Taylor and Maclaurin Series
Some Quiz 2 Problems

Dr. Ronald Koh
ronald.koh@digipen.edu (Teams preferred over email)

AY 23/24 Trimester 1

Table of contents

Power Series

- A power series centered at a is the series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

- A power series centered at a is always convergent at $x = a$

$$x = a \implies \sum_{n=0}^{\infty} c_n(x-a)^n = c_0.$$

- The radius of convergence of $\sum_{n=0}^{\infty} c_n(x-a)^n$ is a number R such that

$a-R$ $\qquad a \qquad$ $a+R$

$$\sum_{n=0}^{\infty} c_n(x-a)^n \text{ converges if } |x-a| < R \text{ and diverges if } |x-a| > R.$$

Power series $\longrightarrow$ function

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \underbrace{\bigcirc}_{} < 1$$

manipulate $\hookrightarrow |x-a| < R$

- We can find R using either the Ratio/Root Test.
- Suppose this power series has a radius of convergence R.
- We can then define a function f as

Convergent

$$\underline{f(x)} = \sum_{n=0}^{\infty} c_n(x-a)^n, \quad \underbrace{|x-a|} < R. \quad (1)$$

- In this case, the function was defined using a power series, and we call equation (1) as the **power series representation** of f.
- **Question**: What about the converse, i.e. given a function f, can we find a power series representation of f?
- We can find an answer using **Taylor/Maclaurin series**. $\swarrow \sum c_n x^n$ (center $a=0$)

$\hookrightarrow \sum c_n(x-a)^n$

Observation about the coefficients

↳ we need to find c_n

Let's suppose that a function f has a power series representation

$$f(x) = \cancel{c_0} + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

$x=a$

We have

$c_0 = f(a)$

- $f(a) = c_0$.
- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \implies c_1 = f'(a)$.
- $f''(x) = 2c_2 + 6c_3(x-a) + 12c_4(x-a)^2 + \cdots \implies c_2 = \dfrac{f''(a)}{2}$.
- $f'''(x) = 6c_3 + 24c_4(x-a) + \cdots \implies c_3 = \dfrac{f'''(a)}{6}$.
- We can continue the rest for $c_4, c_5, \ldots$

Question: What's the general formula for c_n?

General formula for c_n

$$(x-a)^n \xrightarrow{n \text{ times}} n!$$

- We have

$$c_0 = f(a),\ c_1 = f'(a),\ c_2 = \frac{f''(a)}{2},\ c_3 = \frac{f'''(a)}{6}, \ldots$$

- What is the general formula for c_n?

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$(n) \leftarrow$ no. of times f is differentiated.

By defⁿ: $0! = 1$

$f^{(0)}(a) = f(a)$

start $n=0$

Function $\longrightarrow$ power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \rightarrow \text{Taylor Series of } f$$

Theorem

If f has a power series representation centered at $x = a$, i.e. if

$$f(x) = \underline{\sum_{n=0}^{\infty} c_n (x-a)^n}, \ |x-a| < R,$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Taylor series and Maclaurin series

$(-\frac{\pi}{2}, \frac{\pi}{2})$

e^x , $\sin x$, $\cos x$ $\cancel{\tan x}$

- Let f be an **infinitely differentiable** function on an open interval centered at a: $(a-R, a+R)$ for some $R>0$.
- The **Taylor series of f at a** (or about a, or centered at a) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

- The Taylor series centered at $a=0$ is called the **Maclaurin series** of f.

$a=0$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

$a=0$

Remark on Taylor/Maclaurin series

don't write $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

(for now)

- Given a function f defined on $\mathbb{R}$, we do not always have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

for all x.

- The reason is that the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

may not be convergent for all x.

some x that result in the series being divergent.

Example 1

$a = 0.$ $\quad c_n = \frac{f^{(n)}(0)}{n!}$

Find the Maclaurin series for $f(x) = e^x$ and find its radius of convergence.

Find out what exactly $f^{(n)}(x)$ is.

