Lecture 6: Revision

Table of contents

Reminders on Midterm Exam

- Time and locations : Thursday 2-4pm, LT4A and LT4B
- Scope: Weeks 1-5
- Exam format:
  1. Part A: MCQs + fill-in-blank questions
  2. Part B written questions
- Things to bring in
  1. One A4-size cheat sheet
  2. One calculator
- Wifi devices, notes, books, etc. are **not allowed**

Parallelism and orthogonality

Consider $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

- $\vec{a} \parallel \vec{b} \Leftrightarrow \dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$ or $\vec{a} = c\vec{b}$
- $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$

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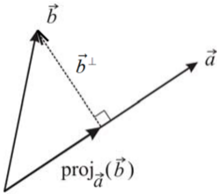
- $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$
- Orthogonal projection

$$\mathrm{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a}$$



Vector operations

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \theta = \angle(\vec{u}, \vec{v}) \in [0°, 180°]$

Vector operations

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- Dot product: $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||||\vec{v}||}$$

Vector operations

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \theta = \angle(\vec{u}, \vec{v}) \in [0^\circ, 180^\circ]$

- Cross product: $[\vec{u}\ \vec{v}] = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix} \Rightarrow \vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$
  1. $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$
  2. $||\vec{u} \times \vec{v}|| =$ area of parallelogram formed by $\vec{u}, \vec{v}$

Lines in $\mathbb{R}^2$

1. The line through $P_0 = (x_0, y_0)$ with direction $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ has vector equation and parametric equation

$$(x, y) = (x_0, y_0) + t \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad \begin{cases} x = x_0 + at \\ y = y_0 + bt \end{cases}$$

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2. The line through $P_0 = (x_0, y_0)$ with normal $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ has equation

$$a(x - x_0) + b(y - y_0) = 0$$

Put $c = ax_0 + by_0$. The general equation and normal equation are

$$ax + by - c = 0 \text{ and } ax + by = c.$$

Lines in $\mathbb{R}^3$

- The line through $P_0 = (x_0, y_0, z_0)$ with direction vector $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is

$$(x, y, z) = (x_0, y_0, z_0) + t \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Leftrightarrow \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

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- The line passing through 2 points P, Q has vector equation

$$(x, y, z) = P + t\overrightarrow{PQ}$$

Planes in $\mathbb{R}^3$

- The plane through $P(x_0, y_0, z_0)$ with normal $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

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- The plane through $P(x_0, y_0, z_0)$ with normal $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has equation

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- The plane through $P(x_0, y_0, z_0)$ with direction vectors $\vec{u}, \vec{v}$ has vector equation and parametric equation

$$(x, y, z) = P + s\vec{u} + t\vec{v} \quad \text{and} \quad \begin{cases} x = x_0 + su_1 + tv_1 \\ y = y_0 + su_2 + tv_2 \\ z = z_0 + su_3 + tv_3 \end{cases}$$

Angles

- $\theta =$ angle between $\vec{u}$ and $\vec{v} \Rightarrow \theta \in [0^0, 180^0]$ and $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{||\vec{u}||||\vec{v}||}$
- The angle a between 2 lines, or between a line and a plane, or between 2 planes is always between 0^0 and 90^0: $0^0 \leq a \leq 90^0$

Angles

- The angle a between 2 lines, or between a line and a plane, or between 2 planes is always between 0^0 and 90^0: $0^0 \leq a \leq 90^0$
    1. Angle between lines l_1 : direction $\vec{d_1}$ and l_2 : direction $\vec{d_2}$

$$a = \min(\theta, 180^0 - \theta) \text{ and } \cos a = \frac{|\vec{d_1} \cdot \vec{d_2}|}{||\vec{d_1}||||\vec{d_2}||}$$

Angles

- The angle a between 2 lines, or between a line and a plane, or between 2 planes is always between 0^0 and 90^0: $0^0 \leq a \leq 90^0$

2. Angle between line l : direction $\vec{d}$ and plane α : normal $\vec{n}$

$$a = |\theta - 90^0| \text{ with } \theta = \text{angle b.w. } \vec{d} \text{ and } \vec{n}$$

Angles

- The angle a between 2 lines, or between a line and a plane, or between 2 planes is always between 0^0 and 90^0: $0^0 \leq a \leq 90^0$

3. Angle between planes α_1 : normal $\vec{n}_1$ and plane α_2 : normal $\vec{n}_2$

$$a = \min(\theta, 180^0 - \theta) \text{ and } \cos a = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{||\vec{n}_1|| ||\vec{n}_2||}$$

Point-line distances

- In $\mathbb{R}^2$:

  Point $P = (x_0, y_0)$ and line $l : ax + by + c = 0$.

$$d(P, l) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

Point-line distances

- In $\mathbb{R}^3$:

  Point P and line $l : (x, y, z) = Q + t\vec{d}$.

$$d(P, l) = \frac{||\overrightarrow{QP} \times \vec{d}||}{||\vec{d}||}$$

Point-plane, plane-plane, line-plane distances

- Point $P_0 = (x_0, y_0, z_0)$ and plane $\alpha : ax + by + cz + d = 0$

$$d(P_0, \alpha) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Point-plane, plane-plane, line-plane distances

