CSD2301 Lecture
15. Angular Momentum

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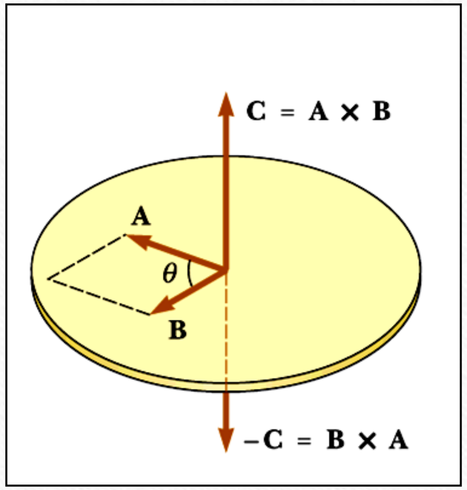
Outline

- Cross product
- Angular momentum
- Conservation of angular momentum

Vector (Cross) Product

- Given two vectors **A and B**, the vector product is another vector **C**. $\boxed{\vec{C} = \vec{A} \times \vec{B}}$
- The magnitude of C is: $\boxed{C = AB\sin\theta}$
- It is equal to the **area of parallelogram** formed by the vectors.
- The direction of C is perpendicular to the plane formed by A and B and is given by the **right-hand rule**.



Some Properties

- If the order of the vectors are changed, the sign of the cross product changes. $\boxed{\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}}$
- If A is parallel to B, then $\boxed{\vec{A} \times \vec{B} = 0}$
- If A is perpendicular to B then $\boxed{|\vec{A} \times \vec{B}| = AB}$
- The vector product obeys the distributive law: $\boxed{\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}}$

Some Properties

$$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$$

$$\hat{\imath} \times \hat{\jmath} = -\hat{\jmath} \times \hat{\imath} = \hat{k}$$

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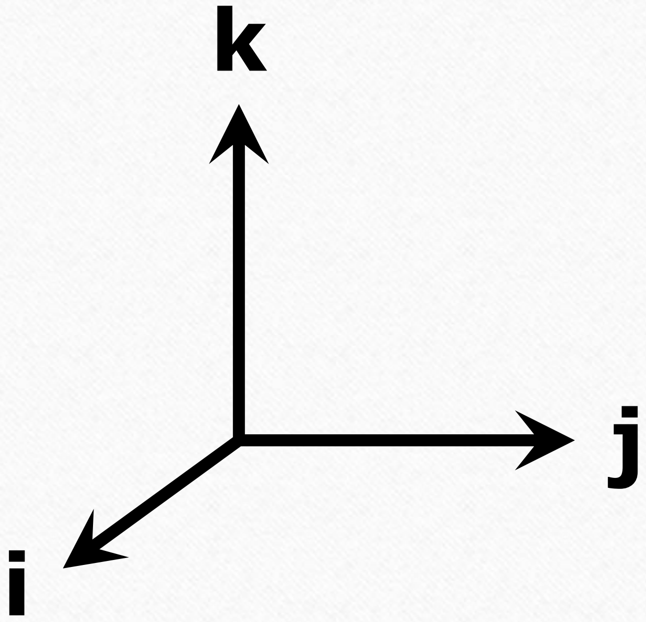


