1(d), 3(f), 3(c)

Derivative at a point a Def$^{\underline{n}}$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad (1)$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad (2)$$

1. Required: Def^{n} of derivative

1(d) $f(x) = \dfrac{1}{\sqrt{2x+2}}$, $a = 1$

1st interpret

$f(x+h) = \dfrac{1}{\sqrt{2(x+h)+2}}$, $f(1) = \dfrac{1}{2}$

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{\frac{2}{2\sqrt{2x+2}} - \frac{\sqrt{2x+2}}{2\sqrt{2x+2}}}{x - 1}$$

$$= \lim_{x \to 1} \frac{\frac{2 - \sqrt{2x+2}}{2\sqrt{2x+2}}}{x - 1}$$

$$\lim_{x \to 1} \frac{(2-\sqrt{2x+2})}{(2\sqrt{2x+2})(x-1)} \cdot \frac{(2+\sqrt{2x+2})}{(2+\sqrt{2x+2})}$$

$$= \lim_{x \to 1} \frac{4-(2x+2)}{(2\sqrt{2x+2})(x-1)(2+\sqrt{2x+2})}$$

$$= \lim_{x \to 1} \frac{-2\cancel{(x-1)}}{(2\sqrt{2x+2})\cancel{(x-1)}(2+\sqrt{2x+2})}$$

$$= \lim_{x \to 1} \frac{-2}{(2\sqrt{2x+2})(2+\sqrt{2x+2})}$$

$$= \frac{-2}{2\sqrt{4} \cdot (2+\sqrt{4})} = \frac{-2}{4 \cdot 4} = -\frac{1}{8}.$$

1(b) $f(x) = 5x^4$, $a = -1$

$f(\underbrace{a+h}_{-1+h}) = 5(h-1)^4$ $\quad f(a) = 5.$

don't need to expand.

$$f'(-1) = \lim_{h\to 0} \frac{5\overbrace{(h-1)^4} - 5}{h}$$

$a = (h-1)^2$

$b = 1$

$a^2 - b^2$

$= (a+b)(a-b)$

$$= 5 \lim_{h\to 0} \frac{(h-1)^4 - 1^4}{h}$$

$(h-1)^4 - 1^4$

$= ((h-1)^2)^2 - (1)^2$

$$= 5 \lim_{h\to 0} \frac{\overbrace{((h-1)^2 - 1)}((h-1)^2+1)}{h}$$

difference of squares

$$= 5 \lim_{h\to 0} \frac{\left((h-1)^2 - 1\right)\left((h-1)^2+1\right)}{h}$$

$$= 5 \lim_{h\to 0} \frac{\left((h-1)-1\right)\cancel{\left((h-1)+1\right)}^{\,h}\left((h-1)^2+1\right)}{\cancel{h}}$$

$$= 5 \lim_{h\to 0} (h-2)\underbrace{\left((h-1)^2+1\right)}_{\left((-1)^2+1\right)}$$

$$= 5 \cdot (-2)(2) = -20.$$

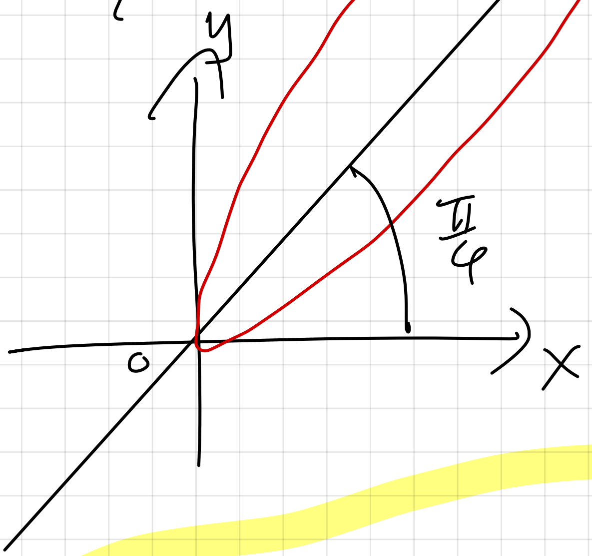
$$f(x) = 5x^4 \rightarrow f'(x) = 20x^3$$

