Game Probability

Dice, Cards, and Random Numbers

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History of Probability

Knucklebones

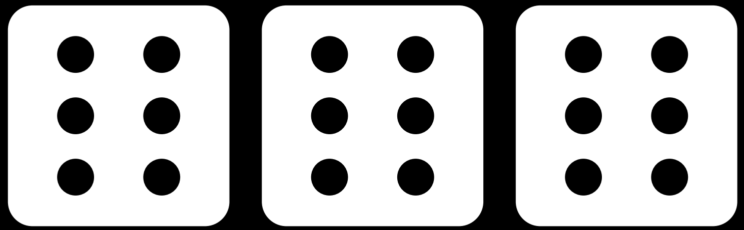
Casting Lots

Lotteries

~5,000+ B.C.

~3,000 – 1,500 B.C.

63 B.C. – 14 A.D.



Bishop Wibold, 960 A.D.

Government Lotteries

Cards in Europe

500 A.D.

14th Century A.D.

Cardano, 1501-76

Galileo, 1613-23

Pascal and Fermat, 1654

15th Century A.D.

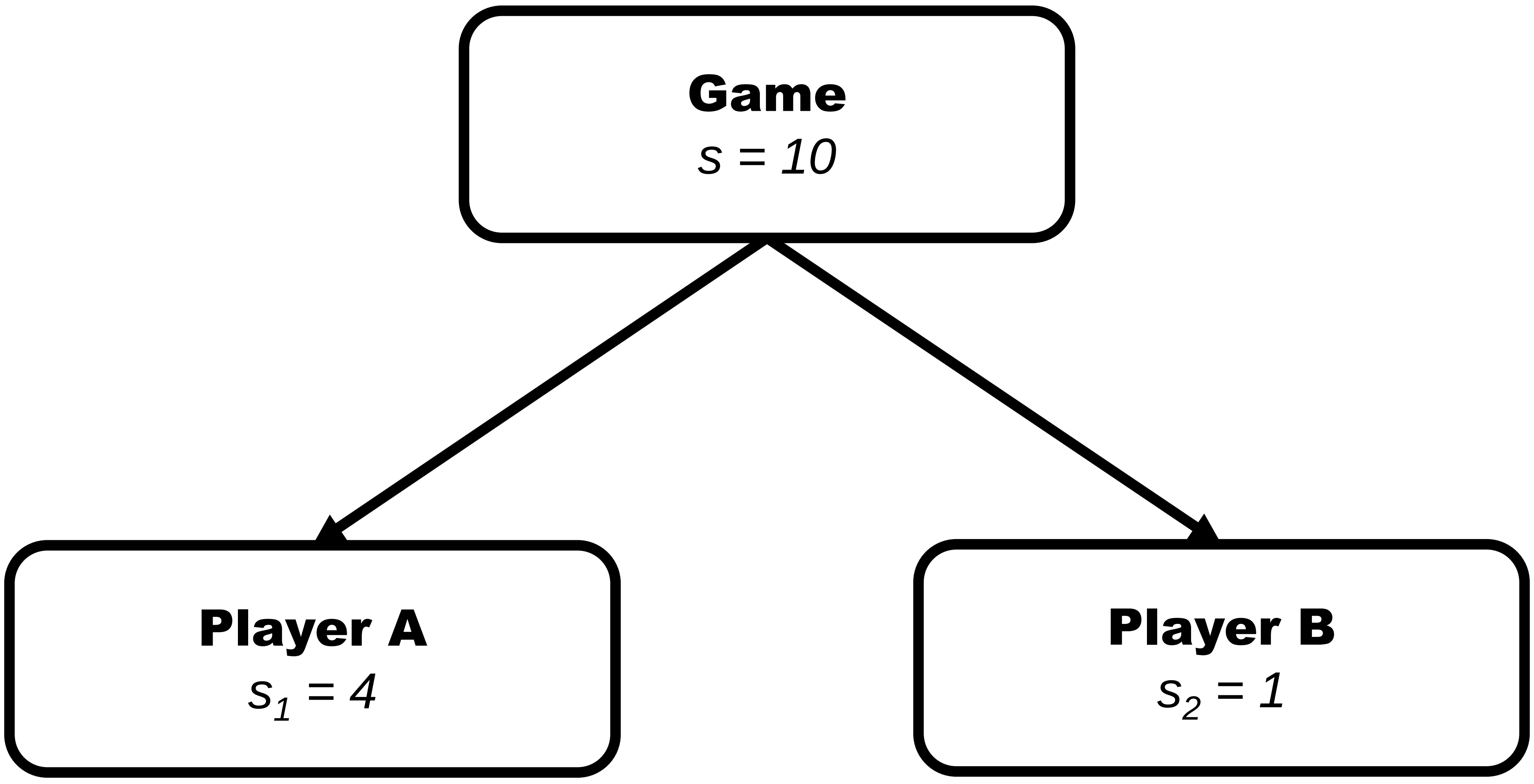
17th Century A.D.

"The direct cause of the new contributions to probability theory was some questions on games of chance from Antoine Gombaud, Chevalier de Mere, to Blaise Pascal in 1654."

Hald, *A History of Probability and Statistics*, 1990, p. 42

"Two players, A and B, agree to play a series of fair games until one of them has won a specified number of games, s, say. For some accidental reason, the play is stopped when A has won s_1 and B s_2 games, s_1 and s_2 being smaller than s. How should the stakes be divided?"

Hald, *A History of Probability and Statistics*, 1990, p. 35


Game
s = 10
Player A
s1 = 4
Player B
s2 = 1

Challenge

How would you solve the problem of points?

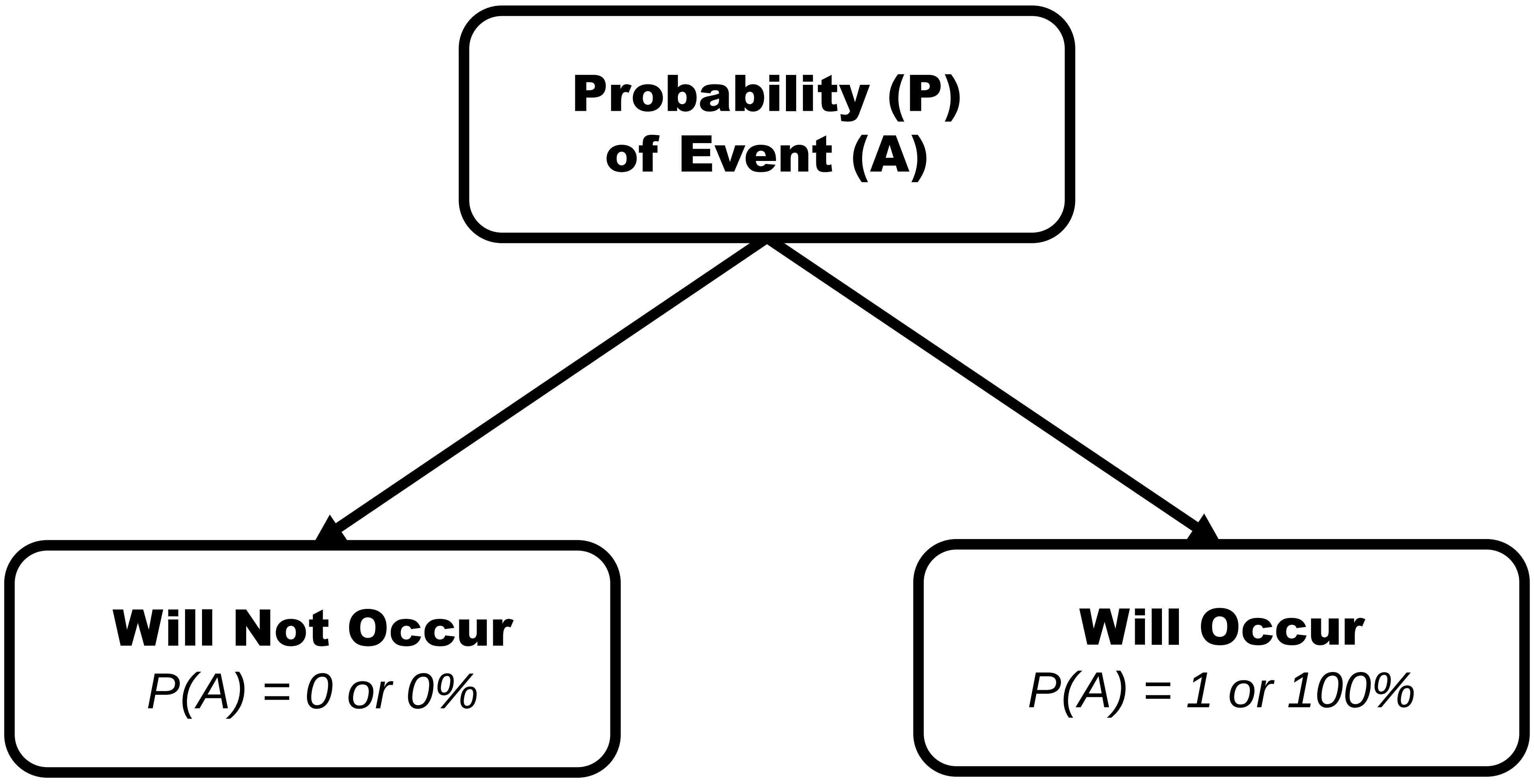
$s = 10$, $s_1 = 4$, $s_2 = 1$

Discuss with a partner.

Basic Probability

"Objective, statistical, or aleatory probabilities are used for describing properties of random mechanisms or experiments, such as games of chance, and for describing chance events in populations, such as the chance of a male birth or the chance of dying at a certain age."

Hald, *A History of Probability and Statistics*, 1990, p. 28


Probability (P)
of Event (A)
Will Not Occur
P(A) = 0 or 0%
Will Occur
P(A) = 1 or 100%

