Lecture 11: Probability Theory

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What is probability?

- Rigorous mathematical theory to analyze events that involve uncertainty
- Almost everything involves uncertainty
- Applications: business, finance, actuarial science, risk management, economics, computer science, quality control, traffic control, and many other areas

Experiments, Sample Spaces and Events

- An **experiment** is a situation with uncertain outcomes.
- A **sample space** of an experiment is the set of all possible outcomes of the experiment.
- An **event** is a subset of the sample space.

Experiments, Sample Spaces and Events

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- A **sample space** of an experiment is the set of all possible outcomes of the experiment.
- An **event** is a subset of the sample space.
- Notations

  Sample space is usually denoted by Ω (pronounce "Omega").

  Events (subsets of Ω) are denoted by capital letters $A, B, C, \ldots$.

Example 1

- Experiment: a commuter passes through 3 traffic lights. At each light, she either stops (s) or continues (c). The sample space is

$$\Omega = \{\text{ccc, ccs, css, csc, sss, ssc, scc, scs}\}$$

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$$A = \{\text{sss, ssc, scc, scs}\}$$

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- Experiment: a commuter passes through 3 traffic lights.
  At each light, she either stops (s) or continues (c).
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$$\Omega = \{\text{ccc}, \text{ccs}, \text{css}, \text{csc}, \text{sss}, \text{ssc}, \text{scc}, \text{scs}\}$$

- Event A: the commuter stops at the 1st light

$$A = \{\text{sss}, \text{ssc}, \text{scc}, \text{scs}\}$$

- What is event B: the commuter stops at the 3rd light?

$$B = \{ccs, css, sss, scs\}$$

Example 2

- Experiment: tossing a coin 3 times.
  The sample space is

$$\Omega = \{\text{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}\}$$

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Example 2

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- Event A: there are exactly 2 heads

$$A = \{\text{HHT, HTH, THH}\}$$

- What is B: there are ≥ 2 heads?

$$B = \{HHT, HTH, THH, HHH\}$$

Exercise 1

Experiment: Choose a letter at random from "probability".

Write down the sample space for this experiment.

Ω = all possible outcomes

$\Omega = \{p, r, o, b, a, i, l, t, y\}$

Remark: Ω doesn't allow repetition of elements.

Exercise 2

Experiment: roll a dice 3 times.

(a) What is the sample space Ω? How many outcomes are there in Ω?

$$\Omega = \{(a,b,c) : a,b,c \in \{1,2,\dots,6\}\}$$

$$|\Omega| = 6 \cdot 6 \cdot 6 = 216$$

(b) Write down the event A that the total score is at least 17.

$$A = \{(a,b,c) : a+b+c \geq 17 \text{ and } a,b,c \in \{1,\dots,6\}\}$$

$$= \{(6,6,5),(6,5,6),(5,6,6),(6,6,6)\}$$

Union, intersection, complement of events

Given events A and B.

- The **union** of A and B is the event $C = A \cup B$.
- The **intersection** of A and B is the event $C = A \cap B$.
  A and B are **disjoint** if $A \cap B = \emptyset$.
- The **complement** $\bar{A}$ of A is the event that A does not occur

$$\bar{A} = \{w \in \Omega : w \notin A\}.$$

Laws of set theory

Given sample space Ω and events A, B, C

- Commutative laws

$$A \cup B = B \cup A \text{ and } A \cap B = B \cap A$$

- Associative laws

$$(A \cup B) \cup C = A \cup (B \cup C) \text{ and } (A \cap B) \cap C = A \cap (B \cap C)$$

- Distributive laws

$$\begin{aligned} (A \cup B) \cap C &= (A \cap C) \cup (B \cap C) \\ (A \cap B) \cup C &= (A \cup C) \cap (B \cup C) \end{aligned}$$

- De Morgan's law (complement interchanges union and intersection)

$$\overline{A \cup B} = \bar{A} \cap \bar{B} \text{ and } \overline{A \cap B} = \bar{A} \cup \bar{B}$$

