Pre-Calculus and Foundations

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AY 22/23 Trimester 2

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Course Logistics

Class schedule:

- Lectures:
  - Mondays 1330 - 1600 HRS, LT6D
    (CSD1251 Groups A/B/C and CSD1250 Group B)
  - Tuesdays 1630 - 1900 HRS, LT6C
    (CSD1251 Groups D/E/F and CSD1250 Group A)
- Tutorials: check the timings on your individual timetables

Instructors:

- Dr. Ronald Koh (Module Coordinator, Lectures and Group F)
- Dr. Wu Yilin (Groups A/E)
- Dr. Teh Yong Liang (Groups B/C/D)

Assessment Tasks

- 5 Homework (10%), tentatively on Weeks 1, 3, 6, 9, 11
- 3 Quizzes (30%), tentatively on Weeks 5, 8, 12
- 1 Midterm Exam (30%)
- 1 Final Exam (30%)

No cheatsheets/formula sheets for quizzes, I may consider it for midterm and finals.

Functions

Definition

A function f is a rule that assigns to each element x in a set D exactly one element $f(x)$ in a set E. $f(x)$ here is read as "f *of* x".

Think of f as a function in a programming language, it takes in variables/parameters as input (x here), and it gives an output/returns $f(x)$. In this course, we only deal with the cases where

1. f is a single variable function (only one parameter x as input),
2. x and $f(x)$ are real numbers, i.e. $x, f(x) \in \mathbb{R}$,

Domain and range of f

Definition

1. The *domain* of f is the subset D of the real numbers $\mathbb{R}$ for which can be input into f.
2. The *range* of f is the set of all outputs of f; i.e. the set of all possible values of f as x varies through its domain.

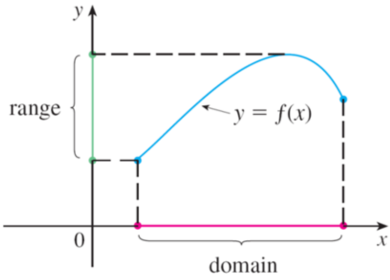
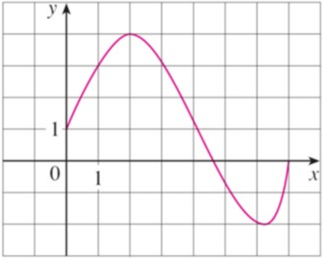


Figure: The domain and range of f visualized in the graph of f.

Exercise 1

The graph of a function f is shown below.

1. Find $f(1), f(2)$, and $f(5)$.
2. What are the domain and range of f?



Exercise 1

Exercise 2

Graph the following functions

1. $f(x) = 2x - 1$,
2. $g(x) = x^2$.

What are the domains and ranges of f and g?

Exercise 2

Exercise 3

Find the domain of each of these functions.

1. $f(x) = \sqrt{x+3}$,
2. $g(x) = \dfrac{1}{x^2 - 3x}$.

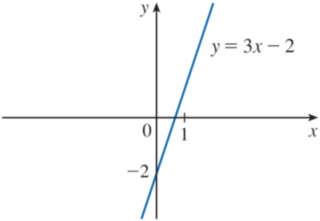
Exercise 3

Linear functions

A *linear* function is a straight line, which can be expressed in the form that we are familiar with:

$$f(x) = mx + c$$

where m is the slope/gradient of the line and c is the y-intercept.



Question: How do we find the x-intercept of this line?

Polynomials

A function P is called a *polynomial* if it has the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a fixed nonnegative integer $(n = 0, 1, 2, \ldots)$, and the constants $a_0, a_1, \ldots, a_n$ are called the *coefficients* of P.

The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then we say that P has *degree* n.

Examples: The functions $f(x) = 2x^3 + 3x + 1$ and $g(x) = 5x^4 + 2x^3 + 2x + 3$ are polynomials of degree 3 and 4 and has leading coefficients 2 and 5 respectively.

Exercise 4

Find the leading coefficients and the degree of the following polynomials.

