Fundamentals of Differentiation Part 3

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Recap

1. Chain Rule

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

2. Implicit and explicit equations, how to differentiate implicitly
3. Explicit equations: Tangent line equation to f at $(a, f(a))$

$y = f(x)$

$$y = f'(a)(x - a) + f(a)$$

Implicit equations: Tangent line equation to graph at (x_0, y_0)

$$y = \underbrace{\frac{dy}{dx}(x_0, y_0)}_{\substack{\text{derivative of } y \\ \text{at } (x_0, y_0)}}(x - x_0) + y_0$$

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Higher order derivatives

$f^{(1)}(x)$ explicit eq^n s

If a function $y = f(x)$ is differentiable, then we have a function $f'(x)$, the derivative of $y = f(x)$.

We also have the derivative of $f'(x)$, which is a function $f''(x)$, called the *second derivative* of $f(x)$, is the derivative of the first derivative

$f^{(2)}(x)$

$$f''(x) = (f')'(x).$$

$f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

The second derivative can also be written as

$y \quad y' \quad y''$

first derivative

second derivatives

$$\underbrace{\frac{d}{dx}}_{\text{derivative of}} \underbrace{\left(\frac{dy}{dx}\right)}_{\text{first derivative}} = \underbrace{\frac{d^2y}{dx^2}}_{\text{second derivative}}.$$

Exercise 1

Find the second derivative for each of the following functions.

1. $f(x) = 3x^2 + 2x + 1$ $\rightarrow$ $f'(x) = 6x+2 \rightarrow f''(x) = 6.$
2. $f(x) = \sin(2x)$ $\longrightarrow$ $f'(x) = \cos(2x)2 = 2\cos(2x)$
3. $f(x) = e^{5x}$
4. $f(x) = \ln(x^2 + 1)$

$f''(x) = -2\sin(2x)\cdot 2$

$= -4\sin(2x)$

$f'(x) = e^{5x}\cdot 5 = 5e^{5x}$

$f''(x) = 5^2 e^{5x}$

$$f(x) = \underline{\ln}\,(\underline{x^2+1})$$

$$f'(x) = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1} = \underline{\underline{2x}} \cdot \underline{\underline{(x^2+1)^{-1}}}$$

$$f''(x) = 2 \cdot (x^2+1)^{-1} + 2x\,(-1)\,(\underline{x^2+1})^{-2} \cdot 2x$$

$$= \frac{2}{x^2+1} - \frac{4x^2}{(x^2+1)^2}$$

$$= \frac{2(x^2+1)}{(x^2+1)^2} - \frac{4x^2}{(x^2+1)^2} = \frac{-2\,(x^2-1)}{(x^2+1)^2}$$

Third derivative onwards

$\to f^{(3)}(x)$

The *third derivative* $f'''(x)$ of $y = f(x)$ is the derivative of the second derivative

$$f'''(x) = (f'')'(x).$$

The *fourth derivative* and onwards are abbreviated slightly differently (for pretty obvious reasons)

$$f^{(4)}(x) = (f''')'(x).$$

In general, the *n-th derivative* $f^{(n)}(x)$ is obtained by differentiating the $(n-1)$-th derivative:

$$f^{(n)}(x) = (f^{(n-1)})'(x).$$

Exercise 2

Find the third derivative for each of the functions. (These were the functions in Exercise 1)

1. $f(x) = 3x^2 + 2x + 1$ $\to f^{(3)}(x) = 0$
2. $f(x) = \sin(2x)$ $\to f^{(2)}(x) = -4\sin(2x) \to f^{(3)}(x) = -4\cos(2x)\cdot 2 = -8\cos(2x)$
3. $f(x) = e^{5x}$ $\to f^{(3)}(x) = 5^3 e^{5x}$
4. $f(x) = \ln(x^2 + 1)$

$$f^{(2)}(x) = \frac{2-2x^2}{(x^2+1)^2} = \underline{(2-2x^2)}\cdot\underline{(x^2+1)^{-2}}$$

denominator

$$\therefore f^{(3)}(x) = -4x(x^2+1)^{-2} + (-2)(x^2+1)^{-3}\cdot 2x\cdot(2-2x^2)$$

$$= \frac{-4x}{(x^2+1)^2} + \frac{8x^3-8x}{(x^2+1)^3}$$

Exercise 2

$$= \frac{-4x}{(x^2+1)^2} + \frac{8x^3-8x}{(x^2+1)^3}$$

$$= \frac{-4x(x^2+1)}{(x^2+1)^3} + \frac{(8x^3-8x)}{(x^2+1)^3} = \frac{4x^3-12x}{(x^2+1)^3}$$

Definitions of increasing and decreasing functions

Definition

Let A be any subset of the domain of a function f. For $x_1, x_2 \in A$,

- f is *increasing* on A if

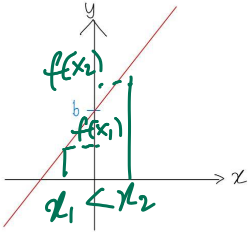
$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2,$$

- f is *decreasing* on A if

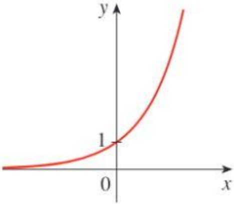
$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2.$$

Important: For a function to be increasing/decreasing, $f(x_1) \leq f(x_2)/f(x_1) \geq f(x_2)$ must hold for all pairs of $x_1 < x_2$!

