Limits and Continuity

Dr. Ronald Koh
ronald.koh@digipen.edu (Teams preferred over email)

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Pre-lecture exercise

Let's take a look at the function

$$f(x) = \frac{x-1}{x^2-1}.$$

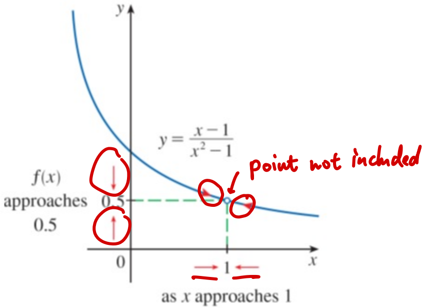
1. Is f defined at $x = 1$? **No.** Cannot divide by 0.
2. What is the domain of f? $\mathbb{R} \setminus \{-1, 1\}$
3. What is the behaviour of f *near* $x = 1$?

x close to 1 but $x \neq 1$

$$\frac{x-1}{x^2-1} = \frac{\cancel{x-1}}{(x+1)\cancel{(x-1)}} = \frac{1}{x+1}$$

Graphical understanding of the limit

Let's have a look at the graph of f from the previous exercise.



f is not defined at $x = 1$, but it can be easily seen that f tends to 0.5 as x tends to 1. We say that the *limit* of f as x tends to 1 is equal to 0.5, and we write it like this:

$$\lim_{x \to 1} \frac{x-1}{x^2-1} = 0.5.$$

Informal definition of the limit

(*)

Definition

Suppose a function $f(x)$ is defined when x is near the number a. We write

$$\lim_{x \to a} f(x) = L$$

and say that "the limit of f, as x tends to a, equals L, if we can make the values of $f(x)$ as close to L as we like, by restricting x to be sufficiently close to a.

If L is a real number ($\pm\infty$ are not real numbers), then we say that the limit of f, as x tends to a, exists. Otherwise, the limit of f, as x tends to a does not exist.

Important: $f(a)$ does not have to be defined for the limit of f as x tends to a to exist. See previous slide for an example.

Exercise 1

Fill in the blanks and circle the correct word/phrase.

1. $$\lim_{x \to 2} x^2 = 4$$

   real no.

   The limit of x^2, as x tends to 2, equals 4.
   This limit exists/does not exist.

2. $$\lim_{x \to 0} \frac{1}{x^4} = \infty$$

   The limit of $\frac{1}{x^4}$, as x tends to 0, equals ∞.
   This limit exists/does not exist.

3. $$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$$

   The limit of $\frac{x^3-1}{x-1}$, as x tends to 1, equals 3.
   This limit exists/does not exist.

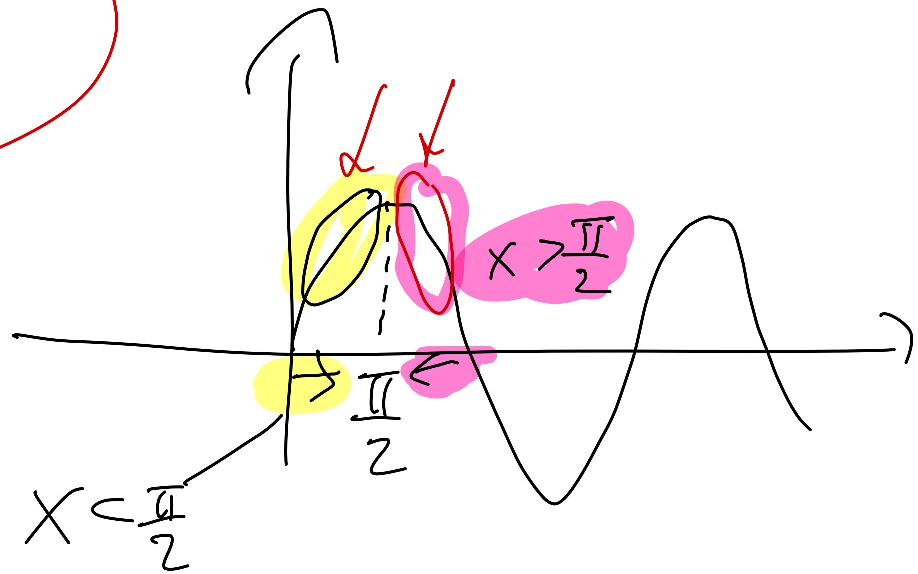
limit of $\frac{x^2-1}{x-1}$, as x tends to 1, is 2

