CSD1241 Tutorial 5 Solutions

Problem 1. Given $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \\ -5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 7 & -8 \\ 9 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \\ 6 & 9 & 10 \\ -2 & 10 & 5 \end{bmatrix}$. Compute

(a) A^TAB (b) $A - B$ (c) $3B - A^TA$ (d) C^TCA

Solution. (a) We have

$$\begin{aligned} A^TA &= \begin{bmatrix} 1 & 3 & -5 \\ -2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 4 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 35 & -20 \\ -20 & 56 \end{bmatrix} \\ A^TAB &= \begin{bmatrix} 35 & -20 \\ -20 & 56 \end{bmatrix} \begin{bmatrix} 7 & -8 \\ 9 & 0 \end{bmatrix} = \begin{bmatrix} 65 & -280 \\ 364 & 160 \end{bmatrix} \end{aligned}$$

(b) Since A have size 3×2 and B has size 2×2, $A - B$ is undefined.
(c) Using the result of A^TA in part a, we have

$$3B - A^TA = \begin{bmatrix} 21 & -24 \\ 27 & 0 \end{bmatrix} - \begin{bmatrix} 35 & -20 \\ -20 & 56 \end{bmatrix} = \begin{bmatrix} -14 & -4 \\ 47 & -56 \end{bmatrix}$$

(d) We have

$$\begin{aligned} C^TC &= \begin{bmatrix} 1 & 3 & 6 & -2 \\ 2 & 0 & 9 & 10 \\ 3 & -1 & 10 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \\ 6 & 9 & 10 \\ -2 & 10 & 5 \end{bmatrix} = \begin{bmatrix} 50 & 36 & 50 \\ 36 & 185 & 146 \\ 50 & 146 & 135 \end{bmatrix} \\ C^TCA &= \begin{bmatrix} 50 & 36 & 50 \\ 36 & 185 & 146 \\ 50 & 146 & 135 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 4 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} -92 & 344 \\ -139 & 1544 \\ -187 & 1294 \end{bmatrix} \end{aligned}$$

□

Problem 2. In this problem, we will learn that in the matrix multiplication AB,

1. the jth column of AB is AB_j, where B_j is the jth column of B
2. the ith row of AB is A_iB, where A_i is the ith row of A

Consider two matrices

$$A = \begin{bmatrix} 1 & 2 & 0 & 6 \\ -2 & -1 & 5 & 0 \\ 7 & 8 & 1 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ 0 & 7 & 1 \\ -1 & -2 & 5 \end{bmatrix}$$

Let A_1, A_2, A_3 be the rows of A and let B_1, B_2, B_3 be the columns of B.
(a) Compute AB.
(b) Verify that

1st, 2nd, 3rd rows of AB are A_1B, A_2B, A_3B, and
1st, 2nd, 3rd columns of AB are AB_1, AB_2, AB_3.

Solution. (a) $AB = \begin{bmatrix} -1 & 0 & 33 \\ -4 & 26 & -1 \\ 22 & 59 & 27 \end{bmatrix}$.

(b) Direct computation verifies that A_1B, A_2B, A_3B are the 1st, 2nd, 3rd rows of AB.

$$\begin{aligned} A_1B &= \begin{bmatrix} 1 & 2 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ 0 & 7 & 1 \\ -1 & -2 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 33 \end{bmatrix} \\ A_2B &= \begin{bmatrix} -2 & -1 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ 0 & 7 & 1 \\ -1 & -2 & 5 \end{bmatrix} = \begin{bmatrix} -4 & 26 & -1 \end{bmatrix} \\ A_3B &= \begin{bmatrix} 7 & 8 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ 0 & 7 & 1 \\ -1 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 22 & 59 & 27 \end{bmatrix} \end{aligned}$$

Similarly, AB_1, AB_2, AB_3 are the 1st, 2nd, 3rd columns of AB.

$$AB_1 = \begin{bmatrix} 1 & 2 & 0 & 6 \\ -2 & -1 & 5 & 0 \\ 7 & 8 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ 22 \end{bmatrix}$$

$$AB_2 = \begin{bmatrix} 1 & 2 & 0 & 6 \\ -2 & -1 & 5 & 0 \\ 7 & 8 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 7 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 26 \\ 59 \end{bmatrix}$$

$$AB_3 = \begin{bmatrix} 1 & 2 & 0 & 6 \\ -2 & -1 & 5 & 0 \\ 7 & 8 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 33 \\ -1 \\ 27 \end{bmatrix}$$

□

Problem 3. Determine which of the following statements are true. Justify your answer (for false statements, you need to give counterexamples).

