Q1 (i), (k), (n), (o), (p), (h), (c)

Q2 (c)

Q3 (g), (f)

Q1(c) $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^4+n^2}$ $\leftarrow$ $n \geq 1$: $0 \leq \tan^{-1} n \leq \frac{\pi}{2}$.

Intuition : $\frac{\tan^{-1} n}{n^4+n^2} = k \cdot \frac{1}{n^4}$ for some constant k

Let $a_n = \frac{\tan^{-1} n}{n^4+n^2}$, $b_n = \frac{1}{n^4}$, $a_n, b_n > 0$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\tan^{-1} n}{n^4+n^2}}{\frac{1}{n^4}} = \lim_{n\to\infty} \frac{n^4 \tan^{-1} n}{n^4+n^2}$$

both limits are convergent

$$= \lim_{n\to\infty} \frac{n^4}{n^4+n^2} \cdot \lim_{n\to\infty} \tan^{-1} n$$

$$= 1 \cdot \frac{\pi}{2} = \frac{\pi}{2} > 0.$$

Since $\sum \frac{1}{n^4} < \infty$ (p-series, $p=4>1$), by the LCT,

$$\sum \frac{\tan^{-1} n}{n^4+n^2} < \infty.$$

$a_n \cdot b_n$

$a_n = (-1)^n \frac{1}{n^2}$ $\rightarrow$ divergent X

divergent

$\sum_{n=3}^{\infty} (-1)^n \tan^{-1} n$

Alternating Series Test

✓ b_n decreasing

Know: $\lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{2} \neq 0$. ✓ $b_n \to 0$

<u>Divergence Test</u>: $\sum a_n$ and $\lim_{n\to\infty} a_n \neq 0$ (or DNE) then $\sum a_n$ divergent

$a_n = (-1)^n \tan^{-1} n$

Can you show that a_n divergent?

$$\left.\begin{aligned} a_{2n} &= \tan^{-1}(2n) \to \frac{\pi}{2} \\ a_{2n+1} &= -\tan^{-1}(2n+1) \to -\frac{\pi}{2} \end{aligned}\right\}$$

Two subsequences of a_n converging to different limits $\Rightarrow$ a_n divergent / $\lim_{n\to\infty} a_n$ D.N.E.

Since $\lim_{n\to\infty} a_n$ D.N.E, $\sum (-1)^n \tan^{-1} n$ is divergent by the Divergence Test.

Generically

$\sum (-1)^n b_n$ If $b_n \to b \neq 0$ $\Rightarrow$ $\sum (-1)^n b_n$ divergent

<u>Proof</u>: Show $\lim_{n\to\infty} (-1)^n b_n$ D.N.E by using even and odd subsequences.

(i) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{2n+1}$

$\cos(n\pi) = (-1)^n$

n odd	-1
n even	1

$= \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{2n+1}$ Alternating series

Let $b_n = \frac{1}{2n+1}$ Let $f(x) = \frac{1}{2x+1}$ $\underbrace{x \geq 1}_{n \geq 1}$

$$f'(x) = -\frac{2}{(2x+1)^2} < 0$$

Alt: $b_{n+1} \leq b_n \quad \forall n \geq 1$

$\frac{1}{2n+3} \leq \frac{1}{2n+1}$ because $2n+3 \geq 2n+1$

$\therefore b_n$ is decreasing

$\lim_{n \to \infty} \frac{1}{2n+1} = 0$ (obvious).

$\therefore$ By AST, $\sum \frac{\cos(n\pi)}{2n+1}$ is convergent

(o) $\sum_{k=1}^{\infty} k e^{-k} = \sum_{k=1}^{\infty} \frac{k}{e^k}$

Don't try this at home

$e^k > k^3$ <u>for large k</u>

$\therefore \frac{1}{e^k} < \frac{1}{k^3}$ for large k

Hence $0 \leq \frac{k}{e^k} < \frac{1}{k^2}$ for large k

$\therefore$ By CT, $\sum_{k=1}^{\infty} k e^{-k}$ converges since $\sum \frac{1}{k^2} < \infty$

Pls try this at home

Ratio Test : $a_k = k e^{-k}$ $a_{k+1} = (k+1)e^{-(k+1)}$

$$\lim_{k \to \infty} \left| \frac{(k+1)\, e^{-(k+1)}}{k \; e^{-k}} \right| = \lim_{k \to \infty} \frac{k+1}{k} \cdot e^{-(k+1)-(-k)}$$

$$= \lim_{k \to \infty} \frac{k+1}{k} \cdot \frac{1}{e} = \frac{1}{e} < 1$$

$\therefore \sum_{k=1}^{\infty} k e^{-k}$ is absolutely convergent by Ratio Test.

(p) $\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^3}$

Ratio Test ✓✓✓ good friends w/ $n!$

Root Test ✓✓ good friends w/ $\bigcirc^n$ ← power

$$a_n = \frac{(3n)!}{(n!)^3} \qquad a_{n+1} = \frac{(3n+3)!}{((n+1)!)^3}$$

$$\lim_{n\to\infty} \frac{(3n+3)!}{((n+1)!)^3} \cdot \frac{(n!)^3}{(3n)!} = \lim_{n\to\infty} \frac{(3n+3)!}{(3n)!} \cdot \frac{(n!)^3}{((n+1)!)^3}$$

$$(3n+3)! = 1 \cdot 2 \cdot \cdots (3n)(3n+1)(3n+2)(3n+3)$$

$$(3n)! = 1 \cdot 2 \cdot \cdots (3n)$$

$$= \lim_{n\to\infty} (3n+1)(3n+2)(3n+3) \cdot \underbrace{\frac{n!}{(n+1)!}}_{\frac{1}{n+1}} \cdot \underbrace{\frac{n!}{(n+1)!}}_{\frac{1}{n+1}} \cdot \underbrace{\frac{n!}{(n+1)!}}_{\frac{1}{n+1}}$$

$$= \lim_{n\to\infty} \frac{(3n+1)(3n+2)(3n+3)}{(n+1)(n+1)(n+1)} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}}$$

$$= \lim_{n\to\infty} \frac{(3+\frac{1}{n})(3+\frac{2}{n})(3+\frac{3}{n})}{(1+\frac{1}{n})(1+\frac{1}{n})(1+\frac{1}{n})} = \frac{3 \cdot 3 \cdot 3}{1 \cdot 1 \cdot 1} = 27 > 1$$

By Ratio Test, $\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^3}$ is divergent.

(k) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ not a p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ p-series, p constant

Intuition

$1+\frac{1}{n} \approx 1$ when n is large $\sum (-1)^n \frac{1}{n^p}$ abs convergent

$\frac{1}{n^{1+1/n}} \approx \frac{1}{n}$ when n is large

$a_n = \frac{1}{n^{1+1/n}}$ $\quad b_n = \frac{1}{n}$ $\quad$ use LCT

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^{1+\frac{1}{n}}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^1}{n^{1+1/n}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = \frac{1}{1} = 1 > 0$$

$(1+\frac{1}{n})^n \to e$ $\quad n^{\frac{1}{n}} \to 1$

lectures

Since $\sum \frac{1}{n}$ diverges (p-series, $p = 1 \leq 1$), by LCT,
$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ diverges.