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Book: DTSP — 3rd

DSP Notes 11/10/23 — We.

Chapter 2: Discrete-Time Signals & Systems.

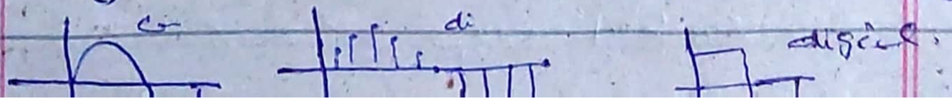
→ The term signal is generally applied to something that carry info. info may be about state or behaviour of the system. Signals are also senthesized for communication purposes.

→ Communication may be in b/w human—human — human—machine or machine—machine.

→ Signal can be represented in many ways, in all cases, the info is contained in some pattern of variation.

→ Mathematically; Signal are represented as function of one or more independent variables.

→ Signals are divide to continuous/analog signals, discrete signals and digital signals.



→ Similarly are classified signal-processing systems; continuous time system, discrete-time system and digital-time system.

2.1 Discrete-Time Signals.

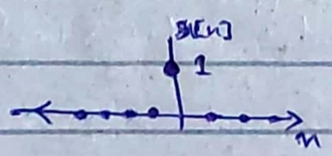
→ DTS are represented mathematically as sequence of numbers. A sequence of number 'x' in which 'n^{th}' number in sequence is denoted by $x[n]$.

$$x = \{x[n]\}, \quad -\infty < n < \infty.$$

→ $x[n]$ is only define for integer value of 'n'. $x[n]$ is not zero for non-integer value of n but simply 'undefine'.

→ unit sample sequence is same in DTS ~~for~~ as unit impulse function for CTS. Also known as DT impulse or simply impulse.

$$\delta[n] = \begin{cases} 0; & n \neq 0. \\ 1; & n = 0. \end{cases}$$



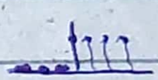
$$* \quad \delta[n] = u[n] - u[n-1]$$

→ Impulse Sequene is used to represent an arbitrary sequence as sum of scaled, delayed impulses.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k].$$

→ The unit step is defined as

$$u[n] = \begin{cases} 1; & n \geq 0 \\ 0; & n < 0 \end{cases}$$



related to unit impulse as

$$u[n] = \sum_{k=-\infty}^{n} \delta[n-k].$$

→ Enponential sequence is generally represented as:

$$x[n] = A\alpha^n$$

- $A \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ then $x[n] \in \mathbb{R}$.
- If $A > 0$ and $0 < \alpha < 1$ then $x[n]$ is decreasing.
- If $A > 0$ and $-1 < \alpha < 0$ the " " " but have opposite sign.
- If $A > 0$ and $|\alpha| > 1$. $x[n]$ grows.

- $A \in \mathbb{C}$ and $\alpha \in \mathbb{C}$ then $x[n]$ has both ~~positive~~ real and ~~negative~~ imaginary values.

(4)

ramp: $m x(t) + b$

Parabola: $a x^2(t) + b x(t) + c$.

Gaussing. $a e^{\frac{-(t-\mu)^2}{2\sigma^2}}$

→ If unit of n to be 'sample' Then ω_0 to be radians per ~~cycle~~. sample.

→ $\cos((\omega_0 + 2\pi r) n + \phi)$
$= \cos(\omega_0 n + \phi) \quad \because n \in Z$

→ A periodic sequence is a sequence for which
$x[n] = x[n+N] \quad \because N \in Z$
$= A \cos(\omega_0 (n+N) + \phi)$
$= A \cos(\omega_0 n + \omega_0 N + \phi)$
which requires
$\omega_0 N = 2\pi k \quad \because k \in Z$

→ Similarly for exponential sequence
$e^{j\omega_0 (n+N)} = e^{j\omega_0 n} \quad \because \omega_0 N = 2\pi k.$

→ For $x[n] = A \cos(\omega_0 n + \phi)$, as ω_0 increase from $\omega_0 = 0$ to $\omega_0 = \pi$, $x[n]$ oscilates more rapidly. As ω_0 increase from $\omega_0 = \pi$ to $\omega_0 = 2\pi$, the oscilation become slower.

