Lecture 3: Cross product and Intersection

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Parallelism and orthogonality

Consider 2 vectors

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

- $\vec{u} \parallel \vec{v} \Leftrightarrow \vec{u} = c\vec{v}$ or $\underbrace{\dfrac{u_1}{v_1} = \dfrac{u_2}{v_2} = \dfrac{u_3}{v_3}}_{\text{where } u_i = 0 \text{ iff } v_i = 0}$

Parallelism and orthogonality

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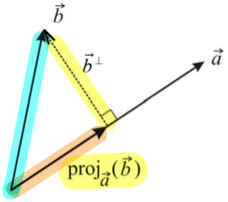
- $\vec{u} \parallel \vec{v} \Leftrightarrow \vec{u} = c\vec{v}$ or $\dfrac{u_1}{v_1} = \dfrac{u_2}{v_2} = \dfrac{u_3}{v_3}$
- $\vec{u} \perp \vec{v} \Leftrightarrow \vec{u} \cdot \vec{v} = 0$

Orthogonal projection

Let $\vec{a}, \vec{b}$ be two vectors with $\vec{a} \neq \vec{0}$.



- The orthogonal projection of $\vec{b}$ onto $\vec{a}$ is

$$\text{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a}$$

- The orthogonal complement of $\vec{b}$ on $\vec{a}$ is

$$\vec{b}^{\perp} = \vec{b} - \text{proj}_{\vec{a}}(\vec{b})$$

Lines in $\mathbb{R}^2$

1. The line through $P_0 = (x_0, y_0)$ with direction $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ has vector equation and parametric equation

$$(x, y) = (x_0, y_0) + t \begin{bmatrix} a \\ b \end{bmatrix} \text{ and } \begin{cases} x = x_0 + at \\ y = y_0 + bt \end{cases}$$

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2. The line through $P_0 = (x_0, y_0)$ with normal $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ has equation

$$a(x - x_0) + b(y - y_0) = 0$$

Lines in $\mathbb{R}^3$

- The line through $P_0 = (x_0, y_0, z_0)$ with direction $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has vector equation and parametric equation

$$(x, y, z) = (x_0, y_0, z_0) + t \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ and } \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

- Remark: There is **no normal vector** for a line in $\mathbb{R}^3$. So there is no normal equation for lines in $\mathbb{R}^3$

Planes

- The plane through $P(x_0, y_0, z_0)$ with normal $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Planes

- The plane through $P(x_0, y_0, z_0)$ with normal $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

- The plane through $P(x_0, y_0, z_0)$ with direction vectors $\vec{u}, \vec{v}$ has vector equation and parametric equation

$$(x, y, z) = P + s\vec{u} + t\vec{v} \text{ and } \begin{cases} x = x_0 + su_1 + tv_1 \\ y = y_0 + su_2 + tv_2 \\ z = z_0 + su_3 + tv_3 \end{cases}$$

Dot product vs. cross product

- The dot product of $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is **a real number**

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

- The **cross product** of $\vec{u}$ and $\vec{v}$, denoted by $\vec{u} \times \vec{v}$, is **vector**.
  To properly define $\vec{u} \times \vec{v}$, we need the concept of **determinant**.

Determinant of 2×2 matrices

The determinant of the 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, denoted by $\det(A)$ (or $|A|$) is

$$\det(A) = |A| = a_{11}a_{22} - a_{12}a_{21}$$

det(A) = (main diagonal) - (anti-diagonal)

Examples

- $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2$
- $\begin{vmatrix} 1 & 5 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 3 \cdot 5 = -11$

Cross product

- Consider $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ in $\mathbb{R}^3$.
- The **cross product** of $\vec{u}$ and $\vec{v}$, denoted by $\vec{u} \times \vec{v}$, is the **vector** defined by

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

Compute $\vec{u} \times \vec{v}$

Form the matrix with "1st column $= \vec{u}$ " and "2nd column $= \vec{v}$ "

$$[\vec{u}\ \vec{v}] = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix}$$

Compute $\vec{u} \times \vec{v}$

Form the matrix with "1st column $= \vec{u}$" and "2nd column $= \vec{v}$"

$$[\vec{u}\ \vec{v}] = \begin{pmatrix} \cancel{u_1} & \cancel{v_1} \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix}$$

1. 1st component of $\vec{u} \times \vec{v}$ = determinant of the matrix obtained by deleting the 1st row

$$\begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} = u_2 v_3 - u_3 v_2$$

Compute $\vec{u} \times \vec{v}$

Form the matrix with "1st column $= \vec{u}$" and "2nd column $= \vec{v}$"

$$[\vec{u}\ \vec{v}] = \begin{pmatrix} u_1 & v_1 \\ \cancel{u_2} & \cancel{v_2} \\ u_3 & v_3 \end{pmatrix}$$

2. 2nd component = **negative** of the determinant of the matrix obtained by deleting the 2nd row.

