Hypothesis Tests for the Means of Two Populations with Equal Standard Deviations, Using Independent Samples

Critical-Value and P-Value Approach

PROCEDURE 10.1 Pooled t-Test

Purpose To perform a hypothesis test to compare two population means, μ_1 and μ_2

Assumptions

1. Simple random samples
2. Independent samples
3. Normal populations or large samples
4. Equal population standard deviations

Step 1 The null hypothesis is H_0: $\mu_1 = \mu_2$, and the alternative hypothesis is

H_a: $\mu_1 \neq \mu_2$ (Two tailed) or H_a: $\mu_1 < \mu_2$ (Left tailed) or H_a: $\mu_1 > \mu_2$ (Right tailed).

Step 2 Decide on the significance level, α.

Step 3 Compute the value of the test statistic

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{(1/n_1) + (1/n_2)}},$$

where

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}.$$

Denote the value of the test statistic t_0.

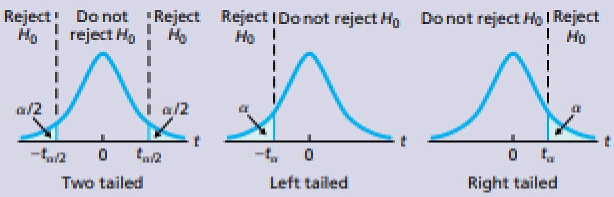
CRITICAL-VALUE APPROACH OR **P-VALUE APPROACH**

CRITICAL-VALUE APPROACH

Step 4 The critical value(s) are

$\pm t_{\alpha/2}$ (Two tailed) or $-t_\alpha$ (Left tailed) or t_α (Right tailed)

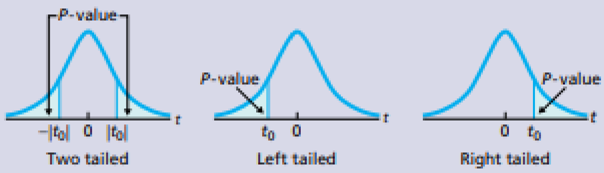
with df $= n_1 + n_2 - 2$. Use Table IV to find the critical value(s).



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

P-VALUE APPROACH

Step 4 The t-statistic has df $= n_1 + n_2 - 2$. Use Table IV to estimate the P-value, or obtain it exactly by using technology.



Step 5 If $P \leq \alpha$, reject H_0; otherwise, do not reject H_0.

Step 6 Interpret the results of the hypothesis test.

Note: The hypothesis test is exact for normal populations and is approximately correct for large samples from nonnormal populations.

Confidence Interval Approach

PROCEDURE 10.2 Pooled *t*-Interval Procedure

Purpose To find a confidence interval for the difference between two population means, μ_1 and μ_2

Assumptions

1. Simple random samples
2. Independent samples
3. Normal populations or large samples
4. Equal population standard deviations

Step 1 **For a confidence level of $1-\alpha$, use Table IV to find $t_{\alpha/2}$ with $df = n_1 + n_2 - 2$.**

Step 2 **The endpoints of the confidence interval for $\mu_1 - \mu_2$ are**

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2} \cdot s_p\sqrt{(1/n_1) + (1/n_2)},$$

where s_p is the pooled sample standard deviation.

Step 3 **Interpret the confidence interval.**

Note: The confidence interval is exact for normal populations and is approximately correct for large samples from nonnormal populations.

Example:

Faculty Salaries Let's return to the salary problem of Example 10.2, in which we want to perform a hypothesis test to decide whether the mean salaries of faculty in private institutions and public institutions are different.

Independent simple random samples of 35 faculty members in private institutions and 30 faculty members in public institutions yielded the data in Table 10.5. At the 5% significance level, do the data provide sufficient evidence to conclude that mean salaries for faculty in private and public institutions differ?

