The Mean of Random Variable

The Mean or Expected Value:

Let X be a random variable with probability distribution $f(x)$. The **mean**, or **expected value**, of X is

$$\mu = E(X) = \sum_x x f(x)$$

if X is discrete, and

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\ dx$$

if X is continuous.

Let X be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is

$$\mu_{g(X)} = E[g(X)] = \sum_x g(x) f(x)$$

if X is discrete, and

$$\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\ dx$$

if X is continuous.

Example 4.1:

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

Solution:

Let X represent the number of good components in the sample. The probability distribution of X is

$$f(x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}, \qquad x = 0, 1, 2, 3.$$

Simple calculations yield $f(0) = 1/35$, $f(1) = 12/35$, $f(2) = 18/35$, and $f(3) = 4/35$. Therefore,

$$\mu = E(X) = (0)\left(\frac{1}{35}\right) + (1)\left(\frac{12}{35}\right) + (2)\left(\frac{18}{35}\right) + (3)\left(\frac{4}{35}\right) = \frac{12}{7} = 1.7.$$

Thus, if a sample of size 3 is selected at random over and over again from a lot of 4 good components and 3 defective components, it will contain, on average, 1.7 good components.

Example 4.2:

A salesperson for a medical device company has two appointments on a given day. At the first appointment, he believes that he has a 70% chance to make the deal, from which he can earn \$1000 commission if successful. On the other hand, he thinks he only has a 40% chance to make the deal at the second appointment, from which, if successful, he can make \$1500. What is his expected commission based on his own probability belief? Assume that the appointment results are independent of each other.

Solution:

First, we know that the salesperson, for the two appointments, can have 4 possible commission totals: \$0, \$1000, \$1500, and \$2500. We then need to calculate their associated probabilities. By independence, we obtain

$$f(\$0) = (1 - 0.7)(1 - 0.4) = 0.18, \qquad f(\$2500) = (0.7)(0.4) = 0.28,$$
$$f(\$1000) = (0.7)(1 - 0.4) = 0.42, \text{ and } f(\$1500) = (1 - 0.7)(0.4) = 0.12.$$

Therefore, the expected commission for the salesperson is

$$\begin{aligned} E(X) &= (\$0)(0.18) + (\$1000)(0.42) + (\$1500)(0.12) + (\$2500)(0.28) \\ &= \$1300. \end{aligned}$$

Example 4.3:

Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is

$$f(x) = \begin{cases} \frac{20{,}000}{x^3}, & x > 100, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected life of this type of device.

Solution:

$$\mu = E(X) = \int_{100}^{\infty} x \frac{20,000}{x^3}\, dx = \int_{100}^{\infty} \frac{20,000}{x^2}\, dx = 200.$$

Therefore, we can expect this type of device to last, *on average*, 200 hours.

Example 4.4:

Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:

x	4	5	6	7	8	9
$P(X = x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Let $g(X) = 2X - 1$ represent the amount of money, in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

Solution:

$$\begin{aligned} E[g(X)] = E(2X-1) &= \sum_{x=4}^{9} (2x-1) f(x) \\ &= (7)\left(\frac{1}{12}\right) + (9)\left(\frac{1}{12}\right) + (11)\left(\frac{1}{4}\right) + (13)\left(\frac{1}{4}\right) \\ &\quad + (15)\left(\frac{1}{6}\right) + (17)\left(\frac{1}{6}\right) = \$12.67. \end{aligned}$$

Example 4.5:

Let X be a random variable with density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of $g(X) = 4X + 3$.

Solution:

$$E(4X + 3) = \int_{-1}^{2} \frac{(4x + 3)x^2}{3}\, dx = \frac{1}{3}\int_{-1}^{2} (4x^3 + 3x^2)\, dx = 8.$$

Example 4.6:

Let X and Y be the random variables with joint probability distribution indicated in Table 3.1 on page 96. Find the expected value of $g(X, Y) = XY$. The table is reprinted here for convenience.

