Quiz 2 collection/review → reply today
Consultation

Series Convergence Tests Part 2
Power Series Fundamentals

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AY 23/24 Trimester 1

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Efficient alternative to Comparison Test

Limit Comparison Test

one of them you have, another you will have to come up yourself

$\frac{\infty}{\infty}$ indeterminates

Theorem (Limit Comparison Test, or the LCT)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with positive terms. If c is a number such that

ratio of series terms

finite

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0,$$

then either both series converge, or both series diverge.

Note: The limit comparison test is NOT the comparison test.

$\frac{a_n}{b_n} \to c > 0$

When n is large

$\frac{a_n}{b_n} \simeq c \Rightarrow a_n \simeq c b_n$

$\sum a_n < \infty \Leftrightarrow \sum b_n < \infty$

Standard steps in the use of LCT

question

We usually start with a series $\sum_{n=1}^{\infty} a_n$, and we are asked to figure out if this series is convergent or divergent.

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\begin{cases} \text{convergent} & p > 1 \\ \text{divergent} & p \leq 1 \end{cases}$

1. Find a series $\sum_{n=1}^{\infty} b_n$ which is *similar* to $\sum_{n=1}^{\infty} a_n$, and whose convergence/divergence is **known**. → got to know if $\sum_{n=1}^{\infty} bn$ converges or diverges
2. Compute $\lim_{n\to\infty} \frac{a_n}{b_n}$.
3. If this limit exists, and is positive, we can apply the LCT.

Example 1

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+2n}$.

$\frac{2}{n^2+n}$ similar to $\frac{1}{n^2}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

p-series $p = 2 > 1$

constant

Let $a_n = \frac{2}{n^2+2n}$ and $b_n = \frac{1}{n^2}$. (all positive)

$$\frac{a_n}{b_n} = \frac{\frac{2}{n^2+2n}}{\frac{1}{n^2}} = \frac{2}{n^2+2n} \cdot \frac{n^2}{1} = \frac{2n^2}{n^2+2n}.$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2}{n^2+2n} = \lim_{n\to\infty} \frac{2}{1+\cancel{\frac{2}{n}}^{\to 0}} = \frac{2}{1} = 2 > 0$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent (p-series, $p=2>1$), by the LCT,

$\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+2n}$ is also convergent.

Exercise 1

'struggled' a lot using CT

Use the LCT to determine the convergence of the following series.

1. $\displaystyle\sum_{n=3}^{\infty} \frac{n}{n^3 - 8}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + 2}$

① $a_n = \dfrac{n}{n^3-8}$, $b_n = \dfrac{1}{n^2}$ > 0 for $n \geq 3$

$$\frac{a_n}{b_n} = \frac{\frac{n}{n^3-8}}{\frac{1}{n^2}} = \frac{n}{n^3-8} \cdot \frac{n^2}{1} = \frac{n^3}{n^3-8}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{1-\frac{8}{n^3}\to 0} = 1 > 0.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ (p-series, $p = 2 > 1$), by LCT, $\displaystyle\sum_{n=3}^{\infty} \frac{n}{n^3-8} < \infty$.

Compare this working with last week's lecture

LCT vs CT essentially

more straightforward

Exercise 1

(2) $a_n = \dfrac{1}{2^n+2}$, $b_n = \dfrac{1}{2^n}$

Divide numerator & denominator by 2^n

$$\frac{a_n}{b_n} = \frac{\frac{1}{2^n+2}}{\frac{1}{2^n}} = \frac{2^n}{2^n+2} \Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{2^n}} = 1 > 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$ (geometric series with $|\frac{1}{2}| < 1$), by LCT,

$$\sum_{n=1}^{\infty} \frac{1}{2^n+2} < \infty.$$

$$\sum_{n=1}^{\infty} \frac{n}{n^2+2}$$

converge/diverge?

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges or converge?

★ Choose b_n as taught in eg.

