Numerical Computing

Under error measurement

4 sig fig

- Absolute (True) error
- Relative & Percentage Relative error → better method as it considers size to be approximated → most cases

→ true Absolute error

E_t = True value − calculated value

→ true Relative errors

$$E = \frac{\text{True value} - \text{calculated value}}{\text{True value}}$$

→ approx Absolute error

E_a = current approx − previous approx

→ approx Relative error

$$E = \frac{\text{current approx} - \text{previous approx}}{\text{present approx}}$$

→ stopping criteria

$E_s = (0.5 \times 10^{2-\underbrace{n}_{n \text{ sig fig}}})\%$

$E_a < E_s$

as iterations increas relative error decreases

keep iterating till stoppin criteria

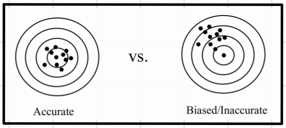
Types of errors

- Truncation error → when no exact value given → like chopping
- Round off error → rounding off

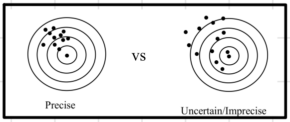
→ lack of accuracy of an estimate

true error: when curren solution compared to true solution

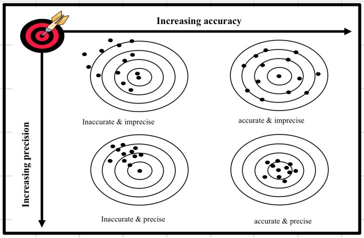
approximate error: current solution compared to the solution obtained in previous itteration



Accuracy: how close to true value



Precision: how close values with eachother



Truncation error

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \frac{x^n}{n!}$$

Step 1: set E_s (stopping criteria)

$E_s = 0.5 \times 10^{2-3}\,\%$ for 3 s.f

$= 0.05\%$

Step 2:

$e^n = 1 + n$ True value: $e^{0.5} = 1.648721$

$n = 0 \rightarrow S = 1$

$n = 0.5 \rightarrow S = 1.5$?

0 0.5 1.5

$$E_t = \left| \frac{1.648721 - 1.5}{1.648721} \right| \times 100 = 9.02\%$$

$$E_a = \left| \frac{1.5 - 1}{1.5} \right| \times 100 = 33.33\%$$

as $E_a < E_s$
hence keep iterating

Terms	Result	ε_t, %	ε_a, %
1	1	39.3	
2	1.5	9.02	33.3
3	1.625	1.44	7.69
4	1.645833333	0.175	1.27
5	1.648437500	0.0172	0.158
6	1.648697917	0.00142	0.0158

?

REPRESENTATION OF REAL NUMBERS

1. Binary machine numbers

sign (MSB)
left most bit

characteristic

mantisa

$$(-1)^{s}\, 2^{c-1023}\,(1+f)$$

1 bit 11 bit 52 bit

0 10000000011 1011100100010000000000000000000000000000000000000000

Sign Characteristic Mantissa

Sign:
$0 = +ve$
$1 = -ve$

right to left

base 2 of binary

$$c = 1\times 2^{10} + 1\cdot 2^{1} + 1\cdot 2^{0} = 1027$$

$$2^{1027-1023} = 2^{4}$$

left to right

$$f = 1\times 0.5^{1} + 1\times 0.5^{3} + 1\times 0.5^{4} + 1\times 0.5^{5} + 1\times 0.5^{8} + 1\times 0.5^{12} =$$

$$(-1)^{s}2^{c-1023}(1+f) = \underbrace{(-1)^{0}}_{\text{sign}} \cdot 2^{\overbrace{1027-1023}^{c}} \underbrace{\left(1+\left(\frac{1}{2}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{256}+\frac{1}{4096}\right)\right)}_{f}$$

$$= 27.56640625.$$

2. Decimal machine numbers

(normalised floating point representation)

$$\pm\, 0.d_1d_2 \ldots d_k \times 10^{n} \qquad 1 \le d_i \le 9$$

4 s.f

$x = 635894$ $y = 0.00218$ $z = 584.63$

methods

rounding

$x = 0.6359 \times 10^{6}$

$y = 0.2180 \times 10^{-2}$

$z = 0.5846 \times 10^{3}$

choping

$x = 0.6358 \times 10^{6}$

$y = 0.2180 \times 10^{-2}$

$z = 0.5846 \times 10^{3}$

absolute error (E_t)

$e_p = |p - p^*|$

true value, calculated value

relative error (E_a)

$\delta_p = \left| \frac{p - p^*}{p} \right|$, $p \neq 0$

Finite Arithmetic

If $x = \frac{5}{7}$ & $y = \frac{1}{3}$, use five digit chopping for calculation

$E_s = 0.5 \times 10^{2-5} = 5 \times 10^{-4}$

1. $x \oplus y$ approximate

chop before adding

$x = 0.71428$

$y = 0.33333$

$x + y = 0.10476 \times 10^1$

True value

$x = 5/7 = 0.714285$

$y = 1/3 = 0.33333$

$x + y = 22/21$

$$E_a = \left| \frac{22/21 - 0.10476 \times 10^1}{22/21} \right| \times 100$$

$= 1.81818 \times 10^{-3}$

$= 0.18181 \times 10^{-2}$

2. $x \ominus y$

after chopping

$x - y = 0.38095$

true value

$x - y = 8/21$

$$E_a = \left| \frac{8/21 - 0.38095}{8/21} \right| \times 100$$

$= 6.24999 \times 10^{-4}$

$= 0.62499 \times 10^{-3}$

3. $x \otimes y$

after chopping

$x \times y = 0.23809$

true value

$x \times y = 5/21$

$$E_a = \left| \frac{5/21 - 0.23809}{5/21} \right| \times 100$$

$= 2.20 \times 10^{-3}$

$= 0.220 \times 10^{-2}$

4. $x \oslash y$

after chopping

$x / y = 0.21428 \times 10^1$

true value

$x / y = 15/7$

$$E_a = \left| \frac{15/7 - 0.21428 \times 10^1}{15/7} \right| \times 100$$

$= 2.66666 \times 10^{-3}$

$= 0.266666 \times 10^{-2}$

Q) $P = 0.54617$, $q = 0.54601$, $P-q$ absolute and relative error

$P-q = 0.00016$ 0.0016

$p^* = 0.5461$ $q^* = 0.5460$ → after chopping

$p^* - q^* = 0.0001$ X → why

$$E_a = \left| \frac{0.00016 - 0.0001}{0.00016} \right|$$

$= 0.376$?

↳ loss of s.f

loss of sig then relative error increases

LOSS OF SF Remedy

when too many sf cancel

↳ rationalise

↳ using series expansion

↳ use trignometric identities

↳ reformulation

```
1 2 3.4 5 6 7
1 2 3.4 5 6 6
0 0 0.0 0 0 1
```

rationalise

Q) $\sqrt{9.01} - 3$ in 3 s.f

$$\sqrt{9.01} - 3 \times \frac{\sqrt{9.01} + 3}{\sqrt{9.01} + 3}$$

$$= \frac{9.01 - 3^2}{\sqrt{9.01} + 3}$$

$$= \frac{0.01}{6}$$

$= 1.67 \times 10^{-3}$

use trignometric identities

Q) $f(x) = \cos^2 x - \sin^2 x$

at $x = \pi/4$ → loss of s.f

$\cos^2 x - \sin^2 x = \cos 2x$

Q1) $x^2 + 62.10x + 1 = 0$

$x_1 = -0.01610723$

$x_2 = -62.08390$

use 4 digit rounding

$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$, $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

$\sqrt{b^2 - 4ac} = \sqrt{(62.10)^2 - (4.0000)(1.0000)(1.0000)}$

$= 62.06$

Nested Arthemetic

$f(x) = x^3 - 6.1x^2 + 3.2x + 1.5$ $x = 4.71$

BINARY SEARCH / BISECTION

↳ if 2 boundaries, where extremese −ve and +ve then a point wher function is 0 exists

Steps

↳ find no of iterations

↳ Repeat till $\int$ n iterations

↳ choose 2 boundaries +ve and −ve

↳ find mid point

↳ get new interval

Check iterations

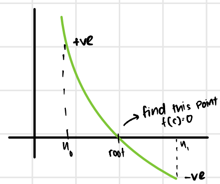
1. absolute relative error

$$\left| \frac{u^{new} - u^{old}}{u^{new}} \right| \times 100 \quad \text{till } E_a < E_s \text{ for each iteration}$$

2. $n = \dfrac{\ln(\Delta n / e)}{\ln 2}$ diff $(u_2 - u_1)$, stopping criteria

Show that $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in $[1, 2]$, and use the Bisection method to determine an approximation to the root that is accurate to at least within 10^{-4}.

no of iterations: $\dfrac{\ln(1/10^{-4})}{\ln 2} = 13.29 = 13$ iterations



↳ $f(1) = -5$ (u_1), $f(2) = 14$ (u_2)

as both opposite sign

1 ↳ mid point: $\dfrac{1+2}{2} = 1.5$

↳ $f(1.5) = 2.735$ +ve so replace u_2

2 ↳ mid point = $\dfrac{1+1.5}{2} = 1.25$

↳ $f(1.25) = -1.796875$ −ve so replace u_1 → do not round off

3 ↳ mid point = $\dfrac{1.25+1.5}{2} = 1.375$

↳ $f(1.375) = 0.162109375$ u_2

4 ↳ mid point: $\dfrac{1.25+1.375}{2} = 1.3125$

↳ $f(1.3125) = -0.848388671\!9$

5 ↳ mid point: $\dfrac{1.3125+1.375}{2} = 1.34315$

↳ $f(1.34315) = -0.35098266$

6 ↳ mid point: $\dfrac{1.34315+1.375}{2} = 1.359375$

do till 13 iterations

SOLVE IN TABULAR FORM FOR PAPER

n	a_n	b_n	p_n	$f(p_n)$
1	1.0	2.0	1.5	2.375
2	1.0	1.5	1.25	−1.79687
3	1.25	1.5	1.375	0.16211
4	1.25	1.375	1.3125	−0.84839
5	1.3125	1.375	1.34375	−0.35098
6	1.34375	1.375	1.359375	−0.09641
7	1.359375	1.375	1.3671875	0.03236
8	1.359375	1.3671875	1.36328125	−0.03215
9	1.36328125	1.3671875	1.365234375	0.000072
10	1.36328125	1.365234375	1.364257813	−0.01605
11	1.364257813	1.365234375	1.364746094	−0.00799
12	1.364746094	1.365234375	1.364990235	−0.00396
13	1.364990235	1.365234375	1.365112305	−0.00194

Question # 1

Use bisection method to find the real root of $f(x) = \sqrt{x} - \cos x$ over $[0,1]$ with absolute approximate error< 0.01

Ans: $x = 0.64844$

$$n = \frac{\ln\left(\frac{1}{0.01}\right)}{\ln 2} = 6.64 \simeq 7 \text{ itterations}$$

n	u_1 (-ve)	u_2 (+ve)	mid	f(u)
1	0	1	0.5	-ve
2	0.5	1	0.75	+ve
3	0.5	0.75	0.625	-ve
4	0.625	0.75	0.6875	+ve
5	0.625	0.6875	0.65625	+ve
6	0.625	0.65625	0.640625	-ve
7	0.640625	0.65625	0.6484375	

0.64844

Question # 3 0,1 both +ve

Solve $x = 2e^{-x}$ by bisection and Regula-Falsi method over $[0,1]$ with percentage relative approximate true error $< 0.1\%$ and comments on the result.

0.001

Ans: $x = 0.85156$ (bisection)

Ans: $x = 0.85396$ (regula-falsi method)

$$\frac{\ln(1/0.01)}{\ln 2} = 6 \text{ itterations}$$

n	u_1 (+ve)	u_2 (-ve)	mid	f(n)
1	0	1	0.5	+ve
2	0.5	1	0.75	+ve
3	0.75	1	0.875	+ve
4	0.875	1	0.9375	+ve
5	0.96875			
6				
7				

?