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = 2.$$

limit of $\sin x$, as x tends to $\frac{\pi}{2}$, is 1

$$\lim_{x \to \frac{\pi}{2}} \sin x = 1$$

two-sided limit



Left and right-handed limits

Definition

We write the *left-handed limit* of f as x tends to a from the left as

$$\lim_{x \to a^-} f(x) = L$$

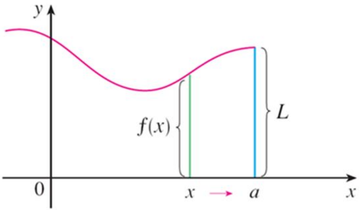
if we can make the values of $f(x)$ as close to L as we like, by restricting x to be sufficiently close to a with x less than a.

The *right-handed limit* of f as x tends to a from the right as

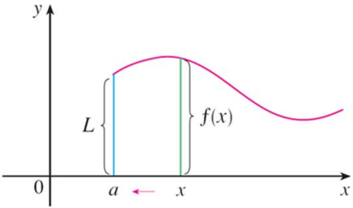
$$\lim_{x \to a^+} f(x) = L$$

if we can make the values of $f(x)$ as close to L as we like, by restricting x to be sufficiently close to a with x greater than a.

Visual understanding of left and right-handed limits



(a) $\lim_{x \to a^-} f(x) = L$



(b) $\lim_{x \to a^+} f(x) = L$

Important theorem

An important characterization of the existence of limits:

Theorem

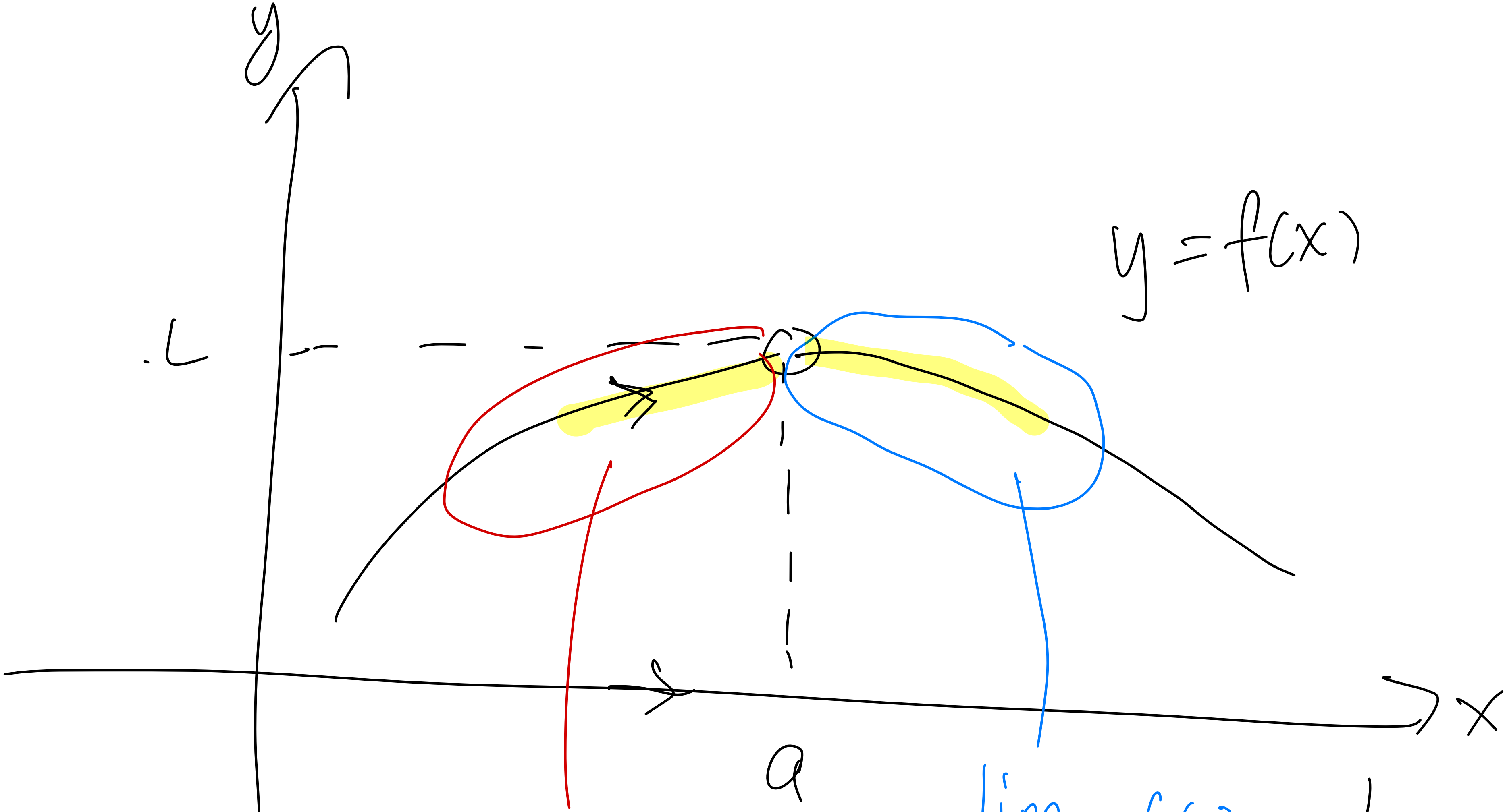
The limit of f as x tends to a exists, and is equal to L, i.e.

