqrt{a^2+b^2+c^2}} & \dfrac{cB}{\sqrt{a^2+b^2+c^2}} \\ \dfrac{aC}{\sqrt{a^2+b^2+c^2}} & \dfrac{bC}{\sqrt{a^2+b^2+c^2}} & 1+\dfrac{cC}{\sqrt{a^2+b^2+c^2}} \end{bmatrix}$$

$$= \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{bmatrix} \sqrt{a^2+b^2+c^2}+aA & bA & cA \\ aB & \sqrt{a^2+b^2+c^2}+bB & cB \\ aC & bC & \sqrt{a^2+b^2+c^2}+cC \end{bmatrix}$$

$$= \frac{1}{\Delta} \begin{bmatrix} \Delta+aA & bA & cA \\ aB & \Delta+bB & cB \\ aC & bC & \Delta+cC \end{bmatrix} \qquad \text{where } \Delta \;=\; \sqrt{a^2+b^2+c^2}$$

When $\|\vec{n}\| = 1$, i.e. $\Delta \;=\; \sqrt{a^2+b^2+c^2} \;=\; 1$ we would get $\begin{bmatrix} 1+aA & bA & cA \\ aB & 1+bB & cB \\ aC & bC & 1+cC \end{bmatrix}$

Example 1: $\vec{n} \;=\; \begin{bmatrix} 3 \\ -4 \\ -12 \end{bmatrix}$ and $\vec{v} \;=\; \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$ then

$$M \;=\; I + \frac{1}{\|\vec{n}\|}\,\vec{v}\,\vec{n}^{\mathsf{T}} \;=\; I \;+\; \frac{1}{13}\begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}\begin{bmatrix} 3 & -4 & -12 \end{bmatrix} \;=\; \frac{1}{13}\begin{bmatrix} 25 & -16 & -48 \\ -9 & 25 & 36 \\ 6 & -8 & -11 \end{bmatrix}$$

Theorem 27.1: A shear with respect to the plane $\alpha: \vec{n}\cdot\vec{x} = 0$, i.e. $ax+by+cz = 0$ where $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, in the direction of the vector $\vec{v}$, parallel to the plane ($\vec{n} \perp \vec{v}$), has the matrix

$$M = I + \frac{1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}}$$

Furthermore

- M is invertible: $M^{-1} = M^{-1}_{\vec{n},\vec{v}} = I_2 + \frac{-1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}} = M_{\vec{n},-\vec{v}}$

  so that $M^{-1} = 2I - M$
- $\det(M) = 1$.
- $M^2 = M^2_{\vec{n},\vec{v}} = I_2 + \frac{2}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}} = M_{\vec{n},2\vec{v}}$ [In general $M^m = M^m_{\vec{n},\vec{v}} = M_{\vec{n},m\vec{v}}$]
- $M \neq M^{\mathsf{T}}$ (unless $\vec{v} = \vec{0}$ in which case $M = I$)
- $\mathrm{Trace}(M) = 3$
- The only fixed points of S is are the points on the plane l. (Unless $\vec{v} = \vec{0}$)

Proof: Most other 'furthermore' points can easily be verified by the reader. We'll prove the first and the last:

$$M^{-1} = M^{-1}_{\vec{n},\vec{v}} = I_2 + \frac{-1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}} = M_{\vec{n},-\vec{v}}$$

follows from

$$\begin{aligned}\left(I + \frac{1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}}\right)\left(I + \frac{-1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}}\right) &= I - \frac{1}{\|\vec{n}\|^2}\vec{v}\,\vec{n}^{\mathsf{T}}\vec{v}\,\vec{n}^{\mathsf{T}} \\ &= I - \frac{1}{\|\vec{n}\|^2}(\vec{v}\,\vec{n}^{\mathsf{T}}\vec{v})\,\vec{n}^{\mathsf{T}} \\ &= I - \frac{1}{\|\vec{n}\|^2}\big((\vec{n}\cdot\vec{v})\vec{v}\big)\,\vec{n}^{\mathsf{T}} \\ &= I - \frac{1}{\|\vec{n}\|^2}\big(0\vec{v}\big)\,\vec{n}^{\mathsf{T}} = I\end{aligned}$$

hence

$$M^{-1} = I + \frac{-1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}} = 2I - (I + \frac{1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}}) = 2I - M$$

Fixed points: $\left(I + \frac{1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}}\right)\vec{x} = \vec{x}$ implies $\vec{v}\,\vec{n}^{\mathsf{T}}\vec{x} = \vec{0}$ i.e. $(\vec{n}\cdot\vec{x})\,\vec{v} = \vec{0}$. This implies $\vec{n}\cdot\vec{x} = 0$, which means that the fixed points are precisely all points on the plane.

[Unless $\vec{v} = \vec{0}$ in which case $M = I$, and all points are fixed.]

Example 2: $M = \frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ -2 & 5 & 2 \\ 1 & -1 & 2 \end{bmatrix}$, then

$$M^{-1} = 2I - M = 2I - \frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ -2 & 5 & 2 \\ 1 & -1 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4 & -1 & -1 \\ 2 & 1 & -2 \\ -1 & 1 & 4 \end{bmatrix}$$

And indeed: $M \cdot M^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ -2 & 5 & 2 \\ 1 & -1 & 2 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 4 & -1 & -1 \\ 2 & 1 & -2 \\ -1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

We call $\vec{v}$ the **shearing vector**. We could call the length of $\vec{v}$ the **shearing factor**
[$\sqrt{A^2 + B^2 + C^2}$]

Note that a shear maps planes parallel to α to themselves: planes parallel to α are 'fixed', be it **not** point-wise but "as a whole". They are translated by the vector $\frac{\vec{n}\cdot\vec{x}}{\|\vec{n}\|}\vec{v}$ where $\frac{\vec{n}\cdot\vec{x}}{\|\vec{n}\|}$ is the signed distance between the two planes.

Also note that $\det(M) = 1$ means that shears preserve volumes.

This definition of a shear generalizes the usual shears with respect to the major planes: the *xy*-plane, the *xz*-plane and the *yz*-plane. When, for example, α is the *xy*-plane, i.e. $\vec{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} A \\ B \\ 0 \end{bmatrix}$ then the shear looks like

$$S(\vec{x}) = \begin{bmatrix} 1 & 0 & A \\ 0 & 1 & B \\ 0 & 0 & 1 \end{bmatrix}\vec{x}$$

Also note that given a shear matrix M, it is easy to find $\vec{n}$ and $\vec{v}$, since:

$$M - I \;=\; \frac{1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}} \;=\; \frac{1}{\|\vec{n}\|}\begin{bmatrix} aA & bA & cA \\ aB & bB & cB \\ aC & bC & cC \end{bmatrix}$$

Notice that the rows are multiples of $\begin{bmatrix} a & b & c \end{bmatrix}$ and the columns are multiples of $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$.

If one of the **rows** of $M - I$ is non-zero then that would give us $\vec{n}$, and

we would find $\vec{v}$ from one of the non-zero **columns** [by dividing by $\frac{a}{\|\vec{n}\|}$, $\frac{b}{\|\vec{n}\|}$ or $\frac{c}{\|\vec{n}\|}$].

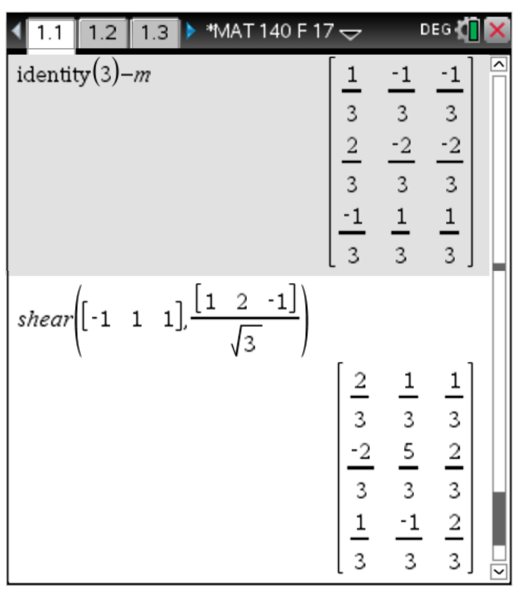
Or one could find $\vec{n}$ by $\boxed{\operatorname{arref}(M - I)}$, and $\vec{v}$ can be found by $\boxed{\vec{v} \;=\; \frac{(M - I)\vec{n}}{\|\vec{n}\|}}$

since $S(\vec{n}) \;=\; M\,\vec{n} \;=\; \vec{n} \;+\; \|\vec{n}\|\cdot\vec{v}$

Example 3: $M = \frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ -2 & 5 & 2 \\ 1 & -1 & 2 \end{bmatrix}$ has trace 3 and $M - I = \frac{1}{3}\begin{bmatrix} -1 & 1 & 1 \\ -2 & 2 & 2 \\ 1 & -1 & -1 \end{bmatrix}$ so that

$\vec{n} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ and thus $\vec{v} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ [indeed $\vec{n}\cdot\vec{v} = 0$].

