Lecture 1: Review on high school geometry

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Course Materials

1. **Lecture handouts**: notes by Professor Boerkoel (Redmond)
2. **Lecture slides** provided by myself.
3. Recommended **textbooks**
   - Practical Linear Algebra: A Geometry Toolbox, 3rd edition by G. Farin and D. Hansford
   - Elementary Linear Algebra with Supplemental Applications, 10th edition by H. Anton & C. Rorres.

Course content

1. Chapter 1: Euclidean Space
2. Chapter 2: Matrices and Determinants
3. Chapter 3: Linear Transformations
4. Chapter 4: Affine Transformations

Assessment tasks (tentative)

Assessment Task	**Weight**	**Tentative date**
Homework assignments (5 or 6)	10%	Weeks 1-13
5 Quizzes	30%	Weeks 3,5,9,11,13
1 Midterm test	30%	Week 6
1 Final test	30%	Week 14

Course structure

- **Online lecture** every Tuesday 3-6pm
- **Physical tutorial** every Thursday
  - Groups A,B,C: taught by myself
  - Groups D,E: taught by Rosa
- **Extra tuitions** (starting from week 4) are provided to weak students

What grades can I expect?

- To pass, you need to
  - Score an overall grade D or above
  - *Tips: Do all homework assignments and do not skip exams*
- To have a higher grade?

Attendance policy

- Student $\geq$ 15 minutes late to class will be marked as absent.
- Student may not leave the class early without the instructor's permission.
- Unexcused absences would result in the following penalty

1 letter grade down for # of unexcused absences	2 letter grade down for # of unexcused absences
4	8

What is CSD1241 about?

- Study the concepts of **linear stuff** in 2D and 3D which include

  points, **lines**, **planes**

- Study the **relation between these linear stuff** by the use of

  matrices and **vectors**

- CSD1241 builds the foundation for *CSD2251 Linear Algebra* and *CSD1201 Introduction to Computer Graphics*

Some applications of Linear Algebra

Linear Algebra has a lot of applications in the real life.

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Linear Algebra has a lot of applications in the real life.

- The **Google PageRank search** algorithm uses the the theory of **Eigenvectors** & **Eigenvalues**.
- **Facial recognition** algorithms are based on **Singular Value Decomposition**.
- Linear algebra is pervasive in **Machine Learning** and **AI**.
- And many more ......

Points

A **point** is a reference to a *location*.

Points are often denoted by capital letters $P, Q, R, \ldots$

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- In **2D plane** (or **xy plane**, or $\mathbb{R}^2$), points are specified by their x-coordinates and y-coordinates

$$P = (a, b) \text{ has x-coordinate} = a, \text{ y-coordinate} = b$$

Points

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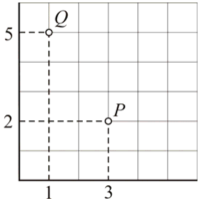
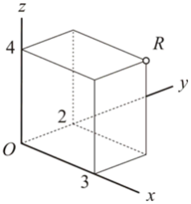
- In **3D space** (or **xyz space**, or $\mathbb{R}^3$), points are specified by their x-coordinates, y-coordinates and z-coordinates

  $P = (a, b, c)$ has x-coordinate $= a$, y-coordinate $= b$, z-coordinate $= c$

Example 1

In the following graphs, what are coordinates of P, Q, R?





Remarks

In this course, we mainly focus on $\mathbb{R}^2$ (xy plane) and $\mathbb{R}^3$ (xyz space).

- $\mathbb{R}^2$ and $\mathbb{R}^3$ are different *coordinate systems*.
  We *cannot sketch points simultaneously* on both these systems.

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In this course, we mainly focus on $\mathbb{R}^2$ (xy plane) and $\mathbb{R}^3$ (xyz space).

- $\mathbb{R}^2$ and $\mathbb{R}^3$ are different *coordinate systems*.
  We *cannot sketch points simultaneously* on both these systems.
- We call $(0,0)$ and $(0,0,0)$ the **origins** in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Example 2

Sketch the following points

(a) $P = (0,1), Q = (1,0), R = (2,0)$ and $S = (2,-1)$ on $\mathbb{R}^2$.

Example 2

(b) $A = (1, 0, 0), B = (0, 1, 0), C = (0, 0, 1), D = (1, 1, 0)$ and $E = (1, 1, 1)$ on $\mathbb{R}^3$.

