# Week 13: Review

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# Linear and Affine Transformations

- The map $T : \mathbb{R}^n \to \mathbb{R}^m$ is a
  1. linear transformation if
     $$T(\vec{x}) = A\vec{x} \text{ for some matrix } A_{m\times n}$$
     from $R^m$ from $R^n$
  2. affine transformation if
     $$T(\vec{x}) = A\vec{x} + \vec{b} \text{ for some matrix } A_{m\times n} \text{ and } \vec{b} \in \mathbb{R}^m$$

$\vec{b} = \vec{0} \Rightarrow T(\vec{x}) = A\vec{x} \rightarrow$ linear transformation.

# Linear and Affine Transformations

- The map $T : \mathbb{R}^n \to \mathbb{R}^m$ is a
  1. linear transformation if
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  2. affine transformation if
  $$T(\vec{x}) = A\vec{x} + \vec{b} \text{ for some matrix } A_{m\times n} \text{ and } \vec{b} \in \mathbb{R}^m$$
- Any map projection, reflection, scaling, rotation, shear
  - is an affine transformation
  - becomes a linear transformation if it involves the origin

# Determine matrix $A$ and vector $\vec{b}$

- Linear transformation: $T(\vec{x}) = A\vec{x}$

$T: \mathbb{R}^n \to \mathbb{R}^m$

$$T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix} \Rightarrow A = \begin{array}{c} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m \times n} \\ \begin{matrix} \underbrace{\hphantom{a_{11}}}_{x_1} & \hphantom{\cdots} & \underbrace{\hphantom{a_{1n}}}_{x_n} \end{matrix} \end{array}$$

# Determine matrix $A$ and vector $\vec{b}$

- Linear transformation: $T(\vec{x}) = A\vec{x}$ $T: \mathbb{R}^n \to \mathbb{R}^m$ ~~for~~ any $m, n$

$$T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix} \Rightarrow A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

- Affine Transformation: $T(\vec{x}) = A\vec{x} + \vec{b}$. If

$$T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n + b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n + b_m \end{pmatrix},$$

then

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

# 3-step algorithm

The 3-step algorithm is used to describe all 2D maps and 3D maps as affine transformations $A\vec{x} - A\vec{x}_0 + \vec{x}_0$

$$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0 \Leftrightarrow T(\vec{x}) = A\vec{x} + \vec{b} \text{ with } \vec{b} = \vec{x}_0 - A\vec{x}_0$$

1st step: translate to O

2nd step: linear trans. about O

3rd step: shift back to correct position

# Projections and reflections in $\mathbb{R}^2$

- If $l : \vec{x} = t\vec{d}$ ($l$ contains O) is the line of projection (or reflection), we have a linear transformation
  1. Projection: $T(\vec{x}) = A\vec{x}$ with $A = \frac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T$
  2. Reflection: $T(\vec{x}) = A\vec{x}$ with $A = \frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_2$

# Projections and reflections in $\mathbb{R}^2$

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  1. Projection: $T(\vec{x}) = A\vec{x}$ with $A = \frac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T$
  2. Reflection: $T(\vec{x}) = A\vec{x}$ with $A = \frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_2$
- If $l : \vec{x} = \vec{x}_0 + t\vec{d}$ with $\vec{x}_0 \neq \vec{0}$, we have an affine transformation

$$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0 \Leftrightarrow T(\vec{x}) = A\vec{x} + \vec{b} \text{ with } \vec{b} = \vec{x}_0 - A\vec{x}_0$$

, where

$\vec{x}_0$ = a point on $l$

$A$ = matrix of projection/reflection wrt to $l' : \vec{x} = t\vec{d}$

# Projections and reflections in $\mathbb{R}^2$

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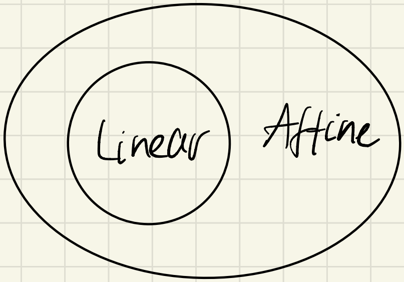
- If $l : \vec{x} = \vec{x}_0 + t\vec{d}$ with $\vec{x}_0 \neq \vec{0}$, we have an affine transformation

$$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0 \Leftrightarrow T(\vec{x}) = A\vec{x} + \vec{b} \text{ with } \vec{b} = \vec{x}_0 - A\vec{x}_0$$

- What are fixed points of projection? Fixed points of reflection?

