limits in Calculus I

$n \to \infty$, $x \to \infty$

Sequence Fundamentals

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AY 23/24 Trimester 1

Table of contents

helps the

What is a sequence?

- A **sequence** is a infinite list of numbers written in order:

index

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

- a_n is the nth term of the sequence, and n is the **index** of a_n; this is akin to indexing lists in coding. → In coding, lists are finite
- We denote the entire sequence as $\{a_n\}_{n=1}^{\infty}$, $\{a_n\}$, or sometimes a_n.
- Two ways of writing an entire sequence:
  - **General formula/Closed form**: e.g.

$$a_n = \frac{\sin(n^2)}{n}.$$

term after is based on previous term(s)

  - **Recursive relation**: e.g. the Fibonacci sequence:

$$a_n = a_{n-1} + a_{n-2}.$$

Convergence of sequences (informal)

a_n let $n \to \infty$

$a_1, a_2, a_3, \ldots, a_n, \ldots$

what happens here

- In a sequence, we have an infinite list of numbers.
- It is thus natural to ask: "Where does this lead to?" or "What number does this list approach as the index n gets **very large**?".
- We refer to this as the **limit** L of a sequence $\{a_n\}$. We say that the sequence $\{a_n\}$ has limit L if the terms are 'close' to L when n gets (arbitrarily) large. It is written as

"arbitrarily large"

$+\infty$ not a number

$$\lim_{n\to\infty} a_n = L.$$

$\lim_{x \to a} f(x)$

what is behavior of $f(x)$ when x close to a

- If such a number L exists, we say that $\{a_n\}$ is **convergent** (or **converges**).
- Otherwise, we say that $\{a_n\}$ is **divergent** (or **diverges**), or the limit of $\{a_n\}$ **does not exist**. DNE

Convergence of sequences (formal, optional)

"behavior of sequence when n is large"

- Formal definition of $\lim_{n\to\infty} a_n = L$:

  For every $\varepsilon > 0$, there is a positive integer N such that

  $$|a_n - L| < \varepsilon \quad \text{whenever} \quad \underline{n > N}.$$

  ✱

- We understand $|a_n - L|$ as the distance from the nth term of the sequence a_n to a number L.
- A layman's way of saying this formal definition is:

  $\{a_n\}$ converges to L if and only if

  for every $\varepsilon > 0$, we go down the list $\{a_n\}$ long enough, **eventually**, after a certain index N, all terms of the sequence with index greater than N would be within ε distance from L.

Example 1 (optional)

Using formal defn of limit to show $\lim_{n\to\infty} \frac{1}{n} = 0$.

Let's try to get an understanding of the formal definition of a sequence limit by considering the sequence $a_n = \frac{1}{n}$. Computing the first few terms, we get

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots$$

At first glance, the sequence looks to be approaching 0. For an example, let's consider $\varepsilon = 1$. For this ε, we can choose $N = 1$, then any term after the index 2 will be within distance of 1 from 0:

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots$$

Example 1 (optional)

For $\varepsilon = \dfrac{1}{2500}$, we can choose $N = 2500$, then any term after index 2500 will be within distance of $\dfrac{1}{2500}$ from 0:

$$\frac{1}{2501}, \frac{1}{2502}, \frac{1}{2503}, \frac{1}{2504} \ldots$$

A suitable index N can actually be chosen for any choice of $\varepsilon > 0$. Thus

$$\lim_{n \to \infty} \frac{1}{n} = 0.$$

Subsequences

original $a_1, a_2, a_3, \dots$

$\to a_2, a_4, a_6, a_8, \dots$ even subsequence (indices are even)

"infinite" part of

- A **subsequence** of $\{a_n\}_{n=1}^{\infty}$ is a sequence of the form

" part of a sequence"

$$a_{n_1}, a_{n_2}, a_{n_3}, \dots$$

index

where $1 \leq n_1 < n_2 < n_3 < \cdots$ → order of the original sequence must be obeyed

