Finals

Week

DSP Theory

Topics covered last week

1) systems
   i) causal vs Non causal.
   ii) Linear vs Non Linear.
   iii) Time invariant / Time variant.
   iv) Memory vs memoryless system.
   v) Periodic vs Aperiodic systems.

Assignment 3 assigned.

Today's agenda

vi) Stable/Unstable systems.

vii) Static /Dynamic systems.

vi

2 methods

1) See input/output of system & decide whether system is stable or not

2) See the system → & decide " " " " "

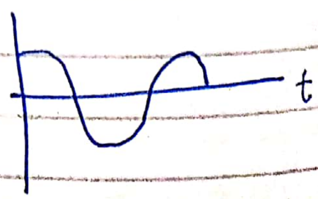
Under what conditions same system is stable, in other conditions, its unstable.

Example

1) $y(t) = \cos(x(t))$

$= \cos(\theta)$

for diff θ's



cos → −1 ↔ 1

y(t) ↗

↳ bounded

↳ Never goes to infinity

↳ unstable.

2) $y(t) = \tan(x(t))$

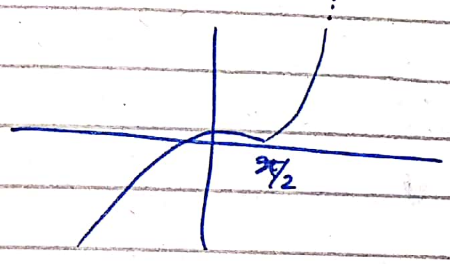
$= \tan(\theta)$.

↳ $\pi/2$, $3\pi/2$

↳ $0 - 2\pi$

↳ $0 - 360^\circ$

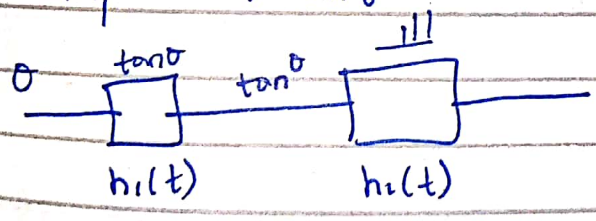
At $\pi/2$ → $\tan\theta$ → infinite.



unstable system

→ Bounded

→ If input → unbounded → output → outbounded.



unbounded unbounded.

<u>Method 2</u>

See the system.

If system is continuous → integrate $-\infty$ — ∞

value of integration $< \infty$ → stable

else unstable.

$$C = \int_{-\infty}^{\infty} |h(t)| < \infty$$

If system is discrete

$$D = \sum_{-\infty}^{\infty} |h[n]| < \infty \rightarrow \text{stable.}$$

Problems:

(a) $h(t) = e^{-t} \sin t \; u(t)$

$$\int_{-\infty}^{\infty} |e^{-t} \sin t \, u(t)| \, dt < \infty \quad \text{— (1)}$$

$$\int_{0}^{\infty} \underset{v}{\overset{\text{II}}{e^{-t}}} \; \underset{u}{\overset{\text{I}}{\sin t}}$$

Integration by parts. ILATE

inverse
Log
Arithmatic
Trignometric
Exponential

$$u\int v - \int u' \int v \; dt$$

$$= \sin t \int e^{-t} - \int -\cos t \int e^{-t}$$

$$= \sin t \frac{e^{-t}}{-t} + \sin t \frac{e^{-t}}{-t}$$

$$= \frac{-\sin t \, e^{-t}}{t} - \frac{\sin t \, e^{-t}}{t}$$

$=$

$$I = 1/2 < \infty$$

Thus stable system.

Problem (2)

Sometimes a system is stable in some range and out of that range its unstable.

$$h(t) = e^{at} u(t) + e^{-bt} u(t).$$

Find the range of a & b for which system is ~~so~~ stable.

$$= \int_0^\infty e^{at}\, dt + \int_0^\infty e^{-bt}\, dt$$

$$= \frac{e^{at}}{a}\Big|_0^\infty - \frac{e^{-bt}}{b}\Big|_0^\infty$$

$$= \frac{1}{a}\left(e^{a\infty} - e^{0}\right) - \frac{1}{b}\left(e^{-b\infty} - e^{0}\right)$$

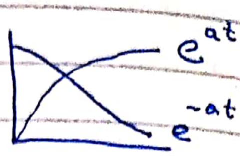
$$= \frac{e^{a}}{a}(\infty - 1) - \frac{e^{-b}}{b}(\infty - 1)$$

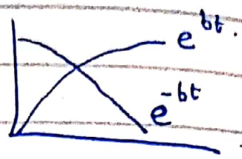
So for $0 \to \infty$ its unstable.

For stable;

$a < 0$

$b > 0$.