Inclusion-exclusion principle

- Two sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- Three sets

$$|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$$

- Inclusion-exclusion principle for n sets

$$\left| \cup_{i=1}^{n} A_i \right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1\le i_1<\cdots<i_k\le n} \left| A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k} \right|$$

Example 1 revisited

- Sample space

$$\Omega = \{\text{ccc, ccs, css, csc, sss, ssc, scc, scs}\}.$$

- Event A: the commuter stops at the 1st light.

$$A = \{\text{sss, ssc, scc, scs}\}.$$

- Event B: the commuter stops at the 3rd light.

$$B = \{\text{sss, scs, ccs, css}\}.$$

Example 1 revisited

(i) $A \cup B$: she stops at the 1st light or the 3rd light

$$A \cup B = \{\text{sss,ssc,scc,scs,ccs,css}\}.$$

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(i) $A \cup B$: she stops at the 1st light or the 3rd light

$$A \cup B = \{\text{sss,ssc,scc,scs,ccs,css}\}.$$

(ii) $A \cap B$: she stops both at the 1st light and the 3rd light

$$A \cap B = \{\text{sss,scs}\}.$$

Example 1 revisited

(i) $A \cup B$: she stops at the 1st light or the 3rd light

$$A \cup B = \{\text{sss,ssc,scc,scs,ccs,css}\}.$$

(ii) $A \cap B$: she stops both at the 1st light and the 3rd light

$$A \cap B = \{\text{sss,scs}\}.$$

(iii) $\bar{A}$: she doesn't stop at the 1st light

$$\bar{A} = \{\text{ccc,ccs,css,csc}\}.$$

(iv) $\bar{B}$: she doesn't stop at the 3rd light

$$\bar{B} = \{\text{ccc,csc,ssc,scc}\}$$

(v) By (iii) and (iv)

$$\bar{A} \cup \bar{B} = \{\text{ccc,ccs,css,csc,ssc,scc}\}$$
$$\bar{A} \cap \bar{B} = \{\text{ccc,csc}\}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

(v) By (iii) and (iv)

$$\begin{aligned}\bar{A} \cup \bar{B} &= \{\text{ccc,ccs,css,csc,ssc,scc}\} \\ \bar{A} \cap \bar{B} &= \{\text{ccc,csc}\}\end{aligned}$$

(vi) By (i) and (ii)

$$A \cap B = \{\text{sss,scs}\} \Rightarrow \overline{A \cap B} = \{\text{ccc,ccs,css,csc,ssc,scc}\} = \bar{A} \cup \bar{B}$$
$$A \cup B = \{\text{sss,ssc,scc,scs,ccs,css}\} \Rightarrow \overline{A \cup B} = \{\text{ccc,csc}\} = \bar{A} \cap \bar{B}$$

Probability measure

A probability measure on Ω is a function

$$P : \{\text{subsets of } \Omega\} \to \mathbb{R}$$

$P: \text{events} \to R$

which satisfies

Probability measure

A probability measure on Ω is a function

$$P : \{\text{subsets of } \Omega\} \to \mathbb{R}$$

$P:$ events $\to [0,1]$

which satisfies

(i) $P(\Omega) = 1$.

(ii) $P(A) \geq 0$ for any $A \subset \Omega$.

$A_i \cap A_j = \emptyset$ for any $i \neq j$

(iii) If $A_1, A_2, \ldots$ are mutually disjoint events, then

$$P\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

$A_1 \cap A_2 = \emptyset \Rightarrow P(A_1 \cup A_2) = P(A_1) + P(A_2)$.

Properties of probability measure

Let P a probability measure on sample space Ω, that is

$$P\colon \{\text{events}\} \to [0,1].$$

Then the following hold

1. $P(\emptyset) = 0$
2. $P(\bar{A}) = 1 - P(A)$: $P(A) + P(\bar{A}) = 1$
3. If $A \subset B$, then $P(A) \le P(B)$.
4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
5. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Example 3

A fair coin is thrown twice.

Event A: head on the first toss.

Event B: head on the second toss.

What is the probability the coin lands on head on one of the tosses?

