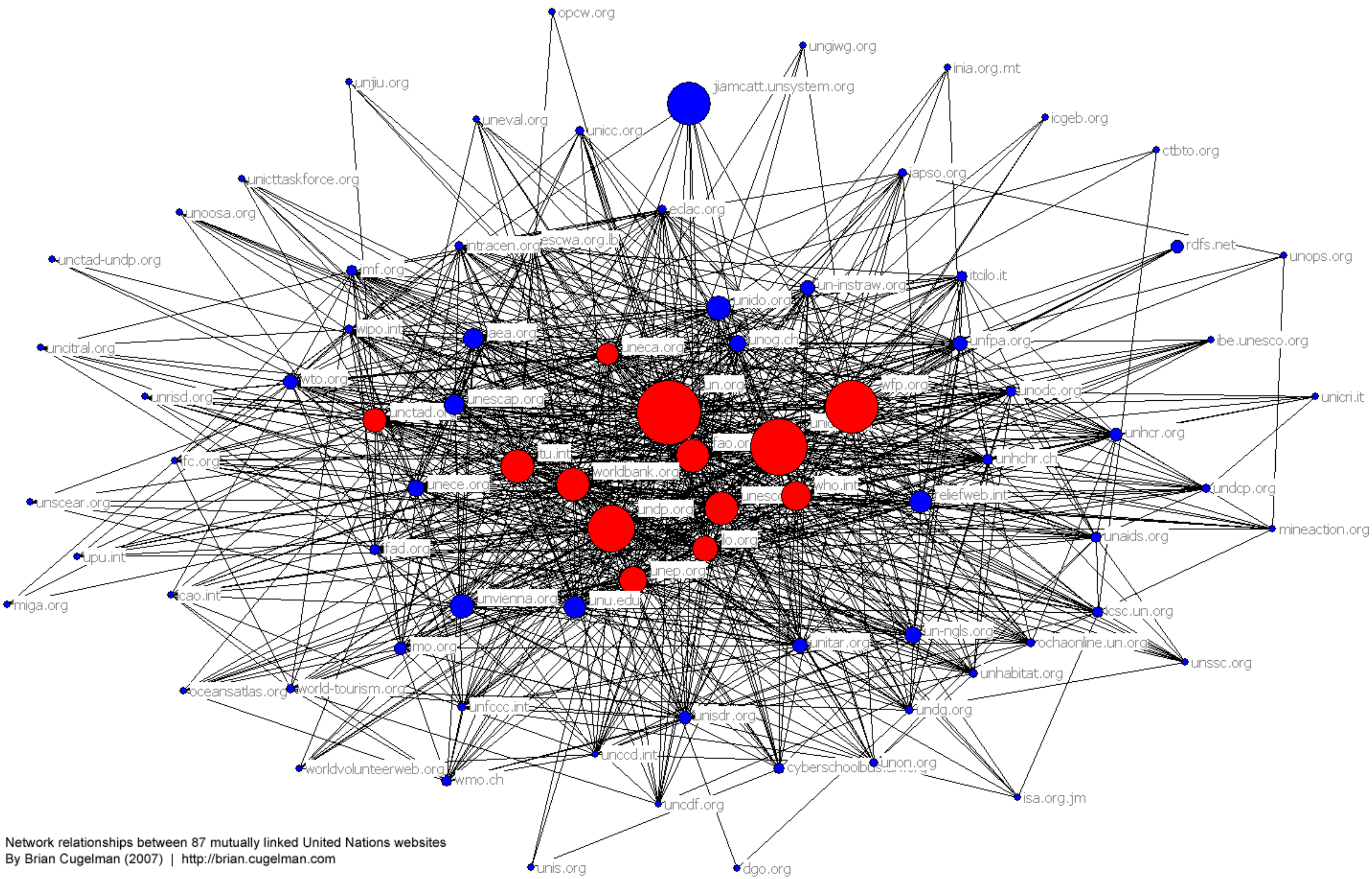
Introduction to Graphs

Outline

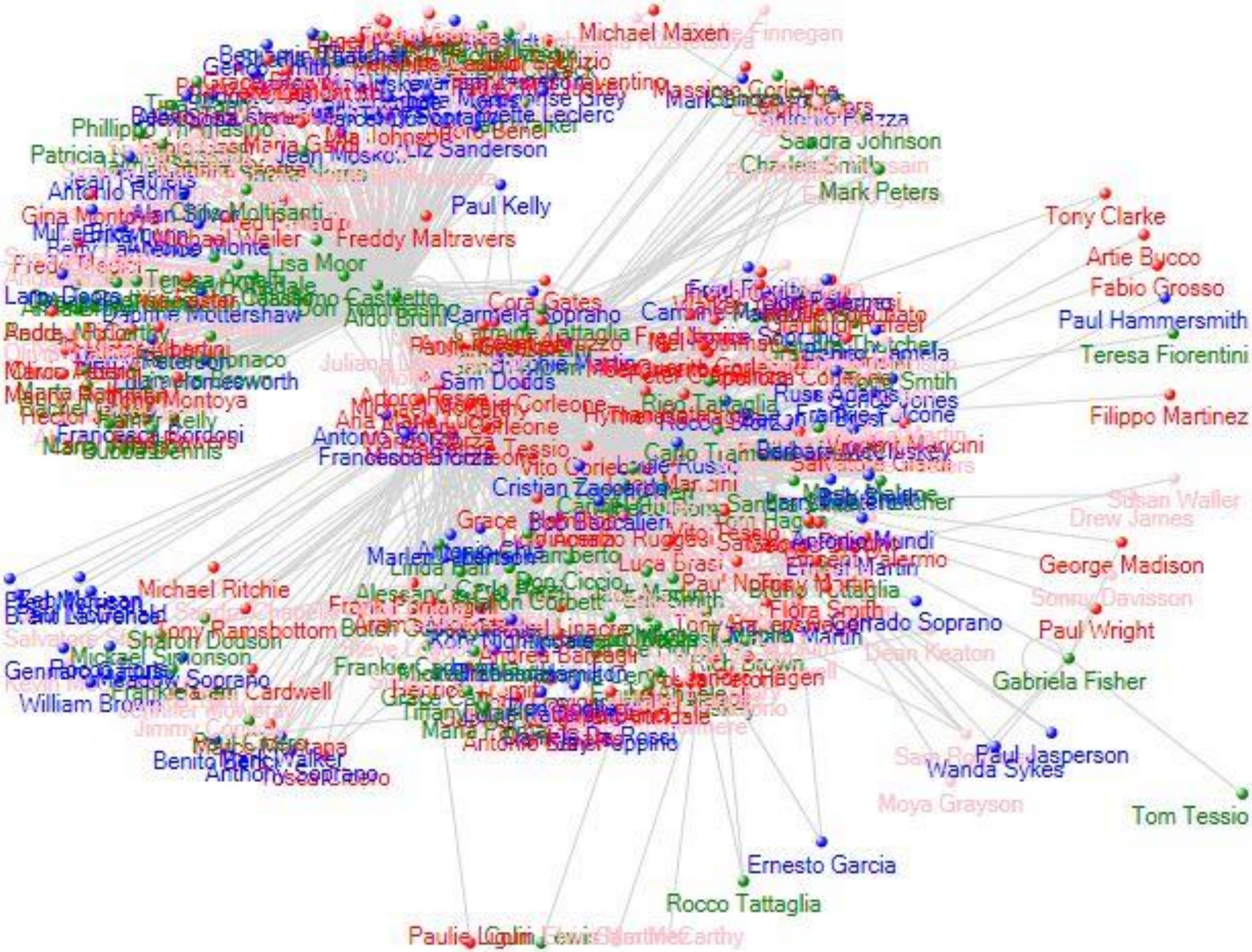
- Introduction & Terminology
- Representing Graphs
- Graph Traversals
- Spanning Trees
- Shortest Path Algorithms

Internet

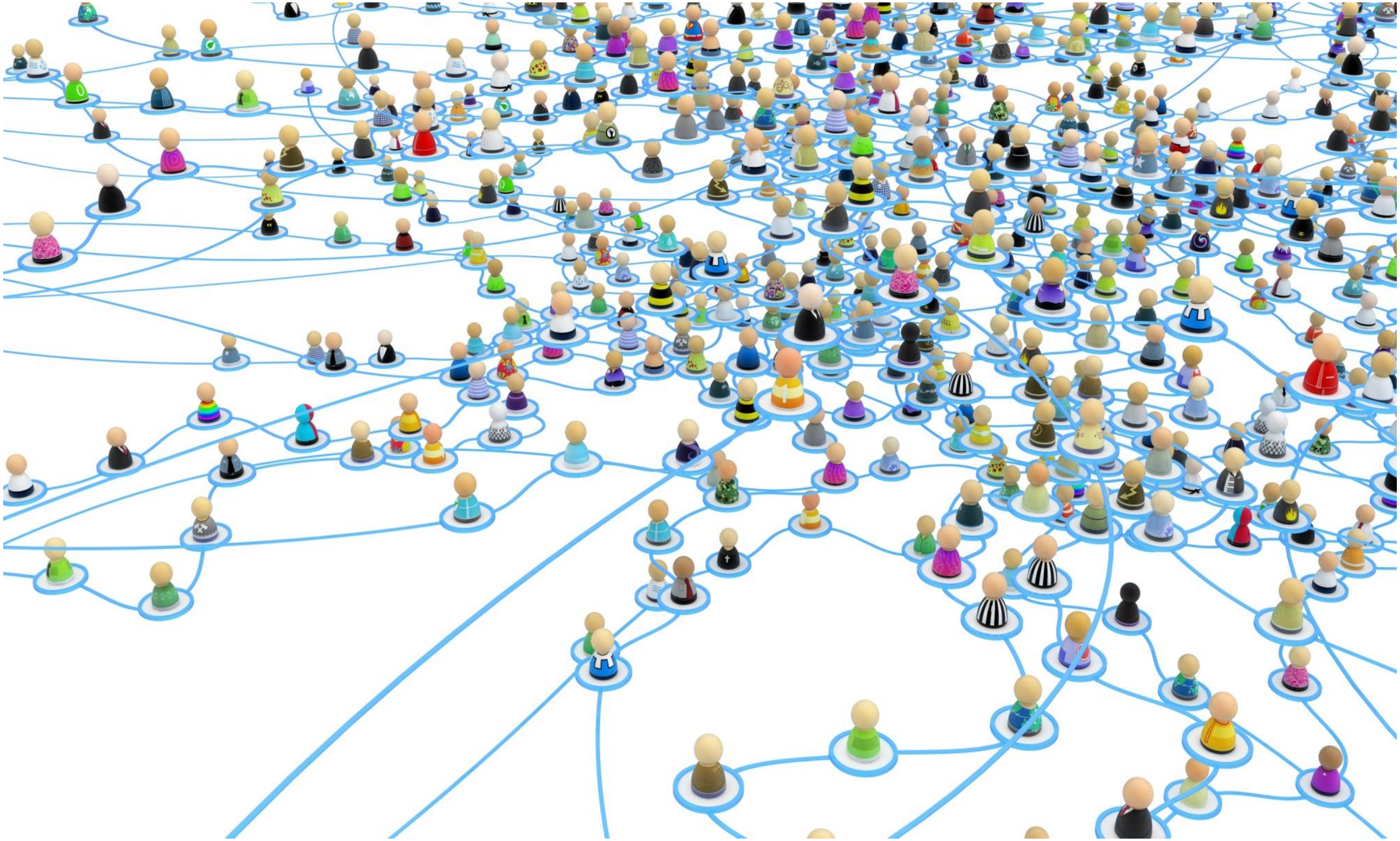


Network relationships between 87 mutually linked United Nations websites
By Brian Cugelman (2007) | http://brian.cugelman.com

