Week 12: Affine transformations

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Questions 3-5

Consider rotation $T : \mathbb{R}^2 \to \mathbb{R}^2$ about O over $120°$.

(a) What is the image m' of the line $m : x - \pi^{100} y = e$ under T?

(b) Let S be the rotation about O over $240°$. What is the image m'' of m' under S?

(a) (1) Write vector equation of m

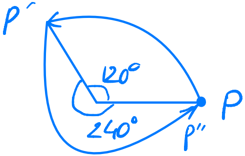
(2) Multiply matrix of T by the vector equation

(b) m' = rotation of m by $120°$

m'' = rotation of m' by $240°$

$\Rightarrow m''$ = rotation of m by $120° + 240° = 360°$

$\Rightarrow m'' = m : x - \pi^{100} y = e$.

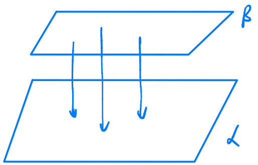


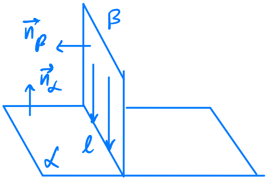
Question 9-10

Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto the plane $\alpha : x + y - z = 0$.

(a) What is the geometric description of the image of $\beta : x - y = 0$?

(b) What is the geometric description of the image of $\gamma : x - y - z = 0$?





$\beta' = \beta$

if $\beta \not\perp \alpha \Leftrightarrow \vec{n}_\alpha \not\perp \vec{n}_\beta$

$\beta' =$ line $l = \alpha \cap \beta$

if $\beta \perp \alpha \Leftrightarrow \vec{n}_\alpha \perp \vec{n}_\beta$

(a) $\vec{n}_\beta = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \perp \vec{n}_\alpha = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \beta' =$ a line

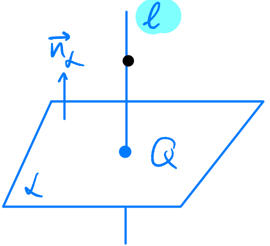
(b) $\vec{n}_\gamma = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \not\perp \vec{n}_\alpha = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \beta' =$ plane α

$Q = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

(c) What points are mapped to the point $(1, 2, 3)$ under P?

$\alpha : x + y - z = 0$

All points that are mapped to Q lie on the line ℓ containing Q & orthogonal to α.



$\ell : \vec{x} = Q + t\,\vec{n}_\alpha$

$= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

(A) ℓ is a plane ×

(B) ℓ is a line on α ×

(C) ℓ is α ×

(D) ℓ is a line not on α ✓

Affine transformation - definition

- The map $T : \mathbb{R}^n \to \mathbb{R}^m$ is an **affine transformation** if there exist an $m \times n$ matrix A and a vector $\vec{b} \in \mathbb{R}^m$ such that

$$T(\vec{x}) = A\vec{x} + \vec{b}$$

- Affine transformation is a generalization of linear transformation. If $\vec{b} = 0$, then we have a linear transformation.

$$T(\vec{x}) = A\vec{x}$$

Find A and $\vec{b}$

- A and $\vec{b}$ can be determined if we have a clear formula for T:

$T: R^n \to R^m$

$$T\begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n + b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n + b_2 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n + b_m \end{pmatrix}$$

- Then $T(\vec{x}) = A\vec{x} + \vec{b}$ with

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Example 1

Which of the following maps are linear transformations? Affine transformations? Or none of these. Find the matrix A and vector $\vec{b}$ of T in $T(\vec{x}) = A\vec{x} + \vec{b}$ for affine transformations.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y^2 + 1 \\ y - x - 2 \end{pmatrix}$ none of these

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ y - x \end{pmatrix}$ Linear

$$T(\vec{x}) = A\vec{x} \quad \text{with } A = \begin{pmatrix} 2 & 1 \\ \underbrace{-1}_{x} & \underbrace{1}_{y} \end{pmatrix}$$

(c) $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - z + 2y - 10 \\ y - x \\ y + z + 1 \end{pmatrix} = \begin{pmatrix} x + 2y - z - 10 \\ -x + y + 0z + 0 \\ 0x + y + z + 1 \end{pmatrix}$