Sample 1 (private institutions)							Sample 2 (public institutions)					
97.3	85.9	118.8	93.9	66.6	109.2	64.9	59.9	115.7	126.1	50.3	133.1	89.3
83.1	100.6	99.3	94.9	94.4	139.3	108.8	82.5	67.1	60.7	79.9	50.1	81.7
158.1	142.4	85.0	108.2	116.3	141.5	51.4	83.9	102.5	109.9	105.1	67.9	107.5
125.6	70.6	74.6	69.9	115.4	84.6	92.0	54.9	41.5	59.5	65.9	76.9	66.9
97.2	55.1	126.6	116.7	76.0	109.6	63.0	85.9	113.9	70.3	90.1	99.7	96.7

Solution First, we find the required summary statistics for the two samples, as shown in Table 10.6. Next, we check the four conditions required for using the pooled t-test, as listed in Procedure 10.1.

- The samples are given as simple random samples and, therefore, Assumption 1 is satisfied.
- The samples are given as independent samples and, therefore, Assumption 2 is satisfied.
- The sample sizes are 35 and 30, both of which are large; furthermore, Figs. 10.2 and 10.3, both on the next page, suggest no outliers for either sample. So, we can consider Assumption 3 satisfied.
- According to Table 10.6, the sample standard deviations are 26.21 and 23.95. These statistics are certainly close enough for us to consider Assumption 4 satisfied, as we also see from the boxplots in Fig. 10.3.

Summary statistics for the samples in Table 10.5

Private institutions	Public institutions
$\bar{x}_1 = 98.19$	$\bar{x}_2 = 83.18$
$s_1 = 26.21$	$s_2 = 23.95$
$n_1 = 35$	$n_2 = 30$

FIGURE 10.2

Normal probability plots of the sample data for faculty in (a) private institutions and (b) public institutions

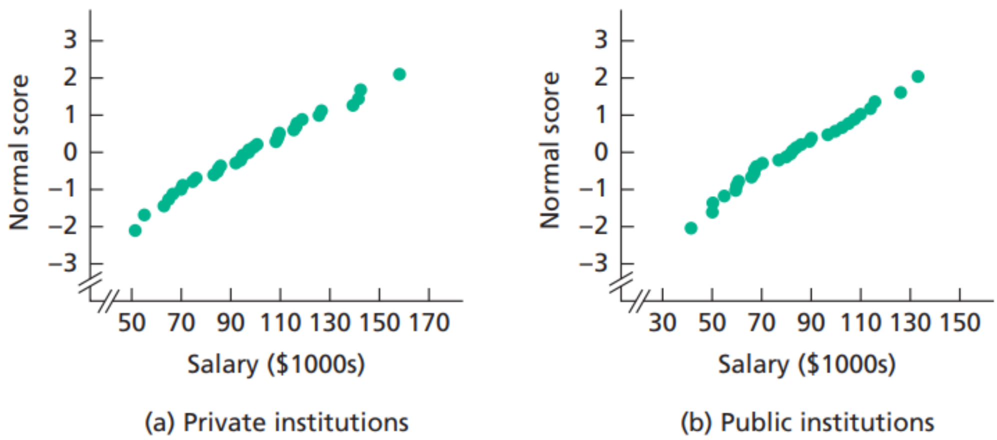
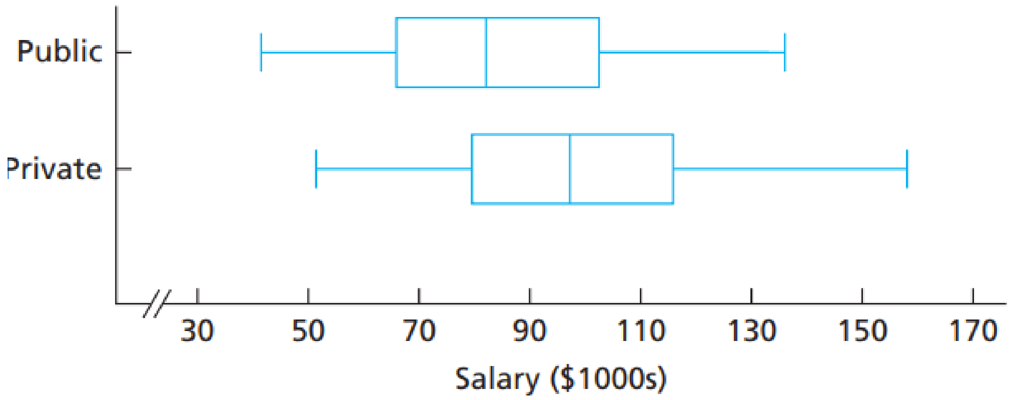


FIGURE 10.3

Boxplots of the salary data for faculty in private institutions and public institutions


Public
Private
30
50
70
90
110
130
150
170
Salary ($1000s)

The preceding items suggest that the pooled t-test can be used to carry out the hypothesis test. We apply Procedure 10.1.

