# Analysis of Variance (ANOVA)

Analysis-of-variance procedures rely on a distribution called the *F-distribution,* named in honor of Sir Ronald Fisher. A variable is said to have an ***F-distribution*** if its distribution has the shape of a special type of right-skewed curve, called an ***F-curve.*** There are infinitely many *F*-distributions, and we identify an *F*-distribution (and *F*-curve) by its number of degrees of freedom, just as we did for *t*-distributions

An *F*-distribution, however, has two numbers of degrees of freedom instead of one. Figure depicts two different *F*-curves; one has df = (10, 2), and the other has df = (9, 50).

The first number of degrees of freedom for an *F*-curve is called the **degrees of freedom for the numerator,** and the second is called the **degrees of freedom for the denominator.** Thus, for the *F*-curve in Fig. with df = (10, 2), we have

Two different F-curves

df = (10, 2)

Degrees of freedom for the numerator — Degrees of freedom for the denominator
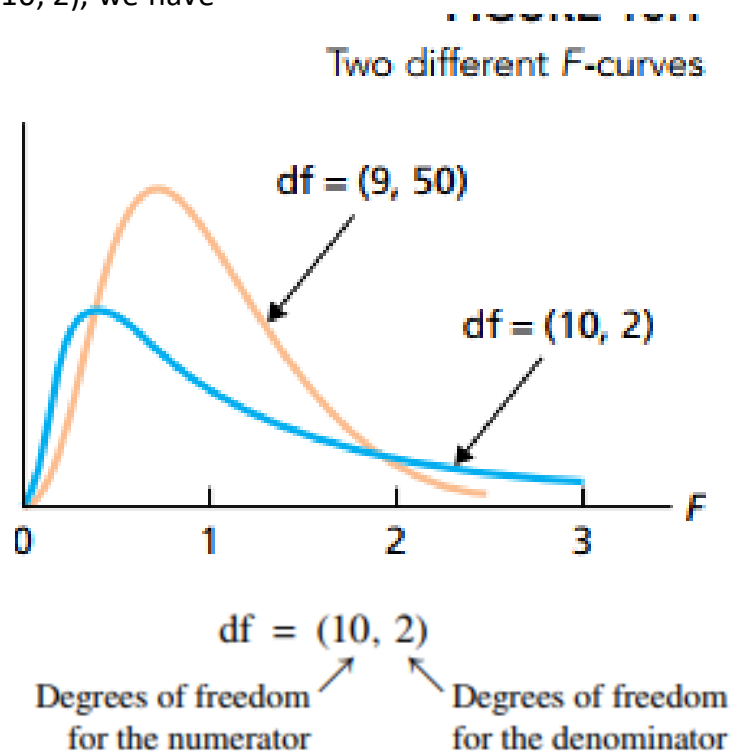
Basic Properties of *F*-Curves

Property 1: The total area under an *F*-curve equals 1.

Property 2: An *F*-curve starts at 0 on the horizontal axis and extends indefinitely to the right, approaching, but never touching, the horizontal axis as it does so.

Property 3: An *F*-curve is right skewed.

One Way ANOVA

## The Logic behind One-Way ANOVA

The reason for the word *variance* in *analysis of variance* is that the procedure for comparing the means analyzes the variation in the sample data. To examine how this procedure works, let's suppose that independent random samples are taken from two populations—say, Populations 1 and 2—with means μ_1 and μ_2. Further, let's suppose that the means of the two samples are $\bar{x}_1 = 20$ and $\bar{x}_2 = 25$. Can we reasonably conclude from these statistics that $\mu_1 \neq \mu_2$, that is, that the population means are different? To answer this question, we must consider the variation within the samples.

Suppose, for instance, that the sample data are as displayed in Table 16.1 and depicted in Fig. 16.3.

Sample from Population 1	21	37	11	20	8	23
Sample from Population 2	24	31	29	40	9	17

FIGURE 16.3

Dotplots for sample data in Table 16.1

Sample from Population 1 ($\bar{x}_1 = 20$)

