Lecture 4: Logics (continued)

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Arguments

- An **argument** is a sequence of compound propositions.
  - All but the final proposition are called **premises**.
  - The final proposition is called **conclusion**.

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  - All but the final proposition are called **premises**.
  - The final proposition is called **conclusion**.
- An argument is **valid** if

  whenever the premises are all true, the conclusion is true.

Example 1

The following is a valid argument

"If you have access to the network, you can change your grade."

"You have access to the network."

$\therefore$ "You can change your grade."

Argument forms

- The **argument form** of an argument is obtained by expressing it by propositional variables $(p, q, r, \dots)$ and logical connectives $(\neg, \wedge, \vee, \dots)$.

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- The notation

$$\therefore = \text{"therefore"}$$

  is used to write the conclusion of an argument (or an argument form).

Example of argument form

Consider the argument

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$\therefore$ " You can change your grade."

- p : You have access to the network.
  q : You can change your grade.
- The argument form is

$$\begin{array}{c} p \to q \\ p \\ \therefore q \end{array}$$

Argument forms

- Now we verify that the argument is valid. Valid argument means

"if all premises are true, the conclusion is true"

$$p \to q,$$
$$p,$$
$$\therefore q.$$

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
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p	q	$p \to q$
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- The last row shows that

  if both premises are true, the conclusion is true.

Test validity of an argument form

We call a row in the truth table of an argument form **critical row** if all premises are true

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1. Identify premises and conclusion.
2. Construct the truth table showing all possible values of the premises and the conclusion.
3. Check the conclusion in each **critical row**
   - If the conclusion is false in any of these rows, the argument form is invalid.
   - If the conclusion is true at all these rows, the argument is valid.

Example 2

Show that the following argument form is invalid.

$$
\begin{aligned}
p &\rightarrow q \vee \neg r \\
q &\rightarrow p \wedge r \\
\therefore p &\rightarrow r.
\end{aligned}
$$

Exercise 1

Show that the following argument form is valid.

$$p \lor (q \lor r)$$
$$\neg r$$
$$\therefore p \lor q.$$

Rules of inference

- A **rule of inference** is a valid argument form
- For example, in the last lecture, we learnt that the argument form

$$p \to q$$
$$p$$
$$\therefore q.$$

is a rule of inference

Modus ponens and modus tollens

- **Modus ponens** is the following valid argument form

$$p \to q$$
$$p$$
$$\therefore q$$

- **Modus tollens** is the following valid argument form

$$p \to q$$
$$\neg q$$
$$\therefore \neg p.$$

Example 3: Validity of modus ponens and modus tollens

Prove that modus tollens is a valid argument form.

$$p \to q$$
$$\neg q$$
$$\therefore \neg p$$

Example 4

Write conclusions for the following arguments

(a) If my CGPA at the end of 4 years is below 2.00, then I cannot graduate in 4 years.

My CGPA is 1.49 at the end of 4 years.

Conclusion:

(b) If my CGPA at the end of 4 years is below 3.00, then I cannot graduate with honours in 4 years.

I graduated with honours in 4 years.

Conclusion:

(c) If my CGPA at the end of 4 years is below 2.00, then I cannot graduate in 4 years.

My CGPA at the end of four years is 3.11.

Conclusion:

Basic rules of inference

Name	Argument form	Example
Generalization	p $\therefore p \vee q$	$x = 3$ $\therefore x = 3$ or $x = -3$
Specialization	$p \wedge q$ $\therefore p$	$y > 0$ and y is an integer $\therefore y > 0$
Elimination	$p \vee q$ $\neg q$ $\therefore p$	$x - 3 = 0$ or $x + 2 = 0$ $x \neq -2$ $\therefore x - 3 = 0$

Basic rules of inference

Transitivity	$p \to q$ $q \to r$ $\therefore p \to r$	If $x > a$, then $x > b$ If $x > b$, then $x > c$ $\therefore$ if $x > a$, then $x > c$
Division into cases	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$	x is positive or x is negative If x is positive, then $x^2 > 0$ If x is negative, then $x^2 > 0$ $\therefore x^2 > 0$
Contradiction	$\neg p \to$ **F** $\therefore p$	If everyone sleeps before 12am, then there is no Covid-19 $\therefore$ Not everyone sleeps before 12am

Example 5

Show that the premises

- It is not sunny this afternoon and it is colder than yesterday,
- We will go swimming only if it is sunny,
- If we do not go swimming, then we will take a canoe trip,
- If we take a canoe trip, then we will be home by sunset,

lead to the conclusion

- We will be home by sunset

Example 5 solution

Exercise 2

Show that the premises

- If you send me an e-mail message, then I will finish writing the program,
- If you do not send me an e-mail message, then I will go to sleep early,
- If I go to sleep early, then I will wake up feeling refreshed

lead to the conclusion

- If I do not finish writing the program, then I will wake up feeling refreshed.

Exercise 2 solution

Predicates and domains

- A **predicate** is a sentence that
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- A **predicate** is a sentence that
  - contains a **finite number of variables** and
  - becomes a proposition when specific values are substituted for the variables.
- The **domain** of a predicate is the set of all values that can be substituted in the variables.

Examples

- "$P(n)$: n is a prime, domain: natural numbers" is a predicate defined on the domain of natural numbers.

$$P(1) \text{ is false, } P(2) \text{ is true.}$$

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$$P(1) \text{ is false, } P(2) \text{ is true.}$$

- "$P(x) : x^2 > x$, domain: $\mathbb{R}$" is a predicate defined on the domain of real numbers.

$$P(2) \text{ is true, } P(0.5) \text{ is false.}$$

Truth sets

Let $P(x)$ be a predicate on domain D.