	$f(x,y)$	x = 0	1	2	Row Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

Solution:

$$\begin{aligned} E(XY) &= \sum_{x=0}^{2}\sum_{y=0}^{2} xyf(x,y) \\ &= (0)(0)f(0,0) + (0)(1)f(0,1) \\ &\quad + (1)(0)f(1,0) + (1)(1)f(1,1) + (2)(0)f(2,0) \\ &= f(1,1) = \frac{3}{14}. \end{aligned}$$

Example 4.7:

Find $E(Y/X)$ for the density function

$$f(x,y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0 < x < 2,\ 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Solution:

We have

$$E\left(\frac{Y}{X}\right) = \int_0^1 \int_0^2 \frac{y(1+3y^2)}{4}\, dx dy = \int_0^1 \frac{y+3y^3}{2}\, dy = \frac{5}{8}.$$

Practice Problems from Book

4.1 The probability distribution of X, the number of imperfections per 10 meters of a synthetic fabric in continuous rolls of uniform width, is given in Exercise 3.13 on page 92 as

x	0	1	2	3	4
$f(x)$	0.41	0.37	0.16	0.05	0.01

Find the average number of imperfections per 10 meters of this fabric.

4.2 The probability distribution of the discrete random variable X is

$$f(x) = \binom{3}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{3-x}, \quad x = 0, 1, 2, 3.$$

Find the mean of X.

4.3 Find the mean of the random variable T representing the total of the three coins in Exercise 3.25 on page 93.

> **3.25** From a box containing 4 dimes and 2 nickels, 3 coins are selected at random without replacement. Find the probability distribution for the total T of the 3 coins. Express the probability distribution graphically as a probability histogram.

4.4 A coin is biased such that a head is three times as likely to occur as a tail. Find the expected number of tails when this coin is tossed twice.

4.6 An attendant at a car wash is paid according to the number of cars that pass through. Suppose the probabilities are 1/12, 1/12, 1/4, 1/4, 1/6, and 1/6, respectively, that the attendant receives \$7, \$9, \$11, \$13, \$15, or \$17 between 4:00 P.M. and 5:00 P.M. on any sunny Friday. Find the attendant's expected earnings for this particular period.

4.10 Two tire-quality experts examine stacks of tires and assign a quality rating to each tire on a 3-point scale. Let X denote the rating given by expert A and Y denote the rating given by B. The following table gives the joint distribution for X and Y.

		y		
	$f(x,y)$	1	2	3
x	1	0.10	0.05	0.02
	2	0.10	0.35	0.05
	3	0.03	0.10	0.20

Find μ_X and μ_Y.

4.11 The density function of coded measurements of the pitch diameter of threads of a fitting is

$$f(x) = \begin{cases} \frac{4}{\pi(1+x^2)}, & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of X.

4.12 If a dealer's profit, in units of \$5000, on a new automobile can be looked upon as a random variable X having the density function

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1, \\ 0, & \text{elsewhere,} \end{cases}$$

find the average profit per automobile.

4.13 The density function of the continuous random variable X, the total number of hours, in units of 100 hours, that a family runs a vacuum cleaner over a period of one year, is given in Exercise 3.7 on page 92 as

$$f(x) = \begin{cases} x, & 0 < x < 1, \\ 2-x, & 1 \le x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the average number of hours per year that families run their vacuum cleaners.

4.14 Find the proportion X of individuals who can be expected to respond to a certain mail-order solicitation if X has the density function

$$f(x) = \begin{cases} \frac{2(x+2)}{5}, & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

4.32 In Exercise 3.13 on page 92, the distribution of the number of imperfections per 10 meters of synthetic fabric is given by

x	0	1	2	3	4
$f(x)$	0.41	0.37	0.16	0.05	0.01

(a) Plot the probability function.

(b) Find the expected number of imperfections, $E(X) = \mu$.

(c) Find $E(X^2)$.