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Quiz3 & Finals

1. Lecture Eg and Ex
2. Homework
3. Tutorials non (*)
(*)

(m), (n), (g), (h)

(j), (k), (o)

(d), (e), (f), (i)

(c) $\sum_{n=1}^{\infty} \frac{3}{\sqrt[4]{n}}$. Let's assume that $\sum_{n=1}^{\infty} \frac{3}{\sqrt[4]{n}}$ is convergent.

Therefore, by the series law,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{3}{\sqrt[4]{n}} \text{ is convergent.}$$

But we know that $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}$ is divergent hence the assumption that $\sum_{n=1}^{\infty} \frac{3}{\sqrt[4]{n}}$ is convergent is wrong.

Therefore $\sum_{n=1}^{\infty} \frac{3}{\sqrt[4]{n}}$ is divergent.

Can use in exam: Constant multiple of a divergent series is divergent.

(e) $\sum_{n=0}^{\infty} 2^{4n} 5^{2-2n}$

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

$* \ |r| < 1$

$$= \sum_{n=0}^{\infty} \frac{2^{4n}}{5^{-2n-2}}$$

a $\quad r = \frac{16}{25}$ $\quad \left|\frac{16}{25}\right| < 1$

$$= \sum_{n=0}^{\infty} \frac{16^n}{25^{n-1}} = \sum_{n=0}^{\infty} 25 \cdot \frac{16^n}{25^n} = \frac{25}{1-\frac{16}{25}} = \frac{25^2}{9} = \frac{625}{9}$$

$\therefore \sum_{n=0}^{\infty} 2^{4n} 5^{2-2n}$ is convergent and its sum is $\frac{625}{9}$.

(f) $\sum_{n=2}^{\infty} \frac{2}{3^{n-1}} = \sum_{n=1}^{\infty} \frac{2}{3^n} = \sum_{n=1}^{\infty} \underbrace{\frac{2}{3}}_{a} \cdot \left(\underbrace{\frac{1}{3}}_{r}\right)^{n-1} = \frac{\frac{2}{3}}{1-\frac{1}{3}} = 1$ $\quad |r|<1$

$\frac{2}{3} \cdot \frac{1}{3^{n-1}}$

n w̄ n+2

(g) $\sum_{n=3}^{\infty} \frac{5^{2n-2}}{4^{3n-3}} = \sum_{\underline{\underline{n=3}}}^{\infty} \frac{25^{n-1}}{64^{n-1}} = \sum_{\underline{n=1}}^{\infty} \frac{25^{n+1}}{64^{n+1}}$

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$= \sum_{n=1}^{\infty} \underbrace{\frac{25^2}{64^2}}_{a} \left(\underbrace{\frac{25}{64}}_{r}\right)^{n-1}$

$= \frac{\frac{25^2}{64^2}}{1-\frac{25}{64}} = \frac{25^2}{64^2} \cdot \frac{64}{39}$

$= \frac{25^2}{64 \cdot 39} = \frac{625}{2496}$

(o) $\sum_{n=3}^{\infty} 4^{4n-2} 3^{4-2n} = \sum_{n=3}^{\infty} \frac{4^{4n-2}}{3^{2n-4}}$ $\quad n \to n+2$, $n \to n+2$

$= \sum_{n=1}^{\infty} \frac{4^{4(n+2)-2}}{3^{2(n+2)-4}} = \sum_{n=1}^{\infty} \frac{4^{4n+8-2}}{3^{2n+4-4}}$

$= \sum_{n=1}^{\infty} \frac{4^{4n+8}}{3^{2n}} \cdot 4^{-2}$

$= \sum_{n=1}^{\infty} \frac{256^{n+2}}{9^n} \cdot \frac{1}{16}$

$= \sum_{n=1}^{\infty} \left(\frac{256}{9}\right)^n \cdot \frac{256^2}{16}$

Alternative

$\lim_{n\to\infty} 4^{4n-2} \cdot 3^{4-2n}$

does not exist

I don't think I will

This series has common ratio $r = \frac{256}{9}$, where $|r|>1$

$\therefore$ This series is divergent.

$$\sum_{n=3}^{\infty} a_n = a_3 + a_4 + a_5 + \dots$$

replace n w/ $n+2$

-2

$$\sum_{n=1}^{\infty} a_{n+2} = a_3 + a_4 + a_5 + \dots$$

(j) $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+2)}{(n+3)^2}$

$$\lim_{n\to\infty} \frac{n(n+2)}{(n+3)^2} \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1\left(1+\frac{2}{n}\right)}{\left(1+\frac{3}{n}\right)^2}$$

$$= \frac{1}{1} = 1 \neq 0.$$

$\therefore$ By the Divergence Test,

$\displaystyle\sum_{n=1}^{\infty} \frac{n(n+2)}{(n+3)^2}$ diverges.

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$, $\quad 4n+3 \geq 0 \Rightarrow n^2 \leq n^2+4n+3$

$$\Rightarrow \frac{1}{n^2} \geq \frac{1}{n^2+4n+3}$$

$$0 \leq \frac{2}{n^2+\underbrace{4n+3}_{\geq 0}} \leq \frac{2}{n^2}$$

$\therefore$ Since $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2} < \infty$ (p-series, $p=2>1$), $\therefore$ by the Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} < \infty$.

(k) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{1}{n}\right)$ $\quad$ Since $\frac{3}{5^n} \geq 0$ for all n

$$0 \leq \frac{1}{n} \leq \frac{3}{5^n} + \frac{1}{n}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ divergent (p-series, $p=1\leq 1$), $\therefore$ By the Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{1}{n}\right)$ is divergent.

