Hypothesis testing for one population Mean σ is known

PROCEDURE 9.1 One-Mean z-Test

Purpose To perform a hypothesis test for a population mean, μ

Assumptions

1. Simple random sample
2. Normal population or large sample
3. σ known

Step 1 The null hypothesis is H_0: $\mu = \mu_0$, and the alternative hypothesis is

$$\underset{\text{(Two tailed)}}{H_a: \mu \neq \mu_0} \quad \text{or} \quad \underset{\text{(Left tailed)}}{H_a: \mu < \mu_0} \quad \text{or} \quad \underset{\text{(Right tailed)}}{H_a: \mu > \mu_0}.$$

Step 2 Decide on the significance level, α.

Step 3 Compute the value of the test statistic

$$z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$$

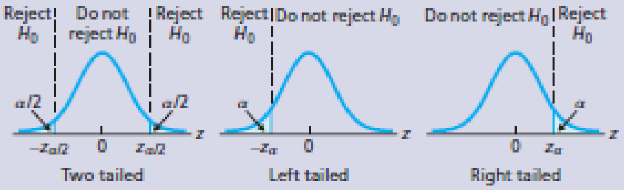
and denote that value z_0.

CRITICAL-VALUE APPROACH

Step 4 The critical value(s) are

$$\underset{\text{(Two tailed)}}{\pm z_{\alpha/2}} \quad \text{or} \quad \underset{\text{(Left tailed)}}{-z_{\alpha}} \quad \text{or} \quad \underset{\text{(Right tailed)}}{z_{\alpha}}.$$

Use Table II to find the critical value(s).

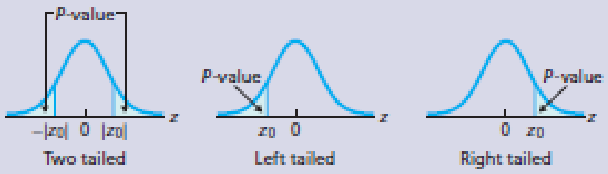


Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

OR

P-VALUE APPROACH

Step 4 Use Table II to obtain the P-value.



Step 5 If $P \leq \alpha$, reject H_0; otherwise, do not reject H_0.

Step 6 Interpret the results of the hypothesis test.

Note: The hypothesis test is exact for normal populations and is approximately correct for large samples from nonnormal populations.

When to Use the One-Mean z-Test†

- For small samples—say, of size less than 15—the z-test should be used only when the variable under consideration is normally distributed or very close to being so.
- For samples of moderate size—say, between 15 and 30—the z-test can be used unless the data contain outliers or the variable under consideration is far from being normally distributed.
- For large samples—say, of size 30 or more—the z-test can be used essentially without restriction. However, if outliers are present and their removal is not justified, you should perform the hypothesis test once with the outliers and once without them to see what effect the outliers have. If the conclusion is affected, use a different procedure or take another sample, if possible.
- If outliers are present but their removal is justified and results in a data set for which the z-test is appropriate (as previously stated), the procedure can be used.

Example:

Taller Young Women In the document *Anthropometric Reference Data for Children and Adults*, C. Fryer et al. present data from the *National Health and Nutrition Examination Survey* on a variety of human body measurements, such as weight, height, and size. Anthropometry is a key component of nutritional status assessment in children and adults.

A half-century ago, the average (U.S.) woman in her 20s was 62.6 inches tall. The heights, in inches, of a random sample of 25 of today's women in their 20s is presented in Table 9.9.

At the 1% significance level, do the data provide sufficient evidence to conclude that the mean height of today's women in their 20s is greater than the mean height of women in their 20s a half-century ago? Assume that the population standard deviation of heights for today's women in their 20s is 2.88 inches.

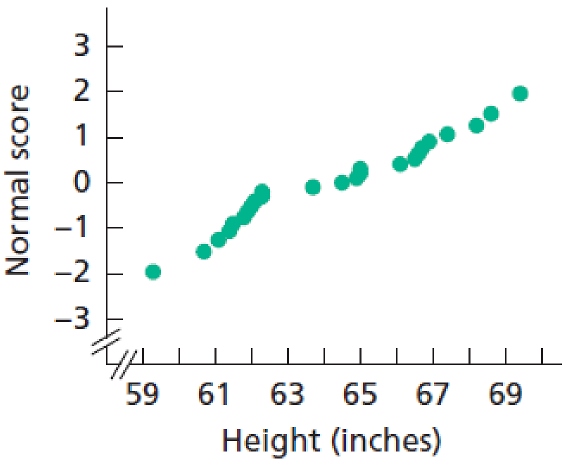
TABLE 9.9

Heights, in inches, of a random sample of 25 of today's women in their 20s

61.8	66.1	60.7	66.6	61.1
66.9	66.5	61.9	64.9	67.4
61.5	59.3	64.5	68.6	63.7
66.7	69.4	62.1	62.3	68.2
65.0	62.3	65.0	62.0	61.4

