CSD2301 Lecture

10. Momentum and Collisions Part 2

LIN QINJIE

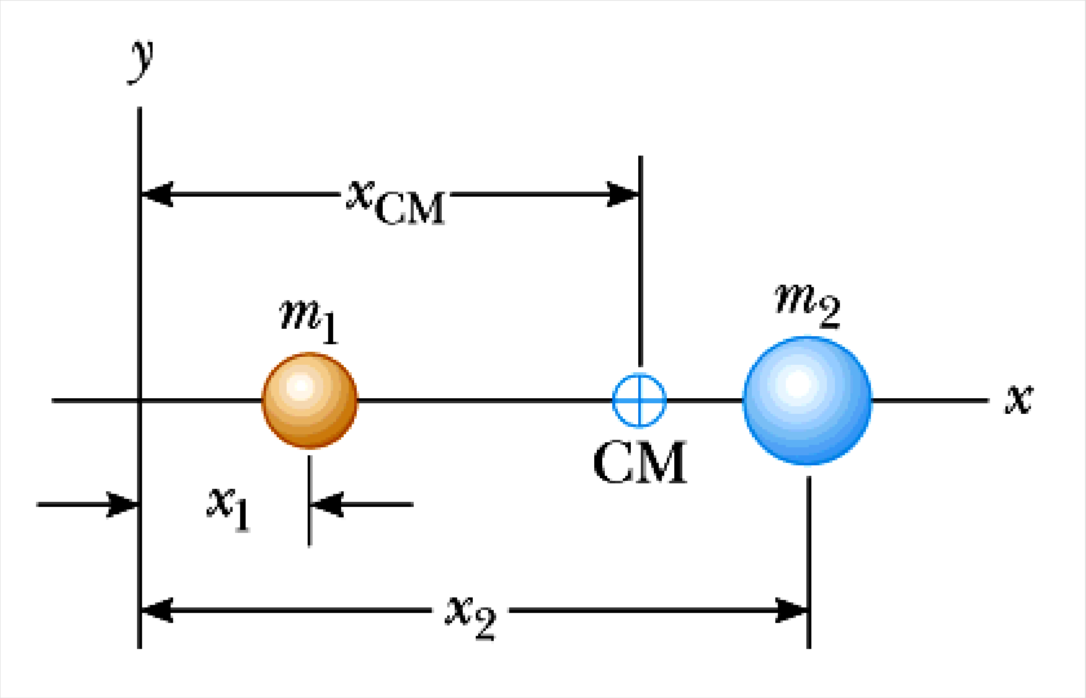
Outline

- Centre of mass
- Motion of a system of particles

Centre of Mass

- A mechanical system moves as if all its mass were concentrated at its **centre of mass** (CM).
- If an external force **F** acts on this system of total mass M, the CM accelerates at $\mathbf{a} = \mathbf{F}/\mathbf{M}$.
- This is independent of other motion, e.g. vibration, rotation of the system
- Provides another way to look at conservation of linear momentum

