scientific calculator

w/ definite integral calculation ✗

$\boxed{\int_{\square}^{\square} dx}$ ✗ $\boxed{\int dx}$ ✗

Trigonometric Integrals Part 2

Method of Partial Fractions Part 1

↳ tedious Integral not tedious

precalculation

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AY 23/24 Trimester 1

Quiz : Concept > Calculation

Table of contents

Integration by parts, sine/cosine integrals

- We have learnt how to integrate the product of functions; i.e. **integration by parts**:

$$\int u\,dv = uv - \int v\,du.$$

- We have learned how to choose u using the **LIATE prioritization tool**, based on the difficulty of integration.
- We have also learned how to integrate $\sin^m x \cos^n x$:
  - m is odd: take out one copy of $\sin x$, convert rest to $\cos x$ using $\sin^2 x = 1 - \cos^2 x$, sub $u = \cos x$.
  - n is odd: take out one copy of $\cos x$, convert rest to $\sin x$ using $\cos^2 x = 1 - \sin^2 x$, sub $u = \sin x$.
  - Both m and n even: use double angle formulae:

$$\sin^2 x = \frac{1-\cos(2x)}{2} \quad \text{and} \quad \cos^2 x = \frac{1+\cos(2x)}{2}.$$

odd even

Example 1

even

$$\sec^4 x = (\sec^2 x)^2 = (1+\tan^2 x)^2$$

Evaluate $\int \sec^4 x \, dx$.

$$= \int \sec^2 x \sec^2 x \, dx$$

(Put 2 copies of secx to one side)

$$= \int (1+\tan^2 x) \underline{\sec^2 x \, dx}$$

(Convert the rest of secx to tanx)

$$= \int 1 + u^2 \, du$$

$u = \tan x$

$du = \sec^2 x \, dx$

$$= u + \frac{u^3}{3} + C = \tan x + \frac{\tan^3 x}{3} + C$$

Example 2

odd at least one copy of Secx

Evaluate $\int \tan^3 x \sec x \, dx$.

$$= \int \tan^2 x \cdot (\sec x \tan x) \, dx$$

(Take out one copy of secx tanx)

$$= \int (\sec^2 x - 1) \cdot (\sec x \tan x) \, dx$$

$u = \sec x$

$du = \sec x \tan x \, dx$

$$= \int u^2 - 1 \, du$$

$$= \frac{u^3}{3} - u + C = \frac{\sec^3 x}{3} - \sec x + C$$

Example 3

$\int \sec^3 x\,dx$

Galaxy brain move

★ base case

Evaluate $\displaystyle\int \sec x\,dx$. $\displaystyle = \int \sec x \cdot \boxed{\frac{\sec x + \tan x}{\sec x + \tan x}}\,dx$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x}\,dx$$

$u = \sec x + \tan x$

$du = \sec x \tan x + \sec^2 x\,dx$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C = \boxed{\ln|\sec x + \tan x| + C}$$

Method for integrating powers of tangent/secant (1)

Method for integrating $\int \tan^m x \sec^n x \, dx$**:**

- If n is **even**, then $n = 2k$ for some integer k. Then

$$\begin{aligned}\int \tan^m x \sec^n x \, dx &= \int \tan^m x \sec^{2k-2} x \sec^2 x \, dx \\ &= \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \, dx \\ &= \int \tan^m x (\tan^2 x + 1)^{k-1} \sec^2 x \, dx.\end{aligned}$$

Then apply substitution $u = \tan x$. See Example 1.

Method for integrating powers of tangent/secant (2)

power of $\tan x$ at least one copy of $\sec x$ $\tan^m x \sec^n x$

- If m is **odd** and $n \geq 1$, then $m = 2k+1$ for some integer k. Then

$$\begin{aligned}
\int \tan^m x \sec^n x \, dx &= \int \tan^{m-1} x \sec^{n-1} x \sec x \tan x \, dx \\
&= \int \tan^{2k} x \sec^{n-1} x \sec x \tan x \, dx \\
&= \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \, dx \\
&= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx.
\end{aligned}$$

Then apply substitution $u = \sec x$. See Example 2.

Method for integrating powers of tangent/secant (3)

power of $\tan x$ is even

There are other cases, e.g.

$$\int \tan^2 x \sec^3 x \, dx,$$

where things are not so "black and white". They usually can be done by converting all the $\tan^2 x$ to $\sec^2 x$ (since m is even), then applying integration by parts on integrals of $\sec^n x$. **See Exercise 1 Q3** for a base example. We will cover more of these cases in Week 5 Tutorial.