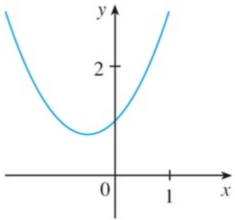
1. $f(x) = 10x^5 + 3x^2 + 2x + 7$
2. $g(x) = x^{1000} - 9$
3. $h(x) = 2x - 1$
4. $p(x) = 3x^2 + 5x + 1$

Quadratic functions

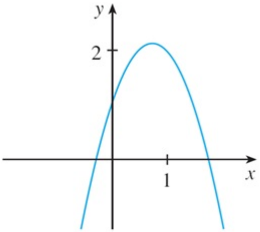
Definition

A *quadratic* function is a polynomial of degree 2, and has the form $P(x) = ax^2 + bx + c$ where $a \neq 0$.

We can obtain the graph of a quadratic function by shifting the parabola $y = ax^2$. The parabola has the shape of a smile when $a > 0$ and has the shape of a frown when $a < 0$.



(a) $y = x^2 + x + 1$



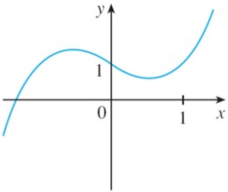
(b) $y = -2x^2 + 3x + 1$

Cubic functions

Definition

A *cubic* function is a polynomial of degree 3, and has the form $P(x) = ax^3 + bx^2 + cx + d$ where $a \neq 0$.

The graphing of a cubic function is slightly more involved than a quadratic function, so we keep it in view first; we will talk more about it around Week 9. The graph below is an example of the graph of a cubic equation.



(a) $y = x^3 - x + 1$

Algebraic functions

Definition

A function f is called an *algebraic function* if it can be constructed using algebraic operations:

1. addition and subtraction,
2. multiplication and division, and
3. raising to a power of some real number,

starting from polynomials.

Examples of algebraic functions include

$$f(x) = \sqrt{x^3 + 2}, \qquad g(x) = \frac{x^4 - 9}{x^2 + \sqrt{x}} - \sqrt[3]{x^4 + 3}.$$

Transcendental functions

Definition

Functions that are not algebraic are called *transcendental* functions.

The transcendental functions which we will study include trigonometric, exponential, and the logarithmic functions.

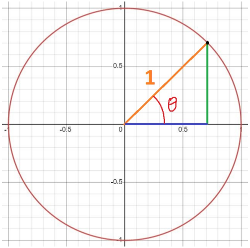
Important: From this point on, and at any point of this course, we will make reference to angles in radians, rather than in degrees. If there is a need to convert, just remember that

$$2\pi = 360^\circ \iff \frac{\pi}{180} = 1^\circ.$$

Thus all you need to do to convert degrees to radians is to multiply by $\frac{\pi}{180}$.

Sine and cosine functions

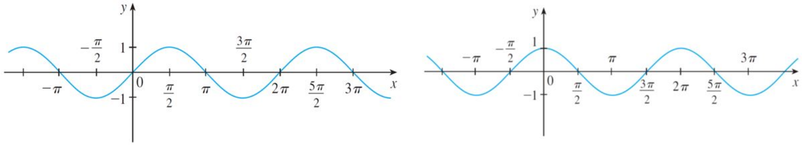
Recall that a circle centered at the origin with radius 1 has the equation $x^2 + y^2 = 1$.



The point in black rotates in an anticlockwise direction, while staying on the circle. The x and y-coordinate of this point, which are dependent on the angle θ, are the $\cos\theta$ and $\sin\theta$ functions respectively.

Sine and cosine functions

Alternatively, you may also define the $\sin\theta$ and $\cos\theta$ functions using the "*TOA CAH SOH*" we learnt in secondary school. The graphs of the sine and cosine functions are as follows.



(a) $f(x) = \sin x$ (b) $g(x) = \cos x$

Both of these functions have domain $\mathbb{R}$.

Important: Two useful properties of the sine and cosine functions:

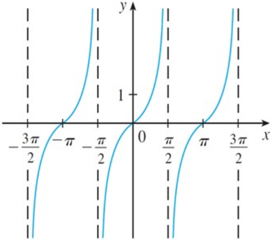
1. $-1 \leq \sin\theta \leq 1$ and $-1 \leq \cos\theta \leq 1$ (Range is $[-1, 1]$),
2. $\sin(\theta) = \sin(\theta + 2\pi)$ and $\cos(\theta) = \cos(\theta + 2\pi)$.