- Some notable subsequences include:
  - $\{a_1, a_3, a_5, \dots\}$ is called the **odd subsequence** of $\{a_n\}_{n=1}^{\infty}$. a_{2n-1}
  - $\{a_2, a_4, a_6, \dots\}$ is called the **even subsequence** of $\{a_n\}_{n=1}^{\infty}$. → a_{2n}
- A subsequence **preserves the order** of its original sequence;

  e.g. for a sequence $\{a_n\}_{n=1}^{\infty}$,
  - $\{a_1, a_5, a_9, \dots\}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$.
  - $\{a_2, a_1, a_3, a_6, a_4, a_5, \dots\}$ is **NOT** a subsequence of $\{a_n\}_{n=1}^{\infty}$.

$2, 1, 3, 6$

$\not< \; < \; \not<$

$\{a_{3n}\} = \{a_3, a_6, a_9 \dots\}$ $a_{4n+1} = \{a_1, a_5, a_9, \dots\}$

Subsequence Test

$a_1, a_2, a_3, a_4, \dots \to L$

odd ss $\to a_1 \quad a_3 \quad a_5 \quad \to L$

even ss $\to a_2, \quad a_4, \dots \to L$

Theorem

If a sequence $\{a_n\}$ converges to L, then **every** subsequence of $\{a_n\}$ converges to L.

Conversely, As a consequence, if there are two subsequences of $\{a_n\}$ that do not converge to the same limit, then $\{a_n\}$ is divergent.

original

Note: This test is usually used to show that a particular sequence is divergent.

Example 2

start $n=1$

Use the Subsequence Test to show that the sequence

$$a_n = (-1)^n$$

very famous eg.

$a_1 = -1$, $a_2 = 1$
$a_3 = -1$, $a_4 = 1$
$a_5 = -1$, $a_6 = 1$
$a_7 = -1$, $a_8 = 1$

is divergent.

First 8 terms of a_n

Consider a_{2n+1}, a_{2n}

$$a_{2n+1} = (-1)^{2n+1} = \underbrace{(-1)^{2n}}_{1} \cdot (-1) = -1$$

$\therefore a_{2n+1} \to -1$

tends to

$$a_{2n} = (-1)^{2n} = 1$$

$\therefore a_{2n} \to 1$

Since there exist two subsequences of a_n that converges to different limits, a_n is divergent.

Exercise 1

a_{4n+1} \to 1

Use the Subsequence Test to show that the sequence

$$a_n = \sin\left(\frac{n\pi}{2}\right)$$

is divergent.

$a_1 = \sin\left(\frac{\pi}{2}\right) = 1$, $a_2 = \sin(\pi) = 0$,

$a_3 = \sin\left(\frac{3\pi}{2}\right) = -1$, $a_4 = \sin(2\pi) = 0$,

$a_5 = \sin\left(\frac{5\pi}{2}\right) = \sin\left(2\pi + \frac{\pi}{2}\right) = 1$ $a_6 = \sin(3\pi) = 0$

$a_7 = \sin\left(\frac{7\pi}{2}\right) = \sin\left(2\pi + \frac{3\pi}{2}\right) = -1$ $a_8 = \sin(4\pi) = 0.$

even SS

$\star$ $a_{2n} = \sin(n\pi) = 0$

1
5
9
13

$4n+1$ $(n=0)$

$4n-3$ $(n=1)$

2π-periodic

$\star$ $a_{4n+1} = \sin\left(\frac{(4n+1)\pi}{2}\right) = \sin\left(2n\pi + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

Limit Laws

Like in functional limits in Calculus I, we also have limit laws for sequences. Let $\{a_n\}$ and $\{b_n\}$ be **convergent** sequences. Then

Limit Laws does not work if either an or bn is divergent

(a) $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$.

(b) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$.

(c) $\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right)$.

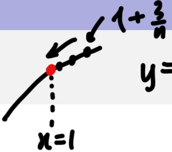
(d) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$.

(e) $\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p$ if $p > 0$ and $a_n > 0$ for all n.

(f) $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$ if f is continuous at $\lim_{n\to\infty} a_n$.

