Lecture 1: Review on high school geometry

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Course Materials

1. **Lecture handouts**: notes by Professor Boerkoel (Redmond)
2. **Lecture slides** provided by myself.
3. Recommended **textbooks**
   - Practical Linear Algebra: A Geometry Toolbox, 3rd edition by G. Farin and D. Hansford
   - Elementary Linear Algebra with Supplemental Applications, 10th edition by H. Anton & C. Rorres.

Course content

1. Chapter 1: Euclidean Space
2. Chapter 2: Matrices and Determinants
3. Chapter 3: Linear Transformations
4. Chapter 4: Affine Transformations

Assessment tasks (tentative)

Assessment Task	**Weight**	**Tentative date**
Homework assignments (5 or 6)	10%	Weeks 1-13
5 Quizzes	30%	Weeks 3,5,9,11,13
1 Midterm test	30%	Week 6
1 Final test	30%	Week 14

Course structure

- **Online lecture** every Tuesday 3-6pm
- **Physical tutorial** every Thursday
  - Groups A,B,C: taught by myself
  - Groups D,E: taught by Rosa
- **Extra tuitions** (starting from week 4) are provided to weak students

What grades can I expect?

- To pass, you need to
  - Score an overall grade D or above
  - *Tips: Do all homework assignments and do not skip exams*
- To have a higher grade?

Attendance policy

- Student $\geq$ 15 minutes late to class will be marked as absent.
- Student may not leave the class early without the instructor's permission.
- Unexcused absences would result in the following penalty

1 letter grade down for # of unexcused absences	2 letter grade down for # of unexcused absences
4	8

What is CSD1241 about?

- Study the concepts of **linear stuff** in 2D and 3D which include

  points, **lines**, **planes**

- Study the **relation between these linear stuff** by the use of

  matrices and **vectors**

- CSD1241 builds the foundation for *CSD2251 Linear Algebra* and *CSD1201 Introduction to Computer Graphics*

Some applications of Linear Algebra

Linear Algebra has a lot of applications in the real life.

- The **Google PageRank search** algorithm uses the the theory of **Eigenvectors** & **Eigenvalues**.

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- **Facial recognition** algorithms are based on **Singular Value Decomposition**.

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Linear Algebra has a lot of applications in the real life.

- The **Google PageRank search** algorithm uses the the theory of **Eigenvectors** & **Eigenvalues**.
- **Facial recognition** algorithms are based on **Singular Value Decomposition**.
- Linear algebra is pervasive in **Machine Learning** and **AI**.
- And many more ......

Points

A **point** is a reference to a *location*.

Points are often denoted by capital letters $P, Q, R, \ldots$

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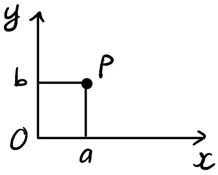
- In **2D plane** (or **xy plane**, or $\mathbb{R}^2$), points are specified by their x-coordinates and y-coordinates

$$P = (a, b) \text{ has x-coordinate} = a, \text{ y-coordinate} = b$$



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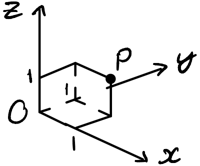
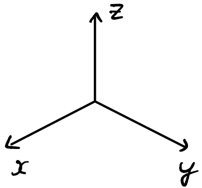


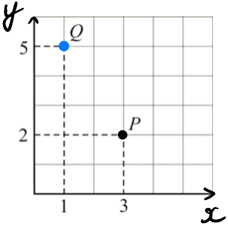


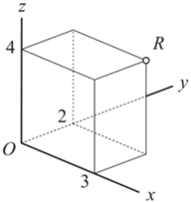
- In **3D space** (or **xyz space**, or $\mathbb{R}^3$), points are specified by their x-coordinates, y-coordinates and z-coordinates

$P = (a, b, c)$ has x-coordinate $= a$, y-coordinate $= b$, z-coordinate $= c$

Example 1

In the following graphs, what are coordinates of P, Q, R?





$P = (3, 2)$

$Q = (1, 5)$

$R = (3, 2, 4)$

Remarks

In this course, we mainly focus on $\mathbb{R}^2$ (xy plane) and $\mathbb{R}^3$ (xyz space).

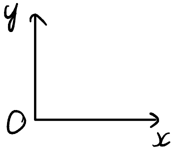
- $\mathbb{R}^2$ and $\mathbb{R}^3$ are different *coordinate systems*.
  We *cannot sketch points simultaneously* on both these systems.

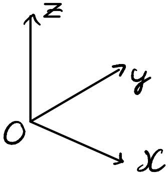
Example : cannot sketch $P=(2,3)$ and $Q=(2,2,2)$ on the same graph.

Remarks

In this course, we mainly focus on $\mathbb{R}^2$ (xy plane) and $\mathbb{R}^3$ (xyz space).

- $\mathbb{R}^2$ and $\mathbb{R}^3$ are different *coordinate systems*.
  We *cannot sketch points simultaneously* on both these systems.
- We call $(0,0)$ and $(0,0,0)$ the **origins** in $\mathbb{R}^2$ and $\mathbb{R}^3$.

