Lecture 3: Logics

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Propositions

- A **proposition** (more precisely **simple proposition**), or a **statement**, is a *declarative sentence* that is **either true or false**, but not both.
- Small letters $p, q, r, \ldots$ are used to denote propositions. These letters are called **propositional variables**.
- 1 (or T) $=$ **truth value** of a *true statement*,
  0 (or F) $=$ truth value of a *false statement*.

Example 1

Which of the following are propositions? Give their truth values.

(a) $2 + 4 = 7$.

Proposition. False.

(b) Julius Caesar was a president of the United States.

Proposition. False.

(c) What time is it?

Not a proposition.

Example 1

(d) Be quiet.

Not a proposition

(e) $2+2=4$.

Proposition. True.

(f) $x+y=z$.

Not a proposition.

(g) $x+2=5$.

Not a proposition.

(h) $x+2=5$ when $x=3$.

Proposition.

Negation

$!p$

- The **negation** of proposition p, denoted by $\neg p$ (read "not p"), is the proposition "it is not the case that p".
- $\neg p$ is true when p is false, and $\neg p$ is false when p is true.

$$\neg p = \begin{cases} 0 \text{ if } p = 1, \\ 1 \text{ if } p = 0. \end{cases}$$

Example 2

Proposition. True!

Let p be the proposition "$\sqrt{2}$ is an irrational number". Which of the following is $\neg p$?

(a) $\sqrt{3}$ is irrational. ✗

(b) Every number is irrational except $\sqrt{2}$. ✗

(c) $\sqrt{2}$ is not an irrational number. ✓

(d) $\sqrt{2}$ is a rational number. ✗

Example 3

Let p be the proposition "there are six stones on the book cover". Which of the following is $\neg p$?

(a) There are seven birds on the book cover.

(b) There are no stones on the book cover.

(c) There are six stones on the table.

Ans. It's not here.

There are not 6 stones on the book cover.

Conjunction, disjunction

- The **conjunction** of p and q, denoted by $p \wedge q$, is the proposition "**p and q**".

$$p \wedge q \text{ is } \begin{cases} \text{true when both p and q are true,} \\ \text{false otherwise.} \end{cases}$$

Conjunction, disjunction

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$$p \wedge q \text{ is } \begin{cases} \text{true when both p and q are true,} \\ \text{false otherwise.} \end{cases}$$

- The **disjunction** of p and q, denoted $p \vee q$, is "p or q".

$$p \vee q \text{ is } \begin{cases} \text{true if } \cancel{\text{either}} \text{ p or q is true,} \\ \text{false otherwise.} \end{cases}$$

Exclusive-or

- The **exclusive-or** (read "ex-or") of p and q, denoted by $p \oplus q$, is the proposition "*exactly one of p or q*".

$$p \oplus q \text{ is } \begin{cases} \text{true if exactly one of p or q is true,} \\ \text{false otherwise.} \end{cases}$$

Exclusive-or

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$$p \oplus q \text{ is } \begin{cases} \text{true if exactly one of p or q is true,} \\ \text{false otherwise.} \end{cases}$$

- Another way to remember $p \oplus q$

%

$$p \oplus q = (p+q) \mod 2$$

In particular,

$$\begin{aligned} 0 \oplus 0 &= 1 \oplus 1 = 0, \\ 0 \oplus 1 &= 1 \oplus 0 = 1 \end{aligned}$$

Operations and names

Notation	Name/Read	Example
$\neg$	negation/not	$\neg p$
$\wedge$	conjunction/and	$p \wedge q$
$\vee$	disjunction/or	$p \vee q$
$\oplus$	exclusive or/ex-or	$p \oplus q$

Compound propositions

- The statements which are formed by *connectives* $\neg, \vee, \wedge, \oplus$ and propositional variables $p, q, \ldots$ are called **compound propositions**:

$$\neg p, \quad p \vee q, \quad p \wedge q, \quad p \oplus q$$

Compound propositions

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$$\neg p,\ p \vee q,\ p \wedge q,\ p \oplus q$$

- The propositions which make up a compound proposition are called **propositional variables**. For examples,
    - p is a propositional variable of $\neg p$.
    - p and q are propositional variables of $p \vee q, p \wedge q, p \oplus q$.

