Fundamental Theorem of Calculus

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AY 23/24 Trimester 1

CSD2201 / CSD2200

Differentiation

Integration

Sequences / series

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Welcome back!

General Information

- Mode of teaching: online lectures, on-campus tutorials
- Instructors:
  - Dr. Ronald Koh (Course Coordinator)
  - Dr. Lin Qinjie
  - Dr. Tay Bee Yen
- Class schedule:
  - Lectures: Mondays 0900 - 1130 HRS, **ONLINE**
  - Tutorials by sections (A) - (D), timings are 1400 - 1630:
    - (A) Fridays at ??, Instructor: Dr. Ronald Koh
    - (B) Tuesdays at Edison (SR2E), Instructor: Dr. Tay Bee Yen
    - (C) Thursdays at LT4B, Instructor: Dr. Lin Qinjie
    - (D) Fridays at ??, Instructor: Dr. Lin Qinjie

    } 2201
  - Tutorials by CSD2200 sections (A) and (B), timings also 1400 - 1630:
    - Fridays at ??, Instructor: Dr. Ronald Koh
    - Thursdays at LT4B, Instructor: Dr. Lin Qinjie

    } 2200
- Consultation: By appointment (Teams is preferred over email)

Course Assessment Tasks

Only non-graphic calculators are allowed

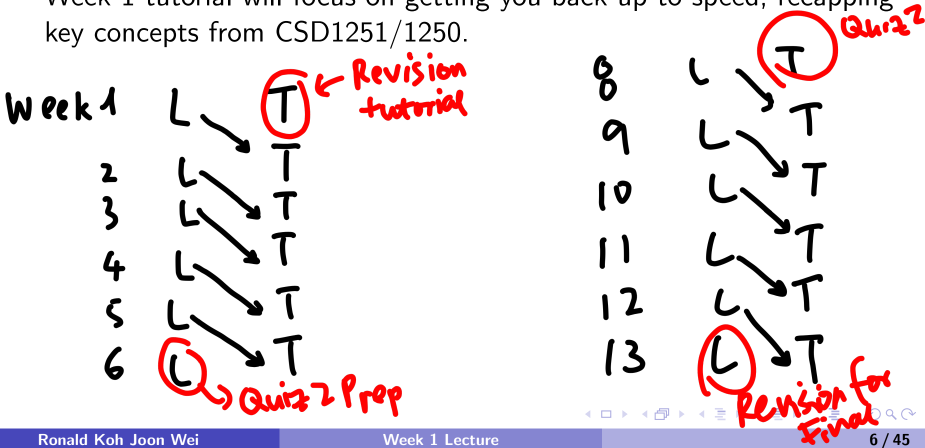
25% open-ended qn

Assessment Task	Weight	Tentative dates
Homeworks (5)	10%	Weeks 2, 3, 5, 9, 11
Quizzes (3)	60%	Weeks 4, 8, 12
Final Exam (1)	30%	Week 14

Like in CSD1251, a list of trigonometric formulae will be provided in quizzes and the final exam whenever required.

What's changed from CSD1251/1250?

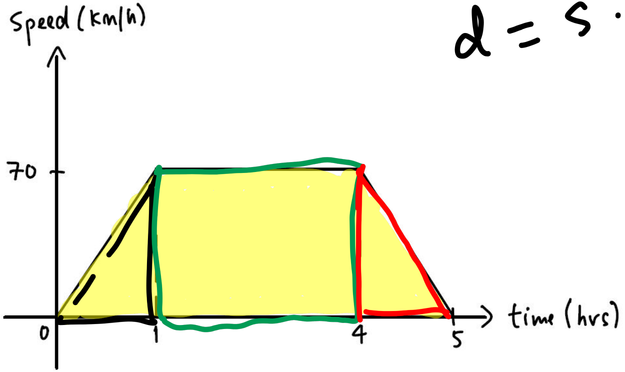
- Tutorial sessions cover the material from the previous week's lecture (ignoring the recess week, Week 8 will be Quiz 2). This gives you more time to revise and consolidate the information learnt in class. Week 1 tutorial will focus on getting you back up to speed; recapping key concepts from CSD1251/1250.



Exercise 1

Below is a speed-time graph of a car travelling on a road.