FIGURE 9.12

Normal probability plot of the heights in Table 9.9



Solution We note that the population standard deviation is known. Because the sample size, $n = 25$, is moderate, we first need to consider questions of normality and outliers. (See the second bulleted item in Key Fact 9.7.) Hence we constructed a normal probability plot for the data, shown in Fig. 9.12. The plot reveals no outliers and falls roughly in a straight line. Thus, we can apply Procedure 9.1 to perform the required hypothesis test.

Step 1 State the null and alternative hypotheses.

Let μ denote the mean height of today's women in their 20s. We obtained the null and alternative hypotheses in Example 9.2. They are

$$H_0: \mu = 62.6 \text{ inches (mean height has not increased)}$$
$$H_a: \mu > 62.6 \text{ inches (mean height has increased)}$$

Note that the hypothesis test is right tailed because a greater-than sign (>) appears in the alternative hypothesis.

Step 2 Decide on the significance level, α.

We are to perform the test at the 1% significance level, or $\alpha = 0.01$.

Step 3 Compute the value of the test statistic

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}.$$

We have $\mu_0 = 62.6$, $\sigma = 2.88$, and $n = 25$. The mean of the sample data in Table 9.9 is $\bar{x} = 64.24$. Thus the value of the test statistic is

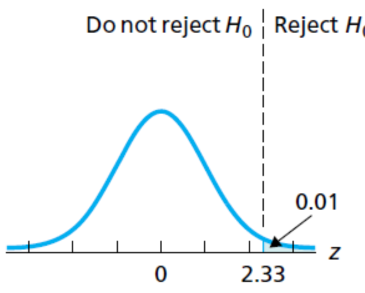
$$z = \frac{64.24 - 62.6}{2.88/\sqrt{25}} = 2.85.$$

CRITICAL-VALUE APPROACH	OR	P-VALUE APPROACH

Step 4 The critical value for a right-tailed test is z_{α}. Use Table II to find the critical value.

Because $\alpha = 0.01$, the critical value is $z_{0.01}$. From Table II (or Table 9.4 on page 402), $z_{0.01} = 2.33$, as shown in Fig. 9.13A.

FIGURE 9.13A



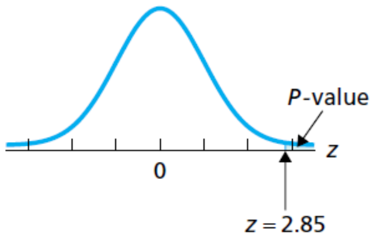
Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

The value of the test statistic found in Step 3 is $z = 2.85$. Figure 9.13A reveals that this value falls in the rejection region, so we reject H_0. The test results are statistically significant at the 1% level.

Step 4 Use Table II to obtain the P-value.

From Step 3, the value of the test statistic is $z = 2.85$. The test is right tailed, so the P-value is the probability of observing a value of z of 2.85 or greater if the null hypothesis is true. That probability equals the shaded area in Fig. 9.13B, which, by Table II, is 0.0022. Hence $P = 0.0022$.

FIGURE 9.13B



Step 5 If $P \leq \alpha$, reject H_0; otherwise, do not reject H_0.

From Step 4, $P = 0.0022$. Because the P-value is less than the specified significance level of 0.01, we reject H_0. The test results are statistically significant at the 1% level and (see Table 9.8 on page 408) provide very strong evidence against the null hypothesis.

Step 6 Interpret the results of the hypothesis test.

Interpretation At the 1% significance level, the data provide sufficient evidence to conclude that the mean height of today's women in their 20s is greater than the mean height of women in their 20s a half-century ago.

