Revision

Dr. Ronald Koh
ronald.koh@digipen.edu (Teams preferred over email)

AY 23/24 Trimester 1

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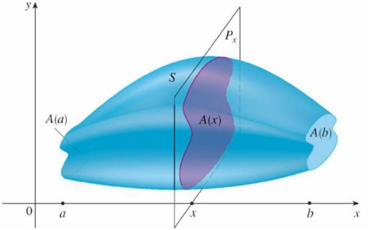
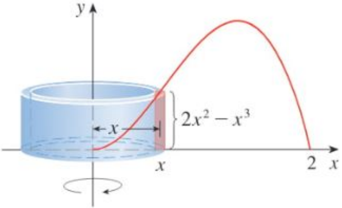
Overview of methods

- Cross-sectional method



- Cylindrical shells method



How to start a volumes of revolution question

1. Sketch the region.
2. **Determine the axis of revolution**.
3. Roughly visualize the 3D solid (don't need to be 100% accurate, just a rough idea is enough).
4. **Determine the method you want to use**; this will determine your **variable of integration**:
   - Cross-sectional method: **same variable** as axis of revolution.
   - Cylindrical shells method: **different variable** from axis of revolution.
5. For cross-sectional method: determine the cross-sectional area $A(x)$ or $A(y)$, depending on the variable in Step 4.
6. For cylindrical shells method: determine the height and radius, depending on the variable in Step 4.

Example 1

Let R be the region enclosed by the curves $x = y^2$ and $x = 4 - y^2$. Find the volume of the solid obtained by rotating R about the y-axis.

Example 1

Example 2

Let R be the region enclosed by the curves $y = \ln x$, $y = 0$ and $x = e$. Find the volume of the solid obtained by rotating R about the y-axis.

Example 2

Recap of sequences

- A **sequence** is a list of numbers written in order:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

- We are interested in finding the **limit** of a sequence

$$\lim_{n\to\infty} a_n.$$

- The limit of a sequence is either a number L or does not exist (oscillating or $\pm\infty$).
- If the limit of a sequence is a number L, $\lim\limits_{n\to\infty} a_n$ is said to **exist** and $\lim\limits_{n\to\infty} a_n = L$, otherwise $\lim\limits_{n\to\infty} a_n$ **does not exist**.

Sequence limit evaluation techniques

- **Limit Laws**: add/subtract/product/quotient (limit of denominator must be non-zero)/power/continuous function.
- **Subsequence Test**: Showing a sequence is **divergent** by finding two subsequences that converge to **two different limits**.
- **Rational/Power functions** in n: dividing by highest power, e.g.

$$\lim_{n\to\infty} \frac{2n^3+3}{3n^3+2n^2+4n} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{2+\frac{3}{n^3}}{3+\frac{2}{n}+\frac{4}{n^2}} = \frac{2}{3}.$$

- **Squeeze Theorem**, e.g.

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \implies \lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

- **L'Hôpital's Rule**: a is a number, or $\pm\infty$.

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \text{ or } \pm\infty \implies \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Example 3

Evaluate the following limits.

1. $$\lim_{n\to\infty} \frac{3n^5 - n^4}{8n^5 + 2n^2 + 3}$$
2. $$\lim_{n\to\infty} (-1)^n \frac{n+1}{n}$$
3. $$\lim_{n\to\infty} n \ln\left(1 + \frac{1}{n}\right)$$

Example 3

Series

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.

- A **series** is an (infinite) sum of all the terms in $\{a_n\}_{n=1}^{\infty}$.
- It is defined as the **limit** of the **sequence of partial sums**:

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} s_N = \lim_{N\to\infty} (a_1 + a_2 + \cdots + a_N).$$

- If this limit exists, we say that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is **convergent**, and

$$S = \sum_{n=1}^{\infty} a_n.$$

Otherwise, $\displaystyle\sum_{n=1}^{\infty} a_n$ is **divergent**.

