Numerical Integration Part 2
Mid-trimester Revision

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AY 23/24 Trimester 1

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Superior to Midpoint & Trapezoidal

Irreducible factors, Midpoint and Trapezoidal Rule

similar to nonrepeated linear factors

- Factorization of denominator $Q(x)$:
  - Non-repeated irreducible factors: one partial fraction for each irreducible factor, numerator is a linear polynomial $Ax + B$. Difference
- Some definite integrals cannot be evaluated exactly, establishes a need for **approximation**; suppose we have a function $f(x)$ on $[a, b]$.

  Let n be a positive integer with $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$, then
  - Midpoint Rule with midpoints $\overline{x}_i = \dfrac{x_i + x_{i-1}}{2}$: mid point of $[x_{i-1}, x_i]$

    Riemann Sum
    $$\int_a^b f(x)\,dx \approx M_n = \Delta x\,[f(\overline{x}_1) + \cdots + f(\overline{x}_n)]$$
  - Trapezoidal Rule:

    $x_i = a + i\Delta x$

    avg of left & right Riemann sum
    $$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\,[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]\,.$$

Error bounds

f, f', f'', f'''

bound for second derivative

Let $|f''(x)| \leq K$ for some constant K. → on $[a,b]$ $f^{(4)}$

- Error bound for Midpoint Rule:

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}.$$

- Error bound for Trapezoidal Rule:

$\cos(x^2)$

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}.$$

These represent "worst-case" scenarios. Methods for finding K:

- Find the point $x \in [a,b]$ which gives the maximum of $|f''(x)|$. In most cases, this is infeasible because it takes too much time to calculate.
- Use the **triangle equality** and knowledge of bounds for certain functions, e.g. $|\sin x| \leq 1$, $|\cos x| \leq 1$. $|x^2|$ on $[0,1]$

Two rectangles at a time, instead of one

Previously in the Midpoint and Trapezoidal rules, we approximated the net area under the graph using one rectangle at a time, for n rectangles.

In these rules, the height of the rectangle/trapezium is dependent on two points, the endpoints of a subinterval $[x_{i-1}, x_i]$.

If we approximate the net area under the graph using **two** rectangles at a time, instead of one, i.e. two subintervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$, we would have three points to work with instead of two.

This considerably increases the accuracy of the approximation as we will see in the next few slides.

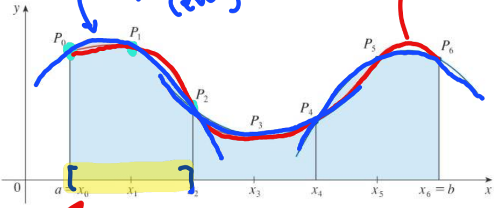
Visualization: Parabolas vs lines

endpoints of $[x_{i-1}, x_i]$

ax^2+bx+c

parabola approximation (every 2 subintervals)

$y=f(x)$

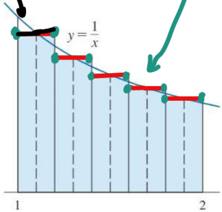


two subintervals

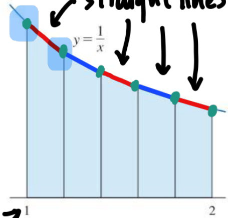
Simpson's

straight lines

Midpoint



straight lines



One subinterval at a time

Trapezoidal

Simpson's Rule

Tested in Quiz 2

every 2 subintervals, one parabola

Let f be a function on $[a, b]$. [redacted], with n **even**. Then

$$\Delta x = \frac{b-a}{n}, \quad \text{and} \quad x_i = a + i\Delta x \quad (i \text{ from } 0 \text{ to } n).$$

