CSD2301 Lecture
13. Rotational Dynamics

LIN QINJIE

Outline

- Torque
- Work, power and energy

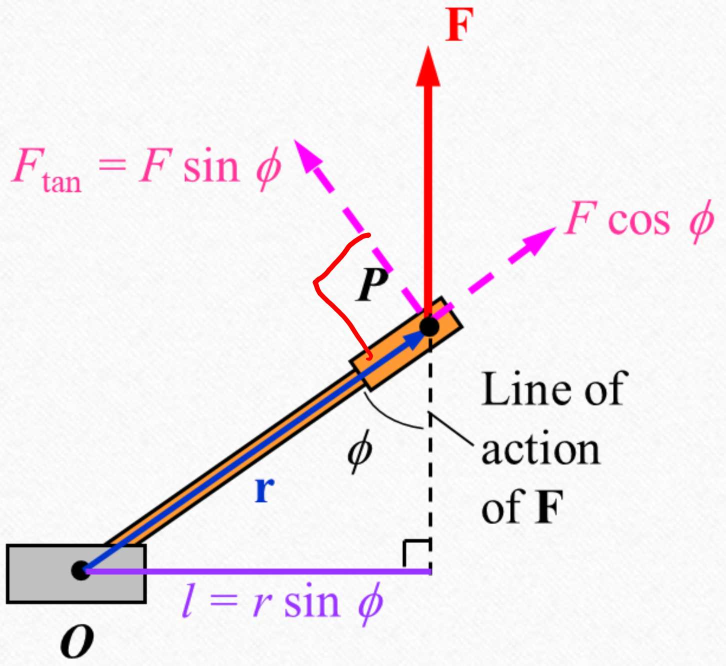
Torque

- Consider a force F acting on a rigid body at point P in the direction as shown. The rod is pivoted so that it can only rotate about the point O. Then the torque t of force F about O, is defined as:

(or ϕ)

$$\boxed{\tau = Fl = rF\sin\theta = F_{\tan}r}$$

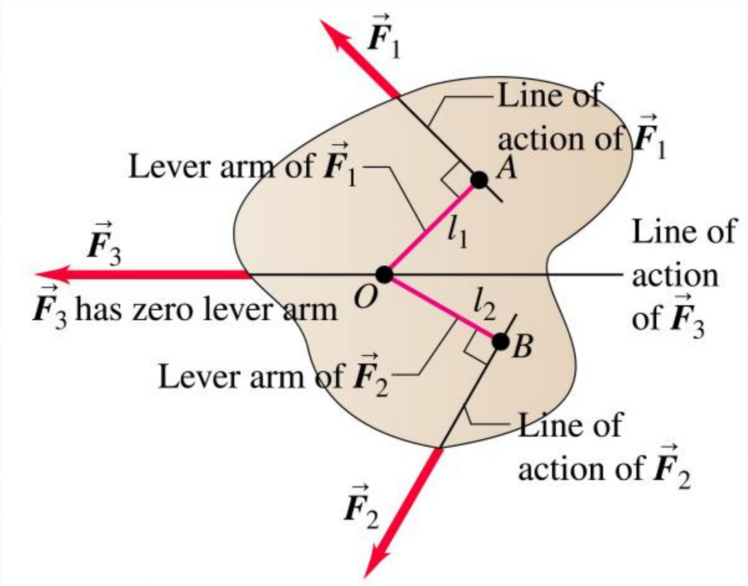
- l is the moment arm (lever arm) of the force, the shortest distance from O to the line of action of the force F.
- Only the component perpendicular to r, $F\sin\phi$, of the force F causes rotation.
- The component along r, $F\cos\phi$ has no effect in rotation



Torque

- Torque is a vector
- If two or more forces are acting on a rigid object, it is assumed that the sign of a torque is positive if the turning tendency of the force is counterclockwise and is negative if the turning tendency is clockwise.

$$\boxed{\sum \tau = \tau_1 + \tau_2 + \tau_3 = F_1 l_1 - F_2 l_2 + 0}$$



Torque vs Force

- Forces can cause a change in linear motion.
- Forces can also cause a change in rotational motion, but the effectiveness of the forces in causing this change depends on both the forces and the moment arms of the forces.
    - Force can result in torque
- The dimension of torque is force times length (unit is newton·meter but not joule).

