Week 12: Bayes' rule and random variables

Password: variable

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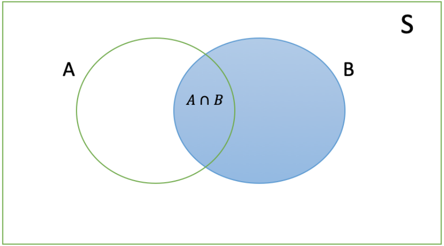
Experiment, sample spaces, events

- An **experiment** is a situation with uncertain outcomes.
- The **sample space** of an experiment is the set Ω of all possible outcomes of the experiment.
- An **event** is a subset of the sample space Ω.

Conditional probability

- Conditional probability: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$



- The sample space for *possible outcomes* is B.
  $P(A|B)$ = probability of event $A \cap B$ in the sample space B.

Independent events

- A and B are **independent** $\Leftrightarrow$ one of the following equations holds $P(A \cap B) = P(A)P(B)$, or $P(A|B) = P(A)$, or $P(B|A) = P(B)$.
- A and B are independent means

  "the information that B occurs does not affect the probability that A occurs, and vice versa"

Multiplication rule for conditional probability

Exercise 1. Let A, B, C be events. Show that

(a) $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$.

In particular if A, B are independent, then $P(A \cap B) = P(A)P(B)$.

(b) $P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C)$.

If A, B, C are **mutually independent**, then

$P(A \cap B \cap C) = P(A)P(B)P(C)$

(a) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A|B)P(B)$

$P(B|A) = \dfrac{P(B \cap A)}{P(A)} \Rightarrow P(A \cap B) = P(B|A)P(A)$

Multiplication rule for conditional probability

Exercise 1. Let A, B, C be events. Show that

(a) $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$.

In particular if A, B are independent, then $P(A \cap B) = P(A)P(B)$.

(b) $P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C)$.

If A, B, C are **mutually independent**, then

$P(A \cap B \cap C) = P(A)P(B)P(C)$

(b) $P(A \cap B \cap C) = P(A \cap (B \cap C))$

$= P(A | B \cap C) P(B \cap C)$ (by (a))

$= P(A|B \cap C) P(B|C) P(C)$ (by (a))

Mutually independent

Three events A, B, C are called **mutually independent** if

1. any two events are independent

$$P(A \cap B) = P(A)P(B),\ P(A \cap C) = P(A)P(C),\ P(B \cap C) = P(B)P(C)$$

2. and $P(A \cap B \cap C) = P(A)P(B)P(C)$

Example 1

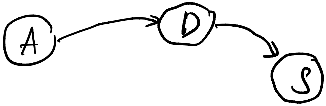
You have a flight from Amsterdam to Sydney with a stopover in Dubai. The probabilities that a luggage is put on the wrong plane at the different airports are

$$\text{Amsterdam} : 0.05, \quad \text{Dubai} : 0.03.$$

What is the probability that your luggage doesn't reach Sydney with you?

Solution

For the luggage to reach Sydney,
it must be on the correct airplane
at both Amsterdam and Dubai.



Example 1

You have a flight from Amsterdam to Sydney with a stopover in Dubai. The probabilities that a luggage is put on the wrong plane at the different airports are

$$\text{Amsterdam} : 0.05, \ \ \text{Dubai} : 0.03.$$

What is the probability that your luggage doesn't reach Sydney with you?

Solution

A=event that the luggage is put on the correct plane at Amsterdam

D=event that the luggage is put on the correct plane at Dubai.

You may assume A and D are independent (why?) $P(D|A) = P(D)$

The info. your luggage is put correctly at Ams doesn't affect the prob. it is put correctly at Dubai!

Example 1

You have a flight from Amsterdam to Sydney with a stopover in Dubai. The probabilities that a luggage is put on the wrong plane at the different airports are

$$\text{Amsterdam} : 0.05, \quad \text{Dubai} : 0.03.$$

What is the probability that your luggage doesn't reach Sydney with you?

Solution

A=event that the luggage is put on the correct plane at Amsterdam

D=event that the luggage is put on the correct plane at Dubai.

You may assume A and D are independent (why?)