2. Let $f(x) = 3(x+1)(x-\frac{1}{2})(x-1)$. Use the Bisection method on the following intervals to find p_3.

a. $[-2, 1.5]$ **b.** $[-1.25, 2.5]$

$f(-2) = -22.5$ $f(1.5) = 3.75$

n	x_1 (−ve)	x_2 (+ve)	mid	f(x)
1	−2	1.5	−0.25	+ve
2	−2	−0.25	−1.125	−ve
3	−1.125	−0.25	−0.6875	−ve

Fixed Point iteration

- let $f(x)=0$
- let $x=x_0$

  transform $f(x)=0$ into $x=\phi(x)$
- Theorem

  1. $|\phi'(x)| < 1$ (derivative)
  2. if all <1 then opt smallest value
- $x_n=\phi(x_{n-1})$
- Question either given

  S.F ↙ ↘ x roots

Steps

- find 2 value where $f(x)$: +ve and -ve
- take its midpoint → x_0

  } OPTIONAL — These step reduce iterations
- make x subject
- check $|\phi'(x)| < 1$ → all +ve values

  <1: converging
  >1: diverging ?
- if all <1, opt smallest
- plug in function $x_n=\phi(x_{n-1})$
- keep iterating till sf same or stopping criteria

ALWAYS IN RADIAN ITTERATIVE METHOD

Q) $f(x)=x^3+x^2-1=0$, correct upto 4 s.f

3 ways to make x a subject

$x^3=1-x^2$	$x^2=1-x^3$	$x^2(x+1)=1$
$x=(1-x^2)^{1/3}$	$x=(1-x^3)^{1/2}$	$x=\dfrac{1}{\sqrt{1+x}}$

give +ve -ve value

$f(0)=-1$, $f(1)=1$

take $x=0.5 \rightarrow x_0$

mid point of extremes values

can take any value but midpoint reduces itterations

$x'=\frac{1}{3}(1-x^2)^{-2/3}(-2x)$	$x'=\frac{1}{2}(1-x^3)^{-1/2}(-3x^2)$	$x'=-\frac{1}{2}(1+x)^{-3/2}(1)$
$x=-0.5291$ (as mod)	$x'=-0.4008$	$x'=-0.2721$

since all the ans <1 choose smallest

use og function

$x=(1+x)^{-1/2}$

$x_n=\phi(x_{n-1})$ for $n=1$ (let $x_0=0.5$)

n	
1	0.8164965809
2	0.7419637843
⋮	
6	0.7548499056
7	0.7548836371

$x_1=\phi(x_0)=(1+0.5)^{-1/2}=0.8164965809$

$x_2=\phi(x_1)=(1+0.8164965809)^{-1/2}=0.7419637843$

Question # 5

Resistance of moving vehicle , $f(x) = x^4 - x - 10$ where x is the displacement. Find the displacement at zero resistance by using fixed point iteration method with absolute approximate error < 0.0001 and $x_0 = 4$

Ans: $x = 1.85558$

↳ midPoint = x_0 = 4

↳ making x subject $x^4 - x - 10 = 0$

3 ways

$x^4 = 10 + x$	$x = x^4 - 10$	$x(x^3 - 1) = 10$
$x = (10+x)^{1/4}$		$x = \dfrac{10}{x^3 - 1}$
		$10(x^3-1)^{-1}$
$x' = \frac{1}{4}(10+x)^{-3/4}$	$x' = 4x^3$	$x' = -10(x^3-1)^{-2}(3x^2)$
$= \underline{0.0502} < 1$	$= 13.5$	$= 11.96$

← insert $x_0 = 4$

↳ $x = (10 + x)^{1/4}$

$x_n = \phi x_{n-1}$

n	
1	1.934336
2	1.858658
3	1.855704
4	1.855589
5	1.855584

$(10+4)^{1/4}$

1.855589 ← 1.85558 → 4sf ? 5sf ?

14. Use a fixed-point iteration method to determine a solution accurate to within 10^{-4} for $x = \tan x$, for x in $[4, 5]$.

15. Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $2\sin\pi x + x = 0$

$\frac{4+5}{2} = 4.5 \rightarrow x_0$

$f(x) = \tan x - x$

$\tan x - x = 0$

when to stop?

$x = \tan x$

$x' = \sec^2 x$

$x' = \frac{1}{\cos^2(x)}$

$= 22.$

$x = \tan^{-1}(x)$

$x' = \frac{1}{1+x^2}$

$x = 0.0470$

$x = \tan^{-1}(x)$

n	
1	1.354
2	0.9340
3	0.75128
4	0.64432
	0.57232
	0.51981
	0.479
	0.44703
	0.42

$4 < 5$

$0.0001 <$

Question # 6

A shell is fired vertically upward and its vertical height x in meters is given by,

$$x = cost - 3t + 3$$

Where t represent time in seconds. Determine the time required for the vertical height will reach 2m by using fixed point iteration method with initial guess=0 and absolute approximate error<0.00001

Ans: $t = 0.60710$

$x_0 = 0$

$f(t) = \cos t - 3t + 1$

making t subject

$\cos t = 3t - 1$

$t = \cos^{-1}(3t-1)$

$3t = \cos t + 1$

$t = \frac{\cos t + 1}{3}$

$\frac{1}{3} \sin n$

$t = \frac{1}{3} \sin t + 1$

0

$t = \frac{\cos t + 1}{3}$ $x_0 = 0$

n
0.66667
0.59524
0.60932
0.60667
0.60718
0.60708
0.60710
0.60710

BEST ↗ FASTEST ↗

LECTURE 6

Newtons Raphsom Method

$y - f(x_0) = f'(x_0)(x - x_0)$

$0 - f(x_0) = f'(x_0)(x - x_0)$

$-\frac{f(x_0)}{f'(x_0)} = x_1 - x_0$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $\therefore f'(x_n) \neq 0$ ↳ denominator never 0

Steps

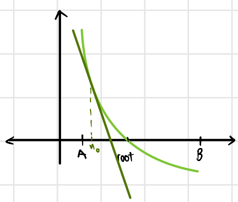
- ↳ find 2 value where $f(x)$ = +ve and -ve
- ↳ take its midpoint

$\}$ OPTIONAL — These step reduce itterations

- ↳ find f'
- ↳ $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $\therefore f'(x_n) \neq 0$ ↳ denominator never 0
- ↳ Keep itterating till sf same or Es

Limitation

Only usable is f' is readily available



Q) $f(x) = x^3 + x^2 - 1 = 0$, correct upto 4 s.f

$f(0) = -1$, $f(1) = 1$ — give +ve -ve value

take $x_0 = 0.5$ → mid point of extremes values — can take any value but midpoint reduces itterations

$f' = 3x^2 + 2x$

n	
0	0.8571428571
1	0.7641369048
2	0.7549634824
3	0.7548776737
4	0.7548776662

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{0.5^3 + 0.5^2 - 1}{3(0.5)^2 + 2(0.5)}$

→ to save time replace 0.5 with ans in calculator

Question # 7

The number of clients in the ABC server is related to time i.e,

$$N(t) = 75e^{-1.5t} + 20e^{-0.075t}$$

Determine the time required for the server will have 15 clients by using Newton-Raphson Method with an initial guess of $t = 6$ and stopping criteria of Absolute approximate percentage error<0.5% ?

Ans: $t = 4.00163$

$n_0 = 6$

$N' = 75(-1.5)e^{-1.5t} + 20(-0.075)e^{-0.075t}$

$N' = -112.5e^{-1.5t} - 1.5e^{-0.075t}$

n	
1	19.15209
2	32.4854
3	45.818762
4	59.15209
5	72.4854
6	85.818762

?

Question # 9

Find the root of $f(x) = x^{\frac{1}{2}} + x^{-\frac{1}{2}} - 3$ by using Newton-Raphson method with absolute approximate error<0.0001 and $x_0 = 2$

Ans: $x = 6.85410$

$x_0 = 2$

$f' = \frac{1}{2}u^{-\frac{1}{2}} - \frac{1}{2}u^{-3/2}$

n	
1	6.970562
2	6.853771
3	6.85410 ✓ ans
4	6.85410
5	
6	

5. Use Newton's method to find solutions accurate to within 10^{-4} for the following problems.

a. $x^3 - 2x^2 - 5 = 0, \quad [1, 4]$ **b.** $x^3 + 3x^2 - 1 = 0, \quad [-3, -2]$

c. $x - \cos x = 0, \quad [0, \pi/2]$ **d.** $x - 0.8 - 0.2\sin x = 0, \quad [0, \pi/2]$

a) $\frac{1+4}{2} = 2.5 \rightarrow x_0$

$f = n^3 - 2n^2 - 5 = 0$

$f' = 3n^2 - 4n$

n
2.71428
2.6909
2.6906 ✓
2.69064

d) $x_0 = 2.5$

$f = n - 0.8 - 0.2\sin n$

$f' = 1 - 0.2\cos n$

n
1.13793
0.967233
0.96437
0.96433 ✓

✓

LECTURE 7

Secant Method → open method

?

$$x_{n+2} = x_n - \frac{f(x_n)(x_n - x_{n+1})}{f(x_{n+1}) - f(x_n)}$$

$$x_{n+2} = \frac{x_n f(x_{n+1}) - x_{n+1} f(x_n)}{f(x_{n+1}) - f(x_n)}$$

$$x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

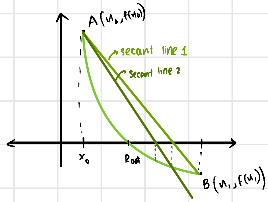
Q) $x^3 + 3x^2 - 1 = 0$ for interval $[\overset{a}{-3}, \overset{b}{-2}]$, find root to 4 s.f.

$$x = \frac{-3(3) - -2(-1)}{3 - -1} = -2.75 \to x_2 \quad [-2, -2.75]$$

$$x = \frac{-2\left(-\frac{911}{64}\right) - -2.75(3)}{-\frac{911}{64} - 3} =$$

Steps

- $x_{n+2} = \frac{x_n f(x_{n+1}) - x_{n+1} f(x_n)}{f(x_{n+1}) - f(x_n)}$
- keep replacing x → sign dosent matter
- keep itterating till sf same or Es



SOLVE IN TABULAR FORM

itteration	x_n x_0	x_{n+1} x_1	x_{n+2} x_2	$f(x)_{n+2}$	$\lvert x_{n+2} - x_{n+1} \rvert$
1	-3				
2	-2				
⋮					
6	-2.877186	-2.879414	-2.879385	0.000000	0.000029

2. become same

1. becomes 0

<0.01 stopping criteria

for Calculator

$C = \frac{af(b) - bf(a)}{f(b) - f(a)}$: function

Press calc

give A, B values

Press = value of C

Press = agin value of f(c)

Question # 8

A particle is moving with the velocity $\boldsymbol{v(t) = t\,cos(t) + sin(t)}$ at time **t**. Find the time at which particle will be at rest by using secant method with an initial guesses $t_0 = 2\ and\ t_1 = 3$ and stopping criteria of absolute approximate error<0.00001

Ans: $t = 0.02876$

n	a (+ve)	b (-ve)	c	f(c)
1	2	3	2.0264	6.10195
2	3	2.0264	2.02859	4.407743
3				2.02649946
4				
5				
6				

?

3. Let $f(x) = x^2 - 6$. With $p_0 = 3$ and $p_1 = 2$, find p_3.

a. Use the Secant method.

b. Use the method of False Position.

c. Which of **a.** or **b.** is closer to $\sqrt{6}$?

a) $a = 3 \qquad b = 2$

Put directly in Calculator

n	a (+ve)	b (-ve)	u
1	3	2	2.4
2	2	2.4	2.454

ans

Regular Falsi

LECTURE 9

False Position / Interpolation method → bracketing method

bracket the root

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n+1})}{f(x_n) - f(x_{n+1})} \Big/ x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

Steps

- $x = \dfrac{a f(b) - b f(a)}{f(b) - f(a)}$
- if $f(x)$ +ve replace b
- if $f(x)$ -ve replace a

sign matters

- keep itterating till sf same or Es

Q) $f(x) = x^2 - 3$, Let $\varepsilon = 0.01$

$x_0 = 1$ $x_1 = 2$

$f(1) = -2$, $f(2) = 1$

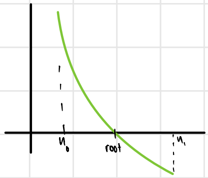
$f(x_0) f(x_1) < 0$

x_0	x_1	x_3
1	2	1.5
+ve	-ve	+ve

closer to 0

$$\frac{x(y^2-3) - y(x^2-3)}{(y^2-3)-(x^2-3)}$$ → PUT IN CALC IN TABLE FORM

itteration	-ve x_0 x_{n-1}	+ve x_1 x_n	x_2 x_{n+1}	$f(x_{n+1})$	$\lvert x_{n+2} - x_{n+1} \rvert$
1	1	2	1.66667 $^{5/3}$	-ve	0.3333
2	1.66667	2	1.72727 $^{19/11}$	-ve	0.272727
3	1.72727	2	1.73170 $^{71/41}$	-ve	0.26829
4	1.73170	2	1.7320	replace	stopping <0.01
⋮					
10					



for Calculator

$C = \dfrac{a f(b) \quad b f(a)}{f(b) - f(a)}$: function

Press calc

give A, B values

Press = value of C

Press = again value of f(c)

Question # 2

Use method of false position to find the real root of $f(x)$ in Question#1 with same interval and absolute approximate error.

Ans: $x = 0.64356$

Ans: $x = 0.65598$ (regula falsi method)

Question # 4

A data base file memory is related to time i.e,

$$N(t) = t^3 - 7t^2 + 14t + 10$$

Where $N(t)$ represent number of bytes and t represents time. Find the time at which file memory reach 16 bytes by using False Position method? Where $a = 0, b = 1$ and absolute true error < 0.001

Ans: $t = 0.58653$

Lagrange Interpolation

↳ no of points = eq^{n-1}

$P(x) = L(x_0)y_0 + L(x_1)\,y_1 \dots L(x_n)y_n$

x_0	x_1	x_2
y_0	y_1	y_2

hide and write the rest

2 Points

$L(x_0) = \dfrac{(x - x_1)}{(x_0 - x_1)}$ $\quad L(x_1) = \dfrac{x - x_0}{x_1 - x_0}$

3 Points

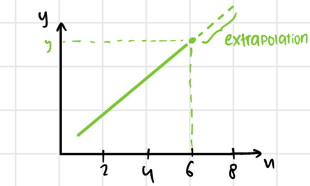
$L(x_0) = \dfrac{(x - x_1)(x - x_2)}{(x_0 - x_1)\,(x_0 - x_2)}$ $\quad L(x_1) = \dfrac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$

n Points

$L(x_n) = \dfrac{(x - x_0)(x - x_1)(x - x_2)\dots(x - x_n)}{(x_n - x_0)(x_n - x_1)(x_n - x_2)\cdots(x_n - x_{n-1})}$

$f(x) =$

x	3	5	7
y	a	6	8



Limitation

↳ Points can't be same (else denominator 0)

Q)

	x_0	x_1
x	2	5
y	4	1

$P(x) = \dfrac{(x-5)}{(2-5)}4 + \dfrac{(x-2)}{(5-2)}1$

$= \dfrac{-4x+20}{3} + \dfrac{x-2}{3}$

$= \dfrac{-3x+18}{3}$

$= 6 - x$

Q)

	x_0	x_1	x_2
x	2	2.75	4
$\frac{1}{x} = y$	0.5	0.364	0.25
	$y = \frac{1}{2}$	$\frac{1}{2.75}$	$\frac{1}{4}$

, find $f(x) = 1/x$ of second order

$P(x) = \underbrace{\dfrac{(x-2.75)(x-4)}{(2-2.75)(2-4)}}_{L(x_0)}\underbrace{(0.5)}_{y_0} + \underbrace{\dfrac{(x-2)(x-4)}{(2.75-2)(2.75-4)}}_{L(x_1)}\underbrace{(0.364)}_{y_1} + \underbrace{\dfrac{(x-2)(x-2.75)}{(4-2)(4-2.75)}}_{L(x_2)}\underbrace{(0.25)}_{y_2}$

$= \dfrac{x^2}{22} - \dfrac{35x}{88} + \dfrac{43}{44}$

$P(3) =$

(a) Use the numbers (called *nodes*) $x_0 = 2$, $x_1 = 2.75$, and $x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = 1/x$.

