# Week 9: Scaling, Rotation, Shear in 2D

# Table of contents

# Linear transformation

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation** if it
  1. preserves addition
  $$T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$$
  2. and preserves scalar multiplication
  $$T(c\vec{x}) = cT(\vec{x})$$
- $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear $\Leftrightarrow$ each component in $T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is a linear combination of $x_1, \ldots, x_n$.

# Matrix representation of linear transformation

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear $\Leftrightarrow$ there exists an $m \times n$ matrix $M$:

$$T(\vec{x}) = M\vec{x}$$

  $M$ is called the **matrix representation** of $T$.

# Matrix representation of linear transformation

- There are 2 ways to determine $M$

1. $$T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix} \Rightarrow M = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$
2. If $\vec{e}_1, \dots, \vec{e}_n$ are *standard unit vectors* of $\mathbb{R}^n$, then

$$M = [T(\vec{e}_1) \ \cdots \ T(\vec{e}_n)]$$

# Exercise 1

Determine whether $T$ is linear. Find its matrix if it is linear.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + y \\ 2x + y \end{pmatrix}$

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - \sqrt{y} \\ 2x + y + 1 \end{pmatrix}$

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ 2x + y \end{pmatrix}$

# Exercise 2

In this exercise, we learn that dot product and cross product can be explained as linear transformations!

(a) Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and let $T_{\vec{u}} : \mathbb{R}^3 \to \mathbb{R}$ be defined by $T_{\vec{u}}(\vec{x}) = \vec{u} \cdot \vec{x}$.

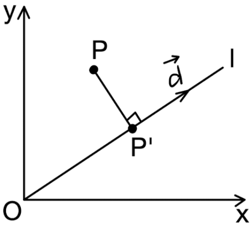
Show that $T_{\vec{u}}$ is a linear transformation. Write out its matrix.

(b) Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and let $C_{\vec{u}} \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $C_{\vec{u}}(\vec{x}) = \vec{u} \times \vec{x}$.

Show that $C_{\vec{u}}$ is a linear transformation. Write out its matrix.

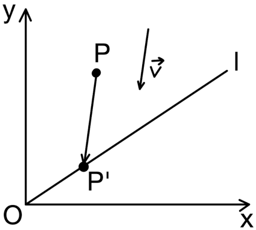
# Projections in $\mathbb{R}^2$

## Orthogonal projection




$$M = \frac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T$$

## Skew projection



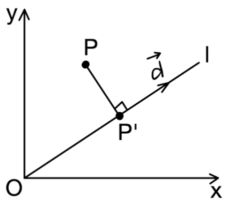

$$M = I_2 - \frac{\vec{v}\vec{n}^T}{\vec{v}\cdot\vec{n}}$$

# Question 1

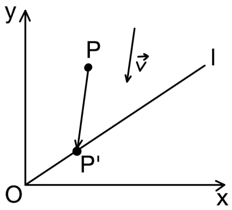
A point $\vec{x}$ is called fixed by a map $T \Leftrightarrow T(\vec{x}) = \vec{x}$.

Which points are fixed by orthogonal projection and skew projection?

**Orthogonal projection**

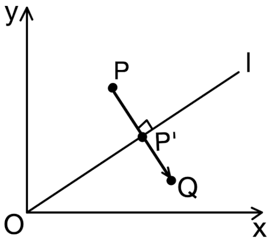



**Skew projection**



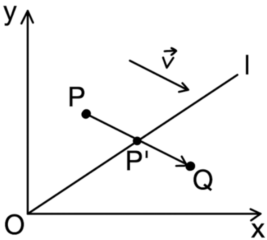
# Reflections in $\mathbb{R}^2$

**Orthogonal reflection**




$$M = \frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_2$$

**Skew reflection**

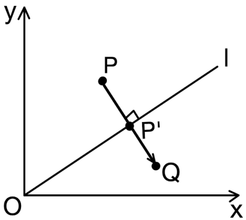



$$M = I_2 - \frac{2}{\vec{v}\cdot\vec{n}}\vec{v}\vec{n}^T$$

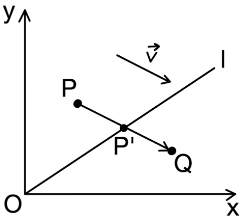
# Question 2

Which points are fixed by reflections?