Step 1 State the null and alternative hypotheses.

The null and alternative hypotheses are, respectively,

$$H_0\text{: } \mu_1 = \mu_2 \text{ (mean salaries are the same)}$$
$$H_a\text{: } \mu_1 \neq \mu_2 \text{ (mean salaries are different),}$$

where μ_1 and μ_2 are the mean salaries of all faculty in private and public institutions, respectively. Note that the hypothesis test is two tailed.

Step 2 Decide on the significance level, α.

The test is to be performed at the 5% significance level, or $\alpha = 0.05$.

Step 3 Compute the value of the test statistic

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{(1/n_1) + (1/n_2)}},$$

where

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}.$$

To find the pooled sample standard deviation, s_p, we refer to Table 10.6:

$$s_p = \sqrt{\frac{(35-1)\cdot(26.21)^2 + (30-1)\cdot(23.95)^2}{35+30-2}} = 25.19.$$

Referring again to Table 10.6, we calculate the value of the test statistic:

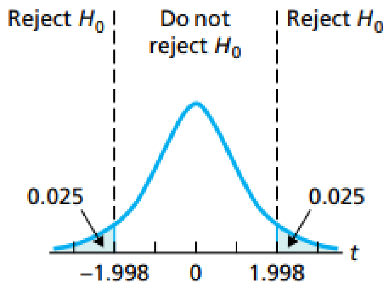
$$t = \frac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{(1/n_1) + (1/n_2)}} = \frac{98.19 - 83.18}{25.19\sqrt{(1/35) + (1/30)}} = 2.395.$$

CRITICAL-VALUE APPROACH	OR	P-VALUE APPROACH

Step 4 The critical values for a two-tailed test are $\pm t_{\alpha/2}$ with $df = n_1 + n_2 - 2$. Use Table IV to find the critical values.

From Table 10.6, $n_1 = 35$ and $n_2 = 30$ and, therefore, $df = 35 + 30 - 2 = 63$. Also, from Step 2, we have $\alpha = 0.05$. In Table IV with $df = 63$, we find that the critical values are $\pm t_{\alpha/2} = \pm t_{0.025} = \pm 1.998$, as shown in Fig. 10.4A.

FIGURE 10.4A



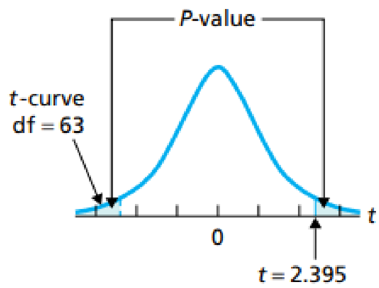
Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

From Step 3, the value of the test statistic is $t = 2.395$, which falls in the rejection region (see Fig. 10.4A). Thus we reject H_0. The test results are statistically significant at the 5% level.

Step 4 The t-statistic has $df = n_1 + n_2 - 2$. Use Table IV to estimate the P-value, or obtain it exactly by using technology.

From Step 3, the value of the test statistic is $t = 2.395$. The test is two tailed, so the P-value is the probability of observing a value of t of 2.395 or greater in magnitude if the null hypothesis is true. That probability equals the shaded area in Fig. 10.4B.

FIGURE 10.4B



From Table 10.6, $n_1 = 35$ and $n_2 = 30$ and, therefore, $df = 35 + 30 - 2 = 63$. Referring to Fig. 10.4B and to Table IV with $df = 63$, we find that $0.01 < P < 0.02$. (Using technology, we obtain $P = 0.0196$.)

Step 5 If $P \le \alpha$, reject H_0; otherwise, do not reject H_0.

