CSD2301 Lecture

12. Rotation and Moment of Inertia Part 2

LIN QINJIE

Outline

- Rotational Energy
- Moment of Inertia
- Parallel Axis Theorem

Rotational Energy

- Consider a rigid body as a collection of small particles rotating about a fixed z-axis at angular speed ω:
  - Kinetic energy of *i*th particle: $$\boxed{K_i = \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i r_i^2 \omega^2}$$
  - Total kinetic energy: $$\boxed{K_R = \sum_i K_i = \frac{1}{2}\sum_i m_i r_i^2 \omega^2}$$

ω is the same for every particle!

Moment of Inertia

- Kinetic energy can be written as: $K_R = \sum_i K_i = \frac{1}{2}\sum_i m_i r_i^2 \omega^2$ $\frac{1}{2}mv^2$

$$\boxed{K_R = \frac{1}{2} I \omega^2}$$

present for rotation only

- Moment of inertia, I, is defined as: $\boxed{I = \sum_i m_i r_i^2}$

- For an extended rigid object (divide into small elements):

$$\boxed{I = \lim_{\Delta m_i \to 0} \sum_i \Delta m_i r_i^2 = \int r^2 dm = \int \rho r^2 dV}$$ $m = \rho \cdot V$

Moment of Inertia

- A measure of the resistance of an object to changes in its rotational motion, just as mass is a measure of the tendency of an object to resists changes in its linear motion.
- Mass is an intrinsic property of an object, but I depends on the physical arrangement of that mass.
- Also depends on the axis of rotation.
- Dimension: ML^2; units $kg.m^2$

Shape matters

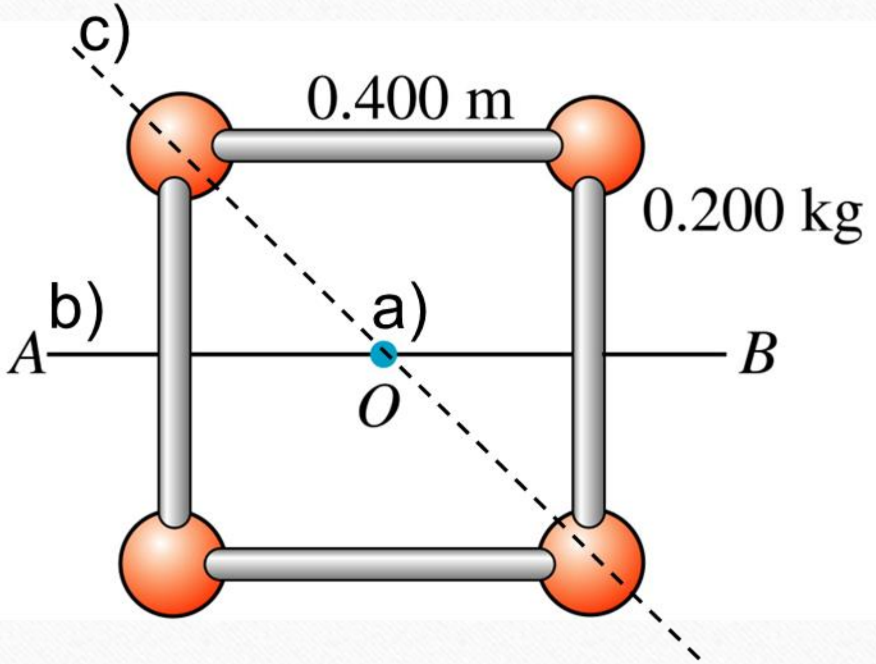
Example: 4 spheres in a square

Four small spheres, each of which may be regarded as a point of mass 0.200 kg, are arranged in a square 0.400 m on a side and connected by light rods. Find the moment of inertia of the system about an axis.

a) through the centre of the square, perpendicular to its plane;

b) bisecting two opposite sides of the square (an axis along line AB);

c) that passes through the centres of the upper left and lower right spheres and through point 0.