Check that with this $\vec{n}$ and $\vec{v}$ indeed $M = I + \frac{1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ -2 & 5 & 2 \\ 1 & -1 & 2 \end{bmatrix}$.



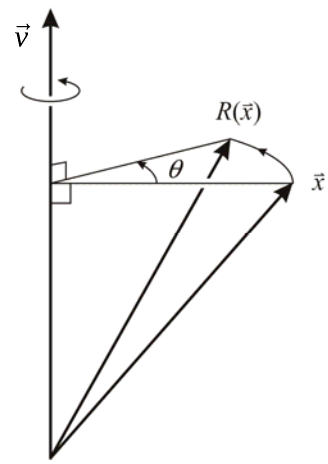
In this chapter we will discuss the following theorem:

Theorem 28.1:

The matrix of a rotation around a vector $\vec{v}$ over the angle θ is given by:

$$M = \boxed{(1-\cos\theta)\frac{\vec{v}\,\vec{v}^{\top}}{\|\vec{v}\|^2} + \cos\theta\, I + \frac{\sin\theta}{\|\vec{v}\|} C_{\vec{v}}}$$

where $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $C_{\vec{v}} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$.



Furthermore:

- $M^{-1} = M^{\top}$
- $\text{Trace}(M) = 1 + 2\cos(\theta)$
- $M - M^{\top} = 2\frac{\sin\theta}{\|\vec{v}\|} C_{\vec{v}}$
- $\det(M) = 1$
- $M_{\theta}^{2} = M_{2\theta}$
- The fixed points of this transformation are $\vec{x} = t\vec{v}$ (unless θ is a multiple of $360°$ in which case $M = I$ so that all points of $\mathbb{R}^3$ are fixed points:)

Up next is the 3D rotation. When we rotated in 2D we did so around a point. In 3-space we have an extra dimension, so we'll rotate around a line (i.e. the direction vector of the line.)

Rotation around a vector

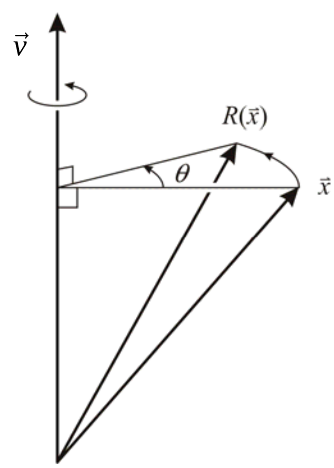
Theorem 28.1: The matrix of a rotation around a vector $\vec{v}$ over the angle θ

is given by: $$M = \boxed{(1-\cos\theta)\frac{\vec{v}\,\vec{v}^{\mathsf{T}}}{\|\vec{v}\|^2} + \cos\theta\, I + \frac{\sin\theta}{\|\vec{v}\|} C_{\vec{v}}}$$

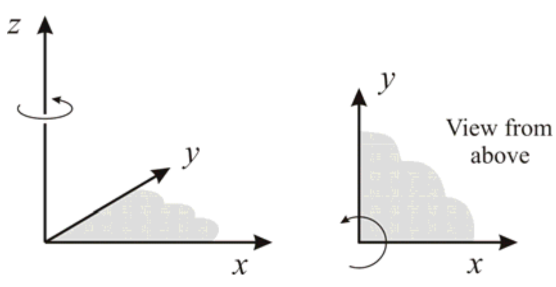
where $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $C_{\vec{v}} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$

Proof: We'll start using a **unit** vector $\vec{v}$, i.e. $\|\vec{v}\| = 1$.

Let $R(\vec{x})$ be the image of $\vec{x}$ after rotation around $\vec{v}$ over an angle θ.



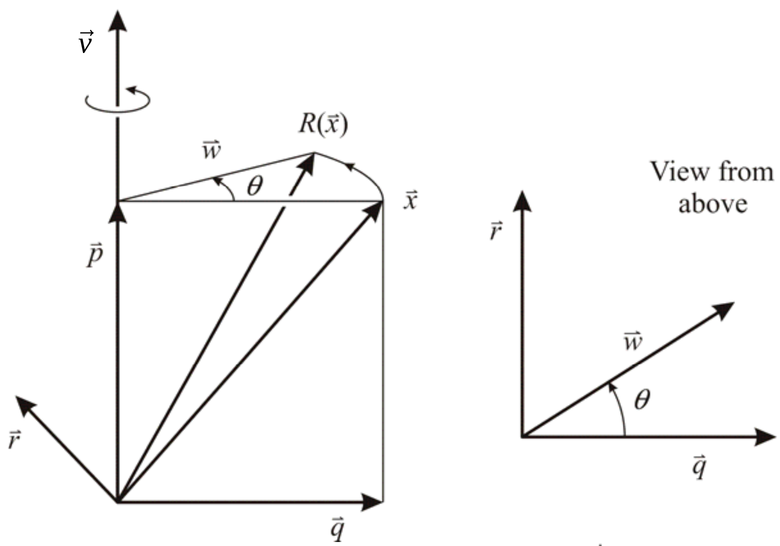
For example if we take $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, i.e. we rotate around the z-axis, it induces a rotation of the *xy*-plane around the origin over the angle θ:

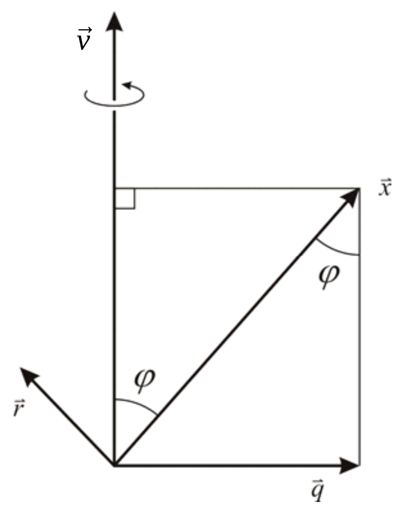


We'll take θ to be *positive* in the usual sense: counter clockwise. This is the same orientation we talked about when we introduced the cross product. When we go around the vector in the positive direction the 'thumb' points in the direction of the vector. When we talk about rotation over e.g. $30°$ we mean $+30°$, with this positive orientation in mind. Otherwise, if we want to turn the other (negative) direction, we would say $-30°$.

To actually derive the matrix we will need to do some work. First let's define the following vectors:

- Let $R(\vec{x})$ be the image of $\vec{x}$ after rotation around $\vec{v}$ over an angle θ.
- Let $\vec{p}$ be the projection of $\vec{x}$ onto $\vec{v}$: $\vec{p} = \dfrac{\vec{v}\cdot\vec{x}}{\vec{v}\cdot\vec{v}}\vec{v} \;\Rightarrow\; \vec{p} = (\vec{v}\cdot\vec{x})\vec{v} = \vec{v}\,\vec{v}^{\mathsf{T}}\vec{x}$
- Let $\vec{q} = \vec{x} - \vec{p}$
- Let $\vec{w} = R(\vec{x}) - \vec{p}$ (so that $R(\vec{x}) = \vec{w} + \vec{p}$)
- and finally let $\vec{r} = \vec{v}\times\vec{x}$.





