Password: sets

Lecture 1: Set theory

Course Materials

1. **Textbooks**:

   Discrete Mathematics and Its Applications, 8th edition, K.Rosen, K. Krithivasan.

2. **Lecture slides** provided by myself.

Course Content

1. Sets and Functions
2. Number Theory and Cryptography
3. Logic and Proofs
4. Combinatorics
5. Discrete Probability

Assessment tasks

Assessment Task	Weight	Tentative dates
Online homeworks	10%	Weeks 1-12
Quizzes (3)	30%	Weeks 3, 9, 12
Midterm test (1)	30%	Week 6
Final test (1)	30%	Week 14/15

Policy for makeup exams

Makeup exams might be given if you have valid reasons. (MC, . . .)

But be aware of the following:

- Makeup exams cover *more materials* (possibly more difficult) than normal exams
- Makeup exams are given *less time* than the normal exams

Course structure

3.30 - 5.30pm

- **Online lecture** every Monday ~~3-5pm~~
- **Physical tutorials**
  - Groups A,B,C: taught by myself
  - Groups D: taught by Dr. Yilin

Grades

LETTER GRADE	GRADE POINT	DESCRIPTION	REMARKS
A+	5.0	Excellent attainment of learning outcomes	
A	5.0		
A-	4.5		
B+	4.0	Very Good attainment of learning outcomes	
B	3.5		
B-	3.0		
C+	2.5	Good attainment of learning outcomes	
C	2.0		
D+	1.5	Adequate attainment of learning outcomes	
D	1.0		Minimum grade required for undergraduate students to earn credit
F	0.0	Failed to attain learning outcomes	

Attendance policy

- Students $\geq$ 15 minutes late to class will be marked as absent.
- Students may not leave the class early without the instructor's permission.
- Unexcused absences would result in the following penalty

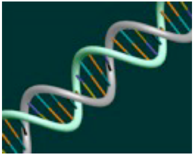
1 letter grade down for # of unexcused absences	2 letter grade down for # of unexcused absences
4	8

What is discrete math?

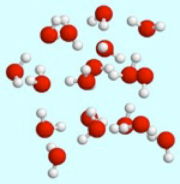
- The study of mathematical structures which are **discrete** - things that are **countable**.
- Intuitively, these are things with *clear gap* between one and another.
- Almost everything in our daily life is discrete.
  - Computers: gates in computer circuits operate via Boolean algebra (0 and 1).

Why study discrete math?

- DNA: DNA base pairs can only be one of the types A, C, G, T.



- Molecules: atomic structure of nature around us



Barber puzzle

- The army captain orders his company barber to *shave all members* of the company *who do not shave themselves*.
- What about the barber himself?

  Solution. Consider 2 cases

  1. The barber doesn't shave himself
     By the order, he needs to shave himself → cannot happen.
  2. The barber shaves himself
     → he doesn't follow the captain's order.

Solutions:
- Have 2 barbers
- Exclude the barber from the members

Basic definitions of sets

- A **set** is an **unordered collection** of distinct objects, called **elements** of the set.
- Let A be a set and let a be an element. Write
  - $a \in A$ if a is an element of A
  - $a \notin A$ if a is not an element of A

$\in$: belongs to

Basic definitions of sets

: such that

- Curly brackets "{ }" are used to write a set
  - $\{1, 2\}$ and $\{-\sqrt{2}, e\}$ are sets with 2 elements
  - Sometimes, we use colon ":" to write sets which follow a rule

such that

$$A = \{x \in \mathbb{R} : x^2 = 2\} \quad \Rightarrow \quad A = \{-\sqrt{2}, \sqrt{2}\}$$

$$B = \{x \in \mathbb{N} : x \text{ is odd and } x \leq 5\} \quad \Rightarrow \quad B = \{1, 3, 5\}$$

Example 1

List all elements of the following sets

$\mathbb{Z}$ = integers = $\{\ldots, -1, 0, 1, \ldots\}$

$\mathbb{N}$ = nonnegative integers $= \{0, 1, 2, \ldots\}$

(a) $A = \{x \in \mathbb{Z} : -2 \leq x \leq 2\}$

$= \{-2, -1, 0, 1, 2\}$

(b) $B = \{x \in \mathbb{R} : x^2 - x = 0\}$

$x^2 - x = 0 \iff x(x-1) = 0 \iff x = 0$ or $x = 1$

$B = \{0, 1\}$

(c) $C = \{x \in \mathbb{N} : x^2 \leq 10\}$

$= \{0, 1, 2, 3\}$

Example 2

List all elements of the following set. Rewrite the sets (if necessary) by deleting repeated elements.

