Next week : Quiz 3 , Difficulty: Quiz 1
Material covered: Week 9, 10, 11
↓
Makeup within same week

Optimization I

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AY 22/23 Trimester 2

Week 13 : Revision → Quiz Qn
→ Midterm Qns

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Optimization Methods

Starting this week, we begin to apply the concepts that you have been taught in the last few weeks to optimize functions, i.e. to find their global extreme values. There will be two core optimization methods we will cover for this course:

1. Closed and Bounded Interval Method, and
2. First Derivative Test for Global Extreme Values. ← Week 12
   (to be covered in Week 12)

Extreme Value Theorem

The Closed and Bounded Interval Method (which I will abbreviate as ICBM), has foundations in the Extreme Value Theorem, which we have learnt in Week 9:

Theorem (Extreme Value Theorem (EVT))

A **continuous** function f with domain $[a, b]$ attains its global maximum and minimum value.

That means that for a continuous function f, its global maximum and minimum values/points are guaranteed to exist.

A significant part of this theorem being true is due to the domain of the function; a closed and bounded interval $[a, b]$. Therefore the ICBM is named after the closed and bounded interval $[a, b]$.

The ICBM

The Closed and Bounded Interval Method can be summed up in three steps:

global max (value, point)
global min (value, point)

Theorem (Closed and Bounded Interval Method)

To find the global extreme values of a continuous function f on a closed and bounded interval $[a, b]$:

exclude

Step 1: Find the values of f at the critical points of f in (a, b).

Step 2: Find the values of f at the endpoints of the interval, at $x = a$ and $x = b$.

plug in critical points/endpoints in f

Step 3: The largest of the values of Steps 1 and 2 is the global maximum value; the smallest of these values is the global minimum value.

$[1, 2]$

Step 1: Critical points of f on $(1, 2)$

Step 2: $x = 1$, $x = 2$

Why the ICBM works

global max $f(c) \geq f(x)$ for all x in domain of f

The global extreme values are the largest/smallest value amongst **all other function values** $f(x)$, where x is in the domain of f. In this case, the domain of f is $[a, b]$.

The endpoints of the interval $[a, b]$ cannot be considered as local extreme points because there are no points to the left of a or right of b to consider, so we exclude them first.

$x \in (a, b)$

If a global extreme point exists in (a, b), then **it must be a local extreme point**. So in Step 1, we consider all critical points in (a, b), which are likely to be local extreme points. ← Fermat's Theorem

We then consider the endpoints $x = a$ and $x = b$ in Step 2, which we have previously omitted. Finally, in Step 3, we consider all critical points and endpoints, and take the largest and smallest function value, which gives us the global extreme values of f.

Example 1

Find the global extreme values and points for the function $f(x) = 3x^4 + 4x^3 - 12x^2$ on the interval $[-3, 2]$.

$$\begin{array}{cc|c} x & +2 & +2x \\ x & -1 & -x \\ \hline x^2 & -2 & x \end{array}$$

Step 1: Find the critical points on $(-3, 2)$, and find the function values at these critical points:

$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2+x-2) = 12x(x+2)(x-1)$

in $(-3,2)$

Therefore the critical points of f in $(-3, 2)$ are $x = -2, 0, 1$,
and the corresponding function values are $f(-2) = -32$,
$f(0) = 0$, $f(1) = -5$.

Step 2: Find the function values at the endpoints:

$f(-3) =$ 27, $f(2) =$ 32.

Step 3: Compare the function values obtained in Step 1 and 2:

Global max value: 32, global max point: $x = 2$.

Global min value: -32, global min point: $x = -2$.

Exercise 1

Find the global extreme values and points for the function $f(x) = x^4 - 14x^2 + 24x$ on the interval $[-3, 2]$.

Step 1: $(-3,2)$

$f'(x) = 4x^3 - 28x + 24$

$= 4(x^3 - 7x + 6)$ factors of 6

$x=1$: $x^3 - 7x + 6 = 1^3 - 7 + 6 = 0 \Rightarrow (x-1)$ is a factor of $x^3 - 7x + 6$.

