CSD2301 Lecture
17. Gravitation

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Outline

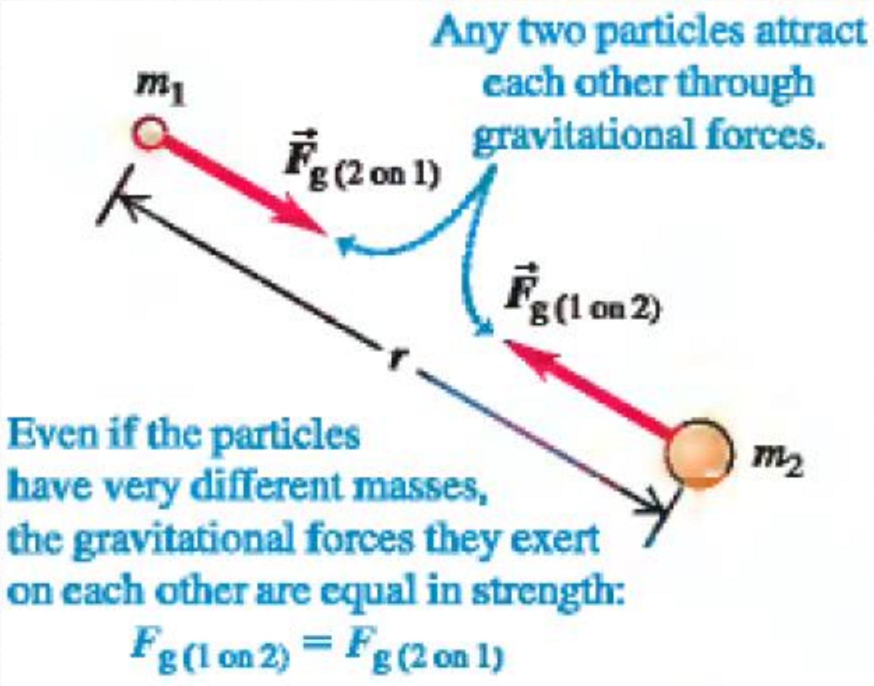
- Newton's law of gravitation
- Gravitational potential energy
- Escape velocity
- Motion of satellites
- Kepler's Laws

Newton's Law of Gravitation

- States that: Every particle of matter in the universe attracts every other particle with a force that is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them.

$$F_g = \frac{Gm_1m_2}{r^2}$$

- Where F_g is the magnitude of gravitational force on either particle, r is the distance between them, and **G is the gravitational constant**.



Newton's Law of Gravitation

- $G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$
- Don't confuse g and G.
  - g can have different values at different locations.
  - G has the same value for any 2 bodies, regardless of location.

Properties of Gravitation

- Gravitation forces always act along the line joining 2 particles, forming an action-reaction pair.
- Force that you exert on earth = force that earth exerts on you

Weight

- More general definition: The weight of a body is the total gravitational force exerted on the body by all other bodies in the universe.
- When body is near surface of earth → can neglect all other gravitational forces.
- If earth has mass m_E, then weight (w) of a small body mass (m) at earth's surface (distance R_E from its center) is:

$$w = F_g = \frac{Gm_E m}{{R_E}^2}$$

- Since w = mg,

$$g = \frac{Gm_E}{{R_E}^2}$$

Weight

- Since $R_E \approx 6380$ km and $m_E \approx 5.98 \times 10^{24}$ kg,

$$g = \frac{Gm_E}{{R_E}^2} = 9.8\, m/s^2$$

- People use this method to calculate the mass of earth. Because mass of earth cannot be measured. R_E and g can be measured.

Example: Gravity of Mars

An unmanned lander is sent to the surface of the planet Mars, which has radius $R_M = 3.40 \times 10^6$ m and mass $m_M = 6.42 \times 10^{23}$ kg. The earth weight of the Mars lander is 3920 N. Calculate its weight F_g and the acceleration g_M due to the gravity of Mars 6.0×10^6 m above the surface of Mars.