Sample space $\Omega = \{HH, HT, TH, TT\}$

$A = \{HH, HT\}$

$B = \{HH, TH\}$

$A \cap B = \{HH\}$

Event that head lands on one of the tosses $A \cup B$

Sol 1: $A \cup B = \{HH, HT, TH\} \rightarrow P(A \cup B) = 3/4$

Sol 2: $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{2}{4} + \frac{2}{4} - \frac{1}{4} = \frac{3}{4}$

Exercise 3

$\Omega = \{0, 1, \dots, 999\}$

Find the probability of the following events.

(a) A randomly chosen integer $x \in \{0, \dots, 999\}$ is divisible by 11.

(b) A randomly chosen integer $x \in \{0, \dots, 999\}$ is divisible by 13.

(c) A randomly chosen integer $x \in \{0, \dots, 999\}$ is divisible by 11 or 13.

Hint. The number of integers in $\{1, \dots, n\}$ divisible by d is $\lfloor \frac{n}{d} \rfloor$.

integers in $\{0, 1, \dots, n\}$ divisible by d = $1 + \lfloor \frac{n}{d} \rfloor$

$\Omega = \{0, 1, \dots, 999\} \to |\Omega| = 1000$

a) A = numbers that are divisible by 11

$|A| = 1 + \lfloor \frac{999}{11} \rfloor = 91$

$P(A) = \frac{91}{1000}$

b) B = numbers that are divisible by 13

$$|B| = 1 + \left\lfloor \frac{999}{13} \right\rfloor = 77$$

$$P(B) = \frac{77}{1000}$$

c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$A \cap B$ = numbers divisible by $\operatorname{lcm}(11, 13) = 143$

$$|A \cap B| = 1 + \left\lfloor \frac{999}{143} \right\rfloor = 7$$

Hence

$$P(A \cup B) = \frac{91}{1000} + \frac{77}{1000} - \frac{7}{1000} = \frac{161}{1000}$$

Uniform distribution

Theorem 1

Let Ω be finite. The function P defined on the subsets of Ω by

$$P(A) = \frac{|A|}{|\Omega|} \text{ for any } A \subset \Omega$$

is a probability measure on Ω.

Proof (sketch). We need to verify 3 properties

1. $P(\Omega) = 1$
2. $P(A) \geq 0$ for any $A \subset \Omega$
3. If $A_1, A_2, \ldots$ are mutually disjoint, then

$$P\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Example 4

$P : \{events\} \to [0,1]$ s.t. $P(A) = \frac{|A|}{|\Omega|}$

Let $\Omega = \{1, 2, 3\}$. The uniform probability P on Ω is

$$\begin{aligned} P(\emptyset) &= 0, \\ P(\{1\}) &= P(\{2\}) = P(\{3\}) = 1/3, \\ P(\{1,2\}) &= P(\{1,3\}) = P(\{2,3\}) = 2/3, \\ P(\{1,2,3\}) &= 1. \end{aligned}$$

Example 5

Let $\Omega = \{0, 1\}$. Write out the uniform probability P on Ω.

$P : \{\text{events}\} \to [0, 1]$ s.t. $P(A) = \dfrac{|A|}{|\Omega|}$

$P(\emptyset) = 0$

$P(\{0\}) = \dfrac{1}{2}$

$P(\{1\}) = \dfrac{1}{2}$

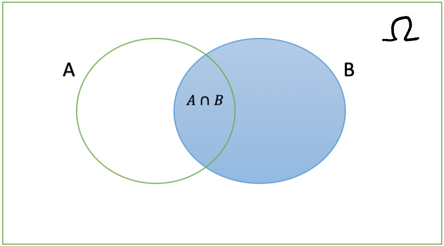
$P(\{0, 1\}) = \dfrac{2}{2} = 1$

Conditional probability

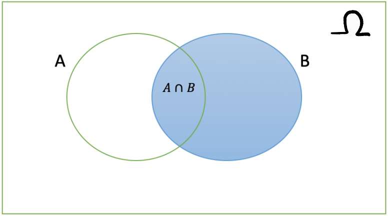
Let A, B be events with $P(B) > 0$.