Exercise 2

$\lim_{n\to\infty} \frac{1}{n} = 0 \leftarrow$ Eg 1



$y = \sin x$

Evaluate the following limits.

1. $\lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty} \frac{1}{n}\cdot\frac{1}{n} = \left(\lim_{n\to\infty}\frac{1}{n}\right)\left(\lim_{n\to\infty}\frac{1}{n}\right) = 0\cdot 0 = 0$
2. $\lim_{n\to\infty} 3 - \frac{1}{n} = \lim_{n\to\infty} 3 - \lim_{n\to\infty}\frac{1}{n} = 3 - 0 = 3$
3. $\lim_{n\to\infty} \cos\left(\frac{4}{n}\right)$

limit is 0 + cos is cts at 0

$\lim_{n\to\infty} \frac{4}{n} = 4 \lim_{n\to\infty} \frac{1}{n} = 4\cdot 0 = 0.$

$\ast$ $\cos x$ is continuous at $x = 0 \leftarrow$ at the limit

'bring in limit'

$\therefore \lim_{n\to\infty} \cos\left(\frac{4}{n}\right) = \cos\left(\lim_{n\to\infty}\frac{4}{n}\right) = \cos(0) = 1.$

$\sin\left(1+\frac{3}{n}\right)$ $\rightarrow 1$

sin cts at 1

$\Rightarrow \lim_{n\to\infty} \sin\left(1+\frac{3}{n}\right)$

$= \sin\left(\lim_{n\to\infty} 1+\frac{3}{n}\right)$

$= \sin 1$

Sequences defined by a function → helpful to evaluate limits of known function

Theorem

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for all n, then

$$\lim_{n\to\infty} a_n = L.$$

$\ln x \longrightarrow \ln n$

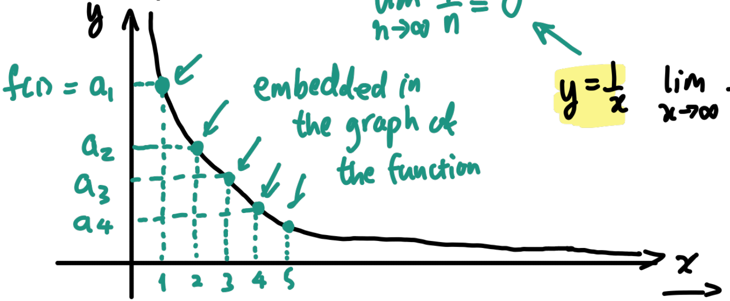
$x \to \infty$

$\ln x \to \infty \to \ln n \to \infty$

Visualization Example:

$\lim_{n\to\infty} \frac{1}{n} = 0$

$y = \frac{1}{x}$ $\lim_{x\to\infty} \frac{1}{x} = 0$



Sequences that diverge to ∞

$(-1)^n$

Eg 2 & Ex 1

- We have seen divergent sequences that have an oscillating behavior.
- There is also another type of divergent sequence:

$$\lim_{n\to\infty} a_n = \infty \quad \text{or} \quad -\infty.$$

$x^p \to \infty$ $\quad \to \infty \quad \to 0^+ \quad \to \infty$

- Examples of such sequences include n^p (for $p > 0$), $\ln n$, $\ln\left(\frac{1}{n}\right)$, 3^n, etc.

$\to -\infty$

- This can be easily observed using the theorem in the previous slide.

Limit Evaluation Technique 1: Squeeze Theorem

infamous

Theorem

If there are three sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ that obey

1. $a_n \leq b_n \leq c_n$ for all $n > N$, where N is a fixed integer, → essentially does not have to start from n=1
   eventually
2. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ → must be same

then

$$\lim_{n\to\infty} b_n = L.$$

TLDR: If b_n is (eventually) sandwiched/squeezed in between two other sequences a_n and c_n, and a_n, c_n both converge to the same limit L, then b_n also converges to L.

