Week 10: Projection and reflection in 3D

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Linear Transformations in $\mathbb{R}^2$

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation.
  Let M be the matrix representation of T

$$T(\vec{x}) = M\vec{x}$$

Linear Transformations in $\mathbb{R}^2$

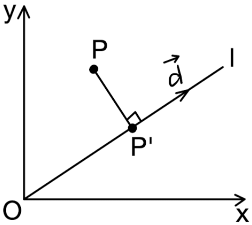
- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation.
  Let M be the matrix representation of T

$$T(\vec{x}) = M\vec{x}$$

- We discussed (found matrix representations) the following maps
  1. Projection
  2. Reflection
  3. Scaling
  4. Rotation
  5. Shear

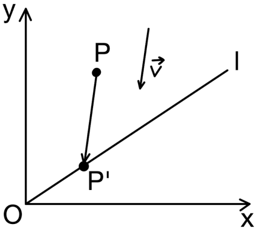
Projections in $\mathbb{R}^2$

Orthogonal projection



$$M = \frac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T$$

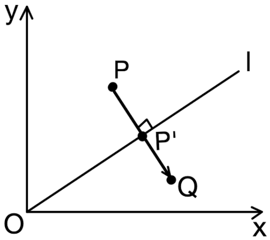
Skew projection



$$M = I_2 - \frac{\vec{v}\vec{n}^T}{\vec{v}\cdot\vec{n}}$$

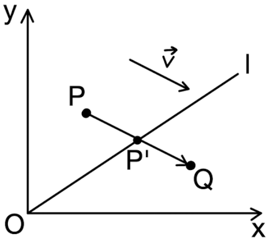
Reflections in $\mathbb{R}^2$

Orthogonal reflection



$$M = \frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_2$$

Skew reflection



$$M = I_2 - \frac{2}{\vec{v}\cdot\vec{n}}\vec{v}\vec{n}^T$$

Scaling and rotation

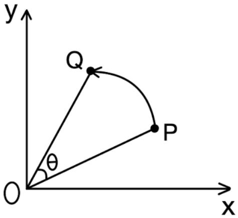
- The scaling $S : \mathbb{R}^2 \to \mathbb{R}^2$ which scales all x-coordinates by a and all y-coordinates by b is defined by

$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

- The counter-clockwise rotation around O over angle θ has matrix

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta\, x - \sin\theta\, y \\ \sin\theta\, x + \cos\theta\, y \end{pmatrix}$$



Shear

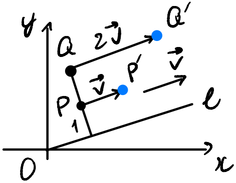
- The **shear** $S : \mathbb{R}^2 \to \mathbb{R}^2$ w.r.t. the line $l : \vec{n} \cdot \vec{x} = 0$ in the direction of **shearing vector** $\vec{v}$ ($\vec{v} \parallel l$) is a map $S : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$S(\vec{x}_0) = \vec{x}_0 + \underbrace{\frac{\vec{n} \cdot \vec{x}_0}{||\vec{n}||}}\vec{v}$$

→ magnitude = distance from $\vec{x}_0$ to l

- S has matrix

$$M = M_{\vec{n},\vec{v}} = I_2 + \frac{1}{||\vec{n}||}\vec{v}\vec{n}^T$$



All fixed points are points on l.

Exercise 1

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be linear with $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\-3\end{pmatrix}$ and $T\begin{pmatrix}1\\4\end{pmatrix} = \begin{pmatrix}18\\-27\end{pmatrix}$.

Find the collection of points $\vec{x}$ that are mapped to the origin.

The matrix of T is $M = [T(\vec{e}_1)\ T(\vec{e}_2)]$. We have

$$\vec{e}_2 = \begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{4}\left(\begin{pmatrix}1\\4\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$T(\vec{e}_2) = \frac{1}{4}T\left(\begin{pmatrix}1\\4\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}\right) = \frac{1}{4}\left(T\begin{pmatrix}1\\4\end{pmatrix} - T\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}4\\-6\end{pmatrix}$$

Hence $M = \begin{pmatrix}2 & 4\\-3 & -6\end{pmatrix}$. Assume $T(\vec{x}) = \vec{0}$. Then

$$\begin{pmatrix}2 & 4\\-3 & -6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Leftrightarrow \begin{pmatrix}2x+4y\\-3x-6y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Leftrightarrow \begin{cases}2x+4y=0\\-3x-6y=0\end{cases}$$

Exercise 1

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be linear with $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\-3\end{pmatrix}$ and $T\begin{pmatrix}1\\4\end{pmatrix} = \begin{pmatrix}18\\-27\end{pmatrix}$.

Find the collection of points $\vec{x}$ that are mapped to the origin.