Example: Gravity of Mars

EXECUTE: (a) The distance r from the *center* of Mars is

$$r = (6.0 \times 10^6\ \text{m}) + (3.40 \times 10^6\ \text{m}) = 9.4 \times 10^6\ \text{m}$$

The mass m of the lander is its earth weight w divided by the acceleration of gravity g on earth:

$$m = \frac{w}{g} = \frac{3920\ \text{N}}{9.8\ \text{m/s}^2} = 400\ \text{kg}$$

$$F_g = \frac{Gm_M m}{r^2} = \frac{(6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2)(6.42 \times 10^{23}\ \text{kg})(400\ \text{kg})}{(9.4 \times 10^6\ \text{m})^2} = 194\ \text{N}$$

Weight of the lander on Mars

The acceleration due to the gravity of Mars at this point is

$$g_M = \frac{F_g}{m} = \frac{194\ \text{N}}{400\ \text{kg}} = 0.48\ \text{m/s}^2$$

Gravity on Mars

Gravitational Potential Energy

- Previously, we assumed that gravitational force on a body is constant in both magnitude and direction.
  - That's why we used $\Delta U = mgh$.
- But more generally, $U = -\dfrac{Gm_E m}{r}$

$$W_{grav} = \int_{r_1}^{r_2} F_r \, dr$$

$$W_{grav} = -Gm_E m \int_{r_1}^{r_2} \frac{dr}{r^2} = \frac{Gm_E m}{r_2} - \frac{Gm_E m}{r_1}$$

$$W_{grav} = U_1 - U_2$$

Gravitational Potential Energy

$$U = -\frac{Gm_E m}{r}$$

- The negative sign is important.
- When body moves away from earth → r increases → U increases (becomes less negative)
  - Since $W_{grav} = U_1 - U_2$, the gravitational force does negative work.
- When body "falls" toward earth → r decreases → U decreases (becomes more negative)
  - Since $W_{grav} = U_1 - U_2$, the gravitational force does positive work.

Escape Velocity (or Escape Speed)

- Definition: The velocity required for a body to escape completely from a planet.

Example to find Escape Speed

Three men were sent to the moon in a shell fired from earth in Florida. (a) Find the muzzle speed needed to shoot the shell straight up to a height above the earth equal to the earth's radius. (b) Find the **escape speed** – that is, the muzzle speed that would allow the shell to escape from earth completely. Neglect air resistance, earth's rotation, and the gravitational pull of the moon. Earth's radius $R_E = 6.38 \times 10^6$ m and mass $m_E = 5.97 \times 10^{24}$ kg.

Example to find Escape Speed

EXECUTE: (a) We can determine v_1 from the energy-conservation equation

$$K_1 + U_1 = K_2 + U_2$$

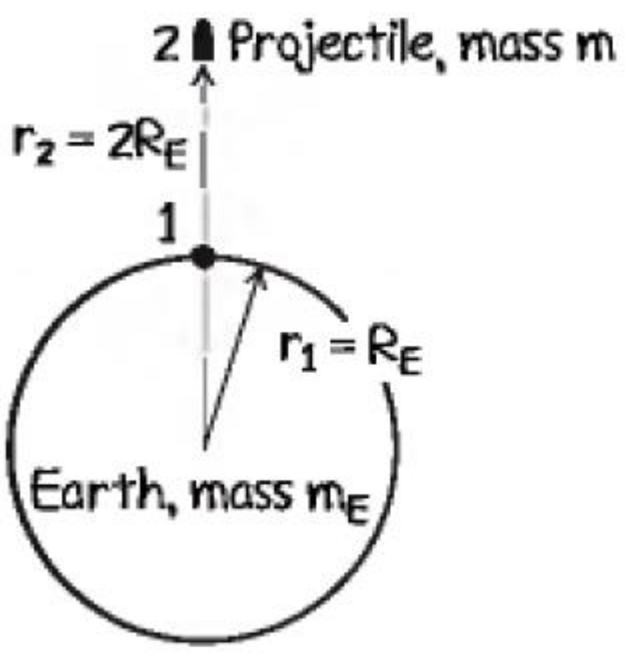
$$\frac{1}{2}mv_1^2 + \left(-\frac{Gm_E m}{R_E}\right) = 0 + \left(-\frac{Gm_E m}{2R_E}\right)$$

