Lecture 4: Angles and distances

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Cross product

- Computation

$$[\vec{u}\ \vec{v}] = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix} \Rightarrow \vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

Cross product

- Computation

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- Geometric properties
  1. $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$
  2. $||\vec{u} \times \vec{v}|| = ||\vec{u}||||\vec{v}|| \sin\theta$ = area of parallelogram formed by $\vec{u}, \vec{v}$

Cross product

- Computation

$$[\vec{u}\ \vec{v}] = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix} \Rightarrow \vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

- Geometric properties
  1. $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$
  2. $||\vec{u} \times \vec{v}|| = ||\vec{u}||||\vec{v}|| \sin\theta =$ area of parallelogram formed by $\vec{u}, \vec{v}$
- Algebraic properties
  1. $\vec{u} \times \vec{v} = \vec{0} \Leftrightarrow \vec{u}$ and $\vec{v}$ are parallel
  2. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u}) \Rightarrow$ same length and opposite directions.

Intersections

We study intersections between lines and planes

1. Two planes
2. A line and a plane
3. Two lines

Intersection of 2 planes

To find the intersection of α, β, we solve for common points on both α, β

- No solution $\Rightarrow \alpha \parallel \beta$

- Infinitely many solutions

  same plane or *intersection = a line*

Intersection of line and plane by solving equations

To find intersection of l and β, we solve for common points on l and β.

1. No solution $\Rightarrow$ l is parallel to β

2. Infinitely many solutions $\Rightarrow$ l lines on β

$$l \cap \beta = l$$

3. 1 solution $\Rightarrow$ l and β intersect at a point

Example 1

Find the intersection of the line l and the plane β in following cases.

$\beta : 3x - 2y - z = -4$ and $l : (x, y, z) = (2, 1, 4) + t \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$

Let $P = (x, y, z)$ be a common point of l & β. Then

$$P \in l \Rightarrow \begin{cases} x = 2 + 2t \\ y = 1 + 3t \\ z = 4 - 2t \end{cases}$$

$$P \in \beta \Rightarrow 3x - 2y - z = -4$$

$$3(2+2t) - 2(1+3t) - (4-2t) = -4 \Rightarrow 2t = -4 \Rightarrow t = -2$$

$\therefore$ l & β intersect at a point $P = (-2, -5, 8)$.

Question

Assume l contains point P and has direction $\vec{d}$. Assume β has normal $\vec{n}_\beta$. When is $l \parallel \beta$? When is l on β? When does l intersect β?

$\vec{d} \perp \vec{n}_\beta$ & $P \notin \beta$

$\vec{d} \perp \vec{n}_\beta$ & $P \in \beta$

$\vec{d} \not\perp \vec{n}_\beta$

Example 2

Given the plane β and the line l:

$$\beta : 3x - 2y - z = -4 \quad \text{and} \quad l : (x, y, z) = (1, 2, 3) + t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Is the intersection of l and β **a point**, or **a line**, or **empty**?

l contains $P = (1,2,3)$ & has direction $\vec{d} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

β has normal $\vec{n} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$.

Since $\vec{n} \cdot \vec{d} = 3 - 4 + 1 = 0$, $\vec{n} \perp \vec{d} \Rightarrow$ l is on β, or l is parallel to β.

Since $3 \cdot 1 - 2 \cdot 2 - 3 = -4$, P is on β.

$\therefore$ l lies on the plane β.

Lines in $\mathbb{R}^3$

- In $\mathbb{R}^2$, two *different lines* either "parallel" or "intersect at a point".
- The situation in $\mathbb{R}^3$ is different .

parallel intersecting skew

Intersection of 2 lines in $\mathbb{R}^3$

Assume l_1, l_2 have directions $\vec{v}_1, \vec{v}_2$.

Solving for common points on l_1 and l_2, we have 3 cases

1 solution $\Rightarrow$ intersecting

No solution & $\vec{v}_1 \parallel \vec{v}_2$

$\Rightarrow$ parallel

No solution & $\vec{v}_1 \nparallel \vec{v}_2$

$\Rightarrow$ skew

Example 3

Find the relative position (parallel, intersecting, skew) and the intersection of

$$k: \begin{cases} x = 1 + t \\ y = 2 + 2t \\ z = 3 - t \end{cases} \quad \text{and} \quad m: \begin{cases} x = 7 + r \\ y = 13 + 4r \\ z = -3 - 3r \end{cases}$$

The lines k & m have directions $\vec{d}_k = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ & $\vec{d}_m = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$.