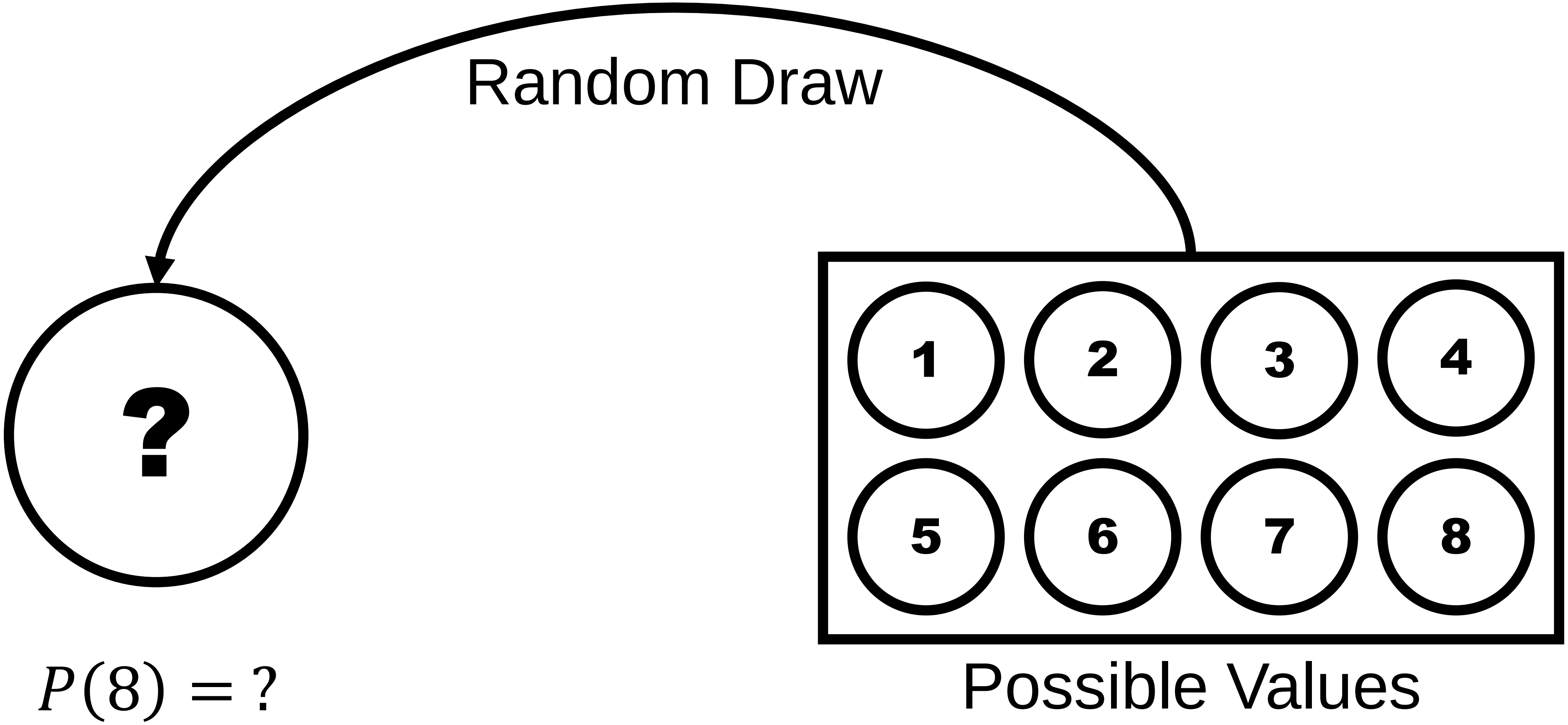
Probability
$P(A)$

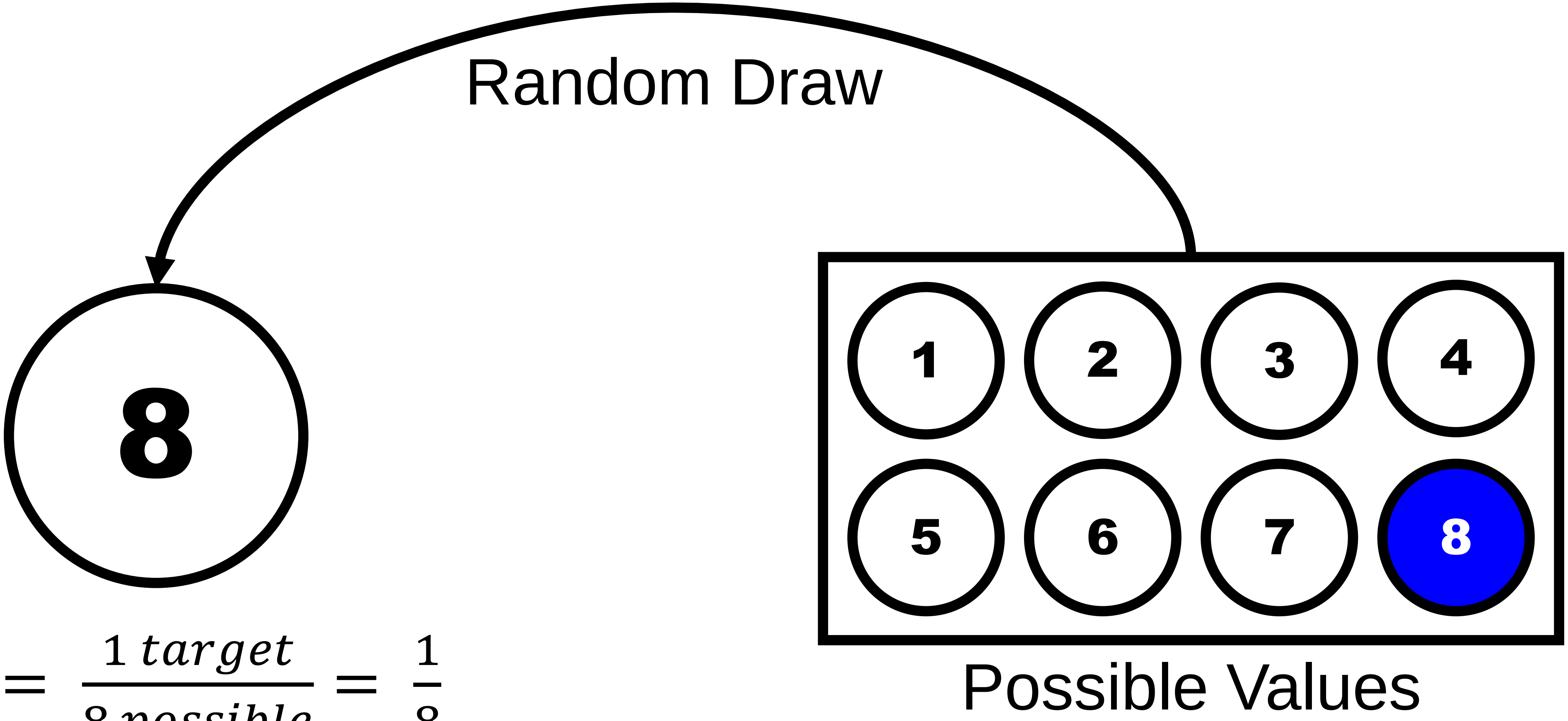
Complement
$1 - P(A)$

Simple

$$P(A) = \frac{k}{n}$$

$\mathrm{k} = $ Number of target outcomes
$\mathrm{n} = $ Number of all possible outcomes


Random Draw
?
1
2
3
4
5
6
7
8
$P(8) = ?$
Possible Values



$$P(8) = \frac{1\,target}{8\,possible} = \frac{1}{8}$$

Intersection (AND)