$d = s \cdot t$



Find the **total distance** covered by the car from $t = 0$ to $t = 5$ hours.

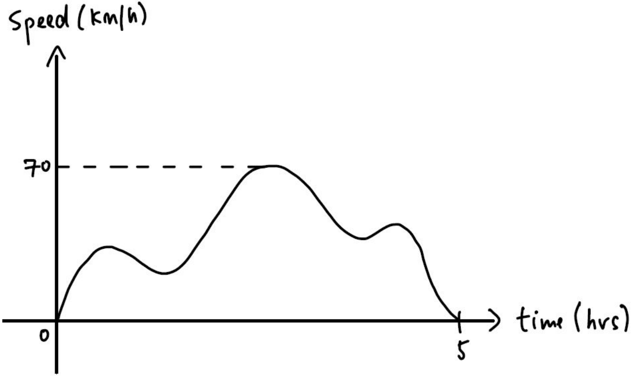
Exercise 1

$$\text{Total Distance} = \frac{1}{2}(70\times 1) + 70\times 3 + \frac{1}{2}(70\times 1)$$

$$= 280\,\text{km}$$

What about a generic speed function?

Realistically, a speed-time graph of a car travelling on a road is more like the graph below, with multiple accelerations and decelerations.



What is the total distance covered by the car?

What about a generic speed function?

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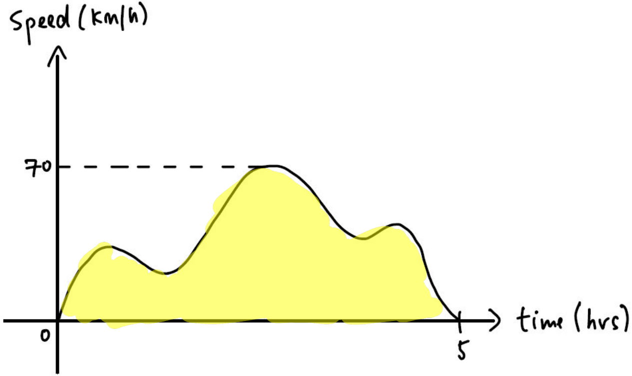
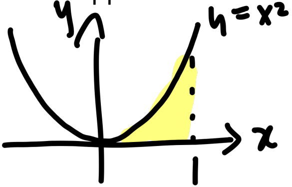
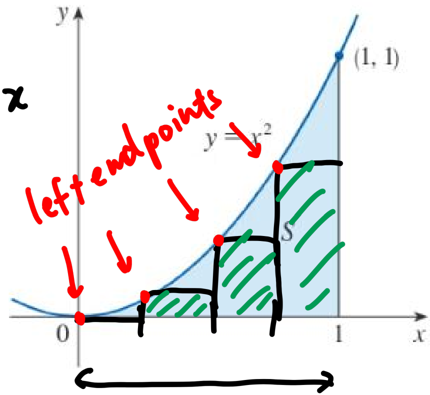


What is the total distance covered by the car? Area under this curve.

Using rectangles as approximation

Finding the area under the graph of a generic function is not straightforward; we use rectangles to approximate this area.
Let's approximate the area under $f(x) = x^2$ on $[0, 1]$ as an example.





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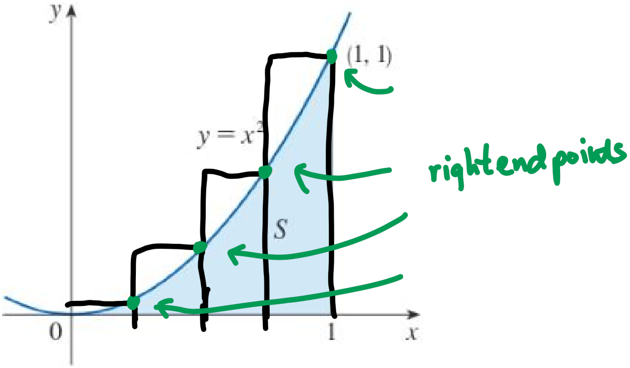
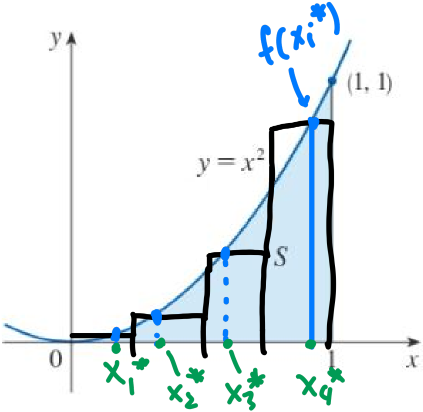


Using rectangles as approximation

Finding the area under the graph of a generic function is not straightforward; we use rectangles to approximate this area.
Let's approximate the area under $f(x) = x^2$ on $[0, 1]$ as an example.

