Lecture 2: Functions

Password : function

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Sets

- The **difference** of A and B is

$$A - B = \{x : x \in A, x \notin B\}.$$

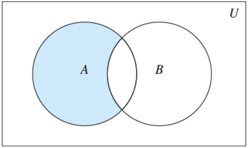
- The **complement** of A in U is

$$\bar{A} = \{x \in U : x \notin A\}.$$

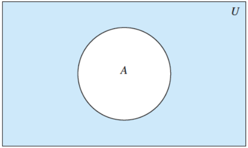
- De Morgan's law:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$



$A - B$ is shaded.



$\bar{A}$ is shaded.

Power set

- $\mathcal{P}(S)$ = all subsets of S. If $|S| = n$, then S has 2^n subsets

$$|\mathcal{P}(S)| = 2^n$$

- **Exercise 1**: Let $A = \{1\}$ and $B = \{2\}$ be two sets.

  (a) Find $\mathcal{P}(A), \mathcal{P}(B), \mathcal{P}(A \times B)$.

  $P(A) = \{\emptyset, \{1\}\}$, $P(B) = \{\emptyset, \{2\}\}$

  $A \times B = \{(a,b) : a \in A, b \in B\} = \{(1,2)\}$

  $P(A \times B) = \{\emptyset, \{(1,2)\}\}$

  (b) Do $\mathcal{P}(A) \times \mathcal{P}(B)$ and $\mathcal{P}(A \times B)$ have same size?

  $|P(A \times B)| = 2$

  $|P(A) \times P(B)| = |P(A)| \cdot |P(B)| = 2 \cdot 2 = 4$

  $\therefore |P(A \times B)| \neq |P(A) \times P(B)|$

Inclusion-exclusion principle

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}|$$

sum of n terms corr. to $k = 1, 2, \ldots, n$

sum of all intersections of k sets among $A_1, \ldots, A_n$

Inclusion-exclusion principle

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}|$$

- $n = 2$

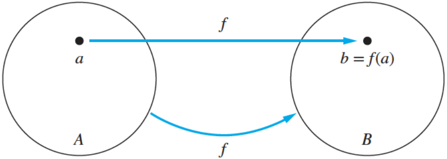
$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

- $n = 3$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

Functions (definition)

- Let A and B be nonempty sets.
- $f : A \to B$ is a function if it assigns each element $a \in A$ to a **unique element** $b \in B$. Write $f(a) = b$.



Functions (definition)

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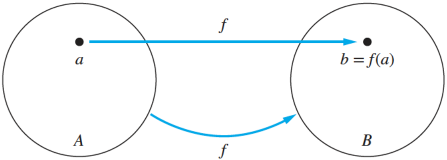
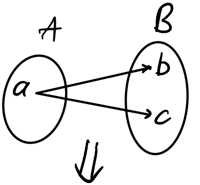


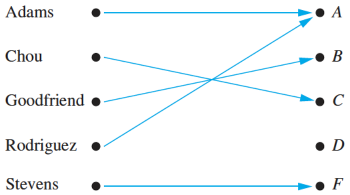


not a function

- A is called the **domain of f** (or the **input set**).
  B is called the **codomain of f** (or the **output set**).
- If $f(a) = b$, we call b the *image* of a and call a a *preimage* of b.

Example 1

The following is an assignment of grades in a discrete mathematics class. This assignment corresponds to a function $f : S \to T$.



(a) Write out the domain S of f.

$S = \{\text{Adams, Chou, Goodfriend, Rodriguez, Stevens}\}$

Example 2

(b) Write out the codomain T of f.

$T = \{A, B, C, D, F\}$

(c) Write out the values of $f(s)$ for all elements $s \in S$.

$f(\text{Adams}) = A$

$f(\text{Chou}) = C$

$f(\text{Goodfriend}) = B$

$f(\text{Rodriguez}) = A$

$f(\text{Stevens}) = F$

Example 3

Let f be the function that assigns the last two bits of a bit string of length 3 to that string. For example, $f(010) = 10.$

(a) Write out the domain A of f.

A = strings of length 3 = $\{000, 001, 010, 011, 100, 101, 110, 111\}$

(b) Write out the codomain B of f.

