CS - Notes 24/10/23 — Tu

Chap # 3: Analysis and Transmission of Signals.

→ $$G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt.$$

→ $$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)\, e^{j\omega t}\, d\omega$$

→ $$g(t) = \mathcal{F}^{-1}\{G(\omega)\}$$
$$G(\omega) = \mathcal{F}\{g(t)\}$$
$$g(t) \Longleftrightarrow G(\omega).$$

→ $$G(\omega) = \underbrace{|G(\omega)|}_{\text{amplitude}}\, e^{j\overbrace{\theta_g(\omega)}^{\text{phase/angle}}}$$

→ $$\left.\begin{aligned} G(-\omega) &= G^*(\omega) \\ |G(-\omega)| &= |G(\omega)| \\ \theta_g(-\omega) &= -\theta_g(\omega) \end{aligned}\right\} \text{Conjugate Symmetric property.}$$

→ Fourier transform is assure only if $g(t)$ satisfy drichlet's condition:

$$\int_{-\infty}^{\infty} |g(t)|\, dt \leq \infty$$

i.e $$|G(\omega)| \leq \int_{-\infty}^{\infty} |g(t)|\, dt.$$

②

→ $g_1(t) \Leftrightarrow G_1(\omega)$, $g_2(t) \Leftrightarrow G_2(\omega)$

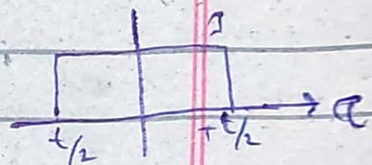
Then

$a_1 g_1(t) + a_2 g_2(t) \Leftrightarrow a_1 G_1(\omega) + a_2 G_2(\omega)$.

This is called Linearity of Fourier Transforms.

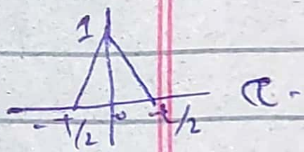
→ rectangular function:

$\mathrm{rect}\left(\frac{\tau}{t}\right)$ → function → width



→ Unit triangle function

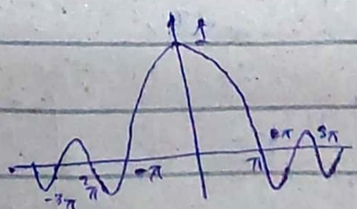
$\Delta\left(\tau/t\right)$ → width



→ $\mathrm{sinc}(x) = \dfrac{\sin \pi x}{\pi x}$

known as filtering or interpolating signal.

- even function
- $\mathrm{sinc}(x) = 0$ for $\sin \pi x = 0$ and $\sin \pi x = 0$ for $\pm n\pi$.



→ $\text{rect}\left(\frac{t}{\tau}\right) \Leftrightarrow \tau \,\text{sinc}\left(\frac{\omega\tau}{2}\right)$

→ Signal bandwidth is the difference between highest (significant) frequency and the lowest (significant) frequency in signal spectrum.

→ $\delta(t) \Leftrightarrow 1$

→ $F^{-1}\{\delta(\omega)\} \Longleftrightarrow \frac{1}{2\pi}$

→ $F\{\cos\omega_0 t\} \Leftrightarrow \pi\left[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right]$

