TUTORIAL 2 - A

1, Which functions are 1-1?

a) $f : \mathbb{R} - \{0\} \to \mathbb{R}$ with $f(x) = \dfrac{3x-1}{x}$

b) $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(x) = \sqrt{x}$

c) $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $h(x, y) = x$

To prove f is 1-1: show $f(x) = f(y) \Leftrightarrow x = y$

To prove f is not 1-1: Give an example of $x \neq y$ and $f(x) = f(y)$

a) $f(x) = \dfrac{3x-1}{x} = \dfrac{3x}{x} - \dfrac{1}{x} = 3 - \dfrac{1}{x}$

$f(x) = f(y) \Leftrightarrow 3 - \dfrac{1}{x} = 3 - \dfrac{1}{y} \Leftrightarrow \dfrac{1}{x} = \dfrac{1}{y} \Leftrightarrow x = y$

$\therefore$ f is 1-1.

b) $g(x) = g(y) \Leftrightarrow \sqrt{x} = \sqrt{y} \Leftrightarrow x = y$

$\therefore$ g is 1-1.

c) $h(x_1, y_1) = h(x_2, y_2) \Leftrightarrow x_1 = x_2$

Consider $(1, 0)$ & $(1, 1)$: $h(1, 0) = h(1, 1) = 1$

$\therefore$ h is not 1-1.

2) Give examples of $f: N \to N$ that is

a) 1-1 but not onto

b) Onto but not 1-1

c) Both 1-1 and onto (not identity function)

$f(x) = x$

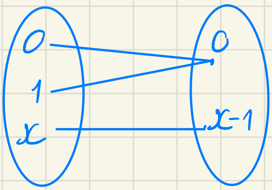
d) Neither 1-1 nor onto

$N = \{0, 1, 2, \dots\}$

a) There are many, for examples

$$f(x) = 2x \text{ or } f(x) = x^2.$$

b) $f: N \to N$ with



$f(0) = 0,$

$f(x) = x - 1$ for $x = 1, 2, \dots$

f is not 1-1 because $f(0) = f(1) = 0$

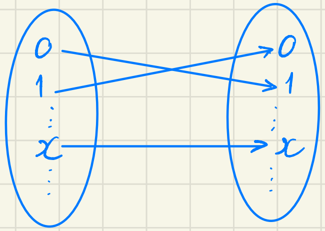
f is onto: For any $y \in N$, we have $f(y+1) = (y+1) - 1 = y$.

d) $f: N \to N$ with $f(x) = 1$.

Not 1-1: $f(0) = f(1) = 1$

Not onto: There is no $x \in N$: $f(x) = 2$.

c) $f: N \to N$ with



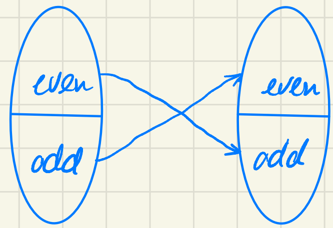
$f(0) = 1$, $f(1) = 0$,

$f(x) = x$ for $x = 2, 3, \dots$

2c) <u>Another example</u>: $f: N \to N$

$f(2m) = 2m+1$ for $m = 0, 1, \dots$

$f(2m+1) = 2m$ for $m = 0, 1, \dots$



3, Which function f is 1-1, onto? Find f^{-1} if it exists.

a) $f: R \to R$ such that $f(x) = x^3 + 1$

b) $f: R \to R_{\geq 0}$ such that $f(x) = |x|$

c) $f: R \times R \to R$ such that $f(x, y) = x$

$x^{2m} = y^{2m} \Leftrightarrow \left[\begin{array}{l} x = y \\ x = -y \end{array}\right.$

$x^{2m+1} = y^{2m+1} \Leftrightarrow x = y$

a) f is 1-1: $f(x) = f(y) \Leftrightarrow x^3 + 1 = y^3 + 1 \Leftrightarrow x = y$

f is onto: For any $y \in R$: $f(x) = y \Rightarrow x^3 + 1 = y \Rightarrow x = \sqrt[3]{y-1}$