"wrong" examples

b_n $\frac{1}{n^2}$ $\to$ $\frac{a_n}{b_n} \to \infty$

$b_n = \frac{1}{\sqrt{n}}$

$\frac{a_n}{b_n} \to 0$

Alternating Series

Convergence will depend on just b_n

An **alternating series** is a series $\sum_{n=1}^{\infty} a_n$ where

alternating factor

$$a_n = (-1)^n b_n \quad \text{or} \quad a_n = (-1)^{n-1} b_n$$

such that $\{b_n\}_{n=1}^{\infty}$ is a **positive** sequence, i.e. $b_n > 0$ for all n.

Examples of such series include

$$\sum_{n=1}^{\infty} (-1)^{n-1} \overbrace{\frac{1}{n}}^{b_n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

$$\rightarrow \quad \sum_{n=1}^{\infty} (-1)^n \overbrace{\frac{n}{n+1}}^{b_n} = -\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots$$

decreasing, $b_n \to 1 \neq 0$.

Alternating Series Test

The convergence of an alternating series will depend on the characteristics of the positive sequence $\{b_n\}$.

Theorem (Alternating Series Test)

If an alternating series

positive b_n

$$\sum_{n=1}^{\infty}(-1)^n b_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n-1} b_n$$

$f'(x) \leq 0$

satisfies the conditions

b_n decreasing

(a) $b_{n+1} \leq b_n$ for $n \geq n_0$

(b) $\lim\limits_{n\to\infty} b_n = 0$, $\rightarrow$ easy to check.

then the series is convergent.

Example 2

What is b_n? $b_n = \frac{1}{n} > 0$ clear.

Is the **alternating harmonic series** $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ convergent?

$= x^{-1} \xrightarrow{\text{Diff}} -1 \cdot x^{-2}$

① b_n decreasing $b_n = \frac{1}{n} \Rightarrow$ Let $f(x) = \frac{1}{x}$ $x \geq 1$

$$f'(x) = -\frac{1}{x^2} < 0 \quad \text{for } x \geq 1.$$

>0

$\therefore$ f is decreasing $\Rightarrow$ b_n decreasing. ✓

② Obviously, $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n} = 0.$

$\therefore$ By AST, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ is convergent.

Exercise 2

Differentiation Rules need to know

Establish the convergence/divergence of the following alternating series.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{n^3+1}$
2. $\displaystyle\sum_{n=1}^{\infty}(-4)^n$

① $b_n = \dfrac{n^2}{n^3+1} > 0$ Let $f(x) = \dfrac{x^2}{x^3+1}$, $x \geq 1$

$$f'(x) = \frac{(x^3+1)\cdot 2x - x^2(3x^2)}{(x^3+1)^2}$$

$$= \frac{2x^4 + 2x - 3x^4}{(x^3+1)^2} = \frac{2x - x^4}{(x^3+1)^2}$$

$$= \frac{\overset{>0}{x}(2 - x^3)}{\underbrace{(x^3+1)^2}_{>0}} < 0$$

$2 - x^3 < 0$ for $x \geq 2$

for $x \geq 2$.

b_n is decreasing for $n \geq 2$. ✓

② $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+1}$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1 + \frac{1}{n^3}} = \frac{0}{1} = 0 \checkmark$$

$\therefore$ By AST, $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{n^3+1}$ is convergent.

Exercise 2

is this limit zero or nonzero/DNE → series is divergent.

② $\sum_{n=1}^{\infty} (-4)^n$

$\lim_{n\to\infty} b_n \neq 0$, can use Divergence Test

$(-1)^n \cdot 4^n$ → clearly does not go to zero. Use Divergence Test on a_n!

Let $a_n = (-4)^n$

$a_{2n} = (-4)^{2n} = ((-4)^2)^n = 16^n$

Since a subsequence of a_n diverges, ∴ a_n must also diverge.

∴ $\lim_{n\to\infty} a_n$ does not exist, hence $\sum_{n=1}^{\infty} (-4)^n$ is divergent by the Divergence Test.

Non-alternating series vs alternating series

absolute value the terms

We have seen two of these series, one of them is convergent and the other is not:

harmonic series divergent. ← $$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$ → convergent alternating harmonic series Example 2

While on the other hand, both of these series are convergent:

p-series $p = 2 > 1$ convergent ← $$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$$

→ ① b_n decreasing ② $b_n \to 0$

$\sum \left| (-1)^n \frac{1}{n} \right| = \sum \frac{1}{n}$

$\left| (-1)^n \frac{1}{n} \right| = \frac{1}{n}$

absolute values.

