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ABELL
BRASELTON

MODERN DIFFERENTIAL EQUATIONS

SECOND EDITION

Harcourt

# MODERN DIFFERENTIAL EQUATIONS

SECOND EDITION

ABELL & BRASELTON

# Table of Integrals

$$\int u^n\,du = \frac{1}{n+1}u^{n+1} + C,\ n \neq -1$$

$$\int \frac{1}{u}\,du = \ln|u| + C$$

$$\int e^u\,du = e^u + C$$

$$\int ue^{au}\,du = \frac{au-1}{a^2}e^{au} + C$$

$$\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$$

$$\int a^u du = \frac{a^u}{\ln a} + C$$

$$\int \sin u\,du = -\cos u + C$$

$$\int \cos u\,du = \sin u + C$$

$$\int \tan u\,du = -\ln|\cos u| + C$$

$$\int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C$$

$$\int \csc u\,du = \ln|\csc u - \cot u| + C$$

$$\int \sec u \tan u\,du = \sec u + C$$

$$\int \sec^2 u\,du = \tan u + C$$

$$\int \sin^2 u\,du = \frac{u}{2} - \frac{1}{4}\sin 2u + C$$

$$\int \cos^2 u\,du = \frac{u}{2} + \frac{1}{4}\sin 2u + C$$

$$\int \sin^3 u\,du = -\frac{1}{3}(2 + \sin^2 u)\cos u + C$$

$$\int \cos^3 u\,du = \frac{1}{3}(2 + \cos^2 u)\sin u + C$$

$$\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$$

$$\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$$

$$\int \tan^2 u\,du = \tan u - u + C$$

$$\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$$

$$\int \sec^3 u\,du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$$

$$\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$$

$$\int \sec^n u\,du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du + C$$

$$\int u \sin u\,du = \sin u - u\cos u + C$$

$$\int u \cos u\,du = \cos u + u\sin u + C$$

$$\int u^n \sin u\,du = -u^n \cos u + n\int u^{n-1}\cos u\,du$$

$$\int u^n \cos u\,du = u^n \sin u - n\int u^{n-1}\sin u\,du$$

$$\int \sin au \sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$$

$$\int \cos au \cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$$

$$\int \sin au \cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$$

$$\int \sin^n u \cos^m u\,du = -\frac{\sin^{n-1} u \cos^{m+1} u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} u \cos^m u\,du = \frac{\sin^{n+1} u \cos^{m-1} u}{n+m} + \frac{m-1}{n+m}\int \sin^n u \cos^{m-2} u\,du$$

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{u^2-a^2}\,du = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$$

$$\int \frac{1}{a^2-u^2}\,du = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$$

$$\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left|u + \sqrt{a^2+u^2}\right| + C$$

$$\int \frac{1}{\sqrt{a^2+u^2}}\,du = \ln\left|u + \sqrt{a^2+u^2}\right| + C$$

$$\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u + \sqrt{u^2-a^2}\right| + C$$

$$\int \frac{1}{\sqrt{u^2-a^2}}\,du = \ln\left|u + \sqrt{u^2-a^2}\right| + C$$

$$\int \frac{1}{(u^2-a^2)^{3/2}}\,du = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$$

$$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$$

$$\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$$

$$\int \sin^{-1} u\,du = u\sin^{-1} u + \sqrt{1-u^2} + C$$

$$\int \cos^{-1} u\,du = u\cos^{-1} u - \sqrt{1-u^2} + C$$

$$\int \tan^{-1} u\,du = u\tan^{-1} u - \frac{1}{2}\ln(1+u^2) + C$$

$$\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$$

$$\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$$

$$\int \ln u\,du = u\ln u - u + C$$

$$\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$$

$$\int u\,dv = uv - \int v\,du$$

# Table of Laplace Transforms

| $f(t)$ | $F(s) = \mathscr{L}\{f(t)\}$ |
|---|---|
| $1$ | $\dfrac{1}{s}$ |
| $t^n$, $n$ a positive integer | $\dfrac{n!}{s^{n+1}}$ |
| $e^{at}$ | $\dfrac{1}{s-a}$ |
| $\sin kt$ | $\dfrac{k}{s^2+k^2}$ |
| $\cos kt$ | $\dfrac{s}{s^2+k^2}$ |
| $\sinh kt$ | $\dfrac{k}{s^2-k^2}$ |
| $\cosh kt$ | $\dfrac{s}{s^2-k^2}$ |
| $e^{at}\sin kt$ | $\dfrac{k}{(s-a)^2+k^2}$ |
| $e^{at}\cos kt$ | $\dfrac{s-a}{(s-a)^2+k^2}$ |
| $e^{at}\sinh kt$ | $\dfrac{k}{(s-a)^2-k^2}$ |
| $e^{at}\cosh kt$ | $\dfrac{s-a}{(s-a)^2-k^2}$ |
| $te^{at}$ | $\dfrac{1}{(s-a)^2}$ |
| $t\sin kt$ | $\dfrac{2ks}{(s^2+k^2)^2}$ |
| $t\cos kt$ | $\dfrac{s^2-k^2}{(s^2+k^2)^2}$ |
| $\sin kt - kt\cos kt$ | $\dfrac{2k^3}{(s^2+k^2)^2}$ |
| $\sin kt + kt\cos kt$ | $\dfrac{2ks^2}{(s^2+k^2)^2}$ |
| $1-\cos kt$ | $\dfrac{k^2}{s(s^2+k^2)}$ |
| $kt - \sin kt$ | $\dfrac{k^3}{s^2(s^2+k^2)}$ |
| $(1+k^2t^2)\sin kt - kt\cos kt$ | $\dfrac{8k^3s^2}{(s^2+k^2)^3}$ |
| $\sin kt \sinh kt$ | $\dfrac{2k^2s}{s^4+4k^4}$ |
| $\cos kt\cosh kt$ | $\dfrac{s^3}{s^4+4k^4}$ |
| $\sin kt\cosh kt$ | $\dfrac{k(s^2+2k^2)}{s^4+4k^4}$ |
| $\cos kt\sinh kt$ | $\dfrac{k(s^2-2k^2)}{s^4+4k^4}$ |
| $e^{at}-e^{bt}$ | $\dfrac{a-b}{(s-a)(s-b)}$ |
| $ae^{at}-be^{bt}$ | $\dfrac{(a-b)s}{(s-a)(s-b)}$ |
| $t^ne^{at}$, $n$ a positive integer | $\dfrac{n!}{(s-a)^{n+1}}$ |
| $t^nf(t)$, $n$ a positive integer | $(-1)^nF^{(n)}(s)$ |
| $f'(t)$ | $sF(s)-f(0)$ |
| $f''(t)$ | $s^2F(s)-sf(0)-f'(0)$ |
| $f^{(n)}(t)$ | $s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$ |
| $e^{at}f(t)$ | $F(s-a)$ |
| $(f*g)(t) = \displaystyle\int_0^t f(t-v)g(v)\,dv$ | $F(s)G(s)$ |
| $\mathscr{U}(t-a)$, $a\ge 0$ | $\dfrac{e^{-as}}{s}$ |
| $f(t-a)\mathscr{U}(t-a)$, $a \ge 0$ | $e^{-as}F(s)$ |
| $f(t)\mathscr{U}(t-a)$, $a\ge 0$ | $e^{-as}\mathscr{L}\{f(t+a)\}$ |
| $\delta(t-t_0)$, $t_0\ge 0$ | $e^{-t_0s}$ |
| $f(t-T)=f(t)$ | $\dfrac{1}{1-e^{-sT}}\displaystyle\int_0^T e^{-st}f(t)\,dt$ |
| $t^{-1/2}$ | $\sqrt{\dfrac{\pi}{s}}$ |
| $t^{1/2}$ | $\dfrac{\sqrt{\pi}}{2s^{3/2}}$ |
| $t^{n-1/2}$, $n=1,2,\ldots$ | $\dfrac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^ns^{n+1/2}}$ |
| $t^\alpha$, $\alpha>-1$ | $\dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ |
| $\dfrac{1}{t}f(t)$ | $\displaystyle\int_s^\infty F(u)\,du$ |
| $\displaystyle\int_0^t f(\alpha)\,d\alpha$ | $\dfrac{F(s)}{s}$ |
| $\dfrac{1}{\sqrt{\pi t}}e^{at}(1+2at)$ | $\dfrac{s}{(s-a)^{3/2}}$ |
| $\dfrac{1}{2\sqrt{\pi t^3}}(e^{bt}-e^{at})$ | $\sqrt{s-a}-\sqrt{s-b}$ |
| $\dfrac{1}{\sqrt{\pi t}} - ae^{a^2t}\operatorname{erfc}(a\sqrt{t})$ | $\dfrac{1}{\sqrt{s}+a}$ |
| $\dfrac{1}{\sqrt{\pi t}} + ae^{a^2t}\operatorname{erf}(a\sqrt{t})$ | $\dfrac{\sqrt{s}}{s-a^2}$ |
| $\dfrac{1}{a}e^{a^2t}\operatorname{erf}(a\sqrt{t})$ | $\dfrac{1}{\sqrt{s}(s-a^2)}$ |
| $e^{a^2t}\operatorname{erfc}(a\sqrt{t})$ | $\dfrac{1}{\sqrt{s}(\sqrt{s}+a)}$ |
| $J_0(kt)$ | $\dfrac{1}{\sqrt{s^2+k^2}}$ |
| $\dfrac{1}{\sqrt{\pi t}}\cos(2\sqrt{kt})$ | $\dfrac{1}{\sqrt{s}}e^{-k/s}$ |
| $\dfrac{1}{\sqrt{\pi k}}\sin(2\sqrt{kt})$ | $\dfrac{1}{s^{3/2}}e^{-k/s}$ |
| $\dfrac{1}{\sqrt{\pi t}}\cosh(2\sqrt{kt})$ | $\dfrac{1}{\sqrt{s}}e^{k/s}$ |
| $\dfrac{1}{\sqrt{\pi k}}\sinh(2\sqrt{kt})$ | $\dfrac{1}{s^{3/2}}e^{k/s}$ |
| $\dfrac{1}{t}(e^{bt}-e^{at})$ | $\ln\dfrac{s-a}{s-b}$ |
| $\dfrac{2}{t}(1-\cos kt)$ | $\ln\dfrac{s^2+k^2}{s^2}$ |
| $\dfrac{2}{t}(1-\cosh kt)$ | $\ln\dfrac{s^2-k^2}{s^2}$ |
| $\dfrac{1}{t}\sin kt$ | $\tan^{-1}\dfrac{k}{s}$ |

# Special Formulas

### Trigonometric Identities

$\cos^2\alpha + \sin^2\alpha = 1$
$1+\tan^2\alpha = \sec^2\alpha$
$1+\cot^2\alpha = \csc^2\alpha$
$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$\sin 2\alpha = 2\sin\alpha\cos\alpha$
$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$\cos^2\alpha = \dfrac{1+\cos 2\alpha}{2}$ $\qquad$ $\sin^2\alpha = \dfrac{1-\cos 2\alpha}{2}$

$\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$ $\qquad$ $\cot\alpha = \dfrac{\cos\alpha}{\sin\alpha}$

$\sec\alpha = \dfrac{1}{\cos\alpha}$ $\qquad$ $\csc\alpha = \dfrac{1}{\sin\alpha}$

$\cos(-\alpha) = \cos\alpha$ $\qquad$ $\sin(-\alpha) = -\sin\alpha$

### Logarithmic and Exponential Function Properties

$\ln a^x = x\ln a$, $a>0$
$\ln ab = \ln a + \ln b$; $a, b > 0$
$\ln\dfrac{a}{b} = \ln a - \ln b$; $a, b>0$
$\ln e^x = x$
$e^{\ln x} = x$, $x>0$
$e^{x+y} = e^xe^y$
$e^{xy} = (e^x)^y$

### Hyperbolic Trigonometric Functions

$\cosh x = \dfrac{1}{2}(e^x+e^{-x})$

$\sinh x = \dfrac{1}{2}(e^x - e^{-x})$

$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

$\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x+e^{-x}}$

$\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x-e^{-x}}$

$\coth x = \dfrac{1}{\tanh x} = \dfrac{e^x+e^{-x}}{e^x-e^{-x}}$

$\cosh^2 x - \sinh^2 x = 1$

### Maclaurin Series

$e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$, $-\infty < x < \infty$

$\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$, $-\infty<x<\infty$

$\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$, $-\infty<x<\infty$

$\dfrac{1}{1-x} = 1 + x + x^2 + \cdots = \displaystyle\sum_{n=0}^{\infty}x^n$, $-1<x<1$

# Modern Differential Equations

# Modern Differential Equations

Second Edition

**Martha L. Abell and James P. Braselton**

Georgia Southern University

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# Contents

# Preface

Computer algebra system and sophisticated graphing calculators have changed the ways in which we learn and teach ordinary differential equations. Instead of focusing students' attention only on a sequence of solution methods, we want them to use their minds to understand what solutions mean and how differential equations can be used to answer pertinent questions.

Interestingly, this metamorphosis in the teaching of differential equations occurred relatively "overnight" and coincided with our professional careers at Georgia Southern University. Our interest in the use of technology in the mathematics classroom began in 1990 when we started to use computer laboratories and demonstrations in our calculus, differential equations, and applied mathematics courses. Over the past ten years we have learned some ways of how to and how not to use technology in the mathematics curriculum. In the early stages, we simply wanted to show students how they could solve more difficult problems by using a computer algebra system so that they could be exposed to the technology. However, we soon realized that we were missing the great opportunity of allowing students to discover aspects of the subject matter on their own. We revised our materials to include experimental problems and thought-provoking questions in which students are asked to make conjectures and investigate supporting evidence. We also developed application projects called *Differential Equations at Work*, not only to emphasize technology, but also to improve the problem-solving and communication skills of our students. To preserve the "wow" aspects of technology, we continue to use it to observe solutions in classroom demonstrations through such things as animating the motion of springs and pendulums. These demonstrations not only grab the attention of students, but also help them to make the connection between a formula and what it represents.

In presenting our findings to colleagues around the country, we quickly found out that others were interested in our work. As a result, we decided to develop a differential equations textbook to share this work with those who share our desire to improve mathematics education. This book is a culmination of years of "trials and tribulation" as the differential equations students at Georgia Southern can attest. Our hope is that its use will inspire students to open their eyes to the exciting discoveries that differential equations offer.

This book is designed to serve as a text for beginning courses in differential equations. Usually, introductory differential equations courses are taken by students who have successfully completed a first-year calculus course, and this text is written at a level readable for them.

## Technology

The benefits of incorporating technology into mathematics courses are well-known. Some of the advantages include enhancing the ability to solve a variety of problems; helping students work examples; supporting varied, realistic, and illuminating applications; exploiting and improving geometric intuition; encouraging mathematical experiments; teaching approximation; showing the mathematical significance of the computer revolution; and making higher-level mathematics accessible to students. In addition, technology is implemented throughout this text to promote the following goals in the learning of differential equations:

1. Solving problems: Using different methods to solve problems and generalize solutions;
2. Reasoning: Exploiting computer graphics to develop spatial reasoning through visualization;
3. Analyzing: Finding the most reasonable solution to real problems or observing changes in the solution under changing conditions;
4. Communicating mathematics: Developing written, verbal, and visual skills to communicate mathematical ideas; and
5. Synthesizing: Making inferences and generalizations, evaluating outcomes, classifying objects, and controlling variables.

Students who develop these skills will succeed not only in differential equations, but also in subsequent courses and in the workforce.

The () icon is used throughout the text to indicate those examples in which technology is used in a nontrivial way to develop or visualize the solution or to indicate the sections of the text, such as those discussing numerical methods, in which the use of appropriate technology is essential.

## Applications

Applications in this text are taken from a variety of fields, especially biology, physics, chemistry, engineering, and economics, and they are documented by references. These applications can be found in many of the examples and exercises, in separate sections and chapters of the text, and in the *Differential Equations at Work* subsections at the end of each chapter. Many of these applications are well-suited to exploration with technology because they incorporate real data. In particular, obtaining closed form solutions is not necessarily "easy" (or always possible). These applications, even if not formally discussed in class, show students that differential equations is an exciting and interesting subject with extensive applications in many fields.

## Style

To keep the text as flexible as possible, addressing the needs of both audiences with different mathematical backgrounds and instructors with varying preferences. *Modern Differential Equations* is written in an easy-to-read, yet mathematically precise, style. It contains all topics usually included in standard differential equations texts. Definitions, theorems, and proofs are concise but worded precisely for mathematical accuracy. Generally, theorems are proved if the proof is instructive or has "teaching value"; these proofs are optional. In other cases, proofs of theorems are developed in the exercises or omitted. Theorems and definitions are boxed for easy reference; key terms are highlighted in boldface. Figures are used frequently to clarify material with a graphical interpretation.



## Second Edition Features

The second edition of *Modern Differential Equations* is an extensive revision of the first edition. Particular features include:

- Over 220 new exercises, for a total of 2567 exercises;
- Seven new *Differential Equations at Work* projects to provide a wider variety of projects to interest, motivate, and challenge classes and students;
- *Many* more computer-generated graphs to help students understand the meaning of the results, especially in the context of applications;
- Greater emphasis on qualitative and graphical aspects of solving differential equations throughout the text (for example, see Section 2.1);
- Greater motivation of material through applications of differential equations throughout the text (for example, see Section 2.2);
- The inclusion of a chapter on Fourier series (Chapter 9) and a chapter on partial differential equations (Chapter 10) to give those who desire to use the text for two semesters or to use the text in an introductory applied mathematics or Fourier series/partial differential equations course the ability to do so.

## Pedagogical Features

### Examples

Throughout the text, numerous examples are given, with thorough explanations and a substantial amount of detail. Solutions to more difficult examples are constructed with the help of graphing calculators or a computer algebra system and are indicated by an icon.

### "Think about it!"

Many examples are followed by a question indicated by a . Generally, basic knowledge about the behavior of functions is sufficient to answer the question. Many of these questions encourage students to use technology. Others focus on the graph of a solution. Thus, "Think about it!" questions help students determine when to use technology and make this text more interactive.

### Technology

Many students entering their first differential course have had substantial experience with various sophisticated calculators and computer algebra systems.

Technology is used throughout the text to explore many of the applications and more difficult examples, especially those marked with , and the problems in the subsections *Differential Equations at Work.* Solutions, partial solutions, or hints to those exercises indicated by an asterisk are contained in the *Student Resource Manual.*

### Differential Equations at Work

*Differential Equations at Work* subsections describe detailed economics, biology, physics, chemistry, and engineering problems documented by references. These problems include real data when available and require students to provide answers based on different conditions. Students must analyze the problem and make decisions about the best way to solve it, including the appropriate use of technology. Each *Differential Equations at Work* project can be assigned as a project requiring a written report, for group work, or for discussion in class.

*Differential Equations at Work* also illustrate how differential equations are used in the real world. Students are often reluctant to believe that the subject matter in calculus, linear algebra, and differential equations classes relates to subsequent courses and to their careers. Each *Differential Equations at Work* subsection illustrates how the material discussed in the course is used in **real life.**

The problems are not connected to a specific section of the text; they require students to draw different mathematical skills and concepts together to solve a problem. Because each *Differential Equations at Work* is cumulative in nature, students must combine mathematical concepts, techniques, and experiences from previous chapters and math courses. Detailed hints regarding the use of technology in solving the problems encountered in *Differential Equations at Work* are included in the *Student Resource Manual.*

### Exercises

Numerous exercises, ranging in level from easy to difficult, are included in each section of the text. In particular, the exercise sets for topics that students find most difficult are rich and varied. The abundant "routine" exercises encourage students to master basic techniques. Most sections also contain interesting *mathematical* and *applied* problems to show that mathematics and its applications are both interesting and relevant. Instructors will find that they can assign a large number of problems, if desired, yet still have plenty for review in addition to those found in the review section at the end of each chapter. Answers to most odd-numbered exercises are included at the end of the text; detailed solutions to those exercises marked with an asterisk are included in the *Student Resource Manual.*

### Chapter Summary and Review Exercises

Each chapter ends with a chapter summary highlighting important concepts, key terms and formulas, and theorems. The Review Exercises following the chapter summary offer students extra practice on the topics in that chapter. These exercises are arranged by section so that students having difficulty can turn to the appropriate material for review.



### Figures

One of technology's benefits is the ease of graphing and plotting. This text provides an abundance of figures and graphs, especially for solutions to examples. In addition, students are encouraged to develop spatial visualization and reasoning skills, to interpret graphs, and to discover and explore concepts from a graphical point of view. To ensure accuracy, the figures and graphs have been completely computer-generated. The *Student Resource Manual* shows typical graphical output using *Mathematica* and *Maple* for exercise solutions.

### Historical Material

Nearly every topic is motivated by either an application or an appropriate historical note. We have also indicated photos of many famous mathematicians and descriptions of the mathematics they discovered.

## Content

The highlights of each chapter are described briefly below.

***Chapter 1*** After introducing preliminary definitions, we discuss direction fields not only for first-order differential equations, but also for systems of equations. In this presentation, we establish a basic understanding of solutions and their graphs. We give an overview of some of the applications covered later in the text to point out the usefulness of the topic and some of the reasons we have for studying differential equations in the exercises.

***Chapter 2*** In addition to discussing the standard techniques for solving several types of first-order differential equations (separable equations, homogeneous equations, exact equations, and linear equations), we introduce several numerical methods (Euler's Method, Improved Euler's Method, Runge-Kutta Method) and discuss the existence and uniqueness of solutions to first-order initial-value problems. Throughout the chapter, we encourage students to build an intuitive approach to the solution process by matching a graph to a solution without actually solving the equation.

***Chapter 3*** Not only do we cover most standard applications of first-order equations in Chapter 3 (orthogonal trajectories, population growth and decay, Newton's law of cooling, free-falling bodies), but we also present many that are not (due to their computational difficulty) in *Differential Equations at Work.*

***Chapter 4*** This chapter emphasizes the methods for solving homogeneous and nonhomogeneous higher order differential equations. It also stresses the Principle of Superposition and the differences between the properties of solutions to linear and nonlinear equations. After discussing Cauchy-Euler equations and introducing series methods of solution to differential equations, the chapter culminates in a discussion of several special equations and the properties of their solutions—equations important in many areas of applied mathematics and physics.

***Chapter 5*** Several applications of higher order equations are presented. The distinctive presentation illustrates the motion of spring–mass systems and pendulums graphically to help students understand what solutions represent and to make the applications more meaningful to them.

***Chapter 6*** The study of systems of differential equations is perhaps the most exciting of all the topics covered in the text. Although we direct most of our attention to solving systems of linear first-order equations with constant coefficients, technology allows us to investigate systems of nonlinear equations and observe phase planes. We also show how to use eigenvalues and eigenvectors to understand the general behavior of systems of linear and nonlinear equations.

***Chapter 7*** Several applications discussed earlier in the text are extended to more than one dimension and solved using systems of differential equations, in an effort to reinforce the understanding of these important problems. Numerous applications involving nonlinear systems are discussed as well.

***Chapter 8*** Laplace transforms are important in many areas of engineering and exhibit intriguing mathematical properties as well. Throughout the chapter, we point out the importance of initial conditions and forcing functions on initial-value problems.

***Chapter 9*** Chapter 9 focuses on boundary value, eigenvalue, and Sturm-Liouville problems and Fourier series, including generalized Fourier series. Technology is utilized to observe the convergence of these series and visualize the Gibbs phenomenon.

***Chapter 10*** In this chapter, we use the method of separation of variables to solve the one-dimensional heat equation, one-dimensional wave equation, Laplace's equation (in a rectangular and circular region), and the wave equation in a circular region. Exercises involving cylindrical and spherical coordinates are also included. Graphics are used so that students can see how the results obtained relate to the applications.

For a one-semester course introducing ordinary differential equations, many instructors will choose to cover topics from Chapters 1 to 7 or from Chapters 1 to 5 and Chapter 8. For a two-semester course, the instructor will easily be able to cover the remaining chapters of the text. In our introductory ordinary differential equations course, we typically cover most of Chapters 1, 2, 4, and 6, and instructors choose a variety of applications from Chapters 3, 5, and 7. In our applied mathematics course, we cover most of the material in Chapters 8 to 10.

## Supplements

The following aids prepared by the authors for instructors and students are available from the publisher:

The ***Instructor's Resource Manual*** contains detailed solutions to the exercises, plus *Mathematica* and *Maple* code and output for the *Differential Equations at Work* problems. Suggestions regarding specific *Differential Equations at Work* are also included.



The ***Student Resource Manual*** contains detailed solutions to selected exercises (those marked with an *) and problems and explanations of prerequisite techniques needed for a standard differential equations course, but not necessarily mastered by students enrolled in the course. The *Student Resource Manual* also contains solutions, partial solutions, or hints to some *Differential Equations at Work* problems and substantial guidance for users of *Mathematica* and *Maple* software.

A website (www.harcourtcollege.com/mathexpress) has specifically been created for the second edition of *Modern Differential Equations*. This website offers additional resources to instructors and students in conjunction with the adoption of the text.

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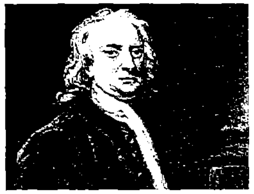
# 1

# Introduction to Differential Equations

The purpose of ***Modern Differential Equations: Theory, Applications, Technology*** is twofold. First, we introduce and discuss the topics covered in an undergraduate course in ordinary differential equations. Second, we indicate how certain technologies such as computer algebra systems and graphics calculators are used to enhance the study of differential equations, not only by eliminating some of the computational difficulties that arise in the study of differential equations but also by overcoming some of the visual limitations associated with the solutions of differential equations.

The advantages of using technology such as graphics calculators and computer algebra systems in the study of differential equations are numerous, but perhaps the most useful is that of being able to produce the graphics associated with solutions of differential equations. This is particularly beneficial in the discussion of applications because many physical situations are modeled with differential equations. For example, in Chapter 5, we see that the motion of a pendulum can be modeled by a differential equation. When we solve the problem of the motion of a pendulum, we use technology to watch the pendulum move. The same is true for the motion of a mass attached to the end of a spring, as

**Gottfried Wilhelm Leibniz**
(1646–1716)
*(North Wind Picture Archives)*



**Isaac Newton**
(1642–1727)
*(North Wind Picture Archives)*

**Leonhard Euler**
(1707–1783)
*(North Wind Picture Archives)*

well as many other problems. In having this ability to use technology, the study of differential equations becomes much more meaningful as well as interesting.

Although Chapter 1 is short in length, the vocabulary introduced here is used throughout the text. Although, to a large extent, this chapter may be read quickly, subsequent chapters will take advantage of the terminology and techniques discussed here.

Any formal introduction to differential equations should begin with German scientist Gottfried Wilhelm Leibniz (1646–1716) and British scientist Isaac Newton (1642–1727), the inventors of calculus. Although we learn in integral calculus that the area under the graph of a smooth positive function is given by a definite integral, both Leibniz and Newton were more concerned with solving differential equations than finding areas. Many of the methods of solution we present in this textbook are from the great Swiss mathematician Leonhard Euler (1707–1783). Subsequently, many problems, such as determining the motion of a string, not only lead to ordinary and partial differential equations but to other areas of mathematics as well. Indeed, differential equations is full of rich and exciting applications; interesting applications are included throughout the textbook to motivate discussions, make the study of differential equations more interesting and pertinent to the real world, and indicate how some people use differential equations beyond this course. However, mathematics is also interesting in its own right; mathematical applications are also included throughout the textbook.

## 1.1 Introduction

We begin our study of differential equations by explaining what a differential equation is. From our experience in calculus, we are familiar with some differential equations. For example, suppose that the acceleration $a(t)$ (measured in ft/s$^2$) of a falling object is $a(t) = -32$. Then, because $a(t) = v'(t)$, where $v(t)$ is the velocity of the object (measured in ft/s), we have $v'(t) = -32$ or

$$\frac{dv}{dt} = -32.$$

An equation like this involving a function of a single variable is called an **ordinary differential equation (ODE).** (If the equation involves partial derivatives, then it is called a **partial differential equation.**) In this case, the function is $v$, which depends on the variable $t$, representing time (measured in seconds). The goal in solving an ODE is to find a function that satisfies the equation. We can solve this ODE through integration:

$$v(t) = \int a(t)dt = \int (-32)dt = -32t + C,$$

where $C$ is an arbitrary constant. This result indicates that $v(t) = -32t + C$ is a **solution** of the ODE for any choice of $C$. (We call this a **general solution** because it involves an arbitrary constant.) In fact, we have found every solution of the ODE because each is expressed as $-32t + C$. Examples of solutions include $v(t) = -32t$ $(C = 0)$, $v(t) = -32t + 32$ $(C = 32)$, and $v(t) = -32t - 8$, $(C = -8)$. This shows that there are an infinite number of solutions.



We can *verify* that $v(t) = -32t + C$ is a general solution of $dv/dt = -32$ through substitution:

$$\frac{dv}{dt} = \frac{d}{dt}(-32t + C) = -32.$$

When we substitute our solution into the left side of the ODE, we obtain $-32$, the value on the right side of the ODE. Therefore, we have verified that $v(t) = -32t + C$ satisfies the ODE for any choice of $C$.

Many times we are given a particular condition that the solution must satisfy. For example, suppose that the object considered earlier has an initial velocity of $-64$ ft/s. In other words, the velocity at time $t = 0$ seconds is represented with the **initial condition** $v(0) = -64$. Therefore, we need to find a solution of the ODE that also satisfies the initial condition. We express this **initial value problem (IVP)** as

$$\frac{dv}{dt} = -32, \ v(0) = -64,$$

where we solve the initial value problem by first *finding a general solution to the ODE* and then by *applying the initial condition to determine the arbitrary constant.* When we substitute $t = 0$ into the general solution $v(t) = -32t + C$, we obtain $v(0) = -32(0) + C = C$. Therefore, $C = -64$ so that $v(0) = -64$ is satisfied. This means that the solution to the IVP is $v(t) = -32t - 64$. Notice that unlike the ODE, the IVP has only one solution.

Another application of differential equations found in calculus is finding a function when given (a) the slope of the line tangent to the graph of the function at any point $(x, y)$, and (b) a point on the graph. For example, suppose that the slope of the tangent line at any point on the graph is given by

$$\frac{dy}{dx} = 3x^2 - 4x,$$

and suppose that the graph passes through the point $(1, 4)$. In this case, the ordinary differential equation is given by $dy/dx = 3x^2 - 4x$, where we try to find the function $y(x)$ that satisfies the initial condition $y(1) = 4$. Therefore, we solve the IVP,

$$\frac{dy}{dx} = 3x^2 - 4x, \ y(1) = 4.$$

As in the previous example, we find a general solution to $dy/dx = 3x^2 - 4x$ through integration. This yields

$$y(x) = x^3 - 2x^2 + C.$$

Now, when we apply the initial condition, we find that

$$y(1) = 1^3 - 2(1)^2 + C = -1 + C = 4$$

so that $C = 5$. Therefore, the solution to the IVP is

$$y(x) = x^3 - 2x^2 + 5.$$

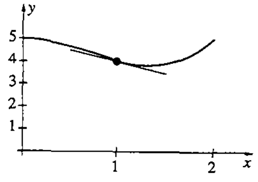



**Figure 1.1** Solution to $dy/dx = 3x^2 - 4x$, $y(1) = 4$ along with the tangent line at (1, 4)

Figure 1.1 shows the graph of the solution together with a portion of the tangent line at the point (1, 4). Notice that the slope of the tangent line is $dy/dx$ evaluated when $x = 1$, $dy/dx = 3(1)^2 - 4(1) = -1$. This observation will be useful in Section 1.2 in helping us better understand the behavior of solutions of differential equations.

The previous two examples are similar in that they each involve an ordinary differential equation in which the highest order derivative is the first derivative. We call equations of this type **first-order** ordinary differential equations because the **order** of a differential equation is the order of the highest order derivative appearing in the equation.

**Example 1**

Determine the order of each of the following differential equations:
**(a)** $dy/dx = x^2/y^2 \cos y$ **(b)** $u_{xx} + u_{yy} = 0$ **(c)** $(dy/dx)^4 = y + x$
**(d)** $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 3x = \sin t$

**Solution** **(a)** This equation is first-order because it includes only one first-order derivative, $dy/dx$. **(b)** This equation is classified as second-order because the highest order derivatives, $u_{xx}$, representing $\partial^2u/\partial x^2$, and $u_{yy}$, representing $\partial^2u/\partial y^2$, are of order two. Hence, Laplace's equation is a second-order partial differential equation. **(c)** This is a first-order equation because the highest order derivative is the first derivative. Raising that derivative to the fourth power does not affect the order of the equation. The expressions

$$(dy/dx)^4 \quad \text{and} \quad d^4y/dx^4$$

do not represent the same quantities: $(dy/dx)^4$ represents the derivative of $y$ with respect to $x$, $dy/dx$, raised to the fourth power; $d^4y/dx^4$ represents the fourth derivative of $y$ with respect to $x$. **(d)** The highest order derivative is $d^2x/dt^2$, so the equation is second order.

**Example 2**

**(a)** Show that $y = c_1 \sin t + c_2 \cos t$ satisfies the second-order ODE $d^2y/dt^2 + y = 0$ where $c_1$ and $c_2$ are arbitrary constants. **(b)** Find the solution to the IVP $d^2y/dt^2 + y = 0$, $y(0) = 0$, $y'(0) = 1$.

**Solution** **(a)** Differentiating, we obtain $dy/dt = c_1 \cos t - c_2 \sin t$ and $d^2y/dt^2 = c_1 \sin t - c_2 \cos t$. Therefore,



$$\frac{d^2y}{dt^2} + y = -c_1 \sin t - c_2 \cos t + c_1 \sin t + c_2 \cos t = 0,$$

so the function satisfies the ODE for any choice of $c_1$ and $c_2$. We graph the solution for several values of these constants in Figure 1.2. Because the solution depends on at least one constant, we say that the functions shown in Figure 1.2 are members of the **family of solutions** of the ODE.
**(b)** Evaluating the function at $t = 0$ yields $y(0) = c_1 \sin 0 + c_2 \cos 0 = c_2$. Then, the initial condition, $y(0) = 0$, indicates that $c_2 = 0$. Similarly, $y'(0) = c_1 \cos 0 - c_2 \sin 0 = c_1$, so $c_1 = 1$ so that the second initial condition, $y'(0) = 1$, is satisfied. Therefore, $y(t) = \sin t$ is the solution to the IVP. We graph this solution in Figure 1.3. Notice that the ODE has an infinite number of solutions while the IVP has only one solution. Also observe that the number of initial conditions in the IVP matches the order of the ODE.

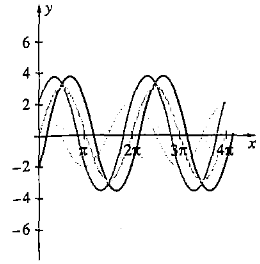



**Figure 1.2** Graph of $y = c_1 \sin t + c_2 \cos t$ for various values of $c_1$ and $c_2$

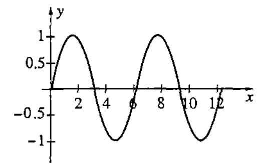



**Figure 1.3** Graph of $y(t) = \sin t$

The next level of classification is based on the following definition. An ordinary differential equation (of order $n$) is called **linear** if it is of the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x),$$

where the functions $a_j(x)$, $j = 0, 1, \ldots, n$, and $f(x)$ are given and $a_n(x)$ is not the zero function.

If the equation under consideration is not of this form, the equation is said to be **nonlinear.** Therefore, some of the properties that lead to classifying an equation as nonlinear are *powers of the dependent variable* (or one of its derivatives) and *functions of the dependent variable.* A similar classification is followed for partial differential equations. In this case, the coefficients in a linear partial differential equation are functions of the independent variables.

**Example 3**

Determine which of the following differential equations are linear: **(a)** $dy/dx = x^3$
**(b)** $\dfrac{d^2u}{dx^2} + u = e^x$ **(c)** $(y - 1)\,dx + x\cos(y)\,dy = 0$ **(d)** $\dfrac{d^3y}{dx^3} + y\dfrac{dy}{dx} = x$
**(e)** $\dfrac{dy}{dx} + x^2y = x$ **(f)** $\dfrac{d^2x}{dt^2} + \sin x = 0$ **(g)** $u_{xx} + yu_y = 0$ **(h)** $u_{xx} + uu_y = 0$

**Solution** **(a)** This equation is linear because the nonlinear term $x^3$ is the function $f(x)$ of the independent variable in the general formula for a linear differential equation. **(b)** This equation is also linear. Using $u$ as the name of the dependent variable does not affect the linearity. **(c)** If $y$ is the *dependent* variable, solving for $dy/dx$ gives us

$$\frac{dy}{dx} = \frac{1-y}{x\cos y}.$$

Because the right side of this equation includes a nonlinear function of $y$, the equation is nonlinear (in $y$). However, if $x$ is the *dependent* variable, solving for $dx/dy$ yields

$$\frac{dx}{dy} = \frac{\cos y}{1-y}x.$$

This equation is linear in the dependent variable $x$. **(d)** The coefficient of the term $dy/dx$ is $y$ instead of an expression involving only the independent variable $x$. Hence, this equation is nonlinear in the dependent variable $y$. **(e)** This equation is linear. The term $x^2$ is the coefficient function. **(f)** This equation, known as the **pendulum equation** because it models the motion of a pendulum, is nonlinear because it involves a nonlinear function of $x$, $\sin x$. Note that $x$ is the dependent variable in this case; $t$ is the independent variable. **(g)** This partial differential equation is linear because the coefficient of $u_y$ is a function of one of the independent variables. **(h)** In this case, there is a product of $u$ and one of its derivatives, so the equation is nonlinear.

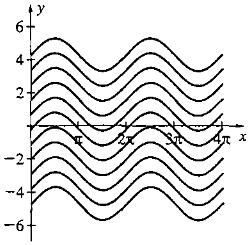


**Figure 1.4** Graph of $y = \sin x + C$ for various values of $C$

If the ODE has the form $dy/dx = f(x)$, we can use integration to determine $y$. The following examples of differential equations of this type illustrate some of the methods of integration that may be encountered.

**Example 4**

Solve the following differential equations.

**(a)** $\dfrac{dy}{dx} = \cos x$ **(b)** $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2+1}}$ **(c)** $\dfrac{dy}{dx} = \dfrac{1}{16+x^2}$ **(d)** $\dfrac{dy}{dx} = xe^x$

**(e)** $\dfrac{dy}{dx} = \dfrac{1}{4-x^2}$

***Solution*** In each case, we integrate the indicated function and graph the solution for several values of the constant of integration. Each solution contains a constant of integration so there are (infinitely) many solutions to each equation.

**(a)** $y = \int \cos x\, dx = \sin x + C$. (See Figure 1.4.)

**(b)** To evaluate $y = \int \dfrac{x}{\sqrt{x^2+1}}\,dx$, we let $u = x^2+1$ so that $du = 2x\,dx$. Then,

$$y = \frac{1}{2}\int \frac{2x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int u^{-1/2}\,du = u^{1/2} + C$$
$$= \sqrt{x^2+1} + C. \quad \text{(See Figure 1.5.)}$$

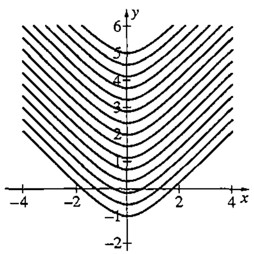


**Figure 1.5** Graph of $y = \sqrt{x^2+1} + C$ for various values of $C$



**(c)** To integrate $y = \int \dfrac{1}{16+x^2}\,dx$, we use a trigonometric substitution. Letting $x = 4\tan\theta$, $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ so that $dx = 4\sec^2\theta\,d\theta$ gives us

$$y = \int \frac{1}{16+(4\tan\theta)^2}\,4\sec^2\theta\,d\theta$$
$$= \int \frac{1}{16+16\tan^2\theta}\,4\sec^2\theta\,d\theta \underbrace{=}_{1+\tan^2\theta=\sec^2\theta} \int \frac{1}{16\sec^2\theta}\,4\sec^2\theta\,d\theta$$
$$= \frac{1}{4}\int d\theta = \frac{1}{4}\theta + C \underbrace{=}_{\frac{x}{4}=\tan\theta \Rightarrow \theta=\tan^{-1}\frac{x}{4}} \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + C.$$

*Graph* $y = \dfrac{1}{4}\tan^{-1}\left(\dfrac{x}{4}\right) + C$ *if* $C = -1, 0,$ *and* $1$.

**(d)** To evaluate $y = \int xe^x\,dx$ by hand, we use the integration by parts formula, $\int u\,dv = uv - \int v\,du$, with $u = x$, $dv = e^x\,dx$, $du = dx$, and $v = e^x$. This gives

$$y = \int xe^x\,dx = \underbrace{xe^x}_{uv} - \int \underbrace{e^x\,dx}_{v\,du} = xe^x - e^x + C.$$

*Find* $C$ *if* $y(1) = 2$.

**(e)** Use partial fractions to evaluate this integral. First, find the partial fraction decomposition of $\dfrac{1}{4-x^2}$, which is determined by finding the constants $A$ and $B$ that satisfy the equation $\dfrac{1}{(2-x)(2+x)} = \dfrac{A}{2-x} + \dfrac{B}{2+x}$. These values are $A = B = \dfrac{1}{4}$. (Why?) Thus,

$$y = \frac{1}{4}\int\left(\frac{1}{2-x} + \frac{1}{2+x}\right)dx = \frac{1}{4}[-\ln|2-x| + \ln|2+x|] + C$$
$$= \frac{1}{4}\ln\left|\frac{2+x}{2-x}\right| + C. \quad \text{(See Figure 1.6.)}$$

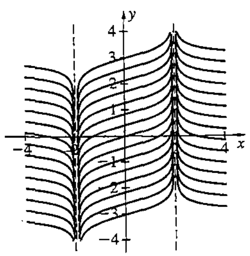


**Figure 1.6** Graph of $y = \frac{1}{4}\ln\left|\frac{2+x}{2-x}\right| + C$

In Example 4, each solution is given as a function $y = y(x)$ of the independent variable. In these cases, the solution is said to be **explicit**. In solving some differential equations, however, we can find only an equation involving $x$ and $y$ that the solution satisfies. In this case, we say that we have found an **implicit** solution.

We will see that given an **arbitrary** differential equation, constructing an explicit or implicit solution is nearly always impossible. Consequently, although mathematicians were first concerned with finding analytic (explicit or implicit) solutions to differential equations, they have since (frequently) turned their attention to addressing properties of the solution and finding algorithms to approximate solutions.

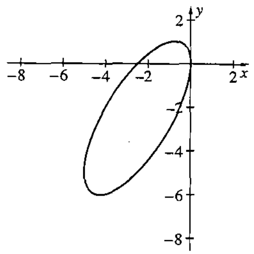



**Figure 1.7** Graph of $2x^2 + y^2 - 2xy + 5x = 0$

Notice that the solutions of $dy/dx = 1/(4 - x^2)$ are undefined when $x = -2$ or $x = 2$. This is because $1/(4 - x^2)$ has vertical asymptotes at these two values of $x$. Later, we will discuss in greater detail the relationship between the differential equation and its solutions.

**Example 5**

Verify that the equation $2x^2 + y^2 - 2xy + 5x = 0$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{2y - 4x - 5}{2y - 2x}.$$

***Solution***

We use implicit differentiation to compute $y' = dy/dx$ if $2x^2 + y^2 - 2xy + 5x = 0$:

$$4x + 2y\frac{dy}{dx} - 2x\frac{dy}{dx} - 2y + 5 = 0$$

$$\frac{dy}{dx}(2y - 2x) = 2y - 4x - 5$$

$$\frac{dy}{dx} = \frac{2y - 4x - 5}{2y - 2x}.$$

The equation satisfies the differential equation $\dfrac{dy}{dx} = \dfrac{2y - 4x - 5}{2y - 2x}$.

Although we cannot solve $2x^2 + y^2 - 2xy + 5x = 0$ for $y$ as a *function* of $x$ (see Figure 1.7), we can determine the corresponding $y$ value(s) for a given value of $x$. For example, if $x = -1$, then

$$2 + y^2 + 2y - 5 = y^2 + 2y - 3 = (y + 3)(y - 1) = 0.$$

Therefore, the points $(-1, -3)$ and $(-1, 1)$ lie on the graph of $2x^2 + y^2 - 2xy + 5x = 0$. (See Figure 1.8.)

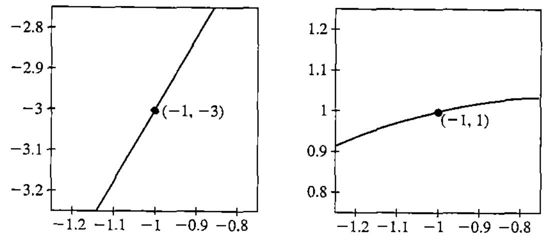



**Figure 1.8** Notice that near the points $(-1, -3)$ and $(-1, 1)$ the implicit solution *looks* like a function. In fact, if we zoom in near the points $(-1, -3)$ and $(-1, 1)$, we see what *appears* to be the graph of a function.



*Find an equation of the line tangent to the graph of $2x^2 + y^2 - 2xy + 5x = 0$ at the points $(-1, -3)$ and $(-1, 1)$.*

In the same manner that we consider systems of equations in algebra, we can also consider systems of differential equations. For example, if $x$ and $y$ represent functions of $t$, we will learn in Chapter 6 to solve the **system of linear equations**

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases},$$

where $a$, $b$, $c$, and $d$ represent constants and differentiation is with respect to $t$. We will see that systems of differential equations arise naturally in many physical situations that are modeled with more than one equation and involve more than one dependent variable. In addition, we will see that it is often useful to write a differential equation with order greater than one as a system of first-order equations.

## EXERCISES 1.1

Determine if each of the following equations is an ordinary differential equation or a partial differential equation. If the equation is an ordinary differential equation, then determine (a) the order of the ordinary differential equation; and (b) if the equation is linear or nonlinear.

1. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = x^3$
2. $y\dfrac{dy}{dx} + y^4 = \sin x$
3. $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}$
4. $y''' - 2y'' + 5y' + y = e^x$

*5. $\left(\dfrac{dy}{dx}\right)^2 + y = 0$

6. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 2y = 0$
7. $\dfrac{1}{c^2}\dfrac{\partial^2 z}{\partial t^2} = \dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}$
8. $uu_x + u_t = 0$

*9. $x\left(\dfrac{d^2y}{dx^2}\right)^4 + 2y = 2x$

10. $\dfrac{d^2x}{dt^2} + 2\sin x = \sin 2t$
11. $u_t + uu_x = \sigma u_{xx}$, $\sigma$ constant
12. $(2x - 1)\,dx - dy = 0$

*13. $(2x - y)\,dx - dy = 0$

14. $\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} = u$
15. $(2x - y)\,dx - y\,dy = 0$
16. Write each of the following second-order equations as a system of first-order equations.
    (a) $\dfrac{d^2x}{dt^2} - \dfrac{dx}{dt} - 6x = 0$
    *(b) $4\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 37x = 0$
    (c) $L\dfrac{d^2x}{dt^2} + g\sin x = 0$
    *(d) $\dfrac{d^2x}{dt^2} - \mu(1 - x^2)\dfrac{dx}{dt} + x = 0$
    (e) $t\dfrac{d^2x}{dt^2} + (b - t)\dfrac{dx}{dt} - ax = 0$

In Exercises 17–28, verify that each of the given functions is a solution to the corresponding differential equation. ($A$, $B$, and $C$ represent constants.)

17. $\frac{dy}{dx} + 2y = 0, y(x) = e^{-2x}, y(x) = 5e^{-2x}$

18. $\frac{dy}{dx} + xy = 0, y(x) = e^{-x^2/2}$

*19. $\frac{dy}{dx} + y = \sin x, y(x) = e^{-x} - \frac{1}{2}\cos x + \frac{1}{2}\sin x$

20. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0, y(x) = e^{4x}, y(x) = e^{-3x}$

21. $\frac{d^2y}{dx^2} + 9\frac{dy}{dx} = 0, y(x) = A + Be^{-9x}$

22. $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - 10x = 0, x(t) = Ae^{2t} + Be^{-5t}$

*23. $\frac{d^2x}{dt^2} + x = t\cos t - \cos t, x(t) = A\cos t + B\sin t + \frac{t^2}{4}\sin t - \frac{t}{2}\sin t + \frac{t}{4}\cos t$

24. $\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 40y = 0, y(x) = e^{6x}\cos 2x, y(x) = e^{6x}\sin 2x$

25. $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 0, y(x) = A + Be^{2x} + Ce^{-2x}$

26. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} = 0, y(x) = A + Bx + Ce^{2x}$

27. $x^2\frac{d^2y}{dx^2} - 12x\frac{dy}{dx} + 42y = 0, y(x) = Ax^6 + Bx^7$

28. $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 5y = 0,$
$y(x) = x^{-1}[A\cos(2\ln x) + B\sin(2\ln x)]$

In Exercises 29–33, verify that the given equation satisfies the differential equation. Use the equation to determine $y$ for the given value of $x$. Graph each equation using a graphing calculator or computer algebra system.

29. $\frac{dy}{dx} = \frac{-x}{y}, x^2 + y^2 = 16; x = 0$

30. $3y(x^2 + y)\,dx + x(x^2 + 6y)\,dy = 0, x^3y + 3xy^2 = 8; x = 2$

*31. $\frac{dy}{dx} = -\frac{2y}{x} - 3, x^3 + x^2y = 100; x = 1$

32. $y\cos x\,dx + (2y + \sin x)\,dy = 0, \quad y^2 + y\sin x = 1; x = 0$

33. $\left(\frac{y}{x} + \cos y\right)dx + (\ln x - x\sin y)\,dy = 0,$
$y\ln x + x\cos y = 0; x = 1$

In Exercises 34–43, use integration to find a solution to the differential equation.

34. $\frac{dy}{dx} = (x^2 - 1)(x^3 - 3x)^3$

35. $\frac{dy}{dx} = x\sin(x^2)$

36. $\frac{dy}{dx} = \frac{x}{\sqrt{x^2 - 16}}$

*37. $\frac{dy}{dx} = \frac{1}{x\ln x}$

38. $\frac{dy}{dx} = x\ln x$

39. $\frac{dy}{dx} = xe^{-x}$

40. $\frac{dy}{dx} = \frac{-2(x + 5)}{(x + 2)(x - 4)}$

*41. $\frac{dy}{dx} = \frac{x - x^2}{(x + 1)(x^2 + 1)}$

42. $\frac{dy}{dx} = \frac{\sqrt{x^2 - 16}}{x}$

43. $\frac{dy}{dx} = (4 - x^2)^{3/2}$

44. $\frac{dy}{dx} = \frac{1}{x^2 - 16}$

In Exercises 45–54, use the indicated conditions with the indicated solution to determine the solution to the given problem.

45. $\frac{dy}{dx} + 2y = 0, y(0) = 2, y(x) = Ae^{-2x}$

46. $\frac{dy}{dx} + y = \sin x, y(0) = -1,$
$y(x) = Ae^{-x} - \frac{1}{2}\cos x + \frac{1}{2}\sin x$

*47. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0, y(0) = 0, y'(0) = 1,$
$y(x) = Ae^{4x} + Be^{-3x}$



48. $\frac{d^2y}{dx^2} + 9\frac{dy}{dx} = 0, y(0) = 2, y'(0) = -1,$
$y(x) = A + Be^{-9x}$

49. $\frac{d^2y}{dx^2} + 9y = 0, y(0) = 0, y'(0) = 1,$
$y(x) = A\cos 3x + B\sin 3x$

50. $\frac{d^2y}{dx^2} + 9y = 0, y(0) = 0, y'(0) = 0,$
$y(x) = A\cos 3x + B\sin 3x$

*51. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} = 0, y(0) = 0, y'(0) = 1,$
$y''(0) = 3, y(x) = A + Bx + Ce^{2x}$

52. $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 0, y(0) = 1, y'(0) = -1,$
$y''(0) = 0, y(x) = A + Be^{2x} + Ce^{-2x}$

53. $x^2\frac{d^2y}{dx^2} - 12x\frac{dy}{dx} + 42y = 0, y(1) = 1,$
$y'(1) = -1, y(x) = Ax^6 + Bx^7$

54. $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 5y = 0, y(1) = 0, y'(1) = 1,$
$y(x) = x^{-1}(A\cos(2\ln x) + B\sin(2\ln x))$

In Exercises 55–58, solve the initial-value problem. Graph the solution on an appropriate interval.

55. $\frac{dy}{dx} = 4x^3 - x + 2, y(0) = 1$

56. $\frac{dy}{dx} = \sin 2x - \cos 2x, y(0) = 0$

*57. $\frac{dy}{dx} = \frac{1}{x^2}\cos\left(\frac{1}{x}\right), y\left(\frac{2}{\pi}\right) = 1$

58. $\frac{dy}{dx} = \frac{\ln x}{x}, y(1) = 0$

59. The velocity of a falling object of mass $m$ that is subjected to air resistance proportional to the instantaneous velocity $v$ of the object is found by solving the initial-value problem

$$\begin{cases} m\frac{dv}{dt} = mg - cv \\ v(0) = v_0 \end{cases}$$ (Note that $c > 0$ is the proportionality constant.)

(a) If a general solution to $m\frac{dv}{dt} = mg - cv$ is $v(t) = \frac{mg}{c} + Ke^{-ct/m}$, find the solution of this initial-value problem.

(b) Determine $\lim_{t\to\infty} v(t)$.

60. The number of cells $P(t)$ in a bacteria colony after $t$ hours is determined by solving the initial-value problem

$$\begin{cases} \frac{dP}{dt} = kP \\ P(0) = P_0 \end{cases}$$

(a) If a general solution of $\frac{dP}{dt} = kP$ is $P(t) = Ce^{kt}$, use the initial condition to find $C$.

(b) Find the value of $k$ so that the population doubles in 8 hours.

*61. In 1840, the Belgian mathematician–biologist Pierre F. Verhulst (1804–1849) developed the **logistic equation** $dP/dt = rP - aP^2$, where $r$ and $a$ are positive constants, to predict the population $P(t)$ in certain countries.

(a) If a general solution to this equation is

$$P(t) = \frac{r}{a + Ce^{-rt}}$$

find the solution that satisfies $P(0) = P_0$.

(b) Determine $\lim_{t\to\infty} P(t)$.

62. The differential equation $\frac{dS}{dt} + \frac{3}{t + 100}S = 0$, where $S(t)$ is the number of pounds of salt in a particular tank at time $t$, is used to approximate the amount of salt in the tank containing a salt–water mixture in which pure water is allowed to flow into the tank while the mixture is allowed to flow out of the tank. If $S(t) = \frac{15{,}000{,}000}{(t + 100)^3}$, show that $S$ satisfies $\frac{dS}{dt} + \frac{3}{t + 100}S = 0$. What is the initial amount of salt in the tank? As $t \to \infty$, what happens to the amount of salt in the tank?

63. The displacement (measured from $x = 0$) of a mass attached to the end of a spring at time $t$ is given by $x(t) = 3\cos 4t + \frac{9}{4}\sin 4t$. Show that $x$ satisfies the ordinary differential equation $x'' + 16x = 0$. What is the initial position of the mass? What is the initial velocity of the mass?

64. Show that $u(x, y) = \ln\sqrt{x^2 + y^2}$ satisfies **Laplace's equation** $u_{xx} + u_{yy} = 0$.

*65. The temperature in a thin rod of length $2\pi$ after $t$ minutes at a position $x$ between 0 and $2\pi$ is given by

$u(x, t) = 3 - e^{-16kt}\cos 4x$. Show that $u$ satisfies $u_t = ku_{xx}$. What is the initial temperature ($t = 0$) at $x = \pi$? What happens to the temperature at each point in the wire as $t \to \infty$?

66. The displacement $u$ of a string of length 1 at time $t$ and position $x$ where $x$ is measured from $x = 0$ is given by $u(x, t) = \sin \pi x \cos t$. Show that $u$ satisfies $\pi^2 u_{tt} = u_{xx}$. What is the value of $u$ at the endpoints $x = 0$ and $x = 1$ for all values of $t$?

*67. Find the value(s) of $m$ so that $y = x^m$ is a solution of
$$x^2y'' - 2xy' + 2y = 0.$$

68. Find the value(s) of $m$ so that $y = e^{mx}$ is a solution of
$$y'' - 3y' - 18y = 0.$$

*69. Use the fact that $\dfrac{d}{dx}(e^{2x}y) = e^{2x}\dfrac{dy}{dx} + 2e^{2x}y$ and integration to solve $e^{2x}\dfrac{dy}{dx} + 2e^{2x}y = e^x$.

70. Use the fact that $\dfrac{d}{dx}(e^x y) = e^x\dfrac{dy}{dx} + e^x y$ and integration to solve $e^x\dfrac{dy}{dx} + e^x y = xe^x$.

71. The **time-independent Schrödinger equation** is given by
$$-\frac{h^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x).$$
If $U(x) = 0$, find conditions of $E$ so that $\psi(x) = A\sin\left(\dfrac{n\pi x}{L}\right)$ is a solution of the time-independent Schrödinger equation.

72. Use implicit differentiation to show that $-\dfrac{1}{x} + \dfrac{2}{x^2} + \dfrac{1}{y} - \dfrac{1}{y^2} = C$ is an implicit solution of the differential equation $\dfrac{dy}{dx} = \dfrac{(x-4)y^3}{x^3(y-2)}$. Is $y = 0$ a solution of this differential equation? Is $y = 0$ a singular solution?

73. Show that $x + \dfrac{x^2}{y} = C$ is an implicit solution of the differential equation $\dfrac{dy}{dx} = \dfrac{y^2 + 2xy}{x^2}$. Is $y = 0$ a solution of this differential equation? Is $y = 0$ a singular solution?

74. The current $I(t)$ in an ***L-R* circuit,** which contains a resistor, an inductor, and a voltage source, satisfies the differential equation $RI + L\dfrac{dI}{dt} = E(t)$, where $R$ and $L$ are constants representing the resistance and the inductance and $E(t)$ is the voltage source. Is this equation linear or nonlinear? Determine the order of the equation.

Throughout *Modern Differential Equations: Theory, Applications, Technology,* we use graphs of solutions of differential equations. In some cases, we are able to predict what the graph of a solution should look like. If the graph of our proposed solution does not appear as predicted, we know that either we made a mistake in constructing our proposed solution or our conjecture about the general shape of the graph of the solution is wrong. In other cases, we will find that it is easier to examine the graph of a solution than it is to examine the solution (if we are able to construct one in the first place).

In Exercises 75–77, **(a)** verify that the indicated function is a solution of the given differential equation, and **(b)** graph the solution on the indicated interval(s).

75. $xy' + y = \cos x$, $y = \dfrac{\sin x}{x}$; $[-2\pi, 0) \cup (0, 2\pi]$

76. $16y'' + 24y' + 153y = 0$, $y = e^{-3x/4}\cos 3x$; $\left[0, \dfrac{3}{2}\pi\right]$

*77. $x^3\dfrac{d^3y}{dx^3} + x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 40y = 0$,
$y = x^{-1}\sin(3\ln x)$; $(0, \pi]$

78. Verify that

**(a)** $$\begin{cases} x(t) = e^{-t}\left[\dfrac{100\sqrt{3}}{3}\sin\sqrt{3}t + 20\cos\sqrt{3}t\right] \\ y(t) = e^{-t}\left[\dfrac{-40\sqrt{3}}{3}\sin\sqrt{3}t + 20\cos\sqrt{3}t\right] \end{cases}$$

is a solution of the system of differential equations $\begin{cases} x'(t) = 4y(t) \\ y'(t) = -x(t) - 2y(t) \end{cases}$.

**(b)** Graph $x(t)$, $y(t)$, and the parametric equations $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ for $0 \le t \le 2\pi$.

*79. **(a)** Show that $(x^2 + y^2)^2 = 5xy$ is an implicit solution of
$$[4x(x^2+y^2) - 5y]\,dx + [4y(x^2+y^2) - 5x]\,dy = 0.$$

**(b)** Graph $(x^2 + y^2)^2 = 5xy$ on the rectangle $[-2, 2] \times [-2, 2]$.

**(c)** Approximate all points on the graph of $(x^2 + y^2)^2 = 5xy$ with $x$-coordinate 1.

**(d)** Approximate all points on the graph of $(x^2 + y^2)^2 = 5xy$ with $y$-coordinate $-0.319$.

Often the calculus and algebra encountered in solving differential equations can be tedious, if not completely overwhelming or impossible. Today many sophisticated calculators and computer algebra systems are capable of performing the integration and algebraic simplification encountered when solving many differential equations. Having access to these tools can be a great advantage: a large number of problems can be solved relatively quickly, we make conjectures as to the general form of a solution to different forms of differential equations, and these tools allow us to check and verify our work.

80. Solve $dy/dx = \sin^4 x$, $y(0) = 0$ and graph the resulting solution on the interval $[0, 4\pi]$.

81. A general solution of $y^{(4)} + \dfrac{25}{2}y'' - 5y' + \dfrac{629}{16}y = 0$ is given by
$$y = e^{-x/2}(C_1\cos 3x + C_2\sin 3x) + e^{x/2}(C_3\cos 2x + C_4\sin 2x),$$
where $C_1$, $C_2$, $C_3$, and $C_4$ are constants. Solve the initial-value problem
$$\begin{cases} y^{(4)} + \dfrac{25}{2}y'' - 5y' + \dfrac{629}{16}y = 0 \\ y(0) = 0,\ y'(0) = 1,\ y''(0) = -1,\ y'''(0) = 1 \end{cases}$$
and graph the resulting solution.

82. A general solution of the system $\begin{cases} x' = 4y \\ y' = -4x \end{cases}$ is $\begin{cases} x = C_2\sin 4t - C_1\cos 4t \\ y = C_1\sin 4t + C_2\cos 4t \end{cases}$, where $C_1$ and $C_2$ are constants. Solve the initial-value problem
$$\begin{cases} x' = 4y \\ y' = -4x \\ x(0) = 4,\ y(0) = 0 \end{cases}$$
and then graph $x(t)$, $y(t)$, and the parametric equations $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$.

83. A general solution of the system $\begin{cases} x' = -5x + 4y \\ y' = 2x + 2y \end{cases}$ is $\begin{cases} x = C_1e^{3t} - 4C_2e^{-6t} \\ y = 2C_1e^{3t} + C_2e^{-6t} \end{cases}$, where $C_1$ and $C_2$ are constants. Solve the initial-value problem
$$\begin{cases} x' = -5x + 4y \\ y' = 2x + 2y \\ x(0) = 4,\ y(0) = 0 \end{cases}$$
and graph $x(t)$, $y(t)$, and the parametric equations $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$.



## 1.2 A Graphical Approach to Solutions: Slope Fields and Direction Fields

- Systems of Ordinary Differential Equations and Direction Fields
- Relationship Between Systems of First-Order and Higher Order Equations

Suppose that we are asked to solve the ordinary differential equation $dy/dx = e^{-x^2}$. In this case, we do not attempt to solve this equation through integration as we did in the previous section because of the presence of the function $f(x) = e^{-x^2}$ on the right side of the ODE. (See Exercise 21.) Instead, we can gain insight into the behavior of solutions of this ODE through a graphical approach by considering the slope of the tangent line to solutions of the ODE. Recall from Section 1.1 that the differential equation gives the slope of the tangent line to solutions of the ODE at the given point $(x, y)$ in the $xy$-plane. Therefore, if we wish to determine the slope of the tangent line to the solution to the ODE that passes through $(0, 1)$ at this point, we substitute $(0, 1)$ into the right side of $dy/dx = e^{-x^2}$. Because the right side only depends on $x$, we have the slope

$$\frac{dy}{dx} = e^{-(0)^2} = 1.$$

In fact, the slope of the line tangent to solutions at all points of the form $(0, y)$ is 1. At the point $(\sqrt{\ln 2}, 4)$, we find that the slope is

$$\frac{dy}{dx} = e^{-(\sqrt{\ln 2})^2} = e^{-\ln 2} = e^{\ln 2^{-1}} = \frac{1}{2}.$$

Again, we obtain the same slope at all points $(\sqrt{\ln 2}, y) \approx (0.832555, y)$ and $(-\sqrt{\ln 2}, y) \approx (-0.832555, y)$.

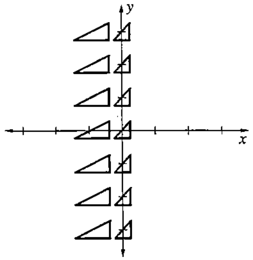


Figure 1.9 Several line segments in the slope field for $dy/dx = e^{-x^2}$

In Figure 1.9, we draw several short line segments using points of the form $(0, y)$ for $y = 0, \pm 1, \pm 2, \pm 3$. Notice that we use a triangle with base 1 and height 1 to assist in sketching the tangent lines with slope 1 at these points. We use a triangle with base 2 and height 1 to help us draw the lines of slope 1/2 that are tangent to solutions at the points $(-\sqrt{\ln 2}, y)$ for $y = 0, \pm 1, \pm 2, \pm 3$. By drawing a set of short line segments representing the tangent lines to solutions of the ODE at numerous points in the plane, we construct the **slope field** of the ODE. We show the slope field for $dy/dx = e^{-x^2}$ on the square $[-2, 2] \times [-2, 2]$ in Figure 1.10. (Note that because constructing a slope field is time consuming, we usually let a computer algebra system do the work for us.) Observe that at each point along the $y$-axis, the slope is 1 as we predicted earlier. Notice also that the slope appears to be zero for larger values of $|x|$. This is because for large values of $x$ (in absolute value), $dy/dx \approx 0$ because $\lim_{x\to\pm\infty} e^{-x^2} = 0$. Therefore, we expect solutions to "flatten out" as $|x|$ increases.

We can use the slope field to investigate the solution to an initial-value problem such as

$$\frac{dy}{dx} = e^{-x^2}, y(-2) = -1$$

by starting at the point $(-2, -1)$ and tracing the solution by following the tangent line slopes. This solution is sketched in Figure 1.11. Thus, although we did not determine a formula for solutions to the ODE or the IVP, we were able to determine some properties of the solutions by using the slope field of the differential equation.

## Systems of Ordinary Differential Equations and Direction Fields

We can also consider systems of differential equations. In Chapter 6, we will learn how to solve systems of first-order ordinary differential equations of the form



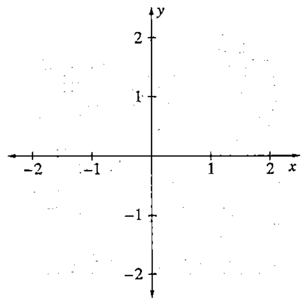


Figure 1.10 Slope field for $dy/dx = e^{-x^2}$

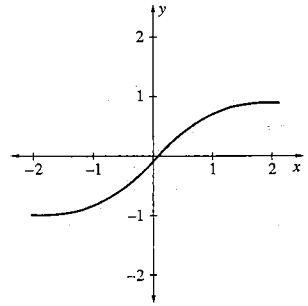


Figure 1.11 Sketching solution to $dy/dx = e^{-x^2}$, $y(-2) = -1$

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases},$$

where $a$, $b$, $c$, and $d$ are given constants. In the case of this system, we solve for $x(t)$ and $y(t)$. For example, if we consider the system

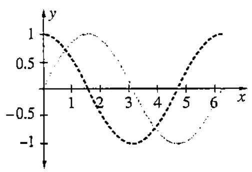


Figure 1.12 Graphs of $x(t) = \sin t$ and $y(t) = \cos t$ over $0 \le t \le 2\pi$.

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x \end{cases},$$

we can verify that the parametric equations

$$\begin{cases} x(t) = \sin t \\ y(t) = \cos t \end{cases}$$

satisfy the system because

$$\frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t = y \quad \text{and} \quad \frac{dy}{dt} = \frac{d}{dt}(\cos t) = -\sin t = -x.$$

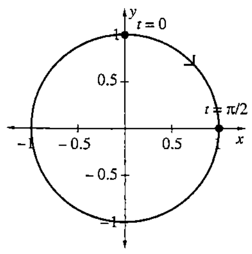



Figure 1.13 Graph of the parametric equations $\begin{cases} x(t) = \sin t \\ y(t) = \cos t \end{cases}$ for $0 \le t \le 2\pi$

We can graph each function separately as we do in Figure 1.12. [The graph of $x(t)$ is the solid curve; that of $y(t)$ is dashed.] Another option is to graph them as a pair of parametric equations as in Figure 1.13. As we recall, the graph of this pair of parametric equations is a circle of radius 1 centered at the origin because $x^2 = \sin^2 t$ and $y^2 = \cos^2 t$, so that $x^2 + y^2 = \sin^2 t + \cos^2 t = 1$. However, we must indicate the *orientation* of the curve (the direction of increasing parameter value $t$). For this pair of parametric equations, we find that at $t = 0$, $x(0) = \sin 0 = 0$ and $y(0) = \cos 0 = 1$.

Therefore, the point (0, 1) corresponds to $t = 0$. Similarly, the point (1, 0) corresponds to $t = \pi/2$ because $x(\pi/2) = \sin(\pi/2) = 1$ and $y(\pi/2) = \cos(\pi/2) = 0$. This means that the solution moves from (0, 1) to (1, 0) as $t$ increases, so the orientation is clockwise. We place an arrow directed from (0, 1) to (1, 0) to indicate this orientation. The parametric equations $\{x(t) = \sin t, y(t) = \cos t\}$ satisfy the *initial-value problem*

$$\begin{cases} \dfrac{dx}{dt} = y,\ x(0) = 0 \\ \dfrac{dy}{dt} = -x,\ y(0) = 1 \end{cases}$$

because they satisfy the system of differential equations as well as the two initial conditions. Notice that in the case of an initial-value problem involving a system of differential equations, an initial condition is given for each of the variables $x$ and $y$ that depend on $t$. In the parametric plot, the solution to this initial-value problem passes through the point $(x(0), y(0)) = (0, 1)$.

Another way to view a system of ordinary differential equations is through the use of a **direction field,** which is similar to a slope field. First, we write the first-order system

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$$

as a first-order equation with

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{cx + dy}{ax + by}.$$

Then, we can consider the slope field associated with this differential equation. For example, if we refer back to the system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x \end{cases},$$

we obtain the first-order equation $dy/dx = -x/y$, so we can determine the slope of tangent lines to solutions at points in the $xy$-plane. At the point $(1/\sqrt{2}, 1/\sqrt{2})$, the solution to this system that passes through this point has slope $\dfrac{-1/\sqrt{2}}{1/\sqrt{2}} = -1$. In a similar manner, we can find the slope at other points in the plane. However, as we mentioned in our earlier discussion, we must indicate the orientation when we graph parametric equations, so we consider the vector $\langle dx/dt, dy/dt\rangle$ or $\dfrac{dx}{dt}\mathbf{i} + \dfrac{dy}{dt}\mathbf{j}$ with com-



ponents from the system of differential equations. In the case of this system, we consider $\langle y, -x\rangle$. At the point $(1/\sqrt{2}, 1/\sqrt{2})$, we obtain the vector $\langle 1/\sqrt{2}, -1/\sqrt{2}\rangle$. This means that the solution through $(1/\sqrt{2}, 1/\sqrt{2})$ has tangent vector $\langle 1/\sqrt{2}, -1/\sqrt{2}\rangle$. The direction field is made up of tangent vectors such as $\langle 1/2, -1/\sqrt{2}\rangle$ to solutions at points in the plane, so it is similar to the slope field for $dy/dx = -x/y$ shown in Figure 1.14, except that vectors are used to indicate the orientation of solutions. In Figure 1.15, we show the direction field for this system. The vectors in the direction field indicate that solutions to this system are circles in the $xy$-plane that are directed clockwise. We graph several solutions along with the direction field in Figure 1.16. A collection of solutions in the $xy$-plane is called the **phase portrait** of the system. Notice that at points in the first quadrant, where $x > 0$ and $y > 0$, $dx/dt = y > 0$ and $dy/dt = -x < 0$. This means that $x(t)$ increases and $y(t)$ decreases along solutions in the first quadrant. In the second quadrant, where $x < 0$ and $y > 0$, $dx/dt = y > 0$ and $dy/dt = -x > 0$. Therefore, $x(t)$ and $y(t)$ increase along solutions in the second quadrant. We can perform a similar analysis for points in the other two quadrants.

## Relationship Between Systems of First-Order and Higher Order Equations

Again, consider the system $\begin{cases} dx/dt = y \\ dy/dt = -x \end{cases}$. If we differentiate the equation $dx/dt = y$ with respect to $t$, we obtain $d^2x/dt^2 = dy/dt$. Therefore, if we equate $dy/dt = -x$ and $dy/dt = d^2x/dt^2$, we have $d^2x/dt^2 = -x$ or $d^2x/dt^2 + x = 0$. We say that the system of two first-order ODEs is equivalent to the second-order ODE $d^2x/dt^2 + x = 0$. Often, how-

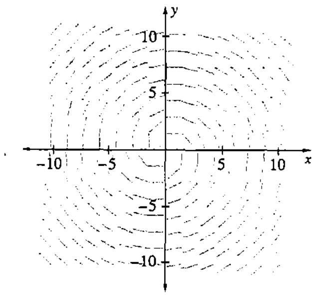



**Figure 1.14** Slope field for $dy/dx = -x/y$

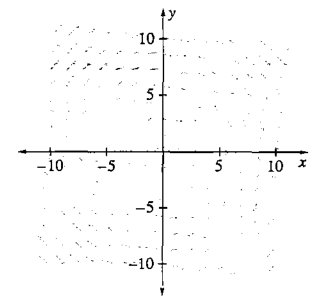



**Figure 1.15** Direction field for $\begin{cases} dx/dt = y \\ dy/dt = -x \end{cases}$

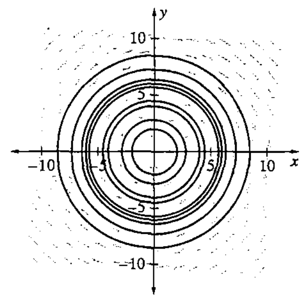



**Figure 1.16** Direction field for $\begin{cases} dx/dt = y \\ dy/dt = -x \end{cases}$ and solution curves

ever, we begin with a second-order ODE of the form $d^2x/dt^2 + b\,dx/dt + cx = f(t)$ and would like to write the equation as a system of first-order ODEs. We do this by letting $dx/dt = y$ and by differentiating this equation with respect to $t$ to obtain $d^2x/dt^2 = dy/dt$. Solving the second-order ODE for $d^2x/dt^2$, we find $d^2x/dt^2 = -b\,dx/dt - cx + f(t)$ so that $dy/dt = -b\,dx/dt - cx + f(t)$. Replacing $dx/dt$ in this equation with $y$, we obtain the following equivalent system of first-order ODEs:

$$\begin{cases} dx/dt = y \\ dy/dt = -by - cx + f(t) \end{cases}.$$

By writing the second-order ODE as a system of first-order ODEs, we investigate the behavior of the solution of the second-order ODE by observing the behavior in the direction field and phase portrait of the corresponding system. A similar procedure is used to transform an ODE of degree greater than two into a system of first-order ODEs.

## EXERCISES 1.2

In Exercises 1–4, use the slope field to determine if the indicated path is that of a solution to the differential equation.

1. $dy/dx = -y/x$

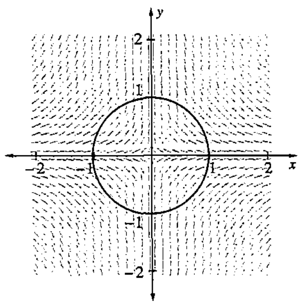


Figure 1.17

2. $dy/dx = x^2 + y^2$

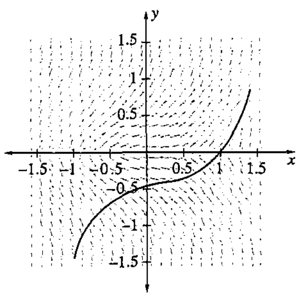


Figure 1.18



3. $dy/dx = x/y$

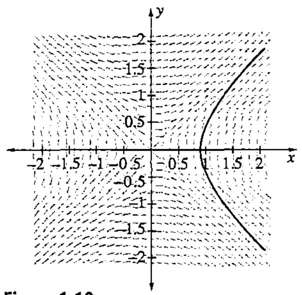


Figure 1.19

4. $dy/dx = x^2 - y$

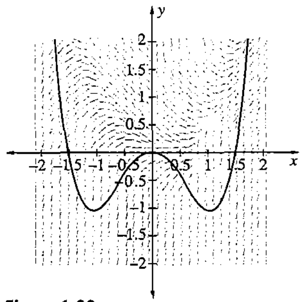


Figure 1.20

In Exercises 5–8, use the slope field to sketch the solutions that pass through the given points.

5. $dy/dx = x/y$

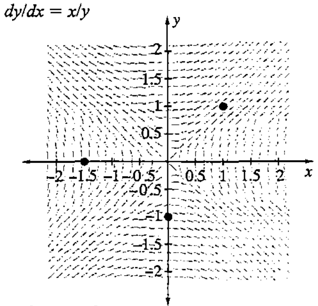


Figure 1.21

6. $dy/dx = x^2 - y$

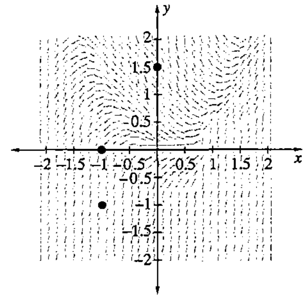


Figure 1.22

7. $dy/dx = x - y$

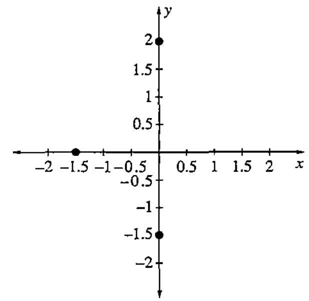


Figure 1.23

8. $dy/dx = y^2 - x^2$

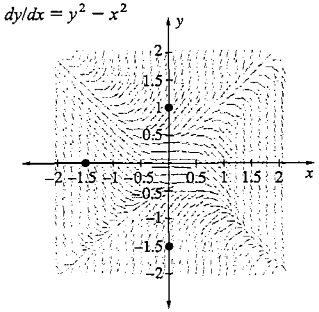


Figure 1.24

**9.** Graph the slope field for $y' = \sin y$. Determine $\lim_{x\to\infty} y(x)$ if $y(0) = y_0$, where **(a)** $y_0 = -3\pi/2$; **(b)** $y_0 = -\pi/2$; **(c)** $y_0 = \pi/2$; **(d)** $y_0 = 3\pi/2$.

**10.** Graph the slope field for $y' = \sin x$. Does $\lim_{x\to\infty} y(x)$ exist for any initial condition $y(0) = y_0$? Solve the ODE and find $\lim_{x\to\infty} y(x)$. Does this match the graphical result?

***11.** Graph the slope field for $y' = e^{-x}$. Does $\lim_{x\to\infty} y(x)$ exist for any initial condition $y(0) = y_0$? Solve the ODE and find $\lim_{x\to\infty} y(x)$. Does this match the graphical result?

**12.** Plot the slope field for $y' = \dfrac{1}{x^2+1}$. Does $\lim_{x\to\infty} y(x)$ exist for any initial condition $y(0) = y_0$? Solve the ODE and find $\lim_{x\to\infty} y(x)$. Does this match the graphical result?

In Exercises 13–14, use the direction field of the given system to sketch the graph of the solution that satisfies the indicated initial conditions. Determine $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$ in each case (if they exist).

**13.** System: $\begin{cases} x' = y \\ y' = x \end{cases}$, **(a)** $\{x(0) = 0, y(0) = 2\}$; **(b)** $\{x(0) = -2; y(0) = 0\}$; **(c)** $\{x(0) = -2, y(0) = 2\}$

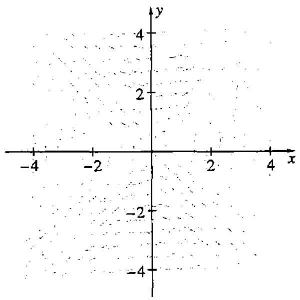


**Figure 1.25**

**14.** System: $\begin{cases} x' = 3x - 2y \\ y' = 4x - y \end{cases}$; $\{x(0) = 0, y(0) = 5\}$.

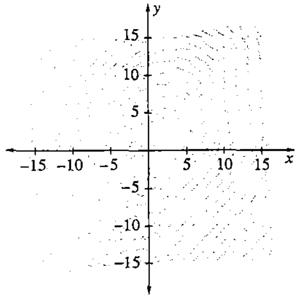


**Figure 1.26**

In Exercises 15–20, write the second-order equation as a system of first-order equations.

**15.** $\dfrac{d^2x}{dt^2} + 4x = 0$

**16.** $\dfrac{d^2x}{dt^2} - 5\dfrac{dx}{dt} = 0$

***17.** $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 13x = 0$

**18.** $\dfrac{d^2x}{dt^2} - 6\dfrac{dx}{dt} + 7x = 0$

**19.** $\dfrac{d^2x}{dt^2} + 16x = \sin t$

**20.** $x'' + 4x' + 13x = e^{-t}$

***21.** **(a)** Use a computer algebra system to solve $dy/dx = e^{-x^2}$; **(b)** Graph the solution to the IVP $dy/dx = e^{-x^2}$, $y(0) = a$ for $a = -2, -1, 0, 1, 2$. **(c)** Graph the slope field of the differential equation together with the solutions in **(b)**. Do the solutions appear to match the results described at the beginning of the section?

**22.** **(a)** Use a computer algebra system to graph the slope field for $dy/dx = \sin(2x - y)$. **(b)** Use the slope field to graph the solution $y(x)$ that satisfies the initial condition $y(0) = 5$. Does $\lim_{x\to\infty} y(x)$ appear to exist?

**23.** Consider the systems

$$\begin{cases} \dfrac{dx}{dt} = -y \\ \dfrac{dy}{dt} = x \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{dx}{dt} = -y \\ \dfrac{dy}{dt} = -x \end{cases}$$



**(a)** For given initial conditions, do you *think* that solutions of the system are similar or different? Why?

**(b)** Figure 1.27 shows the direction field associated with each system. Use the direction fields to graph the solutions that satisfy these initial conditions: **(i)** $x(0) = 0.5, y(0) = 0$; **(ii)** $x(0) = -0.25, y(0) = 0$; **(iii)** $x(0) = 0, y(0) = 0.75$; and **(iv)** $x(0) = 0, y(0) = -0.5$.

**(c)** How do your graphs affect your conjecture in **(a)**?

**24.** Consider the systems

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{2}x \\ \dfrac{dy}{dt} = y \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{dx}{dt} = -\dfrac{1}{2}x \\ \dfrac{dy}{dt} = -y \end{cases}$$

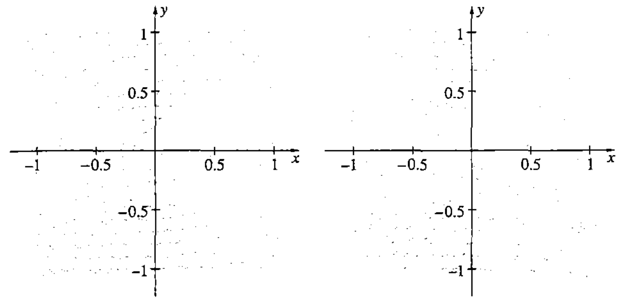


**Figure 1.27**

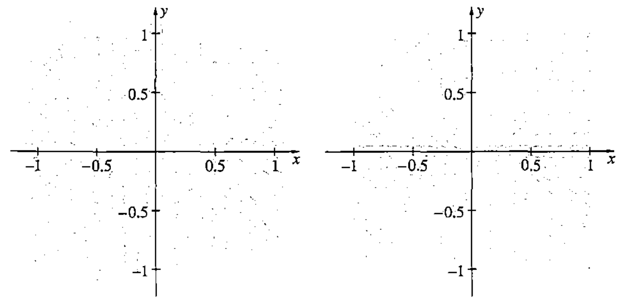


**Figure 1.28**

**(a)** For given initial conditions, do you *think* that solutions of the systems are similar or different? Why?

**(b)** Figure 1.28 shows the direction field associated with each system. Use the direction fields to graph the solutions that satisfy these initial conditions: **(i)** $x(0) = 0.5, y(0) = 0.25$; **(ii)** $x(0) = -0.25, y(0) = -0.5$; **(iii)** $x(0) = -0.5, y(0) = 0.75$; and **(iv)** $x(0) = 0.75, y(0) = -0.5$.

**(c)** How do your graphs affect your conjecture in **(a)**?

**(d)** For each system, use your graphs to calculate $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$, as long as not both $x(t)$ and $y(t)$ are the zero function.

**25.** **(Competing Species)** The system of equations

$$\begin{cases} \dfrac{dx}{dt} = x(a - b_1 x - b_2 y) \\ \dfrac{dy}{dt} = y(c - d_1 x - d_2 y) \end{cases}$$

where $a$, $b_1$, $b_2$, $c$, $d_1$, and $d_2$ represent positive constants that can be used to model the population of two species, represented by $x(t)$ and $y(t)$, competing for common food supply.

**(a)** Figure 1.29 shows the direction field for the system if $a = 1, b_1 = 2, b_2 = 1, c = 1, d_1 = 0.75$, and $d_2 = 2$. **(i)** Use the direction field to graph various solutions if both $x(0)$ and $y(0)$ are not zero. **(ii)** Use the direction field and your graphs to approximate $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$.

**(b)** Figure 1.30 shows the direction field for the system if $a = 1, b_1 = 1, b_2 = 1, c = 0.67, d_1 = 0.75$, and $d_2 = 1$. **(i)** Use the direction field to graph various solutions if both $x(0)$ and $y(0)$ are not zero. **(ii)** Use the direction field and your graphs to determine the fate of the species with population $y(t)$. What happens to the species with population $x(t)$?

**26.** **(a)** Use a computer algebra system to solve $dy/dx = e^{-x^3}$. **(b)** Graph the solution to the IVP $dy/dx = e^{-x^3}$, $y(0) = a$ for $a = -2, -1, 0, 1, 2$. **(c)** Graph the slope field of the differential equation together with the solutions in **(b)**. Do the solutions appear to match the results described at the beginning of the section?

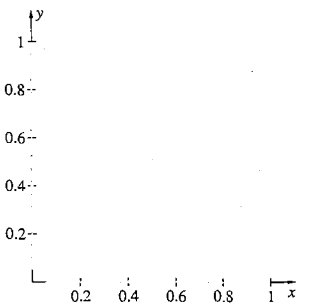



**Figure 1.29**

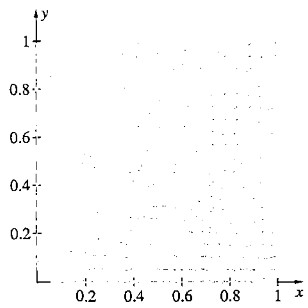



**Figure 1.30**

## *CHAPTER 1 SUMMARY*

### *Concepts & Formulas*

**Section 1.1**

**Differential Equation**
An equation that contains the derivative or differentials of one or more dependent variables with respect to one or more independent variables.

**Ordinary Differential Equation**
If a differential equation contains only ordinary derivatives (of one or more dependent variables) with respect to a single independent variable, the equation is called an **ordinary differential equation.**

**Partial Differential Equation**
A differential equation that contains the partial derivatives or differentials of one or more independent variables with respect to more than one independent variable.

**Linear Ordinary Differential Equation**
A **linear ordinary differential equation** is of the form



$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

**Order of an Equation**
The order of the highest derivative in the differential equation is called the **order** of the equation.

**Solution**
A **solution** of a differential equation on a given interval is a function that is continuous on the interval and has all the necessary derivatives that are present in the differential equation such that when substituted into the equation yields an identity for all values on the interval.

**Explicit Solution**
A solution given as a function of the independent variable.

**Implicit Solution**
A solution given as a relation $f(x, y) = 0$.

**Trivial Solution**
$y(x) = 0$.

**General Solution of an *N*th Order Linear Equation**
A solution that depends on $n$ arbitrary constants and includes all solutions of the equation.

**Section 1.2**

**Slope Field**

**Phase Portrait**
A collection of graphs of solutions to the system in the $xy$-plane.

**Direction Field**
A collection of line segments that indicate the slope of the tangent line to the solution(s) of a differential equation.

## *CHAPTER 1 REVIEW EXERCISES*

In Exercises 1–5, determine **(a)** if the equation is an ordinary differential equation or partial differential equation; **(b)** the order of the ordinary differential equation; and **(c)** if the equation is linear or nonlinear.

1. $y' = y$
2. $au_x + u_t = 0$, $a$ constant
3. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$
4. $m\ddot{x} + kx = \sin t$, $m$ and $k$ constants
5. $\dfrac{\partial\phi}{\partial x}\dfrac{\partial^2\phi}{\partial x^2} = \dfrac{\partial^2\phi}{\partial y^2}$

In Exercises 6–13, verify that the given function is a solution to the corresponding differential equation. ($A$ and $B$ denote constants.)

6. $\dfrac{dy}{dx} + y\cos x = 0$, $y = e^{-\sin x}$
7. $\dfrac{dy}{dx} - y = \sin x$, $y = (e^x - \cos x - \sin x)/2$
8. $y'' + 4y' - 5y = 0$, $y = e^{-5x}$, $y = e^x$

*9. $y'' - 6y' + 45y = 0$, $y = e^{3x}(\cos 6x - \sin 6x)$

10. $x^2y'' - xy' - 16y = 0$, $y = Ax^5 + Bx^{-3}$
11. $x^2y'' + 3xy' + 2y = 0$, $y = x^{-1}(\cos(\ln x) - \sin(\ln x))$
12. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = x$, $y = (x - 1)/2$

*13. $y'' - 7y' + 12y = 2$, $y = Ae^{3x} + Be^{4x} + 1/6$

In Exercises 14 and 15, verify that the given implicit function satisfies the differential equation.

14. $(2x - 3y)\,dx + (2y - 3x)\,dy = 0$, $x^2 - 3xy + y^2 = 1$
15. $(y\cos(xy) + \sin x)\,dx + x\cos(xy)\,dy = 0$, $\sin(xy) - \cos x = 0$

In Exercises 16–19, find a solution of the differential equation.

16. $\dfrac{dy}{dx} = xe^{-x^2}$

17. $\dfrac{dy}{dx} = x^2 \sin x$

18. $\dfrac{dy}{dx} = \dfrac{2x^2 - x + 1}{(x-1)(x^2+1)}$

19. $\dfrac{dy}{dx} = \dfrac{x^2}{\sqrt{x^2-1}}$

In Exercises 20–22, use the indicated initial or boundary conditions with the given general solution to determine the solution(s) to the given problem.

20. $\dfrac{dy}{dx} + 2y = x^2$, $y(0) = 1$,
$y = \frac{1}{4} - \frac{1}{2}x + \frac{1}{2}x^2 + Ae^{-2x}$

21. $y'' + 4y = x$, $y(0) = 1$, $y(\pi/4) = \pi/16$,
$y = x/4 + A\cos 2x + B\sin 2x$

22. $x^2y'' + 5xy' + 4y = 0$, $y(1) = 1$, $y'(1) = 0$,
$y = Ax^{-2} + Bx^{-2}\ln x$

In Exercises 23 and 24, solve the initial-value problem. Graph the solution on an appropriate interval.

23. $\dfrac{dy}{dx} = \cos^2 x \sin x$, $y(0) = 0$

24. $\dfrac{dy}{dx} = \dfrac{4x-9}{3(x-3)^{2/3}}$, $y(0) = 0$

25. The temperature on the surface of a steel ball at time $t$ is given by $u(t) = 70e^{-kt} + 30$ (in °F) where $k$ is a positive constant. Show that $u$ satisfies the first-order equation $du/dt = -k(u - 30)$. What is the initial temperature ($t = 0$) on the surface of the ball? What happens to the temperature as $t \to \infty$?

26. The displacement (measured from $x = 0$) of a mass attached to the end of a spring at time $t$ is given by $x(t) = \frac{1}{4}e^{-t}\left(\cos\sqrt{35}t + \frac{9}{\sqrt{35}}\sin\sqrt{35}t\right)$. Show that $x$ satisfies the ordinary differential equation $x'' + 2x' + 36x = 0$. What is the initial displacement of the mass? What is the initial velocity of the mass?

27. For a particular wire of length 1 unit, the temperature $u$ at time $t$ hours at a position of $x$ feet from the end ($x = 0$) of the wire is estimated by $u(x, t) = e^{-\pi^2 kt}\sin \pi x - e^{-4\pi^2 kt}\sin 2\pi x$. Show that $u$ satisfies the heat equation $u_t = ku_{xx}$. What is the initial temperature ($t = 0$) at $x = 1$? What happens to the temperature at each point in the wire as $t \to \infty$?

28. The height $u$ of a long string at time $t$ and position $x$ where $x$ is measured from the middle of the string ($x = 0$) is given by $u(x, t) = \sin x \cos 2t$. Show that $u$ satisfies the wave equation $u_{tt} = 4u_{xx}$. What is the initial height ($t = 0$) at $x = 0$?

*29. Show that $u(x, y) = \tan^{-1}(y/x)$ satisfies Laplace's equation $u_{xx} + u_{yy} = 0$.

30. The slope field of $y' = 2(y - y^2)$ is shown. Sketch the graph of the solution that satisfies the indicated initial condition. Also, determine if $\lim_{x\to\infty} y(x)$ exists. **(a)** $y(0) = 0.5$; **(b)** $y(0) = 1.5$; **(c)** $y(0) = -0.5$.

31. The direction field for $\{x' = x, y' = 2y\}$ is shown. Is it possible to select initial conditions so that $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = 0$?

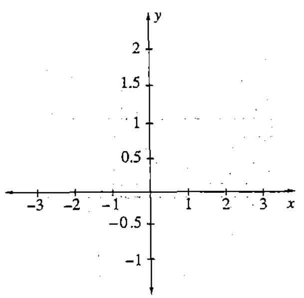



**Figure 1.31**

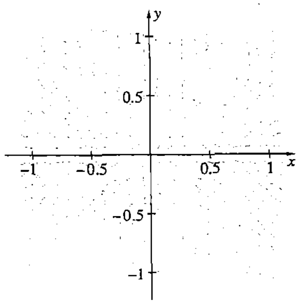



**Figure 1.32**

# 2

# *First-Order Equations*

We will devote a considerable amount of time in this course to developing explicit, implicit, numerical, and graphical solutions of differential equations. In this chapter we discuss first-order ordinary differential equations and some methods used to construct explicit, implicit, numerical, and graphical solutions of them. Several of the equations and methods of solution discussed here will be used in later chapters of the text.

## 2.1 Separable Equations

Many interesting problems involving populations are solved through the use of first-order ordinary differential equations. For example, let $y(t)$ be the fish population size (in tons) at time $t$ (years) and suppose that the population has birth rate $by(t)$ and mortality rate $my(t)$ where we are assuming that these rates are proportional to $y(t)$. If there are no other factors affecting the rate of change in the population, $dy/dt$, then $dy/dt$ equals the rate at which the population increases (the birth rate) minus the rate at which it decreases (the mortality rate), or

$$\frac{dy}{dt} = \text{(Birth Rate)} - \text{(Mortality Rate)}.$$

Mathematically, we represent this relationship with the differential equation

$$\frac{dy}{dt} = by - my = (b - m)y \quad \text{or} \quad \frac{dy}{dt} = ky,$$

where $k = b - m > 0$. If $y(0) = y_0$ is the initial population size, then we find $y(t)$ by solving the initial value problem

$$\frac{dy}{dt} = ky, y(0) = y_0.$$

This is known as the **Malthus model** and is due to the work of the English clergyman and economist Thomas R. Malthus. Notice that the ODE can be written as

$$\frac{dy}{y} = k\,dt.$$

We say that this ODE is **separable** because we are able to collect all of the terms involving $y$ on one side of the equation and all the terms involving $t$ on the other side of the equation.

**Definition 2.1 Separable Differential Equation**

A first-order differential equation that can be written in the form $g(y)y' = f(x)$ or $g(y)\,dy = f(x)\,dx$ is called a **separable differential equation.**

After we place the ODE in this form, we solve through integrating each side with respect to the indicated variable. This gives

$$\int \frac{dy}{y} = \int k\,dt.$$

Integration yields

$$\ln|y| = kt + C_1,$$

where $C_1$ is a constant of integration. When possible, we like to solve for $y(t)$. In this case, we use the exponential function and its properties:

$$e^{\ln|y|} = e^{kt+C_1} = e^{kt}e^{C_1}.$$

Simplifying this result gives us

$$|y| = Ce^{kt},$$

where we replace the constant $e^{C_1}$ with the constant $C$. Finally, because $y(t)$ represents population size, $y(t) \geq 0$ for all $t$, and we eliminate the absolute value to obtain a general solution of the ODE,



$$y(t) = Ce^{kt}.$$

To solve the initial-value problem, we must choose $C$ so that $y(0) = y_0$. Applying the initial condition, we find that $y(0) = Ce^{k\cdot 0} = C$, which indicates that $C = y_0$. Therefore, the solution to the IVP is

$$y(t) = y_0e^{kt}.$$

**Note:** If we had not assumed that $y(t) \geq 0$ for all $t$, then we simplify $|y| = Ce^{kt}$ with $y = \pm Ce^{kt}$. Therefore, by letting $C_2 = \pm C$, we obtain $y(t) = C_2e^{kt}$, which is equivalent to the result obtained above.

The equation $dy/dt = ky$ is a member of a special category of separable equations because it has the form $y' = f(y)$. We find that, although we are able to solve many equations of this form, we can learn much about the behavior of solutions without actually solving the differential equation.

## Equilibrium Solutions of $y' = f(y)$

An **equilibrium solution** of $y' = f(y)$ is a (constant) solution of the ODE that satisfies $f(y) = 0$. Consider the equation $\frac{dy}{dt} = 2y - y^2$ or $\frac{dy}{2y - y^2} = dt$. (This is a nonlinear ODE because it involves the term $y^2$. We learn more about the differences in linear and nonlinear equations in Section 2.3.) Instead of solving by separating variables, let's use a graphical approach to determining properties of solutions. First, we locate the equilibrium solutions of the equation by solving $2y - y^2 = 0$, or $y(2 - y) = 0$. The roots of this equation are $y = 0$ and $y = 2$, so the equilibrium solutions are $y(t) = 0$ and $y(t) = 2$. Notice that each of these functions satisfies $dy/dt = 2y - y^2$ because $dy/dt = 0$ and $y(2 - y) = 0$.

Next, we investigate the behavior of solutions on the intervals $y < 0$, $0 < y < 2$, and $y > 2$ by sketching the **phase line.** After marking the two equilibrium solutions on the vertical line in Figure 2.1, we determine the sign of $dy/dt$ on each of the three intervals listed above. This can be done by substituting a value of $y$ on each interval into $f(y) = 2y - y^2$. For example, $f(-1) = -3$, so $dy/dt < 0$ if $y < 0$. We use an arrow directed downward to indicate that solutions *decrease* on this interval. In a similar manner, we find that $f(1) = 1$ and $f(3) = -3$, so $dy/dt > 0$ if $0 < y < 2$, and $dy/dt < 0$ if $y > 2$. We include arrows on the phase line directed upward and downward, respectively, to indicate this behavior. Based on the orientation, solutions that satisfy $y(0) = a$, where either $0 < a < 2$ or $a > 2$ approaches $y = 2$ as $t \to +\infty$, so we classify $y = 2$ as **asymptotically stable.** Solutions that satisfy $y(0) = a$ where $a < 0$ or $0 < a < 2$ move away from $y = 0$ as $t \to +\infty$. We say that $y = 0$ is **unstable** because there are solutions that begin near $y = 0$ and move away from $y = 0$ as $t \to +\infty$. In Figure 2.2, we graph solutions to this equation (in the $ty$-plane) for $y(0) = a$, $a = -0.5, -0.4, -0.3, -0.2, -0.1, 0.0, 0.25, 0.5, 0.75, 1.0, 1.25, 1.5, 1.75, 2.0, 2.25, 2.5, 2.75, 3.0, 3.25, 3.5, 3.75$, and $4.0$ to illustrate this behavior. Notice that solutions with initial value near $y = 0$ move away from this line while solutions that begin on $0 < y < 2$ or $y > 2$ move toward $y = 2$ as $t \to +\infty$.

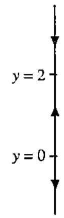



**Figure 2.1** Phase line for $\frac{dy}{dt} = 2y - y^2$

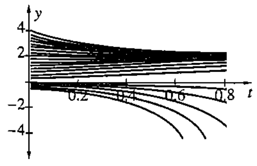



**Figure 2.2** Solutions to $\frac{dy}{dt} = 2y - y^2$

**Note:** Sometimes, we refer to equilibrium solutions as **steady-state solutions,** because over time, the solution approaches a constant value that no longer depends on time $t$. For example, if we interpret the solution $y(t)$ of the differential equation to represent the size of a population, then if $y(0) = a$ where either $0 < a < 2$ or $a > 2$, we expect the population to approach 2 as $t \to +\infty$.

**Example 1**

Show that the equation $dy/dx = -x/y$ is separable and solve this ODE.

**Solution** In this case, we see that the equation is separable by expressing it as

$$x\,dx = -y\,dy.$$

Integration yields

$$\frac{x^2}{2} = -\frac{y^2}{2} + C_1.$$

Multiplying by 2 and simplifying gives us $x^2 + y^2 = 2C_1$. If we let $k = 2C_1$, then the solutions satisfy the equation

$$x^2 + y^2 = k,$$

so that an implicit solution is circles centered at the origin of radius $\sqrt{k}$, as shown in Figure 2.3.

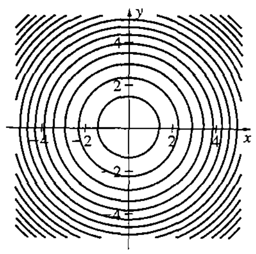



Figure 2.3 Graph of $x^2 + y^2 = k$ for various values of $k$

*Is $x^2 + y^2 = 0$ an implicit solution of the equation? Explain.*

As we have seen, separable differential equations can be stated along with an initial condition. If this is the case, we find a general solution to the differential equation using the separation of variables technique and we then apply the initial condition to determine the unknown constant in the general solution.

**Example 2**

Solve the initial-value problem $dy/dx = e^{2x+y}$, $y(0) = 0$.

**Solution** If we write the equation as $dy/dx = e^{2x}e^{y}$, we see that the variables are separated as

$$e^{-y}dy = e^{2x}dx.$$

Integration gives us $-e^{-y} = e^{2x}/2 + C_1$, and simplifying results in

$$e^{-y} = -\frac{e^{2x}}{2} - C_1.$$



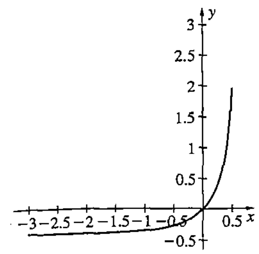



Figure 2.4

Application of the natural logarithm yields

$$-y = \ln\left(-\frac{e^{2x}}{2} - C_1\right) \quad \text{or} \quad y = -\ln\left(-\frac{e^{2x}}{2} - C_1\right).$$

To find the value of $C_1$ so that the solution satisfies the condition $y(0) = 0$, we use the equation $e^{-y} = -e^{2x}/2 - C_1$. Applying $y(0) = 0$ yields $1 = -1/2 - C_1$, so $C_1 = -3/2$ and thus the solution to the initial-value problem is

$$y = -\ln\left(-\frac{e^{2x}}{2} + \frac{3}{2}\right).$$

The domain of the natural logarithm is $(0, +\infty)$, so this solution is valid only for the values of $x$ such that $3 - e^{2x} > 0$, which is equivalent to $e^{2x} < 3$ and has solution $x < \ln(3)/2 \approx 0.5493$. Notice that the graph of the solution shown in Figure 2.4 passes through the point $(0, 0)$. If it had not, we would check for mistakes in our solution procedure.

In some cases, a separable differential equation may be difficult to solve because of the integration that is required.

**Example 3**

Solve $dy/dx = (x^4 + 1)(y^4 + 1)$, $y(0) = 0$.

**Solution** When we separate variables, we obtain

$$\frac{dy}{y^4 + 1} = (x^4 + 1)\,dx.$$

The right side of the equation is easily integrated. However, the left side presents a challenge. Using a computer algebra system, we find that

$$\int \frac{dy}{y^4 + 1} = \frac{1}{2\sqrt{2}}\left[\tan^{-1}\left(\frac{2y - \sqrt{2}}{\sqrt{2}}\right) + \tan^{-1}\left(\frac{2y + \sqrt{2}}{\sqrt{2}}\right)\right] + \frac{1}{4\sqrt{2}}\ln\left|\frac{1 + \sqrt{2}y + y^2}{-1 + \sqrt{2}y - y^2}\right| + C.$$

Therefore, we find that

$$\frac{1}{2\sqrt{2}}\left[\tan^{-1}\left(\frac{2y - \sqrt{2}}{\sqrt{2}}\right) + \tan^{-1}\left(\frac{2y + \sqrt{2}}{\sqrt{2}}\right)\right] + \frac{1}{4\sqrt{2}}\ln\left|\frac{1 + \sqrt{2}y + y^2}{-1 + \sqrt{2}y - y^2}\right| + C = \frac{1}{5}x^5 + x.$$

Substitution of $x = 0$ and $y = 0$ (from the initial condition) into this equation indicates that $C = 0$ because $\tan^{-1}(-\sqrt{2}/\sqrt{2}) = -\pi/4$, $\tan^{-1}(\sqrt{2}/\sqrt{2}) = \pi/4$, and $\ln|-1| = 0$. This means that the solution to the IVP is

$$\frac{1}{2\sqrt{2}}\left[\tan^{-1}\left(\frac{2y-\sqrt{2}}{\sqrt{2}}\right) + \tan^{-1}\left(\frac{2y+\sqrt{2}}{\sqrt{2}}\right)\right] + \frac{1}{4\sqrt{2}}\ln\left|\frac{1+\sqrt{2}y+y^2}{-1+\sqrt{2}y-y^2}\right| = \frac{1}{5}x^5 + x.$$

We can gain insight into the graph of this solution by observing the slope field of the ODE. For example, the slope of the tangent line to the solution passing through (0, 0) at (0, 0) is

$$\frac{dy}{dx} = (0^4 + 1)(0^4 + 1) = 1.$$

The slope at (0, 1) is $dy/dx = (0^4 + 1)(1^4 + 1) = 2$, and that at (0, 2) is $dy/dx = (0^4 + 1)(2^4 + 1) = 17$. As we see, slopes become steeper as the $y$-coordinate increases [or decreases because we obtain the same values for $dy/dx$ at $(0, -1)$ and $(0, -2)$]. Along the $x$-axis, we obtain the same slopes at (1, 0) and (2, 0) as we did at (0, 1) and (0, 2) because $dy/dx$ is symmetric in $x$ and $y$. We generate the slope field with a computer algebra system in Figure 2.5. To understand the solution of the IVP, we begin at the point (0, 0) and follow the tangent line slopes shown in the slope field for $x > 0$ and for $x < 0$. Doing so, we obtain the curve shown in Figure 2.6. We notice that $y$ approaches $+\infty$ as $x$ approaches a value near 0.9, and $y$ approaches $-\infty$ as $x$ approaches a value near $-0.9$. We also note that given any initial condition, the solution to the corresponding IVP becomes unbounded (i.e., $y$ approaches $\pm\infty$) as the $x$-coordinate moves away from 0.

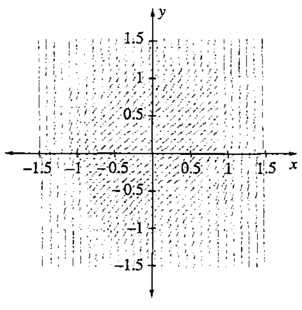



**Figure 2.5** Slope field for $\frac{dy}{dx} = (x^4 + 1)(y^4 + 1)$

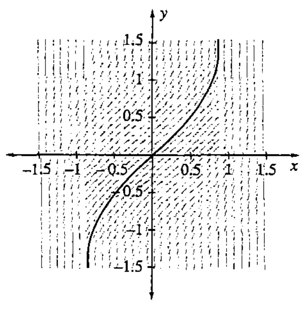



**Figure 2.6** Solution to $\frac{dy}{dx} = (x^4 + 1)(y^4 + 1)$, $y(0) = 0$



## EXERCISES 2.1

In Exercises 1–30, solve each equation. (A computer algebra system or table of integrals may be useful in evaluating some integrals.)

1. $\frac{dy}{dx} = \frac{6x^2}{7y^3}$
2. $\frac{1}{2}x^{-1/2}\,dx + y^2\,dy = 0$
*3. $\frac{dy}{dx} = \frac{3y^7}{x^8}$
4. $\frac{dy}{dx} = \frac{1}{x^2(8 + 9y^2)}$
5. $(6 + 4x^3)\,dx + \left(5 + \frac{9}{y^8}\right)dy = 0$
6. $\left(\frac{6}{x^9} - \frac{6}{x^3} + x^7\right)dx + (9 + y^{-2} - 4y^8)\,dy = 0$
*7. $4\sinh 4y\,dy = 6\cosh 3x\,dx$
8. $\frac{1}{3}x^{-2/3}\,dx = (y^5 - 6y^2 + 8)\,dy$
9. $(x^2 + 2\sqrt{x})\,dx = \frac{-1}{2}y^{-5/2}\,dy$
10. $\frac{3}{x^2}\,dx = \left(\frac{1}{\sqrt{y}} + \sqrt{y}\right)dy$
*11. $3\sin x\,dx - 4\cos y\,dy = 0$
12. $\cos y\,dy = 8\sin 8x\,dx$
13. $(5x^5 - 4\cos x)\,dx + (2\cos 9y + 2\sin 7y)\,dy = 0$
14. $9\cosh 9y\,dy = 4\sinh 4x\,dx$
*15. $(\cosh 6x + 5\sinh 4x)\,dx + 20\sinh y\,dy = 0$
16. $\frac{dy}{dx} = e^{2y}e^{10x}$
17. $(10 + 7e^{-3x})\,dx - (e^y - 8y^3)\,dy = 0$
18. $\sin^2 x\,dx = \cos^2 y\,dy$
*19. $(3\sin x - \sin 3x)\,dx = (\cos 4y - 4\cos y)\,dy$
20. $\frac{dy}{dx} = \frac{\sec^2 x}{\sec y\tan y}$
21. $\left(2 - \frac{5}{y^2}\right)dy + 4\cos^2 x\,dx = 0$
22. $\frac{dy}{dx} = \frac{x^3}{y\sqrt{(1 - y^2)(x^4 + 9)}}$
*23. $\tan y\sec^2 y\,dy + \cos^3 2x\sin 2x\,dx = 0$
24. $\frac{dy}{dx} = \frac{1 + 2e^y}{e^y x\ln x}$
25. $x\sin(x^2)\,dx = \frac{\cos\sqrt{y}}{\sqrt{y}}\,dy$
26. $\frac{x - 2}{x^2 - 4x + 3}\,dx = \left(1 - \frac{1}{y}\right)^2\frac{1}{y^2}\,dy$
*27. $\frac{\cos y}{(1 - \sin y)^2}\,dy = \sin^3 x\cos x\,dx$
28. $\frac{dy}{dx} = \frac{(5 - 2\cos x)^3\sin x\cos^4 y}{\sin y}$
29. $\frac{\sqrt{\ln x}}{x}\,dx = \frac{e^{3/y}}{y^2}\,dy$
30. $\frac{dy}{dx} = \frac{5^{-x}}{y^2}$

In Exercises 31–42, solve the initial-value problem. Graph the solution on an appropriate interval.

31. $\frac{dy}{dx} = x^3$, $y(0) = 4$
32. $\frac{dy}{dx} = \cos x$, $y\left(\frac{\pi}{2}\right) = -1$
*33. $dx = \cos x\,dy$, $y(0) = 2$
34. $\sin^2 y\,dy = dx$, $y(0) = 0$
35. $\frac{dy}{dx} = \frac{\sqrt{x}}{y}$, $y(0) = 2$
36. $\frac{dy}{dx} = \sqrt{\frac{y}{x}}$, $y(1) = 2$
*37. $\frac{dy}{dx} = \frac{e^x}{y + 1}$, $y(0) = -2$
38. $\frac{dy}{dx} = e^{x-y}$, $y(0) = 0$
39. $\frac{dy}{dx} = \frac{y}{\ln y}$, $y(0) = e$
40. $\frac{dy}{dx} = x\sin(x^2)$, $y(\sqrt{\pi}) = 0$
*41. $\frac{dy}{dx} = \frac{1}{1 + x^2}$, $y(0) = 1$
42. $\frac{dy}{dx} = \frac{\sin x}{\cos y + 1}$, $y(0) = 0$

**43.** Find an equation of the curve that passes through the point $(0, 0)$ and has slope $-\dfrac{y-2}{x-2}$ at each point $(x, y)$ on the curve.

**44.** A differential equation of the form $dy/dx = f(ax + by + k)$ is separable if $b = 0$. However, if $b \neq 0$, the substitution $u(x) = ax + by + k$ as the new dependent variable yields a separable equation. Use this transformation to solve **(a)** $dy/dx = (x + y - 4)^2$ and **(b)** $dy/dx = (3y + 1)^4$.

***45.** Let $\omega > 0$. **(a)** Show that the system

$$\begin{cases} \dfrac{dx}{dt} = x(1-r) - \omega y \\ \dfrac{dy}{dt} = y(1-r) + \omega x \end{cases}, \quad r = \sqrt{x^2+y^2}$$

can be rewritten as the system

$$\begin{cases} \dfrac{dr}{dt} = r(1-r) \\ \dfrac{d\theta}{dt} = \omega \end{cases}$$

by changing to polar coordinates $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$.

**(b)** Show that the solution to

$$\begin{cases} \dfrac{dr}{dt} = r(1-r) \\ \dfrac{d\theta}{dt} = \omega \\ r(0) = r_0,\ \theta(0) = \theta_0 \end{cases}$$

is

$$\begin{cases} r(t) = \dfrac{r_0e^t}{(1-r_0) + r_0e^t}, \\ \theta(t) = \omega t + \theta_0 \end{cases}$$

and the solution to

$$\begin{cases} \dfrac{dx}{dt} = x(1-r) - \omega y \\ \dfrac{dy}{dt} = y(1-r) + \omega x \end{cases}, \quad r = \sqrt{x^2+y^2}$$

is

$$\begin{cases} x = r(t)\cos\theta(t) \\ y = r(t)\sin\theta(t) \end{cases}.$$

**(c)** How does the solution change for various initial conditions? (*Hint:* First determine how $\omega$ and $\theta_0$ affect the solution. Then, determine how $r_0$ affects the solution. Try graphing the solution if $\omega = 2$, $\theta_0 = 0$, and $r_0 = \frac{1}{2}$, 1, and $\frac{3}{2}$. What happens if you increase $\omega$? What happens if you increase $\theta_0$?)

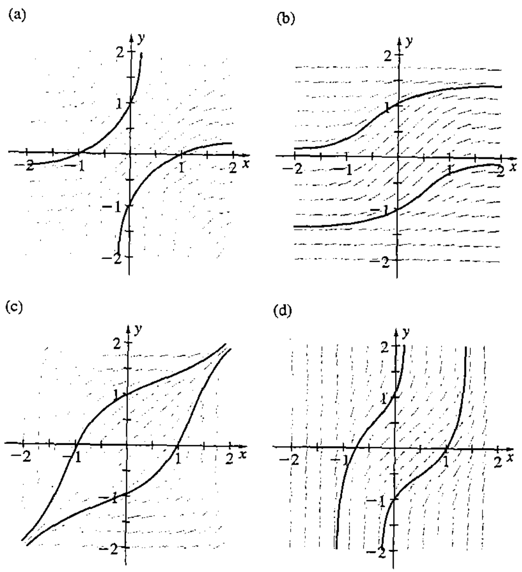
In Exercises 46–49, without actually solving the equations, match each differential equation in Group A with the graph of its direction field and two solutions in Group B.

**Group A**

**46.** $\dfrac{dy}{dx} = \dfrac{y^4+1}{x^4+1}$ **47.** $\dfrac{dy}{dx} = \dfrac{x^4+1}{y^4+1}$

**48.** $\dfrac{dy}{dx} = (x^4+1)(y^4+1)$ **49.** $\dfrac{dy}{dx} = \dfrac{1}{(x^4+1)(y^4+1)}$

**Group B**




In Exercises 50–55, find and classify (as asymptotically stable or unstable) the equilibrium solutions of the first-order equations.



**50.** $\dfrac{dy}{dx} = 3y$ **51.** $\dfrac{dy}{dx} = -y$

**52.** $\dfrac{dy}{dx} = y^2 - y$ ***53.** $\dfrac{dy}{dx} = 16y - 8y^2$

**54.** $\dfrac{dy}{dx} = 12 + 4y - y^2$ **55.** $\dfrac{dy}{dx} = y^2 - 5y + 4$

**56.** **(Lasers)** The variables that characterize a two-level laser are listed in the following table.

| | |
|---|---|
| $\phi$ | Total number of photons in optical resonator |
| $l$ | Length of resonator |
| $N_1$ | Number of atoms per unit volume in Level 1 |
| $N_2$ | Number of atoms per unit volume in Level 2 |
| $V$ | Volume of optical resonator |
| $n$ | Total inversion given by $(N_2 - N_1)V$ |
| $n_i$ | Initial inversion |
| $n_t$ | Total inversion at threshold |
| $t_c$ | Decay time constant for photons in passive resonator |
| $c$ | Phase velocity of light wave |
| $\nu$ | light wave frequency |
| $h$ | magnetic field |

Suppose that a light wave of frequency $\nu$ and intensity $I_\nu$ propagates through an atomic medium with $N_2$ atoms per unit volume in Level 2 and $N_1$ in Level 1. If $N_2 > N_1$, then the medium is amplifying as is the case with lasers. On the other hand, if $N_1 > N_2$, then the medium is absorbing. These two situations are pictured in Figure 2.7.

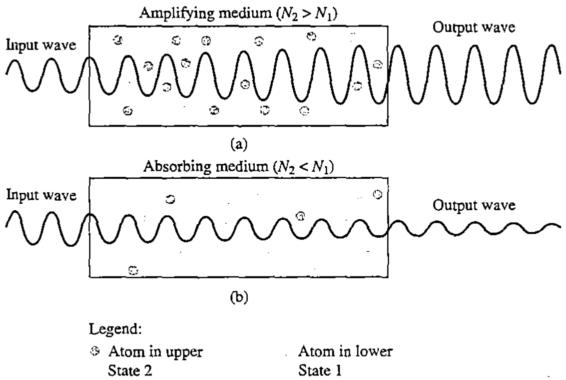



**Figure 2.7**

Optical resonators are used to build up large field intensities with moderate power inputs. A measure of this property is the quality factor $Q$, where

$$Q = \omega \times \frac{\text{field energy stored by resonator}}{\text{power dissipated by resonator}}.$$

A technique called "$Q$-switching" is used to generate short and intense bursts of oscillation from lasers. This is done by lowering the quality factor $Q$ during the pumping so that the inversion $N_2 - N_1$ builds up to a high value without oscillation. When the inversion reaches its peak, the factor $Q$ is suddenly restored to its ordinary value. At this point, the laser medium is well above its threshold, which causes a large buildup of the oscillation and a simultaneous exhaustion of the inversion by simulated transitions from Level 2 to Level 1. This process converts most of the energy that was stored by atoms pumped into Level 2 into photons, which are now in the optical resonator. These photons bounce back and forth between the reflectors with a fraction $1 - R$ escaping from the resonator with each bounce. This causes a decay of the pulse with a photon lifetime

$$t_c \approx \frac{nl}{c(1-R)}.$$

The quantities of $\phi$ and $n$ are related by the differential equation

$$\frac{d\phi}{dn} = \frac{n_t}{2n} - \frac{1}{2}.$$

**(a)** Solve this equation to determine the total number of photons in the optical resonator $\phi$ as a function of $n$ (where $\phi$ depends on an arbitrary constant).

**(b)** Assuming that the initial values of $\phi$ and $n$ are $\phi_0$ and $n_0$ at $t = 0$, show that $\phi$ can be simplified to obtain

$$\phi = \frac{n_t}{2} \ln \frac{n}{n_0} + \frac{1}{2}(n_0 + n) + \phi_0$$

**(c)** If the instantaneous power output of the laser is given by

$$P = \frac{\phi h \nu}{t_c},$$

show that the maximum power output occurs when $n = n_t$. (*Hint:* Solve $\partial P/\partial n = 0$.)

***57. (Destruction of Microorganisms)** Microorganisms can be removed from fluids by mechanical methods such as filtration, centrifugation, or flotation. However, they can also be destroyed by heat, chemical agents, or electromagnetic waves. The fermentation industry is interested in improving this process of sterilizing media. An important component in the design of a sterilizer is the kinetics of the death of microorganisms.

The destruction of microorganisms by heat indicates loss of viability as opposed to physical destruction. This destruction follows the rate of reaction

$$\frac{dN}{dt} = -kN,$$

where $k$ is the reaction constant ($\text{min}^{-1}$) and is a function of temperature, $N$ is the number of viable organisms, and $t$ is time.*

Microbiologists use the term **decimal reduction time** $D$ to indicate the time of exposure to heat during which the original number of viable microbes is reduced by one-tenth. If $N(0) = N_0$, find $N(t)$. Use the fact that $N(D) = \frac{1}{10}N_0$ to find $D$ in terms of $k$.

**58. (Pollution)** Under normal atmospheric conditions, the density of soot particles $N(t)$ satisfies the differential equation

$$\frac{dN}{dt} = -k_c N^2 + k_d N,$$

where $k_c$, called the **coagulation constant,** is a constant that relates how well particles stick together; and $k_d$, called the **dissociation constant,** is a constant that relates to how well particles fall apart. Both of these constants depend on temperature, pressure, particle size, and other external forces.†

**(a)** Rewrite the equation $dN/dt = -k_c N^2 + k_d N$ in the form

$$\frac{1}{-k_c N^2 + k_d N}\, dN = dt$$

and use partial fractions to show that

$$N(t) = \frac{e^{k_d t}}{\dfrac{k_c}{k_d} e^{k_d t} + C},$$

where $C$ represents an arbitrary constant.

**(b)** Find $C$ so that $N(t)$ satisfies the condition $N(t_0) = N_0$.

The following table lists typical values of $k_c$ and $k_d$.

| $k_c$ | $k_d$ |
|---|---|
| 163 | 5 |
| 125 | 26 |
| 95 | 57 |
| 49 | 85 |
| 300 | 26 |

* S. Aiba, A. E. Humphrey, N. F. Millis, *Biochemical Engineering,* Second Edition, Academic Press (1973), pp. 240–242.

† Chr. Feldermann, H. Jander, and H. Gg. Wagner, "Soot Particle Coagulation of Premixed Ethylene/Air Flames at 10 bar," *International Journal of Research in Physical Chemistry and Chemical Physics,* Volume 186, Part II (1994), pp. 127–140.



**(c)** For each pair of values in the previous table, sketch the graph of $N(t)$ if $N(0) = N_0$ for $N_0 = 0.01, 0.05, 0.1, 0.5, 0.75, 1, 1.5$, and 2. Regardless of the initial condition $N(0) = N_0$, what do you notice in each case? Do pollution levels seem to be more sensitive to $k_c$ or $k_d$? Does your result make sense? Why?