Midpoints

The midpoint between 2 points can be obtained by averaging the corresponding coordinates

1. In $\mathbb{R}^2$

$$P = (x_1, y_1), Q = (x_2, y_2) \Rightarrow M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

2. In $\mathbb{R}^3$

$$P = (x_1, y_1, z_1), Q = (x_2, y_2, z_2) \Rightarrow M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Example 3

(a) Sketch the following points: $A = (1, 1),\ B = (4, 0),\ C = (0, 4)$

(b) Find the midpoint M_{AB} of AB and the midpoint M_{AC} of AC

(c) Find the midpoint M of $M_{AB}M_{AC}$

Distances between points

- In $\mathbb{R}^2$, the distance between $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- In $\mathbb{R}^3$, the distance between $R = (x_1, y_1, z_1)$ and $S = (x_2, y_2, z_2)$ is

$$d(R, S) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 4

Find the distances between any of the following 2 points.

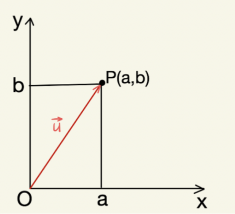
(a) $P = (0, 1), Q = (0, 1), S = (2, -1)$

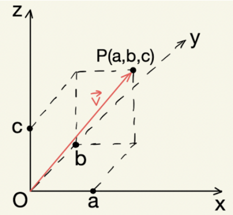
(b) $A = (1, 0, 0), B = (0, 1, 0), C = (2, 3, 1)$

Vectors

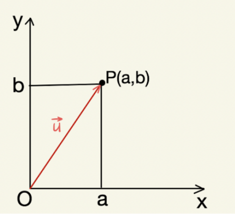
1. In $\mathbb{R}^2$, **vector** $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ is represented by an arrow joining $O = (0,0)$ and $P = (a,b)$

$$\vec{u} = \overrightarrow{OP} = \begin{bmatrix} a \\ b \end{bmatrix}$$



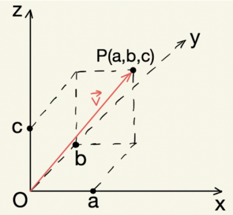


Vectors



2. In $\mathbb{R}^3$, $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is represented by an arrow joining $O = (0,0,0)$ and $Q = (a,b,c)$

$$\vec{v} = \overrightarrow{OQ} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



Zero vector

The **zero vector**, denoted by $\vec{0}$, has all coordinates equal to 0.

- In $\mathbb{R}^2$

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- In $\mathbb{R}^3$

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In this course, we use *small letters with arrows* on top to denote vectors:

$$\vec{a}, \vec{b}, \vec{c}, \ldots, \vec{x}, \vec{y}, \vec{z}$$

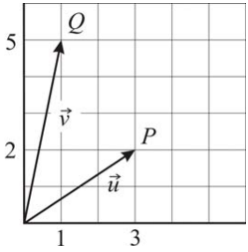
Example

- The vector $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ starts at $O = (0, 0)$ and ends at $P = (3, 2)$

$$\vec{u} = \overrightarrow{OP} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

- Similarly

$$\vec{v} = \overrightarrow{OQ} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$



Geometric representation of vectors

- A vector can be geometrically represented by any arrow as long as it preserves **length** and **direction**

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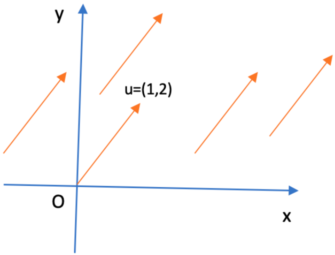
- A vector can be geometrically represented by any arrow as long as it preserves **length** and **direction**
- All the following arrows denote the same vector $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



Algebraic representation of vectors

- A vector is represented by the algebraic values of their coordinates.
- Given 2 vectors $\vec{u}$ and $\vec{v}$

$$\vec{u} = \vec{v} \Leftrightarrow \text{their corresponding coordinates are equal.}$$

Example 5

For what values of a, b the following 2 vectors are equal?

$$\vec{u} = \begin{bmatrix} a+b \\ 2a-3b \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ a-b \end{bmatrix}$$

Vectors forming by endpoints

For any two points P and Q, we can form the vector $\overrightarrow{PQ}$ which starts from P and ends at Q by subtracting $Q - P$.