Since $\frac{1}{1} \neq \frac{2}{4}$, $\vec{d}_k \nparallel \vec{d}_m \Rightarrow k \nparallel m$.

Let $P = (x, y, z)$ be a common point of k & m. We have

$$P \in k \Rightarrow \begin{cases} x = 1 + t \\ y = 2 + 2t \\ z = 3 - t \end{cases} \qquad P \in m \Rightarrow \begin{cases} x = 7 + r \\ y = 13 + 4r \\ z = -3 - 3r \end{cases}$$

Example 3

We have

$$1+t = 7+r \Rightarrow t = r+6 \qquad \Rightarrow t = \frac{13}{2}$$

$$2+2t = 13+4r \Rightarrow 2+2(r+6) = 13+4r \Rightarrow r = \frac{1}{2}$$

$$3-t = -3-3r \Leftrightarrow 3-\frac{13}{2} = -3-\frac{3}{2} \Leftrightarrow -\frac{7}{2} = -\frac{9}{2},$$

impossible.

$\therefore$ h & m are skew.

Angle between intersecting lines

- Let l_1 and l_2 be two lines which intersect at P.
- The angle α between l_1 and l_2 is the smallest angle at P

$\alpha = 0^\circ$

- *Remark.*

$$0^\circ \leq \alpha \leq 90^\circ \text{ and } \alpha = 0^\circ \text{ if } l_1 \parallel l_2$$

Angle between skew lines

1. Pick a point P and draw lines l_1', l_2' parallel to l_1, l_2
2. The angle between l_1, l_2 is equal to the angle between l_1', l_2'

Angle b.w. lines vs angle b.w. direction vectors

- $\vec{d}_1, \vec{d}_2$ = direction vectors of l_1, l_2
- Put $\theta = \angle(\vec{d}_1, \vec{d}_2)$ and $\alpha = \angle(l_1, l_2)$

$$\cos\theta = \frac{\vec{d}_1 \cdot \vec{d}_2}{\|\vec{d}_1\| \, \|\vec{d}_2\|}$$

Case 1: $\theta \le 90^\circ$

$\alpha = \theta$

Case 2: $\theta > 90^\circ$

$\alpha + \theta = 180^\circ \Rightarrow \alpha = 180^\circ - \theta$

- The angle α between l_1, l_2 is

$$\alpha = \min(\theta, 180^\circ - \theta) = \begin{cases} \theta & \text{if } 0 \le \theta \le 90^\circ \\ 180^\circ - \theta & \text{if } \theta > 90^\circ \end{cases}$$

Observation

- Observation

$$\alpha = \theta \quad \Rightarrow \quad \cos\alpha = \cos\theta = \frac{\vec{d}_1 \cdot \vec{d}_2}{||\vec{d}_1||||\vec{d}_2||}$$

$$\alpha = 180^o - \theta \quad \Rightarrow \quad \cos\alpha = -\cos\theta = -\frac{\vec{d}_1 \cdot \vec{d}_2}{||\vec{d}_1||||\vec{d}_2||}$$

Observation

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$$\alpha = \theta \quad \Rightarrow \quad \cos\alpha = \cos\theta = \frac{\vec{d_1} \cdot \vec{d_2}}{||\vec{d_1}||||\vec{d_2}||}$$

$$\alpha = 180^o - \theta \quad \Rightarrow \quad \cos\alpha = -\cos\theta = -\frac{\vec{d_1} \cdot \vec{d_2}}{||\vec{d_1}||||\vec{d_2}||}$$

- In any case $\cos\alpha \geq 0$ (note that $0^\circ \leq \alpha \leq 90^\circ$)

$$\cos\alpha = \frac{|\vec{d_1} \cdot \vec{d_2}|}{||\vec{d_1}||||\vec{d_2}||}$$

Formula for angle between 2 lines

Theorem 1

Assume l_1, l_2 have direction vectors $\vec{d_1}, \vec{d_2}$. Put $\theta = \angle(\vec{d_1}, \vec{d_2})$. Then

