CSD1241 Tutorial 8 Solutions

Problem 1. Let T be the orthogonal projection onto the plane $\alpha : x - 3y + 2z = 0$.
(a) Find the matrix representation of T.
(b) Find the images of the points

$$\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}3\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\8\end{pmatrix}, \begin{pmatrix}3\\-3\\1\end{pmatrix}$$

(c) Find all the points $\vec{x}$ that are fixed under this transformation, that is, $T(\vec{x}) = \vec{x}$.
(d) Find the image of the plane β under T with

$$\beta : \vec{x} = \begin{pmatrix}1\\1\\1\end{pmatrix} + s\begin{pmatrix}1\\-1\\0\end{pmatrix} + t\begin{pmatrix}-8\\0\\1\end{pmatrix}$$

(e) Find the image of $\gamma : x + y + z = 1$ under T.
(f) Find the image of the line $l : \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}6\\-20\\9\end{pmatrix} + t\begin{pmatrix}3\\-3\\1\end{pmatrix}$ under T.

Solution. (a) The plane α has normal $\vec{n} = \begin{bmatrix}1\\-3\\2\end{bmatrix}$. The matrix of T is

$$\begin{aligned} M &= I_3 - \frac{1}{||\vec{n}||^2}\vec{n}\vec{n}^T = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \frac{1}{14}\begin{pmatrix}1\\-3\\2\end{pmatrix}[1\ -3\ 2] \\ &= \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \frac{1}{14}\begin{pmatrix}1&-3&2\\-3&9&-6\\2&-6&4\end{pmatrix} = \frac{1}{14}\begin{pmatrix}13&3&-2\\3&5&6\\-2&6&10\end{pmatrix}\end{aligned}$$

(b) The images of the given 4 points are the 4 columns of the following matrix

$$\frac{1}{14}\begin{pmatrix}13&3&-2\\3&5&6\\-2&6&10\end{pmatrix}\begin{pmatrix}1&3&1&3\\1&1&1&-3\\1&1&8&1\end{pmatrix} = \begin{pmatrix}1&20/7&0&2\\1&10/7&4&0\\1&5/7&6&-1\end{pmatrix}$$

(c) Since T is an orthogonal projection onto α, all points that are fixed under T are all points on $\alpha : x - 3y + 2z = 0$.

(d) The image of β is

$$\begin{aligned}\beta' : \vec{x} &= \frac{1}{14}\begin{pmatrix} 13 & 3 & -2 \\ 3 & 5 & 6 \\ -2 & 6 & 10 \end{pmatrix}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -8 \\ 0 \\ 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{s}{7}\begin{pmatrix} 5 \\ -1 \\ -4 \end{pmatrix} + \frac{t}{7}\begin{pmatrix} -53 \\ -9 \\ 13 \end{pmatrix}\end{aligned}$$

(e) The plane γ has normal vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. So it has 2 direction vectors $\vec{u} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. A vector equation of γ is

$$\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

The image of γ is

$$\begin{aligned}\gamma' : \vec{x} &= \frac{1}{14}\begin{pmatrix} 13 & 3 & -2 \\ 3 & 5 & 6 \\ -2 & 6 & 10 \end{pmatrix}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right) \\ &= \begin{pmatrix} 13/14 \\ 3/14 \\ -2/14 \end{pmatrix} + \frac{s}{7}\begin{pmatrix} 5 \\ -1 \\ -4 \end{pmatrix} + \frac{t}{14}\begin{pmatrix} 5 \\ -1 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 13/14 \\ 3/14 \\ -2/14 \end{pmatrix} + \left(\frac{s}{7} + \frac{t}{14}\right)\begin{pmatrix} 5 \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} 13/14 \\ 3/14 \\ -2/14 \end{pmatrix} + r\begin{pmatrix} 5 \\ -1 \\ -4 \end{pmatrix},\end{aligned}$$

where $r = \frac{s}{7} + \frac{t}{14}$. Note that γ' is a line which is the intersection of α and γ. This happens because α and γ are perpendicular.

(f) The image of l is

$$\begin{aligned} l' : \vec{x} &= \frac{1}{14}\begin{pmatrix} 13 & 3 & -2 \\ 3 & 5 & 6 \\ -2 & 6 & 10 \end{pmatrix}\left(\begin{pmatrix} 6 \\ -20 \\ 9 \end{pmatrix} + t\begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} 0 \\ -2 \\ -3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \end{aligned}$$

Problem 2. Let T be the skew projection onto the plane $\alpha : x - 3y + 2z = 0$ along the vector $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 8 \end{pmatrix}$. Redo (a,c,e,f) of Problem 1.

