Integration by Parts

product rule

Trigonometric Integrals Part 1

↳ rehash Substitution → Chain rule

Dr. Ronald Koh
ronald.koh@digipen.edu (Teams preferred over email)

AY 23/24 Trimester 1

Table of contents

Integration by substitution → Practice Practice Practice

- Integration by substitution deals with the antiderivative of functions that have the form
$$\boxed{f'(g(x))g'(x).}$$
- We learned how to recognize integrands that have the above form.
- For indefinite integrals; with $\underline{u = g(x)}$ as the substitution, → $du = g'(x)\,dx$
$$\int f'(g(x))g'(x)\,dx = \int f'(u)\,du = f(g(x)) + C.$$
expected to be easier
- For definite integrals; with $u = g(x)$ as the substitution and FTC2,
$$\int_a^b f'(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f'(u)\,du = f(g(b)) - f(g(a)).$$

Don't change positions!

Recap of the Product Rule

↳ integration by parts.

Lemma

When u and v are differentiable functions, then uv is also differentiable and

product of u and v

$$(uv)'(x) = u'(x)v(x) + u(x)v'(x),$$

or if the variable is already known, in a more succinct expression,

$$(uv)' = u'v + uv'.$$

Like in integration by substitution, we integrate this expression, but with a **slight modification**; we rearrange this expression to get

$$\boxed{uv' = (uv)' - u'v.}$$

integrate this expression.

'Reversing' the Product Rule

Integrating both sides of the following equation

$$uv' = (uv)' - u'v,$$

we get

$$\boxed{\int uv' = uv - \int u'v.}$$

$$\int \underbrace{\boxed{u\,dv}}_{\text{integrand}}$$

Usually, the above formula is written as

$$\int u\,dv = uv - \int v\,du.$$

LHS $\rightarrow$ RHS

$v' = dv$

$u' = du$

This is known as **integration by parts**.

Chain Rule $\longrightarrow$ Integration by substitution

Product Rule (slight mod) $\rightarrow$ Integration by parts

Integration by parts formula

$x \sin x$

$u\,dv$

When integrating a product of functions uv' we can apply the **integration by parts** formula:

uv'

$$\int u\,dv = uv - \int v\,du.$$

$f'(g(x)) \cdot g'(x)$

Note: There is a significant overlap between integration by substitution and by parts, because integrands that look like $f'(g(x))g'(x)$ are also a product of functions u and dv.

Generally, integration by substitution is easier and less tedious to evaluate compared to integration by parts.

Heuristic/Tip: We only apply integration by parts if the integrand is a product of functions u and dv **BUT** does not have the form $f'(g(x))g'(x)$.

Example 1

$$\int u\,dv = uv - \int v\,du$$

$\sin x$ $\quad x$

Evaluate $\int x \sin x \, dx$.

Since we don't yet know how to choose u and dv, let's just try $u = \sin x$ and $dv = x$. Then $du = \cos x$ and $v = \frac{x^2}{2}$.

$$\boxed{\begin{array}{ll} u = \sin x & dv = x \\ du = \cos x & v = \frac{x^2}{2} \end{array}}$$

Therefore

$$\int x \sin x \, dx = \frac{x^2}{2}\sin x - \underbrace{\int \frac{x^2}{2}\cos x \, dx}_{\text{more complicated!!!}}$$

Choice of u and dv are wrong!

Example 1

We have seen that in the previous choice of u and dv, we end up with an integral which is more difficult to integrate. So let's reverse the choices of u and dv:

$u = x$ and $dv = \sin x$. Then

$du = 1$ and $v = -\cos x$.

$$\begin{array}{ll} u = x & dv = \sin x \\ du = 1 & v = -\cos x \end{array}$$

Now,

$$\int \underline{x \sin x}\, dx = -x\cos x - \int (-\cos x)\, dx$$

$$= -x\cos x + \underbrace{\int \cos x \, dx}_{\text{Can be done!}} = \underline{\sin x - x\cos x + C}$$

Choosing u: LIATE prioritization tool

2 4
u dv

Example 1 strongly suggests that there is a way to choose u and dv so that subsequent applications of the 'by parts' formula will result in integrals that are easier to evaluate.

