Topic:

"Inner Product Space"

- Find eigenvalues, eigen vector and an orthogonal matrix P for which P^TAP is diagonal.

Q # 1

i) $$A = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$$

Sol:

→ For eigen values,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)(1-\lambda) - (2)(2) = 0$$

$$-2 + 2\lambda - \lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$\lambda^2 + 3\lambda - 2\lambda - 6 = 0$$

$$\lambda(\lambda+3) - 2(\lambda+3) = 0$$

$$(\lambda-2)(\lambda+3) = 0$$

$$\lambda + 3 = 0 \qquad \lambda - 2 = 0$$

$$\boxed{\lambda_1 = -3} \qquad \boxed{\lambda_2 = 2}$$

→ For eigen vector,

$$\lambda_1 = -3$$

$$Av_1 = \lambda_1 v_1$$

$$\begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -3\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} -2x + 2y \\ 2x + y \end{bmatrix} = \begin{bmatrix} -3x \\ -3y \end{bmatrix}$$

By comparing,

$$-2x + 2y = -3x$$
$$-2x + 3x = -2y$$
$$x = -2y$$

$$2x + y = -3y$$
$$2x = -3y - y$$
$$2x = -4y$$
$$x = -2y$$

So,

$$v_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2y \\ y \end{bmatrix} = y\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

or $v_1 = \begin{bmatrix} -2 & 1 \end{bmatrix}^t$

$$\lambda_2 = 2$$

$$Av_2 = \lambda_2 v_2$$

$$\begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} -2x + 2y \\ 2x + y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$$-2x + 2y = 2x$$
$$2y = 2x + 2x$$
$$2y = 4x$$
$$y = 2x$$

$$2x + y = 2y$$
$$2x = 2y - y$$
$$y = 2x$$

So,

$$v_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- or $v_2 = [1 \quad 2]^T$

Now,

→ **Normalizing** v_1 and v_2,

$$\vec{v}_1 = \frac{v_1}{\|v_1\|} = \left[\frac{-2}{\sqrt{5}} \quad \frac{1}{\sqrt{5}}\right]^T$$

$$\vec{v}_2 = \frac{v_2}{\|v_2\|} = \left[\frac{1}{\sqrt{5}} \quad \frac{2}{\sqrt{5}}\right]^T$$

Hence, we have:

- $$P = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

Also,

→ $P^T AP$

$$= \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}\begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}\begin{bmatrix} 4/\sqrt{5} + 2/\sqrt{5} & -2/\sqrt{5} + 4/\sqrt{5} \\ -4/\sqrt{5} + 1/\sqrt{5} & 2/\sqrt{5} + 2/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{6}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-3}{\sqrt{5}} & \frac{4}{\sqrt{5}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-12}{5} - \frac{3}{5} & \frac{-4}{5} + \frac{4}{5} \\ \frac{4}{5} - \frac{4}{5} & \frac{2}{5} + \frac{8}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -15/5 & 0 \\ 0 & 10/5 \end{bmatrix}$$

→ So, a matrix which is diagonal to P is:

$$= \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$ Ans.

(ii) $\begin{bmatrix} 5 & 4 \\ 4 & -1 \end{bmatrix}$

Sol:

→ For eigen values:

$$A = \begin{bmatrix} 5 & 4 \\ 4 & -1 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 5-\lambda & 4 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(-1-\lambda) - 16 = 0$$

$$-5 - 5\lambda + \lambda + \lambda^2 - 16 = 0$$

$$\lambda^2 - 4\lambda - 21 = 0$$

$$\lambda^2 - 7\lambda + 3\lambda - 21 = 0$$

$$\lambda(\lambda-7)+3(\lambda-7)=0$$
$$(\lambda+3)(\lambda-7)=0$$

$\lambda+3=0$ | $\lambda-7=0$

$\boxed{\lambda_1=-3}$ | $\boxed{\lambda_2=7}$

→ For eigen vector:

$$\lambda_1=-3$$
$$Av_1=\lambda_1 v_1$$
$$\begin{bmatrix}5 & 4\\ 4 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=-3\begin{bmatrix}x\\ y\end{bmatrix}$$

$$5x+4y=-3x \qquad 4x-y=-3y$$
$$4y=-3x-5x \qquad 4x=-3y+y$$
$$4y=-8x \qquad 4x=-2y$$
$$y=-2x \qquad y=-2x$$

