# Lecture 2: Lines and Planes

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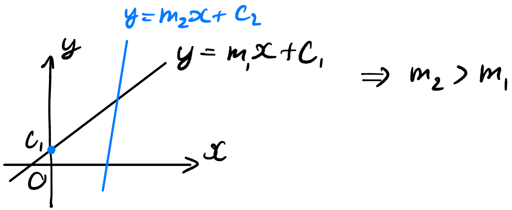
# Line equation (high school)

There are 2 types of line equation

1. Slanted lines

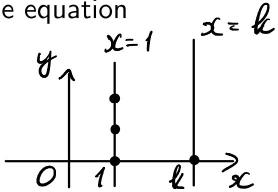
$$y = mx + c$$

with $m$ = **slope** of the line, and $c$ = **y-intercept**.



# Line equation (high school)

There are 2 types of line equation




2. Vertical lines

$$x = k,$$

with $k =$ **x-intercept**.

# Line through 2 points

## Theorem 1

Let $l$ be the line going through $A = (x_1, y_1)$ and $B = (x_2, y_2)$.

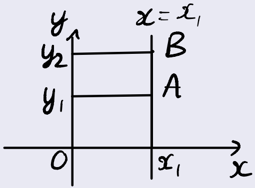
# Line through 2 points

**Theorem 1**

Let $l$ be the line going through $A = (x_1, y_1)$ and $B = (x_2, y_2)$.

(a) If $x_1 = x_2$, then $l$ the *vertical line*

$$x = x_1$$



# Line through 2 points

## Theorem 1

Let $l$ be the line going through $A = (x_1, y_1)$ and $B = (x_2, y_2)$.

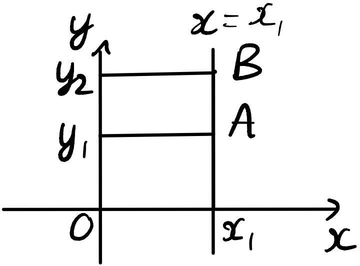
(b) If $x_1 \neq x_2$, then $l$ is the *slanted line*

this is the gradient

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

# Theorem 1 proof

(a) Since $A$ and $B$ have the same $x$-coordinate, the only line going through both both $A$ and $B$ is the vertical line $x = x_1$.



# Theorem 1 proof

- Since $x_1 \neq x_2$, the line going through $A$ and $B$ is a slanted line

$$y = mx + c \tag{1}$$

# Theorem 1 proof

- Since $x_1 \neq x_2$, the line going through $A$ and $B$ is a slanted line

$$y = mx + c \tag{1}$$

- Both $A$ and $B$ are on the line, their coordinates both satisfy (1)

$$\begin{cases} mx_1 + c = y_1 & (1) \\ mx_2 + c = y_2 & (2) \end{cases} \Rightarrow \begin{cases} m = \dfrac{y_2 - y_1}{x_2 - x_1} \\ c = y_1 - \dfrac{y_2 - y_1}{x_2 - x_1} x_1 \end{cases}$$

$$(2) - (1) \Rightarrow m(x_2 - x_1) = y_2 - y_1$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

## Theorem 1 proof

- Since $x_1 \neq x_2$, the line going through $A$ and $B$ is a slanted line

$$y = mx + c \tag{1}$$

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- Conclusion

$$y = \frac{y_2 - y_1}{x_2 - x_1} x + y_1 - \frac{y_2 - y_1}{x_2 - x_1} x_1 \Leftrightarrow y - y_1 == \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

# Example 1

Find the equation of the line going through two points $P$ and $Q$. In each case, write out 2 other points (other than $P, Q$) on the line.

(a) $P = (0, 0)$, $Q = (3, 5)$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

The line going through P & Q is the slanted line

$$y - 0 = \frac{5-0}{3-0}(x - 0)$$

$$y = \frac{5}{3}x$$

2 other points on the line are
$x = 1 \Rightarrow (1, \frac{5}{3})$
$x = 6 \Rightarrow (6, 10)$

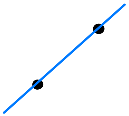
# Example 1

(b) $P = (1, -1)$, $Q = (\pi, 1)$

Exercise!