$$-\begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} = -(u_1 v_3 - u_3 v_1) = u_3 v_1 - u_1 v_3$$

Compute $\vec{u} \times \vec{v}$

3. 3rd component of $\vec{u} \times \vec{v}$ = determinant of the matrix obtained by deleting the 3rd row.

$$\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1$$

$$[\vec{u}\ \vec{v}] = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ \cancel{u_3} & \cancel{v_3} \end{pmatrix}$$

Compute $\vec{u} \times \vec{v}$

3. 3rd component of $\vec{u} \times \vec{v}$ = determinant of the matrix obtained by deleting the 3rd row.

$$\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1$$

Conclusion

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Compute $\vec{u} \times \vec{v}$

Put up the matrix $[\vec{u}\ \vec{v}] = \begin{pmatrix} \cancel{u_1} & \cancel{v_1} \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix}$

1. 1st component of $\vec{u} \times \vec{v}$

$$\text{Delete 1st row} \Rightarrow \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} = u_2 v_3 - u_3 v_2$$

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Put up the matrix $[\vec{u}\ \vec{v}] = \begin{pmatrix} u_1 & v_1 \\ \cancel{u_2} & \cancel{v_2} \\ u_3 & v_3 \end{pmatrix}$

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3. 3rd component of $\vec{u} \times \vec{v}$

$$\text{Delete 3rd row} \Rightarrow \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = u_1v_2 - u_2v_1$$

Example 1

Compute $\vec{w} = \vec{u} \times \vec{v}$ in the following cases.

In each case, check that $\vec{w}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

(a) $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

$\vec{w} \perp \vec{u} \Leftrightarrow \vec{w} \cdot \vec{u} = 0$

$\vec{w} \perp \vec{v} \Leftrightarrow \vec{w} \cdot \vec{v} = 0$

$[\vec{u}\ \vec{v}] = \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ \cancel{-2} & \cancel{1} \end{pmatrix} \Rightarrow \vec{w} = \vec{u} \times \vec{v}$

$w_1 = \begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} = 2 \cdot 1 - 0 \cdot (-2) = 2$

$w_2 = - \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = -(1 \cdot 1 - 3 \cdot (-2)) = -7$

$w_3 = \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = 1 \cdot 0 - 3 \cdot 2 = -6$

$\vec{w} = \vec{u} \times \vec{v} = \begin{pmatrix} 2 \\ -7 \\ -6 \end{pmatrix}$

$\vec{w} \cdot \vec{u} = 2 \cdot 1 + (-7) \cdot 2 + (-6) \cdot (-2) = 0 \Rightarrow \vec{w} \perp \vec{u}$

$\vec{w} \cdot \vec{v} = 0 \Rightarrow \vec{w} \perp \vec{v}$

Example 1

(b) $\vec{u} = \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$.

$$[\vec{u}\ \vec{v}] = \begin{pmatrix} -1 & 2 \\ 4 & 0 \\ \cancel{5} & \cancel{11} \end{pmatrix} \Rightarrow \vec{w} = \vec{u} \times \vec{v} = \begin{pmatrix} 4 \cdot 11 - 0 \cdot 5 \\ -((-1) \cdot 11 - 2 \cdot 5) \\ (-1) \cdot 0 - 2 \cdot 4 \end{pmatrix} = \begin{pmatrix} 44 \\ 21 \\ -8 \end{pmatrix}$$

$$\vec{w} \cdot \vec{u} = 44 \cdot (-1) + 21 \cdot 4 + (-8) \cdot 5 = 0 \Rightarrow \vec{w} \perp \vec{u}$$

$$\vec{w} \cdot \vec{v} = 44 \cdot 2 + 21 \cdot 0 + (-8) \cdot 11 = 0 \Rightarrow \vec{w} \perp \vec{v}$$

Remarks

- $\vec{u} \times \vec{v}$ is only defined for $\vec{u}, \vec{v}$ in $\mathbb{R}^3$.

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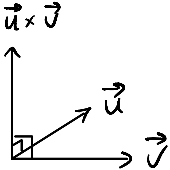
Remarks

- $\vec{u} \times \vec{v}$ is only defined for $\vec{u}, \vec{v}$ in $\mathbb{R}^3$.
- Unlike dot product $\vec{u} \cdot \vec{v}$ which gives a **scalar** (a real number), the cross product $\vec{u} \times \vec{v}$ gives a **vector** in $\mathbb{R}^3$.
- Observation: The vector $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.



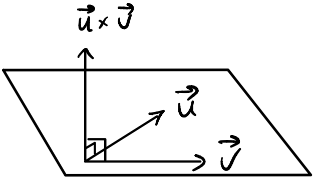
Cross product as normal vector

Theorem 1

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$ and put $\vec{w} = \vec{u} \times \vec{v}$. Then $\vec{w}$ is orthogonal to both $\vec{u}$ and $\vec{v}$, that is,

$$\vec{w} \cdot \vec{u} = \vec{w} \cdot \vec{v} = 0$$

In particular, $\vec{w}$ is a normal vector to any plane which has direction vectors $\vec{u}$ and $\vec{v}$.



