CSD1241 Tutorial 11 Solutions

Remarks. The solution should only be used as guidance for your study. There is no guarantee on errors and typos. Would appreciate if you let me know the errors.

Question 1. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the shear with respect to the line $l : 3x - 4y = 0$ which maps the point $P = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ to $P' = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$.

(a) What is the matrix of S? (*Hint*. Use $S(\vec{x}) = \vec{x} + \dfrac{\vec{n} \cdot \vec{x}}{||\vec{n}||}\vec{v}$ to find $\vec{v}$)

(b) Find the normal equation for the image m' of $m : 2x - 3y = 6$ under S.

(c) Let Q be the intersection of m' and l. Find the image Q' of Q under S.

Solution. (a) Note that S is a linear transformation because l passes through the origin. Further, note that $\vec{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ is the normal vector of l. Since $S(P) = P'$, we have

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \frac{\begin{pmatrix} 3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix}}{5}\vec{v} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} \Rightarrow \vec{v} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

The matrix of S is

$$A = I_2 + \frac{1}{||\vec{n}||}\vec{v}\vec{n}^T = \frac{1}{5}\begin{pmatrix} 17 & -16 \\ 9 & -7 \end{pmatrix}$$

(b) The line m has vector equation $\vec{x} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Its image is

$$m' : \vec{x} = \frac{1}{5}\begin{pmatrix} 17 & -16 \\ 9 & -7 \end{pmatrix}\left(\begin{pmatrix} 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 51/5 \\ 27/5 \end{pmatrix} + s\begin{pmatrix} 19 \\ 13 \end{pmatrix},$$

where $s = t/5$.

(c) First, we find the coordinates of Q. Since Q is on l, there is a real number s such that

$$Q = \begin{pmatrix} 51/5 + 19s \\ 27/5 + 13s \end{pmatrix}$$

Since Q is on l, we have

$$3\left(\frac{51}{5} + 19s\right) - 4\left(\frac{27}{5} + 13s\right) = 0 \Leftrightarrow s = -\frac{9}{5}.$$

Hence $Q = \begin{pmatrix} -24 \\ -18 \end{pmatrix}$. Since Q is a point on l, it is fixed under the shear. We obtain

$$Q' = Q = \begin{pmatrix} -24 \\ -18 \end{pmatrix}.$$

$\square$

Question 2. Consider the point $P = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and the line $l : \begin{cases} x = 2 + t \\ y = 3 - t \\ z = 1 + 2t \end{cases}$

(a) Find the point P' on l which is at the closest distance to P.
(b) Let α be the plane through P and perpendicular to l. Find the point Q' on α that is at the closest distance to $Q = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

Solution. (a) Note that the line l contains $\vec{x}_0 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and has direction $\vec{d} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

The projection onto l can be viewed as an affine transformation $T(\vec{x}) = A\vec{x} + \vec{b}$ with

$$\begin{aligned} A &= \frac{1}{||\vec{d}||^2}\vec{d}\vec{d}^T = \frac{1}{6}\begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & -2 & 4 \end{pmatrix} \\ \vec{b} &= \vec{x}_0 - A\vec{x}_0 = \frac{1}{6}\begin{pmatrix} 11 \\ 19 \\ 4 \end{pmatrix} \end{aligned}$$

The orthogonal projection onto l is

$$T(\vec{x}) = \frac{1}{6}\begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & -2 & 4 \end{pmatrix}\vec{x} + \frac{1}{6}\begin{pmatrix} 11 \\ 19 \\ 4 \end{pmatrix}.$$

The point P' is the image of P

$$P' = \frac{1}{6}\begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & -2 & 4 \end{pmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{6}\begin{pmatrix} 11 \\ 19 \\ 4 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 13 \\ 17 \\ 8 \end{pmatrix}$$

(b) The plane α contains $\vec{x}_0 = P = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and has normal vector $\vec{n} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. The projection onto α is an affine transformation $T(\vec{x}) = A\vec{x} + \vec{b}$ with

$$\begin{aligned} A &= I_3 - \frac{1}{||\vec{n}||^2}\vec{n}\vec{n}^T = \frac{1}{6}\begin{pmatrix} 5 & 1 & -2 \\ 1 & 5 & 2 \\ 2 & -2 & 2 \end{pmatrix} \\ \vec{b} &= \vec{x}_0 - A\vec{x}_0 = \frac{1}{3}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \end{aligned}$$

The orthogonal projection onto l is

$$T(\vec{x}) = \frac{1}{6}\begin{pmatrix} 5 & 1 & -2 \\ 1 & 5 & 2 \\ 2 & -2 & 2 \end{pmatrix}\vec{x} + \frac{1}{3}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

The point Q' is the image of Q

$$Q' = \frac{1}{6}\begin{pmatrix} 5 & 1 & -2 \\ 1 & 5 & 2 \\ 2 & -2 & 2 \end{pmatrix}\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 5/2 \\ 2 \end{pmatrix}.$$

$\square$

Question 3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the shear with respect to the plane $\alpha : z = 3$ in the direction of shearing vector $\vec{v} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$.

(a) Write T in the form of an affine map

$$T(\vec{x}) = A\vec{x} + \vec{b}$$

(b) What is the image of the line $l : \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ under T?

(c) What is the image of the plane $\beta : x - z = 0$ under T?