# Determination of a line in $\mathbb{R}^2$

A line can be determined by one of the following means:

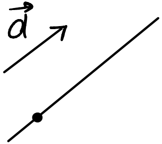


- by 2 points it contain, or
- by **a point** and **an orientation**, which can be

## Determination of a line in $\mathbb{R}^2$

A line can be determined by one of the following means:

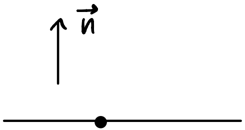
- by **a point** and **an orientation**, which can be
  1. a direction vector $\vec{d}$ (**parallel** to the line), or



# Determination of a line in $\mathbb{R}^2$

A line can be determined by one of the following means:

- by **a point** and **an orientation**, which can be
  2. a **normal vector** $\vec{n}$ (**perpendicular** to the line)



# Lines through a point and a direction vector

## Theorem 2

Let $P = (x_0, y_0)$ be a point and let $\vec{d} = \begin{bmatrix} a \\ b \end{bmatrix}$ be a vector in $\mathbb{R}^2$.

(a) The line going through $P$ with direction $\vec{v}$ has equation

$$(x, y) = P + t\vec{d} = (x_0, y_0) + t\underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}$$

vector equation

for some parameter $t$.

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \end{cases}$$

# Lines through a point and a direction vector

## Theorem 2

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(a) The line going through $P$ with direction $\vec{v}$ has equation

$$(x, y) = P + t\vec{v} = (x_0, y_0) + t \begin{bmatrix} a \\ b \end{bmatrix}$$

(parameter t)

(b) In particular, any point on the line has coordinates

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \end{cases} \quad \text{for some } t \in \mathbb{R}$$

parametric equation

## Comments

- The equation

$$(x, y) = P + t\vec{d}$$

P = point on the line

$\vec{d}$ = direction vector

is called the **vector equation** of the line.

## Comments

- The equation

$$(x, y) = P + t\vec{v}$$

$$(x, y) = (x_0, y_0) + t\begin{pmatrix} a \\ b \end{pmatrix}$$

is called the **vector equation** of the line.

- The equation

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \end{cases}$$

is called the **parametric equation** of the line.

- Each value of $t$ gives one point $(x, y)$ on the line.

## Theorem 2 proof

Assume the line $l$ goes through $P = (x_0, y_0)$ and has direction $\vec{d} = \begin{bmatrix} a \\ b \end{bmatrix}$

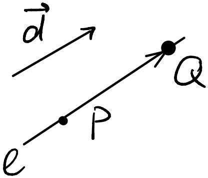
Let $Q$ be any point on $l$.

Then $\vec{PQ}$ is parallel to $\vec{d}$. So

$$\vec{PQ} = t\vec{d}$$

$$Q - P = t\vec{d}$$

$$Q = P + t\vec{d}$$




$\therefore$ Any point $Q = (x, y)$ on $l$ satisfies

$$(x, y) = P + t\vec{d}$$

## Example 2

Find both the *vector equation* and the *parametric equation* of the line going through the point $P$ and having direction $\vec{v}$.
In each case, write out 2 other points (other than $P$) on the line.

(a) $P = (0,0)$ and $\vec{v} = \begin{bmatrix} \pi \\ \sqrt{2} \end{bmatrix}$

Vector equation

$$(x,y) = (0,0) + t\begin{pmatrix} \pi \\ \sqrt{2} \end{pmatrix}$$

Parametric equation

$$\begin{cases} x = 0 + t\pi = t\pi \\ y = 0 + \sqrt{2}t = \sqrt{2}t \end{cases}$$

2 other points on the line are

$$\begin{cases} t = 1 \Rightarrow (x,y) = (\pi, \sqrt{2}) \\ t = 2 \Rightarrow (x,y) = (2\pi, 2\sqrt{2}) \end{cases}$$

Vector equation:

$$(x,y) = P + t\vec{v} = (x_0, y_0) + t\begin{pmatrix} a \\ b \end{pmatrix}$$

Parametric equation:

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \end{cases}$$

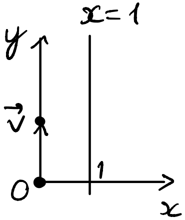
" { " : and

# Example 2

(b) $P = (1, -2)$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Vector equation

$$(x, y) = P + t\vec{v} = (1, -2) + \underbrace{t \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\begin{pmatrix} 0 \\ t \end{pmatrix}}$$




(Vertical line)

Parametric equation

$$\begin{cases} x = 1 + 0 = 1 \\ y = -2 + t \end{cases}$$

2 other points on the line are

$t = 1 \Rightarrow (x, y) = (1, -1)$

$t = 2 \Rightarrow (x, y) = (1, 0)$

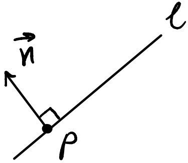
# Lines through a point and a normal vector