$\Rightarrow x^3 - 7x + 6 = (x-1)(ax^2 + bx + c)$

$= ax^3 + bx^2 + cx - ax^2 - bx - c$

$= ax^3 + (b-a)x^2 + (c-b)x - c$

1 no x^2 coeff is 0

$\Rightarrow a = 1$, $-c = +6 \Rightarrow c = -6$, $b - a = 0 \Rightarrow b = 1$

Exercise 1

$$x^3-7x+6=(x-1)(x^2+x-6)$$

$$=(x+3)(x-1)(x-2)$$

$$\begin{array}{cc|c} x & +3 & +3x \\ x & -2 & -2x \\ \hline x^2 & -6 & +x \end{array}$$

Critical points are $x=-3,1,2$

in $(-3,2)$

$f(1)=11$, endpoints: $f(-3)=-117$, $f(2)=8$.

global max value is 11, global max pt : $x=1$

global min value is -117, global min pt : $x=-3$

Exercise 1

Exercise 2

Find the global extreme values and points for the function $f(\theta) = 1 + \cos^2\theta$ on the interval $\left[\frac{\pi}{4}, \pi\right]$.

Read carefully.

$\sin(2\theta) = 2\sin\theta\cos\theta$

$\left(\frac{\pi}{4}, \pi\right)$

$$f'(\theta) = 2\cos\theta \cdot (-\sin\theta)$$
$$= -2\sin\theta\cos\theta = -\sin(2\theta) = 0$$

$\sin\theta = 0 \Leftrightarrow \theta = n\pi \quad n$ integer

$\ldots, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots$

$\sin(2\theta) = 0 \Leftrightarrow 2\theta = n\pi \Leftrightarrow \theta = \frac{n\pi}{2}$

Within $\left(\frac{\pi}{4}, \pi\right)$, $\frac{\pi}{2}$ is a critical point of f.

$$f\left(\frac{\pi}{2}\right) = 1 + \cos^2\left(\frac{\pi}{2}\right) = 1.$$

$\cos\theta = 0.$

Exercise 2

$$f\left(\frac{\pi}{4}\right) = 1 + \cos^2\left(\frac{\pi}{4}\right) = 1 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{3}{2}$$

$$f(\pi) = 1 + \cos^2(\pi) = 1 + (-1)^2 = 2.$$

$\therefore$ global max value is 2, point $x = \pi$

global min value is 1, point $x = \frac{\pi}{2}$.

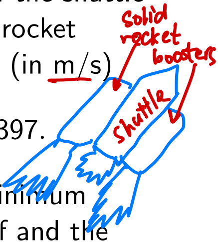
Real-world word problems

Similar to related rates in Week 8, the majority of optimization problems that we will be doing in this 2 weeks are word problems.

We use the information provided in the word problems and convert them into mathematical notation; the function f and the interval $[a, b]$, and apply the 3-step procedure in ICBM to find the global extreme values and points.

Example 2

The Hubble Space Telescope was deployed on April 24, 1990, by the space shuttle *Discovery*. A model for the velocity of the shuttle during this mission, from liftoff at $t = 0$ until the solid rocket boosters were jettisoned at $t = 126$ seconds is given by (in m/s)



$$v(t) = 0.000397t^3 - 0.02752t^2 + 7.196t - 0.9397.$$

Using this model, estimate the global maximum and minimum values of the **acceleration** of the shuttle between liftoff and the jettisoning of the boosters.

$$a(t) = v'(t) = 0.001191t^2 - 0.05504t + 7.196$$

Function to maximise/minimize: $a(t)$

Closed and bounded interval: $[0, 126]$

Example 2

We apply the ICBM to the function along with its domain. We differentiate the function to get

$$a'(t) = \underline{0.002382t - 0.05504}.$$

Step 1: The only critical point in $\underline{(0, 126)}$ occurs where $a'(t) = 0$, i.e.

$$0.002382t - 0.05504 = 0 \Rightarrow t = \frac{0.05504}{0.002382} \approx 23.107\text{ s.}$$

The function value at this critical point is $\underline{6.56 \quad (m/s^2)}$.

