CSD1241 Tutorial 11

Question 1. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the shear with respect to the line $l : 3x - 4y = 0$ which maps the point $P = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ to $P' = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$.

(a) What is the matrix of S? (*Hint.* Use $S(\vec{x}) = \vec{x} + \dfrac{\vec{n} \cdot \vec{x}}{||\vec{n}||}\vec{v}$ to find $\vec{v}$)

(b) Find the normal equation for the image m' of $m : 2x - 3y = 6$ under S.

(c) Let Q be the intersection of m' and l. Find the image Q' of Q under S.

Question 2. Consider the point $P = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and the line $l : \begin{cases} x = 2 + t \\ y = 3 - t \\ z = 1 + 2t \end{cases}$

(a) Find the point P' on l which is at the closest distance to P.

(b) Let α be the plane through P and perpendicular to l. Find the point Q' on α that is at the closest distance to $Q = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

Question 3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the shear with respect to the plane $\alpha : z = 3$ in the direction of shearing vector $\vec{v} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$.

(a) Write T in the form of an affine map

$$T(\vec{x}) = A\vec{x} + \vec{b}$$

(b) What is the image of the line $l : \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ under T?

(c) What is the image of the plane $\beta : x - z = 0$ under T?

Question 4. In this problem, we show that the composition of two reflections in 3D is a rotation.

Let $S \circ T$ be the **composition** of the reflection S through the plane $\alpha : 2x - y + 2z = 0$

and the reflection T through the plane $\beta : x - y = 0$.
(a) Find the matrix M of $S \circ T$ (*Hint*: $M = M_S M_T$).
(b) Find the fixed points of $S \circ T$.
(c) In b, your answer is a line l. Let $\vec{v}$ be the direction of l. Find the angle θ so that M is a rotation matrix, that is,

$$M = (1 - \cos\theta)\frac{\vec{v}\vec{v}^T}{||\vec{v}||^2} + (\cos\theta)I_3 + \frac{\sin\theta}{||\vec{v}||}C_{\vec{v}}$$