Week 10: Projection and reflection in 3D

Table of contents

Linear Transformations in $\mathbb{R}^2$

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation.
  Let M be the matrix representation of T

$$T(\vec{x}) = M\vec{x}$$

Linear Transformations in $\mathbb{R}^2$

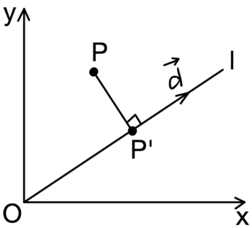
- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation.
  Let M be the matrix representation of T

$$T(\vec{x}) = M\vec{x}$$

- We discussed (found matrix representations) the following maps
  1. Projection
  2. Reflection
  3. Scaling
  4. Rotation
  5. Shear

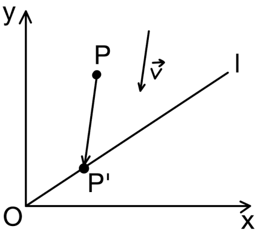
Projections in $\mathbb{R}^2$

Orthogonal projection



$$M = \frac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T$$

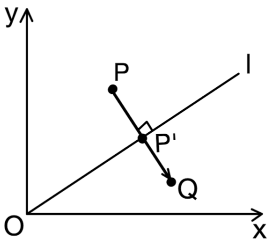
Skew projection



$$M = I_2 - \frac{\vec{v}\vec{n}^T}{\vec{v}\cdot\vec{n}}$$

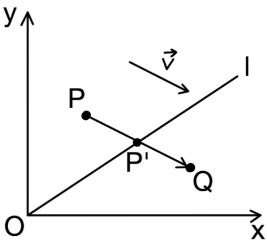
Reflections in $\mathbb{R}^2$

Orthogonal reflection



$$M = \frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_2$$

Skew reflection



$$M = I_2 - \frac{2}{\vec{v}\cdot\vec{n}}\vec{v}\vec{n}^T$$

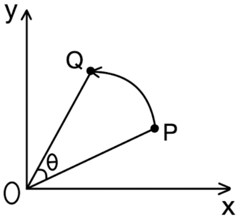
Scaling and rotation

- The scaling $S : \mathbb{R}^2 \to \mathbb{R}^2$ which scales all x-coordinates by a and all y-coordinates by b is defined by

$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- The counter-clockwise rotation around O over angle θ has matrix

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



Shear

- The **shear** $S : \mathbb{R}^2 \to \mathbb{R}^2$ w.r.t. the line $l : \vec{n} \cdot \vec{x} = 0$ in the direction of **shearing vector** $\vec{v}$ ($\vec{v} \parallel l$) is a map $S : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$S(\vec{x}_0) = \vec{x}_0 + \frac{\vec{n} \cdot \vec{x}_0}{||\vec{n}||} \vec{v}$$

- S has matrix

$$M = M_{\vec{n},\vec{v}} = I_2 + \frac{1}{||\vec{n}||} \vec{v} \vec{n}^T$$

Exercise 1

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be linear with $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\-3\end{pmatrix}$ and $T\begin{pmatrix}1\\4\end{pmatrix} = \begin{pmatrix}18\\-27\end{pmatrix}$.

Find the collection of points $\vec{x}$ that are mapped to the origin.

Exercise 2

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal reflection through $l : 2x + y = 0$. Find the line that is mapped to $2x - 4y = \pi^{100}$.

Exercise 3

Exercise 4

Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection through $m : x - y = 0$.

Which of the following is true $R \circ T$?

(A) $R \circ T$ is an orthogonal (or skew) reflection through some line.

(B) $R \circ T$ is an orthogonal (or skew) onto some line.

(C) None of these is true.

Exercise 5

Consider the shear w.r.t. $l : x - y = 0$ in direction $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Sketch the image of the unit and compute its area.