In both cases, fixed points = line $l$.

linear transformations is a subset of affine trans.



# Scalings in $\mathbb{R}^2$

- The scaling $S : \mathbb{R}^2 \to \mathbb{R}^2$ centered at the origin is

$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

- The scaling $S : \mathbb{R}^2 \to \mathbb{R}^2$ centered at the point $\vec{x}_0$ is

$$S = A(\vec{x} - \vec{x}_0) + \vec{x}_0 = A\vec{x} + \vec{b} \text{ with } \vec{b} = \vec{x}_0 - A\vec{x}_0$$

*Remark*: Scaling centered at $\vec{x}_0 \neq \overrightarrow{0}$ is **not tested**.

# Scalings in $\mathbb{R}^2$

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*Remark*: Scaling centered at $\vec{x}_0 \neq \overrightarrow{0}$ is **not tested**.

- What are fixed points of the scaling $S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix}$?

$a = 1 \Rightarrow$ fixed points = $x$-axis.

$b = 1 \Rightarrow$ fixed points = $y$-axis

$a = b = 1 \Rightarrow$ fixed points = $\mathbb{R}^2$

# Rotations in $\mathbb{R}^2$

- The rotation (counter-clockwise) about the origin O over angle $\theta$ is

$$T(\vec{x}) = A\vec{x} \text{ with } A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- The rotation about the point $\vec{x}_0$ over the angle $\theta$ is

$$T(\vec{x}) = A\vec{x} + \vec{b} \text{ with } A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \vec{b} = \vec{x}_0 - A\vec{x}_0$$

- What are fixed points of the rotation about $\vec{x}_0$ over $\theta \in (0^\circ, 360^\circ)$?

Only $\vec{x}_0$.

# Shears in $\mathbb{R}^2$

- The **shear** wrt. $l : \vec{n} \cdot \vec{x} = 0$ in the direction of **shearing vector** $\vec{v}$ is

$$T(\vec{x}) = \vec{x} + \frac{\vec{n} \cdot \vec{x}}{||\vec{n}||} \vec{v}$$

  As a linear transformation, we have

$$T(\vec{x}) = A\vec{x} \text{ with } A = I_2 + \frac{1}{||\vec{n}||} \vec{v}\vec{n}^T$$

- The **shear** w.r.t. $l : \vec{n} \cdot \vec{x} = c$ in the direction of **shearing vector** $\vec{v}$ is

$$T(\vec{x}) = A\vec{x} + \vec{b} \text{ with } \vec{b} = \vec{x}_0 - A\vec{x}_0, \ \vec{x}_0 = \text{ a point on } l$$

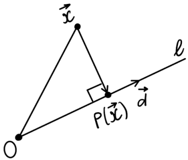
- What are fixed points of a shear?

The line $l$

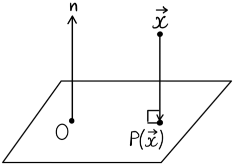
# Projections in $\mathbb{R}^3$

- If the line/plane of projection contains O, it's a linear transformation

$$T(\vec{x}) = A\vec{x} \text{ with}$$




$$A = \frac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T$$




$$A = I_3 - \frac{1}{||\vec{n}||^2}\vec{n}\vec{n}^T$$

# Projections in $\mathbb{R}^3$

- If the line/plane of projection contains O, it's a linear transformation

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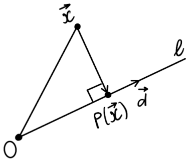
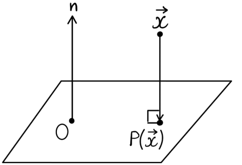



$$A = \frac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T \qquad A = I_3 - \frac{1}{||\vec{n}||^2}\vec{n}\vec{n}^T$$

- If the line/plane doesn't contain O, it's an affine transformation

$$T(\vec{x}) = A\vec{x} + \vec{b} \text{ with } \vec{b} = \vec{x}_0 - A\vec{x}_0$$

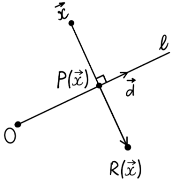
and $\vec{x}_0$ = a point on the line/plane of projection.

