Lecture 2: Functions

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Sets

- The **difference** of A and B is

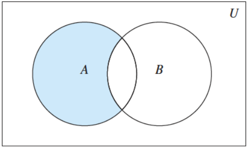
$$A - B = \{x : x \in A, x \notin B\}.$$

- The **complement** of A in U is

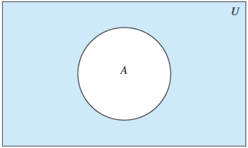
$$\bar{A} = \{x \in U : x \notin A\}.$$

- De Morgan's law:

$$\begin{aligned} \overline{A \cup B} &= \overline{A} \cap \overline{B} \\ \overline{A \cap B} &= \overline{A} \cup \overline{B} \end{aligned}$$



$A - B$ is shaded.



$\bar{A}$ is shaded.

Power set

- $\mathcal{P}(S)$ = all subsets of S. If $|S| = n$, then S has 2^n subsets

$$|\mathcal{P}(S)| = 2^n$$

- **Exercise 1**: Let $A = \{1\}$ and $B = \{2\}$ be two sets.
  (a) Find $\mathcal{P}(A), \mathcal{P}(B), \mathcal{P}(A \times B)$.

  (b) Do $\mathcal{P}(A) \times \mathcal{P}(B)$ and $\mathcal{P}(A \times B)$ have same size?

Inclusion-exclusion principle

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} |A_{i_1} \cap \cdots \cap A_{i_k}|$$

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- $n = 2$

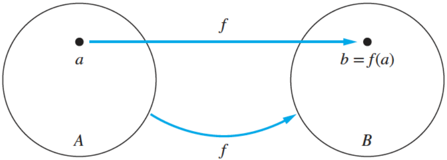
$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

- $n = 3$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| -$$
$$-|A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

Functions (definition)

- Let A and B be nonempty sets.
- $f : A \to B$ is a function if it assigns each element $a \in A$ to a **unique element** $b \in B$. Write $f(a) = b$.



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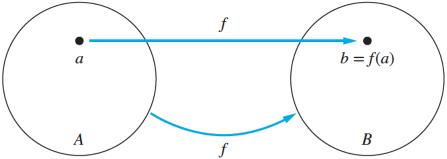
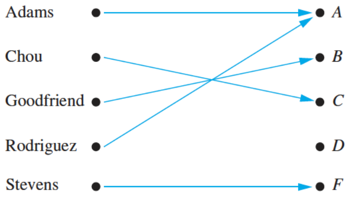


- A is called the **domain of f** (or the **input set**).
  B is called the **codomain of f** (or the **output set**).
- If $f(a) = b$, we call b the *image* of a and call a a *preimage* of b.

Example 1

The following is an assignment of grades in a discrete mathematics class. This assignment corresponds to a function $f : S \to T$.



(a) Write out the domain S of f.

Example 2

(b) Write out the codomain T of f.

(c) Write out the values of $f(s)$ for all elements $s \in S$.

Example 3

Let f be the function that assigns the last two bits of a bit string of length 3 to that string. For example, $f(010) = 10$.

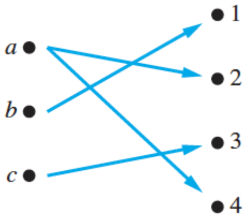
(a) Write out the domain A of f.

(b) Write out the codomain B of f.

(c) Write out the values of $f(a)$ for all elements $a \in A$.

Question 2

Is $f : \{a, b, c\} \to \{1, 2, 3, 4\}$ given by the following rule a function?



Question 3

Let $f : A \to B$ be a function. Is it always true that for any $b \in B$, there exists a unique element $a \in A$ such that $f(a) = b$?

Answer: No. There can be 2 situations.

1. There can be $b \in B$ s.t. there is no $a \in A$ with $f(a) = b$.

Question 3

2. There can be $b \in B$ such that there are more than one element $a \in A$ with $f(a) = b$.

Summary on functions

A function $f : A \to B$ is a rule (or an assignment) that assigns each value $a \in A$ to a unique value $b \in B$, that is,

$$f(a) = b$$

- Given $a \in A$, there is a unique $b \in B$ such that $f(a) = b$.
- Given $b \in B$, there can be more than one $a \in A$ such that $f(a) = b$.