Nm

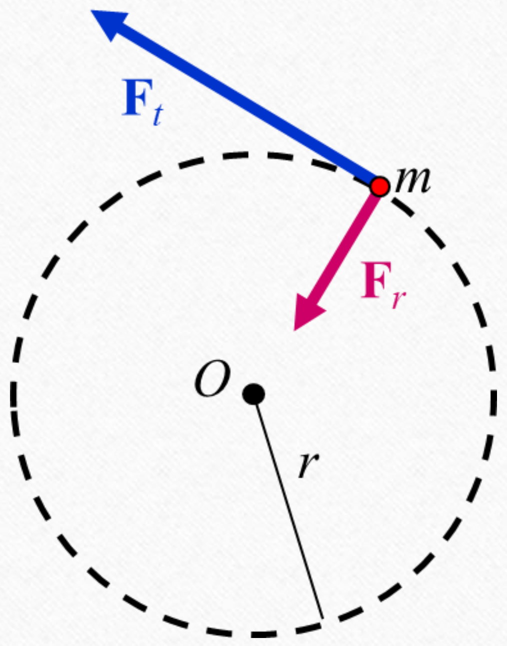
Torque & Angular Acceleration

- Consider a particle mass m rotating in a circle radius r under the influence of a tangential force F_t and a radial force F_r:

$$\begin{aligned}\tau &= rF_t \\ &= r(ma_t) \\ &= r(mr\alpha) \\ &= (mr^2)\alpha\end{aligned}$$

$$\boxed{\tau = I\alpha}$$

F=ma



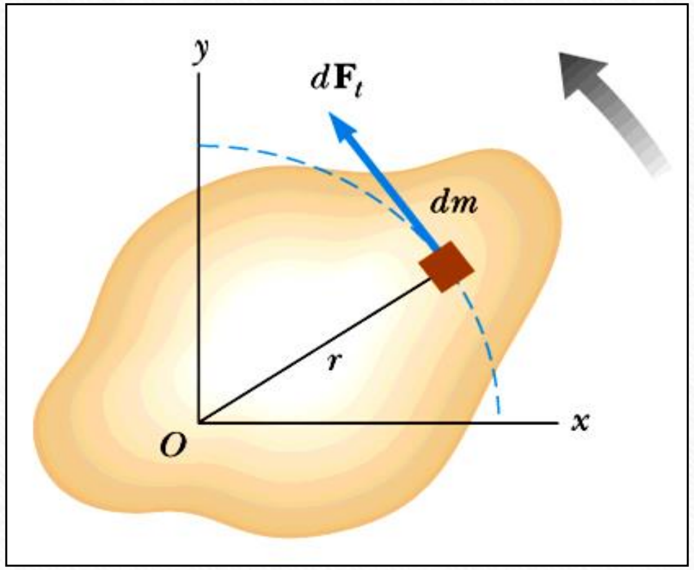
Torque & Angular Acceleration

- For an <u>extended object</u>, each mass element *dm* rotates about the origin (*z*-axis).

$$d\tau = rdF_t = r(dm)a_t = (r^2dm)\alpha$$

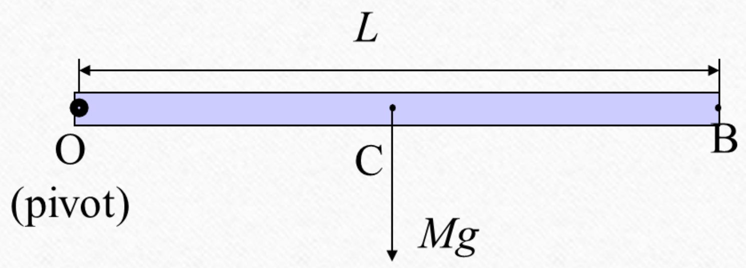
- Integrating: $\tau = \int (r^2dm)\alpha = \alpha \int r^2dm$

