Introduction to Algorithms

Algorithm Analysis

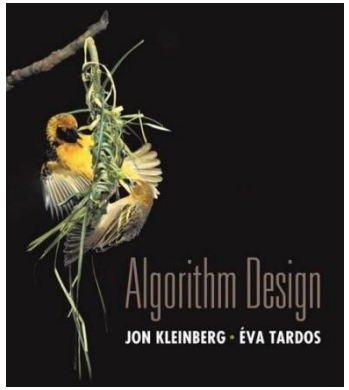
Learning Objectives

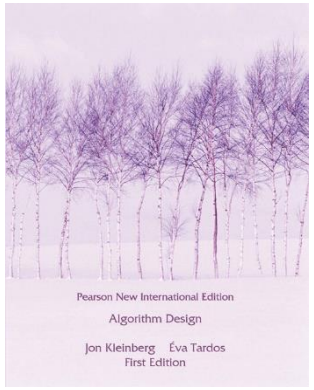
- **Overall**: To promote individual learning and **analytical thinking skills**.

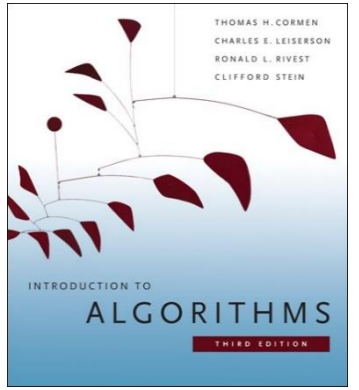
- More detailed objectives:
  1. Be able to **prove correctness** of algorithms.
     - Some people spend their whole career to prove something.
  2. Have deep understanding of differences and advantages of **iterative and recursive algorithms**.
  3. Have a knowledge base of (many) **existing algorithms**.
  4. Understand **speed vs. space trade-off**, importance of **data-structures**, and **data preprocessing**.
     - Search for an item: hash, tree, and traverse all.
  5. Be able to **design new algorithms**.
     - Correctness.
     - Efficiency.

Overview

- Acknowledgement: **Dr. Zhang Wei, Dr. LIU Fang, Dr. Donny** - course materials
- Appointment by email:
  - kan.chen@singaporetech.edu.sg
  - AF_Alfred.Whang@singaporetech.edu.sg
  - Pohkok.Loo@singaporetech.edu.sg
- References:
  - **Algorithm Design**, Jon Kleinberg and Eva Tardos.
  - **Introduction to Algorithms**, Thomas Cormen, Charles Lieserson, *et al*.
  - Introduction to the Design and Analysis of Algorithms, Anany Levitin.
  - Course materials and activities.







Source: https://kwize.com/quote/9797; https://eden.uktv.co.uk/animals/birds/article/weaver-birds/

Tentative Dates and Grading Policy

- **Marks:**
  - 28% - Assignment (group of 5/6 students)
  - 12% - Tutorial participation/quiz – no makeup
  - 30% - Midterm
  - 30% - Final Exam

- Part 1: week 1 to 6 (Loo)
  - 1 assignment (group 14%), week 5, Due on 30/9 23:00
  - 5 tutorial participation/quiz (6%), during each tutorial
  - test 1 (30%), week 6, 8/10 between 13:00-17:00 (1 hour)

- Part 2: week 8 to 13 (Chen Kan)
  - 1 assignment (group 14%), week 12, 18/11 23:00
  - 5 tutorial participation/quiz (6%), during each tutorial
  - Test 2 (30%), week 14, TBA

Tentative Dates and Grading Policy

- Blended learning
  - All lecture content/video/zoom recording will be posted before the tutorial.
  - The key face to face session is during the tutorial where you will work on tutorial questions, ask questions and complete quizzes.

- Form your own team using the excel file I have provided.

  CSD3130,3131 Y2024T1 (team).xlsx

  - Once this is done, I will need your team to enrol yourself in xSit. More information will be provided for the enrolment in xSit.

Tentative Dates and Grading Policy

- Quizzes
  - There will be quiz in each tutorial session in class via xSit with lockdown browser. There is no makeup for the quiz.

- Assignment
  - This is done as a team of 6 students. There will be a self and peer evaluation where you will evaluate yourself and your team members to reflect the contribution of each member after submission. The final score will be scaled according to the evaluation result.

Stable Matching: Example

- History: A self-enforcing college admission process, or job recruiting process.
  - Each **student** has his/her **preference list** of the universities.
  - Each **university** has her **preference list** of the students.
  - Normal that the preferences **do not align** perfectly.
  - **How** to make both sides **satisfied**?