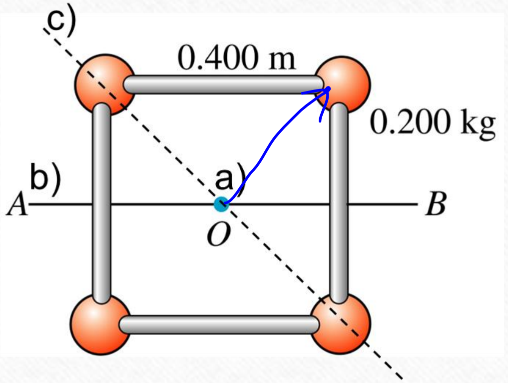


a) Through the centre of the square, perpendicular to its plane;

$a = r$

$$a^2 = \left(\frac{0.4}{2}\right)^2 + \left(\frac{0.4}{2}\right)^2 = 0.08\ \mathrm{m}^2$$

$$I_1 = 4(ma^2) = 4(0.200)0.08 \Rightarrow \boxed{I_1 = 6.40 \times 10^{-2}\ \mathrm{kg\,m}^2}$$



$$I = \sum_i m_i r_i^2$$

b) Bisecting two opposite sides of the square (an axis along line AB);

$$I_2 = 4m\left(\frac{0.4}{2}\right)^2 = 4(0.200)(0.2)^2 \Rightarrow \boxed{I_2 = 3.20 \times 10^{-2}\ \mathrm{kg\,m}^2}$$

c) Passes through the centres of the upper left and lower right spheres and through point 0.

$$I_3 = 2\left(ma^2\right) = 2(0.200)0.08 \Rightarrow \boxed{I_3 = 3.20 \times 10^{-2}\ \mathrm{kg\,m}^2}$$

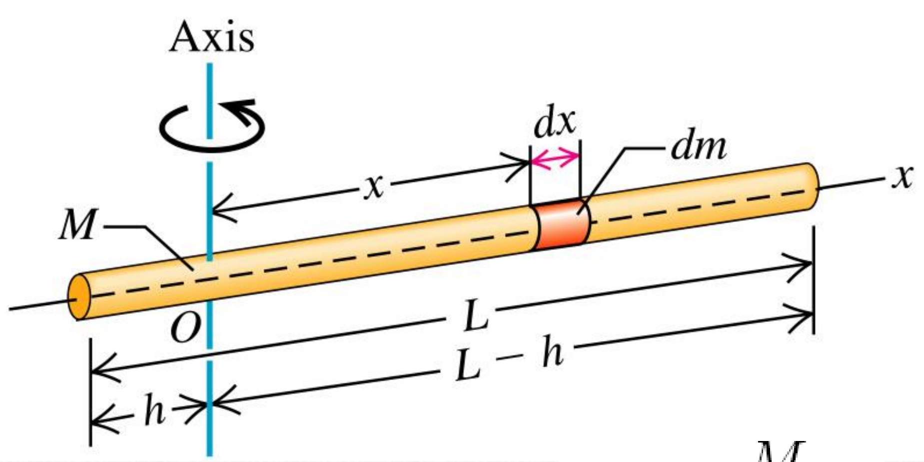
Calculating I: Uniform Rigid Rod

Calculate I of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod and passing through O, at an arbitrary distance h from one end.

$$I = \int r^2\, dm = \int_{-h}^{L-h} x^2 \frac{M}{L}\, dx = \frac{M}{3L}\left[x^3\right]_{-h}^{L-h}$$

$$I = \frac{M}{3L}\left[\left(L^3 - 3L^2h + 3Lh^2 - h^3\right) - \left(-h\right)^3\right]$$

$$\boxed{I = \frac{1}{3}M\left(L^2 - 3Lh + 3h^2\right)}$$



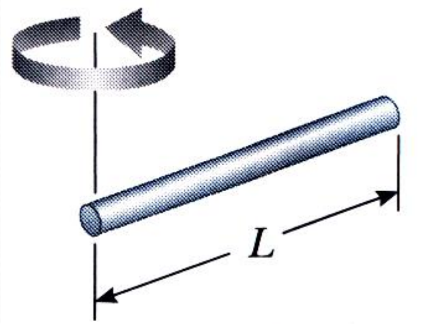
$$dm = \frac{M}{L}dx$$

Calculating *I*: Uniform Rigid Rod

$$\boxed{I = \frac{1}{3}M\left(L^2 - 3Lh + 3h^2\right)}$$

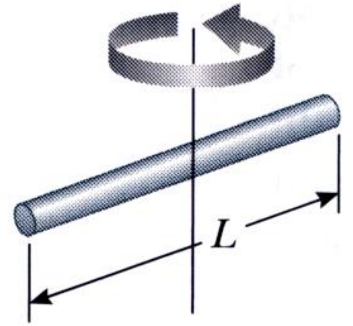
Special cases:

(Axis at one end of rod) ➡ $h = 0$



$$\boxed{I = \frac{1}{3}ML^2}$$

(Axis through CM of rod) ➡ $h = \frac{L}{2}$

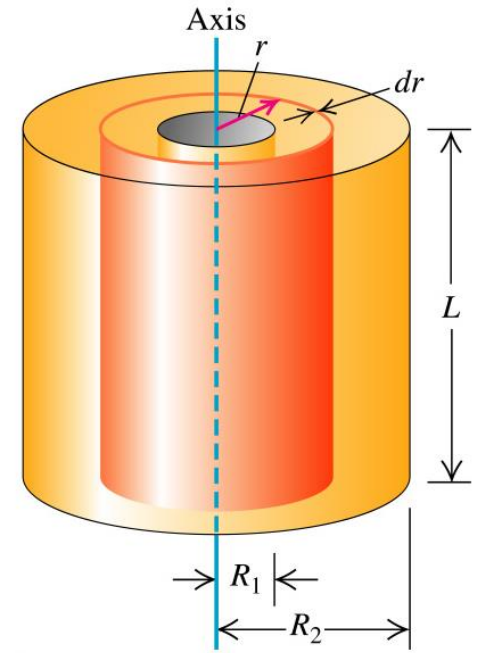


$$\boxed{I = \frac{1}{12}ML^2}$$

I is expected to be highest when the axis is at one end since the mass are now furthest away from the axis of rotation. Lowest is when axis is at the center.

Calculating I: Hollow Solid Cylinder

A hollow cylinder has an inner radius R_1, mass M, outer radius R_2 and length L. Calculate its moment of inertia about its central axis.



$$dm = \rho L(2\pi r)\,dr$$

$$I = \int r^2\,dm = \int_{R_1}^{R_2} r^2 \rho L(2\pi r)\,dr \Rightarrow I = 2\pi\rho L\left[\frac{r^4}{4}\right]_{R_1}^{R_2}$$

$$I = \frac{1}{2}\pi\rho L(R_2{}^4 - R_1{}^4) = \frac{1}{2}\pi\rho L(R_2{}^2 - R_1{}^2)(R_2{}^2 + R_1{}^2)$$

$$M = \rho L(\pi R_2{}^2 - \pi R_1{}^2)$$

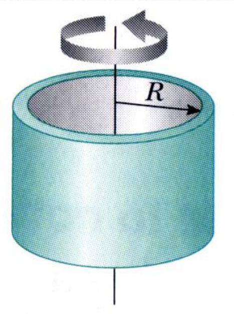
$$\boxed{I = \frac{1}{2}M(R_2{}^2 + R_1{}^2)}$$

