Optimization II

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AY 22/23 Trimester 2

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usually for local extreme points.

The FDT-GEV

$(-\infty, 1)$ $(-2, \infty)$

We finish off this chapter with the last theorem of the course.

Theorem (First Derivative Test for Global Extreme Values)

Suppose that c is a critical point of a continuous function f defined on an interval I. → can be unbounded $(0, \infty)$

1. If $f'(x) > 0$ for **all** $x < c$ and $f'(x) < 0$ for **all** $x > c$, then $f(c)$ is the global maximum value of f on I.
2. If $f'(x) < 0$ for **all** $x < c$ and $f'(x) > 0$ for **all** $x > c$, then $f(c)$ is the global minimum value of f on I.

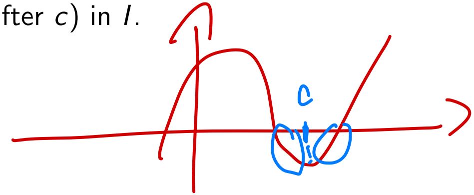
Note that this interval I can be **unbounded**, e.g. $(2, \infty)$, $(-\infty, -1]$, $(-\infty, \infty)$, unlike in the ICBM, where the interval has to be bounded; $[a, b]$.

FDT versus FDT-GEV

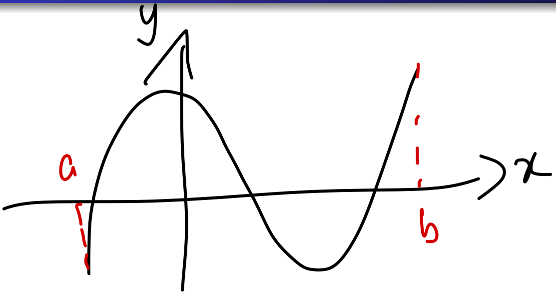
under very strict circumstances

Key differences between the FDT-GEV compared to the FDT:

- FDT shows local max/min, unlike FDT-GEV that shows global max/min.
- FDT only concerns with the function for x that is **near** c, while FDT-GEV looks at the function for all values of x (before and after c) in I.

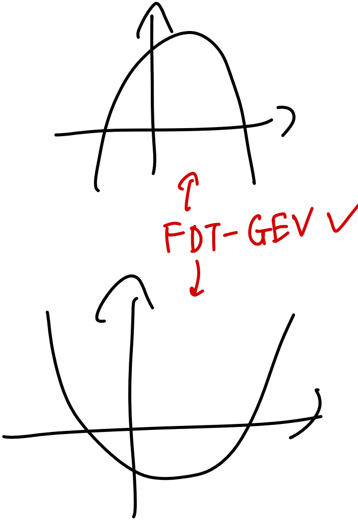


Visualization



ICBM ✓

FDT-GEV ✗



Example 1

A cylindrical can is to be made to hold 1000 cm^3 of oil. Find the dimensions of the can that will minimize the cost of metal to manufacture.