Stable Matching: Example

- **Example** to be more intuitive (in time order):
  - (May 1) Jon accepted the offer from Apple.
  - (May 2) Google offers Jon and Jon likes Google more.
  - (May 3) Jon retracts his acceptance of Apple and accepts Google's offer. (Apple has one free slot now.)
  - (May 4) Apple soon offers David and he likes Apple more.
  - (May 5) David retracks his Facebook offer and accepts Apple's offer. (Facebook has one free slot now.)
  - (May 6) Facebook moves to X, and X turns down Y and accepts Facebook.
  - (May 7) Y moves to Z ... Situation gets **out of control**.

Stable Matching: Example

- Things can be different, possibly even worse.
  - Jon, David and the others have communication channels.
  - They hear something and accordingly do something.
  - The **network** of the communication will be **fully connected** with around $O(n^2)$ links.
    - Suppose 50 students, 50 companies, then we have 10,000 communication links.
    - Each link can trigger a change of the offer distribution.
    - **Chaos** happens. Everyone tired and unhappy.
- Basic issue?
  - People **cannot act in their self-interest**, otherwise system may crash.

Source: http://ictusmarketing.com/blog/2014/5/23/what-happens-when-the-buyer-knows-more-than-the-seller

Stable Matching: Example

- Basic issue?
  - People **cannot act in their self-interest**, otherwise system may crash.
- Our desired process – **stable situation**:
  - Use **intelligence** to **guide** the process.
  - **Idea: self-interest prevents retracting offers.**
    - Company, with its offers all being accepted, gets a call from another applicant
    - -> “No, we prefer each applicant we have accepted than you, so we will not change.” **Chaos avoided**.
    - Student, with an offer, gets a call from a university.
    - -> “No. I am happy where I am.” No chaos.

Source: http://ictusmarketing.com/blog/2014/5/23/what-happens-when-the-buyer-knows-more-than-the-seller

Stable Matching

- Question: given employer E and applicant $A \notin E$, can we have:
  - E **prefers her accepted applicants** than A (another applicant).
  - A **prefers his/her current employer** over working for E.
- Obj: design **algorithm** -> produce **stable situations**.

- Similar application: residents and hospitals matching.
  - Even a decade earlier than the employee and employer one.
  - Still widely used today in USA.
- Intuition: personal interest & system interest.
- **Problem formulation -> algorithm design -> algorithm analysis.**

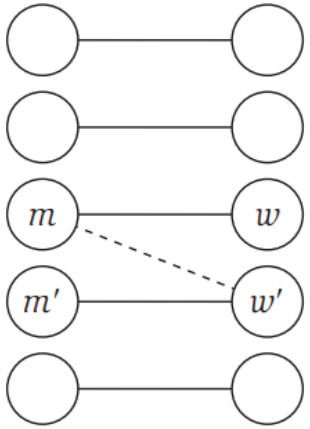
Source: https://www.channelfutures.com/files/2013/12/secretexchange_0.jpg; https://builtin.com/sites/default/files/styles/og/public/2019-04/Machine-Learning-in-Healthcare.png

Problem Formulation

- Problem: As **clean** as possible. As **simple** as possible. **Improve** step by step.
  - Apply to programming as well.
- Complex situation: a company can hire multiple employees.
  - 1-to-many relationship.
- Simple: **marriage problem**.
  - A man can only marry a woman.
  - A woman can only marry a man.
  - **1-to-1 relationship**.
- Math definition.
  - n men: $M = \{m_1, m_2, \cdots, m_n\}$.
  - n women: $W = \{w_1, w_2, \cdots, w_n\}$.
  - **Matching**: S, a set of pairs, pair $(m, w) \in S$ has a man $m \in M$ and a woman $w \in W$.
    - Requirement: each man/woman only appears **at most once** in S.
  - **Perfect matching**: everyone married & married to one person. Or $|M| = |W| = |S| = n$.

Problem Formulation

- **Preference**: each man m ranks all women. The order is m's **preference list**.
  - m prefers w to w' if he ranks w higher than w'.
  - Ties not allowed.
  - Each woman w also ranks all the men, also her preference list.
- Perfect matching all we want? (All men and women in the matching)
- **Unstable pair** (m,w) and (m',w') in a perfect matching S:
  - m prefers w'.
  - w' prefers m.
  - (m,w') is called an **instability**. The pair is not in S.
  - m and w' prefer each other but they are assigned to a different person
  - Objective: find a **perfect matching, without** any **instability** -> **stable matching**.
- Questions:
  - Is stable matching **possible?**
  - If true, **how** to find it fast?