So,

$$v_1=\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}x\\ -2x\end{bmatrix}=x\begin{bmatrix}1\\ -2\end{bmatrix}$$

or $v_1=[1 \quad -2]^T$

Also,

$$\lambda_2=7$$
$$Av_2=\lambda_2 v_2$$
$$\begin{bmatrix}5 & 4\\ 4 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=7\begin{bmatrix}x\\ y\end{bmatrix}$$

$$5x+4y=7x \qquad 4x-y=7y$$
$$4y=7x-5x \qquad 4x=7y+y$$
$$4y=2x \qquad 4x=8y$$

$x = 2y$ | $x = 2y$

So,

$$v_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y \\ y \end{bmatrix} = y\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- $v_2 = [2 \quad 1]^T$

→ **Normalizing** v_1 and v_2:

$$\vec{v}_1 = \frac{v_1}{\|v_1\|} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix}^T$$

$$\vec{v}_2 = \frac{v_2}{\|v_2\|} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}^T$$

Hence, we have:

- $$P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

Also,

→ $P^T A P$

$$= \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} -3/\sqrt{5} & 14/\sqrt{5} \\ 6/\sqrt{5} & 7/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} -15/5 & 0/5 \\ 0/5 & 35/5 \end{bmatrix}$$

So, diagonal matrix is:

$$= \begin{bmatrix} -3 & 0 \\ 0 & 7 \end{bmatrix} \quad \text{Ans}$$

Q # 2

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Sol:

→ For eigen values:

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$2-\lambda\,[(2-\lambda)(2-\lambda)-1] - 1\,(2-\lambda-1) + 1\,(1-2+\lambda) = 0$$

$$(2-\lambda)(4+\lambda^2-4\lambda-1) - 1(1-\lambda) + (-1+\lambda) = 0$$

$$(2-\lambda)(3+\lambda^2-4\lambda) - 1 + \lambda - 1 + \lambda = 0$$

$$6 + 2\lambda^2 - 8\lambda - 3\lambda - \lambda^3 + 4\lambda^2 - 2 + 2\lambda = 0$$

$$-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$$

$$\begin{array}{c|cccc} 1 & -1 & 6 & -9 & 4 \\ & & -1 & 5 & -4 \\ \hline & -1 & 5 & -4 & 0 \end{array}$$

$\boxed{\lambda_1 = 1}$ $\quad -\lambda^2 + 5\lambda - 4 = 0$

$\times(-1) \quad \lambda^2 - 5\lambda + 4 = 0$

$\lambda^2 - 4\lambda - \lambda + 4 = 0$

$\lambda(\lambda - 4) - 1(\lambda - 4) = 0$

$\lambda - 1 = 0 \quad | \quad \lambda - 4 = 0$

$\boxed{\lambda_2 = 1} \quad | \quad \boxed{\lambda_3 = 4}$

→ For eigen vector:

$\lambda_1, \lambda_2 = 1, 1$ (Repeating Value)

$(A - \lambda_1)V = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1$
$R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + y + z = 0$

$x = -y - z$

So,

$$V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y-z \\ y \\ z \end{bmatrix}$$

$$V = \begin{bmatrix} -y \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix}$$

$$v = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So,

- $v_1 = [-1 \quad 1 \quad 0]^T$
- $v_2 = [-1 \quad 0 \quad 1]^T$

Also,

$$\lambda_3 = 4$$

$$(A - \lambda_3) v_3 = 0$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{l} 2R_2 + R_1 \\ 2R_3 + R_1 \end{array} \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

$$R_3 + R_2 \quad \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x + y + z = 0$