**(d)** Show that if $k_d > 0$, $\lim_{t\to\infty} N(t) = k_d/k_c$. Why is the assumption that $k_d > 0$ reasonable?

**(e)** For each pair in the table, calculate $\lim_{t\to\infty} N(t) = k_d/k_c$. Which situation results in the highest pollution levels? How could the situation be changed?

**59. (Kinetic Reactions)** Suppose that substances $A$ and $B$, called the reactants, combine to form substance $C$. The reaction depends on the concentration of the reactants because the rate of the reaction is proportional to the product of their concentrations. When substances $A$ and $B$ are placed in the test tube and begin to combine to form $C$, the amounts of $A$ and $B$ diminish and the reaction stops when either $A$ or $B$ is no longer present. Let $y(t)$ represent the amount of substance $C$ in the test tube at any time. If the initial amounts of substances $A$ and $B$ are $A_0$ and $B_0$ and if the volume in the test tube is $V$, then the concentration of $A$ at any time is $\frac{1}{V}(A_0 - y)$ and that of $B$ is $\frac{1}{V}(B_0 - y)$. Therefore, we determine $y(t)$ by solving

$$\frac{dy}{dt} = K\left[\frac{1}{V}(A_0 - y)\right]\left[\frac{1}{V}(B_0 - y)\right] = \frac{K}{V^2}(A_0 - y)(B_0 - y).$$

**(a)** Identify the equilibrium solutions of the equation. Assuming that $B_0 > A_0$, classify each as stable or unstable. Generate the direction field for the equation with $A_0 = 1$, $B_0 = 2$, $K = 1$, and $V = 1$ to verify your results. What happens when either $y(t) = A_0$ or $y(t) = B_0$?

**(b)** Assume that $y(0) = y_0$ and solve the initial value problem to determine $y(t)$. Calculate $\lim_{t\to\infty} y(t)$. How does this limiting value relate to the physical situation described by the problem?

**(c)** If $B_0$ is much larger than $A_0$, then we may approximate the solution to this problem by solving the simplified equation

$$\frac{dy}{dt} = K\left[\frac{1}{V}(A_0 - y)\right]B_0 = \frac{KB_0}{V^2}(A_0 - y),\ y(0) = y_0.$$

(In other words, the reaction stops when substance $A$ is exhausted.) Solve this initial value problem. Calculate $\lim_{t\to\infty} y(t)$.

**(d)** Suppose that $K = 0.2$, $V = 1$, $A_0 = 1$, $B_0 = 10$, and $y_0 = 0.1$. Compare the solution found in **(b)** to that found in **(c)**. Do the same for the parameter values $K = 0.2$, $V = 1$, $A_0 = 1$, $B_0 = 50$, and $y_0 = 0.1$. How does the value of $B_0$ affect the accuracy of the approximation in **(c)**?

In many cases, computer algebra systems can be used to perform the integration and algebraic simplification associated with a separable differential equation and can sometimes solve them. Similarly, initial-value problems can often be solved quickly with a computer algebra system.

In Exercises 60–66, find a general solution of the given equation and graph various solutions on the indicated interval.

**60.** $\dfrac{dy}{dx} = y\cos x$; $[0, 4\pi]$

**61.** $\dfrac{dy}{dx} = \sqrt{1 - y^2}\sin x$; $[0, 4\pi]$

**62.** $\cos x\, dx = \dfrac{1 + y^2}{y^2}\, dy$; $[0, 10]$

***63.** $e^y \cos y\, dy = \dfrac{x^2}{\sqrt{9 - x^2}}\, dx$; $[-2, 2]$

**64.** Solve the initial-value problem

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{x^2}{\sqrt{9 - x^2}\, e^y \cos y} \\ y(0) = 0 \end{cases}$$

and graph the solution on an appropriate interval.

**65.** How does the graph of the solution to the initial-value problem

$$\begin{cases} y' = \dfrac{cy}{x^2} \\ y(1) = 1 \end{cases}$$

change as $c$ changes from $-2$ to 2?

**66.** **(a)** Assume that $y(t) > 0$ and $\int_1^t f(t)\,dt$ exists for $t \geq 1$. Show that the solution to the initial-value problem

$$\begin{cases} \dfrac{dy}{dt} = yf(t) \\ y(1) = 2 \end{cases}$$

is

$$y(t) = 2e^{\int_1^t f(u)\,du}.$$

**(b)** Find three functions $f(t)$ so that (i) $y(t)$ is periodic; (ii) $\lim_{t\to\infty} y(t) = 0$; and (iii) $\lim_{t\to\infty} y(t) = \infty$. Confirm your results by graphing each solution.

**(c)** If $c > 0$ is given, is it possible to choose $f(t)$ so that $\lim_{t\to\infty} f(t) = c$? Explain.

## 2.2 First-Order Linear Equations

In addition to the assumptions made in the fish population model at the beginning of Section 2.1, suppose that we harvest the population at the rate $h$. In other words, we remove fish from the population by fishing at the rate $h$. By modifying the Malthus model developed in Section 2.1, we find that the rate at which the population size changes satisfies

$$\frac{dy}{dt} = ky - h.$$

Therefore, we determine the population size by solving

$$\frac{dy}{dt} = ky - h,\ y(0) = y_0,$$

where $y_0$ represents the initial population size. In addition to being separable, this differential equation is classified as a **first-order linear differential equation.**

**Definition 2.2 First-Order Linear Differential Equation**

A **first-order linear differential equation** can be written in the form

$$\frac{dy}{dt} + p(t)y = q(t).$$

Writing $dy/dt = ky - h$ in this form, we have

$$\frac{dy}{dt} - ky = -h,$$

so $p(t) = -k$ and $q(t) = -h$ are constant functions.

We solve a first-order linear ODE by using the integrating factor

$$\mu(t) = e^{\int p(t)dt}.$$



Multiplying the ODE by $\mu(t)$, we obtain

$$e^{\int p(t)dt}\frac{dy}{dt} + p(t)e^{\int p(t)dt}y = q(t)e^{\int p(t)dt}.$$

However, notice that by the product rule,

$$\frac{d}{dt}[e^{\int p(t)dt}y] = e^{\int p(t)dt}\frac{dy}{dt} + p(t)e^{\int p(t)dt}y.$$

This expression is the left side of the previous equation, so we can rewrite the equation as

$$\frac{d}{dt}[e^{\int p(t)dt}y] = q(t)e^{\int p(t)dt}.$$

We solve this equation by integrating each side. We illustrate the solution method in the following example.

**Example 1**

Consider the IVP $dy/dt = y - 1/2$, $y(0) = a$. In this case, the fish population grows at the rate of 1 (ton per year) and fish are harvested (or fished) at the rate of 1/2 (ton per year). **(a)** Determine the behavior of solutions by drawing the phase line. **(b)** Find the solution to the IVP. **(c)** Investigate the behavior of solutions of the IVP using $y(0) = 1/4$, $y(0) = 1$, and $y(0) = 1/2$.

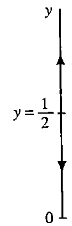



**Figure 2.8** Phase line for $\dfrac{dy}{dt} - y = -\dfrac{1}{2}$

***Solution*** (a) By solving $y - 1/2 = 0$, we find that the equilibrium solution is $y = 1/2$. We draw the phase line in Figure 2.8. We include only nonnegative values on the phase line because we are interested in $y(t) \geq 0$ because $y(t)$ represents population size. Note that if $y > 1/2$, then $dy/dt > 0$ while if $y < 1/2$, then $dy/dt < 0$. Therefore, on the interval $y > 1/2$, the arrow is directed upward and on $0 \leq y < 1/2$, the arrow is directed downward. The phase line indicates that solutions to the IVP $dy/dt = y - 1/2$, $y(0) = a$ with $a > 1/2$ increase as $t \to +\infty$. Conversely, solutions to this IVP with $0 < a < 1/2$ decrease as $t \to +\infty$.

(b) We solve $dy/dt = y - 1/2$ as a first-order linear equation by rewriting the equation as $dy/dt - y = -1/2$. The integrating factor is

$$\mu(t) = e^{\int p(t)dt} = e^{\int(-1)dt} = e^{-t},$$

so the equation can be written as $e^{-t}\dfrac{dy}{dt} - e^{-t}y = -\dfrac{1}{2}e^{-t}$ or

$$\frac{d}{dt}[e^{-t}y] = -\frac{1}{2}e^{-t}.$$

Integration yields $e^{-t}y = \frac{1}{2}e^{-t} + C$, so

$$y(t) = \frac{1}{2} + Ce^t.$$

**(c)** If $y(0) = 1/4$, then $y(0) = 1/2 + C$ implies that $C = -1/4$. Therefore,

$$y(t) = \frac{1}{2} - \frac{1}{4}e^t.$$

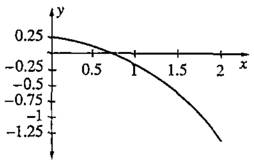



**Figure 2.9** Solution to $\frac{dy}{dt} = y - \frac{1}{2}, y(0) = \frac{1}{4}$

We graph $y(t)$ in Figure 2.9. Notice that the population becomes extinct when $\frac{1}{2} - \frac{1}{4}e^t = 0$, or $t = \ln 2 \approx 0.693147$ year. [We ignore the portion of the solution curve where $y(t) < 0$.] We should note that any solution to this IVP with $y(0) < 1/2$ becomes extinct at some time. If $y(0) = 1$, $1/2 + C = 1$ indicates that $C = 1/2$. Therefore, $y(t) = \frac{1}{2} + \frac{1}{2}e^t$, so $y(t) > 0$ for all $t$, as we see in Figure 2.10. In fact, the population grows without bound. If $y(0) = 1/2$, then $y(t) = 1/2$ as expected because $y(t) = 1/2$ is an equilibrium solution. If the population is initially 1/2 ton, then it remains at that level. (See Figure 2.11.)

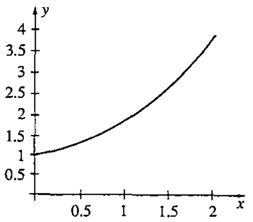



**Figure 2.10** Solution to $\frac{dy}{dt} = y - \frac{1}{2}, y(0) = 1$

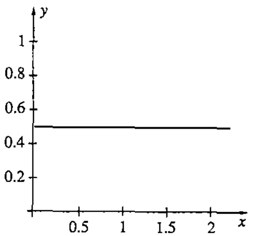



**Figure 2.11** Solution to $\frac{dy}{dt} = y - \frac{1}{2}, y(0) = \frac{1}{2}$

**Example 2**

Solve the initial-value problem $\frac{dy}{dx} + 5x^4y = x^4$, $y(0) = -7$.

***Solution*** We begin by solving the linear equation $\frac{dy}{dx} + 5x^4y = x^4$ using the integrating factor $e^{\int 5x^4dx} = e^{x^5}$. Then, the equation can be written as

$$\frac{d}{dx}(e^{x^5}y) = x^4e^{x^5}$$

so that integration of both sides of the equation yields

$$e^{x^5}y = \frac{1}{5}e^{x^5} + C.$$

A general solution is

$$y = \frac{1}{5} + Ce^{-x^5}.$$

We find the unknown constant $C$ by substituting the initial condition $y(0) = -7$. This gives $-7 = 1/5 + C$, so $C = -36/5$. Therefore, the solution to the initial-value problem is

$$y = \frac{1}{5} - \frac{36}{5}e^{-x^5}.$$



*Figure 2.12 shows the graph of $y = \frac{1}{5} + Ce^{-x^5}$ for various values of C. Identify the graph of the solution that satisfies the condition $y(0) = -7$.*

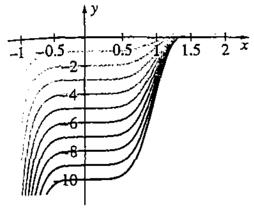



**Figure 2.12**

In some cases, we must rewrite an equation to place it in the form of a linear first-order differential equation.

**Example 3**

Solve $\frac{dt}{dr} = \frac{1}{\sin t - r\tan t}$, $0 < t < \pi/2$.

***Solution*** Notice that if $t$ is the *dependent* variable, the equation is nonlinear (in $t$). (Why?) However, solving the equation for $dr/dt$ yields

$$\frac{dr}{dt} = \sin t - r\tan t \quad \text{or} \quad \frac{dr}{dt} + (\tan t)r = \sin t,$$

which is a *linear* equation in the dependent variable $r$. With $p(t) = \tan t$, the integrating factor is

$$\mu(t) = e^{\int \tan t\,dt} = e^{-\ln|\cos t|} = \frac{1}{\cos t},\ 0 < t < \pi/2.$$

Then,

$$\begin{aligned}
\frac{d}{dt}[\mu(t)r] &= \mu(t)\sin t \\
\frac{d}{dt}\left[\frac{1}{\cos t}r\right] &= \frac{\sin t}{\cos t} \\
\frac{1}{\cos t}r &= -\ln|\cos t| + C \\
r &= -(\cos t)\ln(\cos t) + C\cos t.
\end{aligned}$$

*Solve the initial-value problem*

$$\begin{cases} \dfrac{dt}{dr} = \dfrac{1}{\sin t - r\tan t} \\ t(1) = \pi/4 \end{cases}.$$

*Graph the solution on an appropriate interval.*

**Example 4**

If a drug is introduced into the bloodstream in dosages of $D(t)$ and is removed at a rate proportional to the concentration, the concentration $C(t)$ at time $t$ is given by

$$\begin{cases} \dfrac{dC}{dt} = D(t) - kC \\ C(0) = 0 \end{cases},$$

where $k > 0$ is the constant of proportionality.*

**(a)** Solve this initial-value problem.

**(b)** Suppose that over a 24-hour period, a drug is introduced into the bloodstream at a rate of $24/t_0$ for exactly $t_0$ hours and then stopped so that

$$D_{t_0}(t) = \begin{cases} 24/t_0, & \text{if } 0 \le t \le t_0 \\ 0, & \text{if } t > t_0 \end{cases}.$$

What is the total dosage and average dosage over a 24-hour period?

**(c)** Calculate and then graph $C(t)$ on the interval $[0, 30]$ if $k = 0.05, 0.10, 0.15$, and $0.20$ for $t_0 = 4, 8, 12, 16$, and $20$. How does increasing $t_0$ affect the concentration of the drug in the bloodstream? How does increasing $k$ affect it?

***Solution*** (a) After rewriting the equation as $\dfrac{dC}{dt} + kC = D(t)$, we find the integrating factor to be $\mu(t) = e^{\int k\,dt} = e^{kt}$. Therefore, we obtain $\dfrac{d}{dt}\left[e^{kt}C\right] = e^{kt}D(t)$ so that integration yields

$$e^{kt}C = \int_0^t e^{ks}D(s)\,ds \quad \text{or} \quad C(t) = e^{-kt}\int_0^t e^{ks}D(s)\,ds.$$

(b) In each case, the total dosage over a 24-hour period is $\int_0^{24} D_{t_0}(t)\,dt = \int_0^{t_0} \dfrac{24}{t_0}\,dt = 24$; the average dosage is

$$\frac{1}{24-0}\int_0^{24} D_{t_0}(t)\,dt = \frac{1}{24}\int_0^{t_0} \frac{24}{t_0}\,dt = 1.$$

(c) To compute $C(t) = e^{-kt}\int_0^t e^{ks}D_{t_0}(s)\,ds$, we must keep in mind that $D_{t_0}(t)$ is a piecewise defined function:

$$C(t) = e^{-kt}\int_0^t e^{ks}D_{t_0}(s)\,ds = \begin{cases} e^{-kt}\displaystyle\int_0^t e^{ks}\frac{24}{t_0}\,ds, & \text{if } 0 \le t \le t_0 \\ e^{-kt}\displaystyle\int_0^{t_0} e^{ks}\frac{24}{t_0}\,ds, & \text{if } t > t_0 \end{cases}$$

$$= \begin{cases} \dfrac{24}{kt_0}\left(1 - e^{-kt}\right), & \text{if } 0 \le t \le t_0 \\ \dfrac{24}{kt_0}\left(e^{-k(t-t_0)} - e^{-kt}\right) & \text{if } t > t_0 \end{cases}.$$

* J. D. Murray, *Mathematical Biology*, Springer-Verlag (1990), pp. 645–649.



We graph $C(t)$ on the interval $[0, 30]$ if $k = 0.05, 0.10, 0.15$, and $0.20$ for $t_0 = 4, 8, 12, 16$, and $20$ in Figure 2.13. From the graphs, we see that as $t_0$ is increased, the maximum concentration level decreases and occurs at later times, while increasing $k$ increases the rate at which the drug is removed from the bloodstream.

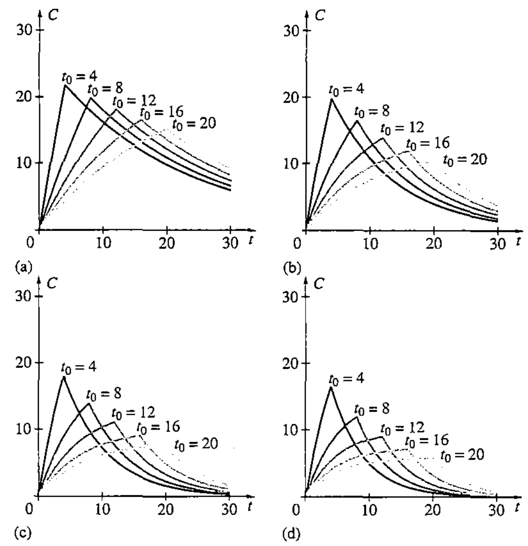



**Figure 2.13** (a) $k = 0.05$ (b) $k = 0.10$ (c) $k = 0.15$ (d) $k = 0.20$

## EXERCISES 2.2

In Exercises 1–24, solve each equation.

1. $\dfrac{dy}{dx} + \dfrac{1}{x}y = x$
2. $\dfrac{dy}{dx} + \dfrac{1}{x}y = \sin x$
*3. $\dfrac{dy}{dx} + \dfrac{1}{x}y = e^x$
4. $\dfrac{dy}{dx} + \dfrac{1}{x}y = xe^{-x}$
5. $\dfrac{dy}{dx} - \dfrac{2x}{1+x^2}y = 2x$
6. $\dfrac{dy}{dx} - \dfrac{2x}{1+x^2}y = x^2$
*7. $dy = \left(2x + \dfrac{xy}{x^2-1}\right)dx$
8. $\dfrac{dy}{dx} + y\cot x = \cos x$
9. $\dfrac{dy}{dx} - y\tan x = e^{-2x}$
10. $dy = \left(2x + \dfrac{2x-6}{x^2-6x+10}y\right)dx$
*11. $\dfrac{dy}{dx} - \dfrac{3x}{x^2-4}y = x^2$
12. $\dfrac{dy}{dx} - \dfrac{10x^2-1}{x(10x^2+7x+1)}y = \dfrac{1}{10x^2+7x+1}$
13. $\dfrac{dy}{dx} - \dfrac{4x}{4x^2-9}y = x^3$

14. $\dfrac{dy}{dx} - \dfrac{16x}{16x^2+25}y = x$

*15. $\dfrac{dy}{dx} - \dfrac{9x}{9x^2+49}y = x$

16. $\dfrac{dy}{dx} + (2\cot x)y = \cos x$

17. $\dfrac{dy}{dx} + xy = x^3$

18. $\dfrac{dy}{dx} - xy = x$

*19. $\dfrac{dy}{dx} = \dfrac{1}{y^2+x}$

20. $\dfrac{dx}{dy} - x = y$

21. $y\,dx - (x+3y^2)\,dy = 0$

22. $\dfrac{dx}{dy} = \dfrac{3xy^2}{1-y^3}$

*23. $\dfrac{dp}{dt} = t^3 + \dfrac{p}{t}$

24. $\dfrac{dv}{ds} + v = e^{-t}$

In Exercises 25–34, solve the initial-value problem. Graph the solution on an appropriate interval.

25. $\dfrac{dy}{dx} - y = 4e^x,\ y(0) = 4$

26. $\dfrac{dy}{dx} + y = e^{-x},\ y(0) = -1$

*27. $\dfrac{dy}{dx} + 3x^2y = e^{-x^3},\ y(0) = 2$

28. $\dfrac{dy}{dx} + 2xy = 2x,\ y(0) = -1$

29. $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{\cos x}{x},\ y\left(\dfrac{\pi}{2}\right) = \dfrac{4}{\pi},\ x > 0$

30. $\dfrac{dy}{dx} + \dfrac{y}{x} = 2e^x,\ y(1) = -1,\ x > 0$

*31. $\dfrac{dy}{dx} + \dfrac{e^x}{e^x+1}y = \dfrac{x}{e^x+1},\ y(0) = 1$

32. $\dfrac{dy}{dx} + \dfrac{2x}{x^2+4}y = \dfrac{2x}{x^2+4},\ y(0) = -4$

33. $\dfrac{dx}{dt} = x + t + 1,\ x(0) = 2$

34. $\dfrac{d\theta}{dt} = e^{2t} + 2\theta,\ \theta(0) = 0$

35. The equation $dy/dx + p(x)y = 0$ is called a **homogeneous first-order linear equation** because $q(x) = 0$. **(a)** Show that $y = 0$ (the trivial solution) is a solution. **(b)** Show that if $y = y_1(x)$ is a solution and $k$ is a constant, $y = ky_1(x)$ is also a solution. **(c)** Show that if $y = y_1(x)$ and $y = y_2(x)$ are solutions, $y = y_1(x) + y_2(x)$ is also a solution.

36. If $y = y_1(x)$ satisfies the homogeneous equation $dy/dx + p(x)y = 0$ and $y = y_2(x)$ satisfies the nonhomogeneous equation $dy/dx + p(x)y = r(x)$, show that $y = y_1(x) + y_2(x)$ satisfies the nonhomogeneous equation $dy/dx + p(x)y = r(x)$.

*37. **(a)** Show that if $y = y_1(x)$ is a solution of $dy/dx + p(x)y = r(x)$ and $y = y_2(x)$ is a solution of $dy/dx + p(x)y = q(x)$, then $y = y_1(x) + y_2(x)$ is a solution of $dy/dx + p(x)y = r(x) + q(x)$.

**(b)** Use the result obtained in **(a)** to solve $\dfrac{dy}{dx} + 2y = e^{-x} + \cos x$.

Without actually solving the problems in Exercises 38–40, match each initial-value problem in Group A with the graph of its solution in Group B.

**Group A**

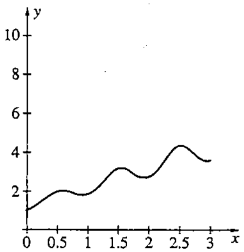
38. $y' - xy = 1,\ y(0) = 1$

39. $y' - \sin(2\pi x)y = 1,\ y(0) = 1$

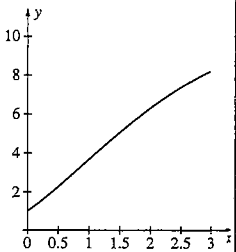
40. $y' - \dfrac{1}{x^2+1}y = 1,\ y(0) = 1$

**Group B**

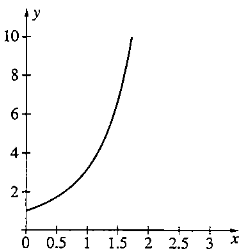
(a)




(b)




(c)






41. Observe Figure 2.13. Describe how the value of $k$ affects the concentration of the drug in the system. Do larger or smaller values cause the drug to remain in the system longer? How would the chemist working for the pharmaceutical company design the drug?

42. $\dfrac{dy}{dx} + y = q(x)$, where $q(x) = \begin{cases} 4, & 0 \le x \le 2 \\ 0, & x > 2 \end{cases},\ y(0) = 0.$

43. $\dfrac{dy}{dx} + y = q(x)$, where $q(x) = \begin{cases} x, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases},\ y(0) = 1.$

44. $\dfrac{dy}{dx} + p(x)y = 0$, where $p(x) = \begin{cases} 1, & 0 \le x \le 2 \\ -1, & x > 2 \end{cases},\ y(0) = 2.$

*45. $\dfrac{dy}{dx} + p(x)y = 0$, where $p(x) = \begin{cases} 2, & 0 \le x \le 1 \\ 4, & x > 1 \end{cases},\ y(0) = 1.$

46. **(Method of Undetermined Coefficients)** Consider the first-order nonhomogeneous equation

$$y' + y = \cos x.$$

We can solve the problem in two parts. First, we can solve the corresponding homogeneous equation $y' + y = 0$. **(a)** Show that $y_h(x) = Ce^{-x}$ is a general solution of this equation. **(b)** Next, we attempt to find a solution of the nonhomogeneous equations by guessing the form of the solution. Suppose that this **particular solution** is $y_p(x) = A\cos + B\sin x$. Substitute $y_p$ into $y' + y = \cos x$ to show that the undetermined coefficients are $A = -1/2$ and $B = 1/2$. Then, a general solution of the nonhomogeneous equation is $y(x) = y_h(x) + y_p(x) = Ce^{-x} - \dfrac{1}{2}\cos x + \dfrac{1}{2}\sin x$. (What integral would need to be evaluated if we used an integrating factor to solve the ODE?)

47. Solve $y' - y = \sin 2x$.

48. Solve $y' + y = 5e^{2x}$ by assuming that $y_p(x) = Ae^{2x}$.

*49. Solve $y' + y = e^{-x}$ by assuming that $y_p(x) = Axe^{-x}$. Why don't we choose $y_p(x) = Ae^{-x}$?

50. Solve $y' + y = 2 - e^{2x}$ by assuming that $y_p(x) = A + Be^{2x}$.

51. Solve $y' - 5y = x$ by assuming that $y_p(x) = Ax + B$.

52. Solve $y' + 3y = 27x^2 + 9$ by assuming that $y_p(x) = Ax^2 + Bx + C$.

*53. Solve $y' - \dfrac{1}{2}y = 5\cos x + 2e^x$ by assuming that $y_p(x) = A\cos x + B\sin x + Ce^x$.

In Exercises 54–59, solve each of the following equations using the Method of Undetermined Coefficients.

54. $y' + 4y = 8\cos 4x$

*55. $y' + 10y = 2e^x$

56. $y' - 3y = 27x^2$

57. $y' - y = 2e^x$

58. $y' + y = 4 + 3e^x$

*59. $y' + y = 2\cos x + x$

60. **(First-Order Linear with Periodic Forcing Function)** Consider the differential equation

$$\frac{dy}{dx} + cy = f(x)$$

where $f(x)$ is a periodic function and $c$ is a constant. The goal of this exercise is to determine if equations of this form have a periodic solution. **(a)** Solve the IVP $\dfrac{dy}{dx} + \dfrac{1}{2}y = \sin x,\ y(0) = a$. For what value of $a$ does the IVP have a periodic solution? Graph the slope field for this ODE. Describe the behavior of the other solutions. Do they approach the periodic solution found in **(a)** as $t \to +\infty$ or as $t \to -\infty$? **(b)** Solve the IVP $\dfrac{dy}{dx} - \dfrac{1}{2}y = \sin x,\ y(0) = a$. For what value of $a$ does the IVP have a periodic solution? Graph the slope field for this ODE. Describe the behavior of the other solutions. Do they approach the periodic solution found in **(a)** as $x \to +\infty$ or as $x \to -\infty$? **(c)** Based on your findings in **(a)** and **(b)**, does the ODE $\dfrac{dy}{dx} + cy = f(x)$, where $f(x)$ is a periodic function and $c$ is a constant, have a periodic solution? How does the value of $c$ affect the other solutions?

61. Suppose that a drug is added to the body at a rate $r(t)$, and let $y(t)$ represent the concentration of the drug in the bloodstream at time $t$ hours. In addition, suppose that the drug is used by the body at the rate $ky$, where $k$ is a positive constant. Then, the net rate of change in $y(t)$ is given by the equation $dy/dt = r(t) - ky$. If at $t = 0$, there is no drug in the body, we determine $y(t)$ by solving the initial-value problem

$$\frac{dy}{dt} = r(t) - ky,\ y(0) = 0.$$

**(a)** Suppose that $r(t) = r$, where $r$ is a positive constant. In this case, the drug is added at a constant rate. Sketch the phase line for $dy/dt = r - ky$.

Solve the IVP. Determine $\lim_{t\to\infty} y(t)$. How does this limit correspond to the phase line?

**(b)** Suppose that $r(t) = 1 + \sin t$ and $k = 1$. In this case, the drug is added at a periodic rate. Solve the IVP. Determine $\lim_{t\to\infty} y(t)$ (if it exists). Describe what happens to the drug concentration over time.

**(c)** Suppose that $r(t) = e^{-t}$ and $k = 1$. In this case, the rate at which the drug is added decreases over time. Solve the IVP. Determine $\lim_{t\to\infty} y(t)$ (if it exists). Describe what happens to the drug concentration over time.

One advantage of using technology is that often when a large number of problems are solved, their solutions are compared and conjectures about general patterns can be discovered and tested. Many computer algebra systems are capable of solving a variety of linear equations, particularly those that are frequently encountered in an elementary differential equations course.

**62.** Find a general solution of the equation $x\dfrac{dy}{dx} + y = x\cos x$. Graph various solutions on the rectangle $[0, 2\pi] \times [-10, 10]$.

**63.** Compare the solutions of $\dfrac{dy}{dx} + y = f(x)$ subject to $y(0) = 0$, where $f(x) = x$, $\sin x$, $\cos x$, and $e^x$.

**64.** Compare the solutions of $\dfrac{dy}{dx} + ky = x$, $y(0) = 1$ for $k = -2, -1, 0, 1, 2$.

***65.** Compare the solutions of $\dfrac{dy}{dx} + y = x$, $y(0) = k$ for $k = -2, -1, 0, 1, 2$.

**66.** **(a)** Graph the direction field for the equation $y' = y - x$. **(b)** Solve and graph the solution to the initial-value problems (i) $\begin{cases} y' = y - x \\ y(0) = 1 \end{cases}$; (ii) $\begin{cases} y' = y - x \\ y(0) = 1.1 \end{cases}$; and (iii) $\begin{cases} y' = y - x \\ y(0) = 0.9 \end{cases}$. **(c)** Comment on this statement: if we slightly change the initial conditions in a linear initial-value problem, the solution also slightly changes.

## 2.3 Substitution Methods and Special Equations

Substitutions are used often in integral calculus to transform integrals into forms that can be computed easily. Similarly, with some differential equations, we can perform substitutions that transform a given differential equation into an equation that is easier to solve. For example, we have considered population problems modeled with the first-order linear IVPs:

$$\frac{dy}{dt} = ky,\ y(0) = y_0 \qquad \text{and} \qquad \frac{dy}{dt} = ky - h,\ y(0) = y_0.$$

However, we can consider a more complicated model that involves a nonlinear equation with

$$\frac{dy}{dt} = ky - ay^2,\ y(0) = y_0.$$

The equation in this model, known as the **logistic equation,** is almost linear because it can be written as $dy/dt - ky = -ay^2$. (The $y^2$ term on the right side causes the equation to be nonlinear.) It can also be expressed as



$$\frac{dy}{dt} = (k - ay)y$$

to show that it differs from $dy/dt = ky$ in that the rate of growth (the coefficient of $y$), $(k - ay)$, is not constant. Instead, this rate depends on $y$. Although this is a separable equation, we can solve the ODE by another approach through the use of the substitution

$$w = y^{1-2} = y^{-1}.$$

We would like to transform the logistic equation from an ODE that depends on $y$ and $t$ into one that depends on $w$ and $t$, so we find $dw/dt$ with the chain rule:

$$\frac{dw}{dt} = \frac{dw}{dy}\frac{dy}{dt} = -y^{-2}\frac{dy}{dt}.$$

Therefore,

$$\frac{dy}{dt} = -y^2\frac{dw}{dt}.$$

Substitution of this expression into $dy/dt - ky = -ay^2$ gives us

$$-y^2\frac{dw}{dt} - ky = -ay^2 \qquad \text{or} \qquad \frac{dw}{dt} + ky^{-1} = a.$$

Finally, with the substitution $w = y^{-1}$, we obtain the first-order linear ODE

$$\frac{dw}{dt} + kw = a,$$

which we solve with the integrating factor $\mu(t) = e^{\int k\,dt} = e^{kt}$. Therefore, $\dfrac{d}{dt}[e^{kt}w] = ae^{kt}$, so

$$e^{kt}w = \frac{a}{k}e^{kt} + C_1 \qquad \text{or} \qquad w = \frac{a}{k} + C_1e^{-kt}.$$

Returning to the original variable, we have

$$y^{-1} = \frac{a}{k} + C_1e^{-kt}.$$

Applying the initial condition, $y(0) = y_0$, we find that $\dfrac{1}{y_0} = \dfrac{a}{k} + C_1$, so that $C_1 = \dfrac{1}{y_0} - \dfrac{a}{k}$, or $C_1 = \dfrac{k - ay_0}{ky_0}$. Therefore, $y^{-1} = \dfrac{a}{k} + \dfrac{k - ay_0}{ky_0}e^{-kt}$. Solving for $y$, we find that

$$y = \frac{ky_0}{ay_0 + (k - ay_0)e^{-kt}}.$$

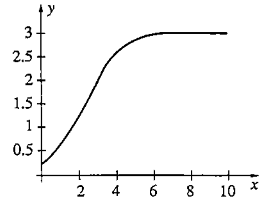



**Figure 2.14** Solution to the logistic equation ($y_0 = 1/4$, $a = 1$, and $k = 3$)

In Figure 2.14, we graph this function using the parameter values $y_0 = 1/4$, $a = 1$, and $k = 3$. Notice that unlike solutions to the Malthus model, solutions to the logistic equation are bounded.

Equations such as the logistic equation are classified as **Bernoulli equations.**

**Definition 2.3 Bernoulli Equation**

A **Bernoulli equation** is a first-order equation of the form

$$\frac{dy}{dt} + p(t)\,y = q(t)\,y^n.$$

Of course, if $n = 0$ or $n = 1$, this equation is linear, so we would not need to make a change of variable to solve the equation. However, for other values of $n$, such as $n = 2$ as we considered in the previous example, we solve these equations with the substitution

$$w = y^{1-n}.$$

In doing so, we transform the nonlinear equation into the linear equation

$$\frac{dw}{dt} + (1-n)p(t)\,w = (1-n)q(t).$$

After solving this equation for $w$ by using an integrating factor, we return to the original variable with the substitution $w = y^{1-n}$ and by solving for $y$. Of course, Bernoulli equations may involve variables other than $y$ or $t$, but we follow a similar solution method.

**Example 1**

Solve $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{1}{xy^2}$, $x > 0$.

***Solution*** In this case, the independent variable is $x$ (instead of $t$ as it was in the previous problem), and the right side of the equation can be written as $y^{-2}/x$, so $p(x) = 1/x$, $q(x) = 1/x$, and $n = -2$. Therefore, we make the substitution $w = y^{1-(-2)} = y^3$, where $dw/dx = 3y^2\, dy/dx$ so that $dy/dx = (y^{-2}/3)\, dw/dx$. With this substitution, we obtain

$$\frac{y^{-2}}{3}\frac{dw}{dx} + \frac{y}{x} = \frac{y^{-2}}{x}.$$

Multiplication by $3y^2$ gives us the first-order ODE $\dfrac{dw}{dx} + \dfrac{3y^3}{x} = \dfrac{3}{x}$, or



$$\frac{dw}{dx} + \frac{3w}{x} = \frac{3}{x}$$

because $w = y^3$. We solve this equation with the integrating factor

$$\mu(x) = e^{\int (3/x)dx} = e^{3\ln x} = x^3,\ x > 0.$$

Multiplying the ODE by $\mu(x)$ and simplifying yields $\dfrac{d}{dx}[x^3 w] = 3x^2$, so that $x^3 w = x^3 + C$. Solving for $w$, we find that $w = 1 + Cx^{-3}$. Substituting $w = y^3$ and solving for $y$ gives us

$$y = (1 + Cx^{-3})^{1/3}.$$

Now, consider the differential equation $\dfrac{dy}{dx} = -\dfrac{x+y}{y-x}$, which can be written as

$$(y-x)\,dy + (x+y)\,dx = 0.$$

Because this equation is not of the form $f_1(x)g_1(y)\,dx + f_2(x)g_2(y)\,dy = 0$, it is not separable. However, if we let $y = ux$, then we use the product rule to obtain $dy = u\,dx + x\,du$. Substituting these expressions into $(y-x)\,dy + (x+y)\,dx = 0$ and simplifying results in

$$(ux - x)(u\,dx + x\,du) + (x + ux)\,dx = 0 \text{ or } x(u^2+1)\,dx + x^2(u-1)\,du = 0.$$

This equation is separable and can be written as

$$\frac{dx}{x} = \frac{1-u}{u^2+1}\,du.$$

Because the right side of this equation is equivalent to $\left(\dfrac{1}{u^2+1} - \dfrac{u}{u^2+1}\right)du$, we find that

$$\ln|x| = \tan^{-1} u - \frac{1}{2}\ln(u^2+1) + C.$$

Notice that the absolute value is not needed in the term $\frac{1}{2}\ln(u^2+1)$ because $u^2 + 1 > 0$ for all $u$. Now, $u = y/x$ because $y = ux$, so resubstitution gives us

$$\ln|x| = \tan^{-1}(y/x) - \frac{1}{2}\ln((y/x)^2 + 1) + C$$

as a general solution of $(y-x)\,dy + (x+y)\,dx = 0$.

The equation $(y-x)\,dy + (x+y)\,dx = 0$ is called a **homogeneous equation.** We can always reduce a homogeneous equation to a separable equation by a suitable substitution.

**Definition 2.4 Homogeneous Differential Equation**

A first-order differential equation that can be written in the form $M(x, y)\,dx + N(x, y)\,dy = 0$, where $M(tx, ty) = t^n M(x, y)$ and $N(tx, ty) = t^n N(x, y)$ is called a **homogeneous differential equation** (of degree $n$).

**Example 2**

Show that the equation $(x^2 + yx)\,dx - y^2\,dy = 0$ is homogeneous.

***Solution*** Let $M(x, y) = x^2 + yx$ and $N(x, y) = -y^2$. The equation

$$(x^2 + yx)\,dx - y^2\,dy = 0$$

is homogeneous of degree 2 because

$$M(tx, ty) = (tx)^2 + (ty)(tx) = t^2(x^2 + yx) = t^2 M(x, y)$$

and

$$N(tx, ty) = -t^2y^2 = t^2 N(x, y).$$

Homogeneous equations are reduced to separable equations by either of the substitutions

$$y = ux \qquad \text{or} \qquad x = vy.$$

Use the substitution $y = ux$ if $N(x, y)$ is less complicated than $M(x, y)$, and use $x = vy$ if $M(x, y)$ is less complicated than $N(x, y)$. If a difficult integration problem is encountered after a substitution is made, try the other substitution to see if it yields an easier problem. As with the separation of variables technique, this technique was also discovered by Leibniz.

**Example 3**

Solve the initial-value problem $\dfrac{dy}{dx} = \dfrac{y}{x + \sqrt{x^2 + y^2}}$, $y(0) = 1$.

***Solution*** When we write the ODE in differential form, $y\,dx - (x + \sqrt{x^2 + y^2})\,dy = 0$, we recognize that the ODE is homogeneous of order one. Then, because $M(x, y) = -y$ is less complicated than $N(x, y) = x + \sqrt{x^2 + y^2}$, we let $x = vy$ so that $dx = v\,dy + y\,dv$. Substitution yields

$$y(v\,dy + y\,dv) - (vy + \sqrt{v^2y^2 + y^2})\,dy = 0$$



$$vy\,dy + y^2\,dv - vy\,dy - y\sqrt{v^2 + 1}\,dy = 0.$$

Notice that by writing $\sqrt{v^2y^2 + y^2} = \sqrt{y^2(v^2 + 1)} = |y|\sqrt{v^2 + 1}$ as $y\sqrt{v^2 + 1}$, we are making the assumption that $y \geq 0$. The initial condition includes a positive value of $y$, so we can solve the IVP with this assumption. We can simplify and separate the variables in this equation to obtain

$$y^2 dv = y\sqrt{v^2 + 1}\,dy \qquad \text{or} \qquad \frac{dv}{\sqrt{v^2 + 1}} = \frac{dy}{y}.$$

Using a table of integrals or a trigonometric substitution to evaluate $\displaystyle\int \frac{dv}{\sqrt{v^2 + 1}}$, we obtain

$$\ln\left|v + \sqrt{v^2 + 1}\right| = \ln|y| + C_1.$$

Returning to the original variables with $v = x/y$, we find that

$$\ln\left|\frac{x}{y} + \sqrt{\left(\frac{x}{y}\right)^2 + 1}\right| = \ln|y| + C_1$$

$$\frac{x}{y} + \sqrt{\left(\frac{x}{y}\right)^2 + 1} = C_2 y$$

$$\frac{x + \sqrt{x^2 + y^2}}{y} = C_2 y$$

$$x + \sqrt{x^2 + y^2} = C_2 y^2$$

$$y = \pm\frac{1}{C_2}\sqrt{1 + 2C_2 x}.$$

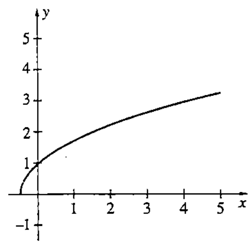



**Figure 2.15** Graph of $\ln\left|\dfrac{x}{y^2} + \dfrac{1}{y}\sqrt{\left(\dfrac{x}{y}\right)^2 + 1}\right| = 0$

Applying the initial conditions gives us $\frac{1}{C_2}\sqrt{1 + 2C_2 \cdot 0} = 1$ so $C_2 = 1$. Therefore, the solution to the IVP is $y = \sqrt{1 + 2x}$. We graph this solution in Figure 2.15. Notice that the curve passes through the point (0, 1) as required by the initial condition.

In addition to verifying that $M(tx, ty) = t^n M(x, y)$ and $N(tx, ty) = t^n N(x, y)$, there are other ways to determine if an equation is homogeneous or not. For example, solving the differential equation $4xy^2\,dx + (x^3 + y^3)\,dy = 0$ for $dy/dx$, we obtain

$$\frac{dy}{dx} = \frac{-4xy^2}{x^3 + y^3} = \frac{\frac{1}{x^3}(-4xy^2)}{\frac{1}{x^3}(x^3 + y^3)} = \frac{-4\left(\frac{y}{x}\right)^2}{1 + \left(\frac{y}{x}\right)^3} = F\left(\frac{y}{x}\right),$$

where $F(t) = -4t^2/(1 + t^3)$. Similarly, we find that

$$\frac{dy}{dx} = \frac{-4xy^2}{x^3 + y^3} = \frac{\frac{1}{y^3}(-4xy^2)}{\frac{1}{y^3}(x^3 + y^3)} = \frac{-4\frac{x}{y}}{\left(\frac{x}{y}\right)^3 + 1} = G\left(\frac{x}{y}\right),$$

where $G(t) = -4t/(t^3 + 1)$. This indicates (although we have not shown it in general) that an equation is homogeneous if we can write it in either of the forms $dy/dx = F(y/x)$ or $dy/dx = G(x/y)$.

**Example 4**

Find values of $a_0$, $a_1$, $a_2$, and $a_3$, none of which are zero, so that the nontrivial solutions to $\dfrac{dy}{dx} = \dfrac{a_0 x + a_1 y}{a_2 x + a_3 y}$ are periodic.

***Solution*** This equation is homogeneous because

$$\frac{dy}{dx} = \frac{a_0 x + a_1 y}{a_2 x + a_3 y} = \frac{a_0 + a_1\frac{y}{x}}{a_2 + a_3\frac{y}{x}} = F\left(\frac{y}{x}\right),$$

where $F(t) = \dfrac{a_0 + a_1 t}{a_2 + a_3 t}$. Letting $y = ux$ leads to

$$-x(a_3u^2 + (a_2 - a_1)u - a_0)\,dx - x^2(a_3u + a_2)\,du = 0$$

$$\frac{a_3u + a_2}{a_3u^2 + (a_2 - a_1)u - a_0}\,du = -\frac{1}{x}\,dx.$$

Integrating and substituting $u = y/x$ yields a general solution of the equation

$$\ln\sqrt{a_0 + (a_1 - a_2)u - a_3u^2} - \frac{a_1 + a_2}{\sqrt{(a_2 - a_1)^2 + 4a_0a_3}}\tanh^{-1}\left(\frac{a_2 - a_1 + 2a_3u}{\sqrt{(a_2 - a_1)^2 + 4a_0a_3}}\right) = -\ln x + C$$

$$\ln\sqrt{a_0 + (a_1 - a_2)\frac{y}{x} - a_3\frac{y^2}{x^2}} - \frac{a_1 + a_2}{\sqrt{(a_2 - a_1)^2 + 4a_0a_3}}\tanh^{-1}\left(\frac{(a_2 - a_1) + 2a_3y}{x\sqrt{(a_2 - a_1)^2 + 4a_0a_3}}\right) = -\ln x + C.$$

This solution is complicated and nearly impossible to analyze analytically, so we proceed graphically. We begin by generating *random* values of $a_0$, $a_1$, $a_2$, and $a_3$, none of which are zero, and then graphing the direction field associated with the



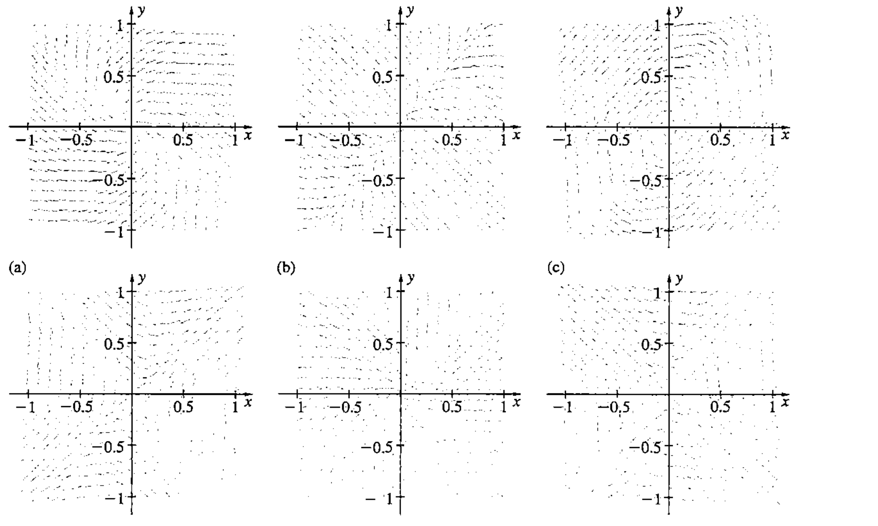



**Figure 2.16** (a) $a_0 = -4$, $a_1 = 3$, $a_2 = 7$, $a_3 = 5$ (b) $a_0 = 4$, $a_1 = -5$, $a_2 = -6$, $a_3 = 3$ (c) $a_0 = 3$, $a_1 = -1$, $a_2 = 1$, $a_3 = -2$ (d) $a_0 = 3$, $a_1 = -1$, $a_2 = 1$, $a_3 = 2$ (e) $a_0 = -2$, $a_1 = -7$, $a_2 = -7$, $a_3 = 2$ (f) $a_0 = -5$, $a_1 = 1$, $a_2 = 2$, $a_3 = -4$

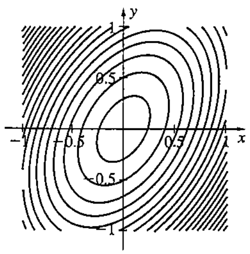


**Figure 2.17** Graph of $\frac{3}{2}x^2 - xy + y^2 = C$ for various values of $C$

equation $\dfrac{dy}{dx} = \dfrac{a_0x + a_1y}{a_2x + a_3y}$. Several of the results we obtain are shown in Figure 2.16. (Use each direction field to graph several solutions to each equation.)

From these graphs, we see that the choices $a_0 = 3$, $a_1 = -1$, $a_2 = 1$, and $a_3 = -2$ *may* lead to periodic solutions. A general solution of the equation $\dfrac{dy}{dx} = \dfrac{3x - y}{x - 2y}$ is $\dfrac{3}{2}x^2 - xy + y^2 = C$ (why?), which is a family of ellipses as shown in Figure 2.17. We see that the choices $a_0 = 3$, $a_1 = -1$, $a_2 = 1$, and $a_3 = -2$ yield periodic solutions to the equation.

*Show that the nontrivial solutions to* $\dfrac{dy}{dx} = \dfrac{a_0x + a_1y}{a_2x + a_3y}$ *are periodic if* $a_2 = -a_1$ *and* $a_1^2 + a_0a_3 < 0$.

## EXERCISES 2.3

In Exercises 1–6, solve the Bernoulli equation.

1. $y' - \frac{1}{2}y = \frac{x}{y}$
2. $y' + y = xy^2$
*3. $y' - \frac{1}{2x}y = y^3 \cos x$
4. $y' - \frac{1}{x}y = y^3 \sin x$
5. $y' - 2y = \frac{\cos x}{\sqrt{y}}$
6. $y' + 3y = \sqrt{y}\sin x$

In Exercises 7–16, determine if the differential equation is homogeneous. If so, determine its degree.

7. $(x + 3y)\,dx - 4x\,dy = 0$
8. $(y^2 - x^2)\,dx + xy\,dy = 0$
*9. $\sqrt{x^2 + xy}\,dy - xy\,dx = 0$
10. $dx + (x + y)\,dy = 0$
11. $\cos\left(\frac{x}{x+y}\right)dx + e^{2y/x}\,dy = 0$
12. $y\ln\left(\frac{x}{y}\right)dx + \frac{x^2}{x+y}\,dy = 0$
*13. $2\ln x\,dx - \ln(4y^2)\,dy = 0$
14. $\left(\frac{2}{x} + \frac{1}{y}\right)dx + \frac{x}{y^2}\,dy = 0$
15. $\frac{\sin 2x}{\cos 2y}\,dx + \left(\frac{\ln y}{\ln x}\right)dy = 0$
16. $\sqrt{x^2 + 1}\,dx + y\,dy = 0$

In Exercises 17–32, solve each equation.

17. $2x\,dx + (y - 3x)\,dy = 0$
18. $(2y - 3x)\,dx + x\,dy = 0$
*19. $(xy - y^2)\,dx + x(x - 3y)\,dy = 0$
20. $(x^2 + xy + y^2)\,dx - xy\,dy = 0$
21. $(x^3 + y^3)\,dx - xy^2\,dy = 0$
22. $\frac{dy}{dx} = \frac{x + 4y}{4x + y}$
*23. $(x - y)\,dx + x\,dy = 0$
24. $y\,dx + (xy + y)\,dy = 0$
25. $(2x^2 - 7xy + 5y^2)\,dx + xy\,dy = 0$
26. $(y + 2\sqrt{x^2 + y^2})\,dx - x\,dy = 0$
*27. $y^2\,dx = (xy - 4x^2)\,dy$
28. $y\,dx - (3\sqrt{xy} + x)\,dy = 0$
29. $(x^2 - y^2)\,dy + (y^2 + xy)\,dx = 0$
30. $xy\,dy - (x^2e^{-y/x} + y^2)\,dx = 0$
*31. $\frac{dx}{dy} = \frac{2y}{x}e^{-x/y} + \frac{x}{y}$
32. $x(\ln x - \ln y)\,dy = y\,dx$

In Exercises 33–39, solve the initial-value problem.

33. $\frac{dy}{dx} = \frac{4y^2 - x^2}{2xy}$, $y(1) = 1$
34. $(x + y)\,dx - x\,dy = 0$, $y(1) = 1$
*35. $x\,dy - (y + \sqrt{x^2 + y^2})\,dx = 0$, $y(1) = 0$
36. $(x^2 + y^2\sqrt{x^2 + y^2})\,dx - xy\sqrt{x^2 + y^2}\,dy = 0$, $y(1) = 1$
37. $(y^3 - x^3)\,dx - xy^2\,dy = 0$, $y(1) = 3$
38. $xy^3\,dx - (x^4 + y^4)\,dy = 0$, $y(1) = 1$
*39. $y^4\,dx + (x^4 - xy^3)\,dy = 0$, $y(1) = 2$
40. First-order equations of the form

$$(a_1x + b_1y + c_1)\,dx + (a_2x + b_2y + c_2)\,dy = 0$$

can be transformed into a homogeneous equation with a transformation. If $a_2/a_1 \neq b_2/b_1$, the transformation $\begin{cases} x = X + h \\ y = Y + k \end{cases}$, where $(h, k)$ satisfies the linear system $\begin{cases} a_1h + b_1k + c_1 = 0 \\ a_2h + b_2k + c_2 = 0 \end{cases}$, reduces this equation to a homogeneous equation in the variables $X$ and $Y$. If $a_2/a_1 = b_2/b_1 = k$, the transformation $z = a_1x + b_1y$ reduces this equation to a homogeneous equation in the variables $x$ and $z$. Use this transformation to solve the following equations.

(a) $(x - 2y + 1)\,dx + (4x - 3y - 6)\,dy = 0$
(b) $(5x + 2y + 1)\,dx + (2x + y + 1)\,dy = 0$
(c) $(3x - y + 1)\,dx - (6x - 2y - 3)\,dy = 0$
(d) $(2x + 3y + 1)\,dx + (4x + 6y + 1)\,dy = 0$

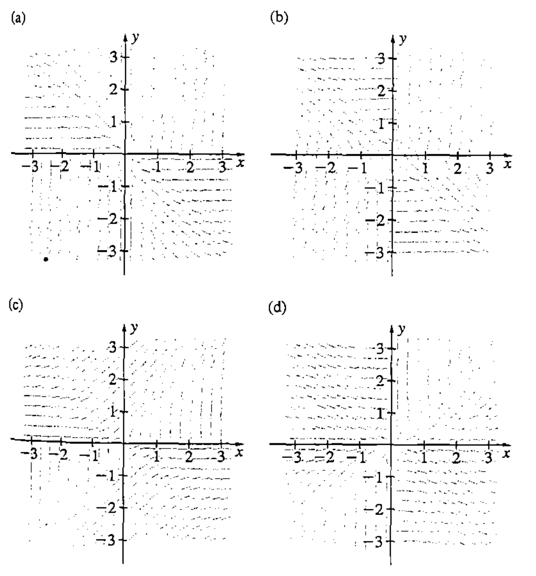
41. Find an equation of the curve that passes through the point $(\sqrt{e}, \sqrt{e})$ and has slope $y/x + x/y$ at each point $(x, y)$ on the curve.

In Exercises 42–45, without actually solving the homogeneous equations, match each equation in Group A with the graph of its direction field in Group B.

**Group A**

42. $\frac{dy}{dx} = \frac{xe^{y/x}}{y}$
43. $\frac{dy}{dx} = \frac{ye^{y/x}}{x}$
44. $\frac{dy}{dx} = \frac{ye^{x/y}}{x}$
45. $\frac{dy}{dx} = \frac{xe^{x/y}}{y}$

**Group B**






46. Show that if the differential equation $dy/dx = f(x, y)$ is homogeneous, the equation can be written as $dy/dx = F(y/x)$. (*Hint:* Let $t = 1/x$.)

*47. If $M(x, y)\,dx + N(x, y)\,dy = 0$ is a homogeneous equation, show that the change of variables $x = r\cos\theta$ and $y = r\sin\theta$ transform the homogeneous equation into a separable equation.

48. Equations of the form

$$f(xy' - y) = g(y')$$

are called **Clairaut equations** after the French mathematician Alexis Clairaut (1713–1765) who studied these equations in 1734. Solutions to this equation are determined by differentiating each side of the equation with respect to $x$.

(a) Use the chain rule to show that the derivative of $f(xy' - y)$ is

$$f'(xy' - y)(xy'' + y' - y') = f'(xy' - y)(xy''),$$

where $'$ denotes differentiation with respect to the argument of the function, $x$.

(b) Show that the equation

$$f'(xy' - y)(xy'') = g'(y')y'',$$

which is equivalent to

$$[f'(xy' - y)x - g'(y')]y'' = 0,$$

is obtained by differentiating both sides of the Clairaut equation with respect to $x$.

(c) This result indicates that $y'' = 0$ or $f'(xy' - y)x - g'(y') = 0$. If $y'' = 0$, $y' = c$ where $c$ is a constant. Substitute $y' = c$ into the differential equation $f(xy' - y) = g(y')$, to find that a general solution is $f(xc - y) = g(c)$. If $f'(xy' - y)x - g'(y') = 0$, this equation can be used along with $f(xy' - y) = g(y')$ to determine another solution by eliminating $y'$. This is called the *singular solution* of the Clairaut equation.

49. Use the following steps to solve the Clairaut equation $xy' - (y')^3 = y$.

(a) Place the equation in the appropriate form to find that $f(x) = x$ and $g(x) = x^3$.

(b) Use the form of a general solution to find that $xc - y = c^3$ and solve this equation for $y$.

(c) Find the singular solution by differentiating $xy' - (y')^3 = y$ with respect to $x$ to obtain

$xy'' + y' - 3(y')^2y'' = y'$, which can be simplified to $[x - 3(y')^2]y'' = 0$. Since $y'' = 0$ was used to find the general solution, solve $x - 3(y')^2 = 0$ for $y'$ to obtain $y' = (x/3)^{1/2}$. Substitute this expression for $y'$ into $xy' - y = (y')^3$ to obtain a relationship between $x$ and $y$.

In Exercises 50–53, solve the Clairaut equation.

**50.** $xy' - y - 2(xy' - y)^2 = y' + 1$

**51.** $xy' - y - 1 = (y')^2 - y'$

**52.** $1 + y - xy' = \ln(y')$

***53.** $1 - 2(xy' - y) = (y')^{-2}$

**54.** Equations of the form $y = xf(y') + g(y')$ are called **Lagrange equations.** These equations are solved by making the substitution

$$p = y'(x).$$

**(a)** Differentiate $y = xf(y') + g(y')$ with respect to $x$ to obtain

$$y' = xf'(y')y'' + f(y') + g'(y')y''.$$

**(b)** Substitute $p$ into the equation to obtain

$$p = xf'(p)\frac{dp}{dx} + f(p) + g'(p)\frac{dp}{dx}$$
$$= f(p) + \frac{dp}{dx}[xf'(p) + g'(p)].$$

**(c)** Solve this equation for $dx/dp$ to obtain the linear equation

$$\frac{dx}{dp} = \frac{xf'(p) + g'(p)}{p - f(p)}$$

which is equivalent to

$$\frac{dx}{dp} + \frac{f'(p)}{f(p) - p}x = \frac{g'(p)}{p - f(p)}.$$

This linear first-order equation can be solved for $x$ in terms of $p$. Then, $x(p)$ can be used with $y = xf(p) + g(p)$ to obtain an equation for $y$.

**55.** Solve the equation $y = -xy' + \frac{1}{5}(y')^5$ using the following steps. In this case, $f(y') = -y'$ and $g(y') = \frac{1}{5}(y')^5$.

**(a)** Differentiate the equation with respect to $x$ to obtain $y' = -y' - xy'' + (y')^4y''$.

**(b)** Substitute $y' = p$ to obtain $p = -p - x\dfrac{dp}{dx} + (p)^4\dfrac{dp}{dx}$. Simplify this equation to obtain $\dfrac{dp}{dx} = \dfrac{2p}{p^4 - x}$.

**(c)** Rewrite this equation as $\dfrac{dx}{dp} = \dfrac{p^4 - x}{2p}$ and solve this first-order linear equation for $x$ to obtain $x = \frac{1}{9}p^4 + Cp^{-1/2}$.

**(d)** Substitute $y' = p$ and $x = \frac{1}{9}p^4 + Cp^{-1/2}$ into the differential equation $y = -xy' + \frac{1}{5}(y')^5$ to obtain a formula for $y$.

**(e)** Graph the solution curves for various values of $C$.

In Exercises 56–58, solve the Lagrange equation.

**56.** $y = x\left(\dfrac{dy}{dx}\right)^2 + 3\left(\dfrac{dy}{dx}\right)^2 - 2\left(\dfrac{dy}{dx}\right)^3$

**57.** $y = x\left(\dfrac{dy}{dx} + 1\right) + \left(2\dfrac{dy}{dx} + 1\right)$

**58.** $y = x\left(2 - \dfrac{dy}{dx}\right) + \left(2\left(\dfrac{dy}{dx}\right)^2 + 1\right)$

**59.** Euler was the first mathematician to take advantage of integrating factors to solve linear differential equations.* Euler used the following steps to solve the equation

$$\frac{dz}{dv} - 2z + \frac{z}{v} = \frac{1}{v}.$$

**(a)** Multiply the equation by the integrating factor $e^{-2v}v$.

**(b)** Show that $\dfrac{d}{dv}(e^{-2v}vz) = e^{-2v}v\dfrac{dz}{dv} - 2e^{-2v}vz + e^{-2v}z$.

**(c)** Express the equation as $\dfrac{d}{dv}(e^{-2v}vz) = e^{-2v}$ and solve this equation for $z$.

* Victor J. Katz, *A History of Mathematics: An Introduction*, HarperCollins (1993), p. 503.



**60.** Consider the solution to the logistic equation $y = \dfrac{ky_0}{ay_0 + (k - ay_0)e^{-kt}}$. Find $\lim_{t\to\infty} y(t)$. Sketch the phase line for $dy/dt = (k - ay)y$ and compare it to the value of $\lim_{t\to\infty} y(t)$.

In the same manner that computer algebra systems and graphing utilities can be used to help find solutions of separable equations, they can also be used to help solve and graph solutions of homogeneous equations.

**61.** Solve the equation $(x^{1/3}y^{2/3} + x)\,dx + (x^{2/3}y^{1/3} + y)\,dy = 0$. Graph several solutions.

**62.** Solve the initial-value problem $\begin{cases} y' = \dfrac{y^2 - x^2}{xy} \\ y(4) = 0 \end{cases}$ and graph the solution for $0 < x \le 4$.

***63. (a)** The **sine integral function,** $\mathrm{Si}(x)$, is defined by

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt.$$

**(i)** Evaluate $\mathrm{Si}'(x)$ and $\lim_{x\to 0}\mathrm{Si}(x)$. **(ii)** Graph $\mathrm{Si}(x)$ on the interval $[0, 6\pi]$. **(iii)** Approximate the maximum value of $\mathrm{Si}(x)$. **(iv)** Can you predict $\lim_{x\to\infty}\mathrm{Si}(x)$?

**(b)** Graph the direction field associated with the equation $y\sin(x/y)\,dx - (x + x\sin(x/y))\,dy = 0$.

**(c)** Solve the initial-value problem

$$\begin{cases} y\sin(x/y)\,dx - (x + x\sin(x/y))\,dy = 0 \\ y(1) = 2 \end{cases}$$

and graph the solution.

## 2.4 Exact Equations

Unlike homogeneous equations, many first-order differential equations cannot be reduced to separable differential equations by a suitable substitution. For example, the first-order differential equation

$$(\sin y + y\cos x)\,dx + (\sin x + x\cos y)\,dy = 0$$

is neither separable nor can be reduced to a separable equation by an appropriate substitution. Nevertheless, a general solution of

$$(\sin y + y\cos x)\,dx + (\sin x + x\cos y)\,dy = 0$$

can be calculated, as we will see in this section.

**Definition 2.5 Exact Differential Equation**

A first-order differential equation that can be written in the form

$$M(x, y)\,dx + N(x, y)\,dy = 0,$$

where

$$M(x, y)\,dx + N(x, y)\,dy = \frac{\partial f}{\partial x}(x, y)\,dx + \frac{\partial f}{\partial y}(x, y)\,dy$$

for some function $f(x, y)$ is called an **exact differential equation.**

In calculus, we learn that the **total differential** of the function $f(x, y)$ is

$$df = \frac{\partial f}{\partial x}(x, y)\, dx + \frac{\partial f}{\partial y}(x, y)\, dy.$$

Therefore, the equation $M(x, y)\, dx + N(x, y)\, dy = 0$ is exact if there exists a function $f(x, y)$ such that $M(x, y)\, dx + N(x, y)\, dy$ is the total differential of $f(x, y)$.

In multivariable calculus, we learned that if $f$, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x \partial y}$, and $\frac{\partial^2 f}{\partial y \partial x}$ are continuous on an open region $R$, then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ on $R$. Hence, if $M(x, y)\, dx + N(x, y)\, dy = 0$ is exact and $M(x, y)\, dx + N(x, y)\, dy$ is the total differential of $f(x, y)$,

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial M}{\partial y}.$$

In fact, we can prove the following. (See Exercise 55 in this section.)

**Theorem 2.1 Test for Exactness**

The first-order differential equation $M(x, y)\, dx + N(x, y)\, dy = 0$ is exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

**Example 1**

Show that the equation $2xy^3\, dx + (1 + 3x^2y^2)\, dy = 0$ is exact and that the equation $x^2y\, dx + 5xy^2\, dy = 0$ is not exact.

***Solution*** The equation $2xy^3\, dx + (1 + 3x^2y^2)\, dy = 0$ is an exact equation because

$$\frac{\partial}{\partial y}(2xy^3) = 6xy^2 = \frac{\partial}{\partial x}(1 + 3x^2y^2).$$

Conversely, the equation $x^2y\, dx + 5xy^2\, dy = 0$ is not exact because

$$\frac{\partial}{\partial y}(x^2y) = x^2 \neq 5y^2 = \frac{\partial}{\partial x}(5xy^2).$$

If the equation $M(x, y)\, dx + N(x, y)\, dy = 0$ is exact, we can find a function $f(x, y)$ such that $M(x, y) = \frac{\partial f}{\partial x}(x, y)$ and $N(x, y) = \frac{\partial f}{\partial y}(x, y)$. Then the differential equation becomes



$$M(x, y)\, dx + N(x, y)\, dy = 0$$
$$df = 0.$$

A general solution of the equation is

$$f(x, y) = C,$$

where $C$ is a constant.

**Example 2**

Find a general solution of $(\sin y + y\cos x)\, dx + (\sin x + x\cos y)\, dy = 0$.

***Solution*** The equation is exact because

$$\frac{\partial}{\partial y}(\sin y + y\cos x) = \cos y + \cos x = \frac{\partial}{\partial x}(\sin x + x\cos y).$$

Let $f(x, y)$ be a function with $\frac{\partial f}{\partial x} = \sin y + y\cos x$ and $\frac{\partial f}{\partial y} = \sin x + x\cos y$. Integrating $\frac{\partial f}{\partial x} = \sin y + y\cos x$ with respect to $x$ results in

$$\int (\sin y + y\cos x)\, dx = x\sin y + y\sin x + g(y),$$

where $g(y)$ denotes an arbitrary function of $y$. We must include this arbitrary function $g(y)$ because the derivative of a function of $y$ with respect to $x$ is zero. That is, the general form of the function $f(x, y)$ whose partial derivative with respect to $x$ is $\partial f/\partial x = \sin y + y\cos x$ is given by

$$f(x, y) = x\sin y + y\sin x + g(y).$$

Because we are looking for a function $f(x, y)$ that satisfies

$$\frac{\partial f}{\partial y} = \sin x + x\cos y,$$

and differentiating $f(x, y) = x\sin y + y\sin x + g(y)$ with respect to $y$ results in

$$\frac{\partial f}{\partial y} = \sin x + x\cos y + g'(y),$$

it must be true that

$$\sin x + x\cos y + g'(y) = \sin x + x\cos y,$$

which indicates that $g'(y) = 0$. This means that $g(y) = k$ for some constant $k$. Thus,

$$f(x, y) = x\sin y + y\sin x + k.$$

Therefore, the implicit function $x \sin y + y \sin x + k = C_1$ or $x \sin y + y \sin x = C$, where $C = C_1 - k$ represents an arbitrary constant, is a general solution of $(\sin y + y \cos x)\,dx + (\sin x + x \cos y)\,dy = 0$. [Note that there is no need to include an arbitrary constant in $f(x, y)$, because it is included in $f(x, y) = C$.] Several members of the family of solutions are graphed in Figure 2.18 by graphing several level curves of the function $f(x, y) = x \sin y + y \sin x$.

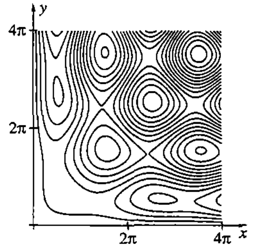



**Figure 2.18** We graph various solutions to the equation by graphing several level curves of the function $f(x, y) = x \sin y + y \sin x$.

**Example 3**

Solve $2x \sin y\,dx + (x^2 \cos y - 1)\,dy = 0$ subject to $y(0) = 1/2$.

***Solution*** The equation is exact because

$$\frac{\partial}{\partial y}(2x \sin y) = 2x \cos y = \frac{\partial}{\partial x}(x^2 \cos y - 1).$$

Let $f(x, y)$ be a function with $\partial f/\partial x = 2x \sin y$ and $\partial f/\partial y = x^2 \cos y - 1$. Integrating $\partial f/\partial x$ with respect to $x$ yields

$$f(x, y) = \int 2x \sin y\,dx = x^2 \sin y + g(y).$$

Notice that the arbitrary function $g(y)$ serves as a "constant" of integration with respect to $x$. From the differential equation, we have

$$\frac{\partial f}{\partial y} = x^2 \cos y - 1,$$

and differentiating $f(x, y) = x^2 \sin y + g(y)$ with respect to $y$ gives us

$$\frac{\partial f}{\partial y} = x^2 \cos y + g'(y).$$

Thus,

$$x^2 \cos y - 1 = x^2 \cos y + g'(y)$$
$$g'(y) = -1$$



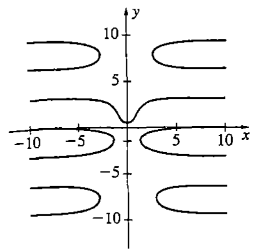



**Figure 2.19**

and $g(y) = -y + C_1$ so that substitution into $f(x, y) = x^2 \sin y + g(y)$ yields

$$f(x, y) = x^2 \sin y - y + C_1.$$

A general solution of the exact equation is then $x^2 \sin y - y + C_1 = C$. Simplifying, we have

$$x^2 \sin y - y = k,$$

where $k = C - C_1$ is a constant. Our solution must satisfy $y(0) = \frac{1}{2}$, so we must find the solution that passes through the point $(0, \frac{1}{2})$. Substituting $x = 0$ and $y = \frac{1}{2}$ into the general solution, we obtain $0^2 \sin(\frac{1}{2}) - \frac{1}{2} = k$ so that $k = -\frac{1}{2}$ and the solution is $x^2 \sin y - y = -\frac{1}{2}$. The solution is graphed in Figure 2.19. We see that the graph passes through the point $(0, \frac{1}{2})$, as required by the initial condition.

**Solving the Exact Differential Equation**
$M(x, y)\,dx + N(x, y)\,dy = 0$

1. Assume that $M(x, y) = \dfrac{\partial f}{\partial x}(x, y)$ and $N(x, y) = \dfrac{\partial f}{\partial y}(x, y)$.
2. Integrate $M(x, y)$ with respect to $x$. (Add an arbitrary function of $y$, $g(y)$.)
3. Differentiate the result in step 2 with respect to $y$ and set the result equal to $N(x, y)$. Solve for $g'(y)$.
4. Integrate $g'(y)$ with respect to $y$ to obtain an expression for $g(y)$. (There is no need to include an arbitrary constant.)
5. Substitute $g(y)$ into the result obtained in step 2 for $f(x, y)$.
6. A general solution is $f(x, y) = C$, where $C$ is a constant.
7. Apply the initial condition if given.

*A similar algorithm can be stated so that in step 2, $N(x, y)$ is integrated with respect to $y$ as we show in Example 4.*

**Example 4**

Solve $\left(e^{y/x} - \dfrac{y}{x} e^{y/x} + \dfrac{1}{1 + x^2}\right) dx + e^{y/x}\,dy = 0.$

***Solution*** This equation is exact because $\dfrac{\partial}{\partial y}\left(e^{y/x} - \dfrac{y}{x} e^{y/x} + \dfrac{1}{1 + x^2}\right) = -\dfrac{y}{x^2} e^{y/x} = \dfrac{\partial}{\partial x}(e^{y/x})$. Let $f(x, y)$ be a function such that $\dfrac{\partial f}{\partial x} = e^{y/x} - \dfrac{y}{x} e^{y/x} +$

$\dfrac{1}{1+x^2}$ and $\dfrac{\partial f}{\partial y} = e^{y/x}$. Integrating $\dfrac{\partial f}{\partial y}$ with respect to $y$ because it is a less complicated expression than $\dfrac{\partial f}{\partial x}$ gives us

$$f(x, y) = \int e^{y/x}\, dy = \frac{1}{1/x} e^{y/x} + g(x) = xe^{y/x} + g(x)$$

where $g(x)$ is an arbitrary function of $x$. Differentiating $f(x, y)$ with respect to $x$ leads to

$$\frac{\partial f}{\partial x} = e^{y/x} + x\left(-\frac{y}{x^2} e^{y/x}\right) + g'(x) = e^{y/x} - \frac{y}{x} e^{y/x} + g'(x),$$

so $g'(x) = \dfrac{1}{1+x^2}$. This implies that $g(x) = \tan^{-1} x$, so $f(x, y) = xe^{y/x} + \tan^{-1} x$. Therefore, a general solution of the exact equation is $xe^{y/x} + \tan^{-1} x = C$.

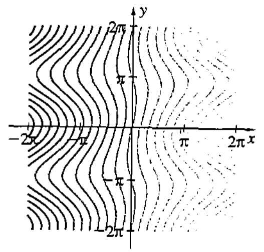



**Figure 2.20** Graph of $x \cos y + 4x - \sin y = C$ for various values of $C$

*Figure 2.20 shows the graph of* $x \cos y + 4x - \sin y = C$ *for various values of* $C$. *Identify the curve(s) corresponding to* $C = 5$.

Up to now, all initial-value problems we have considered have had a *unique* solution. However, this need not be true. (This topic is considered again in Section 2.5.)

**Example 5**

Find a value of $y_0$ so that there is *not* a unique solution to the initial-value problem

$$\begin{cases} (\cos x \cos y - \sin x)\, dx + (\cos y - \sin x \sin y)\, dy = 0 \\ y(0) = y_0 \end{cases}.$$

***Solution*** The equation $(\cos x \cos y - \sin x)\, dx + (\cos y - \sin x \sin y)\, dy = 0$ is exact because

$$\frac{\partial}{\partial y}(\cos x \cos y - \sin x) = -\cos x \sin y = \frac{\partial}{\partial x}(\cos y - \sin x \sin y).$$

A general solution is found to be $\cos x + \sin x \cos y + \sin y = C$, which we graph for various values of $C$ in Figure 2.21.

From the graphs, we see that if $C = 0$, it is possible to find $y_0$ such that there is *not* a unique solution to the initial-value problem. Indeed an exact value of $y_0$ is $3\pi/2$. (Why?) (See Figure 2.22.)



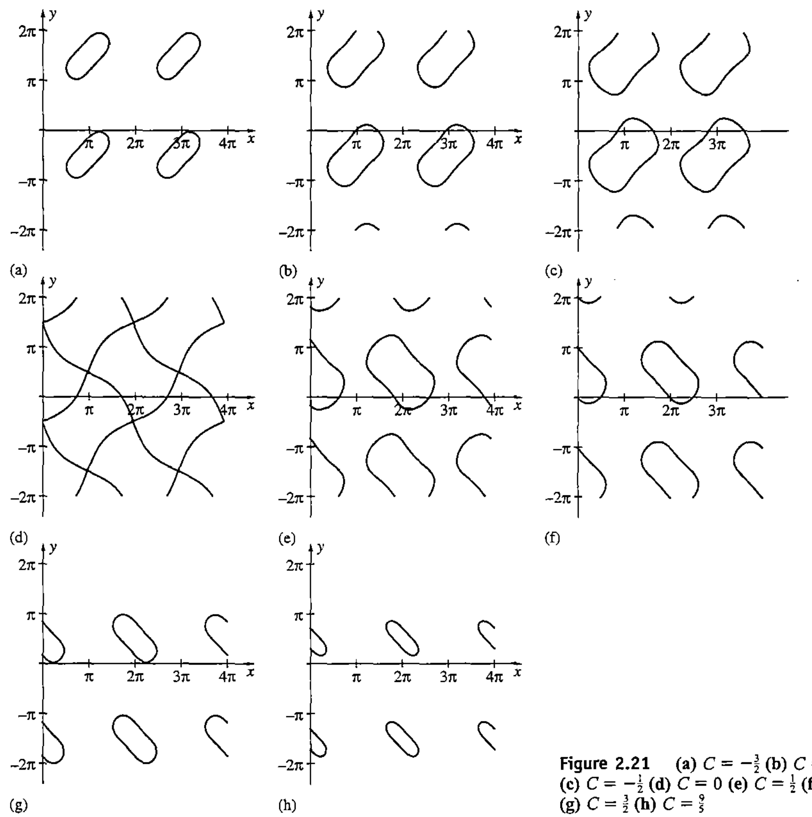



**Figure 2.21** (a) $C = -\frac{3}{2}$ (b) $C = -1$ (c) $C = -\frac{1}{2}$ (d) $C = 0$ (e) $C = \frac{1}{2}$ (f) $C = 1$ (g) $C = \frac{3}{2}$ (h) $C = \frac{9}{5}$

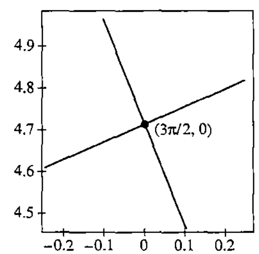



**Figure 2.22** The solution is *not unique* near this point because when we zoom in, we see the graph of more than one function passing through the point.

## EXERCISES 2.4

In Exercises 1–10, determine if the equation is exact.

1. $\left(y^2 - \dfrac{y}{2\sqrt{x}}\right) dx + (2xy - \sqrt{x} + 1)\, dy = 0$
2. $\dfrac{x}{\sqrt{x^2+y^2}}\, dx + \dfrac{y}{\sqrt{x^2+y^2}}\, dy = 0$

*3. $y\cos(xy)\, dx + x\cos(xy)\, dy = 0$

4. $(y\sec^2 x + 2x)\, dx + (\tan x)\, dy = 0$
5. $3xy^2\, dy + y^3\, dx = 0$
6. $(x - y\sin x)\, dx + (y^6 + \cos x)\, dy = 0$

*7. $(y\sin 2x)\, dx - (\sqrt{y} + \cos 2x)\, dy = 0$

8. $(e^{2x} + y)\, dx - (e^y - x)\, dy = 0$
9. $\ln(xy)\, dx + \dfrac{x}{y}\, dy = 0$
10. $e^{xy}\, dx + \dfrac{x}{y} e^{xy}\, dy = 0$

In Exercises 11–30, solve each equation.

11. $3x^2\, dx - dy = 0$
12. $-dx + 3y^2\, dy = 0$

*13. $y^2\, dx + 2xy\, dy = 0$

14. $\dfrac{3x^2}{y}\, dx - \dfrac{x^3}{y^2}\, dy = 0$
15. $(2x + y^3)\, dx + (3xy^2 + 4)\, dy = 0$
16. $-\dfrac{1}{y}\, dx + \left(\dfrac{x}{y^2} + 3y^2\right) dy = 0$

*17. $2xy\, dx + (x^2 + y^2)\, dy = 0$

18. $2xy^3\, dx + (1 + 3x^2y^2)\, dy = 0$
19. $\sin^2 y\, dx + x\sin 2y\, dy = 0$
20. $(3x^2 + 3y^2)\, dx + 6xy\, dy = 0$

*21. $\dfrac{y + y^2}{(y - x)^2}\, dx - \dfrac{x + x^2}{(x - y)^2}\, dy = 0$

22. $(3x^2y + 3y^2 - 1)\, dx + (x^3 + 6xy)\, dy = 0$
23. $-2xy^2\sin(x^2)\, dx + 2y\cos(x^2)\, dy = 0$
24. $(2x - y^2\sin(xy))\, dx + (\cos(xy) - xy\sin(xy))\, dy = 0$

*25. $(1 + y^2\cos(xy))\, dx + (xy\cos(xy) + \sin(xy))\, dy = 0$

26. $ye^{xy}(\cos(xy) + \sin(xy))\, dx + xe^{xy}(\cos(xy) + \sin(xy))\, dy = 0$
27. $((3 + x)\cos(x + y) + \sin(x + y))\, dx + (3 + x)\cos(x + y)\, dy = 0$
28. $\dfrac{2x^2y\cos(x^2) - y\sin(x^2)}{x^2}\, dx + \dfrac{2xy + \sin(x^2)}{x}\, dy = 0$

*29. $\dfrac{e^{y/x}(x - y)}{x}\, dx + e^{y/x}\, dy = 0$

30. $\dfrac{e^{x/y}\left(x^2\cos\left(\frac{y}{x}\right) + y^2\sin\left(\frac{y}{x}\right)\right)}{x^2y}\, dx - \dfrac{e^{x/y}\left(x^2\cos\left(\frac{y}{x}\right) + y^2\sin\left(\frac{y}{x}\right)\right)}{xy^2}\, dy = 0$

In Exercises 31–40, solve the initial-value problem. Graph the solution on an appropriate region.

31. $2xy^2\, dx + 2x^2y\, dy = 0,\ y(1) = 1$
32. $\left(1 + \dfrac{y}{x^2}\right) dx - \dfrac{1}{x}\, dy = 0,\ y(2) = 1$

*33. $(2xy + 3x^2)\, dx + (x^2 - 1)\, dy = 0,\ y(0) = 1$

34. $(1 + 5x - y)\, dx - (x + 2y)\, dy = 0,\ y(0) = 0$
35. $(e^y - 2xy)\, dx + (xe^y - x^2)\, dy = 0,\ y(0) = 0$
36. $(2xe^{x^2} + 2xe^{-y})\, dx + (e^{x^2} - x^2e^{-y} + 1)\, dy = 0,\ y(0) = 0$

*37. $(y^2 - 2\sin 2x)\, dx + (1 + 2xy)\, dy = 0,\ y(0) = 1$

38. $(\cos^2 x - \sin^2 x + y)\, dx + (\sec y\tan y + x)\, dy = 0,\ y(0) = 0$
39. $\left(\dfrac{1}{1 + x^2} - y^2\right) dx - 2xy\, dy = 0,\ y(0) = 0$



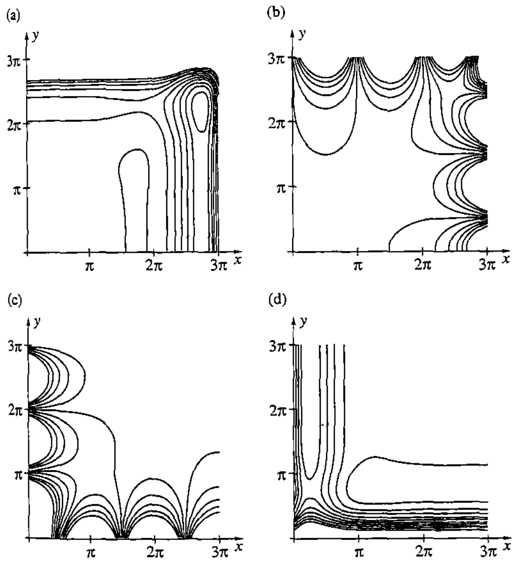
40. $\left(\dfrac{2x}{1 + x^2} + y\right) dx + (e^y + x)\, dy = 0,\ y(0) = 0$

In Exercises 41–44, without actually solving the equations, match each equation in Group A with the graph of some of its solutions in Group B.

**Group A**

41. $\dfrac{dy}{dx} = \dfrac{e^{y-x}(\cos x - \sin x)}{\cos y + \sin y}$
42. $\dfrac{dy}{dx} = \dfrac{-e^{x-y}(\cos x + \sin x)}{\cos y - \sin y}$
43. $\dfrac{dy}{dx} = \dfrac{-e^y\cos x - e^x\cos y}{e^y\sin x - e^x\sin y}$
44. $\dfrac{dy}{dx} = \dfrac{e^x\sin x + e^y\sin y}{e^y\cos y - e^x\cos x}$

**Group B**




45. Show that a separable equation of the form $g(y)\, dy - h(x)\, dx = 0$ is exact.

46. **(Integrating Factors)** If the differential equation $M(x, y)\, dx + N(x, y)\, dy = 0$ is not exact, multiplying it by an *appropriate* function $\mu(x, y)$ yields an exact equation. To find $\mu(x, y)$, we use the fact that if the equation $\mu(x, y)M(x, y)\, dx + \mu(x, y)N(x, y)\, dy = 0$ is exact, $[\mu M]_y = [\mu N]_x$.
   (a) Use the product rule to show that $\mu$ must satisfy the differential equation
   $$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0.$$
   (b) Use this equation to show that if $\mu = \mu(x)$, $\mu$ satisfies the differential equation
   $$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu, \text{ where } \frac{M_y - N_x}{N}$$
   is a function of $x$ only.
   (c) Show that if $\mu = \mu(x)$,
   $$\mu(x) = \exp\left(\int \frac{1}{N(x, y)}\left[\frac{\partial M(x, y)}{\partial y} - \frac{\partial N(x, y)}{\partial x}\right] dx\right).$$
   (d) If $\mu = \mu(y)$, find a differential equation that $\mu$ must satisfy and a restriction on $\dfrac{N_x - M_y}{M}$. Show that
   $$\mu(y) = \exp\left(\int \frac{1}{M(x, y)}\left[\frac{\partial N(x, y)}{\partial x} - \frac{\partial M(x, y)}{\partial y}\right] dy\right).$$

In Exercises 47–54, use an integrating factor to solve each differential equation. (See Exercise 46.)

47. $x^2y\, dx + x^3\, dy = 0$
48. $y(2e^x + 4x)\, dx + 3(e^x + x^2)\, dy = 0$
49. $y\, dx + (2x - ye^y)\, dy = 0$
50. $(2xy + y^2)\, dx - x^2\, dy = 0$
51. $(y + 2x^2)\, dx + (x^2y - x)\, dy = 0$
52. $(5xy + 4y^2 + 1)\, dx + (x^2 + 2xy)\, dy = 0$
53. $(2xy^2 + y)\, dx + (2x^3 - x)\, dy = 0$
54. $(2x + \tan y)\, dx + (x - x^2\tan y)\, dy = 0$

*55. Suppose that $M(x, y)\, dx + N(x, y)\, dy = 0$ is an equation for which $\partial M/\partial y = \partial N/\partial x$.
   (a) Let $g(y) = \int \left(N(x, y) - \dfrac{\partial}{\partial y}\int M(x, y)\, dx\right) dy.$

Show that $g$ is a function of $y$. *Hint:* Show that $N(x, y) - \frac{\partial}{\partial y}\int M(x, y)\,dx$ is a function of $y$ by showing that

$$\frac{\partial}{\partial x}\left(N(x, y) - \frac{\partial}{\partial y}\int M(x, y)\,dx\right) = 0.$$

**(b)** Let $f(x, y) = g(y) + \int M(x, y)\,dx$. Show that

$$M(x, y)\,dx + N(x, y)\,dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

As with other types of equations, technology is useful in solving exact equations or in performing the steps necessary to solve an exact equation.

**56.** Find a general solution of the equation $(2x - y^2\sin(xy))\,dx + (\cos(xy) - xy\sin(xy))\,dy = 0$. Graph various solutions on the rectangle $[0, 3\pi] \times [0, 3\pi]$.

**57.** Find a general solution of the equation

$$(-1 + e^{xy}y + y\cos(xy))\,dx + (1 + e^{xy}x + x\cos(xy))\,dy = 0.$$

Graph several solutions.

**58.** **(a)** Find a general solution for each of the following differential equations:
(i) $(2x + 2y)\,dx + (2x + 2y)\,dy = 0$;
(ii) $(1.8x + 2y)\,dx + (2x + 2y)\,dy = 0$; and
(iii) $(2x + 1.9y)\,dx + (1.9x + 2y)\,dy = 0$.

**(b)** Graph the direction field for each equation along with several solutions.

**(c)** Comment on this statement: "If we slightly change a differential equation, the solutions also slightly change."

***59.** How does the graph of the solution to the initial-value problem

$$\begin{cases} (\sin(cy) - yc\sin(cx))\,dx + (xc\cos(cy) + \cos(cx))\,dy = 0 \\ y(0) = 1 \end{cases}$$

change as $c$ takes on values from $-2$ to $2$?

**60.** Find restrictions on $p$ and $q$ so that the first-order linear equation $\frac{dy}{dx} + p(x)y = q(x)$ is exact.

## 2.5 Theory of First-Order Equations

### Some Differences Between Linear Equations and Nonlinear Equations

To better understand the solution or solutions to an initial-value problem, consider the following:

$$\frac{dy}{dx} = \frac{x}{y},\ y(0) = 0.$$

This equation is solved by separating the variables to obtain $y\,dy = x\,dx$ and by integrating each side of the equation to obtain $y^2/2 = x^2/2 + C$ or

$$y^2 - x^2 = K,$$

where $K = 2C$. if $K \neq 0$, this solution is represented graphically by a family of hyperbolas. For $K > 0$, we have hyperbolas that intersect the $y$-axis (Figure 2.23), and for $K < 0$, the hyperbolas intersect the $x$-axis (Figure 2.24). In the case of this initial-value problem, we are interested in finding the solution or solutions that satisfy $y(0) = 0$. In other words, we require that the solution curve(s) pass through $(0, 0)$. When we substitute $(0, 0)$ into $y^2 - x^2 = K$, we find that $K = 0$, so $y^2 - x^2 = 0$ satisfies the IVP. Factoring indicates that $(y - x)(y + x) = 0$, so the IVP has *two* solutions, $y = x$ and $y = -x$. A question we may ask at this point is whether the IVP has more than one solution if we use another initial condition. By observing the family of hyperbolas, we

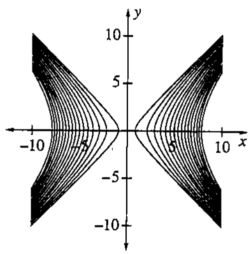



**Figure 2.23** Graph of $y^2 - x^2 = K$ for several values of $K > 0$



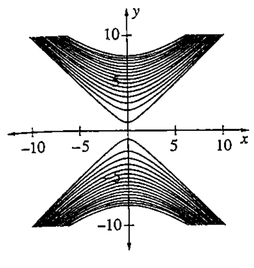



**Figure 2.24** Graph of $y^2 - x^2 = K$ for several values of $K < 0$

would see that no other solution curves intersect. This tells us that if another initial condition were used instead of $y(0) = 0$, then the IVP would have only one solution. Hence, we say that it has a *unique solution.* For example, if we consider

$$\frac{dy}{dx} = \frac{x}{y},\ y(1) = 0,$$

we have the same general solution, $y^2 - x^2 = K$, as before. However, when we apply the initial condition $y(1) = 0$, we find that $(0)^2 - (1)^2 = K$, so $K = -1$. Therefore, the IVP has the unique solution $y^2 - x^2 = -1$ or $x^2 - y^2 = 1$. As we see from this example, an initial-value problem may have one or more solutions depending on how the problem is stated. The following theorem helps us understand the types of initial-value problems that have a unique solution.

**Theorem 2.2 Existence and Uniqueness**

Consider the initial-value problem

$$y' = f(x, y),\ y(x_0) = y_0.$$

If $f$ and $\partial f/\partial y$ are continuous functions on the rectangular region $R$: $a < x < b$, $c < y < d$ containing the point $(x_0, y_0)$, then there exists an interval $|x - x_0| < h$ centered at $x_0$ on which there exists one and only one solution to the differential equation that satisfies the initial condition.

**Example 1**

Does the fact that the IVP $dy/dx = x/y$, $y(0) = 0$ has two solutions contradict the Existence and Uniqueness Theorem?

***Solution*** In this case, $f(x, y) = x/y$ and $(x_0, y_0) = (0, 0)$. The hypothesis of the Existence and Uniqueness Theorem is not satisfied because $f$ is not continuous at $(0, 0)$. Therefore, the fact that the IVP has two solutions does not contradict the Existence and Uniqueness Theorem.

**Example 2**

Verify that the initial-value problem $dy/dx = y$, $y(0) = 1$ has a unique solution.

***Solution*** In this case, $f(x, y) = y$, $x_0 = 0$, and $y_0 = 1$. Hence, both $f$ and $\partial f/\partial y$ are continuous on all rectangular regions containing the point $(x_0, y_0) = (0, 1)$. By the Existence and Uniqueness Theorem, there exists a unique solution to the differential equation that satisfies the initial condition $y(0) = 1$. We verify this by solving the initial-value problem. This equation is separable and equivalent to $dy/y = dx$. A general solution is given by $y = Ce^x$, and the solution that satisfies the initial condition $y(0) = 1$ is $y = e^x$.

The Existence and Uniqueness Theorem gives sufficient, but not necessary, conditions for the existence of a unique solution of an initial-value problem. If an initial-value problem does not satisfy the hypotheses of the theorem, we cannot conclude that a unique solution does not exist. In fact, the problem may have a unique solution, no solution, or many solutions.

**Example 3**

Find a solution to the initial-value problems (a) $\begin{cases} \dfrac{dy}{dx} = \dfrac{x}{y^{2/3}} \\ y(0) = 0 \end{cases}$ and (b) $\begin{cases} 2(xy^3 - x^5y)\,dx + (x^6 - x^2y^2)\,dy = 0 \\ y(0) = 0 \end{cases}$, if possible.

***Solution*** The Existence and Uniqueness Theorem does not guarantee the existence of a solution to either problem because both $\dfrac{x}{y^{2/3}}$ and $-\dfrac{2(xy^3 - x^5y)}{x^6 - x^2y^2}$ are discontinuous at the point (0, 0) specified by the initial condition.

(a) The equation $\dfrac{dy}{dx} = \dfrac{x}{y^{2/3}}$ is separable and has solution

$$\frac{3}{5}y^{5/3} = \frac{1}{2}x^2 + C_1$$

$$y = \left(\frac{5}{6}x^2 + C\right)^{3/5}.$$

Application of the initial condition leads to the *unique* solution $y = (5/6)^{3/5}x^{6/5}$, which is graphed in Figure 2.25.

(b) The equation $2(xy^3 - x^5y)\,dx + (x^6 - x^2y^2)\,dy = 0$ is more difficult to solve. Dividing the equation by $(x^4 + y^2)^2$ leads to an exact equation that has general solution $x^2y = C(x^4 + y^2)$. Applying the initial condition $y(0) = 0$ results in the identity $0 = 0$, which means that $x^2y = C(x^4 + y^2)$ is a solution to the initial-value problem for *any* value of $C$. There are infinitely many solutions to the initial-value problem, so the solution is *nonunique*. Several solutions are graphed in Figure 2.26.

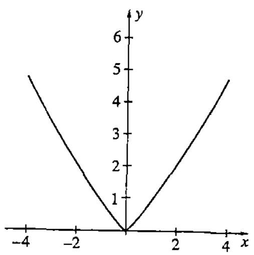


Figure 2.25

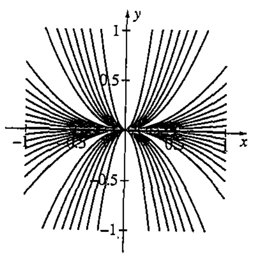


Figure 2.26

*Are the hypotheses of the Existence and Uniqueness Theorem satisfied for the initial-value problem $\dfrac{dy}{dx} = \sqrt{y^2 - 1}$, $y(0) = 1$? If $\displaystyle\int \frac{dy}{\sqrt{y^2 - 1}} = \ln\left|y + \sqrt{y^2 - 1}\right| + C$, does a unique solution to this problem exist? Does more than one solution exist?*



## Some Differences Between Linear Equations and Nonlinear Equations

While solving both linear and nonlinear first-order equations throughout Chapters 1 and 2, you may have noticed some of the differences between the two types of equations. For example, in Exercises 1.1, we considered the ODE $dy/dx = -2xy^2$ with solution $y = 1/(x^2 + k)$ where $k$ is constant. We noted that although $y = 0$ also satisfies the ODE, it cannot be obtained from $y = 1/(x^2 + k)$ for any choice of $k$. (We call $y = 0$ a singular solution.) Therefore, the solution technique used to solve this separable nonlinear ODE does not generate all solutions. This is a difference between nonlinear and linear equations. When we find a general solution of a linear equation, we obtain all possible solutions to the equation.

We can also state an Existence and Uniqueness Theorem for IVPs involving first-order linear equations.

**Theorem 2.3 Existence and Uniqueness: First-Order Linear Equations**

Consider the initial-value problem

$$y' + p(x)y = q(x),\ y(x_0) = y_0.$$

If $p$ and $q$ are continuous on an open interval $I$ that contains $x = x_0$, then there exists one and only one function that satisfies the differential equation for each $x$ in $I$, and this also satisfies the initial condition.

**Example 4**

Find an interval on which the IVP $xy' + 2y = x\cos x$, $y(2) = 4$ has a unique solution.

***Solution*** First, we write the ODE in the form $y' + (2/x)y = \cos x$ to identify $p(x) = 2/x$ and $q(x) = \cos x$. The function $p$ is continuous for $x < 0$ or for $x > 0$; $q$ is continuous for all values of $x$. Therefore, we must determine the interval containing $x_0 = 2$ on which both functions are continuous. This interval is $0 < x < +\infty$. (Of course, if the initial condition were $y(-2) = 4$, then we would have selected $-\infty < x < 0$.) If we solve this IVP, we find that

$$y = \frac{2}{x}\cos x - \frac{2}{x^2}\sin x - \frac{2}{x^2}(\sin 2 + 2\cos 2 - 8) + \sin x.$$

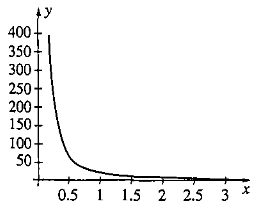


**Figure 2.27** Solution of $xy' + 2y = x\cos x$, $y(2) = 4$

We graph this function in Figure 2.27 to observe that the solution is valid only on $0 < x < +\infty$. The solution becomes unbounded as $x \to 0^+$. It approaches zero as $x \to +\infty$.

As we see, the interval on which the solution to a first-order IVP is unique depends on the functions $p$ and $q$ in the general form of the ODE. This is not necessarily true with an IVP involving a nonlinear ODE. In this case, the interval of definition of the solution may depend on the initial condition, as we illustrate in the following example.

**Example 5**

Determine the interval of definition of the solution to $y' = y^3$, $y(0) = a$, $a > 0$.

***Solution*** This equation is separable. Integrating $\int \frac{dy}{y^3} = \int dx$ yields $-\frac{1}{2}y^{-2} = x + C$, such that simplification gives us $y^{-2} = -2x + K$ where $K = -2C$. Therefore, $y = 1/\sqrt{K - 2x}$. Applying the initial condition $y(0) = a$ indicates that $K = a^{-2}$, so $y = 1/\sqrt{a^{-2} - 2x}$. This solution is defined only if $a^{-2} - 2x > 0$ or $x < (a^{-2})/2$. In Figure 2.28, we graph the solution for $K = a^{-2} = 1, 2, 3$. In this graph, notice that when $a^{-2} = 1$ the solution is defined only for $x < 1/2$. The other two solutions are defined on $x < 1$ and $x < 3/2$, respectively. Notice also that each becomes unbounded as $x$ approaches $(a^{-2})/2$ from the left.

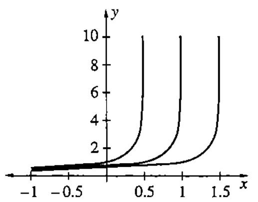



**Figure 2.28** Graph of $y = \dfrac{1}{\sqrt{K - 2x}}$ for $K = 1, 2, 3$

Another difference between linear equations and nonlinear equations that you may have noticed while working problems in this chapter is that a linear equation has an explicit solution because of the way we solve linear ODEs. Conversely, a nonlinear equation often has an implicit solution that cannot be solved analytically to find an explicit form of the solution.

## EXERCISES 2.5

1. Does the Existence and Uniqueness Theorem guarantee a unique solution to the following initial-value problems on some interval? Explain.

   (a) $\frac{dy}{dx} + x^2 = y^2$, $y(0) = 0$

   (b) $\frac{dy}{dx} + x^2 = y^{-2}$, $y(0) = 0$

   (c) $\frac{dy}{dx} = y + \frac{1}{1 - x}$, $y(1) = 0$

2. According to the Existence and Uniqueness Theorem, the initial-value problem $dy/dx = |y|$, $y(1) = 0$ has a solution. **(a)** Must the solution be unique? **(b)** Solve the problem by hand. Is the solution unique? *Hint:* Separate the initial-value problem into two parts. First solve the problem for $y \geq 0$ and then solve it for $y < 0$.

*3. The Existence and Uniqueness Theorem implies that at least one solution to the initial-value problem $dy/dx = y^{1/5}$, $y(0) = 0$ exists. **(a)** Must the solution be unique according to the theorem? **(b)** Solve the problem by hand. Is the solution unique? Note that $y = 0$ satisfies the differential equation as well as the initial condition.

4. Show that $y = 0$ and $y = \frac{x^4}{16}$ both satisfy the initial-value problem $\frac{1}{x}\frac{dy}{dx} = \sqrt{y}$, $y(0) = 0$. Does this contradict the Existence and Uniqueness Theorem?



5. Show that $y = 0$ and $y = x|x|$ both satisfy the initial-value problem $dy/dx = 2\sqrt{|y|}$, $y(0) = 0$. Does this contradict the Existence and Uniqueness Theorem?

6. Does the initial-value problem $dy/dx = 4x^2 - xy^2$, $y(2) = 1$ have a unique solution on an interval containing $x = 2$?

*7. Does the initial-value problem $dy/dx = y\sqrt{x}$, $y(1) = 1$ have a unique solution on an interval containing $x = 1$? Verify your result by solving the problem.

8. Does the initial-value problem $dy/dx = 6y^{2/3}$, $y(1) = 0$ have a unique solution on an interval containing $x = 1$? Solve the problem using separation of variables. Is $y = 0$ obtained through this method? Is $y = 0$ a solution?

9. Does the initial-value problem $y' = \sin y - \cos x$, $y(\pi) = 0$ have a unique solution on an interval containing $x = \pi$?

10. Does the initial-value problem $xy' = y$, $y(0) = 1$ have a unique solution on an interval containing $x = 0$? Verify your response by solving the initial-value problem.

11. Show that $y = \sec x$ satisfies the initial-value problem $y' = y \tan x$, $y(0) = 1$. What is the largest open interval containing $x = 0$ over which $y = \sec x$ is a solution? Explain.

12. Show that $y = \tan x$ satisfies the initial-value problem $y' = 1 + y^2$, $y(0) = 0$. What is the largest open interval containing $x = 0$ over which $y = \tan x$ is a solution? Explain.

13. Using the Existence and Uniqueness Theorem, determine if $dy/dx = \sqrt{y^2 - 1}$ has a unique solution passing through the point **(a)** $(0, 2)$; **(b)** $(4, -1)$; **(c)** $(0, 1/2)$; **(d)** $(2, 1)$ is guaranteed.

14. Using the Existence and Uniqueness Theorem, determine if $dy/dx = \sqrt{25 - y^2}$ has a unique solution passing through the point **(a)** $(-4, 3)$; **(b)** $(0, 5)$; **(c)** $(3, -6)$; **(d)** $(4, -5)$.

In Exercises 15–22, use the Existence and Uniqueness Theorem (for linear IVPs) to determine the largest interval on which the solution is guaranteed to exist.

15. $y' + (1/x)y = x^2$, $y(1) = 0$

16. $x^3y' + x^4y = 2x^3$, $y(0) = 0$

*17. $2y' + xy = \ln x$, $y(e) = 0$

18. $y' + (\sec x)y = x$, $y(0) = 0$

19. $y' + \frac{1}{x - 3}y = \frac{1}{x - 1}$, $y(-1) = 0$

20. $(x - 2)y' + (x^2 - 4)y = \frac{1}{x + 2}$, $y(0) = 3$

*21. $y' + \frac{y}{\sqrt{4 - x^2}} = x$, $y(0) = 0$

22. $y' + \frac{y}{\sqrt{4 - x^2}} = x$, $y(3) = -1$

23. **(a)** Over what interval is the solution to the IVP $y' + (1/x)y = \sin x$, $y(\pi) = 1$ certain to exist? **(b)** Solve this IVP and graph the solution. **(c)** Is the solution valid on a larger interval than what is guaranteed by the Existence and Uniqueness Theorem for Linear IVPs?

24. **(a)** Over what interval is the solution to the IVP $y' + (\tan x)y = \sin x$, $y(0) = 0$ certain to exist? **(b)** Solve this IVP and graph the solution. **(c)** Is the solution valid on a larger interval than what is guaranteed by the Existence and Uniqueness Theorem for Linear IVPs?

In Exercises 25–28, determine the interval of definition of the solution to each IVP. Graph the solution for several values of $a$ to verify your result. Assume that $a$ is positive.

25. $y' = y^2$, $y(0) = a$

26. $y' = xy^2$, $y(0) = a$

*27. $y' = -x/y$, $y(0) = a$

28. $y' = -y^3$, $y(0) = a$

29. Find all points $(x_0, 0)$ in the rectangle $R = \{(x, y) | 0 \leq x \leq 2\pi, -\pi \leq y \leq \pi\}$ such that the initial-value problem
$$\begin{cases} \dfrac{dy}{dx} = \dfrac{\cos x \sin x + \sin x \sin y}{\cos x \cos y - \cos y \sin y} \\ y(x_0) = 0 \end{cases}$$
has *more than one solution.* Confirm your results graphically.

30. Find all points $(x_0, y_0)$ in the rectangle $R = \{(x, y) | 0 \leq x \leq 4\pi, 0 \leq y \leq 4\pi\}$ such that the initial-value problem
$$\begin{cases} \dfrac{dy}{dx} = \dfrac{\cos y - y \cos x}{x \sin y + \sin x - 1} \\ y(x_0) = y_0 \end{cases}$$
has *no* solutions. *Hint:* Graph the direction field associated with the equation.

## 2.6 Numerical Methods for First-Order Equations

Euler's Method · Improved Euler's Method · Errors · Runge-Kutta Method

In many cases, we cannot obtain a formula for the solution to an initial-value problem of the form

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0.$$

For example, we cannot solve

$$\frac{dy}{dx} = \sin(xy),\ y(0) = 1$$

using any of the methods we have discussed to this point. Of course, we can determine certain properties of the solution by looking at the slope field. However, we sometimes would like to find numerical values of $y$ at particular values of $x$. For that reason, we now discuss **numerical methods** for approximating solutions to initial-value problems. We begin with a method based on tangent line approximations.

### Euler's Method

Suppose that we wish to approximate the solution to

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$$

over the interval $x_0 \le x \le X$, where $X$ is a specified value of $x$. For example, we may wish to approximate the solution on $0 \le x \le 1$. This indicates that the initial condition is specified at $x_0 = 0$ and that $X = 1$ is the end of the interval. Now, we partition $x_0 \le x \le X$ into $N$ subintervals of equal length $h$, where

$$h = \frac{X - x_0}{N}.$$

Therefore, the $n$th value of $x$ in the partition is

$$x_n = x_0 + nh,\ n = 1, 2, \ldots, N.$$

This means that $x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots, x_N = x_0 + Nh = x_0 + N\left(\frac{X - x_0}{N}\right) = X$. If we are solving the IVP $dy/dx = f(x, y)$, $y(x_0) = y_0$, then the point $(x_0, y_0)$ is on the solution curve. Therefore, as we discussed in relation to slope fields, we can follow the line tangent to the solution curve at $(x_0, y_0)$ to approximate the value of the solution at the next value of $x$, $x = x_1$. Because the slope of this tangent line is $f(x_0, y_0)$, we find that the equation for the line is $y - y_0 = f(x_0, y_0)(x - x_0)$ or

$$y = f(x_0, y_0)(x - x_0) + y_0.$$



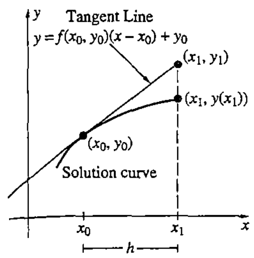



Figure 2.29

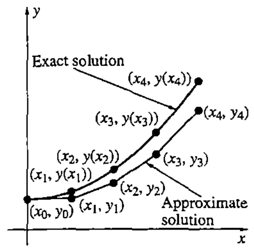



Figure 2.30

Therefore, the approximate value of the solution at $x = x_1$ is

$$y_1 = f(x_0, y_0)(x_1 - x_0) + y_0 = hf(x_0, y_0) + y_0.$$

Notice that we refer to the approximate value of $y$ at $x = x_i$ as $y_i$ while we refer to the exact value of the solution at $x = x_i$ as $y(x_i)$. This means that $y_1$ approximates $y(x_1)$. (See Figure 2.29.) We remember from calculus that this gives a good approximation if $h$ is a small number. Next, we assume that the point $(x_1, y_1)$ is on the solution curve. If we determine the equation of the line with slope $f(x_1, y_1)$ that passes through $(x_1, y_1)$, we obtain

$$y - y_1 = f(x_1, y_1)(x - x_1),$$

so we find the approximate value of $y$ at $x = x_2$ is

$$y_2 = f(x_0, y_0)(x_2 - x_1) + y_1 = hf(x_0, y_0) + y_1.$$

We continue this process until we reach $y_N$, the approximate value of $y(x_N) = y(X)$. In doing so, we obtain a sequence of points of the form $(x_n, y_n)$, $n = 1, 2, \ldots, N$. We show several points of this type along with actual values of $y$ in Figure 2.30.

**Euler's Method**

The solution of the initial-value problem

$$y' = f(x, y),\ y(x_0) = y_0$$

is approximated at the sequence of points $(x_n, y_n)$ $(n = 1, 2, \ldots)$, where $y_n$ is the approximate value of $y(x_n)$ by computing

$$y_n = hf(x_{n-1}, y_{n-1}) + y_{n-1}\ (n = 1, 2, \ldots),$$

where $x_n = x_0 + nh$ and $h$ is the selected stepsize.

**Example 1**

Use Euler's method with $h = 0.1$ and with $h = 0.05$ to approximate the solution of $y' = xy$, $y(0) = 1$ on $0 \le x \le 1$. Determine the exact solution and compare the results.

***Solution*** First, we note that $f(x, y) = xy$, $x_0 = 0$, and $y_0 = 1$. With $h = 0.1$, we have the formula

$$y_n = hf(x_{n-1}, y_{n-1}) + y_{n-1} = 0.1x_{n-1}y_{n-1} + y_{n-1}.$$

Then, for $x_1 = x_0 + h = 0.1$, we have

$$y_1 = 0.1x_0y_0 + y_0 = 0.1(0)(1) + 1 = 1.$$

Similarly, for $x_2 = x_0 + 2h = 0.2$,

$$y_2 = 0.1x_1y_1 + y_1 = 0.1(0.1)(1) + 1 = 1.01.$$

In Table 2.1, we show the results of this sequence of approximations. From this, we see that $y(1)$ is approximately 1.54711.

**TABLE 2.1 Euler's method with $h = 0.1$**

| $x_n$ | $y_n$ | $x_n$ | $y_n$ |
|---|---|---|---|
| 0.0 | 1.0 | 0.6 | 1.15873 |
| 0.1 | 1.0 | 0.7 | 1.22825 |
| 0.2 | 1.01 | 0.8 | 1.31423 |
| 0.3 | 1.0302 | 0.9 | 1.41937 |
| 0.4 | 1.06111 | 1.0 | 1.54711 |
| 0.5 | 1.10355 | | |

For $h = 0.05$, we use

$$y_n = hf(x_{n-1}, y_{n-1}) + y_{n-1} = 0.05x_{n-1}y_{n-1} + y_{n-1}$$

to obtain the values given in Table 2.2. With this stepsize, the approximate value of $y(1)$ is 1.59594.

The exact solution to the initial-value problem, which is found with separation of variables, is $y = e^{x^2/2}$, so the exact value of $y(1)$ is $e^{1/2} \approx 1.64872$. The smaller value of $h$, therefore, yields a better approximation. We graph the approximations obtained with $h = 0.1$ and $h = 0.05$ as well as the graph of $y = e^{x^2/2}$ in Figure 2.31. Notice from these graphs that the approximation is more accurate when $h$ is decreased.

**TABLE 2.2 Euler's method with $h = 0.05$**

| $x_n$ | $y_n$ | $x_n$ | $y_n$ | $x_n$ | $y_n$ |
|---|---|---|---|---|---|
| 0.0 | 1.0 | 0.35 | 1.05361 | 0.70 | 1.2523 |
| 0.05 | 1.0 | 0.40 | 1.07204 | 0.75 | 1.29613 |
| 0.10 | 1.0025 | 0.45 | 1.09348 | 0.80 | 1.34474 |
| 0.15 | 1.00751 | 0.50 | 1.11809 | 0.85 | 1.39853 |
| 0.20 | 1.01507 | 0.55 | 1.14604 | 0.90 | 1.45796 |
| 0.25 | 1.02522 | 0.60 | 1.17756 | 0.95 | 1.52357 |
| 0.30 | 1.03803 | 0.65 | 1.21288 | 1.00 | 1.59594 |



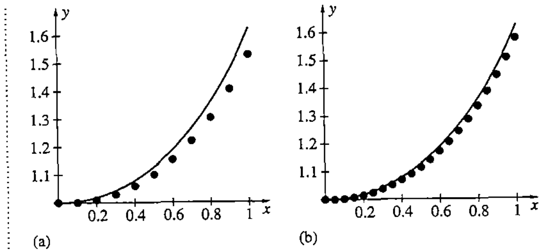



**Figure 2.31** (a) $h = 0.1$ (b) $h = 0.05$

## Improved Euler's Method

Euler's method is improved by using an average slope over each interval. Using the tangent line approximation of the curve through $(x_0, y_0)$, $y = f(x_0, y_0)(x - x_0) + y_0$, we find the approximate value of $y$ at $x = x_1$, which we now call $y_1^*$. Then,

$$y_1^* = hf(x_0, y_0) + y_0,$$

and with the differential equation $y' = f(x, y)$, we find that the approximate slope of the tangent line at $x = x_1$ is $f(x_1, y_1^*)$. The average of the two slopes, $f(x_0, y_0)$ and $f(x_1, y_1^*)$, is $\dfrac{f(x_0, y_0) + f(x_1, y_1^*)}{2}$, and an equation of the line through $(x_0, y_0)$ with slope $\dfrac{f(x_0, y_0) + f(x_1, y_1^*)}{2}$ is

$$y = \frac{f(x_0, y_0) + f(x_1, y_1^*)}{2}(x - x_0) + y_0.$$

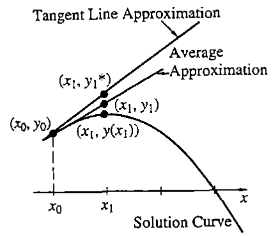



**Figure 2.32**

We illustrate the determination of this average slope and the position of these lines in Figure 2.32.

At $x = x_1$, the approximate value of $y$ is given by

$$y_1 = \frac{f(x_0, y_0) + f(x_1, y_1^*)}{2}(x_1 - x_0) + y_0 = \frac{f(x_0, y_0) + f(x_1, y_1^*)}{2}h + y_0.$$

Continuing in this manner, the approximation in each step in the improved Euler's method depends on the following two calculations:

$$y_n^* = hf(x_{n-1}, y_{n-1}) + y_{n-1}$$

$$y_n = \frac{f(x_{n-1}, y_{n-1}) + f(x_n, y_n^*)}{2}h + y_{n-1},$$

where $x_n = x_0 + nh$.

**Improved Euler's Method**

The solution of the initial-value problem

$$y' = f(x, y),\ y(x_0) = y_0$$

is approximated at the sequence of points $(x_n, y_n)$ $(n = 1, 2, \ldots)$, where $y_n$ is the approximate value of $y(x_n)$ by computing at each step the two calculations:

$$\begin{aligned} y_n^* &= hf(x_{n-1}, y_{n-1}) + y_{n-1} \\ y_n &= \frac{f(x_{n-1}, y_{n-1}) + f(x_n, y_n^*)}{2} h + y_{n-1} \end{aligned} \qquad (n = 1, 2, \ldots),$$

where $x_n = x_0 + nh$ and $h$ is the selected stepsize.

**Example 2**

Use the improved Euler's method to approximate the solution of $y' = xy$, $y(0) = 1$ on $0 \le x \le 1$ for $h = 0.1$. Compare the results to the exact solution.

***Solution*** In this case, $f(x, y) = xy$, $x_0 = 0$, and $y_0 = 1$. Therefore, we use the equations

$$y_n^* = hx_{n-1}y_{n-1} + y_{n-1}$$

$$y_n = \frac{x_{n-1}y_{n-1} + x_n y_n^*}{2} h + y_{n-1}$$

for $n = 1, 2, \ldots, 10$. For example, if $n = 1$, we have

$$y_1^* = hx_0y_0 + y_0 = (0.1)(0)(1) + 1 = 1$$

$$y_1 = \frac{x_0y_0 + x_1y_1^*}{2} h + y_0 = \frac{(0)(1) + (0.1)(1)}{2}(0.1) + 1 = 1.005.$$

Then,

$$y_2^* = hx_1y_1 + y_1 = (0.1)(0.1)(1.005) + 1.005 = 1.01505$$

$$y_2 = \frac{x_1y_1 + x_2y_2^*}{2} h + y_1 = \frac{(0.1)(1.005) + (0.2)(1.01505)}{2}(0.1) + 1.005$$

$$= 1.0201755.$$

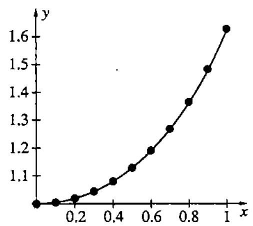



Figure 2.33

In Table 2.3, we list the approximations obtained with this improved method and compare them to those obtained with Euler's method in Example 1 (see Figure 2.33). Which method yields a better approximation?



**TABLE 2.3 Improved Euler's method with $h = 0.1$**

| $x_n$ | $y_n$ (IEM) | $y_n$ (EM) | Actual Value |
|---|---|---|---|
| 0.0 | 1.0 | 1.0 | 1.0 |
| 0.1 | 1.005 | 1.0 | 1.00501 |
| 0.2 | 1.0201755 | 1.01 | 1.0202 |
| 0.3 | 1.0459859 | 1.0302 | 1.04603 |
| 0.4 | 1.083223 | 1.06111 | 1.08329 |
| 0.5 | 1.1330513 | 1.10355 | 1.13315 |
| 0.6 | 1.1970687 | 1.15873 | 1.19722 |
| 0.7 | 1.277392 | 1.22825 | 1.27762 |
| 0.8 | 1.3767731 | 1.31423 | 1.37713 |
| 0.9 | 1.4987552 | 1.41937 | 1.49930 |
| 1.0 | 1.6478813 | 1.54711 | 1.64872 |

## Errors

When approximating the solution of an initial-value problem, there are several sources of error. One of these sources is **round-off error** because computers and calculators use only a finite number of digits in all calculations. Of course, the error is compounded as rounded values are used in subsequent calculations. Therefore, one way to minimize round-off error is to reduce the number of calculations.

Another source of error is **truncation error,** which results from using an approximate formula. We begin our discussion of error by considering the error associated with Euler's method, which uses only the first two terms of the Taylor series expansion.

Recall from calculus the Taylor formula with remainder:

$$y(x) = y(x_0) + \frac{dy}{dx}(x_0)(x - x_0) + \frac{1}{2!}\frac{d^2y}{dx^2}(x_0)(x - x_0)^2 + \cdots$$
$$+ \frac{1}{n!}\frac{d^ny}{dx^n}(x_0)(x - x_0)^n + \frac{1}{(n+1)!}\frac{d^{n+1}y}{dx^{n+1}}(c)(x - x_0)^{n+1},$$

where $c$ is a number between $x$ and $x_0$. The remainder term is

$$R_n(x) = \frac{1}{(n+1)!}\frac{d^{n+1}y}{dx^{n+1}}(c)(x - x_0)^{n+1},$$

so the accuracy of the approximation obtained by using the first $n$ terms of the Taylor series depends on the size of $R_n(x)$. For Euler's method, the remainder is

$$R_1(x) = \frac{1}{2!}\frac{d^2y}{dx^2}(c)(x - x_0)^2,$$

so at $x = x_1 = x_0 + h$, the remainder is

$$R_1(x_1) = R(x_0 + h) = \frac{1}{2!}\frac{d^2y}{dx^2}(c)h^2.$$

Therefore, a **bound** on the error is given by

$$|R_1(x_1)| \leq \frac{h^2}{2}\max_{x_0 \leq x \leq x_1}\left|\frac{d^2y}{dx^2}(x)\right|.$$

**Example 3**

Find a bound for the local truncation error when Euler's method is used to approximate the solution of the initial-value problem $y' = xy$, $y(0) = 1$ at $x_1 = 0.1$.

***Solution*** We found in Example 1 that the exact solution of this problem is $y = e^{x^2/2}$. Therefore, a bound on the error is

$$|R_1(x_1)| \leq \frac{(0.1)^2}{2}\max_{x_0 \leq x \leq x_1}\left|\frac{d^2y}{dx^2}(x)\right|,$$

where $\frac{dy}{dx} = xe^{x^2/2}$ and $\frac{d^2y}{dx^2} = e^{x^2/2} + x^2e^{x^2/2}$.

Then,

$$\begin{aligned}|R_1(x_1)| &\leq \frac{(0.1)^2}{2}\max_{x_0 \leq x \leq x_1}\left|\frac{d^2y}{dx^2}(x)\right| \leq \frac{(0.1)^2}{2}\max_{0 \leq x \leq 0.1}|e^{x^2/2} + x^2e^{x^2/2}|\\ &\leq (0.005)[\max_{0 \leq x \leq 0.1}|e^{x^2/2}| + \max_{0 \leq x \leq 0.1}|x^2e^{x^2/2}|]\\ &\leq (0.005)[e^{(0.1)^2/2} + (0.1)^2e^{(0.1)^2/2}] \approx 0.005075.\end{aligned}$$

Notice that the value of $y = e^{x^2/2}$ at $x = 0.1$ is 1.005012521 and the approximate value obtained with Euler's method is $y_1 = 1$. Therefore, the error is 1.005012521 − 1.0 = 0.005012521, which is less than the bound $|R_1(x_1)| \leq 0.005075$.

## Runge-Kutta Method

In an attempt to improve on the approximation obtained with Euler's method and to avoid the analytic differentiation of the function $f(x, y)$ to obtain $y''$, $y'''$, . . . , we introduce the Runge-Kutta method. We begin with the Runge-Kutta method of order two.