Sample from Population 2 ($\bar{x}_2 = 25$)
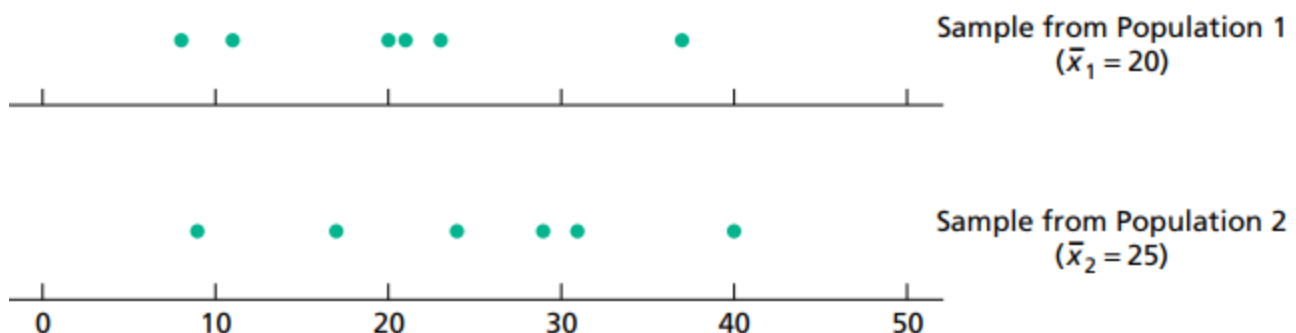
For these two samples, $\bar{x}_1 = 20$ and $\bar{x}_2 = 25$. But here we cannot infer that $\mu_1 \neq \mu_2$ because it is not clear whether the difference between the sample means is due to a difference between the population means or to the variation within the populations.

However, suppose that the sample data are as displayed in Table 16.2 and depicted in Fig. 16.4.

Sample from Population 1	21	21	20	18	20	20
Sample from Population 2	25	28	25	24	24	24
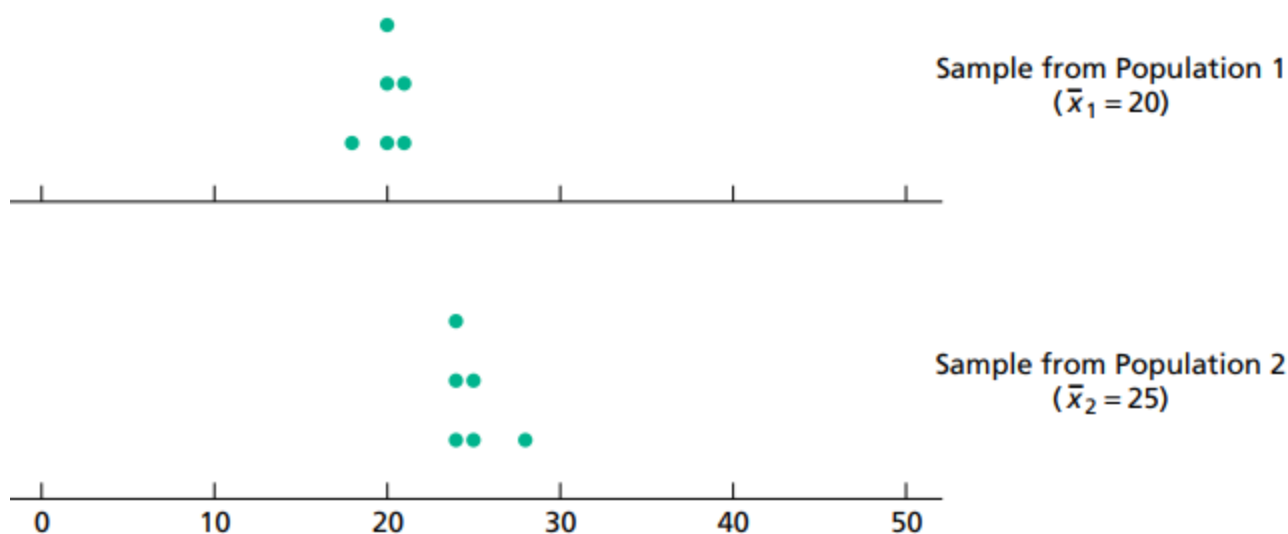
Again, for these two samples, $\bar{x}_1 = 20$ and $\bar{x}_2 = 25$. But this time, we *can* infer that $\mu_1 \neq \mu_2$ because it seems clear that the difference between the sample means is due to a difference between the population means, not to the variation within the populations.
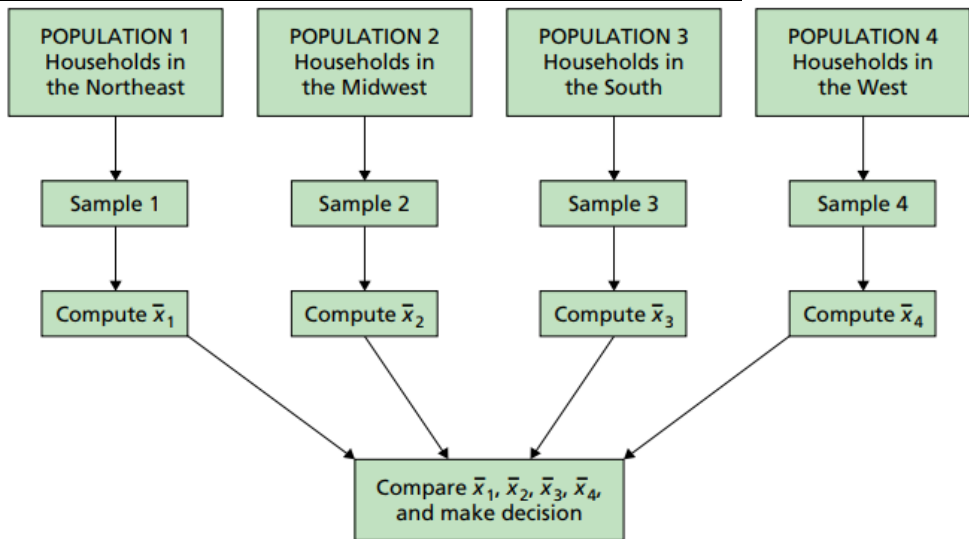
Process for comparing four population means:

