Counting Rule, Factorials, Combinations, and Permutations

Sample space:

The set of all possible outcomes of a statistical experiment is called the **sample space** and is represented by the symbol S.

Example:

Consider the experiment of tossing a die. If we are interested in the number that shows on the top face, the sample space is

$$S_1 = \{1, 2, 3, 4, 5, 6\}.$$

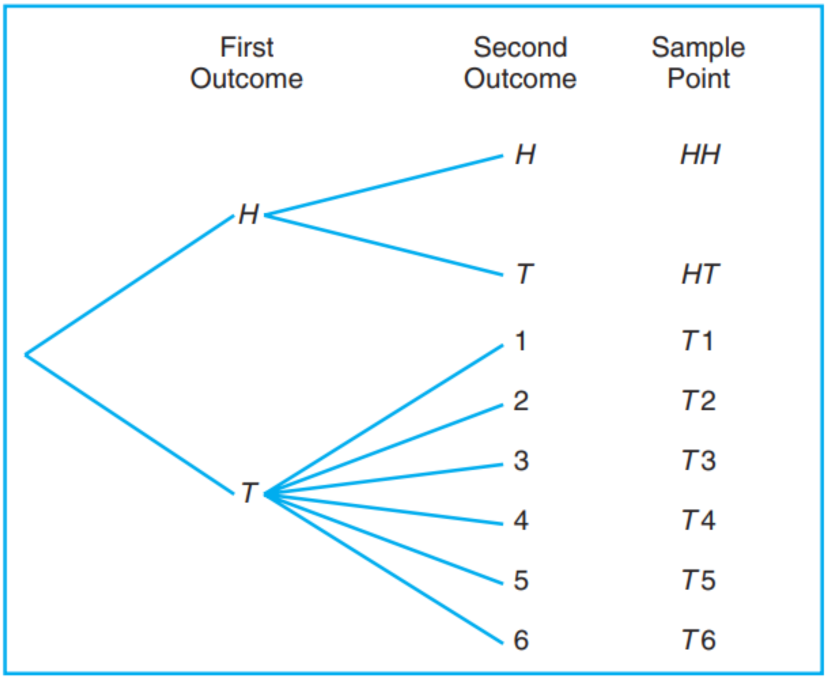
If we are interested only in whether the number is even or odd, the sample space is simply

$$S_2 = \{\text{even, odd}\}.$$

Tree Diagram:

An experiment consists of flipping a coin and then flipping it a second time if a head occurs. If a tail occurs on the first flip, then a die is tossed once. To list the elements of the sample space providing the most information, we construct the tree diagram of Figure 2.1. The various paths along the branches of the tree give the distinct sample points. Starting with the top left branch and moving to the right along the first path, we get the sample point HH, indicating the possibility that heads occurs on two successive flips of the coin. Likewise, the sample point $T3$ indicates the possibility that the coin will show a tail followed by a 3 on the toss of the die. By proceeding along all paths, we see that the sample space is

