CSD1241 Tutorial 1 Solutions

Problem 1. Given 3 vectors $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

(a) Find the coordinates of the vectors $\vec{u} + \vec{v}$, $\vec{v} - \vec{u}$, $-3\vec{w}$ and $\vec{u} + \vec{v} - 3\vec{w}$.

(b) Graph in one picture the vectors $\vec{u} + \vec{v}, \vec{v} - \vec{u}, 2\vec{w}$ and $\vec{v} - \vec{u} + 2\vec{w}$.

(c) Compute the dot products $\vec{u} \cdot \vec{w}, (\vec{u} + \vec{v}) \cdot (-3\vec{w})$ and $(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v} - 3\vec{w})$.

(d) Find the angle between the vectors $\vec{u}$ and $\vec{v}$.

(e) Find the angle between the vectors $\vec{v} - \vec{u}$ and $\vec{w} - \vec{u}$.

Solution. (a) $\vec{u} + \vec{v} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$, $\vec{v} - \vec{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $-3\vec{w} = \begin{bmatrix} -6 \\ 9 \end{bmatrix}$, $\vec{u} + \vec{v} - 3\vec{w} = \begin{bmatrix} -2 \\ 16 \end{bmatrix}$.

(c) $\vec{u} \cdot \vec{w} = 0$, $(\vec{u} + \vec{v}) \cdot (-3\vec{w}) = 39$, $(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v} - 3\vec{w}) = -26$.

(d) $\angle(\vec{u}, \vec{v}) = 45^0$.

(e) $\angle(\vec{v} - \vec{u}, \vec{w} - \vec{u}) = 135^0$. □

Problem 2. Consider the points $P = (2, 5), Q = (4, -1), R = (5, 2)$.

(a) Find the midpoints M_{PQ} and M_{PR} of the line segments PQ and PR.

(b) Find the midpoint M of the line segment $M_{PQ}M_{PR}$.

(c) Find real numbers a, b such that

$$\overrightarrow{PM} = a\overrightarrow{PQ} + b\overrightarrow{PR}.$$

Solution. (a) $M_{PQ} = (3, 2), M_{PR} = (3.5, 3.5)$.

(b) $M = (3.25, -2.75)$.

(c) $a = b = \frac{1}{4}$. □

Problem 3. (a) Graph in one picture the points $P = (3, 2), Q = (5, 0)$ and $R = (2, -1)$.

(b) Compute the distances $d(P, Q), d(P, R), d(Q, R)$.

(c) Compute all three angles of the triangle $\triangle PQR$.

(d) Compute the area of $\triangle PQR$.

Hint for d. Area($\triangle PQR$)=$\frac{1}{2}PQ \times PR \times \sin(\angle P)$

Solution. (b) $d(P, Q) = 2\sqrt{2}, d(P, R) = \sqrt{10}, d(Q, R) = \sqrt{10}$.

(c) $\angle P \approx 63.4^0, Q \approx 63.4^0, \angle R \approx 53.2^0$.

(d) Area(PQR) ≈ 4. □

Problem 4. Consider three points $A = (2,3), B = (-2,4)$, $C = (-3,-2)$.
(a) Find all the lengths of the sides of $\triangle ABC$.
(b) Find all three angles of $\triangle ABC$.
(c) Compute the area of $\triangle ABC$.
(d) From C, draw vertically to AB and let H be the intercept of the vertical line with AB. Find the coordinates of H.

Solution. (a) $AB = \sqrt{17}, AC = 5\sqrt{2}, BC = \sqrt{37}$.
(b) $A \approx 59°,\ \angle B \approx 85°,\ \angle C \approx 36°$.
(c) Area($\triangle ABC$)≈ 12.5.
(d) $\overrightarrow{AH} = \mathrm{proj}_{\overrightarrow{AB}}(\overrightarrow{AC}) = \dfrac{\overrightarrow{AC}\cdot\overrightarrow{AB}}{\overrightarrow{AB}\cdot\overrightarrow{AB}}\overrightarrow{AB} = \dfrac{15}{17}\begin{bmatrix}-4\\1\end{bmatrix} = \begin{bmatrix}-60/17\\15/17\end{bmatrix}$.
Since $\overrightarrow{AH} = H - A$, we obtain

$$H = A + \overrightarrow{AH} = (2,3) + \begin{bmatrix}-60/17\\15/17\end{bmatrix} = (-26/17, 66/17)\,.$$