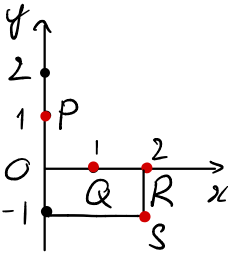




Example 2

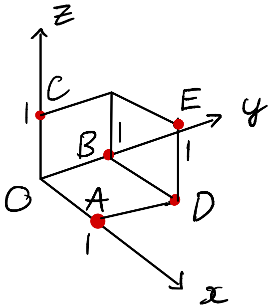
Sketch the following points

(a) $P = (0,1), Q = (1,0), R = (2,0)$ and $S = (2,-1)$ on $\mathbb{R}^2$.



Example 2

(b) $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$, $D = (1, 1, 0)$ and $E = (1, 1, 1)$ on $\mathbb{R}^3$.



Midpoints

The midpoint between 2 points can be obtained by averaging the corresponding coordinates

1. In $\mathbb{R}^2$

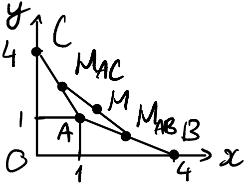
$$P = (x_1, y_1), Q = (x_2, y_2) \Rightarrow M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

2. In $\mathbb{R}^3$

$$P = (x_1, y_1, z_1), Q = (x_2, y_2, z_2) \Rightarrow M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Example 3

(a) Sketch the following points: $A = (1, 1)$, $B = (4, 0)$, $C = (0, 4)$



(b) Find the midpoint M_{AB} of AB and the midpoint M_{AC} of AC

$M_{AB} = \frac{1}{2}(A+B) = \frac{1}{2}(5,1) = (2.5, 0.5)$, $M_{AC} = (0.5, 2.5)$

(c) Find the midpoint M of $M_{AB}M_{AC}$

$M = \frac{1}{2}(M_{AB} + M_{AC}) = (1.5, 1.5)$

Distances between points

- In $\mathbb{R}^2$, the distance between $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- In $\mathbb{R}^3$, the distance between $R = (x_1, y_1, z_1)$ and $S = (x_2, y_2, z_2)$ is

$$d(R, S) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 4

Find the distances between any of the following 2 points.

(a) $P = (0, 1), Q = (1, 0), S = (2, -1)$

$d(P, Q) = \sqrt{(1-0)^2 + (0-1)^2} = \sqrt{2}$

$d(P, S) = \sqrt{(2-0)^2 + (-1-1)^2} = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$

$d(Q, S) = \ldots$

(b) $A = (1, 0, 0), B = (0, 1, 0), C = (2, 3, 1)$

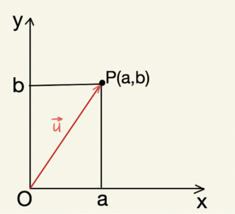
$d(A, B) = \sqrt{(0-1)^2 + (1-0)^2 + (0-0)^2} = \sqrt{2}$

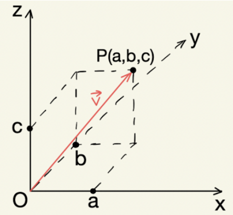
$\vdots$

Vectors

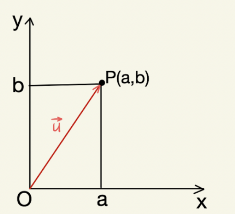
1. In $\mathbb{R}^2$, **vector** $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ is represented by an arrow joining $O = (0, 0)$ and $P = (a, b)$

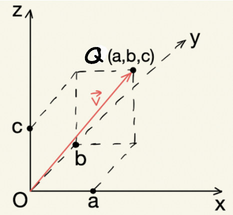
$$\vec{u} = \overrightarrow{OP} = \begin{bmatrix} a \\ b \end{bmatrix}$$





Vectors





2. In $\mathbb{R}^3$, $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is represented by an arrow joining $O = (0,0,0)$ and $Q = (a,b,c)$

$$\vec{v} = \overrightarrow{OQ} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Zero vector

The **zero vector**, denoted by $\vec{0}$, has all coordinates equal to 0.

- In $\mathbb{R}^2$

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- In $\mathbb{R}^3$

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In this course, we use *small letters with arrows* on top to denote vectors:

$$\vec{a}, \vec{b}, \vec{c}, \ldots, \vec{x}, \vec{y}, \vec{z}$$

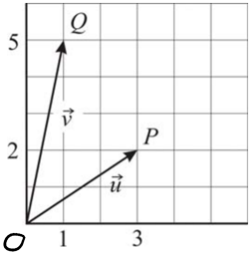
Example

- The vector $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ starts at $O = (0, 0)$ and ends at $P = (3, 2)$

$$\vec{u} = \overrightarrow{OP} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

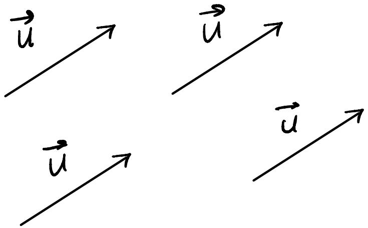
- Similarly

$$\vec{v} = \overrightarrow{OQ} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$