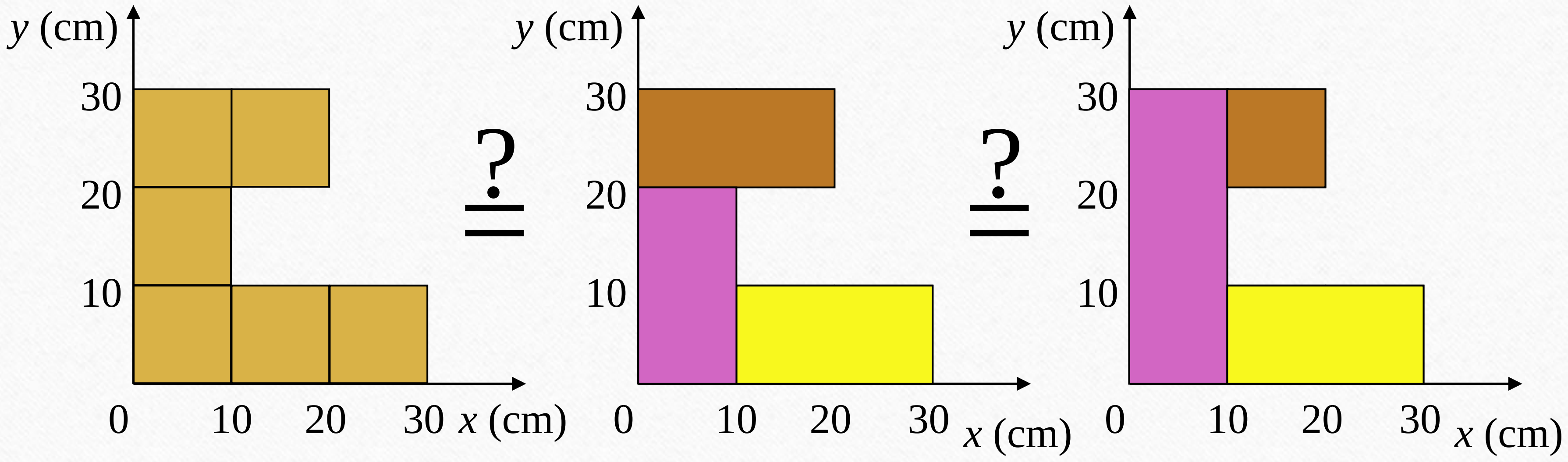
Definition of CM of 2 Particles



$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

CM of Shaped Sheet

- If we want to find the CM of the sheet, can we use the CMs of the different coloured sections to find the CM of the system?



CM of Shaped Sheet

- Consider any first 2 masses m_1 and m_2:

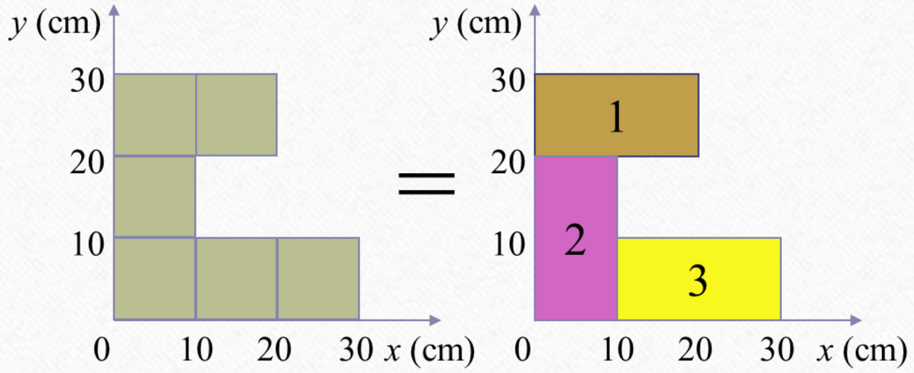
$$\boxed{x_{CM12} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}}$$

- Now add the third mass m_3 treating masses 1 and 2 as a single mass:

$$x_{CM} = \frac{(m_1 + m_2)\frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} + m_3 x_3}{(m_1 + m_2) + m_3} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

- This is the same as finding CM of 3 masses directly. By induction, it will apply to any number of masses, and we can divide the system into any convenient shapes and combinations!

CM of Shaped Sheet



The three coloured sections have equal mass, say m.

CM = center of mass

By symmetry, the CM of the sections are:

$$\mathbf{r}_{CM1} = 10\hat{\imath} + 25\hat{\jmath}$$
$$\mathbf{r}_{CM2} = 5\hat{\imath} + 10\hat{\jmath}$$
$$\mathbf{r}_{CM3} = 20\hat{\imath} + 5\hat{\jmath}$$

$$x_{CM} = \frac{m(10) + m(5) + m(20)}{3m} = 11.7 \text{ cm}$$
$$y_{CM} = \frac{m(25) + m(10) + m(5)}{3m} = 13.3 \text{ cm}$$

CM of a System of Particles

- Extending concept to many particles:

$$x_{CM} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots + m_nx_n}{m_1 + m_2 + m_3 + \cdots + m_n} = \frac{\sum_i m_ix_i}{\sum_i m_i}$$

$$y_{CM} = \frac{\sum_i m_iy_i}{\sum_i m_i}$$

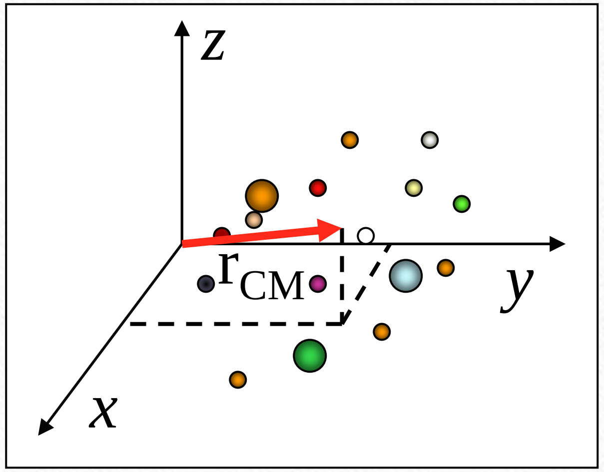
$$z_{CM} = \frac{\sum_i m_iz_i}{\sum_i m_i}$$