$$\lim_{x \to a} f(x) = L$$

if and only if the left and right-handed limits of f as x tends to a and are both equal to L:

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

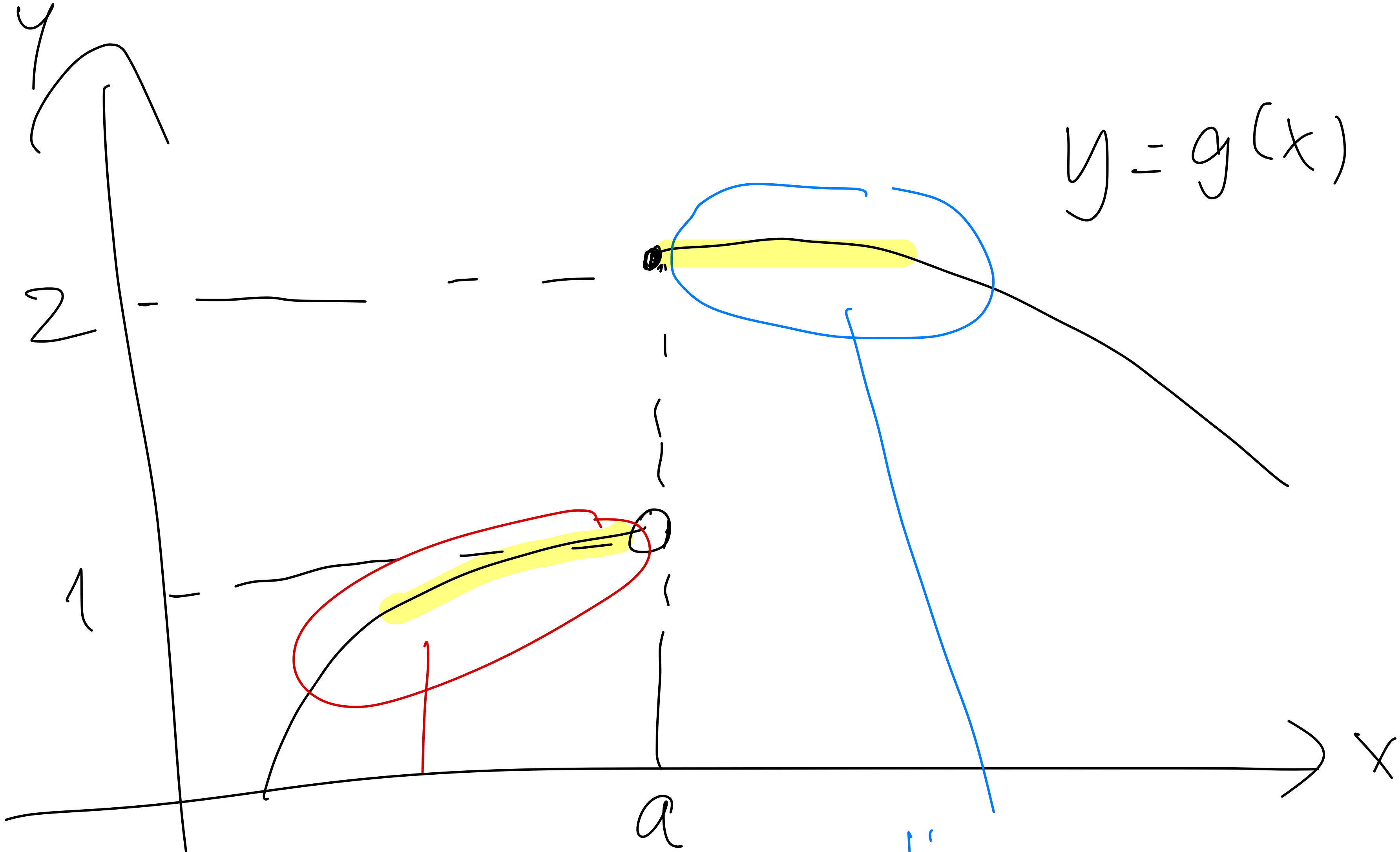
TLDR: This says that checking for the existence of a two-sided limit is the same as checking for the existence and equality of both left and right-handed limits.