→ Value of ω_0 in vicinity of $\omega_0 = 2\pi k$ when $k \in Z$ are refered to as low frequencies (relatively slow oscillations), wher for vicinity of $\omega_0 = (\pi + 2\pi k)$ for $k \in \mathbb{Z}$ is called higher frequencies (rapid oscillations).

1.2 Discrete-Time Systems:

→ Mathematicall, System is a transformation or operator that maps input sequence $x[n]$ into output sequence with values $y[n]$.

$$y[n] = T\{x[n]\}$$

$$x[n] \longrightarrow \boxed{T\{\cdot\}} \longrightarrow y[n].$$

*shifting

→ The ideal delay system shifts the input sequence to the *right/left by n_d sample to form the output

$$y[n] = x[n-n_d] \qquad -\infty < n_d < \infty$$

$n_d \in \mathbb{Z}$

*right when n_d is +ve $\mathbb{Z}$
and left when n_d is −ve $\mathbb{Z}$

→ The general moving-average system is defined by the

else $y[n] \neq y[n-n_0] \Rightarrow$ TV equation:

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

→ Memory Less systems are those systems if the output y[n] at every value of n depends only on the input x[n] at the same value of n.

$$y[n] = f(x[n])$$

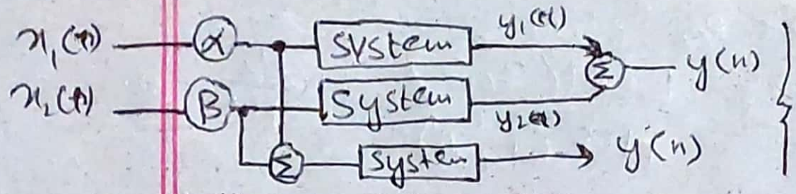
- the delay or advanced systems are't memory less systems.

→ Linear system are those which follows the law of Superposition i.e Law of additivity and law of homogenity/scalability.

- If $x_1[n] \longrightarrow y_1[n]$
  and $x_2[n] \longrightarrow y_2[n]$

  then $x_1[n] + x_2[n] \longrightarrow y_1[n] + y_2[n]$

- $a x_1[n] \longrightarrow a y_1[n]$.



Linear System

- generally we can represent a linear system by LCCDE(linear, constant-coefficient difference equation):

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k].$$

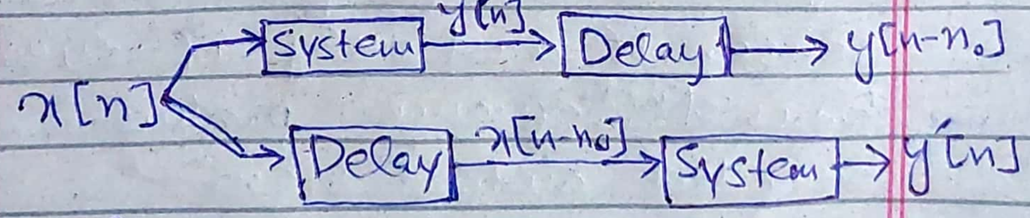
where $x[n]$ is input $y[n]$ is output and a_k and b_k are consfants.

- $T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\}$
  $= y_1[n] + y_2[n]$

  $T\{a x[n]\} = aT\{x[n]\} = a y[n].$

→ Non-linear Systems are those which don't follow law of syperposition.

→ Time-invariant / shift-invariant system is a system for which a time shift in input causes a corresponding shift in output sequence.