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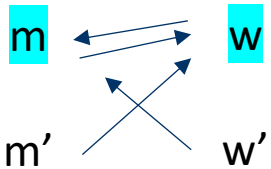
Possible – Proven with Examples

- Two men m, m' and two women w, w'.
  - m prefers w to w'
  - m' prefers w to w'
  - w prefers m to m'
  - w' prefers m to m'



Focus is the pair where they want each other (m,w)

n = 2, n^2 possibilities, 2^2=4
(m,w), (m,w'), (m',w), (m',w')
6 possible combinations
(m,w), (m,w') – not possible
(m,w), (m',w) – not possible
(m,w), (m',w') - ok
(m,w'), (m',w) – ok, but unstable
(m,w'), (m',w') – not possible
(m',w), (m',w') – not possible

- Do we have complete agreement?
  - The men agree on the order of women.
  - The women agree on the order of men.
  - **Unique stable matching**: (m, w) and (m', w').
    - Although m' prefers w, w didn't prefer m'
- Another perfect matching (m, w') and (m', w).
  - Is this stable? No (m and w prefer each other)
- A more complicated example in the next slide.

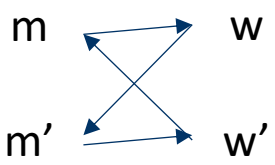
m and w wants each other but now they are both refused their choice and with others

We have a situation where 2 parties want each other but yet they have someone else

https://movies.disney.com/aladdin-2019

Possible – Proven with Examples

- A more complicated example.
  - m prefers w to w'
  - m' prefers w' to w
  - w prefers m' to m
  - w' prefers m to m'



No situation where both parties want each other but yet get someone else

- Men's preferences meshes well -> (m,w) and (m',w') perfect for men.
- Women's preferences meshes well -> (m,w') and (m',w) perfect for women.
- Any problem?
  - Both are **stable matchings**, but they **clash**.
  - No happy ending for all men and women.
- Conclusion: ≥ **1 stable matching possible**.

(m,w), (m,w'), (m',w'), (m',w)
6 possible combinations
(m,w), (m,w') – not possible
(m,w), (m',w') – ok (fr men)
(m,w), (m',w) – not possible
(m,w'), (m',w') – not possible
(m,w'), (m',w) – ok (fr woman)
(m',w'), (m',w) – not possible

https://i.pinimg.com/originals/be/2c/73/be2c73938c39dfa2a8701fedc6492b91.gif

Design the Algorithm

- Our plan:
  - Show that a **stable matching exists**.
  - Propose an **efficient algorithm** accordingly.
- Some basic ideas – 3 Stages.
  - **Free**. In the beginning.
  - **Engagement**. Intermediate state. (Still can change)
  - **Married**. The algorithm or matching terminates. Engagements declared final.
- Why engage?
  - Bob likes Alice the most and proposes to her.
  - Shall Alice accept?
    - Accepts: what if Alice loves Jon the most and Jon proposes to her later? Regret?
    - If not, what if Bob is the best she can get? Lonely in the end?
  - **Engagement** is helpful to deal with the above situation.
- Procedure: all free in the beginning, eventually get engaged, and finally married.

https://www.cartier.sg/?gclid=Cj0KCQjw2PP1BRCiARIsAEqv-pQfx7w6lmf_nHEQkUPGq77fndswenymmP3fzaWULRVlLTxOOsWuo3gaAtVCEALw_wcB&gclsrc=aw.ds

Gale-Shapley Algorithm

```
Initially all m ∈ M and w ∈ W are free
While there is a man m who is free and hasn't proposed to
every woman
   Choose such a man m
   Let w be the highest-ranked woman in m's preference list
      to whom m has not yet proposed
   If w is free then
      (m, w) become engaged
   Else w is currently engaged to m'
      If w prefers m' to m then
         m remains free
      Else w prefers m to m'
         (m, w) become engaged
         m' becomes free
      Endif
   Endif
Endwhile
Return the set S of engaged pairs
```

- Can return a **stable matching?**
- How to **prove** our judgement?