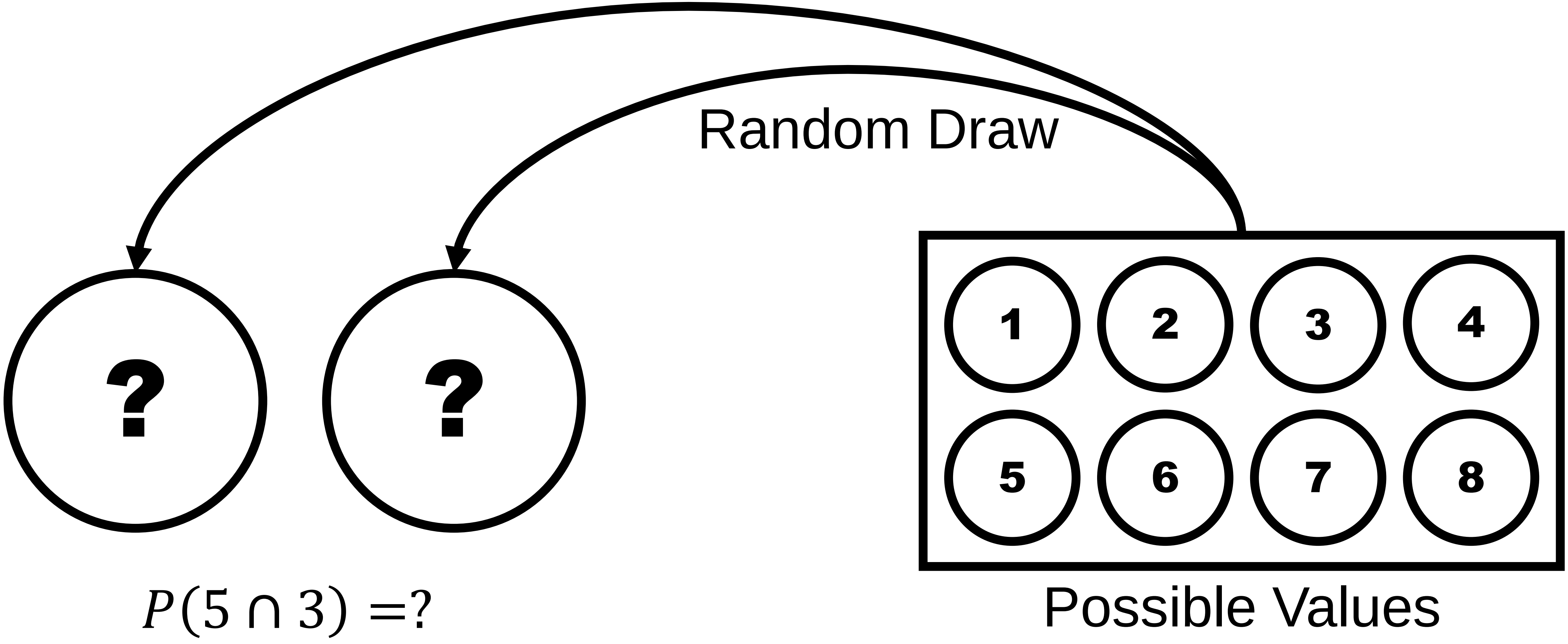
$$P(A \cap B) = P(A) \cdot P(B)$$

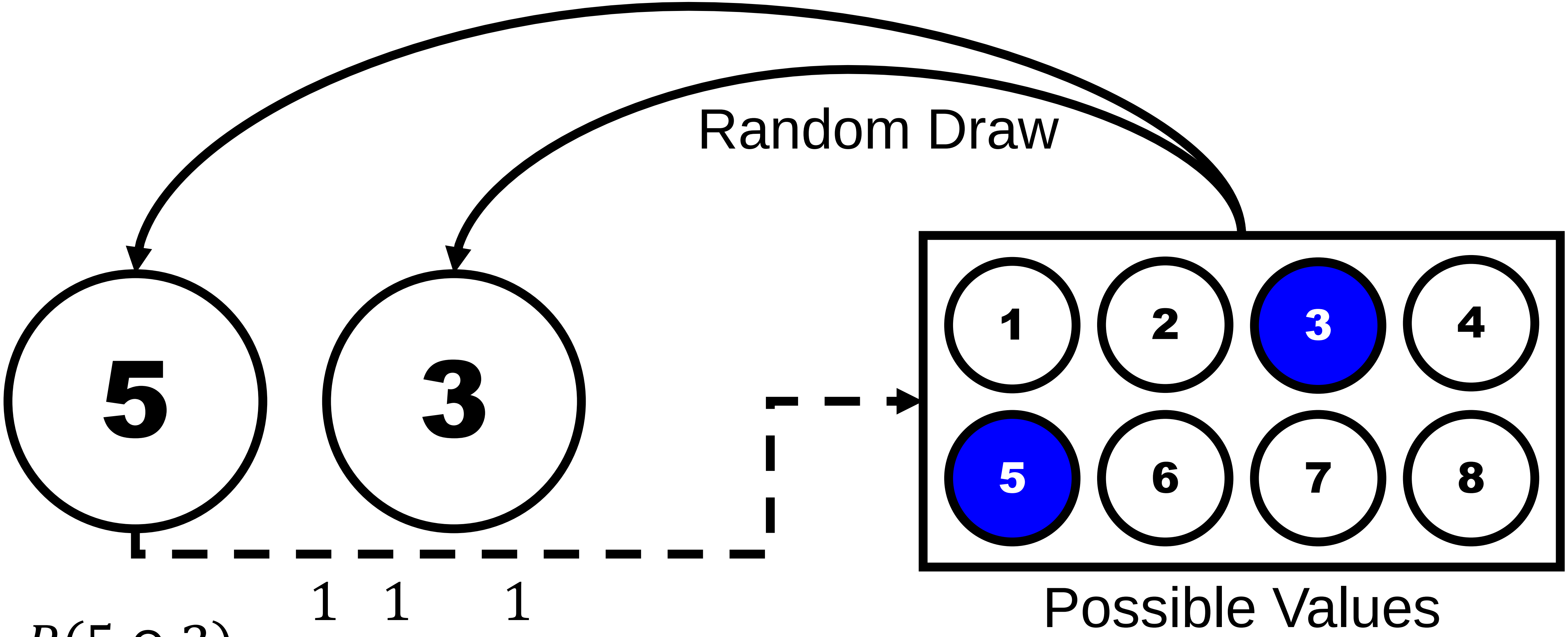
$$P(A \cap B) = P(A) \cdot P(B|A)$$

$\mathrm{P}(A \cap B) =$ Probability of A and B

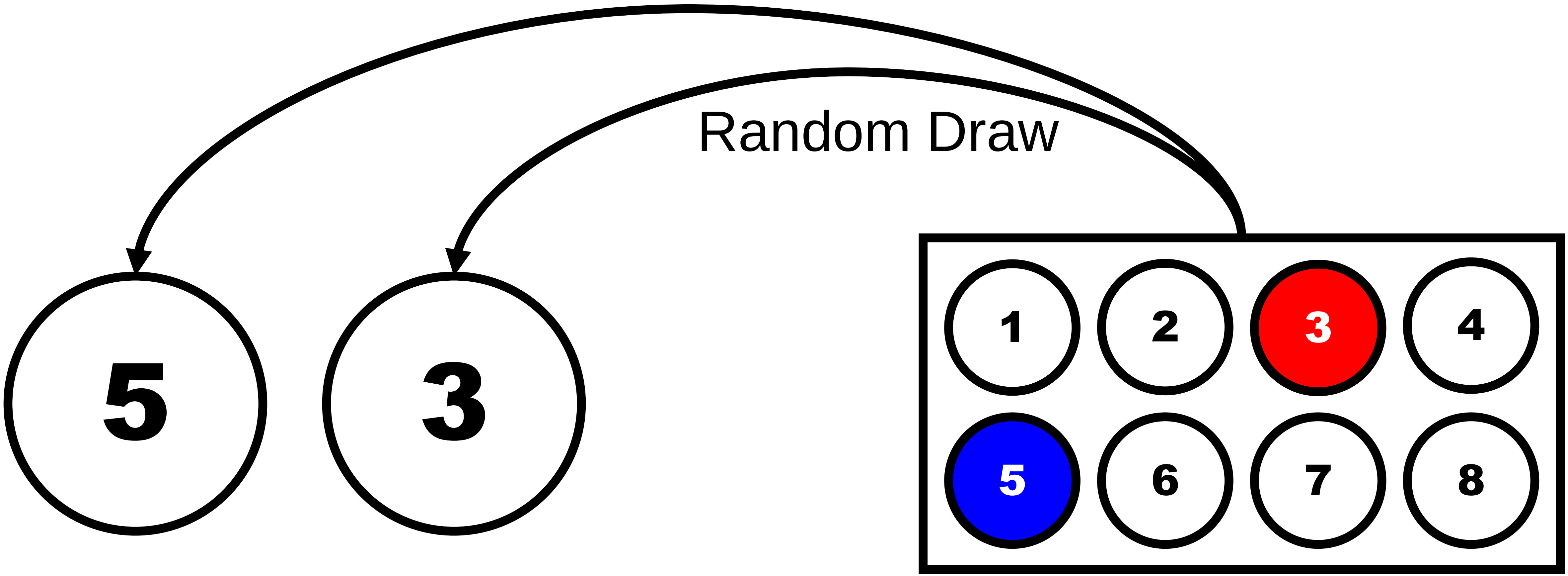
$\cap =$ Intersection symbol, "and"

$\mathrm{P(B|A)} = \mathrm{P(B)}$ given A occurred, for dependent events


Random Draw
?
?
1
2
3
4
5
6
7
8
Possible Values
$P(5 \cap 3) = ?$



$$P(5 \cap 3) = \frac{1}{8} \cdot \frac{1}{8} = \frac{1}{64}$$

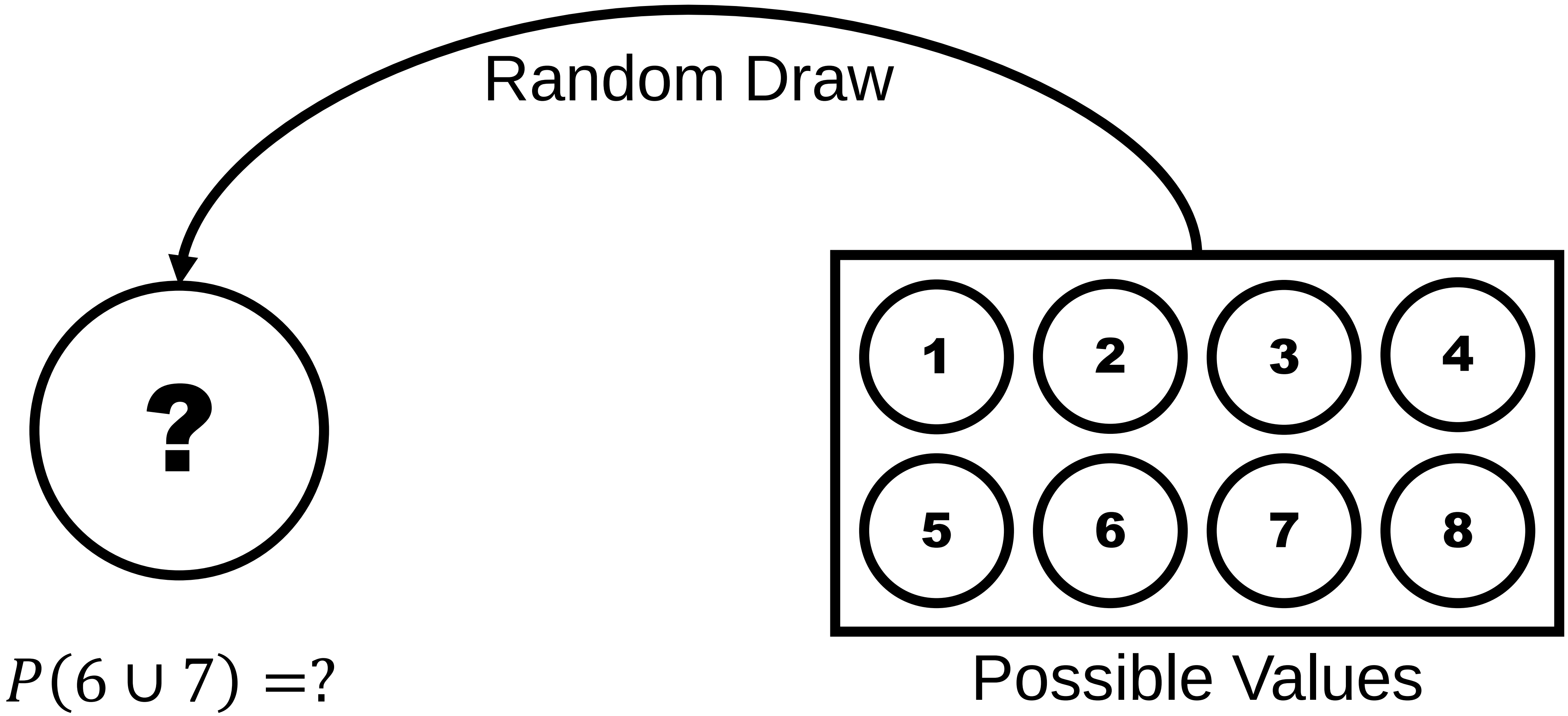


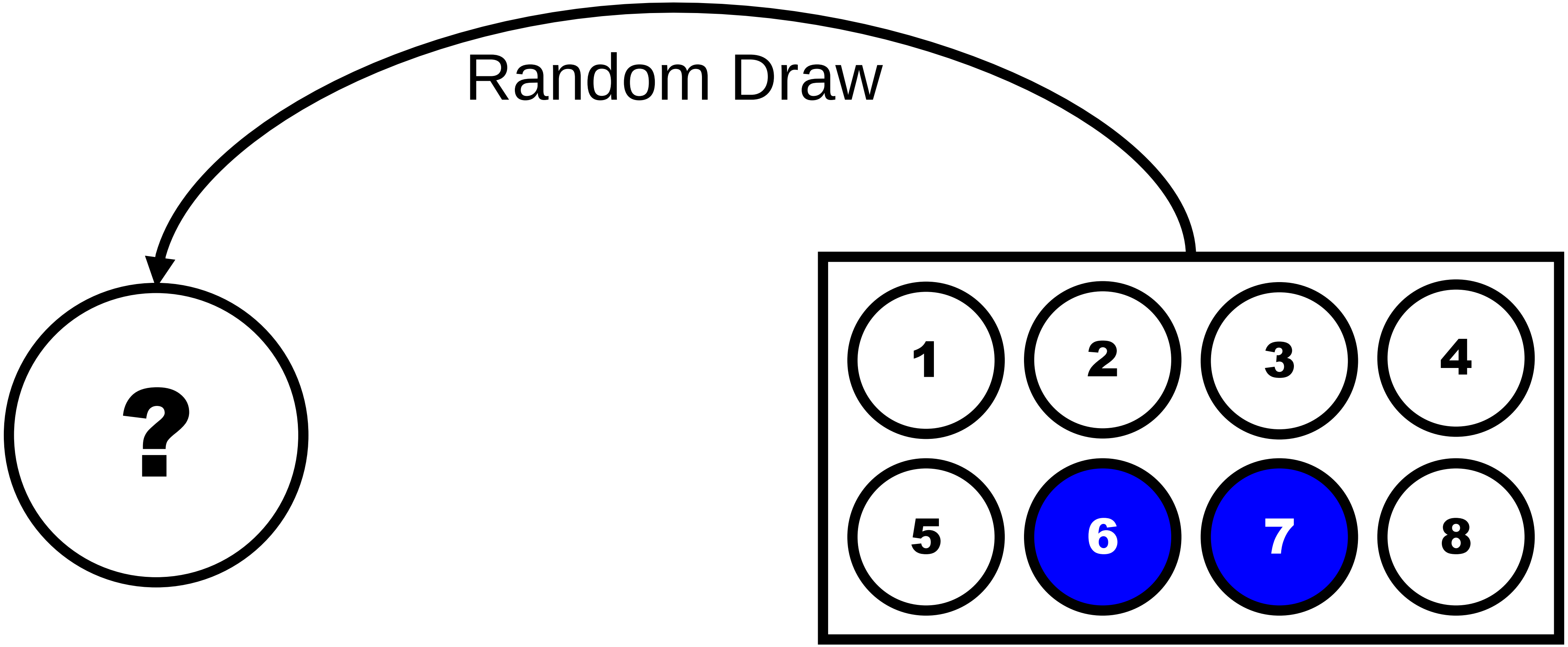
$$P(5 \cap 3) = \frac{1}{8} \cdot \frac{1}{7} = \frac{1}{56}$$