Rearranging this, we find that

$$v_1 = \sqrt{\frac{Gm_E}{R_E}}$$

$$= \sqrt{\frac{(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(5.97 \times 10^{24}\ \mathrm{kg})}{6.38 \times 10^6\ \mathrm{m}}}$$

$$= 7900\ \mathrm{m/s} (= 28{,}400\ \mathrm{km/h} = 17{,}700\ \mathrm{mi/h})$$

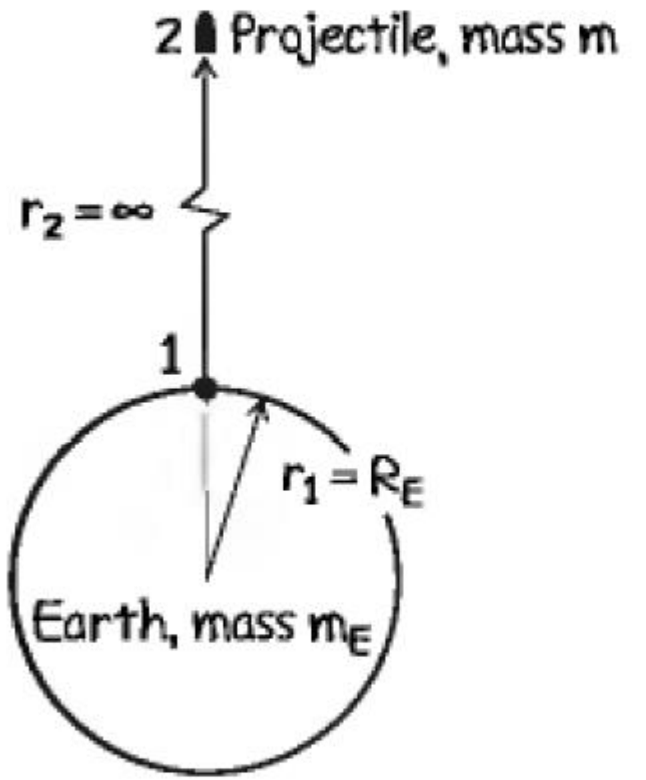


Example to find Escape Speed

(b) We want the shell barely to be able to "reach" point 2 at $r_2 = \infty$, with no kinetic energy left over. Hence $K_2 = 0$ and $U_2 = 0$ (the potential energy goes to zero at infinity; see Fig. 12.11). The total energy is therefore zero, and when the shell is fired its positive kinetic energy K_1 and negative potential energy U_1 must also add to zero:

$$\frac{1}{2}mv_1^2 + \left(-\frac{Gm_E m}{R_E}\right) = 0 + 0$$

$$\begin{aligned} v_1 &= \sqrt{\frac{2Gm_E}{R_E}} \\ &= \sqrt{\frac{2(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(5.97 \times 10^{24}\ \mathrm{kg})}{6.38 \times 10^6\ \mathrm{m}}} \\ &= 1.12 \times 10^4\ \mathrm{m/s} (= 40{,}200\ \mathrm{km/h} = 25{,}000\ \mathrm{mi/h}) \end{aligned}$$