$f(\sqrt[3]{y-1}) = y$

Next, we find $f^{-1}: R \to R$. Let $x \in R$ and put $y = f^{-1}(x)$:

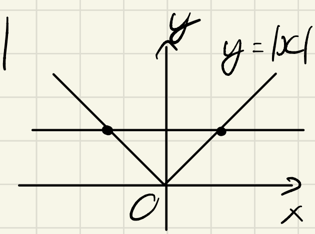
$f(y) = x \Leftrightarrow y^3 + 1 = x \Leftrightarrow y = \sqrt[3]{x-1}$

$\therefore f^{-1}(x) = \sqrt[3]{x-1}$.

b) $f: R \to R_{\geq 0}$ defined by $f(x) = |x|$

f is not 1-1: $f(1) = f(-1) = 1$

f is onto: For $y \in R_{\geq 0}$, $f(y) = |y| = y$.



Since f is not 1-1, f is not invertible.

c) $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = x$

f is not 1-1: $f(1, 0) = f(1, 1) = 1$.

f is onto: let $y \in \mathbb{R}$, we have $f(y, 0) = y$.

4) $f, g : \mathbb{R} \to \mathbb{R}$ with $f(x) = x + 3$, $g(x) = -x^3$

a) Find $f \circ g$, $g \circ f$, f^{-1}, g^{-1}

b) Show that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

a) $f \circ g : \mathbb{R} \to \mathbb{R}$ and $g \circ f : \mathbb{R} \to \mathbb{R}$ with

$$f \circ g(x) = f(g(x)) = f(-x^3) = -x^3 + 3$$

$$g \circ f(x) = g(f(x)) = g(x+3) = -(x+3)^3$$

$f^{-1} : \mathbb{R} \to \mathbb{R}$. let $x \in \mathbb{R}$ and put $y = f^{-1}(x)$:

$$f(y) = x \Leftrightarrow y + 3 = x \Leftrightarrow y = x - 3$$

$\therefore$ $f^{-1}(x) = x - 3$.

$g^{-1} : \mathbb{R} \to \mathbb{R}$. let $x \in \mathbb{R}$ and put $y = g^{-1}(x)$:

$$g(y) = x \Leftrightarrow -y^3 = x \Leftrightarrow y^3 = -x \Leftrightarrow y = \sqrt[3]{-x}$$

$\therefore$ $g^{-1}(x) = \sqrt[3]{-x}$ (same as $g(x) = -\sqrt[3]{x}$)

b) Show that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

$(f \circ g)^{-1} : \mathbb{R} \to \mathbb{R}$. let $x \in \mathbb{R}$ and $y = (f \circ g)^{-1}(x)$:

$$f \circ g(y) = x \Leftrightarrow -y^3 + 3 = x \Leftrightarrow y = \sqrt[3]{3 - x}$$

Hence $(f \circ g)^{-1}(x) = \sqrt[3]{3 - x}$

$$g^{-1} \circ f^{-1}(x) = g^{-1}(f^{-1}(x)) = g^{-1}(x - 3) = \sqrt[3]{-(x-3)}$$ $\Big\}$ $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

5) Find intervals on which $f: \mathbb{R} \to \mathbb{R}$ is increasing / decreasing.