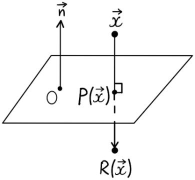
- What are fixed points?

fixed points = the line (or plane)

# Reflections in $\mathbb{R}^3$

- If the line/plane of projection contains O, we have $T(\vec{x}) = A\vec{x}$ with

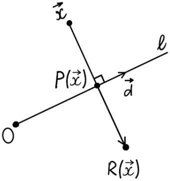


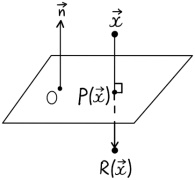




$$M = \frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_3 \qquad M = I_3 - \frac{2}{||\vec{n}||^2}\vec{n}\vec{n}^T$$

# Reflections in $\mathbb{R}^3$

- If the line/plane of projection contains O, we have $T(\vec{x}) = A\vec{x}$ with




$$A = \frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_3 \qquad A = I_3 - \frac{2}{||\vec{n}||^2}\vec{n}\vec{n}^T$$

- If the line/plane doesn't contain O, it's an affine transformation

$$T(\vec{x}) = A\vec{x} + \vec{b} \text{ with } \vec{b} = \vec{x}_0 - A\vec{x}_0$$

and $\vec{x}_0$ = a point on the line/plane of projection.

- What are fixed points?

The line (or the plane)

# Scalings in $\mathbb{R}^3$

- The scaling $S : \mathbb{R}^3 \to \mathbb{R}^3$ centered at the origin is

$$S\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax \\ by \\ cz \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

# Scalings in $\mathbb{R}^3$

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- The scaling $S : \mathbb{R}^3 \to \mathbb{R}^3$ centered at the point $\vec{x}_0$ is

$$S = A(\vec{x} - \vec{x}_0) + \vec{x}_0 = A\vec{x} + \vec{b} \text{ with } \vec{b} = \vec{x}_0 - A\vec{x}_0$$

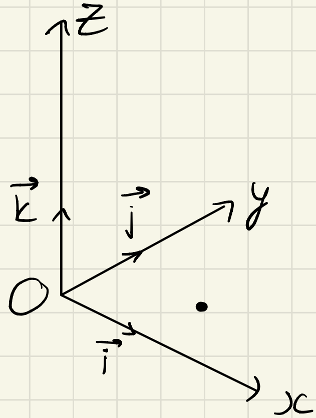
Remark: Scaling centered at $\vec{x}_0 \neq \overrightarrow{0}$ is **not tested**.

- What are fixed points of the scaling $S\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax \\ by \\ cz \end{pmatrix}$?

$a = 1 \Rightarrow$ fixed points $= x$-axis

$b = 1 \Rightarrow$ fixed points $= y$-axis

$c = 1 \Rightarrow$ fixed points $= z$-axis xy plane : $z = 0$ , $\vec{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$




yz plane : $x = 0$ , $\vec{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

xz plane : $y = 0$ , $\vec{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

x-axis : $\vec{x} = \vec{x}_0 + t\vec{d} = t\,\vec{i}$

y-axis : $\vec{x} = t\,\vec{j}$

z-axis : $\vec{x} = t\,\vec{k}$

# Shears in $\mathbb{R}^3$

- The shear w.r.t. $\alpha : \vec{n} \cdot \vec{x} = 0$ in the direction of shearing vector $\vec{v}$ is

$$T(\vec{x}) = \vec{x} + \frac{\vec{n} \cdot \vec{x}}{||\vec{n}||}\vec{v} \Leftrightarrow T(\vec{x}) = A\vec{x} \text{ with } A = I_3 + \frac{1}{||\vec{n}||}\vec{v}\vec{n}^T$$

# Shears in $\mathbb{R}^3$

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$$T(\vec{x}) = \vec{x} + \frac{\vec{n} \cdot \vec{x}}{||\vec{n}||}\vec{v} \Leftrightarrow T(\vec{x}) = A\vec{x} \text{ with } A = I_3 + \frac{1}{||\vec{n}||}\vec{v}\vec{n}^T$$

- If the plane $\alpha : \vec{n} \cdot \vec{x} = c$, then

$$T(\vec{x}) = A\vec{x} + \vec{b} \text{ with } \vec{b} = \vec{x}_0 - A\vec{x}_0$$

  and $\vec{x}_0 =$ a point on $\alpha$.

- What are fixed points of the shear?

All points on $\alpha$.