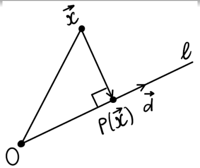
Linear Transformations in $\mathbb{R}^3$

Similar to $\mathbb{R}^2$, we aim to find the **matrix** of following linear transformations

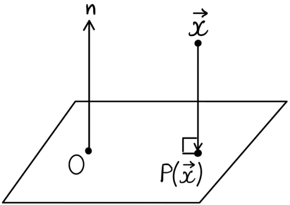
1. Projection
2. Reflection
3. Scaling
4. Rotation
5. Shear

Projections in $\mathbb{R}^3$

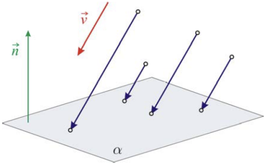
1. Orthogonal projection onto a line through O



2. Orthogonal projection onto a plane through O



3. Skew projection onto a plane through O



Preview of known results

- The line through O with direction $\vec{d}$ has vector equation

$$\vec{x} = t\vec{d}$$

- The plane through O with normal $\vec{n}$ has vector equation

$$\vec{n} \cdot \vec{x} = 0$$

Preview of known results

- Useful identity

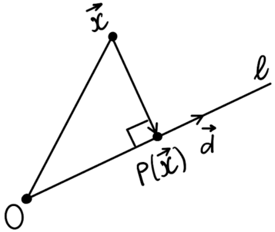
$$(\vec{a} \cdot \vec{x})\vec{b} = M\vec{x} \text{ with } M = \vec{b}\vec{a}^T$$

Orthogonal projection onto a line

Theorem 1

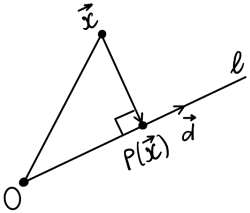
The matrix of the projection onto the line $l : \vec{x} = t\vec{d}$ is

$$M = \frac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T$$



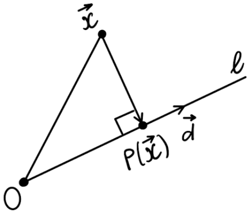
Proof

- Let $\vec{x}$ be any point and let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection.



Proof

- Let $\vec{x}$ be any point and let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection.



- $P(\vec{x}) =$ projection of $\vec{x}$ onto $\vec{d}$

$$P(\vec{x}) = \mathrm{proj}_{\vec{d}}(\vec{x}) = \frac{\vec{x} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \vec{d} = \frac{1}{||\vec{d}||^2} \vec{d}\vec{d}^T \vec{x}$$

Example 1

(a) Find the matrix of orthogonal projection onto $l : \vec{x} = t \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$?

(b) What is the image of the point $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$?

Example 1

(c) Find the image of the line m under P

$$m : \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$$

Example 1

(d) Show that the entire line k is mapped to a point. Explain this.

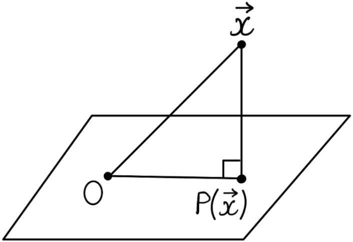
$$k : \vec{x} = \begin{bmatrix} 4 \\ 2 \\ 12 \end{bmatrix} + t \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$$

Orthogonal projection onto a plane

Theorem 2

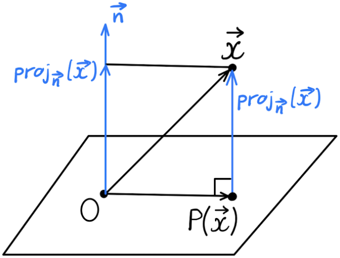
The orthogonal projection $P : \mathbb{R}^3 \to \mathbb{R}^3$ onto the plane $\alpha : \vec{n} \cdot \vec{x} = 0$ has matrix representation

$$M = I_3 - \frac{1}{||\vec{n}||^2}\vec{n}\vec{n}^T$$



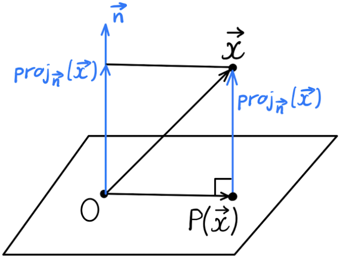
Proof (Sketch)

- Let $\vec{x}$ = any point, $P(\vec{x})$ = projection of $\vec{x}$ onto α.



Proof (Sketch)

- Let $\vec{x}$ = any point, $P(\vec{x})$ = projection of $\vec{x}$ onto α.