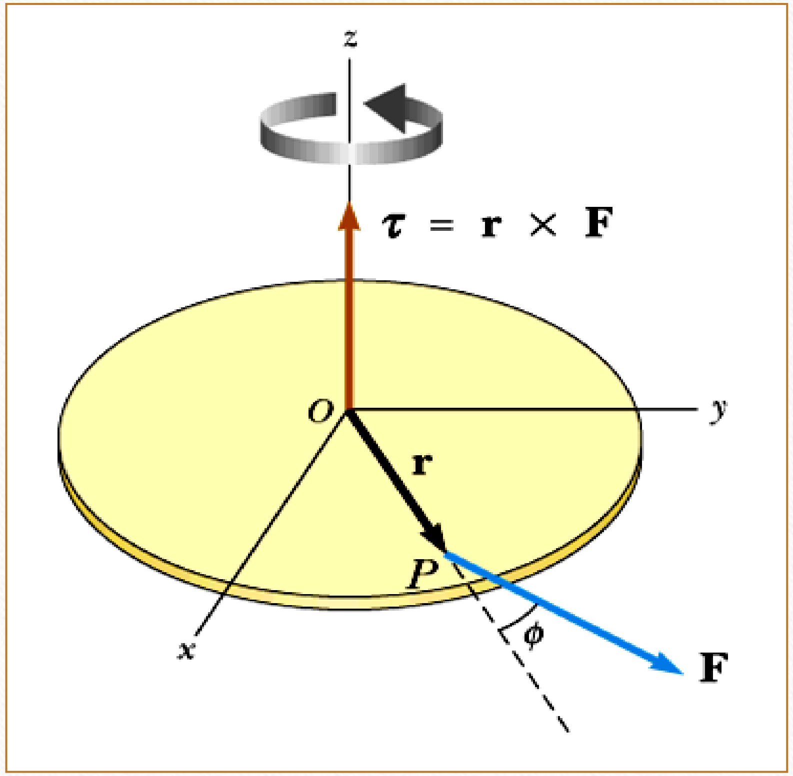
Torque: A Cross Product

- Remember that the torque of a force was earlier defined as:

$$\boxed{\tau = Fl = rF\sin\theta}$$

- This is just the magnitude of torque vector about O: $\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$

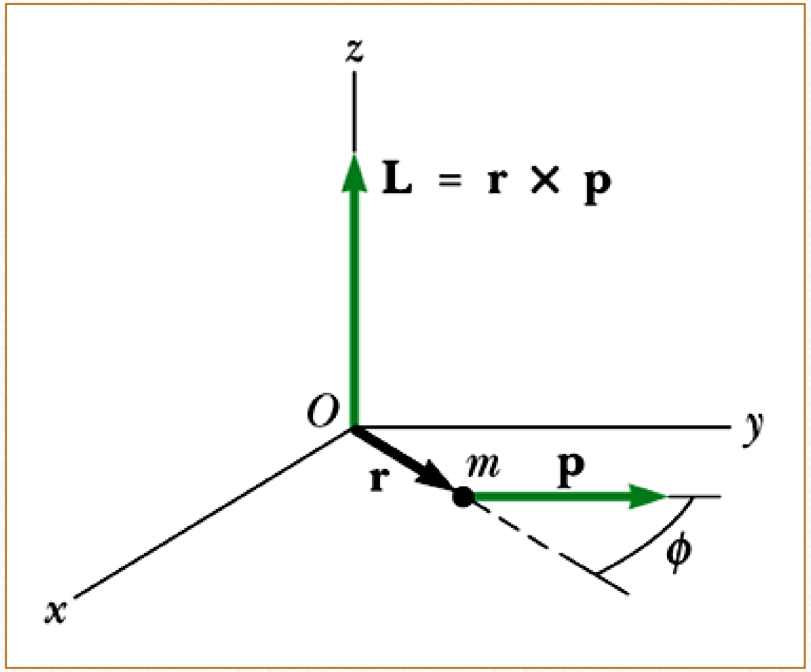
- Direction of vector is normal to the plane formed by **r** and **F**.



Angular Momentum

- The instantaneous angular momentum **L** of a particle relative to the origin *O* is defined by the **cross product** of the instantaneous vector position **r** and its instantaneous linear momentum **p**.