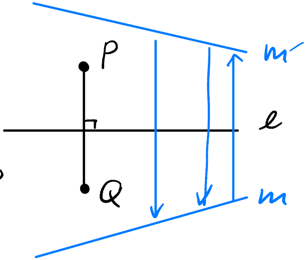
$$\begin{cases} 2x+4y=0 \\ -3x-6y=0 \end{cases} \Leftrightarrow \begin{cases} x+2y=0 \\ x+2y=0 \end{cases} \Leftrightarrow x+2y=0$$

$\therefore$ The points that are mapped to the 0 is the line

$$x+2y=0.$$

Exercise 2

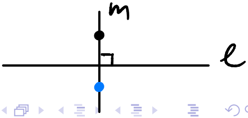
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal reflection through $l : 2x + y = 0$.
Find the line that is mapped to $2x - 4y = \pi^{100}$. (m)



If Q is the image of P under T,
then P is the image of Q under T.

The line that is mapped to $2x - 4y = \pi^{100}$
is the image of $2x - 4y = \pi^{100}$ under T

l & m have direction vectors $\vec{d}_l = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ & $\vec{d}_m = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, which are orthogonal. So the image of m is
$m' = m : 2x - 4y = \pi^{100}$.



Exercise 3

Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection through $m : x - y = 0$.

Which of the following is true $R \circ T$?

(A) $R \circ T$ is an orthogonal (or skew) reflection through some line.

(B) $R \circ T$ is an orthogonal (or skew) onto some line.

projection

(C) None of these is true.

fixed points = a line

$\vec{d}_\ell = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\vec{d}_m = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{d}_\ell \vec{d}_\ell^T = \begin{pmatrix} -1 \\ 2 \end{pmatrix} (-1\ 2) = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$

The matrix of T is

$$M_T = \frac{2}{\|\vec{d}_\ell\|^2} \vec{d}_\ell \vec{d}_\ell^T - I_2 = \frac{2}{5}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -3/5 & -4/5 \\ -4/5 & 3/5 \end{pmatrix}$$

The matrix of R is

$$M_R = \frac{2}{\|\vec{d}_m\|^2} \vec{d}_m \vec{d}_m^T - I_2 = \frac{2}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The matrix of $R \circ T$ is

$$M_{R\circ T} = M_R M_T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -3/5 & -4/5 \\ -4/5 & 3/5 \end{pmatrix} = \begin{pmatrix} -4/5 & 3/5 \\ -3/5 & -4/5 \end{pmatrix}$$

Assume $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ is fixed by $R \circ T$

$$\begin{pmatrix} -4/5 & 3/5 \\ -3/5 & -4/5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{cases} -\frac{4}{5}x + \frac{3}{5}y = x \\ -\frac{3}{5}x - \frac{4}{5}y = y \end{cases}$$

$$\iff \begin{cases} \frac{3}{5}y = \frac{9}{5}x \Rightarrow y = 3x \\ -\frac{3}{5}x = \frac{9}{5}y \Rightarrow x = -3y \end{cases} \Rightarrow y = 3x = 3(-3y) = -9y$$

$$\Rightarrow y = 0,\ x = 0.$$

$\therefore$ The only fixed point is $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0}$. Ans: None of these.

To think about:

$$\begin{cases} \cos\theta = -\frac{4}{5} \\ \sin\theta = -\frac{3}{5} \end{cases} \Rightarrow \theta \simeq 216.87^\circ$$

$$\begin{pmatrix} -4/5 & 3/5 \\ -3/5 & -4/5 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ for } \theta = ?$$

Ans: Yes!

The composition of 2 reflections is a rotation!

Exercise 4

Consider the shear w.r.t. $l : x - y = 0$ in direction $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Sketch the image of the unit square and compute its area.

The shear has matrix $M = I_2 + \frac{1}{\|\vec{n}\|} \vec{v}\,\vec{n}^T$, $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{n} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

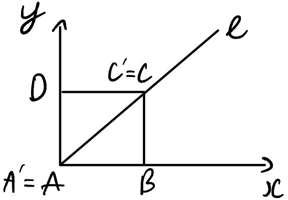
$$M = I_2 + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1 \ \ -1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 - \frac{1}{\sqrt{2}} \end{pmatrix}$$

A & C are fixed because they are on l:

$$A' = A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad C' = C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The images of B & D are