- Hence:

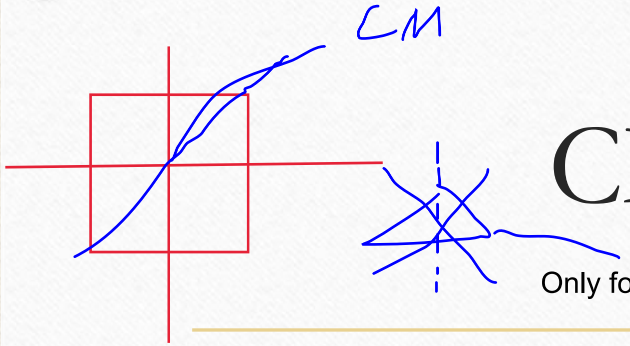
$$\vec{r}_{CM} = x_{CM}\hat{\imath} + y_{CM}\hat{\jmath} + z_{CM}\hat{k}$$

$$\vec{r}_{CM} = \frac{\sum_i m_ix_i\hat{\imath} + m_iy_i\hat{\jmath} + m_iz_i\hat{k}}{\sum_i m_i}$$

$$\vec{r}_{CM} = \frac{\sum_i m_i\vec{r}_i}{M}$$



Type text here



CM of an Extended Object

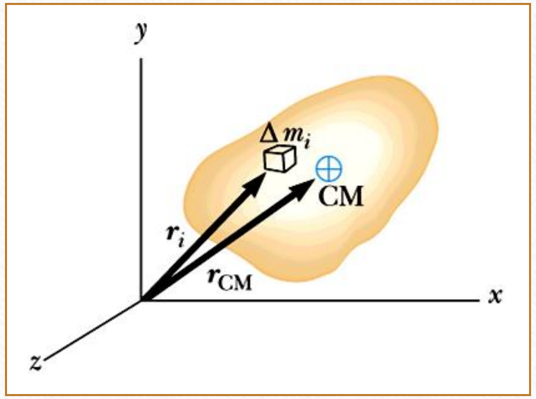
Only for symmetrical shape. The CM is "center"

- Dividing the object into elements of mass Δm_i :

$$x_{CM} \approx \frac{\sum_i x_i \Delta m_i}{M}$$

$$x_{CM} = \lim_{\Delta m_i \to 0} \frac{\sum_i x_i \Delta m_i}{M}$$

$$x_{CM} = \frac{1}{M} \int x dm$$



Similarly for y and z: $y_{CM} = \frac{1}{M} \int y dm$ $\quad z_{CM} = \frac{1}{M} \int z dm$

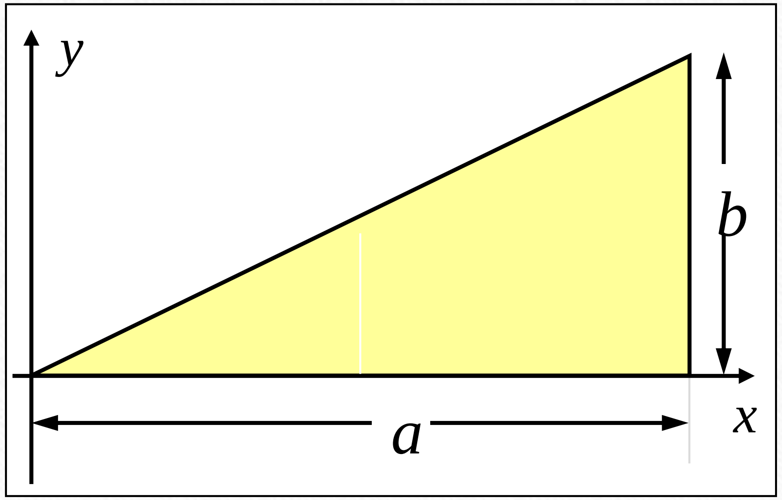
Therefore: $\boxed{\vec{r}_{CM} = \frac{1}{M} \int \vec{r} dm}$

Characteristics of CM

- For a **homogenous** (constant density) body that has a geometric centre, the CM is the geometric centre (eg, solid sphere, cube, and cylinder.)
- The centre of mass of any symmetric object lies on an axis of symmetry and on any plane of symmetry.
- If g is constant over the mass distribution, the centre of gravity coincides with the centre of mass. If an object is hung freely from any point, the vertical line through this point must pass through the centre of mass.
- CM needs not be within the body itself.

