

CSD2301 Lecture

13. Rotational Dynamics

LIN QINJIE

Outline

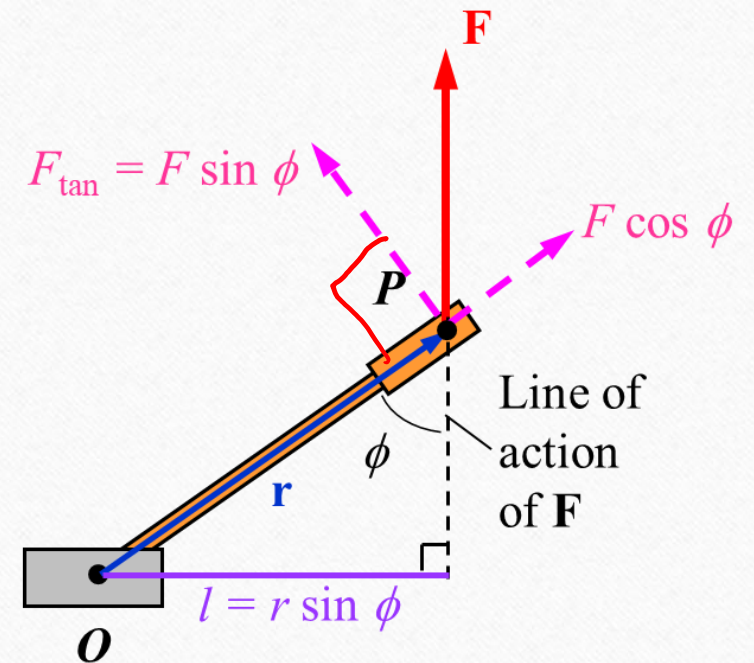
- Torque
- Work, power and energy

Torque

- Consider a force F acting on a rigid body at point P in the direction as shown. The rod is pivoted so that it can only rotate about the point O . Then the **torque** t of force F about O , is defined as:

$$\tau = Fl = rF \sin \theta = F_{\tan} r$$

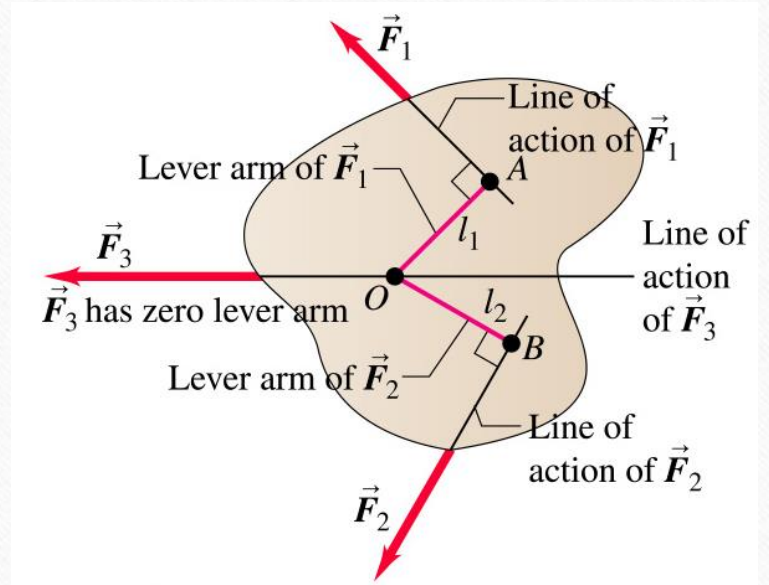
- l is the moment arm (lever arm) of the force, the **shortest distance from O to the line of action** of the force F .
- Only the component perpendicular to r , $F \sin \phi$, of the force F causes rotation.
- The component along r , $F \cos \phi$ has no effect in rotation



Torque

- Torque is a vector
- If two or more forces are acting on a rigid object, it is assumed that the **sign of a torque is positive if the turning tendency of the force is counterclockwise** and is negative if the turning tendency is clockwise.

$$\sum \tau = \tau_1 + \tau_2 + \tau_3 = F_1 l_1 - F_2 l_2 + 0$$



Torque vs Force

- Forces can cause a change in linear motion.
- Forces can also cause a change in rotational motion, but the effectiveness of the forces in causing this change depends on both the forces and the moment arms of the forces.
 - Force can result in torque
- The dimension of torque is force times length (unit is newton·meter but not joule).