Calculating I: Hollow Solid Cylinder

$$\boxed{I = \frac{1}{2} M \left({R_2}^2 + {R_1}^2 \right)}$$

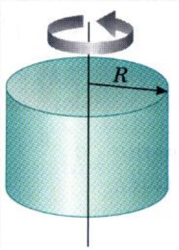
<u>Special cases:</u>

Hoop or thin cylindrical shell



$$\boxed{\begin{aligned} R_1 &= R_2 = R \\ I &= MR^2 \end{aligned}}$$

Disk or solid cylinder



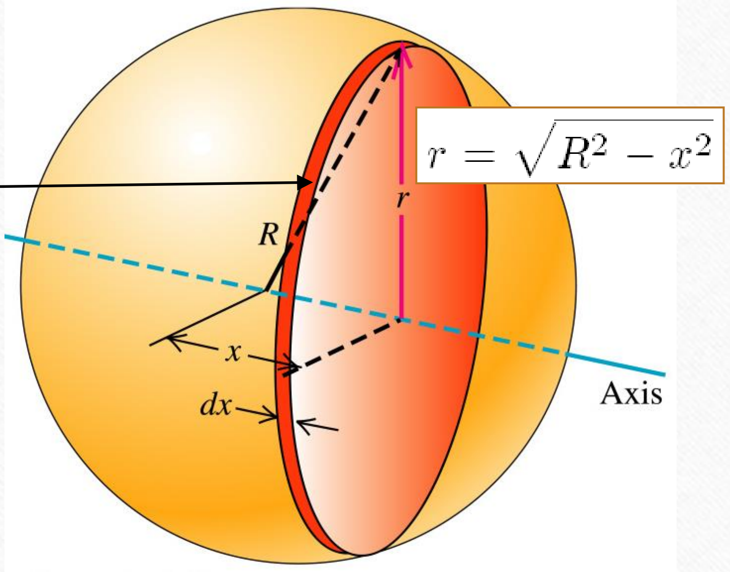
$$\boxed{\begin{aligned} R_1 &= 0 \\ I &= \frac{1}{2} MR^2 \end{aligned}}$$

I is expected to be highest for hoop or cylindrical shell since all the mass are furthest away from the axis of rotation.

Calculating *I*: Uniform Solid Sphere

A uniform solid sphere has a radius *R* and mass *M*.
Calculate its moment of inertia about any axis through its centre

$$dm = \rho\pi r^2\,dx = \rho\pi(R^2 - x^2)dx$$



We will make use of our result for disk:

$$dI = \frac{1}{2}dm\,r^2 = \frac{1}{2}\rho\pi(R^2 - x^2)^2 dx$$

$$I = \int dI = \frac{1}{2}\rho\pi\int_{-R}^{R}\left(R^4 - 2R^2x^2 + x^4\right)dx$$

$$= \frac{1}{2}\rho\pi\left[R^4x - \frac{2R^2x^3}{3} + \frac{x^5}{5}\right]_{-R}^{R} = \frac{8}{15}\rho\pi R^5$$

$$\boxed{I = \frac{2}{5}MR^2}$$

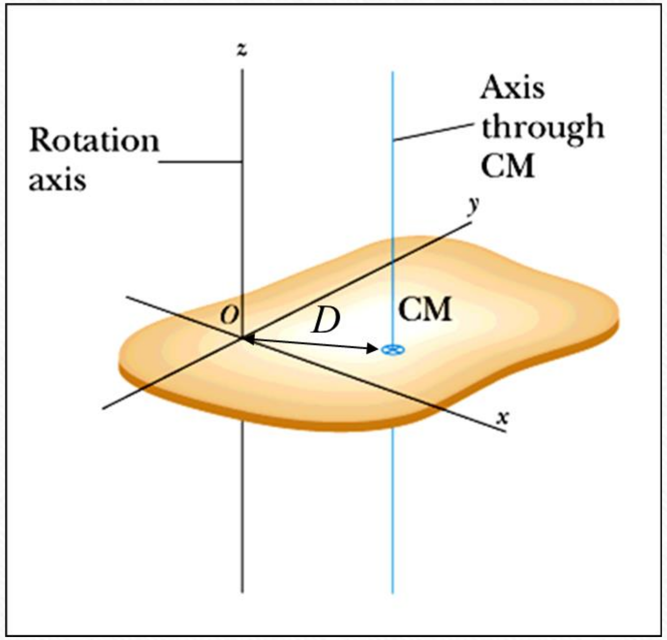
$$\rho = \frac{M}{\text{Volume}} = \frac{3M}{4\pi R^3}$$

Parallel Axis Theorem

- The moment of inertia about any axis parallel to and at distance D away from the axis that passes through the centre of mass is:

$$\boxed{I = I_{\mathrm{CM}} + MD^2}$$

- Theorem works for any solids and shapes.



