MAT 140

Linear Algebra and Affine Geometry

Dr Ton Boerkoel

DigiPen Institute of Technology

Fall 2017

Contents

Introduction

Part 1

An Exploration of 2 and 3 Dimensional Euclidean Space

Part II

Matrices, Systems of Equations and Row reduction

Part III

Linear Transformations

Part IV

Affine Transformations

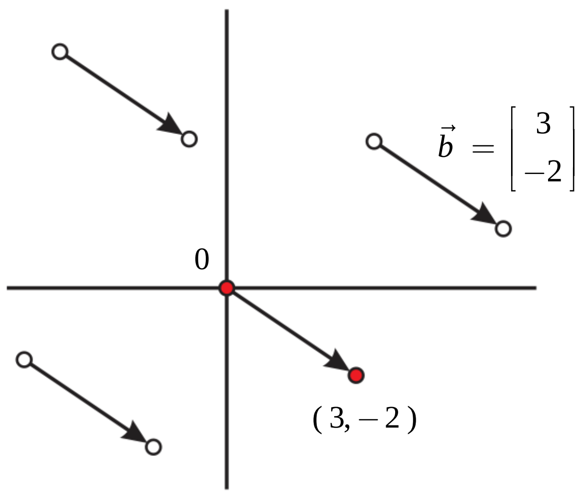
30. Translations

We discovered matrices for a variety of transformations. But there is this one transformation, that we all know very well from high school geometry that somehow we ignored. It is one of the fundamental transformations of geometry: **the translation**. Why did we leave it out? Not because it is so hard to describe. It is very easy to describe: here it is

$$\boxed{T(\vec{x}) = \vec{x} + \vec{b}} \quad \text{i.e.} \quad \boxed{T(\vec{x}) = I\,\vec{x} + \vec{b}}$$

All transformations we discussed so far were **linear**. In particular they kept the origin fixed. All linear transformations keep the origin fixed: $T(\vec{0}) = A\vec{0} = \vec{0}$

But a translation moves the origin! In fact if we translate over $\vec{b} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ every point in the plane moves to another point. There are no fixed points: the origin moves to the point $(3, -2)$.



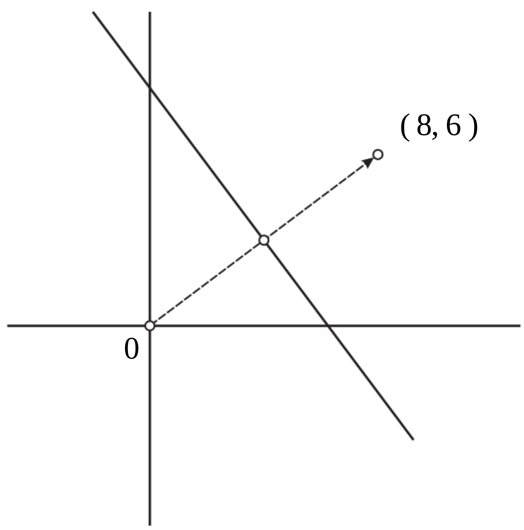
Hence a translation is NOT a linear transformation. We can NOT write it as a matrix multiplication $A\vec{x}$.

So far we discovered how to use linear transformation matrices to rotate **around the origin** or around a vector **through the origin**, project and skew project onto a plane **through the origin**, reflect in a plane **through the origin**, shear parallel to a plane **through the origin**. But every transformation was solidly anchored at the **origin**. In the following sections we will introduce transformations like rotations around **arbitrary points** and **vectors**, projections and skew projections onto **arbitrary lines** and **planes**, reflections in **arbitrary lines** and **planes**, and shears parallel to **arbitrary planes** …

… and none of these leave the origin fixed.

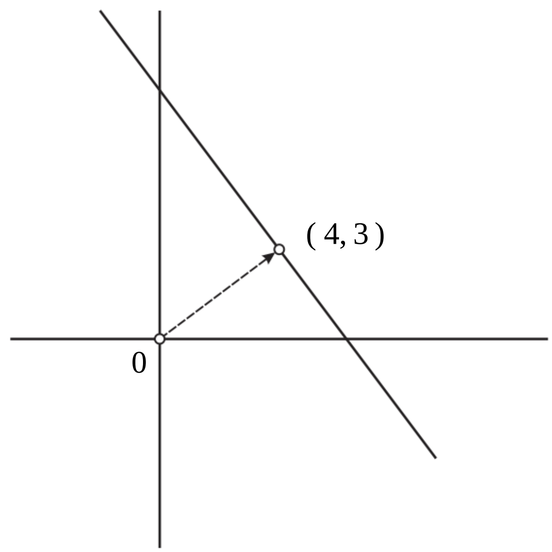
Example 1:

A **reflection** in the line $4x + 3y = 25$ moves the origin to $(\,8,\,6\,)$:



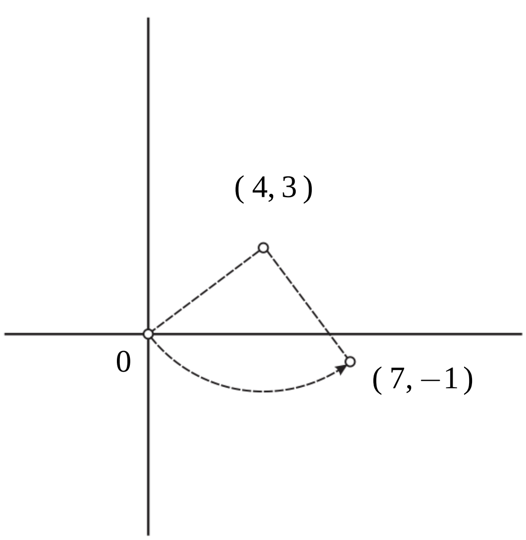
Example 2:

A **projection** in the line $4x + 3y = 25$ moves the origin to $(\,4,\,3\,)$:



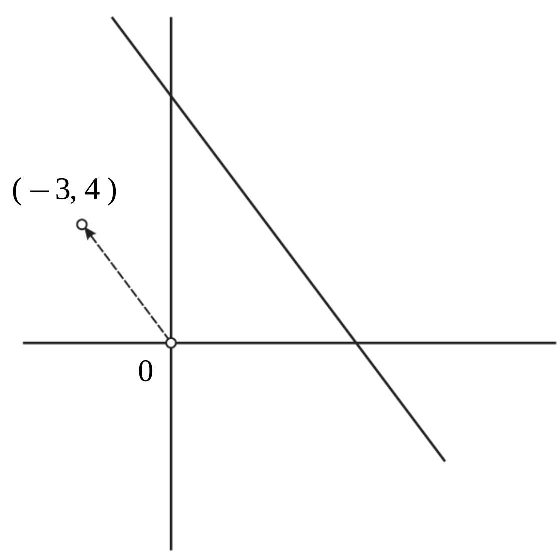
Example 3:

A **rotation** around the point $(\,4,3\,)$ over 90° moves the origin to $(\,7,-1\,)$:



Example 4:

A **shear** along the line $4x + 3y = 25$ with $\vec{v} = \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix}$ moves the origin to $(\,-3,\,4\,)$:



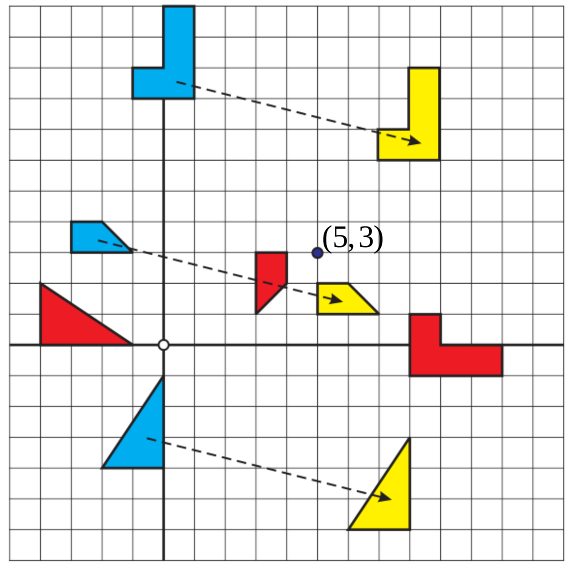
Hence NONE of these transformations is linear.

So in order to describe these transformations we can NOT just use a matrix multiplication.

We are entering the realm of **non-linear** transformations. However we'll see that they differ from the linear transformations—which we described in previous sections—merely by a translation!

Example 5:

Let's examine the **rotation** around the point $(5, 3)$ over $90°$, in particular let's compare it to the rotation around the origin over $90°$:



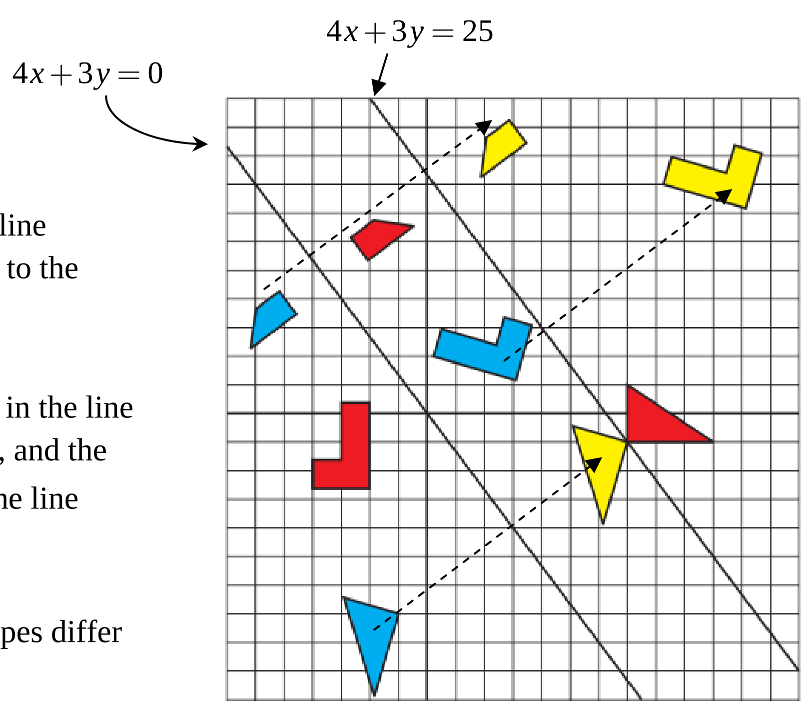
The **red** shapes have been rotated about the point $(5, 3)$ resulting in the **yellow** shapes, and the red shapes rotated around the origin resulting in the **blue** shapes;

The **blue** shapes and the **yellow** shapes differ by a **translation**!

$$R_{C,90°}(\vec{x}) = R_{0,90°}(\vec{x}) + \begin{bmatrix} 8 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

Example 6:

Let's examine the **reflection** in the line $4x+3y=25$, and let's compare it to the reflection in the line $4x+3y=0$:



The **red** shapes have been reflected in the line $4x+3y=25$ to the **yellow** shapes, and the red shapes have been reflected in the line $4x+3y=0$ to the **blue** shapes.

The **blue** shapes and the **yellow** shapes differ by a **translation**!

$$R_{25}(\vec{x}) = R_0(\vec{x}) + \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} -7 & -24 \\ -24 & 7 \end{bmatrix} \vec{x} + \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

Hence we will investigate these transformations that are not quite linear … “almost” linear … that differ from a linear transformation by a translation:

$$T(\vec{x}) = \underbrace{A\vec{x}}_{\text{Linear}} + \underbrace{\vec{b}}_{\text{Translation}}$$

The linear part can be written as a matrix multiplication, which is then followed by a translation.

These transformations are called **Affine Transformations** and will be studied in the next two sections.

31. Affine Transformations

All transformations we discussed so far were linear. In particular they kept the origin fixed. We discovered how to use linear transformation matrices to rotate around the origin or around a vector through the origin, project and skew project onto a plane through the origin, reflect in a plane through the origin, shear parallel to a plane through the origin. But every transformation was solidly anchored at the origin. In this section we will describe the principle of how to perform transformations like rotations around arbitrary points and vectors, projections and skew projections onto arbitrary lines and planes, reflections in arbitrary lines and planes, and shears parallel to arbitrary planes.

Definition: A transformation of $T:\mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T(\vec{x}) = A\vec{x} + \vec{b},$$

where $A \in M_{m\times n}(\mathbb{R})$ and $\vec{b} \in \mathbb{R}^m$ is called an **Affine Transformation**.

Example 1: Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

then

- $T\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ i.e. $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is mapped to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- $T\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ i.e. $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ is mapped to $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$
- $T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ i.e. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is mapped to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

or we could have simplified calculations a bit by noting that $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2-y \end{bmatrix}$.

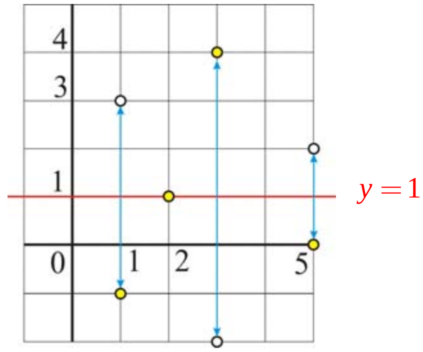
Notice that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a fixed point, since $T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Are there more fixed points?

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ 2-y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{cases} x = x \\ 2-y = y \end{cases} \Rightarrow \begin{cases} x = x \\ y = 1 \end{cases} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

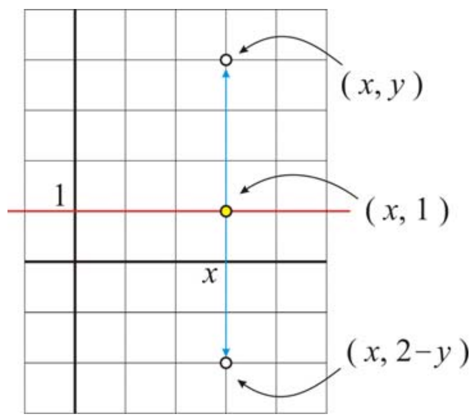
This is the line $y = 1$, so that under this transformation the line $y = 1$ is fixed.

When we plot more points:



we recognize this transformation as the reflection in the line $y = 1$.

This is not too hard to prove: the points $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x \\ 2-y \end{bmatrix}$ are on opposite sides of this line.



Their midpoint is $\frac{1}{2}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x \\ 2-y \end{bmatrix}\right) = \begin{bmatrix} x \\ 1 \end{bmatrix}$, the projection of $\begin{bmatrix} x \\ y \end{bmatrix}$ onto the line $y = 1$.

Example 2: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ -3 \end{bmatrix}$

Then
- $T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ i.e. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is mapped to $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$
- $T\begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 7 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ i.e. $\begin{bmatrix} 7 \\ 1 \end{bmatrix}$ is mapped to $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$
- $T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$ i.e. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is mapped to $\begin{bmatrix} 3 \\ -3 \end{bmatrix}$
- $T\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ i.e. $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is mapped to $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

We could have simplified calculations a bit by rewriting $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y+3 \\ x-3 \end{bmatrix}$.

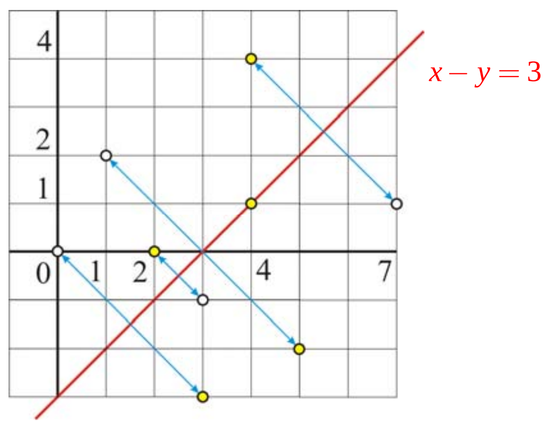
Notice that $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is a fixed point, since $T\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Are there more fixed points?

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \;\Rightarrow\; \begin{bmatrix} y+3 \\ x-3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{cases} y+3 = x \\ x-3 = y \end{cases} \Rightarrow \begin{cases} x-y=3 \\ x-y=3 \end{cases} \Rightarrow \; x-y=3$$

i.e. the line $x-y=3$ is fixed under this transformation.

Let's plot some points:



This seems to be the reflection in the line $x-y=3$.

That this is indeed so, is a bit more work to show: The midpoint of $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} y+3 \\ x-3 \end{bmatrix}$ is

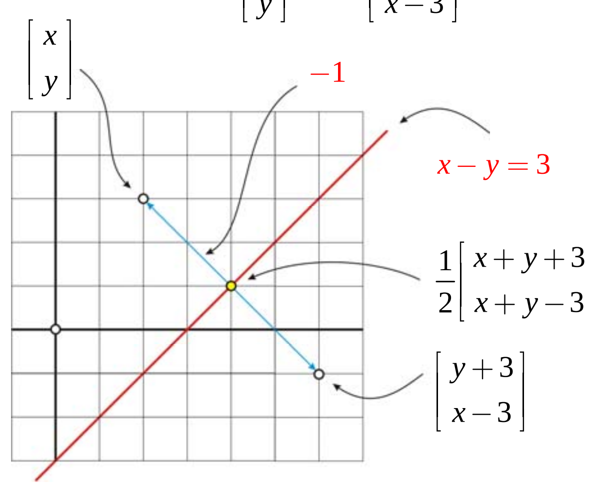
$$\frac{1}{2}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} y+3 \\ x-3 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} x+y+3 \\ x+y-3 \end{bmatrix}.$$

This is a point on the line $x-y=3$

[Check: $\dfrac{x+y+3}{2} - \dfrac{x+y-3}{2} = 3$ ✓]

The slope of the line through

$\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} y+3 \\ x-3 \end{bmatrix}$ is $\dfrac{y-(x-3)}{x-(y+x)} = -1$



Example 3: Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

Then

- $T\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ i.e. $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ is mapped to $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$
- $T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ i.e. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is mapped to $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$
- $T\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ i.e. $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ is mapped to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- $T\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ i.e. $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is mapped to $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

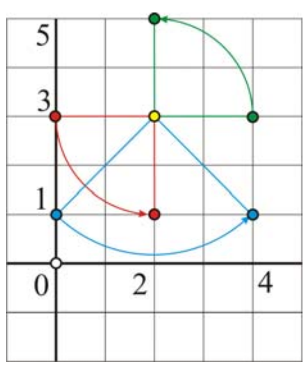
We could have simplified calculations a bit by writing $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5-y \\ x+1 \end{bmatrix}$.

Notice that $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is a fixed point, since $T\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Are there more fixed points?

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} 5-y \\ x+1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\Rightarrow \begin{cases} x+y=5 \\ x-y=-1 \end{cases} \Rightarrow \begin{cases} x=2 \\ y=3 \end{cases}$$

Apparently $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is the *only* fixed point.

Let's plot some points:



Apparently this is the rotation around the point $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ over $+90°$. We'll prove this in a way that allows us to deal with all these transformations.

We *do* know how to rotate around the ***origin*** over 90°. So if we first translate the point $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ to the origin, *and all other points with it,* by the translation

$$\vec{x} - \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

and then rotate over 90° using the usual matrix:

$$\begin{bmatrix} \cos(90^\circ) & -\sin(90^\circ) \\ \sin(90^\circ) & \cos(90^\circ) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We get

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\left(\vec{x} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right)$$

Now we just translate back to $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ making it the center of rotation

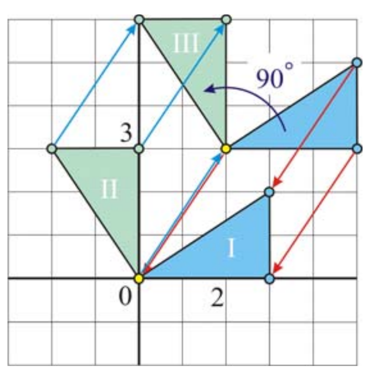
$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\left(\vec{x} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

and this gives us

$$\begin{aligned} T(\vec{x}) &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\left(\vec{x} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\vec{x} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\vec{x} + \begin{bmatrix} 5 \\ 1 \end{bmatrix} \end{aligned}$$

Which is precisely the map we started out with.

Let's illustrate this procedure



(I) Translate to the origin
(II) Perform the transformation around the origin (rotation in this case)
(III) Translate back using the same vector

i.e. algebraically: $T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0$

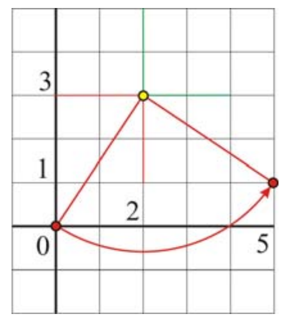
$$\Rightarrow \quad T(\vec{x}) = A\vec{x} - A\vec{x}_0 + \vec{x}_0$$
$$\Rightarrow \quad T(\vec{x}) = A\vec{x} + (I - A)\vec{x}_0$$

Hence we have an affine transformation $T(\vec{x}) = A\vec{x} + \vec{b}$ with $\vec{b} = (I - A)\vec{x}_0$.

Note 1: $\vec{b}$ can be found in two ways: **(a)** $T(\vec{0}) = \vec{b}$

(b) $\vec{b} = (I - A)\vec{x}_0$.

If it is easy to find $T(\vec{0})$ it is easy to find $\vec{b}$. For example in the case of the rotation of example 3, i.e. the rotation over 90° around $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, it is not too hard to see that the origin would rotate to $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$.



Hence $\vec{b} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, as we indeed found by computing $\vec{b} = (I - A)\vec{x}_0$.

We'll use the method $T(\vec{x}) = A\vec{x} + \vec{b}$ with $\vec{b} = (I - A)\vec{x}_0$ to find the basic transformations of projections and reflections in general lines and planes, rotations around arbitrary points etc. in sections to come.

32. More on Affine transformations

Affine vs Linear

Let's issue a **warning**: An affine map $T(\vec{x}) = A\vec{x} + \vec{b}$ is **NOT** linear when $\vec{b} \neq \vec{0}$: e.g.

- $T(\vec{x}) + T(\vec{y}) \neq T(\vec{x} + \vec{y})$
- $T(t\vec{x}) \neq tT(\vec{x})$
- $T(\vec{0}) \neq \vec{0}$

Just check the following:

- $T(\vec{x} + \vec{y}) = A(\vec{x} + \vec{y}) + \vec{b}$ but
  $\begin{cases} T(\vec{x}) = A\vec{x} + \vec{b} \\ T(\vec{y}) = A\vec{y} + \vec{b} \end{cases} \Rightarrow T(\vec{x}) + T(\vec{y}) = A(\vec{x} + \vec{y}) + 2\vec{b}$
- $T(t\vec{x}) = A(t\vec{x}) + \vec{b} = tA\vec{x} + \vec{b}$ but
  $tT(\vec{x}) = t(A\vec{x} + \vec{b}) = tA\vec{x} + t\vec{b}$
- $T(\vec{0}) = A\vec{0} + \vec{b} = \vec{b}$

Hence we can**not** use the common properties of linearity: they are not true for affine transformations.

Example 1: Let $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ -7 \end{bmatrix}$. Suppose we want the image of the line m: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ 11 \end{bmatrix} + t\begin{bmatrix} 10 \\ -9 \end{bmatrix}$ under this to map. Then we have to compute

$$T\begin{bmatrix} x \\ y \end{bmatrix} = T\left(\begin{bmatrix} -8 \\ 11 \end{bmatrix} + t\begin{bmatrix} 10 \\ -9 \end{bmatrix}\right)$$

It is now very tempting—but **wrong**—to do the following:

$$T\left(\begin{bmatrix} -8 \\ 11 \end{bmatrix} + t\begin{bmatrix} 10 \\ 9 \end{bmatrix}\right) = T\begin{bmatrix} -8 \\ 11 \end{bmatrix} + t \cdot T\begin{bmatrix} 10 \\ -9 \end{bmatrix}.$$

T is not linear! The correct way to do it is

$$\begin{aligned} T\begin{bmatrix} x \\ y \end{bmatrix} &= T\begin{bmatrix} -8+10t \\ 11-9t \end{bmatrix} \\ &= A\begin{bmatrix} -8+10t \\ 11-9t \end{bmatrix} + b \end{aligned}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -8+10t \\ 11-9t \end{bmatrix} + \begin{bmatrix} 4 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} 14 \\ 31 \end{bmatrix} + t\begin{bmatrix} -8 \\ -15 \end{bmatrix} + \begin{bmatrix} 4 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} 18 \\ 24 \end{bmatrix} + t\begin{bmatrix} -8 \\ -15 \end{bmatrix}$$

Hence the line $m: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ 11 \end{bmatrix} + t\begin{bmatrix} 10 \\ -9 \end{bmatrix}$ is mapped to $m': \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ 24 \end{bmatrix} + t\begin{bmatrix} 8 \\ 15 \end{bmatrix}$

Affine maps treat points and vectors differently

In the linear case $T(\vec{x}) = A\vec{x}$ we hardly notice any difference between vectors and points:

Let $T(\vec{x}) = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\vec{x}$.

The points $P = (3,2)$ and $Q = (1,3)$ are mapped by the transformation T to the points $P' = (8,5)$ and $Q' = (-2,4)$, since $\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$.

Now let's look at $\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, the vector between the points $Q = (1,3)$ and $P = (3,2)$,

$$\vec{v} = P - Q = (3,2) - (1,3) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

This vector is transformed to the vector

$$\vec{v}' = P' - Q' = (8,5) - (-2,4) = \begin{bmatrix} 10 \\ 1 \end{bmatrix}.$$

If we were to enter this *vector* in the transformation, $T(\vec{v}) = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\vec{v}$, surprisingly we get the correct image vector $T(\vec{v}) = \vec{v}'$:

$$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

The *same* transformation matrix for this linear transformation can be used to transform both *points* and *vectors*! At first this might not be surprising, until one realizes that this is *not* the case for affine transformations. Affine transformations map points to points but when you enter a vector in the transformation you will *not* get out the correct image of the vector:

Let's examine how affine maps treat vectors and points.

For example let $S:\ \mathbb{R}^2 \to \mathbb{R}^2$ be the affine map given by

$$S(\vec{x}) = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}\vec{x} + \begin{bmatrix} 7 \\ 0 \end{bmatrix}$$

S maps the *point* $P = (3,2)$ to the *point* $P' = (8,5)$ and the *point* $Q = (1,3)$ to the *point* $Q' = (-2,4)$, as the following computations verify:

$$\begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 7 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

But S does **NOT** map the vector

$$\vec{v} = P - Q = (3,2) - (1,3) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

to the vector

$$\vec{v}' = P' - Q' = (8,5) - (-2,4) = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

as direct computation shows ,

$$S(\vec{v}) = S\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 17 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 10 \\ 1 \end{bmatrix}.$$

Hence affine transformations treat *points* and *vectors* differently.

$$S(P) = P' \quad \text{but} \quad S(\vec{v}) \neq \vec{v}'$$

If we put a vector $\vec{v} = P - Q$ into the affine transformation $S(\vec{x}) = A\vec{x} + \vec{b}$ we get

$$S(\vec{v}) = A\vec{v} + \vec{b} = A(P-Q) + \vec{b}$$

Whereas $\vec{v}' = P' - Q' = (A\vec{p} + \vec{b}) - (A\vec{q} + \vec{b}) = A(P-Q) = A\vec{v}$.

Hence to get the image under the affine transformation of a vector we *only* need to multiply the vector with the matrix A (the linear part of the transformation) and *not* add $\vec{b}$.

Affine transformations treat points and vectors differently: they transform *points* the way we want them to, but they do not transform *vectors* the way we would like them to.

Example 2: Let $S(\vec{x}) = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\vec{x} + \begin{bmatrix} -1 \\ 5 \end{bmatrix}$, $P = (2, 1)$ and $Q = (3, 3)$ then

$$P' = S(P) = (4, -2) \quad \text{and} \quad Q' = S(Q) = (9, 3)$$

and hence

$$\vec{v} = P - Q = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{v}' = P' - Q' = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

so that indeed

$$\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\vec{v}' = A\vec{v}$$

Example 3: Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be the map defined by

$$S(\vec{x}) = \begin{bmatrix} 1 & -5 & 4 \\ 3 & 2 & 1 \end{bmatrix}\vec{x} + \begin{bmatrix} 6 \\ -7 \end{bmatrix}$$

and the points $P = (1, 3, 2)$ and $Q = (-2, 3, 4)$ then

$$P' = S(P) = (0, 4) \quad \text{and} \quad Q' = S(Q) = (5, -3)$$

and hence

$$\vec{v} = P - Q = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \quad \text{and} \quad \vec{v}' = P' - Q' = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$$

so that indeed

$$\begin{bmatrix} 1 & -5 & 4 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$$

$$\vec{v}' = A\vec{v}$$

The Number of Points that determine an Affine Transformation

A *linear* transformation is completely determined by what happens to a basis. As we will see we can determine an *affine* transformation if we know the images of just a couple of points. How many points are needed?

Example 4: If $T:\mathbb{R}^3 \to \mathbb{R}^3$ is an affine map with

$$T\begin{bmatrix}-1\\1\\0\end{bmatrix}=\begin{bmatrix}1\\4\\6\end{bmatrix},\quad T\begin{bmatrix}1\\1\\-2\end{bmatrix}=\begin{bmatrix}-5\\0\\2\end{bmatrix},\quad T\begin{bmatrix}1\\1\\-1\end{bmatrix}=\begin{bmatrix}-1\\7\\10\end{bmatrix}\quad\text{and}\quad T\begin{bmatrix}1\\0\\-1\end{bmatrix}=\begin{bmatrix}-4\\-2\\8\end{bmatrix}$$

determine T.

Solution: Let $T(\vec{x})=A\vec{x}+\vec{b}$ then

$$\begin{bmatrix}1\\4\\6\end{bmatrix}=A\begin{bmatrix}-1\\1\\0\end{bmatrix}+\vec{b},\quad \begin{bmatrix}-5\\0\\2\end{bmatrix}=A\begin{bmatrix}1\\1\\-2\end{bmatrix}+\vec{b},\quad \begin{bmatrix}-1\\7\\10\end{bmatrix}=A\begin{bmatrix}1\\1\\-1\end{bmatrix}+\vec{b}\quad\text{and}\quad \begin{bmatrix}-4\\-2\\8\end{bmatrix}=A\begin{bmatrix}1\\0\\-1\end{bmatrix}+\vec{b}.$$

When we subtract the *last equation* from each of the first three we get

$$\begin{bmatrix}1\\4\\6\end{bmatrix}-\begin{bmatrix}-4\\-2\\8\end{bmatrix}=A\left(\begin{bmatrix}-1\\1\\0\end{bmatrix}-\begin{bmatrix}1\\0\\-1\end{bmatrix}\right),\quad \begin{bmatrix}-5\\0\\2\end{bmatrix}-\begin{bmatrix}-4\\-2\\8\end{bmatrix}=A\left(\begin{bmatrix}1\\1\\-2\end{bmatrix}-\begin{bmatrix}1\\0\\-1\end{bmatrix}\right),\quad\text{and}$$

$$\begin{bmatrix}-1\\7\\10\end{bmatrix}-\begin{bmatrix}-4\\-2\\8\end{bmatrix}=A\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}-\begin{bmatrix}1\\0\\-1\end{bmatrix}\right)$$

Notice that all the $\vec{b}$ -s are gone, hence we get

$$\begin{bmatrix}5\\6\\-2\end{bmatrix}=A\begin{bmatrix}-2\\1\\1\end{bmatrix},\quad \begin{bmatrix}-1\\2\\-6\end{bmatrix}=A\begin{bmatrix}0\\1\\-1\end{bmatrix},\quad\text{and}\quad \begin{bmatrix}3\\9\\2\end{bmatrix}=A\begin{bmatrix}0\\1\\0\end{bmatrix}$$

Combining this gives us

$$A\begin{bmatrix}-2&0&0\\1&1&1\\1&-1&0\end{bmatrix}=\begin{bmatrix}5&-1&3\\6&2&9\\-2&-6&2\end{bmatrix}$$

So that

$$A = \begin{bmatrix} 5 & -1 & 3 \\ 6 & 2 & 9 \\ -2 & -6 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 9 & 7 \\ 6 & 2 & 8 \end{bmatrix}.$$

What remains is to find $\vec{b}$, but this we can do by picking one of the original equations e.g.

$$T\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 8 \end{bmatrix}$$

would now imply

$$\begin{bmatrix} -4 \\ -2 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 9 & 7 \\ 6 & 2 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \vec{b}$$

So that

$$\vec{b} = \begin{bmatrix} -4 \\ -2 \\ 8 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 4 \\ 5 & 9 & 7 \\ 6 & 2 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 8 \end{bmatrix} - \begin{bmatrix} -3 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 10 \end{bmatrix}$$

And we have found T:

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 9 & 7 \\ 6 & 2 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 10 \end{bmatrix}$$

Why did this method work? There is only one step that could have gone wrong: taking the inverse of a matrix! The matrix has to be invertible. Hence we have to pick our points so that the corresponding determinant is nonzero.