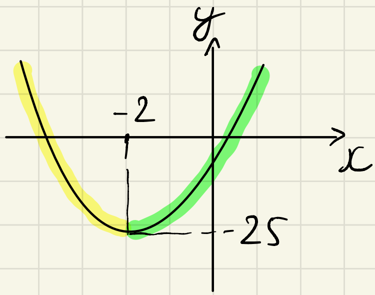
$\quad\quad\quad\quad\quad\quad\quad\quad f'(x) \geq 0 \quad\quad f'(x) \leq 0$

a) $f(x) = x^2 + 4x - 21$

b) $g(x) = 2x^3 + 3x^2 - 36x$

a) $f'(x) = 2x + 4 \; : \; f'(x) = 0 \iff x = -2$

x	$-\infty$		-2		∞
$f'(x)$		$-$	0	$+$	
$f(x)$		$\searrow$	-25	$\nearrow$	



$\therefore$ f is decreasing on $(-\infty, -2]$.

f is increasing on $[-2, \infty)$.

b) $g(x) = 2x^3 + 3x^2 - 36x \Rightarrow g'(x) = 6x^2 + 6x - 36 = 6(x+3)(x-2)$

$g'(x) = 0 \iff x = -3$ or $x = 2$

x	$-\infty$		-3		2		∞
$g'(x)$		$+$	0	$-$	0	$+$	

On $(-\infty, -3]$

$g'(x) = 6\underbrace{(x+3)}_{-}\underbrace{(x-2)}_{-} \geq 0$

On $[-3, 2]$: $g'(x) = 6\underbrace{(x+3)}_{+}\underbrace{(x-2)}_{-} \leq 0$

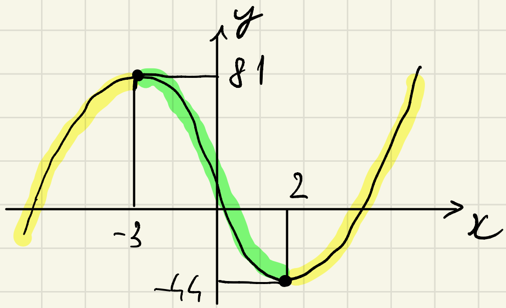
On $[2, \infty)$: $g'(x) = 6\underbrace{(x+3)}_{+}\underbrace{(x-2)}_{+} \geq 0$

$\therefore$ f is increasing on $(-\infty, -3]$ & $[2, \infty)$

f is decreasing on $[-3, 2]$

$$g(x) = 2x^3 + 3x^2 - 36x$$

x	$-\infty$		-3		2		∞
$g'(x)$		$+$	0	$-$	0	$+$	
$g(x)$		$\nearrow$	81	$\searrow$	-44	$\nearrow$	



6) $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation with matrix A if

$$T(x) = Ax \text{ for any } x \in \mathbb{R}^n$$

a) Assume A is invertible. Find T^{-1}.

b) T = rotation about O over $30°$. Find preimages of $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$.

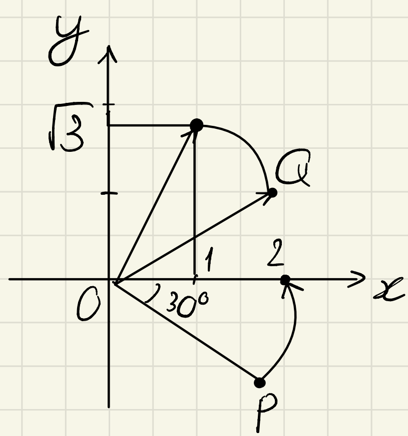
a) $T^{-1}: \mathbb{R}^n \to \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and $y = T^{-1}(x)$:

$$T(y) = x \Leftrightarrow Ay = x \Leftrightarrow y = A^{-1}x$$

$\therefore T^{-1}(x) = A^{-1}x \Rightarrow T^{-1}$ = linear trans. with matrix A^{-1}.

b) Find $P \in \mathbb{R}^2$: $T(P) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$$P = T^{-1}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = A^{-1}\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$



A = matrix of rotation by $30°$

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \underset{=\!=}{\overset{\theta = 30°}{=}} \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

Since A is a rotation matrix, $A^{-1} = A^T$

$$A^{-1} = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}$$

$$P = A^{-1}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}$$

similarly

$$Q = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$$

Since A is a rotation matrix, $A^{-1} = A^T$

$$A^{-1} = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}$$

$$P = A^{-1}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}$$

similarly

$$Q = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$$