based on sequences

Series Fundamentals

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Series: an infinite sum of terms

$a_1, a_2, a_3, a_4, \ldots$

Start off?

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.

$\sum_{n=1}^{\infty} a_n$ is sum of all the terms of this sequence

- A **series** is an infinite sum of all the terms of the sequence $\{a_n\}$

$$\sum_{n=1}^{\infty} a_n.$$

- Sometimes, the series can start summing from a later index, say $n = n_0$. This series is written as

limits

$$\lim_{N \to \infty} s_N = \sum_{n=n_0}^{\infty} a_n.$$

what does it converge to?

- We can sum first N terms of $\{a_n\}_{n=1}^{\infty}$, this gives us a **sequence** $\{s_N\}_{N=1}^{\infty}$ **of partial sums**, i.e.

$$s_N = a_1 + a_2 + \cdots + a_N = \sum_{n=1}^{N} a_n.$$

Sequence $\{a_n\}_{n=1}^{\infty}$

$a_1, a_2, a_3, a_4, \dots$

Series is actually the limit of this sequence

$$\begin{array}{ccccccccc} a_1 & , & a_1+a_2 & , & a_1+a_2+a_3 & , & a_1+a_2+a_3+a_4 & , \dots , & a_1+\dots+a_N, \dots \\ \| & & \| & & \| & & \| & & \| \\ S_1 & , & S_2 & , & S_3 & , & S_4 & , \dots , & S_N \end{array}$$

$\{S_N\}_{N=1}^{\infty}$ sequence of partial sums

By def$^{\underline{n}}$, $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$

Definition of an (infinite) series

- An infinite series is the **limit** of the sequence of partial sums

$$\sum_{n=1}^{\infty} a_n = S = \lim_{N\to\infty} s_N = \lim_{N\to\infty} (a_1 + a_2 + \cdots + a_N).$$

convergent exists

$\sum_{n=1}^{\infty} a_n < \infty$

- If this limit exists, we say that the series $\sum_{n=1}^{\infty} a_n$ is **convergent**, and

$$S = \sum_{n=1}^{\infty} a_n.$$

Series

Sequence

Otherwise, $\sum_{n=1}^{\infty} a_n$ is **divergent**. → S_N oscillating

→ $\pm\infty$

Example 1: Geometric series

related to geometric sequence $a_n = r^n$

Special class of series

The **geometric series** with **starting term** $a \neq 0$ and **common ratio** r is

$\sum_{n=1}^{\infty} a_n \quad a_n = ar^{n-1}$

$$\lim_{N\to\infty} s_N = \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

Find the general formula for the sequence of partial sums s_N.

$\to s_N = a + ar + ar^2 + \ldots + ar^{N-1}$ $\to$ Aim: Find $\lim_{N\to\infty} s_N$

Galaxy brain move: Multiply r to s_N $\leftarrow$

$\to r s_N = ar + ar^2 + \ldots + ar^{N-1} + ar^N$

$s_N - r s_N = a - ar^N \Rightarrow s_N(1-r) = a(1-r^N)$

$\downarrow$ divided by $1-r$, $r \neq 1$.

In most of other series, this is near impossible to find $\to$ $\Rightarrow s_N = \dfrac{a(1-r^N)}{1-r}$

Example 1: Geometric series

Convergence of the Geometric series

$S_N = \dfrac{a(1-r^N)}{1-r}$

Theorem (1)

The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ with starting term $a \neq 0$ and common ratio r is convergent if and only if $|r| < 1$. For $|r| < 1$, its sum is

other values of r: this series is divergent

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

determines everything

$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{a(1-r^N)}{1-r} = \frac{a\left(1-\lim_{N\to\infty} r^N\right)}{1-r}$$

$$\lim_{N\to\infty} r^N = \begin{cases} 0 & \text{if } |r|<1 \\ 1 & \text{if } r=1 \\ \text{divergent} & \text{if } r\leq -1 \text{ or } r>1 \end{cases}$$

last lecture

Proof of Theorem (1)

$|r|<1$

Case 1 If $|r|<1$, $\sum_{n=1}^{\infty} ar^{n-1} = \lim_{N\to\infty} S_N = \dfrac{a(1-\lim_{N\to\infty} r^N)}{1-r}$

$= \dfrac{a(1-0)}{1-r} = \dfrac{a}{1-r}$

eg. $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n-1}$ = exact value.