(b) Use this polynomial to approximate $f(3) = 1/3$.

Solution (a) We first determine the coefficient polynomials $L_0(x)$, $L_1(x)$, and $L_2(x)$. In

	x_0	x_1	x_2
x	2	2.75	4
$f(x) = \frac{1}{x}$	0.5	11/8	0.25

$$P(x) = \frac{(x-2.75)(x-4)}{(2-2.75)(2-4)}(0.5) + \frac{(x-2)(x-4)}{(2.75-2)(2.75-4}\left(\tfrac{11}{8}\right) + \frac{(x-2)(x-2.75)}{(4-2)(4-2.75}(0.25$$

$$= \frac{x^2 - 2.75 - 4 + 11}{-11/4}$$

Divided Difference

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots\ a_n(x-x_0)\dots(x-x_{n-1})$$

$$P_n(x) = f(x_0) + \Delta f(x_0)(x-x_0) + \Delta^2 f(x_0)(x-x_0)(x-x_1) + \Delta^3 f(x_0)(x-x_0)(x-x_1)(x-x_2) + \dots$$

STEPS

↳ make and fill table for $\Delta f(x)\dots$

↳ plug in values

Divided Difference Table

$$\Delta f(x) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$ .....
x_0	2	0.5		
			$\frac{0.364-0.5}{2.75-2} = -0.1813$	
x_1	2.75	0.364		$\frac{-0.0912 - -0.1813}{4-2} = 0.04505$
			$\frac{0.25-0.364}{4-2.75} = -0.0912$	
x_2	4	0.25		

$$P(x) = f(x_0) + \Delta f(x_0)(x-x_0) + \Delta^2 f(x_0)(x-x_0)(x-x_1)$$

$$= 0.5 - 0.1813(x-2) + 0.04505(x-2)(x-2.75)$$

↳ can choose either value

$$P(3) = 0.5 - 0.1813(3-2) + 0.04505(3-2)(3-2.75)$$

above ↑ forward

bottom ↑ backward

↳ step size should be same

↳ order maintain

$$x_0 + sh = \text{given}$$

(h → intervals)

Forward Difference → above

$$P_n(x) = f(x_0) + sh\,\Delta f(x_0) + s(s-1)h^2\,\Delta^2 f(x_0) + s(s-1)(s-2)h^3\,\Delta^3 f(x_0)\dots$$

	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$ .....
x_0	2	0.5		
			$\frac{0.364-0.5}{2.75-2} = -0.1813$	
x_1	2.75	0.364		$\frac{-0.0912 - -0.1813}{4-2} = 0.04505$
			$\frac{0.25-0.364}{4-2.75} = -0.0912$	
x_2	4	0.25		

Backward Difference → bottom

$$P_n(x) = f(x_n) + sh\,\Delta f(x_n) + s(s-1)h^2\,\Delta^2 f(x_n) + s(s-1)(s-2)h^3\,\Delta^3 f(x_n)\dots$$

	x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$ .....
x_0	2	0.5		
			$\frac{0.364-0.5}{2.75-2} = -0.1813$	
x_1	2.75	0.364		$\frac{-0.0912 - -0.1813}{4-2} = 0.04505$
			$\frac{0.25-0.364}{4-2.75} = -0.0912$	
x_2	4	0.25		

↳ sort it

SORT IT

↳ if not mentioned method

closer to x_0 → forward

closer to x_n → backward

MID 1

Bisection

n	x_1	x_2	mid	f(x)

Steps

- find no of iterations $n = \frac{\ln(\Delta x / \varepsilon)}{\ln 2}$ diff $(x_2 - x_1)$, stopping criteria
- Repeat till n iterations
- choose 2 boundaries +ve and -ve
- find mid point
- get new interval

Fixed Point

n	

Steps

- find 2 value where f(x) = +ve and -ve
- take its midpoint → x_0

OPTIONAL — these step reduce iterations

- make x subject
- check $|\phi'(x)| < 1$ → all +ve values — <1 = converging, >1 = diverging ?
- if all <1, opt smallest
- plug in function $x_n = \phi(x_{n-1})$
- keep iterating till sf same or stopping criteria

NEWTON RAPHSON

n	

Steps

- find 2 value where f(x) = +ve and -ve
- take its midpoint

OPTIONAL — these step reduce iterations

- find f'
- $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $\therefore f'(x_n) \neq 0$ → denominator never 0
- keep iterating till sf same or Es

Limitation: only usable is f' is readily available

step size same

Divided
Backward
Forward
ugly

FALSE POSITION / FALSI

n	a	b	c	f(c)

Steps

- $x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$
- if f(x) = +ve replace b
- if f(x) = -ve replace a

sign matters

- keep iterating till sf same or Es

SECANT

n	a	b	c	f(c)

Steps

- $x_{n+2} = \frac{x_n f(x_{n+1}) - x_{n+1} f(x_n)}{f(x_{n+1}) - f(x_n)}$
- keep replacing n → sign dosent matter
- keep iterating till sf same or Es

LANGRAGE

step size diff

Limitation
- Points cant be same (else denominator 0)

$P(x) = L(x_0)y_0 + L(x_1)y_1 \ldots L(x_n)y_n$

x_0	x_1	x_2
y_0	y_1	y_2

wide and write the rest

2 points

$L(x_0) = \frac{(x - x_1)}{(x_0 - x_1)}$ $\quad L(x_1) = \frac{x - x_0}{x_1 - x_0}$

3 points

$L(x_0) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$ $\quad L(x_1) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$

n points

$L(x_n) = \frac{(x - x_0)(x - x_1)(x - x_2) \ldots (x - x_n)}{(x_n - x_0)(x_n - x_1)(x_n - x_2) \cdots (x_n - x_{n-1})}$

absolute error (E_t)

$e_p = |p - p^*|$

p: true value; p^*: calculated value

relative error (E_a)

$\delta_p = \left| \frac{p - p^*}{p} \right| , \quad p \neq 0$

stopping criteria

$E_s = (0.5 \times 10^{2-n})\%$ — n sig fig

$E_a < E_s$

1. Binary machine numbers

$(-1)^s \, 2^{c-1023} (1 + f)$

- s: sign (MSB) left most bit — 1 bit
- c: characteristic — 11 bit
- f: mantisa — 52 bit

LOSS OF SF Remedy

when too many sf cancel

- rationalise
- using series expansion
- use trignometric identities
- reformulation

$$\begin{array}{r} 123.4567 \\ -\,123.4566 \\ \hline 000.0001 \end{array}$$

center difference/stirling Formula

↳ make divided difference table

↳ choose x_0 nearest center value

↳ Δ values x, avg, x, avg, x ....

↳ plug in formula

Formulas

$x_0 + sh = \text{given}$ (h: intervals)

$P_n(x) = f[x_0] + \frac{sh}{2}\left(f[x_{-1},x_0] + f[x_0,x_1]\right) + s^2h^2 f[x_{-1},x_0,x_1]$

↕ same

$P_n(x) = f(x_0) + sh\left(\frac{\Delta + \Delta}{2}\right) + s^2h^2\Delta^2 + s(s^2-1)h^3\left(\frac{\Delta^3+\Delta^3}{2}\right) + s^2(s^2-1)h^4\Delta^4$

Q) $f(1.5)$, $x_0 = 1.6$

$h = 1.3 - 1 = 0.3$

$1.6 + s(0.3) = 1.5$

$s = -1/3$

$f(1.5) =$

Divided Difference Table

i	x_i	$f[x_i]$	$f[x_{i-1},x_i]$	$f[x_{i-2},x_{i-1},x_i]$	$f[x_{i-3},\ldots,x_i]$	$f[x_{i-4},\ldots,x_i]$
0	1.0	0.7651977				
			−0.4837057			
1	1.3	0.6200860		−0.1087339		
			−0.5489460		0.0658784	
2	1.6	0.4554022	avg	−0.0494433	avg	0.0018251
			−0.5786120		0.0680685	
3	1.9	0.2818186		0.0118183		
			−0.5715210			
4	2.2	0.1103623				

9. a. Approximate $f(0.05)$ using the following data and the Newton forward-difference formula:

x	0.0	0.2	0.4	0.6	0.8
$f(x)$	1.00000	1.22140	1.49182	1.82212	2.22554

b. Use the Newton backward-difference formula to approximate $f(0.65)$.

c. Use Stirling's formula to approximate $f(0.43)$.

c. $f(0.43) \approx 1.53725$

	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
x_{-2}	0.0	1				
			$\frac{1.2-1}{0.2} = 1.107$			
x_{-1}	0.2	1.22140		$\frac{1.3-1.1}{0.4} = 0.61275$		
			$\frac{1.4-1.2}{0.2} = 1.3521$		$\frac{0.7-0.6}{0.6} = \frac{181}{800}$	
x_0	0.4	1.49182		$\frac{1.6-1.3}{0.4} = 0.7485$		$\frac{-0.29}{0.8} = \frac{119}{1920}$
			$\frac{1.8-1.4}{0.2} = 1.6515$		$\frac{0.9-0.7}{0.6} = \frac{331}{1200}$	
x_1	0.6	1.82212		$\frac{2-1.6}{0.4} = 0.914$		
			$\frac{2.2-1.8}{0.2} = 2.0171$			
x_2	0.8	2.22554				
			0.2	0.4	0.6	0.8

c) $f(0.43)$

$h = 0.2$, $x_0 = 0.4$

$x_0 + sh = \text{given}$ $\quad s = \frac{\text{given} - x_0}{h}$

$s = \frac{0.43 - 0.4}{0.2} = \frac{3}{20}$

$$f(0.43) = \overset{f(x_0)}{1.49182} + \overset{s}{\frac{3}{10}} \times \overset{h}{0.2}\left(\frac{\overset{\Delta}{1.6515} + \overset{\Delta}{1.3521}}{2}\right) + \overset{s^2}{\left(\frac{3}{10}\right)^2} \overset{h^2}{0.2^2}\overset{\Delta^2}{(0.7485)} + \overset{s}{\left(\frac{3}{10}\right)}\overset{s^2-1}{\left(\left(\frac{3}{10}\right)^2 - 1\right)} \overset{h^3}{0.2^3}\left(\frac{\frac{181}{800} + \frac{331}{1200}}{2}\right)$$

$$+ \overset{s^2}{\left(\frac{3}{10}\right)^2}\overset{s^2-1}{\left(\left(\frac{3}{10}\right)^2 - 1\right)} \overset{h^4}{(0.2)^4} \overset{\Delta^4}{\left(\frac{119}{1920}\right)}$$

$= 1.84425$

?