Let's do it for an arbitrary affine map $T: \mathbb{R}^3 \to \mathbb{R}^3$. Suppose we are given

$$T(\vec{u}) = \vec{a},\ T(\vec{v}) = \vec{b},\ T(\vec{w}) = \vec{c} \quad \text{and} \quad T(\vec{s}) = \vec{d}$$

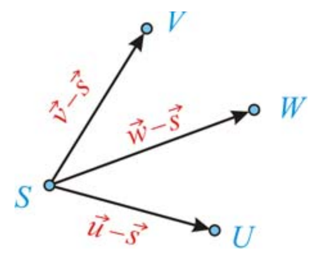
then $\quad T(\vec{u}) - T(\vec{s}) = \vec{a} - \vec{d},\quad T(\vec{v}) - T(\vec{s}) = \vec{b} - \vec{d} \quad$ and $\quad T(\vec{w}) - T(\vec{s}) = \vec{c} - \vec{d}$

i.e. $\quad A(\vec{u} - \vec{s}) = \vec{a} - \vec{d},\quad A(\vec{v} - \vec{s}) = \vec{b} - \vec{d} \quad$ and $\quad A(\vec{w} - \vec{s}) = \vec{c} - \vec{d}$

hence $$A\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{u}-\vec{s} & \vec{v}-\vec{s} & \vec{w}-\vec{s} \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{a}-\vec{d} & \vec{b}-\vec{d} & \vec{c}-\vec{d} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

So now we need $\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{u}-\vec{s} & \vec{v}-\vec{s} & \vec{w}-\vec{s} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$ to be invertible, i.e. $\det\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{u}-\vec{s} & \vec{v}-\vec{s} & \vec{w}-\vec{s} \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \neq 0$

This is exactly the case when we pick our vectors $\vec{u}-\vec{s}$, $\vec{v}-\vec{s}$ and $\vec{w}-\vec{s}$ linearly independent, or expressed in terms of points: if the four points $\vec{u}$, $\vec{v}$, $\vec{w}$ and $\vec{s}$ are not co-planar



When the four points are not coplanar the vectors $\vec{u}-\vec{s}$, $\vec{v}-\vec{s}$ and $\vec{w}-\vec{s}$ linearly independent and hence the matrix is invertible, and we get

$$A = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{a}-\vec{d} & \vec{b}-\vec{d} & \vec{c}-\vec{d} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{u}-\vec{s} & \vec{v}-\vec{s} & \vec{w}-\vec{s} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}^{-1}$$

The same analysis gives the result or an affine map $T:\ \mathbb{R}^2 \to \mathbb{R}^2$, in which we need three non-collinear points.

In general if $T:\ \mathbb{R}^m \to \mathbb{R}^n$ is an affine map than we need $m+1$ points such that we can create m linearly independent vectors, i.e. a basis for $\mathbb{R}^m$.

The Composition of affine transformations

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be the affine transformation $T(\vec{x}) = A\vec{x} + \vec{b}$ and $S:\mathbb{R}^m \to \mathbb{R}^p$ the affine transformation $S(\vec{x}) = B\vec{x} + \vec{c}$ then the composition $S \circ T:\ \mathbb{R}^n \to \mathbb{R}^p$ is

$$S \circ T(\vec{x}) = S(T(\vec{x})) = B \cdot T(\vec{x}) + \vec{c} = B \cdot (A\vec{x} + \vec{b}) + \vec{c} = \underbrace{BA}\vec{x} + \underbrace{B\vec{b} + \vec{c}}$$

which is an affine transformation with matrix BA and translation vector $B\vec{b} + \vec{c}$.
Hence:

Theorem 32.1: The composition of two affine transformations is an affine transformation.

Example 5: $T:\mathbb{R}^3 \to \mathbb{R}^2$ with $T(\vec{x}) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 3 \end{bmatrix}\vec{x} + \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ and $S:\mathbb{R}^m \to \mathbb{R}^p$ with

$S(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}\vec{x} + \begin{bmatrix} 1 \\ 2 \\ -3 \\ 5 \end{bmatrix}$, then

$$\begin{aligned} S \circ T(\vec{x}) &= S(T(\vec{x})) = B \cdot T(\vec{x}) + c \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -2 \\ 3 & 1 \end{bmatrix} T(\vec{x}) + \begin{bmatrix} 1 \\ 2 \\ -3 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -2 \\ 3 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 3 \end{bmatrix}\vec{x} + \begin{bmatrix} 4 \\ -2 \end{bmatrix} \right) + \begin{bmatrix} 1 \\ 2 \\ -3 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 & 4 \\ 1 & 0 & 1 \\ -2 & 2 & -6 \\ 4 & -1 & 6 \end{bmatrix}\vec{x} + \begin{bmatrix} 3 \\ 6 \\ 1 \\ 15 \end{bmatrix} \end{aligned}$$

The Inverse of an Affine Transformation

Theorem 32.2:

An Affine Transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\vec{x}) = A\vec{x} + \vec{b}$ with $A \in M_{m\times n}(\mathbb{R})$ and $\vec{b} \in \mathbb{R}^m$, is **invertible** when A is invertible. $T^{-1}:\mathbb{R}^m \to \mathbb{R}^n$ is given by

$$T^{-1}(\vec{x}) = A^{-1}(\vec{x} - \vec{b})$$

Proof: **(a)** $T\left(T^{-1}(\vec{x})\right) = T\left(A^{-1}(\vec{x}-\vec{b})\right) = A\left(A^{-1}(\vec{x}-\vec{b})\right) + \vec{b} = (\vec{x}-\vec{b}) + \vec{b} = \vec{x}$

(b) $T^{-1}\left(T(\vec{x})\right) = T^{-1}\left(A\vec{x}+\vec{b}\right) = A^{-1}\left((A\vec{x}+\vec{b}) - \vec{b}\right) = A^{-1}A\vec{x} = \vec{x}$

Example 6: **(a)** Let T be the **rotation**

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Then $T^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 5 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 5 \\ 1 \end{bmatrix}\right)$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

(b) Let T be the **reflection** in the plane $\alpha: \ 3x - 1y + 2z = 14$

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -2 & 3 & -6 \\ 3 & 6 & 2 \\ -6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix}$$

Then $T^{-1}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left(\frac{1}{7}\begin{bmatrix} -2 & 3 & -6 \\ 3 & 6 & 2 \\ -6 & 2 & 3 \end{bmatrix}\right)^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix}\right)$

$$= \frac{1}{7}\begin{bmatrix} -2 & 3 & -6 \\ 3 & 6 & 2 \\ -6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix}$$

[Which is not surprising: a reflection is its own inverse!]

Affine transformations and lines

Theorem 32.3: An invertible affine transformation $T(\vec{x}) = A\vec{x} + \vec{b}$, i.e. with A invertible, maps lines to lines.

If the affine transformation is not invertible, it either maps a line to a line or to a point.

Example 7: Let S be the affine transformation from example 2

$$S(\vec{x}) = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\vec{x} + \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

and l the line given by $l: \vec{x} = (2, 1) + t\begin{bmatrix} -4 \\ 5 \end{bmatrix}$ then the image of l is

$$\vec{x}' = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 2-4t \\ 1+5t \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} + t\begin{bmatrix} 6 \\ 32 \end{bmatrix}$$

hence

$$l: \vec{x} = (2, 1) + t\begin{bmatrix} -4 \\ 5 \end{bmatrix} \quad \text{gets mapped to} \quad l': \vec{x} = (3, 3) + t\begin{bmatrix} 6 \\ 32 \end{bmatrix}$$

Proof: The proof of the last theorem is pretty straight forward. If

$$T(\vec{x}) = A\vec{x} + \vec{b}$$

and

$$l: \vec{x} = P + t\vec{v} \quad (\text{with } \vec{v} \neq \vec{0})$$

then

$$\begin{aligned} \vec{x}' = T(P + t\vec{v}) &= A(P + t\vec{v}) + \vec{b} \\ &= (AP + \vec{b}) + t\,A\vec{v} \\ &= P' + t\,A\vec{v} \end{aligned}$$

Hence the image is another line … unless $A\vec{v} = \vec{0}$.

If $A\vec{v} = \vec{0}$ then the line gets mapped to a point. But if the matrix A is invertible then $A\vec{v} \neq \vec{0}$ when $\vec{v} \neq \vec{0}$.

33. Basic Affine Transformations

In this section we will discuss all the transformations we saw earlier but now in a more general setting, e.g. projections and reflections onto *any* line/plane, not just those through the origin, shears parallel to arbitrary lines/planes, rotations around any point or vector etc.

We'll be using the three step procedure discussed in an earlier section

- **(I)** Translate to the origin: $\vec{x} - \vec{x}_0$
- **(II)** Perform the linear transformation: $A(\vec{x} - \vec{x}_0)$
- **(III)** Translate back using the same vector: $A(\vec{x} - \vec{x}_0) + \vec{x}_0$

$$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0$$

i.e.

$$T(\vec{x}) = A\vec{x} + \underbrace{(I - A)\vec{x}_0}_{\vec{b}}$$

It may be useful to realize that $\vec{b}$ can be found in essentially two different ways:

$$\vec{b} = T(\vec{0})$$

and

$$\vec{b} = (I - A)\vec{x}_0$$

Both are useful to know. Sometimes the one is easier than the other.

The three step approach outlined above is basically a composition of three transformations

(1) a translation by $-\vec{x}_0$: $\quad T_{-\vec{x}_0}(\vec{x}) = \vec{x} - \vec{x}_0$

(2) a linear transformation: $\quad L(\vec{x}) = A\vec{x}$

(3) a translation by $+\vec{x}_0$: $\quad T_{\vec{x}_0}(\vec{x}) = \vec{x} + \vec{x}_0$

The resulting composite transformation is an affine transformation and can be written as:

$$\boxed{T_{\vec{x}_0} \circ L \circ T_{-\vec{x}_0}}$$

or

$$T_{\vec{x}_0} \circ L \circ T_{-\vec{x}_0}(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0$$

It would be great if we could perform the composite transformation $T_{\vec{x}_0} \circ L \circ T_{-\vec{x}_0}$ with one matrix multiplication, or maybe initially using three matrix multiplications. Unfortunately a translation is NOT a linear transformation, hence no matrix will do that transformation for us… just yet. In a coming section we will actually be able to solve this dilemma by going up one dimension.

It is important to know this three step approach. We have already discussed all the matrices of the important linear transformations we need to know. So we do **not** need to learn any new matrices. To find the affine cousins of the transformations we have discussed before we only need to follow this three step dance.

Here are the main results (note that at times $\vec{b}$ is given as $(I - M)\vec{x}_0$ but sometimes there is a nice simple expression for it.)

Theorem 33.1: The **projection** onto the line $\vec{x} = \vec{x}_0 + t\vec{v}$ is given by

$$T(\vec{x}) = M\vec{x} + (I - M)\vec{x}_0$$

$$\text{where } M = \frac{1}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}}$$

Theorem 33.2: The **projection** onto $\vec{n} \cdot \vec{x} = c$ is given by

$$T(\vec{x}) = M\vec{x} + \frac{c}{\|\vec{n}\|^2}\vec{n}$$

$$\text{where } M = I - \frac{1}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}}$$

Theorem 33.3 The **skew projection** onto $\vec{n} \cdot \vec{x} = c$ in the direction $\vec{v}$ is given by

$$T(\vec{x}) = M\vec{x} + \frac{c}{\vec{v} \cdot \vec{n}}\vec{v}$$

$$\text{where } M = I - \frac{1}{\vec{v} \cdot \vec{n}}\vec{v} \cdot \vec{n}^{\mathsf{T}}$$

Theorem 33.4 The **reflection** in the line $\vec{x} = \vec{x}_0 + t\vec{v}$ is given by

$$T(\vec{x}) = M\vec{x} + \left(I - M\right)\vec{x}_0$$

$$\text{where } M = \frac{2}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}} - I$$

Theorem 33.5 The **reflection** in $\vec{n}\cdot\vec{x} = c$ is given by

$$T(\vec{x}) = M\,\vec{x} + \frac{2c}{\|\vec{n}\|^2}\vec{n}$$

where $M = I - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}$

Theorem 33.6 The **shear** parallel to $\vec{n}\cdot\vec{x} = c$ in the direction $\vec{v}\perp\vec{n}$ is given by

$$T(\vec{x}) = M\,\vec{x} - \frac{c}{\|\vec{n}\|}\vec{v}$$

where $M = I + \frac{1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}}$

Theorem 33.7 The **scaling** centered at $\vec{x}_0$ with matrix M is given by

$$T(\vec{x}) = M\,\vec{x} + (I - M)\vec{x}_0$$

In 2D: $M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$

In 3D: $M = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

Theorem 33.8 The **rotation** centered at $\vec{x}_0$ with matrix M is given by

$$T(\vec{x}) = M\,\vec{x} + (I - M)\vec{x}_0$$

In 2D: $M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$

In 3D: $M = (1-\cos\theta)\frac{\vec{v}\,\vec{v}^{\mathsf{T}}}{\|\vec{v}\|^2} + \cos\theta\, I + \frac{\sin\theta}{\|\vec{v}\|}C_{\vec{v}}$

where $C_{\vec{v}} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$ when $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

We will start with some examples of each of these transformations. Proofs will come later. It is useful to study these examples and practice computing the various affine transformations. Note that the above theorems work for transformations of both $\mathbb{R}^2$ and $\mathbb{R}^3$. Hence we will give examples from both worlds

Examples

Example 1: Let l be the line $\vec{x} = \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}$ then the **projection** onto l is given by

$$T(\vec{x}) = M\vec{x} + (I - M)\vec{x}_0$$

Note that

$$M = \frac{1}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\top} = \frac{1}{35}\begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & -5 & 3 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 1 & -5 & 3 \\ -5 & 25 & -15 \\ 3 & -15 & 9 \end{bmatrix}$$

and

$$I - M = I - \frac{1}{35}\begin{bmatrix} 1 & -5 & 3 \\ -5 & 25 & -15 \\ 3 & -15 & 9 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 34 & 5 & -3 \\ 5 & 10 & 15 \\ -3 & 15 & 26 \end{bmatrix}$$

Hence

$$T(\vec{x}) = \frac{1}{35}\begin{bmatrix} 1 & -5 & 3 \\ -5 & 25 & -15 \\ 3 & -15 & 9 \end{bmatrix}\vec{x} + \frac{1}{35}\begin{bmatrix} 34 & 5 & -3 \\ 5 & 10 & 15 \\ -3 & 15 & 26 \end{bmatrix}\begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix}$$

$$= \frac{1}{35}\begin{bmatrix} 1 & -5 & 3 \\ -5 & 25 & -15 \\ 3 & -15 & 9 \end{bmatrix}\vec{x} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Example 2: Let $l:\ 3x + y = 10$ be a line in $\mathbb{R}^2$, then the **projection** onto the line l is given by

$$T(\vec{x}) = \left(I - \frac{1}{\|\vec{n}\|^2}\vec{n}\cdot\vec{n}^{\top}\right)\vec{x} + \frac{c}{\|\vec{n}\|^2}\vec{n}$$

$$= \left(I - \frac{1}{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 3 & 1 \end{bmatrix}\right)\vec{x} + \frac{10}{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}\vec{x} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

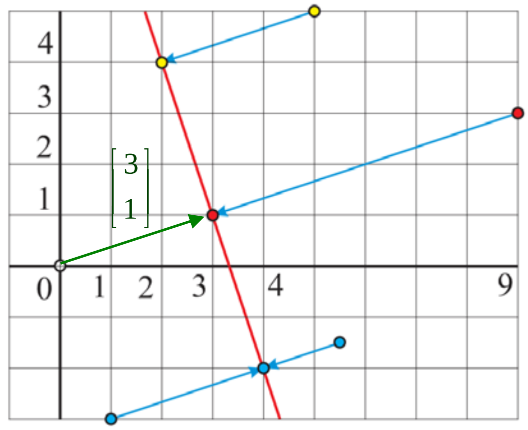
Note that we could have found $\vec{b}$ directly as $T(\vec{0})$: Take the line through the origin perpendicular to l which is given by $l^{\perp}: \begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. This line intersects $l:\ 3x + y = 10$ when $t = 1$ in $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. This is precisely the projection of $\vec{0}$ onto l, i.e. $\vec{b} = T(\vec{0}) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(See picture)

For example $T\begin{bmatrix}5\\5\end{bmatrix} = \frac{1}{10}\begin{bmatrix}1 & -3\\-3 & 9\end{bmatrix}\begin{bmatrix}5\\5\end{bmatrix} + \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}-1\\3\end{bmatrix} + \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$

$$T\begin{bmatrix}9\\3\end{bmatrix} = \frac{1}{10}\begin{bmatrix}1 & -3\\-3 & 9\end{bmatrix}\begin{bmatrix}9\\3\end{bmatrix} + \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$$

$$T\begin{bmatrix}1\\-3\end{bmatrix} = \frac{1}{10}\begin{bmatrix}1 & -3\\-3 & 9\end{bmatrix}\begin{bmatrix}1\\-3\end{bmatrix} + \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}1\\-3\end{bmatrix} + \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}4\\-2\end{bmatrix}$$



Example 3: The **skew projection** onto $3x+2y+5z=12$ in the direction $\vec{v} = \begin{bmatrix}1\\0\\-1\end{bmatrix}$ is

given by $$T(\vec{x}) = \left(I - \frac{1}{\vec{v}\cdot\vec{n}}\vec{v}\cdot\vec{n}^{\mathsf{T}}\right)\vec{x} + \frac{c}{\vec{v}\cdot\vec{n}}\vec{v}$$

$$= \left(I - \frac{1}{-2}\begin{bmatrix}1\\0\\-1\end{bmatrix}\cdot\begin{bmatrix}3 & 2 & 5\end{bmatrix}\right)\vec{x} + \frac{12}{-2}\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

$$= \left(I + \frac{1}{2}\begin{bmatrix}3 & 2 & 5\\0 & 0 & 0\\-3 & -2 & -5\end{bmatrix}\right)\vec{x} + \begin{bmatrix}-6\\0\\6\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}5 & 2 & 5\\0 & 2 & 0\\-3 & -2 & -3\end{bmatrix}\vec{x} + \begin{bmatrix}-6\\0\\6\end{bmatrix}$$

Example 4: The **reflection** in the line $\vec{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is given by

$$T(\vec{x}) = M\,\vec{x} + \left(I - M\right)\vec{x}_0$$

where

$$M = \frac{2}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}} - I = \frac{2}{10}\begin{bmatrix} 3 \\ -1 \end{bmatrix}\begin{bmatrix} 3 & -1 \end{bmatrix} - I = \frac{1}{5}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix} - I = \frac{1}{5}\begin{bmatrix} 4 & -3 \\ -3 & -4 \end{bmatrix}$$

and

$$I - M = I - \frac{1}{5}\begin{bmatrix} 4 & -3 \\ -3 & -4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

so that

$$\begin{aligned} T(\vec{x}) &= \frac{1}{5}\begin{bmatrix} 4 & -3 \\ -3 & -4 \end{bmatrix}\vec{x} + \frac{1}{5}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 4 & -3 \\ -3 & -4 \end{bmatrix}\vec{x} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{aligned}$$

Example 5: The **reflection** in the plane $3x + 2y + 5z = 12$ is given by

$$T(\vec{x}) = M\,\vec{x} + \frac{2c}{\|\vec{n}\|^2}\vec{n}$$

where

$$M = I - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} = I - \frac{2}{38}\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}\begin{bmatrix} 3 & 2 & 5 \end{bmatrix} = \frac{1}{19}\begin{bmatrix} 10 & -6 & -15 \\ -6 & 15 & -10 \\ -15 & -10 & -6 \end{bmatrix}$$

and

$$\frac{2c}{\|\vec{n}\|^2}\vec{n} = \frac{24}{38}\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = \frac{12}{19}\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \quad \textbf{or} \quad (I - M)\vec{x}_0 = \frac{1}{19}\begin{bmatrix} 9 & 6 & 15 \\ 6 & 4 & 10 \\ 15 & 10 & 25 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{19}\begin{bmatrix} 36 \\ 24 \\ 60 \end{bmatrix}$$

so that

$$T(\vec{x}) = \frac{1}{19}\begin{bmatrix} 10 & -6 & -15 \\ -6 & 15 & -10 \\ -15 & -10 & -6 \end{bmatrix}\vec{x} + \frac{12}{19}\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

Example 6: The **shear** $S: \mathbb{R}^2 \to \mathbb{R}^2$ parallel to the line $l: 3x + 4y = 10$ in the direction of the shearing vector $\vec{v} = \begin{bmatrix} -8 \\ 6 \end{bmatrix}$, is given by

$$\begin{aligned} T(\vec{x}) &= \left(I + \frac{1}{\|\vec{n}\|} \vec{v}\,\vec{n}^{\mathsf{T}} \right) \vec{x} - \frac{c}{\|\vec{n}\|} \vec{v} \\ &= \left(I + \frac{1}{5} \begin{bmatrix} -8 \\ 6 \end{bmatrix} \begin{bmatrix} 3 & 4 \end{bmatrix} \right) \vec{x} - \frac{10}{5} \begin{bmatrix} -8 \\ 6 \end{bmatrix} \\ &= \left(I + \frac{1}{5} \begin{bmatrix} -24 & -32 \\ 18 & 24 \end{bmatrix} \right) \vec{x} + \begin{bmatrix} 16 \\ -12 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} -19 & -32 \\ 18 & 29 \end{bmatrix} \vec{x} + \begin{bmatrix} 16 \\ -12 \end{bmatrix} \end{aligned}$$

Example 7: The **scaling** centered at $\vec{x}_0 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ with matrix $M = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ is given by

$$T(\vec{x}) = M\,\vec{x} + (I - M)\vec{x}_0$$

$$\begin{aligned} T(\vec{x}) &= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -4 \end{bmatrix} \vec{x} + \begin{bmatrix} -4 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -4 \end{bmatrix} \vec{x} + \begin{bmatrix} -12 \\ 6 \\ 10 \end{bmatrix} \end{aligned}$$

Example 8: The **rotation** centered at $\vec{x}_0 = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ over $\theta = 30^\circ$ is given by

$$\begin{aligned} T(\vec{x}) &= M\,\vec{x} + (I - M)\vec{x}_0 \\ &= \frac{1}{2} \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \vec{x} + \frac{1}{2} \begin{bmatrix} 2-\sqrt{3} & 1 \\ -1 & 2-\sqrt{3} \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \vec{x} + \begin{bmatrix} 7-2\sqrt{3} \\ 4-3\sqrt{3} \end{bmatrix} \end{aligned}$$

Example 9:

The **rotation** centered at $\vec{x}_0 = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$ about the direction $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ over $\theta = 60°$ has

$$\begin{aligned} M &= (1-\cos\theta)\frac{\vec{v}\,\vec{v}^{\mathsf{T}}}{\|\vec{v}\|^2} + \cos\theta\, I + \frac{\sin\theta}{\|\vec{v}\|} C_{\vec{v}} \\ &= \frac{1}{6}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} + \frac{1}{2} I + \frac{\sqrt{3}/2}{\sqrt{3}}\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 2 & -2 & -1 \\ 1 & 2 & -2 \\ 2 & 1 & 1 \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} T(\vec{x}) &= M\,\vec{x} + (I - M)\vec{x}_0 \\ &= \frac{1}{3}\begin{bmatrix} 2 & -2 & -1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}\vec{x} + \frac{1}{3}\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ -2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 2 & -2 & -1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}\vec{x} + \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} \end{aligned}$$

Next we'll give proofs of the theorems:

Proofs

Theorem 33.1, **33.4**, **33.7** and **33.8** are obvious applications of $T(\vec{x}) = A\vec{x} + \underbrace{(I - A)\vec{x}_0}_{\vec{b}}$

We'll first prove **Theorem 33.3** which then implies **Theorem 33.2** when we take $\vec{v} = \vec{n}$.

Proof of Theorem 33.3: Suppose $\vec{x}_0$ is a point on $\vec{n}\cdot\vec{x} = c$: i.e. $\vec{n}\cdot\vec{x}_0 = c$, then

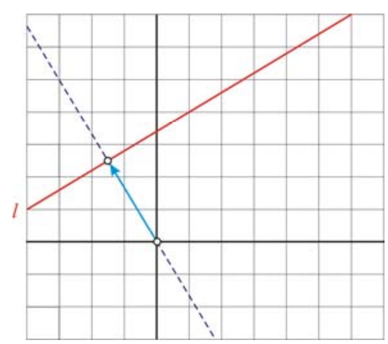
$$\begin{aligned}
T(\vec{x}) &= A(\vec{x}-\vec{x}_0)+\vec{x}_0 \\
&= \left(I - \frac{1}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}\right)(\vec{x} - \vec{x}_0) + \vec{x}_0 \\
&= \left(I - \frac{1}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}\right)\vec{x} - \left(I - \frac{1}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}\right)\vec{x}_0 + \vec{x}_0 \\
&= \left(I - \frac{1}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}\right)\vec{x} + \frac{1}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}\vec{x}_0 \\
&= \left(I - \frac{1}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}\right)\vec{x} + \frac{1}{\vec{v}\cdot\vec{n}}(\vec{n}\cdot\vec{x}_0)\vec{v} \\
&= \left(I - \frac{1}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}\right)\vec{x} + \frac{c}{\vec{v}\cdot\vec{n}}\vec{v}
\end{aligned}$$

Note that with $\vec{v} = \vec{n}$ this becomes

$$T(\vec{x}) = \left(I - \frac{1}{\vec{n}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}\right)\vec{x} + \frac{c}{\vec{n}\cdot\vec{n}}\vec{n}$$

the matrix of **Theorem 33.2**.

Note that we can get the $\vec{b}$ -s of some of the transformations also geometrically as $\vec{b} = T(\vec{0})$:



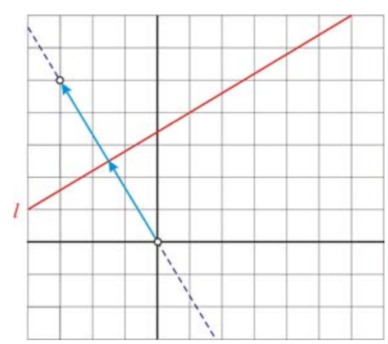
- Projection onto the line l: $\vec{n}\cdot\vec{x} = c$

  The line through $\vec{0}$ perpendicular to l is

  $l^{\perp}: \ \vec{x} = t\vec{n}$

  Intersect with l to get $T(\vec{0})$:

  $$\vec{n}\cdot(t\vec{n}) = c \ \Rightarrow \ t = \frac{c}{\|\vec{n}\|} \ \Rightarrow \ \boxed{\vec{b} = \frac{c}{\|\vec{n}\|}\vec{n}}$$



- Reflection in the line l: $\vec{n}\cdot\vec{x} = c$

  Double previous t to find $T(\vec{0})$:

  $$\boxed{\vec{b} = \frac{2c}{\|\vec{n}\|}\vec{n}}$$

- Skew projection onto the line $l: \ \vec{n}\cdot\vec{x} = c$ in direction $\vec{v}$

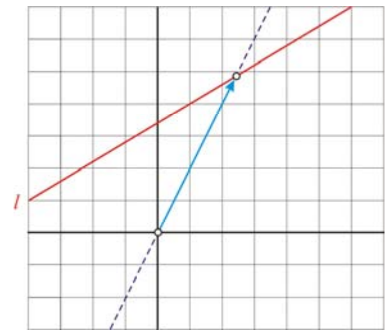
The line through $\vec{0}$ parallel to $\vec{v}$ is

$l^{\parallel}: \ \vec{x} = t\vec{v}$

Intersect with l to get $T(\vec{0})$:

$$\vec{n}\cdot(t\vec{v}) = c \quad \Rightarrow \quad t = \frac{c}{\vec{n}\cdot\vec{v}}$$

$$\Rightarrow \quad \boxed{\vec{b} = \frac{c}{\vec{n}\cdot\vec{v}}\vec{v}}$$



Proof of **Theorem 33.5**: Suppose $\vec{x}_0$ is a point on $\vec{n}\cdot\vec{x} = c$: i.e. $\vec{n}\cdot\vec{x}_0 = c$, then

$$T(\vec{x}) = A\vec{x} + (I - A)\vec{x}_0$$

becomes

$$\begin{aligned} T(\vec{x}) &= \left(I - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right)\vec{x} + \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\vec{x}_0 \\ &= \left(I - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right)\vec{x} + \frac{2}{\|\vec{n}\|^2}(\vec{n}\cdot\vec{x}_0)\vec{n} \\ &= \left(I - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right)\vec{x} + \frac{2}{\|\vec{n}\|^2}c\,\vec{n} \end{aligned}$$

Proof of **Theorem 33.6**: Suppose $\vec{x}_0$ is a point on $\vec{n}\cdot\vec{x} = c$: i.e. $\vec{n}\cdot\vec{x}_0 = c$, then

$$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0$$

with $M = I + \dfrac{1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}}$ becomes

$$\begin{aligned} T(\vec{x}) &= \left(I + \frac{1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}}\right)\vec{x} - \frac{1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}}\vec{x}_0 \\ &= \left(I + \frac{1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}}\right)\vec{x} - \frac{1}{\|\vec{n}\|}(\vec{n}\cdot\vec{x}_0)\vec{v} \\ &= \left(I + \frac{1}{\|\vec{n}\|}\vec{v}\,\vec{n}^{\mathsf{T}}\right)\vec{x} - \frac{1}{\|\vec{n}\|}c\,\vec{v} \end{aligned}$$

Hence all theorems are proven.

34. Barycentric Coordinates

Barycentric coordinates allow us to write points X as weighted sums of other points: e.g. the midpoint M between P and Q as

$$M = \frac{1}{2}(P+Q) = \frac{1}{2}P + \frac{1}{2}Q$$

(1) If X is on a line through 2 points P and Q:

$$X = aP + bQ \quad \text{with} \quad a+b = 1$$

(2) If X is on a plane through 3 points P, Q and R:

$$X = aP + bQ + cR \quad \text{with} \quad a+b+c = 1$$

(3) If X is in a 3D space determined by 4 points P, Q, R and S:

$$X = aP + bQ + cR + dS \quad \text{with} \quad a+b+c+d = 1$$

etc.

2 Points.

Recall that we used to write a line in 2D as follows

$$X = P + t\vec{v}$$

where P is a point on the line, and $\vec{v}$ is a direction vector. If we know two points on the line, say P and Q, then we could take $\vec{v} = Q - P$ and we get

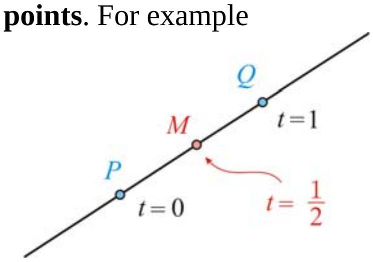
$$X = P + t(Q-P)$$

Now we can write this formally as

$$X = (1-t)P + tQ$$

We see the point X expressed as a weighted sum of **two points**. For example

- $X = P$ when $t=0$
- $X = Q$ when $t=1$
- $M = \frac{1}{2}P + \frac{1}{2}Q$ when $t = \frac{1}{2}$



Note that the total sum of the weights is always one: $(1-t)+t=1$.

On the other hand, suppose $X = aP + bQ$ with $a+b = 1$ then X is a point on the line through P and Q:

$$\begin{aligned} X &= aP + bQ \\ &= P + (a-1)P + bQ \\ &= P - bP + bQ \\ &= P + b(Q-P) \end{aligned}$$

We call a and b the **Barycentric coordinates** of X with respect to the points P and Q, and adopt the notation $[a,b]$, $X = [a,b]$, or $X = [a,b]_{P,Q}$. Of course the points P and Q need to be provided, else $X = [a,b]$ wouldn't make sense.

For example if $P = (2,3)$ and $Q = (5,6)$ then

$$X = [\tfrac{1}{3},\tfrac{2}{3}] = aP+bQ = \tfrac{1}{3}(2,3)+\tfrac{2}{3}(5,6) = (4,5)$$

or

$$X = P+b(Q-P) = (2,3)+\tfrac{2}{3}\big((5,6)-(2,3)\big) = (2,3)+\tfrac{2}{3}(3,3) = (4,5)$$

Another example $P = (1,2)$ and $Q = (3,-2)$ determine the line $l:\ 2x+y=4$.

Note $X = (2,0)\in l$ and hence there are a and b such that $X = aP+bQ$ and $a+b = 1$. Let's do this in vector notation:

$$a\begin{bmatrix}1\\2\end{bmatrix}+b\begin{bmatrix}3\\-2\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$$

and

$$a+b = 1$$

i.e. $\begin{cases} a+3b=2 \\ 2a-2b=0 \\ a+b=1 \end{cases}$ hence: $\operatorname{rref}\begin{bmatrix}1&3&2\\2&-2&0\\1&1&1\end{bmatrix} = \begin{bmatrix}1&0&\frac{1}{2}\\0&1&\frac{1}{2}\\0&0&0\end{bmatrix}$ to find that indeed such a and b exist: $a=b=\dfrac{1}{2}$, i.e.

$$X = (2,0) = \frac{1}{2}(1,2)+\frac{1}{2}(3,-2)$$

Now let's look at $X = (5,-2)$, if we go ahead trying to find a and b in the same way,

$$a\begin{bmatrix}1\\2\end{bmatrix}+b\begin{bmatrix}3\\-2\end{bmatrix} = \begin{bmatrix}5\\-2\end{bmatrix}$$

and

$$a+b = 1$$

i.e. $\begin{cases} a+3b=5 \\ 2a-2b=-2 \\ a+b=1 \end{cases}$ hence: $\text{rref}\begin{bmatrix} 1 & 3 & 5 \\ 2 & -2 & -2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to find that NO such a and b exist: the point $X = (5,-2)$ is NOT on the line through P and Q.