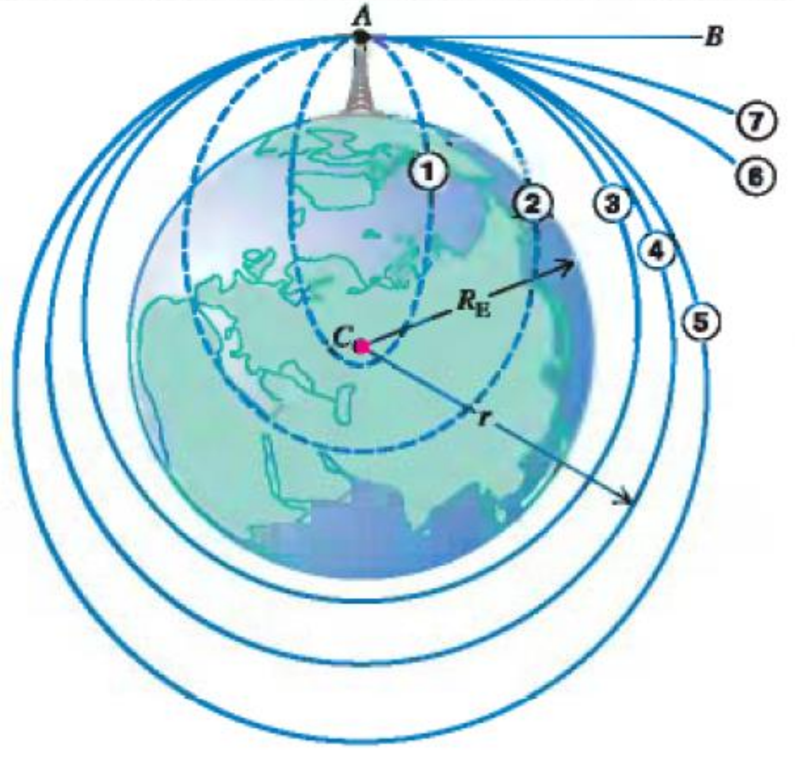
Escape Velocity (or Escape Speed)

- From previous example, generally, the escape (initial) speed v_1 needed for a body to escape from the surface of a spherical mass M with radius R (ignoring air resistance) is:

$$v_1 = \sqrt{\frac{2GM}{R}}$$

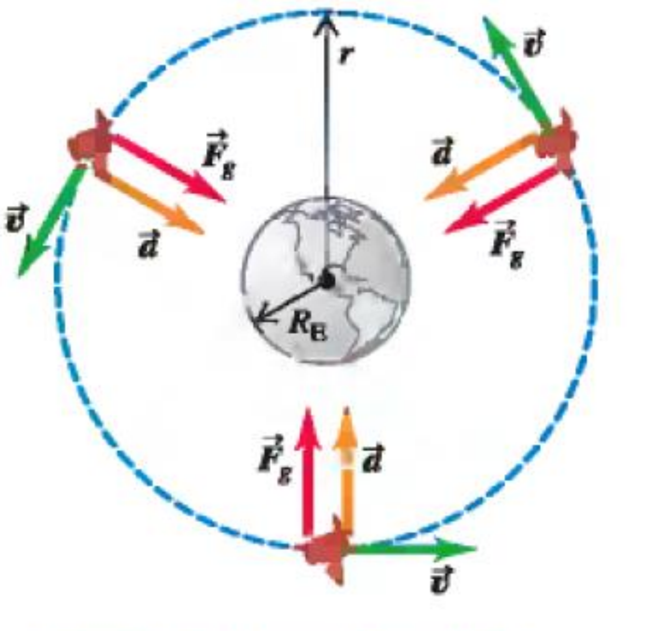
Motion of Satellites

- Imagine launching a projectile from A towards B.
- Trajectories 1 through 7 show effect of increasing speed.
- 1 – 5 close on themselves and are called **closed orbits**.
  - In 3 – 5, the projectile misses earth and becomes a travels back to the starting point.
- 6 – 7 does not return and are called **open orbits**.



Motion of Satellites

- Simplest case: **Circular Orbits**
  - Most artificial satellites travel in a circular motion.
- Only force acting is the earth's gravitational force → Speed is constant.



The satellite is in a circular orbit: Its acceleration $\vec{a}$ is always perpendicular to its velocity $\vec{v}$, so its speed v is constant.