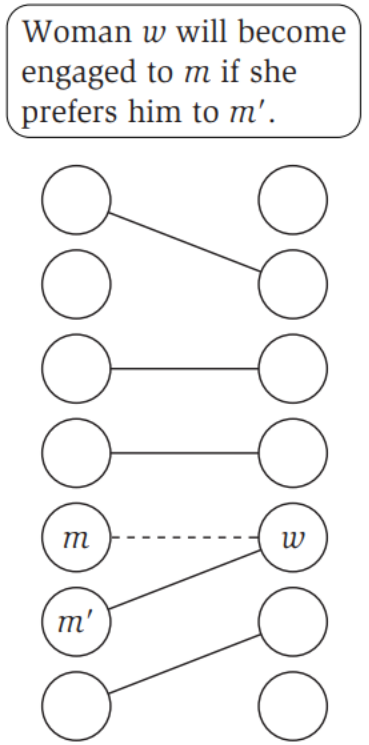


Figure 1.2 An intermediate state of the G-S algorithm when a free man m is proposing to a woman w.

Kleinberg J, Tardos E. Algorithm design. Pearson Education; 2006.

Demo 1

Iteration	Man Proposes to	Woman	Engagement	Free Males	Reason
0				A, B, C	
1					
2					
3					
4					
5					
6					

Man	Preference List	Woman	Preference List
A	w_1, w_2, w_3	w_1	C, B, A
B	w_1, w_2, w_3	w_2	C, B, A
C	w_1, w_2, w_3	w_3	C, B, A

Demo 1 - Result

Male proposes, but woman get to select the most preferred man

Iteration	Man Proposes to	Woman	Engagement	Free Males	Reason
0				A, B, C	
1	A	w_1	(A, w_1)	B, C	1st proposal to w_1
2	B	w_1	(B, w_1)	A, C	w_1 likes B more than A
3	A	w_2	(B, w_1), (A, w_2)	C	1st proposal to w_2
4	C	w_1	(C, w_1), (A, w_2)	B	w_1 likes C more than B
5	B	w_2	(C, w_1), (B, w_2)	A	w_2 likes B more than A
6	A	w_3	(C, w_1), (B, w_2), (A, w_3)	END	1st proposal to w_3

Man	Preference List	Woman	Preference List
A	w_1, w_2, w_3	w_1	C, B, A
B	w_1, w_2, w_3	w_2	C, B, A
C	w_1, w_2, w_3	w_3	C, B, A

Demo 2 (Let Women propose for fairness)

Iteration	Wm Proposes to	Man	Engagement	Free Wms	Reason
0				A, B, C	
1					
2					
3					
4					
5					
6					

Wms	Preference List	Man	Preference List
A	m_1, m_2, m_3	m_1	A, B, C
B	m_1, m_2, m_3	m_2	A, B, C
C	m_1, m_2, m_3	m_3	A, B, C

Demo 2 - Result

Woman proposes, but man get to select the most preferred woman

Iteration	Wm Proposes to	Man	Engagement	Free Wms	Reason
0				A, B, C	
1	A	m_1	(A, m_1)	B, C	1st proposal to m_1
2	B	m_1	(A, m_1)	B, C	m_1 (with A) rejects B
3	B	m_2	(A, m_1), (B, m_2)	C	1st proposal to m_2 (SUC)
4	C	m_1	(A, m_1), (B, m_2)	C	m_1 (with A) rejects C
5	C	m_2	(A, m_1), (B, m_2)	C	m_2 (with B) rejects C
6	C	m_3	(A, m_1), (B, m_2), (C, m_3)	END	1st proposal to m_3 (SUC)

Wms	Preference List	Man	Preference List
A	m_1, m_2, m_3	m_1	A, B, C
B	m_1, m_2, m_3	m_2	A, B, C
C	m_1, m_2, m_3	m_3	A, B, C

Analyze the Algorithm

- (Assume men proposing)
- **Fact 1.1**. **Woman** w remains **engaged** from the point at which she receives her first proposal; and the sequence of **partners** to which she is engaged **gets better and better**.
  - From the woman's perspective.
  - Only changes her mind if a **more preferred man proposes**.
- **Fact 1.2**. The seq. of women to whom **man** m proposes **gets worse** and worse.
  - From the man's perspective.
  - Only proposes again if he is free or his engaged **more preferred woman left** him.
- **Fact 1.3**. The algorithm **terminates** after **at most** $O(n^2)$ **iterations**.
  - Proof **trick**: focus on the **terminate condition** of the while loop.
    - One man proposes to one different woman (he never proposed to) in each iteration.
    - How many men? How many woman? n x n

n = 2, n^2 possibilities, 2^2=4
(m,w), (m,w'), (m',w), (m',w')

Analyze the Algorithm

- **Fact 1.4**. If **man** m is **free** at some point in the execution of the algorithm, then there is a **woman** to whom he has not **yet proposed**.
  - Proof idea: Contradiction analysis. Suppose the man has proposed to all, but now he is free. The only possibility is that all his proposed women are engaged with other men.
  - We have n women, engaged to n men, which does not include the man m. Now, we have $n+1$ men. But it cannot be true.
- **Fact 1.5**. The set S returned at termination is a **perfect matching**.
  - Proof idea: The algorithm terminates only if there is no free man. So, all men and women are in the matching.

