Solution (Mid II) Main Paper

Question # 1

(a)

Let X = number of bugs in the tested modules

$X = 0, 1, 2$

$$P(X=0) = \frac{{}^2C_0 \times {}^6C_4}{{}^8C_4} = \frac{15}{70} = \frac{3}{14} = 0.2142$$

$$P(X=1) = \frac{{}^2C_1 \times {}^6C_3}{{}^8C_4} = \frac{40}{70} = \frac{4}{7} = 0.5714$$

$$P(X=2) = \frac{{}^2C_2 \times {}^6C_2}{{}^8C_4} = \frac{15}{70} = \frac{3}{14} = 0.2142$$

$$P(x) = \frac{{}^2C_x \times {}^6C_{4-x}}{{}^8C_4}$$

X	0	1	2	Total
$P(x)$	3/14	4/7	3/14	1

— PMF (2 marks)

X	$P(x)$	$F(x)$
0	3/14	3/14
1	4/7	11/14
2	3/14	1

— CDF

$$F(x) = \begin{cases} 0 & x < 0 \\ 3/14 & 0 \le x < 1 \\ 11/14 & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$ (2 marks)

(b)

(i) $$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow K\int_0^1 \sqrt{x}\,dx = 1$$

$$\Rightarrow K(0.66667) = 1$$

$$K = 1.5 = \frac{3}{2}$$ (2 marks)

(ii) $$F(x) = \int_0^x \frac{3}{2}\sqrt{x}\,dx = \frac{3}{2}\,\frac{x^{3/2}}{3/2}\Big|_0^x = x^{3/2}$$ (1 mark)

$$P(0.3 < x < 0.6) = F(0.6) - F(0.3)$$

$$= 0.6^{3/2} - 0.3^{3/2} = 0.300441$$

(1 mark)

(c)

i. $n = 18$, $p = 0.0497$, $q = 0.9503$

$$P(X \leq 10) = \sum_{0}^{10} {}^{18}C_x (0.0497)^x (0.9503)^{18-x}$$

$$= 0.99999 \quad (2 \text{ marks})$$

ii. $E(X) = np = 18 \times 0.0497 = 0.8946$ (1 mark)

Let Y = pass test cases

$$\sigma_Y = \sqrt{npq} = \sqrt{18 \times 0.9503 \times 0.0497}$$

$$\sigma_Y = \sqrt{0.85013} = 0.92202$$

(1 mark)

(d)

i. $P(800 < X < 950) = ?$

$$Z_1 = \frac{X_1 - \mu}{\sigma} = \frac{800 - 900}{200} = -0.5$$

$$Z_2 = \frac{X_2 - \mu}{\sigma} = \frac{950 - 900}{200} = 0.25$$

$$P(-0.5 < Z < 0.25) = P(Z < 0.25) - P(Z < -0.5)$$

$$= 0.5987 - 0.3085 = 0.2902$$

(2 marks)

ii. $P(X < x) = 0.05$

$P(Z < z) = 0.05$

$\Rightarrow Z = -1.645$

$$x = \sigma z + \mu = 200 \times (-1.645) + 900$$

$x = 571$ (1 mark)

iii. $P(X > x) = 0.05 \Rightarrow P(X < x) = 0.95$

$P(Z < z) = 0.95 \Rightarrow Z = 1.645$

$$x = \sigma z + \mu = 200 \times 1.645 + 900$$

$x = 1229$ (1 mark)

(e)

y \ x	1	2	3	4	Total (P_y)
1	0	0.06	0.06	0.1	0.22
2	0.1	0.1	0.04	0.04	0.28
3	0.4	0.1	0	0	0.5
Total P_x	0.5	0.26	0.1	0.14	1

i. (1 mark)

X	1	2	3	4	Total
P_x	0.5	0.26	0.1	0.14	1

(1 Mark)

Y	1	2	3	Total
P_y	0.22	0.28	0.5	1

ii. $P(Y=2 \mid X<3) = \dfrac{\sum_{1}^{2} P(x,2)}{\sum_{1}^{2} P_x(x)}$ (1 Mark)

$= \dfrac{P(1,2) + P(2,2)}{P_x(1) + P_x(2)}$ (1 Mark)

$= \dfrac{0.1 + 0.1}{0.5 + 0.26} = \dfrac{0.2}{0.76} = 5/19$

$= 0.26315$ (1 Mark)

iii. Since $P(1,1) \neq P_x(1) P_y(1)$

$0 \neq 0.5 \times 0.22$

$\Rightarrow$ X and Y are dependent (1 mark)

$\rho_{xy} = \dfrac{\sigma_{xy}}{\sigma_x \sigma_y}$, $\sigma_{xy} = E(XY) - E(X) \cdot E(Y)$

$\sigma_x = \sqrt{E(x^2) - (E(x))^2}$

$\sigma_y = \sqrt{E(y^2) - (E(y))^2}$

$E(X) = \sum_{1}^{4} x P_x = 1 \times 0.5 + 2 \times 0.26 + 3 \times 0.1 + 4 \times 0.14$

$= 1.88$ (0.5 mark)

$$E(Y) = \sum_{1}^{3} y P_y = 1 \times 0.22 + 2 \times 0.28 + 3 \times 0.5$$

$$= 2.28 \quad (0.5 \text{ marks})$$

$$E(X^2) = \sum_{1}^{4} x^2 P_x = 1^2 \times 0.5 + 2^2 \times 0.26 + 3^2 \times 0.1 + 4^2 \times 0.14$$

$$= 4.68 \quad (1 \text{ mark})$$

$$E(Y^2) = \sum_{1}^{3} y^2 P_y = 1^2 \times 0.22 + 2^2 \times 0.28 + 3^2 \times 0.5$$

$$= 5.84 \quad (1 \text{ mark})$$

$$E(XY) = \sum_{x=1}^{4} \sum_{y=1}^{3} xy P(x,y) = 0 + 2P(1,2) + 3P(1,3) + 4P(1,4) + 2P(2,1) + 4P(2,2) + 6P(2,3) + 8P(2,4) + 3P(3,1) + 6P(3,2) + 0 + 0$$

$$E(XY) = 2(0.06) + 3(0.06) + 4(0.10) + 2(0.10) + 4(0.10) + 6(0.04) + 8(0.04) + 3(0.4) + 6(0.1)$$

$$E(XY) = 3.66 \quad (1.5 \text{ marks})$$

$$\rho_{xy} = \frac{3.66 - 1.88 \times 2.28}{\sqrt{4.68 - 1.88^2}\sqrt{5.84 - 2.28^2}}$$

$$\rho_{xy} = -0.732 \quad (0.5 \text{ marks})$$

iv. $Z = 2, 3, 4, 6, 8$ (1 mark)

$P(Z=2) = P(1,2) + P(2,1) = 0.1 + 0.06 = 0.16$

$P(Z=3) = P(1,3) + P(3,1) = 0.4 + 0.06 = 0.46$

$P(Z=4) = P(4,1) + P(2,2) = 0.1 + 0.1 = 0.2$

$P(Z=6) = P(2,3) + P(3,2) = 0.1 + 0.04 = 0.14$

$P(Z=8) = P(4,2) = 0.04$

Z	2	3	4	6	8
P(Z)	0.16	0.46	0.2	0.14	0.04

(2 marks)