Week 8: Linear Transformations and 2D Maps

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Vectors and points with the same notation

- From now on, both points and vectors are denoted by *columns*.
  - The point P is identified with vector $\vec{u} = \overrightarrow{OP}$.
  - The vector $\vec{u} = \overrightarrow{OP}$ is identified with the endpoint P.

Vectors and points with the same notation

- For example, $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ can be viewed both as
  1. the point with
     $$\text{x-coordinate} = 0, \text{ y-coordinate} = 1, \text{ z-coordinate} = 2,$$
  2. or a vector *starts at the origin O* and *ends at the point* $P = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

Abbreviation

- In $\mathbb{R}^2$

$$\vec{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

- In $\mathbb{R}^3$

$$\vec{x}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Vector equation of lines in $\mathbb{R}^2$

- Line through $\vec{x}_0$ with direction $\vec{d}$

$$\vec{x} = \vec{x}_0 + t\vec{d}$$

- Line through $\vec{x}_0$ with normal $\vec{n}$

$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$$

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- Special case: Lines through the origin $\mathrm{O} = \tilde{0}$

through the origin with direction $\vec{d}$ through the origin with normal $\vec{n}$

Vector equation of lines and planes in $\mathbb{R}^3$

- Line through $\vec{x}_0$ with direction $\vec{d}$

$$\vec{x} = \vec{x}_0 + t\vec{d}$$

- Planes in $\mathbb{R}^3$

1. Plane through $\vec{x}_0$ with direction vectors $\vec{u}, \vec{v}$

$$\vec{x} = \vec{x}_0 + s\vec{u} + t\vec{v}$$

2. Plane through $\vec{x}_0$ with normal vector $\vec{n}$

$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$$

Linear transformations

A map $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a **linear transformation** if it

1. preserves addition

$$T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y}) \text{ for any } \vec{x}, \vec{y} \in \mathbb{R}^n$$

2. preserves scalar multiplication

$$T(c\vec{x}) = cT(\vec{x}) \text{ for any scalar } c \text{ and } \vec{x} \in \mathbb{R}^n$$

Example 1

(a) Show that the following map $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ 2x + y \\ 3x - 4y \end{bmatrix}$$

Solution. There are 2 things to check

1. T preserves addition

$$T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)?$$

Example 1

2. T preserves scalar multiplication

$$T\left(c\begin{bmatrix}x\\y\end{bmatrix}\right) = cT\left(\begin{bmatrix}x\\y\end{bmatrix}\right)?$$

(b) Verify $T(\vec{x}) = \begin{bmatrix}1 & -1\\2 & 1\\3 & -4\end{bmatrix}\vec{x}$ for any $\vec{x} \in \mathbb{R}^2$

Example 2

Prove that the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as follows is **not linear**

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ y \end{bmatrix}$$

Comment

- Soon, we will learn that $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear $\Leftrightarrow$ each component in $T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is a **linear combination** of $x_1, \ldots, x_n$

$$T\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

Comment

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- Put $M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \Rightarrow T\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = M\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Matrix multiplication $\Rightarrow$ linear map

Theorem 1

Let M be an $m \times n$ matrix. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$T(\vec{x}) = M\vec{x},$$

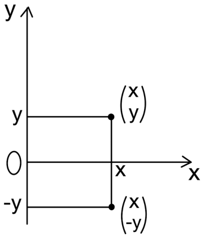
then T is a linear transformation.

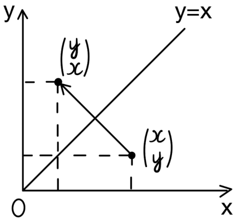
Proof. We need to verify that

1. T preserves addition and
2. T preserves scalar multiplication

Example 3: Reflections in $\mathbb{R}^2$

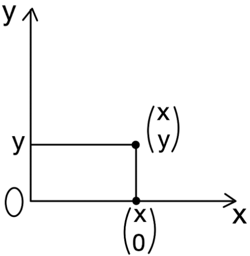
The reflections about the x-axis and about the line $y = x$ are both linear.