$-2x + z + z = 0$

$-2x + 2z = 0$

$2x = 2z$

$x = z$

$-3y + 3z = 0$

$3y = 3z$

$y = z$

So,

$$V_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- $V_3 = [1 \quad 1 \quad 1]^T$

→ Normalizing:

$$\vec{V_1} = \frac{V_1}{\|V_1\|} = \left[\frac{-1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad 0\right]^T$$

$$\vec{V_2} = \frac{V_2}{\|V_2\|} = \left[\frac{-1}{\sqrt{2}} \quad 0 \quad \frac{1}{\sqrt{2}}\right]^T$$

$$\vec{V_3} = \frac{V_3}{\|V_3\|} = \left[\frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}}\right]^T$$

Now, we have:

- $$P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

Also,

→ $P^T A P$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 4/\sqrt{3} \\ 1/\sqrt{2} & 0 & 4/\sqrt{3} \\ 0 & 1/\sqrt{2} & 4/\sqrt{3} \end{bmatrix}$$

i)

→

$$= \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & 1/2 & 0 \\ 1/2 & \frac{1}{2}+\frac{1}{2} & 0 \\ 0 & 0 & \frac{4}{3}+\frac{4}{3}+\frac{4}{3} \end{bmatrix}$$

So, diagonal matrix is:

$$= \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$ Ans.

Q # 3

(i) $$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

Sol:

→ For eigen values:

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$2-\lambda[(2-\lambda)(2-\lambda)-0] - 0 + (-1)[-(-1)(2-\lambda)] = 0$$

$$(2-\lambda)(4+\lambda^2-4\lambda) - 1(2-\lambda) = 0$$

$$8 + 2\lambda^2 - 8\lambda - 4\lambda - \lambda^3 + 4\lambda^2 - 2 + \lambda = 0$$

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0.$$

Using Synthetic division:

$$\begin{array}{c|cccc}
1 & -1 & 6 & -11 & 6 \\
 & & -1 & 5 & -6 \\
\hline
 & -1 & 5 & -6 & \multicolumn{1}{|c}{0}
\end{array}$$

$\boxed{\lambda_1 = 1}$; $-\lambda^2 + 5\lambda - 6 = 0$

$\times(-1)$ $\quad \lambda^2 - 5\lambda + 6 = 0$

$\lambda^2 - 3\lambda - 2\lambda + 6 = 0$

$\lambda(\lambda - 3) - 2(\lambda - 3) = 0$

$(\lambda - 2)(\lambda - 3) = 0$

$\lambda - 2 = 0 \quad | \quad \lambda - 3 = 0$

$\boxed{\lambda_2 = 2} \quad | \quad \boxed{\lambda_3 = 3}$

→ For eigen vectors:

$\lambda_1 = 1$

$(A - \lambda_1)\, v_1 = 0$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$R_3 + R_1 \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$x - z = 0 \quad | \quad y = 0$

$x = z$

So,

$$v_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

• $v_1 = [1 \quad 0 \quad 1]^T$

$\lambda_2 = 2$

$(A - \lambda_2)v_2 = 0$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-z = 0$ | $-x = 0$

$z = 0$ | $x = 0$

So,

$$v_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

• $v_2 = [0 \quad 1 \quad 0]^T$

$\lambda_3 = 3$

$(A - \lambda_3)v_3 = 0$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x - z = 0$ | $-y = 0$

$x = -z$ | $y = 0$

So,

$$v_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

• $v_3 = [-1 \quad 0 \quad 1]^T$

$\rightarrow$ Normalizing vectors,

$$\vec{v_1} = \frac{v_1}{\|v_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$

$$\vec{v_2} = \frac{v_2}{\|v_2\|} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$$

$$\vec{v_3} = \frac{v_3}{\|v_3\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$

So, we have:

- $$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

Also,

$\rightarrow$ $P^T A P$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -3/\sqrt{2} \\ 0 & 2 & 0 \\ 1/\sqrt{2} & 0 & 3/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{3}{2}+\frac{3}{2} \end{bmatrix}$$

So, diagonal matrix,

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$ Ans.

