Midterm Exam Revision

Dr. Ronald Koh
ronald.koh@digipen.edu (Teams preferred over email)

AY 22/23 Trimester 2

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Composition of functions (Variant 1: 29%)

For functions $f(x) = \frac{x^2 - 1}{x^6}$ and $g(x) = \sec(x)$, find $(f \circ g)(x)$.

$$(f \circ g)(x) = \frac{\sec^2 x - 1}{\sec^6 x} = \frac{\tan^2 x}{\sec^2 x} \cdot \frac{1}{\sec^4 x}$$

$$= \frac{\frac{\sin^2 x}{\cancel{\cos^2 x}}}{\frac{1}{\cancel{\cos^2 x}}} \cdot \cos^4 x = \sin^2 x \cos^4 x.$$

$$\sec x = \frac{1}{\cos x}$$

$$\frac{1}{\sec x} = \cos x$$

Composition of functions (Variant 2: 44%)

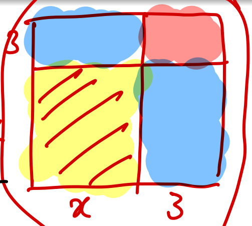
For functions $f(x) = \dfrac{1 - x^2}{x^4}$ and $g(x) = \cos(x)$, find $(f \circ g)(x)$.

$$(f \circ g)(x) = \frac{1 - \cos^2 x}{\cos^4 x} = \frac{\sin^2 x}{\cos^4 x} = \frac{\sin^2 x}{\cos^2 x} \cdot \frac{1}{\cos^2 x}$$

$$= \tan^2 x \sec^2 x.$$

Inverse functions (48%)

Let $f(x) = x^2 + 6x + 4$ for $x \leq -3$. Find $f^{-1}(x)$.



$$y = x^2 + 6x + 4$$

$$x \leq -3 \Rightarrow x+3 \leq 0$$

$$y = x^2 + 6x + 4 + 5 - 5$$

$$= \underbrace{x^2 + 6x + 9}_{(x+3)^2} - 5$$

$$= (x+3)^2 - 5 \Rightarrow y + 5 = (x+3)^2$$

$$\sqrt{(x+3)^2} = |x-3|$$

$$\Rightarrow \sqrt{y+5} = |\underbrace{x+3}_{\leq 0}| = -x-3$$

$$x = -\sqrt{y+5} - 3$$

$$f^{-1}(x) = -\sqrt{x+5} - 3.$$

Limit Techniques: Factorization (Variant 1: 51%)

Evaluate $\lim_{x \to -1} \frac{x^2 + 3x + 2}{x^3 + x^2 + x + 1}$.

$$= \lim_{x \to -1} \frac{(x+1)(x+2)}{x^2(x+1) + 1(x+1)}$$

$$= \lim_{x \to -1} \frac{\cancel{(x+1)}(x+2)}{\cancel{(x+1)}(x^2+1)} = \lim_{x \to -1} \frac{x+2}{x^2+1} = \frac{1}{2}.$$

Limit Techniques: Factorization (Variant 2: 52%)

Evaluate $\displaystyle\lim_{x\to-\frac{1}{3}} \frac{3x^2+4x+1}{3x^3+x^2+3x+1}$.

$$= \lim_{x\to-\frac{1}{3}} \frac{3x^2+4x+1}{x^2(3x+1)+(3x+1)}$$

$$= \lim_{x\to-\frac{1}{3}} \frac{\cancel{(3x+1)}(x+1)}{\cancel{(3x+1)}(x^2+1)} = \frac{\frac{2}{3}}{\frac{10}{9}} = \frac{2}{3}\cdot\frac{9}{10} = \frac{3}{5}.$$

Limits/defn of derivative (Variant 1: 44%)

Evaluate $\lim_{h\to 0} \frac{(h+2)^6 - 64}{h}$. $= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$

$= f'(2)$

$a=2$, $f(x) = x^6$

$\downarrow$

$f'(x) = 6x^5$

$= f'(a)$

$= 6 \cdot 2^5 = 192$

Limits/defn of derivative (Variant 2: 46%)

Evaluate $\lim_{h\to 0} \frac{(h+2)^6 - 64}{2h}$. $= \frac{1}{2} \lim_{h\to 0} \frac{(h+2)^6 - 64}{h}$

$= 96.$

Limits/defn of derivative (Variant 3: 38%)

Evaluate $\lim_{h\to 0} \frac{(h+2)^7 - 128}{h}$. $= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

$a=2$, $f(x) = x^7$

$f'(x) = 7x^6$

$\|$

$7 \cdot 2^6$

$\|$

448

Differentiability (Variant 1: 48%)

⇒ No differentiation rules

Let $f(x) = |x|$. Find $f'(0)$, if it exists.

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}.$$

We covered in Week 2, this limit does not exist. ⇒ $f'(0)$ doesn't exist.