Union (OR)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

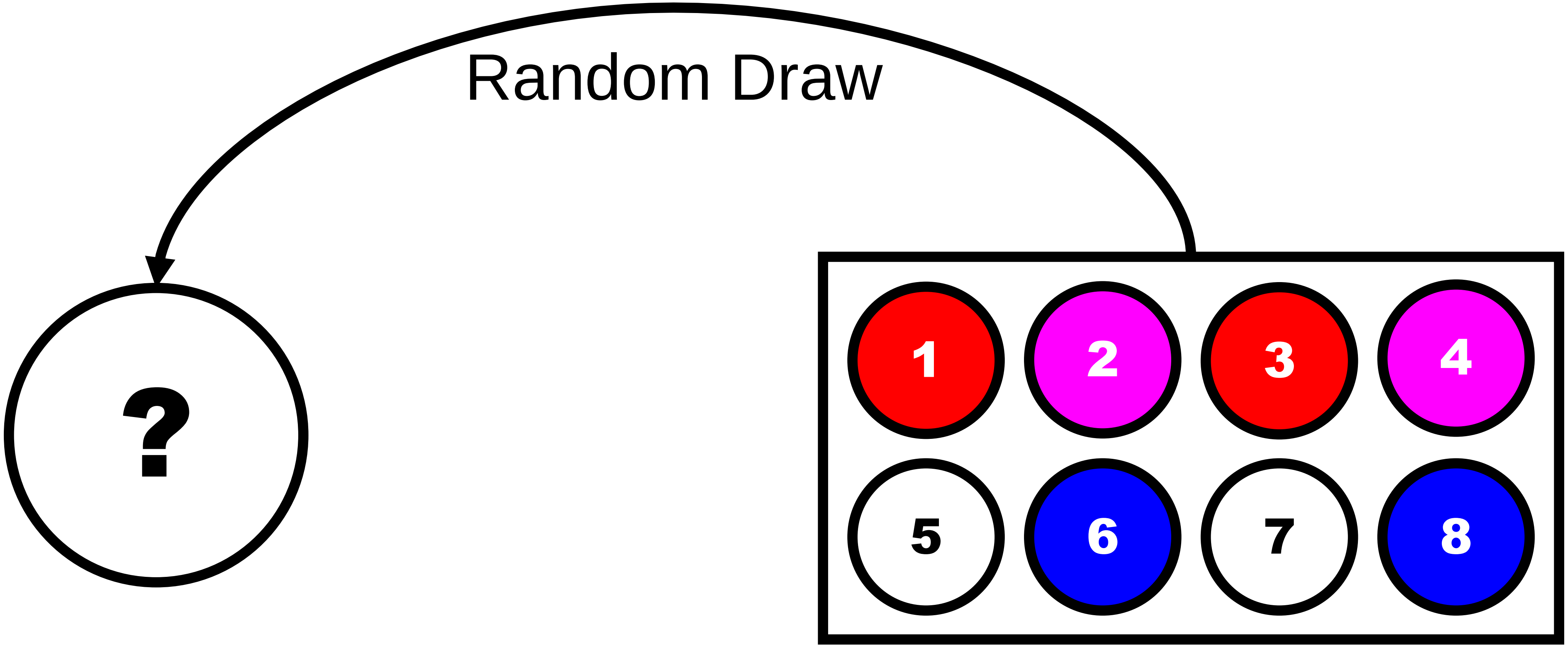
$\mathrm{P(X) = Probability\ of\ A\ or\ B}$
$\cup = \mathrm{Union\ symbol, "or"}$
$\mathrm{P(A \cap B) = 0\ for\ mutually - exclusive\ events}$


Random Draw
?
1
2
3
4
5
6
7
8
Possible Values
$P(6 \cup 7) = ?$



Possible Values

$$P(6 \cup 7) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$



Possible Values

$$P(\leq 4 \cup Even) = \frac{4}{8} + \frac{4}{8} - \frac{2}{8} = \frac{3}{4}$$

Challenge

Roll 2d12. Answer these questions.

1. **Should $P(A \cap B) > P(A \cup B)$? Why?**
2. **What is $P(4 \cap 9)$?**
3. **What is $P(3 \cup 11)$?**
4. **What is $P(4|9)$?**
5. **What is $P(<11 \text{ or Odd})$?**

Combinatorics

	Ordered	Unordered
With Replacement	Ordered sampling with replacement	Unordered sampling with replacement
Without Replacement	Permutation	Combination

	Ordered	Unordered
With Replacement	Draw <u>any</u> 5 lottery numbers 1-9, the winning sequence <u>must match exactly</u>	Draw <u>any</u> 5 lottery numbers 1-9, the winning numbers <u>must be present</u>
Without Replacement	Draw 5 <u>unique</u> lottery numbers 1-9, winning sequence <u>must match exactly</u>	Draw 5 <u>unique</u> lottery numbers 1-9, winning numbers <u>must be present</u>

Challenge

All else equal, rank the 4 sampling methods from highest to lowest in terms of how many valid possibilities they yield.

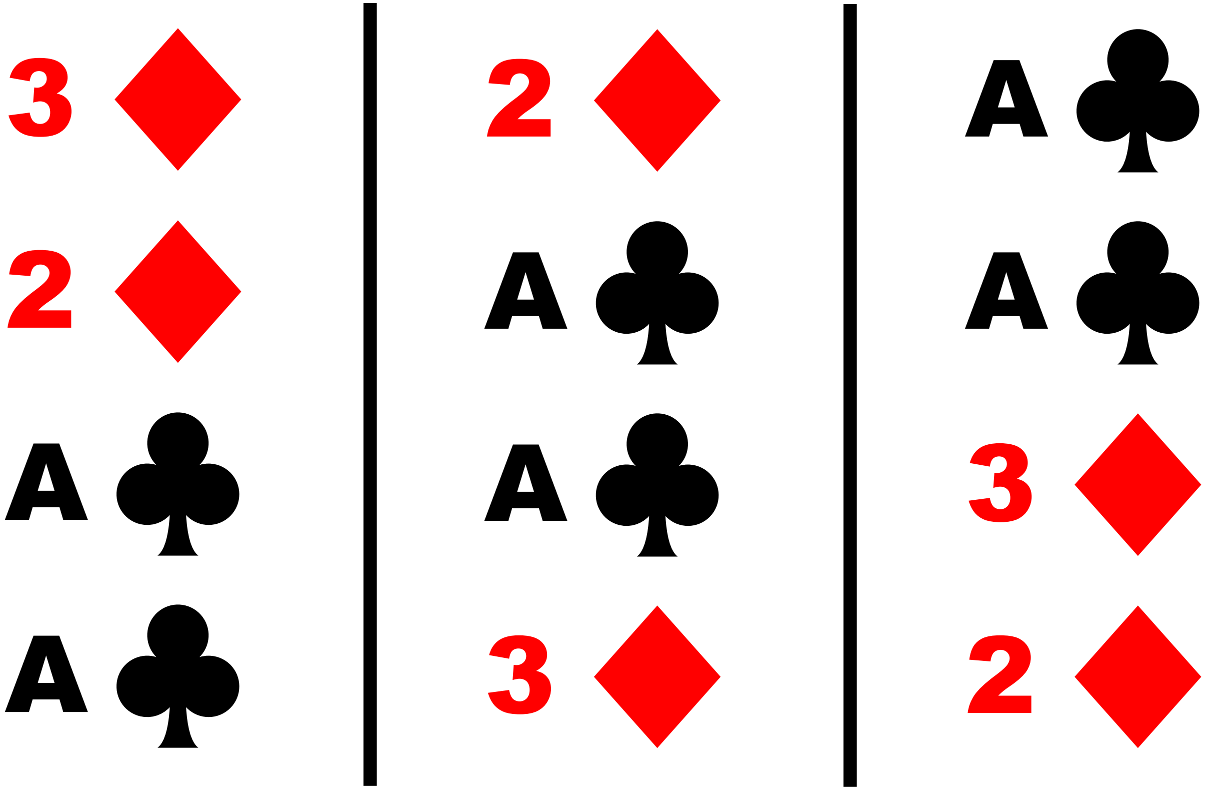
Which lottery is easiest to win?

	Ordered	Unordered
With Replacement	59,049	1,287
Without Replacement	15,120	126

Ordered Sampling With Replacement

$$X = n^k$$

X = total possibilities
n = total number of objects
k = number of selected objects



How many different sets of 4 cards can we draw?

3 sets of the same 4 cards in 3 different positions = 3

Notice duplicates within sets

Unordered Sampling With Replacement

$$X = \binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}$$

$\mathrm{X} = \text{total possibilities}$

$\mathrm{n} = \text{total number of objects}$

$\mathrm{k} = \text{number of selected objects}$

3 ♦	2 ♦	A ♣
2 ♦	A ♣	A ♣
A ♣	A ♣	3 ♦
A ♣	3 ♦	2 ♦

How many different sets of 4 cards can we draw?

3 sets of the same 4 cards ~~in 3 different positions~~ = 1

Notice duplicates within sets

Permutation

Ordered Sampling Without Replacement

$$P_{n,k} = \frac{n!}{(n-k)!}$$

$\mathrm{P}_{\mathrm{n,k}} = \text{total possibilities}$

$\mathrm{n} = \text{total number of objects}$

$\mathrm{k} = \text{number of selected objects}$

3 ♥	2 ♦	1 ♠
2 ♦	1 ♠	A ♣
1 ♠	A ♣	3 ♥
A ♣	3 ♥	2 ♦

How many different sets of 4 cards can we draw?

3 sets of the same 4 cards in 3 different positions = 3

Notice no duplicates within sets

Combination

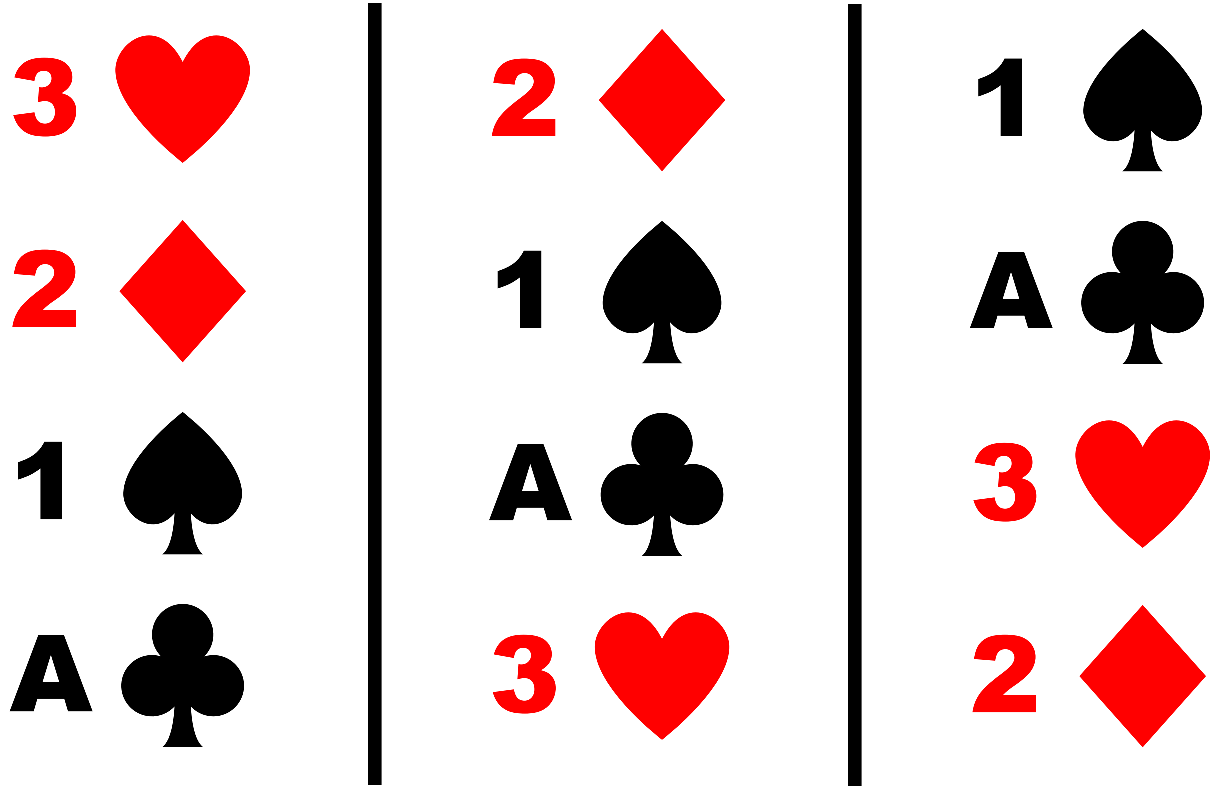
Unordered Sampling Without Replacement

$$C_{n,k} = \frac{n!}{k!(n-k)!}$$

$\mathrm{C_{n,k}} = \text{total possibilities}$

$\mathrm{n} = \text{total number of objects}$

$\mathrm{k} = \text{number of selected objects}$



How many different sets of 4 cards can we draw?