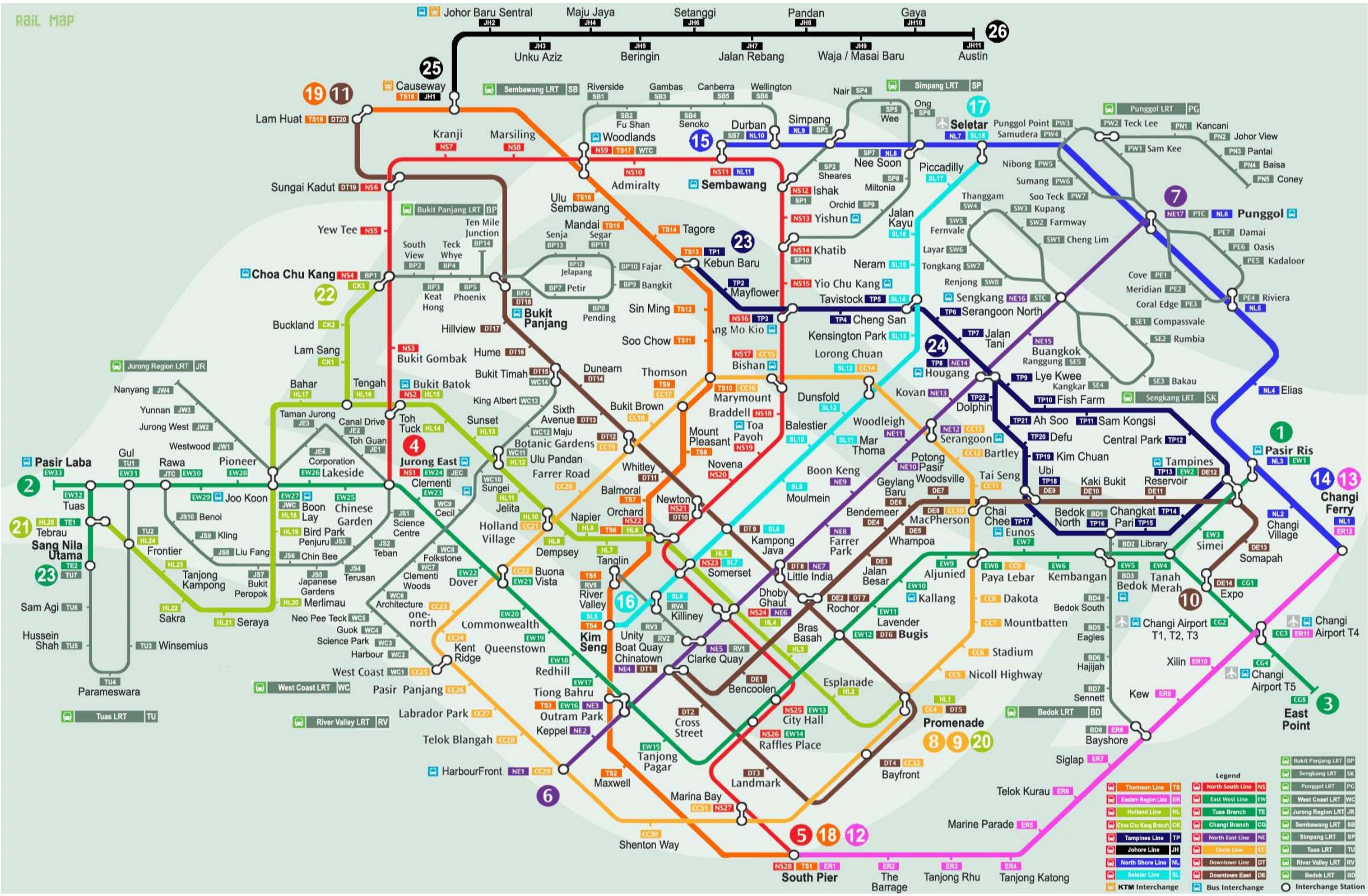
Email Networks



Social Networks

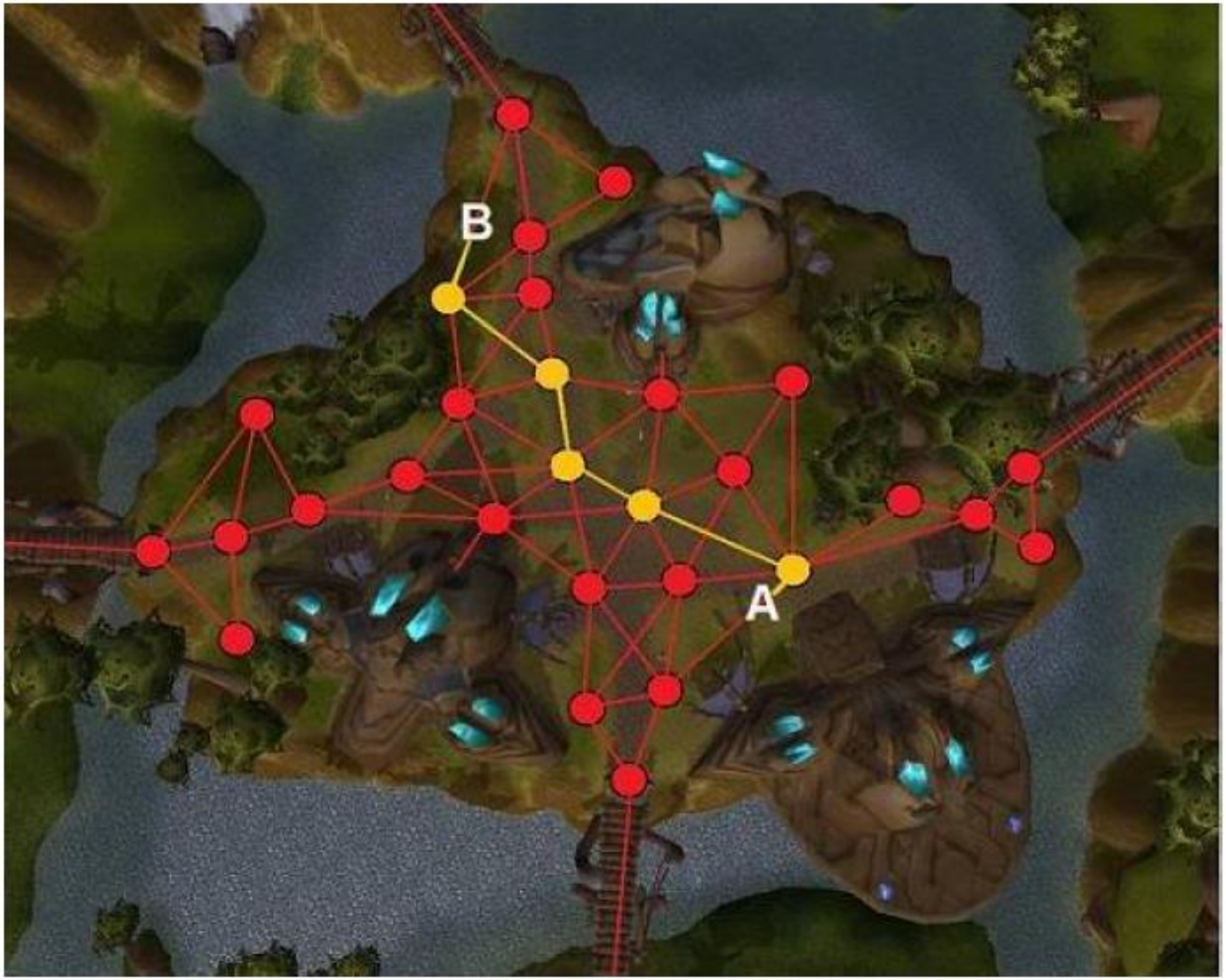


Public Transport



Video Games

Video Games



Introduction & Terminology

Introduction

- One of the most useful data structures (A very large topic).
- Related to trees in that a tree is a special kind of graph (Trees are much simpler).
- Graphs are more general and have a wider range of use. (Generality trades simplicity).
- Represent problems involving interconnected (dependent) objects.
- Graph algorithms are complex. Need to account for cycles; trees have only one path between nodes.

Introduction

- One of the most useful data structures (A very large topic).
- Tree is a special kind of graph (Trees are much simpler).
- Graphs are more general and have a wider range of use. (Generality trades simplicity).
- Represent problems involving interconnected (dependent) objects.
- Graph algorithms are complex. Need to account for cycles; trees have only one path between nodes.

Introduction

- One of the most useful data structures (A very large topic).
- Related to trees in that a tree is a special kind of graph (Trees are much simpler).
- Graphs are more general and have a wider range of use. (Generality trades simplicity).
- Represent problems involving interconnected (dependent) objects.
- Graph algorithms are complex. Need to account for cycles; trees have only one path between nodes.

Introduction

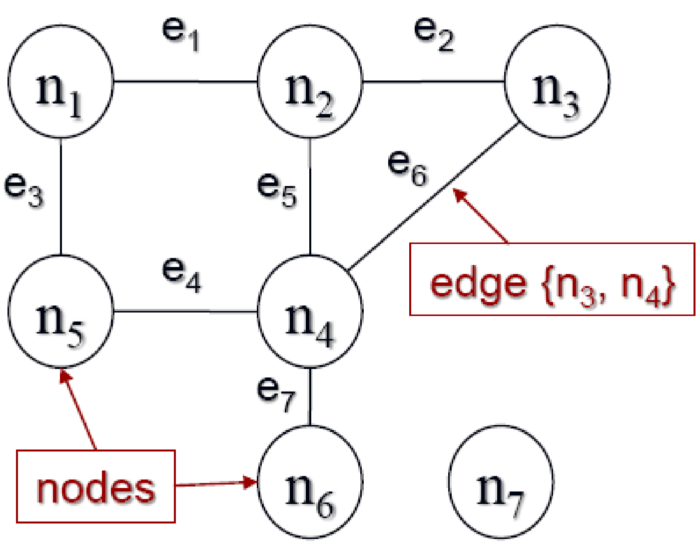
- One of the most useful data structures (A very large topic).
- Related to trees in that a tree is a special kind of graph (Trees are much simpler).
- Graphs are more general and have a wider range of use. (Generality trades simplicity).
- Represent problems involving interconnected (dependent) objects.
- Graph algorithms are complex. Need to account for cycles; trees have only one path between nodes.

Introduction

- One of the most useful data structures (A very large topic).
- Related to trees in that a tree is a special kind of graph (Trees are much simpler).
- Graphs are more general and have a wider range of use. (Generality trades simplicity).
- Represent problems involving interconnected (dependent) objects.
- Graph algorithms are complex. Need to account for cycles; trees have only one path between nodes.

Terminology

- A graph is essentially a collection of points connected by line segments.
- The points are referred to as nodes or vertices; the segments are called edges.

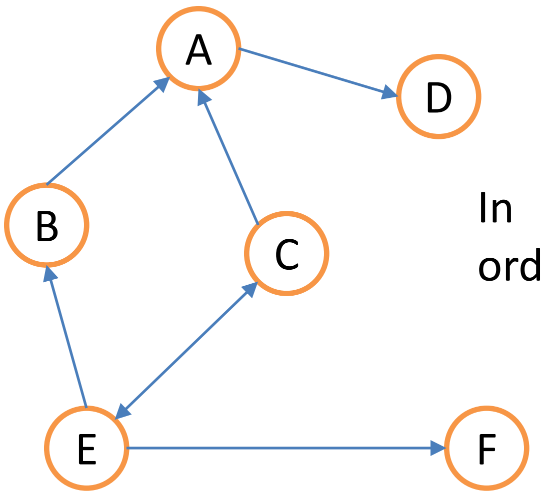


$V = \{n_1, n_2, n_3, n_4, n_5, n_6, n_7\}$

$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$
$= \{(n_1, n_2), (n_2, n_3), (n_1, n_5), (n_4, n_5), (n_2, n_4), (n_3, n_4), (n_4, n_6)\}$

Terminology

- If the edges have a direction (arrowheads in a diagram), the graph is a directed graph, or digraph.

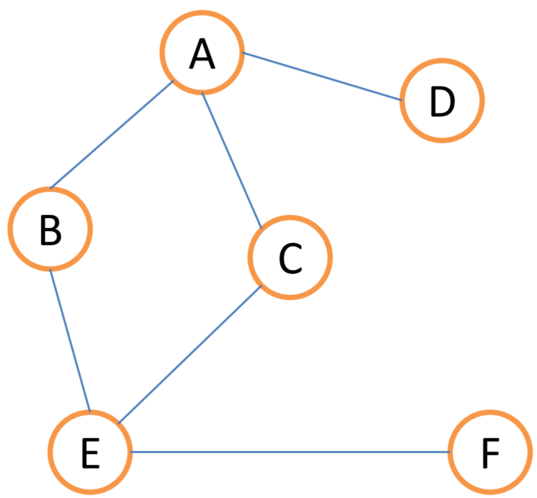


In a directed graph, each edge is an ordered pair: e=(C,A)

C is the origin (source) and A is the terminus (destination).

Terminology

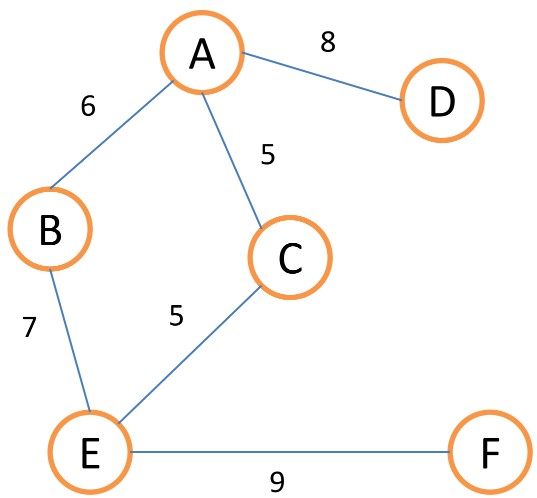
- If the graph edges has no values is called a undirected graph.



In an undirected graph, each edge is an unordered pair: $e=(v_1,v_2)$

Terminology

- If the graph has values (weights or costs) assigned to edges is called a weighted graph.



Notation

- Two vertices, x and y are said to be adjacent if there is an edge connecting them.