$\lim_{x \to a^-} f(x) = L$

$\lim_{x \to a^+} f(x) = L$

What about $\boxed{\lim_{x \to a} f(x)} = L$



$\lim_{x \to a^-} f(x) = 1$

$\lim_{x \to a^+} f(x) =$

$\lim_{x \to a} f(x)$ does not exist

Example

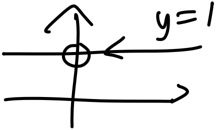
Let $f(x) = \frac{|x|}{x}$. (what is the domain of f?)
Let's examine the left and right-handed limits of f as x tends to 0.
Recall that $|x| = x$ if $x > 0$ and $|x| = -x$ if $x < 0$. Then

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{\cancel{x}}{\cancel{x}} = \lim_{x\to 0^+} 1 = 1.$$

$x>0$

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} -1 = -1.$$

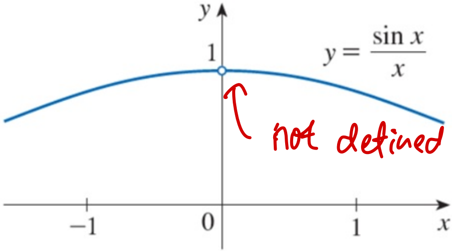
$x<0$



By the theorem above, it is evident now that the limit of f as x tends to 0 does not exist, because the left and right-hand limits are different.

Finding limits using graphs

In some cases, we can find the limit of $f(x)$ as x tends to a can be found by observing the surroundings of $f(x)$ for points x that are near a. An example:



not defined at $x=0$
but

By observing the graph above, determine

$$\lim_{x \to 0} \frac{\sin x}{x}. = 1$$

Limitations to finding limits using graphs

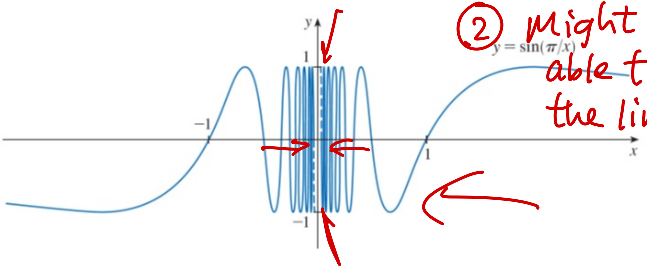
Consider the function $f(x) = \sin\left(\frac{\pi}{x}\right)$. We want to find

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right).$$

① Need to know how to draw graph

② Might not be able to tell the limit

The graph of f is as follows:



Can you find this limit just by observing the graph? Does this limit even exist?

Limitations to finding limits using graphs

In the earlier example, it is difficult to determine limits just by using graphs solely. Therefore, unless otherwise stated, we will be moving away from graphical methods of finding limits, and turning our focus to analytical methods to find limits.

