Rates of Change/Related Rates

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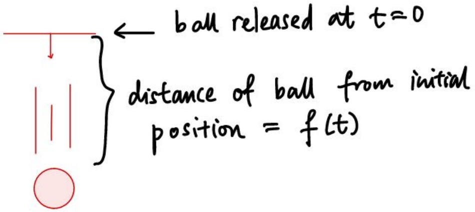
Physical meaning of $f'(x)$

1. The derivative of a function $f(x)$ can be regarded as the **instantaneous rate of change** of f at a certain point x (think: for $y = f(x)$, what is the change in y for at a particular point x? It's $f'(x)$).
2. For example, if t represents time in seconds, and $y = f(t)$ represents the distance of a falling ball from a starting point, then $f'(t)$ is the **instantaneous velocity** of the object at time t in seconds.



Velocity of a free falling object

The function $f(t)$ was discovered by Galileo about four centuries ago, when he discovered that the distance fallen by any freely falling body is proportional to the **square** of the time it has been falling (ignoring air resistance). It turns out that the distance fallen (in meters m) after t seconds is

$$f(t) = 4.9t^2.$$

Exercise 1: Find the instantaneous velocity of the freely falling body at time $t = 5$ seconds.

Exercise 2

Find the **instantaneous acceleration** of the freely falling object at time $t = 5$ seconds.

Introduction to Related Rates: an example

We begin with an example: suppose that we are pumping air into a spherical balloon.

1. The volume of the balloon is increasing with time.
2. The radius of the balloon is also increasing with time.
3. The rate of increase of volume is related to the rate of increase of the radius.
4. It is **easier** to directly measure the rate of increase of the volume, compared to the rate of increase of the radius.

Example 1: How do we find the rate of increase of the radius given that

- the rate of increase of volume is 200 cm^3/s, and
- the diameter of the balloon is 50 cm?

Example 1

Note that these steps are generic; they can be generalized to *any* related rates problem.

- **Step 1:** Find a formula/relation between the volume V and radius r of the **spherical** balloon:

$$V = \frac{4\pi r^3}{3}.$$

- **Step 2:** Since both V and r depend on t, we can use the Chain Rule to differentiate both sides of the equation with respect to t:

$$\frac{dV}{dt} = \frac{4\pi}{3} \cdot 3r^2 \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Example 1

- **Step 3:** We plug in the given information into the equation in Step 2:
$$\frac{dV}{dt} = 200, \;\; r = \frac{50}{2} = 25.$$
This yields
$$200 = 4\pi \cdot 25^2 \cdot \frac{dr}{dt}.$$
- **Step 4:** Making $\frac{dr}{dt}$, **which is the rate of increase of the radius**, the subject of the equation, we can find the rate of increase of the radius:
$$\frac{dr}{dt} = \frac{200}{4 \cdot 25^2 \pi} = \frac{2}{25\pi}.$$
Thus the rate of increase of the radius is $\frac{2}{25\pi}$ cm/s.

Exercise 2

- (a) Let A be the area of a circle with radius r. As r increases, it is understood that A also increases. Form a relation between A and r, and then find a relation between

$$\frac{dA}{dt} \quad \text{and} \quad \frac{dr}{dt}.$$

- (b) Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at constant rate of 2 m/s, how fast is the area of the spill increasing when its radius is 30 m?

Exercise 3

Example 2

The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the **area** of the rectangle increasing?

- **Step 1:** Find a formula/relation between the area of the rectangle A, its length l, and its width w:

$$A = lw.$$

- **Step 2:** As A, l and w all depend on t, use the Product Rule to differentiate both sides of the equation with respect to t:

$$\frac{dA}{dt} = w\frac{dl}{dt} + l\frac{dw}{dt}.$$

Example 2

- **Step 3:** Substitute the given information into the equation in Step 2:

$$\frac{dl}{dt} = 8, \ \frac{dw}{dt} = 3, \ l = 20 \text{ and } w = 10.$$

This yields

$$\frac{dA}{dt} = 10 \cdot 8 + 20 \cdot 3 = 140.$$

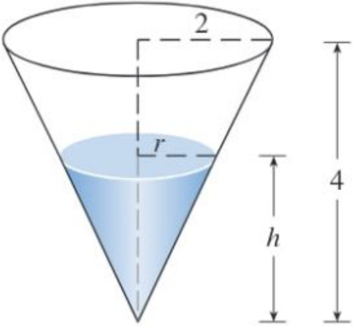
Thus the rate of increase of the area of the rectangle is 140 cm^2/s.

Exercise 4

A particle is moving along a hyperbola $xy = 8$. As it reaches the point $(4, 2)$, the y-coordinate is **decreasing** at a rate of 3 cm/s. How fast is the x-coordinate of the point changing at that instant?

Example 3

A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.



Example 3

- **Step 1:** Let V be the amount of water in the tank, r and h be the radius and height of the surface of the water respectively. We know that the volume of water in the tank is

$$V = \frac{1}{3}\pi r^2 h.$$

($\star$) **Preprocessing step:** Both r and h here depend on t, and thus differentiating the equation above will require the use of the product rule. In order to avoid complicating calculations, we write r in terms of h by using **similar triangles**:

$$(\star)\ \frac{r}{h} = \frac{2}{4} \implies r = \frac{h}{2}.$$

Thus V is expressed in terms of h <u>only</u>:

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3.$$

Example 3

- **Step 2:** Differentiate both sides of the equation with respect to t:
$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}.$$
- **Step 3:** Substitute the given information into the equation in Step 2:
$$\frac{dV}{dt} = 2, \;\; h = 3.$$
This yields
$$2 = \frac{9\pi}{4}\frac{dh}{dt} \implies \frac{dh}{dt} = \frac{8}{9\pi}.$$
Therefore the rate at which the water level is rising is $\frac{8}{9\pi}$ $\mathrm{m}^3/\mathrm{min}$.

Exercise 5

A street light is mounted on top of a 6 m-tall pole. A man 2 m tall walks away from the pole with a speed of 1.5 m/s along a straight path. How fast is the tip of his shadow moving when he is 10 m from the pole?

Exercise 5

Summary/Tips when doing Related Rates problems

1. The most difficult thing to do is **Step 1**; to express a relation between what is given, and what you are required to find. It takes some time and practice, so be patient with yourself at the start, and make sure you have ample practice (you can find more practice in the textbook prescribed in the syllabus).
2. After you have found the relation, try to reduce the number of variables you have, e.g. Example 3 where r can be expressed in terms of h, and Exercise 5 where y can be expressed in terms of x. Note: the technique of similar triangles here! This helps to reduce the difficulty when differentiating the equation.