Lecture 1: Sample spaces, probability measures, Counting rules, conditional probability

Yilin

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MAT340/CSD3240/CSD3241 Probability & Statistics

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Office Hour: Make an appointment or messages on Teams

Course Content

1. Probability Theory
2. Random variables
3. Expected values
4. Survey sampling
5. Distributions derived from the normal distribution
6. Estimation of parameters
7. Hypothesis testing

1H

2H

Assessment Tasks

Assessment Task	Weighting	Tentative date
Homework	10%	Weekly
2 Quizzes	20%	Week 4, 11
5 Pop-Up Quizzes	10%	Week 2, 3, 9, 10, 12
1 Midterm test	30%	Week 6
1 Final test	30%	Week 14

Tutorial: 30 min work on tutorial problems
30 min review together.
1h quizzes. or 30 mins. → 30 min answer questions.
popup quiz

Assessment Detail

- Homework: 10 MCQ questions on Moodle with unlimited attempts.
- Quizzes: 10 MCQ questions on Moodle with single attempts, work individually during tutorial sessions with Safe Exam Browser.
- Pop-Up Quizzes: 5 MCQ questions on Moodle with single attempts, recommend to discuss and work with friends or classmates during tutorial sessions.
- Midterm and Final: 10 MCQ questions and 2 open ended questions. MCQ questions to submit on Moodle and the open ended questions work on the paper and submit the photo on Moodle.

New Rules for Grades on Moodle

- **Only letter grade** is available on Moodle.
- The letter grades for quizzes and midterm test will be released during the trimester
- The letter grades for final test is **not available**.

Plan for the Week 1: Getting Started

$\boxed{\text{E6-04-16}}$

Qinjie.

- Week 1 Lecture
- Week 1 Tutorial
- Week 1 Homework
- No quiz this week :)

What is Probability?

- Rigorous mathematical theory to analyze events that involve uncertainty
- Almost everything involves uncertainty
- Applications: business, finance, actuarial science, risk management, economics, computer science, quality control, and many other areas

Experiments, Sample Spaces and Events

- An **experiment** is a situation with uncertain outcomes.
- A **sample space** of an experiment is the set of all possible outcomes of the experiment.
- An **event** is a subset of the sample space.

Sample space is usually denoted by Ω (pronounce "Omega").

Events (subsets of Ω) are denoted by capital letters $A, B, C, D, \ldots$

Example 1

- Experiment: a commuter passes through 3 traffic lights. At each light, she either stops (s) or continues (c).
- Sample space

$$\Omega = \{\text{ccc}, \text{ccs}, \text{css}, \text{csc}, \boxed{\text{sss}, \text{ssc}, \text{scc}, \text{scs}}\}$$

- Event A: the commuter stops at the first light

$$A = \{\text{sss}, \text{ssc}, \text{scc}, \text{scs}\}$$

Example 2

- Experiment: the number of students attending CSD3240 this trimester.
- Sample space (the number of students is capped at 50)

$$\Omega = \{0, 1, \ldots, 50\}$$

- Event A: there are more than 25 students

$$A = \{26, 27, \ldots, 50\}$$

Example 3

- Experiment: tossing a coin 3 times. H or T
- Sample space

$$\Omega = \{\underset{3}{\text{hhh}},\ \overbrace{\underset{2}{\text{hht}},\ \underset{2}{\text{hth}},\ \underset{2}{\text{thh}}},\ \underset{1}{\text{htt}},\ \underset{1}{\text{tht}},\ \underset{1}{\text{tth}},\ \underset{0}{\text{ttt}}\}$$

- Event A: there are exactly two heads

$$A = \{\text{hht, hth, thh}\}$$

Exercises

Exercise 1. Experiment: Choose a letter at random from the word "probability". Write down the sample space for this experiment.

2 2.

Solution.

$$\Omega = \{p, r, o, b, a, i, l, t, y\}.$$

Exercises

Exercise 2. Experiment: roll a dice 3 times. Write down the event A that the total score is at least 17.

Solution.