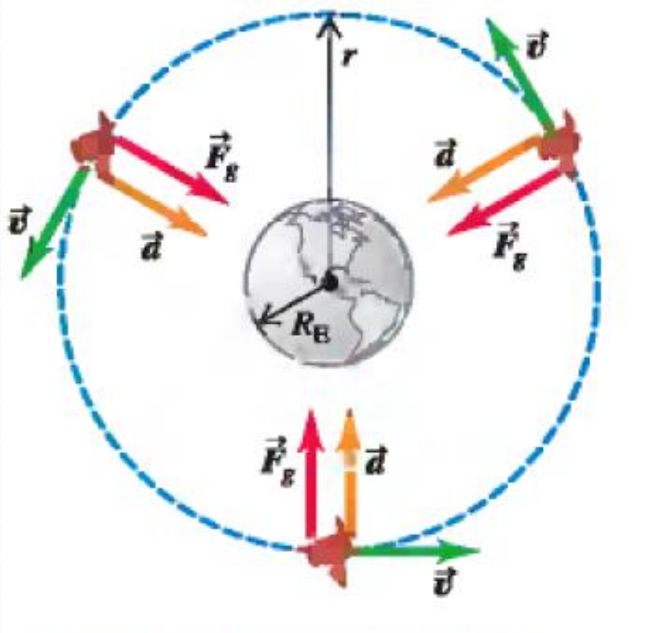
Motion of Satellites

- Use Newton's laws and the law of gravitation to find out satellite motion.
- Since circular motion:

$$\frac{Gm_E m}{r^2} = \frac{mv^2}{r}$$

$$\boxed{v = \sqrt{\frac{Gm_E}{r}}}$$

- Relationship: For a given radius r, the velocity is fixed.



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Motion of Satellites

- To find period, T: $$\boxed{T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{Gm_E}} = \frac{2\pi r^{3/2}}{\sqrt{Gm_E}}}$$
- To find total mechanical energy during orbit, E:

$$E = K + U = \frac{1}{2}mv^2 + \left(-\frac{Gm_E m}{r}\right) = \frac{1}{2}m\left(\frac{Gm_E}{r}\right) - \frac{Gm_E m}{r}$$

$$\boxed{E = -\frac{Gm_E m}{2r}}$$

Example

Suppose you want to place a 1000 kg weather satellite into a circular orbit 300 km above the earth's surface. (a) What speed, period, and radial acceleration must it have? (b) How much work has to be done to place this satellite in orbit? (c) How much additional work would have to be done to make this satellite escape the earth? Earth's radius $R_E = 6380$ km and mass $m_E = 5.97 \times 10^{24}$ kg.

Example

(a) What speed, period, and radial acceleration must it have?

EXECUTE: (a) The radius of the satellite's orbit is

$$r = 6380\text{ km} + 300\text{ km} = 6680\text{ km} = 6.68 \times 10^6\text{ m}$$

From Eq. (12.10), the orbital speed is

$$v = \sqrt{\frac{Gm_E}{r}} = \sqrt{\frac{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.97 \times 10^{24}\text{ kg})}{6.68 \times 10^6\text{ m}}}$$
$$= 7720\text{ m/s}$$

We find the orbital period from Eq. (12.12):

$$T = \frac{2\pi r}{v} = \frac{2\pi(6.68 \times 10^6\text{ m})}{7720\text{ m/s}}$$
$$= 5440\text{ s} = 90.6\text{ min}$$

The radial acceleration is

$$a_{rad} = \frac{v^2}{r} = \frac{(7720\text{ m/s})^2}{6.68 \times 10^6\text{ m}}$$
$$= 8.92\text{ m/s}^2$$

This is the value of g at a height of 300 km above the earth's surface; it is somewhat less than the value of g at the surface.