(a) $A = \{1, 1, 5, 3, 5\}$

$A = \{1, 5, 3\} = \{1, 3, 5\}$? Yes!

A has 3 elements.

(b) $B = \{1, \{1\}, 2\}$

1 and $\{1\}$ are different:

$\{1\}$ = a set with one element equal to 1

1 = a number

B has 3 elements

Basic definitions of sets

- The **empty set** is the set with *no element*, denoted by $\emptyset$.
- The **size** (or **cardinality**) of a set A is the number of elements in A, denoted by $|A|$.
- We call A an **n-set** if $|A| = n$.

Standard sets of numbers

nonnegative integers

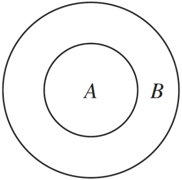
1. $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of all **natural numbers**.
2. $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is the set of all **integers**.
3. $\mathbb{Z}^+ = \{1, 2, \dots\}$ is the set of all **positive integers**.
4. $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$ is the set of all **rational numbers**.
5. $\mathbb{R}$ is the set of all **real numbers.**
6. $\mathbb{R}^+$ is the set of all positive **real numbers**.
7. $\mathbb{C}$ is the set of all **complex numbers**.

Note: 0 is neither positive nor negative.

Subsets

B contains A

- A is a **subset** of B, write $A \subset B$ or $B \supset A$, if any element of A is also an element of B.



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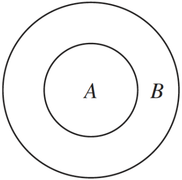

- The empty set $\emptyset$ is a subset of any set

$$\emptyset \subset A \text{ for any } A$$

- If A is not a subset of B, we write $A \not\subset B$.

Example 3

Find all subsets of

(a) $S = \{1,2\}$

$\emptyset, \{1\}, \{2\}, \{1,2\}$

(b) $S = \{1,2,3\}$

$\emptyset,$

$\{1\}, \{2\}, \{3\},$

$\{1,2\}, \{1,3\}, \{2,3\},$

$\{1,2,3\}$

Power set

- The **power set** of S, denoted $\mathcal{P}(S)$, is the set of all subsets of S.

Power set

- The **power set** of S, denoted $\mathcal{P}(S)$, is the set of all subsets of S.
- **Example**: Find $\mathcal{P}(S)$ with

(a) $S = \{1, 2\}$

$P(S) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\} \Rightarrow |P(S)| = 4$

(b) $S = \{1, 2, 3\}$

$P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

$|P(S)| = 8$

(c) Can you guess the size of the power set of $S = \{1, \ldots, n\}$?

Ans. $|P(S)| = 2^n$.

Tuples

- An **ordered n-tuple** (or **n-tuple**) is an ordered collection of n elements, say $(a_1, \ldots, a_n)$, such that a_1 is the 1st element, a_2 is the 2nd element, ... , a_n is the nth element.

Tuples

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- Comparison of ordered tuples

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n) \Leftrightarrow a_i = b_i \text{ for } i = 1, \ldots, n.$$

- The following 3-tuples are pairwise different

$$(1, 2, 3), (1, 3, 2), (2, 1, 3), (3, 1, 2).$$

Remark:

$$\{1,2,3\} = \{1,3,2\} = \{2,1,3\} = \{3,1,2\}$$

Sets vs tuples

set with n elements

What are the similarities and differences between n-set and n-tuple?

$\{a_1, a_2, \ldots, a_n\}$ $(a_1, a_2, \ldots, a_n)$

Similarities

They both have n elements.

Differences

1. Set uses { } and tuple use ( )
2. Sets don't care about "order of elements".
   Tuples take in "order of elements".
3. Tuples can have repeated elements.

Cartesian product

- The **Cartesian product** of A and B is

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

For example, if $A = B = \{1, 2\}$ then

$$A \times B = \{(1,1), (1,2), (2,1), (2,2)\}$$

$$\parallel$$

$$\{(a, b) : a \in A, b \in B\}$$

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- The Cartesian product of $A_1, A_2, \ldots, A_n$ is

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_i \in A_i \text{ for all } i\}.$$

- Special case

$$A^n = A \times \cdots \times A = \{(a_1, a_2, \ldots, a_n) : a_i \in A \text{ for all } i\}.$$

Question 3: Sizes of Cartesian products

(a) If $|A| = a$ and $|B| = b$, what is the size of $A \times B$? Explain your answer.

