CSD1241 Tutorial 9 Solutions

Remarks. The solution should only be used as guidance for your study. There is no guarantee on errors and typos. Would appreciate if you let me know the errors.

Problem 1. Let T be the scaling given by the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(a) Find the images of the points $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$.

(b) Find all points $\vec{x}$ that are fixed under this transformation, that is, $T(\vec{x}) = \vec{x}$.

(c) Find the image of the plane $\beta : 2x + 3y - 4z = 12$ under T.

(d) Find the image of the line $l : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ under T.

(e) Let Q be the intersection of β and l. Find the image of Q under T.

Solution. (a) The images of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$ are given in the columns of the following matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & -3 & -4 \end{pmatrix}$$

(b) Assume $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. We have

$$\begin{pmatrix} 2x \\ y \\ -z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow x = z = 0$$

Therefore, any point that is fixed has form $\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$, that is, the collection of fixed points is the y-axis. This makes sense because the scaling S fixes the y-coordinates.

(c) The plane β has vector equation $\vec{x} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$. Its image β' has vector equation

$$\begin{aligned} \beta' : \vec{x} &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \left(\begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -6 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} + 2s \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}, \end{aligned}$$

which is a plane through $\begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix}$ and having normal vector $\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ -4 \end{pmatrix}$, which is parallel to $\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$. The normal equation of β' is

$$(x - 12) + 3y + 4z = 0 \Leftrightarrow x + 3y + 4z = 12.$$

(d) The image of l is

$$l' : \vec{x} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \left(\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 4 \\ 3 \\ -4 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

(e) Since Q is on l, its coordinates are $Q = \begin{pmatrix} 2+t \\ 3+t \\ 4+t \end{pmatrix}$. Further, since Q is on β, we have

$$2x + 3y - 4z = 12 \Rightarrow 2(2+t) + 3(3+t) - 4(4+t) = 12 \Rightarrow t = 3$$

Hence $Q = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$ and its image is

$$Q' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 10 \\ 6 \\ -7 \end{pmatrix}$$

$\square$

Problem 2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the shear with respect to the xy-plane (equation $z = 0$) and shearing vector $\vec{v} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$ (note that $\vec{v}$ is parallel to the xy-plane).

(a) Find the matrix of T and the images of the points $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$.

(b) Find all points $\vec{x}$ that are fixed by T, that is, $T(\vec{x}) = \vec{x}$.

(c) Find the image β' of the plane $\beta : x - 2y + 3z = 9$ under T.

(d) Find the image of the line $l : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 7 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ under T.

(e) Let Q be the point of intersection of β and l. Find the image of Q.

Solution. (a) Since the xy-plane has equation z=0, a normal vector is $\vec{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The matrix of T is

$$M = I_3 + \frac{1}{||\vec{n}||}\vec{v}\vec{n}^T = I_3 + \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

The images of the given points are the columns of the following matrix

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & -1 & 0 \\ 1 & 0 & 3 & 2 \\ 1 & -4 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & -9 & 5 & 0 \\ -1 & 8 & -1 & 2 \\ 1 & -4 & 2 & 0 \end{pmatrix}$$

(b) The fixed points are all points on the xy-plane.

(c) The plane β has vector equation $\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$. So its image β' has equation

$$\begin{aligned} \beta' : \vec{x} &= \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \right) \\ &= \begin{pmatrix} 9 \\ -6 \\ 3 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix}, \end{aligned}$$

which is a plane through $\begin{pmatrix} 9 \\ -6 \\ 3 \end{pmatrix}$ and having normal vector $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ -8 \end{pmatrix}$, which is parallel to $\begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}$. Its normal equation is

$$\beta' : 1(x-9) - 2(y+6) - 4(z-3) = 0 \Leftrightarrow x - 2y - 4z = 9.$$

(d) The image of l under T is

$$l' : \vec{x} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{bmatrix} 0 \\ 3 \\ 7 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{pmatrix} 21 \\ -11 \\ 7 \end{pmatrix} + t \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$$

(e) Since Q is on l, we have $Q = \begin{pmatrix} t \\ 3+t \\ 7+t \end{pmatrix}$. Further Q is on β, we have

$$t - 2(3+t) + 3(7+t) = 9 \Rightarrow 2t = -6 \Rightarrow t = -3.$$

So the image of $Q = \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$ is

$$Q' = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ -8 \\ 4 \end{pmatrix}.$$

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Problem 3. Let T be the rotation about the vector $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ over 60°.

a) Find the matrix A of this transformation.

b) Find the images $\begin{pmatrix} 1 \\ -1 \\ -7 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$.

c) Find all points $\vec{x}$ that are fixed under this transformation, i.e. $T(\vec{x}) = \vec{x}$.

d) Find the image of the plane $\beta : 3x - 2y - z = 9$ under T.

e) Find the image of the line $l : \vec{x} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ under T.

f) Let Q be the point of intersection of β and l. Find the image of Q.