$\int \sec^5 x \, dx \leftarrow \int \sec^3 x \, dx \leftarrow \underline{\int \sec x \, dx}$ base case

Exercise 1

Evaluate the following integrals.

1. $\int \tan^2 x \sec^4 x \, dx$
2. $\int \tan^3 x \sec^3 x \, dx$
3. $\int \sec^3 x \, dx$

even

odd

$u = \tan x$
$du = \sec^2 x \, dx$

$$= \int \tan^2 x \sec^2 x \cdot \sec^2 x \, dx$$

$$= \int \underbrace{\tan^2 x}_{u^2} (\underbrace{\tan^2 x + 1}_{u^2+1}) \sec^2 x \, dx$$

$$= \int u^4 + u^2 \, du = \frac{u^5}{5} + \frac{u^3}{3} + C$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C$$

$$\int \tan^2 x \sec^2 x (\sec x \tan x) \, dx$$

$u = \sec x$
$du = \sec x \tan x \, dx$

$$= \int (\underbrace{\sec^2 x - 1}_{u^2-1}) \underbrace{\sec^2 x}_{u^2} (\sec x \tan x) dx$$

$$= \int u^4 - u^2 \, du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$$

Exercise 1

more difficult to integrate

Integration by parts

③ $$\int \sec^3 x \, dx = \int \sec x \sec^2 x \, dx$$

$u = \sec x \qquad dv = \sec^2 x$

$du = \sec x \tan x \qquad v = \tan x$

$$= \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$$

$$= \sec x \tan x - \int \sec^3 x - \sec x \, dx$$

$$= \sec x \tan x + \int \sec x \, dx - \int \sec^3 x \, dx$$

$$\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \underbrace{\int \sec x\,dx}_{\text{Example3}}$$

$$\int \sec^3 x\,dx = \frac{1}{2}\left(\sec x \tan x + \int \sec x\,dx\right)$$

$$= \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\,dx$$

$$= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

$$\underbrace{\int \sec x\,dx}_{\text{base case}} \to \int \sec^3 x\,dx \to \int \sec^5 x\,dx$$

$\int e^x \sin x\,dx$ related, technique is same

What are rational functions?

$\int \frac{1}{x} dx$ $\int \frac{1}{1+x^2} dx$

$\int \frac{1}{x^3+x^2} dx$ → split into "smaller" fractions

For the remaining of this lecture, we focus on the integration of **rational functions**. Rational functions are functions of the form

$$f(x) = \boxed{\frac{P(x)}{Q(x)}}$$

$\frac{p}{q}$ $\frac{2}{3}$ $\frac{5}{6}$ rational numbers

where P and Q are polynomials. A polynomial of the form

$$\boxed{P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}$$

improper $\frac{x^2}{x+1}$ ← deg $=2$ ← deg $=1$

with $a_n \neq 0$ has degree n, denoted by $\deg(P) = n$.
If $\deg(P) < \deg(Q)$, we say that the rational function f is **proper**.
If $\deg(P) \geq \deg(Q)$, we say that the rational function f is **improper**.

$\frac{1}{3}$ proper $\frac{5}{3}$ improper

Integration of rational functions

improper

$\int \frac{x^2+1}{x-1}\,dx$

change to proper

If the integrand is an **improper** rational function, then **long division** is required to convert it from improper to proper before integration; we need to find functions S and R such that

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where $\frac{R(x)}{Q(x)}$ is a proper rational function. Then

$$\int f(x)\,dx = \int \underbrace{\frac{P(x)}{Q(x)}}_{\text{improper}}\,dx = \int S(x) + \underbrace{\frac{R(x)}{Q(x)}}_{\text{proper}}\,dx.$$

* Step 1 for integration of any rational function is to check if the rational function is proper/improper

Exercise 2

④ $\frac{x-2}{x-1} = \frac{x-1-1}{x-1} = 1 - \frac{1}{x-1}$

Figure out which of these rational functions are proper or improper. For those that are improper, use long division to convert it to $S(x) + \frac{R(x)}{Q(x)}$ where $\frac{R(x)}{Q(x)}$ is proper.

