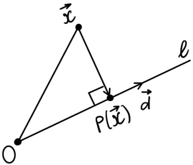
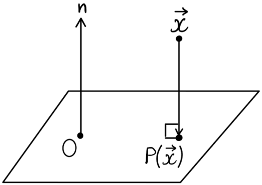
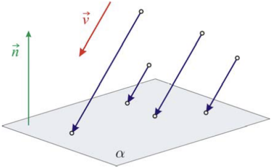
Week 11: Scaling, Shear, Rotation in 3D

Table of contents

Projections in $\mathbb{R}^3$



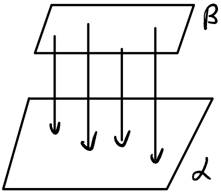


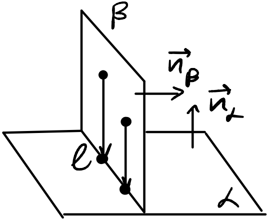


$$M = \frac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T \qquad M = I_3 - \frac{1}{||\vec{n}||^2||}\vec{n}\vec{n}^T \qquad M = I_3 - \frac{1}{\vec{v}\cdot\vec{n}}\vec{v}\vec{n}^T$$

Image of a plane under projection

Question 1: Let T be the orthogonal projection onto plane α. What are the possibilities for the image of a plane β under T?





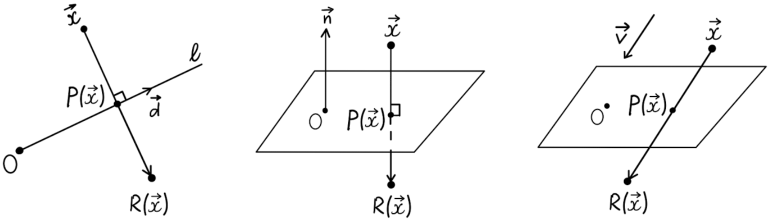
$\beta' = \alpha$

if $\alpha \not\perp \beta \Leftrightarrow \vec{n}_\alpha \not\perp \vec{n}_\beta$

$\beta' =$ the line $\ell = \alpha \cap \beta$

if $\alpha \perp \beta \Leftrightarrow \vec{n}_\alpha \perp \vec{n}_\beta$

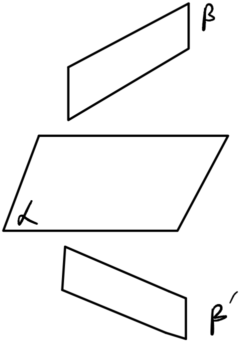
Reflections in $\mathbb{R}^3$



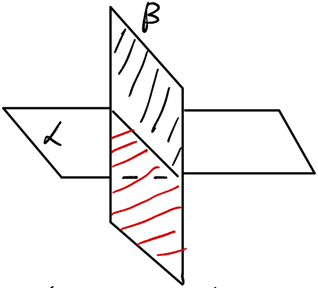
$$M = \frac{2}{||\vec{d}||^2}\vec{d}\vec{d}^T - I_3 \qquad M = I_3 - \frac{2}{||\vec{n}||^2}\vec{n}\vec{n}^T \qquad M = I_3 - \frac{2\vec{v}\vec{n}^T}{\vec{v}\cdot\vec{n}}$$

Image of a plane under reflection

Question 2: Let T be the orthogonal reflection through plane α. What are the possibilities for the image of a plane β under T?



$\beta' \neq \beta$ if $\alpha \not\perp \beta$



$\beta' = \beta$ if $\alpha \perp \beta$

Example 1

Let $\alpha : 3x + 2y - z = 0$ and let $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. $\vec{n} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

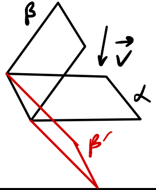
skew

(a) Compute the matrix M of the reflection through α in the direction $\vec{v}$.

