Numerical Integration Part 2
Mid-trimester Revision

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AY 23/24 Trimester 1

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Irreducible factors, Midpoint and Trapezoidal Rule

- Factorization of denominator $Q(x)$:
  - Non-repeated irreducible factors: one partial fraction for each irreducible factor, numerator is a linear polynomial $Ax + B$.
- Some definite integrals cannot be evaluated exactly, establishes a need for **approximation**; suppose we have a function $f(x)$ on $[a, b]$.

  Let n be a positive integer with $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$, then
  - Midpoint Rule with midpoints $\overline{x}_i = \dfrac{x_i + x_{i-1}}{2}$:

$$\int_a^b f(x)\,dx \approx M_n = \Delta x\left[f(\overline{x}_1) + \cdots + f(\overline{x}_n)\right]$$

  - Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

Error bounds

Let $|f''(x)| \leq K$ for some constant K.

- Error bound for Midpoint Rule:

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}.$$

- Error bound for Trapezoidal Rule:

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}.$$

These represent "worst-case" scenarios. Methods for finding K:

- Find the point $x \in [a,b]$ which gives the maximum of $|f''(x)|$. In most cases, this is infeasible because it takes too much time to calculate.
- Use the **triangle equality** and knowledge of bounds for certain functions, e.g. $|\sin x| \leq 1, |\cos x| \leq 1$.

Two rectangles at a time, instead of one

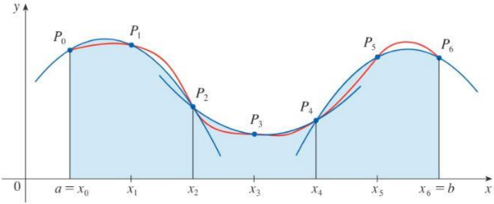
Previously in the Midpoint and Trapezoidal rules, we approximated the net area under the graph using one rectangle at a time, for n rectangles.

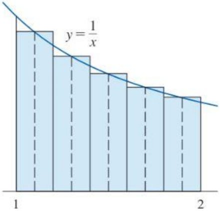
In these rules, the height of the rectangle/trapezium is dependent on two points, the endpoints of a subinterval $[x_{i-1}, x_i]$.

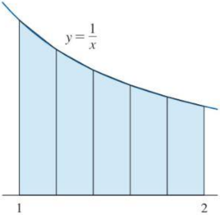
If we approximate the net area under the graph using **two** rectangles at a time, instead of one, i.e. two subintervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$, we would have three points to work with instead of two.

This considerably increases the accuracy of the approximation as we will see in the next few slides.

Visualization: Parabolas vs lines







Simpson's Rule

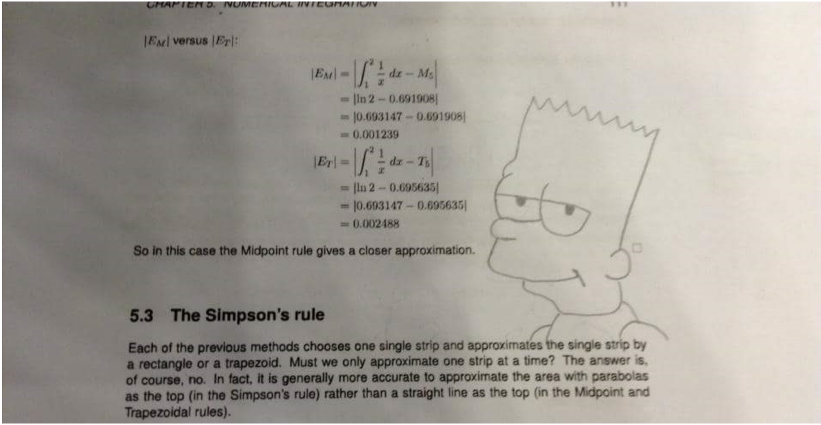
Let f be a function on $[a, b]$. We use n rectangles, with n **even**. Then

$$\Delta x = \frac{b-a}{n}, \quad \text{and} \quad x_i = a + i\Delta x \quad (i \text{ from } 0 \text{ to } n).$$

The **Simpson's Rule** S_n approximation to $\int_a^b f(x)\,dx$ is

$$S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)].$$

Artwork by a former student



CHAPTER 5. NUMERICAL INTEGRATION

$|E_M|$ versus $|E_T|$:

$$
\begin{aligned}
|E_M| &= \left|\int_1^2 \frac{1}{x}\,dx - M_5\right| \\
&= |\ln 2 - 0.691908| \\
&= |0.693147 - 0.691908| \\
&= 0.001239 \\
|E_T| &= \left|\int_1^2 \frac{1}{x}\,dx - T_5\right| \\
&= |\ln 2 - 0.695635| \\
&= |0.693147 - 0.695635| \\
&= 0.002488
\end{aligned}
$$

So in this case the Midpoint rule gives a closer approximation.

5.3 The Simpson's rule

Each of the previous methods chooses one single strip and approximates the single strip by a rectangle or a trapezoid. Must we only approximate one strip at a time? The answer is, of course, no. In fact, it is generally more accurate to approximate the area with parabolas as the top (in the Simpson's rule) rather than a straight line as the top (in the Midpoint and Trapezoidal rules).

Error bound for S_n

The following bound represents the "worst-case" scenario for the error of S_n, the Simpson's Rule approximation of $\int_a^b f(x)\,dx$.

Theorem

Suppose K is a constant where $|f^{(4)}(x)| \leq K$ on $[a, b]$. The **magnitude** of error of the Simpson's Rule (E_S) has the following upper bound:

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}.$$

Example 1

1. Use Simpson's Rule with $n = 10$ to approximate $\displaystyle\int_1^2 \frac{1}{x}\,dx$.
2. How large should we take n in order to guarantee that the Simpson's Rule approximation for $\displaystyle\int_1^2 \frac{1}{x}\,dx$ are accurate to within 0.0001?

Example 1

Exercise 1

1. Use Simpson's Rule with $n = 6$ to approximate $\int_0^1 \cos(x^2)\,dx$.
2. How large should we take n in order to guarantee that the Simpson's Rule approximation for $\int_0^1 \cos(x^2)\,dx$ are accurate to within 0.0001?

Exercise 1

Integration by parts (26% on Tues, 33% on Fri)

Evaluate $\int_0^{\frac{\pi}{4}} x \sec^2 x \tan x \, dx$.

Sine/Cosine integral (26% on Tues, 33% on Fri)

Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^6 x \cos^5 x \, dx$.

Sine/Cosine integral (26% on Tues, 33% on Fri)

Antiderivative of f (33% in Tues, 13% in Thurs)

Find an antiderivative of $f(x) = \ln(2x)$.

(a) $x \ln x - x$

(b) $x \ln(2x)$

(c) $x \ln(2x) - (x - 29)$

(d) $x \ln(2x) - \dfrac{x}{2}$

(e) None of the above

Completing the square (13% on Thurs, 25% on Fri)

Evaluate $\displaystyle\int_1^{\sqrt[3]{2}} \frac{x^2}{x^6 - 2x^3 + 2}\,dx$.

Completing the square (13% on Thurs, 25% on Fri)

Augmented Lecture Exercise (27% on Thurs)

Evaluate $\displaystyle\int_0^1 \frac{3x^5 + 2x^2}{1 + x^6}\, dx.$

Augmented Lecture Exercise (27% on Thurs)