Example:

Poverty and Dietary Calcium Calcium is the most abundant mineral in the human body and has several important functions. Most body calcium is stored in the bones and teeth, where it functions to support their structure. Recommendations for calcium are provided in *Dietary Reference Intakes*, developed by the Institute of Medicine of the National Academy of Sciences. The recommended adequate intake (RAI) of calcium for adults (ages 19–50 years) is 1000 milligrams (mg) per day.

A simple random sample of 18 adults with incomes below the poverty level gives the daily calcium intakes shown in Table 9.10. At the 5% significance level, do the data provide sufficient evidence to conclude that the mean calcium intake of all adults with incomes below the poverty level is less than the RAI of 1000 mg? Assume that $\sigma = 188$ mg.

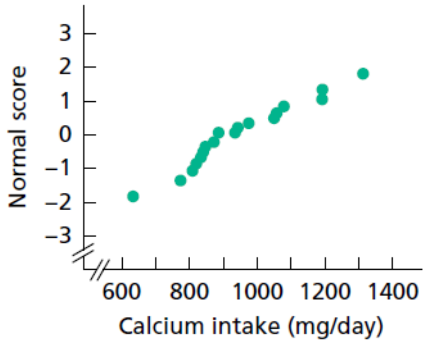
TABLE 9.10

Daily calcium intake (mg) for 18 adults with incomes below the poverty level

886	633	943	847	934	841
1193	820	774	834	1050	1058
1192	975	1313	872	1079	809

FIGURE 9.14

Normal probability plot of the calcium-intake data in Table 9.10



Solution Because the sample size, $n = 18$, is moderate, we first need to consider questions of normality and outliers. (See the second bulleted item in Key Fact 9.7 on page 411.) Hence we constructed a normal probability plot for the data, shown in Fig. 9.14. The plot reveals no outliers and falls roughly in a straight line. Thus, we can apply Procedure 9.1 to perform the required hypothesis test.

Step 1 State the null and alternative hypotheses.

Let μ denote the mean calcium intake (per day) of all adults with incomes below the poverty level. The null and alternative hypotheses, which we obtained in Example 9.3, are, respectively,

$$H_0\colon \mu = 1000 \text{ mg (mean calcium intake is not less than the RAI)}$$

$$H_a\colon \mu < 1000 \text{ mg (mean calcium intake is less than the RAI).}$$

Note that the hypothesis test is left tailed because a less-than sign ($<$) appears in the alternative hypothesis.

Step 2 Decide on the significance level, α.

We are to perform the test at the 5% significance level, or $\alpha = 0.05$.

Step 3 Compute the value of the test statistic

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}.$$

We have $\mu_0 = 1000$, $\sigma = 188$, and $n = 18$. From the data in Table 9.10, we find that $\bar{x} = 947.4$. Thus the value of the test statistic is

$$z = \frac{947.4 - 1000}{188/\sqrt{18}} = -1.19.$$

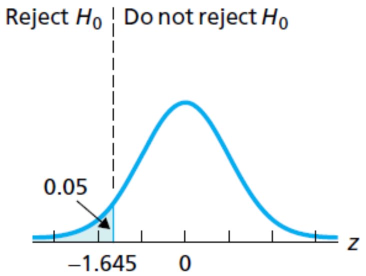
CRITICAL-VALUE APPROACH	OR	P-VALUE APPROACH

Critical-value approach

Step 4 The critical value for a left-tailed test is $-z_\alpha$. Use Table II to find the critical value.

Because $\alpha = 0.05$, the critical value is $-z_{0.05}$. From Table II (or Table 9.4 on page 402), $z_{0.05} = 1.645$. Hence the critical value is $-z_{0.05} = -1.645$, as shown in Fig. 9.15A.

FIGURE 9.15A



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

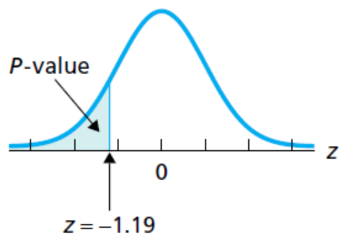
The value of the test statistic from Step 3 is $z = -1.19$. Figure 9.15A reveals that this value does not fall in the rejection region, so we do not reject H_0. The test results are not statistically significant at the 5% level.

P-value approach

Step 4 Use Table II to obtain the P-value.

From Step 3, the value of the test statistic is $z = -1.19$. The test is left tailed, so the P-value is the probability of observing a value of z of -1.19 or less if the null hypothesis is true. That probability equals the shaded area in Fig. 9.15B, which, by Table II, is 0.1170. Hence $P = 0.1170$.