Suppose that we know the value of $y$ at $x_n$. We now use the point $(x_n, y_n)$ to approximate the value of $y$ at a nearby value $x = x_n + h$ by assuming that

$$y_{n+1} = y_n + Ak_1 + Bk_2,$$



where

$$k_1 = hf(x_n, y_n) \quad \text{and} \quad k_2 = hf(x_n + ah, y_n + bk_1).$$

We also use the Taylor series expansion of $y$ to obtain another representation of $y_{n+1} = y(x_n + h)$:

$$y(x_n + h) = y(x_n) + hy'(x_n) + h^2\frac{y''(x_n)}{2!} + \cdots = y_n + hy'(x_n) + h^2\frac{y''(x_n)}{2!} + \cdots.$$

Now, because

$$y_{n+1} = y_n + Ak_1 + Bk_2 = y_n + Ahf(x_n, y_n) + Bhf(x_n + ah, y_n + bhf(x_n, y_n)),$$

we wish to determine values of $A$, $B$, $a$, and $b$ such that these two representations of $y_{n+1}$ agree. Notice that if we let $A = 1$ and $B = 0$, then the relationships match up to order $h$. However, we can choose these parameters more wisely so that agreement occurs up through terms of order $h^2$. This is accomplished by considering the Taylor expansion of a function $F$ of two variables about $(x_0, y_0)$, which is given by

$$F(x, y) = F(x_0, y_0) + \frac{\partial F}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial F}{\partial y}(x_0, y_0)(y - y_0) + \cdots.$$

In our case, we have

$$f(x_n + ah, y_n + bfh(x_n, y_n)) = f(x_n, y_n) + ah\frac{\partial f}{\partial x}(x_n, y_n) + bhf(x_n, y_n)\frac{\partial f}{\partial x}(x_n, y_n) + O(h^2).$$

The power series is then substituted into the following expression and simplified to yield

$$\begin{aligned}y_{n+1} &= y_n + Ahf(x_n, y_n) + Bhf(x_n + ah, y_n + bhf(x_n, y_n))\\ &= y_n + (A + B)hf(x_n, y_n) + aBh^2\frac{\partial f}{\partial x}(x_n, y_n) + bBh^2f(x_n, y_n)\frac{\partial f}{\partial x}(x_n, y_n) + O(h^3).\end{aligned}$$

Comparing the above expression to the following power series obtained directly from the Taylor series of $y$ given by

$$y(x_n + h) = y(x_n) + hf(x_n, y_n) + \frac{h^2}{2}\frac{\partial f}{\partial x}(x_n, y_n) + \frac{h^2}{2}f(x_n, y_n)\frac{\partial f}{\partial y}(x_n, y_n) + O(h^3)$$

or

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2}\frac{\partial f}{\partial x}(x_n, y_n) + \frac{h^2}{2}f(x_n, y_n)\frac{\partial f}{\partial y}(x_n, y_n) + O(h^3),$$

we see that $A$, $B$, $a$, and $b$ must satisfy the following system of nonlinear equations:

$$A + B = 1, \quad aA = \frac{1}{2}, \quad \text{and} \quad bB = \frac{1}{2}.$$

Choosing $a = b = 1$, the Runge-Kutta method of order two uses the equations

$$y_{n+1} = y(x_n + h) = y_n + \frac{1}{2}hf(x_n, y_n) + \frac{1}{2}hf(x_n + h, y_n + hf(x_n, y_n))$$
$$= y_n + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_n, y_n)$ and $k_2 = hf(x_n + h, y_n + k_1)$. Notice that this method is equivalent to the improved Euler method.

**Runge-Kutta Method of Order Two**

The solution of the initial-value problem

$$y' = f(x, y),\ y(x_0) = y_0$$

is approximated at the sequence of points $(x_n, y_n)$ $(n = 1, 2, \ldots)$, where $y_n$ is the approximate value of $y(x_n)$, by computing at each step

$$y_{n+1} = y(x_n + h) = y_n + \frac{1}{2}hf(x_n, y_n) + \frac{1}{2}hf(x_n + h, y_n + hf(x_n, y_n)) \quad (n = 0, 1, \ldots)$$
$$= y_n + \frac{1}{2}(k_1 + k_2),$$

where $k_1 = hf(x_n, y_n)$, $k_2 = hf(x_n + h, y_n + k_1)$, $x_n = x_0 + nh$, and $h$ is the selected stepsize.

**Example 4**

Use the Runge-Kutta method of order two with $h = 0.1$ to approximate the solution of the initial-value problem $y' = xy$, $y(0) = 1$ on $0 \le x \le 1$.

***Solution*** In this case, $f(x, y) = xy$, $x_0 = 0$, and $y_0 = 1$. Therefore, on each step we use the three equations

$$k_1 = hf(x_n, y_n) = 0.1x_ny_n$$
$$k_2 = hf(x_n + h, y_n + k_1) = 0.1(x_n + 0.1)(y_n + k_1)$$
$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2).$$

For example, if $n = 0$, then

$$k_1 = 0.1x_0y_0 = 0.1(0)(1) = 0$$
$$k_2 = 0.1(x_0 + 0.1)(y_0 + k_1) = 0.1(0.1)(1) = 0.01$$
$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2) = 1 + \frac{1}{2}(0.01) = 1.005.$$



Therefore, the Runge-Kutta method of order two approximates that the value of $y$ at $x = 0.1$ is 1.005. Similarly, if $n = 1$, then

$$k_1 = 0.1x_1y_1 = 0.1(0.1)(1.005) = 0.01005$$
$$k_2 = 0.1(x_1 + 0.1)(y_1 + k_1) = 0.1(0.2)(1.01505) = 0.020301$$
$$y_2 = y_1 + \frac{1}{2}(k_1 + k_2) = 1.005 + \frac{1}{2}(0.01005 + 0.020301) = 1.0201755.$$

In Table 2.4, we display the results obtained for the other values on $0 \le x \le 1$ using the Runge-Kutta method of order two.

**TABLE 2.4 Runge-Kutta Method of Order Two with $h = 0.1$**

| $x_n$ | $y_n$ (RK) | Actual Value |
|---|---|---|
| 0.0 | 1.0 | 1.0 |
| 0.1 | 1.005 | 1.00501 |
| 0.2 | 1.0201755 | 1.0202 |
| 0.3 | 1.0459859 | 1.04603 |
| 0.4 | 1.083223 | 1.08329 |
| 0.5 | 1.1330513 | 1.13315 |
| 0.6 | 1.1970687 | 1.19722 |
| 0.7 | 1.277392 | 1.27762 |
| 0.8 | 1.3767731 | 1.37713 |
| 0.9 | 1.4987552 | 1.4993 |
| 1.0 | 1.6478813 | 1.64874 |

The terms of the power series expansions used in the derivation of the Runge-Kutta method of order two can be made to match up to order four. These computations are rather complicated, so they are not discussed here. However, after much work, the approximation at each step is found to be made with

$$y_{n+1} = y_n + \frac{h}{6}[k_1 + 2k_2 + 2k_3 + k_4],\ n = 0, 1, 2, \ldots,$$

where $k_1 = f(x_n, y_n)$, $k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{hk_1}{2}\right)$, $k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{hk_2}{2}\right)$, and $k_4 = f(x_{n+1}, y_n + hk_3)$.

### Runge-Kutta Method of Order Four

The solution of the initial-value problem

$$y' = f(x, y),\ y(x_0) = y_0$$

is approximated at the sequence of points $(x_n, y_n)$ $(n = 1, 2, \ldots)$, where $y_n$ is the approximate value of $y(x_n)$, by computing at each step

$$y_{n+1} = y_n + \frac{h}{6}[k_1 + 2k_2 + 2k_3 + k_4] \ (n = 0, 1, \ldots),$$

where $k_1 = f(x_n, y_n)$, $k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{hk_1}{2}\right)$, $k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{hk_2}{2}\right)$, $k_4 = f(x_{n+1}, y_n + hk_3)$, $x_n = x_0 + nh$, and $h$ is the selected stepsize.

**Example 5**

Use the fourth-order Runge-Kutta method with $h = 0.1$ to approximate the solution of $y' = xy$, $y(0) = 1$ on $0 \leq x \leq 1$.

Detailed discussions of error analysis as well as other numerical methods can be found in most numerical analysis texts.*

***Solution*** With $f(x, y) = xy$, $x_0 = 0$, and $y_0 = 1$, the formulas are

$$k_1 = f(x_n, y_n) = x_n y_n,$$

$$k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{hk_1}{2}\right) = \left(x_n + \frac{0.1}{2}\right)\left(y_n + \frac{0.1k_1}{2}\right),$$

$$k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{hk_2}{2}\right) = \left(x_n + \frac{0.1}{2}\right)\left(y_n + \frac{0.1k_2}{2}\right),$$

$$k_4 = f(x_{n+1}, y_n + hk_3) = x_{n+1}(y_n + 0.1k_3)$$

$$y_{n+1} = y_n + \frac{h}{6}[k_1 + 2k_2 + 2k_3 + k_4] = y_n + \frac{0.1}{6}[k_1 + 2k_2 + 2k_3 + k_4].$$

For $n = 0$, we have $k_1 = x_0 y_0 = (0)(1) = 0$, $k_2 = \left(x_0 + \frac{0.1}{2}\right)\left(y_0 + \frac{0.1k_1}{2}\right) = (0.05)(1) = 0.05$, $k_3 = \left(x_0 + \frac{0.1}{2}\right)\left(y_0 + \frac{0.1k_2}{2}\right) = (0.05)(1 + 0.0025) = 0.050125$, and $k_4 = x_1(y_0 + 0.1k_3) = (0.1)(1 + 0.0050125) = 0.10050125$. Therefore,

* See, for example, Richard L. Burden and J. Douglas Faires. *Numerical Analysis*. Third Edition, PWS Publishers (1985).



$$y_1 = y_0 + \frac{0.1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1 + \frac{0.1}{6}[0 + 2 \cdot 0.05 + 2 \cdot 0.050125 + 0.10050125] = 1.005012521.$$

In Table 2.5, we list the results for the Runge-Kutta method of order four to five decimal places. Notice that this method yields the most accurate approximation of the methods used to this point.

**TABLE 2.5 Fourth-Order Runge-Kutta Method with $h = 0.1$**

| $x_n$ | $y_n$ (RK Order 4) | $y_n$ (IEM) | Actual Value |
|---|---|---|---|
| 0.0 | 1.0 | 1.0 | 1.0 |
| 0.1 | 1.00501 | 1.005 | 1.00501 |
| 0.2 | 1.0202 | 1.0201755 | 1.0202 |
| 0.3 | 1.04603 | 1.0459859 | 1.04603 |
| 0.4 | 1.08329 | 1.083223 | 1.08329 |
| 0.5 | 1.13315 | 1.1330513 | 1.13315 |
| 0.6 | 1.19722 | 1.1970687 | 1.19722 |
| 0.7 | 1.27762 | 1.277392 | 1.27762 |
| 0.8 | 1.37713 | 1.3767731 | 1.37713 |
| 0.9 | 1.4993 | 1.4987552 | 1.4993 |
| 1.0 | 1.64872 | 1.6478813 | 1.64874 |

## EXERCISES 2.6

In Exercises 1–8, use Euler's method with $h = 0.1$ and $h = 0.05$ to approximate the solution at the given value of $x$.

1. $y' = 4y + 3x + 2$, $y(0) = 1$, $x = 1$
2. $y' = 4x - y + 1$, $y(0) = 0$, $x = 1$
*3. $y' - x = y^2 - 1$, $y(0) = 1$, $x = 1$
4. $y' + x = 5y^{1/2}$, $y(0) = 1$, $x = 1$
5. $y' = \sqrt{xy} + 5y$, $y(1) = 1$, $x = 2$
6. $y' = xy^{1/3} - y$, $y(1) = 1$, $x = 2$
*7. $y' = \sin y$, $y(0) = 1$, $x = 1$
8. $y' = \sin(y - x)$, $y(0) = 0$, $x = 1$

In Exercises 9–16, use the improved Euler's method with $h = 0.1$ and $h = 0.05$ to approximate the solution of the corresponding exercise above at the given value of $x$. Compare these results with those obtained in Exercises 1–8.

9. Exercise 1
10. Exercise 2
*11. Exercise 3
12. Exercise 4

13. Exercise 5
14. Exercise 6
*15. Exercise 7
16. Exercise 8

In Exercises 17–24, use the Runge-Kutta method with $h = 0.1$ and $h = 0.05$ to approximate the solution of the corresponding exercise above. Compare these results with those obtained in Exercises 1–16.

17. Exercise 1
18. Exercise 2
*19. Exercise 3
20. Exercise 4
21. Exercise 5
22. Exercise 6
*23. Exercise 7
24. Exercise 8
25. Consider the IVP $dy/dx = -y$, $y(0) = 1$. **(a)** Find the exact solution to the problem. What is $y(1)$? **(b)** Approximate $y(1)$ with Euler's method using $h = 0.1$, $h = 0.05$, and $h = 0.025$. **(c)** Do the values in **(b)** approach the exact value of $y(1)$ as $h$ approaches zero?
26. Repeat Exercise 25 using the improved Euler's method. Do the approximate values of $y(1)$ obtained with the improved Euler's method converge more quickly than those obtained with Euler's method?
27. Repeat Exercise 25 using the Runge-Kutta method of order four. Do the approximate values of y(1) obtained with this method converge more quickly than those obtained with the improved Euler's method and Euler's method?

As indicated in our examples, computers and computer algebra systems are of great use in implementing numerical techniques. In addition to being able to implement the algorithm illustrated earlier, many computer algebra systems contain built-in commands that you can use to implement various numerical methods.

28. Graph the solution to the initial-value problem $\begin{cases} \dfrac{dy}{dx} = \sin(2x - y) \\ y(0) = 0.5 \end{cases}$ on the interval $[0, 15]$.
*29. Graph the solution of $y' = \sin(xy)$ subject to the initial condition $y(0) = i$ on the interval $[0, 7]$ for $i = 0.5, 1.0, 1.5, 2.0$, and $2.5$. In each case, approximate the value of the solution if $x = 0.5$.
30. **(a)** Graph the direction field associated with
$$\frac{dy}{dx} = x^2 + y^2 \text{ for } -1 \le x \le 1 \text{ and } -1 \le y \le 1.$$
**(b)** Graph the solution to the initial-value problem
$$\begin{cases} \dfrac{dy}{dx} = x^2 + y^2 \\ y(0) = 0 \end{cases}$$
on the interval $[-1, 1]$.
**(c)** Approximate $y(1)$.

## CHAPTER 2 SUMMARY

### Concepts & Formulas

**Section 2.1**

**Separable Differential Equation**
A differential equation that can be written in the form $g(y)y' = f(x)$ or $g(y)\,dy = f(x)\,dx$ is called a **separable differential equation.**

**Section 2.2**

**First-Order Linear Differential Equation**
A differential equation that can be written in the form $\dfrac{dy}{dx} + p(x)y = q(x)$ is called a **first-order linear differential equation.**

**Integrating Factor**
An **integrating factor** for the first-order linear equation, $\dfrac{dy}{dx} + p(x)y = q(x)$ is $e^{\int p(x)dx}$.

**Section 2.3**

**Homogeneous Differential Equation (of degree $n$)**
A differential equation that can be written in the form $M(x, y)\,dx + N(x, y)\,dy = 0$, where
$$M(tx, ty) = t^n M(x, y) \quad \text{and} \quad N(tx, ty) = t^n N(x, y)$$
is called a **homogeneous differential equation (of degree $n$).**



**Bernoulli Equation**
A differential equation of the form
$$y' + p(x)y = q(x)y^n.$$

**Section 2.4**

**Exact Differential Equation**
A differential equation that can be written in the form
$$M(x, y)\,dx + N(x, y)\,dy = 0$$
where
$$M(x, y)\,dx + N(x, y)\,dy = \frac{\partial f}{\partial x}(x, y)\,dx + \frac{\partial f}{\partial y}(x, y)\,dy$$
for some function $f(x, y)$ is called an **exact differential equation.**

**Section 2.5**

**Existence and Uniqueness Theorem**
If $f$ and $\dfrac{\partial f}{\partial y}$ are continuous functions on the rectangular region $R$: $a < x < b$, $c < y < d$ containing the point $(x_0, y_0)$, then there exists an interval $|x - x_0| < h$ centered at $x_0$ on which there exists one and only one solution to the initial-value problem
$$y' = f(x, y),\ y(x_0) = y_0.$$

**Section 2.6**

**Euler Method**
The approximate value of $y$ at $x = x_n$ is
$$y_n = hf(x_{n-1}, y_{n-1}) + y_{n-1}$$
where $x_n = x_0 + nh$.

**Improved Euler's Method**
The approximate value of $y$ at $x = x_n$ is computed with
$$y_n^* = hf(x_{n-1}, y_{n-1}) + y_{n-1}$$
$$y_n = \frac{f(x_{n-1}, y_{n-1}) + f(x_n, y_n^*)}{2}h + y_{n-1}.$$

**Runge-Kutta Method of Order Two**
The approximate value of $y$ at $x = x_n$ is $y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$ where $k_1 = hf(x_n, y_n)$ and $k_2 = hf(x_n + h, y_n + k_1)$.

**Runge-Kutta Method of Order Four**
The approximate value of $y$ at $x = x_n$ is
$$y_{n+1} = y_n + \frac{h}{6}[k_1 + 2k_2 + 2k_3 + k_4],\ n = 0, 1, 2, \ldots,$$
where $k_1 = f(x_n, y_n)$, $k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{hk_1}{2}\right)$,
$k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{hk_2}{2}\right)$, and $k_4 = f(x_{n+1}, y_n + hk_3)$.

## CHAPTER 2 REVIEW EXERCISES

In Exercises 1–23, solve each equation.

1. $\dfrac{dy}{dx} = \dfrac{2x^5}{5y^2}$
2. $\cos 4x\,dx - 8\sin y\,dy = 0$
*3. $\dfrac{dy}{dx} = \dfrac{\sinh x}{2\cosh y}$
4. $\dfrac{dy}{dx} = \dfrac{e^{8y}}{x}$
5. $\dfrac{dy}{dx} = \dfrac{e^{5x}}{y^4}$
6. $(-x^{-5} + x^{-3})\,dx - (2y^4 - 6y^9)\,dy = 0$
*7. $\dfrac{dy}{dx} = \dfrac{y}{e^{2x}\ln y}$
8. $\dfrac{dy}{dx} = \dfrac{(4 - 7x)(2y - 3)}{(x - 1)(2x - 5)}$
9. $\dfrac{dy}{dx} = \dfrac{\cos 7x\sin(3x)}{e^y\cos 6y}$
10. $3x\,dx + (x - 4y)\,dy = 0$
*11. $(y - x)\,dx + (x + y)\,dy = 0$
12. $(y^2 - x^2)\,dx + (x + y)\,dy = 0$

13. $(2y - x)\,dx + (4y^2 - 2xy)\,dy = 0$

14. $\dfrac{dy}{dx} = \dfrac{y^2 + x^2}{xy}$

*15. $\dfrac{dy}{dx} = \dfrac{5xy}{y^2 + x^2}$

16. $(x^2 - y)\,dx + (y - x)\,dy = 0$

17. $(x^2y + \sin x)\,dx + \left(\dfrac{1}{3}x^3 - \cos y\right) dy = 0$

18. $(\tan y - x)\,dx + (x \sec^2 y + 1)\,dy = 0$

*19. $(x \ln y)\,dx + \left(\dfrac{x^2}{2y} + 1\right) dy = 0$

20. $y' + y = 5$ *21. $y' + xy = x$

22. $\dfrac{dx}{dy} + \dfrac{x}{y} = y^2$ 23. $y' + \dfrac{1}{x}y = \cos x$

Solve the following Bernoulli equations. (See Section 2.3.)

24. $\dfrac{dy}{dx} - y = xy^3$ *25. $\dfrac{dy}{dx} + y = e^x y^{-2}$

Solve the following Clairaut equations. (See Exercise 48, Section 2.3.)

26. $y = xy' + 3y'^4$ *27. $y - xy' = 2\ln(y')$

Solve the following Lagrange equations. (See Exercise 54, Section 2.3.)

28. $y - 2xy' = -2(y')^3$ *29. $y - 2xy' = -4(y')^2$

In Exercises 30–35, solve each initial-value problem.

30. $(2x - y - 2)\,dx + (2y - x)\,dy = 0$, $y(0) = 1$

31. $\cos(x - y)\,dx + (1 - \cos(x - y))\,dy = 0$, $y(\pi) = \pi$

32. $(ye^{xy} - 2x)\,dx + xe^{xy}\,dy = 0$, $y(0) = 0$

*33. $(\sin y - y\cos x)\,dx + (x\cos y - \sin x)\,dy = 0$, $y(\pi) = 0$

34. $y^2\,dx + (2xy - 2\cos y \sin y)\,dy = 0$, $y(0) = \pi$

35. $\left(\dfrac{y}{x} + \ln y\right) dx + \left(\dfrac{x}{y} + \ln x\right) dy = 0$, $y(1) = 1$

For each of the following initial-value problems, use **(a)** Euler's method, **(b)** the improved Euler's method, and **(c)** the Runge-Kutta method with $h = 0.05$ to approximate the solution to the initial-value problem on the indicated interval.

36. $y' = y^2 - x$, $y(0) = 0$, $[0, 1]$

37. $y' = \sqrt{x - y}$, $y(1) = 1$, $[1, 2]$

38. $y' = x + y^{1/3}$, $y(1) = 1$, $[1, 2]$

*39. $y' = \sin(xy)$, $y(0) = 1$, $[0, 1]$

40. Does the Existence and Uniqueness Theorem guarantee a unique solution to the following initial-value problems?

(a) $\dfrac{dy}{dx} = xy^3$, $y(0) = 0$

*(b) $\dfrac{dy}{dx} = xy^{-3}$, $y(0) = 0$

(c) $\dfrac{dy}{dx} = -\dfrac{y}{x - 2}$, $y(2) = 0$

41. **(Higher-Order Methods with Taylor Series Expansion)** Consider the initial-value problem

$$y' = f(x, y),\ y(x_0) = y_0.$$

Recall that the Taylor series expansion of $y$ about $x = x_0$ is given by

$$y(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \frac{y'''(x_0)}{3!}(x - x_0)^3 + \cdots.$$

We know the value of $y$ at the initial value of $x = x_0$, so we use this value to approximate $y$ at $x_1 = x_0 + h$, which is near $x_0$, in the following manner. We first evaluate the Taylor series at $x_1 = x_0 + h$ to yield

$$y(x_0 + h) = y(x_0) + y'(x_0)h + \frac{y''(x_0)}{2!}h^2 + \frac{y'''(x_0)}{3!}h^3 \cdots.$$

Substituting $y' = f(x, y)$, $y = \dfrac{df}{dx}(x, y)$, and $y''' = \dfrac{d^2f}{dx^2}(x, y)$ into this expansion, using $y(x_0) = y_0$ and calling this new value $y_1$, we have

$$\begin{aligned} y_1 = y(x_0 + h) &= y(x_0) + f(x_0, y_0)h + \frac{1}{2!}\frac{df}{dx}(x_0, y_0)h^2 + \frac{1}{3!}\frac{d^2f}{dx^2}(x_0, y_0)h^3 + \cdots \\ &= y_0 + f(x_0, y_0)h + \frac{1}{2!}\frac{df}{dx}(x_0, y_0)h^2 + \frac{1}{3!}\frac{d^2f}{dx^2}(x_0, y_0)h^3 + \cdots. \end{aligned}$$



Hence, the initial point $(x_0, y_0)$ is used to determine $y_1$. A first-order approximation is obtained from this series by disregarding the terms of order $h^2$ and higher. In other words, we determine $y_1$ from

$$y_1 = y_0 + f(x_0, y_0)h.$$

We next use the point $(x_1, y_1)$ to approximate the value of $y$ at $x_2 = x_1 + h$. Calling this value $y_2$, we have

$$y(x) = y(x_1) + y'(x_1)(x - x_1) + \frac{y''(x_1)}{2!}(x - x_1)^2 + \cdots$$

so that an approximate value of $y(x_2)$ is

$$y_2 = y(x_1) + f(x_1, y_1)h = y_1 + f(x_1, y_1)h.$$

Continuing this procedure, we have that the approximate value of $y$ at $x = x_n = x_{n-1} + h$ is given by

$$\begin{aligned} y_n &= y(x_{n-1}) + f(x_{n-1}, y_{n-1})h \\ &= y_{n-1} + f(x_{n-1}, y_{n-1})h. \end{aligned}$$

If we use the first three terms of the Taylor series expansion instead, then we obtain the approximation

$$\begin{aligned} y_n &= y(x_{n-1}) + f(x_{n-1}, y_{n-1})h + \frac{df}{dx}(x_{n-1}, y_{n-1})\frac{h^2}{2} \\ &= y_{n-1} + f(x_{n-1}, y_{n-1})h + \frac{df}{dx}(x_{n-1}, y_{n-1})\frac{h^2}{2}. \end{aligned}$$

We call the approximation which uses the first three terms of the expansion the **three-term Taylor method.**

**(a)** Use the three-term Taylor method with $h = 0.1$ to approximate the solution of the initial-value problem $y' = xy$, $y(0) = 1$ on $0 \le x \le 1$.

A four-term approximation is derived in a similar manner to be

$$\begin{aligned} y_n &= y(x_{n-1}) + f(x_{n-1}, y_{n-1})h + \frac{df}{dx}(x_{n-1}, y_{n-1})\frac{h^2}{2} + \frac{d^2f}{dx^2}(x_{n-1}, y_{n-1})\frac{h^3}{6} \\ &= y_{n-1} + f(x_{n-1}, y_{n-1})h + \frac{df}{dx}(x_{n-1}, y_{n-1})\frac{h^2}{2} + \frac{d^2f}{dx^2}(x_{n-1}, y_{n-1})\frac{h^3}{6}. \end{aligned}$$

**(b)** Use the four-term method with $h = 0.1$ to approximate the solution of $y' = 1 + y + x^2$, $y(0) = 0$ on $0 \le x \le 1$.

42. **(Running Shoes)** The rate of change of energy absorption in a running shoe is given by

$$\frac{dE}{dt} = F\frac{du}{dt},$$

where

$$F(t) = \frac{F_0}{2}[1 - \cos(\omega t)]$$

represents the force magnitude and pulse duration exerted on the shoe and

$$\frac{du}{dt} = \frac{u_0\omega}{2}\sin(\omega t - \delta)$$

represents the rate of change of the vertical displacement of the midsole.* The constant $F_0$ represents the maximum magnitude of the input force in Newtons (N), $\omega$ represents the frequency of the input profile in radians per second (rad/s), $u_0$ represents the maximum rate of change of the vertical displacement of the midsole in meters (m), $\delta$ represents the phase angle between $F(t)$ and $u(t)$ in radians (rad), and $t$ represents time in seconds (s). Thus, the rate of change of energy absorbed by the shoe is given by

$$\frac{dE}{dt} = \frac{F_0u_0\omega}{4}[1 - \cos(\omega t)]\sin(\omega t - \delta).$$

**(a)** Find the energy absorbed by the shoe from $t = t_1$ to $t = t_2$.

**(b)** The **maximum energy absorbed by the shoe,** *ME*, is found by calculating the energy absorbed if $t_1 = \delta/\omega$ and $t_2 = (\pi + \delta)/\omega$. Show that

$$ME = \frac{F_0u_0}{2}\left(1 + \frac{\pi}{4}\sin\delta\right).$$

43. **(Fermentation)** In the fermentation industry, one of the goals of the molecular biologist is to control the environment and regulate the fermentation. To achieve meaningful environmental control, fermentation research must be carried out on fully monitored environmental systems, the environmental observations must be correlated with existing knowledge of cellu-

* John F. Swigart, Arthur G. Erdman, and Patrick J. Cain, "An Energy-Based Method for Testing Cushioning Durability of Running Shoes," *Journal of Applied Biomechanics*, Volume 9 (1993), pp. 47–65.

lar control mechanisms, and environmental control conditions must be reproduced through continuous computer monitoring, analysis, and feedback control of the fermentation environment.

One component of environmental control is the measurement of dissolved oxygen with a steam-sterilizable dissolved oxygen sensor made up of a polymer membrane-covered electrode. Suppose that the electrode is immersed in a liquid medium and that the oxygen is reduced according to the chemical reaction

$$O_2 + 2H_2O + 4e \rightarrow 4OH^-$$

The rate of change $d\overline{C}/dt$ in dissolved oxygen concentration at a particular point in a fermentor vessel is given by

$$\frac{d\overline{C}}{dt} = k_La(C^* - \overline{C}) - Q_{O_2}X,$$

where $C^*$ is the concentration of dissolved oxygen that is in equilibrium with partial pressure $\overline{p}$ in bulk gas phase, $\overline{C}$ is the concentration of dissolved oxygen in bulk liquid, $k_La$ is the volumetric oxygen transfer coefficient, $Q_{O_2}$ is the specific rate of oxygen uptake (microbial respiration), and $X$ is the cell mass concentration. If there are no cells present, the equation becomes

$$\frac{d\overline{C}}{dt} = k_La(C^* - \overline{C}).^*$$

**(a)** If $\overline{C}(0) = 0$, determine $\overline{C}(t)$ as a function of $C^*$ and $k_La$.

**(b)** Suppose that $\overline{C}_p$ is the concentration of dissolved oxygen that corresponds to the sensor reading and $k_p$ is the sensor constant that depends on the conductance of the membrane and the liquid film outside the sensor. If $\overline{C}_p$ satisfies the equation $d\overline{C}_p/dt = k_p(\overline{C} - \overline{C}_p)$ and $\overline{C}_p(0) = 0$, determine $\overline{C}_p(t)$ as a function of $k_p$, $k_La$, and $C^*$. By knowing $k_p$ in advance and by observing $\overline{C}_p$ experimentally, the value of $k_La$ can be estimated with the solution of this initial-value problem. Thus, the amount of dissolved oxygen can be determined.

## Differential Equations at Work:

### A. Modeling the Spread of a Disease

Suppose that a disease is spreading among a population of size $N$. In some diseases, like chicken pox, once an individual has had the disease, the individual becomes immune to the disease. In other diseases, like most venereal diseases, once an individual has had the disease and recovers from it, the individual does not become immune to the disease. Subsequent encounters can lead to recurrence of the infection.

Let $S(t)$ denote the percentage of the population susceptible to a disease at time $t$, $I(t)$ the percentage of the population infected with the disease, and $R(t)$ the percentage of the population unable to contract the disease. For example, $R(t)$ could represent the percentage of persons who have had a particular disease, recovered, and have subsequently become immune to the disease.

To model the spread of various diseases, we begin by making several assumptions and introducing some notation.

* S. Aiba, A. E. Humphrey, N. F. Millis, *Biochemical Engineering*, Second Edition, Academic Press (1973), pp. 317–336.



1. Susceptible and infected individuals die at a rate proportion to the number of susceptible and infected individuals with proportionality constant $\mu$ called the **daily death removal rate;** the number $1/\mu$ is the **average lifetime** or **life expectancy.**
2. The constant $\lambda$ represents the **daily contact rate.** On average, an infected person will spread the disease to $\lambda$ people per day.
3. Individuals recover from the disease at a rate proportional to the number infected with the disease with proportionality constant $\gamma$. The constant $\gamma$ is called the **daily recovery removal rate;** the **average period of infectivity** is $1/\gamma$.
4. The **contact number** $\sigma = \lambda/(\gamma + \mu)$ represents the average number of contacts an infected person has with both susceptible and infected persons.

If a person becomes susceptible to a disease after recovering from it (such as gonorrhea, meningitis, or streptococcal sore throat), then the percentage of people susceptible to becoming infected with the disease, $S(t)$, and the percentage of people in the population infected with the disease, $I(t)$, can be modeled by the system

$$\begin{cases} S'(t) = -\lambda IS + \gamma I + \mu - \mu S \\ I'(t) = \lambda IS - \gamma I - \mu I \\ S(0) = S_0,\ I(0) = I_0,\ S(t) + I(t) = 1 \end{cases}$$

This model is called an **SIS** (susceptible-infected-susceptible) model because once an individual has recovered from the disease, the individual again becomes susceptible to the disease.*

Since $S(t) = 1 - I(t)$, we can write $I'(t) = \lambda IS - \gamma I + \mu I$ as

$$I'(t) = \lambda I(1 - I) - \gamma I - \mu I$$

and thus we need to solve the initial-value problem

$$\begin{cases} I'(t) = [\lambda - (\gamma + \mu)]I - \lambda I^2 \\ I(0) = I_0 \end{cases}$$

1. Convert the Bernoulli equation $I'(t) = \lambda I(1 - I) - \gamma I - \mu I$ to a linear equation and solve the result with the substitution $y = I^{-1}$.
2. **(a)** Show that the solution to the initial-value problem

$$\begin{cases} I'(t) = [\lambda - (\gamma + \mu)]I - \lambda A^2 \\ I(0) = I_0 \end{cases}$$

is

$$I(t) = \begin{cases} \dfrac{e^{(\lambda+\mu)(\sigma-1)t}}{\sigma[e^{(\lambda+\mu)(\sigma-1)t} - 1]/(\sigma - 1) + 1/I_0}, & \text{if } \sigma \neq 1 \\ \dfrac{1}{\lambda t + 1/I_0}, & \text{if } \sigma = 1 \end{cases}$$

and graph various solutions if **(b)** $\lambda = 3.6$, $\gamma = 2$, and $\mu = 1$; **(c)** $\lambda = 3.6$, $\gamma = 2$,

* Herbert W. Hethcote, "Three Basic Epidemiological Models," *Applied Mathematical Ecology*, edited by Simon A. Levin, Thomas G. Hallan, and Louis J. Gross, Springer-Verlag (1989), pp. 119–143.

and $\mu = 2$. In each case, find the contact number. How does the contact number affect $I(t)$ for large values of $t$?

3. Evaluate $\lim_{t\to\infty} I(t)$.

The incidence of some diseases, such as measles, rubella, and gonorrhea, oscillates seasonally. To model these diseases, we may wish to replace the constant contact rate $\lambda$ by a periodic function $\lambda(t)$.

4. Graph various solutions if **(a)** $\lambda(t) = 5 - 2\sin(6t)$, $\gamma = 1$, and $\mu = 4$; and **(b)** $\lambda(t) = 5 - 2\sin(6t)$, $\gamma = 1$, and $\mu = 2$. In each case, calculate the average contact number. How does the average contact number affect $I(t)$ for large values of $t$?
5. Explain why diseases such as gonorrhea, meningitis, and streptococcal sore throat continue to persist in the population. Do you think there is any way to eliminate these diseases completely from the population? Why or why not?

## B. Linear Population Model with Harvesting

1. Consider: $y' = ay - h$, $y(0) = y_0$, where $a$ and $h$ are positive constants. The value of $a$ represents the growth rate, while that of $h$ represents the fishing (or harvesting) rate. The solution $y(t)$ represents the size of a population at time $t$.
   **(a)** Determine the equilibrium solution.
   **(b)** Sketch the phase line.
   **(c)** Solve the initial-value problem.
   **(d)** If $y_0 > h/a$, determine $\lim_{t\to\infty} y(t)$.
   **(e)** If $y_0 = h/a$, determine $\lim_{t\to\infty} y(t)$.
   **(f)** If $y_0 < h/a$, determine the value of $t$ such that $y(t) = 0$. What does this mean in terms of the population?
2. Consider $y' = \frac{1}{2}y - 1$, $y(0) = y_0$.
   **(a)** Determine the equilibrium solution.
   **(b)** Sketch the phase line.
   **(c)** Solve the initial-value problem.
   **(d)** If $y_0 = 3$, determine $\lim_{t\to\infty} y(t)$.
   **(e)** If $y_0 = 2$, determine $\lim_{t\to\infty} y(t)$.
   **(f)** If $y_0 = 1.5$, determine the value of $t$ such that $y(t) = 0$. What does this mean in terms of the population?
3. Consider the model $y' = y - 1$, $y(0) = 0.5$, and the model $y' = \frac{1}{2}y - 1$, $y(0) = 0.5$. In which model do you predict that the population will become extinct more quickly? Verify your response.
4. Consider the model $y' = \frac{1}{2}y - \frac{1}{2}$, $y(0) = 0.5$, and the model $y' = \frac{1}{2}y - 1$, $y(0) = 0.5$. In which model do you predict that the population will become extinct more quickly? Verify your response.
5. Suppose that you own a "fish farm" and that during the first year, the growth rate of the population is $a = 1/2$ and the harvesting rate is $h = 1/2$. However, during the



second year, instead of harvesting, you decide to add fish at a rate of $r = 1/2$. If $y(0) = 0.5$, find $y(2)$. How does this value compare to the original size of the population? What (approximate) value of $r$ should be used so that $y(2) = y(0)$?
6. Suppose that in problem 5 above, you do not fish or add to the population after the first year. If $y(0) = 0.5$, find $y(2)$. How does this value compare to the original size of the population? How much time $T$ is required so that $y(T) = y(0)$?
7. Suppose that fishing occurs on a seasonal basis so that the situation is modeled with $y' = ay - h\left(1 - \sin\frac{\pi t}{6}\right)$, $y(0) = y_0$. What is the maximum value, minimum value, and period of the harvesting function $h\left(1 - \sin\frac{\pi t}{6}\right)$? Using the indicated parameter values, describe what happens to the size of the population over one year
   **(a)** $a = 1/10$, $h = 1$, $y(0) = 20$
   **(b)** $a = 1/10$, $h = 2$, $y(0) = 20$
   **(c)** $a = 1/10$, $h = 4$, $y(0) = 20$
   **(d)** $a = 1/10$, $h = 10$, $y(0) = 20$
8. Suppose that a situation is modeled with $y' = ay - h\cos\frac{\pi t}{6}$, $y(0) = y_0$. That is, at some times, fishing occurs while at others fish are added. During what time intervals does fishing occur? During what time intervals are fish added? What is the maximum value, minimum value, and period of the function, $h\cos\frac{\pi t}{6}$? Using the indicated parameter values, describe what happens to the size of the population over one year.
   **(a)** $a = 1/10$, $h = 1$, $y(0) = 50$
   **(b)** $a = 1/10$, $h = 4$, $y(0) = 50$
   **(c)** $a = 1/10$, $h = 10$, $y(0) = 50$
   **(d)** $a = 1/10$, $h = 20$, $y(0) = 50$

## C. Logistic Model with Harvesting

1. Consider the logistic model with harvesting, $y' = ay - cy^2 - h$, $y(0) = y_0$, where $a$ represents the growth rate, $c$ the inhibitive factor, and $h$ the harvesting rate. Each of these constants is positive.
   **(a)** Find the equilibrium solutions by solving $ay - cy^2 - h = 0$. (What condition must hold so that there are two real solutions to this equation? What condition must hold so that there is only one equilibrium solution? What condition must hold so that there is no equilibrium solution?)
   **(b)** Sketch the phase line assuming that there are two equilibrium solutions.
   **(c)** Sketch the phase line assuming that there is only one equilibrium solution.
   **(d)** What happens to the population if there is no equilibrium solution?
2. Consider the problem, $y' = \frac{7}{10}y - \frac{1}{10}y^2 - 1$, $y(0) = y_0$.
   **(a)** What are the two equilibrium solutions?
   **(b)** Sketch the phase line.

(c) If $y(0) = 3$, then determine $\lim_{t\to\infty} y(t)$.
(d) If $y(0) = 6$, then determine $\lim_{t\to\infty} y(t)$.
(e) If $y(0) = 1$, what do you expect to happen to the population?

3. Consider the problem, $y' = \frac{7}{10}y - \frac{1}{10}y^2 - h$, $y(0) = y_0$.
(a) What value of $h$, $h = h_0$, causes this model to have only one equilibrium solution? (This value is called a **bifurcation point** because the dynamics of the model change for values slightly larger and smaller than this value.)
(b) Describe solutions to the model with $h = h_0$.
(c) Describe solutions to the model with $h < h_0$.
(d) Describe solutions to the model with $h > h_0$.

4. Suppose that over the first year, the rate of change in the size of a fish population is given by $y' = \frac{7}{10}y - \frac{1}{10}y^2 - \frac{1}{2}$, where $y(0) = 0.5$. After the first year, however, the growth rate remains the same while no fishing is allowed. What is the size of the population after five years? Describe what happens to the population size on the interval $0 \le t \le 5$. Compare this result to population that follows $y' = \frac{7}{10}y - \frac{1}{10}y^2 - \frac{1}{2}$, where $y(0) = 0.5$ over the entire interval $0 \le t \le 5$. (That is, fishing continues after the first year.)

5. Suppose that over the first year, the rate of change in the size of a fish population is given by $y' = \frac{13}{10}y - \frac{1}{10}y^2 - \frac{1}{2}$, where $y(0) = 0.5$. After the first year, however, the growth rate remains the same while no fishing is allowed. What is the size of the population after five years? Describe what happens to the population size on the interval $0 \le t \le 5$. Compare this result to population that follows $y' = \frac{13}{10}y - \frac{1}{10}y^2 - \frac{1}{2}$, where $y(0) = 0.5$ over the entire interval $0 \le t \le 5$. (That is, fishing continues after the first year.)

6. Consider the model $y' = \frac{13}{10}y - \frac{1}{10}y^2 - \frac{1}{2}\sin\frac{\pi t}{6}$, $y(0) = a$, in which fishing occurs at certain times while at other times, fish are added. Investigate the solution to this IVP over $0 \le t \le 12$ for $d = 2, 4, 6, 8, 10, 12, 14$. What eventually happens to the size of the population in each case?

7. Consider the model $y' = \frac{13}{10}y - \frac{1}{10}y^2 - \frac{1}{2}\left(1 - \sin\frac{\pi t}{6}\right)$, $y(0) = a$, in which fishing occurs at different rates. Investigate the solution to this IVP over $0 \le t \le 12$ for $a = 2, 4, 6, 8, 10, 12, 14$. What eventually happens to the size of the population in each case?

8. Consider the model $y' = \frac{7}{10}y - \frac{3}{10}y^2 - \frac{1}{2}\sin\frac{\pi t}{6}$, $y(0) = a$, in which fishing occurs at certain times while at other times, fish are added. Investigate the solution to this IVP over $0 \le t \le 12$ for $a = 2, 4, 6, 8, 10, 12, 14$. What eventually happens to the size of the population in each case?

9. Consider the model $y' = \frac{7}{10}y - \frac{3}{10}y^2 - \frac{1}{2}\left(1 - \sin\frac{\pi t}{6}\right)$, $y(0) = a$, in which fishing occurs at different rates. Investigate the solution to this IVP over $0 \le t \le 10$ for $a = 2, 4, 6, 8, 10, 12, 14$. What eventually happens to the size of the population in each case?



## D. Logistic Model with Predation

One of the more predominant pests in Canadian forests is the spruce budworm. These insects, which are moths in the adult stage, lay eggs in the needles of the trees. After hatching, the larvae eventually tunnel into the old foliage of the tree. Later, they spin a webbing in the needles and devour new growth until they are interrupted. Although they do not destroy the tree, budworms weaken the trees and make them susceptible to disease and forest fires. We can model the rate of change in the size of the budworm population by building on the logistic model. Let $W(t)$ represent the size of the budworm population at time $t$. One factor that helps control the budworm population is predation by birds (that is, birds eat budworms). Consider the model,

$$\frac{dW}{dt} = rW\left(1 - \frac{1}{k}W\right) - P(W),$$

where $P(W)$ is a function of $W$ describing the rate of predation. Notice that for low population levels, the model demonstrates exponential growth rate $r$. Based on the logistic model, $k$ represents the carrying capacity of the forest. In the absence of predation, this gives a maximum size of the budworm population.

**Properties of the Predation Function** To develop a formula for $P(W)$, we make two assumptions. First, if $W = 0$, then $P(W) = 0$ (that is, if there are no budworms, the birds have none to eat). Second, $P(W)$ has a limiting value (that is, the birds have a limited appetite. Even if the budworm population is large, the birds eat only what they need). A function that satisfies both of these properties is $P(W) = (aW^2)/(b^2 + W^2)$. Therefore, our model becomes:

$$\frac{dW}{dt} = rW\left(1 - \frac{1}{k}W\right) - \frac{aW^2}{b^2 + W^2}, \quad W(0) = W_0.$$

1. What is the limiting value of $P(W) = (aW^2)/(b^2 + W^2)$ as $W \to \infty$?
2. Using the indicated parameter values, approximate the equilibrium solutions. Sketch a phase line in each case. **(a)** $r = 1$, $k = 15$, $a = 5$, $b = 2$. **(b)** $r = 1$, $k = 20$, $a = 5$, $b = 2$.
3. Using the values $r = 1$, $a = 5$, $b = 2$, approximate the value of $k$ that leads to three equilibrium solutions. (Do this experimentally by selecting values for $k$ and then plotting. This value is a bifurcation point.) Sketch the phase line.
4. Using various initial conditions, generate numerical solutions to the two situations in problem 2 above and to that in problem 3. Compare these results to the corresponding phase line.
5. Suppose that your job is to determine the carrying capacity of the forest. How important is it that you do a good job? (Use the three cases above for guidance.)

6. Discuss the differences in the following three models:

(a) $y' = 0.48y\left(1 - \frac{1}{15}y\right) - \frac{2y^2}{4+y^2}$; (b) $y' = 0.48y\left(1 - \frac{1}{17}y\right) - \frac{2y^2}{4+y^2}$;

(c) $y' = 0.48y\left(1 - \frac{1}{15.5}y\right) - \frac{2y^2}{4+y^2}$.

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# 3

# Applications of First-Order Differential Equations

When a space shuttle is launched from the Kennedy Space Center, the minimum initial velocity needed for the shuttle to escape the Earth's atmosphere is determined by solving a first-order differential equation. The same can be said for finding the flow of electromagnetic forces, the temperature of a cup of coffee, the population of a species, and numerous other applications. In this chapter, we show how these problems can be expressed as first-order equations. We focus our attention on setting up problems and explaining the meaning of the subsequent solutions because techniques for solving the first-order equations were discussed in Chapter 2.

## 3.1 Population Growth and Decay

Logistic Equation  Population Model with a Threshold

As we discussed in Section 2.1, the **Malthus model,**

$$\begin{cases} \dfrac{dy}{dt} = ky \\ y(0) = y_0 \end{cases},$$

can be used to investigate population growth if $k > 0$. It can also be applied to problems involving the decay of radioactive material if $k < 0$.

Forms of a given element with different numbers of neutrons are called **nuclides.** Some nuclides are not stable. For example, potassium-40 ($^{40}K$) naturally decays to reach argon-40 ($^{40}Ar$). This decay that occurs in some nuclides was first observed, but not understood, by Henri Becquerel (1852–1908) in 1896. Marie Curie, however, began studying this decay in 1898, named it **radioactivity,** and discovered the radioactive substances polonium and radium. Marie Curie (1867–1934), along with her husband Pierre Curie (1859–1906) and Henri Becquerel, received the Nobel Prize in Physics in 1903 for their work on radioactivity. Marie Curie subsequently received the Nobel Prize in Chemistry in 1910 for discovering polonium and radium.

Given a sample of $^{40}K$ of sufficient size, after $1.2 \times 10^9$ years, approximately half of the sample will have decayed to $^{40}Ar$. The **half-life** of a nuclide is the time for half the nuclei in a given sample to decay (see Table 3.1). We see that the rate of decay of a nuclide is proportional to the amount present because the half-life of a given nuclide is constant and independent of the sample size.

**Marie Curie** (1867–1934) and **Pierre Curie** (1859–1906) in their Paris laboratory (1896).
*(AIP Emilio Segrè Visual Archives)*

**Henri Becquerel** (1852–1908) discovered radioactivity in 1896. Marie Curie, Pierre Curie, and Henri Becquerel shared the Nobel Prize in Physics in 1903 for their work on radioactivity.
*(North Wind Picture Archives)*

**TABLE 3.1 Half-Lives of Various Nuclides**

| Element | Nuclide | Half-Life | Element | Nuclide | Half-Life |
|---|---|---|---|---|---|
| Aluminum | $^{26}Al$ | $7.4 \times 10^5$ years | Polonium | $^{209}Po$ | 100 years |
| Beryllium | $^{10}Be$ | $1.51 \times 10^6$ years | Polonium | $^{210}Po$ | 138 days |
| Carbon | $^{14}C$ | 5730 years | Radon | $^{222}Rn$ | 3.82 days |
| Chlorine | $^{36}Cl$ | $3.01 \times 10^5$ years | Radium | $^{226}Ra$ | 1700 years |
| Iodine | $^{131}I$ | 8.05 days | Thorium | $^{230}Th$ | 75,000 years |
| Potassium | $^{40}K$ | $1.2 \times 10^9$ years | Uranium | $^{238}U$ | $4.51 \times 10^9$ years |

**Example 1**

If the half-life of polonium, $^{209}Po$, is 100 years, determine the percentage of the original amount of $^{209}Po$ that remains after 50 years.



***Solution*** Let $y_0$ represent the original amount of $^{209}Po$ that is present. The amount present after $t$ years is $y(t) = y_0e^{kt}$. Using $y(100) = \frac{1}{2}y_0$ and $y(100) = y_0e^{100k}$, we solve $y_0e^{100k} = \frac{1}{2}y_0$ for $e^k$:

$$e^{100k} = \frac{1}{2}$$

$$(e^k)^{100} = \frac{1}{2}$$

$$e^k = \left(\frac{1}{2}\right)^{1/100}.$$

Hence,

$$y(t) = y_0e^{kt} = y_0(e^k)^t = y_0\left(\frac{1}{2}\right)^{t/100}.$$

To determine the percentage of $y_0$ that remains, we evaluate

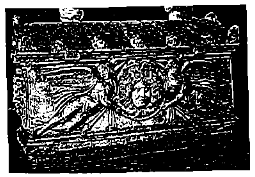
$$y(50) = y_0\left(\frac{1}{2}\right)^{50/100} \approx 0.7071y_0.$$

Therefore, 70.71% of the original amount of $^{209}Po$ remains after 50 years.

In Example 1, we determined the percentage of the original amount of $^{209}Po$ that remains even though we do not know the value of $y_0$, the initial amount of $^{209}Po$. Instead of letting $y(t)$ represent the *amount* of the substance present after time $t$, we can let it represent the *fraction* (or *percentage*) of $y_0$ that remains after time $t$. In doing so, we use the initial condition $y(0) = 1 = 1.00$ to indicate that 100% of $y_0$ is present at $t = 0$.

**Example 2**

The wood of an Egyptian sarcophagus (burial case) is found to contain 63% of the carbon-14, $^{14}C$, found in a present-day sample. What is the age of the sarcophagus?



**Sarcophagus** A Roman sarcophagus in the Antayla Museum, Perge, Turkey.
*(Borys Malkin/Anthro-Photo)*

***Solution*** From Table 3.1, we see that the half-life of $^{14}C$ is 5730 years. Let $y(t)$ be the percent of $^{14}C$ in the sample after $t$ years. Then $y(0) = 1$. Now, $y(t) = y_0e^{kt}$, so $y(5730) = e^{5730k} = 0.5$. Solving for $k$ yields:

$$\ln(e^{5730k}) = \ln(.5)$$

$$5730k = \ln(.5)$$

$$k = \frac{\ln(.5)}{5730} = \frac{-\ln 2}{5730}.$$

Thus, $y(t) = e^{kt} = e^{\frac{-\ln 2}{5730}t} = 2^{-t/5730}$. (An alternate approach to obtain the solution is to solve $e^{5730k} = 0.5$ for $e^k$ instead of for $k$, as we did in Example 1. This yields $e^k = (.5)^{1/5730} = (\frac{1}{2})^{1/5730} = 2^{-1/5730}$. Substitution of this expression into $y(t) = y_0e^{kt} = y_0(e^k)^t$ gives the same solution as found previously.)

In this problem, we must find the value of $t$ for which $y(t) = 0.63$. Solving this equation results in:

$$2^{-t/5730} = 0.63 = \tfrac{63}{100}$$

$$\ln(2^{-t/5730}) = \ln \tfrac{63}{100}$$

$$\frac{-t}{5730}\ln 2 = \ln \tfrac{63}{100}$$

$$t = \frac{-5730 \ln \frac{63}{100}}{\ln 2} = \frac{5730(\ln 100 - \ln 63)}{\ln 2} \approx 3819.48.$$

We conclude that the sarcophagus is approximately 3819 years old.

We can use the Malthus model to predict the size of a population at a given time if the rate of growth of the population is proportional to the present population.

**Example 3**

Suppose that the number of cells in a bacteria culture doubles after three days. Determine the number of days required for the initial population to triple.

***Solution*** In this case, $y(0) = y_0$, so the population is given by $y(t) = y_0e^{kt}$ and $y(3) = 2y_0$ because the population doubles after three days. Substituting this value into $y(t) = y_0e^{kt}$, we have

$$y(3) = y_0e^{3k} = 2y_0.$$

Solving for $e^k$, we find that $e^{3k} = 2$ or $(e^k)^3 = 2$. Therefore, $e^k = 2^{1/3}$. Substitution into $y(t) = y_0e^{kt}$ then yields

$$y(t) = y_0(e^k)^t = y_02^{t/3}.$$

We find when the population triples by solving $y(t) = y_02^{t/3} = 3y_0$ for $t$. This yields $2^{t/3} = 3$ or $\frac{t}{3}\ln 2 = \ln 3$. Therefore, the population triples in $t = \dfrac{3\ln 3}{\ln 2} \approx 4.755$ days.

*Determine the number of days required for the culture of bacteria considered in Example 3 to reach nine times its initial size.*

To observe some of the limitations of the Malthus model, we consider a population problem in which the rate of growth of the population does *not* exclusively depend on the present population.



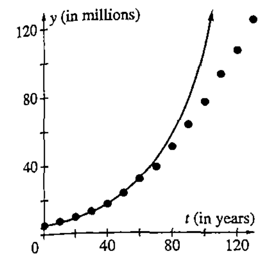



Figure 3.1

**Example 4**

The population of the United States was recorded as 5.3 million in 1800. Use the Malthus model to approximate the population for years after 1800 if $k = 0.03$. Compare these results to the actual population. Is this a good approximation for years after 1800?

***Solution*** In this example, $k = 0.03$, $y_0 = 5.3$, and our model for the population of the United States at time $t$ (where $t$ is the number of years from 1800) is $y(t) = 5.3e^{0.03t}$. To compare this model with the actual population of the United States, census figures for the population of the United States for various years are listed in Table 3.2 along with the corresponding value of $y(t)$. A graph of $y(t)$ with the corresponding points is shown in Figure 3.1.

**TABLE 3.2 U.S. Population and Values of $y(t)$**

| Year ($t$) | Actual Population (in millions) | Value of $y(t) = 5.3e^{0.03t}$ | Year ($t$) | Actual Population (in millions) | Value of $y(t) = 5.3e^{0.03t}$ |
|---|---|---|---|---|---|
| 1800 (0) | 5.30 | 5.30 | 1870 (70) | 38.56 | 43.28 |
| 1810 (10) | 7.24 | 7.15 | 1880 (80) | 50.19 | 58.42 |
| 1820 (20) | 9.64 | 9.66 | 1890 (90) | 62.98 | 78.86 |
| 1830 (30) | 12.68 | 13.04 | 1900 (100) | 76.21 | 106.45 |
| 1840 (40) | 17.06 | 17.60 | 1910 (110) | 92.23 | 143.70 |
| 1850 (50) | 23.19 | 23.75 | 1920 (120) | 106.02 | 193.97 |
| 1860 (60) | 31.44 | 32.06 | 1930 (130) | 123.20 | 261.83 |

Although the model appears to approximate the data for several years after 1800, the accuracy of the approximation diminishes over time because the population of the United States does not exclusively increase at a rate proportional to the population. Another model that better approximates the population would take other factors into account.

## Logistic Equation

Because the approximation obtained with the Malthus model is less than desirable in the previous example, we see that another model is needed. The **logistic equation** (or **Verhulst equation**), which was mentioned in Section 2.3, is the equation

$$y'(t) = (r - ay(t))y(t),$$

where $r$ and $a$ are constants, subject to the condition $y(0) = y_0$. This equation was first introduced by the Belgian mathematician Pierre Verhulst to study population growth. The logistic equation differs from the Malthus model in that the term $(r - ay(t))$ is not constant. This equation can be written as $dy/dt = (r - ay)y = ry - ay^2$ where the term $(-y^2)$ represents an inhibitive factor. Under these assumptions the population is neither allowed to grow out of control nor grow or decay constantly as it was with the Malthus model.

Here, we solve the logistic equation as a separable equation. For convenience, we write the equation as

$$y' = (r - ay)y \quad \text{or} \quad \frac{dy}{dt} = (r - ay)y.$$

Separating variables and using partial fractions to integrate with respect to $y$, we have

$$\frac{dy}{(r - ay)y} = dt$$

$$\left(\frac{a/r}{r - ay} + \frac{1/r}{y}\right) dy = dt$$

$$\left(\frac{a}{r - ay} + \frac{1}{y}\right) dy = r\,dt$$

$$-\ln|r - ay| + \ln|y| = rt + c.$$

We solve this expression for $y$ using the properties of logarithms:

$$\ln\left|\frac{y}{r - ay}\right| = rt + c$$

$$\frac{y}{r - ay} = \pm e^{rt+c} = Ke^{rt} \qquad (K = \pm e^c)$$

$$y = r\left(\frac{1}{K}e^{-rt} + a\right)^{-1}.$$

Applying the initial condition $y(0) = y_0$ and solving for $K$, we find that

$$\frac{y_0}{r - ay_0} = K.$$

After substituting this value into the general solution and simplifying, the solution can be written as

$$y = \frac{ry_0}{ay_0 + (r - ay_0)e^{-rt}}.$$

Notice that $\lim_{t\to\infty} y(t) = r/a$ because $\lim_{t\to\infty} e^{-rt} = 0$ if $r > 0$. This makes the solution to the logistic equation different from that of the Malthus model in that the solution to the logistic equation approaches a finite nonzero limit as $t \to \infty$ while that of the Malthus model approaches either infinity or zero as $t \to \infty$.



*Use a computer algebra system to solve the logistic equation. If the result you obtain is not in the same form as that given above, show (by hand) that the two are the same.*

**Example 5**

Use the logistic equation to approximate the population of the United States using $r = 0.03$, $a = 0.0001$, and $y_0 = 5.3$. Compare this result with the actual census values given in Table 3.2. Use the model obtained to predict the population of the United States in the year 2010.

***Solution*** We substitute the indicated values of $r$, $a$, and $y_0$ into $y = \dfrac{ry_0}{ay_0 + (r - ay_0)e^{-rt}}$ to obtain the approximation of the population of the United States at time $t$, where $t$ represents the number of years since 1800,

$$y(t) = \frac{0.03 \cdot 5.3}{0.0001 \cdot 5.3 + (0.03 - 0.0001 \cdot 5.3)e^{-.03t}} = \frac{0.159}{0.00053 + 0.02947e^{-.03t}}.$$

In Table 3.3, we compare the approximation of the population of the United States given by the approximation $y(t)$ with the actual population obtained from census figures. Note that this model appears to approximate the population more closely over a longer period of time than the Malthus model did (Example 4), as we can

**TABLE 3.3 U.S. Population and Values of $y(t)$**

| Year ($t$) | Actual Population (in millions) | Value of $y(t)$ | Year ($t$) | Actual Population (in millions) | Value of $y(t)$ |
|---|---|---|---|---|---|
| 1800 (0) | 5.30 | 5.30 | 1900 (100) | 76.21 | 79.61 |
| 1810 (10) | 7.24 | 7.11 | 1910 (110) | 92.23 | 98.33 |
| 1820 (20) | 9.64 | 9.52 | 1920 (120) | 106.02 | 119.08 |
| 1830 (30) | 12.68 | 12.71 | 1930 (130) | 123.20 | 141.14 |
| 1840 (40) | 17.06 | 16.90 | 1940 (140) | 132.16 | 163.59 |
| 1850 (50) | 23.19 | 22.38 | 1950 (150) | 151.33 | 185.45 |
| 1860 (60) | 31.44 | 29.44 | 1960 (160) | 179.32 | 205.82 |
| 1870 (70) | 38.56 | 38.42 | 1970 (170) | 203.30 | 224.05 |
| 1880 (80) | 50.19 | 49.63 | 1980 (180) | 226.54 | 239.78 |
| 1890 (90) | 62.98 | 63.33 | 1990 (190) | 248.71 | 252.94 |

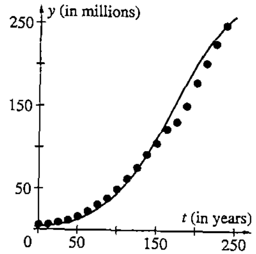



Figure 3.2

see in Figure 3.2. To predict the population of the United States in the year 2010 with this model, we evaluate $y(210)$ to obtain

$$y(210) = \frac{0.159}{0.00053 + 0.02947e^{-0.03 \cdot 210}} \approx 300.$$

Thus, we predict that the population will be approximately 300 million in the year 2010. Note that projections of the population of the United States in the year 2010 made by the Bureau of the Census range from 298.71 million to 300.59 million, so the approximation obtained with the logistic equation seems reasonable.

## Population Model with a Threshold

Suppose that when the population of a species falls below a certain level (or threshold), the species cannot sustain itself, but otherwise, the population follows logistic growth. To describe this situation mathematically, we introduce the differential equation

$$y' = -r\left(1 - \frac{y}{A}\right)\left(1 - \frac{y}{B}\right)y,$$

where $0 < A < B$. In this case, the equilibrium solutions are $y = 0$, $y = A$, and $y = B$. To understand the dynamics of the equation, we generate the corresponding phase line. If $0 < y < A$, then $y' < 0$; if $A < y < B$, then $y' > 0$; and if $y > B$, then $y' < 0$. From the phase line in Figure 3.3, we see that $y = 0$ is asymptotically stable, $y = A$ is unstable, and $y = B$ is asymptotically stable. Notice that we expect the population to die out when $y < A$, so we call $y = A$ the *threshold of the population.* As with the logistic equation, $y = B$ is the *carrying capacity.* In Figure 3.4, we graph several solutions to $y' = -0.25(1 - y)(1 - y/3)y$ using different values for the initial population $y_0$. Notice that if $0 < y_0 < 1$, then $\lim_{t\to\infty} y(t) = 0$, and if $1 < y_0 < 3$ or $y_0 > 3$, then $\lim_{t\to\infty} y(t) = 3$. In this case, the threshold is $y = 1$ and the carrying capacity is $y = 3$.


Figure 3.3 Phase line for $y' = -r\left(1 - \frac{y}{A}\right)\left(1 - \frac{y}{B}\right)y$

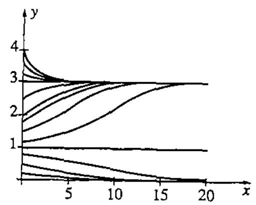



Figure 3.4 Several solutions to $y' = -0.25(1 - y)\left(1 - \frac{y}{3}\right)y$



## EXERCISES 3.1

Solve the following problems. Unless otherwise stated, use the Malthus method.

1. Suppose that a culture of bacteria has initial population of $n = 100$. If the population doubles every three days, determine the number of bacteria present after 30 days. How much time is required for the population to reach 4250 in number?
2. Suppose that the population in a yeast culture triples every seven days. What is the population after 35 days? How much time is required for the population to be 10 times the initial population?
*3. Suppose that two-thirds of the cells in a culture remain after one day. Use this information to determine the number of days until only one-third of the initial population remains.
4. Consider a radioactive substance with half-life 10 days. If there are initially 5000 g of the substance, how much remains after 365 days?
5. Suppose that the half-life of an element is 1000 hours. If there are initially 100 g, how much remains after 1 hour? How much remains after 500 hours?
6. Suppose that the population of a small town is initially 5000. Due to the construction of an interstate highway, the population doubles over the next year. If the rate of growth is proportional to the current population, when will the population reach 25,000? What is the population after five years?
*7. Suppose that mold grows at a rate proportional to the amount present. If there are initially 500 g of mold and 6 hours later there are 600 g, determine the amount of mold present after one day. When is the amount of mold 1000 g?
8. Suppose that the rabbit population on a small island grows at a rate proportional to the number of rabbits present. If this population doubles after 100 days, when does the population triple?
9. In a chemical reaction, chemical A is converted to chemical B at a rate proportional to the amount of chemical A present. If half of chemical A remains after five hours, when does 1/6 of the initial amount of chemical A remain? How much of the initial amount remains after 15 hours?
10. If 90% of the initial amount of a radioactive element remains after one day, what is the half-life of the element?
*11. If $y(t)$ represents the percent of a radioactive element that is present at time $t$ and the values of $y(t_1)$ and $y(t_2)$ are known, show that the half-life $H$ is given by

$$H = \frac{(t_2 - t_1)\ln 2}{\ln[y(t_1)] - \ln[y(t_2)]}.$$

12. The half-life of $^{14}C$ is 5730 years. If the original amount of $^{14}C$ in a particular living organism is 20 g and that found in a fossil of that organism is 0.01 g, determine the approximate age of the fossil.
13. After 10 days, 800 g of a radioactive element remain, and after 15 days, 560 g remain. What is the half-life of this element?
14. After one week, 10% of the initial amount of a radioactive element decays. How much decays after two weeks? When does half of the original amount decay?
*15. Determine the percentage of the original amount of $^{226}Ra$ that remains after 100 years.
16. If an artifact contains 40% of the amount of $^{230}Th$ as a present-day sample, what is the age of the artifact?
17. On an archeological dig, scientists find an ancient tool near a fossilized human bone. If the tool and fossil contain 65% and 60% of the amount of $^{14}C$ as that in present-day samples, respectively, determine if the tool could have been used by the human.
18. A certain group of people with initial population 10,000 grows at a rate proportional to the number present. The population doubles in five years. In how many years will the population triple?
*19. Solve the logistic equation by viewing it as a Bernoulli equation.
20. What is the limiting population, $\lim_{t\to\infty} y(t)$, of the U.S. population using the result obtained in Example 5?
21. Solve the logistic equation if $r = 1/100$ and $a = 1/10^8$ given that $y(0) = 100{,}000$. Find $y(25)$. What is the limiting population?

**22.** Five college students with the flu return to an isolated campus of 2500 students. If the rate at which this virus spreads is proportional to the number of infected students $y$ and to the number not infected $2500 - y$, solve the initial-value problem $dy/dt = ky(2500 - y)$, $y(0) = 5$ to find the number of infected students after $t$ days if 25 students have the virus after one day. How many students have the flu after five days?

***23.** One student in a college organization of 200 members proceeds to spread a rumor. If the rate at which this rumor spreads is proportional to the number of students $y$ that know about the rumor as well as the number that do not know, then solve the initial-value problem to find the number of students informed of the rumor after $t$ days if 50 students are informed after one day. How many students know the rumor after two days? Will all of the students eventually be informed of the rumor? (See Exercise 22.)

**24.** Suppose that glucose enters the bloodstream at the constant rate of $r$ grams per minute while it is removed at a rate proportional to the amount $y$ present at any time. Solve the initial-value problem $dy/dt = r - ky$, $y(0) = y_0$ to find $y(t)$. What is the eventual concentration of glucose in the bloodstream according to this model?

**25.** What is the concentration of glucose in the bloodstream after 10 min if $r = 5$ g per min, $k = 5$, and the initial concentration is $y(0) = 500$? After 20 min? Does the concentration appear to reach its limiting value quickly or slowly? (See Exercise 24.)

**26.** Suppose that we deposit a sum of money in a money market fund that pays interest at an annual rate $k$, and let $S(t)$ represent the value of the investment at time $t$. If the compounding takes place continuously, then the rate at which the value of the investment changes is the interest rate times the value of the investment, $dS/dt = kS$. Use this equation to find $S(t)$ if $S(0) = S_0$.

**27.** Banks use different methods to compound interest. If the interest rate is $k$, and if interest is compounded $m$ times per year, then $S(t) = S_0\left(1 + \dfrac{k}{m}\right)^{mt}$. When interest is compounded continuously, then $m \to \infty$. Compare $\lim_{t\to\infty} S_0\left(1 + \dfrac{k}{m}\right)^{mt}$ to the formula obtained in Exercise 26.

**28. (Dating Works of Art)** We can determine if a work of art is more than 100 years old by determining if the lead-bearing materials contained in the work were manufactured within the last 100 years. The half-life of lead-210 ($^{210}$Pb) is 22 years, while the half-life of radium-226 ($^{226}$Ra) is 1700 years. Let $SF$ denote the ratio of $^{210}$Pb to $^{226}$Ra per unit mass of lead. The approximate value of $SF$ for works of art created in the last 80 years is 100. Then, the quantity of lead $(1 - \text{Ra})/(\text{Po})$ at time $t$ is given by

$$\frac{1 - \text{Ra}}{\text{Po}} = \frac{(SF - 1)e^{-\lambda t}}{(SF - 1)e^{-\lambda t} + 1},$$

where $\lambda$ is the disintegration constant for $^{210}$Pb.* On the other hand, for very old paintings $(1 - \text{Ra})/\text{Po} \approx 0$.

(a) Determine the disintegration $\lambda$ for $^{210}$Pb where the amount of $^{210}$Pb at time $t$ is $y = y_0e^{\lambda t}$.

(b) Graph $(1 - \text{Ra})/(\text{Po})$ for $0 \le t \le 250$ using $SF = 100$.

The table on page 103 shows the ratio of $(1 - \text{Ra})/(\text{Po})$ for various famous paintings.

The last two paintings, *Lace Maker* and *Laughing Girl* were painted by the Dutch painter Jan Vermeer who lived from 1632 to 1675.

(c) Determine if it is likely that the first six paintings were also painted by Vermeer (which would make them very valuable!). If not, approximate when they were painted.

***29.** Consider the differential equation $dy/dt = -r(1 - y/A)y$, where $r$ and $A$ are positive constants. **(a)** Find the equilibrium solutions, sketch the phase line, and classify the equilibrium solutions. **(b)** Describe how this equation differs from the logistic equation. **(c)** If $A = 2$ and $y(0) = 1$, what is $\lim_{t\to\infty} y(t)$? **(d)** If $A = 2$ and $y(0) = 3$, what is $\lim_{t\to\infty} y(t)$?

**30.** Find and classify the equilibrium solutions of the Gompertz equation, $dy/dt = y(r - a \ln y)$, where $r$ and $a$ are positive constants.

***31.** The equilibrium solution $y = c$ is classified as **semistable** if solutions on one side approach $y = c$ as $t \to \infty$, but on the other side they move away from $y = c$ as $t \to \infty$. Use this definition to classify the equilibrium solutions (as asymptotically stable, semistable, or unstable) of the following differential equations.

**(a)** $y' = y(y - 2)^2$ **(b)** $y' = y^2(y - 1)$

**(c)** $y' = y - \sqrt{y}$ **(d)** $y' = y^2(9 - y^2)$

* Bernard Keisch, "Dating Works of Art through Their Natural Radioactivity: Improvements and Applications," *Science*, Volume 160 (April 26, 1968), pp. 413–415.



**32.** Consider the Malthus population model with $k = 0.01$, 0.05, 0.1, 0.5, and 1.0 using $y_0 = 1$. Solve the model, plot the solution with these values, and compare the results. How does the value of $k$ affect the solution?

**33.** Consider the logistic equation with $r = 0.01, 0.05, 0.1$, 0.5, and 1.0 using $y_0 = 1$ and $a = 1$. Solve the model, plot the solution with these values, and compare the results. How does the value of $r$ affect the solution?

**34.** We may use graphing utilities to investigate the behavior of solutions of the logistic equation under different conditions. **(a)** Use a graphing utility to investigate the behavior of the solution if values of $r$, $a$, and $y_0$ are varied. Under what conditions does the limit of the solution approach **(b)** a nonzero number as $t \to \infty$; and **(c)** zero as $t \to \infty$? *Hint:* Consider $0 < r \le 1$, $r > 1$; $0 < a < 1$, $a > 1$.

**35. (Harvesting)** If we wish to model a population of size $P(t)$ at time $t$ and consider a constant harvest rate $h$ (like hunting, fishing, or disease), then we might modify the logistic equation and use the equation

$$\frac{dP}{dt} = rP - aP^2 - h$$

to model the population under consideration. Assume that $h \ge r^2/4a$.

**(a)** Show that if $h \ge r^2/(4a)$, a general solution of $dP/dt = rP - aP^2 - h$ is

$$P(t) = \frac{1}{2a}\left[r + \sqrt{4ah - r^2}\,\tan\left(\frac{1}{2a}(C - at)\sqrt{4ah - r^2}\right)\right].$$

**(b)** Suppose that for a certain species it is found that $r = 0.03$, $a = 0.0001$, $h = 2.26$, and $C = -1501.85$. At what time will the species become extinct?

**(c)** If $r = 0.03$, $a = 0.0001$, and $P(0) = 5.3$, graph $P(t)$ if $h = 0$, $h = 0.5$, $h = 1.0$, $h = 1.5$, $h = 2.0$, $h = 2.25$ and $h = 2.5$.

**(d)** What is the maximum allowable harvest rate to assure that the species survives?

**(e)** Generalize your result from **(d)**. For arbitrary $a$ and $r$, what is the maximum allowable harvest rate that ensures survival of the species?

**36.** Shortly after the Chernobyl accident in the Soviet Union in 1986, several nations reported that the level of $^{131}$I in milk was five times that considered safe for human consumption. Make a table of the level of $^{131}$I in milk as a multiple of that considered safe for human consumption for the first three weeks following

| Painting | $^{210}$Po concentration (dpm/g of Pb) | $^{226}$Ra concentration (dpm/g of Pb) | $\frac{1 - \text{Ra}}{\text{Po}}$ |
|---|---|---|---|
| *Washing of Feet* | 12.6 | 0.26 | 0.98 |
| *Woman Reading Music* | 10.3 | 0.30 | 0.97 |
| *Woman Playing Mandolin* | 8.2 | 0.17 | 0.98 |
| *Woman Drinking* | 8.3 | 0.1 | 0.99 |
| *Disciples of Emmaus* | 8.5 | 0.8 | 0.91 |
| *Boy Smoking* | 4.8 | 0.31 | 0.94 |
| *Lace Maker* | 1.5 | 1.4 | 0.07 |
| *Laughing Girl* | 5.2 | 6.0 | −0.15 |

the accident. After how long did the milk become safe for human consumption?

37. Consider a solution to the logistic equation with initial population $y_0$ where $0 < y_0 < r/a$. Show that this solution is concave up for $0 < y < r/(2a)$ and concave down for $r/(2a) < y < r/a$. *Hint:* Differentiate the right side of the logistic equation and set the result equal to 0. Describe the behavior of $dy/dt$ based on this result.

38. From the early 1800s to the mid-1800s, the passenger pigeon population was thriving. However, due to hunting, the population size was reduced dramatically by the late 1800s. Unfortunately, the passenger pigeon requires a large number of cohorts to achieve successful reproduction. Having fallen below this level, the population size continued to decrease and the bird is now extinct. Which population model should be used to describe this situation?

39. **(a)** Find the solution to the IVP $dy/dt = -r(1 - y/A)y$, $y(0) = y_0$, where $r$ and $A$ are positive constants. **(b)** If $y_0 < A$, then determine $\lim_{t\to\infty} y(t)$. **(c)** If $y_0 > A$, show that $\lim_{t\to\infty} y(t) = +\infty$. **(d)** Show that if $y_0 > A$, then $y(t)$ has a vertical asymptote at $t = \frac{1}{r}\ln\left(\frac{y_0}{y_0 - A}\right)$.

40. Solve the IVP $y' = r\left(1 - \frac{y}{A}\right)\left(1 - \frac{y}{B}\right)y$, $y(0) = y_0$.

## 3.2 Newton's Law of Cooling and Related Problems

First-order linear differential equations can be used to solve a variety of problems that involve temperature. For example, a medical examiner can find the time of death in a homicide case, a chemist can determine the time required for a plastic mixture to cool to a hardening temperature, and an engineer can design the cooling and heating system of a manufacturing facility. Although distinct, each of these problems depends on a basic principle, Newton's law of cooling, which is used to develop the differential equation associated with each problem.

**Newton's Law of Cooling**

The rate at which the temperature $T(t)$ changes in a cooling body is proportional to the difference between the temperature of the body and the constant temperature $T_s$ of the surrounding medium.

This situation is represented as the first-order initial-value problem

$$\frac{dT}{dt} = k(T - T_s),\ T(0) = T_0,$$

where $T_0$ is the initial temperature of the body and $k$ is the constant of proportionality. The equation

$$\frac{dT}{dt} = k(T - T_s)$$



is separable, and separating variables gives us

$$\frac{dT}{T - T_s} = k\,dt,$$

so $\ln|T - T_s| = kt + C$. Using the properties of natural logarithms and simplifying yields

$$T = C_1e^{kt} + T_s,$$

where $C_1 = e^c$. Applying the initial condition implies that $T_0 = C_1 + T_s$, so $C_1 = T_0 - T_s$. Therefore, the solution of the equation is

$$T = (T_0 - T_s)e^{kt} + T_s.$$

Notice that if $k < 0$, $\lim_{t\to\infty} e^{kt} = 0$. Therefore, $\lim_{t\to\infty} T(t) = T_s$, so the temperature of the body approaches that of its surroundings.

To better understand the model, suppose that $k < 0$. If $T > T_s$, then $dT/dt < 0$, so $T(t)$ is a decreasing function. Similarly, if $T < T_s$, then $dT/dt > 0$, so $T(t)$ is an increasing function. Therefore, the object cools off if the surrounding temperature is greater than the object's temperature, whereas it warms up if the temperature of the surroundings is warmer than the object's temperature.

Notice that instead of solving the ODE as a separable equation, we could have solved it as a first-order linear equation.

**Example 1**

A pie is removed from a 350°F oven and placed to cool in a room with temperature 75°F. In 15 min, the pie has a temperature of 150°F. Determine the time required to cool the pie to a temperature of 80°F.

***Solution*** In this example, $T_0 = 350$ and $T_s = 75$. Substituting these values into $T = (T_0 - T_s)e^{kt} + T_s$, we obtain $T(t) = (350 - 75)e^{kt} + 75 = 275e^{kt} + 75$. To solve the problem we must find $k$ or $e^k$. We know that $T(15) = 150$, so $T(15) = 275e^{15k} + 75 = 150$. Solving this equation for $e^k$ gives us

$$\begin{aligned} 275e^{15k} &= 75 \\ e^{15k} &= \frac{3}{11} \\ (e^k)^{15} &= \frac{3}{11} \\ e^k &= \left(\frac{3}{11}\right)^{1/15}. \end{aligned}$$

Thus, $T(t) = 275(e^k)^t + 75 = 275(\frac{3}{11})^{t/15} + 75$.

To find the value of $t$ for which $T(t) = 80$, we solve the equation $275(\frac{3}{11})^{t/15} + 75 = 80$ for $t$:

$$275\left(\frac{3}{11}\right)^{t/15} = 5$$

$$\left(\frac{3}{11}\right)^{t/15} = \frac{1}{55}$$

$$\ln\left(\frac{3}{11}\right)^{t/15} = \ln\left(\frac{1}{55}\right) = -\ln 55$$

$$\frac{t}{15}\ln\left(\frac{3}{11}\right) = -\ln 55$$

$$t = \frac{-15\ln 55}{\ln\left(\frac{3}{11}\right)} = \frac{-15\ln 55}{\ln 3 - \ln 11} \approx 46.264.$$

Thus, the pie will reach a temperature of 80°F after about 46 minutes.

An interesting problem associated with this example is to determine if the pie ever reaches room temperature. From the formula $T(t) = 275(\frac{3}{11})^{t/15} + 75$, we note that $275(\frac{3}{11})^{t/15} > 0$, so $T(t) = 275(\frac{3}{11})^{t/15} + 75 > 75$. Therefore, the pie never actually reaches room temperature according to our model. However, we see its temperature approaches 75° as $t$ increases because $\lim_{t\to\infty}[275(\frac{3}{11})^{t/15} + 75] = 75$.

**Example 2**

In the investigation of a homicide, the time of death is important. Newton's law of cooling can be used to approximate this time. For example, the normal body temperature of most healthy people is 98.6°F. Suppose that when a body is discovered at noon, its temperature is 82°F. Two hours later, it is 72°F. If the temperature of the surroundings is 65°F, what was the approximate time of death?

***Solution*** This problem is solved like the previous example. Let $T(t)$ denote the temperature of the body at time $t$, where $T(0)$ represents the temperature of the body when it is discovered and $T(2)$ represents the temperature of the body two hours after it is discovered. In this case we have $T_0 = 82$ and $T_s = 65$, and substituting these values into $T = (T_0 - T_s)e^{kt} + T_s$ yields $T(t) = (82 - 65)e^{kt} + 65 = 17e^{kt} + 65$. Using $T(2) = 72$, we solve the equation $T(2) = 17e^{2k} + 65 = 72$ for $e^k$ to find $e^k = (\frac{7}{17})^{1/2}$, so $T(t) = 17(e^k)^t + 65 = 17(\frac{7}{17})^{t/2} + 65$. To find the value of $t$ for which $T(t) = 98.6$, we solve the equation $17(\frac{7}{17})^{t/2} + 65 = 98.6$ for $t$ and obtain

$$t = \frac{2\ln(1.97647)}{\ln 7 - \ln 17} = -1.53569.$$



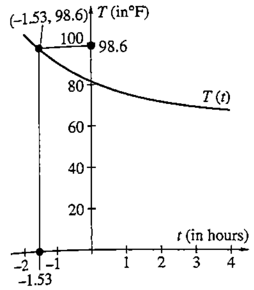



**Figure 3.5** Graph of $T(t) = 17(\frac{7}{17})^{t/2} + 65$

This result means that the time of death occurred approximately 1.53 hours before the body was discovered, as we observe in Figure 3.5. Therefore, the time of death was approximately 10:30 A.M. because the body was discovered at noon.

In each of the previous cases, the temperature of the surroundings was assumed to be constant. However, this does not have to be the case. For example, determining the temperature inside a building over the span of a 24-hour day is more complicated because the outside temperature varies. If we assume that a building has no heating or air conditioning system, the differential equation that needs to be solved to find the temperature $u(t)$ at time $t$ inside the building is

$$\frac{du}{dt} = k(C(t) - u(t))$$

where $C(t)$ is a function that describes the outside temperature and $k > 0$ is a constant that depends on the insulation of the building. According to this equation, if $C(t) > u(t)$, then $du/dt > 0$, which implies that $u$ increases. However, if $C(t) < u(t)$, then $du/dt < 0$, which means that $u$ decreases.

**Example 3**

Suppose that during the month of April in Atlanta, Georgia, the outside temperature in °F is given by $C(t) = 70 - 10\cos(\pi t/12)$, $0 \le t \le 24$. (This implies that the average value of $C(t)$ is 70°F.) Determine the temperature in a building that has an initial temperature of 60°F if $k = 1/4$.

***Solution*** The initial-value problem that we must solve is

$$\begin{cases} \dfrac{du}{dt} = k\left(70 - 10\cos\dfrac{\pi t}{12} - u\right) \\ u(0) = 60 \end{cases}.$$

The differential equation can be solved if we write it as $\frac{du}{dt} + ku = k\left(70 - 10\cos\frac{\pi t}{12}\right)$ and use an integrating factor. This gives us

$$u(t) = \frac{10}{9 + \pi^2}\left(63 + 7\pi^2 - 9\cos\frac{\pi t}{12} - 3\pi\sin\frac{\pi t}{12}\right) + C_1 e^{-t/4}.$$

We then apply the initial condition $u(0) = 60$ to determine the arbitrary constant $C_1$ and obtain the solution

$$u(t) = \frac{10}{9 + \pi^2}\left(63 + 7\pi^2 - 9\cos\frac{\pi t}{12} - 3\pi\sin\frac{\pi t}{12}\right) - \frac{10\pi^2}{9 + \pi^2}e^{-t/4},$$

where $k = 1/4$, which is graphed in Figure 3.6. From the graph, we see that the temperature reaches its maximum (approximately 72°F) near $t = 15.5$ hours, which corresponds to 3:30 P.M.

*At what time during the day is the temperature in the building increasing (decreasing) at the fastest rate?*

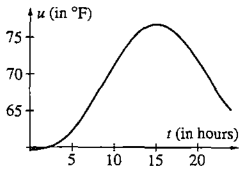



Figure 3.6

In many situations, a heating or cooling system is installed to control the temperature in a building. Another factor in determining the temperature, which we ignored in prior calculations, is the generation of heat from the occupants of the building, including people and machinery. We investigate the inclusion of these factors in the exercises at the end of this section.

An application that involves a differential equation similar to the one encountered with Newton's law of cooling is that associated with **mixture problems.** In this case, suppose that a tank contains $V_0$ gallons of a brine solution, a mixture of salt and water. A brine solution of concentration $S_1$ pounds per gallon is allowed to flow into the tank at a rate $R_1$ gallons per minute while a well-stirred mixture flows out of the tank at the rate of $R_2$ gallons per minute. If $y(t)$ represents the amount (in pounds) of salt in the tank at time $t$ minutes, then the equation that describes the net rate of change in the amount of salt in the tank is

$$\frac{dy}{dt} = (\text{rate salt enters the tank}) - (\text{rate salt leaves the tank})$$

$$= \left(S_1 \frac{\text{lb}}{\text{gal}}\right)\left(R_1 \frac{\text{gal}}{\text{min}}\right) - \left(\frac{y(t)}{V(t)} \frac{\text{lb}}{\text{gal}}\right)\left(R_2 \frac{\text{gal}}{\text{min}}\right),$$

where $V(t)$ is the volume of solution (liquid) in the tank at any time $t$ and $y(t)/V(t)$ represents the salt concentration of the well-stirred mixture. To determine $V(t)$, we solve the equation that gives the net rate of change in the amount of liquid in the tank,

$$\frac{dV}{dt} = (\text{Rate liquid enters the tank}) - (\text{Rate liquid leaves the tank})$$

$$= \left(R_1 \frac{\text{gal}}{\text{min}}\right) - \left(R_2 \frac{\text{gal}}{\text{min}}\right)$$

using the initial condition $V(0) = V_0$. We then use the solution $V(t)$ to solve the previous ODE for $y(t)$ where $y(0) = y_0$ is the initial amount of salt in the tank.

## Example 4

Suppose that $S_1 = 1$, $R_1 = 4$, and $R_2 = 3$. In addition, suppose that the tank initially holds 500 gal of liquid and 250 lb of salt. Find the amount of salt contained in the tank at any time. Determine how much time is required for 1000 lb of salt to accumulate. If the tank holds a maximum of 800 gal, can 1000 lb of salt be present in the tank at any time before the tank reaches its maximum capacity?



***Solution*** We notice that the volume (of liquid) does not remain constant in the tank because $R_1 \neq R_2$. Therefore, we determine $V(t)$ by solving the IVP $dV/dt = 4 - 3 = 1$, $V(0) = 500$ with the general solution $V(t) = t + C_1$. Applying the initial condition yields $V(t) = t + 500$, so that the IVP used to find $y(t)$ is

$$\frac{dy}{dt} = 4 - \frac{3y}{t+500},\ y(0) = 250.$$

Rewriting the ODE as $dy/dt + (3y)/(t + 500) = 4$, we use the integrating factor

$$\mu(t) = e^{\int (3dt)/(t+500)} = e^{3\ln(t+500)} = (t + 500)^3,\ t \geq 0$$

to obtain $\frac{d}{dt}[(t + 500)^3 y] = 4(t + 500)^3$ so that $(t + 500)^3 y = (t + 500)^4 + C_2$. Therefore, $y(t) = t + 500 + C_2/(t + 500)^3$, so that $y(0) = 250$ indicates that $C_2 = -250 \cdot 500^3 = -31{,}250{,}000{,}000$. After graphing $y(t) = t + 500 - 31{,}250{,}000{,}000/(t + 500)^3$, we can use a numerical solver to find that the approximate solution to $t + 500 - 31{,}250{,}000{,}000/(t + 500)^3 = 1000$ is $t = 528.7$ min. However, if $t = 528.7$, then $V(528.7) = 528.7 + 500 = 1028.7$ gal, so the tank would overflow before 1000 lb of salt could be present in the tank.

## EXERCISES 3.2

1. A hot cup of tea is initially 100°C when poured. How long does it take for the tea to reach a temperature of 50°C if it is at 80°C after 15 min and the room temperature is 30°C?
2. Suppose that the tea in Exercise 1 is allowed to cool at room temperature for 20 min. It is then placed in a cooler with temperature 15°C. What is the temperature of the tea after 60 min if it is 60°C after 10 min in the cooler?
*3. A can of orange soda is removed from a refrigerator having temperature 40°F. If the can is at 50°F after 5 min, how long does it take for the can to reach a temperature of 60°F if the surrounding temperature is 75°F?
4. Suppose that a container of tea is placed in a refrigerator at 35°F to cool. If the tea is initially at 75°F and it has a temperature of 70°F after 1 hour, then when does the tea reach 55°F?
5. Determine the time of death if a corpse is at 79°F when discovered at 3:00 P.M. and 68°F 3 hours later. Assume that the temperature of the surroundings is 60°F. (Normal body temperature is 98.6°F.)
6. At the request of his children, a father makes homemade popsicles. At 2:00 P.M., one of the children asks if the popsicles are frozen (0°C), at which time the father tests the temperature of a popsicle and finds it to be 5°C. If the father placed the popsicles with a temperature of 15°C in the freezer at 12:00 P.M. and the temperature of the freezer is −2°C, when will the popsicles be frozen?
*7. A thermometer that reads 90°F is placed in a room with temperature 70°F. After 3 min, the thermometer reads 80°F. What does the thermometer read after 5 min?
8. A thermometer is placed outdoors at 80°F. After 2 min, the thermometer reads 68°F, and after 5 min, it reads 72°F. What was the initial temperature reading of the thermometer?
9. A casserole is placed in a microwave oven to defrost. It is then placed in a conventional oven at 300°F and

bakes for 30 min, at which time its temperature is 150°F. If after baking an additional 30 min its temperature is 200°F, what was the temperature of the casserole when it was removed from the microwave?

10. A bottle of wine at room temperature (70°F) is placed in ice to chill at 32°F. After 20 min, the temperature of the wine is 58°F. When will its temperature be 50°F?

*11. When a cup of coffee is poured its temperature is 200°F. Two minutes later, its temperature is 170°F. If the temperature of the room is 68°F, when is the temperature of the coffee 140°F?

12. After dinner, a couple orders two cups of coffee. Upon being served, the gentleman immediately pours one container of cream into his cup of coffee. His companion waits 4 min before adding the same amount of cream to her cup. Which person has the hotter cup of coffee when they both take a sip of coffee after she adds the cream to her cup? (Assume that the cream's temperature is less than that of the coffee.) Explain.

13. Suppose that during the month of February in Washington, D.C., the outside temperature in °F is given by $C(t) = 40 - 5\cos(\pi t/12)$, $0 \le t \le 24$. Determine the temperature in a building that has an initial temperature of 50° if $k = 1/4$. (Assume that the building has no heating or air conditioning system.)

14. Suppose that during the month of August in Savannah, Georgia, the outside temperature in °F is given by $C(t) = 85 - 10\cos(\pi t/12)$, $0 \le t \le 24$. Determine the temperature in a building that has an initial temperature of 60°F if $k = 1/4$. (Assume that the building has no heating or air conditioning system.)

*15. Suppose that during the month of October in Los Angeles, the outside temperature in °F is given by $C(t) = 70 - 5\cos(\pi t/12)$, $0 \le t \le 24$. Find the temperature in a building that has an initial temperature of 65°F if $k = 1/4$. (Assume that the building has no heat or air conditioning system.)

16. Suppose that during the month of January in Cincinnati, Ohio, the outside temperature in °F is given by $C(t) = 20 - 5\cos(\pi t/12)$, $0 \le t \le 24$. Find the temperature in a building that has an initial temperature of 40°F if $k = 1/4$. (Assume that the building has no heat or air conditioning system.)

17. Find the amount of salt $y(t)$ in a tank with initial volume $V_0$ gallons of liquid and $y_0$ pounds of salt using the given conditions.

**(a)** $R_1 = 4$ gal/min, $R_2 = 4$ gal/min, $S_1 = 2$ lb/gal, $V_0 = 400$, $y_0 = 0$

**(b)** $R_1 = 4$ gal/min, $R_2 = 2$ gal/min, $S_1 = 1$ lb/gal, $V_0 = 500$, $y_0 = 20$

**(c)** $R_1 = 4$ gal/min, $R_2 = 6$ gal/min, $S_1 = 1$ lb/gal, $V_0 = 600$, $y_0 = 100$

Describe how the values of $R_1$ and $R_2$ affect the volume of liquid in the tank.

18. A tank contains 100 gal of a brine solution in which 20 lb of salt is initially dissolved. **(a)** Water (containing no salt) is then allowed to flow into the tank at a rate of 4 gal/min and the well-stirred mixture flows out of the tank at an equal rate of 4 gal/min. Determine the amount of salt $y(t)$ at any time $t$. What is the eventual concentration of the brine solution in the tank? **(b)** If instead of water a brine solution with concentration 2 lb/gal flows into the tank at a rate of 4 gal/min, what is the eventual concentration of the brine solution in the tank?

*19. A tank contains 200 gal of a brine solution in which 10 lb of salt is initially dissolved. A brine solution with concentration 2 lb/gal is then allowed to flow into the tank at a rate of 4 gal/min and the well-stirred mixture flows out of the tank at a rate of 3 gal/min. Determine the amount of salt $y(t)$ at any time $t$. If the tank can hold a maximum of 400 gal, what is the concentration of the brine solution in the tank when the volume reaches this maximum?

20. A tank contains 300 gal of a brine solution in which 30 lb of salt is initially dissolved. A brine solution with concentration 4 lb/gal is then allowed to flow into the tank at a rate of 3 gal/min, and the well-stirred mixture flows out of the tank at a rate of 4 gal/min. Determine the amount of salt $y(t)$ at any time $t$. What is the concentration of the brine solution after 10 min? What is the eventual concentration of the brine solution in the tank? For what values of $t$ is the solution defined? Why?

The temperature $u(t)$ inside a building can be based on three factors: (1) the heat produced by people or machinery inside the building, (2) the heating (or cooling) produced by the furnace (or air conditioning system), and (3) the temperature outside the building based on Newton's law of cooling. If the rate at which these factors affect (increase or decrease) the temperature is given by $A(t)$, $B(t)$, and $C(t)$, respectively, the differential equation that models this situation is

$$\frac{du}{dt} = k(C(t) - u(t)) + A(t) + B(t),$$

where the constant $k > 0$ depends on the insulation of the building.

21. Find the temperature (and the maximum temperature) in the building with $k = 1/4$ if the initial temperature is 70°F, and **(a)** $A(t) = 0.25$, $C(t) = 75$, and $B(t) = 0$; **(b)** $A(t) = 0.25$, $C(t) = 70 - 10\cos(\pi t/12)$, and $B(t) = 0$; **(c)** $A(t) = 1$, $C(t) = 70 - 10\cos(\pi t/12)$, and $B(t) = 0$.

If a heating or cooling system is considered, we must model the system with an appropriate function. Of course, the system could run constantly, but we know that most are controlled by a thermostat. Suppose that the desired temperature is $u_d$. Then $B(t) = k_d(u_d - u(t))$, where $k_d$ is a constant approximately equal to two for most systems.

22. Determine the temperature (and the maximum temperature) in a building with $k = 1/4$ and initial temperature 70° if **(a)** $A(t) = 0.25$, $B(t) = 1.75(68 - u(t))$, and $C(t) = 70 - 10\cos(\pi t/12)$; **(b)** $A(t) = 1$, $B(t) = 1.75(68 - u(t))$, and $C(t) = 70 - 10\cos(\pi t/12)$; **(c)** $A(t) = 0.25$, $B(t) = 1.75(68 - u(t))$, and $C(t) = 80 - 10\cos(\pi t/12)$.

*23. If $A(t) = 0.25$, $B(t) = 1.75(u_d - u(t))$, and $C(t) = 70 - 10\cos(\pi t/12)$; determine the value of $u_d$ needed so that the average temperature in the building over a 24-hour period is 70°F.



## 3.3 Free-Falling Bodies

The motion of some objects can be determined through the solution of a first-order equation. We begin by explaining some of the theory that is needed to set up the differential equation that models the situation.

**Newton's Second Law of Motion**

The rate at which the momentum of a body changes with respect to time is equal to the resultant force acting on the body.

The body's momentum is defined as the product of its mass and velocity, so this statement is modeled as

$$\frac{d}{dt}(mv) = F,$$

where $m$ and $v$ represent the body's mass and velocity, respectively, and $F$ is the sum of the forces acting on the body. The mass $m$ of the body is constant, so differentiation leads to the differential equation

$$m\frac{dv}{dt} = F.$$

If the body is subjected to the force due to gravity, then its velocity is determined by solving the differential equation

$$m\frac{dv}{dt} = mg \quad \text{or} \quad \frac{dv}{dt} = g,$$

where $g \approx 32$ ft/sec$^2$ (English system) or $g \approx 9.8$ m/sec$^2$ (international system). See the summary of units in Table 3.4.

**TABLE 3.4 Units Useful in Solving Problems Associated with Newton's Second Law of Motion**

| | English | International |
|---|---|---|
| Mass | slug $\left(\frac{\text{lb-sec}^2}{\text{ft}}\right)$ | kilogram (kg) |
| Force | pound (lb) | Newton $\left(\frac{\text{m-kg}}{\text{sec}^2}\right)$ |
| Distance | foot (ft) | meter (m) |
| Time | second (sec) | second (sec) |

This differential equation is applicable only when the resistive force due to the medium (such as air resistance) is ignored. If this offsetting resistance is considered, we must discuss all of the forces acting on the object. Mathematically, we write the equation as

$$m\frac{dv}{dt} = \sum \text{(forces acting on the object)},$$

where the direction of motion is taken to be the positive direction.

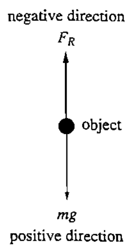



**Figure 3.7** Force diagram

We use a force diagram in Figure 3.7 to set up the differential equation that models the situation. Air resistance acts against the object as it falls and $g$ acts in the same direction of the motion. We state the differential equation in the form:

$$m\frac{dv}{dt} = mg + (-F_R) \quad \text{or} \quad m\frac{dv}{dt} = mg - F_R,$$

where $F_R$ represents this resistive force. Note that down is assumed to be the positive direction. The resistive force is typically proportional to the body's velocity ($v$) or a power of its velocity. Hence, the differential equation is linear or nonlinear based on the resistance of the medium taken into account.

**Example 1**

(a) Determine the velocity and the distance traveled by an object with mass $m = 1$ slug that is thrown downward with an initial velocity of 2 ft/sec from a height of 1000 ft. Assume that the object is subjected to air resistance that is equivalent to the instantaneous velocity of the object. (b) Determine the time at which the object strikes the ground and its velocity when it strikes the ground.



***Solution*** (a) First, we set up the initial-value problem to determine the velocity of the object. The air resistance is equivalent to the instantaneous velocity, so $F_R = v$. The formula $m\frac{dv}{dt} = mg - F_R$ then gives us

$$\frac{dv}{dt} = 32 - v,$$

and imposing the initial velocity $v(0) = 2$ yields the initial-value problem

$$\begin{cases} \dfrac{dv}{dt} = 32 - v \\ v(0) = 2 \end{cases},$$

which can be solved through several methods. We choose to solve it as a linear first-order equation and use an integrating factor. (It also can be solved by separating variables.) With the integrating factor $e^t$, we have $\frac{d}{dt}(e^t v) = 32e^t$. Integrating both sides gives us $e^t v = 32e^t + C$, so

$$v = 32 + Ce^{-t},$$

and applying the initial velocity gives us $v(0) = 32 + Ce^0 = 32 + C = 2$, so $C = -30$. Therefore, the velocity of the object is

$$v = 32 - 30e^{-t}.$$

Notice that the velocity of the object cannot exceed 32 ft/sec, called the **limiting velocity**, which is found by evaluating $\lim_{t\to\infty} v(t)$.

To determine the distance traveled at time $t$, $s(t)$, we solve the first-order equation

$$\frac{ds}{dt} = v = 32 - 30e^{-t}$$

with initial condition $s(0) = 0$. This differential equation is solved by integrating both sides of the equation to obtain $s = 32t + 30e^{-t} + C_2$. Application of $s(0) = 0$ then gives us $s(0) = 32(0) + 30e^0 + C_2 = 30 + C_2 = 0$, so $C_2 = -30$, and the distance traveled by the object is given by

$$s = 32t + 30e^{-t} - 30.$$

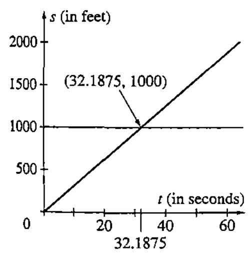



**Figure 3.8** Graph of $s = 32t + 30e^{-t} - 30$

(b) The object strikes the ground when $s(t) = 1000$. Therefore, we must solve the equation $s = 32t + 30e^{-t} - 30 = 1000$ for $t$. The roots of this equation can be approximated with numerical methods like Newton's method. From the graph of this function, shown in Figure 3.8, we see that $s(t) = 1000$ near $t = 35$. Numerical methods show that $s = 32t + 30e^{-t} - 30 = 1000$ when $t \approx 32.1875$.

The velocity at the point of impact is found to be 32.0 ft/sec by evaluating the derivative at the time at which the object strikes the ground, given by $s'(32.1875)$.

**Example 2**

Suppose that the object in Example 1 of mass 1 slug is thrown downward with an initial velocity of 2 ft/sec and that the object is attached to a parachute, increasing this resistance so that it is given by $v^2$. Find the velocity at any time $t$ and determine the limiting velocity of the object.

**Solution** This situation is modeled by the initial-value problem

$$\begin{cases} \dfrac{dv}{dt} = 32 - v^2 \\ v(0) = 2 \end{cases}.$$

We solve the differential equation by separating the variables and using partial fractions:

$$\frac{dv}{32 - v^2} = dt$$

$$\frac{dv}{(4\sqrt{2} + v)(4\sqrt{2} - v)} = dt$$

$$\frac{1}{8\sqrt{2}}\left[\frac{1}{v + 4\sqrt{2}} - \frac{1}{v - 4\sqrt{2}}\right] dv = dt$$

$$\ln|v + 4\sqrt{2}| - \ln|v - 4\sqrt{2}| = 8\sqrt{2}t + C$$

$$\ln\left|\frac{v + 4\sqrt{2}}{v - 4\sqrt{2}}\right| = 8\sqrt{2}t + C$$

$$\left|\frac{v + 4\sqrt{2}}{v - 4\sqrt{2}}\right| = \tilde{C}e^{8\sqrt{2}t} \qquad (\tilde{C} = e^C)$$

$$\frac{v + 4\sqrt{2}}{v - 4\sqrt{2}} = Ke^{8\sqrt{2}t} \qquad (K = \pm\tilde{C}).$$

Solving for $v$, we find that $v + 4\sqrt{2} = Ke^{8\sqrt{2}t}(v - 4\sqrt{2})$ or $(1 - Ke^{8\sqrt{2}t})v = -4\sqrt{2}(Ke^{8\sqrt{2}t} + 1)$, so

$$v = \frac{-4\sqrt{2}(Ke^{8\sqrt{2}t} + 1)}{1 - Ke^{8\sqrt{2}t}}.$$

Application of the initial condition yields $K = \dfrac{1 + 2\sqrt{2}}{1 - 2\sqrt{2}}$. The limiting velocity of the object is found with L'Hopital's rule to be $\lim_{t\to\infty} v(t) = \dfrac{-4\sqrt{2}K}{-K} = 4\sqrt{2}$ ft/sec.

In Example 1, the limiting velocity is 32 ft/sec, so the parachute causes the velocity of the object to be reduced (see Figure 3.9). This shows that the object does not have to endure as great an impact as it would without the help of the parachute.



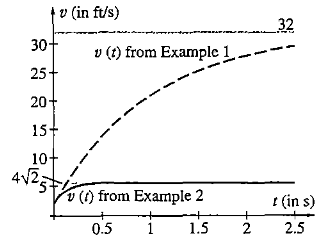



**Figure 3.9** The velocity functions from Example 1 and Example 2. Notice how the different forces due to air resistance affect the velocity of the object.

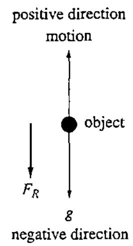



**Figure 3.10** By drawing a force diagram, we see that $g$ and $F_R$ are in the negative direction.

**Example 3**

Determine a solution (for the velocity and the height) of the differential equation that models the motion of an object of mass $m$ when directed upward with an initial velocity of $v_0$ from an initial position $s_0$, assuming that the air resistance equals $cv$ where $c$ is a positive constant.

**Solution** The motion of the object is upward, so $g$ and $F_R$ act against the upward motion of the object as shown in Figure 3.10.

Therefore, the differential equation that must be solved in this case is the linear equation $dv/dt = -g - (c/m)v$. We solve the initial-value problem

$$\begin{cases} \dfrac{dv}{dt} = -g - \dfrac{c}{m}v \\ v(0) = v_0 \end{cases}$$

by first rewriting the equation $\dfrac{dv}{dt} = -g - \dfrac{c}{m}v$ as $\dfrac{dv}{dt} + \dfrac{c}{m}v = -g$ and then calculating the integrating factor $e^{\int \frac{c}{m}dt} = e^{ct/m}$. Multiplying each side of the equation by $e^{ct/m}$ gives us $e^{ct/m}\dfrac{dv}{dt} + \dfrac{c}{m}ve^{ct/m} = -ge^{ct/m}$ so that $\dfrac{d}{dt}(e^{ct/m}v) = -ge^{ct/m}$. Integrating we obtain $e^{ct/m}v = -\dfrac{gm}{c}e^{ct/m} + C$ and, consequently, the general solution is

$$v(t) = -\frac{gm}{c} + Ce^{-ct/m}.$$

Applying the initial condition $v(0) = v_0$ and solving for $C$ yields $C = (cv_0 + gm)/c$ so that the solution to the initial-value problem is

$$v(t) = -\frac{gm}{c} + \frac{cv_0 + gm}{c}e^{-ct/m}.$$

For example, the velocity function for the case with $m = 1/128$ slugs, $c = 1/160$, $g = 32$ ft/sec$^2$ and $v_0 = 48$ ft/sec is $v(t) = 88e^{-4t/5} - 40$. This function is graphed in Figure 3.11. Notice where $v(t) = 0$. This value of $t$ represents the time at which the object reaches its maximum height and begins to fall toward the ground.

Similarly, this function can be used to investigate numerous situations without solving the differential equation each time.

The height function $s(t)$, which represents the distance above the ground at time $t$, is determined by integrating the velocity function:

$$s(t) = \int v(t)\,dt = \int\left(-\frac{gm}{c} + \frac{cv_0 + gm}{c}e^{-ct/m}\right)dt$$

$$= -\frac{gm}{c}t - \frac{cmv_0 + gm^2}{c^2}e^{-ct/m} + C.$$

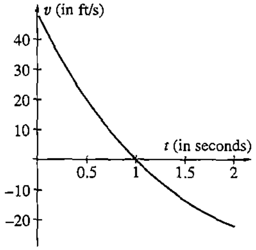


Figure 3.11

If the initial height is given by $s(0) = s_0$, solving for $C$ results in $C = (gm^2 + c^2s_0 + cmv_0)/c^2$, so that

$$s(t) = -\frac{gm}{c}t - \frac{cmv_0 + gm^2}{c^2}e^{-ct/m} + \frac{gm^2 + c^2s_0 + cmv_0}{c^2}.$$

The height and velocity functions are shown in Figure 3.12 using the parameters $m = 1/128$ slugs, $c = 1/160$, $g = 32$ ft/sec$^2$, and $v_0 = 48$ ft/sec as well as $s_0 = 0$.

The time at which the object reaches its maximum height occurs when the derivative of the position is equal to zero. From Figure 3.12, we see that $s'(t) = v(t) = 0$ when $t \approx 1$. Solving $s'(t) = 0$ for $t$ yields the better approximation $t \approx 0.985572$.

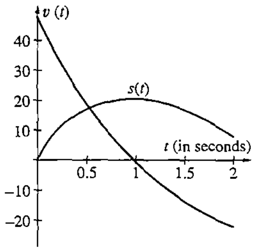


Figure 3.12 Graph of $v(t) = 88e^{-4t/5} - 40$ (in blue) and $s(t) = 110 - 40t - 110e^{-4t/5}$ (in black)

**Weight and Mass** Notice that in the English system, *pounds* describe *force*. Therefore, when the weight $W$ of an object is given, we must calculate its *mass* with the relationship $W = mg$ or $m = W/g$. Conversely, in the International system, the mass of the object (in kilograms) is typically given.

We now combine several of the topics discussed in this section to solve the following problem.

**Example 4**

A 32-lb object is dropped from a height of 50 ft above the surface of a small pond. While the object is in the air, the force due to air resistance is $v$. However, when the object is in the pond it is subjected to a buoyancy force equivalent to $6v$. Determine how much time is required for the object to reach a depth of 25 ft in the pond.

***Solution*** The mass of this object is found using the relationship $W = mg$, where $W$ is the weight of the object. With this, we find that 32 lb $= m(32$ ft/sec$^2)$, so $m = 1$ (lb-sec$^2$)/ft (slug).



This problem must be broken into two parts: an initial-value problem for the object above the pond and an initial-value problem for the object below the surface of the pond. Using techniques discussed in previous examples, the initial-value problem above the pond's surface is found to be

$$\begin{cases} \dfrac{dv}{dt} = 32 - v \\ v(0) = 0 \end{cases}.$$

However, to determine the initial-value problem that yields the velocity of the object beneath the pond's surface, the velocity of the object when it reaches the surface must be known. Hence, the velocity of the object above the surface must be determined first.

The equation $dv/dt = 32 - v$ is separable and rewriting it yields $dv/(32 - v) = dt$. Integrating and applying the initial condition results in $v(t) = 32 - 32e^{-t}$. To find the velocity when the object hits the pond's surface, we must know the time at which the object has fallen 50 ft. Thus, we find the distance traveled by the object by solving $ds/dt = v(t)$, $s(0) = 0$, obtaining $s(t) = 32e^{-t} + 32t - 32$. From the graph of $s(t)$ shown in Figure 3.13, we see that the value of $t$ at which the object has traveled 50 ft appears to be about 2.5 sec.

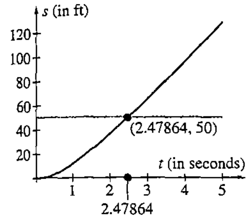


Figure 3.13 Graph of $s(t) = 32e^{-t} + 32t - 32$

A more accurate value of the time at which the object hits the surface is $t \approx 2.47864$. The velocity at this time is then determined by substitution into the velocity function, resulting in $v(2.47864) \approx 29.3166$. Note that this value is the initial velocity of the object when it hits the surface of the pond.

Thus, the initial-value problem that determines the velocity of the object beneath the surface of the pond is given by $dv/dt = 32 - 6v$, $v(0) = 29.3166$. The solution of this initial-value problem is $v(t) = 16/3 + 23.9833e^{-6t}$, and solving $ds/dt = 16/3 + 23.9833e^{-6t}$, $s(0) = 0$ we obtain

$$s(t) = 3.99722 - 3.99722e^{-6t} + \frac{16}{3}t,$$

which gives the depth of the object at time $t$. From the graph of this function, shown in Figure 3.14, we see that the object is 25 ft beneath the surface of the pond after approximately 4 sec.

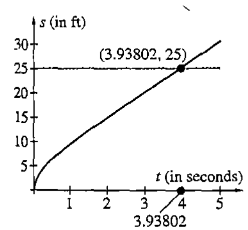


Figure 3.14 Graph of $s(t) = 3.99722 - 3.99722e^{-6t} + \frac{16}{3}t$

A more accurate approximation of the time at which the object is 25 ft beneath the pond's surface is $t \approx 3.93802$.

Finally, the time required for the object to reach the pond's surface is added to the time needed for it to travel 25 ft beneath the surface to see that approximately 6.41667 sec are required for the object to travel from a height of 50 ft above the pond to a depth of 25 ft below the surface.

**Note** The model presented in this section in which the force due to air resistance is proportional to the velocity $m\frac{dv}{dt} = mg - cv$ is a simplified model used to illustrate how differential equations are used to model physical situations. In most cases, the equation in which the force due to air resistance is assumed to be proportional to the square of the velocity $m\frac{dv}{dt} = mg - cv^2$ does a much better job in predicting the velocity of the falling object. In fact, the first model typically works only when an object is dropped in a highly viscous medium or when the object has negligible mass.*

## EXERCISES 3.3

1. A rock that weighs 32 lb is dropped from rest from the edge of a cliff. **(a)** Find the velocity of the rock at time $t$ if the air resistance is equivalent to the instantaneous velocity $v$. **(b)** What is the velocity of the rock at $t = 2$ sec?
2. An object that weighs 4 lb is dropped from the top of a tall building. **(a)** Find the velocity of the object at time $t$ if the air resistance is equivalent to the instantaneous velocity $v$. **(b)** What is the velocity of the object at $t = 2$ sec? How does this compare to the result in Exercise 1?

*3. An object weighing 1 lb is thrown downward with an initial velocity of 8 ft/sec. **(a)** Find the velocity of the object at time $t$ if the air resistance is equivalent to twice the instantaneous velocity. **(b)** What is the velocity of the object at $t = 1$ sec?

4. An object weighing 16 lb is dropped from a tall building. **(a)** Find the velocity of the object at time $t$ if the air resistance is equivalent to twice the instantaneous velocity. **(b)** What is the velocity of the object at $t = 1$ sec? How does this compare to the result in Exercise 3?
5. A ball of weight 4 oz is tossed into the air with an initial velocity of 64 ft/sec. **(a)** Find the velocity of the object at time $t$ if the air resistance is equivalent to 1/16 the instantaneous velocity. **(b)** When does the ball reach its maximum height?
6. A tennis ball weighing 8 oz is hit vertically into the air with an initial velocity of 128 ft/sec. **(a)** Find the velocity of the object at time $t$ if the air resistance is equivalent to half of the instantaneous velocity. **(b)** When does the ball reach its maximum height?

*7. A rock of weight 0.5 lb is dropped (with zero initial velocity) from a height of 300 ft. If the air resistance is equivalent to 1/64 times the instantaneous velocity, find the velocity and distance traveled by the object at time $t$. Does the rock hit the ground before 4 sec elapse?

8. An object of weight 0.5 lb is thrown downward with an initial velocity of 16 ft/sec from a height of 300 ft. If the air resistance is equivalent to 1/64 times the instantaneous velocity, find the velocity of and distance traveled by the object at time $t$. Compare these results to those in Exercise 7.
9. An object of mass 10 kg is dropped from a great height. **(a)** If the object is subjected to air resistance equivalent to 10 times the instantaneous velocity, find the velocity. **(b)** What is the limiting velocity of the object?
10. Suppose that an object of mass 1 kg is thrown with a downward initial velocity of 5 m/sec and is subjected to an air resistance equivalent to the instantaneous velocity. **(a)** Find the velocity of the object and the distance fallen at time $t$. **(b)** How far does the object drop after 5 sec?

*11. A projectile of mass 100 kg is launched vertically from ground level with an initial velocity of 100 m/sec. **(a)** If the projectile is subjected to air resistance equivalent to 1/10 times the instantaneous velocity, determine the velocity of and the height of the projectile at any time $t$. **(b)** What is the maximum height attained by the projectile?

12. In a carnival game, a participant uses a mallet to project an object up a 20-m pole. If the mass of the object is 1 kg and the object is subjected to resistance equivalent to 1/10 times the instantaneous velocity, determine if an initial velocity of 20 m/sec causes the object to reach the top of the pole.
13. Assuming that air resistance is ignored, find the velocity and height functions if an object with mass $m$ is thrown vertically up into the air with an initial velocity $v_0$ from an initial height $s_0$.
14. Use the results of Exercise 13 to find the velocity and height functions if $m = 1/128$, $g = 32$ ft/sec$^2$, $v_0 = 48$ ft/sec, and $s_0 = 0$ ft. What is the shape of the height function? What is the maximum height attained by the object? When does the object reach its maximum height? When does the object hit the ground? How do these results compare to those of Example 3?

*15. Consider the situation described in Exercise 13. What is the velocity of the object when it hits the ground, assuming that the object is thrown from ground level?

16. Consider the situation described in Exercise 13. If the object reaches its maximum height after $T$ sec, when does the object hit the ground, assuming that the object is thrown from ground level?
17. Suppose that an object of mass 10 kg is thrown vertically into the air with an initial velocity $v_0$ m/sec. If the limiting velocity is $-19.6$ m/sec, what can be said about $c$ in the force due to air resistance $F_R = cv$ acting on the object?
18. If the limiting velocity of an object of mass $m$ which is thrown vertically into the air with an initial velocity $v_0$ m/sec is $-9.8$ m/sec, what can be said about the relationship between $m$ and $c$ in the force due to air resistance $F_R = cv$ acting on the object?

*19. A parachutist weighing 192 lb falls from a plane (that is, $v_0 = 0$). When the parachutist's speed is 60 ft/sec, the parachute is opened and the parachutist is then subjected to air resistance equivalent to $F_R = 3v^2$. Find the velocity $v(t)$ of the parachutist. What is the limiting velocity of the parachutist?

20. A parachutist weighing 60 kg falls from a plane (that is, $v_0 = 0$) and is subjected to air resistance equivalent to $F_R = 10v$. After one minute, the parachute is opened so that the parachutist is subjected to air resistance equivalent to $F_R = 100v$. **(a)** What is the parachutist's velocity when the parachute is opened? **(b)** What is the parachutist's velocity $v(t)$ after the parachute is opened? **(c)** What is the parachutist's limiting velocity? How does this compare to the limiting velocity if the parachute does not open?
21. **(Escape Velocity)** Suppose that a rocket is launched from the Earth's surface. At a great (radial) distance $r$ from the center of the Earth, the rocket's acceleration is not the constant $g$. Instead, according to Newton's law of gravitation, $a = k/r^2$, where $k$ is the constant of proportionality ($k > 0$ if the rocket is falling toward the Earth; $k < 0$ if the rocket is moving away from the Earth). **(a)** If $a = -g$ at the Earth's surface (when $r = R$), find $k$ and show that the rocket's velocity is found by solving the initial-value problem $dv/dt = -gR^2/r^2$, $v(0) = v_0$. **(b)** Show that $dv/dt = v\,dv/dr$ so that the solution to the initial-value problem $v\,dv/dr = -gR^2/r^2$, $v(R) = v_0$ is $v^2 = (2gR^2/r) + v_0^2 - 2gR$. **(c)** Compute $\lim_{r\to\infty}$. If $v > 0$ (so that the rocket does not fall to the ground), show that the minimum value of $v_0$ for which this is true (even for very large values of $r$) is $v_0 = \sqrt{2gR}$. This value is called the **escape velocity** and signifies the minimum velocity required so that the rocket does not return to the Earth.
22. If $R \approx 3960$ miles and $g \approx 32$ ft/sec$^2 \approx 0.006$ mi/sec$^2$, use the results of Exercise 21 to compute the escape velocity of the Earth.

*23. *(See Exercise 21)* Determine the minimum initial velocity needed to launch the lunar module (used on early space missions) from the surface of the Earth's moon given that the moon's radius is $R \approx 1080$ miles and the acceleration of gravity of the moon is 16.5% of that of the Earth.

24. *(See Exercise 21)* Compare the Earth's escape velocity to those of Venus and Mars if for Venus $R \approx 3800$ miles and acceleration of gravity is 85% of that of the Earth; and for Mars $R \approx 2100$ and acceleration of gravity is 38% of that of the Earth. Which planet has the largest escape velocity? Which has the smallest?
25. In an electric circuit with one loop that contains a resistor $R$, a capacitor $C$, and a voltage source $E(t)$, the charge $Q$ on the capacitor is found by solving the initial-value problem $R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$, $Q(0) = Q_0$. Solve this problem to find $Q(t)$ if $E(t) = E_0$ where $E_0$ is constant. Find $I(t) = Q'(t)$ where $I(t)$ is the current at any time $t$.
26. If in the $R-C$ circuit described in Exercise 25 $C = 10^{-6}$ farads, $R = 4000$ ohms, and $E(t) = 200$ volts,

* Lyle N. Long and Howard Weiss, "The Velocity Dependence of Aerodynamic Drag: A Primer for Mathematicians," *The American Mathematics Monthly*, Volume 106, Number 2, pp. 127–135.

find $Q(t)$ and $I(t)$ if $Q(0) = 0$. What eventually happens to the charge and the current as $t \to \infty$?

*27. An object that weighs 48 lb is released from rest at the top of a plane metal slide that is inclined 30° to the horizontal. Air resistance (lb) is numerically equal to one-half the velocity (ft/sec), and the coefficient of friction is $\mu = 1/4$. Using Newton's second law of motion by summing the forces along the surface of the slide, we find the following forces:

(a) the component of the weight parallel to the slide: $F_1 = 48 \sin 30° = 24$;

(b) the component of the weight perpendicular to the slide: $N = 48 \cos 30° = 24\sqrt{3}$;

(c) the frictional force (against the motion of the object): $F_2 - \mu N = -\frac{1}{4}(24\sqrt{3}) = -6\sqrt{3}$; and

(d) the force due to air resistance (against the motion of the object): $F_3 = -\frac{1}{2}v$.
Because the mass of the object is $m = 48/32 = 3/2$, we find that the velocity of the object satisfies the initial-value problem $m\,dv/dt = F_1 + F_2 + F_3$ or

$$\frac{3}{2}\frac{dv}{dt} = 24 - 6\sqrt{3} - \frac{1}{2}v,\ v(0) = 0.$$

Solve this problem for $v(t)$. Determine the distance traveled by the object at time $t$, $x(t)$, if $x(0) = 0$.

28. A boat weighing 150 lb with a single rider weighing 170 lb is being towed in a particular direction at a rate of 20 mph. At $t = 0$, the tow rope is cut and the rider begins to row in the same direction, exerting a constant force of 12 lb in the direction that the boat is moving. The resistance is equivalent to twice the instantaneous velocity (ft/sec). The forces acting on the boat are $F_1 = 12$ in the direction of motion and the force due to resistance in the opposite direction, $F_2 = -2v$. Because the total weight (boat and rider) is 320 lb, $m = 320/32 = 10$. Therefore, the velocity satisfies the differential equation $m\,dv/dt = F_1 + F_2$ or $10\,dv/dt = 12 - 2v$ with initial velocity 20 mph, which is equivalent to $v(0) = 20\,\frac{\text{mi}}{\text{hr}} \times \frac{5280\text{ ft}}{1\text{ mi}} \times \frac{1\text{ hr}}{3600\text{ sec}} = \frac{88}{3}$ ft/sec. Find $v(t)$ and the distance traveled by the boat, $x(t)$, if $x(0) = 0$.

29. What is the equilibrium solution of $dv/dt = 32 - v$? How does this relate to the solution $v = 32 + Ce^{-t}$?

30. What is the equilibrium solution of $dv/dt = 32 - v^2$? How does this relate to the solution

$$v = \frac{-4\sqrt{2}(Ke^{8t\sqrt{2}} + 1)}{1 - Ke^{8t\sqrt{2}}}?$$

*31. Find the equilibrium solution to $dv/dt = -g - (c/m)v$. What is the limiting velocity?

32. Find the equilibrium solution to $dv/dt = -g - (c/m)v^2$. What is the limiting velocity?

33. Suppose that a falling body is subjected to air resistance assumed to be $F_R = cv$. Use the values of $c = 0.5$, 1, and 2; and plot the velocity function with $m = 1$, $g = 32$, and $v_0 = 0$. How does the value of $c$ affect the velocity?

34. Compare the effects that air resistance has on the velocity of a falling object of mass $m = 0.5$ that is released with an initial velocity of $v_0 = 16$. Consider $F_R = 16v^2$ and $F_R = 16v$.

*35. Compare the effects that air resistance has on the velocity of a falling object of mass $m = 0.5$ that is released with an initial velocity of $v_0 = 0$. Consider $F_R = 16v^3$ and $F_R = 16\sqrt{v}$.

36. Consider the velocity and height functions found in Example 3 with $m = 1/128$, $c = 1/160$, and $g = 32$ ft/sec$^2$.

(a) Suppose that on the first toss, the object is thrown with $v_0 = 48$ ft/sec from an initial height of $s_0 = 0$ ft and on the second toss with $v_0 = 36$ ft/sec and $s_0 = 6$. On which toss does the object reach the greater maximum height?

