# MAT 140

## Linear Algebra and Affine Geometry

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**Fall 2017**

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## 1. Points and Vectors

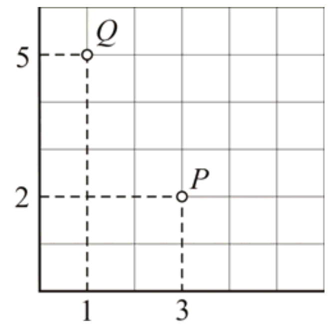
In this course we deal primarily with 2 dimensional and 3 dimensional Euclidean space. The two basic types of objects we will use to describe these spaces are points and vectors.

### Points

**Points** are identified in the usual way with reference to an origin and coordinate axes:

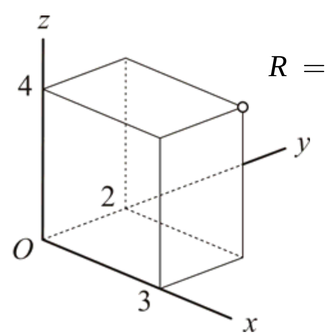
2D notation: $P = (x, y)$

3D notation: $P = (x, y, z)$




**Example 1**: $P = (3, 2)$ and $Q = (1, 5)$ (see picture )

Points in 3D are trickier to graph on a *two*-dimensional piece of paper:




$R = (3, 2, 4)$

**Note**: Recall from high school that we can compute the **midpoint** between two points, by averaging the corresponding coordinates:

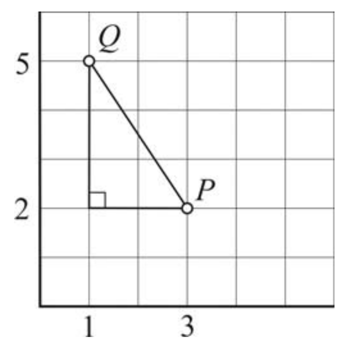
In the 2D case:

$M = \left(\frac{a+A}{2}, \frac{b+B}{2}\right)$ is the midpoint between points $P = (a, b)$ and $Q = (A, B)$

In the 3D case:

$M = \left(\frac{a+A}{2}, \frac{b+B}{2}, \frac{c+C}{2}\right)$ is the midpoint between points $P = (a, b, c)$ and $Q = (A, B, C)$

**Note 2**: We can compute the **distance** between two points, using the theorem of Pythagoras:




$$\text{dist}(P,Q) = \sqrt{(3-1)^2 + (2-5)^2} = \sqrt{4+9} = \sqrt{13}$$

In general:

In 2D when $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ the distance between the points is

$$\text{dist}(P,Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

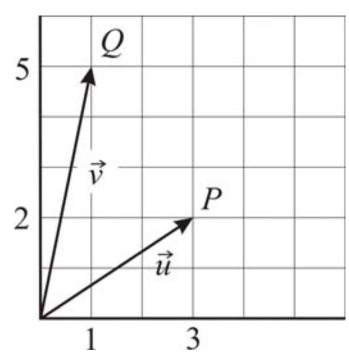
In 3D when $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ the distance between the points is

$$\text{dist}(P,Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

## Vectors

Vectors are different objects than points. They are directed line segments anchored at an origin. Although there is a close connection between points and vectors it is important to be clear that they are different animals. We will make a clear distinction in notation between them. Only later will we change to a purely vector notation of both points and vectors, and the distinction will become less pronounced, but even later, when we use a vector notation with an extra dimension, the difference will reemerge. For now we will use the following distinct notation for points and vectors: $(a, b)$ and $\begin{bmatrix} a \\ b \end{bmatrix}$ (in the 2D case).

**Example 2**: The vectors $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ are shown. Notice the relationship between the vectors $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ and the points $P = (3, 2)$ and $Q = (1, 5)$.




The points are the *endpoints* of the vectors, and the origin is the starting point of the vectors.

In general:

The vector $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ [ or $\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in 3D ] is the directed line segment starting at the origin and ending at the point $P = (a, b)$ [ or $P = (a, b, c)$ in 3D] .

One could also indicate this vector as $\overrightarrow{OP}$ .

## Addition of vectors

**Algebraically**: The vectors $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ can be added as follows

$$\vec{u} + \vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

Notice that the corresponding coordinates were simply added. In general we define the sum of two vectors as follows:

2D

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} a + A \\ b + B \end{bmatrix}$$

3D

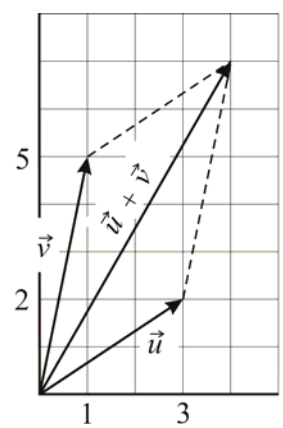
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} a + A \\ b + B \\ c + C \end{bmatrix}$$

**Geometrically**:

**Example 3**: The sum

$$\vec{u} + \vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

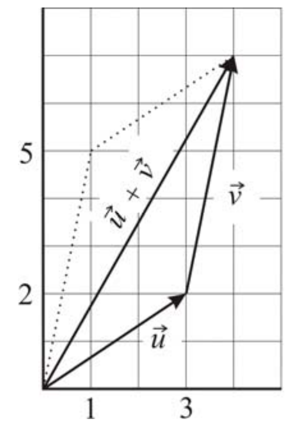
is shown in the picture



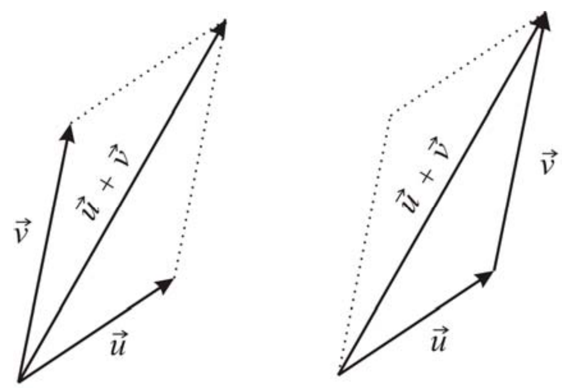

Notice that the **sum** of the two vectors forms the main diagonal of the parallelogram with $\vec{u}$ and $\vec{v}$ as two of its sides.

Strictly speaking vectors are always anchored at the origin, by definition, but sometimes it is useful to see a vector as just having a direction and a length and … ‘floating’ freely.

Using ‘floating’ vectors, putting them *head to tail*, we have another way of looking at vector addition (see picture).




This is how vector addition works, geometrically, in general (in 2D, 3D etc.): the sum $\vec{u}+\vec{v}$ of the two vectors $\vec{u}$ and $\vec{v}$ forms the main diagonal of the parallelogram with $\vec{u}$ and $\vec{v}$ as two of its sides; it is the vector starting at the ‘origin’ and ending at the other corner of the parallelogram:

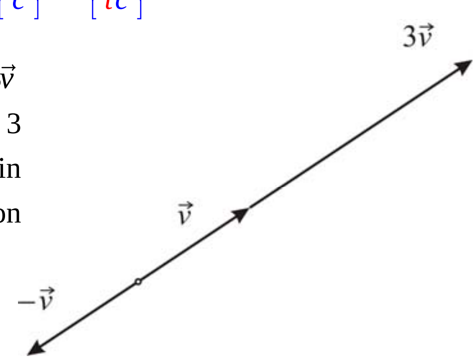



## Scalar multiplication

We can multiply (scale) a vector by any real number.

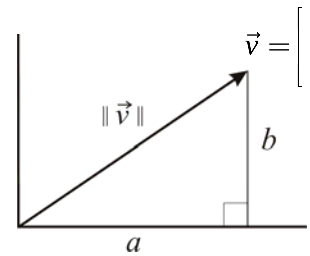
**Algebraically** we have $t\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ta \\ tb \end{bmatrix}$ and $t\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} ta \\ tb \\ tc \end{bmatrix}$

**Geometrically** it is a true scaling: The vector $3\vec{v}$ is the vector in the same direction as $\vec{v}$ but 3 times as long. The vector $-3\vec{v}$ is the vector in the opposite direction of $\vec{v}$ (i.e. in the direction of $-\vec{v}$ ) but 3 times as long.

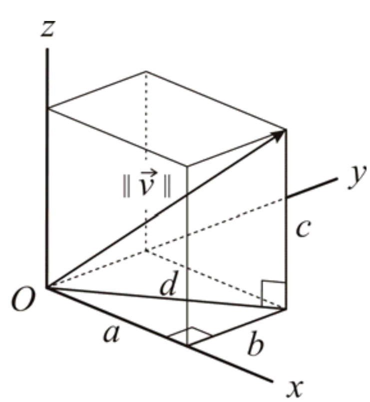


The **length** of a vector $\vec{v}$ is indicated by $\|\vec{v}\|$.

When $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ then, algebraically we have, $\|\vec{v}\| = \sqrt{a^2 + b^2}$ which follows immediately, from geometry, by the theorem of Pythagoras.




When $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ then $\|\vec{v}\| = \sqrt{a^2 + b^2 + c^2}$ which follows immediately from *two* applications of the theorem of Pythagoras:




Let $d$ be the length of the diagonal of the base triangle, then

$$a^2 + b^2 = d^2$$

In the vertical triangle we have

$$\|\vec{v}\|^2 = d^2 + c^2$$

Which combined gives us indeed

$$\|\vec{v}\|^2 = a^2 + b^2 + c^2$$

**Example**: If $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ then $\|\vec{v}\| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$

### Length of a 'scaled' vector $\|t\vec{v}\|$

The length of a scalar multiple of a vector is easily found by

$$\|t\vec{v}\| = |t| \cdot \|\vec{v}\|$$

Note that the absolute values are essential when $t$ is a negative scalar, e.g.

$$\| -3\vec{v} \| = |-3| \cdot \| \vec{v} \| = 3 \| \vec{v} \| .$$

It would be a mistake is to leave out the absolute values: $\| -3\vec{v} \| \neq -3 \| \vec{v} \|$

In 3D the computation would go as follows (2D goes similarly)

$$\| t\vec{v} \| = \sqrt{(ta)^2 + (tb)^2 + (tc)^2} = \sqrt{t^2(a^2 + b^2 + c^2)} = |t| \sqrt{a^2 + b^2 + c^2} = |t| \cdot \| \vec{v} \|$$

At times it is useful to work with a vector with length 1. Such a vector is called a **unit vector**.

Given a vector $\vec{v} \neq \vec{0}$ it is always possible to create a vector $\hat{v}$ that points in the same direction as $\vec{v}$ but has unit length, by scaling the vector $\vec{v}$ as follows

$$\hat{v} = \frac{1}{\| \vec{v} \|} \vec{v}$$

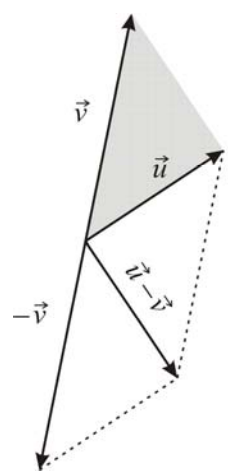
**Example**: $\hat{v} = \frac{1}{\| \vec{v} \|} \vec{v} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

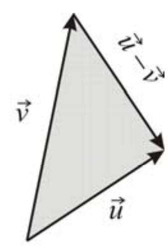
That the scaled vector $\hat{v}$ has unit length is clear from

$$\| \hat{v} \| = \left\| \frac{1}{\| \vec{v} \|} \vec{v} \right\| = \left| \frac{1}{\| \vec{v} \|} \right| \| \vec{v} \| = \frac{1}{\| \vec{v} \|} \| \vec{v} \| = 1$$

## The difference of two vectors

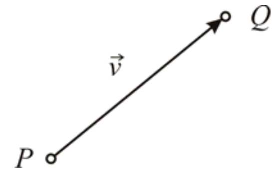
The **difference** of two vectors $\vec{u} - \vec{v}$ also has a nice geometrical interpretation. In particular when we consider 'floating' vectors. The first picture illustrates $\vec{u} - \vec{v}$ as the sum $\vec{u} + (-\vec{v})$, whereas the second shows the same vector but now sitting between the end points of the two vectors $\vec{u}$ and $\vec{v}$ (notice its direction).



## Points *and* vectors

We can translate a point by a given vector.




We will write this shift of a point to a new point as follows:

$$Q = P + \vec{v}$$

Thus we have defined the addition of a point and a vector. (B.t.w. we let $\vec{v} + P = P + \vec{v}$ . )
Note that we use here $\vec{v}$ as a floating vector.

We can define the **difference of two *points*** as a vector

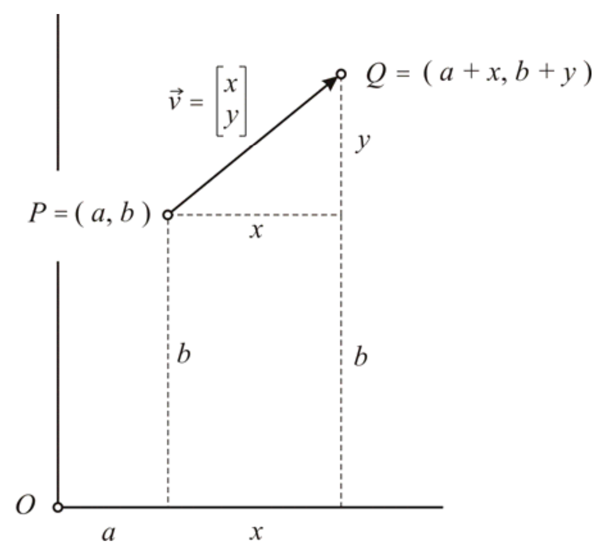
$$\vec{v} = Q - P$$

Algebraically we have:

(2D) If $P = (a, b)$ and $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

then
$$\begin{aligned} Q &= P + \vec{v} \\ &= (a, b) + \begin{bmatrix} x \\ y \end{bmatrix} \\ &= (a + x, b + y) \end{aligned}$$

which agrees with our geometric picture.




Furthermore $\quad Q - P = (a + x, b + y) - (a, b) = \begin{bmatrix} x \\ y \end{bmatrix} = \vec{v}$

(3D) If $P = (a, b, c)$ and $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then

$$Q = P + \vec{v} = (a, b, c) + \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (a + x, b + y, c + z)$$

**A note about the notations of 2D and 3D space**: $\mathbb{R}^n$ vs $\mathbb{E}^n$

We will use $\mathbb{R}^2$ (or $\mathbb{R}^3$) to indicate **both** the space of 2D (or 3D) **points** and the space of 2D (or 3D) **vectors**.

Some authors prefer to use $\mathbb{R}^2$ (or $\mathbb{R}^3$) only when referring to the space of vectors (all of whom are anchored at the origin, i.e. not floating), and may use $\mathbb{E}^2$ (or $\mathbb{E}^3$) when referring to the space of points (the $\mathbb{E}$ here stands for **E**uclidean space.) It may feel like an irrelevant or subtle distinction, but mathematicians can be insistent on proper notation: $\mathbb{R}^n$ for vector spaces, $\mathbb{E}^n$ for Euclidean spaces (of points) and they may even refer to a combination of the two as Affine space $\mathbb{A}^n$ ( The precise definition of such spaces we will not need at this point ) .

At times we may use the second notation, $\mathbb{E}^2$ (or $\mathbb{E}^3$). For example if we want to emphasize that we are more interested in *points* we might refer to the space as $\mathbb{E}^2$ (or $\mathbb{E}^3$).

But by and large when we talk about $\mathbb{R}^2$ (or $\mathbb{R}^3$) we mean **both** the Euclidean space of points and the space of vectors. In fact in this course we will happily be using the wonderful and useful symbiosis of both spaces.