If $y'[n] = y[n-n_0] \Rightarrow$ TIV else $y'[n] \neq y[n-n_0] \Rightarrow$ TV

⑧ $t_1 = 3/7$, $t_2 = 5/2$

$LCM(t_1, t_2) = \frac{LCM(3, 5)}{HCF(7, 2)} = \frac{15}{1} = 15$

→ The compressor System:

$y[n] = x[Mn] \quad -\infty < n < \infty$

with M a +ve Z is called compressor system. It is a subsystem of time scaling.

$y[n] = x[\alpha n] \; ; -\infty < n < \infty$; $\alpha \in Z$

→ $\alpha > 1$ the signal is compressed

→ $\alpha < 1$ the signal is expanded.

→ Amplitude scaling is the process of rescaling the amp of a sig.

$y[n] = \alpha x[n] \quad -\infty < n < \infty, \alpha \in Z$

→ $\alpha > 1$: signal is amplified

→ $\alpha < 1$: signal is attinuated

→ Causal system is a system in which the output only depends only on past and present values of input.

→ A system is stable in bound input - bound output (BIBO) sense. If and only if every bound input sequence produce a bounded output signal.

$$|x[n]| \leq B_x < \infty \quad ; \text{all } n$$

$$|y[n]| \leq B_y < \infty \quad ; \text{all } n.$$

where B_x and a finite input value which produce finite B_y value.

1.3 LTI System:

→ Important class of systems which are both linear and time invariant.

→ If the linearty property is combined with the representation of a general sequence as a linear combination of delayed impulses, it follow that a linear system can be completely characterised by its impulse response:

$$y[n] = T\left\{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right\}$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,T\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]\,h_k[n]$$

↓

from principle of superposition

From time invarian we know that if $h[n]$ is response of $\delta[n]$, then $h[n-k]$ is response of $\delta[n-k]$. The above equation is

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \quad \text{for all } n.$$

→ This equation is the convolution sum and is simply represented as:

$$y[n] = x[n] * h[n].$$

→ Convolution-sum expression is analogous to the convolution integral of continuous-time linear system theory.

→ $h[n-k] = h[-(k-n)]$.

4 Properties of Linear Time-Invariant Systems:

→ Since LTI systems are described by convolution sum, hence its properties are also defined by properties of DT convolution.

→ Convolution operation is commutative:

$$x[n] * h[n] = h[n] * x[n].$$

proof: Let $y[n] = x[n] * h[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Let $m = n-k \Rightarrow k = n-m$

and the limits become

for $k=-\infty \Rightarrow m = n-(-\infty) = \infty$, and $k=\infty \Rightarrow m = n-\infty = -\infty$

hence the $y[n]$ becomes

$$y[n] = \sum_{m=\infty}^{-\infty} x[n-m]\, h[m]$$

$$y[n] = \sum_{m=-\infty}^{\infty} x[n-m]\, h[m] \quad \because \text{tion is comutative}$$

$$= \sum_{m=-\infty}^{\infty} h[m]\, x[n-m] \quad \because \text{xion is comutative}$$

$$= h[n] * x[n].$$

$$= y[n]$$

→ which shows that the order of sequence in convolution operator is unimportant. The output is same if role of system response and input are reversed

→ The convolution operation is also distributive over addition. i.e.

$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$

→ The convolution operation is also satisfy associative property i.e.

$y[n] = (x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$

• The order can be also change because of commutative property.

→ This show that if two systems are cascaded in either order the output will always be h[n],

$h[n] = h_1[n] * h_2[n] = h_2[n] * h_1[n]$.

→ Parallel combinations are distributive while series/cascade combiniations are associative.

→ Stable system is a system for which every bounded input produces a bounded output (BIBO).

→ LTI system are stable if and only if the impulse response is absolutely summable, i.e., if

$$B_h = \sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

As $|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\right|$

$$\leq \sum_{k=-\infty}^{\infty} |h[k]|\, |x[n-k]|$$

If $x[n]$ is bounde i.e $|x[n]| \leq B_x$

The the equation become

$$|y[n]| \leq B_h \cdot B_x$$

→ B_h is sufficient condition for stability. in other word $y[n]$ is bound if equation B_h holds.