# Rotations in $\mathbb{R}^3$

- The rotation centered at O about the vector $\vec{v}$ over angle $\theta$ is $T(\vec{x}) = A\vec{x}$ with

$$A = (1-\cos\theta)\frac{\vec{v}\vec{v}^T}{||\vec{v}||^2} + (\cos\theta)I_3 + \frac{\sin\theta}{||\vec{v}||}C_{\vec{v}}$$

# Rotations in $\mathbb{R}^3$

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$$A = (1 - \cos\theta)\frac{\vec{v}\vec{v}^T}{||\vec{v}||^2} + (\cos\theta)I_3 + \frac{\sin\theta}{||\vec{v}||}C_{\vec{v}}$$

- The rotation centered at $\vec{x}_0$ about the vector $\vec{v}$ over angle $\theta$ is $T(\vec{x}) = A\vec{x} + \vec{b}$ with $\vec{b} = \vec{x}_0 - A\vec{x}_0$
- *Remark*: The rotation centered at $\vec{x}_0 \neq \overrightarrow{0}$ will **not be tested**.

  The phrase "rotation about $\vec{v}$ over angle $\theta$" means the rotation is centered at the origin O.

# Rotations in $\mathbb{R}^3$

- The rotation centered at O about the vector $\vec{v}$ over angle $\theta$ is $T(\vec{x}) = A\vec{x}$ with

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- The rotation centered at $\vec{x}_0$ about the vector $\vec{v}$ over angle $\theta$ is

$$T(\vec{x}) = A\vec{x} + \vec{b} \text{ with } \vec{b} = \vec{x}_0 - A\vec{x}_0$$

- *Remark*: The rotation centered at $\vec{x}_0 \neq \overrightarrow{0}$ will **not be tested**.

  The phrase "rotation about $\vec{v}$ over angle $\theta$" means the rotation is centered at the origin O.

- What are fixed points of the rotation about $\vec{v}$ over angle $\theta$?

Fixed points = line ( contain O, direction $\vec{v}$ : $\vec{x} = t\vec{v}$

# Rotations about the axes

The rotations about the positive $x, y, z$-axes are linear transformations with matrices

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

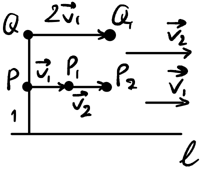
# Problem 1

The shear S wrt $l : 3x - 4y = 0$ maps $P = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ to $P' = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$.

(a) What is the matrix of $S$? (*Hint.* $S(\vec{x}) = \vec{x} + \frac{\vec{n} \cdot \vec{x}}{||\vec{n}||}\vec{v}$ to find $\vec{v}$)

$P' = S(P) = P + \frac{\vec{n} \cdot P}{\|\vec{n}\|}\vec{v}$ with $\vec{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, $P = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $P' = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$

$\begin{pmatrix} 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \frac{5}{5}\vec{v} \Rightarrow \vec{v} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$




The matrix of S is

$$A = I_2 + \frac{1}{\|\vec{n}\|}\vec{v}\,\vec{n}^T = I_2 + \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix}\begin{pmatrix} 3 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 12 & -16 \\ 9 & -12 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 17 & -16 \\ 9 & -7 \end{pmatrix}$$

Note that $S(\vec{x}) = A\vec{x}$

(b) Find the normal equation for the image $m'$ of $m : 2x - 3y = 6$.

m has vector equation $\vec{x} = \vec{x}_0 + t\vec{d} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Its image is

$$m' : \vec{x} = \frac{1}{5}\begin{pmatrix} 17 & -16 \\ 9 & -7 \end{pmatrix}\left(\begin{pmatrix} 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 51/5 \\ 27/5 \end{pmatrix} + \frac{t}{5}\begin{pmatrix} 19 \\ 13 \end{pmatrix}$$

$\Rightarrow$ m' contains $\begin{pmatrix} 51/5 \\ 27/5 \end{pmatrix}$ and has normal $\vec{n}' = \begin{pmatrix} 13 \\ -19 \end{pmatrix}$

$$13\left(x - \frac{51}{5}\right) - 19\left(y - \frac{27}{5}\right) = 0$$

$$13x - 19y = 30$$

(c) Let $Q$ be the intersection of $m'$ and $l$. Find the image $Q'$ of $Q$.

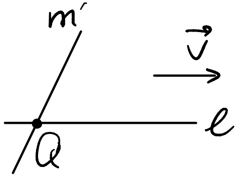
Q has coordinates $x, y$ with

$$\begin{cases} 13x - 19y = 30 \Rightarrow 13x - 19\cdot\frac{3}{4}x = 30 \Rightarrow x = -24 \\ 3x - 4y = 0 \Rightarrow y = \frac{3}{4}x \Rightarrow y = -18 \end{cases}$$

So $Q = \begin{pmatrix} -24 \\ -18 \end{pmatrix}$.

