

## The Product or Multiplication Rule

### Dependent Events

If in an experiment the events  $A$  and  $B$  can both occur, then

$$P(A \cap B) = P(A)P(B|A), \text{ provided } P(A) > 0.$$

### Example:

Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

We shall let  $A$  be the event that the first fuse is defective and  $B$  the event that the second fuse is defective; then we interpret  $A \cap B$  as the event that  $A$  occurs and then  $B$  occurs after  $A$  has occurred. The probability of first removing a defective fuse is  $1/4$ ; then the probability of removing a second defective fuse from the remaining 4 is  $4/19$ . Hence,

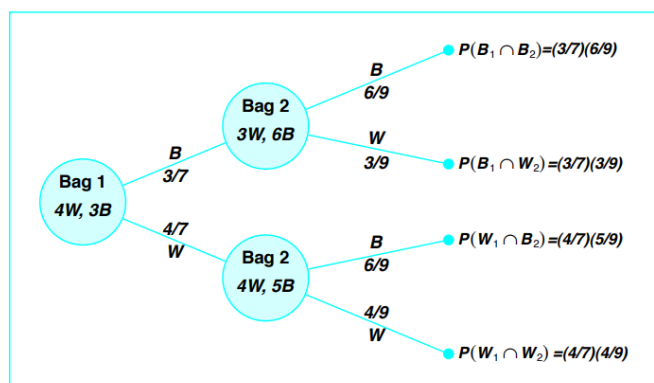
$$P(A \cap B) = \left(\frac{1}{4}\right) \left(\frac{4}{19}\right) = \frac{1}{19}.$$

### Example:

One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

Let  $B_1$ ,  $B_2$ , and  $W_1$  represent, respectively, the drawing of a black ball from bag 1, a black ball from bag 2, and a white ball from bag 1. We are interested in the union of the mutually exclusive events  $B_1 \cap B_2$  and  $W_1 \cap B_2$ . The various possibilities and their probabilities are illustrated in Figure 2.8. Now

$$\begin{aligned} P[(B_1 \cap B_2) \text{ or } (W_1 \cap B_2)] &= P(B_1 \cap B_2) + P(W_1 \cap B_2) \\ &= P(B_1)P(B_2|B_1) + P(W_1)P(B_2|W_1) \\ &= \left(\frac{3}{7}\right) \left(\frac{6}{9}\right) + \left(\frac{4}{7}\right) \left(\frac{5}{9}\right) = \frac{38}{63}. \end{aligned}$$



## Independent Events

Two events  $A$  and  $B$  are independent if and only if

$$P(A \cap B) = P(A)P(B).$$

Therefore, to obtain the probability that two independent events will both occur, we simply find the product of their individual probabilities.

If, in an experiment, the events  $A_1, A_2, \dots, A_k$  can occur, then

$$\begin{aligned} P(A_1 \cap A_2 \cap \dots \cap A_k) \\ = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \dots P(A_k|A_1 \cap A_2 \cap \dots \cap A_{k-1}). \end{aligned}$$

If the events  $A_1, A_2, \dots, A_k$  are independent, then

$$P(A_1 \cap A_2 \cap \dots \cap A_k) = P(A_1)P(A_2) \dots P(A_k).$$

### Example:

A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98, and the probability that the ambulance is available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that both the ambulance and the fire engine will be available, assuming they operate independently.

Let  $A$  and  $B$  represent the respective events that the fire engine and the ambulance are available. Then

$$P(A \cap B) = P(A)P(B) = (0.98)(0.92) = 0.9016.$$

### Example:

An office building has two fire detectors. The probability is .02 that any fire detector of this type will fail to go off during a fire. Find the probability that both of these fire detectors will fail to go off in case of a fire. Assume that these two fire detectors are independent of each other.

**Solution** In this example, the two fire detectors are independent because whether or not one fire detector goes off during a fire has no effect on the second fire detector. We define the following two events:

$A$  = the first fire detector fails to go off during a fire

$B$  = the second fire detector fails to go off during a fire

Then, the joint probability of  $A$  and  $B$  is

$$P(A \text{ and } B) = P(A) \times P(B) = (.02) \times (.02) = .0004$$

**Example:**

Three cards are drawn in succession, without replacement, from an ordinary deck of playing cards. Find the probability that the event  $A_1 \cap A_2 \cap A_3$  occurs, where  $A_1$  is the event that the first card is a red ace,  $A_2$  is the event that the second card is a 10 or a jack, and  $A_3$  is the event that the third card is greater than 3 but less than 7.

First we define the events

$A_1$ : the first card is a red ace,

$A_2$ : the second card is a 10 or a jack,

$A_3$ : the third card is greater than 3 but less than 7.