**The TI-Nspire**

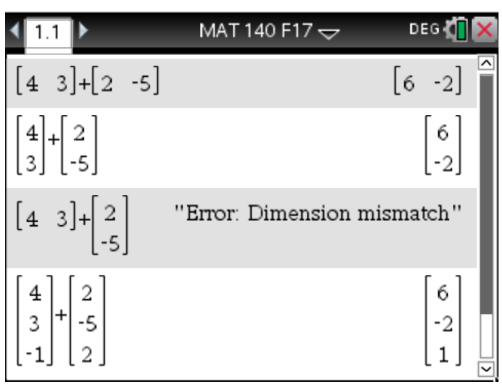
The TI-Nspire doesn't deal with points, just vectors. Hence to perform additions like

$$\begin{aligned} Q &= P + \vec{v} \\ &= (4,3) + \begin{bmatrix} 2 \\ -5 \end{bmatrix} \\ &= (4+2,\ 3-5) = (6,-2) \end{aligned}$$

or

$$\begin{aligned} Q &= P + \vec{v} \\ &= (4,3,-1) + \begin{bmatrix} 2 \\ -5 \\ 2 \end{bmatrix} \\ &= (4+2,\ 3-5,\ -1+2) = (6,-2,\ 1) \end{aligned}$$

you'll have enter points as vectors/matrices … but be sure to use *matching* matrix notation:




You will have to keep track of which is a point and which is a vector yourself ! ( For example, in the above examples the answers are points, not vectors! )

## 2. The Dot Product

**Definition** in 2D:

$$\vec{a}\cdot\vec{b} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\cdot\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} := a_1 b_1 + a_2 b_2$$

**Definition** in 3D:

$$\vec{a}\cdot\vec{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\cdot\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} := a_1 b_1 + a_2 b_2 + a_3 b_3$$

**Note 1:** The dot product takes two vectors as input and produces a ***scalar***.

**Note 2:** This clearly lends itself to be extended to higher dimensions.

**Example 1**:

If $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}$ then $\vec{a}\cdot\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\cdot\begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix} := 1\cdot 4 + 2\cdot(-5) + 3\cdot 1 = -3$

**Note 3:** $\vec{a}\cdot\vec{a} = \|\vec{a}\|^2$ since (in 2D) $\vec{a}\cdot\vec{a} = a_1 a_1 + a_2 a_2 = a_1^2 + a_2^2$

and (in 3D) $\vec{a}\cdot\vec{a} = a_1 a_1 + a_2 a_2 + a_3 a_3 = a_1^2 + a_2^2 + a_3^2$

[ It is **not** a good idea to write $\vec{a}\cdot\vec{a}$ as $\vec{a}^{\,2}$, in which case one might wonder what $\vec{a}^{\,3}$ and $\vec{a}^{\,-1}$ would signify. ]

### Properties of the dot product

**Theorem 2.1:** **(a)** $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ ( Commutative property)

**(b)** $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ ( Distributive property)

**(c)** $(t\,\vec{a})\cdot\vec{b} = \vec{a}\cdot(t\,\vec{b}) = t\,(\vec{a}\cdot\vec{b})$

**(d)** $\vec{a}\cdot\vec{a} = \|\vec{a}\|^2$

**Proof:** **(a)** This is immediate from the definition of the dot product and the fact that the multiplication of real numbers is commutative.

**(b)** The 2D case:

$$\left.\begin{aligned}\vec{a}\cdot(\vec{b}+\vec{c}) &= \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\cdot\left(\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}+\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\cdot\begin{bmatrix} b_1+c_1 \\ b_2+c_2 \end{bmatrix} = a_1(b_1+c_1)+a_2(b_2+c_2) \\ \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c} &= \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\cdot\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}+\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\cdot\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = (a_1b_1+a_2b_2)+(a_1c_1+a_2c_2)\end{aligned}\right\}$$

$$\Rightarrow \quad \vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$$

The 3D case goes similarly.

**(c)** Left as an exercise.

**(d)** See Note 3 $\square$

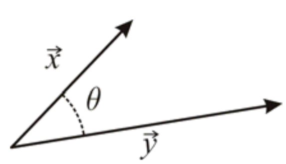
These properties allow us to compute more complex situations, e.g.

**Example 2:**

$$\begin{aligned}(\vec{a}-\vec{b})\cdot(\vec{a}+\vec{b}) &= (\vec{a}-\vec{b})\cdot\vec{a} + (\vec{a}-\vec{b})\cdot\vec{b} && \text{[ by Th 2.1 (b) ]} \\ &= \vec{a}\cdot(\vec{a}-\vec{b}) + \vec{b}\cdot(\vec{a}-\vec{b}) && \text{[ by Th 2.1 (a) ]} \\ &= \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} - \vec{b}\cdot\vec{b} && \text{[ by Th 2.1 (b) ]} \\ &= \|\vec{a}\|^2 - \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{b} - \|\vec{b}\|^2 && \text{[ by Th 2.1 (d) and (a) ]} \\ &= \|\vec{a}\|^2 - \|\vec{b}\|^2\end{aligned}$$

## Geometric significance.

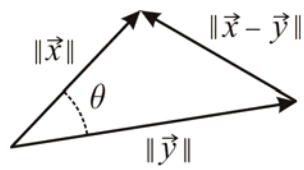
The dot product provides us with a way to measure angles.




**Theorem 2.2:** $\boxed{\vec{x}\cdot\vec{y} = \|\vec{x}\|\|\vec{y}\|\cos\theta}$

**Proof:** The dot product gives us

$$\begin{aligned}\| \vec{x}-\vec{y} \|^2 &= (\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y})\\ &= \|\vec{x}\|^2 - \vec{x}\cdot\vec{y} - \vec{y}\cdot\vec{x} + \|\vec{y}\|^2\\ &= \|\vec{x}\|^2 - \vec{x}\cdot\vec{y} - \vec{x}\cdot\vec{y} + \|\vec{y}\|^2\\ &= \|\vec{x}\|^2 - 2\,\vec{x}\cdot\vec{y} + \|\vec{y}\|^2\end{aligned}$$




And the law of cosines gives us: $\| \vec{x}-\vec{y} \|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\|\vec{x}\|\,\|\vec{y}\|\cos\theta.$

Combining these we get $\vec{x}\cdot\vec{y} = \|\vec{x}\|\|\vec{y}\|\cos\theta$ (after some simplifications).

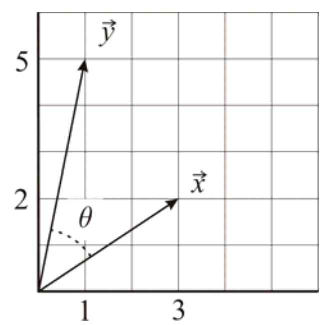
**Application:** We now can calculate angles between vectors ( in 2D, 3D etc.).

To find the angle between the vectors $\vec{x}=\begin{bmatrix}3\\2\end{bmatrix}$ and $\vec{y}=\begin{bmatrix}1\\5\end{bmatrix}$ we'll use the above theorem:

$$\vec{x}\cdot\vec{y} = \|\vec{x}\|\|\vec{y}\|\cos\theta$$

$$\Rightarrow \quad 13 = \sqrt{13}\,\sqrt{26}\,\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta = 45^\circ$$




Of course in this two dimensional case we could have computed the angle in other ways aswell: e.g. $\theta = \tan^{-1}(5) - \tan^{-1}\left(\frac{2}{3}\right)$ [ explain! ]. So let's do a 3D example where it probably would be preferable, certainly easiest, to use the above theorem.

Find the angle between the vectors $\vec{x}=\begin{bmatrix}2\\3\\1\end{bmatrix}$ and $\vec{y}=\begin{bmatrix}3\\1\\-2\end{bmatrix}$.

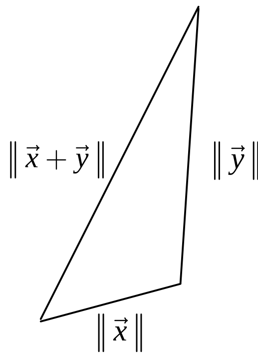
Using the theorem $\vec{x}\cdot\vec{y} = \|\vec{x}\|\|\vec{y}\|\cos\theta$

we get $7 = \sqrt{14}\,\sqrt{14}\,\cos\theta$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = 60^\circ$$

Finally we'll use the dot product in the proof of a well known theorem in mathematics that is usually referred to as the 'triangle inequality'.

**Theorem 2.3**: (The triangle inequality) $\| \vec{x}+\vec{y} \| \leq \| \vec{x} \| + \| \vec{y} \|$



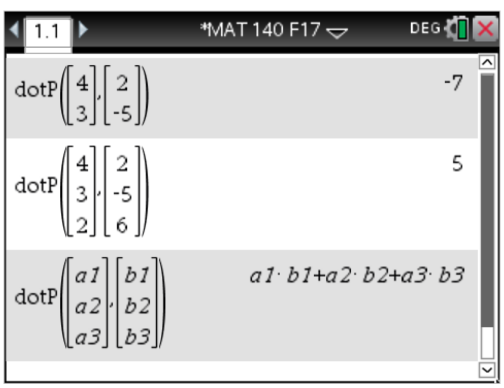

**Proof**:
$$
\begin{aligned}
\| \vec{x}+\vec{y} \|^2 &= (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) \\
&= \vec{x}\cdot(\vec{x}+\vec{y}) + \vec{y}\cdot(\vec{x}+\vec{y}) \\
&= \vec{x}\cdot\vec{x} \;+\; \vec{x}\cdot\vec{y} \;+\; \vec{y}\cdot\vec{x} \;+\; \vec{y}\cdot\vec{y} \\
&= \vec{x}\cdot\vec{x} \;+\; 2\vec{x}\cdot\vec{y} \;+\; \vec{y}\cdot\vec{y} \\
&= \vec{x}\cdot\vec{x} \;+\; 2\| \vec{x} \| \| \vec{y} \| \cos\theta \;+\; \vec{y}\cdot\vec{y} \\
&= \| \vec{x} \|^2 \;+\; 2\| \vec{x} \| \| \vec{y} \| \cos\theta \;+\; \| \vec{y} \|^2 \\
&\leq \| \vec{x} \|^2 \;+\; 2\| \vec{x} \| \| \vec{y} \| \;+\; \| \vec{y} \|^2 \qquad [\text{ since } \cos\theta \leq 1\,] \\
&= \left( \| \vec{x} \| + \| \vec{y} \| \right)^2
\end{aligned}
$$

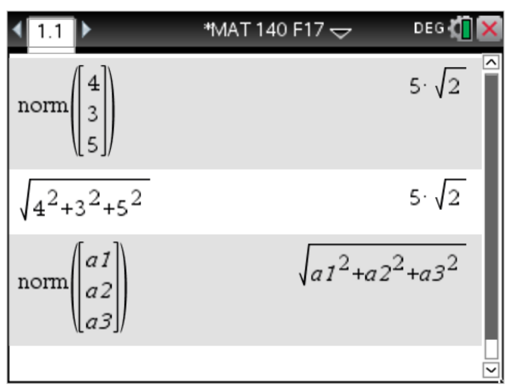
$$\Rightarrow \quad \| \vec{x}+\vec{y} \|^2 \leq \left( \| \vec{x} \| + \| \vec{y} \| \right)^2 \quad \Rightarrow \quad \| \vec{x}+\vec{y} \| \leq \| \vec{x} \| + \| \vec{y} \| \qquad \square$$

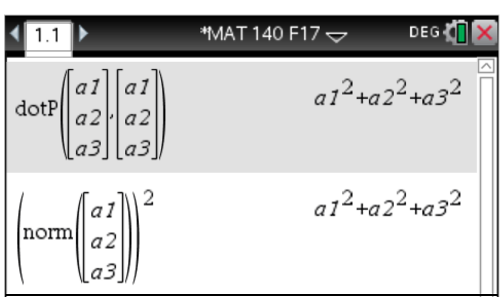
### The TI-Nspire

The TI-Nspire has a *dot product* function built in called: **dotP**

It also has a *length* function called: **norm**




$$\vec{a}\cdot\vec{a} = \| \vec{a} \|^2$$

## 3. A Basic Projection

One of the remarkable facts that follow from $\vec{x}\cdot\vec{y} = \|\vec{x}\|\|\vec{y}\|\cos\theta$ is that it is easy to check if two vectors are **perpendicular**. In that case $\cos\theta = 0$, i.e. $\vec{x}\cdot\vec{y} = 0$.

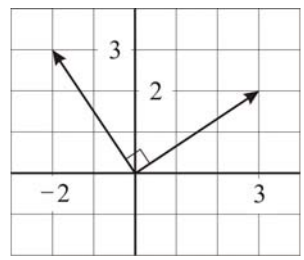
Recall that the zero vector $\vec{0}$ doesn't really have a direction. We adopt the convention that the zero vector is be perpendicular to *any* vector. Hence

**Theorem 3.1:** Two vectors $\vec{x}$ and $\vec{y}$ are perpendicular if and only if $\vec{x}\cdot\vec{y} = 0$.

**Note 1**: This theorem works in 2D and 3D, and in fact in any higher dimension as well.

**Example 1**: Since $\begin{bmatrix}3\\2\end{bmatrix}\cdot\begin{bmatrix}-2\\3\end{bmatrix} = 0$

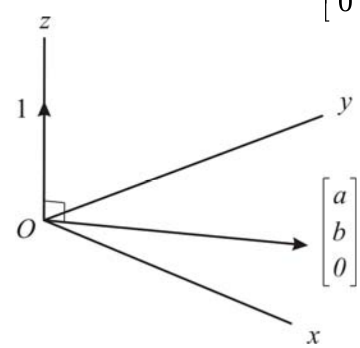
we know that $\begin{bmatrix}3\\2\end{bmatrix}\perp\begin{bmatrix}-2\\3\end{bmatrix}$.




**Example 2**: **(a)** Find a vector that is perpendicular to $\begin{bmatrix}1\\2\\3\end{bmatrix}$. There are many answers possible. For example $\begin{bmatrix}5\\-1\\-1\end{bmatrix}$ would work since $\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}5\\-1\\-1\end{bmatrix} = 0$. Alternatively we can take a vector with zero first component $\begin{bmatrix}0\\?\\?\end{bmatrix}$ and choose the others such that $\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}0\\?\\?\end{bmatrix} = 0$, e.g. $\begin{bmatrix}0\\3\\-2\end{bmatrix}$. Or take for example the third component to be zero: $\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}2\\-1\\0\end{bmatrix} = 0$. Etc.

**(b)** A special case: *Any* vector in the $x\,y-$plane, i.e. $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$, is perpendicular to $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ since

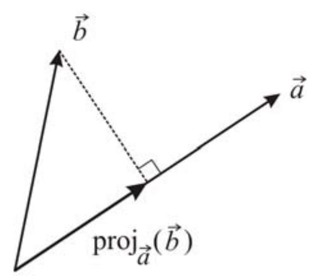
$$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$




Next we'll develop a formula to compute the **orthogonal projection** of one vector onto another, non-zero vector.

Let $\vec{a} \neq \vec{0}$. With $\operatorname{proj}_{\vec{a}}(\vec{b})$ we will denote the (orthogonal) projection of $\vec{b}$ onto $\vec{a}$

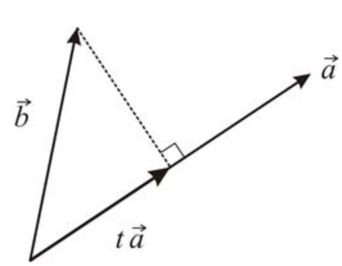
( or rather the projection of $\vec{b}$ onto the *line* of which $\vec{a}$ is a segment.)



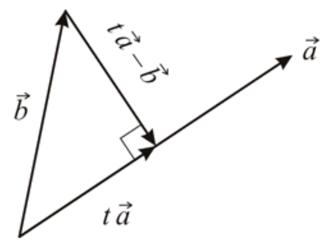

**Theorem 3.2**: $\operatorname{proj}_{\vec{a}}(\vec{b}) = \dfrac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}\,\vec{a}$ [ Of course we assume here that $\vec{a} \neq \vec{0}$ ]

**Proof**: Clearly $\operatorname{proj}_{\vec{a}}(\vec{b})$ is a multiple of $\vec{a}$ ( since it is the projection of $\vec{b}$ **onto** $\vec{a}$ ).

Hence $\operatorname{proj}_{\vec{a}}(\vec{b}) = t\,\vec{a}$ for some real number *t*.



Note that the vector $t\vec{a}-\vec{b}$ is perpendicular to $\vec{a}$: $t\vec{a}-\vec{b}\perp\vec{a}$




Hence $\vec{a}\cdot(t\vec{a}-\vec{b}) = 0 \quad \Rightarrow \quad t\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b} = 0$

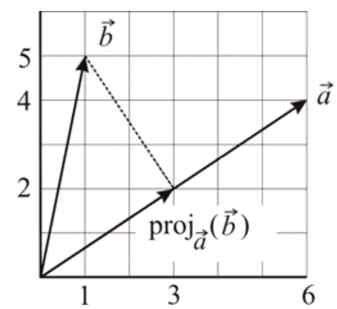
$$\Rightarrow \quad t\vec{a}\cdot\vec{a} = \vec{a}\cdot\vec{b}$$

$$\Rightarrow \quad t = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}} \qquad (\ \vec{a}\cdot\vec{a}\neq 0 \quad \text{so we can divide by it}\ )$$

So that $\operatorname{proj}_{\vec{a}}(\vec{b}) = \dfrac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\,\vec{a}$ $\square$

**Example 3**: Let $\vec{a} = \begin{bmatrix}6\\4\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}1\\5\end{bmatrix}$

then $\operatorname{proj}_{\vec{a}}(\vec{b}) = \dfrac{26}{52}\begin{bmatrix}6\\4\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$




**Example 4**: Let $A=(2,5)$, $B=(8,8)$ and $C=(3,8)$ be the vertices of triangle $ABC$. Find the base $Q$ of the altitude from $B$ using a projection vector.