Analyze the Algorithm

- **Fact 1.6**. The algorithm returns a **stable matching**.
  - Proof idea: at least know it is perfect, so prove there is **no instability**.
    - **Contradiction** analysis again. Suppose we really have one:
    - (m, w) and (m', w') that:
    - Man m prefers w' more && woman w' prefers m more → (m, w') ?
    - If man m is together with w, he must have proposed to w' before, as he like w' more.
    - So w' rejected m for m'? No way, as w' like m more than m'. (Fact 1.1)

Wms	Preference List	Man	Preference List
w		m	w', w
w'	m, m'	m'	

SINGAPORE INSTITUTE OF TECHNOLOGY

Computational Tractability

Computational Tractability

- **Resources** in computer systems?
  - **Time, space, communication.** [https://ieeexplore.ieee.org/document/9063714]
  - Which one is the most important?
    - We **focus** on **time**, and space is also considered.
    - Why do we consider space?
- Obj: **quantify** computational tractability. How it **changes**?
  - Proposing an algorithm is easy.
  - An **efficient** (and correct) algorithm will be challenging.

https://analyticsindiamag.com/wp-content/uploads/2019/10/computer-types-10.jpg

https://i.pcmag.com/imagery/lineups/02SYcRWCQLDk6G1bnereOVv-2.fit_lim.v_1585232946.jpg

https://miro.medium.com/max/600/0*AgiPFgJrm9GQfzKJ.jpg

Computational Tractability

- Informal efficiency def.: **runs quickly** on real input.
  - Problem 1: run the algorithm in what environment?
  - Problem 2: how fast is fast?
    - Special case: a poor-designed algorithm runs super fast with tiny data input
    - Special case: a well-designed algorithm runs super slow with huge data
  - Problem 3: what is real input? Crash with large input?
- We want: **platform-independent** and **instance-independent**.

https://analyticsindiamag.com/wp-content/uploads/2019/10/computer-types-10.jpg

https://i.pcmag.com/imagery/lineups/02SYcRWCQLDk6G1bnereOVv-2.fit_lim.v_1585232946.jpg

https://miro.medium.com/max/600/0*AgiPFgJrm9GQfzKJ.jpg

Worst-Case Running Time

- Quantify the input size: N
  - n men and n women (eg. n=3)
  - $2n$ preference lists, each list of length n (eg. 2x3 plist x 3)
  - $N = 2n \cdot n = 2n^2$
- Task: analyze the **running time as a function of N**.
- **Worst-case** running time: the largest possible running time with input N.
- **Benchmark**: **brute-force** algorithm (brainless design).

Man	Preference List	Woman	Preference List
A	w_1, w_2, w_3	w_1	C, B, A
B	w_1, w_2, w_3	w_2	C, B, A
C	w_1, w_2, w_3	w_3	C, B, A

https://stickershop.line-scdn.net/stickershop/v1/product/1510436/LINEStorePC/main.png;compress=true

Redefine Efficiency

- Informal efficiency def.: achieves **better worst-case** performance, at an **analytical** level, than **brute-force** search.
- **Polynomial** time: quantify "reasonable" running time.
  - **Search space**: often grows **exponentially**.
  - **Reasonable** may be: from n to $n+1$, running time increases by a **constant**.
- Informal efficiency def.: an alg. is **efficient** if runs in **polynomial** time.
  - Extreme cases exists like the worst-case concept, i.e., $n^{10000000}$?
  - Again, in reality, the polynomial concept works well.

Running Times

n is the number of data points

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

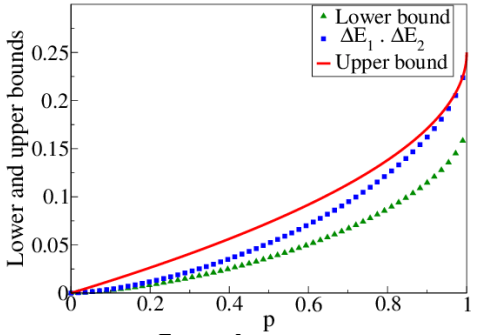
- Assumptions:
  - 1m instructions per second
  - Very long -> runs over 10^{25} years.
- Intelligence does not necessarily produce polynomial algorithms.

