11 Jan 2024
Thu

Communication Systems Notes

chapter # 6: Sampling & Pulse Code Modulation.

→ Analog signal can be digitized through sampling and quantization.

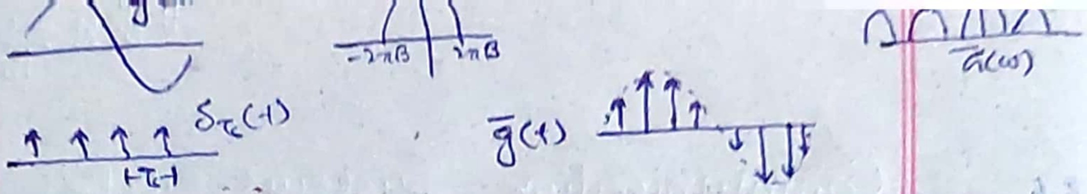
6.1 Sampling Theorm:

→ Sampling theorm is the basis for determing the proper sampling rate for a given signal.

→ sampling rate must be sufficiently large so that the analog signal can be reconstructed from samples with sufficient accuracy.

→ A signal of band limited to B-Hz can be reconstructed exactly from its samples taken uniformly at a rate $R > 2B$ Hz (samples/secon).

→ The minimum sampling frequency is $f_s = 2B$ Hz.



→ To prove the sampling theorm let a signal $g(t)$ whose spectrum of band-limited to B Hz.
We can get the sampled signal $\bar{g}(t)$, by multipling $g(t)$ with impulse train of $\delta_{T_s}(t)$ consist of unit impulses repeating periodically at every $T_s = 1/f_s$ seconds. i.e

$$\bar{g}(t) = g(t)\,\delta_{T_s}(t)$$

The n^{th} impulse located at $t = nT_s$ having strength $g(nT_s)$ is

$$\bar{g}(t) = \sum_n g(nT_s)\,\delta(t - nT_s)$$

As $\delta_{T_s}(t)$ is periodic signal hence by Fourier series.

$$\delta_{T_s}(t) = C_0 + \sum C_n \cos(n\omega_0 t + \theta_n) \qquad \because \omega_s = \frac{2\pi}{T_s}$$

$$C_0 = a_0 = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)\,dt$$

$$= 1/T_s$$

$$a_n = \frac{2}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)\cos n\omega_0 t\,dt$$

$$= 2/T_s$$

$$b_n = \frac{2}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)\sin n\omega_0 t\,dt = 0$$

③ $\because$ $g(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$ $t_1 \le t \le t_1 + T_0$

where $C_0 = a_0 = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} g(t)\,dt$; $a_n = \frac{2}{T_0}\int_0^{T_0} g(t)\cos n\omega_0 t\,dt$, $b_n = \frac{2}{T_0}\int_0^{T_0} g(t)\sin n\omega_0 t\,dt$

$C_n = \sqrt{a_n^2 + b_n^2}$, $\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$.

hence $C_n = \sqrt{(2/T_s)^2} = 2/T_s$

and $\theta_n = 0$

$$\delta_{T_s}(t) = \frac{1}{T_s}\left(1 + 2\sum \cos n\omega_s t\right)$$

Therefor $\bar{g}(t) = g(t)\,\delta_{T_s}(t)$

$$= \frac{1}{T_s}\left(g(t) + 2g(t)\sum \cos n\omega_s t\right).$$

$$= \frac{1}{T_s}\left(g(t) + 2g(t)\cos\omega_s t + 2g(t)\cos 3\omega_s t + \cdots\right)$$

By Fourier transform of $\bar{g}(t)$ we get

$$\bar{G}(\omega) = \frac{1}{T_s}\left[G(\omega) + G(\omega \pm \omega_s) + G(\omega \pm 2\omega_s) + \cdots\right]$$

$$= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} G(\omega - n\omega_s)$$

This equation shows that we can recover $g(t)$ from $\bar{g}(t)$ if there is no overlap between successive cycle of $\bar{G}(\omega)$. This show that this require $f_s \geq 2B$

$$\frac{1}{f_s} \leq \frac{1}{2B}$$

$$T_s \leq \frac{1}{2B}$$

(4) $*\ \frac{W}{\pi} \operatorname{sinc}(W t) \Leftrightarrow \operatorname{rect}\left(\frac{\omega}{2W}\right)$.