B = strings of length 2 = $\{00, 01, 10, 11\}$

(c) Write out the values of $f(a)$ for all elements $a \in A$.

$f(000) = 00$

$f(001) = 01$

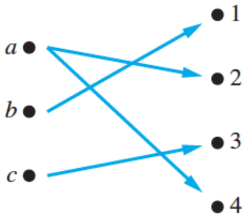
$\vdots$

$f(110) = 10$

$f(111) = 11$

Question 2

Is $f : \{a, b, c\} \to \{1, 2, 3, 4\}$ given by the following rule a function?



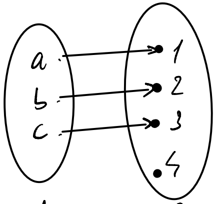
No : a is assigned to two values 2 & 4.

Question 3

Let $f : A \to B$ be a function. Is it always true that for any $b \in B$, there exists a unique element $a \in A$ such that $f(a) = b$?

Answer: No. There can be 2 situations.

1. There can be $b \in B$ s.t. there is no $a \in A$ with $f(a) = b$.



$f : A \longrightarrow B$

$f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^2$$

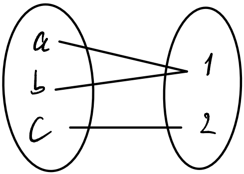
Let $b = -1$, there is no $a \in \mathbb{R}$:

$$f(a) = b \Leftrightarrow a^2 = -1.$$

There is no $a \in A$: $f(a) = 4$

Question 3

2. There can be $b \in B$ such that there are more than one element $a \in A$ with $f(a) = b$.



$f : A \to B$

$f(a) = f(b) = 1$

$f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$

$f(1) = f(-1) = 1$

Summary on functions

A function $f : A \to B$ is a rule (or an assignment) that assigns each value $a \in A$ to a unique value $b \in B$, that is,

$$f(a) = b$$

- Given $a \in A$, there is a unique $b \in B$ such that $f(a) = b$.
- Given $b \in B$, there can be more than one $a \in A$ such that $f(a) = b$, or there can be no $a \in A$ s.t. $f(a) = b$.

One-to-one, onto, bijective

- A function $f : A \to B$ is **one-to-one** (also write **1-1**), or **injective**, if for any $a, b \in A$

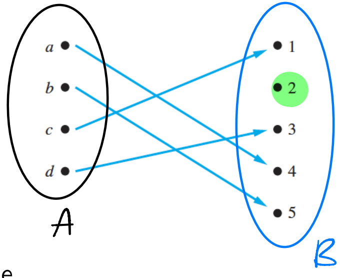
$$f(a) = f(b) \Leftrightarrow a = b$$

- f is **onto**, or **surjective**, if for any $b \in B$ there exists $a \in A$ such that $f(a) = b$.
- f is **bijective** (or a **bijection**) if it is both 1-1 and onto.

Example 4

Which of the following functions is 1-1, onto, or bijective?

(a) f given by the rule

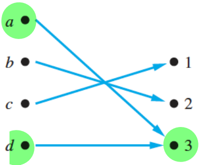


1-1? ✓

Onto? ✗

bijective? ✗

(b) $f : \{a, b, c, d\} \to \{1, 2, 3\}$ given by the rule



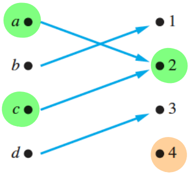
1-1? ✗

Onto? ✓

bijective? ✗

Example 4

(c) $f : \{a, b, c, d\} \to \{1, 2, 3, 4\}$ given by the rule



1-1? ✗

Onto? ✗

bijective? ✗

(d) $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x^2$. $\mathbb{N} = \{0, 1, 2, \ldots\}$

1-1? ✓

Onto? ✗ There is no $x \in \mathbb{N}$: $x^2 = 3$.

bijective? ✗

Example 4

(e) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 5$.

f is 1-1: Assume $x, y \in \mathbb{R}$: $f(x) = f(y)$

$2x+5 = 2y+5 \Rightarrow 2x = 2y \Rightarrow x = y$.

f is onto: let $b \in \mathbb{R}$ be any output $\to$ need to show that there is $a \in A$: $f(a) = b$

$2a + 5 = b \Rightarrow a = \dfrac{b-5}{2}$.