4.1 LECTURE 19, 20, 21

Numerical Differentiation

use any one
- ↳ differentiate
- ↳ gradient $\frac{f(b)-f(a)}{b-a}$ ($\frac{y_2-y_1}{x_2-x_1}$) → basically gradient
- ↳ identify x_0
- ↳ given formulas → all given in paper
- ↳ error: differential − given formula

FORMULAS

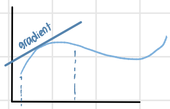
ans will be approx equal to derivative

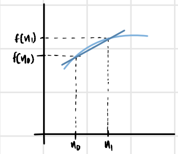
$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h}$$

h → interval

aim to get biggest error

Trancuiation Error ↑ $= \frac{|h|}{2(\xi)^2}$ ↓





Q) $x_0 = 1.8$, $f(x) = \ln x$, $h=0.1$, $h=0.05$, $h=0.01$, forward difference

$f'(1.8) = ?$

$h = 0.1$

formula → $f'(1.8) = \frac{f(1.8+h) - f(1.8)}{h} = \frac{f(1.9) - f(1.8)}{0.1} = 0.54067$

differentiate → $f'(1.8) = \frac{1}{x} = \frac{1}{1.8} = 0.555$

difference → − = 0.01488 error

almost close

Error → $\frac{0.1}{2(1.8)^2}$ ↓ $= \uparrow 0.0154$ error

(1.8 / 1.9)

to get bigger error, take 1.8

*Midpoint formulas always have least errors

Three Point Formula → forward

$$f'(x_0) = \frac{1}{2h}\left[3f(x_0) - 4f(x_0-h) + f(x_0-2h)\right]$$

Three Point Endpoint Formula → backward

$$f'(x_0) = \frac{1}{2h}\left[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right]$$

Three Point Midpoint Formula

$$f'(x_0) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right]$$

Five Point Midpoint Formula → best

$$f'(x_0) = \frac{1}{12h}\left[f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)\right]$$

Five Point Endpoint Formula

$$f'(x_0) = \frac{1}{12h}\left[-25f(x_0) + 48f(x_0+h) - 36f(x_0+2h) + 16(x_0+3h) - 3f(x_0+4h)\right]$$

Second Derivative Midpoint formula

$$f''(x_0) = \frac{1}{h^2}\left[f(x_0-h) - 2f(x_0) + f(x_0+h)\right]$$

Q) $f(x) = xe^x$, $h = 0.1$ interval , $f'(2) = ?$

ALWAYS ARRANGE IN ORDER FIRST!

	x	$f(x)$	
$x_0 - 2h / x_{-2}$	1.8	10.889365	$f(x_{-2}) / f(x_0 - 2h)$
$x_0 - h / x_{-1}$	1.9	12.703199	$f(x_{-1}) / f(x_0 - h)$
x_0	2.0	14.778112	$f(x_0) / f(x_0)$
$x_0 + h / x_1$	2.1	17.148957	$f(x_1) / f(x_0 + h)$
$x_0 + 2h / x_2$	2.2	19.855030	$f(x_2) / f(x_0 + 2h)$

Since Q said f(2.0) $\rightarrow x_0$

Direct Differentiation

$f'(x) = xe^x + e^x$

$f'(2) = 2e^2 + e^2$

$= 22.1671683$

$f''(x) = xe^x + e^x + e^x$

$f''(2) = 29.556224$

Three Point Formula

→ forward

$$f'(x_0) = \frac{1}{2h}\left[3f(x_0) - 4f(x_0 - h) + f(x_0 - 2h)\right]$$

$$f'(2) = \frac{1}{2(0.1)}\left[3(14.778112) - 4(12.703199) + 10.889365\right] = 22.056125$$

direct differentiation

$$22.056125 - 22.1671683 = 0.111$$

Three Point EndPoint Formula

→ backward

$$f'(x_0) = \frac{1}{2h}\left[-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)\right]$$

$$f'(2) = \frac{1}{2(0.1)}\left[-3(14.778112) + 4(17.148957) - 19.855030\right] = 22.03231$$

direct differentiation

$$22.03231 - 22.1671683 = 0.135$$

Three Point MidPoint Formula

$$f'(x_0) = \frac{1}{2h}\left[f(x_0 + h) - f(x_0 - h)\right]$$

$$= \frac{1}{2(0.1)}\left[17.148957 - 12.703199\right] = 22.22879$$

direct differentiation

$$22.22879 - 22.1671683 = 0.144$$

Five Point MidPoint Formula

→ best

$$f'(x_0) = \frac{1}{12h}\left[f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)\right]$$

$$= \frac{1}{12(0.1)}\left[10.889365 - 8(12.703199) + 8(17.148957) - 19.855030\right] = 22.16699917$$

direct differentiation

$$22.16699917 - 22.1671683 = 1.691 \times 10^{-4}$$

Second Derivative MidPoint formula

$$f''(x_0) = \frac{1}{h^2}\left[f(x_0 - h) - 2f(x_0) + f(x_0 + h)\right]$$

$$= \frac{1}{(0.1)^2}\left[12.703199 - 2(14.778112) + 17.148957\right] = 29.5932$$

direct differentiation

$$29.5932 - 29.556224 = 0.037$$

NUMERICAL INTEGRATION

↳ integrate

↳ identify x_0

↳ given formulas → all given in paper

↳ error: integral − given formula

bounded error / truncation error

if derivative dosent exist then exact answer

boundary

$0 < x < 2$

$0, 1, 2$ use these values for denominator

$$= \frac{h^3}{12} f''(\xi)$$

↳ use smallest denominator

Trapezoidal Rule

$n = 1$ ← no of nodes → $n = 2$

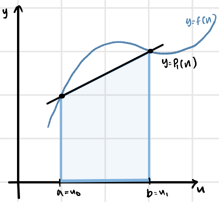
↳ 2 points → 2 nodes

↳ $h = b - a$

$x_0 = a$

$x_1 = b$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0) + f(x_1)\right]$$



Simpsons Rule

→ approximation more exact

↳ n 3 points → 3 nodes

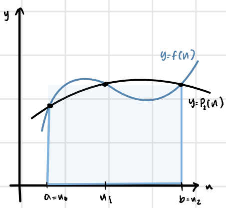
↳ $h = \frac{b-a}{2}$

$x_0 = a$

$x_1 = a + h$

$x_2 = b$

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$



as interval increases the error increases

↓

SOLUTION

COMPOSITE NUMERICAL INTEGRATION

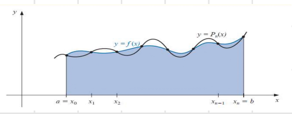
Closed Newton-Cotes Formulas

↳ called closed as end points of closed interval [a,b] are included as nodes

↳ $x_0 = a$, $x_n = b$

↳ $h = \frac{b-a}{n}$

same method for Simpsons and Trapezoidal NO changes



Open Newton-Cotes Formulas

↳ do not include endpoints as of [a,b] as nodes

↳ $x_0 = a + h$, $x_n = b - h$, $x_{-1} = a$, $x_{n+1} = b$

↳ $h = \frac{b-a}{n+2}$

same method

only use $x_0 = a + h$, $x_n = b - h$

$h = \frac{b-a}{n+2}$

for Simpsons and trapezoidal

Q) Solve $\int_{a\,0}^{b\,2} x^2\,dx$ using Trapezoial and Simpsons

Trapezoidal

↳ $h = b-a \quad x_0 = a \quad x_1 = b$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0) + f(x_1)\right]$$

$f(x) = x^2$

$f(a = x_0) = 0^2 = 0$

$f(b = x_1) = 2^2 = 4$

$h = 2$

$\frac{2}{2}[0+4]$

$= 4$

Simpsons

↳ $h = \frac{b-a}{2} \quad x_0 = a \quad x_1 = a+h \quad x_2 = b$

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$f(x) = x^2$

$f(a = x_0) = 0^2$

$f(a+h = x_1) = 0+1 = 1^2 = 1$

$f(b = x_2) = 2^2 = 4$

$h = \frac{2}{2} = 1$

$\frac{1}{3}[0 + 4(1) + 4]$

$= \frac{8}{3}$

Direct Integration

$\int_0^2 x^2$

$\left|\frac{x^3}{3}\right|_0^2$

$= \frac{8}{3}$

Q) Solve $\int_0^{\frac{\pi}{4}} \sin x\,dx$ using closed and open newtons cotes

closed

↳ $x_0 = a \quad , x_n = b$

↳ $h = \frac{b-a}{n}$

open

↳ $x_0 = a+h \quad , x_n = b-h \quad , x_{-1} = a \, , \, x_{n+1} = b$

↳ $h = \frac{b-a}{n+2}$

take $x_0 = a+h$,

4.4 LECTURE 25-27

Composite Numerical Integrations → most imp

↳ Subdivide the interval [a,b] into n sub intervals
always even for simpsons

ALWAYS IN RADIAN

Composite Simpsons rule

f(a), f(b) not repeated

↳ $h = \frac{b-a}{n}$ → no of sub intervals

$$\int y\,dx = \frac{h}{3}\Big(\underbrace{y_0}_{f(a)} + \underbrace{y_n}_{f(b)} + 2(\underbrace{y_2+y_4+y_6 \ldots y_{n-2}}_{even}) + 4(\underbrace{y_1+y_3+y_5+\ldots y_{n-1}}_{odd})\Big)$$

Composite Trapezoidal rule

$h = \frac{b-a}{n}$ → no of sub intervals

$$\int y\,dx = \frac{h}{2}\Big(\underbrace{y_0}_{f(a)} + \underbrace{y_n}_{f(b)} + 2(y_1+y_2+y_3+\ldots y_{n-1})\Big)$$

Q) Solve $\int_1^2 x \ln x \, dx$, $n=4$ using composite Simpsons and Trapezoidal

$= 0.6362943611$ → direct integration

$$h = \frac{2-1}{4} = 0.25$$

	x_0	x_1	x_2	x_3	x_n
x	1	1.25	1.5	1.75	2
$f(x)$	0				1.386
	y_0	y_1	y_2	y_3	y_n

Composite Simpsons Rule

$$\frac{0.25}{3}\Big[0 + 2\ln 2 + 2(1.5\ln 1.5) + 4(1.25\ln 1.25 + 1.75\ln 1.75)\Big]$$

$= 0.63630 - \underset{\text{direct integration}}{0.6362943611} = 1.549 \times 10^{-5}$

Composite Trapezoidals rule

$$\frac{0.25}{2}\Big(0 + 2\ln 2 + 2(1.25\ln 1.25 + 1.5\ln 1.5 + 1.75\ln 1.75)\Big)$$

$= 0.639904 - \underset{\text{direct integration}}{0.6362943611} = 3.606 \times 10^{-3}$

		$f(x)$	
0.1667 $y_0 = f(a)$	0	$e^0 - 1$	$= f(x_0) = y_0$
	0.166667	$e^{0.166667}$	$= f(x_1) = y_1$
			$f(x_2) = y_2$
			$f(x_3) = y_3$
			$f(x_4) = y_4$
			$f(x_5) = y_5$
$f(b) =$	1	$e^1 = 2.718\ldots$	$= f(x_n) = y_n$

Eulers Method

→ 4 marks → 2 wtg

↳ Diffrential equation

↳ ans in terms of constant value

↳ $h = \frac{b-a}{N}$ → usually given

$$\boxed{y(t_i+1) = y(t_i) + hf(t_i, y(t_i))}$$

e.g $y' = \underset{\downarrow\, y t_i}{y} - t^2 + 1$

Q) $y' = y - t^2 + 1$, $0 \leq t \leq 2$, $\overbrace{\underset{t}{y(0)} = \underset{y(t_i)}{0.5}}^{\text{initial value}}$, $h = 0.5$

$t = 0$ $y = 0.5$

$t = 2$ $y = ?$

$$y(t_i+1) = y(t_i) + hf(t_i, y(t_i))$$

$y(0.5) = \underset{y(t_i)}{0.5} + \underset{h}{0.5}(\underset{y}{0.5} - (0)^2 + 1) = 1.25$

$y(1) = 1.25 + 0.5(1.25 - (0.5)^2 + 1) = 2.25$

$y(1.5) = 2.25 + 0.5(2.25 - 1^2 + 1) = 3.375$

$y(2) = 3.375 + 0.5(3.375 - 1.5^2 + 1) = 4.4375$

	t	y(ti)	
	0	0.5	→ given
ti	0.5	1.25	→ ti
ti+1	1	2.25	→ ti
	1.5	3.375	→ ti
	2	4.4375	

Q2) $y' = 1 + \frac{y}{t}$, $1 \leq t \leq 2$, $y(1) = 2$, $h = 0.25$

IN CALCULATOR

$Y + 0.5(Y - X^2 + 1)$

$$y(t_i+1) = y(t_i) + hf(t_i, y(t_i))$$

$y(1.25) = 2 + 0.25(1 + 2/1) = 2.75$

$y(1.5) = 2.75 + 0.25(1 + 2.75/1.25) = 3.55$

$y(1.75) = 3.55 + 0.25(1 + 3.55/1.5) = 4.39$

$y(2) = 4.39 + 0.25(1 + 4.39/1.75) = 5.18$

	t	y(ti)
	1	2
	1.25	2.75
	1.5	3.55
ti →	1.75	4.39
ti+1	2	5.1

ASSIGNMENT

Task 1

at $n=50$, Langrage Polynomial

n	0	20	40	60	80	100
y	26	-48.6	61.6	-71.2	74.8	-75.2

$$P(n)= \underbrace{\frac{(n-20)(n-40)(n-60)(n-80)(n-100)}{(0-20)(0-40)(0-60)(0-80)(0-100)}}_{L(n_0)} \underbrace{(26)}_{y_0} + \underbrace{\frac{(n-0)(n-40)(n-60)(n-80)(n-100)}{(20-0)(20-40)(20-60)(20-80)(20-100)}}_{L(n_1)} \underbrace{(-48.6)}_{y_1} + \ldots$$

$$\frac{(n-0)(n-20)(n-60)(n-80)(n-100)}{(40-0)(40-20)(40-60)(40-80)(40-100)}(61.6) + \frac{(n-0)(n-20)(n-40)(n-80)(n-100)}{(60-0)(60-20)(60-40)(60-80)(60-100)}(-71.2)$$

$$\frac{(n-0)(n-20)(n-40)(n-60)(n-100)}{(80-0)(80-20)(80-40)(80-60)(80-100)}(74.8) + \frac{(n-0)(n-20)(n-40)(n-60)(n-80)}{(100-0)(100-20)(100-40)(100-60)(100-80)}(-75.2)$$

$$P(n) = -5.329\times10^{-6}n^5 + 0.001313n^4 - 0.1132n^3 + 3.985n^2 - 47.81n + 26 \ ?$$

$$P(50) = \frac{39}{128} + \frac{1215}{256} + \frac{1155}{32} - \frac{1335}{32} - \frac{935}{128} - \frac{141}{60} \ ?$$

$$P(50) = -8.760$$

Task 3

↳ Divided dif table

n	$f(n)$ (y)	$\Delta f(n)$	$\Delta^2 f(n)$	$\Delta^3 f(n)$	$\Delta^4 f(n)$	$\Delta^5 f(n)$
0	26					
		1.13				
20	48.6		-0.012			
		0.65		0.0001		
40	61.6		-0.0042		0	
		0.48		0		0
60	71.2		-0.0075		0	
		0.18		0		
80	74.8		-0.004			
		0.02				
100	75.2					