(A) a (B) b (C) $a+b$ (D) ab ✓ (E) a^b

$A \times B = \{(x,y) : x \in A, y \in B\}$

To form an element of $A \times B$

① choose $x \in A \Rightarrow a$ choices for x

② choose $y \in B \Rightarrow b$ choices for y

$\Rightarrow ab$ choices for $(x,y) \Rightarrow |A \times B| = ab$.

(b) If $|A| = a$, what is the size of A^n?

(A) 0 (B) 1 (C) a (D) a^n ✓

$$A^n = A \times A \times \dots \times A = \{\underbrace{(a_1, a_2, \dots, a_n)}_{\text{a choices, a choices, a choices}} : a_i \in A\}$$

a choices a choices a choices

$\therefore |A^n| = a^n$.

(c) Let A be any set. What is the size of $A \times \emptyset$?

(A) 0 (B) 1 (C) a

$$A \times \emptyset = \{(x, y) : x \in A, y \in \emptyset\} \Rightarrow |A \times \emptyset| = 0.$$

no choice for y

In fact,

$$A \times \emptyset = \emptyset.$$

Exercise 1

Define $A = \{0, 1\}$, $B = \{1, 2\}$, $C = \{0, 1, 2\}$. List all elements of

(a) $A \times B$, $B \times A$, $A \times B \times C$ and $(A \times B) \times C$.

$$A \times B = \{(0,1), (0,2), (1,1), (1,2)\}$$

$$A \times B \times C = \{(a,b,c) : a \in A, b \in B, c \in C\}$$

$$= \{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)\}$$

$$(A \times B) \times C = \{(x,y) : x \in A \times B, y \in C\}$$

$$= \{((0,1),0), ((0,1),1), ((0,1),2), ((0,2),0), ((0,2),1), ((0,2),2), ((1,1),0), ((1,1),1), ((1,1),2), ((1,2),0), ((1,2),1), ((1,2),2)\}$$

(a)

(b) Show that $A \times B \neq B \times A$ and $(A \times B) \times C \neq A \times B \times C$.

(exercise)

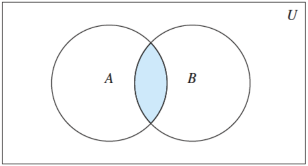
$A \times B \times C$ = set of triples

$(A \times B) \times C$ = set of pairs

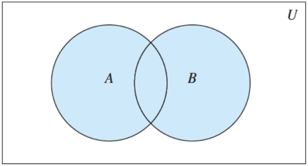
$\Rightarrow A \times B \times C \neq (A \times B) \times C$

Intersection and union

Let A and B be subsets of U. We call U the **universal set**, that is, the set of all elements which are under consideration.



$A \cap B$ is shaded.



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Intersection and union

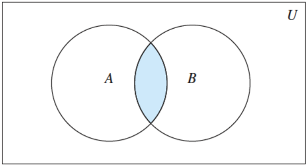
Let A and B be subsets of U. We call U the **universal set**, that is, the set of all elements which are under consideration.

- The **intersection** of A and B is

  such that

  $$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

  A and B are **disjoint** if $A \cap B = \emptyset$.



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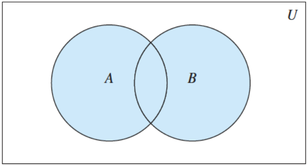

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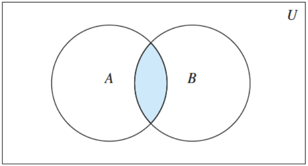
  A and B are **disjoint** if $A \cap B = \emptyset$.

- The **union** of A and B is

  $$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$



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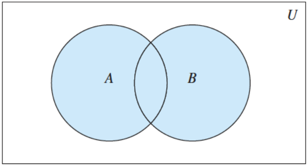
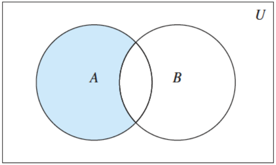


$A \cup B$ is shaded.

Difference and complement

- The **difference** of A and B is

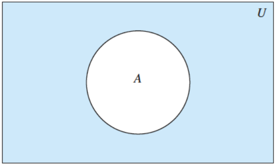
$$A - B = \{x : x \in A, x \notin B\}$$



$A - B$ is shaded.

- The **complement** of A in U is

$$\bar{A} = \{x \in U : x \notin A\}$$



$\bar{A}$ is shaded.