$\therefore$ f is both 1-1 & onto $\Rightarrow$ f is a bijection.

Sums and products of functions

- $f : A \to B$ is called **real-valued** if its codomain is $B = \mathbb{R}$, and it is called **integer-valued** if its codomain is $B = \mathbb{Z}$.

Sums and products of functions

- $f : A \to B$ is called **real-valued** if its codomain is $B = \mathbb{R}$, and it is called **integer-valued** if its codomain is $B = \mathbb{Z}$.
- Let $f_1, f_2 : A \to B$ be real-valued (or integer-valued) functions. Then $f_1 + f_2$ and $f_1 f_2$ are functions from A to B defined by

$$\begin{aligned} (f_1 + f_2)(x) &= f_1(x) + f_2(x) \\ f_1 f_2(x) &= f_1(x) f_2(x) \end{aligned}$$

Increasing and decreasing functions

Let $f : A \to B$ be a ~~function~~ real-valued function

- f is called **increasing** if

$$f(x) \leq f(y) \text{ whenver } x < y$$

- f is called **strictly increasing** if

$$f(x) < f(y) \text{ whenver } x < y$$

Increasing and decreasing functions

Let $f : A \to B$ be a function

- f is called **decreasing** if

$$f(x) \geq f(y) \text{ whenver } x < y$$

- f is called **strictly decreasing** if

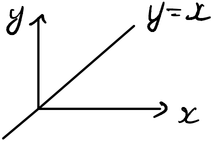
$$f(x) > f(y) \text{ whenver } x < y$$

Example 5

(a) Is $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ increasing or decreasing?

If $x < y$, then $f(x) = x < f(y) = y$

$\therefore$ f is increasing, even strictly increasing.



(b) Is $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ increasing or decreasing?

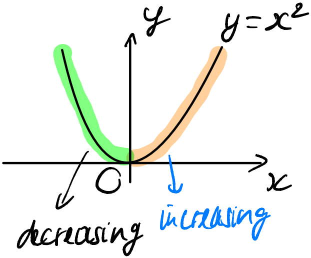
Not increasing:

$-2 < -1$, but $f(-2) = 4 > f(-1) = 1$

Not decreasing:

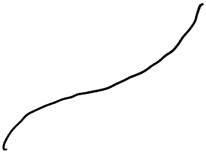
$1 < 2$, but $f(1) = 1 < f(2) = 4$

$\therefore$ f is neither increasing nor decreasing.

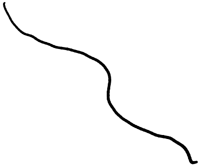


Graphs of increasing and decreasing functions

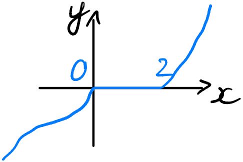
Strictly Increasing



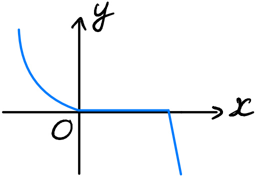
Strictly decreasing



Increasing



Decreasing



Derivative test

Theorem 1

Assume that $f : A \to B$ is a differentiable function.

(a) If $f'(x) \geq 0$ for all $x \in A$, then f is increasing.
Further if $f'(x) > 0$ for all $x \in A$, f is strictly increasing.

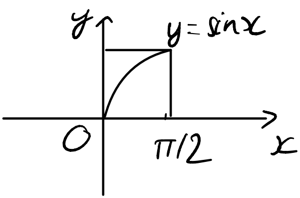
(b) If $f'(x) \leq 0$ for all $x \in A$, then f is decreasing.
Further if $f'(x) < 0$ for all $x \in A$, f is strictly decreasing.

Example 6

Determine whether following functions are increasing or decreasing. Support your claim by drawing graphs of these functions.

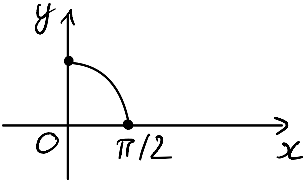
(a) $f : [0, \pi/2] \to [0, 1]$ given by $f(x) = \sin x$.

