CSD1241 Tutorial 10

Question 1. Determine which of the following maps are affine transformations. Further find the matrix A and the vector $\vec{b}$ of the affine transformations.
(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+1 \\ \sqrt{x}+y+1 \end{pmatrix}$$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+1 \\ y \\ z-x-y \end{pmatrix}$$

(c) $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+1 \\ z-x^2-y \end{pmatrix}$$

(d) $T : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ 2y+2 \\ y+3x \end{pmatrix}$$

Solution. (a) and (c) are not affine transformations.

(b) is an affine transformation with $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

(d) is an affine transformation with $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \\ 3 & 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$.

□

In the next problems, we use the 3-step approach

- Translating to the origin
- Performing the transformation around the origin (rotation, reflection, etc.)
- Translating back using the same vector

The 3 steps can be summarized by the formula

$$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0$$

Equivalently, we have

$$T(\vec{x}) = A\vec{x} + \vec{b} \text{ with } \vec{b} = \vec{x}_0 - A\vec{x}_0$$

Question 2. Consider the line $l : x - 2y = 5$ in $\mathbb{R}^2$.

(a) Write the reflection T through l as an affine map

$$T(\vec{x}) = A\vec{x} + \vec{b}$$

Hint. Take $\vec{x}_0 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ and $A =$ matrix of reflection through $l' : x - 2y = 0$, which is a line through O and parallel to l.

(b) Find the image of the points $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \end{pmatrix}$ under T.

(c) Find the image of the line $m : x + y = 3$ under T.

Solution. (a) Using $\vec{x}_0 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ and $\vec{d} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, we have $T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0 = A\vec{x} + \vec{b}$ with

$$\begin{aligned} A &= I_2 - \frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \\ \vec{b} &= \vec{x}_0 - A\vec{x}_0 = \begin{pmatrix} 6 \\ 0 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} \end{aligned}$$

We obtain

$$T(\vec{x}) = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\vec{x} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

(b) The images of the given points are

$$\begin{aligned} \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix} &= \begin{pmatrix} 21/5 \\ -22/5 \end{pmatrix} \\ \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix} &= \begin{pmatrix} 23/5 \\ -11/5 \end{pmatrix} \\ \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 5 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix} &= \begin{pmatrix} 21/5 \\ 3/5 \end{pmatrix} \end{aligned}$$

(c) The line m has vector equation $\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The image of the line m is

$$m' : \vec{x} = \frac{1}{5} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 21/5 \\ -22/5 \end{pmatrix} + \frac{t}{5} \begin{pmatrix} 1 \\ -7 \end{pmatrix},$$

which is a line through $\begin{pmatrix} 21/5 \\ -22/5 \end{pmatrix}$ and having normal vector $\begin{pmatrix} 7 \\ 1 \end{pmatrix}$. The normal equation of m' is

$$7\left(x - \frac{21}{5}\right) + 1\left(y + \frac{2}{5}\right) = 0 \Leftrightarrow ...$$

$\square$

Question 3. Consider the line $l : x + y = 3$ in $\mathbb{R}^2$.

(a) Write the projection T through l as an affine map

$$T(\vec{x}) = Ax + \vec{b}$$

(b) Find the image of the line $m : x - y = 5$ under T.

Solution. (a) Taking $\vec{x}_0 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and $\vec{d} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, we have

$$\begin{aligned} A &= \frac{1}{||\vec{d}||^2} \vec{d}\vec{d}^T = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ \vec{b} &= \vec{x}_0 - A\vec{x}_0 = \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix} \end{aligned}$$

We obtain

$$T(\vec{x}) = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix}$$

(b) The line m has vector equation $\vec{x} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So the image of m is

$$m' : \vec{x} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \left(\begin{pmatrix} 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix},$$

which is a point. This makes sense because the line m is orthogonal to the line l. $\square$

Question 4. (a) Write the rotation T around the point $Q = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ over $\theta = -45°$ as

$$T(\vec{x}) = A\vec{x} + \vec{b}.$$

(b) Find the image of the line $m : x + y = 3$ under T.

Solution. (a) Take $\vec{x}_0 = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$. The rotation matrix about O over $\theta = -45°$ is

$$A = \begin{pmatrix} \cos(-45°) & -\sin(-45°) \\ \sin(-45°) & \cos(-45°) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

The vector $\vec{b}$ is

$$\vec{b} = \vec{x}_0 - A\vec{x}_0 = \begin{pmatrix} -1 \\ 5 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 - 2\sqrt{2} \\ 5 - 3\sqrt{2} \end{pmatrix}.$$

We obtain

$$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -1 - 2\sqrt{2} \\ 5 - 3\sqrt{2} \end{pmatrix}$$

(b) The line $m : x + y = 3$ has vector equation $\vec{x} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The image of m is

$$\begin{aligned} m' : \vec{x} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \left(\begin{pmatrix} 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} -1 - 2\sqrt{2} \\ 5 - 3\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} -1 - \sqrt{2}/2 \\ 5 - 9\sqrt{2}/2 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

Note that m' has direction vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. So it is a vertical line (a line with equation $x = k$). Its normal equation is

$$m' : x = -1 - \frac{\sqrt{2}}{2}.$$

□

Question 5. Write the shear parallel to the line $l : x + y = 3$ in the direction $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ as an affine map $T(\vec{x}) = A\vec{x} + \vec{b}$.