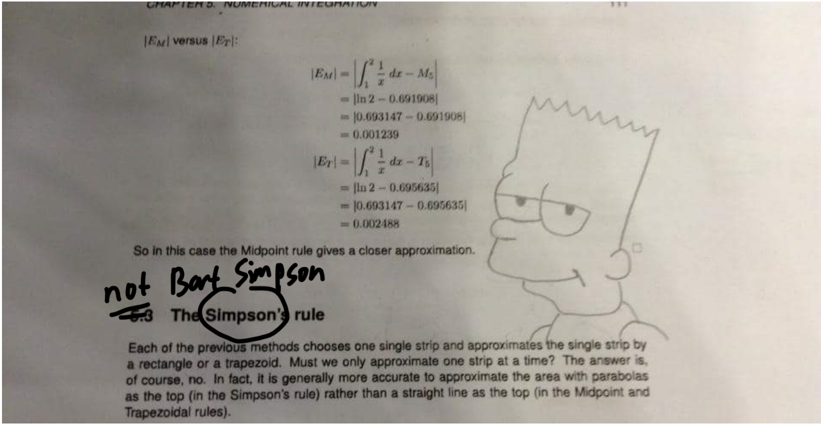
The **Simpson's Rule** S_n approximation to $\int_a^b f(x)\,dx$ is

odd

$$S_n = \frac{\Delta x}{3}\underbrace{[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]}_{n+1 \text{ points}}.$$

odd

Artwork by a former student



CHAPTER 5. NUMERICAL INTEGRATION

$|E_M|$ versus $|E_T|$:

$$\begin{aligned}|E_M| &= \left|\int_1^2 \frac{1}{x}\,dx - M_5\right| \\ &= |\ln 2 - 0.691908| \\ &= |0.693147 - 0.691908| \\ &= 0.001239 \\ |E_T| &= \left|\int_1^2 \frac{1}{x}\,dx - T_5\right| \\ &= |\ln 2 - 0.695635| \\ &= |0.693147 - 0.695635| \\ &= 0.002488\end{aligned}$$

So in this case the Midpoint rule gives a closer approximation.

not Bart Simpson

~~5.3~~ The Simpson's rule

Each of the previous methods chooses one single strip and approximates the single strip by a rectangle or a trapezoid. Must we only approximate one strip at a time? The answer is, of course, no. In fact, it is generally more accurate to approximate the area with parabolas as the top (in the Simpson's rule) rather than a straight line as the top (in the Midpoint and Trapezoidal rules).

Error bound for S_n

The following bound represents the "worst-case" scenario for the error of S_n, the Simpson's Rule approximation of $\int_a^b f(x)\,dx$.

no longer $f''(x)$

Theorem

Suppose K is a constant where $|f^{(4)}(x)| \leq K$ on $[a, b]$. The **magnitude** of error of the Simpson's Rule (E_S) has the following upper bound:

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}.$$

Example 1

$a=1, b=2, \quad \Delta x = \frac{b-a}{n} = \frac{2-1}{10} = \frac{1}{10} = 0.1$

even

$f(x) = \frac{1}{x}$

1. Use Simpson's Rule with $n = 10$ to approximate $\int_1^2 \frac{1}{x}\, dx$.
2. How large should we take n in order to guarantee that the Simpson's Rule approximation for $\int_1^2 \frac{1}{x}\, dx$ are accurate to within 0.0001?

$x_0 = 1, x_1 = 1.1, x_2 = 1.2, \ldots, x_9 = 1.9, x_{10} = 2$

$$\int_1^2 \frac{1}{x}\, dx \approx S_{10} = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 4f(x_9) + f(x_{10})\right]$$

$$= \frac{1}{30}\left[\frac{1}{1} + \frac{4}{1.1} + \frac{2}{1.2} + \ldots + \frac{4}{1.9} + \frac{1}{2}\right]$$

≈ 0.693150 (6 d.p)

Example 1

$f(x) = \frac{1}{x}$ on $[1,2]$

(b) $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$, $f'''(x) = -\frac{6}{x^4}$

$f^{(4)}(x) = \frac{24}{x^5}$ $\quad [1,2]$ $x^5 > 0$ $\quad \frac{24}{x^5}$ is decreasing $[1,2]$

$$|f^{(4)}(x)| = \left|\frac{24}{x^5}\right| = \frac{24}{x^5} \le \frac{24}{1} = 24 \qquad K = 24$$

this is the worst case scenario

$$|E_S| \le \frac{K(b-a)^5}{180n^4} = \frac{24}{180n^4} \le 0.0001$$

in order to make error within 0.0001

$$\Rightarrow \frac{24}{180n^4} \le 0.0001 \Rightarrow n^4 \ge \frac{24}{0.018} \Rightarrow n \ge \sqrt[4]{\frac{24}{0.018}}$$

$$= 6.042751$$

Take $n = 8$ (not 7)

$\frac{24}{x^5}$ ↙ first derivative of this is $-\frac{120}{x^6} < 0$

is decreasing on $[1, 2]$.