Theorem 1 proof

Put $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Then $[\vec{u}\ \ \vec{v}] = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix}$

$$\vec{w} = \boxed{\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}}$$

$$\vec{w} \cdot \vec{u} = u_1(u_2 v_3 - u_3 v_2) + u_2(u_3 v_1 - u_1 v_3) + u_3(u_1 v_2 - u_2 v_1)$$

$$= \cancel{u_1 u_2} v_3 - \cancel{u_1 u_3} v_2 + \cancel{u_2 u_3} v_1 - \cancel{u_2 u_1} v_3 + \cancel{u_3 u_1} v_2 - \cancel{u_3 u_2} v_1$$

$$= 0 \Rightarrow \vec{w} \perp \vec{u}$$

Similarly $\vec{w} \perp \vec{v}$.

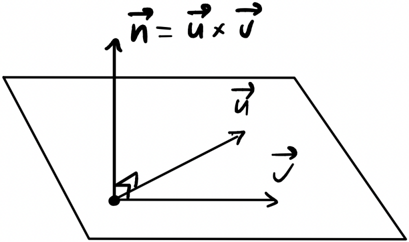
Cross product as normal vector

- If $\vec{n} = \vec{u} \times \vec{v}$, then $\vec{n}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

Cross product as normal vector

- If $\vec{n} = \vec{u} \times \vec{v}$, then $\vec{n}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.
- A plane with direction vectors $\vec{u}, \vec{v}$ has normal vector

$$\vec{n} = \vec{u} \times \vec{v}$$

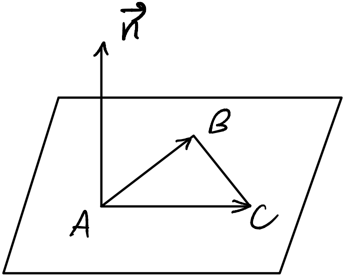


Exercise 1

(a) Find the general equation of the plane α passing through

$$A = (0,1,1),\ B = (2,3,4),\ C = (-1,1,5)$$

(b) Find the distance from the origin O to α.



(a) The plane has direction vectors

$$\vec{AB} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \quad \vec{AC} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$$

A normal vector to α is $\vec{n} = \vec{AB} \times \vec{AC}$

$$[\vec{AB}\ \vec{AC}] = \begin{pmatrix} 2 & -1 \\ 2 & 0 \\ \cancel{3} & \cancel{4} \end{pmatrix} \Rightarrow \vec{n} = \begin{pmatrix} 2\cdot 4 - 0\cdot 3 \\ -(2\cdot 4 - (-1)\cdot 3) \\ 2\cdot 0 - (-1)\cdot 2 \end{pmatrix} = \begin{pmatrix} 8 \\ -11 \\ 2 \end{pmatrix}$$

Exercise 1

(a) Find the general equation of the plane α passing through

$$A = (0, 1, 1),\ B = (2, 3, 4),\ C = (-1, 1, 5)$$

(b) Find the distance from the origin O to α.

(a) The general equation of α is

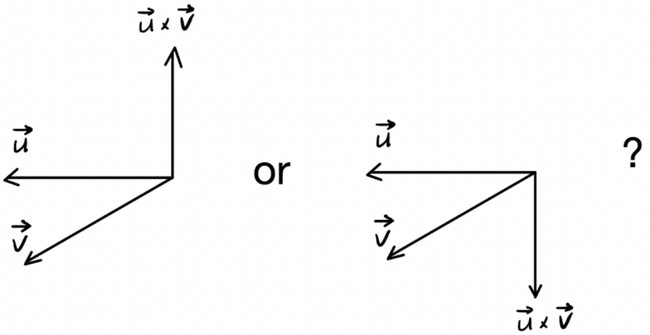
$$8x - 11(y-1) + 2(z-1) = 0$$

$$8x - 11y + 2z + 9 = 0$$

(b) $$d(O, \alpha) = \frac{|9|}{\sqrt{8^2 + (-11)^2 + 2^2}} = \frac{9}{\sqrt{189}}$$

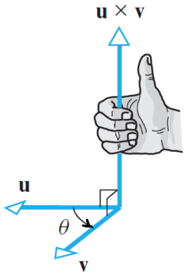
Direction of $\vec{u} \times \vec{v}$

$\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v} \Rightarrow$ two possible directions for $\vec{u} \times \vec{v}$.
Which one is correct?



Right-hand rule

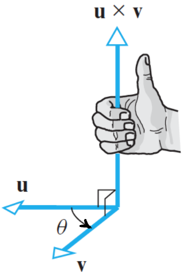
Put $\theta = \angle(\vec{u}, \vec{v})$ and assume $\vec{u}$ is rotated through θ until it coincides $\vec{v}$.