Since $Q \in l$, it is fixed by $S$:

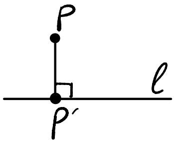
$$Q' = Q = \begin{pmatrix} -24 \\ -18 \end{pmatrix}$$



# Problem 2

Given the point $P = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and the line $l : \begin{cases} x = 2 + t \\ y = 3 - t \\ z = 1 + 2t \end{cases}$ contains $\vec{x}_0 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, direction $\vec{d} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

(a) Using affine trans., find the point $P'$ on $l$ that is closest to $P$.




Let $T$ = projection onto $l$. We have

$$T(\vec{x}) = A\vec{x} + \vec{b} \text{ with}$$

$$A = \frac{1}{\|\vec{d}\|^2} \vec{d}\,\vec{d}^T = \frac{1}{6}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix}$$

$$\vec{b} = \vec{x}_0 - A\vec{x}_0 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 11 \\ 19 \\ 4 \end{pmatrix}$$

Thus $T(\vec{x}) = \frac{1}{6}\begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix}\vec{x} + \frac{1}{6}\begin{pmatrix} 11 \\ 19 \\ 4 \end{pmatrix} \Rightarrow P' = \frac{1}{6}\begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 11 \\ 19 \\ 4 \end{pmatrix}$

$$= \begin{pmatrix} 13/6 \\ 17/6 \\ 8/6 \end{pmatrix}$$

(b) $\alpha =$ plane through $P$ and perpendicular to $l$. Using affine transformation, find the point $Q'$ on $\alpha$ that is closest to $Q = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

$\alpha$ through $P = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, normal $\vec{n} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$$\left( \begin{array}{c} 1(x-1) - 1(y-1) + 2(z-1) = 0 \\ \alpha: \; x - y + 2z = 2 \end{array} \right)$$

$T(\vec{x}) = A\vec{x} + \vec{b}$ with $A = I_3 - \frac{1}{\|\vec{n}\|^2} \vec{n}\vec{n}^T$

$$A = I_3 - \frac{1}{6}\begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 5 & 1 & -2 \\ 1 & 5 & 2 \\ -2 & 2 & 2 \end{pmatrix}$$

$$\vec{b} = \vec{x}_0 - A\vec{x}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 5 & 1 & -2 \\ 1 & 5 & 2 \\ -2 & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

We obtain

$$Q' = \frac{1}{6}\begin{pmatrix} 5 & 1 & -2 \\ 1 & 5 & 2 \\ -2 & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 5/2 \\ 2 \end{pmatrix}$$

# Problem 3

$T =$ the shear with respect to the plane $\alpha : z = 3$ in the direction of shearing vector $\vec{v} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$.

(a) Write $T$ in the form of an affine map $T(\vec{x}) = A\vec{x} + \vec{b}$.

(b) Find the image of $l : \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ under $T$?

(c) Find the image of the plane $\beta : x - z = 0$ under $T$.

# Problem 4

$S$ and $T$ = reflections through $\alpha : 2x - y + 2z = 0$ and $\beta : x - y = 0$.

(a) Find the matrix $M$ of $S \circ T$ (*Hint*: $M = M_S M_T$).

$$\vec{n}_\alpha = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \quad \vec{n}_\beta = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$M_S = I_3 - \frac{2}{\|\vec{n}_\alpha\|^2} \vec{n}_\alpha \vec{n}_\alpha^T = \frac{1}{9} \begin{pmatrix} 1 & 4 & -8 \\ 4 & 7 & 4 \\ -8 & 4 & 1 \end{pmatrix}$$

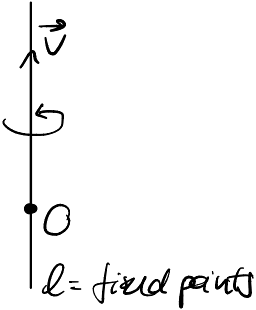
$$M_T = I_3 - \frac{2}{\|\vec{n}_\beta\|^2} \vec{n}_\beta \vec{n}_\beta^T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M = M_S M_T = \frac{1}{9} \begin{pmatrix} 1 & 4 & -8 \\ 4 & 7 & 4 \\ -8 & 4 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 4 & 1 & -8 \\ 7 & 4 & 4 \\ 4 & -8 & 1 \end{pmatrix}$$

(b) Find the fixed points of $S \circ T$.