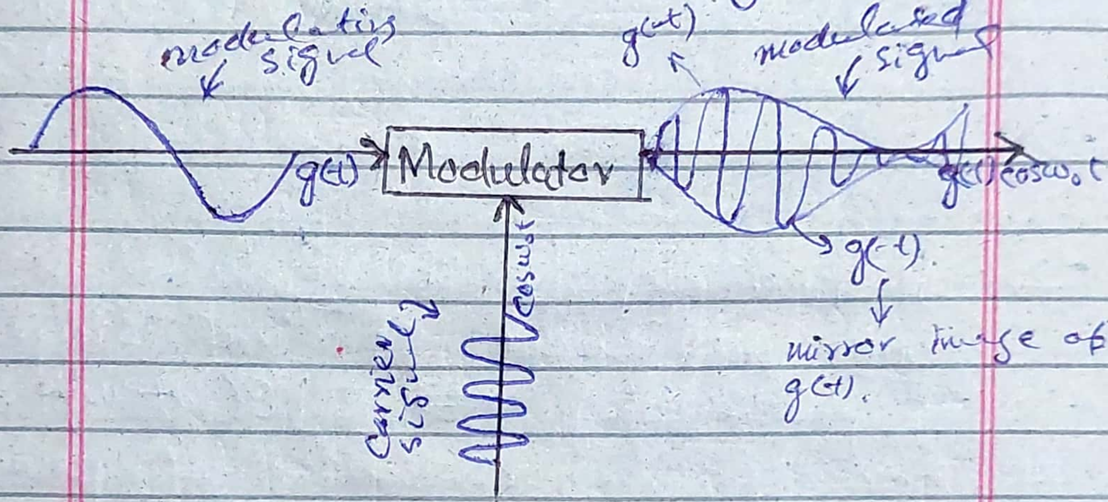
→ $g(t-t_0) \Leftrightarrow G(\omega)e^{-j\omega t_0}$

$g(t)e^{j\omega_0 t} \Leftrightarrow G(\omega-\omega_0)$

→ $g(at) \Leftrightarrow \frac{1}{|a|} G(\omega/a)$

→ ~~g(t-t₀)~~) Multiplication of a signal $g(t)$ by a sinusoid of frequency ω_0 shifts the spectrum $G(\omega)$ by $\pm\omega_0$. Multiplication of sinusoid $\cos\omega_0 t$ by $g(t)$ amount to modulating the sinusoid amplitude. This modulation is called amplitud modulation.

→ The $\cos\omega_0 t$ is called carrier signal, $g(t)$ is called modulating signal and $g(t)\cos\omega_0 t$ is called the modulated signal.



→ We can shift the phase of each spectral component of a modulated signal by θ_0 merely by using carrier $\cos(\omega_0 t + \theta_0)$.

→ $\sin\omega_0 t$ is $\cos\omega_0 t$ with a phase delay of $\pi/2$ i.e

$$g(t)\cos(\omega_0 t + \theta_0) \longrightarrow \frac{1}{2}\left[G(\omega-\omega_0)e^{j\theta_0} + G(\omega+\omega_0)e^{-j\theta_0}\right]$$

for $\theta_0 = \pi/2$

then $g(t)\cos(\omega t + \pi/2) = g(t)\sin\omega_0 t$

→ If several signals, each occupying the same frequency band are transmitted simultaneously over same transmission medium, they will interfere; and will be imposible to seperate or retrive them at receiver. This problem is salved by modulation.

→ Modulation and demodulation both implement spectral shifting.

→ For effective radiation of power over a radio link, the antena size must be on the order of the wavelength of signal to be radiated.

→ For audio signals frequencie are so low (λ are very large) that a very large antenal will be required which is impractical. This is solved by shifting a signal to high frequencie via modulation.

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→ Convalution is define as below for two functions.

$$g(t) * h(t) = \int_{-\infty}^{\infty} g(\tau) h(t-\tau)\, d\tau$$

→ $g(t) * h(t) \Longleftrightarrow G(\omega) H(\omega)$

→ $g(t) \cdot h(t) \Longleftrightarrow \frac{1}{2\pi} G(\omega) * H(\omega)$

→ If: $g(t) \Leftrightarrow G(\omega)$

Then

- $\frac{dg}{dt} = j\omega\, G(\omega)$
- $\frac{d^n g}{dt^n} = (j\omega)^n\, G(\omega)$
- $\int_{-\infty}^{t} g(\tau)\, d\tau \Leftrightarrow \frac{G(\omega)}{j\omega} + \pi G(0)\delta(\omega)$

3.4 Signal Transmission Through linear Systems:

→ • $\underbrace{y(t)}_{\text{output}} = \underbrace{g(t)}_{\text{input}} * \underbrace{h(t)}_{\text{unit impulse response of LTI}}$

• $Y(\omega) = G(\omega)\, H(\omega)$

• $|Y(\omega)| e^{j\theta_y(\omega)} = |G(\omega)||H(\omega)| e^{j[\theta_g(\omega) + \theta_h(\omega)]}$

→ During transmission amplitude spectrum $|G(\omega)|$ is changed to $|G(\omega)||H(\omega)|$ and phase spectrum $\theta_g(\omega)$ is changed to $\theta_g(\omega) + \theta_h(\omega)$

So the output waveform is different from the input.