Increasing!



(b) $f : [0, \pi/2] \to [0, 1]$ given by $f(x) = \cos x$.

Decreasing !



Example 7

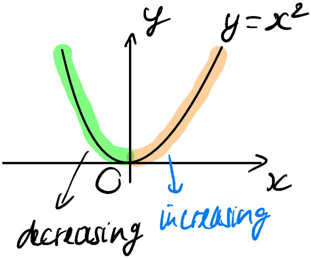
Determine the intervals on which $f(x) = x^2$ is increasing and decreasing. Support your claim by drawing the graph of $f(x)$.

$f: \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$.

$f'(x) = 2x$

$f'(x) \geq 0 \Leftrightarrow x \geq 0$

$f'(x) \leq 0 \Leftrightarrow x \leq 0$

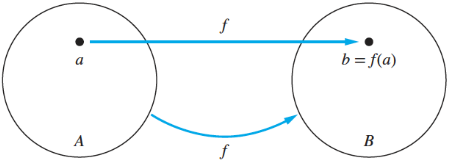


$\therefore$ f is increasing on $[0, \infty)$.

f is decreasing on $(-\infty, 0]$.

Question 1

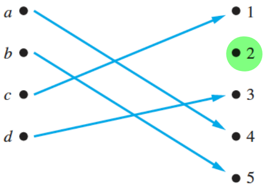
- A function $f : A \to B$ is a rule which assigns each $a \in A$ to $b \in B$, that is, $f(a) = b$.

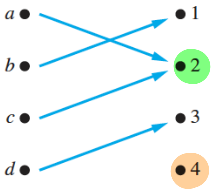


- Can this assignment be reverse, i.e. each element $b \in B$ is assigned to $a \in A$ if $f(a) = b$?

Question 1 answer

- The reverse assignment doesn't always work. Consider examples





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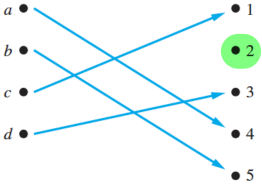
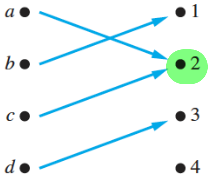
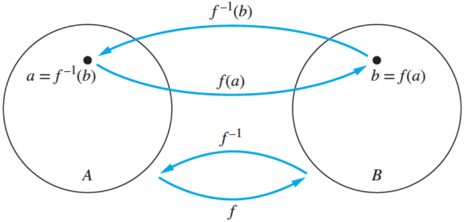




- For the reverse assignment to work, f needs to be both $1-1$ and onto, that is, f is a bijection.



Inverse function - definition

- Let $f : A \to B$ be $1-1$ and onto.
- The **inverse function** of f is $f^{-1} : B \to A$ that assigns $b \in B$ to $a \in A$ if $f(a) = b$:

$$f^{-1}(b) = a \Leftrightarrow f(a) = b.$$

apply f to both sides : $f(f^{-1}(b)) = f(a)$

$b = f(a)$

Inverse function - definition

- Let $f : A \to B$ be $1-1$ and onto.
- The **inverse function** of f is $f^{-1} : B \to A$ that assigns $b \in B$ to $a \in A$ if $f(a) = b$:

$$f^{-1}(b) = a \Leftrightarrow f(a) = b.$$

- We call f **invertible** if its inverse exists, that is,

$$f \text{ is both } 1-1 \text{ and onto.}$$

Invertible $\Leftrightarrow$ bijective

Remarks

1. f^{-1} and $\dfrac{1}{f}$ are different functions.
2. Difference in notation
   - When writing f, we usually use x to denote its input: $f(x)$.
   - When writing f^{-1}, we usually use y to denote input: $f^{-1}(y)$.

Example 8

Given $f : \{a, b, c\} \to \{1, 2, 5\}$ defined by $f(a) = 1, f(b) = 2, f(c) = 5$.
Find f^{-1} if it exists.

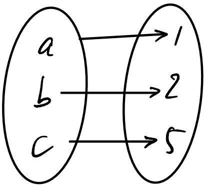
f is both 1-1 & onto $\to$ f^{-1} exists.