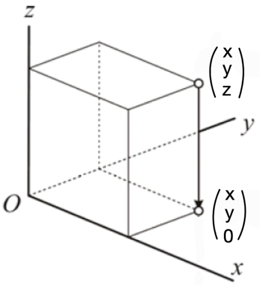




Example 4: Orthogonal projections

The orthogonal projections onto the x-axis in $\mathbb{R}^2$ and the orthogonal projection onto xy-plane in $\mathbb{R}^3$ are both linear

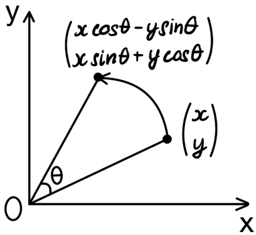




Example 5: Rotation in $\mathbb{R}^2$

The counter-clockwise rotation by angle θ is linear

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$



Linear transformation $\Leftrightarrow$ matrix multiplication

Theorem 2

The map $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if there exists an $m \times n$ matrix M such that

$$T(\vec{x}) = M\vec{x} \text{ for all } \vec{x} \in \mathbb{R}^n$$

The matrix M is called the **matrix representation** of T.

Comments

There are 2 parts in the statement of Theorem 2.

1. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $T(\vec{x}) = M\vec{x}$, then T is linear.
2. If T is linear, there is a matrix $M \in M_{m\times n}(\mathbb{R})$ such that

$$T(\vec{x}) = M\vec{x}$$

Matrix of linear transformation

Lemma 1

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix},$$

then $T\vec{x} = M\vec{x}$ with

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Standard unit vectors

In $\mathbb{R}^n$, there are n standard unit vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \vec{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

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- In $\mathbb{R}^2$

$$\vec{e}_1 = \vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- In $\mathbb{R}^3$

$$\vec{e}_1 = \vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{e}_3 = \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Matrix of linear transformation

Lemma 2

Assume $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Let M be the $m \times n$ matrix with columns $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$

$$M = [T(\vec{e}_1) \; T(\vec{e}_2) \; \cdots \; T(\vec{e}_n)]$$

Then

$$T(\vec{x}) = M\vec{x} \text{ for all } \vec{x} \in \mathbb{R}^n.$$

Comment on Lemma 2

There are 2 steps in finding the matrix of linear $T : \mathbb{R}^n \to \mathbb{R}^m$

1. Let $\vec{e}_1, \ldots, \vec{e}_n$ be standard unit vectors of $\mathbb{R}^n$. Compute

$$T(\vec{e}_1), \ldots, T(\vec{e}_n)$$

2. Form the matrix M having $T(\vec{e}_1), \ldots, T(\vec{e}_n)$ as columns

$$M = [T(\vec{e}_1) \ \cdots \ T(\vec{e}_n)].$$

Example 6

(a) Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix},\ T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$$

(b) Find $T\begin{pmatrix}1\\2\end{pmatrix}$ and $T\begin{pmatrix}3\\5\end{pmatrix}$?

Example 7

(a) Find the matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ with

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \; T \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

Example 7

(b) Find $T\begin{pmatrix} 3 \\ 5 \\ -7 \end{pmatrix}$?

Summary on matrix of linear transformation

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear $\Leftrightarrow$ there exists an $m \times n$ matrix M:

$$T(\vec{x}) = M\vec{x}$$

M is called the **matrix representation** of T.

Summary on matrix of linear transformation

- There are 2 ways to determine M

  1. $T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix} \Rightarrow M = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

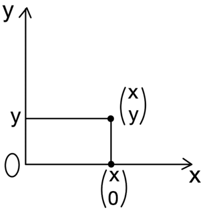
Summary on matrix of linear transformation

- There are 2 ways to determine M

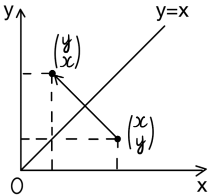
  1. $T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix} \Rightarrow M = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$
  2. If $\vec{e}_1, \ldots, \vec{e}_n$ are *standard unit vectors* of $\mathbb{R}^n$, then

$$M = [T(\vec{e}_1)\ T(\vec{e}_2)\ \cdots\ T(\vec{e}_n)]$$

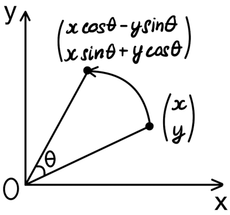
Summary on linear transformations in 2D



$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$



$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$



$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

A useful identity

Lemma 3

If $\vec{a}, \vec{x}, \vec{b}$ are in $\mathbb{R}^n$, then

$$(\vec{a} \cdot \vec{x})\vec{b} = \vec{b}\vec{a}^T\vec{x}$$

In particular we have

$$(\vec{a} \cdot \vec{x})\vec{b} = M\vec{x} \quad \text{with} \quad M = \vec{b}\vec{a}^T$$