Note that if

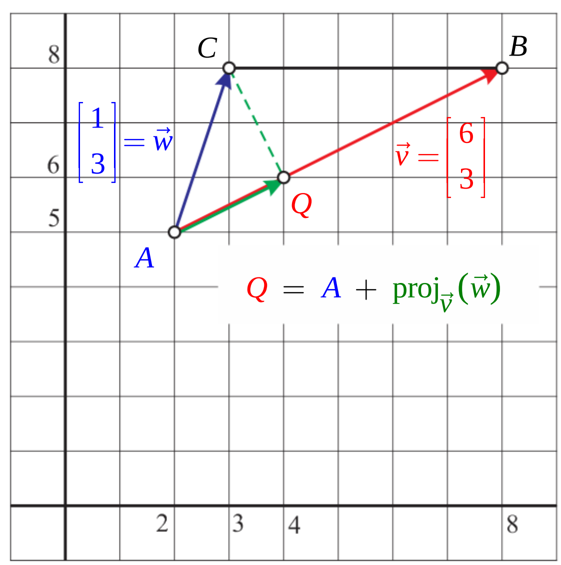
$$\vec{v} = \overrightarrow{AB} = B-A = (8,8)-(2,5) = \begin{bmatrix}6\\3\end{bmatrix} \quad \text{and}$$

$$\vec{w} = \overrightarrow{AC} = C-A = (3,8)-(2,5) = \begin{bmatrix}1\\3\end{bmatrix} \quad \text{then}$$

$$\operatorname{proj}_{\vec{v}}(\vec{w}) = \frac{15}{45}\begin{bmatrix}6\\3\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

So that

$$Q = A+\operatorname{proj}_{\vec{v}}(\vec{w}) = (2,5) + \begin{bmatrix}2\\1\end{bmatrix} = (4,6).$$



Note that the **length** of the projection vector is

$$\left\| \operatorname{proj}_{\vec{a}}(\vec{b}) \right\| = \left\| \frac{\vec{b}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\,\vec{a} \right\| = \frac{|\vec{b}\cdot\vec{a}|}{\|\vec{a}\|^2}\|\vec{a}\| = \frac{|\vec{b}\cdot\vec{a}|}{\|\vec{a}\|}$$

Alternatively this is also immediately clear after using some trigonometry:

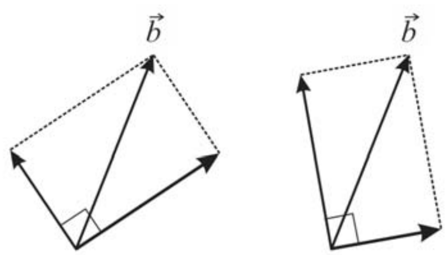
$$\left\| \operatorname{proj}_{\vec{a}}(\vec{b}) \right\| = \|\vec{b}\|\cdot|\cos(\theta)| = \frac{\|\vec{a}\|\cdot\|\vec{b}\|\cdot|\cos(\theta)|}{\|\vec{a}\|} = \frac{|\vec{b}\cdot\vec{a}|}{\|\vec{a}\|}$$

In particular we have that when $\vec{a}$ is a *unit* vector ( i.e. $\|\vec{a}\|=1$) then

$$\left\| \operatorname{proj}_{\vec{a}}(\vec{b}) \right\| = |\vec{a}\cdot\vec{b}|$$

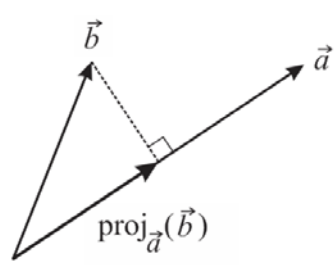
## The orthogonal complement

Sometimes it is useful to decompose a vector into the sum of two orthogonal vectors.




Clearly this can be done in many ways. Usually one particular direction is given (or needed).

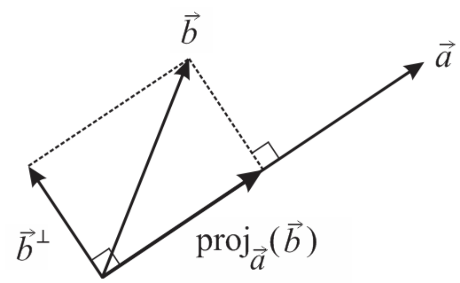
The component of the vector $\vec{b}$ in a given direction $\vec{a}$ is simply: $\operatorname{proj}_{\vec{a}}(\vec{b})$ .




The other component, called the *orthogonal complement,* is denoted by $\vec{b}^{\perp}$ ( maybe a better notation would be $\vec{b}^{\perp\vec{a}}$, to indicate the other—given—direction as well). It can be found by

$$\vec{b}^{\perp} = \vec{b} - \operatorname{proj}_{\vec{a}}(\vec{b})$$

which is obvious because only then we would have $\vec{b}^{\perp} + \mathrm{proj}_{\vec{a}}(\vec{b}) = \vec{b}$.

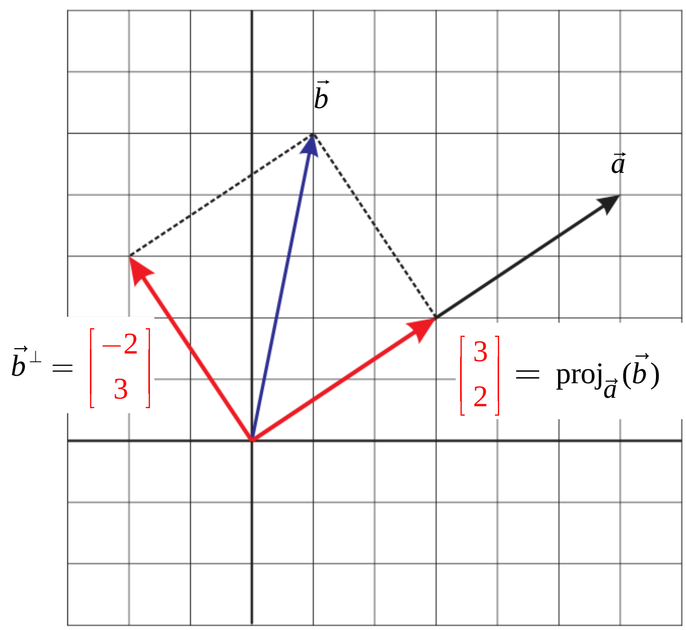



**Example 5**: Let $\vec{a} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ then

$$\mathrm{proj}_{\vec{a}}(\vec{b}) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

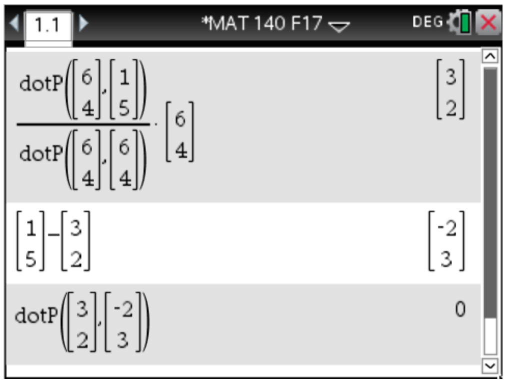
and

$$\vec{b}^{\perp} = \vec{b} - \mathrm{proj}_{\vec{a}}(\vec{b}) = \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$




It is easy to check: $\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \perp \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

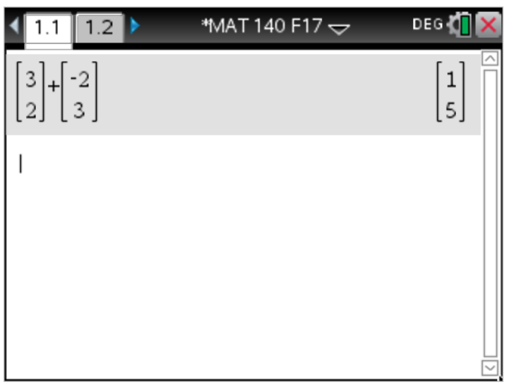
Here are the computations using the TI-Nspire:




$$\text{proj}_{\vec{a}}(\vec{b}) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\vec{b}^{\perp} = \vec{b} - \text{proj}_{\vec{a}}(\vec{b}) = \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\vec{b}^{\perp} \perp \text{proj}_{\vec{a}}(\vec{b}): \quad \begin{bmatrix} -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 0$$




$$\vec{b}^{\perp} + \text{proj}_{\vec{a}}(\vec{b}) = \begin{bmatrix} -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \vec{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

## 4. Vector equations of lines

In high school we are introduced to the various forms with which we can represent lines in the 2D plane:

**Non-vertical lines**:

**a)** Slope-intercept form: $y = mx + b$

**b)** Slope-point form: $y - y_0 = m(x - x_0)$

**c)** Two points form: $y - y_0 = \dfrac{y_1 - y_0}{x_1 - x_0}(x - x_0)$ $\quad [x_1 \neq x_0]$

**Vertical lines**:

**d)** Vertical line: $x = c$

**Standard, or normal form:**

**e)** Normal form: $ax + by = c$

**f)** Normal-point form: $ax + by = ax_0 + by_0$
[ or $a(x - x_0) + b(y - y_0) = 0$ ]

**g)** Normal Two points form: $(x_1 - x_0)(y - y_0) = (y_1 - y_0)(x - x_0)$

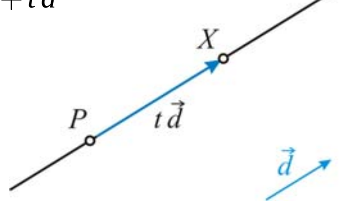
(The last two forms are probably unfamiliar to most.)

All of these forms are useful in their own right. The form $y = mx + b$ is by far the first choice of most high school students … but quite often—depending on the situation—other forms might actually be easier to use.

Even with this abundance of algebraic descriptions of a line there is a real need for yet another. The problem with the above forms is that they do not easily extend to give us formulas for lines in higher dimensional space. For example one could venture that $ax + by + cz = d$ might possibly be an equation of a line in 3D … yet it is not. It is the equation of a plane. We need to modify the concept of 'slope' to generalize to 3D space. This can be done using vectors, in this case called *direction* vectors.

**Direction vectors** give us an easy way to describe lines in any dimensional space!

$X = P + t\vec{d}$




or $(x, y) = (a, b) + t\begin{bmatrix} A \\ B \end{bmatrix}$

The expression $X = P + t\vec{d}$ we will refer to as the **vector equation** of the line.

In 2D this looks like $(x, y) = (a,b) + t\begin{bmatrix} A \\ B \end{bmatrix}$

In 3D this looks like $(x, y, z) = (a,b,c) + t\begin{bmatrix} A \\ B \\ C \end{bmatrix}$

**Example 1**: The line $l:\ (x, y) = (1,1) + t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ goes through the point $(1,1)$ and in the direction $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

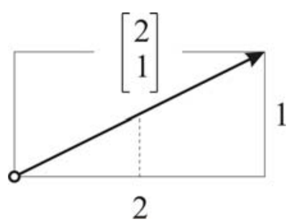
When $t = 1$ then $(x, y) = (1,1) + 1\cdot\begin{bmatrix} 2 \\ 1 \end{bmatrix} = (3, 2)$

When $t = 3$ then $(x, y) = (1,1) + 3\cdot\begin{bmatrix} 2 \\ 1 \end{bmatrix} = (7, 4)$

When $t = -1$ then $(x, y) = (1,1) - 1\cdot\begin{bmatrix} 2 \\ 1 \end{bmatrix} = (-1, 0)$ etc.

So by choosing different values of $t$ we get different points on the line. When $t$ runs through all real numbers, we get all points on the line.

We could rewrite the line in one of the another forms as follows: since the line goes in the direction of the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ we know that the slope is $m = \frac{1}{2}$,




and since we know $(1,1)$ is a point on the line we have

$$l:\ y - 1 = \frac{1}{2}(x-1) \qquad \text{or} \qquad l:\ y = \frac{1}{2}x + \frac{1}{2} \qquad \text{or} \qquad l:\ x - 2y = -1.$$

Alternatively we could find this algebraically equation from the vector equation of the line:

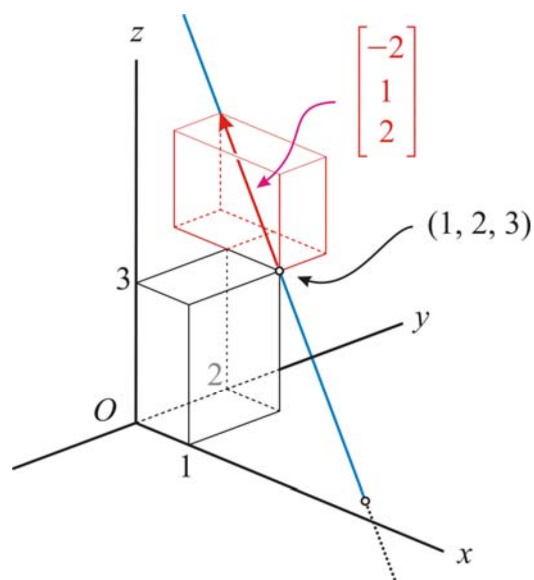
$$(x,y) = (1,1)+t\begin{bmatrix}2\\1\end{bmatrix} \Rightarrow (x,y) = (1+2t, 1+t) \Rightarrow \begin{cases} x = 1+2t \\ y = 1+t \end{cases}$$

Eliminating $t$ gives us

$$x = 1+2(y-1) \Rightarrow y-1 = \frac{1}{2}(x-1).$$

**Example 2**: A line $l$ through $(1,2,3)$ in the direction of $\begin{bmatrix}-2\\1\\2\end{bmatrix}$ can be described as

$$l: \ (x,y,z) = (1,2,3)+t\begin{bmatrix}-2\\1\\2\end{bmatrix}$$




Note that we could also write this line as $\begin{cases} x = 1-2t \\ y = 2+t \\ z = 3+2t \end{cases}$ since

$$(x,y,z) = (1,2,3)+t\begin{bmatrix}-2\\1\\2\end{bmatrix} = (1-2t,\ 2+t,\ 3+2t)$$

There is one more description of this line in 3D that is sometimes (rarely) used

$$\frac{x-1}{-2} = \frac{y-2}{1} = \frac{z-3}{2} \quad (= t)$$

[ Do you see how this follows from $\begin{cases} x = 1-2t \\ y = 2+t \\ z = 3+2t \end{cases}$ ? ]

In general one could write the line through $(a, b, c)$ in the direction of $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$ as

$$\frac{x-a}{A} = \frac{y-b}{B} = \frac{z-c}{C} \qquad [\text{ provided } A \neq 0 \textbf{ and } B \neq 0 \textbf{ and } C \neq 0 \ ]$$

But we will not use this way of describing a line in this course.

**In summary: lines in 2D and 3D**

The two most important forms of 2D and 3D lines that we will use in this course are:

**(1)** The **vector equation** of a line:

$$(x, y) = (a, b) + t\begin{bmatrix} A \\ B \end{bmatrix} \qquad \text{and} \qquad (x, y, z) = (a, b, c) + t\begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

**(2)** The **parametric equations** of a line

$$\begin{cases} x = a + At \\ y = b + Bt \end{cases} \qquad \text{and} \qquad \begin{cases} x = a + At \\ y = b + Bt \\ z = c + Ct \end{cases}$$

where the **parameter** $t$ runs through all real values.

## 5. The distance of a point to a line in $\mathbb{R}^2$

### The Normal

The reason why we like to refer to the equation $a\,x + b\,y = c$ as the *normal* equation of a line in 2D is that the coefficients in front of the $x$ and the $y$ give us a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ which is **normal**, i.e. perpendicular, to the line. There are various ways to see this. Here is one:

Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two points on the line $ax + by = c$ then

$$\begin{aligned} a\,x_1 + b\,y_1 &= c \\ a\,x_2 + b\,y_2 &= c \end{aligned} \qquad -$$

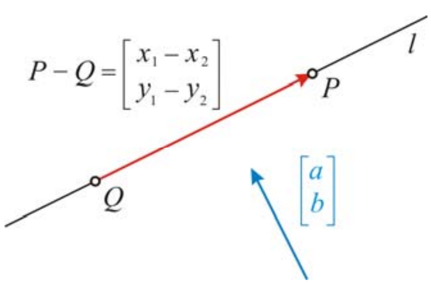
Hence:
$$a(x_1 - x_2) + b(y_1 - y_2) = 0$$

Notice that we can write this as a dot product:

$$\begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

But this tells us that the vectors $\begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix}$ and $\begin{bmatrix} a \\ b \end{bmatrix}$ are perpendicular.