Example 3 (Geometric Sequence)

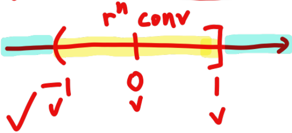
Let r be a fixed number. The sequence $\{a_n\}$ defined by

$$a_n = r^n$$

$\lim_{n\to\infty} r^n = ?$

is called the **geometric sequence with rate** r. We show that

1. $\{a_n\}$ is convergent for $-1 < r \leq 1$ with



$$r^n = \begin{cases} 0 & \text{if } -1 < r < 1, \\ 1 & \text{if } r = 1. \end{cases}$$

Conclusion

2. $\{a_n\}$ is divergent for $r <= -1$ and $r > 1$.

$(-1)^n$

We do this by cases. We have already shown in Example 2 that when

2 $r = -1$, r^n is divergent.

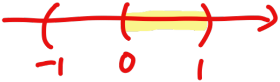
When $r = 1$, $r^n = 1^n = 1$, so r^n converges to 1.

Also, when $r = 0$, $r^n = 0$, so r^n converges to 0.

Example 3 (Geometric Sequence: Case 1)

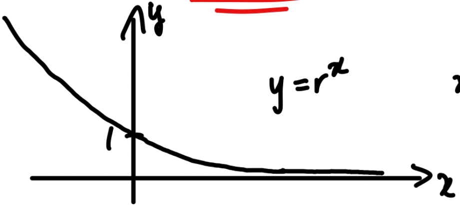
The remaining cases are

1. $-1 < r < 1,\ r \neq 0$
2. $r > 1$
3. $r < -1$



$0 < |r| < 1$

For case 1, we first consider $0 < r < 1$. Consider the graph $f(x) = r^x$:



$\lim_{x \to \infty} r^x = 0$

We have $\lim_{x \to \infty} r^x = 0$. As $a_n = r^n = f(n)$, by the theorem on slide 14 (sequences defined by a function),

$$\lim_{n \to \infty} r^n = 0.$$

Example 3 (Geometric Sequence: Case 1)

(previous slide)

$-1 < r < 0 \qquad r \to 0 < |r| < 1 \qquad |r|^n \to 0$

For $-1 < r < 0$, we first note that $r^n \leq |r^n| = ||r|^n|$, so

$a_n \to 0 \qquad c_n \to 0$

$$-|r|^n \leq r^n \leq |r|^n.$$

$b_n \to 0$ by Squeeze

$r^n = |r^n|$

$= ||r|^n|$

$x \le |a|$

$\Updownarrow$

$-|a| \le x \le |a|$

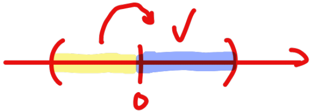
$a = |r|^n$

Since $-1 < r < 0$, we have $0 < |r| < 1$, so by the last slide,

$$\lim_{n\to\infty} -|r|^n = \lim_{n\to\infty} |r|^n = 0.$$

Thus by the Squeeze Theorem,

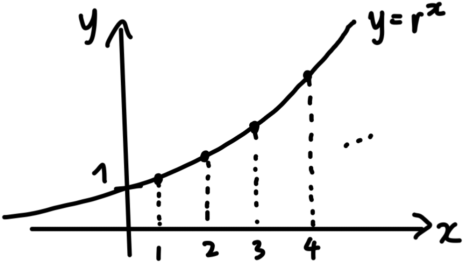
$$\lim_{n\to\infty} r^n = 0.$$



Case 1 is thus proved.

Example 3 (Geometric Sequence: Case 2)

For $r > 1$, we consider, again, the graph $f(x) = r^x$.



Clearly, $\lim_{x\to\infty} f(x) = \infty$, thus by the Theorem on slide 14, $\lim_{n\to\infty} r^n = \infty$, and r^n is divergent.

Example 3 (Geometric Sequence: Case 3)

$r \to |r|$ $r < -1 \Rightarrow |r| > 1$

When $r < -1$, consider the even subsequence of r^n:

$$|r|^2, |r|^4, |r|^6, \ldots \to \infty$$

Since $r < -1$, we have $|r| > 1$.