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In particular we have

$$(\vec{a} \cdot \vec{x})\vec{b} = M\vec{x} \quad \text{with} \quad M = \vec{b}\vec{a}^T$$

- $\vec{a} \cdot \vec{x}$ is a number $\Rightarrow (\vec{a} \cdot \vec{x})\vec{b}$ is a scalar multiple of $\vec{b}$
- $M\vec{x}$ is multiplication of the matrix $M = \vec{b}\vec{a}^T$ by the vector $\vec{x}$

Example

Given $\vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Express the orthogonal projection $\text{proj}_{\vec{b}}(\vec{x})$ as a matrix multiplication, that is, find the matrix M such that

$$\text{proj}_{\vec{b}}(\vec{x}) = M\vec{x}$$

2D maps

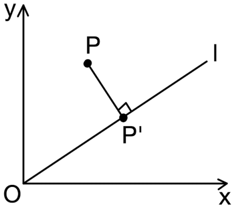
- We will discuss the following 2D maps
  1. Projection
  2. Reflection
  3. Scaling
  4. Rotation
  5. Shear
- All these are linear transformations.

  We aim to find the **matrix representations** of these maps.

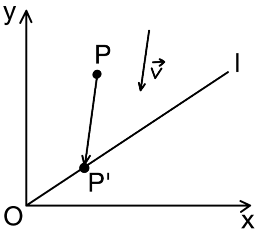
Projections in $\mathbb{R}^2$

Let l be a line through the origin. There are 2 types of projections onto l

1. Orthogonal projection



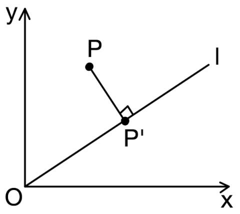
2. Skew projection along $\vec{v}$



Orthogonal projection

Theorem 3

Let l be a line in $\mathbb{R}^2$ which *passes through the origin*.



If l has direction $\vec{d}$, the orthogonal projection onto l has matrix

$$M = \frac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T$$

What Theorem 3 says?

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal projection onto l. The matrix of T is

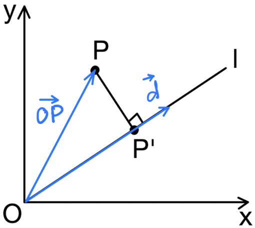
$$M = \frac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T$$

- Any point $P = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is projected to the point P' with coordinates

$$P' = T(P) = M \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Proof

- Assume $P = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \vec{x}_0$. We find coordinates of its projection P'.



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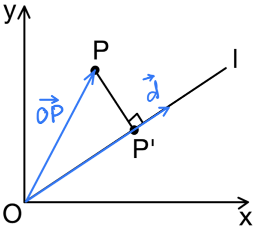


- P' has the same coordinates as $\overrightarrow{OP'}$, which is

$$\mathrm{proj}_{\vec{d}}(\overrightarrow{OP}) = \frac{\vec{d} \cdot \overrightarrow{OP}}{\vec{d} \cdot \vec{d}} \vec{d} = \frac{1}{||\vec{d}||^2} (\vec{d} \cdot \vec{x}_0) \vec{d} = \frac{1}{||\vec{d}||^2} \vec{d} \vec{d}^T \vec{x}_0$$

Exercise 1

Assume $\vec{d} = \begin{bmatrix} A \\ B \end{bmatrix}$. Find the projection matrix in A, B.

Example 8

Find the orthogonal projection P' of the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ onto the line l.

(a) $l : \vec{x} = t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

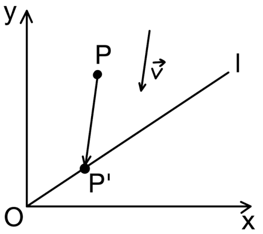
Example 8

(b) $l : 2x - 3y = 0.$

Skew projection

Theorem 4

Let $\vec{n}, \vec{v}$ be nonzero vectors. Let l be a line through O having normal $\vec{n}$.

