Lecture 10: Permutation and Combination

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Permutation

- A **permutation** of S is an **ordered arrangement** of its elements.
- Example: $S = \{1, 2, 3\}$ has 6 permutations

$$(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1).$$

Permutation

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- **Recall:** $(a_1, \ldots, a_n)$ is an n-tuple, that is, an ordered arrangement of $a_1, \ldots, a_n$ such that

$$a_1 = \text{1st element}, \ a_2 = \text{2nd element}, \ a_n = \text{last element}.$$

Two tuples are the same $\Leftrightarrow$ they have the same length and the same corresponding elements

$$(a_1, \ldots, a_n) = (b_1, \ldots, b_m) \Leftrightarrow n = m \text{ and } a_i = b_i \text{ for all } i$$

Number of permutations of a set

Theorem 1

If S is a set of n elements, then S has $n!$ permutations.

Proof. Assume $S = \{1, 2, \ldots, n\}$.

We count the number of choices for permutation $(a_1, \ldots, a_n)$ of S.

$\{a_1, \ldots, a_n\} = \{1, 2, \ldots, n\}$

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- n choices for a_1 $(a_1 \in S)$.
- $n - 1$ choices for a_2 $(a_2 \in S - \{a_1\})$.
- $n - 2$ choices for a_3 $(a_2 \in S - \{a_1, a_2\})$

. . . .

- 1 choice for a_n $(a_n \in S - \{a_1, \ldots, a_{n-1}\})$.

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  ....

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  The number choices for $(a_1, \ldots, a_n)$ is

$$n(n-1)\ldots 1 = n!$$

r-permutation

- Let S be a set with n elements.
- An **r-permutation** of S is an **ordered selection** of r elements in S, say

  $(a_1, \ldots, a_r)$ with all $a_i \in S$ and $a_i \neq a_j$ whenever $i \neq j$.
- The number of r-permutations of S is denoted by $\mathbf{P(n, r)}$.

  $P(n,r) = {}_nP_r = P_r^n = {}^nP_r$

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- The number of r-permutations of S is denoted by $\mathbf{P(n, r)}$.
- Example: All 2-permutations of $S = \{a, b, c\}$ are

$$(a, b), (b, a), (a, c), (c, a), (b, c), (c, b)$$

So $P(3, 2) = 6$.

n r

Number of r-permutations

Theorem 2

Let n be a positive integer and let r be a nonnegative integer. Then

$$P(n,r) = \frac{n!}{(n-r)!}.$$

Proof. Assume $S = \{1, 2, \ldots, n\}$.

We count the choices for r-permutations $(x_1, \ldots, x_r)$ of S.

$x_i \in S$ for $i = 1, \ldots, r$

$x_i \neq x_j$ whenever $i \neq j$

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  . . . .

- $n-r+1$ choices for x_r $(x_r \in S - \{x_1, \ldots, x_{r-1}\})$

  $n - (r-1)$

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- $n-r+1$ choices for x_r $(x_r \in S - \{x_1, \ldots, x_{r-1}\})$

  The number of choices for $(x_1, \ldots, x_r)$ is

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = n!/(n-r)!$$

Special cases of $P(n, r)$

$$P(n,r) = \#\ \text{ordered selections of } r \text{ elements in } S = \frac{n!}{(n-r)!}$$

Special cases of $P(n, r)$

$r \in \{0, 1, \ldots, n\}$

$$P(n,r) = \#\text{ ordered selections of } r \text{ elements in } S = \frac{n!}{(n-r)!}$$

- $P(n, n) = \#$ n-permutations of S, which is simply the number of permutations of S

$$P(n, n) = n!$$

$r = n$: $P(n,n) = \dfrac{n!}{0!} = \dfrac{n!}{1} = n!$

<u>Remark</u> : $0! = 1$

Special cases of $P(n, r)$

$$P(n,r) = \#\text{ ordered selections of } r \text{ elements in } S = \frac{n!}{(n-r)!}$$

- $P(n,n) = \#$ n-permutations of S, which is simply the number of permutations of S

$$P(n,n) = n!$$

- The only 0-permutation of S is $\emptyset$. So

$$P(n,0) = 1$$

$r = 0$: $P(n,0) = \frac{n!}{n!} = 1$

Example 1

How many ways are there to select a first-prize winner, a second-prize winner and a third-prize winner from 100 different people who are in a race?

