The Binomial Probability Distribution

An experiment often consists of repeated trials, each with two possible outcomes that may be labeled **success** or **failure**. The most obvious application deals with the testing of items as they come off an assembly line, where each trial may indicate a defective or a nondefective item. We may choose to define either outcome as a success. The process is referred to as a **Bernoulli process**. Each trial is called a **Bernoulli trial**. Observe, for example, if one were drawing cards from a deck, the probabilities for repeated trials change if the cards are not replaced. That is, the probability of selecting a heart on the first draw is 1/4, but on the second draw it is a conditional probability having a value of 13/51 or 12/51, depending on whether a heart appeared on the first draw: this, then, would no longer be considered a set of Bernoulli trials.

The Bernoulli Process:

Strictly speaking, the Bernoulli process must possess the following properties:

1. The experiment consists of repeated trials.
2. Each trial results in an outcome that may be classified as a success or a failure.
3. The probability of success, denoted by p, remains constant from trial to trial.
4. The repeated trials are independent.

Consider the set of Bernoulli trials where three items are selected at random from a manufacturing process, inspected, and classified as defective or nondefective. A defective item is designated a success. The number of successes is a random variable X assuming integral values from 0 through 3. The eight possible outcomes and the corresponding values of X are

Outcome	NNN	NDN	NND	DNN	NDD	DND	DDN	DDD
$\boldsymbol{x}$	0	1	1	1	2	2	2	3

Since the items are selected independently and we assume that the process produces 25% defectives, we have

$$P(NDN) = P(N)P(D)P(N) = \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right) = \frac{9}{64}.$$

Similar calculations yield the probabilities for the other possible outcomes. The probability distribution of X is therefore

x	0	1	2	3
$f(x)$	$\frac{27}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

Binomial Distribution:

The number X of successes in n Bernoulli trials is called a **binomial random variable**. The probability distribution of this discrete random variable is called the **binomial distribution**, and its values will be denoted by $b(x; n, p)$ since they depend on the number of trials and the probability of a success on a given trial. Thus, for the probability distribution of X, the number of defectives is

$$P(X = 2) = f(2) = b\left(2; 3, \frac{1}{4}\right) = \frac{9}{64}.$$

Let us now generalize the above illustration to yield a formula for $b(x; n, p)$. That is, we wish to find a formula that gives the probability of x successes in n trials for a binomial experiment. First, consider the probability of x successes and $n - x$ failures in a specified order. Since the trials are independent, we can multiply all the probabilities corresponding to the different outcomes. Each success occurs with probability p and each failure with probability $q = 1 - p$. Therefore, the probability for the specified order is $p^x q^{n-x}$. We must now determine the total number of sample points in the experiment that have x successes and $n-x$ failures. This number is equal to the number of partitions of n outcomes into two groups with x in one group and $n-x$ in the other and is written $\binom{n}{x}$ as introduced in Section 2.3. Because these partitions are mutually exclusive, we add the probabilities of all the different partitions to obtain the general formula, or simply multiply $p^x q^{n-x}$ by $\binom{n}{x}$.

Binomial Distribution A Bernoulli trial can result in a success with probability p and a failure with probability $q = 1 - p$. Then the probability distribution of the binomial random variable X, the number of successes in n independent trials, is

$$b(x; n, p) = \binom{n}{x} p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n.$$

Example:

The probability that a certain kind of component will survive a shock test is $3/4$. Find the probability that exactly 2 of the next 4 components tested survive.
Assuming that the tests are independent and $p = 3/4$ for each of the 4 tests, we obtain

$$b\left(2; 4, \frac{3}{4}\right) = \binom{4}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^2 = \left(\frac{4!}{2!\ 2!}\right)\left(\frac{3^2}{4^4}\right) = \frac{27}{128}.$$

Example:

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) at least 10 survive, (b) from 3 to 8 survive, and (c) exactly 5 survive?

Let X be the number of people who survive.

(a) $$P(X \geq 10) = 1 - P(X < 10) = 1 - \sum_{x=0}^{9} b(x; 15, 0.4) = 1 - 0.9662$$
$$= 0.0338$$

(b) $$P(3 \leq X \leq 8) = \sum_{x=3}^{8} b(x; 15, 0.4) = \sum_{x=0}^{8} b(x; 15, 0.4) - \sum_{x=0}^{2} b(x; 15, 0.4)$$
$$= 0.9050 - 0.0271 = 0.8779$$

(c) $$P(X = 5) = b(5; 15, 0.4) = \sum_{x=0}^{5} b(x; 15, 0.4) - \sum_{x=0}^{4} b(x; 15, 0.4)$$
$$= 0.4032 - 0.2173 = 0.1859$$

Example:

A large chain retailer purchases a certain kind of electronic device from a manufacturer. The manufacturer indicates that the defective rate of the device is 3%.

