CSD2258 Tutorial 2

Problem 1. Which of the following functions are 1-1? Justify your answers.
(a) $f : \mathbb{R} - \{0\} \to \mathbb{R}$ with $f(x) = \frac{3x-1}{x}$.
(b) $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(x) = \sqrt{x}$.
(c) $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $h(x, y) = x$.

Problem 2. Give an example of a function from $\mathbb{N}$ to $\mathbb{N}$ that is
(a) one-to-one but not onto.
(b) onto but not one-to-one.
(c) both onto and one-to-one (but different from the identity function).
(d) neither one-to-one nor onto.

Problem 3. Which of the following functions f are 1-1, or onto, or a bijection, or none of these? Find f^{-1} in the case f is a bijection.
(a) $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x) = x^3 + 1.$$
(b) $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that
$$f(x) = |x|,$$
where $\mathbb{R}_{\geq 0}$ is the set of nonnegative real numbers.
(c) $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with
$$f(x, y) = x.$$

Problem 4. Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ which are defined by
$$f(x) = x + 3, \;\; g(x) = -x^3$$
(a) Find $f \circ g$, $g \circ f$, f^{-1}, g^{-1}
(b) Show that
$$(f \circ g)^{-1} = g^{-1} \circ f^{-1} \text{ and } (g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Problem 5. In each of the following cases of the the function $f : \mathbb{R} \to \mathbb{R}$, state the intervals on which f is increasing and the intervals on which f is decreasing.

(a) $f(x) = x^2 + 4x - 21$ (b) $g(x) = 2x^3 + 3x^2 - 36x$

Problem 6. In this exercise, consider the vector space

$$\mathbb{R}^n = \{x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : \ x_i \in \mathbb{R} \text{ for all } i\}.$$

A **linear transformation** of $\mathbb{R}^n$ is a function $T : \mathbb{R}^n \to \mathbb{R}^n$ such that there is an $n \times n$ matrix A with $T(x) = Ax$ for all $x \in \mathbb{R}^n$ (note that Ax is a matrix multiplication). We call A the **matrix** or T.

(a) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with the matrix, that is, A^{-1} exists. Show that T^{-1} is linear transformation with matrix A^{-1}.
(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation about the origin over 30°. Using the result of part a, find the preimages P and Q of the points $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$, respectively.

Remarks. The rotation matrix is $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Further, the inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Hints and Instructions.

1. f, g are 1-1. h is not 1-1.

2. Try it.

3a. f is a bijection.
3b. f is onto, but not $1 - 1$.
3c. f is onto, but not $1 - 1$.

4b. You need to show that for any $y \in R$,

$$(f \circ g)^{-1}(y) = g^{-1} \circ f^{-1}(y) \text{ and } (g \circ f)^{-1}(y) = f^{-1} \circ g^{-1}(y).$$

5c. Solving $f'(x) = 0$, you will get 2 solutions $x = c$ and $x = d$. Three intervals $(-\infty, c]$, $[c, d]$ and $[d, \infty)$ should be considered.

6a. You need to show $T^{-1}(x) = A^{-1}x$.
6b. The preimage of a point A is any point A' such that $T(A') = A$. So $A' = T^{-1}(A)$.