(b) If $s_0 = 0$ ft, compare the effect that the initial velocities $v_0 = 48$ ft/sec, $v_0 = 64$ ft/sec, and $v_0 = 80$ ft/sec have on the height function.

(c) If $v_0 = 48$ ft/sec, compare the effect that the initial heights $s_0 = 0$ ft, $s_0 = 10$ ft, and $s_0 = 20$ ft have on the height function.

37. A woman weighing 125 lb falls from an airplane at an altitude of 4000 ft and opens her parachute after 5 sec. If the force due to air resistance is $F_R = v$ before she opens her parachute and $F_R = 10v$ afterward, how long does it take for the woman to reach the ground?

38. Consider the problem discussed in Example 4. Instead of a buoyancy force equivalent to $6v$, suppose that when the object is in the pond, it is subjected to a buoyancy force equivalent to $6v^2$. Determine how much time is required for the object to reach a depth of 25 ft in the pond.



## CHAPTER 3 SUMMARY

### Concepts & Formulas

**Section 3.1**

**Malthus model**
The initial-value problem $dy/dt = ky$, $y(0) = y_0$ has the solution

$$y = y_0e^{kt}.$$

**Logistic equation (or Verhulst equation)**
The initial-value problem $y'(t) = (r - ay(t))y(t)$, $y(0) = y_0$ has the solution

$$y = \frac{ry_0}{ay_0 + (r - ay_0)e^{-rt}}.$$

**Section 3.2**

**Newton's law of cooling**
The initial-value problem $dT/dt = k(T - T_s)$, $T(0) = T_0$ has the solution

$$T = (T_0 - T_s)e^{kt} + T_s.$$

**Section 3.3**

**Newton's second law of motion**
The rate at which the momentum of a body changes with respect to time is equal to the resultant force acting on the body: $\frac{d}{dt}(mv) = F$.
The velocity of the falling body is found by solving the differential equation determined with

$$m\frac{dv}{dt} = \sum \text{(forces acting on the object)}.$$

## CHAPTER 3 REVIEW EXERCISES

In Exercises 1–4, classify the equilibrium solutions.

1. $y' = y(1 - 2y)$
2. $y' = -y$

*3. $y' = -\frac{1}{4}y(y - 4)$

4. $y' = -y\left(1 - \frac{y}{2}\right)\left(1 - \frac{y}{4}\right)$

5. The initial population in a bacteria culture is $y_0$. Suppose that after four days the population is $3y_0$. When is the population $5y_0$?

6. Suppose that a culture contains 200 cells. After one day the culture contains 600 cells. How many cells does the culture contain after two days?

*7. What percentage of the original amount of the element $^{226}$Ra remains after 50 years?

8. If an artifact contains 10% of the amount of $^{14}$C as that of a present-day sample, how old is the artifact?

9. Suppose that in an isolated population of 1000 people, 250 initially have a virus. If after one day, 500 have the virus, how many days are required for three-fourths of the population to acquire the virus?

10. The **Gompertz equation** given by $dy/dt = y(r - a \ln y)$ is used by actuaries to predict certain populations. If $y(0) = y_0$, then find $y(t)$. Find $\lim_{t\to\infty} y(t)$ if $a > 0$.

*11. A bottle that contains water with a temperature of 40°F is placed on a tennis court with temperature 90°F. After 20 min, the water is 65°F. What is the water's temperature after 30 min?

12. A can of diet cola at room temperature of 70°F is placed in a cooler with temperature 40°F. After 30 min the can is at 60°F. When is the can at 45°F?

13. A frozen turkey breast is placed in a microwave oven to defrost. It is then placed in a conventional oven at 325°F and bakes for 1 hour, at which time its temperature is 100°F. If after baking an additional 45 min its temperature is 150°F, what was the temperature of the turkey when it was removed from the microwave?

14. Suppose that during the month of July in Statesboro, Georgia, the outside temperature in °F is given by $C(t) = 85 - 10\cos(\pi t/12)$, $0 \le t \le 24$. Find the temperature in a parked car that has an initial temperature of 70°F if $k = 1/4$. (Assume that the car has no heat or air conditioning system.)

*15. A rock weighing 4 lb is dropped from rest from a large height and is subjected to air resistance equivalent to $F_R = v$. Find the velocity $v(t)$ of the rock at any time $t$. What is the velocity of the rock after 3 sec? How far has the rock fallen after 3 sec?

16. A container of waste weighing 6 lb is accidentally released from an airplane at an altitude of 1000 ft with an initial velocity of 6 ft/sec. If the container is subjected to air resistance equivalent to $F_R = 2v/3$, find the velocity $v(t)$ of the container at any time $t$. What is the velocity of the container after 5 sec? How far has the container fallen after 5 sec? Approximately when does the container hit the ground?

17. An object of mass 5 kg is thrown vertically in the air from ground level with an initial velocity of 40 m/sec. If the object is subjected to air resistance equivalent to $F_R = 5v$, find the velocity of the object at any time $t$. When does the object reach its maximum height? What is its maximum height?

18. A ball weighing 0.75 lb is thrown vertically in the air with an initial velocity of 20 ft/sec. If the ball is subjected to air resistance equivalent to $F_R = v/64$, find the velocity of the object at any time $t$. When does the object reach its maximum height? What is its maximum height if it is thrown from an initial height of 5 ft?

*19. A parachutist weighing 128 lb falls from a plane ($v_0 = 0$). When the parachutist's speed is 30 ft/sec, the parachute is opened and the parachutist is then subjected to air resistance equivalent to $F_R = 2v^2$. Find the velocity $v(t)$ of the parachutist. What is the limiting velocity of the parachutist?

20. A relief package weighing 256 lb is dropped from a plane ($v_0 = 0$) over a war-ravaged area and is subjected to air resistance equivalent to $F_R = 16v$. After 2 sec the parachute opens and the package is then subjected to air resistance equivalent to $F_R = 4v^2$. Find the velocity $v(t)$ of the package. What is the limiting velocity of the package? Compare this to the limiting velocity if the parachute does not open.

21. Atomic waste is placed in sealed canisters and dumped in the ocean. It has been determined that the seal will not break and leak the waste when the canister hits the bottom of the ocean as long as the velocity of the canister is less than 12 m/sec when it hits the bottom. Using Newton's second law, the velocity satisfies the equation $m(dv/dt) = W - B - kv$, where $v(0) = 0$, $W$ is the weight of the canister, $B$ is the buoyancy force, and the drag is given by $-kv$. Solve this first-order linear equation for $v(t)$ and then integrate to find the position $y(t)$. If $W = 2254$ Newtons, $B = 2090$ Newtons, and $k = 0.637$ kg/sec, determine the time at which the velocity is 12 m/s. Determine the depth $H$ of the ocean so that the seal will not break when the canister hits the bottom.

22. According to the **law of mass action,** if the temperature is constant, then the velocity of a chemical reaction is proportional to the product of the concentrations of the substances that are reacting. The reaction $A + B \to M$ combines $a$ moles per liter of substance $A$ and $b$ moles per liter of substance $B$. If $y(t)$ is the number of moles per liter that have reacted after time $t$, the reaction rate is $dy/dt = k(a - y)(b - y)$. Find $y(t)$ if $y(0) = 0$. Find $\lim_{t\to\infty} y(t)$ if $a > b$ and if $b > a$.

23. Solve the following equations for $r(\theta)$. Graph the polar equation that results.

(a) $r\dfrac{dr}{d\theta} + 4\sin 2\theta = 0$, $r(0) = 2$



*(b) $\dfrac{dr}{d\theta} - 2\sec\theta\tan\theta = 0$, $r(0) = 4$

(c) $\dfrac{dr}{d\theta} - 6\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2} = 0$, $r(0) = 0$

24. A cylindrical tank 1.50 m high stands on its circular base of radius $r = 0.50$ m and is initially filled with water. At the bottom of the tank, there is a hole of radius $r = 0.50$ cm that is opened at some instant so that draining starts due to gravity. According to **Torricelli's law,** $v = 0.600\sqrt{2gh}$, where $g = 980$ cm/sec$^2$ and $h$ is the height of the water. By determining the rate at which the volume changes, we find that $\dfrac{dh}{dt} = -\dfrac{0.600A\sqrt{2g}}{B}\sqrt{h}$, where $A$ is the cross-sectional area of the outlet and $B$ is the cross-sectional area of the tank. In this case, $A = 0.500^2\pi$ cm$^2$ and $B = 50.0^2\pi$ cm$^2$, so $dh/dt = -0.00266\sqrt{h}$. Find $h(t)$ if $h(0) = 150$.

25. **(Fishing)** Consider a population of fish with size at time $t$ given by $x(t)$. Suppose the fish are harvested at a rate of $h(t)$. If the fish are sold at price $p$ and $\delta$ is the interest rate, the present value $P$ of the harvest is given by the improper integral.

$$P = \int_0^\infty e^{-\delta t}\left[p - \frac{c}{qx(t)}\right]h(t)\,dt,^*$$

where $c$ and $q$ are constants related to the cost of the effort of catching fish ($q$ is called the **catchability**).

(a) Evaluate $P$ if $\delta = 0.05$, $x(t) = 1$, and $h(t) = \frac{1}{2}$.

If we assume that the harvesting rate $h(t)$ is proportional to the population of the fish, then

$$h(t) = qEx(t),$$

where $E$ represents the **effort** in catching the fish, and, under certain assumptions, the size of the population of the fish $x(t)$ satisfies the differential equation

$$\frac{dx}{dt} = (r - ax(t))x(t) - h(t)$$

$$\frac{dx}{dt} = (r - ax(t))x(t) - qEx(t).$$

This equation can be rewritten in the form

$$\frac{dx}{dt} = ((r - qE) - ax(t))x(t).$$

(b) Solve the equation $dx/dt = ((r - qE) - ax(t))x(t)$ and find the solution that satisfies the initial condition $x(0) = x_0$. What is $h(t)$?

Suppose that $x_0 = 1$, $r = 1$, and $a = 1/2$.

(c) Graph $x(t)$ if there is no harvesting for $0 \le t \le 10$. *Hint:* If there is no harvesting, $qE = 0$.

(d) Graph $x(t)$ and $h(t)$ for $0 \le t \le 20$ using $qE = 0$, 0.1, 0.2, . . . , 2. What is the maximum sustainable harvest rate? In other words, what is the highest rate at which the fish can be harvested without becoming extinct? At what rate should the fish be harvested to produce the largest overall harvest? How does this result compare to **(a)**?

In 1965, the values of $r$, $a$, $p$, $c$, and $q$ for the Antarctic whaling industry, were determined to be $r = 0.05$, $a = 1.25 \times 10^{-7}$, $p = 7000$, $c = 5000$, and $q = 1.3 \times 10^{-5}$. Assume that $t = 0$ corresponds to the year 1965 and that $x(0) = 78{,}000$. Assume that a typical firm expects a return of 10% on their investment, so that $\delta = 0.10$.

(e) Approximate

$$P = \int_0^\infty e^{-\delta t}\left[p - \frac{c}{qx(t)}\right]h(t)\,dt$$

if (i) $E = 5000$ and (ii) $E = 7000$. What happens to the whale population in each case? What advice would you give to the whaling industry?

(f) Approximate

$$P = \int_0^\infty e^{-\delta t}\left[p - \frac{c}{qx(t)}\right]h(t)\,dt$$

using the values in the following table.

* J. N. Kapur, "Some Problems in Biomathematics," *International Journal of Mathematical Education in Science and Technology,* Volume 9, Number 3 (August 1978), pp. 287–306; and Colin W. Clark, "Bioeconomic Modeling and Resource Management," *Applied Mathematical Ecology,* edited by Simon A. Levin, Thomas G. Hallam, and Louis J. Goss, Springer-Verlag, New York (1980), pp. 11–57. For more information, see Robert M. May, John R. Beddington, Colin W. Clark, Sidney J. Holt, and Richard M. Laws, "Management of Multispecies Fisheries," *Science,* Volume 205, Number 4403 (July 20, 1979), pp. 267–277.

| $E$ | Approximation of $P = \int_0^\infty e^{-\delta t}\left[p - \frac{c}{qx(t)}\right]h(t)\,dt$ |
|---|---|
| 1000 | |
| 1500 | |
| 2000 | |
| 2500 | |
| 3000 | |
| 3500 | |
| 4000 | |
| 4500 | |

What value of $E$ produces the maximum profit? What happens to the whale population in this case?

(g) Some reports have indicated that the optimal stock level of whales should be about 227,500. How does this number compare to the maximum number of whales that the environment can sustain? *Hint:* Evaluate $\lim_{t\to\infty} x(t)$ if there is no harvesting. How can the whaling industry make a profit and maintain this number of whales in the ocean?

26. **(Family of Orthogonal Trajectories)** Two curves $C_1$ and $C_2$ are orthogonal at a point of intersection if their respective tangent lines to the curves at that point are orthogonal. For example, consider the curves $y = x$ and $y = \sqrt{1-x^2}$. **(a)** Show that the derivatives of these functions are $y' = 1$ and $y' = -x/\sqrt{1-x^2}$, respectively. **(b)** Show that the curves are orthogonal at the point $(\sqrt{2}/2, \sqrt{2}/2)$ because the derivatives evaluated at this point are 1 and $-1$, respectively. Therefore, the tangent lines are perpendicular at $(\sqrt{2}/2, \sqrt{2}/2)$.

27. Given a family of curves $F$, we would like to find a set of curves that are orthogonal to each curve in $F$. We refer to this set of orthogonal curves as the *family of orthogonal trajectories*. Suppose that a family of curves is defined as $F(x, y) = c$ and that the slope of the tangent line at any point on these curves is $dy/dx = f(x, y)$ when it is obtained by differentiating $F(x, y) = c$ with respect to $x$ and solving for $dy/dx$. Then the slope of the tangent line of the orthogonal trajectory is $dy/dx = -1/f(x, y)$. Therefore, the family of orthogonal trajectories is found by solving $dy/dx = -1/f(x, y)$. Find the family of orthogonal trajectories of the set of curves $y = cx^2$ by carrying out the following steps. **(a)** Show that the slope of the tangent line at any point on the parabola $y = cx^2$ is $dy/dx = 2cx$. **(b)** Solve $y = cx^2$ for $c$ and substitute this result into $dy/dx = 2cx$ to obtain $dy/dx = 2y/x$. **(c)** Solve the equation $dy/dx = -1/f(x, y) = -x/(2y)$ to find that the family of orthogonal trajectories is $x^2/2 + y^2 = k$, a family of ellipses.

28. Let $T(x, y)$ represent the temperature at the point $(x, y)$. The curves given by $T(x, y) = c$, where $c$ is a constant, are called **isotherms.** The orthogonal trajectories are curves along which heat flows. Show that the isotherms are $(x^2/2) + xy - (y^2/2) = c$ if the curves of heat flow are given by $y^2 + 2xy - x^2 = k$. *Hint:* On the heat flow curves, $dy/dx = (x - y)/(x + y)$, so we must solve $dy/dx = (x + y)/(x - y)$, which is an exact differential equation.

In Exercises 29–32, determine the orthogonal trajectories of the given family of curves. (Graph the orthogonal trajectories and curves simultaneously.)

29. $y + 2x = c$

30. $y = e^{cx}$

*31. $y^2 = x^2 + c$

32. $y^2 = x^2 + cx$

33. A family of curves is **self-orthogonal** if the family of orthogonal trajectories is the same as the original family of curves. Is $y^2 - 2cx = c^2$ a self-orthogonal family of curves (parabolas)?

34. Find a value of $c$ so that the two families of curves $y = k_1x^2 + c$ and $x^2 + 2y^2 = k_2 + y$, where $k_1$ and $k_2$ are constants, are orthogonal.

35. Suppose that an electrical current is flowing in a wire along the $z$-axis. Then, the equipotential lines in the $xy$-plane are concentric circles centered at the origin. If the electric lines of force are the orthogonal trajectories of these circles, find the electric lines of force.

36. The path along which a fluid particle flows is called a **streamline,** and the orthogonal trajectories are called **equipotential lines.** If the streamlines are $y = k/x$, find the equipotential lines.

*37. **(Oblique Trajectories)** Let $\ell_1$ and $\ell_2$ denote two lines, not perpendicular to each other, with slopes $m_1$ and $m_2$, respectively; and let $\theta$ denote the angle between them as shown in Figure 3.15. Then,

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2m_1}.$$

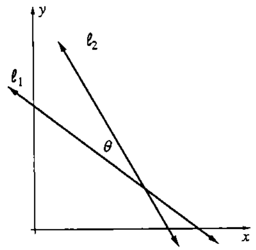



**Figure 3.15**

**(a)** Show that $m_2 = \dfrac{m_1 + \tan\theta}{1 - m_1\tan\theta}$.

**(b)** Suppose we are given a family of curves that satisfies the differential equation $dy/dx = f(x, y)$. Use **(a)** to show that if we want to find a family of curves that intersects this family at a constant angle $\theta$, we must solve the differential equation

$$\frac{dy}{dx} = \frac{f(x, y) \pm \tan\theta}{1 \mp f(x, y)\tan\theta}.$$

*Hint:* Studying Figure 3.16 might help.

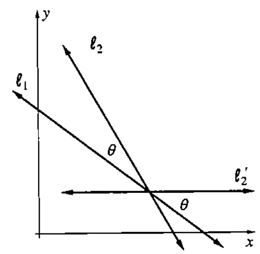



**Figure 3.16**

**(c)** Find a family of curves that intersects the family of curves $x^2 - y^2 = c$ at an angle of $\pi/4$. Graph several members of both families to confirm your result.

**(d)** Find a family of curves that intersects the family of curves $x^2 + y^2 = c^2$ at an angle of $\pi/6$. Graph several members of both families to confirm your result.

38. **(a)** Determine the orthogonal trajectories of the family of curves $y^2 = 2cx + 2c^2$. **(b)** Graph several members of both families of curves on the same set of axes. **(c)** What is your reaction to the graphs?



## Differential Equations at Work:

### A. Mathematics of Finance

Suppose that $P$ dollars are invested in an account at an annual rate of $r$% compounded continuously. To find the balance of the account $x(t)$ at time $t$, we must solve the initial-value problem

$$\begin{cases} \dfrac{dx}{dt} = rx \\ x(0) = P \end{cases}$$

for $x$.

1. Show that $x(t) = Pe^{rt}$.

2. If \$1000 is deposited into an account with an annual interest rate of 8% compounded continuously, what is the balance of the account at the end of 5, 10, 15, and 20 years?

If we allow additions or subtractions of sums of money from the account, the problem becomes more complicated. Suppose that an account like a savings account, home mortgage loan, student loan or car loan, has an initial balance of $P$ dollars and $r$ denotes the interest rate per compounding period. Let $p(t)$ denote the money flow per unit time and $\delta = \ln(1 + r)$. Then the balance of the account at time $t$, $x(t)$, satisfies the initial-value problem

$$\begin{cases} \dfrac{dx}{dt} - \delta x = p(t) \\ x(0) = P \end{cases}.^{*}$$

3. Show that the balance of the account at time $t_0$ is given by

$$x(t_0) = Pe^{\delta t_0} + e^{\delta t_0} \int_0^{t_0} p(t)e^{-\delta t}\, dt.$$

4. Suppose that the initial balance of a student loan is \$12,000 and that monthly payments are made in the amount of \$130. If the annual interest rate, compounded monthly, is 9%, then $P = 12{,}000$, $p(t) = -130$, $r = 0.09/12 = 0.0075$, and $\delta = \ln(1 + r) = \ln(1.0075)$.

   **(a)** Show that the balance of the loan at time $t$, in months, is given by

   $$x(t) = 17398.3 - 5398.25 \cdot 1.0075^t.$$

   **(b)** Graph $x(t)$ on the interval $[0, 180]$, corresponding to the loan balance for the first 15 years. **(c)** How long will it take to pay off the loan?

5. Suppose that the initial balance of a home mortgage loan is \$80,000 and that monthly payments are made in the amount of \$599. If the annual interest rate, compounded monthly, is 8%, how long will it take to pay off the mortgage loan?

6. Suppose that the initial balance of a home mortgage loan is \$80,000 and that monthly payments are made in the amount of \$599. If the annual interest rate, compounded monthly, is 8%, how long will it take to pay off the mortgage loan if the monthly payment is increased at an annual rate of 3%, which corresponds to a monthly increase of $\frac{1}{4}$%?

7. If an investor invests \$250 per month in an account paying an annual interest rate of 10%, compounded monthly, how much will the investor have accumulated at the end of 10, 20, and 30 years?

* Thoddi C. T. Kotiah, "Difference and Differential Equations in the Mathematics of Finance," *International Journal of Mathematics in Education, Science, and Technology,* Volume 22, Number 5 (1991), pp. 783–789.

Edward W. Herold, "Inflation Mathematics for the Professional," *Mathematical Modeling: Classroom Notes in Applied Mathematics,* Edited by Murray S. Klamkin, SIAM (1987), pp. 206–209.



8. Suppose an investor begins by investing \$250 per month in an account paying an annual interest rate of 10%, compounded monthly, and in addition increases the amount invested by 6% each year for an increase of $\frac{1}{2}$% each month. How much will the investor have accumulated at the end of 10, 20, and 30 years?

9. Suppose that a 25-year-old investor begins investing \$250 per month in an account paying an annual interest rate of 10%, compounded monthly. If at the age of 35 the investor stops making monthly payments, what will be the account balance when the investor reaches 45, 55, and 65 years of age? Suppose that a 35-year-old friend begins investing \$250 per month in an account paying an annual interest rate of 10%, compounded monthly, at the same time the first investor stops. Who has a larger investment at the age of 65?

10. If you are given a choice between saving \$150 a month beginning when you first start working and continuing until you retire, or saving \$300 per month beginning 10 years after you first start working and continuing until you retire, which should you do to help ensure a financially secure retirement?

From Exercises 8 and 9, we see that consistent savings beginning at an early age can help assure that a large sum of money will accumulate by the age of retirement. How much money does a person need to have accumulated to help ensure a financially secure retirement? For example, corporate pension plans and social security generally provide a relatively small portion of living expenses, especially for those with above-average incomes. In addition, inflation erodes the buying power of the dollar; large sums of money today will not necessarily be large sums of money tomorrow.

As an illustration, we see that if inflation were to average a modest 3% per year for the next 20 years and a person has living expenses of \$20,000 annually, then after 20 years the person would have living expenses of

$$\$20{,}000 \cdot 1.03^{20} \approx \$36{,}122.$$

Let $t$ denote years and suppose that a person's after-tax income as a function of years is given by $I(t)$ and $E(t)$ represents living expenses. Here $t = 0$ might represent the year a person enters the work force. Generally, during working years, $I(t) > E(t)$; during retirement years, when $I(t)$ represents income from sources such as corporate pension plans and social security, $I(t) < E(t)$.

Suppose that an account has an initial balance of $S_0$ and the after-tax return on the account is $r$%. We assume that the amount deposited into the account each year is $I(t) - E(t)$. What is the balance of the account at year $t$, $S(t)$? $S$ must satisfy the initial-value problem

$$\begin{cases} \dfrac{dS}{dt} = rS(t) + I(t) - E(t) \\ S(0) = S_0 \end{cases}.$$

11. Show that the balance of the account at time $t = t_0$ is

$$S(t_0) = e^{rt_0}\left\{S_0 + \int_0^{t_0} [I(t) - E(t)]e^{-rt}\, dt\right\}.$$

Assume that inflation averages an annual rate of $i$. Then, in terms of $E(0)$,

$$E(t) = E(0)e^{it}.$$

Similarly, during working years we assume that annual raises are received at an annual rate of $j$. Then, in terms of $I(0)$,

$$I(t) = I(0)e^{jt}.$$

However, during retirement years, we assume that $I(t)$ is given by a fixed sum $F$, such as a corporate pension or annuity, and a portion indexed to the inflation rate $V$, such as social security. Thus, during retirement years

$$I(t) = F + Ve^{i(t-T)},$$

where $T$ denotes the number of working years. Therefore,

$$I(t) = \begin{cases} I(0)e^{jt}, \ 0 \le t \le T \\ F + Ve^{i(t-T)}, \ t > T \end{cases}.$$

12. Suppose that a person has an initial income of $I(0) = 20{,}000$ and receives annual average raises of 5%, so that $j = 0.05$ and initial living expenses are $E(0) = 18{,}000$. Further, we assume that inflation averages 3%, so that $i = 0.03$, while the after-tax return on the investment is 6%, so that $r = 0.06$. Upon retirement after $T$ years of work, we assume that the person receives a fixed pension equal to 20% of his living expenses at that time, so that

$$F = 0.2 \cdot 18{,}000 \cdot e^{0.03T},$$

while social security provides 30% of his living expenses at that time, so that

$$V = 0.3 \cdot 18{,}000 \cdot e^{0.03T}.$$

(a) Find the smallest value of $T$ so that the balance in the account is zero after 30 years of retirement. Sketch a graph of $S$ for this value of $T$.
(b) Find the smallest value of $T$ so that the balance in the account is never zero. Sketch a graph of $S$ for this value of $T$.

13. What is the relationship between the results you obtained in Exercises 9 and 10 and that obtained in Exercise 12?
14. (a) How would you advise a person 22 years of age first entering the work force to prepare for a financially secure retirement?
(b) How would you advise a person 50 years of age with no savings who hopes to retire at 65 years of age?
(c) When should you start saving for retirement?

## B. Algae Growth

When wading in a river or stream, you may notice that microorganisms like algae are frequently found on rocks. Similarly, if you have a swimming pool, you may notice that without maintaining appropriate levels of chlorine and algaecides, small patches of algae take over the pool surface, sometimes overnight. Underwater surfaces are attractive environments for microorganisms because water movement removes wastes and



provides a continuous supply of nutrients. The organisms, however, must spread over the surface without being washed away. If conditions become unfavorable, they must be able to free themselves from the surface and recolonize on a new surface.

The rate at which cells accumulate on a surface is proportional to the rate of growth of the cells and the rate at which the cells attach to the surface. An equation describing this situation is given by

$$\frac{dN}{dt} = \mu(N + A),$$

where $N$ represents the cell density, $\mu$ the growth rate, $A$ the attachment rate, and $t$ time.*

1. If the attachment rate $A$ is constant, solve the initial-value problem

$$\begin{cases} \dfrac{dN}{dt} = \mu(N + A) \\ N(0) = 0 \end{cases}$$

for $N$ and then solve the result for $\mu$.

2. In a colony of cells it was observed that $A = 3$. The number of cells $N$ at the end of $t$ hours is shown in the following table. Estimate the growth rate at the end of each hour.

| $t$ | $N$ | $\mu$ |
|---|---|---|
| 1 | 3 | |
| 2 | 9 | |
| 3 | 21 | |
| 4 | 45 | |

3. Using the growth rate obtained in Problem 2, estimate the number of cells at the end of 24 hours and 36 hours.

## C. Dialysis

The primary purpose of the kidney is to remove waste products like urea, creatinine, and excess fluid from blood. When the kidneys are not working properly, wastes accumulate in the blood; when toxic levels are reached, death is certain. The leading causes of chronic kidney failure in the United States are hypertension (high blood pressure) and diabetes mellitus. In fact, one-fourth of all patients requiring **kidney**

* Douglas E. Caldwell, "Microbial Colonization of Solid-Liquid Interfaces," *Biochemical Engineering V*, Annals of the New York Academy of Sciences, Volume 56, New York Academy of Sciences (1987), pp. 274–280.

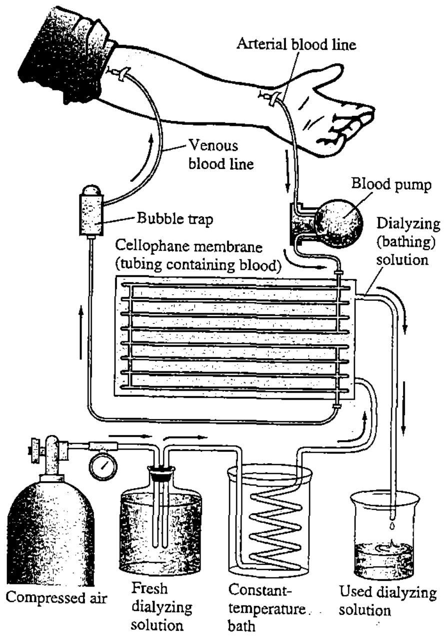



Diagram of a kidney dialysis machine

**dialysis** have diabetes. Fortunately, kidney dialysis removes waste products from the blood of patients with improperly working kidneys. During the hemodialysis process, the patient's blood is pumped through a **dialyser,** usually at a rate of 1–3 deciliters/min. The patient's blood is separated from the "cleaning fluid" by a semipermeable membrane, which permits wastes (but not blood cells) to diffuse to the cleaning fluid. The cleaning fluid contains some substances beneficial to the body which diffuse to the blood. The cleaning fluid, called the **dialysate,** is flowing in the **opposite** direction as the blood, usually at a rate of 2–6 dL/min. Waste products from the blood diffuse to the dialysate through the membrane at a rate proportional to the difference in concentration of the waste products in the blood and dialysate. If we let $u(x)$ represent the concentration of wastes in blood and $v(x)$ represent the concentration of wastes in the dialysate, where $x$ is the distance along the dialyser, $Q_D$ is the flow rate of the dialysate through the machine, and $Q_B$ is the flow rate of the blood through the machine, then

$$\begin{cases} Q_B u' = -k(u - v) \\ -Q_D v' = k(u - v) \end{cases},$$



where $k$ is the proportionality contant.*

We let $L$ denote the length of the dialyser, and the initial concentration of wastes in the blood is $u(0) = u_0$, while the initial concentration of wastes in the dialysate is $v(L) = 0$. Then, we must solve the initial-value problem

$$\begin{cases} Q_B u' = -k(u - v) \\ -Q_D v' = k(u - v) \\ u(0) = u_0,\ v(L) = 0 \end{cases}.$$

1. Show that the solution to

$$\begin{cases} Q_B u' = -k(u - v) \\ -Q_D v' = k(u - v) \\ u(0) = u_0,\ v(L) = 0 \end{cases}$$

is

$$u(x) = u_0 \frac{Q_B e^{\alpha x} - Q_D e^{\alpha L}}{e^{\alpha x}(Q_B - Q_D e^{\alpha L})}$$

$$v(x) = u_0 \frac{e^{\alpha L} - e^{\alpha x}}{e^{\alpha x}\dfrac{Q_D}{Q_B}(e^{\alpha L} - 1)},$$

where $\alpha = \dfrac{k}{Q_B} - \dfrac{k}{Q_D}$. First add the equations $\begin{cases} Q_B u' = -k(u - v) \\ -Q_D v' = k(u - v) \end{cases}$ to obtain the linear equation $\dfrac{d}{dx}(u - v) = -\dfrac{k}{Q_B}(u - v) + \dfrac{k}{Q_D}(u - v)$, then let $z = u - v$. Next solve for $z$ and subsequently solve for $u$ and $v$.

In healthy adults, typical urea nitrogen levels are 11–23 mg/dL (1 dL = 100 mL), while serum creatinine levels range from 0.6–1.2 mg/dL, and the total volume of blood is 4 to 5 L (1 L = 1000 mL).

2. Suppose that hemodialysis is performed on a patient with urea nitrogen level of 34 mg/dL and serum creatinine level of 1.8 using a dialyser with $k = 2.25$ and $L = 1$. If the flow rate of blood, $Q_B$, is 2 dL/min while the flow rate of the dialysate, $Q_D$, is 4 dL/min, will the level of wastes in the patient's blood reach normal levels after dialysis is performed? For what waste levels would dialysis have to be performed twice?
3. The **amount of waste removed** is given by $\int_0^L k[u(x) - v(x)]\,dx$. Show that

$$\int_0^L k[u(x) - v(x)]\,dx = Q_B[u_0 - u(L)].$$

4. The **clearance of a dialyser,** $CL$, is given by $CL = \dfrac{Q_B}{u_0}[u_0 - u(L)]$. Use the solution obtained in **(a)** to show that

* D. N. Burghess and M. S. Borrie, *Modeling with Differential Equations,* Ellis Horwood Limited, pp. 41–45. Joyce M. Black and Esther Matassarin-Jacobs, *Luckman and Sorensen's Medical-Surgical Nursing: A Psychophysiologic Approach,* Fourth Edition, W. B. Saunders Company (1993), pp. 1509–1519, 1775–1808.

$$CL = Q_B \frac{1 - e^{-\alpha L}}{1 - \frac{Q_B}{Q_D} e^{-\alpha L}}.$$

Typically, hemodialysis is performed 3–4 hours at a time three or four times per week. In some cases, a kidney transplant can free patients from the restrictions of dialysis. Of course, transplants have other risks not necessarily faced by those on dialysis; the number of available kidneys also affects the number of transplants performed. For example, in 1991 over 130,000 patients were on dialysis while only 7000 kidney transplants had been performed.

## D. Antibiotic Production

When you are injured or sick, your doctor may prescribe antibiotics to prevent or cure infections. In the journal article "Changes in the Protein Profile of *Streptomyces griseus* during a Cycloheximide Fermentation," we see that production of the antibiotic cycloheximide by *Streptomyces* is typical of antibiotic production. During the production of cycloheximide, the mass of *Streptomyces* grows relatively quickly and produces little cycloheximide. After approximately 24 hours, the mass of *Streptomyces* remains relatively constant and cycloheximide accumulates. However, once the level of cycloheximide reaches a certain level, extracellular cycloheximide is degraded **(feedback inhibited).** One approach to alleviating this problem and to maximize cycloheximide production is to remove extracellular cycloheximide continuously. The rate of growth of *Streptomyces* can be described by the equation

$$\frac{dX}{dt} = \mu_{\max}\left(1 - \frac{X}{X_{\max}}\right)X,$$

where $X$ represents the mass concentration in g/L, $\mu_{\max}$ is the maximum specific growth rate, and $X_{\max}$ represents the maximum mass concentration.*

1. Find the solution to the initial-value problem

$$\begin{cases} \dfrac{dX}{dt} = \mu_{\max}\left(1 - \dfrac{X}{X_{\max}}\right)X \\ X(0) = 1 \end{cases}$$

by first converting the equation $\frac{dX}{dt} = \mu_{\max}\left(1 - \frac{X}{X_{\max}}\right)X$ to a linear equation with the substitution $y = X^{-1}$.

Experimental results have shown that $\mu_{\max} = 0.3\ \text{hr}^{-1}$ and $X_{\max} = 10$ g/L.

* Kevin H. Dykstra and Henry Y. Wang, "Changes in the Protein Profile of *Streptomyces griseus* during a Cycloheximide Fermentation," *Biochemical Engineering V*, Annals of the New York Academy of Sciences, Volume 56, New York Academy of Sciences (1987), pp. 511–522.



2. Substitute these values into the result obtained in Problem 1. **(a)** Graph $X(t)$ on the interval [0, 24]. **(b)** Find the mass concentration at the end of 4, 8, 12, 16, 20, and 24 hours.

   The rate of accumulation of cycloheximide is the difference between the rate of synthesis and the rate of degradation:

$$\frac{dP}{dt} = R_s - R_d.$$

It is known that $R_d = K_d P$, where $k_d \approx 5 \times 10^{-3}\ \text{h}^{-1}$, so $dP/dt = R_s - R_d$ is equivalent to $dP/dt = R_s - K_d P$. Furthermore,

$$R_s = Q_{\text{po}} EX\left(1 + \frac{P}{K_I}\right)^{-1},$$

where $Q_{\text{po}}$ represents the specific enzyme activity with value $Q_{\text{po}} \approx 0.6$ g CH/g protein · hr and $K_I$ represents the inhibition constant. $E$ represents the intracellular concentration of an enzyme, which we will assume is constant. For large values of $K_I$ and $t$, $X(t) \approx 10$ and $(1 + (P/K_I))^{-1} \approx 1$. Thus, $R_s \approx 10Q_{\text{po}}E$, so

$$\frac{dP}{dt} = 10Q_{\text{po}}E - K_d P.$$

3. Solve the initial-value problem

$$\begin{cases} \dfrac{dP}{dt} = 10Q_{po}E - K_d P \\ p(24) = 0 \end{cases}$$

4. Graph $\frac{1}{E}P(t)$ on the interval [24, 1000].
5. What happens to the net accumulation of the antibiotic as time increases?
6. If instead the antibiotic is removed from the solution so that no degradation occurs, what happens to the net accumulation of the antibiotic as time increases?

# 4

# Higher Order Equations

In Chapters 2 and 3, we saw that first-order differential equations can be used to model a variety of physical situations. However, there are many physical situations that need to be modeled by higher order differential equations.

For example, in 1735, Daniel Bernoulli's (1700–1782) study of the vibration of an elastic beam led to the fourth-order differential equation

$$k^4 \frac{d^4y}{dx^4} = y,$$

which describes the displacement of the *simple modes*. This equation can be rewritten in the form

$$y - k^4 \frac{d^4y}{dx^4} = 0.$$

Both Bernoulli and Euler realized that a solution to the equation is $y = e^{x/k}$ but that other solutions to the equation must also exist.





## 4.1 Second-Order Equations: An Introduction

The Second-Order Linear Homogeneous Equation with Constant Coefficients · The General Case · Reduction of Order

In this chapter, we focus our attention on solving linear ordinary differential equations of order two or higher. There are many instances in which we encounter equations of this type. For example, we see them in the study of vibrations and sound, and as we mentioned in Chapter 1, there is a relationship between a system of two first-order ODEs and a second-order ODE.

Let's consider the second-order equation $y'' + y = 0$ or $y'' = -y$. To find a solution, we must determine a function with the property that the second derivative of the function is the negative of the function itself. When we consider functions familiar to us, such as polynomials, trigonometric functions, exponential functions, and natural logarithms, we can conclude that $y_1(t) = \sin t$ is a solution because

$$y_1' = \cos t \text{ and } y_1'' = -\sin t; \qquad y_1'' = -y_1$$

Similarly, $y_2(t) = \cos t$ is a solution because

$$y_2' = -\sin t \text{ and } y_2'' = -\cos t; \qquad y_2'' = -y_2$$

Of course, we are not always able to determine solutions of a differential equation by inspection as we have in this case. Therefore, we now investigate a more general approach to solving second-order equations. We begin this discussion by considering a problem studied in physics.

### The Second-Order Linear Homogeneous Equation with Constant Coefficients

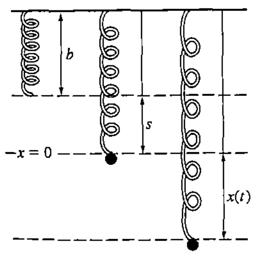



**Figure 4.1** A spring–mass system

Suppose that we attach an object to the end of a spring that is mounted to a horizontal rod (see Figure 4.1). The object comes to rest. We call this the equilibrium position. Let $x(t)$ represent the displacement of the object from equilibrium. If $x(t) > 0$, then the object is below equilibrium; if $x(t) < 0$, it is above equilibrium. If we assume there is a damping force acting on the object that impedes motion, then we can find $x(t)$ by solving a second-order linear ODE such as

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0.$$

This is a **linear second-order ODE** because it has the form

$$a_2(t)\frac{d^2x}{dt^2} + a_1(t)\frac{dx}{dt} + a_0(t)x = f(t),$$

and the highest order derivative in the equation is of order two. Of course in this equation, $a_2(t) = 1$, $a_1(t) = 3$, and $a_0(t) = 2$, so the coefficient functions are constant. In addition, $f(t) = 0$ for all $t$, so this equation is **homogeneous.** (We give more details on this spring–mass problem in Chapter 5.) For now, we attempt to find a solution of this

ODE by assuming that solutions have the form $x(t) = e^{rt}$ because $e^{rt}$ and its derivatives are constant multiples of one another. We find the value(s) of $r$ that lead to a solution by substituting $x(t) = e^{rt}$ into the ODE. Notice that $dx/dt = re^{rt}$ and $d^2x/dt^2 = r^2e^{rt}$. Substitution into $d^2x/dt^2 + 3\,(dx/dt) + 2x = 0$ then gives us $r^2e^{rt} + 3re^{rt} + 2e^{rt} = 0$ or

$$e^{rt}(r^2 + 3r + 2) = 0.$$

This algebraic equation tells us that either $e^{rt} = 0$ or $r^2 + 3r + 2 = 0$. We know that there are no values of $t$ so that $e^{rt} = 0$. Therefore, we solve the *characteristic equation*

$$r^2 + 3r + 2 = 0$$

to find the values of $r$ that satisfy the equation. In the case of a second-order equation, the characteristic equation is quadratic, so we can either factor or use the quadratic formula. Factoring yields $(r + 1)(r + 2) = 0$, so $r = -1$ or $r = -2$. This means that $x(t) = e^{-t}$ and $x(t) = e^{-2t}$ each satisfy the ODE. We verify this by substituting these two functions into the ODE. If $x(t) = e^{-t}$, then $dx/dt = -e^{-t}$ and $d^2x/dt^2 = e^{-t}$. Therefore,

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = e^{-t} + 3(-e^{-t}) + 2e^{-t} = 0.$$

Because simplification of the left side of the ODE yields zero (the right side of the equation), $x(t) = e^{-t}$ satisfies the ODE. In a similar manner, we verify that $x(t) = e^{-2t}$ also satisfies the ODE. Notice that these two functions are not constant multiples of one another, so we say that $x(t) = e^{-t}$ and $x(t) = e^{-2t}$ are *linearly independent*. Notice also that for arbitrary constants $c_1$ and $c_2$, the function $x(t) = c_1e^{-t} + c_2e^{-2t}$, called the *linear combination* of $x(t) = e^{-t}$ and $x(t) = e^{-2t}$, is also a solution of the ODE. We verify this through substitution as well. If $x(t) = c_1e^{-t} + c_2e^{-2t}$, then $dx/dt = -c_1e^{-t} - 2c_2e^{-2t}$ and $d^2x/dt^2 = c_1e^{-t} + 4c_2e^{-2t}$. Therefore,

$$\begin{aligned}\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x &= (c_1e^{-t} + 4c_2e^{-2t}) + 3(-c_1e^{-t} - 2c_2e^{-2t}) + 2(c_1e^{-t} + c_2e^{-2t})\\ &= c_1(e^{-t} - 3e^{-t} + 2e^{-t}) + c_2(4e^{-2t} - 6e^{-2t} + 2e^{-2t})\\ &= 0\end{aligned}$$

This property, called the *Principle of Superposition*, tells that if we have two linearly independent solutions of a linear homogeneous ODE, then any linear combination of the two solutions is also a solution. We call $x(t) = c_1e^{-t} + c_2e^{-2t}$ a *general solution* of this second-order ODE because all solutions are obtained from it. In Figure 4.2, we graph this solution for several different choices for $c_1$ and $c_2$.

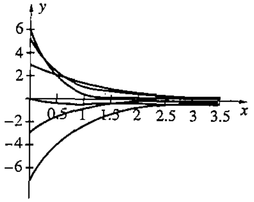



**Figure 4.2** Graph of general solution for various values of $c_1$ and $c_2$

If we are given additional information about the spring–mass system such as the initial displacement and the initial velocity of the object, then we have a *second-order initial-value problem* (IVP). For example, suppose that the object is released from equilibrium ($x = 0$) with initial velocity $dx/dt\,(0) = -1$. Therefore, we need to solve

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0, \qquad x(0) = 0,\ \frac{dx}{dt}(0) = -1.$$



After finding a general solution of the ODE, we apply the two initial conditions. In this case, we have $x(0) = c_1 + c_2$ and $dx/dt\,(0) = -c_1 - 2c_2$, so we solve the system

$$\begin{cases} c_1 + c_2 = 0 \\ -c_1 - 2c_2 = -1 \end{cases}$$

for $c_1$ and $c_2$. Adding these equations, we obtain $-c_2 = -1$, so $c_2 = 1$. Substituting this value into $c_1 + c_2 = 0$ then indicates that $c_1 = -1$. Therefore, the unique solution to this IVP is

$$x(t) = e^{-2t} - e^{-t},$$

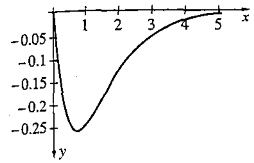



**Figure 4.3** Graph of solution $x(t) = e^{-2t} - e^{-t}$ to the IVP

which we graph in Figure 4.3. Notice that the curve passes through (0, 0) because of the initial condition $x(0) = 0$. The curve then moves in the downward direction because $dx/dt\,(0) = -1$. Notice also that $\lim_{t\to\infty} x(t) = 0$ (as with any general solution shown in Figure 4.2). This means that the object eventually comes to rest.

In the previous problem, a general solution is $x(t) = c_1e^{-t} + c_2e^{-2t}$. Notice that if we select $c_1 = c_2 = 0$, we obtain the *trivial solution* $x(t) = 0$. This shows us that the trivial solution always satisfies a linear homogeneous ODE. Therefore, when solving equations of this type, we seek *nontrivial solutions*.

## The General Case

Now that we have discussed a representative problem considered in this chapter, we can state more precisely several theorems and definitions that were mentioned earlier. Let's begin with the definition of linear dependence and linear independence.

**Definition 4.1 Linearly Dependent**

Let $S = \{f_1(t), f_2(t)\}$ be a set of functions. The set $S$ is **linearly dependent** on an interval $I$ if there are constants $c_1$ and $c_2$ not both zero, so that

$$c_1f_1(t) + c_2f_2(t) = 0$$

for every value of $t$ in the interval $I$. The set $S$ is **linearly independent** if $S$ is not linearly dependent (that is, if $c_1 = c_2 = 0$).

*Note:* We can rewrite $c_1f_1(t) + c_2f_2(t) = 0$ as $f_2(t) = -(c_1/c_2)f_1(t)$, so we say that the *two* functions $f_1(t)$ and $f_2(t)$ are linearly dependent if they are constant multiples. They are linearly *in*dependent otherwise.

**Example 1**

Determine if the following sets of functions are linearly dependent or linearly independent. **(a)** $\{2t, 4t\}$; **(b)** $\{e^t, e^{-t}\}$; **(c)** $\{\sin 2t, 5\sin t\cos t\}$; **(d)** $\{t, 0\}$.

**Solution** (a) Notice that $4t = 2(2t)$, so the functions are linearly dependent. (b) These functions are not constant multiples of one another, so they are linearly independent. (c) We use the identity $\sin 2t = 2 \sin t \cos t$ to show that $\frac{5}{2} \sin 2t = \frac{5}{2}(2 \sin t \cos t) = 5 \sin t \cos t$, so the functions are linearly dependent. (d) This set contains the zero function. Because we can obtain the zero function by multiplying the second function by 0, $0 \cdot t = 0$, the functions are linearly dependent. Therefore, if 0 is a member of a set of two functions, the set is linearly dependent.

Consider the second-order linear ODE

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = F(t),$$

where the coefficient functions $a_2(t)$, $a_1(t)$, $a_0(t)$, and $F(t)$ are continuous on the open interval $I$. Also, assume that $a_2(t) \neq 0$ for each value of $t$ in $I$ so that we may divide the equation by $a_2(t)$ to write the equation in normal form given by

$$y'' + p(t)y' + q(t)y = f(t),$$

where $p(t) = a_1(t)/a_2(t)$, $q(t) = a_0(t)/a_2(t)$, and $f(t) = F(t)/a_2(t)$. If $f(t) \equiv 0$, that is, if $f(t)$ is identically the zero function, then the equation is **homogeneous** and is represented by

$$y'' + p(t)y' + q(t)y = 0.$$

If $f(t)$ is not identically the zero function, the equation, $y'' + p(t)y' + q(t)y = f(t)$, is **nonhomogeneous.**

**Theorem 4.1 Principle of Superposition**

If $f_1(t)$ and $f_2(t)$ are solutions of the linear homogeneous ODE $y'' + p(t)y' + q(t)y = 0$ on the interval $I$, and if $c_1$ and $c_2$ are arbitrary constants, then $y = c_1 f_1(t) + c_2 f_2(t)$ is also a solution of $y'' + p(t)y' + q(t)y = 0$ on $I$.

***PROOF OF THEOREM 4.1***

Suppose that $f_1(t)$ and $f_2(t)$ are solutions of the linear homogeneous ODE $y'' + p(t)y' + q(t)y = 0$ on the interval $I$. Then, $f_1''(t) + p(t)f_1'(t) + q(t)f_1(t) = 0$ and $f_2''(t) + p(t)f_2'(t) + q(t)f_2(t) = 0$. Let $y = c_1 f_1(t) + c_2 f_2(t)$. Then, $y' = c_1 f_1'(t) + c_2 f_2'(t)$ and $y'' = c_1 f_1''(t) + c_2 f_2''(t)$. Substitution into the left side of the ODE yields

$$\begin{aligned} y'' + p(t)y' + q(t)y &= \left(c_1 f_1''(t) + c_2 f_2''(t)\right) + p(t)\left(c_1 f_1'(t) + c_2 f_2'(t)\right) \\ &\quad + q(t)\left(c_1 f_1(t) + c_2 f_2(t)\right) \\ &= c_1\left(f_1''(t) + p(t)f_1'(t) + q(t)f_1(t)\right) \\ &\quad + c_2\left(f_2''(t) + p(t)f_2'(t) + q(t)f_2(t)\right) \\ &= c_1(0) + c_2(0) = 0 \end{aligned}$$

Therefore, $y = c_1 f_1(t) + c_2 f_2(t)$ is also a solution of $y'' + p(t)y' + q(t)y = 0$.



Another way to express Theorem 4.1 is to say that any linear combination of two or more solutions of $y'' + p(t)y' + q(t)y = 0$ is also a solution of this ODE.

**Example 2**

Show that $y_1(t) = \cos 2t$ and $y_2(t) = \sin 2t$ are solutions of the second-order linear homogeneous ODE $y'' + 4y = 0$. Use the Principle of Superposition to find another solution of this ODE.

**Solution** For $y_1(t) = \cos 2t$, we have $y_1'(t) = -2 \sin 2t$ and $y_1''(t) = -4 \cos 2t$. Then, $y_1'' + 4y_1 = (-4 \cos 2t) + 4(\cos 2t) = 0$, so $y_1$ satisfies the ODE. In a similar manner, we can show that $y_2$ is a solution of the ODE. (We leave this task to you as an exercise.) Therefore, by the Principle of Superposition, any linear combination of these two functions, $y(t) = c_1 \cos 2t + c_2 \sin 2t$, is also a solution of the ODE. Notice that the two functions $y_1(t) = \cos 2t$ and $y_2(t) = \sin 2t$ are linearly independent because they are not constant multiples of each other. Then, because the ODE is second-order and we know two linearly independent solutions, a *general solution* of the ODE is the linear combination of these two functions, $y(t) = c_1 \cos 2t + c_2 \sin 2t$.

As with first-order initial value problems, we can state an existence and uniqueness theorem for an initial-value problem involving a second-order linear ordinary differential equation.

**Theorem 4.2 Existence and Uniqueness**

Suppose that $p(t)$, $q(t)$, and $f(t)$ are continuous functions on an open interval $I$ that contains $t = t_0$. Then, the initial-value problem

$$y'' + p(t)y' + q(t)y = f(t),\ y(t_0) = a,\ y'(t_0) = b$$

has a unique solution on $I$.

**Example 3**

Find the solution to the IVP $y'' + 4y = 0$, $y(0) = 4$, $y'(0) = -4$.

**Solution** We found a general solution of the ODE, $y(t) = c_1 \cos 2t + c_2 \sin 2t$, in Example 2. Application of the initial condition $y(0) = 4$ gives us $y(0) = c_1 \cos 0 + c_2 \sin 0 = c_1$, so $c_1 = 4$. To apply the condition $y'(0) = -4$, we compute $y'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t$, so that $y'(0) = -2c_1 \sin 0 + 2c_2 \cos 0 =$

$2c_2$. Therefore, $2c_2 = -4$, so $c_2 = -2$. The unique solution to the IVP is $y(t) = 4\cos 2t - 2\sin 2t$. This solution is unique over the interval $(-\infty, \infty)$ because the coefficient functions $p(t) = 0$, $q(t) = 4$, and $f(t) = 0$ are continuous on $(-\infty, \infty)$, and the Existence and Uniqueness Theorem guarantees that the solution is as well.

Now we discuss why we can write every solution of the ODE, $y'' + p(t)y' + q(t)y = 0$ with linearly independent solutions $y_1(t)$ and $y_2(t)$, as the linear combination $y(t) = c_1y_1(t) + c_2y_2(t)$. Generally, we call the form $y(t) = c_1y_1(t) + c_2y_2(t)$ a *general solution* of this ODE. We begin by stating the following definition.

**Definition 4.2 Wronskian of a Set of Two Functions**

Let $S = \{f_1(t), f_2(t)\}$ be a set of differentiable functions. The **Wronskian of** $S$, denoted by $W(S) = W(f_1(t), f_2(t))$, is the determinant

$$W(S) = \begin{vmatrix} f_1(t) & f_2(t) \\ f_1'(t) & f_2'(t) \end{vmatrix} = f_1(t)f_2'(t) - f_1'(t)f_2(t).$$

**Example 4**

Compute the Wronskian of each of the following sets of functions. **(a)** $S = \{\sin t, \cos t\}$; **(b)** $S = \{e^t, te^t\}$; **(c)** $S = \{2t, 4t\}$.

***Solution*** We compute a $2 \times 2$ determinant with $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

**(a)** $$W(S) = \begin{vmatrix} \sin t & \cos t \\ \frac{d}{dt}(\sin t) & \frac{d}{dt}(\cos t) \end{vmatrix} = \begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix}$$
$$= -\sin^2 t - \cos^2 t = -(\sin^2 t + \cos^2 t) = -1.$$

**(b)** $$W(S) = \begin{vmatrix} e^t & te^t \\ \frac{d}{dt}(e^t) & \frac{d}{dt}(te^t) \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & te^t + e^t \end{vmatrix}$$
$$= e^t(te^t + e^t) - e^t(te^t) = te^{2t} + e^{2t} - te^{2t} = e^{2t}.$$

**(c)** $$W(S) = \begin{vmatrix} 2t & 4t \\ 2 & 4 \end{vmatrix} = 8t - 8t = 0$$

In part **(c)** of Example 4, we found that $W(S) = 0$. Recall also that we concluded in Example 1 that the functions in $S = \{2t, 4t\}$ are linearly dependent.



**Theorem 4.3 Wronkians of Solutions**

Consider the second-order linear homogeneous equation $y'' + p(t)y' + q(t)y = 0$, where $p(t)$ and $q(t)$ are continuous on the open interval $I$. Suppose that $y_1(t)$ and $y_2(t)$ are solutions of this ODE on $I$. If $y_1(t)$ and $y_2(t)$ are linearly dependent, then $W(y_1(t), y_2(t)) = 0$ on $I$. If $y_1(t)$ and $y_2(t)$ are linearly independent, then $W(y_1(t), y_2(t)) \neq 0$ for all values on $I$.

We omit the proof of this theorem here. We leave it as part of the section exercises (see Abel's formula).

**Example 5**

Show that $y_1(t) = t^{2/3}$ and $y_2(t) = t$ are solutions of $3t^2y'' - 2ty' + 2y = 0$. Compute the Wronskian of $S = \{t^{2/3}, t\}$. Does your result contradict Theorem 4.3?

***Solution*** First, we find $y_1'(t) = 2t^{-1/3}/3$ and $y_1''(t) = -2t^{-4/3}/9$. Then, $3t^2y'' - 2ty' + 2y = 3t^2(-2t^{-4/3}/9) - 2t(2t^{-1/3}/3) + 2t^{2/3} = (-\frac{2}{3} - \frac{4}{3} + 2)t^{2/3} = 0$, so $y_1(t) = t^{2/3}$ satisfies the ODE. In a similar manner, we can show that $y_2(t) = t$ is also a solution. (We leave this to you as an exercise.) Now,

$$W(S) = \begin{vmatrix} t^{2/3} & t \\ \frac{2t^{-1/3}}{3} & 1 \end{vmatrix} = t^{2/3} - \frac{2}{3}t^{2/3} = \frac{1}{3}t^{2/3}.$$

Notice that $W(S) \neq 0$ for all values of $t$ except $t = 0$. However, this does not contradict Theorem 4.3. After writing the ODE in normal form, $y'' - 2/(3t)y' + 2/(3t^2)y = 0$, we see that the coefficient functions $p(t) = -2/(3t)$ and $q(t) = 2/(3t^2)$ are not continuous at $t = 0$. The theorem holds only on an open interval $I$ where these functions are continuous.

**Theorem 4.4 General Solution**

Consider the second-order linear homogeneous equation $y'' + p(t)y' + q(t)y = 0$, where $p(t)$ and $q(t)$ are continuous on the open interval $I$. Suppose that $y_1(t)$ and $y_2(t)$ are *linearly independent solutions* of this ODE on $I$. Then, if $Y$ is any solution of this ODE, there are constants $c_1$ and $c_2$ so that $Y(t) = c_1y_1(t) + c_2y_2(t)$ for all $t$ on $I$.

This theorem states that if we have two linearly independent solutions of a second-order linear homogeneous equation, then we have all solutions, called a **gen-**

**eral solution,** given by the linear combination of the two solutions. In addition, we call $\{y_1(t), y_2(t)\}$ a **fundamental set of solutions,** and we note that a fundamental set of solutions for $y'' + p(t)y' + q(t)y = 0$ must contain two linearly independent solutions. The number of functions in the set equals the order of the ODE, and these functions must be linearly independent. We will generalize this result to higher-order equations later in this chapter.

***PROOF OF THEOREM 4.4***

Let $t = t_0$ be a value on $I$ where $W(y_1(t), y_2(t)) \neq 0$. Then, $Y(t_0) = c_1y_1(t_0) + c_2y_2(t_0)$ and $Y'(t_0) = c_1y_1'(t_0) + c_2y_2'(t_0)$. We can write this system of two equations as

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} Y(t_0) \\ Y'(t_0) \end{pmatrix},$$

where the determinant of the matrix of coefficients at $t = t_0$, $W(y_1(t_0), y_2(t_0)) = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)$, is not zero because $y_1(t)$ and $y_2(t)$ are linearly independent. This means that the system has a nontrivial solution. That is, $(c_1, c_2) \neq (0, 0)$, or equivalently, at least one of the values $c_1$ or $c_2$ is not zero. These values can be found through elimination to be

$$c_1 = \frac{Y(t_0)y_2'(t_0) - Y'(t_0)y_2(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)} \quad \text{and} \quad c_2 = \frac{-Y(t_0)y_1'(t_0) + Y'(t_0)y_1(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}.$$

Notice that these values make sense only if $y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0$, as we mentioned earlier in connection to the linear independence of $y_1(t)$ and $y_2(t)$. Using these values of $c_1$ and $c_2$, let $Z(t) = c_1y_1(t) + c_2y_2(t)$ be a solution of $y'' + p(t)y' + q(t)y = 0$. Then, in addition to satisfying the ODE, $Z(t_0) = c_1y_1(t_0) + c_2y_2(t_0) = Y(t_0)$ and $Z'(t_0) = c_1y_1'(t_0) + c_2y_2'(t_0) = Y'(t_0)$. By the Existence and Uniqueness Theorem, the IVP

$$y'' + p(t)y' + q(t)y = 0, y(t_0) = Y(t_0), y'(t_0) = Y'(t_0)$$

has a unique solution. Because $Z(t)$ and $Y(t)$ each satisfy the ODE and have the same initial conditions, they must be equal (that is, $Z(t) \equiv Y(t)$ on $I$). There is only one solution to the IVP.

**Example 6**

Show that $y = c_1e^{-5t} + c_2e^{2t}$ is a general solution of $y'' + 3y' - 10y = 0$.

***Solution*** First, we must verify that $y_1(t) = e^{-5t}$ and $y_2(t) = e^{2t}$ satisfy the ODE. With $y_1'(t) = -5e^{-5t}$ and $y_1''(t) = 25e^{-5t}$, we have

$$y_1'' + 3y_1' - 10y_1 = 25e^{-5t} + 3(-5e^{-5t}) - 10e^{-5t} = 0,$$



so $y_1(t) = e^{-5t}$ is a solution. In a similar manner, we can show that $y_2(t) = e^{2t}$ is also a solution. Next, we verify that these functions are linearly independent. We can establish this by noting that the two are not constant multiples, or we can compute

$$W(y_1, y_2) = \begin{vmatrix} e^{-5t} & e^{2t} \\ -5e^{-5t} & 2e^{2t} \end{vmatrix} = 2e^{-3t} - (-5e^{-3t}) = 7e^{-3t}.$$

The Wronskian is not identically zero, so the functions are linearly independent. Therefore, because we have two linearly independent solutions to the second-order ODE, a general solution is the linear combination of the solutions, $y = c_1e^{-5t} + c_2e^{2t}$. In addition, we say that $\{e^{-5t}, e^{2t}\}$ is a fundamental set of solutions of $y'' + 3y' - 10y = 0$.

## Reduction of Order

In the next section, we learn how to find solutions of homogeneous equations with constant coefficients. In doing so, we will find it necessary to determine a second linearly independent solution from a known solution. We illustrate this procedure, called **reduction of order,** by considering a second-order equation.

Suppose we have the equation

$$y'' + p(t)y' + q(t)y = 0,$$

and that $y = f(t)$ is a solution. We know from our previous discussion that to find a general solution of this second-order equation, we must have two linearly independent solutions. We must determine a second linearly independent solution. We accomplish this by attempting to find a solution of the form

$$y = v(t)f(t)$$

and solving for $v(t)$. Differentiating with the product rule, we obtain

$$y' = f'v + v'f \quad \text{and} \quad y'' = f''v + 2v'f' + fv''.$$

For convenience, we have omitted the argument of these functions. We now substitute $y$, $y'$, and $y''$ into the equation $y'' + p(t)y' + q(t)y = 0$. This gives us

$$\begin{aligned} y'' + p(t)y' + q(t)y &= f''v + 2v'f' + fv'' + p(t)(f'v + fv') + q(t)vf \\ &= \underbrace{[f'' + p(t)f' + q(t)f]}_{=0}v + fv'' + 2v'f' + p(t)fv' \\ &= fv'' + (2f' + p(t)f)v'. \end{aligned}$$

Therefore, we have the equation

$$fv'' + (2f' + p(t)f)v' = 0,$$

which can be written as a first-order equation by letting $w = v'$. Making this substitution gives us the linear first-order equation

$$fw' + (2f' + p(t)f)w = 0 \quad \text{or} \quad f\frac{dw}{dt} + (2f' + p(t)f)w = 0$$

which is separable, so we obtain the separated equation

$$\frac{dw}{w} = \left(-\frac{2f'}{f} - p\right)dt.$$

We solve this equation by integrating both sides of the equation to yield

$$\ln|w| = \ln\left(\frac{1}{f^2}\right) - \int p(t)\,dt.$$

This means that $w = \frac{1}{f^2}e^{-\int p(t)dt}$, so we have the formula $\frac{dv}{dt} = \frac{1}{f^2}e^{-\int p(t)dt}$ or

$$v(t) = \int \frac{e^{-\int p(t)dt}}{[f(t)]^2}\,dt.$$

If $y = f(t)$ is a known solution of the differential equation $y'' + p(t)y' + q(t)y = 0$, we can obtain a second linearly independent solution of the form $y = f(t)v(t)$, where $v(t) = \int \frac{e^{-\int p(t)dt}}{[f(t)]^2}\,dt$.

We leave the proof that $y_1(t) = f(t)$ and $y_2(t) = f(t)v(t) = f(t)\int \frac{e^{-\int p(t)dt}}{[f(t)]^2}\,dt$ are linearly independent as an exercise.

**Example 7**

Determine a second linearly independent solution to the differential equation $y'' + 6y' + 9y = 0$ given that $y = e^{-3t}$ is a solution.

**Solution** First we identify the functions $p(t) = 6$ and $f(t) = e^{-3t}$. Then we determine the function $v(t)$ such that $y(t) = f(t)v(t)$ with the formula

$$v(t) = \int \frac{e^{-\int p(t)dt}}{[f(t)]^2}\,dt = \int \frac{e^{-\int 6\,dt}}{[e^{-3t}]^2}\,dt = \int \frac{e^{-6t}}{e^{-6t}}\,dt = \int dt = t.$$

A second linearly independent solution is $y = f(t)v(t) = te^{-3t}$.

**Example 8**

Determine a second linearly independent solution to the differential equation $4t^2\frac{d^2y}{dt^2} + 8t\frac{dy}{dt} + y = 0$, $t > 0$, if $y = t^{-1/2}$ is a solution.



**Solution** In this case, we must divide by $4t^2$ to obtain an equation of the form $y'' + p(t)y' + q(t)y = 0$. This gives us the equation $\frac{d^2y}{dt^2} + \frac{2}{t}\frac{dy}{dt} + \frac{1}{4t^2}y = 0$. Therefore, $p(t) = 2/t$ and $f(t) = t^{-1/2}$. Using the formula for $v$, we obtain

$$v(t) = \int \frac{e^{-\int p(t)dt}}{[f(t)]^2}\,dt = \int \frac{e^{-\int 2/t\,dt}}{[t^{-1/2}]^2}\,dt = \int \frac{e^{-2\ln(t)}}{t^{-1}}\,dt = \int t^{-1}\,dt = \ln t,\ t > 0.$$

A second solution is $y = f(t)v(t) = t^{-1/2}\ln t$.

## EXERCISES 4.1

In Exercises 1–6, calculate the Wronskian of the indicated set of functions. Classify each set of functions as linearly independent or linearly dependent.

1. $S = \{t, 4t - 1\}$
2. $S = \{t, e^t\}$
*3. $S = \{e^{-6t}, e^{-4t}\}$
4. $S = \{\cos 2t, \sin 2t\}$
*5. $S = \{e^{-3t}\cos 3t, e^{-3t}\sin 3t\}$
6. $S = \{e^{5t}\cos 4t, e^{5t}\sin 4t\}$

In Exercises 7–12, show that $y = c_1y_1(t) + c_2y_2(t)$ satisfies the given differential equation and that $\{y_1(t), y_2(t)\}$ is a linearly independent set.

7. $y = c_1e^t + c_2e^{-t}$, $y'' - y = 0$
8. $y = c_1e^t + c_2e^{-3t}$, $y'' + 2y' - 3y = 0$
*9. $y = c_1e^{-t} + c_2te^{-t}$, $y'' + 2y' + y = 0$
10. $y = c_1e^{2t} + c_2te^{2t}$, $y'' - 4y' + 4y = 0$
11. $y = c_1\cos 2t + c_2\sin 2t$, $y'' + 4y = 0$
12. $y = e^{3t}(c_1\cos 4t + c_2\sin 4t)$, $y'' - 6y' + 25y = 0$

In Exercises 13–17, use the given solution on the indicated interval to find the solution to the initial-value problem.

13. $\begin{cases} y'' - y' - 2y = 0 \\ y(0) = -1, y'(0) = -5 \end{cases}$, $y = c_1e^{-t} + c_2e^{2t}$, $-\infty < t < \infty$
14. $\begin{cases} y'' + 4y = 0 \\ y(0) = -1, y'(0) = 2 \end{cases}$, $y = c_1\cos 2t + c_2\sin 2t$, $-\infty < t < \infty$
*15. $\begin{cases} y'' - 8y' + 16y = 0 \\ y(1) = 0, y'(1) = -e^4 \end{cases}$, $y = c_1e^{4t} + c_2te^{4t}$, $-\infty < t < \infty$
16. $\begin{cases} t^2y'' + 7ty' - 7y = 0 \\ y(1) = 2, y'(1) = -22 \end{cases}$, $y = c_1t^{-7} + c_2t$, $t > 0$
17. $\begin{cases} y'' + y = 2\cos t \\ y(0) = 1, y'(0) = 1 \end{cases}$, $y = c_1\cos t + c_2\sin t + t\sin t$, $-\infty < t < \infty$
18. Use the Wronskian to show that if $y_1$ and $y_2$ are solutions of the first-order differential equation
$$y' + p(t)y = 0,$$
then $y_1$ and $y_2$ are linearly dependent.
19. Consider the hyperbolic trigonometric functions $\cosh t = (e^t + e^{-t})/2$ and $\sinh t = (e^t - e^{-t})/2$. Show that
    (a) $d/dt(\cosh t) = \sinh t$.
    (b) $d/dt(\sinh t) = \cosh t$.
    (c) $\cosh^2 t - \sinh^2 t = 1$.
    (d) $\cosh t$ and $\sinh t$ are linearly independent functions.

In Exercises 20–23, show that $S$ is a fundamental set of solutions for the given equation. (See Exercises 1–6 to verify linear independence.)

**20.** $S = \{e^{-6t}, e^{-4t}\}$; $y'' + 10y' + 24y = 0$

**21.** $S = \{\cos 2t, \sin 2t\}$; $y'' + 4y = 0$

**22.** $S = \{e^{-3t}\cos 3t, e^{-3t}\sin 3t\}$; $y'' + 6y' + 18y = 0$

***23.** $S = \{e^{5t}\cos 4t, e^{5t}\sin 4t\}$; $y'' - 10y' + 41y = 0$

**24.** Let $ay'' + by' + cy = 0$ be a homogeneous second-order equation with constant coefficients and let $m_1$ and $m_2$ be the solutions of the equation $am^2 + bm + c = 0$.

**(a)** If $m_1 \neq m_2$ and both $m_1$ and $m_2$ are real, show that $\{e^{m_1t}, e^{m_2t}\}$ is a fundamental set of solutions of $ay'' + by' + cy = 0$.

**(b)** If $m_1 = m_2$, show that $\{e^{m_1t}, xe^{m_1t}\}$ is a fundamental set of solutions of $ay'' + by' + cy = 0$.

**(c)** If $m_1 = \alpha + i\beta$, $\beta \neq 0$, and $m_2 = \overline{m_1} = \alpha - i\beta$, show that $\{e^{\alpha t}\cos \beta t, e^{\alpha t}\sin \beta t\}$ is a fundamental set of solutions of $ay'' + by' + cy = 0$.

In Exercises 25–34, use reduction of order with the solution $y_1(t)$ to find a second linearly independent solution of the given differential equation.

**25.** $y_1(t) = e^{3t}$, $y'' - 5y' + 6y = 0$

**26.** $y_1(t) = e^{-2t}$, $y'' + 6y' + 8y = 0$

***27.** $y_1(t) = e^{2t}$, $y'' - 4y' + 4y = 0$

**28.** $y_1(t) = e^{-5t}$, $y'' + 10y' + 25y = 0$

**29.** $y_1(t) = \cos 2t$, $y'' + 4y = 0$

**30.** $y_1(t) = \sin 7t$, $y'' + 49y = 0$

***31.** $y_1(t) = t^{-4}$, $t^2y'' + 4ty' - 4y = 0$

**32.** $y_1(t) = t^{-2}$, $t^2y'' + 6ty' + 6y = 0$

**33.** $y_1(t) = t$, $t^2y'' + 3ty' + y = 0$

**34.** $y_1(t) = t^{-1}$, $t^2y'' - ty' + y = 0$

**35.** Find $a$ and $b$ so that $S = \{e^{-t}\cos 2t, e^{-t}\sin 2t\}$ is a fundamental set of solutions for $y'' + ay' + by = 0$.

**36.** Is the Principle of Superposition ever valid for linear nonhomogeneous equations? Explain.

***37.** Suppose that $f(t)$ is a solution to the equation $y'' + p(t)y' + q(t)y = 0$. Show that $f(t)$ and the solution $f(t)\int \frac{e^{-\int p(t)dt}}{[f(t)]^2}\,dt$ obtained by reduction of order are linearly independent. *Hint:* Use the Wronskian.

**38.** Show that a general solution of $y'' - k^2y = 0$ is $y = c_1\cosh kt + c_2\sinh kt$, where $\cosh kt = (e^{kt} + e^{-kt})/2$ and $\sinh kt = (e^{kt} - e^{-kt})/2$.

**39.** **(Abel's Formula)** Suppose that $y_1$ and $y_2$ are two solutions of $y'' + p(t)y' + q(t)y = 0$ on an open interval $I$, where $p(t)$ and $q(t)$ are continuous. Then, the Wronskian of $y_1$ and $y_2$ is

$$W(y_1, y_2) = Ce^{-\int p(t)dt}.$$

Prove Abel's formula using the following steps. Because $y_1$ and $y_2$ are solutions, begin with the system

$$\begin{cases} y_1'' + p(t)y_1' + q(t)y_1 = 0 \\ y_2'' + p(t)y_2' + q(t)y_2 = 0 \end{cases}.$$

**(a)** Multiply the first equation by $(-y_2)$ and the second by $y_1$. Add the resulting equations to obtain $(y_1y_2'' - y_1''y_2) + p(y_1y_2' - y_1'y_2) = 0$. **(b)** Show that $(d/dt)\ W(y_1, y_2) = y_1y_2'' - y_1''y_2$. **(c)** Use the results in **(a)** and **(b)** to obtain the first-order ODE $dW/dt + pW = 0$. Solve this equation for $W$.

**40.** Can the Wronskian be zero at only one value of $t$ on $I$? *Hint:* Use Abel's formula.

***41.** **(a)** Find conditions on the constants $c_1$, $c_2$, $c_3$, and $c_4$ so that $y(t) = c_1 + c_2\tan(c_3 + c_4\ln t)$ is a solution of the nonlinear second-order equation $ty'' - 2yy' = 0$. **(b)** Is the Principle of Superposition valid for this equation? Explain.

**42.** Find a general solution of $4t^2y'' + 4ty' + (36t^2 - 1)y = 0$ given that $y = (\cos 3t)/\sqrt{t}$ is one solution.

***43.** Given that $y = (\sin 4t)/t$ is a solution of $ty'' + 2y' + 16y = 0$, find and graph the solution of the equation that satisfies $y(\pi/8)$ and $y'(\pi/8) = -32/\pi$.

**44.** Find a general solution of $y'' + b(t)y' + c(t)y = 0$ given that $y = (\sin t)/t^2$ is one solution.

***45.** Given that $y = (\cos t)/t^3$ is a solution of $y'' + b(t)y' + c(t)y = 0$, find and graph the solution(s), if any, that satisfy the boundary conditions $y(\pi) = y(2\pi) = 0$.



## 4.2 Solutions of Second-Order Linear Homogeneous Equations with Constant Coefficients

Two Distinct Real Roots · Repeated Roots · Complex Conjugate Roots

In Section 4.1, we started to investigate the method by which we solve second-order linear homogeneous equations. Here, we restrict our attention to equations with constant coefficients,

$$ay'' + by' + cy = 0,$$

and we solve the equation by assuming that $y = e^{rt}$ is a solution for some value(s) of $r$. We find $r$ by substituting this solution into the ODE. With $y' = re^{rt}$ and $y'' = r^2e^{rt}$, we obtain $ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$, or

$$e^{rt}(ar^2 + br + c) = 0.$$

Therefore, we solve the **characteristic equation**

$$ar^2 + br + c = 0$$

to determine $r$ (because $e^{rt} \neq 0$ for all values of $t$). Of course, the roots are

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

so the roots of this quadratic equation depend on the values of $a$, $b$, and $c$. There are three possibilities: two real distinct roots when $b^2 - 4ac > 0$, one real repeated root when $b^2 - 4ac = 0$, and two complex conjugate roots when $b^2 - 4ac < 0$. We now consider each case.

### Two Distinct Real Roots

Suppose that the characteristic equation of an ODE has two roots $r_1$ and $r_2$, where $r_1 \neq r_2$, so two solutions to the ODE are $y_1(t) = e^{r_1t}$ and $y_2(t) = e^{r_2t}$. We show that these solutions are linearly independent by computing the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} e^{r_1t} & e^{r_2t} \\ r_1e^{r_1t} & r_2e^{r_2t} \end{vmatrix} = r_2e^{(r_1+r_2)t} - r_1e^{(r_1+r_2)t} = (r_2 - r_1)e^{(r_1+r_2)t}.$$

These functions are linearly independent because $r_2 - r_1 \neq 0$, so that $W(y_1, y_2) \neq 0$. Therefore, a general solution is $y(t) = c_1y_1(t) + c_2y_2(t) = c_1e^{r_1t} + c_2e^{r_2t}$.

| **Theorem 4.5 Distinct Real Roots** |
| --- |
| Let $r_1$ and $r_2$ be the real solutions of $ar^2 + br + c = 0$. If $r_1 \neq r_2$, then a general solution of $ay'' + by' + cy = 0$ is $y(t) = c_1e^{r_1t} + c_2e^{r_2t}$. |

**Example 1**

Solve $y'' + 3y' - 4y = 0$.

**Solution** The characteristic equation for this ODE is $r^2 + 3r - 4 = (r + 4)(r - 1) = 0$, with roots $r_1 = -4$ and $r_2 = 1$. Therefore, a general solution is $y(t) = c_1e^{-4t} + c_2e^t$.

## Repeated Root

Suppose that the characteristic equation of an ODE has a repeated root $r = r_1 = r_2$. In other words, $b^2 - 4ac = 0$ in the quadratic formula so that the root is $r = -b/(2a)$ or $(r + b/(2a))^2$ is a factor in the characteristic equation. Because $r = -b/(2a)$ is repeated, we only have one solution of the ODE, $y_1(t) = e^{rt} = e^{-bt/(2a)}$. We use this solution to obtain a second linearly independent solution through reduction of order with the formula

$$y_2(t) = y_1(t)\int \frac{e^{-\int p(t)dt}}{[y_1(t)]^2}\,dt,$$

where $p(t)$ appears in the general equation $y'' + p(t)y' + q(t)y = 0$. In this case, the ODE is $ay'' + by' + cy = 0$. Dividing by $a$ yields $y'' + (b/a)y' + (c/a)y = 0$, so $p(t) = b/a$. Returning to the reduction of order formula, we find that

$$y_2(t) = e^{-bt/(2a)}\int \frac{e^{-\int (b/a)dt}}{[e^{-bt/(2a)}]^2}\,dt = e^{-bt/(2a)}\int \frac{e^{-bt/a}}{e^{-bt/a}}\,dt = e^{-bt/(2a)}\int dt = te^{-bt/(2a)}.$$

Therefore, a general solution is $y(t) = c_1e^{-bt/(2a)} + c_2te^{-bt/(2a)} = c_1e^{rt} + c_2te^{rt}$.

**Theorem 4.6 Repeated Root**

Let $r = r_1 = r_2$ be the repeated (real) solution of $ar^2 + br + c = 0$. A general solution of $ay'' + by' + cy = 0$ is $y(t) = c_1e^{rt} + c_2te^{rt}$.

**Example 2**

Solve $y'' + 2y' + y = 0$.

**Solution** The characteristic equation for this ODE is $r^2 + 2r + 1 = (r + 1)^2 = 0$, with roots $r_1 = r_2 = -1$. Therefore, a general solution is $y(t) = c_1e^{-t} + c_2te^{-t}$.



## Complex Conjugate Roots

Suppose that the characteristic equation of an ODE has the complex conjugate roots $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$ where $\alpha$ and $\beta$ ($\beta > 0$) are real numbers and $i = \sqrt{-1}$. *Note:* Sometimes, we denote the roots as $r_{1,2} = \alpha \pm \beta i$.

To construct a real-valued general solution, we use Euler's formula,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

which can be obtained through the use of the Maclaurin series:

$$e^t = \sum_{k=0}^{\infty}\frac{t^k}{k!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \cdots,$$

$$\cos t = \sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{(2k)!} = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots,$$

and

$$\sin t = \sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+1}}{(2k+1)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots.$$

We derive Euler's formula using these Maclaurin series and substitution:

$$\begin{aligned} e^{i\theta} &= \sum_{k=0}^{\infty}\frac{(i\theta)^k}{k!} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) = \cos\theta + i\sin\theta. \end{aligned}$$

Using the properties of sine and cosine, Euler's formula also implies that $e^{-i\theta} = \cos\theta - i\sin\theta$. The roots of the characteristic equation are $m_{1,2} = \alpha \pm \beta i$, so $y_1 = e^{(\alpha+\beta i)t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$ and $y_2 = e^{(\alpha-\beta i)t} = e^{\alpha t}(\cos\beta t - i\sin\beta t)$ are both solutions to the differential equation. By the Principle of Superposition, any linear combination of $y_1$ and $y_2$ is also a solution. For example, $z_1 = \frac{1}{2}(y_1 + y_2) = e^{\alpha t}\cos\beta t$ and $z_2 = -\frac{i}{2}(y_1 - y_2) = e^{\alpha t}\sin\beta t$ are both solutions. You should verify that $\{e^{\alpha t}\cos\beta t,\ e^{\alpha t}\sin\beta t\}$ is linearly independent, so a general solution is $y = e^{\alpha t}(c_1\cos\beta t + c_2\sin\beta t)$.

**Theorem 4.7 Complex Conjugate Roots**

Let $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$, where $\alpha$ and $\beta$ ($\beta > 0$) are real numbers, be the solutions of $ar^2 + br + c = 0$. A general solution of $ay'' + by' + cy = 0$ is $y(t) = c_1e^{\alpha t}\cos\beta t + c_2e^{\alpha t}\sin\beta t = e^{\alpha t}(c_1\cos\beta t + c_2\sin\beta t)$.

**Example 3**

Solve $y'' + 4y' + 20y = 0$, $y(0) = 3$, $y'(0) = -1$.

**Solution** The characteristic equation is $r^2 + 4r + 20 = 0$ with roots

$$r_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4(20)}}{2} = \frac{-4 \pm \sqrt{-64}}{2} = \frac{-4 \pm 8i}{2} = -2 \pm 4i.$$

Therefore, in our earlier notation, $\alpha = -2$ and $\beta = 4$, so that a general solution of the ODE is

$$y(t) = e^{-2t}(c_1 \cos 4t + c_2 \sin 4t),$$

because the solutions of the characteristic equation are complex conjugates. To find the solution for which $y(0) = 3$ and $y'(0) = -1$, we first calculate

$$y' = 2e^{-2t}(-c_1 \cos 4t + 2c_2 \cos 4t - 2c_1 \sin 4t - c_2 \sin 4t)$$

with the product rule and then evaluate both

$$y(0) = c_1 \quad \text{and} \quad y'(0) = 2(2c_2 - c_1)$$

obtaining the system of equations $\begin{cases} c_1 = 3 \\ 2(2c_2 - c_1) = -1 \end{cases}$. Substituting $c_1 = 3$ into the second equation results in $c_2 = 5/4$. Thus, the solution of $y'' + 4y' + 20y = 0$ for which $y(0) = 3$ and $y'(0) = -1$ is $y(t) = e^{-2t}(3 \cos 4t + \frac{5}{4} \sin 4t)$.

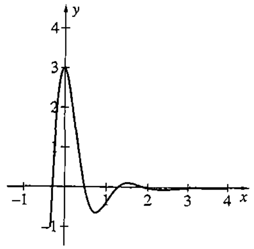



**Figure 4.4** Graph of $y(t) = e^{-2t}\left(3\cos 4t + \frac{5}{4}\sin 4t\right)$

*In Example 3, evaluate $\lim_{t\to\infty} y(t)$. Does this limit depend on the initial conditions? Does this result agree with the graph in Figure 4.4?*

Theorems 4.5 through 4.7 can be summarized as follows:

**Solving Second-Order Equations with Constant Coefficients**

Let $ay'' + by' + cy = 0$ be a homogeneous second-order equation with constant real coefficients and let $m_1$ and $m_2$ be the solutions of the equation $am^2 + bm + c = 0$.

① If $m_1 \neq m_2$ and both $m_1$ and $m_2$ are real, a general solution of $ay'' + by' + cy = 0$ is

$$y = c_1 e^{m_1 t} + c_2 e^{m_2 t}.$$

② If $m_1 = m_2$, a general solution of $ay'' + by' + cy = 0$ is

$$y = c_1 e^{m_1 t} + c_2 t e^{m_1 t}.$$

③ If $m_1 = \alpha + i\beta$, $\beta \neq 0$, and $m_2 = \overline{m_1} = \alpha - i\beta$, a general solution of $ay'' + by' + cy = 0$ is

$$y = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t).$$

$\overline{m_1}$ is the **complex conjugate** of $m_1$:

$$\overline{m_1} = \overline{\alpha + i\beta} = \alpha - i\beta.$$



## EXERCISES 4.2

In Exercises 1–15, find a general solution of each equation.

1. $y'' + 8y' + 12y = 0$
2. $y'' - 4y' - 12y = 0$
*3. $y'' + 3y' - 4y = 0$
4. $y'' + y' - 72y = 0$
5. $y'' + 16y = 0$
6. $4y'' + 9y = 0$
*7. $y'' + 7y = 0$
8. $y'' + 8y = 0$
9. $y'' - 6y' + 25y = 0$
10. $y'' - 8y' + 20y = 0$
*11. $y'' + 6y' + 18y = 0$
12. $4y'' + 21y' + 5y = 0$
13. $7y'' + 4y' - 3y = 0$
14. $y'' + 4y' + 4y = 0$
*15. $y'' - 6y' + 9y = 0$

In Exercises 16–27, solve the initial-value problem. Graph the solution on an appropriate interval.

16. $y'' - y' = 0$, $y(0) = 3$, $y'(0) = 2$
17. $3y'' - y' = 0$, $y(0) = 0$, $y'(0) = 7$
18. $y'' + y' - 12y = 0$, $y(0) = 0$, $y'(0) = 7$
*19. $y'' - 7y' + 12y = 0$, $y(0) = 3$, $y'(0) = -2$
20. $2y'' - 7y' - 4y = 0$, $y(0) = 0$, $y'(0) = 1$
21. $y'' - 7y' + 10y = 0$, $y(0) = 1$, $y'(0) = 5$
22. $y'' + 36y = 0$, $y(0) = 2$, $y'(0) = -6$
*23. $y'' + 100y = 0$, $y(0) = 1$, $y'(0) = 10$
24. $y'' - 2y' + y = 0$, $y(0) = 4$, $y'(0) = 0$
25. $y'' + 4y' + 4y = 0$, $y(0) = 1$, $y'(0) = 3$
26. $y'' + 2y' + 5y = 0$, $y(0) = 1$, $y'(0) = 0$
*27. $y'' + 4y' + 20y = 0$, $y(0) = 2$, $y'(0) = 0$
28. A **Ricatti equation,** named for the Italian mathematician Jacopo Francesco Ricatti (1676–1754), is a nonlinear first-order equation of the form

$$y' + a(t)y^2 + b(t)y + c(t) = 0.$$

We make the substitution $y(t) = \frac{w'(t)}{w(t)}\frac{1}{a(t)}$ to solve Ricatti equations.

(a) Show that

$$y'(t) = \frac{w''(t)}{a(t)w(t)} - \frac{(w'(t))^2}{a(t)(w(t))^2} - \frac{a'(t)w'(t)}{(a(t))^2 w(t)}.$$

(b) Substitute $y$ and $y'$ into the Ricatti equation to obtain the second-order equation

$$\frac{w''(t)}{a(t)w(t)} - \frac{(w'(t))^2}{a(t)(w(t))^2} - \frac{a'(t)w'(t)}{(a(t))^2 w(t)} + \frac{(w'(t))^2}{a(t)(w(t))^2} + \frac{b(t)w'(t)}{a(t)w(t)} + c(t) = 0.$$

Multiply this equation by $a(t)w(t)$ to obtain $w''(t) - \frac{a'(t)w'(t)}{a(t)} + b(t)w'(t) + a(t)c(t)w(t) = 0$ and simplify the result to obtain the second-order equation

$$w'' - \left(\frac{a'(t)}{a(t)} - b(t)\right)w' + a(t)c(t)w = 0.$$

29. Convert the Ricatti equation $y' + (t^4 + t^2 + 1)y^2 + \frac{2(1 - t + t^2 - 2t^3 + t^4)}{1 + t^2 + t^4}y + \frac{1}{t^4 + t^2 + 1} = 0$ to a second-order equation by using the following steps.

(a) Note that $a(t) = t^4 + t^2 + 1$, $b(t) = \frac{2(1 - t + t^2 - 2t^3 + t^4)}{1 + t^2 + t^4}$, and $c(t) = \frac{1}{t^4 + t^2 + 1}$. Let $y(t) = \frac{w'(t)}{w(t)}\frac{1}{a(t)}$ so that the second-order equation is

$$w'' - \left(\frac{4t^3 + 2t}{t^4 + t^2 + 1} - \frac{2(1 - t + t^2 - 2t^3 + t^4)}{1 + t^2 + t^4}\right)w' + (t^4 + t^2 + 1)\frac{1}{t^4 + t^2 + 1}w = 0,$$

which simplifies to $w'' - 2w' + w = 0$.

(b) Show that $w(t) = C_1 e^t + C_2 t e^t$ is a general solution of $w'' - 2w' + w = 0$.

(c) Use $y(t) = \frac{w'(t)}{w(t)}\frac{1}{a(t)}$ to show that $y(t) = \frac{C_1 e^t + C_2(e^t + te^t)}{(t^4 + t^2 + 1)(C_1 e^t + C_2 t e^t)}$ satisfies the Ricatti equation.

In Exercises 30–35, solve the given Ricatti equation. (See Exercises 28 and 29.)

30. $y' + \dfrac{1}{t^2+1}y^2 - \dfrac{2(-6+t-6t^2)}{t^2+1}y + 45(t^2+1) = 0.$

31. $y' + \sin t y^2 - (2 - \cot t)y + \csc t = 0$

32. $y' + t^2 \cos t y^2 - \dfrac{t \tan t + 2t - 2}{t}y + \dfrac{2 \sec t}{t^2} = 0$

33. $y' + \dfrac{\sin t}{t}y^2 - \dfrac{1 - 2t - t\cot t}{t}y + 26t \csc t = 0$

34. $y' + \dfrac{t}{t^2+4}y^2 - \dfrac{t^2-4}{t(t^2+4)}y + \dfrac{4(t^2+4)}{t} = 0$

35. $y' + t \tan 4t + \dfrac{1 + 10t + 4t \csc 4t \sec 4t}{t}y + \dfrac{41 \cot 4t}{t} = 0$

36. Consider the homogeneous equation $y'' + p(t)y' + q(t)y = 0$. **(a)** If $y_1(t)$ satisfies the equation and $c$ is a constant, show that $y(t) = cy_1(t)$ satisfies the equation. **(b)** If $y_1(t)$ and $y_2(t)$ are both solutions of the equation, show that $y(t) = y_1(t) + y_2(t)$ is a solution.

37. Use the substitution $t = e^x$ to solve **(a)** $3t^2y'' - 2ty' + 2y = 0$, $t > 0$ and **(b)** $t^2y'' - ty' + y = 0$, $x > 0$. *Hint:* Show that if $t = e^x$, $\dfrac{dy}{dt} = \dfrac{1}{t}\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dt^2} = \dfrac{1}{t^2}\left(\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx}\right)$.

38. **(a)** Use the Maclaurin series $e^t = \sum_{k=0}^{\infty} \dfrac{t^k}{k!}$ and the Maclaurin series for $\sin t$ and $\cos t$ to prove $e^{-i\theta} = \cos\theta - i \sin\theta$. **(b)** Use **(a)** and trigonometric identities to prove that $e^{i\theta} = \cos\theta + i \sin\theta$.

39. Show that a general solution of the differential equation $ay'' + 2by' + cy = 0$ where $b^2 - ac > 0$ can be written as

$$y = e^{-bt/a}\left[c_1 \cosh \frac{t\sqrt{b^2-ac}}{a} + c_2 \sinh \frac{t\sqrt{b^2-ac}}{a}\right].$$

40. Express the solution to each differential equation in terms of the hyperbolic trigonometric function (see Exercise 39).

**(a)** $y'' + 6y' + 2y = 0$

**(b)** $y'' - 5y' + 6y = 0$

**(c)** $y'' - 6y' - 16y = 0$

**(d)** $y'' - 16y = 0$

41. Show that the boundary-value problem $\begin{cases} y'' + 2y' + 5y = 0 \\ y(0) = 0,\ y(\pi/2) = 0 \end{cases}$ has infinitely many solutions, the boundary-value problem $\begin{cases} y'' + 2y' + 5y = 0 \\ y(0) = 0,\ y(\pi/4) = 0 \end{cases}$ has no nontrivial solutions, and that the boundary-value problem $\begin{cases} y'' + 2y' + 5y = 0 \\ y(0) = 1,\ y(\pi/4) = 0 \end{cases}$ has one (nontrivial) solution.

*42. Use factoring to solve each of the following nonlinear equations. For which equations, if any, is the Principle of Superposition valid? Explain

**(a)** $(y'')^2 - 5y''y + 4y^2 = 0$

**(b)** $(y'')^2 - 2y''y + y^2 = 0$

43. Find conditions on $a$ and $b$, if possible, so that the solution to the initial-value problem

$$\begin{cases} y'' + 4y' + 3y = 0 \\ y(0) = a,\ y'(0) = b \end{cases}$$

has **(a)** neither local maxima nor local minima; **(b)** exactly one local maximum; and **(c)** exactly one local minimum on the interval $[0, \infty)$.

44. **(a)** Show that the roots of the characteristic equation of $\dfrac{d^2x}{dt^2} + a_1\dfrac{dx}{dt} + a_0x = 0$ are

$$m_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}.$$

**(b)** Graph $a_0 = \frac{1}{4}a_1^2$ and $a_1 = 0$, using the horizontal axis to represent $a_0$ and the vertical axis to represent $a_1$. *Randomly* generate three points on the graph of $a_0 = \frac{1}{4}a_1^2$, three points in the region $a_0 < \frac{1}{4}a_1^2$, and three points in the region $a_0 > \frac{1}{4}a_1^2$.

**(c)** Write the second-order homogeneous equation $\dfrac{d^2x}{dt^2} + a_1\dfrac{dx}{dt} + a_0x = 0$ as a system by letting $y = x' = \dfrac{dx}{dt}$.

**(d)** For each pair of points obtained in (b), graph the phase plane associated with the system for $-1 \le x \le 1$ and $-1 \le y \le 1$. Compare your results with your classmates.

**(e)** How does the phase plane associated with the system change as the roots of the characteristic equation change?



## 4.3 Higher Order Equations: An Introduction

The ideas presented for second-order linear homogeneous equations in Sections 4.1 and 4.2 can be extended to those of order three or higher. We begin with the general form of an $n$th-order linear equation.

**Definition 4.3 *n*th-order Ordinary Linear Differential Equation**

An ordinary differential equation of the form

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = g(t),$$

where $a_n(t) \not\equiv 0$, is called an ***n*th-order ordinary linear differential equation.** If $g(t)$ is identically the zero function, the equation is said to be **homogeneous;** if $g(t)$ is not the zero function, the equation is said to be **nonhomogeneous;** and if the functions $a_i(t)$, $i = 0, 1, 2, \ldots, n$ are constants, the equation is said to have **constant coefficients.** An $n$th-order equation accompanied by the conditions

$$y(t_0) = y_0,\ y'(t_0) = y_0',\ \ldots,\ y^{(n-1)}(t_0) = y_0^{(n-1)}$$

where $y_0, y_0', \ldots, y_0^{(n-1)}$ are constants is called an ***n*th-order initial-value problem.**

On the basis of this definition, the equation $y''' - 8y'' + 10y' - 3y = \cos t$ is a third-order nonhomogeneous equation with constant coefficients, while $y^{(4)} - t^2y = 0$ is a fourth-order homogeneous equation. We assume that the coefficient functions $a_n(t), \ldots, a_0(t)$ and $g(t)$ in the equation

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_0(t)y(t) = g(t)$$

are continuous on an open interval $I$. Therefore, if we also assume that $a_n(t) \neq 0$ for all $t$ on $I$, we can divide by $a_n(t)$ to place the ODE in normal form,

$$y^{(n)}(t) + p_n(t)y^{(n-1)}(t) + \cdots + p_1(t)y(t) = f(t),$$

where $p_n(t) = a_{n-1}(t)/a_n(t), \ldots, p_1(t) = a_0(t)/a_n(t)$, and $f(t) = g(t)/a_n(t)$.

In Sections 4.1 and 4.2, we discussed how we needed two linearly independent solutions to solve a second-order linear homogeneous ODE. Therefore, we gave a definition for the linear dependence or linear independence of a set of two functions. We now define this concept for a set of $n$ functions because we need to understand this property to solve the higher order equations.

**Definition 4.4 Linearly Dependent and Linearly Independent**

Let $S = \{f_1(t), f_2(t), f_3(t), \ldots, f_{n-1}(t), f_n(t)\}$ be a set of $n$ functions. $S$ is **linearly dependent** on an interval $I$ if there are constants $c_1, c_2, \ldots, c_n$, not all zero, so that

$$c_1f_1(t) + c_2f_2(t) + \cdots + c_{n-1}f_{n-1}(t) + c_nf_n(t) = 0$$

for every value of $t$ in the interval $I$.
$S$ is **linearly independent** if $S$ is not linearly dependent.

Thus, $S = \{f_1(t), f_2(t), f_3(t), \ldots, f_{n-1}(t), f_n(t)\}$ is linearly independent if

$$c_1f_1(t) + c_2f_2(t) + \cdots + c_{n-1}f_{n-1}(t) + c_nf_n(t) = 0$$

implies that $c_1 = c_2 = \cdots = c_n = 0$.

**Example 1**

Classify the sets of functions as linearly independent or linearly dependent.
**(a)** $S = \{t, 3t - 6, 1\}$ **(b)** $S = \{\cos 2t, \sin 2t, \sin t \cos t\}$ **(c)** $S = \{1, t, t^2\}$

***Solution*** **(a)** Here, we must find constants $c_1$, $c_2$, and $c_3$ such that

$$c_1t + c_2(3t - 6) + c_3(1) = (c_1 + 3c_2)t + (c_3 - 6c_2) = 0.$$

Equating each of the coefficients to zero leads to the system of equations $\{c_1 + 3c_2 = 0, c_3 - 6c_2 = 0\}$. This system has infinitely many solutions of the form $\{c_1 = -3c_2, c_3 = 6c_2, c_2 \text{ arbitrary}\}$. With the choice $c_1 = -3$, $c_2 = 1$, and $c_3 = 6$, the equation is satisfied. Therefore, the set $S = \{t, 3t - 6, 1\}$ is linearly dependent.
**(b)** For this set of functions, we apply the identity $\sin 2t = 2 \sin t \cos t$. Doing this, we consider the equation

$$c_1 \cos 2t + c_2 \sin 2t + c_3 \sin t \cos t = c_1 \cos 2t + 2c_2 \sin t \cos t + c_3 \sin t \cos t = 0.$$

The choices of $c_1 = 0$, $c_2 = 1$, and $c_3 = -2$ lead to a solution. At least one of the constants is not zero, so the set $S = \{\cos 2t, \sin 2t, \sin t \cos t\}$ is linearly dependent.
**(c)** For the set $S = \{1, t, t^2\}$, we consider the equation $c_1(1) + c_2t + c_3t^2 = 0$. The only constants that satisfy the equation are $c_1 = c_2 = c_3 = 0$, so the set is linearly independent.

We also extend the Principle of Superposition to include more than two solutions.

**Theorem 4.8 Principle of Superposition**

If $f_1(t), f_2(t), \ldots, f_n(t)$ are solutions of the linear homogeneous ODE $y^{(n)}(t) + p_n(t)y^{(n-1)}(t) + \cdots + p_1(t)y(t) = 0$ on the interval $I$, and if $c_1, c_2, \ldots, c_n$ are arbitrary constants, then $y = c_1f_1(t) + c_2f_2(t) + \cdots + c_nf_n(t)$ is also a solution of this ODE on $I$.



Another way to express Theorem 4.8 is to say that any linear combination of solutions of $y^{(n)}(t) + p_n(t)y^{(n-1)}(t) + \cdots + p_1(t)y(t) = 0$ is also a solution of this ODE.

**Example 2**

Show that $y_1(t) = e^{-t}$, $y_2(t) = \cos t$, and $y_3(t) = \sin t$ are solutions of the third-order linear homogeneous ODE $y''' + y'' + y' + y = 0$. Use the Principle of Superposition to find another solution to this ODE.

***Solution*** For $y_1(t) = e^{-t}$, we have $y_1'(t) = -e^{-t}$, $y_1''(t) = e^{-t}$, and $y_1'''(t) = -e^{-t}$. Then, $y_1''' + y_1'' + y_1' + y_1 = -e^{-t} + e^{-t} - e^{-t} + e^{-t} = 0$, so $y_1$ satisfies the ODE. In a similar manner, we can show that $y_2$ and $y_3$ are solutions of the ODE. (We leave this to the reader.) Therefore by the Principle of Superposition, the linear combination of these three functions, $y(t) = c_1e^{-t} + c_2 \cos t + c_3 \sin t$, is also a solution of the ODE.

As with first-order and second-order initial-value problems, we can state an existence and uniqueness theorem for an initial-value problem involving a higher order linear ordinary differential equation.

**Theorem 4.9 Existence and Uniqueness**

Suppose that $p_n(t), \ldots, p_1(t)$, and $f(t)$ are continuous functions on an open interval $I$ that contains $t = t_0$. Then, the initial value problem

$$y^{(n)}(t) + p_n(t)y^{(n-1)}(t) + \cdots + p_1(t)y(t) = f(t),\ y(t_0) = b_0,\ y'(t_0) = b_1, \ldots, y^{(n-1)}(t_0) = b_{n-1}$$

has a unique solution on $I$.

The proof of this theorem is well beyond the scope of this text but can be found in advanced differential equations textbooks.*

At this point, we need to discuss why we can write every solution of the ODE $y^{(n)}(t) + p_n(t)y^{(n-1)}(t) + \cdots + p_1(t)y(t) = 0$ as a linear combination of solutions if we have $n$ linearly independent solutions of the $n$th-order equation. (We call this a *general solution* of the ODE.) Before moving on, we must define the Wronskian of a set of $n$ functions. This definition is particularly useful because determining whether a set of functions is linearly independent or dependent becomes more difficult as the number of functions in the set under consideration increases.

* For example, see Chapter 2 of C. Corduneanu, *Principles of Differential and Integral Equations*, Chelsea Publishing Company (1971).

**Definition 4.5 Wronskian**

Let $S = \{f_1(t), f_2(t), f_3(t), \ldots, f_{n-1}(t), f_n(t)\}$ be a set of $n$ functions for which each is differentiable at least $n-1$ times. The **Wronskian** of $S$, denoted by

$$W(S) = W(f_1(t), f_2(t), f_3(t), \ldots, f_{n-1}(t), f_n(t)),$$

is the determinant

$$W(S) = \begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{vmatrix}.$$

Although most computer algebra systems can quickly find and simplify the Wronskian of a set of functions, the following example illustrates how to compute the Wronskian for a set of three functions by hand.

**Example 3**

Compute the Wronskian of each of the following sets of functions. **(a)** $S = \{t, 3t-6, 1\}$; **(b)** $S = \{1, t, t^2\}$.

***Solution*** We compute a $3 \times 3$ determinant with

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

**(a)** $W(S) = \begin{vmatrix} t & 3t-6 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{vmatrix}$. We can carry out the step given above for computing a $3 \times 3$ determinant. However, the row of zeros indicates that $W(S) = 0$. (You should verify this.)

**(b)** $W(S) = \begin{vmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix} - t\begin{vmatrix} 0 & 2t \\ 0 & 2 \end{vmatrix} + t^2\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 2$. In addition, we could have observed that all nonzero entries lie on the diagonal or above, so the determinant is the product of the numbers on the diagonal. *Note:* This approach also works if the nonzero entries lie on the diagonal or below. Matrices having these properties are called *upper triangular* and *lower triangular*, respectively.



In part **(a)** of Example 3, we found that $W(S) = 0$. Recall that we concluded that the functions in $S = \{t, 3t-6, 1\}$ are linearly dependent because $3t - 6 = 3(t) - 6(1)$, which means that one of the functions in the set can be written as a linear combination of other members of the set. This is no coincidence. *The Wronskian of a set of linearly dependent functions is identically zero, whereas the Wronskian of a set of linearly independent functions is not equal to zero for at least one value.* In part **(b)**, we found that $W(S) = 2 \neq 0$. Therefore, the functions are linearly independent as we discussed earlier.

**Theorem 4.10 Wronskian of Solutions**

Consider the $n$th-order linear homogeneous equation $y^{(n)}(t) + p_n(t)y^{(n-1)}(t) + \cdots + p_1(t)y(t) = 0$, where $p_n(t), \ldots, p_1(t)$ are continuous functions on an open interval $I$. Suppose that $y_1(t), y_2(t), \ldots, y_n(t)$ are solutions of this ODE on $I$. If $y_1(t), y_2(t), \ldots, y_n(t)$ are linearly dependent, then $W(y_1(t), y_2(t), \ldots, y_n(t)) \equiv 0$ on $I$. If $y_1(t), y_2(t), \ldots, y_n(t)$ are linearly independent, then $W(y_1(t), y_2(t), \ldots, y_n(t)) \neq 0$ for all values of $t$ on $I$.

We omit the proof of this theorem here. We leave it as part of the section exercises (see Abel's formula).

**Example 4**

The functions $y_1(t) = t^{-4}$, $y_2(t) = t^{-1}$, $y_3(t) = t$, and $y_4(t) = t \ln t$ are solutions of $t^4y^{(4)} + 9t^3y''' + 11t^2y'' - 4ty' + 4y = 0$. Show that these solutions are linearly independent on the interval $t > 0$.

***Solution*** Notice that we have assumed that $t > 0$, so we can divide each term by $t^4$ to place the ODE in normal form,

$$y^{(4)} + \frac{9}{t}y''' + \frac{11}{t^2}y'' - \frac{4}{t^3}y' + \frac{4}{t^4}y = 0,$$

for use with Theorem 4.10. (Note that we assumed $t > 0$ so that the coefficient functions in the ODE are continuous.) Then, if $S = \{t^{-4}, t^{-1}, t, t\ln t\}$, we find with the help of a computer algebra system that

$$W(S) = \begin{vmatrix} t^{-4} & t^{-1} & t & t\ln t \\ -4/t^5 & -1/t^2 & 1 & 1+\ln t \\ 20/t^6 & 2/t^3 & 0 & 1/t \\ -120/t^7 & -6/t^4 & 0 & -1/t^2 \end{vmatrix} = \frac{300}{t^9},$$

so $W(S) \neq 0$ for $t > 0$. Therefore, the functions are linearly independent on the interval $t > 0$.

**Theorem 4.11 General Solution**

Consider the $n$th-order linear homogeneous equation $y^{(n)}(t) + p_n(t)y^{(n-1)}(t) + \cdots + p_1(t)y(t) = 0$, where $p_n(t), \ldots, p_1(t)$ are continuous functions on an open interval $I$. Suppose that $y_1(t), y_2(t), \ldots, y_n(t)$ are *linearly independent* solutions of this ODE on $I$. Then, if $y$ is any solution of this ODE, there are constants $c_1, c_2, \ldots, c_n$ so that $y(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t)$ for all $t$ on $I$.

Theorem 4.11 tells us that if we have $n$ linearly independent solutions of an $n$th-order linear homogeneous equation, then we have all solutions, called a **general solution**, given by the linear combination of the solutions. In addition, we call $\{y_1(t), y_2(t), \ldots, y_n(t)\}$ a **fundamental set of solutions,** and we note that a fundamental set of solutions for $y^{(n)}(t) + p_n(t)y^{(n-1)}(t) + \cdots + p_1(t)y(t) = 0$ must contain $n$ linearly independent solutions. (In other words, the number of functions in the set equals the order of the $n$th-order linear homogeneous ODE, and these functions must be linearly independent.)

We omit the proof of this theorem because it follows the same steps as those used in the proof of the case for second-order equations discussed in Section 4.1.

**Example 5**

Determine a general solution of $t^4y^{(4)} + 9t^3y''' + 11t^2y'' - 4ty' + 4y = 0$ using the solutions given in Example 4.

***Solution*** We showed in Example 4 that the four solutions $y_1(t) = t^{-4}$, $y_2(t) = t^{-1}$, $y_3(t) = t$, and $y_4(t) = t \ln t$ are linearly independent. Therefore, because we have four linearly independent solutions of a fourth-order linear homogeneous ODE, we can write a general solution as the linear combination

$$y(t) = c_1y_1(t) + c_2y_2(t) + c_3y_3(t) + c_4y_4(t) = c_1t^{-4} + c_2t^{-1} + c_3t + c_4t \ln t.$$

The set $\{t^{-4}, t^{-1}, t, t \ln t\}$ is a fundamental set of solutions for the ODE.

## EXERCISES 4.3

In Exercises 1–8, calculate the Wronskian of the indicated set of functions. Classify each set of functions as linearly independent or linearly dependent.

*1. $S = \{3t^2, t, 2t - 2t^2\}$

2. $S = \{\cos 2t, \sin t, 1\}$

3. $S = \{e^{-t}, e^{3t}, te^{3t}\}$

4. $S = \{e^t, e^{-2t}, e^{-4t}\}$

*5. $S = \{e^t, e^{-5t}, \sin t, e^{-5t} \cos t\}$

6. $S = \{e^{3t}, e^{-2t}, e^{5t}, te^{5t}\}$

7. $S = \{e^{-3t}, e^{-t}, e^{-4t} \cos 3t, e^{-4t} \sin 3t\}$



8. $S = \{e^{-t} \cos 2t, e^{-t} \sin 2t, e^{2t} \cos 5t, e^{2t} \sin 5t\}$

In Exercises 9–14, show that $S$ is a fundamental set of solutions for the given equation.

9. $S = \{e^{-t}, e^{3t}, te^{3t}\}$; $y''' - 5y'' + 3y' + 9y = 0$

10. $S = \{e^{-t}, e^{-2t}, e^{-4t}\}$; $y''' + 7y'' + 14y' + 8y = 0$

*11. $S = \{e^t, e^{-5t} \sin t, e^{-5t} \cos t\}$; $y''' + 9y'' + 16y' - 26y = 0$

12. $S = \{e^{3t}, e^{-2t}, e^{5t}, te^{5t}\}$; $y^{(4)} - 11y''' + 29y'' + 35y' - 150y = 0$

13. $S = \{e^{-3t}, e^{-t}, e^{-4t} \cos 3t, e^{-4t} \sin 3t\}$; $y^{(4)} + 12y''' + 60y'' + 124y' + 75y = 0$

14. $S = \{e^{-t} \cos 2t, e^{-t} \sin 2t, e^{2t} \cos 5t, e^{2t} \sin 5t\}$; $y^{(4)} - 2y''' + 26y'' + 38y' + 145y = 0$

15. **(Abel's Formula for Higher Order Equations)** Consider the $n$th-order linear homogeneous equation $y^{(n)}(t) + p_n(t)y^{(n-1)}(t) + \cdots + p_1(t)y(t) = 0$ with solutions $y_1(t), y_2(t), \ldots, y_n(t)$. We can show that the Wronskian of these $n$ solutions satisfies the same identity as that presented in Exercises 4.1. We do this for the third-order ODE $y^{(3)} + p_3(t)y'' + p_2(t)y' + p_1(t)y(t) = 0$, with solutions $y_1$, $y_2$, and $y_3$.
   **(a)** Show that
   $$\frac{d}{dt} W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \end{vmatrix}.$$
   **(b)** Use the ODE to solve for $y_1^{(3)}$, $y_2^{(3)}$, and $y_3^{(3)}$. Substitute these values to obtain $dW/dt + p_3(t)W = 0$.
   **(c)** Solve this ODE to find that $W(y_1, y_2, \ldots, y_n) = Ce^{-\int p_3(t)dt}$.

Using traditional pencil and paper to compute the Wronskian of a set of functions quickly becomes tedious and time-consuming. Fortunately, computer algebra systems are capable of computing derivatives and evaluating the determinant encountered when calculating the Wronskian of a set of functions. Moreover, the graphing capabilities can be used to help determine whether the resulting expression is or is not zero. Compute the Wronskian of each of the following sets $S$ to classify each set as linearly independent or linearly dependent. If necessary, graph the Wronskian to see if it is the zero function.

16. $S = \{1 - 2 \sin^2 t, \cos 2t\}$

17. $S = \{1, t, \frac{1}{2}(3t^2 - 1), \frac{1}{2}(5t^3 - 3t), \frac{1}{8}(35t^4 - 30t^2 + 3)\}$

18. $S = \{\sin t, \sin 2t, \sin 3t, \sin 4t\}$

19. $S = \{e^t, te^t, t^2e^t, t^3e^t, t^4e^t\}$

20. Is it possible to choose values of $c_1, \ldots, c_7$, not all zero, so that
   $$f(x) = c_1 + c_2t^{-5/2} + c_3t^{-3/2} + c_4t^{-1/2} + c_3t^{1/2} + c_6t^{3/2} + c_7t^{5/2}$$
   is the zero function?

21. Use the symbolic manipulation capabilities of a computer algebra system to help you prove the following theorem. Suppose that $f(t)$ is a differentiable function on an interval $I$ and that $f(t) \neq 0$ for all $x$ in $I$. **(a)** Prove that $f(t)$ and $tf(t)$ are linearly independent on $I$. **(b)** Prove that $f(t)$, $tf(t)$, and $t^2f(t)$ are linearly independent on $I$. **(c)** Generalize your result.

## 4.4 Solutions to Higher Order Linear Homogeneous Equations with Constant Coefficients

Distinct Real Roots  Repeated Real Roots  Complex Conjugate Roots

Suppose that we wish to solve

$$a_ny^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1y'(t) + a_0y(t) = 0,$$

where $a_n, a_{n-1}, \ldots, a_0$ are real constants with $a_n \neq 0$. Following the same procedure we used to solve $ay'' + by' + cy = 0$, we assume that $y(t) = e^{rt}$ is a solution, and we substitute this function into the ODE to determine the value(s) of $r$ that lead to solutions. Taking derivatives, we find that $y'(t) = re^{rt}$, $y''(t) = r^2e^{rt}$, and $y'''(t) = r^3e^{rt}$. Following this pattern, we see that the $k$th derivative is $y^{(k)}(t) = r^k e^{rt}$. Therefore, substitution yields

$$a_n r^n e^{rt} + a_{n-1} r^{(n-1)} e^{rt} + \cdots + a_1 r e^{rt} + a_0 e^{rt} = 0 \quad \text{or} \quad e^{rt}(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0) = 0,$$

so the *characteristic equation* is

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

because $e^{rt} \neq 0$ for all values of $t$.

## Distinct Real Roots

Suppose that the characteristic equation has the roots $r_1, r_2, \ldots, r_n$ and they are distinct (that is, no two are equal). Then we have the $n$ solutions $y_1(t) = e^{r_1 t}$, $y_2(t) = e^{r_2 t}$, $\ldots, y_n(t) = e^{r_n t}$. Later in Exercise 39, we show that these solutions are linearly independent. Therefore, we obtain a general solution by taking the linear combination of the $n$ solutions.

**Theorem 4.12 Distinct Real Roots**

If the roots $r_1, r_2, \ldots, r_n$ of $a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$ are real and distinct, then a general solution of $a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = 0$ is $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}$.

**Example 1**

Solve $y''' - y' = 0$.

**Solution** In this case, the characteristic equation is $r^3 - r = 0$. Factoring, we have $r(r^2 - 1) = r(r - 1)(r + 1) = 0$. The three distinct roots are $r = 0$, $r = 1$, and $r = -1$ with corresponding solutions $e^0 = 1$, $e^t$, and $e^{-t}$, respectively. Therefore, a general solution is the linear combination of these functions, $y(t) = c_1 + c_2 e^t + c_3 e^{-t}$.

**Example 2**

Solve $y''' - y' = 0$, $y(0) = 0$, $y'(0) = -2$, $y''(0) = 2$.

**Solution** (Notice that an initial-value problem involving a third-order ODE requires three initial conditions.) Typically, we solve an initial-value problem by first finding a general solution. However, we found this in the previous example to be $y(t) = c_1 + c_2 e^t + c_3 e^{-t}$. Next, we apply the initial conditions where $y'(t) = c_2 e^t - c_3 e^{-t}$ and $y''(t) = c_2 e^t + c_3 e^{-t}$. Then, $y(0) = c_1 + c_2 + c_3$, $y'(0) = c_2 - c_3$, and $y''(0) = c_2 + c_3$, so we obtain the system of three equations and three unknowns:

$$\begin{cases} c_1 + c_2 + c_3 = 0 \\ c_2 - c_3 = -2 \\ c_2 + c_3 = 2 \end{cases}.$$

Adding the last two equations indicates that $c_2 = 0$. Substitution of this value into either of the last two equations gives $c_3 = 2$. Then, substitution of $c_2 = 0$ and $c_3 = 2$ into the first equation gives us $c_1 = -2$. Therefore, the unique solution to this IVP is $y(t) = -2 + (0)e^t + (2)e^{-t} = -2 + 2e^{-t}$.



## Repeated Real Roots

When solving second-order equations in Section 4.2, we encountered equations in which the root of the characteristic equation was repeated. For example, the characteristic equation for $y'' - 2y' + y = 0$ is $r^2 - 2r + 1 = 0$ or $(r - 1)^2 = 0$. In this case, we say that the root $r = 1$ has *multiplicity* two, and we found a general solution to be $y(t) = c_1 e^t + c_2 t e^t$. As we increase the order of the ODE, we can have roots of multiplicity greater than two, so we present the following theorem.

**Theorem 4.13 Repeated Real Roots**

Suppose that the root $r = R$ of the characteristic equation $a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$ has multiplicity $k$ (that is, $(r - R)^k$ is a factor in the characteristic equation). Then, the $k$ linearly independent solutions of $a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = 0$ associated with $r = R$ are

$$e^{Rt}, te^{Rt}, t^2 e^{Rt}, \ldots, t^{k-1} e^{Rt}.$$

For example, applying this theorem to a second-order equation with $k = 2$ and $R = -1$, we obtain $y(t) = c_1 e^{-t} + c_2 t e^{-t} = (c_1 + c_2 t)e^{-t}$ as we found in Example 2 in Section 4.2.

**Example 3**

Solve $2y^{(6)} - 7y^{(5)} - 4y^{(4)} = 0$.

**Solution** The characteristic equation is $2r^6 - 7r^5 - 4r^4 = 0$, or

$$r^4(2r^2 - 7r - 4) = r^4(2r + 1)(r - 4) = 0.$$

Therefore, the roots are $r = 0$ (of multiplicity $k = 4$), $r = -1/2$ (of multiplicity $k = 1$), and $r = 4$ (of multiplicity $k = 1$). Then, the *four* linearly independent solutions corresponding to $r = 0$ are $e^{0\cdot t}$, $te^{0\cdot t}$, $t^2e^{0\cdot t}$, and $t^3e^{0\cdot t}$, which are simplified to 1, $t$, $t^2$, and $t^3$.

Notice that the number of solutions associated with $r = 0$ equals the multiplicity of $r = 0$. The solution associated with $r = -1/2$ is $e^{-t/2}$ and that associated with $r = 4$ is $e^{4t}$ (Table 4.1). We find a general solution of $2y^{(6)} - 7y^{(5)} - 4y^{(4)} = 0$ by taking the linear combination of the six (linearly independent) solutions: 1, $t$, $t^2$, $t^3$, $e^{-t/2}$, $e^{4t}$. This solution is

$$y(t) = c_1 + c_2t + c_3t^2 + c_4t^3 + c_5e^{-t/2} + c_6e^{4t}.$$

**TABLE 4.1 Linearly Independent Solutions of $2y^{(6)} - 7y^{(5)} - 4y^{(4)} = 0$**

| Root | Multiplicity | Corresponding Solution(s) |
|---|---|---|
| $r = 0$ | $k = 4$ | $y = 1, y = t, y = t^2, y = t^3$ |
| $r = -1/2$ | $k = 1$ | $y = e^{-t/2}$ |
| $r = 4$ | $k = 1$ | $y = e^{4t}$ |

Note that in Example 3, a fundamental set of solutions for the ODE is $\{1, t, t^2, t^3, e^{-t/2}, e^{4t}\}$. The number of solutions in the set equals the order of the ODE. *Therefore, to form a general solution of an nth-order linear homogeneous ODE, we take the linear combination of the n functions in the fundamental set.*

## Complex Conjugate Roots

**Theorem 4.14 Complex Conjugate Roots**

Suppose that $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$ ($\alpha$ and $\beta > 0$ are real) are roots of the characteristic equation $a_nr^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0 = 0$. Then, two linearly independent solutions associated with this pair of roots are

$$e^{\alpha t}\cos\beta t \quad \text{and} \quad e^{\alpha t}\sin\beta t.$$

**Example 4**

Solve $y^{(4)} - y = 0$.

***Solution*** In this case, the characteristic equation is

$$r^4 - 1 = (r^2 - 1)(r^2 + 1) = (r - 1)(r + 1)(r^2 + 1) = 0,$$

so the roots are $r = 1$, $r = -1$, $r = i$, and $r = -i$. The solutions corresponding to the roots $r = 1$ and $r = -1$ are $e^t$ and $e^{-t}$, respectively, while the solutions corresponding to the complex conjugate pair $r = \pm i$ (where $\alpha = 0$ and $\beta = 1$) are $e^{0\cdot t}\cos t = \cos t$ and $e^{0\cdot t}\sin t = \sin t$ (see Table 4.2). Therefore, a general solution is the linear combination of these four linearly independent functions given by

$$y(t) = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t.$$



**TABLE 4.2 Linearly Independent Solutions of $y^{(4)} - y = 0$**

| Roots | Multiplicity | Corresponding Solution(s) |
|---|---|---|
| $r = 1$ | $k = 1$ | $y = e^t$ |
| $r = -1$ | $k = 1$ | $y = e^{-t}$ |
| $r = \pm i$ | $k = 1, k = 1$ | $y = \cos t, y = \sin t$ |

*Figure 4.5 shows the graph of $y = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t$ for various values of $c_1$, $c_2$, $c_3$, and $c_4$. Identify those graphs for which **(a)** $c_1 = c_3 = c_4 = 0$; **(b)** $c_1 = c_2 = 0$; and (c) $c_3 = c_4 = 0$.*

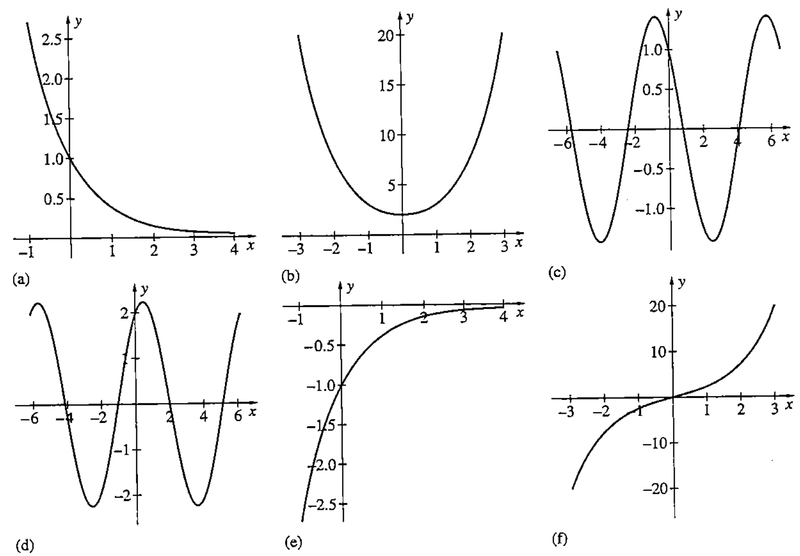



**Figure 4.5** (a)–(f)

We state the following rules for finding a general solution of an $n$th-order linear homogeneous equation with constant coefficients because of the many situations that are encountered.

### Determining a General Solution of a Higher Order Equation

① Let $r$ be a real root of the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

of an $n$th-order homogeneous linear differential equation with real constant coefficients. Then, $e^{rt}$ is the solution associated with the root $r$.