The volume of the can is

$$V = \underline{\pi r^2 h},$$

$$\pi r^2 h = 1000$$

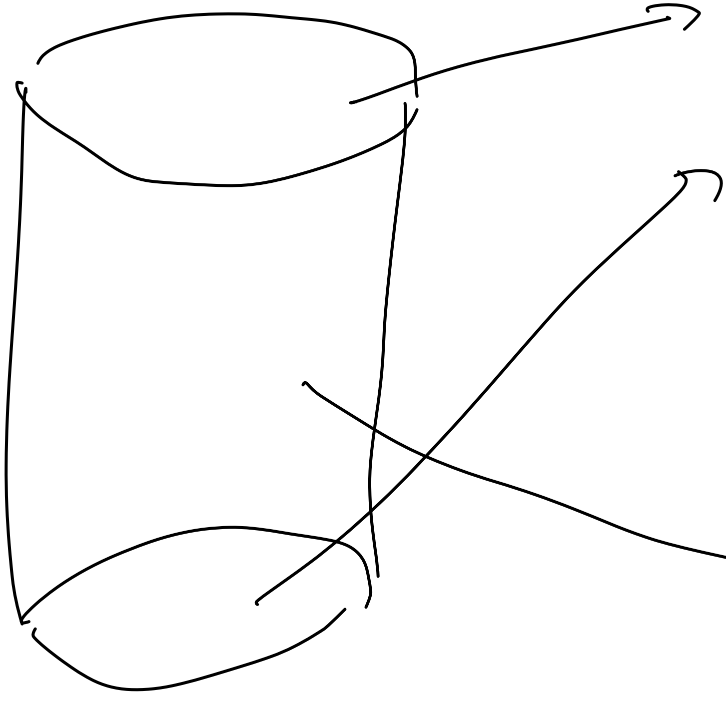
$$h = \frac{1000}{\pi r^2}$$

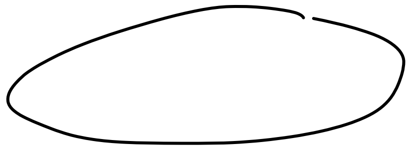
where r and h is the radius and the height (in cm) of the can respectively. The function that we are aiming to minimize is the surface area of the can, which is

$$A = \underline{2\pi r^2 + 2\pi r h}.$$

We need to express A in terms of one variable, thus we use the volume equation to express h in terms of r:

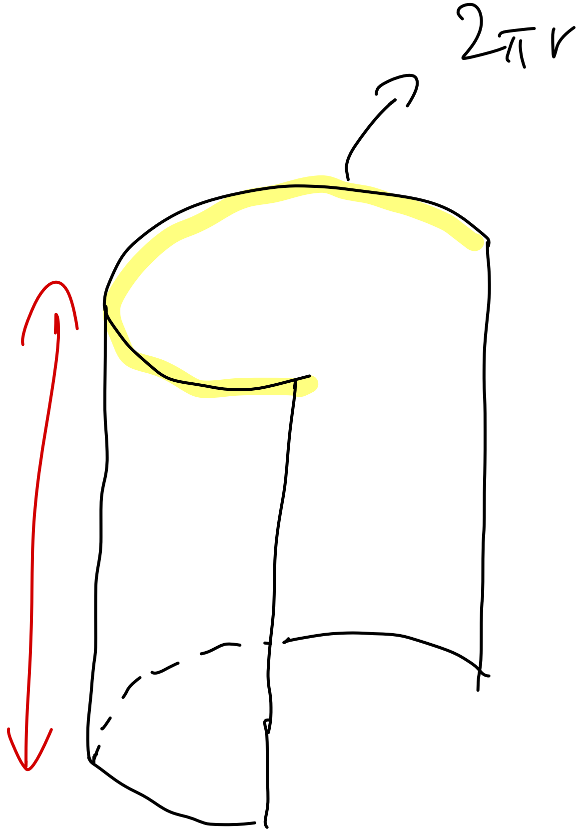
$$h = \underline{\frac{1000}{\pi r^2}}..$$