Example: CM of a right-angle triangle

- An object of mass M is in the shape of a right-angle triangle whose dimensions are shown in the figure. Locate the coordinates of the centre of mass, assuming the object has a uniform mass per unit area.

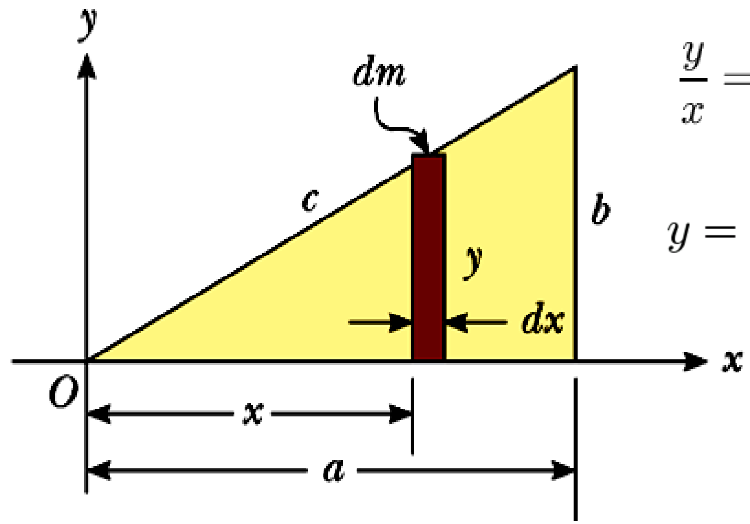


$$x_{CM} = \frac{1}{M}\int x dm$$

$$y_{CM} = \frac{1}{M}\int y dm$$

$$x_{CM} = \frac{1}{M}\int x dm$$

$$M = \rho\left(\frac{1}{2}ab\right) \Rightarrow \rho = \frac{2M}{ab}$$



$$\frac{y}{x} = \frac{b}{a}$$

$$y = \frac{b}{a}x$$

ρ = <u>mass per unit area</u>

$$dm = \rho(y\,dx) = \rho\left(\frac{b}{a}x\right)\,dx$$

$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{\rho}{M}\int_0^a x\left(\frac{b}{a}x\right)\,dx$$

$$x_{CM} = \frac{1}{M}\frac{2M}{ab}\left[\frac{b}{a}\frac{x^3}{3}\right]_0^a \Rightarrow x_{CM} = \frac{2}{ab}\left(\frac{b}{a}\frac{a^3}{3}\right) \Rightarrow \boxed{x_{CM} = \frac{2}{3}a}$$

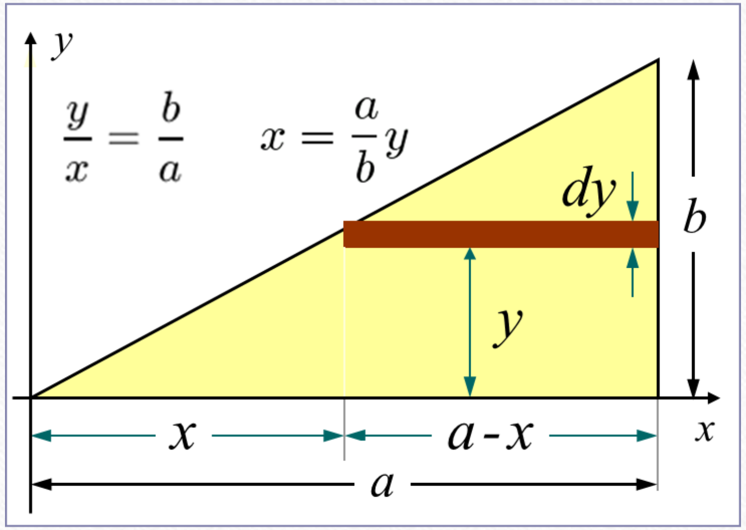
$$y_{CM} = \frac{1}{M}\int y\,dm$$

$$dm = \rho(a-x)\,dy = \rho\left(a - \frac{a}{b}y\right)\,dy$$

$$y_{CM} = \frac{1}{M}\int y\,dm = \frac{\rho}{M}\int_0^b y\left(a - \frac{a}{b}y\right)\,dy$$

$$y_{CM} = \frac{1}{M}\frac{2M}{ab}\left[\frac{ay^2}{2} - \frac{ay^3}{3b}\right]_0^b$$