P.S. (you don't need to know this for exams) It turns out that

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$$

does not exist, as there are infinitely many values of x near 0 such that $f(x)$ are 1 and -1.

Limit Laws

Theorem

($\star$) Suppose c is a constant and the limits

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

exist. Then

$$(1)\ \lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$$

$$(2)\ \lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

$$(3)\ \lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

$$(4)\ \lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \text{ if } \lim_{x \to a} g(x) \neq 0$$

Exercise 2

Given that

$$\lim_{x\to\frac{\pi}{2}} \sin x = 1, \quad \lim_{x\to\frac{\pi}{2}} \cos x = 0, \quad \text{and} \quad \lim_{x\to\frac{\pi}{2}} 2x = \pi,$$

use the Limit Laws to find

1. $\lim_{x\to\frac{\pi}{2}} 3\sin x$ $= 3\lim_{x\to\frac{\pi}{2}} \sin x$ (Law (1)) $= 3\cdot 1 = 3$
2. $\lim_{x\to\frac{\pi}{2}} [2x\cos x - 3\sin x]$ $= \lim_{x\to\frac{\pi}{2}} 2x\cos x - \underbrace{\lim_{x\to\frac{\pi}{2}} 3\sin x}_{\text{pt (1)}}$ (2)
   $= \lim_{x\to\frac{\pi}{2}} 2x\cos x - 3$
   $= \left(\lim_{x\to\frac{\pi}{2}} 2x\right)\cdot\left(\lim_{x\to\frac{\pi}{2}} \cos x\right) - 3$
   $= \pi \cdot 0 - 3 = -3.$
3. $\lim_{x\to\frac{\pi}{2}} \dfrac{2\sin x}{2x}$

$$\lim_{x \to \frac{\pi}{2}} \frac{2\sin x}{2x}$$

check $\lim_{x \to \frac{\pi}{2}} 2x = \pi \neq 0.$

$$= \frac{\lim_{x \to \frac{\pi}{2}} 2\sin x}{\lim_{x \to \frac{\pi}{2}} 2x} \quad (4)$$

$$= \frac{2 \lim_{x \to \frac{\pi}{2}} \sin x}{\pi} \quad (1) \quad = \frac{2 \cdot 1}{\pi} = \boxed{\frac{2}{\pi}}$$

Other limit laws

Here are some other limit laws:

Theorem

Suppose c is a constant and the limit

$$\lim_{x \to a} f(x)$$

exists. Then

(5) $\lim_{x \to a} c = c$

(6) $\lim_{x \to a} x = a$

(7) $\lim_{x \to a} x^n = a^n$ where n is a positive integer

(8) $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer

Exercise 3

Use the limit laws (1) - (8) to find the following limits.

1. $\lim_{x\to 2} x^2 + 3x + 5$ $= \lim_{x\to 2} x^2 + 3\underbrace{\lim_{x\to 2} x} + \lim_{x\to 2} \underline{5}$ (1),(2)
   $= 2^2 + 3 \cdot 2 + 5 = 15$
   Law(7) Law(6) Law(5)
2. $\lim_{x\to 7}(x-5)^3$
3. $\lim_{x\to -2} \dfrac{x^3+2x-1}{5-3x}$

Can we use limit laws to find

$$\lim_{x\to 1} \frac{x-1}{x^2-1}?$$

(2) $\lim_{x\to 7}(x-5)^3 = \left(\lim_{x\to 7}(x-5)\right)^3 = \left(\lim_{x\to 7} x - \lim_{x\to 7} 5\right)^3$ Law (2)

Law (8)

$= (7-5)^3$ Law(6)

$= 2^3 = 8$

Exercise 3

③ $$\lim_{x\to -2} \frac{x^3+2x+1}{5-3x}$$

Check $\lim_{x\to -2} 5-3x = \lim_{x\to -2} 5 - 3\lim_{x\to -2} x$ (Limit Laws (1),(2))

$= 5 - 3\cdot(-2)$ (Limit Laws (5),(6))

