Attendance: ykdag9

Week 7 HW: tuned towards Quiz 2

Additional problems.

submission one week after Week 5 HW submission

Ideally should finish before Quiz 2

Week 8 tutorials: Quiz 2 1hr

6 MCQ + 2 open-ended

Q1 (c), (d), (e)

Q2, Q3

(c) $\displaystyle\int \frac{4x}{x^3+x^2+x+1}\,dx.$ ← proper rational function

$$x^3+x^2+x+1 = x^2(x+1)+1(x+1)$$

Similar in lecture slides

$$= (x+1)(x^2+1).$$

ax^2+bx+c, Check $b^2-4ac = -4(1)(1) = -4 < 0$

$$\frac{4x}{x^3+x^2+x+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

numerator must be linear

← irr factor

$x=-1$

$$4x = A(x^2+1) + (Bx+C)(x+1)$$

Sub $x=-1$: $-4 = 2A \Rightarrow A = -2$

$$4x = -2(x^2+1) + (Bx+C)(\underline{\underline{x+1}})$$

Sub $x=0$: $0 = -2 + C$

$$\Rightarrow C = 2$$

$$4x = -2(x^2+1) + (Bx+2)(x+1)$$

Compare coefficient of x^2:

$$0 = -2 + B \Rightarrow B = 2.$$

$$\frac{4x}{x^3+x^2+x+1} = -\frac{2}{x+1} + \frac{2x+2}{x^2+1}$$

$$\therefore \int \frac{4x}{x^3+x^2+x+1}\,dx = -2\int \frac{1}{x+1}\,dx + \int \frac{2x+2}{x^2+1}\,dx$$

$$= -2\ln|x+1| + \underbrace{\int \frac{2x}{x^2+1}\,dx}_{u=x^2+1} + \int \frac{2}{x^2+1}\,dx$$

$$= -2\ln|x+1| + \ln(x^2+1) + 2\tan^{-1}x + C.$$

(d) $\displaystyle\int \frac{2}{x^3+2x^2+2x}\,dx$

$$x^3+2x^2+2x = x\underbrace{(x^2+2x+2)}_{\text{irreducible ✓}}$$

$$\frac{2}{x^3+2x^2+2x} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+2}$$

$$2 = A(x^2+2x+2) + (Bx+C)x$$

$x=0$: $2 = 2A \Rightarrow \underline{A=1}$

$$2 = (x^2+2x+2) + (Bx+C)x$$

Comparing coefficient of x^2:

$$0 = 1 + B \Rightarrow \underline{B=-1}$$

Comparing coefficient of x:

$$0 = 2 + C \Rightarrow \underline{C=-2.}$$

$$\therefore \frac{2}{x^3+2x^2+2x} = \frac{1}{x} - \frac{(x+2)}{x^2+2x+2}$$

ideally I want $2x+2$

$$\int \frac{2}{x^3+2x^2+2x}\,dx = \int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{2(x+2)}{\underbrace{x^2+2x+2}_{2x+2}}\,dx.$$

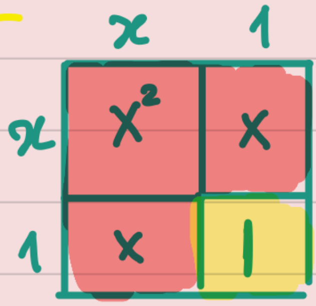
$$= \ln|x| - \frac{1}{2}\int \frac{2x+2+2}{x^2+2x+2}\,dx$$

$$= \ln|x| - \frac{1}{2}\int \frac{2x+2}{x^2+2x+2}\,dx - \frac{1}{2}\int \frac{2}{x^2+2x+2}\,dx$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+2x+2) - \int \frac{1}{x^2+2x+2}\,dx$$

$$x^2+2x+2 = x^2+2x+1+1 = (x+1)^2+1$$

$x \qquad 1$



$$x^2+2x+1 = (x+1)^2$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+2x+2) - \int \frac{1}{1+(x+1)^2}\,dx$$