FIGURE 9.15B



Step 5 If $P \leq \alpha$, reject H_0; otherwise, do not reject H_0.

From Step 4, $P = 0.1170$. Because the P-value exceeds the specified significance level of 0.05, we do not reject H_0. The test results are not statistically significant at the 5% level and (see Table 9.8 on page 408) provide at most weak evidence against the null hypothesis.

Step 6 Interpret the results of the hypothesis test.

Interpretation At the 5% significance level, the data do not provide sufficient evidence to conclude that the mean calcium intake of all adults with incomes below the poverty level is less than the RAI of 1000 mg per day.

Example:

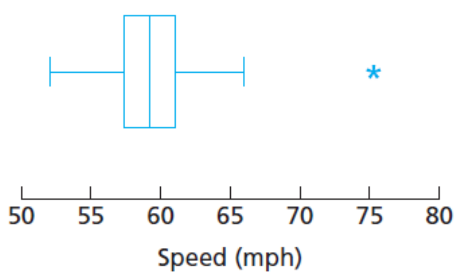
Clocking the Cheetah The cheetah is the fastest land mammal and is highly specialized to run down prey. According to the online document "Cheetah Conservation in Southern Africa" (*Trade & Environment Database (TED) Case Studies*, Vol. 8, No. 2) by J. Urbaniak, the cheetah is capable of speeds up to 72 mph.

One common estimate of mean top speed for cheetahs is 60 mph. Table 9.11 gives the top speeds, in miles per hour, for a sample of 35 cheetahs.

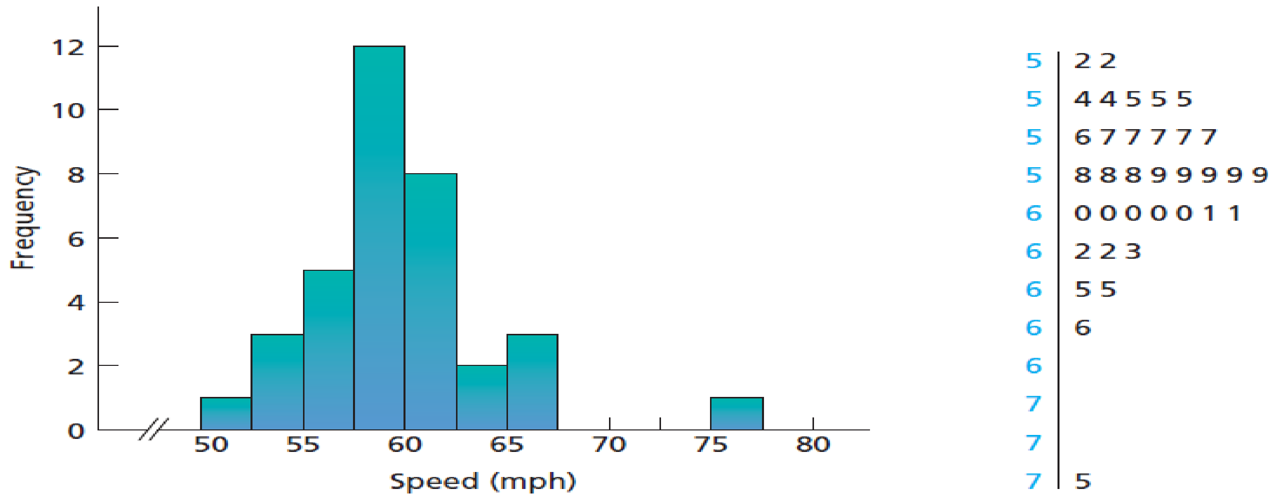
At the 5% significance level, do the data provide sufficient evidence to conclude that the mean top speed of all cheetahs differs from 60 mph? Assume that the population standard deviation of top speeds is 3.2 mph.

Top speeds, in miles per hour, for a sample of 35 cheetahs

57.3	57.5	59.0	56.5	61.3
57.6	59.2	65.0	60.1	59.7
62.6	52.6	60.7	62.3	65.2
54.8	55.4	55.5	57.8	58.7
57.8	60.9	75.3	60.6	58.1
55.9	61.6	59.6	59.8	63.4
54.7	60.2	52.4	58.3	66.0



Box Plot



Histogram and stem-and-leaf diagram

Solution We note that the population standard deviation is known. Because the sample size is 35, which is large, we need only check for outliers in the speed data before applying Procedure 9.1. (See the third bulleted item in Key Fact 9.7 on page 411.)