Goal : choose (1^{st} prize, 2^{nd} prize, 3^{rd} prize) from 100 people.

Solution 1: This is the number of 3-permutations of a set with 100 people $\rightarrow$ $P(\underbrace{100}_{n},\underbrace{3}_{r}) = \dfrac{100!}{(100-3)!} = \dfrac{100!}{97!}$

$$\frac{100!}{97!} = \frac{1\cdot 2\cdots 97\cdot 98\cdot 99\cdot 100}{1\cdot 2\cdots 97} = 98\cdot 99\cdot 100$$

Solution 2:

\# choices for 1^{st} prize = 100

\# choices for 2^{nd} prize = 99

\# choices for 3^{rd} prize = 98

$\therefore$ # choices $= 100\cdot 99\cdot 98 = \dfrac{100!}{97!}$

Example 2

(a) How many ways can the letters in the word COMPUTER be arranged in a row?

\# permutations of the letters C, O, M, P, U, T, E, R

$8! = 40320$

(b) How many ways can the letters in the word COMPUTER be arranged if the letters CO must be next to each other as a block?

\# arrangements of CO, M, P, U, T, E, R

$7! = 5040$

r-combination

- Let S be a set with n elements.
- An **r-combination** of S is an **unordered selection** of r elements from S = a subset of S with r elements.

r-combination

- Let S be a set with n elements.
- An $\mathbf{r}$**-combination** of S is an **unordered selection** of r elements from S - a subset of S with r elements.
- The number of r-combinations of S is denoted by $\binom{\mathbf{n}}{\mathbf{r}}$

$$\binom{n}{r} = nCr = {}^{n}C_{r} = C_{r}^{n} = C(n, r)$$

Example

Let $S = \{a, b, c\}$. Then

- The only 3-combination of S is $\{a, b, c\}$.

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- The only 3-combination of S is $\{a, b, c\}$.
- All 2-combinations of S are

$$\{a, b\}, \{a, c\}, \{b, c\}.$$

Note : $\{a, b\} = \{b, a\}$

Example

Let $S = \{a, b, c\}$. Then

- The only 3-combination of S is $\{a, b, c\}$.
- All 2-combinations of S are

$$\{a, b\}, \{a, c\}, \{b, c\}.$$

- All 1-combinations of S are

$$\{a\}, \{b\}, \{c\}.$$

- The only 0-combination of S is $\emptyset$.

Number of r-combinations of a set

Theorem 3

Let n and r be nonnegative integers with $0 \leq r \leq n$. Then

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Proof. Optional. See textbook.

$$\binom{n}{r} = \binom{n}{n-r} = \frac{n!}{r!\,(n-r)!}$$

$\binom{n}{r}$ is also called binomial coefficients

Example 3

(a) How many poker hands of 5 cards can be dealt from a standard deck of 52 cards?

$$\binom{52}{5} = \frac{52!}{5!\,47!} = \cdots \qquad (nCr)$$

(b) How many ways are there to select 47 cards from a standard deck of 52 cards?

$$\binom{52}{47} = \frac{52!}{47!\,5!} = \binom{52}{5}$$

Example 4

Two members A and B of a group of 12 people insist on working as a pair. So any team must contain either both or neither. How many 5-person teams can be formed?

Type 1: Both A & B are chosen — choose 3 more ppl from remaining 10 people

$$\# \text{choices} = \binom{10}{3}$$

Type 2: Both A & B are not chosen — choose 5 ppl from 10 ppl

$$\# \text{choices} = \binom{10}{5}$$

$$\therefore \# \text{choices} = \binom{10}{3} + \binom{10}{5}.$$

Permutation with repetition allowed

- Let S be a set with n elements.
- An **r-permutation of S with repetition allowed** is an ordered r-tuple $(a_1, \ldots, a_r)$ such that $a_i \in S$ for all i. (*Remark*: a_i's don't need to be different).