Questions to ask:

1. For each pair of series, what is the relation between their terms?
2. How do we reconcile the similarities/differences between their convergence? Absolute & Conditional Convergence (2nd pair) (1st pair)

Absolute Convergence

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ is absolutely convergent because $\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

- A series $\sum_{n=1}^{\infty} a_n$ is said to be **absolutely convergent** if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ converges. → NOT $\left|\sum_{n=1}^{\infty} a_n\right|$!

conditionally convergent ← $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ → $\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$

- A series $\sum_{n=1}^{\infty} a_n$ is said to be **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges, but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ diverges.
- Can you think of examples of absolutely convergent and conditionally convergent series? → In this slide

Absolute Convergence Test

For the 'few' cases where the LCT/CT not applicable (got negative terms)

Slide 13 gives us an example of a series that is conditionally convergent. Conversely, the following test tells us that absolutely convergent series must be convergent.

Another way to test convergence.

Theorem (Absolute Convergence Test)

If not absolutely convergent, move on to other test

If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is also convergent.

$a_n \leq |a_n|$

$0 \leq a_n + |a_n| \leq 2|a_n|$

Can prove Absolute convergence test using this inequality & Comparison Test.

Example 3

Some negative terms, cannot apply CT/LCT

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is convergent or divergent.

Consider $\displaystyle\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$ ← show that this series is convergent

$0 \le |\sin n| \le 1$

$\Rightarrow 0 \le \dfrac{|\sin n|}{n^2} \le \dfrac{1}{n^2}$

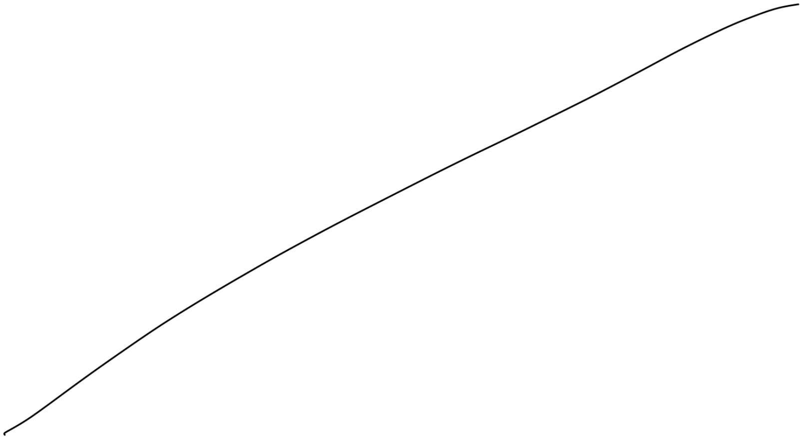
Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ (p-series, $p=2>1$)

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} < \infty$ by Comparison Test.

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is absolutely convergent

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is convergent by the Absolute Convergence Test.

Example 3



Exercise 3

$\sum |a_n| < \infty$ $\quad$ $\sum a_n < \infty$ but $\sum |a_n|$ divergent

Determine if the following series is absolutely convergent, conditionally convergent, or divergent.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ $\rightarrow$ $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$ convergent (p-series, $p = 3 > 1$)

   $\therefore \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ is absolutely convergent.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ $\rightarrow$ $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ divergent (p-series, $p = \frac{1}{2} \le 1$)

$b_n = \frac{1}{\sqrt{n}}$

① $f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} \Rightarrow f'(x) = -\frac{1}{2} \underbrace{x^{-3/2}}_{>0} < 0$

$x \geq 1$ $\quad$ $\therefore$ b_n decreasing $\rightarrow$ by ① & ②, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

② $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ $\rightarrow$ is convergent by AST.

Hence $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is conditionally convergent.