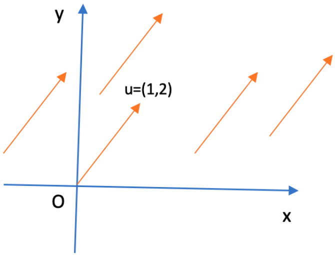
Geometric representation of vectors

- A vector can be geometrically represented by any arrow as long as it preserves **length** and **direction**



Geometric representation of vectors

- A vector can be geometrically represented by any arrow as long as it preserves **length** and **direction**
- All the following arrows denote the same vector $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



Algebraic representation of vectors

In R^2: $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$

In R^3: $\vec{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

- A vector is represented by the algebraic values of their coordinates.
- Given 2 vectors $\vec{u}$ and $\vec{v}$

$$\vec{u} = \vec{v} \Leftrightarrow \begin{cases} \text{their corresponding coordinates are equal.} \\ \vec{u} \,\&\, \vec{v} \text{ are from the same coordinate systems} \end{cases}$$

Example 5

For what values of a, b the following 2 vectors are equal?

$$\vec{u} = \begin{bmatrix} a+b \\ 2a-3b \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ a-b \end{bmatrix}$$

$$\vec{u} = \vec{v} \iff \begin{cases} a+b = 2 \\ 2a-3b = a-b \end{cases} \iff \begin{cases} a+b = 2 & (1) \\ a-2b = 0 & (2) \end{cases}$$

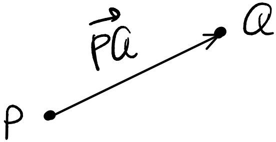
Subtract (1) $-$ (2): $3b = 2 \Rightarrow b = \frac{2}{3}$.

Hence $a = 2 - b = 2 - \frac{2}{3} = \frac{4}{3}$.

$\therefore a = \frac{4}{3}, \; b = \frac{2}{3}$.

Vectors forming by endpoints

For any two points P and Q, we can form the vector $\overrightarrow{PQ}$ which starts from P and ends at Q by subtracting $Q - P$.



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- In $\mathbb{R}^3$

$$P = (x_1, y_1, z_1), Q = (x_2, y_2, z_2) \Rightarrow \overrightarrow{PQ} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$$

Vector addition

Let $\vec{u}, \vec{v}$ be in the **same space** (both in $\mathbb{R}^2$ or both in $\mathbb{R}^3$). We can form $\vec{u} + \vec{v}$ or $\vec{u} - \vec{v}$

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- In $\mathbb{R}^2$:

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \vec{u}+\vec{v} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}, \ \vec{u}-\vec{v} = \begin{bmatrix} u_1-v_1 \\ u_2-v_2 \end{bmatrix}$$

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- In $\mathbb{R}^3$:

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \Rightarrow \vec{u}+\vec{v} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}, \quad \vec{u}-\vec{v} = \begin{bmatrix} u_1-v_1 \\ u_2-v_2 \\ u_3-v_3 \end{bmatrix}$$

Question 1

Let $\vec{u}$ be any vector. What is $\vec{u} + \vec{0}$?

Guess: $\vec{u} + \vec{0} = \vec{u}$!

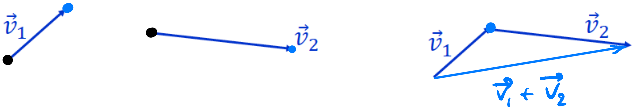
In $\mathbb{R}^2$: $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Rightarrow \vec{u} + \vec{0} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \vec{u}$

In $\mathbb{R}^2$: $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \Rightarrow \vec{u} + \vec{0} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \vec{u}$

Geometrical interpretation of vector addition

To add two vectors $\vec{v_1}, \vec{v_2}$ geometrically, we do the following

- Take 2 arrows which correspond to $\vec{v}_1, \vec{v}_2$ and arrange them so that the ending point of $\vec{v}_1$ lies at the starting point of v_2.



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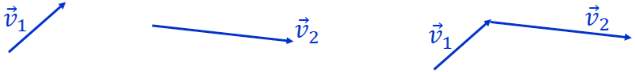
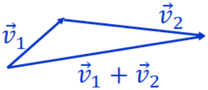
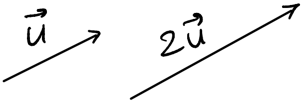


- $\vec{v_1} + \vec{v_2}$ is the arrow which goes from the starting point of $\vec{v}_1$ to the ending point of $\vec{v}_2$



Scalar multiplication

- Any real constant c is called a **scalar**.



Scalar multiplication

- Any real constant c is called a **scalar**.
- The *scalar multiplication* of c by $\vec{u}$, denoted by $c\vec{u}$, is another vector formed by multiplying c into each coordinate of $\vec{u}$.

  (i) In $\mathbb{R}^2$

$$c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}$$

  (ii) In $\mathbb{R}^3$

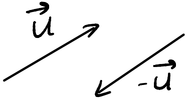
$$c\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}$$

Geometric interpretation of scalar multiplication

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \qquad |2| = 2\ , \ |-2| = 2$$

Geometrically, $c\vec{u}$ is the **scaling** of the vector $\vec{u}$ by the factor c.