Right-hand rule

Put $\theta = \angle(\vec{u}, \vec{v})$ and assume $\vec{u}$ is rotated through θ until it coincides $\vec{v}$.

1. Put the fingers of the right hand so that they are cupped in the direction of rotation.
2. The thumb indicates the direction of $\vec{u} \times \vec{v}$.

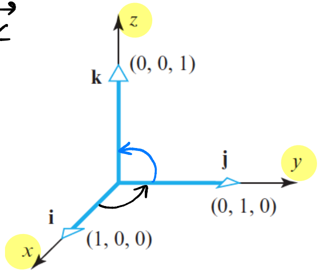


Example 2

$$\vec{i} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \vec{j} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \vec{k} = \begin{bmatrix}0\\0\\1\end{bmatrix} = \text{unit vectors along x-axis, y-axis, z-axis.}$$

(a) Verify that $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$

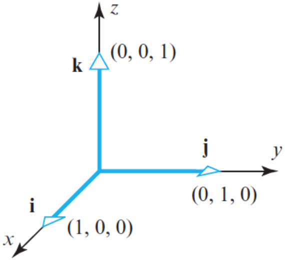
$$[\vec{i}\ \vec{j}] = \begin{pmatrix}1 & 0\\0 & 1\\0 & 0\end{pmatrix} \Rightarrow \vec{i} \times \vec{j} = \begin{pmatrix}0\\0\\1\end{pmatrix} = \vec{k}$$



Example 2

(b) Practice right-hand rule with the vectors i, j, k in part a.

$$\vec{i} \times \vec{j} = \vec{k},\ \vec{j} \times \vec{k} = \vec{i},\ \vec{k} \times \vec{i} = \vec{j}$$



Exercise

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$. Prove that

(a) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

Put $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$$[\vec{u}\ \vec{v}] = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ \cancel{u_3} & \cancel{v_3} \end{pmatrix} \Rightarrow \vec{u} \times \vec{v} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

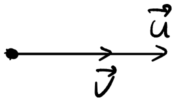
$$[\vec{v}\ \vec{u}] = \begin{pmatrix} v_1 & u_1 \\ v_2 & u_2 \\ \cancel{v_3} & \cancel{u_3} \end{pmatrix} \Rightarrow \vec{v} \times \vec{u} = \begin{pmatrix} u_3 v_2 - u_2 v_3 \\ u_1 v_3 - u_3 v_1 \\ u_2 v_1 - u_1 v_2 \end{pmatrix}$$

$$\Bigg\} \ \vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

Exercise

(b) If $\vec{u} \parallel \vec{v}$, then $\vec{u} \times \vec{v} = \vec{0}$.

Put $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\vec{v} = c\,\vec{u} = \begin{pmatrix} cu_1 \\ cu_2 \\ cu_3 \end{pmatrix}$



$$[\vec{u}\ \vec{v}] = \begin{pmatrix} u_1 & cu_1 \\ u_2 & cu_2 \\ u_3 & cu_3 \end{pmatrix}$$

$$\vec{u} \times \vec{v} = \begin{pmatrix} u_2 cu_3 - u_3 cu_2 \\ -(u_1 cu_3 - u_3 cu_1) \\ u_1 cu_2 - u_2 cu_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0}$$

Properties of cross product

Let $\vec{u}, \vec{v}, \vec{w}$ be any vectors in $\mathbb{R}^3$ and let c be a scalar. Then

(a) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

Properties of cross product

Let $\vec{u}, \vec{v}, \vec{w}$ be any vectors in $\mathbb{R}^3$ and let c be a scalar. Then

(a) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

(b) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

(c) $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

} distributivity

Properties of cross product

Let $\vec{u}, \vec{v}, \vec{w}$ be any vectors in $\mathbb{R}^3$ and let c be a scalar. Then

(a) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

(b) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

(c) $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

(d) $\vec{u} \times \vec{v} = \vec{0}$ whenever $\vec{u} \parallel \vec{v}$. In particular $\vec{u} \times \vec{u} = \vec{0}$.

Properties of cross product

Let $\vec{u}, \vec{v}, \vec{w}$ be any vectors in $\mathbb{R}^3$ and let c be a scalar. Then

(a) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

(b) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

(c) $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

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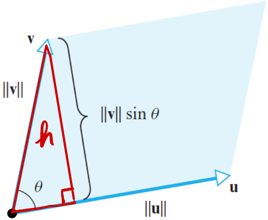
(e) $\vec{u} \times \vec{0} = \vec{0}$

(f) $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$

Area of parallelogram

- Consider the parallelogram formed by two vectors $\vec{u}$ and $\vec{v}$.



$\theta = \angle(\vec{u}, \vec{v})$

$\sin\theta = \frac{h}{\|\vec{v}\|}$

$h = \|\vec{v}\| \sin\theta$

- What is its area?