Notice that the vector $P - Q = \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix}$ is in the direction of the line




hence it follows that $\begin{bmatrix} a \\ b \end{bmatrix}$ is perpendicular to the line.

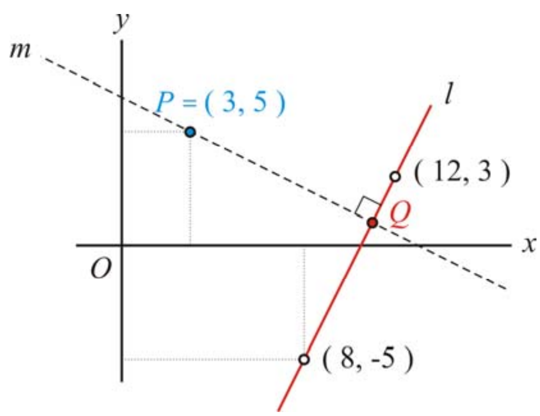
[ Alternatively one could argue that the slope of the line is $-\dfrac{a}{b}$ ( provided $b \neq 0$ ) and hence the slope of a perpendicular line would have to be $\frac{b}{a}$, which means we could take $\begin{bmatrix} a \\ b \end{bmatrix}$ as a direction vector for the perpendicular line. (What if $b = 0$ ?) ]

## Distance from a line

Let $P$ be a point and $l$ be a line, how do we compute the distance from the point to the line?

**Example 1**: Let $P = (3, 5)$ and $l:\ 2x - y = 21$, then $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is a normal to the line, hence the line $m$ through $P$ but *perpendicular* to $l$ has the vector equation

$$m:\quad (x, y) \;=\; (3, 5) + t\begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \text{or (parametrically)} \qquad m:\ \begin{cases} x = 3 + 2t \\ y = 5 - t \end{cases}$$




To find the point $Q = (x_0, y_0)$ as the intersection of the lines $l$ and $m$, we use the parametric form of $m$ and 'substitute' in $l$, i.e. we find the $t_0$ that gives us $Q$:

$$\left.\begin{array}{l} \begin{cases} x_0 = 3 + 2t_0 \\ y_0 = 5 - t_0 \end{cases} \\ 2x_0 - y_0 = 21 \end{array}\right\} \;\Rightarrow\; 2(3 + 2t_0) - (5 - t_0) = 21 \;\Rightarrow\; t_0 = 4$$

Hence $Q \;=\; (3, 5) + 4\begin{bmatrix} 2 \\ -1 \end{bmatrix} \;=\; (11, 1)$

And therefore the distance of $P$ to the line $l$ can be computed as follows:

$$\operatorname{dist}(P, l) \;=\; \operatorname{dist}(P, Q) \;=\; \sqrt{(11 - 3)^2 + (1 - 5)^2} \;=\; 4\sqrt{5}$$

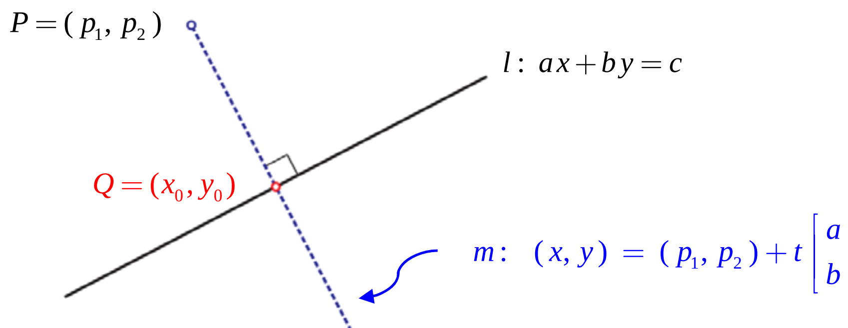
This method can be used in general to find the following formula

**Theorem 5.1**: The distance of the point $P=(p_1,\, p_2)$ to the line $l:\ ax+by=c$ is given by

$$\operatorname{dist}(P,l) \;=\; \frac{|a\,p_1+b\,p_2-c|}{\sqrt{a^2+b^2}}$$

**Proof**: Let $m$ be the line through $P$ and perpendicular to $l$, i.e.

$$m:\ \ (x,y)\;=\;(p_1,\,p_2)+t\begin{bmatrix} a\\ b\end{bmatrix} \qquad \text{or (parametrically)} \qquad m:\ \begin{cases} x = p_1 + a\,t\\ y = p_2 + b\,t\end{cases}$$




Hence the point $Q=(x_0,y_0)$, the intersection of lines $l$ and $m$, is found as follows:

$$\left.\begin{cases} x_0 = p_1+a\,t_0\\ y_0 = p_2+b\,t_0\end{cases}\atop a\,x_0+b\,y_0=c\right\} \;\Rightarrow\; a(p_1+a\,t_0)+b(p_2+b\,t_0)=c \;\Rightarrow\; t_0=\frac{c-a\,p_1-b\,p_2}{a^2+b^2}$$

Thus $\quad Q\;=\;(x_0,\,y_0)\;=\;(p_1,\,p_2)+t_0\begin{bmatrix} a\\ b\end{bmatrix}\;=\;P+t_0\begin{bmatrix} a\\ b\end{bmatrix}\qquad$ i.e. $\qquad Q\;=\;P+t_0\begin{bmatrix} a\\ b\end{bmatrix}$

And therefore

$$\operatorname{dist}(P,l)\;=\;\|\,Q-P\,\|\;=\;\left\|\,t_0\begin{bmatrix} a\\ b\end{bmatrix}\right\|\;=\;|\,t_0\,|\left\|\begin{bmatrix} a\\ b\end{bmatrix}\right\|\;=\;|\,t_0\,|\sqrt{a^2+b^2}$$

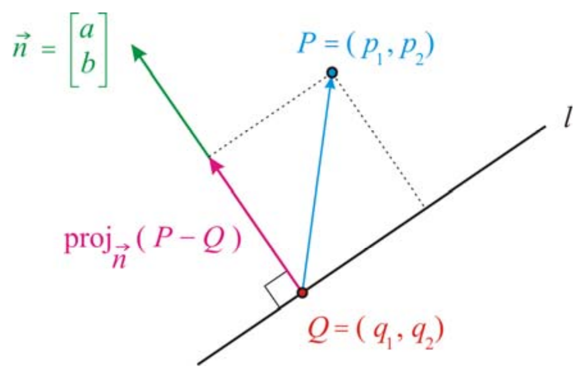
So that

$$\operatorname{dist}(P,l)\;=\;\left|\frac{c-a\,p_1-b\,p_2}{a^2+b^2}\right|\sqrt{a^2+b^2}\;=\;\frac{|c-a\,p_1-b\,p_2|}{\sqrt{a^2+b^2}} \qquad \square$$

Even though this is a nice method we could do this more elegantly using a projection:

**Alternative Proof**:

Since $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ is the normal of the line, the length of the projection of $P-Q$ onto $\vec{n}$ is the same as the distance of $P$ to $l$. ( See picture )




$$\|\text{proj}_{\vec{n}}(P-Q)\| = \left\| \frac{\vec{n}\cdot(P-Q)}{\vec{n}\cdot\vec{n}}\,\vec{n} \right\|$$

$$= \left| \frac{\vec{n}\cdot(P-Q)}{\vec{n}\cdot\vec{n}} \right| \|\vec{n}\| \qquad \text{[ 'moving out' a scalar } \|t\vec{n}\| = |t|\cdot\|\vec{n}\| \text{ ]}$$

$$= \frac{\left| \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \end{bmatrix} \right|}{a^2+b^2} \sqrt{a^2+b^2}$$

$$= \frac{|a(p_1-q_1)+b(p_2-q_2)|}{\sqrt{a^2+b^2}}$$

$$= \frac{|a\,p_1+b\,p_2-(a\,q_1+b\,q_2)|}{\sqrt{a^2+b^2}}$$

$$= \frac{|a\,p_1+b\,p_2-c|}{\sqrt{a^2+b^2}} \qquad [\ a\,q_1+b\,q_2 = c \quad \text{since } Q\in l\ ]$$

Redoing **Example 1**: $P=(3,5)$ and $l:\ 2x-y=21$, using this formula we find

$$\frac{|a\,p_1+b\,p_2-c|}{\sqrt{a^2+b^2}} = \frac{|2\cdot 3-1\cdot 5-21|}{\sqrt{2^2+1^2}} = \frac{|-20|}{\sqrt{5}} = 4\sqrt{5}$$

which is the answer we found before.

# 6. Lines and Planes in 3D

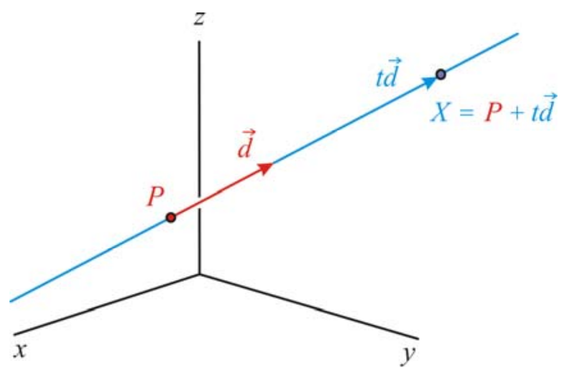
## Lines

We introduced vector equations for lines in 2D with the idea to use them in higher dimensional spaces as well. We'll do that now in 3D:

A **line** through the point $P = (p_1, p_2, p_3)$ in the direction $\vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ can be described by

the **vector equation** $\quad l: \quad X = P + t\vec{d}$

i.e. $\quad l: \quad (x, y, z) = (p_1, p_2, p_3) + t\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$




where $X = (x, y, z)$, the generic/arbitrary point, runs through all point of $l$ when we let the parameter/the free variable $t$ run through all real numbers. This one degree of freedom of the parameter results in the one dimensional line through $P$ in the direction of $\vec{d}$.

Of course we can rewrite this vector equation as **parametric equations**:

$$l: \begin{cases} x = p_1 + t\,d_1 \\ y = p_2 + t\,d_2 \\ z = p_3 + t\,d_3 \end{cases} \quad \text{since} \quad (x, y, z) = (p_1, p_2, p_3) + t\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = (p_1 + t\,d_1,\ p_2 + t\,d_2,\ p_3 + t\,d_3)$$

**Example 1**: $\quad l: \quad (x, y, z) = (1, 2, 3) + t\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$. This is the line through $(1, 2, 3)$ in the direction of $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$. When we take $t = 1$ we get the point $(5, 3, 6)$, or when $t = -1.5$ we get the point $(-5,\ 0.5,\ -1.5)$ etc. The parametric form of the line is $l: \begin{cases} x = 1 + 4t \\ y = 2 + t \\ z = 3 + 3t \end{cases}$.

Where does the line $l$ intersect the $xy$-plane?

Since all points on the $xy$-plane have $z = 0$ the line $l$ hits the $xy$-plane when $t = -1$ i.e. at the point $(-3, 1, 0)$.

## Planes

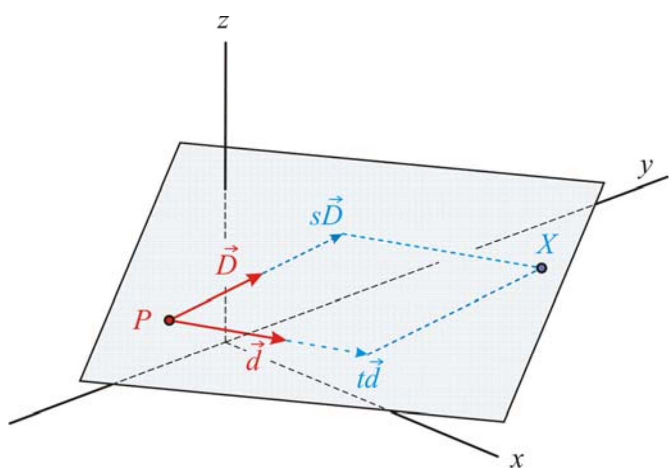
Again we can use vectors to describe planes. This time though, since they are two dimensional objects we need to specify two direction vectors and two parameters.

The point $P = (p_1, p_2, p_3)$ and the two *independent* directions $\vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ and $\vec{D} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}$

determine the **vector equation** of the plane $\alpha: \; X = P + t\vec{d} + s\vec{D}$

i.e. $$\alpha: \; (x, y, z) = (p_1, p_2, p_3) + t\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} + s\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}$$

When the two free variables/parameters $t$ and $s$ independently run through all real numbers we get every point on the plane $\alpha$.




Of course the direction vectors $\vec{d}$ and $\vec{D}$ have to be **independent** of each other, i.e. they should not be multiples of each other, otherwise $X = P + t\vec{d} + s\vec{D}$ would be a line, e.g. if $\vec{d}$ *were* a multiple of $\vec{D}$ $(\neq \vec{0})$, e.g. $\vec{d} = m\vec{D}$, then $X = P + t\vec{d} + s\vec{D}$ would really just represent the **line** $X = P + (tm + s)\vec{D}$. Or worse: just a point when both $\vec{d} = \vec{0}$ and $\vec{D} = \vec{0}$.

The **parametric equations** of the plane would be $\begin{cases} x = p_1 + t\,d_1 + s\,D_1 \\ y = p_2 + t\,d_2 + s\,D_2 \\ z = p_3 + t\,d_3 + s\,D_3 \end{cases}$

**Example 2**: $\alpha: \ (x, y, z) = (1,2,3) + t\begin{bmatrix}4\\5\\2\end{bmatrix} + s\begin{bmatrix}3\\1\\7\end{bmatrix}$

This is the vector equation of the plane through e.g. the points $(1,2,3)$, $(5,7,5)$ [ take $t=1$ and $s=0$ ] and $(4,3,10)$ [ take $t=0$ and $s=1$ ]. Its parametric form is

$$\alpha: \begin{cases} x = 1+4t+3s \\ y = 2+5t+s \\ z = 3+2t+7s \end{cases}$$

Does this plane intersect to $z$-axis?

Since a point on the $z$-axis has $x=0$ and $y=0$ we need $\begin{cases} 4t+3s=-1 \\ 5t+s=-2 \end{cases}$. Solving this system of equations we get $\begin{cases} t=-5/11 \\ s=\ \ 3/11 \end{cases}$ so that $z = 3+\dfrac{-10}{11}+\dfrac{21}{11} = 4$ hence the plane intersects the $z$-axis at $(0,0,4)$.

Now observe the following interesting rewrite of the vector equation of the plane:

$$(x, y, z)-(1,2,3) = t\begin{bmatrix}4\\5\\2\end{bmatrix} + s\begin{bmatrix}3\\1\\7\end{bmatrix}$$

and recall that the *difference* of two points is a vector: $(x, y, z)-(1,2,3) = \begin{bmatrix}x-1\\y-2\\z-3\end{bmatrix}$ so that

$$\begin{bmatrix}x-1\\y-2\\z-3\end{bmatrix} = t\begin{bmatrix}4\\5\\2\end{bmatrix} + s\begin{bmatrix}3\\1\\7\end{bmatrix}$$

If we now take the dot product with $\begin{bmatrix}3\\-2\\-1\end{bmatrix}$ on *both* sides of this equation we obtain

$$\begin{bmatrix}3\\-2\\-1\end{bmatrix}\cdot\begin{bmatrix}x-1\\y-2\\z-3\end{bmatrix} = \begin{bmatrix}3\\-2\\-1\end{bmatrix}\cdot\left(t\begin{bmatrix}4\\5\\2\end{bmatrix} + s\begin{bmatrix}3\\1\\7\end{bmatrix}\right)$$

Which gives us a **normal equation** of the plane

$$3(x-1)-2(y-2)-(z-3)=0$$

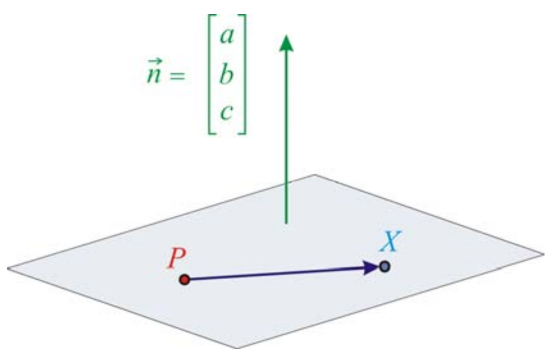
or equivalently $\alpha: \; 3x - 2y - z = -4$

The right hand side of the vector equation became zero because both $\begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$ are perpendicular to $\begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$. This means that that $\begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$ is a **normal** of the plane ( since it is perpendicular to both direction vectors of the plane ).

