Tutorial 3.

Question 1: 20 questions in total, at least 12 questions right to pass. probability of getting a correct answer is $1/3$

(a) X = number of questions the student answered correctly.

$X \sim \text{Binomial}(20, \frac{1}{3})$.

$$P(X=x) = \binom{20}{x}\left(\frac{1}{3}\right)^x\left(1-\frac{1}{3}\right)^{20-x} \quad x \in \{0,1,2,3,\ldots 20\}$$

$$P(X \geq 12) = P(X=12) + P(X=13) + P(X=14) + \cdots + P(X=20).$$
$$= \sum_{x=12}^{20}\binom{20}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{20-x} \approx 0.013$$

(b) $X \sim \text{Binomial}(20, \frac{1}{2})$

$$P(X \geq 12) = P(X=12) + P(X=13) + P(X=14) + \cdots + P(X=20)$$
$$= \sum_{x=12}^{20}\binom{20}{x}\underbrace{\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{20-x}}$$
$$= \sum_{x=12}^{20}\binom{20}{x}\left(\frac{1}{2}\right)^{20} \approx 0.2517.$$

Question 2:

(a) The chance of getting a double sixes is $\frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36}$.

$X \sim \text{Binomial}(100, \frac{1}{36})$

(b) $n = 100 \geq 50$ and $np = 100 \times \frac{1}{36} \approx 2.78 \leq 10$.

So, it is suitable to approximate Binomial $(100, \frac{1}{36})$

by poisson (2.78).

(c). $X \sim \text{Binomial}(100, \frac{1}{36})$

$$P(X=x) = \binom{n}{x}(p)^x(1-p)^{n-x}, \quad n=100, \quad p=\frac{1}{36}.$$

$$P(X=x) = \binom{100}{x}\left(\frac{1}{36}\right)^x\left(\frac{35}{36}\right)^{n-x}$$

$$P(X\geq 3) = P(X=3)+P(X=4)+\cdots+P(X=100)$$
$$= \sum_{x=3}^{100}\binom{100}{x}\left(\frac{1}{36}\right)^x\left(\frac{35}{36}\right)^{n-x}$$

$$= 1-P(X=0)-P(X=1)-P(X=2).$$
$$= 1-\binom{100}{0}\left(\frac{1}{36}\right)^0\left(\frac{35}{36}\right)^{100}-\binom{100}{1}\left(\frac{1}{36}\right)^1\left(\frac{35}{36}\right)^{99}$$
$$-\binom{100}{2}\left(\frac{1}{36}\right)^2\left(\frac{35}{36}\right)^{98} \approx 0.5279$$

$X \sim \text{poi}(2.78)$.

$$P(X=x) = \frac{\lambda^x}{x!}e^{-\lambda} \Rightarrow P(X=x) = \frac{(2.78)^x}{x!}e^{-2.78}.$$

$$P(X\geq 3) = 1-P(X=0)-P(X=1)-P(X=2)$$
$$= 1-\frac{(2.78)^0}{0!}e^{-2.78}-\frac{(2.78)^1}{1!}e^{-2.78}-\frac{(2.78)^2}{2!}e^{-2.78}$$
$$\approx 0.5279.$$

The approximation is quite good!

Question 3 $\lambda = rt$

(a) $\begin{cases} r: 5 \text{ hits per second.} \\ t = 2. \end{cases}$

$\lambda = rt = 10.$

$X \sim poi(10)$

$$P(X=k) = \frac{10^k}{k!} e^{-10}.$$

$$P(X=0) = \frac{10^0}{0!} e^{-10} = e^{-10}.$$

(b) $\begin{cases} r = 5 \text{ hits per second} \\ t = 1 \end{cases}$

$\lambda = 5$

$X \sim poi(5)$

$$P(X=k) = \frac{5^k}{k!} e^{-5}$$

$$P(X \geq 1) = 1 - P(X=0)$$

$$= 1 - \frac{5^0}{0!} e^{-5}$$

$$= 1 - e^{-5}$$

Question 4 :

$X \sim geom(0.2)$

(a) ① trials are independent of one another.