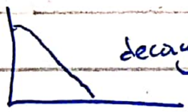
rising exp → unstable

falling exp → stable.

$$= \frac{e^{a\infty}}{a} - \frac{1}{a} - \frac{e^{-b\infty}}{b} + \frac{1}{b}$$

$$= \frac{e^{a\infty}}{a}$$

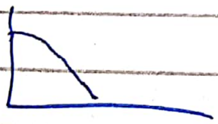
$a < 0$



decaying exp.

$$= \frac{e^{-b\infty}}{b}$$

$b > 0$



For discrete system

$$h[n] = \begin{cases} b^n ; & n < 0 . \\ a^n ; & n \geq 0 . \end{cases}$$

range of a & b = ?

$$= \sum_{n=-\infty}^{\infty} h[n] < \infty$$

$$= \sum_{n=-\infty}^{-1} b^n + \sum_{n=0}^{\infty} a^n$$

since $\sum_{0}^{\infty} c^n = \frac{1}{1-|c|}$

$$\sum_{-\infty}^{-1} = \sum_{-1}^{-\infty}$$

$$= \sum_{-1}^{-\infty} b^n + \sum_{0}^{\infty} a^n.$$

$$= \sum_{1}^{\infty} b^{-n} + \sum_{0}^{\infty} a^n.$$

$$= \sum_{0}^{\infty} b^{-n} + \sum_{0}^{\infty} a^n - 1$$

$$= \sum_{0}^{\infty} (b^{-1})^n + \sum_{0}^{\infty} (a)^n - 1$$

$$= \frac{1}{1-|1/b|} + \frac{1}{1-|a|} - 1$$

if $a = 1$

system $\rightarrow$ unstable $\rightarrow \infty$.

$$\left.\begin{array}{ll} \Rightarrow 0 < \frac{1}{b} < 1 & b > 1 \\ \Rightarrow a < a < 1 & a < 1 \end{array}\right\} \text{stability}$$

Static vs Dynamic system

↓ Memoryless.   ↓ Memory.

Problem

→ $y(t) = a x(t)$. → expression of system.

$y(0) = a x(0)$

↳ present

$y(-1) = a x(-1)$

↳ past

$y(1) = a x(1)$

↳ future

Thus static.

→ $y(t) = b x(t^2)$.

↗ present

$y(0) = b x(0)$

↗ past   ↗ future

$y(-1) = b x(1)$

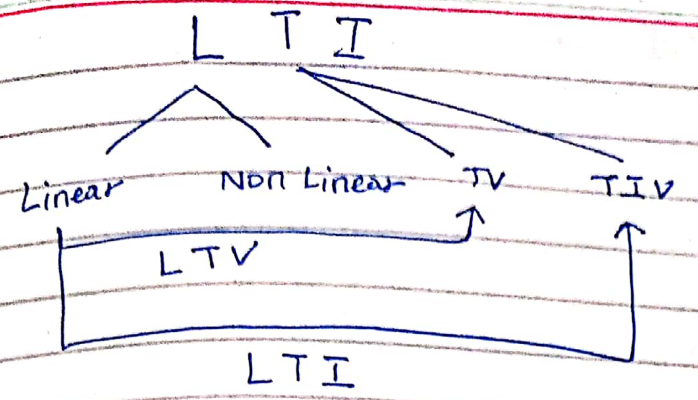
$y(-2) = b x(4)$

$y(3) = b x(9)$.

Thus Dynamic.

↳ bcz depending upon some shifted value.

DSP



Linear System

$$x[n] \longrightarrow \boxed{LTI} \longrightarrow y[n]$$

condition

$$y[n] = x[n] * h[n] \qquad DT$$

$$y(t) = x(t) * h(t) \qquad CT$$

After LTI, we'll study convolution.

- Convolution is not simple to calculate but MATLAB can make it easy for us
- Since convolution is difficult so if we multiply simply by changing it to freq domain

① $Y[\omega] = x[\omega] \cdot H(\omega)$.

$\omega \rightarrow$ freq domain

↳ we need Fourier Transform

② $Y[s] = x(s) \cdot H(s)$

↳ Laplace transform

→ Transfer function

$$H(\ ) = \frac{Y(\ )}{x(\ )}$$

Law of superposition

1. Additivity.
2. Scaling.

Time variancy

① system → delay by k units

② Replace $n \rightarrow n-k$
$t \rightarrow t-k$

① = ② TIV
① ≠ ② TV.

Consider a system with inputs $x_1(t)$ and $x_2(t)$ & output

Problem ①

$$y(t) = t\,x(t).$$

→ input $x_1(t)$ $\qquad y_1(t) = t\,x_1(t)$.

→ $x_2(t)$ $\qquad y_2(t) = t\,x_2(t)$

$$y(t) = y_1(t) + y_2(t)$$
$$= t\left[x_1(t) + x_2(t)\right] \longrightarrow ①$$

Add inputs

$x_1(t) + x_2(t) \longrightarrow y(t) = t\left[x_1(t) + x_2(t)\right] — ②$

$\boxed{1 = 2}$ → bcz both additivity & homogeneity varified

Linear system.