Notice that indeed $(1,2,3)$, $(5,7,5)$ and $(4,3,10)$ are points on the plane since they satisfy the equation $3x - 2y - z = -4$. Also notice that when $x = 0$ and $y = 0$ it follows that $z = 4$ hence the point $(0,0,4)$ on the $z$-axis is on the plane too.

## Normal equation of a plane

Suppose $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a vector that is perpendicular to a plane $\alpha$ which goes through the point $P = (p_1, p_2, p_3)$ and let $X = (x, y, z)$ be an arbitrary point on the plane.




It then follows that the vector $X - P$ is perpendicular to $\vec{n}$, i.e. $\vec{n} \cdot (X - P) = 0$

so that $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x - p_1 \\ y - p_2 \\ z - p_3 \end{bmatrix} = 0$, which means $a(x - p_1) + b(y - p_2) + c(z - p_3) = 0$

or $$a\,x + b\,y + c\,z = a\,p_1 + b\,p_2 + c\,p_3$$

or $$a\,x + b\,y + c\,z = d$$

The equation $\alpha: \quad a\,x + b\,y + c\,z = d$ is called a **normal equation** of a plane, and as in the 2D case for the line, we can read off a *normal* $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ for the plane.

## From normal equation to vector equation

It is not hard at all to convert a normal equation to a vector equation

**Example 3**: If we start out with the normal equation $\alpha: \quad 3x - 2y - z = -4$ then we know that the vector $\begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$ is *normal* to the plane, i.e. any vector perpendicular to this normal will automatically be a direction vector for the plane:

e.g. $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$ etc. We just need two independent ones. It is also not too hard to read off points on the plane: e.g. $(0, 0, 4)$ or $(0, 2, 0)$ or $(1, 2, 3)$ or $(5, 7, 5)$ or $(4, 3, 10)$ etc.

We can now write down *many* vector equations of the same plane $\alpha$

$$\alpha: \quad (x, y, z) \;=\; (1, 2, 3) + t\begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} + s\begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$$

$$\alpha: \quad (x, y, z) \;=\; (0, 0, 4) + t_1\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + s_1\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\alpha: \quad (x, y, z) \;=\; (0, 2, 0) + t_2\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + s_2\begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$$

$$\alpha: \quad (x, y, z) \;=\; (5, 7, 5) + t_3\begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} + s_3\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \qquad \text{etc.}$$

If you have any doubt that these represent the same plane we can show that they share the same three points $(0, 0, 4)$ or $(0, 2, 0)$ or $(1, 2, 3)$ hence must be the same plane:

To produce these points from $(x, y, z) = (1, 2, 3) + t\begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} + s\begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$ we need

$t = -5/11$ and $s = 3/11$ for $(0, 0, 4)$

$t = 1/11$ and $s = -5/11$ for $(0, 2, 0)$

$t = 0$ and $s = 0$ for $(1, 2, 3)$

To produce these points from $(x, y, z) = (0, 0, 4) + t_1\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + s_1\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ we need

$t_1 = 0$ and $s_1 = 0$ for $(0, 0, 4)$

$t_1 = 2/3$ and $s_1 = -4/3$ for $(0, 2, 0)$

$t_1 = 2/3$ and $s_1 = -1/3$ for $(1, 2, 3)$

To produce these points from $(x, y, z) = (0, 2, 0) + t_2\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + s_2\begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$ we need

$t_2 = 6$ and $s_2 = -2$ for $(0, 0, 4)$

$t_2 = 0$ and $s_2 = 0$ for $(0, 2, 0)$

$t_2 = 1$ and $s_2 = 0$ for $(1, 2, 3)$

We'll let you figure out the values of $t_3$ and $s_3$ for the last equation.

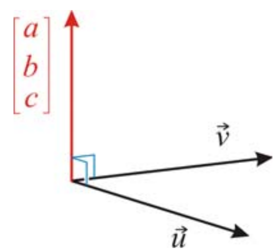
## From vector equation to normal equation I

To go from vector equation to normal equation is easy once you can find a normal to the plane. This can be done with a cross product which we will discuss in the next section.

## 7. The Cross Product

As we saw in the previous section that the normal equation of a plane $ax + by + cz = d$ conveniently provides us a **normal** vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ (perpendicular) to the plane.

How do we find a normal to the plane when we have the plane given as a vector equation or put it in a slightly different way: how do we find a vector normal to two given (independent) vectors?




**Definition**: The **Cross Product** $\vec{u} \times \vec{v}$

If $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ then $\vec{u} \times \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} := \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$

is called the cross product of $\vec{u}$ and $\vec{v}$.

**Note 1**: The cross product is a three dimensional phenomena ( although it is on occasion adapted for use in two dimensional space as well ). It takes two vectors in $\mathbb{R}^3$ and produces another *vector* in $\mathbb{R}^3$.

**Example 1:** $\vec{u} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$ then

$$\vec{u} \times \vec{v} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} \times \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} := \begin{bmatrix} 4 \cdot 5 - 7(-3) \\ 7 \cdot 1 - 2 \cdot 5 \\ 2(-3) - 4 \cdot 1 \end{bmatrix} = \begin{bmatrix} 41 \\ -3 \\ -10 \end{bmatrix}$$

Note that this cross product is perpendicular to both $\vec{u}$ and $\vec{v}$:

$$\begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 41 \\ -3 \\ -10 \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 41 \\ -3 \\ -10 \end{bmatrix} = 0$$

**Theorem 7.1**: **(a)** $(\vec{u}\times\vec{v})\cdot\vec{u} = 0$ and $(\vec{u}\times\vec{v})\cdot\vec{v} = 0$

**(b)** $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$

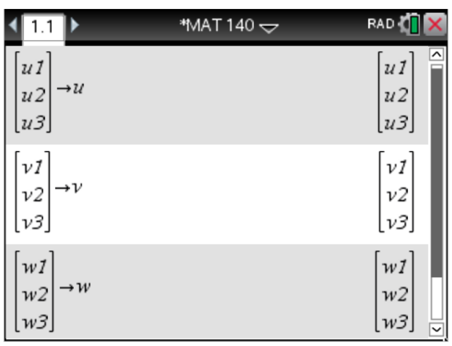
**(c)** $t(\vec{u}\times\vec{v}) = (t\vec{u})\times\vec{v} = \vec{u}\times(t\vec{v})$

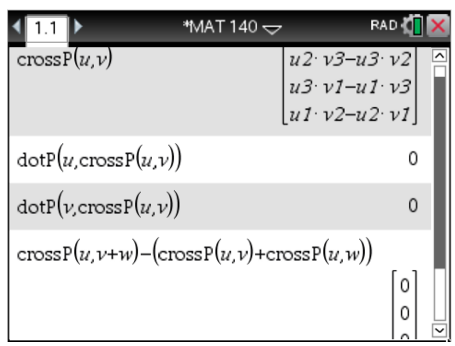
**(d)** $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v}+\vec{u}\times\vec{w}$

**Proof**: **(a)** This requires some algebraic work, but it is essentially trivial. (The TI-Nspire with CAS has no problems with such a simple algebraic tasks.)

$$\begin{aligned}(\vec{u}\times\vec{v})\cdot\vec{u} &= \begin{bmatrix} u_2v_3-u_3v_2 \\ u_3v_1-u_1v_3 \\ u_1v_2-u_2v_1\end{bmatrix}\cdot\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} \\ &= (u_2v_3-u_3v_2)u_1+(u_3v_1-u_1v_3)u_2+(u_1v_2-u_2v_1)u_3 \\ &= u_1u_2v_3-u_1u_3v_2+u_2u_3v_1-u_1u_2v_3+u_1u_3v_2-u_2u_3v_1 \\ &= 0\end{aligned}$$

The other equality is proved similarly. Here is the TI-Nspire's work:




Proofs of **(b)** **(c)** and **(d)** are left as exercises for the reader.

**Theorem 7.2**: $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta$ ($\theta$ is the angle between $\vec{u}$ and $\vec{v}$: $0\le\theta\le\pi$)

**Proof**:
$$\begin{aligned}\|\vec{u}\times\vec{v}\|^2 &= (u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2 \\ &= u_2^2v_3^2-2u_2u_3v_2v_3+u_3^2v_2^2+u_3^2v_1^2-2u_1u_3v_1v_3+u_1^2v_3^2+u_1^2v_2^2-2u_1u_2v_1v_2+u_2^2v_1^2 \\ &= u_2^2v_3^2+u_3^2v_2^2+u_3^2v_1^2+u_1^2v_3^2+u_1^2v_2^2+u_2^2v_1^2-2u_1u_2v_1v_2-2u_2u_3v_2v_3-2u_1u_3v_1v_3 \\ &= (u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1v_1+u_2v_2+u_3v_3)^2 \\ &= \|\vec{u}\|^2\|\vec{v}\|^2-(\vec{u}\cdot\vec{v})^2 \\ &= \|\vec{u}\|^2\|\vec{v}\|^2-\left(\|\vec{u}\|\|\vec{v}\|\cos\theta\right)^2 \\ &= \|\vec{u}\|^2\|\vec{v}\|^2-\|\vec{u}\|^2\|\vec{v}\|^2\cos^2\theta \\ &= \|\vec{u}\|^2\|\vec{v}\|^2(1-\cos^2\theta) \\ &= \|\vec{u}\|^2\|\vec{v}\|^2\sin^2\theta\end{aligned}$$

and the result follows □

The definition of the cross product comes a bit out of nowhere. How would one find such an expression for a normal?

Assume that we want the vector $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to be normal to $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ *and* $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. We need $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0$ and $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$ i.e. $\begin{cases} a u_1 + b u_2 + c u_3 = 0 \\ a v_1 + b v_2 + c v_3 = 0 \end{cases}$. Multiplying the first equation with $v_3$ and the second with $u_3$ and subtracting we get:

$$\begin{array}{c} \begin{cases} a u_1 v_3 + b u_2 v_3 + c u_3 v_3 = 0 \\ a u_3 v_1 + b u_3 v_2 + c u_3 v_3 = 0 \end{cases} \\ \hline a(u_1 v_3 - u_3 v_1) + b(u_2 v_3 - u_3 v_2) = 0 \end{array} \; -$$

or equivalently
$$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} u_1 v_3 - u_3 v_1 \\ u_2 v_3 - u_3 v_2 \end{bmatrix} = 0 .$$

Hence let us take $\begin{cases} a = u_2 v_3 - u_3 v_2 \\ b = -(u_1 v_3 - u_3 v_1) \end{cases}$ then the above equation is true. What remains is to find the corresponding $c$:

$$\begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ c \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0 \quad \Rightarrow \quad (u_2 v_3 - u_3 v_2) u_1 + (u_3 v_1 - u_1 v_3) u_2 + c u_3 = 0$$

$$\begin{aligned} &\Rightarrow \quad u_1 u_2 v_3 - u_1 u_3 v_2 + u_2 u_3 v_1 - u_1 u_2 v_3 + c u_3 = 0 \\ &\Rightarrow \quad -u_1 u_3 v_2 + u_2 u_3 v_1 \qquad + c u_3 = 0 \\ &\Rightarrow \quad c u_3 = (u_1 v_2 - u_2 v_1) u_3 \end{aligned}$$

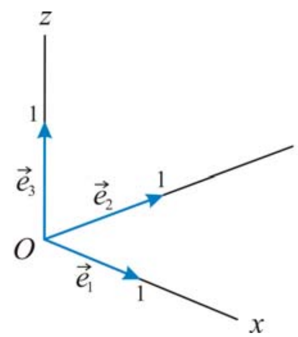
Thus if we take $c = u_1 v_2 - u_2 v_1$ then the last equation is also true, and we have found

$$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

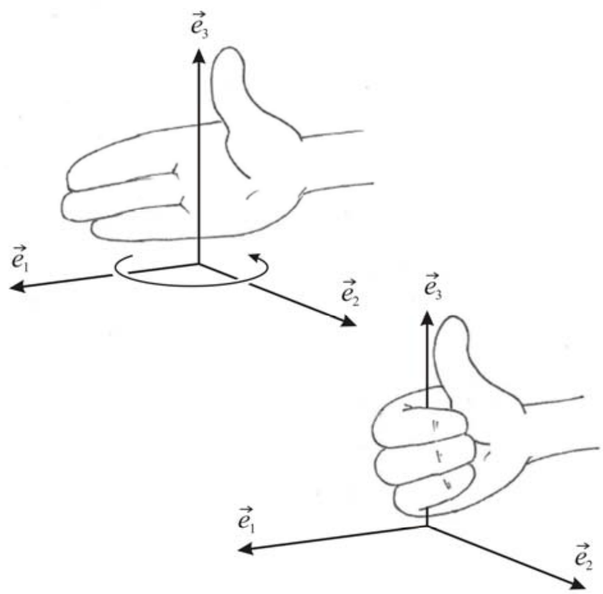
which is a vector perpendicular to both $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, and which is exactly what we defined the cross product $\vec{u} \times \vec{v}$ to be.

## Orientation

Note that $\begin{bmatrix}1\\0\\0\end{bmatrix}\times\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$ i.e. the standard basis vectors $\vec{e}_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{e}_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\vec{e}_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$ are related: $\vec{e}_3=\vec{e}_1\times\vec{e}_2$. Also notice their relative position in space, i.e. their **orientation** in space:




We call this orientation of $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3=\vec{e}_1\times\vec{e}_2$ **right handed**: if you grab the $\vec{e}_3$ vector with your right hand, thumb up in the $\vec{e}_3$ direction, then your fingers move from $\vec{e}_1$ to $\vec{e}_2$.




The orientation of the triple of vectors $\vec{u}$, $\vec{v}$ and $\vec{u}\times\vec{v}$ is called right handed as they have the same orientation, i.e. same relative position in space as $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3=\vec{e}_1\times\vec{e}_2$.

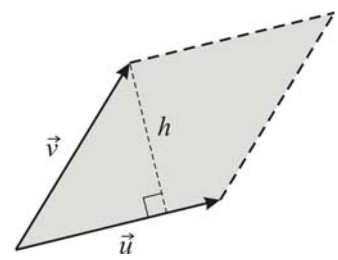
We could say that the configuration $\vec{e}_1$, $\vec{e}_2$ and $-\vec{e}_3$ has a left handed orientation.

Some people like to compare the motion to that of a screw. If you turn a screw *in*, that corresponds to our right handed orientation: if we turn from $\vec{e}_1$ to $\vec{e}_2$ we go up (the screw goes in.)

## Area

There is a wonderful geometric interpretation of the length of the cross product.

**Theorem 7.3**: $\|\vec{u}\times\vec{v}\|$ is equal to the area of the parallelogram 'spanned' by $\vec{u}$ and $\vec{v}$.




**Proof**: Recall that the area of a parallelogram is 'base' times 'height': $\mathcal{A} = h\cdot b$

Hence if we take $\|\vec{u}\|$ as our base, then the height is $h = \|\vec{v}\|\sin\theta$, and therefore

$$\mathcal{A} = h\cdot b = \|\vec{u}\|\,\|\vec{v}\|\sin\theta$$

So that by Theorem 7.2 we see that $\mathcal{A}$, the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$, is indeed the length of the cross product $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\sin\theta$.

## From vector equation to normal equation II

Now that we can compute a normal vector to two given vectors we can easily find a normal equation of a plane from a vector equation: Suppose we start with the following vector equation

$$\alpha: \quad X = P + t\vec{u} + s\vec{v}$$

i.e.
$$\alpha: \quad (x, y, z) = (p_1, p_2, p_3) + t\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + s\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

First compute $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{u}\times\vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\times\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ then the plane has equation

$$a\,x + b\,y + c\,z = d$$

Where $d$ can be found by plugging in the point $P$: $\quad d = a\,p_1 + b\,p_2 + c\,p_1$.

[ We could have also used the form $a(x-p_1)+b(y-p_2)+c(z-p_3)=0$ ]

**Example 2**: Let a plane be given by $\vec{x} = (1,-2,4) + t\begin{bmatrix}2\\4\\7\end{bmatrix} + s\begin{bmatrix}1\\-3\\5\end{bmatrix}$.