$= 11 \neq 0$ ✓ $\therefore$ Limit Law (4) ok to use

$\lim_{x\to -2} x^3+2x+1 = \lim_{x\to -2} x^3 + 2\lim_{x\to -2} x + \lim_{x\to -2} 1$ Laws (1),(2)

$= (-2)^3 + 2\cdot(-2) + 1$ Laws (7),(6),(5)

$= -8 - 4 + 1 = -11$

Exercise 3

$$\therefore \lim_{x\to -2} \frac{x^3+2x+1}{5-2x} = \frac{\lim_{x\to -2} x^3+2x+1}{\lim_{x\to -2} 5-2x} \qquad \text{Law (4)}$$

$$= \frac{-11}{11} = -1.$$

Continuous functions

continuous: forming an unbroken whole, without interruption.

The layman understanding of continuous functions are that they have graphs that do not "break"; you can draw a line through the graph without ever lifting your pen.

Definition

($\star$) A function f is *continuous at a point a* if these three things are satisfied.

1. $f(a)$ is defined, i.e. a is in the domain of f, ← behavior of f at $x=a$
2. $\lim_{x\to a} f(x)$ exists, ← behavior of f near a
3. Both of the two things above agree, i.e.

$$\lim_{x\to a} f(x) = f(a).$$

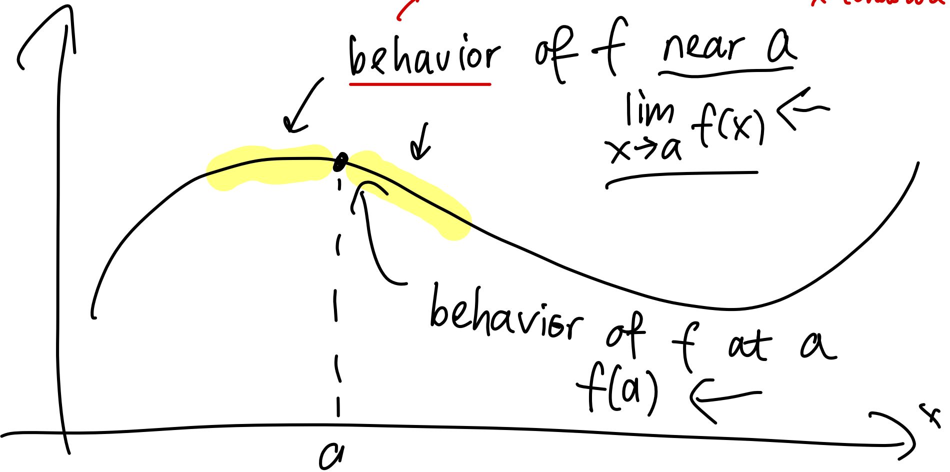
Understanding continuity at a point a

In reference to the definition in the previous slide, let us assume that we are using a pen to draw the curve of a continuous function.

1. $f(a)$ is defined means that the pen will touch the point $(a, f(a))$ in the graph.
2. $\lim_{x \to a} f(x)$ exists means that I can also draw some *surroundings of the point* $(a, f(a))$.
3. $\lim_{x \to a} f(x) = f(a)$ means that the surroundings of the point $(a, f(a))$ are "close" to the point $(a, f(a))$, thus the tracing through the portion of the curve near $(a, f(a))$, and also through the point $(a, f(a))$ will not require us to lift the pen up, i.e. no breaks in the curve.

Demonstration

what number does $f(x)$ approach as x tends to a?



Continuous on its domain

Definition

A function f is *continuous on its domain* if it is continuous at every point in its domain.

$(\star)$ For this course, you may assume that the following functions are continuous on its domain:

1. Square root function $f(x) = \sqrt{x}$ is continuous on $x \geq 0$.
2. Algebraic functions
3. Trigonometric functions
4. Exponential functions
5. Natural log function
6. Composite functions of any of the above

Exercise 4

Using continuity, evaluate the following limits.