$$\boxed{\tau = I\alpha}$$



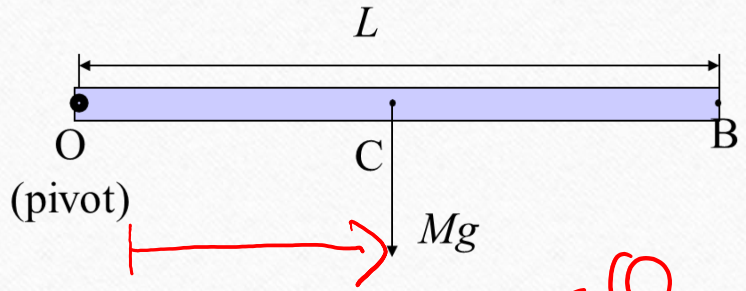
Example: Rotating Rod

- A uniform rod of mass M and length L is attached to a frictionless pivot. When released from rest from horizontal position, what is its angular acceleration? What are the linear accelerations of points B and C?



Example: Rotating Rod

- About pivot, torque is: $\tau = Mg\left(\frac{L}{2}\right)$



$s = r\theta$

$v = r\omega$

$a = r\alpha$

$$I = \frac{1}{3}ML^2 \quad \text{(I of rod about its end)}$$

$$\alpha = \frac{\tau}{I} = \frac{MgL}{2} \times \frac{3}{ML^2}$$

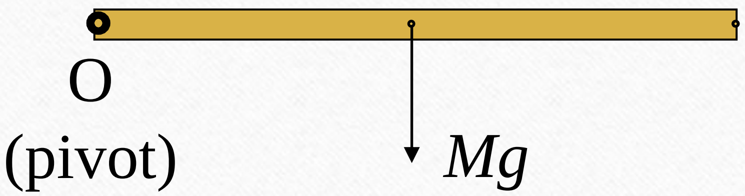
$$\boxed{\alpha = \frac{3g}{2L}}$$

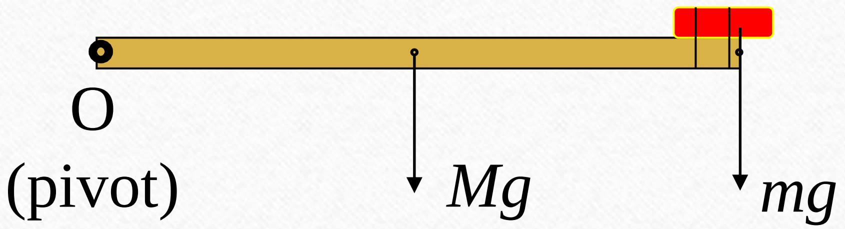
$$\boxed{a_B = L \times \frac{3g}{2L} = \frac{3}{2}g}$$

$$\boxed{a_C = \frac{L}{2} \times \frac{3g}{2L} = \frac{3}{4}g}$$

Example: Rotating Rod

- Riddle: Tie another mass to the rod at the end as shown. Which rod will take a shorter time to swing to the vertical position?





$\tau = Mg\left(\frac{L}{2}\right)$ & $I = \frac{1}{3}ML^2$

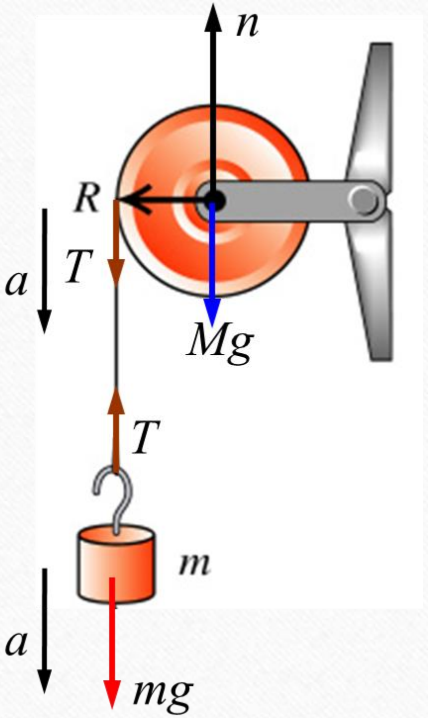
$\tau = Mg\left(\frac{L}{2}\right) + mgL$ & $I = \frac{1}{3}ML^2 + mL^2$

$$\boxed{\alpha = \frac{3g}{2L}}$$

$$\boxed{\alpha = \frac{3g}{2L}\left(\frac{M+2m}{M+3m}\right)}$$

Example: An unwinding cable

Want to find a, α and T in terms of I, R and M: (may assume $I = 1/2\, MR^2$ if wheel is a uniform cylinder).