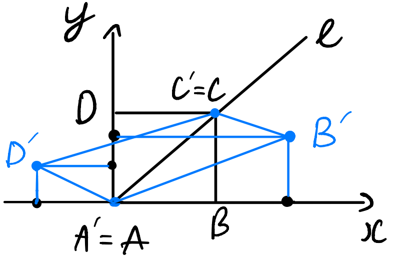
$$B' = \begin{pmatrix} 1 + \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 - \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{pmatrix} \simeq \begin{pmatrix} 1.7 \\ 0.7 \end{pmatrix}$$



$$D' = \begin{pmatrix} 1+\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1-\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1-\frac{1}{\sqrt{2}} \end{pmatrix} \simeq \begin{pmatrix} -0.7 \\ 0.3 \end{pmatrix}$$

Exercise: Verify that

$$\left| \det \left(\vec{A'B'} \; \vec{A'D'} \right) \right| = 1$$



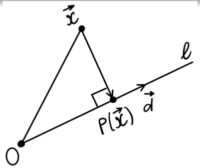
Linear Transformations in $\mathbb{R}^3$

Similar to $\mathbb{R}^2$, we aim to find the **matrix** of following linear transformations

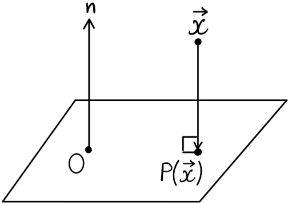
1. Projection
2. Reflection
3. Scaling
4. Rotation
5. Shear

Projections in $\mathbb{R}^3$

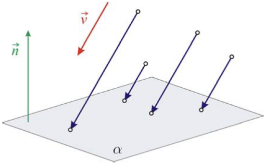
1. Orthogonal projection onto a line through O



2. Orthogonal projection onto a plane through O



3. Skew projection onto a plane through O



Preview of known results

- The line through O with direction $\vec{d}$ has vector equation

$\vec{x} = \vec{x}_0 + t\vec{d} = \vec{0} + t\vec{d} = t\vec{d}$

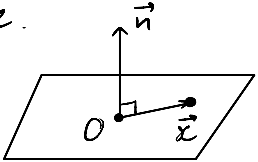
$$\vec{x} = t\vec{d}$$

- The plane through O with normal $\vec{n}$ has ~~vector~~ equation

$$\vec{n} \cdot \vec{x} = 0$$

Let $P = \vec{x}$ be any point on the plane.
Note that $\vec{OP} = \vec{x}$ is orthogonal to $\vec{n}$. So

$\vec{n} \cdot \vec{x} = 0$



Preview of known results

- Useful identity

$$(\vec{a} \cdot \vec{x})\vec{b} = M\vec{x} \text{ with } M = \vec{b}\vec{a}^T$$

Orthogonal projection onto a line

Theorem 1

The matrix of the projection onto the line $l : \vec{x} = t\vec{d}$ is

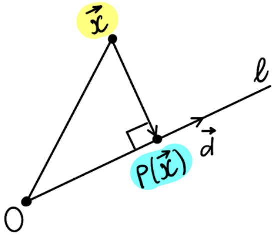
$$M = \frac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T$$

Proof (idea)

$$P(\vec{x}) = \text{proj}_{\vec{d}}(\vec{x})$$

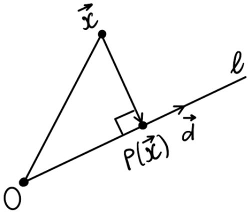
$$= \frac{\vec{x}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\vec{d}$$

$$= \dots = \left(\frac{1}{\|\vec{d}\|^2}\vec{d}\,\vec{d}\right)\vec{x} = Mx$$



Proof

- Let $\vec{x}$ be any point and let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection.



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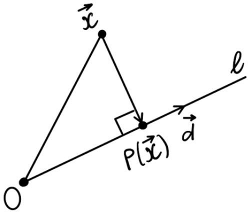


- $P(\vec{x}) =$ projection of $\vec{x}$ onto $\vec{d}$

$$P(\vec{x}) = \text{proj}_{\vec{d}}(\vec{x}) = \frac{\vec{x} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \vec{d} = \frac{1}{||\vec{d}||^2} \vec{d}\vec{d}^T \vec{x}$$

Example 1

(a) Find the matrix of orthogonal projection onto $l : \vec{x} = t \underbrace{\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}}_{\vec{d}}$?

(b) What is the image of the point $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$?

