Method of Partial Fractions Part 2
Numerical Integration Part 1

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AY 23/24 Trimester 1

Table of contents

Tangent/secant integrals, Partial fractions (1) and (2)

- We learnt how to integrate $\tan^m x \sec^n x$:
  - n is even: take out a copy of $\sec^2 x$, convert rest of $\sec^2 x$ to $\tan^2 x + 1$, sub $u = \tan x$.
  - m is odd and $n \geq 1$: take out one copy of $\sec x \tan x$, convert rest of $\tan^2 x$ to $\sec^2 x - 1$, sub $u = \sec x$.
  - m is even and n is odd: convert all $\tan^2 x$ to $\sec^2 x - 1$, reduce the power of sec by 2 every integration by parts iteration.
- Partial fraction decomposition: only works on **proper** fractions; use long division to convert to proper otherwise.
- Factorization of denominator $Q(x)$:
  - Non-repeated linear factors: one partial fraction for each factor.
  - Repeated (power m) linear factors: for each factor that is repeated, one partial fraction for each power $k = 1, \ldots, m$.

Exercise 3 from last week's lecture

Evaluate the following integrals.

1. $$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\, dx$$

Exercise 3 from last week's lecture

Exercise 4 from last week's lecture

Evaluate the following integrals.

1. $$\int \frac{1}{x^3 + x^2}\, dx$$
2. $$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx$$

Exercise 4 from last week's lecture

Exercise 4 from last week's lecture

Method for Non-repeating Irreducible Quadratic Factors

Definition

A quadratic polynomial $ax^2 + bx + c$ is said to be **irreducible** (in the reals) if $b^2 - 4ac < 0$.

If $Q(x)$ factors into a non-repeated, irreducible factor $ax^2 + bx + c$, then the partial fraction decomposition of $\dfrac{R(x)}{Q(x)}$ must contain

$$\frac{Ax + B}{ax^2 + bx + c},$$

where A and B are constants to be determined.

Example 1

Evaluate $\displaystyle\int \frac{x+1}{x^4+x^2}\,dx$.

Example 1

Exercise 1

Use the substitution $x = au$ to show that

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C.$$

Exercise 2

Evaluate the following integral.

1. $\displaystyle\int \frac{x+2}{x^3+4x}\,dx$

Exercise 2

Not every function has simple antiderivatives

It turns out that not every function has simple antiderivatives. For example,

$$\int_0^1 e^{x^2}\,dx \quad \text{and} \quad \int_0^1 \cos(x^2)\,dx$$

cannot be evaluated exactly because there is no *simple* antiderivative for e^{x^2} and $\cos(x^2)$.

We instead turn to **approximations** to help us get (close to) the answer.

The Midpoint Rule

The **Midpoint Rule** is a Riemann sum, with sample points as the **midpoints** of the subintervals. Let f be a function on the interval $[a, b]$. We use n rectangles, so

$$\Delta x = \frac{b-a}{n}, \quad \text{and} \quad x_i = a + i\Delta x \quad (i \text{ from } 0 \text{ to } n).$$

The n subintervals are $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$. The sample points are the midpoints of these intervals:

$$\overline{x_i} = \frac{x_i + x_{i-1}}{2} = a + \left(\frac{2i-1}{2}\right)\Delta x, \quad (i \text{ from } 1 \text{ to } n).$$

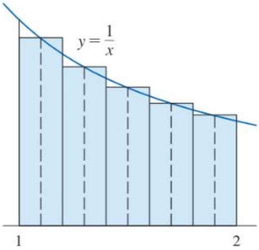
The **Midpoint Rule** M_n is

$$\int_a^b f(x)\,dx \approx M_n = \Delta x\left[f(\overline{x_1}) + f(\overline{x_2}) + \cdots + f(\overline{x_n})\right].$$

Example 2

Use the Midpoint Rule with $n = 5$ to approximate the integral $\int_1^2 \frac{1}{x}\,dx$.

Give your final answer in 6 decimal places.



The Trapezoidal Rule

The **Trapezoidal Rule** is the average of the left Riemann sum L_n (left endpoints as sample points) and the right Riemann sum R_n (right endpoints as sample points). Recall that the left and right endpoints on $[a, b]$ with $x_i = a + i\Delta x$ are

$$\begin{aligned}\text{Left endpoints} &: x_0, x_1, \dots, x_{n-1},\\ \text{Right endpoints} &: x_1, x_2, \dots, x_n.\end{aligned}$$

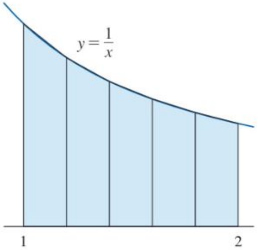
The Trapezoidal Rule T_n is the **average** of L_n and R_n:

$$\begin{aligned}T_n = \frac{L_n + R_n}{2} = \frac{\Delta x}{2}[f(x_0) + f(x_1) + \dots + f(x_{n-1})\\ + f(x_1) + \dots + f(x_{n-1}) + f(x_n)]\end{aligned}$$

$$\implies \int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n)].$$

Example 3

Use the Trapezoidal Rule with $n = 5$ to approximate the integral $\int_1^2 \frac{1}{x}\,dx$.
Give your final answer in 6 decimal places.



Approximations and errors

As with most approximations, we would expect some **error**. The error of the Midpoint Rule is

$$E_M = \int_a^b f(x)\,dx - M_n,$$

and the error of the Trapezoidal Rule is

$$E_T = \int_a^b f(x)\,dx - T_n.$$

Most of the time, we are unable to evaluate the error exactly, because we don't know the exact value of the definite integral. But we know that it cannot exceed a certain value, i.e. it is **bounded** by a certain value.

Error bounds

Theorem

Suppose K is a constant where $|f''(x)| \leq K$ on $[a, b]$. The **magnitude** of errors of the Trapezoidal (E_T) and Midpoint Rule (E_M) have the following upper bounds:

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \leq \frac{K(b-a)^3}{24n^2}.$$

In other words, even if we do not know the exact error of the approximation, we know that the magnitude of the error can **never** exceed a certain number. A helpful inequality that you can use to find the bound K for $|f''(x)|$ is the **triangle inequality**:

Theorem (Triangle Inequality)

For any $a, b \in \mathbb{R}$,

$$|a+b| \leq |a| + |b|.$$

Example 4

1. Find error bounds for the Midpoint and Trapezoidal Rule approximations in Examples 2 and 3.
2. How large should we take n in order to guarantee that the Midpoint and Trapezoidal Rule approximations for $\int_1^2 \frac{1}{x}\, dx$ are accurate to within 0.0001?

Example 4

Exercise 3

1. Use both the Midpoint and Trapezoidal Rule with $n = 10$ to approximate the integral $\displaystyle\int_0^1 \cos(x^2)\, dx$.
2. How large should we take n in order to guarantee that these approximations for this integral are accurate to within 0.0001?

Exercise 3