- We have

$$\begin{aligned} P(\vec{x}) &= \vec{x} - \mathrm{proj}_{\vec{n}}(\vec{x}) = \vec{x} - \frac{\vec{n}\cdot\vec{x}}{\vec{n}\cdot\vec{n}}\vec{n} \\ &= \vec{x} - \frac{1}{||\vec{n}||^2}(\vec{n}\cdot\vec{x})\vec{n} = \vec{x} - \frac{1}{||\vec{n}||^2}\vec{n}\vec{n}^T\vec{x} \\ &= \left(I_3 - \frac{1}{||\vec{n}||^2}\vec{n}\vec{n}^T\right)\vec{x} \end{aligned}$$

Example 2

(a) Find the matrix of the projection onto $\alpha : 3x + 2y - z = 0$.

(b) What is the image of the point $\begin{pmatrix} 0 \\ -4 \\ 6 \end{pmatrix}$?

(c) What is the image of the line $l : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -4 \\ 6 \end{pmatrix}$?

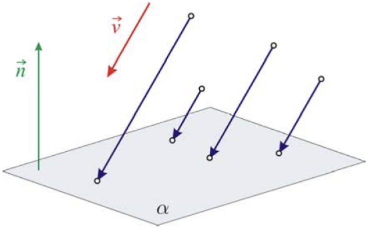
(d) What is the image of the plane $\beta : 4x - 9y - 6z = 7$?

Skew projection onto a plane

Theorem 3

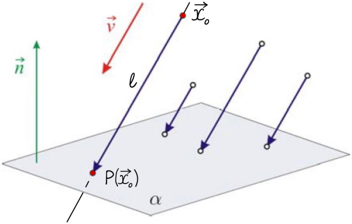
The skew projection $P : \mathbb{R}^3 \to \mathbb{R}^3$ onto the plane $\alpha : \vec{n} \cdot \vec{x} = 0$ in the direction of $\vec{v}$ has matrix representation

$$M = I_3 - \frac{\vec{v}\vec{n}^T}{\vec{v} \cdot \vec{n}}$$



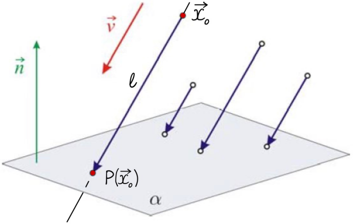
Proof(Sketch)

- Let $\vec{x}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ = any point, $P(\vec{x}_0)$ = projection of $\vec{x}_0$ on α. Let l be the line through $\vec{x}_0$ with direction $\vec{v}$.



Proof(Sketch)

- Let $\vec{x}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ = any point, $P(\vec{x}_0)$ = projection of $\vec{x}_0$ on α. Let l be the line through $\vec{x}_0$ with direction $\vec{v}$.



- $P(\vec{x}_0)$ is the intersection of l and α. So

$$\begin{cases} P(\vec{x}_0) = \vec{x}_0 + t\vec{v} \\ \vec{n} \cdot P(\vec{x}_0) = 0 \end{cases} \Rightarrow \vec{n} \cdot (\vec{x}_0 + t\vec{v}) = 0 \Rightarrow t = -\frac{\vec{n} \cdot \vec{x}_0}{\vec{n} \cdot \vec{v}}$$

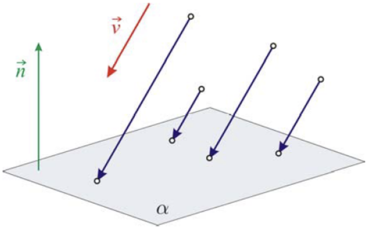
Proof (Sketch)

We obtain

$$\begin{aligned} P(\vec{x}_0) &= \vec{x}_0 - \frac{\vec{n}\cdot\vec{x}_0}{\vec{n}\cdot\vec{v}}\vec{v} \\ &= \vec{x}_0 - \frac{1}{\vec{n}\cdot\vec{v}}(\vec{n}\cdot\vec{x}_0)\vec{v} \\ &= \vec{x}_0 - \frac{1}{\vec{n}\cdot\vec{v}}\vec{v}\vec{n}^T\vec{x}_0 \\ &= \left(I_3 - \frac{1}{\vec{n}\cdot\vec{v}}\vec{v}\vec{n}^T\right)\vec{x}_0 \end{aligned}$$

Question

What points are fixed by the skew projection?



Exercise 6

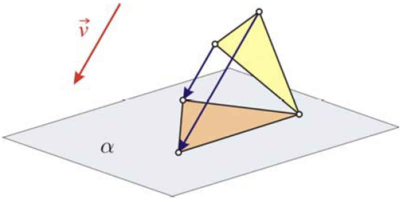
Let $\alpha : 3x + 2y - z = 0$ and $\vec{v} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$.