Example 4

Construct $p \wedge q$, $p \vee q$ and $p \oplus q$ and find their truth values.

(a) $p : 5 > 9$, $q : 9 > 7$

$p = 0$, $q = 1$

$p \wedge q$: $5 > 9$ and $9 > 7$ $\Rightarrow p \wedge q = 0$

$p \vee q$: $5 > 9$ or $9 > 7$ $\Rightarrow p \vee q = 1$

$p \oplus q$: Either $5 > 9$ or $9 > 7$ $\Rightarrow p \oplus q = 1$

(b) p : Today is Wednesday, q : It is raining

$p = 0$, $q = 0$

$p \wedge q = 0$

$p \vee q = 0$

$p \oplus q = 0$

Order of operations

- If a compound proposition consists of $\neg$, $\vee$ and $\wedge$, the operation $\neg$ is always performed first. For example,

$$\neg p \vee q = (\neg p) \vee q$$

- When $\oplus$ is used, we use brackets to clearly indicate its order of precedence. For example,

$$(p \oplus q) \vee r, \;\; p \wedge (q \oplus r)$$

- When $\vee$ and $\wedge$ are used, we use brackets to indicate the order. For example

$$(p \wedge q) \vee r, \;\; p \wedge (q \vee r)$$

Example 5

Find the truth value of $\neg(p \wedge q) \vee r$

p : Today is Wednesday.

q : $2 + 1 = 3$.

r : There is no Covid-19 infection in Singapore.

$p = 0, \; q = 1, \; r = 0$

$q \wedge q = 0 \Rightarrow \neg(p \wedge q) = 1$

$\neg(p \wedge q) \vee r = 1$

Truth table

- The **truth table** of a compound proposition is the table of all possible truth values for that proposition.
- We can make a single table for many propositions that share the same propositional variables: $\neg p, p \wedge q, p \vee q, p \oplus q$

columns of propositional variables

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$
0	0	1	0	0	0
0	1	1	0	1	1
1	0	0	0	1	1
1	1	0	1	1	0

  - The first 2 columns are the propositional variables p and q.
  - Starting from the 2nd row, the table contains all possible combinations of p and q (4 in total).

Exercise 1

Construct the truth table (one single table) for

$$(p \oplus q) \oplus r, (p \oplus q) \wedge r.$$

3 variables p, q, r
\# possibilities = 2^3

p	q	r	$p \oplus q$	$(p \oplus q) \oplus r$	$(p \oplus q) \wedge r$
0	0	0	0	0	0
0	0	1	0	1	0
0	1	0	1	1	0
0	1	1	1	0	1
1	0	0	1	1	0
1	0	1	1	0	1
1	1	0	0	0	0
1	1	1	0	1	0

Tautology, contradiction, contingency

- A **tautology** is a compound proposition that is always true.
- A **contradiction** is a compound proposition that is always false.
- A **contingency** is a compound proposition that is neither a tautology nor a contradiction.
- Notations

$$\mathbf{T} = \text{tautology}, \ \mathbf{F} = \text{contradiction}.$$

Example 6

- Tautology

  "Today is Sunday or today is not Sunday"

- Contradiction

  "Today is Sunday and today is not Sunday"

- Contingency

  "I am going to get an A for CSD2258"

Exercise 2

Let p, q be propositions. Using truth table, show that

(a) Using truth table, show that $p \vee \neg p$ is a tautology and $p \wedge \neg p$ is a contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
0	1	1	0
1	0	1	0

$\therefore$ $p \vee \neg p$ is a tautology : $p \vee \neg p = T$

$p \wedge \neg p$ is a contradiction : $p \wedge \neg p = F$

Exercise 2

(b) Construct the truth table for $(p \wedge q) \vee (\neg p \vee \neg q)$. Determine if this proposition is a tautology, a contradiction or a contingency.