If φ is the angle between $\vec{x}$ and $\vec{v}$ then

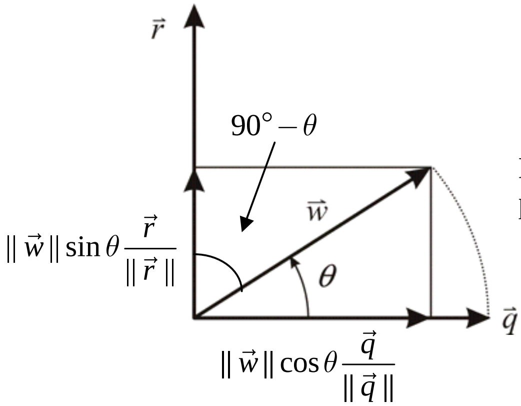
$$\begin{aligned}\|\vec{r}\| &= \|\vec{v}\times\vec{x}\| \\ &= \|\vec{v}\|\|\vec{x}\|\sin\varphi \\ &= \|\vec{x}\|\sin\varphi \\ &= \|\vec{q}\|\end{aligned}$$

i.e. $\|\vec{r}\| = \|\vec{q}\|$

Also note that $\|\vec{w}\| = \|\vec{q}\|$ so that

$$\|\vec{r}\| = \|\vec{q}\| = \|\vec{w}\|$$

Next we'll express $\vec{w}$ in terms of $\vec{q}$, $\vec{r}$ and θ:



$$\begin{aligned}\vec{w} &= \|\vec{w}\|\cos\theta\frac{\vec{q}}{\|\vec{q}\|} + \|\vec{w}\|\sin\theta\frac{\vec{r}}{\|\vec{r}\|}\\ &= \cos\theta\,\vec{q} + \sin\theta\,\vec{r}\end{aligned}$$

Here is an alternative derivation using projections

$$\begin{aligned}\vec{w} &= \operatorname{proj}_{\vec{q}}(\vec{w}) + \operatorname{proj}_{\vec{r}}(\vec{w})\\ &= \frac{\vec{w}\cdot\vec{q}}{\|\vec{q}\|^2}\vec{q} + \frac{\vec{w}\cdot\vec{r}}{\|\vec{r}\|^2}\vec{r}\\ &= \frac{\|\vec{w}\|\cdot\|\vec{q}\|\cos\theta}{\|\vec{q}\|^2}\vec{q} + \frac{\|\vec{w}\|\cdot\|\vec{r}\|\cos(90°-\theta)}{\|\vec{r}\|^2}\vec{r}\\ &= \cos\theta\,\vec{q} + \sin\theta\,\vec{r}\end{aligned}$$

We need one more item: the **cross product matrix**

If $\vec{v} = \begin{bmatrix} a\\ b\\ c\end{bmatrix}$ then $C_{\vec{v}} = \begin{bmatrix} 0 & -c & b\\ c & 0 & -a\\ -b & a & 0\end{bmatrix}$ which we will call the **cross product** matrix.

Observe the following

$$\vec{v}\times\vec{x} = \begin{bmatrix} a\\ b\\ c\end{bmatrix}\times\begin{bmatrix} x\\ y\\ z\end{bmatrix} = \begin{bmatrix} zb-cy\\ cx-az\\ ay-bx\end{bmatrix} = \begin{bmatrix} 0 & -c & b\\ c & 0 & -a\\ -b & a & 0\end{bmatrix}\begin{bmatrix} x\\ y\\ z\end{bmatrix} = C_{\vec{v}}\,\vec{x}$$

hence

$$\vec{v}\times\vec{x} = C_{\vec{v}}\,\vec{x}$$

So that $C_{\vec{v}}$ is the matrix that 'computes' the cross product of $\vec{v}$ with any other vector $\vec{x}$.

We are now ready to derive the rotation matrix:

We have

$$\begin{aligned} R(\vec{x}) &= \vec{p} + \vec{w} \\ &= \vec{p} + \cos\theta\,\vec{q} + \sin\theta\,\vec{r} \\ &= \vec{p} + \cos\theta\,(\vec{x}-\vec{p}) + \sin\theta\,(\vec{v}\times\vec{x}) \\ &= (1-\cos\theta)\,\vec{p} + \cos\theta\,\vec{x} + \sin\theta\, C_{\vec{v}}\,\vec{x} \\ &= (1-\cos\theta)\vec{v}\,\vec{v}^{\mathsf{T}}\vec{x} + \cos\theta\,\vec{x} + \sin\theta\; C_{\vec{v}}\;\vec{x} \\ &= \Big((1-\cos\theta)\vec{v}\,\vec{v}^{\mathsf{T}} + \cos\theta\, I + \sin\theta\; C_{\vec{v}}\Big)\,\vec{x} \end{aligned}$$

Hence

$$M = (1-\cos\theta)\vec{v}\,\vec{v}^{\mathsf{T}} + \cos\theta\; I + \sin\theta\; C_{\vec{v}}$$

In case $\vec{v}$ was **not** of unit length, we first normalize: i.e. replace $\vec{v}$ with $\dfrac{\vec{v}}{\|\vec{v}\|}$ to get

$$M = (1-\cos\theta)\frac{\vec{v}\,\vec{v}^{\mathsf{T}}}{\|\vec{v}\|^2} + \cos\theta\, I + \frac{\sin\theta}{\|\vec{v}\|}\, C_{\vec{v}}$$

If we take $\vec{v} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ and let $\Delta = \sqrt{A^2+B^2+C^2}$ the matrix would look like

$$\begin{bmatrix} (1-\cos\theta)\dfrac{A^2}{\Delta^2}+\cos\theta & (1-\cos\theta)\dfrac{AB}{\Delta^2}-\dfrac{C}{\Delta}\sin\theta & (1-\cos\theta)\dfrac{AC}{\Delta^2}+\dfrac{b}{\Delta}\sin\theta \\ (1-\cos\theta)\dfrac{AB}{\Delta^2}+\dfrac{C}{\Delta}\sin\theta & (1-\cos\theta)\dfrac{B^2}{\Delta^2}+\cos\theta & (1-\cos\theta)\dfrac{BC}{\Delta^2}-\dfrac{a}{\Delta}\sin\theta \\ (1-\cos\theta)\dfrac{AC}{\Delta^2}-\dfrac{B}{\Delta}\sin\theta & (1-\cos\theta)\dfrac{BC}{\Delta^2}+\dfrac{A}{\Delta}\sin\theta & (1-\cos\theta)\dfrac{C^2}{\Delta^2}+\cos\theta \end{bmatrix}$$

or

$$\frac{1}{\Delta^2}\begin{bmatrix} (1-\cos\theta)\,A^2+\Delta^2\cos\theta & (1-\cos\theta)\,AB-\Delta C\sin\theta & (1-\cos\theta)\,AC+\Delta\,B\sin\theta \\ (1-\cos\theta)\,AB+\Delta C\sin\theta & (1-\cos\theta)\,B^2+\Delta^2\cos\theta & (1-\cos\theta)\,BC-\Delta\,A\sin\theta \\ (1-\cos\theta)\,AC-\Delta\,B\sin\theta & (1-\cos\theta)\,BC+\Delta\,A\sin\theta & (1-\cos\theta)C^2+\Delta^2\cos\theta \end{bmatrix}$$

Clearly the better way to remember the matrix is as

$$(1-\cos\theta)\frac{\vec{v}\,\vec{v}^{\mathsf{T}}}{\|\vec{v}\|^2} + \cos\theta\, I + \frac{\sin\theta}{\|\vec{v}\|}\, C_{\vec{v}}$$

Example 1: **(a)** If $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ then the matrix becomes: $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which is related to the 2D matrix we found before. When we only look at what happens in the *xy*–plane this rotation induces precisely the rotation around the origin we saw in the previous chapter.