↳ Polynomial

$$P(n) = 26 + n(1.13) + n(n-20)(-0.012) + n(n-20)(n-40)(0.0001) + (0) + (0)$$

$$= 1.118n^2 - 1.353n + 26.23$$

INTERPOLATION LABS

Practice to make a polynmial in numpy.

```
import numpy as np
x=[1,2,3]
np.poly1d(x)
```

```
[ ] import numpy as np
    x=[1,2,3] # here 1,2,3 are coefficients of the polynomial in descending order
    Poly1=np.poly1d(x)
    print(Poly1)

    Poly2=np.poly1d(x,True) #another format to print polynomial
    print(Poly2)
```

```
   2
1 x + 2 x + 3
   3     2
1 x - 6 x + 11 x - 6
```

Function for getting Lagrange Polynmial

1. List item
2. List item

```
[ ]
    x = [0, 20,40,60, 80, 100]
    y = [26.0, -48.6, 61.6, -71.2, 74.8, -75.2]

    import numpy as np
    # Function to calculate Lagrange polynomial
    def lagrange_poly(x, y):
        n = len(x)
        p = np.poly1d(0.0)
        for i in range(n):
            L = np.poly1d(y[i])
            for j in range(n):
                if j != i:
                    L *= np.poly1d([1.0, -x[j]]) / (x[i] - x[j])
            p += L
        return p

    # Calculate Lagrange polynomial
    p = lagrange_poly(x, y)
    print(p)
```

```
            5           4          3         2
-5.329e-06 x + 0.001313 x - 0.1132 x + 3.985 x - 47.81 x + 26
```

For Interpolating at a specific point

```
[ ] # Interpolate at a specific point
    point = float(input("Enter x-coordinate to interpolate: "))
    interp_value = p(point)

    # Print Lagrange polynomial and interpolated value
    print("Lagrange polynomial is:")
    print(p)
    print("Interpolated value at x =", point, "is:", interp_value)
```

```
Enter x-coordinate to interpolate: 45
Lagrange polynomial is:
            5           4          3         2
-5.329e-06 x + 0.001313 x - 0.1132 x + 3.985 x - 47.81 x + 26
Interpolated value at x = 45.0 is: 31.29079589843832
```

TASK # 01

Solve the above problem manually (Hand written & verify the polynomial & Interpolated value at x = 50 Show all the necessary steps and submitted in the form of PDF along with the project file at GCR

Plotting of Lagrange Polynomial

```
[ ] import matplotlib.pyplot as plt
    xi=45
    yi=31.29079589843832
    p = lagrange_poly(x[0:6], y[0:6])
    print(p)
    xp=np.linspace(0,x[5],100)
    yp=p(xp)

    plt.plot(xp, yp, label='Lagrange Poly')
    plt.plot(xi, yi, 'bo', label='Interpolated Point')
    plt.plot(x[0:6], y[0:6], 'ro', label='Data Points')
    plt.xlabel('x')
    plt.ylabel('y')
    plt.legend()
    plt.show()
```

```
            5           4          3         2
```

Scipy Implimentation of Lagrange Polynomial

Instead we calculate everything from scratch, in scipy, we can use the lagrange function directly to interpolate the data. Let's see the above example

```
[ ] import numpy as np
    import matplotlib.pyplot as plt
    from scipy.interpolate import lagrange

    # Define the data points
    x = np.array([0, 20, 40, 60, 80, 100])
    y = np.array([26.0, -48.6, 61.6, -71.2, 74.8, -75.2])

    # Define the Lagrange Polynomial
    f = lagrange(x, y)

    # Find P(50) by evaluating the polynomial at x=50
    p_45 = f(45)
    print("P(45) =", p_45)

    # Plot the Lagrange Polynomial and the data points
    x_new = np.linspace(0, 100, 100)
    fig = plt.figure(figsize = (10,8))
    plt.plot(x_new, f(x_new), 'b', x, y, 'ro')
    plt.plot(45, p_45, 'go', markersize=10)
    plt.title('Lagrange Polynomial')
    plt.grid()
    plt.xlabel('x')
    plt.ylabel('y')
    plt.show()
```

Task # 02

Use the above code and apply the following alteration and show them along with plot

(i) Take input from user and show interpolation at that point along with its plot.

(ii) Also add a code that will display the polynomial too.

Solution of Task # 02

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.interpolate import lagrange

# Define the data points
x = np.array([0, 20, 40, 60, 80, 100])
y = np.array([26.0, -48.6, 61.6, -71.2, 74.8, -75.2])

# Define the Lagrange Polynomial
f = lagrange(x, y)

# Find P(50) by evaluating the polynomial at x=50
#p_45 = f(45)
#print("P(45) =", p_45)

# Interpolate at a specific point
point = float(input("Enter x-coordinate to interpolate: "))
p = f(point)
print("The value interpolated value is",p )

# Print the polynomial coefficients
print("Lagrange Polynomial:", np.poly1d(f).coefficients)

# Plot the Lagrange Polynomial and the data points
x_new = np.linspace(0, 100, 100)
fig = plt.figure(figsize = (10,8))
plt.plot(x_new, f(x_new), 'b', x, y, 'ro')
plt.plot(45, p, 'go', markersize=10)
plt.title('Lagrange Polynomial')
plt.grid()
plt.xlabel('x')
plt.ylabel('y')
plt.show()
```

Code for Newton divided difference Method

+ Code + Text

```
[ ] import numpy as np
def divided_difference_table(x, y):
    n = len(x)
    F = [[0] * n for i in range(n)]
    for i in range(n):
        F[i][0] = y[i]
    for j in range(1, n):
        for i in range(j, n):
            F[i][j] = (F[i][j-1] - F[i-1][j-1]) / (x[i] - x[i-j])
    return F
def newton_div_dif_poly(x,y,xi):
   F=divided_difference_table(x,y) # Saving divided difference in a variable F
   n=len(x)
   prod=np.poly1d(1)
   N=np.poly1d(F[0][0])
   for i in range(1,n):
     prod=np.poly1d(x[0:i],True)
     N+=np.poly1d(F[i][i]*(prod.c))

   print(N)
   print(N(xi))
   return

x = [0, 20,40,60, 80, 100]
y = [26.0, -48.6, 61.6, -71.2, 74.8, -75.2]
newton_div_dif_poly(x, y,45)
```

```
            5            4          3         2
-5.329e-06 x + 0.001313 x - 0.1132 x + 3.985 x - 47.81 x + 26
31.29079589843672
```

Task # 03 (A)

Use the above code by adding divided difference table code i.e. it will show divided difference table.

Task # 03 (B)

With the help of **"pandas"** as shown at the starting of this lab session, read code from provided csv file & Write a code for Newton's forward divided difference. Print the polynomial and plot the interpolating point too.

Task # 03 (C)

Do part B manually (Mentioned all steps and verify the result.

3 (A) Solution)

```
[ ] from tabulate import tabulate
    import numpy as np

    def divided_difference_table(x, y):
        n = len(x)
        F = [[0] * n for i in range(n)]
        for i in range(n):
            F[i][0] = y[i]
        for j in range(1, n):
            for i in range(j, n):
                F[i][j] = (F[i][j-1] - F[i-1][j-1]) / (x[i] - x[i-j])
        return F

    def print_table(x, y):
        F = divided_difference_table(x, y)
        headers = ['x', 'y'] + ['F[{},{}]'.format(i, j) for i in range(1, len(x)) for j in range(i)]
        data = [[x[i], y[i]] + [F[i][j] for j in range(1, i+1)] + [''] * (len(x) - i - 1) for i in range(len(x))]
        print(tabulate(data, headers=headers))

        # Compute the polynomial
        n = len(x)
        prod = np.poly1d(1)
        N = np.poly1d(F[0][0])
        for i in range(1,n):
            prod = np.poly1d(x[0:i], True)
            N += np.poly1d(F[i][i]*(prod.c))
        print('Polynomial: \n', N)

    x = [0, 20, 40, 60, 80, 100]
    y = [26.0, -48.6, 61.6, -71.2, 74.8, -75.2]
    print_table(x, y)
```

```
  x      y  F[1,0]                F[2,0]                F[2,1]                 F[3,0]                  F[3,1]
---  -----  --------------------  --------------------  ---------------------  ----------------------  ----------------------
  0   26
 20  -48.6  -3.7299999999999995
 40   61.6  5.51                  0.23099999999999996
 60  -71.2  -6.640000000000001    -0.30375              -0.008912499999999999
 80   74.8  7.3                   0.34850000000000003   0.010870833333333333   0.0002472916666666666
100  -75.2  -7.5                  -0.37                 -0.011975000000000001  -0.0002855729166666667  -5.3286458333333335e-06
Polynomial:
            5            4          3         2
-5.329e-06 x + 0.001313 x - 0.1132 x + 3.985 x - 47.81 x + 26
```

3 (B)

```
[1] import pandas as pd
    from tabulate import tabulate
    import numpy as np

    # Read X and Y values from CSV file
    df = pd.read_csv("interpolation.csv")
    x = df["x"].values
    y = df["y"].values

    # Divided Difference Table
    n = len(x)
    F = [[0] * n for i in range(n)]
    for i in range(n):
        F[i][0] = y[i]
    for j in range(1, n):
         for i in range(j, n):
             F[i][j] = (F[i][j-1] - F[i-1][j-1]) / (x[i] - x[i-j])

     # Print the divided difference table using tabulate
     headers = ["X"] + ["f[x{0}]".format(i) for i in range(1, n)]
     table = []
     for i in range(n):
         row = [x[i]] + F[i]
         table.append(row)
     print(tabulate(table, headers, floatfmt=".4f"))

     # Newton's Divided Difference Polynomial
     def newton_div_dif_poly(x,y,xi):
         F = [[0] * n for i in range(n)]
         for i in range(n):
             F[i][0] = y[i]
         for j in range(1, n):
             for i in range(j, n):
                 F[i][j] = (F[i][j-1] - F[i-1][j-1]) / (x[i] - x[i-j])

         prod = np.poly1d(1)
         N = np.poly1d(F[0][0])
         for i in range(1,n):
             prod = np.poly1d(x[0:i], True)
             N += np.poly1d(F[i][i]*(prod.c))

         print("Newton's Divided Difference Polynomial: ", N)
         print("Interpolated value at xi = {0}: ".format(xi), N(xi))
         return