Now

$$P(A_1) = \frac{2}{52}, \quad P(A_2|A_1) = \frac{8}{51}, \quad P(A_3|A_1 \cap A_2) = \frac{12}{50},$$

and hence, by Theorem 2.12,

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3) &= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \\ &= \left(\frac{2}{52}\right) \left(\frac{8}{51}\right) \left(\frac{12}{50}\right) = \frac{8}{5525}. \end{aligned}$$

## Conditional probability

The conditional probability of  $B$ , given  $A$ , denoted by  $P(B|A)$ , is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad \text{provided } P(A) > 0.$$

Two events  $A$  and  $B$  are **independent** if and only if

$$P(B|A) = P(B) \quad \text{or} \quad P(A|B) = P(A),$$

assuming the existences of the conditional probabilities. Otherwise,  $A$  and  $B$  are **dependent**.

### Example:


The concept of conditional probability has countless uses in both industrial and biomedical applications. Consider an industrial process in the textile industry in which strips of a particular type of cloth are being produced. These strips can be defective in two ways, length and nature of texture. For the case of the latter, the process of identification is very complicated. It is known from historical information on the process that 10% of strips fail the length test, 5% fail the texture test, and only 0.8% fail both tests. If a strip is selected randomly from the process and a quick measurement identifies it as failing the length test, what is the probability that it is texture defective?

Consider the events

$L$ : length defective,       $T$ : texture defective.

Given that the strip is length defective, the probability that this strip is texture defective is given by

$$P(T|L) = \frac{P(T \cap L)}{P(L)} = \frac{0.008}{0.1} = 0.08.$$

Thus, knowing the conditional probability provides considerably more information than merely knowing  $P(T)$ . 

**Example:**

The probability that a regularly scheduled flight departs on time is  $P(D) = 0.83$ ; the probability that it arrives on time is  $P(A) = 0.82$ ; and the probability that it departs and arrives on time is  $P(D \cap A) = 0.78$ . Find the probability that a plane

(a) arrives on time, given that it departed on time, and (b) departed on time, given that it has arrived on time.

Using Definition 2.10, we have the following.

- (a) The probability that a plane arrives on time, given that it departed on time, is

$$P(A|D) = \frac{P(D \cap A)}{P(D)} = \frac{0.78}{0.83} = 0.94.$$

- (b) The probability that a plane departed on time, given that it has arrived on time, is

$$P(D|A) = \frac{P(D \cap A)}{P(A)} = \frac{0.78}{0.82} = 0.95.$$



## Practice Problems

**2.78** A manufacturer of a flu vaccine is concerned about the quality of its flu serum. Batches of serum are processed by three different departments having rejection rates of 0.10, 0.08, and 0.12, respectively. The inspections by the three departments are sequential and independent.

- (a) What is the probability that a batch of serum survives the first departmental inspection but is rejected by the second department?
- (b) What is the probability that a batch of serum is rejected by the third department?

**2.84** The probability that the head of a household is home when a telemarketing representative calls is 0.4. Given that the head of the house is home, the probability that goods will be bought from the company is 0.3. Find the probability that the head of the house is home and goods are bought from the company.

**2.85** The probability that a doctor correctly diagnoses a particular illness is 0.7. Given that the doctor makes an incorrect diagnosis, the probability that the patient files a lawsuit is 0.9. What is the probability that the doctor makes an incorrect diagnosis and the patient sues?

**2.76** In an experiment to study the relationship of hypertension and smoking habits, the following data are collected for 180 individuals:

	<b>Nonsmokers</b>	<b>Moderate Smokers</b>	<b>Heavy Smokers</b>
$H$	21	36	30
$NH$	48	26	19

where  $H$  and  $NH$  in the table stand for *Hypertension* and *Nonhypertension*, respectively. If one of these individuals is selected at random, find the probability that the person is

- experiencing hypertension, given that the person is a heavy smoker;
- a nonsmoker, given that the person is experiencing no hypertension.

**2.83** The probability that a vehicle entering the Luray Caverns has Canadian license plates is 0.12; the probability that it is a camper is 0.28; and the probability that it is a camper with Canadian license plates is 0.09. What is the probability that

- a camper entering the Luray Caverns has Canadian license plates?
- a vehicle with Canadian license plates entering the Luray Caverns is a camper?
- a vehicle entering the Luray Caverns does not have Canadian plates or is not a camper?

**2.80** The probability that an automobile being filled with gasoline also needs an oil change is 0.25; the probability that it needs a new oil filter is 0.40; and the probability that both the oil and the filter need changing is 0.14.

- (a) If the oil has to be changed, what is the probability that a new oil filter is needed?
- (b) If a new oil filter is needed, what is the probability that the oil has to be changed?