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$$P = (x_1, y_1), Q = (x_2, y_2) \Rightarrow \overrightarrow{PQ} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$

- In $\mathbb{R}^3$

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Vector addition

Let $\vec{u}, \vec{v}$ be in the **same space** (both in $\mathbb{R}^2$ or both in $\mathbb{R}^3$). We can form $\vec{u} + \vec{v}$ or $\vec{u} - \vec{v}$

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- In $\mathbb{R}^2$:

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \vec{u}+\vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}, \ \vec{u}-\vec{v} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \end{bmatrix}$$

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- In $\mathbb{R}^3$:

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \Rightarrow \vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}, \quad \vec{u} - \vec{v} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{bmatrix}$$

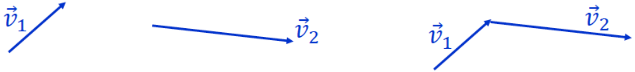
Question 1

Let $\vec{u}$ be any vector. What is $\vec{u} + \vec{0}$?

Geometrical interpretation of vector addition

To add two vectors $\vec{v_1}, \vec{v_2}$ geometrically, we do the following

- Take 2 arrows which correspond to v_1, v_2 and arrange them so that the ending point of v_1 lies at the starting point of v_2.



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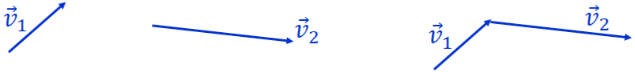
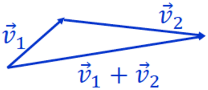


- $\vec{v_1} + \vec{v_2}$ is the arrow which goes from the starting point of $\vec{v}_1$ to the ending point of $\vec{v}_2$



Scalar multiplication

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- The *scalar multiplication* of c by $\vec{u}$, denoted by $c\vec{u}$, is another vector formed by multiplying c into each coordinate of $\vec{u}$.
  (i) In $\mathbb{R}^2$
  $$c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}$$
  (ii) In $\mathbb{R}^3$
  $$c\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}$$

Geometric interpretation of scalar multiplication

Geometrically, $c\vec{u}$ is the **scaling** of the vector $\vec{u}$ by the factor c.

- The length of $c\vec{u}$ is $|c|$ times bigger than the length of $\vec{u}$.

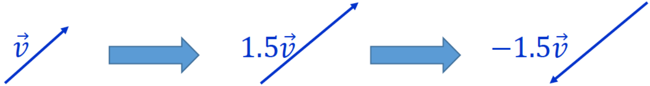
Geometric interpretation of scalar multiplication

Geometrically, $c\vec{u}$ is the **scaling** of the vector $\vec{u}$ by the factor c.

- The length of $c\vec{u}$ is $|c|$ times bigger than the length of $\vec{u}$.
- If $c > 0$, $c\vec{u}$ and $\vec{u}$ point to the same direction.
  If $c < 0$, $c\vec{u}$ and $\vec{u}$ point to opposite directions.

Example

The vector $(-1.5)\vec{u}$ is obtained as follows.



Example 6

Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Find the following vectors

(a) $\vec{u} + \vec{v}$

(b) $2\vec{u} - 3\vec{v}$

(c) $a\vec{u} + b\vec{v}$ for any real constants a, b

Length of vectors

The **length** (or **norm**) of a vector $\vec{u}$, denoted by $||\vec{u}||$, is the squareroot of the sum of squares of the coordinates of $\vec{u}$.

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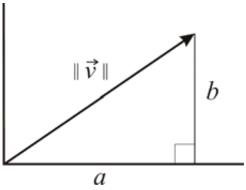
- In $\mathbb{R}^2$
$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow ||\vec{u}|| = \sqrt{u_1^2 + u_2^2}$$
- In $\mathbb{R}^3$
$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \Rightarrow ||\vec{u}|| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

Explanation for length of vectors

- In the following, the vector $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ is the hypothenuse of a triangle with side lengths a and b.



- By the Pythagoras theorem, its length is $||\vec{v}|| = \sqrt{a^2 + b^2}$.

Example 7

Find the length of the following vectors

$$\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \ \vec{w} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

Exercise 1 (properties of length)

(a) What is the length of the zero vector $\vec{0}$?

(b) Show that for any $\vec{u} \in \mathbb{R}^2$, we have

$$||\vec{u}|| \geq 0$$

Further, show that the only vector in $\mathbb{R}^2$ with length 0 is $\vec{0}$.

Exercise 1

(c) Let $\vec{u}$ be any vector in $\mathbb{R}^2$. Prove the following

(i) $||2\vec{u}|| = 2||\vec{u}||$

(ii) For any *positive number* c, $||c\vec{u}|| = c||\vec{u}||$.

