Quiz 1 Results out in Week 6

Week 5 HW out this Saturday, deadline 15 Oct

go through in Week 6

# Method of Partial Fractions Part 2
# Numerical Integration Part 1


Dr. Ronald Koh

ronald.koh@digipen.edu (Teams preferred over email)


AY 23/24 Trimester 1

Quiz 2 : $\geq 2$ open-ended questions, less MCQ

1 hr

Week 1-6

# Table of contents

# Tangent/secant integrals, Partial fractions (1) and (2)

$$\frac{x}{x^2(x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

$n$ even $\rightarrow \sec^2 x$

- We learnt how to integrate $\tan^m x \sec^n x$:
  - $n$ is even: take out a copy of $\sec^2 x$, convert rest of $\sec^2 x$ to $\tan^2 x + 1$, sub $u = \tan x$.
  - $m$ is odd and $n \geq 1$: take out one copy of $\sec x \tan x$, convert rest of $\tan^2 x$ to $\sec^2 x - 1$, sub $u = \sec x$.
  - ✱ $m$ is even and $n$ is odd: convert all $\tan^2 x$ to $\sec^2 x - 1$, reduce the power of sec by 2 every integration by parts iteration. $\rightarrow du = \sec^2 x$
- Partial fraction decomposition: only works on **proper** fractions; use long division to convert to proper otherwise. $\frac{P(x)}{Q(x)}$ $\deg P < \deg Q$
- Factorization of denominator $Q(x)$:
  - Non-repeated linear factors: one partial fraction for each factor.
  - Repeated (power $m$) linear factors: for each factor that is repeated, one partial fraction for each power $k = 1, \ldots, m$.

p.f.d entirely depends on factorization of $Q(x)$

# Exercise 3 from last week's lecture

proper rational function

Evaluate the following integrals.

1. $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

$$2x^3+3x^2-2x = x(2x^2+3x-2)$$

| | | | |
|---|---|---|---|
| $2x$ | | $-1$ | $-x$ |
| $x$ | | $2$ | $+4x$ |
| $2x^2$ | | $-2$ | $+3x$ |

$$x(2x-1)(x+2)$$

Case 1: Non repeated linear factors

$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

$0 \qquad \frac{1}{2} \qquad -2$

$\times\, Q(x)$

$$x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

$$x=\tfrac{1}{2} \Rightarrow \tfrac{1}{4}+1-1 = B\cdot\tfrac{1}{2}\left(\tfrac{1}{2}+2\right) \Rightarrow \tfrac{1}{4} = \tfrac{5}{2}\cdot\tfrac{1}{2}B$$

$$B = \tfrac{1}{5}$$

$$x=-2 \Rightarrow 4-4-1 = C(-2)(-4-1) \Rightarrow C = -\tfrac{1}{10}$$

$$x=0 \Rightarrow -1 = A(-1)(2) \Rightarrow A = \tfrac{1}{2}$$

# Exercise 3 from last week's lecture

$$\int \frac{1}{2x-1}\,dx$$

$$u = 2x-1$$

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{1}{2x} + \frac{1}{5(2x-1)} - \frac{1}{10(x+2)}$$

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \frac{1}{2}\int \frac{1}{x}\,dx + \frac{1}{5}\int \frac{1}{2x-1}\,dx - \frac{1}{10}\int \frac{1}{x+2}\,dx$$

$$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C$$

$$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln\left|\frac{2x-1}{x+2}\right| + C$$

# Exercise 4 from last week's lecture

$$x^3+x^2 = x^2(x+1)$$

Evaluate the following integrals.