Ratio Test ✱✱

$\sum_{n=1}^{\infty} ar^{n-1} < \infty$ if $|r|<1$ $\quad |r| = \left|\frac{ar^n}{ar^{n-1}}\right| < 1$

$n+1$ term

$a_n \rightarrow a_n = ar^{n-1}$

nth term

Theorem (Ratio Test)

$a_{n+1} = ar^{n+1-1} = ar^n$

$\left|\frac{a_{n+1}}{a_n}\right| \rightarrow L<1$

Let $\sum_{n=1}^{\infty} a_n$ be a series, and let

$$\rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L.$$

parallels → Geometric series

$|r|<1$ $\sum a_n$ absolutely convergent

- If $L<1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$|r|>1$ $\sum a_n$ divergent

- If $L>1$ or $L=\infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

$|r|=1$ inconclusive

- If $L=1$, then the Ratio Test is inconclusive; no conclusion can be drawn about the convergence or divergence of this series.

Root Test ✱✱

Geometric series $\sum_{n=0}^{\infty} ar^n$ $\sqrt[n]{|ar^n|} = \sqrt[n]{a} \cdot |r|$

Theorem

Let $\sum_{n=1}^{\infty} a_n$ be a series, and let

Ratio Test & Root Test are essential for power series

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L.$$

Same as Ratio Test

- If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- If $L > 1$ or $L = \infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $L = 1$, then the Root Test is inconclusive; no conclusion can be drawn about the convergence or divergence of this series.

Example 4

Determine the convergence of the following series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ $\rightarrow$ $a_n = \frac{n!}{n^n}$ $\rightarrow$ $\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \right|$

$$= \frac{\cancel{(n+1)!}}{\cancel{n!}} \frac{1}{\cancel{n+1}} \frac{n^n}{(n+1)^n}$$

$$= \left(\frac{n+1-1}{n+1}\right)^n = \left(1 + \frac{(-1)}{n+1}\right)^n \overset{*}{\rightarrow} \frac{1}{e} < 1$$

2. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n+2}{4n+5}\right)^n$

$$* \left(1 - \frac{1}{n+1}\right)^n = e^{n\ln\left(1 - \frac{1}{n+1}\right)}$$

$$\lim_{n\to\infty} n\ln\left(1 - \frac{1}{n+1}\right) = \lim_{n\to\infty} \frac{\ln\left(1-\frac{1}{n+1}\right)}{\frac{1}{n}} \left(\frac{0}{0}\right) \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{1-\frac{1}{n+1}} \cdot \frac{1}{(n+1)^2}}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \underbrace{\frac{1}{1-\frac{1}{n+1}}}_{\rightarrow 1} \cdot \underbrace{\left(-\frac{n^2}{(n+1)^2}\right)}_{\rightarrow -1} = -1, \quad \therefore \left(1 - \frac{1}{n+1}\right)^n \rightarrow e^{-1} = \boxed{\frac{1}{e}}$$

Example 4

$$\sum_{n=1}^{\infty} \left(\frac{3n+2}{4n+5}\right)^n \longrightarrow \text{Root Test}$$

$$\sqrt[n]{\left|\left(\frac{3n+2}{4n+5}\right)^n\right|} = \sqrt[n]{\left(\frac{3n+2}{4n+5}\right)^n} = \frac{3n+2}{4n+5} \to \frac{3}{4} < 1$$

$\therefore \sum_{n=1}^{\infty} \left(\frac{3n+2}{4n+5}\right)^n$ is absolutely convergent by Root Test.

Power Series

$\sum a_n$ there is only n

center of a power series

A **power series** centered at a is a series of the form

minus

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

akin to coefficients of a polynomial

where x is a **variable** and c_n are constants called the **coefficients** of this power series.

Power series convergence depends on the variable x

Example: If $a = 0$ and $c_n = 1$ for all $n \geq 0$, we have

$$\sum_{n=0}^{\infty} \overset{1}{c_n}(x-\overset{0}{a})^n = \underbrace{\sum_{n=0}^{\infty}}_{\text{replace } r \text{ with } x} x^n = \begin{cases} \dfrac{1}{1-x} & \text{if } |x| < 1 \\ \text{divergent} & \text{if } |x| \geq 1. \end{cases}$$

This is a geometric series that we saw in last lecture. Thus, a geometric series is also a power series.