(a) The angle α between l_1 and l_2 is

$$\alpha = \min(\theta, 180^o - \theta)$$

(b) α can be computed by

$$\cos a = \frac{|\vec{d_1} \cdot \vec{d_2}|}{||\vec{d_1}||||\vec{d_2}||}$$

Example 4

Find the angle between any two of the following lines

(a) In $\mathbb{R}^2$:

$$l_1 : 2x + y + 1 = 0, \quad l_2 : (x, y) = (1, 0) + t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

l_1 & l_2 have direction vectors

$$\vec{d}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \text{ \& } \vec{d}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\left(ax + by + c = 0 \Downarrow \vec{d} = \begin{pmatrix} -b \\ a \end{pmatrix} \right)$$

The angle α b.w. l_1 & l_2 is

$$\cos\alpha = \frac{|\vec{d}_1 \cdot \vec{d}_2|}{\|\vec{d}_1\| \|\vec{d}_2\|} = \frac{|-5|}{\sqrt{5}\sqrt{5}} = 1 \Rightarrow \alpha = 0^\circ$$

(Note : $l_1 \parallel l_2$)

Example 4

(b) In $\mathbb{R}^3$:

$$l_1 : (x, y, z) = (1, 0, 1) + t \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad l_2 : \begin{cases} x = 1 \\ y = 2 + 5t \\ z = 3 + 6t \end{cases}$$

l_1 & l_2 have directions $\vec{d}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ & $\vec{d}_2 = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$.

The angle α between l_1 & l_2 is

$$\cos\alpha = \frac{|\vec{d}_1 \cdot \vec{d}_2|}{\|\vec{d}_1\| \|\vec{d}_2\|} = \frac{|-5|}{\sqrt{5}\sqrt{61}} \Rightarrow \alpha \cong 73.36^\circ.$$

Angle between a line and a plane

- Let l be a line and let α be a plane.
- The angle a between l and α is the angle between l and *its orthogonal projection* l' onto α.

- *Remark*

$$0^\circ \leq a \leq 90^\circ \text{ and } a = 0^\circ \text{ if } l \parallel \alpha$$

Angle between a line and a plane

Case 1: $\theta \le 90^\circ$

$a + \theta = 90^\circ$

$a = 90^\circ - \theta$

Case 2: $\theta > 90^\circ$

$a + 90^\circ = \theta$

$a = \theta - 90^\circ$

$\Downarrow$

$a = |\theta - 90^\circ|$

- Put $\vec{d}$ = direction vector of l, $\vec{n}$ = normal vector of α, $\theta = \angle(\vec{d}, \vec{n})$

Angle between a line and a plane

- Put $\vec{d}$ = direction vector of l, $\vec{n}$ = normal vector of α, $\theta = \angle(\vec{d}, \vec{n})$
- The angle between l and α is $a = \begin{cases} 90^\circ - \theta \text{ if } \theta \leq 90^0 \\ \theta - 90^\circ \text{ if } \theta > 90^\circ \end{cases}$. So

$$a = \left|\theta - 90^0\right|$$

Angle between a line and a plane

Theorem 2

Let l be a line with direction $\vec{d}$ and let α be a plane with normal $\vec{n}$. Put $\theta = \angle(\vec{n}, \vec{d})$. The angle $a \in [0^0, 90^0]$ between l and α is

$$a = |\theta - 90^o|$$

Example 5

Find the angle a between l and α in the following case

$$l : \begin{cases} x = 1 + 2t \\ y = 3 - 5t \\ z = 2 + 10t \end{cases} \quad , \quad \alpha : (x, y, z) = (1, 2, -1) + s \underbrace{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}_{\vec{u}} + t \underbrace{\begin{bmatrix} 3 \\ -5 \\ -1 \end{bmatrix}}_{\vec{v}}$$

l has direction $\vec{d} = \begin{pmatrix} 2 \\ -5 \\ 10 \end{pmatrix}$.

A normal vector of α is $\vec{n} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ 7 \\ -5 \end{pmatrix}$.

The angle between $\vec{d}$ & $\vec{n}$ is

$$\cos\theta = \frac{\vec{d} \cdot \vec{n}}{\|\vec{d}\| \|\vec{n}\|} = \frac{-75}{\sqrt{129}\sqrt{174}} \cong 120^\circ$$

The angle between l & α is $a = |\theta - 90^\circ| = 30^\circ$.