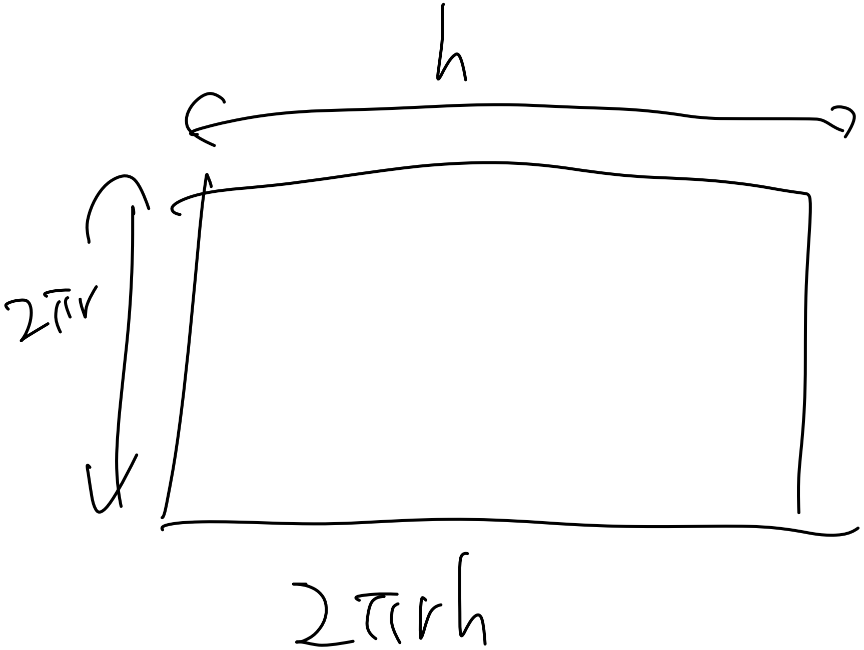




$\pi r^2 \quad \times 2$

$$A = 2\pi r^2 + 2\pi r h$$





Example 1

Then A can be written in terms of r only:

$$A(r) = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}.$$

The domain of this function is $(0, \infty)$.

$r^3 = \frac{500}{\pi}$ $4\pi r^3 = 2000$ $\pi r^3 = 500$

We find the critical points of this function by first differentiating A with respect to r:

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4\pi r^3 - 2000}{r^2}. = 0$$

Critical points are $r = 0$ (ignored), and $r = \sqrt[3]{\frac{500}{\pi}}$.

Observe from the expression of A' that $A'(r)$ < 0 when $r < \sqrt[3]{\frac{500}{\pi}}$, and $A'(r)$ > 0 when $r > \sqrt[3]{\frac{500}{\pi}}$. Therefore $r = \sqrt[3]{\frac{500}{\pi}}$ is a global minimum point.

$r = 5$ $\quad$ $\sqrt[3]{\frac{500}{\pi}}$ $\quad$ $r = 6$

$A' < 0$ $\quad$ $5.41\ldots$ $\quad$ $A' > 0.$

Example 1

We need the dimensions r and h, we already know r, so we can compute h because we have a relation between h and r:

$$h = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = \frac{1000}{\pi\left(\frac{500}{\pi}\right)^{2/3}} = \frac{1000}{\pi}\cdot\left(\frac{\pi}{500}\right)^{2/3} = 2\cdot\frac{500}{\pi}\cdot\frac{\pi^{2/3}}{500^{2/3}}$$

$$= 2\sqrt[3]{\frac{500}{\pi}}.$$

Hence, the dimensions of the cylindrical can that minimizes the cost of metal to manufacture are

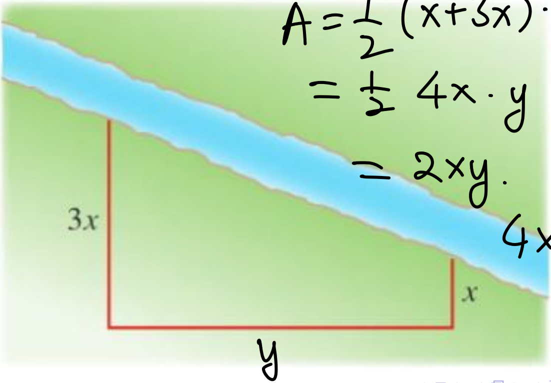
radius $= r = \sqrt[3]{\frac{500}{\pi}}$ cm and height $= h = 2\sqrt[3]{\frac{500}{\pi}}$ cm.

Observation: In this case, the height is twice the size of the radius.

Example 2

A farmer has 400 m of fencing of negligible thickness for enclosing a trapezoidal field along a river as shown below. One of the parallel sides is three times longer than the other. No fencing is needed along the river. Find the largest area the farmer can enclose.

