Week 13: Review

Table of contents

Linear and Affine Transformations

- The map $T : \mathbb{R}^n \to \mathbb{R}^m$ is a
  1. linear transformation if
  $$T(\vec{x}) = A\vec{x} \text{ for some matrix } A_{m\times n}$$
  2. affine transformation if
  $$T(\vec{x}) = A\vec{x} + \vec{b} \text{ for some matrix } A_{m\times n} \text{ and } \vec{b} \in \mathbb{R}^m$$

Linear and Affine Transformations

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     $$T(\vec{x}) = A\vec{x} + \vec{b} \text{ for some matrix } A_{m \times n} \text{ and } \vec{b} \in \mathbb{R}^m$$
- Any map projection, reflection, scaling, rotation, shear
  - is an affine transformation
  - becomes a linear transformation if it involves the origin

Determine matrix A and vector $\vec{b}$

- Linear transformation: $T(\vec{x}) = A\vec{x}$

$$T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix} \Rightarrow A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

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- Affine Transformation: $T(\vec{x}) = A\vec{x} + \vec{b}$. If

$$T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n + b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n + b_m \end{pmatrix},$$

then

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

3-step algorithm

The 3-step algorithm is used to describe all 2D maps and 3D maps as affine transformations

$$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0 \Leftrightarrow T(\vec{x}) = A\vec{x} + \vec{b} \text{ with } \vec{b} = \vec{x}_0 - A\vec{x}_0$$

Projections and reflections in $\mathbb{R}^2$

- If $l : \vec{x} = t\vec{d}$ (l contains O) is the line of projection (or reflection), we have a linear transformation
  1. Projection: $T(\vec{x}) = A\vec{x}$ with $A = \dfrac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T$
  2. Reflection: $T(\vec{x}) = A\vec{x}$ with $A = \dfrac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_2$

Projections and reflections in $\mathbb{R}^2$

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- If $l : \vec{x} = \vec{x}_0 + t\vec{d}$ with $\vec{x}_0 \neq \overrightarrow{0}$, we have an affine transformation

$$T(\vec{x}) = A(\vec{x} - \vec{x}_0) + \vec{x}_0 \Leftrightarrow T(\vec{x}) = A\vec{x} + \vec{b} \text{ with } \vec{b} = \vec{x}_0 - A\vec{x}_0$$

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- What are fixed points of projection? Fixed points of reflection?

Scalings in $\mathbb{R}^2$

- The scaling $S : \mathbb{R}^2 \to \mathbb{R}^2$ centered at the origin is

$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

- The scaling $S : \mathbb{R}^2 \to \mathbb{R}^2$ centered at the point $\vec{x}_0$ is

$$S = A(\vec{x} - \vec{x}_0) + \vec{x}_0 = A\vec{x} + \vec{b} \text{ with } \vec{b} = \vec{x}_0 - A\vec{x}_0$$

Remark: Scaling centered at $\vec{x}_0 \neq \overrightarrow{0}$ is **not tested**.

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Remark: Scaling centered at $\vec{x}_0 \neq \overrightarrow{0}$ is **not tested**.

- What are fixed points of the scaling $S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix}$?

Rotations in $\mathbb{R}^2$

- The rotation (counter-clockwise) about the origin O over angle θ is

$$T(\vec{x}) = A\vec{x} \text{ with } A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- The rotation about the point $\vec{x}_0$ over the angle θ is

$$T(\vec{x}) = A\vec{x} + \vec{b} \text{ with } A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \ \vec{b} = \vec{x}_0 - A\vec{x}_0$$

- What are fixed points of the rotation about $\vec{x}_0$ over $\theta \in (0^\circ, 360^\circ)$?