One-to-one, onto, bijective

- A function $f : A \to B$ is **one-to-one** (also write **1-1**), or **injective**, if for any $a, b \in A$

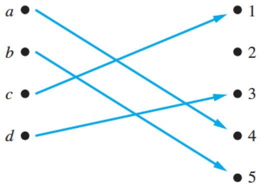
$$f(a) = f(b) \Leftrightarrow a = b$$

- f is **onto**, or **surjective**, if for any $b \in B$ there exists $a \in A$ such that $f(a) = b$.
- f is **bijective** (or a **bijection**) if it is both 1-1 and onto.

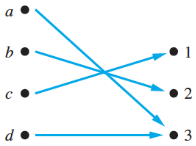
Example 4

Which of the following functions is 1-1, onto, or bijective?

(a) f given by the rule

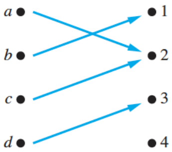


(b) $f : \{a, b, c, d\} \to \{1, 2, 3\}$ given by the rule



Example 4

(c) $f : \{a, b, c, d\} \to \{1, 2, 3, 4\}$ given by the rule



(d) $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x^2$.

Example 4

(e) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 5$.

Sums and products of functions

- $f : A \to B$ is called **real-valued** if its codomain is $B = \mathbb{R}$, and it is called **integer-valued** if its codomain is $B = \mathbb{Z}$.

Sums and products of functions

- $f : A \to B$ is called **real-valued** if its codomain is $B = \mathbb{R}$, and it is called **integer-valued** if its codomain is $B = \mathbb{Z}$.
- Let $f_1, f_2 : A \to B$ be real-valued (or integer-valued) functions. Then $f_1 + f_2$ and $f_1 f_2$ are functions from A to B defined by

$$\begin{aligned}(f_1 + f_2)(x) &= f_1(x) + f_2(x)\\ f_1 f_2(x) &= f_1(x) f_2(x)\end{aligned}$$

Increasing and decreasing functions

Let $f : A \to B$ be a function

- f is called **increasing** if

$$f(x) \leq f(y) \text{ whenver } x < y$$

- f is called **strictly increasing** if

$$f(x) < f(y) \text{ whenver } x < y$$

Increasing and decreasing functions

Let $f : A \to B$ be a function

- f is called **decreasing** if

$$f(x) \geq f(y) \text{ whenver } x < y$$

- f is called **strictly decreasing** if

$$f(x) > f(y) \text{ whenver } x < y$$

Example 5

(a) Is $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ increasing or decreasing?

(b) Is $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ increasing or decreasing?

Graphs of increasing and decreasing functions

Derivative test

Theorem 1

Assume that $f : A \to B$ is a differentiable function.

(a) If $f'(x) \geq 0$ for all $x \in A$, then f is increasing.
Further if $f'(x) \geq 0$ for all $x \in A$, f is strictly increasing.

(b) If $f'(x) \leq 0$ for all $x \in A$, then f is decreasing.
Further if $f'(x) < 0$ for all $x \in A$, f is strictly decreasing.

Example 6

Determine whether following functions are increasing or decreasing. Support your claim by drawing graphs of these functions.

(a) $f : [0, \pi/2] \to [0, 1]$ given by $f(x) = \sin x$.

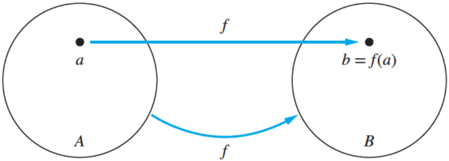
(b) $f : [0, \pi/2] \to [0, 1]$ given by $f(x) = \cos x$.

Example 7

Determine the intervals on which $f(x) = x^2$ is increasing and decreasing. Support your claim by drawing the graph of $f(x)$.

Question 1

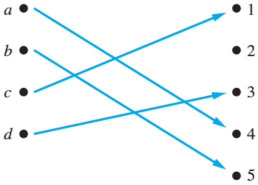
- A function $f : A \to B$ is a rule which assigns each ~~b~~ $a \in A$ to ~~a~~ $b \in B$, that is, $f(a) = b$.

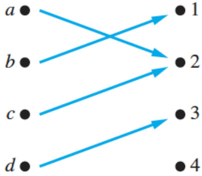


- Can this assignment be reverse, i.e. each element $b \in B$ is assigned to $a \in A$ if $f(a) = b$?

