Lecture 11: Probability Theory

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What is probability?

- Rigorous mathematical theory to analyze events that involve uncertainty
- Almost everything involves uncertainty
- Applications: business, finance, actuarial science, risk management, economics, computer science, quality control, traffic control, and many other areas

Experiments, Sample Spaces and Events

- An **experiment** is a situation with uncertain outcomes.
- A **sample space** of an experiment is the set of all possible outcomes of the experiment.
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- A **sample space** of an experiment is the set of all possible outcomes of the experiment.
- An **event** is a subset of the sample space.
- Notations

  Sample space is usually denoted by Ω (pronounce "Omega").

  Events (subsets of Ω) are denoted by capital letters $A, B, C, \ldots$

Example 1

- Experiment: a commuter passes through 3 traffic lights. At each light, she either stops (s) or continues (c). The sample space is

$$\Omega = \{\text{ccc, ccs, css, csc, sss, ssc, scc, scs}\}$$

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- What is event B: the commuter stops at the 3rd light?

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- Event A: there are exactly 2 heads

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- What is B: there are $\geq$ 2 heads?

Exercise 1

Experiment: Choose a letter at random from “probability”.

Write down the sample space for this experiment.

Exercise 2

Experiment: roll a dice 3 times.

(a) What is the sample space Ω? How many outcomes are there in Ω?

(b) Write down the event A that the total score is at least 17.

Union, intersection, complement of events

Given events A and B.

- The **union** of A and B is the event $C = A \cup B$.
- The **intersection** of A and B is the event $C = A \cap B$.
  A and B are **disjoint** if $A \cap B = \emptyset$.
- The **complement** $\bar{A}$ of A is the event that A does not occur

$$\bar{A} = \{w \in \Omega : w \notin A\}.$$

Laws of set theory

Given sample space Ω and events A, B, C

- Commutative laws

$$A \cup B = B \cup A \text{ and } A \cap B = B \cap A$$

- Associative laws

$$(A \cup B) \cup C = A \cup (B \cup C) \text{ and } (A \cap B) \cap C = A \cap (B \cap C)$$

- Distributive laws

$$\begin{aligned} (A \cup B) \cap C &= (A \cap C) \cup (B \cap C) \\ (A \cap B) \cup C &= (A \cup C) \cap (B \cup C) \end{aligned}$$

- De Morgan's law (complement interchanges union and intersection)

$$\overline{A \cup B} = \bar{A} \cap \bar{B} \text{ and } \overline{A \cap B} = \bar{A} \cup \bar{B}$$

Inclusion-exclusion principle

- Two sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- Three sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

- Inclusion-exclusion principle for n sets

$$\left| \cup_{i=1}^{n} A_i \right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \left| A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k} \right|$$

Example 1 revisited

- Sample space

$$\Omega = \{\text{ccc, ccs, css, csc, sss, ssc, scc, scs}\}.$$

- Event A: the commuter stops at the 1st light.

$$A = \{\text{sss, ssc, scc, scs}\}.$$

- Event B: the commuter stops at the 3rd light.

$$B = \{\text{sss, scs, ccs, css}\}.$$

Example 1 revisited

(i) $A \cup B$: she stops at the 1st light or the 3rd light

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(ii) $A \cap B$: she stops both at the 1st light and the 3rd light

$$A \cap B = \{\text{sss,scs}\}.$$

(iii) $\bar{A}$: she doesn't stop at the 1st light

$$\bar{A} = \{\text{ccc,ccs,css,csc}\}.$$

(iv) $\bar{B}$: she doesn't stop at the 3rd light

$$\bar{B} = \{\text{ccc,csc,ssc,scc}\}$$

(v) By (iii) and (iv)

$$\begin{aligned} \bar{A} \cup \bar{B} &= \{\text{ccc,ccs,css,csc,ssc,scc}\} \\ \bar{A} \cap \bar{B} &= \{\text{ccc,csc}\} \end{aligned}$$