$f^{-1}: \{1,2,5\} \to \{a,b,c\}$

$f^{-1}(1) = a$

$f^{-1}(2) = b$

$f^{-1}(5) = c$



How to find f^{-1}?

Assume that $f : A \to B$ is given by a formula.

To find $f^{-1} : B \to A$, we follow 3 steps

1. Let $x \in B$ and put $y = f^{-1}(x)$. → apply f on both sides: $f(y) = f(f^{-1}(x)) = x$
2. Solve for y (in terms of x) based on the equation

$$\mathbf{f(y) = x}.$$

3. Give conclusion.

Example 9

In the following cases, determine whether f is invertible and find its inverse if it exists.

(a) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x + 1$.

1. Note that $f^{-1} : \mathbb{Z} \to \mathbb{Z}$. Let $x \in \mathbb{Z}$ and put $y = f^{-1}(x)$.
2. We have
$$f(y) = x \Rightarrow y + 1 = x \Rightarrow y = x - 1.$$
3. Conclusion
$$f^{-1}(x) = x - 1.$$

Example 9

(b) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

f is not 1-1 : $f(1) = f(-1) = 1$. So f is not invertible $\Rightarrow f^{-1}$ doesn't exist.

(c) $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f(x) = x^2$.

(real positive numbers) $\to x, y \in \mathbb{R}^+$

f is invertible

1-1 : $f(x) = f(y) \Leftrightarrow x^2 = y^2 \Leftrightarrow \left[\begin{array}{l} x = y \\ x = -y \end{array}\right.$ (cannot happen because $x, y \in \mathbb{R}^+$)

onto : For any $y \in \mathbb{R}^+$, $f(\sqrt{y}) = y$

(1) $f^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$. Take $x \in \mathbb{R}^+$ and put $y = f^{-1}(x)$

(2) $f(y) = x \Rightarrow y^2 = x \Rightarrow \left[\begin{array}{l} y = \sqrt{x} \\ y = -\sqrt{x} \ \times \end{array}\right.$

(3) $\therefore f^{-1}(x) = \sqrt{x}$.

Example 9

(d) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$. Further, find the preimages of $1, 27, 64$ using f^{-1}.

1-1: $f(x) = f(y) \Leftrightarrow x^3 = y^3 \Leftrightarrow x = y$

(i) f is invertible < onto: For any $y \in R$ we have $f(\sqrt[3]{y}) = y$

(1) $f^{-1} : R \to R$. Put $x \in R$ and $y = f^{-1}(x)$

(2) $f(y) = x \Rightarrow y^3 = x \Rightarrow y = \sqrt[3]{x}$

(3) $\therefore$ $f^{-1}(x) = \sqrt[3]{x}$.

(ii) The preimage of b (under f) is any a :

$f(a) = b \Leftrightarrow a = f^{-1}(b)$ (definition of f^{-1})

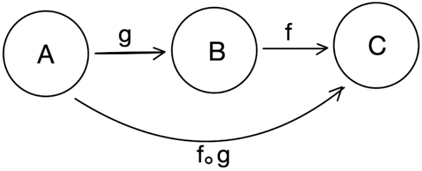
The preimage of 1 is $f^{-1}(1) = \sqrt[3]{1} = 1$.

__________ 64 is $f^{-1}(64) = \sqrt[3]{64} = 4$.

Composition of functions

- Let $g : A \to B$ and let $f : B \to C$ be functions.
- The **composition** of f and g, denoted by $f \circ g$, is the function $f \circ g : A \to C$ defined by

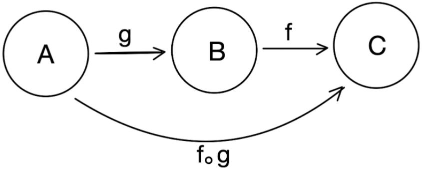
$$(f \circ g)(a) = f(g(a))$$



Composition of functions

- Let $g : A \to B$ and let $f : B \to C$ be functions.
- The **composition** of f and g, denoted by $f \circ g$, is the function $f \circ g : A \to C$ defined by

$$(f \circ g)(a) = f(g(a))$$



- **Remark**: $f \circ g \neq fg$

$$f \circ g(x) = f(g(x)) \text{ and } fg(x) = f(x)g(x)$$

Example 11

Find $f \circ g$ and $g \circ f$ in following cases

(a) $f, g : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 2x + 1$ and $g(x) = 3x + 2$.