→ The minimum sampling rate $f_s = 2B$ required to recover $g(t)$ from its samples $\bar{g}(t)$ is called Nyquist rate for $g(t)$ and the corresponding sampling interval $T_s = 1/2B$ is called Nyquist Interval for $g(t)$.

6.1.1 Signal Reconstruction : The interpolation Formula.

→ The process of reconstructing a continuous-time signal $g(t)$ from its samples is known as interpolation.

→ As the sampled signal contains a component $\frac{1}{T_s} g(t)$, and to recover $g(t)$ [or $G(\omega)$], the sampled signal must be passed through an ideal low pass folter of bandwidth B Hz and gain T_s. i.e:

(★) — $$H(\omega) = T_s \operatorname{rect}\left(\frac{\omega}{4\pi B}\right)$$

→ Now lets examin it in time domain. Let we have an interpolating filter of response $h(t)$. If we pass $\bar{g}(t)$

$$\bar{g}(t) \longrightarrow \boxed{h(t)} \longrightarrow g(t).$$

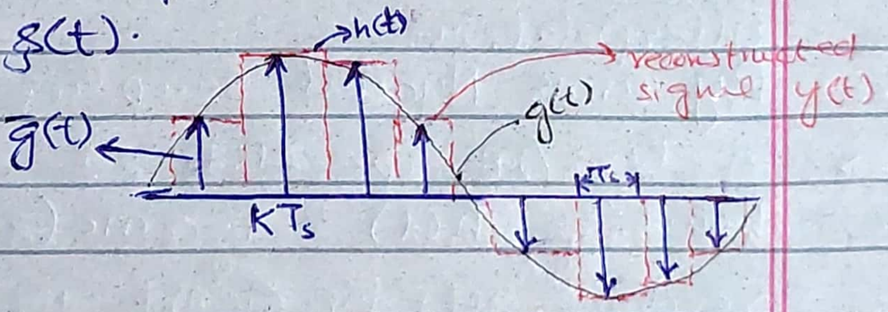
through this filter its output response would be $g(t)$.

The filter is a gate pulse of unit height, centered at origin and of width T_s. i.e $h(t) = \text{rect}(t/T_s)$

- When a K^{th} sample $[g(KT_s)\,\delta(t-KT_s)]$ of $\bar{g}(t)$ is passed throug the filter it generates $g(KT_s)\,\text{rect}(t/T_s)$ a gate pulse of height $g(KT_s)$ and centered at $t = KT_s$. Each sample in $\bar{g}(t)$ will produce

$$y = \sum_K g(KT_s)\,\text{rect}(t/T_s).$$

This output is a staircase approximation of $g(t)$.



The fourier Transform of $h(t)$ is ~~given~~ find by assuming Nyquist rate ($T_s = 1/2B$)

$$h(t) = \text{rect}\left(\frac{t}{T_s}\right) = \text{rect}(2B\cdot t).$$

$$\Rightarrow H(\omega) = T_s\,\text{sinc}\left(\frac{\omega T_s}{2}\right) = \frac{1}{2B}\,\text{sinc}\left(\frac{\omega}{4B}\right)$$

→ This filter is known as

zero-order hold filter which is poor approximation of ideal LPF required for exact interpolation.

→ We can improve on the zero-order hold filter by using first-order hold filter, which result in a linear interpolation instead of staircase interpolation.