Thus by the previous slide, r^n diverges to ∞.

Since a subsequence of r^n diverges to ∞, it follows that r^n is divergent.

subsequence test

We have thus completed this proof. $\square$

Exercise 3

4^n

$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1<r<1 \\ 1 & \text{if } r=1 \\ \text{divergent} & \text{otherwise} \end{cases}$

Evaluate the following limits.

1. $\lim_{n\to\infty} 3^n = \infty$
2. $\lim_{n\to\infty} (-0.5)^n$
3. $\lim_{n\to\infty} \frac{1}{2^n}$

$\lim_{n\to\infty} 3^n$ $\quad |3|>1$

$= \infty$

$\lim_{n\to\infty} (-0.5)^n = 0 \quad |-0.5|<1$

$\lim_{n\to\infty} \frac{1}{2^n} = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0 \quad \left|\frac{1}{2}\right|<1$

$\lim_{n\to\infty} \frac{1}{2^n} = 0$

because $\lim_{n\to\infty} 2^n = \infty$

$\frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 2^n}$ should not write like this

Limit Evaluation Technique 2: Rational Functions

We start off with polynomials in n, say of degree 2, $n^2 - 2n + 3$. We know that when n is large ($n > 1$), $n^2 >> n >>$ constant. Thus the limit

$$\lim_{n\to\infty} n^2 - 2n + 3$$

is **fully dependent** on the 'highest power term' n^2. Since

highest growth rate / dominant term

$$\lim_{n\to\infty} n^2 = \infty,$$

it follows that

$$\lim_{n\to\infty} n^2 - 2n + 3 = \infty.$$

Limit Evaluation Technique 2: Rational Functions

We move on to limits of rational functions in n, for example

$$\lim_{n\to\infty} \frac{n^2 - 2n + 3}{2n^4 + 3n^3 + 2n^2}.$$

The numerator and denominator both tend to ∞. How do we handle these kind of limits? We **divide** both the numerator and denominator by the highest power of n in the fraction; here being n^4. We get

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^2 - 2n + 3}{2n^4 + 3n^3 + 2n^2} &= \lim_{n\to\infty} \frac{n^2 - 2n + 3}{2n^4 + 3n^3 + 2n^2} \cdot \frac{\frac{1}{n^4}}{\frac{1}{n^4}} \\
&= \lim_{n\to\infty} \frac{\overset{0}{\frac{1}{n^2}} - \overset{0}{\frac{2}{n^3}} + \overset{0}{\frac{3}{n^4}}}{2 + \overset{0}{\frac{3}{n}} + \overset{0}{\frac{2}{n^2}}} \\
&= \frac{\lim_{n\to\infty} \frac{1}{n^2} - \frac{2}{n^3} + \frac{3}{n^4}}{\lim_{n\to\infty} 2 + \frac{3}{n} + \frac{2}{n^2}} = \frac{0}{\cancel{1}\ 2} = 0.
\end{aligned}$$

Example 4

Evaluate the following limits.

1. $\lim_{n\to\infty} \frac{n^3 - 2n^2 + 4n}{3n^3 + n^2 + 1} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{1 - \frac{2}{n} + \frac{4}{n^2}}{3 + \frac{1}{n} + \frac{1}{n^3}} = \frac{1}{3}$
2. $\lim_{n\to\infty} \frac{-n^4 + n^2}{n^3 + n} \cdot \frac{\frac{1}{n^4}}{\frac{1}{n^4}} = \lim_{n\to\infty} \frac{-1 + \frac{1}{n^2}}{\underbrace{\frac{1}{n} + \frac{1}{n^3}}_{\to 0^+}} = -\infty.$ $\frac{1}{n^2} < 1$ after some n
3. $(*) \lim_{n\to\infty} \frac{3^n + 2^n}{4^n + 5^n}$

$$\lim_{n\to\infty} \frac{3^n + 2^n}{4^n + 5^n} \cdot \frac{\frac{1}{5^n}}{\frac{1}{5^n}}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{3}{5}\right)^n + \left(\frac{2}{5}\right)^n}{\left(\frac{4}{5}\right)^n + 1} = \frac{0+0}{0+1} = 0$$

$$\left.\begin{aligned} \left(\tfrac{3}{5}\right)^n &\to 0 \\ \left(\tfrac{2}{5}\right)^n &\to 0 \\ \left(\tfrac{4}{5}\right)^n &\to 0 \end{aligned}\right\} |\text{rate}| < 1$$

Example 4

Exercise 4

Evaluate the following limits.