One could argue also as follows:

$$a\begin{bmatrix} 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}$$

So that $a = \frac{1}{2}$ and $b = \frac{3}{2}$ would be the only values to satisfy the first two equations, but unfortunately for these a and b we have: $a+b \neq 1$.

Hence we cannot find a and b such that $X = aP + bQ$ *and* $a+b = 1$.

This we could have realized earlier by checking if $X = (5,-2)$ is on the line or not!
[which is it not]

Any point on the line through $P = (1,2)$ and $Q = (3,-2)$ would look like

$$X = P + t(Q-P)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

When $t = \frac{1}{2}$ we get $(2, 0)$ on the line, and: $X = (2,\ 0) = \frac{1}{2}(1,2) + \frac{1}{2}(3,-2)$

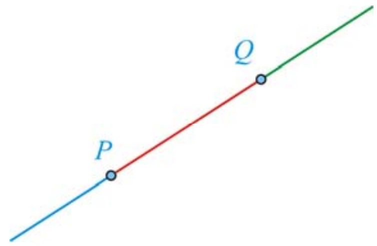
But $(5,-2)$ is not on the line, so that $X = (5,\ -2) \neq a\cdot(1,2) + b\cdot(3,-2)$.

Geometrically

(I) Information about the **signs** of a and b gives us an indication about the position of the point

$$X = aP + bQ\,, \quad \text{with} \quad a + b = 1$$

on the line through P and Q :



(1) When $a = 0$ then $X = aP + bQ = Q$

(2) When $a > 0$ then

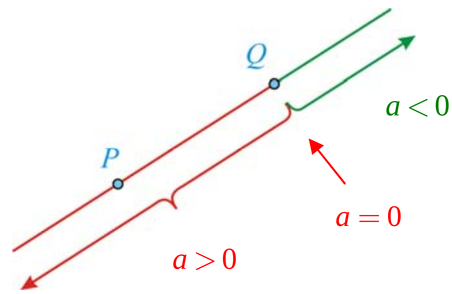
$$X = aP + bQ = Q + a\cdot(P - Q)\,,$$

and hence X lies on the half-line with P on it.
[See picture]

(3) When $a < 0$ then

$$X = aP + bQ = Q + a\cdot(P - Q)\,,$$

and hence X lies on the other half-line—with**out** P on it. [See picture]



(4) When $b = 0$ then $X = aP + bQ = P$

(5) When $b > 0$ then

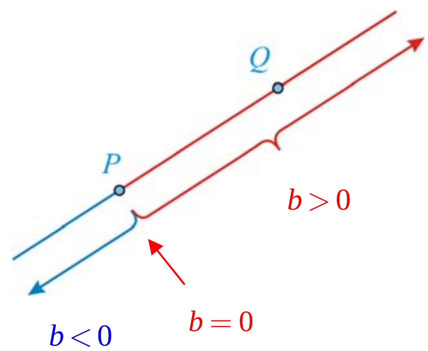
$$X = aP + bQ = P + b\cdot(Q - P)\,,$$

and hence X lies on the half-line with Q on it.
[See picture]

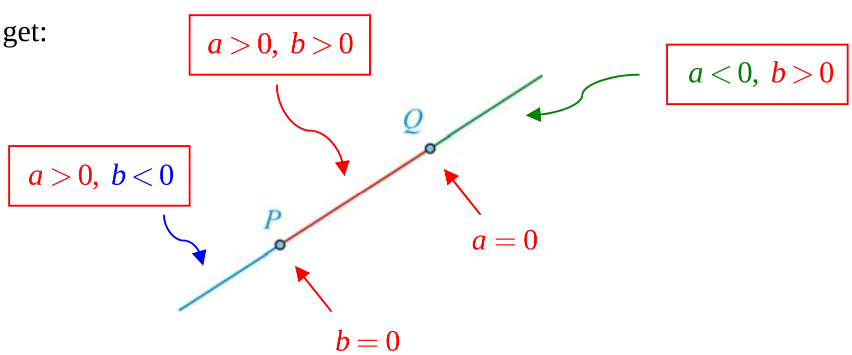
(6) When $b < 0$ then

$$X = aP + bQ = P + b\cdot(Q - P)\,,$$

and hence X lies on the other half-line—with**out** Q on it. [See picture]

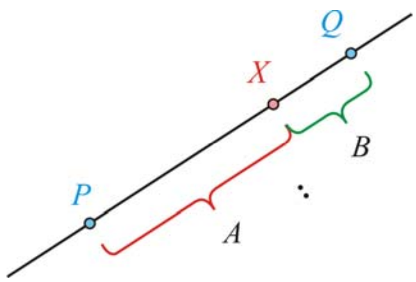


Put together we get:



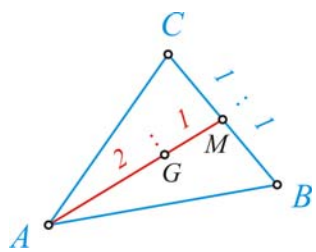
Note that since $a + b = 1$ the case with both $a < 0$ and $b < 0$ does **not** exist.
Also note that the points with **all** barycentric coordinates **positive** lie **inside** the segment PQ.

(II) Suppose a point lies inside the line segment connecting points P and Q, i.e. its barycentric coordinates with respect to P and Q are both positive, and suppose we know the ratio $|PX|:|XQ| = A:B$



Then $X = P + \frac{A}{A+B}(Q-P)$ i.e. $\boxed{X = \frac{B}{A+B}P + \frac{A}{A+B}Q}$

For example in triangle ΔABC the midpoint of side B is $M = \frac{1}{2}B+\frac{1}{2}C$, since the ratio $|BM|:|MC| = 1:1$.



A well known fact from geometry is that the centroid, or center of gravity of the triangle lies on the line AM, with the ratio $|AG|:|GM| = 2:1$.

Hence

$$G = \frac{1}{3}A+\frac{2}{3}M = \frac{1}{3}A+\frac{2}{3}\left(\frac{1}{2}B+\frac{1}{2}C\right)$$
$$= \frac{1}{3}A+\frac{1}{3}B+\frac{1}{3}C$$

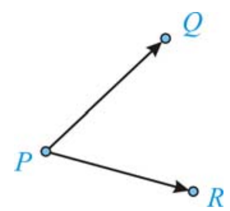
We have written the center of gravity as a barycentric combination of the three points A, B and C [barycentric beause the sum of the coefficients is 1: $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1$]

Let's do this for three points in general.

3 Points

If we take three points P, Q and R, not all on one line, then we can write any point X in the plane through P, Q and R, as

$$X = P + b(Q-P) + c(R-P)$$



since the vector $Q-P$ and $R-P$ are linearly independent:

Hence

$$\begin{aligned} X &= (1-b-c)P + bQ + cR \\ &= aP + bQ + cR \qquad \text{where } a+b+c=1 \end{aligned}$$

We call a, b and c the **barycentric coordinates** of X with respect to the points P, Q and R, and write $[a, b, c]$, $X = [a, b, c]$, or $X = [a, b, c]_{P,Q,R}$.

For example if $P = (2,3,6)$, $Q = (5,6,-1)$ and $R = (5,-6,3)$ then

$$\begin{aligned} X &= [\tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{6}] = \tfrac{1}{3}P + \tfrac{1}{2}Q + \tfrac{1}{6}R \\ &= \tfrac{1}{3}(2,3,6) + \tfrac{1}{2}(5,6,-1) + \tfrac{1}{6}(5,6,3) \\ &= (4,3,2) \end{aligned}$$

Note that $X = (4,3,2)$ is on the plane $\alpha: \; 6x + y + 3z = 33$ through P, Q and R.

How do we find the barycentric coordinates of $X = (1,2,3)$ with respect to $P = (2,3,6)$, $Q = (5,6,-1)$ and $R = (5,6,3)$?

$$\begin{aligned} X &= [a,b,c] = aP + bQ + cR \\ &= a(2,3,6) + b(5,6,-1) + c(5,6,3) = (1,2,3) \end{aligned}$$

which implies

$$\begin{cases} 2a + 5b + 5c = 1 \\ 3a + 6b + 6c = 2 \\ 6a - b + 3c = 3 \end{cases}$$

and we want

$$a + b + c = 1$$

and thus $\begin{cases} 2a+5b+5c=1 \\ 3a+6b+6c=2 \\ 6a-b+3c=3 \\ a+b+c=1 \end{cases}$ so that $\operatorname{rref}\begin{bmatrix} 2 & 5 & 5 & 1 \\ 3 & 6 & 6 & 2 \\ 6 & -1 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

or we could look at the first three equations and solve these as follows

$$\begin{cases} 2a+5b+5c=1 \\ 3a+6b+6c=2 \\ 6a-b+3c=3 \end{cases} \Rightarrow \begin{bmatrix} 2 & 5 & 5 \\ 3 & 6 & 6 \\ 6 & -1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 & 5 & 5 \\ 3 & 6 & 6 \\ 6 & -1 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 1 \\ -4/3 \end{bmatrix}$$

and then check that for these a, b and c indeed $a+b+c=1$ so that

$$X = \frac{4}{3}(2,3,6)+1\cdot(5,6,-1)-\frac{4}{3}(5,6,3) = (1,2,3)$$

When we try this for $X = (5,12,3)$ with respect to $P = (2,3,6)$, $Q = (5,6,-1)$ and $R = (5,-6,3)$

Then we would get $X = [a,b,c] = aP+bQ+cR$
$= a(2,3,6)+b(5,6,-1)+c(5,6,3) = (5,12,3)$

$$\begin{cases} 2a+5b+5c=5 \\ 3a+6b+6c=12 \\ 6a-b+3c=3 \end{cases}$$

and $a+b+c=1$

then

$\begin{cases} 2a+5b+5c=5 \\ 3a+6b+6c=12 \\ 6a-b+3c=3 \\ a+b+c=1 \end{cases}$ so that $\operatorname{rref}\begin{bmatrix} 2 & 5 & 5 & 1 \\ 3 & 6 & 6 & 2 \\ 6 & -1 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Or we could have solved the first three equations (using an inverse matrix)

$$\begin{bmatrix} 2 & 5 & 5 \\ 3 & 6 & 6 \\ 6 & -1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \\ 3 \end{bmatrix} \quad\Rightarrow\quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 & 5 & 5 \\ 3 & 6 & 6 \\ 6 & -1 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 12 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ -15 \end{bmatrix}$$

and *then* check if these are Barycentric coordinates: **BUT** $a+b+c=7\neq 1$ which means that there is **no** solution such that also $a+b+c=1$.

Notice that $X = (5,12,3)$ is **not on** the plane $\alpha:\ x-y=-1$ through P, Q and R, so we canNOT write it as a Barycentric combination of P, Q and R.

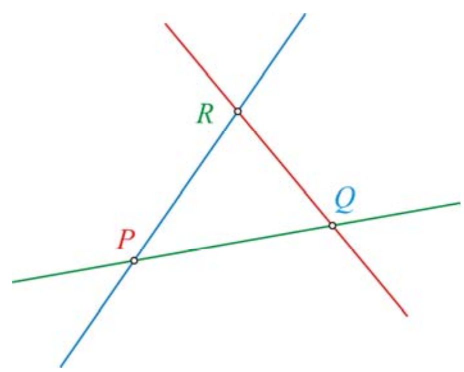
But $X = (1,2,3)$ is on this plane, and hence we were able to write it as a Barycentric combination of P, Q and R.

Geometrically

Again information about the **signs** of a, b and c gives us an indication about the position of the point

$$X = aP+bQ+cR, \quad \text{with} \quad a+b+c = 1$$

on the plane through P, Q and R :



(1) When $a=0$ then

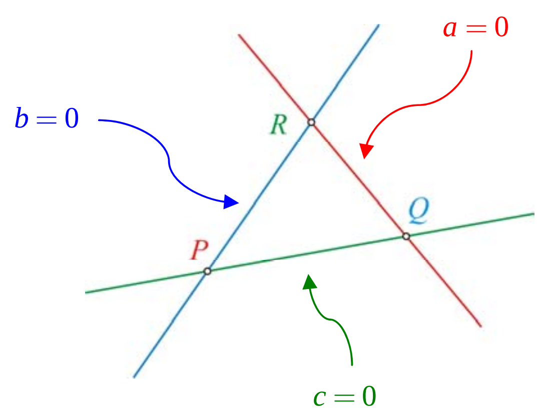
$$X = bQ+cR \quad \text{with} \quad b+c = 1$$

i.e. the line through Q and R.

(2) When $b=0$ then

$$X = aP+cR \quad \text{with} \quad a+c = 1$$

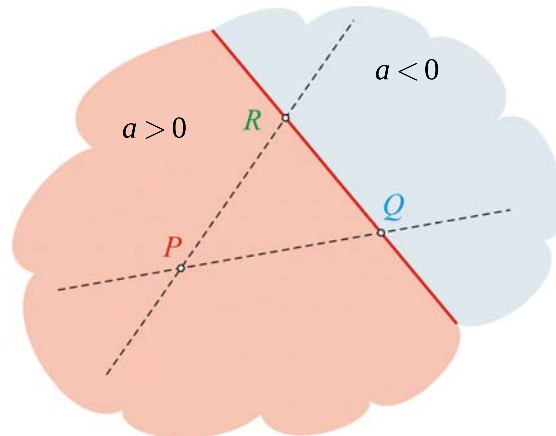
i.e. the line through P and R.



(3) When $c=0$ then

$$X = aP+bQ \quad \text{with} \quad a+b = 1$$

i.e. the line through P and Q.

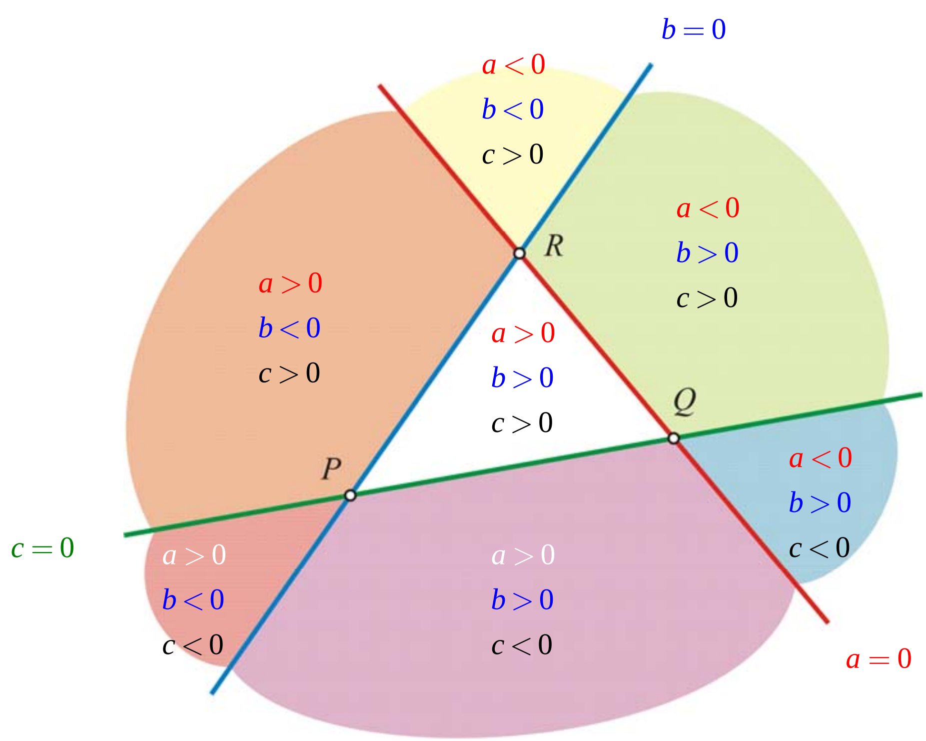


(4) When $a>0$ then X is on the half-plane with P in it, and when $a<0$ then X is on the half-plane with**out** P in it.
[See picture]
similarly

(5) When $b>0$ then X is on the half-plane with Q in it, and when $b<0$ then X is on the half-plane with**out** Q in it.

(6) When $c>0$ then X is on the half-plane with R in it, and when $c<0$ then X is on the half-plane with**out** R in it.

When we put all these together we get 7 regions

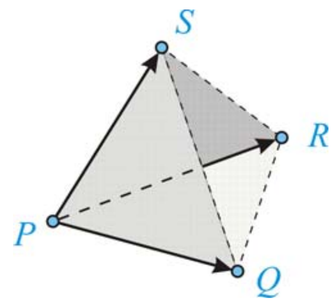


Note again that the case where all of them are negative ($a<0,\ b<0$ and $c<0$) does **not** exist since $a+b+c\ =\ 1$

Also note that the points with **all** barycentric coordinates **positive** lie **inside** the triangle.

4 Points

To describe any point $X \in \mathbb{R}^3$ using barycentric coordinates we need four points P, Q, R and S, **not** all on the same plane:



$$X = P + b(Q-P) + c(R-P) + d(S-P) \in \mathbb{R}^3$$

Hence

$$X = (1-b-c-d)P + bQ + cR + dS$$

i.e. $$X = aP + bQ + cR + dS \quad \text{with} \quad a+b+c+d = 1.$$

For example with respect to $P = (2,3,6)$, $Q = (5,6,-1)$, $R = (5,-6,3)$ and $S = (1,1,1)$ we have

$$X = [\tfrac{1}{3}, 0, \tfrac{1}{6}, \tfrac{1}{2}] = \tfrac{1}{3}(2,3,6) + 0(5,6,-1) + \tfrac{1}{6}(5,-6,3) + \tfrac{1}{2}(1,1,1)$$

$$= (2, \tfrac{1}{2}, 3)$$

Or when for example $X = (5,12,3)$ with respect to $P = (2,3,6)$, $Q = (5,6,-1)$, $R = (5,-6,3)$ and $S = (1,1,1)$

Then we would get $X = [a,b,c,d] = aP + bQ + cR + dS$
$$= a(2,3,6) + b(5,6,-1) + c(5,6,3) + d(1,1,1) = (1,2,3)$$

$$\begin{cases} 2a + 5b + 5c + d = 5 \\ 3a + 6b + 6c + d = 12 \\ 6a - b + 3c + d = 3 \end{cases}$$

and $a + b + c + d = 1$

Those four equations written as one matrix equation gives us

$$\begin{bmatrix} 2 & 5 & 5 & 1 \\ 3 & 6 & 6 & 1 \\ 6 & -1 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \\ 3 \\ 1 \end{bmatrix}$$

i.e.

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & 5 & 5 & 1 \\ 3 & 6 & 6 & 1 \\ 6 & -1 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \quad = \begin{bmatrix} 8 \\ 9 \\ -10 \\ -6 \end{bmatrix}$$

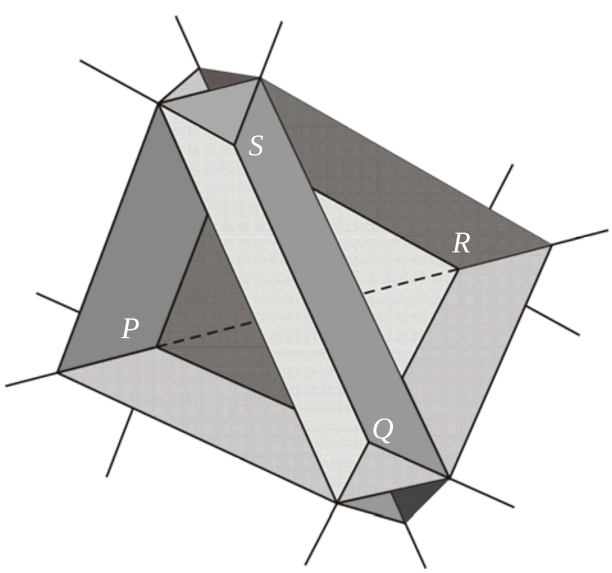
i.e. $$X = 8(2,3,6)+9(5,6,-1)-10(5,6,3)-6(1,1,1) = (5,12,3).$$

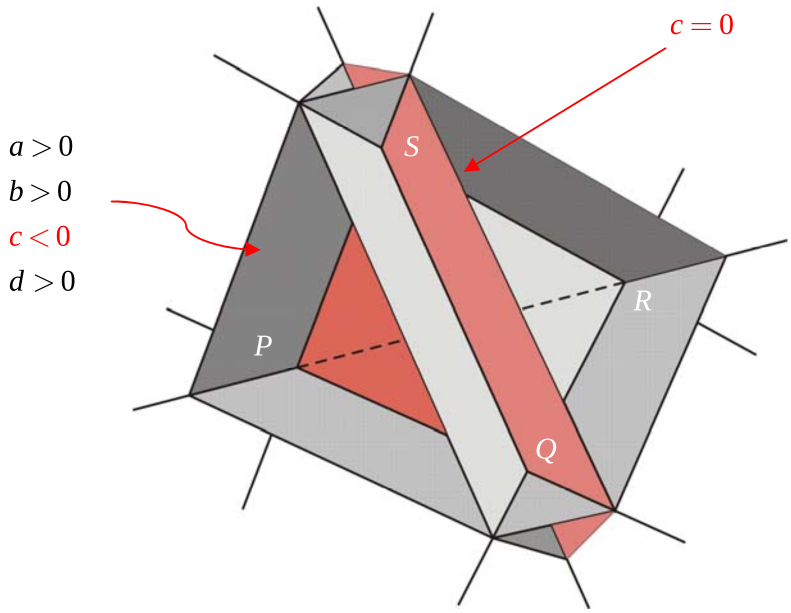
Note that here $a+b+c+d=1$ was built in as the fourth equation.

Geometrically

There are 4 planes through various combinations of three of the points P, Q, R, S. [PQR, PQS, PRS and QRS] These planes divide this 3 dimensional space into 15 regions. Again we can locate a point $X = aP+bQ+cR+dS$ (with $a+b+c+d=1$) in one of these regions, solely based on the signs of a, b, c, and d.

The following picture illustrates the situation.





The plane through PQS (**no** R) has "barycentric equation" $c = 0$

$$X = aP + bQ + 0R + dS \quad \text{with} \quad a + b + 0 + d = 1$$

It divides 3-space in two regions: where $c > 0$ and $c < 0$. The region with $c > 0$ has R in it, the region with $c < 0$ doesn't have R in it.

Note again that the case where all of barycentric coordinates are negative ($a < 0$, $b < 0$, $c < 0$ and $d < 0$) does **not** exist since $a + b + c + d = 1$

Also note that the points whose barycentric coordinates are all **positive** are **inside** the tetrahedron *PQRS*.

Uniqueness of Barycentric Coordinates

The following theorems summarize the facts on uniqueness of barycentric coordinates.

Theorem 34.1: (Barycentric coordinates of points on a line)

(1) Let P and Q be two distinct points on a line l, then any point X on l can be written *uniquely* as $X = aP + bQ$ with $a + b = 1$

(2) Let P, Q and R be three distinct points on a line l, then any point X on l can be written as $X = aP + bQ + cR$ with $a + b + c = 1$, but this description is no longer unique.

Theorem 34.2: (Barycentric coordinates of points on a plane)

(1) Let P, Q and R be three distinct, non collinear points on a plane α, then any point X on α can be written *uniquely* as $X = aP + bQ + cR$ with $a + b + c = 1$

(2) Let P, Q, R and S be four distinct, non collinear points on a plane α, then any point X on α can be written as $X = aP + bQ + cR + dS$ with $a + b + c + d = 1$ but this description is no longer unique.

Theorem 34.3: (Barycentric coordinates of points in $\mathbb{R}^3$)

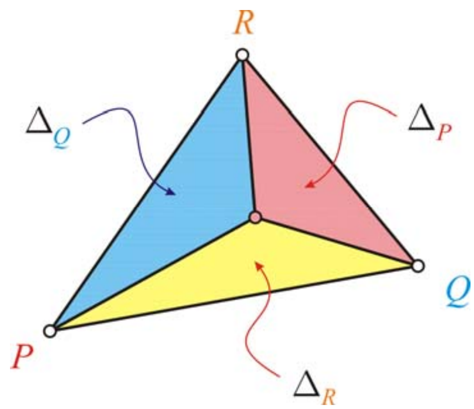
(1) Let P, Q, R and S be four distinct, non planar points in $\mathbb{R}^3$, then any point X in $\mathbb{R}^3$ can be written *uniquely* as $X = aP + bQ + cR + dS$ with $a + b + c + d = 1$

(2) Let P, Q, R, S and T be five distinct, non planar points in $\mathbb{R}^3$, then any point X in $\mathbb{R}^3$ can be written as $X = aP + bQ + cR + dS + eT$ with $a + b + c + d + e = 1$ but this description is no longer unique.

This of course extends to higher dimensional space. The proofs of these theorems are left to the reader. They rely on whether the points form an affine basis of the space. If they do then linearly independency of the vectors guarantees uniqueness.

A geometric interpretation

Here is an interesting geometric interpretation of barycentric coordinates with respect to three points.



Theorem 34.4: Let X be a point inside triangle ΔPQR and

$$X = aP + bQ + cR \quad \text{with} \quad a + b + c = 1$$

and let

$$\begin{aligned} \Delta &= \text{Area}[\Delta PQR] \\ \Delta_P &= \text{Area}[\Delta XQR] \\ \Delta_Q &= \text{Area}[\Delta PXR] \\ \Delta_R &= \text{Area}[\Delta PQX] \end{aligned}$$

then $$a = \frac{\Delta_P}{\Delta}, \quad b = \frac{\Delta_Q}{\Delta}, \quad c = \frac{\Delta_R}{\Delta}$$

The proof of this theorem is not too hard, after one realizes the similarity with Cramer's rule. Note with adding the appropriate $\pm$ signs we need not restrict X to the inside of ΔPQR.

Note this interpretation can be extended to barycentric coordinates with respect to more (or fewer) points. Of course areas will be replaced by volumes corresponding to the appropriate dimensions, and can be formulated in terms of determinants.

Affine Transformations and barycentric coordinates

Affine transformations preserve barycentric coordinates:

Theorem 34.5: Let $T(\vec{x}) = A\vec{x} + \vec{b}$ be an affine transformation and

$\vec{x} = a\,\vec{p} + b\vec{q} + c\vec{r}$ with $a+b+c = 1$ then

$T(\vec{x}) = aT(\vec{p}) + bT(\vec{q}) + cT(\vec{r})$.

Proof:
$$\begin{aligned}
T(\vec{x}) &= T(a\,\vec{p} + b\vec{q} + c\vec{r}) \\
&= A(a\,\vec{p} + b\vec{q} + c\vec{r}) + \vec{b} \\
&= a\,A\vec{p} + b\,A\vec{q} + c\,A\vec{r} + \vec{b} \\
&= a\,A\vec{p} + b\,A\vec{q} + c\,A\vec{r} + 1\vec{b} \\
&= a\,A\vec{p} + b\,A\vec{q} + c\,A\vec{r} + (a+b+c)\vec{b} \\
&= a\,A\vec{p} + a\vec{b} + b\,A\vec{q} + b\vec{b} + c\,A\vec{r} + c\vec{b} \\
&= a(A\vec{p} + \vec{b}) + b(A\vec{q} + \vec{b}) + c(A\vec{r} + \vec{b}) \\
&= a\,T(\vec{p}) + bT(\vec{q}) + c\,T(\vec{r}) \qquad \square
\end{aligned}$$

Note: Barycentric coordinates make an affine transformation ***appear*** to behave **linear**!

If $a+b+c = 1$ then $T(a\,\vec{p} + b\vec{q} + c\vec{r}) = a\,T(\vec{p}) + bT(\vec{q}) + c\,T(\vec{r})$

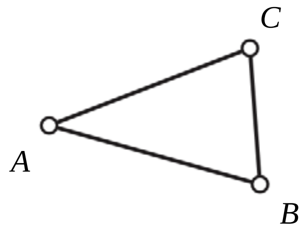
Note: This theorem can be stated in general with a weighted sum of many points as long as the weights add up to 1:

If $\sum_{i=1}^{n} a_i = 1$ then $T\left(\sum_{i=1}^{n} a_i\,\vec{p}_i\right) = \sum_{i=1}^{n} a_i\,T(\vec{p}_i)$

35. More on Barycentric Coordinates

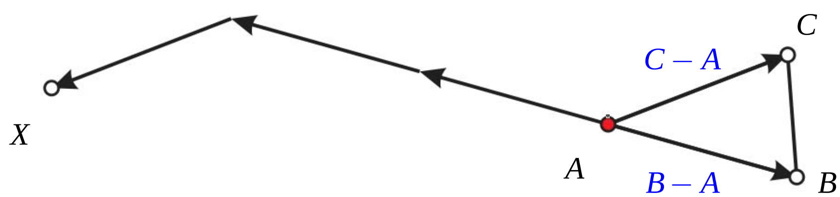
The Geometry of Barycentric Coordinates.

Let the three points A, B and C be given, such that they do not lie all on one line.



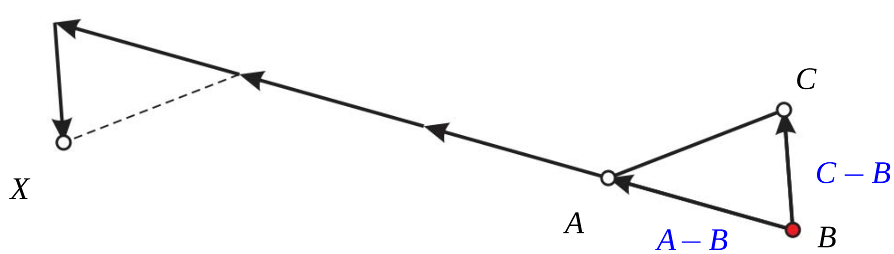
Example 1: Suppose we want to locate the points $X = 4A - 2B - C$. Notice that this is indeed a Barycentric combination of points: $4 - 2 - 1 = 1$.

We could rewrite X as follows: $$X = 4A - 2B - C = \underset{\substack{\uparrow\\ \text{a point}}}{A} - 2\underset{\substack{\uparrow\\ \text{a vector}}}{(B - A)} - \underset{\substack{\uparrow\\ \text{a vector}}}{(C - A)}$$

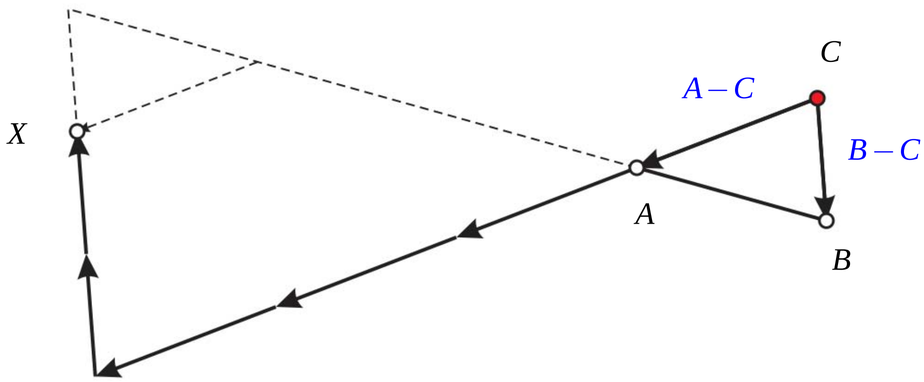


Note that this is not the only way to get to this point. We also have

$$X = 4A - 2B - C = \underset{\substack{\uparrow\\ \text{a point}}}{B} + 4\underset{\substack{\uparrow\\ \text{a vector}}}{(A - B)} - \underset{\substack{\uparrow\\ \text{a vector}}}{(C - B)}$$



We can also write it as: $$X = 4A - 2B - C = \underset{\text{a point}}{\underset{\uparrow}{C}} + 4\underset{\text{a vector}}{\underset{\uparrow}{(A-C)}} - 2\underset{\text{a vector}}{\underset{\uparrow}{(B-C)}}$$

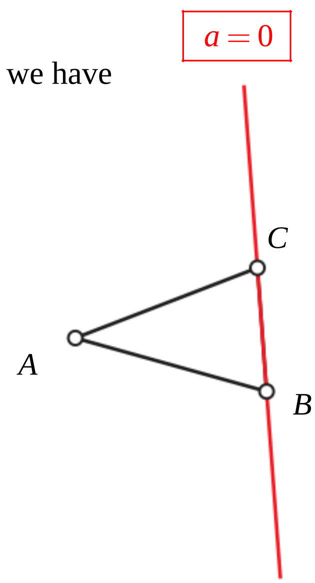


Lines

All points in the plane can be written as: $X = aA + bB + cC$ with $a+b+c = 1$.