Generically



$\frac{1}{3}$

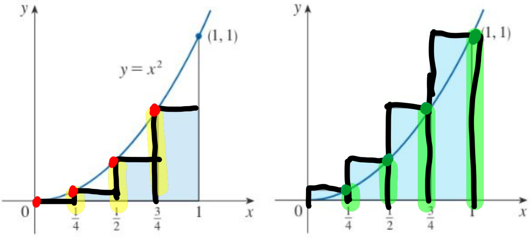
We will also use $n = 4$ rectangles to approximate the area under this graph. Divide $[0, 1]$ into $n = 4$ subintervals with equal length, each has length $\frac{1}{4}$. Each rectangle has a base length of $\frac{1}{4}$.

Since we already have the base length of each rectangle, we need the height. There are many ways to do this, but commonly, we use *left* and *right endpoints*:

6

$n = 4$

right endpoints



L_4 R_4

Exercise 2

$y = x^2$ $\quad f(x_i^*)$ left, right

Compute L_4 and R_4, the total area of the 4 rectangles using left (L) and right (R) endpoints respectively.

$$L_4 \text{ (left)} = 0 + \frac{1}{4} \times \left(\frac{1}{4}\right)^2 + \frac{1}{4} \times \left(\frac{1}{2}\right)^2 + \frac{1}{4} \times \left(\frac{3}{4}\right)^2$$

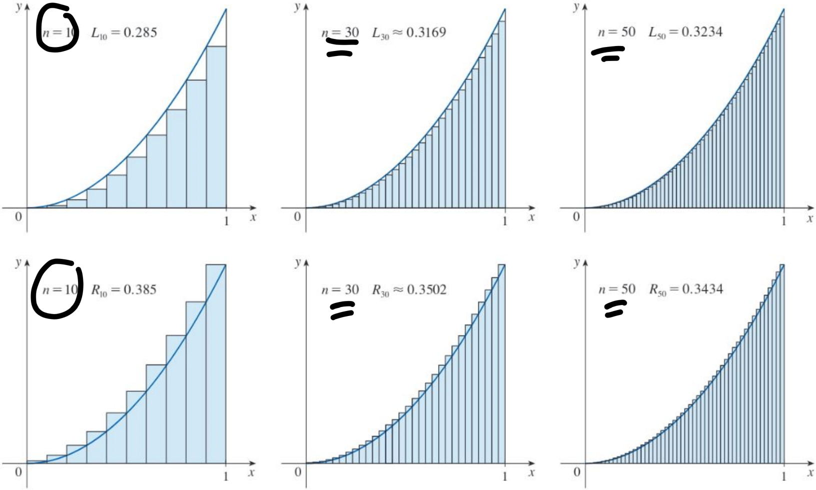
$$= \dots = \frac{7}{32} = 0.21875$$

$$R_4 \text{ (right)} = \frac{1}{4} \times \left(\frac{1}{4}\right)^2 + \frac{1}{4} \times \left(\frac{1}{2}\right)^2 + \frac{1}{4} \times \left(\frac{3}{4}\right)^2 + \frac{1}{4} \times 1^2$$

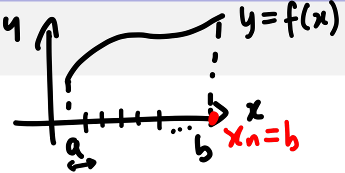
$$= \frac{15}{32} = 0.46875$$

What happens when we use more rectangles? $n \uparrow$

Notice we get closer and closer to the area under the graph! $n \to \infty$



Generic Riemann Sums



What happens in a general context, i.e. $y = f(x)$ on $[a, b]$? We follow these steps:

1. Divide $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. Let $a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b$ where