(b) If $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\theta = 30^\circ$ then the matrix becomes: $\frac{1}{3}\begin{bmatrix} 1+\sqrt{3} & 1-\sqrt{3} & 1 \\ 1 & 1+\sqrt{3} & 1-\sqrt{3} \\ 1-\sqrt{3} & 1 & 1+\sqrt{3} \end{bmatrix}$

(c) If $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\theta = 60^\circ$ then the matrix becomes: $\frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}$

When we compare this with the matrix from **(b)**

$$\left(\frac{1}{3}\begin{bmatrix} 1+\sqrt{3} & 1-\sqrt{3} & 1 \\ 1 & 1+\sqrt{3} & 1-\sqrt{3} \\ 1-\sqrt{3} & 1 & 1+\sqrt{3} \end{bmatrix}\right)^2 = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}$$

(d) If $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\theta = 120^\circ$ then the matrix becomes: $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

This maps permutes the standard bases vectors: $\vec{e}_1 \xrightarrow{R} \vec{e}_2 \xrightarrow{R} \vec{e}_3 \xrightarrow{R} \vec{e}_1$

Notice that the matrix from **(c)** would give us the matrix from **(d)** by squaring it (two rotations over 60° in a row)

$$\left(\frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}\right)^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and from (b)

$$\left(\frac{1}{3}\begin{bmatrix} 1+\sqrt{3} & 1-\sqrt{3} & 1 \\ 1 & 1+\sqrt{3} & 1-\sqrt{3} \\ 1-\sqrt{3} & 1 & 1+\sqrt{3} \end{bmatrix}\right)^4 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Also note that $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ i.e. we rotate basically over 360°.

(e) If $\vec{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ and $\theta = 30°$ then the matrix becomes:

$$\begin{bmatrix} 1.8+4.1\sqrt{2} & -2.6-1.2\sqrt{2} & 7-1.5\sqrt{2} \\ 7.4-1.2\sqrt{2} & 3.2+3.4\sqrt{2} & 1-2\sqrt{2} \\ -1-1.5\sqrt{2} & 7-2\sqrt{2} & 5+2.5\sqrt{2} \end{bmatrix} \approx \begin{bmatrix} 0.89014083 & -0.32139949 & 0.32303509 \\ 0.38570729 & 0.90889727 & -0.1585422 \\ -0.24265033 & 0.26572187 & 0.9330127 \end{bmatrix}$$

The Inverse

Note that the **inverse** of the matrix would be the rotation around the same vector but over $-\theta$

$$\begin{aligned} M^{-1} &= \left(1-\cos(-\theta)\right)\frac{\vec{v}\,\vec{v}^{\mathsf{T}}}{\|\vec{v}\|^2} + \cos(-\theta)\,I + \frac{\sin(-\theta)}{\|\vec{v}\|}C_{\vec{v}} \\ &= (1-\cos\theta)\frac{\vec{v}\,\vec{v}^{\mathsf{T}}}{\|\vec{v}\|^2} + \cos\theta\, I - \frac{\sin\theta}{\|\vec{v}\|}C_{\vec{v}} \\ &= (1-\cos\theta)\frac{\vec{v}\,\vec{v}^{\mathsf{T}}}{\|\vec{v}\|^2} + \cos\theta\, I + \frac{\sin\theta}{\|\vec{v}\|}C_{\vec{v}}^{\mathsf{T}} \end{aligned}$$

since $-C_{\vec{v}} = -\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} = C_{\vec{v}}^{\mathsf{T}}$.

But also note that $(\vec{v}\,\vec{v}^{\mathsf{T}})^{\mathsf{T}} = \vec{v}\,\vec{v}^{\mathsf{T}}$ and $I^{\mathsf{T}} = I$, so that

$$\boxed{M^{-1} = M^{\mathsf{T}}}$$

which makes it really easy to find the inverse of a rotation matrix.

Example 2: We computed the following rotation matrix $\frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}$ earlier.

Hence we now know that

$$\left(\frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}\right)^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}$$

In fact, it is easy to check this

$$\frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The Trace

Let's also check the trace of the rotation matrix:

$$\begin{aligned}
\text{Trace}(M) &= \text{Trace}\left((1-\cos\theta)\frac{\vec{v}\,\vec{v}^{\intercal}}{\|\vec{v}\|^2} + \cos\theta\, I + \frac{\sin\theta}{\|\vec{v}\|} C_{\vec{v}}\right) \\
&= \text{Trace}\left((1-\cos\theta)\frac{\vec{v}\,\vec{v}^{\intercal}}{\|\vec{v}\|^2}\right) + \text{Trace}(\cos\theta\, I) + \text{Trace}\left(\frac{\sin\theta}{\|\vec{v}\|} C_{\vec{v}}\right) \\
&= \frac{1-\cos\theta}{\|\vec{v}\|^2}\text{Trace}\left(\vec{v}\,\vec{v}^{\intercal}\right) + \cos\theta\,\text{Trace}(I) + \frac{\sin\theta}{\|\vec{v}\|}\text{Trace}\left(C_{\vec{v}}\right) \\
&= \frac{1-\cos\theta}{\|\vec{v}\|^2}\left(a^2+b^2+c^2\right) + 3\cos\theta + 0 \\
&= (1-\cos\theta) + 3\cos\theta \\
&= 1 + 2\cos\theta
\end{aligned}$$

$$\boxed{\text{Trace}(M) = 1 + 2\cos\theta}$$

Example 3: The rotation matrix we computed earlier was $M = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}$

it's trace is: $\text{Trace}(M) = 2.$

Hence we know that: $2 = 1 + 2\cos\theta$

$$\Rightarrow \quad \cos\theta = \frac{1}{2}$$

Recall that indeed the rotation was over $\theta = 60^\circ$.

Hence the trace allows us to find the angle over which we rotate.

To figure out what the direction was around which we rotated we *could* look at fixed points of the matrix, but there is a nicer way of doing it:

$$\boxed{M - M^{\mathsf{T}} \;=\; 2\frac{\sin\theta}{\|\vec{v}\|} C_{\vec{v}}}$$

[We'll leave the easy proof of this fact to the readers]

Hence $M - M^{\mathsf{T}}$ is a scalar times the matrix $C_{\vec{v}}$. Which allows us to read off $\vec{v}$.

Example 4: The rotation matrix we computed earlier was $M = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}$

Hence

$$\begin{aligned} M - M^{\mathsf{T}} &= \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{aligned}$$

So that when we have $\theta = 60^\circ$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ we find that indeed

$$M - M^{\mathsf{T}} = 2\frac{\sin\theta}{\|\vec{v}\|} C_{\vec{v}} = 2\frac{\sqrt{3}/2}{\sqrt{3}} C_{\vec{v}} = C_{\vec{v}} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

and we can read off $\vec{v}$

$$\begin{bmatrix} 0 & -\boxed{1} & \boxed{1} \\ 1 & 0 & -\boxed{1} \\ -1 & 1 & 0 \end{bmatrix} \longrightarrow \vec{v} = \begin{bmatrix} \boxed{1} \\ \boxed{1} \\ \boxed{1} \end{bmatrix}$$

Example 5: Suppose a rotation matrix had $\theta = 35^\circ$ and

$$M - M^\intercal = k \cdot \begin{bmatrix} 0 & -3 & -7 \\ 3 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix}$$

We can now deduce that we are rotating around the vector $\vec{v} = \begin{bmatrix} -8 \\ -7 \\ 3 \end{bmatrix}$

Or—which is the same—we are rotating around $\begin{bmatrix} 8 \\ 7 \\ -3 \end{bmatrix}$ over -35°!