(a) The (i, j)-entry of AB can be computed by multiplying the ith row of A by the jth column of B.

(b) For every matrix A, it is true that $(A^T)^T = A$.

(c) If A and B are square matrices of the same order, then

$$AB = BA.$$

(d) If A is a 6×4 matrix and B is an $m \times n$ matrix such that $B^T A^T$ is a 2×6 matrix, then $m = 4$ and $n = 2$.

(e) If B has a column of zeros, then so does AB if this product is defined.

(f) If A has a row of zeros, then so does AB if this product is defined.

Solution. (a) **True**.

(b) **True**.

(c) **False**. Here is a counter example (check yourself that $AB \neq BA$)

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

(d) **True**. Since B^T has size $n \times m$ and A^T has size 4×6, the product $B^T A^T$ is defined only if $m = 4$. Now for $m = 4$, $B^T A^T$ has size $n \times 6$. So $n = 2$.

(e) **True**. Same reasoning as part e. Assume A has size $m \times n$ and B has size $n \times p$ with the first column of B consisting of all zeros. The entries in the first column of AB are all zeros.

$$\begin{aligned}
(AB)_{11} &= \text{(1st row of A)} \cdot \text{(1st column of B)} = 0 \\
(AB)_{21} &= \text{(2nd row of A)} \cdot \text{(1st column of B)} = 0 \\
&\dots \\
(AB)_{m1} &= \text{(m-th row of A)} \cdot \text{(1st column of B)} = 0
\end{aligned}$$

(f) **True**. Assume $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times p}$ with the first row of A consisting of all zeros, i.e. $a_{11} = a_{12} = \cdots = a_{1n} = 0$. The entries in the first row of AB are all zeros.

$$\begin{aligned}
(AB)_{11} &= \text{(1st row of A)} \cdot \text{(1st column of B)} = a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} = 0 \\
(AB)_{12} &= \text{(1st row of A)} \cdot \text{(2nd column of B)} = a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1n}b_{n2} = 0 \\
&\dots \\
(AB)_{1p} &= \text{(1st row of A)} \cdot \text{(p-th column of B)} = a_{11}b_{1p} + a_{12}b_{2p} + \cdots + a_{1n}b_{np} = 0
\end{aligned}$$

$\square$

Problem 4. Find λ so that $\det(A) = 0$.

(a) $A = \begin{bmatrix} \lambda - 2 & 1 \\ -5 & \lambda + 4 \end{bmatrix}$ (b) $A = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$ (c) $A = \begin{bmatrix} 1 & 1 & 2 \\ \lambda & -1 & -2 \\ 2 & 3 & 7 \end{bmatrix}$

Solution. (a) We have

$$\det(A) = (\lambda - 2)(\lambda - 4) + 5 = \lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1).$$

So $\lambda = -3$ or $\lambda = 1$.

(b) Note the following formula for 3×3 matrices

$$\begin{aligned}
\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} &= \text{main diagonals} - \text{anti diagonals} \\
&= (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32})
\end{aligned}$$

We have

$$\det(A) = (\lambda - 4)\lambda(\lambda - 1) - (\lambda - 4) \cdot 2 \cdot 3 = (\lambda - 4)(\lambda - 3)(\lambda + 2)$$

So $\lambda = 4$ or $\lambda = 3$ or $\lambda = -2$.

(c) We have

$$\begin{aligned} \det(A) &= (1 \cdot (-1) \cdot 7 + 1 \cdot (2-) \cdot 2 + 2 \cdot \lambda \cdot 3) - (2 \cdot (-1) \cdot 2 + 1 \cdot \lambda \cdot 7 + 1 \cdot (-2) \cdot 3) \\ &= -\lambda - 1 \end{aligned}$$

So $\lambda = -1$. $\square$

Problem 5. Consider 3 vectors $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ \lambda - 1 \\ 3 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ -1 \\ \lambda + 1 \end{bmatrix}$. Let V be the volume of the parallelepiped formed by $\vec{u}, \vec{v}, \vec{w}$.

(a) What is V for $\lambda = 2$?

(b) What is the value of λ so that V has the smallest possible value?

Solution. Note that

$$V = \det \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda - 1 & -1 \\ 0 & 3 & \lambda + 1 \end{bmatrix} = \lambda^2 + 2$$

(a) For $\lambda = 2$, we have

$$V = \lambda^2 + 2 = 2^2 + 2 = 6.$$

(b) Since $V = \lambda^2 + 2 \geq 2$, the smallest possible value of V is $V = 2$. The value $V = 2$ can be achieved at $\lambda = 0$. $\square$