→ By using the ideal filter from page 4 eq ★. taking inverse fourier transform of $H(\omega)$ we get:

$$h(t) = 2BT_s \text{ sinc}(2\pi B \cdot t)$$

By using Nyquist sampling rate

$$T_s = 1/2B \Rightarrow 2BT_s = 1$$

hence $\quad h(t) = 1 \cdot \text{sinc}(2\pi B \cdot t)$

→ This show that $h(t) = 0$ at all Nyquest sampling instants except at $t=0$ i.e

$$2\pi B \cdot t = \pm n\pi$$

$$t = \pm n/2B$$

→ The K^{th} impulse of input $\overline{g}(t)$ is the impulse $g(KT_s)\,\delta(t-KT_s)$;

The filtered out put of this impulse is $g(KT_s)\,h(t-KT_s)$.
hence $g(t)$ produced from $\overline{g}(t)$ can be expressed as sum

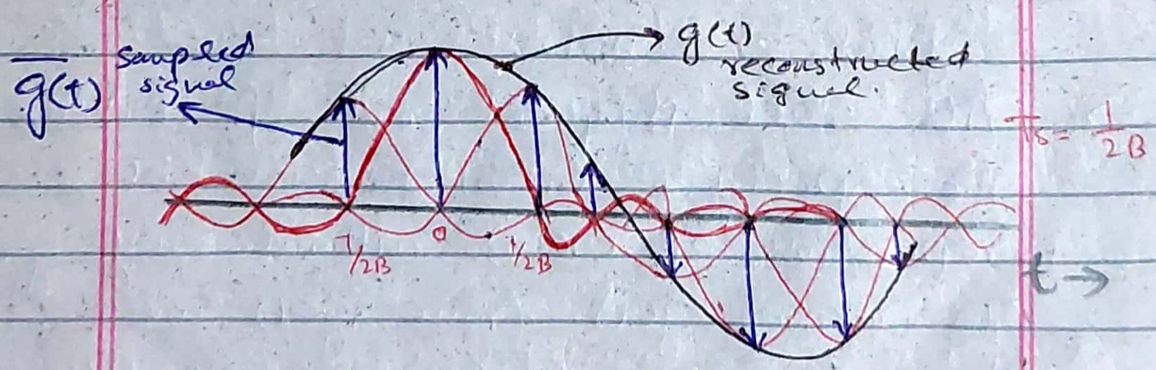
$$g(t) = \sum_K g(KT_s)\,h(t-KT_s)$$

$$= \sum_K g(KT_s)\,\text{sinc}[2\pi B(t-KT_s)]$$

$$= \sum_K g(KT_s)\,\text{sinc}\left[2\pi Bt - \cancel{2}\pi \cancel{B}\cdot K\frac{1}{\cancel{2B}}\right]$$

$$g(t) = \sum_K g(KT_s)\,\text{sinc}[2\pi Bt - K\pi]$$

→ This equation is the interpolation formula, which yields values of $g(t)$ between samples as a weighted sum of all the sample values.



Example 6.1: $g(t) = ?$ of B-Hz

$$\overline{g}(t) = [\underset{0}{1},\ \underset{\pm T_s}{0},\ \underset{\pm 2T_s}{0},\ \underset{\pm 3T_s}{0},\ \cdots].$$

Solution: By using interpolation formula we have

$$g(t) = \sum_{k} g(kT_s)\,\text{sinc}(2\pi B.t - k\pi)$$

as we can see only at $k=0$
$g(0)=1$ and at $k \neq 0$ $g(t)=0$
hence the above formula become

$$g(t) = 1 \cdot \text{sinc}(2\pi B.t - 0\pi) + 0 + 0 + \cdots$$
$$= \text{sinc}(2\pi B t)$$

hence this is the reconstructed signal of bandwidth B Hz, $g(0)=1$ and $g(nT_s)=0$ (at $n \neq 0$).

6.1.2 Practical Dificulties in signal reconstruction.

→ At exact Nyquest rate $f_s = 2B$ Hz the spectrum $\overline{G}(\omega)$ consists of repetition of $G(\omega)$ without any gap. To recover $g(t)$ from $\overline{g}(t)$ we have to pass it through LPF discussed above which is ideal filter. such a filter is unrealizable.

