Q1

(a) $f(x) = (x^4+5)^{77}$

$f'(x) = 77(x^4+5)^{77-1}(x^4+5)'$

$= 77(x^4+5)^{76}(4x^3) = 308x^3(x^4+5)^{76}$

(b) $g(x) = e^{\sin x}$

$g'(x) = e^{\sin x} \cdot (\sin x)' = \cos x \, e^{\sin x}$

(c) $h(\theta) = \ln(\tan(3\theta))$

$h'(\theta) = \frac{1}{\tan(3\theta)} \cdot [\tan(3\theta)]' = \frac{1}{\tan(3\theta)} \cdot \sec^2(3\theta) \cdot (3\theta)'$

$= \frac{3\sec^2(3\theta)}{\tan(3\theta)}$

(d) $f(x) = 5^{x^3}$

$f'(x) = 5^{x^3} \ln 5 (x^3)' = 5^{x^3} \ln 5 \cdot 3x^2 = (3\ln 5) x^2 5^{x^3}$

(e) $u(x) = \sin(\cos(\tan x))$

$u'(x) = \cos(\cos(\tan x)) \cdot (\cos(\tan x))'$

$= \cos(\cos(\tan x)) \cdot -\sin(\tan x) \cdot (\tan x)'$

$= \cos(\cos(\tan x)) \cdot -\sin(\tan x) \cdot \sec^2 x$

$= -\cos(\cos(\tan x)) \cdot \sin(\tan x) \cdot \sec^2 x$

(f) $t(x) = \sqrt{4x-1}$

$t'(x) = [(4x-1)^{\frac{1}{2}}]' = \frac{1}{2}(4x-1)^{-\frac{1}{2}} \cdot (4x-1)'$

$$= 2(4x-1)^{-\frac{1}{2}} = \frac{2}{\sqrt{4x-1}}$$

(g) $r(\theta) = \cos(\theta^2)$

$r'(\theta) = -\sin(\theta^2) \cdot (\theta^2)' = -2\theta \sin(\theta^2)$

(h) $f(t) = e^{2t} \sin(3t)$ product rule + chain rule.

$f'(t) = (e^{2t})' \sin(3t) + (e^{2t})(\sin(3t))'$

$= 2e^{2t} \sin(3t) + \cos(3t) \cdot 3 \cdot e^{2t}$

$= e^{2t}(2\sin(3t) + 3\cos(3t))$

(i) $G(z) = (1-4z)^2 (z^2-2z+5)^4$

$G'(z) = [(1-4z)^2]' (z^2-2z+5)^4 + (1-4z)^2 [(z^2-2z+5)^4]'$

$= 2(1-4z)\underbrace{(1-4z)'}_{-4}(z^2-2z+5)^4$

$+ (1-4z)^2 4(z^2-2z+5)^3 \underbrace{(z^2-2z+5)'}_{2z-2}$

$= -8(1-4z)(z^2-2z+5)^4 + 4(2z-2)(1-4z)^2(z^2-2z+5)^3.$

$= -8(1-4z)(z^2-2z+5)^3(5z^2-7z+6)$

(j) $g(t) = \frac{1}{(2t+1)^3} = (2t+1)^{-3}$

$g'(t) = -3(2t+1)^{-3-1} (2t+1)'$

$= -3(2t+1)^{-4} \cdot 2 = -6(2t+1)^{-4} = -\frac{6}{(2t+1)^4}$

(k) $f(x) = \frac{1}{\sqrt[3]{x^2-1}} = (x^2-1)^{-\frac{1}{3}}$

$f'(x) = -\frac{1}{3}(x^2-1)^{-\frac{1}{3}-1} (x^2-1)' = -\frac{2x}{3(x^2-1)^{4/3}}$

(d) $S(t)=\sqrt{\dfrac{1+\sin t}{1+\cos t}} = \left(\dfrac{1+\sin t}{1+\cos t}\right)^{\frac{1}{2}}$.