(v) By (iii) and (iv)

$$\begin{aligned}\bar{A} \cup \bar{B} &= \{\text{ccc,ccs,css,csc,ssc,scc}\} \\ \bar{A} \cap \bar{B} &= \{\text{ccc,csc}\}\end{aligned}$$

(vi) By (i) and (ii)

$$A \cap B = \{\text{sss,scs}\} \Rightarrow \overline{A \cap B} = \{\text{ccc,ccs,css,csc,ssc,scc}\} = \bar{A} \cup \bar{B}$$
$$A \cup B = \{\text{sss,ssc,scc,scs,ccs,css}\} \Rightarrow \overline{A \cup B} = \{\text{ccc,csc}\} = \bar{A} \cap \bar{B}$$

Probability measure

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$$P : \{\text{subsets of } \Omega\} \to \mathbb{R}$$

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A probability measure on Ω is a function

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which satisfies

(i) $P(\Omega) = 1$.

(ii) $P(A) \geq 0$ for any $A \subset \Omega$.

(iii) If $A_1, A_2, \ldots$ are mutually disjoint events, then

$$P\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

Properties of probability measure

Let P a probability measure on sample space Ω, that is

$$P\colon \{\text{events}\} \to [0,1].$$

Then the following hold

1. $P(\emptyset) = 0$
2. $P(\bar{A}) = 1 - P(A)$
3. If $A \subset B$, then $P(A) \leq P(B)$.
4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
5. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Example 3

A fair coin is thrown twice.

Event A: head on the first toss.

Event B: head on the second toss.

What is the probability the coin lands on head on one of the tosses?

Exercise 3

Find the probability of the following events.

(a) A randomly chosen integer $x \in \{0, \ldots, 999\}$ is divisible by 11.

(b) A randomly chosen integer $x \in \{0, \ldots, 999\}$ is divisible by 13.

(c) A randomly chosen integer $x \in \{0, \ldots, 999\}$ is divisible by 11 or 13.

Hint. The number of integers in $\{1, \ldots, n\}$ divisible by d is $\lfloor \frac{n}{d} \rfloor$.

Uniform distribution

Theorem 1

Let Ω be finite. The function P defined on the subsets of Ω by

$$P(A) = \frac{|A|}{|\Omega|} \text{ for any } A \subset \Omega$$

is a probability measure on Ω.

Proof (sketch). We need to verify 3 properties

1. $P(\Omega) = 1$
2. $P(A) \geq 0$ for any $A \subset \Omega$
3. If $A_1, A_2, \ldots$ are mutually disjoint, then

$$P\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Example 4

Let $\Omega = \{1, 2, 3\}$. The uniform probability P on Ω is

$$
\begin{aligned}
P(\emptyset) &= 0, \\
P(\{1\}) &= P(\{2\}) = P(\{3\}) = 1/3, \\
P(\{1,2\}) &= P(\{1,3\}) = P(\{2,3\}) = 2/3, \\
P(\{1,2,3\}) &= 1.
\end{aligned}
$$

Example 5

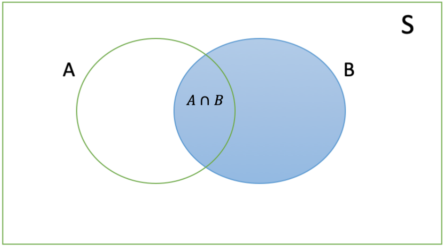
Let $\Omega = \{0, 1\}$. Write out the uniform probability P on Ω.

Conditional probability

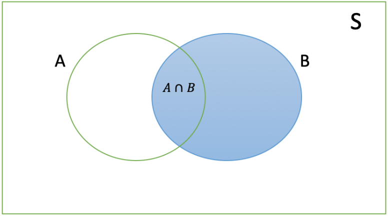
Let A, B be events with $P(B) > 0$.