     # Sample usage
     newton_div_dif_poly(x, y, 45)
```

we have actual solution meaning when we solve differential eq the y give we just have to aply rk mehtod to find y value

$y' = 4e^{0.8t} - 0.5y$

$t_i = 0,\ y_i = 2$

$\rightarrow k_1 = f(t_i, y_2) = 4e^{0.8(0)} - 0.5(2)$

$k_1 = 3$

$\rightarrow k_2 = f(t_i + \frac{h}{2}, y_2 + \frac{1}{2}k_1 h)$

$= f(0.5, 2 + \frac{3(1)}{2})$

$= 4e^{0.8(0.5)} - 0.5(3.5)$

$= 4.21729$

$\rightarrow k_3 = f(t_i + \frac{h}{2}, 2 + \frac{4.21798}{2}(1))$

$= f(0.5, 4.1086)$

$k_3 = 3.9129$

increment function

$y_{i+1} = y_i + \phi h$

$\phi = a_1 k_1 + a_2 k_2 + \dots + a_n k_n$

constants

$k_1 = f(t_i, y_i)$

$k_2 = f(t_i, P_1 h, y_i + q_{11} k_1 h)$

$k_3 = f(t_i, P_2 h, y_i + q_{21} k_1 h + q_{22} k_2 h)$

second order Runge-Kutta method

$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2) h$

$k_1 = f(t_i, y_i)$

$k_2 = f(t_i, P_1 h, y_i + q_{11} k_1 h)$

$a_1 + a_2 = 1$

$a_2 P_1 = 1/2$

$a_2 P_{11} = 1/2$

$a_1 = 1 - a_2$

$P_1 = q_{11} = \frac{1}{2a_2}$

RUNGE KUTTA METHODS

↳ used to find increment k of y corresponding to an increment h of x

$$\frac{dy}{dx} = f(x,y) \quad , \quad y(x_0) = y_0$$

RUNGE KUTTA FORMULAS

1st order

Eulers method

$$y_{n+1} = y_n + h f(x_n, y_n)$$

2nd order

Heun Method (without iteration)

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left[x_n + h, y_n + k_1 h\right)$$

$$y_{n+1} = y_n + \tfrac{1}{2}h\left[k_1 + k_2\right]$$

Mid Point method

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left[x_n + \tfrac{h}{2}, y_n + k_1 \tfrac{h}{2}\right)$$

$$y_{n+1} = y_n + k_2 h$$

4th order

most popular

$$k_1 = f(x_0, y_0)$$

$$k_2 = f\left(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_1}{2}h\right)$$

$$k_3 = f\left(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_2}{2}h\right)$$

$$k_4 = f(x_0 + h, y_0 + k_3 h)$$

$$y = y_0 + \tfrac{1}{6}h\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

Summary of all formulas

Euler's Method: $y_{i+1} = y_i + hf(t_i, y_i)$	Heun's Method: (Special Case of RK-2 methods) $y_{i+1} = y_i + \frac{h}{2}(f(t_i, y_i) + f(t_{i+1}, y_i + hf(t_i, y_i))$
Midpoint Method: (Special case of RK-2 methods) $y_{i+1} = y_i + h\left(f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}f(t_i, y_i)\right)\right)$	RK-4 Classical approach method (special case of RK-4 Methods) $y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$ Where, $k_1 = f(t_i, y_i)$ $k_2 = f\left(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$ $k_3 = f\left(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$ $k_4 = f(t_i + h, y_i + k_3 h)$

Q) $y' = te^{3t} - 2y$, $0 \le t \le 1$

$h = 0.5$, $t = 0 \to y = 0$, $t = 1 \to y = ?$

actual solution $= y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$

$y(1) = 3.21909$

a) Heun's Method

$k_1 = f(x_n, y_n)$

$k_2 = f[x_n + h, y_n + k_1 h)$

$y_{n+1} = y_n + \frac{1}{2}h[k_1 + k_2]$

$k_1 = f(0,0) = 0$

$k_2 = f(0.5, 0) \cdot 0.5e^{3(0.5)} - 2(0) = 2.2408$

$y = 0 + \frac{1}{2}(0.5)[0 + 2.2408] = 0.56021$

$y(0.5) = 0.56021$

iteration 1

$k_1 = (0.5, 0.56021) = 0.5e^{3(0.5)} - 2(0.56021) = 1.1204$

$k_2 = f(1, 0.56021 + 1.1204(0.5))$

$= f(1, 1.12041) = e^{3} - 2(1.12041) = 17.8447$

$y = 0.56021 + \frac{1}{2}(0.5)(1.1204 + 17.8447) = 5.3015$

$y(1) = 5.3015$

iteration 2

Error $= 5.3015 - 3.21909 = 2.08241$

b) Midpoint

$k_1 = f(x_n, y_n)$

$k_2 = f[x_n + \frac{h}{2}, y_n + k_1\frac{h}{2})$

$y_{n+1} = y_n + k_2 h$

$k_1 = f(0,0) = 0$

$k_2 = f(0 + \frac{0.5}{2}, 0 + 0(\frac{0.5}{2}))$

$= f(0.25, 0) = 0.25e^{3(0.25)} - 2(0) = 0.52925$

$y = 0 + 0.52925(0.5) = 0.264625$

$y(0.5) = 0.264625$

iteration 1

$k_1 = f(0.5, 0.264625) = 0.5e^{3(0.5)} - 2(0.264625) = 1.71159$

$k_2 = f(0.5 + \frac{0.5}{2}, 0.264625 + (1.71195)(\frac{0.5}{2}))$

$= f(0.75, 0.6925) = 0.75e^{3(0.75)} - 2(0.6925)) = 5.73075$

$y = 0.264625 + 5.73075(0.5) = 3.13000$

$y(1) = 3.13000$

iteration 2

Error $= 3.13000 - 3.21909 = 0.08909$

c) RK4 method

$k_1 = f(x_0, y_0)$
$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1 h}{2}\right)$
$k_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2 h}{2}\right)$
$k_4 = f(x_0 + h, y_0 + k_3 h)$
$y = y_0 + \frac{1}{6}h[k_1 + 2k_2 + 2k_3 + k_4]$

iteration 1

$k_1 = f(0, 0) = 0$

$k_2 = f\left(0 + \frac{0.5}{2}, 0 + \frac{0}{2}(0.5)\right)$

$= f(0.25, 0) = 0.25e^{3(0.25)} - 2(0) = 0.52925$

$k_3 = f\left(0 + \frac{0.5}{2}, 0 + \frac{0.52925(0.5)}{2}\right)$

$f(0.25, 0.1323125) = 0.25e^{3(0.25)} - 2(0.1323125) = 0.264625$

$k_4 = f(0.5, 0 + 0.264625(0.5)) =$

$(0.5, 0.1323125) = 0.5e^{3(0.5)} - 2(0.1323125) = 1.97622$

$y = 0 + \frac{1}{6}(0.5)[0 + 2(0.52925) + 2(0.264625) + 1.97622) = 0.2969915$

$y(0.5) = 0.2969915$

iteration 2

$k_1 = (0.5, 0.2969915) = 0.5e^{3(0.5)} - 2(0.2969915) = 1.646849$

$k_2 = \left(0.5 + \frac{0.5}{2}, 0.2969915 + \frac{1.646849}{2}(0.5)\right)$

$= (0.75, 0.7087098) = 0.75e^{3(0.75)} - 2(0.7087098) = 5.69838211$

$k_3 = \left(0.5 + \frac{0.5}{2}, 0.2969915 + \frac{5.69838211}{2}(0.5)\right)$

$= (0.75, 2.394570) = 0.75e^{3(0.75)} - 2(2.394570) = 2.3266608$

$k_4 = (0.5 + 0.5, 0.2969915 + 2.3266608(0.5))$

$= (1, 1.4603279) = e^{3} - 2(1.4603279) = 17.1648811$

$y = 0.2969915 + \frac{1}{6}(0.5)[1.646849 + 2(5.69838211) + 2(2.3266608) + 17.1648811 = 2.905151$

$y(1) = 2.905151$

Error $= 2.905151 - 3.21909 = 0.313939$

Matrix Factorization (LU Decomposition)

Directs Methods
- assume exact solution exists
- final number of steps give precise solution

Gaus Jordan → simple and accurate

Gaus elimination → calculatively efficient

Cholesky → in case of non singular symmetric matrix

Iterative Methods
- start from some inial appoximation value
- converge to exact solution of a system

Jacobi

Gauss-Seidel

SOR (Succesive Over Relaxation)

1. $y = Ux$, $Ly = b$ → $O(n^2)$ operations
2. $Ux = y$ → $O(n^2)$

$Ax = b$ → $O(2n^2)$

LU DECOMPOSITION

1. determiant $\neq 0$
2. $Ax = B$
3. $A = \underset{\text{multipliers of gaus elimination}}{L}\ \underset{\text{gaus Elimination}}{U}$

$$\begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x & x & 1 \end{bmatrix}\begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix} \longrightarrow \text{Doolittle form}$$

any real number

4. $LY = B$ and solve simultaneously
5. $UX = Y$ and solve simultaneously

Question

①

$$\underset{A}{\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix}}\underset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}}$$

② Forward Gaus Elimination

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \quad R_2 - R_1$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 2 & 3 & 1 \end{bmatrix} \quad R_3 - 2R_1$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 1 & 3 \end{bmatrix} \quad R_3 + \tfrac{1}{3}R_2$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -\tfrac{1}{3} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & \tfrac{13}{3} \end{bmatrix}$$

no need to make diagonals 1

OR

② Dolittle form

dosent need to be Positive definite

$$\underset{A}{\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}}\underset{U}{\begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & i \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} d & e & f \\ ad & ae+g & af+h \\ bd & be+cg & bf+ch+i \end{bmatrix}$$

$\boxed{d=1}$ $\boxed{e=1}$ $\boxed{f=-1}$ $ad=1$ $\boxed{a=1}$ $be+cg=3$ $\boxed{c=-1/3}$

$ae+g=-2$ $\boxed{g=-3}$ $af+h=3$ $\boxed{h=4}$ $bd=2$ $\boxed{b=2}$ $bf+ch+i=1$ $\boxed{i=13/3}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1/3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & 13/3 \end{bmatrix}$$

③

$$L = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -\tfrac{1}{3} & 1 \end{bmatrix}}\underset{Y}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}}$$

Solve simultaneously $\longrightarrow$ forward substitution

$y_1 = 4,\ y_2 = -10,\ y_3 = \frac{-13}{3}$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ -13/3 \end{bmatrix}$$

④

$$\underset{U}{\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & 13/3 \end{bmatrix}}\underset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{Y}{\begin{bmatrix} 4 \\ -10 \\ -13/3 \end{bmatrix}}$$

Solve simultaneously $\longrightarrow$ backward substitution

$x_1 = 1,\ x_2 = 2,\ x_3 = -1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Permutation matrix → in LA we called it elementary matrix

$A = n \times n$

each row has one 1

each column has one 1

↳ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

↳ rearranging permutation matrice changes order not values

↳ $A = (P^t L) U$

Q) $A = \begin{bmatrix} 0 & 2 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ → identity matrice

↳ doesnt have leading 1s

↳ first entry should be non zero

so swap R_1 and R_2

$\begin{bmatrix} 5 & 6 & 7 \\ 0 & 2 & 4 \\ 8 & 9 & 10 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ → Permutation matrice

Ex 6.5

4. Consider the following matrices. Find the permutation matrix P so that PA can be factored into the product LU, where L is lower triangular with 1s on its diagonal and U is upper triangular for these matrices.

a. $A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & -1 & 4 \end{bmatrix}$ **b.** $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 7 \\ -1 & 2 & 5 \end{bmatrix}$

a) $A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & -1 & 4 \end{bmatrix}$ $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 1 & -1 & 4 \end{bmatrix} R_3 - R_1$

$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

b). $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 7 \\ -1 & 2 & 5 \end{bmatrix} 2R_1 - R_2$

$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & -9 \\ -1 & 2 & 5 \end{bmatrix} R_1 + R_3$

$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & -9 \\ 0 & 4 & 4 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 4 \\ 0 & 0 & -9 \end{bmatrix}$ $U =$

5. Factor the following matrices into the LU decomposition using the LU Factorization Algorithm with $l_{ii} = 1$ for all i.

a. $\begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{bmatrix}$ **b.** $\begin{bmatrix} 1.012 & -2.132 & 3.104 \\ -2.132 & 4.096 & -7.013 \\ 3.104 & -7.013 & 0.014 \end{bmatrix}$

convert to U

$A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{bmatrix} R_2 - 1.5R_1$ $U = \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}$ $L = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x & x & 1 \end{bmatrix}$

$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 4.5 & 7.5 \\ 3 & 3 & 5 \end{bmatrix} R_3 - 1.5R_1$

$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 4.5 & 7.5 \\ 0 & 4.5 & 3.5 \end{bmatrix} R_3 - R_2$

$U = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 4.5 & 7.5 \\ 0 & 0 & -4 \end{bmatrix}$ $L = \begin{bmatrix} 1 & 0 & 0 \\ 1.5 & 1 & 0 \\ 1.5 & 1 & 1 \end{bmatrix}$

7. Modify the *LU* Factorization Algorithm so that it can be used to solve a linear system, and then solve the following linear systems.

a.
$$2x_1 - x_2 + x_3 = -1,$$
$$3x_1 + 3x_2 + 9x_3 = 0,$$
$$3x_1 + 3x_2 + 5x_3 = 4.$$

A X = B

b.
$$1.012x_1 - 2.132x_2 + 3.104x_3 = 1.984,$$
$$-2.132x_1 + 4.096x_2 - 7.013x_3 = -5.049,$$
$$3.104x_1 - 7.013x_2 + 0.014x_3 = -3.895.$$

steps done above

$$U = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 4.5 & 7.5 \\ 0 & 0 & -4 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 1.5 & 1 & 0 \\ 1.5 & 1 & 1 \end{bmatrix}$$

$LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1.5 & 1 & 0 \\ 1.5 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$$

$y_1 \; -1$

$y_2 : 1.5$

$y_3 \; 4$

$UX = Y$

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 4.5 & 7.5 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1.5 \\ 4 \end{bmatrix}$$

$x_1 \; 1$

$x_2 : 2$

$x_3 \; -1$

9. Obtain factorizations of the form $A = P^tLU$ for the following matrices.

a. $A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ 2

b. $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 3 \\ 2 & -1 & 4 \end{bmatrix}$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