Problem 2

$y(t) = 10x(t) + 5$

$x_1(t)$ $\quad y_1(t) = 10x_1(t) + 5$

$x_2(t)$ $\quad y_2(t) = 10x_2(t) + 5$

$y(t) = y_1(t) + y_2(t)$

$= 10x_1(t) + 10x_2(t) + 10. \quad — ①$

$x_1(t) + x_2(t)$

$y(t) = 10(x_1(t) + x_2(t)) + 5$

$= 10x_1(t) + 10x_2(t) + 5. \quad — ②$

$1 \neq 2.$

Non Linear

Problem 3

$y[n] = n\,x[n]$

$x_1[n]$ $\quad y_1[n] = n x_1[n]$

$x_2[n]$ $\quad y_2[n] = n x_2[n]$

$y[n] = y_1[n] + y_2[n]$

$= n x_1[n] + n x_2 n$

$= n\{x_1[n] + x_2[n]\} \quad — ①$

$x_1[n] + x_2[n]$

$y[n] = n\{x_1[n] + x_2[n]\} \quad — ②$

$1 = 2 \rightarrow$ Linear

TIME VARIANT / INVARIANT

↳ characteristics changes with time.

step 1

delay by k units

step 2

Eq → replacing $n \to n-k$.

$1 = 2 \to$ TIV

Problem 1

$y[n] = x[n] + x[n-2]$

① Delay $x(n)$ by k units

$y[n,k] = x[n-k] + x[n-k-2]$ — ①

② replace n by $n-k$

$y[n-k] = x[n-k] + x[n-k-2]$ — ②

$1 = 2 \to$ TIV.

Problem 2

$y[n] = x[n] + nx[n-3]$

① Delay in $y[n]$

$y[n,k] = x[n-k] + (n)x[n-k-3]$ — ①

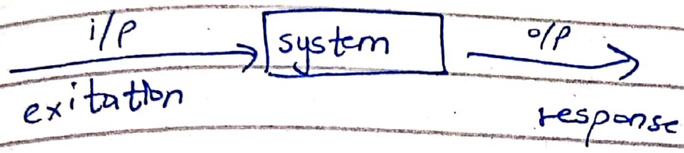
② replace

$$y[n-k] = x[n-k] + (n-k)x[n-k-3] \quad — ②$$

$1 \neq 2$

TV.

$\Rightarrow$ LTI system.



Problem ①

$$y[n] = n x^2[n]$$

check Linearity

$x_1[n]$ $\qquad y_1[n] = n x_1^2[n]$

$x_2[n]$ $\qquad y_2[n] = n x_2^2[n]$

$$y[n] = y_1[n] + y_2[n]$$
$$= n[x_1^2[n] + x_2^2[n]] \quad — ①$$

$x_1[n] + x_2[n]$

$$y[n] = n(x_1[n] + x_2[n])^2$$
$$= n x_1^2[n] + n x_2^2[n] + 2x_1[n]x_2[n] \quad — ②$$

No Linear

Problem 2

$y[n] = n\,x(n^2)$

Linearity

① $x_1[n]$ $\qquad y_1[n] = n\,x_1[n^2]$

$x_2[n]$ $\qquad y_2[n] = n\,x_2[n^2]$

$$y[n] = y_1[n] + y_2[n]$$

$$= n\,[x_1(n^2) + x_2(n^2)]$$

② $x_1[n] + x_2[n]$

$$y[n] = n\left(x_1(n^2) + x_2(n^2)\right)$$

$1 = 2 \quad \rightarrow$ Linear.

TIV

$\downarrow$

① $y[n,k] = n\,x(n-k)^2$

② $y[n-k] = [n-k]\,x(n-k)^2 \qquad \neq$

TV

$\boxed{\text{LTV}}$

Invertable & Noninvertable System

A system is said to be invertible if the input of the system appears at the output.

$x(t) \longrightarrow \boxed{h(t)} \longrightarrow \boxed{\text{filter}} \longrightarrow y(t) = x(t)$

invertable.

Tatari

DSP Theory

Shift invariance

↳ ~~if space in place of time~~

If system is not LTI → we cant use convolution.

even if system is LTI → but initial conditions are not zero. → still we cant use convolution.

