5 fowns

5 (d), (j), (n), (q) *

(g), (p)

Q1 → Q2 → Q3 → Q4

Q1 $a_n = a_{n-1} + a_{n-2}$ with $a_1 = 1$, $a_2 = 1$

$a_3 = a_2 + a_1 = 1+1 = 2$, $a_4 = a_3 + a_2 = 2+1 = 3$,

$a_5 = a_4 + a_3 = 3+2 = 5$, $a_6 = a_5 + a_4 = 8$,

$a_7 = a_6 + a_5 = 8+5 = 13$, $a_8 = a_7 + a_6 = 13+8 = 21$.

Q2 $a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a_n \text{ is even} \\ 3a_n + 1 & \text{if } a_n \text{ is odd.} \end{cases}$

Collatz

Demonstration → see code provided

Q3 $a_n = \cos\left(\frac{n\pi}{2}\right)$

Tip: Not sure ⇒ compute first few terms of sequence.

$a_1 = \cos\left(\frac{\pi}{2}\right) = 0$, $a_2 = \cos(\pi) = -1$,

$a_3 = \cos\left(\frac{3\pi}{2}\right) = 0$, $a_4 = \cos(2\pi) = 1$,

$a_5 = \cos\left(\frac{5\pi}{2}\right) = 0$, $a_6 = \cos(3\pi) = -1$,

$a_7 = \cos\left(\frac{7\pi}{2}\right) = 0$, $a_8 = \cos(4\pi) = 1$.

a_{4n-2}

a_{2n-1} odd subsequence

$$a_{2n-1} = \cos\left(\frac{(2n-1)\pi}{2}\right) = \cos\left(n\pi - \frac{\pi}{2}\right) = 0.$$

$$a_{4n-2} = \cos\left(\frac{(4n-2)\pi}{2}\right) = \cos(2n\pi - \pi) = \cos(-\pi) = -1$$

$a_{2n-1} \to 0$ but $a_{4n-2} \to -1$

Since we have two subsequences of a_n converging to different values, a_n must be divergent (Subsequence Test).

Q4 $a_n = (-1)^n$ divergent

$$b_n = (-1)^n \frac{1}{n}$$

$\frac{1}{n} \to 0$

$\exists$ there exists

$\therefore$ Therefore $\because$ Because

$$-1 \le (-1)^n \le 1$$

$$\Rightarrow \underset{a_n}{\underset{\shortparallel}{-\frac{1}{n}}} \le \underset{b_n}{\underset{\shortparallel}{(-1)^n \frac{1}{n}}} \le \underset{c_n}{\underset{\shortparallel}{\frac{1}{n}}}$$

(Multiply $\frac{1}{n}$ to all sides)

$\because \lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} c_n \quad \therefore$ By Squeeze Theorem, $\lim_{n\to\infty} b_n = 0.$

bread

face

$(-1)^n \frac{1}{n^2} \to 0$

$(-1)^n \frac{1}{e^n} \to 0$ $\quad$ $(-1)^n \frac{1}{\ln n} \to 0$

$(-1)^n \left(\frac{n^2+1}{2n^2+2}\right)$ divergent.

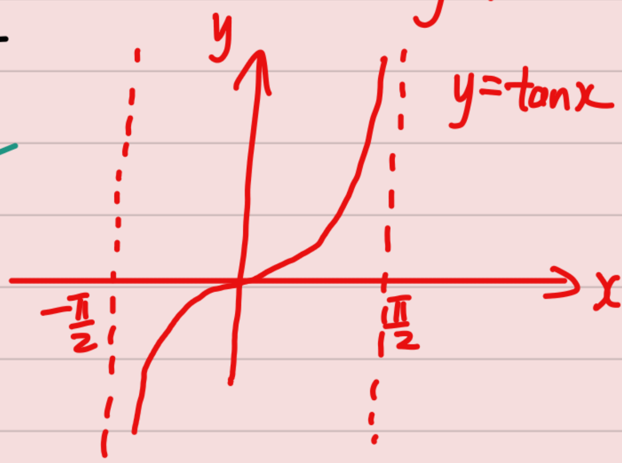
5 ~~(d)~~, ~~(j)~~, (n), (q) ☆

~~(g)~~, (p)

$y = \tan^{-1} x$
$y = \tan x$
inverse
graph is reflection about $y = x$

5(d) $\lim_{n\to\infty} \frac{\tan^{-1} n}{n} = \lim_{x\to\infty} \frac{\tan^{-1} x}{x}$

$y = \tan^{-1} x$



$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$.

$$-\frac{\pi}{2} \le \tan^{-1} n \le \frac{\pi}{2}$$

$$\Rightarrow -\frac{\pi}{2}\cdot\frac{1}{n} \le \frac{\tan^{-1} n}{n} \le \frac{\pi}{2}\cdot\frac{1}{n}$$

$\because -\frac{\pi}{2}\cdot\frac{1}{n} \to 0$ and $\frac{\pi}{2}\frac{1}{n} \to 0$ $\therefore$ By the Squeeze Theorem,

$$\lim_{n\to\infty} \frac{\tan^{-1} n}{n} = 0.$$

L'H : $\frac{\pm\infty}{\pm\infty}$ $\frac{0}{0}$

(g) $\lim_{n\to\infty} \frac{\sin n}{n^2}$

For all n, $-1 \le \sin n \le 1$

Multiply by $\frac{1}{n^2}$ $\Rightarrow$ $-\frac{1}{n^2} \le \frac{\sin n}{n^2} \le \frac{1}{n^2}$