Procedure of One-Way ANOVA:

Purpose To perform a hypothesis test to compare k population means, $\mu_1, \mu_2, \ldots, \mu_k$

Assumptions
1. Simple random samples
2. Independent samples
3. Normal populations
4. Equal population standard deviations

Step 1 The null and alternative hypotheses are, respectively,

$$H_0: \mu_1 = \mu_2 = \cdots = \mu_k$$
$$H_a: \text{Not all the means are equal.}$$

Step 2 Decide on the significance level, α.

Step 3 Compute the value of the test statistic

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Computed f
Treatments	SSA	$k-1$	$s_1^2 = \dfrac{SSA}{k-1}$	$\dfrac{s_1^2}{s^2}$
Error	SSE	$k(n-1)$	$s^2 = \dfrac{SSE}{k(n-1)}$	
Total	SST	$kn-1$		

CRITICAL-VALUE APPROACH

Step 4 The critical value is F_α with df $=(k-1, n-k)$. Use Table VIII to find the critical value.

Do not reject H_0 | Reject H_0
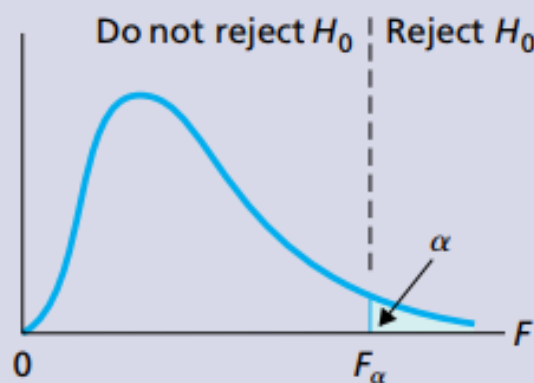
Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

Step 6 Interpret the results of the hypothesis test.

Formulas:

Table 13.2: k Random Samples

Treatment:	1	2	$\cdots$	i	$\cdots$	k	
	y_{11}	y_{21}	$\cdots$	y_{i1}	$\cdots$	y_{k1}	
	y_{12}	y_{22}	$\cdots$	y_{i2}	$\cdots$	y_{k2}	
	$\vdots$	$\vdots$		$\vdots$		$\vdots$	
	y_{1n}	y_{2n}	$\cdots$	y_{in}	$\cdots$	y_{kn}	
Total	$Y_{1.}$	$Y_{2.}$	$\cdots$	$Y_{i.}$	$\cdots$	$Y_{k.}$	$Y_{..}$
Mean	$\bar{y}_{1.}$	$\bar{y}_{2.}$	$\cdots$	$\bar{y}_{i.}$	$\cdots$	$\bar{y}_{k.}$	$\bar{y}_{..}$

Sum-of-Squares Identity

$$\sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{..})^2 = n\sum_{i=1}^{k}(\bar{y}_{i.} - \bar{y}_{..})^2 + \sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{i.})^2$$

$$SST = \sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{..})^2 = \text{total sum of squares,}$$

$$SSA = n\sum_{i=1}^{k}(\bar{y}_{i.} - \bar{y}_{..})^2 = \text{treatment sum of squares,}$$

$$SSE = \sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{i.})^2 = \text{error sum of squares.}$$