- We use the notation sGd to mean that *s* is adjacent to d. With a digraph, sGd implies direction. (xGy is not the same as yGx).

- The set of nodes adjacent to s is called the adjacency set of s or neighbors of s.
  - This set is fundamental to many graph algorithms.

Notation

- Two vertices, x and y are said to be adjacent if there is an edge connecting them.

- We use the notation sGd to mean that s is adjacent to d. With a digraph, sGd implies direction. (xGy is not the same as yGx).

- The set of nodes adjacent to s is called the adjacency set of s or neighbors of s.
  - This set is fundamental to many graph algorithms.

Notation

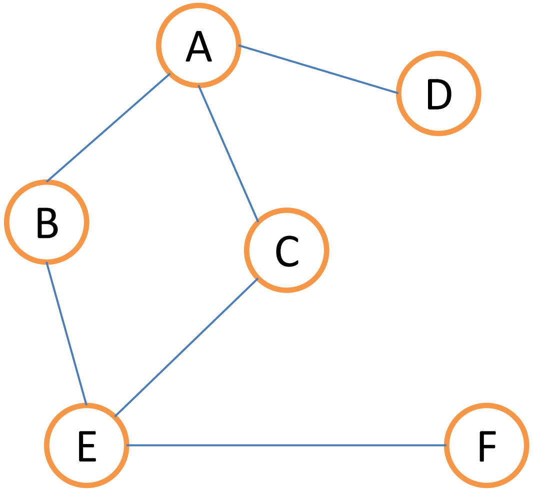
- Two vertices, x and y are said to be adjacent if there is an edge connecting them.

- We use the notation sGd to mean that s is adjacent to d. With a digraph, sGd implies direction. (xGy is not the same as yGx).

- The set of nodes adjacent to s is called the adjacency set of s or neighbors of s.
  - This set is fundamental to many graph algorithms.

Paths and Connectivity

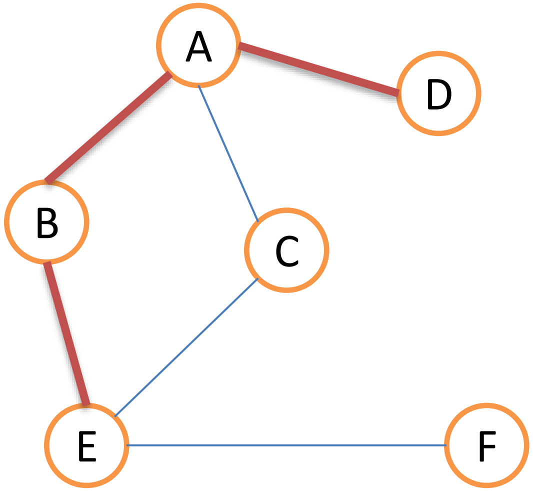
- A (contiguous) sequence of edges is a path



Paths and Connectivity

- A (contiguous) sequence of edges is a path

  path D to E : {D, A, B,E}
- If there is a path from x to y, y is reachable from x
- The length of a path is the number of edges on the path: 3 for the example
- Two vertices are connected if there is a path from one to the other
- A connected component is a subset, S, of vertices that are all connected

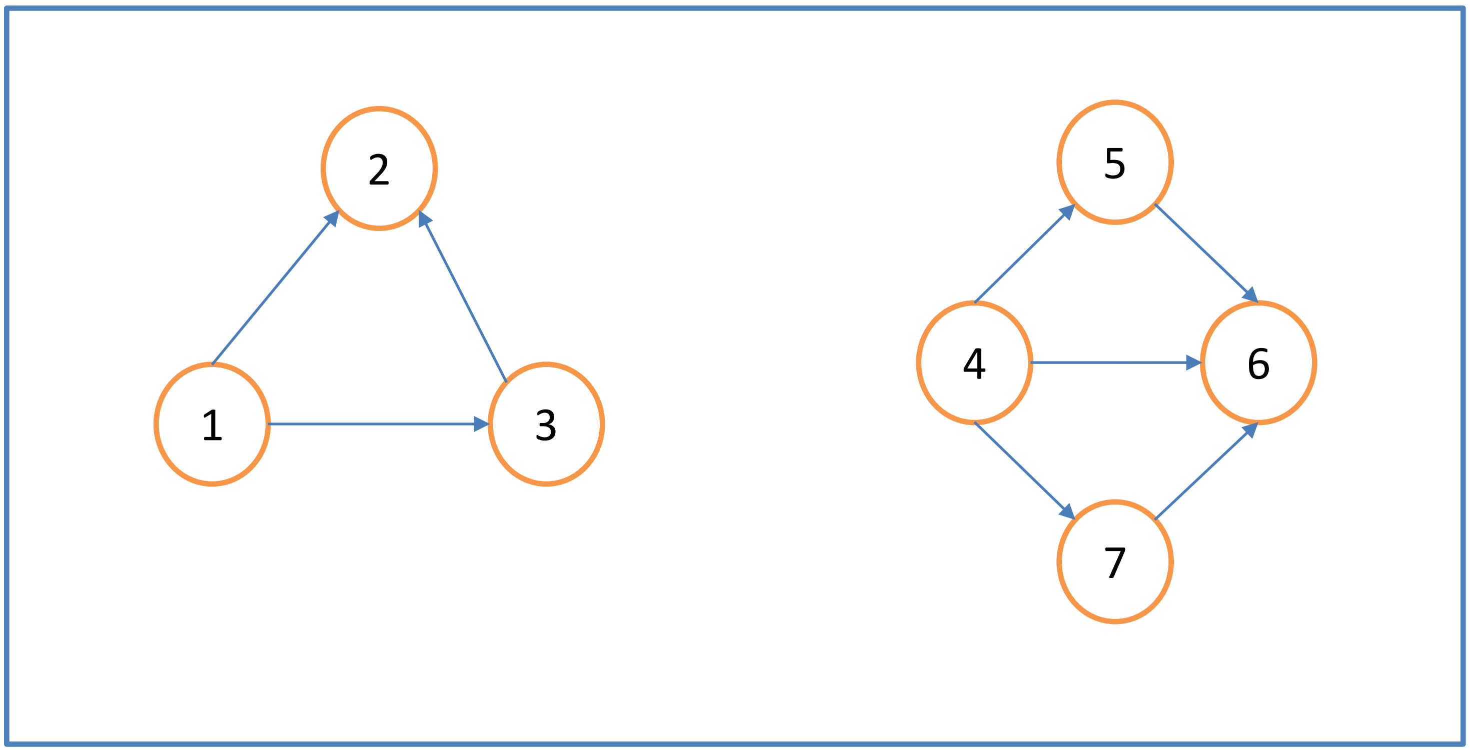


Connectedness

- Connectedness is an equivalence relation on the node set of a graph
  - Reflexive: every node is in a path of length 0 with itself
  - Symmetric: if $(n_i, n_j) \in$ path, then $(n_j, n_i) \in$ path
  - Transitive: if $(n_i, n_j) \in$ path and $(n_j, n_w) \in$ path, then $(n_i, n_w) \in$ path.

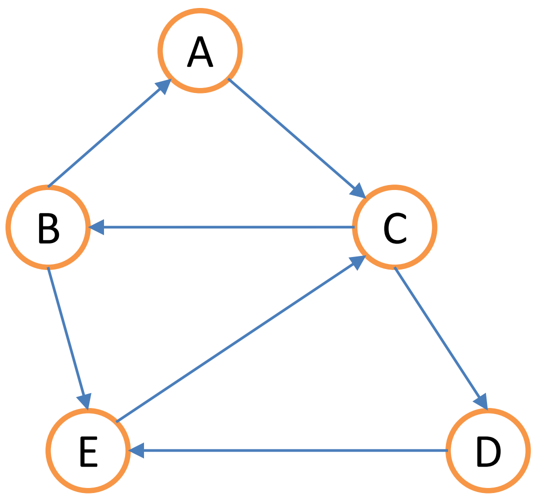
Connected Component: Example

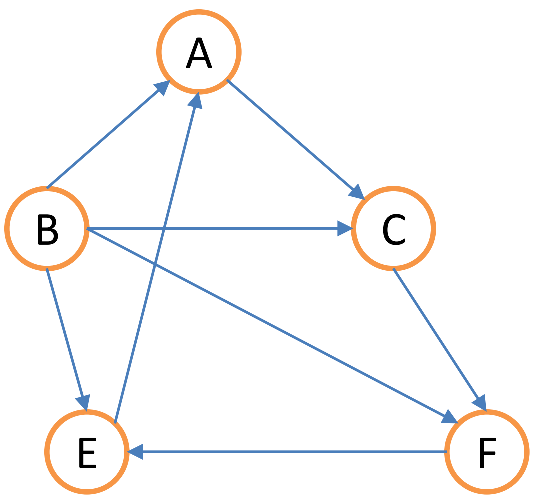
- A single directed graph with two components:



Connection Types

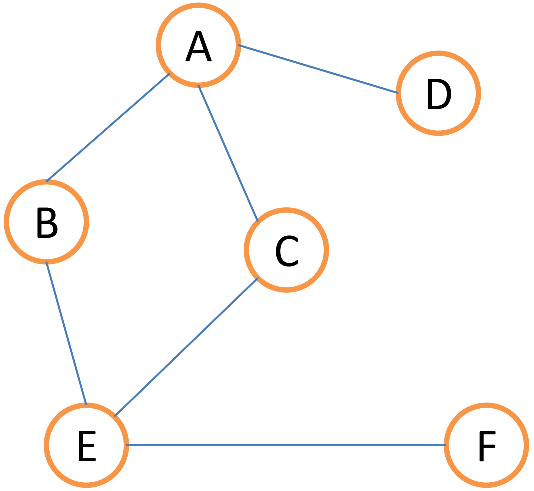
- Digraphs can be strongly connected or weakly connected.
  - **Strongly connected**: There is a path from every node to every other node
  - **Weakly connected**: There is **NOT** a path from each node to every other node (see Node C)





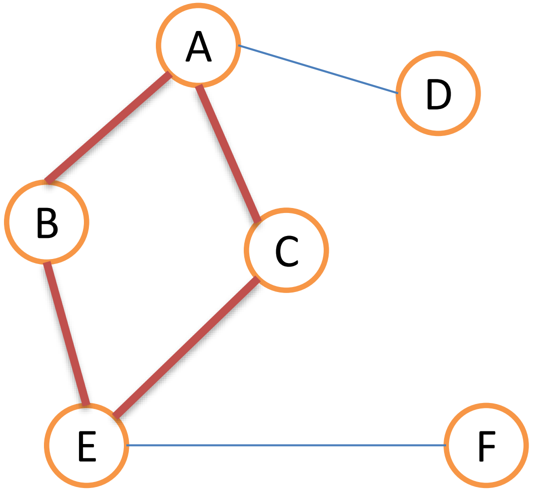
Cycles

- A cycle is a path whose source and destination node are the same
- A cycle is simple if all nodes on the path are distinct (with the exception of the first and last). A simple cycle must include at least 3 vertices



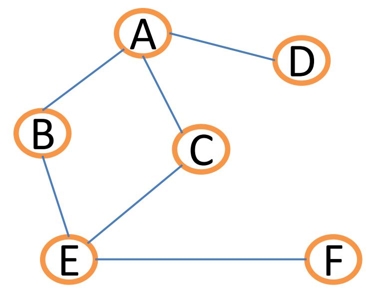
Cycles

- A cycle is a path whose source and destination node are the same
- A cycle is simple if all nodes on the path are distinct (with the exception of the first and last). A simple cycle must include at least 3 vertices
- If a graph has no cycles, it is acyclic. A directed acyclic graph is called a DAG

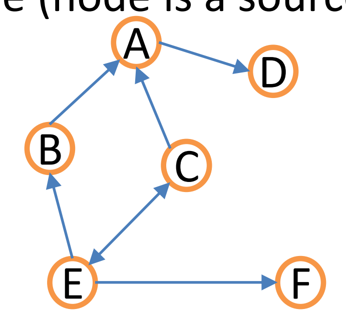


Degree

- For an undirected graph, the degree is the number of edges connecting a node
  - A: 3
  - B: 2



- For directed graph:
  - <u>In-degree:</u> Is the number of incoming edges into a node(node is a destination).
    - A:2
    - B:1
  - <u>Out-degree:</u> Is the number of outgoing edges from a node (node is a source).
    - A:1
    - B:1