1,000,000 instructions per second (1mins = 60secs)
$n^3 = 1{,}000^3 = 1{,}000{,}000{,}000$
1,000,000,000 / 1,000,000 = 1,000secs x 1/60 = 17mins

Kleinberg J, Tardos E. Algorithm design. Pearson Education; 2006.

Asymptotic Order of Growth – Upper Bound

- Target: **worst-case** time, given input n, grows at a rate **at most proportional to** $f(n)$.
  - We also call $f(n)$ a **bound**.
- **Too specific expressions unnecessary**, i.e., runs in $1.62n^2 + n + 4$ steps.
  - Constants vary for different architectures
  - Determined by the highest order (e.g. n^2)
- How do you plan to quantify the above terms concisely?
  - We **ignore** the **constants** and We **ignore** the **lower order** items.
- Asymptotic Upper Bounds $O(\cdot)$:
  - The **worst-case** running time $T(n)$ of a certain algorithm on an input of size n.
    - Expresses only an **upper bound**.
  - $f(n) = pn^2 + qn + r$ for constants p, q, and r → $O(n^2)$.

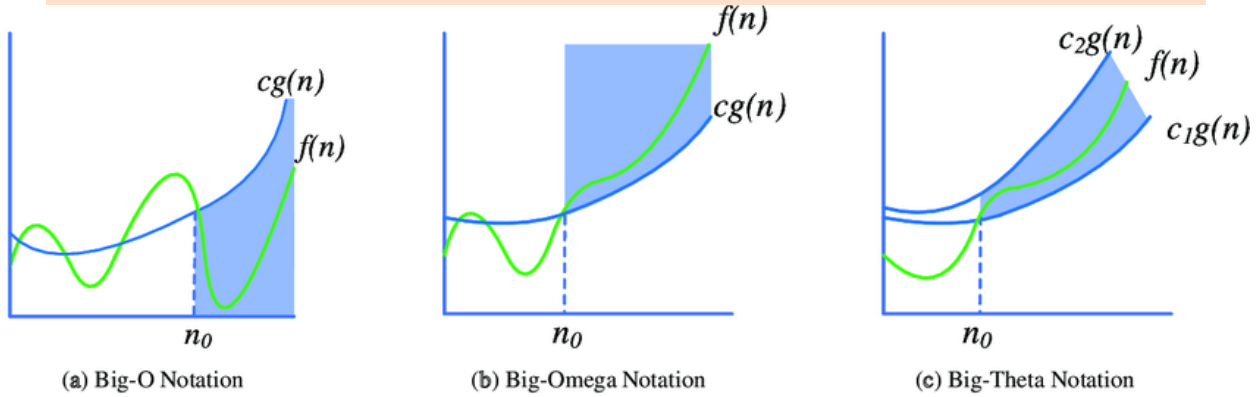


https://www.researchgate.net/profile/Uttam_Singh7/publication/263048941/figure/fig3/AS:667047482630161@1536047921910/Color-online-The-lower-and-upper-bounds-on-E1E2-Here-both-the-axes-are.png (Tue. Till Here)