Example

(b) How much work has to be done to place this satellite in orbit?

$$E_2 = -\frac{Gm_E m}{2r}$$
$$= -\frac{(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(5.97 \times 10^{24}\ \mathrm{kg})(1000\ \mathrm{kg})}{2(6.38 \times 10^6\ \mathrm{m})}$$
$$= -2.99 \times 10^{10}\ \mathrm{J}$$

At rest on the earth's surface $(r = R_E)$, the kinetic energy is zero:

$$E_1 = K_1 + U_1 = 0 + \left(-\frac{Gm_E m}{R_E}\right)$$
$$= -\frac{(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(5.97 \times 10^{24}\ \mathrm{kg})(1000\ \mathrm{kg})}{6.38 \times 10^6\ \mathrm{m}}$$
$$= -6.25 \times 10^{10}\ \mathrm{J}$$

and so

$$W_{\mathrm{required}} = E_2 - E_1 = -2.99 \times 10^{10}\ \mathrm{J} - (-6.25 \times 10^{10}\ \mathrm{J})$$
$$= 3.26 \times 10^{10}\ \mathrm{J}$$

Example

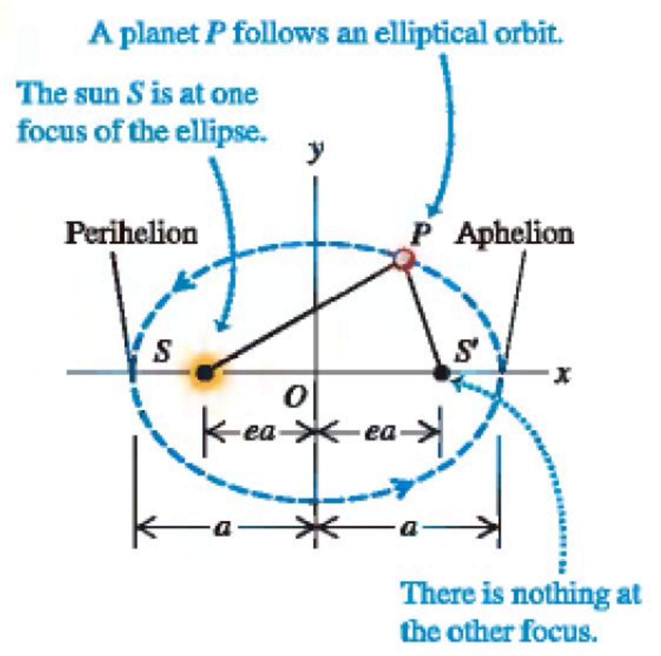
(c) How much additional work would have to be done to make this satellite escape the earth?

ANS:

To escape, total mechanical energy must be zero. Energy (E_2) in orbit is -2.99×10^{10} J. To increase this to zero, an amount of work equal to $\underline{2.99 \times 10^{10}\ \text{J}}$ would have to be done.

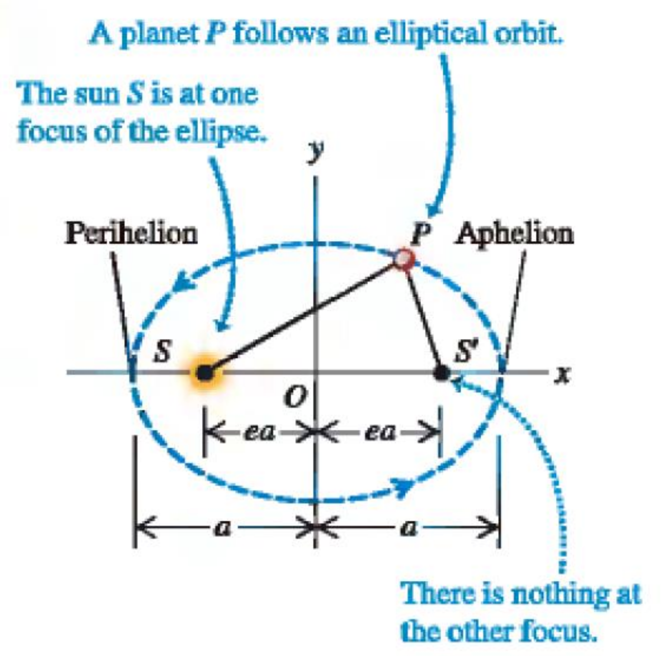
Kepler's Laws and Observations

- Describe the motion of planets around the sun.
- **Kepler's first law** – Each planet moves in an elliptical orbit, with the sun at one focus of the orbit.
- The sum of distances SP and S'P is the same for every point on the curve.
- Longest dimension is the major axis, so the half-length (a) is the **semi-major axis**.
    - The other shorter dimension is the minor axis with half-length (b) being the **semi-minor axis**.



