CSD1241 Tutorial 7

Problem 1. Let $a, b, c \in \mathbb{R}$ be constants and let $\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ be a vector. Define the cross-product map $T_{\vec{u}} : \mathbb{R}^3 \to \mathbb{R}^3$ as follows

$$T_{\vec{u}}(\vec{x}) = \vec{u} \times \vec{x}$$

(a) Is T linear? Justify your answer.
(b) If T is linear, write out its matrix.

Problem 2. Let $a, b > 0$. In this exercise, we learn that the scaling $S : \mathbb{R}^2 \to \mathbb{R}^2$

$$S \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix}$$

scales the area of a region in $\mathbb{R}^2$ by the factor ab.
(a) What is the matrix representation of S?
(b) Find the image $A'B'C'$ of the triangle ABC with

$$A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \; C = \begin{pmatrix} 7 \\ 10 \end{pmatrix}$$

(c) Compare the areas of the triangles $\triangle ABC$ and $\triangle A'B'C'$.

Problem 3. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the counter-clockwise rotation around O over the angle $\theta = 120^0$.
(a) Find the matrix representation of R.
(b) Find the images of the points $\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$.
(c) Find the image of the line $m : x - \sqrt{3}y = 0$ under T (find general equation).
(d) Find the image of the line $n : y = 2$ under T (find general equation).

Problem 4. In this problem, we will learn that a **shear** not only transforms a square into a parallelogram, it also preserves the area of the square.

Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the shear with respect to $l : 2x - 5y = 0$ in the direction of $\vec{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

(a) Find the matrix representation of S.
(b) Find the image $A'B'C'D'$ (under S) of the unit square $ABCD$ with

$$A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, D = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(c) Verify that $A'B'C'D'$ is a parallelogram, that is, $\vec{A'B'} = \vec{D'C'}$. Further, verify that the area of $A'B'C'D'$ is equal to 1 (the same as the area of $ABCD$).

In the last problem, we study **composition of linear transformations**.
Let $\mathbb{R}^m \xrightarrow{S} \mathbb{R}^n \xrightarrow{T} \mathbb{R}^k$ be a sequence of linear transformations. The composition $T \circ S : \mathbb{R}^m \to \mathbb{R}^k$ is another linear transformation defined by

$$T \circ S(\vec{x}) = T(S(\vec{x}))$$

Further, if M_T, M_S are matrices of T, S, then the matrix of $T \circ S$ is

$$M_{T \circ S} = M_T M_S.$$

Problem 5. Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto the line $l : \sqrt{3}x - y = 0$ and let R be the reflection through the line $m : x - \sqrt{3}y = 0$.
(a) Find the matrices of $M_P, M_R, M_{P \circ R}, M_{R \circ P}$ of $P, R, P \circ R, R \circ P$.
(b) Describe $P \circ R$ and $R \circ P$, that is, find $P \circ R \begin{pmatrix} x \\ y \end{pmatrix}$ and $R \circ P \begin{pmatrix} x \\ y \end{pmatrix}$.
(c) Find the points which are fixed by $P \circ R$.