Bcz if initial conditions ≠ 0, system is not linear so how LTI.

$$y[n] = x(n) * h(n)$$ → mathematical

real ↓

$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$$= h(n) * x(n)$$

commulative

Convolution

Youtube video lecture

Properties of convolution

1) commutative :

$$y(n) = x(n) * h(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

Let $n-k=m$
$n-m=k$ } change in variable

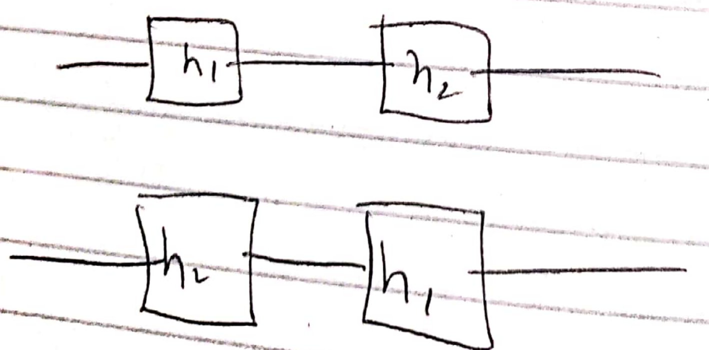
If $k \to -\infty$ then: $m \to +\infty$
$k \to \infty$ then $m \to -\infty$ } change in limits

$$= \sum_{m=\infty}^{\infty} h(m)\; x(n-m)$$

$$= \sum_{m=-\infty}^{\infty} h(m)\, x(n-m)$$

$$= h(n) * x(n)$$

2) Distributive :



$$\sum h(m)\,x(n-m)$$

DSP

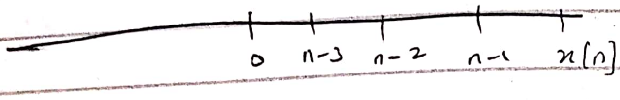
$$y[n] = \{\underbrace{h[0]\,x[n]}_{\text{Present}} + \underbrace{h[1]\,x(n-1)}_{\text{Past}} ---- \infty\} +$$

$$[h(-1)\,\underbrace{x(n-(-1))}_{\text{future value}} + h(-2)\,x(n-(-2)) + h(-3)\,x(n-(-3)) ---- \infty]$$

masla hai !

Present

$x[n] \rightarrow$ present



Present + past values ✓

future value X

so $y(n] \rightarrow$ output $\rightarrow$ here output find krenge ?

Nahi kr skte

Solution:

$x(n-(-1)) \rightarrow$ control me nhi hai.

$h[-1] \rightarrow$ control me hai.

$h[k],\ k\ -1, -2, -3, ---- \infty$

so now only calculate

from $\underbrace{0 \rightarrow \infty}$

$\underbrace{\text{Present + past value.}}_{\text{causal}}$

causal $\rightarrow$ (Pr Pa)

Non causal $\rightarrow$ $\underbrace{(\text{Pa}, \text{Fut})}$

$\hookrightarrow$ image processing.

Anti causal $\rightarrow$ (Fut).

If we want stability in convolution?

$$y[n] = h(n) * x(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k) \rightarrow \text{for causal.}$$

signal may be real or complex.

$\boxed{\text{for stability}}$

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

$|y(n)|$

$|x(n)|$

$$|z_1 z_2| = |z_1|\,|z_2|$$

$$y[n] \leq \sum_{k=-\infty}^{\infty} |h(k) \cdot x(n-k)|$$

$$\leq \sum_{k=-\infty}^{\infty} |h(k)|\, \underbrace{|x(n-k)|}_{\text{let } x(l)}$$

For any l (index), given

$$|x(l)| \leq M < \infty$$

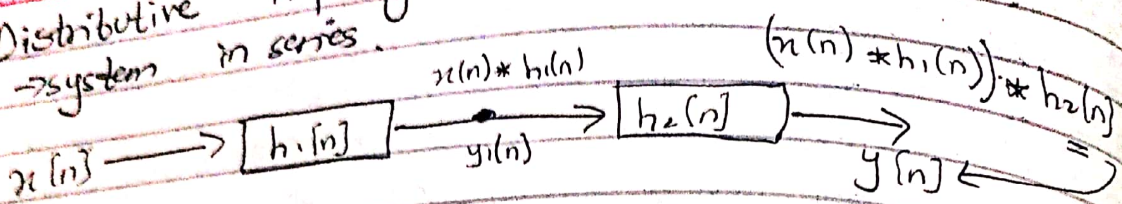
$$-M \leq x(l) \leq M$$

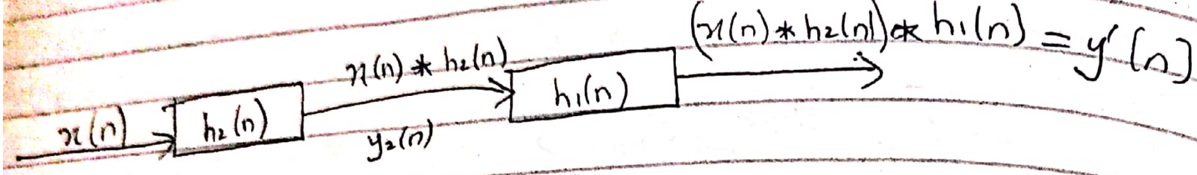
$$y[n] \leq M \sum_{k=-\infty}^{\infty} |h(k)|$$

BIBO ✓

② Distributive Property.

