Fundamentals of Differentiation Part 2

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Recap

Definition

1. Defn of the derivative of f at a point a

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

1st   2nd

2. Defn of the derivative of f (or the derivative function of f)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}$$

3. Differentiation rules: constant, power rule, trigo, expo, and log. ← natural log
4. Algebraic differentiation rules: constant multiple, addition, difference, product, and quotient.

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Differentiating composite functions

$(f \circ g)(x) = f(g(x))$

$\sin(x^2)$

The differentiation rules we have learnt in the last lecture cover most of the functions, with the exception of composite functions, for example

$$f(x) = \ln(\cos x), \quad g(x) = \sqrt{1 - x^2}.$$

How do we differentiate such functions? Using the *Chain Rule*.

Chain Rule

Theorem

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $f \circ g$ is differentiable at x and the derivative of $f \circ g$, $(f \circ g)'$ is given by

$$\boxed{(f \circ g)'(x) = f'(g(x)) \cdot g'(x).}$$

An alternative form of the chain rule: if $y = f(u)$ and $u = g(x)$ are differentiable, then

$y = f(g(x))$

differentiating $y \longrightarrow$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

TLDR: Differentiate outer function f, sub in inner function g, then multiply by the derivative of the inner function.

Example 1

We differentiate $f(x) = \ln(\cos x)$. Set $g(x) = \ln x$ (outer function) and $h(x) = \cos x$ (inner function). Note that $f = g \circ h$. Then $g'(x) = \dfrac{1}{x}$ and $h'(x) = -\sin x$. Therefore, by the Chain Rule,

$h(x) = \cos x$
$h'(x) = -\sin x.$

$$
\begin{aligned}
f'(x) = \underline{(g \circ h)'(x)} &= g'(h(x)) \cdot \boxed{h'(x)} \\
&= \frac{1}{h(x)} \cdot (-\sin x) \\
&= \frac{1}{\cos x} \cdot (-\sin x) \\
&= -\frac{\sin x}{\cos x} \\
&= \boxed{-\tan x.}
\end{aligned}
$$

$g'(x) = \frac{1}{x}$
$g'(h(x)) = \frac{1}{h(x)}$

$\tan x = \dfrac{\sin x}{\cos x}.$

Example 2

We differentiate $f(x) = \sqrt{1-x^2}$. Set $g(x) = \sqrt{x}$ and $h(x) = 1 - x^2$. Then $g'(x) = \dfrac{1}{2\sqrt{x}}$ and $h'(x) = -2x$. By the Chain Rule,

$$
\begin{aligned}
f'(x) = (g \circ h)'(x) &= g'(h(x)) \cdot \boxed{h'(x)} \\
&= \frac{1}{2\sqrt{h(x)}} \cdot (-2x) \\
&= \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) \\
&= -\frac{-2x}{2\sqrt{1-x^2}} \\
&= \boxed{-\frac{x}{\sqrt{1-x^2}}}.
\end{aligned}
$$

$$g'(x) = \frac{1}{2\sqrt{x}}$$

$$g'(h(x)) = \frac{1}{2\sqrt{h(x)}}$$

Exercise 1

Differentiate the following functions.

1. $\sin^2(x)$
2. $\sin(x^2)$
3. $(x^2+1)^6$
4. $(\star)$ $e^{\sin(x^2)}$
5. $(\star)$ $x^2\ln(\tan x)$

① $\sin^2 x = (\sin x)^2$

$f(x) = x^2$ outer function

$g(x) = \sin x$ inner function

$$[\sin^2 x]' = \underbrace{2\sin x}_{f'(g(x))} \cdot \underbrace{\cos x}_{g'(x)}$$

$$= \sin(2x)$$

② let $h(x) = \sin(x^2)$

$$h'(x) = \underbrace{\cos(x^2)}_{f'(g(x))} \cdot \underbrace{2x}_{g'(x)}$$

$$= \underline{2x\cos(x^2)}$$

Exercise 1

(3) $h(x) = (x^2+1)^6$

$f(x) = x^6$ outer

$g(x) = x^2+1$ inner

$h'(x) = 6(x^2+1)^5 \cdot 2x = 12x(x^2+1)^5.$

(4) $h(x) = e^{\sin(x^2)}$

$f(x) = e^x$

$g(x) = \sin(x^2)$ → in part (2)