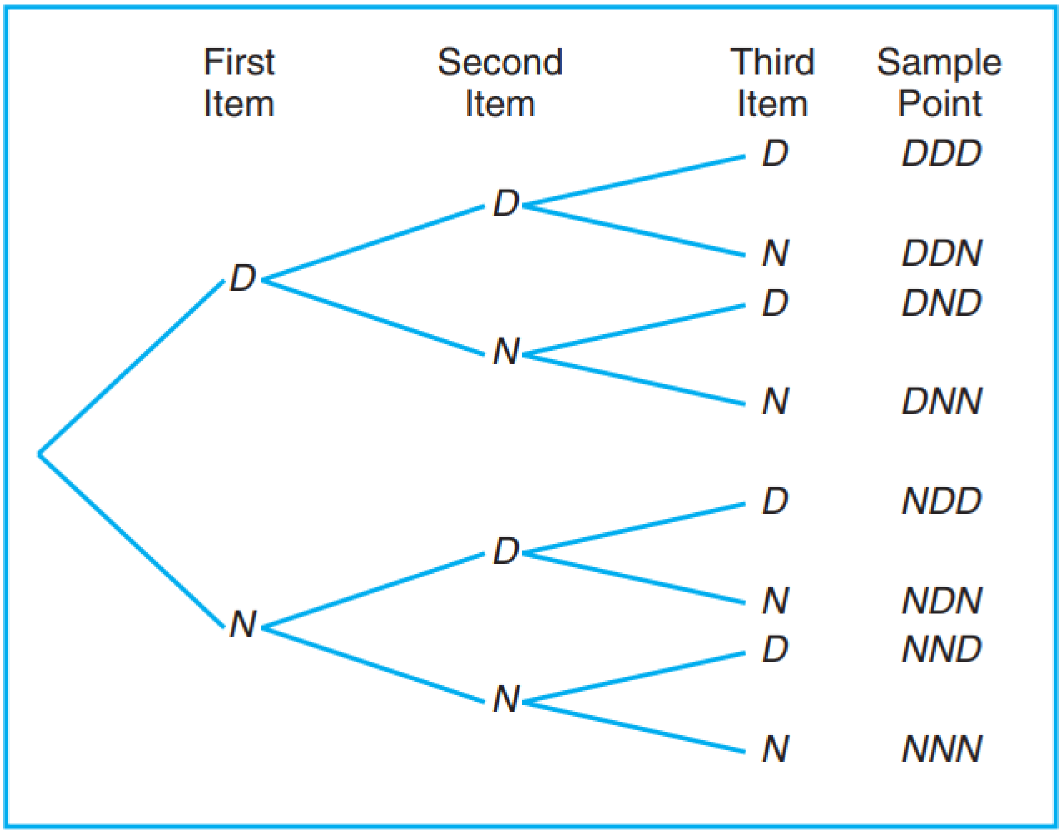
$$S = \{HH,\ HT,\ T1,\ T2,\ T3,\ T4,\ T5,\ T6\}.$$



Example:

Suppose that three items are selected at random from a manufacturing process. Each item is inspected and classified defective, D, or nondefective, N. To list the elements of the sample space providing the most information, we construct the tree diagram of Figure 2.2. Now, the various paths along the branches of the tree give the distinct sample points. Starting with the first path, we get the sample point DDD, indicating the possibility that all three items inspected are defective. As we proceed along the other paths, we see that the sample space is

$$S = \{DDD, DDN, DND, DNN, NDD, NDN, NND, NNN\}.$$



Event:

An **event** is a subset of a sample space.

The **complement** of an event A with respect to S is the subset of all elements of S that are not in A. We denote the complement of A by the symbol A'.

Example:

Consider the sample space

$$S = \{\text{book, cell phone, mp3, paper, stationery, laptop}\}.$$

Let $A = \{\text{book, stationery, laptop, paper}\}$. Then the complement of A is $A' = \{\text{cell phone, mp3}\}$.

The **intersection** of two events A and B, denoted by the symbol $A \cap B$, is the event containing all elements that are common to A and B.

Example:

Let $V = \{a, e, i, o, u\}$ and $C = \{l, r, s, t\}$; then it follows that $V \cap C = \phi$. That is, V and C have no elements in common and, therefore, cannot both simultaneously occur.

Two events A and B are **mutually exclusive**, or **disjoint**, if $A \cap B = \phi$, that is, if A and B have no elements in common.

The **union** of the two events A and B, denoted by the symbol $A \cup B$, is the event containing all the elements that belong to A or B or both.

Example:

If $M = \{x \mid 3 < x < 9\}$ and $N = \{y \mid 5 < y < 12\}$, then

$$M \cup N = \{z \mid 3 < z < 12\}.$$

The relationship between events and the corresponding sample space can be illustrated graphically by means of **Venn diagrams**. In a Venn diagram we let the sample space be a rectangle and represent events by circles drawn inside the rectangle. Thus, in Figure 2.3, we see that

$$\begin{aligned} A \cap B &= \text{regions 1 and 2,} \\ B \cap C &= \text{regions 1 and 3,} \end{aligned}$$

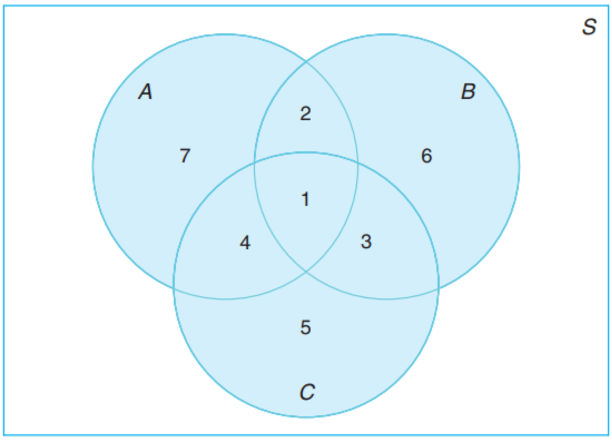


Figure 2.3: Events represented by various regions.

$$\begin{aligned} A \cup C &= \text{regions 1, 2, 3, 4, 5, and 7,} \\ B' \cap A &= \text{regions 4 and 7,} \\ A \cap B \cap C &= \text{region 1,} \\ (A \cup B) \cap C' &= \text{regions 2, 6, and 7,} \end{aligned}$$

and so forth.