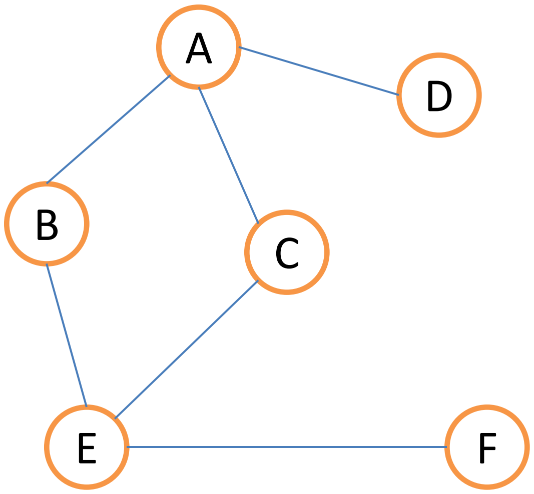
Tree vs Graph

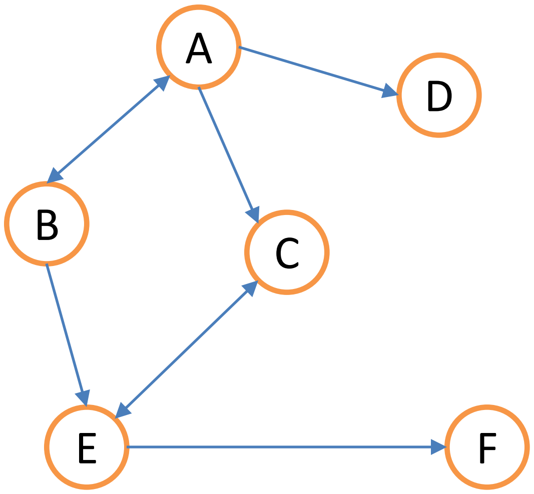
Characteristics	Tree	Graph
Cycle	X	✓
Root	✓	X
Reachability of Nodes	✓	X
Direction	✓	Optional

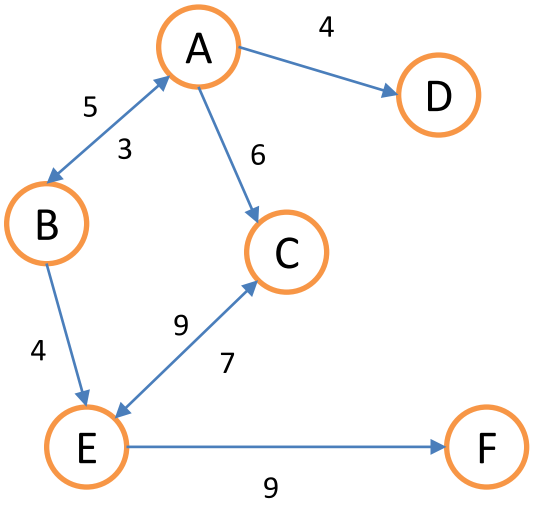
Representing Graphs

Adjacency Matrix

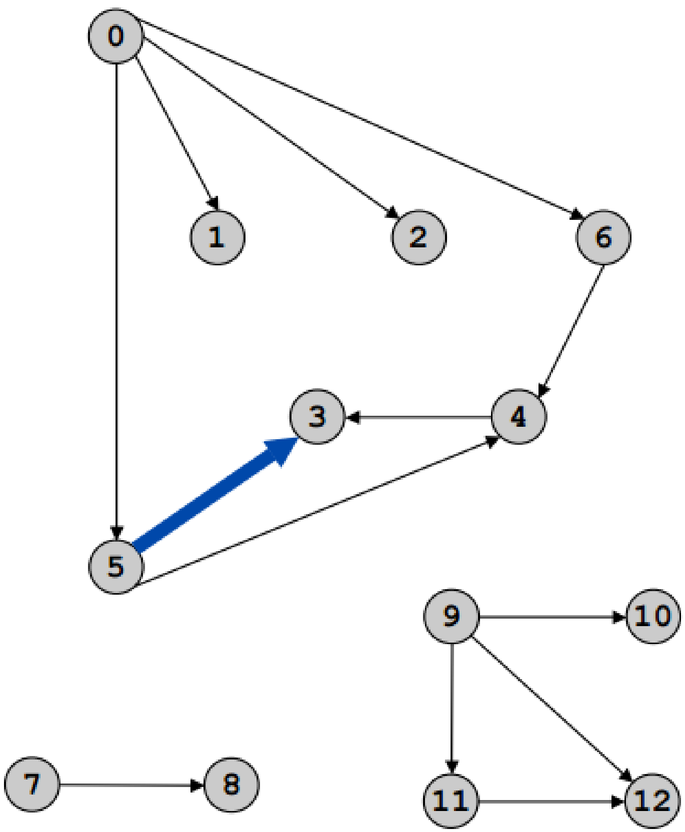
- A graph G with N nodes represented by an $N \times N$ boolean array (matrix).
- For each x and y, $G(x,y) = \text{TRUE}$ if xGy, otherwise false.







Adjacency Matrix - Example



to

from	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	1	0	0	1	1	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	1	0	0	0	0	0	0	0	0	0
5	0	0	0	1	1	0	0	0	0	0	0	0	0
6	0	0	0	0	1	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	1	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	1	1	1
10	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	1
12	0	0	0	0	0	0	0	0	0	0	0	0	0

Adjacency Matrix

- Space required is $\mathrm{O}(\mathrm{N}^2)$
  - A sparse graph has few edges
    - Sparse graphs will have many matrix entries of 0.
  - a dense graph has many edges.
    - Dense graphs will have many matrix entries of 1.
- Determining if two nodes are adjacent is $\mathrm{O}(1)$
- The size of the matrix is **independent** of the number of edges.
- An adjacency matrix may be a more desirable representation for dense graphs.

Adjacency Matrix

- Space required is $\mathrm{O}(\mathrm{N}^2)$
  - A sparse graph has few edges
    - Sparse graphs will have many matrix entries of 0.
  - a dense graph has many edges.
    - Dense graphs will have many matrix entries of 1.
- Determining if two nodes are adjacent is $\mathrm{O}(1)$
- The size of the matrix is **independent** of the number of edges.
- An adjacency matrix may be a more desirable representation for dense graphs.

Adjacency Matrix

- Space required is $O(N^2)$
  - A sparse graph has few edges
    - Sparse graphs will have many matrix entries of 0.
  - a dense graph has many edges.
    - Dense graphs will have many matrix entries of 1.
- Determining if two nodes are adjacent is $O(1)$
- The size of the matrix is **independent** of the number of edges.
- An adjacency matrix may be a more desirable representation for dense graphs.

Adjacency Matrix

- Space required is $O(N^2)$
  - A sparse graph has few edges
    - Sparse graphs will have many matrix entries of 0.
  - a dense graph has many edges.
    - Dense graphs will have many matrix entries of 1.
- Determining if two nodes are adjacent is $O(1)$
- The size of the matrix is **independent** of the number of edges.
- An adjacency matrix may be a more desirable representation for dense graphs.

Adjacency Matrix

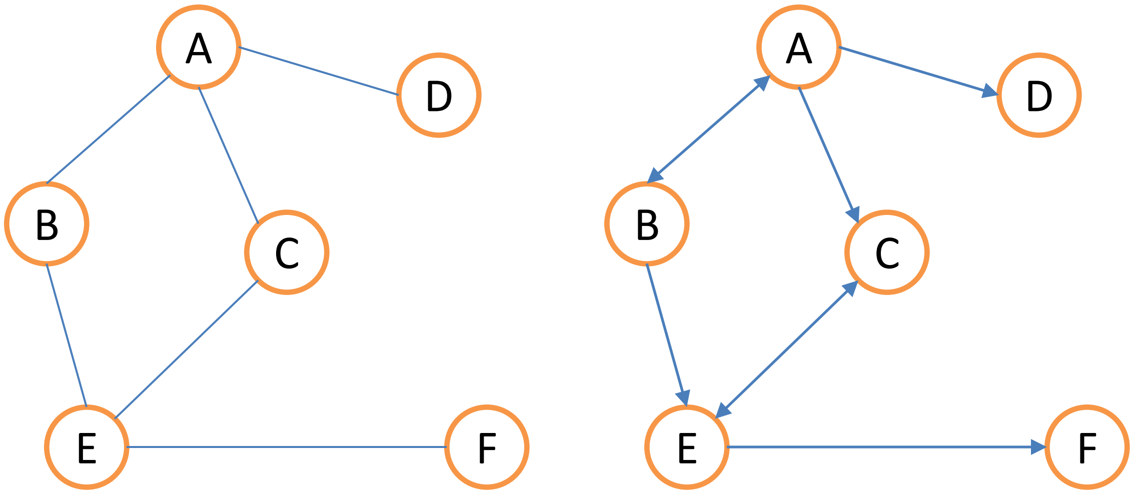
- Space required is $O(N^2)$
  - A sparse graph has few edges
    - Sparse graphs will have many matrix entries of 0.
  - a dense graph has many edges.
    - Dense graphs will have many matrix entries of 1.
- Determining if two nodes are adjacent is $O(1)$
- The size of the matrix is **independent** of the number of edges.
- An adjacency matrix may be a more desirable representation for dense graphs.

Adjacency Matrix

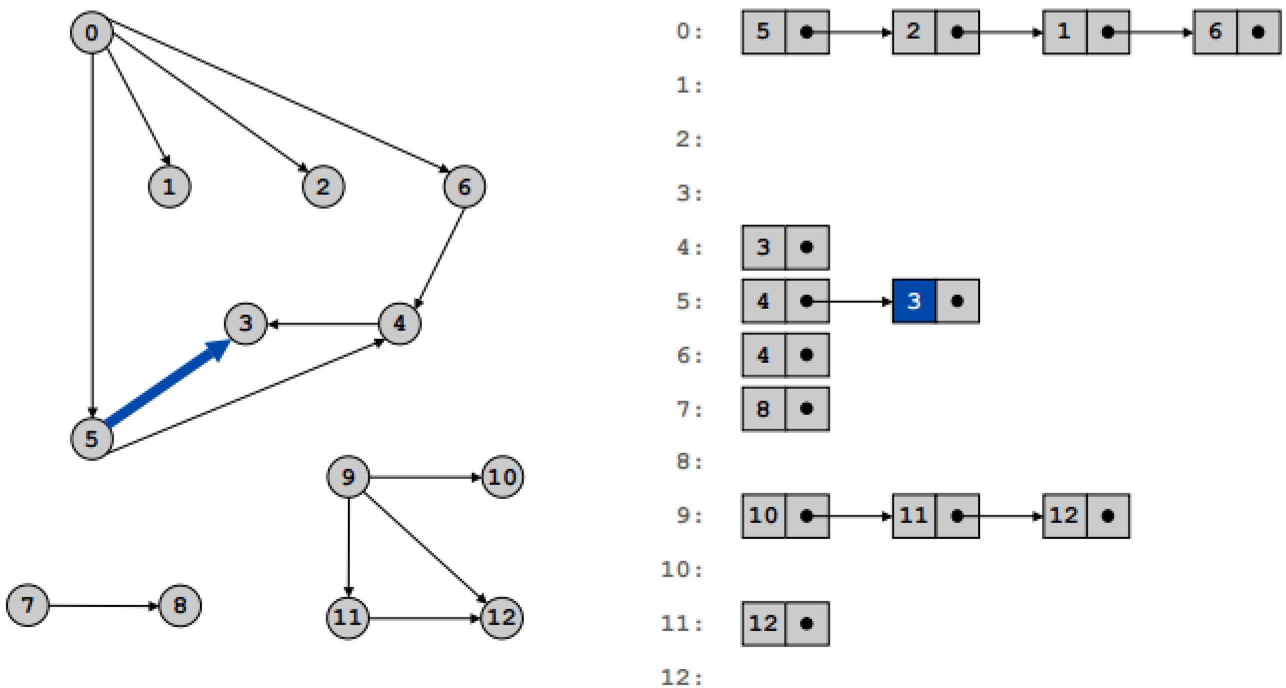
- Space required is $O(N^2)$
  - A sparse graph has few edges
    - Sparse graphs will have many matrix entries of 0.
  - a dense graph has many edges.
    - Dense graphs will have many matrix entries of 1.
- Determining if two nodes are adjacent is $O(1)$
- The size of the matrix is **independent** of the number of edges.
- An adjacency matrix may be a more desirable representation for dense graphs.

Adjacency Lists

- A graph G with N nodes represented by an array of N linked lists.
- For each x and y, if xGy is TRUE, y is on x's list.



Adjacency Lists - Example



Adjacency Lists

- Space required is $O(N^2)$
- Density affects the lists:
  - Sparse graphs will have shorter lists.
  - Dense graphs will have longer lists.
- The order of the nodes in a list may be arbitrary.
  - A weighted graph may order them by weight
- Determining if two nodes are adjacent is O(N) in the worst case. Could be much less if there are few edges.
- The number of nodes in the lists is **dependent** on the number of edges.
- An adjacency list may be a more desirable representation for sparse graphs.

Adjacency Lists

- Space required is $O(N^2)$
- Density affects the lists:
  - Sparse graphs will have shorter lists.
  - Dense graphs will have longer lists.
- The order of the nodes in a list may be arbitrary.
  - A weighted graph may order them by weight
- Determining if two nodes are adjacent is O(N) in the worst case. Could be much less if there are few edges.
- The number of nodes in the lists is **dependent** on the number of edges.
- An adjacency list may be a more desirable representation for sparse graphs.