If $r$ is a root of multiplicity $k$ where $k \geq 2$ of the characteristic equation, then the $k$ solutions associated with $r$ are

$$e^{rt},\ te^{rt},\ t^2e^{rt},\ \ldots,\ t^{k-1}e^{rt}.$$

② Suppose that $r$ and $\bar{r}$ represent the complex conjugate pair $\alpha \pm \beta i$. Then the two solutions associated with these two roots are

$$e^{\alpha t}\cos\beta t \quad \text{and} \quad e^{\alpha t}\sin\beta t.$$

If the values $\alpha \pm \beta i$ are each a root of multiplicity $k$ of the characteristic equation, then the other solutions associated with this pair are

$$te^{\alpha t}\cos\beta t,\ te^{\alpha t}\sin\beta t,\ t^2e^{\alpha t}\cos\beta t,\ t^2e^{\alpha t}\sin\beta t,\ \ldots,$$
$$t^{k-1}e^{\alpha t}\cos\beta t,\ t^{k-1}e^{\alpha t}\sin\beta t.$$

A general solution to the $n$th-order differential equation is the linear combination of the solutions obtained for all values of $r$.

Note that if $r_1, r_2, \ldots, r_j$ are the roots of the equation of multiplicity $k_1, k_2, \ldots, k_j$, respectively, then $k_1 + k_2 + \cdots + k_j = n$, where $n$ is the order of the differential equation.

**Example 5**

Determine a general solution of the eighth-order homogeneous linear ODE with constant coefficients if the roots and corresponding multiplicities of the characteristic equation are $r = 2 - 3i$, $k = 2$; $r = 2 + 3i$, $k = 2$; $r = -5$, $k = 1$; and $r = 2$, $k = 3$.

***Solution*** The solutions that correspond to the complex conjugate pair $2 \pm 3i$ are $y = e^{2t}\cos 3t$ and $y = e^{2t}\sin 3t$. These roots are repeated so the two other solutions corresponding to the pair are $y = te^{2t}\cos 3t$ and $y = te^{2t}\sin 3t$. For the single root $r = -5$, the corresponding solution is $y = e^{-5t}$. Finally, the root of multiplicity three $r = 2$ yields the three solutions $y = e^{2t}$, $y = te^{2t}$, and $y = t^2e^{2t}$. Therefore, a general solution is



$$y(t) = e^{2t}(c_1\cos 3t + c_2\sin 3t + c_3 t\cos 3t + c_4 t\sin 3t) + c_5e^{-5t} + e^{2t}(c_6 + c_7t + c_8t^2).$$

*Find an eighth-order homogeneous linear ODE with constant coefficients that has general solution*

$$y(t) = e^{2t}(c_1\cos 3t + c_2\sin 3t + c_3 t\cos 3t + c_4 t\sin 3t) + c_5e^{-5t} + e^{2t}(c_6 + c_7t + c_8t^2).$$

**Example 6**

**(a)** If $a$ is a positive constant, find conditions on the constant $b$ so that $y(t)$ satisfies

$$\begin{cases} 5.69889y''' + 5.21742y'' - 3.39914y' - 5.6932y = 0 \\ y(0) = 0,\ y'(0) = a,\ y''(0) = b \end{cases}$$

and $\lim_{t\to\infty} y(t) = 0$. **(b)** For this function, find and classify the first critical point on the interval $[0, +\infty)$.

***Solution*** A general solution of the equation

$$5.69889y''' + 5.21742y'' - 3.39914y' - 5.69232y = 0$$

is

$$y(x) = e^{-0.91693t}(c_1\cos 0.496889t + c_2\sin 0.496889t) + c_3e^{0.918345t},$$

while the solution to the initial-value problem is

$$y(t) = e^{-0.91693t}[(-0.507273a - 0.276615b)\cos 0.496889t - (-0.138893a + 1.02168b)\sin 0.496889t] + (0.507273a + 0.276615b)e^{0.918345t}.$$

We see that $\lim_{t\to\infty} y(t) = 0$ if $0.507273a + 0.276615b = 0$, which leads to $b = -1.83386a$.

To find and classify the first critical point of $y(t)$, we compute

$$y'(t) = ae^{-0.91693t}(\cos 0.496889t - 1.84534\sin 0.496889t)$$

and graph $e^{-0.91693t}(\cos 0.496889t - 1.84534\sin 0.496889t)$ in Figure 4.6 to locate the first zero of $y'$.

From the graph, we see that $y'(t) = 0$ near 1 and, numerically, we obtain the critical number $t = 0.999432$. At this critical number $y(0.999432) = 0.383497a$, so by the first derivative test, $(0.999432, 0.383497a)$ is a local maximum. To see that $(0.999432, 0.383497a)$ is the *absolute* maximum, we graph $y(t)$ for various choices of $a$ in Figure 4.7.

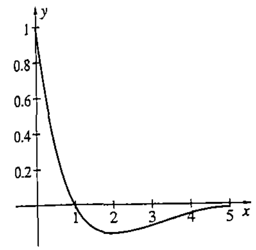



**Figure 4.6** Graph of $\frac{1}{a}y'(t) = e^{-0.91693t}(\cos 0.496889t - 1.84534\sin 0.496889t)$

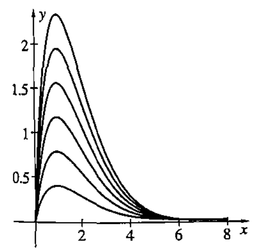



**Figure 4.7** Graph of $y(t)$ for various choices of $a$

## EXERCISES 4.4

In Exercises 1–10, given the list of roots and multiplicities of the characteristic equation, form a general solution. What is the order of the corresponding ODE?

1. $r = -2, k = 3; r = 2, k = 1$
2. $r = 1, k = 2; r = -10, k = 2$
*3. $r = 0, k = 2; r = 3i, k = 1; r = -3i, k = 1$
4. $r = 4, k = 3; r = 1 + i, k = 1; r = 1 - i, k = 1$
5. $r = -3 + 4i, k = 2; r = -3 - 4i, k = 2; r = -5, k = 1; r = -1/3, k = 1$
6. $r = i/2, k = 3; r = -i/2, k = 3; r = -1, k = 2$
7. Can $\{1, t, t^2, 2t - 8\}$ be a fundamental set of solutions for a fourth-order linear ODE?
8. Can $\{e^{-t}, e^t, \cos t, \sin t, \cos 2t\}$ be a fundamental set of solutions for a fifth-order linear ODE?
*9. Can $\{e^{2t}, e^{t/4}, e^{-t} \cos t, e^{-t} \sin t\}$ be a fundamental set of solutions for a fifth-order linear ODE?
10. Can $\{e^{2t}, e^{t/4}, e^{-t}, te^{-t}, t^2e^{-t}\}$ be a fundamental set of solutions for a fourth-order linear ODE?

In Exercises 11–30, find a general solution of each equation. (A computer algebra system may be useful in solving some of the equations.)

11. $y''' - 10y'' + 25y' = 0$
12. $8y''' + y'' = 0$
*13. $y''' + 7y'' + 17y' + 15y = 0$
14. $y''' + 7y'' + 24y' - 32y = 0$
15. $y''' - 8y'' + 5y' + 14y = 0$
16. $y''' - 6y'' - 7y' + 60y = 0$
*17. $y''' + y'' - 16y' + 20y = 0$
18. $y''' + 12y'' + 36y' + 32y = 0$
19. $y''' + 11y'' + 24y' - 36y = 0$
20. $y''' + 2y'' - 15y' - 36y = 0$
*21. $y^{(4)} - 9y'' = 0$
22. $8y^{(4)} + y' = 0$
23. $y^{(4)} - 16y = 0$
24. $9y^{(4)} + 4y''' = 0$
*25. $y^{(4)} - 6y''' - y'' + 54y' - 72y = 0$
26. $y^{(4)} - 5y''' - 10y'' + 80y' - 96y = 0$
27. $y^{(4)} + 7y''' + 6y'' - 32y' - 32y = 0$
28. $y^{(4)} + y''' + 10y'' - 52y' + 40y = 0$
*29. $y^{(4)} + 2y''' - 2y'' + 8y = 0$
30. $y^{(4)} + 32y'' + 256y = 0$

In Exercises 31–38, solve the initial-value problem. Graph the solution on an appropriate interval.

31. $y''' - 2y'' = 0, y(0) = 1, y'(0) = 2, y''(0) = 0$
32. $y''' + 49y' = 0, y(0) = -1, y'(0) = 0, y''(0) = -2$
*33. $y''' - y = 0, y(0) = 0, y'(0) = 0, y''(0) = 3$
34. $y^{(4)} - 16y = 0, y(0) = 0, y'(0) = -8, y''(0) = 0, y'''(0) = 0$
35. $y^{(4)} - 8y'' + 16y = 0, y(0) = 0, y'(0) = 0, y''(0) = 8, y'''(0) = 0$
36. $y^{(5)} - 4y''' = 0, y(0) = 0, y'(0) = 0, y''(0) = 0, y'''(0) = 0, y^{(4)}(0) = 32$
*37. $y^{(5)} + 8y^{(4)} = 0, y(0) = 8, y'(0) = 4, y''(0) = 0, y'''(0) = 48, y^{(4)}(0) = 0$
38. $y^{(6)} - 3y^{(4)} + 3y'' - y = 0, y(0) = 16, y'(0) = 0, y''(0) = 0, y'''(0) = 0, y^{(4)}(0) = 0, y^{(5)}(0) = 0$
39. Suppose that the roots $r_1$, $r_2$, and $r_3$ of the characteristic equation of a third-order linear homogeneous ODE with constant coefficients are real and distinct, and consider a general solution $y = c_1e^{r_1t} + c_2e^{r_2t} + c_3e^{r_3t}$. Show that the functions $e^{r_1t}$, $e^{r_2t}$, and $e^{r_3t}$ are linearly independent using the following steps: **(a)** Assume that the functions are linearly dependent. Then, there are constants $c_1$, $c_2$, and $c_3$ (not all zero) such that $c_1e^{r_1t} + c_2e^{r_2t} + c_3e^{r_3t} = 0$. **(b)** Multiply this equation by $e^{-r_1t}$ to obtain $c_1 + c_2e^{(r_2-r_1)t} + c_3e^{(r_3-r_1)t} = 0$. **(c)** Differentiate this equation with respect to $t$ to obtain $(r_2 - r_1)c_2e^{(r_2-r_1)t} + c_3(r_3 - r_1)e^{(r_3-r_1)t} = 0$. **(d)** Multiply this equation by $e^{-(r_2-r_1)t}$ and differentiate the resulting equation to obtain $c_3(r_3 - r_1)(r_2 - r_1)e^{(r_3-r_1)t} = 0$. This indicates that $c_3 = 0$. **(e)** Follow similar steps to show that $c_1 = c_2 = 0$, which contradicts the assumption. **(f)** How could you apply a similar argument to show that the functions $e^{r_1t}, e^{r_2t}, \ldots, e^{r_nt}$ are linearly independent if the $n$ roots $r_1, r_2, \ldots, r_n$ are real and distinct?



40. **(Operator Notation)** The $n$th-order derivative of a function $y$ is given in **operator notation** by $D^ny = d^ny/dt^n$. Then, the left side of the $n$th-order linear homogeneous ODE with constant coefficients,

$$a_ny^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1y'(t) + a_0y(t) = 0,$$

can be expressed as

$$a_nD^ny + a_{n-1}D^{n-1}y + \cdots + a_1Dy + a_0y \quad \text{or} \quad (a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)y,$$

where $p(D) = a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0$ is called an ***n*th-order linear differential operator** with constant coefficients. Therefore, the ODE can be written as $(a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)y = 0$. For example, we can write $y'' + 2y' - 8y = 0$ as $(D^2 - 2D - 8)y = 0$ or in either of the forms $(D + 2)(D - 4)y = 0$ or $(D - 4)(D + 2)y = 0$. Write each of the following equations in differential operator notation.

(a) $y'' + y' + 7y = 0$
(b) $y'' + 16y = 0$
(c) $y''' + 8y' - y = 0$
(d) $16y^{(4)} + y = 0$

41. Consider the linear differential operator $p(D) = (2D - 1)$. When we apply $p(D)$ to the function $f(t) = t^2$, we obtain $(2D - 1)(x^2) = 2D(x^2) - x^2 = 2(2x) - x^2 = 4x - x^2$. Apply the given differential operator to the given function.

(a) $p(D) = 3D, f(t) = t$
(b) $p(D) = (1 - D), f(t) = t$
(c) $p(D) = D^2, f(t) = t^2$
(d) $p(D) = (D^2 + 1), f(t) = \cos t$

42. The linear differential operator $p(D)$ is said to **annihilate** a function $f(t)$ if $p(D)[f(t)] = 0$ for all $t$. In fact, if $P_1 = p_1(D)$ annihilates $f(x)$ and if $P_2 = p_2(D)$ is another linear differential operator, then $P_2P_1 = P_1P_2$ annihilates $f(t)$. For example, $(D - 1)$ annihilates $e^t$ because $(D - 1)(e^t) = e^t - e^t = 0$. Therefore, if we apply another linear differential operator such as $D$, we have $D(D - 1)(e^t) = D(e^t - e^t) = D(0) = 0$. Show that $(D - 1)D(e^t) = 0$. In terms of ODEs, this means that a solution to $(D - 1)y = y' - y = 0$ is also a solution to $D(D - 1)y = y'' - y' = 0$.

43. **(Repeated Roots)** Suppose that the characteristic equation $a_nr^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0 = 0$ has two distinct roots, $r_1$ of multiplicity one and $r_2$ of multiplicity $k = n - 1$. Then, the characteristic equation can be written as $(r - r_1)(r - r_2)^k = 0$. Similarly, the ODE can be written in differential operator notation as $(D - r_1)(D - r_2)^ky = 0$. Therefore, every solution of $(D - r_2)^ky = 0$ is also a solution of $(D - r_1)(D - r_2)^ky = 0$ (see Exercise 42). To find solutions of $(D - r_2)^ky = 0$, assume that $y = v(t)e^{r_2t}$. **(a)** Use the product rule to show that $(D - r_2)(v(t)e^{r_2t}) = (Dv)e^{r_2t}$ so that, by induction, $(D - r_2)^k(v(t)e^{r_2t}) = (D^kv)e^{r_2t}$ for any function $v(t)$. **(b)** Therefore, a solution of $(D - r_2)^ky = 0$ must satisfy $(D^kv)e^{r_2t} = 0$ or $D^kv = 0$. Solve $D^kv = 0$ (or $v^{(k)} = 0$) to find $v(t)$ and to conclude that the portion of the solution corresponding to $r = r_2$ is $y = v(t)e^{r_2t} = (c_1 + c_2t + \cdots + c_kt^{k-1})e^{r_2t}$.

Computer algebra systems have built-in functions that can be used to find exact solutions of $n$th-order linear homogeneous differential equations with constant coefficients as long as $n$ is smaller than 5. In cases when the roots of the characteristic equation are symbolically complicated, when approximations are desired, or when $n$ is greater than 4, the roots of the characteristic equation can be approximated by taking advantage of the numerical approximation capabilities of the system being used.

44. Find a general solution of

(a) $y''' + 2y'' + 5y' - 26y = 0$
(b) $0.9y''' + 18.78y' - 0.2987y = 0$
(c) $8.9y^{(4)} - 2.5y'' + 32.0y' + 0.773y = 0$.

Graph the solution for various initial conditions.

45. Solve each of the following initial-value problems. Verify that your result satisfies the initial conditions by graphing it on an appropriate interval.

(a) $\begin{cases} y''' + 3y'' + 2y' + 6y = 0 \\ y(0) = 0, y'(0) = 1, y''(0) = -1 \end{cases}$

(b) $\begin{cases} y^{(4)} - 8y''' + 30y'' - 56y' + 49y = 0 \\ y(0) = 1, y'(0) = 2, y''(0) = -1, y'''(0) = -1 \end{cases}$

(c) $\begin{cases} 0.31y''' + 11.2y'' - 9.8y' + 5.3y = 0 \\ y(0) = -1, y'(0) = -1, y''(0) = 0 \end{cases}$

46. Use a computer algebra system to find a general solution of

$$y - k^4 \frac{d^4y}{dx^4} = 0.$$

Show that the result you obtain is equivalent to

$$y = Ae^{-x/k} + Be^{x/k} + C\sin\frac{x}{k} + D\cos\frac{x}{k}.$$

has **(a)** neither local maxima nor local minima; **(b)** exactly one local maximum; and **(c)** exactly one local minimum on the interval $[0, \infty)$.

*47. Show that if $y_0 \neq 0$ the problem

$$\begin{cases} 4.02063y''' - 0.224975y'' + 4.486y' - 2.48493y = 0 \\ y(0) = 0,\ y'(0) = y_0,\ y(4) = 0 \end{cases}$$

has a unique solution.

48. Complete the following table.

| Differential Equation | Characteristic Equation | Roots of Characteristic Equation | General Solution |
|---|---|---|---|
| $y^{(n)} = 0$ | | | |
| | $(r-k)^n$ | | |
| | | $r_{1,2} = \alpha \pm i\beta$, | |
| | | $\beta \neq 0$ | |
| | | | $y = e^{\alpha t}[(c_{1,1} + c_{1,2}t + \cdots c_{1,n-1}t^{n-1})\cos\beta t + (c_{2,1} + c_{2,2}t + \cdots c_{2,n-1}t^{n-1})\sin\beta t]$ |

# 4.5 Introduction to Solving Nonhomogeneous Equations with Constant Coefficients: Method of Undetermined Coefficients

General Solution of a Nonhomogeneous Equation · Method of Undetermined Coefficients

## General Solution of a Nonhomogeneous Equation

Suppose that we encounter the homogeneous equation $y'' + y = 0$. In previous sections, we learned that this equation has the general solution $y(t) = c_1\cos t + c_2\sin t$. Now we investigate ways to solve nonhomogeneous equations. For example, consider the equation $y'' + y = 2e^{-t}$. Notice that if we substitute the function $y(t) = e^{-t}$ with derivatives $y'(t) = -e^{-t}$ and $y''(t) = e^{-t}$ into this equation, we obtain

$$y'' + y = e^{-t} + e^{-t} = 2e^{-t}.$$

Therefore, $y(t) = e^{-t}$ satisfies the nonhomogeneous equation $y'' + y = 2e^{-t}$. In fact, this solution is called a *particular solution* of the nonhomogeneous equation.

**Definition 4.6 Particular Solution**

A **particular solution,** $y_p(t)$, of the nonhomogeneous differential equation $a_ny^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1y'(t) + a_0y(t) = g(t)$ is a specific function that contains no arbitrary constants and satisfies the differential equation.



**Example 1**

Verify that $y_p(t) = \frac{3}{5}\sin t$ is a particular solution of $y'' - 4y = -3\sin t$.

***Solution*** First we compute $y_p'(t) = \frac{3}{5}\cos t$ and $y_p''(t) = -\frac{3}{5}\sin t$. Substituting into $y'' - 4y$ results in

$$y_p'' - 4y_p = -\frac{3}{5}\sin t - 4\cdot\frac{3}{5}\sin t = -3\sin t.$$

We conclude that $y_p(t) = \frac{3}{5}\sin t$ is a particular solution of $y'' - 4y = -3\sin t$ because $y_p$ satisfies the equation $y'' - 4y = -3\sin t$ and contains no arbitrary constants.

*Show that $y_p(t) = \frac{1}{5}e^{-2t}(15 - 10e^{4t} + 3e^{2t}\sin t)$ is also a particular solution of $y'' - 4y = -3\sin t$.*

In this example, we see that $y_p(t) = \frac{3}{5}\sin t$ is a particular solution of $y'' - 4y = -3\sin t$. The **corresponding homogeneous equation** of $y'' - 4y = -3\sin t$ is $y'' - 4y = 0$ with general solution $y_h(t) = c_1e^{-2t} + c_2e^{2t}$. Let $y(t) = y_h(t) + y_p(t)$. Then,

$$\begin{aligned} y'' - 4y &= (y_h(t) + y_p(t))'' - 4(y_h(t) + y_p(t)) \\ &= y_h'' + y_p'' - 4y_h - 4y_p \\ &= \underbrace{y_h'' - 4y_h}_{0} + \underbrace{y_p'' - 4y_p}_{-3\sin t} = -3\sin t. \end{aligned}$$

We see that $y(t) = y_h(t) + y_p(t)$ is a solution of the nonhomogeneous equation $y'' - 4y = -3\sin t$. In fact, it is not hard to show that if $y(t)$ is *any* solution to the equation, then there are constants $c_1$ and $c_2$ so that $y(t) = y_h(t) + y_p(t)$ (see Exercise 65).

More generally, if $y_p(t)$ is a particular solution of the nonhomogeneous equation

$$a_ny^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1y'(t) + a_0y(t) = g(t)$$

and

$$y_h(t) = c_1f_1(t) + c_2f_2(t) + \cdots + c_nf_n(t)$$

is a general solution of the corresponding homogeneous equation

$$a_ny^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1y'(t) + a_0y(t) = 0,$$

every solution, $y(t)$, of the nonhomogeneous equation can be written in the form

$$y(t) = y_h(t) + y_p(t),$$

for some choice of $c_1, c_2, \ldots, c_n$ (see Exercise 65).

To prove this theorem for second-order equations, consider

$$a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = g(t)$$

and assume that $\{f_1(t), f_2(t)\}$ is a fundamental set of solutions for the corresponding homogeneous equation

$$a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0.$$

If both $y(t)$ and $y_p(t)$ are solutions to

$$a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = g(t),$$

let $y_h(t) = y(t) - y_p(t)$ so that substitution into the nonhomogeneous equation yields

$$\begin{aligned} &a_2(t)y_h''(t) + a_1(t)y_h'(t) + a_0(t)y_h(t) \\ &\quad = a_2(t)[y''(t) - y_p''(t)] + a_1(t)[y'(t) - y_p'(t)] + a_0(t)[y(t) - y_p(t)] \\ &\quad = [a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t)] - [a_2(t)y_p''(t) + a_1(t)y_p'(t) + a_0(t)y_p(t)] \\ &\quad = g(t) - g(t) = 0. \end{aligned}$$

Thus, $y_h(t) = y(t) - y_p(t)$ is a solution to the homogeneous equation $a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0$. There are constants $c_1$ and $c_2$ so that $y_h(t) = c_1 f_1(t) + c_2 f_2(t)$, which indicates that

$$y(t) - y_p(t) = \underbrace{c_1 f_1(t) + c_2 f_2(t)}_{y_h(t)} \quad \text{or} \quad y(t) = \underbrace{c_1 f_1(t) + c_2 f_2(t)}_{y_h(t)} + y_p(t).$$

This means that every solution to the nonhomogeneous equation can be written as the sum of a general solution to the corresponding homogeneous equation and a particular equation of the nonhomogeneous equation. This leads us to the following definition.

**Definition 4.7 General Solution of a Nonhomogeneous Equation**

**A general solution to the linear nonhomogeneous $n$th-order differential equation**

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = g(t)$$

is

$$y(t) = y_h(t) + y_p(t),$$

where $y_h(t)$ is a general solution of the corresponding homogeneous equation

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = 0,$$

and $y_p(t)$ is a particular solution to the nonhomogeneous equation.

**Example 2**

Let $y_p(t) = \frac{1}{2}t^3 - \frac{3}{2}t^2 + \frac{3}{2}t$ and $y_h(t) = e^{-t}(c_1 \cos t + c_2 \sin t)$. Show that $y(t) = y_h(t) + y_p(t)$ is a general solution of $y'' + 2y' + 2y = t^3$.



**Solution** We first show that $y_p(t) = \frac{1}{2}t^3 - \frac{3}{2}t^2 + \frac{3}{2}t$ is a particular solution of $y'' + 2y' + 2y = t^3$. Calculating $y_p' = \frac{3}{2}t^2 - 3t + \frac{3}{2}$, $y_p'' = 3t - 3$, and $y_p'' + 2y_p' + 2y_p$ gives us

$$\begin{aligned} y_p'' + 2y_p' + 2y_p &= (3t - 3) + 2\left(\frac{3}{2}t^2 - 3t + \frac{3}{2}\right) + 2\left(\frac{1}{2}t^3 - \frac{3}{2}t^2 + \frac{3}{2}t\right) \\ &= 3t - 3 + 3t^2 - 6t + 3 + t^3 - 3t^2 + 3t = t^3. \end{aligned}$$

The corresponding homogeneous equation of $y'' + 2y' + 2y = t^3$ is $y'' + 2y' + 2y = 0$ with characteristic equation $r^2 + 2r + 2 = 0$ and roots $r = -1 \pm i$. Thus, a general solution of $y'' + 2y' + 2y = 0$ is $y_h(t) = e^{-t}(c_1 \cos t + c_2 \sin t)$, so that

$$\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ &= e^{-t}(c_1 \cos t + c_2 \sin t) + \frac{1}{2}t^3 - \frac{3}{2}t^2 + \frac{3}{2}t \end{aligned}$$

is a general solution of $y'' + 2y' + 2y = t^3$.

*If $y_1(t)$ and $y_2(t)$ are nontrivial solutions of $y'' + 2y' + 2y = t^3$, is $y_1(t) + y_2(t)$ also a solution?*

### Method of Undetermined Coefficients

Now that we understand how to form a general solution to a nonhomogeneous equation, we present a method, called the *Method of Undetermined Coefficients,* that can be used to find a particular solution in some cases. Consider the nonhomogeneous second-order linear differential equation

$$y'' + 4y' + 3y = e^t.$$

A general solution to this nonhomogeneous equation is $y(t) = y_h(t) + y_p(t)$, where $y_h(t)$ is a solution of the corresponding homogeneous equation $y'' + 4y' + 3y = 0$. This equation has characteristic equation $r^2 + 4r + 3 = (r + 1)(r + 3) = 0$ so $y_h(t) = c_1e^{-t} + c_2e^{-3t}$.

Now we must select the proper form of a particular solution $y_p(t)$. Because $g(t) = e^t$, we are safe to assume that a particular solution is a multiple of $e^t$. Notice that the function we choose for $y_p(t)$ must satisfy $y'' + 4y' + 3y = e^t$, so we need to select a function that includes $e^t$ in the function and its derivatives. We let $y_p(t) = Ae^t$ and attempt to find $A$. Substituting this function into $y'' + 4y' + 3y = e^t$ yields

$$y_p'' + 4y_p' + 3y_p = Ae^t + 4Ae^t + 3Ae^t = 8Ae^t = e^t.$$

Equating the coefficients of $e^t$ gives us $8A = 1$ or $A = \frac{1}{8}$. Therefore, $y_p(t) = \frac{1}{8}e^t$, and a general solution is

$$y(t) = y_h(t) + y_p(t) = c_1e^{-t} + c_2e^{-3t} + \frac{1}{8}e^t.$$

*Find a particular solution of $y'' + 4y = \sin t$ of the form $y_p(t) = A \cos t + B \sin t$.*

Now consider the equation

$$y'' + 4y' + 3y = e^{-3t}.$$

We just saw that the solution of the corresponding homogeneous equation

$$y'' + 4y' + 3y = 0$$

is $y_h(t) = c_1e^{-t} + c_2e^{-3t}$. Therefore, if we assume that $y_p(t) = Ae^{-3t}$ as we did in the previous example, we have the derivatives $y_p'(t) = -3Ae^{-3t}$ and $y_p''(t) = 9Ae^{-3t}$. Substitution into $y'' + 4y' + 3y = e^{-3t}$ then yields

$$y'' + 4y' + 3y = 9Ae^{-3t} - 12Ae^{-3t} + 3Ae^{-3t} = 0 \neq e^{-3t}$$

which does *not* lead to determining the value of $A$. Of course, this should not surprise us because $e^{-3t}$ is a solution of the corresponding homogeneous equation. Therefore, we *cannot* include this function in the particular solution. However, we can obtain a function that resembles $e^{-3t}$ and includes $e^{-3t}$ in $y_p(t)$, $y_p'(t)$, and $y_p''(t)$, by multiplying $e^{-3t}$ by $t$. We let $y_p(t) = Ate^{-3t}$. The derivatives of this function are

$$y_p'(t) = Ae^{-3t} - 3Ate^{-3t} \quad \text{and} \quad y_p''(t) = -6Ae^{-3t} + 9Ate^{-3t}.$$

Substitution into $y'' + 4y' + 3y = e^{-3t}$ gives us

$$\begin{aligned} y_p'' + 4y_p' + 3y_p &= -6Ae^{-3t} + 9Ate^{-3t} + 4(Ae^{-3t} - 3Ate^{-3t}) + 3Ate^{-3t} \\ &= -2Ae^{-3t} = e^{-3t}. \end{aligned}$$

Equating the coefficients of $e^{-3t}$ then yields $-2A = 1$ or $A = -\frac{1}{2}$. Hence, $y_p(t) = -\frac{1}{2}te^{-3t}$ and a general solution is

$$y(t) = y_h(t) + y_p(t) = c_1e^{-t} + c_2e^{-3t} - \frac{1}{2}te^{-3t}.$$

*Show that it is **not** possible to find a particular solution of $y'' + 4y = \sin 2t$ of the form $y_p(t) = A\cos 2t + B\sin 2t$, but that it **is** possible to find a particular solution of the form $y_p(t) = t(A\cos 2t + B\sin 2t)$.*

These two examples illustrate the guesswork involved in choosing the form of a particular solution. To eliminate some of the guessing required to apply this method, we introduce an algorithm that can be used to determine the form of $y_p(t)$.

Consider the linear nonhomogeneous $n$th-order differential equation with constant coefficients

$$a_ny^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1y'(t) + a_0y(t) = g(t)$$

where $a_n \neq 0$ and $g(t)$ is a linear combination of the functions $1, t, t^2, \ldots, e^{kt}, te^{kt}, t^2e^{kt}, \ldots, e^{\alpha t}\cos\beta t, te^{\alpha t}\cos\beta t, t^2e^{\alpha t}\cos\beta t, \ldots, e^{\alpha t}\sin\beta t, te^{\alpha t}\sin\beta t, t^2e^{\alpha t}\sin\beta t, \ldots$. A general solution of this differential equation is

$$y(t) = y_h(t) + y_p(t),$$

where $y_h(t)$ is a solution of the corresponding homogeneous equation

$$a_ny^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1y'(t) + a_0y(t) = 0,$$



and $y_p(t)$ is a particular solution involving no arbitrary constants of the nonhomogeneous equation

$$a_ny^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1y'(t) + a_0y(t) = g(t).$$

We learned how to solve homogeneous equations in Sections 4.2 and 4.4, so we must learn how to find the form of a particular solution to solve nonhomogeneous equations with the method of undetermined coefficients.

**Outline of the Method of Undetermined Coefficients to Solve**

$$a_ny^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1y'(t) + a_0y(t) = g(t),$$

**where $g(t)$ is a linear combination of the functions**

$1, t, t^2, \ldots, e^{kt}, te^{kt}, t^2e^{kt}, \ldots, e^{\alpha t}\cos\beta t, te^{\alpha t}\cos\beta t, t^2e^{\alpha t}\cos\beta t, \ldots,$
$e^{\alpha t}\sin\beta t, te^{\alpha t}\sin\beta t, t^2e^{\alpha t}\sin\beta t, \ldots.$

① Solve the corresponding homogeneous equation for $y_h(t)$.

② Determine the form of a particular solution $y_p(t)$ (see **Determining the Form of $y_p(t)$**).

③ Determine the unknown coefficients in $y_p(t)$ by substituting $y_p(t)$ into the nonhomogeneous equation and equating the coefficients of like terms.

④ Form a general solution with $y(t) = y_h(t) + y_p(t)$.

**Determining the Form of $y_p(t)$ (step 2):**

Suppose that $g(t) = b_1g_1(t) + b_2g_2(t) + \cdots + b_jg_j(t)$ where $b_1, b_2, \ldots, b_j$ are constants and each $g_i(t)$, $i = 1, 2, \ldots, j$ is a function of the form $t^m$, $t^me^{kt}$, $t^me^{\alpha t}\cos\beta t$, or $t^me^{\alpha t}\sin\beta t$.

**(A)** If $g_i(t) = t^m$, the associated set of functions is

$$S = \{t^m, t^{m-1}, \ldots, t^2, t, 1\}.$$

**(B)** If $g_i(t) = t^me^{kt}$, the associated set of functions is

$$S = \{t^me^{kt}, t^{m-1}e^{kt}, \ldots, t^2e^{kt}, te^{kt}, e^{kt}\}.$$

**(C)** If $g_i(t) = t^me^{\alpha t}\cos\beta t$, or $g_i(t) = t^me^{\alpha t}\sin\beta t$, the associated set of functions is

$$\begin{aligned} S = \{&t^me^{\alpha t}\cos\beta t, t^{m-1}e^{\alpha t}\cos\beta t, \ldots, t^2e^{\alpha t}\cos\beta t, te^{\alpha t}\cos\beta t, e^{\alpha t}\cos\beta t, \\ &t^me^{\alpha t}\sin\beta t, t^{m-1}e^{\alpha t}\sin\beta t, \ldots, t^2e^{\alpha t}\sin\beta t, te^{\alpha t}\sin\beta t, e^{\alpha t}\sin\beta t\} \end{aligned}$$

For each function in $g(t)$, determine the associated set of functions $S$. If any of the functions in $S$ appears in the homogeneous solution $y_h(t)$, multiply each function in $S$ by $t^r$ to obtain a new set $S'$, where $r$ is the smallest positive integer so that each function in $S'$ is not a function in $y_h(t)$.

The correct form of a particular solution is obtained by taking the linear combination of all functions in the associated sets where repeated functions should appear only once in the form of the particular solution.

**Example 3**

For each of the following functions, determine the associated set of functions: **(a)** $f(t) = t^4$, **(b)** $f(t) = t^3e^{-2t}$, and **(c)** $f(t) = t^2e^{-t}\cos 4t$.

**Solution**
**(a)** Using **(A)**, we have $S = \{t^4, t^3, t^2, t, 1\}$.
**(b)** In this case we use **(B)**:

$$S = \{t^3e^{-2t}, t^2e^{-2t}, te^{-2t}, e^{-2t}\}.$$

**(c)** According to **(C)**,

$$S = \{t^2e^{-t}\cos 4t, te^{-t}\cos 4t, e^{-t}\cos 4t, t^2e^{-t}\sin 4t, te^{-t}\sin 4t, e^{-t}\sin 4t\}.$$

**Example 4**

Solve the nonhomogeneous equations **(a)** $y'' + 5y' + 6y = 2e^t$, **(b)** $y'' + 5y' + 6y = 2t^2 + 3t$, **(c)** $y'' + 5y' + 6y = 3e^{-2t}$, and **(d)** $y'' + 5y' + 6y = 4\cos t$.

**Solution** **(a)** The corresponding homogeneous equation $y'' + 5y' + 6y = 0$ has general solution $y_h(t) = c_1e^{-2t} + c_2e^{-3t}$. (Why?) Next, we determine the form of $y_p(t)$. Because $g(t) = 2e^t$, we choose $S = \{e^t\}$.

Notice that $e^t$ is not a solution to the homogeneous equation, so we take $y_p(t)$ to be the linear combination of the functions in $S$. Therefore,

$$y_p(t) = Ae^t.$$

Substituting this solution into $y'' + 5y' + 6y = 2e^t$, we have

$$Ae^t + 5Ae^t + 6Ae^t = 12Ae^t = 2e^t.$$

Equating the coefficients of $e^t$ then gives us $A = \frac{1}{6}$. A particular solution is $y_p(t) = \frac{1}{6}e^t$, so a general solution of $y'' + 5y' + 6y = 2e^t$ is

$$y(t) = y_h(t) + y_p(t) = c_1e^{-2t} + c_2e^{-3t} + \tfrac{1}{6}e^t.$$

**(b)** Here, we see that $g(t) = b_1g_1(t) + b_2g_2(t) = 2t^2 + 3t$. The set of functions associated with $g_1(t) = 2t^2$ is $S_1 = \{t^2, t, 1\}$ and that associated with $g_2(t) = 3t$ is $S_2 = \{t, 1\}$. Notice that none of these functions are solutions of the corresponding homogeneous equation. Notice also that $S_1$ and $S_2$ have two elements in common. If we take the linear combination of the functions $t^2$, $t$, and 1, then

$$y_p(t) = At^2 + Bt + C.$$



Substitution of the derivatives $y_p'(t) = 2At + B$ and $y_p''(t) = 2A$ into $y'' + 5y' + 6y = 2t^2 + 3t$ yields

$$y_p'' + 5y_p' + 6y_p = 6At^2 + (10A + 6B)t + (2A + 5B + 6C) = 2t^2 + 3t$$

Therefore, $6A = 2$, $10A + 6B = 3$, and $2A + 5B + 6C = 0$. This system of equations has the solution $A = \frac{1}{3}$, $B = -\frac{1}{18}$, and $C = -\frac{7}{108}$, so

$$y_p(t) = \frac{1}{3}t^2 - \frac{1}{18}t - \frac{7}{108}.$$

A general solution is

$$y(t) = y_h(t) + y_p(t) = c_1e^{-2t} + c_2e^{-3t} + \frac{1}{3}t^2 - \frac{1}{18}t - \frac{7}{108}.$$

**(c)** In this case, we see that $g(t) = 3e^{-2t}$. Then the associated set is $S = \{e^{-2t}\}$. However, because $e^{-2t}$ is a solution to the corresponding homogeneous equation, we must multiply this function by $t^r$ so that it is no longer a solution. We multiply the element of $S$ by $t^1 = t$ to obtain $S' = \{te^{-2t}\}$ because $te^{-2t}$ is not a solution of $y'' + 5y' + 6y = 0$. Hence, $y_p(t) = Ate^{-2t}$. Differentiating $y_p(t)$ twice and substituting into the equation yields:

$$\begin{aligned} y_p'' + 5y_p' + 6y_p &= -4Ae^{-2t} + 4Ate^{-2t} + 5(Ae^{-2t} - 2Ate^{-2t}) + 6Ate^{-2t} \\ &= Ae^{-2t} = 3e^{-2t}. \end{aligned}$$

Thus, $A = 3$ so $y_p(t) = 3te^{-2t}$, and a general solution of $y'' + 5y' + 6y = 3e^{-2t}$ is

$$y(t) = y_h(t) + y_p(t) = c_1e^{-2t} + c_2e^{-3t} + 3t^{-2t}.$$

**(d)** In this case, we see that $g(t) = 4\cos t$, so the associated set is $S = \{\cos t, \sin t\}$. Because neither of these functions is a solution of the homogeneous equation, we take the particular solution to be the linear combination

$$y_p(t) = A\cos t + B\sin t.$$

We determine $A$ and $B$ by differentiating $y_p(t)$ twice and substituting into the nonhomogeneous equation

$$\begin{aligned} y_p'' + 5y_p' + 6y_p &= -A\cos t - B\sin t - 5A\sin t + 5B\cos t + 6A\cos t + 6B\sin t \\ &= (5A + 5B)\cos t + (-5A + 5B)\sin t \\ &= 4\cos t. \end{aligned}$$

Hence, $5A + 5B = 4$ and $-5A + 5B = 0$, so $A = B = \frac{2}{5}$. Therefore,

$$y_p(t) = \frac{2}{5}\cos t + \frac{2}{5}\sin t,$$

and a general solution is

$$y(t) = y_h(t) + y_p(t) = c_1e^{-2t} + c_2e^{-3t} + \frac{2}{5}\cos t + \frac{2}{5}\sin t.$$

**Example 5**

Solve the following nonhomogeneous equations: **(a)** $y'' + 5y' + 6y = 4te^{-2t} + e^{-2t}$ and **(b)** $y'' + 4y' + 13y = 4e^{-2t}\sin 3t$.

***Solution*** (a) As we saw in Example 4, $y_h(t) = c_1e^{-2t} + c_2e^{-3t}$ is a general solution of the corresponding homogeneous equation $y'' + 5y' + 6y = 0$. Next, we notice that $g(t) = b_1g_1(t) + b_2g_2(t) = 4te^{-2t} + e^{-2t}$. The set associated with $g_1(t) = te^{-2t}$ is $S_1 = \{te^{-2t}, e^{-2t}\}$ and that associated with $g_2(t) = e^{-2t}$ is $S_2 = \{e^{-2t}\}$. Notice that both sets contain a function that appears in the corresponding homogeneous solution $y_h(t) = c_1e^{-2t} + c_2e^{-3t}$. We multiply both sets by $t$ to obtain $S_1' = \{t^2e^{-2t}, te^{-2t}\}$ and $S_2' = \{te^{-2t}\}$ so that neither set contains a solution to the corresponding homogeneous set. However, because $S_1'$ and $S_2'$ contain the common element $te^{-2t}$, we let

$$y_p(t) = At^2e^{-2t} + Bte^{-2t}.$$

Notice that if we include the repeated function twice, we arrive at the same result:

$$y_p(t) = c_1t^2e^{-2t} + c_2te^{-2t} + c_3te^{-2t} = c_1t^2e^{-2t} + (c_2 + c_3)te^{-2t}.$$

Differentiating $y_p(t)$ twice and substituting into $y'' + 5y' + 6y = 4te^{-2t} + e^{-2t}$ yields

$$y_p'' + 5y_p' + 6y_p = (2A + B)e^{-2t} + 2Ate^{-2t} = 4te^{-2t} + e^{-2t}.$$

Therefore, $2A + B = 1$ and $2A = 4$ so $A = 2$ and $B = -3$. This results in the particular solution

$$y_p(t) = 2t^2e^{-2t} - 3te^{-2t}$$

and the general solution

$$y(t) = y_h(t) + y_p(t) = c_1e^{-2t} + c_2e^{-3t} + 2t^2e^{-2t} - 3te^{-2t}.$$

**(b)** As in **(a)**, first find a general solution of the corresponding homogeneous equation $y'' + 4y' + 13y = 0$. Verify that a general solution of the corresponding homogeneous equation is

$$y_h(t) = c_1e^{-2t}\cos 3t + c_2e^{-2t}\sin 3t.$$

Because $g(t) = 4e^{-2t}\sin 3t$, the associated set of functions is $S = \{e^{-2t}\cos 3t, e^{-2t}\sin 3t\}$, but both of the functions in $S$ are solutions to the homogeneous equation, so we multiply by $t$ so that they are no longer solutions. Hence, $S' = \{te^{-2t}\cos 3t, te^{-2t}\sin 3t\}$ and we let

$$y_p(t) = Ate^{-2t}\cos 3t + Bte^{-2t}\sin 3t.$$

This function has first and second derivatives



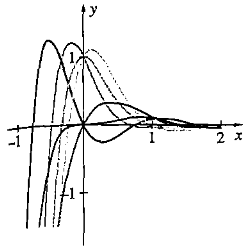



**Figure 4.8** Graph of $y(t) = y_h(t) + y_p(t)$ for various values of $c_1$ and $c_2$

$$y_p'(t) = Ae^{-2t}\cos 3t + (3B - 2A)te^{-2t}\cos 3t + Be^{-2t}\sin 3t + (-3A - 2B)te^{-2t}\sin 3t$$

and

$$y_p''(t) = (-4A + 6B)e^{-2t}\cos 3t + (-5A - 12B)te^{-2t}\cos 3t + (-6A - 4B)e^{-2t}\sin 3t + (12A - 5B)te^{-2t}\sin 3t,$$

which when substituted into the equation $y'' + 4y' + 13y = 4e^{-2t}\sin 3t$ yield

$$y_p'' + 4y_p' + 13y_p = 6Be^{-2t}\cos 3t - 6Ae^{-2t}\sin 3t = 4e^{-2t}\sin 3t.$$

Equating the coefficients of like terms, we have $-6A = 4$ and $6B = 0$. Hence, $A = -\frac{2}{3}$ and $B = 0$, so

$$y_p(t) = -\frac{2}{3}te^{-2t}\cos 3t.$$

Therefore, a general solution is

$$y(t) = y_h(t) + y_p(t) = c_1e^{-2t}\cos 3t + c_2e^{-2t}\sin 3t - \frac{2}{3}te^{-2t}\cos 3t.$$

*What is the limit as $t \to \infty$ of* ***every*** *solution? Why doesn't the choice of $c_1$ and $c_2$ matter? Hint: First, identify and use the graph of $y_p(t)$ in Figure 4.8 (or generate your own) and then apply L'Hopital's rule, if necessary.*

To solve an initial-value problem, first determine a general solution and then use the initial conditions to solve for the unknown constants in the general solution.

**Example 6**

Solve the initial-value problem $y''' + y'' = 12t^2$, $y(0) = 0$, $y'(0) = 1$, $y''(0) = 25$.

***Solution*** First, we solve the corresponding homogeneous equation $y''' + y'' = 0$, which has characteristic equation $r^3 + r^2 = r^2(r + 1) = 0$ and thus, general solution

$$y_h(t) = c_1 + c_2t + c_3e^{-t}.$$

Next, we set up the set of functions associated with $g(t) = 12t^2$, which is $S = \{t^2, t, 1\}$. However, both of the functions $t$ and $1$ are solutions of $y''' + y'' = 0$, so we multiply the functions in $S$ by $t^2$ so that the functions that result do not appear in $y_h(t)$. Hence, $S' = \{t^4, t^3, t^2\}$, so

$$y_p(t) = At^4 + Bt^3 + Ct^2.$$

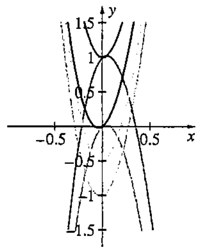



Figure 4.9

The derivatives of $y_p(t)$ are $y_p'(t) = 4At^3 + 3Bt^2 + 2Ct$, $y_p''(t) = 12At^2 + 6Bt + 2C$, and $y_p'''(t) = 24At + 6B$. Substitution into $y''' + y'' = 12t^2$ then gives us

$$y_p''' + y_p'' = 12At^2 + (24A + 6B)t + (2C + 6B) = 12t^2,$$

so $12A = 12$, $24A + 6B = 0$, and $2C + 6B = 0$. This system of equations has the solution $A = 1$, $B = -4$, and $C = 12$, so

$$y_p(t) = t^4 - 4t^3 + 12t^2,$$

and a general solution is

$$y(t) = y_h(t) + y_p(t) = c_1 + c_2t + c_3e^{-t} + t^4 - 4t^3 + 12t^2.$$

*Figure 4.9 shows the graph of $y(t) = c_1 + c_2t + c_3e^{-t} + t^4 - 4t^3 + 12t^2$ for various values of $c_1$, $c_2$, and $c_3$. Identify the graph of the solution that satisfies the initial conditions $y(0) = 0$, $y'(0) = 1$ and $y''(0) = 25$.*

To determine the unknown coefficients $c_1$, $c_2$, and $c_3$, we apply the initial conditions. First, we compute $y'(t) = c_2 - c_3e^{-t} + 4t^3 - 12t^2 + 24t$ and $y''(t) = c_3e^{-t} + 12t^2 - 24t + 24$. Applying the initial conditions, we have $y(0) = c_1 + c_3 = 0$, $y'(0) = c_2 - c_3 = 1$, and $y''(0) = c_3 + 24 = 25$, so $c_1 = -1$, $c_2 = 2$, and $c_3 = 1$. Therefore, the solution of the initial-value problem is

$$y(t) = y_h(t) + y_p(t) = -1 + 2t + e^{-t} + t^4 - 4t^3 + 12t^2 \text{ (see Figure 4.10).}$$

## Example 7

If $\omega \geq 0$, solve the initial-value problem $\begin{cases} y'' + y = \cos \omega t \\ y(0) = 0, y'(0) = 1 \end{cases}$.

***Solution*** A general solution of the corresponding homogeneous equation $y'' + y = 0$ is $y_h(t) = c_1 \cos t + c_2 \sin t$. We see that if $\omega \neq 1$ we can find a particular solution of the nonhomogeneous equation of the form $y_p(t) = A \cos \omega t + B \sin \omega t$, while if $\omega = 1$ we can find a particular solution of the form $y_p(t) = t(A \cos t + B \sin t)$. Solving for a particular solution, forming a general solution, and applying the initial conditions yields the solution

$$y(t) = \begin{cases} \dfrac{1}{\omega^2 - 1}(\cos t - \cos \omega t) + \sin t, & \text{if } \omega \neq 1 \\ \dfrac{1}{2}(t + 2) \sin t, & \text{if } \omega = 1 \end{cases}.$$

Notice that the behavior of the solution changes dramatically when $\omega = 1$: if $0 \leq \omega < 1$ or $\omega > 1$, the solution is periodic and bounded (why?), but if $\omega = 1$, the solution is unbounded (see Figure 4.11).

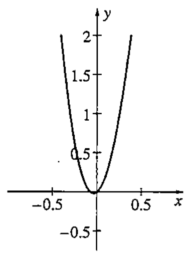



Figure 4.10 Graph of $y(t) = -1 + 2t + e^{-t} + t^4 - 4t^3 + 12t^2$



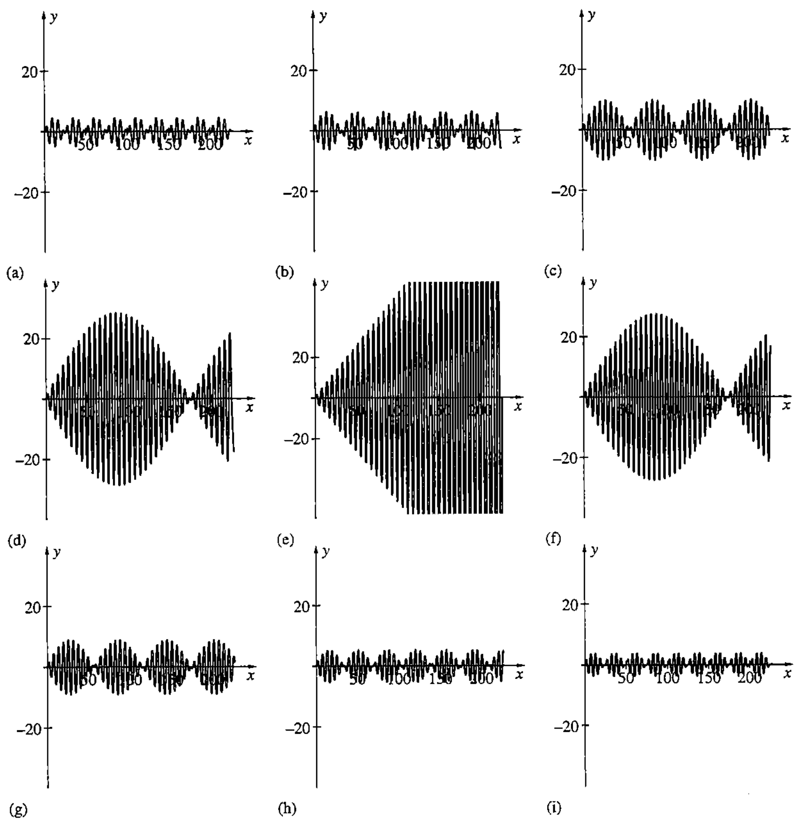



Figure 4.11 Graph of the solution of $\begin{cases} y'' + y = \cos \omega t \\ y(0) = 0, y'(0) = 1 \end{cases}$ for various values of $\omega$

**(a)** $\omega = 3/4$ **(b)** $\omega = 23/28$ **(c)** $\omega = 25/28$ **(d)** $\omega = 27/28$ **(e)** $\omega = 1$ **(f)** $\omega = 29/28$ **(g)** $\omega = 31/28$ **(h)** $\omega = 33/28$ **(i)** $\omega = 5/4$

## EXERCISES 4.5

For each function in Exercises 1–12, determine the associated set of functions.

1. $f(t) = 2t + 5$
2. $f(t) = 3 - 8t^2$
3. $f(t) = 7e^{2t} + 2$
4. $f(t) = -2te^{-t}$

*5. $f(t) = 4e^{-t} - 2t^4 + t$

6. $f(t) = 4t - 8t^2e^{-3t}$
7. $f(t) = 6 \sin 2t + 3e^{-4t}$
8. $f(t) = 2t \sin t$
9. $f(t) = 2 \sin 3t + 4 \sin 3t - 8 \cos 2t$
10. $f(t) = 2te^t \sin t - 7t^3 + 8$

*11. $f(t) = e^{-t} \cos 2t + 1$

12. $f(t) = te^{2t} \sin t + \cos 4t + 3t^2 - 8e^{-4t}$

In Exercises 13–24, determine the form of a particular solution for the given nonhomogeneous equation. (Do not solve for the unknown coefficients.)

13. $y'' + y = 8e^{2t}$
14. $y'' - 4y' + 3y = -e^{-9t}$

*15. $y'' - 4y' + 3y = 2e^{3t}$

16. $y'' - y = 2t - 4$
17. $y'' - 2y' + y = t^2$
18. $y'' + 2y' = 3 - 4t$

*19. $y'' + y = \cos 2t$

20. $y'' + 4y = 4 \cos t - \sin t$
21. $y'' + 4y = \cos 2t + t$
22. $y'' + 4y = 3te^{-t}$

*23. $y'' = 3t^4 - 2t$

24. $y'' - 4y' + 13y = 2te^{-2t} \sin 3t$

In Exercises 25–45, find a general solution of each equation.

25. $y'' + y' - 2y = -1$
26. $5y'' + y' - 4y = -3$

*27. $y'' - 2y' - 8y = 32t$

28. $16y'' - 8y' - 15y = 75t$
29. $y'' + 2y' + 26y = -338t$
30. $y'' + 3y' - 4y = -32t^2$

*31. $8y'' + 6y' + y = 5t^2$

32. $y'' - 6y' + 8y = -256t^3$
33. $y'' - 2y' = 52 \sin 3t$
34. $y'' - 6y' + 13y = 25 \sin 2t$

*35. $y'' - 9y = 54t \sin 3t$

36. $y'' - 5y' + 6y = -78 \cos 3t$
37. $y'' + 4y' + 4y = -32t^2 \cos 2t$
38. $y'' - y' - 20y = -2e^t$

*39. $y'' - 4y' - 5y = -648t^2e^{5t}$

40. $y'' - 7y' + 12y = -2t^3e^{4t}$
41. $y''' + 6y'' + 11y' + 6y = 2e^{-3t} - te^{-t}$
42. $y''' + 10y'' + 34y' + 40y = te^{-4t} + 2t^{-3t} \cos t$

*43. $y''' + 6y'' - 14y' - 104y = -111e^t$

44. $y^{(4)} - 6y''' + 13y'' - 24y' + 36y = 108t$
45. $y^{(4)} - 10y''' + 38y'' - 64y' + 40y = 153e^{-t}$

In Exercises 46–58, solve the initial-value problem.

46. $y'' + 3y' = 18, y(0) = 0, y'(0) = 3$
47. $y'' - y = 4, y(0) = 0, y'(0) = 0$
48. $y'' - 4y = 32t, y(0) = 0, y'(0) = 6$

*49. $y'' + 2y' - 3y = -2, y(0) = \frac{2}{3}, y'(0) = 8$

50. $y'' + y' - 6y = 3t, y(0) = \frac{23}{12}, y'(0) = -\frac{3}{2}$
51. $y'' + 8y' + 16y = 4, y(0) = \frac{5}{4}, y'(0) = 0$
52. $y'' + 7y' + 10y = te^{-t}, y(0) = -\frac{15}{48}, y'(0) = \frac{9}{16}$

*53. $y'' + 6y' + 25y = -1, y(0) = -\frac{1}{25}, y'(0) = 7$

54. $y'' - 3y' = -e^{3t} - 2t, y(0) = 0, y'(0) = \frac{8}{9}$
55. $y'' - y' = -3t - 4t^2e^{2t}, y(0) = -\frac{7}{2}, y'(0) = 0$
56. $y'' - 2y' = 2t^2, y(0) = 3, y'(0) = \frac{3}{2}$

*57. $y''' + 5y'' = 125t, y(0) = 0, y'(0) = 0, y''(0) = 0$

58. $y''' + 25y' = 325e^{-t}, y(0) = 0, y'(0) = 0, y''(0) = 0$

In Exercises 59–61, without actually solving the equations, match each initial-value problem in Group A with the graph of its solution in Group B.



**Group A**

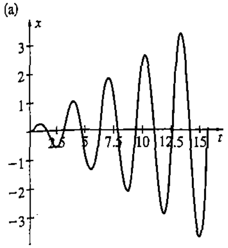
59. $\begin{cases} x'' + 4x = \cos t \\ x(0) = 0, x'(0) = 0 \end{cases}$

60. $\begin{cases} x'' + 4x = \cos 2t \\ x(0) = 0, x'(0) = 0 \end{cases}$

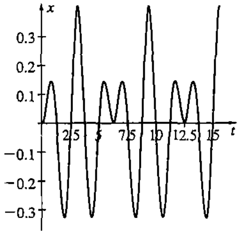
61. $\begin{cases} x'' + 4x = \cos 3t \\ x(0) = 0, x'(0) = 0 \end{cases}$

**Group B**

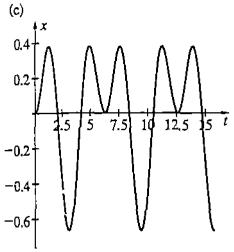
(a)




(b)




(c)




62. The method of undetermined coefficients can be used to solve first-order nonhomogeneous equations. Use this method to solve the following problems.
    (a) $y' - 4y = t^2$
    (b) $y' + y = \cos 2t$
    (c) $y' - y = e^{4t}$
63. (a) Suppose that $y_1(t)$ and $y_2(t)$ are solutions of $a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t)$, where $a$, $b$, and $c$ are positive constants. Show that
    $$\lim_{t\to\infty} [y_2(t) - y_1(t)] = 0.$$
    (b) Is the result of (a) true if $b = 0$?
    (c) Suppose that $f(t) = k$ where $k$ is a constant. Show that $\lim_{t\to\infty} y(t) = k/c$ for every solution $y(t)$ of
    $$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = k.$$
    (d) Determine the solution of $a\dfrac{d^2y}{dt^2} + b\dfrac{dy}{dt} = k$ and find $\lim_{t\to\infty} y(t)$.
    (e) Determine the solution of $a\, d^2y/dt^2 = k$ and find $\lim_{t\to\infty} y(t)$.
64. Let $y_p(t) = \frac{3}{5} \sin t$. Show that if $y(t)$ is any solution of $y'' - 4y = -3 \sin t$ then $y(t) - y_p(t)$ is a solution of $y'' - 4y = 0$. Explain why there are constants $c_1$ and $c_2$ so that $y(t) - y_p(t) = c_1e^{-2t} + c_2e^{2t}$ and, thus, $y(t) = c_1e^{-2t} + c_2e^{2t} + y_p(t)$.
65. Show that if $y_p(t)$ is a particular solution of the nonhomogeneous equation
    $$a_ny^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1y'(t) + a_0y(t) = g(t)$$
    and
    $$y_h(t) = c_1f_1(t) + c_2f_2(t) + \cdots + c_nf_n(t)$$
    is a general solution of the corresponding homogeneous equation
    $$a_ny^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1y'(t) + a_0y(t) = 0$$
    (assume that $\{f_1(t), f_2(t), \ldots, f_n(t)\}$ is a fundamental set of solutions for the corresponding homogeneous equation), then every solution of the nonhomogeneous equation, $y(t)$, can be written in the form
    $$y(t) = y_h(t) = y_p(t),$$
    for some choice of $c_1, c_2, \ldots, c_n$.
66. (a) Find conditions on $\omega$ so that $y_p(t) = A \cos \omega t + B \sin \omega t$, where $A$ and $B$ are constants to be determined, is a particular solution of $y'' + y = \cos \omega t$.
    (b) For the value(s) of $\omega$ obtained in (a), show that every solution of the nonhomogeneous equation $y'' + y = \cos \omega t$ is bounded (and periodic).
67. (a) Find conditions on $\omega$ so that $y_p(t) = t(A \cos \omega t + B \sin \omega t)$, where $A$ and $B$ are constants to be determined, is a particular solution of $y'' + 4y = \sin \omega t$.
    (b) For the value(s) of $\omega$ obtained in (a), show that every solution of the nonhomogeneous equation $y'' + 4y = \sin \omega t$ is unbounded.

As when calculating solutions of homogeneous equations, technology is useful in helping us calculate and visualize solutions of nonhomogeneous equations, particularly when the calculations encountered are symbolically complicated.

**68.** Find a general solution of each equation.

(a) $y''' - y'' - 7y' + 15y = t^2e^{-3t} + e^{2t}\cos 3t$

(b) $y^{(4)} - 9y''' + 24y'' - 36y' + 80y = \cos 2t + 4\sin 2t$

**69.** Solve the following initial-value problems. Graph the solution.

(a) $\begin{cases} 2y''' - 6y'' + 18y' - 56y = e^{-2t} + 4e^{3t} \\ y''(0) = -1, y'(0) = 0, y(0) = 1 \end{cases}$

(b) $\begin{cases} y^{(4)} - \frac{1}{4}y''' - 21y'' - \frac{50}{4}y' + 5y = t \\ y'''(0) = 0, y''(0) = 0, y'(0) = 0, y(0) = 1 \end{cases}$

**70.** Consider the initial-value problem

$$\begin{cases} y''' + y'' + 4y' + 4y = e^{-t}\cos 2t \\ y(0) = a, y'(0) = b, y''(0) = c \end{cases}$$

(a) Find conditions on $a$, $b$, and $c$, if any, so that $y(t) \to 0$ as $t \to \infty$. Confirm your results by graphing a (the) solution that satisfies the conditions.

(b) Is it possible to choose $a$, $b$, and $c$ so that the solution behaves like $\cos 2t$ for positive large values of $t \sin 2t$?

**71.** Determine if it is possible to find values of $a_3$, $a_2$, $a_1$, and $a_0$ so that a general solution of

$$y^{(4)} + a_3y''' + a_2y'' + a_1y' + a_0y = \cos t$$

has the indicated form. In each case, either find such values (and confirm your results) or explain why no such numbers exist.

(a) $y(t) = c_1e^{-2t} + c_2e^{-t} + c_3e^t + c_4e^{2t} + \frac{1}{10}\cos t$

(b) $y(t) = c_1e^{-t} + c_2e^t + c_3\cos t + c_4\sin t - \frac{3}{8}\cos t - \frac{1}{4}t\sin t$

(c) $y(t) = c_1\cos t + c_2\sin t + c_3t\cos t + c_4t\sin t + \frac{3}{8}\cos t + \frac{3}{8}\sin t - \frac{1}{8}t^2\cos t$

(d) $y(t) = c_1\cos t + c_2\sin t + c_3t\cos t + c_4t\sin t + t^3\sin t$

**72.** Show that the solution to the initial-value problem

$$\begin{cases} y'' + y' - 2y = f(t) \\ y(0) = 0, y'(0) = a \end{cases}$$

is

$$y(t) = \frac{1}{3}ae^t - \frac{1}{3}ae^{-2t} + \frac{1}{3}e^t\int_0^t f(x)e^{-t}\,dx - \frac{1}{3}e^{-2t}\int_0^t f(x)e^{2t}\,dx$$

(as long as the integrals can be evaluated).

(a) Use the Fundamental Theorem of Calculus to verify this result.

(b) If possible, find a function $f(t)$ and a value of $a$ so that (i) the solution is periodic; (ii) the solution approaches 0 as $x$ approaches infinity; and (iii) the solution has no limit as $x$ approaches infinity. Confirm your results graphically.

**73.** How does the solution to the initial-value problem

$$\begin{cases} x'' + 9x = \sin(ct) \\ x(0) = 0, x'(0) = 0 \end{cases}$$

change as $c$ assumes value from 0 to 6?

**74.** If we consider a differential equation over a certain interval and are given conditions that must be satisfied at the endpoints of the interval, then we call this a **boundary-value problem** (BVP).

(a) Show that the boundary-value problem

$$\begin{cases} 4y'' + 4y' + 37y = \cos 3t \\ y(0) = y(\pi) \end{cases}$$

has infinitely many solutions. Confirm your results graphically.

(b) If possible, find conditions on $y(0)$ and $y(\pi)$ so that the solution to the boundary-value problem is unique.



## 4.6 Nonhomogeneous Equations with Constant Coefficients: Variation of Parameters

Second-Order Equations · Higher Order Nonhomogeneous Equations · Green's Function

### Second-Order Equations

For the nonhomogeneous equation $y'' + \frac{1}{4}y = \sec(t/2) + \csc(t/2)$, $0 < t < \pi$, the nonhomogeneous function is $g(t) = \sec(t/2) + \csc(t/2)$, which is not a function of terms of the form $t^m$, $t^me^{kt}$, $t^me^{\alpha t}\cos\beta t$ or $t^me^{\alpha t}\sin\beta t$. Because the method of undetermined coefficients or the annihilator method are limited to equations involving these functions, we need to introduce another method, called **variation of parameters,** discovered by Lagrange, that can be used to find a particular solution of other equations such as this one.

A general solution to the corresponding homogeneous equation $y'' + \frac{1}{4}y = 0$ is $y_h(t) = c_1\cos(t/2) + c_2\sin(t/2)$.

In the method of variation of parameters, we try to find a particular solution of the form

$$y_p(t) = u_1(t)\cos\frac{t}{2} + u_2(t)\sin\frac{t}{2},$$

where we replace the constants $c_1$ and $c_2$ in $y_h(t)$ with the unknown functions $u_1(t)$ and $u_2(t)$. We arrive at the name of the method from this replacement because we *vary the parameters* $c_1$ and $c_2$ by allowing them to be *functions* of $t$ instead of *constants.* (Notice that there should be many choices for $u_1(t)$ and $u_2(t)$ because we have two unknown functions and only one equation, the nonhomogeneous differential equation, to use in finding them.) We find possible choices for $u_1(t)$ and $u_2(t)$ by substitution of $y_p(t)$ into the nonhomogeneous equation $y'' + \frac{1}{4}y = \sec(t/2) + \csc(t/2)$. Differentiating $y_p(t)$, we find that

$$y_p'(t) = -\frac{1}{2}u_1(t)\sin\frac{t}{2} + u_1'(t)\cos\frac{t}{2} + \frac{1}{2}u_2(t)\cos\frac{t}{2} + u_2'(t)\sin\frac{t}{2}.$$

To simplify the process of finding $u_1(t)$ and $u_2(t)$, we assume that

$$u_1'(t)\cos\frac{t}{2} + u_2'(t)\sin\frac{t}{2} = 0,$$

which is our *first restriction* on $u_1(t)$ and $u_2(t)$. (In other words, of all functions $u_1(t)$ and $u_2(t)$ that lead to a particular solution to the nonhomogeneous differential equation, we look for functions that satisfy this condition.) Eliminating this expression from $y_p'(t)$ gives us

$$y_p'(t) = -\frac{1}{2}u_1(t)\sin\frac{t}{2} + \frac{1}{2}u_2(t)\cos\frac{t}{2},$$

so that the second derivative is

$$y_p''(t) = -\frac{1}{4}u_1(t)\cos\frac{t}{2} - \frac{1}{2}u_1'(t)\sin\frac{t}{2} - \frac{1}{4}u_2(t)\sin\frac{t}{2} + \frac{1}{2}u_2'(t)\cos\frac{t}{2}.$$

Next, we substitute $y_p(t)$ into the nonhomogeneous equation $y'' + \frac{1}{4}y = \sec(t/2) + \csc(t/2)$ to obtain

$$\begin{aligned} y_p'' + \frac{1}{4}y_p &= -\frac{1}{4}u_1(t)\cos\frac{t}{2} - \frac{1}{2}u_1'(t)\sin\frac{t}{2} - \frac{1}{4}u_2(t)\sin\frac{t}{2} \\ &\quad + \frac{1}{2}u_2'(t)\cos\frac{t}{2} + \frac{1}{4}\left[u_1(t)\cos\frac{t}{2} + u_2(t)\sin\frac{t}{2}\right] \\ &= -\frac{1}{2}u_1'(t)\sin\frac{t}{2} + \frac{1}{2}u_2'(t)\cos\frac{t}{2}. \end{aligned}$$

Then, because $y_p(t)$ satisfies the nonhomogeneous equation, we have a *second restriction* on $u_1(t)$ and $u_2(t)$,

$$-\frac{1}{2}u_1'(t)\sin\frac{t}{2} + \frac{1}{2}u_2'(t)\cos\frac{t}{2} = \sec\frac{t}{2} + \csc\frac{t}{2}.$$

This gives us the system of two equations

$$\begin{cases} u_1'(t)\cos\frac{t}{2} + u_2'(t)\sin\frac{t}{2} = 0 \\ -\frac{1}{2}u_1'(t)\sin\frac{t}{2} + \frac{1}{2}u_2'(t)\cos\frac{t}{2} = \sec\frac{t}{2} + \csc\frac{t}{2} \end{cases}$$

that we solve for the two functions $u_1'(t)$ and $u_2'(t)$. Multiplying the first equation by $\frac{1}{2}\sin(t/2)$, the second by $\cos(t/2)$, and adding the resulting equations yields

$$u_2'(t) = \frac{\cos\frac{t}{2}\left[\sec\frac{t}{2} + \csc\frac{t}{2}\right]}{\frac{1}{2}},$$

so that one choice for $u_2(t)$ is obtained through integration to be

$$u_2(t) = \int \frac{\cos\frac{t}{2}\left(\sec\frac{t}{2} + \csc\frac{t}{2}\right)}{\frac{1}{2}} = 2\int\left(1 + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}\right)dt = 2t + 4\ln\left|\sin\frac{t}{2}\right|.$$

(Notice that any antiderivative of $u_2'(t)$ is a possible choice for $u_2(t)$. Here, we assume that the constant of integration is zero.) Similarly, if we multiply the first equation in the system by $-\frac{1}{2}\cos(t/2)$, the second by $\sin(t/2)$, and add the equations that result, we have

$$u_1'(t) = \frac{-\sin\frac{t}{2}\left(\sec\frac{t}{2} + \csc\frac{t}{2}\right)}{\frac{1}{2}},$$



so one possibility for $u_1(t)$ is

$$u_1(t) = \int \frac{-\sin\frac{t}{2}\left(\sec\frac{t}{2} + \csc\frac{t}{2}\right)}{\frac{1}{2}}\,dt = -2\int\left(\frac{\sin\frac{t}{2}}{\cos\frac{t}{2}} + 1\right)dt = -2t + 4\ln\left|\cos\frac{t}{2}\right|.$$

Again, we have selected the antiderivative with constant of integration equal to zero.

Then by *variation of parameters*,

$$\begin{aligned} y_p(t) &= u_1(t)y_1(t) + u_2(t)y_2(t) \\ &= \cos\frac{t}{2}\left[-2t + 4\ln\left|\cos\frac{t}{2}\right|\right] + \sin\frac{t}{2}\left[2t + 4\ln\left|\sin\frac{t}{2}\right|\right] \end{aligned}$$

is a particular solution of $y'' + \frac{1}{4}y = \sec(t/2) + \csc(t/2)$ and

$$\begin{aligned} y &= y_h(t) + y_p(t) \\ &= c_1\cos\frac{t}{2} + c_2\sin\frac{t}{2} + \cos\frac{t}{2}\left[-2t + 4\ln\left|\cos\frac{t}{2}\right|\right] + \sin\frac{t}{2}\left[2t + 4\ln\left|\sin\frac{t}{2}\right|\right] \end{aligned}$$

is a general solution. Note that $\cos(t/2) > 0$ and $\sin(t/2) > 0$ on $0 < x < \pi$, so the absolute value signs can be eliminated from the arguments of the natural logarithm functions.

In general, to solve the second-order linear nonhomogeneous differential equation

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = g(t),$$

where $y_h(t) = c_1y_1(t) + c_2y_2(t)$ is a general solution of the corresponding homogeneous equation $a_2(t)y'' + a_1(t)y' + a_0(t)y = 0$, we first divide the equation by $a_2(t)$ to rewrite it in the form

$$y'' + p(t)y' + q(t)y = f(t),$$

assume that a particular solution has a form similar to the general solution by varying the parameters $c_1$ and $c_2$, and let

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t).$$

We need two equations to determine the two unknown functions $u_1(t)$ and $u_2(t)$, and we obtain them by substituting $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ into the nonhomogeneous differential equation $y'' + p(t)y' + q(t)y = f(t)$. Differentiating $y_p(t)$, we obtain

$$y_p'(t) = u_1(t)y_1'(t) + u_1'(t)y_1(t) + u_2(t)y_2'(t) + u_2'(t)y_2(t),$$

which can be simplified to

$$y_p'(t) = u_1(t)y_1'(t) + u_2(t)y_2'(t)$$

with the assumption that $u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$. Notice that this assumption is made to eliminate the derivatives of $u_1(t)$ and $u_2(t)$ from this expression for $y_p'(t)$, which simplifies the process for finding $u_1(t)$ and $u_2(t)$. Therefore, the *first equation* satisfied by $u_1(t)$ and $u_2(t)$ is

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0.$$

With our simplified expression for $y_p'(t)$, the second derivative is

$$y_p''(t) = u_1(t)y_1''(t) + u_1'(t)y_1'(t) + u_2(t)y_2''(t) + u_2'(t)y_2'(t).$$

Substitution into $y'' + p(t)y' + q(t)y = f(t)$ then yields

$$\begin{aligned} y_p'' + p(t)y_p' + q(t)y_p &= u_1(t)[y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)] \\ &\quad + u_2(t)[y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)] \\ &\quad + u_1'(t)y_1'(t) + u_2'(t)y_2'(t) \\ &= u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = f(t) \end{aligned}$$

because $y_1(t)$ and $y_2(t)$ are solutions of the corresponding homogeneous equation. Therefore, our *second equation* for determining $u_1(t)$ and $u_2(t)$ is $u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = f(t)$, so we have the system

$$\begin{cases} u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0 \\ u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = f(t)' \end{cases}$$

which is written in matrix form as $\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}\begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$. In linear algebra, we learn that this system has a unique solution if and only if $\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \neq 0$.

Notice that this determinant is the Wronskian of the set $S = \{y_1(t), y_2(t)\}$, $W(S)$. We stated in Section 4.1 that $W(S) \neq 0$ if the functions $y_1(t)$ and $y_2(t)$ in the set $S$ are linearly independent. Because $S = \{y_1(t), y_2(t)\}$ represents a fundamental set of solutions of the corresponding homogeneous equation, $W(S) \neq 0$. Hence, this system has a unique solution that can be found by Cramer's rule to be

$$u_1'(t) = \frac{\begin{vmatrix} 0 & y_2(t) \\ f(t) & y_2'(t) \end{vmatrix}}{W(S)} = \frac{-y_2(t)f(t)}{W(S)}$$

and

$$u_2'(t) = \frac{\begin{vmatrix} y_1(t) & 0 \\ y_1'(t) & f(t) \end{vmatrix}}{W(S)} = \frac{y_1(t)f(t)}{W(S)}.$$

*Note:* If you are not familiar with Cramer's rule, these formulas can be found with elimination as we did in solving $y'' + \frac{1}{4}y = \sec(t/2) + \csc(t/2)$ at the beginning of the section. The functions $u_1(t)$ and $u_2(t)$ are then found through integration.



**Summary of Variation of Parameters for Second-Order Equations**

Given the second-order equation $y'' + p(t)y' + q(t)y = f(t)$:

① Find a general solution $y_h(t) = c_1y_1(t) + c_2y_2(t)$ and fundamental set of solutions $S = \{y_1(t), y_2(t)\}$ of the corresponding homogeneous equation $y'' + p(t)y' + q(t)y = 0$.

② Let $u_1'(t) = \dfrac{-y_2(t)f(t)}{W(S)}$ and $u_2'(t) = \dfrac{y_1(t)f(t)}{W(S)}$.

③ Integrate to obtain $u_1(t)$ and $u_2(t)$.

④ A particular solution of $y'' + p(t)y' + q(t)y = f(t)$ is given by $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$.

⑤ A general solution of $y'' + p(t)y' + q(t)y = f(t)$ is given by $y = y_h(t) + y_p(t)$.

**Example 1**

Solve $y'' - 2y' + y = e^t \ln t$, $t > 0$.

***Solution*** The corresponding homogeneous equation has the characteristic equation $r^2 - 2r + 1 = (r-1)^2 = 0$ so $y_h(t) = c_1 \underbrace{e^t}_{y_1(t)} + c_2 \underbrace{te^t}_{y_2(t)}$ and $S = \{e^t, te^t\}$. Therefore, we compute

$$W(S) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t} = e^{2t}.$$

Through the use of integration by parts, a table of integrals, or a computer algebra system,

$$u_1(t) = \int \frac{-te^t(e^t \ln t)}{e^{2t}}\,dt = -\int t \ln t\,dt = \frac{1}{4}t^2 - \frac{1}{2}t^2 \ln t$$

and

$$u_2(t) = \int \frac{e^t(e^t \ln t)}{e^{2t}}\,dt = \int \ln t\,dt = t\ln t - t.$$

Then a particular solution, which is assumed to have the form

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

is

$$y_p(t) = \left(\frac{1}{4}t^2 - \frac{1}{2}t^2 \ln t\right)e^t + (t\ln t - t)te^t = \frac{1}{2}t^2e^t \ln t - \frac{3}{4}t^2e^t.$$

Therefore, a general solution is

$$y(t) = y_h(t) + y_p(t) = c_1e^t + c_2te^t + \frac{1}{2}t^2e^t \ln t - \frac{3}{4}t^2e^t,\ t > 0.$$

Of course, the problems that were solved in the preceding sections by the method of undetermined coefficients can be solved by variation of parameters as well.

**Example 2**

Solve the initial-value problem $y'' + 4y = \sin 2t$, $y(0) = 0$, $y'(0) = 1$ using variation of parameters.

**Solution** The characteristic equation of the corresponding homogeneous equation has the complex conjugate roots $r = \pm 2i$, so $y_h(t) = c_1 \underbrace{\cos 2t}_{y_1(t)} + c_2 \underbrace{\sin 2t}_{y_2(t)}$ and $S = \{\cos 2t, \sin 2t\}$. Thus, $W(S) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2$. Through the use of trigonometric identities, we obtain

$$u_1(t) = \int -\frac{1}{2}\sin^2 2t\,dt = -\frac{1}{4}t + \frac{1}{16}\sin 4t$$

and

$$u_2(t) = \int \frac{1}{2}\cos 2t \sin 2t\,dt = -\frac{1}{8}\cos^2 2t.$$

Simplifying and using the identity $\sin 4t = 2\sin 2t\cos 2t$ leads to the particular solution

$$\begin{aligned} y_p(t) &= u_1(t)y_1(t) + u_2(t)y_2(t) \\ &= \left(-\frac{1}{4}t + \frac{1}{16}\sin 4t\right)\cos 2t - \frac{1}{8}\cos^2 2t \cdot \sin 2t \\ &= -\frac{1}{4}t\cos 2t + \frac{2}{16}\sin 2t\cos 2t \cdot \cos 2t - \frac{1}{8}\cos^2 2t \cdot \sin 2t \\ &= -\frac{1}{4}t\cos 2t \end{aligned}$$

and, thus, a general solution is

$$y(t) = y_h(t) + y_p(t) = c_1\cos 2t + c_2\sin 2t - \frac{1}{4}t\cos 2t.$$

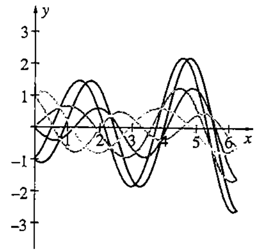



Figure 4.12

*Figure 4.12 shows the graph $y(t) = c_1\cos 2t + c_2\sin 2t - \frac{1}{4}t\cos 2t$ for various values of $c_1$ and $c_2$. Identify the graph of the solution that satisfies the initial conditions (a) $y(0) = 0$ and $y'(0) = 1$; and (b) $y(0) = 0$ and $y'(0) = -1$.*

To solve the initial-value problem, we compute

$$y'(t) = -2c_1\sin 2t + 2c_2\cos 2t - \frac{1}{4}\cos 2t + \frac{1}{2}t\sin 2t.$$



We then evaluate $y(0) = c_1 = 0$ and $y'(0) = 2c_2 - 1/4 = 1$. Hence, $c_1 = 0$ and $c_2 = 5/8$, so the solution is

$$y(t) = \frac{5}{8}\sin 2t - \frac{1}{4}t\cos 2t.$$

*Note:* If we had used the method of undetermined coefficients, we should not expect a particular solution of the same form as that obtained through variation of parameters unless we simplify the original functions $u_1(t)$ and $u_2(t)$ obtained through integration. (Why?)

In the following example, a condition is given at each of the endpoints of the interval $0 \le t \le \pi$. Solutions to problems of this type, called *boundary value problems,* differ from those of initial-value problems in that there may be no solution or more than one solution.

**Example 3**

For what value(s) of $\lambda > 0$, if any, does the boundary-value problem $\begin{cases} y'' + \lambda^2 y = \sin 2t \\ y(0) = 0, y(\pi) = 0 \end{cases}$ have **(a)** one solution, **(b)** no solutions, and **(c)** infinitely many solutions?

**Solution** A fundamental set of solutions of the corresponding homogeneous equation is $S = \{\cos\lambda t, \sin\lambda t\}$ with $W(s) = \lambda$. By variation of parameters,

$$u_1(t) = \int \frac{-\sin\lambda t\sin 2t}{\lambda}\,dt = \begin{cases} \dfrac{(2+\lambda)\sin((2-\lambda)t) - (2-\lambda)\sin((2+\lambda)t)}{2\lambda(\lambda^2-4)}, & \text{if } \lambda \neq 2 \\ \dfrac{\sin 4t - 4t}{16}, & \text{if } \lambda = 2 \end{cases}$$

and

$$u_2(t) = \int \frac{\cos\lambda t\sin 2t}{\lambda}\,dt$$

$$= \begin{cases} \dfrac{(2+\lambda)\cos((2-\lambda)t) + (2-\lambda)\cos((2+\lambda)t)}{2\lambda(\lambda^2-4)}, & \text{if } \lambda \neq 2, \\ -\dfrac{\cos^2 2t}{8}, & \text{if } \lambda = 2 \end{cases}$$

so a general solution of the nonhomogeneous equation is

$$y(t) = c_1 \cos \lambda t + c_2 \sin \lambda t + u_1(t)y_1(t) + u_2(t)y_2(t)$$

$$= \begin{cases} c_1 \cos \lambda t + c_2 \sin \lambda t + \dfrac{1}{\lambda^2 - 4} \sin 2t, \text{ if } \lambda \neq 2 \\ c_1 \cos 2t + c_2 \sin 2t - \dfrac{1}{4} t \cos 2t, \text{ if } \lambda = 2. \end{cases}$$

Applying the boundary conditions yields the system of equations

$$\begin{cases} c_1 = 0, \\ c_1 \cos \lambda\pi + c_2 \sin \lambda\pi = 0 \end{cases} \text{if } \lambda \neq 2$$

and

$$\begin{cases} c_1 = 0 \\ c_1 - \dfrac{\pi}{4} = 0 \end{cases}, \text{ if } \lambda = 2.$$

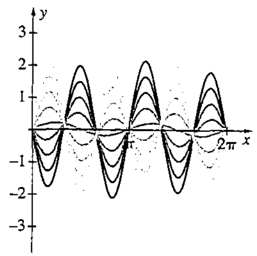


Figure 4.13

The system $\begin{cases} c_1 = 0 \\ c_1 - \pi/4 = 0 \end{cases}$ does not have a solution (why?), so the boundary-value problem does not have a solution if $\lambda = 2$. However, the solution to the system $\begin{cases} c_1 = 0 \\ c_1 \cos \lambda\pi + c_2 \sin \lambda\pi = 0 \end{cases}$ is $c_1 = 0$ and $c_2 = 0$, provided that $\lambda$ is not an integer, because $\sin \lambda\pi = 0$ if $\lambda$ is an integer. But, if $\lambda \neq 2$ is an integer, then *any* value of $c_2$ is a solution to this system, so there are infinitely many solutions to the boundary-value problem if $\lambda \neq 2$ is an integer.

Thus, the boundary-value problem has no solution if $\lambda = 2$, no solution if $\lambda$ is not an integer, and infinitely many solutions if $\lambda \neq 2$ is an integer (see Figure 4.13).

An advantage that the method of variation of parameters has over the method of undetermined coefficients is that nonhomogeneous equations with *nonconstant coefficients* can be considered.

**Example 4**

A general solution of $t^2y'' + ty' = 0$, $t > 0$ is $y_h(t) = c_1 + c_2 \ln t$. Solve the nonhomogeneous equation $t^2y'' + ty' = 2t^2$, $t > 0$.

***Solution*** In this case, $S = \{y_1(t), y_2(t)\} = \{1, \ln t\}$, so $W(S) = \begin{vmatrix} 1 & \ln t \\ 0 & 1/t \end{vmatrix} = 1/t.$ Recall that we divided the second-order equation $a_2(t)y'' + a_1(t)y' + a_0(t)y = g(t)$ by the coefficient function of $y''$ to place it in the form $y'' + p(t)y' + q(t)y = f(t)$ before deriving the formulas for $u_1(t)$ and $u_2(t)$. Doing this, we have $y'' + \dfrac{1}{t}y' = 2$,



so $f(t) = 2$. Therefore, using integration by parts, we obtain

$$u_1(t) = \int \frac{-\ln t\,(2)}{\frac{1}{t}}\,dt = -2\int t \ln t\,dt = -t^2 \ln t + \frac{1}{2}t^2$$

and

$$u_2(t) = \int \frac{(1)(2)}{\frac{1}{t}}\,dt = 2\int t\,dt = t^2,$$

so $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = (1)\left(-t^2 \ln t + \frac{1}{2}t^2\right) + (\ln t)(t^2) = \frac{1}{2}t^2$, and a general solution to the nonhomogeneous equation is

$$y(t) = y_h(t) + y_p(t) = c_1 + c_2 \ln t + \frac{1}{2}t^2,\ t > 0.$$

## Higher Order Nonhomogeneous Equations

Higher order nonhomogeneous linear equations can be solved through variation of parameters as well. In general, if we are given the nonhomogeneous equation

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = g(t)$$

and a fundamental set of solutions $\{y_1(t), y_2(t), \ldots, y_n(t)\}$ of the associated homogeneous equation

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = 0,$$

we generalize the method for second-order equations to find $u_1(t), u_2(t), \ldots, u_n(t)$ such that

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t)$$

is a particular solution of the nonhomogeneous equation.

If

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) + \cdots + u_n'(t)y_n(t) = 0,$$

then

$$y_p^{(m)}(t) = u_1(t)y_1^{(m)}(t) + u_2(t)y_2^{(m)}(t) + \cdots + u_n(t)y_n^{(m)}(t)$$

for $m = 0, 1, 2, \ldots, n-1$, and if

$$u_1'(t)y_1^{(m-1)}(t) + u_2'(t)y_2^{(m-1)}(t) + \cdots + u_n'(t)y_n^{(m-1)}(t) = 0$$

for $m = 1, 2, \ldots, n-1$, then

$$y_p^{(n)}(t) = \left(u_1(t)y_1^{(n)}(t) + u_2(t)y_2^{(n)}(t) + \cdots + u_n(t)y_n^{(n)}(t)\right) + \left(u_1'(t)y_1^{(n-1)}(t) + u_2'(t)y_2^{(n-1)}(t) + \cdots + u_n'(t)y_n^{(n-1)}(t)\right).$$

Following a method similar to that of a second-order differential equation, we obtain the system of $n$ equations

$$\begin{cases} u_1'(t)y_1(t) + u_2'(t)y_2(t) + \cdots + u_n'(t)y_n(t) = 0 \\ u_1'(t)y_1'(t) + u_2'(t)y_2'(t) + \cdots + u_n'(t)y_n'(t) = 0 \\ \vdots \\ y_1^{(n-1)}(t)u_1'(t) + y_2^{(n-1)}(t)u_2'(t) + \cdots + y_n^{(n-1)}(t)u_n'(t) = g(t) \end{cases},$$

which can be solved for $u_1'(t), u_2'(t), \ldots, u_n'(t)$ using Cramer's rule.

Let $W_m(y_1(t), y_2(t), \ldots, y_n(t))$ denote the determinant of the matrix obtained by replacing the $m$th column of

$$\begin{pmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{pmatrix}$$

by the column $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{pmatrix}$. Then, by Cramer's rule,

$$u_i'(t) = \frac{g(t)W_i(y_1(t), y_2(t), \ldots, y_n(t))}{W(y_1(t), y_2(t), \ldots, y_n(t))}$$

for $i = 1, 2, \ldots, n$, and

$$u_i(t) = \int \frac{g(t)W_i(y_1(t), y_2(t), \ldots, y_n(t))}{W(y_1(t), y_2(t), \ldots, y_n(t))}\, dx,$$

so

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t)$$

is a particular solution of the nonhomogeneous equation. A general solution of the nonhomogeneous equation is given by $y(t) = y_h(t) + y_p(t)$, where $y_h(t)$ is a general solution of the corresponding homogeneous equation.

**Example 5**

Solve $y''' + 3y'' + 2y' = \cos t$.

***Solution*** You should verify that a general solution of the corresponding homogeneous equation is $y_h(t) = c_1 + c_2e^{-t} + c_3e^{-2t}$ and a fundamental set of solutions is $S = \{1, e^{-t}, e^{-2t}\}$. Therefore, we must solve the system



$$\begin{pmatrix} 1 & e^{-t} & e^{-2t} \\ 0 & -e^{-t} & -2e^{-2t} \\ 0 & e^{-t} & 4e^{-2t} \end{pmatrix}\begin{pmatrix} u_1'(t) \\ u_2'(t) \\ u_3'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cos t \end{pmatrix},$$

where

$$W(S) = \begin{vmatrix} 1 & e^{-t} & e^{-2t} \\ 0 & -e^{-t} & -2e^{-2t} \\ 0 & e^{-t} & 4e^{-2t} \end{vmatrix} = -2e^{-3t}.$$

Using this with Cramer's rule, we have

$$u_1'(t) = \frac{\begin{vmatrix} 0 & e^{-t} & e^{-2t} \\ 0 & -e^{-t} & -2e^{-2t} \\ \cos t & e^{-t} & 4e^{-2t} \end{vmatrix}}{-2e^{-3t}} = \frac{-e^{-3t}\cos t}{-2e^{-3t}} = \frac{1}{2}\cos t,$$

$$u_2'(t) = \frac{\begin{vmatrix} 1 & 0 & e^{-2t} \\ 0 & 0 & -2e^{-2t} \\ 0 & \cos t & 4e^{-2t} \end{vmatrix}}{-2e^{-3t}} = \frac{2e^{-2t}\cos t}{-2e^{-3t}} = -e^{t}\cos t,$$

and

$$u_3'(t) = \frac{\begin{vmatrix} 1 & e^{-t} & 0 \\ 0 & -e^{-t} & 0 \\ 0 & e^{-t} & \cos t \end{vmatrix}}{-2e^{-3t}} = \frac{-e^{-t}\cos t}{-2e^{-3t}} = \frac{1}{2}e^{2t}\cos t.$$

Integration then gives us $u_1(t) = \frac{1}{2}\sin t$, $u_2(t) = -\frac{1}{2}e^{t}(\cos t + \sin t)$, and $u_3(t) = \frac{1}{10}e^{2t}(2\cos t + \sin t)$.

Because

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + u_3(t)y_3(t),$$

we find through substitution into this equation and simplification that $y_p(t) = -\frac{3}{10}\cos t + \frac{1}{10}\sin t$ is a particular solution of the nonhomogeneous equation. A general solution is

$$y(t) = y_h(t) + y_p(t) = c_1 + c_2e^{-t} + c_3e^{-2t} - \frac{3}{10}\cos t + \frac{1}{10}\sin t.$$

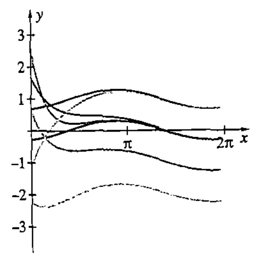



Figure 4.14

*Figure 4.14 shows the graph of $y(t) = y_h(t) + y_p(t)$ for various values of $c_1$, $c_2$, and $c_3$. Identify the graph of $y_p(t)$.*

## EXERCISES 4.6

In Exercises 1–38, solve each differential equation using variation of parameters.

1. $y'' - 7y' + 10y = e^{3t}$
2. $y'' + 16y = 2\cos 4t$
*3. $y'' + 4y' + 4y = e^{-2t}$
4. $y'' - 4y' + 13y = 8e^{2t}\cos 3t$
5. $y'' + 4y' + 20y = 2te^{-2t}$
6. $y'' + 4y = \sec 2t$
*7. $y'' + 16y = \csc 4t$
8. $y'' + 16y = \cot 4t$
9. $y'' + 2y' + 50y = e^{-t}\csc 7t$
10. $y'' + 6y' + 25y = e^{-3t}(\sec 4t + \csc 4t)$
*11. $y'' - 2y' + 26y = e^{t}(\sec 5t + \csc 5t)$
12. $y'' + 12y' + 37y = \dfrac{e^{-6t}\sin t}{t}$, $t > 0$
13. $y'' - 6y' + 34y = e^{3t}\tan 5t$
14. $y'' - 10y' + 34y = \dfrac{e^{-5t}\sin 3t}{t^2}$, $t > 0$
*15. $y'' - 12y' + 37y = e^{6t}\sec t$
16. $y'' - 8y' + 17 = e^{4t}\sec t$
17. $y'' - 9y = \dfrac{1}{1 + e^{3t}}$
18. $y'' - 25y = \dfrac{1}{1 - e^{5t}}$
*19. $y'' - y = 2\sinh t$
20. $y'' - 2y' + y = \dfrac{e^{t}}{t}$, $t > 0$
21. $y'' - 4y' + 4y = \dfrac{e^{2t}}{t^2}$, $t > 0$
22. $y'' + 8y' + 16y = \dfrac{e^{-4t}}{t^4}$, $t > 0$
*23. $y'' + 6y' + 9y = \dfrac{1}{te^{3t}}$, $t > 0$
24. $y'' + 6y' + 9y = e^{-3t}\ln t$, $t > 0$
25. $y'' + 3y' + 2y = \cos(e^{t})$
26. $y'' + 4y' + 4y = e^{-2t}\sqrt{1 - t^2}$, $-1 \le t \le 1$
*27. $y'' - 2y' + y = e^{t}\sqrt{1 - t^2}$, $-1 \le t \le 1$
28. $y'' - 10y' + 25y = e^{5t}\ln 2t$
29. $y'' - 4y' + 4y = e^{2t}\tan^{-1} t$
30. $y'' + 8y' + 16y = \dfrac{e^{-4t}}{1 + t^2}$
*31. $y'' + \dfrac{1}{4}y = \sec\dfrac{t}{2} + \csc\dfrac{t}{2}$
32. $y''' + y' = -2\sin t - \dfrac{\sin t}{\cos^2 t}$
33. $y''' + 4y' = \sec 2t$
34. $y''' - 2y'' = -\dfrac{1 + 2t}{t^2}$, $t > 0$
*35. $y''' - 3y'' + 3y' - y = \dfrac{e^{t}}{t}$, $t > 0$
36. $y''' - 4y'' - 11y' + 30y = e^{4t}$
37. $y''' + 3y'' - 10y' - 24y = e^{-3t}$
38. $y''' - 13y' + 12y = \cos t$

In Exercises 39–45, solve the initial-value problem.

39. $y'' + y = \tan t$, $y(0) = 2$, $y'(0) = 1$, $-\pi/2 < t < \pi/2$
40. $y'' + 9y = \frac{1}{2}\csc t$, $y(\pi/4) = \sqrt{2}$, $y'(\pi/4) = 0$
*41. $y'' + 5y' + 6y = -3 - \sin 4t$, $y(0) = -23/50$, $y'(0) = 27/25$
42. $y'' - 16y = \dfrac{16t}{e^{4t}}$, $y(0) = 0$, $y'(0) = 0$
43. $y'' - 2y' + y = -e^{-2t}\cos 3t$, $y(0) = -\dfrac{18}{13}$, $y'(0) = -\dfrac{10}{13}$
44. $y'' - 6y' + 18y = -e^{3t}\sin 3t$, $y(0) = -\dfrac{1}{6}$, $y'(0) = 0$
*45. $y''' - y'' = 3t^2$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$
46. Solve $y'' + 4y' + 3y = 65\cos 2t$ by **(a)** the method of undetermined coefficients; **(b)** the method of variation of parameters. Which method is more easily applied?
47. Show that the solution of the initial-value problem
$$\begin{cases} y'' + a_1(t)y' + a_0(t)y = f(t) \\ y(t_0) = y_0,\ y'(t_0) = y'_0 \end{cases}$$
can be written as $y(t) = u(t) + v(t)$ where $u$ is the solution of
$$\begin{cases} u'' + a_1(t)u' + a_0(t)u = 0 \\ u(t_0) = y_0,\ u'(t_0) = y'_0 \end{cases}$$
and $v$ is the solution of
$$\begin{cases} v'' + a_1(t)v' + a_0(t)v = f(t) \\ v(t_0) = 0,\ v'(t_0) = 0 \end{cases}.$$
48. Let $y_1(t)$ and $y_2(t)$ be two linearly independent solutions of the second-order homogeneous linear differential equation
$$a_2(t)y'' + a_1(t)y' + a_0(t)y = 0.$$
What is a general solution of $a_2(t)y'' + a_1(t)y' + a_0(t)y = g(t)$? (*Hint:* In order to apply the formulas of variation of parameters, the lead coefficient must be 1; divide by $a_2(t)$ and apply the variation of parameters formula.)



### Green's Functions

Suppose that we are trying to solve the initial-value problem $y'' + p(t)y' + q(t)y = f(t)$, $y(t_0) = 0$, $y'(t_0) = 0$. Based on the Method of Variation of Parameters, we write the solution of this equation as $y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$, where $u_1(t) = \int_{t_0}^{t} \dfrac{-y_2(x)f(x)}{W(S)}\,dx$ and $u_2(t) = \int_{t_0}^{t} \dfrac{y_1(x)f(x)}{W(S)}\,dx$; $y_1$ and $y_2$ satisfy $y'' + p(t)y' + q(t)y = 0$. Here $W(S)$ represents the Wronskian of $y_1$ and $y_2$ as a function of $x$. Therefore,

$$\begin{aligned} y(t) &= u_1(t)y_1(t) + u_2(t)y_2(t) \\ &= y_1(t)\int_{t_0}^{t} \frac{-y_2(x)f(x)}{W(S)}\,dx + y_2(t)\int_{t_0}^{t} \frac{y_1(x)f(x)}{W(S)}\,dx \\ &= \int_{t_0}^{t} \frac{y_2(t)y_1(x) - y_1(t)y_2(x)}{W(S)} f(x)dx. \end{aligned}$$

Thus, we can solve the nonhomogeneous problem using an integral involving the function $f(t)$. The function

$$G(t, x) = \frac{y_2(t)y_1(x) - y_1(t)y_2(x)}{W(S)}$$

is called the **Green's function.**

49. Show that $u_1(t) = \int_{t_0}^{t} \dfrac{-y_2(x)f(x)}{W(S)}\,dx$ and $u_2(t) = \int_{t_0}^{t} \dfrac{y_1(x)f(x)}{W(S)}\,dx$ indicate that $u_1(t_0) = 0$ and $u_2(t_0) = 0$. Recall that the Method of Variation of Parameters is based on the two equations $y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ and $y'(t) = u_1'(t)y_1(t) + u_2'(t)y_2(t)$. Show that $y(t_0) = y'(t_0) = 0$, so that the solution obtained with the Green's function satisfies the IVP.

Suppose that we wish to solve $y'' + 4y = f(t)$, $y(0) = 0$, $y'(0) = 0$. In this case, $y_h(t) = c_1\cos 2t + c_2\sin 2t$, so $y(t) = \int_0^t \dfrac{\sin 2t\cos 2x - \cos 2t\sin 2x}{2} f(x)\,dx = \int_0^t \dfrac{\sin 2(t - x)}{2} f(x)\,dx$. Therefore, $G(t, x) = \dfrac{\sin 2(t - x)}{2}$. For the IVP, $y'' + 4y = \sin t$, $y(0) = 0$, $y'(0) = 0$, the solution is $y(t) = \int_0^t \dfrac{\sin 2(t - x)}{2}\sin x\,dx = \dfrac{1}{3}\sin t - \dfrac{1}{3}\sin 2t$.

50. Use the Green's function obtained above to solve $y'' + 4y = \sin\omega t$, $y(0) = 0$, $y'(0) = 0$ for **(a)** $\omega = 3$; **(b)** $\omega = 2.5$; **(c)** $\omega = 2.1$; **(d)** $\omega = 2$. What happens to the solution as $\omega \to 2$?
51. Use a Green's function to solve $y'' + k^2y = f(t)$, $y(0) = 0$, $y'(0) = 0$.
52. If we use the solution in Exercise 51 to solve $y'' + k^2y = \sin\omega t$, $y(0) = 0$, $y'(0) = 0$, what value of $\omega$ leads to the behavior seen in Exercise 50 **(d)**?
*53. Use a Green's function to solve $y'' - k^2y = f(t)$, $y(0) = 0$, $y'(0) = 0$.
54. Use the solution obtained in Exercise 53 to solve the IVP $y'' - 4y = f(t)$, $y(0) = 0$, $y'(0) = 0$ for **(a)** $f(t) = 1$; **(b)** $f(t) = e^{t}$; **(c)** $f(t) = e^{2t}$; **(d)** $f(t) = \cos t$; **(e)** $f(t) = e^{-2t}\cos t$; **(f)** $f(t) = e^{2t}\cos t$. What type of function $f(t)$ (if any) yields a periodic solution to this IVP?
55. Given that $y = c_1t^{-1} + c_2t^{-1}\ln t$ is a general solution to $t^2y'' + 3ty' + y = 0$, $t > 0$, use variation of parameters to solve $t^2y'' + 3ty' + y = \ln t$, $t > 0$.
*56. Given that $y = c_1\sin(2\ln t) + c_2\cos(2\ln t)$ is a general solution to $t^2y'' + ty' + 4y = 0$, $t > 0$, use variation of parameters to solve $t^2y'' + ty' + 4y = t$, $t > 0$.
57. Given that $y = c_1t^6 + c_2t^{-1}$ is a general solution to $t^2y'' - 4ty' - 6y = 0$, $t > 0$, use variation of parameters to solve $t^2y'' - 4ty' - 6y = 2\ln t$, $t > 0$.
58. Consider the initial-value problem
$$\begin{cases} 4y'' + 4y' + y = e^{-t/2} \\ y(0) = a,\ y'(0) = b \end{cases}.$$
**(a)** Show that regardless of the choices of $a$ and $b$, the limit as $t \to \infty$ of every solution is 0.
**(b)** Find conditions on $a$ and $b$, if possible, so that $y(t)$ has **(i)** no local minima or maxima; **(ii)** exactly

one local minimum or local maximum; and **(iii)** two distinct local minima and maxima.

**(c)** If possible, find $a$ and $b$ so that $y(t)$ has local extrema if $t = 1$ and $t = 3$. Graph $y(t)$ on the interval $[0, 5]$.

**(d)** If possible, find $a$ and $b$ so that $y(t)$ has exactly one local extremum at $t = 3$. Graph $y(t)$ on the interval $[0, 6]$.

**(e)** Is it possible to determine $a$ and $b$ so that $y(t)$ has local extrema if $t = t_0$ and $t = t_1$ ($t_0 \neq t_1$ both arbitrary)? Support your conclusion with several randomly generated values of $t_0$ and $t_1$.

**59.** Solve the equation

$$e^{-2t}\left[y\frac{d^2y}{dt^2} - \left(\frac{dy}{dt}\right)^2\right] - 2t(1 + t)y^2 = 0$$

by making the substitution $y = e^{u(t)}$. Graph the solution for various values of the constants.

**60.** Show that the solution to the initial-value problem

$$\begin{cases} y'' + 4y = f(t) \\ y(0) = 0, y'(0) = 2 \end{cases}$$

is

$$y(t) = \sin(2t) + \frac{1}{2}\sin(2t)\int_0^t f(t)\cos 2t\, dt$$
$$- \frac{1}{2}\cos(2t)\int_0^t f(t)\sin 2t\, dt$$

(as long as the integrals can be evaluated).

**(a)** Verify this result by using the Fundamental Theorem of Calculus to show that

$$y(t) = \sin(2t) + \frac{1}{2}\sin(2t)\int_0^t f(x)\cos 2x\, dx$$
$$- \frac{1}{2}\cos(2t)\int_0^t \cos(2t)\int_0^t f(x)\sin 2x\, dx$$

is the solution to

$$\begin{cases} y'' + 4y = f(t) \\ y(0) = 0, y'(0) = 2 \end{cases}.$$

**(b)** Show that if $f(t)$ is constant, the resulting solution is periodic and find its period. Confirm your result with a graph.

**(c)** If $f(t)$ is periodic, is the resulting solution periodic? As in **(b)**, confirm your result with a graph.

***61.** One solution of $t^2y'' - 4ty' + (t^2 + 6)y = 0$ is $y = t^2\cos t$. **(a)** Solve $t^2y'' - 4ty' + (t^2 + 6)y = t^3 + 2t$. **(b)** Find the solutions, if any, that satisfy the initial conditions $y(0) = 0$ and $y'(0) = 1$. **(c)** Does the result found in **(b)** contradict the Existence and Uniqueness theorem? Explain.

**62.** One solution of $ty'' + 2y' + ty = 0$ is $y = \dfrac{\cos t}{t}$. **(a)** Solve $ty'' + 2y' + ty = -t$. **(b)** Find the solution that satisfies the initial conditions $y(\pi) = -1$ and $y'(\pi) = -1/\pi$. **(c)** Find the solutions, if any, that satisfy the initial conditions $y(0) = y'(0) = 0$. **(d)** Does the result found in **(c)** contradict the Existence and Uniqueness theorem? Explain.

**63.** One solution of $4t^2y'' + 4ty' + (16t^2 - 1)y = 0$ is $y = (\sin 2t)/\sqrt{t}$. **(a)** Solve $4t^2y'' + 4ty' + (16t^2 - 1)y = 16t^{3/2}$. **(b)** Find the solutions, if any, that satisfy the boundary conditions $y(\pi) = y(3\pi/2) = 0$. **(c)** Under what conditions, if any, do infinitely many solutions satisfy the boundary conditions $y(\pi) = y'(2\pi) = 0$?

## 4.7 Cauchy-Euler Equations

**Second-Order Cauchy-Euler Equations** · **Nonhomogeneous Cauchy-Euler Equations** · **Higher Order Cauchy-Euler Equations**

In previous sections, we solved linear differential equations with constant coefficients. Generally, solving an arbitrary differential equation is a formidable if not impossible task, particularly when the coefficients are not constants. However, we are able to solve certain equations with variable coefficients using techniques similar to those discussed previously. We begin by considering differential equations of the form



$$a_nx^ny^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + a_1xy' + a_0y = g(x),$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants, called **Cauchy-Euler** equations. (Notice that in Cauchy-Euler equations, the order of each derivative in the equation equals the power of $x$ in the corresponding coefficient.) Euler observed that equations of this type are reduced to linear equations with constant coefficients with the substitution $x = e^t$.*

**Definition 4.8 Cauchy-Euler Equation**

A **Cauchy-Euler differential equation** is an equation of the form

$$a_nx^ny^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + a_1xy' + a_0y = g(x),$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants.

### Second-Order Cauchy-Euler Equations

Consider the second-order homogeneous Cauchy-Euler equation

$$3x^2y'' - 2xy' + 2y = 0,\ x > 0.$$

Here, the solution $y$ is a function of $x$. We assume that $x > 0$ because we are guaranteed solutions where the coefficient functions in $y'' - \dfrac{2}{3x}y' + \dfrac{2}{3x^2}y = 0$ are continuous. (Similarly, we can solve the equation for $x < 0$.)

Suppose that a solution of this differential equation is of the form $y = x^r$ for some constant(s) $r$. Substitution of $y = x^r$ with derivatives $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$ into this ODE yields

$$3x^2[r(r-1)x^{r-2}] - 2x[rx^{r-1}] + 2x^r = 0$$
$$3r(r-1)x^r - 2rx^r + 2x^r = 0$$
$$[3r(r-1) - 2r + 2]x^r = 0$$
$$(3r^2 - 5r + 2)x^r = 0.$$

We are seeking a nontrivial solution, so we solve the quadratic equation $3r^2 - 5r + 2 = 0$. Factoring gives us $(3r - 2)(r - 1) = 0$, so the roots are $r_1 = 2/3$ and $r_2 = 1$. Therefore, two solutions of this equation are $x^{2/3}$ and $x$. These two functions are linearly independent because they are not constant multiples of one another, so a general solution is $y = c_1x^{2/3} + c_2x$.

This approach works nicely if the roots of the quadratic equation are real and distinct. However, if we obtain a repeated root or complex conjugate roots, additional work is required to determine a general solution (see Exercises 43–45). Therefore, we refer back to Euler's idea by using the substitution

* Carl B. Boyer, *A History of Mathematics*, Princeton University Press (1985), p. 496.

$$x = e^t.$$

In doing this, the chain rule must be used to transform the derivatives with respect to $x$ into derivatives with respect to $t$. If $x = e^t$, $t = \ln x$, so

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} + \frac{d}{dx}\left(\frac{1}{x}\right)\frac{dy}{dt} = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}$$
$$= \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

Substitution of these derivatives into a second-order Cauchy-Euler equation yields a second-order linear differential equation with constant coefficients.

**Example 1**

Solve $x^2y'' - xy' + y = 0$, $x > 0$.

***Solution*** Substituting the derivatives above, we have

$$x^2\left[\frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right] - x\left[\frac{1}{x}\frac{dy}{dt}\right] + y = 0$$
$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - \frac{dy}{dt} + y = 0$$
$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 0.$$

This equation has constant coefficients, so solutions are of the form $y = e^{rt}$. In this case, the characteristic equation is $r^2 - 2r + 1 = (r-1)^2 = 0$ with repeated root $r_1 = r_2 = 1$. Therefore, a general solution of this equation is $y(t) = c_1e^t + c_2te^t$. Returning to the original variable, $x = e^t$ (or $t = \ln x$), we have $y(x) = c_1x + c_2x \ln x$. In Figure 4.15, we graph the solution for several values of $c_1$ and $c_2$. Notice that each function approaches zero as $x \to 0^+$.

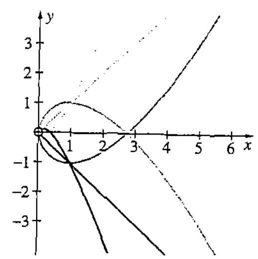



Figure 4.15

*Graphs of the solutions corresponding to various values of $c_1$ and $c_2$ are shown in Figure 4.15. Regardless of the choice of $c_1$ and $c_2$, calculate $\lim_{x\to 0^+}(c_1x + c_2x\ln x)$. Why is a right-hand limit necessary? Are the solutions in Figure 4.15 bounded as $x \to \infty$?*



**Example 2**

Solve $x^2y'' - 5xy' + 10y = 0$, $y(1) = 1$, $y'(1) = 0$.

***Solution*** After substituting the appropriate derivatives, we obtain the equation

$$x^2\left[\frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right] - 5x\left[\frac{1}{x}\frac{dy}{dt}\right] + 10y = 0 \quad \text{or} \quad \frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 10y = 0.$$

The characteristic equation is $r^2 - 6r + 10 = 0$ with complex conjugate roots $r = \frac{1}{2}(6 \pm \sqrt{36-40}) = 3 \pm i$. A general solution of the transformed differential equation is $y(t) = e^{3t}(c_1\cos t + c_2 \sin t)$. We can transform the initial conditions as well. If $x = 1$ as indicated in the two initial conditions, then $t = \ln 1 = 0$. Therefore, in terms of $t$, the initial conditions are $y(0) = 1$ and $y'(0) = 0$, and the transformed initial-value problem is

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 10y = 0,\ y(0) = 1,\ y'(0) = 0.$$

The characteristic equation is $r^2 - 6r + 10 = 0$ with complex conjugate roots $r = \frac{1}{2}(6 \pm \sqrt{36-40}) = 3 \pm i$, so the general solution of the transformed differential equation is $y(t) = e^{3t}(c_1\cos t + c_2\sin t)$. Then, $y(0) = c_1 = 1$. With

$$y'(t) = 3e^{3t}(c_1\cos t + c_2\sin t) + e^{3t}(-c_1\sin t + c_2\cos t),$$

we find that $y'(0) = 3c_1 + c_2 = 0$, so $c_2 = -3c_1 = -3$. Therefore, $y(t) = e^{3t}(\cos t - 3\sin t)$. In terms of $x$, this solution is $y(x) = x^3(\cos(\ln x) - 3\sin(\ln x))$, which is graphed in Figure 4.16.

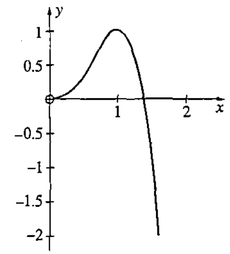



Figure 4.16

*What is $\lim_{x\to 0^+} y(x)$?*

**Theorem 4.14 Solving Second-Order Homogeneous Cauchy-Euler Equations**

Consider the second-order Cauchy-Euler equation $ax^2y'' + bxy' + cy = 0$, $x > 0$. With the substitution $x = e^t$, this equation is transformed into the second-order differential equation with constant coefficients, $a(d^2y/dt^2) + (b-a)(dy/dt) + cy = 0$. Let $r_1$ and $r_2$ be solutions of the equation $ar^2 + (b-a)r + c = 0$.

**(a)** If $r_1 \neq r_2$ are real, a general solution of the Cauchy-Euler equation is $y(x) = c_1x^{r_1} + c_2x^{r_2}$.

**(b)** If $r_1, r_2$ are real and $r_1 = r_2$, a general solution is $y(x) = c_1x^{r_1} + c_2x^{r_1}\ln x$.

**(c)** If $r_1 = \overline{r_2} = \alpha + \beta i$, $\beta \neq 0$, a general solution is $y(x) = x^{\alpha}(c_1\cos(\beta\ln x) + c_2\sin(\beta\ln x))$.

## Nonhomogeneous Cauchy-Euler Equations

Just as we encountered in nonhomogeneous linear differential equations with constant coefficients, the methods of variation of parameters and undetermined coefficients can be used to solve nonhomogeneous Cauchy-Euler equations. The following two examples illustrate two different approaches to solving nonhomogeneous Cauchy-Euler equations: (1) solving the transformed equation with the method of undetermined coefficients, and (2) solving the equation in its original form using variation of parameters.

**Example 3**

Solve $x^2y'' - 3xy' + 13y = 4 + 3x$, $x > 0$.

***Solution*** After substitution of the appropriate derivatives and functions of $t$, we obtain the differential equation

$$x^2\frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 3x\frac{1}{x}\frac{dy}{dt} + 13y = 4 + 3e^t,$$

which in simplified form is $d^2y/dt^2 - 4(dy/dt) + 13y = 4 + 3e^t$. The characteristic equation of the corresponding homogeneous equation is $r^2 - 4r + 13 = 0$, which has complex conjugate roots $r_{1,2} = 2 \pm 3i$ and general solution $y_h(t) = e^{2t}(c_1 \cos 3t + c_2 \sin 3t)$. Using the method of undetermined coefficients to solve the equation

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 13y = 4 + 3e^t,$$

we assume that the particular solution is $y_p(t) = A + Be^t$. The derivatives of this function are $(dy_p/dt)(t) = Be^t$ and $(d^2y_p/dt^2)(t) = Be^t$; thus, substitution into $d^2y/dx^2 - 4(dy/dt) + 13y = 4 + 3e^t$ yields

$$Be^t - 4Be^t + 13A + 13Be^t = 10Be^t + 13A = 4 + 3e^t.$$

Therefore, $B = \frac{3}{10}$ and $A = \frac{4}{13}$, so $y_p(t) = \frac{4}{13} + \frac{3}{10}e^t$. A general solution in the variable $t$ is given by

$$y(t) = e^{2t}(c_1 \cos 3t + c_2 \sin 3t) + \frac{4}{13} + \frac{3}{10}e^t.$$

Returning to the original variable $x = e^t$ (or $t = \ln x$), we have

$$y(x) = x^2(c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x)) + \frac{4}{13} + \frac{3}{10}x.$$

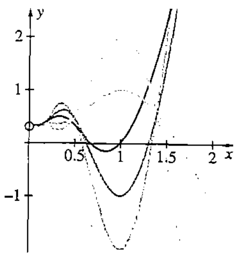


Figure 4.17

*This general solution is plotted for various values of the arbitrary constants $c_1$ and $c_2$ in Figure 4.17. Identify the graph of the solution that satisfies the initial conditions $y(1) = 2$ and $y'(1) = 0$. Show that the solution that satisfies these initial conditions is given by $y(x) = x^2(\frac{181}{130}\cos(3 \ln x) - \frac{401}{390}\sin(3 \ln x)) + \frac{4}{13} + \frac{3}{10}x$ and then approximate the first value of $x$ greater than 10 for which (a) $y'(x) = 0$ and (b) $y(x) = 0$.*



*When solving nonhomogeneous Cauchy-Euler equations, the method of undetermined coefficients should be used **only** when the equation is transformed to a constant coefficient equation. On the other hand, the method of variation of parameters can be used with the original equation in standard form **or** with the transformed equation.*

**Example 4**

Solve $x^2y'' + 4xy' + 2y = \ln x$, $y(1) = 2$, $y'(1) = 0$.

***Solution*** We begin by determining a solution of the corresponding homogeneous equation $x^2y'' + 4xy' + 2y = 0$, which is transformed into $d^2y/dt^2 + 3(dy/dt) + 2y = 0$ with characteristic equation $r^2 + 3r + 2 = (r + 1)(r + 2) = 0$. Therefore, $y_h(t) = c_1e^{-t} + c_2e^{-2t}$, so $y_h(x) = c_1x^{-1} + c_2x^{-2}$. Assuming that $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) = u_1(x)x^{-1} + u_2(x)x^{-2}$, we find using variation of parameters with the equation $y'' + (4/x)y' + (2/x^2)y = (\ln x)/x^2$ that

$$u_1'(x) = \frac{-x^{-4}\ln x}{-x^{-4}} = \ln x \quad \text{and} \quad u_2'(x) = \frac{x^{-3}\ln x}{-x^{-4}} = -x \ln x.$$

Then, $u_1(x) = \int \ln x\,dx = x \ln x - x$ and $u_2(x) = -\int x \ln x\,dx = 1/4\,x^2 - 1/2\,x^2 \ln x$, so

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) = (x \ln x - x)x^{-1} + \left(\frac{1}{4}x^2 - \frac{1}{2}x^2 \ln x\right)x^{-2}$$
$$= \frac{1}{2}\ln x - \frac{3}{4}.$$

A general solution is

$$y(x) = y_h(x) + y_p(x) = c_1x^{-1} + c_2x^{-2} + \frac{1}{2}\ln x - \frac{3}{4},$$

where $y'(x) = -c_1x^{-2} - 2c_2x^{-3} + 1/(2x)$. Applying the initial conditions, we find that $y(1) = c_1 + c_2 - 3/4$ and $y'(1) = -c_1 - 2c_2 + 1/2$. Solving the system $\{c_1 + c_2 - 3/4 = 2, -c_1 - 2c_2 + 1/2 = 0\}$ indicates that $c_1 = 5$ and $c_2 = -9/4$, so the solution to the IVP is

$$y(x) = y_h(x) + y_p(x) = 5x^{-1} - \frac{9}{4}x^{-2} + \frac{1}{2}\ln x - \frac{3}{4}.$$

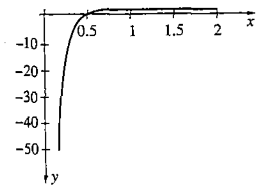


Figure 4.18 Graph of $y(x) = 5x^{-1} - \frac{9}{4}x^{-2} + \frac{1}{2}\ln x - \frac{3}{4}$

We graph this function in Figure 4.18. The solution becomes unbounded as $x \to 0^+$.

### Higher Order Cauchy-Euler Equations

Solutions of higher order homogeneous Cauchy-Euler equations are determined in the same manner as solutions of higher order homogeneous differential equations with constant coefficients.

**Example 5**

Solve $2x^3y''' - 4x^2y'' - 20xy' = 0$, $x > 0$.

**Solution** In this case, if we assume that $y = x^r$ for $x > 0$, we have the derivatives $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$, and $y''' = r(r-1)(r-2)x^{r-3}$. Substitution into the differential equation and simplification yields $(2r^3 - 10r^2 - 12r)x^r = 0$. Because $x^r \neq 0$, we must solve

$$(2r^3 - 10r^2 - 12r) = r(2r+2)(r-6) = 0$$

for $r$. Hence, $r_1 = 0$, $r_2 = -1$, and $r_3 = 6$, and a general solution is $y = c_1 + c_2x^{-1} + c_3x^6$.

*Find and classify all relative extrema of $y(x)$ in the interval $(1, 4)$ if $y$ satisfies the boundary conditions $y(1) = y(4) = 0$ (see Figure 4.19).*

If a root $r$ of the auxiliary equation is repeated $m$ times, the $m$ linearly independent solutions that correspond to $r$ are $x^r$, $x^r \ln x$, $x^r(\ln x)^2, \ldots, x^r(\ln x)^{m-1}$ because in the transformed variable $t$, the solutions are $e^{rt}, te^{rt}, t^2e^{rt}, \ldots, t^{r-1}e^{rt}$.

**Example 6**

Solve $x^3y''' + xy' - y = 0$, $x > 0$.

**Solution** As in Example 5, we substitute $y = x^r$ into the equation. This gives us $x^r(r^3 - 3r^2 + 3r - 1) = 0$, where $r^3 - 3r^2 + 3r - 1 = (r-1)^3$, so the roots are $r_1 = r_2 = r_3 = 1$. One solution to this equation is $x$, a second is $x \ln x$, and a third is $x(\ln x)^2$. Therefore, a general solution is $y = c_1x + c_2x \ln x + c_3x(\ln x)^2$.

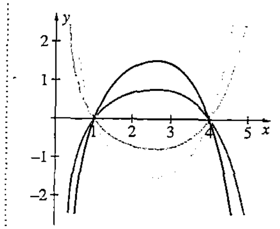



**Figure 4.19** Several solutions to the $2x^3y''' - 4x^2y'' - 20xy' = 0$ equation that satisfy the boundary conditions $y(1) = y(4) = 0$



## EXERCISES 4.7

In Exercises 1–20, find a general solution of the Cauchy-Euler equation.

1. $4x^2y'' - 8xy' + 5y = 0$
2. $3x^2y'' - 4xy' + 2y = 0$
*3. $2x^2y'' - 8xy' + 8y = 0$
4. $2x^2y'' - 7xy' + 7y = 0$
5. $4x^2y'' + 17y = 0$
6. $9x^2y'' - 9xy' + 10y = 0$
7. $2x^2y'' - 2xy' + 20y = 0$
8. $x^2y'' - 5xy' + 10y = 0$
*9. $x^2y'' + 2xy' - 6y = 0$
10. $x^2y'' + xy' - 16y = 0$
11. $x^2y'' + xy' + 4y = 0$
12. $x^2y'' + xy' + 36y = 0$
13. $x^2y'' + 5xy' + 13y = 0$
14. $x^2y'' + 2xy' + y = 0$
*15. $x^3y''' + 22x^2y'' + 124xy' + 140y = 0$
16. $x^3y''' - 4x^2y'' - 46xy' + 100y = 0$
17. $x^3y''' + 2x^2y'' - 4xy' + 4y = 0$
18. $x^3y''' + 4x^2y'' + 6xy' + 4y = 0$
*19. $x^3y''' + 2xy' - 2y = 0$
20. $x^4y^{(4)} + 9x^3y''' + 11x^2y'' - 4xy' + 4y = 0$

In Exercises 21–30, solve the nonhomogeneous Cauchy-Euler equation.