Differentiability (Variant 2: 37%)

Let $f(x) = x|x|$. Find $f'(0)$, if it exists.

$$\cancel{f'(x) = |x| + x(|x|)'}$$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x|x| - 0}{x - 0}$$

$$= \lim_{x \to 0} \frac{\cancel{x}|x|}{\cancel{x}}$$

Proven in Tutorial 2.

$$= \lim_{x \to 0} |x| = 0$$

Question 1

Differentiate $f(x) = e^{\sin^2(x^2)}$.

$$\overbrace{\sin^2(x^2)}^{\to (\sin(x^2))^2}$$

$$f'(x) = e^{\sin^2(x^2)} \cdot \left[\sin^2(x^2)\right]'$$

$$= e^{\sin^2(x^2)} \cdot 2\sin(x^2) \cdot \left[\sin(x^2)\right]'$$

$$= e^{\sin^2(x^2)} \cdot 2\sin(x^2) \cdot \cos(x^2) \cdot \left[x^2\right]'$$

$$= e^{\sin^2(x^2)} \cdot \underbrace{2\sin(x^2) \cdot \cos(x^2)} \cdot 2x$$

$$= 2x \sin(2x^2)\, e^{\sin^2(x^2)}$$

$$\sin(2x^2) = 2\sin(x^2)\cos(x^2)$$

Question 2

Find an equation of the tangent line to the function $f(x) = 10xe^{-x^2}$ at the point $(\overset{a}{0}, \overset{f(a)}{0})$.

$$f'(x) = 10xe^{-x^2} \cdot (-2x) + 10e^{-x^2}.$$
$$= 10e^{-x^2} - 20x^2e^{-x^2}$$

$$f'(0) = 10.$$

Tangent line: $y = f'(a)(x - \overset{0}{a}) + \overset{0}{f(a)}$
$= f'(a)x = 10x.$

Question 3

Find an equation of the tangent line to the graph of $y^2 = x^3 + 3x^2$ at the point $(1, -2)$.

$$y^2 = x^3 + 3x^2$$

Diff wrt x: $2y \frac{dy}{dx} = 3x^2 + 6x$

$$\frac{dy}{dx} = \frac{3x^2 + 6x}{2y}$$

$$\frac{dy}{dx}(1,-2) = \frac{3+6}{-4} = -\frac{9}{4}$$

Tangent line:

$$y = -\frac{9}{4}(x - 1) - 2 = -\frac{9}{4}x + \frac{9}{4} - 2$$

$$= -\frac{9}{4}x + \frac{1}{4}.$$

Question 4

Find $\dfrac{dy}{dx}$ for the following equation.

$$\cos(x^2 + 2y) + xe^{y^2} = 1$$

Differentiate wrt x:

$$-\sin(x^2+2y)\cdot\left(2x+2\frac{dy}{dx}\right) + e^{y^2} + xe^{y^2}\cdot 2y\cdot\frac{dy}{dx} = 0.$$

$$\Rightarrow -2x\sin(x^2+2y) - 2\sin(x^2+2y)\frac{dy}{dx} + e^{y^2} + 2xye^{y^2}\frac{dy}{dx} = 0$$

$$\Rightarrow -2\sin(x^2+2y)\frac{dy}{dx} + 2xye^{y^2}\frac{dy}{dx} = 2x\sin(x^2+2y) - e^{y^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x\sin(x^2+2y) - e^{y^2}}{2xye^{y^2} - 2\sin(x^2+2y)}$$

Question 5

There is only one critical point c of the function $f(x) = x^2 + x$. Find c.

$f'(c) = 0$

$f'(x) = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$.

$f' < 0$ $\quad -\frac{1}{2} \quad$ $f' > 0$

decreasing $(-\infty, -\frac{1}{2})$

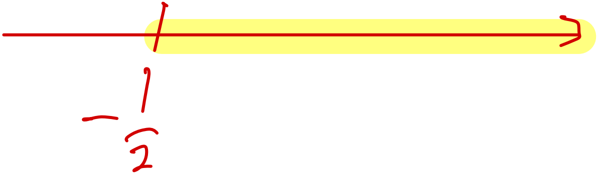
$(-\frac{1}{2}, \infty)$ increasing

Question 6

For the function f in Question 5, find **an** interval where f is increasing.

(a) $(-1, \infty)$ (b) $(-\infty, 0)$ (c) $(0, 1)$

(d) $(-2, \infty)$ (e) None of the above



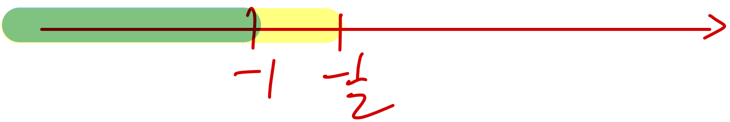
are any of these intervals a subset of $(-\frac{1}{2}, \infty)$

Question 7

For the function f in Question 5, find **an** interval where f is decreasing.

(a) $(1, \infty)$ (b) $\cancel{(-\infty, 1)}$ (c) $(0, 1)$

(d) $(0, \infty)$ (e) None of the above



decreasing

$(-\infty, -\frac{1}{2})$