Adjacency Lists

- Space required is $O(N^2)$
- Density affects the lists:
  - Sparse graphs will have shorter lists.
  - Dense graphs will have longer lists.
- The order of the nodes in a list may be arbitrary.
  - A weighted graph may order them by weight
- Determining if two nodes are adjacent is O(N) in the worst case. Could be much less if there are few edges.
- The number of nodes in the lists is **dependent** on the number of edges.
- An adjacency list may be a more desirable representation for sparse graphs.

Adjacency Lists

- Space required is $O(N^2)$
- Density affects the lists:
  - Sparse graphs will have shorter lists.
  - Dense graphs will have longer lists.
- The order of the nodes in a list may be arbitrary.
  - A weighted graph may order them by weight
- Determining if two nodes are adjacent is $O(N)$ in the worst case. Could be much less if there are few edges.
- The number of nodes in the lists is **dependent** on the number of edges.
- An adjacency list may be a more desirable representation for sparse graphs.

Adjacency Lists

- Space required is $O(N^2)$
- Density affects the lists:
  - Sparse graphs will have shorter lists.
  - Dense graphs will have longer lists.
- The order of the nodes in a list may be arbitrary.
  - A weighted graph may order them by weight
- Determining if two nodes are adjacent is $O(N)$ in the worst case. Could be much less if there are few edges.
- The number of nodes in the lists is **dependent** on the number of edges.
- An adjacency list may be a more desirable representation for sparse graphs.

Adjacency Lists

- Space required is $O(N^2)$
- Density affects the lists:
  - Sparse graphs will have shorter lists.
  - Dense graphs will have longer lists.
- The order of the nodes in a list may be arbitrary.
  - A weighted graph may order them by weight
- Determining if two nodes are adjacent is $O(N)$ in the worst case. Could be much less if there are few edges.
- The number of nodes in the lists is **dependent** on the number of edges.
- An adjacency list may be a more desirable representation for sparse graphs.

Graph Traversals

Graph Traversals

- Unlike tree traversals, there is no "starting" (i.e. root) node in a graph.

- Choosing an arbitrary starting node will not guarantee that all nodes are visited.

- The search must systematically traverse all of the edges in order to discover all of the vertices.

- Although it sounds like a lot of redundant work, it can be accomplished in $O(N)$ time, where N is the number of vertices.

Graph Traversals

- Unlike tree traversals, there is no "starting" (i.e. root) node in a graph.
- Choosing an arbitrary starting node will not guarantee that all nodes are visited.
- The search must systematically traverse all of the edges in order to discover all of the vertices.
- Although it sounds like a lot of redundant work, it can be accomplished in O(N) time, where N is the number of vertices.

Graph Traversals

- Unlike tree traversals, there is no "starting" (i.e. root) node in a graph.

- Choosing an arbitrary starting node will not guarantee that all nodes are visited.

- The search must systematically traverse all of the edges in order to discover all of the vertices.

- Although it sounds like a lot of redundant work, it can be accomplished in $O(N)$ time, where N is the number of vertices.

Graph Traversals

- Unlike tree traversals, there is no "starting" (i.e. root) node in a graph.
- Choosing an arbitrary starting node will not guarantee that all nodes are visited.
- The search must systematically traverse all of the edges in order to discover all of the vertices.
- Although it sounds like a lot of redundant work, it can be accomplished in $\mathrm{O(N)}$ time, where N is the number of vertices.

Graph Traversals

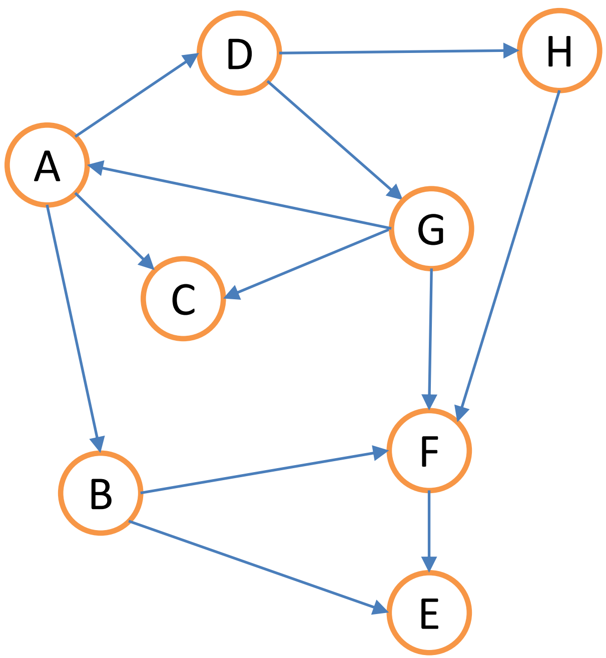
- Breath-first traversal
- Depth-first traversal

Pseudo-code for Graphs Traversals

```
GraphSearch(G is the graph to search, v is the starting vertex){
   Put v into container C;
   while (container C is not empty){
     Remove a vertex, x, from container C;
     if (x has not been visited){
       Visit x;
       Set x.visited to TRUE;
       for (each vertex, w, adjacent to x){
         if (w has not been visited)
           Put w into container C;
       }//end for
     }//end if
   }//end while
} //end GraphSearch
```

Example

- Given this graph, determine the sequence of nodes that are visited from different starting nodes. Starting at A/G, using Stack/Queue

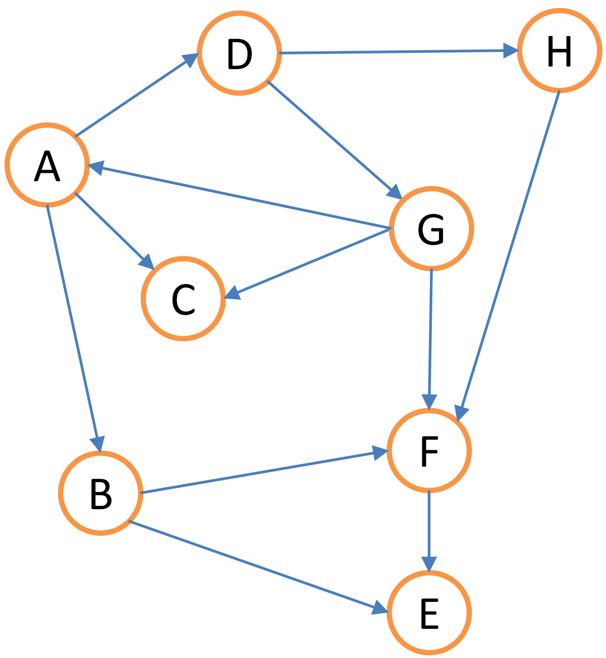


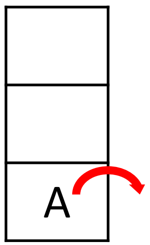
	A	B	C	D	E	F	G	H
A	0	3	9	7	0	0	0	0
B	0	0	0	0	6	5	0	0
C	0	0	0	0	0	0	0	0
D	0	0	0	0	0	0	4	2
E	0	0	0	0	0	0	0	0
F	0	0	0	0	8	0	0	0
G	5	0	1	0	0	4	0	0
H	0	0	0	0	0	8	0	0