(a) Find the matrix of the projection onto α along $\vec{v}$.

(b) Find the images of the points $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

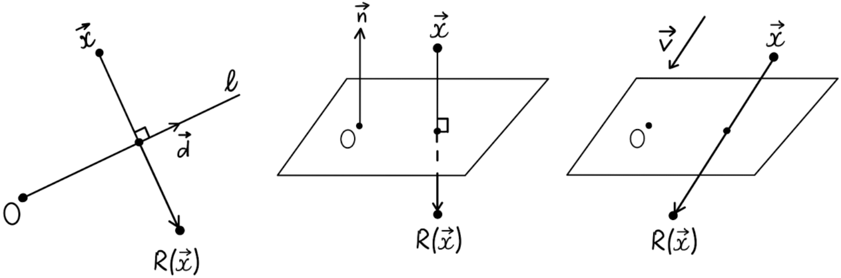
(c) Find images of $l: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$ and $m: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

(d) Find image $\triangle A'B'C'$ of $\triangle ABC$ with $A = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}, C = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$ (note $C \in \alpha$). Compare areas of $\triangle ABC$ and $\triangle A'B'C'$.



Reflections in $\mathbb{R}^3$

1. Orthogonal reflection through a line
2. Orthogonal reflection through a plane
3. Skew reflection through a plane

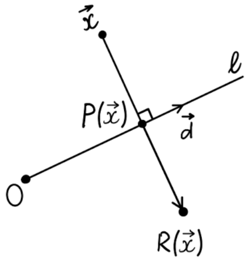


Orthogonal reflection through a line

Theorem 4

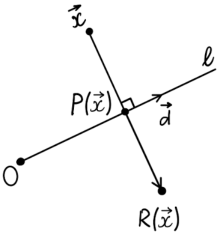
The orthogonal reflection through the line $l : \vec{x} = t\vec{d}$ has matrix representation

$$M = \frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_3$$



Proof(Sketch)

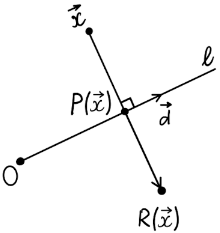
- $\vec{x}$ = any point, $R(\vec{x})$ = image of $\vec{x}$, $P(\vec{x})$ = projection of $\vec{x}$ onto l.



- We knew $P(x) = \dfrac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T\vec{x}$

Proof(Sketch)

- $\vec{x}$ = any point, $R(\vec{x})$ = image of $\vec{x}$, $P(\vec{x})$ = projection of $\vec{x}$ onto l.

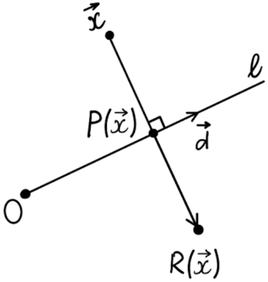


- We knew $P(x) = \dfrac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T\vec{x}$
- $P(\vec{x})$ = midpoint of $\vec{x}$ and $R(\vec{x}) \Rightarrow P(x) = \frac{1}{2}\left(\vec{x} + R(\vec{x})\right)$

$$R(\vec{x}) = 2P(\vec{x}) - \vec{x} = \left(\frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_3\right)\vec{x}$$

Question

What points are fixed the reflection through a line?



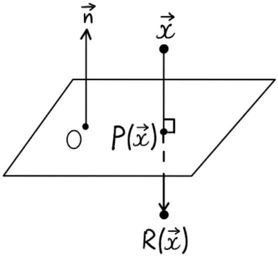
Example 2

(a) Compute the matrix of the reflection through $l : \vec{x} = t \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$.

(b) Let $P = \begin{pmatrix} 4 \\ 2 \\ 12 \end{pmatrix}$. Find the reflection Q of P through l .

(c) Check that the midpoint M_{PQ} of PQ is on l. Compute $d(P, l)$.

Orthogonal reflection through a plane



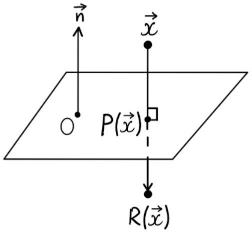
Theorem 5

The orthogonal reflection through the plane $\alpha : \vec{n} \cdot \vec{x} = 0$ has matrix representation

$$M = I_3 - \frac{2}{||\vec{n}||^2} \vec{n}\vec{n}^T$$

Proof

- $\vec{x}$ = any point, $R(\vec{x})$ = image of $\vec{x}$, $P(\vec{x})$ = projection of $\vec{x}$ onto α.