The **conditional probability of A given B**, denoted $P(A|B)$, is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$



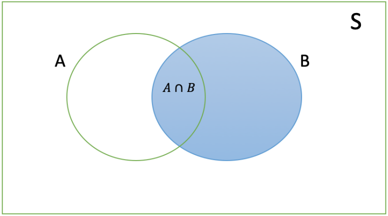
Explanation of conditional probability



Equation $P(A|B) = \frac{P(A \cap B)}{P(B)}$ can be explained as follows.

↓
probability that A happens, given that B happens

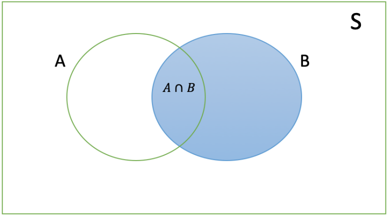
Explanation of conditional probability



Equation $P(A|B) = \frac{P(A \cap B)}{P(B)}$ can be explained as follows.

- It is **given** that B happens $\Rightarrow$ space for *possible outcomes* is B.

Explanation of conditional probability



Equation $P(A|B) = \frac{P(A \cap B)}{P(B)}$ can be explained as follows.

- It is **given** that B happens $\Rightarrow$ space for *possible outcomes* is B.
- A happens only if $A \cap B$ happens.
  $P(A|B) =$ probability of event $A \cap B$ in the sample space B

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Example 6

Roll a fair dice twice. You know that one of the rolls gave the value of 6. What is the probability that the other roll also gave 6?

Intuition: The chance to get 6 in the other roll is $\frac{1}{6}$?

Example 6 solution

The intuition is wrong!

$$\Omega = \{ab : a, b \in \{1, 2, \ldots, 6\}\} \to |\Omega| = 36$$

B = event that one of the rolls is 6

$$= \{61, 62, 63, 64, 65, 66, 16, 26, 36, 46, 56\}$$

A = event that both rolls give 6

$$A \cap B = A = \{66\}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{11/36} = \frac{1}{11}$$

Exercise 4

A bit string of length 4 is generated at random so that each of the 16 bit strings of length four is equally likely.

(a) What is the sample space Ω?

$$\Omega = \{abcd : a, b, c, d \in \{0,1\}\} \Rightarrow |\Omega| = 16$$

(b) What is the probability that it contains at least two consecutive 0's, given that its first bit is 0? $P(A \mid B)$

B = event that 1st bit = 0

$$B = \{0bcd : b, c, d \in \{0,1\}\} \rightarrow |B| = 8$$

A = event that there are ≥ 2 consecutive 0's

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$A \cap B$ = event that there are ≥ 2 consecutive 0's and 1st bit = 0

(1) 00** $\Rightarrow$ 0000, 0001, 0010, 0011

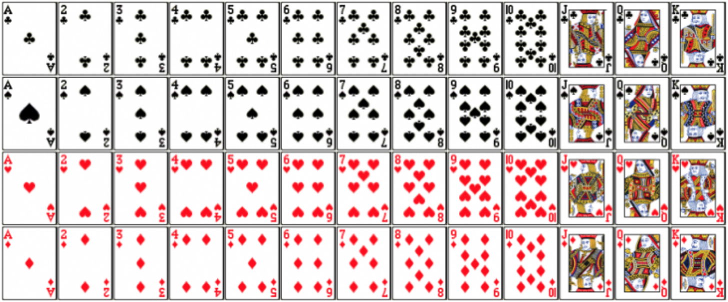
(2) 01** $\Rightarrow$ 0100

$A \cap B = \{0000, 0001, 0010, 0011, 0100\}$

$$\therefore \quad P(A \mid B) = \frac{5/16}{8/16} = \frac{5}{8}.$$

Exercise 5

Given that a bridge player's hand of 13 cards contains at least one ace. What is the probability that it contains exactly one ace?