1. $\frac{x^3+x}{x-1}$ improper
2. $\frac{x+5}{x^2+x-2}$ proper
3. $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}$ improper

$$
\begin{array}{r|l}
 & x^2 + x + 2 \\
\hline
x-1 & \cancel{x^3} \quad + x \\
 & -(\cancel{x^3} - x^2) \downarrow \\
\hline
 & +x^2 + x \\
 & -(x^2 - x) \\
\hline
 & 2x \\
 & -(2x-2) \\
\hline
 & +2
\end{array}
$$

Partial fractions:
- preprocess is hard
- integration is easy

$$\frac{x^3+x}{x-1} = x^2+x+2 + \frac{2}{x-1}$$

Exercise 2

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}$$

$$
\begin{array}{r|l}
 & \qquad\qquad\qquad\qquad x+1 \\
\hline
x^3-x^2-x+1 & x^4 \qquad -2x^2+4x+1 \\
 & -(x^4-x^3-x^2+x) \\
\hline
 & \qquad +x^3-x^2+3x+1 \\
 & \qquad -(x^3-x^2-x+1) \\
\hline
 & \qquad\qquad\qquad 4x \\
\hline
\end{array}
$$

$$\therefore \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\underbrace{\frac{4x}{x^3-x^2-x+1}}_{\text{partial fractions decomp}}$$

Partial fraction decomposition

Assumption

From this point on, we will assume that you have converted the improper rational function to a proper one. The whole idea of partial fraction decomposition is

$\frac{R(x)}{Q(x)}$ proper

Everything here is based on factorization of $Q(x)$

1. Factorizing $Q(x)$ into linear and irreducible quadratic factors.
   E.g. if $Q(x) = x^4 - 16$, then

   $(x^2)^2 - 4^2$

   $a^2 - b^2 = (a+b)(a-b)$

   $$Q(x) = (x^2 - 4)(x^2 + 4) = \underbrace{(x+2)}_{\text{linear}}\underbrace{(x-2)}_{\text{linear}}\underbrace{(x^2+4)}_{\text{irreducible}}.$$

   next week

2. Writing $\frac{R(x)}{Q(x)}$ as a decomposition into different fractions, each fraction tagged to a linear/irreducible quadratic factor of Q.
   E.g. for the example $Q(x) = x^4 - 16$,

   $$\frac{R(x)}{Q(x)} = \frac{A}{x+2} + \frac{B}{x-2} + \frac{Cx+D}{x^2+4}$$

   irreducible

Partial fraction decomposition

Depending on the factors of Q, we have different partial fraction decompositions. There are **four** different cases:

1. Q is a product of distinct, non-repeating linear factors.
   E.g. $Q(x) = (x-2)(x+2)$.
   ↳ all factors power 1
2. Q contains repeated linear factors.
   E.g. $Q(x) = \underbrace{(x-2)}^{3}(2x+2)\underbrace{(x-3)}^{2}$.
3. Q contains a non-repeated irreducible quadratic factor.
   E.g. $Q(x) = (x-2)^3(6x-3)\underbrace{(x^2+9)}$.
   $b^2 - 4ac < 0$
   $ax^2 + bx + c$
4. ✗ Q contains repeated irreducible quadratic factors.
   E.g. $Q(x) = (x-1)^2(2x^2+1)^2$.

In this course, we only cover cases (1), (2) and (3).

Method for distinct, non-repeating linear factors

The first case is when $Q(x)$ factors only into distinct, non-repeating factors $(a_1x + b_1)$, $(a_2x + b_2)$,..., $(a_nx + b_n)$. Then there exist constants $A_1, A_2, \ldots, A_n$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{(a_1x + b_1)} + \frac{A_2}{(a_2x + b_2)} + \cdots + \frac{A_n}{(a_nx + b_n)}.$$

We can find these constants A_i by multiplying $Q(x)$ to both sides of the equation, substituting the roots corresponding to these linear factors to solve for A_i.

$$\frac{A_1}{(a_1x+b_1)}$$

Example 4

Let a be a constant. Evaluate $\displaystyle\int \frac{1}{x^2 - a^2}\,dx$.