→ The causal system are those for which the output $y[n_0]$ depends only on the input samples $x[n]$, for $n \leq n_0$. ~~i.e h[n] = 0 for n < 0~~.

→ For LTI system the above definition implies the condition $h[n] = 0$ for $n < 0$.

→ Somtimes a sequence i.e 0 for $n < 0$ is refer to as causal sequence meaning that it could be the impulse response of a causal system.

→ Convolution of a shifted impulse sequence with any signal $x[n]$ is easily evaluated by simply shifting $x[n]$ by the displacement of the impulse. i.e

$$y[n] = x[n] * h[n]$$
$$= x[n] * \delta[n-n_d] = \delta[n-n_d] * x[n]$$
$$= x[n-n_d].$$

→ Generally, any non-causal FIR system can be made causal by cascading it with a sufficient long delay.

1.6 Frequency-domain representation of Discrete-time signals and Systems:

→ Complex exponential sequences are eignfunctions of LTI systems and the response to a sinusoidal input is sinusoidal with same frequency as input and with amplitude and phase determined

$X(e^{j\omega}) = \underset{\text{eignvalue}}{\lambda}\ \underset{\text{eignfunction}}{x(t)}$

Gaussian function

$g(t) = e^{-t^2/2}$ — similar

$\Updownarrow$

$G(j\omega) = \sqrt{2\pi}\, e^{-\omega^2/2}$

$= \sqrt{2\pi}\, g(\omega)$

by the system.

→ Specificall with input $x[n] = e^{j\omega n}$ for $-\infty < n < \infty$ the corresponding output of LTI with impulse response $h[n]$ is shown to be

$$y[n] = H(e^{j\omega}) \cdot e^{j\omega n}$$

where,

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

here $x[n] = e^{j\omega n}$ is eignfunction and associated eignvalue is $H(e^{j\omega})$.

→ $H(e^{j\omega})$ describe changes in complex amplitude of complex exponential input signal as function of frequency ω.

→ As $H(e^{j\omega}) \in \mathbb{C} \Rightarrow$

$$H(e^{j\omega}) = H_R(e^{j\omega}) + j\,\underset{\text{imaginary part}}{H_I(e^{j\omega})}$$

or $$\underset{\text{real part.}}{H(e^{j\omega})} = \underset{\text{amp}}{|H_R(e^{j\omega})|}\, e^{j\angle \underset{\text{phase}}{H(e^{j\omega})}}$$

16

Example 1.14 (p# 45):

$$y[n] = x[n-n_d]$$

Sol: Let put $x[n] = e^{j\omega n}$

$$y[n] = e^{j\omega(n-n_d)} = e^{j\omega n} \cdot e^{-j\omega n_d}$$

$$y[n] = x[n]\, e^{-j\omega n_d}$$

which show that $H(e^{j\omega}) = e^{-j\omega n_d}$

$H_R(e^{j\omega}) = \cos\omega n_d$, $H_I(e^{j\omega}) = -\sin\omega n_d$

$|H(e^{j\omega})| = 1$, $\angle H(e^{j\omega}) = -\omega n_d$.

→ Frequency response of discrete-time LTI system is always a periodic function of the frequency variable ω with period 2π.

$$H(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j(\omega+2\pi)n}$$

as $e^{\pm j2\pi n} = 1$ for $n \in Z$.

$$e^{-j(\omega+2\pi)n} = e^{-j\omega n} \cdot \cancel{e^{-j2\pi n}} = e^{-j\omega n}.$$

Therefor $H(e^{j(\omega+2\pi)}) = H(e^{j\omega})$

More generally

$$H(e^{j(\omega+2\pi r)}) = H(e^{j\omega})$$

for $r \in Z$.