(a) $$M = \frac{1}{\|\vec{d}\|^2} \vec{d}\,\vec{d}^T = \frac{1}{30} \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \begin{pmatrix} 1 & -2 & 5 \end{pmatrix} = \frac{1}{30} \begin{pmatrix} 1 & -2 & 5 \\ -2 & 4 & -10 \\ 5 & -10 & 25 \end{pmatrix}$$

(b) The image is

$$\frac{1}{30} \begin{pmatrix} 1 & -2 & 5 \\ -2 & 4 & -10 \\ 5 & -10 & 25 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/10 \\ 1/5 \\ -1/2 \end{pmatrix}$$

Example 1

(c) Find the image of the line m under P

$$m : \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$$

The image of m is

$$m' : \vec{x} = \frac{1}{30}\begin{pmatrix} 1 & -2 & 5 \\ -2 & 4 & -10 \\ 5 & -10 & 25 \end{pmatrix}\left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}\right)$$

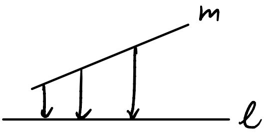
$$= \begin{pmatrix} -1/10 \\ 1/5 \\ -1/2 \end{pmatrix} + \frac{t}{30}\begin{pmatrix} 19 \\ -38 \\ 95 \end{pmatrix} = \begin{pmatrix} -1/10 \\ 1/5 \\ -1/2 \end{pmatrix} + \frac{19t}{30}\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

$$m' : \vec{x} = \begin{pmatrix} -1/10 \\ 1/5 \\ -1/2 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

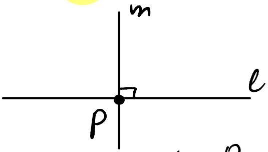
Example 1

(c) Find the image of the line m under P

$$m : \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$$



$m' = l$



$m' = P$

Note: m' is the same as l : $\vec{x} = t \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$

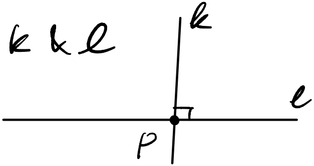
(Assign $t = -\frac{1}{10} \Rightarrow \vec{x} = \begin{pmatrix} -1/10 \\ 1/5 \\ -1/2 \end{pmatrix}$)

Example 1

(d) Show that the entire line k is mapped to a point. Explain this.

$$k : \vec{x} = \begin{bmatrix} 4 \\ 2 \\ 12 \end{bmatrix} + t \begin{bmatrix} 4 \\ -3 \\ -2 \end{bmatrix}$$

k has direction $\vec{d}_k = \begin{pmatrix} 4 \\ -3 \\ -2 \end{pmatrix}$ which is orthogonal to $\vec{d}_\ell = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$ $(\vec{d}_k \cdot \vec{d}_\ell = 0)$. So $k \perp \ell$, which implies that $k' =$ intersection of k & ℓ



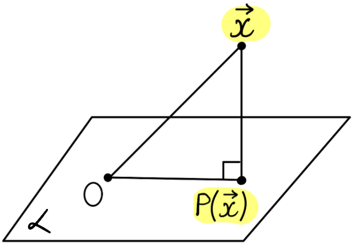
Orthogonal projection onto a plane

Theorem 2

The orthogonal projection $P : \mathbb{R}^3 \to \mathbb{R}^3$ onto the plane $\alpha : \vec{n} \cdot \vec{x} = 0$ has matrix representation

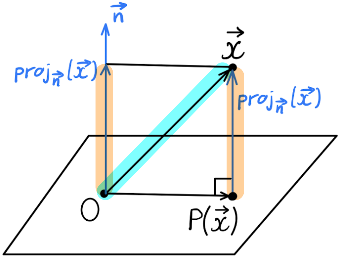
$$M = I_3 - \frac{1}{||\vec{n}||^2} \vec{n}\vec{n}^T$$

$P(\vec{x}) = M\vec{x}$



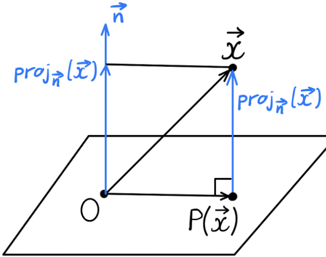
Proof (Sketch)

- Let $\vec{x}$ = any point, $P(\vec{x})$ = projection of $\vec{x}$ onto α.



Proof (Sketch)

- Let $\vec{x}$ = any point, $P(\vec{x})$ = projection of $\vec{x}$ onto α.



- We have

$$
\begin{aligned}
P(\vec{x}) &= \vec{x} - \text{proj}_{\vec{n}}(\vec{x}) = \vec{x} - \frac{\vec{n}\cdot\vec{x}}{\vec{n}\cdot\vec{n}}\vec{n} \\
&= \vec{x} - \frac{1}{||\vec{n}||^2}(\vec{n}\cdot\vec{x})\vec{n} = \vec{x} - \frac{1}{||\vec{n}||^2}\vec{n}\vec{n}^T\vec{x} \\
&= \left(I_3 - \frac{1}{||\vec{n}||^2}\vec{n}\vec{n}^T\right)\vec{x}
\end{aligned}
$$