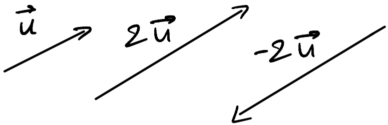
- The length of $c\vec{u}$ is $|c|$ times bigger than the length of $\vec{u}$.



Geometric interpretation of scalar multiplication

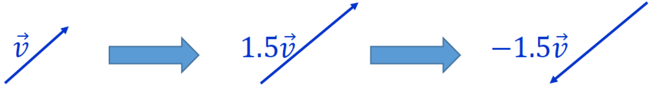
Geometrically, $c\vec{u}$ is the **scaling** of the vector $\vec{u}$ by the factor c.

- The length of $c\vec{u}$ is $|c|$ times bigger than the length of $\vec{u}$.
- If $c > 0$, $c\vec{u}$ and $\vec{u}$ point to the same direction.
  If $c < 0$, $c\vec{u}$ and $\vec{u}$ point to opposite directions.



Example

The vector $(-1.5)\vec{v}$ is obtained as follows.



Example 6

Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Find the following vectors

(a) $\vec{u} + \vec{v}$

$$\vec{u} + \vec{v} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

(b) $2\vec{u} - 3\vec{v}$

$$2\vec{u} - 3\vec{v} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} -3 \\ 9 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$$

(c) $a\vec{u} + b\vec{v}$ for any real constants a, b

$$a\vec{u} + b\vec{v} = \begin{pmatrix} a \\ 2a \end{pmatrix} + \begin{pmatrix} -b \\ 3b \end{pmatrix} = \begin{pmatrix} a-b \\ 2a+3b \end{pmatrix}$$

Length of vectors

(not $|\vec{u}|$)

The **length** (or **norm**) of a vector $\vec{u}$, denoted by $||\vec{u}||$, is the squareroot of the sum of squares of the coordinates of $\vec{u}$.

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- In $\mathbb{R}^2$

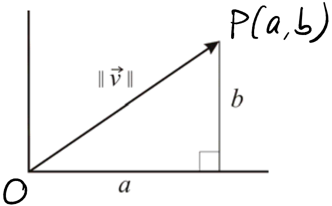
$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow ||\vec{u}|| = \sqrt{u_1^2 + u_2^2}$$

- In $\mathbb{R}^3$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \Rightarrow ||\vec{u}|| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

Explanation for length of vectors

- In the following, the vector $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ is the hypothenuse of a triangle with side lengths a and b.



- By the Pythagoras theorem, its length is $||\vec{v}|| = \sqrt{a^2 + b^2}$.

$P=(x_1, y_1)$, $Q=(x_2, y_2)$

$$d(P,Q) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$\vec{PQ} = \begin{pmatrix} x_2-x_1 \\ y_2-y_1 \end{pmatrix} \Rightarrow \|\vec{PQ}\| = d(P,Q) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Example 7

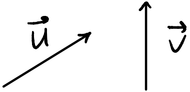
Find the length of the following vectors

$$\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \ \vec{w} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$\|\vec{u}\| = \sqrt{3^2 + 4^2} = 5$$

$$\|\vec{v}\| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$$

$$\|\vec{w}\| = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$$



$$\|\vec{u}\| = \|\vec{v}\|$$

$$\times \quad \vec{u} \neq \vec{v}$$

Remark: Different vectors can have the same length!

Exercise 1 (properties of length)

(a) What is the length of the zero vector $\vec{0}$?

In $\mathbb{R}^2$: $\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \|\vec{0}\| = \sqrt{0^2+0^2} = 0$

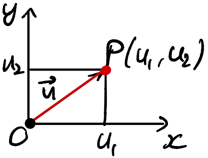
In $\mathbb{R}^3$: $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \|\vec{0}\| = \sqrt{0^2+0^2+0^2} = 0$

$\in$: belong to

$\vec{u} \in \mathbb{R}^2$: $\vec{u}$ is in $\mathbb{R}^2$

(b) Show that for any $\vec{u} \in \mathbb{R}^2$, we have

$$||\vec{u}|| \geq 0$$



Further, show that the only vector in $\mathbb{R}^2$ with length 0 is $\vec{0}$.

$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Rightarrow \|\vec{u}\| = \sqrt{u_1^2+u_2^2} \geq 0$

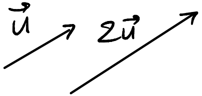
$\|\vec{u}\| = 0 \Leftrightarrow \sqrt{u_1^2+u_2^2} = 0 \Leftrightarrow \underbrace{u_1^2}_{\geq 0} + \underbrace{u_2^2}_{\geq 0} = 0 \Leftrightarrow u_1^2 = u_2^2 = 0$

$\Leftrightarrow u_1 = u_2 = 0$

$\therefore \quad \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0}$.