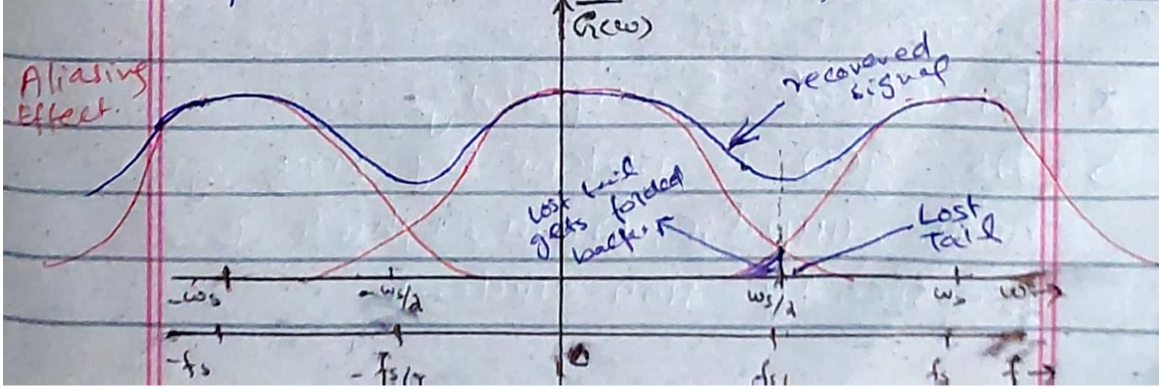
→ A practical solution to this problem is to use a heigher rate for sampling than Nyquest rate ($f_s > 2B$ or $\omega_s > 4\pi B$).
Thus we can use LPF with gradual cuttoff.

* The Treachery of Aliasing.

→ All practical signals are time-limited.

→ If a signal is time-limited, it cann't be bandlimited, and vice versa. (but it can be simultaneously non-time-limited and non-band-limited.

→ Because of infinite bandwidth the spectrum overlaps regardless of sampling rate. Because of overlaping tails $\overline{G}(\omega)$ no longer has completer information about $G(\omega)$, and it



is no longer possible to recover, even theoretically $g(t)$ from $\bar{g}(t)$. If this is passed throug LPF the signal output is portion/version of $G(\omega)$ not exact $G(\omega)$. due to
- loss of tail beyond $|f| > f_s/2$ Hz
- reappearance of this tail inverted or folded onto the spectrum.

→ $f_s/2 = \frac{1}{2} T_s$ Hz where the spectra crosses is called the folding frequency.

→ The tail inversion is known as spectral folding or aliasing.

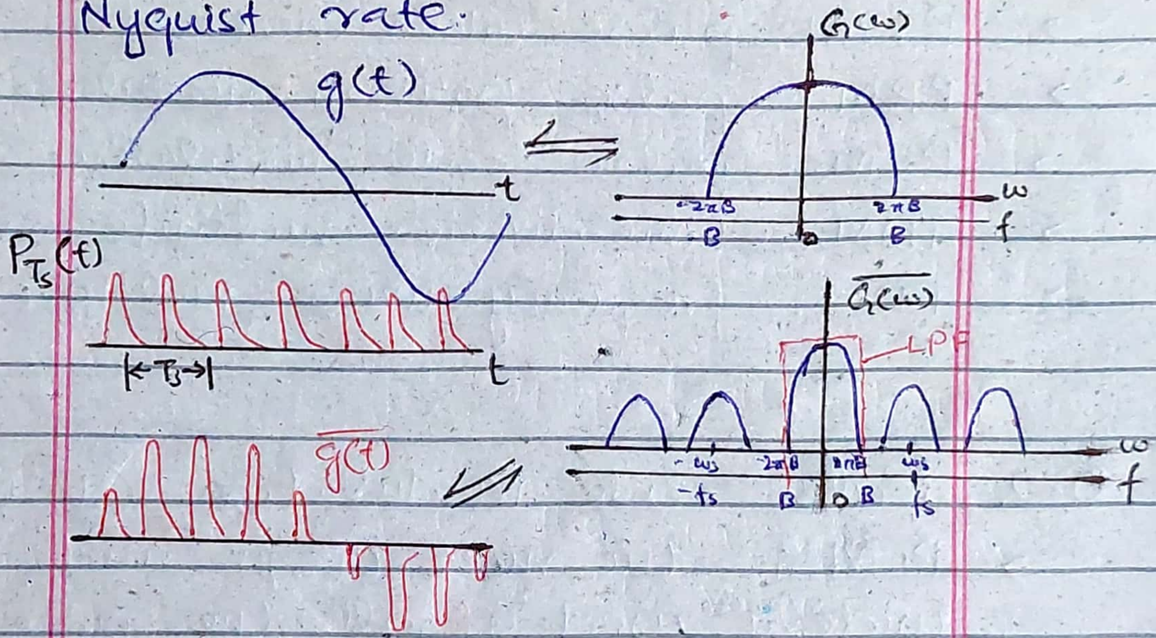
→ Solution to this aliasing is to pass the signal before sampling throug an ideal LPF of bandwidth $f_s/2$ Hz (folding frequency). This filter is called antialiasing filter.