Example 2

(a) Find the matrix of the projection onto $\alpha : 3x + 2y - z = 0.$ $\vec{n} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

$$M = I_3 - \frac{1}{\|\vec{n}\|^2} \vec{n}\,\vec{n}^T = I_3 - \frac{1}{14}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}\begin{pmatrix} 3 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{14}\begin{pmatrix} 9 & 6 & -3 \\ 6 & 4 & -2 \\ -3 & -2 & 1 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{pmatrix} - \frac{1}{14}\begin{pmatrix} 9 & 6 & -3 \\ 6 & 4 & -2 \\ -3 & -2 & 1 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 5 & -6 & 3 \\ -6 & 10 & 2 \\ 3 & 2 & 13 \end{pmatrix}$$

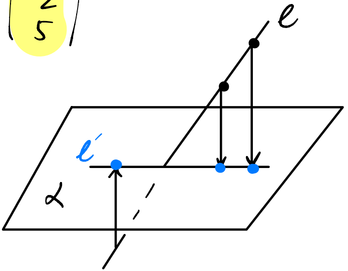
(b) What is the image of the point $\begin{pmatrix} 0 \\ -4 \\ 6 \end{pmatrix}$?

$$\frac{1}{14}\begin{pmatrix} 5 & -6 & 3 \\ -6 & 10 & 2 \\ 3 & 2 & 13 \end{pmatrix}\begin{pmatrix} 0 \\ -4 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$$

(c) What is the image of the line $l : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -4 \\ 6 \end{pmatrix}$?

$$\ell' : \vec{x} = \frac{1}{14}\begin{pmatrix} 5 & -6 & 3 \\ -6 & 10 & 2 \\ 3 & 2 & 13 \end{pmatrix}\left(\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -4 \\ 6 \end{pmatrix}\right)$$

$$\ell' : \vec{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$$



(d) What is the image of the plane $\beta : 4x - 9y - 6z = 7$?

point $\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$

normal $\vec{n}_\beta = \begin{pmatrix} 4 \\ -9 \\ -6 \end{pmatrix}$

β has vector equation

$$\vec{x} = \vec{x}_0 + s\vec{u} + t\vec{v} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ -4 \\ 6 \end{pmatrix} + t\begin{pmatrix} 6 \\ 0 \\ 4 \end{pmatrix}$$

The image of β is

$$\beta' : \vec{x} = \frac{1}{14}\begin{pmatrix} 5 & -6 & 3 \\ -6 & 10 & 2 \\ 3 & 2 & 13 \end{pmatrix}\left(\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + s\underbrace{\begin{pmatrix} 0 \\ -4 \\ 6 \end{pmatrix}}_{\vec{u}} + t\underbrace{\begin{pmatrix} 6 \\ 0 \\ 4 \end{pmatrix}}_{\vec{v}}\right)$$

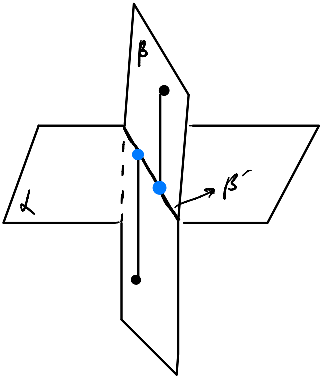
$$= \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + s\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$$

$$\beta' : \vec{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + (s+t)\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + r\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$$

$\boxed{\text{a line}}$

(d) What is the image of the plane $\beta : 4x - 9y - 6z = 7$?

Why $\beta' =$ a line?



$\beta \perp \alpha$

$\beta' =$ a line which is the intersection b/w α & β!

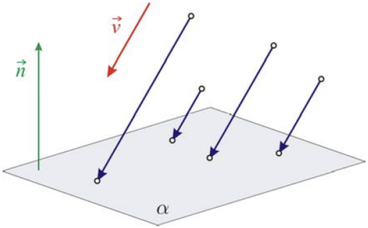
Skew projection onto a plane

Theorem 3

The skew projection $P : \mathbb{R}^3 \to \mathbb{R}^3$ onto the plane $\alpha : \vec{n} \cdot \vec{x} = 0$ in the direction of $\vec{v}$ has matrix representation (α contain O)

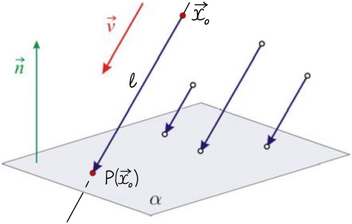
$$M = I_3 - \frac{\vec{v}\vec{n}^T}{\vec{v} \cdot \vec{n}}$$

($\vec{v} \not\parallel \alpha$, or equivalently $\vec{v} . \vec{n} \neq 0$)



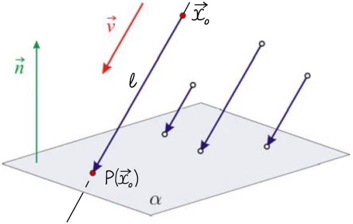
Proof(Sketch)

- Let $\vec{x}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ = any point, $P(\vec{x}_0)$ = projection of $\vec{x}_0$ on α. Let l be the line through $\vec{x}_0$ with direction $\vec{v}$.