The **conditional probability of $\mathbf{A}$ given $\mathbf{B}$**, denoted $P(A|B)$, is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$



Explanation of conditional probability



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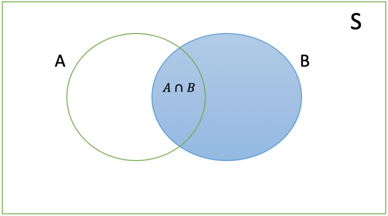

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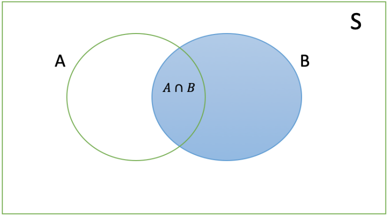


Equation $P(A|B) = \frac{P(A \cap B)}{P(B)}$ can be explained as follows.

- It is **given** that B happens $\Rightarrow$ space for *possible outcomes* is B.
- A happens only if $A \cap B$ happens.
  $P(A|B)$ = probability of event $A \cap B$ in the sample space B

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Example 6

Roll a fair dice twice. You know that one of the rolls gave the value of 6. What is the probability that the other roll also gave 6?

Intuition: The chance to get 6 in the other roll is $\frac{1}{6}$?

Example 6 solution

The intuition is wrong!

Exercise 4

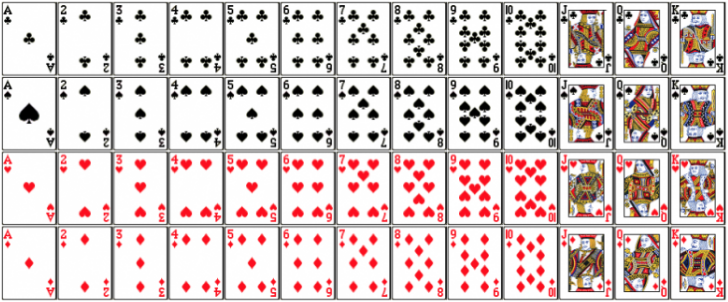
A bit string of length 4 is generated at random so that each of the 16 bit strings of length four is equally likely.

(a) What is the sample space Ω?

(b) What is the probability that it contains at least two consecutive 0's, given that its first bit is 0?

Exercise 5

Given that a bridge player's hand of 13 cards contains at least one ace. What is the probability that it contains exactly one ace?



(standard 52 card deck used for bridge)

Independent events

- Two events A and B are **independent** if and only if

$$P(A \cap B) = P(A)P(B). \tag{1}$$

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- If $P(A) > 0$ and $P(B) > 0$, (1) is equivalent to either

$$\begin{aligned} P(A|B) &= P(A) \text{ or} \qquad (2) \\ P(B|A) &= P(B). \qquad (3) \end{aligned}$$

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- If $P(A) > 0$ and $P(B) > 0$, (1) is equivalent to either

$$P(A|B) = P(A) \text{ or} \tag{2}$$
$$P(B|A) = P(B). \tag{3}$$

- To prove the independence of A and B, we only need to prove one of the equations (1) or (2) or (3).

Explanation of independent events

- The independence of A and B means
  "the information that B occurs does not affect the probability that A occurs, and vice versa".

Explanation of independent events

- The independence of A and B means
  "the information that B occurs does not affect the probability that A occurs, and vice versa".
- Do not use any other definitions of independence such as "A and B have no influence on each other" or "A and B are disjoint". They are simply **incorrect**.

Question

Let A and B be disjoint events. Are A and B independent? If the answer is not, find a counterexample.

Example 7

A fair dice is rolled two times.

E_1: the 1st role gives 1.

E_2: the 2nd role gives 1.

Are E_1 and E_2 independent events?

Example 8

A number is chosen at random from $S = \{1, 2, \dots, 9\}$.

A: the number is a prime.

B: the number is smaller than 5.

Are A and B independent?