- Planes α and β with normal $\vec{n}_\alpha$ and $\vec{n}_\beta$
  1. $\vec{n}_\alpha \nparallel \vec{n}_\beta \Rightarrow \alpha$ and β intersect
  $$d(\alpha, \beta) = 0$$
  2. $\vec{n}_\alpha \parallel \vec{n}_\beta \Rightarrow \alpha \parallel \beta$
  $$d(\alpha, \beta) = d(P, \beta) \text{ for any point } P \text{ on } \alpha$$

Point-plane, plane-plane, line-plane distances

- Line $l : (x, y, z) = P + t\vec{d}$ and plane α with normal vector $\vec{n}$.
  1. $\vec{n} \cdot \vec{d} \neq 0 \Rightarrow l$ and α intersect $\Rightarrow d(l, \alpha) = 0$
  2. $\vec{n} \cdot \vec{d} = 0 \Rightarrow l \parallel \alpha \Rightarrow d(l, \alpha) = d(P, \alpha)$

Line-line distance

Line $l_1 : (x, y, z) = Q_1 + t\vec{d_1}$ and $l_2 : Q_2 + t\vec{d_2}$

- $l_1 \parallel l_2$ $(\vec{d_1} \parallel \vec{d_2})$

$$\begin{aligned} d(l_1, l_2) &= d(Q_1, l_2) = \frac{||\overrightarrow{Q_2Q_1} \times \vec{d_2}||}{||\vec{d_2}||}, \quad \text{or} \\ d(l_1, l_2) &= d(Q_2, l_1) = \frac{||\overrightarrow{Q_1Q_2} \times \vec{d_1}||}{||\vec{d_1}||} \end{aligned}$$

Line-line distance

Line $l_1 : (x, y, z) = Q_1 + t\vec{d_1}$ and $l_2 : Q_2 + t\vec{d_2}$

- l_1 and l_2 are skew or intersecting ($\vec{d_1} \nparallel \vec{d_2}$)

$$d(l_1, l_2) = ||\operatorname{proj}_{\vec{d_1} \times \vec{d_2}}(\overrightarrow{Q_1 Q_2})||$$

Matrix multiplication and determinant

- $A = (a_{ij})_{m \times n}$ means that A has size $m \times n$ in which the (i, j)th entry of A is a_{ij}.
- A is called a squared matrix if and only if $m = n$.
- If $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{n \times p}$, then AB has size $m \times p$.

$$(AB)_{ij} = (i\text{th row of } A) \times (j\text{th column of } B)$$

- For a 2×2 or a 3×3 matrix A

$$\det(A) = \text{min diagonal - anti diagonal}$$

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13}) - (a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{23}a_{32}a_{11})$$

Area of parallelogram and volume of parallelepiped

- Area of the parallelogram spanned by $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is

$$\left| \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \right|$$

- Volume of parallelepiped spanned by $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

$$\left| \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \right|$$

Exercise 1

Consider two vectors $\vec{u} = \begin{bmatrix} 2 \\ -4 \\ c \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ c \\ -1 \end{bmatrix}$. Find c such that

(a) $\vec{u}$ is parallel to $\vec{v}$.

(b) $\vec{u}$ is perpendicular to $\vec{v}$. Further, compute the area of the parallelogram formed by $\vec{u}, \vec{v}$.

Exercise 2

The parallelogram formed by 2 vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$ of equal length is called a **rhombus**. Prove that the diagonals of this rhombus are perpendicular.

Exercise 3

Consider 2 planes $\alpha : x - y + 2z = 1$ and $\beta : 2x - y = 0$.

(a) Find the line l which is the intersection of α and β.

(b) Find the plane γ containing the point $(1, 3, 5)$ and perpendicular to l.

Exercise 3

(c) Find the intersection l_1 of γ and α and the intersection l_2 of γ and β.

(d) Find the angle a between l_1 and l_2. Verify that $a = \angle(\alpha, \beta)$.

Exercise 4

Given 3 points $A = (1, 2, 0), B = (0, 3, 1), C = (-1, 0, 1)$ and the line

$$l : (x, y, z) = (3, 5, -1) + t \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}.$$

(a) Let m be the line containing A, B. Find the intersection, the angle and the distance between l and m.

Exercise 4

(b) Let α be the plane through A, B, C. Find the intersection and the angle between l and α.

Exercise 5

Consider 2 lines $l_1 : \begin{cases} x = -1 + s \\ y = 10 + 2s \\ z = 10 \end{cases}$ and $l_2 : \begin{cases} x = t \\ y = 4 + 2t \\ z = 5 - 3t \end{cases}$.

(a) Find the distance $d(l_1, l_2)$ between l_1 and l_2.

(b) Find the equation of the plane β containing l_2 and parallel to l_1.

(c) Find the point Q on l_2 which at the closet distance to l_1.

(d) Find the point P on l_1 which at the closet distance to l_2.

Exercise 6

(a) Given 3 points A, B, C in $\mathbb{R}^2$. How to determine whether they lie on a line?

Exercise 6

(b) Given 4 points A, B, C, D in $\mathbb{R}^3$. How to determine whether they line on a plane?

HW3 Problem 6-8

- Let A be a **square matrix**.
  A matrix B is called the inverse of A, denoted $B = A^{-1}$, if $AB = I$.
- Our question: Compute A^2, A^4 with $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$.

Which matrix among A^3, A^7, A^{11} could be A^{-1}?