→ During transmission some frequency components may be bosted in amplitud while others may be attenuated.

→ Transmission are said to be distortionless if input and output have identical wave shapes. with in a multiplicative constant. A delayed output that retains the input is also considerd distortionless.

$$y(t) = K\, g(t - t_d)$$

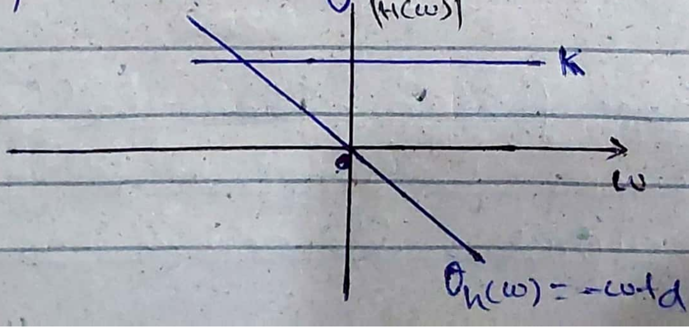
$$Y(\omega) = K\, G(\omega)\, e^{-j\omega t_d}$$

$$\therefore H(\omega) = K\, e^{-j\omega t_d}$$

amp response

→ For distortionless $|H(\omega)| = K$ must be a constant and

phase response ← $\theta_h(\omega) = -\omega t_d$ must be a linear function of ω



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→ $t_d(\omega) = -\frac{d}{d\omega}\theta_h(\omega)$

time delay is the -ve of the slope of system phase response θ_h.

- If slope is constant, td is linear and all components are delayed by same interval td.
- If slope is not constant, time delay varies with frequency and each componenet is delayed with different intervals.

→ The human ear is insensitive to phase distortion but sensitive to amplitud distortion while the eye the situation is exactly the opposite.

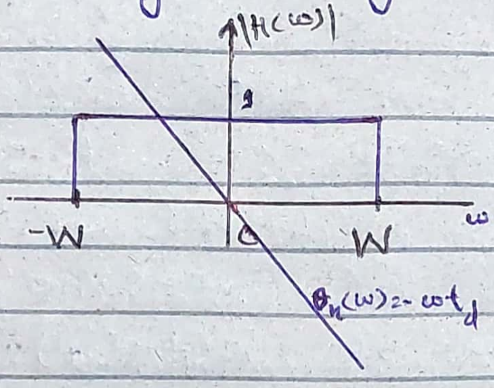
3.5 Ideal and practical filters:

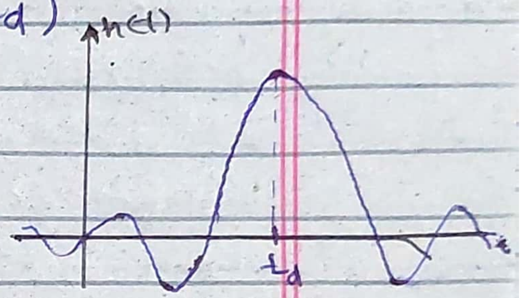
→ Ideal filter allow distortionless transmission of a certain band of frequencies and supress all the remaining other.