What you have to find.

$$\frac{1}{x^2 - a^2} = \frac{1}{\underbrace{(x+a)}_{\text{linear}}\underbrace{(x-a)}_{\text{linear}}} = \frac{A}{x+a} + \frac{B}{x-a}$$

non-repeating

Multiply denominator to both sides of equation

$$1 = A\underbrace{(x-a)}_{x=a} + B\underbrace{(x+a)}_{x=-a}$$

$x = a$: $1 = B \cdot 2a \Rightarrow B = \frac{1}{2a}$

$x = -a$: $1 = A(-2a) \Rightarrow A = -\frac{1}{2a}$

Example 4

$$\therefore \frac{1}{x^2-a^2} = \frac{A}{x+a} + \frac{B}{x-a} = -\frac{1}{2a(x+a)} + \frac{1}{2a(x-a)}$$

$$\therefore \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\int \frac{1}{x-a}\,dx - \frac{1}{2a}\int \frac{1}{x+a}\,dx$$

$$= \frac{1}{2a}\left(\ln|x-a| - \ln|x+a|\right) + C$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

$\ln a - \ln b$

$= \ln\left(\frac{a}{b}\right)$

Method for Repeated Linear Factors

$$\frac{x+1}{x^3(x-2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-2} + \frac{E}{(x-2)^2}$$

When $Q(x)$ has a repeated linear factor, i.e. $(ax+b)^m$ where m is the **multiplicity** of the factor, then there must be m repeating terms in the partial fraction decomposition of $\dfrac{R(x)}{Q(x)}$; there exists constants $B_1, B_2, \ldots, B_m$ such that

$$\frac{B_1}{(ax+b)} + \frac{B_2}{(ax+b)^2} + \cdots + \frac{B_m}{(ax+b)^m}.$$

Example 5

proper

Evaluate $\displaystyle\int \frac{x^2 + 2x}{x^3 - x^2 - x + 1}\, dx.$

typo

$$x^3 - x^2 - x + 1 = x^2(x-1) - (x-1) = (x-1)(x^2-1)$$
$$= (x-1)(x-1)(x+1)$$
$$= (x-1)^2(x+1)$$

$$\therefore \frac{x^2+2x}{x^3-x^2-x+1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\Rightarrow x^2+2x = A(x-1)^2 + B(x-1)(x+1) + C(x+1)$$

Example 5

$$x^2+2x = A(x-1)^2 + B(x-1)(x+1) + C(x+1)$$

$x=-1$: $(-1)^2-2 = A(-2)^2 \Rightarrow -1 = 4A \Rightarrow A = -\frac{1}{4}$

$x=1$: $1+2 = C\cdot 2 \Rightarrow C = \frac{3}{2}$

$$x^2+2x = -\frac{1}{4}(x-1)^2 + B(x-1)(x+1) + \frac{3}{2}(x+1)$$

Compare the coefficients of x^2

$$1 = -\frac{1}{4} + B \Rightarrow B = \frac{5}{4}$$

$$\frac{x^2+2x}{x^3-x^2-x+1} = -\frac{1}{4(x+1)} + \frac{5}{4(x-1)} + \frac{3}{2(x-1)^2}$$

$$\frac{x^2+2x}{x^3-x^2-x+1} = -\frac{1}{4(x+1)} + \frac{5}{4(x-1)} + \frac{3}{2(x-1)^2}$$

$$\int \frac{x^2+2x}{x^3-x^2-x+1}\,dx = -\frac{1}{4}\int \frac{1}{x+1}\,dx + \frac{5}{4}\int \frac{1}{x-1}\,dx$$

$$+\frac{3}{2}\int \frac{1}{(x-1)^2}\,dx$$

$(x-1)^{-2}$

$\downarrow$ integrade

$-(x-1)^{-1}$

$$= -\frac{1}{4}\ln|x+1| + \frac{5}{4}\ln|x-1|$$

$$-\frac{3}{2}\,\frac{1}{x-1} + C$$

Exercise 3

$$x^2 + x - 2 = (x+2)(x-1)$$

$$\begin{array}{ccc|c} x & & +2 & +2x \\ x & & -1 & -x \\ \hline x^2 & & -2 & +x \end{array}$$

Evaluate the following integrals.

1. $\displaystyle\int \frac{x+5}{x^2+x-2}\,dx$
2. $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

$$\rightarrow \frac{x+5}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$\Rightarrow x+5 = A(x-1) + B(x+2)$$

$$x=1: \quad 6 = 3B \Rightarrow B = 2$$

$$x=-2: \quad 3 = -3A \Rightarrow A = -1$$

$$\therefore \int \frac{x+5}{x^2+x-2}\,dx = \int -\frac{1}{x+2} + \frac{2}{x-1}\,dx$$

$$= 2\ln|x-1| - \ln|x+2| + C$$

Exercise 3

Exercise 4

Evaluate the following integrals.

1. $$\int \frac{1}{x^3 + x^2}\, dx$$
2. $$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx$$

Exercise 4

Exercise 4