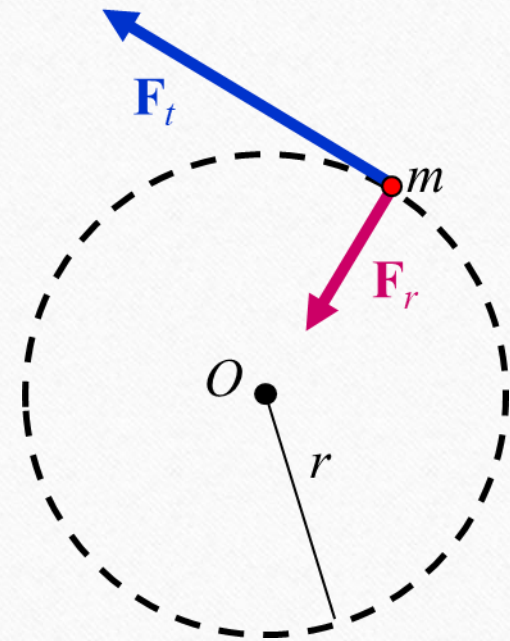
Nm

Torque & Angular Acceleration

- Consider a particle mass m rotating in a circle radius r under the influence of a tangential force F_t and a radial force F_r :

$$\begin{aligned}\tau &= rF_t \\ &= r(ma_t) \\ &= r(mr\alpha) \\ &= (mr^2)\alpha\end{aligned}$$

$$\boxed{\tau = I\alpha} \Rightarrow F = ma$$



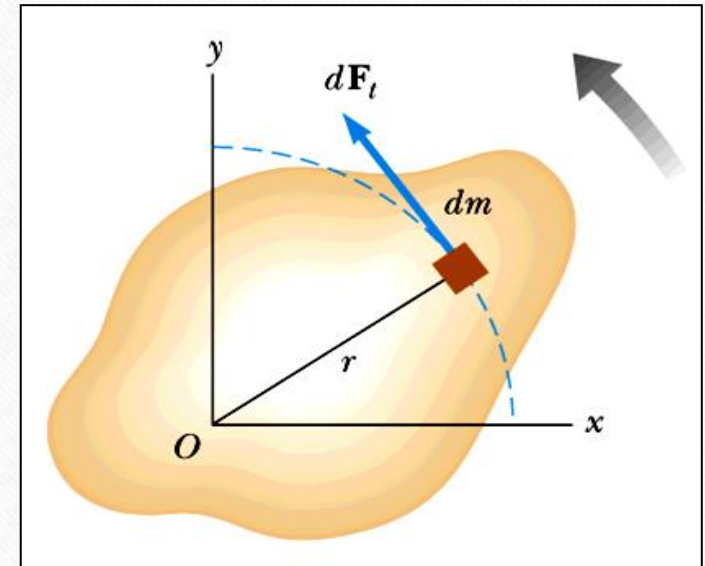
Torque & Angular Acceleration

- For an extended object, each mass element dm rotates about the origin (z -axis).

$$d\tau = r dF_t = r(dm)a_t = (r^2 dm)\alpha$$

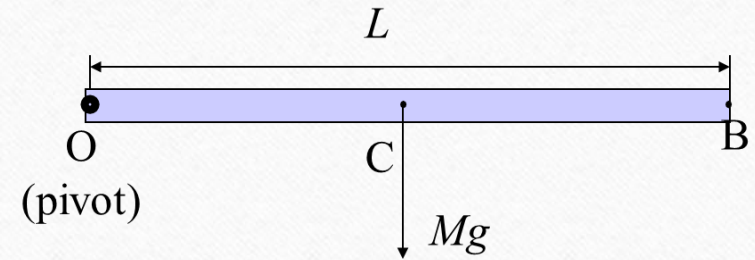
- Integrating: $\tau = \int (r^2 dm)\alpha = \alpha \int r^2 dm$

$$\tau = I\alpha$$



Example: Rotating Rod

- A uniform rod of mass M and length L is attached to a frictionless pivot. When released from rest from horizontal position, what is its angular acceleration? What are the linear accelerations of points B and C?