$\overline{A \cap D}$ = event that your luggage doesn't reach Sydney is

$$P\left(\overline{A \cap D}\right) = 1 - P(A \cap D)$$

$$P(\overline{A \cap D}) = 1 - P(A \cap D)$$

$$= 1 - P(A)P(D), \quad P(\bar{A}) = 0.05, \; P(\bar{D}) = 0.03$$

$$= 1 - (1 - P(\bar{A}))(1 - P(\bar{D}))$$

$$= 1 - 0.95 \times 0.97$$

$$= 0.0785 = \boxed{7.85\%}$$

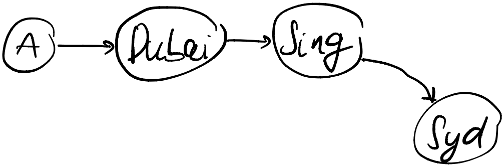
Example 2

You have a flight from Amsterdam to Sydney with stopovers in Dubai and Singapore. The probabilities that a luggage is put on the wrong plane at the different airports are

$$\text{Amsterdam} : 0.05, \; \text{Dubai} : 0.03, \; \text{Singapore} : 0.01.$$

What is the probability that your luggage doesn't reach Sydney with you?

Solution



Example 2

You have a flight from Amsterdam to Sydney with stopovers in Dubai and Singapore. The probabilities that a luggage is put on the wrong plane at the different airports are

$$\text{Amsterdam} : 0.05, \ \text{Dubai} : 0.03, \ \text{Singapore} : 0.01.$$

What is the probability that your luggage doesn't reach Sydney with you?

Solution

A=event that the luggage is put on the correct plane at Amsterdam

D=event that the luggage is put on the correct plane at Dubai.

S= event that the luggage is put on the correct plane at Singapore.

You may assume A, D, S are mutually independent.

The probability that your luggage doesn't reach Sydney is

$$P\left(\overline{A \cap D \cap S}\right) = 1 - P(A \cap D \cap S)$$

Given:

$P(\bar{A}) = 0.05$,

$P(\bar{D}) = 0.03$,

$P(\bar{S}) = 0.01$

$= 1 - P(A)P(D)P(S)$

$= 1 - (1-0.05)(1-0.03)(1-0.01)$

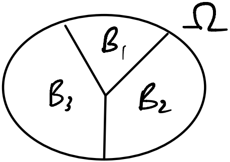
$= 0.087715$

$\approx 8.77\%$

Partition - Definition

- $B_1, \ldots, B_n$ is a **partition** of Ω if
  1. $\cup_{i=1}^{n} B_i = \Omega$ and
  2. $B_1, \ldots, B_n$ are pairwise disjoint

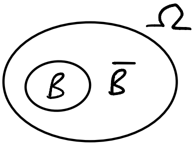
     $B_i \cap B_j = \phi$



B_1, B_2, B_3 is a partition of Ω

Partition - Definition

- $B_1, \ldots, B_n$ is a **partition** of Ω if
  1. $\cup_{i=1}^{n} B_i = \Omega$ and
  2. $B_1, \ldots, B_n$ are pairwise disjoint
- Examples
  1. $\{1\}, \{2, 3\}$ is a partition of $\{1, 2, 3\}$.
  2. $\{1, 2\}, \{2, 3\}$ is not a partition of $\{1, 2, 3\}$.
  3. B and $\bar{B}$ is a partition of Ω.



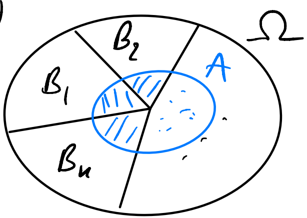
Law of total probability

Theorem 1

Let P be a probability measure on Ω. Assume that $B_1, \ldots, B_n$ is a partition of Ω. Then for any event A, we have

$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i).$$

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots + P(A \cap B_n)$$
$$= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \ldots + P(A|B_n)P(B_n)$$



$$\boxed{P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)}$$

Corollary of Theorem 1

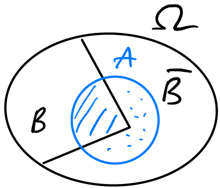
Corollary 1

Let A and B be events in the sample space Ω. Then

$$P(A) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B}).$$

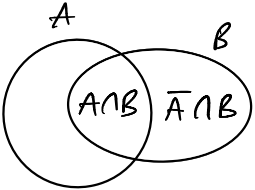
$$P(A) = P(A \cap B) + P(A \cap \bar{B})$$

$$P(A) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B})$$



Complement of conditional events

- A **conditional event** is an event of the form $A|B$, read as "A given B".
- The complement of $A|B$ is $\bar{A}|B$.