Asymptotic Order of Growth – More bounds

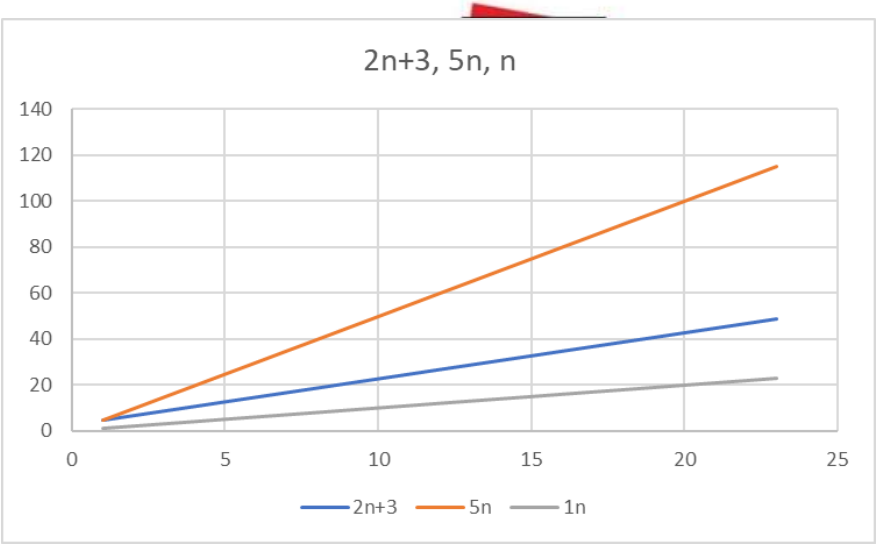
- How to justify our proven bound $O(\cdot)$ is good?
  - i.e., we are almost always true if we say it is $O(n^{100000n})$.
- Asymptotic Lower Bounds : $\Omega(n)$.
  - The func. $T(n)$ is **at least a constant "c" multiple** of $f(n) = c.g(n)$.
  - $T(n) = pn^2 + qn + r$ for constants p, q, and r → $\Omega(n^2)$.
- Asymptotically Tight Bounds $\Theta(n)$: $f(n) = \Omega(g(n)) = O(g(n))$ (optimal bound)
  - i.e., $f(n) = pn^2 + qn + r = \Omega(n^2) = O(n^2) = \Theta(n^2)$.
- Note: consider $n \to \infty$.
  - Only matters when n is large.
  - $f(n) = \Theta(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$.
  - c is a positive constant.

c.g(n) is the estimated bound



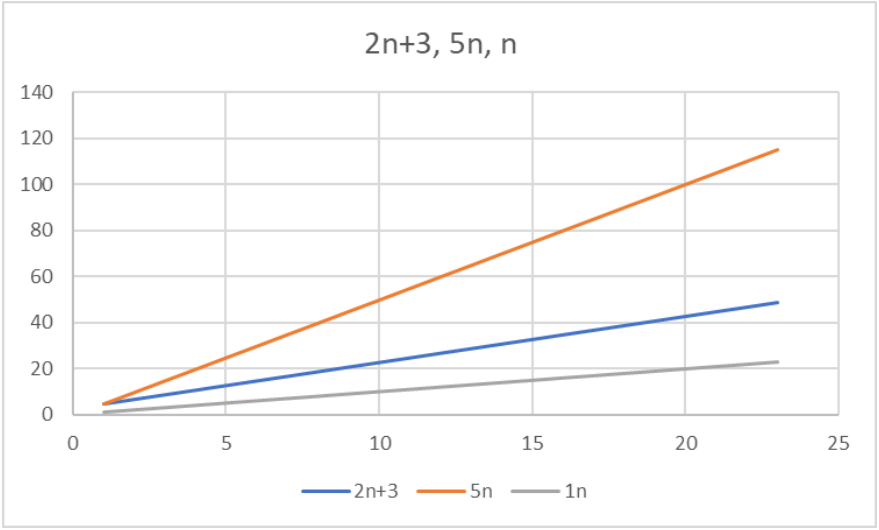
(a) Big-O Notation (b) Big-Omega Notation (c) Big-Theta Notation

https://www.researchgate.net/publication/329846158/figure/fig2/AS:706473407565824@1545447794620/Asymptotic-upper-lower-and-tight-bounds-of-a-function-from-a-c.png



- f(n) = 2n + 3
- f(n) = O(g(n)) where g(n) = n
- f(n) <= c_1.g(n) = c_1. n where n>n_o, c_1>0, n_o>=1
- 2n+3 <= c_1.n where the RHS, big-O is larger than actual function
- If n_o=1 and c_1 = 5, then 5 <= 5: ok
- If n_o=2 and c_1 = 5, then 7 <= 10: ok
- Conclude that the big-oh of the function 2n+3 is O(n) where c_1=5, n_o>=1. This is the best worst case of this algorithm
- f(n) = Ω(g(n)) where g(n) = n
- f(n) >= c_2.g(n) = c_2. n where n>n_o, c_2>0, n_o>=1
- 2n+3 >= c_2.n where RHS, big-omega is smaller than the actual function
- If n_o=1 and c_2 = 1, then 5 >= 1: ok
- If n_o=2 and c_2 = 1, then 7 >= 2: ok
- Conclude that the big-omega of the function 2n+3 is Ω(n) where c_2=1, n_o>=1. This is the best case of this algorithm (the algo will never be better than this)

