Integration by Parts
Trigonometric Integrals Part 1

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AY 23/24 Trimester 1

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Integration by substitution

- Integration by substitution deals with the antiderivative of functions that have the form
$$f'(g(x))g'(x).$$
- We learned how to recognize integrands that have the above form.
- For indefinite integrals; with $u = g(x)$ as the substitution,
$$\int f'(g(x))g'(x)\,dx = \int f'(u)\,du = f(g(x)) + C.$$
- For definite integrals; with $u = g(x)$ as the substitution and FTC2,
$$\int_a^b f'(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f'(u)\,du = f(g(b)) - f(g(a)).$$

Recap of the Product Rule

Lemma

When u and v are differentiable functions, then uv is also differentiable and

$$(uv)'(x) = u'(x)v(x) + u(x)v'(x),$$

or if the variable is already known, in a more succinct expression,

$$(uv)' = u'v + uv'.$$

Like in integration by substitution, we integrate this expression, but with a **slight modification**; we rearrange this expression to get

$$uv' = (uv)' - u'v.$$

'Reversing' the Product Rule

Integrating both sides of the following equation

$$uv' = (uv)' - u'v,$$

we get

$$\int uv' = uv - \int u'v.$$

Usually, the above formula is written as

$$\int u\,dv = uv - \int v\,du.$$

This is known as **integration by parts**.

Integration by parts formula

When integrating a product of functions uv', we can apply the **integration by parts** formula:

$$\int u\,dv = uv - \int v\,du.$$

Note: There is a significant overlap between integration by substitution and by parts, because integrands that look like $f'(g(x))g'(x)$ are also a product of functions u and dv.

Generally, integration by substitution is easier and less tedious to evaluate compared to integration by parts.

Heuristic/Tip: We only apply integration by parts if the integrand is a product of functions u and dv **BUT** <u>does not</u> have the form $f'(g(x))g'(x)$.

Example 1

Evaluate $\int x \sin x \, dx$.

Since we don't yet know how to choose u and dv, let's just try

$u = \sin x$ and $dv = x$. Then

$du = \cos x$ and $v = \frac{x^2}{2}$.

Therefore

$$\int x \sin x \, dx =$$

Example 1

We have seen that in the previous choice of u and dv, we end up with an integral which is more difficult to integrate. So let's reverse the choices of u and dv:

$u = x$ and $dv = \sin x$. Then

$du = 1$ and $v = -\cos x$.

Now,

$$\int x \sin x \, dx =$$

Choosing u: LIATE prioritization tool

Example 1 strongly suggests that there is a way to choose u and dv so that subsequent applications of the 'by parts' formula will result in integrals that are easier to evaluate.

The **LIATE prioritization tool** below allows you to choose u based on the **difficulty of integration** (1 for most difficult, 5 for easiest):

1. **L**ogarithmic functions, e.g. $\ln x$.
2. **I**nverse trigonometric functions, e.g. $\sin^{-1} x$, $\tan^{-1} x$.
3. **A**lgebraic functions, e.g. x^2, $2x$, x^{-1}, etc.
4. **T**rigonometric functions, e.g. $\sin x$, $\sec^2 x$, $\cos x$, etc.
5. **E**xponential functions, e.g. e^x, e^{2x}, etc.

In Example 1, x is ranked 3, and $\sin x$ is ranked 4, so we choose $u = x$, and $dv = \sin x$.

Example 2

Evaluate $\displaystyle\int te^t\,dt$.

Example 3

Evaluate $\int \ln x \, dx$.

Exercise 1

Evaluate the following integrals.

1. $\displaystyle\int t^2 e^t \, dt$
2. $\displaystyle\int \sin^{-1} x \, dx$

Exercise 1

Integration by parts for definite integrals

The integration by parts formula for definite integrals can be obtained by applying the FTC2 to the formula for indefinite integrals:

Theorem

If u' and v' are continuous on $[a, b]$, then

$$\int_a^b u(x)v'(x)\,dx = [u(x)v(x)]_a^b - \int_a^b v(x)u'(x)\,dx.$$

Example 4

Evaluate $\int_0^\pi x \cos x \, dx$.

Example 5

Evaluate $\displaystyle\int_1^4 \frac{\ln x}{x^3}\, dx$.

Example 5

Exercise 2

Evaluate the following integrals.

1. $\displaystyle\int_0^1 \tan^{-1} x \, dx$
2. $\displaystyle\int_1^2 x \ln x \, dx$

Example 6

Evaluate $\int \sin^2 x \cos^3 x \, dx$.

Example 7

Evaluate $\displaystyle\int \sin^3 x \cos^2 x \, dx$.

Example 8

Evaluate $\int \sin^2 x \, dx$.

Method for integrating powers of sine and cosine (1)

Method for integrating $\int \sin^m x \cos^n x \, dx$**:**

- If n **is odd**, then $n = 2k + 1$ for some integer k. Then

$$\begin{aligned}\int \sin^m x \cos^n x \, dx &= \int \sin^m x \, (\cos^2 x)^k \cos x \, dx \\ &= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx\end{aligned}$$

Then apply substitution $u = \sin x$. See Example 5.

Method for integrating powers of sine and cosine (2)

- If m **is odd**, then $m = 2k + 1$ for some integer k. Then

$$\begin{aligned}\int \sin^m x \cos^n x \, dx &= \int (\sin^2 x)^k \cos^n x \sin x \, dx \\ &= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx\end{aligned}$$

  Then apply substitution $u = \cos x$. See Example 6.

- If **both** m **and** n **are even**, we can use the double angle formulae (will be provided in assessments, see Example 8):

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos(2x)}{2}.$$

Exercise 2

Evaluate the following integrals.

1. $$\int \sin^4 x \cos^3 x \, dx$$
2. $$\int \sin^5 x \cos^4 x \, dx$$
3. $$\int \cos^2 x \, dx$$

Exercise 2