(a) The inspector randomly picks 20 items from a shipment. What is the probability that there will be at least one defective item among these 20?

(b) Suppose that the retailer receives 10 shipments in a month and the inspector randomly tests 20 devices per shipment. What is the probability that there will be exactly 3 shipments each containing at least one defective device among the 20 that are selected and tested from the shipment?

(a) Denote by X the number of defective devices among the 20. Then X follows a $b(x; 20, 0.03)$ distribution. Hence,

$$P(X \geq 1) = 1 - P(X = 0) = 1 - b(0; 20, 0.03)$$
$$= 1 - (0.03)^0(1 - 0.03)^{20-0} = 0.4562.$$

(b) In this case, each shipment can either contain at least one defective item or not. Hence, testing of each shipment can be viewed as a Bernoulli trial with $p = 0.4562$ from part (a). Assuming independence from shipment to shipment and denoting by Y the number of shipments containing at least one defective item, Y follows another binomial distribution $b(y; 10, 0.4562)$. Therefore,

$$P(Y = 3) = \binom{10}{3} 0.4562^3(1 - 0.4562)^7 = 0.1602.$$

The Mean and Variance

The mean and variance of the binomial distribution $b(x; n, p)$ are

$$\mu = np \text{ and } \sigma^2 = npq.$$

Example:

It is conjectured that an impurity exists in 30% of all drinking wells in a certain rural community. In order to gain some insight into the true extent of the problem, it is determined that some testing is necessary. It is too expensive to test all of the wells in the area, so 10 are randomly selected for testing.

(a) Using the binomial distribution, what is the probability that exactly 3 wells have the impurity, assuming that the conjecture is correct?

(b) What is the probability that more than 3 wells are impure?

(a) We require

$$b(3; 10, 0.3) = \sum_{x=0}^{3} b(x; 10, 0.3) - \sum_{x=0}^{2} b(x; 10, 0.3) = 0.6496 - 0.3828 = 0.2668.$$

(b) In this case, $P(X > 3) = 1 - 0.6496 = 0.3504$.

Example:

Find the mean and variance of the binomial random variable of Example 5.2, and then use Chebyshev's theorem (on page 137) to interpret the interval $\mu \pm 2\sigma$.

Since Example 5.2 was a binomial experiment with $n = 15$ and $p = 0.4$, by Theorem 5.1, we have

$$\mu = (15)(0.4) = 6 \text{ and } \sigma^2 = (15)(0.4)(0.6) = 3.6.$$

Taking the square root of 3.6, we find that $\sigma = 1.897$. Hence, the required interval is $6 \pm (2)(1.897)$, or from 2.206 to 9.794. Chebyshev's theorem states that the number of recoveries among 15 patients who contracted the disease has a probability of at least 3/4 of falling between 2.206 and 9.794 or, because the data are discrete, between 2 and 10 inclusive.

Example:

Consider the situation of Example 5.4. The notion that 30% of the wells are impure is merely a conjecture put forth by the area water board. Suppose 10 wells are randomly selected and 6 are found to contain the impurity. What does this imply about the conjecture? Use a probability statement.

We must first ask: "If the conjecture is correct, is it likely that we would find 6 or more impure wells?"

$$P(X \geq 6) = \sum_{x=0}^{10} b(x; 10, 0.3) - \sum_{x=0}^{5} b(x; 10, 0.3) = 1 - 0.9527 = 0.0473.$$

As a result, it is very unlikely (4.7% chance) that 6 or more wells would be found impure if only 30% of all are impure. This casts considerable doubt on the conjecture and suggests that the impurity problem is much more severe.

Problems

Problem 1: Which of the following are binomial experiments? Explain why.

1. Selecting a few households from New York City and observing whether or not they own stocks when it is known that 28% of all households in New York City own stocks.
2. Rolling a die 10 times and observing the number of spots
3. In a group of 12 students at a college, 9 use Instagram. Five students are selected from this group of 12 and are asked whether or not they use Instagram. Is this experiment a binomial experiment?

