Question 1. Let X and Y be jointly distributed random variables whose joint PMF $p(x, y)$ is given in the follow table.

$X \setminus Y$	1	2	3	4
1	0.1	0.05	0.02	0.02
2	0.05	$\boxed{0.2}$	0.05	0.02
3	0.02	0.05	0.2	0.04
4	0.02	0.02	0.04	0.1

(a) Write marginal PMFs $p_X(x)$ of X and $p_Y(y)$ of Y in the same table above.

(b) Are X and Y independent?

(c) Find $\mathrm{Cov}(X, Y)$.

(a) $P(X=1) = 0.1 + 0.05 + 0.02 + 0.02 = 0.19 = P(Y=1)$

$P(X=2) = 0.05 + 0.2 + 0.05 + 0.02 = 0.32 = P(Y=2)$

$P(X=3) = 0.02 + 0.05 + 0.2 + 0.04 = 0.31 = P(Y=3)$

$P(X=4) = 0.02 + 0.02 + 0.04 + 0.1 = 0.18 = P(Y=4)$

(b) $P(X=2 \text{ and } Y=2) = 0.2.$

$P(X=2) = 0.32 = P(Y=2)$

$P(X=2, Y=2) \neq P(X=2)P(Y=2)$ Hence X and Y are dependent.

(c) $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$

$E(X) = E(Y) = 1 \times 0.19 + 2 \times 0.32 + 3 \times 0.31 + 4 \times 0.18 = 2.48.$

$XY = \{1, 2, 3, 4, 6, 8, 9, 12, 16\}$

$P(XY=1) = 0.1$

$P(XY=2) = P(X=1, Y=2) + P(X=2, Y=1) = 0.1$

$P(XY=3) = P(X=1, Y=3) + P(X=3, Y=1) = 0.04$

$P(XY=4) = P(X=1, Y=4) + P(X=2, Y=2) + P(X=4, Y=1) = 0.24$

$P(XY=6) = P(X=2, Y=3) + P(X=3, Y=2) = 0.1$

$P(XY=8) = P(X=2, Y=4) + P(X=4, Y=2) = 0.04$

$P(XY=9) = P(X=3, Y=3) = 0.2$

$P(XY=12) = P(X=3, Y=4) + P(X=4, Y=3) = 0.08$

$P(XY=16) = P(X=4, Y=4) = 0.1$

$E(XY) = 6.66$, $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = 0.5096$

independent is wrong.

Question 2. Let X be a random variable that takes on values $-2, -1, 0, 1, 2$, each with probability $1/5$. Let $Y = X^2$.

(a) Express the values of $p(x, y)$, $p_X(x)$, $p_Y(y)$ in the joint probability table.

(b) Show that $\mathrm{Cov}(X, Y) = 0$.

(c) Are X and Y independent?

X:	-2	-1	0	1	2
P	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

$Y = X^2$

Y	0	1	4
P	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$

(a)

$X \backslash Y$	0	1	4	
-2	0	0	$\frac{1}{5}$	$\frac{1}{5}$
-1	0	$\frac{1}{5}$	0	$\frac{1}{5}$
0	$\frac{1}{5}$	0	0	$\frac{1}{5}$
1	0	$\frac{1}{5}$	0	$\frac{1}{5}$
2	0	0	$\frac{1}{5}$	$\frac{1}{5}$
	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	

(b) $E(X) = (-2)(\frac{1}{5}) + (-1)(\frac{1}{5}) + (0)(\frac{1}{5}) + (1)(\frac{1}{5}) + (2)(\frac{1}{5})$

$= 0$

$E(Y) = (0)(\frac{1}{5}) + (1)(\frac{2}{5}) + (4)(\frac{2}{5}) = 2.$

$E(XY) = E(X \cdot X^2) = E(X^3)$

$= (-2)^3(\frac{1}{5}) + (-1)^3(\frac{1}{5}) + (0^3)(\frac{1}{5})$

$+ (1)^3(\frac{1}{5}) + (2)^3(\frac{1}{5}) = 0$

$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0$

$P(X=-1, Y=1) = \frac{1}{5} \neq \underbrace{P(X=-1)}_{\frac{1}{5}} \underbrace{P(Y=1)}_{\frac{2}{5}}$

So X and Y are not independent.

Question 3. Let X and Y be jointly distributed variables with joint PDF

$$f(x, y) = \begin{cases} 2x + 2y - 4xy & \text{if } 0 \le x, y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal PDFs $f_X(x)$ of X and $f_Y(y)$ of Y.

(b) Compute $E(X)$ and $E(Y)$.

(c) Compute $\mathrm{Cov}(X, Y)$.

(a) If $x < 0$ or $x > 1$ $\quad f_X(x, y) = \int_{-\infty}^{+\infty} f(x, y)\, dy = 0.$

If $0 \le x \le 1$

$$f_X(x,y) = \int_0^1 (2x+2y-4xy)\, dy$$

$$= \left(2xy + y^2 - 2xy^2\right)\Big|_0^1 = 2x+1-2x = 1$$

$$f_X(x,y) = \begin{cases} 0 & \text{if } x<0 \text{ or } x>1 \\ \underline{1} & \text{if } 0 \le x \le 1 \end{cases}$$

Similarly

$$f_Y(x,y) = \begin{cases} 0 & \text{if } y<0 \text{ or } y>1 \\ 1 & \text{if } 0 \le y \le 1 \end{cases}$$

(b) $E(X) = \int_{-\infty}^{+\infty} x \cdot f_X(x)\, dx = \int_0^1 x\, dx = \frac{1}{2}$.