Example: Rotating Rod

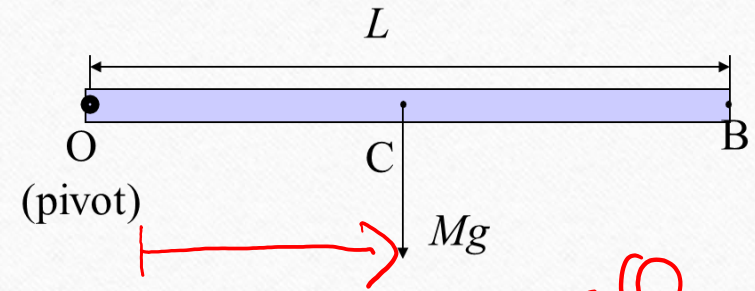
- About pivot, torque is: $\tau = Mg\left(\frac{L}{2}\right)$

$$I = \frac{1}{3}ML^2 \quad (\text{I of rod about its end})$$

$$\alpha = \frac{\tau}{I} = \frac{MgL}{2} \times \frac{3}{ML^2} \Rightarrow \alpha = \frac{3g}{2L}$$

$$a_B = L \times \frac{3g}{2L} = \frac{3}{2}g$$

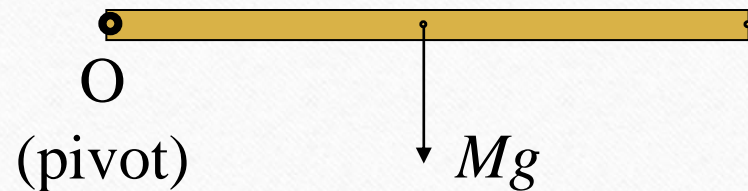
$$a_C = \frac{L}{2} \times \frac{3g}{2L} = \frac{3}{4}g$$



$$\begin{aligned} s &= r\theta \\ v &= r\omega \\ a &= r\alpha \end{aligned}$$

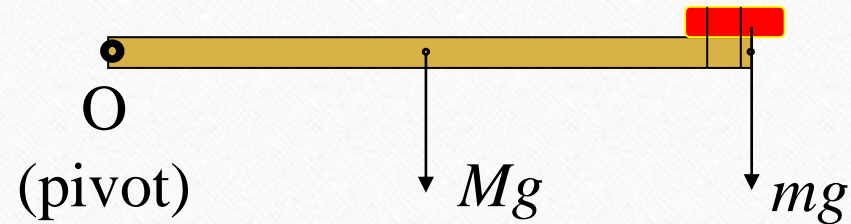
Example: Rotating Rod

- Riddle: Tie another mass to the rod at the end as shown. Which rod will take a shorter time to swing to the vertical position?



$$\tau = Mg\left(\frac{L}{2}\right) \quad \& \quad I = \frac{1}{3}ML^2$$

$$\alpha = \frac{3g}{2L}$$

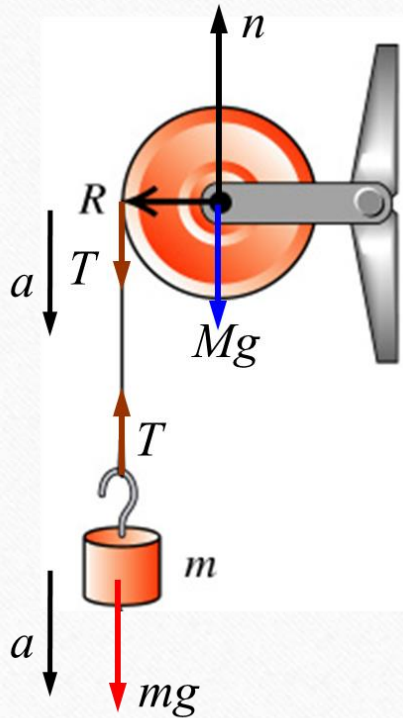


$$\tau = Mg\left(\frac{L}{2}\right) + mgL \quad \& \quad I = \frac{1}{3}ML^2 + mL^2$$

$$\alpha = \frac{3g}{2L} \left(\frac{M + 2m}{M + 3m} \right)$$

Example: An unwinding cable

Want to find a , α and T in terms of I , R and M : (may assume $I = 1/2 MR^2$ if wheel is a uniform cylinder).