The sum-of-squares identity can then be represented symbolically by the equation

$$SST = SSA + SSE.$$

The identity above expresses how between-treatment and within-treatment variation add to the total sum of squares. However, much insight can be gained by investigating the **expected value of both *SSA* and *SSE***

Treated Mean Square:

$$s_1^2 = \frac{SSA}{k-1}$$

Error Mean Square:

$$s^2 = \frac{SSE}{k(n-1)}$$

Example:

Suppose in an industrial experiment that an engineer is interested in how the mean absorption of moisture in concrete varies among 5 different concrete aggregates. The samples are exposed to moisture for 48 hours. It is decided that 6 samples are to be tested for each aggregate, requiring a total of 30 samples to be tested.

Table 13.1: Absorption of Moisture in Concrete Aggregates

Aggregate:	1	2	3	4	5	
	551	595	639	417	563	
	457	580	615	449	631	
	450	508	511	517	522	
	731	583	573	438	613	
	499	633	648	415	656	
	632	517	677	555	679	
Total	3320	3416	3663	2791	3664	16,854
Mean	553.33	569.33	610.50	465.17	610.67	561.80

Solution:

The model for this situation may be set up as follows. There are 6 observations taken from each of 5 populations with means $\mu_1, \mu_2, \ldots, \mu_5$, respectively. We may wish to test

$$H_0: \mu_1 = \mu_2 = \cdots = \mu_5,$$
$$H_1: \text{At least two of the means are not equal.}$$
$$\alpha = 0.05.$$

Critical region: $f > 2.76$ with $v_1 = 4$ and $v_2 = 25$ degrees of freedom. The sum-of-squares computations give

$$SST = 209{,}377, \quad SSA = 85{,}356,$$
$$SSE = 209{,}377 - 85{,}356 = 124{,}021.$$

$$(MSA)\ s_1^2 = \frac{SSA}{k-1} = \frac{85356}{4} = 21339.1167$$

$$(MSE)s^2 = \frac{SSE}{k(n-1)} = \frac{209376.8}{5(6-1)} = 4960.8133$$

$$f = \frac{21339.1167}{4960.8133} = 4.30$$

Reject H_0 and conclude that the aggregates do not have the same mean absorption.
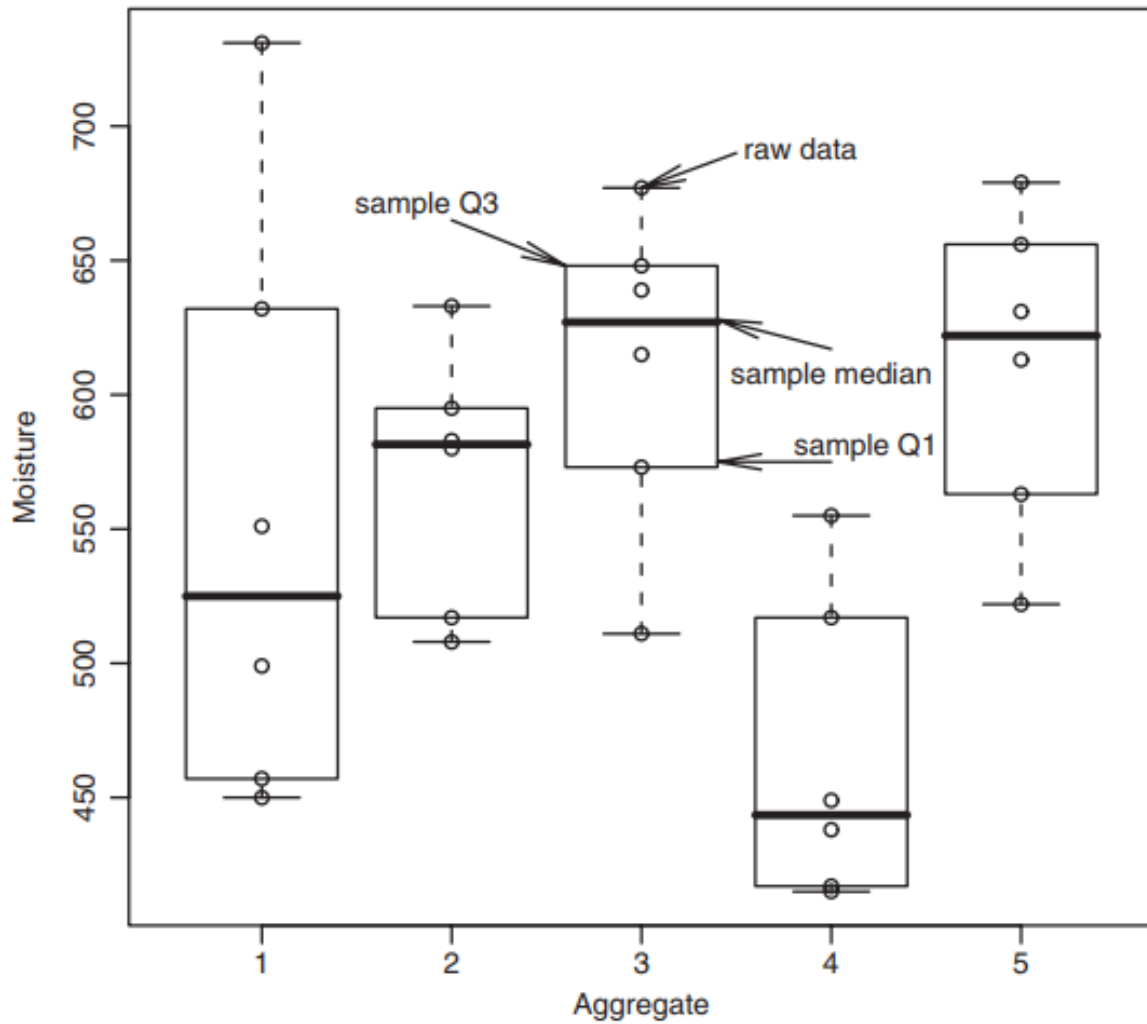
Figure 13.2: Box plots for the absorption of moisture in concrete aggregates.