Example 4

$A = \{a, b, c, d, e\}$, $B = \{a, b, f\}$, $U = \{a, b, c, d, e, f, g\}$. Find

(a) $A \cup B$ and $A \cap B$

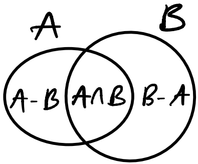
$A \cup B = \{a, b, c, d, e, f\}$

$A \cap B = \{a, b\}$

(b) $A - B$ and $B - A$

$A - B = \{c, d, e\}$

$B - A = \{f\}$



(c) $\bar{A}$ and $\bar{B}$

$\bar{A} = \{f, g\}$

$\bar{B} = \{c, d, e, g\}$

De Morgan's laws for sets

Lemma 1

Let U be the universal set and let A and B be subsets of U. Then

(a) $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

(b) $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

De Morgan's laws say that the complement operation

reverses intersection and union

Same sets

To prove $A = B$, we prove 2 things:

1. Prove $A \subset B$ by showing that

   any $x \in A$ is also in B.

2. Prove $B \subset A$ by showing that

   any $x \in B$ is also in A.

Proof of (a): $\overline{A \cup B} = \overline{A} \cap \overline{B}$

- Claim 1: $\overline{A \cup B} \subset \overline{A} \cap \overline{B}$. Let $x \in \overline{A \cup B}$. By def

$$x \notin A \cup B \Rightarrow \begin{cases} x \notin A \Rightarrow x \in \overline{A} \\ x \notin B \Rightarrow x \in \overline{B} \end{cases} \Rightarrow x \in \overline{A} \cap \overline{B}$$

$$\therefore \overline{A \cup B} \subset \overline{A} \cap \overline{B}$$

- Claim 2: $\overline{A} \cap \overline{B} \subset \overline{A \cup B}$. Let $x \in \overline{A} \cap \overline{B}$. By def

$$\begin{cases} x \in \overline{A} \Rightarrow x \notin A \\ x \in \overline{B} \Rightarrow x \notin B \end{cases} \Rightarrow x \notin A \cup B \Rightarrow x \in \overline{A \cup B}$$

$$\therefore \overline{A} \cap \overline{B} \subset \overline{A \cup B}$$

$$\therefore \overline{A \cup B} = \overline{A} \cap \overline{B}$$

Properties of union and intersection

Let A, B, C be three sets. Then the following hold.

(a) $A \cup B = B \cup A$ and (Commutativity)
$A \cap B = B \cap A$

(b) $A \cup (B \cup C) = (A \cup B) \cup C$ and (Associativity)
$A \cap (B \cap C) = (A \cap B) \cap C$

(c) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and (Distributivity)
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$a \times (b + c) = a \times b + a \times c$

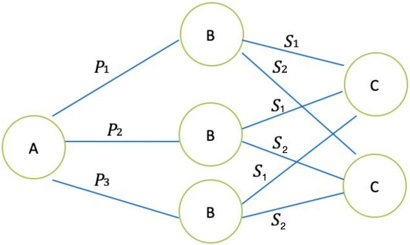
Question 2

- Consider 3 cities A,B,C.
- There are 3 ways from A to B, and 2 ways from B to C.
- How many ways are there from city A to city C via city B?

Question 2

Assume that

- $A \to B$ through P_1, P_2, P_3
- $B \to C$ through S_1, S_2



To go $A \to B \to C$

$P_1 - S_1$, $P_1 - S_2$, $P_2 - S_1$, $P_2 - S_2$, $P_3 - S_1$, $P_3 - S_2$

$\Rightarrow$ 6 ways

Product rule

If a procedure can be divided into a sequence of k tasks $T_1, \dots, T_k$:

- T_1 can be performed in n_1 different ways,
- T_2 can be performed in n_2 different ways (regardless of how T_1 was performed), ...
- T_k can be performed in n_k different ways (regardless of how $T_1, \dots, T_{k-1}$ were performed),

then the entire procedure can be performed in $n_1 n_2 \cdots n_k$ ways.

Exercise 2

How many different license plates are available if each plate contains a sequence of 3 letters followed by 3 digits?

capital

Solution.

Each plate has form $\underbrace{L_1}_{26}\underbrace{L_2}_{26}\underbrace{L_3}_{26}\underbrace{D_1}_{10}\underbrace{D_2}_{10}\underbrace{D_3}_{10}$

$\Rightarrow 26^3 10^3$ choices.

Exercise 3

How many subsets does the set $S = \{1, \ldots, n\}$ have?