$$\tau = TR = I\alpha$$
$$a = R\alpha$$
$$a = \frac{TR^2}{I}$$

From FBD of weight:

$$mg - T = ma$$

$$mg - T = m\left(\frac{TR^2}{I}\right)$$

$$mg = T\left[1 + \frac{mR^2}{I}\right]$$

$$\boxed{T = \frac{mg}{1 + (mR^2/I)}}$$

$$a = \frac{mg}{1 + (mR^2/I)} \times \frac{R^2}{I} = \frac{mg}{(I/R^2) + m}$$

$$\boxed{a = \frac{g}{1 + (I/mR^2)}}$$

$$\boxed{\alpha = \frac{g}{R + (I/mR)}}$$

Work, Power and Energy

- Consider a point rotating through distance *ds*.

$$dW = \vec{F} \cdot d\vec{s} = (F \sin\phi) r d\theta$$

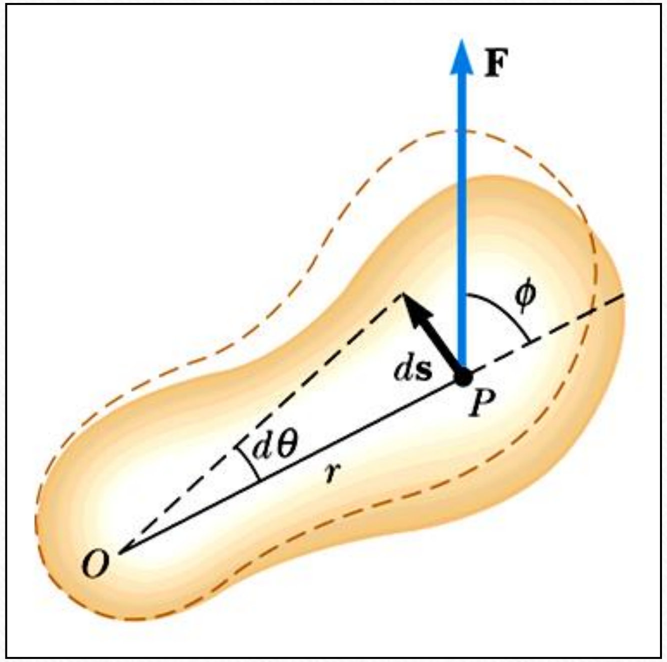
$$r F \sin\phi = \tau$$

work

$$\boxed{dW = \tau d\theta}$$

$$\boxed{P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega}$$

$P = FV$



Work and Energy

- The net work done by external forces in rotating a rigid object about a fixed axis equals the change in the object's rotational energy.

$$\tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\frac{d\omega}{d\theta}\omega$$

$$dW = \tau d\theta = I\omega d\omega$$

$$W = \int_{\theta_0}^{\theta} \tau d\theta = I\int_{\omega_0}^{\omega} \omega d\omega$$