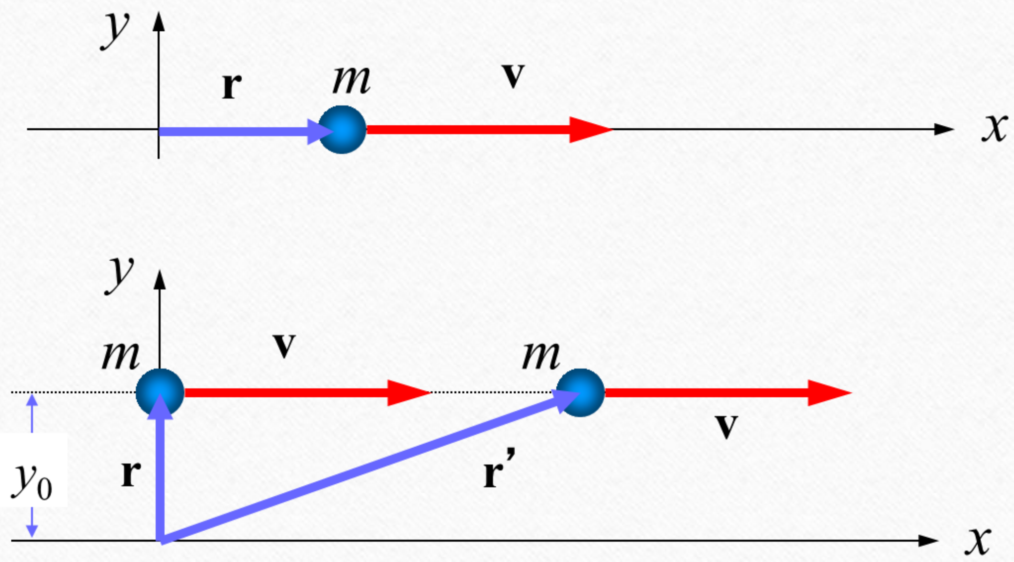
$$\boxed{\vec{L} = \vec{r} \times \vec{p}}$$



- SI unit: kg·m^2/s
- Both magnitude and direction depends on the choice of origin.

Concept Question

- Does a particle moving along a straight line have any angular momentum?



Torque & Angular Momentum

- We know that: $\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}$
- And: $\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$
- But: $\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = 0$
- So: $\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$ $\Rightarrow$ $\boxed{\vec{\tau} = \frac{d\vec{L}}{dt}}$

Torque & Angular Momentum

- The **torque** acting on a particle is equal to the time rate of change of the particle's angular momentum.

$$\boxed{\vec{\tau} = \frac{d\vec{L}}{dt}}$$

- The time rate of change of the total angular momentum of the system about some origin in an inertial frame equals the net external torque acting on the system about that origin.

$$\boxed{\sum \vec{\tau}_{\text{ext}} = \sum_{i=1}^{n} \frac{d\vec{L}_i}{dt} = \frac{d}{dt}\sum_{i=1}^{n} \vec{L}_i = \frac{d\vec{L}_{\text{tot}}}{dt}}$$