Question 1 answer

- The reverse assignment doesn't always work. Consider examples





Question 1 answer

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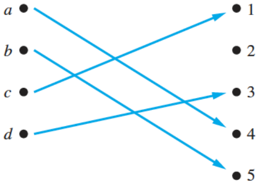
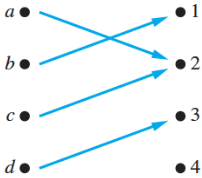
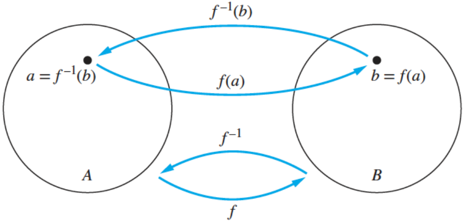




- For the reverse assignment to work, f needs to be both $1-1$ and onto, that is, f is a bijection.



Inverse function - definition

- Let $f : A \to B$ be $1 - 1$ and onto.
- The **inverse function** of f is $f^{-1} : B \to A$ that assigns $b \in B$ to $a \in A$ if $f(a) = b$:

$$f^{-1}(b) = a \Leftrightarrow f(a) = b.$$

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$$f^{-1}(b) = a \Leftrightarrow f(a) = b.$$

- We call f **invertible** if its inverse exists, that is,

f is both $1-1$ and onto.

Remarks

1. f^{-1} and $\dfrac{1}{f}$ are different functions.
2. Difference in notation
   - When writing f, we usually use x to denote its input: $f(x)$.
   - When writing f^{-1}, we usually use y to denote input: $f^{-1}(y)$.

Example 8

Given $f : \{a, b, c\} \to \{1, 2, 5\}$ defined by $f(a) = 1, f(b) = 2, f(c) = 5$. Find f^{-1} if it exists.

How to find f^{-1}?

Assume that $f : A \to B$ is given by a formula.

To find $f^{-1} : B \to A$, we follow 3 steps

1. Let $x \in B$ and put $y = f^{-1}(x)$.
2. Solve for y (in terms of x) based on the equation
$$\mathbf{f(y) = x.}$$
3. Give conclusion.

Example 9

In the following cases, determine whether f is invertible and find its inverse if it exists.

(a) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x + 1$.

1. Note that $f^{-1} : \mathbb{Z} \to \mathbb{Z}$. Let $x \in \mathbb{Z}$ and put $y = f^{-1}(x)$.
2. We have
$$f(y) = x \Rightarrow y + 1 = x \Rightarrow y = x - 1.$$
3. Conclusion
$$f^{-1}(x) = x - 1.$$

Example 9

(b) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

(c) $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f(x) = x^2$.

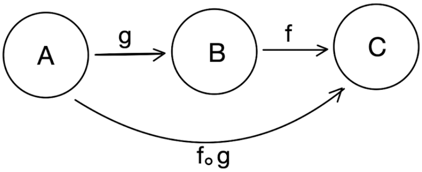
Example 9

(d) $f : \mathbb{R} \to \mathbb{R}^+$ defined by $f(x) = x^3$. Further, find the preimages of $1, 27, 64$ using f^{-1}.

Composition of functions

- Let $g : A \to B$ and let $f : B \to C$ be functions.
- The **composition** of f and g, denoted by $f \circ g$, is the function $f \circ g : A \to C$ defined by

$$(f \circ g)(a) = f(g(a))$$



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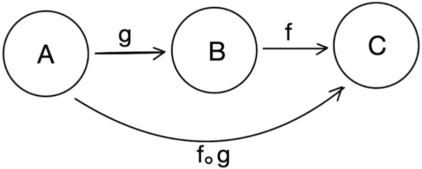

- **Remark**: $f \circ g \neq fg$

$$f \circ g(x) = f(g(x)) \text{ and } fg(x) = f(x)g(x)$$

Example 11

Find $f \circ g$ and $g \circ f$ in following cases

(a) $f, g : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 2x + 1$ and $g(x) = 3x + 2$.

(b) $f : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ with $f(x) = x^2$, $g : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ with $g(x) = \sqrt{x}$.

Example 12

Find $f^{-1}, g^{-1}, g \circ f, (g \circ f)^{-1}$ for

$$f, g : \mathbb{R} \to \mathbb{R} \text{ defined by } f(x) = 2x + 1 \text{ and } g(x) = 3x + 2.$$

Exercise 2

Let $f : A \to B$ be both $1 - 1$ and onto. Show that

$f^{-1} \circ f(x) = x$ for any $x \in A$ and $f \circ f^{-1}(y) = y$ for any $y \in B$.