A normal of the plane would then be:

$$\vec{n} = \begin{bmatrix}2\\4\\7\end{bmatrix} \times \begin{bmatrix}1\\-3\\5\end{bmatrix} = \begin{bmatrix}4\cdot 5-7(-3)\\7\cdot 1-2\cdot 5\\2(-3)-4\cdot 1\end{bmatrix} = \begin{bmatrix}41\\-3\\-10\end{bmatrix}$$

so that the normal equation of the plane is

$$41x-3y-10z=d\ .$$

To figure out $d$ we substitute a point of the plane into this equation. In this case the obvious choice would be $(1,-2,4)$:

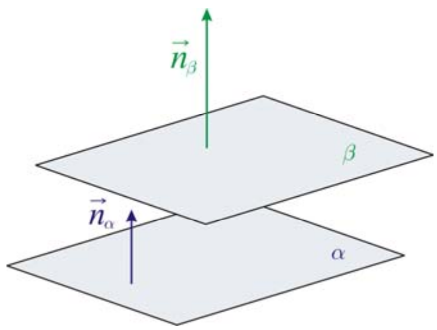
$$41\cdot 1-3\cdot(-2)-10\cdot 4 = d = 7$$

Hence the plane has *normal* equation: $41x-3y-10z=7$

## 8. Intersections

### Intersection of two planes

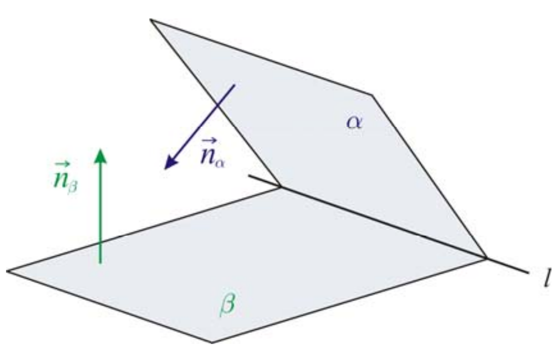
**(a) Parallel planes**: Two planes in 3D are parallel when their normals point in the same or opposite direction ( i.e. when their normals are parallel, i.e. when they are multiples of each other ). Parallel planes don't intersect *unless* they are the same plane.



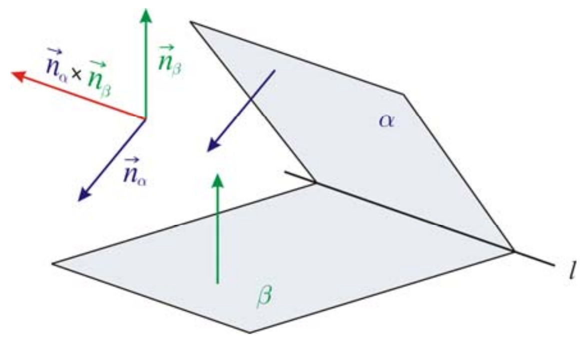

**Example 1**: $\alpha: \ 3x-2y-z=-4$ and $\beta: \ -6x+4y+2z=5$ are parallel since their normals $\vec{n}_\alpha = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$ and $\vec{n}_\beta = \begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix} = -2\vec{n}_\alpha$ are multiples of each other.

In this example $\alpha$ and $\beta$ are different planes [ e.g. $(0,2,0)$ is on $\alpha$ but not on $\beta$ ] hence they don't intersect.

**(b) Non parallel planes**: Two planes in 3D whose normals are *not* multiples of each other (i.e. they point in different directions) *have* to intersect, and do so in a line.




The cross product of their normals gives us a direction vector for that line.




To find one point in the intersection you could select $x = 0$ (or $y = 0$ or $z = 0$ or $y = 1$ etc.) and solve the resulting system of two equations with two unknowns:

**Example 2**: $\alpha: \ 3x - 2y - z = -4$ and $\beta: \ 3x + 2y - z = 4$ do intersect.

$\begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 12 \end{bmatrix}$ so we could take $\vec{n} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ as a direction vector for the line.

To find a point on the line select $x = 0$ then $\begin{cases} -2y - z = -4 \\ 2y - z = 4 \end{cases}$ which gives us $\begin{cases} y = 2 \\ z = 0 \end{cases}$.

Hence the point $(0, 2, 0)$ is on both planes, so that a vector equation of the line would be

$$l: \ (x, y, z) = (0, 2, 0) + t\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

## Other methods

Here is another way of doing this: substitute the *parametric* equations of one plane into the *normal* equation of the other

**Example 3** : $\alpha: \ 3x - 2y - z = 15$ and $\beta: \ 3x + 2y + 7z = 3$.

Rewrite $\alpha$ as $\vec{x} = (5, 0, 0) + t\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + s\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ so that $\alpha: \begin{cases} x = 5 + t + 2s \\ y = 3s \\ z = 3t \end{cases}$ and

substitute this in $\beta: \ 3x + 2y + 7z = 3$ to get

$$3(5+t+2s)+2(3s)+7(3t)=3 \quad\Rightarrow\quad 12s+24t=-12 \quad\Rightarrow\quad s=-2t-1$$

hence we find

$$\begin{cases} x=5+t+2(-2t-1) \\ y=3(-2t-1) \\ z=3t \end{cases} \quad\Rightarrow\quad \begin{cases} x=3-3t \\ y=-3-6t \\ z=3t \end{cases}$$

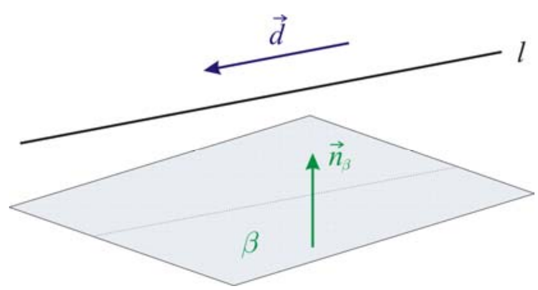
i.e. the line

$$(x,y,z) = (3,-3,0)+\tilde{t}\begin{bmatrix}1\\2\\-1\end{bmatrix}$$

where $\tilde{t}=-3t$. We'll discuss yet another method using **row reduction** of matrices later in this course (see chapters 14 and 15).

## Intersection of plane and line

**(a) Line parallel to plane**: A line does not intersect a plane when it is parallel to the plane *unless* the line is contained in the plane (in which case the entire line would be the intersection). To know if a line is parallel to a plane we need to check if the direction vector of the line is perpendicular to the normal of the plane (a simple dot product will do).




**Example 4**: **(i)** $\alpha: \ 3x-2y-z=-4$ and $l: \ (x,y,z) = (1,2,3)+t\begin{bmatrix}1\\2\\-1\end{bmatrix}$ are parallel since $\begin{bmatrix}3\\-2\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\2\\-1\end{bmatrix} = 0$. Clearly $(1,2,3)$ is on both $l$ and $\alpha$, so the entire

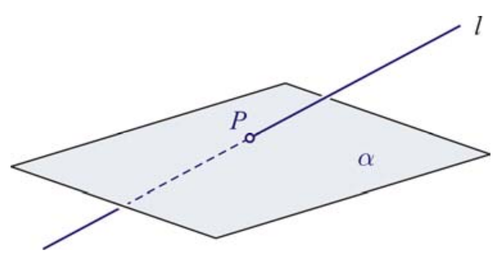
line $l$ must be contained in $\alpha$. An easy check, by substituting $l: \begin{cases} x = 1 + t \\ y = 2 + 2t \\ z = 3 - t \end{cases}$ in $\alpha$ gives us

$$3(1+t) - 2(2+2t) - (3-t) = -4 \quad \Rightarrow \quad -4 = -4$$

which is always true ( for any $t$ ), so that every point of $l$ indeed lies in $\alpha$.

**(ii)** $\alpha: \ 3x - 2y - z = -4$ and $m: \ (x, y, z) = (1,1,1) + t\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ again this line $m$ is parallel to the plane, but this time the point $(1,1,1)$ which is on the line is *not* on the plane, hence these two do *not* intersect.

**(b) Line not parallel to plane**: In 3D a line that is not parallel to a plane must intersect the plane in one point.




To find that point one substitutes the *parametric* equations of the line into the *normal* equation of the plane and solves for the $t$ value of the point of intersection.

**Example 5**: Let $\alpha: \ 3x - 2y - z = -4$ and $k: \ (x, y, z) = (2,1,4) + t\begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$.

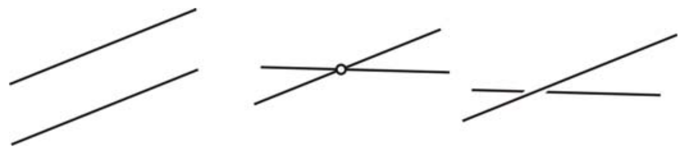
Substituting $k: \begin{cases} x = 2 + 2t \\ y = 1 + 3t \\ z = 4 - 2t \end{cases}$ into $\alpha: \ 3x - 2y - z = -4$ gives us

$$3(2+2t) - 2(1+3t) - (4-2t) = -4 \quad \Rightarrow \quad t = -2$$

Hence the point of intersection is $(-2, -5, 8)$ ( substituting $t = -2$ in $k$ ).

## Intersection of two lines

The situation in 3D is a bit different than in 2D. In the 2 dimensional case two lines are either parallel or intersect each other in one point. In 3 dimensional space we have an extra degree of freedom, and two lines can be non parallel and yet not intersect.




**(a) Parallel lines**: Two parallel lines do not intersect, unless they are the same line. To determine if two lines are parallel we only need to check if their direction vectors are multiples of each other (i.e. if they point in the same or opposite directions).

**(b) Intersecting lines**: To find the point of intersection of two lines we need to solve a system of 3 equations with 2 unknowns. We will discuss this again later, when we talk about solving systems of equations in general (chapters 14 and 15). But since there are here only two unknowns it is not too hard to do. The fact that there are three equations makes it very possible that there are *no* solutions [ i.e. that we are really in case **(c)** ]

**(c) Crossing lines**: The third possibility allows for non parallel lines to cross without intersecting.

**Example 6**: Consider the lines

$$k:\begin{cases} x=1+t\\ y=2+2t\\ z=3-t\end{cases}\quad l:\begin{cases} x=2+2s\\ y=1+4s\\ z=4-2s\end{cases}\quad m:\begin{cases} x=7+r\\ y=13+4r\\ z=-3-3r\end{cases}\quad n:\begin{cases} x=7+u\\ y=13+4u\\ z=5-3u\end{cases}$$

Note that we took a ***different*** parameter—*t*, *s*, *r* and *u*—for each line! There is no reason why the parameters of different lines should be linked at all, let alone be equal to each other.

**(1)** $k$ and $l$ are parallel lines since $\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. It is also clear that $(2,1,4)$ is on $l$ but not on $k$. Hence these lines do not intersect.

**(2)** $l$ and $m$ are clearly not parallel ( their direction vectors are pointing in different directions). To see if the intersect we solve:

$$\begin{cases} 2+2s = 7+r \\ 1+4s = 13+4r \\ 4-2s = -3-3r \end{cases} \Rightarrow \begin{cases} 2s-r=5 \\ 4s-4r=12 \end{cases} \Rightarrow \begin{cases} s=2 \\ r=-1 \end{cases}$$

Note that we just used the first **two** equations to solve $r$ and $s$, we still need to check that these two values satisfy the third equation! … and they do. Hence using e.g. $s=2$ in $l: \begin{cases} x=2+2s \\ y=1+4s \\ z=4-2s \end{cases}$ we find the point of intersection $(6,9,0)$;

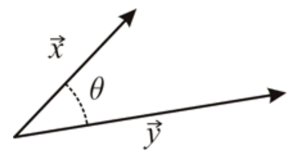
of course $r=-1$ in $m: \begin{cases} x=7+r \\ y=13+4r \\ z=-3-3r \end{cases}$ gives the same point $(6,9,0)$.

**(3)** $l$ and $n$. This case is *almost* identical to the previous case. Again the lines are not parallel. To find the point of intersection (if it exists) we need to solve:

$$\begin{cases} 2+2s = 7+u \\ 1+4s = 13+4u \\ 4-2s = 5-3u \end{cases} \Rightarrow \begin{cases} 2s-u=5 \\ 4s-4u=12 \end{cases} \Rightarrow \begin{cases} s=2 \\ u=-1 \end{cases}$$

but this time when we check the last equation we get a false statement ( $0=8$ ), and hence the system is not solvable: the first two equations would force $s$ to be 2, and $u$ to be $-1$, but these values do not satisfy the last equation. Hence there are no solutions, and thus no point of intersection.

The relationship $\vec{x}\cdot\vec{y} = \|\vec{x}\|\|\vec{y}\|\cos\theta$ between the dot product $\vec{x}\cdot\vec{y}$ and the angle $\theta$ between the vectors $\vec{x}$ and $\vec{y}$ is true in 2D and any higher dimensional Euclidean space. In fact it is essentially a 2 dimensional phenomena: it only depends on the law of cosines in 2 dimensions – in the triangle spanned by $\vec{x}$ and $\vec{y}$.




The wonderful thing about it is that it allows us to compute angles *algebraically* in higher dimensional spaces, without having to worry about being able to visually grasp the geometry.

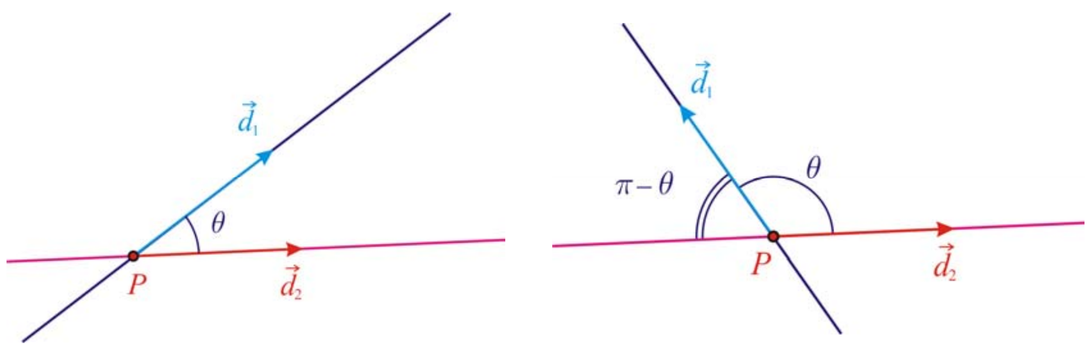
$$\theta = \cos^{-1}\left(\frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\|\vec{y}\|}\right)$$

## Angles between two lines

**Theorem 9.1**: Let $l_1: \ X = P + t\vec{d}_1$ and $l_2: \ X = P + t\vec{d}_2$ be two intersecting lines, and let $\theta$ be the angle between the two direction vectors $\vec{d}_1$ and $\vec{d}_2$, i.e.

$$\theta = \cos^{-1}\left(\frac{\vec{d}_1\cdot\vec{d}_2}{\|\vec{d}_1\|\|\vec{d}_2\|}\right)$$

then the angle between the two lines is: $\min(\theta, 180^\circ - \theta)$
[or in radians: $\min(\theta, \pi - \theta)$ ]



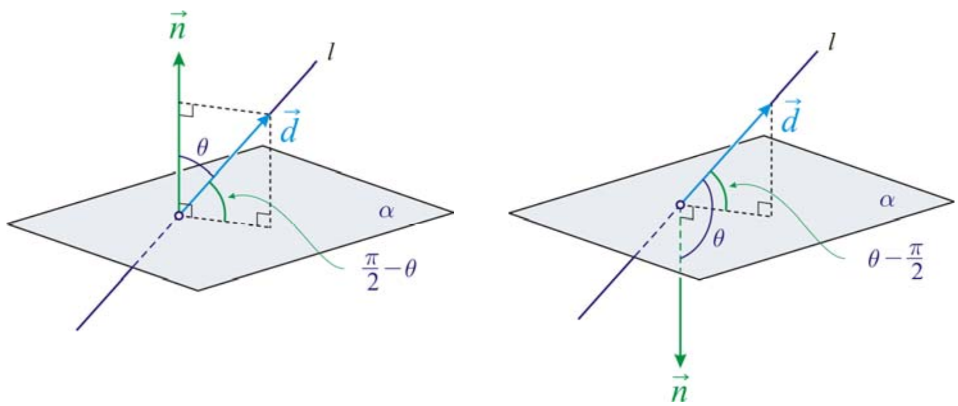

**Proof**: Since the angle between two lines is the 'smaller' of the two possible angles it is either $\theta = \cos^{-1}\left(\frac{\vec{d}_1\cdot\vec{d}_2}{\|\vec{d}_1\|\|\vec{d}_2\|}\right)$ ***or*** its supplement $180^\circ - \theta$ (or: $\pi - \theta$). $\square$