Clearly $f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = e^0 = 1.$

Maclaurin series for e^x is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \underbrace{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

Example 1

$a_n = \frac{x^n}{n!} \rightarrow a_{n+1} = \frac{x^{n+1}}{(n+1)!}$

Radius of convergence, $x \neq 0$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

$$= \lim_{n\to\infty} \frac{|x|}{n+1} = |x| \underbrace{\lim_{n\to\infty} \frac{1}{n+1}}_{\to 0} = 0 < 1$$

no matter what x is

$\therefore \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent for all x.

$\Rightarrow$ Radius of convergence $R = \infty$.

Extra info: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$

Exercise 1

$a=3$

Find the Taylor series for $f(x) = e^x$ centered at $a = 3$ and find its radius of convergence.

$$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(3) = e^3$$

The Taylor series of $f(x) = e^x$ centered at $a=3$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

Exercise 1

$$a_n = \frac{e^3}{n!}(x-3)^n$$

$$\frac{n!}{(n+1)!} = \frac{1}{n+1}$$

For $x \neq 3$,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{e^3}(x-3)^{\cancel{n+1}}}{(n+1)\cancel{!}}\cdot\frac{\cancel{n!}}{\cancel{e^3}\cancel{(x-3)^n}}\right|$$

$$= \lim_{n\to\infty}\frac{|x-3|}{n+1} = |x-3|\lim_{n\to\infty}\frac{1}{n+1} = 0 < 1$$

$\therefore \sum_{n=0}^{\infty}\frac{e^3}{n!}(x-3)^n$ is abs. convergent for all x

$\Rightarrow$ Radius of convergence $R = \infty$.

Exercise 2

$f^{(n)}(x)$ not so simple

$\frac{f^{(n)}(0)}{n!} = 0$

Find the Maclaurin series for $f(x) = \sin x$ and find its radius of convergence.

$f(x) = \sin x \rightarrow f^{(0)}(0) = 0$ 0   0

$f'(x) = \cos x \rightarrow f^{(1)}(0) = 1$ 1   1   ...

$f''(x) = -\sin x \rightarrow f^{(2)}(0) = 0$ 0   0

$f'''(x) = -\cos x \rightarrow f^{(3)}(0) = -1$ -1 -1

The Maclaurin Series of $\sin x$ is

$$0 + \frac{1}{1!}x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + \dots$$

$f^{(0)}(0)$ $f^{(1)}(0)$ $f^{(2)}(0)$ $f^{(3)}(0)$

Exercise 2

$$= \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

$(-1)^n$

$2n+1$

$(2n+1)!$

Additional info:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

for all x

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad a_n = (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Let $x \neq 0$: $\displaystyle \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right|$

$$= \lim_{n\to\infty} \frac{|x|^2}{(2n+2)(2n+3)} = |x|^2 \underbrace{\left(\lim_{n\to\infty}\frac{1}{2n+2}\right)}_{\to 0} \underbrace{\left(\lim_{n\to\infty}\frac{1}{2n+3}\right)}_{\to 0} = 0 < 1$$

$\Rightarrow$ Radius of convergence is $R = \infty$.

$$\frac{(2n+1)!}{(2n+3)!} = \frac{\cancel{1 \cdot 2 \cdot \cdots \cdot (2n+1)}}{\cancel{1 \cdot 2 \cdot \cdots (2n+1)}(2n+2)(2n+3)}$$

Question 2

$a=0,\ b=2,\quad \Delta x = \frac{2-0}{1} = 2$ Trapezoidal $n=1$ Midpoint $n=1$

For the integral $\int_0^2 f(x)\,dx$, the approximations T_1 and M_1 give the values 5 and 4 respectively. Determine S_2. → Simpson's Rule $n=2$ → $\Delta x' = 1$

$n=1:\ x_0=0,\ x_1=2 \rightarrow \bar{x}_1 = 1$

$n=2:\ x_0=0,\ x_1=1,\ x_2=2$

$M_1 = \Delta x\,[f(\bar{x}_1)] = 2f(1) = 4$

$T_1 = \frac{\Delta x}{2}[f(x_0)+f(x_1)] = \frac{2}{2}[f(0)+f(2)] = f(0)+f(2) = 5$