21. $x^2y'' + 5xy' + 4y = x^{-5}$
22. $x^2y'' - 5xy' + 9y = x^3$
*23. $x^2y'' + xy' + y = x^{-2}$
24. $x^2y'' + xy' + 4y = x^{-2}$
25. $x^2y'' + 2xy' - 6y = 2x$
26. $x^2y'' + xy' - 16y = \ln x$
*27. $x^2y'' + xy' + 4y = 8$
28. $x^2y'' + xy' + 36y = x^2$
*29. $x^3y''' + 3x^2y'' - 11xy' + 16y = x^{-3}$
30. $x^3y''' + 16x^2y'' + 70xy' + 80y = x^{-13}$

In Exercises 31–42, solve the initial-value problem.

31. $\begin{cases} 3x^2y'' - 4xy' + 2y = 0 \\ y(1) = 2, y'(1) = 1 \end{cases}$
32. $\begin{cases} 2x^2y'' - 7xy' + 7y = 0 \\ y(1) = -1, y'(1) = 1 \end{cases}$
*33. $\begin{cases} x^2y'' + xy' + 4y = 0 \\ y(1) = 1, y'(1) = 0 \end{cases}$
34. $\begin{cases} x^2y'' + xy' + 2y = 0 \\ y(1) = 0, y'(1) = 2 \end{cases}$
35. $\begin{cases} x^3y''' + 10x^2y'' - 20xy' + 20y = 0 \\ y(1) = 0, y'(1) = -1, y''(1) = 1 \end{cases}$
36. $\begin{cases} x^3y''' + 15x^2y'' + 54xy' + 42y = 0 \\ y(1) = 5, y'(1) = 0, y''(1) = 0 \end{cases}$
*37. $\begin{cases} x^3y''' - 2x^2y'' + 5xy' - 5y = 0 \\ y(1) = 0, y'(1) = -1, y''(1) = 0 \end{cases}$
38. $\begin{cases} x^3y''' - 6x^2y'' + 17xy' - 17y = 0 \\ y(1) = -2, y'(1) = 0, y''(1) = 0 \end{cases}$
39. $\begin{cases} 2x^2y'' + 3xy' - y = x^{-2} \\ y(1) = 0, y'(1) = 2 \end{cases}$
40. $\begin{cases} x^2y'' + 4xy' + 2y = \ln x \\ y(1) = 2, y'(1) = 0 \end{cases}$
*41. $\begin{cases} 4x^2y'' + y = x^3 \\ y(1) = 1, y'(1) = -1 \end{cases}$
42. $\begin{cases} 9x^2y'' + 27xy' + 10y = x^{-1} \\ y(1) = 0, y'(1) = -1 \end{cases}$
43. Suppose that a second-order Cauchy-Euler equation has solutions $x^{r_1}$ and $x^{r_2}$, where $r_1$ and $r_2$ are real and unequal. Calculate the Wronskian of these two solutions to show that they are linearly independent. Therefore, $y = c_1x^{r_1} + c_2x^{r_2}$ is a general solution.
44. **(a)** Show that the transformed equation for the Cauchy-Euler equation $ax^2y'' + bxy' + cy = 0$ is $a(d^2y/dt^2) + (b-a)(dy/dt) + cy = 0$. **(b)** Show that if the characteristic equation for a transformed Cauchy-Euler equation has a related root, then this root is $r = (-(b-a)/2a)$. **(c)** Use reduction of order to show that a second solution to this equation is $x^r \ln x$. **(d)** Calculate the Wronskian of $x^r$ and $x^r \ln x$ to

show that these two solutions are linearly independent.

**45.** Suppose the roots of the characteristic equation for the transformed Cauchy-Euler equation are $r = \alpha \pm \beta i$, so that two solutions are $x^{\alpha+\beta i}$ and $x^{\alpha-\beta i}$. Use Euler's formula and the Principle of Superposition to show that $x^{\alpha}\cos(\beta \ln x)$ and $x^{\alpha}\sin(\beta \ln x)$ are solutions of the original equation. *Hint:* $x^{\beta i} = (e^{\ln x})^{\beta i} = \cos(\beta \ln x) + i\sin(\beta \ln x)$.

**46.** The solution in Example 4 is $y(x) = 5x^{-1} - \frac{9}{4}x^{-2} + \frac{1}{2}\ln x - \frac{3}{4} = (-9 + 20x - 3x^2 + 2x^2 \ln x)/4x^2$. Use L'Hopital's Rule to show that $\lim_{x\to\infty} y(x) = +\infty$.

**47.** Consider the equation $x^2y'' + Axy' + By = 0$, $x > 0$, where $A$ and $B$ are constants. Solve the equation for each of the following. Investigate $\lim_{x\to 0^+} y(x)$ and $\lim_{x\to\infty} y(x)$.

(a) $A = -1, B = 2$

(b) $A = 4, B = 2$

(c) $A = 1, B = 1$

**48.** Use the results of Exercise 47 to show that solutions to $x^2y'' + Axy' + By = 0, x > 0$ (a) approach zero as $x \to 0^+$ if $A < 1$ and $B > 0$; (b) approach zero as $x \to \infty$ if $A > 1$ and $B > 0$; (c) are bounded as $x \to 0^+$ and $x \to \infty$ if $A = 1$ and $B > 0$.

**49.** Show that if $x = e^t$ and $y = y(x)$, then $\dfrac{d^3y}{dx^3} = \dfrac{1}{x^3}\left(\dfrac{d^3y}{dt^3} - 3\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt}\right)$.

In Exercises 50–53, use the substitution in Exercise 49 to solve the following equations.

**50.** $x^3y''' + 3x^2y'' + 37xy' = 0, x > 0$

**51.** $x^3y''' + 3x^2y'' - 3xy' = 0, x > 0$

**52.** $x^3y''' + 3x^2y'' + 7xy' + y = 0, x > 0$

**53.** $x^3y''' + 3x^2y'' - 3xy' = -8, x > 0$

**54.** Let $x = -e^t$. (a) Show that $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \dfrac{1}{x}\dfrac{dy}{dt}$ and $\dfrac{d^2y}{dx^2} = \dfrac{1}{x^2}\left(\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}\right)$. (b) Show that the differential equation $ax^2y'' + bxy' + cy = f(x)$ is transformed into $a\dfrac{d^2y}{dt^2} + (b-a)\dfrac{dy}{dt} + cy = f(-e^t)$.

In Exercises 55–58, use the substitution described in Exercise 54 to solve the indicated equation or initial-value problem.

**55.** $x^2y'' + 4xy' + 2y = 0, x < 0$

**56.** $x^2y'' + xy' + y = x^2, x < 0$

**57.** $x^2y'' + xy' + 4y = 0, y(-1) = 0, y'(-1) = 2$

**58.** $x^2y'' - xy' + y = 0, y(-1) = 0, y'(-1) = 1$

**59.** Consider the second-order Cauchy-Euler equation $x^2y'' + Bxy' + y = 0$, $x > 0$, where $B$ is a constant.

(a) Find $\lim_{x\to\infty} y(x)$ where $y(x)$ is a general solution using the given restriction on $B$.

(b) Determine if the solution is bounded as $x \to \infty$ in each case: (i) $B = 1$; (ii) $B > 1$; (iii) $B < 1$.

**60.** Consider the second-order Cauchy-Euler equation stated in Exercise 59. Determine $\lim_{x\to 0^+} y(x)$ as well as if $y(x)$ is bounded as $x \to 0^+$ in each case: (a) $B = 1$; (b) $B > 1$; (c) $B < 1$.

**61.** Use variation of parameter to solve the nonhomogeneous differential equation that was solved in Example 3 with undetermined coefficients. Which method is easier?

**62.** Use a computer algebra system to assist in finding a general solution of the following Cauchy-Euler equations.

(a) $x^3y''' + 16x^2y'' + 79xy' + 125y = 0$

(b) $x^4y^{(4)} + 5x^3y''' - 12x^2y'' - 12xy' + 48y = 0$

(c) $x^4y^{(4)} + 14x^3y''' + 55x^2y'' + 65xy' + 15y = 0$

(d) $x^4y^{(4)} + 8x^3y''' + 27x^2y'' + 35xy' + 45y = 0$

(e) $x^4y^{(4)} + 10x^3y''' + 27x^2y'' + 21xy' + 4y = 0$

**63.** (a) Solve the initial-value problem

$$\begin{cases} x^3y''' + 9x^2y'' + 44xy' + 58y = 0 \\ y(1) = 2, y'(1) = 10, y''(1) = -2 \end{cases}$$

(b) Graph the solution on the interval [0.2, 1.8] and approximate all local minima and maxima of the solution on this interval.

**64.** (a) Solve the initial-value problem

$$\begin{cases} 6x^2y'' + 5xy' - y = 0 \\ y(1) = a, y'(1) = b \end{cases}$$

(b) Find conditions on $a$ and $b$ so that $\lim_{x\to 0^+} y(x) = 0$. Graph several solutions to confirm your results.

(c) Find conditions on $a$ and $b$ so that $\lim_{x\to\infty} y(x) = 0$. Graph several solutions to confirm your results.

(d) If both $a$ and $b$ are not zero, is it possible to find $a$ and $b$ so that both $\lim_{x\to 0^+} y(x) = 0$ and $\lim_{x\to\infty} y(x) = 0$? Explain.

**65.** An equation of the form $f(x, y, y')y'' + g(x, y, y') = 0$ that satisfies the system

$$\begin{cases} f_{xx} + 2pf_{xy} + p^2f_{yy} = g_{xp} + pg_{yp} - g_y \\ f_{xp} + pf_{yp} + 2f_y = g_{pp} \end{cases},$$

where $p = y'$ is called an **exact second-order differential equation.** (a) Show that the equation

$$x(y')^2 + yy' + xyy'' = 0$$

is an exact second-order equation. (b) Show that $\phi(x, y, p) = h(x, y) + \int f(x, y, p)\, dp$ is a solution of the exact equation

$$f(x, y, y')y'' + g(x, y, y') = 0$$

and that a general solution of

$$x(y')^2 + yy' + xyy'' = 0$$

is $y^2/2 = C_1 + C_2 \ln x$.

**66.** (a) Find a general solution of the nonlinear equation

$$\tfrac{4}{5}(y' - xy'')^2 = 2yy'' - (y')^2$$

by differentiating both sides of the equation, setting the result equal to zero, and factoring. (b) Is the Principle of Superposition valid for this nonlinear equation? Explain.



## 4.8 Series Solutions of Ordinary Differential Equations

Series Solutions About Ordinary Points · Legendre's Equation · Regular and Irregular Singular Points and the Method of Frobenius · Gamma Function · Bessel's Equation

In calculus, we learned that Maclaurin and Taylor polynomials can be used to approximate *functions*. This idea can be extended to finding or approximating the *solution of a differential equation*.

### Series Solutions About Ordinary Points

Consider the equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ and let $p(x) = a_1(x)/a_2(x)$ and $q(x) = a_0(x)/a_2(x)$. Then, $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ is equivalent to $y'' + p(x)y' + q(x)y = 0$, which is called the **standard form** of the equation. A number $x_0$ is an **ordinary point** of the differential equation if both $p(x)$ and $q(x)$ are analytic at $x_0$. If $x_0$ is not an ordinary point, $x_0$ is called a **singular point.** *Note:* A function is analytic at $x_0$ if its power series centered at $x = x_0$ has a positive radius of convergence.

If $x_0$ is an ordinary point of the differential equation $y'' + p(x)y' + q(x)y = 0$, we can write $p(x) = \sum_{n=0}^{\infty} b_n(x - x_0)^n$, where $b_n = \dfrac{p^{(n)}(x_0)}{n!}$, and $q(x) = \sum_{n=0}^{\infty} c_n(x - x_0)^n$, where $c_n = \dfrac{q^{(n)}(x_0)}{n!}$. Substitution into the equation $y'' + p(x)y' + q(x)y = 0$ results in

$$y'' + y'\sum_{n=0}^{\infty} b_n(x - x_0)^n + y\sum_{n=0}^{\infty} c_n(x - x_0)^n = 0.$$

If we assume that $y$ is analytic at $x_0$, we can write $y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$. Because a power series can be differentiated term by term, we can compute the first and second derivatives of $y$ and substitute back into the equation to calculate the coefficients $a_n$. Thus, we obtain a power series solution of the equation.

**Example 1**

**(a)** Find a general solution of $(4-x^2)\,dy/dx + y = 0$ and **(b)** solve the initial-value problem $\begin{cases}(4-x^2)\,dy/dx + y = 0\\ y(0)=1\end{cases}$.

***Solution*** **(a)** Because $x = 0$ is an ordinary point of the equation, we assume that $y = \sum_{n=0}^{\infty} a_n x^n$. Substitution of this function and its derivatives into the equation gives us

$$(4-x^2)\frac{dy}{dx} + y = (4-x^2)\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=1}^{\infty} 4na_n x^{n-1} - \sum_{n=1}^{\infty} na_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Note that the first terms in these three series involve $x^0$, $x^2$, and $x^0$, respectively. If we pull off the first two terms in the first and third series, all three series will begin with $x^2$. Doing so, we have

$$(4a_1 + a_0) + (8a_2 + a_1)x + \sum_{n=3}^{\infty} 4na_n x^{n-1} - \sum_{n=1}^{\infty} na_n x^{n+1} + \sum_{n=2}^{\infty} a_n x^n = 0.$$

Now the indices of these three series do not match, so we change two of the three to match the third. Substitution of $(n+1)$ for $n$ in $\sum_{n=3}^{\infty} 4na_n x^{n-1}$ yields

$$\sum_{n+1=3}^{\infty} 4(n+1)a_{n+1}x^{n+1-1} = \sum_{n=2}^{\infty} 4(n+1)a_{n+1}x^n.$$

Similarly, substitution of $(n-1)$ for $n$ in $\sum_{n=1}^{\infty} na_n x^{n+1}$ yields

$$\sum_{n-1=1}^{\infty} (n-1)a_{n-1}x^{n-1+1} = \sum_{n=2}^{\infty} (n-1)a_{n-1}x^n.$$

After combining the three series, we have the equation

$$(4a_1 + a_0) + (8a_2 + a_1)x + \sum_{n=2}^{\infty} (a_n + 4(n+1)a_{n+1} - (n-1)a_{n-1})x^n = 0.$$

Equating the coefficients of $x^0$ and $x$ to zero yields

$$a_1 = -\frac{a_0}{4} \quad \text{and} \quad a_2 = -\frac{a_1}{8} = \frac{a_0}{32}.$$

When the coefficient of $x^n$, $a_n + 4(n+1)a_{n+1} - (n-1)a_{n-1}$, $n \ge 2$, is set to zero, we obtain the recurrence relation $a_{n+1} = ((n-1)a_{n-1} - a_n)/4(n+1)$ for the



indices in the series, $n \ge 2$. We use this formula to determine the values of $a_n$ for $n = 2, 3, \ldots, 10$, and give these values in Table 4.3.

**TABLE 4.3**

| $n$ | $a_n$ | $n$ | $a_n$ |
|---|---|---|---|
| 0 | $a_0$ | 6 | $\frac{69}{65536}a_0$ |
| 1 | $-\frac{1}{4}a_0$ | 7 | $-\frac{187}{262144}a_0$ |
| 2 | $\frac{1}{32}a_0$ | 8 | $\frac{1843}{8388608}a_0$ |
| 3 | $-\frac{3}{128}a_0$ | 9 | $-\frac{4859}{33554432}a_0$ |
| 4 | $\frac{11}{2048}a_0$ | 10 | $\frac{12767}{268435456}a_0$ |
| 5 | $-\frac{31}{8192}a_0$ | | |

Therefore,

$$y = a_0 - \frac{1}{4}a_0x + \frac{1}{32}a_0x^2 - \frac{3}{128}a_0x^3 + \frac{11}{2048}a_0x^4 - \frac{31}{8192}a_0x^5 + \frac{69}{65536}a_0x^6$$
$$- \frac{187}{262144}a_0x^7 + \frac{1843}{8388608}a_0x^8 - \frac{4859}{33554432}a_0x^9 + \frac{12767}{268435456}a_0x^{10} + \cdots$$

**(b)** When we apply the initial condition $y(0) = 1$, we substitute $x = 0$ into the solution obtained in **(a)**. Hence, $a_0 = 1$, so the series solution of the initial-value problem is

$$y = 1 - \frac{1}{4}x + \frac{1}{32}x^2 - \frac{3}{128}x^3 + \frac{11}{2048}x^4 - \frac{31}{8192}x^5 + \frac{69}{65536}x^6$$
$$- \frac{187}{262144}x^7 + \frac{1843}{8388608}x^8 - \frac{4859}{33554432}x^9 + \frac{12767}{268435456}x^{10} + \cdots$$

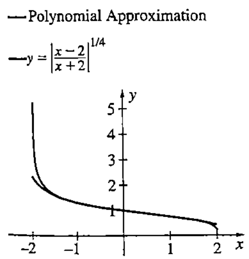



**Figure 4.20** Comparison of exact and approximate solutions to the initial-value problem $\begin{cases}(4-x^2)\,dy/dx + y = 0\\ y(0) = 1\end{cases}$

The equation $(4-x^2)\,dy/dx + y = 0$ is a separable first-order equation. Separating, integrating, and applying the initial condition yields $y = \left|\frac{x-2}{x+2}\right|^{1/4}$. We can approximate the solution of the problem by taking a finite number of terms of the series solution. The graph of the polynomial of degree ten is shown in Figure 4.20 along with the solution obtained through separation of variables. Notice that the accuracy of the approximation decreases near $x = 2$ and $x = -2$, which are singular points of the differential equation. (Why?) The reason for this is discussed in the following theorem.

The following theorem explains where a series solution of a differntial equation is valid.

**Theorem 4.15 Convergence of a Power Series Solution**

Let $x = x_0$ be an ordinary point of the differential equation

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0,$$

and suppose that $R$ is the distance from $x = x_0$ to the closest singular point of the equation. Then the power series solution $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges *at least* on the interval $(x_0 - R, x_0 + R)$.*

Series solutions have been used since the days of Newton. At that time, however, mathematicians did not concern themselves with such issues as convergence, which is essential to successful application of the method.

This theorem indicates that a polynomial approximation may not be accurate near singular points of the equation. Now we understand why the approximation in Example 1 breaks down near $x = 2$ and $x = -2$; these are the closest singular points to the ordinary point $x = 0$.

Of course, $x = 0$ is not an ordinary point for every differential equation. However, because the series $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ is easier to work with if $x_0 = 0$, we can always make a transformation so that we can use $y = \sum_{n=0}^{\infty} a_n x^n$ to solve any linear equation. For example, if $x = x_0$ is an ordinary point of a linear equation and we make the change of variable $t = x - x_0$, then $t = 0$ corresponds to $x = x_0$. Hence, $t = 0$ is an ordinary point of the transformed equation.

## Legendre's Equation

**Legendre's equation** is

$$(1 - x^2)y'' - 2xy' + k(k + 1)y = 0,$$

where $k$ is a constant, named after the French mathematician Adrien Marie Legendre (1752–1833). The **Legendre polynomials,** solutions of Legendre's equation, were introduced by Legendre in his three-volume work *Traité des fonctions elliptiques et des intégrales euleriennes* (1825–1832). Legendre encountered these polynomials while trying to determine the gravitational potential associated with a point mass.

* A proof of this theorem can be found in more advanced texts, such as *Introduction to Ordinary Differential Equations* by Albert L. Rabenstein, Academic Press (1966), pp. 103–107.



**Example 2**

Find a general solution of Legendre's equation $(1 - x^2)y'' - 2xy' + k(k + 1)y = 0$.

**Solution** In standard form, the equation is $y'' - \dfrac{2x}{1 - x^2}y' + \dfrac{k(k + 1)}{1 - x^2}y = 0$. Because $x = 0$ is an ordinary point, there is a solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$. This solution will converge at least on the interval $(-1, 1)$, because the closest singular points to $x = 0$ are $x = \pm 1$. Substitution of this function and its derivatives into the differential equation yields

$$\sum_{n=2}^{\infty} n(n - 1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n - 1)a_n x^n - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} k(k + 1)a_n x^n = 0$$

$$[2a_2 + k(k + 1)a_0]x^0 + [-2a_1 + k(k + 1)a_1 + 6a_3]x$$
$$+ \sum_{n=4}^{\infty} n(n - 1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n - 1)a_n x^n - \sum_{n=2}^{\infty} 2na_n x^n + \sum_{n=2}^{\infty} k(k + 1)a_n x^n = 0.$$

After substituting $(n + 2)$ for each occurrence of $n$ in the first series and simplifying, we have

$$[2a_2 + k(k + 1)a_0]x^0 + [-2a_1 + k(k + 1)a_1 + 6a_3]x$$
$$+ \sum_{n=2}^{\infty} \{(n + 2)(n + 1)a_{n+2} + [-n(n - 1) - 2n + k(k + 1)]a_n\}x^n = 0.$$

Equating the coefficients to zero, we find that

$$a_2 = -\frac{k(k + 1)}{2}a_0, \quad a_3 = -\frac{k(k + 1) - 2}{6}a_1 = -\frac{(k - 1)(k + 2)}{6}a_1,$$

and

$$a_{n+2} = \frac{n(n - 1) + 2n - k(k + 1)}{(n + 2)(n + 1)}a_n = \frac{(n - k)(n + k + 1)}{(n + 2)(n + 1)}a_n, \; n \geq 2.$$

Using this formula, we find the following coefficients:

$$a_4 = \frac{(2 - k)(3 + k)}{4 \cdot 3}a_2 = -\frac{(2 - k)(3 + k)k(k + 1)}{4 \cdot 3 \cdot 2}a_0$$

$$a_5 = \frac{(3 - k)(4 + k)}{5 \cdot 4}a_3 = -\frac{(3 - k)(4 + k)(k - 1)(k + 2)}{5 \cdot 4 \cdot 3 \cdot 2}a_1$$

$$a_6 = \frac{(4 - k)(5 + k)}{6 \cdot 5}a_4 = -\frac{(4 - k)(5 + k)(2 - k)(3 + k)k(k + 1)}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}a_0$$

$$a_7 = \frac{(5 - k)(6 + k)}{7 \cdot 6}a_5 = -\frac{(5 - k)(6 + k)(3 - k)(4 + k)(k - 1)(k + 2)}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}a_1.$$

We have the two linearly independent solutions

$$y_1(x) = a_0\left(1 - \frac{k(k+1)}{2!}x^2 - \frac{(2-k)(3+k)k(k+1)}{4!}x^4 - \frac{(4-k)(5+k)(2-k)(3+k)k(k+1)}{6!}x^6 - \cdots\right)$$

and

$$y_2(x) = a_1\left(x - \frac{(k-1)(k+2)}{6}x^3 - \frac{(3-k)(4+k)(k-1)(k+2)}{5!}x^5 - \frac{(5-k)(6+k)(3-k)(4+k)(k-1)(k+2)}{7!}x^7 - \cdots\right),$$

so a general solution is

$$y = a_0\left(1 - \frac{k(k+1)}{2!}x^2 - \frac{(2-k)(3+k)k(k+1)}{4!}x^4 - \frac{(4-k)(5+k)(2-k)(3+k)k(k+1)}{6!}x^6 - \cdots\right)$$
$$+ a_1\left(x - \frac{(k-1)(k+2)}{6}x^3 - \frac{(3-k)(4+k)(k-1)(k+2)}{5!}x^5 - \frac{(5-k)(6+k)(3-k)(4+k)(k-1)(k+2)}{7!}x^7 - \cdots\right).$$

An interesting observation from the general solution to Legendre's equation given here is that the series solutions terminate for integer values of $k$. If $k$ is an even integer, then the first series terminates, while if $k$ is an odd integer the second series terminates. Therefore, polynomial solutions are found for integer values of $k$. We list several of these polynomials for suitable choices of $a_0$ and $a_1$ in Table 4.4 and graph them in Figure 4.21. Because these polynomials are useful and are encountered in numerous applications, we have a special notation for them: $P_n(x)$ is called the **Legendre polynomial of degree $n$** and represents the $n$th-degree polynomial solution to Legendre's equation.

**TABLE 4.4** $P_n(x)$ for $n = 0, 1, \ldots, 5$

| |
|---|
| $P_0(x) = 1$ |
| $P_1(x) = x$ |
| $P_2(x) = \frac{1}{2}(3x^2 - 1)$ |
| $P_3(x) = \frac{1}{2}(5x^3 - 3x)$ |
| $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$ |
| $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$ |

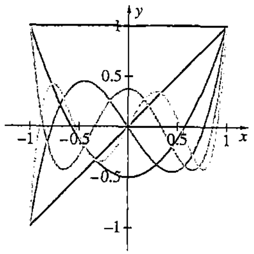



**Figure 4.21** $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$, $P_4(x)$, and $P_5(x)$

*Match each Legendre polynomial to the appropriate graph in Figure 4.21.*

## Regular and Irregular Singular Points and the Method of Frobenius

Let $x_0$ be a singular point of $y'' + p(x)y' + q(x)y = 0$. Then $x_0$ is a **regular singular point** of the equation if both $(x - x_0)p(x)$ and $(x - x_0)^2q(x)$ are analytic at $x = x_0$. If $x_0$ is not a regular singular point, then $x_0$ is called an **irregular singular point** of the equation.



**Theorem 4.16 (Method of Frobenius)**

Let $x_0$ be a regular singular point of $y'' + p(x)y' + q(x)y = 0$. Then this differential equation has at least one solution of the form

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^{n+r},$$

where $r$ is a constant that must be determined. This solution is convergent at least on some interval $|x - x_0| < R$, $R > 0$.

Suppose that $x = 0$ is a regular singular point of the differential equation $y'' + p(x)y' + q(x)y = 0$. Then the functions $x\,p(x)$ and $x^2q(x)$ are analytic, which means that both of these functions have a power series in $x$ with a positive radius of convergence. Hence,

$$x\,p(x) = p_0 + p_1x + p_2x^2 + \cdots \quad \text{and} \quad x^2q(x) = q_0 + q_1x + q_2x^2 + \cdots.$$

Therefore,

$$p(x) = \frac{p_0}{x} + p_1 + p_2x + p_3x^2 + p_4x^3 + \cdots$$

and

$$q(x) = \frac{q_0}{x^2} + \frac{q_1}{x} + q_2 + q_3x + q_4x^2 + q_5x^3 + \cdots.$$

Substitution of these series into the differential equation $y'' + p(x)y' + q(x)y = 0$ and multiplying through by the first term in the power series for $p(x)$ and $q(x)$, we see that the lowest term in the series involves $x^{n+r-2}$:

$$\left(\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}\right) + \left(\sum_{n=0}^{\infty} a_np_0(n+r)x^{n+r-2}\right)$$
$$+ (p_1 + p_2x + p_3x^2 + p_4x^3 + \cdots)\left(\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_nq_0x^{n+r-2}\right)$$
$$+ \left(\frac{q_1}{x} + q_2 + q_3x + q_4x^2 + q_5x^3 + \cdots\right)\left(\sum_{n=0}^{\infty} a_nx^{n+r}\right) = 0.$$

Then, with $n = 0$, we find that the coefficient of $x^{r-2}$ is

$$-ra_0 + r^2a_0 + ra_0p_0 + a_0q_0 = a_0(r^2 + (p_0 - 1)r + q_0) = a_0(r(r-1) + p_0r + q_0).$$

Thus, for any equation of the form $y'' + p(x)y' + q(x)y = 0$ with regular singular point $x = 0$, we have the **indicial equation**

$$r(r-1) + p_0r + q_0 = 0.$$

The values of $r$ that satisfy this equation are called the **exponents** or **indicial roots** and are

$$r_1 = \frac{1}{2}\left(1 - p_0 + \sqrt{1 - 2p_0 + p_0^2 - 4q_0}\right)$$

and

$$r_2 = \frac{1}{2}\left(1 - p_0 - \sqrt{1 - 2p_0 + p_0^2 - 4q_0}\right).$$

Note that $r_1 \geq r_2$ and $r_1 - r_2 = \sqrt{1 - 2p_0 + p_0^2 - 4q_0}$.

Several situations can arise when finding the roots of the indicial equation.

1. If $r_1 \neq r_2$ and $r_1 - r_2 = \sqrt{1 - 2p_0 + p_0^2 - 4q_0}$ is not an integer, then there are two linearly independent solutions of the equation of the form $y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n$ and $y_2(x) = x^{r_2}\sum_{n=0}^{\infty} b_n x^n$.
2. If $r_1 \neq r_2$ and $r_1 - r_2 = \sqrt{1 - 2p_0 + p_0^2 - 4q_0}$ is an integer, then there are two linearly independent solutions of the equation of the form $y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n$ and $y_2(x) = cy_1(x)\ln x + x^{r_2}\sum_{n=0}^{\infty} b_n x^n$.
3. If $r_1 - r_2 = \sqrt{1 - 2p_0 + p_0^2 - 4q_0} = 0$, then there are two linearly independent solutions of the problem of the form $y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n$ and $y_2(x) = y_1(x)\ln x + x^{r_1}\sum_{n=0}^{\infty} b_n x^n$.

In any case, if $y_1(x)$ is a solution of the equation, a second linearly independent solution is given by

$$y_2(x) = y_1(x)\int \frac{e^{-\int p(x)\,dx}}{[y_1(x)]^2}\,dx,$$

which can be obtained through reduction of order.

When solving a differential equation in Case 2, first attempt to find a general solution using $y = x^{r_2}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r_2}$, where $r_2$ is the smaller of the two roots. A general solution can sometimes be found with this procedure. However, if the contradiction $a_0 = 0$ is reached, find solutions of the form $y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n$ and $y_2(x) = cy_1(x)\ln x + x^{r_2}\sum_{n=0}^{\infty} b_n x^n$.

**Example 3**

Find a general solution of $xy'' + 3y' - y = 0$ using a series expansion about the regular singular point $x = 0$.



***Solution*** In standard form, this equation is $y'' + (3/x)y' - (1/x)y = 0$. Hence, $xp(x) = x(3/x) = 3$ and $x^2q(x) = x^2(-1/x) = -x$, so $p_0 = 3$ and $q_0 = 0$. Thus, the indicial equation is $r(r-1) + 3r = r^2 + 2r = r(r+2) = 0$ with roots $r_1 = 0$ and $r_2 = -2$. (Notice that we always use $r_1$ to denote the larger root.) Therefore, we attempt to find a solution of the form $y = \sum_{n=0}^{\infty} a_n x^{n-2}$ with derivatives $y' = \sum_{n=0}^{\infty}(n-2)a_n x^{n-3}$ and $y'' = \sum_{n=0}^{\infty}(n-2)(n-3)a_n x^{n-4}$. Substitution into the differential equation yields

$$\sum_{n=0}^{\infty}(n-2)(n-3)a_n x^{n-3} + \sum_{n=0}^{\infty}3(n-2)a_n x^{n-3} - \sum_{n=0}^{\infty}a_n x^{n-2} = 0$$

$$(6a_0 - 6a_0)x^{-3} + \sum_{n=1}^{\infty}(n-2)(n-3)a_n x^{n-3} + \sum_{n=1}^{\infty}3(n-2)a_n x^{n-3} - \sum_{n=0}^{\infty}a_n x^{n-2} = 0$$

$$\sum_{n=1}^{\infty}(n-2)(n-3)a_n x^{n-3} + \sum_{n=1}^{\infty}3(n-2)a_n x^{n-3} - \sum_{n=1}^{\infty}a_{n-1}x^{n-3} = 0$$

$$\sum_{n=1}^{\infty}\{[(n-2)(n-3) + 3(n-2)]a_n - a_{n-1}\}x^{n-3} = 0.$$

Equating the coefficients to zero, we have

$$a_n = \frac{a_{n-1}}{(n-2)(n-3+3)} = \frac{a_{n-1}}{n(n-2)},\ n \geq 1,\ n \neq 2.$$

Notice that from this formula, $a_1 = -a_0$. When $n = 2$, we refer to the recurrence relation $n(n-2)a_n - a_{n-1} = 0$ obtained from the coefficient in the series solution. When $n = 2$, $2(0)a_2 - a_1 = 0$, which indicates that $a_1 = 0$. Because $a_1 = -a_0$, $a_0 = 0$. However, $a_0 \neq 0$ by assumption, so there is no solution of this form.

Because there is no series solution corresponding to $r_2 = -2$, we assume there is a solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ corresponding to $r_1 = 0$ with derivatives $y' = \sum_{n=1}^{\infty} na_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$. Substitution into the differential equation yields

$$\sum_{n=2}^{\infty}n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty}3na_n x^{n-1} - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$(3a_1 - a_0)x^0 + \sum_{n=2}^{\infty}n(n-1)a_n x^{n-1} + \sum_{n=2}^{\infty}3na_n x^{n-1} - \sum_{n=1}^{\infty}a_n x^n = 0$$

$$(3a_1 - a_0)x^0 + \sum_{n=2}^{\infty}\{[n(n-1) + 3n]a_n - a_{n-1}\}x^{n-1} = 0.$$

Equating the coefficients to zero, we have $a_1 = \frac{1}{3}a_0$ and

$$a_n = \frac{a_{n-1}}{n(n-1) + 3n} = \frac{a_{n-1}}{n^2 + 2n},\ n \geq 2.$$

**TABLE 4.5**

| $n$ | $a_n$ |
|---|---|
| 0 | $a_0$ |
| 1 | $\frac{1}{3}a_0$ |
| 2 | $\frac{1}{24}a_0$ |
| 3 | $\frac{1}{360}a_0$ |

We use this formula to calculate several coefficients in Table 4.5 and use them to form $y_1(x) = a_0\left(1 + \frac{x}{3} + \frac{x^2}{24} + \frac{x^3}{360} + \cdots\right)$.

To determine a second linearly independent solution, we assume that

$$y_2(x) = cy_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n-2}$$

and substitute this function into the differential equation to find the coefficients $b_n$. Because the derivatives of $y_2(x)$ are

$$y_2'(x) = \frac{cy_1(x)}{x} + cy_1'(x)\ln x + \sum_{n=0}^{\infty}(n-2)b_n x^{n-3}$$

and

$$y_2''(x) = \frac{-cy_1(x)}{x^2} + \frac{2cy_1'(x)}{x} + cy_1''(x)\ln x + \sum_{n=0}^{\infty}(n-2)(n-3)b_n x^{n-4},$$

substitution into the differential equation yields

$$x\left[\frac{-cy_1(x)}{x^2} + \frac{2cy_1'(x)}{x} + cy_1''(x)\ln x + \sum_{n=0}^{\infty}(n-2)(n-3)b_n x^{n-4}\right]$$

$$+ 3\left[\frac{cy_1(x)}{x} + cy_1'(x)\ln x + \sum_{n=0}^{\infty}(n-2)b_n x^{n-3}\right] - cy_1(x)\ln x - \sum_{n=0}^{\infty} b_n x^{n-2} = 0$$

$$\frac{-cy_1(x)}{x} + 2cy_1'(x) + cxy_1''(x)\ln x + \sum_{n=0}^{\infty}(n-2)(n-3)b_n x^{n-3}$$

$$+ \frac{3cy_1(x)}{x} + 3cy_1'(x)\ln x + \sum_{n=0}^{\infty}3(n-2)b_n x^{n-3} - cy_1(x)\ln x - \sum_{n=0}^{\infty} b_n x^{n-2} = 0$$

$$\frac{2cy_1(x)}{x} + 2cy_1'(x) + \sum_{n=0}^{\infty}(n-2)(n-3)b_n x^{n-3} + \sum_{n=0}^{\infty}3(n-2)b_n x^{n-3}$$

$$- \sum_{n=0}^{\infty} b_n x^{n-2} + \underbrace{c[xy_1''(x) + 3y_1'(x) - y_1(x)]}_{\substack{= 0 \text{ because } y_1 \text{ is a solution} \\ \text{of the differential equation}}}\ln x = 0.$$

Simplifying this expression gives us

$$\frac{2cy_1(x)}{x} + 2cy_1'(x) + 6b_0x^{-3} - 6b_0x^{-3} + \sum_{n=1}^{\infty}(n-2)(n-3)b_n x^{n-3} +$$

$$\sum_{n=1}^{\infty}3(n-2)b_n x^{n-3} - \sum_{n=0}^{\infty} b_n x^{n-2} = 0$$

$$\frac{2cy_1(x)}{x} + 2cy_1'(x) + \sum_{n=1}^{\infty}[(n-2)nb_n - b_{n-1}]x^{n-3} = 0.$$



**TABLE 4.6**

| $n$ | $b_n$ |
|---|---|
| 0 | $-2$ |
| 1 | $2$ |
| 2 | $b_2$ |
| 3 | $\frac{3b_2 - 4}{9}$ |
| 4 | $\frac{12b_2 - 25}{288}$ |
| 5 | $\frac{60b_2 - 157}{21600}$ |

Now we *choose* $a_0 = 1/c$, so

$$y_1(x) = \frac{1}{c}\left(1 + \frac{x}{3} + \frac{x^2}{24} + \frac{x^3}{360} + \cdots\right) \text{ and } y_1'(x) = \frac{1}{c}\left(\frac{1}{3} + \frac{x}{12} + \frac{x^2}{120} + \cdots\right).$$

Substitution into the previous equation then yields

$$\frac{2}{x}\left[1 + \frac{x}{3} + \frac{x^2}{24} + \frac{x^3}{360} + \cdots\right] + 2\left[\frac{1}{3} + \frac{x}{12} + \frac{x^2}{120} + \cdots\right]$$

$$+ \sum_{n=1}^{\infty}[(n-2)nb_n - b_{n-1}]x^{n-3} = 0$$

$$\left(\frac{2}{x} + \frac{4}{3} + \frac{x}{4} + \frac{x^2}{45} + \cdots\right) + \sum_{n=1}^{\infty}[(n-2)nb_n - b_{n-1}]x^{n-3} = 0$$

$$\left(\frac{2}{x} + \frac{4}{3} + \frac{x}{4} + \frac{x^2}{45} + \cdots\right) + (-b_1 - b_0)x^{-2} - b_1x^{-1}$$

$$+ (3b_3 - b_2)x^0 + (8b_4 - b_3)x + (15b_5 - b_4)x^2 = 0,$$

so we have the sequence of equations $-b_1 - b_0 = 0$, $-b_1 + 2 = 0$, $3b_3 - b_2 + \frac{4}{3} = 0$, $8b_4 - b_3 + \frac{1}{4} = 0$, $15b_5 - b_4 + \frac{1}{45} = 0, \ldots$. Solving these equations, we see that $b_1 = 2$ and $b_0 = -2$. However, the other coefficients depend on the value of $b_2$. We give these values in Table 4.6. Hence, a second linearly independent solution is given by

$$y_2(x) = cy_1(x)\ln x + x^{-2}\left(-2 + 2x + b_2x^2 + \frac{3b_2 - 4}{9}x^3 + \frac{12b_2 - 25}{288}x^4 + \frac{60b_2 - 157}{21600}x^5 + \cdots\right)$$

$$= \left(1 + \frac{x}{3} + \frac{x^2}{24} + \frac{x^3}{360} + \cdots\right)\ln x$$

$$+ x^{-2}\left(-2 + 2x + b_2x^2 + \frac{3b_2 - 4}{9}x^3 + \frac{12b_2 - 25}{288}x^4 + \frac{60b_2 - 157}{21600}x^5 + \cdots\right),$$

where $b_2$ is arbitrary. In particular, two linearly independent solutions of the equation are ($c = 1/a_0 = 1$)

$$y_1(x) = 1 + \frac{x}{3} + \frac{x^2}{24} + \frac{x^3}{360} + \cdots$$

and ($b_2 = 0$)

$$y_2(x) = \left(1 + \frac{x}{3} + \frac{x^2}{24} + \frac{x^3}{360} + \cdots\right)\ln x$$

$$+ x^{-2}\left(-2 + 2x - \frac{4}{9}x^3 - \frac{25}{288}x^4 - \frac{157}{21600}x^5 + \cdots\right).$$

A general solution is therefore given by $y(x) = c_1y_1(x) + c_2y_2(x)$, where $c_1$ and $c_2$ are arbitrary.

*Explain why the choice $a_0 = 1/c$ does not affect the general solution obtained in Example 3. Hint: If $c = 0$, $b_0 = 0$, which is impossible (why?), so $c$ cannot be zero.*

## Gamma Function

One of the more useful functions, which we will use shortly to solve *Bessel's equation,* is the **Gamma function,** first introduced by Euler in 1768, which is defined as follows.

Notice that because integration is with respect to $u$, the result is a function of $x$.

**Definition 4.9 Gamma Function**

The **Gamma function,** denoted $\Gamma(x)$, is given by

$$\Gamma(x) = \int_0^\infty e^{-u}u^{x-1}\, du,\ x > 0. \text{ (See Figure 4.22.)}$$

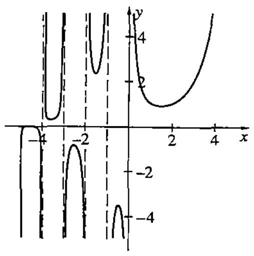



**Figure 4.22** Although we have only defined $\Gamma(x)$ for $x > 0$, $\Gamma(x)$ can be defined for all real numbers *except* $x = 0, -1, -2, \ldots$.*

**Example 4**

Evaluate $\Gamma(1)$.

***Solution***

$$\Gamma(1) = \int_0^\infty e^{-u}u^{1-1}\, du = \int_0^\infty e^{-u}\, du = \lim_{b\to\infty} [-e^{-u}]_0^b$$
$$= \lim_{b\to\infty} (-e^{-b} + 1) = 1.$$

* This topic is discussed in most complex analysis books such as *Functions of One Complex Variable,* Second Edition, by John B. Conway, Springer-Verlag (1978), pp. 176–185.



A useful property associated with the Gamma function is

$$\Gamma(x + 1) = x\Gamma(x).$$

If $x$ is an integer, using this property we have $\Gamma(2) = \Gamma(1 + 1) = 1 \cdot \Gamma(1) = \Gamma(1)$,

$$\Gamma(3) = \Gamma(2 + 1) = 2\Gamma(2) = 2,\ \Gamma(4) = \Gamma(3 + 1) = 3\Gamma(3) = 3 \cdot 2,\ \Gamma(5) = \Gamma(4 + 1) = 4\Gamma(4) = 4 \cdot 3 \cdot 2, \ldots.$$

and for the integer $n$,

$$\Gamma(n + 1) = n!$$

This property is used in solving the following equation.

## Bessel's Equation

Another important equation is **Bessel's equation (of order $\mu$),** named after the German astronomer Friedrich Wilhelm Bessel (1784–1846), who was a friend of Gauss. Bessel's equation is

$$x^2y'' + xy' + (x^2 - \mu^2)y = 0,$$

where $\mu \geq 0$ is a constant. Bessel determined several representations of $J_\mu(x)$, the **Bessel function of order $\mu$,** which is a solution of Bessel's equation, and noticed some of the important properties associated with the Bessel functions. The equation received its name due to Bessel's extensive work with $J_\mu(x)$, even though Euler solved the equation before Bessel.

To use a series method to solve Bessel's equation, first write the equation in standard form as

$$y'' + \frac{1}{x}y' + \frac{x^2 - \mu^2}{x^2}y = 0,$$

so $x = 0$ is a regular singular point. Using the Method of Frobenius, we assume that there is a solution of the form $y = \sum_{n=0}^\infty a_nx^{n+r}$. We determine the value(s) of $r$ with the indicial equation. Because $xp(x) = x\frac{1}{x} = 1$ and $x^2q(x) = x^2\left(\frac{x^2 - \mu^2}{x^2}\right) = x^2 - \mu^2$, $p_0 = 1$ and $q_0 = -\mu^2$. Hence, the indicial equation is

$$r(r - 1) + p_0r + q_0 = r(r - 1) + r - \mu^2 = r^2 - \mu^2 = 0$$

with roots $r_1 = \mu$ and $r_2 = -\mu$. We assume that $y = \sum_{n=0}^\infty a_nx^{n+\mu}$ with derivatives $y' = \sum_{n=0}^\infty (n + \mu)a_nx^{n+\mu-1}$ and $y'' = \sum_{n=0}^\infty (n + \mu)(n + \mu - 1)a_nx^{n+\mu-2}$. Substitution into Bessel's equation yields

$$x^2 \sum_{n=0}^\infty (n + \mu)(n + \mu - 1)a_nx^{n+\mu-2} + x \sum_{n=0}^\infty (n + \mu)a_nx^{n+\mu-1} + (x^2 - \mu^2) \sum_{n=0}^\infty a_nx^{n+\mu} = 0$$

$$\sum_{n=0}^{\infty}(n+\mu)(n+\mu-1)a_nx^{n+\mu}+\sum_{n=0}^{\infty}(n+\mu)a_nx^{n+\mu}+\sum_{n=0}^{\infty}a_nx^{n+\mu+2}$$
$$-\sum_{n=0}^{\infty}\mu^2a_nx^{n+\mu}=0$$

$$\mu(\mu-1)a_0x^{\mu}+(1+\mu)\mu a_1x^{\mu+1}+\sum_{n=2}^{\infty}(n+\mu)(n+\mu-1)a_nx^{n+\mu}$$
$$+\mu a_0x^{\mu}+(1+\mu)a_1x^{\mu+1}+\sum_{n=2}^{\infty}(n+\mu)a_nx^{n+\mu}+\sum_{n=0}^{\infty}a_nx^{n+\mu+2}$$
$$-\mu^2a_0x^{\mu}-\mu^2a_1x^{\mu+1}-\sum_{n=2}^{\infty}\mu^2a_nx^{n+\mu}=0$$

$$[\mu(\mu-1)+\mu-\mu^2]a_0x^{\mu}+[(1+\mu)\mu+(1+\mu)-\mu^2]a_1x^{\mu+1}$$
$$+\sum_{n=2}^{\infty}\{[(n+\mu)(n+\mu-1)+(n+\mu)-\mu^2]a_n+a_{n-2}\}x^{n+\mu}=0.$$

Notice that the coefficient of $a_0x^{\mu}$ is zero because $r_1=\mu$ is a root of the indicial equation. After simplifying the other coefficients and equating them to zero, we have $(1+2\mu)a_1=0$ and

$$a_n=-\frac{a_{n-2}}{(n+\mu)(n+\mu-1)+(n+\mu)-\mu^2}=-\frac{a_{n-2}}{n(n+2\mu)},\ n\geq 2.$$

From the first equation, $a_1=0$. Therefore, from $a_n=-(a_{n-2}/n(n+2\mu))$, $n\geq 2$, $a_n=0$ for all odd $n$. The coefficients that correspond to even indices are given by

$$a_2=-\frac{a_0}{2(2+2\mu)}=-\frac{a_0}{2^2(1+\mu)},\ a_4=-\frac{a_2}{4(4+2\mu)}=\frac{a_0}{2^4\cdot 2(2+\mu)(1+\mu)},$$

$$a_6=-\frac{a_4}{6(6+2\mu)}=-\frac{a_0}{2^6\cdot 3\cdot 2(3+\mu)(2+\mu)(1+\mu)},$$

$$a_8=-\frac{a_6}{8(8+2\mu)}=\frac{a_0}{2^8\cdot 4\cdot 3\cdot 2(4+\mu)(3+\mu)(2+\mu)(1+\mu)}.$$

A general formula for these coefficients is given by $a_{2n}=\dfrac{(-1)^na_0}{2^{2n}n!(1+\mu)(2+\mu)(3+\mu)\cdots(n+\mu)}$, $n\geq 2$. Our solution can then be written as

$$y=\sum_{n=0}^{\infty}a_{2n}x^{2n+\mu}=\sum_{n=0}^{\infty}\frac{a_0(-1)^nx^{2n+\mu}}{2^{2n}n!(1+\mu)(2+\mu)(3+\mu)\cdots(n+\mu)}$$
$$=\sum_{n=0}^{\infty}\frac{a_0(-1)^n2^{\mu}}{n!(1+\mu)(2+\mu)(3+\mu)\cdots(n+\mu)}\left(\frac{x}{2}\right)^{2n+\mu}.$$



Using the Gamma function, $\Gamma(x)$, we write this solution as

$$y=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\mu+n)}\left(\frac{x}{2}\right)^{2n+\mu},\qquad\text{where } a_0=\frac{1}{2^{\mu}\mu!}.$$

This function, denoted $J_\mu(x)$, is called the **Bessel function of the first kind of order $\mu$.** For the other root $r_2=-\mu$, a similar derivation yields a second solution

$$y=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1-\mu+n)}\left(\frac{x}{2}\right)^{2n-\mu},$$

which is the **Bessel function of the first kind of order $-\mu$** and is denoted $J_{-\mu}(x)$.

Now, we must determine if the functions $J_\mu(x)$ and $J_{-\mu}(x)$ are linearly independent.

Notice that if $\mu=0$, these two functions are the same. If $\mu>0$, $r_1-r_2=\mu-(-\mu)=2\mu$. If $2\mu$ is not an integer, by the Method of Frobenius the two solutions $J_\mu(x)$ and $J_{-\mu}(x)$ are linearly independent. Also, we can show that if $2\mu$ is an odd integer, then $J_\mu(x)$ and $J_{-\mu}(x)$ are linearly independent. In both of these cases, a general solution is given by $y=c_1J_\mu(x)+c_2J_{-\mu}(x)$. The graphs of the functions $J_\mu(x)$, $\mu=0$, 1, 2, 3 are shown in Figure 4.23. Notice that these functions have numerous zeros.

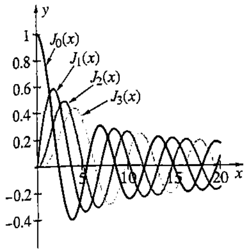


Figure 4.23

*What happens to the maximum value of Jm(x) as m increases?*

If $\mu$ is not an integer, we define the **Bessel function of the second kind of order $\mu$** as the linear combination of the functions $J_\mu(x)$ and $J_{-\mu}(x)$. This function, denoted by $Y_\mu(x)$, is given by

$$Y_\mu(x)=\frac{\cos\mu\pi J_\mu(x)-J_{-\mu}(x)}{\sin\mu\pi}.$$

We can show that $J_\mu(x)$ and $Y_\mu(x)$ are linearly independent solutions of Bessel's equation of order $\mu$, so a general solution of the equation is $y=c_1J_\mu(x)+c_2Y_\mu(x)$. We show the graphs of the functions $Y_\mu(x)$, $\mu=0,1,2$ in Figure 4.24. Notice that $\lim_{x\to 0^+}Y_\mu(x)=-\infty$. We can show that if $m$ is an integer and if $Y_m(x)=\lim_{\mu\to m}Y_\mu(x)$, then $J_m(x)$ and $Y_m(x)$ are linearly independent. Therefore, $y=c_1J_\mu(x)+c_2Y_\mu(x)$ is a general solution to $x^2y''+xy''+(x^2-\mu^2)y=0$ for any value of $\mu$.

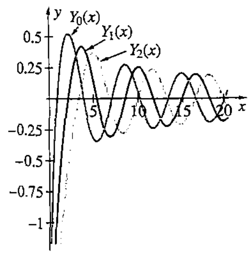


Figure 4.24

A more general form of Bessel's equation is expressed in the form

$$x^2y''+xy'+(\lambda^2x^2-\mu^2)y=0.$$

Through a change of variables, we can show that a general solution of this parametric Bessel equation is

$$y=c_1J_\mu(\lambda x)+c_2Y_\mu(\lambda x).$$

**Example 5**

Find a general solution of each of the following equations:
**(a)** $x^2y''+xy'+(x^2-16)y=0$; **(b)** $x^2y''+xy'+(x^2-\frac{1}{25})y=0$;
**(c)** $x^2y''+xy'+(9x^2-4)y=0$.

**Solution** (a) In this case, $\mu = 4$. Hence, $y = c_1 J_4(x) + c_2 Y_4(x)$. We graph this solution on [1, 20] for various choices of the arbitrary constants in Figure 4.25(a). Notice that we must avoid graphing near $x = 0$ because of the behavior of $Y_4(x)$. (b) Because $\mu = \frac{1}{5}$, $y = c_1 J_{1/5}(x) + c_2 Y_{1/5}(x)$. This solution is graphed for several values of the arbitrary constants in Figure 4.25(b). (c) Using the parametric Bessel's equation with $\lambda = 3$ and $\mu = 2$, we have $y = c_1 J_2(3x) + c_2 Y_2(3x)$. We graph this solution for several choices of the arbitrary constants in Figure 4.25(c).

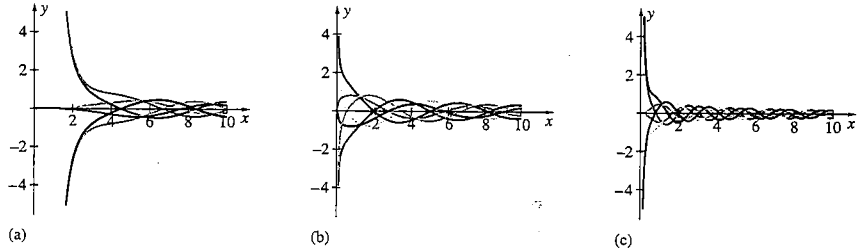



Figure 4.25 (a) $y = c_1 J_4(x) + c_2 Y_4(x)$ (b) $y = c_1 J_{1/5}(x) + c_2 Y_{1/5}(x)$ (c) $y = c_1 J_2(3x) + c_2 Y_2(3x)$

## EXERCISES 4.8

In Exercises 1 and 2, determine the singular points of the equations. Use these points to find an upper bound on the radius of convergence of a series solution about $x_0$.

1. $x^2 y'' - 2xy' + 7y = 0$, $x_0 = 1$
2. $(x - 2)y'' + y' - y = 0$, $x_0 = -2$

In Exercises 3–8, solve the differential equation with a power series expansion about $x = 0$. Write out at least the first five nonzero terms of each series.

3. $y'' - 11y' + 30y = 0$
4. $y'' + y = 0$
*5. $y'' - y' - 2y = e^{-x}$
6. $(1 + 3x)y'' - 3y' - 2y = 0$
7. $(2 + 3x)y'' + 3xy' = 0$
8. $(2 - x^2)y'' + 2(x - 1)y' + 4y = 0$

In Exercises 9 and 10, determine at least the first five nonzero terms in a power series expansion about $x = 0$ for the solution of each initial-value problem.

9. $y'' - 4x^2 y = 0$, $y(0) = 1$, $y'(0) = 0$
10. $(-1 + 2x^2)y'' + 2xy' - 3y = 0$, $y(0) = -2$, $y'(0) = 2$

In Exercises 11 and 12, solve the equation with a power series expansion about $t = 0$ by making the indicated change of variable.

11. $4xy'' + y' = 0$, $x = t + 1$
12. $4x^2 y'' + (x + 1)y' = 0$, $x = t + 2$

In Exercises 13 and 14, determine at least the first five nonzero terms in a power series expansion about $x = 0$ of the solution to each nonhomogeneous initial-value problem.



13. $y'' + xy' = \sin x$, $y(0) = 1$, $y'(0) = 0$
14. $y'' + y' + xy = \cos x$, $y(0) = 0$, $y'(0) = 1$
15. (a) If $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$, what are the first three nonzero terms of the power series for $y^2$? (b) Use this series to find the first three nonzero terms in the power series solution about $x = 0$ to **Van-der-Pol's equation,**
$$y'' + (y^2 - 1)y' + y = 0$$
if $y(0) = 0$ and $y'(0) = 1$.
*16. Use a method similar to that in Exercise 15 to find the first three nonzero terms of the power series solution about $x = 0$ to **Rayleigh's equation,**
$$y'' + \left(\frac{1}{3}(y')^2 - 1\right)y' + y = 0$$
if $y(0) = 1$ and $y'(0) = 0$, an equation that arises in the study of the motion of a violin string.

In Exercises 17 and 18, determine the singular point of each equation. In each case, classify the point as regular or irregular.

*17. $x(x + 1)y'' - \frac{1}{x^2}y' + 5y = 0$
18. $(x^2 - 25)^2 y'' - (x + 5)y' + 10y = 0$

In Exercises 19–24, use the Method of Frobenius to obtain two linearly independent solutions about the regular singular point $x = 0$.

*19. $2xy'' - 5y' - 3y = 0$
20. $9xy'' + 14y' + (x - 1)y = 0$
21. $x^2 y'' + xy' + (x - 1)y = 0$
22. $y'' + \frac{8}{3x}y' - \left(\frac{2}{3x^2} - 1\right)y = 0$
*23. $y'' + \left(\frac{1}{2x} - 2\right)y' - \frac{35}{16x^2}y = 0$
24. $y'' - \left(\frac{1}{x} + 2\right)y' + \left(\frac{1}{x^2} + x\right)y = 0$

In Exercises 25 and 26, solve the differential equation with a series expansion about $x = 0$. Compare these results with the solution obtained by solving the problem as a Cauchy-Euler equation.

25. $x^2 y'' + 7xy' - 7y = 0$
26. $x^2 y'' + 3xy' + y = 0$
27. The differential equation $d^2y/dx^2 + p(x)(dy/dx) + q(x)y = 0$ has a **singular point at infinity** if after substitution of $w = 1/x$ the resulting equation has a singular point at $w = 0$. Similarly, the equation has an **ordinary point at infinity** if the transformed equation has an ordinary point at $w = 0$. Use the chain rule and the substitution $w = 1/x$ to show that the differential equation $d^2y/dx^2 + p(x)(dy/dx) + q(x)y = 0$ is equivalent to
$$\frac{d^2y}{dw^2} + \left(\frac{2}{w} + \frac{p(1/w)}{w^2}\right)\frac{dy}{dw} + \frac{q(1/w)}{w^4}y = 0.$$
28. Use the definition in Exercise 42 to determine if infinity is an ordinary point or a singular point of the given differential equation.
(a) $d^2y/dx^2 + xy = 0$
(b) $x^2\, d^2y/dx^2 + x\, dy/dx + (x^2 - n^2)y = 0$
(c) $(1 - x^2)\, d^2y/dx^2 - 2x\, dy/dx + n(n + 1)y = 0$
29. **Hermite's equation** is given by
$$y'' - 2xy' + 2ky = 0,\ k \ge 0.$$
Using a power series expansion about the ordinary point $x = 0$, obtain a general solution of this equation for (a) $k = 1$ and (b) $k = 3$. Show that if $k$ is a nonnegative integer, then one of the solutions is a polynomial of degree $k$.
30. **Chebyshev's equation** is given by
$$(1 - x^2)y'' - xy' + k^2 y = 0,\ n \ge 0.$$
Using a power series expansion about the ordinary point $x = 0$, obtain a general solution of this equation for (a) $k = 1$ and (b) $k = 3$. Show that if $k$ is a nonnegative integer, then one of the solutions is a polynomial of degree $k$.
31. The **hypergeometric equation** is given by
$$x(1 - x)y'' + [c - (a + b + 1)x]y' - aby = 0.$$
where $a$, $b$, and $c$ are constants. (a) Show that $x = 0$ and $x = 1$ are regular singular points. (b) Show that the roots of the indicial equation for the series $\sum_{n=0}^{\infty} a_n x^{n+r}$ are $r = 0$ and $r = 1 - c$. (c) Show that for $r = 0$, the solution obtained with the Method of Frobenius is
$$y_1(x) = 1 + \frac{ab}{1!c}x + \frac{a(a+1)b(b+1)}{2!c(c+1)}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)}x^3 + \cdots$$

where $c \neq 0, -1, -2, \ldots$ . This series is called the **hypergeometric series.** Its sum, denoted $F(a, b, c; x)$, is called the **hypergeometric function.** **(d)** Show that $F(1, b, b; x) = 1/(1 - x)$. **(e)** Find the solution that corresponds to $r = 1 - c$.

32. **Laguerre's equation** is given by

$$xy'' + (1 - x)y' + ky = 0.$$

**(a)** Show that $x = 0$ is a regular singular point of Laguerre's equation. **(b)** Use the Method of Frobenius to determine one solution of Laguerre's equation. **(c)** Show that if $k$ is a positive integer, then the solution is a polynomial. This polynomial, denoted $L_k(x)$, is called the Laguerre polynomial of order $k$.

*33. Show that Legendre's equation can be written as $\dfrac{d}{dx}[(1 - x^2)y'] + k(k + 1)y = 0$. Use this equation to show that $P_m(x)$ and $P_n(x)$ are **orthogonal** on the interval $[-1, 1]$ by showing that

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0,\ m \neq n.$$

*Hint:* $P_m(x)$ and $P_n(x)$ satisfy the differential equations $\dfrac{d}{dx}[(1 - x^2)P_n'(x)] + n(n + 1)P_n(x) = 0$ and $\dfrac{d}{dx}[(1 - x^2)P_m'(x)] + m(m + 1)P_m(x) = 0$, respectively. Multiply the first equation by $P_n(x)$ and the second by $P_m(x)$, and subtract the results. Then, integrate from $-1$ to 1.

34. **(Relationships among Bessel functions) (a)** Using $J_\mu(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1 + \mu + n)}\left(\frac{x}{2}\right)^{2n+\mu}$ and $\Gamma(x + 1) = x\Gamma(x)$, show that $\dfrac{d}{dx}[x^\mu J_\mu(x)] = x^\mu J_{\mu-1}(x)$.
   **(b)** Using $J_\mu(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1 + \mu + n)}\left(\frac{x}{2}\right)^{2n+\mu}$, show that $\dfrac{d}{dx}[s^{-\mu}J_\mu(x)] = -x^{-\mu}J_{\mu+1}(x)$.
   **(c)** Using the results of parts **(a)** and **(b)**, show that $J_{\mu-1}(x) - J_{\mu+1}(x) = 2J'_\mu(x)$.
   **(d)** Evaluate $\int x^\mu J_{\mu-1}(x)\,dx$.

35. Show that $y = J_0(kx)$ where $k$ is a constant is a solution of the parametric Bessel equation of order zero, $xy'' + y' + k^2xy = 0$.

36. Show that the equation $xy'' + y' + k^2xy = 0$ is equivalent to $\dfrac{d}{dx}(xy') + k^2xy = 0$. Show that $\displaystyle\int_0^1 xJ_0(k_mx)J_0(k_nx)\,dx = 0,\ m \neq n$. (This shows that $J_0(k_mx)$ and $J_0(k_nx)$ are *orthogonal* on the interval [0, 1].) *Hint:* Follow a procedure like that described in Exercise 33. Assume that $J_0(k_m) = J_0(k_n) = 0$.

37. Find a general solution of each equation.
   **(a)** $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$
   **(b)** $x^2y'' + xy' + (16x^2 - 25)y = 0$

In Exercises 38 and 39, solve each hypergeometric equation. Express all solutions in terms of the function $F(a, b, c; x)$. (Notice that when either $a$ or $b$ is a negative integer, the solution is a polynomial.) (See Exercise 31.)

38. $x(1 - x)y'' + y' + 2y = 0$

*39. $x(1 - x)y'' + (1 - 2x)y' = 0$

40. Use the power series of expansion of the Bessel function of the first kind of order $n$ ($n$ an integer),

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n + m)!}\left(\frac{x}{2}\right)^{2n+m}$$

to verify that $J_0'(x) = -J_1(x)$.

41. Use the change of variables $y = \dfrac{v(x)}{\sqrt{x}}$ to transform Bessel's equation $x^2y'' + xy' + (x^2 - n^2)y = 0$ into the equation $v'' + \left[1 + \dfrac{\frac{1}{4} - n^2}{x^2}\right]v = 0$. By substituting $n = \frac{1}{2}$ into the transformed equation, derive the solution to Bessel's equation with $n = 1/2$.

42. Show that a solution of $x(d^2y/dx^2) + dy/dx + y/v = 0$ is $y = J_0(2\sqrt{x/v})$ where $\nu$ is a constant.

*43. Use integration by parts to show that $\Gamma(p + 1) = p\Gamma(p)$, $p > 0$. *Note:* $p$ is any real number.

44. **(a)** Show that $\Gamma(\frac{1}{2}) = 2\displaystyle\int_0^\infty e^{-x2}\,dx$. (*Hint:* Let $u = x^2$.) **(b)** Use polar coordinates to evaluate $\left(\displaystyle\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right) = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$.
   **(c)** Use the results of **(a)** and **(b)** to evaluate $\Gamma(\frac{1}{2})$ and then $\Gamma(\frac{3}{2})$.

45. Consider the initial-value problem $y'' + f(x)y' + y = 0$, $y(0) = 1, y'(0) = -1$, where $f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$. **(a)** Show that $x = 0$ is an ordinary point of the equation. **(b)** Find a power series solution of the equation and graph an approximation of the solution on an interval. **(c)** Generate a numerical solution of the equation. Explain any unexpected results.



46. **(a)** Use power series to solve $\begin{cases} y'' - (\cos x)y = \sin x \\ y(0) = 1, y'(0) = 0 \end{cases}$.
   **(b)** Compare the polynomial approximations of degree 4, 7, 10, and 13 to the numerical solution obtained with a computer algebra system.

47. The differential equation $y'' - xy = 0$ is called **Airy's equation** and arises in electromagnetic theory and quantum mechanics. Two linearly independent solutions to Airy's equation, denoted by $Ai(x)$ and $Bi(x)$, are called the **Airy functions.** The function $Ai(x) \to 0$ as $x \to \infty$ while $Bi(x) \to +\infty$ as $x \to \infty$. (Most computer algebra systems contain built-in definitions of the Airy functions.)

   **(a)** If your computer algebra system contains built-in definitions of the Airy functions, graph each on the interval $[-15, 5]$. **(b)** Find a series solution of Airy's equation and obtain formulas for both $Ai(x)$ and $Bi(x)$. **(c)** Graph the polynomial approximation of degree $n$ of $Ai(x)$ for $n = 6$, 15, 30, and 45 on the interval $[-15, 5]$. *Compare your results to* ***(a)****, if applicable.* **(d)** Graph the polynomial approximation of degree $n$ of $Bi(x)$ for $n = 6$, 15, 30, and 45 on the interval $[-15, 5]$. *Compare your results to* ***(a)****, if applicable.*

48. **(a)** Use a series solution to approximate the solution to the initial-value problem

$$xy'' + (2 - x)y' + (1/4x)y = 0,\quad y(1) = 1, y'(1) = -1.$$

   **(b)** Use a computer algebra system to generate a numerical solution of the initial-value problem and compare these results with those obtained in **(a)** by graphing the two approximations simultaneously.

49. **Laguerre's equation** is $xy'' + (1 - x)y' + ny = 0$. **(a)** Show that $x = 0$ is a regular singular point of Laguerre's equation. **(b)** Use the Method of Frobenius to show that one solution of Laguerre's equation is $L_n(x) = \displaystyle\sum_{m=0}^{n} \frac{(-1)^m}{m!}\frac{n!}{m!(n - m)!}x^m$, where $L_n(x)$ is called the **Laguerre polynomial of order** $\boldsymbol{n}$. **(c)** Calculate the first eight Laguerre polynomials with this formula (or find them using a built-in computer algebra system command). **(d)** Show that the Laguerre polynomials satisfy the formula $L_n(x) = \dfrac{e^x}{n!}\dfrac{d^n(x^ne^{-x})}{dx^n}$ for $n = 1, 2, 3, \ldots, 8$. **(e)** Show that $\int_0^\infty e^{-x}L_n(x)L_m(x)\,dx = 0$ for $n \neq m$, $n, m = 1, 2, 3, \ldots, 8$. (This indicates that the Laguerre polynomials are **orthogonal.**) **(f)** Determine the value of $\int_0^\infty e^{-x}[L_n(x)]^2\,dx$ by experimenting with $n = 1, 2, 3, \ldots, 8$. *Note:* All of the properties mentioned in this problem hold for all $n$. We simply chose the values $n = 1, 2, 3, \ldots, 8$ so that the calculations can be carried out with technology instead of by hand.

50. The values of $x$ that satisfy the equation $J_0(x) = 0$ are useful in many applications in applied mathematics. **(a)** Approximate the first ten zeros (or roots) of the Bessel function of the first kind of order zero, $J_0(x)$, which is graphed in Figure 4.26. **(b)** Approximate the first nine zeros of $J_\mu(x)$ for $\mu = 1, 2, \ldots, 8$.

51. **(a)** Verify that the Legendre polynomials given in Table 4.4 satisfy the relationship

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0,\ m \neq n$$

   (called an **orthogonality condition**).
   **(b)** Evaluate $\displaystyle\int_{-1}^{1}[P_n(x)]^2\,dx$ for $n = 0, 1, \ldots, 5$.
   How do these results compare to the value of $2/(2n + 1)$, for $n = 0, 1, \ldots, 5$?

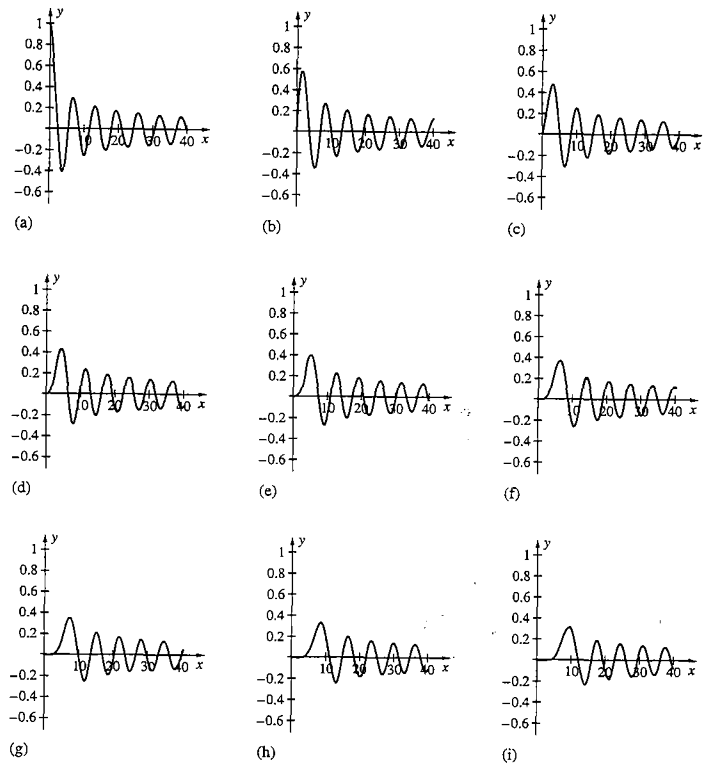


**Figure 4.26** (a) $J_0(x)$ (b) $J_1(x)$ (c) $J_2(x)$ (d) $J_3(x)$ (e) $J_4(x)$ (f) $J_5(x)$ (g) $J_6(x)$ (h) $J_7(x)$ (i) $J_8(x)$



# *CHAPTER 4 SUMMARY*

## *Concepts & Formulas*

### Section 4.1

**Second-order ordinary linear differential equation**

$$a_2(t)y^{(2)}(t) + a_1(t)y'(t) + a_0(t)y(t) = F(t)$$

**Homogeneous**

$$a_2(t)y^{(2)}(t) + a_1(t)y'(t) + a_0(t)y(t) = 0$$

**Constant coefficients**

$$a_2y^{(2)}(t) + a_1y'(t) + a_0y(t) = F(t)$$

**Linearly dependent and independent set of functions**
If $S = \{f_1(t), f_2(t)\}$, $S$ is **linearly dependent** if there are constants $c_1$ and $c_2$ not both zero, so that

$$c_1f_1(t) + c_2f_2(t) = 0.$$

$S$ is **linearly independent** if $S$ is not linearly dependent.

**Wronskian**
The **Wronskian** of

$$S = \{f_1(t), f_2(t)\},$$

is the determinant

$$W(S) = \begin{vmatrix} f_1(t) & f_2(t) \\ f_1'(t) & f_2'(t) \end{vmatrix}.$$

If $W(S) \neq 0$ for at least one value of $t$ in the interval $I$, $S$ is linearly independent.

**Principle of Superposition**
Any linear combination of a set of solutions of the second-order linear homogeneous equation is also a solution.

**Fundamental set of solutions**
A set $S = \{f_1(t), f_2(t)\}$ of two linearly independent solutions of the second-order linear homogeneous equation. Every second-order linear homogeneous equation has a fundamental set of solutions from which a general solution can be obtained.

**General solution of a homogeneous equation**
If $S = \{f_1(t), f_2(t)\}$ is a fundamental set of solutions of the second-order linear homogeneous equation

$$a_2(t)y^{(2)}(t) + a_1(t)y'(t) + a_0(t)y(t) = 0,$$

a **general solution** of the equation is

$$f(t) = c_1f_1(t) + c_2f_2(t)$$

where $\{c_1, c_2\}$ is a set of two arbitrary constants.

**Reduction of Order**
If $y = f(t)$ is one solution of $y'' + p(t)y' + q(t)y = 0$, a second linearly independent solution is $y = f(t)\,v(t)$, where

$$v(t) = \int \frac{1}{[f(t)]^2} e^{-\int p(t)\,dt}\,dt.$$

### Section 4.2

**Characteristic equation**
The equation

$$ar^2 + br + c = 0$$

is the **characteristic equation** of the second-order linear homogeneous differential equation with constant coefficients

$$ay'' + by' + cy = 0.$$

**Solving second-order homogeneous linear equations with constant coefficients**
If $ay'' + by' + cy = 0$ and $r_1$ and $r_2$ are the solutions of the characteristic equation $ar^2 + br + c = 0$, then a general solution is

(a) $y = c_1e^{r_1t} + c_2e^{r_2t}$, if $r_1 \neq r_2$ and both $r_1$ and $r_2$ are real;
(b) $y = y = c_1e^{r_1t} + c_2te^{r_1t}$, if $r_1 = r_2$; and
(c) $y = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$, if $r_1 = \alpha + i\beta$, $\beta \neq 0$ and $r_2 = \bar{r}_1 = \alpha - i\beta$.

### Section 4.3

***n*th-order ordinary linear differential equation**
$a_n(t)y^{(n)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = g(t)$

**Homogeneous**
$a_n(t)y^{(n)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = 0$

**Constant coefficients**
$a_ny^{(n)}(t) + \cdots + a_1y'(t) + a_0y(t) = g(t)$

**Linearly dependent and independent set of functions**
If $S = \{f_1(t), f_2(t), f_3(t), \ldots, f_{n-1}(t), f_n(t)\}$, $S$ is **linearly dependent** if there are constants $c_1, c_2, \ldots, c_n$, not all zero, so that

$$c_1f_1(t) + c_2f_2(t) + \cdots + c_nf_n(t) = 0.$$

$S$ is **linearly independent** if $S$ is not linearly dependent.

**Wronskian**
The **Wronskian** of

$$S = \{f_1(t), f_2(t), f_3(t) + \cdots + f_{n-1}(t), f_n(t)\},$$

is the determinant

$$W(S) = \begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{vmatrix}.$$

If $W(S) \neq 0$ for at least one value of $t$ in the interval $I$, $S$ is linearly independent

**Principle of Superposition**
Any linear combination of a set of solutions of the $n$th-order linear homogeneous equation is also a solution.

**Fundamental set of solutions**
A set $S = \{f_1(t), f_2(t), f_3(t), \ldots, f_{n-1}(t), f_n(t)\}$ of $n$ linearly independent solutions of the $n$th-order linear homogeneous equation. Every $n$th-order linear homogeneous equation has a fundamental set of solutions from which a general solution can be obtained.

**General solution of a homogeneous equation**
If $S = \{f_1(t), f_2(t), \ldots, f_{n-1}(t), f_n(t)\}$ is a fundamental set of solutions of the $n$th-order linear homogeneous equation

$$a_n(t)y^{(n)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = 0,$$

a **general solution** of the equation is

$$f(t) = c_1f_1(t) + c_2f_2(t) + \cdots + c_{n-1}f_{n-1}(t) + c_nf_n(t),$$

where $\{c_1, c_2, \ldots, c_{n-1}, c_n\}$ is a set of $n$ arbitrary constants.

## Section 4.4

**Characteristic equation**
The equation

$$a_nr^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0 = 0$$

is the **characteristic equation** of the $n$th-order linear homogeneous differential equation with constant coefficients

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0.$$

**Solving higher-order homogeneous linear equations with constant coefficients**

1. Let $r$ be a real root of the characteristic equation

$$a_nr^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0 = 0$$

of an $n$th-order homogeneous linear differential equation with real constant coefficients. Then, $e^{rt}$ is the solution associated with the root $r$.

If $r$ is a real root of multiplicy $k$ where $k \geq 2$ of the characteristic equation, then the $k$ solutions associated with $r$ are

$$e^{rt}, te^{rt}, t^2e^{rt}, \ldots, t^{k-1}e^{rt}.$$

2. Suppose that $r$ and $\bar{r}$ represent the complex conjugate pair $\alpha \pm \beta i$. Then, the two solutions associated with these two roots are

$$e^{\alpha t}\cos\beta t \quad \text{and} \quad e^{\alpha t}\sin\beta t.$$

If the values $\alpha \pm \beta i$ are each a root of multiplicity $k$ of the characteristic equation, then the other solutions associated with this pair are

$$te^{\alpha t}\cos\beta t,\ te^{\alpha t}\sin\beta t,\ t^2e^{\alpha t}\cos\beta t,\ t^2e^{\alpha t}\sin\beta t, \ldots, t^{k-1}e^{\alpha t}\cos\beta t,\ t^{k-1}e^{\alpha t}\sin\beta t.$$

A general solution to the $n$th-order differential equation is the linear combination of the solutions obtained for all values of $r$. Note that if $r_1, r_2, \ldots, r_j$ are the roots of the equation of multiplicity $k_1, k_2, \ldots, k_j$, respectively, then $k_1 + k_2 + \cdots + k_j = n$, where $n$ is the order of the differential equation.

## Section 4.5

**Particular solution**
A **particular solution,** $y_p(t)$, of the differential equation $a_ny^{(n)}(t) + \cdots + a_1y'(t) + a_0y(t) = g(t)$ is a specific function, containing no arbitrary constants, that satisfies the equation.

**General solution of a nonhomogeneous equation**
A general solution to the nonhomogeneous equation is

$$y(t) = y_h(t) + y_p(t),$$



where $y_h(t)$ is a general solution of the corresponding homogeneous equation

$$a_ny^{(n)}(t) + \cdots + a_1y'(t) + a_0y(t) = 0,$$

and $y_p(t)$ is a particular solution of the nonhomogeneous equation.

**Method of undetermined coefficients**
Method used to find a particular solution:

1. If $g_i(t) = t^m$, the associated set of functions is

$$S = \{t^m, t^{m-1}, \ldots, t^2, t - 1\}.$$

2. If $g_i(t) = t^me^{kt}$, the associated set of functions is

$$S = \{t^me^{kt}, t^{m-1}e^{kt}, \ldots, t^2e^{kt}, te^{kt}, e^{kt}\}.$$

3. If $g_i(t) = t^me^{\alpha t}\cos\beta t$, or $g_i(t) = t^me^{\alpha t}\sin\beta t$, the associated set of functions is

$$S = \{t^me^{\alpha t}\cos\beta t, \ldots, te^{\alpha t}\cos\beta t, e^{\alpha t}\cos\beta t, t^me^{\alpha t}\sin\beta t, \ldots, te^{\alpha t}\sin\beta t, e^{\alpha t}\sin\beta t].$$

For each function in $S$, determine the associated set of functions. If any of the functions in $S$ appear in the homogeneous solution $y_h(t)$, multiply each function in $S$ by $t^r$ to obtain a new set $S'$ ($r$ is the smallest positive integer so that each function in $S'$ is not a function in $y_h(t)$). A particular solution is obtained by taking the linear combination of all functions in the associated sets where repeated functions should appear only once in the particular solution.

## Section 4.6

**Variation of parameters**
A particular solution of $y'' + p(t)y' + q(t)y = f(t)$ is

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

where $u_1(t) = \int \frac{-y_2(t)f(t)}{W(S)}\,dt$ and $u_2(t) = \int \frac{y_1(t)f(t)}{W(S)}\,dt$;
$S = \{y_1(t), y_2(t)\}$ is a fundamental set of solutions of the corresponding homogeneous equation.

## Section 4.7

**Cauchy-Euler Equation**
A **Cauchy-Euler** differential equation is an equation of the form

$$a_nx^ny^{(n)} + \cdots + a_1xy' + a_0y = g(x),$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants.

**General solution of a second-order Cauchy-Euler equation**
$y = c_1x^{m_1} + c_2x^{m_2}$, if $m_1 \neq m_2$ are real;
$y = c_1x^{m_1} + c_2x^{m_1}\ln x$, if $m_1 = m_2$; and
$y = x^{\alpha}[c_1\cos(\beta\ln x) + c_2\sin(\beta\ln x)]$ if $m_1 = \overline{m_2} = \alpha + i\beta$, $\beta \neq 0$.

**Nonhomogeneous Cauchy-Euler Equations**

**Higher-order Cauchy-Euler equations**

## Section 4.8

**Ordinary and singular points**
$x_0$ is an **ordinary point** of $y'' + p(x)y' + q(x)y = 0$ if both $p(x)$ and $q(x)$ are analytic at $x_0$. If $x_0$ is not an ordinary point, $x_0$ is called a **singular point.**

**Power Series Solution method about an ordinary point**

1. Assume that $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$.
2. After taking the appropriate derivatives, substitute $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ into the differential equation.
3. Find the unknown series coefficients $a_n$ through an equation relating the coefficients.
4. Apply any given initial conditions, if applicable.

**Convergence of the power series solution**

**Legendre's equation**

$$(1 - x^2)y'' - 2xy' + k(k+1)y = 0$$

**Regular and irregular singular points**
Let $x_0$ be a singular point of $y'' + p(x)y' + q(x)y = 0$. $x_0$ is a **regular singular point** of the equation if both $(x - x_0)p(x)$ and $(x - x_0)^2q(x)$ are analytic at $x_0$. If $x_0$ is not a regular singular point, $x_0$ is called an **irregular singular point** of the equation.

**Method of Frobenius**

**Indicial equation**
$r(r-1) + p_0r + q_0 = 0$

**Gamma function**
$\Gamma(x) = \int_0^{\infty} e^{-u}u^{x-1}\,du, x > 0$

**Bessel's equation**
$x^2y'' + xy' + (x^2 - \mu^2)y = 0$

## CHAPTER 4 REVIEW EXERCISES

In Exercises 1–6, determine if the given set is linearly independent or linearly dependent.

1. $S = \{e^{5t}, 1\}$
2. $S = \{\cos^2 t, \sin^2 t\}$
*3. $S = \{t, t \ln t\}$
4. $S = \{t, t-1, 3t\}$
5. $S = \{t, \cos t, \sin t\}$
6. $S = \{e^t, e^{-2t}, e^{-t}\}$

In Exercises 7–9, verify that $y$ is a general solution of the given differential equation.

7. $y'' - 7y' + 10y = 0$; $y = c_1e^{5t} + c_2e^{2t}$
8. $y'' - y' - 2y = 0$; $y = c_1e^{2t} + c_2e^{-t}$
9. $y'' - 2y' + 2y = 0$; $y = e^t(c_1 \sin t + c_2 \cos t)$

In Exercises 10 and 11, show that the function $y_1(t)$ satisfies the differential equation and find a second linearly independent solution.

10. $y_1(x) = t + 1$, $y'' - \dfrac{2}{t+1}y' + \dfrac{2}{(t+1)^2}y = 0$
11. $y_1(x) = \dfrac{\sin t}{t}$, $y'' + \dfrac{2}{t}y' + y = 0$

In Exercises 12–37, find a general solution for each equation.

12. $y'' + 7y' + 10y = 0$
*13. $6y'' + 5y' - 4y = 0$
14. $y'' + 2y' + y = 0$
*15. $y'' + 3y' + 2y = 0$
16. $y'' - 10y' + 34y = 0$
*17. $2y'' - 5y' + 2y = 0$
18. $15y'' - 11y' + 2y = 0$
*19. $20y'' + y' - y = 0$
20. $12y'' + 8y' + y = 0$
*21. $2y''' + 3y'' + y' = 0$
22. $9y'' + 36y' + 40y = 0$
*23. $9y'' + 12y' + 13y = 0$
24. $y'' - 2y' - 8y = -t$
*25. $y'' + 5y' = 5t^2$
26. $y'' - 4y' = -3 \sin t$
*27. $y'' + 2y' + 5y = 3 \sin 2t$
28. $y'' - 9y = \cos 3t$
29. $y'' - 2y' = 2 \cos 4t$
30. $y'' - 3y' + 2y = -4e^{-2t}$
*31. $y'' - 6y' + 13y = 3e^{-2t}$
32. $y'' + 9y' + 20y = -2te^t$
33. $y'' + 7y' + 12y = 3t^2e^{-4t}$
34. $y''' + 3y'' - 9y' + 5y = e^t$
*35. $y''' - 12y' - 16y = e^{4t} - e^{-2t}$
36. $y^{(4)} + 6y''' + 18y'' + 30y' + 25y = e^{-t} \cos 2t + e^{-2t} \sin t$
37. $y^{(4)} + 4y''' + 14y'' + 20y' + 25y = t^2$

In Exercises 38–45, solve the initial-value problem.

38. $y'' + 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 0$
39. $y'' + 10y' + 16y = 0$, $y(0) = 0$, $y'(0) = 4$
40. $y'' + 16y = 0$, $y(0) = 0$, $y'(0) = -8$
*41. $y'' + 25y = 0$, $y(0) = 1$, $y'(0) = 0$
42. $y'' - 4y = x$, $y(0) = 2$, $y'(0) = 0$
43. $y'' + 3y' - 4y = e^t$, $y(0) = 0$, $y'(0) = 4$
44. $y'' + 9y = \sin 3t$, $y(0) = 6$, $y'(0) = 0$
45. $y'' + y = \cos t$, $y(0) = 0$, $y'(0) = 0$

In Exercises 46–51, use variation of parameters to solve the indicated differential equations and initial-value problems.

46. $y'' + 4y = \tan 2t$
47. $y'' + y = \csc t$
48. $y'' - 8y' + 16y = t^{-3}e^{4t}$
*49. $y'' - 8y' + 16y = t^{-3}e^{4t}$, $y(1) = 0$, $y'(1) = 0$
50. $y'' - 2y' + y = e^t \ln t$
51. $y'' - 2y' + y = e^t \ln t$, $y(1) = 0$, $y'(1) = 0$
52. Show that the substitution $u = y'/y$ converts the equation $p_0(t)y'' + p_1(t)y' + p_2(t)y = 0$ to $p_0(t)(u' + u^2) + p_1(t)u + p_2(t) = 0$.



53. Use the substitution in the previous problem to solve $y'' - 2ty' + t^2y = 0$. *Hint:* You should obtain the differential equation $du/dt = -(u-t)^2$. Make the substitution $v = u - t$ to solve this equation.
*54. **(Abel's formula)** Suppose that $y_1$ and $y_2$ are two solutions of $y'' + p(t)y' + q(t)y = 0$ on an interval $I$ where $p(t)$ and $q(t)$ are continuous functions. Then, the Wronskian of $y_1$ and $y_2$ is
$$W(y_1, y_2)(x) = Ce^{-\int p(t)dt}.$$
Prove Abel's formula by computing $\dfrac{d}{dt}W(y_1, y_2)(t)$ and using the relationship $y'' = -p(t)y' - q(t)y$ for $y_i''$ to obtain a first-order ordinary differential equation for $W$.
55. Can the Wronskian be zero at only one value of $x$ on $I$? *Hint:* Use Abel's formula.
56. Use Abel's formula to find the Wronskian (within a constant multiple) associated with the following differential equations. Also, obtain a fundamental set of solutions and compute the Wronskian directly. Compare the results.
   (a) $y'' + 3y' - 4y = 0$
   *(b) $y'' + 4y' + 13y = 0$
   (c) $y'' + 4y' + 4y = 0$
   (d) $y'' + 9y = 0$
57. Use Abel's formula to find the Wronskian (within a constant multiple) associated with the following differential equations.
   (a) $ty'' + y' + ty = 0$
   (b) $t^2y'' - 5ty' + 5y = 0$
   (c) $t^2y'' - ty' + 5y = 0$
58. Solve each of the following boundary-value problems.
   (a) $\begin{cases} y'' - y = 0 \\ y'(0) + 3y(0) = 0, y'(1) + y(1) = 1 \end{cases}$
   *(b) $\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0, y(p) = 0 \end{cases}, p > 0$
   *Hint:* Consider three cases: $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.
   (c) $\begin{cases} y'' + \lambda y = 0 \\ y'(0) = 0, y'(p) = 0 \end{cases}, p > 0$
   *Hint:* Consider three cases: $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.
   (d) $\begin{cases} y'' + 2y' - (\lambda - 1)y = 0 \\ y(0) = 0, y(2) = 0 \end{cases}$
   *Hint:* Consider three cases: $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.

In Exercises 59–65, solve the Cauchy-Euler equation.

59. $x^2y'' - 4xy' + 6y = 0$
*60. $x^2y'' + 7xy' + 8y = 0$
61. $\begin{cases} 2x^2y'' + 5xy' + y = 0 \\ y(1) = 1, y'(1) = 0 \end{cases}$
*62. $x^2y'' + xy' + y = 0$
63. $x^2y'' + 7xy' + 25y = 0$
*64. $5x^2y'' - xy' + 2y = 0$
*65. $x^2y'' - 7xy' + 15y = 8x$

In Exercises 66–68, solve the differential equation with a series expansion about $x = 0$. Write out at least the first five nonzero terms of each series.

66. $y'' - 4y' + 4y = 0$
*67. $y'' + 2y' - 3y = xe^x$
68. $(-1 + 2x^2)y'' + 2xy' - 3y = 0$

In Exercises 69–71, use the Method of Frobenius to obtain two linearly independent solutions about the regular singular point $x = 0$.

69. $3xy'' + 11y' - y = 0$
*70. $2x^2y'' + 5xy' - 2y = 0$
71. $y'' - \dfrac{7}{x}y' + \left(\dfrac{7}{x^2} - 2\right)y = 0$

In Exercises 72 and 73, find a solution to each hypergeometric equation. Express the solution in terms of the function $F(a, b, c; x)$. (See Exercise 31 in Section 4.8).

72. $x(1-x)y'' + (1 + 2x)y' + 10y = 0$
73. $x(1+x)y'' + \left(\dfrac{1}{3} - 12x\right)y' - 10y = 0$
74. **(Simple Modes of a Vibrating Chain)** The equation that describes the simple modes of a vibrating chain of length $\ell$ is
$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{y}{v} = 0,$$

where $y$ is the displacement and $x$ is the distance from the bottom of the chain. This equation was studied extensively by Daniell Bernoulli around 1727.

**(a)** If the chain is fixed at the top so that $y(\ell) = 0$ and $y(0) = 1$, show that a solution to this equation is $J_0(2\sqrt{x/v})$.

**(b)** Convince yourself that for any value of $\ell$, the equation $J_0(2\sqrt{\ell/v}) = 0$ has infinitely many solutions.

**(c)** If $\ell = 1$, graph $J_0(2\sqrt{\ell/v})$ and approximate the last ten solutions of $J_0(2\sqrt{\ell/v}) = 0$.

## Differential Equations at Work:

### A. Testing for Diabetes

Diabetes mellitus affects approximately 12 million Americans; approximately one-half of these people are unaware that they have diabetes. Diabetes is a serious disease: it is the leading cause of blindness in adults and the leading cause of renal failure, and it is responsible for about one-half of all nontraumatic amputations in the United States. In addition, people with diabetes have an increased rate of coronary artery disease and strokes. People at risk for developing diabetes include those who are obese; those suffering from excessive thirst, hunger, urination, and weight loss; women who have given birth to a baby of weight greater than 9 lb; those with a family history of diabetes; and those who are over 40 years of age.

People with diabetes cannot metabolize glucose because their pancreas produces an inadequate or ineffective supply of insulin. Subsequently, glucose levels rise. The body attempts to remove the excess glucose through the kidneys; the glucose acts as a diuretic, resulting in increased water consumption. Since some cells require energy, which is not being provided by glucose, fat, and protein are broken down and ketone levels rise. Although there is no cure for diabetes at this time, many cases can be effectively managed by a balanced diet and insulin therapy in addition to maintaining an optimal weight.

Diabetes can be diagnosed by several tests. In the **fasting blood sugar** test, a patient fasts for at least four hours and the glucose level is measured. In a fasting state, the glucose level in normal adults ranges from 70–110 mg/mL. An adult in a fasting state with consistent readings of over 150 mg probably has diabetes. However, individuals vary greatly, so people with mild cases of diabetes might have fasting state glucose levels within the normal range. A highly accurate test that is frequently used to diagnose mild diabetes is the **glucose tolerance test** (GTT), which was developed by Rosevear and Molnar of the Mayo Clinic and Ackerman and Gatewood of the University of Minnesota. During the GTT, a blood and urine sample are taken from a patient in a fasting state to measure the glucose ($G_0$), hormone ($H_0$), and glycosuria levels, respectively. We assume that these values are equilibrium values. The patient is then given 100 g of glucose. Blood and urine samples are then taken at 1-, 2-, 3-, and 4-hour intervals. In a person without diabetes, glucose levels return to normal after two hours; in diabetics the blood sugar levels either take longer or never return to normal levels.



Let $G$ denote the cumulative level of glucose in the blood, $g = G - G_0$, $H$ the cumulative level of hormones that affect insulin production (such as glucagon, epinephrine, cortisone, and thyroxin), and $h = H - H_0$. Notice that $g$ and $h$ represent the fluctuation of the cumulative levels of glucose and hormones from their equilibrium values. The relationship between the rate of change of glucose in the blood and the rate of change of the cumulative levels of the hormones in the blood which affect insulin production is

$$\begin{cases} g' = f_1(g, h) + J(t) \\ h' = f_2(g, h) \end{cases},$$

where $J(t)$ represents the **external** rate at which the blood glucose concentration is being increased.* If we assume that $f_1$ and $f_2$ are linear functions, then this system of equations becomes

$$\begin{cases} g' = -ag - bh + J(t) \\ h' = -ch + dg \end{cases},$$

where $a$, $b$, $c$, and $d$ represent positive numbers.

**(a)** Show that if $g' = -ag - bh + J(t)$ then

$$h = \frac{1}{b}(-g' - ag + J) \quad \text{and} \quad h' = \frac{1}{b}(-g'' - ag' + J').$$

**(b)** Substitute $h = (1/b)(-g' - ag + J)$ and $h' = (1/b)(-g'' - ag' + J')$ into $h' = -ch + dg$ to obtain the second-order equation

$$\frac{1}{b}(-g'' - ag' + J') = -\frac{c}{b}(-g' - ag + J) + dg$$

$$g'' + (a + c)g' + (ac + bd)g = J' + cJ.$$

Since the glucose solution is consumed at $t = 0$, for $t > 0$ we have that

$$g'' + (a + c)g' + (ac + bd)g = 0.$$

**(c)** Show that the solutions of the characteristic equation of $g'' + (a + c)g' + (ac + bd)g = 0$ are

$$\frac{1}{2}\left(-a - c - \sqrt{(a - c)^2 - 4bd}\right) \quad \text{and} \quad \frac{1}{2}\left(-a - c + \sqrt{(a - c)^2 - 4bd}\right).$$

**(d)** Explain why it is reasonable to assume that glucose levels are periodic and subsequently that $(a - c)^2 - 4bd < 0$.

**(e)** If $(a - c)^2 - 4bd < 0$ and $t > 0$, show that a general solution of

$$g'' + (a + c)g' + (ac + bd)g = 0$$

is

* D. N. Burghess and M. S. Borrie, *Modeling with Differential Equations,* Ellis Horwood Limited, pp. 113–116. Joyce M. Black and Esther Matassarin-Jacobs, *Luckman and Sorensen's Medical-Surgical Nursing: A Psychophysiologic Approach,* Fourth Edition, W. B. Saunders Co., Philadelphia (1993), pp. 1775–1808.

$$g(t) = e^{-(a+c)t/2}\left[A \cos\left(\frac{t}{2}\sqrt{4bd - (a-c)^2}\right) + B \sin\left(\frac{t}{2}\sqrt{4bd-(a-c)^2}\right)\right]$$

and that

$G(t) =$

$$G_0 + e^{-(a+c)t/2}\left[A \cos\left(\frac{t}{2}\sqrt{4bd - (a-c)^2}\right) + B \sin\left(\frac{t}{2}\sqrt{4bd-(a-c)^2}\right)\right].$$

Let $\alpha = \frac{1}{2}(a + c)$ and $\omega = \frac{1}{2}\sqrt{4bd - (a-c)^2}$. Then we can rewrite the general solution obtained here as

$$G(t) = G_0 + e^{-\alpha t}[A \cos \omega t + B \sin \omega t].$$

Research has shown that lab results of $2\pi/\omega > 4$ indicate a mild case of diabetes.

(f) Suppose that you have given the GTT to four patients you suspect of having a mild case of diabetes. The results for each patient are shown in the following table. Which patients, if any, have a mild case of diabetes?

| | Patient 1 | Patient 2 | Patient 3 | Patient 4 |
|---|---|---|---|---|
| $G_0$ | 80.00 | 90.00 | 100.00 | 110.00 |
| $t = 1$ | 85.32 | 91.77 | 103.35 | 114.64 |
| $t = 2$ | 82.54 | 85.69 | 98.26 | [illegible] |
| $t = 3$ | 78.25 | 92.39 | 96.59 | [illegible]8.14 |
| $t = 4$ | 76.61 | 91.13 | 99.47 | 113.76 |

## B. Modeling the Motion of a Skier

During a sporting event, an athlete loses strength because the at[illegible]e has to perform work against many physical forces. Some of the forces acting [illegible] an athlete include gravity (the athlete must usually work against gravity when m[illegible]ng body parts; friction is created between the ground and the athlete) and aerod[illegible]mic drag (a pressure gradient exists between the front and the back of the athlete; [illegible] athlete must overcome the force of air friction). In downhill skiing, friction and dr[illegible] affect the skier most because the skier is not working against gravity (why?). Th[illegible]stance traveled by a skier moving down a slope is given by

$$m\frac{d^2s}{dt^2} = F_g - F_\mu - D.$$

where $m$ is the mass of the skier, $s$ is the distance traveled by the skier at time $t$, $F_g$ is the gravitational force, $F_\mu$ is the friction force between ski and snow, and $D$ is the aerodynamic drag.



If the slope has constant angle $\alpha$, we can rewrite the equation as

$$\frac{d^2s}{dt^2} = g[\sin\alpha - \mu\cos\alpha] - \frac{C_D A\rho}{2m}\left(\frac{ds}{dt}\right)^2,$$

where $\mu$ is the coefficient of friction, $C_D$ the drag coefficient, $A$ the projected area of the skier, $\rho$ the air density, and $g \cong 9.81\ \text{m}\cdot\text{s}^{-2}$ the gravitational constant.* Because $g$, $\alpha$, $\mu$, $m$, $C_D$, $A$, and $\rho$ are constants, $g[\sin\alpha - \mu\cos\alpha]$ and $C_DA\rho/2m$ are constants. Thus, if we let $k^2 = g[\sin\alpha - \mu\cos\alpha]$ (assuming that $g[\sin\alpha - \mu\cos\alpha]$ is nonnegative) and $h^2 = C_DA\rho/2m$, we can rewrite this equation in the simpler form

$$\frac{d^2s}{dt^2} = k^2 - h^2\left(\frac{ds}{dt}\right)^2.$$

Remember that the relationship between displacement, $s$, and velocity, $v$, is

$$v = \frac{ds}{dt}.$$

Thus, we can find displacement by integrating velocity, if the velocity is known.

1. Use the substitution $v = ds/dt$ to rewrite $d^2s/dt^2 = k^2 - h^2\ (ds/dt)^2$ as a first-order equation and find the solution that satisfies $v(0) = v_0$. *Hint:* Use the method of partial fractions.
2. Find $s(t)$ if $s(0) = 0$.

Thus, in this case, we see that we are able to express both $s$ and $v$ as functions of $t$. Typical values of the constants $m$, $\rho$, $\mu$, and $C_DA$ are shown in the following table.

| Constant | Typical Value |
|---|---|
| $m$ | 75 kg |
| $\rho$ | $1.29\ \text{kg}\cdot\text{m}^{-3}$ |
| $\mu$ | 0.06 |
| $C_DA$ | $0.16\ \text{m}^2$ |

3. Use the values given in the previous table to complete the entries in the following table. For each value of $\alpha$, graph $v(t)$ and $s(t)$ if $v(0) = 0$, 5, and 10 on the interval [0, 40]. In each case, calculate the maximum velocity achieved by the skier. What is the limit of the velocity achieved by the skier? How does changing the initial velocity affect the maximum velocity?

* Sauli Savolainen and Reijo Visuri, "A Review of Athletic Energy Expenditure, Using Skiing as a Practical Example," *Journal of Applied Biomechanics*, Volume 10, Number 3 (August 1994), pp. 253–269.

| $\alpha$ | $k^2 = g[\sin\alpha - \mu\cos a]$ | $h^2 = \frac{C_DA\rho}{2m}$ |
|---|---|---|
| $\alpha = 30° = \pi/6$ rad | | |
| $\alpha = 40° = 2\pi/9$ rad | | |
| $\alpha = 45° = \pi/4$ rad | | |
| $\alpha = 50° = 5\pi/18$ rad | | |

4. For each value of $\alpha$ in the previous table, graph $v(t)$ and $s(t)$ if $v(0) = 80$ and 100 on the interval [0, 40], if possible. Interpret your results.

Sometimes, it is more useful to express $v$ (velocity) as a function of a different variable, like $s$ (displacement). To do so, we write $dt/ds = 1/v$ because $ds/dt = v$. Multiplying $dv/dt = k^2 - h^2v^2$ by $1/v$ and simplifying leads to

$$\frac{1}{v}\frac{dv}{dt} = \frac{1}{v}(k^2 - h^2v^2)$$

$$\frac{dv}{dt}\frac{dt}{ds} = k^2v^{-1} - h^2v$$

$$\frac{dv}{ds} + h^2v = k^2v^{-1}.$$

Now let $w = v^2$. Then $\frac{dw}{[illegible]} = 2$ [illegible] $\frac{dv}{[illegible]}$ [illegible] $\frac{1}{[illegible]}\frac{dw}{[illegible]}$ [illegible] equation becomes

$$\frac{1}{2v}\frac{dw}{dx} + h^2v = k^2v^{-1}.$$

Multiplying this equation by $2v$ and applying the substitution $w = v^2$ leads us to the first-order linear differential equation

$$\frac{dw}{ds} + 2h^2w = 2k^2.$$

5. Show that a solution of this differential equation is given by

$$w(s) = (k/h)^2 + Ce^{-2h^2s},$$

where $C$ is an arbitrary constant. Resubstitute $w = v^2$ to obtain

$$[v(s)]^2 = (k/h)^2 + Ce^{-2h^2s}.$$

6. Show that the solution that satisfies the initial condition $v(0) = v_0$ is

$$[v(s)]^2 = \frac{-k^2 + k^2e^{2h^2s} + h^2v_0^2}{h^2e^{2hs}} = \left(\frac{k}{h}\right)^2(1 - e^{-2h^2s}) + v_0^2e^{-2h^2s}.$$

7. For each value of $\alpha$ in the previous table, graph $v(s)$ if $v(0) = 0$, 5, and 10 on the interval (0, 1500]. In each case, calculate the maximum velocity achieved by the skier. How does changing the initial velocity affect the maximum velocity? Compare your results to (3). Are your results consistent?



8. How do these results change if $\mu$ is increased or decreased? if $C_DA$ is increased or decreased? How much does a 10% increase or decrease in friction decrease or increase the skier's velocity?
9. The values of the constants considered here for three skiers are listed in the following table.

| Constant | Skier 1 | Skier 2 | Skier 3 |
|---|---|---|---|
| $m$ | 75 kg | 75 kg | 75 kg |
| $\rho$ | 1.29 kg · m$^{-3}$ | 1.29 kg · m$^{-3}$ | 1.29 kg · m$^{-3}$ |
| $\mu$ | 0.05 | 0.06 | 0.07 |
| $C_DA$ | 0.16 m$^2$ | 0.14 m$^2$ | 0.12 m$^2$ |

Suppose that these three skiers are racing down a slope with constant angle $\alpha = 30° = \pi/6$ rad. Who wins the race if the length of the slope if 400 m? Who loses? Who wins the race if the length of the slope is 800 or 1200 m? Who loses? Does the length of the slope matter as to who wins or loses? What if the angle of the slope is increased or decreased? Explain.

10. Which of the variables considered here affects the velocity of the skier the most? Which variable do you think is the easiest to change? How would you advise a group of skiers who want to increase their maximum attainable velocity by 10%?

## C. The Schrödinger Equation

The time independent **Shrödinger equation,** proposed by the Austrian physicist Erwin Schrödinger (1887–1961) in 1926, describes the relationship between particles and waves. The **Shrödinger equation in spherical coordinates** is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + \frac{2\mu}{h^2}(E - V)\psi = 0.*$$

If we assume that a solution to this partial differential equation of the form

$$\psi(r, \theta, \phi) = R(r)\Theta(\theta)g(\phi)$$

exists, then by a technique called **separation of variables,** the Schrödinger equation can be rewritten as three second-order ordinary differential equations:

**Azimuthal Equation** $\quad \frac{d^2g}{d\phi^2} = -m_\ell^2 g$

* Stephen T. Thornton and Andrew Rex, *Modern Physics for Scientists and Engineers*, Saunders College Publishing (1993), pp. 248–253.

**Radial Equation** $$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{2\mu}{h^2}\left(E-V-\frac{h^2}{2\mu}\frac{\ell(\ell+1)}{r^2}\right)R=0$$

**Angular Equation** $$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[\ell(\ell+1)-\frac{m_\ell^2}{\sin^2\theta}\right]\Theta=0.$$

The *orbital angular momentum quantum number* $\ell$ must be a nonnegative integer: $\ell = 0, \pm1, \pm2, \ldots$ and the *magnetic quantum number* $m_\ell$ depends on $\ell$: $m_\ell = -\ell, -\ell + 1, \ldots, -2, -1, 0, 1, 2, \ldots, \ell, \ell + 1$.

If $\ell = 0$, the radial equation is $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{2\mu}{h^2}(E-V)R=0$. Using $V(r) = -e^2/4\pi\epsilon_0 r$ (the constants $\mu$, $h$, $e$, and $\epsilon_0$ are described later), the equation becomes $\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\frac{2\mu}{h^2}\left(E+\frac{e^2}{4\pi\epsilon_0 r}\right)R=0$.

1. Show that if $R = Ae^{-r/a_0}$ is a solution of this equation, $a_0 = 4\pi\epsilon_0 h^2/\mu e^2$ and $E = -h^2/2\mu a_0^2$.

   The constants in the expression $4\pi\epsilon_0 h^2/\mu e^2$ are described as follows: $h \approx 1.054572 \times 10^{-34}$ J · s is the *Planck constant* ($6.6260755 \times 10^{-34}$ J · s divided by $2\pi$), $\epsilon_0 \approx 8.854187817 \times 10^{-12}$ F/m is the *permitivity of vacuum*, and $e \approx 1.60217733 \times 10^{-19}$ C is the *elementary charge*. For the hydrogen atom, $\mu = 9.104434 \times 10^{-31}$ kg is the *reduced mass of the proton and electron*.
2. Calculate $a_0$, $2a_0$, and $E$ for the hydrogen atom.
3. The diameter of a hydrogen atom is approximately $10^{-10}$ m while the energy, $E$, is known to be approximately 13.6 eV. How do the results you obtained in problem 2 compare to these?

   For the hydrogen atom, the radial equation can be written in the form

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left[2E+\frac{2Z}{r}-\frac{\ell(\ell+1)}{r^2}\right]R=0,$$

where $E$ is the energy of the hydrogen atom and $Z$ is a constant.*

When $E < 0$ and we let $p = \sqrt{-2E}$, a solution of

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left[2E+\frac{2Z}{r}-\frac{\ell(\ell+1)}{r^2}\right]R=0$$

has the form

$$R(r) = e^{-pr}u(r).$$

When this solution is substituted into the equation, we obtain the equation

$$\frac{d^2u}{dr^2}+2\left(\frac{1}{r}-p\right)\frac{du}{dr}+\left[2\left(Z-\frac{p}{r}\right)-\frac{\ell(\ell+1)}{r^2}\right]u=0.$$

* Yi-Hsin Liu and Wai-Ning Mei, "Solution of the radial equation for hydrogen atom: series solution or Laplace transform?" *International Journal of Mathematical Education in Science and Technology*, Volume 21, Number 6, (1990), pp. 913–918.



4. Show that the solutions of the indicial equation of

$$\frac{d^2u}{dr^2}+2\left(\frac{1}{r}-p\right)\frac{du}{dr}+\left[2\left(Z-\frac{p}{r}\right)-\frac{\ell(\ell+1)}{r^2}\right]u=0$$

   are $\ell$ and $-(\ell + 1)$.
5. Find a power series solution to the equation of the form $u(r) = \sum_{n=0}^{\infty} a_n r^{n+\ell}$.
6. Find a series solution of

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left[2E+\frac{2Z}{r}-\frac{\ell(\ell+1)}{r^2}\right]R=0.$$

# 5

# Applications of Higher Order Differential Equations

In Chapter 4, we discussed several techniques for solving higher order differential equations. In this chapter, we illustrate how those methods can be used to solve some initial-value problems that model physical situations.

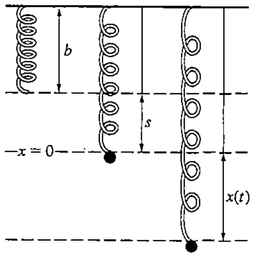



**Figure 5.1** A spring–mass system.

## 5.1 Simple Harmonic Motion

Suppose that an object of mass $m$ is attached to an elastic spring that is suspended from a rigid support such as a ceiling or a horizontal rod. The object causes the spring to stretch a distance $s$ from its *natural length.* The position at which it comes to rest is called the *equilibrium position.* According to Hooke's law, the spring exerts a restoring force in the upward (opposite) direction that is proportional to the distance $s$ that the spring is stretched. Mathematically, this is stated as

$$F = ks,$$

where $k > 0$ is the constant of proportionality or **spring constant.** In Figure 5.1 we see that the spring has natural length $b$. When the object is attached to the spring, it is





stretched $s$ units past its natural length to the equilibrium position $x = 0$. When the system is in motion, the displacement from $x = 0$ at time $t$ is given by $x(t)$.

By Newton's second law of motion,

$$F = ma = m\,\frac{d^2x}{dt^2},$$

where $m$ represents mass of the object and $a$ represents acceleration. If we assume that there are no forces other than the force as a result of gravity acting on the mass, we determine the differential equation that models this situation by summing the forces acting on the **spring–mass system** with

$$\begin{aligned} m\,\frac{d^2x}{dt^2} &= \Sigma\ (\text{forces acting on the system}) \\ &= -k(s + x) + mg \\ &= -ks - kx + mg. \end{aligned}$$

At equilibrium $ks = mg$, so after simplification we obtain the differential equation

$$m\,\frac{d^2x}{dt^2} = -kx \quad \text{or} \quad m\,\frac{d^2x}{dt^2} + kx = 0.$$

The two initial conditions that are used with this problem are the *initial position* $x(0) = \alpha$ and the *initial velocity* $x'(0) = \beta$. The function $x(t)$, which describes the *displacement* of the object with respect to the equilibrium position at time $t$, is found by solving the initial-value problem

$$\begin{cases} m\,\dfrac{d^2x}{dt^2} + kx = 0 \\ x(0) = \alpha,\ x'(0) = \beta. \end{cases}$$

Based on the assumption made in deriving the differential equation, *positive values of $x(t)$ indicate that the mass is below the equilibrium position* and *negative values of $x(t)$ indicate that the mass is above the equilibrium position.* The units that are encountered in these problems are summarized in Table 5.1.

**TABLE 5.1 Units Encountered when Solving Spring–Mass Systems**

| System | Force | Mass | Length | $k$ (Spring Constant) | Time |
|---|---|---|---|---|---|
| English | pounds (lb) | slugs (lb-s$^2$/ft) | feet (ft) | lb/ft | seconds (s) |
| Metric | Newton (N) | kilograms (kg) | meters (m) | N/m | seconds (s) |

**Example 1**

Determine the spring constant of the spring with natural length 10 in. that is stretched to a distance of 13 in. by an object weighing 5 lb.

***Solution*** Because the mass weighs 5 lb, $F = 5$ lb, and because displacement from the equilibrium position is $13 - 10 = 3$ in., $s = 3 \text{ in.} \times 1 \text{ ft}/12 \text{ in.} = 1/4$ ft. Therefore,

$$F = ks$$

$$5 = \frac{1}{4}k$$

$$k = 20.$$

Notice that the spring constant is given in the units lb/ft because $F = 5$ lb and $s = 1/4$ ft.

**Example 2**

An object of weight 16 lb stretches a spring 3 in. Determine the initial-value problem that models this situation if **(a)** the object is released from a point 4 in. below the equilibrium position with an upward initial velocity of 2 ft/s; **(b)** the object is released from rest 6 in. above the equilibrium position; **(c)** the object is released from the equilibrium position with a downward initial velocity of 8 ft/s.

***Solution*** We first determine the differential equation that models the spring–mass system (we use the same equation for all three parts of the problem). The information is given in English units, so we use $g = 32$ ft/s$^2$. We must convert all measurements given in inches to feet. The object stretches the spring 3 in., so $s = 3 \text{ in.} \times 1 \text{ ft}/12 \text{ in.} = 1/4$ ft. Also, because the mass weighs 16 lb, $F = 16$. According to Hooke's law, $16 = k \cdot 1/4$, so $k = 64$ lb/ft. We then find the mass $m$ of the object with $F = mg$ to find that $16 = m \cdot 32$ or $m = 1/2$ slug. The differential equation used to find the displacement of the object at time $t$ is

$$\frac{1}{2}\frac{d^2x}{dt^2} + 64x = 0.$$

The initial conditions for parts (a), (b), and (c) are then found.
**(a)** Because $4 \text{ in.} \times 1 \text{ ft}/12 \text{ in.} = 1/3$ ft and down is the positive direction, $x(0) = 1/3$. Notice, however, that the initial velocity is 2 ft/s in the upward (negative) direction. Hence, $x'(0) = -2$.
**(b)** A position 6 in. (or 1/2 ft) above the equilibrium position (in the negative direction) corresponds to initial position $x(0) = -1/2$. Being released from rest indicates that $dx/dt(0) = 0$.



**(c)** Because the mass is released from the equilibrium position, $x(0) = 0$. Also, the initial velocity is 8 ft/s in the downward (positive) direction, so $x'(0) = 8$.

**Example 3**

An object weighing 60 lb stretches a spring 6 in. Determine the function $x(t)$ that describes the displacement of the object if it is released from rest 12 in. below the equilibrium position.

***Solution*** First, the spring constant $k$ is determined from the supplied information. By Hooke's law, $F = ks$, so we have $60 = k \cdot 0.5$. Therefore, $k = 120$ lb/ft. Next, the mass $m$ of the object is determined using $F = mg$. In this case, $60 = m \cdot 32$, so $m = 15/8$ slugs. Because $k/m = 64$, and 12 in. is equivalent to 1 ft, the initial-value problem that models the situation is

$$\begin{cases} \dfrac{d^2x}{dt^2} + 64x = 0 \\ x(0) = 1,\ x'(0) = 0. \end{cases}$$

The characteristic equation that corresponds to the differential equation is $r^2 + 64 = 0$. It has solutions $r = \pm 8i$, so a general solution of the equation is $x(t) = c_1 \cos 8t + c_2 \sin 8t$.

To find the values of $c_1$ and $c_2$ that satisfy the initial conditions, we calculate $x'(t) = -8c_1 \sin 8t + 8c_2 \cos 8t$. Then $x(0) = c_1 = 1$ and $x'(0) = 8c_2 = 0$. So $c_1 = 1$, $c_2 = 0$, and $x(t) = \cos 8t$.

Notice that [illegible] indicates that the spring–mass system never comes [illegible] it is set into motion. The solution is periodic, so the mass moves vertically, retracing its motion, as shown in Figure 5.2. Motion of this type is called **simple harmonic motion.**

*What is the maximum displacement of the object in Example 3 from the equilibrium position?*

**Example 4**

An objective weighing 2 lb stretches a spring 1.5 in. **(a)** Determine the function $x(t)$ that describes the displacement of the object if it is released with a downward initial velocity of 32 ft/s from 12 in. above the equilibrium position. **(b)** At what value of $t$ does the object first pass through the equilibrium position?

***Solution*** **(a)** We begin by determining the spring constant. Because the force $F = 2$ stretches the spring $3/2 \text{ in.} \times 1 \text{ ft}/12 \text{ in.} = 1/8$ ft, $k$ is found by solving $2 = k \cdot 1/8$. Hence $k = 16$ lb/ft. With $F = mg$, we have $m = 2/32 = 1/16$ slug. The dif-

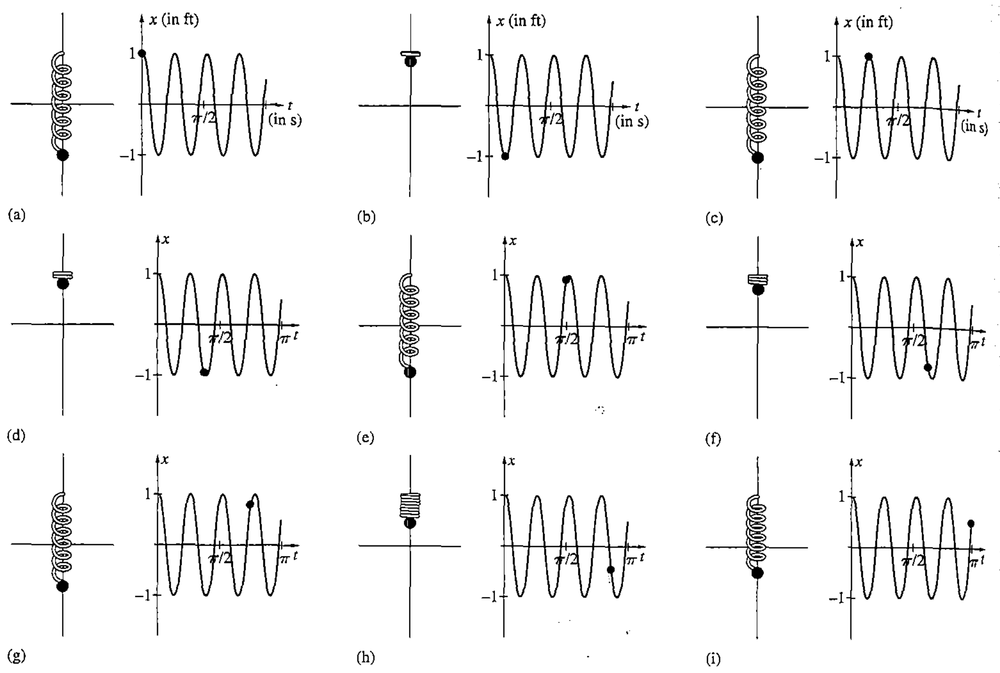



Figure 5.2

ferential equation that models this situation is $1/16\, d^2x/dt^2 + 16x = 0$ or $d^2x/dt^2 + 256x = 0$. Because 12 in. is equivalent to 1 ft, the initial position above the equilibrium (in the negative direction) is $x(0) = -1$. The downward initial velocity (in the positive direction) is $x'(0) = 32$. Therefore, we must solve the initial-value problem

$$\begin{cases} \dfrac{d^2x}{dt^2} + 256x = 0 \\ x(0) = -1,\ x'(0) = 32. \end{cases}$$

Because the characteristic equation is $r^2 + 256 = 0$, the roots are $r = \pm 16i$. A general solution is

$$x(t) = c_1 \cos 16t + c_2 \sin 16t,$$



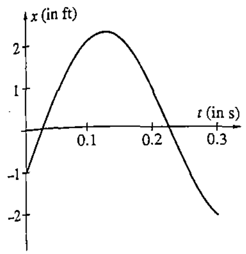



Figure 5.3

with derivative $x'(t) = -16c_1 \sin 16t + 16c_2 \cos 16t$. Application of the initial conditions then yields $x(0) = c_1 \cos 0 + c_2 \sin 0 = c_1 = -1$ and

$$x'(0) = -16c_1 \sin 0 + 16c_2 \cos 0 = 16c_2 = 32, \qquad c_2 = 2.$$

The position function is given by $x(t) = -\cos 16t + 2 \sin 16t$. (See Figure 5.3.)
**(b)** To determine when the object first passes through its equilibrium position, we solve the equation $x(t) = -\cos 16t + 2 \sin 16t = 0$ or $\tan 16t = 1/2$. Therefore, $t = 1/16 \tan^{-1}(.5) \approx 0.03$ s, which appears to be a reasonable approximation based on the graph of $x(t)$.

*Approximate the second time the mass considered in Example 4 passes through the equilibrium position.*

A general formula for the solution of the initial-value problem

$$\begin{cases} m\dfrac{d^2x}{dt^2} + kx = 0 \\ x(0) = \alpha,\ \dfrac{dx}{dt}(0) = \beta \end{cases}$$

is

$$x(t) = \alpha \cos \omega t + \frac{\beta}{\omega} \sin \omega t \qquad \text{where} \qquad \omega = \sqrt{\frac{k}{m}}.$$

Through the use of the trigonometric identity $\cos(a + b) = \cos a \cos b - \sin a \sin b$, we can write $x(t)$ in terms of a cosine function with a phase shift. First, let

$$x(t) = A \cos(\omega t - \phi).$$

Th... ...$\phi + A \sin \omega t \sin \phi$. Comparing the functions

$$x(t) = \alpha \cos \omega t + \frac{\beta}{\omega} \sin \omega t \qquad \text{and} \qquad x(t) = A \cos \omega t \cos \phi + A \sin \omega t \sin \phi$$

indicates that

$$A \cos \phi = \alpha \qquad \text{and} \qquad A \sin \phi = \frac{\beta}{\omega}.$$

Thus

$$\cos \phi = \frac{\alpha}{A} \qquad \text{and} \qquad \sin \phi = \frac{\beta}{A\omega}.$$

Because $\cos^2 \phi + \sin^2 \phi = 1$, $(\alpha/A)^2 + [\beta/(A\omega)]^2 = 1$. Therefore, the amplitude of the solution is

$$A = \sqrt{\alpha^2 + \frac{\beta^2}{\omega^2}}$$

and

$$x(t) = \sqrt{\alpha^2 + \frac{\beta^2}{\omega^2}} \cos(\omega t - \phi),$$

where $\phi = \cos^{-1}\left(\frac{\alpha}{\sqrt{\alpha^2 + \beta^2/\omega^2}}\right)$ and $\omega = \sqrt{k/m}$. Note that the period of $x(t)$ is

$$T = 2\frac{\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}.$$

In many cases, questions about the displacement function are more easily answered if the solution is written in this form.

*How does an increase in the magnitude (absolute value) of the initial position and initial velocity affect the amplitude of the resulting motion of the spring–mass system? From experience, does this agree with the actual physical situation?*

**Example 5**

A 4-kg mass stretches a spring 0.392 m. **(a)** Determine the displacement function if the mass is released from 1 m below the equilibrium position with a downward initial velocity of 10 m/s. **(b)** What is the maximum displacement of the mass? **(c)** What is the approximate period of the displacement function?