(b) Find the image of the plane $\beta : 4x - 9y - 6z = 7$

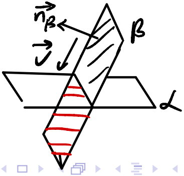
(a) $M = I_3 - \frac{2}{\vec{v}\cdot\vec{n}} \vec{v}\,\vec{n}^T = I_3 - \frac{2}{5}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}(3 \ \ 2 \ \ -1)$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{2}{5}\begin{pmatrix} 3 & 2 & -1 \\ 3 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1/5 & -4/5 & 2/5 \\ -6/5 & 1/5 & 2/5 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) 

$\beta' \neq \beta$ if

$\beta \not\parallel \vec{v} \Leftrightarrow \vec{n}_\beta \not\perp \vec{v}$



$\beta' = \beta$ if

$\beta \parallel \vec{v}$

$\Leftrightarrow \vec{n}_\beta \perp \vec{v}$

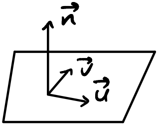
$\beta : 4x - 9y - 6z = 7$

$\vec{n}_\beta = \begin{pmatrix} 4 \\ -9 \\ -6 \end{pmatrix}$

point $\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$

direction vectors $\vec{u} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$

$\beta : \vec{x} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} + t \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$



The image of β is

$\beta' : \vec{x} = M \left(\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} + t \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \right)$

$= \ldots$

$= \begin{pmatrix} -8/5 \\ -23/5 \\ 0 \end{pmatrix} + \frac{-s}{5} \begin{pmatrix} 14 \\ 4 \\ 15 \end{pmatrix} + \frac{t}{5} \begin{pmatrix} 1 \\ -14 \\ 10 \end{pmatrix}$

Scaling in $\mathbb{R}^3$

- The scaling $S : \mathbb{R}^3 \to \mathbb{R}^3$ that scales all $x, y, z-$coordinates by factors a, b, c is

$$S\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax \\ by \\ cz \end{pmatrix}$$

- The matrix of S is

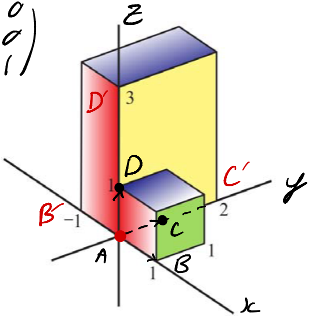
$$M = M_{a,b,c} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

Example 2

(a) Verify that the map $S\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ 2y \\ 3z \end{pmatrix}$ maps the unit cube to the rectangular cuboid.

$A = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, D = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$A' = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, B' = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, C' = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, D' = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$



(b) Compute the volume of the rectangular cuboid.

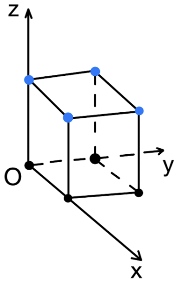
Unit cube has volume $= 1 \times 1 \times 1 = 1$

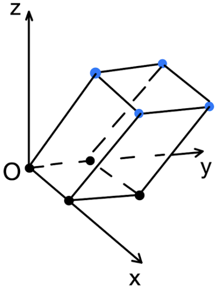
Rectangular cuboid has volume $= 1 \times 2 \times 3 = 6$

Remark. The scaling that scales x, y, z-coordinates by factors a, b, c will scale the volume of any object by the factor $|abc|$

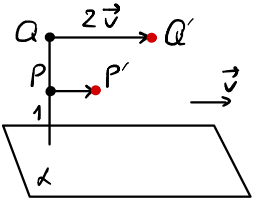
Shear

- The shear in $\mathbb{R}^3$ maps a cube to a parallelepiped





Shear



α: $\vec{n} \cdot \vec{x} = 0$, point $\vec{x}_0$

$d(\vec{x}_0, \alpha) = \dfrac{|\vec{n} \cdot \vec{x}_0|}{||\vec{n}||}$

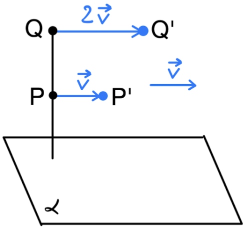
α contain O   normal $\vec{n}$

- The shear $S : \mathbb{R}^3 \to \mathbb{R}^3$ w.r.t. a plane $\alpha : \vec{n} \cdot \vec{x} = 0$ in the direction of **shearing vector** $\vec{v}$ ($\vec{v} \parallel \alpha$) is defined by