Adjacency Matrix

Graph Traversals: Depth First

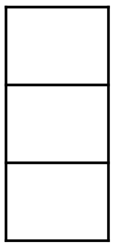
Container: Stack
Start at A

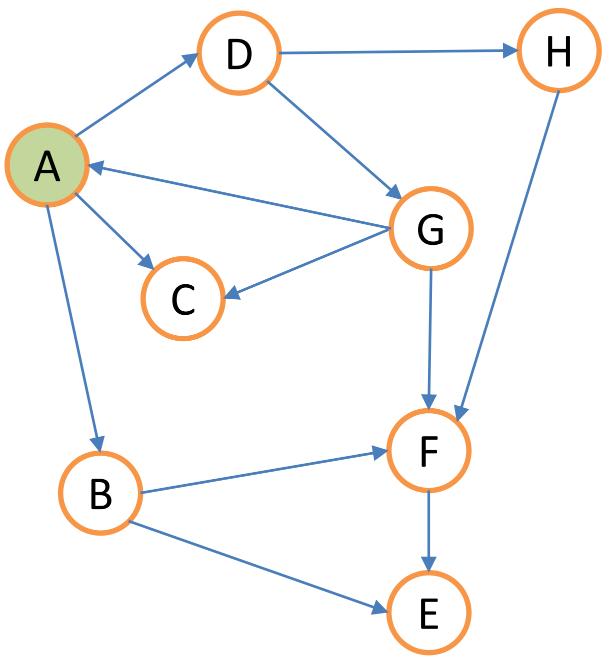




Graph Traversals: Depth First

Container: Stack
Start at A

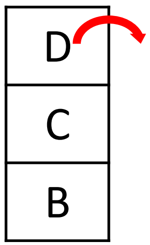


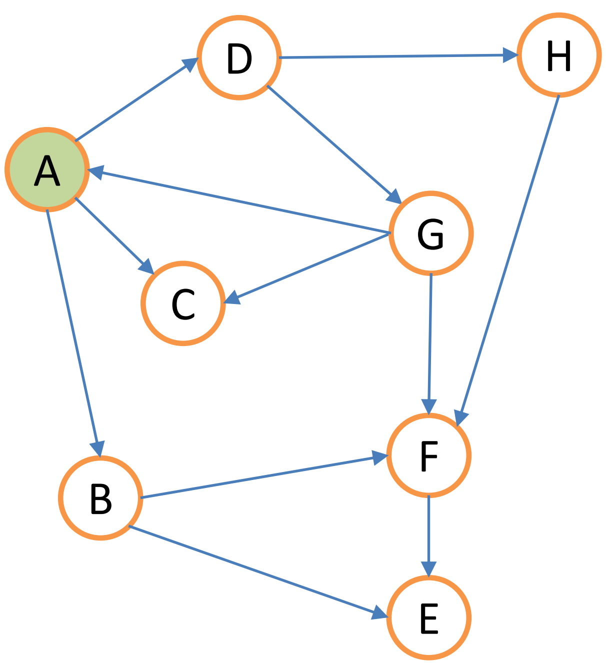


Traversal Order
A

Graph Traversals: Depth First

Container: Stack
Start at A

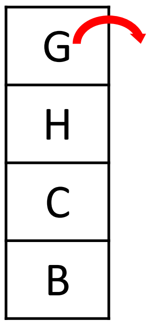


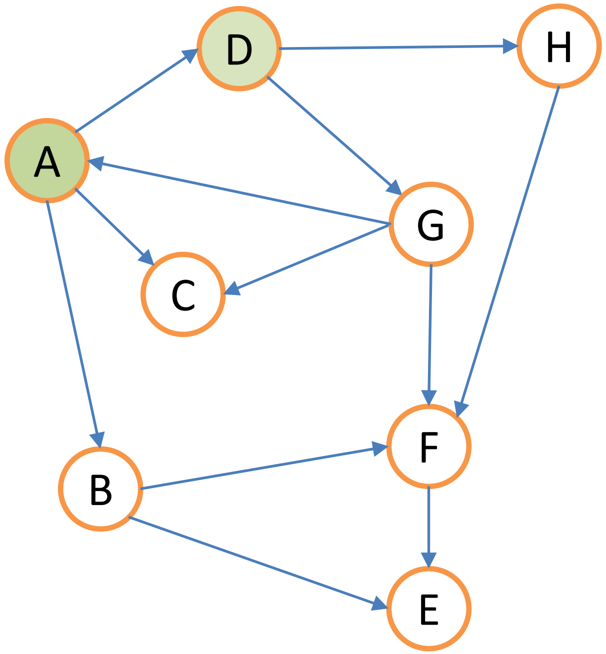


Traversal Order
A

Graph Traversals: Depth First

Container: Stack
Start at A

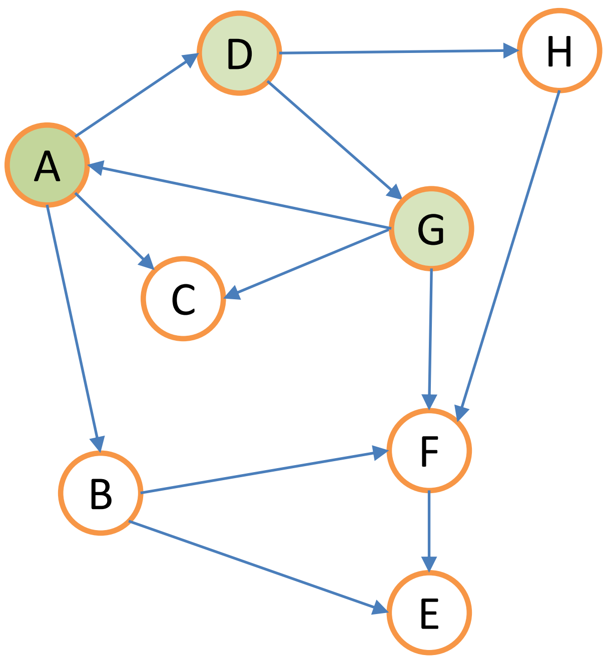


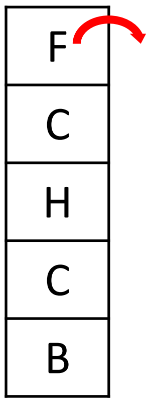


Traversal Order
AD

Graph Traversals: Depth First

Container: Stack
Start at A

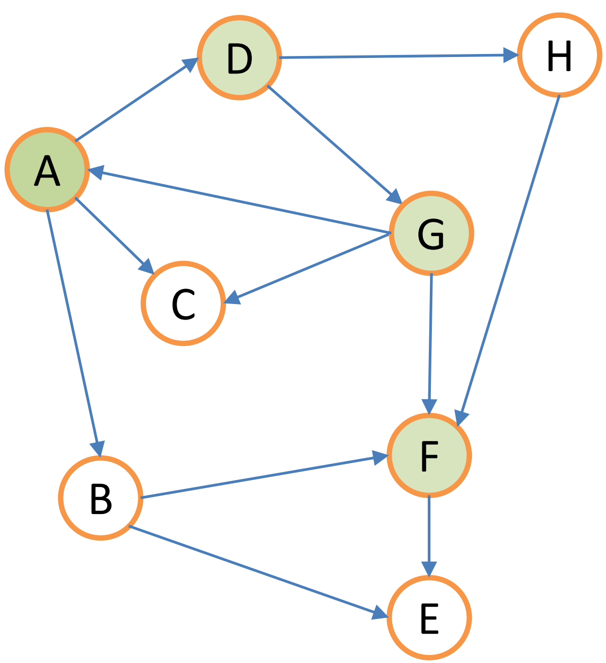




Traversal Order
ADG

Graph Traversals: Depth First

Container: Stack
Start at A

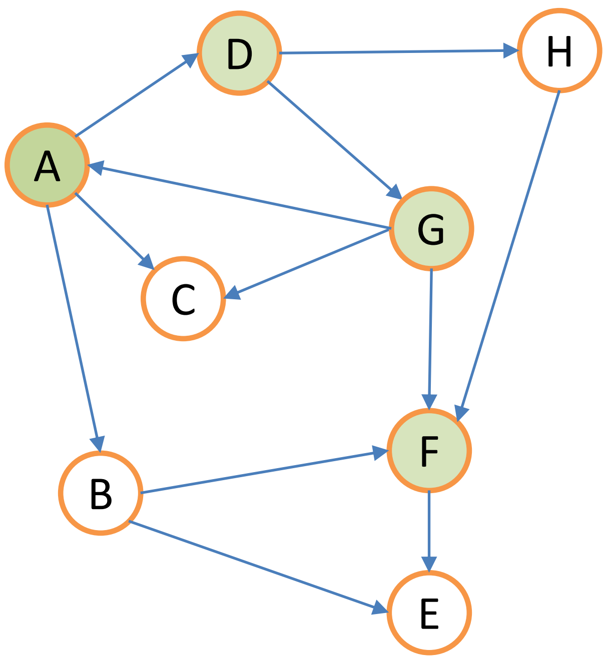


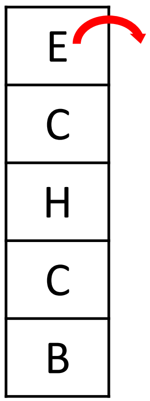
E
C
H
C
B

Traversal Order
ADGF

Graph Traversals: Depth First

Container: Stack
Start at A

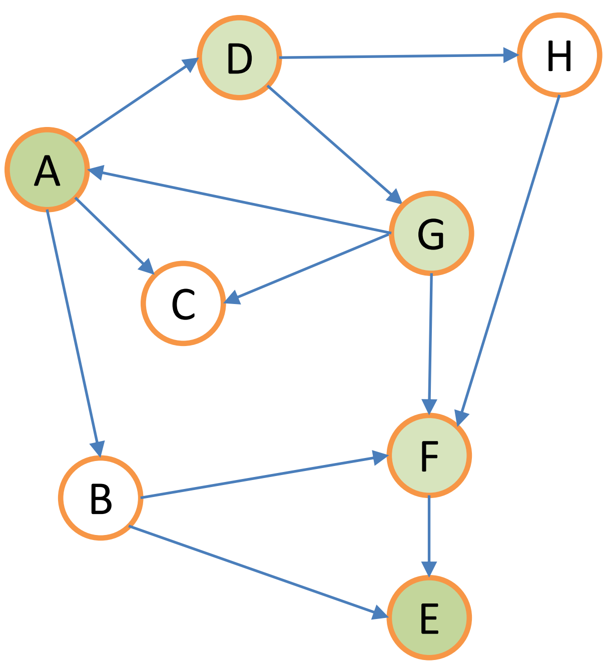


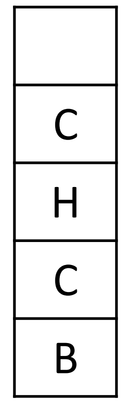


Traversal Order
ADGF

Graph Traversals: Depth First

Container: Stack
Start at A

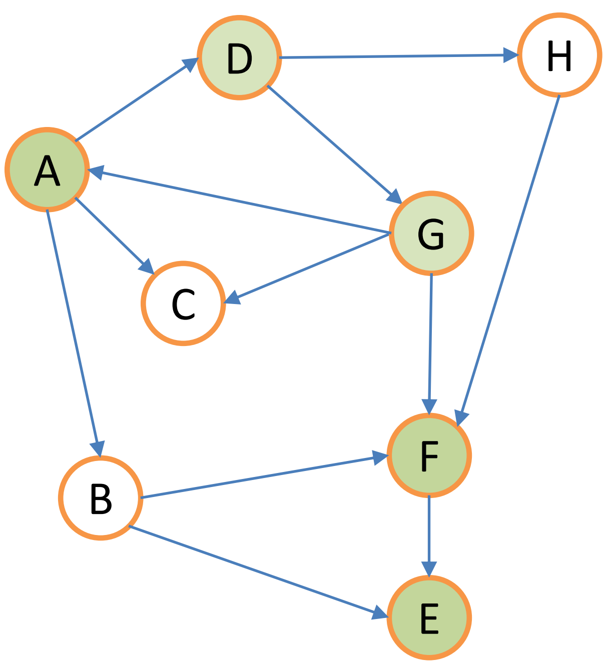


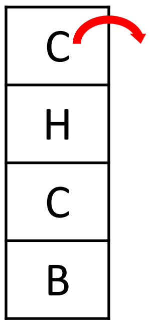


Traversal Order
ADGFE

Graph Traversals: Depth First

Container: Stack
Start at A

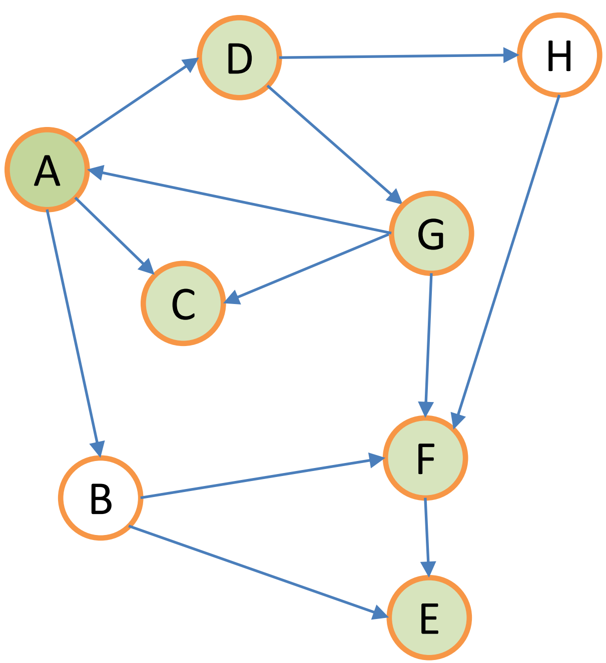


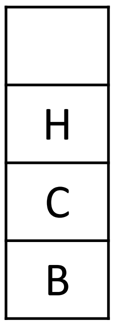


Traversal Order
ADGFE

Graph Traversals: Depth First

Container: Stack
Start at A

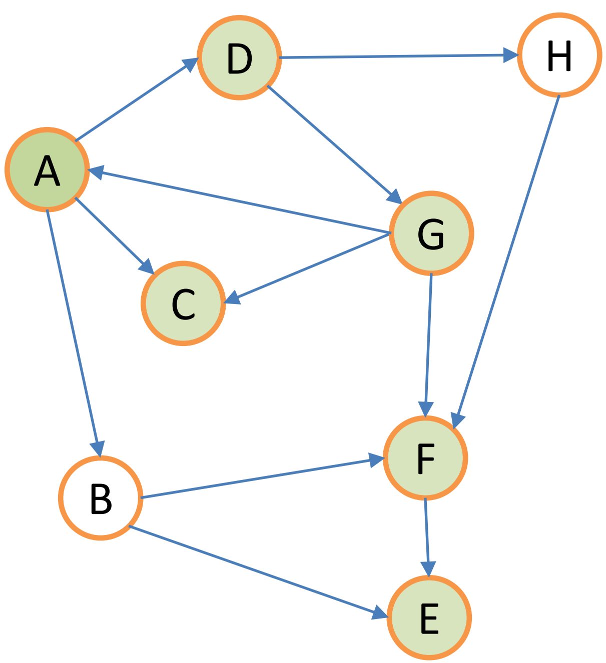


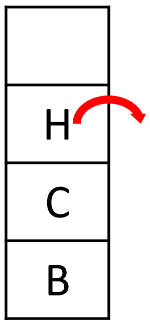


Traversal Order
ADGFEC

Graph Traversals: Depth First

Container: Stack
Start at A

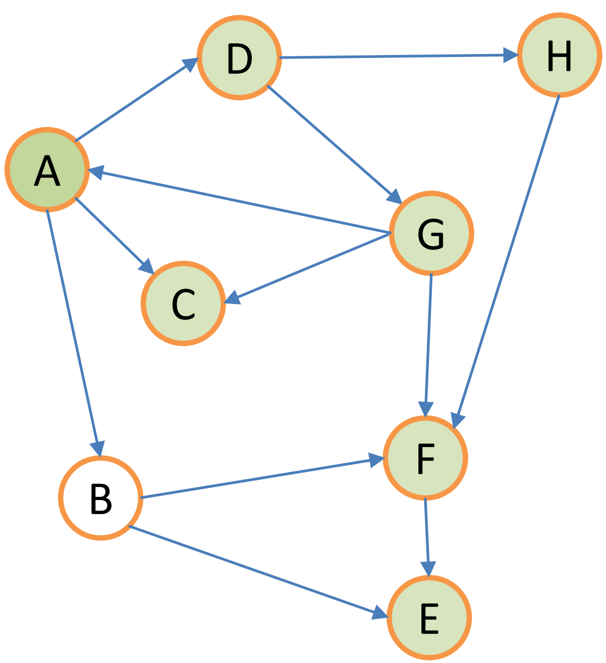