From Step 4, $0.01 < P < 0.02$. Because the P-value is less than the specified significance level of 0.05, we reject H_0. The test results are statistically significant at the 5% level and (see Table 9.8 on page 408) provide strong evidence against the null hypothesis.

Step 6 Interpret the results of the hypothesis test.

Interpretation At the 5% significance level, the data provide sufficient evidence to conclude that a difference exists between the mean salaries of faculty in private and public institutions.

Confidence interval Approach

Step 1 For a confidence level of $1 - \alpha$, use Table IV to find $t_{\alpha/2}$ with $df = n_1 + n_2 - 2$.

For a 95% confidence interval, $\alpha = 0.05$. From Table 10.6, $n_1 = 35$ and $n_2 = 30$, so $df = n_1 + n_2 - 2 = 35 + 30 - 2 = 63$. In Table IV, we find that with $df = 63$, $t_{\alpha/2} = t_{0.05/2} = t_{0.025} = 1.998$.

Step 2 **The endpoints of the confidence interval for $\mu_1 - \mu_2$ are**

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2} \cdot s_p\sqrt{(1/n_1) + (1/n_2)},$$

where s_p is the pooled sample standard deviation.

From Step 1, $t_{\alpha/2} = 1.998$. Also, $n_1 = 35$, $n_2 = 30$, and, from Example 10.3, we know that $\bar{x}_1 = 98.19$, $\bar{x}_2 = 83.18$, and $s_p = 25.19$. Hence the endpoints of the confidence interval for $\mu_1 - \mu_2$ are

$$(98.19 - 83.18) \pm 1.998 \cdot 25.19\sqrt{(1/35) + (1/30)},$$

or 15.01 ± 12.52. Thus the 95% confidence interval is from 2.49 to 27.53.

Step 3 **Interpret the confidence interval.**

Interpretation We can be 95% confident that the difference between the mean salaries of faculty in private institutions and public institutions is somewhere between \$2,490 and \$27,530. In other words (see page 465), we can be 95% confident that the mean salary of faculty in private institutions exceeds that of faculty in public institutions by somewhere between \$2,490 and \$27,530. ■

Hypothesis Tests for the Means of Two Populations with un-equal Equal Standard Deviations Using Independent Samples

Critical-Value and P-Value Approach

PROCEDURE 10.3 Nonpooled t-Test

Purpose To perform a hypothesis test to compare two population means, μ_1 and μ_2

Assumptions

1. Simple random samples
2. Independent samples
3. Normal populations or large samples

Step 1 The null hypothesis is H_0: $\mu_1 = \mu_2$, and the alternative hypothesis is

H_a: $\mu_1 \neq \mu_2$ (Two tailed) or H_a: $\mu_1 < \mu_2$ (Left tailed) or H_a: $\mu_1 > \mu_2$ (Right tailed).

Step 2 Decide on the significance level, α.

Step 3 Compute the value of the test statistic

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{(s_1^2/n_1) + (s_2^2/n_2)}}.$$

Denote the value of the test statistic t_0.

CRITICAL-VALUE APPROACH	OR	P-VALUE APPROACH

CRITICAL-VALUE APPROACH

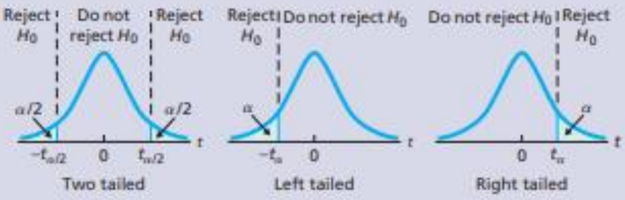
Step 4 The critical value(s) are

$\pm t_{\alpha/2}$ (Two tailed) or $-t_\alpha$ (Left tailed) or t_α (Right tailed)

with df = Δ, where

$$\Delta = \frac{\left[(s_1^2/n_1) + (s_2^2/n_2)\right]^2}{\dfrac{(s_1^2/n_1)^2}{n_1 - 1} + \dfrac{(s_2^2/n_2)^2}{n_2 - 1}}$$

rounded down to the nearest integer. Use Table IV to find the critical value(s).