1. $\lim_{x\to 3} \frac{x^3 - 3x + 9}{x + 6}$
2. $\lim_{x\to \frac{\pi}{4}} \sin^2 x$
3. $\lim_{x\to 4} \ln\left(x^2 + 9\right)$

$$= \frac{3^3 - 3\cdot 3 + 9}{9} = \frac{27}{9} = 3$$

function is cts $x=3$

$$\sin^2\left(\frac{\pi}{4}\right) = \left(\sin\left(\frac{\pi}{4}\right)\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2$$

$$\lim_{x\to 4} \ln(x^2+9) = \ln(25) = \ln 5^2$$

$$= 2\ln 5.$$

Introduction of some limit evaluation techniques

As seen previously in the limit

$$\lim_{x \to 1} \frac{x-1}{x^2-1},$$

we are unable to evaluate it using the limit laws since

$$\lim_{x \to 1} x^2 - 1 = 0,$$

so techniques must be employed to reduce it to a form where limit laws may be used. The two techniques we will cover are:

1. Factorization
2. Rationalization using conjugate pairs

Factorization method

The factorization method applies to limits whose functions f are consists of *rational functions*, functions which consist of two polynomials in both the numerator and denominator in a fraction, i.e.

$$\lim_{x \to a} \frac{p(x)}{q(x)}$$

$p(a) = 0$

$\Leftrightarrow (x-a)$ factor of p

where p and q are polynomials. If both

$$\lim_{x \to a} p(x) = \lim_{x \to a} q(x) = 0,$$

which also means $p(a) = q(a) = 0$, since p and q are both continuous. Then it must mean that $\star$ $(x - a)$ is a factor of both $p(x)$ and $q(x)$.

By cancelling out $(x - a)$, we can cancel this term off to reduce this fraction to (hopefully) a limit we can evaluate by limit laws.

$$p(x) = x^2 + 2x + 1$$

$x = -1$: $p(-1) = 0$.

↳ can factor $(x+1)$ from $x^2 + 2x + 1$

$$= (x+1)(x+1)$$

Factorization example

Let's demonstrate the factorization method using the limit

$$\lim_{x \to 1} \frac{x-1}{x^2-1}.$$

$\star\ a^2 - b^2$
$= (a-b)(a+b)$

Note that $x^2 - 1 = (x-1)(x+1)$, thus

$$\begin{aligned}
\lim_{x \to 1} \frac{x-1}{x^2-1} &= \lim_{x \to 1} \frac{x-1}{(x-1)(x+1)} \quad \text{(factoring out } (x-1)\text{)} \\
&= \lim_{x \to 1} \frac{1}{x+1} \quad \text{(after cancelling out the } (x-1) \text{ factor)} \\
&= \frac{\lim_{x \to 1} 1}{\lim_{x \to 1} (x+1)} \quad \text{(limit laws)} = \frac{1}{2}.
\end{aligned}$$

Exercise 5

Use factorization to find the following limits

1. $\lim_{x\to 2} \frac{x^2-4}{x-2}$ $= \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = \lim_{x\to 2}(x+2) = 4.$
2. $\lim_{x\to 3} \frac{x^2-4x+3}{x-3}$
3. $\lim_{x\to 1} \frac{x^3-1}{x-1}$

$$x^2 - 4x + 3$$

$$\begin{array}{ccc|c} x & & -1 & -x \\ x & & -3 & -3x \\ \hline x^2 & & +3 & -4x \end{array}$$

$$\lim_{x\to 3} \frac{\cancel{(x-3)}(x-1)}{\cancel{x-3}}$$

$$= \lim_{x\to 3}(x-1) = 2$$

$$\lim_{x\to 1} \frac{x^3-1}{x-1}$$

$$\frac{x^3-1}{x-1} = ax^2+bx+c$$

$(x-1)$ factor of x^3-1

$$(x^3-1) = (x-1)(ax^2+bx+c)$$

$$= ax^3+bx^2+cx-ax^2-bx-c$$

$$= ax^3+(b-a)x^2+(c-b)x-c$$

$a=1$, $c=1$, $b=1$

$$\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{x-1}} = \lim_{x\to 1} x^2+x+1$$

$$=3$$

Rationalization method

$a^2-b^2=(a-b)(a+b)$

Definition

For any fixed real numbers a and b, the factors $(a+b)$ and $(a-b)$ are called *conjugate pairs*.

The usual application of this method are on fractions whose numerator and denominators with a sum or difference of square roots, e.g.

$$\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2}.$$

We cannot apply the limit laws directly because

$$\lim_{x\to 2} x-2=0.$$

Also, there does not look like there is anything to factorize here.