$\text{Area} = \text{Base} \times \text{height} = \|\vec{u}\| h = \|\vec{u}\| \|\vec{v}\| \sin\theta$

<u>Rmk</u>: $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

Area of parallelogram

Theorem 2

Let θ be the angle between $\vec{u}, \vec{v} \in \mathbb{R}^3$. Then the following hold.

(a) The vector $\vec{u} \times \vec{v}$ has norm $||\vec{u}||||\vec{v}|| \sin\theta$:

$$||\vec{u} \times \vec{v}|| = ||\vec{u}||||\vec{v}|| \sin\theta$$

Area of parallelogram

Theorem 2

Let θ be the angle between $\vec{u}, \vec{v} \in \mathbb{R}^3$. Then the following hold.

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$$||\vec{u} \times \vec{v}|| = ||\vec{u}||||\vec{v}|| \sin\theta$$

(b) $||\vec{u} \times \vec{v}||$ = area of the parallelogram determined by $\vec{u}$ and $\vec{v}$.

Example 3

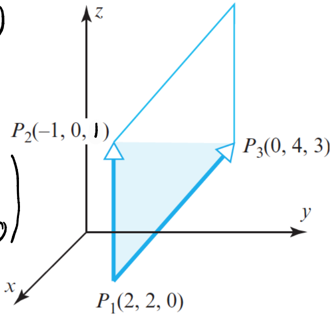
Find the area of the triangles with three vertices

(a) $P_1 = (2, 2, 0), P_2 = (-1, 0, 1), P_3 = (0, 4, 3)$.

$$\text{Area } P_1P_2P_3 = \frac{1}{2}(\text{Area of parallelogram})$$

$$= \frac{1}{2}\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\|$$

$$[\overrightarrow{P_1P_2}\ \overrightarrow{P_1P_3}] = \begin{pmatrix} -3 & -2 \\ -2 & 2 \\ \cancel{1} & \cancel{3} \end{pmatrix} \Rightarrow \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{pmatrix} -8 \\ 7 \\ -10 \end{pmatrix}$$



$$\text{Area } P_1P_2P_3 = \frac{1}{2}\sqrt{(-8)^2 + 7^2 + (-10)^2}$$

$$\cong 7.3$$

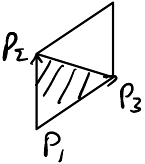
Question

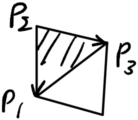
In the solution, we use Area$(\triangle P_1P_2P_3) = \frac{1}{2}||\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}||$. Is the following true?

$$\text{Area}(\triangle P_1P_2P_3) = \frac{1}{2}||\overrightarrow{P_2P_1} \times \overrightarrow{P_2P_3}||? \text{ Or}$$

$$\text{Area}(\triangle P_1P_2P_3) = \frac{1}{2}||\overrightarrow{P_3P_1} \times \overrightarrow{P_3P_2}||?$$

Yes. You can pick any pair of vectors





Exercise 2

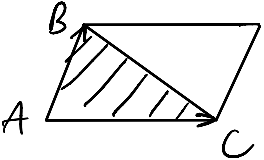
(b) $A = (2,0),\ B = (3,4),\ C = (-1,2)$. (in $\mathbb{R}^2$)

Put A, B, C into xyz-space by amending $z = 0$

$A = (2,0,0)$, $B = (3,4,0)$, $C = (-1,2,0)$

Area $ABC = \frac{1}{2}$ (Area of parallelogram)

$= \frac{1}{2} \|\vec{AB} \times \vec{AC}\|$



. . .

Exercise 3: True/False statements

(a) If c is a scalar and $\vec{v}$ is a vector, then $\vec{v}$ is always parallel to $c\vec{v}$.

True

Exercise 3: True/False statements

(a) If c is a scalar and $\vec{v}$ is a vector, then $\vec{v}$ is always parallel to $c\vec{v}$.

(b) The zero vector $\vec{0}$ is parallel to any vector $\vec{v}$. True

$$\vec{0} = 0\vec{v}$$

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$$\vec{0} \cdot \vec{v} = 0$$

Exercise 3: True/False statements

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(d) If each component of a vector in $\mathbb{R}^3$ is doubled, the length of that vector is doubled. True

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(c) The zero vector $\vec{0}$ is orthogonal to any vector $\vec{v}$.

(d) If each component of a vector in $\mathbb{R}^3$ is doubled, the length of that vector is doubled.

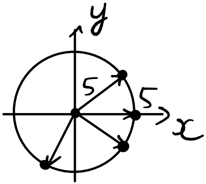
(e) Any vector in $\mathbb{R}^2$ has positive (> 0) length. False!

$$\|\vec{0}\| = \sqrt{0^2+0^2} = 0$$

Exercise 3: True/False statements

(f) In $\mathbb{R}^2$, the vectors of length 5 whose initial points at the origin O have terminal points lying on a circle of radius 5.

