Volumes of Revolution

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AY 23/24 Trimester 1

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1. We slice the solid S vertically using a plane.
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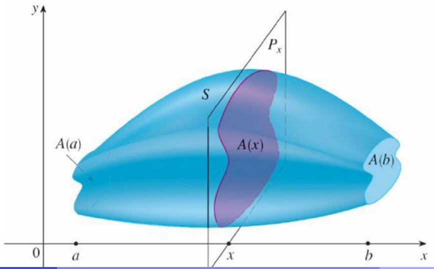
3. Summing all these cross-sections means integrating $A(x)$ from $x = a$ to $x = b$, which gives us the volume of the solid, i.e.

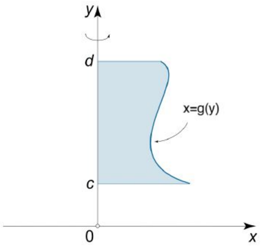
$$\text{Volume of solid } S = \int_a^b \underbrace{A(x)}_{\text{area}} \underbrace{dx}_{\text{thickness}} .$$

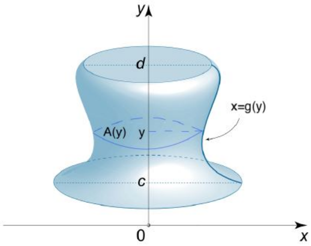


Horizontal plane slicing

1. We slice the solid S horizontally using a plane.
2. We calculate, **for every** $y \in [c, d]$, the **cross-sectional area function** $A(y)$.
3. Summing all these cross-sections means integrating $A(y)$ from $y = c$ to $y = d$, which gives us the volume of the solid, i.e.

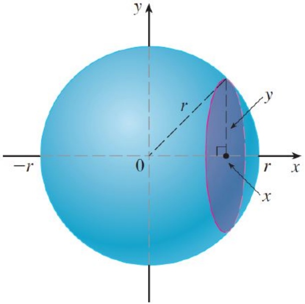
$$\text{Volume of solid } S = \int_c^d \underbrace{A(y)}_{\text{area}} \; \underbrace{dy}_{\text{thickness}} .$$



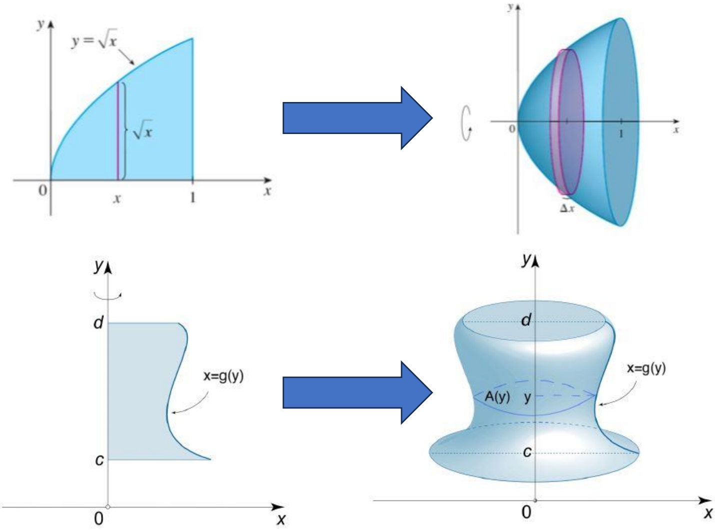


Example 1

Show that the volume V of a sphere of radius $r > 0$ is $\dfrac{4}{3}\pi r^3$.



Solids by revolving a curve about an axis

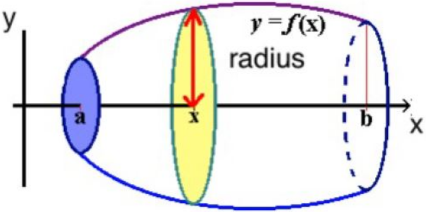


Cross-sectional area

To find these volumes of revolution, the key step is to find the cross-sectional area (either $A(x)$ or $A(y)$ depending on vertical or horizontal planar slicing). We focus on two cases:

- Cross-section is a **disk**.
- Cross-section is a **washer**.