$$\boxed{W = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2}$$

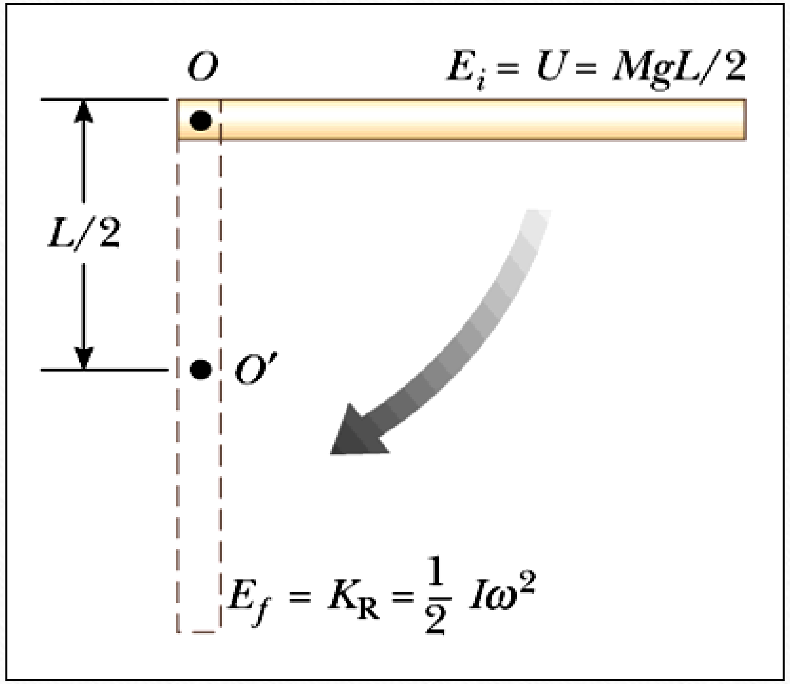
$\frac{1}{2}mv^2$

Example: Rotating rod

A uniform rod of length L and mass M is free to rotate on a frictionless pin passing through one end. The rod is released from rest in the horizontal position.

(a) What is its angular speed when it reaches its lowest position?

(b) Determine the linear speed of the centre of mass and the linear speed of the lowest point on the rod when it is in the vertical position.



Example: Rotating rod

(a) Using conservation of energy:

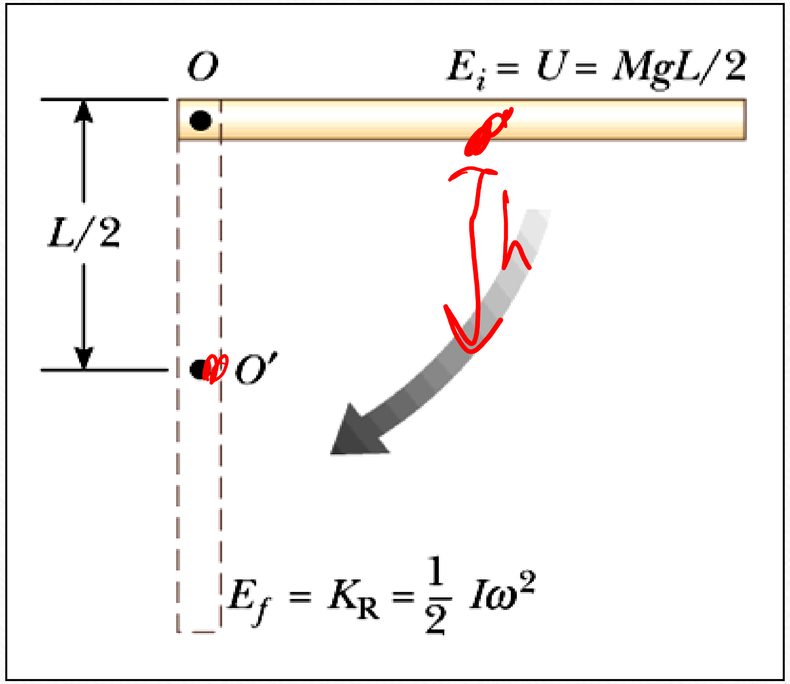
$$K_f + U_f = K_i + U_i$$

$$\frac{1}{2}I\omega^2 + 0 = 0 + Mg\left(\frac{L}{2}\right)$$

$$\frac{1}{2}\left(\frac{1}{3}ML^2\right)\omega^2 = \frac{1}{2}MgL \Rightarrow \boxed{\omega = \sqrt{\frac{3g}{L}}}$$