The Determinant

Since $M^{-1} = M^\intercal$ we find that

$$1 = \det(MM^{-1}) = \det(M)\det(M^{-1}) = \det(M)\det(M^\intercal) = \det(M)\det(M)$$

i.e.

$$\det(M)^2 = 1$$

Hence

$$\det(M) = \pm 1$$

Computing the determinant the hard way (which can be done by algebraically fearless souls, or using the TI-Nspire ☺) we find

$$\det\left(\frac{1}{\Delta^2} \begin{bmatrix} (1-\cos\theta)A^2 + \Delta^2\cos\theta & (1-\cos\theta)AB - \Delta C\sin\theta & (1-\cos\theta)AC + \Delta B\sin\theta \\ (1-\cos\theta)AB + \Delta C\sin\theta & (1-\cos\theta)B^2 + \Delta^2\cos\theta & (1-\cos\theta)BC - \Delta A\sin\theta \\ (1-\cos\theta)AC - \Delta B\sin\theta & (1-\cos\theta)BC + \Delta A\sin\theta & (1-\cos\theta)C^2 + \Delta^2\cos\theta \end{bmatrix} \right) = 1$$

so that

$$\boxed{\det(M) = 1}$$

We have basically proven most parts of the 'furthermore' points on page 201:

- $M^{-1} = M^{\mathsf{T}}$
- $\text{Trace}(M) = 1 + 2\cos(\theta)$
- $M - M^{\mathsf{T}} = 2\dfrac{\sin\theta}{\|\vec{v}\|} C_{\vec{v}}$
- $\det(M) = 1$
- $M_\theta^2 = M_{2\theta}$
- The fixed points of this transformation are $\vec{x} = t\vec{v}$
  (unless θ is a multiple of $360°$ in which case $M = I$ so that all points of $\mathbb{R}^3$ are fixed points:)

Example 6: (A rotation as a composition of two reflections)

Let $R_l : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection in the plane $x - y + 2z = 0$
and $R_m : \mathbb{R}^3 \to \mathbb{R}^3$ the reflection in the plane $4x - 5y + 3z = 0$

We'll show that $R_m \circ R_l$ is a rotation.

The matrix for R_l is $M_l = I - \frac{2}{6}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ -2 & 2 & -1 \end{bmatrix}$ and

the matrix for R_m is $M_l = I - \frac{2}{50}\begin{bmatrix} 4 \\ -5 \\ 3 \end{bmatrix}\begin{bmatrix} 4 & -5 & 3 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 9 & 20 & -12 \\ 20 & 0 & 15 \\ -12 & 15 & 16 \end{bmatrix}$

hence the matrix of $R_m \circ R_l$ is

$$M_m M_l = \frac{1}{75}\begin{bmatrix} 9 & 20 & -12 \\ 20 & 0 & 15 \\ -12 & 15 & 16 \end{bmatrix}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ -2 & 2 & -1 \end{bmatrix} = \frac{1}{75}\begin{bmatrix} 62 & 25 & 34 \\ 10 & 50 & -55 \\ -41 & 50 & 38 \end{bmatrix}$$

Let's call the matrix M :

$$M = \frac{1}{75}\begin{bmatrix} 62 & 25 & 34 \\ 10 & 50 & -55 \\ -41 & 50 & 38 \end{bmatrix}$$

To see if M is indeed a rotation, let's look at $M - M^{\mathsf{T}}$

$$\begin{aligned} M - M^{\mathsf{T}} &= \frac{1}{75}\begin{bmatrix} 62 & 25 & 34 \\ 10 & 50 & -55 \\ -41 & 50 & 38 \end{bmatrix} - \frac{1}{75}\begin{bmatrix} 62 & 10 & -41 \\ 25 & 50 & 50 \\ 34 & -55 & 38 \end{bmatrix} \\ &= \frac{1}{75}\begin{bmatrix} 0 & 15 & 75 \\ -15 & 0 & -105 \\ -75 & 105 & 0 \end{bmatrix} \end{aligned}$$

Hence since $M - M^{\mathsf{T}} = 2\frac{\sin\theta}{\|\vec{v}\|}C_{\vec{v}}$ (provided this is indeed a rotation) then

$$2\frac{\sin\theta}{\|\vec{v}\|}C_{\vec{v}} = \frac{1}{75}\begin{bmatrix} 0 & 15 & 75 \\ -15 & 0 & -105 \\ -75 & 105 & 0 \end{bmatrix}$$

Therefore we can take $\vec{v}$ to be a multiple of $\begin{bmatrix} 105 \\ 75 \\ -15 \end{bmatrix}$:

Let's take $\vec{v} = \begin{bmatrix} 7 \\ 5 \\ -1 \end{bmatrix}$.

Notice that this is a positive multiple of the vector in the cross product matrix, so that we are taking $\sin\theta$ to be a positive quantity!

Next we'll find the angle using the trace (again provided this is indeed a rotation):

$$\text{Trace}(M) = \text{Trace}\left(\frac{1}{75}\begin{bmatrix} 62 & 25 & 34 \\ 10 & 50 & -55 \\ -41 & 50 & 38 \end{bmatrix}\right) = \frac{150}{75} = 2$$

Hence since $\text{Trace}(M) = 1 + 2\cos(\theta)$ we find

$$\begin{aligned} 2 &= 1 + 2\cos(\theta) \\ \Rightarrow \quad \cos(\theta) &= \frac{1}{2} \end{aligned}$$

And thus $\theta = 60^\circ$ (since $\sin\theta$ is positive!).

So apparently the composition is a rotation around the vector $\vec{v} = \begin{bmatrix} 7 \\ 5 \\ -1 \end{bmatrix}$ over 60°.

Let's check:

$$M = (1-\cos\theta)\frac{\vec{v}\,\vec{v}^{\mathsf{T}}}{\|\vec{v}\|^2} + \cos\theta\, I + \frac{\sin\theta}{\|\vec{v}\|}C_{\vec{v}}$$

$$= \frac{1-\frac{1}{2}}{75}\begin{bmatrix} 7 \\ 5 \\ -1 \end{bmatrix}\begin{bmatrix} 7 & 5 & -1 \end{bmatrix} + \frac{1}{2}I + \frac{\sqrt{3}/2}{\sqrt{75}}\begin{bmatrix} 0 & 1 & 5 \\ -1 & 0 & -7 \\ -5 & 7 & 0 \end{bmatrix}$$

$$= \frac{1}{150}\begin{bmatrix} 49 & 35 & -7 \\ 35 & 25 & -5 \\ -7 & -5 & 1 \end{bmatrix} + \frac{1}{2}I + \frac{1}{10}\begin{bmatrix} 0 & 1 & 5 \\ -1 & 0 & -7 \\ -5 & 7 & 0 \end{bmatrix}$$

$$= \frac{1}{150}\begin{bmatrix} 49 & 35 & -7 \\ 35 & 25 & -5 \\ -7 & -5 & 1 \end{bmatrix} + \frac{1}{150}\begin{bmatrix} 75 & 0 & 0 \\ 0 & 75 & 0 \\ 0 & 0 & 75 \end{bmatrix} + \frac{1}{150}\begin{bmatrix} 0 & 15 & 75 \\ -15 & 0 & -105 \\ -75 & 105 & 0 \end{bmatrix}$$

$$= \frac{1}{150}\begin{bmatrix} 124 & 50 & 68 \\ 20 & 100 & -110 \\ -82 & 100 & 76 \end{bmatrix}$$

$$= \frac{1}{75}\begin{bmatrix} 62 & 25 & 34 \\ 10 & 50 & -55 \\ -41 & 50 & 38 \end{bmatrix}$$

Which is indeed the matrix we found. Hence we have demonstrated that this is a rotation.

From a geometric perspective we also know that the composition of two reflections in planes is a rotation, in fact a rotation around the line of intersection over an angle double the angle between the two planes:

This can readily be checked

(1) The line of intersection of the two planes is $\vec{x} = t\begin{bmatrix} 7 \\ 5 \\ -1 \end{bmatrix}$ since $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \times \begin{bmatrix} 4 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ -1 \end{bmatrix}$

(2) The angle between the two planes is

$$\cos^{-1}\frac{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\cdot\begin{bmatrix} 4 \\ -5 \\ 3 \end{bmatrix}}{\sqrt{6}\sqrt{50}} = \cos^{-1}\left(\frac{15}{10\sqrt{3}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^\circ$$

29. Traces, Determinants and Eigenvalues

(Optional … but very illuminating)

The **trace** of a matrix and the **determinant** of a matrix are closely related to the **eigenvalues** of the matrix.

Eigenvectors and Eigenvalues

An **eigenvector** of a matrix M is a non-zero vector such that $M\vec{x} = t\vec{x}$ for some scalar t. The corresponding scalar t is called the **eigenvalue** of this eigenvector.

Example 1: Let $M = \begin{bmatrix} 8 & -2 \\ 15 & -3 \end{bmatrix}$.

Note that $\begin{bmatrix} 8 & -2 \\ 15 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 2\cdot\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ hence $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector with eigenvalue 2.

Also $\begin{bmatrix} 8 & -2 \\ 15 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = 3\cdot\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ hence $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ is an eigenvector with eigenvalue 3.