## Angles between a line and a plane

**Theorem 9.2**: Let the line $l$ with direction vector $\vec{d}$ intersect the plane $\alpha$ with normal $\vec{n}$, and let $\theta = \cos^{-1}\left(\dfrac{\vec{n}\cdot\vec{d}}{\|\vec{n}\|\|\vec{d}\|}\right)$ be the angle between $\vec{n}$ and $\vec{d}$, then the angle between the line and the plane is

$$|\theta - 90°| \qquad \left(\text{ or in radians: } \left|\theta - \frac{\pi}{2}\right|\right)$$

**Proof**: Let $\theta = \cos^{-1}\left(\dfrac{\vec{n}\cdot\vec{d}}{\|\vec{n}\|\|\vec{d}\|}\right)$ be the angle between $\vec{n}$ and $\vec{d}$, then depending on what direction the normal faces ('up' or 'down': see pictures ), the angle between the line




and the plane is either $90° - \theta$ (in radians: $\dfrac{\pi}{2} - \theta$ ) *or* when $\theta$ is obtuse $\theta - 90°$ (in radians: $\theta - \dfrac{\pi}{2}$ ) :

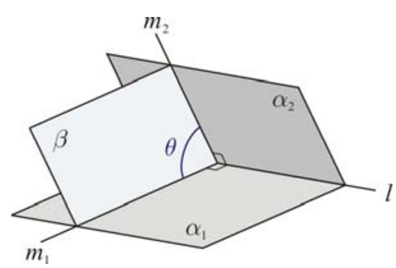
i.e. $\quad |\theta - 90°| \quad \left(\text{ or in radians: } \left|\theta - \frac{\pi}{2}\right|\right)$

## Angles between two planes

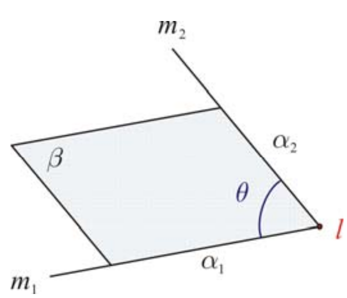
**Theorem 9.3**: Let $\alpha_1$, the plane with normal $\vec{n}_1$, and $\alpha_2$, the plane with normal $\vec{n}_2$, be intersecting planes, and let $\theta = \cos^{-1}\left(\dfrac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\|\|\vec{n}_2\|}\right)$ be the angle between $\vec{n}_1$ and $\vec{n}_2$, then the angle between the two planes is given by $\theta$ or its supplement $180° - \theta$ whichever is smaller $\left(\text{in radians: } \theta \text{ or } \pi - \theta\right)$, i.e.

$$\min(\theta, 180° - \theta) \qquad [\,\min(\theta, \pi - \theta)\,]$$

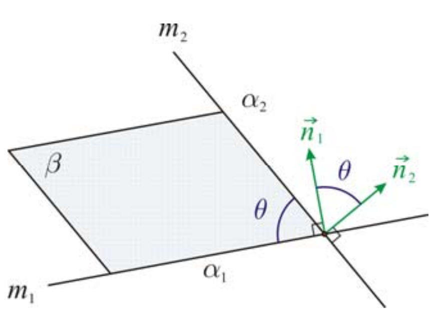
**Proof**: Let $l$ be the line of intersection of the two planes $\alpha_1$ and $\alpha_2$, and let $\beta$ be a plane perpendicular to $l$. Let $m_1$ and $m_2$ be the lines of intersection of plane $\beta$ and planes $\alpha_1$ and $\alpha_2$ resp. ( see picture )

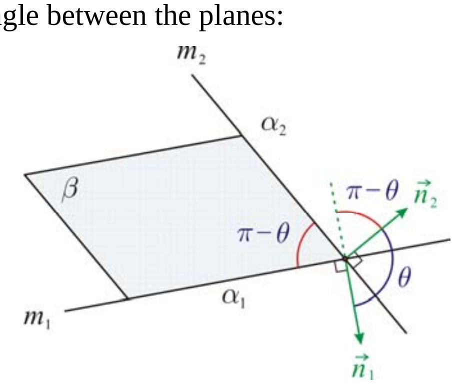



The angle between the two planes is defined to be the same as the angle between $m_1$ and $m_2$. A side view is very instructive here, basically looking in the direction of line *l*:




If we also put the normals $\vec{n}_1$ and $\vec{n}_2$ of $\alpha_1$ and $\alpha_2$ in this picture it becomes clear that the angle between the normals is related to the angle between the planes:



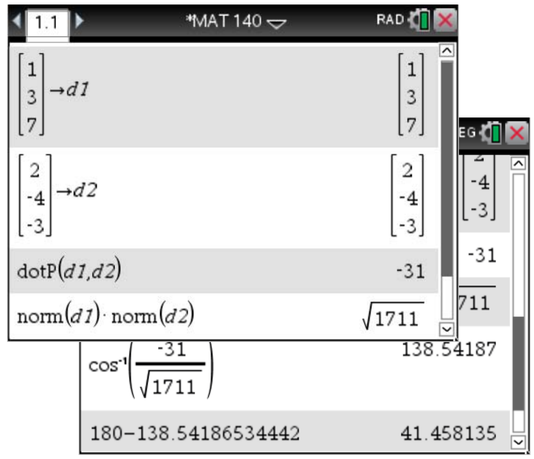



Depending on the direction of the normals the angle between the planes is either the angle between the normals or its supplement: $\min(\theta, 180° - \theta)$ $[\min(\theta, \pi - \theta)]$

**Example 1**: $l_1:\ (x, y, z) = (6, 0, 8) + t\begin{bmatrix}1\\3\\7\end{bmatrix}$ and $l_2:\ (x, y, z) = (6, 0, 8) + t\begin{bmatrix}2\\-4\\-3\end{bmatrix}$ do clearly intersect in $(6, 0, 8)$, and

$$\theta = \cos^{-1}\left(\frac{\vec{d}_1 \cdot \vec{d}_2}{\|\vec{d}_1\|\|\vec{d}_2\|}\right) = 138.5419°$$

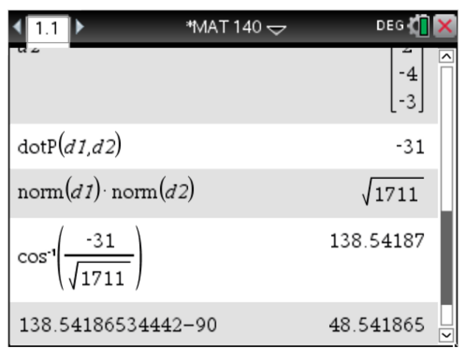
hence the angle between the lines is $41.4581°$




**Example 2**: $l:\ (x, y, z) = (2, -4, 5) + t\begin{bmatrix}1\\3\\7\end{bmatrix}$ and $\alpha:\ 2x - 4y - 3z = 5$ intersect and

$$\theta = \cos^{-1}\left(\frac{\vec{n} \cdot \vec{d}}{\|\vec{n}\|\|\vec{d}\|}\right) = 138.5419°$$ hence the

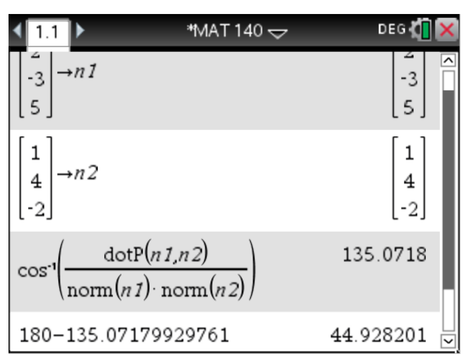
angle between the line and the plane is $48.5419°$.




**Example 3**: $\alpha_1:\ 2x - 3y + 5z = 4$ and $\alpha_2:\ x + 4y - 2z = 5$ intersect, and

$$\theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\|\|\vec{n}_2\|}\right) = 135.0718°$$

hence the angle between the planes is $44.9282°$



# 10. Distances in 3D

There are several distances we would like to be able to compute:

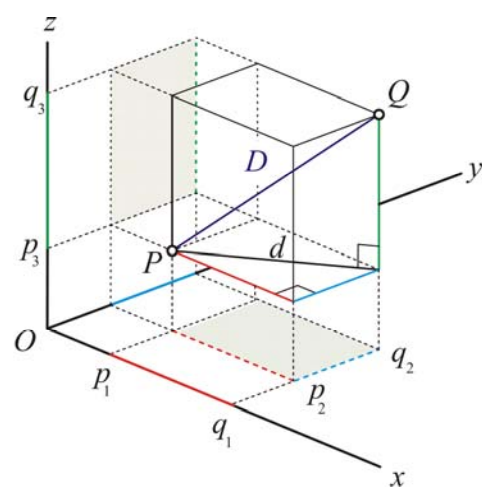
- **(a)** Distance between two points
- **(b)** Distance between a point and a line
- **(c)** Distance between a point and a plane
- **(d)** Distance between two lines
- **(e)** Distance between a line and a plane
- **(f)** Distance between two planes

## (a) Distance between two points

**Theorem 10.1**: Let $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ then

$$\operatorname{dist}(P,Q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2}$$

**Proof**: Use the theorem of Pythagoras twice:




$$d^2 = (p_1 - q_1)^2 + (p_2 - q_2)^2$$

and

$$D^2 = d^2 + (p_3 - q_3)^2$$

Combining these gives us

$$\begin{aligned} \operatorname{dist}(P,Q)^2 &= D^2 = d^2 + (p_3 - q_3)^2 \\ &= (p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2 \end{aligned} \qquad \square$$

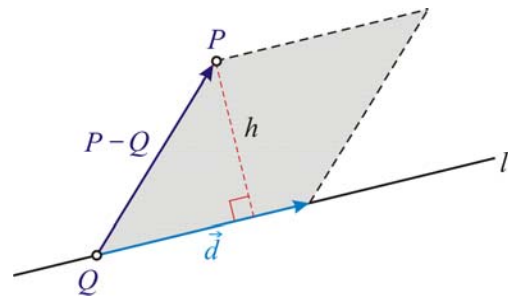
**Example 1**: Let $P = (6, 8, 9)$ and $Q = (3, 4, -3)$ then

$$\operatorname{dist}(P,Q) = \sqrt{(6-3)^2 + (8-4)^2 + (9-(-3))^2} = \sqrt{9+16+144} = 13$$

## (b) Distance between a point and a line

**Theorem 10.2**: The distance of point $P$ to line $l: \quad X = Q + t\vec{d}$ is given by

$$\operatorname{dist}(P,l) = \frac{\|(P-Q)\times\vec{d}\,\|}{\|\vec{d}\,\|}$$




**Proof**: The parallelogram spanned by $P-Q$ and $\vec{d}$ has area $\|(P-Q)\times\vec{d}\,\|$ [ see chapter 7 ]. But the area of a parallelogram is height times base, hence

$$\|(P-Q)\times\vec{d}\,\| = h\cdot b = h\cdot\|\vec{d}\,\| \quad\Rightarrow\quad h = \frac{\|(P-Q)\times\vec{d}\,\|}{\|\vec{d}\,\|}.$$

Since $h$ is precisely the $\operatorname{dist}(P,l)$, so where done. $\square$

**Example 2**: Let $P=(2,5,8)$ and $l: \ (x,y,z) = (2,3,6)+t\begin{bmatrix}-3\\4\\5\end{bmatrix}$

Then

$$P-Q = (2,5,8)-(2,3,6) = \begin{bmatrix}0\\2\\2\end{bmatrix}$$

so that

$$(P-Q)\times\vec{d} = \begin{bmatrix}0\\2\\2\end{bmatrix}\times\begin{bmatrix}-3\\4\\5\end{bmatrix} = \begin{bmatrix}4\\-8\\8\end{bmatrix}$$

and thus $\qquad \|(P-Q)\times\vec{d}\,\| = \sqrt{4^2+(-8)^2+8^2} = 12$

and $\qquad \|\vec{d}\,\| = \sqrt{(-3)^2+4^2+5^2} = \sqrt{50}$

hence

$$\operatorname{dist}(P,l) = \frac{\|(P-Q)\times\vec{d}\,\|}{\|\vec{d}\,\|} = \frac{12}{\sqrt{50}} = \frac{6\sqrt{2}}{5} \approx 1.6971$$

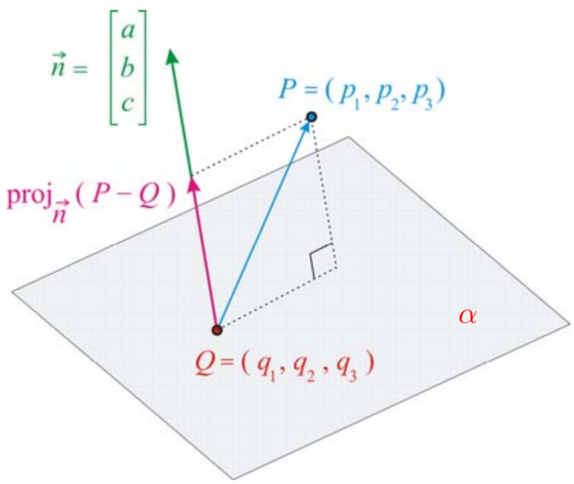
[ Note that there are many other methods to compute the distance of a point to a line! ]

### (c) Distance between a point and a plane

**Theorem 10.3**: The distance of a point $P = (p_1, p_2, p_3)$ to the plane $\alpha$ through $Q$ and with normal $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, i.e. with $\alpha\colon\ ax + by + cz = d$ is given by

$$\text{dist}(P,\alpha) = \left\| \text{proj}_{\vec{n}}(P-Q) \right\| = \frac{|\, a\,p_1 + b\,p_2 + c\,p_3 - d \,|}{\sqrt{a^2+b^2+c^2}}$$

**Proof**: This proof is very much like the proof of the distance formula of a point to a line in 2D (see chapter 5).




Clearly $\text{dist}(P,\alpha) = \left\| \text{proj}_{\vec{n}}(P-Q) \right\|$ (see picture)

$$\begin{aligned}
\left\| \text{proj}_{\vec{n}}(P-Q) \right\| &= \left\| \frac{\vec{n}\cdot(P-Q)}{\vec{n}\cdot\vec{n}}\,\vec{n} \right\| \\
&= \left| \frac{\vec{n}\cdot(P-Q)}{\vec{n}\cdot\vec{n}} \right| \left\| \vec{n} \right\| && [\text{ moving out the scalar } \|t\,\vec{n}\| = |t|\cdot\|\vec{n}\| \text{ ]} \\
&= \frac{\left| a(p_1-q_1)+b(p_2-q_2)+c(p_3-q_3) \right|}{a^2+b^2+c^2}\sqrt{a^2+b^2+c^2} \\
&= \frac{\left| a\,p_1+b\,p_2+c\,p_3-(a\,q_1+b\,q_2+c\,q_3) \right|}{\sqrt{a^2+b^2+c^2}} \\
&= \frac{\left| a\,p_1+b\,p_2+c\,p_3-d \right|}{\sqrt{a^2+b^2+c^2}} && [\ a\,q_1+b\,q_2+c\,q_3 = d \quad \text{since } Q\in\alpha\ ]
\end{aligned}$$

$\square$

**Example 3**: Let $P=(6,9,7)$ and $\alpha:\ (x,y,z) = (2,3,5)+t\begin{bmatrix}-1\\4\\2\end{bmatrix}+s\begin{bmatrix}3\\-5\\1\end{bmatrix}$

First we'll write $\alpha$ in normal form: $\begin{bmatrix}-1\\4\\2\end{bmatrix}\times\begin{bmatrix}3\\-5\\1\end{bmatrix} = \begin{bmatrix}14\\7\\-7\end{bmatrix}$. Hence $\vec{n} = \begin{bmatrix}2\\1\\-1\end{bmatrix}$ is a good choice for the normal.