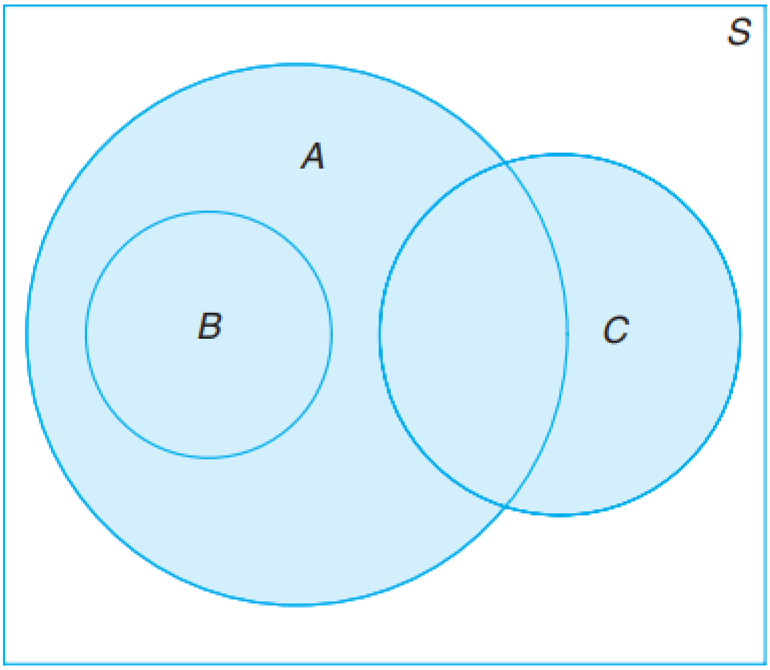


Figure 2.4: Events of the sample space S.

In Figure 2.4, we see that events A, B, and C are all subsets of the sample space S. It is also clear that event B is a subset of event A; event $B \cap C$ has no elements and hence B and C are mutually exclusive; event $A \cap C$ has at least one element; and event $A \cup B = A$. Figure 2.4 might, therefore, depict a situation where we select a card at random from an ordinary deck of 52 playing cards and observe whether the following events occur:

A: the card is red,

B: the card is the jack, queen, or king of diamonds,

C: the card is an ace.

Clearly, the event $A \cap C$ consists of only the two red aces.

Several results that follow from the foregoing definitions, which may easily be verified by means of Venn diagrams, are as follows:

1. $A \cap \phi = \phi$.
2. $A \cup \phi = A$.
3. $A \cap A' = \phi$.
4. $A \cup A' = S$.
5. $S' = \phi$.
6. $\phi' = S$.
7. $(A')' = A$.
8. $(A \cap B)' = A' \cup B'$.
9. $(A \cup B)' = A' \cap B'$.

Multiplication Rule:

If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways.

Example:

How many sample points are there in the sample space when a pair of dice is thrown once?

The first die can land face-up in any one of $n_1 = 6$ ways. For each of these 6 ways, the second die can also land face-up in $n_2 = 6$ ways. Therefore, the pair of dice can land in $n_1 n_2 = (6)(6) = 36$ possible ways.

A developer of a new subdivision offers prospective home buyers a choice of Tudor, rustic, colonial, and traditional exterior styling in ranch, two-story, and split-level floor plans. In how many different ways can a buyer order one of these homes?