u dv

The **LIATE prioritization tool** below allows you to choose u based on the **difficulty of integration** (1 for most difficult, 5 for easiest):

1. **L**ogarithmic functions, e.g. $\ln x$. arc sin x arc tan x
2. **I**nverse trigonometric functions, e.g. $\sin^{-1} x$, $\tan^{-1} x$.
3. **A**lgebraic functions, e.g. x^2, $2x$, x^{-1}, etc. $\sqrt{x}$ ✓
4. **T**rigonometric functions, e.g. $\sin x$, $\sec^2 x$, $\cos x$, etc. Trigo integrals
5. **E**xponential functions, e.g. e^x, e^{2x}, etc.

In Example 1, x is ranked 3, and $\sin x$ is ranked 4, so we choose $u = x$, and $dv = \sin x$.

$\frac{x}{3}$ $\frac{\sin x}{4}$

Example 2

t ranked 3 $\quad e^t$ ranked 5

$u \quad\quad dv$

Evaluate $\int te^t\,dt$.

$u = t \quad\quad dv = e^t$

$du = 1 \quad\quad v = e^t$

$$\int te^t\,dt = te^t - \int e^t\,dt$$

$$= te^t - e^t + C$$

$t^2 e^t$

$t^3 e^t$

Example 3

Galaxy brain move

Evaluate $\int \ln x \, dx$.

$$\ln x = \underbrace{1}_{\text{rank3}} \cdot \underbrace{\ln x}_{\text{rank1}}$$

$$u = \ln x \qquad dv = 1$$

$$du = \frac{1}{x} \qquad v = x$$

$$\int \ln x \, dx = x \ln x - \int \frac{1}{x} \cdot x \, dx$$

$$= x \ln x - \int 1 \, dx = x \ln x - x + C.$$

Exercise 1

Example 1

$u = t^2 \qquad dv = e^t$

$du = 2t \qquad v = e^t$

Evaluate the following integrals.

1. $\displaystyle\int t^2 e^t\,dt$ $= t^2e^t - \int 2t\cdot e^t\,dt$

$$= t^2e^t - 2\underbrace{\int te^t\,dt}_{\text{Example 1}}$$

2. $\displaystyle\int \sin^{-1} x\,dx$

$$= t^2e^t - 2(te^t - e^t) + C$$

$$= t^2e^t - 2te^t + 2e^t + C$$

$$= e^t(t^2 - 2t + 2) + C$$

Exercise 1

$$\sin^{-1}x = \underset{3}{\underline{1}} \cdot \underset{2}{\underline{\sin^{-1}x}}$$

$$\int \sin^{-1}x \, dx$$

$u = \sin^{-1}x \qquad dv = 1$

$du = \frac{1}{\sqrt{1-x^2}} \qquad v = x$

$$= x\sin^{-1}x + \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx$$

Hint: Substitution

$w = 1 - x^2$

$dw = -2x \, dx$

$\frac{1}{\sqrt{w}} = w^{-\frac{1}{2}}$

$$= x\sin^{-1}x + \frac{1}{2}\int \frac{1}{\sqrt{w}}\,dw$$

$$= x\sin^{-1}x + \cancel{\frac{1}{2}}\left[\frac{w^{\frac{1}{2}}}{\cancel{1/2}}\right] + C = x\sin^{-1}x + w^{\frac{1}{2}} + C$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + C$$

Integration by parts for definite integrals

The integration by parts formula for definite integrals can be obtained by applying the FTC2 to the formula for indefinite integrals:

Theorem

If u' and v' are continuous on $[a, b]$, then

$$\int_a^b u(x)v'(x)\,dx = \underbrace{[u(x)v(x)]_a^b}_{= u(b)v(b) - u(a)v(a)} - \int_a^b v(x)u'(x)\,dx.$$

Example 4

Evaluate $\int_0^{\pi} x\cos x\,dx$.