$\boxed{a=0}$ If we only look at the points for which $a=0$ then we have

$$X = bB + cC \quad \text{with} \quad b+c = 1$$



This is the line through B and C, which is easy to see when we realize that

$$\begin{aligned} X &= bB + cC \\ &= (1-c)B + cC \\ &= \underset{\text{a point}}{\underset{\uparrow}{B}} + c\underset{\text{a vector}}{\underset{\uparrow}{(C-B)}} \end{aligned}$$

which is a vector equation of the line through B and C.

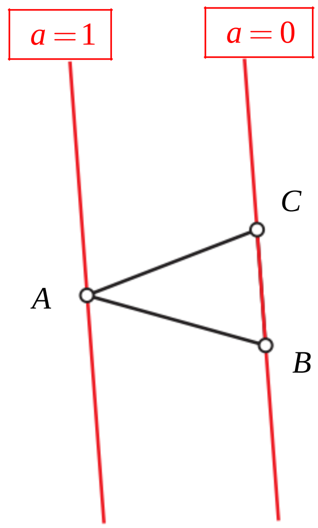
Note that the line through A parallel to this line has as vector equation:

$$X = A + t(C - B)$$

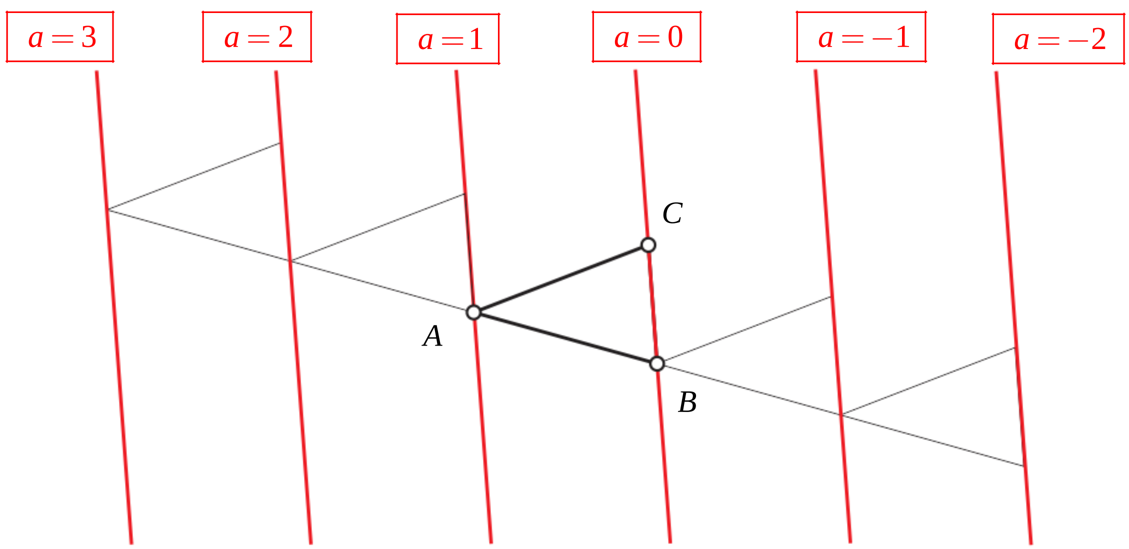
i.e. these points look like: $X = A - tB + tC$

Note that this is indeed a Barycentric combination since $1 - t + t = 1$.

But the points $X = A - tB + tC$ all have $\boxed{a = 1}$



Let's include also the lines $a = 3$, $a = 2$, $a = -1$ and $a = -2$



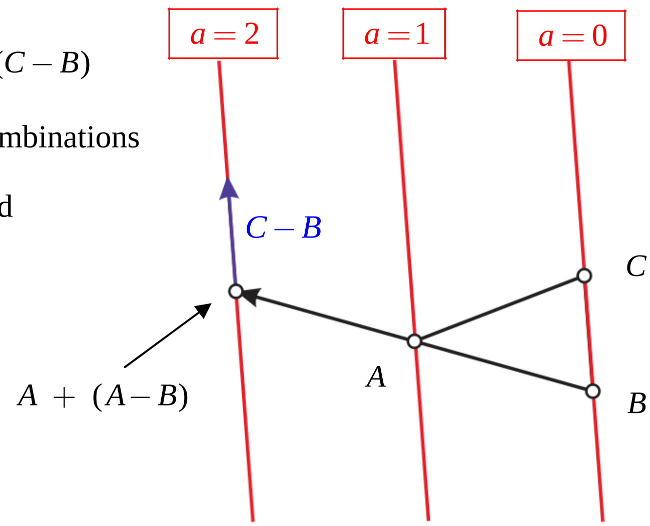
Example 2: The line $a = 2$ is the line with vector equation $X = A + (A - B) + t(C - B)$

Note that indeed these are barycentric combinations since $X = 2A - (1+t)B + tC$ and

$$2 - (1+t) + t = 1$$

and all these points have

$$a = 2$$

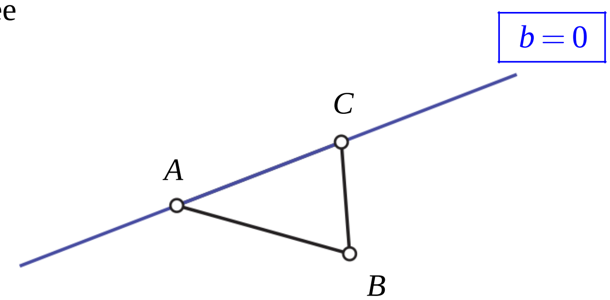


$\boxed{b=0}$ Similarly, if we look at the points for which $b = 0$ then we have

$$X = aA + cC \quad \text{with} \quad a + c = 1$$

This is the line through A and C, which is easy to see when we realize that

$$\begin{aligned} X &= aA + cC \\ &= (1-c)A + cC \\ &= \underset{\substack{\uparrow \\ \text{a point}}}{A} + c\underset{\substack{\uparrow \\ \text{a vector}}}{(C-A)} \end{aligned}$$



which is a vector equation of the line through A and C.

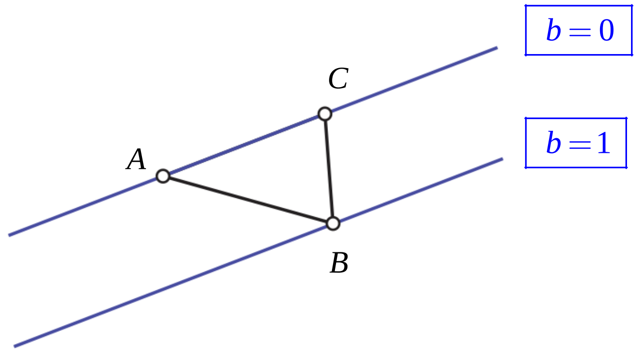
Next note that the line through B parallel to this line has as vector equation:

$$X = B + t(C - A)$$

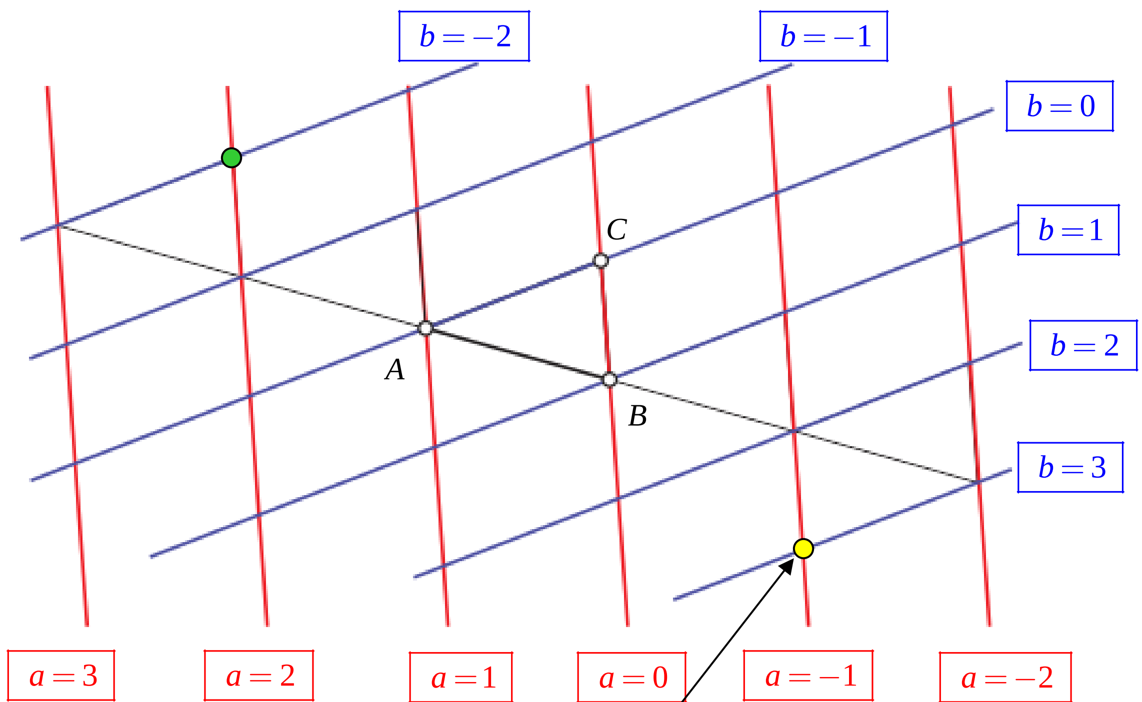
i.e. these points look like: $X = -tA + B + tC$

Note that this is yet another Barycentric combination since $-t + 1 + t = 1$.

and the points $X = -t\,A + B + t\,C$ all have $\boxed{b=1}$



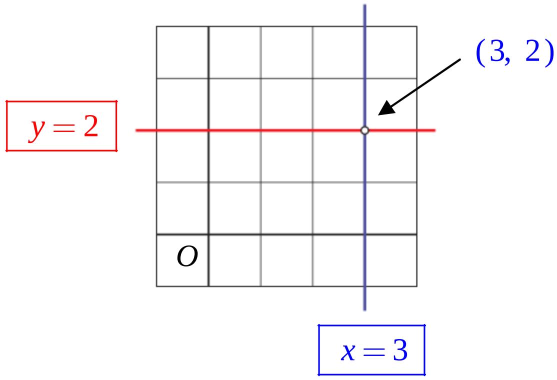
Summarizing what we have so far:



Example 3: the point $X = -A+3B-C$ is located here.

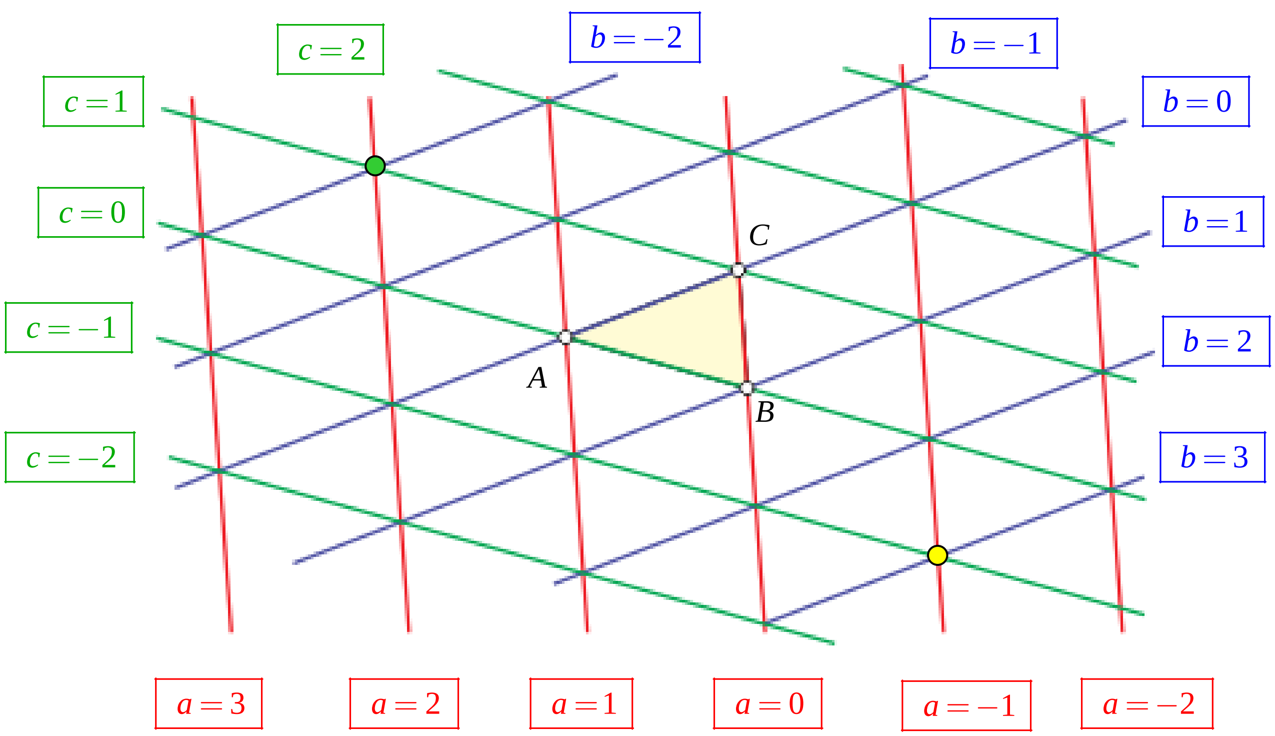
The location of the point $X = 2A-2B+C$ is indicated by the green dot in the picture.

Note this is similar to the situation for Euclidean coordinates:



The line $y = 2$ is just the set of all points $(x,\ 2)$, and the line $x = 3$ is just the set of all points $(3,\ y)$.

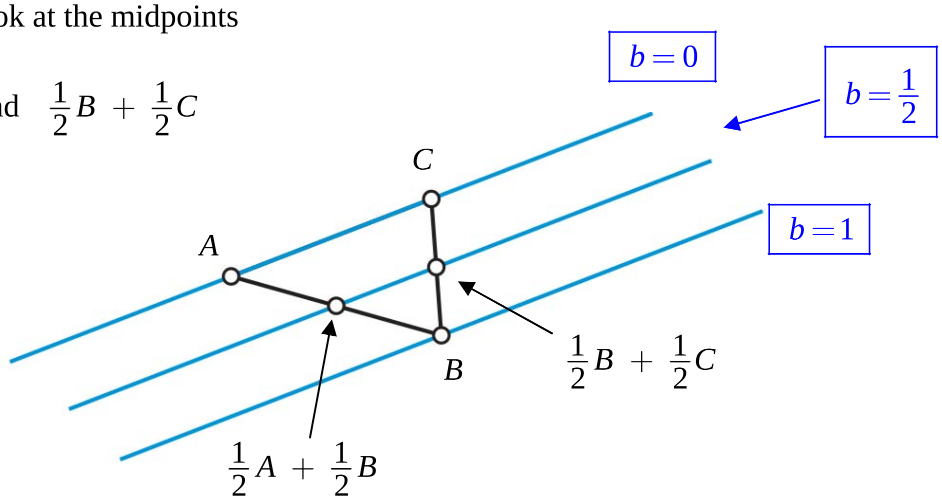
Returning to Barycentric coordinates: let's also include the lines where $c = -2, -1,\ 0,\ 1,\ 2$



This can be done for non-integer values too.

Example 4: look at the midpoints

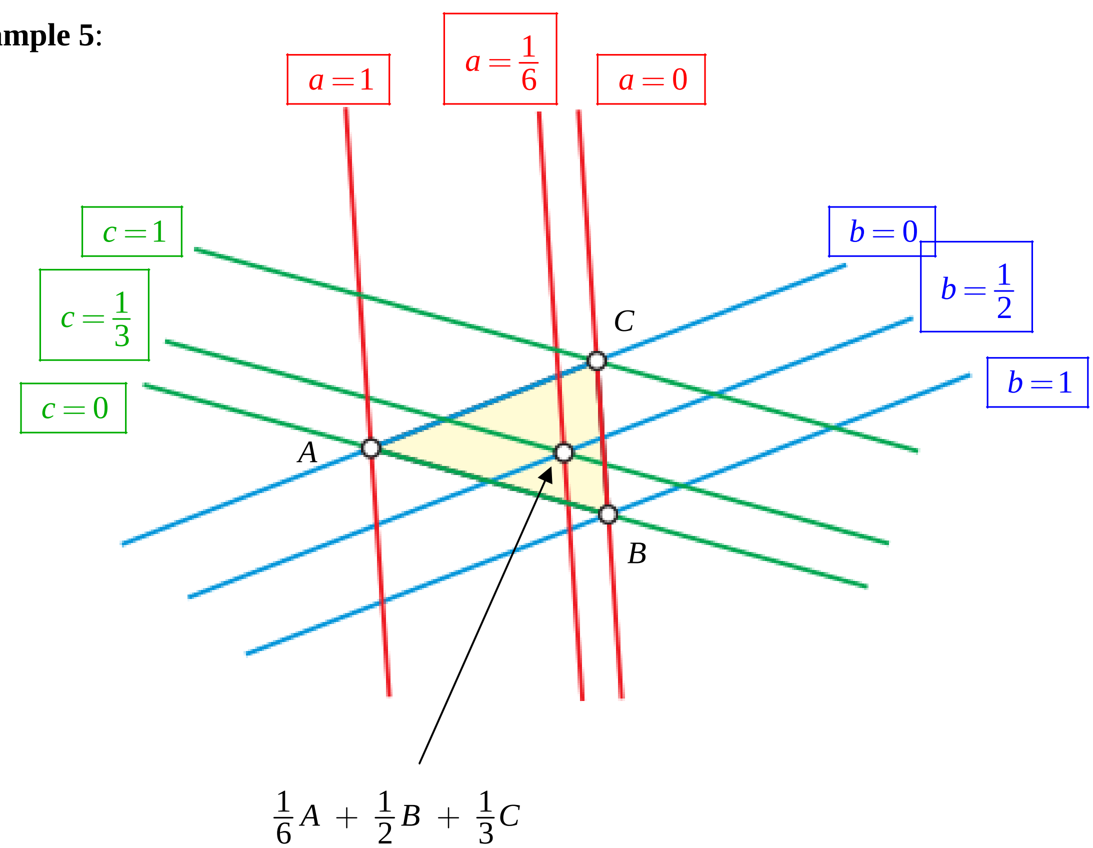
$\frac{1}{2}A + \frac{1}{2}B$ and $\frac{1}{2}B + \frac{1}{2}C$



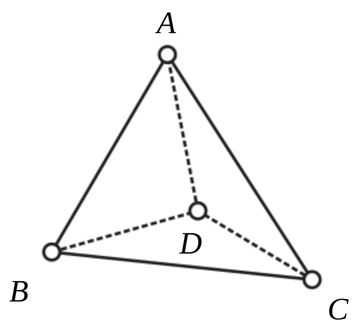
Note that both points have $b = \frac{1}{2}$, hence they lie on the line: $b = \frac{1}{2}$.

The line $b = \frac{1}{2}$ is exactly in the middle between the lines $b = 0$ and $b = 1$.

Example 5:

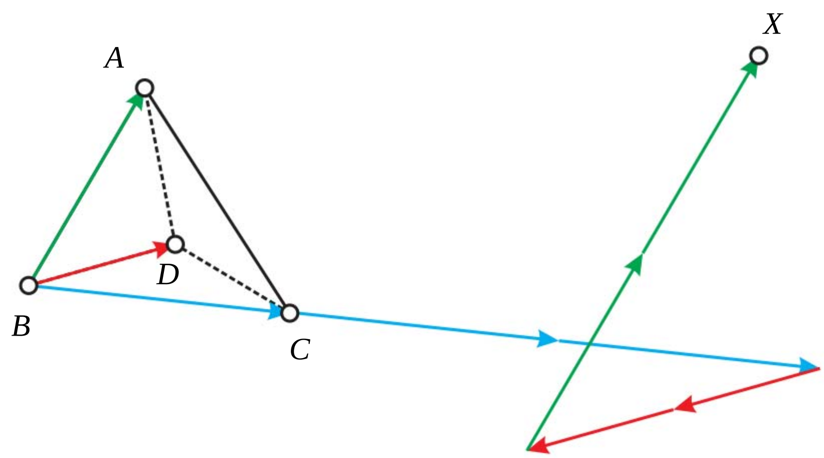


Of course a similar analysis also applies to higher dimensional Barycentric coordinate systems. The only difficulty is trying to draw pictures. Even 3D, on a 2D piece of paper is a challenge. Let A, B, C and D be four points not all on one plane. Then any point X can be written as $X = aA + bB + cC + dD$ with $a+b+c+d=1$

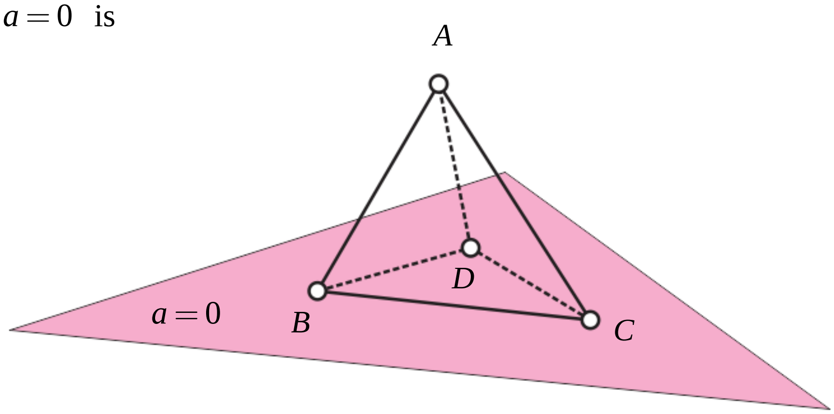


For example the point $X = 2A - 2B + 3C - 2D$

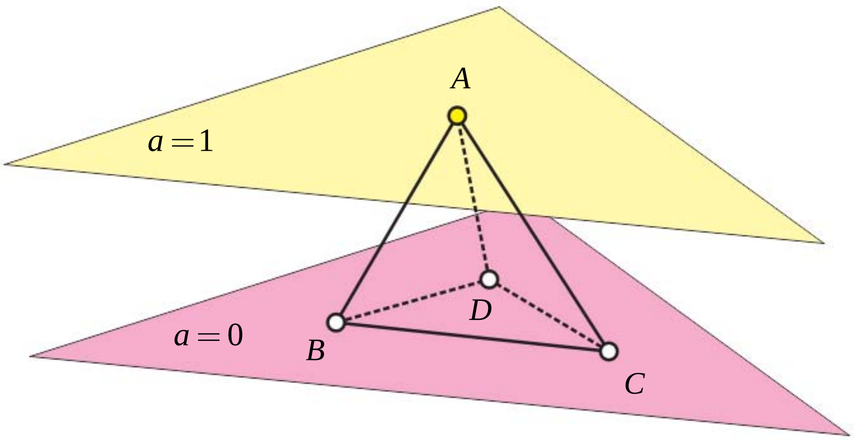
$$= B + 3(C-B) - 2(D-B) + 2(A-B)$$



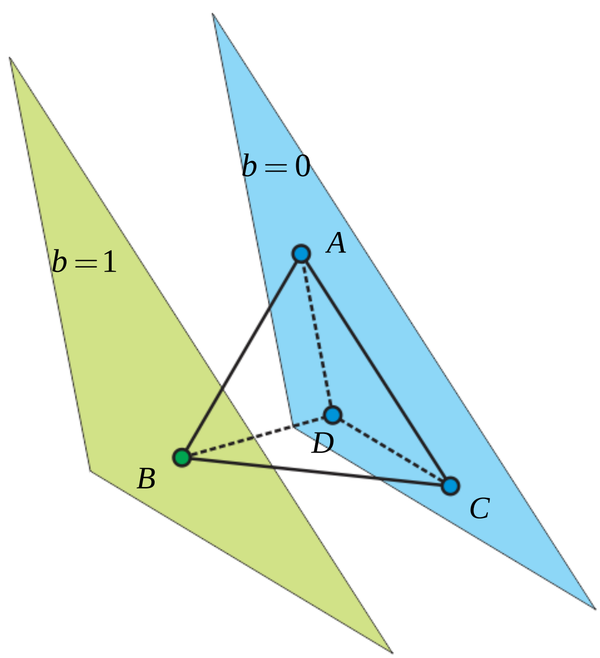
The plane $a=0$ is



and the plane $a = 1$ is



The parallel planes $b = 0$ and $b = 1$ are



Etc. A similar 3D grid is produced, but now with planes (instead of lines in the 2D case). A bit harder to visualize, but we can essentially extend/extrapolate from the 2D case.

Affine Transformations and barycentric coordinates

We'll prove the following properties

- **(1)** Invertible Affine transformations map lines to lines.
- **(2)** Invertible Affine transformations map parallel lines to parallel lines.
- **(3)** Affine transformations preserve Barycentric coordinates.
- **(4)** Affine transformations preserve ratios on lines.

Theorem 35.1: Let $T(\vec{x}) = A\vec{x} + \vec{b}$ be an invertible affine transformation, and l the line $\vec{x} = \vec{p} + t\vec{v}$ then the image of this line is another line: $\vec{x}' = \vec{p}' + t\vec{v}'$ where $\vec{p}' = A\vec{p} + \vec{b}$ and $\vec{v}' = A\vec{v}$.

Proof: The image of l is $\vec{x}' = T(\vec{x}) = A\big(\vec{p} + t\vec{v}\big) + \vec{b} = \underbrace{A\vec{p} + \vec{b}}_{\vec{p}'} + t\underbrace{A\vec{v}}_{\vec{v}'}$

This is another line through the point $\vec{p}'$ and with direction vector $\vec{v}'$.

Notice that $\vec{v}' = A\vec{v} \neq \vec{0}$ since T is invertible [A is invertible.]

If T were not invertible a line *could* be mapped to a point when $\vec{p}' + t\underbrace{\vec{0}}_{A\vec{v}} = \vec{p}'$!

Example 6: Let $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 & -12 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 18 \\ -6 \end{bmatrix}$. Note that this is an invertible Affine transformation. Let $l: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} + t\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ then the image of this line is

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 & -12 \\ 2 & 5 \end{bmatrix}\left(\begin{bmatrix} 4 \\ -1 \end{bmatrix} + t\begin{bmatrix} -3 \\ 2 \end{bmatrix}\right) + \begin{bmatrix} 18 \\ -6 \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \end{bmatrix} + t\begin{bmatrix} -9 \\ 4 \end{bmatrix}$$

Hence the image of l is $l': \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \end{bmatrix} + t\begin{bmatrix} -9 \\ 4 \end{bmatrix}$ i.e. another line.

Example 7: Let $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & -6 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 9 \\ -3 \end{bmatrix}$. Note that this is NOT an invertible Affine transformation.

The line $l: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} + t\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ gets mapped to the line $l': \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix} + t\begin{bmatrix} -6 \\ 3 \end{bmatrix}$, but the line $m: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} + t\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ gets mapped to the point $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$. [Check!]

Theorem 35.2: Let $T(\vec{x}) = A\vec{x} + \vec{b}$ be an invertible affine transformation, l the line $\vec{x} = \vec{p} + t\vec{v}$ and $m \parallel l$ with vector equation $\vec{x} = \vec{q} + t\vec{v}$.
then $l' \parallel m'$

Proof: The image of l is $\vec{x}' = T(\vec{x}) = A\big(\vec{p} + t\vec{v}\big) + \vec{b} = \underbrace{A\vec{p} + \vec{b}}_{\vec{p}'} + t\underbrace{A\vec{v}}_{\vec{v}'}$

The image of m is $\vec{x}' = T(\vec{x}) = A\big(\vec{q} + t\vec{v}\big) + \vec{b} = \underbrace{A\vec{q} + \vec{b}}_{\vec{q}'} + t\underbrace{A\vec{v}}_{\vec{v}'}$

Notice that the images of the two lines have the same non-zero slope, hence are parallel.

Example 8: Let $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 & -12 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 18 \\ -6 \end{bmatrix}$. Note that this is an invertible Affine transformation. Let $l: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} + t\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $m: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \end{bmatrix} + s\begin{bmatrix} -3 \\ 2 \end{bmatrix}$

The image of l is $l': \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \end{bmatrix} + t\begin{bmatrix} -9 \\ 4 \end{bmatrix}$

parallel

The image of m is $m': \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ -5 \end{bmatrix} + s\begin{bmatrix} -9 \\ 4 \end{bmatrix}$

Example 9: Parallelograms are mapped to parallelograms.

Theorem 35.3: Let $T(\vec{x}) = A\vec{x} + \vec{b}$ be an affine transformation and $\vec{x} = a\,\vec{p} + b\vec{q} + c\vec{r}$ with $a+b+c = 1$ then $T(\vec{x}) = aT(\vec{p}) + bT(\vec{q}) + cT(\vec{r})$.

Proof:
$$\begin{aligned} T(\vec{x}) &= T(a\,\vec{p} + b\vec{q} + c\vec{r}) \\ &= A(a\,\vec{p} + b\vec{q} + c\vec{r}) + \vec{b} \\ &= a\,A\vec{p} + b\,A\vec{q} + c\,A\vec{r} + \vec{b} \\ &= a\,A\vec{p} + b\,A\vec{q} + c\,A\vec{r} + 1\vec{b} \\ &= a\,A\vec{p} + b\,A\vec{q} + c\,A\vec{r} + (a+b+c)\vec{b} \\ &= a\,A\vec{p} + a\vec{b} + b\,A\vec{q} + b\vec{b} + c\,A\vec{r} + c\vec{b} \\ &= a(A\vec{p} + \vec{b}) + b(A\vec{q} + \vec{b}) + c(A\vec{r} + \vec{b}) \\ &= aT(\vec{p}) + bT(\vec{q}) + cT(\vec{r}) \end{aligned} \qquad \square$$

Note: Barycentric coordinates make an affine transformation ***appear*** to behave **linear**!

If $a+b+c = 1$ then $T(a\,\vec{p} + b\vec{q} + c\vec{r}) = aT(\vec{p}) + bT(\vec{q}) + cT(\vec{r})$

Note: This theorem can be stated in general with a weighted sum of many points as long as the weights add up to 1:

$$\text{If} \quad \sum_{i=1}^{n} a_i = 1 \quad \text{then} \quad T\left(\sum_{i=1}^{n} a_i\,\vec{p}_i\right) = \sum_{i=1}^{n} a_i\,T(\vec{p}_i) \quad \cdots (*)$$

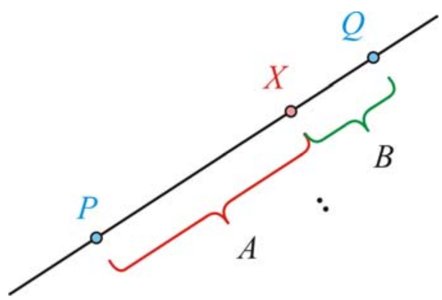
This could be taken as the definition of Affine transformations.

Example 10: The centroid of a triangle ΔABC is $G = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$. The triangle is mapped to another triangle $\Delta A'B'C'$ (Since lines are mapped to lines by Theorem 36.1). The centroid of the new triangle is $G' = \frac{1}{3}A' + \frac{1}{3}B' + \frac{1}{3}C'$. Theorem 35.3 shows that the centroid G gets mapped to the new centroid G'. One could say that affine transformations preserve centroids.

Theorem 35.4: Affine transformations preserve ratios on lines.

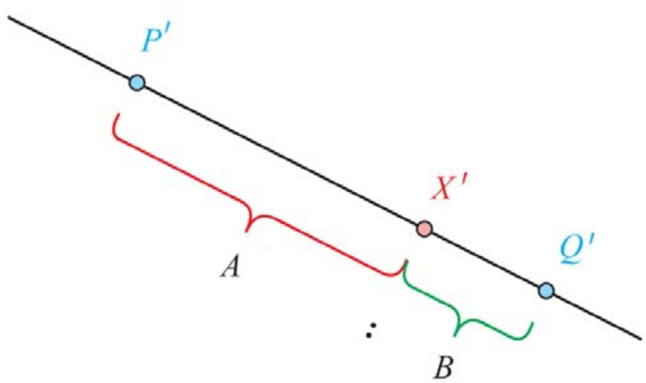
Proof: This follows immediately from Theorem 35.3.

Suppose we have three points on a line with the given ratios



Then $X = P + \frac{A}{A+B}(Q-P)$ i.e. $X = \frac{B}{A+B}P + \frac{A}{A+B}Q$ (Barycentric!)

But then $X' = \frac{B}{A+B}P' + \frac{A}{A+B}Q'$ by Theorem 35.3 which means that



So that the affine transformation preserved the ratios.

Example 11:

Midpoints are mapped to midpoints. Medians to medians.

36. An Extra Dimension

In this section we will dramatically change the way we look at our world, which will clarify and simplify things.

(I) Points and Vectors … as we knew them.

When we started the journey of exploring 2D and 3D space we made the point of distinguishing between points and vectors. In fact we started off writing them differently:

For example the *vector* $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ was pointing at the *point* $P = (3,2)$.