**Theorem 3**

Let $P = (x_0, y_0)$ be a point and let $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ be a vector in $\mathbb{R}^2$. The line which goes through $P$ and has normal vector $\vec{n}$ has equation

$$a(x - x_0) + b(y - y_0) = 0$$

normal vector is perpendicular to the line



# Lines through a point and a normal vector

## Theorem 3

Let $P = (x_0, y_0)$ be a point and let $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ be a vector in $\mathbb{R}^2$. The line which goes through $P$ and has normal vector $\vec{n}$ has equation

$$a(x - x_0) + b(y - y_0) = 0$$

$ax + by - ax_0 - by_0 = 0$

- Putting $c = -ax_0 - by_0$, we obtain the **general equation** of the line

$$ax + by + c = 0$$

right side = 0

- Putting $d = ax_0 + by_0$, we obtain the **normal equation** of the line

$$ax + by = d$$

left side contains x, y
right side is some number

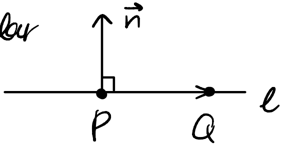
# Theorem 2 proof

Assume the line $l$ goes through $P = (x_0, y_0)$ and has normal $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$

Let $Q = (x, y)$ be any point on $\ell$.

The vector $\vec{PQ} = \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$ is perpendicular

to $\vec{n} = \begin{pmatrix} a \\ b \end{pmatrix}$. So




$\vec{PQ} \cdot \vec{n} = 0 \Leftrightarrow a(x - x_0) + b(y - y_0) = 0$.

# Example 3

Find the **normal equation** of line going through $P$ and having normal vector $\vec{n}$. In each case, find 2 other points (other than $P$) on the line.

(a) $P = (0,0)$ and $\vec{n} = \begin{bmatrix} \pi \\ \sqrt{2} \end{bmatrix}$

$\left(y = -\frac{\pi}{\sqrt{2}}x\right)$

normal equation

$ax + by = c$

$P = (x_0, y_0)$ & $\vec{n} = \begin{pmatrix} a \\ b \end{pmatrix}$

$a(x - x_0) + b(y - y_0) = 0$

$\pi(x - 0) + \sqrt{2}(y - 0) = 0 \iff \pi x + \sqrt{2}y = 0$

2 other points $\begin{cases} x = 1 \Rightarrow (x,y) = \left(1, -\frac{\pi}{\sqrt{2}}\right) \\ x = \sqrt{2} \Rightarrow (x,y) = (\sqrt{2}, -\pi) \end{cases}$

(b) $P = (1,2)$ and $\vec{n} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$2(x-1) + 5(y-2) = 0 \iff 2x - 2 + 5y - 10 = 0 \iff 2x + 5y = 12$

2 other points $\begin{cases} x = 0 \Rightarrow (x,y) = \left(0, \frac{12}{5}\right) \\ x = 6 \Rightarrow (x,y) = (6,0) \end{cases}$

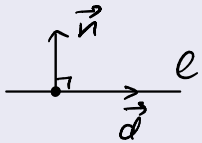
# Normal vector and direction vector of a line

## Lemma 1

A line with general equation $ax + by + c = 0$ (or normal equation $ax + by = -c$) has

(a) a normal vector

$$\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$$




(b) and a direction vector

$$\vec{d} = \begin{bmatrix} -b \\ a \end{bmatrix}$$

# Example 4

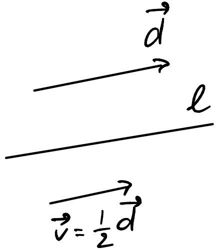
Find the direction vector $\vec{d}$ and the normal vector $\vec{n}$ in following cases. In each case, check that $\vec{d} \perp \vec{n}$.

(a) $x - y + 3 = 0$ $\quad a = 1,\ b = -1$

$$\vec{n} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},\quad \vec{d} = \begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\quad \vec{n}\cdot\vec{d} = 0 \Rightarrow \vec{n} \perp \vec{d}$$

(c) $x - y = -1$ $\quad a = 1,\ b = -1$

$$\vec{n} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},\quad \vec{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\quad \vec{n}\cdot\vec{d} = 0$$




(b) $3y + 1 = 0$ $\quad a = 0,\ b = 3$

$$\underbrace{3y+1}_{}$$

$$0x + 3y + 1 = 0$$

$$\vec{n} = \begin{pmatrix} 0 \\ 3 \end{pmatrix},\quad \vec{d} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

# Summary on types of line equations in $\mathbb{R}^2$

- Vector equation and parametric equation → obtained by { a point, a direction vec.