Power series is also a function of x

- A power series can be seen as a function of x.
- By substituting different values of x, we get different series. Hence, a power series may converge for some values of x and diverge for other values of x.
- The power series $\sum_{n=0}^{\infty} c_n(\overset{a}{x} - a)^n$ **always** converges at $x = a$, that is

always converges at its center a.

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots \quad x=a$$

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 \text{ if } x = a.$$

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- For x values for which the power series is **convergent**, we can let

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n.$$

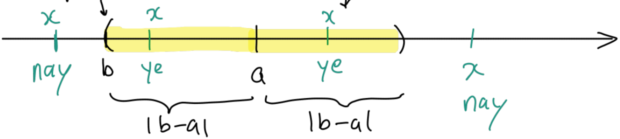
Convergence of Power Series

Some point other than the center

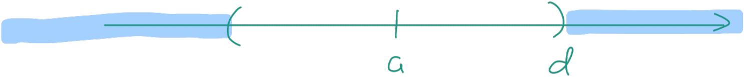
- If $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges at $x = b$ where $b \neq a$, then it converges whenever $|x-a| < |b-a|$.

power series convergent.

values highlight in yellow is where the series is convergent

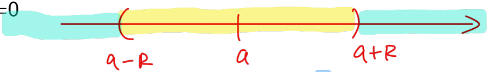


- If $\sum_{n=0}^{\infty} c_n(x-a)^n$ diverges at $x = d$ where $d \neq a$, then it diverges whenever $|x-a| > |d-a|$.



Radius of convergence

The **radius of convergence** of $\sum_{n=0}^{\infty} c_n(x-a)^n$ is **a number** R such that



$$\sum_{n=0}^{\infty} c_n(x-a)^n \text{ converges if } |x-a| < R \text{ and diverges if } |x-a| > R.$$

coefficient of x must be 1.

$|2x+3|$
$= 2|x+\frac{3}{2}|$

$a = -\frac{3}{2}$

There are **three** cases for R:

- R is a positive number. In this case, $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $|x-a| < R$ and diverges for $|x-a| > R$.
- $R = 0$. In this case, $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges only at $x = a$.
- $R = \infty$. In this case, $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for all $x \in \mathbb{R}$.

every x value.

Example 5: Calculation of radius of convergence

Find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n+1}}$.

$a=0$

$a_n = \dfrac{(-1)^n x^n}{\sqrt{n+1}}$

Ratio Test / Root Test

For $x=0$, series is convergent

For $x \neq 0$

$\boxed{|x-a| < R}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-1)^n x^n}\right|$$

$|x-0| < 1 = R$

$$= |x| \left|-\frac{\sqrt{n+1}}{\sqrt{n+2}}\right| = |x| \frac{\sqrt{n+1}}{\sqrt{n+2}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x| \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n+2}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = |x| \lim_{n\to\infty} \overbrace{\frac{\sqrt{1+\frac{1}{n}}}{\sqrt{1+\frac{2}{n}}}}^{1} = |x| < 1$$

Radius of convergence is 1.

Example 5

Exercise 4

$\left|x+\frac{2}{3}\right|$

Find the radius of convergence for the following power series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n(3x+2)^n}{4^n}$
2. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

① For $x = -\frac{2}{3}$, series is convergent.

For $x \neq -\frac{2}{3}$, $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(3x+2)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{n(3x+2)^n}\right|$

$= \frac{|3x+2|}{4} \cdot \underbrace{\frac{n+1}{n}}_{\to 1} \to \frac{|3x+2|}{4} < 1$

$\Rightarrow |3x+2| < 4$

$\Rightarrow \left|x+\frac{2}{3}\right| < \frac{4}{3} \Rightarrow$ Radius of convergence $R = \frac{4}{3}$

* coefficient 1

Exercise 4

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

For $x=0$, series is convergent.

For $x \neq 0$, $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = |x| \cdot \frac{1}{n+1} \to 0 < 1$ as $n \to \infty$

Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$ no matter what x,

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent for all x

$\Rightarrow R = \infty$.