Solution. Ans : 2^n.

To form a subset A of S, for each element $1, 2, \ldots, n$, we have 2 choices

(1) Put it in A

(2) Do not put it in A

1
2
3
⋮
n

A

$\Rightarrow$ there are $2 \times 2 \times \ldots \times 2 = 2^n$ ways to form A.

Exercise 4

Let $A_1, \ldots, A_k$ be finite sets with $|A_i| = n_i$ for $i = 1, \ldots, k$.
Find the cardinality of $A_1 \times \cdots \times A_k$.

Solution.

Ans. $|A_1 \times \ldots \times A_k| = n_1 n_2 \cdots n_k$

$$A_1 \times A_2 \times \ldots \times A_k = \{ (\underbrace{a_1}_{n_1 \text{ choices}}, \underbrace{a_2}_{n_2 \text{ choices}}, \ldots, \underbrace{a_k}_{n_k \text{ choices}}) : a_i \in A_i \text{ for all } i \}$$

$$\Rightarrow |A_1 \times A_2 \times \ldots \times A_k| = n_1 n_2 \ldots n_k$$

Two sets

Theorem 1

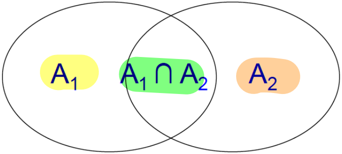
Let A_1 and A_2 be two finite sets. Then

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

This equation is called the inclusion-exclusion principle for two sets.

Proof

Partition $A_1 \cup A_2$ into 3 disjoint pieces



$A_1 - (A_1 \cap A_2)$, $A_1 \cap A_2$, $A_2 - (A_1 \cap A_2)$

$$|A_1 \cup A_2| = |A_1 - A_1 \cap A_2| + |A_1 \cap A_2| + |A_2 - A_1 \cap A_2|$$

$$= |A_1| - \cancel{|A_1 \cap A_2|} + \cancel{|A_1 \cap A_2|} + |A_2| - |A_1 \cap A_2|$$

$$= |A_1| + |A_2| - |A_1 \cap A_2|$$

Example 5

Every student in a class is either a math major, a biology major or a joint major in these two subjects. How many students are in the class if there are 38 math majors (including joint majors), 23 biology majors (including joint majors), and 7 joint majors?

Denote

A & B = sets of math major students & biology major

$|A| = 38$, $|B| = 23$, $|A \cap B| = 7$

Hence

$$\# \text{ students} = |A \cup B| = |A| + |B| - |A \cap B|$$

$$= 38 + 23 - 7 = 54.$$

Exercise 5

(a) Let d and n be positive integers. How many integers are there in $\{1, \dots, n\}$ that are divisible by d?

Answer: $\lfloor \frac{n}{d} \rfloor$

Solution. Let md be a multiple of d in the set $\{1, 2, \dots, n\}$:

$md \geq 1 \Rightarrow m \geq \frac{1}{d} \Rightarrow m \geq 1$ (1)

$md \leq n \Rightarrow m \leq \frac{n}{d} \Rightarrow m \leq \lfloor \frac{n}{d} \rfloor$ (2)

By (1) & (2), there are $\lfloor \frac{n}{d} \rfloor$ choices for m which are $1, 2, \dots, \lfloor \frac{n}{d} \rfloor$.

$\therefore$ There are $\lfloor \frac{n}{d} \rfloor$ multiples of d in $\{1, 2, \dots, n\}$.

Floor function

$\lfloor x \rfloor$ = largest integer $\leq x$

Example,

$\lfloor 2 \rfloor = \lfloor 2.1 \rfloor = \lfloor 2.9 \rfloor = 2$

$\lfloor -1.1 \rfloor = \lfloor -1.2 \rfloor = -2$

Exercise 6

(b) How many integers are there in $\{1, \ldots, 1000\}$ that can be divisible by either 2 or 3?

A, B = sets of integers in $\{1, \ldots, 1000\}$ divisible by $2, 3$

$|A| = \lfloor \frac{1000}{2} \rfloor = 500$, $|B| = \lfloor \frac{1000}{3} \rfloor = 333$, $|A \cap B| = \lfloor \frac{1000}{6} \rfloor = 166$

Hence

$$|A \cup B| = |A| + |B| - |A \cap B| = 500 + 333 - 166 = 667.$$

(c) How many integers are there in $\{1, \ldots, 1000\}$ that can be divisible by either 4 or 6?

Exercise!