  1. Vector equation $(x, y) = P + t\vec{v} = (x_0, y_0) + t \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$
  2. Parametric equation $\begin{cases} x = x_0 + tv_1 \\ y = y_0 + tv_2 \end{cases}$

- General equation and normal equation → obtained by { a point, a normal vector

  3. General equation $ax + by + c = 0$
  4. Normal equation $ax + by = d$

# Summary on line equations in $\mathbb{R}^2$

A line can be determined by **a point** and **an orientation**

1. Line through point $P = (x_0, y_0)$ with direction vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$
   - Vector equation
     $$(x, y) = P + t\vec{v} = (x_0, y_0) + t \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
   - Parametric equation $\begin{cases} x = x_0 + tv_1 \\ y = y_0 + tv_2 \end{cases}$

# Summary on line equations in $\mathbb{R}^2$

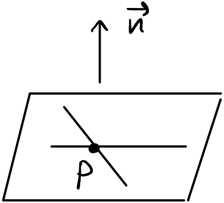
A line can be determined by **a point** and **an orientation**

2. Line through point $P = (x_0, y_0)$ with normal vector $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$

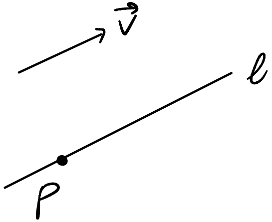
$$a(x - x_0) + b(y - y_0) = 0$$

# Determination of lines in $\mathbb{R}^3$

- Unlike in $\mathbb{R}^2$, there is *no normal vector* for a line in $\mathbb{R}^3$.
  Can you see that the *geometric object* passing through a point $P$ and perpendicular to $\vec{n}$ is a plane?



# Determination of lines in $\mathbb{R}^3$




- A line in $\mathbb{R}^3$ is determined by

$$\text{A point } P = (x_0, y_0, z_0) \text{ and a direction vector } \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

# Lines through a point and a direction vector

## Theorem 4

Let $P = (x_0, y_0, z_0)$ be a point and let $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a vector in $\mathbb{R}^3$.

(a) The line through $P$ with direction $\vec{v}$ has **vector equation**

$$(x, y, z) = P + t\vec{v}$$

# Lines through a point and a direction vector

## Theorem 4

Let $P = (x_0, y_0, z_0)$ be a point and let $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a vector in $\mathbb{R}^3$.

(a) The line through $P$ with direction $\vec{v}$ has **vector equation**

$$(x, y, z) = P + t\vec{v}$$
$$= (x_0, y_0, z_0) + t\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

(b) In particular, any point on the line has coordinates $\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + ct \end{cases}$

which is called the **parametric equation** of the line.

## Theorem 4 proof

Assume $l$ goes through $P = (x_0, y_0, z_0)$ and has direction $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

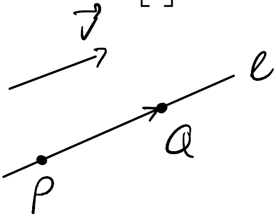
Let Q be any point on P. Then

$\vec{PQ} \parallel \vec{v}$

$Q - P = \vec{PQ} = t\vec{v}$

$Q = P + t\vec{v}$

$(x, y, z) = (x_0, y_0, z_0) + t\begin{pmatrix} a \\ b \\ c \end{pmatrix}$



# Example 5

Find the line $l$ going through a point $P$ and having direction $\vec{v}$. List 2 other points (other than $P$) on the line

$$P = (1, 2, 1) \text{ and } \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Vector equation

$$(x, y, z) = (1, 2, 1) + t \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Parametric equation

$$\begin{cases} x = 1 + t \\ y = 2 \\ z = 1 + 2t \end{cases}$$

$$t = 1 \Rightarrow (x, y, z) = (2, 2, 3)$$

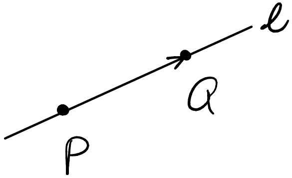
$$t = 2 \Rightarrow (x, y, z) = (3, 2, 5)$$

# Exercise 1

Find both the *vector equation* and the *parametric equation* of the line $l$ going through $P, Q$. List 2 other points (other than $P$ and $Q$) on the line.