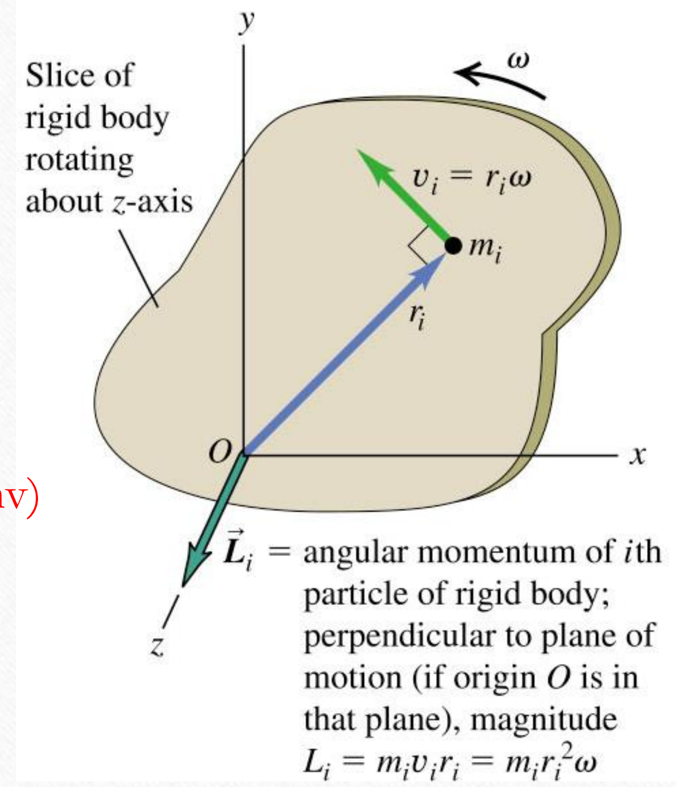
Rotating Rigid Object

- Consider rotation of a rigid object about a fixed symmetric axis.
- For a particle with mass m_i: $\boxed{L_i = m_i r_i^2 \omega}$

L = (r)(mv)

- For the whole object: $\boxed{L_z = \sum_i m_i r_i^2 \omega}$

$$\boxed{L_z = I\omega}$$



Rotating Rigid Object

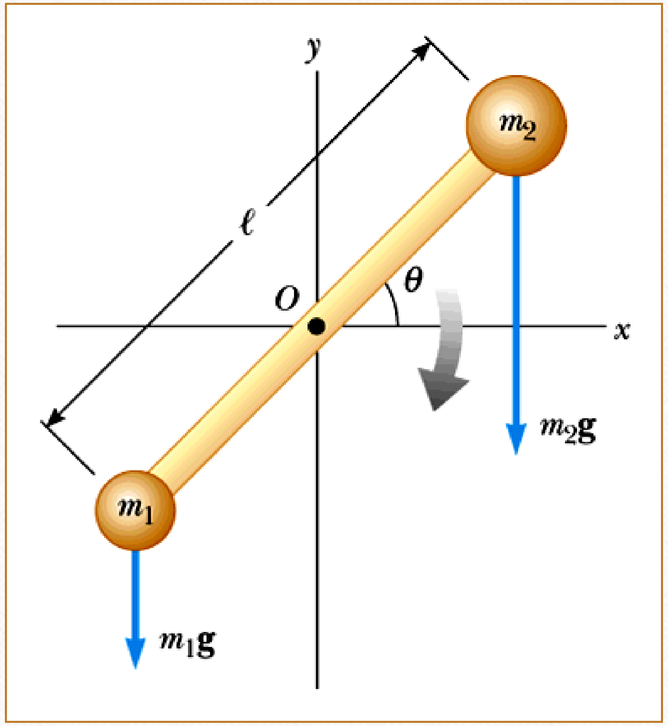
- Differentiate with respect to time:

$$\boxed{\frac{dL_z}{dt} = I\frac{d\omega}{dt} = I\alpha}$$

$$\boxed{\sum \tau_{\text{ext}} = \frac{dL_z}{dt} = I\alpha}$$

Example: Rotating Rod

- A rigid rod of mass M and length l is pivoted without friction. Two particles of masses m_1 and m_2 are connected to its ends. The combination rotates in a vertical plane with an angular speed ω. Find the expression for the magnitude of angular momentum and acceleration of the system.



Example: Rotating Rod

$$\tau_1 = m_1 g \frac{l}{2} \cos\theta$$

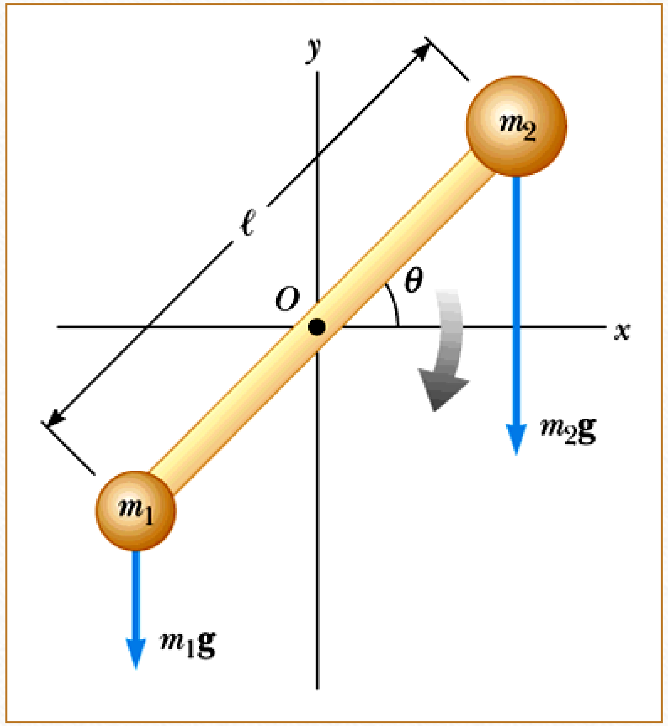
$$\tau_2 = -m_2 g \frac{l}{2} \cos\theta$$