Case 2 $r=1 \to$ geometric series divergent

$S_N = \dfrac{a(1-r^N)}{1-r}$ doesn't hold for $r=1$

$\sum_{n=1}^{\infty} ar^{n-1} \xrightarrow{r=1} \sum_{n=1}^{\infty} a = \lim_{N\to\infty} S_N = \lim_{N\to\infty}(\underset{a_1}{a}+\underset{a_2}{a}+\underset{a_3}{a}+\dots+\underset{a_N}{a})$

$a_n = a$

$= \lim_{N\to\infty} N\cdot a = +\infty \to a>0$

$-\infty \to a<0$

Case 3 $r\le -1$, $r>1$,

$S_N = \dfrac{a(1-r^N)}{1-r}$ divergent $\Rightarrow S_N$ is divergent.

Exercise 1

"normal punch" $3^{1-n} = 3^{-(n-1)} = \frac{1}{3^{n-1}}$

Determine if the following series are convergent. If they are, find their sum.

1. $\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty} 4^n 3^{1-n} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} \underbrace{4}_{a} \cdot \underbrace{\left(\frac{4}{3}\right)^{n-1}}_{r^{n-1}}$

   $r = 4/3$ Series is divergent because $|r| \geq 1$

   $$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r|<1$$

2. $\sum_{n=0}^{\infty} r^n, |r| < 1$

   replace n with n-1

   +1

   $= r^0 + r^1 + r^2 + r^3 + \dots$

   $= \sum_{n=1}^{\infty} r^{n-1}$

   decrease by the same amount

   increase starting index

   starting term is $a=1$

   ratio r, $|r|<1$

   $$\sum_{n=0}^{\infty} r^n = \sum_{n=1}^{\infty} r^{n-1} = \boxed{\frac{1}{1-r}, \ |r|<1}$$

   Answer

Exercise 1

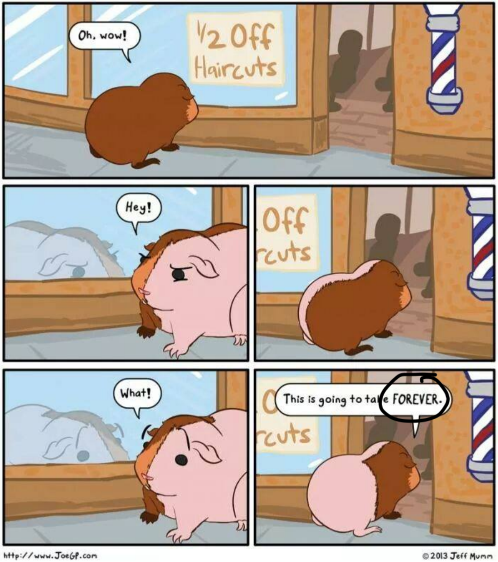
Surprise Exercise

Find the sum $\sum_{n=1}^{\infty} \frac{1}{2^n}$, if it is convergent.

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 1$$

$$\sum_{n=1}^{\infty} \underbrace{\frac{1}{2}}_{a} \cdot \Big(\underbrace{\frac{1}{2}}_{r}\Big)^{n-1} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1.$$

Never-ending haircut



$\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

$\vdots$

cut off all the hair

Why is there a need for convergence/divergence tests?

$S_N = \frac{a(1-r^N)}{1-r}$ → Find limit of S_N is "simple".

- To evaluate the sum $S = \sum_{n=1}^{\infty} a_n$, we need to find the limit of partial sums:

$$S = \lim_{N\to\infty} s_N = \lim_{N\to\infty} a_1 + a_2 + \cdots + a_N.$$

- Finding this limit is no easy feat, e.g. for the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, the sequence of partial sums $\{s_N\}_{N=1}^{\infty}$ is

$$s_N = \overbrace{1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{N^2}}^{\text{no. of terms depend on } N}.$$

→ Not easy to represent into one term like in the geometric series

Common mistake

$$\lim_{N\to\infty} 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{N^2} \not\Rightarrow \lim_{N\to\infty} 1 + \lim_{N\to\infty} \frac{1}{2^2} + \dots + \lim_{N\to\infty} \frac{1}{N^2}$$

Why is there a need for convergence/divergence tests?

- The limit of partial sums is then

$$S = \lim_{N\to\infty} \overbrace{1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{N^2}}^{\text{rational}} = \frac{\pi^2}{6}.$$

irrational

- This is difficult to evaluate, and the difficulty is representative of most limits of partial sums, with some exceptions like geometric series.
- Instead of finding this limit S, we focus on showing if this limit exists or not (convergence or divergence).
- This is done by **convergence/divergence tests**.