* Practical Sampling:

→ As $\bar{g}(t)$ is obtain by xing $g(t)$ with impulse train, which is physically non-existent.

→ In practice, we multiply a signal $g(t)$ by a train of pulses of finite width.

→ We can reconstruct $g(t)$ from sampled signal $\overline{g}(t)$ if provided sampling rate is not below Nyquist rate.



→ As sampling pulse train $P_{T_s}(t)$ is periodic. by Fourier series.

$$P_{T_s}(t) = C_0 + \sum_{n=1}^{\infty} C_n (\cos n\omega_s t + \theta_n).$$

$$\because \omega_s = \frac{2\pi}{T_s}$$

$$\overline{g}(t) = P_{T_s}(t) \cdot g(t) = g(t) C_0 + \sum_{n=1}^{\infty} g(t) \cos(n\omega_s t + \theta_n)$$

$$= C_0 g(t) + C_1 g(t) \cos(\omega_s t + \theta_1) + C_2 g(t) \cos(2\omega_s t + \theta_2) + C_3 g(t) \cos(3\omega_s t + \theta_3) + \cdots$$

→ The first term is desired signal which can be recovered by LPF, the other terms are signals with spectra centered at $\pm\omega_s$, $\pm 2\omega_s$, $\pm 3\omega_s$, ...

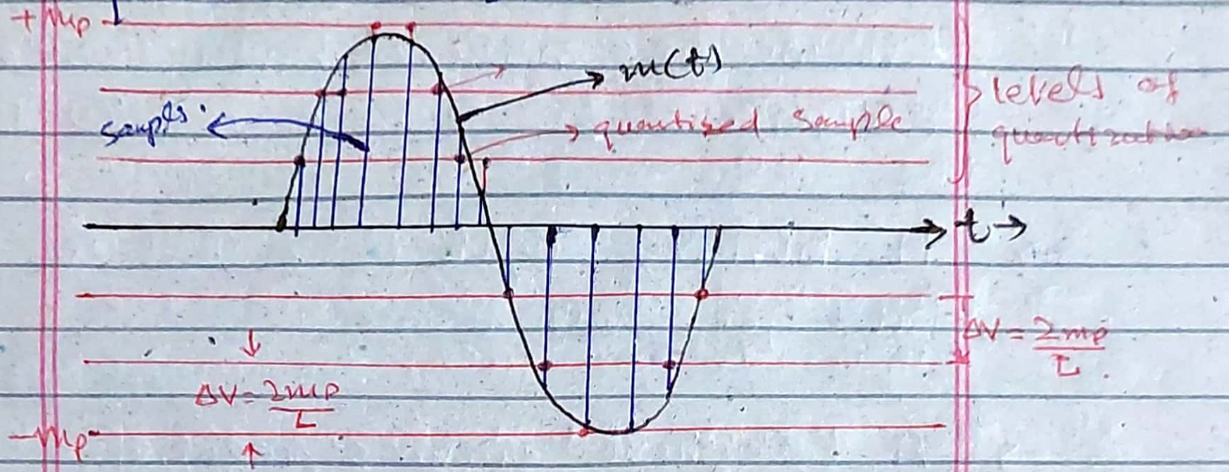
→ PAM : pulse amplitude modulation.
→ PWM : pulse width modulation.
→ PPM : pulse position modulation
→ PCM : pulse code modulation.
→ TDM : time-division multiplexing
→ FDM : frequency-division multiplexing.

