TUTORIAL 1 Solutions

1) a) Find power sets of $A = \{a, b\}$ and $B = \{\emptyset, \{\emptyset\}\}$

$$B = \{\underbrace{\emptyset}_{a}, \underbrace{\{\emptyset\}}_{b}\}$$

$$P(A) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$$

$$P(B) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$$

b) Which of the following sets can be a power set? → size $= 2^n$

(A) $\emptyset$ ✗ (B) $\{\emptyset, \{a\}\}$ ✓ (C) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$ ✗ (D) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ ✓

(A) size $= 0$

(B) size $= 2$, power set of $\{a\}$

(C) size $= 3$

(D) size $= 4$, power set of $\{a, b\}$

2) How many numbers in $\{1, 2, \dots, n\}$ are divisible by d?

Floor function: $\lfloor x \rfloor$ = biggest integer $\leq x$

$$\lfloor 1.2 \rfloor = \lfloor 1.99 \rfloor = 1$$

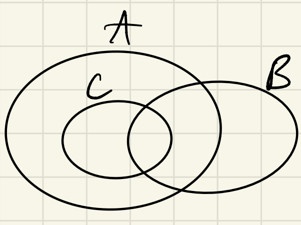
$$\lfloor -1.2 \rfloor = \lfloor -1.9 \rfloor = -2$$

Answer: $\left\lfloor \frac{n}{d} \right\rfloor$

a) How many numbers in $\{1, 2, \dots, 100\}$ are divisible by 3 or 4 or 6?

A, B, C = sets of integers in $\{1, \dots, 100\}$ divisible by $3, 4, 6$.

→ need to compute $|A \cup B \cup C|$.



Since any number divisible by 6 is also divisible by 3, $C \subset A \Rightarrow |A \cup B \cup C| = |A \cup B|$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= \left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{4} \right\rfloor - \left\lfloor \frac{100}{12} \right\rfloor$$

$$= 33 + 25 - 8 = 50.$$

b) How many numbers in $\{1, 2, \ldots, 100\}$ are divisible by 4 or 5 or 6?

A, B, C = sets of numbers that are divisible by 4, 5, 6

$\mathrm{lcm}(a,b)$ = least common multiple of a, b

$\gcd(a,b)$ = greatest common divisor of a, b

$$\boxed{\mathrm{lcm}(a,b) = \frac{ab}{\gcd(a,b)}}$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= \left\lfloor \frac{100}{4} \right\rfloor + \left\lfloor \frac{100}{5} \right\rfloor + \left\lfloor \frac{100}{6} \right\rfloor - \left\lfloor \frac{100}{\mathrm{lcm}(4,5)} \right\rfloor - \left\lfloor \frac{100}{\mathrm{lcm}(4,6)} \right\rfloor$$

$$- \left\lfloor \frac{100}{\mathrm{lcm}(5,6)} \right\rfloor + \left\lfloor \frac{100}{\mathrm{lcm}(4,5,6)} \right\rfloor$$

$$= 25 + 20 + 16 - \left\lfloor \frac{100}{20} \right\rfloor - \left\lfloor \frac{100}{12} \right\rfloor - \left\lfloor \frac{100}{30} \right\rfloor + \left\lfloor \frac{100}{\mathrm{lcm}(4,30)} \right\rfloor$$

$$= 61 - 5 - 8 - 3 + \left\lfloor \frac{100}{60} \right\rfloor = 46.$$

3) A president, a treasurer, a secretary are chosen from Alice, Bob, Cyd, Dan

a) How many ways are there?

b) How many ways if Bob cannot be treasurer and Cyd cannot be secretary?