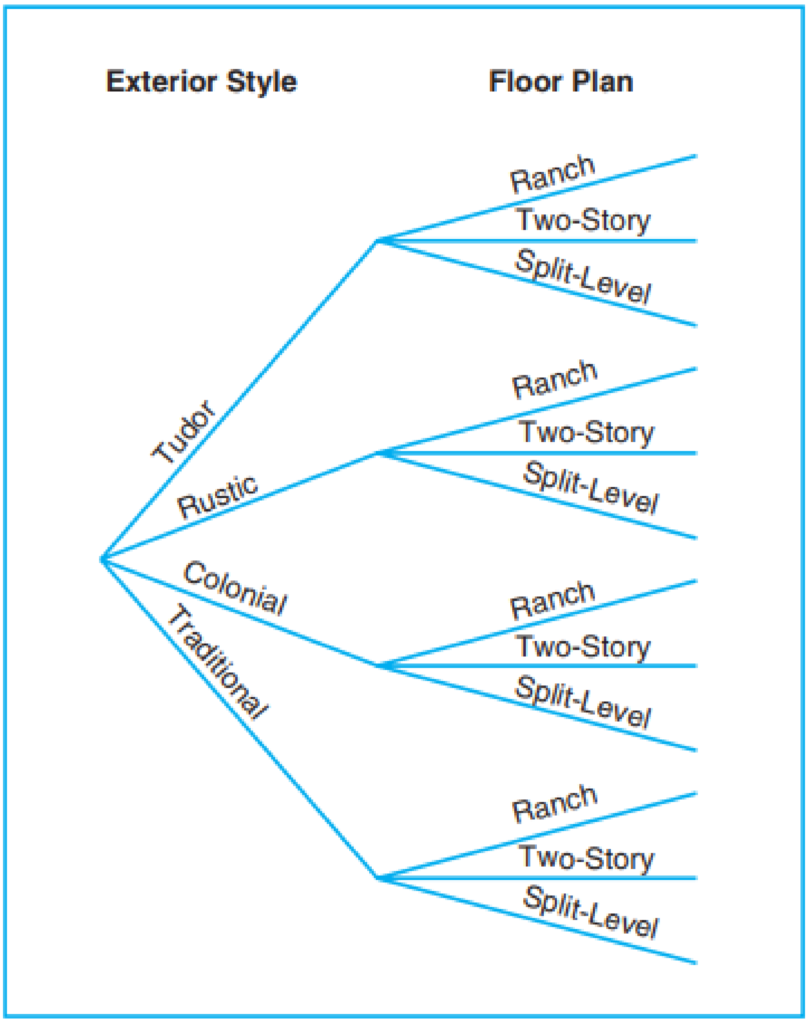


Figure 2.6: Tree diagram for Example 2.14.

Since $n_1 = 4$ and $n_2 = 3$, a buyer must choose from

$$n_1 n_2 = (4)(3) = 12 \text{ possible homes.}$$

Example:

If a 22-member club needs to elect a chair and a treasurer, how many different ways can these two to be elected?

For the chair position, there are 22 total possibilities. For each of those 22 possibilities, there are 21 possibilities to elect the treasurer. Using the multiplication rule, we obtain $n_1 \times n_2 = 22 \times 21 = 462$ different ways.

> If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \cdots n_k$ ways.

Example:

Sam is going to assemble a computer by himself. He has the choice of chips from two brands, a hard drive from four, memory from three, and an accessory bundle from five local stores. How many different ways can Sam order the parts?

Since $n_1 = 2$, $n_2 = 4$, $n_3 = 3$, and $n_4 = 5$, there are

$$n_1 \times n_2 \times n_3 \times n_4 = 2 \times 4 \times 3 \times 5 = 120$$

different ways to order the parts.

Example:

How many even four-digit numbers can be formed from the digits 0, 1, 2, 5, 6, and 9 if each digit can be used only once?

Since the number must be even, we have only $n_1 = 3$ choices for the units position. However, for a four-digit number the thousands position cannot be 0. Hence, we consider the units position in two parts, 0 or not 0. If the units position is 0 (i.e., $n_1 = 1$), we have $n_2 = 5$ choices for the thousands position, $n_3 = 4$ for the hundreds position, and $n_4 = 3$ for the tens position. Therefore, in this case we have a total of

$$n_1 n_2 n_3 n_4 = (1)(5)(4)(3) = 60$$

even four-digit numbers. On the other hand, if the units position is not 0 (i.e., $n_1 = 2$), we have $n_2 = 4$ choices for the thousands position, $n_3 = 4$ for the hundreds position, and $n_4 = 3$ for the tens position. In this situation, there are a total of

$$n_1 n_2 n_3 n_4 = (2)(4)(4)(3) = 96$$

even four-digit numbers.

Since the above two cases are mutually exclusive, the total number of even four-digit numbers can be calculated as $60 + 96 = 156$.

Permutations:

A **permutation** is an arrangement of all or part of a set of objects.