***Solution*** (a) Because the mass of the object (in metric units) is $m = 4$ kg, we use this with $F = mg$ to determine the force. We first compute

$$F = (4)(9.8) = 39.2 \text{ N}.$$

We then find the spring constant with $39.2 = k \cdot 0.392$, so $k = 100$ N/m. The differential equation that models this spring–mass system is $4\, d^2x/dt^2 + 100x = 0$ or $d^2x/dt^2 + 25x = 0$. The initial position is $x(0) = 1$, and the initial velocity is $x'(0) = 10$. We must solve the initial-value problem

$$\begin{cases} \dfrac{d^2x}{dt^2} + 25x = 0 \\ x(0) = 1,\ x'(0) = 10 \end{cases}$$

either directly or with the general formula obtained previously. Using the general formula with $\alpha = 1$, $\beta = 10$, and $\omega = \sqrt{100/4} = 5$, we have

$$x(t) = \sqrt{\alpha^2 + \frac{\beta^2}{\omega^2}} \cos(\omega t - \phi) = \sqrt{1^2 + \frac{10^2}{5^2}} \cos(5t - \phi)$$
$$= \sqrt{5}\cos(5t - \phi),$$

where $\phi = \cos^{-1}(1/\sqrt{5}) \approx 1.11$ rad, which we graph in Figure 5.4.

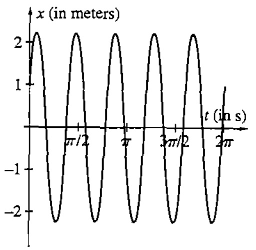



**Figure 5.4** Simple harmonic motion.



**(b)** From our knowledge of trigonometric functions, we know that the maximum value of $x(t) = \sqrt{5}\cos(5t - \phi)$ is $x = \sqrt{5}$. Therefore, the maximum displacement of the mass from its equilibrium position is $\sqrt{5} \approx 2.236$ meters.
**(c)** The period of this trigonometric function is $T = 2\pi/\omega = 2\pi/5$. The mass returns to its initial position every $2\pi/5 \approx 1.257$ s.

## EXERCISES 5.1

In Exercises 1–4, determine the mass $m$ and the spring constant $k$ for the given spring–mass system.

$$\begin{cases} m\dfrac{d^2x}{dt^2} + kx = 0 \\ x(0) = \alpha,\ \dfrac{dx}{dt}(0) = \beta. \end{cases}$$

Interpret the initial conditions. (Assume the English system.)

1. $4\dfrac{d^2x}{dt^2} + 9x = 0,\ x(0) = -1,\ x'(0) = 0$
2. $2\dfrac{d^2x}{dt^2} + 128x = 0,\ x(0) = -0.5,\ x'(0) = 1$

*3. $\dfrac{1}{4}\dfrac{d^2x}{dt^2} + 16x = 0,\ x(0) = 0.75,\ x'(0) = -2$

4. $\dfrac{1}{25}\dfrac{d^2x}{dt^2} + 4x = 0,\ x(0) = -\dfrac{1}{4},\ x'(0) = 1$

In Exercises 5–10, express the solution of the initial-value problem in the form $x(t) = \sqrt{\alpha^2 + \beta^2/\omega^2}\cos(\omega t - \phi)$. What is the period and amplitude of the solution?

5. $\dfrac{d^2x}{dt^2} + x = 0,\ x(0) = 3,\ x'(0) = -4$
6. $\dfrac{d^2x}{dt^2} + 4x = 0,\ x(0) = 1,\ x'(0) = 1$

*7. $\dfrac{1}{16}\dfrac{d^2x}{dt^2} + x = 0,\ x(0) = -2,\ x'(0) = 1$

8. $\dfrac{d^2x}{dt^2} + 256x = 0,\ x(0) = 2,\ x'(0) = 4$
9. $\dfrac{d^2x}{dt^2} + 9x = 0,\ x(0) = \dfrac{1}{3},\ x'(0) = -1$
10. $10\dfrac{d^2x}{dt^2} + \dfrac{1}{10}x = 0,\ x(0) = -5,\ x'(0) = 1$
11. A 16-lb object stretches a spring 6 in. If the object is lowered 1 ft below the equilibrium position and released, determine the displacement of the object. What is the maximum displacement of the object? When does it occur?
12. A 4-lb weight stretches a spring 1 ft. A 16-lb weight is then attached to the spring, and it comes to rest in its equilibrium position. If it is then put into motion with a downward velocity of 2 ft/s, determine the displacement of the mass. What is the maximum displacement of the object? When does it occur?

*13. A 6-lb object stretches a spring 6 in. If the object is lifted 3 in. above the equilibrium position and released, det[illegible]quired for the mass to return to [illegible] equilibrium position. What is the displacement of the object at $t = 5$ seconds? If the object is released from its equilibrium position with a downward initial velocity of 1 ft/s, determine the time required for the object to return to its equilibrium position.

14. A 16-lb weight stretches a spring 8 in. If the weight is lowered 4 in. below the equilibrium position and released, find the time required for the weight to return to the equilibrium position. What is the displacement of the weight at $t = 4$ s? If the weight is released from its equilibrium position with an upward initial velocity of 2 ft/s, determine the time required for the weight to return to the equilibrium position.
15. Solve the initial-value problem $d^2x/dt^2 + kx = 0$, $x(0) = -1$, $x'(0) = 0$ for values of $k = 1$, 4, and 9. Comment on the effect that $k$ has on the resulting motion.

16. Solve the initial-value problem $d^2x/dt^2 + \frac{4}{m}x = 0$, $x(0) = -1$, $x'(0) = 0$ for values of $m = 1, 4$, and 9. Comment on the effect that $m$ has on the resulting motion.

*17. Suppose that a 1-lb object stretches a spring 1/8 ft. The object is pulled downward and released from a position $b$ ft beneath its equilibrium with an upward initial velocity of 1 ft/s. Determine the value of $b$ so that the maximum displacement is 2 ft.

18. Suppose that a 70-gram mass stretches a spring 5 cm and that the mass is pulled downward and released from a position $b$ units beneath its equilibrium with an upward initial velocity of 10 cm/s. Determine the value of $b$ so that the mass first returns to its equilibrium at $t = 1$ s. ($g \approx 980$ cm/s$^2$)

19. The period of the motion of an undamped spring–mass system is $\pi/2$ seconds. Find the mass $m$ if the spring constant is $k = 32$ lb/ft.

20. Find the period of the motion of an undamped spring–mass system if the mass of the object is 4 kg and the spring constant is $k = 0.25$ N/m.

21. If the motion of an object satisfies the initial-value problem

$$\begin{cases} m\dfrac{d^2x}{dt^2} + kx = 0 \\ x(0) = \alpha, \dfrac{dx}{dt}(0) = \beta, \end{cases}$$

find the maximum velocity of the object.

22. Show that the solution to the initial-value problem

$$\begin{cases} m\dfrac{d^2x}{dt^2} + kx = 0 \\ x(0) = \alpha, \dfrac{dx}{dt}(0) = \beta \end{cases}$$

can be written as $x(t) = u(t) + v(t)$, where $u$ and $v$ satisfy the same differential equation as $x$, $u$ satisfies the initial conditions $u(0) = \alpha$, $du/dt\,(0) = 0$, and $v$ satisfies the initial conditions $v(0) = 0$, $dv/dt\,(0) = \beta$.

23. **(Archimedes' principle)** Suppose that an object of mass $m$ is submerged (either partially or totally) in a liquid of density $\rho$. Archimedes' principle states that a body in liquid experiences a buoyant upward force equal to the weight of the liquid displaced by the body. The object is in equilibrium when the buoyant force of the displaced liquid equals the force of gravity on the object. (See Figure 5.5.) Consider the cylinder of radius $r$ and height $H$ of which $h$ units of the height is submerged at equilibrium.

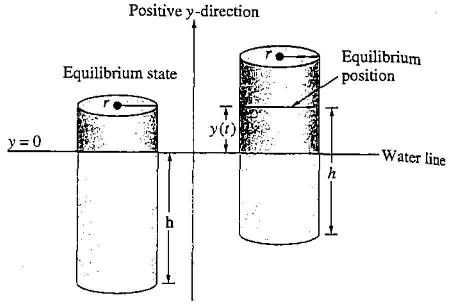



**Figure 5.5**

(a) Show that the weight of liquid displaced at equilibrium is $\pi r^2 h\rho$. Therefore, at equilibrium $\pi r^2 h\rho = mg$.

(b) Let $y(t)$ represent the vertical displacement of the cylinder from equilibrium. Show that when the cylinder is raised out of the liquid, the downward force is $\pi r^2[h - y(t)]\rho$.

(c) Use Newton's second law of motion to show that $m\,d^2y/dt^2 = \pi r^2[h - y(t)]\rho - mg$. Simplify this equation to obtain a second-order equation that models this situation.

24. Determine if the cylinder can float in a deep pool of water ($\rho \approx 62.4$ lb/ft$^3$) using the given radius $r$, height $H$, and weight $W$: **(a)** $r = 3$ in., $H = 12$ in., $W = 5$ lb; **(b)** $r = 4$ in., $H = 8$ in., $W = 20$ lb; **(c)** $r = 6$ in., $H = 9$ in., $W = 50$ lb.

*25. Determine the motion of the cylinder of weight 512 lb, radius $r = 1$ ft, and height $H = 4$ ft if it is released with 3 ft of its height above the water ($\rho = 62.5$ lb/ft$^3$) with a downward initial velocity of 3 ft/s. What is the maximum displacement of the cylinder from its equilibrium?

26. Consider the cylinder of radius $r = 3$ in., height $H = 12$ in., and weight 10 lb. Show that the portion of the cylinder submerged in water of density ($\rho$) 62.5 lb/ft$^3$ is $h \approx 0.815$ ft $\approx 9.78$ in. Find the motion of the cylinder if it is released with 1.22 in. of its height above the water with no initial velocity. (*Hint:* At $t = 0$, $h \approx 9.78$ in., so 0.22 in. is above the water. Therefore, the initial position is $y(0) = 1$.)

*27. An object with mass $m = 1$ slug is attached to a spring with spring constant $k = 4$ lb/ft. **(a)** Determine the displacement function of the object if $x(0) = \alpha$ and $x'(0) = 0$. Graph the solution for $\alpha = 1, 4, -2$. How does varying the value of $\alpha$ affect the solution? Does it change the values of $t$ at which the mass passes through the equilibrium position? **(b)** Determine the displacement function of the object if $x(0) = 0$ and $x'(0) = \beta$. Graph the solution for $\beta = 1, 4, -2$. How does varying the value of $\beta$ affect the solution? Does it change the values of $t$ at which the mass passes through the equilibrium position?

28. An object of mass $m = 4$ slugs is attached to a spring with spring constant $k = 20$ lb/ft. If the object is released from 7 in. above its equilibrium with a downward initial velocity of 2.5 ft/s, find **(a)** the maximum displacement from the equilibrium position; **(b)** the time at which the object first passes through its equilibrium position; **(c)** the period of the motion.

29. If the spring in Problem 28 has the spring constant $k = 16$ lb/ft, what is the maximum displacement from the equilibrium position? How does this compare to the result in the previous exercise? Determine the maximum displacement if $k = 24$ lb/ft. Do these results agree with those that would be obtained with the general formula $A = \sqrt{\alpha^2 + \beta^2/\omega^2}$?

30. An object of mass $m = 3$ slugs is attached to a spring with spring constant $k = 15$ lb/ft. If the object is released from 9 in. below its equilibrium with a downward initial velocity of 1 ft/s, find **(a)** the maximum displacement from the equilibrium position; **(b)** the time at which the object first passes through its equilibrium position; **(c)** the period of the motion.

31. If the mass in Problem 30 is $m = 4$ slugs, find the maximum displacement from the equilibrium position. Compare this result to that obtained with $m = 3$ slugs.



## 5.2 Damped Motion

Because the differential equation derived in Section 5.1 disregarded all retarding forces acting on the motion of the mass, a more realistic model is needed. Studies in mechanics reveal that the resistive force due to **damping** is a function of the velocity of the motion. For $c > 0$, functions such as

$$F_R = c\,\text{[illegible]}\left(\frac{}{dt}\right), \text{ and } F_R = c\,\text{sgn}\left(\frac{dx}{dt}\right), \text{ where } \text{sgn}\left(\frac{dx}{dt}\right) = \begin{cases} 1, & x' > 0 \\ 0, & x' = 0 \\ -1, & x' < 0 \end{cases}$$

can be used to represent the damping force. We follow procedures similar to those used in Section 5.1 to model simple harmonic motion and to determine a differential equation that models the spring–mass system, which includes damping. Assuming that $F_R = c\,dx/dt$, after summing the forces acting on the spring–mass system, we have

$$m\frac{d^2x}{dt^2} = -c\frac{dx}{dt} - kx \quad \text{or} \quad m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.$$

The displacement function is found by solving the initial-value problem

$$\begin{cases} m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx = 0 \\ x(0) = \alpha, x'(0) = \beta. \end{cases}$$

From our experience with second-order ordinary differential equations with constant coefficients in Chapter 4, the solutions to initial-value problems of this type depend on the values of $m$, $k$, and $c$. Suppose we assume (as we did in Section 5.1) that solutions of the differential equation have the form $x(t) = e^{rt}$. Then the characteristic equation is $mr^2 + cr + k = 0$ with solutions

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.$$

The solution depends on the value of the quantity $c^2 - 4mk$. In fact, problems of this type are classified by the value of $c^2 - 4mk$ as follows.

**Case 1:** $c^2 - 4mk > 0$

This situation is said to be **overdamped,** because the damping coefficient $c$ is large in comparison with the spring constant $k$.

**Case 2:** $c^2 - 4mk = 0$

This situation is described as **critically damped,** because the resulting motion is oscillatory, with a slight decrease in the damping coefficient $c$.

**Case 3:** $c^2 - 4mk < 0$

This situation is called **underdamped,** because the damping coefficient $c$ is small in comparison with the spring constant $k$.

**Example 1**

An 8-lb object is attached to a spring of length 4 ft. At equilibrium, the spring has length of 6 ft. Determine the displacement function $x(t)$ if $F_R = 2\, dx/dt$ and **(a)** the object is released from the equilibrium position with a downward initial velocity of 1 ft/s; **(b)** the object is released 6 in. above the equilibrium position with an initial velocity of 5 ft/s in the downward direction.

***Solution*** Notice that $s = 6 - 4 = 2$ ft and that $F = 8$ lb. We find the spring constant with $8 = k \cdot 2$, so $k = 4$ lb/ft. Also, the mass of the object is $m = 8/32 = 1/4$ slug. The differential equation that models this spring–mass system is

$$\frac{1}{4}\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 4x = 0 \quad \text{or} \quad \frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0.$$

The corresponding characteristic equation is

$$r^2 + 8r + 16 = (r+4)^2 = 0,$$

so $r = -4$ is a root of multiplicity two. A general solution is

$$x(t) = c_1 e^{-4t} + c_2 t e^{-4t}.$$

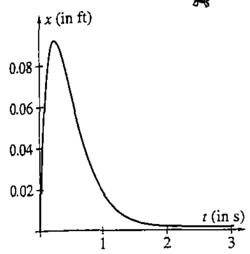



**Figure 5.6** Critically damped motion.

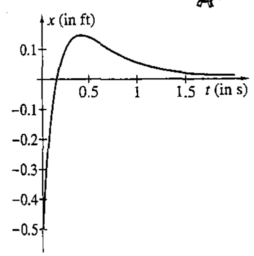



**Figure 5.7** Critically damped motion.



Differentiating yields

$$x'(t) = (-4c_1 + c_2)e^{-4t} - 4c_2 t e^{-4t}.$$

**(a)** The initial conditions in this case are $x(0) = 0$ and $x'(0) = 1$, so $c_1 = 0$ and $-4c_1 + c_2 = 1$. Thus, $c_2 = 1$ and the solution is $x(t) = te^{-4t}$, which is shown in Figure 5.6. Notice that $x(t)$ is always *positive,* so the object is always *below* the equilibrium position and approaches zero (the equilibrium position) as $t$ approaches infinity. Because of the resistive force due to damping, the object is not allowed to pass through its equilibrium position.

*What is the maximum displacement from the equilibrium position? If you were looking at this spring, would you perceive its motion?*

**(b)** In this case, $x(0) = -1/2$ and $x'(0) = 5$. When we apply these initial conditions, we find that $x(0) = c_1 = -1/2$ and $x'(0) = -4c_1 + c_2 = -4(-1/2) + c_2 = 5$. Hence $c_2 = 3$, and the solution is

$$x(t) = -\frac{1}{2}e^{-4t} + 3te^{-4t}.$$

This function, which is graphed in Figure 5.7, indicates the importance of the initial conditions on the resulting motion. In this case, the displacement is negative (above the equilibrium position) initially, but the positive initial velocity causes the function to become positive (below the equilibrium position) before approaching zero.

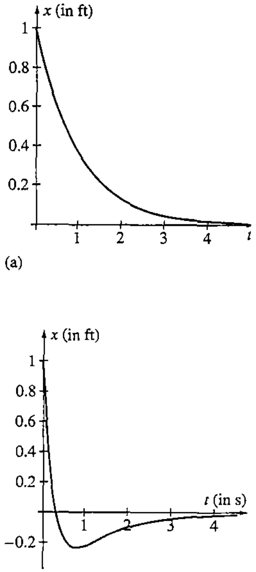
*If the object in Example 1 is released from any point below its equilibrium position with an upward initial velocity of 1 ft/s, can it possibly pass through its equilibrium position? If the object is released from below its equilibrium position with any upward initial velocity, can it possibly pass through the equilibrium position?*

**Examp**

A 32-lb object stretches a spring 8 ft. If the resistive force due to damping is $F_R = 5\, dx/dt$, determine the displacement function if the object is released from 1 ft below the equilibrium position with **(a)** an upward velocity of 1 ft/s; **(b)** an upward velocity of 6 ft/s.

***Solution*** **(a)** Because $F = 32$ lb, the spring constant is found with $32 = k(8)$, so $k = 4$ lb/ft. Also, $m = 32/32 = 1$ slug. The differential equation that models this situation is $d^2x/dt^2 + 5\, dx/dt + 4x = 0$. The initial position is $x(0) = 1$, and the initial velocity in (a) is $x'(0) = -1$. The characteristic equation of the differential equation is

$$r^2 + 5r + 4 = (r+1)(r+4) = 0,$$



(a)

(b)

Figure 5.8

with roots $r_1 = -1$ and $r_2 = -4$. A general solution is

$$x(t) = c_1e^{-t} + c_2e^{-4t} \quad \text{and} \quad \frac{dx}{dt}(t) = -c_1e^{-t} - 4c_2e^{-4t}.$$

(Because $c^2 - 4mk = 5^2 - 4(1)(4) = 9 > 0$, the system is overdamped.) Application of the initial conditions yields the system of equations $\begin{cases} c_1 + c_2 = 1 \\ -c_1 - 4c_2 = -1 \end{cases}$ with solution $\{c_1 = 1, c_2 = 0\}$, so the solution to the initial-value problem is $x(t) = e^{-t}$. The graph of $x(t)$ is shown in Figure 5.8(a). Notice that this solution is always positive and, due to the damping, approaches zero as $t$ approaches infinity. Therefore, the object is always below its equilibrium position.

**(b)** Using the initial velocity $x'(0) = -6$, we solve the system $\begin{cases} c_1 + c_2 = 1 \\ -c_1 - 4c_2 = -6 \end{cases}$, which has solution $\{c_1 = -2/3, c_2 = 5/3\}$. The solution of the initial-value problem is

$$x(t) = \tfrac{5}{3}e^{-4t} - \tfrac{2}{3}e^{-t}.$$

The graph of this function is shown in Figure 5.8(b). As in Example 1, these results indicate the importance of the initial conditions on the resulting motion. In this case, the displacement is positive (below its equilibrium) initially, but the larger negative initial velocity causes the function to become negative (above its equilibrium) before approaching zero. Therefore, we see that the initial velocity in part (b) causes the mass to pass through its equilibrium position.

*Can you find a minimum value for the upward initial velocity $v_0$ in Example 2 so that the object passes through its equilibrium position for all values greater than $v_0$?*

## Example 3

A 16-lb object stretches a spring 2 ft. Determine the displacement $x(t)$ if the resistive force due to damping is $F_R = (1/2)\, dx/dt$ and the object is released from the equilibrium position with a downward velocity of 1 ft/s.

***Solution*** Because $F = 16$ lb, the spring constant is determined with $16 = k \cdot 2$. Hence, $k = 8$ lb/ft. Also, $m = 16/32 = 1/2$ slug. Therefore, the differential equation is

$$\frac{1}{2}\frac{d^2x}{dt^2} + \frac{1}{2}\frac{dx}{dt} + 8x = 0 \quad \text{or} \quad \frac{d^2x}{dt^2} + \frac{dx}{dt} + 16x = 0.$$

The initial position is $x(0) = 0$ and the initial velocity is $x'(0) = 1$. We must solve the initial-value problem



$$\begin{cases} \dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + 16x = 0 \\ x(0) = 0,\ x'(0) = 1. \end{cases}$$

A general solution of $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + 16x = 0$ is

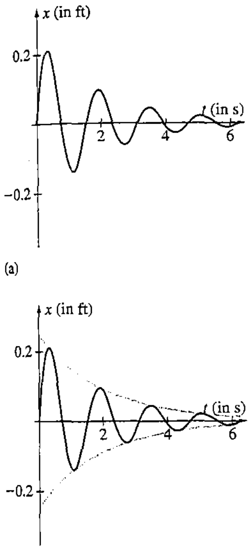
$$x(t) = e^{-t/2}\left(c_1 \cos\frac{3\sqrt{7}t}{2} + c_2 \sin\frac{3\sqrt{7}t}{2}\right).$$

Notice that $c^2 - 4mk = (1/2)^2 - 4(1/2)(8) = -63/4 < 0$, so the spring–mass system is underdamped. Because

$$x'(t) = -\frac{1}{2}e^{-t/2}\left(c_1 \cos\frac{3\sqrt{7}t}{2} + c_2 \sin\frac{3\sqrt{7}t}{2}\right) + e^{-t/2}\frac{3\sqrt{7}}{2}\left(-c_1 \sin\frac{3\sqrt{7}t}{2} + c_2 \cos\frac{3\sqrt{7}t}{2}\right),$$

application of the initial conditions yields $\{c_1 = 0, c_2 = 2/(3\sqrt{7})\}$. Therefore, the solution is

$$x(t) = \frac{2}{3\sqrt{7}}e^{-t/2}\sin\left(\frac{3\sqrt{7}t}{2}\right).$$



(a)

(b)

Figure 5.9

Solutions of this type have several interesting properties. First, the trigonometric component of the solution causes the motion to *oscillate*. Also, the exponential portion forces the solution to approach zero as $t$ approaches infinity. These qualities are illustrated in the graph of $x(t)$ in Figure 5.9(a). Physically, the position of the mass in this case oscillates about the equilibrium position and eventually comes to rest in the equilibrium position. Of course, with our model the displacement function $x(t) \to 0$ as $t \to \infty$, but there are no values $t = T$ such that $x(t) = 0$ for $t > T$ as we might expect from the physical situation. Our model only approximates the behavior of the mass. Notice also that the solution is bounded above and below by the ex ... through the horizontal axis, $-e^{-t/2}$. This is illustrated with the simultaneous display of these functions in Figure 5.9(b); the motion of the spring is illustrated in Figure 5.10.

Notice that when the system is underdamped as in Example 3, the amplitude (or **damped amplitude**) of the solution decreases as $t \to \infty$. The time interval between two successive local maxima (or minima) of $x(t)$ is called the **quasiperiod** of the solution.

*Approximate the quasiperiod of the solution in Example 3.*

In Section 5.1 we developed a general formula for the displacement function. We can do the same for systems that involve damping. Assuming that the spring–mass system is underdamped, the differential equation $m\, d^2x/dt^2 + c\, dx/dt + kx = 0$ has the

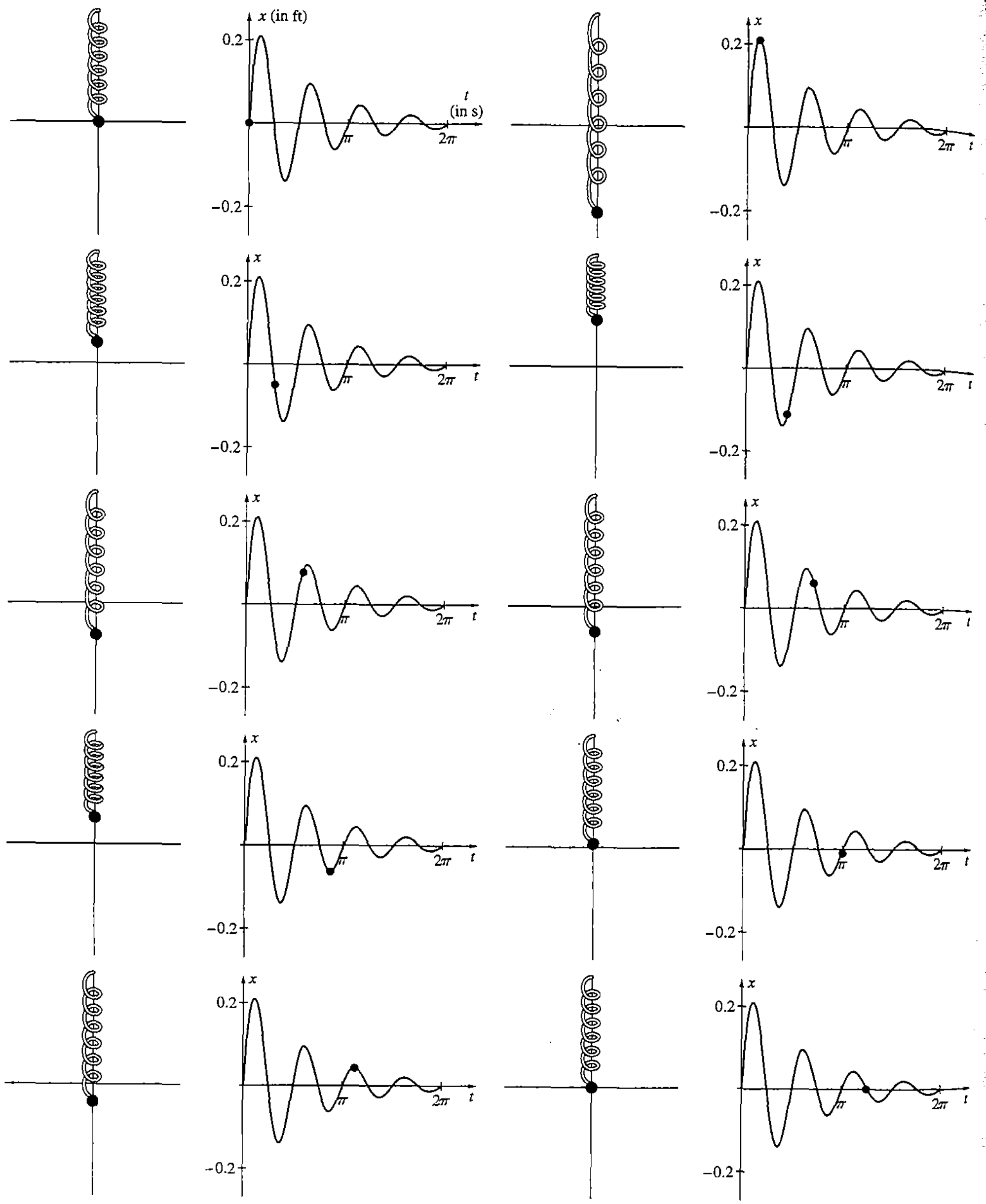



**Figure 5.10**



characteristic equation $mr^2 + cr + k = 0$ with roots $r = (-c \pm i\sqrt{4mk - c^2})/(2m)$. If we let $\rho = c/(2m)$ and $\mu = \sqrt{4mk - c^2}/(2m)$, then $r = -\rho \pm \mu i$ and a general solution is $x(t) = e^{-\rho t}(c_1 \cos \mu t + c_2 \sin \mu t)$. Applying the initial conditions $x(0) = \alpha$ and $x'(0) = \beta$ yields the solution $x(t) = e^{-\rho t}\left(\alpha \cos \mu t + \frac{\beta + \alpha\rho}{\mu} \sin \mu t\right)$, which can be written as

$$x(t) = Ae^{-\rho t} \cos(\mu t - \phi).$$

Then, $x(t) = e^{-\rho t}(A \cos \mu t \cos \phi + A \sin \mu t \sin \phi)$. Comparing the functions

$$x(t) = e^{-\rho t}\left(\alpha \cos \mu t + \frac{\beta + \alpha\rho}{\mu} \sin \mu t\right)$$

and

$$x(t) = e^{-\rho t}(A \cos \mu t \cos \phi + A \sin \mu t \sin \phi),$$

we have

$$A \cos \phi = \alpha \quad \text{and} \quad A \sin \phi = \frac{\beta + \alpha\rho}{\mu},$$

which indicates that

$$\cos \phi = \frac{\alpha}{A} \quad \text{and} \quad \sin \phi = \frac{\beta + \alpha\rho}{A\mu}.$$

Now, $\cos^2 \phi + \sin^2 \phi = 1$, so $\alpha/A^2 + [(\beta + \alpha\rho)/(A\mu)]^2 = 1$. Therefore, the decreasing amplitude of the solution is

$$A = \sqrt{\alpha^2 + \left[\frac{\beta + \alpha\rho}{\mu}\right]^2}$$

and

$$x(t) = e^{-\rho t}\sqrt{\alpha^2 + \left[\frac{\beta + \alpha\rho}{\mu}\right]^2} \cos(\mu t - \phi),$$

where $\phi = \cos^{-1}\left(\frac{\alpha}{\sqrt{\alpha^2 + [(\beta + \alpha\rho)/\mu]^2}}\right)$.

The quantity

$$\frac{2\pi}{\mu} = \frac{4\pi m}{\sqrt{4km - c^2}}$$

is called the **quasiperiod** of the function. (Note that functions of this type are *not* periodic.) We can also determine the times at which the mass passes through the equilibrium position from the general formula given here. We do this by setting the argument equal to odd multiples of $\pi/2$, because the cosine function is zero at these values. The mass passes through the equilibrium position at

$$t = \frac{\left[(2n+1)\frac{\pi}{2} + \phi\right]}{\mu} = \frac{[m(2n+1)\pi + 2m\phi]}{\sqrt{4mk - c^2}},\ n = 0, \pm 1, \pm 2, \ldots$$

*Use this formula to find the quasiperiod for the solution in Example 3.*

**Example 4**

An object of mass 1 slug is attached to a spring with spring constant $k = 13$ lb/ft and is subjected to a resistive force of $F_R = 4\,dx/dt$ due to damping. If the initial position is $x(0) = 1$ and the initial velocity is $x'(0) = 1$, determine the quasiperiod of the solution and find the values of $t$ at which the object passes through the equilibrium position.

***Solution*** The initial-value problem that models this situation is

$$\begin{cases} \dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 13x = 0 \\ x(0) = 1,\ x'(0) = 1. \end{cases}$$

The characteristic equation is $r^2 + 4r + 13 = 0$ with roots $r_{1,2} = -2 \pm 3i$. A general solution is $x(t) = e^{-2t}(c_1 \cos 3t + c_2 \sin 3t)$ with derivative

$$\frac{dx}{dt}(t) = -2e^{-2t}(c_1 \cos 3t + c_2 \sin 3t) + 3e^{-2t}(-c_1 \sin 3t + c_2 \cos 3t).$$

Application of the initial conditions yields the solution $x(t) = e^{-2t}(\cos 3t + \sin 3t)$, which can be written as

$$x(t) = e^{-2t}\sqrt{\alpha^2 + \left[\frac{(\beta + \alpha p)}{\mu}\right]^2} \cos(\mu t - \phi) =$$

$$e^{-2t}\sqrt{1^2 + \left[\frac{1+2}{3}\right]^2} \cos(3t - \phi) = \sqrt{2}e^{-2t}\cos(3t - \phi).$$

The quasiperiod is

$$\frac{2\pi}{\mu} = \frac{4\pi m}{\sqrt{4km - c^2}} = \frac{4\pi}{\sqrt{4(13) - 4^2}} = \frac{2\pi}{3}.$$

The values of $t$ at which the mass passes through the equilibrium are

$$t = \frac{m(2n+1)\pi + 2m\phi}{\sqrt{4mk - c^2}} = \frac{(2n+1)\pi + 2\phi}{6},\ n = 0, 1, 2, \ldots$$

where $\phi = \cos^{-1}(1/\sqrt{2}) = \pi/4$. (See Figure 5.11.)

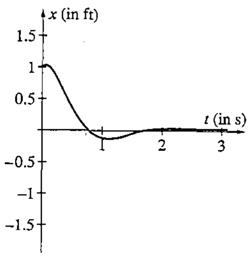



**Figure 5.11** $x(t) = \sqrt{2}e^{-2t} \cos(3t - \phi)$, $\phi = \pi/4$.



## EXERCISES 5.2

In Exercises 1–4, determine the mass $m$ (slugs), spring constant $k$ (lb/ft), and damping coefficient $c$ in $F_R = c\,dx/dt$ for the given spring–mass system

$$\begin{cases} m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx = 0 \\ x(0) = \alpha,\ x'(0) = \beta. \end{cases}$$

Describe the initial conditions. (Assume the English system.)

1. $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 3x = 0,\ x(0) = 0,\ x'(0) = -4$

2. $\dfrac{1}{32}\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + x = 0,\ x(0) = 1,\ x'(0) = 0$

*3. $\dfrac{1}{4}\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + x = 0,\ x(0) = -0.5,\ x'(0) = 1$

4. $4\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 8x = 0,\ x(0) = 0,\ x'(0) = 2$

In Exercises 5–8, express the solution of the initial-value problem in the form

$$x(t) = e^{-pt}\sqrt{\alpha^2 + \left[\frac{\beta + \alpha p}{\mu}\right]^2} \cos(\mu t - \phi).$$

In each case, find the quasiperiod and the time at which the mass first passes through its equilibrium position.

5. $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 13x = 0,\ x(0) = 1,\ x'(0) = -1$

6. $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 20x = 0,\ x(0) = 1,\ x'(0) = 2$

*7. $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 26x = 0,\ x(0) = 1,\ x'(0) = 1$

8. $\dfrac{d^2x}{dt^2} + 10\dfrac{dx}{dt} + 41x = 0,\ x(0) = 3,\ x'(0) = -2$

In Exercises 9–16, solve the initial-value problem. Classify each as overdamped or critically damped. Determine if the mass passes through its equilibrium position and, if so, when. Determine the maximum displacement of the object from the equilibrium position.

9. $\dfrac{d^2x}{dt^2} + 8\dfrac{dx}{dt} + 15x = 0,\ x(0) = 0,\ x'(0) = 1$

10. $\dfrac{d^2x}{dt^2} + 7\dfrac{dx}{dt} + 12x = 0,\ x(0) = -1,\ x'(0) = 4$

*11. $\dfrac{d^2x}{dt^2} + \dfrac{3}{2}\dfrac{dx}{dt} + \dfrac{1}{2}x = 0,\ x(0) = -1,\ x'(0) = 2$

12. $\dfrac{d^2x}{dt^2} + 5\dfrac{dx}{dt} + 4x = 0,\ x(0) = 0,\ x'(0) = 5$

13. $\dfrac{d^2x}{dt^2} + 8\dfrac{dx}{dt} + 16x = 0,\ x(0) = 4,\ x'(0) = -2$

14. $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 9x = 0,\ x(0) = 3,\ x'(0) = -3$

*15. $\dfrac{d^2x}{dt^2} + 10\dfrac{dx}{dt} + 25x = 0,\ x(0) = -5,\ x'(0) = 1$

16. $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + \dfrac{1}{4}x = 0,\ x(0) = -1,\ x'(0) = 2$

17. Suppose that an object with $m = 1$ is attached to the end of a spring with spring constant $k = 1$. After reaching its equilibrium position, the object is pulled one unit above the equilibrium and released with an initial velocity $v_0$. If the spring–mass system is critically damped, what is the value of $c$?

18. A weight having mass $m = 1$ is attached to the end of a spring with $k = 5/4$ and $F_R = 2\,dx/dt$. Determine the displacement of the mass if the object is released from the equilibrium position with an initial velocity of 3 units/s in the downward direction.

*19. A 32-lb weight is attached to the end of a spring with spring constant $k = 24$ lb/ft. If the resistive force is $F_R = 10\,dx/dt$, determine the displacement of the mass if it is released with no initial velocity from a position

mass passes through its equilibrium position and, if so, when. Determine the maximum displacement of the object from the equilibrium position.

20. An object weighing 8 lb stretches a spring 6 in. beyond its natural length. If the resistive force is $F_R = 4\,dx/dt$, find the displacement of the mass if it is set into motion from its equilibrium position with an initial velocity of 1 ft/s in the downward direction.

**21.** An object of mass $m = 70$ kg is attached to the end of a spring and stretches the spring 0.25 m beyond its natural length. If the restrictive force is $F_R = 280\, dx/dt$, find the displacement of the object if it is released from a position 3 m above its equilibrium position with no initial velocity. Does the object pass through its equilibrium position at any time?

**22.** Suppose that an object of mass $m = 1$ slug is attached to a spring with spring constant $k = 25$ lb/ft. If the resistive force is $F_R = 6\, dx/dt$, determine the displacement of the object if it is set into motion from its equilibrium position with an upward velocity of 2 ft/s. What is the quasiperiod of the motion?

***23.** An object of mass $m = 4$ slugs is attached to a spring with spring constant $k = 64$ lb/ft. If the resistive force is $F_R = c\, dx/dt$, find the value of $c$ so that the motion is critically damped. For what values of $c$ is the motion underdamped?

**24.** An object of mass $m = 2$ slugs is attached to a spring with spring constant $k$ lb/ft. If the resistive force is $F_R = 8\, dx/dt$, find the value of $k$ so that the motion is critically damped. For what values of $k$ is the motion underdamped? For what values of $k$ is the motion overdamped?

**25.** If the quasiperiod of the underdamped motion is $\pi/6$ seconds when a 1/13 slug mass is attached to a spring with spring constant $k = 13$ lb/ft, find the damping constant $c$.

**26.** If a mass of 0.2 kg is attached to a spring with spring constant $k = 5$ N/m that undergoes damping equivalent to $\frac{6}{5}\, dx/dt$, find the quasiperiod of the resulting motion.

**27.** Show that the solution $x(t)$ of the initial-value problem

$$\begin{cases} m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx = 0 \\ x(0) = \alpha,\ \dfrac{dx}{dt}(0) = \beta \end{cases}$$

can be written as $x(t) = u(t) + v(t)$, where $u$ and $v$ satisfy the same differential equation as $x$, $u$ satisfies the initial conditions $u(0) = \alpha$, $u'(0) = 0$, and $v$ satisfies the initial conditions $v(0) = 0$, $v'(0) = \beta$.

**28.** If the spring–mass system $m\, d^2x/dt^2 + c\, dx/dt + kx = 0$ is either critically damped or overdamped, show that the mass can pass through its equilibrium position at most one time (independent of the initial conditions).

***29.** Suppose that the spring–mass system

$$\begin{cases} m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx = 0 \\ x(0) = \alpha,\ x'(0) = \beta \end{cases}$$

is critically damped. If $\beta = 0$, show that $\lim_{t\to\infty} x(t) = 0$, but that there is no value $t = t_0$ such that $x(t_0) = 0$. (The mass approaches but never reaches its equilibrium position.)

**30.** Suppose that the spring–mass system

$$\begin{cases} m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx = 0 \\ x(0) = \alpha,\ x'(0) = \beta \end{cases}$$

is critically damped. If $\beta > 0$, find a condition on $\beta$ so that the mass passes through its equilibrium position after it is released.

**31.** In the case of underdamped motion, show that the amount of time between two successive times at which the mass passes through its equilibrium is one-half of the quasiperiod.

**32.** In the case of underdamped motion, show that the amount of time between two successive positive maxima of the position function is $4\pi m/\sqrt{4km - c^2}$.

**33.** In the case of underdamped motion, show that the ratio between two consecutive maxima (or minima) is $e^{2c\pi/\sqrt{4mk-c^2}}$.

**34.** The natural logarithm of the ratio in Exercise 33, called the **logarithmic decrement**, is

$$d = \frac{2c\pi}{\sqrt{4mk - c^2}}$$

and indicates the rate at which the motion dies out because of damping. Notice that because $m$, $k$, and $d$ are all quantities that can be measured in the spring–mass system, the value of $d$ is useful in measuring the damping constant $c$. Compute the logarithmic decrement of the system in **(a)** Example 3 and **(b)** Example 4.

**35.** Determine how the value of $c$ affects the solution of the initial-value problem

$$\begin{cases} \dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + 6x = 0 \\ x(0) = 0,\ x'(0) = 1, \end{cases}$$

where $c = 2\sqrt{6}$, $4\sqrt{6}$, and $\sqrt{6}$.



**36.** In Problem 35, consider the solution that results using the damping coefficient that produces critical damping with $x'(0) = -1, 0, 1$, and 2. In which case does the object pass through its equilibrium position? When does it pass through the equilibrium position?

**37.** Using the values of $x'(0)$ and $c$ in Problem 36, in addition suppose that the equilibrium position is 1 unit above the floor. Does the object come into contact with the floor in any of the cases? If so, when?

**38.** Solve the initial-value problem $4x'' + cx' + 5x = 0$, $x(0) = 3$, $x'(0) = 0$ using $c = 1, -4$, and 0. Plot the solutions and compare them.

**39.** Solve the initial-value problem $x'' + cx' + x = 0$, $x(0) = -1$, $x'(0) = 3$ for values of $c = 2$ and $c = \sqrt{8}$. (Notice the effect of the coefficient of damping on the solution to the corresponding critically damped and overdamped equation.)

## 5.3 Forced Motion

In some cases, the motion of a spring is influenced by an external driving force, $f(t)$. Mathematically, this force is included in the differential equation that models the situation by

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt} + f(t).$$

The resulting initial-value problem is

$$\begin{cases} m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx = f(t) \\ x(0) = \alpha,\ x'(0) = \beta. \end{cases}$$

Therefore, differential equations modeling forced motion are nonhomogeneous and require the method of undetermined coefficients or variation of parameters for solution. We first consider forced motion that is undamped.

**Example 1**

An object of mass $m = 1$ slug is attached to a spring with spring constant $k = 4$ lb/ft. Assuming there is no damping and that the object begins from rest in the equilibrium position, determine the position function of the object if it is subjected to an external force of **(a)** [illegible] $= 0$; **(b)** $f(t) = 1$; **(c)** $f(t) = \cos t$.

***Solution*** First, we note that we must solve [illegible] problem

$$\begin{cases} \dfrac{d^2x}{dt^2} + 4x = f(t) \\ x(0) = 0,\ x'(0) = 0 \end{cases}$$

for each of the forcing functions in **(a)**, **(b)**, and **(c)**. A general solution of the homogeneous problem $d^2x/dt^2 + 4x = 0$ is $x_h(t) = c_1 \cos 2t + c_2 \sin 2t$.

**(a)** With $f(t) = 0$, the equation is homogeneous, so we apply the initial conditions to $x(t) = c_1 \cos 2t + c_2 \sin 2t$ with derivative $x'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t$. Because $x(0) = c_1 = 0$ and $x'(0) = 2c_2 = 0$, $c_1 = c_2 = 0$, the solution is $x(t) = 0$. This solution indicates that the object does not move from the equilibrium position because there is no forcing function, no initial displacement from the equilibrium position, and no initial velocity.

**(b)** Using the method of undetermined coefficients with a particular solution of the form $x_p(t) = A$, substitution into the differential equation $d^2x/dt^2 + 4x = 1$ yields $4A = 1$, so $A = \frac{1}{4}$. Hence,

$$x(t) = x_h(t) + x_p(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{4},$$

with derivative $x'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t$. With $x(0) = c_1 + \frac{1}{4} = 0$ and $x'(0) = 2c_2 = 0$, we have $c_1 = -\frac{1}{4}$ and $c_2 = 0$, so

$$x(t) = -\frac{1}{4} \cos 2t + \frac{1}{4}.$$

Notice from the graph of this function in Figure 5.12(a) that the object never moves above the equilibrium position. (Positive values of $x$ indicate that the mass is below the equilibrium position.)

**(c)** In this case, we assume that $x_p(t) = A \cos t + B \sin t$. Substitution into $d^2x/dt^2 + 4x = \cos t$ yields $3A \cos t + 3B \sin t = \cos t$, so $A = \frac{1}{3}$ and $B = 0$. Therefore,

$$x(t) = x_h(t) + x_p(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{3} \cos t,$$

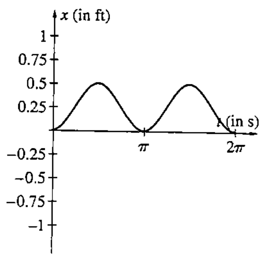
with derivative

$$x'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t - \frac{1}{3} \sin t.$$

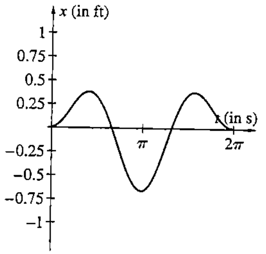
Applying the initial conditions then gives us $x(0) = c_1 + \frac{1}{3} = 0$ and $x'(0) = 2c_2 = 0$, so $c_1 = -\frac{1}{3}$ and $c_2 = 0$. Thus,

$$x(t) = -\frac{1}{3} \cos 2t + \frac{1}{3} \cos t.$$

The graph of one period of this solution is shown in Figure 5.12(b). In this case, the mass passes through the equilibrium position twice (near $t = 2$ and $t = 4$) over the period, and it returns to the equilibrium position without passing through it at $t = 2\pi$. (*Can you predict other values of t where this occurs?*)



(a)



(b)

**Figure 5.12** (a) Graph of $x(t) = -\frac{1}{4} \cos 2t + \frac{1}{4}$. (b) Graph of $x(t) = -\frac{1}{3} \cos 2t + \frac{1}{3} \cos t$.

*How does changing the initial position to $x(0) = 1$ affect the solution in (c)? How does changing the initial velocity to $x'(0) = 1$ affect the solution? Is the maximum displacement affected in either case?*

When we studied nonhomogeneous equations, we considered equations in which the nonhomogeneous (right-hand side) function was a solution of the corresponding



homogeneous equation. We consider this type of situation with the initial-value problem

$$\begin{cases} \dfrac{d^2x}{dt^2} + \omega^2 x = F_1 \cos \omega t + F_2 \sin \omega t + G(t) \\ x(0) = \alpha,\ x'(0) = \beta, \end{cases}$$

where $F_1$ and $F_2$ are constants and $G$ is any function of $t$. (Note that one of the constants $F_1$ and $F_2$ can equal zero and $G$ can be identically the zero function.) In this case, we say that $\omega$ is the **natural frequency of the system** because a general solution of the corresponding homogeneous equation is $x_h(t) = c_1 \cos \omega t + c_2 \sin \omega t$. In the case of this initial-value problem, the **forced frequency**, the frequency of the trigonometric functions in $F_1 \cos \omega t + F_2 \sin \omega t + G(t)$, equals the natural frequency.

**Example 2**

Investigate the effect that the forcing function $f(t) = \cos 2t$ has on the solution of the initial-value problem

$$\begin{cases} \dfrac{d^2x}{dt^2} + 4x = f(t) \\ x(0) = 0,\ x'(0) = 0. \end{cases}$$

***Solution*** As we saw in Example 1, $x_h(t) = c_1 \cos 2t + c_2 \sin 2t$. Because $f(t) = \cos 2t$ is contained in this solution, we assume that $x_p(t) = At \cos 2t + Bt \sin 2t$, with first and second derivatives

$$x_p'(t) = A \cos 2t - 2At \sin 2t + B \sin 2t + 2Bt \cos 2t$$

and

$$x_p''(t) = -4A \sin 2t - 4At \cos 2t + 4B \cos 2t - 4Bt \sin 2t.$$

Substitution into $d^2x/dt^2 + 4x = \cos 2t$ yields $-4A \sin 2t + 4B \cos 2t = \cos 2t$. Thus $A = 0$ and $B = 1/4$, so $x_p(t) = \frac{1}{4}t \sin 2t$, and

$$x(t) = x_h(t) + x_p(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{4} t \sin 2t.$$

This function has derivative

$$x'(t) = -2c_1 \sin 2t + [illegible] 2t + \frac{1}{4} \sin 2t + \frac{1}{2} t \cos 2t,$$

so application of the initial conditions yields $x$[illegible] $= 0$ and $x'(0) = 2c_2 = 0$. Therefore, $c_1 = c_2 = 0$, so

$$x(t) = \frac{1}{4} t \sin 2t.$$

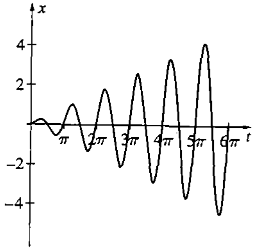


Figure 5.13 Resonance.

The graph of this solution is shown in Figure 5.13. Notice that the amplitude increases as $t$ increases. This indicates that the spring–mass system will encounter a serious problem; either the spring will break or the mass will eventually hit its support (like a ceiling or beam) or a lower boundary (like the ground or floor).

The phenomenon illustrated in Example 2 is called **resonance** and can be extended to other situations such as vibrations in an aircraft wing, a skyscraper, a glass, or a bridge. Some of the sources of excitation that lead to the vibration of these structures include unbalanced rotating devices, vortex shedding, strong winds, rough surfaces, and moving vehicles. Many engineers must overcome problems caused when structures and machines are subjected to forced vibrations.

*Over a sufficient amount of time, do changes in the initial conditions affect the motion of a spring–mass system? Experiment by changing the initial conditions in the initial-value problem in Example 2.*

Let us investigate in detail initial-value problems of the form

$$\begin{cases} \dfrac{d^2x}{dt^2} + \omega^2 x = F\cos\beta t,\ \omega \neq \beta \\ x(0) = 0,\ x'(0) = 0. \end{cases}$$

A general solution of the corresponding homogeneous equation is $x_h(t) = c_1\cos\omega t + c_2\sin\omega t$. Using the method of undetermined coefficients, a particular solution is given by $x_p(t) = A\cos\beta t + B\sin\beta t$. The corresponding derivatives of this solution are

$$x_p'(t) = -A\beta\sin\beta t + B\beta\cos\beta t \quad \text{and} \quad x_p''(t) = -A\beta^2\cos\beta t - B\beta^2\sin\beta t.$$

Substituting into the nonhomogeneous equation $d^2x/dt^2 + \omega^2 x = F\cos\beta t$ and equating the corresponding coefficients yields

$$A = \frac{F}{\omega^2 - \beta^2} \quad \text{and} \quad B = 0.$$

Therefore, a general solution is

$$x(t) = c_1\cos\omega t + c_2\sin\omega t + \frac{F}{\omega^2 - \beta^2}\cos\beta t.$$

Application of the initial conditions yields the solution

$$x(t) = \frac{F}{\omega^2 - \beta^2}(\cos\beta t - \cos\omega t).$$

Using the trigonometric identity $\frac{1}{2}[\cos(A - B) - \cos(A + B)] = \sin A \sin B$, we have

$$x(t) = \frac{2F}{\omega^2 - \beta^2}\sin\frac{(\omega + \beta)t}{2}\sin\frac{(\omega - \beta)t}{2}.$$



Notice that the solution can be represented as

$$x(t) = A(t)\sin\frac{(\omega + \beta)t}{2},$$

where

$$A(t) = \frac{2F}{\omega^2 - \beta^2}\sin\frac{(\omega - \beta)t}{2}.$$

Therefore, when the quantity $(\omega - \beta)$ is small, $(\omega + \beta)$ is relatively large in comparison. The function $\sin(\omega + \beta)t/2$ oscillates quite frequently because it has period $4\pi/(\omega + \beta)$. Meanwhile, the function $\sin(\omega - \beta)t/2$ oscillates relatively slowly because it has period $4\pi/(\omega - \beta)$. When we graph $x(t)$, we see that the functions $\pm\dfrac{2F}{\omega^2 - \beta^2}\sin\dfrac{(\omega - \beta)t}{2}$ form an **envelope** for the solution.

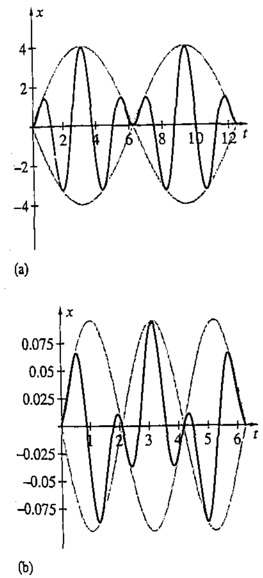


Figure 5.14

## Example 3

Solve the initial-value problem

$$\begin{cases} \dfrac{d^2x}{dt^2} + 4x = f(t) \\ x(0) = 0,\ x'(0) = 0, \end{cases}$$

with **(a)** $f(t) = \cos 3t$ and **(b)** $f(t) = \cos 5t$.

***Solution*** **(a)** A general solution of the corresponding homogeneous equation is $x_h(t) = c_1\cos 2t + c_2\sin 2t$. By the method of undetermined coefficients, we assume that $x_p(t) = A\cos 3t + B\sin 3t$. Substitution into $x'' + 4x = \cos 3t$ yields $-5A\cos 3t - 5B\sin 3t = \cos 3t$, so $A = -1/5$ and $B = 0$. Thus $x_p(t) = -\frac{1}{5}\cos 3t$ and

$$x(t) = x_h(t) + x_p(t) = c_1\cos 2t + c_2\sin 2t - \frac{1}{5}\cos 3t.$$

Because the derivative of this function is

$$x'(t) = -2c_1\sin 2t + 2c_2\cos 2t + \frac{3}{5}\sin 3t,$$

application of the initial conditions yields $x(0) = c_1 - 1/5 = 0$, and $x'(0) = 2c_2 = 0$. Therefore, $c_1 = 1/5$ and $c_2 = 0$, so

$$x(t) = \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t.$$

The graph of this function is shown in Figure 5.14(a) along with the envelope functions

$$\frac{2}{5}\sin\frac{t}{2} \quad \text{and} \quad -\frac{2}{5}\sin\frac{t}{2}.$$

**(b)** In a similar manner, the solution of the initial-value problem

$$\begin{cases} \dfrac{d^2x}{dt^2} + 4x = \cos 5t \\ x(0) = 0,\ x'(0) = 0 \end{cases}$$

is

$$x(t) = \frac{1}{21}\cos 2t - \frac{1}{21}\cos 5t.$$

The graph of the solution is shown in Figure 5.14(b) along with the envelope functions

$$\frac{2}{21}\sin\frac{3t}{2} \quad \text{and} \quad -\frac{2}{21}\sin\frac{3t}{2}.$$

Some computer algebra systems contain commands that allow us to "play" functions. If this is possible with your technology, see Exercises 27 and 28.

Oscillations like those illustrated in Example 3 are called **beats** because of the periodic variation of amplitude. This phenomenon is commonly encountered when two musicians try to simultaneously tune their instruments or when two tuning forks with almost equivalent frequencies are played at the same time.

*Consider the problem*

$$\begin{cases} \dfrac{d^2x}{dt^2} + 4x = \cos \beta t \\ x(0) = 0,\ x'(0) = 0 \end{cases}$$

*for $\beta = 6, 8,$ and $10$. What happens to the amplitude of the beats as $\beta$ increases?*

**Example 4**

Investigate the effect that the forcing function $f(t) = e^{-t}\cos 2t$ has on the initial-value problem

$$\begin{cases} \dfrac{d^2x}{dt^2} + 4x = f(t) \\ x(0) = 0,\ x'(0) = 0. \end{cases}$$

***Solution*** Using the method of undetermined coefficients, a general solution of the equation is



$$x(t) = x_h(t) + x_p(t) = c_1\cos 2t + c_2\sin 2t + \frac{1}{17}e^{-t}\cos 2t - \frac{4}{17}e^{-t}\sin 2t.$$

Applying the initial conditions with $x(t)$ and

$$x'(t) = -2c_1\sin 2t + 2c_2\cos 2t - \frac{9}{17}e^{-t}\cos 2t + \frac{2}{17}e^{-t}\sin 2t,$$

we have

$$x(0) = c_1 + \frac{1}{17} = 0 \quad \text{and} \quad x'(0) = 2c_2 - \frac{9}{17} = 0.$$

Therefore, $c_1 = -1/17$ and $c_2 = 9/34$, so

$$x(t) = -\frac{1}{17}\cos 2t + \frac{9}{34}\sin 2t + \frac{1}{17}e^{-t}\cos 2t - \frac{4}{17}e^{-t}\sin 2t,$$

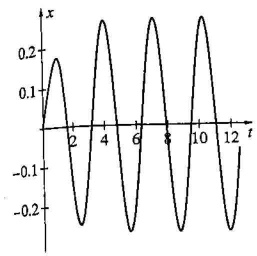



**Figure 5.15**

which is graphed in Figure 5.15. Notice that the effect of terms involving the exponential function diminishes as $t$ increases. In this case, the forcing function $f(t) = e^{-t}\cos 2t$ approaches zero as $t$ increases. Over time, the solution of the nonhomogeneous problem approaches that of the corresponding homogeneous problem, so we observe simple harmonic motion as $t \to \infty$.

We now consider spring problems that involve forces due to damping as well as external forces. In particular, consider the initial-value problem

$$\begin{cases} m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx = \rho\cos\lambda t \\ x(0) = \alpha,\ x'(0) = \beta, \end{cases}$$

which has a solution of the form

$$x(t) = h(t) + s(t),$$

where $\lim_{t\to\infty} h(t) = 0$ and $s(t) = c_1\cos\lambda t + c_2\sin\lambda t$.

The function $h(t)$ is called the **transient** solution, and $s(t)$ is known as the **steady-state** solution. Therefore, as $t$ approaches infinity, the solution $x(t)$ approaches the steady-state solution. (Why?) Note that the steady-state solution corresponds to a particular solution obtained through the method of undetermined coefficients or variation of parameters.

**Example 5**

Solve the initial-value problem

$$\begin{cases} \dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 13x = \cos t \\ x(0) = 0,\ x'(0) = 1 \end{cases}$$

that models the motion of an object of mass $m = 1$ slug attached to a spring with spring constant $k = 13$ lb/ft that is subjected to a resistive force of $F_R = 4\,dx/dt$ and an external force of $f(t) = \cos t$. Identify the transient and steady-state solutions.

**Solution** A general solution of $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 13x = 0$ is $x_h(t) = e^{-2t}(c_1 \cos 3t + c_2 \sin 3t)$. We assume that a particular solution has the form $x_p(t) = A\cos t + B\sin t$, with derivatives $x_p'(t) = -A\sin t + B\cos t$ and $x_p''(t) = -A\cos t - B\sin t$. After substitution into $d^2x/dt^2 + 4\,dx/dt + 13x = \cos t$, we see that $\begin{cases} 12A + 4B = 1 \\ -4A + 12B = 0 \end{cases}$ must be satisfied. Hence $A = 3/40$ and $B = 1/40$, so

$$x_p(t) = \frac{3}{40}\cos t + \frac{1}{40}\sin t.$$

Therefore,

$$x(t) = x_h(t) + x_p(t) = e^{-2t}(c_1 \cos 3t + c_2 \sin 3t) + \frac{3}{40}\cos t + \frac{1}{40}\sin t,$$

with derivative

$$x'(t) = -2e^{-2t}(c_1 \cos 3t + c_2 \sin 3t) + 3e^{-2t}(-c_1 \sin 3t + c_2 \cos 3t) - \frac{3}{40}\sin t + \frac{1}{40}\cos t.$$

Application of the initial conditions yields $x(0) = c_1 + (3/40) = 0$ and $x'(0) = -2c_1 + 3c_2 + (1/40) = 1$. Therefore, $c_1 = -3/40$ and $c_2 = 11/40$, so

$$x(t) = e^{-2t}\left(-\frac{3}{40}\cos 3t + \frac{11}{40}\sin 3t\right) + \frac{3}{40}\cos t + \frac{1}{40}\sin t.$$

This indicates that the transient solution is $e^{-2t}(-\frac{3}{40}\cos 3t + \frac{11}{40}\sin 3t)$ and the steady-state solution is $\frac{3}{40}\cos t + \frac{1}{40}\sin t$. We graph this solution in Figure 5.16(a). The solution and the steady-state solution are graphed together in Figure 5.16(b). Notice that the two curves appear identical for $t > 2.5$. The reason for this is shown in the subsequent plot of the transient solution (in Figure 5.16(c)), which becomes quite small near $t = 2.5$.

*Over a sufficient amount of time, do changes in the initial conditions in Example 5 affect the motion of the spring–mass system? Experiment by changing the initial conditions in the initial-value problem in Example 5.*



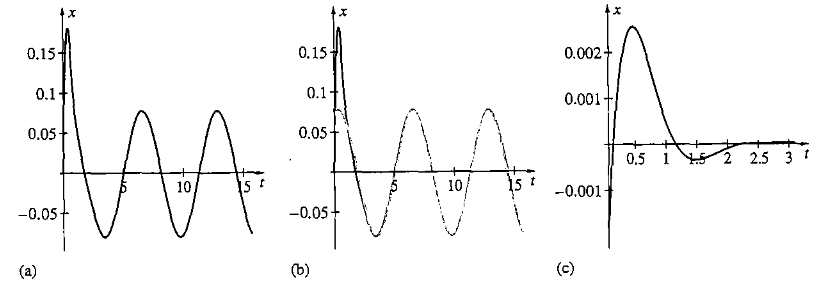



Figure 5.16

## EXERCISES 5.3

1. An 8-lb weight stretches a spring 1 ft. If a 16-lb weight is attached to the spring, it comes to rest in its equilibrium position. If it is then put into motion with a downward initial velocity of 2 ft/s, determine the displacement of the mass if there is no damping and an external force $f(t) = \cos 3t$. What is the natural frequency of the spring–mass system?
2. A 16-lb weight stretches a spring 6 in. If the mass is lowered 1 ft below its equilibrium position and released, determine the displacement of the mass if there is no damping and an external force of $f(t) = 2\cos t$. What is the natural frequency of the spring–mass system?
*3. A 16-lb weight stretches a spring 8 in. If the mass is lowered 4 in. below its equilibrium position and released, determine the displacement of the mass if there is no damping and an external force of $f(t) = 2\cos t$.
4. A 6-lb weight stretches a spring 6 in. The mass is raised 3 in. above its equilibrium position and released. Determine the displacement of the mass if there is no damping and an external force of $f(t) = 2\cos 5t$.
5. An object of mass $m = 1$ kg is attached to a spring with spring constant $k = 9$ kg/m. If there is no damping and the external force is $f(t) = 4\cos \omega t$, find the displacement of the object if $x(0) = 0$ and $x'(0) = 0$. What must the value of $\omega$ be for resonance to occur?
6. An object of mass $m = 2$ kg is attached to a spring with spring constant $k = 1$ kg/m. If the resistive force is $F_R = 3\,dx/dt$ and the external force is $f(t) = 2\cos \omega t$, find the displacement of the object if $x(0) = 0$ and $x'(0) = 0$. Will resonance occur for any values of $\omega$?
*7. An object of mass $m = 1$ slug is attached to a spring with spring constant $k = 25$ lb/ft. If the resistive force is $F_R = 8\,dx/dt$ and the external force is $f(t) = \cos t - \sin t$, find the displacement of the object if $x(0) = 0$ and $x'(0) = 0$.
8. An object of mass $m = 2$ slug is attached to a spring with spring constant $k = 6$ lb/ft. If the resistive force is $F_R = 6\,dx/dt$ and the external force is $f(t) = 2\sin 2t + \cos t$, find the displacement of the object if $x(0) = 0$ and $x'(0) = 0$.
9. Suppose that an object of mass 1 slug is attached to a spring with spring constant $k = 4$ lb/ft. If the motion of the object is undamped and subjected to an external force of $f(t) = \cos t$, determine the displacement of the object if $x(0) = 0$ and $x'(0) = 0$. What functions envelope this displacement function? What is the maximum displacement of the object? If the external force is changed to $f(t) = \cos(t/2)$, does the maximum displacement increase or decrease?

10. An object of mass 1 slug is attached to a spring with spring constant $k = 25$ lb/ft. If the motion of the object is undamped and subjected to an external force of $f(t) = \cos \beta t$, $\beta \neq 5$, determine the displacement of the object if $x(0) = 0$ and $x'(0) = 0$. What is the value of $\beta$ if the maximum displacement of the object is $2/11\sqrt{2}$ ft?

*11. An object of mass 4 slugs is attached to a spring with spring constant $k = 26$ lb/ft. It is subjected to a resistive force of $F_R = 4\,dx/dt$ and an external force $f(t) = 250 \sin t$. Determine the displacement of the object if $x(0) = 0$ and $x'(0) = 0$. What is the transient solution? What is the steady-state solution?

12. An object of mass 1 slug is attached to a spring with spring constant $k = 4000$ lb/ft. It is subjected to a resistive force of $F_R = 40\,dx/dt$ and an external force $f(t) = 600 \sin t$. Determine the displacement of the object if $x(0) = 0$ and $x'(0) = 0$. What is the transient solution? What is the steady-state solution?

13. Find the solution of the differential equation

$$m\frac{d^2x}{dt^2} + kx = F \sin \omega t,\ \omega \neq \sqrt{k/m}$$

that satisfies the initial conditions: **(a)** $x(0) = \alpha$, $x'(0) = 0$; **(b)** $x(0) = 0$, $x'(0) = \beta$; **(c)** $x(0) = \alpha$, $x'(0) = \beta$.

14. Find the solution of the differential equation

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F \sin \omega t,\ c^2 - 4mk < 0$$

that satisfies the initial conditions: **(a)** $x(0) = \alpha$, $x'(0) = 0$; **(b)** $x(0) = 0$, $x'(0) = \beta$; **(c)** $x(0) = \alpha$, $x'(0) = \beta$.

*15. Find the solution to the initial-value problem

$$\frac{d^2x}{dt^2} + x = \begin{cases} 1, & 0 \le t \le \pi \\ 0, & t > \pi \end{cases},\ x(0) = 0,\ x'(0) = 0.$$

(*Hint:* Solve the initial-value problem over each interval. Choose constants appropriately so that the functions $x$ and $x'$ are continuous.)

16. Find the solution to the initial-value problem

$$\frac{d^2x}{dt^2} + x = \begin{cases} \cos t, & 0 \le t \le \pi \\ 0, & t > \pi \end{cases},\ x(0) = 0,\ x'(0) = 0.$$

17. Find the solution to the initial-value problem $\frac{d^2x}{dt^2} + x = f(t)$, $x(0) = 0$, $x'(0) = 0$, where

$$f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2 - t, & 1 < t \le 2 \\ 0, & t > 2. \end{cases}$$

(*Hint:* Solve the initial-value problem over each interval. Choose constants appropriately so that the functions $x$ and $x'$ are continuous.)

18. Find the solution to the initial-value problem

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 13x = f(t),\quad x(0) = 0,\quad x'(0) = 0$$

$$\text{where } f(t) = \begin{cases} 1, & 0 \le t \le \pi \\ 1 - t, & \pi < 1 \le 2\pi \\ 0, & t > 2\pi. \end{cases}$$

19. Show that a general solution of

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F \sin \gamma t$$

is

$$x(t) = Ae^{-\lambda t}\sin\left(\sqrt{\omega^2 - \lambda^2}\,t + \phi\right) + \frac{F}{\sqrt{(\omega^2 - \gamma^2)^2 + 4\lambda^2\gamma^2}}\sin(\gamma t + \theta),$$

where $A = \sqrt{c_1^2 + c_2^2}$, $\lambda < \omega$, and the phase angles $\phi$ and $\theta$ are found with $\sin \phi = c_1/A$, $\cos \phi = c_2/A$,

$$\sin \theta = \frac{-2\lambda\gamma}{\sqrt{(\omega^2 - \gamma^2)^2 + 4\lambda^2\gamma^2}},$$

and

$$\cos \theta = \frac{\omega^2 - \gamma^2}{\sqrt{(\omega^2 - \gamma^2)^2 + 4\lambda^2\gamma^2}}.$$

20. The steady-state solution (the approximation of the solution for large values of $t$) of the differential equation in Exercise 19 is $x_h(t) = g(\gamma)\sin(\gamma t + \theta)$, where $g(\gamma) = F/\sqrt{(\omega^2 - \gamma^2)^2 + 4\lambda^2\gamma^2}$. By differentiating $g$ with respect to $\gamma$, show that the maximum value of $g(\gamma)$ occurs when $\gamma = \sqrt{\omega^2 - 2\lambda^2}$. The quantity $\sqrt{\omega^2 - 2\lambda^2}/2\pi$ is called the **resonance frequency** for the system. Describe the motion if the external force has frequency $\sqrt{\omega^2 - 2\lambda^2}/2\pi$.

21. Solve the initial-value problem $x'' + x = \cos t$, $x(0) = 0$, $x'(0) = b$ using $b = 0$ and $b = 1$. Graph the solutions simultaneously to determine the effect that the nonhomogeneous initial velocity has on the solution to the second initial-value problem as $t$ increases.



22. Solve the initial-value problem $x'' + x = \cos \omega t$, $x(0) = 0$, $x'(0) = 1$ using values $\omega = 0.9$ and $\omega = 0.7$. Graph the solutions simultaneously to determine the effect that the value of $\omega$ has on each solution.

*23. Investigate the effect that the forcing functions **(a)** $f(t) = \cos 1.9t$ and **(b)** $f(t) = \cos 2.1t$ have on the initial-value problem

$$\begin{cases} \dfrac{d^2x}{dt^2} + 4x = f(t) \\ x(0) = 0,\ \dfrac{dx}{dt}(0) = 0. \end{cases}$$

How do these results differ from those of Example 3? Are there more or fewer beats with these two functions?

24. Solve the initial-value problem

$$x'' + 0.1x' + x = 3\cos 2t,\ x(0) = 0,\ x'(0) = 0$$

using the method of undetermined coefficients and compare the result with the forcing function $3\cos 2t$. Determine the phase difference between these two functions.

25. Solve the following initial-value problem involving a piecewise defined forcing function over $[0, 2]$:

$$\begin{cases} \dfrac{d^2x}{dt^2} + x = \begin{cases} t, & 0 \le t \le 1 \\ 2 - t, & 1 < t \le 2 \\ 0, & t > 2. \end{cases} \\ x(0) = a,\ x'(0) = b \end{cases}$$

Graph the solution using the initial conditions $a = 1$, $b = 1$; $a = 0$, $b = 1$; and $a = 1$; $b = 0$.

26. Consider the function $g(\gamma) = F/\sqrt{(\omega^2 - \gamma^2)^2 + 4\lambda^2\gamma^2}$ defined in Exercise 20. **(a)** Graph this function for $k = 4$, $m = 1$, $F = 2$, and the damping coefficient: $c = 2\lambda = 2$, 1, 0.75, 0.50, 0.25. **(b)** Graph the function for $k = 49$, $m = 10$, $F = 20$, and the damping coefficient: $c = 2\lambda = 2$, 1, 0.75, 0.50, 0.25. In each case, describe what happens to the maximum magnitude of $g(\gamma)$ as $c \to 0$. Also, as $c \to 0$, how does the resonance frequency relate to the natural frequency of the corresponding undamped system?

*27. To hear beats, solve the initial-value problem

$$\begin{cases} \dfrac{d^2x}{dt^2} + \omega^2 x = F\cos\beta t,\ \omega \neq \beta \\ x(0) = 0,\ x'(0) = 0 \end{cases}$$

using $\omega^2 = 6000$, $\beta = 5991.62$, and $F = 2$. In each case, plot and, if possible, play the solution. (*Note:* The purpose of the high frequencies is to assist in hearing the solutions when they are played.)

28. To hear resonance, solve the initial-value problem

$$\begin{cases} \dfrac{d^2x}{dt^2} + \omega^2 x = F\cos\beta t,\ \omega \neq \beta \\ x(0) = 0,\ x'(0) = 0 \end{cases}$$

using $\omega^2 = 6000$, $\beta = 6000$, and $F = 2$. In each case, plot and, if possible, play the solution. (*Note:* The purpose of the high frequencies is to assist in hearing the solutions when they are played.)

## 5.4 Other Applications

*L-R-C* Circuits · Deflection of a Beam

### *L-R-C* Circuits

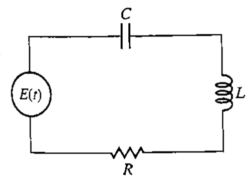
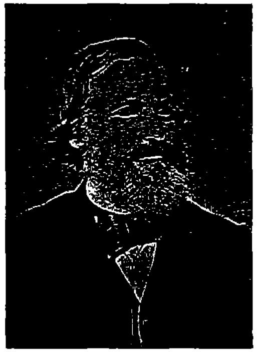



**Figure 5.17** An *L-R-C* circuit.

Second-order nonhomogeneous linear ordinary differential equations arise in the study of electrical circuits after the application of *Kirchhoff's law.* Suppose that $I(t)$ is the current in the *L-R-C* series electrical circuit (shown in Figure 5.17) where $L$, $R$, and $C$ represent the inductance, resistance, and capacitance of the circuit, respectively.

The voltage drops across the circuit elements in Table 5.2 have been obtained from experimental data, where $Q$ is the charge of the capacitor and $dQ/dt = I$.



**Gustav Robert Kirchhoff** (1824–1887). German physicist; worked in spectrum analysis, optics, and electricity. (*Northwind Picture Archives*)

**TABLE 5.2**

| Circuit Element | Voltage Drop |
|---|---|
| Inductor | $L\dfrac{dI}{dt}$ |
| Resistor | $RI$ |
| Capacitor | $\dfrac{1}{C}Q$ |

Our goal is to model this physical situation with an initial-value problem so that we can determine the current and charge in the circuit. For convenience, the terminology used in this section is summarized in Table 5.3.

**TABLE 5.3**

| Electrical Quantities | Units |
|---|---|
| Inductance ($L$) | Henrys (H) |
| Resistance ($R$) | Ohms (Ω) |
| Capacitance ($C$) | Farads (F) |
| Charge ($Q$) | Coulombs (C) |
| Current ($I$) | Amperes (A) |

The physical principle needed to derive the differential equation that models the *L-R-C* series circuit is **Kirchhoff's law.**

> **Kirchhoff's law:**
>
> The sum of the voltage drops across the circuit elements is equivalent to the voltage $E(t)$ impressed on the circuit.

Applying Kirchhoff's law with the voltage drops in Table 5.2 yields the differential equation $L\,dI/dt + RI + (1/C)Q = E(t)$. Using the fact that $dQ/dt = I$, we also have $d^2Q/dt^2 = dI/dt$. Therefore, the equation becomes $L\,d^2Q/dt^2 + R\,dQ/dt + (1/C)Q = E(t)$, which can be solved by the method of undetermined coefficients or the method of variation of parameters. If the initial charge and current are $Q(0) = Q_0$ and $I(0) = Q'(0) = I_0$, we solve the initial-value problem



$$\begin{cases} L\dfrac{d^2Q}{dt^2} + R\dfrac{dQ}{dt} + \dfrac{1}{C}Q = E(t) \\ Q(0) = Q_0,\ I(0) = Q'(0) = I_0 \end{cases}$$

for the charge $Q(t)$. The solution is differentiated to find the current $I(t)$.

**Example 1**

Consider the *L-R-C* circuit with $L = 1$ henry, $R = 40$ ohms, $C = 1/4000$ farads, and $E(t) = 24$ volts. Determine the current in this circuit if there is zero initial current and zero initial charge.

***Solution*** Using the indicated values, the initial-value problem that we must solve is

$$\begin{cases} \dfrac{d^2Q}{dt^2} + 40\dfrac{dQ}{dt} + 4000Q = 24 \\ Q(0) = 0,\ I(0) = Q'(0) = 0. \end{cases}$$

The characteristic equation of the corresponding homogeneous equation is $r^2 + 40r + 4000 = 0$ with roots $r_{1,2} = -20 \pm 60i$, so a general solution of the corresponding homogeneous equation is $Q_h(t) = e^{-20t}(c_1 \cos 60t + c_2 \sin 60t)$. Because the voltage is the constant function $E(t) = 24$, we assume that the particular solution has the form $Q_p(t) = A$. Substitution into $d^2Q/dt^2 + 40\,dQ/dt + 4000Q = 24$ yields $4000A = 24$ or $A = 3/500$. Therefore, a general solution of the nonhomogeneous equation is

$$Q(t) = Q_h(t) + Q_p(t) = e^{-20t}(c_1 \cos 60t + c_2 \sin 60t) + \frac{3}{500},$$

with derivative

$$Q'(t) = -20e^{-20t}(c_1 \cos 60t + c_2 \sin 60t) + 60e^{-20t}(-c_1 \sin 60t + c_2 \cos 60t).$$

After application of the initial conditions, we have

$$Q(0) = c_1 + \frac{3}{500} = 0 \quad \text{and} \quad \frac{dQ}{dt}(0) = -20c_1 + 60c_2 = 0.$$

Therefore, $c_1 = -3/500$ and $c_2 = -1/500$, so the charge in the circuit is

$$Q(t) = e^{-20t}\left(-\frac{3}{500}\cos 60t - \frac{1}{500}\sin 60t\right) + \frac{3}{500},$$

and the current is given by

$$\begin{aligned} Q'(t) &= -\frac{20}{500}e^{-20t}(-3\cos 60t - \sin 60t) + \frac{60}{500}e^{-20t}(3\sin 60t - \cos 60t) \\ &= \frac{2}{5}e^{-20t}\sin 60t. \end{aligned}$$

These results indicate that in time the charge approaches the constant value of 3/500, which is known as the **steady-state charge.** Also, because of the exponential term, the current approaches zero as $t$ increases. We show the graphs of $Q(t)$ and $I(t)$ in Figures 5.18(a) and (b) to verify these observations.

*In Example 1, how is the charge $Q(t)$ affected if $E(t) = 48$ volts? What happens to $Q(t)$ if $R = 40\sqrt{10}$ ohms?*

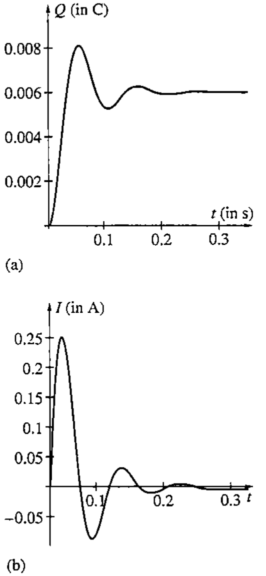



**Figure 5.18** (a) $Q(t)$ (b) $I(t)$.

## Deflection of a Beam

An important mechanical model involves the deflection of a long beam that is supported at one or both ends, as shown in Figure 5.19. Assuming that in its undeflected form the beam is horizontal, then the deflection of the beam can be expressed as a function of $x$. Suppose that the shape of the beam when it is deflected is given by the graph of the function $y(x) = -s(x)$, where $x$ is the distance from the left end of the beam and $s$ the measure of the vertical deflection from the equilibrium position. The boundary value problem that models this situation is derived as follows.

Let $m(x)$ equal the turning moment of the force relative to the point $x$, and $w(x)$ represent the weight distribution of the beam. These two functions are related by the equation

$$\frac{d^2m}{dx^2} = w(x).$$

Also, the turning moment is proportional to the curvature of the beam. Hence

$$m(x) = \frac{EI}{\left(\sqrt{1 + \left(\frac{ds}{dx}\right)^2}\right)^3}\frac{d^2s}{dx^2},$$

where $E$ and $I$ are constants related to the composition of the beam and the shape and size of a cross section of the beam, respectively. Notice that this equation is nonlinear. This difficulty is overcome with an approximation. For small values of $s$, the denominator of the right-hand side of the equation can be approximated by the constant 1. Therefore, the equation is simplified to

$$m(x) = EI\frac{d^2s}{dx^2}.$$

This equation is linear and can be differentiated twice to obtain

$$\frac{d^2m}{dx^2} = EI\frac{d^4s}{dx^4},$$

which is then used with the equation above relating $m(x)$ and $w(x)$ to obtain the single fourth-order linear nonhomogeneous differential equation

$$EI\frac{d^4s}{dx^4} = w(x).$$

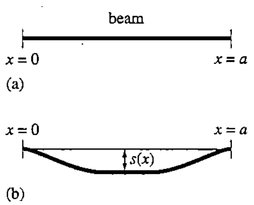



**Figure 5.19**



Boundary conditions for this problem may vary. In most cases, two conditions are given for each end of the beam. Some of these conditions, which are specified in pairs at $x = \rho$, where $\rho = 0$ or $\rho = a$, include $s(\rho) = 0$, $s'(\rho) = 0$ (fixed end); $s''(\rho) = 0$, $s'''(\rho) = 0$ (free end); $s(\rho) = 0$, $s''(\rho) = 0$ (simple support); and $s'(\rho) = 0$, $s'''(\rho) = 0$ (sliding clamped end).

**Example 2**

Solve the beam equation over the interval $0 \le x \le 1$ if $E = I = 1$, $w(x) = 48$, and the following boundary conditions are used: $s(0) = 0$, $s'(0) = 0$ (fixed end at $x = 0$); and

**(a)** $s(1) = 0$, $s''(1) = 0$ (simple support at $x = 1$)
**(b)** $s''(1) = 0$, $s'''(1) = 0$ (free end at $x = 1$)
**(c)** $s'(1) = 0$, $s'''(1) = 0$ (sliding clamped end at $x = 1$)
**(d)** $s(1) = 0$, $s'(1) = 0$ (fixed end at $x = 1$).

***Solution*** We begin by noting that the differential equation is $d^4s/dx^4 = 48$, which is a separable equation that can be solved by integrating each side four times to yield

$$s(x) = 2x^4 + c_1x^3 + c_2x^2 + c_3x + c_4,$$

with derivatives $s'(x) = 8x^3 + 3c_1x^2 + 2c_2x + c_3$, $s''(x) = 24x^2 + 6c_1x + 2c_2$, and $s'''(x) = 48x + 6c_1$. (Note that we could have used the method of undetermined coefficients or variation of parameters to find this solution.) We next determine the arbitrary constants for the pair of boundary conditions at $x = 0$. Because $s(0) = c_4 = 0$ and $s'(0) = c_3 = 0$,

$$s(x) = 2x^4 + c_1x^3 + c_2x^2.$$

**(a)** Because $s(1) = 2 + c_1 + c_2 = 0$ and $s''(1) = 24 + 6c_1 + 2c_2 = 0$, $c_1 = -5$ and $c_2 = 3$. Hence $s(x) = 2x^4 - 5x^3 + 3x^2$. We can visualize the shape of the beam by graphing $y = -s(x)$ as shown in Figure 5.20(a). **(b)** In this case, $s''(1) = 24 + 6c_1 + 2c_2 = 0$ and $s'''(1) = 48 + 6c_1 = 0$, so $c_1 = -8$ and $c_2 = 12$. The deflection of the beam is given by $s(x) = 2x^4 - 8x^3 + 12x^2$. We graph $y = -s(x)$ in Figure 5.20(b). We see from the graph that the end is free at $x = 1$. **(c)** Because $s'(1) = 8 + 3c_1 + 2c_2 = 0$ and $s'''(1) = 48 + 6c_1 = 0$, $c_1 = -8$ and $c_2 = 8$. Therefore, $s(x) = 2x^4 - 8x^3 + 8x^2$. From the shape graph of $y = -s(x)$ in Figure 5.20(c), we see that the end at $x = 1$ is clamped as compared to the free end in (b). **(d)** In this instance $s(1) = 2 + c_1 + c_2 = 0$ and $s'(1) = 8 + 3c_1 + 2c_2 = 0$, so $c_1 = -4$ and $c_2 = 2$. Thus, $s(x) = 2x^4 - 4x^3 + 2x^2 \cdot y = -s(x)$ is graphed in Figure 5.20(d). Notice that both ends are fixed. Finally, all four graphs are shown together in Figure 5.21 to compare the different boundary conditions.

*If we had used free ends at both $x = 0$ and $x = 1$ in Example 2, what is the displacement? Is this what we should expect from the physical problem?*

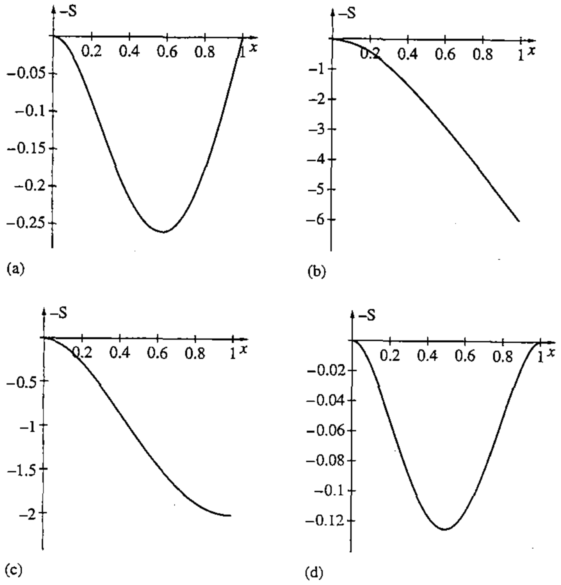



Figure 5.20

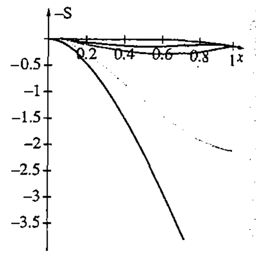



Figure 5.21

## EXERCISES 5.4

In Exercises 1–4, find the charge on the capacitor and the current in the $L$-$C$ series circuit (in which $R = 0$) assuming that $Q(0) = 0$ and $I(0) = Q'(0) = 0$.

1. $L = 2$ henry, $C = 1/32$ farad, and $E(t) = 220$ volts.
2. $L = 2$ henry, $C = 1/50$ farad, and $E(t) = 220$ volts.

*3. $L = \frac{1}{4}$ henry, $C = 1/64$ farad, and $E(t) = 16t$ volts.

4. $L = \frac{1}{4}$ henry, $C = 1/64$ farad, and $E(t) = 16 \sin 4t$ volts.
5. Find the charge $Q(t)$ on the capacitor in an $L$-$R$-$C$ series circuit if $L = 0.2$ henry, $R = 25$ ohms, $C = 0.001$ farad, $E(t) = 0$, $Q(0) = 0$ coulombs, and $I(0) = Q'(0) = 4$ amps. What is the maximum charge on the capacitor?
6. Consider the $L$-$R$-$C$ circuit given in Exercise 5 with $E(t) = 1$. Determine the value of $Q(t)$ as $t$ approaches infinity.

*7. Consider the solution to the $L$-$R$-$C$ series circuit indicated in Exercise 5. In this case, let $E(t) = 126 \cos t + 5000 \sin t$. Determine the solution to this initial-value problem. At what time does the charge first equal zero? What are the steady-state charge and current?

8. If the resistance, $R$, is changed in Exercise 5 to $R = 8$ ohms, what is the resulting charge on the capacitor? What is the maximum charge attained, and when does the charge first equal zero?



9. A beam of length 10 is fixed at both ends. Determine the shape of the beam if the weight distribution is the constant function $w(x) = 8$, with constants $E$ and $I$ such that $EI = 100$, $EI = 10$, and $EI = 1$. What is the displacement of the beam from $s = 0$ in each case? How does the value of $EI$ affect the solution?
10. Suppose that the beam in Exercise 9 is fixed at $x = 0$ and has simple support at $x = 10$. Determine the maximum displacement using constants $E$ and $I$ such that $EI = 100$, $EI = 10$, and $EI = 1$.

*11. Consider Exercise 9 with simple support at $x = 0$ and $x = 10$. How does the maximum displacement compare to that found in each case in Problems 9 and 10?

12. Determine the shape of the beam of length 10 with constants $E$ and $I$ such that $EI = 1$, weight distribution $w(x) = x^2$, and boundary conditions:
    (a) $s(0) = 0$, $s'(0) = 0$, (fixed end at $x = 0$); $s(10) = 0$, $s''(10) = 0$ (simple support at $x = 10$)
    (b) $s(0) = 0$, $s'(0) = 0$, (fixed end at $x = 0$); $s''(10) = 0$, $s'''(10) = 0$ (free end at $x = 10$)
    (c) $s(0) = 0$, $s'(0) = 0$, (fixed end at $x = 0$); $s'(10) = 0$, $s'''(10) = 0$ (sliding clamped end at $x = 10$)
    (d) $s(0) = 0$, $s'(0) = 0$, (fixed end at $x = 0$); $s(10) = 0$, $s'(10) = 0$ (fixed end at $x = 10$)

    Discuss the differences brought about by these conditions. (See Exercise 21.)

*13. Determine the shape of the beam of length 10 with constants $E$ and $I$ such that $EI = 1$, weight distribution $w(x) = x^2$, and boundary conditions:
    (a) $s(0) = 0$, $s''(0) = 0$, (simple support at $x = 0$); $s(10) = 0$, $s''(10) = 0$ (simple support at $x = 10$)
    (b) $s(0) = 0$, $s''(0) = 0$, (simple support at $x = 0$); $s''(10) = 0$, $s'''(10) = 0$ (free end at $x = 10$)
    (c) $s(0) = 0$, $s''(0) = 0$, (simple support at $x = 0$); $s'(10) = 0$, $s'''(10) = 0$ (sliding clamped end at $x = 10$)
    (d) $s(0) = 0$, $s''(0) = 0$, (simple support at $x = 0$); $s(10) = 0$, $s'(10) = 0$ (fixed end at $x = 10$)

    Discuss the differences brought about by these conditions. (See Exercise 22.)

14. Determine the shape of the beam of length 10 with constants $E$ and $I$ such that $EI = 1$, weight distribution $w(x) = 48 \sin \pi x/10$ and boundary conditions:
    (a) $s(0) = 0$, $s''(0) = 0$, (simple support at $x = 0$); $s(10) = 0$, $s''(10) = 0$ (simple support at $x = 10$)
    (b) $s(0) = 0$, $s''(0) = 0$, (simple support at $x = 0$); $s''(10) = 0$, $s'''(10) = 0$ (free end at $x = 10$)
    (c) $s(0) = 0$, $s''(0) = 0$, (simple support at $x = 0$); $s'(10) = 0$, $s'''(10) = 0$ (sliding clamped end at $x = 10$)
    (d) $s(0) = 0$, $s''(0) = 0$, (simple support at $x = 0$); $s(10) = 0$, $s'(10) = 0$ (fixed end at $x = 10$)

    Discuss the differences brought about by these conditions. (See Exercise 23.)

15. Consider the $L$-$C$ series circuit in which $E(t) = 0$ modeled by the initial-value problem
$$\begin{cases} L\dfrac{d^2Q}{dt^2} + \dfrac{1}{C}Q = 0 \\ Q(0) = Q_0,\ I(0) = Q'(0) = 0. \end{cases}$$
Find the charge $Q$ and the current $I$. What is the maximum charge? What is the maximum current?
16. Consider the $L$-$C$ series circuit in which $E(t) = 0$ modeled by the initial-value problem
$$\begin{cases} L\dfrac{d^2Q}{dt^2} + \dfrac{1}{C}Q = 0 \\ Q(0) = 0,\ I(0) = Q'(0) = I_0. \end{cases}$$
Find the charge $Q$ and the current $I$. What is the maximum charge? What is the maximum current? How do these results compare to those of Exercise 15?

*17. Consider the $L$-$R$-$C$ circuit modeled by
$$\begin{cases} L\dfrac{d^2Q}{dt^2} + R\dfrac{dQ}{dt} + \dfrac{1}{C}Q = E_0 \sin \omega t \\ Q(0) = Q_0,\ I(0) = Q'(0) = I_0. \end{cases}$$
Find the **steady-state current**, $\lim_{t\to\infty} I(t)$, of this problem. Note that in this formula, $L\omega - 1/C\omega$ is called the **reactance** of the circuit, and $\sqrt{(L\omega - 1/C\omega)^2 + R^2}$ is called the **impedance** of the circuit, where both of these quantities are measured in ohms.

18. Compute the reactance and the impedance of the circuit in Example 1 if $E(t) = 24 \sin 4t$.
19. Show that the maximum amplitude of the steady-state current found in Exercise 17 occurs when $\omega = 1/\sqrt{LC}$. (In this case, we say that **electrical resonance** occurs.)

20. **(Elastic shaft)** The differential equation that models the torsional motion of a weight suspended from an elastic shaft is $I\,d^2\theta/dt^2 + c\,d\theta/dt + k\theta = T(t)$, where $\theta$ represents the amount that the weight is twisted at time $t$, $I$ is the moment of inertia, $c$ is the damping constant, $k$ is the elastic shaft constant (similar to the spring constant), and $T(t)$ is the applied torque. Consider the differential equation with $I = 1$, $c = 4$ and $k = 13$. Find $\theta(t)$ if **(a)** $T(t) = 0$, $\theta(0) = \theta_0$, and $\theta'(0) = 0$; **(b)** $T(t) = \sin \pi x$, $\theta(0) = \theta_0$, and $\theta'(0) = 0$. Describe the motion that results in each case.

To better understand the solutions to the elastic beam problem, we can use a graphics device to determine the shape of the beam under different boundary conditions.

21. Graph the solution of the beam equation if the beam has length 10, the constants $E$ and $I$ are such that $EI = 1$, the weight distribution is $w(x) = x^2$, and the boundary conditions are
   **(a)** $s(0) = 0$, $s'(0) = 0$, (fixed end at $x = 0$); $s(10) = 0$, $s''(10) = 0$ (simple support at $x = 10$)
   **(b)** $s(0) = 0$, $s'(0) = 0$, (fixed end at $x = 0$); $s''(10) = 0$, $s'''(10) = 0$ (free end at $x = 10$)
   **(c)** $s(0) = 0$, $s'(0) = 0$, (fixed end at $x = 0$); $s'(10) = 0$, $s'''(10) = 0$ (sliding clamped end at $x = 10$)
   **(d)** $s(0) = 0$, $s'(0) = 0$, (fixed end at $x = 0$); $s(10) = 0$, $s'(10) = 0$ (fixed end at $x = 10$)

   Discuss the differences brought about by these conditions.
22. Repeat Exercise 21 with the following boundary conditions:
   **(a)** $s(0) = 0$, $s''(0) = 0$, (simple support at $x = 0$); $s(10) = 0$, $s''(10) = 0$ (simple support at $x = 10$)
   **(b)** $s(0) = 0$, $s''(0) = 0$, (simple support at $x = 0$); $s''(10) = 0$, $s'''(10) = 0$ (free end at $x = 10$)
   **(c)** $s(0) = 0$, $s''(0) = 0$, (simple support at $x = 0$); $s'(10) = 0$, $s'''(10) = 0$ (sliding clamped end at $x = 10$)
   **(d)** $s(0) = 0$, $s''(0) = 0$, (simple support at $x = 0$); $s(10) = 0$, $s'(10) = 0$ (fixed end at $x = 10$)
23. Graph the solution of the beam equation if the beam has length 10, the constants $E$ and $I$ are such that $EI = 1$, the weight distribution is $w(x) = 48 \sin(\pi x/10)$, and the boundary conditions are the same as in Problem 22.
24. Repeat Exercise 21 with $EI = 10$.
25. Repeat Exercise 22 with $EI = 100$. How do these solutions compare to those in Exercise 22?
26. Repeat Exercise 23 with $EI = 100$. How do these solutions compare to those in Exercise 23?
27. Attempt to find solutions of the beam equation in Exercises 21–23 using other combinations of boundary conditions. How do the differing boundary conditions affect the solution?

## 5.5 The Pendulum Problem

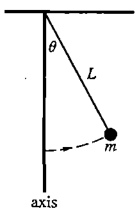


**Figure 5.22** A swinging pendulum.

Suppose that a mass $m$ is attached to the end of a rod of length $L$, the weight of which is negligible. (See Figure 5.22.) We want to determine an equation that describes the motion of the mass in terms of the displacement $\theta(t)$, which is measured counterclockwise in radians from the vertical axis shown in Figure 5.22. This is possible if we are given an initial position and an initial velocity of the mass. A force diagram for this situation is shown in Figure 5.23.

Notice that the forces are determined with trigonometry using the diagram in Figure 5.23. In this instance, $\cos\theta = mg/x$ and $\sin\theta = mg/y$, so we obtain the forces

$$x = mg\cos\theta \quad \text{and} \quad y = mg\sin\theta,$$

which are indicated in Figure 5.24.



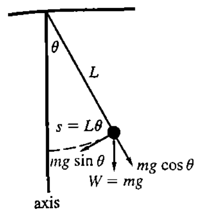


**Figure 5.23** A force diagram for the swinging pendulum.

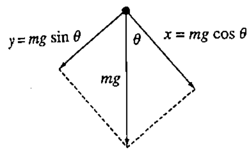


**Figure 5.24**

The momentum of the mass is given by $m\,ds/dt$, so the rate of change of the momentum is

$$\frac{d}{dt}\left(m\frac{ds}{dt}\right) = m\frac{d^2s}{dt^2},$$

where $s$ represents the length of the arc formed by the motion of the mass. Then, because the force $mg\sin\theta$ acts in the opposite direction of the motion of the mass, we have the equation

$$m\frac{d^2s}{dt^2} = -mg\sin\theta.$$

(Notice that the force $mg\cos\theta$ is offset by the force of constraint in the rod, so $mg$ and $mg\cos\theta$ cancel each other in the sum of the forces.) Using the relationship from geometry between the length of the arc, the length of the rod, and the angle $\theta$, $s = L\theta$, we have the relationship

$$\frac{d^2s}{dt^2} = \frac{d^2}{dt^2}(L\theta) = L\frac{d^2\theta}{dt^2}.$$

The displacement $\theta(t)$ satisfies $mL\dfrac{d^2\theta}{dt^2} = -mg\sin\theta$ or

$$mL\frac{d^2\theta}{dt^2} + mg\sin\theta = 0,$$

which is a *nonlinear* equation. However, because we are only concerned with small displacements, we note from the Maclaurin series for $\sin\theta$,

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots,$$

and that for small values of $\theta$, $\sin\theta \approx \theta$. Therefore, we obtain the linear equation $mL\,d^2\theta/dt^2 + mg\,\theta = 0$ or $d^2\theta/dt^2 + (g/L)\,\theta = 0$, which approximates the original problem. If the initial displacement (position of the mass) is given by $\theta(0) = \theta_0$ and the initial velocity (the velocity with which the mass is set into motion) is given by $\theta'(0) = v_0$, we have the initial-value problem

$$\begin{cases} \dfrac{d^2\theta}{dt^2} + \dfrac{g}{L}\theta = 0 \\ \theta(0) = \theta_0,\ \theta'(0) = v_0 \end{cases}$$

to find the displacement function $\theta(t)$.

Suppose that $\omega^2 = g/L$ so that the differential equation becomes $d^2\theta/dt^2 + \omega^2\theta = 0$. Therefore, functions of the form

$$\theta(t) = c_1\cos\omega t + c_2\sin\omega t,$$

where $\omega = \sqrt{g/L}$, satisfy the equation $d^2\theta/dt^2 + g/L\ \theta = 0$. When we use the conditions $\theta(0) = \theta_0$ and $\theta'(0) = v_0$, we find that the function

$$\theta(t) = \theta_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t$$

satisfies the equation as well as the initial displacement and velocity conditions. As we did with the position function of spring–mass systems, we can write this function as a cosine function that includes a phase shift with

$$\theta(t) = \sqrt{\theta_0^2 + \frac{v_0^2}{\omega^2}} \cos (\omega t - \phi),$$

where $\phi = \cos^{-1}(\theta_0/\sqrt{\theta_0^2 + v_0^2/\omega^2})$ and $\omega = \sqrt{g/L}$.

Note that the approximate period of $\theta(t)$ is $T = 2\pi/\omega = 2\pi\sqrt{L/g}$.

**Example 1**

Determine the displacement of a pendulum of length $L = 8$ feet if $\theta(0) = 0$ and $\theta'(0) = 2$. What is the period? If the pendulum is part of a clock that ticks once for each time the pendulum makes a complete swing, how many ticks does the clock make in one minute?

***Solution*** Because $g/L = 32/8 = 4$, the initial-value problem that models this situation is

$$\begin{cases} \dfrac{d^2\theta}{dt^2} + 4\theta = 0 \\ \theta(0) = 0,\ \theta'(0) = 2. \end{cases}$$

A general solution of the differential equation is $\theta(t) = c_1 \cos 2t + c_2 \sin 2t$, so application of the initial conditions yields the solution $\theta(t) = \sin 2t$. The period of this function is

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{8 \text{ ft}}{32 \text{ ft/s}^2}} = \pi s.$$

(Notice that we can use our knowledge of trigonometry to compare the period with $T = 2\pi/2 = \pi$.) Therefore, the number of ticks made by the clock per minute is calculated with the conversion (1 rev/$\pi$ sec) $\times$ (1 tick/1 rev) $\times$ (60 s/1 min) $\approx$ 19.1 ticks/min. Hence the clock makes approximately 19 ticks in one minute.

*How is motion affected if the length of the pendulum in Example 1 is changed to $L = 4$?*

If the pendulum undergoes a damping force that is proportional to the instantaneous velocity, the force due to damping is given by

$$F_R = b\frac{d\theta}{dt}.$$



Incorporating this force into the sum of the forces acting on the pendulum, we obtain the nonlinear equation $L\, d^2\theta/dt^2 + b\, d\theta/dt + g \sin \theta = 0$. Again, using the approximation $\sin \theta \approx \theta$ for small values of $\theta$, we use the linear equation $L\, d^2\theta/dt^2 + b\, d\theta/dt + g\theta = 0$ to approximate the situation. Therefore, we solve the initial-value problem

$$\begin{cases} L\dfrac{d^2\theta}{dt^2} + b\dfrac{d\theta}{dt} + g\theta = 0 \\ \theta(0) = \theta_0,\ \theta'(0) = v_0 \end{cases}$$

to find the displacement function $\theta(t)$.

**Example 2**

A pendulum of length $L = 8/5$ ft is subjected to the resistive force $F_R = (32/5)\, d\theta/dt$ due to damping. Determine the displacement function if $\theta(0) = 1$ and $\theta'(0) = 2$.

***Solution*** The initial-value problem that models this situation is

$$\begin{cases} \dfrac{8}{5}\dfrac{d^2\theta}{dt^2} + \dfrac{32}{5}\dfrac{d\theta}{dt} + 32\theta = 0 \\ \theta(0) = 1,\ \theta'(0) = 2. \end{cases}$$

Simplifying the differential equation, we obtain $d^2\theta/dt^2 + 4\, d\theta/dt + 20\theta = 0$, which has characteristic equation $r^2 + 4r + 20 = 0$ with roots $r_{1,2} = -2 \pm 4i$. A general solution is $\theta(t) = e^{-2t}(c_1 \cos 4t + c_2 \sin 4t)$. Application of the initial conditions yields the solution $\theta(t) = e^{-2t}(\cos 4t + \sin 4t)$. We graph this solution in Figure 5.25. Notice that the damping causes the displacement of the pendulum to decrease over time.

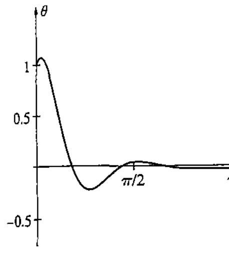



Figure 5.25

*When does the object in Example 2 first pass through its equilibrium position? What is the maximum displacement from equilibrium?*

In many cases, we can use computer algebra systems to obtain accurate approximations of nonlinear problems.

**Example 3**

Use a computer algebra system to approximate the solutions of the nonlinear problems

(a) $\begin{cases} \dfrac{d^2\theta}{dt^2} + 4 \sin \theta = 0 \\ \theta(0) = \theta,\ \theta'(0) = 2 \end{cases}$ and (b) $\begin{cases} \dfrac{8}{5}\dfrac{d^2\theta}{dt^2} + \dfrac{32}{5}\dfrac{d\theta}{dt} + 32 \sin \theta = 0 \\ \theta(0) = 1,\ \theta'(0) = 2 \end{cases}$

Compare the results to the approximations obtained in Examples 1 and 2.

**Solution** We show the results obtained with a typical computer algebra system in Figure 5.26. We see that as $t$ increases, the approximate solution obtained in Example 1 becomes less accurate. However, for small values of $t$, the results are nearly identical.

We show the results obtained for (b) with a typical computer algebra system in Figure 5.27. In this case, we see that the error diminishes as $t$ increases. (Why?)

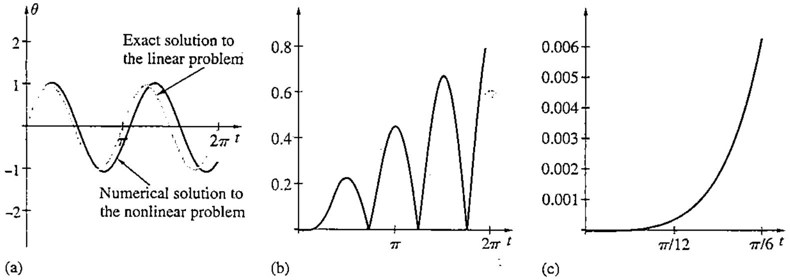



**Figure 5.26** (a) The approximations are quite similar for small values of $t$. (b) The absolute value of the difference between the two approximations. (c) The absolute value of the difference between the two approximations.

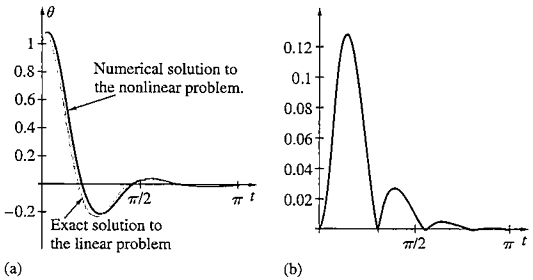



**Figure 5.27** (a) The approximations are quite similar for nearly all values of $t$. (b) The absolute value of the difference between the two approximations.



## EXERCISES 5.5

1. Use the linear approximation of the model of the simple pendulum to determine the motion of a pendulum with rod length $L = 2$ ft subject to the following sets of initial conditions:
   **(a)** $\theta(0) = 0.05$, $\theta'(0) = 0$
   **(b)** $\theta(0) = 0.05$, $\theta'(0) = 1$
   **(c)** $\theta(0) = 0.05$, $\theta'(0) = -1$
   In each case, determine the maximum displacement (in absolute value).
2. Consider the situation indicated in Exercise 1. However, use the initial velocity $\theta'(0) = 2$. How does the maximum displacement (in absolute value) differ from that in Exercise 1**(b)**–**(c)**?

*3. Suppose that the pendulum in Exercise 1 is subjected to a resistive force with damping coefficient $b = 4\sqrt{7}$. Solve the initial-value problems given in Exercise 1, and compare the resulting motion to the undamped case.

4. Verify that $\theta(t) = C_1 \cos \omega t + C_2 \sin \omega t$, where $\omega = \sqrt{g/L}$ satisfies the equation $d^2\theta/dt^2 + (g/L)\theta = 0$.
5. Show that $\theta(t) = \theta_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t$, where $\omega = \sqrt{g/L}$ is the solution of the initial-value problem
$$\begin{cases} \dfrac{d^2\theta}{dt^2} + \dfrac{g}{L}\,\theta = 0 \\ \theta(0) = \theta_0,\ \theta'(0) = v_0. \end{cases}$$
6. Let $\theta(t) = A \cos(\omega t - \phi)$. Use $\cos(a + b) = \cos a \cos b - \sin a \sin b$ with the solution in Exercise 5 to find $A$ so that $\theta(t)$ satisfies the initial-value problem in Exercise 5.

*7. Show that the phase angle in $\theta(t) = A \cos(\omega t - \phi)$ is $\phi = \cos^{-1}(\theta_0/\sqrt{\theta_0^2 + v_0^2/\omega^2})$.

In Exercises 8–11, approximate the period of the motion of the pendulum using the given length.

8. $L = 1$ m
9. $L = 2$ m
10. $L = 2$ ft

*11. $L = 8$ ft

12. If $L = 1$ m, how many ticks does the clock make in 1 min if it ticks once for each time the pendulum makes a complete swing?
13. Assuming that a clock ticks once each time the pendulum makes a complete swing, how long (in meters) does the pendulum need to be for the clock to tick once per second?
14. In Exercise 13, how long (in feet) does the pendulum need to be for the clock to tick once per second?

*15. For the undamped problem
$$\begin{cases} \dfrac{d^2\theta}{dt^2} + \dfrac{g}{L}\theta = 0 \\ \theta(0) = \theta_0,\ \theta'(0) = v_0, \end{cases}$$
what is the maximum value of $\theta(t)$? For what values of $t$ does the maximum occur?

16. For what values of $t$ is the pendulum vertical in Exercise 15?
17. Solve the initial-value problem
$$\begin{cases} L\dfrac{d^2\theta}{dt^2} + b\dfrac{d\theta}{dt} + g\theta = 0 \\ \theta(0) = \theta_0,\ \theta'(0) = v_0. \end{cases}$$
Determine restrictions on the parameters $L$, $b$, and $g$ that correspond to overdamping, critical damping, and underdamping. Describe the physical situation in each case.
18. Solve the initial-value problem
$$\begin{cases} L\dfrac{d^2\theta}{dt^2} + b\dfrac{d\theta}{dt} + g\theta = F\cos\gamma t \\ \theta(0) = 0,\ \theta'(0) = 0, \end{cases}$$
assuming that $b^2 - 4gL < 0$. Describe the motion of the pendulum as $t \to \infty$.

*19. Consider **Van-der-Pol's equation,**
$$\frac{d^2x}{dt^2} + \varepsilon(x^2 - 1)\frac{dx}{dt} + x = 0,$$
where $\varepsilon$ is a small positive number. Notice that this equation has a nonconstant damping coefficient. Because $\varepsilon$ is small, we can approximate a solution of

Van-der-Pol's equation with $x(t) = A\cos\omega t$, a solution of $d^2x/dt^2 + x = 0$ (the equation obtained when $\varepsilon = 0$). This method of approximation is called **harmonic balance.**

(a) Substitute $x(t) = A\cos\omega t$ into the nonlinear term in Van-der-Pol's equation to obtain

$$\varepsilon(x^2-1)\frac{dx}{dt} = -\varepsilon A\omega\left(\frac{1}{4}A^2 - 1\right)\sin\omega t - \frac{1}{4}A^3\sin 3\omega t.$$

(b) If we ignore the term involving the higher harmonic $\sin 3\omega t$, we have $\varepsilon(x^2+1)\,dx/dt \approx -\varepsilon A\omega(\frac{1}{4}A^2-1)\sin\omega t = \varepsilon(\frac{1}{4}A^2-1)\,dx/dt$. Substitute this expression into Van-der-Pol's equation to obtain the linear equation $d^2x/dt^2 + \varepsilon(\frac{1}{4}A^2-1)\,dx/dt + x = 0$.

(c) If $A = 2$ in the linear equation in **(b)**, is the approximate solution periodic?

(d) If $A \neq 2$ in the linear equation in **(b)**, is the approximate solution periodic?

**20.** Comment on the behavior of solutions obtained in Exercise 19 if **(a)** $A < 2$ and **(b)** $A > 2$.

**21.** Use a computer algebra system to solve the initial-value problem

$$\begin{cases} \dfrac{d^2\theta}{dt^2} + \theta = 0 \\ \theta(0) = \theta_0,\ \theta'(0) = v_0 \end{cases}$$

subject to the following initial conditions:

**(a)** $\theta(0) = 0,\ \theta'(0) = 2$
**(b)** $\theta(0) = 2,\ \theta'(0) = 0$
**(c)** $\theta(0) = -2,\ \theta'(0) = 0$
**(d)** $\theta(0) = 0,\ \theta'(0) = -1$
**(e)** $\theta(0) = 0,\ \theta'(0) = -2$
**(f)** $\theta(0) = 1,\ \theta'(0) = -1$
**(g)** $\theta(0) = -1,\ \theta'(0) = 1$.

Plot each solution individually and plot the seven solutions simultaneously. Explain the physical interpretation of these solutions.

**22.** Solve the initial-value problem

$$\begin{cases} \dfrac{d^2\theta}{dt^2} + \dfrac{1}{2}\dfrac{d\theta}{dt} + \theta = 0 \\ \theta(0) = \theta_0,\ \theta'(0) = v_0 \end{cases}$$

subject to the initial conditions:

**(a)** $\theta(0) = 1,\ \theta'(0) = 0$
**(b)** $\theta(0) = -1,\ \theta'(0) = 0$
**(c)** $\theta(0) = 0,\ \theta'(0) = 1$
**(d)** $\theta(0) = 0,\ \theta'(0) = -1$
**(e)** $\theta(0) = 1,\ \theta'(0) = 1$
**(f)** $\theta(0) = 1,\ \theta'(0) = -1$
**(g)** $\theta(0) = -1,\ \theta'(0) = 1$
**(h)** $\theta(0) = -1,\ \theta'(0) = -1$
**(i)** $\theta(0) = 1,\ \theta'(0) = 2$
**(j)** $\theta(0) = 1,\ \theta'(0) = 3$
**(k)** $\theta(0) = -1,\ \theta'(0) = 2$
**(l)** $\theta(0) = -1,\ \theta'(0) = 3$.

**23.** Plot the solutions obtained in Exercise 22 individually and then plot them simultaneously. Give a physical interpretation of the results.

**24.** If the computer algebra system you are using has a built in function that approximates the solution of nonlinear differential equations, solve the problem

$$\begin{cases} \dfrac{d^2\theta}{dt^2} + \dfrac{1}{2}\dfrac{d\theta}{dt} + \sin\theta = 0 \\ \theta(0) = \theta_0,\ \theta'(0) = v_0, \end{cases}$$

with the initial conditions stated in Exercise 22. Compare the results you obtain with each.

***25.** Use a built-in computer algebra system function to approximate the solution of the pendulum problem with a variable damping coefficient

$$\begin{cases} \dfrac{d^2\theta}{dt^2} + \dfrac{1}{2}(\theta^2 - 1)\dfrac{d\theta}{dt} + \theta = 0 \\ \theta(0) = \theta_0,\ \theta'(0) = v_0 \end{cases}$$

using the initial conditions stated in Exercise 22. Compare these results with those of Exercise 24.

**26.** Repeat Exercise 25 using

$$\begin{cases} \dfrac{d^2\theta}{dt^2} + \dfrac{1}{2}(\theta^2 - 1)\dfrac{d\theta}{dt} + \sin\theta = 0 \\ \theta(0) = \theta_0,\ \theta'(0) = v_0. \end{cases}$$



## CHAPTER 5 SUMMARY

### Concepts & Formulas

**Section 5.1**

**Hooke's Law**
$F = ks$

**Simple harmonic motion**
The initial-value problem

$$\begin{cases} m\dfrac{d^2x}{dt^2} + kx = 0 \\ x(0) = \alpha,\ x'(0) = \beta \end{cases}$$

has the solution

$$x(t) = \alpha\cos\omega t + \frac{\beta}{\omega}\sin\omega t,$$

where $\omega = \sqrt{k/m}$; the amplitude of the solution is

$$A = \sqrt{\alpha^2 + \beta^2/\omega^2}.$$

**Section 5.2**

**Damped motion**

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.$$

**Overdamped**
$c^2 - 4mk > 0.$

**Critically damped**
$c^2 - 4mk = 0.$

**Underdamped**
$c^2 - 4mk < 0.$

**Section 5.3**

**Forced motion**

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t).$$

**Section 5.4**

**Kirchhoff's law**
The sum of the voltage drops across the circuit elements is equivalent to the voltage $E(t)$ impressed on the circuit.

***L-R-C* Circuit**

$$\begin{cases} L\dfrac{d^2Q}{dt^2} + R\dfrac{dQ}{dt} + \dfrac{1}{C}Q = E(t) \\ Q(0) = Q_0,\ I(0) = Q'(0) = I_0. \end{cases}$$

**Deflection of a Beam**

$$EI\frac{d^4s}{dx^4} = w(x).$$

**Section 5.5**

**Motion of a Pendulum**
The initial-value problem

$$\begin{cases} \dfrac{d^2\theta}{dt^2} + \dfrac{g}{L}\theta = 0 \\ \theta(0) = \theta_0,\ \theta'(0) = v_0 \end{cases}$$

has solution

$$\theta(t) = \theta_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t.$$

## CHAPTER 5 REVIEW EXERCISES

1. An object weighing 32 lb stretches a spring 6 in. If the object is lowered 4 in. below the equilibrium and released from rest, determine the displacement of the object, assuming there is no damping. What is the maximum displacement of the object from equilibrium? When does the object first pass through the equilibrium position? How often does the object return to the equilibrium position?

2. If the object in Exercise 1 is released from a point 3 in. above equilibrium with a downward initial velocity of 1 ft/s, determine the displacement of the object, assuming there is no damping. What is the maximum displacement of the object from equilibrium? When does the object first pass through the equilibrium position? How often does the object return to the equilibrium position?

*3. An object of mass 5 kg is attached to the end of a spring with spring constant $k = 65$ N/m. If the object is released from the equilibrium position with an upward initial velocity of 1 m/s, determine the displacement of the object assuming the force due to damping is $F_R = 20\ dx/dt$. Find $\lim_{t\to\infty} x(t)$ and the quasiperiod. What is the maximum displacement of the object from equilibrium? When does the object first pass through the equilibrium position?

4. If the object in Exercise 3 is released from a point 1 m below equilibrium with zero initial velocity, determine the displacement. Find $\lim_{t\to\infty} x(t)$ and the quasiperiod. What is the maximum displacement of the object from equilibrium? When does the object first pass through the equilibrium position?

5. An object of mass 4 slugs is attached to a spring with spring constant $k = 16$ lb/ft. If there is no damping and the object is subjected to the forcing function $f(t) = 4$, determine the displacement function $x(t)$ if $x(0) = x'(0) = 0$. What is the maximum displacement of the object from equilibrium? When does the object first pass through the equilibrium position?

6. If the object in Exercise 5 is subjected to the forcing function $f(t) = 4\cos 2t$, determine the displacement. Find $\lim_{t\to\infty} x(t)$ if it exists. Describe the physical phenomenon that occurs.

*7. If the object in Exercise 5 is subjected to the forcing function $f(t) = 4\cos t$, determine the displacement. Describe the physical phenomenon that occurs. Find the enveloping functions.

8. An object of mass 2 slugs is attached to a spring with spring constant $k = 5$ lb/ft. If the resistive force is $F_R = 6\ dx/dt$ and the external force is $f(t) = 12\cos 2t$, determine the displacement if $x(0) = x'(0) = 0$. What is the steady-state solution? What is the transient solution?

9. Find the charge and current in the $L$-$R$-$C$ circuit if $L = 4$ H, $R = 80\ \Omega$, $C = 1/436$ farad, and $E(t) = 100$ if $Q(0) = Q'(0) = 0$. Find $\lim_{t\to\infty} Q(t)$ and $\lim_{t\to\infty} I(t)$.

10. Find the charge and current in the $L$-$R$-$C$ circuit in Exercise 9 if $E(t) = 100\sin 2t$. Find $\lim_{t\to\infty} Q(t)$ and $\lim_{t\to\infty} I(t)$. How do these limits compare to those in Exercise 9?

*11. Find the charge and current in the $L$-$R$-$C$ circuit if $L = 1$ H, $R = 0\ \Omega$, $C = 10^{-4}$ farad, and $E(t) = 220$ if $Q(0) = Q'(0) = 0$. Find $\lim_{t\to\infty} Q(t)$ and $\lim_{t\to\infty} I(t)$.

12. Find the charge and current in the $L$-$R$-$C$ circuit in Exercise 11 if $E(t) = 100\sin 10t$. Find $\lim_{t\to\infty} Q(t)$ and $\lim_{t\to\infty} I(t)$.

13. Determine the shape of the beam of length 10 with constants $E$ and $I$ such that $EI = 1$, weight distribution $w(x) = x(10 - x)$, and fixed-end boundary conditions at $x = 0$ and $x = 10$.

14. Determime the shape of the beam in Exercise 13 if there are fixed-end boundary conditions at $x = 0$ and a sliding clamped end at $x = 10$.

*15. Determine the shape of the beam in Exercise 13 if there are fixed-end boundary conditions at $x = 0$ and a free end at $x = 10$.

16. Determine the shape of the beam in Exercise 13 if there are fixed-end boundary conditions at $x = 0$ and simple support at $x = 10$.

17. Use the linear approximation of the model of the simple pendulum to determine the motion of a pendulum with rod length $L = \frac{1}{2}$ ft subject to the initial conditions $\theta(0) = 1$ and $d\theta/dt(0) = 0$. What is the maximum displacement of the pendulum from the vertical position? When does the pendulum first pass through the vertical position?



18. If the initial conditions in Exercise 17 are $\theta(0) = 0$ and $\theta'(0) = -1$, what is the maximum displacement of the pendulum from the vertical position? When does the pendulum first return to the vertical position?

*19. How does the motion of the pendulum in Exercise 17 differ if it undergoes the damping force $F_R = 8\ d\theta/dt$?

20. Solve the model in Exercise 17 with the damping force $F_R = 8\sqrt{3}\ d\theta/dt$. How does the motion differ from that in Exercise 19?

21. Undamped torsional vibrations (rotations back and forth) of a wheel attached to a thin elastic rod or wire satisfy the differential equation $I_0\theta'' + k\theta = 0$, where $\theta$ is the angle measured from the state of equilibrium, $I_0$ is the polar moment of inertia of the wheel about its center, and $k$ is the torsional stiffness of the rod. Solve this equation if $k/I_0 = 13.69\ \text{s}^{-2}$, the initial angle is $15° \approx 0.2168$ rad, and the initial angular velocity is $10°\ \text{s}^{-1} \approx 0.1745\ \text{rad}\cdot\text{s}^{-1}$.

22. Determine the displacement of the spring–mass system with mass 0.250 kg, spring constant $k = 2.25\ \text{kg/s}^2$, and driving force $f(t) = \cos t - 4\sin t$ if there is no damping, zero initial position, and zero initial velocity. For what frequency of the driving force would there be resonance?

23. The differential equation

$$y'' + y = \begin{cases} 1 - t^2, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}, \quad y(0) = y'(0) = 0$$

can be thought of as an undamped system in which a force $F$ acts during the interval of time $0 \le t \le 1$. This is the situation that occurs in a gun barrel when a shell is fired. The barrel is braked with heavy springs (see Figure 5.28). Solve this initial-value problem.

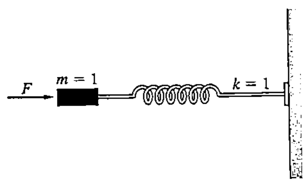



**Figure 5.28**

24. Consider a buoy in the shape of a cylinder of radius $r$, height $h$, and density $\rho$ where $\rho \le 0.5\ \text{g/cm}^3$ (the density of water is $1\ \text{g/cm}^3$). Initially, the buoy sits with its base on the surface of the water. It is then released so that it is acted on by two forces: the force of gravity (in the downward direction) equal to the weight of the buoy, $F_1 = mg = \rho\pi r^2 hg$; and the force of buoyancy (in the upward direction) equal to the weight of the displaced water, $F_2 = \pi r^2 xg$, where $x = x(t)$ is the depth of the base of the cylinder from the surface of the water at time $t$. Using Newton's second law of motion with $m = \rho\pi r^2 h$, we have the differential equation $\rho\pi r^2 hx'' = -\pi r^2 xg$ or $\rho hx'' + gx = 0$. Find the displacement of the buoy if $x(0) = 1$ and $x'(0) = 0$. What is the period of the solution? What is the amplitude of the solution?