**Orthogonal reflection**




**Skew reflection**



# Scaling

- A **scaling** (centered at the origin) is a map $S : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

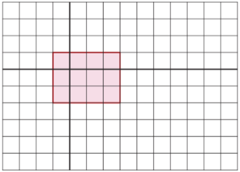
$$S \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix}$$

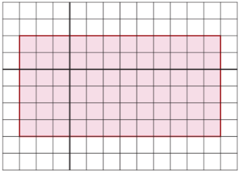
  for some constants $a, b \in \mathbb{R}$.

- All $x$-coordinates are scaled by $a$, all $y$-coordinates are scaled by $b$.

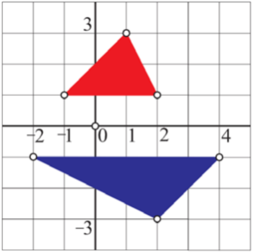
# Example

- The small rectangle is scaled into large rectangle





- The red triangle is scaled into blue triangle



# Scaling matrix

## Theorem 1

The representation matrix of the scaling $S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix}$ is

$$M = M_{a,b} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

# Example 1

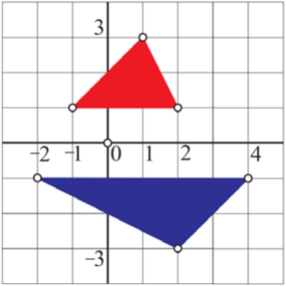
Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the scaling which scale all $x$-coordinates by $2$ and scale all $y$-coordinates by -$1$.

(a) What is the matrix representation $M$ of $S$?

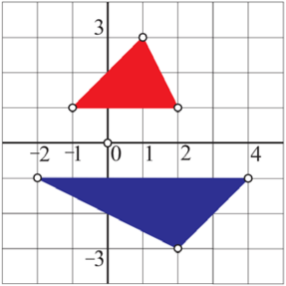
(b) What are the images of the points $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

# Example 1

(c) Check that $S$ maps the red triangle into the blue triangle below.



(d) Can you compare the areas of these 2 triangles?

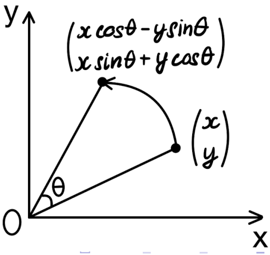


# Matrix of rotation

**Theorem 2**

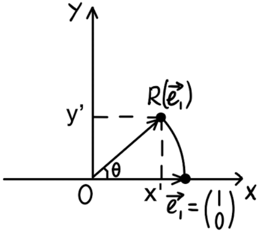
The counter-clockwise rotation $R : \mathbb{R}^2 \to \mathbb{R}^2$ around the origin over the angle $\theta$ has matrix representation

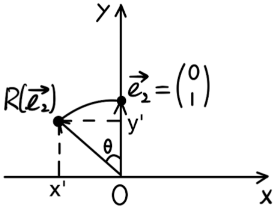
$$M = M(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



# Proof (sketch)

The matrix is $M = [R(\vec{e}_1)\ R(\vec{e}_2)]$



# Example 2

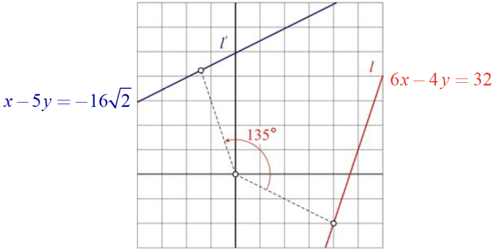
(a) What is the counter-clockwise rotation (around O) matrix over $90^0$?

(b) What is the counter-clockwise rotation (around O) matrix over $135^0$?

# Example 2

(c) Find the images of $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ by $135^\circ$ rotation about O.

(d) Find the image of the line $3x - 2y = 16$ by $135^\circ$ rotation about O.



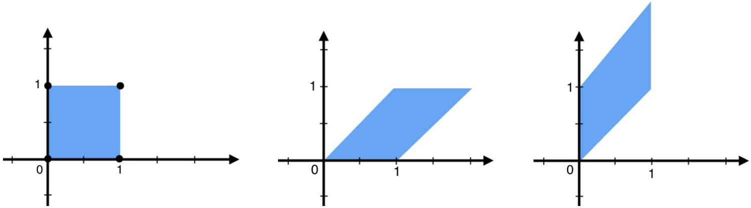
# Example 3

(a) What is matrix $M$ of the rotation by $60^0$?