Alternative Sol^n

WRONG! $\cancel{\sum_{n=1}^{\infty} \frac{3}{5^n}} + \sum_{n=1}^{\infty} \frac{1}{n}$ divergent

Assume that $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{1}{n}\right)$ is convergent.

We know that $\sum_{n=1}^{\infty} \frac{3}{5^n}$ is convergent (geometric series $r = \frac{1}{5}$, $|r| < 1$)

$\therefore$ By Series Law,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{1}{n}\right) - \sum_{n=1}^{\infty} \frac{3}{5^n} \text{ is convergent.}$$

This is a contradiction because $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (p-series, $p = 1 \leq 1$). Therefore the assumption that

$$\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{1}{n}\right) < \infty$$

is false. Hence this sum must be divergent.

(d) $\sum_{n=1}^{\infty} \frac{n^2+n}{n^3+2}$

Informal: $\frac{n^2+n}{n^3+2} \approx \frac{n^2}{n^3} = \frac{1}{n}$

Solution 1:

$$\frac{n^2+n}{n^3+2} \geq \frac{n^2}{\underline{\underline{n^3}}+2} \geq \frac{2}{3n} \geq 0$$

dominating

For $n \geq 2$,

$$\frac{n^3}{2} \geq 2 \Rightarrow \frac{3n^3}{2} = n^3 + \frac{n^3}{2} \geq n^3+2$$

$$\Rightarrow \frac{3n^3}{2} \geq n^3+2$$

$$\Rightarrow \frac{2}{3n^3} \leq \frac{1}{n^3+2} \Rightarrow \frac{2n^2}{3n^3} \leq \frac{n^2}{n^3+2}$$

$\frac{2}{3n}$ $\parallel$

$$\frac{n^2+n}{n^3+2} \geq \frac{2}{3n} \geq 0$$

Since $\sum_{n=1}^{\infty} \frac{2}{3n}$ divergent (p-series, $p=1\leq 1$) by Comparison Test, $\sum_{n=1}^{\infty} \frac{n^2+n}{n^3+2}$ divergent.

<u>Solution 2</u> LCT. $a_n = \frac{n^2+n}{n^3+2}$, $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2+n}{n^3+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3+n^2}{n^3+2} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n^3}} = 1 > 0$$

$\therefore$ Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series, $p=1\leq 1$), by LCT,

$\sum_{n=1}^{\infty} \frac{n^2+n}{n^3+2}$ diverges.

(m) $\sum_{n=3}^{\infty} \frac{1}{n^2-8}$ expect to be convergent.

<u>Ex</u> use LCT to show $\sum_{n=3}^{\infty} \frac{1}{n^2-8} < \infty$

Let $a_n = \frac{1}{n^2-8}$, $b_n = \frac{1}{n^2}$ >0 for $n \geq 3$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2-8}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-8} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1}{1-\frac{8}{n^2}} = 1 > 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ (p-series, $p=2>1$), therefore, by LCT

$$\sum_{n=3}^{\infty} \frac{1}{n^2-8} < \infty.$$

replace non-dominating terms with $\frac{n^2}{k}$ $k>1$

$$\sum_{n=1}^{\infty} a_n \quad a_n = \begin{cases} 0 & \text{for } n \text{ odd} \\ \frac{1}{n^2} & \text{for } n \text{ even} \end{cases}$$

$$a_n \le \max\{0, \tfrac{1}{n^2}\} = \frac{1}{n^2}$$

CT: $\frac{n^2}{2} \geq 8$ $\boxed{n \geq 4}$

$$\Rightarrow -\frac{n^2}{2} \leq -8$$

$$\Rightarrow \frac{n^2}{2} = n^2 - \frac{n^2}{2} \leq n^2 - 8$$

$$\Rightarrow \frac{n^2}{2} \leq n^2 - 8$$

$$\Rightarrow \frac{2}{n^2} \geq \frac{1}{n^2-8} \geq 0.$$

$$\frac{1}{n^2 - n}$$ non-dominating terms

$$\frac{n^2}{2} \geq n$$

$\vdots$

Since $\sum_{n=1}^{\infty} \frac{2}{n^2} < \infty$ (p-series, $p=2>1$), by CT,

$$\sum_{n=3}^{\infty} \frac{1}{n^2-8} < \infty.$$

(n) $\sum_{n=1}^{\infty} \frac{n^2}{n^3+23}$ LCT: $a_n = \frac{n^2}{n^3+23}$, $b_n = \frac{1}{n}$

CT: $\frac{n^3}{2} \geq 23$ $\underline{n \geq 4}$

$$\Rightarrow \frac{3n^3}{2} = n^3 + \frac{n^3}{2} \geq n^3 + 23$$

$$\Rightarrow \frac{3n^3}{2} \geq n^3 + 23$$

$$\Rightarrow \frac{2}{3n^3} \leq \frac{1}{n^3+23} \Rightarrow \frac{2n^2}{3n^3} \leq \frac{n^2}{n^3+23}$$

$$\frac{2n^2}{3n^3} = \frac{2}{3n}$$

$$\Rightarrow \quad 0 \leq \frac{2}{3n} \leq \frac{n^2}{n^3+23}$$

Since $\sum_{n=1}^{\infty} \frac{2}{3n}$ divergent (p-series, $p=1\leq 1$) by CT,

$\sum_{n=1}^{\infty} \frac{n^2}{n^3+23}$ is diverges.