$A = \frac{1}{2}(a+b)h$



$$A = \frac{1}{2}(x+3x)\cdot y$$
$$= \frac{1}{2}\, 4x \cdot y$$
$$= 2xy.$$

$$4x + y = 400$$

Example 2

Let x be the shorter end on one of the widths. Let y be the length of the field. Then the area A of the trapezoidal field in terms of x and y is

$$A = \underline{\quad 2xy \quad}.$$

Since the farmer has 400 m of fencing, we have another relation between x and y:

$$4x + y = 400.$$

We want A to be in terms of x only, so $y = 400 - 4x$, and thus

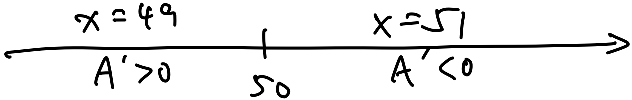
$$A(x) = \underline{\quad 2x(400-4x) = 800x - 8x^2 \quad}.$$

Differentiating A with respect to x gives

$$A'(x) = \underline{\quad 800 - 16x \quad}.$$

Example 2

Thus the only critical point of A is $x =$ 50.



We also notice that $A'(x)$ > 0 for $x <$ 50 and $A'(x)$ < 0 for $x >$ 50. So by the FDT-GEV, $x =$ 50 is a global maximum point. Thus $y =$ 200, and the largest area the farmer can enclose is

$$A(50) = 2 \cdot 50 \cdot 200 = 20000\,m^2$$

Note: This example can also be done using the ICBM. You only need to realize that the interval for x is $[0, 100]$.

Exercise 1

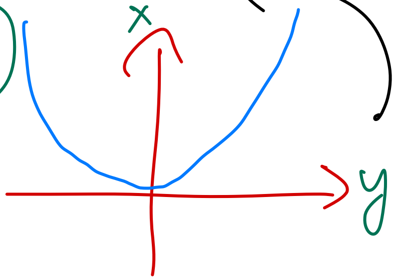
Find the ~~point~~ coordinates on the parabola $y^2 = 2x$ that is closest to the ~~point~~ coordinates (1, 4)?

(x_0, y_0) distance to (x_1, y_1)

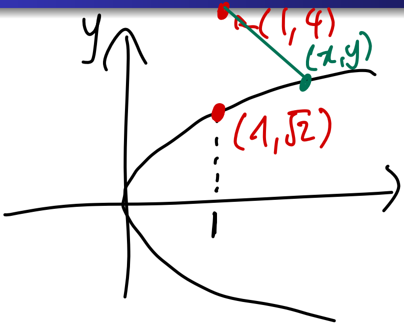
$$d = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}$$

$$y^2 = 2x \iff x = \frac{y^2}{2}$$

$$y = \frac{x^2}{2}$$



Exercise 1



$y^2 = 2x$

where is (1, 4)

let (x, y) be any coordinate on $y^2 = 2x$.

Distance between (x, y) & (1, 4) $\Rightarrow \sqrt{(x-1)^2 + (y-4)^2}$

square

✓ $(x-1)^2 + (y-4)^2$

Exercise 1

Let $D = (x-1)^2 + (y-4)^2$

change everything to y (or x)?

$y^2 = 2x \longrightarrow x = \frac{y^2}{2}$ ✓

$\searrow y = \pm\sqrt{2x}$ ✗

$D'(y) = \frac{2(y^2-2)\cdot 2y}{4} + 2(y-4)$

$D(y) = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2$

$= \frac{(y^2-2)^2}{4} + (y-4)^2$

Exercise 1

$$D'(y) = y(y^2-2) + 2(y-4)$$

$$= y^3 - 2y + 2y - 8 = y^3 - 8 = 0$$

$$\Rightarrow y = 2, \quad x = \frac{y^2}{2} = \frac{2^2}{2} = 2$$

$y=1$	2	$y=3$
$D' < 0$		$D' > 0$

Coordinates on the graph $y^2 = 2x$ that is closest to $(1,4)$ is $(2,2)$.