$T(\vec{x}) = A\vec{x} + \vec{b} \Rightarrow A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ & $\vec{b} = \begin{pmatrix} -10 \\ 0 \\ 1 \end{pmatrix}$

(d) $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - z + 2y - 10 \\ y - x \end{pmatrix} = \begin{pmatrix} x + 2y - z - 10 \\ -x + y + 0z + 0 \end{pmatrix}$

$T(\vec{x}) = A\vec{x} + \vec{b}$ with $A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 0 \end{pmatrix}$ & $\vec{b} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$

Projection, reflection, shear, rotation

- Linear transformations fixed the origin O.
  For projection, reflection, shear, rotation to be linear transformation, the line/plane under consideration must go through O.
- When the line/plane under consideration doesn't go through O, we have an affine transformation.

3-step approach

Idea: Shift everything to the origin, do the linear map with respect to the origin, and then shift everything back.

3-step approach

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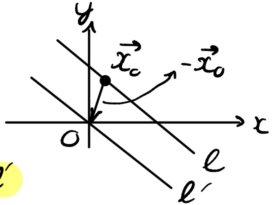


1. Translation by $-\vec{x}_0$, that is

$$\vec{x} \mapsto \vec{x} - \vec{x}_0$$

2. Perform linear transformation by matrix A of ℓ'

$$\vec{x} \mapsto \underbrace{\vec{x} - \vec{x}_0}_{1^{st}\text{ step}} \mapsto \underbrace{?\ A(\vec{x} - \vec{x}_0)}_{2^{nd}\text{ step}}$$

3. Translation by $\vec{x}_0$

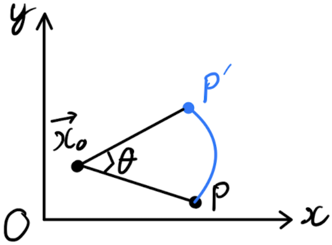
$$A(\vec{x}-\vec{x}_0)$$

$$\vec{x} \mapsto \vec{x} - \vec{x}_0 \mapsto ? \mapsto ?\ A(\vec{x}-\vec{x}_0)+\vec{x}_0$$

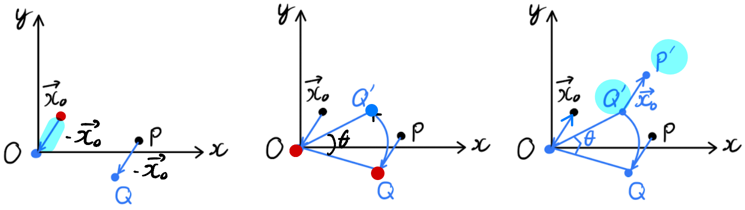
Summary: $\vec{x} \mapsto A(\vec{x}-\vec{x}_0)+\vec{x}_0$

Example 2

Assume we want to find the image P' of P by rotation about $\vec{x}_0$ over angle θ.



Example 2: 3-step approach



1. Shift $\vec{x}_0$ to O by the vector $-\vec{x}_0$

$$\vec{x}_0 \mapsto \vec{x}_0 - \vec{x}_0 = \vec{0} \text{ and } \; P \mapsto P - \vec{x}_0 = Q$$

2. Find the image Q' of Q under rotation about O over θ

$$Q' = AQ \text{ with } A = \text{matrix of rotation}$$

3. Shift everything back by $\vec{x}_0$

$$P' = Q' + \vec{x}_0$$

Example 2

Find the image of $\begin{pmatrix}1\\2\end{pmatrix}$ under rotation about $\vec{x}_0 = \begin{pmatrix}10\\1\end{pmatrix}$ over $\theta = 30°$.