1. $\lim_{n\to\infty} \frac{-2n^3+4n}{3n^3+3n^2} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{-2+\frac{4}{n^2}}{3+\frac{3}{n}} = \frac{-2+0}{3+0} = -\frac{2}{3}.$
2. $\lim_{n\to\infty} \frac{n^3+2n^2}{n^2-2n}$ $\lim_{n\to\infty} \frac{1+\frac{2}{n}}{\frac{1}{n}-\frac{2}{n^2}} = +\infty$

   $\frac{1}{n}-\frac{2}{n^2} \to 0^+$

   $\frac{1}{n} >> \frac{2}{n^2}$

   n is large
3. $(*) \lim_{n\to\infty} \frac{1+2^n}{6^n+2^n} \cdot \frac{\frac{1}{6^n}}{\frac{1}{6^n}}$

$$= \lim_{n\to\infty} \frac{\frac{1}{6^n}+\left(\frac{2}{6}\right)^n}{1+\left(\frac{2}{6}\right)^n} = \frac{0+0}{1+0} = 0.$$

Exercise 4

Motivation for L'Hôpital's Rule: Indeterminate Cases

L' Hospital

- Consider $\lim_{n\to\infty} \frac{a_n}{b_n}$.
- Under the limit laws, if $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ with $b \neq 0$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

cannot be determined

- **Indeterminate cases**:

$$\frac{a_n}{b_n} \to \frac{\pm\infty}{\pm\infty} \text{ or } \frac{a_n}{b_n} \to \frac{0}{0}.$$

$\frac{\pm\infty}{\pm\infty}$ or $\frac{0}{0}$

- We can use the Theorem in slide 14 (sequences defined by functions) along with **L'Hôpital's Rule** to solve these indeterminate cases.

L'Hôpital's Rule

Theorem (L'Hôpital's Rule)

Let a be any number, or $\pm\infty$. Assume f and g are differentiable functions with $g'(x) \neq 0$ on an open interval containing a. If

1. $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, and
2. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\pm\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

$\frac{+\infty}{-\infty}$ or $\frac{-\infty}{+\infty}$ OK

Example 5

n discrete values

x continuous / differentiable

Evaluate the following limits.

1. $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} \overset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$
2. (∗) $\lim_{n\to\infty} n^{\frac{1}{n}}$ ∞^0 indeterminate case

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln x^{\frac{1}{x}}} = \lim_{x\to\infty} e^{\frac{1}{x}\cdot\ln x} = \lim_{x\to\infty} e^{\frac{\ln x}{x}}$$

$\frac{\ln x}{x} \to 0$

galaxy brain move

$x = e^{\ln x}$

$\text{musk} = e^{\ln(\text{musk})}$

e^x is continuous at $x=0$, therefore

$$= e^{\lim_{x\to\infty}\frac{\ln x}{x}} = e^0 = 1$$

Example 5

Exercise 5

DIY

Evaluate the following limits.

1. $\lim_{n\to\infty} \frac{n^2+n}{e^n}$ 0
2. $(*)$ $\lim_{n\to\infty} \ln(n^2+2) - \ln(3n^2-1)$ $\ln \frac{1}{3}$

polynomial asked exponential for a race.

exponential said "you can have a head start".

Exercise 5

Food for thought

Should we use L'Hôpital's Rule to solve

$$\lim_{n\to\infty} \frac{n^{2023} + n^{2021}}{3n^{2023} - 3n^{2019}}?$$

L'Hôpital's Rule is not a magic pill.