Solution. Taking $\vec{x}_0 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and $\vec{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we have

$$\begin{aligned} A &= I_2 + \frac{1}{||\vec{n}||}\vec{v}\vec{n}^T = I_2 + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}+1 & 1 \\ -1 & \sqrt{2}-1 \end{pmatrix} \\ \vec{b} &= \vec{x}_0 - A\vec{x}_0 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix} 1+\sqrt{2} & 1 \\ -1 & \sqrt{2}-1 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -3/\sqrt{2} \\ 3/\sqrt{2} \end{pmatrix}, \end{aligned}$$

we obtain

$$T(\vec{x}) = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}+1 & 1 \\ -1 & \sqrt{2}-1 \end{pmatrix}\vec{x} + \begin{pmatrix} -3/\sqrt{2} \\ 3/\sqrt{2} \end{pmatrix}.$$

□

Question 6. Consider the plane the plane $\alpha : 2x - 4y + 3z = 12$ in $\mathbb{R}^3$.
(a) Write the reflection T through α as an affine map $T(\vec{x}) = A\vec{x} + \vec{b}$.
(*Hint.* Take $\vec{x}_0 = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$) .
(b) Find the image of the points $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}$ under T.
(c) Find the image of the plane $\beta : x + y + z = 3$ under T.

Solution. (a) Take $\vec{x}_0 = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$. The matrix A and vector $\vec{b}$ are

$$\begin{aligned} A &= I_2 - \frac{2}{||\vec{n}||^2}\vec{n}\vec{n}^T = \frac{1}{29}\begin{pmatrix} 21 & 16 & -12 \\ 16 & -3 & 24 \\ -12 & 24 & 11 \end{pmatrix} \\ \vec{b} &= \vec{x}_0 - A\vec{x}_0 = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{29}\begin{pmatrix} 21 & 16 & -12 \\ 16 & -3 & 24 \\ -12 & 24 & 11 \end{pmatrix}\begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{29}\begin{pmatrix} 48 \\ -96 \\ 72 \end{pmatrix} \end{aligned}$$

we obtain

$$T(\vec{x}) = A\vec{x} + \vec{b} = \frac{1}{29}\begin{pmatrix} 21 & 16 & -12 \\ 16 & -3 & 24 \\ -12 & 24 & 11 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \frac{1}{29}\begin{pmatrix} 48 \\ -96 \\ 72 \end{pmatrix}$$

(b) The images of the given points are given in the columns of the following matrix

$$\frac{1}{29}\begin{pmatrix} 21 & 16 & -12 \\ 16 & -3 & 24 \\ -12 & 24 & 11 \end{pmatrix}\begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} + \frac{1}{29}\begin{pmatrix} 48 \\ -96 \\ 72 \end{pmatrix} = \begin{pmatrix} 89/29 & 127/29 & 165/29 \\ -62/29 & -51/29 & -40/29 \\ 119/29 & 60/29 & 1/29 \end{pmatrix}$$

(c) The plane β has vector equation $\vec{x} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. Its image is

$$\begin{aligned} m' : \vec{x} &= \frac{1}{29}\begin{pmatrix} 21 & 16 & -12 \\ 16 & -3 & 24 \\ -12 & 24 & 11 \end{pmatrix}\left(\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right) + \frac{1}{29}\begin{pmatrix} 48 \\ -96 \\ 72 \end{pmatrix} \\ &= \begin{pmatrix} 111/29 \\ -48/29 \\ 36/29 \end{pmatrix} + \frac{s}{29}\begin{pmatrix} 5 \\ 19 \\ -36 \end{pmatrix} + \frac{t}{29}\begin{pmatrix} 28 \\ -27 \\ 13 \end{pmatrix}. \end{aligned}$$

The line m' contains the point $\begin{pmatrix} 111/29 \\ -48/29 \\ 36/29 \end{pmatrix}$ and has normal vector

$$\vec{n} = \begin{pmatrix} 5 \\ 19 \\ -36 \end{pmatrix} \times \begin{pmatrix} 28 \\ -27 \\ 13 \end{pmatrix} = \begin{pmatrix} -725 \\ -1073 \\ -667 \end{pmatrix} = (-29)\begin{pmatrix} 25 \\ 37 \\ 23 \end{pmatrix} \parallel \begin{pmatrix} 25 \\ 37 \\ 23 \end{pmatrix}$$

The normal equation of m' is

$$25\left(x - \frac{11}{29}\right) + 37\left(y + \frac{48}{29}\right) + 23\left(z - \frac{36}{29}\right) = 0 \Leftrightarrow 25x + 37y + 23z = -\frac{673}{29}$$

$\square$