Example

all pairs (x, y) s.t. $x \in S$, $y \in S$

- $S = \{a, b, c\}$.
- All 2-permutations of S with repetition allowed are

$$(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c).$$

Number of permutations with repetition allowed

Theorem 4

If S has n elements, the number of r-permutations with repetition allowed of S is n^r.

Number of permutations with repetition allowed

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If S has n elements, the number of r-permutations with repetition allowed of S is n^r.

- Let $(a_1, \ldots, a_r)$ be an r-permutation with repetition allowed.
- Each a_i has n choices.
- The number of choices in total is

$$n \times n \times \cdots \times n = n^r$$

Combination with repetition allowed

- Let S be a set with n elements.
- An **r-combination of S with repetition allowed** is an *unordered set* $\{a_1, \ldots, a_r\}$ such that $a_i \in S$ for all i.
  Remark: a_i's don't need to be different.

Combination with repetition allowed

- Let S be a set with n elements.
- An **r-combination of S with repetition allowed** is an *unordered set* $\{a_1, \ldots, a_r\}$ such that $a_i \in S$ for all i.
  Remark: a_i's don't need to be different.

  $\{a_1, a_2\}$
- All 2-combinations of with repetitions allowed of $\{a, b, c\}$ are

$$\{a, a\}, \{a, b\}, \{a, c\}, \{b, b\}, \{b, c\}, \{c, c\}.$$

Why is $\{b, a\}$ not in the list?

$\{b, a\} = \{a, b\}$

Remarks

- $\{a_1, \ldots, a_r\}$ is an r-combination with repetition allowed of S if $a_i \in S$ for all $i \in S$.

  In particular, $a_1, \ldots, a_r$ don't need to be different.

Remarks

- $\{a_1, \ldots, a_r\}$ is an r-combination with repetition allowed of S if $a_i \in S$ for all $i \in S$.
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- The curly brackets $\{\ \}$ are used to denote combination with repetition allowed.
  To **avoid confusion with sets**, we always clearly say "combination with repetition allowed" whenever $\{\ \}$ is used.

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- $\{a_1, \ldots, a_r\}$ is an r-combination with repetition allowed of S if $a_i \in S$ for all $i \in S$.
  In particular, $a_1, \ldots, a_r$ don't need to be different.
- The curly brackets $\{\ \}$ are used to denote combination with repetition allowed.
  To **avoid confusion with sets**, we always clearly say "combination with repetition allowed" whenever $\{\ \}$ is used.
- To choose an r-combination with repetition allowed, we do
  1. Pick r terms $a_1, \ldots, a_r$ (don't need to be distinct) from S
  2. Put them in between a pair of curly brackets $\{a_1, \ldots, a_r\}$

Example 5

(a) List all 3-combinations with repetition allowed of $S = \{a, b, c\}$

(b) Is there a 4-combinations with repetition allowed of S?

(a) List all choices $\{a_1, a_2, a_3\}$: $a_1, a_2, a_3 \in S$

10 in total

(1) All choices containing a :

$\{a,a,a\}$, $\{a,a,b\}$, $\{a,a,c\}$, $\{a,b,b\}$, $\{a,b,c\}$, $\{a,c,c\}$

(2) All choices not containing a : $\{a_1, a_2, a_3\}$ with $a_i \in \{b,c\}$

$\{b,b,b\}$, $\{b,b,c\}$, $\{b,c,c\}$, $\{c,c,c\}$

(b) $\{a_1, a_2, a_3, a_4\}$: $a_i \in \{a, b, c\}$

Yes : $\{a,a,a,a\}$, $\{a,a,b,c\}$, . . .

Example 6

How many ways are there to select 4 pieces of fruit from a bowl containing

apples, oranges, pears

if the order in which the pieces are selected does not matter, only the type of fruit and not the individual piece matters?

Example 6 solutions

Answer: 15 ways in total.

4 A	4 O	4 P
3 A, 1 O	3 A, 1 P	3 O, 1 A
3 O, 1 P	3 P, 1 A	3 P, 1 O
2 A, 2 O	2 A, 2 P	2 O, 2 P
2 A, 1 O, 1 P	2 O, 1 A, 1 P	2 P, 1 A, 1 O

Example 7

How many 5-combinations of $S = \{1, 2, \ldots, 7\}$ with repetition allowed are there?