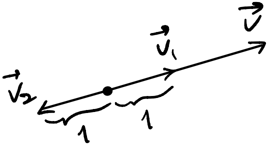
True !



Exercise 3: True/False statements

(f) In $\mathbb{R}^2$, the vectors of length 5 whose initial points at the origin O have terminal points lying on a circle of radius 5.

(g) If $\vec{v}$ is a nonzero vector in $\mathbb{R}^3$, there are exactly 2 unit vectors (vectors of length 1) that are parallel to $\vec{v}$.



$\vec{v}_2 = -\vec{v}_1$

Exercise 3: True/False statements

(f) In $\mathbb{R}^2$, the vectors of length 5 whose initial points at the origin O have terminal points lying on a circle of radius 5.

(g) If $\vec{v}$ is a nonzero vector in $\mathbb{R}^n$, there are exactly 2 unit vectors (vectors of length 1) that are parallel to $\vec{v}$.

(h) If $||\vec{u}|| = 2, \ ||\vec{v}|| = 1$ and $\vec{u} \cdot \vec{v} = 1$, then the angle between $\vec{u}$ and $\vec{v}$ is 60^o.

True !

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| \, ||\vec{v}||} = \frac{1}{2 \cdot 1} \Rightarrow \theta = 60^\circ$$

Exercise 3: True/False statements

(f) In $\mathbb{R}^2$, the vectors of length 5 whose initial points at the origin O have terminal points lying on a circle of radius 5.

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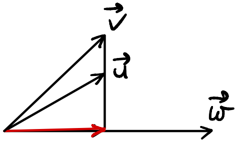
(h) If $||\vec{u}|| = 2, \;\; ||\vec{v}|| = 1$ and $\vec{u} \cdot \vec{v} = 1$, then the angle between $\vec{u}$ and $\vec{v}$ is 60^o.

(i) If $\mathrm{proj}_{\vec{w}}(\vec{u}) = \mathrm{proj}_{\vec{w}}(\vec{v})$ some nonzero vector $\vec{w}$, then $\vec{u} = \vec{v}$.

False !



Intersection of 2 geometrical objects

To find the intersection between any two geometrical objects, we often

1. start with algebraic descriptions of the objects (often in equations),
2. find the points whose coordinates satisfy equations of both objects

Intersection of 2 geometrical objects

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1. start with algebraic descriptions of the objects (often in equations),
2. find the points whose coordinates satisfy equations of both objects

Example: the intersection of 2 planes $x + y - z + 1 = 0$ and $x + \pi z = 0$ is the set of points (x, y, z) which satisfy

$$\begin{cases} x + y - z + 1 = 0 \\ x + \pi z = 0 \end{cases}$$

Intersection of 2 planes

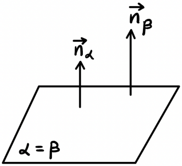
Given two planes α, β, there are 3 possibilities for the intersection

α: point P, normal $\vec{n}_\alpha$

β: point Q, normal $\vec{n}_\beta$

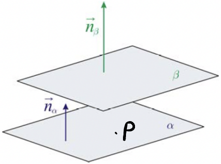
1. Same plane

   $\vec{n}_\alpha \parallel \vec{n}_\beta$ and P is on β



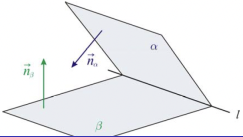
2. α and β are parallel

   $\vec{n}_\alpha \parallel \vec{n}_\beta$ and $P \notin \beta$



3. α and β intersect at a line

   $\vec{n}_\alpha \nparallel \vec{n}_\beta$



Intersection of 2 planes

Theorem 3

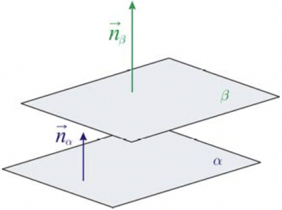
Let α and β be 2 *different* planes with normal vectors $\vec{n}_\alpha$ and $\vec{n}_\beta$. Put

$$\vec{v} = \vec{n}_\alpha \times \vec{n}_\beta.$$

(a) If $\vec{v} = \vec{0}$, then α and β are parallel.
(b) If $\vec{v} \neq \vec{0}$, then α and β intersect at a line whose direction is $\vec{v}$.

(a) $\vec{v} = \vec{0} \rightarrow \vec{n}_\alpha \parallel \vec{n}_\beta \Rightarrow \alpha \parallel \beta$.

Theorem 3 proof



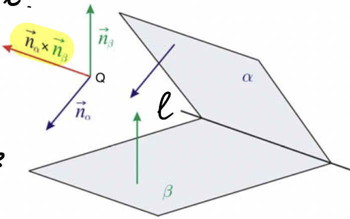
(a) If $\vec{v} = \vec{n}_\alpha \times \vec{n}_\beta = \vec{0}$, then $\vec{n}_\alpha \parallel \vec{n}_\beta$.
So α and β are parallel.