$h'(x) = e^{\sin(x^2)} \cdot [\sin(x^2)]'$

$= e^{\sin(x^2)} \cdot \cos(x^2) \cdot 2x$

$= 2x\cos(x^2)\, e^{\sin(x^2)}.$

⑤ $h(x) = \underline{x^2}\,\underline{\ln(\tan x)}$

$$h'(x) = 2x\ln(\tan x) + x^2\left[\ln(\tan x)\right]'$$

$$= 2x\ln(\tan x) + x^2\left(\frac{1}{\tan x}\cdot\sec^2 x\right)$$

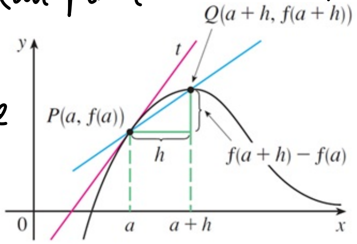
$$= 2x\ln(\tan x) + \frac{x^2\sec^2 x}{\tan x}.$$

Tangent line to f

(a, b) ← set (all pts from a to b, non-inclusive of a, b)

(a,b) ← point in 2D space

x-coordinate y-coordinate



The magenta line here is called the *tangent line to the function f at the point* $(a, f(a))$. It has several properties:

1. It has the **same gradient** as the function f at the point $(a, f(a))$, i.e. its gradient is $f'(a)$.
2. It intersects the graph of $y = f(x)$ at only at the point $(a, f(a))$.

Tangent line equation

Using the information above, we can find the equation of the tangent line to f at $(a, f(a))$. Let

$$y = mx + c$$

be the equation of this tangent line, where m and c are unknown constants.

Since the gradient of this line is $f'(a)$, $m = f'(a)$, we have

$$y = f'(a)x + c.$$

This line contains the point $(a, f(a))$, so

$$f(a) = f'(a)a + c \implies c = f(a) - f'(a)a.$$

Therefore the equation of the tangent line to f at $(a, f(a))$ is

$$y = f'(a)x + f(a) - f'(a)a = f'(a)(x - a) + f(a).$$

Tangent line to f at $\underbrace{(a, f(a))}_{x \quad y}$

① gradient $m = f'(a)$

General eq^n of a line : $y = mx + c$ ← find c

$= f'(a)x + c$

② $f(a) = f'(a)a + c$

$\Rightarrow c = f(a) - f'(a) \cdot a$

$\Rightarrow$ (sub back) $y = \underline{f'(a)x} + f(a) - \underline{f'(a) \cdot a}$

$= f'(a)(x-a) + f(a)$

Tangent line equation

Theorem

The equation of *tangent line to f at $(a, f(a))$* is

$$y = f'(a)(x - a) + f(a).$$

Example 3

a $f(a)$

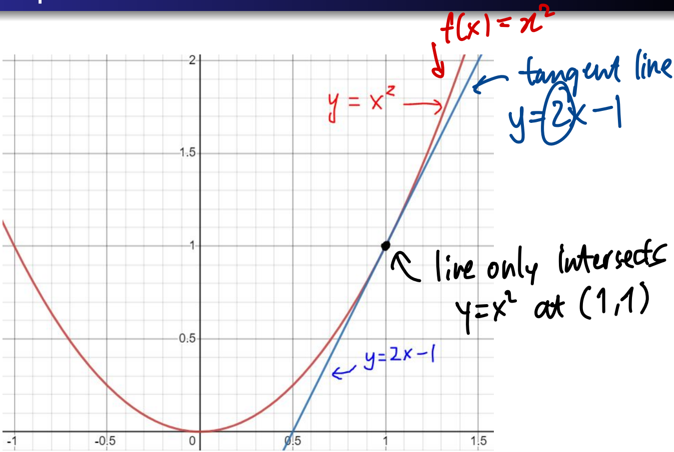
We find the tangent line to $f(x) = x^2$ at the point $(1, 1)$.

We have $f(1) = 1$. Note that $f'(x) = 2x$, therefore $f'(1) = 2$. Putting this together, the equation of the tangent line to $f(x) = x^2$ at $(1, 1)$ is

$f'(a)$ a $f(a)$

$$\begin{aligned} y &= f'(1)(x - 1) + f(1) \\ &= 2(x - 1) + 1 \\ &= 2x - 1. \end{aligned}$$

Graph of Example 3



The black point is the point $(1, 1)$. One can observe that the line $y = 2x - 1$ is tangent to the graph of $f(x) = x^2$ at $a = 1$.