$\{a_1, a_2, \ldots, a_5\} : a_i \in \{1, 2, \ldots, 7\}$

Discussion.

Example 7

How many 5-combinations of $S = \{1, 2, \ldots, 7\}$ with repetition allowed are there?

Discussion.

- Consider a box of 7 cells which are labelled by the numbers $1, \ldots, 7$. The box has **6 bars** (in red) which separate the cells.

- Each cell can be viewed as having infinitely many numbers (all equal to the label) inside it.

Example 7 (discussion)

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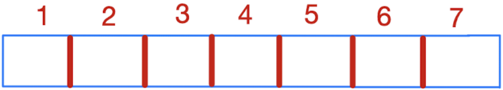
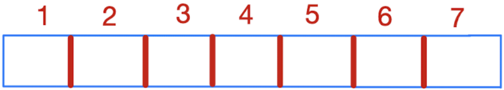


- To choose a 5-combination with repetition allowed, we choose
  - terms coming from cell 1,
  - terms coming from cell 2,

  . . . .
  - terms coming from cell 7,

  such that the total number of chosen terms is 5.

Example 7 (discussion)



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  - terms coming from cell 1,
  - terms coming from cell 2,

  . . . .
  - terms coming from cell 7,

  such that the total number of chosen terms is 5.
- Examples
  - Four terms from cell 1 $+$ one term from cell 2 $+$ no other term $\Rightarrow$ 5-combination $\{1, 1, 1, 1, 2\}$
  - Three terms from cell 1 $+$ one term from cell 2 $+$ one term from cell 3 $+$ no other term $\Rightarrow$ 5-combination $\{1, 1, 1, 2, 3\}$

Example 7 solution

Problem: Choose **5 elements**, all coming from $\{1, ..., 7\}$

1. Consider 5 elements as **5 stars**
2. Use **6 vertical bars** to separate these stars into 7 cells $1, 2, \dots, 7$

 # stars in each cell $=$ *# times* the *corresponding number appears*

1	2	3	4	5	6	7	
$\star\star$	$\star$	$\star$				$\star$	$\Rightarrow \{1,1,2,3,7\}$
	$\star\star$		$\star$		$\star$	$\star$	$\Rightarrow \{2,2,4,6,7\}$
$\star$		$\star$	$\star$	$\star$	$\star$		$\Rightarrow \{1,3,4,5,6\}$

Example 7 solution (continued)

1	2	3	4	5	6	7	
$\star\star$	$\star$	$\star$				$\star$	$\Rightarrow \{1,1,2,3,7\}$
	$\star\star$		$\star$		$\star$	$\star$	$\Rightarrow \{2,2,4,6,7\}$
$\star$		$\star$	$\star$	$\star$	$\star$		$\Rightarrow \{1,3,4,5,6\}$

- Each arrangement of 5 stars and 6 bars gives one 5-combination.

Example 7 solution (continued)

1 2 3 4 5 6 7

$$\star\star \mid \star \mid \star \mid \quad \mid \quad \mid \quad \mid \star \qquad \Rightarrow \{1,1,2,3,7\}$$

$$\mid \star\star \mid \quad \mid \star \mid \quad \mid \star \mid \star \qquad \Rightarrow \{2,2,4,6,7\}$$

$$\star \mid \quad \mid \star \mid \star \mid \star \mid \star \mid \qquad \Rightarrow \{1,3,4,5,6\}$$

- Each arrangement of 5 stars and 6 bars gives one 5-combination.
- $\#$ 5-combinations $=$ $\#$ arrangements of 5 stars and 6 bars in a row with 11 positions. We simply choose positions for 6 bars

$$\binom{11}{6} = \frac{11!}{5!6!} = 462$$

Number of r-combinations with repetition allowed

Theorem 5

If S has n elements, then the number of r-combinations with repetition allowed of S is

$$\binom{r+n-1}{n-1} = \frac{(r+n-1)!}{(n-1)!r!}$$

Number of r-combinations with repetition allowed

$S = \{1, 2, \ldots, n\}$

Theorem 5

If S has n elements, then the number of r-combinations with repetition allowed of S is

$$\binom{r+n-1}{n-1} = \frac{(r+n-1)!}{(n-1)!r!}$$

$\{a_1, \ldots, a_r\} : a_1, \ldots, a_r \in S$

- Consider a sequence of r stars
- We use $n-1$ vertical bars to separate these stars into n cells