LECTURE 38-40

DIAGONALLY DOMINANT

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\left.\begin{array}{l} |a_{11}| \geq |a_{12}| + |a_{13}| \\ |a_{22}| \geq |a_{21}| + |a_{23}| \\ |a_{33}| \geq |a_{31}| + |a_{32}| \end{array}\right\} \text{if all 3 satisfy then Diagonally Dominant}$$

Positive Definite matrices $\xrightarrow{\text{if true}}$ LD^TL, LL^T

$$\left.\begin{array}{l} \hookrightarrow \text{leading principal submatrices determiants} > 0 \\ \hookrightarrow u^TAu > 0 \end{array}\right\} \text{check either}$$

if diagonally dominant then also positive definite

Using determiants

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\left.\begin{array}{l} \begin{vmatrix} 2 \end{vmatrix} = 2 > 0 \\ \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \; 3 > 0 \\ \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} \; 2(4-1)+1(-2) = 4 > 0 \end{array}\right\} \text{all determiants} > 0 \text{ hence positive definite}$$

Using u^TAu

$$\mathbf{x}'A\mathbf{x} = [x_1, x_2, x_3] \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [x_1, x_2, x_3] \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{bmatrix}$$

$$= 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2.$$

$$= 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2.$$

Rearranging the terms gives

$$\mathbf{x}'A\mathbf{x} = x_1^2 + (x_1^2 - 2x_1x_2 + x_2^2) + (x_2^2 - 2x_2x_3 + x_3^2) + x_3^2$$
$$= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2,$$

which implies that

$$x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 > 0$$

unless $x_1 = x_2 = x_3 = 0$.

symmetric

$A = A^T$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

hence symmetric

singular

$\det = 0$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad d = 2(4-1)+1(-2) = 4 \neq 0$$

hence not singular

Positive definite

LDL^t Factorization

$$\underset{A}{\begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}} \underset{L^T}{\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix} = \begin{bmatrix} d_1 & d_1a & d_1f \\ d_1a & d_2+d_1a^2 & d_2c+d_1ab \\ d_1b & d_1ab+d_2c & d_1b^2+d_2c^2+d_3 \end{bmatrix}$$

$\boxed{d_1 = 4}$ $\quad d_1a = -1$, $\boxed{a = -1/4}$ $\quad d_1b = 1$, $\boxed{b = 1/4}$ $\quad d_2 + d_1a^2 = 4.25$, $\boxed{d_2 = 4}$

$d_2c + d_1ab = 2.75$, $\boxed{c = 0.75}$ $\quad d_1b^2 + d_2c^2 + d_3 = 3.5$, $\boxed{d_3 = 1}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.25 & 0.75 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

LL^T Factorization

$$\underset{A}{\begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix}} = \underset{L}{\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}} \cdot \underset{L^T}{\begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{bmatrix}}$$

$$\begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix} = \begin{bmatrix} a^2 & ab & ad \\ ab & b^2+c^2 & bd+ce \\ ad & bd+ec & d^2+e^2+f^2 \end{bmatrix}$$

$a^2 = 4$, $\boxed{a = 2}$ $\quad ab = 10$, $b = 10/2$, $\boxed{b = 5}$ $\quad ad = 8$, $d = 10/2$, $\boxed{d = 4}$

$b^2 + c^2 = 26$, $c^2 = 26 - 25$, $\boxed{c = 1}$ $\quad bd + ce = 26$, $e = 26 - 20$, $\boxed{e = 6}$ $\quad d^2 + e^2 + f^2 = 61$, $f^2 = 61 - 16 - 36$, $\boxed{f = 3}$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 6 & 3 \end{bmatrix}$$

Band Matrices

↳ $n \times n$ matrice

↳ $p > 1$, $q < n$

bandwith

$$W = p + q - 1$$

no of diagonals above ↳ including main diagonal ↲ no of diagonals below

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & -5 & -6 \end{bmatrix}$$

$p = 2$

$q = 2$

$W = 2 + 2 - 1 = 3$

Tridiagonal matrices

↳ matrices of bandwith 3

↳ $p = 2$, $q = 2$

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \ddots & \vdots \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{n-1,n} \\ 0 & \cdots & 0 & a_{n,n-1} & a_{nn} \end{bmatrix}. \quad L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & l_{n,n-1} & l_{nn} \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & u_{12} & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & u_{n-1,n} \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

CHOLESKY

1. A is symmetric
2. A is positive definite

both conditions need to be satisfied

1. $Ax = B$
2. Check symmetry
3. Check positive definite
4. $A = LL^T$
5. $LY = B$
6. $L^TX = Y$

Question

$$\overset{A}{\begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix}} \overset{X}{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}} = \overset{B}{\begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}}$$

① check symmetry

$$A^T = \begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix}$$

hence symmetric

② check positive definite

$$|4| = 4 > 0$$

$$\begin{vmatrix} 4 & 10 \\ 10 & 26 \end{vmatrix} > 0$$

$$\begin{vmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{vmatrix} > 0$$

all determiants > 0 hence positive definite

③ $A = LL^T$

$$\begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix} = \begin{bmatrix} a^2 & ab & ad \\ ab & b^2+c^2 & bd+ce \\ ad & bd+ec & d^2+e^2+f^2 \end{bmatrix}$$

$a^2 = 4$ $\boxed{a = 2}$ $ab = 10$ $\boxed{b = 5}$ $ad = 8$ $\boxed{d = 4}$

$b^2+c^2 = 26$ $\boxed{c = 1}$ $bd+ce = 26$ $\boxed{e = 6}$ $d^2+e^2+f^2 = 61$ $\boxed{f = 3}$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 6 & 3 \end{bmatrix}, \quad L^T = \begin{bmatrix} 2 & 5 & 4 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

④ $LY = B$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 6 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}$$

solve simultaneously

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 18 \\ 6 \end{bmatrix}$$

⑤ $L^TX = Y$

$$L^T = \begin{bmatrix} 2 & 5 & 4 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 18 \\ 6 \end{bmatrix}$$

solve simultaneously

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 6 \\ 2 \end{bmatrix}$$

Positive definite

1. determinant $\neq 0$
2. $AX = B$
3. $A = L\ U$

$$\begin{bmatrix} \times & 0 & 0 \\ \times & \times & 0 \\ \times & \times & \times \end{bmatrix}\begin{bmatrix} 1 & \times & \times \\ 0 & 1 & \times \\ 0 & 0 & 1 \end{bmatrix}$$ → L and U are diff in this method

4. $LY = B$ and solve simultaneously
5. $UX = Y$ and solve simultaneously

Crout Factorization

↳ L and U are diff in this method

↳ the rest same as LU Decomposition

Question

1.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 18 \\ 22 \end{bmatrix}$$

2. $A = LU$

$$\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}\begin{bmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} a & ag & ah \\ b & bg+c & bh+ci \\ d & dg+e & dh+ie+f \end{bmatrix}$$

$\boxed{a=1}$ $ag=2$ $\boxed{g=2}$ $ah=3$ $\boxed{h=3}$ $\boxed{b=2}$ $dg+e=2$ $\boxed{e=-4}$

$\boxed{d=3}$ $bg+c=5$ $\boxed{c=1}$ $bh+ci=2$ $\boxed{i=-4}$ $dh+ie+f=5$ $\boxed{f=-20}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -4 & -20 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

3. $LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -4 & -20 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 18 \\ 22 \end{bmatrix}$$

solve simultaneously

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -10 \\ 3 \end{bmatrix}$$

4. $UX = Y$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -10 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 25 \\ -10 \\ 3 \end{bmatrix}$$

6.6

1. Determine which of the following matrices are (i) symmetric, (ii) singular, (iii) strictly diagonally dominant, (iv) positive definite.

a. $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ **b.** $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 4 \end{bmatrix}$ $A^T = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ not symmetric

strickly diognal

$|2| > 0$

$\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix}$ $6-1 = 4 > 0$

$\begin{vmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 4 \end{vmatrix} = 2(12)$
$24 > 0$ Positive definite

not singular

3. Use the LDL^t Factorization Algorithm to find a factorizaton of the form $A = LDL^t$ for the following matrices:

a. $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ **b.** $A = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$

$$\underbrace{A}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}}_{L^t}$$

$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \begin{bmatrix} d_1 & d_1 a & d_1 b \\ 0 & d_2 & d_2 c \\ 0 & 0 & d_3 \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} d_1 & d_1 a & d_1 b \\ d_1 a & d_1 a^2 + d_2 & d_1 ab + d_2 c \\ d_1 b & d_1 ab + d_2 c & d_1 b^2 + d_2 c^2 + d_3 \end{bmatrix}$$

$\boxed{d_1 = 2}$ $2a = -1$ $\boxed{a = -1/2}$ $2b = 0$ $\boxed{b = 0}$

$2(1/4) + d_2 = 2$ $\boxed{d_2 = 3/2}$

$2(-1/2)(0) + (3/2)c = -1$ $\boxed{c = -2/3}$

$2(0) + \cancel{\frac{3}{2}}\left(\cancel{\frac{4}{9}}\right) + d_3 = 2$

$\boxed{d_3 = 4/3}$

$\Rightarrow$ $L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}$

5. Use the Cholesky Algorithm to find a factorization of the form $A = LL^t$ for the matrices in Exercise 3.

$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ $A^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ symmetric

$|2| > 0$

$4 - 1 = 3 > 0$

$2(4-1) + (-2) = 6 - 2 = 4 > 0$

hence positive definite

$$\underset{L}{\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}} \cdot \underset{L^T}{\begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{bmatrix}}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} a^2 & ab & ad \\ ab & b^2+c^2 & bd+ce \\ ad & bd+ec & d^2+e^2+f^2 \end{bmatrix}$$

$a^2 = 2$ $\sqrt{2}\, b = -1$ $c =$

$\boxed{a = \sqrt{2}}$ $b = \boxed{-1/\sqrt{2}}$

11. Use Crout factorization for tridiagonal systems to solve the following linear systems.

a.
$$
\begin{aligned}
x_1 - x_2 \qquad\quad &= 0,\\
-2x_1 + 4x_2 - 2x_3 &= -1,\\
- x_2 + 2x_3 &= 1.5.
\end{aligned}
$$

b.
$$
\begin{aligned}
3x_1 + x_2 \qquad\quad &= -1,\\
2x_1 + 4x_2 + x_3 &= 7,\\
2x_2 + 5x_3 &= 9.
\end{aligned}
$$

JACOBIS METHOD

1. check diagonally dominant → if not then we cant solve
2. input formula and itterate
$$x_1^{(k+1)} = \frac{1}{a_{11}}\left(d_1 - a_{12}x_2^k - a_{13}x_3^k\right)$$
$$x_2^{(k+1)} = \frac{1}{a_{22}}\left(d_2 - a_{21}x_1^k - a_{23}x_3^k\right)$$
$$x_3^{(k+1)} = \frac{1}{a_{33}}\left(d_3 - a_{31}x_1^k - a_{32}x_2^k\right)$$
3. keep iterating till ans same/close (x_1, x_2, x_3)
   ↳ may stop after 5 or 6 iteration if answer dosent come ??

Question

$4x_1 + x_2 - x_3 = 5$
$-x_1 + 3x_2 - x_3 = -4$
$2x_1 + 2x_2 + 5x_3 = 1$

① check diagonally dominant

$4 > 2,\ 3 > 2,\ 5 > 4$

hence diagonally dominant

② get equation

$x_1^{k+1} = \frac{1}{4}(5 - x_2^k + x_3^k)$
$x_2^{k+1} = \frac{1}{3}(-4 + x_1^k + x_3^k)$
$x_3^{k+1} = \frac{1}{5}(1 - 2x_1^k - 2x_2^k)$

↳ make subject then add the extras

iteration 1 $k=0,\ x_1^0, x_2^0, x_3^0 = 0$

$x_1^1 = \frac{1}{4}(5 - x_2^0 + x_3^0) =$
$= \frac{1}{4}(5 - 0 + 0) = 5/4$

$x_2^1 = \frac{1}{3}(-4 + x_1^0 + x_3^0)$
$= \frac{1}{3}(-4 + 0 + 0) = -4/3$

$x_3^1 = \frac{1}{5}(1 - 2x_1^0 - 2x_2^0)$
$= \frac{1}{5}(1 - 0 - 0) = 1/5$

iteration 2 $k=1,\ x_1^1 = 5/4,\ x_2^1 = -4/3,\ x_3^1 = 1/5$

$x_1^2 = \frac{1}{4}(5 - x_2^1 + x_3^1)$
$= \frac{1}{4}(5 - (-4/3) + 1/5) = 1.633$

$x_2^2 = \frac{1}{3}(-4 + x_1^1 + x_3^1)$
$= \frac{1}{3}(-4 + 5/4 + 1/5) = -0.983$

$x_3^2 = \frac{1}{5}(1 - 2x_1^1 - 2x_2^1)$
$= \frac{1}{5}(1 - 2(5/4) - 2(-4/3)) = -0.8332$

iteration 3 $k=2,\ x_1^2 = 1.633,\ x_2^2 = 0.983,\ x_3^2 = -0.8332$

$x_1^2 = \frac{1}{4}(5 - x_2^2 + x_3^2)$
$= \frac{1}{4}(5 - 0.983 - 0.8332) = 1.2875$

$x_2^2 = \frac{1}{3}(-4 + x_1^2 + x_3^2)$
$= \frac{1}{3}(-4 + 1.633 - 0.8332) = -0.5112$

$x_3^2 = \frac{1}{5}(1 - 2x_1^2 - 2x_2^2)$
$= \frac{1}{5}(1 - 2(1.633) - 2(0.983)) = -0.06$

on calculator

$x_1 \to A,\ x_2 \to B,\ x_3 \to C$

$X = \frac{1}{4}(5 + B + C)$
$: Y = \frac{1}{3}(-4 + A + C)$
$: M = \frac{1}{5}(1 - 2A - 2B)$
calc

Gauss-Seidel method

1. check diagonally dominant → if not then we cant solve
2. input formula and itterate
   - ↳ make subject