Complement of conditional events

- A **conditional event** is an event of the form $A|B$, read as "A given B".
- The complement of $A|B$ is $\bar{A}|B$.
- Question: What is the relation between $P(A|B)$ and $P(\bar{A}|B)$?

$$
\begin{aligned}
P(A|B) + P(\bar{A}|B) &= P(A)? \quad \times \\
P(A|B) + P(\bar{A}|B) &= P(B)? \quad \times \\
P(A|B) + P(\bar{A}|B) &= 1? \quad \checkmark
\end{aligned}
$$

Probability of the complement of a conditional event

Lemma 2

Let A, B be two events with $P(B) > 0$. Then

$$P(A|B) + P(\bar{A}|B) = 1$$

Probability of the complement of a conditional event

Lemma 2

Let A, B be two events with $P(B) > 0$. Then

$$P(A|B) + P(\bar{A}|B) = 1$$

- Since A and $\bar{A}$ for a partition for Ω

$$P(B) = P(B \cap A) + P(B \cap \bar{A}) = P(A \cap B) + P(\bar{A} \cap B),$$

- Hence

$$\begin{aligned} P(B) &= P(A|B)P(B) + P(\bar{A}|B)P(B) \\ 1 &= P(A|B) + P(\bar{A}|B) \end{aligned}$$

Bayes' rule (simplified version)

Theorem 2

Let A, B be events with $P(A) > 0$, $P(B) > 0$. Then

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}$$

Bayes' rule (simplified version)

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- $P(B|A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{P(A \cap B)}{P(A)} = \dfrac{P(A|B)P(B)}{P(A)}$

Bayes' rule (simplified version)

Theorem 2

Let A, B be events with $P(A) > 0$, $P(B) > 0$. Then

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- $P(B|A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{P(A \cap B)}{P(A)} = \dfrac{P(A|B)P(B)}{P(A)}$
- Since B and $\bar{B}$ form a partition of Ω,

$$P(A) = P(A \cap B) + P(A \cap \bar{B}) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B})$$

Bayes' rule (simplified version)

Theorem 2

Let A, B be events with $P(A) > 0$, $P(B) > 0$. Then

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}$$

- $P(B|A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{P(A \cap B)}{P(A)} = \dfrac{P(A|B)P(B)}{P(A)}$
- Since B and $\bar{B}$ form a partition of Ω,

$$P(A) = P(A \cap B) + P(A \cap \bar{B}) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B})$$

- $P(B|A) = \dfrac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}$

Example 3

1 in $100,000$ people has a rare disease for which there is a fairly accurate diagnostic test. This test is correct 99.0% of the time when given to a person selected at random who has the disease, and correct 99.5% of the time when given to a person selected at random who does not have the disease. Find

(a) The probability that a person who tests positive actually has the disease?

given event B — event A

(b) The probability that a person who tests negative does not have the disease?

given event $\bar{B}$ — event $\bar{A}$

(a) $P(A|B)$ (b) $P(\bar{A}|\bar{B})$

Solution

A=event that a randomly selected person has the disease.

B=event that a randomly selected person tests positive.

Need to compute $P(A|B)$ and $P(\bar{A}|\bar{B})$.

$$P(A)=\frac{1}{100000}=10^{-5},\ P(B|A)=0.99,\ P(\bar{B}|\bar{A})=0.995$$

a) $$P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{P(B\cap A)}{P(B\cap A)+P(B\cap\bar{A})}=\frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|\bar{A})P(\bar{A})}$$

$$=\frac{0.99\times 10^{-5}}{0.99\times 10^{-5}+(1-0.995)\times(1-10^{-5})}$$

$$\simeq 0.002=0.2\%$$

Solution

A=event that a randomly selected person has the disease.