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \wedge q$	$(p \wedge q) \vee (\neg p \vee \neg q)$
0	0	1	1	1	0	1
0	1	1	0	1	0	1
1	0	0	1	1	0	1
1	1	0	0	0	1	1

$\therefore (p \wedge q) \vee (\neg p \vee \neg q) = T.$

Logical equivalence

- Two compound propositions are **equivalent** if they have exactly the same truth values *under all possibilities*.
- If p and q are equivalent, we write

$$p \equiv q$$

- If p and q are not equivalent, we write

$$p \not\equiv q$$

Logical equivalence rules

Theorem 1

Let p, q, r be simple propositions. **T**=tautology, **F**=contradiction. The following logical equivalences hold.

(a) Identity laws

$$p \wedge \mathbf{T} \equiv p$$

$$p \vee \mathbf{F} \equiv p$$

(b) Negation laws

$$p \vee \neg p \equiv \mathbf{T}$$

$$p \wedge \neg p \equiv \mathbf{F}$$

Theorem 1 continued

(c) Domination laws

$$
\begin{aligned}
p \vee \mathbf{T} &\equiv \mathbf{T} \\
p \wedge \mathbf{F} &\equiv \mathbf{F}
\end{aligned}
$$

(d) Commutative laws

$$
\begin{aligned}
p \wedge q &\equiv q \wedge p \\
p \vee q &\equiv q \vee p
\end{aligned}
$$

(e) Associative laws

$$
\begin{aligned}
(p \wedge q) \wedge r &\equiv p \wedge (q \wedge r) \\
(p \vee q) \vee r &\equiv p \vee (q \vee r).
\end{aligned}
$$

Theorem 1 continued

$$a \times (b + c) = a \times b + a \times c$$

(f) Distributive laws

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

(g) Double negation

$$\neg(\neg p) \equiv p$$

De Morgan's laws

Remind: Sets: $\overline{A \cup B} = \bar{A} \cap \bar{B}$

$\overline{A \cap B} = \bar{A} \cup \bar{B}$

Theorem 2

Let p, q be propositions. Then

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$
$$\neg(p \wedge q) \equiv \neg p \vee \neg q.$$

The negation operator $\neg$ reverses $\wedge$ and $\vee$

Proof: $\neg(p \vee q) \equiv \neg p \wedge \neg q$

We prove by truth table.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

$\therefore \neg(p \vee q) \equiv \neg p \wedge \neg q$

Example 7

Use De Morgan's law to write negations for the following.

(a) I am dreaming or today is Sunday.

p q

$p \vee q$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

I am not dreaming and today is not Sunday.

(b) $-3 \leq x \leq 5$. → same as $x \geq -3$ and $x \leq 5$

p q

$p \wedge q$

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg p : x < -3$ $\neg q : x > 5$

$\therefore x < -3$ or $x > 5$.

Exercise 3

Is $\neg(p \vee q) \equiv \neg p \vee \neg q$? Justify your answer.

De Morgan's law:

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Ans. No

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
0	0	0	1	1	1	1
0	1	1	0	1	0	1
1	0	1	0	0	1	1
1	1	1	0	0	0	0

$\therefore \neg(p \vee q) \not\equiv \neg p \vee \neg q$

Exercise 4

Prove that

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

$$\neg(\neg p \wedge q) \wedge (p \vee q) \equiv p.$$

Solution 1: Truth table.

Solution 2:

$\neg(\neg p \wedge q) \wedge (p \vee q) \equiv (\neg(\neg p) \vee \neg q) \wedge (p \vee q)$ (De Morgan)

$\equiv (p \vee \neg q) \wedge (p \vee q)$ (double negation)

$\equiv p \vee (\neg q \wedge q)$ (distributivity)

$\equiv p \vee F$ (negation law)

$\equiv p$ (identity)

Exercise 5

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

Prove that the following is a tautology

$$(p \wedge q) \vee (\neg p \vee \neg q)$$

$(p \wedge q) \vee (\neg p \vee \neg q) \equiv (p \wedge q) \vee \neg(p \wedge q)$ (De Morgan)

$\equiv T$ (negation)

Conditional statement

The **conditional statement** $p \to q$ is the proposition

"if p, then q" or "p implies q" or "p only if q"

- p is called **hypothesis** (or **premise**), q is called **conclusion**.