$\therefore$ max of $\frac{24}{x^5}$ occurs when $x = 1$

max value $= \frac{24}{1^5} = 24$

Exercise 1

1. Use Simpson's Rule with $n = 6$ to approximate $\int_0^1 \cos(x^2)\,dx$.
2. How large should we take n in order to guarantee that the Simpson's Rule approximation for $\int_0^1 \cos(x^2)\,dx$ are accurate to within $\underline{0.0001}$?

① $a=0$, $b=1$, $n=6$, $\Delta x = \frac{b-a}{n} = \frac{1}{6}$

$x_0 = 0$, $x_1 = \frac{1}{6}$, $x_2 = \frac{1}{3}$, $x_3 = \frac{1}{2}$, $x_4 = \frac{2}{3}$, $x_5 = \frac{5}{6}$, $x_6 = 1$

$$S_6 = \frac{1}{18}\left[\cos(0) + 4\cos\left(\frac{1}{36}\right) + 2\cos\left(\frac{1}{9}\right) + \dots + 4\cos\left(\frac{25}{36}\right) + \cos(1)\right] \approx 0.904523.$$

Exercise 1

$f(x) = \cos(x^2)$, $f'(x) = -2x\sin(x^2)$

not minus

$f''(x) = -4x^2\cos(x^2) - 2\sin(x^2)$

$|a+b| \leq |a| + |b|$

$f'''(x) = 8x^3\sin(x^2) - 12x\cos(x^2)$

modulus of sum $\leq$ sum of modulus

$$\boxed{f^{(4)}(x) = 16x^4\cos(x^2) + 48x^2\sin(x^2) - 12\cos(x^2)}$$

$$|f^{(4)}(x)| = |16x^4\cos(x^2) + 48x^2\sin(x^2) + (-12\cos(x^2))|$$

$$\leq |16x^4\cos(x^2)| + |48x^2\sin(x^2)| + |-12\cos(x^2)|$$

triangle inequality

$$= 16|x^4||\cos(x^2)| + 48|x^2||\sin(x^2)| + 12|\cos(x^2)|$$

$|-12| = 12$

$$|a+b+c| = |(a+b)+c|$$

$$\leq |a+b| + |c| \leq |a| + |b| + |c|$$

$$16\underbrace{|x^4|}_{\leq 1}\underbrace{|\cos(x^2)|}_{\leq 1} + 48\underbrace{|x^2|}_{\leq 1}\underbrace{|\sin(x^2)|}_{\leq 1} + 12\underbrace{|\cos(x^2)|}_{\leq 1} \qquad [0,1]$$

$$\leq 16 + 48 + 12 = 76 = K$$

$$|E_S| \leq \frac{K(b-a)^5}{180n^4} = \frac{76}{180n^4} = \frac{19}{45n^4} \leq 0.0001$$

$$n^4 \geqslant \frac{19}{0.0045} \implies n \geqslant \sqrt[4]{\frac{19}{0.0045}}$$

$$\approx 8.0609$$

Need to choose $n = 10$.

Integration by parts (26% on Tues, 33% on Fri)

Evaluate $\int_0^{\frac{\pi}{4}} \underbrace{x}_{u} \underbrace{\sec^2 x \tan x \, dx}_{dv}$.

