Fundamentals of Differentiation Part 3

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AY 22/23 Trimester 2

Recap

1. Chain Rule
$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$
2. Implicit and explicit equations, how to differentiate implicitly
3. Explicit equations: Tangent line equation to f at $(a, f(a))$
$$y = f'(a)(x - a) + f(a)$$
Implicit equations: Tangent line equation to graph at (x_0, y_0)
$$y = \underbrace{\frac{dy}{dx}(x_0, y_0)}_{\substack{\text{derivative of } y \\ \text{at } (x_0, y_0)}} (x - x_0) + y_0$$

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Higher order derivatives

If a function $y = f(x)$ is differentiable, then we have a function $f'(x)$, the derivative of $y = f(x)$.

We also have the derivative of $f'(x)$, which is a function $f''(x)$, called the *second derivative* of $f(x)$, is the derivative of the first derivative

$$f''(x) = (f')'(x).$$

The second derivative can also be written as

$$\underbrace{\frac{d}{dx}}_{\substack{\text{derivative}\\ \text{of}}} \underbrace{\left(\frac{dy}{dx}\right)}_{\substack{\text{first}\\ \text{derivative}}} = \underbrace{\frac{d^2y}{dx^2}}_{\substack{\text{second}\\ \text{derivative}}}.$$

Exercise 1

Find the second derivative for each of the following functions.

1. $f(x) = 3x^2 + 2x + 1$
2. $f(x) = \sin(2x)$
3. $f(x) = e^{5x}$
4. $f(x) = \ln(x^2 + 1)$

Third derivative onwards

The *third derivative* $f'''(x)$ of $y = f(x)$ is the derivative of the second derivative

$$f'''(x) = (f'')'(x).$$

The *fourth derivative* and onwards are abbreviated slightly differently (for pretty obvious reasons)

$$f^{(4)}(x) = (f''')'(x).$$

In general, the *n-th derivative* $f^{(n)}(x)$ is obtained by differentiating the $(n-1)$-th derivative:

$$f^{(n)}(x) = (f^{(n-1)})'(x).$$

Exercise 2

Find the third derivative for each of the functions. (These were the functions in Exercise 1)

1. $f(x) = 3x^2 + 2x + 1$
2. $f(x) = \sin(2x)$
3. $f(x) = e^{5x}$
4. $f(x) = \ln(x^2 + 1)$

Exercise 2

Definitions of increasing and decreasing functions

Definition

Let A be any subset of the domain of a function f. For $x_1, x_2 \in A$,

- f is *increasing* on A if

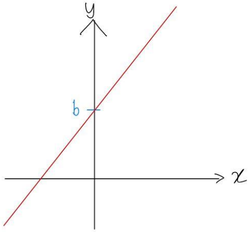
$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2,$$

- f is *decreasing* on A if

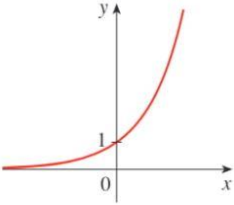
$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2.$$

Important: For a function to be increasing/decreasing, $f(x_1) \leq f(x_2)/f(x_1) \geq f(x_2)$ must hold for all pairs of $x_1 < x_2$!

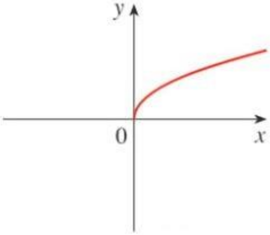
Examples of increasing functions



$y = mx + b \ (m > 0)$



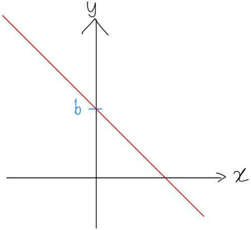
$y = a^x \ (a > 0)$



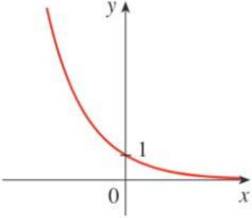
$y = \sqrt{x}$

What do you observe about the gradient of the functions here?

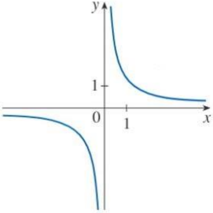
Examples of decreasing functions



$y = mx + b \ (m < 0)$



$y = a^x \ (0 < a < 1)$



$y = \frac{1}{x}$

What do you observe about the gradient of the functions here?

Increasing/Decreasing Test

By observing the sign (positive/negative) of the gradient of a differentiable function f, we can tell if f is increasing or decreasing.

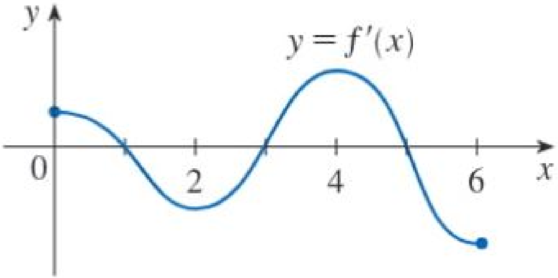
Theorem (Increasing/Decreasing Test or I/D Test)

Let I be an interval which is a subset of the domain of f. If f is differentiable on I, then

- f is increasing on I if and only if $f'(x) > 0$ on A,
- f is decreasing on I if and only if $f'(x) < 0$ on A.

Example 1

The graph of the *derivative* f' of a function f is shown below. Determine the intervals for which f is increasing or decreasing.



Critical points of f

Definition

$(\star)$ A *critical point* of a function f is a point c where either

1. $f'(c) = 0$, or
2. f is not differentiable at c.

Critical points play an important role in the identification of intervals where a function is increasing or decreasing, and they also play a big role in optimization (later in the course).

Exercise 3

Let $f(x) = \dfrac{x}{x^2 + 1}$.

1. What is the domain of f?
2. Find the intervals for which f is increasing or decreasing.

Exercise 3