   $a \quad a + \frac{b-a}{n} \quad a + 2 \cdot \frac{b-a}{n}$

   $$\boxed{x_i = x_0 + i\Delta x.}$$

   $x_n = \overset{a}{x_0} + \cancel{n} \cdot \frac{b-a}{\cancel{n}}$

   $= a + (b-a)$

   $= b$

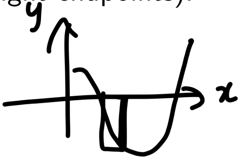
   The n subintervals are

   $$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$$

Generic Riemann Sums

$f(x_i^*)$ height of ith rectangle

2. Within each of these subintervals, choose a **sample point** x_i^* (previously, our sample points were either left or right endpoints).
   For $i = \{1, \ldots, n\}$,
   - Left endpoints $x_i^* = x_{i-1}$,
   - Right endpoints $x_i^* = x_i$.



3. Construct the *Riemann sum*:

   possibly negative

   $$S_n = \underline{f(x_1^*)}\Delta x + \underline{f(x_2^*)}\Delta x + \cdots \underline{f(x_n^*)}\Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

   n here represents the number of rectangles used.

4. When n gets larger and larger, we get the **$\boxed{\text{definite}}$ integral of f from a to b**:

   can be negative

   $$\boxed{\int_a^b f(x)\,dx} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

   background

Definition

The **definite integral of** f **from** a **to** b is the following expression

$$\int_a^b f(x)\,dx.$$

- The symbol $\int$ is an elongated S, because the definite integral is the limit of *sums*.
- The function $f(x)$ here is called the **integrand**.
- a and b are the **lower** and **upper limits** of integration.
- dx refers to the **variable of integration**, in this case, x.

function 'within' integral / function that we are integrating

a

b

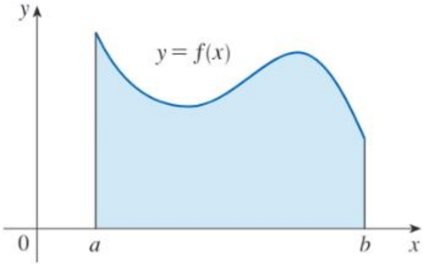
Meaning of the definite integral

above x-axis

(1) When $f(x) \geq 0$ on $[a, b]$, i.e. the graph of f is above the x-axis for $x \in [a, b]$, then

$$\int_a^b f(x)\,dx$$

measures the area under the graph of f from a to b.



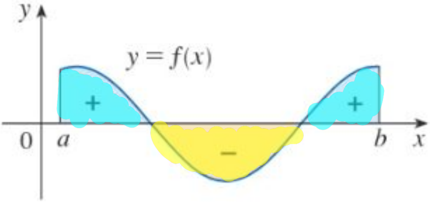
Meaning of the definite integral

(2) If f takes on both negative and positive values on $[a, b]$, then

$$\int_a^b f(x)\,dx$$

measures the **net area** $A_1 - A_2$ where

- A_1 is the total area of the region **above** the x-axis and **below the graph** of f, and
- A_2 is the total area of the region **below** the x-axis and **above the graph** of f.

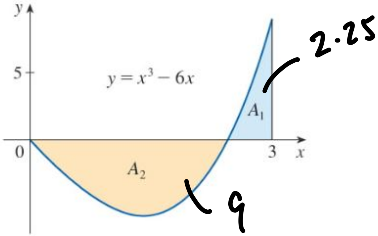


Example 2

The graph of a function $y = x^3 - 6x$ on the interval $[0, 3]$ can be found below. Given that $A_1 = 2.25$ and $A_2 = 9$, evaluate

$$\int_0^3 x^3 - 6x\, dx. = A_1 - A_2$$

$$= 2.25 - 9 = -6.75$$



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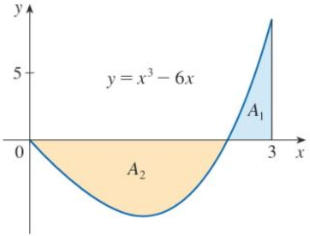
The graph of a function $y = x^3 - 6x$ on the interval $[0, 3]$ can be found below. Given that $A_1 = 2.25$ and $A_2 = 9$, evaluate

