Week 11 Tutorial

Question 1:

(a)

$$L(p) = P(8 \text{ heads} \mid p)$$

$$= \binom{10}{8} p^8 (1-p)^{10-8}$$

$$= \binom{10}{8} p^8 (1-p)^2 = 45 p^8 (1-p)^2 .$$

(b) $$L'(p) = \left[45 p^8 (1-p)^2\right]'$$

$$= 45 \left[p^8 (1-p)^2 \right]'$$

product rule
chain rule.

$$= 45 \left[8p^7 (1-p)^2 + p^8 \cdot 2(1-p) \cdot (-1) \right]$$

$$= 45 p^7 (1-p) \left[8(1-p) - 2p \right]$$

$$= 45 p^7 (1-p) \left[8 - 10p \right] = 0$$

$p = 0$ or $1 - p = 0$ or $8 - 10p = 0$

$p = 0$ or $p = 1$ or $p = \frac{4}{5} = 0.8$

(c) Find the MLE. $\hat{p}$

$L(0) = 0$; $L(1) = 0$;

$$L(0.8) = 45(0.8)^8(0.2)^2 \approx \boxed{0.302}.$$

Since 0.302 is greater than 0, $\hat{p} = 0.8$.

<u>Question 2</u> :

4 in one minutes.

2 in another minute

7 in another minute.

$$P(N=k) = \frac{e^{-\Lambda}\Lambda^k}{k!}$$

(a).

$$L(\Lambda) = P(N=4)\,P(N=2)\,P(N=7)$$

$$= \left(\frac{e^{-\Lambda}\Lambda^4}{4!}\right)\left(\frac{e^{-\Lambda}\Lambda^2}{2!}\right)\left(\frac{e^{-\Lambda}\Lambda^7}{7!}\right)$$

$$= \frac{e^{-3\Lambda}\Lambda^{13}}{(4!)(2!)(7!)} \leftarrow$$

(b). Find the critical points of L.

$$L'(\Lambda) = \frac{-3e^{-3\Lambda}\Lambda^{13} + e^{-3\Lambda} \cdot 13\Lambda^{12}}{(4!)(2!)(7!)}$$

$$= \frac{e^{-3\Lambda}\Lambda^{12}(-3\Lambda+13)}{(4!)(2!)(7!)} = 0$$

$$e^{-3\Lambda} \cdot \Lambda^{12}(-3\Lambda+13)=0$$

so $L'(\Lambda)=0$ only when $\Lambda=0$ or $\Lambda=\frac{13}{3}$

(3) MLE $\hat{\Lambda}$

$L(0)=0$

$$L\left(\frac{13}{3}\right)=\frac{e^{-3\cdot\frac{13}{3}}\cdot\left(\frac{13}{3}\right)^{13}}{(4!)(2!)(7!)}$$

$$=\frac{e^{-13}\,13^{13}}{3^{13}(4!)(2!)(7!)} \approx 0.0018$$

Since 0.0018 is greater than 0, $\hat{\Lambda}=13/3$

Question 3

(a) let T be the first time winning.

$$L(q)=P(T=2\mid q)$$

$$=(1-q)q=q-q^2.$$

(b) Find the critical points of L

$$L'(q)=(q-q^2)'=1-2q=0$$

$$q=\frac{1}{2}.$$

(c) Find the MLE $\hat{q}$

Since only one value of q maximize it.

(the endpoint $q=0$ yields 0)

we have $\hat{q} = (\frac{1}{2})$

$$L(\hat{q}) = \frac{1}{2} - (\frac{1}{2})^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

(d) Compute the upper limit P_h of the 95% confidence interval for q.

q_h:

$$P(\underbrace{T \geq 2.}\,|\, q_h) = 2.5\%$$

$$1 - q_h = 0.025$$

$$q_h = 0.975$$

(e) Compute the lower limit P_l of the 95% confidence interval. for q

q_l.

$$P(\underbrace{T \leq 2}\,|\, q_l) = 0.025$$

$$P(T=1\,|\, q_l) + P(T=2\,|\, q_l) = 0.025$$

$$q_l + (1 - q_l)\, q_l = 0.025$$

$$q_l + q_l - q_l^2 = 0.025$$

$$\boxed{-q_l^2 + 2q_l = 0.025}$$

$$\underbrace{-}_{=a} q_l^2 + \underbrace{2}_{=b} q_l \underbrace{- 0.025}_{=c} = 0 .$$

$$q_l = \frac{-2 \pm \sqrt{4 - 0.1}}{-2}$$

$$q_l \approx 0.0126 .$$

$[0.0126, 0.975] \rightarrow$ 95% confidence interval.