Exercise 1

(c) Let $\vec{u}$ be any vector in $\mathbb{R}^2$. Prove the following



(i) $||2\vec{u}|| = 2||\vec{u}||$

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \rightarrow 2\vec{u} = \begin{pmatrix} 2u_1 \\ 2u_2 \end{pmatrix}$$

$$||2\vec{u}|| = \sqrt{(2u_1)^2 + (2u_2)^2} = \sqrt{4(u_1^2 + u_2^2)} = 2\sqrt{u_1^2 + u_2^2} = 2||\vec{u}||$$

(ii) For any *positive number* c, $||c\vec{u}|| = c||\vec{u}||$.

Similar!

Exercise 2: Points vs vectors

List the *similarities* and *differences* between points and vectors in $\mathbb{R}^2$.

Similarities

- They are both algebraically represented by coordinate values.
- The length of a vector = distance b/w its 2 endpoints
- We can form a vector from any 2 points

$$P = (x_1, y_1),\ Q = (x_2, y_2) \Rightarrow \vec{PQ} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

Differences

(1) Point = a fixed dot, vector = any arrow $\begin{cases} \text{same length} \\ \text{same direction} \end{cases}$

(2) Points cannot be scaled, vectors can be scaled

(3) Vectors have direction, but points don't have

(4) Vectors have length, but points don't have

(5) Notation difference: points = capital letters, vectors = arrows

Dot product

The **dot product** of two vectors $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \cdot \vec{v}$, is a *scalar* (a real number) defined as follows.

- In $\mathbb{R}^2$

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1 v_1 + u_2 v_2$$

Dot product

The **dot product** of two vectors $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \cdot \vec{v}$, is a *scalar* (a real number) defined as follows.

- In $\mathbb{R}^3$

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Example 8

Compute the dot product of any 2 vectors among the following 3 vectors

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ \vec{u} = \begin{bmatrix} 1 \\ 2 \\ \pi \end{bmatrix}, \ \vec{v} = \begin{bmatrix} -3 \\ 7 \\ 1/\pi \end{bmatrix}$$

$$\vec{0} \cdot \vec{u} = 0 \cdot 1 + 0 \cdot 2 + 0 \cdot \pi = 0$$

$$\vec{0} \cdot \vec{v} = \underline{\hspace{4cm}} = 0$$

$$\vec{u} \cdot \vec{v} = 1 \cdot (-3) + 2 \cdot 7 + \pi \cdot \frac{1}{\pi} = 12$$

Exercise 3

(a) Show that for any vector $\vec{u} \in \mathbb{R}^2$

$$\vec{u} \cdot \vec{u} = ||\vec{u}||^2$$

$$\left.\begin{array}{l} \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Rightarrow \vec{u} \cdot \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1^2 + u_2^2 \\ ||\vec{u}|| = \sqrt{u_1^2 + u_2^2} \Rightarrow ||\vec{u}||^2 = u_1^2 + u_2^2 \end{array}\right\} \Rightarrow \vec{u} \cdot \vec{u} = ||\vec{u}||^2$$

(b) Prove (a) for $\vec{u} \in \mathbb{R}^3$

Similar!

Exercise 3

(c) Show that for any $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

look at the handout!

(d) Prove (c) for $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$

look at the handout!

Properties of dot product

Theorem 1

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in the same space. Then the following hold

(a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (Commutativity)

(b) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (Distributivity)

(c) $\vec{u} \cdot (c\vec{w}) = c(\vec{u} \cdot \vec{w})$ for any scalar c.

(d) $\vec{u} \cdot \vec{u} = ||\vec{u}||^2$.

Exercise 4

Compare:

$(x-y)(x+y) = x^2 - y^2$

Using the properties of dot products, prove that

$$(\vec{u}-\vec{v})\cdot(\vec{u}+\vec{v}) = ||\vec{u}||^2 - ||\vec{v}||^2$$

Proof.

$$\begin{aligned}(\vec{u}-\vec{v})\cdot(\vec{u}+\vec{v}) &= \vec{u}\cdot(\vec{u}+\vec{v}) - \vec{v}\cdot(\vec{u}+\vec{v}) \\ &= \vec{u}\cdot\vec{u} + \cancel{\vec{u}\cdot\vec{v}} - (\cancel{\vec{v}\cdot\vec{u}} + \vec{v}\cdot\vec{v}) \\ &= \vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{v} \\ &= ||\vec{u}||^2 - ||\vec{v}||^2\end{aligned}$$

Angle between 2 vectors

Theorem 2

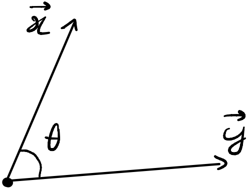
Let $\vec{x}, \vec{y}$ be 2 vectors in the same space and let $\theta \in [0, 180^o]$ be the angle between $\vec{x}$ and $\vec{y}$. Then

$$\vec{x} \cdot \vec{y} = ||\vec{x}|| \, ||\vec{y}|| \cos\theta$$

$$\theta = \angle(\vec{x}, \vec{y})$$

$$\vec{x} \cdot \vec{y} = ||\vec{x}|| \, ||\vec{y}|| \cos\theta$$

$$\cos\theta = \frac{\vec{x} \cdot \vec{y}}{||\vec{x}|| \, ||\vec{y}||} \Rightarrow \theta = \cos^{-1}\left(\frac{\vec{x} \cdot \vec{y}}{||\vec{x}|| \, ||\vec{y}||}\right)$$



Example 9

Compute the angle between $\vec{x}$ and $\vec{y}$ in the following cases.