(ii)
$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Sol:

→ For eigen values:

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[(3-\lambda)(2-\lambda)] - 0 + 1(-1(3-\lambda)) = 0$$

$$(2-\lambda)(6-3\lambda-2\lambda+\lambda^2) - 3 + \lambda = 0$$

$$(2-\lambda)(6-5\lambda+\lambda^2) - 3 + \lambda = 0$$

$$12 - 10\lambda + 2\lambda^2 - 6\lambda + 5\lambda^2 - \lambda^3 - 3 + \lambda = 0$$

$$-\lambda^3 + 7\lambda^2 - 15\lambda + 9 = 0$$

By synthetic division,

3	-1	7	-15	9
		-3	12	-9
	-1	4	-3	0

$\boxed{\lambda_1 = 3}$; $-\lambda^2 + 4\lambda - 3 = 0$

$\times(-1) \Rightarrow \quad \lambda^2 - 4\lambda + 3 = 0$

$\lambda^2 - 3\lambda - \lambda + 3 = 0$

$\lambda(\lambda - 3) - 1(\lambda - 3) = 0$

$(\lambda - 1)(\lambda - 3) = 0$

$\boxed{\lambda_2 = 1}$; $\boxed{\lambda_3 = 3}$

→ For eigen vector.

$\lambda_1, \lambda_3 = 3$ (Repeating Value)

$(A - \lambda I) v = 0$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$R_3 + R_1 \quad \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-x + z = 0 \qquad x = z$

So,

$$v_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

- $v_1 = [1 \quad 0 \quad 1]^T$

Let, $$v_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$v_3 = [0 \quad 1 \quad 0]^T$

$$\lambda_2 = 1$$

$$(A - \lambda_2 I)\, v_2 = 0$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$R_3 - R_1 \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$x + z = 0$ | $2y = 0$

$x = -z$ | $y = 0$

So,

$$v_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- $v_2 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T$

→ **Normalizing vectors:**

$$\vec{v_1} = \frac{v_1}{\|v_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$

$$\vec{v_2} = \frac{v_2}{\|v_2\|} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$$

$$\vec{v_3} = \frac{v_3}{\|v_3\|} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$

So,

$$P = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

Also,

$\rightarrow$ $P^T A P$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 3 & 0 \\ 3/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2}+\frac{3}{2} & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2}+\frac{1}{2} \end{bmatrix}$$

So, diagonal matrix,

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ Ans.

Q # 4

(i) $A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$

Sol:

→ For eigen values,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} a-\lambda & a \\ a & a-\lambda \end{vmatrix} = 0$$

$$(a-\lambda)^2 - a^2 = 0$$

$$\cancel{a^2} + \lambda^2 - 2a\lambda - \cancel{a^2} = 0$$

$$\lambda^2 - 2a\lambda = 0$$

$$\lambda(\lambda - 2a) = 0$$

$\boxed{\lambda_1 = 0}$ $\boxed{\lambda_2 = 2a}$

→ For eigen vector:

$$\lambda_1 = 0$$

$$Av_1 = \lambda_1 v_1$$

$$\begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0\begin{bmatrix} x \\ y \end{bmatrix}$$

$ax + ay = 0$	$ax + ay = 0$
$ax = -ay$	$ax = -ay$
$x = -y$	$x = -y$

So,

$$v_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- $v_1 = [-1 \quad 1]^T$

$\lambda_2 = 2a$

$Av_2 = \lambda_2 v_2$

$$\begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2a\begin{bmatrix} x \\ y \end{bmatrix}$$

$ax + ay = 2ax$	$ax + ay = 2ay$
$ay = 2ax - ax$	$ax = 2ay - ay$
$ay = ax$	$ax = ay$
$x = y$	$x = y$

So,

$$v_2 = \begin{bmatrix} y \\ y \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- $v_2 = [1 \quad 1]^T$

→ Normalizing vectors:

$$\vec{v_1} = \frac{v_1}{\|v_1\|} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$

$$\vec{v_2} = \frac{v_2}{\|v_2\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

So, we have,

- $$P = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Now,

$\rightarrow$ $P^T A P$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 2a/\sqrt{2} \\ 0 & 2a/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & \frac{2a}{2} + \frac{2a}{2} \end{bmatrix}$$

So, diagonal matrix is:

$$= \begin{bmatrix} 0 & 0 \\ 0 & 2a \end{bmatrix}$$ Ans.