$u = x \qquad dv = \cos x$

$du = 1 \qquad v = \sin x$

$$= \left[x\sin x\right]_0^{\pi} - \int_0^{\pi} \sin x\,dx$$

$$= \underbrace{\pi\sin\pi - 0\cdot\sin 0}_{0} - \left[-\cos x\right]_0^{\pi}$$

$$= -\left[\underbrace{-\cos\pi} - \underbrace{(-\cos 0)}\right] = -\left[-(-1) - (-1)\right]$$

$$= -2.$$

Example 5

Evaluate $\displaystyle\int_1^4 \frac{\ln x}{x^3}\, dx$.

$$\frac{\ln x}{x^3} = \frac{x^{-3}}{3}\ \frac{\ln x}{1}$$

$u = \ln x \qquad dv = x^{-3}$

$du = \frac{1}{x} \qquad v = \frac{x^{-2}}{-2} = -\frac{1}{2x^2}$

$$= \left[-\frac{1}{2x^2}\ln x\right]_1^4 - \int_1^4 -\frac{1}{2x^3}\,dx$$

$$= \left[-\frac{1}{2\cdot 16}\ln 4 - \frac{1}{2\cdot 1^2}\ln 1\right] + \frac{1}{2}\int_1^4 x^{-3}\,dx$$

$$= -\frac{1}{32}\ln 4 + \frac{1}{2}\left[\frac{x^{-2}}{-2}\right]_1^4 \qquad -\frac{1}{32}\ln 4 - \frac{1}{4}\left[-\frac{15}{16}\right]$$

$$= -\frac{1}{32}\ln 4 - \frac{1}{4}\left[\frac{1}{16} - 1\right] \qquad = -\frac{1}{32}\ln 4 + \frac{15}{64}.$$

Exercise 2

$1 \cdot \tan^{-1} x \quad u = \tan^{-1} x \quad dv = 1$

$du = \frac{1}{1+x^2} \quad v = x$

substitution

Evaluate the following integrals.

1. $\displaystyle\int_0^1 \tan^{-1} x \, dx = \left[x \tan^{-1} x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\, dx$

$w = 1 + x^2$

$dw = 2x\, dx$

$x = 0 \Rightarrow w = 1$

$x = 1 \Rightarrow w = 2$

$$= \left[\tan^{-1} 1\right] - \frac{1}{2}\int_1^2 \frac{1}{w}\, dw$$

$$= \frac{\pi}{4} - \frac{1}{2}\left[\ln|w|\right]_1^2$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

2. $\displaystyle\int_1^2 x \ln x \, dx$

$u = \ln x \quad dv = x$

$du = \frac{1}{x} \quad v = \frac{x^2}{2}$

$$= \left[\frac{x^2}{2}\ln x\right]_1^2 - \int_1^2 \frac{x^2}{2}\cdot\frac{1}{x}\, dx$$

$$= 2\ln 2 - \frac{1}{2}\int_1^2 x\, dx = 2\ln 2 - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^2$$

$$= 2\ln 2 - \frac{1}{2}\left[2 - \frac{1}{2}\right] = 2\ln 2 - \frac{3}{4}.$$

Example 6

Evaluate $\int \sin^2 x \cos^3 x \, dx$.

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$\cos^2 x = 1 - \sin^2 x$

$$= \int \sin^2 x \cdot \underbrace{\cos^2 x}_{\text{convert to } \sin^2 x} \cdot \underline{\cos x \, dx}$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int \sin^2 x \cdot (1 - \sin^2 x) \cdot \underline{\cos x \, dx}$$

$$= \int u^2(1-u^2) \, du = \int u^2 - u^4 \, du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

Example 7

odd powers

Evaluate $\int \sin^3 x \cos^2 x \, dx$.

$$= \int \underbrace{\sin^2 x}_{\text{convert to } \cos^2 x} \cos^2 x \cdot \sin x \, dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$= -\int (1 - \cos^2 x) \cdot \cos^2 x \cdot (-\sin x) \, dx$$

$$= -\int (1 - u^2) u^2 \, du$$

$$= -\int u^2 - u^4 \, du = \int u^4 - u^2 \, du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

Example 8

all even powers

$\sin^2 x = \frac{1-\cos(2x)}{2}$ $\cos^2 x = \frac{1+\cos(2x)}{2}$

Evaluate $\int \sin^2 x \, dx$.