6.2 Pulse-code Modulation (PCM):

→ PCM is the most useful and widely used of all the pulse modulations mentioned.

→ PCB is a method of converting an analog signal into digital signal (A/D conversion), by means of sampling and quantizing.

→ By quantizing we mean rounding off its value to the closest permisible numbers or quantized levels.



→ The figure shows that amplitudes of m(t) lies in range $(-m_p, m_p)$ which is partitioned into L subintervals, each of magnitude $\Delta v = 2m_p/L$.

→ Each sample is now approximated to one of the L numbers. thus signal is digitized, with quantized samples taking on any of the L-values. Such a signal is known as L-ary digital signal.

→ We can convert an L-ary signal into binary signal by using pulse coding.

→ The code, formed by binary representation of the 16 decimals digits from 0 to 15, is known as natural binary code (NBC).

→ A binary digit is called a bit for convenience

*E → A telephone signals require how much binary pulses per second?

As audio frequency is
$f = 15\,kHz$ to $3.4\,kHz$
So the sampling rate is $f_s = 8\,kHz$
which is greater than $f_s \geq 6.8\,kHz$ — nyquist criteria
As this signal is represented with 256 levels hence in binary $256 = 2^8$ we can represent each level with 8 bits
so,

$$8 \times 8\,kHz = 64k \text{ bits per second}$$

are required to represent a telephonic signal.

16.2.0.1 Advantages of digital communication

→ Digital communication can withstand channel noise and distortion if these are within limits.

→ In analog the signal grows progressively weaker whereas signal noise and distortion become progressively ~~to~~ stronger. Amplification is of little help because it enhance the signal and noise in same proportion.
In a digital signal a repeter station are placed which detect, and new clean pulse are transmitted to next repeater station.

→ device hardware implementation is flexible and permits the use of processer;

→ Can be coded to extremely low error rates, high fidelity and high privacy.

→ Can be multiplexed easily and efficiently.

→ more efficient than analog in relazing the exchange of SNR ~~and~~ for bandwidth.

→ Reproduction of digital signal is extremely reliable without deterioration.

→ Digital hardware continues to halve every two or three years ~~and~~, while performance or capacity doubles over the same period.

6.2.1 Quantizing:

→ PCM has various aspects such as: quantizing, encoding, synchronizing, the required transmission bandwidth, the SNR and so on.

→ For quantizing, we limit the amplitude of $m(t)$ to rang $(-m_p, m_p)$. The m_p is not necessary the peak amplitude of $m(t)$. it is't a parameter of $m(t)$ but a constant of the quantizer. Amplitudes of $m(t)$ beyond $\pm m_p$ are chopped off.

→ The amplitude range $(-m_p, m_p)$ is divided into L uniformly spaced interval each of width $\Delta v = 2m_p/L$.

→ A sample value is approximated by the mid point of interval in which it lies.

→ There are two sources of errors in this scheme;

- quantization error: This can be reduced ~~by de~~ as much as desired by increasing the number of quantizing levels, the price of which is paid in an increased bandwidth of the transmission medium (channel). The error in the received signal is caused exclusively by this error.

- pulse detection error: This error is quite small and can be ignored.