$$\int_0^3 x^3 - 6x \, dx.$$

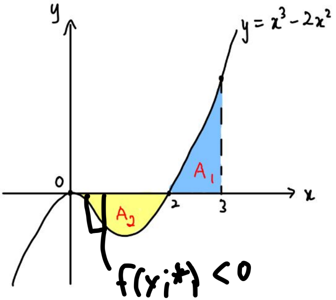


Answer: $\int_0^3 x^3 - 6x \, dx = A_1 - A_2 = 2.25 - 9 = -6.75.$

Exercise 3

The graph of a function $y = x^3 - 2x^2$ on the interval $[0, 3]$ can be found below. Given that $A_1 = \frac{43}{12}$ and $A_2 = \frac{4}{3}$, evaluate

$$\int_0^3 x^3 - 2x^2 \, dx.$$



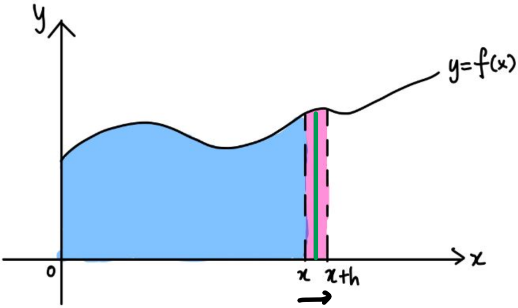
$$= A_1 - A_2 = \frac{43}{12} - \frac{4}{3}$$

$$= \frac{43}{12} - \frac{16}{12}$$

$$= \frac{27}{12}$$

Thought experiment

Let $A(x)$ be the area under the graph of $y = f$[redacted] from 0 to x, shaded in blue. Then for a small $h > 0$, $A(x+h)$ is the area from 0 to $x+h$.



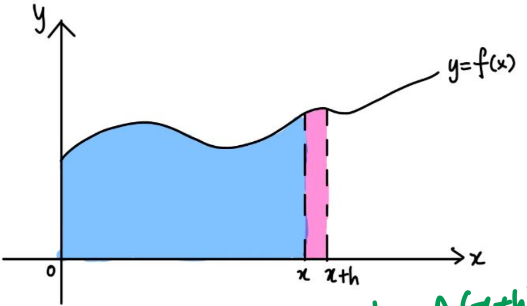
Thus, $A(x+h) - A(x)$ is the area shaded in pink. When h is very small, this area is approximately $f(x) \cdot h$. This implies that

height

$$\longrightarrow \quad A(x+h) - A(x) \approx f(x) \cdot h.$$

$h \to 0$ equality

Thought experiment



$$\lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

We rearrange this to get

$$\frac{A(x+h) - A(x)}{h} \approx f(x). = A'(x)$$

Letting $h \to 0$, we get equality (details omitted)

original $\quad$ integral

$$f(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = A'(x)$$

Conclusions drawn from the thought experiment

s = 0;
for x = 1 to x
s = s + x
end

s = 0;
x = pre defined value
for t = 1 to x
s = s + t
end

The area under the graph of $y = f(x)$ from 0 to x is

does not make sense $\int_0^x f(x)dx$

$$A(x) = \int_0^x f(t)\,dt.$$

The previous slide tells us that differentiating both sides of this equation gives us

$$A'(x) = f(x)$$

i.e. after first integrating f, followed by differentiating, we get back f.

This means that differentiation and integration are inverse processes.

The Fundamental Theorem of Calculus

$$F(x) \underset{\text{diff}}{\downarrow} \overset{\text{integrate}}{\uparrow} f(x)$$

Definition

A function F is an **antiderivative** of another function f if $F'(x) = f(x)$.

Theorem (Fundamental Theorem of Calculus 1 and 2)

FTC

If f is continuous on $[a, b]$, then

1. the function g defined by

FTC 1

$$g(x) = \int_a^x f(t)\,dt \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $\boxed{g'(x) = f(x).}$

2.

FTC 2

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

output is number

definite integral

where F is *any* antiderivative of f.

FTC 2 and indefinite integrals

output $\int f(x)\,dx$ is antiderivative function F

FTC 2 tells us that we can evaluate any definite integral of f just by finding *one* antiderivative of f. We introduce the concept of antiderivatives using **indefinite integrals**.

We have an important fact about antiderivatives:

Lemma

If F and G are antiderivatives of a function f, then F and G differ by a constant, i.e.

$$F(x) - G(x) = C$$

for a fixed constant C.