$S_2 = \frac{\Delta x'}{3}[f(0)+4f(1)+f(2)]$

$= \frac{1}{3}[f(0)+4f(1)+f(2)] = \frac{1}{3}\cdot[5+2\cdot 4] = \frac{13}{3}.$

Question 3

$a=-1,\ b=3,\quad b-a=4$

Let f be defined on $[-1,3]$. Suppose $-8 \leq f^{(4)}(x) \leq 1$ for all $x \in [-1,3]$. What is the error bound for S_4, as an approximation for the integral

$n=4$

$$\int_{-1}^{3} f(x)\,dx?$$

$|f^{(4)}(x)| \leq K$

magnitude

Note: About 90 - 95% of students got this question wrong.

$$|E_S| \leq \frac{K(b-a)^5}{180n^4} = \frac{K \cdot 4^5}{180 \cdot 4^4} = \frac{K}{45} = \frac{8}{45}$$

$\frac{1}{45}$ X

Question 3

Question 4/5

Partial fractions

Find α, where

$$\int_0^2 \frac{5x+15}{x^3+x^2+4x+4}\,dx = \alpha + \ln\left(\frac{9}{2}\right).$$

linear

irr quadratic

$$x^3+x^2+4x+4 = x^2(x+1) + 4(x+1) = (x+1)(x^2+4)$$

$$\frac{5x+15}{x^3+x^2+4x+4} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$$

$$\Rightarrow 5x+15 = A(x^2+4) + (Bx+C)(x+1)$$

$$A=2, \quad B=-2, \quad C=7.$$

Question 4/5

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\frac{5x+15}{x^3+x^2+4x+4} = \frac{2}{x+1} + \frac{7-2x}{x^2+4}$$

$$\int_0^2 \frac{5x+15}{x^3+x^2+4x+4}\,dx = \int_0^2 \frac{2}{x+1}\,dx - \int_0^2 \frac{2x}{x^2+4}\,dx + 7\int_0^2 \frac{1}{x^2+4}\,dx$$

$$= 2\left[\ln|x+1|\right]_0^2 - \left[\ln(x^2+4)\right]_0^2 + \frac{7}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= 2[\ln 3] - [\ln(8) - \ln 4] + \frac{7}{2}[\tan^{-1} 1 - \tan^{-1} 0]$$

Question 4/5

$$= 2[\ln 3] - [\ln(8) - \ln 4] + \frac{7}{2}[\tan^{-1} 1 - \tan^{-1} 0]$$

$$= \ln 9 - \ln 8 + \ln 4 + \frac{7}{2}\frac{\pi}{4}$$

$$\frac{36}{8} = \frac{9}{2}$$

$$= \ln\left(\frac{9 \times 4}{8}\right) + \frac{7\pi}{8}$$

$$= \underbrace{\ln\left(\frac{9}{2}\right)}_{\text{given}} + \frac{7\pi}{8}.$$

$$\alpha = \frac{7\pi}{8}.$$

Question 7

$u = x^7$

or $u = 1 - \frac{1}{x^7}$ efficient

Evaluate $\int \frac{1}{x^8 - x}\, dx$. *Hint:* Recommended to use substitution first.

$$\int \frac{1}{x^8 - x}\, dx = \int \frac{1}{x^8\left(1 - \frac{1}{x^7}\right)}\, dx$$

x^{-8}

$u = 1 - \frac{1}{x^7}$

$du = x^{-8}$

$$= \int \frac{1}{u}\, du = \ln\left|1 - \frac{1}{x^7}\right| + C$$

Alternatively,

$$\int \frac{1}{x^8 - x}\, dx = \int \frac{1 \cdot x^6}{x^6 \cdot x(x^7 - 1)}\, dx = \frac{1}{7}\int \frac{7x^6}{x^7(x^7 - 1)}\, dx$$

Question 7

partial fractions

$u = x^7 \quad du = 7x^6$

$$= \frac{1}{7}\int \frac{1}{u(u-1)}\,du. = \frac{1}{7}\int \frac{1}{u-1} - \frac{1}{u}\,du$$

$$= \frac{1}{7}\ln|u-1| - \frac{1}{7}\ln|u| + C$$

$$= \frac{1}{7}\ln|x^7-1| - \frac{1}{7}\ln|x^7| + C$$

$$= \frac{1}{7}\ln|x^7-1| - \ln|x| + C$$