$$\tau = TR = I\alpha$$

$$a = R\alpha$$

$$a = \frac{TR^2}{I}$$

From FBD of weight:

$$mg - T = ma$$

$$mg - T = m\left(\frac{TR^2}{I}\right)$$

$$mg = T\left[1 + \frac{mR^2}{I}\right]$$

$$T = \frac{mg}{1 + (mR^2/I)}$$

$$a = \frac{mg}{1 + (mR^2/I)} \times \frac{R^2}{I} = \frac{mg}{(I/R^2) + m}$$

$$a = \frac{g}{1 + (I/mR^2)}$$

$$\alpha = \frac{g}{R + (I/mR)}$$

Work, Power and Energy

- Consider a point rotating through distance ds .

$$dW = \vec{F} \cdot d\vec{s} = (F \sin \phi) r d\theta$$

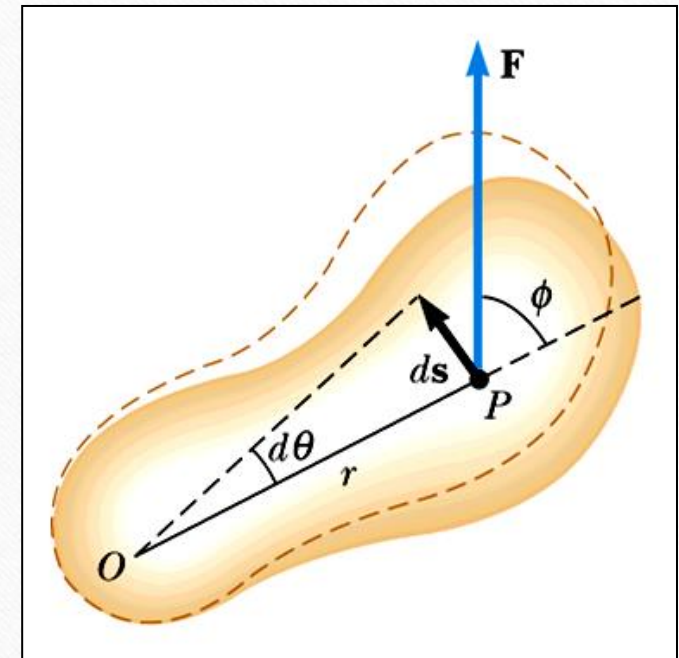
$$r F \sin \phi = \tau$$

work \rightarrow

$$dW = \tau d\theta$$


$$P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau \omega$$

$$P = Fv$$



Work and Energy

- The net work done by external forces in rotating a rigid object about a fixed axis equals the change in the object's rotational energy.


$$\tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\frac{d\omega}{d\theta}\omega$$

$$dW = \tau d\theta = I\omega d\omega$$

$$W = \int_{\theta_0}^{\theta} \tau d\theta = I \int_{\omega_0}^{\omega} \omega d\omega$$