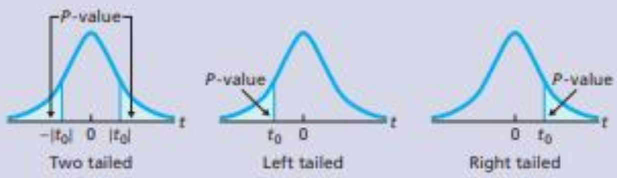
Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

P-VALUE APPROACH

Step 4 The t-statistic has df = Δ, where

$$\Delta = \frac{\left[(s_1^2/n_1) + (s_2^2/n_2)\right]^2}{\dfrac{(s_1^2/n_1)^2}{n_1 - 1} + \dfrac{(s_2^2/n_2)^2}{n_2 - 1}}$$

rounded down to the nearest integer. Use Table IV to estimate the P-value, or obtain it exactly by using technology.



Step 5 If $P \leq \alpha$, reject H_0; otherwise, do not reject H_0.

Step 6 Interpret the results of the hypothesis test.

Confidence Interval Approach

PROCEDURE 10.4 Nonpooled t-Interval Procedure

Purpose To find a confidence interval for the difference between two population means, μ_1 and μ_2

Assumptions

1. Simple random samples
2. Independent samples
3. Normal populations or large samples

Step 1 For a confidence level of $1 - \alpha$, use Table IV to find $t_{\alpha/2}$ with df $= \Delta$, where

$$\Delta = \frac{\left[(s_1^2/n_1) + (s_2^2/n_2)\right]^2}{\dfrac{(s_1^2/n_1)^2}{n_1 - 1} + \dfrac{(s_2^2/n_2)^2}{n_2 - 1}}$$

rounded down to the nearest integer.

Step 2 The endpoints of the confidence interval for $\mu_1 - \mu_2$ are

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2} \cdot \sqrt{(s_1^2/n_1) + (s_2^2/n_2)}.$$

Step 3 Interpret the confidence interval.

Example:

Neurosurgery Operative Times Several neurosurgeons wanted to determine whether a dynamic system (Z-plate) reduced the operative time relative to a static system (ALPS plate). R. Jacobowitz, Ph.D., an Arizona State University professor, along with G. Vishteh, M.D., and other neurosurgeons obtained the data displayed in Table 10.7 on operative times, in minutes, for the two systems. At the 5% significance level, do the data provide sufficient evidence to conclude that the mean operative time is less with the dynamic system than with the static system?

Dynamic							Static		
370	360	510	445	295	315	490	430	445	455
345	450	505	335	280	325	500	455	490	535

Solution First, we find the required summary statistics for the two samples, as shown in Table 10.8. Because the two sample standard deviations are considerably different, as seen in Table 10.8 or Fig. 10.6, the pooled t-test is inappropriate here.

Next, we check the three conditions required for using the nonpooled t-test. These data were obtained from a randomized comparative experiment, a type of designed experiment. Therefore, we can consider Assumptions 1 and 2 satisfied.

To check Assumption 3, we refer to the normal probability plots and boxplots in Figs. 10.5 and 10.6, respectively. These graphs reveal no outliers and, given that the nonpooled t-test is robust to moderate violations of normality, show that we can consider Assumption 3 satisfied.

TABLE 10.8

Summary statistics for the samples in Table 10.7

Dynamic	Static
$\bar{x}_1 = 394.6$	$\bar{x}_2 = 468.3$
$s_1 = 84.7$	$s_2 = 38.2$
$n_1 = 14$	$n_2 = 6$

FIGURE 10.5

Normal probability plots of the sample data for the (a) dynamic system and (b) static system