Example 4

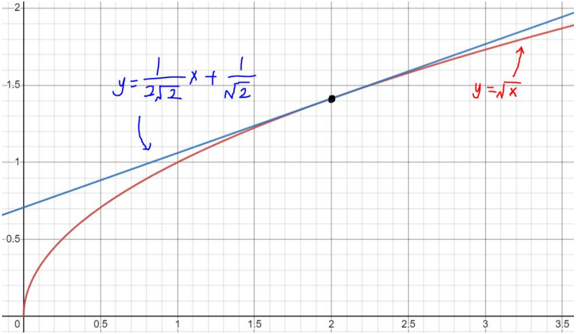
We find the tangent line to $f(x) = \sqrt{x}$ at the point $(\overset{a}{2}, \overset{f(a)}{\sqrt{2}})$.

We have $f(2) = \sqrt{2}$. Note that $f'(x) = \dfrac{1}{2\sqrt{x}}$, hence $f'(2) = \dfrac{1}{2\sqrt{2}}$.

We put this together to get the equation of the tangent line to $f(x) = \sqrt{x}$ at $(2, \sqrt{2})$:

$$\begin{aligned}
y &= f'(2)(x - \overset{a}{2}) + \overset{f(a)}{f(2)} \\
&= \frac{1}{2\sqrt{2}}(x-2) + \sqrt{2} \\
&= \frac{1}{2\sqrt{2}}x - \frac{1}{\sqrt{2}} + \sqrt{2} \\
&= \underbrace{\frac{1}{2\sqrt{2}}}_{m}x + \underbrace{\frac{1}{\sqrt{2}}}_{c}.
\end{aligned}$$

Graph of Example 4



Exercise 2

Find the tangent line for each of the following functions at the given points.

$a \quad f(a)$

1. $f(x) = \dfrac{x+2}{x-3}$, at $(2, -4)$
2. $f(x) = \sqrt{1-3x}$, at $(-1, 2)$

① $f(x) = \dfrac{x+2}{x-3} = \dfrac{(x-3)+5}{x-3}$

$= 1 + \dfrac{5}{x-3} = 5(x-3)^{-1}$

Calculate $f'(a)$.

$f'(x) = 0 - 5(x-3)^{-2}$

$= \dfrac{-5}{(x-3)^2}$ $\quad x = 2 \Rightarrow f'(2) = \dfrac{-5}{\underbrace{(2-3)^2}_{1}} = -5$

$y = f'(2)(x-2) + f(2) = -5(x-2) - 4$

$= -5x + 10 - 4 = -5x + 6.$

Exercise 2

(2) $f(x) = \sqrt{1-3x}$ at $(-1, 2)$

$a \quad f(a)$

$$f'(x) = \frac{1}{2\sqrt{1-3x}} \cdot (-3) = -\frac{3}{2\sqrt{1-3x}}$$

$x = -1$

$$f'(-1) = -\frac{3}{2\sqrt{1+3}} = -\frac{3}{4}$$

The tangent line is

$$y = -\frac{3}{4}(x-(-1)) + 2$$
$$= -\frac{3}{4}x - \frac{3}{4} + 2 = -\frac{3}{4}x + \frac{5}{4}.$$

Explicit and implicit functions

Most of the functions so far which we have seen are written in an *explicit* form, where a variable y is expressed explicitly in terms of another variable x, called *explicit functions*, for example,

independent

$y = f(x)$

dependent

$$y = x^2 + 1, \quad \text{or} \quad y = \sqrt{1 - 3x}$$

or in general, $y = f(x)$. On the other hand, there are functions y in terms of x which are defined *implicitly*, called *implicit functions*, for example

$$x^2 + y^2 = 1, \quad \text{or} \quad \sin(y^2 + x) + \cos(x^2 + y) = 0.$$

y depends on x

unit circle centred at $(0,0)$

Explicit and implicit functions

in terms of x.

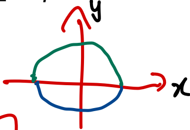
It is not always possible to feasibly find an explicit formula for y given an implicit definition.

1. Let $x^2 + y^2 = 1$, where y is defined implicitly here. We can make y the subject of this equation, which yields <u>two</u> *explicit* functions:

$y^2 = 1 - x^2$

$y = \pm\sqrt{1-x^2}$

$$y = \sqrt{1 - x^2}, \quad \text{or} \quad y = -\sqrt{1 - x^2}.$$



2. On the other hand, let $\sin(y^2 + x) + \cos(x^2 + y) = 0$, where y is also defined implicitly here. It is clearly not obvious nor is feasible to try to make y the subject of this equation.

Implicit differentiation

Problem: The derivatives which we have found so far are for explicit functions.

Question: Can we still find $\dfrac{dy}{dx}$ for implicit functions?

The answer to this question is yes. We can differentiate an implicit function y, provided y is a differentiable function.

$(\star)$ In this course, with respect to implicit differentiation, it is always assumed that y is a differentiable function.