(b) Claim: $\vec{n}_\alpha \times \vec{n}_\beta$ is a direction vector of ℓ.

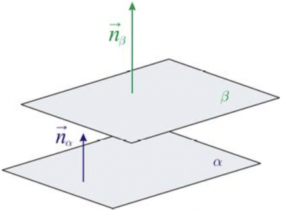
$\left.\begin{array}{l} \vec{n}_\alpha \text{ is normal to } \alpha \Rightarrow \vec{n}_\alpha \perp \ell \\ \vec{n}_\beta \text{ is normal to } \beta \Rightarrow \vec{n}_\beta \perp \ell \end{array}\right\}$

$\Rightarrow \ell$ is perpendicular to both $\vec{n}_\alpha$ & $\vec{n}_\beta$

$\Rightarrow \ell$ is parallel to $\vec{n}_\alpha \times \vec{n}_\beta$

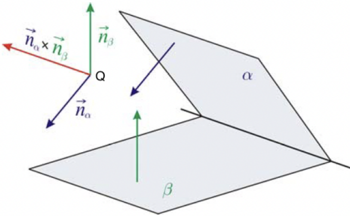


Theorem 3 proof



(a) If $\vec{v} = \vec{n}_\alpha \times \vec{n}_\beta = \vec{0}$, then $\vec{n}_\alpha \parallel \vec{n}_\beta$.
So α and β are parallel.

(b) Assume $\vec{v} = \vec{n}_\alpha \times \vec{n}_\beta = \vec{0}$.
Arrange $\vec{n}_\alpha$ and $\vec{n}_\beta$ so that they start at the same point, say Q.



Example 4

Consider the planes α and β.

$$\alpha : (x, y, z) = (1, 2, 3) + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \text{ and } \beta : 2x - y = 1.$$

(a) Find the intersection of α and β by solving equations directly.

Let $P = (x, y, z)$ be a common point of α & β. Then

$$P \in \alpha \Rightarrow \begin{cases} x = 1 + s + 2t \\ y = 2 + 3t \\ z = 3 + s + 2t \end{cases}$$

$$P \in \beta \Rightarrow 2x - y = 1$$

$$2(1+s+2t) - (2+3t) = 1$$

$$2s + t = 1 \Rightarrow t = 1 - 2s$$

So

$$\begin{cases} x = 1 + s + 2(1 - 2s) = 3 - 3s \\ y = 2 + 3(1 - 2s) = 5 - 6s \\ z = 3 + s + 2(1 - 2s) = 5 - 3s \end{cases}$$

Example 4

Consider the planes α and β.

$$\alpha : (x, y, z) = (1, 2, 3) + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \text{ and } \beta : 2x - y = 1.$$

(a) Find the intersection of α and β by solving equations directly.

The point P has coordinates

$$(x, y, z) = (3-3s, 5-6s, 5-3s) = (3, 5, 5) + s \begin{pmatrix} -3 \\ -6 \\ -3 \end{pmatrix} \qquad (1)$$

$\therefore$ α & β intersect at the line ℓ having equation (1).

Example 4

(b) Find the intersection of α and β by Theorem 3. $\cap$: intersect

$\alpha \cap \beta$ at the line ℓ which has direction

$\vec{v} = \vec{n}_\alpha \times \vec{n}_\beta$ $\quad \vec{n}_\alpha = \begin{pmatrix}1\\0\\1\end{pmatrix} \times \begin{pmatrix}2\\3\\2\end{pmatrix} = \begin{pmatrix}-3\\0\\3\end{pmatrix}$, $\quad \vec{n}_\beta = \begin{pmatrix}2\\-1\\0\end{pmatrix}$

$$\vec{v} = \begin{pmatrix}-3\\0\\3\end{pmatrix} \times \begin{pmatrix}2\\-1\\0\end{pmatrix} = \begin{pmatrix}3\\6\\3\end{pmatrix}$$

Next, we find a point on both $\begin{cases} \alpha : -3(x-1)+3(z-3)=0 \\ \quad \Leftrightarrow -x+z=2 \\ \beta : 2x-y=1 \end{cases}$

Assign $z=0 \Rightarrow \begin{cases} -x=2 \\ 2x-y=1 \end{cases} \Rightarrow x=-2,\ y=2x-1=-5$

So $P=(-2,-5,0)$ is a point on ℓ.

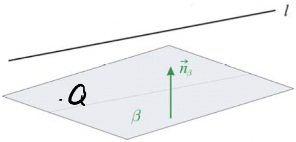
$\therefore$ ℓ has equation $(x,y,z)=(-2,-5,0)+t\begin{pmatrix}3\\6\\3\end{pmatrix}$.