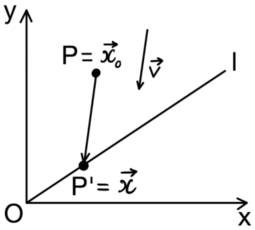


The projection onto l along the direction $\vec{v}$ has matrix representation

$$M = I_2 - \frac{\vec{v}\vec{n}^T}{\vec{v} \cdot \vec{n}}$$

Proof

- The line l has vector equation $\vec{n} \cdot \vec{x} = 0$
- Let $P = \vec{x}_0$ and $P' = \vec{x}$ be skew projection along $\vec{v}$ of P onto l.



Proof

- The line l has vector equation $\vec{n} \cdot \vec{x} = 0$
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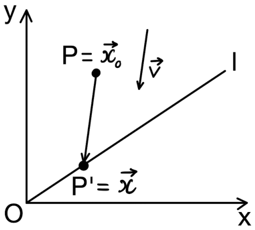
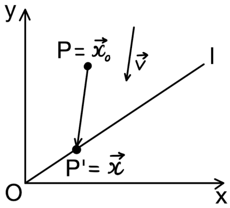


- Since $\overrightarrow{PP'} \parallel \vec{v}$, we have $\overrightarrow{PP'} = t\vec{v}$

$$\vec{x} - \vec{x}_0 = t\vec{v} \Rightarrow \vec{x} = \vec{x}_0 + t\vec{v}$$

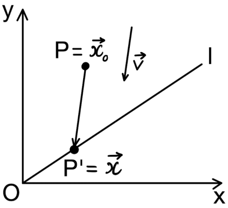
Proof



- P' is on $l \Rightarrow \vec{n} \cdot \vec{x} = 0$

$$\vec{n} \cdot (\vec{x}_0 + t\vec{v}) = 0 \Rightarrow t = -\frac{\vec{n} \cdot \vec{x}_0}{\vec{n} \cdot \vec{v}}$$

Proof



- P' is on $l \Rightarrow \vec{n} \cdot \vec{x} = 0$

$$\vec{n} \cdot (\vec{x}_0 + t\vec{v}) = 0 \Rightarrow t = -\frac{\vec{n} \cdot \vec{x}_0}{\vec{n} \cdot \vec{v}}$$

- We obtain

$$\begin{aligned}
\vec{x} &= \vec{x}_0 - \frac{\vec{n} \cdot \vec{x}_0}{\vec{n} \cdot \vec{v}}\vec{v} = \vec{x}_0 - \frac{1}{\vec{n} \cdot \vec{v}}(\vec{n} \cdot \vec{x}_0)\vec{v} \\
&= \vec{x}_0 - \frac{1}{\vec{n} \cdot \vec{v}}\vec{v}\vec{n}^T\vec{x}_0 = \left(I_2 - \frac{\vec{v}\vec{n}^T}{\vec{n} \cdot \vec{v}}\right)\vec{x}_0
\end{aligned}$$

Exercise 2

Assume $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} A \\ B \end{bmatrix}$. Write out M in Theorem 3 in a, b, A, B.

Example 9

(a) Find the images of the points $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$ with

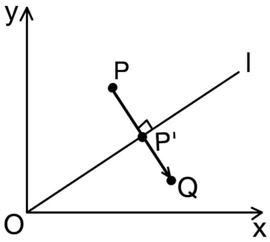
$$l : x - 2y = 0, \ \vec{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

Example 9

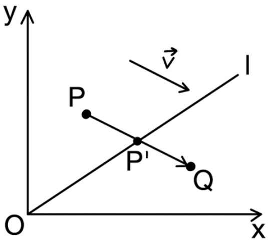
(b) Show that any point on the line $l' : x + 3y = 10$ is projected to a fixed point on the line l. Can you explain this?

Reflections in $\mathbb{R}^2$

Let l be a line through the origin. We discuss 2 types of reflection through l



Orthogonal reflection

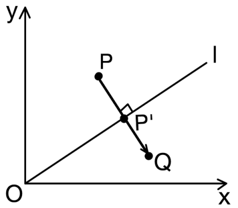


Skew reflection

Orthogonal reflection

Theorem 5

Let l be a line though O with direction $\vec{d}$.