→ low pass filter allow all frequencies below $\omega = W$ rad/s and supress all above $\omega = W$ rad/se

→ Ideal LPF has a linear phase of slop $-t_d$, which delay all the input signal below $\omega = W$ rad/sec i.e

$$y(t) = g(t - t_d)$$





as $|H(\omega)| = \text{rect}\left(\frac{\omega}{W}\right)$ & $\theta_h(\omega) = -\omega t_d$

$$H(\omega) = \text{rect}(\omega / W)\, e^{-j\omega t_d}$$

$$h(t) = \frac{W}{\pi} \text{sinc}[W(t - t_d)];$$

As this is a non-causal system to make it causal $h(t) = 0;\ t < 0$
In frequency domain this condition is equivalent to well-known Paley-Wiener criterion, $|H(\omega)|$ to be relizable in must follow:

$$\int_{-\infty}^{\infty} \frac{|\ln|H(\omega)||}{1+\omega^2}\, d\omega < \infty$$

→ One practical approach to filter design is to cutoff $h(t)$ for $t < 0$. ie

$$\hat{h}(t) = h(t)u(t)$$

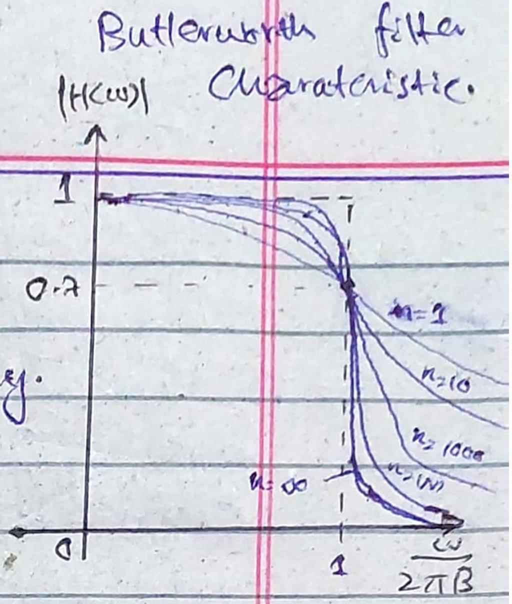
→ theoretically a delay $t_d = \infty$ will make an ideal filter which is inpractical. Hoewer t_d of 3 to 4 times of π/ω will make $\hat{h}(t)$ reasonably close version of $h(t-t_d)$.

→ To approach the ideal character- of filter Butterworth aproach is one of many.

$$|H(\omega)| = \frac{1}{\sqrt{1+(\omega/2\pi B)^{2n}}}$$

$n \to \infty$, $H(\omega) \to$ ideal filter.

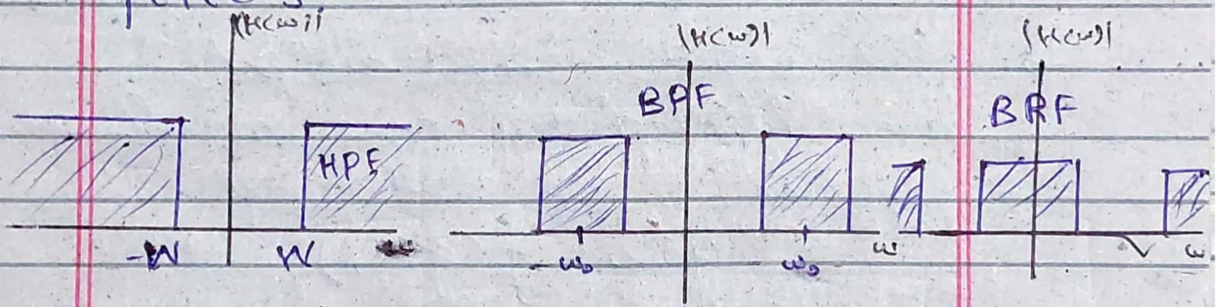
→ half power bandwidth is the bandwidth over which the amp resp $|H(\omega)|$ remain constant within variation of 3dB (or a ratio of $1/\sqrt{2} = 0.707$).



→ Half-power bandwidth of LPF is also called cuttoff frequency.

→ If we try to perfect |H(ω)| more, $\theta_n(\omega)$ deviates more from ideal and vice-versa.

→ Some other filters are high-pass, band-pass and band-reject filters:



3.6 Signal distortion over a Communication Channel:

Linear distortion: → Signal distortion can be caused over such a channel by non-ideal characteristic of either the magnitude, the phase or both.