$f(x) = x$, $F(x) = \frac{x^2}{2} + 100$

FTC 2 and indefinite integrals

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for a fixed constant C.

$f(x) = x$ what differentiates into this?

$\frac{x^2}{2} - 2$

Indefinite integrals

An **indefinite integral** is a definite integral without the limits a and b:

$$\int f(x)\,dx.$$

This indefinite integral is usually used to denote ~~a~~ all antiderivatives for f, i.e.

$$\int f(x)\,dx = F(x) \quad \text{means} \quad F'(x) = f(x).$$

Example

1. $$\int x^4\,dx = \overbrace{\frac{x^5}{5} + \underbrace{C}_{\text{arbitrary constant}}}^{\text{all possible antiderivatives of } x^4} \quad \text{because} \quad \frac{d}{dx}\left(\frac{x^5}{5} + \underline{C}\right) = x^4.$$

2. $$\int \cos x\,dx = \sin x + C \quad \text{because} \quad \frac{d}{dx}(\sin x + C) = \cos x.$$

Table of antiderivatives/indefinite integrals (1)

Let c and k be constants.

integral operator is linear

- $\int c f(x)\, dx = c \int f(x)\, dx$
- $\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$
- $\int k\, dx = kx + C$
- $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
- $\int e^x\, dx = e^x + C$
- $\int \frac{1}{x}\, dx = \ln|x| + C$

Table of antiderivatives/indefinite integrals (2)

diff

- $\int \sin x \, dx = -\cos x + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \sec^2 x \, dx = \tan x + C$
- $\int \sec x \tan x \, dx = \sec x + C$
- $\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$
- $\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$

Note: These antiderivatives are only valid on an interval where the integrand is continuous!

Tutorial 1
implicit differentiation

Example 3 (Indefinite integrals)

Evaluate the following integrals.

1. $\int \underline{3x^2 + 4x + 2}\, dx$

$$= \int 3x^2\, dx + \int 4x\, dx + \int 2\, dx$$

$$= 3\int x^2\, dx + 4\int x\, dx + 2\int 1\, dx$$

$$= 3 \cdot \frac{x^3}{3} + 4 \cdot \frac{x^2}{2} + 2 \cdot x + C$$

$$= \underline{x^3 + 2x^2 + 2x + C}$$

diff

Example 4 (Indefinite integrals)

Evaluate the following integrals.

2. $\displaystyle\int 6x^{-1} + 4\sin x - \frac{7}{1+x^2}\,dx$

$$= 6\underline{\int x^{-1}\,dx} + 4\underline{\int \sin x\,dx} - 7\underline{\int \frac{1}{1+x^2}\,dx}$$

$$= 6\ln|x| + 4(-\cos x) - 7\tan^{-1}x + C$$

$$= 6\ln|x| - 4\cos x - 7\tan^{-1}x + C$$

Exercise 4

Evaluate the following integrals.

1. $\int 2x^4 + 3x^2 + 5x\, dx$ $= 2\int x^4 dx + 3\int x^2 dx + 5\int x\, dx$

   $= 2\cdot\frac{x^5}{5} + 3\frac{x^3}{3} + 5\frac{x^2}{2} + C$

   $= \frac{2x^5}{5} + x^3 + \frac{5x^2}{2} + C$

2. $\int \frac{3\tan x}{\sec x} - \frac{2}{\sqrt{1-x^2}}\, dx$

reduce

$\frac{\tan x}{\sec x} = \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} = \sin x$

Exercise 4

$$= \int 3\sin x - \frac{2}{\sqrt{1-x^2}}\,dx$$

$$= 3\int \sin x\,dx - 2\underline{\int \frac{1}{\sqrt{1-x^2}}\,dx}$$

$$= 3(-\cos x) - 2\sin^{-1}x + C$$

$$= -3\cos x - 2\sin^{-1}x + C$$

How to evaluate definite integrals

table

The most basic rule for evaluation of definite integral of a function f is the FTC2; we find an antiderivative F (see table of integrals on slide 29 and 30, just set $C = 0$ because the C cancels each other below):

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Rules governing definite integrals (1)

The rules governing definite integrals include

1. $\int_a^a f(x)\,dx = 0$ $\Delta x = 0$
2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
3. $\int_a^b c\,dx = c(b-a)$, where c is any constant