→ If $m(KT_s)$ is K^{th} sample of $m(t)$ then $\hat{m}(KT_s)$ is corresponding quantized ~~sample~~ sample of reconstructed signal $\hat{m}(t)$ from quantized samples then by interpolation formula (p# 7)

$$\hat{m}(t) = \sum_K \hat{m}(KT_s) \operatorname{sinc}(2\pi B t - K\pi).$$

The distortion componet $q(t)$ is

$$q(t) = \hat{m}(t) - m(t)$$

$$= \sum_K [\hat{m}(KT_s) - m(KT_s)] \operatorname{sinc}(\text{''})$$

$$= \sum_K q(KT_s) \operatorname{sinc}(\text{''})$$

where $q(KT_s)$ is quantization

(18)

error in K^{th} sample. the $q(t)$ act is noise known as quantization noise. The power of $q(t)$ is

$$\overline{q^2(t)} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} q^2(t)\, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[\sum q(kT_s) \operatorname{sinc}(\text{''}) \right]^2 dt$$

→ As we know $\operatorname{sinc}(2\pi Bt - m\pi)$ is orthogonal to $\operatorname{sinc}(2\pi Bt - n\pi)$ hence,

$$\int_{-\infty}^{\infty} \operatorname{sinc}(2\pi Bt - m\pi) \operatorname{sinc}(2\pi Bt - n\pi) = \begin{cases} 0, & m \neq n \\ 1/2B, & m = n \end{cases}$$

$$\overline{q^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \sum_k q^2(kT_s)\, \mathrm{sinc}^2(2\pi Bt - k\pi)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \sum_k q^2(kT_s) \int_{-T/2}^{T/2} \mathrm{sinc}^2(2\pi Bt - k\pi)\, dt$$

$1/2B$

average/mean square of quantization error

$$= \lim_{T\to\infty} \frac{1}{T} \sum_k q^2(kT_s) \cdot \frac{1}{2B}$$

$$\overline{q^2(t)} = \lim_{T\to\infty} \frac{1}{2BT} \sum_k q^2(kT_s)$$

As sampling rate is 2B hence total samples over the interval T is 2BT.
as $\Delta V = 2m_p/L$, the maximum error in quantization is $\pm \Delta V/2$. thus its range lies from $-\Delta V/2$ to $\Delta V/2$.
So the mean square quantizing error q^2 is :

$$q^2 = \frac{1}{\Delta V} \int_{-\Delta V/2}^{\Delta V/2} q^2\, dq$$

$$= \frac{1}{\Delta V} \frac{1}{3} q^3 \Big|_{-\Delta V/2}^{\Delta V/2}$$

$$= \frac{1}{\Delta V} \frac{1}{3} \left[\left(\frac{\Delta V}{2}\right)^3 + \left(\frac{\Delta V}{2}\right)^3 \right] = \frac{1}{3\Delta V} \left(\frac{\Delta V^3}{8} + \frac{\Delta V^3}{8} \right)$$

$$= \frac{1}{3\cancel{\Delta V}} \cdot \frac{\cancel{2}\Delta V^{\cancel{3}}}{\cancel{8}4} = \frac{\Delta V^2}{12}$$

Now putting Δv we get

$$\tilde{q}^2 = \frac{4 m_p^2}{\cancel{12} L^2}$$

$$\tilde{q}^2 = \frac{1}{3}\left(\frac{m_p}{L}\right)^2$$

As $q^2(t)$ is the mean square value hence

$$N_q = \overline{q^2(t)} = \tilde{q}^2 = \frac{1}{3}\frac{m_p^2}{L^2}$$

The detected pulse in which detection error is negligible, then

$$\hat{m}(t) = m(t) + q(t)$$

As the power of $m(t)$ is $m^2(t)$

then $S_o = m^2(t)$

and $N_o = N_q = \frac{m_p^2}{3L^2}$

$$\frac{S_o}{N_o} = 3L^2 \frac{m^2(t)}{m_p^2}$$

As m_p is peak amplitude of quantizer, a constant. This mean

S_o/N_o, is a linear function of the message power $\widetilde{m^2(t)}$.

6.2.2 Non-uniform Quantization:

→ SNR is an indication of the quality of the received signal.