Shears in $\mathbb{R}^2$

- The **shear** wrt. $l : \vec{n} \cdot \vec{x} = 0$ in the direction of **shearing vector** $\vec{v}$ is

$$T(\vec{x}) = \vec{x} + \frac{\vec{n} \cdot \vec{x}}{||\vec{n}||}\vec{v}$$

  As a linear transformation, we have

$$T(\vec{x}) = A\vec{x} \text{ with } A = I_2 + \frac{1}{||\vec{n}||}\vec{v}\vec{n}^T$$

- The **shear** w.r.t. $l : \vec{n} \cdot \vec{x} = c$ in the direction of **shearing vector** $\vec{v}$ is

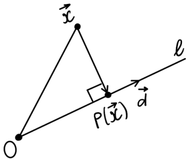
$$T(\vec{x}) = A\vec{x} + \vec{b} \text{ with } \vec{b} = \vec{x}_0 - A\vec{x}_0, \ \vec{x}_0 = \text{ a point on } l$$

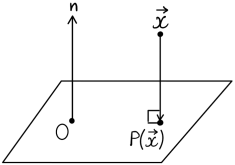
- What are fixed points of a shear?

Projections in $\mathbb{R}^3$

- If the line/plane of projection contains O, it's a linear transformation

$$T(\vec{x}) = A\vec{x} \text{ with}$$





$$A = \frac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T \qquad A = I_3 - \frac{1}{||\vec{n}||^2}\vec{n}\vec{n}^T$$

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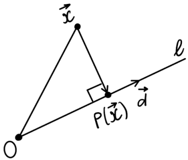

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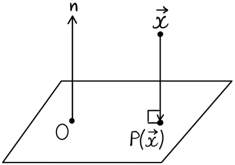


$$A = I_3 - \frac{1}{||\vec{n}||^2}\vec{n}\vec{n}^T$$

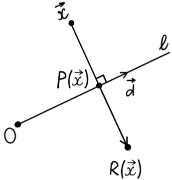
- If the line/plane doesn't contain O, it's an affine transformation

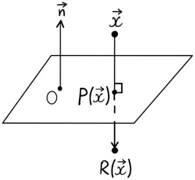
$$T(\vec{x}) = A\vec{x} + \vec{b} \text{ with } \vec{b} = \vec{x}_0 - A\vec{x}_0$$

  and $\vec{x}_0 =$ a point on the line/plane of projection.
- What are fixed points?

Reflections in $\mathbb{R}^3$

- If the line/plane of projection contains O, we have $T(\vec{x}) = A\vec{x}$ with





$$M = \frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_3 \qquad M = I_3 - \frac{2}{||\vec{n}||^2}\vec{n}\vec{n}^T$$

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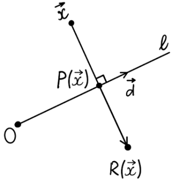

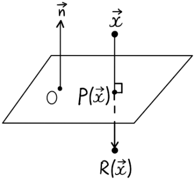

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- If the line/plane doesn't contain O, it's an affine transformation

$$T(\vec{x}) = A\vec{x} + \vec{b} \text{ with } \vec{b} = \vec{x}_0 - A\vec{x}_0$$

and $\vec{x}_0$ = a point on the line/plane of projection.

- What are fixed points?

Scalings in $\mathbb{R}^3$

- The scaling $S : \mathbb{R}^3 \to \mathbb{R}^3$ centered at the origin is

$$S \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax \\ by \\ cz \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

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Remark: Scaling centered at $\vec{x}_0 \neq \overrightarrow{0}$ is **not tested**.

- What are fixed points of the scaling $S\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax \\ by \\ cz \end{pmatrix}$?

Shears in $\mathbb{R}^3$

- The shear w.r.t. $\alpha : \vec{n} \cdot \vec{x} = 0$ in the direction of shearing vector $\vec{v}$ is

$$T(\vec{x}) = \vec{x} + \frac{\vec{n} \cdot \vec{x}}{||\vec{n}||}\vec{v} \Leftrightarrow T(\vec{x}) = A\vec{x} \text{ with } A = I_3 + \frac{1}{||\vec{n}||}\vec{v}\vec{n}^T$$

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- If the plane $\alpha : \vec{n} \cdot \vec{x} = c$, then

$$T(\vec{x}) = A\vec{x} + \vec{b} \text{ with } \vec{b} = \vec{x}_0 - A\vec{x}_0$$

  and $\vec{x}_0 =$ a point on α.