Angle between 2 planes

- Let α_1, α_2 be two planes which intersect at the line l.
  Let β be any plane which is *perpendicular* to l.

$m_1 = \beta \cap \alpha_1 \qquad m_2 = \beta \cap \alpha_2$

$\angle(m_1, m_2) = \angle(\alpha_1, \alpha_2)$

Angle between 2 planes

- Let α_1, α_2 be two planes which intersect at the line l.
  Let β be any plane which is *perpendicular* to l.

- $m_1, m_2 =$ intersections between β and α_1, α_1.
  The angle θ between α_1 and α_2 is the angle between m_1 and m_2.

Angle between 2 planes

- $\vec{n}_1, \vec{n}_2 =$ normal vectors of α_1, α_2. Put $\theta = \angle(\vec{n}_1, \vec{n}_2)$

$\alpha = 180^0 - \theta$
if $\theta > 90^0$

$\alpha = \theta$
if $\theta \leq 90^0$

- The angle between α_1, α_2 is

$$a = \min\left(\theta, 180^0 - \theta\right) = \begin{cases} \theta & \text{if } \theta \leq 90^0 \\ 180^0 - \theta & \text{if } \theta > 90^0 \end{cases}$$

Angle between 2 planes

- $\vec{n}_1, \vec{n}_2 =$ normal vectors of α_1, α_2. Put $\theta = \angle(\vec{n}_1, \vec{n}_2)$

- The angle between α_1, α_2 is

$$\alpha = \min\left(\theta, 180^0 - \theta\right)$$

- a can be computed by $\cos\alpha = \dfrac{|\vec{n}_1 \cdot \vec{n}_2|}{||\vec{n}_1||||\vec{n}_2||}$

Angle between 2 planes

Theorem 3

Let α_1, α_2 be planes with normal vectors $\vec{n}_1, \vec{n}_2$. Put $\theta = \angle(\vec{n}_1, \vec{n}_2)$. Then

(a) The angle between α_1 and α_2 is

$$a = \min\left(\theta, 180^o - \theta\right)$$

(b) a can be computed by

$$\cos a = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{||\vec{n}_1||||\vec{n}_2||}$$

Example 6

Find the angle between 2 planes

$$\alpha_1 : x + 3y - z = 1 \text{ and } \alpha_2 : (x, y, z) = (1, 0, -1) + s \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

α_1 has a normal vector $\vec{n}_1 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

α_2 has a normal vector $\vec{n}_2 = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \\ -8 \end{pmatrix}$

The angle between α_1 & α_2 is

$$\cos a = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \, \|\vec{n}_2\|} = \frac{26}{\sqrt{11}\sqrt{154}} \Rightarrow a \cong 50.82^\circ.$$

Distance

- The **distance** between any two geometrical objects (points, lines, planes) is the *shortest direct path* which connect one object to the other.
- Question: What if the two objects have a common point?

distance = 0

Zero distance

In any of the following cases, the distance is 0

- Distance from a point to a line containing it
- Distance from a point to a plane containing it
- Distance between two intersecting lines
- Distance between two intersection planes
- Distance between a line and a plane intersecting it

Distance in $\mathbb{R}^2$

In $\mathbb{R}^2$, there are 3 distances all of which we knew how to compute.

1. The distance between $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

2. The distance from $P = (x_0, y_0)$ to $l : ax + by + c = 0$ is

$$d(P, l) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

3. The distance between 2 different lines l_1 and l_2

- l_1 and l_2 intersect $\Rightarrow d(l_1, l_2) = 0$
- l_1 and l_2 are parallel

$d(l_1, l_2) = d(P, l_2)$ with $P =$ any point on l_1.

Geometric objects and distances in $\mathbb{R}^3$

We would like to measure the distances between

Points, Lines, Planes

Point-point and point-plane distances

- Points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ is

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- Point $P = (x_0, y_0, z_0)$ and plane $\alpha : ax + by + cz + d = 0$

$$d(P, \alpha) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Distance between 2 planes

Consider 2 planes α and β. There are 2 cases

- α and β intersect

$$d(\alpha, \beta) = 0$$

- α and β are parallel

$$d(\alpha, \beta) = d(P, \beta) \text{ with } P = \text{any point on } \alpha.$$

When α and β are parallel?