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- The **truth set** of $P(x)$ is the set of all $x \in D$ which makes $P(x)$ true.
- Example: The predicate "$P(x) : x > 0$ on the domain of integers $\mathbb{Z}$" has truth set $\mathbb{Z}^+$.

Example 6

Consider predicates

$P(x) : |x| < 4$ and $Q(x) : x^2 = 8$ both defined on the domain of integers.

Find truth sets of $P(x)$ and $Q(x)$.

Universal quantifier

Let $P(x)$ be a predicate on domain D.

- The **universal quantifier** $\forall$ is the notation for **for all**.
- A **universal statement** is a statement of the form

$$\forall\ x \in D\ P(x), \text{ or } \forall\ x\ P(x) \tag{1}$$

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- (1) is true if $P(x)$ is true for all $x \in D$.
- (1) is false if there exists $x \in D$ for which $P(x)$ is false.
  The value x for which $P(x)$ is false is called a **counterexample**.

Example 7

(a) Let $D = \{2, 3, 4\}$. Show that the following statement is true

$$\forall\ x \in D, x > \frac{1}{x}.$$

(b) Is the statement "$\forall\ x \in \mathbb{Z} - \{0\}, x \geq \frac{1}{x}$" true?

Existential quantifier

Let $P(x)$ be a predicate on domain D.

- The **existential quantifier** $\exists$ is the notation for **there exists**.
- An **existential statement** is a statement of the form

$$\exists\ \mathbf{x} \in \mathbf{D}\ \mathbf{P}(\mathbf{x}), \text{ or } \exists\ \mathbf{x}\ \mathbf{P}(\mathbf{x}). \tag{2}$$

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- (2) is true if $P(x)$ is true for at least one $x \in D$.
- (2) is false if $P(x)$ is false for all $x \in D$.

Example 8

Which of the following statements are true? Justify your answer.

(a) $\exists\ x \in \mathbb{R}$ such that $x^4 < x^2$.

(b) Let $D = \{3, 4, 5\}$. Then

$$\exists\ x \in D \text{ such that } x^4 < x^2.$$

Universal quantifier vs existential quantifier

Predicate $P(x)$, domain D.

- $\forall$ = for all, $\exists$ = there exists.
- Universal statement:

$$\forall\ x \in D\ P(x), \text{ or } \forall\ x\ P(x)$$

Universal quantifier vs existential quantifier

Predicate $P(x)$, domain D.

- $\forall$ = for all, $\exists$ = there exists.
- Universal statement:

$$\forall\ x \in D\ P(x),\ \text{ or } \forall\ x\ P(x)$$

- Existential statement:

$$\exists\ \mathbf{x} \in \mathbf{D}\ \mathbf{P}(\mathbf{x}),\ \text{ or } \exists\ \mathbf{x}\ \mathbf{P}(\mathbf{x}).$$

Precedence of quantifiers

- $\forall$ and $\exists$ have higher precedence than $\neg, \wedge, \vee, \oplus$.
- For example,

$$
\begin{aligned}
\forall\ x\ P(x) \vee Q(x) &= (\forall x P(x)) \vee Q(x) \\
\exists\ x\ P(x) \wedge Q(x) &= (\exists x P(x)) \wedge Q(x).
\end{aligned}
$$

Precedence of quantifiers and logical operators

Operators	Precedence
$\forall$ $\exists$	1
$\neg$	2
$\wedge$ $\vee$	3
$\rightarrow$ $\leftrightarrow$	4

Quantified statements, logical equivalence

- A **quantified statement** is a statement which involves predicates $P(x), Q(x), \ldots$ and quantifiers $\forall, \exists$.

Quantified statements, logical equivalence

- A **quantified statement** is a statement which involves predicates $P(x), Q(x), \ldots$ and quantifiers $\forall, \exists$.
- 2 quantified statements are **logically equivalent** if they have the same truth values under all situations.
- We use $\equiv$ and $\not\equiv$ to denote **equivalent** and **non-equivalent**.

Universal quantifier and conjunction

Lemma 1

Let $P(x)$ and $Q(x)$ be predicates defined on the same domain D. Then

(a) $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$.

(b) $\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$

Proof. Optional. See textbook.

De Morgan's laws for quantifiers

Lemma 2

Let $P(x)$ be a predicate defined on domain D. Then

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$
$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

Proof. Optional. See textbook.

The law says that

$\forall$ and $\exists$ are negation of each other

Example 9

Write negations of the following statements.

- There is an honest politician.
- All Americans eat cheeseburgers.
- No students sleep before 12am.
- $\exists\, x \in \mathbb{Z}^+$ such that $x^2 + 1$ is a square.

Truth table for predicates?

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  - The truth table for $P(x)$ would have $|D|$ propositions $\Rightarrow 2^{|D|}$ rows $\Rightarrow$ often too large (even infinite if $|D|$ is infinite, say $D = \mathbb{Z}$ or $D = \mathbb{R}$).
- To prove the equivalence between quantified statements, we often don't use truth table (D can be infinite).
  We use logical equivalence rules and Lemma 1+2.

Exercise 3

Let $P(x), Q(x)$ be defined on the same domain D. Show that

$$\neg\forall x(P(x) \to Q(x)) \equiv \exists x(P(x) \wedge \neg Q(x)).$$

Hint: $P(x) \to Q(x) \equiv \neg P(x) \vee Q(x)$.

Example 10 (Lewis Carroll)

Consider the following argument.

"All lions are fierce."

"Some lions do not drink coffee."

$\therefore$ "Some fierce creatures do not drink coffee."

Let $P(x), Q(x), R(x)$ be "x is a lion", "x is fierce", "x drinks coffee".

Assuming that the *domain consists of all creatures*, express the argument in its argument form using quantifiers and $P(x), Q(x)$, and $R(x)$.