$$f'(-1) = -20$$

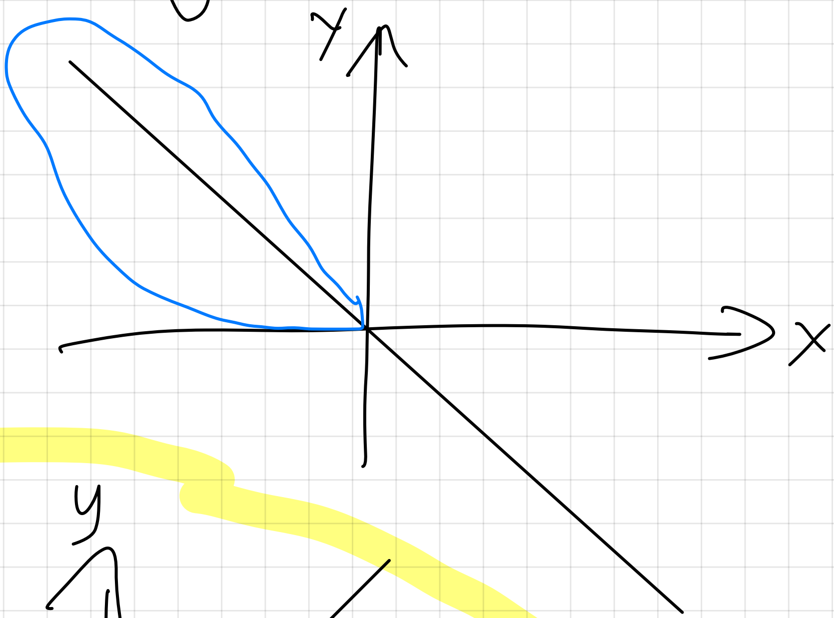
Q2

(a) ? → $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

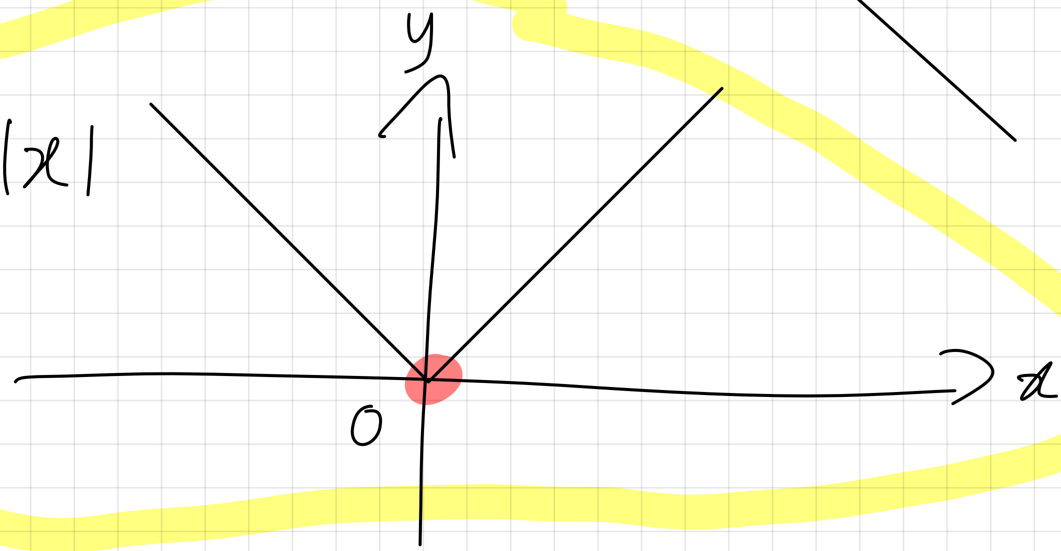
$y = x$



$y = -x$.



$f(x) = |x|$



(b) f not differentiable at 0.

$f(x) = |x|$

limit $f'(0)$

$$= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{|x|}{x}$$

$x > 0$: $\frac{x}{x}$

$x < 0$: $\frac{-x}{x}$

Done in lecture 2

This limit doesn't exist.

$\hookrightarrow$ f is not differentiable at 0.