$$S'(t)= \frac{1}{2}\left(\frac{1+\sin t}{1+\cos t}\right)^{-\frac{1}{2}} \left(\frac{1+\sin t}{1+\cos t}\right)'$$

$$= \frac{1}{2}\left(\frac{1+\sin t}{1+\cos t}\right)^{-\frac{1}{2}} \cdot \frac{(1+\sin t)'(1+\cos t)-(1+\sin t)(1+\cos t)'}{(1+\cos t)^2}$$

$$= \frac{1}{2}\left(\frac{1+\sin t}{1+\cos t}\right)^{-\frac{1}{2}} \cdot \frac{\cos t(1+\cos t)+(1+\sin t)\sin t}{(1+\cos t)^2}$$

$\boxed{\sin^2 t+\cos^2 t=1}$

$$= \frac{1}{2}\left(\frac{1+\cos t}{1+\sin t}\right)^{\frac{1}{2}} \cdot \frac{\cos t+\sin t+1}{(1+\cos t)^2}.$$

$$= \frac{1+\cos t+\sin t}{2\sqrt{1+\sin t}\cdot(1+\cos t)^{3/2}}.$$

Q2: show $\dfrac{d}{dx}x^n=nx^{n-1}$

let $y=x^n$ take "ln" on both sides.

$\ln y=\ln x^n$ $\;= n\ln x$ take derivative on both sides.

$\dfrac{1}{y}\cdot\dfrac{dy}{dx}=\dfrac{1}{x^n}\cdot nx^{n-1} \Rightarrow \dfrac{1}{y}\dfrac{dy}{dx}=n\cdot\dfrac{1}{x}.$

$\dfrac{dy}{dx}=\underbrace{\dfrac{y}{x^n}}_{=1}\cdot nx^{n-1} \Rightarrow \dfrac{dx^n}{dx}=nx^{n-1}$

$\dfrac{dy}{dx}=\dfrac{yn}{x}=\dfrac{x^n n}{x}=nx^{n-1}$

Q3. (a) $x^2 - 4xy + y^2 = 4$

$$2x - 4(xy)' + 2y\frac{dy}{dx} = 0$$

$$2x - 4\left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0.$$

$$2x - 4y - 4x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2y - 4x) = 4y - 2x$$

$$\frac{dy}{dx} = \frac{4y-2x}{2y-4x} = \frac{2y-x}{y-2x}.$$

(b) $x^4 + x^2y^2 + y^3 = 5$

$$4x^3 + 2xy^2 + 2x^2y\cdot\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0.$$

$$\frac{dy}{dx}(2x^2y + 3y^2) = -(4x^3 + 2xy^2)$$

$$\frac{dy}{dx} = -\frac{4x^3 + 2xy^2}{2x^2y + 3y^2} = -\frac{2x(2x^2+y^2)}{2x^2y+3y^2}.$$

(c) $x^3 - xy^2 + y^3 = 1$

$$3x^2 - \left(y^2 + 2yx\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0.$$

$$\frac{dy}{dx}(3y^2 - 2yx) = y^2 - 3x^2$$

$$\frac{dy}{dx} = \frac{y^2-3x^2}{3y^2-2yx} = \frac{y^2-3x^2}{y(3y-2x)}$$

(d) $\sin(x+y) = \cos x + \cos y$.

$\cos(x+y)(x+y)' = -\sin x - \sin y \frac{dy}{dx}$.

$\cos(x+y)(1+\frac{dy}{dx}) = -\sin x - \sin y \frac{dy}{dx}$.

$\cos(x+y) + \cos(x+y)\frac{dy}{dx} + \sin x + \sin y \frac{dy}{dx} = 0$.

$\frac{dy}{dx}(\cos(x+y) + \sin y) = -\sin x - \cos(x+y)$.

$$\frac{dy}{dx} = -\frac{\sin x + \cos(x+y)}{\cos(x+y) + \sin y}.$$

(e) $\tan(x-y) = 2xy^3 + 1$

$\sec^2(x-y)(x-y)' = 2y^3 + 2x \cdot 3y^2 \cdot \frac{dy}{dx}$

$\sec^2(x-y)(1-\frac{dy}{dx}) = 2y^3 + 6xy^2\frac{dy}{dx}$

$(6xy^2 + \sec^2(x-y))\frac{dy}{dx} = -2y^3 + \sec^2(x-y)$

$$\frac{dy}{dx} = \frac{-2y^3 + \sec^2(x-y)}{6xy^2 + \sec^2(x-y)}.$$

(f) $xy = \sqrt{x^2+y^2} = (x^2+y^2)^{\frac{1}{2}}$.