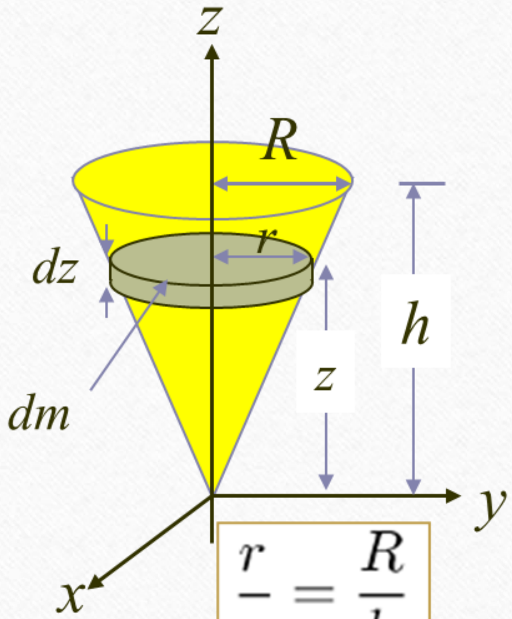
$$y_{CM} = \frac{2}{ab}\left(\frac{ab^2}{2} - \frac{ab^3}{3b}\right) \Rightarrow \boxed{y_{CM} = \frac{1}{3}b}$$



Consistent with CM in the x-direction: 1/3 from the base !

Example: CM of a cone

- Determine the CM of a uniform cone of height h and radius R.



$$\frac{r}{z} = \frac{R}{h}$$

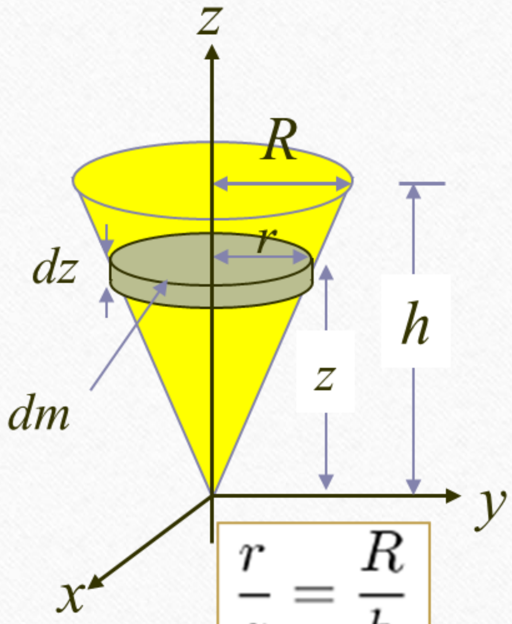
$$dm = \overset{\text{density}}{\rho}\,\overset{\text{volume}}{\boxed{\left(\pi r^2\right) dz}} = \rho\pi\frac{R^2}{h^2}z^2 dz$$

$$M = \int dm = \int_0^h \rho\pi\frac{R^2}{h^2}z^2 dz$$

$$M = \rho\pi\frac{R^2}{h^2}\frac{z^3}{3}\Bigg|_0^h = \rho\pi R^2\frac{h}{3}$$

Example: CM of a cone

- Determine the CM of a uniform cone of height h and radius R.



$$\frac{r}{z} = \frac{R}{h}$$

$$z_{CM} = \frac{1}{M}\int z\,dm \quad = \frac{1}{M}\int_0^h z\,\rho\pi\frac{R^2}{h^2}z^2 dz$$

$$z_{CM} = \frac{\rho\pi}{M}\frac{R^2}{h^2}\frac{z^4}{4}\bigg|_0^h = \frac{3}{\rho\pi R^2 h}\rho\pi\frac{R^2}{h^2}\frac{h^4}{4}$$

$$\boxed{z_{CM} = \frac{3}{4}h}$$

$$\boxed{\begin{aligned} x_{CM} &= 0 \\ y_{CM} &= 0 \end{aligned}}$$

Motion of a System of Particles

$$\boxed{\vec{r}_{CM} = \frac{\sum_i m_i \vec{r}_i}{M}}$$

$$\boxed{\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\vec{r}_i}{dt} = \frac{\sum_i m_i \vec{v}_i}{M}}$$

Velocity of the CM

$$\boxed{M\vec{v}_{CM} = \sum_i m_i \vec{v}_i = \sum_i \vec{p}_i = \vec{p}_{\rm tot}}$$

Total momentum of a system of particles

The total linear momentum of the system is equal to that of a single particle of mass M moving with a velocity $\mathbf{v}_{CM}$.