- Sample space $\Omega = \{\boxed{(a, b, c)} : a, b, c \in \{1, 2, \ldots, 6\}\}$.
- Event $A = \{(6, 6, 5), (6, 5, 6), (5, 6, 6), (6, 6, 6)\}$.

Union, Intersection, Complement of Events

Given events A and B.

union .

intersect

- The **union** of A and B is the event $C = A \cup B$.
- The **intersection** of A and B is the event $C = A \cap B$.
- A and B are **disjoint** if $A \cap B = \emptyset$. empty set
- The complement of A, denoted A^c, is the event that A does not occur

$\bar{A} = A^c$

$$A^c = \{w \in \Omega : w \notin A\}.$$

Union, Intersection, Complement of Events

Venn diagrams are useful in illustrating relation between events.

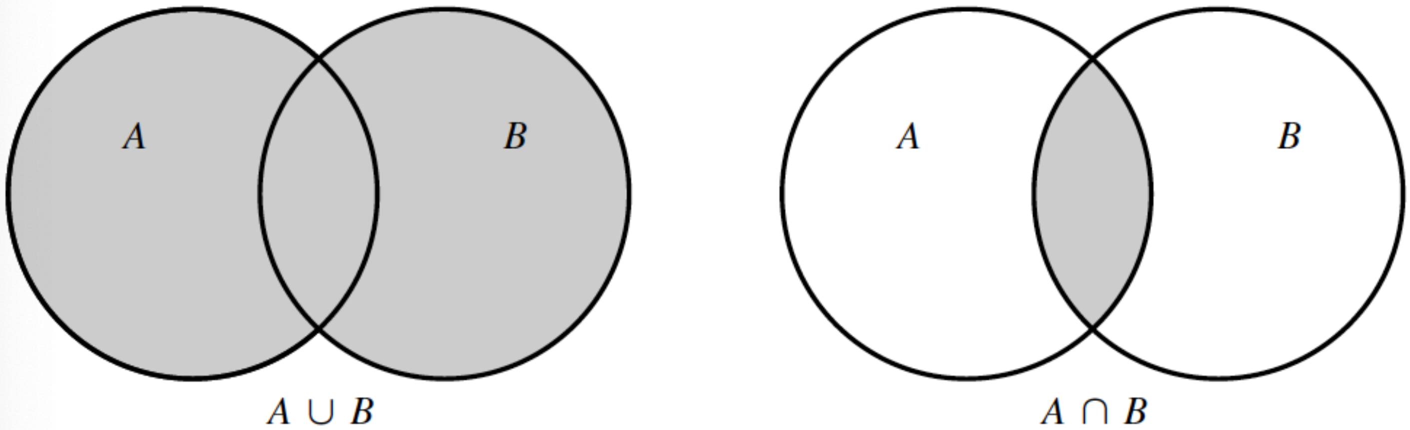


FIGURE **1.1** Venn diagrams of $A \cup B$ and $A \cap B$.

Laws of Set Theory

Given sample space Ω and events A, B, C

- Commutative laws

$$\begin{aligned} A \cup B &= B \cup A \\ A \cap B &= B \cap A \end{aligned}$$

- Associative laws

$$\begin{aligned} (A \cup B) \cup C &= A \cup (B \cup C) \\ (A \cap B) \cap C &= A \cap (B \cap C) \end{aligned}$$

Laws of Set Theory

- Distributive laws

$a(b+c) = ab + ac$

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

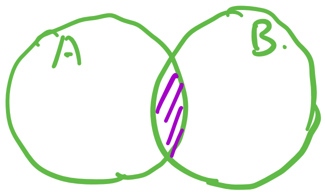
- De Morgan's law

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

reverse "union" and "intersection"

Laws of Set Theory

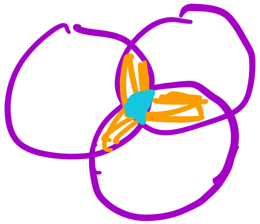
- Inclusion-exclusion principle for two sets



$$|A \cup B| = |A| + |B| - |A \cap B|$$

- Inclusion-exclusion principle for three sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



