Q1 (a) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.

This is the Product Rule. ← Used in integration by parts

(b) $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$.

This is the Chain Rule. ← used in integration by substitution

Q2 (a) $f(x) = x^3 + 2x^2 - 6x$

$\Rightarrow \quad f'(x) = 3x^2 + 4x - 6$.

(b) $f(x) = \cos(2x) \Rightarrow f'(x) = -\sin(2x) \cdot 2$

$= -2\sin(2x)$.

(c) $g(t) = \tan^2(2t) \Rightarrow g'(t) = 2\tan(2t) \cdot \sec^2(2t) \cdot 2$

$= 4\tan(2t) \cdot \sec^2(2t)$

(d) $f(x) = \sqrt{x^2 + 2x} \Rightarrow f'(x) = \dfrac{1}{2\sqrt{x^2+2x}}(2x+2)$

$$= \frac{x+1}{\sqrt{x^2+2x}}$$

(e) $g(n) = \dfrac{n}{n^2+1} \Rightarrow g'(n) = \dfrac{(n^2+1)\cdot 1 - n(2n)}{(n^2+1)^2}$ Quotient Rule

$$= \frac{1-n^2}{(n^2+1)^2}.$$

(f) $h(x) = \sec^2(x) \Rightarrow h'(x) = 2\sec(x)\cdot \sec(x)\cdot \tan(x)$

$$= 2\sec^2(x)\tan(x).$$

(g) $u(t) = \dfrac{\ln t}{t^2} \Rightarrow u'(t) = \dfrac{t^2\cdot\frac{1}{t} - \ln t\cdot 2t}{t^4}$

$$= \frac{t - 2t\ln t}{t^4} = \frac{1-2\ln t}{t^3}$$

(h) $v(t) = \sin t + \cos^2(2t)$

$v'(t) = \cos t + 2\cos(2t)\cdot(-\sin(2t))\cdot 2$

$= \cos t - 4\sin(2t)\cos(2t)$

$= \cos t - 2\sin(4t)$ $\left(\sin(2x) = 2\sin x\cos x\right)$

(i) $f(x) = \dfrac{1}{(1-x)^2} = (1-x)^{-2}$

$f'(x) = -2(1-x)^{-3}\cdot(-1) = \dfrac{2}{(1-x)^3}$

Questions that use Quotient Rule: can alternatively use Chain Rule, can be easier sometimes.

Q3 $f(x)=\frac{1}{x} \Rightarrow f'(x)=-\frac{1}{x^2}<0$ on $(0,\infty)$

$\therefore f$ is decreasing on $(0,\infty)$.

Q4 $f(x)=\frac{\ln x}{x} \Rightarrow f'(x)=\frac{x\cdot\frac{1}{x}-\ln x\cdot 1}{x^2}$

$$=\frac{1-\ln x}{x^2}$$

for $f'(x)<0$, we must have $1-\ln x<0 \Leftrightarrow \ln x>1$

$\Leftrightarrow x>e$

$\therefore f$ is decreasing on (e,∞).

$$g(x)=-\frac{\ln x}{x} \Rightarrow g'(x)=-\underbrace{\frac{(1-\ln x)}{x^2}}_{<0 \text{ on } (e,\infty)}>0$$

$\therefore g$ is increasing on (e,∞)

Definition of derivative of f at a:

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

Q5 (a) Let $f(x)=x^7 \Rightarrow f'(x)=7x^6$

$$\lim_{x\to 1}\frac{x^7-1}{x-1}=\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}=f'(1)=7\cdot 1^6=7$$

(b) Let $f(x)=e^{4x}\sin(3x) \Rightarrow f(\frac{\pi}{2})=e^{2\pi}\sin(\frac{3\pi}{2})=-e^{2\pi}$

$\Rightarrow f'(x)=4e^{4x}\sin(3x)+3e^{4x}\cos(3x)$.

$$\lim_{x\to\frac{\pi}{2}} \frac{e^{4x}\sin(3x)+e^{2\pi}}{x-\frac{\pi}{2}} = \lim_{x\to\frac{\pi}{2}} \frac{f(x)-f(\frac{\pi}{2})}{x-\frac{\pi}{2}}$$

$$= f'(\tfrac{\pi}{2}) = -4e^{2\pi} + 3e^{2\pi}\cdot 0$$

$$= -4e^{2\pi}$$

(c) By limit laws and part (b)

$$\lim_{x\to\frac{\pi}{2}} \frac{e^{4x}\sin(3x)+e^{2\pi}}{\sqrt{x}-\sqrt{\frac{\pi}{2}}} \cdot \frac{(\sqrt{x}+\sqrt{\frac{\pi}{2}})}{(\sqrt{x}+\sqrt{\frac{\pi}{2}})}$$

Conjugation method
a^2-b^2
$=(a+b)(a-b)$

$$= \lim_{x\to\frac{\pi}{2}} \frac{e^{4x}\sin(3x)+e^{2\pi}}{x-\frac{\pi}{2}} \cdot \lim_{x\to\frac{\pi}{2}} \sqrt{x}+\sqrt{\frac{\pi}{2}}$$

$$= -4e^{2\pi}\cdot 2\cdot\sqrt{\frac{\pi}{2}} = -4\sqrt{2\pi}\, e^{2\pi}.$$

<u>Q6</u> (a) Differentiate both sides wrt x for $\sin y = x$

$$\Rightarrow \underbrace{\cos y\cdot\frac{dy}{dx}}_{} = 1$$

Chain Rule as y is a function of x

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}}$$

(positive square root as $-\frac{\pi}{2} < y < \frac{\pi}{2}$ so $\cos y > 0$)

$$= \frac{1}{\sqrt{1-x^2}}.$$

(b) Same for $\tan(y) = x$

$$\Rightarrow \sec^2 y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$