a) choose president → 4 choices

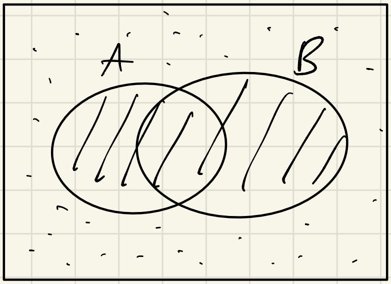
choose treasurer → 3 choices

choose secretary → 2 choices

$\#$ choices $= 4 \times 3 \times 2 = 24$

b) A = choices with Bob = treasurer

U



$|A| < \begin{array}{l} \text{president} \to 3 \text{ choices} \\ \text{secretary} \to 2 \text{ choices} \end{array} \Rightarrow |A| = 6$

B = choices with Cyd = secretary

$|B| < \begin{array}{l} \text{president} \to 3 \text{ choices} \\ \text{treasurer} \to 2 \text{ choices} \end{array} \Rightarrow |B| = 6$

$A \cap B$ = choices with Bob = treasurer and Cyd = secretary

choose president → 2 choices → $|A \cap B| = 2$

$|A \cup B| = 6 + 6 - 2 = 10 \Rightarrow \#$ choices $= 24 - |A \cup B| = 14$.

4, Report on 1000 people:

818 like candy, 723 ice cream, 645 cake

562 candy + ice cream, 463 candy + cake, 470 ice cream + cake

310 candy + ice cream + cake

A, B, C = sets of people who like candy, ice cream, cake

$|A| = 818$, $|B| = 723$, $|C| = 645$,

$|A \cap B| = 562$, $|A \cap C| = 463$, $|B \cap C| = 470$, $|A \cap B \cap C| = 310$

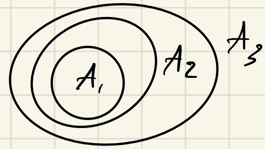
$|A \cup B \cup C| = 818 + 723 + 645 - 562 - 463 - 470 + 310 = 1001$

Since $\underbrace{|A \cup B \cup C| \leq 1000}$, this is impossible.

(the report is on 1000 people)

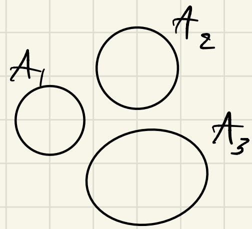
5) $|A_1| = 100$, $|A_2| = 1000$, $|A_3| = 10000$

a) $A_1 \subset A_2$ and $A_2 \subset A_3$



$$|A_1 \cup A_2 \cup A_3| = |A_3| = 10000$$

b) $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = \emptyset$



$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$$

$$= 11100$$

c) $|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| = 2$

$$|A_1 \cap A_2 \cap A_3| = 1$$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

$$= 11100 - 2 - 2 - 2 + 1 = 11095.$$

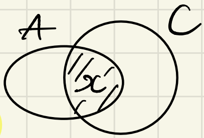
6) $A \oplus B$ = all elements in A or B, but not both.

Given $A \oplus C = B \oplus C$. Show that $A = B$.

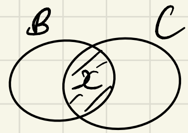
Claim 1: $A \subset B$

Let $x \in A$ be an arbitrary element. We need to prove $x \in B$.

Case 1: $x \in C$



Since $x \in A$, $x \in A \cap C \Rightarrow x \notin A \oplus C \Rightarrow x \notin B \oplus C$

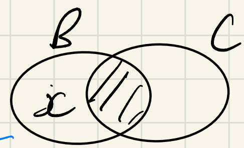


Now $x \notin B \oplus C$ and $x \in C \Rightarrow x \in B \cap C$

$\Rightarrow x \in B$

Case 2: $x \notin C$

Since $x \in A$, $x \in A \oplus C \Rightarrow x \in B \oplus C$



Now $x \in B \oplus C$ and $x \notin C \Rightarrow x \in B$

In any case $x \in B \Rightarrow A \subset B$

Claim 2: $B \subset A$

By Claim 1, if $A \oplus C = B \oplus C$, then $A \subset B$.

Since $B \oplus C = A \oplus C$, we have $B \subset A$.

$\therefore A = B$.