Example 2: Let $M = \begin{bmatrix} 6 & -6 & -2 \\ 13 & -13 & -6 \\ -24 & 24 & 13 \end{bmatrix}$ then

- $\begin{bmatrix} 6 & -6 & -2 \\ 13 & -13 & -6 \\ -24 & 24 & 13 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0\cdot\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ hence $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue 0.
- $\begin{bmatrix} 6 & -6 & -2 \\ 13 & -13 & -6 \\ -24 & 24 & 13 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = 1\cdot\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ hence $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ is an eigenvector with eigenvalue 1.
- $\begin{bmatrix} 6 & -6 & -2 \\ 13 & -13 & -6 \\ -24 & 24 & 13 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = 5\cdot\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ hence $\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ is an eigenvector with eigenvalue 5.

Hence eigenvectors get mapped by the transformation $T(\vec{x}) = M\vec{x}$ to a multiple of themselves.

Notes: **(1)** Eigen*vectors* are non-zero, but eigen*values* can be zero.

(2) The eigenvectors with eigenvalue 1 we call fixed points.

(3) The eigenvectors with eigenvalue 0 are points in the ‘nullspace’.

How to find eigenvalues and eigenvectors.

To find eigenvectors we basically need to solve $M\vec{x} = t\vec{x}$:

$$\begin{aligned} M\vec{x} = t\vec{x} \quad &\Leftrightarrow \quad M\vec{x} - t\vec{x} = \vec{0} \\ &\Leftrightarrow \quad (M - tI)\vec{x} = \vec{0} \end{aligned}$$

Clearly $\vec{x} = \vec{0}$ is a solution of this equation. But for $(M - tI)\vec{x} = \vec{0}$ to have a *non-zero* solution $\vec{x}$ the matrix $M - tI$ has to be **non-invertible**, i.e.

$$\det(M - tI) = 0$$

Example 3: Let $M = \begin{bmatrix} 8 & -2 \\ 15 & -3 \end{bmatrix}$ then

$$\begin{aligned} \det(M - tI) &= \det\left(\begin{bmatrix} 8 & -2 \\ 15 & -3 \end{bmatrix} - tI\right) = \det\left(\begin{bmatrix} 8 & -2 \\ 15 & -3 \end{bmatrix} - \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 8-t & -2 \\ 15 & -3-t \end{bmatrix}\right) \\ &= (8-t)(-3-t)+30 \\ &= t^2 - 5t + 6 \\ &= (t-2)(t-3) \end{aligned}$$

So clearly: $\det(M - tI) = 0 \quad \Leftrightarrow \quad t = 2,3$

Hence these are the only two eigenvalues of M. Once we have found the eigenvalues we can set out finding eigenvectors:

(1) $t = 2$: Solve $M\vec{x} = 2\vec{x}$ i.e. $\begin{bmatrix} 8 & -2 \\ 15 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$

$$\Rightarrow \begin{cases} 8x - 2y = 2x \\ 15x - 3y = 2y \end{cases}$$

$$\Rightarrow \begin{cases} 6x - 2y = 0 \\ 15x - 5y = 0 \end{cases} \Rightarrow 3x - y = 0 \Rightarrow 3x = y \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 3x \end{bmatrix} = x\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Hence all nonzero multiples of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ are the eigenvectors with eigenvalue 2.

(2) $t = 3$: Solve $M\vec{x} = 3\vec{x}$ i.e. $\begin{bmatrix} 8 & -2 \\ 15 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 3\begin{bmatrix} x \\ y \end{bmatrix}$

$$\Rightarrow \begin{cases} 8x - 2y = 3x \\ 15x - 3y = 3y \end{cases}$$

$$\Rightarrow \begin{cases} 5x - 2y = 0 \\ 15x - 6y = 0 \end{cases} \Rightarrow 5x - 2y = 0 \Rightarrow 5x = 2y \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ \frac{5}{2}x \end{bmatrix} = \frac{x}{2}\begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Hence all nonzero multiples of $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ are the eigenvectors with eigenvalue 3.

Compare these results with Example 1!

Example 4: Let $M = \begin{bmatrix} 6 & -6 & -2 \\ 13 & -13 & -6 \\ -24 & 24 & 13 \end{bmatrix}$ then

$$\det(M - tI) = \det\left(\begin{bmatrix} 6 & -6 & -2 \\ 13 & -13 & -6 \\ -24 & 24 & 13 \end{bmatrix} - tI\right) = \det\left(\begin{bmatrix} 6 & -6 & -2 \\ 13 & -13 & -6 \\ -24 & 24 & 13 \end{bmatrix} - \begin{bmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 6-t & -6 & -2 \\ 13 & -13-t & -6 \\ -24 & 24 & 13-t \end{bmatrix}\right)$$

$$= -t^3 + 6t^2 - 5t$$

$$= -t(t-1)(t-5)$$

So clearly: $\det(M - tI) = 0 \Leftrightarrow t = 0, 1, 5$

Hence these are the only three eigenvalues of M. Once we have found the eigenvalues we can set out finding eigenvectors:

(1) $t = 0$: Solve $M\vec{x} = 0\vec{x}$ i.e. $\begin{bmatrix} 6 & -6 & -2 \\ 13 & -13 & -6 \\ -24 & 24 & 13 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\text{rref}\begin{bmatrix} 6 & -6 & -2 & 0 \\ 13 & -13 & -6 & 0 \\ -24 & 24 & 13 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x = y \\ z = 0 \end{cases}$$

i.e $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ x \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Hence all the nonzero multiples of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ are the eigenvectors corresponding to the eigenvalue 0.

(2) $t = 1$: Solve $M\vec{x} = \vec{x}$ i.e. $\begin{bmatrix} 6 & -6 & -2 \\ 13 & -13 & -6 \\ -24 & 24 & 13 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\text{i.e.} \begin{cases} 6x - 6y - 2z = x \\ 13x - 13y - 6z = y \\ -24x + 24y + 13z = z \end{cases} \Rightarrow \begin{cases} 5x - 6y - 2z = 0 \\ 13x - 14y - 6z = 0 \\ -24x + 24y + 12z = 0 \end{cases}$$

$$\text{rref}\begin{bmatrix} 5 & -6 & -2 & 0 \\ 13 & -14 & -6 & 0 \\ -24 & 24 & 12 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x = z \\ y = \frac{1}{2}z \end{cases}$$

Note $M\vec{x} = \vec{x}$ implies $(M - I)\vec{x} = \vec{0}$ so that we could have just arref-ed $M - I$:

$$\text{a rref}(M - I) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ (recall arref augments the column of zeros).

Either way we get $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ \frac{1}{2}z \\ z \end{bmatrix} = \frac{1}{2}z \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, so that the eigenvectors corresponding to the eigenvalue 1 are all the nonzero multiples of $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

(3) $t = 5$: Solve $M\vec{x} = 5\vec{x}$ i.e. $\begin{bmatrix} 6 & -6 & -2 \\ 13 & -13 & -6 \\ -24 & 24 & 13 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 5\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\text{i.e.} \begin{cases} 6x - 6y - 2z = 5x \\ 13x - 13y - 6z = 5y \\ -24x + 24y + 13z = 5z \end{cases} \Rightarrow \begin{cases} x - 6y - 2z = 0 \\ 13x - 18y - 6z = 0 \\ -24x + 24y + 8z = 0 \end{cases}$$

$$\text{rref}\begin{bmatrix} 1 & -6 & -2 & 0 \\ 13 & -18 & -6 & 0 \\ -24 & 24 & 8 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x = 0 \\ y = -\frac{1}{3}z \end{cases}$$

Note $M\vec{x} = 5\vec{x}$ implies $(M-5I)\vec{x} = \vec{0}$ i.e. we could have just arref-ed $M-5I$:

$$\operatorname{arref}(M-5I) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Either way we get $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{3}z \\ z \end{bmatrix} = -\frac{1}{3}z \cdot \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$, so that the eigenvectors

corresponding to the eigenvalue 5 are all the nonzero multiples of $\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$.

Compare these results with Example 2!