Our different notation stressed the fact that in reality points and vectors, although closely linked, are two entirely different objects. Nevertheless we blurred the distinction when we entered the realm of transformations. There it was more convenient to work with just *vectors*. For example $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ could refer to either a point or a vector. The context usually indicated which interpretation was meant. The reason it turned out to be convenient to work with vectors was the fact we could then use matrix multiplication to transform one vector into another vector:

For example the transformation $T(\vec{x}) = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\vec{x}$ which maps the *vectors* $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ to $\begin{bmatrix} 8 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 4 \end{bmatrix}$ respectively, as follows:

$$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Yet we use the ‘vector’ notation to transform **points**: so the above matrix multiplications transform the *point* $P = (3,2)$ to the *point* $P' = (8,5)$ and the *point* $Q = (1,3)$ to the *point* $Q' = (-2,4)$.

In the linear case $T(\vec{x}) = A\vec{x}$ we hardly notice any difference between vectors and points:

The vector $\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ between the points $Q = (1,3)$ and $P = (3,2)$,

$$\vec{v} = P - Q = (3,2) - (1,3) = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

is transformed to the vector

$$\vec{v}' = P' - Q' = (8,5) - (-2,4) = \begin{bmatrix} 10 \\ 1 \end{bmatrix}.$$

If we were to enter this *vector* in the transformation, $T(\vec{v}) = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\vec{v}$, surprisingly we get the correct image vector $T(\vec{v}) = \vec{v}'$:

$$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

The ***same*** transformation matrix can be used to accomplish the transformation of both ***points*** and ***vectors***! At first this might not be surprising, until one realizes that this is *not* the case for affine transformations. Affine transformations, as we discussed before, map points to points but if you enter a vector in the transformation you will *not* get out the correct image of the vector:

For example the affine map

$$S(\vec{x}) = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}\vec{x} + \begin{bmatrix} 7 \\ 0 \end{bmatrix}$$

also maps the *point* $P = (3,2)$ to the *point* $P' = (8,5)$ and the *point* $Q = (1,3)$ to the *point* $Q' = (-2,4)$, as seen by the computations (using *vectors*!)

$$\begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 7 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

But $S(\vec{v}) \neq \vec{v}'$, since $\vec{v}' = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$ but

$$S(\vec{v}) = S\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 17 \\ 1 \end{bmatrix}.$$

Hence affine transformations treat *points* and *vectors* differently. If we put a vector $\vec{v} = P - Q$ into the affine transformation $S(\vec{x}) = A\vec{x} + \vec{b}$ we get

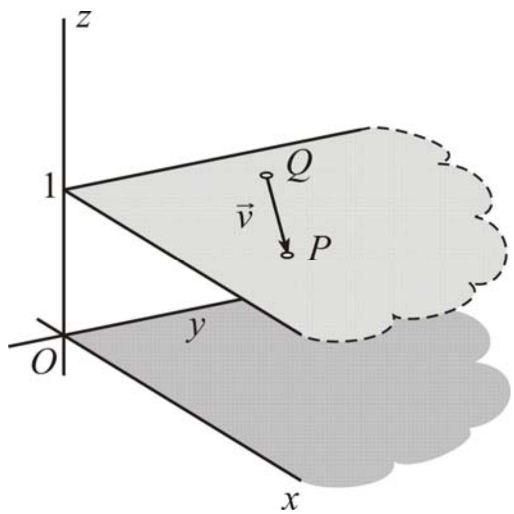
$$S(\vec{v}) = A\vec{v} + \vec{b} = A(P-Q) + \vec{b}$$

Whereas $\quad \vec{v}' = P' - Q' = (A\vec{p} + \vec{b}) - (A\vec{q} + \vec{b}) = A(P-Q) = A\vec{v}$.

Hence to get the image under the affine transformation of a vector we *only* need to multiply the vector with the matrix A. Hence affine transformations treat points and vectors differently: it transforms *points* the way we want it to, but does not transform *vectors* the way we would like. This can be fixed.

(II) Another perspective on Points and Vectors … from a higher dimension.

Imagine for a moment that we live on a two dimensional plane. And suppose we are only aware of those two dimensions. But unbeknownst to us, our familiar, homey plane is in reality a two dimensional plane imbedded in a three dimensional world. We just don't perceive the third dimension. Let's say our plane is actually the plane $z = 1$ in three-space.



What *we* see as the origin $0 = (0,0)$ is identified by the creatures in the three dimensional world, who can actually see all three dimensions, as $(0,0,1)$ or, as they decided to call it, the 'point' $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. They too decided to write *points* using a *vector* notation. Similarly the points $P = (3,2)$ and $Q = (1,3)$, as we denoted them, are actually the points $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

All our points actually have a third coordinate (equal to 1): $R = (a,b)$ really is $\begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$.

What about our *vectors*? Let's look, for example, at the vector $\vec{v} = P - Q = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Note that this vector is ***in*** our plane, i.e. parallel to our plane. Hence it is actually $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ with the third coordinate being 0. All the vectors in our plane are parallel to our plane. Hence in the three dimensional world they have a zero as third coordinate!

All *our* vectors now look like $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$. In particular: they do **not** look like points! Points in the new way of looking at our world would have as third coordinate a 1. Vectors have as third coordinate a 0.

Notice that this works out beautifully *algebraically*, since a difference of points *is* a vector:

$$\vec{v} = P - Q = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

The third coordinate becomes zero when we take the difference of two points algebraically. The difference of two points gives a vector, exactly as we were used to.

From this perspective of an extra dimension *points* and *vectors* now really look (and are) different:

$\begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$ is a **point** in our world and $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ is a **vector** in our world.

In and of itself---even though pretty---this is not the main reason why we would want to go up one dimension. One of the really nice features is that the *affine* transformations of *our* world now become part of a *linear* transformation of the *higher* dimensional world. The particular linear transformations of the new world we will discuss, in effect induce an affine transformation when restricted to our plane. This we will discuss next.

(III) One matrix for linear *and* affine transformations

Let's look at the affine transformation: $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$, which maps the point $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ to $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} ax_0 + by_0 + e \\ cx_0 + dy_0 + f \end{bmatrix}$.

In our 3D world it would mean the *point* $\begin{bmatrix} x_0 \\ y_0 \\ 1 \end{bmatrix}$ gets mapped to the *point* $\begin{bmatrix} ax_0 + by_0 + e \\ cx_0 + dy_0 + f \\ 1 \end{bmatrix}$,

i.e $\tilde{T}\begin{bmatrix} x_0 \\ y_0 \\ 1 \end{bmatrix} = \begin{bmatrix} ax_0 + by_0 + e \\ cx_0 + dy_0 + f \\ 1 \end{bmatrix}$. But notice that this can be written as

$$\tilde{T}\begin{bmatrix} x_0 \\ y_0 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \\ 1 \end{bmatrix}$$

This is a linear transformation of the 3D world! But when we restrict it to the points of our world it represents an *affine* transformation. We could say that the affine transformation is realized as a slice of a linear transformation: the linear transformation $\tilde{T}\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ when restricted to the plane where $z = 1$ transforms this plane affinely!

For example the affine map: $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 7 \\ 0 \end{bmatrix}$ can be written as

$$\tilde{T}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 7 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Note the position of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the translation vector $\begin{bmatrix} e \\ f \end{bmatrix}$ of the transformation $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$ in the 2D format and the transformation in the 3D format:

$$\tilde{T}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

In general, if the transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(\vec{x}) = A\vec{x} + \vec{b}$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} e \\ f \end{bmatrix}$ then the 3×3 matrix needed to express T as a map from $\mathbb{R}^3$ to $\mathbb{R}^3$ looks like:

$$\begin{bmatrix} \multicolumn{2}{c}{\boxed{A}} & \boxed{\vec{b}} \\ 0 & 0 & 1 \end{bmatrix}$$

Example:

The map $\tilde{T}(\vec{x}) = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix}\vec{x}$ could simply refer to the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$, i.e. it maps every vector in $\mathbb{R}^3$ to another vector in $\mathbb{R}^3$. But if we restrict to only the points $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ on the plane $z = 1$ then this map can be interpreted as describing the *affine* map

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Note that we can use the matrix $\begin{bmatrix} \multicolumn{2}{c}{\boxed{A}} & \boxed{\vec{b}} \\ 0 & 0 & 1 \end{bmatrix}$ for *linear* transformations as well: $T(\vec{x}) = A\vec{x}$, because we simply have $\vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, i.e. $\begin{bmatrix} \multicolumn{2}{c}{\boxed{A}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 & 0 & 1 \end{bmatrix}$.

This discussion, of course, also applies to linear and affine transformations of $\mathbb{R}^3$. For example the affine transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -4 & 7 \\ 1 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$$

can be realized as the transformation

$$\tilde{T}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 7 & -1 \\ 1 & 1 & 0 & 2 \\ -3 & 2 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

So for example T transforms $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}$ as follows

$$T\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 7 \\ 1 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}$$

or as follows $\tilde{T}\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 7 & -1 \\ 1 & 1 & 0 & 2 \\ -3 & 2 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 2 \\ 1 \end{bmatrix}$ i.e. $\tilde{T}: \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 4 \\ 6 \\ 2 \\ 1 \end{bmatrix}$.

If the transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ is given by $T(\vec{x}) = A\vec{x} + \vec{b}$ where $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} k \\ l \\ m \end{bmatrix}$ then the 4×4 matrix needed to express T as a map from $\mathbb{R}^4$ to $\mathbb{R}^4$ looks like:

$$\begin{bmatrix} \boxed{A} & \boxed{\vec{b}} \\ 0\ \ 0\ \ 0 & 1 \end{bmatrix}$$

Again this works for both linear and affine transformations: $\vec{b} = \vec{0}$ or $\vec{b} \neq \vec{0}$.

We will call matrices of the form $\begin{bmatrix} \boxed{A} & \boxed{\vec{b}} \\ 0\ \cdots\ 0 & 1 \end{bmatrix}$ **Affine matrices**. Some people refer to them as homogeneous matrices and the representation of our points with an extra coordinate of 1 as homogeneous coordinates. We will revisit homogeneous coordinates later.

Affine Matrices transform both vectors and points correctly.

Finally notice that this one matrix, with the extra dimension, transforms both points and vectors correctly:

As observed earlier if $S(\vec{x}) = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}\vec{x} + \begin{bmatrix} 7 \\ 0 \end{bmatrix}$ then S maps the point $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ to the point $\begin{bmatrix} 8 \\ 5 \end{bmatrix}$

$$S\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

But a vector, like e.g. $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is not mapped by S: $S(\vec{v}) \neq \vec{v}'$. Instead we have to multiply $\vec{v}$ with the matrix A **only** to get $\vec{v}'$

$$A\vec{v} = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \end{bmatrix} = \vec{v}'$$

[In general: if $\vec{v} = P - Q$ then

$$\vec{v}' = P' - Q' = S(P) - S(Q) = (AP + \vec{b}) - (AQ + \vec{b}) = A(P - Q) = A\vec{v}$$]

Our extra dimensional approach takes care of this

$$\begin{bmatrix} 3 & -4 & 7 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & -4 & 7 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 0 \end{bmatrix}$$

The new matrix of the transformation—using the extra dimension—treats **both** points and vectors correctly! [Due to the 1 and 0 in the last coordinate]

$$\begin{bmatrix} \boxed{\quad A \quad} & \boxed{\vec{b}} \\ 0 \quad 0 & 1 \end{bmatrix}\begin{bmatrix} \vec{x} \\ 1 \end{bmatrix} = \begin{bmatrix} A\vec{x} + \vec{b} \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \boxed{\quad A \quad} & \boxed{\vec{b}} \\ 0 \quad 0 & 1 \end{bmatrix}\begin{bmatrix} \vec{v} \\ 0 \end{bmatrix} = \begin{bmatrix} A\vec{v} \\ 0 \end{bmatrix}$$

37. The extra dimension at work

Recall that the basic affine transformations were found using translations and linear transformations:

$$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0$$

(1) a translation by $-\vec{x}_0$: $\quad T_{-\vec{x}_0}(\vec{x}) = \vec{x} - \vec{x}_0$

(2) a linear transformation: $\quad L(\vec{x}) = A\vec{x}$

(3) a translation by $+\vec{x}_0$: $\quad T_{\vec{x}_0}(\vec{x}) = \vec{x} + \vec{x}_0$

$$T_{\vec{x}_0} \circ L \circ T_{-\vec{x}_0}$$

We are now able to perform all these transformations with matrices of one degree higher.

$$\begin{bmatrix} \multicolumn{2}{c}{\boxed{\ A\ }} & \boxed{\vec{b}} \\ 0 & 0 & 1 \end{bmatrix}$$

Recall that we named these matrices **affine matrices**, or homogeneous matrices.

Example 1: The translation $T(\vec{x}) = \vec{x} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ has as affine matrix:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 2: The linear transformation $L(\vec{x}) = \begin{bmatrix} 2 & -4 \\ 5 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 & -4 \\ 5 & 3 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has as affine matrix:

$$\begin{bmatrix} 2 & -4 & 0 \\ 5 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 3: As we saw in a previous section the projection onto the line $l: \ 3x + y = 10$ in $\mathbb{R}^2$ is given by

$$T(\vec{x}) = \frac{1}{10}\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}\vec{x} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

It's affine matrix would be

$$\begin{bmatrix} 0.1 & -0.3 & 3 \\ -0.3 & 0.9 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

We could find this affine transformation using $T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0$ or equivalently

$$T_{\vec{x}_0} \circ L \circ T_{-\vec{x}_0}$$

Using only affine matrices we immediately find the affine matrix of the composition, as follows:

Take $\vec{x}_0 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0.1 & -0.3 & 0 \\ -0.3 & 0.9 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.1 & -0.3 & 3 \\ -0.3 & 0.9 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 4: Let's compute the affine matrix of the skew projection onto the plane $x + 2y - 3z = 5$ in the direction $\vec{v} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$. Note that $\vec{x}_0 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ is on the plane:

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 8 & -12 & 0 \\ -1 & -1 & 3 & 0 \\ 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 8 & -12 & -20 \\ -1 & -1 & 3 & 5 \\ 1 & 2 & -2 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where the matrix of the linear part comes from

$$I - \frac{1}{\vec{v}\cdot\vec{n}}\vec{v}\cdot\vec{n}^{\mathsf{T}} = I - \frac{1}{-1}\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 5 & 8 & -12 \\ -1 & -1 & 3 \\ 1 & 2 & -2 \end{bmatrix}$$

Of course $T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0$ gives us the same information:

$$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0 = \begin{bmatrix} 5 & 8 & -12 \\ -1 & -1 & 3 \\ 1 & 2 & -2 \end{bmatrix}\left(\vec{x} - \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}\right) + \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & 8 & -12 \\ -1 & -1 & 3 \\ 1 & 2 & -2 \end{bmatrix}\vec{x} + \begin{bmatrix} -20 \\ 5 \\ -5 \end{bmatrix}$$

We can now do all basic transformations with affine matrix multiplications!

Example 5:

Let $R:\ \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation around $(4,1)$ and over $\theta = 90^\circ$ and $L:\ \mathbb{R}^2 \to \mathbb{R}^2$ a reflection in the line $3x+2y=14$.

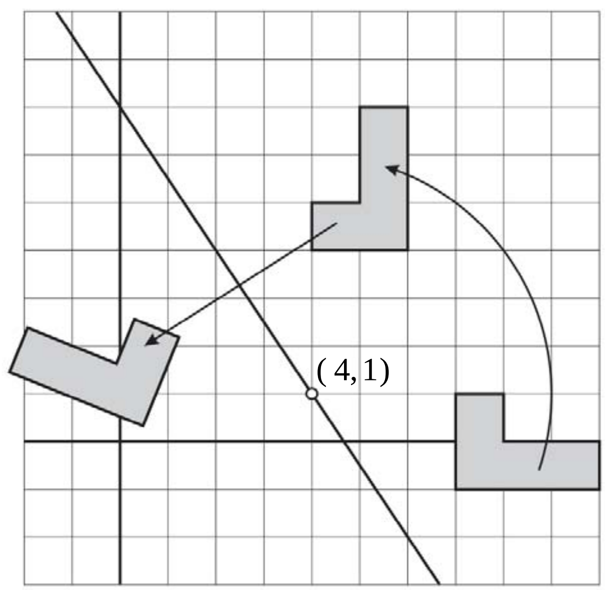
The affine matrix for R is
$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 5 \\ 1 & 0 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_{\vec{x}_0} \circ L \circ T_{-\vec{x}_0}$$

The affine matrix for L is
$$\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -5/13 & -12/13 & 0 \\ -12/13 & 5/13 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5/13 & -12/13 & 84/13 \\ -12/13 & 5/13 & 56/13 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_{\vec{x}_0} \circ L \circ T_{-\vec{x}_0}$$

or $$\frac{1}{13}\begin{bmatrix} -5 & -12 & 84 \\ -12 & 5 & 56 \\ 0 & 0 & 13 \end{bmatrix}$$



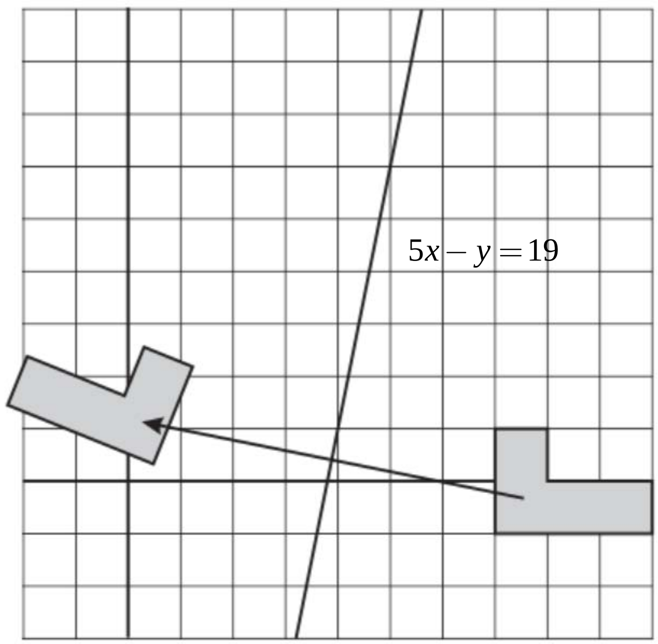
$3x+2y=14$

Hence the composition matrix of $L \circ R$ is
$$\frac{1}{13}\begin{bmatrix} -5 & -12 & 84 \\ -12 & 5 & 56 \\ 0 & 0 & 13 \end{bmatrix}\begin{bmatrix} 0 & -1 & 5 \\ 1 & 0 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \frac{1}{13}\begin{bmatrix} -12 & 5 & 95 \\ 5 & 12 & -19 \\ 0 & 0 & 13 \end{bmatrix}$$

Note that this is exactly the reflection in the line $5x - y = 19$

$$\underbrace{\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{T_{\vec{x}_0}} \underbrace{\frac{1}{13}\begin{bmatrix} -12 & 5 & 0 \\ 5 & 12 & 0 \\ 0 & 0 & 13 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{T_{-\vec{x}_0}} = \frac{1}{13}\begin{bmatrix} -12 & 5 & 95 \\ 5 & 12 & -19 \\ 0 & 0 & 13 \end{bmatrix}$$

$$T_{\vec{x}_0} \circ L \circ T_{-\vec{x}_0}$$



Of course in the previous problem the center of rotation was **on** the line of reflection. What do we get when we have the center of rotation not on the line of reflection? [This is done in the next example]

Example 6:

Let $R:\ \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation around $(4,1)$ and over $\theta = 90^\circ$ and $L:\ \mathbb{R}^2 \to \mathbb{R}^2$ a reflection in the line $2x - 3y = 31$.

The affine matrix for R is $\underbrace{\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{T_{\vec{x}_0}} \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{T_{-\vec{x}_0}} = \begin{bmatrix} 0 & -1 & 5 \\ 1 & 0 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

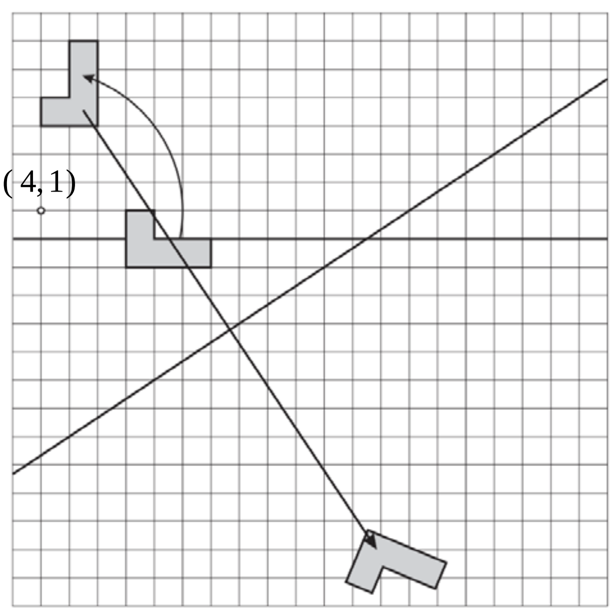
$$T_{\vec{x}_0} \circ L \circ T_{-\vec{x}_0}$$

The affine matrix for L is

$$\underbrace{\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5/13 & 12/13 & 0 \\ 12/13 & -5/13 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}}_{T_{\vec{x}_0} \circ L \circ T_{-\vec{x}_0}} = \frac{1}{13}\begin{bmatrix} 5 & 12 & 124 \\ 12 & -5 & -186 \\ 0 & 0 & 13 \end{bmatrix}$$

Hence the composition matrix of $L \circ R$ is

$$\frac{1}{13}\begin{bmatrix} 5 & 12 & 124 \\ 12 & -5 & -186 \\ 0 & 0 & 13 \end{bmatrix}\begin{bmatrix} 0 & -1 & 5 \\ 1 & 0 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 12 & -5 & 133 \\ -5 & -12 & -111 \\ 0 & 0 & 13 \end{bmatrix}$$



We'll show that this is the same as the composition of the following reflection and translation: Let $L_1:\ \mathbb{R}^2 \to \mathbb{R}^2$ be translation in the line $x+5y=-17$ and $T:\ \mathbb{R}^2 \to \mathbb{R}^2$ the translation over the vector $\begin{bmatrix} 10 \\ -2 \end{bmatrix}$ note that this translation is parallel to the line L_1 reflects in:

The affine matrix for L_1 is

$$\underbrace{\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}}_{T_{\vec{x}_0}}\underbrace{\begin{bmatrix} 12/13 & -5/13 & 0 \\ -5/13 & -12/13 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}}_{T_{-\vec{x}_0}} = \frac{1}{13}\begin{bmatrix} 12 & -5 & -17 \\ -5 & -12 & -85 \\ 0 & 0 & 13 \end{bmatrix}$$

$$T_{\vec{x}_0} \circ L \circ T_{-\vec{x}_0}$$

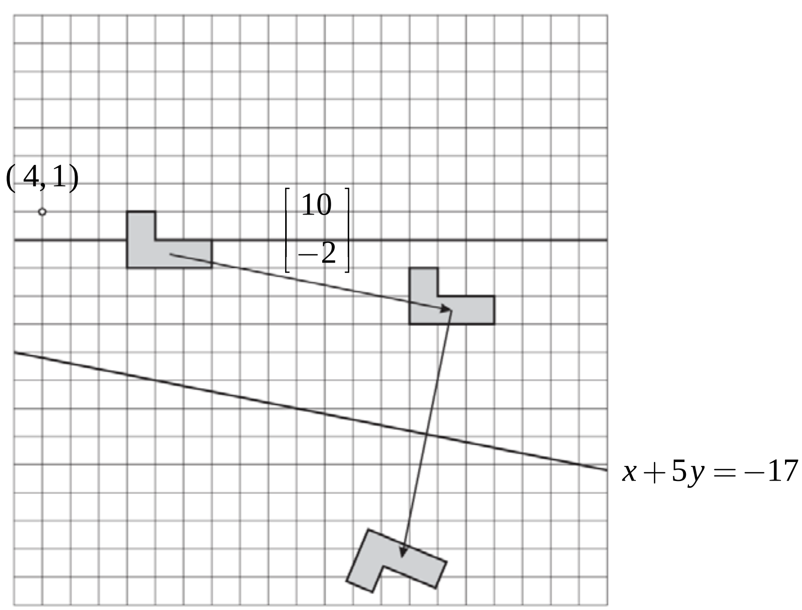
and the affine matrix for the translation is

$$\begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

So that the composition matrix is

$$\begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}\frac{1}{13}\begin{bmatrix} 12 & -5 & -17 \\ -5 & -12 & -85 \\ 0 & 0 & 13 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 12 & -5 & 133 \\ -5 & -12 & -111 \\ 0 & 0 & 13 \end{bmatrix}$$

This transformation is called a **glide reflection**.



Example 7:

Let T be the **shear** parallel to the plane $\alpha:\ 2x-y+2z=3$ with $\vec{v}=\begin{bmatrix}6\\6\\-3\end{bmatrix}$

(a) Find the images of the points $(1,2,3)$ $(2,0,1)$ $(0,1,2)$

(b) Find the image of the line $l:\ \vec{x}=\begin{bmatrix}2\\-3\\1\end{bmatrix}+t\begin{bmatrix}4\\5\\-6\end{bmatrix}$ under T.

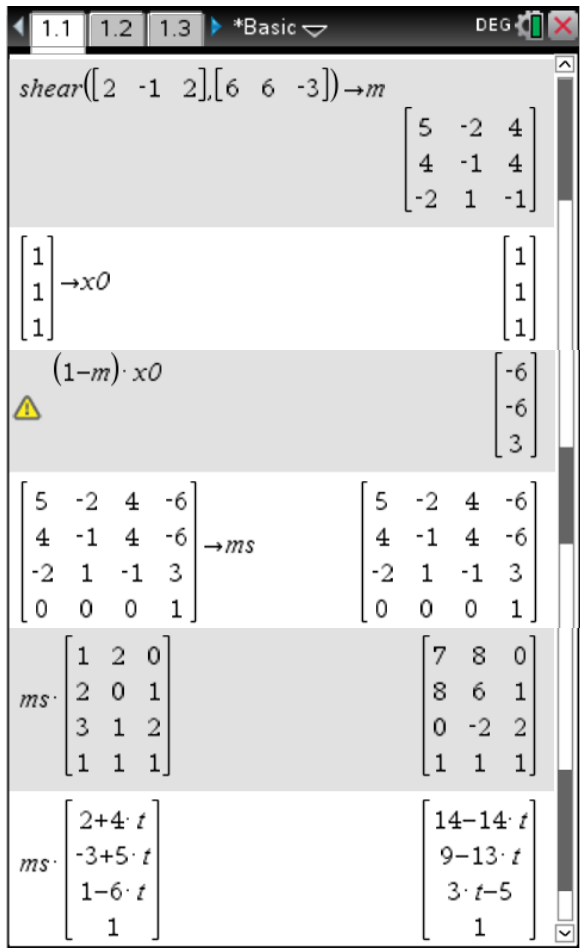
(c) Find the image of the plane $\beta:\ x-y+z=4$ under T.

Solution:

(a) $$T\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}5&-2&4\\4&-1&4\\-2&1&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}-6\\-6\\3\end{bmatrix}$$

hence we can write it, using an extra dimension as:

$$\tilde{T}\begin{bmatrix}x\\y\\z\\1\end{bmatrix}=\begin{bmatrix}5&-2&4&-6\\4&-1&4&-6\\-2&1&-1&3\\0&0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\\1\end{bmatrix}$$



Hence the points

$(1,2,3)$ $(2,0,1)$ $(0,1,2)$

get mapped to

$(7,8,0)$ $(8,6,-2)$ $(0,1,2)$

Since $$\begin{bmatrix}5&-2&4&-6\\4&-1&4&-6\\-2&1&-1&3\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&2&0\\2&0&1\\3&1&2\\1&1&1\end{bmatrix}=\begin{bmatrix}7&8&0\\8&6&1\\0&-2&2\\1&1&1\end{bmatrix}$$

(b) The points on the line l look like:

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 4 \\ 5 \\ -6 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+4t \\ -3+5t \\ 1-6t \\ 1 \end{bmatrix}$$

(a *point* on the line) (a ‘direction’ *vector* of the line)

These points get transformed to the points

$$\begin{bmatrix} 5 & -2 & 4 & -6 \\ 4 & -1 & 4 & -6 \\ -2 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2+4t \\ -3+5t \\ 1-6t \\ 1 \end{bmatrix} = \begin{bmatrix} 14-14t \\ 9-13t \\ -5+3t \\ 1 \end{bmatrix} \quad \text{i.e. the line} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 9 \\ -5 \end{bmatrix} + t \begin{bmatrix} -14 \\ -13 \\ 3 \end{bmatrix}$$

(c) To find the image of plane $\beta: \ x - y + z = 4$ let’s first rewrite the plane as a vector equation:

$$\beta: \ \vec{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

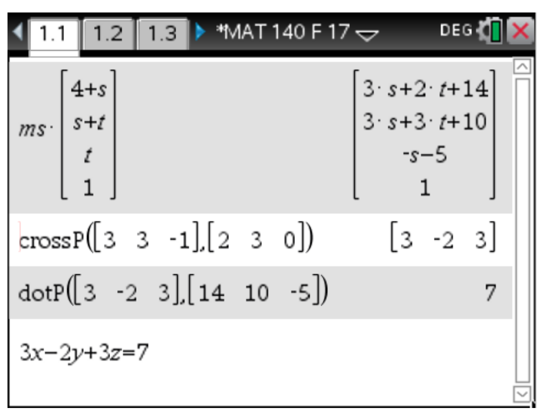
Hence the points of this plane β get mapped to

$$\begin{bmatrix} 5 & -2 & 4 & -6 \\ 4 & -1 & 4 & -6 \\ -2 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4+s \\ s+t \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} 14+3s+2t \\ 10+3s+3t \\ -5-s \\ 1 \end{bmatrix}$$

i.e. the image of β is another plane:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \\ -5 \end{bmatrix} + s \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

with normal equation: $3x - 2y + 3z = 7$.



Example 8: Let P be the **skew projection** onto the plane $\alpha : 3x + 4y + 6z = 5$ in the direction of $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and let R be the **rotation** around the vector $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ over $60°$.

What is the affine matrix of the composition $P \circ R$?

Solution:

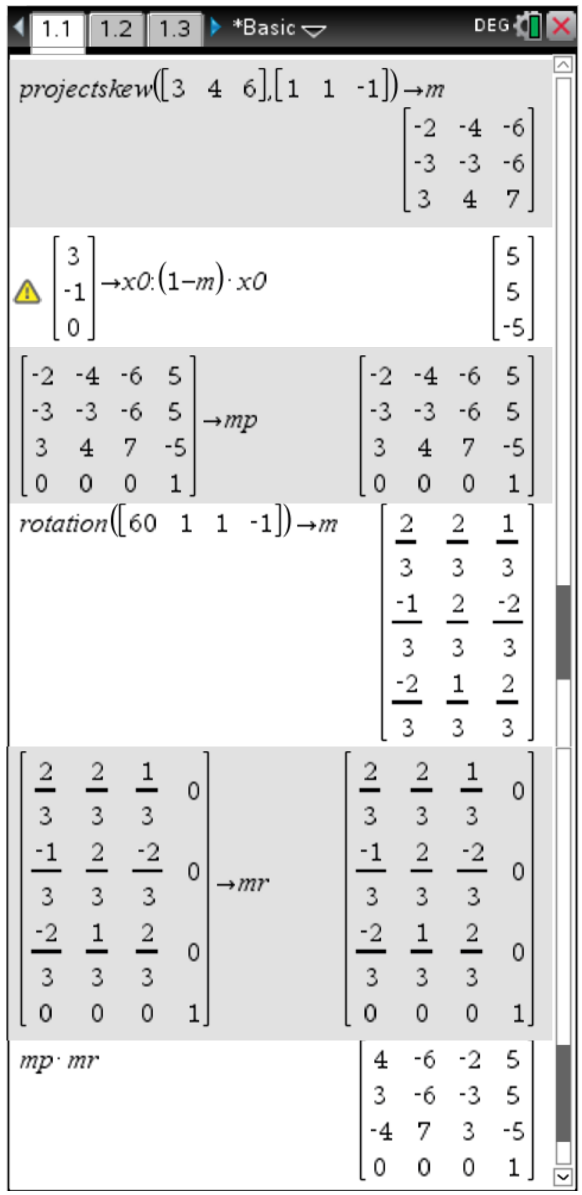
The affine matrices for P and R are (respectively)

$$\tilde{M}_P = \begin{bmatrix} -2 & -4 & -6 & 5 \\ -3 & -3 & -6 & 5 \\ 3 & 4 & 7 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{M}_R = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 & 0 \\ -1 & 2 & -2 & 0 \\ -2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

So that

$$\tilde{M}_P \cdot \tilde{M}_R = \begin{bmatrix} 4 & -6 & -2 & 5 \\ 3 & -6 & -3 & 5 \\ -4 & 7 & 3 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Example 9: Let R_1 be the **rotation** around the line $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ over $60°$, and R_2 be the **rotation** around the line $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ over $90°$.

What is the affine matrix of the composition $R_2 \circ R_1$?