Motion of a System of Particles

$$\boxed{\vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\vec{v}_i}{dt} = \frac{1}{M}\sum_i m_i \vec{a}_i}$$

→ Acceleration of the CM

$$\boxed{M\vec{a}_{CM} = \sum_i m_i \vec{a}_i = \sum_i \vec{F}_i}$$

→ Newton's second law for a system of particles

$$\boxed{\sum \vec{F}_{\text{ext}} = M\vec{a}_{CM} = \frac{d\vec{p}_{\text{tot}}}{dt}}$$

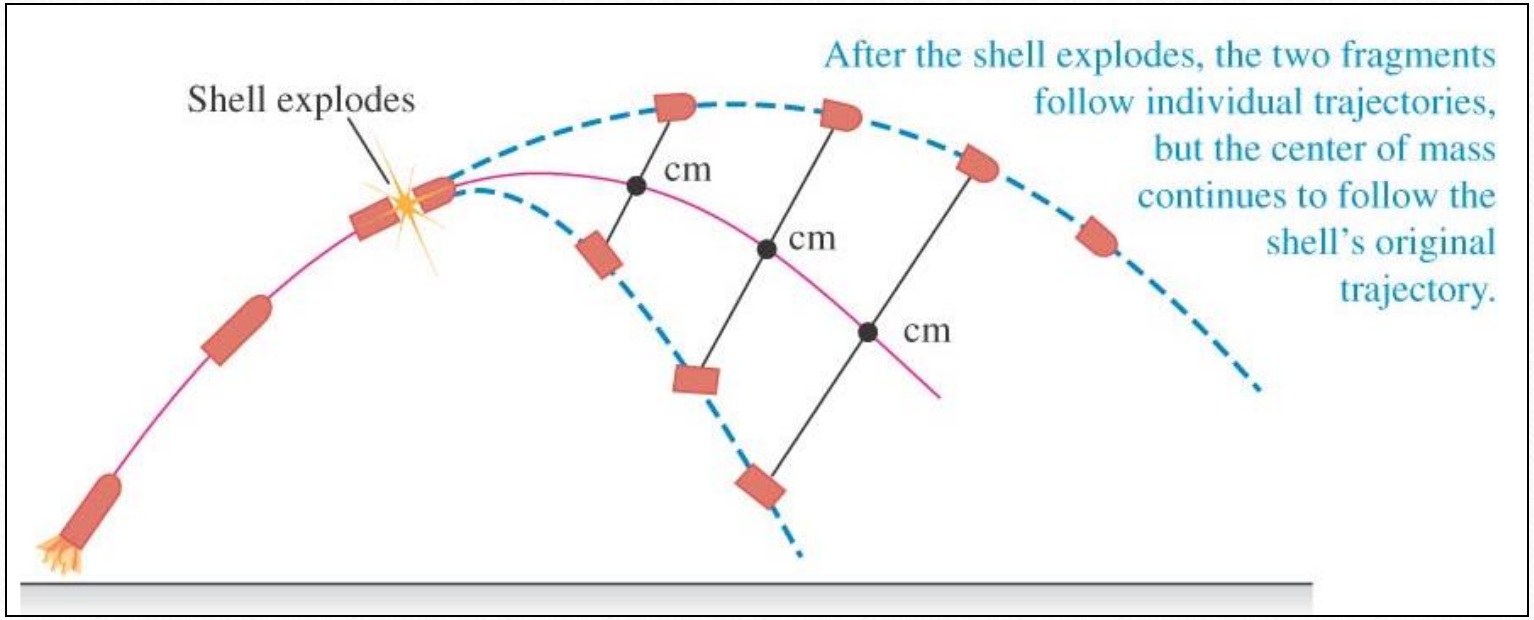
→ F_i includes int. & ext. forces
Internal forces cancel out.

The CM moves like an imaginary particle of mass M under the influence of the external resultant force on the system.