(1) The image of $\vec{x}$ is $T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0$ with

$\vec{x}_0 = \begin{pmatrix}10\\1\end{pmatrix}$ & A = rotation matrix about O over $30°$

$$A = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} = \begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix}$$

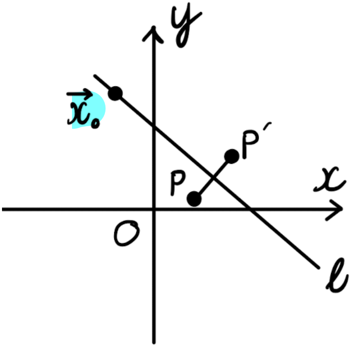
(2) The image of $\begin{pmatrix}1\\2\end{pmatrix}$ is

$$T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix}\left(\begin{pmatrix}1\\2\end{pmatrix} - \begin{pmatrix}10\\1\end{pmatrix}\right) + \begin{pmatrix}10\\1\end{pmatrix}$$

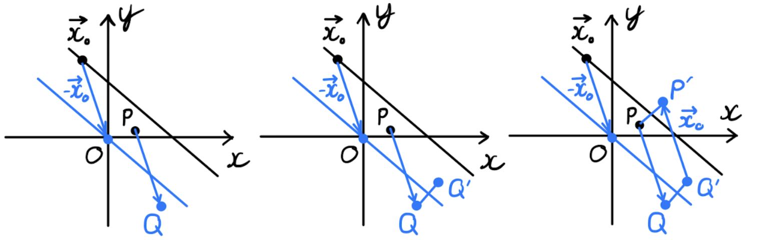
$$= \begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix}\begin{pmatrix}-9\\1\end{pmatrix} + \begin{pmatrix}10\\1\end{pmatrix} \approx \begin{pmatrix}1.7\\-2.3\end{pmatrix}$$

Example 3

Assume we want to find the image P' of P by reflection through a line l which goes through $\vec{x}_0$.



Example 3: 3-step approach



1. Shift l to a line through O by $-\vec{x}_0$

$$l \mapsto l' \text{ and } P \mapsto Q$$

2. Find the image Q' of Q under reflection through l'

$$Q' = AQ \text{ with } A = \text{matrix of reflection through } l'$$

3. Shift everything back by $\vec{x}_0$

$$P' = Q' + \vec{x}_0$$

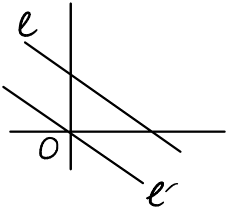
Example 3

Find the image of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ under reflection through $l : x + 2y = 3.$

(1) The image of $\vec{x}$ is $T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0$ with

$\vec{x}_0$ = a point on l = $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

A = matrix of reflection through $l' : x+2y=0$



$A = \frac{2}{\|\vec{d}\|^2} \vec{d}\vec{d}^T - I_2$ with $\vec{d} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$A = \frac{2}{5} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \end{pmatrix} - I_2 = \frac{2}{5} \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3/5 & -4/5 \\ -4/5 & -3/5 \end{pmatrix}$

(2) The image of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is

Example 3

Find the image of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ under reflection through $l : x + 2y = 3$.

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3/5 & -4/5 \\ -4/5 & -3/5 \end{pmatrix}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3/5 & -4/5 \\ -4/5 & -3/5 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/5 \\ 2/5 \end{pmatrix}$$

Comments

- Using 3-step approach, we can find the matrix A and the vector $\vec{b}$ for the cases of projections, reflections, shear, rotations.
- Here is the key formula

$$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0 = A\vec{x} + (\vec{x}_0 - A\vec{x}_0)$$

- We will give a summary on the maps in $\mathbb{R}^2$. The maps in $\mathbb{R}^3$ can be done similarly.

$$T(\vec{x}) = A\vec{x} + \vec{b} \quad \text{with} \quad \vec{b} = \vec{x}_0 - A\vec{x}_0$$

Projection in $\mathbb{R}^2$

Theorem 1

(a) The orthogonal projection onto $l : \vec{x} = \vec{x}_0 + t\vec{d}$ is

$$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0 \text{ with } A = \frac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T$$

(b) The skew projection onto $l : \vec{n} \cdot \vec{x} = c$ in the direction $\vec{v}$ is

$$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0$$

with $A = I_2 - \frac{1}{\vec{v} \cdot \vec{n}}\vec{v}\vec{n}^T$ and $\vec{x}_0 =$ a point on l.