Cross-section is a disk



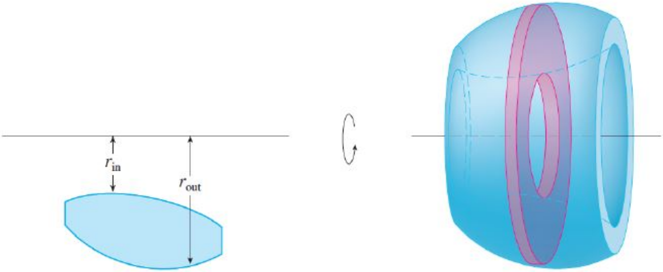
- We use vertical slicing (since function $y = f(x)$ is in variable x).
- Cross-section is a disk with radius $f(x)$. Thus the cross-sectional area function is

$$A(x) = \pi(f(x))^2.$$

- By summing all cross-sectional areas from $x = a$ to $x = b$, we obtain the volume V of the solid. Thus

$$V = \int_a^b A(x)\,dx = \int_a^b \pi(f(x))^2\,dx.$$

Cross-section is a washer



- The cross-sectional area is the difference of the area of two disks:

$$A(x) = \pi r_{\text{out}}^2 - \pi r_{\text{in}}^2.$$

- The volume V of the solid is

$$V = \int_a^b A(x)\,dx.$$

Example 2

Find the volume of the solid obtained by rotating about the x-axis, the region under the curve $y = \sqrt{x}$ from $x = 0$ to $x = 1$.

Example 2

Exercise 1

The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

Exercise 1

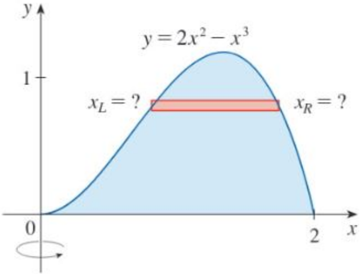
Exercise 2

Find the volume of the solid generated by rotating the region bounded by $y = x^{\frac{1}{3}}$ and $y = \dfrac{x}{4}$ that lies in the first quadrant, about the y-axis.

Exercise 2

Cross-sectional method can be difficult

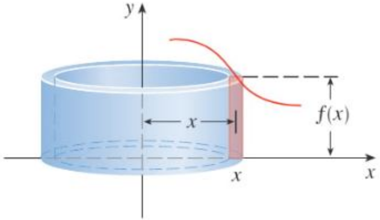
- Some volumes of revolution can be difficult to handle with the cross-sectional method.
- Let's suppose we wish to find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$ (see figure below).
- If we use the cross-sectional method, we would need to find the area of the washer, and as we can see, it is not easy to find r_{out} and r_{in}: x in terms of y.



Cylindrical Shells

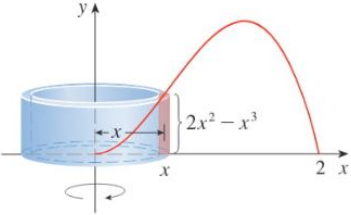
- For each x, we rotate the height $f(x)$ around the y-axis, the resulting **cylindrical shell** has a radius x, with thickness dx.
- Each of these shells has a volume of $2\pi x f(x)\,dx$ (see figure below).
- Summing these shells from $x = a$ to $x = b$, we get the volume V of the solid is

$$V = \int_a^b 2\pi x f(x) dx.$$



Example 3

Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.



Example 4

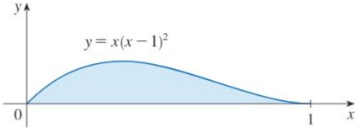
Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis, the region under the curve $y = \sqrt{x}$ from $x = 0$ to $x = 1$.

Example 4

Exercise 3

Let S be the solid obtained by rotating the region shown in the figure below about the y-axis.

- Briefly explain why it might be awkward to find the volume V of S using the washer method.
- Find V using cylindrical shells.



Exercise 3