(a) $P = (1, 2, 1)$ and $Q = (36, 7, 9)$

The line goes through $P = (1,2,1)$ & has direction $\vec{PQ} = \begin{pmatrix} 35 \\ 5 \\ 8 \end{pmatrix}$.




Vector equation

$$(x, y, z) = (1, 2, 1) + t \begin{pmatrix} 35 \\ 5 \\ 8 \end{pmatrix}$$

Parametric equation

$$\begin{cases} x = 1 + 35t \\ y = 2 + 5t \\ z = 1 + 8t \end{cases}$$

## Exercise 1

(b) $P = (0, 0, 1)$ and $Q = (\pi, 1, 4)$.

Exercise!

# Summary on lines in $\mathbb{R}^3$

The line through $P = (x_0, y_0, z_0)$ with direction $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ has

(a) Vector equation

$$(x, y, z) = P + t\vec{v} = (x_0, y_0, z_0) + t \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

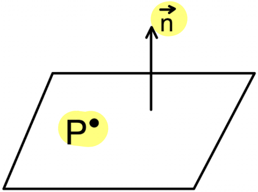
(b) Parametric equation

$$\begin{cases} x = x_0 + tv_1 \\ y = y_0 + tv_2 \\ z = z_0 + tv_3 \end{cases}$$

# Determination of planes

A plane can be determined by **a point** and **an orientation** which can be

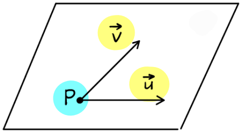
1. a vector perpendicular to the plane, called **normal vector**,



# Determination of planes

A plane can be determined by **a point** and **an orientation** which can be

2. or a pair of non-parallel vectors, called **direction vectors**.

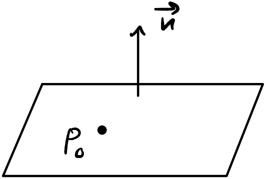


# Plane by a point and a normal vector

**Theorem 5**

A plane passing through the point $P_0 = (x_0, y_0, z_0)$ and having normal vector $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$



# Plane by a point and a normal vector

## Theorem 5

A plane passing through the point $P_0 = (x_0, y_0, z_0)$ and having normal vector $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has equation

$ax+by+cz-ax_0-by_0-cz_0=0$

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

- Putting $d = -ax_0 - by_0 - cz_0$, we obtain the **general equation**

$$ax+by+cz+d=0$$

- Putting $e = ax_0 + by_0 + cz_0$, we obtain the **normal equation**

$$ax+by+cz=e$$

# Theorem 1 proof

Assume the plane contains $P_0 = (x_0, y_0, z_0)$ and has normal $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

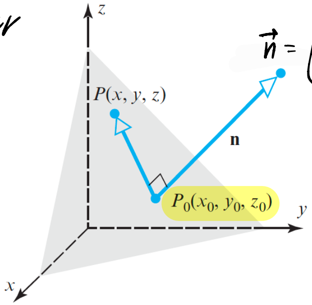
Let $P = (x, y, z)$ be any point on the plane.

Then $\overrightarrow{P_0P} = \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix}$ is perpendicular

to $\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

$\overrightarrow{P_0P} \cdot \vec{n} = 0 \Leftrightarrow a(x-x_0) + b(y-y_0) +$

$+ c(z - z_0) = 0$




$\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

# Example 6

Find the plane passing through $P_0$ and having normal vector $\vec{n}$ with

(a) $P_0 = (1, 2, 2)$, $\vec{n} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$ (b) $P_0 = (1, 2, 3)$, $\vec{n} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$0(x-1)+3(y-2)+4(z-2)=0$

$3y+4z-14=0$ ; general equation

$3y+4z=14$ ; normal equation

$1(x-1)+(-1)(y-2)+2(z-3)=0$

$x-y+2z-5=0$ ; general equation

$x-y+2z=5$ ; normal equation

## Normal vector of a plane

**Lemma 2**

Any plane with general equation $ax + by + cz + d = 0$ (or normal equation $ax + by + cz = -d$) has normal vector

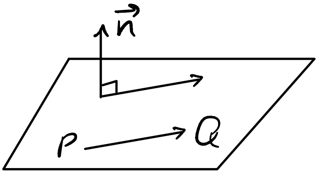
$$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

# Example 7

In each following case, find the normal $\vec{n}$ of the plane with given equation. Further, find 2 points $P, Q$ on the plane and check that $\overrightarrow{PQ} \perp \vec{n}$.