Reflection in $\mathbb{R}^2$

Theorem 2

(a) The orthogonal reflection through $l : \vec{x} = \vec{x}_0 + t\vec{d}$ is

$$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0 \text{ with } A = \frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_2$$

(b) The skew ~~projection~~ reflection through $l : \vec{n} \cdot \vec{x} = c$ in the direction $\vec{v}$ is

$$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0$$

with $A = I_2 - \dfrac{2}{\vec{v} \cdot \vec{n}}\vec{v}\vec{n}^T$ and $\vec{x}_0 =$ a point on l.

Example 4

Consider the line $l : x + 3y = 5$ in $\mathbb{R}^2$.

(a) Describe the projection onto l in the form $T(\vec{x}) = A\vec{x} + \vec{b}$.

Find the image of the line $x - y = 1$ under T.

$\vec{d} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

(a) $T(\vec{x}) = A(\underbrace{\vec{x} - \vec{x}_0}_{\text{step 1}}) + \vec{x}_0$

step 2 step 1 step 3

$\vec{x}_0$ = a point on l $= \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

A = matrix of projection onto $l' : x + 3y = 0$

$A = \frac{1}{\|\vec{d}\|^2} \vec{d}\,\vec{d}^T = \frac{1}{10}\begin{pmatrix} -3 \\ 1 \end{pmatrix}\begin{pmatrix} -3 & 1 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 9 & -3 \\ -3 & 1 \end{pmatrix}$

$T(\vec{x}) = A\vec{x} + (\vec{x}_0 - A\vec{x}_0) = A\vec{x} + \begin{pmatrix} 5 \\ 0 \end{pmatrix} - \frac{1}{10}\begin{pmatrix} 9 & -3 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 0 \end{pmatrix}$

$T(\vec{x}) = \frac{1}{10}\begin{pmatrix} 9 & -3 \\ -3 & 1 \end{pmatrix}\vec{x} + \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix}$

Example 4

Consider the line $l : x + 3y = 5$ in $\mathbb{R}^2$.

(a) Describe the projection onto l in the form $T(\vec{x}) = A\vec{x} + \vec{b}$.

Find the image of the line $x - y = 1$ under T.

The line $m: x - y = 1$ has vector equation $m: \vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Any point on the line has form $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for some t.

Such point is mapped to

$$T(\vec{x}) = \frac{1}{10}\begin{pmatrix} 9 & -3 \\ -3 & 1 \end{pmatrix}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) + \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix}$$

$$T(\vec{x}) = \begin{pmatrix} 9/10 \\ -3/10 \end{pmatrix} + \frac{t}{10}\begin{pmatrix} 6 \\ -2 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix} = \begin{pmatrix} 7/5 \\ 6/5 \end{pmatrix} + \frac{t}{5}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$\Longrightarrow m': \vec{x} = \begin{pmatrix} 7/5 \\ 6/5 \end{pmatrix} + s\begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$\ell : x+3y=5$

(b) Describe the reflection through l in the form $T(\vec{x}) = A\vec{x} + \vec{b}$.
Find the image of the line $x - y = 1$ under T.

$\vec{d} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0$ with $\vec{x}_0 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

A = reflection through $\ell' : x+3y=0$

$$A = \frac{2}{\|\vec{d}\|^2}\vec{d}\,\vec{d}^T - I_2 = \frac{1}{5}\begin{pmatrix} 9 & -3 \\ -3 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ -3 & -4 \end{pmatrix}$$

Hence

$$T(\vec{x}) = A\vec{x} + (\vec{x}_0 - A\vec{x}_0) = A\vec{x} + \begin{pmatrix} 5 \\ 0 \end{pmatrix} - \left(\frac{1}{5}\right)\begin{pmatrix} 4 & -3 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$$T(\vec{x}) = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ -3 & -4 \end{pmatrix}\vec{x} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

(b) Describe the reflection through l in the form $T(\vec{x}) = A\vec{x} + \vec{b}$.
Find the image of the line $x - y = 1$ under T.