The normal equation $\alpha:\ ax+by+cz = d$ then becomes

$$\alpha:\ 2x+y-z = 2 \qquad [\text{ of course here } d = 2\cdot 2+3-5 = 2\ ]$$

So that:

$$\mathrm{dist}(P,\alpha) = \frac{|\,2\cdot 6+9-7-2\,|}{\sqrt{2^2+1^2+(-1)^2}} = \frac{12}{\sqrt{6}} = 2\sqrt{6}$$

Alternatively we could use $\mathrm{dist}(P,\alpha) = \left\|\mathrm{proj}_{\vec{n}}(P-Q)\right\|$

Since $P-Q = (6,9,7)-(2,3,5) = \begin{bmatrix}4\\6\\2\end{bmatrix}$ and $\vec{n} = \begin{bmatrix}2\\1\\-1\end{bmatrix}$

we find that

$$\mathrm{proj}_{\vec{n}}(P-Q) = \frac{4\cdot 2+6\cdot 1+2(-1)}{2^2+1^2+(-1)^2}\begin{bmatrix}2\\1\\-1\end{bmatrix} = \frac{12}{6}\begin{bmatrix}2\\1\\-1\end{bmatrix} = \begin{bmatrix}4\\2\\-2\end{bmatrix}$$

and thus

$$\mathrm{dist}(P,\alpha) = \left\|\mathrm{proj}_{\vec{n}}(P-Q)\right\| = \sqrt{16+4+4} = \sqrt{24} = 2\sqrt{6}$$

**Note 1**: When the point $P=(p_1,p_2,p_3)$ is *on* the plane $\alpha:\ ax+by+cz=d$,

i.e. when $a\,p_1+b\,p_2+c\,p_3=d$, the formula indeed gives a zero distance:

$$\mathrm{dist}(P,\alpha) = \frac{|\,a\,p_1+b\,p_2+c\,p_3-d\,|}{\sqrt{a^2+b^2+c^2}} = \frac{|\,0\,|}{\sqrt{a^2+b^2+c^2}} = 0$$

**Note 2**: The distance between the origin and a plane is given by

$$\mathrm{dist}(O,\alpha) = \frac{|\,a\cdot 0+b\cdot 0_2+c\cdot 0-d\,|}{\sqrt{a^2+b^2+c^2}} = \frac{|\,d\,|}{\sqrt{a^2+b^2+c^2}} \qquad [\text{ Compare Th 10.7 }]$$

## (d) Distance between two lines

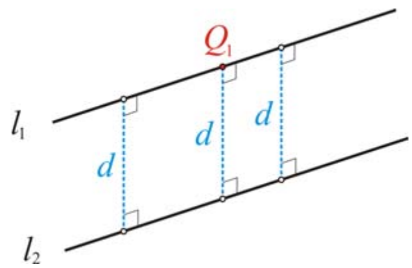
### (1) Parallel lines:

**Theorem 10.4**: Let $l_1$ and $l_2$ be parallel lines with common direction vector $\vec{d}$ and points $Q_1$ and $Q_2$ on $l_1$ and $l_2$ respectively, then

$$\operatorname{dist}(l_1,l_2) = \frac{\|(Q_1-Q_2)\times\vec{d}\,\|}{\|\vec{d}\,\|}$$

**Proof**: If two lines $l_1$ and $l_2$ are parallel, all points of $l_1$ have the same distance to $l_2$ (and vice versa).

We could just take $Q_1$ on $l_1$ and compute its distance to the other line $l_2$ using Theorem 10.2:




$$\begin{aligned}\operatorname{dist}(l_1,l_2) &= \operatorname{dist}(Q_1,l_2)\\ &= \frac{\|(Q_1-Q_2)\times\vec{d}\,\|}{\|\vec{d}\,\|}\end{aligned}$$

[ Here $\vec{d}$ is the common direction vector of both $l_1$ and $l_2$. ]

**Example 4**:

Let $l_1:\ (x,y,z) = (7,5,11)+t\begin{bmatrix}4\\-3\\12\end{bmatrix}$ and $l_1:\ (x,y,z) = (4,3,5)+t\begin{bmatrix}4\\-3\\12\end{bmatrix}$ then

$Q_1-Q_2 = (7,5,11)-(4,3,5) = \begin{bmatrix}3\\2\\6\end{bmatrix}$ and thus $(Q_1-Q_2)\times\vec{d} = \begin{bmatrix}3\\2\\6\end{bmatrix}\times\begin{bmatrix}4\\-3\\12\end{bmatrix} = \begin{bmatrix}42\\-12\\-17\end{bmatrix}$

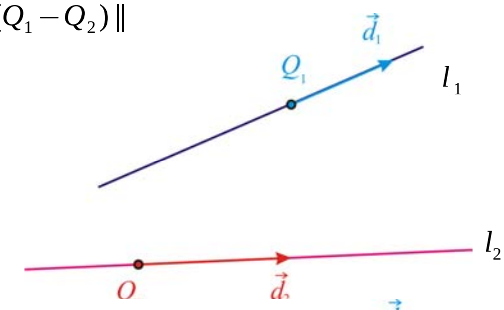
which gives is that

$$\operatorname{dist}(l_1,l_2) = \frac{\|(Q_1-Q_2)\times\vec{d}\,\|}{\|\vec{d}\,\|} = \frac{\sqrt{42^2+(-12)^2+(-17)^2}}{\sqrt{4^2+(-3)^2+12^2}} = \sqrt{13}$$

### (2) Non parallel lines

**Theorem 10.5**: The distance between the lines $l_1: \quad X = Q_1 + t\vec{d}_1$ and $l_2: \quad X = Q_2 + t\vec{d}_2$, where $\vec{d}_1$ is not parallel to $\vec{d}_2$, is given by

$$\text{dist}(l_1, l_2) = \| \text{proj}_{\vec{d}_1 \times \vec{d}_2} (Q_1 - Q_2) \|$$

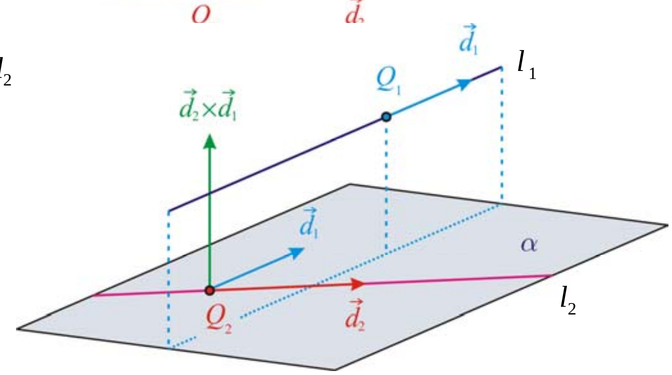



**Proof**:

Let the two lines be given as above.

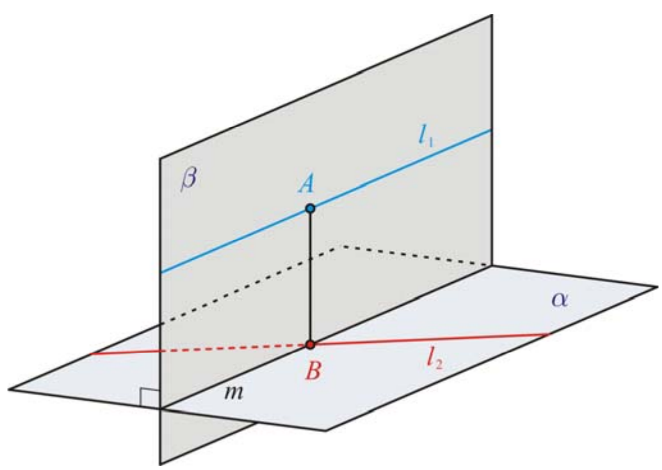
Let's introduce a plane $\alpha$ *through* $l_2$ *parallel* to $l_1$

$$\alpha: \quad X = Q_2 + t\vec{d}_2 + s\vec{d}_1$$




Note that $\vec{d}_1 \times \vec{d}_2$ is a normal of plane $\alpha$. Also note that $l_1$ is parallel to $\alpha$, so that every point on the line $l_1$ has the same distance to the plane $\alpha$.
In fact this common distance we claim is precisely the distance between the two lines. This may need a bit of explaining

Let $\beta$ be the plane *through* $l_1$ *perpendicular* to the plane $\alpha$. If you fancy an equation for $\beta$, here it is $\beta: \quad X = Q_1 + r\vec{d}_1 + u(\vec{d}_1 \times \vec{d}_2)$. Furthermore let $m$ be the intersection of $\alpha$ and $\beta$, and let $B$ be the intersection of $l_2$ and $m$ (see picture).

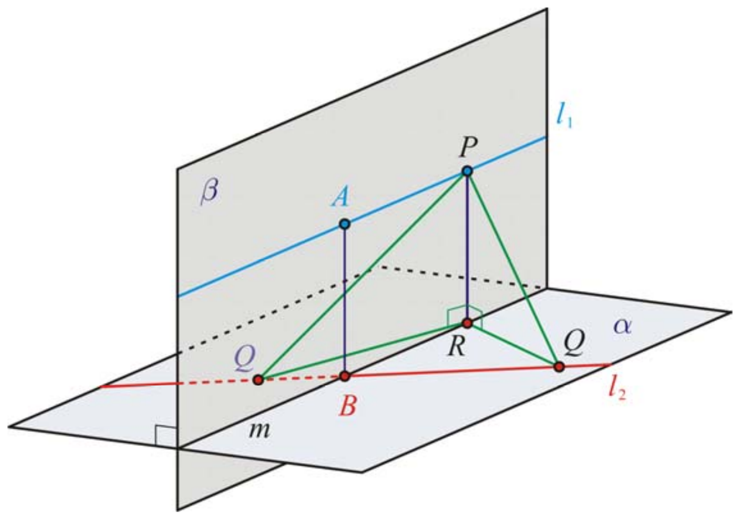



Note $l_1 \| m$.

Finally, let $A$ be the point on $l_1$ straight above $B$.

We claim that dist( $A$, $B$ ) is the distance between the lines $l_1$ and $l_2$: it is the shortest distance amongst all the distances between any point $P$ on $l_1$ to any point $Q$ on $l_2$.

(see next picture)




If $P$ is an arbitrary point on $l_1$ and $R$ is the point on $\alpha$ plumb down from $P$ on $m$ and $Q$ any point on $l_2$ then $\text{dist}(P,Q) \geq \text{dist}(P,R) = \text{dist}(A,B)$ [ The hypotenuse in a right triangle is longer than either of the other sides (or equal) ]. Hence by Theorem 10.3

$$\text{dist}(l_1,l_2) = \text{dist}(A,B) = \text{dist}(Q_1,\alpha) = \|\text{proj}_{\vec{d}_1\times\vec{d}_2}(Q_1-Q_2)\| \qquad \square$$

**Example 5**: $l_1: \ (x,y,z) = (7,5,11)+t\begin{bmatrix}3\\4\\0\end{bmatrix}$ and $l_2: \ (x,y,z) = (4,3,5)+t\begin{bmatrix}0\\4\\1\end{bmatrix}$

then $Q_1-Q_2 = (7,5,11)-(4,3,5) = \begin{bmatrix}3\\2\\6\end{bmatrix}$ and $\vec{d}_1\times\vec{d}_2 = \begin{bmatrix}3\\4\\0\end{bmatrix}\times\begin{bmatrix}0\\4\\1\end{bmatrix} = \begin{bmatrix}4\\-3\\12\end{bmatrix}$.

Hence $\text{proj}_{\vec{d}_1\times\vec{d}_2}(Q_1-Q_2) = \dfrac{78}{169}\begin{bmatrix}4\\-3\\12\end{bmatrix} = \dfrac{6}{13}\begin{bmatrix}4\\-3\\12\end{bmatrix}$ so that $\|\text{proj}_{\vec{d}_1\times\vec{d}_2}(Q_1-Q_2)\| = 6$.

**Alternatively**: The plane through $l_2$ parallel to $l_1$ is $4x-3y+12z=67$ (why?) hence

$$\text{dist}(Q_1,\alpha) = \frac{|\,4\cdot7-3\cdot5+12\cdot11-67\,|}{\sqrt{4^2+(-3)^2+12^2}} = \frac{78}{13} = 6$$

## (e) Distance between a line and a plane

### (1) Line not parallel to plane

In this case the line has to intersect the plane and hence the distance (i.e. shortest distance) is zero.

### (2) Line parallel to plane

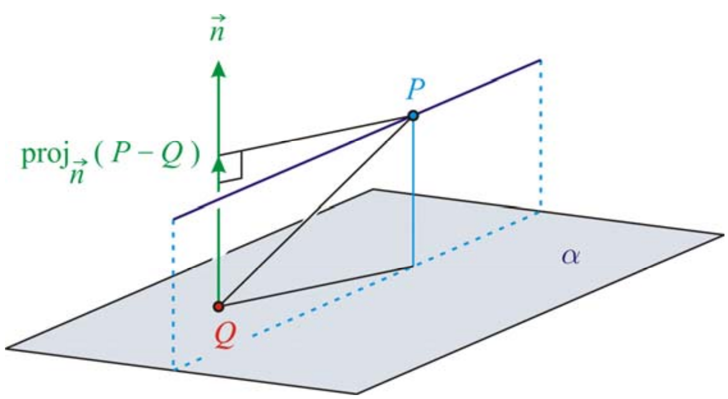
**Theorem 10.6**: Let line $l: \quad X = P + t\vec{D}$ be parallel to the plane $\alpha$, through $Q$, with

normal $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, i.e. $\alpha: \; ax + by + cz = d$ and $\vec{n} \times \vec{D} = 0$, then

$$\operatorname{dist}(l, \alpha) = \left\| \operatorname{proj}_{\vec{n}}(P - Q) \right\| = \frac{| a p_1 + b p_2 + c p_3 - d |}{\sqrt{a^2 + b^2 + c^2}}$$

**Proof**: In this case we can take the point $P = (p_1, p_2, p_3)$ on the line and compute its distance to the plane $\alpha$ as in Theorem 10.3:

$$\operatorname{dist}(P, \alpha) = \left\| \operatorname{proj}_{\vec{n}}(P - Q) \right\| = \frac{| a p_1 + b p_2 + c p_3 - d |}{\sqrt{a^2 + b^2 + c^2}}$$




**Example 6**: Let $l: \; (x, y, z) = (6, 9, 7) + t \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$ and $\alpha: \; 2x + y - z = 2$

[ Note that $l$ is parallel to the plane! ] $\operatorname{dist}(l, \alpha) = \dfrac{| 2 \cdot 6 + 9 - 7 - 2 |}{\sqrt{2^2 + 1^2 + (-1)^2}} = \dfrac{12}{\sqrt{6}} = 2\sqrt{6}$

## (f) Distance between two planes

### (1) Non parallel planes:

When the planes are not parallel they intersect in 3D, so that their distance is zero.

### (2) Parallel planes:

**Theorem 10.7**: Let two parallel planes be given by

$$\alpha_1 : \; ax+by+cz=d_1 \quad \text{and} \quad \alpha_2 : \; ax+by+cz=d_2$$

then

$$\text{dist}(\alpha_1,\alpha_2) = \frac{|\, d_1 - d_2 \,|}{\sqrt{a^2+b^2+c^2}}$$

[ Note that the two planes have the same normal here! If the normals are non-zero multiple of each other, we would have to divide first by this multiple to make the normals the same. ]

**Proof**: Again we can pick any point from one of the planes and compute its distance to the other plane using Theorem 10.3:

$$P = (p_1, p_2, p_3) \quad \text{on} \quad \alpha_1 : \; ax+by+cz=d_1 \quad \Rightarrow \quad a\,p_1 + b\,p_2 + c\,p_3 = d_1$$

and thus

$$\text{dist}(\alpha_1,\alpha_2) = \text{dist}(P,\alpha_2) = \frac{\left|\, a\,p_1 + b\,p_2 + c\,p_3 - d_2 \,\right|}{\sqrt{a^2+b^2+c^2}} = \frac{\left|\, d_1 - d_2 \,\right|}{\sqrt{a^2+b^2+c^2}} \quad \square$$

**Example 7**: Let $\alpha_1 : \; 4x-3y+12z=12$ and $\alpha_2 : \; 4x-3y+12z=-14$ then

$$\text{dist}(\alpha_1,\alpha_2) = \frac{\left|\, 12-(-14) \,\right|}{\sqrt{a^2+b^2+c^2}} = \frac{26}{13} = 2$$

**Example 8**: Let $\alpha_1 : \; x-3y+4z=2$ and $\alpha_2 : \; -2x+6y-8z=6$. We first have to make the normals the same, i.e. divide e.g. $-2x+6y-8z=6$ by $-2$ to get $\alpha_2 : \; x-3y+4z=-3$. Then

$$\text{dist}(\alpha_1,\alpha_2) = \frac{\left|\, 2-(-3) \,\right|}{\sqrt{a^2+b^2+c^2}} = \frac{5}{\sqrt{26}}$$