TUTORIAL 10

1) A = event that Joe is a heroin user,

B = event that Joe tests positive

Given: $P(A) = 0.03$, $P(B|A) = 0.95$, $P(\bar{B}|\bar{A}) = 0.9$

a) The prob. Joe tests positive given that he is not a heroin user

$$P(B|\bar{A}) = 1 - P(\bar{B}|\bar{A}) = 0.1$$

b) The prob. that Joe is a heroin user given that he tested positive

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B \cap A)}{P(B \cap A) + P(B \cap \bar{A})}$$

$$= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})}$$

$$= \frac{0.95 \times 0.03}{0.95 \times 0.03 + 0.1 \times (1 - 0.03)} \approxeq 0.23 = 23\%$$

Remark: The test says that compared to random person (3%), Joe has higher chance (23%) of being a heroin user.

c) The prob. Joe is not a heroin user given negative test result:

$$P(\bar{A}|\bar{B}) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})} = \frac{P(\bar{B} \cap \bar{A})}{P(\bar{B} \cap A) + P(\bar{B} \cap \bar{A})}$$

$$P(\bar{A}|\bar{B}) = \frac{P(\bar{B}|\bar{A})P(\bar{A})}{P(\bar{B}|A)P(A) + P(\bar{B}|\bar{A})P(\bar{A})}$$

<u>Given</u>: $P(A) = 0.03$, $P(B|A) = 0.95$, $P(\bar{B}|\bar{A}) = 0.9$

$$P(\bar{A}|\bar{B}) = \frac{P(\bar{B}|\bar{A})\,P(\bar{A})}{P(\bar{B}|A)\,P(A) + P(\bar{B}|\bar{A})\,P(\bar{A})}$$

$$= \frac{0.9 \times 0.97}{(1-0.95)\times 0.03 + 0.9 \times 0.97}$$

$$\approx 0.9987 = 99.87\%$$

<u>Remark</u>: You can be sure that Joe is not a heroin user.

2) A = event that a randomly chosen person has the disease

B = ______________________ tests positive

<u>Given</u>: $P(A) = 10^{-4}$, $P(B|A) = 0.999$, $P(B|\bar{A}) = 0.0002$

a) $$P(A|B) = \frac{P(B|A)\,P(A)}{P(B|A)\,P(A) + P(B|\bar{A})\,P(\bar{A})}$$

$$= \frac{0.999 \times 0.0001}{0.999 \times 0.0001 + 0.0002 \times (1-0.0001)}$$

$$= 0.3331 \approx 33.31\%$$

b) $$P(\bar{A}|\bar{B}) = \frac{P(\bar{B}|\bar{A})\,P(\bar{A})}{P(\bar{B}|\bar{A})\,P(\bar{A}) + P(\bar{B}|A)\,P(A)}$$

$$= \frac{(1-0.0002)(1-0.0001)}{(1-0.0002)(1-0.0001) + (1-0.999)\times 0.0001}$$

$$\approx 0.9999\ldots \approx 100\%$$

3, $X =$ # rolls until a six shows up

Possible values for X are $1, 2, 3, \ldots$

Define $S =$ event of getting six in a roll

$F =$ event of not getting six in a roll $\Rightarrow P(S) = \frac{1}{6}$, $P(F) = \frac{5}{6}$

a, $P(X = x) = P(\underbrace{FF\ldots F}_{x-1 \text{ rolls}}S)$

$$= \left(\frac{5}{6}\right)^{x-1} \cdot \frac{1}{6} = \frac{5^{x-1}}{6^{x-1}} \cdot \frac{1}{6} = \frac{5^{x-1}}{6^x}$$

b) $P(5 \le X \le 10) = P(X=5) + P(X=6) + \ldots + P(X=10)$

$$= \frac{5^4}{6^5} + \frac{5^5}{6^6} + \frac{5^6}{6^7} + \frac{5^7}{6^8} + \frac{5^8}{6^9} + \frac{5^9}{6^{10}}$$

$$\cong 0.3207 = 32.07\%$$

4, $X =$ # hearts in 5 cards drawn randomly from 52 cards

13 hearts

39 other types

a, Possible values for X are $0, 1, 2, 3, 4, 5$.

b) Given $x \in \{0, 1, 2, 3, 4, 5\}$, find $P(X = x)$:

How to choose 5 cards with x hearts

x hearts from 13 hearts

$5-x$ cards from 39 cards

$$P(X = x) = \frac{\binom{13}{x}\binom{39}{5-x}}{\binom{52}{5}}$$

x	$p(x) = P(X=x)$	$F(x) = P(X \le x)$
0	0.2215	0.2215
1	0.4114	0.6329
2	...	...
3	...	...
4	...	...
5	...	0.9999

5) X = # questions answered correctly in 10 questions

→ prob. of passing is $P(X \geq 6) = P(X=6) + \ldots + P(X=10)$

a) For 1 question: $P(\text{correct}) = \frac{1}{3}$, $P(\text{wrong}) = \frac{2}{3}$

We compute $P(X=x) \iff$ getting exactly x correct answers

(1) Choose x questions → $\binom{10}{x}$ choices

(2) probability of these x questions correct & $10-x$ remaining question wrong → $\left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{10-x}$

Hence $P(X=x) = \binom{10}{x}\left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{10-x}$

$$P(X \geq 6) = P(X=6) + P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$\cong 0.0766 = 7.66\%$$

b) For each question: $P(\text{correct}) = \frac{1}{2}$, $P(\text{wrong}) = \frac{1}{2}$

$$P(X=x) = \binom{10}{x}\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{10-x} = \binom{10}{x}\frac{1}{2^{10}}$$

$$P(X \geq 6) = P(X=6) + \dots + P(X=10)$$

$$= \frac{1}{2^{10}}\left(\binom{10}{6} + \binom{10}{7} + \dots + \binom{10}{10}\right)$$

$$\cong 0.377 = 37.7\%$$

6) X = # failures in 9 transmissions (possible values are $0, 1, \dots, 9$)

For a single transmission: $P(F) = 0.05$, $P(S) = 0.95$

The prob of incorrect decoding is

$$P(X \geq 5) = P(X=5) + \dots + P(X=9)$$

Let $x \in \{0, 1, \dots, 9\}$, we compute $P(X=x)$:

$$P(X=x) = \binom{9}{x} 0.05^{x} 0.95^{9-x}$$

Hence

$$P(X \geq 5) = \binom{9}{5}0.05^{5}0.95^{4} + \binom{9}{6}0.05^{6}0.95^{3} + \dots + \binom{9}{9}0.05^{9}$$
$$\cong 3.32 \times 10^{-5}.$$

<u>Remark</u>: If the bit is transmitted only once, the chance of incorrect decoding is 0.05. By transmitting 9 times, the chance is reduced to 3.32×10^{-5}