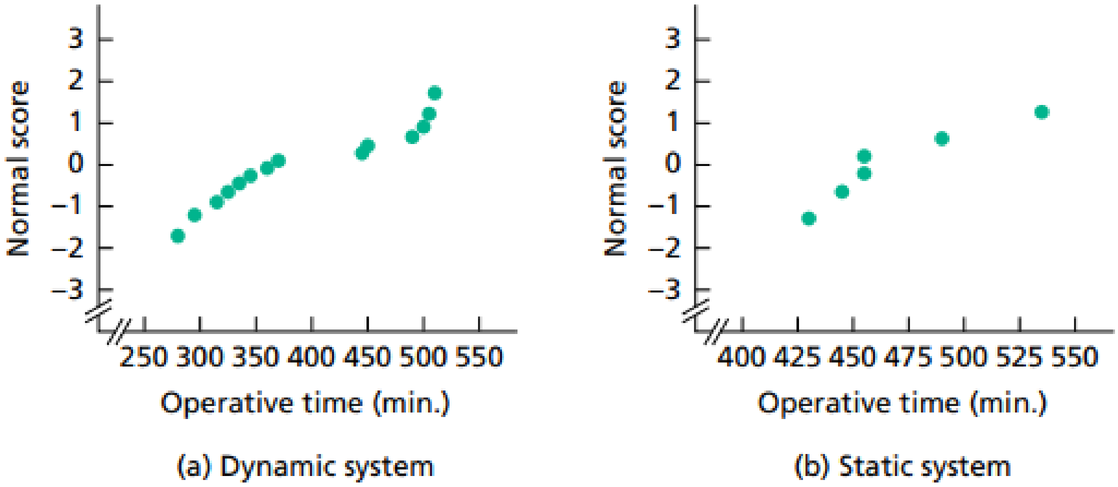
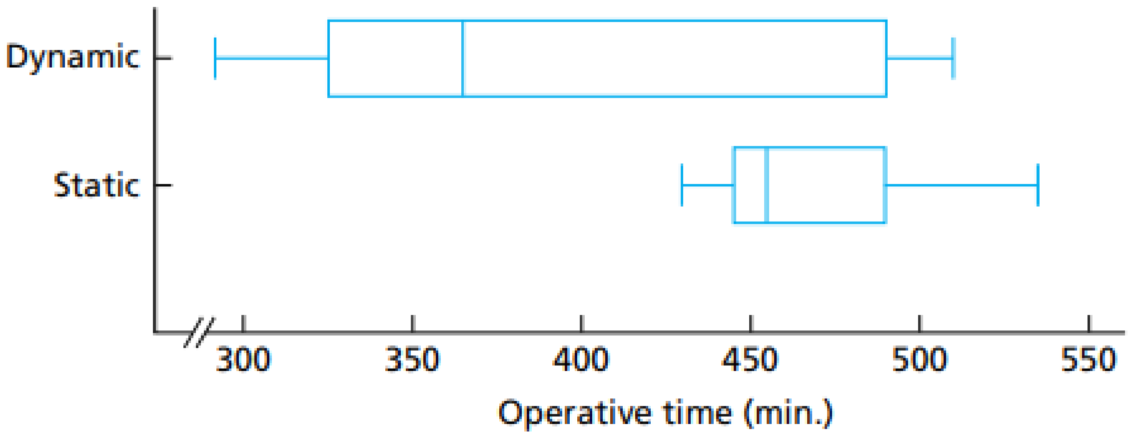


FIGURE 10.6

Boxplots of the operative times for the dynamic and static systems



The preceding two paragraphs suggest that the nonpooled t-test can be used to carry out the hypothesis test. We apply Procedure 10.3.

Step 1 State the null and alternative hypotheses.

Let μ_1 and μ_2 denote the mean operative times for the dynamic and static systems, respectively. Then the null and alternative hypotheses are, respectively,

H_0: $\mu_1 = \mu_2$ (mean dynamic time is not less than mean static time)

H_a: $\mu_1 < \mu_2$ (mean dynamic time is less than mean static time).

Note that the hypothesis test is left tailed.

Step 2 Decide on the significance level, α.

The test is to be performed at the 5% significance level, or $\alpha = 0.05$.

Step 3 Compute the value of the test statistic

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{(s_1^2/n_1) + (s_2^2/n_2)}}.$$

Referring to Table 10.8, we get

$$t = \frac{394.6 - 468.3}{\sqrt{(84.7^2/14) + (38.2^2/6)}} = -2.681.$$

CRITICAL-VALUE APPROACH OR ***P*-VALUE APPROACH**

Step 4 The critical value for a left-tailed test is $-t_\alpha$ with df $= \Delta$. Use Table IV to find the critical value.