B=event that a randomly selected person tests positive.

Need to compute $P(A|B)$ and $P(\bar{A}|\bar{B})$.

$P(A) = \frac{1}{100000} = 10^{-5}$, $P(B|A) = 0.99$, $P(\bar{B}|\bar{A}) = 0.995$

(b) $P(\bar{A}|\bar{B}) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})} = \frac{P(\bar{B} \cap \bar{A})}{P(\bar{B} \cap A) + P(\bar{B} \cap \bar{A})} = \frac{P(\bar{B}|\bar{A}) P(\bar{A})}{P(\bar{B}|A) P(A) + P(\bar{B}|\bar{A}) P(\bar{A})}$

$= \frac{0.995 \times (1 - 10^{-5})}{(1 - 0.99) \times 10^{-5} + 0.995 \times (1 - 10^{-5})} \approx 0.99999\ldots$

$\approx 100\%$

Bayes' rule (general version)

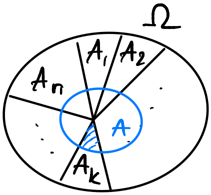
Theorem 3

Let $A_1, A_2, \ldots, A_n$ be a partition of Ω and let A be an event. Assume $P(A) > 0$ and $P(A_i) > 0$ for all i. Then for any $k \in \{1, \ldots, n\}$, we have

$$P(A_k|A) = \frac{P(A|A_k)P(A_k)}{\sum_{i=1}^{n} P(A|A_i)P(A_i)}.$$

Interpretation of Bayes' rule

$$P(A_k|A) = \frac{P(A|A_k)P(A_k)}{\sum_{i=1}^{n} P(A|A_i)P(A_i)}$$



- A_i's are possible causes for the occurrence of A.
- The Bayes' formula computes the probability that A_k caused A, given that A occurred.

Proof of Theorem 2

Writing $P(A_k \cap A)$ in two different ways, we have

$$P(A_k|A)P(A) = P(A|A_k)P(A_k) \Rightarrow P(A_k|A) = \frac{P(A|A_k)P(A_k)}{P(A)}$$

$$P(A_k|A) = \frac{P(A_k \cap A)}{P(A)} = \frac{P(A \cap A_k)}{P(A \cap A_1) + \dots + P(A \cap A_n)}$$

$$= \frac{P(A|A_k)P(A_k)}{P(A|A_1)P(A_1) + \dots + P(A|A_n)P(A_n)} = \frac{P(A|A_k)P(A_k)}{\sum_{i=1}^{n} P(A|A_i)P(A_i)}$$

Summary on Bayes' rules

- Partition $\Omega = B \cup \bar{B}$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}$$

- Partition $\Omega = A_1 \cup A_2 \cup A_3$

$$P(A_1|A) = \frac{P(A|A_1)P(A_1)}{P(A|A_1)P(A_1) + P(A|A_2)P(A_2) + P(A|A_3)P(A_3)}$$

- General partition $\Omega = A_1 \cup \cdots \cup A_n$

$$P(A_1|A) = \frac{P(A|A_1)P(A_1)}{P(A|A_1)P(A_1) + \cdots + P(A|A_n)P(A_n)}$$

Example 4

A factory uses 3 machines M_1, M_2, M_3 to produce certain items.

- M_1 produces 50% of the items, of which 3% are defective.
- M_2 produces 30% of the items, of which 4% are defective.
- M_3 produces 20% of the items, of which 5% are defective.

Suppose that a defective item is found. What is the probability that it came from M_2?

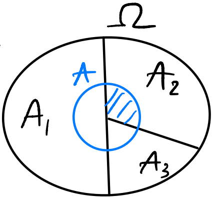
A_1, A_2, A_3 = events that a given item comes from M_1, M_2, M_3.

A = event that a given item is defective.

Need to find $P(A_2 | A)$.