Example 5

Consider $x^2 + y^2 = 1$. We differentiate both sides of the equation, with respect to x
taking into account that y is a function of x:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$$
$$\implies \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$
$$\implies 2x + \frac{d}{dx}(y^2) = 0.$$

Now, since y is a function of x, y^2 is therefore a composite function of x, thus the Chain Rule applies (then make $\dfrac{dy}{dx}$ the subject of the equation):

$$2x + 2y \cdot \frac{dy}{dx} = 0 \implies \boxed{\frac{dy}{dx} = -\frac{x}{y}.}$$

$$\frac{d}{dx}y^2 = \left(\frac{d}{dy}y^2\right)\cdot\frac{dy}{dx} = 2y\cdot\frac{dy}{dx}$$

$$f(x) = x^2 \to 2x$$

$$g(x) = y$$

$$\underline{f'(g(x))}\cdot\underline{g'(x)}$$

Exercise 3

For the following equations, find $\dfrac{dy}{dx}$.

1. $x^3 + y^3 = 6xy$
2. $2x^2 + xy - y^2 = 2$
3. $e^x \sin(y) = x + y$

① $x^3 + y^3 = 6xy$

Differentiate both sides wrt x

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx} 6xy.$$

chain

product

$$\Rightarrow \frac{d}{dx}x^3 + \underline{\frac{d}{dx}y^3} = 6\frac{d}{dx}xy$$

$$\Rightarrow 3x^2 + 3y^2\cdot\frac{dy}{dx} = 6\left(\underline{y + x\overbrace{\frac{d}{dx}y}^{\frac{dy}{dx}}}\right)$$

$$\Rightarrow x^2 + y^2\cdot\frac{dy}{dx} = 2y + 2x\frac{dy}{dx}$$

Exercise 3

$$x^2 + y^2 \cdot \frac{dy}{dx} = 2y + 2x\frac{dy}{dx}$$

$$\Rightarrow y^2\frac{dy}{dx} - 2x\frac{dy}{dx} = 2y - x^2$$

$$\Rightarrow \frac{dy}{dx}(y^2 - 2x) = 2y - x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$

② $2x^2 + xy - y^2 = 2.$

Differentiate both sides wrt x:

$$\Rightarrow \frac{d}{dx}(2x^2 + xy - y^2) = \frac{d}{dx}(2)$$

$$\Rightarrow \frac{d}{dx}(2x^2) + \frac{d}{dx}(xy) - \frac{d}{dx}y^2 = 0$$

$$\Rightarrow 4x + \left[y + x\frac{dy}{dx}\right] - 2y\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow x\frac{dy}{dx} - 2y\frac{dy}{dx} = -4x - y$$

$$\Rightarrow \frac{dy}{dx}[x - 2y] = -4x - y$$

$$\Rightarrow \frac{dy}{dx} = \frac{-4x - y}{x - 2y} = \frac{4x + y}{2y - x}$$

(3) $\underbrace{e^{x}\sin(y)}_{\text{product rule + chain rule}} = x+y$

Differentiate both sides wrt x

$$\Rightarrow \frac{d}{dx}\left(e^{x}\sin(y)\right) = \frac{d}{dx}(x+y)$$

$$\Rightarrow e^{x}\sin(y) + e^{x}\cdot\frac{d}{dx}\sin(y) = 1+\frac{dy}{dx}$$

$$\Rightarrow e^{x}\sin(y) + e^{x}\cos(y)\cdot\frac{dy}{dx} = 1+\frac{dy}{dx}$$

$$\Rightarrow e^{x}\cos(y)\frac{dy}{dx} - \frac{dy}{dx} = 1-e^{x}\sin(y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-e^{x}\sin(y)}{e^{x}\cos(y)-1} = \frac{e^{x}\sin(y)-1}{1-e^{x}\cos(y)}$$

Exercise 4

ellipse

Find the equation of the tangent line to the graph of $x^2 + 3y^2 = 16$ at the point $(2, 2)$.

Implicitly defined y a

y value at a

Differentiate both sides wrt x.

$$\Rightarrow \frac{d}{dx}(x^2+3y^2) = \frac{d}{dx}(16)$$

$$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(3y^2) = 0$$

chain rule

$$\Rightarrow 2x + 6y \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{6y} = -\frac{x}{3y}$$

$$\frac{dy}{dx}(2,2) = -\frac{2}{3\cdot 2} = -\frac{1}{3}$$

Tangent line to graph is

$$y = -\frac{1}{3}(x-2)+2 = -\frac{1}{3}x + \frac{2}{3} + 2 = -\frac{1}{3}x + \frac{8}{3}.$$