Lecture 1: Set theory

Course Materials

1. **Textbooks**:

   Discrete Mathematics and Its Applications, 8th edition, K.Rosen, K. Krithivasan.

2. **Lecture slides** provided by myself.

Course Content

1. Sets and Functions
2. Number Theory and Cryptography
3. Logic and Proofs
4. Combinatorics
5. Discrete Probability

Assessment tasks

Assessment Task	Weight	Tentative dates
Online homeworks	10%	Weeks 1-12
Quizzes (3)	30%	Weeks 3, 9, 12
Midterm test (1)	30%	Week 6
Final test (1)	30%	Week 14/15

Policy for makeup exams

Makeup exams might be given if you have valid reasons.

But be aware of the following:

- Makeup exams cover *more materials* (possibly more difficult) than normal exams
- Makeup exams are given *less time* than the normal exams

Course structure

- **Online lecture** every Monday 3-5pm
- **Physical tutorials**
  - Groups A,B,C: taught by myself
  - Groups D: taught by Dr. Yilin

Grades

LETTER GRADE	GRADE POINT	DESCRIPTION	REMARKS
A+	5.0	Excellent attainment of learning outcomes	
A	5.0		
A-	4.5		
B+	4.0	Very Good attainment of learning outcomes	
B	3.5		
B-	3.0		
C+	2.5	Good attainment of learning outcomes	
C	2.0		
D+	1.5	Adequate attainment of learning outcomes	
D	1.0		Minimum grade required for undergraduate students to earn credit
F	0.0	Failed to attain learning outcomes	

Attendance policy

- Students $\geq$ 15 minutes late to class will be marked as absent.
- Students may not leave the class early without the instructor's permission.
- Unexcused absences would result in the following penalty

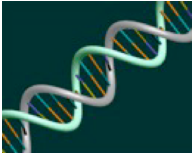
1 letter grade down for # of unexcused absences	2 letter grade down for # of unexcused absences
4	8

What is discrete math?

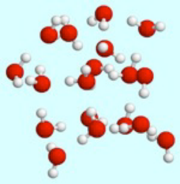
- The study of mathematical structures which are **discrete** - things that are **countable**.
- Intuitively, these are things with *clear gap* between one and another.
- Almost everything in our daily life is discrete.
  - Computers: gates in computer circuits operate via Boolean algebra (0 and 1).

Why study discrete math?

- DNA: DNA base pairs can only be one of the types A, C, G, T.



- Molecules: atomic structure of nature around us



Barber puzzle

- The army captain orders his company barber to *shave all members* of the company *who do not shave themselves*.
- What about the barber himself?

  Solution. Consider 2 cases

  1. The barber doesn't shave himself

  2. The barber shaves himself

Basic definitions of sets

- A **set** is an **unordered collection** of distinct objects, called **elements** of the set.
- Let A be a set and let a be an element. Write
    - $a \in A$ if a is an element of A
    - $a \notin A$ if a is not an element of A

Basic definitions of sets

- Curly brackets "{ }" are used to write a set
  - $\{1, 2\}$ and $\{-\sqrt{2}, e\}$ are sets with 2 elements
  - Sometimes, we use colon ":" to write sets which follow a rule

$$A = \{x \in \mathbb{R} : x^2 = 2\} \quad \Rightarrow \quad A = \{-\sqrt{2}, \sqrt{2}\}$$

$$B = \{x \in \mathbb{N} : x \text{ is odd and } x \leq 5\} \quad \Rightarrow \quad B = \{1, 3, 5\}$$

Example 1

List all elements of the following sets

(a) $A = \{x \in \mathbb{Z} : -2 \leq x \leq 2\}$

(b) $B = \{x \in \mathbb{R} : x^2 - x = 0\}$

(c) $C = \{x \in \mathbb{N} : x^2 \leq 10\}$

Example 2

List all elements of the following set. Rewrite the sets (if necessary) by deleting repeated elements.