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* $\frac{W}{\pi} \text{Sinc } Wt = \text{rect}\left(\frac{\omega}{2W}\right)$

* $\frac{W}{2\pi} \text{Sinc}^2 Wt = \Delta\left(\frac{\omega}{2W}\right).$

3.8.2 Distortion Caused by Channel non-linearities:

→ linear distortion approximation are valid only for small signals: For large amplitude non linearities cannot be ignored.

→ Amplitud Modulated signals are badly affected by non-linear distorton hower FM signals are not affected.

→ let input g and output are related by some non-linear function

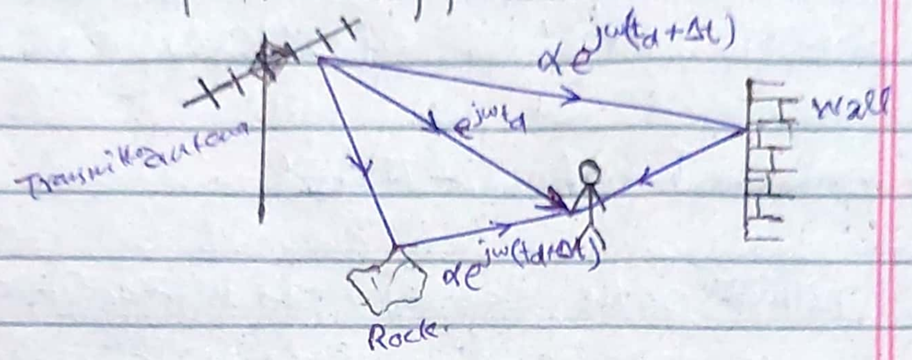
$$y(t) = f(g(t))$$

RHS can be written in McLaurin's series as:

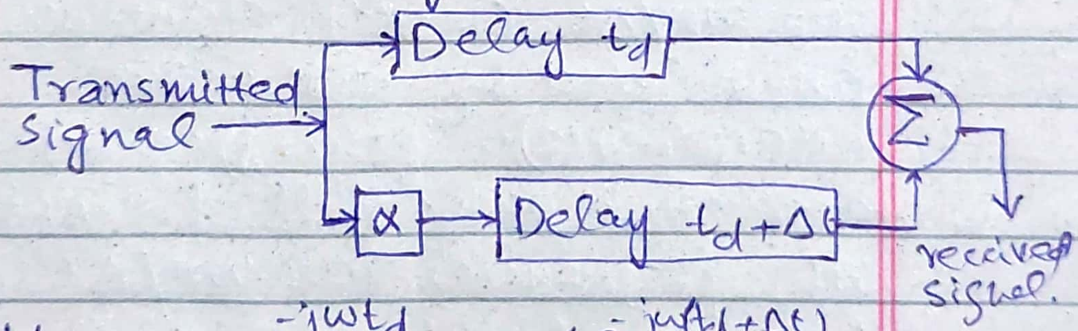
$$y(t) = a_0 + a_1 g(t) + a_2 g^2(t) + \cdots a_k g^k(t) + \cdots$$

If bandwidth of $g(t)$ is B Hz then that of $g^k(t)$ is KB Hz also $y(t)$ = KB Hz.

3.6.3 Distortion caused by Multipath Effects:



→ A multipath transmission takes place when a transmitted signal arrives at the receiver by more than one path of different delays.



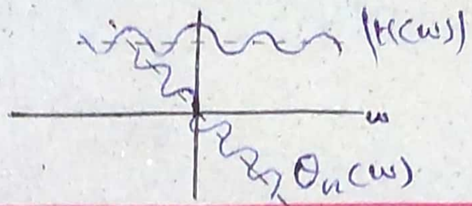
$$H(\omega) = e^{-j\omega t_d} + \alpha e^{-j\omega(t_d+\Delta t)}$$

$$= e^{-j\omega t_d}\left(1 + \alpha e^{-j\omega \Delta t}\right)$$

$$= e^{-j\omega t_d}\left(1 + \alpha \cos\omega\Delta t - j\alpha \sin\omega\Delta t\right)$$

$$= \underbrace{\sqrt{1+\alpha^2+2\alpha\cos\omega\Delta t}}_{|H(\omega)|}\ \underbrace{e^{-j\left[\omega t_d + \tan^{-1}\frac{\alpha\sin\omega\Delta t}{1+\alpha\cos\omega\Delta t}\right]}}_{\theta_h(\omega)}$$

(15)



Which shows that both the magnitude and phase characteristics of $H(\omega)$ are periodic in ω with a period of $2\pi/\Delta t$.