$$S(\vec{x}_0) = \vec{x}_0 + \frac{\vec{n} \cdot \vec{x}_0}{||\vec{n}||}\vec{v}$$

scalar magnitude $= d(\vec{x}_0, \alpha)$

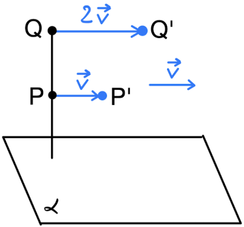
Comments



- S moves $\vec{x}_0$ in the direction of $\vec{v}$ by the factor $\dfrac{\vec{n}\cdot\vec{x}_0}{||\vec{n}||}$

$$S(\vec{x}_0) = \vec{x}_0 + \frac{\vec{n}\cdot\vec{x}_0}{||\vec{n}||}\vec{v}$$

Comments



- S moves $\vec{x}_0$ in the direction of $\vec{v}$ by the factor $\dfrac{\vec{n}\cdot\vec{x}_0}{||\vec{n}||}$

$$S(\vec{x}_0) = \vec{x}_0 + \frac{\vec{n}\cdot\vec{x}_0}{||\vec{n}||}\vec{v}$$

- The factor $\dfrac{\vec{n}\cdot\vec{x}_0}{||\vec{n}||}$ has *magnitude* equal to $d(\vec{x}_0,\alpha)$: $d(\vec{x}_0,\alpha) = \dfrac{|\vec{n}\cdot\vec{x}_0|}{||\vec{n}||}$

Question

$$S(\vec{x}_0) = \vec{x}_0 + \frac{\vec{n}\cdot\vec{x}_0}{\|\vec{n}\|}\vec{v}$$

Consider the shear S with respect to the plane α and the shearing vector $\vec{v}$. What points are fixed by S?

A point P is shifted by $c\vec{v}$ with

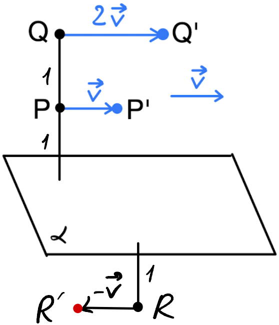
$$|c| = d(P, \alpha)$$

A point P is fixed $\Leftrightarrow c = 0$

$\Leftrightarrow d(P,\alpha) = 0$

$\Leftrightarrow P \in \alpha$

$\therefore$ All fixed points are on α.

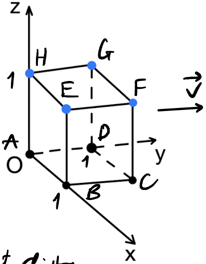


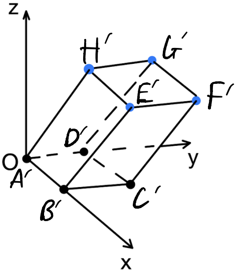
Example 3

Verify that the shear w.r.t. the xy-plane in the direction of the shearing vector $\vec{j}$ maps the unit cube to the parallelepiped.

α : xy-plane

Shearing vector $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$





A, B, C, D are on α:

$A' = A,\ B' = B,\ C' = C,\ D' = D$

All points E, F, G, H are at distance 1 from α $\Rightarrow$ they are all shifted by $1\vec{v}$

$E' = E + \vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $F' = F + \vec{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $G' = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, $H' = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Matrix of shear in $\mathbb{R}^3$

Theorem 1

The shear with respect to the plane $\alpha : \vec{n} \cdot \vec{x} = 0$ in the direction of the vector $\vec{v}$ ($\vec{v} \perp \vec{n}$) has matrix representation

$$M = M_{\vec{n},\vec{v}} = I_3 + \frac{1}{||\vec{n}||}\vec{v}\vec{n}^T$$

Proof

- Let $\vec{x}_0$ be any point.
- The image of $\vec{x}_0$ is

$$
\begin{aligned}
S(\vec{x}_0) &= \vec{x}_0 + \frac{\vec{n}\cdot\vec{x}_0}{||\vec{n}||}\vec{v} = \vec{x}_0 + \frac{1}{||\vec{n}||}(\vec{n}\cdot\vec{x}_0)\vec{v} \\
&= \vec{x}_0 + \frac{1}{||\vec{n}||}\vec{v}\vec{n}^T\vec{x}_0 = \left(I_3 + \frac{\vec{v}\vec{n}^T}{||\vec{n}||}\right)\vec{x}_0
\end{aligned}
$$