$f \circ g : \mathbb{Z} \to \mathbb{Z}$ with

$f \circ g(x) = f(g(x)) = f(3x+2)$
$= 2(3x+2)+1 = 6x+5$

$g \circ f : \mathbb{Z} \to \mathbb{Z}$ with

$g \circ f(x) = g(f(x)) = g(2x+1)$
$= 3(2x+1)+2 = 6x+5$

(b) $f : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ with $f(x) = x^2$, $g : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ with $g(x) = \sqrt{x}$.

$f \circ g : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$

$$\begin{tikzcd} \mathbb{R}^+ \cup \{0\} \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & \mathbb{R} \arrow[r, "f"] & \mathbb{R}^+ \cup \{0\} \end{tikzcd}$$

$f \circ g(x) = f(g(x)) = f(\sqrt{x}) = x$

$g \circ f : \mathbb{R} \to \mathbb{R}$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & \mathbb{R}^+ \cup \{0\} \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

$g \circ f(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = |x|$

Example 12

Find $f^{-1}, g^{-1}, g \circ f, (g \circ f)^{-1}$ for

$$f, g : \mathbb{R} \to \mathbb{R} \text{ defined by } f(x) = 2x + 1 \text{ and } g(x) = 3x + 2.$$

(i) $f^{-1} : \mathbb{R} \to \mathbb{R}$. let $x \in \mathbb{R}$ and put $y = f^{-1}(x)$

$$f(y) = x \Rightarrow 2y + 1 = x \Rightarrow y = \frac{x-1}{2}$$

$$\therefore f^{-1}(x) = \frac{x-1}{2}$$

(ii) $g^{-1} : \mathbb{R} \to \mathbb{R}$. let $x \in \mathbb{R}$ and put $y = g^{-1}(x)$

$$g(y) = x \Rightarrow 3y + 2 = x \Rightarrow y = \frac{x-2}{3}$$

$$\therefore g^{-1}(x) = \frac{x-2}{3}$$

Example 12

Find $f^{-1}, g^{-1}, g \circ f, (g \circ f)^{-1}$ for

$$f, g : \mathbb{R} \to \mathbb{R} \text{ defined by } f(x) = 2x + 1 \text{ and } g(x) = 3x + 2.$$

(iii) By 11a, $g \circ f(x) = 6x + 5$

$(g \circ f)^{-1} : \mathbb{R} \to \mathbb{R}$. let $x \in \mathbb{R}$ and put $y = (g \circ f)^{-1}(x)$

$g \circ f(y) = x \Rightarrow 6y + 5 = x \Rightarrow y = \dfrac{x-5}{6}$

$\therefore f^{-1}(x) = \dfrac{x-5}{6}$

Exercise 2

Let $f : A \to B$ be both $1-1$ and onto. Show that

$f^{-1} \circ f(x) = x$ for any $x \in A$ and $f \circ f^{-1}(y) = y$ for any $y \in B$.

(i) We prove $f^{-1} \circ f(x) = x$ for any $x \in A$. Note that

$$A \xrightarrow{f} B \xrightarrow{f^{-1}} A \Rightarrow f^{-1} \circ f : A \to A$$

To compute $f^{-1} \circ f(x) = f^{-1}(f(x))$, we first compute $y = f(x)$.

Since $y = f(x)$, we have $x = f^{-1}(y)$ (def of inverse function):

$$f^{-1} \circ f(x) = f^{-1}(f(x)) = f^{-1}(y) = x.$$

(ii) Proving $f \circ f^{-1}(y) = y$ for any $y \in B$ is similar.

<u>Conclusion</u>: f & f^{-1} cancel out each other $\left\langle \begin{array}{l} f \circ f^{-1}(x) = x \\ f^{-1} \circ f(x) = x \end{array} \right.$