② trials have constant probability of successes

$$P(X=x) = (1-0.2)^{x-1}(0.2)$$

(b). $P(X=3) = (1-0.2)^2(0.2) = (0.8)^2(0.2) \approx 0.128$

$P(X>8) = P(X \geq 9) = (0.8)^8 \approx 0.1677$

(c). $P(5 \leq X < 13) = P(5 \leq X \leq 12)$

$$= \underline{P(X \leq 12)} - P(X \leq 4).$$

$$= [1 - P(X \geq 13)] - [1 - P(X \geq 5)]$$

$$= P(X \geq 5) - P(X \geq 13)$$

$$= (0.8)^4 - (0.8)^{12} \approx 0.3409$$

Question 5 $\quad \int_0^{+\infty} e^{-x}\,dx.$

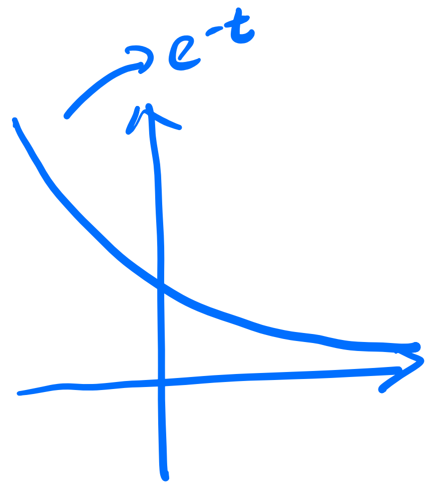
$$\int_0^{\infty} e^{-x}\,dx = \lim_{t\to\infty} \boxed{\int_0^{t} e^{-x}\,dx}$$

$$= \lim_{t\to\infty} \left(-e^{-x}\right)\Big|_0^t$$

$$= \lim_{t\to\infty} \left(-e^{-t}\right) - \left(-e^{-0}\right)$$

$$= \lim_{t\to\infty} \left(-e^{-t}\right) + 1$$

$$= 1 - \underbrace{\lim_{t\to\infty}\left(-e^{-t}\right)}_{\to 0} = 1 - 0 = 1$$



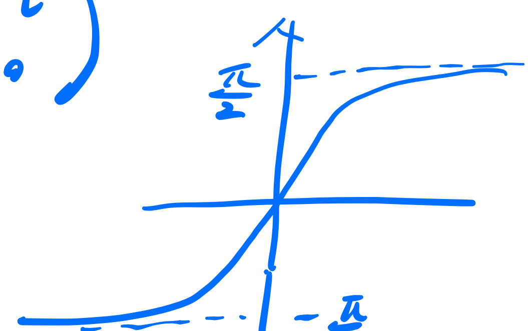
Question 6: $\quad$ Find $\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx.$

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^{+\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\,dx + \lim_{t\to+\infty} \int_0^t \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty} \left(\arctan x\Big|_t^0\right) + \lim_{t\to+\infty} \left(\arctan x\Big|_0^t\right)$$

$$= 0 - \lim_{t\to-\infty} \arctan t + \lim_{t\to+\infty} \arctan t - 0.$$



$= \frac{\pi}{2} + \frac{\pi}{2} = \pi.$

Question 7: $\int 4x \cos(2-3x)\,dx$ $\qquad\qquad \int u\,dv = uv - \int v\,du.$

$u = 4x \qquad\qquad dv = \cos(2-3x)\,dx$

$du = 4\,dx \qquad\qquad v = -\frac{1}{3}\sin(2-3x)$

$$= -\frac{4x}{3}\sin(2-3x) - \int -\frac{1}{3}\cdot 4\sin(2-3x)\,dx$$

$$= -\frac{4x}{3}\sin(2-3x) + \frac{4}{3}\int \sin(2-3x)\,dx$$

$$= -\frac{4x}{3}\sin(2-3x) + \frac{4}{3}\cdot\frac{1}{3}\cos(2-3x) + C$$

$$= -\frac{4x}{3}\sin(2-3x) + \frac{4}{9}\cos(2-3x) + C.$$