$E(Y) = \int_{-\infty}^{+\infty} y\, f_y(x)\, dy = \int_0^1 y\, dy = \frac{1}{2}$.

(c) $E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f(x,y)\, dy\, dx$

$$= \int_0^1 \int_0^1 xy\,(2x+2y-4xy)\, dy\, dx$$

$$= \int_0^1 \int_0^1 (2x^2y + 2xy^2 - 4x^2y^2)\, dy\, dx.$$

$$= \int_0^1 \left(2x^2\frac{y^2}{2} + 2x\frac{y^3}{3} - 4x^2\frac{y^3}{3}\right)\Big|_0^1 dx$$

$$= \int_0^1 \left(x^2 + \frac{2x}{3} - \frac{4x^2}{3}\right) dx = \int_0^1 \left(-\frac{x^2}{3} + \frac{2x}{3}\right) dx$$

$$= \left(-\frac{1}{3}\frac{x^3}{3} + \frac{2}{3}\frac{x^2}{2}\right)\Big|_0^1 = \frac{2}{9}$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{2}{9} - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \underline{-\frac{1}{36}}.$$

Question 4. Let c be a real number and let $f(x,y)$ be the following function

$$f(x,y)=\begin{cases} c(x+y)^2 \text{ if } 0\le x\le 1, 0\le y\le 1\\ 0 \text{ otherwise.}\end{cases}$$

(a) Find c so that $f(x,y)$ is the joint PDF of two random variables X and Y.

(b) Find the marginal PDFs $f_X(x)$ of X and $f_Y(y)$ of Y.

(c) Are X and Y independent?

(d) Compute $\mathrm{Cov}(X,Y)$.

(a) $1 = c\int_0^1\int_0^1 (x+y)^2\,dy\,dx$

$= c\int_0^1\int_0^1 (x^2+2xy+y^2)\,dy\,dx$

$= c\int_0^1 \left(x^2y + \cancel{2}x\frac{y^2}{\cancel{2}} + \frac{y^3}{3}\right)\Big|_0^1 dx$

$= c\int_0^1 (x^2+\frac{1}{3}+x)\,dx$

$= c\left(\frac{x^3}{3}+\frac{1}{3}x+\frac{x^2}{2}\right)\Big|_0^1 = c\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{2}\right) = \frac{7c}{6}$

$c = \frac{6}{7}$

(b) $f(x,y)=\begin{cases}\frac{6}{7}(x+y)^2 & 0\le x,y\le 1\\ 0 & \text{otherwise.}\end{cases}$

If $x<0$ or $x>1$

$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = 0.$

If $0\le x\le 1$

$f_X(x) = \frac{6}{7}\int_0^1 (x+y)^2\,dy$

$= \frac{6}{7}\int_0^1 (x^2+2xy+y^2)\,dy$

$= \frac{6}{7}\left(x^2y + 2x\frac{y^2}{2} + \frac{y^3}{3}\right)\Big|_0^1$

$= \frac{6}{7}\left(x^2+\frac{1}{3}+x\right) = \frac{6}{7}x^2+\frac{2}{7}+\frac{6}{7}x$

We obtain

$$f_X(x) = \begin{cases} \frac{6}{7}x^2+\frac{6}{7}x+\frac{2}{7} & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we have

$$f_Y(y) = \begin{cases} \frac{6}{7}y^2+\frac{6}{7}y+\frac{2}{7} & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(c) $f(x,y) \overset{\text{No.}}{=} f_X(x)\, f_Y(y)$. ?

$f(0,0) = 0.$

$f_X(0)\cdot f_Y(0) = \left(\frac{2}{7}\right)^2$

$f(x,y) \neq f_X(x) \times f_Y(y)$. Not independent.

(d) $Cov(X,Y)$

$E(X) = E(Y) = \int_0^1 x\, f_X(x)\,dx.$

$= \int_0^1 x\left(\frac{6}{7}x^2+\frac{6}{7}x+\frac{2}{7}\right)dx$

$= \int_0^1 \left(\frac{6}{7}x^3+\frac{6}{7}x^2+\frac{2}{7}x\right)dx$

$= \frac{6}{7}\frac{x^4}{4}+\frac{6}{7}\frac{x^3}{3}+\frac{2}{7}\frac{x^2}{2}\Big|_0^1 = \frac{9}{14}.$

$$E(XY) = \frac{6}{7}\int_0^1\int_0^1 xy\,(x+y)^2\,dy\,dx$$

$$= \frac{6}{7}\int_0^1\int_0^1 xy(x^2+y^2+2xy)\,dy\,dx$$

$$= \frac{6}{7}\int_0^1\int_0^1 x^3y + xy^3 + 2x^2y^2\,dy\,dx$$

$$= \frac{6}{7}\int_0^1\left(x^3\frac{y^2}{2} + x\frac{y^4}{4} + 2x^2\frac{y^3}{3}\Big|_0^1\right)dx$$

$$= \frac{6}{7}\int_0^1\left(\frac{x^3}{2} + \frac{2x^2}{3} + \frac{x}{4}\right)dx$$

$$= \frac{6}{7}\left(\frac{x^4}{8} + \frac{2}{3}\cdot\frac{x^3}{3} + \frac{1}{4}\,\frac{x^2}{2}\right)\Big|_0^1 = \frac{17}{42}$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{17}{42} - \left(\frac{9}{14}\right)^2 \approx -0.0085$$