$$\sum \tau_{\text{ext}} = \tau_1 + \tau_2 = I\alpha$$

$$I = \frac{1}{12} M l^2 + m_1 \left(\frac{l}{2}\right)^2 + m_2 \left(\frac{l}{2}\right)^2$$

$$\boxed{\alpha = \frac{2(m_1 - m_2) g \cos\theta}{l(M/3 + m_1 + m_2)}}$$

$$\boxed{L = I\omega = \frac{l^2}{4}\left(\frac{M}{3} + m_1 + m_2\right)\omega}$$



Conservation of Angular Momentum

- The total angular momentum of a system is constant if the resultant external torque acting on the system is zero.

$$\boxed{\sum \vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt} = 0} \Rightarrow \boxed{\vec{L} = \text{const.}}$$

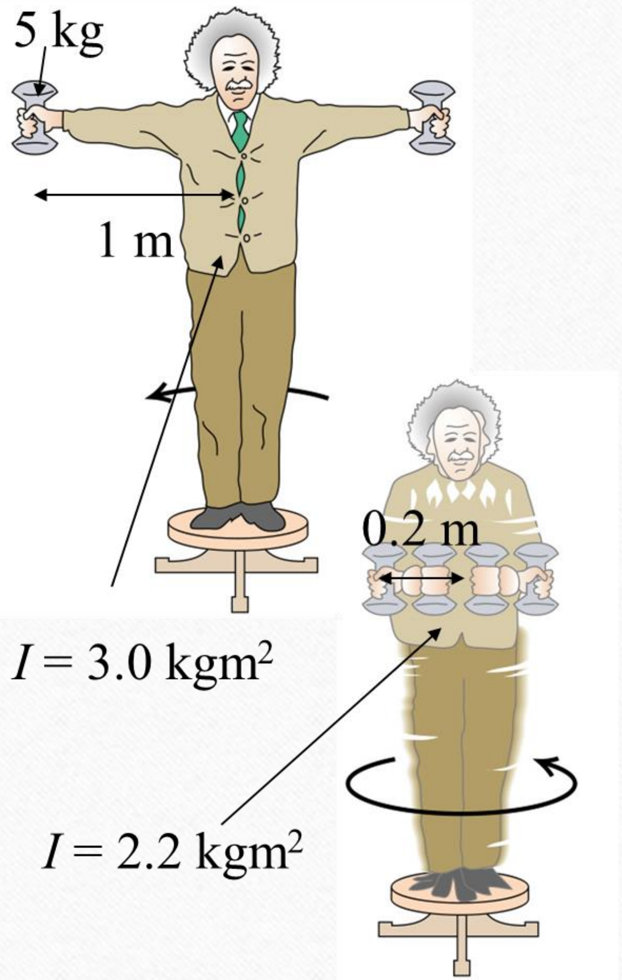
- Conservation of angular momentum: $\boxed{\vec{L}_i = \vec{L}_f}$

$$\boxed{I_i\omega_i = I_f\omega_f}$$

Valid for rotation about a fixed axis or axis through the CM of a moving system as long as that axis remains fixed in direction.