Summary: To find eigenvectors we first have to find the eigenvalues, i.e. the solutions to the equation $\det(M - tI) = 0$, and then for each of those eigenvalues t_0 we need to rref $(M - t_0 I)$ to find the corresponding eigenvectors: i.e. perform the following steps

(1) Solve: $\det(M - tI) = 0$. You get the eigenvalues t_i.

(2) For each of the eigenvalues t_i solve $(M - t_i I)\vec{x} = \vec{0}$ by arref-ing $(M - t_i I)$.

The polynomial $p(t) = \det(M - tI)$ is called the **characteristic polynomial** of the matrix *M*. The zeros, the roots of the polynomial are the eigenvalues of the matrix.

Example 5: If $M = \begin{bmatrix} 8 & -2 \\ 15 & -3 \end{bmatrix}$ then its characteristic polynomial is

$p(t) = \det(M - tI) = t^2 - 5t + 6 = (t-2)(t-3)$,
and the eigenvalues are $t_1 = 2$ and $t_2 = 3$

Example 6: If $M = \begin{bmatrix} 6 & -6 & -2 \\ 13 & -13 & -6 \\ -24 & 24 & 13 \end{bmatrix}$ then its characteristic polynomial is

$p(t) = \det(M - tI) = -t^3 + 6t^2 - 5t = -t(t-1)(t-5)$
and the eigenvalues are $t_1 = 0$, $t_2 = 1$ and $t_3 = 5$

The Determinant

(1) Suppose M is a 2×2 matrix and has t_1 and t_2 as eigenvalues then its characteristic polynomial looks like

$$p(t) = \det(M - tI) = (t-t_1)(t-t_2) = t^2 - (t_1+t_2)t + t_1\cdot t_2$$

Note that $p(0) = \det(M) = t_1\cdot t_2$

(2) Suppose M is a 3×3 matrix and has t_1, t_2 and t_3 as eigenvalues then its characteristic polynomial looks like

$$\begin{aligned} p(t) &= \det(M - tI) = -(t-t_1)(t-t_2)(t-t_3) \\ &= -t^3 + (t_1+t_2+t_3)t^2 - (t_1t_2+t_1t_3+t_2t_3)t^2 + t_1\cdot t_2\cdot t_3 \end{aligned}$$

Note that $p(0) = \det(M) = t_1\cdot t_2\cdot t_3$

The determinant is the product of the eigenvalues

The Trace

What follows relies on results from Linear algebra which will be covered in the next Linear Algebra course. So what follows is a just an outline. In that future Linear Algebra class we'll see that a change of basis transformation does not change the characteristic polynomial, i.e. that the matrices M and $Q^{-1}M\,Q$ have the same characteristic polynomial. This amounts to showing that

$$\begin{aligned} \det(Q^{-1}M\,Q - tI) &= \det(Q^{-1}(M - tI)Q) \\ &= \det(Q(M - tI)Q^{-1}) \\ &= \det(Q)\det(M - tI)\det(Q^{-1}) \\ &= \det(M - tI) \\ &= p(t) \end{aligned}$$

If you didn't follow this completely, that is fine. It will be explained in the near future.

Hence the matrices M and $Q^{-1}MQ$ have the same eigenvalues.

A little more work is needed to show that the matrices M and $Q^{-1}MQ$ also have the same **trace**. For this we'll also refer to the future more advanced Linear Algebra class.

When a matrix has a 'basis' of eigenvectors we can change to this basis and produce an **diagonal** matrix. We call such a matrix **diagonalizable**.

In general if M is diagonalizable, and Q is the eigenvector matrix, i.e. a matrix with as columns linearly independent eigenvectors, then

$$Q^{-1}M\,Q = \begin{bmatrix} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_n \end{bmatrix}$$

where on the main diagonal we get the eigenvalues and zeros elsewhere.

Hence: $\text{Trace}\left(Q^{-1}M\,Q\right) = t_1 + t_n + \cdots + t_n = \text{Trace}(M)$

Example 7: If $M = \begin{bmatrix} 8 & -2 \\ 15 & -3 \end{bmatrix}$ then as we saw in examples 1, 3 and 5, that the matrix has two linearly independent eigenvectors: $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$. If we let $Q = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ then

$$Q^{-1}MQ = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 8 & -2 \\ 15 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Notice that the eigenvectors appear on the main diagonal!

Hence indeed the trace of $\text{Trace}(Q^{-1}MQ) = t_1 + t_2 = 2+3 = 5 = \text{Trace}(M)$.

Example 8: If $M = \begin{bmatrix} 6 & -6 & -2 \\ 13 & -13 & -6 \\ -24 & 24 & 13 \end{bmatrix}$ then as we saw in examples 2,4 and 6, that the matrix has three linearly independent eigenvectors: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$. If we let $Q = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & -3 \end{bmatrix}$ then

$$Q^{-1} M\, Q = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 6 & -6 & -2 \\ 13 & -13 & -6 \\ -24 & 24 & 13 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Notice that the eigenvectors appear on the main diagonal!

Hence: $\text{Trace}(M) = \text{Trace}(Q^{-1} M\, Q) = t_1 + t_2 + t_3 = 0 + 1 + 5 = 6$

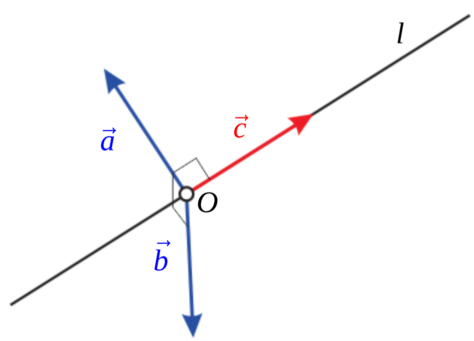
The trace is the sum of the eigenvalues

Applications

We'll only look at the 3D matrices, although similar arguments can be used in 2D or higher dimensions.

(1) Projection onto a line in 3D

$$M = \frac{1}{\|\vec{v}\|^2} \vec{v}\, \vec{v}^{\mathsf{T}}$$



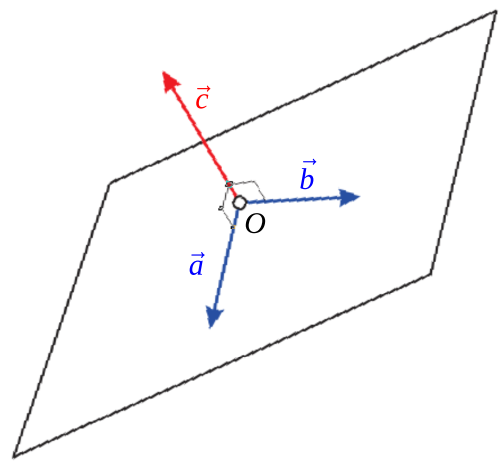
Take two independent vectors $\vec{a}$ and $\vec{b}$ both perpendicular to the line, and one vector $\vec{c}$ in the direction of the line:

then $M\vec{a} = \vec{0}$ i.e. $\vec{a}$ is an eigenvector with eigenvalue 0.
$M\vec{b} = \vec{0}$ i.e. $\vec{b}$ is also an eigenvector with eigenvalue 0.
$M\vec{c} = \vec{c}$ i.e. $\vec{c}$ is an eigenvector with eigenvalue 1.

Hence: $\det(M) = 0 \cdot 0 \cdot 1 = 0$

and $\mathrm{Trace}(M) = 0+0+1 = 1$

(2) Projection onto a plane in 3D



$$M = I - \frac{1}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}$$

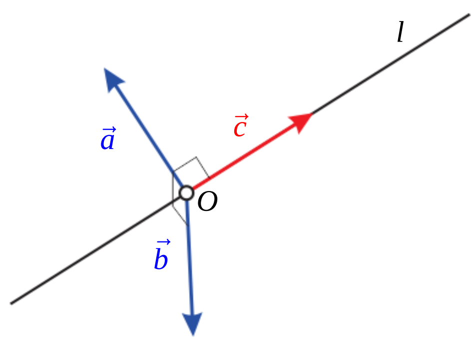
Take two independent vectors $\vec{a}$ and $\vec{b}$ both in the plane, and one vector $\vec{c}$ perpendicular to the plane:

then $M\vec{a} = \vec{a}$ i.e. $\vec{a}$ is an eigenvector with eigenvalue 1.
$M\vec{b} = \vec{b}$ i.e. $\vec{b}$ is also an eigenvector with eigenvalue 1.
$M\vec{c} = \vec{0}$ i.e. $\vec{c}$ is an eigenvector with eigenvalue 0.