1 2 3 .... n-2 n-1 n

$\star\star \mid \star \mid \star \mid \ldots\ldots \mid \quad \mid \quad \mid \star \quad \Rightarrow \{1,1,2,3,\ldots,n\}$

Theorem 5 proof

- *# stars in each cell = # times the corresponding number appears*

1	2	3	…….	n-2	n-1	n	
⋆ ⋆ |	⋆ |	⋆ |	……. |	|	|	⋆	$\Rightarrow \{1,1,2,3,\ldots,n\}$

Theorem 5 proof

- *# stars in each cell = # times the corresponding number appears*

1	2	3	....	n-2	n-1	n	
⋆ ⋆ |	⋆ |	⋆ |	.... |	|	|	⋆	$\Rightarrow \{1,1,2,3,\ldots,n\}$

- # r-combinations = # arrangements of r stars and $n-1$ bars in a row with $r+n-1$ positions, which is

$$\binom{r+n-1}{n-1} = \frac{(r+n-1)!}{r!(n-1)!}$$

Example 6 revisited

- Problem: Choose 4 pieces of fruit from "apples, oranges, pears".
  We need to choose 4 items (with repetition allowed) from

  $$S = \{\text{apple}, \text{orange}, \text{pear}\}$$

  $n = |S| = 3$

  $r = 4$

  This is 4-combination with repetition allowed of S.

  $$\#\text{ choices} = \binom{r+n-1}{n-1} = \binom{4+3-1}{3-1} = \binom{6}{2} = 15$$

Example 6 revisited

- Problem: Choose 4 pieces of fruit from "apples, oranges, pears". We need to choose 4 items (with repetition allowed) from

$$S=\{\text{apple}, \text{orange}, \text{pear}\}$$

This is 4-combination with repetition allowed of S.

$$\#\text{ choices} = \binom{r+n-1}{n-1} = \binom{4+3-1}{3-1} = \binom{6}{2} = 15$$

- You can do it directly:

4 pieces $\rightarrow$ 4 stars, 3 types of fruits $\rightarrow$ 2 bars

$$\#\text{ choices} = \binom{6}{2} = 15$$

A O P
$** | ** |$ $\rightarrow \{A, A, O, O\}$
$* | ** | *$ $\rightarrow \{A, O, O, P\}$

choices = # arrangements for 4 stars and 2 bars

Example 8

How many ways are there to select 5 pieces of fruit from a bowl containing 3 types of fruits

apples, oranges, pears?

Choose 5 items from apples, oranges, pears

5 stars   2 vertical bars

Use 2 vertical bars to separate 5 stars:

Each arrangement of 2 bars and 5 stars corresponds to one combination:

Apple Orange Pear

$* * * \mid * * \mid$

$\rightarrow \{A, A, A, O, O\}$

$$\# \text{arrangements} = \binom{7}{2} = 21$$

Example 9

How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2, x_3 are **nonnegative integers**?

Remark. A few examples of solutions are $(x_1, x_2, x_3) = (11, 0, 0), (8, 2, 1)$. The triple $(x_1, x_2, x_3) = (-1, 12, 0)$ is not a solution because $x_1 = -1 < 0$.

Example 9 (Fruit bowl analogy)

- Put S={apple, orange, pear}. Consider a choice of 11 fruits such that

$$x_1 \text{ are apples}, x_2 \text{ are oranges}, x_3 \text{ are pears}$$

- Each solution to $x_1 + x_2 + x_3 = 11$ gives one choice to choose 11 fruits from S.