$= \int \frac{1-\cos(2x)}{2} dx$ can do $u=2x$

$= \int \frac{1}{2} - \frac{\cos(2x)}{2} dx$

$[\sin(2x)]' = 2\cos(2x)$

$= \int \frac{1}{2} dx - \frac{1}{2}\int \cos(2x) dx$ $\rightarrow$ $\left[\frac{\sin(2x)}{2}\right]' = \cos(2x)$

$= \frac{x}{2} - \frac{1}{2}\frac{\sin(2x)}{2} + C = \frac{x}{2} - \frac{\sin(2x)}{4} + C$

Method for integrating powers of sine and cosine (1)

$\cos^n x = \cos^{(2k+1)} x$

$= \cos^{2k} x \cdot \cos x$

Method for integrating $\int \sin^m x \cos^n x \, dx$**:**

- If n **is odd**, then $n = 2k + 1$ for some integer k. Then

cosine power

$$\int \sin^m x \cos^n x \, dx = \int \sin^m x \left(\cos^2 x\right)^k \cos x \, dx$$

$\cos^2 x = 1 - \sin^2 x$

$$= \int \sin^m x \left(1 - \sin^2 x\right)^k \cos x \, dx$$

Then apply substitution $u = \sin x$. See Example 6.

Method for integrating powers of sine and cosine (2)

$\sin^m x = \sin^{(2k+1)} x = \sin^{2k} x \cdot \sin x$

- If m **is odd**, then $m = 2k+1$ for some integer k. Then

sine power

$$\int \sin^m x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \sin x \, dx$$

$\sin^2 x = 1 - \cos^2 x$

$$= \int (\underline{1 - \cos^2 x})^k \cos^n x \sin x \, dx$$

Then apply substitution $\underline{u = \cos x}$. See Example 7.

- If **both m and n are even**, we can use the double angle formulae (will be provided in assessments, see Example 8):

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos(2x)}{2}.$$

double angle

Exercise 2

Evaluate the following integrals.

1. $\int \sin^4 x \cos^3 x \, dx$ $= \int \sin^4 x \overbrace{\cos^2 x}^{1-\sin^2 x} \cos x \, dx$

   $u = \sin x$
   $du = \cos x \, dx$

   $= \int \sin^4 x (1 - \sin^2 x) \underline{\cos x \, dx}$

   $= \int u^4 (1 - u^2) \, du = \int u^4 - u^6 \, du$

   $= \frac{u^5}{5} - \frac{u^7}{7} + C$

   $= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$

2. $\int \sin^5 x \cos^4 x \, dx$

3. $\int \cos^2 x \, dx$

$\int \frac{1 + \cos(2x)}{2} \, dx$

$= \int \frac{1}{2} \, dx + \frac{1}{2} \int \cos(2x) \, dx$

antiderivative $\frac{\sin(2x)}{2}$

$= \frac{x}{2} + \frac{\sin(2x)}{4} + C$

Exercise 2

$$\int \sin^5 x \cos^4 x \, dx = \int \sin^4 x \cos^4 x \cdot \sin x \, dx$$

$$= -\int (1-\cos^2 x)^2 \cos^4 x \cdot (-\sin x)\, dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$= -\int \underbrace{(1-u^2)^2}\, u^4 \, du$$

$$= -\int (1-2u^2+u^4)\, u^4 \, du$$

$$= \int -u^8 + 2u^6 - u^4 \, du = -\frac{u^9}{9} + \frac{2u^7}{7} - \frac{u^5}{5} + C$$

$$= -\frac{\cos^9 x}{9} + \frac{2\cos^7 x}{7} - \frac{\cos^5 x}{5} + C$$

Week 4 Quiz Material Week 1 - Week 3

10 MCQ

Same format as in CSD1251/1250

Quiz Paper + Answer Sheet

Circle answers for both
and answers have to be tallied

* Discrepancies btw Quiz Paper and Answer Sheet will result in that Qn being wrong AND -1 to total score