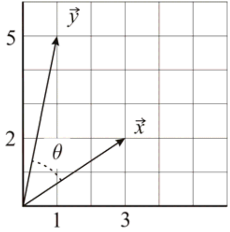
(a) $\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \, \|\vec{y}\| \cos\theta$$

$$\cos\theta = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \, \|\vec{y}\|} = \frac{3\times 1 + 2\times 5}{\sqrt{3^2+2^2}\sqrt{1^2+5^2}}$$

$$\cos\theta = \frac{13}{\sqrt{13}\sqrt{26}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\theta = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^\circ.$$



θ : theta

Example 9

(b) $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$

$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \, \|\vec{y}\| \cos\theta$$

$$\cos\theta = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \, \|\vec{y}\|} = \frac{2 \cdot 3 + 3 \cdot 1 + 1 \cdot (-2)}{\sqrt{2^2+3^2+1^2} \sqrt{3^2+1^2+(-2)^2}} = \frac{7}{14} = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ.$$

Types of angles

$$\theta = \angle(\vec{x}, \vec{y}) \Rightarrow \theta \in [0^\circ, 180^\circ]$$

There are three types of angles between two vectors $\vec{x}$ and $\vec{y}$.

$$\cos\theta = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}$$

1. Right angle: $\theta = 90^o$

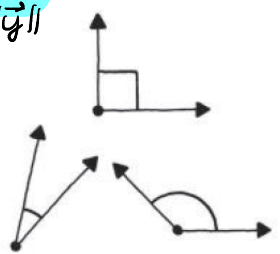
   $\cos\theta = 0 \Leftrightarrow \vec{x} \cdot \vec{y} = 0$

2. Acute angle: $\theta < 90^o$

   $\cos\theta > 0 \Leftrightarrow \vec{x} \cdot \vec{y} > 0$

3. Obtuse angle: $\theta > 90^o$

   $\cos\theta < 0 \Leftrightarrow \vec{x} \cdot \vec{y} < 0$



Exercise 5

Find the type of angle (right, acute, obtuse) between $\vec{x}$ and $\vec{y}$

(a) $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{y} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\pi = 3.14\ldots$

Exercise

(b) $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ \pi \end{bmatrix}, \vec{y} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$\vec{x} \cdot \vec{y} = 2 \cdot 0 + 1 \cdot (-1) + \pi \cdot 1 = \pi - 1 > 0$

$\therefore \angle(\vec{x}, \vec{y})$ is acute!

Exercise 5

(c) $\vec{x} = \begin{bmatrix} 1 \\ e \\ \pi \end{bmatrix}, \vec{y} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Exercise.

Exercise 6 (Triangle inequality)

Let $\vec{x}$ and $\vec{y}$ be vectors in the same space. Prove that

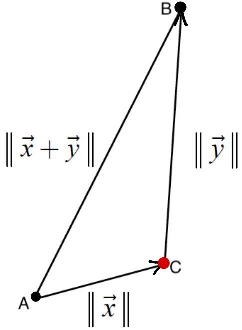
$$||\vec{x}+\vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||.$$

Geometric interpretation of triangle inequality

Put $\vec{x} = \overrightarrow{AC}, \vec{y} = \overrightarrow{CB} \Rightarrow \vec{x} + \vec{y} = \overrightarrow{AB}$.

$$||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}|| \Leftrightarrow AB \leq AC + CB$$

Imagine you have to travel from A to B. Consider 2 paths



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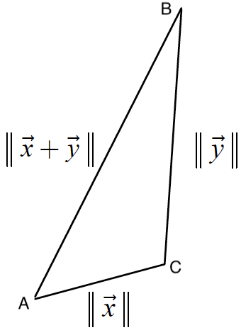
Imagine you have to travel from A to B. Consider 2 paths

1. Direct path $\Rightarrow$ shortest way and

$$\text{length} = ||\overrightarrow{AB}|| = ||\vec{x} + \vec{y}||$$



Geometric interpretation of triangle inequality

Put $\vec{x} = \overrightarrow{AC}, \vec{y} = \overrightarrow{CB} \Rightarrow \vec{x} + \vec{y} = \overrightarrow{AB}$.

$$||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}|| \Leftrightarrow AB \leq AC + CB$$

Imagine you have to travel from A to B. Consider 2 paths

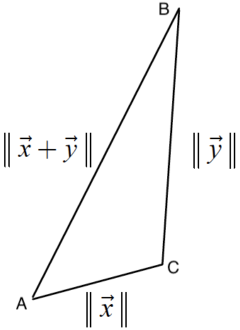
1. Direct path $\Rightarrow$ shortest way and

$$\text{length} = ||\overrightarrow{AB}|| = ||\vec{x} + \vec{y}||$$

2. Move A to C, then C to B. The length is

$$||\overrightarrow{AC}|| + ||\overrightarrow{CB}|| = ||\vec{x}|| + ||\vec{y}||$$