Exercise 2: Points vs vectors

List the *similarities* and *differences* between points and vectors in $\mathbb{R}^2$.

Dot product

The **dot product** of two vectors $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \cdot \vec{v}$, is a *scalar* (a real number) defined as follows.

- In $\mathbb{R}^2$

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1 v_1 + u_2 v_2$$

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Example 8

Compute the dot product of any 2 vectors among the following 3 vectors

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ \vec{u} = \begin{bmatrix} 1 \\ 2 \\ \pi \end{bmatrix}, \ \vec{v} = \begin{bmatrix} -3 \\ 7 \\ 1/\pi \end{bmatrix}$$

Exercise 3

(a) Show that for any vector $\vec{u} \in \mathbb{R}^2$

$$\vec{u} \cdot \vec{u} = ||\vec{u}||^2$$

(b) Prove (a) for $\vec{u} \in \mathbb{R}^3$

Exercise 3

(c) Show that for any $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

(d) Prove (c) for $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$

Properties of dot product

Theorem 1

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in the same space. Then the following hold

(a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (Commutativity)

(b) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (Distributivity)

(c) $\vec{u} \cdot (c\vec{w}) = c(\vec{u} \cdot \vec{w})$ for any scalar c.

(d) $\vec{u} \cdot \vec{u} = ||\vec{u}||^2$.

Exercise 4

Using the properties of dot products, prove that

$$(\vec{u} - \vec{v}) \cdot (\vec{u} + \vec{v}) = ||\vec{u}||^2 - ||\vec{v}||^2$$

Angle between 2 vectors

Theorem 2

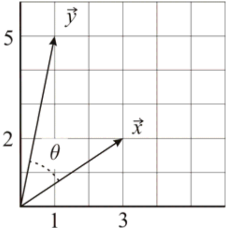
Let $\vec{x}, \vec{y}$ be 2 vectors in the same space and let $\theta \in [0, 180^o]$ be the angle between $\vec{x}$ and $\vec{y}$. Then

$$\vec{x} \cdot \vec{y} = ||\vec{x}||||\vec{y}|| \cos\theta$$

Example 9

Compute the angle between $\vec{x}$ and $\vec{y}$ in the following cases.

(a) $\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$



Example 9

(b) $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \vec{y} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$

Types of angles

There are three types of angles between two vectors $\vec{x}$ and $\vec{y}$.

1. Right angle: $\theta = 90^o$

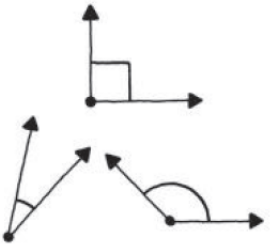
$$\cos\theta = 0 \Leftrightarrow \vec{x}\cdot\vec{y} = 0$$

2. Acute angle: $\theta < 90^o$

$$\cos\theta > 0 \Leftrightarrow \vec{x}\cdot\vec{y} > 0$$

3. Obtuse angle: $\theta > 90^o$

$$\cos\theta < 0 \Leftrightarrow \vec{x}\cdot\vec{y} < 0$$



Exercise 5

Find the type of angle (right, acute, obtuse) between $\vec{x}$ and $\vec{y}$

(a) $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

(b) $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ \pi \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Exercise 5

(c) $\vec{x} = \begin{bmatrix} 1 \\ e \\ \pi \end{bmatrix}, \vec{y} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Exercise 6 (Triangle inequality)

Let $\vec{x}$ and $\vec{y}$ be vectors in the same space. Prove that

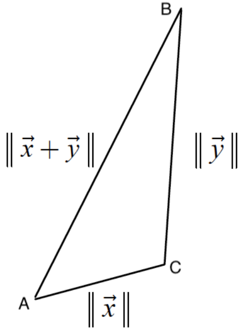
$$||\vec{x}+\vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||.$$

Geometric interpretation of triangle inequality

Put $\vec{x} = \overrightarrow{AC}, \vec{y} = \overrightarrow{CB} \Rightarrow \vec{x} + \vec{y} = \overrightarrow{AB}$.

$$||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}|| \Leftrightarrow AB \leq AC + CB$$