Solution. (a) Note that $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 8 \end{pmatrix}$ and $\vec{n} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$. The matrix of T is

$$\begin{aligned} M = I_3 - \frac{1}{\vec{v}\cdot\vec{n}}\vec{v}\vec{n}^T &= I_3 - \frac{1}{14}\begin{pmatrix} 1 \\ 1 \\ 8 \end{pmatrix}\begin{pmatrix} 1 & -3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{14}\begin{pmatrix} 1 & -3 & 2 \\ 1 & -3 & 2 \\ 8 & -24 & 16 \end{pmatrix} \\ &= \frac{1}{14}\begin{pmatrix} 13 & 3 & -2 \\ -1 & 17 & -2 \\ -8 & 24 & -2 \end{pmatrix} \end{aligned}$$

(c) All fixed points are the points on $\alpha : x - 3y + 2z = 0$.
(e,f) can be done the same way as Problem 1.

Problem 3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal reflection through the line

$$l : \vec{x} = t\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

(a) Find the matrix of T.

(b) Find the image of the line $k : \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ under T.

Solution. (a) The line l has direction $\vec{d} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. The matrix of T is

$$\begin{aligned} M = \frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_3 &= \frac{2}{6}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \end{pmatrix} - I_3 \\ &= \frac{1}{3}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} -2 & 2 & -1 \\ 2 & 1 & -2 \\ -1 & -2 & -2 \end{pmatrix} \end{aligned}$$

(b) The image of k is the line

$$k' : \vec{x} = \frac{1}{3}\begin{pmatrix} -2 & 2 & -1 \\ 2 & 1 & -2 \\ -1 & -2 & -2 \end{pmatrix}\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} -1/3 \\ -2/3 \\ -11/3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

□

Problem 4. Let T be the orthogonal projection onto the plane $\alpha : x - 2y + z = 0$.

(a) Find the matrix M of T.

(b) Find the images of the points $\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$.

(c) Find the image of $\beta : x - z = 6$ under T.

(d) Find the image of $l : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ under T.

(e) Let Q be the intersection of β and l. Find the image of Q under T.

Solution. (a) The plane α has normal $\vec{n} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. The matrix of T is

$$M = I_3 - \frac{1}{||\vec{n}||^2}\vec{n}\vec{n}^T = \frac{1}{6}\begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$$

(b) The images of the given points are the last 3 columns of the following matrix

$$\frac{1}{6}\begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}\begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & -1 \\ 5 & -5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 3 & -1 & 1 \\ 4 & -4 & 2 \end{pmatrix}$$

(c) The plane β has normal $\vec{n}_\beta = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. So its direction vectors are $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Hence β has vector equation

$$\beta : \vec{x} = \begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

The image of β is β' which has equation

$$\begin{aligned}
\vec{x} &= \frac{1}{6}\begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}\left(\begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} + \frac{s}{6}\begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} + \frac{t}{6}\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} + \left(\frac{2s}{3} + \frac{t}{3}\right)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} + r\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},
\end{aligned}$$

which is a line through the point $\begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$ and having direction vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(d) The image of l is the line l' which has equation

$$\vec{x} = \frac{1}{6}\begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}\left(\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

(e) First, we find the point $Q = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Since Q is on both l and β, we have

$$\begin{cases} x = 3 + t \\ y = 1 - t \\ z = 5 + 3t \end{cases} \quad \text{and} \quad x - z = 6$$

We have

$$(3 + t) - (5 + 3t) = 6 \Rightarrow -2 - 2t = 6 \Rightarrow t = -4$$

Thus $Q = \begin{pmatrix} 3+t \\ 1-t \\ 5+3t \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -7 \end{pmatrix}$. The image of Q is

$$Q' = \frac{1}{6}\begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}\begin{pmatrix} -1 \\ 5 \\ -7 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}.$$

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Problem 5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection in the xz-plane, and let $S : \mathbb{R}^3 \to \mathbb{R}^3$ the reflection in the plane $x - y = 0$.

The composition $T \circ S : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T \circ S(\vec{x}) = T(S(\vec{x}))$.

(a) Find the matrix K of the composition $T \circ S$.

Hint: M, N = matrices of $T, S \Rightarrow$ matrix of $T \circ S$ is $K = MN$.

(b) Find the matrix L of the composition $S \circ T$. (*Hint*: $L = NM$).

(c) Check that K and L are inverses of each other, that is,

$$KL = LK = I_3.$$

Solution. The xz-plane has equation $y = 0$, so a normal vector is $\vec{n}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. The matrix of T is

$$M_T = I_3 - \frac{2}{||\vec{n}_1||^2}\vec{n}_1\vec{n}_1^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The plane $x - y = 0$ has normal vector $\vec{n}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. The matrix of S is

$$M_S = I_3 - \frac{2}{||\vec{n}_1||^2}\vec{n}_1\vec{n}_1^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{2}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) The matrix of $T \circ S$ is

$$K = M_{T\circ S} = M_T M_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) The matrix of $S \circ T$ is

$$L = M_{S\circ T} = M_S M_T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) It is straightforward to verify that $KL = LK = I_3$

$$\begin{aligned} KL &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 \\ LK &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 \end{aligned}$$

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