- Assume

$$\alpha : \text{through } P \text{ and normal vector } \vec{n}_\alpha$$

$$\beta : \text{through } Q \text{ and normal vector } \vec{n}_\beta$$

When α and β are parallel?

- Assume

$$\alpha : \text{through } P \text{ and normal vector } \vec{n}_\alpha$$

$$\beta : \text{through } Q \text{ and normal vector } \vec{n}_\beta$$

- α and β are parallel if and only if

$$\vec{n}_\alpha \parallel \vec{n}_\beta$$

$d(\alpha, \beta) = d(P, \beta)$, where P = a point on α.

Example 7

Find the distance between two planes α and β in following cases.

(a) $\alpha : 3x - 2y - z = -4$ and $\beta : 3x + 2y + z = 4$

α & β have normal vectors $\vec{n}_\alpha = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$ & $\vec{n}_\beta = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

Since $\frac{3}{3} \neq \frac{-2}{2}$, $\vec{n}_\alpha \nparallel \vec{n}_\beta \Rightarrow \alpha \nparallel \beta$. Thus

$d(\alpha, \beta) = 0$

Example 7

(b) $\alpha : (x, y, z) = (1, 2, 3) + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$ and $\beta : x - z = 1$.

$\beta : x - z - 1 = 0$

β has normal vector $\vec{n}_\beta = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

α has a normal vector $\vec{n}_\alpha = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}$.

Since $\vec{n}_\alpha = (-3)\vec{n}_\beta$, $\vec{n}_\alpha \parallel \vec{n}_\beta \Rightarrow \alpha \parallel \beta$.

Let $P = (1, 2, 3)$ be a point on α. We have

$$d(\alpha, \beta) = d(P, \beta) = \frac{|1 - 3 - 1|}{\sqrt{1^2 + 0^2 + (-1)^2}} = \frac{3}{\sqrt{2}}.$$

Summary on point-point, point-plane and plane-plane distances

- The distance between $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ is

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- The distance from $P_0 = (x_0, y_0, z_0)$ to $\mathcal{P} : ax + by + cz + d = 0$ is

$$d(P_0, \mathcal{P}) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

- Consider 2 planes α and β
  1. α and β intersect $\Rightarrow d(\alpha, \beta) = 0$
  2. α and β are parallel

$$d(\alpha, \beta) = d(P, \beta) \text{ for any point } P \text{ on } \alpha$$

Distance between a line and a plane

Consider the line l and the plane α

1. l and α intersect:

$$d(l, \alpha) = 0$$

2. l parallel to α:

$$d(l, \alpha) = d(P, \alpha) \text{ for any point } P \text{ on } l.$$

Question

When are l and β parallel?

$$l \parallel \beta \Leftrightarrow \vec{d} \perp \vec{n}_\beta \Leftrightarrow \vec{d} \cdot \vec{n}_\beta = 0$$

Distance between a line and a plane

Theorem 4

Let $l : (x, y, z) = (x_0, y_0, z_0) + t\vec{d}$ be a line and let

$\alpha : ax + by + cz + d = 0$ be a plane. Let $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a normal of α.

Distance between a line and a plane

Theorem 4

Let $l : (x, y, z) = (x_0, y_0, z_0) + t\vec{d}$ be a line and let

P

$\alpha : ax + by + cz + d = 0$ be a plane. Let $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a normal of α.

(a) If $\vec{n} \cdot \vec{d} = 0$ (that is $l \parallel \alpha$), then

$$d(l, \alpha) = \underbrace{\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}}_{d(l,\alpha) = d(P,\alpha)}$$

Distance between a line and a plane

Theorem 4

Let $l : (x, y, z) = (x_0, y_0, z_0) + t\vec{d}$ be a line and let $\alpha : ax + by + cz + d = 0$ be a plane. Let $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a normal of α.

(a) If $\vec{n} \cdot \vec{d} = 0$ (that is $l \parallel \alpha$), then

$$d(l, \alpha) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

(b) If $\vec{n} \cdot \vec{d} \neq 0$ (that is $l \nparallel \alpha$), then

$$d(l, \alpha) = 0.$$

Example 8

Find the distance between the line l and the plane α in the following cases

(a) $l : (x, y, z) = \underbrace{(2, 1, 4)}_{P} + t \underbrace{\begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}}_{\vec{d}}$ and $\alpha : 3x - 2y - z = -4.$

$\vec{n} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$

We have $\vec{d} \cdot \vec{n} = 2 \neq 0 \Rightarrow l \nparallel \alpha$. Thus

$d(l, \alpha) = 0.$

$\alpha : 3x - 2y - z + 4 = 0$

(b) $l : (x, y, z) = \underbrace{(1, 2, 3)}_{P} + t \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}_{\vec{d}}$ and $\alpha : \underbrace{3x - 2y - z = -4.}_{\vec{n} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}}$

Since $\vec{d} \cdot \vec{n} = 0$, we have $\vec{d} \perp \vec{n}$, that is, $l \parallel \alpha$.