Solution:

$$\mathit{rotation}([60 \ \ 1 \ \ 1 \ \ 1]) \rightarrow m1 \qquad \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

$$(1-m1)\cdot\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{-2}{3} \end{bmatrix}$$

$$\mathit{rotation}([90 \ \ 1 \ \ 2 \ \ -2]) \rightarrow m2 \qquad \begin{bmatrix} \frac{1}{9} & \frac{8}{9} & \frac{4}{9} \\ \frac{-4}{9} & \frac{4}{9} & \frac{-7}{9} \\ \frac{-8}{9} & \frac{-1}{9} & \frac{4}{9} \end{bmatrix}$$

$$(1-m2)\cdot\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} \frac{4}{3} \\ \frac{-1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{9} & \frac{8}{9} & \frac{4}{9} & \frac{4}{3} \\ \frac{-4}{9} & \frac{4}{9} & \frac{-7}{9} & \frac{-1}{3} \\ \frac{-8}{9} & \frac{-1}{9} & \frac{4}{9} & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{-1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} & \frac{2}{3} & \frac{-2}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \frac{14}{27} & \frac{23}{27} & \frac{2}{27} & \frac{37}{27} \\ \frac{7}{27} & \frac{-2}{27} & \frac{-26}{27} & \frac{5}{27} \\ \frac{-22}{27} & \frac{14}{27} & \frac{-7}{27} & \frac{-8}{27} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{M}_1 = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 & 1 \\ 2 & 2 & -1 & 1 \\ -1 & 2 & 2 & -2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\tilde{M}_2 = \frac{1}{9}\begin{bmatrix} 1 & 8 & 4 & 12 \\ -4 & 4 & -7 & -3 \\ -8 & -1 & 4 & 3 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

$$\tilde{M}_2 \tilde{M}_1 = \boxed{\frac{1}{27}\begin{bmatrix} 14 & 23 & 2 & 37 \\ 7 & -2 & -26 & 5 \\ -22 & 14 & -7 & -8 \\ 0 & 0 & 0 & 27 \end{bmatrix}}$$

Affine matrices and Barycentric coordinates

Here are some applications of affine matrices in the arena of Barycentric coordinates:

(I) Barycentric coordinates.

Recall that a point can be written in terms of Barycentric coordinates with respect to a given set of points: For example, we can express the point

$$X = (4,3,2) \text{ in } 3\text{-space},$$

with respect to the three points

$$P = (2,3,6), \quad Q = (5,6,-1) \text{ and } R = (5,-6,3)$$

as

$$\begin{aligned} X &= [\tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{6}] \\ &= \tfrac{1}{3}P + \tfrac{1}{2}Q + \tfrac{1}{6}R \\ &= \tfrac{1}{3}(2,3,6) + \tfrac{1}{2}(5,6,-1) + \tfrac{1}{6}(5,6,3) \\ &= (4,3,2) \end{aligned}$$

Note that we need $a+b+c = 1$ to express $X = aP + bQ + cR$ Barycentrically.

With our new way of writing points this works like a charm:

$$X = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \text{ with } P = \begin{bmatrix} 2 \\ 3 \\ 6 \\ 1 \end{bmatrix}, \; Q = \begin{bmatrix} 5 \\ 6 \\ -1 \\ 1 \end{bmatrix} \text{ and } R = \begin{bmatrix} 5 \\ -6 \\ 3 \\ 1 \end{bmatrix}$$

gives us

$$X = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \tfrac{1}{3}\begin{bmatrix} 2 \\ 3 \\ 6 \\ 1 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} 5 \\ 6 \\ -1 \\ 1 \end{bmatrix} + \tfrac{1}{6}\begin{bmatrix} 5 \\ -6 \\ 3 \\ 1 \end{bmatrix}$$

Notice that the last row gives us $1 = \tfrac{1}{3} + \tfrac{1}{2} + \tfrac{1}{6}$. The last row basically encodes the Barycentric property:

$$X = aP + bQ + cR \quad \text{i.e.} \quad \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = a\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix} + b\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} + c\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ 1 \end{bmatrix}$$

Here again, the last row gives us $1 = a+b+c$.

In fact we see in this new way of looking at our world the *need* for this Barycentric property: if for example $a+b+c = 2$ then $aP+bQ+cR$ would never give us a point in our plane, we would get

$$a\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix} + b\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} + c\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ a+b+c \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 2 \end{bmatrix}$$

But this is *not* a point in our world (the last coordinate is not 1).

(II) Finding an affine transformation given points and their images.

Recall that in a previous section we computed the affine map $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by the info

$$T\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}, \quad T\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 2 \end{bmatrix}, \quad T\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 10 \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 8 \end{bmatrix}$$

How did we determine T ?

Just for fun let's report the entire solution again:

Solution (the old way): Let $T(\vec{x}) = A\vec{x}+\vec{b}$ then

$$\begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} = A\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}+\vec{b}, \quad \begin{bmatrix} -5 \\ 0 \\ 2 \end{bmatrix} = A\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}+\vec{b}, \quad \begin{bmatrix} -1 \\ 7 \\ 10 \end{bmatrix} = A\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}+\vec{b} \quad \text{and} \quad \begin{bmatrix} -4 \\ -2 \\ 8 \end{bmatrix} = A\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}+\vec{b}.$$

When we subtract the last equation from each of the first three we get

$$\begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} -4 \\ -2 \\ 8 \end{bmatrix} = A\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right), \qquad \begin{bmatrix} -5 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} -4 \\ -2 \\ 8 \end{bmatrix} = A\left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right), \qquad \text{and}$$

$$\begin{bmatrix} -1 \\ 7 \\ 10 \end{bmatrix} - \begin{bmatrix} -4 \\ -2 \\ 8 \end{bmatrix} = A\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$$

Notice that all the $\vec{b}$ -s are gone, hence we get

$$\begin{bmatrix} 5 \\ 6 \\ -2 \end{bmatrix} = A\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 2 \\ -6 \end{bmatrix} = A\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 3 \\ 9 \\ 2 \end{bmatrix} = A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Combining this gives us

$$A\begin{bmatrix} -2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 3 \\ 6 & 2 & 9 \\ -2 & -6 & 2 \end{bmatrix}$$

So that

$$A = \begin{bmatrix} 5 & -1 & 3 \\ 6 & 2 & 9 \\ -2 & -6 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 9 & 7 \\ 6 & 2 & 8 \end{bmatrix}.$$

What remains is to find $\vec{b}$, but this we can do by picking one of the original equations e.g.

$$T\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 8 \end{bmatrix}$$

would now imply

$$\begin{bmatrix} -4 \\ -2 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 9 & 7 \\ 6 & 2 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \vec{b}$$

So that

$$\vec{b} = \begin{bmatrix} -4 \\ -2 \\ 8 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 4 \\ 5 & 9 & 7 \\ 6 & 2 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 8 \end{bmatrix} - \begin{bmatrix} -3 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 10 \end{bmatrix}$$

And we have found T:

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 9 & 7 \\ 6 & 2 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 10 \end{bmatrix}$$

This method was actually pretty elegant in and of itself … yet using the extra dimensional world view it becomes easier to find T.

Solution (the new way):

Now let's do it the 'right' way using our extra-dimension: Let $\tilde{T}:\mathbb{R}^4 \to \mathbb{R}^4$ with

$$\tilde{T}(\vec{x}) = \begin{bmatrix} \boxed{A} & \boxed{\vec{b}} \\ 0 \quad 0 & 1 \end{bmatrix}\vec{x}$$

then

$$\tilde{T}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \\ 1 \end{bmatrix}, \quad \tilde{T}\begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad \tilde{T}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 10 \\ 1 \end{bmatrix} \text{ and } \tilde{T}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 8 \\ 1 \end{bmatrix}$$

gives us immediately that

$$\begin{bmatrix} \boxed{A} & \boxed{\vec{b}} \\ 0 \quad 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -5 & -1 & -4 \\ 4 & 0 & 7 & -2 \\ 6 & 2 & 10 & 8 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

so that we get $\begin{bmatrix} \boxed{A} & \boxed{\vec{b}} \\ 0 \quad 0 & 1 \end{bmatrix}$ with a simple matrix inversion:

$$\begin{bmatrix} \boxed{A} & \boxed{\vec{b}} \\ 0 \quad 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -5 & -1 & -4 \\ 4 & 0 & 7 & -2 \\ 6 & 2 & 10 & 8 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 3 & 4 & -1 \\ 5 & 9 & 7 & 0 \\ 6 & 2 & 8 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Done! $A = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 9 & 7 \\ 6 & 2 & 8 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -1 \\ 0 \\ 10 \end{bmatrix}$ It almost looks like a miracle ☺ .

38. The Affine Matrices and Homogeneous Coordinates

We now have a method to find the Affine (or homogeneous) matrices of the main affine transformations, based on the fact that we can get the affine transformations from the linear transformations, in a three step process

$$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0$$

i.e.

(1) a translation by $-\vec{x}_0$: $\quad T_{-\vec{x}_0}(\vec{x}) = \vec{x} - \vec{x}_0$

(2) a linear transformation: $\quad L(\vec{x}) = A\vec{x}$

(3) a translation by $+\vec{x}_0$: $\quad T_{\vec{x}_0}(\vec{x}) = \vec{x} + \vec{x}_0$

$$T_{\vec{x}_0} \circ L \circ T_{-\vec{x}_0}$$

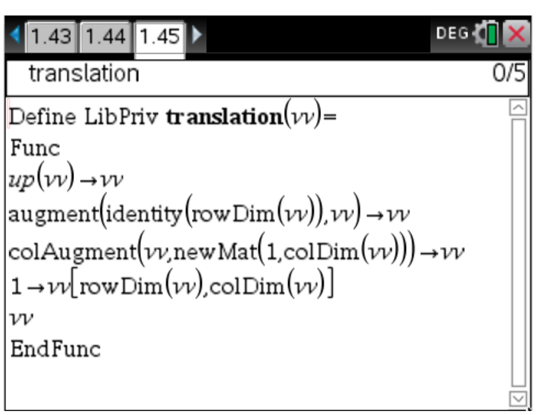
And this can all be done with Affine (homogeneous) matrices:

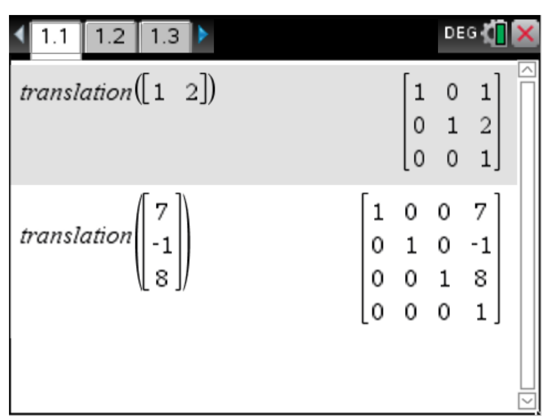
$$\begin{bmatrix} \boxed{I} & \boxed{\vec{x}_0} \\ 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} \boxed{A} & \boxed{\vec{0}} \\ 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} \boxed{I} & \boxed{-\vec{x}_0} \\ 0 \quad 0 & 1 \end{bmatrix}$$

The **TI-Nspire** can be easily programmed to do this:

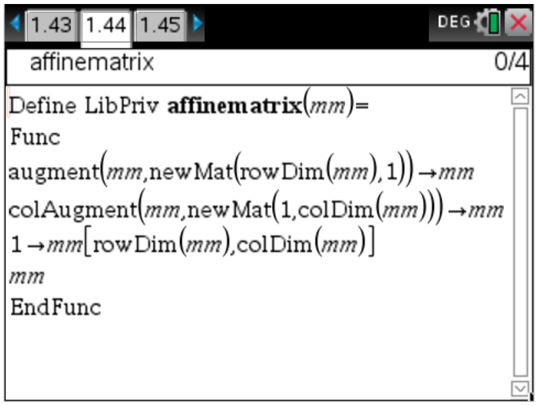
Here are the functions: **translation**, **affinematrix** and **affinemap**.

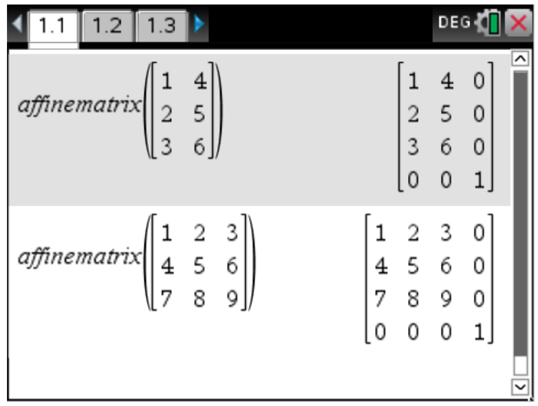
(1) **translation**, takes a vector $\vec{v}$ and produces the affine translation matrix $\begin{bmatrix} \boxed{I} & \boxed{\vec{v}} \\ 0 \cdots 0 & 1 \end{bmatrix}$





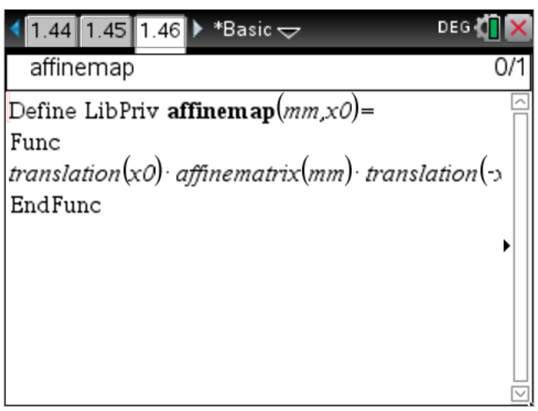
(2) Affinematrix, transforms a matrix M into an affine matrix $\begin{bmatrix} \boxed{M} & \boxed{\vec{0}} \\ 0 \cdots 0 & 1 \end{bmatrix}$

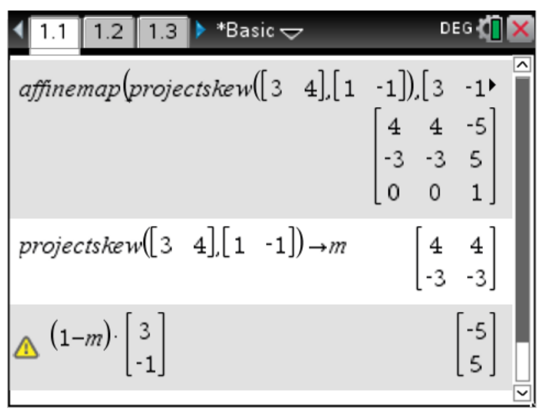




(3) Affinemap, which takes A and $\vec{x}_0$ and creates the affine matrix of $T(\vec{x}) = A(\vec{x}-\vec{x}_0)+\vec{x}_0$,

by computing $\begin{bmatrix} \boxed{I} & \boxed{\vec{x}_0} \\ 0 \quad 0 & 1 \end{bmatrix}\begin{bmatrix} \boxed{A} & \boxed{\vec{0}} \\ 0 \quad 0 & 1 \end{bmatrix}\begin{bmatrix} \boxed{I} & \boxed{-\vec{x}_0} \\ 0 \quad 0 & 1 \end{bmatrix}$





Check

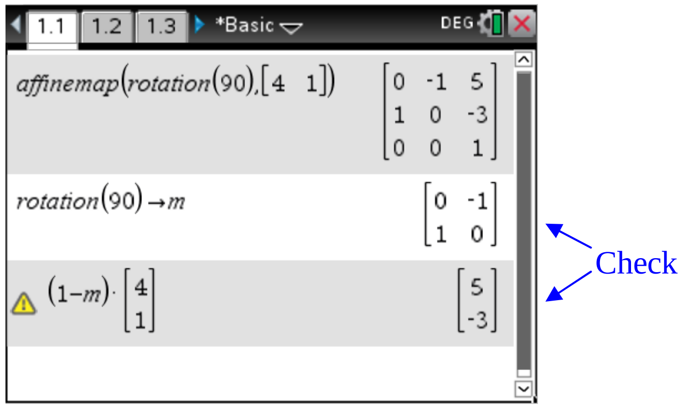
Example 1:

Let $R: \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation around $(4,1)$ and over $\theta = 90°$.

The affine matrix for R is $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 5 \\ 1 & 0 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

$T_{\vec{x}_0} \circ L \circ T_{-\vec{x}_0}$

This can now be produced by

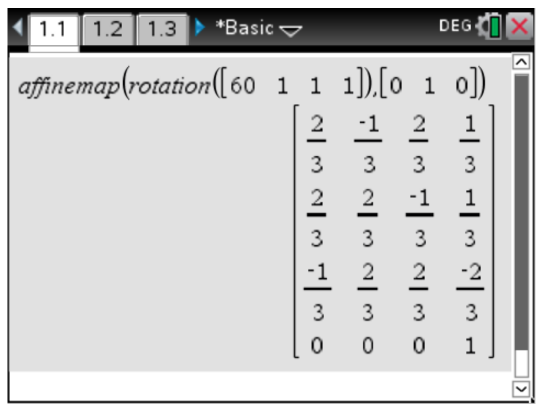


Example 2:

Let R_l be the **rotation** around the line $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ over $60°$, then

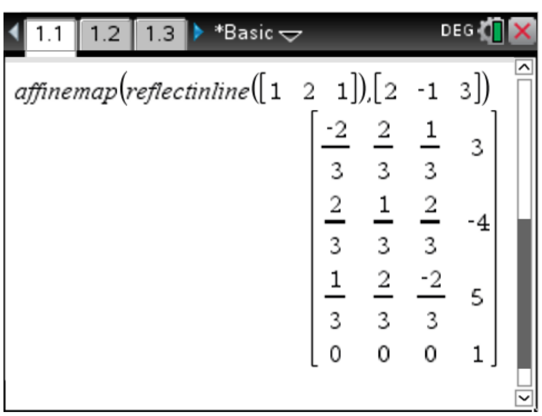
affinemap(rotation([60, 1, 1, 1] , [0, 1, 0])

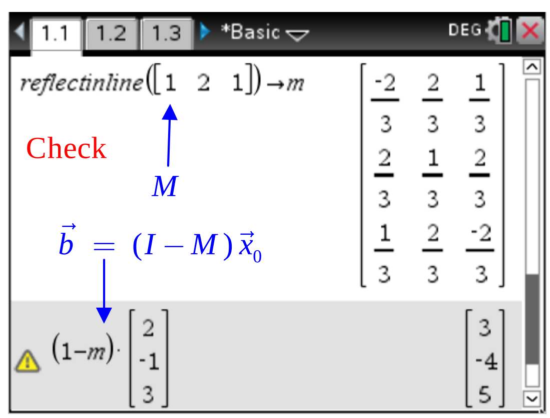
produces the affine matrix of this transformation



Example 3:

We can compute the reflection in the line $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ as follows, using the TI-Nspire:





We will write separate programs for each of the affine transformations matrices:

- Projection onto the line $\vec{x} = \vec{p} + t\vec{v}$
- Reflection in the line $\vec{x} = \vec{p} + t\vec{v}$
- Projection onto the plane $\vec{n}\cdot\vec{x} = k$
- Reflection in the plane $\vec{n}\cdot\vec{x} = k$
- Skew projection onto the plane $\vec{n}\cdot\vec{x} = k$ in the direction $\vec{v}$
- Skew reflection onto the plane $\vec{n}\cdot\vec{x} = k$ in the direction $\vec{v}$
- Shear along plane $\vec{n}\cdot\vec{x} = k$ in the direction $\vec{v}$
- Scaling centered at point $\vec{p}$
- 2D Rotation around point $\vec{p}$ over angle θ

  3D Rotation around line $\vec{x} = \vec{p} + t\vec{v}$ over angle θ

Note that the names of TI-Nspire functions can have only 15 characters.

The Transformations involving the hyper-planes $\vec{n}\cdot\vec{x} = k$ are special. They can actually be done without the three step process! Hence the projections and skew projections onto a plane, reflections in a plane, shears along a plane can all be done more directly and relatively easily as we will see.

They have actual simple matrix expressions using an augmented normal, which we will discuss next.

The 'Tilde' notation indicates the switch to an extra-dimension

We will use a 'tilde' notation to indicate the switch to the extra-dimensional way of describing our world. By lack of a better name we can refer to the vector $\tilde{v}$ and point $\tilde{P}$ as the 'affine' or 'homogeneous' coordinates of the vector $\vec{v}$ and the point P :

The point $P = (x, y)$ can be written as the point $\tilde{P} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$.

The vector $\vec{v} = \begin{bmatrix} A \\ B \end{bmatrix}$ can be written vector $\tilde{v} = \begin{bmatrix} A \\ B \\ 0 \end{bmatrix}$.

The point $P = (x,\ y,\ z)$ can be written as the point $\tilde{P} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$.

The vector $\vec{v} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ can be written vector $\tilde{v} = \begin{bmatrix} A \\ B \\ C \\ 0 \end{bmatrix}$ etc.

Recall that $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ is a normal of the line $a\,x + b\,y = c$ in 2D : we let $\tilde{n} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$

and $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a normal of the plane $a\,x + b\,y + c\,y = d$ in 3D: we let $\tilde{n} = \begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix}$.

We'll introduce a new notation for what we refer to as the **augmented normal of a plane**

In 2D: we'll call $\tilde{N} = \begin{bmatrix} a \\ b \\ -c \end{bmatrix}$ the **augmented normal** of the line $a\,x + b\,y = c$

In 3D: we'll call $\tilde{N} = \begin{bmatrix} a \\ b \\ c \\ -d \end{bmatrix}$ the **augmented normal** of the plane $a\,x + b\,y + c\,y = d$.

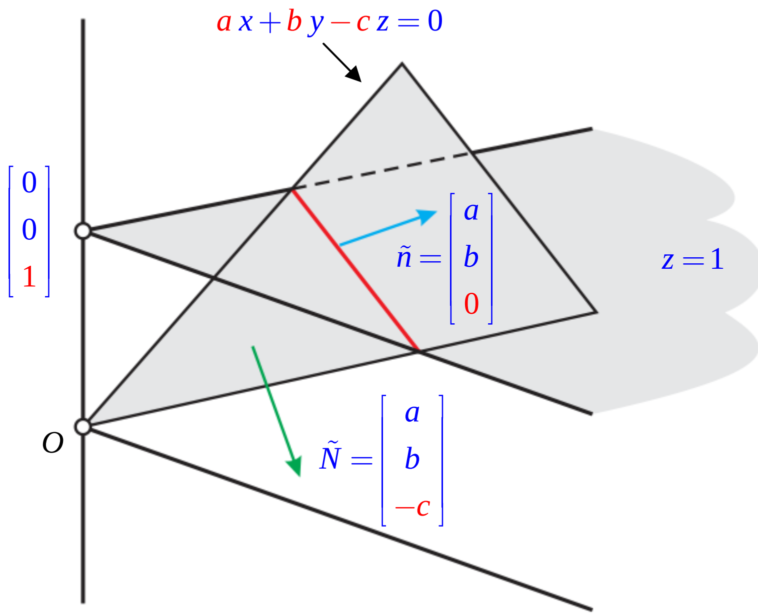
In fact the augmented normal $\tilde{N} = \begin{bmatrix} a \\ b \\ -c \end{bmatrix}$ is the actual normal of a plane (in $\mathbb{R}^3$)

$$a\,x + b\,y - c\,z = 0$$

This is the plane that goes through (0, 0, 0) *and* contains all points $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ on the original line

(which in the 2D world of $z = 1$ was described as $a\,x + b\,y = c$).

A picture to illustrate this might be useful here:



Note that the line which in 2D is described by $a\,x + b\,y = c$ in 3D is now described as the **intersection** of $a\,x+b\,y-c\,z=0$ **and** $z=1$. [We don't have a 'normal form' of a line in 3D]

Also note that in the 2D plane $z=1$ the normal $\vec{n}=\begin{bmatrix}a\\b\end{bmatrix}$ of the line $a\,x + b\,y = c$ is now called

$\tilde{n}=\begin{bmatrix}a\\b\\0\end{bmatrix}$ and that $\tilde{n}$ is just one of the vectors normal to this line in 3D. In fact $\tilde{N}=\begin{bmatrix}a\\b\\-c\end{bmatrix}$ which is the actual normal of the plane $a\,x+b\,y-c\,z=0$, and which we called the augmented normal of

$a\,x+b\,y=c$ is ***also*** normal to this line! [After all $\tilde{v}=\begin{bmatrix}b\\-a\\0\end{bmatrix}$ is a direction vector of the line.]

Recall that a **hyper-plane** is a $(n-1)$-dimensional affine subspace of an n-dimensional space.
In 2D space we usually call a hyper-plane a **line**. In 3D space we usually call a hyper-plane simply a **plane**. In 4D we call e.g. $3x+4y-z+2w=12$ a hyper-plane, etc.

Augmented normals can be used to describe hyper-planes rather nicely:

$$\boxed{\tilde{N}\cdot\tilde{x} = 0}$$

For example

In 2D the line $a\,x + b\,y = c$ becomes $\tilde{N}\cdot\tilde{x} = 0$, since

$$a\,x + b\,y = c \quad\Leftrightarrow\quad \tilde{N}\cdot\tilde{x} = \begin{bmatrix} a \\ b \\ -c \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0.$$

and

In 3D the plane $a\,x + b\,y + c\,z = d$ becomes $\tilde{N}\cdot\tilde{x} = 0$ since

$$a\,x + b\,y + c\,z = d \quad\Leftrightarrow\quad \tilde{N}\cdot\tilde{x} = \begin{bmatrix} a \\ b \\ c \\ -d \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

In 4D the hyper-plane $a\,x + b\,y + c\,z + d\,w = e$ also becomes $\tilde{N}\cdot\tilde{x} = 0$ since

$$a\,x + b\,y + c\,z + d\,w = e \quad\Leftrightarrow\quad \tilde{N}\cdot\tilde{x} = \begin{bmatrix} a \\ b \\ c \\ d \\ -e \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ z \\ w \\ 1 \end{bmatrix} = 0$$

etc.

Hence from now on we can simply describe any hyper-plane by $\boxed{\tilde{N}\cdot\tilde{x} = 0}$

We can use this notation to find the affine/homogeneous matrix of the projection onto a plane, a skew projection onto a plane, a reflection in a plane, a shear parallel to a plane:

Theorem 38.1 The affine/homogeneous matrix of the **projection** onto the plane $\tilde{N} \cdot \tilde{x} = 0$ is given by

$$\boxed{\tilde{M} = I - \frac{\tilde{n}\,\tilde{N}^{\mathsf{T}}}{\tilde{n} \cdot \tilde{N}}}$$

The **Proof** of this theorem is not that hard. It uses the same arguments we have seen before, but we will postpone it till the end of the chapter.

Example 4: Let $2x - y = 6$ then $\vec{n} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ hence $\tilde{n} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\tilde{N} = \begin{bmatrix} 2 \\ -1 \\ -6 \end{bmatrix}$ so that

$$\tilde{M} = I - \frac{1}{5}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}\begin{bmatrix} 2 & -1 & -6 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{12}{5} \\ \frac{2}{5} & \frac{4}{5} & -\frac{6}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

Check:

$$M = I - \frac{\vec{n}\,\vec{n}^{\mathsf{T}}}{\vec{n} \cdot \vec{n}} = I - \frac{1}{5}\begin{bmatrix} 2 \\ -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

and

$$(I - M)\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 12 \\ -6 \end{bmatrix}$$

Notice that this is really the **skew** projection in $\mathbb{R}^3$ onto the plane $2x - y - 6z = 0$

[the plane $\tilde{N} \cdot \vec{x} = 0$ where $\tilde{N} = \begin{bmatrix} 2 \\ -1 \\ -6 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$] in the direction: $\tilde{n} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$,

but when restricted to the plane $z = 1$, i.e. when we only look at points $\vec{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \in \mathbb{R}^3$, it is in effect the orthogonal projection onto a line.

Example 5: Let $3x - 2y + z = 6$ then $\vec{n} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ hence $\tilde{n} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\tilde{N} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ -6 \end{bmatrix}$

so that $\tilde{M} = I - \frac{1}{14}\begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 3 & -2 & 1 & -6 \end{bmatrix} = \begin{bmatrix} \frac{5}{14} & \frac{6}{14} & \frac{-3}{14} & \frac{18}{14} \\ \frac{6}{14} & \frac{10}{14} & \frac{2}{14} & \frac{-12}{14} \\ \frac{-3}{14} & \frac{2}{14} & \frac{13}{14} & \frac{6}{14} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Check:

$$M = I - \frac{\vec{n}\,\vec{n}^{\mathsf{T}}}{\vec{n}\cdot\vec{n}} = I - \frac{1}{14}\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & -2 & 1 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 5 & 6 & -3 \\ 6 & 10 & 2 \\ -3 & 2 & 13 \end{bmatrix}$$

and

$$(I - M)\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 18 \\ -12 \\ 6 \end{bmatrix}$$

Notice that this is really the **skew** projection in $\mathbb{R}^4$ onto the plane

$3x - 2y + z - 6w = 0$ [i.e. $\tilde{N}\cdot\vec{x} = 0$ where $\tilde{N} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ -6 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4$]

in the direction: $\tilde{n} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$, but when restricted to the plane $w = 1$, i.e. when we only look at

points $\vec{x} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \in \mathbb{R}^4$, it is in effect the orthogonal projection on the 'plane'

$3x - 2y + z = 6$.

Theorem 38.2 The affine/homogeneous matrix of the **reflection** in the plane $\tilde{N}\cdot\tilde{x} = 0$ is given by

$$\boxed{\tilde{M} = I - 2\frac{\tilde{n}\,\tilde{N}^{\mathsf{T}}}{\tilde{n}\cdot\tilde{N}}}$$

The **Proof** of this theorem is not that hard. It uses the same arguments we have seen before, but we will postpone it till the end of the section.

Example 6: Let $3x - y = 6$ then $\vec{n} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ hence $\tilde{n} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$ and $\tilde{N} = \begin{bmatrix} 3 \\ -1 \\ -6 \end{bmatrix}$ so that

$$\tilde{M} = I - \frac{2}{10}\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}\begin{bmatrix} 3 & -1 & -6 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} & \frac{18}{5} \\ \frac{3}{5} & \frac{4}{5} & -\frac{6}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

Check:

$$M = I - 2\frac{\vec{n}\,\vec{n}^{\mathsf{T}}}{\vec{n}\cdot\vec{n}} = I - \frac{2}{10}\begin{bmatrix} 3 \\ -1 \end{bmatrix}\begin{bmatrix} 3 & -1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix}$$

and

$$(I - M)\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 18 \\ -6 \end{bmatrix}$$

Notice that $I - \frac{2}{10}\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}\begin{bmatrix} 3 & -1 & -6 \end{bmatrix}$ in $\mathbb{R}^3$ is really the skew reflection in $\mathbb{R}^3$ in the plane

$3x - y - 6z = 0$ [i.e. $\tilde{N}\cdot\vec{x} = 0$ where $\tilde{N} = \begin{bmatrix} 3 \\ -1 \\ -6 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$]

in the direction: $\tilde{n} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$.

Example 7: Let $3x - 2y + z = 6$ then $\vec{n} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ hence $\tilde{n} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\tilde{N} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ -6 \end{bmatrix}$

so that $\tilde{M} = I - \dfrac{2}{14}\begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 3 & -2 & 1 & -6 \end{bmatrix} = \begin{bmatrix} \frac{-2}{7} & \frac{6}{7} & \frac{-3}{7} & \frac{18}{7} \\ \frac{6}{7} & \frac{3}{7} & \frac{2}{7} & \frac{-12}{7} \\ \frac{-3}{7} & \frac{2}{7} & \frac{6}{7} & \frac{6}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Check:

$$M = I - 2\frac{\vec{n}\,\vec{n}^{\mathsf{T}}}{\vec{n}\cdot\vec{n}} = I - \frac{2}{14}\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & -2 & 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -2 & 6 & -3 \\ 6 & 3 & 2 \\ -3 & 2 & 6 \end{bmatrix}$$

and

$$(I - M)\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 18 \\ -12 \\ 6 \end{bmatrix}$$

Notice that $I - \dfrac{2}{14}\begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 3 & -2 & 1 & -6 \end{bmatrix}$ is really the skew reflection in $\mathbb{R}^4$ in the plane

$3x - 2y + z - 6w = 0$ [i.e. $\tilde{N}\cdot\vec{x} = 0$ where $\tilde{N} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ -6 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4$]

in the direction: $\tilde{n} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

Theorem 38.3 The affine/homogeneous matrix of the **skew projection** in the plane $\tilde{N}\cdot\tilde{x} = 0$ in the direction $\vec{v}$ is given by

$$\boxed{\tilde{M} = I - \frac{\tilde{v}\,\tilde{N}^{\mathsf{T}}}{\tilde{v}\cdot\tilde{N}}}$$

The **Proof** of this theorem is not that hard. It uses the same arguments we have seen before, but we will postpone it till the end of the section.