3 sets of the same 4 cards ~~in 3 different positions~~ = 1

Notice no duplicates within sets

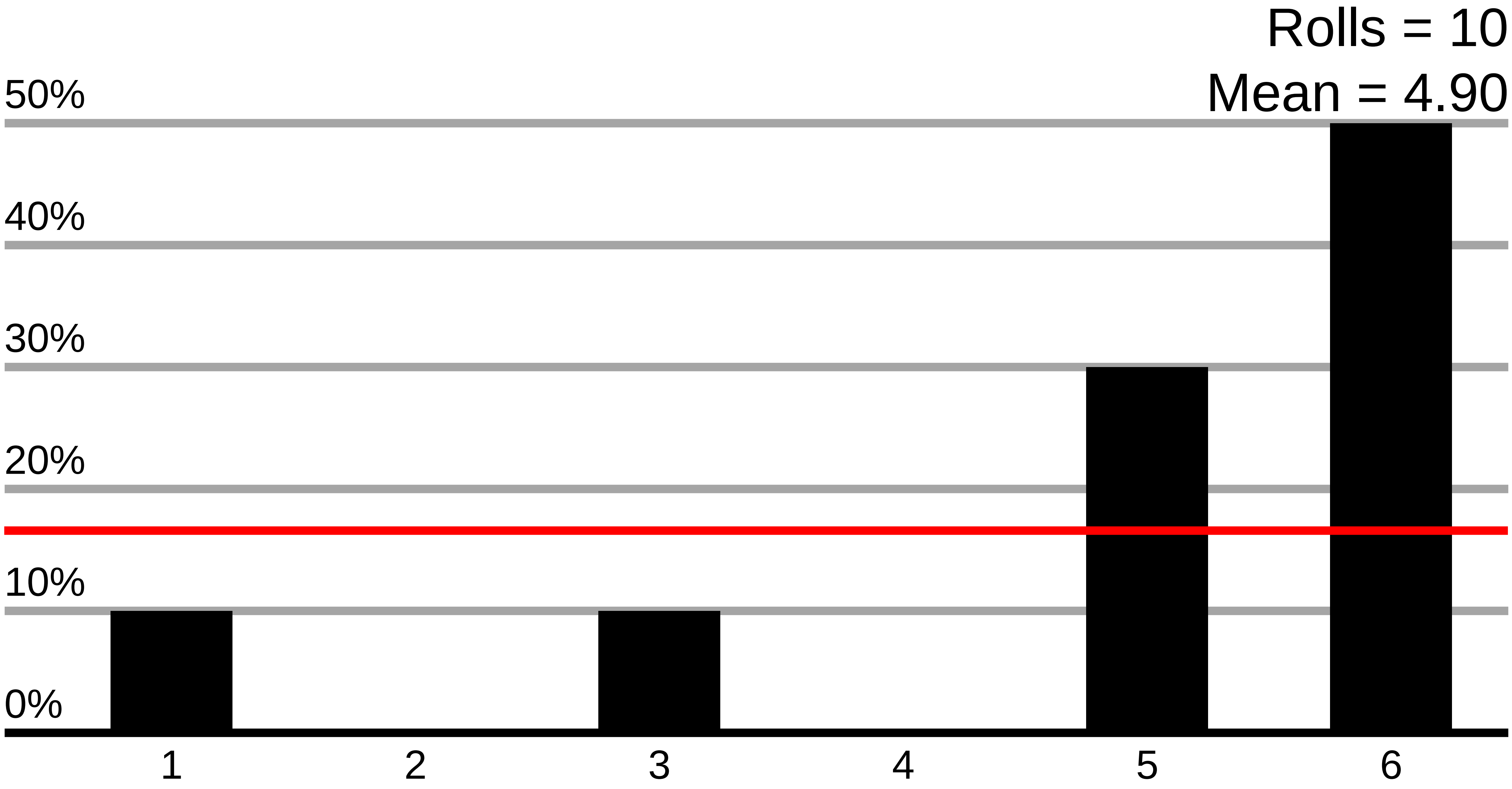
	Ordered	Unordered
With Replacement	$X = n^k$	$X = \frac{(n+k-1)!}{k!(n-1)!}$
Without Replacement	$P_{n,k} = \frac{n!}{(n-k)!}$	$C_{n,k} = \frac{n!}{k!(n-k)!}$

Challenge

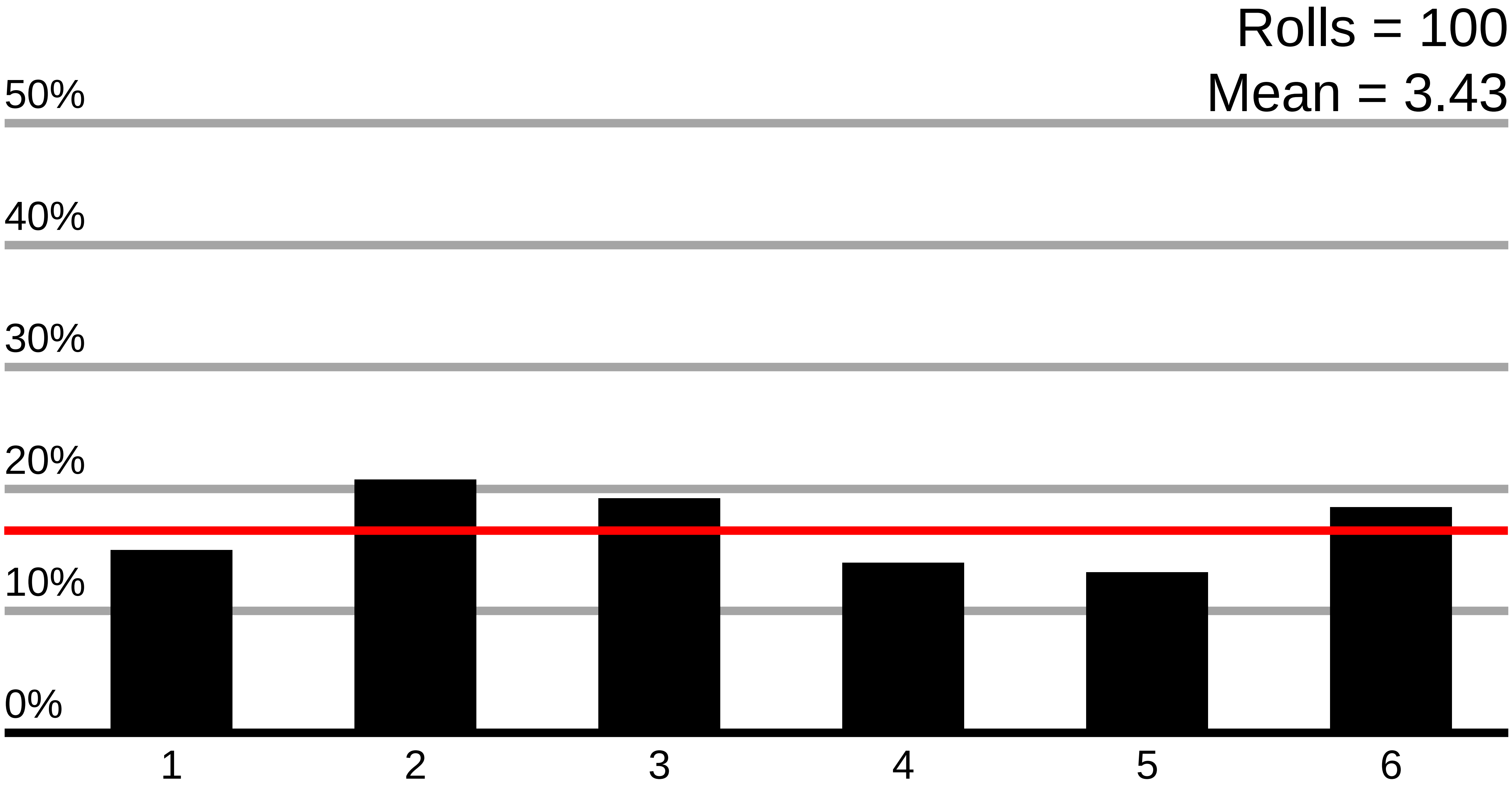
How many sets of 4 cards can we draw from a 52-card deck?

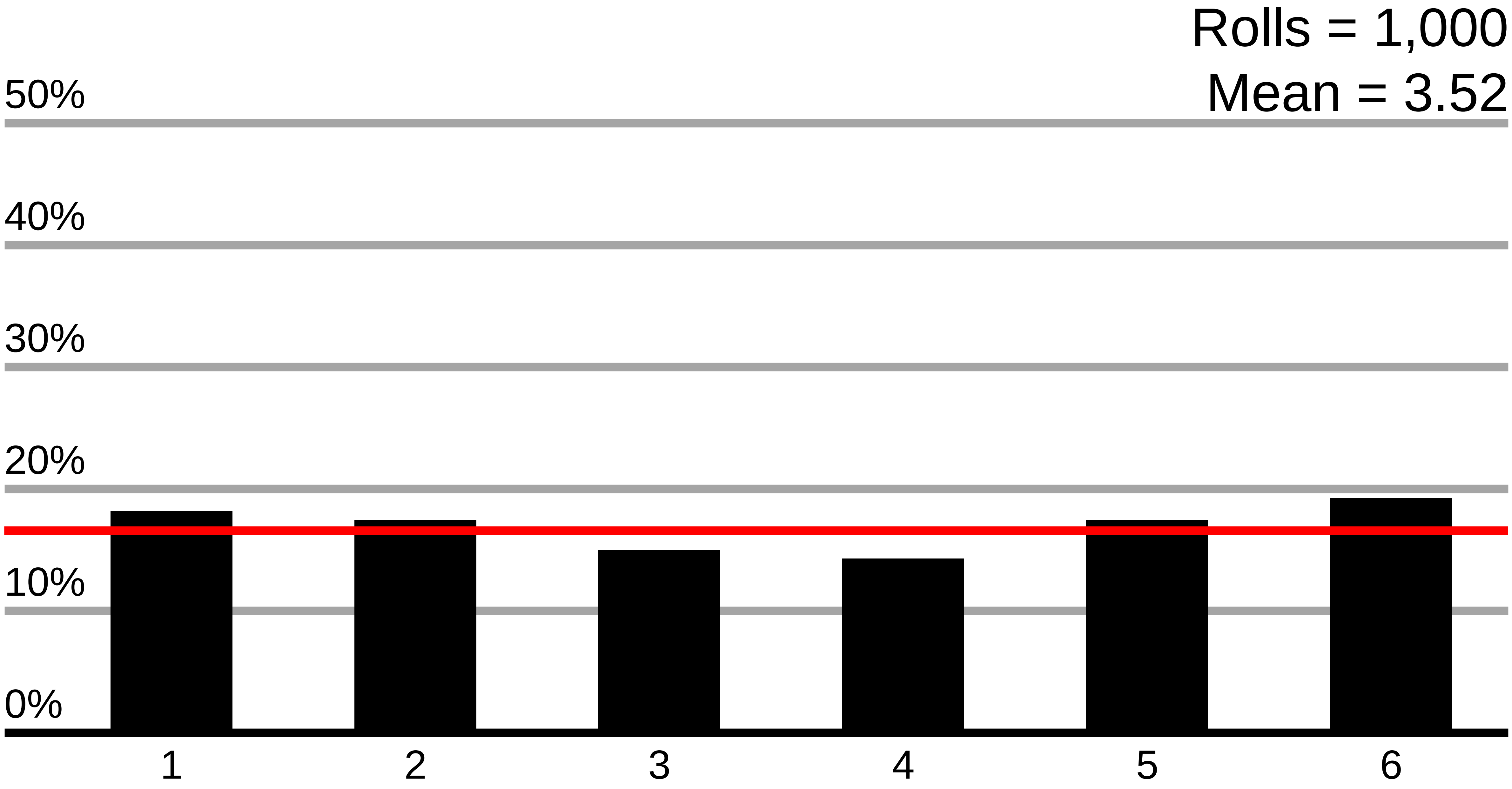
Solve the problem using each sampling method.