(ii)

$$A = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}$$

Sol:

$\rightarrow$ For eigen values:

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} a-\lambda & a & a \\ a & a-\lambda & a \\ a & a & a-\lambda \end{vmatrix} = 0$$

$$a-\lambda\left[(a-\lambda)^2 - a^2\right] - a(\cancel{a^2} - a\lambda - \cancel{a^2})$$
$$+ a(\cancel{a^2} - \cancel{a^2} + a\lambda) = 0$$

$$a-\lambda(\cancel{a^2} + \lambda^2 - 2a\lambda - \cancel{a^2}) - a(-a\lambda) + a(a\lambda) = 0$$

$$(a-\lambda)(\lambda^2 - 2a\lambda) + a^2\lambda + a^2\lambda = 0$$

$$a\lambda^2 - \cancel{2a^2\lambda} - \lambda^3 + 2a\lambda^2 + \cancel{2a^2\lambda} = 0$$

$$-\lambda^3 + 3a\lambda^2 = 0$$

$\times(-1)\quad \lambda^3 - 3a\lambda^2 = 0$

$$\lambda^2(\lambda - 3a) = 0$$

$\lambda^2 = 0$ | $\lambda - 3a = 0$

$\boxed{\lambda_1 = \lambda_2 = 0}$ | $\boxed{\lambda = 3a}$

→ For eigen vector:

$$\lambda_1 = \lambda_2 = 0$$

$$(A - \lambda I)\,v = 0$$

$$\begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array}\begin{bmatrix} a & a & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$ax + ay + az = 0$$

or $\quad x + y + z = 0$

$$x = -y - z$$

So,

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y-z \\ y \\ z \end{bmatrix} \Rightarrow \begin{bmatrix} -y \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix}$$

So,

- $v_1 = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}^T$

and,

- $v_2 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T$

$\lambda_3 = 3a$

$(A - \lambda_3 I) v_3 = 0$

$$\begin{bmatrix} -2a & a & a \\ a & -2a & a \\ a & a & -2a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{array}{l} 2R_2 + R_1 \\ 2R_3 + R_1 \end{array} \begin{bmatrix} -2a & a & a \\ 0 & -3a & 3a \\ 0 & 3a & -3a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-2ax + ay + az = 0$

$(\div a)\quad -2x + y + z = 0$

$-2x + z + z = 0$

$-2x + 2z = 0$

$2x = 2z$

$\boxed{x = z}$

$-3ay + 3az = 0$

$3ay = 3az$

$\boxed{y = z}$

So, $v_3 = [x \quad y \quad z]^t$

$v_3 = [z \quad z \quad z]^t$

$v_3 = [1 \quad 1 \quad 1]^t$

→ **Normalizing vectors:**

$\vec{V_1} = \dfrac{v_1}{\|V_1\|} = \left[\dfrac{-1}{\sqrt{2}} \quad \dfrac{1}{\sqrt{2}} \quad 0\right]^t$

$\vec{V_2} = \dfrac{V_2}{\|V_2\|} = \left[\dfrac{-1}{\sqrt{2}} \quad 0 \quad \dfrac{1}{\sqrt{2}}\right]^t$

$\vec{V_3} = \dfrac{V_3}{\|V_3\|} = \left[\dfrac{1}{\sqrt{3}} \quad \dfrac{1}{\sqrt{3}} \quad \dfrac{1}{\sqrt{3}}\right]^t$

So,

$$P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

Now,

$$P^T A P$$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 0 & 0 & 3a/\sqrt{3} \\ 0 & 0 & 3a/\sqrt{3} \\ 0 & 0 & 3a/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{3a+3a+3a}{3} \end{bmatrix}$$

So, diagonal matrix is:

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3a \end{bmatrix}$$ Ans