1. $\displaystyle\int \frac{1}{x^3+x^2}\,dx$
2. $\displaystyle\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$

$$\frac{1}{x^3+x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

0    $-1$

$$1 = Ax(x+1) + B(x+1) + Cx^2$$

$x=-1$: $\quad 1 = C(-1)^2 \Rightarrow C = 1$

$x=0$: $\quad 1 = B$

$$1 = Ax(x+1) + (x+1) + x^2$$

1

Compare coefficients of $x^2$

$$0 = A+1 \Rightarrow A = -1$$

Compare coefficients of $x^2$ or $x$

See what $A$ is tagged to, both are possible

# Exercise 4 from last week's lecture

$A=-1,\ B=1,\ C=1$

$$\frac{1}{x^3+x^2} = -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$$

$$\therefore \int \frac{1}{x^3+x^2}\,dx = -\int \frac{1}{x}\,dx + \int \frac{1}{x^2}\,dx + \int \frac{1}{x+1}\,dx$$

$$= -\ln|x| - \frac{1}{x} + \ln|x+1| + C$$

Last week:

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}$$

partial fraction this

$$x^3-x^2-x+1 = x^2(x-1)-1(x-1) = (x^2-1)(x-1)$$

$$= (x+1)(x-1)(x-1) = (x+1)(x-1)^2$$

# Exercise 4 from last week's lecture

$$\int \frac{2}{(x-1)^2}\,dx = -\frac{2}{(x-1)}$$

$$\frac{4x}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$4x = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

sub $x=-1 \Rightarrow -4 = A(-2)^2 \Rightarrow A=-1$

$x=1 \Rightarrow 4 = C\cdot 2 \Rightarrow C=2$

$$4x = -(x-1)^2 + B(x+1)(x-1) + 2(x+1)$$

tagged to constant, $x$, $x^2$

Compare coefficients of $x^2$: $0 = -1+B \Rightarrow B=1$

$$\int x+1+\frac{4x}{x^3-x^2-x+1}\,dx = \frac{x^2}{2}+x+\int\frac{4x}{(x+1)(x-1)^2}\,dx$$

$$= \frac{x^2}{2}+x-\ln|x+1|+\ln|x-1|-\frac{2}{x-1}+C$$

# Method for Non-repeating Irreducible Quadratic Factors

✓ · non-repeated linear
✓ · repeated linear
✓ · non-repeated irreducible quadratic

discriminant

**Definition**

A quadratic polynomial $ax^2 + bx + c$ is said to be **irreducible** (in the reals) if $b^2 - 4ac < 0$.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

cannot be factored into linear factors

If $Q(x)$ factors into a non-repeated, irreducible factor $ax^2 + bx + c$, then the partial fraction decomposition of $\frac{R(x)}{Q(x)}$ must contain

$$\frac{Ax + B}{ax^2 + bx + c},$$

linear polynomial, no longer just constant

where $A$ and $B$ are constants to be determined.

Eg. $$\frac{x+1}{x(x+1)^2(x^2+1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{Dx+E}{x^2+1}$$

## Example 1

proper

$x^4+x^2 = \underbrace{x^2}_{\text{repeated linear}}\underbrace{(x^2+1)}_{\text{non-repeated irr quadratic}}$

$x^2+1$ irr

$a=1,\ b=0,\ c=1$

$b^2-4ac=-4<0$

Evaluate $\displaystyle\int \frac{x+1}{x^4+x^2}\,dx.$

$$\frac{x+1}{x^4+x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$(-1)^{\text{odd}} = -1$

$$x+1 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$$

$x=0 \Rightarrow 1 = B(0^2+1) \Rightarrow B=1$

$x+1 = Ax(x^2+1) + (x^2+1) + (Cx+D)x^2$

Find the power of $x$ that has lowest amount of terms to compare coefficients

Compare coefficients of $x^3$: $0 = A + C$

Compare coefficients of $x$: $1 = A$

$\Rightarrow C=-1$

Compare coefficients of $x^2$: $0 = 1 + D \Rightarrow D=-1$

# Example 1

$$\frac{x+1}{x^2(x^2+1)} = \frac{1}{x} + \frac{1}{x^2} - \frac{x+1}{x^2+1}$$

$$\int \frac{x+1}{x^2(x^2+1)}\,dx = \int \frac{1}{x}\,dx + \int \frac{1}{x^2}\,dx - \int \frac{x+1}{x^2+1}\,dx$$

$$= \ln|x| - \frac{1}{x} - \left(\frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx\right)$$

$$= \ln|x| - \frac{1}{x} - \frac{1}{2}\ln(x^2+1) - \tan^{-1}x + C$$

# Exercise 1

$dx = a\,du$

$u = \dfrac{x}{a}$

Use the substitution $x = au$ to show that

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$$

$$= \int \frac{1}{a^2 + a^2u^2} \cdot a\,du = \int \frac{a}{a^2(1+u^2)}\,du$$

$$= \frac{1}{a}\int \frac{1}{1+u^2}\,du$$

$$= \frac{1}{a}\tan^{-1}(u) + C$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

## Exercise 2

$$\underbrace{x^3+4x = x(x^2+4)}$$

$a=1, b=0, c=4$

$b^2-4ac = -16 < 0$

$\therefore x^2+4$ irr

Evaluate the following integral.