Given: $P(A_1) = 0.5$, $P(A_2) = 0.3$, $P(A_3) = 0.2$

$P(A|A_1) = 0.03$, $P(A|A_2) = 0.04$, $P(A|A_3) = 0.05$



We have

$$P(A_2|A) = \frac{P(A_2 \cap A)}{P(A)} = \frac{P(A \cap A_2)}{P(A \cap A_1) + P(A \cap A_2) + P(A \cap A_3)}$$

$$= \frac{P(A|A_2)P(A_2)}{P(A|A_1)P(A_1) + P(A|A_2)P(A_2) + P(A|A_3)P(A_3)} = \cdots = \frac{12}{37} \cong 32\%$$

Discussion

- A dice is thrown three times

$$\Omega = \{abc : a, b, c \in \{1, \ldots, 6\}\}$$

- Usually we are not interested in the whole Ω (too complex), but only extract information of interests, for examples,

Discussion

- A dice is thrown three times

$$\Omega = \{abc : a, b, c \in \{1, \dots, 6\}\}, \quad |\Omega| = 6^3 = 216$$

- Usually we are not interested in the whole Ω (too complex), but only extract information of interests, for examples,
  - the sum of all numbers that show up, or
  - the number of sixes, or
  - the number of ones
- Each of these quantities is a random variable.

Random variables

- A **random variable** on the sample space Ω is a function

$$X : \Omega \to \mathbb{R},$$

  that is, X assigns a real number to each possible outcome.
- Capital letters $X, Y, Z, \ldots$ denote random variables.
  Small letters $x, y, z, \ldots$ denote **possible values** of X, Y, Z.

Example 5

- $X = \#$ heads in 3 coin tosses.

$$\Omega = \{\text{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}\}.$$

$X : \Omega \to \mathbb{R}$

Example 5

- $X = \#$ heads in 3 coin tosses.

$$\Omega = \{\text{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}\}.$$

- X is a function $X : \Omega \to \mathbb{R}$

$$\begin{aligned} X(\text{HHH}) &= 3, \\ X(\text{HHT}) &= X(\text{HTH}) = X(\text{THH}) = 2, \\ X(\text{HTT}) &= X(\text{THT}) = X(\text{TTH}) = 1, \\ X(\text{TTT}) &= 0 \end{aligned}$$

Example 5

- $X = \#$ heads in 3 coin tosses.

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X(\text{HTT}) &= X(\text{THT}) = X(\text{TTH}) = 1, \\
X(\text{TTT}) &= 0
\end{aligned}$$

- The set of possible values of X is $\{0, 1, 2, 3\}$.

Example 6

$X = \#$ number of heads in 3 consecutive fair-coin tosses.

Find $P(X=3)$, $P(X \leq 1)$ and $P(X \neq 2)$.

$$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

$$P(X=3) = P(\{HHH\}) = \frac{1}{8}$$

$$P(X \leq 1) = P(X=0) + P(X=1)$$

$$= P(\{TTT\}) + P(\{HTT, THT, TTH\}) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$$

$$P(X \neq 2) = 1 - P(X=2) = 1 - P(\{HHT, HTH, THH\})$$

$$= 1 - \frac{3}{8} = \frac{5}{8}$$

Example 7

$p \in [0,1]$. A calibrated coin has chance of landing head is p.
$X =$ # tosses until a head comes up.
Given $n \in \mathbb{Z}^+$. Find $P(X=n)$ and $P(X \leq n)$.

$P(H) = p$
$P(T) = 1-p$

Possible values for X are 1, 2, 3, ...

$P(X=1) = P(H) = p$

$P(X=2) = P(TH) = (1-p)p$

$P(X=3) = P(TTH) = (1-p)^2 p$

$\vdots$

$P(X=n) = P(\underbrace{T\ldots T}_{n-1 \text{ tosses}}H) = (1-p)^{n-1}p$

$P(X \leq n) = P(X=1) + \ldots + P(X=n) = p + (1-p)p + \ldots + (1-p)^{n-1}p$

Discrete random variables

A random variable is **discrete** if it takes on only *countably many values*, that is, the set of possible values of X is countable.

- Countable means there is an order to list out everything. Examples
  1. Any finite subset $S = \{a_1, a_2, \ldots, a_n\}$ of $\mathbb{R}$ is countable.
  2. $\mathbb{N} = \{0, 1, 2, \ldots\}$ is countable.
     $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ is countable.