Proof(Sketch)

- Let $\vec{x}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ = any point, $P(\vec{x}_0)$ = projection of $\vec{x}_0$ on α. Let l be the line through $\vec{x}_0$ with direction $\vec{v}$.



- $P(\vec{x}_0)$ is the intersection of l and α. So

$$\begin{cases} P(\vec{x}_0) = \vec{x}_0 + t\vec{v} \\ \vec{n} \cdot P(\vec{x}_0) = 0 \end{cases} \Rightarrow \vec{n} \cdot (\vec{x}_0 + t\vec{v}) = 0 \Rightarrow t = -\frac{\vec{n} \cdot \vec{x}_0}{\vec{n} \cdot \vec{v}}$$

Proof (Sketch)

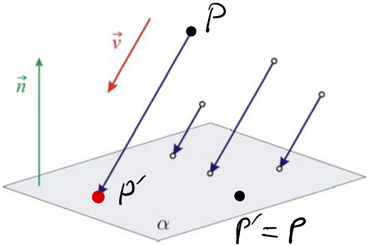
We obtain

$$\begin{aligned}
P(\vec{x}_0) &= \vec{x}_0 - \frac{\vec{n}\cdot\vec{x}_0}{\vec{n}\cdot\vec{v}}\vec{v} \\
&= \vec{x}_0 - \frac{1}{\vec{n}\cdot\vec{v}}(\vec{n}\cdot\vec{x}_0)\vec{v} \\
&= \vec{x}_0 - \frac{1}{\vec{n}\cdot\vec{v}}\vec{v}\vec{n}^T\vec{x}_0 \\
&= \left(I_3 - \frac{1}{\vec{n}\cdot\vec{v}}\vec{v}\vec{n}^T\right)\vec{x}_0
\end{aligned}$$

Question

What points are fixed by the skew projection?

All points on α !



Exercise 5

Let $\alpha : 3x + 2y - z = 0$ and $\vec{v} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$. $\vec{n} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

skew

(a) Find the matrix of the projection onto α along $\vec{v}$.

(b) Find the images of the points $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. (Exercise)

(a) $$M = I_3 - \frac{1}{\vec{v}\cdot\vec{n}} \vec{v}\,\vec{n}^T = I_3 - \frac{1}{6}\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}\begin{pmatrix} 3 & 2 & -1 \end{pmatrix}$$

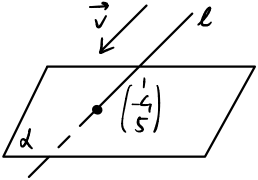
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 3 & 2 & -1 \\ 12 & 8 & -4 \\ 15 & 10 & -5 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 3 & -2 & 1 \\ -12 & -2 & 4 \\ -15 & -10 & 11 \end{pmatrix}$$

(c) Find images of $l : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$ and $m : \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}}_{\text{(Exercise)}}$.

The image of l is

$$l' : \vec{x} = \frac{1}{6} \begin{pmatrix} 3 & -2 & 1 \\ -12 & -2 & 4 \\ -15 & -10 & 11 \end{pmatrix} \left(\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 1 \\ -4 \\ -5 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$$



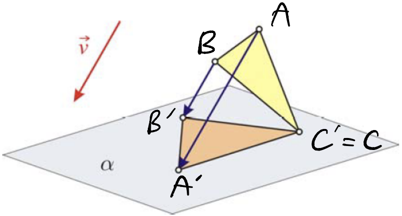
$$\vec{d}_l = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} = \vec{v}$$

(d) Find image $\triangle A'B'C'$ of $\triangle ABC$ with $A = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}, C = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$ (note $C \in \alpha$). Compare areas of $\triangle ABC$ and $\triangle A'B'C'$.

$$\frac{1}{6}\begin{pmatrix} 3 & -2 & 1 \\ -12 & -2 & 4 \\ -15 & -10 & 11 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3 & 5 \\ 3 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ -1 & 1 \\ -2 & 2 \end{pmatrix} = [A' \; B']$$

$$C' = C = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$$



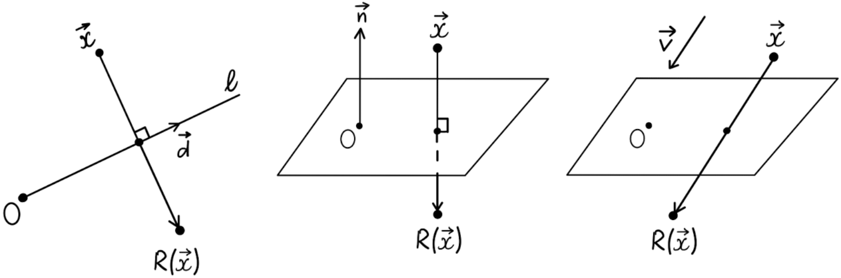
$\alpha : 3x + 2y - z = 0$

Area of $ABC = \frac{1}{2}\|\vec{AB} \times \vec{AC}\|$; (compute it yourself)!

