TUTORIAL 9 SOLUTIONS

1) Events A, B, C

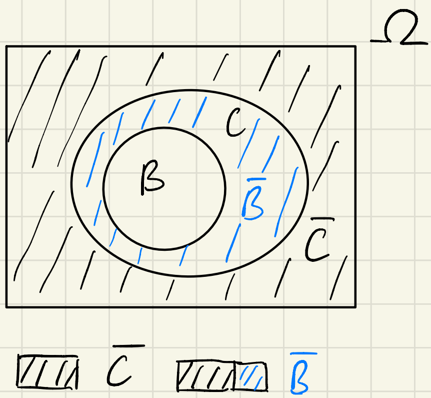
a) Given $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$, $P(A \cup B) = \frac{5}{9}$. Find $P(A \cap B)$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$= \frac{1}{3} + \frac{1}{4} - \frac{5}{9} = \frac{1}{36}$$

b) $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{4}$, $P(A \cap \bar{B}) = \frac{3}{8}$, $B \subseteq C$. Find $P(A \cup \bar{B} \cup \bar{C}) = ?$



Since $B \subseteq C$, $\bar{C} \subset \bar{B}$:

$$A \cup \bar{B} \cup \bar{C} = A \cup \bar{B}$$

$$P(A \cup \bar{B}) = P(A) + P(\bar{B}) - P(A \cap \bar{B})$$

$$= \frac{1}{2} + \left(1 - \frac{1}{4}\right) - \frac{3}{8} = \frac{7}{8}.$$

c) $P(A) = P(B) = P(C) = \frac{9}{10}$, $P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{7}{10}$,

$P(A \cap B \cap C) = \frac{5}{10}$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$$= 3 \cdot \frac{9}{10} - 3 \cdot \frac{7}{10} + \frac{5}{10} = \frac{11}{10} > 1 \rightarrow \text{impossible.}$$

2a) Ω = all possibilities for 2 children = $\{BB, BG, GB, GG\}$

B = event that one of the children is boy = $\{BB, BG, GB\}$

A = event that both children are boys = $\{BB\}$

$A \cap B = \{BB\}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{3}$$

3) Ω = all 13-combinations of 52 cards

4 aces   48 remaining cards

$$|\Omega| = \binom{52}{13}$$

a) B = event that 13 cards contain ≥ 1 ace

$$|B| = |\Omega| - |\overline{B}| = \binom{52}{13} - \binom{48}{13}$$

A = event that 13 cards contain exactly 2 aces

$A \cap B$ = event that 13 cards contain exactly 2 aces

$$|A \cap B| = \binom{4}{2}\binom{48}{11}$$

(1) choose 2 aces from 4 aces
(2) choose 11 cards from 48 cards

$$\therefore P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\binom{4}{2}\binom{48}{11} / \cancel{\binom{52}{13}}}{\left(\binom{52}{13} - \binom{48}{13}\right) / \cancel{\binom{52}{13}}} \simeq 0.31$$

b) B = 13 cards contain the ace of hearts

Take out the ace of hearts → choose 12 cards from 51 cards

$$|B| = \binom{51}{12}$$

A = 13 cards contain exactly 2 aces

$A \cap B$ = 13 cards contain exactly 2 aces with the ace of hearts

Take out the ace of hearts — choose 1 ace from 3 aces

choose 11 cards from 48 cards

$$|A \cap B| = \binom{3}{1}\binom{48}{11}$$

$$\therefore P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\binom{3}{1}\binom{48}{11} \Big/ \cancel{\binom{52}{13}}}{\binom{51}{12} \Big/ \cancel{\binom{52}{13}}} \cong 0.43$$

4) Ω = all binary strings of length 4

$= \{abcd : a,b,c,d \in \{0,1\}\} \Rightarrow |\Omega| = 2^4 = 16$

A = event that a string starts with 1

$= \{1bcd : b,c,d \in \{0,1\}\} \Rightarrow |A| = 2^3 = 8$

B = event that a string contain an even number of 1's

$$|B| = \binom{4}{0} + \binom{4}{2} + \binom{4}{4} = 8$$

$A \cap B$ = strings that start with 1 and contain even number of 1's

1*** two 1's → $\binom{3}{1}$ choices : 1100, 1010, 1001

four 1's → 1111

$A \cap B = \{1111, 1100, 1010, 1001\} \rightarrow |A \cap B| = 4$

$$P(A \cap B) = \frac{4}{16} = P(A)\,P(B) = \frac{8}{16} \cdot \frac{8}{16}$$

$P(A|B) = P(A)$

$P(B|A) = P(B)$

$\therefore$ A & B are independent.

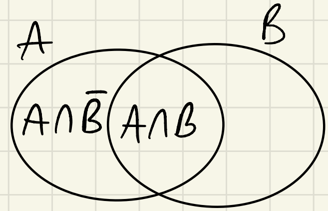
<u>Remark</u>: It's evident that A and B influence each other. However, they are independent! Again, do not use any other explanation of independent events such as they are disjoint, or they do not influence each other.

5) Given that A, B are independent

$P(A \cap B) = P(A)\,P(B)$ (or $P(A|B) = P(A)$, or $P(B|A) = P(B)$)

a) Prove that A & $\bar{B}$ are independent : $P(A|\bar{B}) = P(A)$?

$\rightarrow$ need to prove $P(A \cap \bar{B}) = P(A)\,P(\bar{B})$.



$P(A \cap \bar{B}) + P(A \cap B) = P(A)$

$P(A \cap \bar{B}) = P(A) - P(A \cap B)$

$P(A \cap B) = P(A)\,P(B)$

$P(A) - P(A \cap B) = P(A) - P(A)\,P(B)$

$P(A \cap \bar{B}) = P(A)\,(1 - P(B))$

$P(A \cap \bar{B}) = P(A)\,P(\bar{B})$

$\therefore$ A & $\bar{B}$ are independent.

b) Prove that any following 2 events are independent

A & $\bar{B}$, $\bar{A}$ & B, $\bar{A}$ & $\bar{B}$

By part (a): X & Y are independent $\rightarrow$ X & $\bar{Y}$ are independent

B & A are independent $\rightarrow$ B & $\bar{A}$ are independent

$\bar{A}$ & B are independent $\rightarrow$ $\bar{A}$ & $\bar{B}$ are independent