We have

$$d(l, \alpha) = d(P, \alpha) = \frac{|3 \cdot 1 - 2 \cdot 2 - 3 + 4|}{\sqrt{3^2 + (-2)^2 + (-1)^2}} = 0.$$

Remark: l is on α.

Remaining distances

1. Distance between a point to a line
2. Distance between lines

Distance from a point to a line

- Point P, line l : through Q and direction $\vec{v}$

$$\|\vec{QP} \times \vec{v}\| = \text{Area of parallelogram} = \|\vec{v}\|\, h$$

$$h = d(P, \ell) = \frac{\|\vec{QP} \times \vec{v}\|}{\|\vec{v}\|}$$

Distance from a point to a line

- Point P, line l : through Q and direction $\vec{v}$

- $d(P, l) =$ height h of parallelogram

$$d(P,l) = \frac{\text{Area of parallelogram}}{||\vec{v}||} = \frac{||\overrightarrow{QP} \times \vec{v}||}{||\vec{v}||}$$

Distance from a point to a line in $\mathbb{R}^3$

Theorem 5

The distance from the point P to the line $l : (x, y) = Q + t\vec{v}$ is

$$d(P, l) = \frac{||\overrightarrow{QP} \times \vec{v}||}{||\vec{v}||}$$

Remark. To use this formula, we need

1. A point Q on l and
2. A direction vector $\vec{v}$ of l

Example 9

Find the distance from $P=(1,1,1)$ to the line $l:\begin{cases} x = 1+5t \\ y = -6+7t \\ z = 2+2t \end{cases}$

l contains $Q=(1,-6,2)$ and has direction $\vec{d} = \begin{pmatrix} 5 \\ 7 \\ 2 \end{pmatrix}$.

The distance from P to l is $d(P,l) = \dfrac{\|\vec{QP} \times \vec{d}\|}{\|\vec{d}\|}$.

We have

$$\vec{QP} \times \vec{d} = \begin{pmatrix} 0 \\ 7 \\ -1 \end{pmatrix} \times \begin{pmatrix} 5 \\ 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 21 \\ -5 \\ -35 \end{pmatrix}.$$

Hence

$$d(P,l) = \frac{\sqrt{21^2+(-5)^2+(-35)^2}}{\sqrt{5^2+7^2+2^2}} \cong 4.66.$$

Distance between 2 lines in $\mathbb{R}^3$

l_1 : through Q_1 and direction $\vec{d_1}$, l_2 : through Q_2 and direction $\vec{d_2}$

1. l_1 and l_2 intersect $\Rightarrow d(l_1, l_2) = 0$

Distance between 2 lines in $\mathbb{R}^3$

l_1 : through Q_1 and direction $\vec{d_1}$, l_2 : through Q_2 and direction $\vec{d_2}$

1. l_1 and l_2 intersect $\Rightarrow d(l_1, l_2) = 0$
2. $l_1 \parallel l_2 \Rightarrow d(l_1, l_2) = d(Q_1, l_2)$