$\because -\frac{1}{n^2}$ and $\frac{1}{n^2} \to 0$, by the Squeeze Theorem, $\lim_{n\to\infty} \frac{\sin n}{n^2} = 0$

(j) $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$ $\quad 0^0 \quad \infty^{\infty} \quad \frac{\infty}{\infty} \quad \frac{0}{0}$

$\infty - \infty$ indeterminate form

Conjugation / Rationalization

$$\lim_{n\to\infty} \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$

$\because \lim_{n\to\infty} \sqrt{n+1} + \sqrt{n} = \infty.$

(n) $\lim_{n\to\infty} \left(1 + \frac{3}{n}\right)^n$ $\qquad \lim_{n\to\infty} n^{\frac{1}{n}}$

$$\left(1+\frac{3}{n}\right)^n = e^{\ln\left(1+\frac{3}{n}\right)^n}$$

$x = e^{\ln x}$

$\rightarrow n^{\frac{1}{n}} = e^{\ln n^{\frac{1}{n}}} = e^{\frac{\ln n}{n}}$

$\lim_{n\to\infty} \ln\left(1+\frac{3}{n}\right)^n$ $\qquad \ln a^b = b \ln a$

$= \lim_{n\to\infty} n \ln\left(1+\frac{3}{n}\right)$ $\quad \infty \cdot 0 \rightarrow$ indeterminate

$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{3}{n}\right)}{\frac{1}{n}}$ $\qquad n = \frac{1}{\frac{1}{n}}$ $\quad$ Galaxy brain move

$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}}$ $\quad$ function $\left(\frac{0}{0}\right)$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+3/x} \cdot 3\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{3}{1+\frac{3}{x}} = 3.$$

$$\lim_{n\to\infty} \left(1+\frac{3}{n}\right)^n = e^{\lim_{n\to\infty} \ln\left(1+\frac{3}{n}\right)^n} = e^3$$

Food for thought $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$

e is the constant where $\int_1^e \frac{1}{x}\,dx = 1$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad x \in (-\infty, \infty)$ $\Leftarrow$

Week12 material

(q) $\lim_{n\to\infty} \frac{7^n}{n!}$

Limit of a sequence does not depend on the first few finite terms.

$$\frac{7^n}{n!} = \frac{\overbrace{7\cdot 7\cdot \cdots \cdot 7}^{n \text{ copies}}}{\underbrace{1\cdot 2\cdot \cdots\cdot n}_{n \text{ copies}}}$$

Let $n \geq 8$.

$$0 \leq \frac{7^n}{n!} = \underbrace{\frac{7}{1}\cdot\frac{7}{2}\cdots\frac{7}{7}}_{\text{constant } C = \frac{7^7}{7!}}\cdot\underbrace{\overset{\leq 1}{\frac{7}{8}}\cdot\overset{\leq 1}{\cdots}\cdot\frac{7}{n-1}}_{\leq 1}\cdot\frac{7}{n} \leq \frac{7^7}{7!}\cdot 1\cdot 1\cdots\frac{7}{n}$$

$$= \frac{7^8}{7!}\cdot\frac{1}{n}$$

Since $\lim_{n\to\infty} \frac{7^8}{7!}\frac{1}{n} = 0$, $\lim_{n\to\infty}\frac{7^n}{n!} = 0$ by the Squeeze Theorem.

(p) $\lim_{n\to\infty} \frac{n^2\cos(n\pi)}{n^2+1} = \lim_{n\to\infty} \underbrace{\cos(n\pi)}_{\text{bounded}}\cdot\underbrace{\frac{n^2}{n^2+1}}_{\to 1}$

$\cos(2n\pi) = 1$

$\cos((2n+1)\pi) = -1$

Let $a_n = \frac{n^2\cos(n\pi)}{n^2+1}$

$$a_{2n} = \frac{(2n)^2 \cos(2n\pi)}{(2n)^2+1} \Rightarrow \lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \frac{(2n)^2}{(2n)^2+1} \cdot \frac{\frac{1}{(2n)^2}}{\frac{1}{(2n)^2}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{(2n)^2}} = 1$$

$$a_{2n+1} = \frac{(2n+1)^2 \cos((2n+1)\pi)}{(2n+1)^2+1} = -\frac{(2n+1)^2}{(2n+1)^2+1}$$

$$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} \frac{-1}{1+\frac{1}{(2n+1)^2}} = -1$$

$$\lim_{n\to\infty} \frac{n\cos(n\pi)}{n^2+1} = \lim_{n\to\infty} \cos(n\pi) \cdot \underbrace{\frac{n}{n^2+1}}_{\to 0}$$

$$\lim_{n\to\infty} \frac{2n^3 \cos(n\pi)}{n^3+n^2} \rightarrow$$ same as part (a)

$$= \lim_{n\to\infty} \cos(n\pi) \cdot \underbrace{\frac{2n^3}{n^3+n}}_{\to 2}$$