The line $m: x - y = 1$ has vector equation $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Its image is

$$m': \vec{x} = \underbrace{\frac{1}{5}\begin{pmatrix} 4 & -3 \\ -3 & -4 \end{pmatrix}}_{A}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) + \underbrace{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}_{\vec{b}}$$

$$m': \vec{x} = \begin{pmatrix} 4/5 \\ -3/5 \end{pmatrix} + \frac{t}{5}\begin{pmatrix} 1 \\ -7 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 9/5 \\ 12/5 \end{pmatrix} + \frac{t}{5}\begin{pmatrix} 1 \\ -7 \end{pmatrix}$$

Shear in $\mathbb{R}^2$

Theorem 3

The shear with respect to the line $l : \vec{n} \cdot \vec{x} = c$ in the direction of shearing vector $\vec{v}$ is given by

$$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0$$

with $A = I_2 + \frac{1}{||\vec{n}||}\vec{v}\vec{n}^T$ and $\vec{x}_0 =$ a point on l.

matrix of shear wrt $l' : \vec{n} . \vec{x} = 0$

Example 5

Describe the shear with respect to $l : 3x + 4y = 10$ in the direction of the shearing vector $\vec{v} = \begin{pmatrix} -8 \\ 6 \end{pmatrix}$. $\vec{n} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$\vec{x}_0$ = a point on ℓ = $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$

$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0$

A = shear wrt ℓ': $3x + 4y = 0$

$$A = I_2 + \frac{1}{\|\vec{n}\|} \vec{v}\,\vec{n}^T = I_2 + \frac{1}{5}\begin{pmatrix} -8 \\ 6 \end{pmatrix}\begin{pmatrix} 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} -24 & -32 \\ 18 & 24 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -19 & -32 \\ 18 & 29 \end{pmatrix}$$

$$T(\vec{x}) = A\vec{x} + (\vec{x}_0 - A\vec{x}_0) = A\vec{x} + \begin{pmatrix} 6 \\ -2 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} -19 & -32 \\ 18 & 29 \end{pmatrix}\begin{pmatrix} 6 \\ -2 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} -19 & -32 \\ 18 & 29 \end{pmatrix}\vec{x} + \begin{pmatrix} 16 \\ -12 \end{pmatrix}$$

Rotation in $\mathbb{R}^2$

Theorem 4

The rotation about the point $\vec{x}_0$ over the angle θ is given by

$$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0 \text{ with } A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Example 6

(a) Describe the rotation about $\vec{x}_0 = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ over $\theta = 30°$.

(b) Find the image of the point $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

(b) Find the image of the line $x - 2y = 2$.

(a) $T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0$ with $\vec{x}_0 = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$, $A = \begin{pmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{pmatrix}$

$= \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$

$T(\vec{x}) = A\vec{x} + (\vec{x}_0 - A\vec{x}_0)$

$= A\vec{x} + \begin{pmatrix} 4 \\ 6 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}\begin{pmatrix} 4 \\ 6 \end{pmatrix}$

$T(\vec{x}) = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}\vec{x} + \begin{pmatrix} -2\sqrt{3} + 7 \\ -3\sqrt{3} + 4 \end{pmatrix}$

(b) Image of $\begin{pmatrix}2\\0\end{pmatrix}$ is

$$T\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}-2\sqrt{3}+7\\ -3\sqrt{3}+4\end{pmatrix} = \begin{pmatrix}7-\sqrt{3}\\ 5-3\sqrt{3}\end{pmatrix}$$

(c) The $\ell: x-2y=2$ has vector equation $\ell: \vec{x} = \begin{pmatrix}2\\0\end{pmatrix} + t\begin{pmatrix}2\\1\end{pmatrix}$

Its image

$$\ell': \vec{x} = \begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix}\left(\begin{pmatrix}2\\0\end{pmatrix} + t\begin{pmatrix}2\\1\end{pmatrix}\right) + \begin{pmatrix}7-2\sqrt{3}\\ 4-3\sqrt{3}\end{pmatrix}$$

$$\vec{x} = \begin{pmatrix}7-\sqrt{3}\\ 5-3\sqrt{3}\end{pmatrix} + t\begin{pmatrix}\sqrt{3}-1/2\\ 1+\sqrt{3}/2\end{pmatrix}$$

Homework exercise

Write each map **projection, reflection, shear, rotation** in 3D as affine transformation

$$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0$$

by specifying the matrix A and the point $\vec{x}_0$ (look at Theorems 1-4).