______ $A'B'C' = \frac{1}{2}\|\vec{A'B'} \times \vec{A'C'}\|$

Reflections in $\mathbb{R}^3$

1. Orthogonal reflection through a line
2. Orthogonal reflection through a plane
3. Skew reflection through a plane



Orthogonal reflection through a line

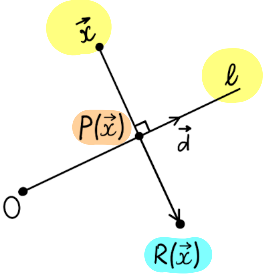
Theorem 4

The orthogonal reflection through the line $l : \vec{x} = t\vec{d}$ has matrix representation

$$M = \frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_3$$

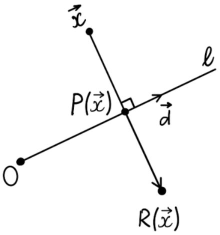
$$P(\vec{x}) = \frac{1}{2}(\vec{x} + R(\vec{x}))$$

$$R(\vec{x}) = 2P(\vec{x}) - \vec{x}$$



Proof(Sketch)

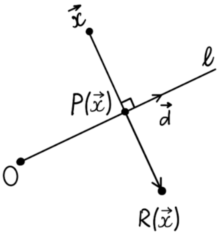
- $\vec{x}$ = any point, $R(\vec{x})$ = image of $\vec{x}$, $P(\vec{x})$ = projection of $\vec{x}$ onto l.



- We knew $P(x) = \dfrac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T\vec{x}$

Proof(Sketch)

- $\vec{x}$ = any point, $R(\vec{x})$ = image of $\vec{x}$, $P(\vec{x})$ = projection of $\vec{x}$ onto l.



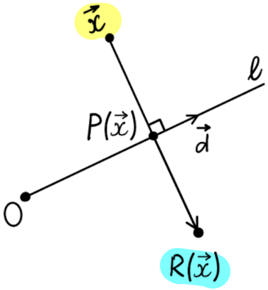
- We knew $P(x) = \dfrac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T\vec{x}$
- $P(\vec{x})$ = midpoint of $\vec{x}$ and $R(\vec{x}) \Rightarrow P(x) = \frac{1}{2}\left(\vec{x} + R(\vec{x})\right)$

$$R(\vec{x}) = 2P(\vec{x}) - \vec{x} = \left(\frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_3\right)\vec{x}$$

Question

What points are fixed the reflection through a line?

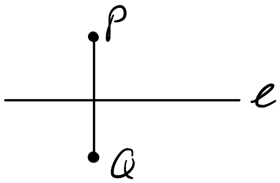
Answers : All points on ℓ.



Example 2

(a) Compute the matrix of the reflection through $l : \vec{x} = t \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$.

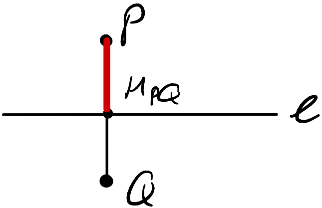
(b) Let $P = \begin{pmatrix} 4 \\ 2 \\ 12 \end{pmatrix}$. Find the reflection Q of P through l .



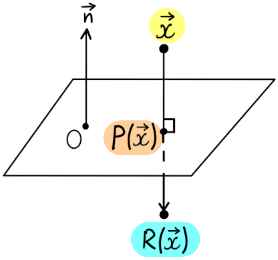
(c) Check that the midpoint M_{PQ} of PQ is on l. Compute $d(P, l)$.

$$M_{PQ} = \frac{1}{2}(P+Q)$$

$$d(P, l) = \| \overrightarrow{PM_{PQ}} \|$$



Orthogonal reflection through a plane



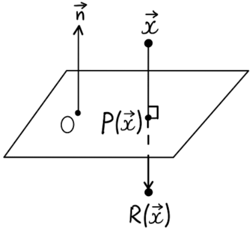
Theorem 5

The orthogonal reflection through the plane $\alpha : \vec{n} \cdot \vec{x} = 0$ has matrix representation

$$M = I_3 - \frac{2}{||\vec{n}||^2} \vec{n}\vec{n}^T$$

Proof

- $\vec{x}$ = any point, $R(\vec{x})$ = image of $\vec{x}$, $P(\vec{x})$ = projection of $\vec{x}$ onto α.