Example 4

Consider $\alpha : 3x - 4y - 12z = 0$ and $\vec{v} = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$. $\vec{n} = \begin{pmatrix} 3 \\ -4 \\ -12 \end{pmatrix}$

(a) Compute the matrix of the shear wrt to α and shearing vector $\vec{v}$.

(b) What are the images of $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$?

(a) $$M = I_3 + \frac{1}{\|\vec{n}\|} \vec{v}\, \vec{n}^T = I_3 + \frac{1}{13} \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & -4 & -12 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{13} \begin{pmatrix} 12 & -16 & -48 \\ -9 & 12 & 36 \\ 6 & -8 & -24 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 25 & -16 & -48 \\ -9 & 25 & 36 \\ 6 & -8 & -11 \end{pmatrix}$$

(b) Exercise!

(c) Show that the image of $m : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$ is parallel to m'.

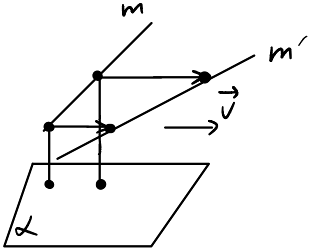
The image of m is:

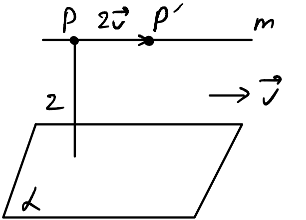
$$m' : \vec{x} = \frac{1}{13}\begin{pmatrix} 25 & -16 & -48 \\ -9 & 25 & 36 \\ 6 & -8 & -11 \end{pmatrix}\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}\right)$$

$$m' : \vec{x} = \frac{1}{13}\begin{pmatrix} -151 \\ 149 \\ -43 \end{pmatrix} + t\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$$

$m \parallel m'$ because they have the same direction vector.

(d) What is the image of $n : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$? Exercise





$m' \neq m$ if $m \nparallel \vec{v}$

$m' = m$ if $m \parallel \vec{v}$

(e) What is the image of $\beta : x + y - z = 1$?

contain $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

normal $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

β has vector equation

$$\beta : \vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

the image of β is

$$\beta' : \vec{x} = \frac{1}{13}\begin{pmatrix} 25 & -16 & -48 \\ -9 & 25 & 36 \\ 6 & -8 & -11 \end{pmatrix}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 25/13 \\ -9/13 \\ 6/13 \end{pmatrix} + \frac{s}{13}\begin{pmatrix} 41 \\ -34 \\ 14 \end{pmatrix} + \frac{t}{13}\begin{pmatrix} -23 \\ 27 \\ -5 \end{pmatrix}$$

Example 5

Consider the same shear as in the previous example. Find the images $\vec{i}', \vec{j}', \vec{k}'$ of the unit cube formed by $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$$\vec{i}' = \frac{1}{13}\begin{pmatrix} 25 & -16 & -48 \\ -9 & 25 & 36 \\ 6 & -8 & -11 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 25 \\ -9 \\ 6 \end{pmatrix}$$

$$\vec{j}' = \frac{1}{13}\begin{pmatrix} 25 & -16 & -48 \\ -9 & 25 & 36 \\ 6 & -8 & -11 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} -16 \\ 25 \\ -8 \end{pmatrix}$$

$$\vec{k}' = \frac{1}{13}\begin{pmatrix} 25 & -16 & -48 \\ -9 & 25 & 36 \\ 6 & -8 & -11 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} -48 \\ 36 \\ -11 \end{pmatrix}$$

Example 5

(b) The unit cube is mapped to a parallelepiped formed by $\vec{i}', \vec{j}', \vec{k}'$. What is the volume of this parallelepiped?