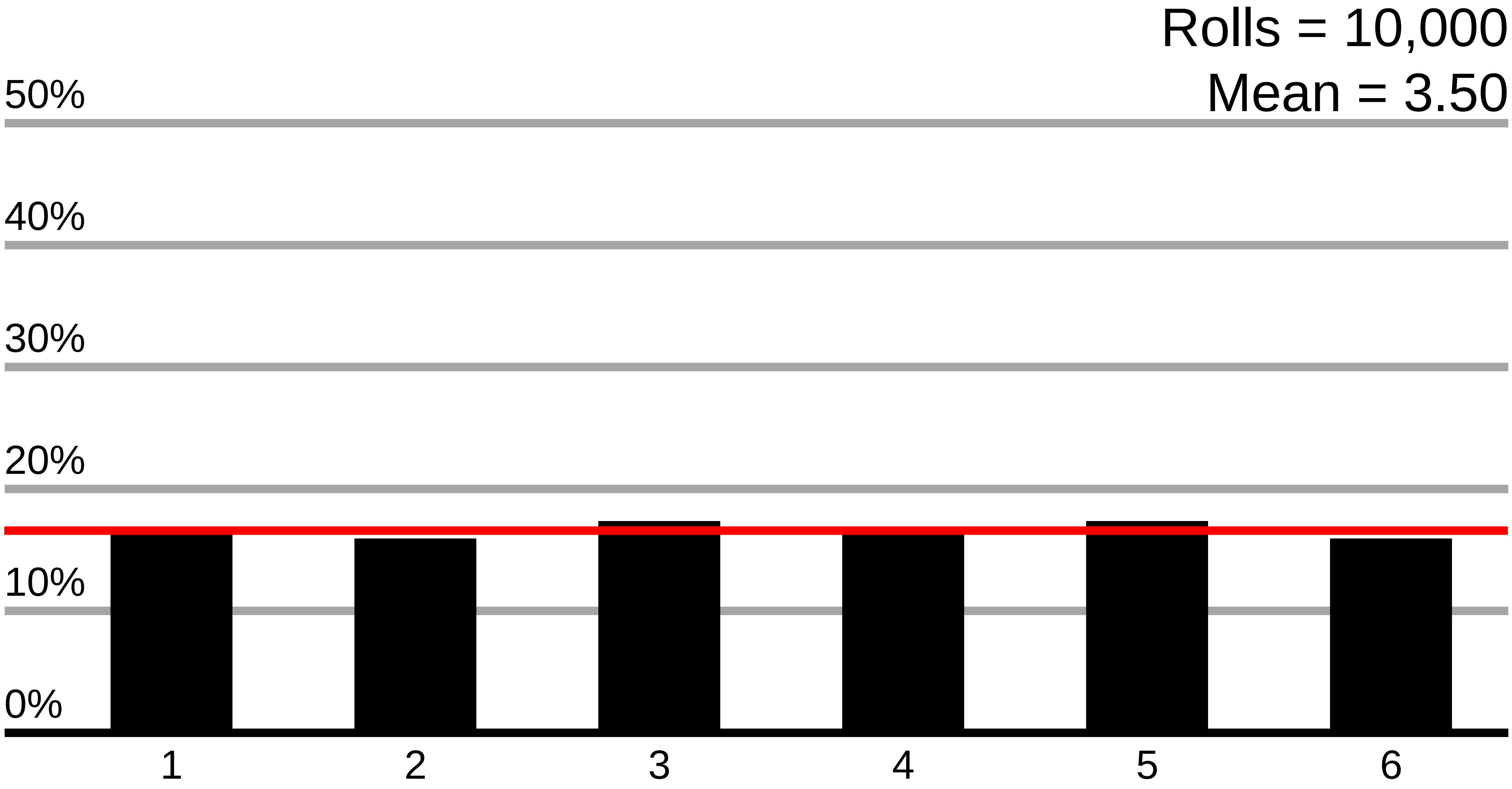
Law of Large Numbers



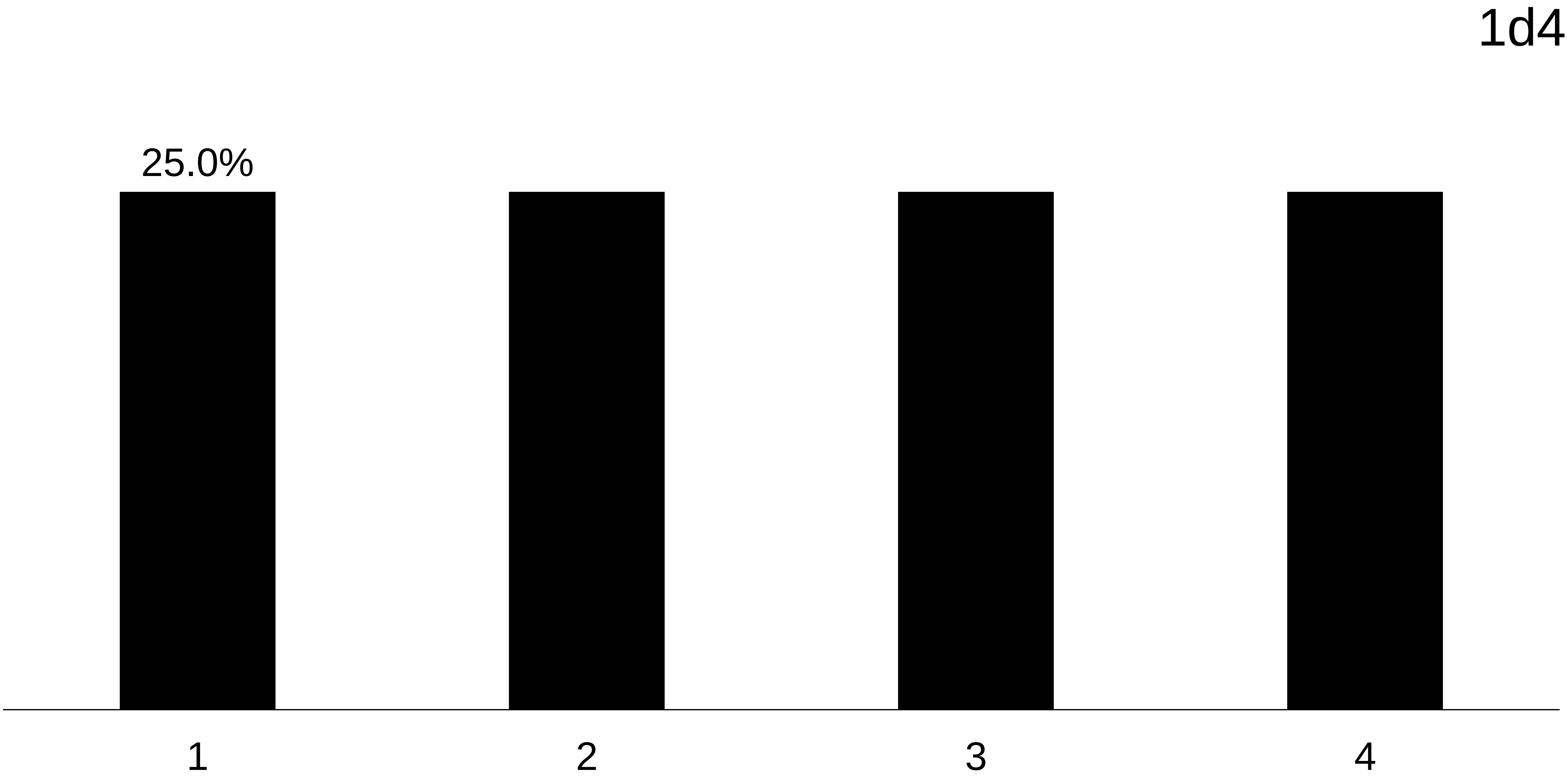
Rolls = 10
Mean = 4.90
50%
40%
30%
20%
10%
0%
1
2
3
4
5
6