By length comparison,
$||\overrightarrow{AB}|| \leq ||\overrightarrow{AC}|| + ||\overrightarrow{CB}||$

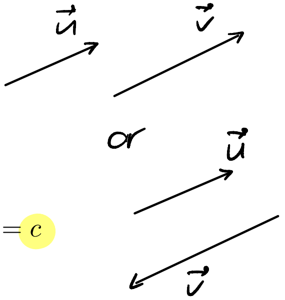


Parallel vectors

Two vectors $\vec{u}$ and $\vec{v}$ are called **parallel**, denoted $\vec{u} \parallel \vec{v}$, if there exists a scalar c such that

$$\vec{u} = c\vec{v}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} cv_1 \\ cv_2 \end{pmatrix} \Rightarrow \begin{cases} u_1 = cv_1 \\ u_2 = cv_2 \end{cases}$$



- In $\mathbb{R}^2$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Leftrightarrow \frac{u_1}{v_1} = \frac{u_2}{v_2} = c$$

- In $\mathbb{R}^3$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = c \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \Leftrightarrow \frac{u_1}{v_1} = \frac{u_2}{v_2} = \frac{u_3}{v_3} = c$$

Characterization of parallel vectors

Theorem 3

(a) In $\mathbb{R}^2$, $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ are parallel if and only if

$$\frac{u_1}{v_1} = \frac{u_2}{v_2},$$

where we use the convention that $u_i = 0$ whenever $v_i = 0$.

Characterization of parallel vectors

Theorem 3

(a) In $\mathbb{R}^2$, $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ are parallel if and only if

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where we use the convention that $u_i = 0$ whenever $v_i = 0$.

(b) In $\mathbb{R}^3$, $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ are parallel if and only if

$$\frac{u_1}{v_1} = \frac{u_2}{v_2} = \frac{u_3}{v_3},$$

where we use the convention that $u_i = 0$ whenever $v_i = 0$.

Example 10

$\nparallel$: not parallel

Which of the following pairs of vectors are parallel?

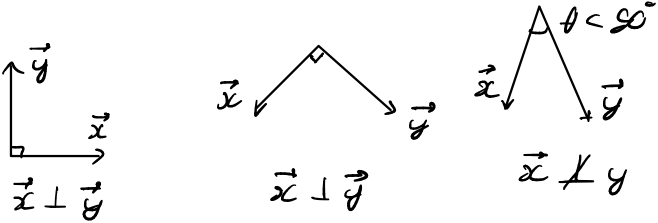
(a) $\vec{u} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ $\quad \frac{2}{1} = \frac{4}{2} = \frac{10}{5} \Rightarrow \vec{u} \parallel \vec{v}$

(b) $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\quad \frac{1}{2} \neq \frac{2}{3} \Rightarrow \vec{x} \nparallel \vec{y}$

(c) $\vec{u} = \begin{bmatrix} -2 \\ -4 \\ -8 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ $\quad \frac{-2}{1} = \frac{-4}{2} = \frac{-8}{4} \Rightarrow \vec{u} \parallel \vec{v}$

Orthogonal vectors

- $\vec{x}$ and $\vec{y}$ are **orthogonal** (or perpendicular), denoted $\vec{x} \perp \vec{y}$, if the angle θ between $\vec{x}$ and $\vec{y}$ is 90^o.
- If $\theta \neq 90^\circ$, we write $\vec{x} \not\perp \vec{y}$.



Orthogonal vectors

- $\vec{x}$ and $\vec{y}$ are **orthogonal** (or perpendicular), denoted $\vec{x} \perp \vec{y}$, if the angle θ between $\vec{x}$ and $\vec{y}$ is 90^o.
- If $\theta \neq 90^\circ$, we write $\vec{x} \not\perp \vec{y}$.
- Question: When is $\theta = 90^0$?

$$\cos\theta = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \, \|\vec{y}\|}$$

$$\theta = 90^\circ \Leftrightarrow \cos\theta = 0 \Leftrightarrow \vec{x} \cdot \vec{y} = 0$$

Orthogonal vectors

- $\vec{x}$ and $\vec{y}$ are **orthogonal** (or perpendicular), denoted $\vec{x} \perp \vec{y}$, if the angle θ between $\vec{x}$ and $\vec{y}$ is 90^o.
- If $\theta \neq 90^\circ$, we write $\vec{x} \not\perp \vec{y}$.
- Remark: The zero vector $\vec{0}$ is orthogonal to any vector.

Why?

Because $\vec{0} \cdot \vec{x} = 0$.

Example 11

Which of the following pairs of vectors are orthogonal?

(a) $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$

(b) $\vec{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\vec{c} \cdot \vec{d} = -1 \neq 0 \Rightarrow \vec{c} \not\perp \vec{d}$

(c) $\vec{e} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{f} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

$\vec{e} \cdot \vec{f} = 0 \Rightarrow \vec{e} \perp \vec{f}$

Exercise 7

(a) Find the condition for real numbers a, b so that

$$\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix} \text{ is orthogonal to } \vec{y} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

(b) Give 3 examples of the vector $\vec{x}$ in part a.