1. $\displaystyle\int \frac{x+2}{x^3+4x}\,dx$

$$\frac{x+2}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$x+2 = A(x^2+4) + (Bx+C)x$$

$$x=0 : \quad 2 = 4A \Rightarrow A = \frac{1}{2}$$

$$\Rightarrow x+2 = \frac{1}{2}(x^2+4) + (Bx+C)x$$

Comparing coeff of $x^2$ : $0 = \frac{1}{2} + B \Rightarrow B = -\frac{1}{2}$

Comparing coeff of $x$ : $1 = C$

$$\frac{x+2}{x^3+4x} = \frac{1}{2x} + \frac{\left(-\frac{1}{2}\right)x+1}{x^2+4} = \frac{1}{2x} + \frac{2-x}{2(x^2+4)}$$

# Exercise 2

$$\int \frac{x+2}{x^3+4x}\,dx = \int \frac{1}{2x} + \frac{2-x}{2(x^2+4)}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{2-x}{x^2+4}\,dx$$

Exercise 1

$a=2$

$$= \frac{1}{2}\int \frac{1}{x}\,dx + \frac{1}{2}\left[-\frac{1}{2}\int \frac{2x}{x^2+4}\,dx + 2\int \frac{1}{x^2+4}\,dx\right]$$

$$= \frac{1}{2}\ln|x| + \frac{1}{2}\left[-\frac{1}{2}\ln(x^2+4) + 2\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right] + C$$

$$= \frac{1}{2}\ln|x| - \frac{1}{4}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

# Not every function has simple antiderivatives

TLDR: If something cannot be evaluated exactly, the next best thing is to approximate.

It turns out that not every function has simple antiderivatives. For example,

$$\int_0^1 e^{x^2}\,dx \quad \text{and} \quad \int_0^1 \cos(x^2)\,dx$$

cannot be evaluated exactly because there is no *simple* antiderivative for $e^{x^2}$ and $\cos(x^2)$.

We instead turn to **approximations** to help us get (close to) the answer.

Midpoint Rule
Trapezoidal Rule

Simpson's Rule
Next week

# The Midpoint Rule

left and right Riemann sum

The **Midpoint Rule** is a Riemann sum, with sample points as the **midpoints** of the subintervals. Let $f$ be a function on the interval $[a, b]$. We use $n$ rectangles, so

$x_0, x_1, x_2, \ldots, x_{n-1}, x_n$

$\overline{x_1}$ $\overline{x_2}$ $\overline{x_n}$ ← Midpoint

$$\Delta x = \frac{b-a}{n}, \quad \text{and} \quad x_i = a + i\Delta x \quad (i \text{ from } 0 \text{ to } n).$$

The $n$ subintervals are $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$. The sample points are the midpoints of these intervals:

$$\overline{x_i} = \frac{x_i + x_{i-1}}{2} = a + \left(\frac{2i-1}{2}\right)\Delta x, \quad (i \text{ from } 1 \text{ to } n).$$

The **Midpoint Rule** $M_n$ is

$$\int_a^b f(x)\,dx \approx M_n = \Delta x \left[ f(\overline{x_1}) + f(\overline{x_2}) + \cdots + f(\overline{x_n}) \right].$$

# Example 2

$[a,b]=[1,2]$ $f(x)=\frac{1}{x}$ $n=5 \Rightarrow \Delta x=\frac{2-1}{5}=\frac{1}{5}$

$a=1,\ b=2$

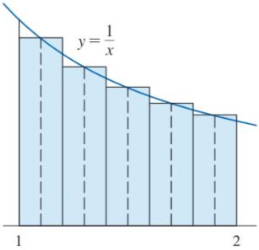
Use the Midpoint Rule with $n = 5$ to approximate the integral $\int_1^2 \frac{1}{x}\,dx$.