Intersection of line and plane

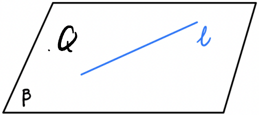
ℓ : point P, direction $\vec{d}$

β : point Q, normal $\vec{n}_\beta$

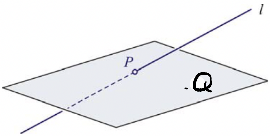
1. l is parallel to β (no common point)



2. l is on β (infinitely many common points)



3. l and β intersect at a point P



Find intersection by solving systems of equations

To find the intersection of l and β, one way is to solve the equations for the points which lie on both l and β.

Find intersection by solving systems of equations

To find the intersection of l and β, one way is to solve the equations for the points which lie on both l and β.

1. No solution $\Rightarrow$ l and β are parallel.
2. One solution $\Rightarrow$ l and β intersect at a point.
3. Infinitely many solutions $\Rightarrow$ l is a line on β, that is, l intersects β at the line l itself.

Example 5

Find the intersection of the line l and the plane β in following cases.

(a) $\beta : 3x - 2y - z = -4$ and $l : (x, y, z) = (2, 1, 4) + t \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$

Example 5

(b) $\beta : 3x - 2y - z = -4$ and $l : (x, y, z) = (1, 2, 3) + t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Intersection by direction vector and normal vector

Theorem 4

Let l be a line through a point P and having direction $\vec{v}$.

Let β be a plane with normal vector $\vec{n}$. Then

(a) l is a line on $\beta \Leftrightarrow P$ is on β and $\vec{v} \cdot \vec{n} = 0$.

Intersection by direction vector and normal vector

Theorem 4

Let l be a line through a point P and having direction $\vec{v}$.

Let β be a plane with normal vector $\vec{n}$. Then

(b) l is parallel to $\beta \Leftrightarrow P$ is not on β and $\vec{v} \cdot \vec{n} = 0$.

Intersection by direction vector and normal vector

Theorem 4

Let l be a line through a point P and having direction $\vec{v}$.

Let β be a plane with normal vector $\vec{n}$. Then

(c) l intersects β at a unique point $\Leftrightarrow \vec{v} \cdot \vec{n} \neq 0$.

Example 6

Find the *number of common points* between l and β in following cases.

(a) $\beta : 3x - 2y - z = -4$ and $l : (x, y, z) = (2, 1, 4) + t \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$

Example 6

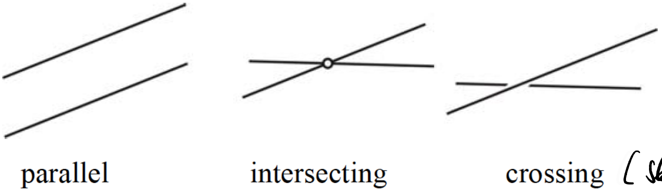
(b) $\beta : 3x - 2y - z = -4$ and $l : (x, y, z) = (1, 2, 3) + t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Lines in $\mathbb{R}^3$

- In $\mathbb{R}^2$, two *different lines* either "parallel" or "intersect at a point".

Lines in $\mathbb{R}^3$

- In $\mathbb{R}^2$, two *different lines* either “parallel” or “intersect at a point”.
- The situation in $\mathbb{R}^3$ is different .



(skew)

Intersection of lines in $\mathbb{R}^3$

Let $\vec{v}_1$ and $\vec{v}_2$ be the directions of lines l_1 and l_2.

- Case 1: $\vec{v}_1$ is parallel to $\vec{v}_2 \Rightarrow l_1$ is parallel to l_2.

Intersection of lines in $\mathbb{R}^3$

Let $\vec{v}_1$ and $\vec{v}_2$ be the directions of lines l_1 and l_2.

- Case 2. $\vec{v}_1$ is not parallel to $\vec{v}_2 \Rightarrow l_1$ is not parallel to l_2.
  Solve the equations for points on both l_1 and $l_2 \Rightarrow$ 2 possibilities

Intersection of lines in $\mathbb{R}^3$

Let $\vec{v}_1$ and $\vec{v}_2$ be the directions of lines l_1 and l_2.

- Case 2. $\vec{v}_1$ is not parallel to $\vec{v}_2 \Rightarrow l_1$ is not parallel to l_2.
  Solve the equations for points on both l_1 and $l_2 \Rightarrow$ 2 possibilities
  1. Exactly one solution $\Rightarrow l_1$ and l_2 intersect at a point
  2. No solution $\Rightarrow l_1$ and l_2 are skew.

Example 7

Find the relative position (parallel, intersecting, skew) and the intersection between any two lines.

$$k: \begin{cases} x = 1 + t \\ y = 2 + 2t \\ z = 3 - t \end{cases} \quad l: \begin{cases} x = 2 + 2s \\ y = 1 + 4s \\ z = 4 - 2s \end{cases} \quad m: \begin{cases} x = 7 + r \\ y = 13 + 4r \\ z = -3 - 3r \end{cases}$$

Example 7