Example 8: Let $3x - 2y = 6$ then $\vec{n} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ hence $\tilde{n} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$ and $\tilde{N} = \begin{bmatrix} 3 \\ -2 \\ -6 \end{bmatrix}$ so that with $\vec{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ so that $\tilde{M} = I - \frac{1}{-5}\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}\begin{bmatrix} 3 & -2 & -6 \end{bmatrix} = \begin{bmatrix} \frac{8}{5} & -\frac{2}{5} & -\frac{6}{5} \\ \frac{12}{5} & -\frac{3}{5} & -\frac{24}{5} \\ 0 & 0 & 1 \end{bmatrix}$

Check:

$$M = I - \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}} = I - \frac{1}{-5}\begin{bmatrix} 1 \\ 4 \end{bmatrix}\begin{bmatrix} 3 & -2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 8 & -2 \\ 12 & -3 \end{bmatrix}$$

and

$$(I - M)\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -6 \\ -24 \end{bmatrix}$$

Notice that this is the skew projection in $\mathbb{R}^3$ onto the plane $3x - 2y - 6z = 0$

[i.e. $\tilde{N}\cdot\vec{x} = 0$ where $\tilde{N} = \begin{bmatrix} 3 \\ -2 \\ -6 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$] in the direction: $\tilde{v} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$.

Example 9: Let $3x - 2y + z = 6$ then $\vec{n} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ hence $\tilde{n} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\tilde{N} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ -6 \end{bmatrix}$ and

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ so that } \tilde{M} = I - \frac{1}{4}\begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \end{bmatrix}\begin{bmatrix} 3 & -2 & 1 & -6 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & \frac{4}{7} & \frac{-2}{7} & \frac{12}{7} \\ \frac{-3}{7} & \frac{9}{7} & \frac{-1}{7} & \frac{6}{7} \\ \frac{-9}{7} & \frac{6}{7} & \frac{4}{7} & \frac{18}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Check:

$$M = I - \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\vec{v}\cdot\vec{n}} = I - \frac{1}{7}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}\begin{bmatrix} 3 & -2 & 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 1 & 4 & -2 \\ -3 & 9 & -1 \\ -9 & 6 & 4 \end{bmatrix}$$

and

$$(I - M)\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix}$$

Notice that this is the skew projection in $\mathbb{R}^4$ onto the plane $3x - 2y + z - 6w = 0$

[i.e. $\tilde{N}\cdot\vec{x} = 0$ where $\tilde{N} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ -6 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4$] in the direction: $\tilde{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \end{bmatrix}$.

Theorem 38.4 The affine/homogeneous matrix of the **shear** in the plane $\tilde{N}\cdot\tilde{x} = 0$ in the direction $\vec{v}$ is given by

$$\boxed{\tilde{M} = I + \frac{\tilde{v}\,\tilde{N}^{\mathsf{T}}}{\|\tilde{n}\|}}$$

The **Proof** of this theorem is not that hard. It uses the same arguments we have seen before, but we will postpone it till the end of the section.

Example 10: Let $3x - 4y = 6$ then $\vec{n} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ hence $\tilde{n} = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$ and $\tilde{N} = \begin{bmatrix} 3 \\ -4 \\ -6 \end{bmatrix}$ and

$\vec{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ so that $\tilde{M} = I + \frac{1}{5}\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}\begin{bmatrix} 3 & -4 & -6 \end{bmatrix} = \begin{bmatrix} \frac{17}{5} & -\frac{16}{5} & -\frac{24}{5} \\ \frac{9}{5} & -\frac{7}{5} & -\frac{18}{5} \\ 0 & 0 & 1 \end{bmatrix}$

Check:

$$M = I + \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\|\vec{n}\|} = I + \frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 17 & -16 \\ 9 & -7 \end{bmatrix}$$

and

$$(I - M)\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -24 \\ -18 \end{bmatrix}$$

Note this is **not** quite a shear in $\mathbb{R}^3$! It is actually a ‘skew’ shear ☺.

The shear with respect to $\tilde{N}\cdot\vec{x} = 0$ where $\tilde{N} = \begin{bmatrix} 3 \\ -4 \\ -6 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ in the direction

$\vec{v} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$ would have matrix: $M = I + \frac{\tilde{v}\,\tilde{N}^{\mathsf{T}}}{\|\tilde{N}\|}$.

The fact of the matter is that here we would actually be measuring the signed orthogonal distance to the plane $\tilde{N}\cdot\vec{x} = 0$, whereas in example 10 the ‘skew’ distance to that plane in the direction $\tilde{n}$ is measured.

Example 11: Let $2x - 2y + z = 6$ then $\vec{n} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ hence $\tilde{n} = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\tilde{N} = \begin{bmatrix} 2 \\ -2 \\ 1 \\ -6 \end{bmatrix}$

and $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$ so that $\tilde{M} = I + \frac{1}{3}\begin{bmatrix} 3 \\ 1 \\ -4 \\ 0 \end{bmatrix}\begin{bmatrix} 2 & -2 & 1 & -6 \end{bmatrix} = \begin{bmatrix} \frac{9}{3} & \frac{-6}{3} & \frac{3}{3} & -6 \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & -2 \\ \frac{-8}{3} & \frac{8}{3} & \frac{-1}{3} & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Check:

$$M = I + \frac{\vec{v}\,\vec{n}^{\mathsf{T}}}{\|\vec{n}\|} = I + \frac{1}{3}\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}\begin{bmatrix} 2 & -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 & -6 & 3 \\ 2 & 1 & 1 \\ -8 & 8 & -1 \end{bmatrix}$$

and

$$(I - M)\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \\ 8 \end{bmatrix}$$

Note that the remaining transformations, which do not involve a hyper-plane, are

- Projection onto the line $\vec{x} = \vec{p} + t\vec{v}$
- Reflection in the line $\vec{x} = \vec{p} + t\vec{v}$
- Scaling centered at point $\vec{p}$
- 2D Rotation around point $\vec{p}$ over angle θ

  3D Rotation around line $\vec{x} = \vec{p} + t\vec{v}$ over angle θ

These need to be computed using $\begin{bmatrix} \boxed{I} & \boxed{\vec{x}_0} \\ 0 \quad 0 & 1 \end{bmatrix}\begin{bmatrix} \boxed{A} & \boxed{\vec{0}} \\ 0 \quad 0 & 1 \end{bmatrix}\begin{bmatrix} \boxed{I} & \boxed{-\vec{x}_0} \\ 0 \quad 0 & 1 \end{bmatrix}$ and the functions **affinemap**, **affinematrix** and **translation**, as discussed at the beginning of this section.

Proofs of the Theorems

Theorem 38.1 The affine/homogeneous matrix of the **projection** onto the plane $\tilde{N}\cdot\tilde{x} = 0$ is given by

$$\boxed{\tilde{M} = I - \frac{\tilde{n}\,\tilde{N}^{\mathsf{T}}}{\tilde{n}\cdot\tilde{N}}}$$

Proof: The equation $\tilde{N}\cdot\tilde{x} = 0$ is the same as $\vec{n}\cdot\vec{x} = k$. Let $\vec{x}_0$ be the point we want to project onto $\vec{n}\,\vec{x} = k$. The line perpendicular to $\vec{n}\,\vec{x} = k$ is given by $\vec{x} = \vec{x}_0 + t\,\vec{n}$. When we intersect this line with $\vec{n}\,\vec{x} = k$ we get the projection of $\vec{x}_0$.

Of course we'll do the computations in affine/homogeneous coordinates (one dimension up)

$$\left.\begin{array}{lll} \vec{n}\cdot\vec{x} = k & \text{becomes} & \tilde{N}\cdot\tilde{x} = 0 \\ \vec{x} = \vec{x}_0 + t\,\vec{n} & \text{becomes} & \tilde{x} = \tilde{x}_0 + t\,\tilde{n} \end{array}\right\} \Rightarrow \tilde{N}\cdot(\tilde{x}_0 + t\,\tilde{n}) = 0 \Rightarrow t = -\frac{\tilde{N}\cdot\tilde{x}_0}{\tilde{N}\cdot\tilde{n}}$$

So that

$$\operatorname{Proj}(\tilde{x}_0) = \tilde{x}_0 - \frac{\tilde{N}\cdot\tilde{x}_0}{\tilde{N}\cdot\tilde{n}}\tilde{n} = \tilde{x}_0 - \frac{\tilde{n}\,\tilde{N}^{\mathsf{T}}}{\tilde{N}\cdot\tilde{n}}\tilde{x}_0 = \underbrace{\left(I - \frac{\tilde{n}\,\tilde{N}^{\mathsf{T}}}{\tilde{N}\cdot\tilde{n}}\right)}_{\tilde{M}}\tilde{x}_0$$

Theorem 38.2 The affine/homogeneous matrix of the **reflection** onto the plane $\tilde{N}\cdot\tilde{x} = 0$ is given by

$$\boxed{\tilde{M} = I - 2\frac{\tilde{n}\,\tilde{N}^{\mathsf{T}}}{\tilde{n}\cdot\tilde{N}}}$$

Proof: The computations are as in the previous proof.

$$\left.\begin{array}{lll} \vec{n}\cdot\vec{x} = k & \text{becomes} & \tilde{N}\cdot\tilde{x} = 0 \\ \vec{x} = \vec{x}_0 + t\,\vec{n} & \text{becomes} & \tilde{x} = \tilde{x}_0 + t\,\tilde{n} \end{array}\right\} \Rightarrow \tilde{N}\cdot(\tilde{x}_0 + t\,\tilde{n}) = 0 \Rightarrow t = -\frac{\tilde{N}\cdot\tilde{x}_0}{\tilde{N}\cdot\tilde{n}}$$

And of course we get the reflection—as we have seen before—by inserting the factor 2

$$\operatorname{Refl}(\tilde{x}_0) = \tilde{x}_0 - 2\frac{\tilde{N}\cdot\tilde{x}_0}{\tilde{N}\cdot\tilde{n}}\tilde{n} = \tilde{x}_0 - 2\frac{\tilde{n}\,\tilde{N}^{\mathsf{T}}}{\tilde{N}\cdot\tilde{n}}\tilde{x}_0 = \underbrace{\left(I - 2\frac{\tilde{n}\,\tilde{N}^{\mathsf{T}}}{\tilde{N}\cdot\tilde{n}}\right)}_{\tilde{M}}\tilde{x}_0$$

Theorem 38.3 The affine/homogeneous matrix of the **skew projection** in the plane $\tilde{N}\cdot\tilde{x} = 0$ in the direction $\vec{v}$ is given by

$$\boxed{\tilde{M} = I - \frac{\tilde{v}\,\tilde{N}^{\mathsf{T}}}{\tilde{v}\cdot\tilde{N}}}$$

Proof: The equation $\tilde{N}\cdot\tilde{x} = 0$ is the same as $\vec{n}\cdot\vec{x} = k$. Let $\vec{x}_0$ be the point we want to project onto $\vec{n}\cdot\vec{x} = k$ in the direction $\vec{v}$. The line through $\vec{x}_0$ in the direction $\vec{v}$ is given by $\vec{x} = \vec{x}_0 + t\,\vec{v}$. When we intersect this line with $\vec{n}\cdot\vec{x} = k$ we get the projection of $\vec{x}_0$.

Of course we'll do the computations in affine/homogeneous coordinates (one dimension up)

$$\left.\begin{array}{lll} \vec{n}\cdot\vec{x} = k & \text{becomes} & \tilde{N}\cdot\tilde{x} = 0 \\ \vec{x} = \vec{x}_0 + t\,\vec{v} & \text{becomes} & \tilde{x} = \tilde{x}_0 + t\,\tilde{v} \end{array}\right\} \Rightarrow \tilde{N}\cdot(\tilde{x}_0 + t\,\tilde{v}) = 0 \Rightarrow t = -\frac{\tilde{N}\cdot\tilde{x}_0}{\tilde{N}\cdot\tilde{v}}$$

So that

$$\mathrm{Proj}(\tilde{x}_0) = \tilde{x}_0 - \frac{\tilde{N}\cdot\tilde{x}_0}{\tilde{N}\cdot\tilde{n}}\tilde{n} = \tilde{x}_0 - \frac{\tilde{n}\,\tilde{N}^{\mathsf{T}}}{\tilde{N}\cdot\tilde{v}}\tilde{x}_0 = \underbrace{\left(I - \frac{\tilde{n}\,\tilde{N}^{\mathsf{T}}}{\tilde{N}\cdot\tilde{v}}\right)}_{\tilde{M}}\tilde{x}_0$$

Theorem 38.4 The affine/homogeneous matrix of the **shear** in the plane $\tilde{N}\cdot\tilde{x} = 0$ in the direction $\vec{v}$ is given by

$$\boxed{\tilde{M} = I + \frac{\tilde{v}\,\tilde{N}^{\mathsf{T}}}{\|\vec{n}\|}}$$

Proof:

Let $\vec{n}\cdot\vec{x} = k$ be the plane we shear along in the direction $\vec{v}$. Let $\vec{x}_0$ be a point we want to shear. Recall that its shear depends on the signed distance of the point $\vec{x}_0$ to the plane $\vec{n}\cdot\vec{x} = k$

The distance $= \dfrac{|\vec{n}\cdot\vec{x}_0 - k|}{\|\vec{n}\|}$ and the signed distance $= \dfrac{\vec{n}\cdot\vec{x}_0 - k}{\|\vec{n}\|}$.

In affine/homogeneous coordinates this becomes: $\dfrac{\vec{n}\cdot\vec{x}_0 - k}{\|\vec{n}\|} = \dfrac{\tilde{N}\cdot\tilde{x}_0}{\|\tilde{n}\|}$

in 2D : $$\frac{\vec{n}\cdot\vec{x}_0 - c}{\|\vec{n}\|} = \frac{a\,x_0 + b\,y_0 - c}{\sqrt{a^2+b^2}} = \frac{\begin{bmatrix} a \\ b \\ -c \end{bmatrix}\cdot\begin{bmatrix} x_0 \\ y_0 \\ 1 \end{bmatrix}}{\sqrt{a^2+b^2}} = \frac{\tilde{N}\,\tilde{x}_0}{\|\tilde{n}\|}$$

in 3D: $$\frac{\vec{n}\cdot\vec{x}_0 - d}{\|\vec{n}\|} = \frac{a\,x_0 + b\,y_0 + c\,z_0 - d}{\sqrt{a^2+b^2+c^2}} = \frac{\begin{bmatrix} a \\ b \\ c \\ -d \end{bmatrix}\cdot\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}}{\sqrt{a^2+b^2+c^2}} = \frac{\tilde{N}\,\tilde{x}_0}{\|\tilde{n}\|} \quad \text{etc.}$$

Hence

$$S(\tilde{x}_0) = \tilde{x}_0 + \frac{\tilde{N}\cdot\tilde{x}_0}{\|\tilde{n}\|}\tilde{v} = \tilde{x}_0 + \frac{\tilde{v}\,\tilde{N}^{\mathsf{T}}}{\|\tilde{n}\|}\tilde{x}_0 = \underbrace{\left(I + \frac{\tilde{v}\,\tilde{N}^{\mathsf{T}}}{\|\tilde{n}\|}\right)}_{\tilde{M}}\tilde{x}_0$$

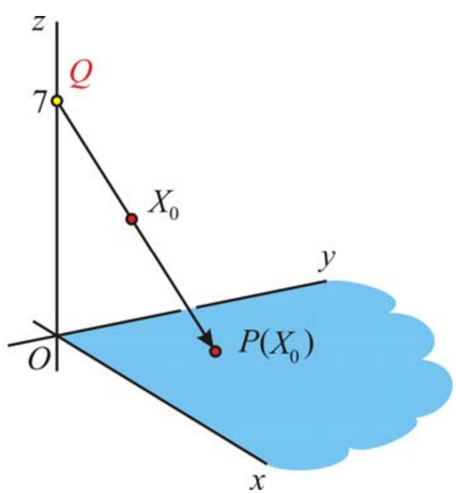
39. Perspective Projection

In this section we will examine a very non-linear transformation: perspective projection onto a plane. There are various names we could use to refer to this map: point projection, stereographic projection, perspective projection. Stereographic projection is by some reserved for the map that maps a sphere by projection from a point (usually its northpole) to a plane.

At first it seems that this map cannot be performed with a matrix multiplication, but with a very clever use of homogeneous coordinates we once again produce a matrix that transforms all the points for us.

Projection onto the *xy* - plane from a given point

Example 1: Assume we have a light source at the point $Q = (0,\ 0,\ 7)$ and we want to find the shadow of the point $X_0 = (x_0,\ y_0,\ z_0)$ onto the xy – plane [Let's assume $0 \le z_0 < 7$].



The line through Q and X_0 is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} + t \begin{bmatrix} x_0 \\ y_0 \\ z_0 - 7 \end{bmatrix}$$

When we intersect this with the xy – plane, whose equation is $z = 0$, we find

$$t = \frac{7}{7 - z_0}$$

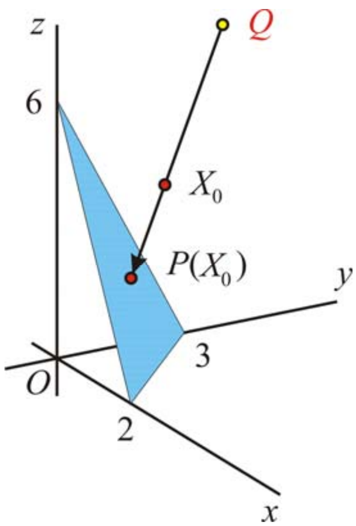
so that $P(X_0) = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} + \dfrac{7}{7 - z_0} \begin{bmatrix} x_0 \\ y_0 \\ z_0 - 7 \end{bmatrix}$

Which means $P \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} \dfrac{7x_0}{7 - z_0} \\ \dfrac{7y_0}{7 - z_0} \\ 0 \end{bmatrix}$, i.e. $P \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dfrac{7}{7 - z} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$

Note that this is highly non-linear due to the $7 - z$ in the *denominator*.

Another example.

Example 2: Let's look at a less trivial example. Let the light source be at $Q = (4, 1, 8)$ and the plane be $3x + 2y + z = 6$. Again we want to find the 'shadow' of the point $X_0 = (x_0, y_0, z_0)$ onto the plane.



The line through Q and X_0 is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} + t \begin{bmatrix} x_0 - 4 \\ y_0 - 1 \\ z_0 - 8 \end{bmatrix}$$

When we intersect this line with the xy – plane, we find

$$3\big(4 + t(x_0 - 4)\big) + 2\big(1 + t(y_0 - 1)\big) + \big(8 + t(z_0 - 8)\big) = 6$$

i.e. $$t = \frac{-16}{3x_0 + 2y_0 + z_0 - 22}$$

so that $$P(X_0) = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} - \frac{16}{3x_0 + 2y_0 + z_0 - 22} \begin{bmatrix} x_0 - 4 \\ y_0 - 1 \\ z_0 - 8 \end{bmatrix}$$

Which means: $$P \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} \dfrac{-4x_0 + 8y_0 + 4z_0 - 24}{3x_0 + 2y_0 + z_0 - 22} \\ \dfrac{3x_0 - 14y_0 + z_0 - 6}{3x_0 + 2y_0 + z_0 - 22} \\ \dfrac{24x_0 + 16y_0 - 8z_0 - 48}{3x_0 + 2y_0 + z_0 - 22} \end{bmatrix}.$$

It looks like there is no way to write this as a matrix multiplication, with all the x_0, y_0 and z_0 in the denominators. But amazingly enough there is a way of achieving this.

A marvelous insight:

(1) First let's reformulate everything using our **extra dimension**:

$$\tilde{P}\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix} = \begin{bmatrix} \dfrac{-4x_0+8y_0+4z_0-24}{3x_0+2y_0+z_0-22} \\ \dfrac{3x_0-14y_0+z_0-6}{3x_0+2y_0+z_0-22} \\ \dfrac{24x_0+16y_0-8z_0-48}{3x_0+2y_0+z_0-22} \\ 1 \end{bmatrix}$$

(2) Next, a marvelous insight, a truly amazing stroke of genius: observe what happens when we multiply the entire vector by the denominator $3x_0+2y_0+z_0-22$.

$$\begin{bmatrix} -4x_0+8y_0+4z_0-24 \\ 3x_0-14y_0+z_0-6 \\ 24x_0+16y_0-8z_0-48 \\ 3x_0+2y_0+z_0-22 \end{bmatrix}$$

This we ***can*** write as a matrix multiplication

$$\begin{bmatrix} -4x_0+8y_0+4z_0-24 \\ 3x_0-14y_0+z_0-6 \\ 24x_0+16y_0-8z_0-48 \\ 3x_0+2y_0+z_0-22 \end{bmatrix} = \begin{bmatrix} -4 & 8 & 4 & -24 \\ 3 & -14 & 1 & -6 \\ 24 & 16 & -8 & -48 \\ 3 & 2 & 1 & -22 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}$$

The only thing we did was making the image vector bigger by a factor $3x_0+2y_0+z_0-22$, which now shows up as the fourth entry. So to find the actual image, we just need to divide the entire vector by that fourth entry. For example:

if $X_0 = (2, 1, 4)$ then $\begin{bmatrix} -4 & 8 & 4 & -24 \\ 3 & -14 & 1 & -6 \\ 24 & 16 & -8 & -48 \\ 3 & 2 & 1 & -22 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ -10 \\ -16 \\ -10 \end{bmatrix}$. This is *almost* the perspective projection we want except it is -10 times too big, hence $\tilde{P}\begin{bmatrix} 2 \\ 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 1 \\ 1.6 \\ 1 \end{bmatrix}$.

We could decide to call all points that are a nonzero multiple of $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}$ the **same** point. Of course we are only interested in the coordinates of the point when the last coordinate is 1. But we could call every point $\begin{bmatrix} t\,x_0 \\ t\,y_0 \\ t\,z_0 \\ t \end{bmatrix}$ with $t \neq 0$, the ***same as*** or ***equivalent*** to $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}$.

Your mind might object, but you could remind yourself that we have done things like this before: e.g. we have no problem seeing the two fractions $\frac{3}{5}$ and $\frac{6}{10}$ as the same fraction, after all $\frac{6}{10} = \frac{2\cdot 3}{2\cdot 5}$. We just multiplied both numerator and denominator by a non-zero constant.

What we basically do is we create equivalence classes of objects. All elements in one such class we call equivalent. Some of you may have seen the use these equivalence classes in projective geometry and some may have encountered them as **homogeneous coordinates**.

In this light we could write

$$\tilde{P}\begin{bmatrix} 2 \\ 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 & 8 & 4 & -24 \\ 3 & -14 & 1 & -6 \\ 24 & 16 & -8 & -48 \\ 3 & 2 & 1 & -22 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ -10 \\ -16 \\ -10 \end{bmatrix} \cong \begin{bmatrix} 0.8 \\ 1 \\ 1.6 \\ 1 \end{bmatrix},$$

where the last step is just picking that element in the equivalence class with fourth coordinate 1, i.e. the point we were looking for.

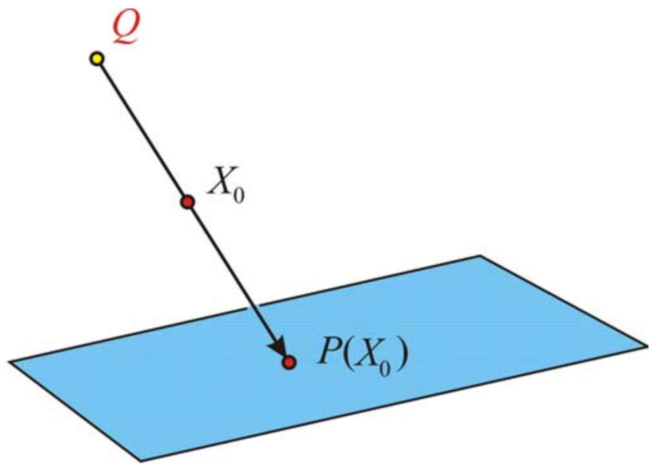
The 3D perspective projection in general: the algebraically messy way

Let's do the 3D perspective projection in general: We will follow exactly the same steps as before but now with a generic point and a generic plane. It is a bit messy, algebraically … but we will also do it in a very slick and more general way later.

From a given point onto a given plane.

Let the point $Q = (q_1, q_2, q_3)$ and the plane $\alpha: \; a\,x + b\,y + c\,z = d$ be given.

[We'll assume the point Q is not on the plane: i.e. $d-(a\,q_1+b\,q_2+c\,q_3)\neq 0$, we'll also assume the point X_0 is 'between' Q and the plane (more on that later).]



The line through $X_0 = (x_0, y_0, z_0)$ and

$Q = (q_1, q_2, q_3)$ is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + t \begin{bmatrix} x_0-q_1 \\ y_0-q_2 \\ z_0-q_3 \end{bmatrix}$$

When we intersect this line with the plane α, we find $P(X_0)$:

$$a\big(q_1+t(x_0-q_1)\big)+b\big(q_2+t(y_0-q_2)\big)+c\big(q_3+t(z_0-q_3)\big)=d$$

$$\Rightarrow \quad t = \frac{d-(a\,q_1+b\,q_2+c\,q_3)}{a(x_0-q_1)+b(y_0-q_2)+c(z_0-q_3)}$$

$$\Rightarrow \quad t = \frac{d-a\,q_1-b\,q_2-c\,q_3}{a\,x_0+b\,y_0+c\,z_0-a\,q_1-b\,q_2-c\,q_3}$$

Hence

$$P\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \frac{d-a\,q_1-b\,q_2-c\,q_3}{a\,x_0+b\,y_0+c\,z_0-a\,q_1-b\,q_2-c\,q_3}\begin{bmatrix} x_0-q_1 \\ y_0-q_2 \\ z_0-q_3 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} q_1(a\,x_0+b\,y_0+c\,z_0-a\,q_1-b\,q_2-c\,q_3)+(d-a\,q_1-b\,q_2-c\,q_3)(x_0-q_1) \\ q_2(a\,x_0+b\,y_0+c\,z_0-a\,q_1-b\,q_2-c\,q_3)+(d-a\,q_1-b\,q_2-c\,q_3)(y_0-q_2) \\ q_3(a\,x_0+b\,y_0+c\,z_0-a\,q_1-b\,q_2-c\,q_3)+(d-a\,q_1-b\,q_2-c\,q_3)(z_0-q_3) \end{bmatrix}}{a\,x_0+b\,y_0+c\,z_0-a\,q_1-b\,q_2-c\,q_3}$$

$$= \frac{\begin{bmatrix} (d-b\,q_2-c\,q_3)\,x_0+b\,q_1\,y_0+c\,q_1\,z_0-d\,q_1 \\ a\,q_2\,x_0+(d-a\,q_1-c\,q_3)\,y_0+c\,q_2\,z_0-d\,q_2 \\ a\,q_3\,x_0+b\,q_3\,y_0+(d-a\,q_1-b\,q_2)\,z_0-d\,q_3 \end{bmatrix}}{a\,x_0+b\,y_0+c\,z_0-a\,q_1-b\,q_2-c\,q_3}$$

This gives us

$$P\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} \dfrac{(d-b\,q_2-c\,q_3)\,x_0+b\,q_1\;y_0+c\,q_1\;z_0-d\,q_1}{a\,x_0+b\,y_0+c\,z_0-a\,q_1-b\,q_2-c\,q_3} \\ \dfrac{a\,q_2\,x_0+(d-a\,q_1-c\,q_3)\,y_0+c\,q_2\;z_0-d\,q_2}{a\,x_0+b\,y_0+c\,z_0-a\,q_1-b\,q_2-c\,q_3} \\ \dfrac{a\,q_3\,x_0+b\,q_3\,y_0+(d-a\,q_1-b\,q_2)\,z_0-d\,q_3}{a\,x_0+b\,y_0+c\,z_0-a\,q_1-b\,q_2-c\,q_3} \end{bmatrix}$$

Again there seems no way to write this as a matrix multiplication. We first use the extra dimension and write it as:

$$\tilde{P}\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix} = \begin{bmatrix} \dfrac{(d-b\,q_2-c\,q_3)\,x_0+b\,q_1\;y_0+c\,q_1\;z_0-d\,q_1}{a\,x_0+b\,y_0+c\,z_0-a\,q_1-b\,q_2-c\,q_3} \\ \dfrac{a\,q_2\,x_0+(d-a\,q_1-c\,q_3)\,y_0+c\,q_2\;z_0-d\,q_2}{a\,x_0+b\,y_0+c\,z_0-a\,q_1-b\,q_2-c\,q_3} \\ \dfrac{a\,q_3\,x_0+b\,q_3\,y_0+(d-a\,q_1-b\,q_2)\,z_0-d\,q_3}{a\,x_0+b\,y_0+c\,z_0-a\,q_1-b\,q_2-c\,q_3} \\ 1 \end{bmatrix}$$

Again it looks like it's impossible to write this as a matrix multiplication, with all the x_0, y_0 and z_0 in the denominators. But again …the amazing insight to shift to homogeneous coordinates changes everything

Note what happens when we multiply this image by $a\,x_0+b\,y_0+c\,z_0-a\,q_1-b\,q_2-c\,q_3$

$$\begin{bmatrix} (d-b\,q_2-c\,q_3)\,x_0+b\,q_1\;y_0+c\,q_1\;z_0-d\,q_1 \\ a\,q_2\,x_0+(d-a\,q_1-c\,q_3)\,y_0+c\,q_2\;z_0-d\,q_2 \\ a\,q_3\,x_0+b\,q_3\,y_0+(d-a\,q_1-b\,q_2)\,z_0-d\,q_3 \\ a\,x_0+b\,y_0+c\,z_0-a\,q_1-b\,q_2-c\,q_3 \end{bmatrix}$$

This **can** be written in as a matrix multiplication!

$$\begin{bmatrix} d-b\,q_2-c\,q_3 & b\,q_1 & c\,q_1 & -d\,q_1 \\ a\,q_2 & d-a\,q_1-c\,q_3 & c\,q_2 & -d\,q_2 \\ a\,q_3 & b\,q_3 & d-a\,q_1-b\,q_2 & -d\,q_3 \\ a & b & c & -a\,q_1-b\,q_2-c\,q_3 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}$$

But notice that

$$\begin{bmatrix} d-b\,q_2-c\,q_3 & b\,q_1 & c\,q_1 & -d\,q_1 \\ a\,q_2 & d-a\,q_1-c\,q_3 & c\,q_2 & -d\,q_2 \\ a\,q_3 & b\,q_3 & d-a\,q_1-b\,q_2 & -d\,q_3 \\ a & b & c & -a\,q_1-b\,q_2-c\,q_3 \end{bmatrix}$$

$$= \begin{bmatrix} a\,q_1 & b\,q_1 & c\,q_1 & -d\,q_1 \\ a\,q_2 & b\,q_2 & c\,q_2 & -d\,q_2 \\ a\,q_3 & b\,q_3 & c\,q_3 & -d\,q_3 \\ a & b & c & -d \end{bmatrix} + (d-a\,q_1-b\,q_2-c\,q_3)\,I$$

$$= \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} \begin{bmatrix} a & b & c & -d \end{bmatrix} + (d-a\,q_1-b\,q_2-c\,q_3)\,I$$

$$= \boxed{\tilde{q}\,\tilde{N}^{\mathsf{T}} - (\tilde{q}\cdot\tilde{N})\,I}$$

Note that in similar fashion the 2D perspective projection from $Q = (q_1, q_2)$ onto the line $l: \ a\,x + b\,y = c$ has the matrix

$$\begin{bmatrix} q_1 \\ q_2 \\ 1 \end{bmatrix} \begin{bmatrix} a & b & -c \end{bmatrix} - (aq_1 + bq_2 - c)\,I = \begin{bmatrix} c-bq_2 & bq_1 & -cq_1 \\ aq_2 & c-aq_1 & -cq_2 \\ a & b & -aq_1-bq_2 \end{bmatrix}$$

$$= \boxed{\tilde{q}\,\tilde{N}^{\mathsf{T}} - (\tilde{q}\cdot\tilde{N})\,I}$$

Which suggests the following theorem (applicable to higher dimensions as well):

Theorem 39.1: Let $\tilde{N}\cdot\tilde{x} = 0$ be a hyper-plane and $\vec{q}$ a point not on this plane. The perspective projection from $\vec{q}$ onto this plane is given by

$$\tilde{P}(\tilde{x}) \cong \left(\tilde{q}\,\tilde{N}^{\mathsf{T}} - (\tilde{N}\cdot\tilde{q})I\right)\tilde{x}_0$$

i.e. its matrix is $\boxed{\tilde{q}\,\tilde{N}^{\mathsf{T}} - (\tilde{N}\cdot\tilde{q})I}$ where $\tilde{N} = \begin{bmatrix} \vec{n} \\ -k \end{bmatrix}$ and $\tilde{q} = \begin{bmatrix} \vec{q} \\ 1 \end{bmatrix}$.

To find $\tilde{P}(\tilde{x})$ we still have to 'normalize' the vector $\left(\tilde{q}\,\tilde{n}^{\mathsf{T}} - (\tilde{n}\cdot\tilde{q})I\right)\tilde{x}_0$ so that its last component is a 1 (since the result is given in homogeneous coordinates).