To check for outliers, we constructed a boxplot of the speed data, as shown in Fig. 9.16(a). The boxplot indicates that the top speed of 75.3 mph (third entry of the fifth row of Table 9.11) is a potential outlier. To decide whether 75.3 mph is in fact an outlier, we also constructed a normal probability plot, histogram, and stem-and-leaf diagram of the speed data, as shown in Figs. 9.16(b)–(d). *Note:* For simplicity in constructing the stem-and-leaf diagram, we first removed the decimal parts of the speeds.

From the four graphs in Fig. 9.16, we see that the top speed of 75.3 mph is indeed an outlier. Thus, as suggested in the third bulleted item in Key Fact 9.7, we apply Procedure 9.1 first to the full data set in Table 9.11 and then to that data set with the outlier removed.

Step 1 **State the null and alternative hypotheses.**

The null and alternative hypotheses are, respectively,

$$H_0\colon \mu = 60 \text{ mph (mean top speed of cheetahs is 60 mph)}$$

$$H_a\colon \mu \neq 60 \text{ mph (mean top speed of cheetahs is not 60 mph)},$$

where μ denotes the mean top speed of all cheetahs. Note that the hypothesis test is two tailed because a does-not-equal sign ($\neq$) appears in the alternative hypothesis.

Step 2 **Decide on the significance level, α.**

We are to perform the hypothesis test at the 5% significance level, or $\alpha = 0.05$.

Step 3 **Compute the value of the test statistic**

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}.$$

We have $\mu_0 = 60$, $\sigma = 3.2$, and $n = 35$. From the data in Table 9.11, we find that $\bar{x} = 59.526$. Thus the value of the test statistic is

$$z = \frac{59.526 - 60}{3.2/\sqrt{35}} = -0.88.$$

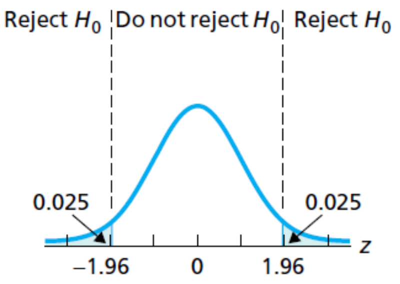
CRITICAL-VALUE APPROACH	OR	*P*-VALUE APPROACH

CRITICAL-VALUE APPROACH

Step 4 The critical values for a two-tailed test are $\pm z_{\alpha/2}$. Use Table II to find the critical values.

Because $\alpha = 0.05$, we find from Table II (or Table 9.4) the critical values of $\pm z_{0.05/2} = \pm z_{0.025} = \pm 1.96$, as shown in Fig. 9.17A.

FIGURE 9.17A



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

The value of the test statistic from Step 3 is $z = -0.88$. Figure 9.17A reveals that this value does not fall in the rejection region, so we do not reject H_0. The test results are not statistically significant at the 5% level.

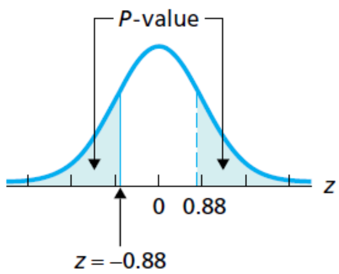
OR

***P*-VALUE APPROACH**

Step 4 Use Table II to obtain the *P*-value.

From Step 3, the value of the test statistic is $z = -0.88$. The test is two tailed, so the P-value is the probability of observing a value of z of 0.88 or greater in magnitude if the null hypothesis is true. That probability equals the shaded area in Fig. 9.17B, which, by Table II, is $2 \cdot 0.1894$ or 0.3788. Hence $P = 0.3788$.

FIGURE 9.17B



Step 5 If $P \leq \alpha$, reject H_0; otherwise, do not reject H_0.

From Step 4, $P = 0.3788$. Because the P-value exceeds the specified significance level of 0.05, we do not reject H_0. The test results are not statistically significant at the 5% level and (see Table 9.8 on page 408) provide at most weak evidence against the null hypothesis.

Step 6 Interpret the results of the hypothesis test.

Interpretation At the 5% significance level, the (unabridged) data do not provide sufficient evidence to conclude that the mean top speed of all cheetahs differs from 60 mph.