$u = x \qquad dv = \sec^2 x \tan x$

$du = 1 \qquad v = \dfrac{\sec^2 x}{2}$

tutorial 3 qn

$= \left[\dfrac{x \sec^2 x}{2} \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \dfrac{\sec^2 x}{2} \, dx$

$= \dfrac{\frac{\pi}{4} \cdot 2}{2} - \dfrac{1}{2} \left[\tan x \right]_0^{\frac{\pi}{4}}$

$= \dfrac{\pi}{4} - \dfrac{1}{2}$

Sine/Cosine integral (26% on Tues, 33% on Fri)

Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^6 x \cos^5 x \, dx$.

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^6 x (\cos^2 x)^2 \cdot \cos x \, dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^6 x \, (1 - \sin^2 x)^2 \cos x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$x = -\frac{\pi}{2}, \; u = -1$

$x = \frac{\pi}{2}, \; u = 1$

$$= \int_{-1}^{1} u^6 (1 - u^2)^2 \, du$$

$$= \int_{-1}^{1} u^6 (u^4 - 2u^2 + 1) \, du$$

$$= \int_{-1}^{1} \underline{u^{10} - 2u^8 + u^6} \, du = 2\int_0^1 u^{10} - 2u^8 + u^6 \, du$$

$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

f even

tutorial

even function

Sine/Cosine integral (26% on Tues, 33% on Fri)

$$2\int_0^1 u^{10} - 2u^8 + u^6 \, du$$

$$= 2\left[\frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7}\right]_0^1 = 2\left[\frac{1}{11} - \frac{2}{9} + \frac{1}{7}\right]$$

$$= 2 \cdot \frac{8}{693} = \frac{16}{693}$$

Antiderivative of f (33% in Tues, 13% in Thurs)

$F'(x) = f(x)$

Find an antiderivative of $f(x) = \ln(2x)$.

(a) $x \ln x - x$ (b) $x \ln(2x)$ (c) $x \ln(2x) - (x - 29)$

(d) $x \ln(2x) - \frac{x}{2}$ (e) None of the above

$$F(x) = x\ln(2x) - (x-29)$$

$$F'(x) = \ln(2x) + x \cdot \frac{1}{2x} \cdot 2 - 1 = \ln(2x).$$

Completing the square (13% on Thurs, 25% on Fri)

Evaluate $\displaystyle\int_1^{\sqrt[3]{2}} \frac{x^2}{x^6 - 2x^3 + 2}\,dx.$

$$x^6 - 2x^3 + 2 = (x^3-1)^2 + 1$$

Complete the square

$$= \frac{1}{3}\int_1^{\sqrt[3]{2}} \frac{3x^2}{(x^3-1)^2+1}\,dx$$

$u = x^3 - 1$

$du = 3x^2\,dx$

$x = 1,\ u = 0$

$x = \sqrt[3]{2},\ u = 1$

$$= \frac{1}{3}\int_0^1 \frac{1}{1+u^2}\,du$$

$$= \frac{1}{3}\left[\tan^{-1} u\right]_0^1 = \frac{1}{3}\cdot\frac{\pi}{4} = \frac{\pi}{12}$$

Completing the square (13% on Thurs, 25% on Fri)

Augmented Lecture Exercise (27% on Thurs)

Evaluate $\displaystyle\int_0^1 \frac{3x^5+2x^2}{1+x^6}\,dx$.

$$= \frac{3}{6}\int_0^1 \frac{6x^5}{1+x^6}\,dx + \frac{2}{3}\int_0^1 \frac{3x^2}{1+x^6}\,dx$$

$u = 1+x^6$ $\quad x=0, u=1$

$du = 6x^5\,dx$ $\quad x=1, u=2$

$t = x^3$ $\quad x=0, t=0$

$dt = 3x^2\,dx$ $\quad x=1, t=1$

$$= \frac{1}{2}\int_1^2 \frac{1}{u}\,du + \frac{2}{3}\int_0^1 \frac{1}{1+t^2}\,dt$$

$$= \frac{1}{2}\left[\ln|u|\right]_1^2 + \frac{2}{3}\left[\tan^{-1} t\right]_0^1 = \frac{1}{2}\ln 2 + \frac{2}{3}\cdot\frac{\pi}{4}$$

$$= \frac{1}{2}\ln 2 + \frac{\pi}{6}.$$

Augmented Lecture Exercise (27% on Thurs)