3(c) $f(x) = 4 + 8x - 5x^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad (1)$$

$$= \lim_{y \to x} \frac{f(y) - f(x)}{y - x} \quad (2)$$

$$f(y) = 4 + 8y - 5y^2$$

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}$$

$$= \lim_{y \to x} \frac{\cancel{4} + 8y - 5y^2 - (\cancel{4} + 8x - 5x^2)}{y - x}.$$

$$= \lim_{y \to x} \frac{8y - 8x - 5y^2 + 5x^2}{y - x}$$

$$= \lim_{y \to x} \frac{8(y-x) - 5(y^2 - x^2)}{y - x}$$

$$= \lim_{y \to x} \frac{8(y-x) - 5(y-x)(y+x)}{y - x}$$

$$= \lim_{y \to x} \frac{\cancel{(y-x)}\,(8 - 5(y+x))}{\cancel{y-x}}$$

$$= \lim_{y \to x} 8 - 5(y+x) = 8 - 10x.$$

3(f) $g(x) = \dfrac{1}{1+\sqrt{x}}$ $\rightarrow$ quotient
$\rightarrow$ Chain Rule.

$$g(x+h) = \frac{1}{1+\sqrt{x+h}}$$

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{1}{1+\sqrt{x+h}} - \dfrac{1}{1+\sqrt{x}}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{1}+\sqrt{x} - (\cancel{1}+\sqrt{x+h})}{h(1+\sqrt{x+h})(1+\sqrt{x})}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x} - \sqrt{x+h})}{h(1+\sqrt{x+h})(1+\sqrt{x})} \cdot \frac{(\sqrt{x} + \sqrt{x+h})}{(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h \to 0} \frac{x - (x+h) \;\; -h}{\cancel{h}(1+\sqrt{x+h})(1+\sqrt{x})(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h \to 0} \frac{-1}{(1+\sqrt{x+h})(1+\sqrt{x})(\sqrt{x} + \sqrt{x+h})}$$

$$= \frac{-1}{(1+\sqrt{x})(1+\sqrt{x})\, 2\sqrt{x}} = -\frac{1}{2\sqrt{x}(1+\sqrt{x})^2}$$

$$g(x) = \frac{1}{1+\sqrt{x}}$$

$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

Quotient Rule:

$$g'(x) = \frac{(1+\sqrt{x})\cdot 0 - 1\cdot\frac{1}{2\sqrt{x}}}{(1+\sqrt{x})^2}$$

$$= -\frac{1}{2\sqrt{x}\,(1+\sqrt{x})^2}$$

✓

Q4 (a) $\underbrace{\left(\frac{f}{g}\right)'(x)}_{\text{differentiating } \frac{f(x)}{g(x)}} = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

$g(0) \neq 0$. Write quotient rule for $x=0$.

$$\left(\frac{f}{g}\right)'(0) = \frac{g(0)f'(0) - f(0)g'(0)}{[g(0)]^2}$$

(b) $$R(x) = \frac{x - 3x^3 + 5x^5}{1 + 3x^3 + 6x^6 + 9x^9}$$

Find $R'(0)$ in 2 different ways.

(1) Quotient Rule directly

$R'(x) =$

$$\frac{(1+3x^3+6x^6+9x^9)\cdot(1-9x^2+25x^4) - (x-3x^3+5x^5)(9x^2+36x^5+81x^8)}{(1+3x^3+6x^6+9x^9)^2}$$

$$R'(0) = \frac{1\cdot 1 - 0\cdot 0}{1^2} = 1.$$

$f(x) = x - 3x^3 + 5x^5$

$g(x) = 1 + 3x^3 + 6x^6 + 9x^9$

$f'(x) = 1 - 9x^2 + 25x^4$

$g'(x) = 9x^2 + 36x^5 + 81x^8.$

$f(0)$,	$g(0)$,	$f'(0)$,	$g'(0)$	
0	1	1	0	Obv

(2)

$$R'(0) = \frac{g(0) \cdot f'(0) - f(0)\, g'(0)}{[g(0)]^2}$$

easier

no need to write taylor exp

$$= \frac{1 \cdot 1 - 0 \cdot 0}{1^2} = 1.$$

5. (f), (j) onwards

(f) $R(t) = 4\sqrt[3]{t} = 4t^{\frac{1}{3}}$

$R'(t) = 4 \cdot \frac{1}{3} t^{-\frac{2}{3}}$

$= \frac{4}{3} \cdot \frac{1}{t^{2/3}} = \frac{4}{3} \frac{1}{(t^2)^{\frac{1}{3}}} = \frac{4}{3\sqrt[3]{t^2}}$