Focus on proving a series is convergent/divergent is the first thing, finding the exact sum S is just a by-product.

Convergence of p-series ✱ Impt!

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ $p=2$

Theorem (p-series)

A p-series is a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for some fixed real number p.

This series is convergent if $p > 1$ and divergent if $p \le 1$.

Some examples:

- ✱ $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the **harmonic series**. It is divergent.
- $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. $(p = 2)$
- $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent. $\left(p = \frac{1}{2}\right)$

① $\sum_{n=1}^{\infty} \frac{1}{n^4}$ convergent

② $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

Divergence Test

$\sum_{n=1}^{\infty} a_n$ ① Check if $\lim_{n\to\infty} a_n \neq 0$.

It turns out that one of the indicators of a divergent series is the **limit of the sequence** $\{a_n\}_{n=1}^{\infty}$ (NOT the limit of partial sums $\{s_N\}_{n=1}^{\infty}$).

Theorem (Divergence Test)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. If

$$\lim_{n\to\infty} a_n \neq 0 \text{ (or does not exist)},$$

then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Note: If $\lim_{n\to\infty} a_n = 0$, the test is **inconclusive**; there are examples of series $\sum_{n=1}^{\infty} a_n$ which are convergent/divergent. (Can you come up with examples?)

$\sum \frac{1}{n}$

Example 2

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{2n^2-1}$.

Divergence Test: Limit of the TERMS of the series

$$\lim_{n\to\infty} \frac{n^2+1}{2n^2-1} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1+\cancel{\frac{1}{n^2}}^{\to 0}}{2-\cancel{\frac{1}{n^2}}^{\to 0}} = \frac{1}{2} \neq 0.$$

$\therefore$ By the Divergence Test, $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{2n^2-1}$ diverges.

Exercise 2

Determine the convergence of the following series.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^n$ $\rightarrow$ $\lim_{n\to\infty}(-1)^n$ does not exist $\Rightarrow \sum_{n=1}^{\infty}(-1)^n$ diverges. by the Divergence Test.

2. $\displaystyle\sum_{n=1}^{\infty}\frac{n^3+n}{\sqrt{n^6+1}}$

$$\lim_{n\to\infty}\frac{n^3+n}{\sqrt{n^6+1}}\,\frac{\frac{1}{n^3}}{\frac{1}{n^3}}$$

$|x| = \sqrt{x^2}$

$$= \lim_{n\to\infty}\frac{1+\frac{1}{n^2}}{\frac{1}{n^3}\sqrt{n^6+1}}$$

$\boxed{|n^3| = \sqrt{n^6}}$

$\parallel$

n^3 $n>0$

(if $n\to-\infty$, then $|n^3| = -n^3$) *

$$\overset{*}{=} \lim_{n\to\infty}\frac{1+\frac{1}{n^2}}{\frac{1}{\sqrt{n^6}}\sqrt{n^6+1}} = \lim_{n\to\infty}\frac{1+\cancel{\frac{1}{n^2}}^{\,0}}{\sqrt{1+\cancel{\frac{1}{n^6}}^{\,0}}}$$

$$= \frac{1}{1} = 1$$

Since $\lim_{n\to\infty} \frac{n^3+n}{\sqrt{n^6+1}} = 1 \neq 0$.

$\therefore$ By the Divergence Test, $\sum_{n=1}^{\infty} \frac{n^3+n}{\sqrt{n^6+1}}$ diverges.

$|x| = \sqrt{x^2} \leftarrow$

$x \neq \sqrt{x^2} \quad x = -1$

$n \to -\infty \qquad |n^3| = \sqrt{n^6}$ ✓

$n < 0$.

$\downarrow \qquad \underline{-n}^3 = \sqrt{n^6}$

$n^3 < 0$.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ \underline{-}x & \text{if } \underline{x < 0}. \end{cases}$$

Series Laws

$$\sum_{n=1}^{\infty} a_n b_n \neq \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$$

Like the limit laws for sequences, some of these laws extend to series, **except** for product, quotient and power laws.

$$\sum_{n=1}^{\infty} \frac{a_n}{b_n} \neq \frac{(\Sigma a_n)}{(\Sigma b_n)}$$

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are **convergent**. Then

$$\text{(a)} \sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n.$$

$$\text{(b)} \sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n.$$

Exercise 3

similar to an earlier exercise

Find the sum $\displaystyle\sum_{n=1}^{\infty}\left[\underbrace{\left(\frac{3}{4}\right)^n}+\underbrace{\underline{2^{2n}6^{1-n}}}\right]$.

