Week 12: Affine transformations

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Affine transformation - definition

- The map $T : \mathbb{R}^n \to \mathbb{R}^m$ is an **affine transformation** if there exist an $m \times n$ matrix A and a vector $\vec{b} \in \mathbb{R}^m$ such that

$$T(\vec{x}) = A\vec{x} + b$$

- Affine transformation is a generalization of linear transformation. If $\vec{b} = 0$, then we have a linear transformation.

$$T(\vec{x}) = A\vec{x}$$

Find A and $\vec{b}$

- A and $\vec{b}$ can be determined if we have a clear formula for T:

$$T\begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n + b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n + b_2 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n + b_2 \end{pmatrix}$$

- Then $T(\vec{x}) = A\vec{x} + \vec{b}$ with

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Example 1

Which of the following maps are linear transformations? Affine transformations? Or none of these. Find the matrix A and vector $\vec{b}$ of T in $T(\vec{x}) = A\vec{x} + \vec{b}$ for affine transformations.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y^2 + 1 \\ y - x - 2 \end{pmatrix}$

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ y - x \end{pmatrix}$

(c) $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - z + 2y - 10 \\ y - x \\ y + z + 1 \end{pmatrix}$

(d) $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - z + 2y - 10 \\ y - x \end{pmatrix}$

Projection, reflection, shear, rotation

- Linear transformations fixed the origin O.
  For any projection, or reflection, or shear, or rotation to be a linear transformation, the line/plane under consideration must go through O.
- When the line/plane under consideration doesn't go through O, we have an affine transformation.

3-step approach

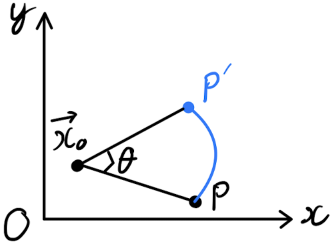
Idea: Shift everything to the origin, do the linear map with respect to the origin, and then shift everything back.

1. Translation by $-\vec{x}_0$, that is, map any point $\vec{x}$ to $\vec{x} - \vec{x}_0$
2. Perform linear transformation by matrix A
3. Translation by $\vec{x}_0$

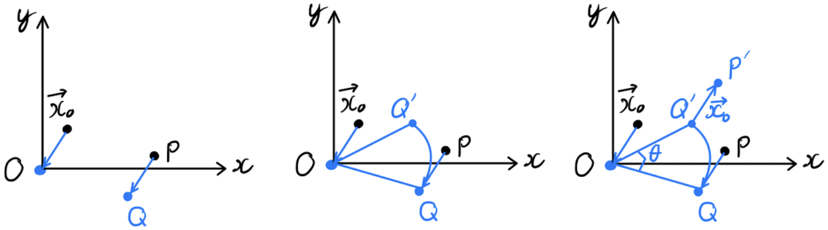
Question: What is the image of $\vec{x}$ when applying the 3-approach algorithm to $\vec{x}$?

Example 2

Assume we want to find the image P' of P by rotation about $\vec{x}_0$ over angle θ.



Example 2: 3-step approach



1. Shift $\vec{x}_0$ to O by the vector $-\vec{x}_0$

$$\vec{x}_0 \mapsto \vec{x}_0 - \vec{x}_0 = \vec{0} \text{ and } P \mapsto P - \vec{x}_0 = Q$$