- What are fixed points of the shear?

Rotations in $\mathbb{R}^3$

- The rotation centered at O about the vector $\vec{v}$ over angle θ is $T(\vec{x}) = A\vec{x}$ with

$$A = (1 - \cos\theta)\frac{\vec{v}\vec{v}^T}{||\vec{v}||^2} + (\cos\theta)I_3 + \frac{\sin\theta}{||\vec{v}||}C_{\vec{v}}$$

Rotations in $\mathbb{R}^3$

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- The rotation centered at $\vec{x}_0$ about the vector $\vec{v}$ over angle θ is
$$T(\vec{x}) = A\vec{x} + \vec{b} \text{ with } \vec{b} = \vec{x}_0 - A\vec{x}_0$$
- *Remark*: The rotation centered at $\vec{x}_0 \neq \overrightarrow{0}$ will **not be tested**.

  The phrase "rotation about $\vec{v}$ over angle θ" means the rotation is centered at the origin O.

Rotations in $\mathbb{R}^3$

- The rotation centered at O about the vector $\vec{v}$ over angle θ is $T(\vec{x}) = A\vec{x}$ with

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- The rotation centered at $\vec{x}_0$ about the vector $\vec{v}$ over angle θ is

$$T(\vec{x}) = A\vec{x} + \vec{b} \text{ with } \vec{b} = \vec{x}_0 - A\vec{x}_0$$

- *Remark*: The rotation centered at $\vec{x}_0 \neq \overrightarrow{0}$ will **not be tested**.

  The phrase "rotation about $\vec{v}$ over angle θ" means the rotation is centered at the origin O.
- What are fixed points of the rotation about O over angle θ?

Rotations about the axes

The rotations about the positive x, y, z-axes are linear transformations with matrices

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 1

The shear S wrt $l : 3x - 4y = 0$ maps $P = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ to $P' = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$.

(a) What is the matrix of S? (*Hint.* $S(\vec{x}) = \vec{x} + \dfrac{\vec{n} \cdot \vec{x}}{||\vec{n}||}\vec{v}$ to find $\vec{v}$)

(b) Find the normal equation for the image m' of $m : 2x - 3y = 6$.

(c) Let Q be the intersection of m' and l. Find the image Q' of Q.

Problem 2

Given the point $P = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and the line $l : \begin{cases} x = 2 + t \\ y = 3 - t \\ z = 1 + 2t \end{cases}$

(a) Using affine trans., find the point P' on l that is closest to P.

(b) $\alpha =$ plane through P and perpendicular to l. Using affine transformation, find the point Q' on α that is closest to $Q = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

Problem 3

$T =$ the shear with respect to the plane $\alpha : z = 3$ in the direction of shearing vector $\vec{v} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$.

(a) Write T in the form of an affine map $T(\vec{x}) = A\vec{x} + \vec{b}$.

(b) Find the image of $l : \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ under T?

(c) Find the image of the plane $\beta : x - z = 0$ under T.

Problem 4

S and T = reflections through $\alpha : 2x - y + 2z = 0$ and $\beta : x - y = 0$.

(a) Find the matrix M of $S \circ T$ (*Hint*: $M = M_S M_T$).

(b) Find the fixed points of $S \circ T$.

(c) In b, your answer is a line l. Let $\vec{v}$= direction of l. Find the angle θ so that M is a rotation matrix, that is,

$$M = (1 - \cos\theta)\frac{\vec{v}\vec{v}^T}{||\vec{v}||^2} + (\cos\theta)I_3 + \frac{\sin\theta}{||\vec{v}||}C_{\vec{v}}$$

Reminders on final exam

- Date and Time: Wednesday, November 30, 10am-12pm
- Scope: Weeks 8-12 materials
- The following are **not tested**
    1. All skew maps which include skew projection and reflection
    2. Scalings (both 2D and 3D) centered at a point $\vec{x}_0 \neq O$
    3. 3D rotation centered at a point $\vec{x}_0 \neq O$
- Allowed to bring in: 1 A4-size cheat sheet + 1 calculator


Thank you