- Inclusion-exclusion principle for n sets

$$\left| \cup_{i=1}^{n} A_i \right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \left| A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k} \right|$$

Example 1 Revisited

In example 1, the sample space is

$$\Omega = \{\text{ccc, ccs, css, csc, sss, ssc, scc, scs}\}.$$

Event A: the commuter stops at the 1st light.
Event B: the commuter stops at the 3rd light. We have

$$A = \{\text{sss, ssc, scc, scs}\},$$
$$B = \{\text{sss, scs, ccs, css}\}.$$

Example 1 revisited

i. $A \cup B$: she stops either at the 1st light or the 3rd light

$$A \cup B = \{\text{sss,ssc,scc,scs,ccs,css}\}.$$

ii. $A \cap B$: she stops both at the 1st light and the 3rd light

$$A \cap B = \{\text{sss,scs}\}.$$

iii. A^c: she doesn't stop at the 1st light

$$A^c = \{\text{ccc,ccs,css,csc}\}.$$

iv. B^c: she doesn't stop at the 3rd light

$$B^c = \{\text{ccc,csc,ssc,scc}\}.$$

v. By (iii) and (iv)

$$\begin{aligned} A^c \cup B^c &= \{\text{ccc,ccs,css,csc,ssc,scc}\} \\ A^c \cap B^c &= \{\text{ccc,csc}\} \end{aligned}$$

vi. By (i) and (ii)

$$\begin{aligned}(A \cap B)^c &= \{\text{ccc,ccs,css,csc,ssc,scc}\}\\(A \cup B)^c &= \{\text{ccc,csc}\}\end{aligned}$$

- From (v) and (vi), the De Morgan's laws hold in this case

$$(A \cup B)^c = A^c \cap B^c \text{ and } (A \cap B)^c = A^c \cup B^c.$$

Probability Measure

Definition 1. A probability measure on Ω is a function $P : \{\text{subsets of } \Omega\} \to \mathbb{R}$ which satisfies

(i) $P(\Omega) = 1.$

subset

(ii) $P(A) \geq 0$ for any $A \subset \Omega$.

(iii) If $A_1, A_2, \ldots$ are mutually disjoint events, then

$$P\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

Properties of Probability Measure

Lemma 1. P a probability measure on sample space Ω. Then

1. $P(\emptyset) = 0$.
   $\emptyset$ and Ω are disjoint, so $P(\emptyset \cup \Omega) = P(\emptyset) + P(\Omega)$, which implies $P(\emptyset) = 0$.
2. If A and B are disjoint, then $P(A \cup B) = P(A) + P(B)$
   This is a direct consequence of (iii).
3. $P(A^c) = 1 - P(A)$.
   A and A^c **partition** Ω, that is, $A \cap A^c = \emptyset$ and $A \cup A^c = \Omega$.
   So $1 = P(\Omega) = P(A) + P(A^c) \Rightarrow P(A^c) = 1 - P(A)$.

Properties of Probability Measure (Continued)

4. If $A \subset B$, then $P(A) \leq P(B)$.
    - For any two sets X, Y, define $X \setminus Y = \{x \in X : x \notin Y\}$.
    - Since A and $B \setminus A$ partition B, we have $P(B) = P(A) + \boxed{P(B \setminus A)} \geq P(A)$. non-negative.
5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. ← inclusion-exclusion principle.

$A \setminus (A \cap B)$, $A \cap B$ and $B \setminus (A \cap B)$ partition $A \cup B$. We have

$$\begin{aligned} P(A \cup B) &= P(A \setminus (A \cap B)) + P(A \cap B) + P(B \setminus (A \cap B)) \\ &= P(A) - P(A \cap B) + P(A \cap B) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

Properties of Probability Measure (continued)

inclusion-exclusion principle for the probability of 3 events.

6. $P(A \cup B \cup C) =$
$P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$

Example 4

A fair coin is thrown twice.

Event A: head on the first toss.