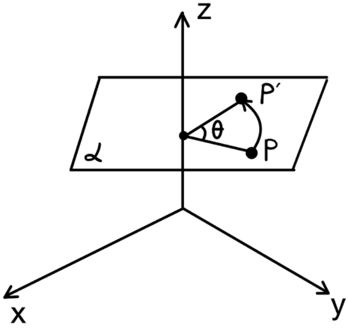
$$\text{Volume} = \left|\det\left(\vec{i}' \ \vec{j}' \ \vec{k}'\right)\right| = \left|\det\begin{pmatrix} 25/13 & -16/13 & -48/13 \\ -9/13 & 25/13 & 36/13 \\ 6/13 & -8/13 & -11/13 \end{pmatrix}\right|$$

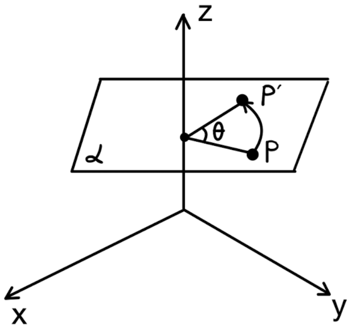
$$= \text{main diagonals} - \text{anti-diagonals}$$

$$= \ldots = 1$$

Rotation about positive z-axis

- Assume we want to rotate a point P counter-clockwise about the positive z-axis over the angle θ.
- This means when we *look down* from above, the image P' of P is shifted by the angle θ.

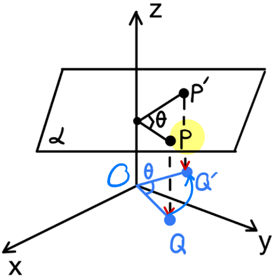




1. Draw the horizontal plane α through P
2. On α, rotate P counter-clockwise by the angle θ to get P'

Coordinates of P'

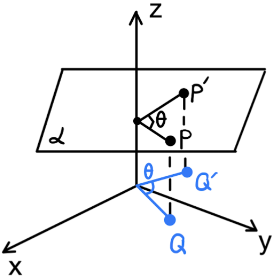
Assume $P = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \Rightarrow z$-coordinate of $P' = z_0$.



Coordinates of P'

Assume $P = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \Rightarrow z$-coordinate of $P' = z_0$.

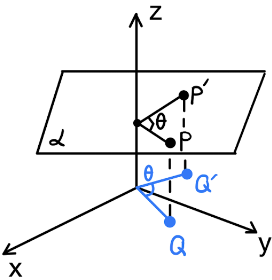
- Project P, P' onto xy-plane to get Q, Q'.



Coordinates of P'

Assume $P = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \Rightarrow z$-coordinate of $P' = z_0$.

- Project P, P' onto xy-plane to get Q, Q'.
- As points on xy-plane,
  $Q = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and $Q' =$ rotation of Q over θ
- $Q' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0\cos\theta - y_0\sin\theta \\ x_0\sin\theta + y_0\cos\theta \end{bmatrix}$



Rotation about positive z-axis

- In summary, P' has coordinates

$$P' = \begin{bmatrix} x_0 \cos\theta - y_0 \sin\theta \\ x_0 \sin\theta + y_0 \cos\theta \\ z_0 \end{bmatrix} = \overbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{M} \overbrace{\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}}^{P}$$

- The matrix for counter-clockwise rotation about the positive z-axis over angle θ is

$$M_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotations about the axes

Theorem 2

The counter-clockwise rotation over the angle θ about

positive

(a) ✓ x-axis has matrix $R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

Positive

(b) ✓ y-axis has matrix $R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

Positive

(c) ✓ z-axis has matrix $R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example 6

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation about the x-axis over 30^0.

(a) Find the matrix A of this transformation.

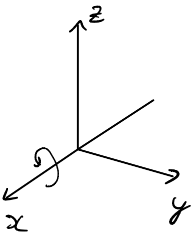
(b) Find the images of the points $\begin{bmatrix} 1 \\ 1 \\ \sqrt{3} \end{bmatrix}$ and $\begin{bmatrix} 11 \\ \sqrt{3} \\ 1 \end{bmatrix}$ under T.