(standard 52 card deck used for bridge)

Ω = all possible combinations of 13 cards in 52 cards

4 aces　　48 remain.

$$|\Omega| = \binom{52}{13}$$

B = event that 13 cards contain ≥ 1 ace

$\bar{B}$ = 13 cards contain no ace $\rightarrow |\bar{B}| = \binom{48}{13} \rightarrow |B| = \binom{52}{13} - \binom{48}{13}$

A = event that 13 cards contain exactly 1 ace

$A \cap B$ = event that 13 cards contain exactly 1 ace

1 ace from 4 aces

12 cards from 48 cards

$$|A \cap B| = \binom{4}{1}\binom{48}{12}$$

$$\therefore P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\binom{4}{1}\binom{48}{12} / \cancel{\binom{52}{13}}}{\left(\binom{52}{13} - \binom{48}{13}\right) / \cancel{\binom{52}{13}}} \simeq 0.63$$

Independent events

- Two events A and B are **independent** if and only if

$$P(A \cap B) = P(A)P(B). \tag{1}$$

Independent events

- Two events A and B are **independent** if and only if

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- If $P(A) > 0$ and $P(B) > 0$, (1) is equivalent to either

$$P(A|B) = P(A) \text{ or} \tag{2}$$
$$P(B|A) = P(B). \tag{3}$$

Independent events

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- If $P(A) > 0$ and $P(B) > 0$, (1) is equivalent to either

$$P(A|B) = P(A) \text{ or} \tag{2}$$
$$P(B|A) = P(B). \tag{3}$$

- To prove the independence of A and B, we only need to prove one of the equations (1) or (2) or (3).

Explanation of independent events

$P(A|B) = P(A)$ $\qquad$ $P(B|A) = P(B)$

- The independence of A and B means "the information that B occurs does not affect the probability that A occurs, and vice versa".

Explanation of independent events

- The independence of A and B means
  "the information that B occurs does not affect the probability that A occurs, and vice versa".
- Do not use any other definitions of independence such as "A and B have no influence on each other" or "A and B are disjoint". They are simply **incorrect**.

Question

Let A and B be disjoint events. Are A and B independent? If the answer is not, find a counterexample.

Experiment : Flip a coin $\to \Omega = \{H, T\}$

$A =$ event that head comes up $= \{H\} \Rightarrow P(A) = \frac{1}{2}$

$B =$ event that tail comes up $= \{T\} \Rightarrow P(B) = \frac{1}{2}$

$A \cap B = \emptyset \Rightarrow P(A \cap B) = 0 \neq P(A)P(B) = \frac{1}{4}$

$\therefore$ A and B are not independent.

Example 7

A fair dice is rolled two times.

E_1: the 1st roll gives 1.

E_2: the 2nd roll gives 1.

Are E_1 and E_2 independent events?

$P(E_1 \cap E_2) = P(E_1)P(E_2)$

$\Omega = \{ab : a, b \in \{1, 2, \ldots, 6\}\} \to |\Omega| = 36$

$E_1 = \{11, 12, 13, 14, 15, 16\} \Rightarrow P(E_1) = \frac{6}{36} = \frac{1}{6}$

$E_2 = \{11, 21, 31, 41, 51, 61\} \Rightarrow P(E_2) = 1/6$

$E_1 \cap E_2 = \{11\} \to P(E_1 \cap E_2) = \frac{1}{36} = P(E_1)P(E_2)$

$\therefore$ E_1 & E_2 are independent!

Example 8

A number is chosen at random from $\Omega = \{1, 2, \ldots, 9\}$.

A: the number is a prime.

B: the number is smaller than 5.

Are A and B independent? $P(A \cap B) = P(A)P(B)$?

$A = \{2, 3, 5, 7\} \rightarrow P(A) = \frac{4}{9}$

$B = \{1, 2, 3, 4\} \rightarrow P(B) = \frac{4}{9}$

$\Rightarrow P(A)P(B) = \frac{16}{81}$

$A \cap B = \{2, 3\} \rightarrow P(A \cap B) = \frac{2}{9} \neq P(A)P(B)$

$\therefore$ A and B are not independent!