CSD1241 Tutorial 6

Problem 1. The **matrix representation** of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is an $m \times n$ matrix M such that $T(\vec{x}) = M\vec{x}$ for any $\vec{x} \in \mathbb{R}^n$.
Find the matrix representation of T in the following cases. In each case, find the points $\vec{x}$ that are fixed by T, that is, $T(\vec{x}) = \vec{x}$.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ y \end{pmatrix}$$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ with

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + y - z \\ y + z \\ x - y - z \end{pmatrix}$$

(c) Based on the results from a,b, we see that the map T always fixes the origin $O = \vec{0}$. Show that this property is true for any linear map, that is, $T(\vec{0}) = \vec{0}$ whenever $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.
(d) Let l be a line in $\mathbb{R}^2$ which doesn't go through the origin. Using the result in c, could you explain that the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ which is the reflection through l is not a linear transformation?

Problem 2. Another way to find the matrix representation M of $T : \mathbb{R}^n \to \mathbb{R}^m$ is to use the standard unit vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ of $\mathbb{R}^n$

$$M = [T(\vec{e}_1) \ \ T(\vec{e}_2) \ \ \cdots \ \ T(\vec{e}_n)]$$

In the following cases, find the matrix representation of the linear transformation T by the method described above. In each case, find the points $\vec{x}$ that are mapped to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, that is, $T(\vec{x}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ with

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ T\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \ T\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$$

Problem 3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal projection onto the line $l : x - 2y = 0$.
(a) Find the matrix M of T.
(b) Find the image of the points $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 5 \end{pmatrix}$.
(c) Find the image of the line $m : x + y = 3$ under this map (find normal equation).
Hint. Use the vector equation of m to express its coordinates.
(d) Find the image of the line $n : 2x + y = 15$ under this map (find normal equation).

Problem 4. Redo Problem 3 with T be the skew projection onto $l : x - 2y = 0$ along the direction $\vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Problem 5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal reflection through the line $l : x - 2y = 0$.
(a) Find the matrix M of T.
(b) Find the image of the points $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.
(c) Find the image of the line $m : x + y = 3$ under this map (find normal equation).
(d) Find the image of the line $n : 2x + y = 5$ under this map (find normal equation).

Problem 6. Redo Problem 5 with T be the skew reflection through $l : x - 2y = 0$ along the direction $\vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.