→ system in series.





$$h(n) * y_1(n)$$

$$y(n) = \sum_{r=-\infty}^{\infty} h_2(r) \underbrace{y_1(n-r)}$$

$$= \sum_{r=-\infty}^{\infty} h_2(r) \left[\sum_{m=-\infty}^{\infty} h_1(m) x(n-r-m) \right]$$

$$= \sum_{r\to-\infty}^{\infty} \sum_{m\to\infty}^{\infty} h_2(r)\, h_1(m)\, x(n-r-m)$$

$$= \sum_{m=-\infty}^{\infty} h_1(m) \sum_{r=-\infty}^{\infty} h_2(r)\, x(n-r-m).$$

$=$

$$= \sum_{m=-\infty}^{\infty} h_1(m) \quad y_2(n-m)$$

$$y(n) = y'(n)$$

DSP

commutative, distributive & associative properties hold for convolution.

Graphical Representation

let $x[n] = \{1, 2, -1, 0.5\}$
$h[n] = \{2, 1, 0.5\}$

ye nhi pta ye kaha se kaha th hai.
0 indexes ka nhi pata.

$$x[n] = \{1, \underset{\uparrow}{2}, -1, 0.5\} \rightarrow \text{length} = 4$$
$$\quad -1 \quad 0 \quad 1 \quad 2$$

$$h[n] = \{\underset{\uparrow}{2}, 1, 0.5\} \quad \text{length} = 3$$
$$0 \quad 1 \quad 2$$

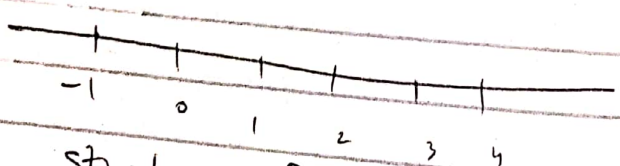
$$y[n] = \underbrace{0 + (-1)}_{\text{sum of } 1^{st} \text{ indexes}} = -1$$

length of $y[n] = \{\text{length } x(n) + \text{length } h(n)\} - 1$

$= (4 + 3) - 1$

$= 6$

↳ 6 samples



Structure of $y[n]$

① fix   shift } decide

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

② change index ↗ $x(n)\ h(n) \rightarrow x(k)\ h(k)$

To find $h[n-k]$, we need $h[-k]$

③ ↓

folding/mirroring

↳ $h[k] \rightarrow h[-k]$

④ Now shift $n \rightarrow n-k$

$h[n-k]$

④ Now multiply with $x(k)$

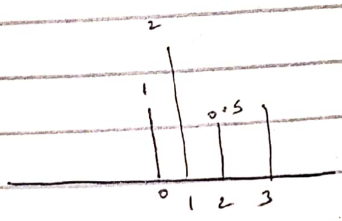
This will be $y[-1]$ → pehla index

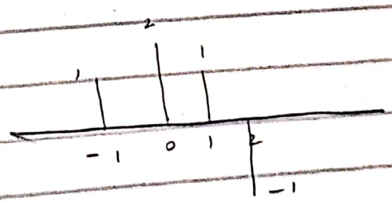
⑤ Now sum all values of product sequence.

Problem

$x[n] = \{\underset{\uparrow}{1}, 2, 0.5, 1\}$, $h[n] = \{1, \underset{\uparrow}{2}, 1, -1\}$

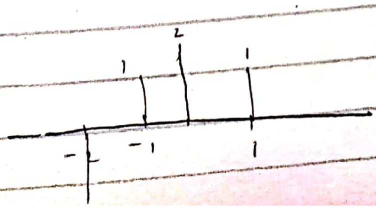
0 1 2 3   −1 0 1 2





② $x(k)$ $h(k)$

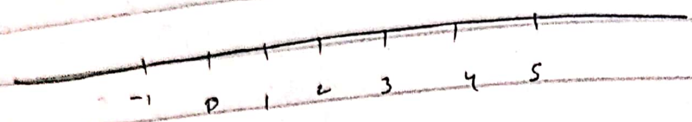
③ folding → $h[-k]$



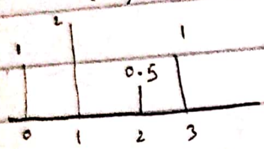
starting index of y(n)