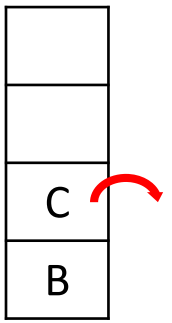


Traversal Order
ADGFEC

Graph Traversals: Depth First

Container: Stack
Start at A

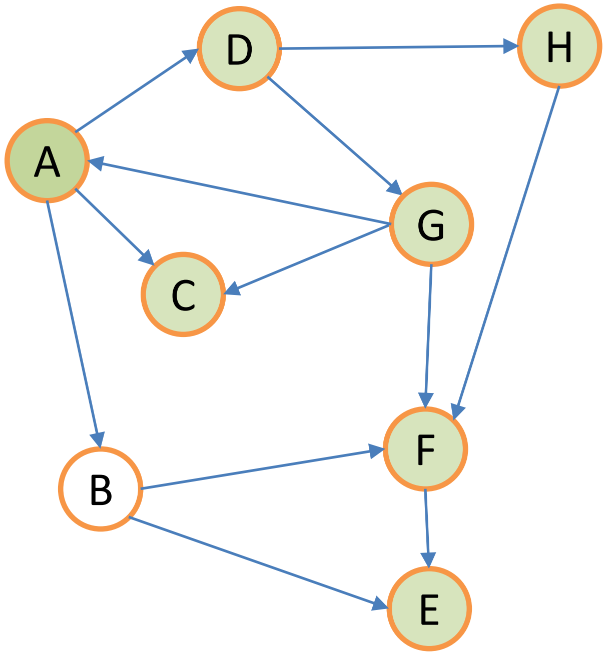


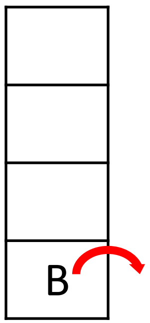


Traversal Order
ADGFECH

Graph Traversals: Depth First

Container: Stack
Start at A

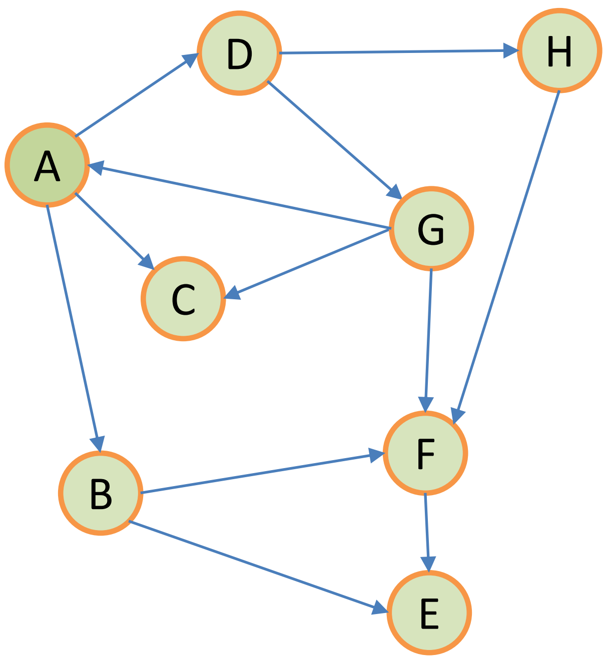


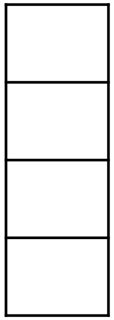


Traversal Order
ADGFECH

Graph Traversals: Depth First

Container: Stack
Start at A



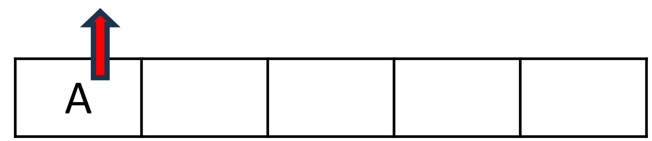


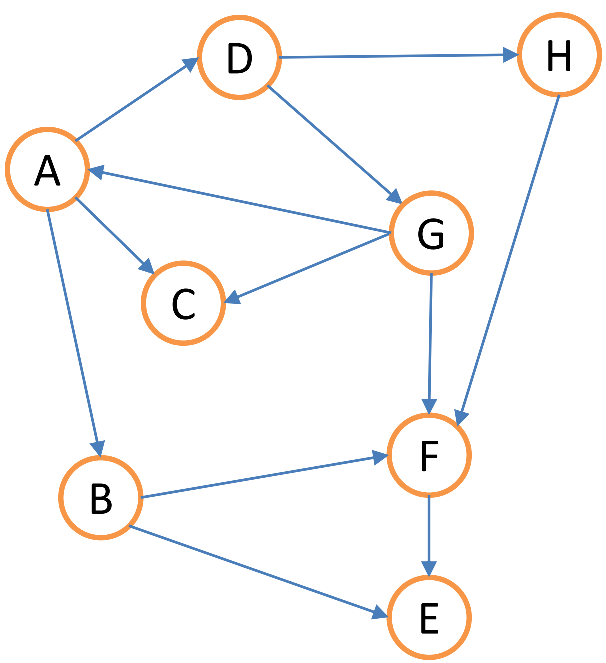
Traversal Order
ADGFECHB

Graph Traversals: Depth First

Container: Queue

Start at A

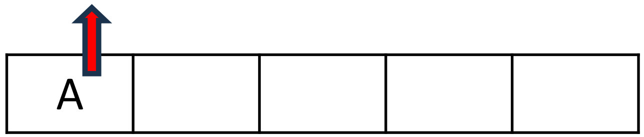


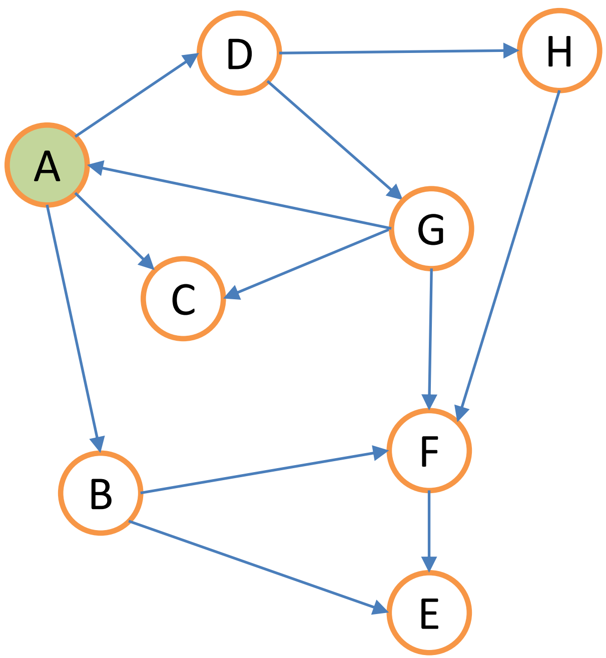


Graph Traversals: Depth First

Container: Queue

Start at A



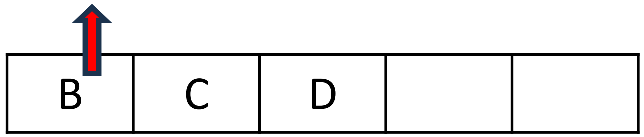


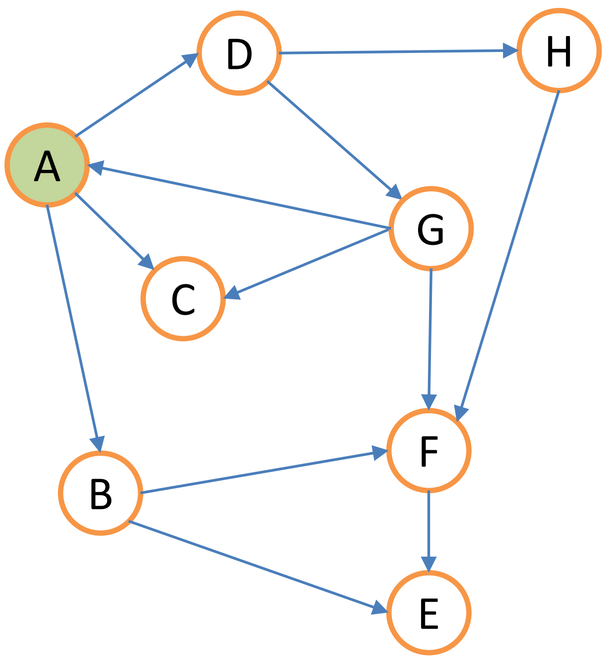
Traversal Order

A

Graph Traversals: Depth First

Container: Queue
Start at A

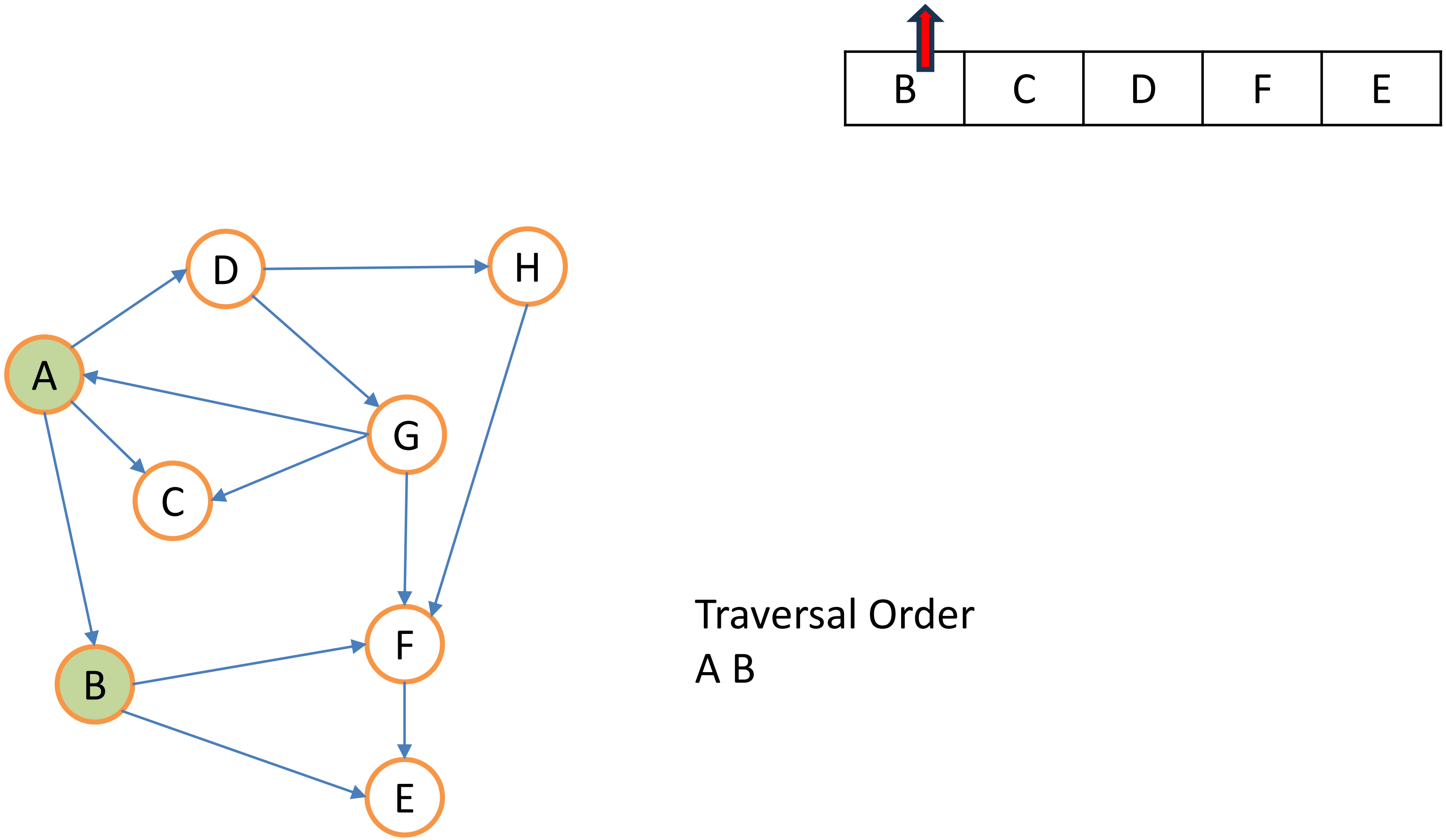




Traversal Order
A

Graph Traversals: Depth First

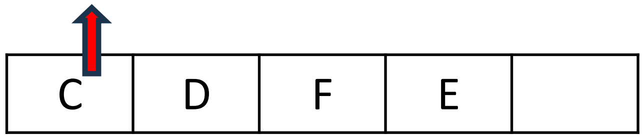
Container: Queue
Start at A

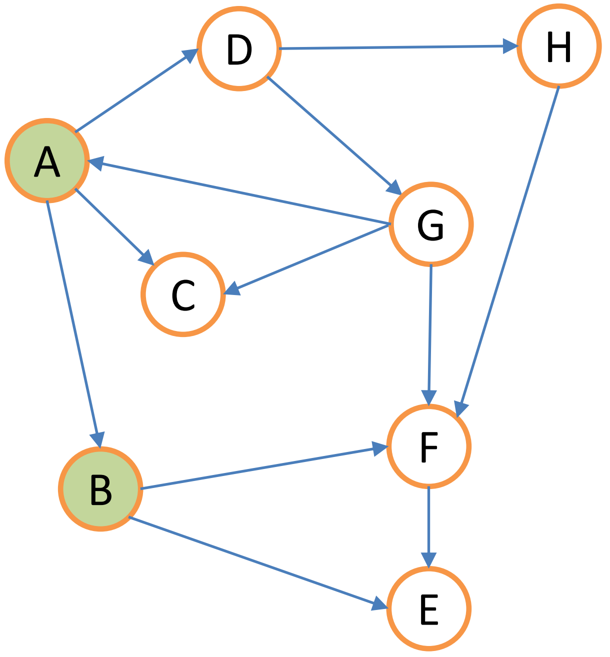


Traversal Order
A B

Graph Traversals: Depth First

Container: Queue
Start at A

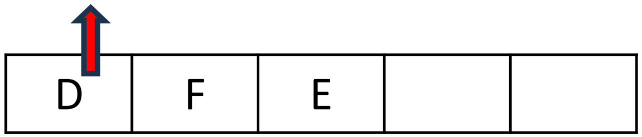


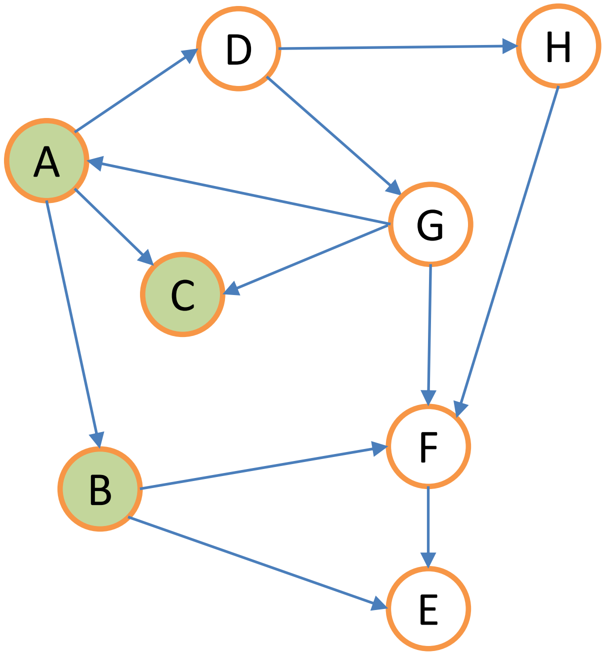


Traversal Order
A B

Graph Traversals: Depth First

Container: Queue
Start at A

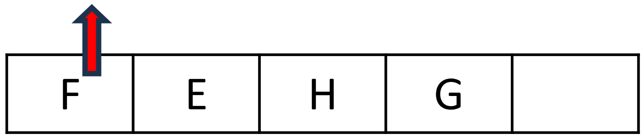


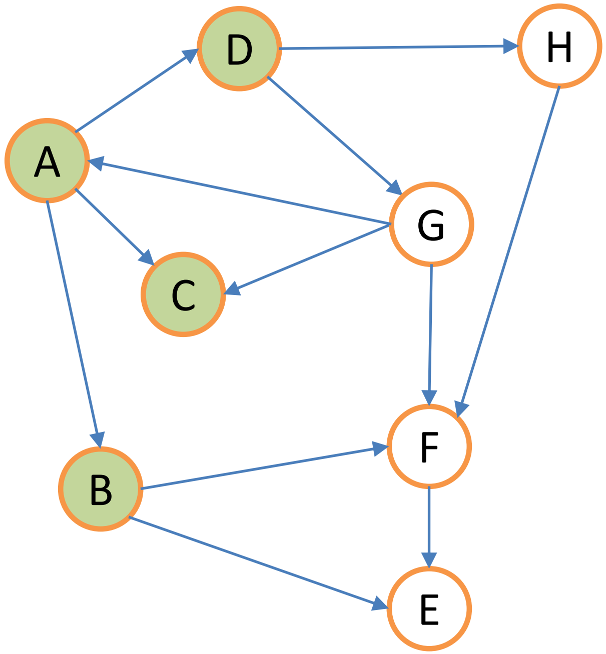


Traversal Order
A B C

Graph Traversals: Depth First

Container: Queue
Start at A

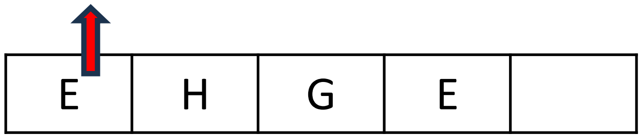


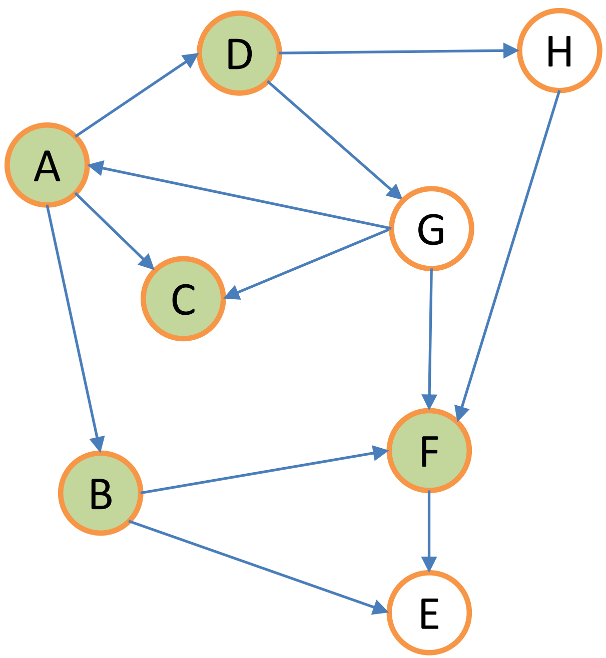


Traversal Order
A B C D

Graph Traversals: Depth First

