Combinatorics and Probability By Muhammad Mustafa

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1 Combinatorics

- The Basics of Counting
- The Product Rule (Ex: Cartesian Product)
- The Sum Rule (The union of pairwise disjoint sets)
- The Subtraction Rule (Inclusion-Exclusion for Two Sets)
- The Division Rule
  - (For every way w, There are d ways corresponds to w)
- Tree Diagrams (Visualize it as tree)
- Inclusion-Exclusion Principle $\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset \neq J \subseteq \{1,2,\ldots,n\}} (-1)^{|J|-1} \left|\bigcap_{j \in J} A_j\right|.$
- The Pigeonhole Principle
- Permutations
- The index of the Permutation
- *jth* Permutaion
- Combinations
- Permutations with repetition
- Stars and bars (Combinations with repetition)
- Permutations with indistinguishable objects (SUCCESS string) = $\frac{n!}{n_1!n_2!...n_k!}$
- Distributing a set of n objects
  - Into k **ordered** subsets of $r_1, r_2, ..., r_k$ elements (where $r_1, r_2, ..., r_k = n$) = $\frac{n!}{r_1!r_2!...r_k!}$
  - Into k **unordered** subsets of $r_1, r_2, ..., r_k$ elements (where $r_1, r_2, ..., r_k = n$) = $\frac{1}{x} \cdot \frac{n!}{r_1!r_2!...r_k!}$ where x can be computed easily by tracking duplicates
  - Into k **unordered** *non-empty* subsets of **non-restricted sizes** (where the sum of all cardinalities $= n$)
    - The answer is *Stirling numbers of the second kind*
  - Into *whatever* number of **unordered** *non-empty* subsets of **non-restricted sizes** (where the sum of all cardinalities $= n$)
    - The answer is *Bell sequence* $1, 1, 2, 5, 15, 52, 203, 877, 4140, ...$
- Fibonacci Numbers
  - $F[0] = 0, F[1] = 1, F[n] = F[n1] + F[n2]$
  - $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...$

- Catalan Numbers
  - $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!} = \prod_{k=2}^{n} \frac{n+k}{k} \qquad \text{for } n \geq 0.$
  - $1, 1, 2, 5, 14, 42, 132, 429, 1430,$
- Burnside's Lemma

2 Probability Theory

- A and B are mutually exclusive if $P(A \cap B) = 0$

  A and B are independent if $P(A \mid B) = P(A)$ or $P(B \mid A) = P(B)$

  Addition rule : $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

  If A and B are mutually exclusive: $P(A \cup B) = P(A) + P(B)$

  Multiplication rule : $P(A \cap B) = P(A) * P(B \mid A)$ or $P(B) * P(A \mid B)$

  If A and B are independent: $P(A \cap B) = P(A) * P(B)$

- **Complement rule :** $P(A^C) = 1 - P(A)$

  Law of Total Probability : $P(B) = P(A) * P(B \mid A) + P\left(A^C\right) * P\left(B \mid A^C\right)$

- **Bayes' Law (or Bayes' Theorem) :** $P(A \mid B) = \dfrac{P(A) * P(B \mid A)}{P(A) * P(B \mid A) + P\left(A^C\right) * P\left(B \mid A^C\right)}$