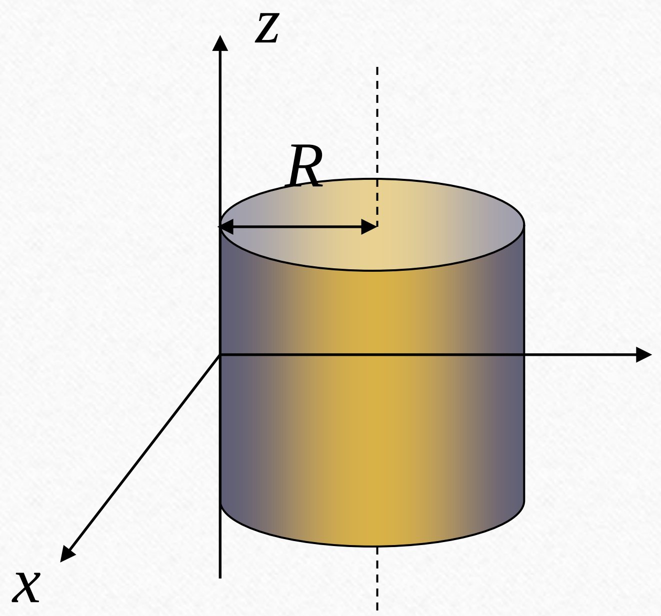
Applying Parallel Axis Theorem

- What is the moment of inertia of a cylinder about an axis on its edge?

$$I_{CM} = \frac{1}{2}MR^2 \qquad \text{(for a cylinder)}$$

$$I_z = I_{CM} + MD^2 = \frac{1}{2}MR^2 + MR^2$$

$$\boxed{I_z = \frac{3}{2}MR^2}$$



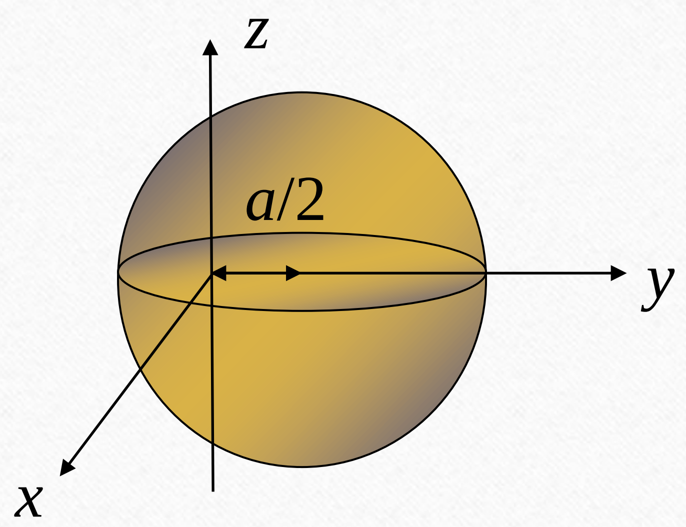
Applying Parallel Axis Theorem

- What is I of a sphere of radius a about the axis $a/2$ from its centre?

$$I_{CM} = \frac{2}{5}Ma^2 \qquad \text{(for a sphere)}$$

$$I_z = I_{CM} + MD^2 = \frac{2}{5}Ma^2 + M\left(\frac{a}{2}\right)^2$$

$$\boxed{I_z = \frac{13}{20}Ma^2}$$



Test Your Understanding

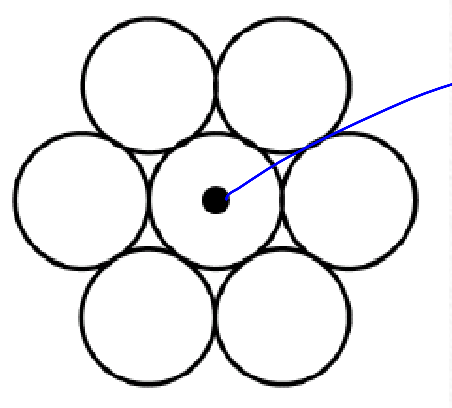
- Seven coins are arranged in a hexagonal, planar pattern so as to touch each neighbour, as shown in the figure below. Each coin is a uniform disk of mass *m* and radius *r*. What is the moment of inertia of the system of seven coins about an axis that passes through the centre of the central coin and is normal to the plane of the coins?