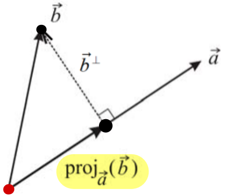
(a) $\vec{x} \perp \vec{y} \Leftrightarrow \vec{x} \cdot \vec{y} = 0$

$\Leftrightarrow a - 2b = 0$

$\Leftrightarrow a = 2b$

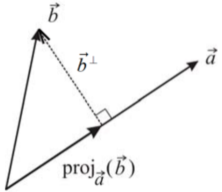
(b) Chose $b = 1, 2, 3 \Rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 3 \end{pmatrix}$.

Orthogonal projection



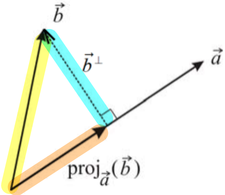
- The *orthogonal projection* of $\vec{b}$ onto a nonzero vector $\vec{a}$, denoted $\mathrm{proj}_{\vec{a}}(\vec{b})$, is formed by

Orthogonal projection



- The *orthogonal projection* of $\vec{b}$ onto a nonzero vector $\vec{a}$, denoted $\text{proj}_{\vec{a}}(\vec{b})$, is formed by
  1. Arrange $\vec{a}$ and $\vec{b}$ so that they have the **same starting point**.
  2. Project the **endpoint** of $\vec{b}$ orthogonally into $\vec{a}$.

Orthogonal projection



- The *orthogonal projection* of $\vec{b}$ onto a nonzero vector $\vec{a}$, denoted $\mathrm{proj}_{\vec{a}}(\vec{b})$, is formed by
  1. Arrange $\vec{a}$ and $\vec{b}$ so that they have the **same starting point**.
  2. Project the **endpoint** of $\vec{b}$ orthogonally into $\vec{a}$.
- The **orthogonal complement** of $\vec{b}$ onto $\vec{a}$ is

$$\vec{b}^{\perp} = \vec{b} - \mathrm{proj}_{\vec{a}}(\vec{b})$$

Note : $\vec{b}^{\perp} \perp \vec{a}$

Formula for orthogonal projection

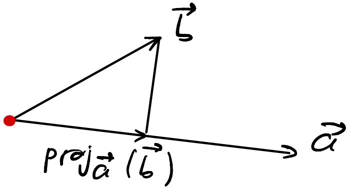
Theorem 4

Let $\vec{a}, \vec{b}$ be two vectors in the same space with $\vec{a} \neq \vec{0}$. The *orthogonal projection* of $\vec{b}$ onto $\vec{a}$ is

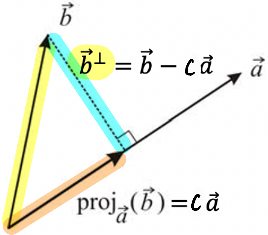
$$\mathrm{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a}$$

$\mathrm{proj}_{\vec{a}}(\vec{b}) \parallel \vec{a}$

$\mathrm{proj}_{\vec{a}}(\vec{b}) = c\,\vec{a}$



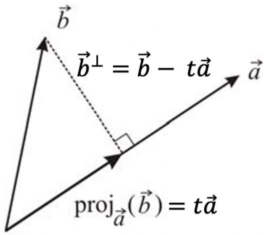
Theorem 3 proof



- Since $\text{proj}_{\vec{a}}(\vec{b})$ and $\vec{a}$ are parallel, there is $c \in \mathbb{R}$:

$$\text{proj}_{\vec{a}}(\vec{b}) = c\vec{a} \Rightarrow \vec{b}^{\perp} = \vec{b} - c\vec{a}$$

Theorem 3 proof



- Since $\mathrm{proj}_{\vec{a}}(\vec{b})$ and $\vec{a}$ are parallel, there is $c \in \mathbb{R}$:

$$\mathrm{proj}_{\vec{a}}(\vec{b}) = c\vec{a} \Rightarrow \vec{b}^{\perp} = \vec{b} - c\vec{a}$$

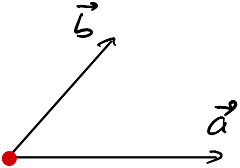
- $\vec{b}^{\perp} = \vec{b} - c\vec{a}$ is orthogonal to $\vec{a}$. So

$$0 = (c\vec{a} - \vec{b}) \cdot \vec{a} = c\vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{a} \Rightarrow c = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}$$

Summary on orthogonal projection

1. The orthogonal projection $\mathrm{proj}_{\vec{a}}(\vec{b})$ is only defined if $\vec{a} \neq \vec{0}$.
2. $\mathrm{proj}_{\vec{a}}(\vec{b})$ is a scalar multiple of $\vec{a}$, say

$$\mathrm{proj}_{\vec{a}}(\vec{b}) = c\vec{a}$$



with the scale

$$c = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}$$

$$\mathrm{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a}$$

Example 12

Find $\text{proj}_{\vec{a}}(\vec{b})$ and $\vec{b}^{\perp}$. Verify that $\vec{b}^{\perp}$ is orthogonal (perpendicular) to $\vec{a}$.

(a) $\vec{a} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

Example 12

(b) $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$