(b) Computation the matrix of rotation by $120^0, 180^0, 360^0$ by computing $M^2, M^3, M^6$. Check that $M^6 = I_2$.

# Shear

- A **shear** is a map which transforms a square into a parallelogram.



# Shear

- The **shear** with respect to the line $l : \vec{n} \cdot \vec{x} = 0$ in the direction of the **shearing vector** $\vec{v}$ ( $\vec{v} \parallel l$) is a map $S : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$S(\vec{x}_0) = \vec{x}_0 + \frac{\vec{n} \cdot \vec{x}_0}{||\vec{n}||}\vec{v}$$

# Remark

$$S(\vec{x}_0) = \vec{x}_0 + \frac{\vec{n} \cdot \vec{x}_0}{||\vec{n}||}\vec{v}$$

- $\vec{v}$ must be parallel to $l$ for the **shear S to be defined**. So

$$\vec{v} \cdot \vec{n} = 0$$

# Remark

$$S(\vec{x}_0) = \vec{x}_0 + \frac{\vec{n} \cdot \vec{x}_0}{||\vec{n}||}\vec{v}$$

- $l$ has equation $\vec{n} \cdot \vec{x} = 0$. The distance from the point $\vec{x}_0$ to $l$ is

$$d(\vec{x}_0, l) = \frac{|\vec{n} \cdot \vec{x}_0|}{||\vec{n}||}$$

# Remark

$$S(\vec{x}_0) = \vec{x}_0 + \frac{\vec{n} \cdot \vec{x}_0}{||\vec{n}||}\vec{v}$$

- $l$ has equation $\vec{n} \cdot \vec{x} = 0$. The distance from the point $\vec{x}_0$ to $l$ is

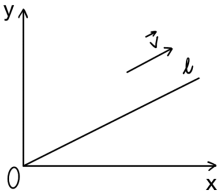
$$d(\vec{x}_0, l) = \frac{|\vec{n} \cdot \vec{x}_0|}{||\vec{n}||}$$

- The shear $S$ shifts $\vec{x}_0$

  in the direction $\vec{v}$ by the factor $\pm d(\vec{x}_0, l)$.

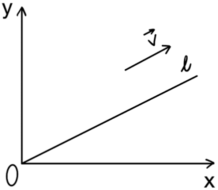
# Example

- A shear is defined based on a line $l$ through O and a vector $\vec{v} \parallel l$.

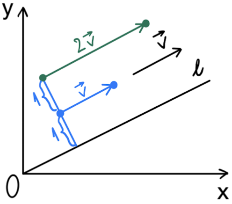


# Example

- A shear is defined based on a line $l$ through O and a vector $\vec{v} \parallel l$.




- Points at distance $1$ are shifted by $\vec{v}$. Points at distance $2$ are shifted by $2\vec{v}$.



# Example 4

Consider the shear $S$ w.r.t. the **x-axis** in the direction of $\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

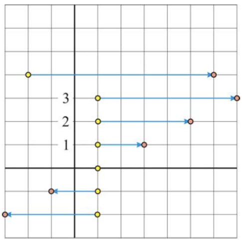
(a) Check that

$$S\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + y_0 \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

that is, $S$ shifts any point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ parallel to the $x$-axis by $y_0\vec{v}$.

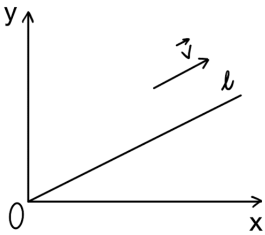
# Example 4

(b) Check that in the figure below, all “yellow” points are shifted to “red” points. Could you explain why the points above x-axis are shifted to the right and the points below x-axis are shifted to the left?



# Question

For what points $\vec{x}_0$ do we have $S(\vec{x}_0) = \vec{x}_0$, that is, $\vec{x}_0$ is fixed by the shear?