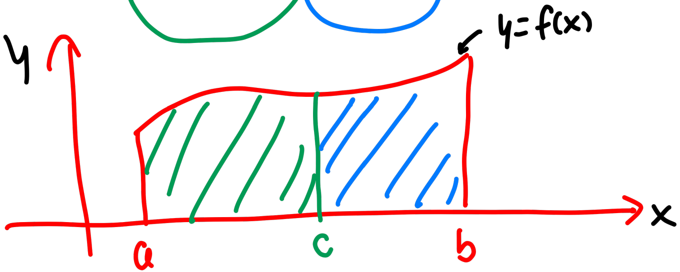
$\Delta x = \frac{b-a}{n}$ $\Delta x = \frac{a-b}{n}$

$-$

Rules governing definite integrals (2)

4. $\int_a^b cf(x) = c\int_a^b f(x)\,dx$, where c is any constant
5. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
6. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for any c

Same as indefinite



Example 5 (Definite integrals)

Evaluate the following integrals.

1. $\int_1^3 3x^2 + 4x + 2\,dx$

$F(x) = x^3 + 2x^2 + 2x + C$

$= \left[x^3 + 2x^2 + 2x + C \right]_1^3$

$F(3) - F(1)$

$= \left[3^3 + 2\cdot 3^2 + 2\cdot 3 + \underline{C} \right] - \left[1^3 + 2\cdot 1^2 + 2\cdot 1 + \underline{C} \right]$

$= (27 + 18 + 6) - (1 + 2 + 2) = 46$

Example 6 (Definite integrals)

Evaluate the following integrals.

2. $\displaystyle\int_0^1 x^2 + \frac{1}{\sqrt{1-x^2}}\,dx$

$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$y = \sin^{-1} x$

$\sin y = x = 1$

$= \left[\frac{x^3}{3} + \sin^{-1} x \right]_0^1$

$= \frac{1}{3} + \sin^{-1} 1 - \left(0 + \sin^{-1} 0\right)$

$= \frac{1}{3} + \frac{\pi}{2} - (0+0) = \frac{1}{3} + \frac{\pi}{2}$

Exercise 5

Evaluate the following integrals.

1. $\int_{-1}^{1} 4x^3 + 2x\,dx$ $= \left[x^4 + x^2\right]_{-1}^{1} = (1+1) - ((-1)^4 + (-1)^2)$
   $= 2 - 2 = 0$
2. $\int_{0}^{2} 2x^3 - 6x + \frac{3}{1+x^2}\,dx$

$= \left[\frac{x^4}{2} - 3x^2 + 3\tan^{-1} x\right]_0^2$

tan 0 = 0

$= \left(\frac{2^4}{2} - 3\cdot 2^2 + 3\tan^{-1} 2\right) - \left(0 - 0 + \underbrace{3\tan^{-1} 0}_{0}\right)$

no common angle,
just leave this as it is

$= 3\tan^{-1} 2 - 4.$

Common mistake

The output of an indefinite integral vs the output of a definite integral:

- $\int f(x)\,dx$ outputs **a (family of) functions**.
- $\int_a^b f(x)\,dx$ outputs a **number**.

Summary

- We can use rectangles of equal base length $\Delta x = \dfrac{b-a}{n}$ and **sample points** (most commonly left/right endpoints x_i^* to approximate the **net area** under the graph. This approximation is known as a **Riemann Sum**:

$$S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

- Letting n (the number of rectangles) $\to \infty$, we get the net area under the graph:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Summary

- If $f(x) \geq 0$, the definite integral of f from a to b measures the area under the graph and above the x-axis.
- If f takes both positive and negative values on $[a, b]$, the definite integral of f from a to b measures the net area $A_1 - A_2$ where
  - A_1 is the total area of the region **above** the x-axis and **below the graph** of f, and
  - A_2 is the total area of the region **below** the x-axis and **above the graph** of f.

Summary

- By FTC1, integration and differentiation are inverse processes; to evaluate integrals of a function f, one must know an **antiderivative** of f.
- We can find antiderivatives of standard functions using the table of indefinite integrals.
- By FTC2, we can evaluate definite integrals of f from a to b using an antiderivative F of f:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$