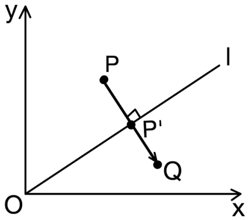


Then the orthogonal reflection through l has matrix representation

$$M = \frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_2.$$

Proof

- Assume $P = \vec{x}_0$. We find its reflection Q.



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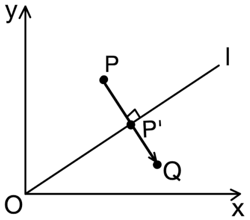


- P' is the midpoint of $PQ \Rightarrow P' = \frac{1}{2}(P+Q)$

$$Q = 2P' - P = \frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T\vec{x}_0 - \vec{x}_0 = \left(\frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_2\right)\vec{x}_0$$

Remark

- The result works for lines in both $\mathbb{R}^2$ and $\mathbb{R}^3$.
- The line l needs to go through the origin, that is, $\vec{x} = t\vec{d}$
- The orthogonal reflection through l has matrix

$$M = \frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I$$

Exercise 3

Assume $\vec{d} = \begin{bmatrix} A \\ B \end{bmatrix}$. Find the reflection matrix in A, B.

Example 10

(a) Let $l : \vec{x} = t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ be a line. Find the matrix of reflection through l.

(b) Find the image of the point $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

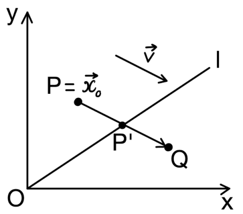
Example 10

(c) Find the image of the line $m : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Skew reflection

Theorem 6

Let l be a line through O with normal vector $\vec{n}$. Let $\vec{v}$ be a nonzero vector.

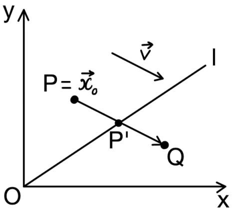


The skew reflection through l in the direction $\vec{v}$ has matrix

$$M = I_2 - \frac{2}{\vec{v} \cdot \vec{n}} \vec{v}\vec{n}^T.$$

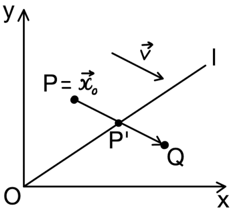
Proof (Sketch)

- $P = \vec{x}_0$ a point, $P' =$ skew projection along $\vec{v}$ of P onto l, $Q =$ skew reflection of P.



Proof (Sketch)

- $P = \vec{x}_0$ a point, $P' =$ skew projection along $\vec{v}$ of P onto l, $Q =$ skew reflection of P.

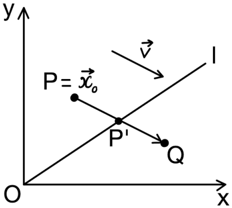


- We knew how to compute P' from the previous result

$$P' = \left(I_2 - \frac{1}{\vec{v} \cdot \vec{n}} \vec{v}^T \vec{n} \right) \vec{x}_0$$

Proof (Sketch)

- $P = \vec{x}_0$ a point, P' = skew projection along $\vec{v}$ of P onto l, Q = skew reflection of P.



- We knew how to compute P' from the previous result

$$P' = \left(I_2 - \frac{1}{\vec{v} \cdot \vec{n}} \vec{v}^T \vec{n} \right) \vec{x}_0$$

- Since P' is the midpoint of PQ, we have $P' = \frac{1}{2}(P + Q)$

$$Q = 2P' - P = \left(I_2 - \frac{2}{\vec{v} \cdot \vec{n}} \vec{v} \vec{n}^T \right) \vec{x}_0$$

Exercise 4

Assume $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} A \\ B \end{bmatrix}$. Find the matrix of the skew reflection through l in the direction $\vec{v}$ in a, b, A, B.

Example 11

Consider $l : x - 2y = 0$ and $\vec{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

(a) Find the matrix of the skew reflection through l in the direction $\vec{v}$.

(b) Find the images of the points $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$.

(c) What is the image of the line $x - 2y = 1$?

(d) Show that the image of the line $m : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ is a line.