   - ↳ any new value found during an itteration will be used for the next variable
3. keep itterating till ans same/close (x, y, z)

Question

$$3x - y + z = -1$$
$$-x + 3y - z = 7$$
$$x - y + 3z = -7$$

① check diagonally dominant

$3 > 2$, $3 > 2$, $3 > 2$

hence diagonally dominant

② get equation

$$x = \tfrac{1}{3}(-1 + y - z)$$
$$y = \tfrac{1}{3}(7 + x + z)$$
$$z = \tfrac{1}{3}(-7 - x + y)$$

<u>iteration 1</u> $\quad x, y, z = 0$

$$x' = \tfrac{1}{3}(-1 + 0 - 0) = -\tfrac{1}{3}$$
$$y' = \tfrac{1}{3}(7 - \tfrac{1}{3} + 0) = \tfrac{20}{9}$$
$$z' = \tfrac{1}{3}\left(-7 - (-\tfrac{1}{3}) + \tfrac{20}{9}\right) =$$

<u>on calculator</u>

$y \to B$, $z \to C$ →old values

$$X = \tfrac{1}{3}(-1 + B - C)$$
$$: Y = \tfrac{1}{3}(7 + \overset{\text{as new X}}{X} + C)$$
$$: M = \tfrac{1}{3}(-7 - X + \overset{\text{as new Y}}{Y})$$

calc

<u>to check if ans correct</u>

mode + 5 + 2

put matrice

7.3

1. Find the first two iterations of the Jacobi method for the following linear systems, using $\mathbf{x}^{(0)} = \mathbf{0}$:

a.
$$3x_1 - x_2 + x_3 = 1,$$
$$3x_1 + 6x_2 + 2x_3 = 0,$$
$$3x_1 + 3x_2 + 7x_3 = 4.$$

b.
$$10x_1 - x_2 = 9,$$
$$-x_1 + 10x_2 - 2x_3 = 7,$$
$$-2x_2 + 10x_3 = 6.$$

a) $x_1^{k+1} = \frac{1}{3}(1 + x_2^k - x_3^k)$

$x_2^{k+1} = \frac{1}{6}(-3x_1^k - 2x_3^k)$

$x_3^{k+1} = \frac{1}{7}(4 - 3x_1^k - 3x_2^k)$

diagonally dominant

n	A	B	C	K
0	0	0	0	0
1	1/3	0	4/7	1
2	1/7	-5/14	3/7	2

3. Repeat Exercise 1 using the Gauss-Seidel method.

$x_1 = \frac{1}{3}(1 + x_2 - x_3)$

$x_2 = \frac{1}{6}(-3x_1 - 2x_3)$

$x_3 = \frac{1}{7}(4 - 3x_1 - 3x_2)$

n	A	B	C
0	0	0	0
1	1/3	-1/6	1/2
2	1/9	-2/9	13/21

on calculator

y→B, z→C →old values

X = 1/3(-1 + B - C)

: Y = 1/3(7 + X + C) as new X

: M = 1/3(-7 - X + Y) as new Y

calc

5. Use the Jacobi method to solve the linear systems in Exercise 1, with $TOL = 10^{-3}$ in the l_∞ norm.

4 5

n	A	B	C	K
0	0	0	0	0
1	$1/3$	0	$4/7$	1
2	$1/7$	$-5/14$	$3/7$	2

x_1

x_2

x_3

$\frac{1}{4}$ $\frac{5}{14}$ $\frac{1}{7}$

0.25 0.357 0.14

?

$\dfrac{0.25 \rightarrow \text{biggest diff b/w itteration 2 and 1}}{3/7 \rightarrow \text{biggest value in 2 itteration}} = 0.58$

Vector Norm and Matrix Norms

1. $|A| \geq 0$
2. $|kA| = k|A|$
3. $|A+B| \leq |A| + |B|$
4. $|AB| \leq |A||B|$

Q) find l_2 and l_∞ norm

vector $x = (-1, 1, -2)^t$

$|x|_1 = |-1| + |1| + |-2| = 4$

$|x|_2 = \sqrt{(-1)^2 + (1)^2 + (-2)^2} = \sqrt{6}$

$|x|_\infty = \max\{(|-1|, |1|, |-2|)\} = 2$

Power method

↳ finds dominant eigen value

↳ $n \times n$

↳ λ values distinct

↳ $|\lambda_1| = |\lambda_2|$ then dont solve

Q) find larger eigen value of matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \quad x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

IN CAL

mode + 6

↳ A , 3x3

shift + 4 + 2

↳ B , 3x1

press AC to save

shift 4

iteration 1: $Ax_0 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 14 \end{bmatrix}$ take largest common $= \underset{\text{Eigen value}}{14} \underbrace{\begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix}}_{\text{Eigen Vector}}$

iteration 2: $Ax_0 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \\ 6.5 \end{bmatrix} = \underset{\text{Eigen value}}{6.5} \begin{bmatrix} 0.076 \\ 0.153 \\ 1 \end{bmatrix}$

iteration 3: $Ax_0 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.076 \\ 0.153 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.645 \\ -0.076 \\ 5.923 \end{bmatrix} = \underset{\text{Eigen value}}{5.923} \begin{bmatrix} 0.0769 \\ -0.1538 \\ 1 \end{bmatrix}$

keep repeating till

↳ itterations if mentioned

↳ Eigen value becomes same

9.3

1. Find the first three iterations obtained by the Power method applied to the following matrices.

a. $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$; $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$
Use $\mathbf{x}^{(0)} = (1, -1, 2)^t$.

b. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$;
Use $\mathbf{x}^{(0)} = (-1, 0, 1)^t$.

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{matrix} 2-1+2 = \\ 1-2+2 = \\ 1-1+4 = \end{matrix} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \rightarrow 4 \cdot \begin{bmatrix} 3/4 \\ 1/4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3/4 \\ 1/4 \\ 1 \end{bmatrix} = \begin{bmatrix} 11/4 \\ 9/4 \\ 3 \end{bmatrix} \rightarrow 3 \cdot \begin{bmatrix} 11/12 \\ 9/12 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 11/12 \\ 9/12 \\ 1 \end{bmatrix} = \begin{bmatrix} 43/12 \\ 41/12 \\ 11/3 \end{bmatrix} \quad \frac{11}{3} = \begin{bmatrix} 43/44 \\ 41/44 \\ 1 \end{bmatrix}$$

Gradient Descent

Q)

x	y
4	7
5	8
6	9

$m=3$ → as 3 Pairs , $\alpha = 0.05$

$$h_{\theta_0,\theta_1}(x) = \theta_0 + \theta_1 x_1$$

$$J(\theta_0,\theta_1) = \frac{1}{2m}\sum_{i=1}^{m}(\theta_0 + \theta_1 x_1 - y_i)^2$$

$$= 1/6\,((\theta_0 + 4\theta_1 - 7)^2 + (\theta_0 + 5\theta_1 - 8)^2 + (\theta_0 + 6\theta_1 - 9)^2$$

$$= 1/6\,[\qquad a^2 + 2a(b-c) - 2bc + b^2 - c^2$$

$$= 1/6\,(3\theta_0^2 + 77\theta_1^2 + 300\theta_1\theta_0 - 244\theta_1 - 480\theta_0 - 194)$$

$$\frac{\delta J}{\delta \theta_0} = \frac{1}{m}\sum_{i=1}^{m}(\theta_0 + \theta_1 x_1 - y_i)$$

$$= 1/3\,[(\theta_0 + 4\theta_1 - 7) + (\theta_0 + 5\theta_1 - 8) + (\theta_0 + 6\theta_1 - 9)]$$

$$= 1/3\,(3\theta_0 + 15\theta_1 - 24)$$

$$\frac{\delta J}{\delta \theta_1} = \frac{1}{m}\sum_{i=1}^{m}(x_{1i}(\theta_0 + \theta_1 x_1 - y_i))$$

$$= 1/3\,[4(\theta_0 + 4\theta_1 - 7) + 5(\theta_0 + 5\theta_1 - 8) + 6(\theta_0 + 6\theta_1 - 9)]$$

$$= 1/3\,(4\theta_0 + 16\theta_1 - 28 + 5\theta_0 + 25\theta_1 - 40 + 6\theta_0 + 36\theta_1 - 54)$$

$$= 1/3\,(15\theta_0 + 77\theta_1 - 122$$

itteration 0

$\theta_0 = 0 \qquad \theta_1 = 0$

new $\theta_0 : \theta_0 - \alpha \left(\frac{\delta J}{\delta \theta_0} \right)$

new $\theta_1 : \theta_1 - \alpha \left(\frac{\delta J}{\delta \theta_1} \right)$

Gradient Descent

Monday, May 15, 2023 3:00 PM

X	Y
4	7
5	8
6	9

Fit a line $\hat{y} = \theta_0 + \theta_1 x$ on given data. Using loss function

$J = \frac{1}{2m} \sum_{i=1}^{m} (\hat{y} - y)^2$ where $\hat{y} = \theta_0 + \theta_1 x$.

Take initial value of $\theta_0 = 0$ and $\theta_1 = 0$ and update the parameters 3 times by gradient descent algorith taking $\alpha = 0.05$.

Solution :-

As $m = 3$ ← no of data points

$$J = \frac{1}{2 \times 3} \left[(\theta_0 + 4\theta_1 - 7)^2 + (\theta_0 + 5\theta_1 - 8)^2 + (\theta_0 + 6\theta_1 - 9)^2 \right]$$

$$J = \frac{1}{6} (3\theta_0^2 + 77\theta_1^2 + 30\theta_0\theta_1 - 244\theta_1 - 48\theta_0 + 194)$$

$$\frac{\partial J}{\partial \theta_0} = \frac{1}{6} \left[6\theta_0 + 30\theta_1 - 48 \right]$$

for $\alpha = 0.05$

$$\frac{\partial J}{\partial \theta_1} = \frac{1}{6} \left[154\theta_1 + 30\theta_0 - 244 \right]$$

Iter. no	θ_0	θ_1	J	$\partial J / \partial \theta_0$	$\partial J / \partial \theta_1$
1.	0	0	32.333	8	40.67
2.	0.4000	2.033	3.6498	2.5667	13.5222
3-	0.2717	1.3572	0.4864	-0.9422	-4.4729
4-	0.3188	1.5809	0.1374	0.2231	1.5029

$\hat{y} = 0.3188 + 1.5809x$ ← required line

For more practice questions change values of data points in code and generate sol table to match your answers.

RUNGE KUTTA FORMULAS

1^{st} order

Eulers method

$$y_{n+1} = y_n + h f(x_n, y_n)$$

2^{nd} order

Heun Method
(without iteration)

$$k_1 = f(x_n, y_n)$$

$$k_2 = f[x_n + h, y_n + k_1 h)$$

$$y_{n+1} = y_n + \tfrac{1}{2}h[k_1 + k_2]$$

Mid Point method

$$k_1 = f(x_n, y_n)$$

$$k_2 = f[x_n + \tfrac{h}{2}, y_n + k_1 \tfrac{h}{2})$$

$$y_{n+1} = y_n + k_2 h$$

4^{th} order

most popular

$$k_1 = f(x_0, y_0)$$

$$k_2 = f\left(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_1}{2}h\right)$$

$$k_3 = f\left(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_2}{2}h\right)$$

$$k_4 = f(x_0 + h, y_0 + k_3 h)$$

$$y = y_0 + \tfrac{1}{6}h[k_1 + 2k_2 + 2k_3 + k_4]$$

LU Decomposition

det $\neq 0$

1. $Ax = B$
2. $A = L\ U$

multipliers of gaus elimination — gaus Elimination

$$\begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x & x & 1 \end{bmatrix}\begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}$$

→ Doolittle form

any real number

3. $LY = B$ and solve simultaneously
4. $UX = Y$ and solve simultaneously

Cholesky

1. $Ax = B$
2. Check symmetry
3. Check positive definite
4. $A = LL^T$
5. $LY = B$
6. $L^T X = Y$

Crout factorization

det $\neq 0$

1. $Ax = B$
2. $A = L\ U$

$$\begin{bmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{bmatrix}\begin{bmatrix} 1 & x & x \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}$$

→ L and U are diff in this method

3. $LY = B$ and solve simultaneously
4. $UX = Y$ and solve simultaneously

Jacobis

1. Check diagonally dominant → if not then we cant solve
2. input formula and iterate

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left(d_1 - a_{12}x_2^k - a_{13}x_3^k\right)$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left(d_2 - a_{21}x_1^k - a_{23}x_3^k\right)$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}\left(d_3 - a_{31}x_1^k - a_{32}x_2^k\right)$$

3. keep iterating till ans same/close (x_1, x_2, x_3)

↳ may stop after 5 or 6 iteration if answer dosent come ??

Gaus seidel

1. Check diagonally dominant → if not then we cant solve
2. input formula and iterate
   - ↳ make subject
   - ↳ any new value found during an iteration will be used for the next variable
3. keep iterating till ans same/close (x, y, z)

Positive definite

LDL^t Factorization

$$\underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}}\underset{D}{\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}}\underset{L^T}{\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}}$$

LL^T Factorization

$$\underset{L}{\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}} \cdot \underset{L^T}{\begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{bmatrix}}$$

DO Little

$$\underset{A}{\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}}\underset{U}{\begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & i \end{bmatrix}}$$

ishma hafeez notes

MID 1

Bisection

n	x_1	x_2	mid	f(x)

steps

- ↳ find no of iterations $n = \frac{\ln(\Delta x / \epsilon)}{\ln 2}$ — diff $(a-b)$, stopping criteria
- ↳ Repeat till n iterations
- ↳ choose 2 boundaries +ve and -ve
- ↳ find mid point
- ↳ get new interval

Fixed Point

n	

steps

- ↳ find 2 value where f(x) = +ve and -ve
- ↳ take its midpoint → x_0

OPTIONAL — These step reduce iterations

- ↳ make x subject
- ↳ check $|\phi'(x)| < 1$ → all +ve values — <1: converging, >1: diverging ?
- ↳ if all <1, opt smallest
- ↳ plug in function $x_n = \phi(x_{n-1})$
- ↳ keep iterating till sf same or stopping criteria

NEWTON RAPHSON

n	

steps

- ↳ find 2 value where f(x) = +ve and -ve
- ↳ take its midpoint

OPTIONAL — These step reduce iterations

- ↳ find f'
- ↳ $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $\therefore f'(x_n) \neq 0$ → denominator never 0
- ↳ keep iterating till sf same or Es

Limitation
only usable is f' is readily available

FALSE POSITION / FALSI

a	b	c	f(c)

steps

- ↳ $x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$
- ↳ if f(x) +ve replace b
- ↳ if f(x) = -ve replace a

sign matters

- ↳ keep iterating till sf same or Es

SECANT

n	a	b	c	f(c)

steps

- ↳ $x_{n+1} = \frac{x_n f(x_{n+1}) - x_{n+1} f(x_n)}{f(x_{n+1}) - f(x_n)}$
- ↳ keep replacing x → sign dosent matter
- ↳ keep iterating till sf same or Es

---

absolute error (E_t)

$e_p = |p - p^*|$

p: true value; p^*: calculated value

relative error (E_a)

$\delta p = \left| \frac{p - p^*}{p} \right|$, $p \neq 0$

stopping criteria

$E_s = (0.5 \times 10^{2-n})\%$ → n sig fig

$E_a < E_s$

1. Binary machine numbers

$(-1)^s \, 2^{c-1023} \, (1+f)$

- s: sign (MSB) left most bit — 1 bit
- c: characteristic — 11 bit
- f: mantisa — 52 bit

LOSS OF SF Remedy

when too many sf cancel

- ↳ rationalise
- ↳ using series expansion
- ↳ use trignometric identities
- ↳ reformulation

$$\begin{array}{r} 123.4567 \\ -\ 123.4566 \\ \hline 000.0001 \end{array}$$

ishma hafeez notes

repshtreet