$= \frac{4}{3} \frac{1}{(t^{\frac{1}{3}})^2} = \frac{4}{3(\sqrt[3]{t})^2}$

(j) $h(\theta) = e^{\theta}\left(\theta + \theta\,\theta^{\frac{1}{2}}\right)$

$= e^{\theta}\left(\theta + \theta^{\frac{3}{2}}\right)$

$$h'(\theta) = \underline{e^{\theta}(\theta + \theta^{3/2})} + \underline{e^{\theta}\left(1 + \frac{3}{2}\theta^{\frac{1}{2}}\right)}$$

$$= e^{\theta}\left(1 + \frac{3}{2}\theta^{\frac{1}{2}} + \theta + \theta^{\frac{3}{2}}\right)$$

$$= e^{\theta}\left(1 + \frac{3}{2}\sqrt{\theta} + \theta + \theta\sqrt{\theta}\right)$$

(k) $$h(x) = \frac{e^x \sin x}{x^2+1}$$

$$h'(x) = \frac{(x^2+1)(e^x \sin x + e^x \cos x) - e^x \sin x\,(2x)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)(e^x \sin x + e^x \cos x) - e^x \sin x (2x)}{(x^2+1)^2}$$

$$= \frac{e^x\left((x^2+1)(\sin x+\cos x) - 2x\sin x\right)}{(x^2+1)^2}$$

$$= \frac{e^x\left(x^2\sin x + x^2\cos x + \sin x + \cos x - 2x\sin x\right)}{(x^2+1)^2}$$

$(x-1)^2$

$$= \frac{e^x\left(\sin x(x^2-2x+1) + \cos x(x^2+1)\right)}{(x^2+1)^2}$$

(l) $$g(u) = \frac{3\sin u + 1}{\cos u}$$

$$= 3\,\frac{\sin u}{\cos u} + \frac{1}{\cos u}$$

$$= 3\tan u + \sec u.$$

$$g'(u) = 3\sec^2 u + \sec u \tan u.$$

(m) $$f(x) = \frac{\tan x}{\sec x} = \frac{\frac{\sin x}{\cancel{\cos x}}}{\frac{1}{\cancel{\cos x}}} = \sin x.$$

$$f'(x) = \cos x.$$

(n) $g(t) = \dfrac{5^t \tan t}{t^5}$

diff

$$g'(t) = \frac{t^5(5^t \ln 5 \tan t + 5^t \sec^2 t) - 5^t \tan t \cdot 5t^4}{t^{10}}$$

$$= \frac{5^t(t^5(\ln 5 \tan t + \sec^2 t) - \tan t \cdot 5t^4)}{t^{10}}$$

$$= \frac{5^t \cdot \cancel{t^4}(t(\ln 5 \tan t + \sec^2 t) - 5\tan t)}{\cancel{t^{10}}\ t^6}$$

$$= \frac{5^t(\ln 5 \cdot t \cdot \tan t + t\sec^2 t - 5\tan t)}{t^6}$$

$$= \frac{5^t(\tan t(t\ln 5 - 5) + t\sec^2 t)}{t^6},$$

(o) $$r(\theta) = \frac{\theta^2+2}{\theta^3+4\theta}$$

$$r'(\theta) = \frac{(\theta^3+4\theta)(2\theta) - (\theta^2+2)(3\theta^2+4)}{(\theta^3+4\theta)^2}$$

$$= \frac{(\theta^3+4\theta)(2\theta) - (\theta^2+2)(3\theta^2+4)}{(\theta^3+4\theta)^2}$$

$$= \frac{2\theta^4 + 8\theta^2 - (3\theta^4 + 4\theta^2 + 6\theta^2 + 8)}{(\theta^3+4\theta)^2}$$

$$= \frac{-\theta^4 - 2\theta^2 - 8}{(\theta^3+4\theta)^2}$$

$$= -\frac{\theta^4+2\theta^2+8}{(\theta^3+4\theta)^2} = -\frac{\theta^4+2\theta^2+8}{(\theta(\theta^2+4))^2}$$

factor θ

$$= -\frac{\theta^4+2\theta^2+8}{\theta^2(\theta^2+4)^2}$$