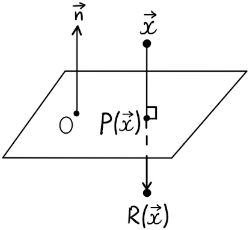


- We knew $P(x)$ from last lecture

$$P(\vec{x}) = \left(I_3 - \frac{1}{||\vec{n}||^2}\vec{n}\vec{n}^T\right)\vec{x} = \vec{x} - \frac{\vec{n}\vec{n}^T}{||\vec{n}||^2}\vec{x}$$

Proof

- $\vec{x}$ = any point, $R(\vec{x})$ = image of $\vec{x}$, $P(\vec{x})$ = projection of $\vec{x}$ onto α.



- We knew $P(x)$ from last lecture

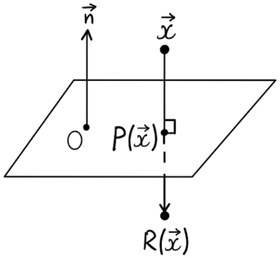
$$P(\vec{x}) = \left(I_3 - \frac{1}{||\vec{n}||^2}\vec{n}\vec{n}^T\right)\vec{x} = \vec{x} - \frac{\vec{n}\vec{n}^T}{||\vec{n}||^2}\vec{x}$$

- $P(\vec{x})$ = midpoint of $\vec{x}$ and $R(\vec{x}) \Rightarrow P(x) = \frac{1}{2}\left(\vec{x} + R(\vec{x})\right)$

$$R(\vec{x}) = 2P(\vec{x}) - \vec{x} = \left(I_3 - \frac{2}{||\vec{n}||^2}\vec{n}\vec{n}^T\right)\vec{x}$$

Question

Which points are fixed by the reflection through α?



Example 3

Let $\alpha : 3x + 2y - z = 0$ be a plane in $\mathbb{R}^3$.

(a) Compute the matrix M of the reflection through α.

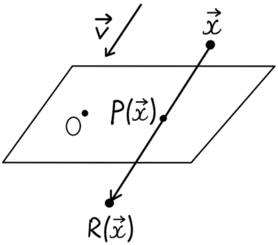
(b) Find the images of the points $\begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

(c) Find the image of $m : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix} + t \begin{bmatrix} 12 \\ -2 \\ 4 \end{bmatrix}$.

(d) Show that the image of $k : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ is itself.

(e) Find the image of the plane $\beta : 4x - 9y - 6z = 7$.

Skew reflection through a plane



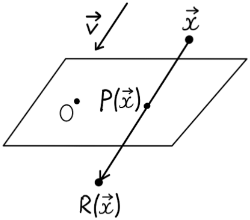
Theorem 6

Let $\alpha : \vec{n} \cdot \vec{x} = 0$ be a plane in $\mathbb{R}^3$. Let $\vec{v}$ be a vector such that $\vec{v} \not\perp \vec{n}$. The skew reflection through α in the direction $\vec{v}$ has matrix representation

$$M = I_3 - \frac{2\vec{v}\vec{n}^T}{\vec{v} \cdot \vec{n}}$$

Remark

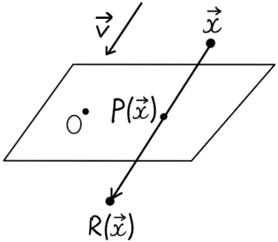
- The skew reflection is only *meaningful* when $\vec{v} \not\perp \vec{n}$, that is, $\vec{v} \nparallel \alpha$.



- What happens if $\vec{v} \parallel \alpha$?

Question

Which points are fixed by the skew projection along $\vec{v}$ through α?



Example 4

Let $\alpha : 3x + 2y - z = 0$ and let $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

(a) Compute the matrix M of the reflection through α in the direction $\vec{v}$.

(b) Find the images of the points $\begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

(c) Find the image of the line $m : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix} + t \begin{bmatrix} 12 \\ -2 \\ 4 \end{bmatrix}$

(d) Find the image of the plane $\beta : 4x - 9y - 6z = 7$