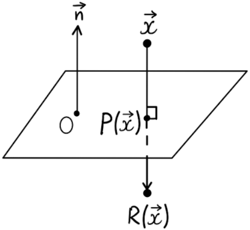


- We knew $P(x)$ from last lecture

$$P(\vec{x}) = \left(I_3 - \frac{1}{||\vec{n}||^2}\vec{n}\vec{n}^T\right)\vec{x} = \vec{x} - \frac{\vec{n}\vec{n}^T}{||\vec{n}||^2}\vec{x}$$

Proof

- $\vec{x}$ = any point, $R(\vec{x})$ = image of $\vec{x}$, $P(\vec{x})$ = projection of $\vec{x}$ onto α.



- We knew $P(x)$ from last lecture

$$P(\vec{x}) = \left(I_3 - \frac{1}{||\vec{n}||^2}\vec{n}\vec{n}^T\right)\vec{x} = \vec{x} - \frac{\vec{n}\vec{n}^T}{||\vec{n}||^2}\vec{x}$$

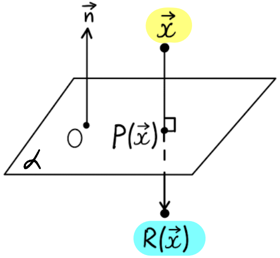
- $P(\vec{x})$ = midpoint of $\vec{x}$ and $R(\vec{x}) \Rightarrow P(x) = \frac{1}{2}\left(\vec{x} + R(\vec{x})\right)$

$$R(\vec{x}) = 2P(\vec{x}) - \vec{x} = \left(I_3 - \frac{2}{||\vec{n}||^2}\vec{n}\vec{n}^T\right)\vec{x}$$

Question

Which points are fixed by the reflection through α?

All points on α.



Example 3

Let $\alpha : 3x + 2y - z = 0$ be a plane in $\mathbb{R}^3$.

(a) Compute the matrix M of the reflection through α.

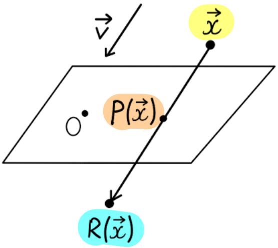
(b) Find the images of the points $\begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

(c) Find the image of $m : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix} + t \begin{bmatrix} 12 \\ -2 \\ 4 \end{bmatrix}$.

(d) Show that the image of $k : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ is itself.

(e) Find the image of the plane $\beta : 4x - 9y - 6z = 7$.

Skew reflection through a plane



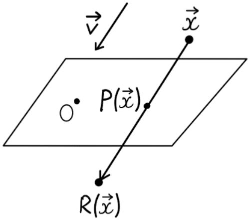
Theorem 6

Let $\alpha : \vec{n} \cdot \vec{x} = 0$ be a plane in $\mathbb{R}^3$. Let $\vec{v}$ be a vector such that $\vec{v} \not\perp \vec{n}$. The skew reflection through α in the direction $\vec{v}$ has matrix representation

$$M = I_3 - \frac{2\vec{v}\vec{n}^T}{\vec{v} \cdot \vec{n}}$$

Remark

- The skew reflection is only *meaningful* when $\vec{v} \not\perp \vec{n}$, that is, $\vec{v} \nparallel \alpha$.

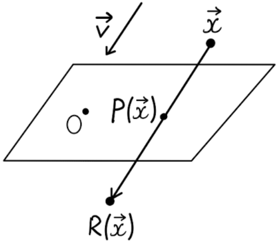


- What happens if $\vec{v} \parallel \alpha$?

Question

Which points are fixed by the skew ~~projection~~ reflection along $\vec{v}$ through α?

All points on α.



Example 4

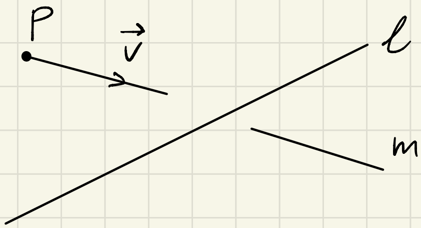
Let $\alpha : 3x + 2y - z = 0$ and let $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

(a) Compute the matrix M of the reflection through α in the direction $\vec{v}$.

(b) Find the images of the points $\begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

(c) Find the image of the line $m : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix} + t \begin{bmatrix} 12 \\ -2 \\ 4 \end{bmatrix}$

Why there is no skew projection onto a line ?



ℓ & m are skew

What is the skew projection of P onto ℓ along $\vec{v}$?

Image $= \ell \cap m =$ empty !

(d) Find the image of the plane $\beta : 4x - 9y - 6z = 7$