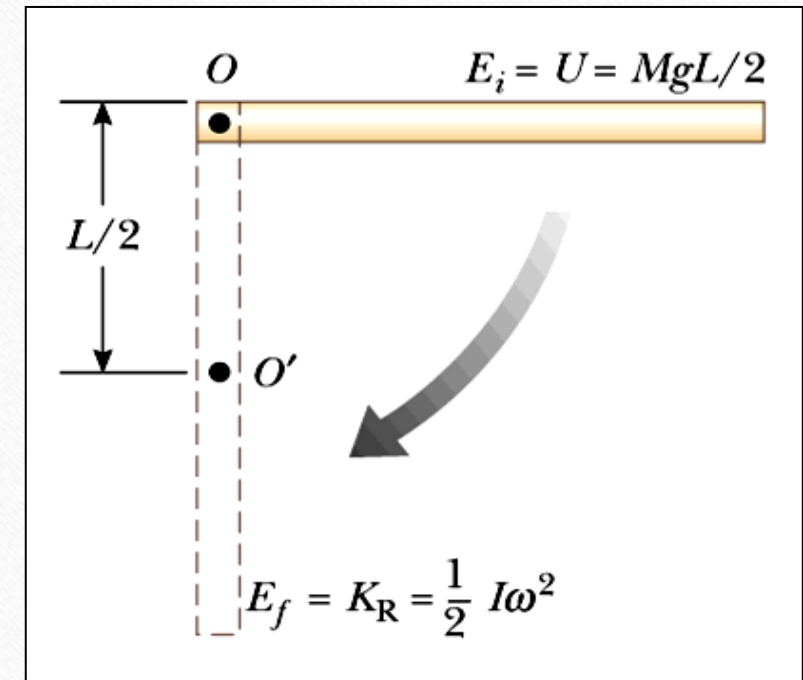
$$W = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$$

$$\frac{1}{2}mV^2$$

Example: Rotating rod

A uniform rod of length L and mass M is free to rotate on a frictionless pin passing through one end. The rod is released from rest in the horizontal position.

- (a) What is its angular speed when it reaches its lowest position?
- (b) Determine the linear speed of the centre of mass and the linear speed of the lowest point on the rod when it is in the vertical position.



Example: Rotating rod

(a) Using conservation of energy:

$$K_f + U_f = K_i + U_i$$

$$\frac{1}{2}I\omega^2 + 0 = 0 + Mg\left(\frac{L}{2}\right)$$

$$\frac{1}{2}\left(\frac{1}{3}ML^2\right)\omega^2 = \frac{1}{2}MgL$$

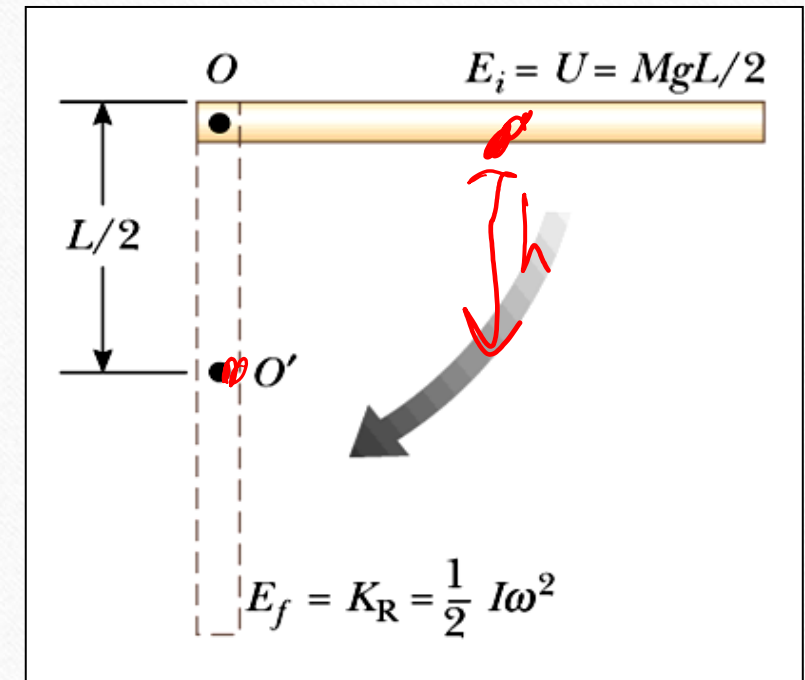


$$\omega = \sqrt{\frac{3g}{L}}$$

(b) $v_{CM} = \frac{L}{2}\omega = \frac{1}{2}\sqrt{3gL}$

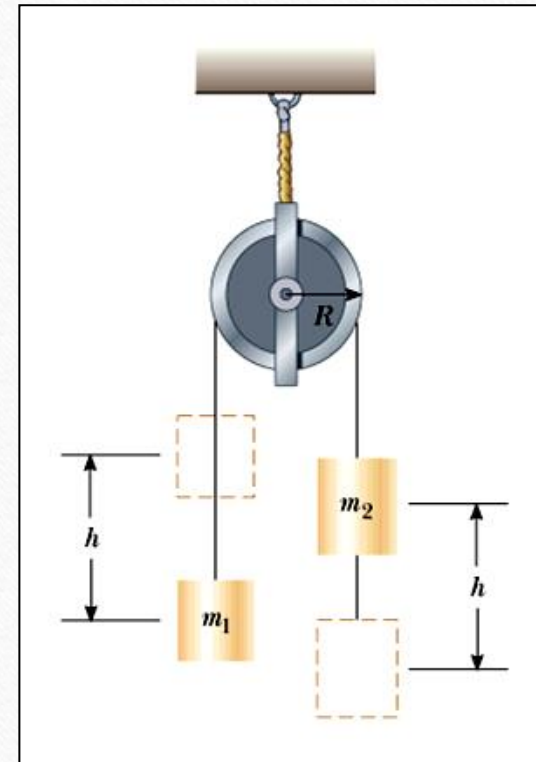
$$v_{tip} = L\omega = \sqrt{3gL}$$

$$v = r\omega$$



Example: Rotating rod

Consider two cylinders having masses m_1 and m_2 , where $m_1 \neq m_2$, connected by a string passing over a pulley. The pulley has a radius R and moment of inertia I about its axis of rotation. The string does not slip on the pulley, and the system is released from rest. Find the linear speeds of the cylinders after cylinder 2 descends through a distance h , and the angular speed of the pulley at this time.



Example: Rotating rod

$$K_f + U_f = K_i + U_i$$

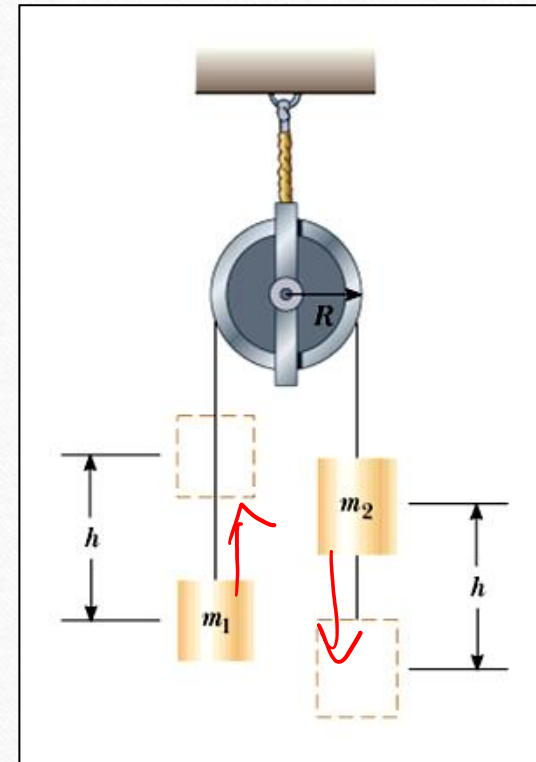
$$\frac{1}{2}m_1v_f^2 + \frac{1}{2}m_2v_f^2 + \frac{1}{2}I\omega_f^2 + m_1gh - m_2gh = 0 + 0$$

$$v_f = R\omega_f$$

$$\frac{1}{2}m_1v_f^2 + \frac{1}{2}m_2v_f^2 + \frac{1}{2}I\frac{v_f^2}{R^2} = (m_2 - m_1)gh$$

$$v_f = \sqrt{\frac{2(m_2 - m_1)gh}{m_1 + m_2 + (I/R^2)}}$$

$$\omega_f = \frac{1}{R} \sqrt{\frac{2(m_2 - m_1)gh}{m_1 + m_2 + (I/R^2)}}$$



The End