$y + x\frac{dy}{dx} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(x^2+y^2)'$

$y + x\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+y^2}}(2x + 2y\frac{dy}{dx})$.

$\left(x - \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}}\right)\frac{dy}{dx} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} - y$

$$\frac{dy}{dx} = \frac{\frac{x}{\sqrt{x^2+y^2}} - y}{x - \frac{y}{\sqrt{x^2+y^2}}} = \frac{x - y\sqrt{x^2+y^2}}{x\sqrt{x^2+y^2} - y}$$

$$= \frac{y\sqrt{x^2+y^2} - x}{y - x\sqrt{x^2+y^2}}.$$

Q4. (a) $y=2^x$ at $(0,1)$ $\quad (a,b) \Rightarrow y-b = k(x-a)$

$\uparrow$ slope. solved by differentiation.

$$\frac{dy}{dx} = y' = 2^x \ln 2 = 2^0 \ln 2 = \ln 2.$$

$$y-1 = \ln 2(x-0) \Rightarrow y = (\ln 2)x + 1$$

(b) $y = \sqrt{1+x^3}$ at $(2,3)$

$$\frac{dy}{dx} = y' = \frac{1}{2}(1+x^3)^{-\frac{1}{2}} \cdot 3x^2 = \frac{3}{2}x^2 \frac{1}{\sqrt{1+x^3}} = \frac{3}{2} \cdot 4 \cdot \frac{1}{\sqrt{9}} = 2.$$

$$y-3 = 2(x-2) \Rightarrow y = 2x-1$$

(c) $y = \frac{2}{1+e^{-x}}$ at $(0,1)$

$$\frac{dy}{dx} = y' = \frac{\cancel{(2)'(1+e^{-x})} - 2(1+e^{-x})'}{(1+e^{-x})^2}$$

$$= \frac{-2(0-e^{-x})}{(1+e^{-x})^2} = \frac{2e^{-x}}{(1+e^{-x})^2} = \frac{2}{4} = \frac{1}{2}$$

$$y-1 = \frac{1}{2}(x-0) \Rightarrow y = \frac{1}{2}x+1$$

(d) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ at $(-3\sqrt{3}, 1)$

$$\cancel{\tfrac{2}{3}}x^{-\frac{1}{3}} + \cancel{\tfrac{2}{3}}y^{-\frac{1}{3}} \cdot \frac{dy}{dx} = 0$$

$$(-3\sqrt{3})^{-\frac{1}{3}} = (-3 \cdot 3^{\frac{1}{2}})^{-\frac{1}{3}} = (-3^{\frac{3}{2}})^{-\frac{1}{3}} = -3^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\frac{(-3^{\frac{3}{2}})^{-\frac{1}{3}}}{1} = 3^{-\frac{1}{2}} = \boxed{\frac{1}{\sqrt{3}}}$$

$$y - 1 = \frac{1}{\sqrt{3}}(x + 3\sqrt{3})$$

$$y = \frac{1}{\sqrt{3}}x + 3 + 1 = \frac{1}{\sqrt{3}}x + 4$$

(e) $x^2y^2 = (y+1)^2(4-y^2)$ at $(2\sqrt{3}, 1)$

$$2xy^2 + 2x^2y\frac{dy}{dx} = 2(y+1)\frac{dy}{dx}(4-y^2) + (y+1)^2\left(0 - 2y\frac{dy}{dx}\right)$$

$$2 \cdot 2\sqrt{3} \cdot 1 + 2(2\sqrt{3})^2\frac{dy}{dx} = 2(2)\frac{dy}{dx}(4-1) + 2^2\left(-2\frac{dy}{dx}\right)$$

$$4\sqrt{3} + 24\frac{dy}{dx} = 12\frac{dy}{dx} - 8\frac{dy}{dx}$$

$$20\frac{dy}{dx} = -4\sqrt{3} \Rightarrow \frac{dy}{dx} = -\frac{4\sqrt{3}}{20} = -\frac{\sqrt{3}}{5}$$

$$y - 1 = -\frac{\sqrt{3}}{5}(x - 2\sqrt{3})$$

$$y = -\frac{\sqrt{3}}{5}x + \frac{11}{5}$$