From Step 2, $\alpha = 0.05$. Also, from Table 10.8, we see that

$$\text{df} = \Delta = \frac{\left[(84.7^2/14) + (38.2^2/6)\right]^2}{\dfrac{(84.7^2/14)^2}{14-1} + \dfrac{(38.2^2/6)^2}{6-1}},$$

Step 4 The t-statistic has df $= \Delta$. Use Table IV to estimate the P-value, or obtain it exactly by using technology.

From Step 3, the value of the test statistic is $t = -2.681$. The test is left tailed, so the P-value is the probability of observing a value of t of -2.681 or less if the null hypothesis is true. That probability equals the shaded area shown in Fig. 10.7B.

which equals 17 when rounded down. From Table IV with df = 17, we determine that the critical value is $-t_\alpha = -t_{0.05} = -1.740$, as shown in Fig. 10.7A.

FIGURE 10.7A

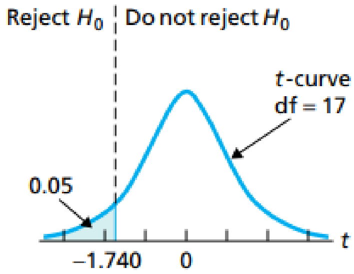
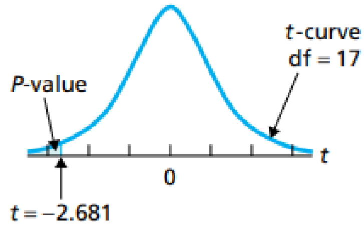


FIGURE 10.7B



From Table 10.8, we find that

$$\text{df} = \Delta = \frac{[(84.7^2/14) + (38.2^2/6)]^2}{\dfrac{(84.7^2/14)^2}{14-1} + \dfrac{(38.2^2/6)^2}{6-1}},$$

which equals 17 when rounded down. Referring to Fig. 10.7B and Table IV with df = 17, we determine that $0.005 < P < 0.01$. (Using technology, we find that $P = 0.00789$.)

Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

From Step 3, the value of the test statistic is $t = -2.681$, which, as we see from Fig. 10.7A, falls in the rejection region. Thus we reject H_0. The test results are statistically significant at the 5% level.

Step 5 If $P \le \alpha$, reject H_0; otherwise, do not reject H_0.

From Step 4, $0.005 < P < 0.01$. Because the P-value is less than the specified significance level of 0.05, we reject H_0. The test results are statistically significant at the 5% level and (see Table 9.8 on page 408) provide very strong evidence against the null hypothesis.

Confidence interval Approach

Step 1 For a confidence level of $1 - \alpha$, use Table IV to find $t_{\alpha/2}$ with df = Δ.

For a 90% confidence interval, $\alpha = 0.10$. From Example 10.6, df = 17. In Table IV, with df = 17, $t_{\alpha/2} = t_{0.10/2} = t_{0.05} = 1.740$.

Step 2 The endpoints of the confidence interval for $\mu_1 - \mu_2$ are

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2} \cdot \sqrt{(s_1^2/n_1) + (s_2^2/n_2)}.$$

From Step 1, $t_{\alpha/2} = 1.740$. Referring to Table 10.8 on page 482, we conclude that the endpoints of the confidence interval for $\mu_1 - \mu_2$ are

$$(394.6 - 468.3) \pm 1.740 \cdot \sqrt{(84.7^2/14) + (38.2^2/6)}$$

or −121.5 to −25.9.

Step 3 Interpret the confidence interval.

Interpretation We can be 90% confident that the difference between the mean operative times of the dynamic and static systems is somewhere between −121.5 minutes and −25.9 minutes. In other words (see page 465), we can be 90% confident that the dynamic system, relative to the static system, reduces the mean operative time by somewhere between 25.9 minutes and 121.5 minutes.

Choosing between a Pooled and a Nonpooled t-Procedure

Suppose you want to use independent simple random samples to compare the means of two populations. To decide between a pooled t-procedure and a nonpooled t-procedure, follow these guidelines: If you are reasonably sure that the populations have nearly equal standard deviations, use a pooled t-procedure; otherwise, use a nonpooled t-procedure.