Example: Spinning Scientist

- A scientist stands at center of turntable carrying 5-kg dumbbell in each hand. $I = 3.0$ kgm^2 with arms extended and $I = 2.2$ kgm^2 when dumbbells at stomach. $r_1 = 1.0$ m, $r_2 = 0.20$ m. If period of revolution is initially $T_1 = 2.0$ s, what is his final angular velocity. Compare initial and final KEs and explain why they are different.



No external torques: $\boxed{I_1\omega_1 = I_2\omega_2}$

Example: Spinning Scientist

- With arms extended:

$$I_1 = 3.0 + 2(5.0)(1.0)^2 = 13 \text{ kg m}^2$$

$$\omega_1 = \frac{2\pi}{T} = \frac{2\pi}{2.0} = 3.14 \text{ rad/s}$$

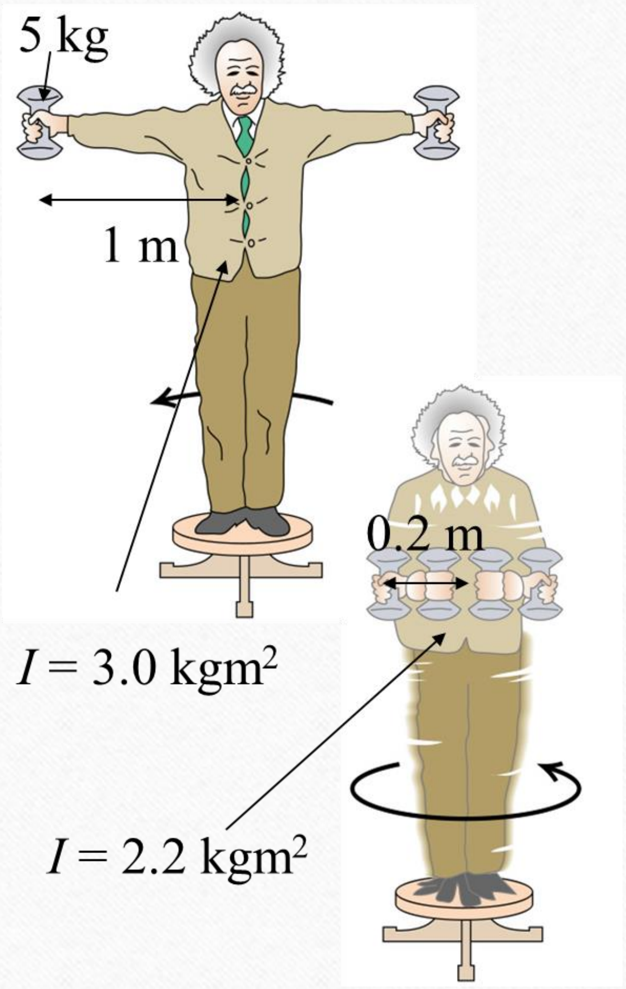
$$\boxed{K_1 = \frac{1}{2} I_1 \omega_1^2 = \frac{1}{2}(13)(3.14)^2 = 64.2 \text{ J}}$$

- With arms pulled in:

$$I_2 = 2.2 + 2(5.0)(0.2)^2 = 2.6 \text{ kg m}^2$$

$$\boxed{\omega_2 = \frac{I_1}{I_2}\omega_1 = \frac{13}{2.6} 3.14 = 15.7 \text{ rad/s}}$$