(a) $x - y + 2z = 0$ $\quad a = 1, b = -1, c = 2$

$$\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$




$$\boxed{x = y - 2z}$$

$$y = z = 1 \Rightarrow P = (-1, 1, 1)$$

$$y = 10, z = 1 \Rightarrow Q = (8, 10, 1)$$

$$\Rightarrow \overrightarrow{PQ} = \begin{pmatrix} 9 \\ 9 \\ 0 \end{pmatrix}$$

$$\overrightarrow{PQ} \cdot \vec{n} = 9 \cdot 1 + 9 \cdot (-1) + 0 \cdot 2 = 0.$$

# Example 7

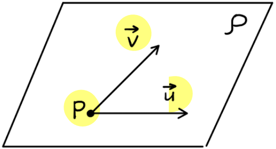
(b) $x - y = 2$ $\quad a = 1 \quad b = -1 \quad c = 0$

$$\vec{n} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Exercise !

# Direction vectors of a plane

- A pair of *non-parallel vectors* $\vec{u}, \vec{v}$ is called **direction vectors** of a plane $\mathcal{P}$ if both $\vec{u}$ and $\vec{v}$ are parallel to $\mathcal{P}$.




- A plane is uniquely determined if we know *a point* $P$ it contains and a pair of *direction vectors* $\vec{u}$ and $\vec{v}$

# Planes by a point and direction vectors

## Theorem 6

The plane through $P_0 = (x_0, y_0, z_0)$ with direction $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ has

(a) vector equation

$$(x, y, z) = P_0 + s\vec{u} + t\vec{v} = (x_0, y_0, z_0) + s \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + t \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

for some $s, t \in \mathbb{R}$.

# Planes by a point and direction vectors

## Theorem 6

The plane through $P_0 = (x_0, y_0, z_0)$ with direction $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ has

(a) vector equation

$$(x, y, z) = P_0 + s\vec{u} + t\vec{v} = (x_0, y_0, z_0) + s \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + t \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

(b) and parametric equation

parameters $s, t$.

$$\begin{cases} x = x_0 + su_1 + tv_1 \\ y = y_0 + su_2 + tv_2 \\ z = z_0 + su_3 + tv_3 \end{cases}$$

# Example 8

In each of the following cases, find both the *vector equation* and the *parametric equation* of the plane

(a) Through $P = (0, 0, 0)$ with direction $\vec{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$

Vector equation

$$(x, y, z) = P + s\vec{u} + t\vec{v} = (0,0,0) + \underbrace{s\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix}} + \underbrace{t\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}}_{\begin{pmatrix} 2t \\ t \\ 5t \end{pmatrix}}$$

Parametric equation

$$\begin{cases} x = 0 + 0 + 2t = 2t \\ y = 0 + 0 + t = t \\ z = 0 + s + 5t = s + 5t \end{cases}$$

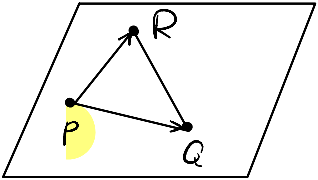
2 other points
- $s = 1, t = 0 \Rightarrow (x, y, z) = (0, 0, 1)$
- $s = t = 1 \Rightarrow (x, y, z) = (2, 1, 6)$

# Example 8

(b) Passing through $P = (1, -1, 2)$, $Q = (0, 1, 5)$, $R = (2, 3, -1)$

The plane passes through $P$ and has direction vectors

$$\vec{PQ} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \text{ \& } \vec{PR} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$




Vector equation:

$$(x, y, z) = P + s\vec{PQ} + t\vec{PR} = (1, -1, 2) + s\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

Parametric equation:

$$\begin{cases} x = 1 - s + t \\ y = -1 + 2s + 4t \\ z = 2 + 3s - 3t \end{cases}$$

## Summary on types of plane equations

- General equation and normal equation
  1. General equation $ax + by + cz + d = 0$
  2. Normal equation $ax + by + cz = d$