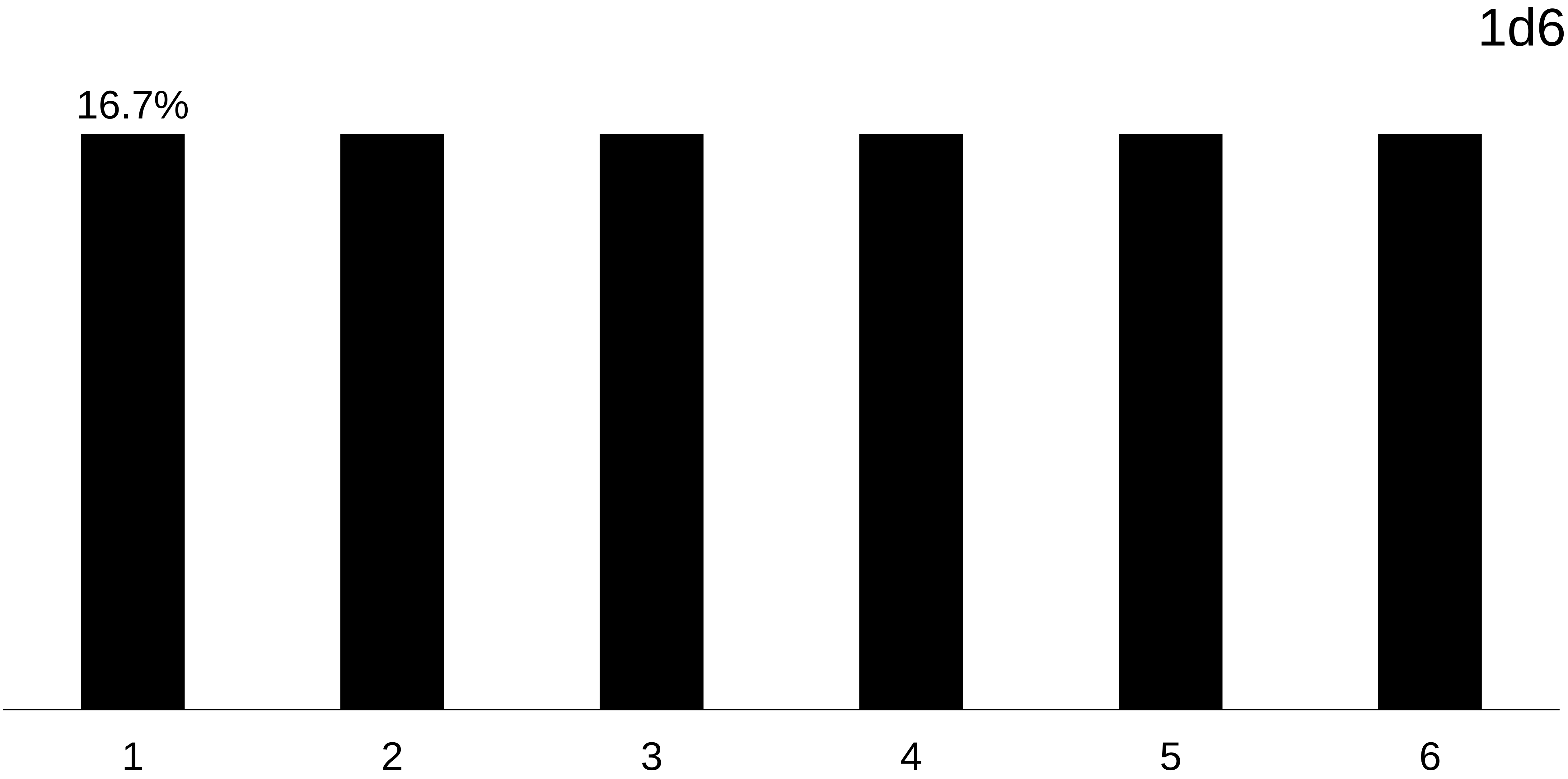

Rolls = 100
Mean = 3.43
50%
40%
30%
20%
10%
0%
1
2
3
4
5
6

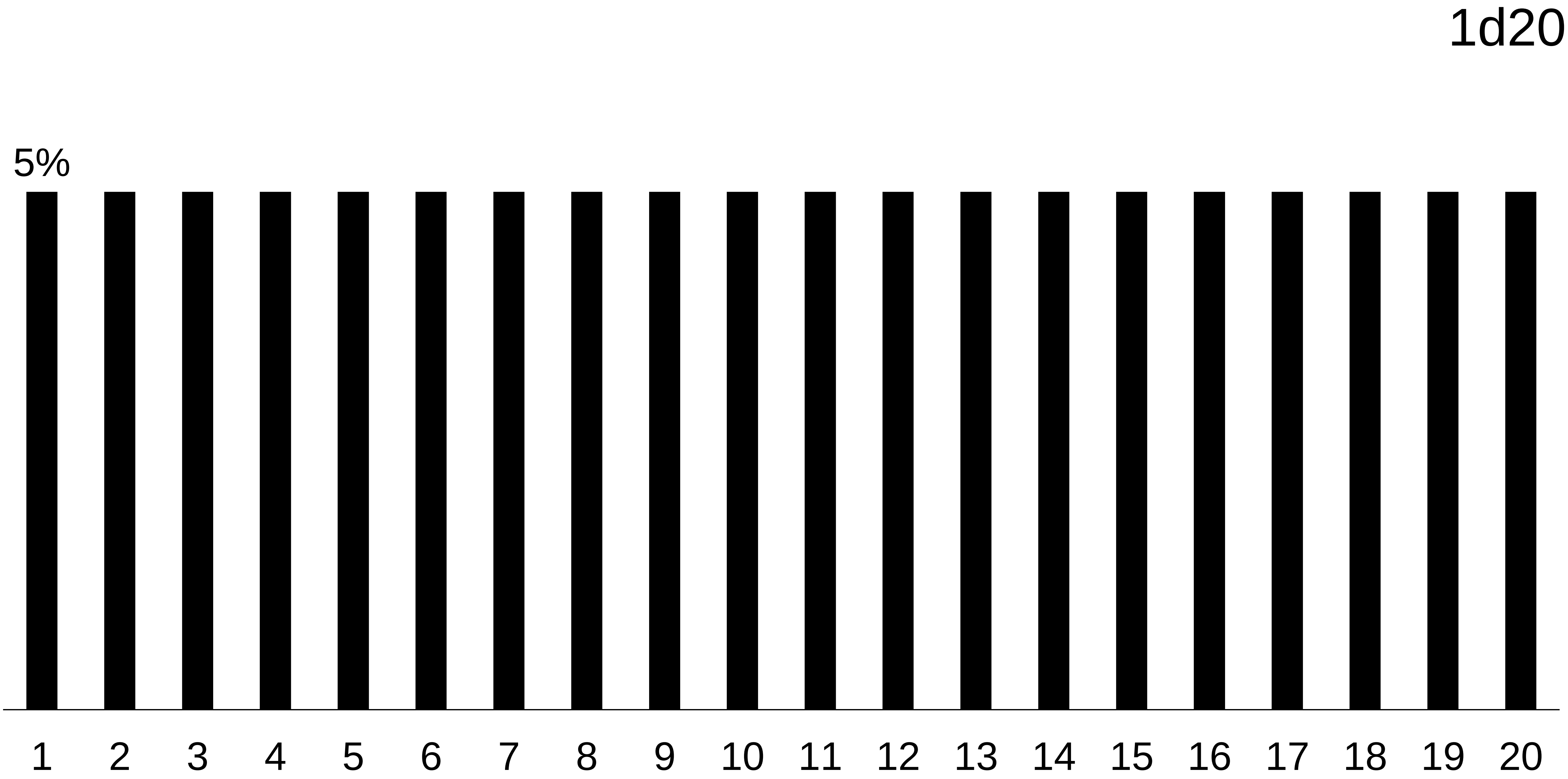

Rolls = 1,000
Mean = 3.52
50%
40%
30%
20%
10%
0%
1
2
3
4
5
6


Rolls = 10,000
Mean = 3.50
50%
40%
30%
20%
10%
0%
1
2
3
4
5
6

Distributions


1d4
25.0%
1
2
3
4

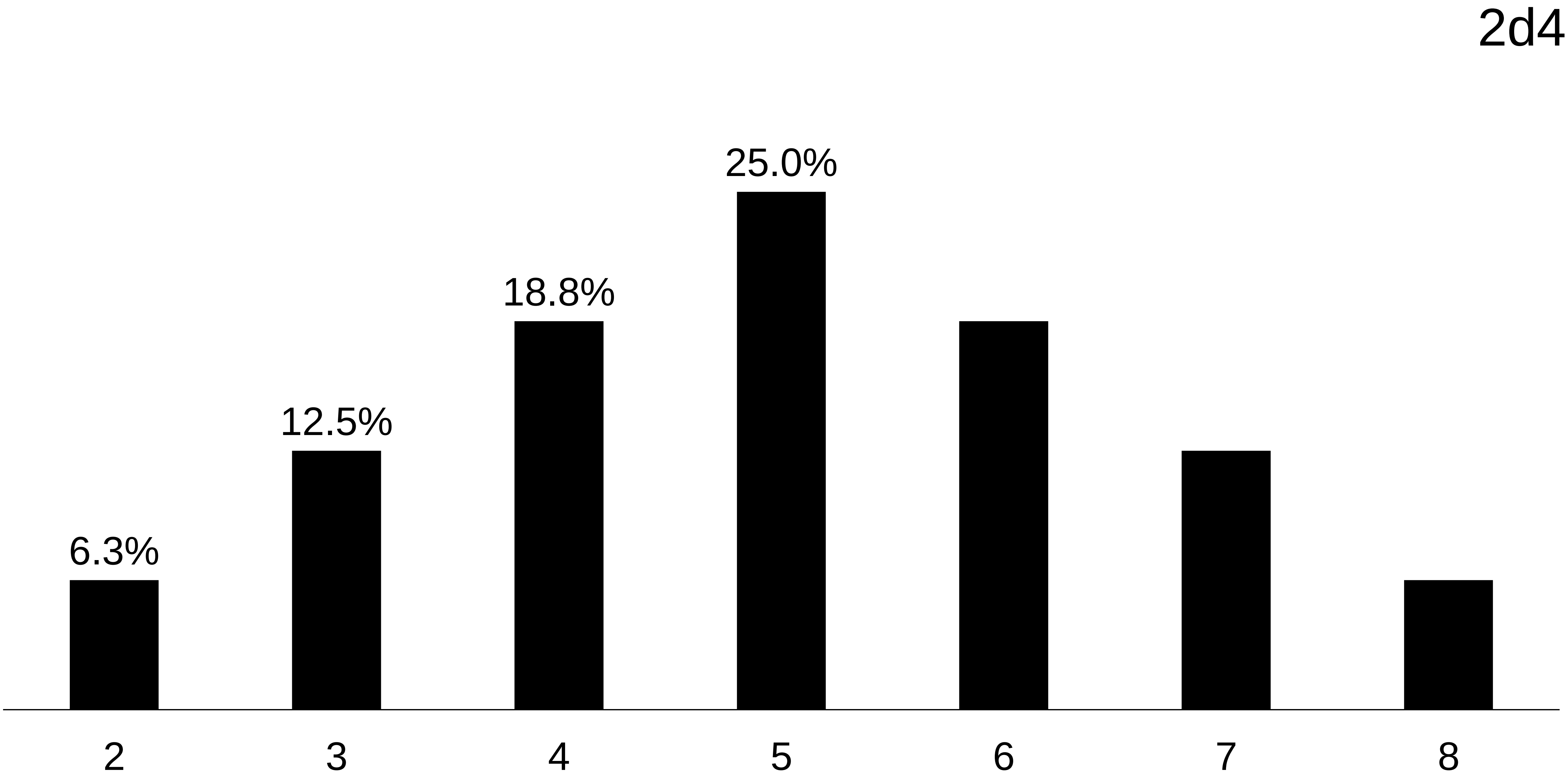

1d6
16.7%
1
2
3
4
5
6

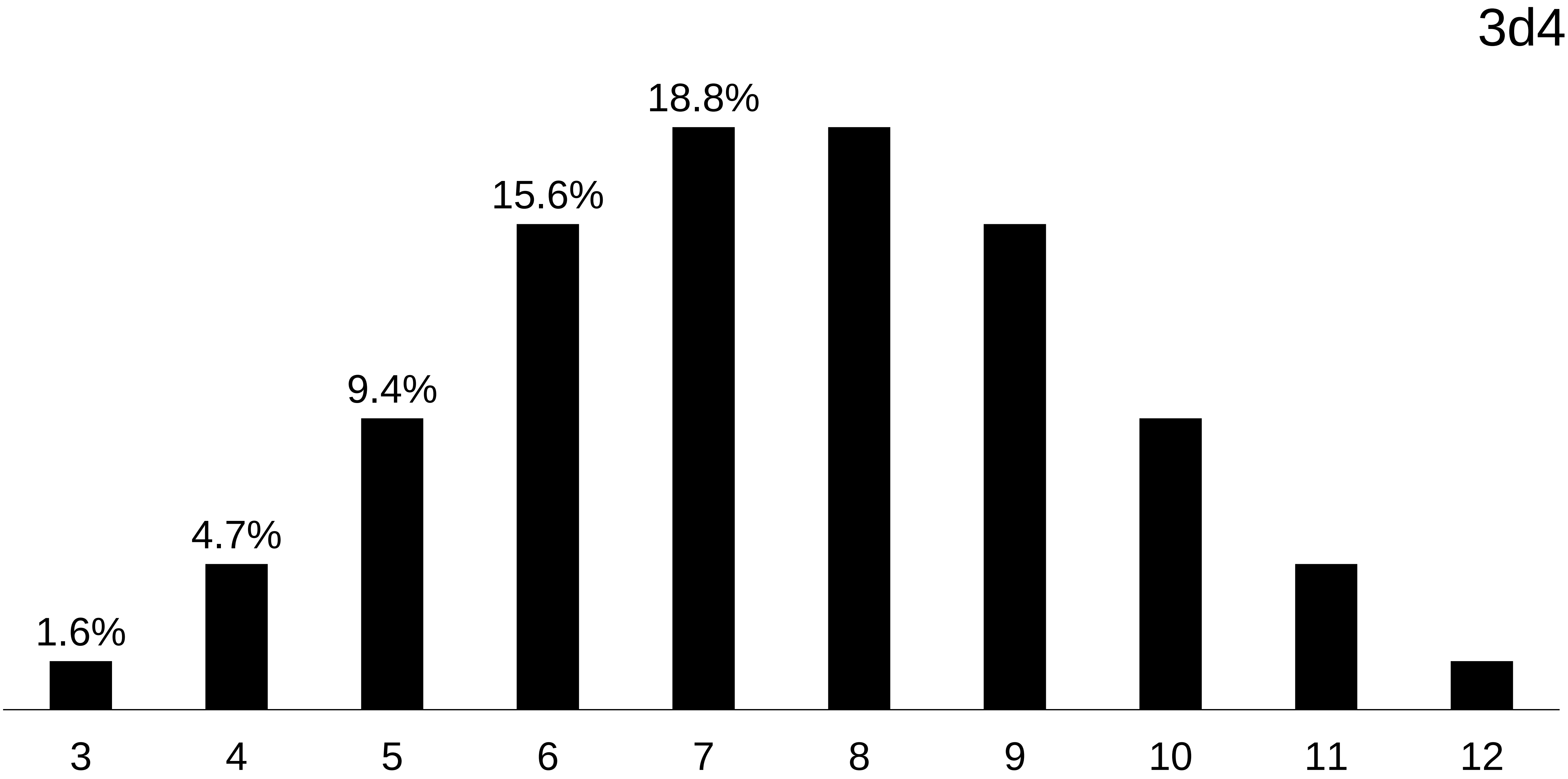

1d20
5%
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20

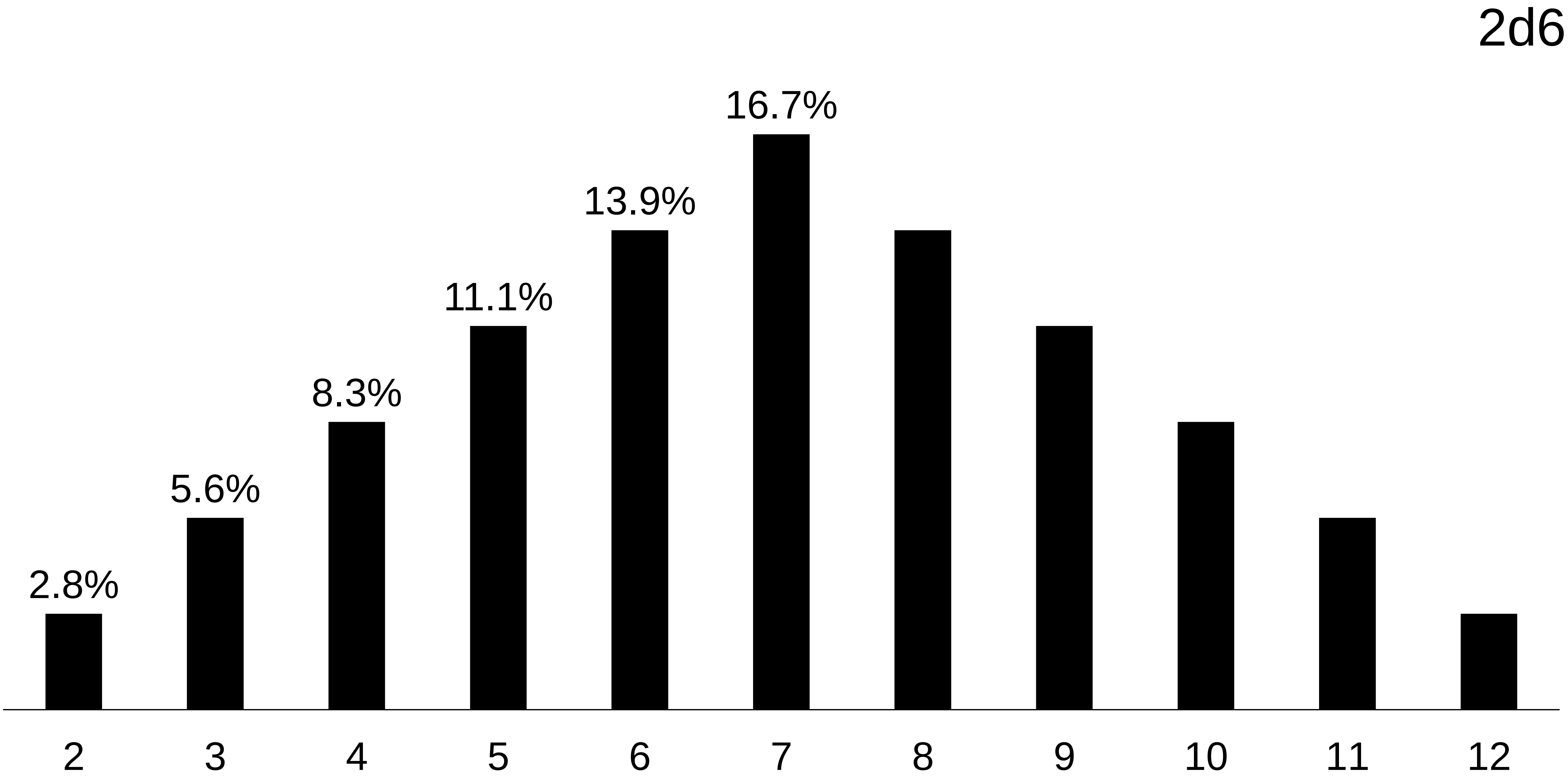
Challenge

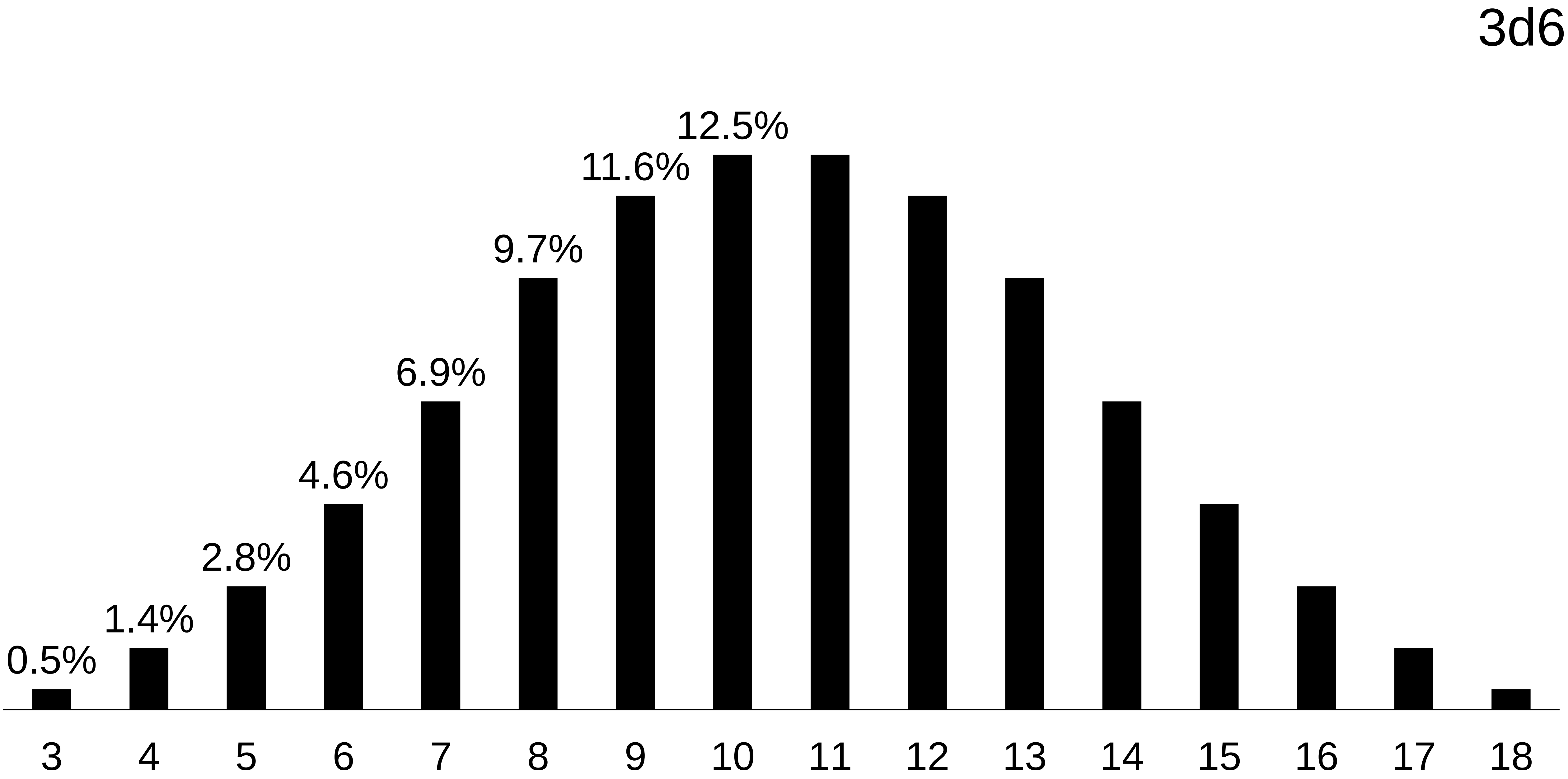
Why use a flat distribution?

What is the difference between 1d4, 1d6, and 1d20?


2d4
25.0%
18.8%
12.5%
6.3%
2
3
4
5
6
7
8