(a) $A = \{1, 1, 5, 3, 5\}$

(b) $B = \{1, \{1\}, 2\}$

Basic definitions of sets

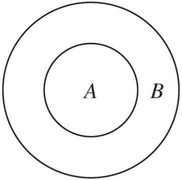
- The **empty set** is the set with *no element*, denoted by $\emptyset$.
- The **size** (or **cardinality**) of a set A is the number of elements in A, denoted by $|A|$.
- We call A an $\mathbf{n}$**-set** if $|A| = n$.

Standard sets of numbers

1. $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of all **natural numbers**.
2. $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is the set of all **integers**.
3. $\mathbb{Z}^+ = \{1, 2, \dots\}$ is the set of all **positive integers**.
4. $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$ is the set of all **rational numbers**.
5. $\mathbb{R}$ is the set of all **real numbers.**
6. $\mathbb{R}^+$ is the set of all positive **real numbers**.
7. $\mathbb{C}$ is the set of all **complex numbers**.

Subsets

- A is a **subset** of B, write $A \subset B$ or $B \supset A$, if any element of A is also an element of B.



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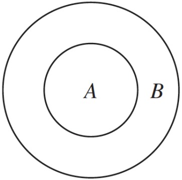

- The empty set $\emptyset$ is a subset of any set

$$\emptyset \subset A \text{ for any } A$$

- If A is not a subset of B, we write $A \not\subset B$.

Example 3

Find all subsets of

(a) $S = \{1, 2\}$

(b) $S = \{1, 2, 3\}$

Power set

- The **power set** of S, denoted $\mathcal{P}(S)$, is the set of all subsets of S.

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- **Example**: Find $\mathcal{P}(S)$ with

(a) $S = \{1, 2\}$

(b) $S = \{1, 2, 3\}$

(c) Can you guess the size of the power set of $S = \{1, \ldots, n\}$?

Tuples

- An **ordered n-tuple** (or **n-tuple**) is an ordered collection of n elements, say $(a_1, \ldots, a_n)$, such that a_1 is the 1st element, a_2 is the 2nd element, ... , a_n is the nth element.

Tuples

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- Comparison of ordered tuples

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n) \Leftrightarrow a_i = b_i \text{ for } i = 1, \ldots, n.$$

- The following 3-tuples are pairwise different

$$(1, 2, 3), (1, 3, 2), (2, 1, 3), (3, 1, 2).$$

Sets vs tuples

What are the similarities and differences between n-set and n-tuple?

Cartesian product

- The **Cartesian product** of A and B is

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

For example, if $A = B = \{1, 2\}$ then

$$A \times B = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

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- The Cartesian product of $A_1, A_2, \dots, A_n$ is

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \dots, a_n) : a_i \in A_i \text{ for all } i\}.$$

- Special case

$$A^n = A \times \cdots \times A = \{(a_1, a_2, \dots, a_n) : a_i \in A \text{ for all } i\}.$$

Question 3: Sizes of Cartesian products

(a) If $|A| = a$ and $|B| = b$, what is the size of $A \times B$? Explain your answer.

(A) a (B) b (C) $a + b$ (D) ab (E) a^b

(b) If $|A| = a$, what is the size of A^n?

(A) 0 (B) 1 (C) a (D) a^n

(c) Let A be any set. What is the size of $A \times \emptyset$?

(A) 0 (B) 1 (C) a

Exercise 1

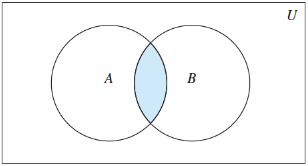
Define $A = \{0, 1\}$, $B = \{1, 2\}$, $C = \{0, 1, 2\}$. List all elements of
(a) $A \times B$, $B \times A$, $A \times B \times C$ and $(A \times B) \times C$.