$y(n) = 0 + (-1) = -1$

length of $y(n) = 4 + 4 - 1 = 7$



$x(k)$



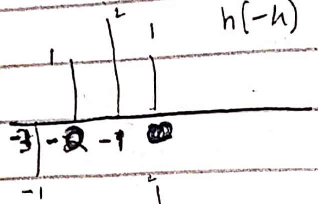
$\times$

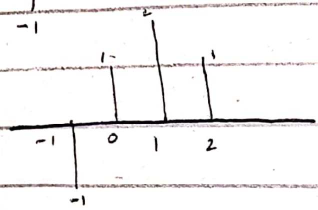
$h(n-k)$

-1

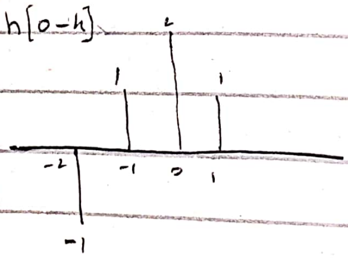
$h(-1-k)$

$h(-k)$

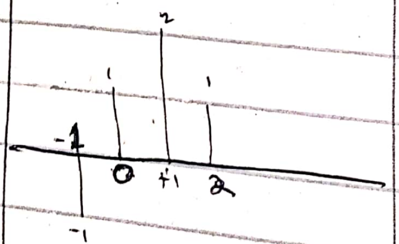




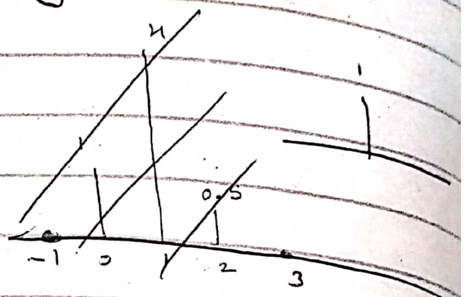
$h[0-k]$



$h[1-k]$



$y(k)$



1

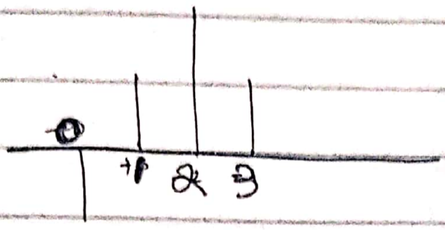
2 2

$2+2=4$

4

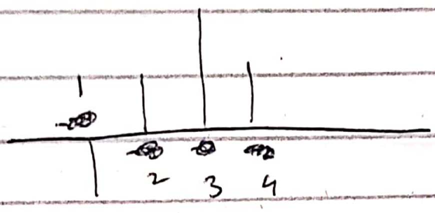
$1+4+0.5=5.5$

$h[2-k]$



$-1+2+1+1$
$= 3$

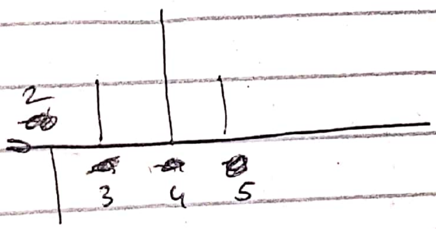
$h[3-k]$



$-2+0.5+2 = 0.5$

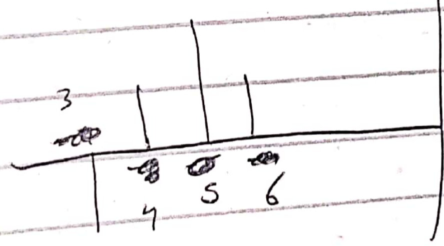
$h[4-k]$

$-0.5+1 = 0.5$

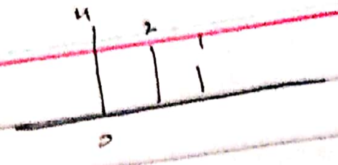


-1

$h[5-k]$

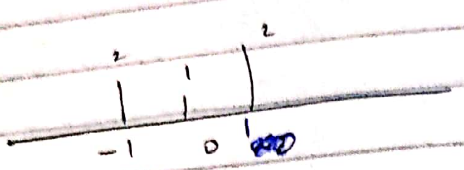


$x(n) =$



$x(n) = [\underset{\uparrow}{4}, 2, 1]$

$h(n)$

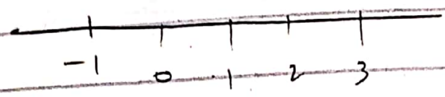


