Integration by Substitution

Dr. Ronald Koh
ronald.koh@digipen.edu (Teams preferred over email)

AY 23/24 Trimester 1

Table of contents

Antiderivatives, FTC 1 and 2

- An antiderivative F of a function f satisfies $F'(x) = f(x)$.
- Any two antiderivatives of a function f differ by a constant.
- An indefinite integral $\displaystyle\int f(x)\,dx$ is a common way of writing (a family of) antiderivatives of f.
- FTC1: Differentiation and integration are inverse processes.
- FTC2 allows us to evaluate a definite integral of f from a to b by using **any** antiderivative F of f (always the using the antiderivative with $C = 0$):
$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Inverse of sine/tangent versus reciprocal of sine/tangent

We covered the inverse trigonometric functions $\sin^{-1} x$ and $\tan^{-1} x$. These functions are the **inverses** of $\sin x$ and $\tan x$ respectively.

One common confusion is the subtle similarity to the **reciprocal** of sin and tan, $\dfrac{1}{\sin x}$ and $\dfrac{1}{\tan x}$. For clarification,

$$\sin^{-1} x \neq \frac{1}{\sin x} \quad \text{and} \quad \tan^{-1} x \neq \frac{1}{\tan x}.$$

If at the end, you're still confused, you may choose to write the inverse trigonometric functions $\sin^{-1} x$ and $\tan^{-1} x$ as

$$\arcsin x \quad \text{and} \quad \arctan x,$$

both forms will be accepted for this course.

Recap of the Chain Rule

Theorem (Chain Rule)

When g is differentiable at x and f is differentiable at $g(x)$, then $f \circ g$ is differentiable at x and the derivative of $f \circ g$, $(f \circ g)'$ is given by

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

What happens if we integrate both sides of this equation?

'Reversing' the Chain Rule

When we integrate the LHS of the following equation

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x),$$

we get $(f \circ g)(x)$ (we set the constant to be 0) because differentiation and integration are inverse processes. For the RHS, integrating gives

$$\int f'(g(x)) \cdot g'(x)\, dx.$$

'Reversing' the Chain Rule

This yields

$$(f \circ g)(x) = \int f'(g(x)) \cdot g'(x)\, dx.$$

This chapter revolves around being able to observe the structure of the integrand and applying appropriate substitutions to get from RHS to LHS.

Advice: Being able to recognize that the integrand is of the structure $f'(g(x)) \cdot g'(x)$ is **NOT EASY.** Tonnes of practice are needed to learn how to recognize. Tutorial 3 will contain a large amount of practice. Should you feel it is not enough, you can attempt the rest of the exercises (1 - 80) in Chapter 5.5 of the textbook.

Observational exercise 1

For the following examples, choose suitable functions $f'(x)$ and $g(x)$ so that the example looks exactly like $f'(g(x)) \cdot g'(x)$.

$$2x(x^2 + 1)^2$$

Observational exercise 2

For the following examples, choose suitable functions $f'(x)$ and $g(x)$ so that the example looks exactly like $f'(g(x)) \cdot g'(x)$.

$$x^2 \sin(x^3 + 3)$$

Observational exercise 3

For the following examples, choose suitable functions $f'(x)$ and $g(x)$ so that the example looks exactly like $f'(g(x)) \cdot g'(x)$.

$$x^4 e^{x^5}$$

Observational exercise 4

For the following examples, choose suitable functions $f'(x)$ and $g(x)$ so that the example looks exactly like $f'(g(x)) \cdot g'(x)$.

$$\frac{1}{x(\ln(x))^2}$$

u - substitution

We can use a substitution to get from RHS to LHS.
Let $u = g(x)$, then $du = g'(x)\,dx$ so that

$$\begin{aligned} \text{RHS} = \int f'(\underbrace{g(x)}_{u}) \cdot \underbrace{g'(x)\,dx}_{du} &= \int f'(u)\,du \\ &= f(u) \\ &= f(g(x)) = \text{LHS}. \end{aligned}$$

This is why integration by substitution is often referred to as u–substitution, because this technique requires us to identify an appropriate u for substitution. After performing a substitution, you should expect that the integral in u should be **easier** to evaluate than the integral in x previously.

Substitution rule for indefinite integrals

Theorem

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f'(g(x)) \cdot g'(x)\, dx = \int f'(u)\, du.$$

Example 1

Evaluate $\int 3x^2 \cos(x^3 + 2)\, dx$.

Example 2

Evaluate $\displaystyle\int \frac{x^3}{1+x^4}\,dx.$

Example 3

Evaluate $\displaystyle\int \sin^2 x \cos x \, dx$.

Example 4

Evaluate $\displaystyle\int \sec^2 x \cdot e^{\tan x}\, dx$.

Exercise 1

Evaluate the following integrals.

1. $\displaystyle\int 4x^2 \sin(x^3 + 5)\, dx$
2. $\displaystyle\int \frac{\cos x}{\sin x}\, dx$
3. $(*) \displaystyle\int \frac{x^2}{\sqrt{1 - x^6}}\, dx$

Exercise 1

Definite integrals

The substitution rule for definite integrals work similarly to indefinite integrals, but also we have limits of integration to care about now.

$$\int_a^b f'(g(x)) \cdot g'(x)\, dx$$

In an earlier slide, $f(g(x))$ is an antiderivative for $f'(g(x)) \cdot g(x)$, so we apply FTC2 to get

$$\int_a^b f'(g(x)) \cdot g'(x)\, dx = f(g(b)) - f(g(a)).$$

But the RHS of this equation can also be interpreted as

$$\int_{g(a)}^{g(b)} f'(u)\, du,$$

thus we have the substitution rule for definite integrals.

Substitution rule for definite integrals

Theorem

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f'(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f'(u)\,du.$$

Common mistake: Do not change the positions of $g(a)$ and $g(b)$!

$$\int_0^\pi \cos^2 x \sin x\,dx$$

Example 5

Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} e^{\sin x} \cos x \, dx.$

Example 6

Evaluate $\displaystyle\int_0^3 \frac{x^2}{1+x^3}\,dx$.

Example 7

Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tan^4 x \sec^2 x \, dx$.

Example 8

Evaluate $\displaystyle\int_{-1}^{1} \frac{x^2}{\sqrt{1-x^6}}\, dx.$

Exercise 2

Evaluate the following integrals.

1. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^6 x \cos x \, dx$
2. $\displaystyle\int_0^1 \frac{e^x}{1+e^{2x}} \, dx$
3. $(*) \displaystyle\int_0^1 \frac{2x+3}{1+x^2} \, dx$

Exercise 2