Exercise 7

How many $0-1$ bit strings of length eight either start with a bit 1 or end with the two bits 00?

A = strings of length 8 starting with 1

$= \{1b_2 \dots b_8 : b_i \in \{0,1\}\} \Rightarrow |A| = 2^7 = 128$

B = strings of length 8 ending with 00

$= \{b_1 \dots b_6 00 : b_i \in \{0,1\}\} \Rightarrow |B| = 2^6 = 64$

$A \cap B$ = strings starting with 1 & ending with 00

$= \{1b_2 b_3 b_4 b_5 b_6 00 : b_i \in \{0,1\}\} \Rightarrow |A \cap B| = 2^5 = 32$

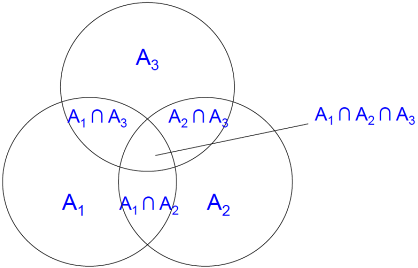
$\therefore |A \cup B| = |A| + |B| - |A \cap B| = 128 + 64 - 32 = 160.$

Three sets

Theorem 2

Let A_1, A_2 and A_3 be three finite sets. Then

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - \\ - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$



Proof

By using $A_1 \cup A_2 \cup A_3 = (A_1 \cup A_2) \cup A_3$, we can apply the case of 2 sets

$$|A_1 \cup A_2 \cup A_3| = |(A_1 \cup A_2) \cup A_3| = |A_1 \cup A_2| + |A_3| - |(A_1 \cup A_2) \cap A_3|$$

distributivity

$$= |A_1| + |A_2| - |A_1 \cap A_2| + |A_3| - |(A_1 \cap A_3) \cup (A_2 \cap A_3)|$$

$$= |A_1| + |A_2| - |A_1 \cap A_2| + |A_3| - (|A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|)$$

$$= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

Example 6

1232 students take a course in Spanish, 879 take French, 114 take Russian, 103 take both Spanish and French, 23 take both Spanish and Russian, and 14 take both French and Russian.
If 2092 students take at least one of these three courses, how many students take all three?

Denote

A, B, C = sets of students taking Spanish, French, Russian

We are given

$|A| = 1232, |B| = 879, |C| = 114$

$|A \cap B| = 103, |A \cap C| = 23, |B \cap C| = 14, |A \cup B \cup C| = 2092$

We need to find $|A \cap B \cap C|$.

Example 7

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |A \cap B \cap C|$$

$$|A \cap B \cap C| = 7$$

Exercise 8

How many integers are there the interval $[1, 1000]$ that can be divisible by one of the numbers 2 or 3 or 5?

A, B, C = sets of integers in $[1, 1000]$ that are divisible by $2, 3, 5$

$|A| = \left\lfloor \frac{1000}{2} \right\rfloor = 500, \ |B| = \left\lfloor \frac{1000}{3} \right\rfloor = 333, \ |C| = \left\lfloor \frac{1000}{5} \right\rfloor = 200,$

$|A \cap B| = \left\lfloor \frac{1000}{6} \right\rfloor = 166, \ |A \cap C| = \left\lfloor \frac{1000}{10} \right\rfloor = 100, \ |B \cap C| = \left\lfloor \frac{1000}{15} \right\rfloor = 66$

$|A \cap B \cap C| = \left\lfloor \frac{1000}{30} \right\rfloor = 33$

We obtain

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$= 734$$

General case

Theorem 3

Let n be a positive integer and let $A_1, A_2, \ldots, A_n$ be n finite sets. Then

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}|.$$

General case

The right-hand side can be written in more details as follows.

$$
\begin{aligned}
|A_1 \cup \cdots \cup A_n| \quad &= \quad \sum_{1 \le i_1 \le n} |A_{i_1}| - \sum_{1 \le i_1 < i_2 \le n} |A_{i_1} \cap A_{i_2}| + \\
&+ \sum_{1 \le i_1 < i_2 < i_3 \le n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \\
&\cdots + (-1)^{n-1} |A_1 \cap A_2 \ldots \cap A_n|
\end{aligned}
$$

Exercise 9

Give a formula for inclusion-exclusion for 4 sets.

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4|$$
$$- |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| -$$
$$|A_2 \cap A_4| - |A_3 \cap A_4|$$
$$+ |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| +$$
$$|A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|$$
$$- |A_1 \cap A_2 \cap A_3 \cap A_4|$$