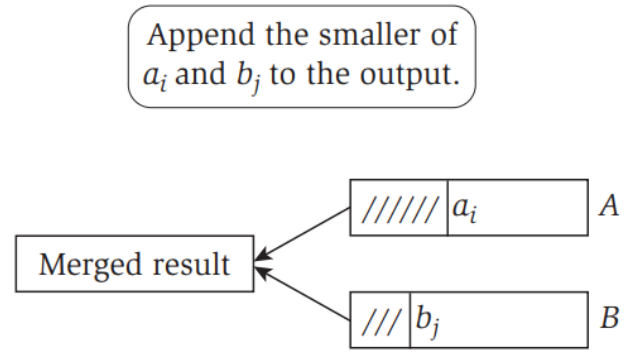
- f(n) = 2n + 3
- f(n) = θ(g(n)) where g(n) = n
- c_2. n = c_2.g(n) <= f(n) <= c_1.g(n) = c_1. n where n>n_o, c_1,c_2 >0, n_o>=1
- 2n+3 <= c_1.n where the RHS is the worst case
- Conclude that the worst case of the function 2n+3 is θ(n) where c_1=5, n_o>=1. This is the best worst case of this algorithm
- c_2.n <= 2n+3 where LHS, is the best case
- Conclude that the best case of the function 2n+3 is θ(n) where c_2=1, n_o>=1. This is the best case of this algorithm (the algo will never be better than this)
- theta is the average case of the algorithm



Common Times - $O(n)$

- Long-time goal: recognize **common styles** of analyzing the running time.
- **Linear time**: $\boldsymbol{O(n)}$, time scales by a constant factor times the input size n.
- **Find the maximum**: $a_1, a_2, \cdots, a_n$.
  - Each needs a comparison, in constant time.
  - $O(n) * O(1) = O(n)$.
- **Merge sorted lists**. The same $O(n)$, but more complex.
  - List $a_1, a_2, \cdots, a_n$ and list $b_1, b_2, \cdots, b_n$, both in ascending order.
  - Objective: $c_1, c_2, \cdots, c_{2n}$.
    - i.e., 2, 3, 11, 19 and 4, 9, 16, 25 → 2, 3, 4, 9, 11, 16, 19, 25.
  - A non-intelligent solution: concatenate the lists → sort all the items.
  - Record pa, pb, and pc, where $pa = pb = pc = 1$ in the beginning.
  - While $pa \leq n$ and $pb \leq n$:
    - If $a_{pa} > b_{pb}$ → smaller b_{pb} to c_{pc} → $pb++$ and $pc++$.
    - If $a_{pa} \leq b_{pb}$ → smaller a_{pa} to c_{pc} → $pa++$ and $pc++$.
  - Whatever left, append to c.

```
max = a_1
For i = 2 to n
  If a_i > max then
    set max = a_i
  Endif
Endfor
```



Common Times - $O(n \log n)$, $O(n^2)$, $O(n^3)$, $O(n^k)$

- $O(n \log n)$: the time of any alg. **splits input** into **equal-sized pieces**, **solves** each **recursively**, and **combines** the two **solutions** in linear time.
  - Popular example: **sorting**.
  - Analyze the basic idea: sorting is expensive -> sort less -> combine.
    - i.e., $100^2 = 10{,}000$ -> $2 \times 50^2 = 5{,}000$ -> $5{,}000 + 100 = 5{,}100$, around half time.
- $O(n^2)$: i.e., n points in a plane, what the **nearest points**?
  - Brute-force sol.: for each pt, compute the dist. between it and another pt.
  - $O(n)$ points, each $O(n)$ distances, each distance $O(1)$ -> $O(n^2)$.
- $O(n^3)$: nested loops, i.e., 3-variable solver: $2x + 5y + 9z = 100$.
  - For x -> for y -> for z -> check the sum.
- $O(n^k)$: Any idea on getting $O(n^k)$?
  - Independent sets in a graph with k nodes.
  - Brute-force, choose k from n: $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdots k} \leq \frac{n^k}{k!} \leq n^k$.

Beyond Polynomial Time

- $\boldsymbol{O(2^n)}$ vs. $\boldsymbol{O(n!)}$. Which one is faster?
  - $\boldsymbol{O(2^n)}$: the number of the subsets of n nodes.
  - $\boldsymbol{O(n!)}$: the number of perfect matchings; traveling salesman problem.
- **Sublinear time**: better than linear, i.e., $O(\log n)$.
  - Binary search.
  - **Query** the inputs instead of traversing them.
  - Half → half → half → half …