Imagine you have to travel from A to B. Consider 2 paths



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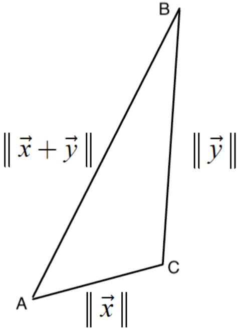
1. Direct path $\Rightarrow$ shortest way and

$$\text{length} = ||\overrightarrow{AB}|| = ||\vec{x} + \vec{y}||$$



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Imagine you have to travel from A to B. Consider 2 paths

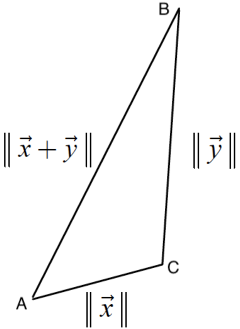
1. Direct path $\Rightarrow$ shortest way and

$$\text{length} = ||\overrightarrow{AB}|| = ||\vec{x} + \vec{y}||$$

2. Move A to C, then C to B. The length is

$$||\overrightarrow{AC}|| + ||\overrightarrow{CB}|| = ||\vec{x}|| + ||\vec{y}||$$

By length comparison,
$||\overrightarrow{AB}|| \leq ||\overrightarrow{AC}|| + ||\overrightarrow{CB}||$



Parallel vectors

Two vectors $\vec{u}$ and $\vec{v}$ are called **parallel**, denoted $\vec{u} \parallel \vec{v}$, if there exists a scalar c such that

$$\vec{u} = c\vec{v}$$

- In $\mathbb{R}^2$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Leftrightarrow \frac{u_1}{v_1} = \frac{u_2}{v_2} = c$$

- In $\mathbb{R}^3$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = c \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \Leftrightarrow \frac{u_1}{v_1} = \frac{u_2}{v_2} = \frac{u_3}{v_3} = c$$

Characterization of parallel vectors

Theorem 3

(a) In $\mathbb{R}^2$, $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ are parallel if and only if

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(b) In $\mathbb{R}^3$, $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ are parallel if and only if

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Example 10

Which of the following pairs of vectors are parallel?

(a) $\vec{u} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

(b) $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(c) $\vec{u} = \begin{bmatrix} -2 \\ -4 \\ -8 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

Orthogonal vectors

- $\vec{x}$ and $\vec{y}$ are **orthogonal** (or perpendicular), denoted $\vec{x} \perp \vec{y}$, if the angle θ between $\vec{x}$ and $\vec{y}$ is 90^o.
- If $\theta \neq 90^\circ$, we write $\vec{x} \not\perp \vec{y}$.

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- Question: When is $\theta = 90^0$?

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- Remark: The zero vector $\vec{0}$ is orthogonal to any vector.

Example 11

Which of the following pairs of vectors are orthogonal?

(a) $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(b) $\vec{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(c) $\vec{e} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{f} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

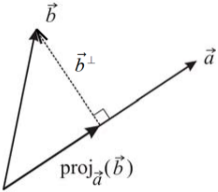
Exercise 7

(a) Find the condition for real numbers a, b so that

$$\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix} \text{ is orthogonal to } \vec{y} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

(b) Give 3 examples of the vector $\vec{x}$ in part a.

Orthogonal projection



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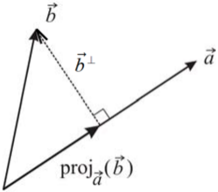

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  1. Arrange $\vec{a}$ and $\vec{b}$ so that they have the **same starting point**.
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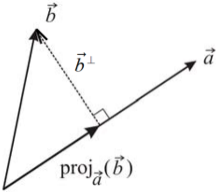

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  1. Arrange $\vec{a}$ and $\vec{b}$ so that they have the **same starting point**.
  2. Project the **endpoint** of $\vec{b}$ orthogonally into $\vec{a}$.
- The **orthogonal complement** of $\vec{b}$ onto $\vec{a}$ is

$$\vec{b}^{\perp} = \vec{b} - \text{proj}_{\vec{a}}(\vec{b})$$

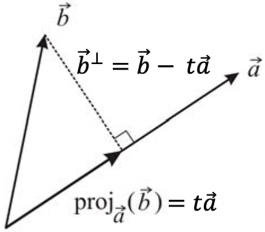
Formula for orthogonal projection

Theorem 4

Let $\vec{a}, \vec{b}$ be two vectors in the same space with $\vec{a} \neq \vec{0}$. The *orthogonal projection* of $\vec{b}$ onto $\vec{a}$ is

$$\mathrm{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a}$$

Theorem 3 proof



- Since $\text{proj}_{\vec{a}}(\vec{b})$ and $\vec{a}$ are parallel, there is $c \in \mathbb{R}$:

$$\text{proj}_{\vec{a}}(\vec{b}) = c\vec{a} \Rightarrow \vec{b}^{\perp} = \vec{b} - c\vec{a}$$