Event B: head on the second toss.

Find the probability the coin lands on head on only one of the tosses.

$\Omega = \{HH, HT, TH, TT\}$

$A = \{HH, HT\}$; $B = \{HH, TH\}$.

Solution

$$(A \cup B) \setminus (A \cap B) = \{HH, HT, TH\} \setminus \{HH\}$$

$$= \{HT, TH\}.$$

$$|(A \cup B) \setminus (A \cap B)| = 2.$$

$$\frac{|(A \cup B) \setminus A \cap B|}{|\Omega|} = \frac{2}{4} = 50\%$$

Exercise 3

Total # of integers: 1000

Find the probability of the following events.

A a. A randomly chosen integer $x \in [0, 999]$ is divisible by 11.

B b. A randomly chosen integer $x \in [0, 999]$ is divisible by 13.

C c. A randomly chosen integer $x \in [0, 999]$ is divisible by 11 or 13.

$$|A| = \left\lfloor \frac{999}{11} \right\rfloor + 1 = 91$$

$$|B| = \left\lfloor \frac{999}{13} \right\rfloor + 1 = 77$$

Solution

$|C| = |A \cup B| = |A| + |B| - |A \cap B| = 91 + 77 - 7 = 161$

$|A \cap B| = \lfloor \frac{999}{143} \rfloor + 1 = 6 + 1 = 7$

$\lfloor \ \rfloor$: floor.

$\lfloor 9.5 \rfloor$: 9

$\lfloor 9.999 \rfloor = 9$

Uniform Distribution

A uniform distribution is a type of probability distribution in which all outcomes are equally likely. In a uniform distribution, every value within a given range has an equal probability of occurring. Example: If the distribution is over a finite set of discrete outcomes, each outcome has the same probability. For example, rolling a fair six-sided die results in a discrete uniform distribution where each side (1 through 6) has a probability of $1/6$.

Example 5

$\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$

Let $\Omega = \{\cancel{0}, 1, 2\}$, 3. The uniform probability P on Ω is

$$
\begin{aligned}
P(\emptyset) &= 0, \\
\underline{P(\{1\})} &= \underline{P(\{2\})} = \underline{P(\{3\})} = \underline{1/3}, \\
P(\{1, 2\}) &= P(\{1, 3\}) = P(\{2, 3\}) = 2/3, \\
P(\underline{\{1, 2, 3\}}) &= \underline{1}.
\end{aligned}
$$

Product Rule

A procedure can be divided into a sequence of k tasks $T_1, \ldots, T_k$:

- T_1 can be performed in n_1 different ways,
- T_2 can be performed in n_2 different ways (regardless of how T_1 was performed), ...
- T_k can be performed in n_k different ways (regardless of how preceding tasks are performed).

Then the procedure can be performed in $n_1 n_2 \cdots n_k$ different ways.

Home $\xrightarrow{\text{MRT}}$ / Walk / Bus DG $\xrightarrow{\text{MRT}}$ / Bus. punggol $\xrightarrow{\text{MRT}}$ / Bus / Walk SIT.

$3 \times 2 \times 3 = 18$

3 2 3

Example 1

A class has 12 boys and 18 girls. In how many ways can the teacher choose 1 boy and 1 girl as representatives to the student government?

Solution.

$$12 \times 18 = 216.$$

Example 2

An 8-bit binary word is a sequence of 8 digits, of which each may be either a 0 or a 1. How many different 8-bit words are there?

Solution.

$$\underset{1}{\underline{2}} \times \underset{2}{\underline{2}} \times \underline{2} \times \underline{2} \times \underline{2} \times \underline{2} \times \underline{2} \times \underset{8}{\underline{2}} = 2^8.$$

$$\underline{9} \times \underline{10} \times \underline{10} = \boxed{900}$$

Permutations

- A **permutation** of a set S is an *ordered arrangement* of its elements.
- An **r-permutation** of S an *ordered selection* of r elements from S (with no repetitions allowed).
  These are r-tuples $(a_1, \dots, a_r)$ such that a_i's are pairwise distinct and $a_i \in S$ for all i.