→ This is obviously true for the ideal delay system, since $e^{-j(\omega+2\pi)n_d} = e^{-j\omega n_d}$ when $n_d \in \mathbb{Z}$.

→ Since $H(e^{j\omega})$ is periodic with period 2π, and the frequency ω and $\omega+2\pi$ are indistinguishable, it follow that we only need $H(e^{j\omega})$ over an interval of length 2π eg. $0 \le \omega \le 2\pi$ or $-\pi \le \omega \le \pi$.

→ This shows that low frequencies are that are close to zero (or even multiple of π) and high frequencies are close to $\pm\pi$ (or odd multiple of π).

1.6.2 Suddenly applied Complex exponential inputs:

→ Input of the form $e^{j\omega n}$ for $-\infty < n < \infty$ produce output of the form $H(e^{j\omega})e^{j\omega n}$ for LTI systems.

→ We can gain insights into LTI system by considering input of the form

$$x[n] = e^{j\omega n} u[n],$$

i.e complex exponentials that are suddenly applied at arbitrary time.

2.7 Representation of sequence by Fourier transform.

→ Many sequences can be represeted by fourier integral of the form-

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \quad (*)$$

where $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}. \quad (**)$$

→ Eq (*) is "inverse Fourier transform", is the synthesis formula. It represent $x[n]$ as a superposition of infinitesimally small complex sinusoids of form

$$\frac{1}{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega.$$

with ω ranging over an interval of 2π, and $X(e^{j\omega})$ determine the relative amount of each complex sinusoidal component.

→ Eq (**), the Fourier Transform, (often refered to as discrete-time Fourier transform [DTFT], to distinguish it from continuous-time FT), is an expression for computing $X(e^{j\omega})$ from sequence $x[n]$, i.e, for analyzing the sequence $x[n]$ to determine how much of each frequency component is required to synthesize $x[n]$.

→ As $X(e^{j\omega}) \in \mathbb{C}$, hence in rectangular form

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$$

and in polar form

$$X(e^{j\omega}) = |X(e^{j\omega})| \, e^{j\angle X(e^{j\omega})}.$$

→ If $x[n]$ is absalutely summable, the $X(e^{j\omega})$ exists. It shows that all stable systems/sequences are transformable.

→ Any stable system i.e one having an absalutly summable impulse response will have a finite and continuous frequency response.

*E Example 2.17:

$$x[n] = a^n u[n] \quad , \quad FT\{x[n]\} = ?$$

Sol?. as $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \, e^{-j\omega n}$.

$$X(e^{j\omega}) = \sum_{n=\infty}^{\infty} a^n u[n] \, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

as this is a G.S hence its infinite sum is $\frac{a_1}{1-r}$ $|r|<1$

here $a_1 = 1$ and $r = \frac{ae^{-j\omega}}{1}$ for $|r|$

hence

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \quad \text{for } |ae^{-j\omega}| < 1 \text{ or } |a| < 1$$

It shows that $|a| < 1$ is the condition for absolute summability.

*E Example 2.18: Ideal LPF.

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

Sol: $h_{LP}(t)$ can be found using

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} \, d\omega$$

$$h_{LP}[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 \cdot e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi j n} \cdot \left. e^{j\omega n} \right|_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi j n} \left[e^{j\omega_c n} - e^{-j\omega_c n} \right]$$

$$= \frac{1}{\pi n} \left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j} \right] \quad \because \frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$$

$$= \frac{\sin \omega_c n}{\pi n}$$

For explaination of this example see book p# 55.

2.8 Symmetry Properties of Fourier Transform

→ A conjugate-symmetric sequence $x_e[n]$ is a sequence for which $x_e[n] = x_e^*[-n]$ and a conjugate-antisymmetric sequence $x_o[n]$ is defined as $x_o[n] = -x^*[-n]$

→ Any sequence $x[n]$ can be expressed as sum of conjugate-symmetric and conjugate-antisymmetric sequence.

$$x[n] = x_e[n] + x_o[n]$$

where

$$x_e[n] = \frac{1}{2}\left(x[n] + x^*[-n]\right) = x_e^*[-n]$$

and

$$x_o[n] = \frac{1}{2}\left(x[n] - x^*[-n]\right) = -x_o^*[-n]$$

also $x_e[n]$ is even and $x_o[n]$ is odd sequences.