$u = x+1$
$du = dx$

$$= \ln|x| - \frac{1}{2}\ln(x^2+2x+2) - \int \frac{1}{1+u^2}\,du$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+2x+2) - \tan^{-1}u + C$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+2x+2) - \tan^{-1}(x+1) + C$$

(e) $x^3+4x^2+5x = x\underbrace{(x^2+4x+5)}_{b^2-4ac=-4<0 \text{ irr}}$

Complete the square

$$x^2+4x+5 = (x+2)^2+1$$

$$\frac{x-1}{x^3+4x^2+5x} = \frac{A}{x} + \frac{Bx+C}{x^2+4x+5}$$

$$x-1 = A(x^2+4x+5) + (Bx+C)x$$

$x=0: \quad -1 = 5A \Rightarrow A = -\frac{1}{5}$

$\Rightarrow x-1 = -\frac{1}{5}(x^2+4x+5) + (Bx+C)x$

Comparing coefficient of x^2

$0 = -\frac{1}{5} + B \Rightarrow B = \frac{1}{5}$

Compare coefficient of x

$1 = -\frac{4}{5} + C \Rightarrow C = \frac{9}{5}.$

$\therefore \frac{x-1}{x^3+4x^2+5x} = -\frac{1}{5x} + \frac{x+9}{5(x^2+4x+5)}$

$\therefore \int \frac{x-1}{x^3+4x^2+5x}\,dx = -\frac{1}{5}\int \frac{1}{x}\,dx + \frac{1}{10}\int \frac{2(x+9)}{x^2+4x+5}\,dx$

$= -\frac{1}{5}\ln|x| + \frac{1}{10}\int \frac{2x+4+14}{x^2+4x+5}\,dx$

$= -\frac{1}{5}\ln|x| + \frac{1}{10}\int \frac{2x+4}{x^2+4x+5}\,dx + \frac{1}{10}\int \frac{14}{x^2+4x+5}\,dx$

$u = x^2+4x+5$

$= -\frac{1}{5}\ln|x| + \frac{1}{10}\ln(x^2+4x+5) + \frac{7}{5}\int \frac{1}{1+(x+2)^2}\,dx$

$= -\frac{1}{5}\ln|x| + \frac{1}{10}\ln(x^2+4x+5) + \frac{7}{5}\tan^{-1}(x+2) + C.$

Q2 $\quad a=0,\ b=1,\ n=10 \quad f(x) = \sin(x^2) \quad \Delta x = \frac{1}{10}$

Used in Trapezoidal / Simpson's

$x_0 = 0,\ x_1 = 0.1,\ x_2 = 0.2,\ \ldots,\ x_8 = 0.8,\ x_9 = 0.9,\ x_{10} = 1$

$\bar{x}_1 = 0.05 \quad \bar{x}_2 = 0.15 \quad \bar{x}_8 = 0.75 \quad \bar{x}_9 = 0.85 \quad \bar{x}_{10} = 0.95$

$$M_{10} = \Delta x \left[f(\bar{x}_1) + f(\bar{x}_2) + \ldots + f(\bar{x}_9) + f(\bar{x}_{10}) \right]$$

$$= \frac{1}{10} \left[\sin(0.05^2) + \sin(0.15^2) + \ldots + \sin(0.85^2) + \sin(0.95^2) \right]$$

$$\approx 0.309816.$$

$$\int \sin^2 x \, dx$$

$$\int \sin(x^2) \, dx \quad \times$$

$$T_{10} = \frac{\Delta x}{2} \left[f(x_0) + 2f(x_1) + \ldots + 2f(x_9) + f(x_{10}) \right]$$

$$= \frac{1}{20} \left[\sin(0^2) + 2\sin(0.1^2) + \ldots + 2\sin(0.9^2) + \sin(1^2) \right]$$