Example 9 (Fruit bowl analogy)

- Put S={apple, orange, pear}. Consider a choice of 11 fruits such that

$$x_1 \text{ are apples, } x_2 \text{ are oranges, } x_3 \text{ are pears}$$

- Each solution to $x_1 + x_2 + x_3 = 11$ gives one choice to choose 11 fruits from S.
- $\#$ solutions to $x_1 + x_2 + x_3 = 11$ is equal to the number of 11-combinations with repetition allowed of S

$$\binom{r+n-1}{n-1} = \binom{3+11-1}{3-1} = \binom{13}{2} = 78.$$

$n = |S| = 3, \; r = 11$

Linear integral equation

Theorem 6

Let r, n be positive integers. Then the number of solutions to the equations

$x_i \in \{0, 1, 2, \ldots\}$

$$x_1 + x_2 + \cdots + x_n = r \text{ such that } x_i \in \mathbb{N} \text{ for } i = 1, \ldots, n$$

is

$$\binom{r+n-1}{n-1} = \frac{(r+n-1)!}{(n-1)!r!}$$

Theorem 6 proof

- Let S be a set of n items, say

$$S = \{1, \ldots, n\}$$

Theorem 6 proof

- Let S be a set of n items, say

$$S = \{1, \ldots, n\}$$

- Each choice of r-combination with repetition allowed of S corresponds to a choice of

$$x_1 \text{ items } 1, \; x_2 \text{ items } 2, \quad \ldots, \; x_n \text{ items } n,$$

such that $x_1 + \cdots + x_n = r$.

Theorem 6 proof

- Let S be a set of n items, say

$$S = \{1, \ldots, n\}$$

- Each choice of r-combination with repetition allowed of S corresponds to a choice of

$$x_1 \text{ items } 1,\ x_2 \text{ items } 2,\ \ \ldots,\ x_n \text{ items } n,$$

such that $x_1 + \cdots + x_n = r$.

- $\#$ solutions to $x_1 + x_2 + \cdots + x_n = r$ is equal to the number of r-combinations with repetition allowed of S, which is

$$\binom{r+n-1}{n-1} = \frac{(r+n-1)!}{(n-1)!r!}$$

Example 10

What is the number of solutions to

solutions to $x_1 + \dots + x_n = r$ is $\binom{r+n-1}{n-1}$

(a) $x_1 + x_2 + x_3 = 9$ with $x_1, x_2, x_3 \in \mathbb{N}$?

$n = 3, \; r = 9$

$$\# \text{ solutions} = \binom{r+n-1}{n-1} = \binom{9+3-1}{3-1} = \binom{11}{2} = 55$$

(b) $x_1 + x_2 + x_3 + x_4 = 10$ with $x_1, x_2, x_3, x_4 \in \mathbb{N}$?

$n = 4, \; r = 10$

$$\# \text{ solutions} = \binom{r+n-1}{n-1} = \binom{13}{3} = 286$$

Exercise 1

How many ways are there to pick 11 fruits from a fruit bowl containing 3 types of fruits

apples, oranges, pears

so that there are at least 2 apples and 1 orange?

Take out 2 apples and 1 orange

→ choose 8 fruits from 3 types of fruits $\begin{cases} n = 3 \\ r = 8 \end{cases}$

$$\binom{r+n-1}{n-1} = \binom{8+3-1}{3-1} = \binom{10}{2} = 45$$

Exercise 2

Find the number of solutions to the equation $x_1 + x_2 + x_3 = 11$, where x_1, x_2, x_3 are nonnegative integers with $x_1 \geq 2$, $x_2 \geq 1$, $x_3 \geq 1$.

solutions to $x_1 + \dots + x_n = r$ with $x_i \geq 0$ is $\binom{r+n-1}{n-1}$

Put $y_1 = x_1 - 2 \geq 0$, $y_2 = x_2 - 1 \geq 0$, $y_3 = x_3 - 1 \geq 0$

$$y_1 + y_2 + y_3 = x_1 - 2 + x_2 - 1 + x_3 - 1$$

$$y_1 + y_2 + y_3 = x_1 + x_2 + x_3 - 4 = 7$$

solutions $= \binom{r+n-1}{n-1} = \binom{7+3-1}{3-1} = \binom{9}{2} = 36$.