Give your final answer in 6 decimal places.

$x_0$ $x_1$ $x_2$ $x_3$ $x_4$ $x_5$

1 1.2 1.4 1.6 1.8 2

$\bar{x}_1=1.1$ $\bar{x}_2=1.3$ $\bar{x}_3=1.5$ $\bar{x}_4=1.7$ $\bar{x}_5=1.9$




$$\int_1^2 \frac{1}{x}dx \approx M_5 = \Delta x\left[f(\bar{x}_1)+\dots+f(\bar{x}_5)\right]$$

$$= \frac{1}{5}\left[\frac{1}{1.1}+\frac{1}{1.3}+\frac{1}{1.5}+\frac{1}{1.7}+\frac{1}{1.9}\right]$$

$$\approx 0.691908.$$

# The Trapezoidal Rule

$\frac{L_n + R_n}{2}$

The **Trapezoidal Rule** is the average of the left Riemann sum $L_n$ (left endpoints as sample points) and the right Riemann sum $R_n$ (right endpoints as sample points). Recall that the left and right endpoints on $[a, b]$ with $x_i = a + i\Delta x$ are

$L_n = \Delta x\,[f(x_0) + f(x_1) + \ldots + f(x_{n-1})]$

$$\text{Left endpoints}: x_0, x_1, \ldots, x_{n-1},$$
$$\text{Right endpoints}: x_1, x_2, \ldots, x_n.$$

$R_n = \Delta x\,[f(x_1) + f(x_2) + \ldots + f(x_n)]$

The Trapezoidal Rule $T_n$ is the **average** of $L_n$ and $R_n$:

$$T_n = \frac{L_n + R_n}{2} = \frac{\Delta x}{2}[f(x_0) + f(x_1) + \ldots + f(x_{n-1}) + f(x_1) + \ldots + f(x_{n-1}) + f(x_n)]$$

$$\implies \int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)].$$

# Example 3


$\frac{1}{2}(a+b)h$

$\Delta x = \frac{1}{5}$

Use the Trapezoidal Rule with $n = 5$ to approximate the integral $\displaystyle\int_1^2 \frac{1}{x}\,dx$.

Give your final answer in 6 decimal places.

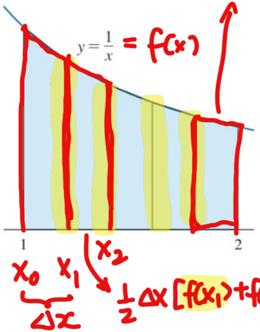
1 1.2 1.4 1.6 1.8 2

$$\int_1^2 \frac{1}{x}\,dx \approx T_5 = \frac{\Delta x}{2}\left[f(1) + 2f(1.2) + \dots + 2f(1.8) + f(2)\right]$$

$$= \frac{1}{10}\left[\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right]$$

$$\approx 0.695635.$$

$\frac{1}{2}\Delta x\,[f(x_4) + f(x_5)]$




$\frac{1}{2}\Delta x\,[f(x_1) + f(x_2)]$

$\frac{1}{2}\Delta x\,[f(x_0) + f(x_1)]$

# Approximations and errors

As with most approximations, we would expect some **error**. The error of the Midpoint Rule is

$$E_M = \int_a^b f(x)\,dx - M_n,$$

and the error of the Trapezoidal Rule is

magnitude of error cannot exceed a certain value (a bound)

$$E_T = \int_a^b f(x)\,dx - T_n.$$

Most of the time, we are unable to evaluate the error exactly, because we don't know the exact value of the definite integral. But we know that it cannot exceed a certain value, i.e. it is **bounded** by a certain value.