not in axis, thus need use parallel axis therom

$\frac{1}{2}mr^2 + m(2r)^2 = 4.5mr^2$

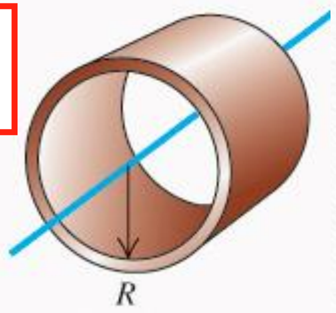
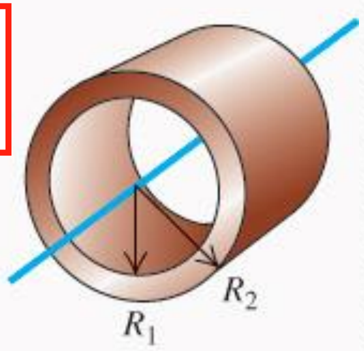
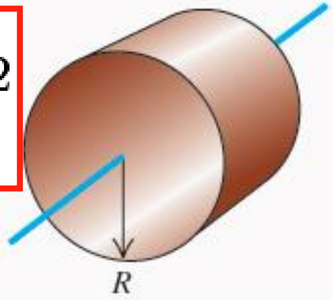
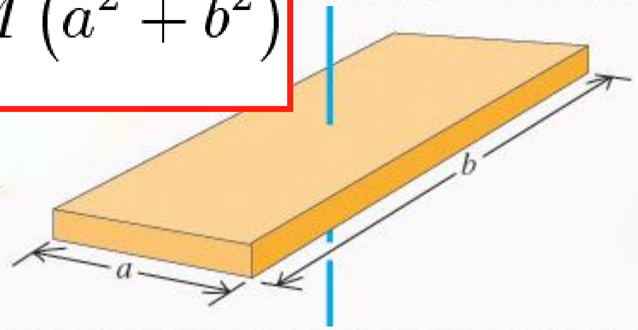
$6 \cdot (4.5)mr^2 + \frac{1}{2}mr^2$

A. 3.5 mr^2
B. 6.5 mr^2
C. 14.5 mr^2
D. 24.5 mr^2
E. 27.5 mr^2

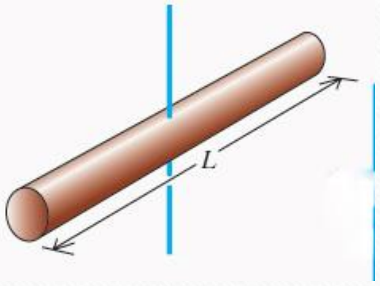
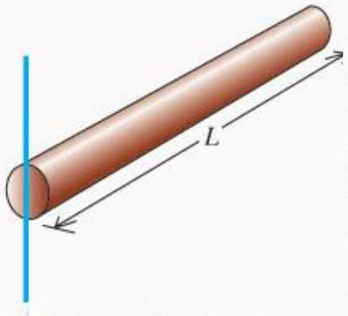
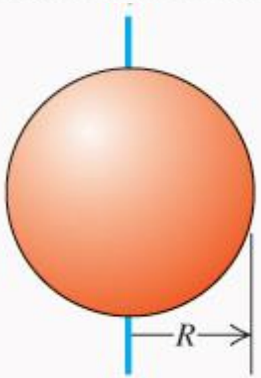
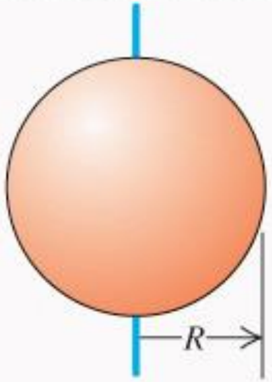


Axis ?

Moment of Inertia for different shapes

Hoop or thin cylindrical shell $\boxed{I_{\mathrm{CM}} = MR^2}$ 	Hollow cylinder $\boxed{I_{\mathrm{CM}} = \frac{1}{2}M\left(R_1^2 + R_2^2\right)}$ 
Solid cylinder or disk $\boxed{I_{\mathrm{CM}} = \frac{1}{2}MR^2}$ 	Rectangular plate $\boxed{I_{\mathrm{CM}} = \frac{1}{12}M\left(a^2 + b^2\right)}$ 

Moment of Inertia for different shapes

Long thin rod (about centre) $I_{\mathrm{CM}} = \frac{1}{12}ML^2$ 	Long thin rod (about end) $I = \frac{1}{3}ML^2$ 
Solid sphere $I_{\mathrm{CM}} = \frac{2}{5}MR^2$ 	Thin spherical shell $I_{\mathrm{CM}} = \frac{2}{3}MR^2$ 

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