Assume $S \circ T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\frac{1}{9}\begin{pmatrix} 4 & 1 & -8 \\ 7 & 4 & 4 \\ 4 & -8 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$




$$\begin{pmatrix} 4x+y-8z \\ 7x+4y+4z \\ 4x-8y+z \end{pmatrix} = \begin{pmatrix} 9x \\ 9y \\ 9z \end{pmatrix} \iff \begin{cases} -5x+y-8z=0 & (1) \\ 7x-5y+4z=0 & (2) \\ x-2y-2z=0 & (3) \end{cases}$$

$(1) \Rightarrow y = 5x+8z$

$(3) \Rightarrow x = 2y+2z = 2z+2(5x+8z) = 10x+18z$

(b) Find the fixed points of $S \circ T$.

$x = 10x + 18z \Rightarrow -9x = 18z \Rightarrow x = -2z$

Hence $y = 5x + 8z = -10z + 8z = -2z$

Substitute into (2):

$7 \cdot (-2z) - 5 \cdot (-2z) + 4z = 0 \Leftrightarrow 0 = 0$.

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2z \\ -2z \\ z \end{pmatrix} = z \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \overset{z=t}{=\!=} t \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$

Fixed points is a line $\ell$ $\begin{cases} \text{through } O \\ \text{direction } \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \end{cases}$

(c) In b, your answer is a line $l$. Let $\vec{v}=$ direction of $l$. Find the angle $\theta$ so that $M$ is a rotation matrix, that is,

$$M = (1-\cos\theta)\frac{\vec{v}\vec{v}^T}{||\vec{v}||^2} + (\cos\theta)I_3 + \frac{\sin\theta}{||\vec{v}||}C_{\vec{v}}$$

Take $\vec{v} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$. Then $C_{\vec{v}} = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$ and $\vec{v}\vec{v}^T = \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$

$$M = \frac{1-\cos\theta}{9}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix} + \cos\theta\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\sin\theta}{3}\begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 4+5\cos\theta & 4-4\cos\theta-3\sin\theta & -2+2\cos\theta-6\sin\theta \\ 4-4\cos\theta+3\sin\theta & 4+5\cos\theta & -2+2\cos\theta+6\sin\theta \\ -2+2\cos\theta+6\sin\theta & -2+2\cos\theta-6\sin\theta & 1+8\cos\theta \end{pmatrix}$$

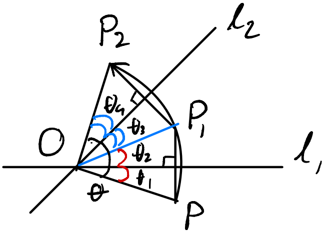
Comparing 2 matrices, we have

$$\frac{1}{9}\begin{pmatrix} 4+5\cos\theta & 4-4\cos\theta-3\sin\theta & -2+2\cos\theta-6\sin\theta \\ 4-4\cos\theta+3\sin\theta & 4+5\cos\theta & -2+2\cos\theta+6\sin\theta \\ -2+2\cos\theta+6\sin\theta & -2+2\cos\theta-6\sin\theta & 1+8\cos\theta \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 4 & 1 & -8 \\ 7 & 4 & 4 \\ 4 & -8 & 1 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} \cos\theta = 0 \\ \sin\theta = 1 \end{cases} \Leftrightarrow \theta = 90^\circ$$

$\therefore$ So T is a rotation about $\vec{v} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$ over $\theta = 90^\circ$.

Reflections in 2D

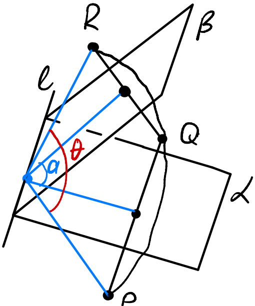



$P_1$ = reflection of $P$ through $\ell_1$

$P_2$ = ——— $P_1$ ——— $\ell_2$

$P_2$ = rotation of $P$ over $\theta = 2\angle(\ell_1, \ell_2)$

Reflection in 3D




$\ell = \alpha \wedge \beta$

$R$ = rotation of $P$ about $\ell$ over

$\theta = 2\angle(\alpha, \beta)$

# Reminders on final exam

- Date and Time: Wednesday, November 30, 10am-12pm
- Scope: Weeks 8-12 materials
- The following are **not tested**
  1. All skew maps which include skew projection and reflection
  2. Scalings (both 2D and 3D) centered at a point $\vec{x}_0 \neq O$
  3. 3D rotation centered at a point $\vec{x}_0 \neq O$

  only linear maps
- Allowed to bring in: 1 A4-size cheat sheet + 1 calculator


Thank you