$h(n) = [2, \underset{\uparrow}{1}, 2]$

starting index $= 0 - 1 = -1$

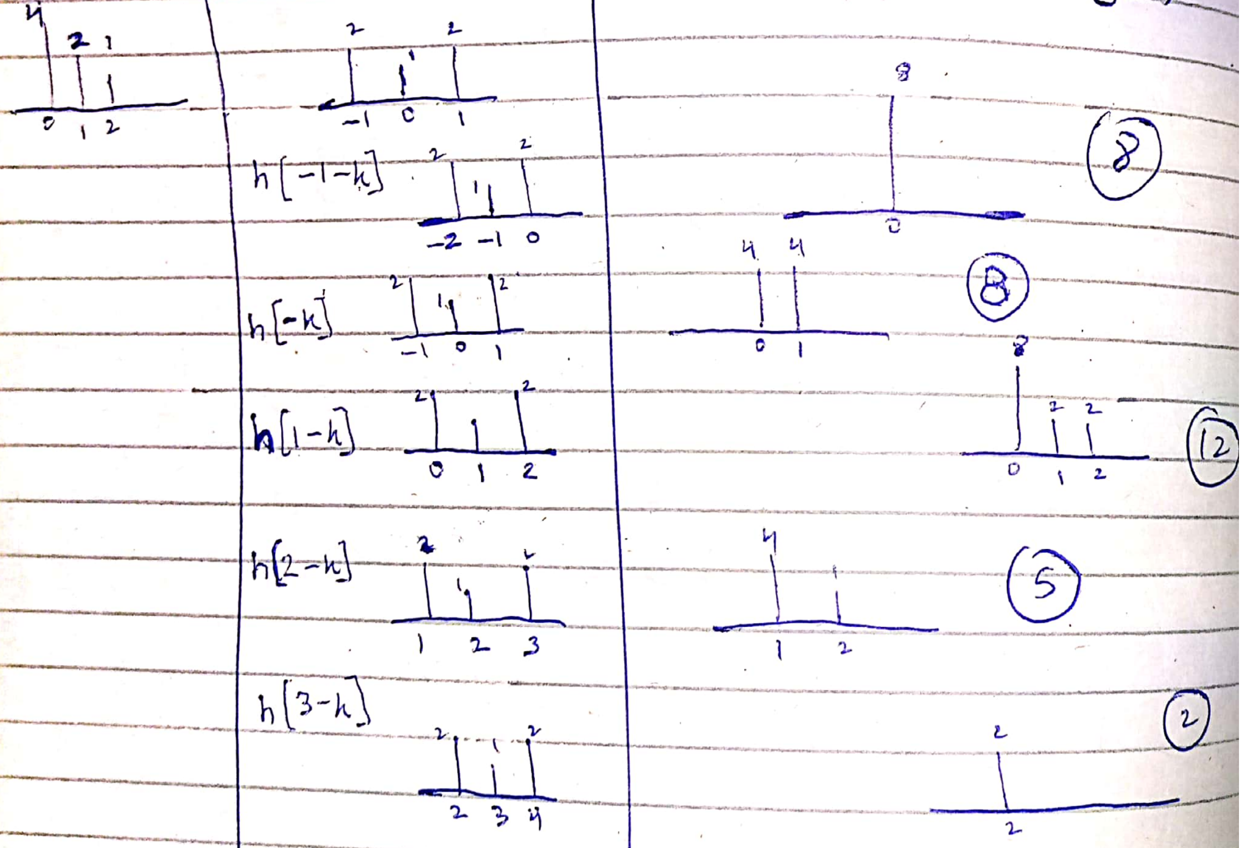
length of $y(n) = 3 + 3 - 1 = 5$

indexes



Solve

$x(k)$	$h[k] \to h[-k]$	$x(k) * h(k) = y(k)$



$[8, \underset{\uparrow}{8}, 12, 5, 2]$

DSP

Time domain | **freq domain** | **T-F domain**

① Continuous
② discrete

periodic → F.S →
Aperiodic → FT → FT
x(n)

→ DTFT

$x(n) \xrightarrow{FT} Xe^{jw}$

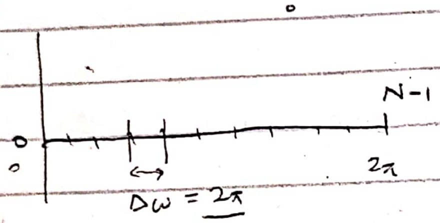
FT^{-1}

↳ Discrete

but w → continuous
↳ $0 \to 2\pi$

$\sum_{-\infty}^{\infty}$ → for convenience

$\sum_{0}^{\infty}$



$\Delta w = \frac{2\pi}{N}$

$= \frac{2\pi}{N} \times k$

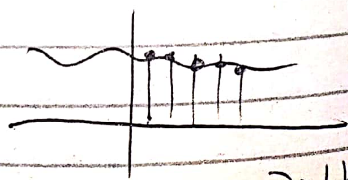
$k \to 0, 1, 2, \ldots, N-1$

$Xe^{jw} \to X(k)$
DFT
↳ best algo → fft

DTF & its derivation

$x(t) \xrightarrow{\text{disrete}} x(n) \xrightarrow{FT} Xe^{jw}$ (DTFT)