Exploding Projectile

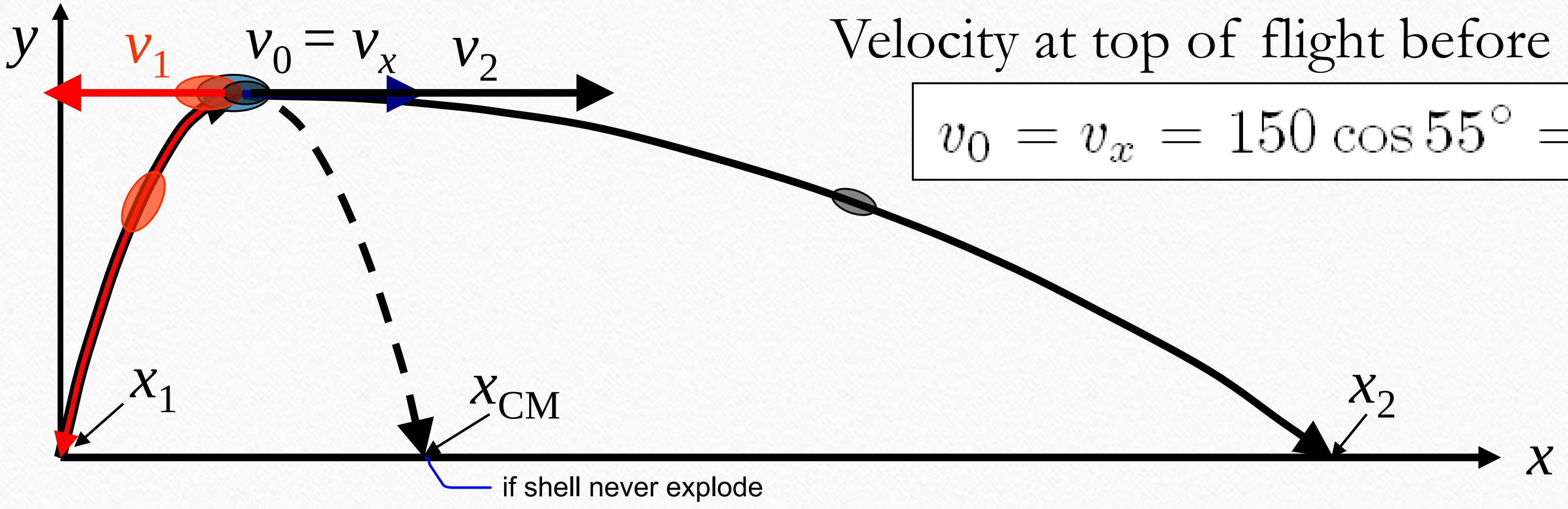
A projectile fired into the air suddenly explodes into two or more fragments. What can be said about the motion of the centre of mass of the system made up of all the fragments after the explosion?

Type text here



Example: Shell exploding in midair

- A 12.0-kg shell is launched at an angle of 55.0° above the horizontal with an initial speed of 150 m/s. When it is at its highest point, the shell exploded into two fragments, one three times as heavy as the other. The two fragments reach the ground at the same time. Assume that air resistance can be ignored. If the heavier fragment lands back at the same point from which the shell was launched, where will the lighter fragment land and how much energy was released in the explosion?



Velocity at top of flight before explosion is:

$$v_0 = v_x = 150 \cos 55^\circ = 86.0 \text{ m/s}$$

We can use the fact that CM travels along the original parabola to find the where the lighter fragment lands. To find where the CM would land, we first find the time to reach the top:

u + gt = v

$$v_y = 150 \sin 55^\circ - gt = 0 \quad \Rightarrow \quad t = 12.5 \text{ s}$$

$$x_{CM} = 2 \times t \times v_x = 2(12.5)(86.0) = 2150 \text{ m}$$

Using $$x_{CM} = \frac{(9)(0) + 3(x_2)}{12} \quad \Rightarrow \quad \boxed{x_2 = 4x_{CM} = 8600 \text{ m}}$$

1st frag

2nd frag

Since the 9-kg fragment landed back at the original position (ouch!), its velocity must be the negative that of the shell, that is:

$$\boxed{v_1 = -v_0 = -86.0 \text{ m/s}}$$

Using conservation of linear momentum:

$$\boxed{(m_1 + m_2)v_0 = m_1 v_1 + m_2 v_2}$$

$$\boxed{12(86.0) = 9(-86.0) + 3v_2} \Rightarrow \boxed{v_2 = 7(86.0) = 602 \text{ m/s}}$$