$$\approx 0.311171$$

(b) $f(x) = \sin(x^2)$

$$f'(x) = 2x\cos(x^2)$$

$$f''(x) = 2\cos(x^2) + 2x(-\sin(x^2) \cdot 2x)$$

$$= 2\cos(x^2) - 4x^2 \sin(x^2)$$

$$|a+b| \le |a| + |b|$$

$$|ab| = |a| \cdot |b|$$

$$|f''(x)| = |\underbrace{2\cos(x^2)}_{a} + \underbrace{(-4x^2\sin(x^2))}_{b}|$$

$$\le |2\cos(x^2)| + |-4x^2\sin(x^2)|$$ triangle inequality

$$= 2\underbrace{|\cos(x^2)|}_{\le 1} + 4\underbrace{|x^2|}_{\le 1}\underbrace{|\sin(x^2)|}_{\le 1} \le 2 \cdot 1 + 4 \cdot 1 \cdot 1 = 6.$$

$$K = 6$$

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}$$

$$= \frac{6}{12 \cdot 10^2}$$

$$= \frac{1}{200}$$

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}$$

$$= \frac{6}{24 \cdot 10^2}$$

$$= \frac{1}{400}$$

(c) $10^{-5} = \frac{1}{10^5} = 0.00001$

n variable, $K=6$

$$|E_T| \leq \frac{6}{12n^2} < 0.00001$$

$$\Rightarrow n^2 > \frac{6}{0.00012} \Rightarrow n > \sqrt{\frac{6}{0.00012}}$$

$$\approx 223.607$$

T_n

Take $n=224$ terms to get error $< 10^{-5}$.

$$|E_M| \leq \frac{6}{24n^2} < 0.00001$$

$$\Rightarrow n^2 > \frac{6}{0.00024} \Rightarrow n > \sqrt{\frac{6}{0.00024}}$$

$$\approx 158.114.$$

M_n: Take $n=159$ terms.

$\underline{Q3}$ $\int_0^1 e^{x^2} dx$ $a=0$, $b=1$, $n=8$

$f(x) = e^{x^2}$

$x_0 = 0, x_1 = \frac{1}{8}, x_2 = \frac{1}{4}, \ldots, x_7 = \frac{7}{8}, x_8 = 1$ $\Delta x = \frac{1-0}{8} = \frac{1}{8}$

$$S_8 = \frac{\Delta x}{3}\left[f(0) + 4f\left(\tfrac{1}{8}\right) + 2f\left(\tfrac{1}{4}\right) + \ldots + 4f\left(\tfrac{7}{8}\right) + f(1)\right]$$

$$= \frac{1}{24}\left[e^{0^2} + 4e^{\frac{1}{64}} + 2e^{\frac{1}{16}} + \ldots + 4e^{\frac{49}{64}} + e\right]$$

$$\approx 1.462723$$

(b) $f(x) = e^{x^2}$

$f'(x) = 2xe^{x^2} \geq 0 \to f$ increasing $[0,1]$ $|e^{x^2}| \leq e^{1^2} = e$

$f''(x) = (4x^2+2)e^{x^2}$

$f'''(x) = (8x^3+12x)e^{x^2}$

$f^{(4)}(x) = (16x^4+48x^2+12)e^{x^2}$

$$|f^{(4)}(x)| = |(16x^4+48x^2+12)e^{x^2}|$$

$$= |16x^4+48x^2+12|\,\underbrace{|e^{x^2}|} \leq 76e = K$$

$$|16x^4+48x^2+12| \leq |16x^4| + |48x^2| + |12|$$

$$= 16\underbrace{|x^4|}_{\leq 1} + 48\underbrace{|x^2|}_{\leq 1} + 12 \quad [0,1]$$

$$\leq 16 + 48 + 12 = 76$$

$$|E_s| \le \frac{K(b-a)^5}{180\,n^4} = \frac{76e}{180n^4} = \frac{76}{180\cdot 8^4} = \frac{19e}{184320}$$

$$\approx 0.000280205$$

(c) n unknown

$$|E_s| \le \frac{76e}{180\cdot n^4} \le 0.0001$$

$$n^4 \ge \frac{76e}{0.018} \Rightarrow n \ge \sqrt[4]{\frac{76e}{0.018}}$$

$$\approx 10.35044$$

n=12 (not 11 because n even