Rationalization example

The first and second steps of this example constitutes multiplying both the numerator and denominator by the conjugate of $\sqrt{x}-\sqrt{2}$, which is $\sqrt{x}+\sqrt{2}$. This process is known as *rationalization*.

$a - b$ $\quad a+b\,(\sqrt{x}+\sqrt{2})$

$$
\begin{aligned}
\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2} &= \lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2}\cdot 1 \\
&= \lim_{x\to 2}\frac{(\sqrt{x}-\sqrt{2})}{(x-2)}\cdot\frac{(\sqrt{x}+\sqrt{2})}{(\sqrt{x}+\sqrt{2})} \\
&= \lim_{x\to 2}\frac{\cancel{x-2}}{\cancel{(x-2)}(\sqrt{x}+\sqrt{2})} \\
&= \lim_{x\to 2}\frac{1}{\sqrt{x}+\sqrt{2}} = \frac{1}{2\sqrt{2}}.
\end{aligned}
$$

$(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})$
$= x-2$

$\sqrt{x}$ continuous

In the third step, rationalization converts this limit to a limit where factorization may be used. In the last step, a simple application of limit laws gives the answer.

Exercise 6

Use rationalization to find the following limits.

1. $\lim_{x\to 4} \dfrac{x-4}{\sqrt{x}-2}$
2. $\lim_{x\to\sqrt{2}} \dfrac{x^2-2}{x-\sqrt{2}}$
3. $\lim_{x\to 7^+} \dfrac{(\sqrt{x+2}-3)}{(x-7)} \cdot \dfrac{(\sqrt{x+2}+3)}{(\sqrt{x+2}+3)}$

$$= \lim_{x\to 7^+} \frac{(x+2)-9}{(x-7)(\sqrt{x+2}+3)} = \lim_{x\to 7^+} \frac{\cancel{x-7}}{\cancel{(x-7)}(\sqrt{x+2}+3)}$$

$$= \frac{1}{\sqrt{7+2}+3} = \frac{1}{6}.$$

Exercise 6

Use rationalization to find the following limits.

1. $\lim_{x \to 4} \dfrac{x-4}{\sqrt{x}-2}$
2. $\lim_{x \to \sqrt{2}} \dfrac{x^2-2}{x-\sqrt{2}}$
3. $\lim_{x \to 7^+} \dfrac{\sqrt{x+2}-3}{x-7}$

→ * has indeterminate form " $\frac{0}{0}$ " ← when sub in x=a

① $\lim_{x \to 4} \dfrac{x-4}{\sqrt{x}-2}$

with square roots, multiply numerator and denominator by conjugate pair $(\sqrt{x}+2)$

$$= \lim_{x \to 4} \frac{(x-4)}{(\sqrt{x}-2)} \boxed{\frac{(\sqrt{x}+2)}{(\sqrt{x}+2)}}$$

$$= \lim_{x \to 4} \frac{\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{(x-4)}} = \lim_{x \to 4} (\sqrt{x}+2) = \sqrt{4}+2 = 2+2 = 4.$$

$(\sqrt{x}-2)(\sqrt{x}+2)$
$= (x-4)$

Exercise 6

Use rationalization to find the following limits.

1. $\lim_{x \to 4} \frac{x-4}{\sqrt{x}-2}$
2. $\lim_{x \to \sqrt{2}} \frac{x^2-2}{\boxed{x-\sqrt{2}}}$
3. $\lim_{x \to 7^+} \frac{\sqrt{x+2}-3}{x-7}$

(2) Denominator has square root

So multiply numerator and denominator

by $\boxed{x+\sqrt{2}}$

conjugate pairs

$$= \lim_{x \to \sqrt{2}} \frac{(x^2-2)}{(x-\sqrt{2})} \cdot \frac{(x+\sqrt{2})}{(x+\sqrt{2})}$$

$$= \lim_{x \to \sqrt{2}} \frac{\cancel{(x^2-2)}(x+\sqrt{2})}{\cancel{(x^2-2)}} = \lim_{x \to \sqrt{2}} x+\sqrt{2} = \sqrt{2}+\sqrt{2} = 2\sqrt{2}.$$