## Summary on types of plane equations

- General equation and normal equation
  a point
  a normal vector
  1. General equation $ax + by + cz + d = 0$
  2. Normal equation $ax + by + cz = d$
- Vector equation and parametric equation
  a point
  direction vectors $\vec{u}, \vec{v}$
  3. Vector equation $(x, y, z) = P_0 + s\vec{u} + t\vec{v}$
  4. Parametric equation $\begin{cases} x = x_0 + su_1 + tv_1 \\ y = y_0 + su_2 + tv_2 \\ z = z_0 + su_3 + tv_3 \end{cases}$

## Summary on plane equations

- The plane through $P(x_0, y_0, z_0)$ with normal $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

# Summary on plane equations

- The plane through $P(x_0, y_0, z_0)$ with normal $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

- The plane through $P(x_0, y_0, z_0)$ with direction vectors $\vec{u}$ and $\vec{v}$ has
  1. vector equation

$$(x, y, z) = P + s\vec{u} + t\vec{v}$$

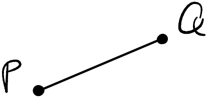
  2. parametric equation

$$\begin{cases} x = x_0 + su_1 + tv_1 \\ y = y_0 + su_2 + tv_2 \\ z = z_0 + su_3 + tv_3 \end{cases}$$

# Distances

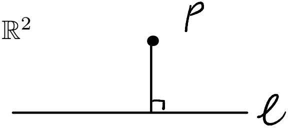
We will discuss three types of distances

1. Distance between 2 points

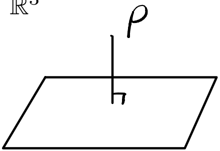



$d(P,Q) = \|\vec{PQ}\|$

2. Distance from a point to a line in $\mathbb{R}^2$




3. Distance from a point to a plane in $\mathbb{R}^3$



# Point-line distance in $\mathbb{R}^2$

## Theorem 7

The distance from the point $P_0 = (x_0, y_0)$ to the line $l : ax + by + c = 0$ is

sub $(x_0, y_0)$ into $l$

$$d(P_0, l) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

$\vec{n} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \|\vec{n}\| = \sqrt{a^2+b^2}$

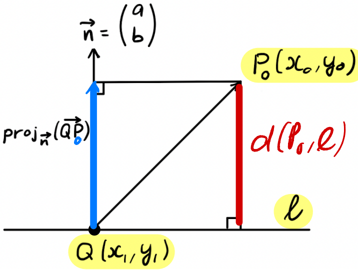
# Theorem 3 proof (sketch)

Let $Q$ be a point on $l$ : $ax + by + c = 0$.

Let $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ be a normal vector of $l$ starting at $Q$.

Observation

$d(P_0, \ell) = \text{length of } \text{proj}_{\vec{n}}(\vec{QP_0})$



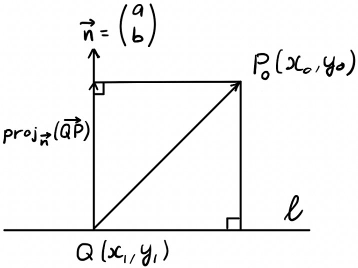
# Theorem 3 proof (sketch)

Let $Q$ be a point on $l : ax + by + c = 0$.

Let $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ be a normal vector of $l$ starting at $Q$.

- Observation

$$d(P_0, l) = ||\,\mathrm{proj}_{\vec{n}}(\overrightarrow{QP_0})||$$



# Theorem 3 proof (sketch)

Let $Q$ be a point on $l : ax + by + c = 0$.

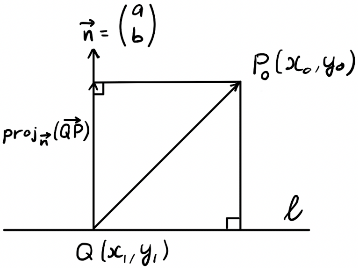
Let $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ be a normal vector of $l$ starting at $Q$.

- Observation

$$d(P_0, l) = ||\operatorname{proj}_{\vec{n}}(\overrightarrow{QP_0})||$$

- $\operatorname{proj}_{\vec{n}}(\overrightarrow{QP_0}) = \dfrac{\overrightarrow{QP_0} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n}$ implies

$$||\operatorname{proj}_{\vec{n}}(\overrightarrow{QP_0})|| = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$



# Remark on Theorem 3

- To use $d(P_0, l) = \dfrac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$, we need the *general equation* of $l$

$$ax + by + c = 0$$

- This can be achieved by a point $P = (x_1, y_1)$ and a normal $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$

$$a(x - x_1) + b(y - y_1) = 0$$

# Example 9

Find the distance from $P$ to the line $l$ in the following cases

$$d(P,l) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}}$$

(a) $P = (1, 2)$, $l : 3x + 4y - 5 = 0$.