Step 2: The function values at the endpoints of the interval are 0.0...

$$a(0) = 7.196, \quad a(126) = 19.16$$

3sf, 2/3dp

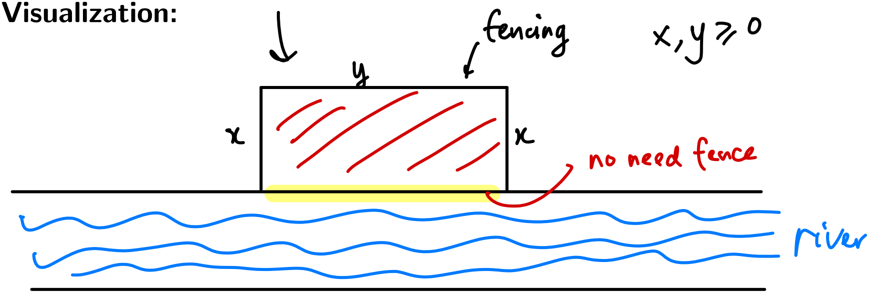
Therefore, the maximum acceleration is $\underline{19.16}$ m/s^2, and the minimum acceleration is $\underline{6.56}$ m/s^2.

Example 3

A farmer has 1200 m of fencing of negligible thickness and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Visualization:

$x, y \geq 0$



Restriction: $2x + y = 1200$

Smallest x : 0

largest x : 600

Example 3

Let x be the width of the rectangular field and y be the length of the rectangular field. Therefore, the area of the rectangular field is

$y = 1200 - 2x$

$$A = \underline{\quad xy \quad}.$$

We can reduce this expression to a single variable, instead of 2, by noting that $2x + y = 1200$. Therefore, the area A in terms of x (and also the function to maximise is) is

$$A = \underline{x(1200-2x)} = -2x^2 + 1200x$$

The smallest possible value of x is $\underline{\quad 0 \quad}$, while the largest possible value of x is $\underline{\ 600\ }$. Therefore the interval is $\underline{[0, 600]}$

Example 3

We differentiate A with respect to x to get

$$A'(x) = \underline{-4x + 1200}.$$

plug into A

The only critical point is $x = \underline{300}$. Hence, the function value at this critical point is $\underline{300 \cdot 600} = 180{,}000$.

The function values at the endpoints of the interval are

$A(0) = 0$ $\quad$ $A(600) = 0$

Therefore the dimensions of the field that has the largest area is $\underline{\text{width} = 300\text{m}, \text{ length} = 600\text{m}}$.

Exercise 3

The water level (measured in m), above sea level, of Lake Lanier in Georgia, USA, during 2012 can be modelled by the function

$$L(t) = 0.00439t^3 - 0.1273t^2 + 0.8239t + 323.1$$

where t is measured in months since January 1, 2012. Estimate when the water level was the highest during 2012.

point

global max

Jan 1 2012 — $t=0$

Jan 1 2013 — $t=12$

Interval : $[0, 12]$.

Exercise 3

$$L'(t) = \underbrace{0.01317}_{a}t^2 \underbrace{-0.2546}_{b}t + \underbrace{0.8239}_{c} = 0.$$

$$t = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{0.2546 \pm \sqrt{(-0.2546)^2 - 4(0.01317)(0.8239)}}{2(0.01317)}$$

$t_0 = 4.10974\ldots$ $L(t_0) = 324.64\ldots$

or $t = 15.222067\ldots$ not $(0, 12)$ ignore

$t = 4.11$ months

$L(0) = 323.1$, $L(12) = 322.24\ldots$

→ Apr 3^{rd} 2012 or 4^{th}

Rough date