→ Similarly a Fourier transform $X(e^{j\omega})$ can be decomposed into sum of conjugate-symmetric and conjugate-antisymmetric functions

$$X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega})$$

→ see table in book p# 59.

2.9 Fourier Transform Theorms:

→ Fourier transform is linear.

If $x_1[n] \xrightarrow{\mathcal{F}} X_1(e^{j\omega})$

and $x_2[n] \xrightarrow{\mathcal{F}} X_2(e^{j\omega})$

Then

$$a x_1[n] + b x_2[n] \xrightarrow{\mathcal{F}} a X_1(e^{j\omega}) + b X_2(e^{j\omega}).$$

→ Time shifting and frequency shifting:

If $x[n] \xrightarrow{\mathcal{F}} X(e^{j\omega})$

Then

$$x[n-n_d] \xrightarrow{\mathcal{F}} e^{j\omega n_d} X(e^{j\omega})$$

$$x[n]e^{j\omega_0 n} \xrightarrow{\mathcal{F}} X(e^{j(\omega-\omega_0)}).$$

→ Time reversal

If $x[n] \xrightarrow{\mathcal{F}} X(e^{j\omega})$

then $x[-n] \xrightarrow{\mathcal{F}} X(e^{-j\omega})$

If $x[n]$ is real then

$$x[-n] \xrightarrow{\mathcal{F}} X^*(e^{j\omega})$$

→ Differentiation in frequency

If $x[n] \xrightarrow{\mathcal{F}} X(e^{j\omega})$

Then

$$n\,x[n] \longrightarrow j\frac{d}{d\omega} X(e^{j\omega})$$

→ Parseval theorm, the function $|X(e^{j\omega})|^2$ is called the "energy density spectrum", since it determines how energy is distributed in the frequency domain.

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega.$$

Energy density spectrum is defined only for finite-energy signals.

→ Convolution of sequences implies multiplication of the corresponding Fourier transform.

If $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \underbrace{x[n]}_{\xrightarrow{F} X(e^{j\omega})} * \underbrace{h[n]}_{\xrightarrow{F} H(e^{j\omega})}$

Then $Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$.

→ The modulation / windowing theorm

If $y[n] = \underbrace{x[n]}_{\xrightarrow{F} X(e^{j\omega})} \; \underbrace{w[n]}_{\xrightarrow{F} W(e^{j\omega})}$

then

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta}) W(e^{j(\omega-\theta)}) \, d\theta.$$

This is a periodic convolution, i.e., a convolution of two periodic functions with the limits of integration extending over only one period.

→ For continuous-time, convalution in time domain is represented by multiplication in frequency domain and vice vers.
For discrete-time, the convolution sum is equivalent to multiplication of corresponding periodic Fourier transforms, and multiplication of sequences is equivalent to periodic convolution of corresponding Fourier transforms.

2.10 Discrete-time Random Signals:

→ Untill now, we have assumed that signals are deterministic, i.e, each value of a sequence is uniqly determined by a mathematical expression, a tabl of data, or a rule of some type.

→ A random signal (stochastic) is considered to be a member of an ensemble of DT signals that is characterised by a set of probability density functions.

→ Each individual sample ~~of~~ $x[n]$ of a particular signal is assume to be an outcome of some underlying random variable x_n. This collection of random variables is reffered to as random process.

→ Since they are not summable, they do not directly have FT (some have but not all).

→ Many of the properties ~~have~~ of such signals can be summerized in term of averages such as the autocorrelation or autocovariance sequences for which the FT often exists.