Kepler's Laws and Observations

- Distance from S from centre of ellipse is **ea**, where e is the **eccentricity** and has a dimensionless number between 0 and 1.
  - If e = 0, the ellipse is a circle.
  - Actual orbits of planets are fairly circular. For e.g. earth has e = 0.017

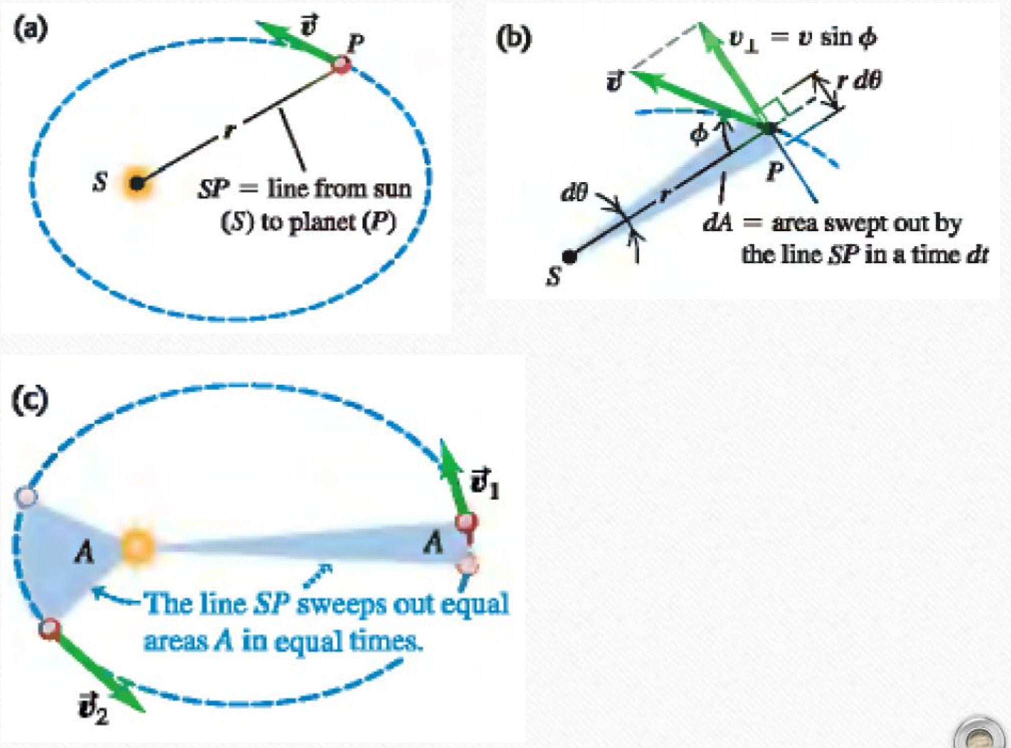


Kepler’s Laws and Observations

- **Kepler’s second law** – a line from the sun to a given planet sweeps out equal areas at any equal time interval.
  - In other words, **sector velocity** is the same throughout.
- Sector velocity is the rate at which area is swept out:

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}$$

where height is r, base length is $rd\theta$



Kepler's Laws and Observations

- **Kepler's third law** – The periods of the planets are proportional to the 3/2 powers of the major axis lengths of their orbits.
- Hence, period: $T = \frac{2\pi a^{3/2}}{\sqrt{Gm_s}}$

where m_s is the mass of sun.

The End