"worst case scenario"

# Error bounds

Worst case scenario

don't need max

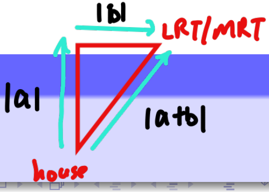
### Theorem

Suppose $K$ is a constant where $|f''(x)| \leq K$ on $[a, b]$. The **magnitude** of errors of the Trapezoidal ($E_T$) and Midpoint Rule ($E_M$) have the following upper bounds:

worst case scenario for Trapezoidal Rule

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

In other words, even if we do not know the exact error of the approximation, we know that the magnitude of the error can **never** exceed a certain number. A helpful inequality that you can use to find the bound $K$ for $|f''(x)|$ is the **triangle inequality**:




### Theorem (Triangle Inequality)

For any $a, b \in \mathbb{R}$,

$$|a+b| \leq |a| + |b|.$$

## Example 4

$\int_1^2 \frac{1}{x}$ $n=5$, $a=1, b=2$ $f(x)=\frac{1}{x}$ $[1,2]$

1. Find error bounds for the Midpoint and Trapezoidal Rule approximations in Examples 2 and 3.

$\frac{1}{|x^3|}$

2. How large should we take $n$ in order to guarantee that the Midpoint and Trapezoidal Rule approximations for $\int_1^2 \frac{1}{x}\,dx$ are accurate to within 0.0001?

$f'(x)=-\frac{1}{x^2}$ $f''(x)=\frac{2}{x^3} \Rightarrow |f''(x)|=\left|\frac{2}{x^3}\right|=\frac{2}{|x^3|} \leq 2 = K$

$(x=1)$

$|E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{2}{12\cdot 25} = 0.00\bar{6}$

$|E_M| \leq \frac{K(b-a)^3}{24\cdot n^2} = \frac{2}{24\cdot 25} = 0.00\bar{3}$

## Example 4

$n$ unknown

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \leq 0.0001$$

$$\frac{2}{12n^2} = \frac{1}{6n^2} \leq 0.0001$$

$$\Rightarrow \quad n^2 \geq \frac{1}{0.0006}$$

$$\Rightarrow \quad n \geq \sqrt{\frac{1}{0.0006}} = 40.8248$$

$n = 41$

$$|E_M| \leq \frac{K(b-a)^3}{24n^2} \leq 0.0001$$

$$\frac{2}{24n^2} = \frac{1}{12n^2} \leq 0.0001$$

$$n^2 \geq \frac{1}{0.0012}$$

$$n \geq \sqrt{\frac{1}{0.0012}} = 28.8675$$

$n = 29$

## Exercise 3

$T_{10} = 0.903122$ $\quad$ $M_{10} = 0.905225$

1. Use both the Midpoint and Trapezoidal Rule with $n = 10$ to approximate the integral $\int_0^1 \cos(x^2)\,dx$.
2. How large should we take $n$ in order to guarantee that these approximations for this integral are accurate to within 0.0001?

$$f(x) = \cos(x^2)$$

$$f'(x) = -\sin(x^2)\cdot 2x$$

$$f''(x) = -\cos(x^2)\cdot 2x \cdot 2x - 2\sin(x^2)$$

$$= -4x^2\cos(x^2) - 2\sin(x^2)$$

$$|f''(x)| = |\underbrace{-4x^2\cos(x^2)}_{a} + \underbrace{(-2\sin(x^2))}_{b}|$$

$$\leq |-4x^2\cos(x^2)| + |-2\sin(x^2)|$$

don't sub just any $x \in [0,1]$, you most likely will won't get a bound $K$.

# Exercise 3

not clear cut if $x=0$ or $x=1$ gives largest value.

(2)

$x \in [0,1]$

$$|f''(x)| = |\overbrace{\underbrace{-4x^2\cos(x^2)}_{a} + \underbrace{(-2\sin(x^2))}_{b}}|$$

$$\leq |-4x^2\cos(x^2)| + |-2\sin(x^2)|$$

triangle inequality

$$= 4\underbrace{|x^2|}_{|x^2|\leq 1}\underbrace{|\cos(x^2)|}_{\leq 1} + 2\underbrace{|\sin(x^2)|}_{\leq 1} \leq 4\cdot 1\cdot 1 + 2\cdot 1 = 6.$$

$K=6$.

$|E_M|$

$n=50$

$|E_T|$

$n=71$