Before we proceed with a new and more elegant proof we'll give some examples:

Example 3: $Q = (1,\ 5)$ and $l:\ 2x - 3y = 0$

$$\begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 0 \end{bmatrix} - (2 - 15 - 0)\, I = \begin{bmatrix} 15 & -3 & 0 \\ 10 & -2 & 0 \\ 2 & -3 & 13 \end{bmatrix}$$

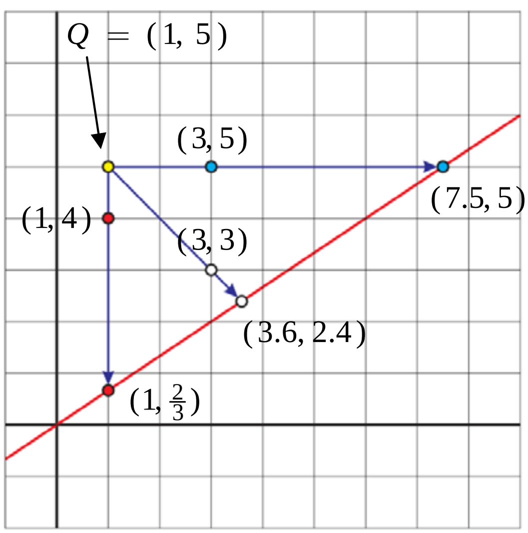
$$\begin{bmatrix} 15 & -3 & 0 \\ 10 & -2 & 0 \\ 2 & -3 & 13 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 30 \\ 20 \\ 4 \end{bmatrix} \cong \begin{bmatrix} 7.5 \\ 5 \\ 1 \end{bmatrix} \quad \text{hence} \quad (3, 5) \xrightarrow{P} (7.5, 5)$$

and

$$\begin{bmatrix} 15 & -3 & 0 \\ 10 & -2 & 0 \\ 2 & -3 & 13 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \cong \begin{bmatrix} 1 \\ 2/3 \\ 1 \end{bmatrix} \quad \text{hence} \quad (1, 4) \xrightarrow{P} (1, 2/3)$$

and

$$\begin{bmatrix} 15 & -3 & 0 \\ 10 & -2 & 0 \\ 2 & -3 & 13 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 36 \\ 24 \\ 10 \end{bmatrix} \cong \begin{bmatrix} 3.6 \\ 2.4 \\ 1 \end{bmatrix} \quad \text{hence} \quad (3, 3) \xrightarrow{P} (3.6, 2.4)$$



Example 4: Let $\alpha: \quad x-2y+3z=5$ and $Q = (4,\ 7,\ 8)$ then the matrix becomes

$$\begin{bmatrix} 4 \\ 7 \\ 8 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 & -5 \end{bmatrix} - 9I = \begin{bmatrix} -5 & -8 & 12 & -20 \\ 7 & -23 & 21 & -35 \\ 8 & -16 & 15 & -40 \\ 1 & -2 & 3 & -14 \end{bmatrix}$$

So what is the image of the point $P=(1,\ 3,\ 5)$, i.e. $\tilde{P} = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 1 \end{bmatrix}$?

$$\tilde{P}\begin{bmatrix} 1 \\ 3 \\ 5 \\ 1 \end{bmatrix} \cong \begin{bmatrix} -5 & -8 & 12 & -20 \\ 7 & -23 & 21 & -35 \\ 8 & -16 & 15 & -40 \\ 1 & -2 & 3 & -14 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 8 \\ -5 \\ -4 \end{bmatrix} \cong \begin{bmatrix} -11/4 \\ -2 \\ 5/4 \\ 1 \end{bmatrix}, \quad \text{i.e. } P\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -2.75 \\ -2 \\ 1.25 \end{bmatrix}$$

Example 5: If $Q = (0,\ 0,\ 0)$ and the plane $\alpha: \quad z=5$ than the matrix becomes

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & -5 \end{bmatrix} + 5I = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

So the image of $(1,2,3)$ is $(\frac{5}{3}, \frac{10}{3}, 5)$ since

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 15 \\ 3 \end{bmatrix} \cong \begin{bmatrix} 5/3 \\ 10/3 \\ 5 \\ 1 \end{bmatrix}$$

Example 6:

The matrix of example 1 would be $\begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 7 \end{bmatrix}$ which we could have gotten from

$$P\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{7}{7-z}\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \Rightarrow \quad \tilde{P}\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix} = \frac{1}{7-z_0}\begin{bmatrix} 7x_0 \\ 7y_0 \\ 0 \\ 7-z_0 \end{bmatrix} \cong \begin{bmatrix} 7x_0 \\ 7y_0 \\ 0 \\ 7-z_0 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 7 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}$$

or from $\tilde{q}\,\tilde{N}^{\mathsf{T}} - (\tilde{N}\cdot\tilde{q})I$.

Of course $\tilde{q}\,\tilde{N}^{\mathsf{T}} - (\tilde{N}\cdot\tilde{q})I$ doesn't give us the actual image: it doesn't have a 1 as fourth coordinate.

The fourth coordinate happens to be $7-z_0$. In fact the entire vector is $7-z_0$ times too big. So the only thing left to do is divide it by the fourth coordinate … producing a 1 as fourth coordinate *and* giving the correct image, i.e. we picked the element in the equivalence class with fourth coordinate 1.

For example $$\tilde{P}\begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} \cong \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 21 \\ 0 \\ 3 \end{bmatrix} \cong \begin{bmatrix} \frac{14}{3} \\ 7 \\ 0 \\ 1 \end{bmatrix}, \quad \text{i.e.} \quad \tilde{P}\begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{14}{3} \\ 7 \\ 0 \\ 1 \end{bmatrix}$$

Next we'll prove the main theorem. It illustrates nicely how a theorem in a general form with the right notation can become wonderfully simple and short. (Just compare it to the earlier algebraic mess of the 3D case.)

We'll prove the main theorem using the 'tilde' notation we introduced.

Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ then $\tilde{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$, **or** if $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then $\tilde{x} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$ etc. Symbolically

$$\tilde{x} = \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix}$$

similarly $\tilde{x}_0 = \begin{bmatrix} \vec{x}_0 \\ 1 \end{bmatrix}$ and $\tilde{q} = \begin{bmatrix} \vec{q} \\ 1 \end{bmatrix}$.

And we'll also use the **augmented normal** $\tilde{N}$, of an hyper-plane $\vec{n} \cdot \vec{x} = k$:

i.e. $\tilde{N} = \begin{bmatrix} a \\ b \\ -c \end{bmatrix}$ in case of the 2D line $a\,x + b\,y = c$

and $\tilde{N} = \begin{bmatrix} a \\ b \\ c \\ -d \end{bmatrix}$ in case of the 3D plane $a\,x + b\,y + c\,z = d$ etc.

Recall that the equations $a\,x + b\,y = c$ and $a\,x + b\,y + c\,z = d$ now all can be written as simply: $\tilde{N} \cdot \tilde{x} = 0$.

Proof of the theorem:

Let Q be a point and the (hyper) plane $\vec{n}\cdot\vec{x} = d$. We want to map a point $\vec{x}_0$ onto the plane by a perspective projection from Q:

We'll formulate everything immediately in terms of affine coordinates (i.e. using the extra dimension):

The plane is $\tilde{N}\cdot\tilde{x} = 0$

The line through $\tilde{q}$ and $\tilde{x}_0$ is $\tilde{x} = \tilde{q} + t(\tilde{x}_0 - \tilde{q})$

Then the intersection of the plane and the line satisfies $\tilde{N}\cdot\big(\tilde{q} + t(\tilde{x}_0 - \tilde{q})\big) = 0$

i.e. $t = -\dfrac{\tilde{N}\cdot\tilde{q}}{\tilde{N}\cdot(\tilde{x}_0 - \tilde{q})}$

so that $\tilde{P}(\tilde{x}_0) = \tilde{q} - \dfrac{\tilde{N}\cdot\tilde{q}}{\tilde{N}\cdot(\tilde{x}_0 - \tilde{q})}(\tilde{x}_0 - \tilde{q})$

Now if we go to homogeneous coordinates and multiply by the scalar $\tilde{n}\cdot(\tilde{x}_0 - \tilde{q})$ we get

$$\tilde{P}(\tilde{x}_0) \cong \big(\tilde{N}\cdot(\tilde{x}_0 - \tilde{q})\big)\tilde{q} - (\tilde{N}\cdot\tilde{q})(\tilde{x}_0 - \tilde{q})$$

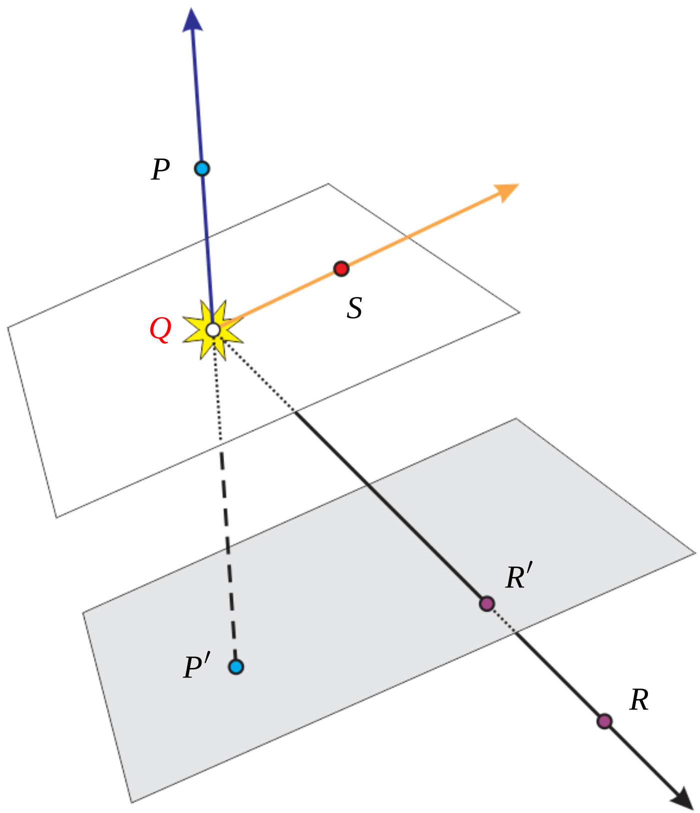
so that

$$\begin{aligned}
\tilde{P}(\tilde{x}_0) &\cong \big(\tilde{N}\cdot\tilde{x}_0\big)\tilde{q} - \big(\tilde{N}\cdot\tilde{q}\big)\tilde{q} - (\tilde{N}\cdot\tilde{q})\,\tilde{x}_0 + (\tilde{N}\cdot\tilde{q})\,\tilde{q} \\
&= \big(\tilde{N}\cdot\tilde{x}_0\big)\tilde{q} - (\tilde{N}\cdot\tilde{q})\,\tilde{x}_0 \\
&= \tilde{q}\,\tilde{N}^{\mathsf{T}}\tilde{x}_0 - (\tilde{N}\cdot\tilde{q})\,\tilde{x}_0 \\
&= \big(\tilde{q}\,\tilde{N}^{\mathsf{T}} - (\tilde{N}\cdot\tilde{q})I\big)\tilde{x}_0
\end{aligned}$$

Hence the homogeneous matrix we were looking for is $\boxed{\tilde{q}\,\tilde{N}^{\mathsf{T}} - (\tilde{q}\cdot\tilde{N})I}$ □

Which points cast shadows? Which points are projected?

Of course the way we introduced the transformation, as "casting shadows", would imply that if the point is behind the plane (on the other side of the light source Q) "casting a shadow" wouldn't make sense, nor if the point is on the other side of the light (away from the plane) then there would be no casting of a shadow either. Nevertheless in all these cases the projection operation we found does result in a point on the plane. Except if we start out with a point on the plane that goes through Q and is parallel to the plane of projection; in that case we do not get a projected point.



.

The point R is behind the plane (on the other side of the plane than Q) so even if the light goes through the plane there would not be a shadow cast, nevertheless the projection operator will assign R' on the plane as the image of R.

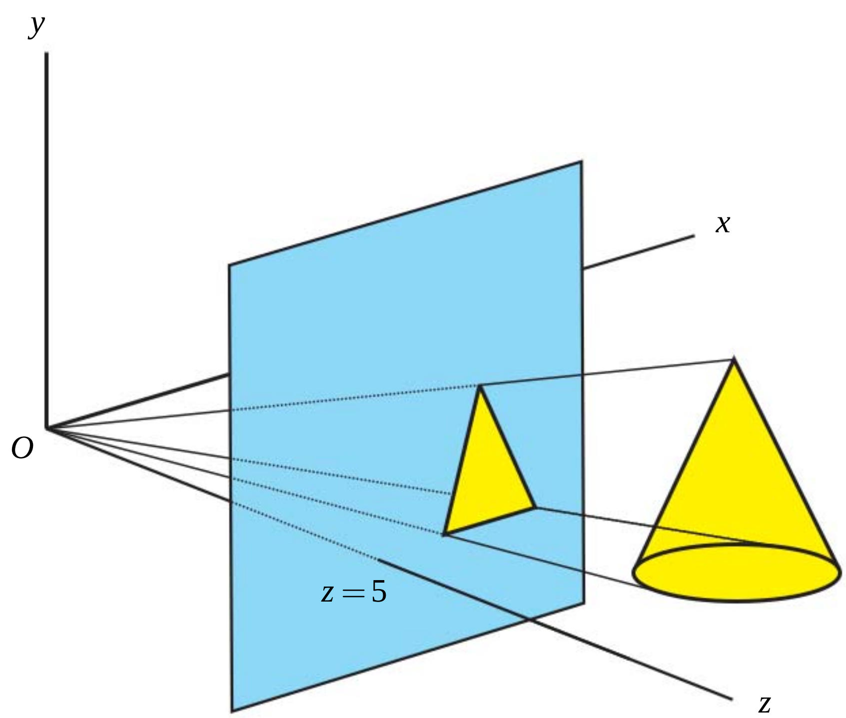
The point P is behind the light source Q, farther away from the plane. Here too there would be no shadow cast on the plane, nevertheless the projection operator will assign P' on the plane as the image of P.

The point S is on the plane through Q parallel to the plane of projection. Here there would be no shadow cast either, *and* the projection operator would not assign any point.

If the purpose of this map were "shadow casting" it would be reasonable to only allow points between the two planes.

The perspective projection operator has in computer graphics another use:

Q might not be taken as a light source, but a camera or the eye of an observer. For example if Q is a camera at the origin pointing in the direction of the positive *z*-axis, and the plane we project onto is $z = 5$, and is taken as the computer screen onto which the entire 3D world (at least that part that is visible by the camera) is perspectively projected:



The cone is seen perspectively projected on the screen. Any 3D object on this side of the camera (both in front and behind the plane) is projected onto this screen.

The projection operator for this is $\tilde{T}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \cong \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 5x \\ 5y \\ 5z \\ z \end{bmatrix} \cong \begin{bmatrix} 5x/z \\ 5y/z \\ 5 \\ 1 \end{bmatrix}$

Of course in the coming CS computer graphics courses many other aspects need to be addressed as well, such as for example if an object is partially behind another solid object only part of it will be visible, only part of its projection will show up on the screen. Or if an object is partially outside of the viewing screen it needs to be clipped. Or how do we create the illusion of three dimensionality on a 2D screen: The perspective projection does give a correct "perspective" rendition of the 3D world but that is not enough, objects may still look flat. Color variations (brighter colors closer, softer colors farther away), shadow casting, highlights etc. are also needed.

The Determinant and Eigenvalues of Perspective Projection Matrix

Of course the determinant of the perspective projection matrix is zero, since it is a projection:

many points get mapped to the same point on the plane. For example take a point $\vec{p}$ not on the plane through $\tilde{q}$ [i.e. $\tilde{N}\cdot\tilde{p} \neq \tilde{N}\cdot\tilde{q}$], and its image $\vec{p}' = \tilde{M}\,\vec{p}$. Then both $\vec{p}$ and $\frac{1}{2}(\vec{p}+\vec{p}')$ get mapped to $\vec{p}'$. Hence $\tilde{M}$ cannot be invertible, i.e. $\boxed{\det(\tilde{M}) = 0}$.

Note that the projections we saw before all had the property: $\tilde{M}^2 = \tilde{M}$. This is **Not** true in the homogeneous case though: $\boxed{\tilde{M}^2 \neq \tilde{M}}$:

Take for example the matrix from example 4:

$$\tilde{M}^2 = \begin{bmatrix} -5 & -8 & 12 & -20 \\ 7 & -23 & 21 & -35 \\ 8 & -16 & 15 & -40 \\ 1 & -2 & 3 & -14 \end{bmatrix}^2 = \begin{bmatrix} 45 & 75 & -108 & 180 \\ -63 & 207 & -189 & 315 \\ -72 & 144 & -135 & 360 \\ -9 & 18 & -27 & 126 \end{bmatrix} \neq \tilde{M}$$

Nevertheless for any $\tilde{p}$: $\boxed{\tilde{M}^2\tilde{p} \cong \tilde{M}\,\tilde{p}}$ (That is: for any $\tilde{p}$ for which $\tilde{N}\cdot\tilde{p} \neq \tilde{N}\cdot\tilde{q}$)

Proof: For any $\tilde{p}$ on the plane, i.e. for which $\tilde{N}\cdot\tilde{p} = 0$, we find that

$$\begin{aligned} \tilde{M}\,\tilde{p} &= \left(\tilde{q}\,\tilde{N}^{\mathsf{T}} - (\tilde{N}\cdot\tilde{q})I\right)\tilde{p} \\ &= \tilde{q}\,\tilde{N}^{\mathsf{T}}\tilde{p} - (\tilde{N}\cdot\tilde{q})\tilde{p} \\ &= (\tilde{N}\cdot\tilde{p})\tilde{q} - (\tilde{N}\cdot\tilde{q})\tilde{p} \\ &= -(\tilde{N}\cdot\tilde{q})\tilde{p} \\ &\cong \tilde{p} \qquad [\text{ since } \tilde{N}\cdot\tilde{q} \neq \tilde{N}\cdot\tilde{p} = 0\,] \end{aligned}$$

Hence for any $\tilde{p}$ [with $\tilde{N}\cdot\tilde{p} \neq \tilde{N}\cdot\tilde{q}$]: $\tilde{M}^2\tilde{p} \cong \tilde{M}\,\tilde{p}$, since $\tilde{M}\,\tilde{p}$ is on the plane.

Note that all these $\tilde{p}$ are **eigenvectors** with **eigenvalue** $-(\tilde{N}\cdot\tilde{q})$.

Also note that
$$\begin{aligned}\tilde{M}\,\tilde{q} &= \left(\tilde{q}\,\tilde{N}^{\mathsf{T}} - (\tilde{N}\cdot\tilde{q})I\right)\tilde{q}\\ &= \tilde{q}\,\tilde{N}^{\mathsf{T}}\tilde{q} - (\tilde{N}\cdot\tilde{q})\,\tilde{q}\\ &= (\tilde{N}\cdot\tilde{q})\,\tilde{q} - (\tilde{N}\cdot\tilde{q})\,\tilde{q}\\ &= \vec{0}\end{aligned}$$

So that the characteristic polynomial of $\tilde{M}$ is $\det(\tilde{M}-t\,I) = (-1)^{n+1}t\,(t+\tilde{N}\cdot\tilde{q})^n$ and $\tilde{M}$ is diagonalizable, i.e. there is a basis of eigenvectors such that:

$$Q^{-1}\tilde{M}\,Q = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & -(\tilde{N}\cdot\tilde{q}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -(\tilde{N}\cdot\tilde{q}) \end{bmatrix}$$

The Trace of the Perspective Projection Matrix

One might expect that the trace of the perspective projection matrix has little value in the world of homogeneous coordinates. Nevertheless it is worth looking at:

$$\begin{aligned}\operatorname{Trace}\left(\tilde{q}\,\tilde{N}^{\mathsf{T}} - (\tilde{N}\cdot\tilde{q})I\right) &= \operatorname{Trace}\left(\begin{bmatrix} q_1 \\ \vdots \\ q_n \\ 1 \end{bmatrix}\begin{bmatrix} a_1 & \cdots & a_n & -k \end{bmatrix} - (\tilde{N}\cdot\tilde{q})I\right)\\ &= (a_1q_1+\cdots a_nq_n-k)-(n+1)(\tilde{N}\cdot\tilde{q})\\ &= (\tilde{N}\cdot\tilde{q})-(n+1)(\tilde{N}\cdot\tilde{q})\\ &= -n(\tilde{N}\cdot\tilde{q})\end{aligned}$$

$$\operatorname{Trace}(M) = \boxed{-n\cdot(\tilde{N}\cdot\tilde{q})}$$

where n is the dimension of the space in which $\vec{q}$ lives (i.e. one less than the dimension of $\tilde{M}$)

So it is an interesting quantity after all! It is n times the nonzero eigenvalue we found in the above. This allows us to find Q and the plane of projection from $\tilde{M}$.

Fixed points

When one tries to find fixed points of the matrix $\tilde{M}$ in the usual way with $\operatorname{arref}(\tilde{M}-I)$ we do not get too far:

$$\operatorname{arref}(\tilde{M}-I) = \operatorname{arref}\left(\begin{bmatrix} -5 & -8 & 12 & -20 \\ 7 & -23 & 21 & -35 \\ 8 & -16 & 15 & -40 \\ 1 & -2 & 3 & -14 \end{bmatrix} - I\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The fact is that a fixed point doesn't get mapped to itself but to a **multiple** of itself. It is an eigenvector of $\tilde{M}$, and therefore the previous remarks give us a clue how to proceed, since:

$$\tilde{M}\,\tilde{p} = -(\tilde{N}\cdot\tilde{q})\,\tilde{p}$$

for points on the plane. Hence we need to

$$\operatorname{arref}\left(\tilde{M} + (\tilde{N}\cdot\tilde{q})\,I\right)$$

But if it is the plane of fixed points we are after, i.e. we do not know $\tilde{N}$, nor do we know $\tilde{q}$, so how could we proceed? Recall that we discovered that $\operatorname{Trace}(\tilde{M}) = -n(\tilde{N}\cdot\tilde{q})$, so that to find fixed points all we need to do is

$$\operatorname{rref}\left(\tilde{M} - \frac{\operatorname{Trace}(\tilde{M})}{n} I\right)$$

Example: If $\tilde{M} = \begin{bmatrix} -5 & -8 & 12 & -20 \\ 7 & -23 & 21 & -35 \\ 8 & -16 & 15 & -40 \\ 1 & -2 & 3 & -14 \end{bmatrix}$ then $\operatorname{Trace}(\tilde{M}) = -27$, and thus

$$\operatorname{arref}\left(\tilde{M} - \frac{\operatorname{Trace}(\tilde{M})}{n} I\right) = \operatorname{arref}\left(\tilde{M} - \frac{-27}{3} I\right) = \operatorname{arref}\left(\tilde{M} + 9I\right)$$

$$= \begin{bmatrix} 1 & -2 & 3 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So that the fixed points are all the points on the plane: $x - 2y + 3z = 5$

How to find Q

Recall that $\tilde{q}$ is an eigenvector with eigenvalue 0 as we saw earlier, i.e.

$$\tilde{M}\,\tilde{q} = \vec{0}$$

So that to find Q all we need to do is arref($\tilde{M}$).

For example:

$$\operatorname{rref}(M) = \operatorname{rref}\begin{bmatrix} -5 & -8 & 12 & -20 \\ 7 & -23 & 21 & -35 \\ 8 & -16 & 15 & -40 \\ 1 & -2 & 3 & -14 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{so that} \quad \tilde{q} = \begin{bmatrix} 4 \\ 7 \\ 8 \\ 1 \end{bmatrix}$$

The 3D case in general looks like:

$$\operatorname{arref}(\tilde{M}) = \operatorname{arref}\begin{bmatrix} d - b\,q_2 - c\,q_3 & b\,q_1 & c\,q_1 & -d\,q_1 \\ a\,q_2 & d - a\,q_1 - c\,q_3 & c\,q_2 & -d\,q_2 \\ a\,q_3 & b\,q_3 & d - a\,q_1 - b\,q_2 & -d\,q_3 \\ a & b & c & -a\,q_1 - b\,q_2 - c\,q_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -q_1 & 0 \\ 0 & 1 & 0 & -q_2 & 0 \\ 0 & 0 & 1 & -q_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The TI-Nspire has no problem computing this:

$$\begin{bmatrix} q1 \\ q2 \\ q3 \\ 1 \end{bmatrix} \to q \qquad \begin{bmatrix} q1 \\ q2 \\ q3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ -d \end{bmatrix} \to n \qquad \begin{bmatrix} a \\ b \\ c \\ -d \end{bmatrix}$$

$$q \cdot n^{T} - \mathrm{dotP}(q,n) \qquad \begin{bmatrix} -(b \cdot q2 + c \cdot q3 - d) & b \cdot q1 & c \cdot q1 & -d \cdot q1 \\ a \cdot q2 & -a \cdot q1 - c \cdot q3 + d & c \cdot q2 & -d \cdot q2 \\ a \cdot q3 & b \cdot q3 & -a \cdot q1 - b \cdot q2 + d & -d \cdot q3 \\ a & b & c & -a \cdot q1 - b \cdot q2 - c \cdot q3 \end{bmatrix}$$

$$\mathrm{rref}\left(\begin{bmatrix} -(b \cdot q2 + c \cdot q3 - d) & b \cdot q1 & c \cdot q1 & -d \cdot q1 \\ a \cdot q2 & -a \cdot q1 - c \cdot q3 + d & c \cdot q2 & -d \cdot q2 \\ a \cdot q3 & b \cdot q3 & -a \cdot q1 - b \cdot q2 + d & -d \cdot q3 \\ a & b & c & -a \cdot q1 - b \cdot q2 - c \cdot q3 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & 0 & -q1 \\ 0 & 1 & 0 & -q2 \\ 0 & 0 & 1 & -q3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We could have done this by hand using some basic row operations:

For example after $\mathrm{mRowAdd}(-q_3, M, 4, 3)$, followed by $\mathrm{mRowAdd}(-q_2, M, 4, 2)$, followed by $\mathrm{mRowAdd}(-q_1, M, 4, 1)$ the matrix has reduced to

$$\begin{bmatrix} -\tilde{N}\cdot\tilde{q} & 0 & 0 & (\tilde{N}\cdot\tilde{q})\,q_1 \\ 0 & -\tilde{N}\cdot\tilde{q} & 0 & (\tilde{N}\cdot\tilde{q})\,q_2 \\ 0 & 0 & -\tilde{N}\cdot\tilde{q} & (\tilde{N}\cdot\tilde{q})\,q_3 \\ a & b & c & -\tilde{n}\cdot\tilde{q} \end{bmatrix}$$

which reduces immediately to $\begin{bmatrix} 1 & 0 & 0 & -q_1 \\ 0 & 1 & 0 & -q_2 \\ 0 & 0 & 1 & -q_3 \\ a & b & c & -\tilde{n}\cdot\tilde{q} \end{bmatrix}$ after three row multiplications by $-\dfrac{1}{\tilde{N}\cdot\tilde{q}}$

which after $\mathrm{mRowAdd}(-a, M, 1, 4)$, $\mathrm{mRowAdd}(-b, M, 2, 4)$ and $\mathrm{mRowAdd}(-c, M, 3, 4)$

clearly reduces to $\begin{bmatrix} 1 & 0 & 0 & -q_1 \\ 0 & 1 & 0 & -q_2 \\ 0 & 0 & 1 & -q_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

We can do this in general as follows:

$$\tilde{q}\,\tilde{N}^{\mathsf{T}} - (\tilde{N}\cdot\tilde{q})I = \begin{bmatrix} q_1 \\ \vdots \\ q_n \\ 1 \end{bmatrix}\begin{bmatrix} a_1 & \cdots & a_n & -k \end{bmatrix} - (\tilde{N}\cdot\tilde{q})I$$

$$= \begin{bmatrix} a_1 q_1 & a_2 q_1 & \dots & a_n q_1 & -k\,q_1 \\ a_1 q_2 & a_2 q_2 & \dots & a_n q_2 & -k\,q_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 q_n & a_2 q_n & \dots & a_n q_n & -k\,q_n \\ a_1 & a_2 & \cdots & a_n & -k \end{bmatrix} - \begin{bmatrix} \tilde{N}\cdot\tilde{q} & 0 & \cdots & 0 & 0 \\ 0 & \tilde{N}\cdot\tilde{q} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \tilde{N}\cdot\tilde{q} & 0 \\ 0 & 0 & \cdots & 0 & \tilde{N}\cdot\tilde{q} \end{bmatrix}$$

After $\mathrm{mRowAdd}(-q_1, M, n+1, 1)$, followed by $\mathrm{mRowAdd}(-q_2, M, n+1, 2)$, followed by $\mathrm{mRowAdd}(-q_3, M, n+1, 3)$ etc. the matrix has reduced to

$$\begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ a_1 & a_2 & \cdots & a_n & -k \end{bmatrix} - \begin{bmatrix} \tilde{N}\cdot\tilde{q} & 0 & \cdots & 0 & -(\tilde{N}\cdot\tilde{q})\,q_1 \\ 0 & \tilde{N}\cdot\tilde{q} & \cdots & 0 & -(\tilde{N}\cdot\tilde{q})\,q_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \tilde{N}\cdot\tilde{q} & -(\tilde{N}\cdot\tilde{q})\,q_n \\ 0 & 0 & \cdots & 0 & \tilde{N}\cdot\tilde{q} \end{bmatrix}$$

which, after dividing the first n lines by $-\tilde{N}\cdot\tilde{q}$ becomes:

$$\begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ a_1 & a_2 & \cdots & a_n & -k \end{bmatrix} - \begin{bmatrix} -1 & 0 & \cdots & 0 & q_1 \\ 0 & -1 & \cdots & 0 & q_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & q_n \\ 0 & 0 & \cdots & 0 & \tilde{N}\cdot\tilde{q} \end{bmatrix}$$

and hence

$$\begin{bmatrix} 1 & 0 & \dots & 0 & -q_1 \\ 0 & 1 & \dots & 0 & -q_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -q_n \\ a_1 & a_2 & \cdots & a_n & -k-\tilde{N}\cdot\tilde{q} \end{bmatrix}$$

So that the final step of using the pivots to eliminate the final row of a-s gives us:

$$\begin{bmatrix} 1 & 0 & \dots & 0 & -q_1 \\ 0 & 1 & \dots & 0 & -q_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -q_n \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Of course, even though this is a nice review of row operations, it is much easier to first show that $\tilde{M}\,\tilde{q} = \vec{0}$, from which it immediately follows.

Summary

We have shown that:

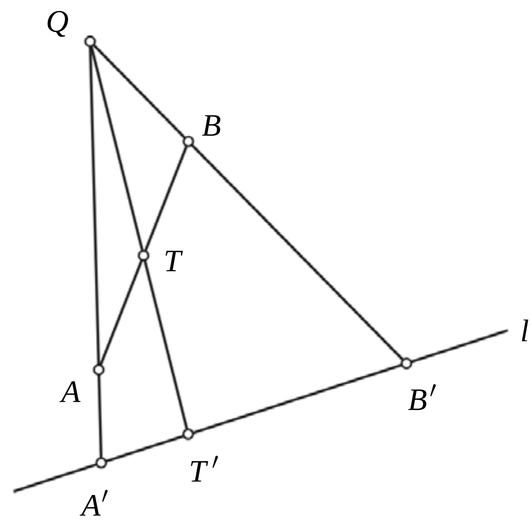
If $\boxed{\tilde{M} = \tilde{q}\,\tilde{N}^{\mathsf{T}} - (\tilde{q}\cdot\tilde{N})\,I}$ then

- $\boxed{\tilde{M}\,\tilde{p} = -(\tilde{N}\cdot\tilde{q})\,\tilde{p}}$ for $\tilde{p}$ on the plane, i.e. with $\tilde{N}\cdot\tilde{p} = 0$
- $\boxed{\mathrm{Trace}\left(\tilde{M}\right) = -n(\tilde{N}\cdot\tilde{q})}$ where n is the dimension of $\vec{q}$, i.e. n is one less than the dimension of the $(n+1)\times(n+1)$ square matrix $\tilde{M}$
- $\boxed{\mathrm{arref}\left(\tilde{M} - \dfrac{\mathrm{Trace}(\tilde{M})}{n}\,I\right)}$ gives us the fixed points of the transformation
- $\boxed{\mathrm{arref}\,(\tilde{M}) = \begin{bmatrix} I & -\vec{q} & 0 \\ 0\cdots0 & 0 & 0 \end{bmatrix}}$ gives us Q.

Barycentric Coordinates?

Do perspective projections preserve Barycentric coordinates? **NO**.

Midpoints are not even mapped to midpoints:
e.g. A and B are mapped to A' and B' (resp.) and the midpoint T of A and B gets mapped to T' which is ***not*** the midpoint of A' and B'



We'll close with an interesting exercise:

A perspective projection P maps the points $(1,1,2)$, $(2,2,2)$, $(0,1,1)$, as follows

$$P(1,1,2)=(1,0,-2), \quad P(2,2,2)=(3,2,-2) \quad \text{and} \quad P(0,1,1)=(-2,-1,-9)$$

(a) What plane is P mapping onto? i.e. Find α. [Hint: plane through 3 points]
(b) What point is being projected from? i.e. Find Q. [Hint: Intersection of lines.]
(c) What is the 4×4 matrix that describes the map.
(d) Find the image of $(1,2,3)$ i.e. $P(1,2,3)$.