## Matrix of shear

### Theorem 3

The shear w.r.t. $l : \vec{n} \cdot \vec{x} = 0$ in the direction of the shearing vector $\vec{v}$ for which $\vec{n} \cdot \vec{v} = 0$ has matrix

$$M = M_{\vec{n},\vec{v}} = I_2 + \frac{1}{||\vec{n}||} \vec{v} \vec{n}^T$$

# Matrix of shear

**Theorem 3**

The shear w.r.t. $l : \vec{n} \cdot \vec{x} = 0$ in the direction of the shearing vector $\vec{v}$ for which $\vec{n} \cdot \vec{v} = 0$ has matrix

$$M = M_{\vec{n},\vec{v}} = I_2 + \frac{1}{||\vec{n}||}\vec{v}\vec{n}^T$$

**Question**. Why is there condition $\vec{n} \cdot \vec{v} = 0$?

# Proof

Let $\vec{x}_0$ be any point

$$\begin{aligned} S(\vec{x}_0) &= \vec{x}_0 + \frac{\vec{n}\cdot\vec{x}_0}{||\vec{n}||}\vec{v} \\ &= \vec{x}_0 + \frac{1}{||\vec{n}||}(\vec{n}\cdot\vec{x}_0)\vec{v} \\ &= \vec{x}_0 + \frac{1}{||\vec{n}||}\vec{v}\vec{n}^T\vec{x}_0 \\ &= \left(I_2 + \frac{\vec{v}\vec{n}^T}{||\vec{n}||}\right)\vec{x}_0 \end{aligned}$$

# Example 5

Let $l : 3x + 4y = 0$ and $\vec{v} = \begin{bmatrix} 8 \\ -6 \end{bmatrix}$.

(a) What is $M_{\vec{n},\vec{v}}$?

(b) What are the images of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$?

# Example 5

(c) What is the image of the line $n : 3x - y = 5$?

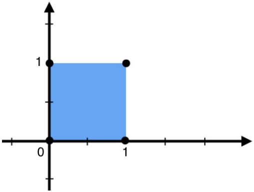
## Example 5

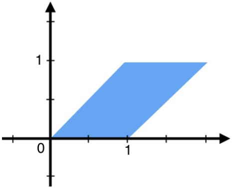
(d) Show that the image of $m : 3x + 4y = 5$ is itself (note that $m \parallel l$).

# Exercise 2 (Horizontal shear)

(a) Consider the shear w.r.t. $l : y = 0$ (the x-axis) in direction $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

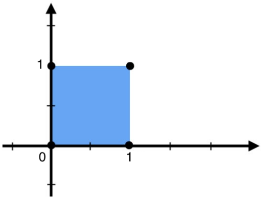
Show that the image of the unit square is the parallelogram as below. What is the area of the resulting parallelogram?

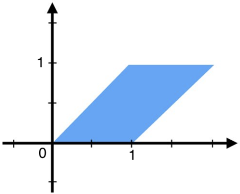


# Exercise 2 (Vertical shear)

(b) Consider the shear w.r.t. y-axis in direction $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Show that the image of the unit square is the parallelogram as below. What is the area of the resulting parallelogram?



# Exercise 2

(c) Consider the shear w.r.t. $l : x - y = 0$ in direction $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Sketch the image of the unit square in part a and compute its area.

# Composition of linear transformation

- Let $\mathbb{R}^m \xrightarrow{S} \mathbb{R}^n \xrightarrow{T} \mathbb{R}^k$ be a sequence of linear transformations. The composition $T \circ S : \mathbb{R}^m \to \mathbb{R}^k$ is defined by

$$T \circ S(\vec{x}) = T(S(\vec{x}))$$

- We will see that $T \circ S$ is another linear transformation. Further, if $M_T, M_S$ are matrices of $T, S$, the matrix of $T \circ S$ is

$$M_{T \circ S} = M_T M_S.$$

# Exercise 3

Let $P$ be the projection onto $l : \sqrt{3}x - y = 0$ and let $R$ be the reflection through $m : x - \sqrt{3}y = 0$.

(a) Describe $P \circ R$, that is, find a formula for $P \circ R\left(\vec{x}\right)$.

(b) Find the matrices of $M_P, M_R, M_{P\circ R}$ of $P, R, P \circ R$.

(c) Find the points which are fixed by $P \circ R$.