3d4
18.8%
15.6%
9.4%
4.7%
1.6%
3
4
5
6
7
8
9
10
11
12


2d6
16.7%
13.9%
11.1%
8.3%
5.6%
2.8%
2
3
4
5
6
7
8
9
10
11
12

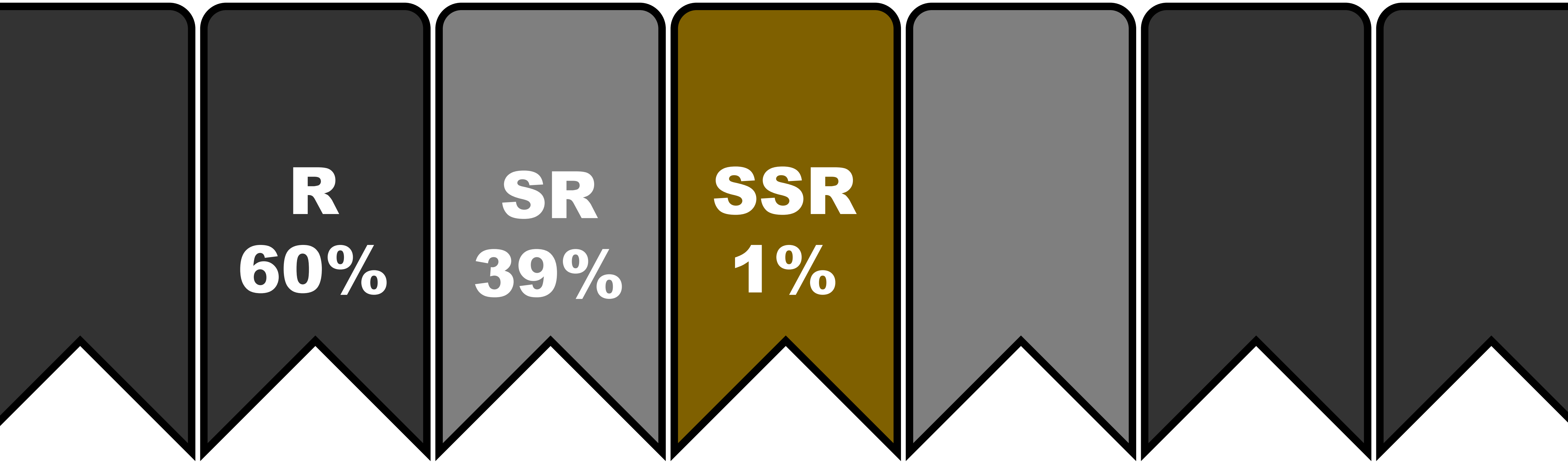

3d6
0.5%
1.4%
2.8%
4.6%
6.9%
9.7%
11.6%
12.5%
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18

Challenge

Why use a curve distribution?

What happens as we add dice?

Bonus Challenge


R
60%
SR
39%
SSR
1%

Challenge

How many SSR do you expect to get after 50, 100, and 200 pulls?

How many pulls do you need to almost guarantee an SSR?

How many pulls do you need to guarantee an SSR?

References

Hald, Anders. *A History of Probability and Statistics and Their Applications before 1750*. Hoboken: John Wiley & Sons, 1990.