→ Multipath causes non-idealities and will cause linear distortion (pulse dispersion).

→ gain is too close i.e $\alpha \approx 1$.

→ This equation shows that $\omega = n\pi/\Delta t$ (n odd) signal annihilated by destructive interference). These frequencies are multipath null frequencies. At frequencies $\omega = n\pi/\Delta t$ (n even) two signal interfere constructively to enhance the gain.

3.7 Signal Energy and Energy spectral density:

→ The energy E_g of a signal $g(t)$ is defined as the area under $|g(t)|^2$.

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 \, dt.$$

→ Parseval's Theorm: is used to determine energy of signal in its fourier transform $G(\omega)$.

$$E_g = \int_{-\infty}^{\infty} g(t) g^*(t) \, dt = \int_{-\infty}^{\infty} g^*(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) \, d\omega \right] dt$$

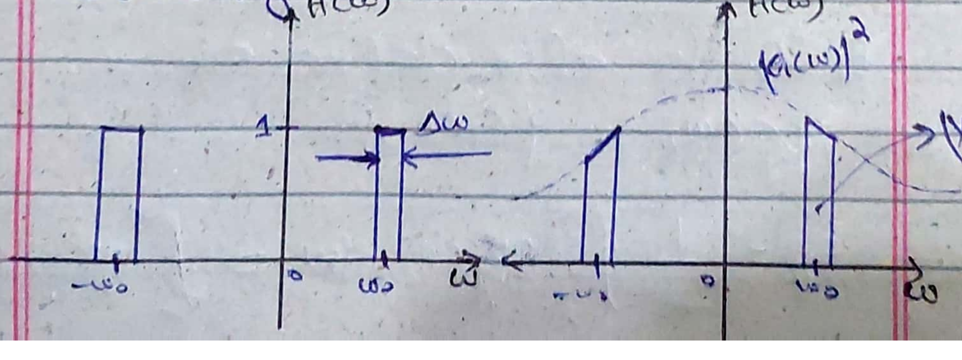
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) \left[\int_{-\infty}^{\infty} g^*(t) e^{-j\omega t} \, dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) G^*(\omega) \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2 \, d\omega$$

3.7.1 Energy Spectral Density (ESD):

→ Energy per unit of small bandwidth is energy spectral density



This filter supress all frequencies except a small band of $\Delta\omega$ ($\Delta\omega \rightarrow 0$). centered at ω_0. whose transfer function is $H(\omega)$.
If filter output is $y(t)$ then
$Y(\omega) = G(\omega)\, H(\omega)$

$$E_y = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega) H(\omega)|^2 \, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2 d\omega$$ $\because H(\omega) = 1$ at $\Delta\omega$ and zero everywhere else

$$= 2 \frac{1}{2\pi} |G(\omega)|^2 d\omega$$ $\because$ sum of '2'

$$= 2 \frac{1}{\cancel{2\pi}} |G(\omega)|^2 \, d(\cancel{2\pi} f)$$

$$= 2 |G(\omega)|^2 df.$$

which is contributed energy of spectral component within two narrow band each of width Δf Hz, centered at $\pm\omega_0$

→ ESD is also define as
$\Psi_g(\omega) = |G(\omega)|^2$

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_g(\omega)\, d\omega = \int_{\infty}^{\infty} \Psi(\omega)\, df.$$

3.7.2 Essential Bandwidth of a signal

→ The spectra of most signal extends to infinity, However most of energy is contained within a certain band of B Hz, and other than that component energy is negligible. The bandwidth B is called essential bandwidth of signal.

→ Select B to be that bandwidth which contain 95% of the signal energy.