Solution. (a) Note that $\vec{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a normal vector of α and $\vec{x}_0 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$ is a point on α. We have

$$\begin{aligned} A &= I_3 - \frac{1}{||\vec{n}||}\vec{v}\vec{n}^T = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \\ \vec{b} &= \vec{x}_0 - A\vec{x}_0 = \begin{pmatrix} -9 \\ 6 \\ 0 \end{pmatrix} \end{aligned}$$

We obtain

$$T(\vec{x}) = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} -9 \\ 6 \\ 0 \end{pmatrix}$$

(b) The image of l is

$$l' : \vec{x} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right) + \begin{pmatrix} -9 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ 5 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

(c) The plane β has vector equation $\vec{x} = s \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Its image under T is

$$\beta' : \vec{x} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \left(s \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 9 \\ -6 \\ 0 \end{pmatrix} = \begin{pmatrix} -9 \\ 6 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$$

A normal vector of β' is $\vec{n}' = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$. The normal equation of β' is

$$1(x+9) - 4(z-0) = 0 \Leftrightarrow x - 4z = -9.$$

□

Question 4. In this problem, we show that the composition of two reflections in 3D is a rotation.

Let $S \circ T$ be the **composition** of the reflection S through the plane $\alpha : 2x - y + 2z = 0$ and the reflection T through the plane $\beta : x - y = 0$.

(a) Find the matrix M of $S \circ T$ (*Hint*: $M = M_S M_T$).

(b) Find the fixed points of $S \circ T$.

(c) In b, your answer is a line l. Let $\vec{v}$ be the direction of l. Find the angle θ so that M is a rotation matrix, that is,

$$M = (1 - \cos\theta)\frac{\vec{v}\vec{v}^T}{||\vec{v}||^2} + (\cos\theta)I_3 + \frac{\sin\theta}{||\vec{v}||}C_{\vec{v}}$$

Solution. (a) The matrices of S, T and $S \circ T$ are

$$\begin{aligned} M_S &= I_3 - \frac{2}{||\vec{n}_\alpha||^2}\vec{n}_\alpha\vec{n}_\alpha^T = \frac{1}{9}\begin{pmatrix} 1 & 4 & -8 \\ 4 & 7 & 4 \\ -8 & 4 & 1 \end{pmatrix} \\ M_T &= I_3 - \frac{2}{||\vec{n}_\beta||^2}\vec{n}_\beta\vec{n}_\beta^T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ M &= M_{S\circ T} = M_S M_T = \frac{1}{9}\begin{pmatrix} 4 & 1 & -8 \\ 7 & 4 & 4 \\ 4 & -8 & 1 \end{pmatrix} \end{aligned}$$

(b) Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be a fixed point of $S \circ T$. We have

$$\frac{1}{9}\begin{pmatrix} 4 & 1 & -8 \\ 7 & 4 & 4 \\ 4 & -8 & 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{pmatrix} 4x + y - 8z \\ 7x + 4y + 4z \\ 4x - 8y + z \end{pmatrix} = \begin{pmatrix} 9x \\ 9y \\ 9z \end{pmatrix} \Leftrightarrow \begin{cases} -5x + y - 8z = 0 \\ 7x - 5y + 4z = 0 \\ 4x - 8y - 8z = 0 \end{cases}$$

The 3rd equation implies $x = 2y + 2z$. Substituting this into the 1st equation, we have

$$y = 5x + 8z = 5(2y + 2z) + 8z \Rightarrow y = -2z.$$

Now the simultaneous equations become

$$\begin{cases} y = -2z \\ x = 2y + 2z \\ 7x - 5y + 4z = 0 \end{cases} \Leftrightarrow \begin{cases} y = -2z \\ x = 2y + 2z = -2z \\ 7(-2z) - 5(-2z) + 4z = 0 \Leftrightarrow 0 = 0 \end{cases}$$

Therefore, the fixed points of $S \circ T$ are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2z \\ -2z \\ z \end{pmatrix} = z \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix},$$

which is a line going through the origin and having direction $\vec{v} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$.

(c) Taking $\vec{v} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$ from part b, we have

$$\begin{aligned} M &= \frac{1-\cos\theta}{||\vec{v}||^2}\vec{v}\vec{v}^T + (\cos\theta)I_3 + \frac{\sin\theta}{||\vec{v}||}C_{\vec{v}} \\ &= \frac{1-\cos\theta}{9}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix} + \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{pmatrix} + \frac{\sin\theta}{3}\begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix} \\ &= \frac{1}{9}\begin{pmatrix} 4+5\cos\theta & 4-4\cos\theta-3\sin\theta & -2+2\cos\theta-6\sin\theta \\ 4-4\cos\theta+3\sin\theta & 4+5\cos\theta & -2+2\cos\theta+6\sin\theta \\ -2+2\cos\theta+6\sin\theta & -2+2\cos\theta-6\sin\theta & 1+8\cos\theta \end{pmatrix} \end{aligned}$$

Comparing this matrix with the matrix $M = \frac{1}{9}\begin{pmatrix} 4 & 1 & -8 \\ 7 & 4 & 4 \\ 4 & -8 & 1 \end{pmatrix}$ in part a, we obtain

$$\cos\theta = 0, \ \sin\theta = 1,$$

which implies $\theta = 90°$. Therefore, $S \circ T$ is the rotation about $\vec{v} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$ over $90°$.

Remark. If you choose $\vec{v} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$, the angle θ is $\theta = -90° = 270°$. This is because the positive (counter-clockwise) rotation with respect to a vector $\vec{v}$ becomes negative (clockwise) rotation with respect to $-\vec{v}$. $\square$