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- $p \to q$ is false only when "p is true and q is false".

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Conditional statement

The **conditional statement** $p \to q$ is the proposition

"if p, then q" or "p implies q" or "p only if q"

- p is called **hypothesis** (or **premise**), q is called **conclusion**.
- $p \to q$ is false only when "p is true and q is false".
  - True hypothesis $\to$ true conclusion.
  - False hypothesis $\to$ conclusion can be anything: **true by default** (or **vacuously true**). For example, "If I am superman, you are spiderman".

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Example 8

(a) Using truth table, prove that $p \to q \equiv \neg p \vee q$.

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

(b) Using (a), rewrite the following sentence in if-then form.

"~~Either~~ you get to work on time or you are fire".

$\neg p$ $\quad$ q

p: You do not get to work on time, q: You are fire

$p \to q$: If you do not get to work on time, then you are fire.

Exercise 6: Negation of a conditional statement

(a) Prove that $\neg(p \to q) \equiv p \wedge \neg q$.

$$\neg(p \to q) \equiv \neg(\neg p \vee q) \overset{D'M}{\equiv} p \wedge \neg q$$

(b) Using (a), write negations for the following statements

Caution. Negation of an if-then statement *does not start with if*.

"If $\underbrace{\text{you attend all lectures}}_{p}$, then $\underbrace{\text{you will get an A}}_{q}$". $\neg(p \to q) \equiv p \wedge \neg q$

You attend all lectures and you will not get an A.

"If $\underbrace{\text{you are caught littering}}_{p}$, then $\underbrace{\text{you will be fined}}_{q}$".

You are caught littering and you will not be fined.

Converse, inverse and contrapositive

- The **converse** of $p \to q$ is $q \to p$.
- The **inverse** of $p \to q$ is $\neg p \to \neg q$.
- The **contrapositive** of $p \to q$ is $\neg q \to \neg p$.

$$\begin{tikzcd}
p \to q \arrow[r, "\text{converse}"] \arrow[rr, bend right, "\text{contrapositive}"'] & q \to p \arrow[r, "\text{inverse}"] & \neg q \to \neg p
\end{tikzcd}$$

Example 9

Write the converse, inverse and contrapositive of

"If today is Friday, then I have a test today". $p \to q$

p q

Converse : $q \to p$

If I have a test today, then today is Friday.

Inverse : $\neg p \to \neg q$

If today is not Friday, then I don't have a test today.

Contrapositive : $\neg q \to \neg p$

If I don't have a test today, then today is not Friday.

Exercise 7

Using a truth table, prove the following

(a) $p \to q \not\equiv q \to p$.

(b) $p \to q \not\equiv \neg p \to \neg q$.

(c) $p \to q \equiv \neg q \to \neg p$.

$p \to q$

$q \to p$

$\neg p \to \neg q$

$\neg q \to \neg p$

p	q	$\neg p$	$\neg q$	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
0	0	1	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	1	0	0	1	1	1	1

Conditional statement vs converse, inverse, contrapositive

- Neither the converse nor the inverse of a conditional statement is equivalent to the statement itself.

$$p \to q \quad \not\equiv \quad q \to p$$

$$p \to q \quad \not\equiv \quad \neg p \to \neg q$$

Conditional statement vs converse, inverse, contrapositive

- Neither the converse nor the inverse of a conditional statement is equivalent to the statement itself.

$$p \to q \quad \not\equiv \quad q \to p$$
$$p \to q \quad \not\equiv \quad \neg p \to \neg q$$

- The contrapositive of a conditional statement is equivalent to the statement itself.

$$p \to q \equiv \neg q \to \neg p.$$

Biconditional statement

$$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$$

- The **biconditional** statement $p \leftrightarrow q$ is the proposition

  "p if and only if q"

- It is true if both p and q have the same truth value, and it is false otherwise.

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

Example 10

Using truth table to show that $p \leftrightarrow q$ is logically equivalent to the conjunction of $p \to q$ and $q \to p$, that is,

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p).$$

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$	$p \leftrightarrow q$
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	1	1	1	1

$\therefore \quad (p \to q) \wedge (q \to p) \equiv p \leftrightarrow q$