Examples

order matters

Let $S = \{1, 2, 3\}$. Then

- $(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2)$ and $(3, 2, 1)$ are all permutations of S.
- $(1, 2)$, $(2, 1)$ and $(1, 3)$ are all 2-permutations of S.
- $(1), (2)$ and (3) are all 1-permutations of S.

Number of permutations

The number of r-permutations of a set of size n is

$$\boxed{P(n,r) = \frac{n!}{(n-r)!}.}$$

In particular, the number of permutations of a set of size n is $n!$.

$$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

Exercise 2

order matters

a) How many ways can five children be lined up in a vertical line?

b) Suppose that from ten children, five are to be chosen and lined up. How many different lines are possible?

Solution.

(a) $5!$

(b). $P(10,5) = \dfrac{10!}{(10-5)!} = \dfrac{10!}{5!}$

$= 10 \times 9 \times 8 \times 7 \times 6$

Combinations

order doesn't matter.

- An **r-combination** of a set S is a subset of size r of S.
- The number of r-combinations of a set of size n is denoted by $\binom{n}{r}$, read as "n choose r".

$C(n,r) = C^n_r = \binom{n}{r}$

Example 3

List all 2-combinations of the set $\{1, 2, 3, 4\}$.

Solution.

$\{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$

Number of combinations

Lemma 2. The number of r-combinations of a set of size n is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Proof. Denote the set of n elements by S. An r-permutation can be chosen in two steps

1. Choose a subset of r elements $\{a_1, \dots, a_r\}$.
2. Choose an ordering for the subset chosen in Step 1.

Binomial coefficients

- The numbers $\binom{n}{k}$ are called **binomial coefficients**.
- They occur in

Binomial Theorem :

$$(a+b)^n = \boxed{\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}}. \quad (1)$$

$(a+b)^2 = a^2+2ab+b^2$. 1; 2; 1 $\to \binom{2}{0}; \binom{2}{1}, \binom{2}{2}$

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$. 1; 3, 3, 1 $\to \binom{3}{0}; \binom{3}{1}, \binom{3}{2}, \binom{3}{3}$

Exercise 4

What is the probability that a bridge player's hand of 13 cards contains at least two Aces (standard 52-card deck is used)?

Solution. at least $\boxed{2 \text{ Aces}}$, 2 aces, 3 aces, 4 aces.

$$\frac{\binom{4}{2}\binom{48}{11}}{\binom{52}{13}} + \frac{\binom{4}{3}\binom{48}{10}}{\binom{52}{13}} + \frac{\binom{4}{4}\binom{48}{9}}{\binom{52}{13}}$$

Conditional probability

Events A and B with $P(B) > 0$.

The **conditional probability of $\mathbf{A}$ given $\mathbf{B}$**, denoted $P(A|B)$, is

formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Explanation of conditional probability