Distance between skew lines

l_1 : through Q_1 and direction $\vec{d_1}$,

l_2 : through Q_2 and direction $\vec{d_2}$

2. l_1 and l_2 are skew

contain Q_2

α ← direction $\vec{d_2}$

$\alpha \parallel l_1 \Rightarrow$ direction $\vec{d_1}$

- $\alpha =$ plane containing l_2 and parallel to l_1

$$\alpha : (x, y, z) = Q_2 + s\vec{d_1} + t\vec{d_2}$$

Distance between skew lines

l_1 : through Q_1 and direction $\vec{d}_1$,

l_2 : through Q_2 and direction $\vec{d}_2$

2. l_1 and l_2 are skew

- $\alpha =$ plane containing l_2 and parallel to l_1

$$\alpha : (x, y, z) = Q_2 + s\vec{d}_1 + t\vec{d}_2$$

- $d(l_1, l_2) = d(l_1, \alpha) = d(Q_1, \alpha)$

Summary on distance between lines

l_1 : through Q_1 and direction $\vec{d_1}$,

l_2 : through Q_2 and direction $\vec{d_2}$

- Case 1: $\vec{d_1} \parallel \vec{d_2} \Rightarrow l_1 \parallel l_2$. So

$$d(l_1, l_2) = d(Q_1, l_2) = \frac{||\overrightarrow{Q_2Q_1} \times \vec{d_2}||}{||\vec{d_2}||}$$

Summary on distance between lines

l_1 : through Q_1 and direction $\vec{d_1}$,
l_2 : through Q_2 and direction $\vec{d_2}$

- Case 2: $\vec{d_1} \nparallel \vec{d_2} \Rightarrow l_1$ and l_2 are not parallel
  1. Find the plane α through ℓ_2 and parallel to ℓ_1

     α goes through Q_2 and has normal $\vec{n} = \vec{d_1} \times \vec{d_2}$
  2. $d(l_1, l_2) = d(Q_1, \alpha)$

Example 10

Find the distance between

$$l_1 : \begin{cases} x = 2 - 4t \\ y = 3 + 4t \\ z = 3t \end{cases} \quad \text{and} \quad l_2 : (x, y, z) = \underbrace{(1, 1, 2)}_{Q_2} + t \begin{bmatrix} 1 \\ -1 \\ -3/4 \end{bmatrix}$$

l_1 & l_2 has direction vectors $\vec{d_1} = \begin{pmatrix} -4 \\ 4 \\ 3 \end{pmatrix} \parallel \vec{d_2} = \begin{pmatrix} 1 \\ -1 \\ -3/4 \end{pmatrix}$.

So

$$d(l_1, l_2) = d(Q_1, l_2) \text{ with } Q_1 = (2, 3, 0)$$

$$d(Q_1, l_2) = \frac{\|\vec{Q_2Q_1} \times \vec{d_2}\|}{\|\vec{d_2}\|} \text{ with } \vec{Q_2Q_1} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$$= \dots$$

Distance between 2 lines in $\mathbb{R}^3$ (another way)

Theorem 6

Let $l_1 : (x, y, z) = Q_1 + t\vec{d_1}$ and $l_2 : (x, y, z) = Q_2 + t\vec{d_2}$ be 2 lines.

(a) If l_1 and l_2 are parallel ($\vec{d_1} \parallel \vec{d_2}$), then

$$d(l_1, l_2) = \frac{||\overrightarrow{Q_2Q_1} \times \vec{d_1}||}{||\vec{d_1}||}$$

(b) If l_1 and l_2 are not parallel (intersect, or skew), then

$$d(l_1, l_2) = ||\operatorname{proj}_{\vec{d_1} \times \vec{d_2}}(\overrightarrow{Q_1Q_2})||$$

Example 11

~~Using Theorem 6~~, find the distance between

$$l_1 : \begin{cases} x = 2 - 4t \\ y = 3 + 4t \\ z = 3 \end{cases} \quad \text{and} \quad l_2 : (x, y, z) = (1, 1, 2) + t \begin{bmatrix} 1 \\ -1 \\ -3/4 \end{bmatrix}$$

l_1 contains $Q_1 = (2,3,3)$ & direction $\begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix} \parallel \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \vec{d}_1$

l_2 contains $Q_2 = (1,1,2)$ & direction $\begin{pmatrix} 1 \\ -1 \\ -3/4 \end{pmatrix} \parallel \begin{pmatrix} 4 \\ -4 \\ -3 \end{pmatrix} = \vec{d}_2$

Since $\frac{-1}{4} = \frac{1}{-4} \neq \frac{0}{-3}$, $\vec{d}_1 \nparallel \vec{d}_2 \Rightarrow l_1 \nparallel l_2$.