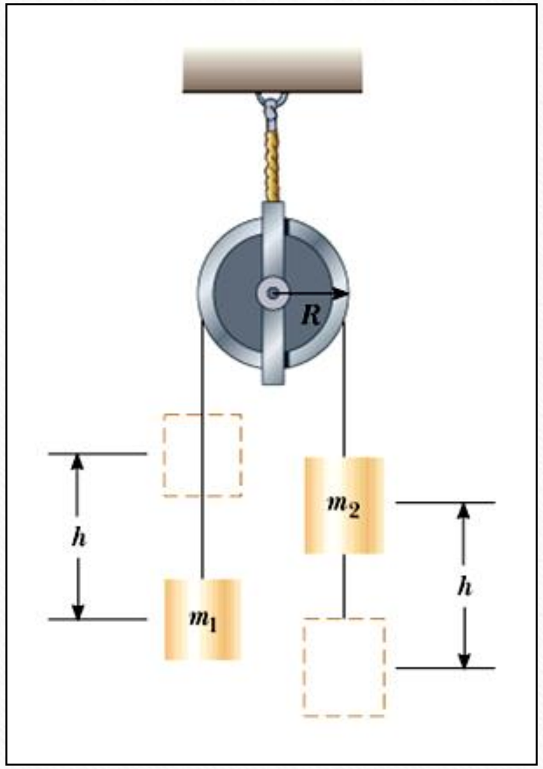
(b) $\boxed{v_{CM} = \frac{L}{2}\omega = \frac{1}{2}\sqrt{3gL}}$ $\boxed{v_{tip} = L\omega = \sqrt{3gL}}$

$v = r\omega$



Example: Rotating rod

Consider two cylinders having masses m_1 and m_2, where $m_1 \neq m_2$, connected by a string passing over a pulley. The pulley has a radius R and moment of inertia I about its axis of rotation. The string does not slip on the pulley, and the system is released from rest. Find the linear speeds of the cylinders after cylinder 2 descents through a distance h, and the angular speed of the pulley at this time.



Example: Rotating rod

$$K_f + U_f = K_i + U_i$$

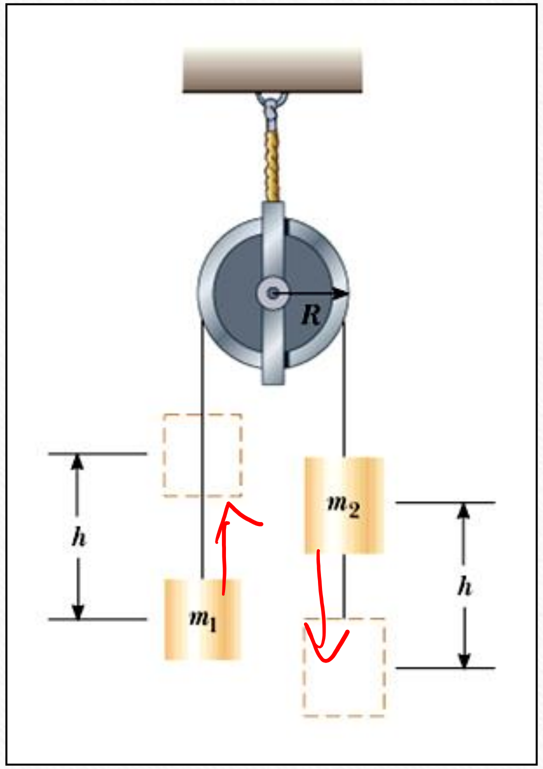
$$\frac{1}{2}m_1 v_f^2 + \frac{1}{2}m_2 v_f^2 + \frac{1}{2}I\omega_f^2 + m_1 gh - m_2 gh = 0 + 0$$

$$v_f = R\omega_f$$

$$\frac{1}{2}m_1 v_f^2 + \frac{1}{2}m_2 v_f^2 + \frac{1}{2}I\frac{v_f^2}{R^2} = (m_2 - m_1)gh$$

$$\boxed{v_f = \sqrt{\frac{2(m_2 - m_1)gh}{m_1 + m_2 + (I/R^2)}}}$$

$$\boxed{\omega_f = \frac{1}{R}\sqrt{\frac{2(m_2 - m_1)gh}{m_1 + m_2 + (I/R^2)}}}$$



The End