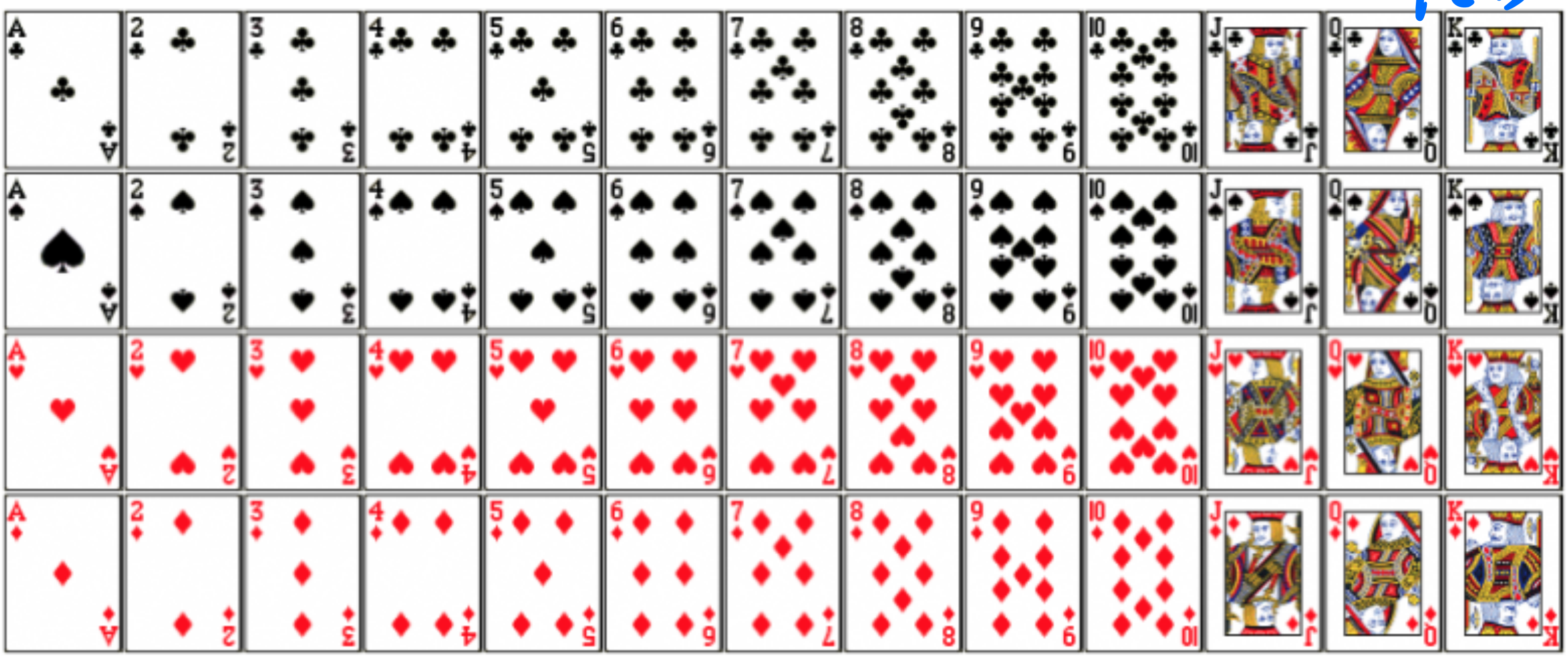
Equation $P(A|B) = \frac{P(A \cap B)}{P(B)}$ can be explained as follows.

- It is **given** that B happens $\Rightarrow$ sample space for *possible outcomes* is B.
- A happens only if $A \cap B$ happens.
- $P(A|B)$ is the probability of the event $A \cap B$ in the sample space B

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Example 4

Given that a bridge player's hand of 13 cards contains at least one ace. What is the probability that it contains exactly one ace? P(B) P(A)



(standard 52 card deck used for bridge)

Example 4 solution

$$P(B) = \overset{2}{\downarrow}\frac{\binom{4}{2}\binom{48}{11}}{\binom{52}{13}} + \overset{3}{\downarrow}\frac{\binom{4}{3}\binom{48}{10}}{\binom{52}{13}} + \overset{4.}{\downarrow}\frac{\binom{4}{4}\binom{48}{9}}{\binom{52}{13}} + \overset{1}{\downarrow}\frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}}$$

$$= 1 - \frac{\binom{4}{0}\binom{48}{13}}{\binom{52}{13}} = 1 - P(\bar{B})$$

Example 4 solution (continued)

Numerator:

$$\frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}} = P(A \cap B)$$

$$P(A|B) = \frac{\dfrac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}}}{1 - \dfrac{\binom{4}{0}\binom{48}{13}}{\binom{52}{13}}}$$

answer

Exercise 4

Roll a fair dice twice. You know that one of the rolls gave the value of 6. What is the probability that the other roll also gave 6?

Intuition: The chance to get 6 in the other roll is $\frac{1}{6}$?

Exercise 4 solution

The intuition is wrong!

Solution.

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	$\boxed{(6,6)}$

$\frac{1}{11}$