Problem 2: According to a survey, 18% of the car owners said that they get the maintenance service done on their cars according to the schedule recommended by the auto company. Suppose that this result is true for the current population of car owners.
a. Let x be a binomial random variable that denotes the number of car owners in a random sample of 12 who get the maintenance
service done on their cars according to the schedule recommended by the auto company. What are the possible values that x can assume?
b. Find the probability that exactly 3 car owners in a random sample of 12 get the maintenance service done on their cars according to the schedule recommended by the auto company. Use the binomial probability distribution formula.

Problem 3: According to a survey conducted at the local DMV, 50% of drivers who drive to work stated that they regularly exceed the posted speed limit on their way to work. Suppose that this result is true for the population of drivers who drive to work. A random sample of 13 drivers who drive to work is selected. Use the binomial probabilities table to find the probability that the number of drivers in this sample of 13 who regularly exceed the posted speed limit on their way to work is,
a. at most 5 **b.** 6 to 9 **c.** at least 7

Problem 4: Johnson Electronics makes calculators. Consumer satisfaction is one of the top priorities of the company's management. The company guarantees a refund or a replacement for any calculator that malfunctions within 2 years from the date of purchase. It is known from past data that despite all efforts, 5% of the calculators manufactured by the company malfunction within a 2-year period.
The company mailed a package of 10 randomly selected calculators to a store.
a. Let x denote the number of calculators in this package of 10 that will be returned for refund or replacement within a 2-year period. Using the binomial probabilities table, obtain the probability distribution of x and draw a histogram of the probability distribution. Determine the mean and standard deviation of x.

		p										
n	*x*	.05	.10	.20	.30	.40	.50	.60	.70	.80	.90	.95
10	0	.5987	.3487	.1074	.0282	.0060	.0010	.0001	.0000	.0000	.0000	.0000
	1	.3151	.3874	.2684	.1211	.0403	.0098	.0016	.0001	.0000	.0000	.0000
	2	.0746	.1937	.3020	.2335	.1209	.0439	.0106	.0014	.0001	.0000	.0000
	3	.0105	.0574	.2013	.2668	.2150	.1172	.0425	.0090	.0008	.0000	.0000
	4	.0010	.0112	.0881	.2001	.2508	.2051	.1115	.0368	.0055	.0001	.0000
	5	.0001	.0015	.0264	.1029	.2007	.2461	.2007	.1029	.0264	.0015	.0001
	6	.0000	.0001	.0055	.0368	.1115	.2051	.2508	.2001	.0881	.0112	.0010
	7	.0000	.0000	.0008	.0090	.0425	.1172	.2150	.2668	.2013	.0574	.0105
	8	.0000	.0000	.0001	.0014	.0106	.0439	.1209	.2335	.3020	.1937	.0746
	9	.0000	.0000	.0000	.0001	.0016	.0098	.0403	.1211	.2684	.3874	.3151
	10	.0000	.0000	.0000	.0000	.0001	.0010	.0060	.0282	.1074	.3487	.5987
13	0	.5133	.2542	.0550	.0097	.0013	.0001	.0000	.0000	.0000	.0000	.0000
	1	.3512	.3672	.1787	.0540	.0113	.0016	.0001	.0000	.0000	.0000	.0000
	2	.1109	.2448	.2680	.1388	.0453	.0095	.0012	.0001	.0000	.0000	.0000
	3	.0214	.0997	.2457	.2181	.1107	.0349	.0065	.0006	.0000	.0000	.0000
	4	.0028	.0277	.1535	.2337	.1845	.0873	.0243	.0034	.0001	.0000	.0000
	5	.0003	.0055	.0691	.1803	.2214	.1571	.0656	.0142	.0011	.0000	.0000
	6	.0000	.0008	.0230	.1030	.1968	.2095	.1312	.0442	.0058	.0001	.0000
	7	.0000	.0001	.0058	.0442	.1312	.2095	.1968	.1030	.0230	.0008	.0000
	8	.0000	.0000	.0011	.0142	.0656	.1571	.2214	.1803	.0691	.0055	.0003
	9	.0000	.0000	.0001	.0034	.0243	.0873	.1845	.2337	.1535	.0277	.0028
	10	.0000	.0000	.0000	.0006	.0065	.0349	.1107	.2181	.2457	.0997	.0214
	11	.0000	.0000	.0000	.0001	.0012	.0095	.0453	.1388	.2680	.2448	.1109
	12	.0000	.0000	.0000	.0000	.0001	.0016	.0113	.0540	.1787	.3672	.3512
	13	.0000	.0000	.0000	.0000	.0000	.0001	.0013	.0097	.0550	.2542	.5133