(1) Find plane α through l_2 & parallel to l_1

α contains $Q_2 = (1,1,2)$ & has normal $\vec{n} = \vec{d}_1 \times \vec{d}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 4 \\ -4 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix}$

Example 11

~~Using Theorem 6~~, find the distance between

$$l_1 : \begin{cases} x = 2 - 4t \\ y = 3 + 4t \\ z = 3 \end{cases} \quad \text{and} \quad l_2 : (x, y, z) = (1, 1, 2) + t \begin{bmatrix} 1 \\ -1 \\ -3/4 \end{bmatrix}$$

The general equation for α is

$$(-3)(x-1) + (-3)(y-1) = 0 \Leftrightarrow x + y - 2 = 0$$

(2) We obtain

$$d(l_1, l_2) = d(l_1, \alpha) = d(A_1, \alpha) = \frac{|2+3-2|}{\sqrt{1^2+1^2}} = \frac{3}{\sqrt{2}}.$$

Example 11

Using Theorem 6, find the distance between

$$l_1 : \begin{cases} x = 2 - 4t \\ y = 3 + 4t \\ z = 3 \end{cases} \quad \text{and} \quad l_2 : (x, y, z) = (1, 1, 2) + t \begin{bmatrix} 1 \\ -1 \\ -3/4 \end{bmatrix}$$

$$d(l_1, l_2) = \| \text{proj}_{\vec{d}_1 \times \vec{d}_2} (\overrightarrow{Q_1 Q_2}) \|$$

l_1 contains $Q_1 = (2,3,3)$ & direction $\begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix} \parallel \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \vec{d}_1$

l_2 contains $Q_2 = (1,1,2)$ & direction $\begin{pmatrix} 1 \\ -1 \\ -3/4 \end{pmatrix} \parallel \begin{pmatrix} 4 \\ -4 \\ -3 \end{pmatrix} = \vec{d}_2$

Since $\frac{-1}{4} = \frac{1}{-4} \neq \frac{0}{-3}$, $\vec{d}_1 \nparallel \vec{d}_2 \Rightarrow l_1 \nparallel l_2$.

We have

$$\vec{d} = \vec{d}_1 \times \vec{d}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 4 \\ -4 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix}$$

Example 11

Using Theorem 6, find the distance between

$$l_1 : \begin{cases} x = 2 - 4t \\ y = 3 + 4t \\ z = 3 \end{cases} \quad \text{and} \quad l_2 : (x, y, z) = (1, 1, 2) + t \begin{bmatrix} 1 \\ -1 \\ -3/4 \end{bmatrix}$$

Note that $\overrightarrow{Q_1Q_2} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$ and $\vec{d} = \begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix}$. So

$$\text{proj}_{\vec{d}}(\overrightarrow{Q_1Q_2}) = \frac{\overrightarrow{Q_1Q_2} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \vec{d} = \frac{9}{18} \vec{d} = \begin{pmatrix} -3/2 \\ -3/2 \\ 0 \end{pmatrix}$$

Thus

$$d(l_1, l_2) = \| \text{proj}_{\vec{d}}(\overrightarrow{Q_1Q_2}) \| = \sqrt{\left(-\frac{3}{2}\right)^2 + \left(-\frac{3}{2}\right)^2} = \frac{3}{\sqrt{2}}.$$

Summary on distance between 2 lines

Assume $l_1 : (x, y, z) = Q_1 + t\vec{d_1}$ and $l_2 : (x, y, z) = Q_2 + t\vec{d_2}$

- Method 1

1. $\vec{d_1} \parallel \vec{d_2} \Rightarrow d(l_1, l_2) = d(Q_1, l_2)$
2. $\vec{d_1} \nparallel \vec{d_2}$

   $\alpha =$ plane containing l_2 & parallel to $l_1 \Rightarrow d(l_1, l_2) = d(Q_1, \alpha)$

Summary on distance between 2 lines

Assume $l_1 : (x, y, z) = Q_1 + t\vec{d_1}$ and $l_2 : (x, y, z) = Q_2 + t\vec{d_2}$

$(d(Q_2, \ell_1) = d(Q_1, \ell_2))$

- Method 2 (use Theorem 6)

$$d(l_1, l_2) = \begin{cases} \dfrac{||\overrightarrow{Q_2Q_1} \times \vec{d_1}||}{||\vec{d_1}||} & \text{if } \vec{d_1} \parallel \vec{d_2} \\ ||\text{proj}_{\vec{d_1} \times \vec{d_2}}(\overrightarrow{Q_1Q_2})|| & \text{if } \vec{d_1} \nparallel \vec{d_2} \end{cases}$$