2. Find the image Q' of Q under rotation about O over θ

$$Q' = AQ \text{ with } A = \text{matrix of rotation}$$

3. Shift everything back by $\vec{x}_0$

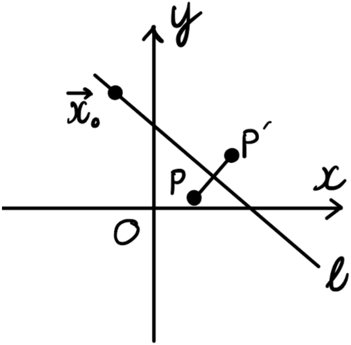
$$P' = Q' + \vec{x}_0$$

Example 2

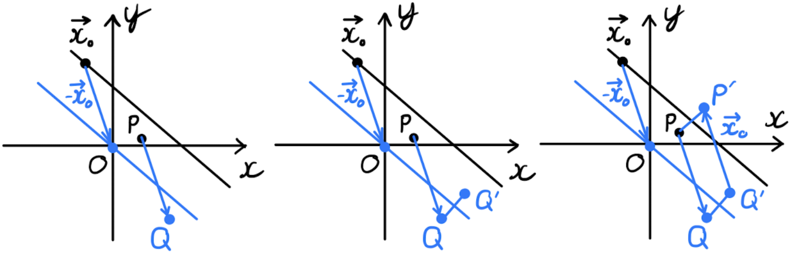
Find the image of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ under rotation about $\vec{x}_0 = \begin{pmatrix} 10 \\ 1 \end{pmatrix}$ over $\theta = 30^\circ$.

Example 3

Assume we want to find the image P' of P by reflection through a line l which goes through $\vec{x}_0$.



Example 3: 3-step approach



1. Shift l to a line through O by $-\vec{x}_0$

$$l \mapsto l' \text{ and } P \mapsto Q$$

2. Find the image Q' of Q under reflection through l'

$$Q' = AQ \text{ with } A = \text{matrix of reflection through } l'$$

3. Shift everything back by $\vec{x}_0$

$$P' = Q' + \vec{x}_0$$

Example 3

Let $l : x + 2y = 3$ be a line in $\mathbb{R}^2$. Find the image of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ under reflection through l.

Comments

Using the above idea, we will find the matrix A and the vector $\vec{b}$ for the cases of projections, reflections, shear, rotations in $\mathbb{R}^2$.

Projection in $\mathbb{R}^2$

Theorem 1

(a) The orthogonal projection onto $\vec{x} = \vec{x}_0 + t\vec{d}$ is

$$T(\vec{x}) = M\vec{x} + (I_2 - M)\vec{x}_0 \text{ with } M = \frac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T$$

(b) The skew projection onto $ax + by = c$ in the direction $\vec{v}$ is

$$T(\vec{x}) = M\vec{x} + \frac{c}{\vec{v}\cdot\vec{n}}\vec{v} \text{ with } M = I_2 - \frac{1}{\vec{v}\cdot\vec{n}}\vec{v}\vec{n}^T$$

Reflection in $\mathbb{R}^2$

Theorem 2

(a) The orthogonal reflection through $\vec{x} = \vec{x}_0 + t\vec{d}$ is

$$T(\vec{x}) = M\vec{x} + (I_2 - M)\vec{x}_0 \text{ with } M = \frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_2$$

(b) The skew projection through $ax + by = c$ in the direction $\vec{v}$ is

$$T(\vec{x}) = M\vec{x} + \frac{c}{\vec{v}\cdot\vec{n}}\vec{v} \text{ with } M = I_2 - \frac{2}{\vec{v}\cdot\vec{n}}\vec{v}\vec{n}^T$$

Example 4

Consider the line $l : x + 3y = 5$ in $\mathbb{R}^2$.

(a) Describe the projection onto l in the form $T(\vec{x}) = A\vec{x} + \vec{b}$.

Find the image of the line $x - y = 1$ under T.

(b) Describe the reflection through l in the form $T(\vec{x}) = A\vec{x} + \vec{b}$.
Find the image of the line $x - y = 1$ under T.

Shear in $\mathbb{R}^2$

Theorem 3

The shear with respect to the line $ax + by = c$ in the direction of shearing vector $\vec{v}$ is given by

$$T(\vec{x}) = M\vec{x} - \frac{c}{||\vec{n}||}\vec{v} \text{ with } M = I_2 + \frac{1}{||\vec{n}||}\vec{v}\vec{n}^T$$

Example 5

Describe the shear with respect to $l : 3x + 4y = 10$ in the direction of the shearing vector $\vec{v} = \begin{pmatrix} 8 \\ -6 \end{pmatrix}$.

Rotation in $\mathbb{R}^2$

Theorem 4

The rotation about the point $\vec{x}_0$ over the angle θ is given by

$$T(\vec{x}) = M\vec{x} + (I_2 - M)\vec{x}_0 \text{ with } M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Example 6

(a) Describe the rotation about $\vec{x}_0 = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ over $\theta = 30°$.

(b) Find the image of the line $x - 2y = 2$.