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  1. Any finite subset $S = \{a_1, a_2, \ldots, a_n\}$ of $\mathbb{R}$ is countable.
  2. $\mathbb{N} = \{0, 1, 2, \ldots\}$ is countable.
     $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ is countable.
  3. $\mathbb{R}$ is not countable.
     $[0, 1]$ is not countable.
  4. We only focus on discrete random variables in this course

Examples of discrete random variables

- $X =$ number of heads in 3 coin tosses
  The set of possible values of X is $\{0, 1, 2, 3\} \Rightarrow$ countable.

Examples of discrete random variables

- $X =$ number of heads in 3 coin tosses
  The set of possible values of X is $\{0, 1, 2, 3\} \Rightarrow$ countable.
- $X =$ number of coin tosses until a head comes up
  The set of possible values of X is $\mathbb{Z}^+ = \{1, 2, 3, \dots\} \Rightarrow$ countable.

Probability mass function (PMF)

The **probability mass function (PMF)** of a discrete random variable X is a function $p : \mathbb{R} \to [0, 1]$ defined by

$$p(x) = P(X = x)$$

Example 8

- $X = \#$ heads in 3 independent fair-coin tosses.
  The set of possible values of X is $\{0, 1, 2, 3\}$ and

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$$p(0) = P(X = 0) = P(\{\text{TTT}\}) = 1/8$$
$$p(1) = P(X = 1) = P(\{\text{HTT,THT,TTH}\}) = 3/8$$
$$p(2) = P(X = 2) = P(\{\text{HHT,HTH,THH}\}) = 3/8$$
$$p(3) = P(X = 3) = P(\{\text{HHH}\}) = 1/8$$

Example 8

- $X = \#$ heads in 3 independent fair-coin tosses.
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$$
\begin{aligned}
p(0) &= P(X = 0) = P(\{\text{TTT}\}) = 1/8 \\
p(1) &= P(X = 1) = P(\{\text{HTT,THT,TTH}\}) = 3/8 \\
p(2) &= P(X = 2) = P(\{\text{HHT,HTH,THH}\}) = 3/8 \\
p(3) &= P(X = 3) = P(\{\text{HHH}\}) = 1/8
\end{aligned}
$$

- Note that

$$p(0) + p(1) + p(2) + p(3) = 1.$$

Properties of PMF

Lemma 3

If $X : \Omega \to R$ is a discrete random variable with PMF $p(x)$. Then

$$\sum_{\text{all x}} p(x) = 1$$

Cumulative distribution function (CDF)

- The **cumulative distribution function (CDF)** of a random variable $X : \Omega \to \mathbb{R}$ is a function $F : \mathbb{R} \to [0, 1]$ defined by

$$F(x) = P(X \leq x).$$

Remark

$$P(X \leq n) = F(n)$$

Cumulative distribution function (CDF)

- The **cumulative distribution function (CDF)** of a random variable $X : \Omega \to \mathbb{R}$ is a function $F : \mathbb{R} \to [0, 1]$ defined by

$$F(x) = P(X \leq x).$$

(increasing)

- F is a *nondecreasing* function, that is,

$$F(a) \leq F(b) \text{ whenever } a \leq b.$$

Example 9

$X = \#$ heads in 3 independent fair-coin tosses.

Find $p(x)$ and $F(x)$ for all possible values x of X.

0, 1, 2, 3

$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$x = 0$: $p(0) = P(X=0) = P(\{TTT\}) = \frac{1}{8}$

$F(0) = P(X \le 0) = P(X=0) = p(0) = \frac{1}{8}$

$x = 1$: $p(1) = P(X=1) = P(\{HTT, THT, TTH\}) = 3/8$

$F(1) = P(X \le 1) = P(X=0) + P(X=1) = p(0) + p(1) = \frac{1}{8} + \frac{3}{8} = \frac{4}{8}$

$x = 2$: ...

$x = 3$: ...

Example 9

$X = \#$ heads in 3 independent fair-coin tosses.
Find $p(x)$ and $F(x)$ for all possible values x of X.

x	$p(x)$	$F(x)$
0	1/8	1/8
1	3/8	4/8
2	3/8	7/8
3	1/8	1

$$F(x) = P(X \leq x) = \sum_{k \leq x} P(X = k) = \sum_{k \leq x} p(k)$$