□

Problem 5. Consider three points $A = (1,1,2), B = (0,1,4), C = (2,3,5)$.
(a) Find the projection of $\overrightarrow{AC}$ onto $\overrightarrow{AB}$, that is, $\mathrm{proj}_{\overrightarrow{AB}}(\overrightarrow{AC})$.
(b) Find the orthogonal complement of $\overrightarrow{AC}$ on $\overrightarrow{AB}$, that is,

$$\overrightarrow{AC}^{\perp} = \overrightarrow{AC} - \mathrm{proj}_{\overrightarrow{AB}}(\overrightarrow{AC}).$$

Further, check that $\overrightarrow{AC}^{\perp}$ and $\overrightarrow{AB}$ are orthogonal.
(c) Let D be another point such that $ABCD$ is a parallelogram. Find D.
Hint. $\overrightarrow{AD} = \overrightarrow{BC}$.

Solution. (a) $\mathrm{proj}_{\overrightarrow{AB}}(\overrightarrow{AC}) = \begin{bmatrix}-1\\0\\2\end{bmatrix}$.

(b) $\overrightarrow{AC}^{\perp} = \overrightarrow{AC} - \mathrm{proj}_{\overrightarrow{AB}}(\overrightarrow{AC}) = \begin{bmatrix}2\\2\\1\end{bmatrix}$, $\overrightarrow{AC}^{\perp}\cdot\overrightarrow{AB} = \begin{bmatrix}2\\2\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\0\\2\end{bmatrix} = 0$.

(c) $D = (3,3,3)$. □

Problem 6. (a) Find the condition for the coordinates a, b of $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ such that $\vec{x}$ is orthogonal to $\vec{u} = \begin{bmatrix} -7 \\ 2 \end{bmatrix}$. Could you give 3 examples of such vectors $\vec{y}$? Could you give a geometric interpretation for all vectors which are orthogonal to $\vec{u}$?

(b) Find the condition for the coordinates a, b, c of $\vec{y} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $\vec{y}$ is orthogonal to $\vec{v} = \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}$. Could you give 3 examples of such vectors $\vec{y}$? Do you know the geometric description for all these vectors $\vec{y}$?

Solution. (a) $\vec{x}$ and $\vec{u}$ are orthogonal if and only if $\vec{x} \cdot \vec{u} = 0$, which implies

$$-7a + 2b = 0 \Rightarrow b = \frac{7}{2}a.$$

To give examples for $\vec{x}$, we simply assign values for (a, b) so that $b = \frac{7}{2}a$..

$$a = 0 \Rightarrow \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; a = 1 \Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 7/2 \end{bmatrix}, \; a = 2 \Rightarrow \vec{x} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

Geometric description 1. Any vector $\vec{x}$ perpendicular to $\vec{u}$ has coordinates $\vec{x} = \begin{bmatrix} a \\ 7/2a \end{bmatrix} = a \begin{bmatrix} 1 \\ 7/2 \end{bmatrix}$, which is a scalar multiple of the vector $\begin{bmatrix} 1 \\ 7/2 \end{bmatrix}$. Therefore, any vector perpendicular to $\vec{u}$ is parallel to $\begin{bmatrix} 1 \\ 7/2 \end{bmatrix}$.

Geometric description 2. Any vector $\vec{x}$ perpendicular to $\vec{u}$ has coordinates satisfying the equation of the line $l : y = \frac{7}{2}x$. So $\vec{x}$ can be represented by an arrow starting at the origin O and ending at any point on the line l.

(b) We have

$$\vec{y} \perp \vec{v} \Leftrightarrow \vec{y} \cdot \vec{v} = 0 \Leftrightarrow 2a - 3b + 7c = 0.$$

To give examples of $\vec{y}$, we simply assign any 2 values for b, c and find a by a by $a =$

$\frac{1}{2}(3b - 7c)$.

$$b = c = 1 \Rightarrow a = \frac{1}{2}(3b - 7c) = -2 \quad \Rightarrow \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$b = 2, c = 0 \Rightarrow a = \frac{1}{2}(3b - 7c) = 3 \quad \Rightarrow \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

$$b = 0, c = 2 \Rightarrow a = \frac{1}{2}(3b - 7c) = -7 \quad \Rightarrow \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ 2 \end{bmatrix}$$

Now we describe $\vec{y}$ geometrically. Fix the startpoint of $\vec{y}$ at the origin O. The endpoint of $\vec{y}$ has coordinates a, b, c which satisfies

$$2a - 3b + 7c = 0,$$

that is, the equation of the plane $2x - 3y + 7z = 0$. Therefore, the collection of vectors $\vec{y}$ has startpoint O and endpoints lying on the plane $2x - 3y + 7z = 0$. $\square$