Tangent function

The tangent function is based on the sine and cosine function:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

and its domain is the set of θ for which $\cos\theta \neq 0$, i.e. $\mathbb{R} \setminus \left\{\frac{\pi}{2} + n\pi : n \in \mathbb{Z}\right\}$. It is also π-periodic: $\tan(\theta+\pi) = \tan(\theta)$.



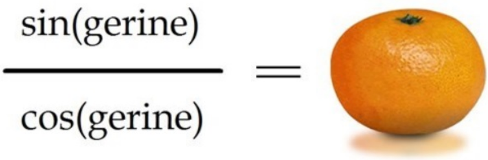


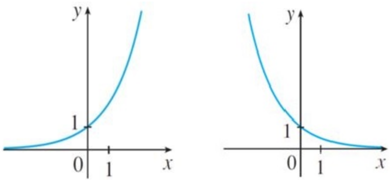
Figure: Left: Graph of $\tan\theta$, Right: Tangential meme. :)

Exponential functions

Definition

Exponential functions are functions of the form $f(x) = a^x$, where the base a is a positive constant.

The graphs of $f(x) = a^x$ differ in the two cases $a > 1$ (left) and $0 < a < 1$ (right), as shown below:



Exponential functions are essential in modelling many naturally occurring things, like disease spread, population growth, radioactive decay, just to name a few.

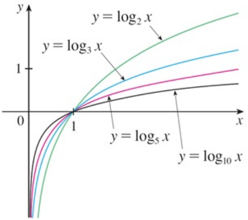
Logarithmic functions

Definition

Logarithmic functions are functions of the form $f(x) = \log_a x$, where the base a is a positive constant. The logarithmic function is related to the exponential function a^x:

$$y = \log_a x \iff a^y = x$$

For each $a > 0$, the domain of $f(x) = \log_a x$ is the set of positive real numbers, i.e. $(0, \infty)$, and its range is $\mathbb{R}$.



Natural logarithmic function

In this course and other than this lecture, we will only deal with the case where $a = e = 2.71828\ldots$, then we get the *natural logarithmic function*

$$f(x) = \log_e x = \ln x.$$

This natural logarithmic function is related to the exponential function $g(x) = e^x$; it's the *inverse* of $g(x) = e^x$. We will talk more about inverse functions in the upcoming slides.

Exercise 5

Determine which functions are algebraic and which are transcendental.

1. $f(x) = \dfrac{\ln^2 x}{x^2}$
2. $g(x) = \dfrac{\sqrt{\sin x}}{x^5 + 1}$
3. $h(x) = \dfrac{\sin^2(x^4) + \cos^2(x^4)}{x^2 + 1}$
4. $q(x) = \dfrac{x^3 - 1}{x - 1}$

Composite functions

Definition

Let f and g be functions such that the range of f is a subset of the domain of g. The *composite function* $g \circ f$ is the function such that

$$(g \circ f)(x) = g(f(x)).$$

Example

Let $f(x) = \sin x$ and $g(x) = x^2$.

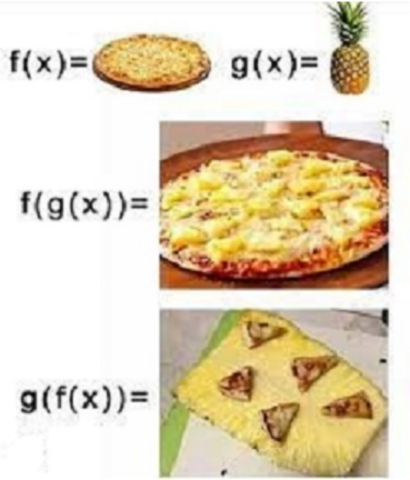
Note that the domain and range of f are $\mathbb{R}$ and $[-1, 1]$ respectively, while g has domain $\mathbb{R}$ and range $[0, \infty)$.

The range of f is a subset of the domain of g, and the range of g is a subset of the domain of f, thus both $f \circ g$ and $g \circ f$ are well-defined. Thus,

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) = g(\sin x) = \sin^2 x,\\ (f \circ g)(x) &= f(g(x)) = f(x^2) = \sin(x^2).\end{aligned}$$

Note that $g \circ f$ is generally not the same function as $f \circ g$.