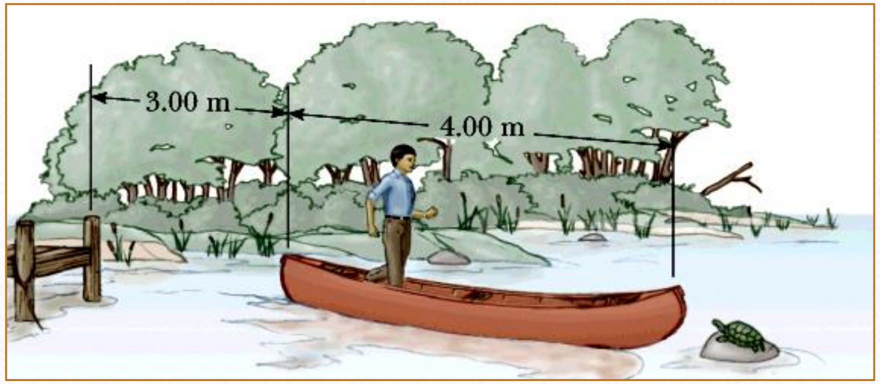
Energy released in the explosion:

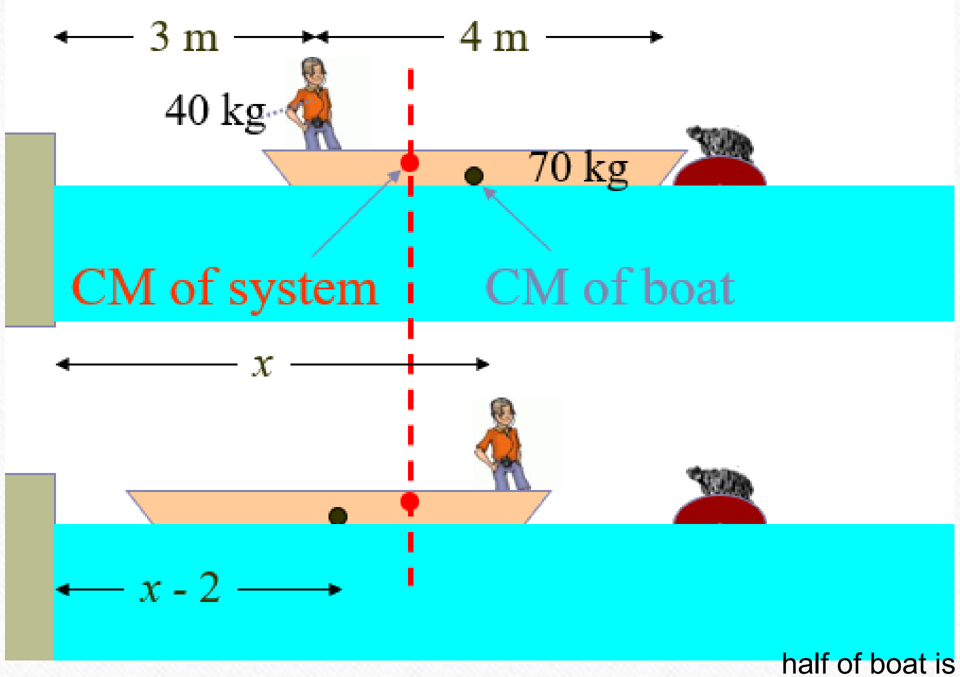
$$\boxed{\Delta E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{1}{2}(m_1 + m_2)v_0^2}$$

$$\boxed{\Delta E = \frac{1}{2}(3)(602)^2 + \frac{1}{2}(9)(86)^2 - \frac{1}{2}(12)(86)^2} \Rightarrow \boxed{\Delta E = 5.33 \times 10^5 \text{ J}}$$

Example: Boy on boat and turtle

- A 40-kg boy stands at one end of a 70-kg boat that is 4.00 m in length. The boat is initially 3.00 m from the pier. The child notices a turtle on a rock near the far end of the boat and proceeds to walk to that end to catch the turtle. Neglecting friction, (a) where is the boy relative to the pier when he reaches the far end of the boat? (b) Will he catch the turtle? (Assume he can reach out 1.00 m from the end of the boat)





System = boat + boy. When the boy walks to the right, the boat moves to the left, so that CM remains at the same place.

Taking the pier as origin:

$\frac{4\,m}{2}$

$$\boxed{x_{CM} = \frac{40(3.00) + 70(3.00 + 2.00)}{40 + 70}\ \mathrm{m}}$$

half of boat is 2

Let the boy be x m from the pier after he moves 4 m w.r.t the boat.
The CM of the boat is $(x - 2)$ m from the pier after that.

$$\boxed{x_{CM} = \frac{40x + 70(x-2)}{40+70}} \Rightarrow \boxed{\frac{40(3)+70(5)}{110} = \frac{40x+70(x-2)}{110}} \Rightarrow \boxed{x = 5.5\ \mathrm{m}}$$

<u>No</u>, can only reach (5.5 + 1) = 6.5 m from pier (turtle is 7.0 m away).

The End