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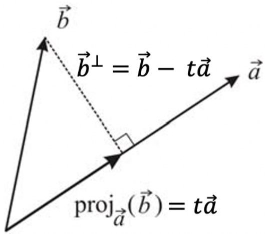

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- $\vec{b}^{\perp} = \vec{b} - c\vec{a}$ is orthogonal to $\vec{a}$. So

$$0 = (c\vec{a} - \vec{b}) \cdot \vec{a} = c\vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{a} \Rightarrow c = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}$$

Summary on orthogonal projection

1. The orthogonal projection $\mathrm{proj}_{\vec{a}}(\vec{b})$ is only defined if $\vec{a} \neq \vec{0}$.
2. $\mathrm{proj}_{\vec{a}}(\vec{b})$ is a scalar multiple of $\vec{a}$, say

$$\mathrm{proj}_{\vec{a}}(\vec{b}) = c\vec{a}$$

with the scale

$$c = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}$$

Example 12

Find $\mathrm{proj}_{\vec{a}}(\vec{b})$ and $\vec{b}^{\perp}$. Verify that $\vec{b}^{\perp}$ is orthogonal (perpendicular) to $\vec{a}$.

(a) $\vec{a} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

Example 12

(b) $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$

Question

Assume $\vec{a} \perp \vec{b}$. What is $\mathrm{proj}_{\vec{a}}(\vec{b})$?

Exercise 8

Let $A = (2,5), B = (8,8), C = (3,8)$ be three vertices of a triangle. From A, draw a line perpendicular to BC at the point Q.

(a) Find the coordinates of Q.

(b) What is the area of $\triangle ABC$?

Exercise 8

General equation and normal equation of lines

- The **general equation** of lines in $\mathbb{R}^2$ has form $ax + by + c = 0$

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  The **normal equation** of lines in $\mathbb{R}^2$ has form $ax + by = c$
- More commonly known is the following 2 types of equation
  1. Slanted lines

     $$y = mx + c$$

     with $m =$ **slope** of the line, and $c =$ **y-intercept**.

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2. Vertical lines

$$x = k,$$

with $k =$ **x-intercept**.

Lines through 2 points

Theorem 5

Let $A(x_1, y_1)$ and $B = (x_2, y_2)$ be any two points in $\mathbb{R}^2$.

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Let $A(x_1, y_1)$ and $B = (x_2, y_2)$ be any two points in $\mathbb{R}^2$.

(a) If $x_1 = x_2$, then the line going through A and B is the *vertical line*

$$x = x_1$$

Lines through 2 points

Theorem 5

Let $A(x_1, y_1)$ and $B = (x_2, y_2)$ be any two points in $\mathbb{R}^2$.

(b) If $x_1 \neq x_2$, then the line going through A and B is the *slanted line*

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Theorem 4 proof

(a) Since A and B have the same x-coordinate, the only line going through both both A and B is the vertical line $x = x_1$.

Theorem 4 proof

- Since $x_1 \neq x_2$, the line going through A and B is a slanted line

$$y = mx + c \tag{1}$$

Theorem 4 proof

- Since $x_1 \neq x_2$, the line going through A and B is a slanted line

$$y = mx + c \tag{1}$$

- Both A and B are on the line, their coordinates both satisfy (1)

$$\begin{cases} mx_1 + c = y_1 \\ mx_2 + c = y_2 \end{cases} \Rightarrow \begin{cases} m = \dfrac{y_2 - y_1}{x_2 - x_1} \\ c = y_1 - \dfrac{y_2 - y_1}{x_2 - x_1} x_1 \end{cases}$$

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- Conclusion

$$y = \frac{y_2 - y_1}{x_2 - x_1} x + y_1 - \frac{y_2 - y_1}{x_2 - x_1} x_1 \Leftrightarrow y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

Example 13

Find the equation of the line going through two points P and Q. In each case, write out 2 other points (other than P, Q) on the line.

(a) $P = (0, 1)$, $Q = (3, 5)$

Example 13

(b) $P = (1, -1)$, $Q = (1, \pi)$