Examples of increasing functions

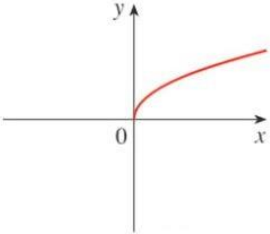


$y = mx + b\ (m > 0)$



$y = a^x\ (a > \cancel{0})$

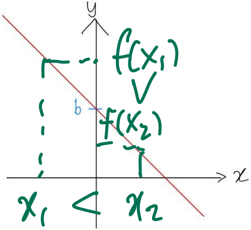
1



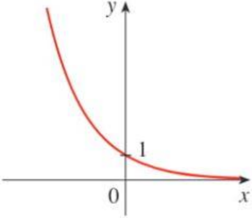
$y = \sqrt{x}$

What do you observe about the gradient of the functions here?

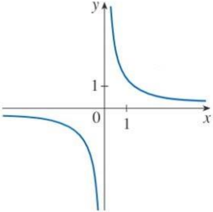
Examples of decreasing functions



$y = mx + b \ (m < 0)$



$y = a^x \ (0 < a < 1)$



$y = \dfrac{1}{x}$

What do you observe about the gradient of the functions here?

Increasing/Decreasing Test

By observing the sign (positive/negative) of the gradient of a differentiable function f, we can tell if f is increasing or decreasing.

Theorem (Increasing/Decreasing Test or I/D Test)

Let I be an interval which is a subset of the domain of f. If f is differentiable on I, then

- f is increasing on I if and only if $f'(x) > 0$ on A,
- f is decreasing on I if and only if $f'(x) < 0$ on A.

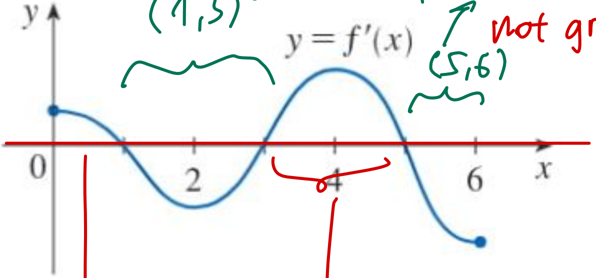
Example 1

The graph of the *derivative* f' of a function f is shown below. Determine the intervals for which f is increasing or decreasing.

$(1,3)$ $f'(x)<0 \rightarrow f$ decreasing

not graph of f

$(5,6)$



$(0,1)$ $(3,5)$

$f'(x)>0 \rightarrow f$ increasing on these intervals

Critical points of f

Definition

($\star$) A *critical point* of a function f is a point c where either

1. $f'(c) = 0$, or
2. f is not differentiable at c.

Critical points play an important role in the identification of intervals where a function is increasing or decreasing, and they also play a big role in optimization (later in the course).

Exercise 3

Let $f(x) = \dfrac{x}{x^2+1}$. > 0

1. What is the domain of f? $\rightarrow \mathbb{R}.$
2. Find the intervals for which f is increasing or decreasing.

Step 1 : Domain $= \mathbb{R}$ ✓ $\rightarrow$

Step 2 : $f'(x) = \dfrac{(x^2+1)\cdot 1 - x(2x)}{(x^2+1)^2}$

$$= \frac{1-x^2}{(x^2+1)^2} \leftarrow \text{never zero}$$

Find Critical points: $f'(x) = 0$.

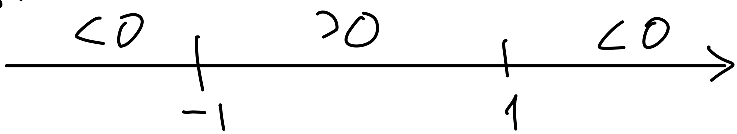
Exercise 3

$$\frac{1-x^2}{(1+x^2)^2} = 0 \iff 1-x^2=0.$$

$$\iff \underbrace{(1-x)(1+x)}=0$$

$x=-1,\ x=1$ $\quad$ $x=-2,\ x=0,\ x=2$

Step 3:



Sub $x=-2,\ x=0,\ x=2$

in $f'(x)$

Step 4

$\therefore$ f increasing $(-1, 1)$

f decreasing $(-\infty, -1)$ and $(1, \infty)$