Container: Queue
Start at A

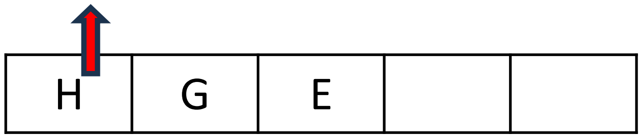


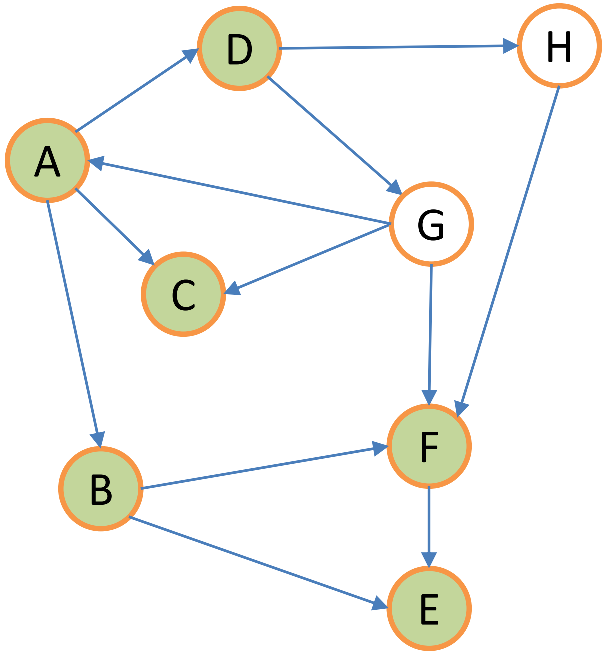


Traversal Order
A B C D F

Graph Traversals: Depth First

Container: Queue
Start at A

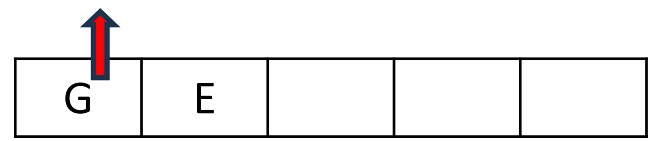


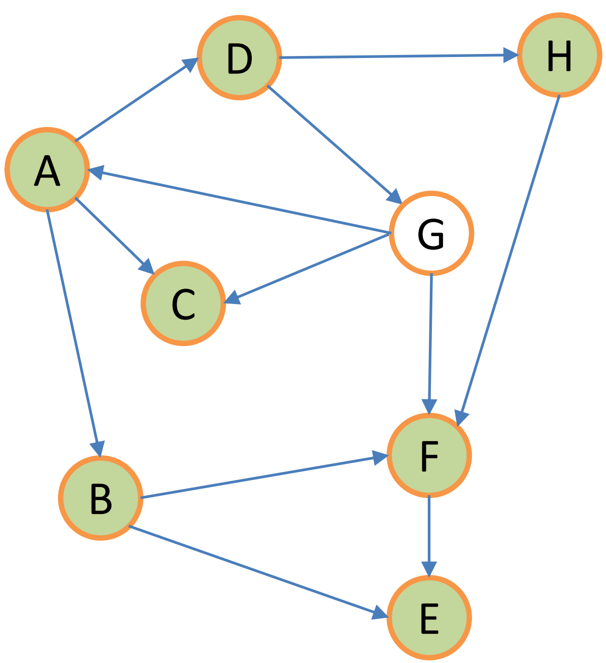


Traversal Order
A B C D F E

Graph Traversals: Depth First

Container: Queue
Start at A

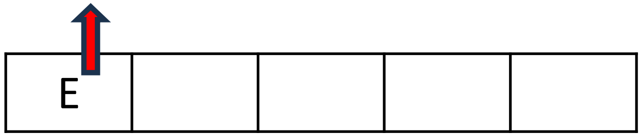


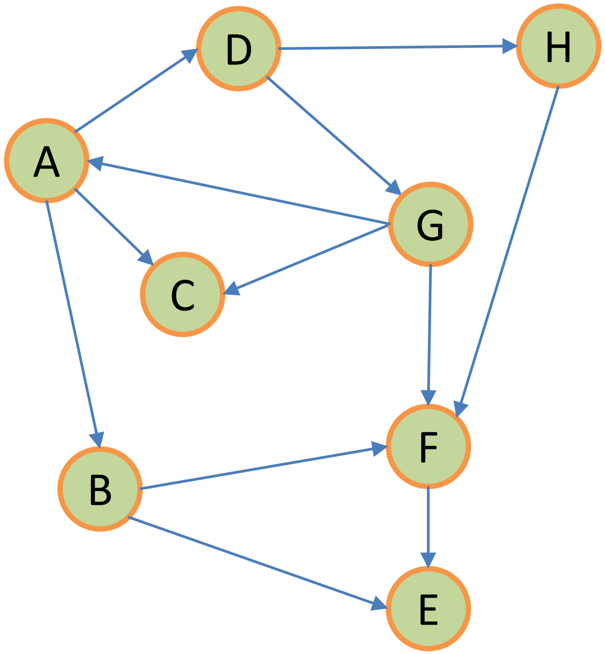


Traversal Order
A B C D F E H

Graph Traversals: Depth First

Container: Queue
Start at A

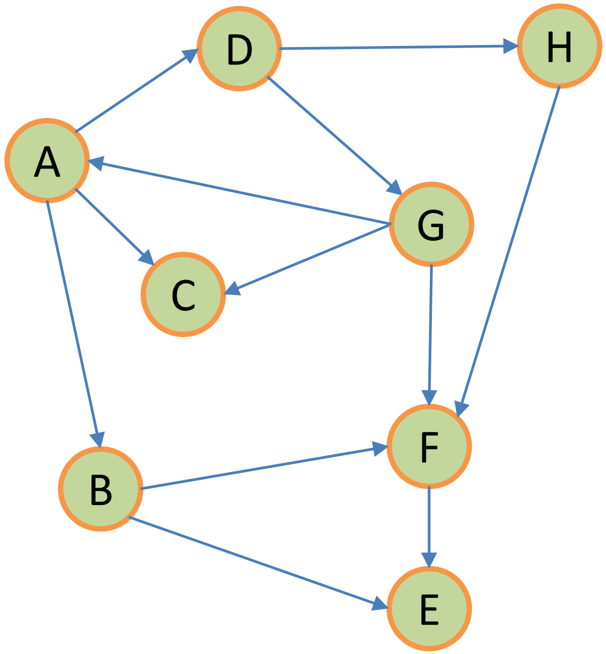




Traversal Order
A B C D F E H G

Graph Traversals: Depth First

Container: Queue
Start at A



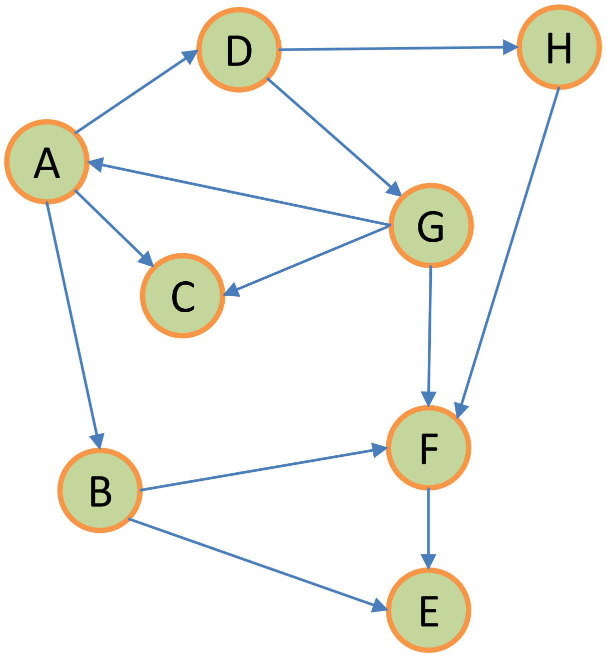
Traversal Order
A B C D F E H G

Graph Traversals

- Breath-first traversal
- Depth-first traversal
- **Example 1: Starting at A**
  - If *C is a Stack, one order of traversal is:*
    - *A, D, H, F, E, G, C, B*
    - Another traversal is: A, B, E, F, C, D, G, H
  - If *C is a Queue, one order of traversal is:*
    - *A, B, C, D, E, F, G, H*
    - Another traversal is: A, D, C, B, H, G, F, E

Graph Traversals

- Exercise: **Starting at G**
  - If *C is a Stack, one order of traversal is:*
  - If *C is a Queue, one order of traversal is:*



Notes

- Depth-first: Descendants are visited before siblings.
  - To traverse depth-first, use a Stack.
- Breadth-first: siblings are visited before descendants.
  - To traverse breadth-first, use a Queue.
- For all vertices to be visited from any node, the graph must be strongly connected.
- For weakly connected graphs, you'd need to exhaustively traverse from every vertex.

```
for each vertex, v, in graph, G
    GraphSearch(G, v)
```