Solution. (a) Note that the cross-product matrix induced by $\vec{v}$ is

$$C_{\vec{v}} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

The matrix of T is

$$\begin{aligned} M &= (1-\cos\theta)\frac{\vec{v}\vec{v}^T}{||\vec{v}||^2} + (\cos\theta)I_3 + \frac{\sin\theta}{||\vec{v}||}C_{\vec{v}} \\ &= \frac{1-\cos 60^\circ}{3}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 \end{pmatrix} + \cos 60^\circ I_3 + \frac{\sin 60^\circ}{\sqrt{3}}\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \\ &= \frac{1}{6}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix} \end{aligned}$$

(b) The images of the given points are the columns of the following matrix

$$\frac{1}{3}\begin{pmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 & 0 & -3 \\ -1 & 0 & 1 & -2 \\ -7 & 0 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 2 & -1 & 2 \\ 1 & -2 & 0 & -3 \\ -5 & 1 & 2 & -1 \end{pmatrix}$$

(c) Assume $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. We have

$$\frac{1}{3}\begin{pmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{cases} 2x + y - 2z = 3x \\ -2x + 2y - z = 3y \\ x + 2y + 2z = 3z \end{cases} \Leftrightarrow \begin{cases} x = y - 2z \\ 2x = -y - z \\ x = -2y + z \end{cases}$$

By the 1st and the 3rd equations, we have $y - 2z = -2y + z \Rightarrow y = z$. Substituting $y = z$ into the 1st and 2nd equations, we have $2x = -2z$ and $x = -z$, that is, $x = -z$. Therefore, all points fixed by T has coordinates

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ z \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + (-z) \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix},$$

which is a line through the origin and having direction $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$. Note that T is the rotation about the vector $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$. So all fixed points line on the line around which T rotates.

(d) The plane β has vector equation $\vec{x} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$. Its image β' has vector equation

$$\begin{aligned} \beta' : \vec{x} &= \frac{1}{3} \begin{pmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix} \left(\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right) \\ &= \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + \frac{s}{3} \begin{pmatrix} 7 \\ 2 \\ 8 \end{pmatrix} + \frac{t}{3} \begin{pmatrix} -4 \\ -5 \\ 7 \end{pmatrix}. \end{aligned}$$

The plane β' has normal equation

$$2x - 3y - z = 9.$$

(e) The image of l is

$$l' : \vec{x} = \frac{1}{3} \begin{pmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix} \left(\begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

(f) We have $Q = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ and its image is

$$Q' = \frac{1}{3}\begin{pmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

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Problem 4. In this problem, we learn that the composition of two reflections is a rotation. The following maps S, T were used in Tutorial 8 (Problem 5).
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection through the xz-plane, and let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection through the plane $x - y = 0$.
(a) Find the matrices M and N of the composition $T \circ S$ and $S \circ T$.
(b) Show that both $T \circ S$ and $S \circ T$ are rotations.

Solution. (a) Using Tutorial 5 problem 5, we have

$$M = M_{T\circ S} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } N = M_{S\circ T} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) First, consider the matrix $M = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ of $T \circ S$. Comparing this with the matrix of rotation about the positive z-axis $R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we have $\theta = 270°$.
Thus, $T \circ S$ is a rotation about the positive z-axis over the angle $270°$.
Similarly, comparing the matrix $N = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we have $\theta = 90°$. Thus, $S \circ T$ is a rotation about the positive z-axis over the angle $90°$.
Note that the composition of $T \circ S$ and $S \circ T$ give a rotation over $270° + 90° = 360°$, that is, the identity map which maps any point to itself. The matrix of the identity map is I_3, i.e. $I_3\vec{x} = \vec{x}$. A simple computation verifies that $MN = I_3$.