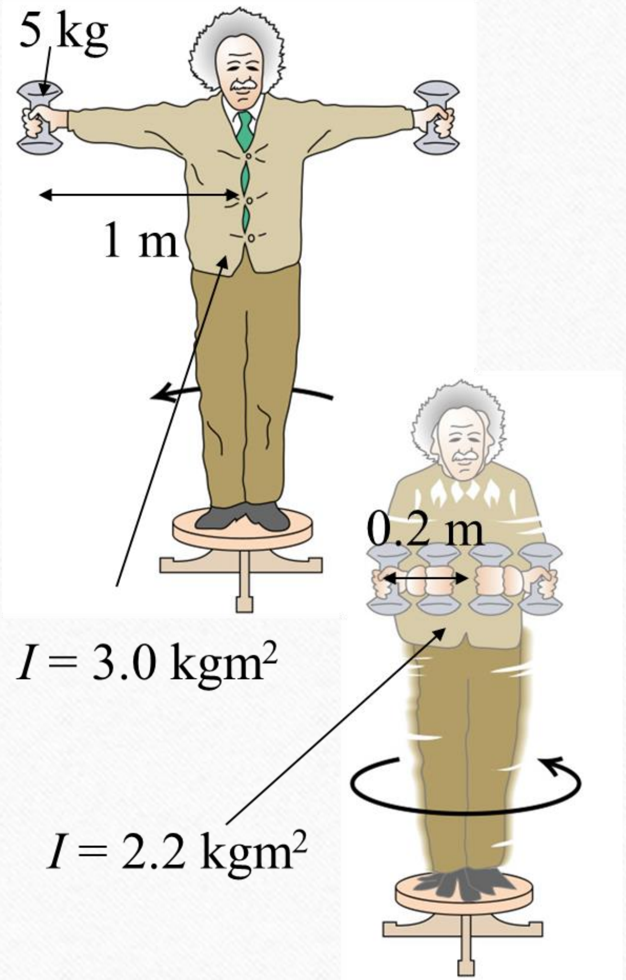
$$\boxed{K_2 = \frac{1}{2} I_2 \omega_2^2 = \frac{1}{2}(2.6)(15.7)^2 = 321 \text{ J}}$$



Example: Spinning Scientist

$$\boxed{K_2 - K_1 > 0}$$

- Scientist has to do +ve work!
- In his frame, there are forces pulling the dumbbells outwards (centrifugal force), and he must apply an inward force to pull the dumbbells in.
- Force he exerted on dumbbells is in the direction of displacement of dumbbells $\Rightarrow$ +ve work.



Example: Neutron Star

- A star rotates with a period of 30 days about an axis through its centre. After the star undergoes a supernova explosion, the stellar core, which had a radius of 1.0 x 10^4 km, collapses into a neutron star of radius 3.0 km. Determine the period of rotation of the neutron star.

<u>Assumptions:</u>

1. Mass remains the same.
2. No external torque.
3. Sphere with uniform density before and after explosion.

Angular momentum is conserved:

$$I_f \omega_f = I_i \omega_i$$

$$\frac{2}{5} M R_f^2 \left(\frac{2\pi}{T_f} \right) = \frac{2}{5} M R_i^2 \left(\frac{2\pi}{T_i} \right)$$

$$T_f = \left(\frac{R_f}{R_i} \right)^2 T_i = \left(\frac{3}{1 \times 10^4} \right)^2 \times 30 \text{ days}$$

$$\boxed{T_f = 0.23 \text{ s}}$$

Example: Disk and Stick

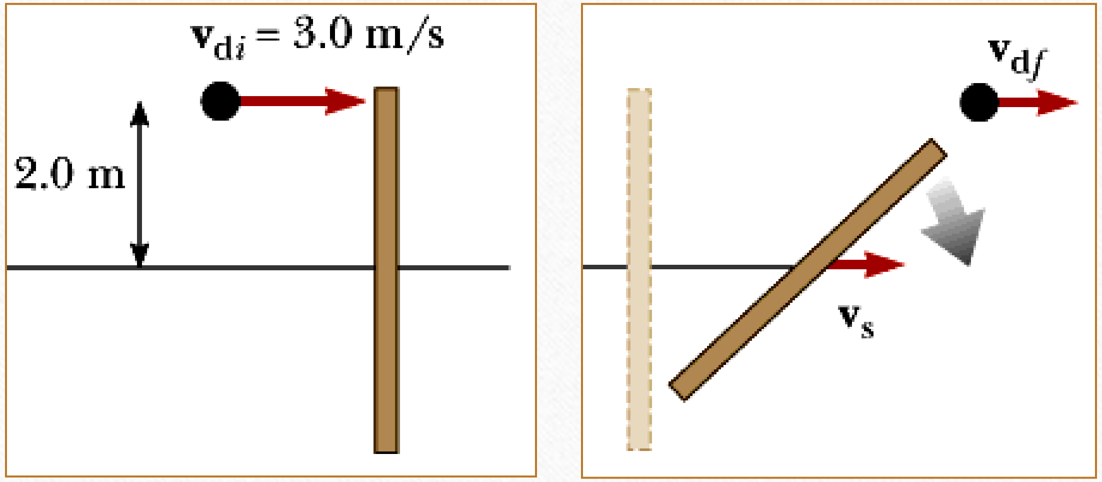
A 2.0-kg disk travelling at 3.0 m/s strikes a 1.0-kg stick of length 4.0 m that is lying flat on frictionless ice. Assume elastic collision and that the disk does not deviate from its original line of motion, find