Now sample



DFT → $x(k)$

Derivation of DFT

$$Xe^{j\omega} = X(\omega)$$

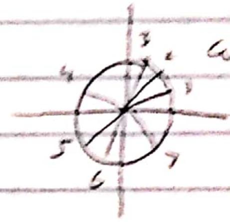
$$DTFT\{x(n)\} = \sum_{n=0}^{\infty} x(n)\, e^{-j\omega n}$$

$n \longrightarrow \omega$

Discrete   Continuous

sampling interval $= \frac{2\pi}{N}$

here $N=8$



$$\omega_k = \frac{2\pi}{N} \times k$$

$$\sum_{n=0}^{N-1} x(n)\, e^{-j\omega_k n}$$

$Xe^{j\omega}$

$r = 1$

magnitude

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

Example

$x(n) = a^n$ $0 \leq n \leq N-1$

$$x(k) = \sum_{n=0}^{N-1} a^n\, e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} \left(a\, e^{-j\frac{2\pi}{N}k}\right)^n$$

Eg $\Rightarrow$ $\sum_{n=0}^{N} a^n \rightarrow a^0 + a^1 + a^2 + a^3 \cdots$

geometric series

Finite $\dfrac{a(1-r^n)}{1-r}$

infinite $\dfrac{a}{1-r}$

$$= \frac{\left(a e^{-j\frac{2\pi}{N}}\right)^N - 1}{a e^{-j\frac{2\pi}{N}k} - 1}$$

$$= \frac{a^N e^{-j2\pi k} - 1}{a e^{-j\frac{2\pi}{N}} - 1}$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$= \cos 2\pi k - j\sin 2\pi k$$

$2\pi, 3\pi, 4\pi$ $\qquad$ $\sin 0$

$$= \frac{a^N - 1}{a e^{-j\frac{2\pi}{N}} - 1}$$

$$x(n) \xrightarrow{DTFT} X(\omega) = X e^{-j\omega}$$

IDTFT

↳ integration bcz its continuous

$$x(n) = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega$$

$$\text{if } x(k) = \sum_{n=0}^{n-1} x(n)\, e^{-j\frac{2\pi n k}{n}}$$

IDTFT

summation

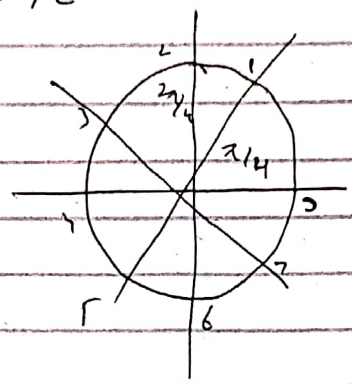
bcz its discrete

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{j\frac{2\pi nk}{N}}$$

Relation b/w DTFT & DFT

DTFT

$$X(\omega) = \sum_{n=0}^{\infty} x(n) e^{-j\omega n}$$



$$2 \times \pi/4 = 2\left(2\pi/8\right)$$

$$\pi/4 = \frac{2\pi}{8}$$

$$2\pi/4 = 2\left(\frac{2\pi}{8}\right)$$

$$3\left(2\pi/8\right)$$

$$7\pi/4 = 7\left(2\pi/8\right)$$

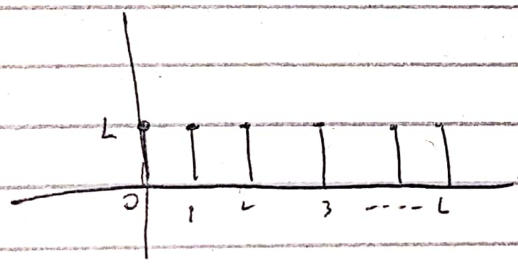
$k = 0, 1, 2, 3$

N = 8 here

$k \quad N$

$$k = \frac{2\pi}{N}$$

$$x(n) = \begin{cases} A & \text{for } 0 \leq n \leq L \\ 0 & \text{otherwise} \end{cases}$$



$X(\omega) = ?$

DTFT = ?

$$X(\omega) = \sum_{n=0}^{L} x(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{L} A e^{-j\omega n}$$

$$= A \sum_{n=0}^{L} e^{-j\omega n}$$

Recall $\sum_{n=0}^{4} a^n =$

$$X(\omega) = A\left(\frac{1 - e^{-j\omega(L+1)}}{1 - e^{-j\omega n}}\right)$$

Example 2

$$X(n) = \begin{cases} 1/4 & 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$x(k) \to$ DFT ?

$$x(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j\pi 2 n k}{N}}$$

$N-1 = 2$

$N = 3$

$$= \sum_{n=0}^{2} x(n) e^{-j\frac{2\pi n}{3}k}$$

3 → term

$$= x(0)e^{0} + x(1)e^{-j\frac{2\pi}{3}k} + x(2)e^{-j\frac{4\pi}{3}k}$$

put $k = 0, 1, 2 \ldots$

$x(0) = 3/4$

$x(1) =$

$x(2) =$

Plot