Series Convergence Tests Part 2
Power Series Fundamentals

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Limit Comparison Test

Theorem (Limit Comparison Test, or the LCT)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with positive terms. If c is a number such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0,$$

then either both series converge, or both series diverge.

Note: The limit comparison test is NOT the comparison test.

Standard steps in the use of LCT

We usually start with a series $\sum_{n=1}^{\infty} a_n$, and we are asked to figure out if this series is convergent or divergent.

1. Find a series $\sum_{n=1}^{\infty} b_n$ which is *similar* to $\sum_{n=1}^{\infty} a_n$, and whose convergence/divergence is **known**.
2. Compute $\lim_{n\to\infty} \frac{a_n}{b_n}$.
3. If this limit exists, and is positive, we can apply the LCT.

Example 1

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2 + 2n}$.

Exercise 1

Use the LCT to determine the convergence of the following series.

1. $$\sum_{n=3}^{\infty} \frac{n}{n^3 - 8}$$
2. $$\sum_{n=1}^{\infty} \frac{1}{2^n + 2}$$

Exercise 1

Alternating Series

An **alternating series** is a series $\sum_{n=1}^{\infty} a_n$ where

$$a_n = (-1)^n b_n \quad \text{or} \quad a_n = (-1)^{n-1} b_n$$

such that $\{b_n\}_{n=1}^{\infty}$ is a **positive** sequence, i.e. $b_n > 0$ for all n.

Examples of such series include

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

$$\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+1} = -\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots$$

Alternating Series Test

The convergence of an alternating series will depend on the characteristics of the positive sequence $\{b_n\}$.

Theorem (Alternating Series Test)

If an alternating series

$$\sum_{n=1}^{\infty}(-1)^n b_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n-1} b_n$$

satisfies the conditions

$$\text{(a) } b_{n+1} \leq b_n \quad \text{for } n \geq n_0$$
$$\text{(b) } \lim_{n\to\infty} b_n = 0,$$

then the series is convergent.

Example 2

Is the **alternating harmonic series** $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ convergent?

Exercise 2

Establish the convergence/divergence of the following alternating series.

1. $$\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{n^3+1}$$
2. $$\sum_{n=1}^{\infty}(-4)^n$$

Exercise 2

Non-alternating series vs alternating series

We have seen two of these series, one of them is convergent and the other is not:

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

While on the other hand, both of these series are convergent:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$$

Questions to ask:

1. For each pair of series, what is the relation between their terms?
2. How do we reconcile the similarities/differences between their convergence?

Absolute Convergence

- A series $\sum_{n=1}^{\infty} a_n$ is said to be **absolutely convergent** if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ converges.
- A series $\sum_{n=1}^{\infty} a_n$ is said to be **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges, but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ diverges.
- Can you think of examples of absolutely convergent and conditionally convergent series?

Absolute Convergence Test

Slide 13 gives us an example of a series that is conditionally convergent. Conversely, the following test tells us that absolutely convergent series must be convergent.

Theorem (Absolute Convergence Test)

If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is also convergent.

Example 3

Determine whether the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is convergent or divergent.

Example 3

Exercise 3

Determine if the following series is absolutely convergent, conditionally convergent, or divergent.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Ratio Test

Theorem (Ratio Test)

Let $\sum_{n=1}^{\infty} a_n$ be a series, and let

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

- If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- If $L > 1$ or $L = \infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $L = 1$, then the Ratio Test is inconclusive; no conclusion can be drawn about the convergence or divergence of this series.

Root Test

Theorem

Let $\sum_{n=1}^{\infty} a_n$ be a series, and let

$$\lim_{n \to \infty} \sqrt[n]{|a_n|} = L.$$

- If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- If $L > 1$ or $L = \infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $L = 1$, then the Root Test is inconclusive; no conclusion can be drawn about the convergence or divergence of this series.

Example 4

Determine the convergence of the following series.

1. $$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$
2. $$\sum_{n=1}^{\infty} \left(\frac{3n+2}{4n+5}\right)^n$$

Example 4

Power Series

A **power series** centered at a is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where x is a **variable** and c_n are constants called the **coefficients** of this power series.

Example: If $a = 0$ and $c_n = 1$ for all $n \geq 0$, we have

$$\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} x^n = \begin{cases} \dfrac{1}{1-x} & \text{if } |x| < 1 \\ \text{divergent} & \text{if } |x| \geq 1. \end{cases}$$

This is a geometric series that we saw in last lecture. Thus, a geometric series is also a power series.

Power series is also a function of x

- A power series can be seen as a function of x.
- By substituting different values of x, we get different series. Hence, a power series may converge for some values of x and diverge for other values of x.
- The power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ **always** converges at $x = a$, that is

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 \text{ if } x = a.$$

- For x values for which the power series is **convergent**, we can let

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n.$$

Convergence of Power Series

- If $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges at $x = b$ where $b \neq a$, then it converges whenever $|x-a| < |b-a|$.

- If $\sum_{n=0}^{\infty} c_n(x-a)^n$ diverges at $x = d$ where $d \neq a$, then it diverges whenever $|x-a| > |d-a|$.

Radius of convergence

The **radius of convergence** of $\sum_{n=0}^{\infty} c_n(x-a)^n$ is **a number** R such that

$$\sum_{n=0}^{\infty} c_n(x-a)^n \text{ converges if } |x-a| < R \text{ and diverges if } |x-a| > R.$$

There are **three** cases for R:

- R is a positive number. In this case, $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $|x-a| < R$ and diverges for $|x-a| > R$.
- $R = 0$. In this case, $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges only at $x = a$.
- $R = \infty$. In this case, $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for all $x \in \mathbb{R}$.

Example 5: Calculation of radius of convergence

Find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n+1}}$.

Example 5

Exercise 4

Find the radius of convergence for the following power series.

1. $$\sum_{n=1}^{\infty} \frac{n(3x+2)^n}{4^n}$$
2. $$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Exercise 4