(a)

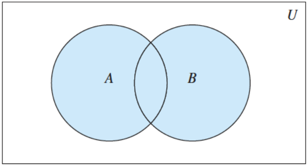
(b) Show that $A \times B \neq B \times A$ and $(A \times B) \times C \neq A \times B \times C$.

Intersection and union

Let A and B be subsets of U. We call U the **universal set**, that is, the set of all elements which are under consideration.



$A \cap B$ is shaded.



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Intersection and union

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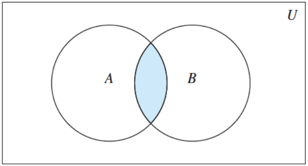
- The **intersection** of A and B is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

A and B are **disjoint** if $A \cap B = \emptyset$.



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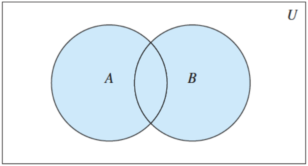

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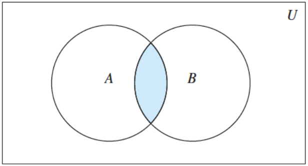

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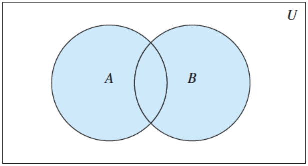
- The **union** of A and B is

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

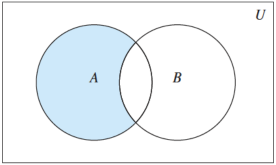


$A \cup B$ is shaded.

Difference and complement

- The **difference** of A and B is

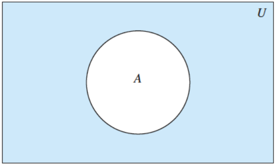
$$A - B = \{x : x \in A, x \notin B\}$$



$A - B$ is shaded.

- The **complement** of A in U is

$$\bar{A} = \{x \in U : x \notin A\}$$



$\bar{A}$ is shaded.

Example 4

$A = \{a, b, c, d, e\}$, $B = \{a, b, f\}$, $U = \{a, b, c, d, e, f, g\}$. Find

(a) $A \cup B$ and $A \cap B$

(b) $A - B$ and $B - A$

(c) $\bar{A}$ and $\bar{B}$

De Morgan's laws for sets

Lemma 1

Let U be the universal set and let A and B be subsets of U. Then

(a) $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

(b) $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

De Morgan's laws say that the complement operation

reverses intersection and union

Same sets

To prove $A = B$, we prove 2 things:

1. Prove $A \subset B$ by showing that

   any $x \in A$ is also in B.

2. Prove $B \subset A$ by showing that

   any $x \in B$ is also in A.

Proof of (a): $\overline{A \cup B} = \overline{A} \cap \overline{B}$

- Claim 1: $\overline{A \cup B} \subset \overline{A} \cap \overline{B}$. Let $x \in \overline{A \cup B}$.

- Claim 2: $\overline{A} \cap \overline{B} \subset \overline{A \cup B}$. Let $x \in \overline{A} \cap \overline{B}$.

Properties of union and intersection

Let A, B, C be three sets. Then the following hold.

(a) $A \cup B = B \cup A$ and (Commutativity)
$A \cap B = B \cap A$

(b) $A \cup (B \cup C) = (A \cup B) \cup C$ and (Associativity)
$A \cap (B \cap C) = (A \cap B) \cap C$

(c) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and (Distributivity)
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

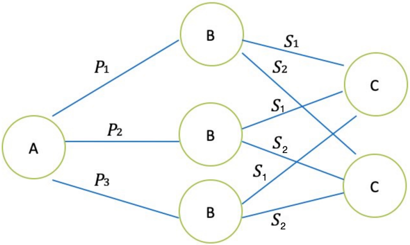
Question 2

- Consider 3 cities A,B,C.
- There are 3 ways from A to B, and 2 ways from B to C.
- How many ways are there from city A to city C via city B?