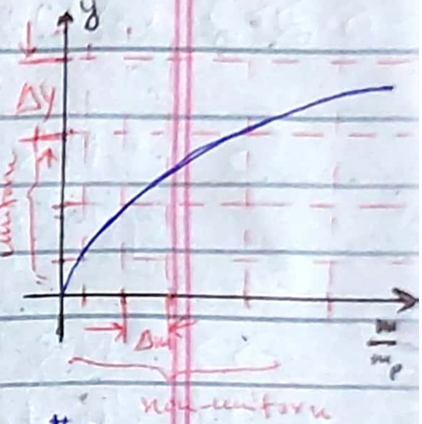
→ SNR can vary widely depends on talker to talker and connecting circuits.

→ Statistically, it is found that smaller amplitudes predominate in speech and larger amplitudes are much less frequent.

→ The root of this difficulty lies in the fact that the quantizing steps are of uniform value $\Delta v = 2m_p/L$.

→ This problem can be solved by using smaller steps for smaller amplitudes (non-uniform quantizing).

→ An approximately logarithmic compression characteristic yields a quantization noise nearly proportional to signal power $\widetilde{m^2(t)}$, thus making the SNR practically independent of input signal power over a large dynamic range.



→ Among several choices; two compression laws have been accepted as desirable standards by CCITT. Both of these laws have odd symmetry about the vertical axis.

- The μ-law is used in America and Japan and for positive amplitudes is given by:

$$y = \frac{1}{\ln(1+\mu)} \ln\left(1 + \frac{\mu m}{m_p}\right) \; ; \quad 0 \leq \frac{m}{m_p} \leq 1$$

- The A-law is used in Europe and rest of world, and international routes, and for +ve amps is given by:

$$y = \begin{cases} \dfrac{A}{1+\ln A}\left(\dfrac{m}{m_p}\right) \; ; & 0 \leq \dfrac{m}{m_p} \leq \dfrac{1}{A} \\[2ex] \dfrac{1}{1+\ln A}\left(1 + \ln \dfrac{Am}{m_p}\right) \; ; & \dfrac{1}{A} \leq \dfrac{m}{m_p} \leq 1 \end{cases}$$

→ The compression parameter (μ or A)

determine the degree of compression.

→ A value of $\mu = 100$ (for 128 encoding) 7-bit and $\mu = 255$ (for 8-bits) is used in America for optimum/constant S_o/N_o.
And a value of $A = 87.6$ gives comparable results and has been standarized by CCITT.

→ The compressor and expander together are called compander.

*E Example 6.2:

$B_m = 3\,kHz$, $f_s = 33\frac{1}{3}\%$ > Nyquist rate.

error $\leq 0.5\%$ of m_p.

Sol:

Nyquest sampling rate $R_N = 2 f_m = 2 \times 3K = 6\,kHz$

Actual sampling rate $R_A = R_N \times 33\frac{1}{3}\%$

$$R_A = 6000 \times \left(1 + \frac{33.\bar{3}}{100}\right) = 7999.80$$

$$= 8000\ Hz$$

As maximum quantization error is $\pm \Delta V/2$ (0.5% of mp)

$$\frac{\Delta V}{2} = \frac{1}{2} \frac{2m_p}{L} = \frac{m_p}{L}$$

$$0.5\%\ m_p = \frac{m_p}{L} \Rightarrow$$

$$\frac{0.5}{100} \cancel{m_p} = \frac{\cancel{m_p}}{L} \Rightarrow L = \frac{100}{0.5} = 200$$

For binary coding, L must be power of 2. Hence, the next L i.e a power of 2 is $L = 256$.

No. of bits $n = \log_2 256 = 8$-bit/sample.

hence $C = 8 \times 8000 = 64k$ bit/s

As we can transmit 2 bit/s per Hz of bandwidth, the minimum transmision bandwidth $B_T = \frac{C}{2} = \frac{64k}{2} = 32\,KHz$.

Now as multiplexed signal has 24 signal:

$C_m = 24 \times 64k = 1.536$ Mbits/s

which require minimum of

$\frac{C_m}{2} = 0.768$ MHz of transmission bandwidth.