Hence: $\det(M) = 1 \cdot 1 \cdot 0 = 0$

and $\mathrm{Trace}(M) = 1+1+0 = 2$

(3) Reflection in a line in 3D



$$M = \frac{2}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}} - I$$

Take two independent vectors $\vec{a}$ and $\vec{b}$ both perpendicular to the line, and one vector $\vec{c}$ in the direction of the line:

then $M\vec{a} = -\vec{a}$ i.e. $\vec{a}$ is an eigenvector with eigenvalue -1.

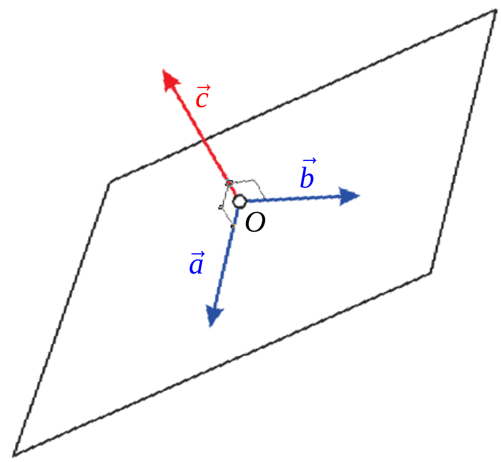
$M\vec{b} = -\vec{b}$ i.e. $\vec{b}$ is also an eigenvector with eigenvalue -1.

$M\vec{c} = \vec{c}$ i.e. $\vec{c}$ is an eigenvector with eigenvalue 1.

Hence: $\det(M) = (-1)\cdot(-1)\cdot 1 = 1$

and $\mathrm{Trace}(M) = (-1)+(-1)+1 = -1$

(4) Reflection in a plane in 3D



$$M = I - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}$$

Take two independent vectors $\vec{a}$ and $\vec{b}$ both in the plane, and one vector $\vec{c}$ perpendicular to the plane:

then $M\vec{a} = \vec{a}$ i.e. $\vec{a}$ is an eigenvector with eigenvalue 1.

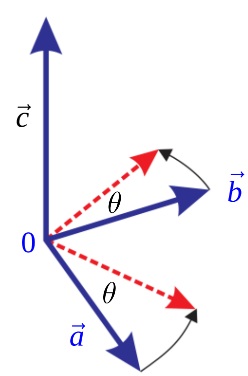
$M\vec{b} = \vec{b}$ i.e. $\vec{b}$ is also an eigenvector with eigenvalue 1.

$M\vec{c} = -\vec{c}$ i.e. $\vec{c}$ is an eigenvector with eigenvalue -1.

Hence: $\det(M) = 1\cdot 1\cdot(-1) = -1$

and $\mathrm{Trace}(M) = 1+1+(-1) = 1$

(5) A 3D **rotation** around the vector $\vec{v}$ over θ:



$$M = (1-\cos\theta)\frac{\vec{v}\,\vec{v}^{\mathsf{T}}}{\|\vec{v}\|^2} + \cos\theta\, I + \frac{\sin\theta}{\|\vec{v}\|}C_{\vec{v}}$$

Since there is only one real eigenvalue here, we have to argue differently: we'll select another basis

Let $\vec{c} = \dfrac{\vec{v}}{\|\vec{v}\|}$, and let $\vec{a}$ be a unit vector perpendicular to $\vec{v}$ and finally let $\vec{b} = \vec{c} \times \vec{a}$. Note: $\vec{a}$, $\vec{b}$ and $\vec{c}$ are all unit vectors.

Then the transformation with respect to the basis $\{\vec{a}, \vec{b}, \vec{c}\}$ looks like

$$\tilde{M} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since from the perspective of the orthogonal unit vectors $\{\vec{a}, \vec{b}, \vec{c}\}$ is looks just like a rotation over the z - axis.

Since this is the same rotation but described with respect to a different basis, we have that $\tilde{M} = Q^{-1} M Q$ for some change of basis transformation matrix $Q = [\vec{a}, \vec{b}, \vec{c}]$.

But as we argued before $Q^{-1} M Q$ and M have the same determinant and trace.

Hence: $$\det(M) = \det(Q^{-1} M Q) = \det \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \cos^2\theta + \sin^2\theta = 1$$

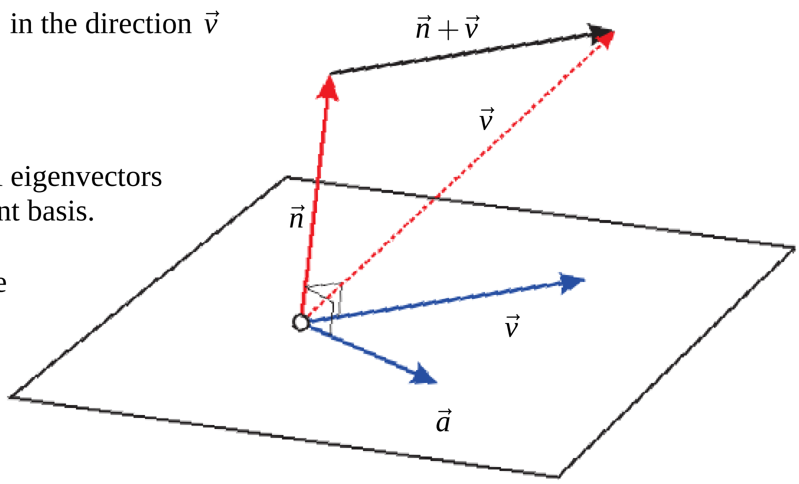
and $$\mathrm{Trace}(M) = 1 + \cos\theta + \cos\theta = 1 + 2\cos\theta .$$

(6) Shears.

The shear with respect to $\vec{n} \cdot \vec{x} = 0$ in the direction $\vec{v}$ has matrix $M = I + \frac{1}{\|\vec{n}\|} \vec{v}\, \vec{n}^{\mathsf{T}}$

Again we don't have a basis of real eigenvectors here, so we'll argue using a different basis.

Take $\vec{a}$ to be a vector in the plane independent of $\vec{v}$ and if we take $\|\vec{n}\| = 1$ then



$$M\vec{a} = \vec{a}$$
$$M\vec{v} = \vec{v}$$
$$M\vec{n} = \vec{n} + \vec{v}$$

so that $M[\vec{a}, \vec{v}, \vec{n}]) = [\vec{a}, \vec{v}, \vec{n} + \vec{v}]$

i.e.

$$\det(M[\vec{a},\vec{v},\vec{n}\,]) = \det([\vec{a},\vec{v},\vec{n}+\vec{v}\,])$$

$$\Rightarrow \quad \det(M)\det([\vec{a},\vec{v},\vec{n}\,]) = \det([\vec{a},\vec{v},\vec{n}+\vec{v}\,])$$

But one of the properties of determinants is that adding a row (or a column) to another row (or column) doesn't change the determinant, i.e. $\det([\vec{a},\vec{v},\vec{n}\,]) = \det([\vec{a},\vec{v},\vec{n}+\vec{v}\,])$, and since this determinant is non-zero we can divide by it and get

$$\det(M) = 1$$

To compute the trace(M) we look at the matrix $\tilde{M}$ of this transformation with respect to the basis $\{\vec{a},\vec{v},\vec{n}\}$:

$$\tilde{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $Q\tilde{M}Q^{-1} = M$ using the change of basis matrix $Q = [\vec{a},\vec{v},\vec{n}\,]$.

But then $\mathrm{Trace}(M) = \mathrm{Trace}(\tilde{M}) = \mathrm{Trace}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 3$.

We might as well compute the determinant this way too:

$$\det(M) = \det(\tilde{M}) = \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 1$$

[since $\det(M) = \det(Q\tilde{M}Q^{-1}) = \det(Q)\det(\tilde{M})\det(Q^{-1}) = \det(\tilde{M})$]