For any non-negative integer n, $n!$, called "n factorial," is defined as

$$n! = n(n-1)\cdots(2)(1),$$

with special case $0! = 1$.

The number of permutations of n distinct objects taken r at a time is

$${}_nP_r = \frac{n!}{(n-r)!}.$$

Example:

In one year, three awards (research, teaching, and service) will be given to a class of 25 graduate students in a statistics department. If each student can receive at most one award, how many possible selections are there?

Since the awards are distinguishable, it is a permutation problem. The total number of sample points is

$${}_{25}P_3 = \frac{25!}{(25-3)!} = \frac{25!}{22!} = (25)(24)(23) = 13,800.$$

Example:

A president and a treasurer are to be chosen from a student club consisting of 50 people. How many different choices of officers are possible if

(a) there are no restrictions;

(b) A will serve only if he is president;

(c) B and C will serve together or not at all;

(d) D and E will not serve together?

(a) The total number of choices of officers, without any restrictions, is

$${}_{50}P_2 = \frac{50!}{48!} = (50)(49) = 2450.$$

(b) Since A will serve only if he is president, we have two situations here: (i) A is selected as the president, which yields 49 possible outcomes for the treasurer's position, or (ii) officers are selected from the remaining 49 people without A, which has the number of choices ${}_{49}P_2 = (49)(48) = 2352$. Therefore, the total number of choices is $49 + 2352 = 2401$.

(c) The number of selections when B and C serve together is 2. The number of selections when both B and C are not chosen is ${}_{48}P_2 = 2256$. Therefore, the total number of choices in this situation is $2 + 2256 = 2258$.

(d) The number of selections when D serves as an officer but not E is $(2)(48) = 96$, where 2 is the number of positions D can take and 48 is the number of selections of the other officer from the remaining people in the club except E. The number of selections when E serves as an officer but not D is also $(2)(48) = 96$. The number of selections when both D and E are not chosen is ${}_{48}P_2 = 2256$. Therefore, the total number of choices is $(2)(96) + 2256 = 2448$. This problem also has another short solution: Since D and E can only serve together in 2 ways, the answer is $2450 - 2 = 2448$.

> The number of distinct permutations of n things of which n_1 are of one kind, n_2 of a second kind, $\ldots$, n_k of a kth kind is
>
> $$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Example:

In a college football training session, the defensive coordinator needs to have 10 players standing in a row. Among these 10 players, there are 1 freshman, 2 sophomores, 4 juniors, and 3 seniors. How many different ways can they be arranged in a row if only their class level will be distinguished?

Directly using Theorem 2.4, we find that the total number of arrangements is

$$\frac{10!}{1!\ 2!\ 4!\ 3!} = 12,600.$$

Example:

In how many ways can 7 graduate students be assigned to 1 triple and 2 double hotel rooms during a conference?

The total number of possible partitions would be

$$\binom{7}{3,2,2} = \frac{7!}{3!\ 2!\ 2!} = 210.$$

Combinations

Combinations Notation *Combinations* give the number of ways x elements can be selected from n elements. The notation used to denote the total number of combinations is

$${}_nC_x$$

which is read as "the number of combinations of n elements selected x at a time."

Note that some calculators use r instead of x, so that the combinations notation then reads ${}_nC_r$.

Suppose there are a total of n elements from which we want to select x elements. Then,

n denotes the total number of elements

${}_nC_x$ = the number of combinations of n elements selected x at a time

x denotes the number of elements selected per selection

The number of combinations of n distinct objects taken r at a time is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Example:

A young boy asks his mother to get 5 Game-Boy™ cartridges from his collection of 10 arcade and 5 sports games. How many ways are there that his mother can get 3 arcade and 2 sports games?

The number of ways of selecting 3 cartridges from 10 is

$$\binom{10}{3} = \frac{10!}{3!\ (10-3)!} = 120.$$

The number of ways of selecting 2 cartridges from 5 is

$$\binom{5}{2} = \frac{5!}{2!\ 3!} = 10.$$

Example:

How many different letter arrangements can be made from the letters in the word *STATISTICS*?

Using the same argument as in the discussion for Theorem 2.6, in this example we can actually apply Theorem 2.5 to obtain

$$\binom{10}{3, 3, 2, 1, 1} = \frac{10!}{3!\ 3!\ 2!\ 1!\ 1!} = 50,400.$$

Here we have 10 total letters, with 2 letters (S, T) appearing 3 times each, letter I appearing twice, and letters A and C appearing once each. On the other hand,