$$\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n = \sum_{n=1}^{\infty}\frac{3}{4}\cdot\left(\frac{3}{4}\right)^{n-1} = \frac{\frac{3}{4}}{1-\frac{3}{4}} = \frac{3}{4}\cdot\frac{4}{1} = 3.$$

$$\sum_{n=1}^{\infty}2^{2n}6^{1-n} = \sum_{n=1}^{\infty}\frac{4^n}{6^{n-1}} = \sum_{n=1}^{\infty}4\cdot\left(\frac{2}{3}\right)^{n-1} = \frac{4}{1-\frac{2}{3}} = 12.$$

$\therefore$ By series law,

$$\sum_{n=1}^{\infty}\left[\left(\frac{3}{4}\right)^n+2^{2n}6^{1-n}\right] = \sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n+\sum_{n=1}^{\infty}2^{2n}6^{1-n}$$

$$= 3+12 = 15.$$

Comparison Test

$\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty$ when $p > 1$
divergent when $p \leq 1$

Theorem (Comparison Test)

Let n_0 be a fixed positive integer. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with $0 \leq a_n \leq b_n$ for $n \geq n_0$ (i.e. the terms of each series **eventually** obey this inequality; doesn't have to start from $n = 1$). Then

- If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
- If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is divergent.

The Comparison Test can be intuitively summed up in two sentences. If we have two non-negative series, with the larger of the two series being convergent, then the smaller series must also be convergent. Also, if the smaller series is divergent, then the larger series must also be divergent.

Example 3

$\frac{1}{n-1} \sim \frac{1}{n}$

Determine the convergence of $\sum_{n=2}^{\infty} \frac{1}{n-1}$.

For $n \geq 2$

$0 < n-1 \leq n.$

$\Rightarrow \frac{1}{n-1} \geq \frac{1}{n} > 0$

$\sum_{n=2}^{\infty} \frac{1}{n}$ divergent (p-series, $p=1$)

So by Comparison Test,

$\sum_{n=2}^{\infty} \frac{1}{n-1}$ diverges.

Flip inequality "$\geq$" to "$\leq$" or vice-versa

- reciprocal $\frac{1}{\text{something}}$ (>0 also)
- Multiply by negative number

$3 \leq 5$ } multiply by -1

$\Rightarrow -3 \geq -5$

Exercise 4

Compare with $\frac{1}{n}$

Determine the convergence of the following series.

☆ 1. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2-2}$

Difficult 2. $\displaystyle\sum_{n=3}^{\infty} \frac{n}{n^3-8}$

Expect that this series is divergent

$0 < n^2-2 \leq n^2$

$\Rightarrow \dfrac{1}{n^2-2} \geq \dfrac{1}{n^2} > 0$

$\Rightarrow \dfrac{n}{n^2-2} \geq \dfrac{n}{n^2} = \dfrac{1}{n} > 0$

Comparison Test only works with nonnegative series.

$\dfrac{n}{n^2-2} \geq \dfrac{1}{n} > 0 \Rightarrow$ Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ divergent (p-series, $p=1$)

by Comparison Test, $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2-2}$ diverges.

Exercise 4

Predictions? Convergent or divergent?

$$\sum_{n=3}^{\infty} \frac{n}{n^3-8}$$

$$\frac{n}{n^3-8} \approx \frac{1}{n^2}$$

change to $-\frac{n^3}{k}$ for some constant k

$n \geqslant 3$

$n^3 \geqslant 27$

$$\Rightarrow \frac{n^3}{3} \geqslant 9 > 8 \Rightarrow \frac{n^3}{3} > 8 \Rightarrow \boxed{-\frac{n^3}{3} < -8}$$

$$\Rightarrow n^3 - \frac{n^3}{3} < n^3 - 8$$

$$\Rightarrow \frac{2n^3}{3} < n^3 - 8$$

$$\Rightarrow \frac{3}{2n^3} > \frac{1}{n^3-8}$$

$$\Rightarrow \frac{3n}{2n^3} > \frac{n}{n^3-8} > 0$$

$$\frac{3}{2n^2} > \frac{n}{n^3-8} > 0$$

$\sum_{n=3}^{\infty} \frac{3}{2n^2}$ (p-series, $p=2$) is convergent, $\therefore$ by CT, $\sum_{n=3}^{\infty} \frac{n}{n^3-8}$ converges

$$0 \le \frac{n}{n^3+8} \le \frac{n}{n^3} = \frac{1}{n^2}$$