(a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{pmatrix}$

(b) Images of the given points are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 & 11 \\ 1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 11 \\ 0 & 1 \\ 2 & \sqrt{3} \end{pmatrix}$$

(c) Find all points $\vec{x}$ that are fixed under T, that is, $T(\vec{x}) = \vec{x}$.



Guess: Any point on x-axis, i.e. $\vec{x} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$

Proof. Assume $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is fixed:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{cases} x = x \\ \frac{\sqrt{3}}{2}y - \frac{1}{2}z = y \\ \frac{1}{2}y + \frac{\sqrt{3}}{2}z = z \end{cases}$$

$$\Leftrightarrow \begin{cases} \left(\frac{\sqrt{3}}{2} - 1\right) y = \frac{1}{2} z \Rightarrow z = (\sqrt{3} - 2)\, y \\ \frac{1}{2} y = \left(1 - \frac{\sqrt{3}}{2}\right) z \Rightarrow y = (2 - \sqrt{3}) z = (2-\sqrt{3})(\sqrt{3}-2)\, y \end{cases}$$

$$y = -(2-\sqrt{3})^2 y \Rightarrow y\left(1 + (2-\sqrt{3})^2\right) = 0 \Rightarrow y = 0,\ z = 0$$

$\therefore$ Any fixed point is $\vec{x} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$, i.e. any point on x-axis.

(d) Find the image of $\beta : x - 3y - z = 11$ under T.

Exercise!

Answer:

$$\beta' : \vec{x} = \begin{pmatrix} 11 \\ 0 \\ 0 \end{pmatrix} + \frac{s}{2}\begin{pmatrix} 6 \\ \sqrt{3} \\ 1 \end{pmatrix} + \frac{t}{2}\begin{pmatrix} 2 \\ -1 \\ \sqrt{3} \end{pmatrix}$$

(e) Find the image of the line $l : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{3} \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ \sqrt{3} \end{bmatrix}$ under T.

Exercise!

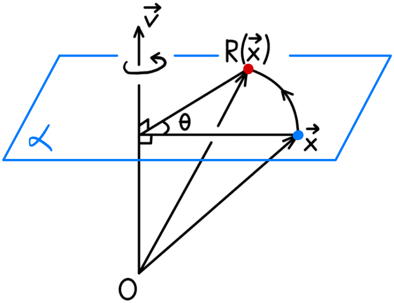
Answer:

$$l' : \vec{x} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Rotation about any vector

The rotation in $\mathbb{R}^3$ can be defined around any nonzero vector $\vec{v}$. Assume we want to find the image of $\vec{x}$ when rotating about $\vec{v}$ over angle θ.

1. Draw a plane α through $\vec{x}$ and perpendicular to $\vec{v}$



2. On α, rotate $\vec{x}$ by angle θ to obtain its image $R(\vec{x})$.

Rotation about any vector

Theorem 3

The matrix of the counter clockwise rotation around the vector $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ over the angle θ is

$$M = \frac{(1 - \cos\theta)}{\|\vec{v}\|^2} \vec{v}\vec{v}^T + (\cos\theta) I_3 + \frac{\sin\theta}{||\vec{v}||} C_{\vec{v}},$$

where $C_{\vec{v}} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$ is the cross-product matrix induced by $\vec{v}$.

Example 7

Find the matrix of the rotation about the vector $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ over $\theta = 60^\circ$.

$$M = \frac{1-\cos\theta}{\|\vec{v}\|^2} v v^T + \cos\theta\, I_3 + \frac{\sin\theta}{\|\vec{v}\|} C_{\vec{v}} \quad \text{with} \quad C_{\vec{v}} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

$$= \frac{1-\frac{1}{2}}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} + \frac{1}{2} I_3 + \frac{\sqrt{3}/2}{\sqrt{3}} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix}$$

Example 7

Find the matrix of the rotation about the vector $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ over $\theta = 60^\circ$.

(b) Image of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is $\frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(c) Question: What points are fixed by this rotation?

All fixed points lie on the line through O & having direction $\vec{v}$, i.e.,

$$\vec{x} = \vec{O} + t\vec{v} = t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$