Convergence/Divergence Tests for Series (1)

- Often, the sum S cannot be found easily (exception: geometric series), thus we only test for convergence/divergence.
- For **geometric series**, starting term a and common ratio r:

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \dfrac{a}{1-r} & \text{if } |r| < 1, \\ \text{divergent} & \text{if } |r| \geq 1. \end{cases}$$

- p-**series** (**IMPORTANT!**):

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{is} \quad \begin{cases} \text{convergent} & \text{if } p > 1, \\ \text{divergent} & \text{if } p \leq 1. \end{cases}$$

- First test to do everytime we test the convergence of a series - **Divergence Test**:

$$\lim_{n\to\infty} a_n \neq 0 \implies \sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

Convergence/Divergence Tests for Series (2)

- **Comparison Test**: If $0 \leq a_n \leq b_n$ for $n \geq n_0$,
  - $\sum b_n$ converges $\implies \sum a_n$ converges.
  - $\sum a_n$ diverges $\implies \sum b_n$ diverges.
- **Limit Comparison Test**: $a_n, b_n > 0$ with $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, c finite:

$$\sum a_n \quad \text{and} \quad \sum b_n \text{ both converge or both diverge.}$$

- **Alternating Series Test**: $b_n > 0$, decreasing, $\lim_{n\to\infty} b_n = 0$:

$$\sum (-1)^n b_n \text{ converges.}$$

- **Absolute Convergence Test**:

$$\sum |a_n| \text{ converges} \implies \sum a_n \text{ converges.}$$

Convergence/Divergence Tests for Series (3)

Let $\sum_{n=1}^{\infty} a_n$ be a series.

- **Ratio Test**: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$:
- **Root Test**: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$:

$$\sum_{n=1}^{\infty} a_n \text{ is } \begin{cases} \text{absolutely convergent} & \text{if } L < 1, \\ \text{divergent} & \text{if } L > 1, \\ \text{inconclusive} & \text{if } L = 1. \end{cases}$$

Tips for testing for convergence

- When unsure, use Divergence Test first.
- For series with rational function terms, use LCT with

$$b_n = \frac{\text{dominating term in numerator}}{\text{dominating term in denominator}}.$$

- For alternating series $\sum(-1)^n b_n$, if $\lim\limits_{n\to\infty} b_n = b \neq 0$, use the Subsequence Test; even and odd subsequences of $a_n = (-1)^n b_n$, and Divergence Test to show $\sum(-1)^n b_n$ is divergent.
- If terms of series has mostly $n!$, heavily consider Ratio Test.
- If terms of series has mostly powers of n, heavily consider Root Test.
- If terms of series has both $n!$ and powers of n, priortize the Ratio Test over the Root Test.

Example 4

Determine the convergence of the following series.

1. $$\sum_{n=1}^{\infty}(-1)^n \frac{n+1}{n}$$
2. $$\sum_{n=1}^{\infty} \frac{3n^2}{n^4+1}$$
3. $$\sum_{n=1}^{\infty} \frac{n!2^n}{n^n}$$

Example 4

Power series

- A **power series** centered at a is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

  where x is a **variable** and c_n are constants called the **coefficients** of this power series.

- The **radius of convergence** of $\sum_{n=0}^{\infty} c_n(x-a)^n$ is a **number** R such that

$$\sum_{n=0}^{\infty} c_n(x-a)^n \text{ converges if } |x-a| < R \text{ and diverges if } |x-a| > R.$$

- R can be found using either the **Ratio Test** or the **Root Test**.

Example 5

Find the radius of convergence for the following power series.

1. $$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n\,9^n}$$
2. $$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Example 5

Taylor and Maclaurin series

- Let f be an **infinitely differentiable** function on an open interval centered at a: $(a-R, a+R)$ for some $R>0$.
- The **Taylor series of f at** a (or about a, or centered at a) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

- The Taylor series centered at $a=0$ is called the **Maclaurin series** of f.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

Final exam details

- Thursday, 30 November 2023, 1400 - 1530 HRS.
- LT6C or LT6D depending on your surname in **alphabetical order**
  - LT6C: Surname/last name AARON SANUSI to LIONG
  - LT6D: Surname/last name LIU to ZHUO
- Material covered: Week 8 to Week 12; this may indirectly include material from the first half of the trimester.
- Format: 6/7 MCQ + 3 Open-ended
- Closed-book, no formula list will be provided.
- Graphic calculators or scientific calculators with definite integral calculation capabilities are **strictly not allowed**.