Composite functions TLDR



Exercise 6

Given the following pairs of functions, determine both $g \circ f$ and $f \circ g$.

1. $f(x) = \sqrt{x}$, $g(x) = x^2 + 1$
2. $f(x) = 2x$, $g(x) = \cos x$
3. $f(x) = \ln x$, $g(x) = e^x$

Exercise 6

Inverse functions

In the last exercise, we saw how some functions could give rise to $(f \circ g)(x) = (g \circ f)(x) = x$.

Definition

A function f is said to be an *inverse* of g (and vice versa) if

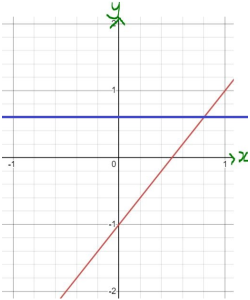
$$(f \circ g)(x) = (g \circ f)(x) = x.$$

Here, g is denoted by f^{-1}. **Important**: This f^{-1} is not to be confused with the reciprocal of f, $\frac{1}{f}$!

Not every function has an inverse. One way to check this is the horizontal line test.

Horizontal line test example 1

Let's start with a few examples. Let $f(x) = 2x - 1$. We have drawn the graph earlier:



If we draw **any** line (in blue) parallel to the x-axis, and if the line intersects the graph **exactly once**, then the inverse of the function exists.

Finding inverse

If the horizontal line test gives the green light on the inverse, we may apply this algorithm to get the inverse function f^{-1}:

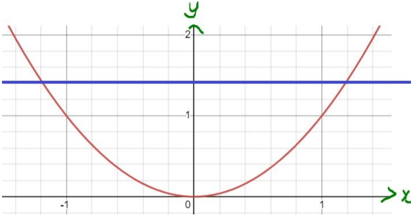
1. Set $y = f(x)$, so $y = 2x - 1$, with domain $\mathbb{R}$.
2. Make x the subject of the equation, so $x = \dfrac{y+1}{2}$.
3. Replace x with y in the RHS of the equation, so the RHS becomes $\dfrac{x+1}{2}$.
4. Set $g(x) =$ RHS of the equation, so $g(x) = \dfrac{x+1}{2}$.
5. Check that the function g is indeed an inverse of f:

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x+1}{2}\right) = 2\left(\frac{x+1}{2}\right) - 1 = x, \checkmark$$

$$(g \circ f)(x) = g(f(x)) = g(2x-1) = \frac{(2x-1)+1}{2} = x. \checkmark$$

Horizontal line test example 2

Let $f(x) = x^2$, with domain $\mathbb{R}$.



The horizontal line test fails here; there is a blue line for which the line intersects the graph more than once! Thus there is no inverse for this $f(x) = x^2$ with domain $\mathbb{R}$.

Plot twist

Let $f(x) = x^2$ with a different domain, so that the horizontal line test does not fail. We can choose $(-\infty, 0]$ as its domain.

Note that **any** horizontal line intersects the graph exactly once, thus this function has an inverse by the horizontal line test!

Finding inverse

We follow the same algorithm. Pay attention to the second step.

1. Let $y = x^2$ with domain $(-\infty, 0]$.
2. Taking square roots on both sides, we get

$$\sqrt{y} = \sqrt{x^2}.$$

   Now, $\sqrt{x^2}$ does not necessarily equal to x. In general,

$$\sqrt{x^2} = |x|.$$

   Since x is nonpositive, $|x| = -x$, thus

$$\sqrt{y} = -x \iff x = -\sqrt{y}.$$

You can finish the rest of the steps as an exercise.

Exercise 7

(a) Use the horizontal line test to check if each of these functions has an inverse. If the inverse exists, find it.

1. $g(x) = 1 - x^2$, $x \in \mathbb{R}$
2. $h(x) = 2 + \sqrt{x+1}$ (if we don't tell you the domain, we expect you to find it first, so take the largest possible domain)

(b) Each of the functions below has an inverse. Use the algorithm (Step 2 to 4) specified previously to find an inverse of the function.

1. $h(x) = e^{1-x}$
2. $g(x) = x^2 - 2x$, $x \geq 1$

Exercise 7

Exercise 7