Alternate Formula:

$$SST = \sum_{i=1}^{k}\sum_{j=1}(y_{ij})^2 - \frac{y_{..}^2}{N}$$

$$SSA = \frac{1}{n}\sum_{i=1}^{k}(y_{i.})^2 - \frac{y_{..}^2}{N}$$

$$SSE = SST - SSA$$

Question: Six different machines are being considered for use in manufacturing rubber seals. The machines are being compared with respect to tensile strength of the product. A random sample of four seals from each machine is used to determine whether the mean tensile strength varies from machine to machine. The following are the tensile-strength measurements in kilograms per square centimeter $* 10^{-1}$:

Machine

1	2	3	4	5	6
17.5	16.4	20.3	14.6	17.5	18.3
16.9	19.2	15.7	16.7	19.2	16.2
15.8	17.7	17.8	20.8	16.5	17.5
18.6	15.4	18.9	18.9	20.5	20.1

Perform the analysis of variance at the 0.05 level of significance and indicate whether or not the mean tensile strengths differ significantly for the six machines.

$$f_{0.05}(v_1, v_2)$$

v_2	v_1									
	10	12	15	20	24	30	40	60	120	∞
1	241.88	243.91	245.95	248.01	249.05	250.10	251.14	252.20	253.25	254.31
2	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

$$f_{0.01}(v_1, v_2)$$

v_2					v_1				
	1	2	3	4	5	6	7	8	9
1	4052.18	4999.50	5403.35	5624.58	5763.65	5858.99	5928.36	5981.07	6022.47
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41

Table A.6 (continued) Critical Values of the F-Distribution

$$f_{0.01}(v_1, v_2)$$

v_2	v_1									
	10	12	15	20	24	30	40	60	120	∞
1	6055.85	6106.32	6157.28	6208.73	6234.63	6260.65	6286.78	6313.03	6339.39	6365.86
2	99.40	99.42	99.43	99.45	99.46	99.47	99.47	99.48	99.49	99.50
3	27.23	27.05	26.87	26.69	26.60	26.50	26.41	26.32	26.22	26.13
4	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65	13.56	13.46
5	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
6	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
7	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
8	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
9	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
10	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
11	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
12	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
13	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
14	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.00
15	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
16	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75
17	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65
18	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
19	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49
20	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
21	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36
22	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31
23	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26
24	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
25	3.13	2.99	2.85	2.70	2.62	2.54	2.45	2.36	2.27	2.17
26	3.09	2.96	2.81	2.66	2.58	2.50	2.42	2.33	2.23	2.13
27	3.06	2.93	2.78	2.63	2.55	2.47	2.38	2.29	2.20	2.10
28	3.03	2.90	2.75	2.60	2.52	2.44	2.35	2.26	2.17	2.06
29	3.00	2.87	2.73	2.57	2.49	2.41	2.33	2.23	2.14	2.03
30	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
40	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
60	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
120	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
∞	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00