$$d(P,l) = \frac{|3\cdot 1 + 4\cdot 2 - 5|}{\sqrt{3^2+4^2}} = \frac{|6|}{5} = \frac{6}{5}$$

(b) $P = (2, -1)$, $l$ passes through $A(0, 1)$ and has normal $\vec{n} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$l$ has general equation

$$1(x-0) + (-1)(y-1) = 0 \iff x - y + 1 = 0$$

The distance is

$$d(P,l) = \frac{|2-(-1)+1|}{\sqrt{1^2+(-1)^2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

# Example 9

(c) $P = (0, -1)$, $l$ passes through $A(0, 1)$ and has direction $\vec{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

Exercise !

# Distance from a point to a plane in $\mathbb{R}^3$

## Theorem 8

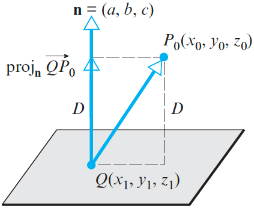
The distance from the point $P_0 = (x_0, y_0, z_0)$ to the plane $\alpha : ax + by + cz + d = 0$ is

$$d(P_0, \alpha) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

# Theorem 4 proof (sketch)

Let $Q(x_1, y_1, z_1)$ be any point on $\mathcal{P}$.

Let $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a normal vector that is positioned at $Q$.




Distance from $P_0$ to plane.

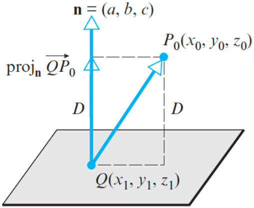
# Theorem 4 proof (sketch)

Let $Q(x_1, y_1, z_1)$ be any point on $\mathcal{P}$.

Let $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a normal vector that is positioned at $Q$.

- Observation

$$d(P_0, \alpha) = ||\operatorname{proj}_{\vec{n}}(\overrightarrow{QP_0})||$$




Distance from $P_0$ to plane.

# Theorem 4 proof (sketch)

Let $Q(x_1, y_1, z_1)$ be any point on $\mathcal{P}$.

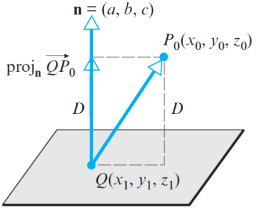
Let $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a normal vector that is positioned at $Q$.

- Observation

$$d(P_0, \alpha) = ||\,\mathrm{proj}_{\vec{n}}(\overrightarrow{QP_0})||$$

- $\mathrm{proj}_{\vec{n}}(\overrightarrow{QP_0}) = \dfrac{\overrightarrow{QP_0} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n}$ implies

$$||\,\mathrm{proj}_{\vec{n}}(\overrightarrow{QP_0})|| = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$



Distance from $P_0$ to plane.

## Remark

- To use $d(P_0, \mathcal{P}) = \dfrac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$, we need general equation

$$ax + by + cz + d = 0$$

- This is achieved by a point $Q = (x_1, y_1, z_1)$ and a normal $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

# Example 10

Find the distance from $A(1,2,0)$ to the plane $\mathcal{P}$ in the following cases.

(a) $\mathcal{P} : 2x + y - z + 1 = 0$

$$d(A,P) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$$

$$d(A,P) = \frac{|2\cdot 1 + 2 - 0 + 1|}{\sqrt{2^2+1^2+(-1)^2}} = \frac{5}{\sqrt{6}}$$

(b) $\mathcal{P}$ passes through $(2,1,-1)$ and has normal vector $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

The general equation of P is

$$1(x-2) + 2(y-1) + 3(z+1) = 0$$

$$x + 2y + 3z - 1 = 0$$

$$d(A,P) = \frac{|1 + 2\cdot 2 + 3\cdot 0 - 1|}{\sqrt{1^2+2^2+3^2}} = \frac{|4|}{\sqrt{14}} = \frac{4}{\sqrt{14}}$$

## Summary on distances

- In $\mathbb{R}^2$, the distance from $P_0(x_0, y_0)$ to the line $l : ax + by + c = 0$ is

$$d(P_0, l) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

- In $\mathbb{R}^3$, the distance from $P(x_0, y_0, z_0)$ to the plane $\mathcal{P} : ax + by + cz + d = 0$ is

$$d(P_0, \mathcal{P}) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$