Question 2

Assume that

- $A \to B$ through P_1, P_2, P_3
- $B \to C$ through S_1, S_2



Product rule

If a procedure can be divided into a sequence of k tasks $T_1, \ldots, T_k$:

- T_1 can be performed in n_1 different ways,
- T_2 can be performed in n_2 different ways (regardless of how T_1 was performed), ...
- T_k can be performed in n_k different ways (regardless of how $T_1, \ldots, T_{k-1}$ were performed),

then the entire procedure can be performed in $n_1 n_2 \cdots n_k$ ways.

Exercise 2

How many different license plates are available if each plate contains a sequence of 3 letters followed by 3 digits?

Solution.

Exercise 3

How many subsets does the set $S = \{1, \ldots, n\}$ have?

Solution.

Exercise 4

Let $A_1, \ldots, A_k$ be finite sets with $|A_i| = n_i$ for $i = 1, \ldots, k$.

Find the cardinality of $A_1 \times \cdots \times A_k$.

Solution.

Two sets

Theorem 1

Let A_1 and A_2 be two finite sets. Then

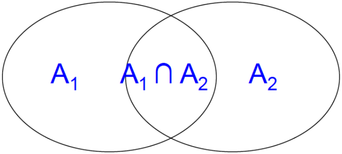
$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

This equation is called the inclusion-exclusion principle for two sets.

Proof

Partition $A_1 \cup A_2$ into 3 disjoint pieces

$$A_1 - (A_1 \cap A_2), A_1 \cap A_2, A_2 - (A_1 \cap A_2)$$



Example 5

Every student in a class is either a math major, a biology major or a joint major in these two subjects. How many students are in the class if there are 38 math majors (including joint majors), 23 biology majors (including joint majors), and 7 joint majors?

Exercise 5

(a) Let d and n be positive integers. How many integers are there in $\{1, \ldots, n\}$ that are divisible by d?

Exercise 6

(b) How many integers are there in $\{1, \ldots, 1000\}$ that can be divisible by either 2 or 3?

(c) How many integers are there in $\{1, \ldots, 1000\}$ that can be divisible by either 4 or 6?

Exercise 7

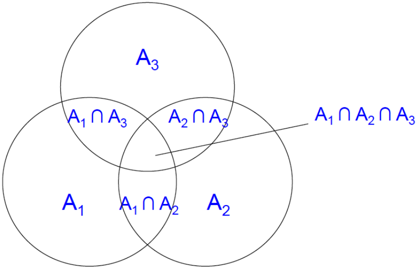
How many $0-1$ bit strings of length eight either start with a bit 1 or end with the two bits 00?

Three sets

Theorem 2

Let A_1, A_2 and A_3 be three finite sets. Then

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - \\ -|A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$



Proof

Example 6

1232 students take a course in Spanish, 879 take French, 114 take Russian, 103 take both Spanish and French, 23 take both Spanish and Russian, and 14 take both French and Russian.

If 2092 students take at least one of these three courses, how many students take all three?

Example 7

Exercise 8

How many integers are there the interval $[1, 1000]$ that can be divisible by one of the numbers 2 or 3 or 5?

General case

Theorem 3

Let n be a positive integer and let $A_1, A_2, \ldots, A_n$ be n finite sets. Then

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} |A_{i_1} \cap \cdots \cap A_{i_k}|.$$

General case

The right-hand side can be written in more details as follows.

$$\begin{aligned}
|A_1 \cup \cdots \cup A_n| &= \sum_{1 \le i_1 \le n} |A_{i_1}| - \sum_{1 \le i_1 < i_2 \le n} |A_{i_1} \cap A_{i_2}| + \\
&\quad + \sum_{1 \le i_1 < i_2 < i_3 \le n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \\
\cdots &\quad + (-1)^{n-1} \sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_n}|
\end{aligned}$$

Exercise 9

Give a formula for inclusion-exclusion for 4 sets.