(a) the translational speed of the disk, v_{df}

(b) the translational speed of the stick, v_s

(c) the angular speed of the stick, ω

after the collision.



Example: Disk and Stick

The three conservation laws apply.

$$m_d v_{di} = m_d v_{df} + m_s v_s$$ Linear Momentum

$$-r m_d v_{di} = -r m_d v_{df} + I\omega$$ Angular Momentum

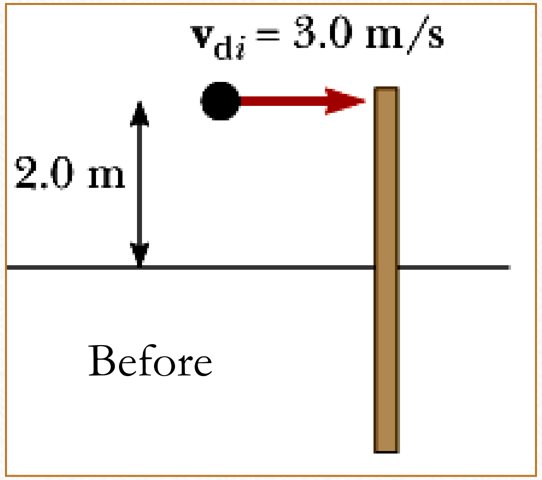
$$\frac{1}{2} m_d v_{di}^2 = \frac{1}{2} m_d v_{df}^2 + \frac{1}{2} m_s v_s^2 + \frac{1}{2} I\omega^2$$ Energy

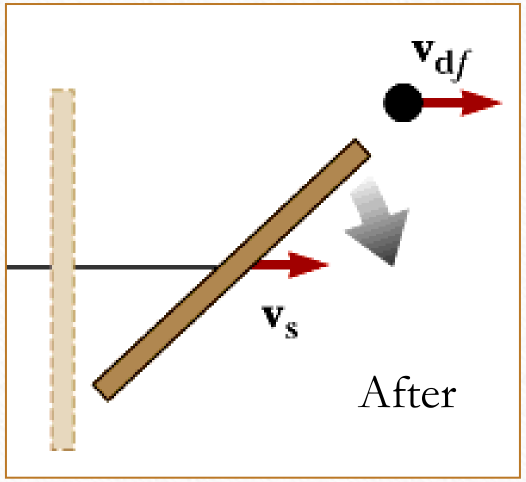
$$2(3) = (2)v_{df} + (1)v_s$$

$$-2(2)(3) = -2(2)v_{df} + \frac{1}{12}(1)(4)^2\omega$$

$$\frac{1}{2}(2)(3)^2 = \frac{1}{2}(2)v_{df}^2 + \frac{1}{2}(1)v_s^2 + \frac{1}{2}\left[\frac{1}{12}(1)(4)^2\right]\omega^2$$

$\boxed{v_s = 1.3\ \mathrm{m/s}}$ $\boxed{\omega = -2.0\ \mathrm{rad/s}}$





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