25. A cube-shaped buoy of side length $\ell$ and mass density $\rho$ per unit volume is floating in a liquid of mass density $\rho_0$ per unit volume, where $\rho_0 > \rho$. If the buoy is slightly submerged into the liquid and released, it oscillates up and down. If there is no damping and no air resistance, the buoy is acted on by two forces: the force of gravity (in the downward direction), which is equal to the weight of the buoy, $F_1 = mg = \rho\ell^3 g$; and the force of buoyancy (in the upward direction), which is equal to the weight of the displaced water, $F_2 = \ell^2 xg\rho_0$, where $x = x(t)$ is the depth of the base of the buoy from the surface of the water at time $t$. Then by Newton's second law with $m = \rho\ell^3$, we have the differential equation $\rho\ell^2 x'' + g\rho_0 x = 0$. Determine the amplitude of the motion if $\rho_0 = 1\ \text{g/cm}^3$, $\rho = 0.25\ \text{g/cm}^3$, $\ell = 100$ cm, $g = 980\ \text{cm/s}^2$, $x(0) = 25$ cm and $x'(0) = 0$.

26. A rabbit starts at the origin and runs with speed $a$ due north toward a hole in a fence located at the point $(0, d)$ on the $y$-axis. At the same time, a dog starts at the point $(c, 0)$ on the $x$-axis, running at speed $b$ in pursuit of the rabbit. (*Note:* The dog runs directly toward the rabbit.) The slope of the tangent line to the dog's path is $dy/dx = -(at - y)/x$, which can be written as $xy' = y - at$. Differentiate both sides of this equation with respect to $t$ to obtain $xy'' = -a\ dt/dx$. If $s$ is the length of the arc from $(c, 0)$ along the dog's path, then $ds/dt = b$ is the dog's speed. Also, $ds/dx = -\sqrt{1 + y'^2}$, where the negative sign indicates that $s$ increases as $x$ decreases. By the chain rule,

$$\frac{dt}{dx} = \frac{dt}{ds}\frac{ds}{dx} = -\frac{1}{b}\sqrt{1 + y'^2},$$

so substitution into the equation $xy'' = -a\,dt/dx$ yields the equation $xy'' = k\sqrt{1+y'^2}$, where $k = a/b$. Make the substitution

$$p = y' \quad \text{or} \quad y'' = \frac{dp}{dx}$$

to find the path of the dog with the initial condition $y'(c) = 0$. If $a < b$, when does the dog catch the rabbit? Does the problem make sense if $a = b$? (See Figure 5.29.)

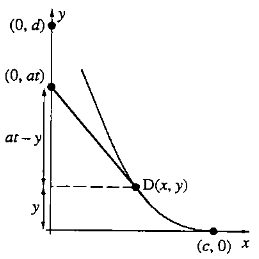



**Figure 5.29**

27. **(Coulomb damping)** Damping that results from dry friction (as when an object slides over a dry surface) is called **Coulomb damping** or **dry-friction damping.** On the other hand, **viscous damping** is damping that can be represented by a term proportional to the velocity (as when an object such as a vibrating spring vibrates in air or when an object slides over a lubricated surface).

Suppose that the kinetic coefficient of friction is $\mu$. Friction forces oppose motion. Therefore, the force as a result of friction $F$ is shown opposite the direction of motion in Figure 5.30. However, because $F$ is a discontinuous function, we cannot use a single differential equation to model the motion as was done with previous damping problems. Instead, we have one equation for motion to the right and one equation for motion to the left:

$$\textbf{Motion to right} \qquad x'' + \omega_n^2 x = -\frac{F}{m}.$$

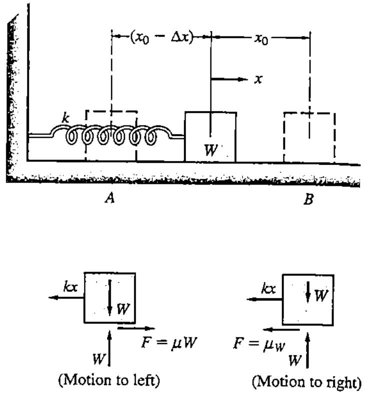



**Figure 5.30** A system with coulomb (dry-friction) damping.

and

$$\textbf{Motion to left} \qquad x'' + \omega_n^2 x = \frac{F}{m}.$$

where $F$ represents the friction force, $m$ the mass, and $\omega_n^2 = k/m$.*

(a) Find a general solution of $x'' + \omega_n^2 x = F/m$.

(b) Solve the initial-value problem

$$\begin{cases} x'' + \omega_n^2 x = \dfrac{F}{m} \\ x(0) = x_0,\ x'(0) = 0. \end{cases}$$

(c) Find the values of $t$ for which the solution in (b) is valid.

(d) At the right-endpoint of the interval obtained in (c), what is the displacement of the object?

(e) What must the force $F$ be to guarantee that the object does not move under the given initial conditions?

* M. L. James, G. M. Smith, J. C. Wolford, and P. W. Whaley, *Vibration of Mechanical and Structural Systems with Microcomputer Applications,* Harper & Row, New York (1989), pp. 70–72.



28. **(Self-excited vibrations)** The differential equation that describes the motion of a spring–mass system with a single degree of freedom excited by the force $Px'$ is

$$mx'' + cx' + kx = Px',$$

which can be rewritten as

$$x'' + \frac{c-P}{m}x' + \frac{k}{m}x = 0.*$$

(a) Show that the roots of the characteristic equations of $x'' + [(c-P)/m]\,x' + (k/m)\,x = 0$ are

$$\frac{P-c}{2m} \pm \sqrt{\left(\frac{P-c}{2m}\right)^2 - \frac{k}{m}}.$$

(b) Show that if $P > c$ the motion of the system diverges **(dynamically unstable),** if $P = c$ the solution is the solution for a free undamped system, and if $P < c$ the solution is the solution for a free damped system.

## Differential Equations at Work:

### A. Rack-and-Gear Systems

Consider the rack-and-gear system shown in Figure 5.31. Let $T$ represent the kinetic energy of the system and $U$ the change in potential energy of the system from its potential energy in the static-equilibrium position. The kinetic energy of a system is a function of the velocities of the system masses. The potential energy of a system consists of the strain energy $U_e$ stored in elastic elements and the energy $U_g$, which is a function of the vertical distances between system masses.

The rack-and-gear system consists of two identical gears of pitch radius $r$ and centroidal mass moment of inertia $\bar{I}$, a rack of weight $W$, and a linear spring of stiffness $k$, length $l$, and a mass of $\gamma$ per unit length. To determine the differential equation to model the motion, we differentiate the law of conservation, $T + U =$ constant, to obtain

$$\frac{d}{dt}(T+U) = 0.$$

We then use $T_{\max} = U_{\max}$ to determine the natural circular frequency of the system, where $T_{\max}$ represents the maximum kinetic energy and $U_{\max}$ the maximum potential energy.

Because the static displacement $x_s$ of any point on the spring is proportional to its distance $y$ from the spring support, we can write

$$x_s = \frac{yx}{l}.$$

* Robert K. Vierck, *Vibration Analysis,* Second Edition, HarperCollins, New York (1979), pp. 137–139.

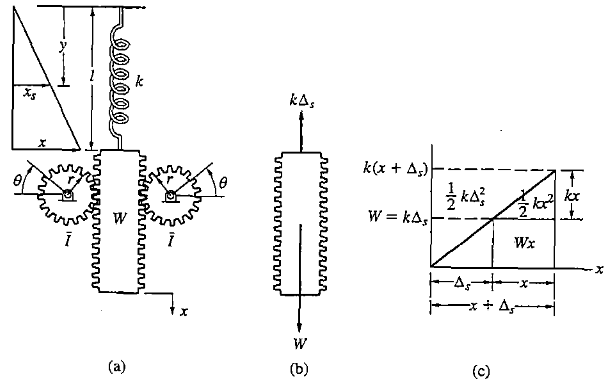


Figure 5.31 (a) A rack-and-gear system. (b) Static-equilibrium position. (c) Spring-force diagram.

When the rack is displaced a distance $x$ below the equilibrium position, the gears have the angular displacements $\theta$, which are related to the displacement of the rack by

$$\theta = \frac{x}{r}.$$

Then the angular velocity of each gear is given (through differentiation of $\theta$ with respect to $t$) by

$$\dot{\theta} = \frac{\dot{x}}{r}.$$

(Notice that we use a "dot" to indicate differentiation with respect to $t$. This is a common practice in physics and engineering.) Because $x = r\theta$, we have

$$x_s = \frac{yr\theta}{l},$$

so the velocity at any point on the spring is

$$\dot{x}_s = \frac{yr\dot{\theta}}{l}.$$

The kinetic energy of a differential element of length $dy$ of the spring is

$$dT = \frac{1}{2}\gamma(\dot{x}_s)^2\, dy.$$

The total kinetic energy of the system in terms of $\theta$ is



$$T = \underbrace{\bar{I}\dot{\theta}^2}_{\text{Gears}} + \underbrace{\frac{1}{2}\frac{W}{g}(r\dot{\theta})^2}_{\text{Rack}} + \underbrace{\frac{\gamma}{2}\int_0^l \left(\frac{yr\dot{\theta}}{l}\right)^2 dy}_{\text{Spring}}$$

or

$$T = \left[\bar{I} + \frac{1}{2}\frac{W}{g}r^2 + \frac{m_3 r^2}{2\cdot 3}\right]\dot{\theta}^2,$$

where $m_3 = \gamma l$ is the total mass of the spring.

The change in the strain energy $U_e$ for a positive downward displacement $x$ of the rack from the static-equilibrium position where the spring is already displaced $\Delta_s$ from its free length [see Figure 5.30(b)] is the area of the shaded region [see Figure 5.30(c)] given by

$$U_e = \frac{1}{2}kx^2 + Wx.$$

Similarly, the change in potential energy $U_g$ as the rack moves a distance $x$ below the static-equilibrium position is

$$U_g = -Wx$$

and the total change in the potential energy is

$$U = U_e + U_g = \left(\frac{1}{2}kx^2 + Wx\right) + (-Wx) = \frac{1}{2}kx^2$$

or

$$U = \frac{1}{2}k(r\theta)^2.$$

Substitution of these expressions into the equation $T + U =$ constant yields

$$\left[\bar{I} + \frac{1}{2}\frac{W}{g}r^2 + \frac{m_3 r^2}{2(3)}\right]\dot{\theta}^2 + \frac{1}{2}k(r\theta)^2 = \text{constant}$$

Differentiating with respect to $t$ then gives us the differential equation

$$\ddot{\theta} + \left[\frac{kr^2}{2\bar{I} + \frac{W}{g}r^2 + \frac{m_3 r^2}{3}}\right]\theta = 0.*$$

* M. L. James, G. M. Smith, J. C. Wolford, and P. W. Whaley, *Vibration of Mechanical and Structural Systems with Microcomputer Applications*, Harper & Row, New York (1989), pp. 82–86.

1. Determine the natural frequency $\omega_n$ of the system.
2. Determine $\theta(t)$ if $\theta(0) = 0$ and $\dot{\theta}(0) = \theta_0\omega_n$.
3. Find $T_{\max}$ and $U_{\max}$. Determine the natural frequency with these two quantities. Compare this value of $\omega_n$ with that obtained above.
4. How does the natural frequency change as $r$ increases?

## B. Soft Springs

In the case of a **soft spring**, the spring force weakens with compression or extension. For springs of this type, we model the physical system with the nonlinear equation

$$\begin{cases} m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx - jx^3 = f(t) \\ x(0) = \alpha,\ x'(0) = \beta, \end{cases}$$

where $j$ is a positive constant.

1. Approximate the solution to

$$\begin{cases} \dfrac{d^2x}{dt^2} + 0.2\dfrac{dx}{dt} + 10x - 0.2x^3 = -9.8 \\ x(0) = \alpha,\ x'(0) = \beta \end{cases}$$

   for various values of $\alpha$ and $\beta$ in the initial conditions.
2. Find conditions on $\alpha$ and $\beta$ so that $\lim_{t\to\infty} x(t)$ (a) converges and (b) becomes unbounded.
3. If $\lim_{t\to\infty} x(t)$ exists, does the motion of the spring converge to its equilibrium position? Explain.

## C. Hard Springs

In the case of a **hard spring**, the spring force strengthens with compression or extension. For springs of this type, we model the physical system with

$$\begin{cases} m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx + jx^3 = f(t) \\ x(0) = \alpha,\ x'(0) = \beta, \end{cases}$$

where $j$ is a positive constant.

1. Approximate the solution to



$$\begin{cases} \dfrac{d^2x}{dt^2} + 0.3x + 0.04x^3 = 0 \\ x(0) = \alpha,\ x'(0) = \beta \end{cases}$$

   for various values of $\alpha$ and $\beta$ in the initial conditions.
2. How does increasing the initial amplitude affect the period of the motion?

## D. Aging Springs

In the case of an **aging spring**, the spring constant weakens with time. For springs of this type, we model the physical system with

$$\begin{cases} m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + k(t)x = f(t) \\ x(0) = \alpha,\ x'(0) = \beta. \end{cases}$$

where $k(t) \to 0$ as $t \to \infty$.

1. Approximate the solution to

$$\begin{cases} \dfrac{d^2x}{dt^2} + 4e^{-t/4}x = 0 \\ x(0) = \alpha,\ x'(0) = \beta \end{cases}$$

   for various values of $\alpha$ and $\beta$ in the initial conditions.
2. What happens to the period of the oscillations as time increases? Is the motion oscillatory for large values of $t$? Why?

## E. Bodé Plots

Consider the differential equation $d^2x/dt^2 + 2c\,dx/dt + k^2x = F_0 \sin \omega t$, where $c$ and $k$ are positive constants such that $c < k$. Therefore, the system is underdamped. To find a particular solution, we can consider the complex exponential form of the forcing function, $F_0e^{i\omega t}$, which has imaginary part $F_0 \sin \omega t$. Assuming a solution of the form $z_p(t) = Ae^{i\omega t}$, substitution into the differential equation yields $A[-\omega^2 + 2ic\omega + k^2] = F_0e^{i\omega t}$. Because $k^2 - \omega^2 + 2ic\omega = 0$ only when $k = \omega$ and $c = 0$, we find that $A = F_0/(k^2 - \omega^2 + 2ic\omega)$ or $A = F_0/(k^2 - \omega^2 + 2ic\omega) \cdot (k^2 - \omega^2 - 2ic\omega)/(k^2 - \omega^2 - 2ic\omega) = (k^2 - \omega^2 - 2ic\omega)/[(k^2 - \omega^2)^2 + 4c^2\omega^2]\ F_0 = H(i\omega)F_0$. Therefore, a particular solution is $z_p(t) = H(i\omega)F_0e^{i\omega t}$. Now we can write $H(i\omega)$ in polar form as $H(i\omega) = M(\omega)e^{i\phi(\omega)}$, where $M(\omega) = 1/\sqrt{(k^2 - \omega^2)^2 + 4c^2\omega^2}$ and $\phi(\omega) = \cot^{-1}((\omega^2 - k^2)/2c\omega)$ $(-\pi \le \phi \le 0)$. A particular solution can then be written as $z_p(t) = M(\omega)F_0e^{i\omega t}e^{i\phi(\omega)} = M(\omega)F_0e^{i(\omega t + \phi(\omega))}$ with imaginary part $M(\omega)F_0 \sin(\omega t + \phi(\omega))$, so we take the particular solution to be $x_p(t) = M(\omega)F_0 \sin(\omega t + \phi(\omega))$. Comparing the forcing function to $x_p$, we see that the two functions have the same form but different amplitudes and phase shifts. The ratio of the amplitude of the particular solution (or steady state), $M(\omega)F_0$, to that of the forcing function, $F_0$, is $M(\omega)$ and is called the **gain.** Also, $x_p$ is shifted in time by $|\phi(\omega)|/\omega$ radians to the right, so $\phi(\omega)$ is called the **phase shift.** When we graph the gain and the phase shift against $\omega$ (using a $\log_{10}$ on the $\omega$ axis) we obtain the **Bodé plots.** Engineers refer to the value of $20 \log_{10} M(\omega)$ as the gain in **decibels.**

1. Solve the initial-value problem

$$\begin{cases} \dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 4x = \sin 2t \\ x(0) = \dfrac{1}{2},\ x'(0) = 1. \end{cases}$$

2. Graph the solution simultaneously with the forcing function $f(t) = \sin 2t$.
3. Approximate $M(2)$ and $\phi(2)$ using the graph in Number 2.
4. Graph the corresponding Bodé plots.
5. Compare the values of $M(2)$ and $\phi(2)$ with those obtained in Number 2.

# 6

# Systems of Differential Equations

## 6.1 Introduction

Suppose that we consider the path of an object in the $xy$-plane. For example, if we launch a model rocket or kick a ball, we would describe the object's position in the plane with the vector-valued function $\mathbf{r}(t) = \langle x(t), y(t)\rangle = x(t)\mathbf{i} + y(t)\mathbf{j}$. In other words, the coordinates of the object at time $t = T$ are given at the terminal point of the vector $\mathbf{r}(T) = \langle x(T), y(T)\rangle$. Following a method similar to that discussed in Chapter 3 concerning falling bodies, we can assume that the only force acting on the object is caused by gravitational acceleration, $\mathbf{g} = \langle 0, -g\rangle = -g\mathbf{j}$. (Note that this force acts entirely in the vertical direction as expected.) Then if we let the velocity of the object be represented by the function $\mathbf{v}(t) = \langle v_x(t), v_y(t)\rangle$, where $v_x(t)$ is the horizontal component of the velocity vector and $v_y(t)$ is the vertical component, and the acceleration function with $\mathbf{a}(t) = \mathbf{v}'(t) = \langle v_x'(t), v_y'(t)\rangle$, then we can make use of Newton's Second Law, $\mathbf{F} = m\mathbf{a}$, to model the situation. In this case, we have $m\mathbf{g} = m\mathbf{a}$ so that the acceleration must satisfy $\mathbf{a}(t) = \mathbf{g}$ or $\langle v_x'(t), v_y'(t)\rangle = \langle 0, -g\rangle$. Equating the corresponding components of these two vectors, we obtain the first-order system of differential equations

$$\begin{cases} v_x'(t) = 0 \\ v_y'(t) = -g \end{cases}.$$

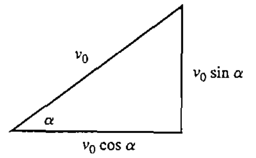



**Figure 6.1** Sketch to determine components of $\mathbf{v}(0)$.

The equations in this system are uncoupled in that $v_x$ does not appear in the equation involving $v_y'$ and $v_y$ does not appear in that describing $v_x'$. Therefore, we can solve this system by finding a solution of each individual first-order equation. If we assume that the object is launched with an initial speed $v_0$ making an angle $\alpha$ with the horizontal (Figure 6.1), the initial velocity vector is

$$\mathbf{v}(0) = \langle v_x(0), v_y(0)\rangle = \langle v_0 \cos \alpha, v_0 \sin \alpha\rangle.$$

This means that we determine the components of $\mathbf{v}(t) = \langle v_x(t), v_y(t)\rangle$ by solving the IVP

$$\begin{cases} v_x'(t) = 0,\ v_x(0) = v_0 \cos \alpha \\ v_y'(t) = -g,\ v_y(0) = v_0 \sin \alpha \end{cases}.$$

Through integration, we find that $v_x(t) = C_1$ and $v_y(t) = -gt + C_2$, so that application of the initial conditions $v_x(0) = C_1$ and $v_y(0) = C_2$ indicates that $C_1 = v_0 \cos \alpha$ and $C_2 = v_0 \sin \alpha$. Therefore, the solution is

$$\mathbf{v}(t) = \langle v_0 \cos \alpha, -gt + v_0 \sin \alpha\rangle$$

with components $v_x(t) = v_0 \cos \alpha$ and $v_y(t) = -gt + v_0 \sin \alpha$. We determine the position function $\mathbf{r}(t)$ using the relationship $\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle = \mathbf{v}(t)$. Equating the corresponding components as we did earlier, we obtain the system of first-order equations

$$\begin{cases} x'(t) = v_0 \cos \alpha \\ y'(t) = -gt + v_0 \sin \alpha \end{cases}.$$

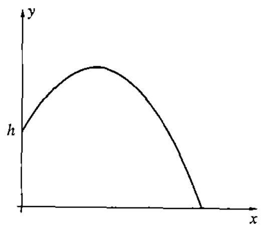



**Figure 6.2** Sketch to determine $\mathbf{r}(0)$.

If we also assume that the object is launched from the position $\mathbf{r}(0) = \langle 0, h\rangle$ (Figure 6.2), then we consider the IVP

$$\begin{cases} x'(t) = v_0 \cos \alpha,\ x(0) = 0 \\ y'(t) = -gt + v_0 \sin \alpha,\ y(0) = h \end{cases}.$$

As in the previous system, we can solve the individual differential equations. Integration yields $x(t) = (v_0 \cos \alpha)t + C_3$ and $y(t) = -\frac{1}{2}gt^2 + (v_0 \sin \alpha)t + C_4$. Then, application of the initial conditions $x(0) = C_3$ and $y(0) = C_4$ indicates that $C_3 = 0$ and $C_4 = h$, so

$$\mathbf{r}(t) = \langle (v_0 \cos \alpha)t, -\frac{1}{2}gt^2 + (v_0 \sin \alpha)t + h\rangle$$

with components $x(t) = (v_0 \cos \alpha)t$ and $y(t) = -\frac{1}{2}gt^2 + (v_0 \sin \alpha)t + h$.

For example, if $v_0 = 64$ ft/s, $\alpha = \pi/6$, $g \approx 32$ ft/s$^2$, and $h = 0$, then $\mathbf{r}(t) = \langle 32t\sqrt{3}, -16t^2 + 32t\rangle$. We graph this function (parametrically) in Figure 6.3 indicating the orientation of the curve. The object moves in the direction of the orientation as $t$ increases.

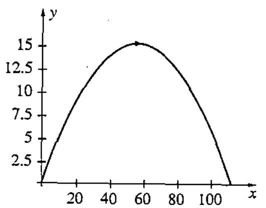



**Figure 6.3** Path of projectile with orientation.

Of course, we may encounter systems of differential equations that are more difficult to solve than the previous system. For example, we can consider a population problem involving the interaction between two populations, predator and prey. In this case, if we let $x(t)$ represent the size of the prey population and $y(t)$ that of the predator population, we can then model the situation with the IVP



$$\begin{cases} \dfrac{dx}{dt} = ax - bxy,\ x(0) = x_0 \\ \dfrac{dy}{dt} = -cy + dxy,\ y(0) = y_0 \end{cases},$$

where the initial populations are $x(0) = x_0$ and $y(0) = y_0$, and $a$, $b$, $c$, and $d$ are positive constants. Notice that when we have a system of two first-order equations, we must specify an initial condition for each of the unknowns, $x(t)$ and $y(t)$. The terms $xy$ indicate an interaction between the two populations. For example, if the two populations are foxes, denoted by $y$, and rabbits, denoted by $x$, then $xy$ denotes a rabbit being eaten by a fox. This interaction hurts the rabbit population, as indicated by the negative coefficient of $xy$, $-b$, in the $dx/dt$ equation, while it helps the fox population because the coefficient $d$ of the $xy$ term is positive in the $dy/dt$ equation.

We may also encounter systems that involve second-order differential equations. For example, we could consider a spring–mass system in which an object of mass $m_1$ is attached to the end of a spring with spring constant $k_1$ that is mounted on a support (as we did in Chapter 5). Then, if we attach a second spring with spring constant $k_2$ and mass $m_2$ to the end of the first spring, we can determine the displacement of the first and second springs, $x(t)$ and $y(t)$, respectively, by solving the system

$$\begin{cases} m_1\dfrac{d^2x}{dt^2} = -k_1x + k_2(y - x) \\ m_2\dfrac{d^2y}{dt^2} = -k_2(y - x) \end{cases}.$$

Later, we discuss how this system is modeled using Hooke's Law and Newton's Second Law. In the case of a system of second-order equations, we must have two initial conditions for each unknown to define an initial-value problem. For example, we could have $x(0) = 0$, $x'(0) = -1$, $y(0) = 1$, and $y'(0) = 0$.

Although we can solve and approximate solutions of many different types of systems of differential equations, we will focus much of our attention on **systems of first-order linear differential equations.** These systems can be written as

$$\begin{cases} x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\ x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\ \vdots \\ x_n' = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t) \end{cases}.$$

In this system, there are $n$ unknowns $x_1(t), x_2(t), \ldots, x_n(t)$ and $n$ differential equations. If $f_j(t) \equiv 0, j = 1, 2, \ldots, n$, then we say that the system is **homogeneous.** Otherwise, the system is **nonhomogeneous.** The system

$$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 \end{cases}$$

is an example of a first-order linear homogeneous system, while

$$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 + \sin t \end{cases}$$

is a first-order linear nonhomogeneous system because $f_2(t) = \sin t$. A system such as

$$\begin{cases} x_1' = x_1 - x_1^2 - x_1x_2 \\ x_2' = -x_2 + x_1x_2 \end{cases}$$

is **nonlinear** because of the nonlinear terms $x_1^2$ and $x_1x_2$. Of course, variable names other than $x_1(t), x_2(t), \ldots, x_n(t)$ can be used. For example, the previous system can be written in terms of $x(t)$ and $y(t)$ with

$$\begin{cases} x' = x - x^2 - xy \\ y' = -y + xy \end{cases}.$$

We mentioned in Chapter 1 that there is a relationship between a single higher order equation and a system of first-order equations. We revisit that topic now by considering the second-order linear ODE with constant coefficients $x'' - 4x' + 13x = 2\sin t$. We can write this equation as a system of first-order equations by making the substitution

$$x' = y$$

so that differentiation with respect to $t$ yields

$$x'' = y'.$$

Solving $x'' - 4x' + 13x = 2\sin t$ for $x''$ gives us $x'' = 4x' - 13x + 2\sin t$. Substitution of $x'' = y'$ and $x' = y$ into this equation then gives us $y' = 4y - 13x + 2\sin t$. Therefore, the second-order equation $x'' - 4x' + 13x = 2\sin t$ is equivalent to the system of first-order linear differential equations

$$\begin{cases} x' = y \\ y' = 4y - 13x + 2\sin t \end{cases}.$$

We can also begin with a system of first-order equations and transform it into a higher order equation. For example, consider the system

$$\begin{cases} x' = x + 5y \\ y' = -x - y \end{cases}.$$

If we differentiate $x' = x + 5y$ with respect to $t$, we obtain

$$x'' = x' + 5y'.$$

Substituting $y' = -x - y$ into this equation then gives us

$$x'' = x' + 5(-x - y) = x' - 5x - 5y.$$

Solving $x' = x + 5y$ for $y$ (or in this case, $5y$, yields $5y = x' - x$ so that substitution into $x'' = x' - 5x - 5y$ gives us the second-order equation $x'' = x' - 5x - (x' - x) = -4x$, or

$$x'' + 4x = 0,$$



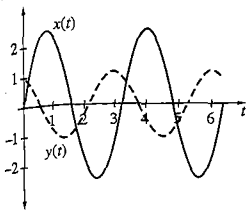


**Figure 6.4** Graphs of $x(t) = \frac{5}{2}\sin 2t$ and $y(t) = \cos 2t - \frac{1}{2}\sin 2t$.

which is equivalent to the system. A general solution of this equation is $x(t) = c_1\cos 2t + c_2\sin 2t$. We can use $y = \frac{1}{5}(x' - x)$ to determine $y(t)$. This gives us

$$y(t) = \frac{1}{5}[(-2c_1\sin 2t + 2c_2\cos 2t) - (c_1\cos 2t + c_2\sin 2t)]$$

$$= \frac{1}{5}(-c_2 - 2c_1)\sin 2t + \frac{1}{5}(2c_2 - c_1)\cos 2t.$$

Therefore, we have solved the system of first-order equations by solving the equivalent second-order equation. Now, if we had considered the initial-value problem

$$\begin{cases} x' = x + 5y,\ x(0) = 0 \\ y' = -x - y,\ y(0) = 1 \end{cases},$$

then we would apply the initial conditions to a general solution of the system. In our case, we have $x(0) = c_1$, so $c_1 = 0$. Then, $y(0) = \frac{1}{5}(2c_2)$, so $c_2 = \frac{5}{2}$, and the solution to the IVP is

$$\left\{x(t) = \frac{5}{2}\sin 2t,\ y(t) = \cos 2t - \frac{1}{2}\sin 2t\right\}.$$

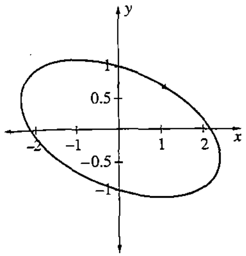


**Figure 6.5** Parametric plot of $\{x(t) = \frac{5}{2}\sin 2t,\ y(t) = \cos 2t - \frac{1}{2}\sin 2t\}$.

We can graph these two functions of $t$ as done in Figure 6.4. The graph of $x(t)$ is the solid curve, and that of $y(t)$ is the dashed curve. (We can recognize the functions by referring to the initial conditions.) We can also plot them parametrically in the $xy$-plane as done in Figure 6.5. *Note:* We should include the orientation of this curve. The value of $t = 0$ corresponds to the point $(0, 1)$ while $t = \pi/4$ corresponds to $(5/2, -1/2)$. Therefore, the orientation is clockwise because we move from $(0, 1)$ to $(5/2, -1/2)$ as $t$ increases.

Nonlinear differential equations can often be written as a system of first-order equations as well.

**Example 1**

The **Van-der-Pol equation** is the nonlinear ordinary differential equation

$$w'' - \mu(1 - w^2)w' + w = 0.$$

Write the Van-der-Pol equation as a system. Notice that this equation is similar to those encountered when studying spring–mass systems, except that the coefficient of $w'$ that corresponds to damping is not constant. Therefore, we refer to this situation as *variable damping* and can observe its effect on the system.

***Solution*** Let $x_1 = w$ and $x_2 = w' = x_1'$. Then, $x_2' = w'' = \mu(1 - w^2)w' - w = \mu(1 - x_1^2)x_2 - x_1$, so the Van-der-Pol equation is equivalent to the nonlinear system

$$\begin{cases} x_1' = x_2 \\ x_2' = \mu(1 - x_1^2)x_2 - x_1 \end{cases}.$$

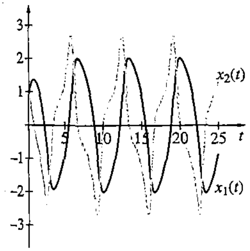


**Figure 6.6** Graph of $x_1(t)$ and $x_2(t)$.

In Figure 6.6, we show graphs of the solution in the initial-value problem

$$\begin{cases} x_1' = x_2 \\ x_2' = (1 - x_1^2)x_2 - x_1 \\ x_1(0) = 1, x_2(0) = 1 \end{cases}$$

on the interval [0, 25]. Because we let $x_2 = x_1'$, notice that $x_2(t) > 0$ when $x_1(t)$ is increasing and $x_2(t) < 0$ when $x_1(t)$ is decreasing. Note that these functions appear to become periodic as $t$ increases.

The observation that these solutions become periodic is further confirmed by a graph of $x_1(t)$ (the horizontal axis) versus $x_2(t)$ (the vertical axis) shown in Figure 6.7, called the *phase plane*. We see that as $t$ increases, the solution approaches a certain fixed path, called a *limit cycle*. We will find that nonlinear equations are more easily studied when they are written as a system of equations.

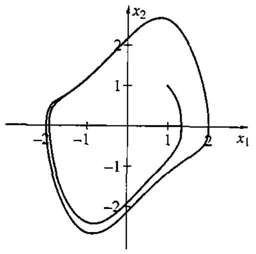



**Figure 6.7** The graph of $\begin{cases} x_1(t) \\ x_2(t) \end{cases}$ indicates that the solution approaches an isolated periodic solution, which is called a *limit cycle*.

*Graph the solution to the initial-value problem* $\begin{cases} w'' + (w^2 - 1)w' + w = 0 \\ w(0) = 1,\ w'(0) = 1 \end{cases}$ *on the interval* [0, 25].

## EXERCISES 6.1

Solve each of the following uncoupled systems of equations.

1. $\begin{cases} x' = t \\ y' = \cos t \end{cases}$ 2. $\begin{cases} x' = x \\ y' = 1 \end{cases}$

*3. $\begin{cases} x' = 0 \\ y' = -2y \end{cases}$ 4. $\begin{cases} x' = x^2 \\ y' = e^t \end{cases}$

Solve each of the following initial-value problems.

5. $\begin{cases} x_1' = -3x_1, x_1(0) = -1 \\ x_2' = 1, x_2(0) = 1 \end{cases}$

6. $\begin{cases} x_1' = -x_1 + 1, x_1(0) = 0 \\ x_2' = x_2, x_2(0) = 1 \end{cases}$

Solve each system by solving the equivalent second-order equation.

7. $\begin{cases} x' = -3x + 6y \\ y' = 4x - y \end{cases}$ 8. $\begin{cases} x' = 8x - y \\ y' = x + 6y \end{cases}$

*9. $\begin{cases} x' = -x - 2y \\ y' = x + y \end{cases}$ 10. $\begin{cases} x' = 4x + 2y \\ y' = -x + 2y \end{cases}$



11. $\begin{cases} x' = y \\ y' = -x + 1 \end{cases}$ 12. $\begin{cases} x' = y \\ y' = -x + \sin 2t \end{cases}$

In Exercises 13–18, write each equation as an equivalent system of first-order equations.

13. $x'' - 3x' + 4x = 0$
14. $x'' + 6x' + 9x = 0$
*15. $x'' + 16x = t \sin t$
16. $x'' + x = e^t$
17. $y''' + 3y'' + 6y' + 3y = x$
18. $y^{(4)} + y'' = 0$

*19. **Rayleigh's equation** is

$$x'' + \mu\left[\frac{1}{3}(x')^2 - 1\right]x' + x = 0,$$

where $\mu$ is a constant.

(a) Write Rayleigh's equation as a system.

(b) Show that differentiating Rayleigh's equation and setting $x' = z$ reduces Rayleigh's equation to the Van-der-Pol equation.

20. **(Operator Notation)** Recall the differential operator $D = d/dt$, which was introduced in Exercises 4.4. Using this notation, we can write the system $\begin{cases} y'' + 2y' - x' + 4x = \cos t \\ x'' + x + y'' - y' - 2y = e^t \end{cases}$ as

$$\begin{cases} (D^2 + 2D)y - (D - 4)x = \cos t \\ (D^2 + 1)x + (D^2 - D - 2)y = e^t \end{cases}$$

Operator notation can be used to solve a system. For example, if we consider the system $\begin{cases} Dx = y \\ Dy = -x \end{cases}$ or $\begin{cases} Dx - y = 0 \\ Dy + x = 0 \end{cases}$, we can eliminate one of the variables by applying the operator $(-D)$ to the second equation to obtain $\begin{cases} Dx - y = 0 \\ -Dx - D^2y = 0 \end{cases}$. When these two equations are added, we have the second-order ODE $-D^2y - y = 0$ or $D^2y + y = 0$. This equation has the characteristic equation $r^2 + 1 = 0$ with roots $r = \pm i$. Therefore, $y(t) = c_1 \cos t + c_2 \sin t$. At this point, we can repeat the elimination procedure to solve for $x$, or we can use the equation $Dy + x = 0$ or $x = -Dy$ to find $x$. Applying $(-D)$ to $y$ yields

$$\begin{aligned} x(t) &= -Dy(t) = -D(c_1 \cos t + c_2 \sin t) \\ &= c_1 \sin t - c_2 \cos t. \end{aligned}$$

Therefore, a general solution of the system is

$$\begin{cases} x(t) = c_1 \sin t - c_2 \cos t \\ y(t) = c_1 \cos t + c_2 \sin t \end{cases}$$

In Exercises 21–27, solve the system using the operator method.

21. $\begin{cases} x' = -2x - 2y + 4 \\ y' = -5x + y \end{cases}$ 22. $\begin{cases} x' = x + 2y + \cos t \\ y' = 3x + 2y \end{cases}$

23. $\begin{cases} x' = -3x + 2y - e^t \\ y' = -3y + 1 \end{cases}$ 24. $\begin{cases} x'' = 2y - 5x \\ y'' = 2x - 2y \end{cases}$

*25. $\begin{cases} x'' + y'' + x + y = 0 \\ y'' + 2x' - 2x - y = 0 \end{cases}$

26. $\begin{cases} x' = -3x + 2y + 2z \\ y' = -2x + 3y + 2z \\ z' = x + y \end{cases}$

27. $\begin{cases} x' = x + 2y - 2z + \cos t \\ y' = x - y + 2z \\ z' = -x - y \end{cases}$

In Exercises 28–30, solve the initial-value problem. For each problem, graph $x(t)$, $y(t)$, and the parametric equations $\begin{cases} x(t) \\ y(t) \end{cases}$.

28. $x' - x - 2y = 0$, $y' - 2x - y = 0$, $x(0) = 0$, $y(0) = -1$

*29. $x' - 3x + 2y = 0$, $y' - 2x + y = 10$, $x(0) = 0$, $y(0) = 0$

30. $x'' = y - x$, $y'' = y - x + \sin t$, $x(0) = 0$, $x'(0) = 0$, $y(0) = 0$, $y'(0) = 0$

**(Degenerate Systems)** Suppose that the system of linear differential equations has the form $\begin{cases} L_1x + L_2y = f_1(t) \\ L_3x + L_4y = f_2(t) \end{cases}$, where $L_1$, $L_2$, $L_3$, and $L_4$ are linear differential operators. For example, in the system

$$\begin{cases} (D^2 + 2D)y - (D - 4)x = \cos t \\ (D^2 + 1)x + (D^2 - D - 2)y = e^t \end{cases}$$

$L_1 = -(D - 4)$, $L_2 = (D^2 + 2D)$, $L_3 = (D^2 + 1)$, and $L_4 = (D^2 - D - 2)$. If $\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} = 0$, we say the system is degenerate, which means that the system has infinitely many solutions or no solutions. If $\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} = \begin{vmatrix} f_1(t) & L_2 \\ f_2(t) & L_4 \end{vmatrix} = \begin{vmatrix} L_1 & f_1(t) \\ L_3 & f_2(t) \end{vmatrix} = 0$, then the system has *infinitely many solutions*. If at least one of the other two determinants is not zero, the system has *no solutions*.

In Exercises 31 and 32, show that the system has no solution.

**31.** $\begin{cases} x' + x + y' = e^t \\ x' + x + y' = 2e^t \end{cases}$

**32.** $\begin{cases} x'' + y' = \cos t \\ 3x'' + 3y' = \sin t \end{cases}$

In Exercises 33 and 34, show that the system has infinitely many solutions.

**33.** $\begin{cases} x'' + y'' = t^2 \\ 4x'' + 4y'' = 4t^2 \end{cases}$ **34.** $\begin{cases} x'' + y'' - y' = te^t \\ 2x'' + 2y'' - 2y' = 2te^t \end{cases}$

In Exercises 35 and 36, determine the value of $k$ so that the system has infinitely many solutions.

***35.** $\begin{cases} x' + x - y' = t \\ 4y' - 4x' - 4x = kt \end{cases}$ **36.** $\begin{cases} y'' - x = \cos t \\ 4y'' + kx = 4\cos t \end{cases}$

In Exercises 37 and 38, determine restrictions on $c$ so that the system has no solutions.

***37.** $\begin{cases} x + x' - cy = e^{-t} \\ 2x' - 8y + 2x = 2e^{-t} \end{cases}$

**38.** $\begin{cases} x'' + 4y' = \cos t \\ 2x'' + 8y' = c\cos t \end{cases}$

**39.** Suppose that we have a system of second-order equations of the form

$$\begin{cases} x'' = F_1(t, x, y, x', y') \\ y'' = F_2(t, x, y, x', y') \end{cases}.$$

If we let $x_1 = x$, $x_2 = x' = x_1'$, $y_1 = y$, and $y_2 = y' = y_1'$, then we can transform the original system to a system of first-order linear equations. For example, with these substitutions, we transform the system

$$\begin{cases} x'' = -x + 2y \\ y'' = 3x - y + 5\sin t \end{cases}$$

into the first-order system

$$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 + 2y_1 \\ y_1' = y_2 \\ y_2' = 3x_1 - y_1 + 5\sin t \end{cases}.$$

Transform each of the following systems of second-order equations into a system of first-order equations.

**(a)** $\begin{cases} x'' = -3x + y \\ y'' = -x - 2y \end{cases}$

**(b)** $\begin{cases} x'' = -x + 2y + \cos 3t \\ y'' = -2x + y \end{cases}$

**40.** Consider the initial-value problem

$$\begin{cases} x' = -2x - y \\ y' = \dfrac{5}{4}x \\ x(0) = 1, y(0) = 0 \end{cases}.$$

The solution obtained with one computer algebra system was

$$\begin{cases} x(t) = (\frac{1}{2} - i)e^{(-1-i/2)t} + (\frac{1}{2} + i)e^{(-1+i/2)t} \\ y(t) = \frac{5}{4}ie^{(-1-i/2)t} - \frac{5}{4}ie^{(-1+i/2)t} \end{cases},$$

while the solution obtained with another was

$$\begin{cases} x(t) = e^{-t}(\cos(\frac{1}{2}t) - 2\sin(\frac{1}{2}t)) \\ y(t) = \frac{5}{2}e^{-t}\sin(\frac{1}{2}t) \end{cases}.$$

Show that these solutions are equivalent.

Often, we can use a computer algebra system to generate numerical solutions to a system of equations, which is particularly useful if the system under consideration is nonlinear. (Numerical techniques for systems are discussed in more detail in Section 6.8.)

***41.** Figure 6.8 shows the direction field associated with the nonlinear system of equations

$$\begin{cases} \dfrac{dx}{dt} = 2x - xy \\ \dfrac{dy}{dt} = -3y + xy \end{cases}$$

for $0 \le x \le 6$ and $0 \le y \le 6$. Use the direction field to sketch the graphs of the solutions $\begin{cases} x(t) \\ y(t) \end{cases}$ that satisfy the initial conditions **(a)** $x(0) = 2$ and $y(0) = 3$; and **(b)** $x(0) = 3$ and $y(0) = 2$. **(c)** How are the solutions alike? How are they different?

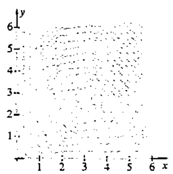


**Figure 6.8**



## 6.2 Review of Matrix Algebra and Calculus

Basic Operations · Determinants and Inverses · Eigenvalues and Eigenvectors · Matrix Calculus

When we encounter a system of linear first-order differential equations such as

$$\begin{cases} x' = -x + 2y \\ y' = 4x - 3y \end{cases},$$

we often prefer to write the system in terms of matrices. Because of their importance in the study of systems of linear equations, we now briefly review matrices and the basic operations associated with them. Detailed discussions of the definitions and properties discussed here are found in most introductory linear algebra texts.

### Basic Operations

**Definition 6.1 $n \times m$ Matrix**

An $n \times m$ **matrix** is an array of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

with $n$ rows and $m$ columns. This matrix can be denoted $\mathbf{A} = (a_{ij})$.

We generally call an $n \times 1$ matrix $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ a **column vector** and a $1 \times n$ matrix $(v_1 \ v_2 \ \cdots \ v_n)$ a **row vector.**

**Definition 6.2 Transpose**

The **transpose** of the $n \times m$ matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

is the $m \times n$ matrix

$$\mathbf{A}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{pmatrix}.$$

Hence, $\mathbf{A}^T = (a_{ji})$.

**Definition 6.3 Scalar Multiplication, Matrix Addition**

Let $\mathbf{A} = (a_{ij})$ be an $n \times m$ matrix and $c$ a scalar. The **scalar multiple** of $\mathbf{A}$ by $c$ is the $n \times m$ matrix given by $c\mathbf{A} = (ca_{ij})$.

If $\mathbf{B} = (b_{ij})$ is also an $n \times m$ matrix, then the **sum** of matrices $\mathbf{A}$ and $\mathbf{B}$ is the $n \times m$ matrix $\mathbf{A} + \mathbf{B} = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$.

Hence, $c\mathbf{A}$ is the matrix obtained by multiplying each element of $\mathbf{A}$ by $c$; $\mathbf{A} + \mathbf{B}$ is obtained by adding corresponding elements of the matrices $\mathbf{A}$ and $\mathbf{B}$ that have the same dimensions.

**Example 1**

Compute $3\mathbf{A} - 9\mathbf{B}$ if $\mathbf{A} = \begin{pmatrix} -1 & 4 & -2 \\ 6 & 2 & -10 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -4 & 8 \\ 7 & 4 & 2 \end{pmatrix}$.

**Solution** Because $3\mathbf{A} = \begin{pmatrix} -3 & 12 & -6 \\ 18 & 6 & -30 \end{pmatrix}$ and

$$-9\mathbf{B} = \begin{pmatrix} -18 & 36 & -72 \\ -63 & -36 & -18 \end{pmatrix}, \quad 3\mathbf{A} - 9\mathbf{B} = 3\mathbf{A} + (-9\mathbf{B}) = \begin{pmatrix} -21 & 48 & -78 \\ -45 & -30 & -48 \end{pmatrix}.$$

**Definition 6.4 Matrix Multiplication**

If $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} \\ a_{21} & a_{22} & \cdots & a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} \end{pmatrix}$ is an $n \times j$ matrix and



$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{j1} & b_{j2} & \cdots & b_{jm} \end{pmatrix}$ is a $j \times m$ matrix,

$$\mathbf{AB} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} \\ a_{21} & a_{22} & \cdots & a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{j1} & b_{j2} & \cdots & b_{jm} \end{pmatrix}$$

is the unique $n \times m$ matrix

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{21} & c_{22} & \cdots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nm} \end{pmatrix}$$

where

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1j}b_{j1} = \sum_{k=1}^{j} a_{1k}b_{k1},$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1j}b_{j2} = \sum_{k=1}^{j} a_{1k}b_{k2},$$

and

$$c_{uv} = a_{u1}b_{1v} + a_{u2}b_{2v} + \cdots + a_{uj}b_{jv} = \sum_{k=1}^{j} a_{uk}b_{kv}.$$

In other words, the element $c_{uv}$ is obtained by multiplying each member of the $u$th row of $\mathbf{A}$ by the corresponding entry in the $v$th column of $\mathbf{B}$ and adding the result.

**Example 2**

Compute $\mathbf{AB}$ if $\mathbf{A} = \begin{pmatrix} 0 & 4 & 5 \\ -5 & -1 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -3 & 4 \\ -5 & -4 \\ 1 & -4 \end{pmatrix}$.

**Solution** Because $\mathbf{A}$ is a $2 \times 3$ matrix and $\mathbf{B}$ is a $3 \times 2$ matrix, $\mathbf{AB}$ is the $2 \times 2$ matrix:

$$\mathbf{AB} = \begin{pmatrix} 0 & 4 & 5 \\ -5 & -1 & 5 \end{pmatrix}\begin{pmatrix} -3 & 4 \\ -5 & -4 \\ 1 & -4 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \cdot -3 + 4 \cdot -5 + 5 \cdot 1 & 0 \cdot 4 + 4 \cdot -4 + 5 \cdot -4 \\ -5 \cdot -3 + -1 \cdot -5 + 5 \cdot 1 & -5 \cdot 4 + -1 \cdot -4 + 5 \cdot -4 \end{pmatrix}$$
$$= \begin{pmatrix} -15 & -36 \\ 25 & -36 \end{pmatrix}.$$

**Definition 6.5 Identity Matrix**

The $n \times n$ matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is called the **$n \times n$ identity matrix,** denoted by $\mathbf{I}$ or $\mathbf{I}_n$.

If $\mathbf{A}$ is an $n \times m$ matrix (an $n \times n$ is called a **square matrix**), then $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$.

## Determinants and Inverses

**Definition 6.6 Determinant**

If $\mathbf{A} = (a_{11})$, the **determinant** of $\mathbf{A}$, denoted by $\det(\mathbf{A})$ or $|\mathbf{A}|$, is $\det(\mathbf{A}) = a_{11}$; if $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

More generally, if $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ is an $n \times n$ matrix and $\mathbf{A}_{ij}$ is the $(n-1) \times (n-1)$ matrix obtained by deleting the $i$th row and $j$th column from $\mathbf{A}$, then

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(\mathbf{A}_{ij}) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |\mathbf{A}_{ij}|.$$

The number $(-1)^{i+j} a_{ij} \det(\mathbf{A}_{ij}) = (-1)^{i+j} a_{ij} |\mathbf{A}_{ij}|$ is called the **cofactor** of $a_{ij}$. The **cofactor matrix, $\mathbf{A}^c$,** of $\mathbf{A}$ is the matrix obtained by replacing each element of $\mathbf{A}$ by its cofactor. Hence,

$$\mathbf{A}^c = \begin{pmatrix} |\mathbf{A}_{11}| & -|\mathbf{A}_{12}| & \cdots & (-1)^{n+1}|\mathbf{A}_{1n}| \\ -|\mathbf{A}_{21}| & |\mathbf{A}_{22}| & \cdots & (-1)^{n}|\mathbf{A}_{2n}| \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1}|\mathbf{A}_{n1}| & (-1)^{n}|\mathbf{A}_{n2}| & \cdots & |\mathbf{A}_{nn}| \end{pmatrix}.$$



**Example 3**

Calculate $|\mathbf{A}|$ if $\mathbf{A} = \begin{pmatrix} -4 & -2 & -1 \\ 5 & -4 & -3 \\ 5 & 1 & -2 \end{pmatrix}$.

***Solution***

$$|\mathbf{A}| = \begin{vmatrix} -4 & -2 & -1 \\ 5 & -4 & -3 \\ 5 & 1 & -2 \end{vmatrix}$$
$$= (-1)^2(-4)\begin{vmatrix} -4 & -3 \\ 1 & -2 \end{vmatrix} + (-1)^3(-2)\begin{vmatrix} 5 & -3 \\ 5 & -2 \end{vmatrix} + (-1)^4(-1)\begin{vmatrix} 5 & -4 \\ 5 & 1 \end{vmatrix}$$
$$= -4((-4)(-2) - (-3)(1)) + 2((5)(-2) - (-3)(5)) - ((5)(1) - (-4)(5))$$
$$= -59.$$

**Definition 6.7 Adjoint and Inverse**

$\mathbf{B}$ is an **inverse** of the $n \times n$ matrix $\mathbf{A}$ means that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$. The **adjoint,** $\mathbf{A}^a$, of an $n \times n$ matrix $\mathbf{A}$ is the transpose of the cofactor matrix: $\mathbf{A}^a = (\mathbf{A}^c)^T$. If $|\mathbf{A}| \neq 0$ and $\mathbf{B} = \frac{1}{|\mathbf{A}|}\mathbf{A}^a$, then $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$. Therefore, if $|\mathbf{A}| \neq 0$, the inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathbf{A}^a.$$

**Example 4**

Find $\mathbf{A}^{-1}$ if $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix}$.

**Solution** In this case, $|\mathbf{A}| = \begin{vmatrix} 2 & -1 \\ -3 & 1 \end{vmatrix} = 2 - 3 = -1 \neq 0$, so $\mathbf{A}^{-1}$ exists. Moreover, $\mathbf{A}^c = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$, so $\mathbf{A}^a = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$ and $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathbf{A}^a = \begin{pmatrix} -1 & -1 \\ -3 & -2 \end{pmatrix}$.

**Example 5**

Find $\mathbf{A}^{-1}$ if $\mathbf{A} = \begin{pmatrix} -2 & -1 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & -1 \end{pmatrix}$.

**Solution** We begin by finding $|\mathbf{A}|$, which is given by

$$|\mathbf{A}| = (-1)^2(-2)\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + (-1)^3(-1)\begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix} + (-1)^4(1)\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} =$$
$$(-2)(-1) + (1)(-2) + (1)(-1) = -1.$$

We then calculate the cofactors:

$$A_{11} = (-1)^2\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = -1, \quad A_{12} = (-1)^3\begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix} = 2,$$
$$A_{13} = (-1)^4\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = -1$$
$$A_{21} = (-1)^3\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 0, \quad A_{22} = (-1)^4\begin{vmatrix} -2 & 1 \\ 3 & -1 \end{vmatrix} = -1,$$
$$A_{23} = (-1)^5\begin{vmatrix} -2 & -1 \\ 3 & 1 \end{vmatrix} = -1$$
$$A_{31} = (-1)^4\begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad A_{32} = (-1)^5\begin{vmatrix} -2 & 1 \\ 2 & 0 \end{vmatrix} = 2,$$
$$A_{33} = (-1)^6\begin{vmatrix} -2 & -1 \\ 2 & 1 \end{vmatrix} = 0.$$

Then

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathbf{A}^a = \frac{1}{-1}\begin{pmatrix} -1 & 0 & -1 \\ 2 & -1 & 2 \\ -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & -2 \\ 1 & 1 & 0 \end{pmatrix}.$$



For convenience, we state the following theorem. The proof is left as an exercise.

**Theorem 6.1 Inverse of a 2 × 2 Matrix**

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $|\mathbf{A}| = ad - bc \neq 0$, then

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

**Example 6**

Find $\mathbf{A}^{-1}$ if $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 2 & 3 \end{pmatrix}$.

**Solution** Because $|\mathbf{A}| = (5)(3) - (2)(-1) = 17$,

$$\mathbf{A}^{-1} = \frac{1}{17}\begin{pmatrix} 3 & 1 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} \frac{3}{17} & \frac{1}{17} \\ -\frac{2}{17} & -\frac{5}{17} \end{pmatrix}.$$

We almost always take advantage of a computer algebra system to perform operations on matrices. In addition, if you have taken linear algebra, you can use techniques such as row reduction to find the inverse of a matrix or solve systems of equations.

The inverse $\mathbf{A}^{-1}$ can be used to solve the linear system of equations $\mathbf{Ax} = \mathbf{b}$. For example, to solve $\begin{pmatrix} 5 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -34 \\ 17 \end{pmatrix}$ in which $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -34 \\ 17 \end{pmatrix}$, we find $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{pmatrix} \frac{3}{17} & \frac{1}{17} \\ -\frac{2}{17} & \frac{5}{17} \end{pmatrix}\begin{pmatrix} -34 \\ 17 \end{pmatrix} = \begin{pmatrix} -5 \\ 9 \end{pmatrix}$. We can find several uses for the inverse in solving systems of differential equations as well.

## Eigenvalues and Eigenvectors

**Definition 6.8 Eigenvalues and Eigenvectors**

A *nonzero* vector $\mathbf{x}$ is an **eigenvector** of the square matrix $\mathbf{A}$ if there is a number $\lambda$, called an **eigenvalue** of $\mathbf{A}$, so that

$$\mathbf{Ax} = \lambda\mathbf{x}.$$

*Note:* By definition, an eigenvector of a matrix is *never* the zero vector.

**Example 7**

Show that $\begin{pmatrix}-1\\2\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$ are eigenvectors of $\begin{pmatrix}-1&2\\4&-3\end{pmatrix}$ with corresponding eigenvalues $-5$ and $1$, respectively.

**Solution** Because $\begin{pmatrix}-1&2\\4&-3\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix}=\begin{pmatrix}5\\-10\end{pmatrix}=-5\begin{pmatrix}-1\\2\end{pmatrix}$ and $\begin{pmatrix}-1&2\\4&-3\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}=1\begin{pmatrix}1\\1\end{pmatrix}$, $\begin{pmatrix}-1\\2\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$ are eigenvectors of $\begin{pmatrix}-1&2\\4&-3\end{pmatrix}$ with corresponding eigenvalues $-5$ and $1$, respectively.

If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ with corresponding eigenvalue $\lambda$, then $\mathbf{Ax}=\lambda\mathbf{x}$. Because this equation is equivalent to the equation $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$, $\mathbf{x}\neq\mathbf{0}$ is an eigenvector if and only if $\det(\mathbf{A}-\lambda\mathbf{I})=0$.

*If* $\det(\mathbf{A}-\lambda\mathbf{I})\neq 0$, *what is the solution of* $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$? *Can this solution (vector) be an eigenvector of* $\mathbf{A}$?

**Definition 6.9 Characteristic Polynomial**

The equation $\det(\mathbf{A}-\lambda\mathbf{I})=0$ is called the **characteristic equation** of $\mathbf{A}$; $\det(\mathbf{A}-\lambda\mathbf{I})$ is called the **characteristic polynomial** of $\mathbf{A}$.

Notice that the roots of the characteristic polynomial of $\mathbf{A}$ are the eigenvalues of $\mathbf{A}$.

**Example 8**

Calculate the eigenvalues and corresponding eigenvectors of $\mathbf{A}=\begin{pmatrix}4&-6\\3&-7\end{pmatrix}$.

**Solution** The characteristic polynomial of $\mathbf{A}=\begin{pmatrix}4&-6\\3&-7\end{pmatrix}$ is

$$\begin{vmatrix}4-\lambda&-6\\3&-7-\lambda\end{vmatrix}=\lambda^2+3\lambda-10=(\lambda+5)(\lambda-2).$$

Because the eigenvalues are found by solving $(\lambda+5)(\lambda-2)=0$, the eigenvalues are $\lambda=-5$ and $\lambda=2$. Let $\begin{pmatrix}x_1\\y_1\end{pmatrix}$ denote the eigenvectors corresponding to the eigenvalue $\lambda=-5$. Then,



When finding an eigenvector $\mathbf{v}$ corresponding to the eigenvalue $\lambda$, we see that there is actually a collection (or family) of eigenvectors corresponding to $\lambda$. In the study of differential equations, we find that we only need to find one member of the collection of eigenvectors. Therefore, as we did in Example 8, we usually eliminate the arbitrary constants when we encounter them in eigenvectors by selecting particular values for the constants.

$$\left\{\begin{pmatrix}4&-6\\3&-7\end{pmatrix}-(-5)\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\}\begin{pmatrix}x_1\\y_1\end{pmatrix}=\mathbf{0}.$$

Simplifying yields the system of equations $\begin{cases}9x_1-6y_1=0\\3x_1-2y_1=0\end{cases}$, so $y_1=\frac{3}{2}x_1$. Therefore, if $x_1$ is any real number, then $\begin{pmatrix}x_1\\\frac{3}{2}x_1\end{pmatrix}$ is an eigenvector. In particular, if $x_1=2$, then $\begin{pmatrix}2\\3\end{pmatrix}$ is an eigenvector of $\mathbf{A}=\begin{pmatrix}4&-6\\3&-7\end{pmatrix}$ with corresponding eigenvalue $\lambda=-5$. Similarly, if we let $\begin{pmatrix}x_2\\y_2\end{pmatrix}$ denote the eigenvectors corresponding to $\lambda=2$, then $\left\{\begin{pmatrix}4&-6\\3&-7\end{pmatrix}-2\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\}\begin{pmatrix}x_2\\y_2\end{pmatrix}=\mathbf{0}$, which yields the system $\begin{cases}2x_2-6y_2=0\\3x_2-9y_2=0\end{cases}$, so $y_2=\frac{1}{3}x_2$. If $x_2=3$, then $\begin{pmatrix}3\\1\end{pmatrix}$ is an eigenvector of $\mathbf{A}=\begin{pmatrix}4&-6\\3&-7\end{pmatrix}$ with corresponding eigenvalue $\lambda=2$. (Notice that constant (scalar) multiples of the eigenvectors $\begin{pmatrix}2\\3\end{pmatrix}$ and $\begin{pmatrix}3\\1\end{pmatrix}$ are also eigenvectors corresponding to $\lambda=-5$ and $\lambda=2$, respectively.)

**Example 9**

Find the eigenvalues and corresponding eigenvectors of $\mathbf{A}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$.

**Solution** In this case, the characteristic polynomial is $\begin{vmatrix}-\lambda&1\\-1&-\lambda\end{vmatrix}=\lambda^2+1$, so the eigenvalues are the roots of the equation $\lambda^2+1=0$. These are the imaginary numbers $\lambda=i$ and $\lambda=-i$ where $i=\sqrt{-1}$. The corresponding eigenvectors are found by substituting the eigenvalues into the equation $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$ and solving for $\mathbf{x}$. For $\lambda=i$, this equation is $\begin{pmatrix}-i&1\\-1&-i\end{pmatrix}\begin{pmatrix}x_1\\y_1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, which is equivalent to the system $\begin{cases}-ix_1+y_1=0\\-x_1-iy_1=0\end{cases}$. Notice that the second equation of this system is a constant multiple ($i$) of the first equation. Hence, an eigenvector $\begin{pmatrix}x_1\\y_1\end{pmatrix}$ must satisfy $y_1=ix_1$. Therefore, $\begin{pmatrix}x_1\\ix_1\end{pmatrix}=\begin{pmatrix}1\\i\end{pmatrix}x_1$ is an eigenvector for any value of $x_1$. For example, if $x_1=1$, then $\begin{pmatrix}1\\i\end{pmatrix}$ is an eigenvector. For $\lambda=-i$, the system of equations is $\begin{pmatrix}i&1\\-1&i\end{pmatrix}\begin{pmatrix}x_2\\y_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, which is equivalent to $\begin{cases}ix_2+y_2=0\\-x_2+iy_2=0\end{cases}$. Because the second equation equals $i$ times the first equation, the eigenvector $\begin{pmatrix}x_2\\y_2\end{pmatrix}$ must satisfy

$y_2 = -ix_2$. Hence, $\begin{pmatrix} x_2 \\ -ix_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} x_2$ is an eigenvector for any value of $x_2$. Therefore, if $x_2 = 1$, then $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector.

Recall that the complex conjugate of the complex number $z = a + bi$ is $\bar{z} = a - bi$. Similarly, the complex conjugate of the vector $\mathbf{x} = \begin{pmatrix} a_1 + b_1 i \\ a_2 + b_2 i \\ \vdots \\ a_n + b_n i \end{pmatrix}$ is the vector $\bar{\mathbf{x}} = \begin{pmatrix} a_1 - b_1 i \\ a_2 - b_2 i \\ \vdots \\ a_n - b_n i \end{pmatrix}$. Notice that the eigenvectors corresponding to the complex conjugate eigenvalues $\lambda = i$ and $\lambda = -i$ in the previous example are $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -i \end{pmatrix}$, which are complex conjugates. This is no coincidence. We can prove that the *eigenvectors that correspond to complex eigenvalues are themselves complex conjugates.*

**Example 10**

Calculate the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$.

***Solution*** We begin by finding the characteristic polynomial of $\mathbf{A}$ with

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 3-\lambda & 1 \\ 0 & 2 & 4-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 7\lambda + 10)$$
$$= (1-\lambda)(\lambda - 2)(\lambda - 5).$$

The eigenvalues of $\mathbf{A}$ are $\lambda = 1$, $\lambda = 2$, and $\lambda = 5$. For each eigenvalue, we find the corresponding eigenvectors by substituting the eigenvalue $\lambda$ into the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ and solving for the vector $\mathbf{x}$. If $\lambda = 1$, we obtain

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 1 \\ 0 & 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which is reduced to $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, so $x_1 = z_1$ and $y_1 = -\frac{3}{2}z_1$. Hence,



$\begin{pmatrix} z_1 \\ -\frac{3}{2}z_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{pmatrix} z_1$ is an eigenvector for any value of $z_1$.

If $\lambda = 2$, we have $\begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, which is equivalent to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ in reduced form. Thus, $x_2 = 0$ and $y_2 = -z_2$, so $\begin{pmatrix} 0 \\ -z_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} z_2$ is an eigenvector for any value of $z_2$.

If $\lambda = 5$, then $\begin{pmatrix} -4 & 0 & 0 \\ 2 & -2 & 1 \\ 0 & 2 & -1 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, which is equivalent to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Therefore, $x_3 = 0$ and $z_3 = 2y_3$, so $\begin{pmatrix} 0 \\ y_3 \\ 2y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} y_3$ is an eigenvector for any value of $y_3$.

**Definition 6.10 Eigenvalue of Multiplicity *m***

Suppose that $(\lambda - \lambda_1)^m$ where $m$ is a positive integer is a factor of the characteristic polynomial of the $n \times n$ matrix $\mathbf{A}$, while $(\lambda - \lambda_1)^{m+1}$ is not a factor of this polynomial. Then $\lambda = \lambda_1$ is an **eigenvalue of multiplicity *m*.**

We often say that the eigenvalue of an $n \times n$ matrix $\mathbf{A}$ is **repeated** if it is of multiplicity $m$ where $m \geq 2$ and $m \leq n$. When trying to find the eigenvector(s) corresponding to an eigenvalue of multiplicity $m$, two situations may be encountered: either $m$ or fewer than $m$ linearly independent eigenvectors can be found that correspond to $\lambda$.

**Example 11**

Find the eigenvalues and corresponding eigenvectors of **(a)** $\mathbf{A} = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$; and **(b)** $\mathbf{B} = \begin{pmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{pmatrix}$.

**Solution** (a) The eigenvalues are found with

$$\begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -5-\lambda & 3 \\ -6 & 4-\lambda \end{vmatrix} - (-3)\begin{vmatrix} 3 & 3 \\ 6 & 4-\lambda \end{vmatrix} +$$

$$3\begin{vmatrix} 3 & -5-\lambda \\ 6 & -6 \end{vmatrix} = 16 + 12\lambda - \lambda^3 = 0.$$

Note that $\lambda = -2$ is a root of the characteristic polynomial. Using this root with synthetic or long division, we find that $\lambda^3 - 12\lambda - 16 = (\lambda + 2)^2(\lambda - 4) = 0$. Hence, $\lambda = -2$ is an eigenvalue of multiplicity 2. When we try to find an eigenvector $\mathbf{v} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ corresponding to $\lambda = -2$, we see that

$$\begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

must be satisfied. After the necessary row operations, we find that this system is equivalent to

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which indicates that $x_1 - y_1 + z_1 = 0$ or $x_1 = y_1 - z_1$. If $z_1 = 0$, then $x_1 = y_1$. Hence, $\mathbf{v}_1 = \begin{pmatrix} y_1 \\ y_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} y_1$ is an eigenvector for any choice of $y_1$. On the other hand, if in $x_1 = y_1 - z_1, y_1 = 0$, then $x_1 = -z_1$. Therefore, $\mathbf{v}_2 = \begin{pmatrix} -z_1 \\ 0 \\ z_1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} z_1$ (for any choice of $z_1$) is another linearly independent eigenvector corresponding to $\lambda = -2$ because $\mathbf{v}_2$ is not a constant multiple of $\mathbf{v}_1$. Therefore, we have found two linearly independent eigenvectors that correspond to the eigenvalue $\lambda = -2$ of multiplicity 2. We leave it to you to find an eigenvector corresponding to $\lambda = 4$.

(b) The eigenvalues of $\mathbf{B}$ are determined by solving

$$\begin{vmatrix} 5-\lambda & -4 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & 5-\lambda \end{vmatrix} = (5-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} - (-4)\begin{vmatrix} 1 & 2 \\ 0 & 5-\lambda \end{vmatrix}$$

$$= -(5-\lambda)^2\lambda = 0.$$



Hence, $\lambda = 5$ is an eigenvalue of multiplicity 2. In this case, when we find a corresponding eigenvector $\mathbf{v} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$, we solve $\begin{pmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, which is equivalent to $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The components of $\mathbf{v}$ must satisfy $x_1 = -2z_1$ and $y_1 = 0$. We find only one eigenvector $\mathbf{v} = \begin{pmatrix} -2z_1 \\ 0 \\ z_1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} z_1$ that corresponds to the eigenvalue $\lambda = 5$ of multiplicity 2. We leave it to you to find an eigenvector corresponding to $\lambda = 0$.

## Matrix Calculus

**Definition 6.11 Derivative and Integral of a Matrix**

The **derivative** of the $n \times m$ matrix $\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1m}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nm}(t) \end{pmatrix}$, where $a_{ij}(t)$ is differentiable for all values of $i$ and $j$, is

$$\frac{d}{dt}\mathbf{A}(t) = \begin{pmatrix} \frac{d}{dt}a_{11}(t) & \frac{d}{dt}a_{12}(t) & \cdots & \frac{d}{dt}a_{1m}(t) \\ \frac{d}{dt}a_{21}(t) & \frac{d}{dt}a_{22}(t) & \cdots & \frac{d}{dt}a_{2m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d}{dt}a_{n1}(t) & \frac{d}{dt}a_{n2}(t) & \cdots & \frac{d}{dt}a_{nm}(t) \end{pmatrix}.$$

The **integral** of $\mathbf{A}(t)$, where $a_{ij}(t)$ is integrable for all values of $i$ and $j$, is

$$\int \mathbf{A}(t)\,dt = \begin{pmatrix} \int a_{11}(t)\,dt & \int a_{12}(t)\,dt & \cdots & \int a_{1m}(t)\,dt \\ \int a_{21}(t)\,dt & \int a_{22}(t)\,dt & \cdots & \int a_{2m}(t)\,dt \\ \vdots & \vdots & \ddots & \vdots \\ \int a_{n1}(t)\,dt & \int a_{n2}(t)\,dt & \cdots & \int a_{nm}(t)\,dt \end{pmatrix}.$$

**Example 12**

Find $\frac{d}{dt}\mathbf{A}(t)$ and $\int \mathbf{A}(t)\,dt$ if $\mathbf{A}(t) = \begin{pmatrix} \cos 3t & \sin 3t & e^{-t} \\ t & t\sin(t^2) & e^t \end{pmatrix}$.

**Solution** We find $\frac{d}{dt}\mathbf{A}(t)$ by differentiating each element of $\mathbf{A}(t)$. This yields

$$\frac{d}{dt}\mathbf{A}(t) = \begin{pmatrix} -3\sin 3t & 3\cos 3t & -e^{-t} \\ 1 & \sin(t^2) + 2t^2\cos(t^2) & e^t \end{pmatrix}.$$

Similarly, we find $\int \mathbf{A}(t)\,dt$ by integrating each element of $\mathbf{A}(t)$.

$$\int \mathbf{A}(t)\,dt = \begin{pmatrix} \frac{1}{3}\sin 3t + c_{11} & \frac{-1}{3}\cos 3t + c_{12} & -e^{-t} + c_{13} \\ \frac{1}{2}t^2 + c_{21} & \frac{-1}{2}\cos(t^2) + c_{22} & e^t + c_{23} \end{pmatrix}$$

where each $c_{ij}$ represents an arbitrary constant.

## EXERCISES 6.2

In Exercises 1–4, perform the indicated calculation if
$\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 0 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 3 & 6 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 4 & 1 \\ -2 & 7 \end{pmatrix}$.

1. $\mathbf{B} - \mathbf{A}$
2. $2\mathbf{A} + \mathbf{C}$
3. $(\mathbf{B} + \mathbf{C}) - 4\mathbf{A}$
4. $(\mathbf{A} - 3\mathbf{C}) + (\mathbf{C} - 5\mathbf{B})$

In Exercises 5–8, perform the indicated calculation if
$\mathbf{A} = \begin{pmatrix} 0 & 3 & 2 \\ 5 & -1 & 4 \\ 2 & -1 & -3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 & -2 & -3 \\ 5 & 5 & 1 \\ 4 & -5 & 2 \end{pmatrix}$, and
$\mathbf{C} = \begin{pmatrix} -2 & -4 & 0 \\ 5 & -1 & 1 \\ -1 & -3 & -3 \end{pmatrix}$.

5. $7\mathbf{A} + 3\mathbf{B}$
6. $8\mathbf{B} - 9\mathbf{A}$
7. $2\mathbf{A} - 4(\mathbf{B} + \mathbf{C})$
8. $4\mathbf{C} + 2(2\mathbf{A} - 5\mathbf{B})$

In Exercises 9–14, compute **AB** and **BA**, when defined, using the given matrices.

9. $\mathbf{A} = \begin{pmatrix} 1 & -3 & -4 & 3 \\ -5 & -1 & -2 & -2 \\ 4 & -1 & 0 & -4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -5 & 2 & 4 \\ -5 & 0 & -5 \\ 0 & 4 & 5 \\ 4 & 3 & -4 \end{pmatrix}$

10. $\mathbf{A} = \begin{pmatrix} -2 & 4 & 3 & 3 \\ -3 & 2 & 4 & -3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & -2 & 2 \\ -2 & -4 & 1 \\ -2 & 0 & 4 \\ -1 & -4 & 5 \end{pmatrix}$

*11. $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -1 & -5 \end{pmatrix}$

12. $\mathbf{A} = \begin{pmatrix} -4 & 5 \\ 1 & -5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 2 \\ 3 & 5 \end{pmatrix}$

13. $\mathbf{A} = \begin{pmatrix} 1 & -1 & -1 \\ -4 & 5 & -2 \\ 5 & -4 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -3 & -3 & -2 \\ -1 & 4 & -1 \\ -1 & 5 & 3 \end{pmatrix}$



14. $\mathbf{A} = \begin{pmatrix} 4 & 2 & 2 & 0 \\ -2 & -3 & 0 & -2 \\ -1 & -2 & -5 & 4 \\ 3 & -5 & -3 & -3 \end{pmatrix}$ and
$\mathbf{B} = \begin{pmatrix} 4 & 1 & -3 & -5 \\ 5 & -2 & -2 & 3 \\ -3 & -5 & 2 & 0 \\ 5 & 3 & -3 & -5 \end{pmatrix}$

In Exercises 15–19, find the determinant of the square matrix.

15. $\mathbf{A} = \begin{pmatrix} -1 & -4 \\ 5 & 3 \end{pmatrix}$

16. $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -5 & 4 \end{pmatrix}$

*17. $\mathbf{A} = \begin{pmatrix} 3 & -2 & 0 \\ -1 & 3 & 0 \\ 3 & -2 & 0 \end{pmatrix}$

18. $\mathbf{A} = \begin{pmatrix} 0 & 3 & 3 \\ 1 & 1 & -2 \\ -3 & 2 & -3 \end{pmatrix}$

*19. $\mathbf{A} = \begin{pmatrix} 2 & 0 & -2 & -1 \\ -3 & -2 & 0 & 1 \\ -3 & -1 & -3 & -3 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

In Exercises 20–23, find the inverse of each square matrix.

20. $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

*21. $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 2 & 2 \end{pmatrix}$

22. $\mathbf{A} = \begin{pmatrix} -3 & 3 & 3 \\ 3 & -2 & -2 \\ 3 & 1 & 0 \end{pmatrix}$

*23. $\mathbf{A} = \begin{pmatrix} -1 & -1 & -2 \\ 3 & 2 & 2 \\ 0 & 0 & -2 \end{pmatrix}$

In Exercises 24–34, find the eigenvalues and corresponding eigenvectors of the matrix.

24. $\mathbf{A} = \begin{pmatrix} -7 & -2 \\ 4 & -1 \end{pmatrix}$

*25. $\mathbf{A} = \begin{pmatrix} -6 & 1 \\ -2 & -3 \end{pmatrix}$

26. $\mathbf{A} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

27. $\mathbf{A} = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$

28. $\mathbf{A} = \begin{pmatrix} -1 & 3 \\ -3 & -7 \end{pmatrix}$

*29. $\mathbf{A} = \begin{pmatrix} -7 & 4 \\ -1 & -3 \end{pmatrix}$

30. $\mathbf{A} = \begin{pmatrix} -1 & 3 \\ -6 & -7 \end{pmatrix}$

31. $\mathbf{A} = \begin{pmatrix} -1 & -1 \\ 5 & -3 \end{pmatrix}$

32. $\mathbf{A} = \begin{pmatrix} -4 & 0 & -3 \\ 9 & 1 & 9 \\ 2 & 0 & 1 \end{pmatrix}$

*33. $\mathbf{A} = \begin{pmatrix} -41 & -38 & 18 \\ 48 & 44 & -21 \\ 8 & 6 & -3 \end{pmatrix}$

34. $\mathbf{A} = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$

In Exercises 35–40, find $d/dt\,\mathbf{A}(t)$ and $\int \mathbf{A}(t)\,dt$.

35. $\mathbf{A} = \begin{pmatrix} e^{2t} \\ e^{-5t} \end{pmatrix}$

36. $\mathbf{A} = \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}$

*37. $\mathbf{A} = \begin{pmatrix} \cos t & t\cos t \\ \sin t & t\sin t \end{pmatrix}$

38. $\mathbf{A} = \begin{pmatrix} e^{-t} & te^{-t} \\ t & t^2 \end{pmatrix}$

*39. $\mathbf{A} = \begin{pmatrix} e^{4t} \\ \cos 3t \\ \sin 3t \end{pmatrix}$

40. $\mathbf{A} = \begin{pmatrix} \frac{\ln t}{t} \\ t\ln t \\ \ln t \end{pmatrix}$

*41. If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $|\mathbf{A}| = ad - bc \neq 0$, then show that $\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Many computer software packages and calculators contain built-in functions for working with matrices. Use a computer or calculator to perform the following calculations to become familiar with these functions.

42. If $\mathbf{A} = \begin{pmatrix} 0 & 0 & -2 & 1 \\ 3 & -1 & 7 & 2 \\ -6 & 0 & 5 & -1 \\ -6 & 0 & 1 & -2 \end{pmatrix}$ and
$\mathbf{B} = \begin{pmatrix} -7 & -6 & -3 & -7 \\ 2 & -3 & 0 & 4 \\ 3 & 4 & 1 & 2 \\ 5 & 6 & 3 & 6 \end{pmatrix}$, compute **(a)** $3\mathbf{A} - 2\mathbf{B}$; **(b)** $\mathbf{B}^T$; **(c)** $\mathbf{AB}$; **(d)** $|\mathbf{A}|$, $|\mathbf{B}|$, and $|\mathbf{AB}|$; **(e)** $\mathbf{A}^{-1}$.

*43. Find the eigenvalues and corresponding eigenvectors of the matrices
$\mathbf{A} = \begin{pmatrix} -4 & 4 & -4 \\ 2 & 3 & -4 \\ 5 & 0 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -3 & 5 \\ 5 & 5 & 0 \\ -5 & -2 & 3 \end{pmatrix}$.

44. Find $\frac{d}{dt}\mathbf{A}(t)$ and $\int \mathbf{A}(t)\,dt$ if

$$\mathbf{A}(t) = \begin{pmatrix} te^{-t} & t^2 \sin 2t & \frac{1}{9+4t^2} \\ \cos^6 t & \sec^3 2t & \frac{1}{t(t-1)} \\ \frac{4}{\sqrt{1-t^2}} & \sin^5 t \cos t & t^3 \sin^2 t \end{pmatrix}.$$

*45. Calculate the eigenvalues of $\begin{pmatrix} 0 & -1-k^2 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1-k^2 \\ 1 & 2k \end{pmatrix}$. How do the eigenvalues change for $-\infty < k < +\infty$?

46. Nearly every computer algebra system can compute exact values of the eigenvalues and corresponding eigenvectors of an $n \times n$ matrix for $n = 2$, 3, and 4. (a) Use a computer algebra system to compute the exact values of the eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} 3 & 1 & 1 & 1 \\ -2 & -3 & 2 & -3 \\ -3 & 3 & -2 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$. Write down a portion of the result (or print it and staple it to your homework). (b) Compute approximations of the eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} 3 & 1 & 1 & 1 \\ -2 & -3 & 2 & -3 \\ -3 & 3 & -2 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$. (c) Which results are more meaningful to you?

## 6.3 Preliminary Definitions and Notation

We first encounter systems of equations in elementary algebra courses. For example,

$$\begin{cases} 3x - 5y = -13 \\ -3x + 6y = 15 \end{cases}$$

is a system of two linear equations in two variables with solution $(x, y) = (-1, 2)$. In the same manner, we can consider a system of differential equations.

We begin our study of systems of ordinary differential equations by introducing several definitions along with some convenient notation. Let

$$\mathbf{X} = \mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \quad \text{and} \quad \mathbf{F}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}.$$

Then, the homogeneous system of first-order linear differential equations

$$\begin{cases} x_1'(t) = a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) \\ x_2'(t) = a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) \\ \quad\vdots \\ x_n'(t) = a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) \end{cases}$$

is equivalent to $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$, and the nonhomogeneous system



$$\begin{cases} x_1'(t) = a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + f_1(t) \\ x_2'(t) = a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + f_2(t) \\ \quad\vdots \\ x_n'(t) = a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) + f_n(t) \end{cases}$$

is equivalent to $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{F}(t)$.

**Example 1**

(a) Write the homogeneous system $\begin{cases} x' = -5x + 5y \\ y' = -5x + y \end{cases}$ in matrix form. (b) Write the nonhomogeneous system $\begin{cases} x' = x + 2y - \sin t \\ y' = 4x - 3y + t^2 \end{cases}$ in matrix form.

***Solution*** (a) The homogeneous system $\begin{cases} x' = -5x + 5y \\ y' = -5x + y \end{cases}$ is equivalent to the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -5 & 5 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

(b) The nonhomogeneous system $\begin{cases} x' = x + 2y - \sin t \\ y' = 4x - 3y + t^2 \end{cases}$ is equivalent to $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\sin t \\ t^2 \end{pmatrix}$.

At this point, given a system of ordinary differential equations, our goal is to construct an explicit, numerical, or graphical solution of the system.

We now state the theorems and terminology used in establishing the fundamentals of solving systems of differential equations. All proofs are omitted but can be found in advanced differential equations textbooks. In each case, we assume that the matrix $\mathbf{A}(t)$ in the systems $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{F}(t)$ and $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ is an $n \times n$ matrix.

**Definition 6.12 Solution Vector**

A **solution vector** of the system $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{F}(t)$ on the interval $I$ is an $n \times 1$ matrix of the form

$$\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix},$$

where the $x_i(t)$ are differentiable functions on $I$, that satisfies $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{F}(t)$.

**Example 2**

Show that $\mathbf{X}(t) = \begin{pmatrix} -2e^{2t} \\ e^{2t} \end{pmatrix}$ is a solution of $\mathbf{X}' = \begin{pmatrix} 1 & -2 \\ 2 & 6 \end{pmatrix}\mathbf{X}$.

**Solution** Notice that $\mathbf{X}'(t) = \begin{pmatrix} -4e^{2t} \\ 2e^{2t} \end{pmatrix}$ and $\begin{pmatrix} 1 & -2 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} -2e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} -2e^{2t} - 2e^{2t} \\ -4e^{2t} + 6e^{2t} \end{pmatrix} = \begin{pmatrix} -4e^{2t} \\ 2e^{2t} \end{pmatrix}$. Then, because $\mathbf{X}' = \mathbf{A}\mathbf{X}$, $\mathbf{X}(t) = \begin{pmatrix} -2e^{2t} \\ e^{2t} \end{pmatrix}$ is a solution of the system.

**Theorem 6.2 Principle of Superposition**

Suppose that $\mathbf{X}_1(t), \mathbf{X}_2(t), \ldots, \mathbf{X}_n(t)$ are $n$ solutions of the system of first-order linear homogeneous differential equations $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ on the open interval $I$. Then, the linear combination

$$\mathbf{X}(t) = c_1\mathbf{X}_1(t) + c_2\mathbf{X}_2(t) + \cdots + c_n\mathbf{X}_n(t),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants, is also a solution of $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ on $I$.

**Example 3**

Show that $\mathbf{X}(t) = \begin{pmatrix} -e^{5t} \\ 2e^{5t} \end{pmatrix}$ and $\mathbf{X}(t) = c_1\begin{pmatrix} -2e^{2t} \\ e^{2t} \end{pmatrix} + c_2\begin{pmatrix} -e^{5t} \\ 2e^{5t} \end{pmatrix}$ are solutions of $\mathbf{X}' = \begin{pmatrix} 1 & -2 \\ 2 & 6 \end{pmatrix}\mathbf{X}$.

**Solution** We let the reader follow the procedure used in Example 2 to show that $\mathbf{X}(t) = \begin{pmatrix} -e^{5t} \\ 2e^{5t} \end{pmatrix}$ satisfies $\mathbf{X}' = \begin{pmatrix} 1 & -2 \\ 2 & 6 \end{pmatrix}\mathbf{X}$. By the Principle of Superposition, the linear combination of the two solutions $\begin{pmatrix} -2e^{2t} \\ e^{2t} \end{pmatrix}$ and $\begin{pmatrix} -e^{5t} \\ 2e^{5t} \end{pmatrix}$ is also a solution. We verify this now by first writing $\mathbf{X}(t)$ as $\mathbf{X}(t) = \begin{pmatrix} -2c_1e^{2t} - c_2e^{5t} \\ c_1e^{2t} + 2c_2e^{5t} \end{pmatrix}$. Then, $\mathbf{X}'(t) = \begin{pmatrix} -4c_1e^{2t} - 5c_2e^{5t} \\ 2c_1e^{2t} + 10c_2e^{5t} \end{pmatrix}$ and



$$\begin{pmatrix} 1 & -2 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} -2c_1e^{2t} - c_2e^{5t} \\ c_1e^{2t} + 2c_2e^{5t} \end{pmatrix} = \begin{pmatrix} (-2c_1e^{2t} - c_2e^{5t}) - 2(c_1e^{2t} + 2c_2e^{5t}) \\ 2(-2c_1e^{2t} - c_2e^{5t}) + 6(c_1e^{2t} + 2c_2e^{5t}) \end{pmatrix}$$
$$= \begin{pmatrix} -4c_1e^{2t} - 5c_2e^{5t} \\ 2c_1e^{2t} + 10c_2e^{5t} \end{pmatrix}.$$

Therefore, the linear combination of the solutions is also a solution.

We define linear dependence and independence of a set of vector-valued functions $\{\mathbf{X}_1(t), \mathbf{X}_2(t), \ldots, \mathbf{X}_n(t)\}$ in the same way we defined linear dependence and independence of sets of real-valued functions. Then, the set $\{\mathbf{X}_1(t), \mathbf{X}_2(t), \ldots, \mathbf{X}_n(t)\}$ is **linearly dependent** on an interval $I$ if there is a set of constants $\{c_1, c_2, \ldots, c_n\}$ not all zero such that

$$c_1\mathbf{X}_1(t) + c_2\mathbf{X}_2(t) + \cdots + c_n\mathbf{X}_n(t) = \mathbf{0};$$

otherwise, the set is **linearly independent**. (Note that $\mathbf{0}$ is the zero vector with the same dimensions as each of the $\mathbf{X}_j(t)$, $j = 1, 2, \ldots, n$.) As with two real-valued functions, two vector-valued functions are linearly dependent if they are scalar multiples of each other. Otherwise, the functions are linearly independent. For more than two vector-valued functions, we often use the Wronskian to determine if the functions are linearly independent or dependent.

**Definition 6.13 Wronskian of a Set of Vector-Valued Functions**

The **Wronskian** of $\{\mathbf{X}_1(t), \mathbf{X}_2(t), \ldots, \mathbf{X}_n(t)\}$ is defined as the determinant of the matrix with columns $\mathbf{X}_1(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}$, $\mathbf{X}_2(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \\ \vdots \\ x_{n2}(t) \end{pmatrix}$, $\ldots$, $\mathbf{X}_n(t) = \begin{pmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix}$:

$$W(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}.$$

**Theorem 6.3 The Wronskian of Solutions**

Suppose that $\mathbf{X}_1(t), \mathbf{X}_2(t), \ldots, \mathbf{X}_n(t)$ are $n$ solutions of the system of first-order linear homogeneous differential equations $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ on the open interval $I$, where each component of $\mathbf{A}(t)$ is continuous on $I$. If the set $\{\mathbf{X}_1(t), \mathbf{X}_2(t), \ldots, \mathbf{X}_n(t)\}$ is linearly dependent, then $W(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) \equiv 0$ on $I$. If the set $\{\mathbf{X}_1(t), \mathbf{X}_2(t), \ldots, \mathbf{X}_n(t)\}$ is linearly independent, then $W(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) \neq 0$ for all values on $I$.

**Example 4**

Verify that $\mathbf{X}_1(t) = \begin{pmatrix} -2e^{2t} \\ e^{2t} \end{pmatrix}$ and $\mathbf{X}_2(t) = \begin{pmatrix} -e^{5t} \\ 2e^{5t} \end{pmatrix}$ are linearly independent solutions of $\mathbf{X}' = \begin{pmatrix} 1 & -2 \\ 2 & 6 \end{pmatrix}\mathbf{X}$.

**Solution** We showed earlier that $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$ are solutions of the given system. Therefore, we calculate $W(\mathbf{X}_1, \mathbf{X}_2) = \begin{vmatrix} -2e^{2t} & -e^{5t} \\ e^{2t} & 2e^{5t} \end{vmatrix} = -3e^{7t}$. The vector-valued functions are linearly independent because $W(\mathbf{X}_1, \mathbf{X}_2) = -3e^{7t} \neq 0$ for all values of $t$.

**Definition 6.14 Fundamental Set of Solutions**

Any set $\{\mathbf{X}_1(t), \mathbf{X}_2(t), \ldots, \mathbf{X}_n(t)\}$ of $n$ linearly independent solutions of $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ on an open interval $I$ is called a **fundamental set of solutions** on $I$.

**Example 5**

Which of the following is a fundamental set of solutions for $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -2 & -8 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$? **(a)** $\left\{\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}, \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}\right\}$ **(b)** $\left\{\begin{pmatrix} -2\sin 2t + 2\cos 2t \\ \sin 2t \end{pmatrix}, \begin{pmatrix} 4\cos 2t \\ \sin 2t - \cos 2t \end{pmatrix}\right\}$

**Solution** We first remark that the equation $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -2 & -8 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is equivalent to the system

$$\begin{cases} x' = -2x - 8y \\ y' = x + 2y \end{cases}.$$

**(a)** Differentiating we see that

$$\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}' = \begin{pmatrix} -2\sin 2t \\ 2\cos 2t \end{pmatrix} \neq \begin{pmatrix} -2\cos 2t - 8\sin 2t \\ \cos 2t + 2\sin 2t \end{pmatrix}$$

which shows us that $\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}$ is not a solution to the system. Therefore, $\left\{\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}, \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}\right\}$ is not a fundamental set of solutions. **(b)** You should verify that both $\begin{pmatrix} -2\sin 2t + 2\cos 2t \\ \sin 2t \end{pmatrix}$ and $\begin{pmatrix} 4\cos 2t \\ \sin 2t - \cos 2t \end{pmatrix}$ are solutions to the system. Computing the Wronskian, we have

$$\begin{aligned} \begin{vmatrix} -2\sin 2t + 2\cos 2t & 4\cos 2t \\ \sin 2t & \sin 2t - \cos 2t \end{vmatrix} &= (-2\sin 2t + 2\cos 2t)(\sin 2t - \cos 2t) - (4\cos 2t)(\sin 2t) \\ &= -2\cos^2 2t - 2\sin^2 2t = -2. \end{aligned}$$

Thus, the set $\left\{\begin{pmatrix} -2\sin 2t + 2\cos 2t \\ \sin 2t \end{pmatrix}, \begin{pmatrix} 4\cos 2t \\ \sin 2t - \cos 2t \end{pmatrix}\right\}$ is linearly independent and is consequently a fundamental set of solutions.

*Show that any linear combination of* $\begin{pmatrix} -2\sin 2t + 2\cos 2t \\ \sin 2t \end{pmatrix}$ *and* $\begin{pmatrix} 4\cos 2t \\ \sin 2t - \cos 2t \end{pmatrix}$ *is also a solution of the system.*



**Theorem 6.4 General Solution of a Homogeneous System**

Suppose that $\mathbf{X}_1(t), \mathbf{X}_2(t), \ldots, \mathbf{X}_n(t)$ are $n$ solutions of the system of first-order linear homogeneous differential equations $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ on the open interval $I$, where each component of $\mathbf{A}(t)$ is continuous on $I$. Then every solution of $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ is a linear combination of these solutions. Therefore, a **general solution** of $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ on $I$ is

$$\mathbf{X}(t) = c_1\mathbf{X}_1(t) + c_2\mathbf{X}_2(t) + \cdots + c_n\mathbf{X}_n(t).$$

**Example 6**

Find a general solution of $\mathbf{X}' = \begin{pmatrix} 1 & -2 \\ 2 & 6 \end{pmatrix}\mathbf{X}$.

**Solution** In previous examples, we have shown that $\mathbf{X}_1(t) = \begin{pmatrix} -2e^{2t} \\ e^{2t} \end{pmatrix}$ and $\mathbf{X}_2(t) = \begin{pmatrix} -e^{5t} \\ 2e^{5t} \end{pmatrix}$ are linearly independent solutions of $\mathbf{X}' = \begin{pmatrix} 1 & -2 \\ 2 & 6 \end{pmatrix}\mathbf{X}$. Because $\mathbf{A}(t)$ is a $2 \times 2$ matrix, we need two linearly independent solutions to form a general solution. Therefore,

$$\mathbf{X}(t) = c_1\begin{pmatrix} -2e^{2t} \\ e^{2t} \end{pmatrix} + c_2\begin{pmatrix} -e^{5t} \\ 2e^{5t} \end{pmatrix} = \begin{pmatrix} -2c_1e^{2t} - c_2e^{5t} \\ c_1e^{2t} + 2c_2e^{5t} \end{pmatrix}$$

is a general solution.

**Definition 6.15 Fundamental Matrix**

Suppose that $\mathbf{X}_1(t), \mathbf{X}_2(t), \ldots, \mathbf{X}_n(t)$ are $n$ solutions of the system of first-order linear homogeneous differential equations $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ on the open interval $I$, where each component of $\mathbf{A}(t)$ is continuous on $I$. The matrix

$$\mathbf{\Phi}(t) = (\mathbf{X}_1 \quad \mathbf{X}_2 \quad \cdots \quad \mathbf{X}_n)$$

is called a **fundamental matrix** of the system $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ on $I$. Thus, a general solution can be written as $\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{C}$, where $\mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$.

**Example 7**

Show that $\mathbf{\Phi}(t) = \begin{pmatrix} e^{-2t} & -3e^{5t} \\ 2e^{-2t} & e^{5t} \end{pmatrix}$ is a fundamental matrix for the system $X'(t) = \begin{pmatrix} 4 & -3 \\ -2 & -1 \end{pmatrix}\mathbf{X}(t)$. Use the matrix to find a general solution of $\mathbf{X}'(t) = \begin{pmatrix} 4 & -3 \\ -2 & -1 \end{pmatrix}\mathbf{X}(t)$.

***Solution*** Because

$$\begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix}' = \begin{pmatrix} -2e^{-2t} \\ -4e^{-2t} \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} -3e^{5t} \\ e^{5t} \end{pmatrix}' = \begin{pmatrix} -15e^{5t} \\ 5e^{5t} \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} -3e^{5t} \\ e^{5t} \end{pmatrix},$$

both $\begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix}$ and $\begin{pmatrix} -3e^{5t} \\ e^{5t} \end{pmatrix}$ are solutions to the system $\mathbf{X}'(t) = \begin{pmatrix} 4 & -3 \\ -2 & -1 \end{pmatrix}\mathbf{X}(t)$. The solutions are linearly independent because

$$W\left(\begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix}, \begin{pmatrix} -3e^{5t} \\ e^{5t} \end{pmatrix}\right) = \begin{vmatrix} e^{-2t} & -3e^{5t} \\ 2e^{-2t} & e^{5t} \end{vmatrix} = 7e^{3t} \neq 0.$$

A general solution is given by

$$\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{C} = \begin{pmatrix} e^{-2t} & -3e^{5t} \\ 2e^{-2t} & e^{5t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} c_1e^{-2t} - 3c_2e^{5t} \\ 2c_1e^{-2t} + c_2e^{5t} \end{pmatrix} = c_1\begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix} + c_2\begin{pmatrix} -3e^{5t} \\ e^{5t} \end{pmatrix}.$$

The theorems and definitions introduced in the section indicate that when solving an $n \times n$ homogeneous system of linear equations $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$, we find $n$ linearly independent solutions. After finding these solutions, we form a fundamental matrix that can be used to form a general solution or solve an initial-value problem.



*Graphs of several solutions are shown in Figure 6.9. (a) Identify the graph of the solution that satisfies the initial conditions $x(0) = 0$ and $y(0) = 1$. (b) Find $c_1$ and $c_2$ so that $\mathbf{X}(t) = c_1\begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix} + c_2\begin{pmatrix} -3e^{5t} \\ e^{5t} \end{pmatrix}$ satisfies $\mathbf{X}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.*

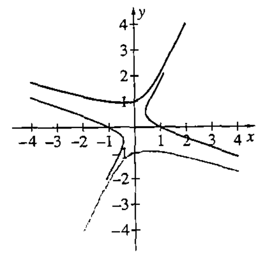



**Figure 6.9**

## EXERCISES 6.3

In Exercises 1–8, write the system of first-order equations in matrix form.

1. $\begin{cases} x' = x + 4y \\ y' = 2x - y \end{cases}$

2. $\begin{cases} x' = 2x - 3y \\ y' = x + y \end{cases}$

3. $\begin{cases} x' = y + 2x \\ y' = x - 5y \end{cases}$

4. $\begin{cases} x' = y - 4x \\ y' = y - x \end{cases}$

5. $\begin{cases} x' - x = e^t \\ y' = x + y \end{cases}$

6. $\begin{cases} x' = y \\ y' = x + \cos t \end{cases}$

7. $\begin{cases} x' - y = 0 \\ y' + x = 2 \sin t \end{cases}$

8. $\begin{cases} x' + x - 3y = 0 \\ y' - x = e^{-t} \end{cases}$

In Exercises 9–14, determine if the given vectors are linearly independent.

9. $\begin{pmatrix} e^t \\ 2e^t \end{pmatrix}, \begin{pmatrix} 3e^{-t} \\ e^{-t} \end{pmatrix}$

10. $\begin{pmatrix} t \\ 1 \end{pmatrix}, \begin{pmatrix} -t \\ t \end{pmatrix}$

*11. $\begin{pmatrix} \cos 2t \\ -2 \sin 2t \end{pmatrix}, \begin{pmatrix} \sin 2t \\ 2 \cos 2t \end{pmatrix}$

12. $\begin{pmatrix} \cos 2t \\ -2 \sin 2t \end{pmatrix}, \begin{pmatrix} 1 - 2 \sin^2 t \\ -4 \sin t \cos t \end{pmatrix}$

*13. $\begin{pmatrix} e^{2t} \\ e^{2t} \\ 2e^{2t} \end{pmatrix}, \begin{pmatrix} e^t \\ -e^t \\ 3e^t \end{pmatrix}, \begin{pmatrix} e^{-t} \\ e^{-t} \\ e^{-t} \end{pmatrix}$

14. $\begin{pmatrix} 6e^{4t} \\ 6e^{4t} \\ 3e^{4t} \end{pmatrix}, \begin{pmatrix} e^{-2t} \\ e^{-2t} \\ -e^{-2t} \end{pmatrix}, \begin{pmatrix} 2e^{4t} \\ 2e^{4t} \\ e^{4t} \end{pmatrix}$

In Exercises 15–20, determine whether the given matrix is a fundamental matrix for the system.

15. $\mathbf{\Phi}(t) = \begin{pmatrix} -2e^{-8t} & 5e^{-t} \\ e^{-8t} & e^{-t} \end{pmatrix}$ and $\mathbf{X}'(t) = \begin{pmatrix} -3 & 10 \\ 1 & -6 \end{pmatrix}\mathbf{X}(t)$

16. $\mathbf{\Phi}(t) = \begin{pmatrix} e^{-10t} & -3e^{5t} \\ 2e^{-10t} & e^{5t} \end{pmatrix}$ and $\mathbf{X}'(t) = \begin{pmatrix} 5 & 0 \\ -5 & -10 \end{pmatrix}\mathbf{X}(t)$

*17. $\Phi(t) = \begin{pmatrix} \cos t + 2\sin t & 2\cos t - \sin t \\ \sin t & \cos t \end{pmatrix}$ and $\mathbf{X}'(t) = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{X}(t)$

18. $\Phi(t) = \begin{pmatrix} e^t & 3e^{3t} \\ e^t & 5e^{3t} \end{pmatrix}$ and $\mathbf{X}'(t) = \begin{pmatrix} -2 & 3 \\ -5 & 6 \end{pmatrix}\mathbf{X}(t)$

*19. $\Phi(t) = \begin{pmatrix} 27e^{-6t} & -e^t & 0 \\ -29e^{-6t} & -e^{-t} & e^{3t} \\ 36e^{-6t} & e^{-t} & 0 \end{pmatrix}$ and

$\mathbf{X}'(t) = \begin{pmatrix} -2 & 0 & -3 \\ 3 & 3 & 5 \\ -4 & 0 & -3 \end{pmatrix}\mathbf{X}(t)$

20. $\Phi(t)$

$= \begin{pmatrix} 5e^{3t} & -2e^{-t}\cos t & 2e^{-t}\sin t \\ -5e^{3t} & e^{-t}(\sin t - 2\cos t) & e^{-t}(\cos t + 2\sin t) \\ e^{3t} & 3e^{-t}\cos t & -3e^{-t}\sin t \end{pmatrix}$

and $\mathbf{X}'(t) = \begin{pmatrix} 1 & -2 & 0 \\ -5 & -3 & -5 \\ 3 & 3 & 3 \end{pmatrix}\mathbf{X}(t)$

In Exercises 21–23, use the given fundamental matrix to obtain a general solution to the system of first-order homogeneous equations $\mathbf{X}' = \mathbf{AX}$. Also, find the solution that satisfies the given initial condition.

21. $\Phi(t) = \begin{pmatrix} 2e^{4t} & e^{-t} \\ 3e^{4t} & -e^{-t} \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$, $\mathbf{X}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

22. $\Phi(t) = \begin{pmatrix} 3e^{4t} & e^{3t} \\ 2e^{4t} & e^{3t} \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix}$, $\mathbf{X}(0) = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$

23. $\Phi(t) = \begin{pmatrix} 2\cos 2t + 2\sin 2t & 2\cos 2t - 2\sin 2t \\ -\sin 2t & -\cos 2t \end{pmatrix}$,

$\mathbf{A} = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix}$, $\mathbf{X}(0) = \begin{pmatrix} -8 \\ 2 \end{pmatrix}$

24. **(Modeling the Motion of Spiked Volleyball)** Under certain assumptions, the position of a spiked volleyball can be modeled by the system of two second-order nonlinear equations

$$\begin{cases} X'' = \dfrac{1}{m}\left(C_M\omega^a Y'([X']^2 + [Y']^2)^{(b-1)/2} - \dfrac{1}{2}C_D\rho A X'\sqrt{[X']^2 + [Y']^2}\right) \\ Y'' = -g - \dfrac{1}{m}\left(C_M\omega^a X'([X']^2 + [Y']^2)^{(b-1)/2} + \dfrac{1}{2}C_D\rho A Y'\sqrt{[X']^2 + [Y']^2}\right) \end{cases}.^{*}$$

Write this system of two second-order nonlinear equations as a system of four first-order equations.

25. **(Principle of Superposition) (a)** Show that any linear combination of solutions of the homogeneous system $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ is also a solution of the homogeneous system. **(b)** Is the Principle of Superposition ever valid for nonhomogeneous systems of equations? Explain.

26. Show that each matrix is a fundamental matrix for the given system. In each case, use the fundamental matrix to construct a general solution to the system.

**(a)** $\begin{pmatrix} \cos 3t - \sin 3t & -\sin 3t - \cos 3t \\ \sin 3t & \cos 3t \end{pmatrix}$; $\mathbf{X}'(t) = \begin{pmatrix} -3 & -6 \\ 3 & 3 \end{pmatrix}\mathbf{X}(t)$

**(b)** $\begin{pmatrix} e^t(\sin 3t - 3\cos 3t) & 5e^t\cos 3t \\ 2e^t\sin 3t & e^t(\cos 3t - 3\sin 3t) \end{pmatrix}$; $\mathbf{X}'(t) = \begin{pmatrix} 0 & 5 \\ -2 & 2 \end{pmatrix}\mathbf{X}(t)$

**(c)** $\begin{pmatrix} e^t(\sin 3t + 3\cos 3t) & 5e^{-t}\cos 3t \\ 2e^{-t}\sin 3t & e^{-t}(\cos 3t + 3\sin 3t) \end{pmatrix}$; $\mathbf{X}'(t) = \begin{pmatrix} 0 & -5 \\ 2 & -2 \end{pmatrix}\mathbf{X}(t)$

**(d)** Use the results of (a), (b), and (c) to solve (a), (b), and (c) subject to $\mathbf{X}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In each case, graph the resulting solution for $0 \le t \le 2\pi$. What similarities and differences between the graphs do you observe?

* Shawn S. Kao, Richard W. Sellens, and Joan M. Stevenson, "A Mathematical Model for the Trajectory of a Spiked Volleyball and Its Coaching Applications," *Journal of Applied Biomechanics*, Human Kinetics Publishers, Inc. (1994), pp. 95–109.



## 6.4 First-Order Linear Homogeneous Systems with Constant Coefficients

**Distinct Real Eigenvalues • Complex Conjugate Eigenvalues • Alternate Method for Solving Initial-Value Problems • Repeated Eigenvalues**

Now that we have covered the necessary terminology, we turn our attention to solving linear systems with constant coefficients. Let $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{22} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ be an $n \times n$ matrix with real components. In this section, we will see that a general solution to the homogeneous system $\mathbf{X}' = \mathbf{AX}$ is determined by the eigenvalues and corresponding eigenvectors of $\mathbf{A}$. We begin by considering the cases when the eigenvalues of $\mathbf{A}$ are distinct and real or the eigenvalues of $\mathbf{A}$ are distinct and complex, and then consider the case when $\mathbf{A}$ has repeated eigenvalues (eigenvalues of multiplicity greater than one).

### Distinct Real Eigenvalues

In Section 6.3, we verified that a general solution of the $2 \times 2$ system $\mathbf{X}' = \begin{pmatrix} 4 & -3 \\ -2 & -1 \end{pmatrix}\mathbf{X}$, where the eigenvalues of $\begin{pmatrix} 4 & -3 \\ -2 & -1 \end{pmatrix}$ are $\lambda_1 = -2$ and $\lambda_2 = 5$ and corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, respectively, is

$$\mathbf{X}(t) = \Phi(t)\mathbf{C} = c_1\begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix} + c_2\begin{pmatrix} -3e^{5t} \\ e^{5t} \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{-2t} + c_2\begin{pmatrix} -3 \\ 1 \end{pmatrix}e^{5t}.$$

More generally, if the eigenvalues of the $n \times n$ matrix $\mathbf{A}$ are distinct, we may expect a general solution of the linear homogeneous system $\mathbf{X}' = \mathbf{AX}$ to have the form

$$\mathbf{X}(t) = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t} + \cdots + c_n\mathbf{v}_ne^{\lambda_n t},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the $n$ distinct real eigenvalues of $\mathbf{A}$ with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, respectively. We investigate this claim by assuming that $\mathbf{X} = \mathbf{v}e^{\lambda t}$ is a solution of $\mathbf{X}' = \mathbf{AX}$. Then, $\mathbf{X}' = \lambda\mathbf{v}e^{\lambda t}$ must satisfy the system of differential equations, which implies that

$$\lambda\mathbf{v}e^{\lambda t} = \mathbf{A}\mathbf{v}e^{\lambda t}.$$

Using $\mathbf{Iv} = \mathbf{v}$ yields

$$\lambda\mathbf{I}\mathbf{v}e^{\lambda t} = \mathbf{A}\mathbf{v}e^{\lambda t}$$

$$\mathbf{A}\mathbf{v}e^{\lambda t} - \lambda\mathbf{I}\mathbf{v}e^{\lambda t} = \mathbf{0}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}e^{\lambda t} = \mathbf{0}.$$

Then, because $e^{\lambda t} \neq 0$, we have $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$. In order for this system of equations to have a solution other than $\mathbf{v} = \mathbf{0}$ (remember that eigenvectors are never the zero vector), we must have

$$|\mathbf{A} - \lambda\mathbf{I}| = 0.$$

A solution $\lambda$ to this equation is an eigenvalue of $\mathbf{A}$ while a nonzero vector $\mathbf{v}$ satisfying $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ is an eigenvector that corresponds to $\lambda$. Hence, if $\mathbf{A}$ has $n$ distinct eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, we can find a set of $n$ linearly independent eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. From these eigenvalues and corresponding eigenvectors, we form the $n$ linearly independent solutions

$$\mathbf{X}_1 = \mathbf{v}_1 e^{\lambda_1 t},\ \mathbf{X}_2 = \mathbf{v}_2 e^{\lambda_2 t},\ \ldots,\ \mathbf{X}_n = \mathbf{v}_n e^{\lambda_n t}.$$

Therefore, if $\mathbf{A}$ is an $n \times n$ matrix with $n$ distinct real eigenvalues $\{\lambda_k\}_{k=1}^{n}$, a general solution of $\mathbf{X}' = \mathbf{AX}$ is the linear combination of the set of solutions $\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n\}$,

$$\mathbf{X}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{v}_n e^{\lambda_n t}.$$

**Example 1**

Find a general solution of $\mathbf{X}' = \begin{pmatrix} 5 & -1 \\ 0 & 3 \end{pmatrix}\mathbf{X}$.

***Solution*** The eigenvalues of $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 0 & 3 \end{pmatrix}$ are found with $\begin{vmatrix} 5-\lambda & -1 \\ 0 & 3-\lambda \end{vmatrix} = (5-\lambda)(3-\lambda) = 0$. The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 3$ and $\lambda_2 = 5$. An eigenvector $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ corresponding to $\lambda_1 = 3$ satisfies the system $\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so solutions to this system satisfy $2x_1 - y_1 = 0$ or $y_1 = 2x_1$. Choosing $x_1 = 1$, we obtain the eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Similarly, an eigenvector $\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ corresponding to $\lambda_2 = 5$ satisfies $\begin{pmatrix} 0 & -1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which indicates that $y_2 = 0$. Hence, $\mathbf{v}_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}$ is an eigenvector for any value of $x_2$. If we let $x_2 = 1$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Therefore, a general solution of the system $\mathbf{X}' = \begin{pmatrix} 5 & -1 \\ 0 & 3 \end{pmatrix}\mathbf{X}$ is

$$\mathbf{X}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{5t}.$$

Remember that the system $\mathbf{X}' = \begin{pmatrix} 5 & -1 \\ 0 & 3 \end{pmatrix}\mathbf{X}$ is the same as the system $\begin{cases} x' = 5x - y \\ y' = 3y \end{cases}$. Thus, we can write the general solution obtained here as $\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1e^{3t} + c_2e^{5t} \\ 2c_1e^{3t} \end{pmatrix}$ or $\begin{cases} x(t) = c_1e^{3t} + c_2e^{5t} \\ y(t) = 2c_1e^{3t} \end{cases}$. Several solutions along with the direction field for the system are shown in Figure 6.10. (Notice that each curve corresponds to the parametric plot of the pair $\begin{cases} x(t) \\ y(t) \end{cases}$ for particular values of the constants $c_1$ and $c_2$.) Because both eigenvalues are positive, all solutions move away from the origin as $t$ increases. The arrows on the vectors in the direction field show this behavior.



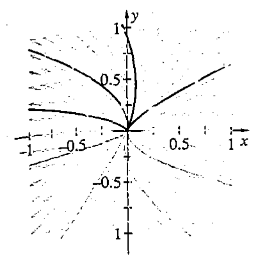



Figure 6.10

You may notice that general solutions you obtain when solving systems do not agree with those of your classmates or those given in the Exercise 6.4 solutions. Before you become alarmed, realize that your solutions may be correct. You may have simply selected different values for the arbitrary constants in finding eigenvectors.

## Complex Conjugate Eigenvalues

If $\mathbf{A}$ has complex conjugate eigenvalues $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$ with corresponding eigenvectors $\mathbf{v}_1 = \mathbf{a} + \mathbf{b}i$ and $\mathbf{v}_2 = \mathbf{a} - \mathbf{b}i$, one solution of $\mathbf{X}' = \mathbf{AX}$ is

$$\begin{aligned} \mathbf{X}(t) &= \mathbf{v}_1 e^{\lambda_1 t} = (\mathbf{a} + \mathbf{b}i)e^{(\alpha+\beta i)t} = e^{\alpha t}(\mathbf{a} + \mathbf{b}i)e^{i\beta t} = e^{\alpha t}(\mathbf{a} + \mathbf{b}i)(\cos \beta t + i \sin \beta t) \\ &= e^{\alpha t}(\mathbf{a} \cos \beta t - \mathbf{b} \sin \beta t) + ie^{\alpha t}(\mathbf{a} \sin \beta t + \mathbf{b} \cos \beta t) \\ &= \mathbf{x}_1(t) + i\mathbf{x}_2(t). \end{aligned}$$

Because $\mathbf{X}$ is a solution to the system $\mathbf{X}' = \mathbf{AX}$, we have that $\mathbf{x}_1'(t) + i\mathbf{x}_2'(t) = \mathbf{A}\mathbf{x}_1(t) + i\mathbf{A}\mathbf{x}_2(t)$. Equating the real and imaginary parts of this equation yields $\mathbf{x}_1'(t) = \mathbf{A}\mathbf{x}_1(t)$ and $\mathbf{x}_2'(t) = \mathbf{A}\mathbf{x}_2(t)$. Therefore, $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions to $\mathbf{X}' = \mathbf{AX}$, and any linear combination of $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ is also a solution. We can show that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent (see Exercise 41), so this linear combination forms a portion of a general solution of $\mathbf{X}' = \mathbf{AX}$ where $\mathbf{A}$ is a square matrix of *any size.*

**Theorem 6.5**

Let $\mathbf{A}$ be a square matrix. If $\mathbf{A}$ has complex conjugate eigenvalues $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$, $\beta \neq 0$, and corresponding eigenvectors $\mathbf{v}_1 = \mathbf{a} + \mathbf{b}i$ and $\mathbf{v}_2 = \mathbf{a} - \mathbf{b}i$, two linearly independent solutions of $\mathbf{X}' = \mathbf{AX}$ are $\mathbf{x}_1(t) = e^{\alpha t}(\mathbf{a} \cos \beta t - \mathbf{b} \sin \beta t)$ and $\mathbf{x}_2(t) = e^{\alpha t}(\mathbf{a} \sin \beta t + \mathbf{b} \cos \beta t)$.

Notice that in the case of complex conjugate eigenvalues, we are able to obtain two linearly independent solutions from knowing one of the eigenvalues and an eigenvector that corresponds to it.

If $\mathbf{A}$ is a $2 \times 2$ matrix with complex conjugate eigenvalues $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$ and corresponding eigenvectors $\mathbf{v}_1 = \mathbf{a} + \mathbf{b}i$ and $\mathbf{v}_2 = \mathbf{a} - \mathbf{b}i$, a general solution of $\mathbf{X}' = \mathbf{AX}$ is

$$\mathbf{X}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1e^{\alpha t}(\mathbf{a} \cos \beta t - \mathbf{b} \sin \beta t) + c_2e^{\alpha t}(\mathbf{a} \sin \beta t + \mathbf{b} \cos \beta t).$$

**Example 2**

Find a general solution to $\mathbf{X}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}\mathbf{X}$.

**Solution** The eigenvalues of $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$ are $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. An eigenvector $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ that corresponds to $\lambda_1 = 1 + 2i$ satisfies $\begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The first equation, $(2-2i)x_1 - 2y_1 = 0$, is equivalent to $\dfrac{2-2i}{4}$ times the second equation, $4x_1 + (-2-2i)y_1 = 0$. We can write the system as $\begin{pmatrix} 1-i & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which indicates that $y_1 = (1-i)x_1$. Choosing $x_1 = 1$, we obtain $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}$. Therefore, in the notation used in Theorem 6.5,

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

With $\alpha = 1$ and $\beta = 2$ from the eigenvalues, a general solution to the system is

$$\mathbf{X}(t) = c_1 e^t\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin 2t\right] + c_2 e^t\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin 2t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\cos 2t\right]$$

$$= \begin{pmatrix} c_1 e^t \cos 2t + c_2 e^t \sin 2t \\ c_1 e^t(\cos 2t + \sin 2t) + c_2 e^t(\sin 2t - \cos 2t) \end{pmatrix}.$$

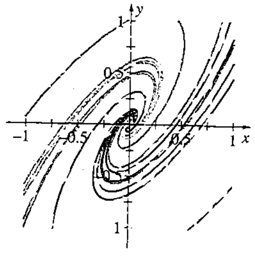



**Figure 6.11** We see that all (nontrivial) solutions spiral away from the origin.

Figure 6.11 shows the graph of several solutions along with the direction field for the equation. Notice the spiraling motion of the vectors in the direction field. This is due to the product of exponential and trigonometric functions in the solution; the exponential functions cause $x(t)$ and $y(t)$ to increase, while the trigonometric functions lead to rotation about the origin.

*Sketch the graphs of $x(t)$, $y(t)$, and $\begin{cases} x(t) \\ y(t) \end{cases}$ in Example 2 if $x(0) = 0$ and $y(0) = 1$. Calculate $\lim_{t\to\infty} x(t)$, $\lim_{t\to\infty} y(t)$, $\lim_{t\to-\infty} x(t)$, and $\lim_{t\to-\infty} y(t)$.*

Initial-value problems can be solved through the use of eigenvalues and eigenvectors as well.

**Example 3**

Solve $\mathbf{X}' = \begin{pmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{pmatrix}\mathbf{X}$ subject to $\mathbf{X}(0) = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$.

**Solution** The eigenvalues of $\mathbf{A} = \begin{pmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{pmatrix}$ satisfy



$$\begin{vmatrix} 5-\lambda & 5 & 2 \\ -6 & -6-\lambda & -5 \\ 6 & 6 & 5-\lambda \end{vmatrix}$$

$$= (5-\lambda)\begin{vmatrix} -6-\lambda & -5 \\ 6 & 5-\lambda \end{vmatrix} - 5\begin{vmatrix} -6 & -5 \\ 6 & 5-\lambda \end{vmatrix} + 2\begin{vmatrix} -6 & -6-\lambda \\ 6 & 6 \end{vmatrix}$$

$$= (5-\lambda)(\lambda^2 + \lambda) - 5(6\lambda) + 2(6\lambda) = -\lambda(\lambda^2 - 4\lambda + 13) = 0.$$

Hence, $\lambda_1 = 0$ and $\lambda = (4 \pm \sqrt{16 - 52})/2 = 2 \pm 3i$, so $\lambda_2 = 2 + 3i$ and $\lambda_3 = 2 - 3i$. An eigenvector $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ corresponding to $\lambda_1 = 0$ satisfies the system $\begin{pmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, which, after row operations, is equivalent to $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Thus, $x_1 + y_1 = 0$ and $z_1 = 0$. If we choose $y_1 = 1$, $\mathbf{v}_1 = \begin{pmatrix} -y_1 \\ y_1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. One solution of the system of differential equations is $\mathbf{X}_1 = \mathbf{v}_1 e^{\lambda_1 t} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{(0)t} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. We find two solutions that correspond to the complex conjugate pair of eigenvalues by finding an eigenvector $\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ corresponding to $\lambda_2 = 2 + 3i$. This vector satisfies the system $\begin{pmatrix} 3-3i & 5 & 2 \\ -6 & -8-3i & -5 \\ 6 & 6 & 3-3i \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, which can be reduced to $\begin{pmatrix} 1 & 0 & -\frac{1}{2}(1+i) \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Therefore, the components of $\mathbf{v}_2$ must satisfy $x_2 = \frac{1}{2}(1+i)z_2$ and $y_2 = -z_2$. If we let $z_2 = 2$, then $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1+i \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} i = \mathbf{a} + \mathbf{b}i$. Thus, two linearly independent solutions that correspond to the eigenvalues $\lambda = 2 \pm 3i$ are

$$\mathbf{X}_2 = e^{2t}\left[\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}\cos 3t - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\sin 3t\right] = \begin{pmatrix} e^{2t}(\cos 3t - \sin 3t) \\ -2e^{2t}\cos 3t \\ 2e^{2t}\cos 3t \end{pmatrix}$$

and

$$\mathbf{X}_3 = e^{2t}\left[\begin{pmatrix}1\\-2\\2\end{pmatrix}\sin 3t + \begin{pmatrix}1\\0\\0\end{pmatrix}\cos 3t\right] = \begin{pmatrix}e^{2t}(\sin 3t + \cos 3t)\\ -2e^{2t}\sin 3t\\ 2e^{2t}\sin 3t\end{pmatrix},$$

so a general solution is

$$\mathbf{X}(t) = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + c_3\mathbf{X}_3$$

$$= c_1\begin{pmatrix}-1\\1\\0\end{pmatrix} + c_2\begin{pmatrix}e^{2t}(\cos 3t - \sin 3t)\\ -2e^{2t}\cos 3t\\ 2e^{2t}\cos 3t\end{pmatrix} + c_3\begin{pmatrix}e^{2t}(\sin 3t + \cos 3t)\\ -2e^{2t}\sin 3t\\ 2e^{2t}\sin 3t\end{pmatrix}$$

$$= \begin{pmatrix}-c_1 + c_2e^{2t}(\cos 3t - \sin 3t) + c_3e^{2t}(\sin 3t + \cos 3t)\\ c_1 - 2c_2e^{2t}\cos 3t - 2c_3e^{2t}\sin 3t\\ 2c_2e^{2t}\cos 3t + 2c_3e^{2t}\sin 3t\end{pmatrix}.$$

Application of the initial condition $\mathbf{X}(0) = \begin{pmatrix}0\\0\\2\end{pmatrix}$ yields

$\mathbf{X}(0) = \begin{pmatrix}-c_1 + c_2 + c_3\\ c_1 - 2c_2\\ 2c_2\end{pmatrix} = \begin{pmatrix}0\\0\\2\end{pmatrix}$, which gives us the system of equations $\begin{cases}-c_1 + c_2 + c_3 = 0\\ c_1 - 2c_2 = 0\\ 2c_2 = 2\end{cases}$ with solution $c_2 = 1$, $c_1 = 2$, and $c_3 = 1$. Therefore, the solution of the initial-value problem is

$$\mathbf{X}(t) = \begin{pmatrix}-2 + e^{2t}(\cos 3t - \sin 3t) + e^{2t}(\sin 3t + \cos 3t)\\ 2 - 2e^{2t}\cos 3t - 2e^{2t}\sin 3t\\ 2e^{2t}\cos 3t + 2e^{2t}\sin 3t\end{pmatrix}$$

$$= \begin{pmatrix}-2 + 2e^{2t}\cos 3t\\ 2 - 2e^{2t}\cos 3t - 2e^{2t}\sin 3t\\ 2e^{2t}\cos 3t + 2e^{2t}\sin 3t\end{pmatrix}$$

In Figure 6.12, we graph $x(t)$, $y(t)$, and $z(t)$ and we show the parametric plot of $\{x(t), y(t), z(t)\}$ in three dimensions. Notice that $\lim_{t\to\infty} x(t) = \infty$, $\lim_{t\to\infty} y(t) = \infty$, and $\lim_{t\to\infty} z(t) = \infty$, so the solution is directed away from the initial point $(0, 0, 2)$.

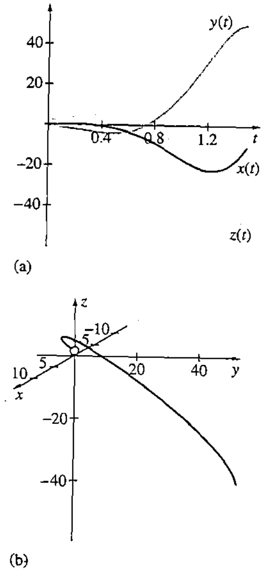



**Figure 6.12** (a) Graph of $x(t)$, $y(t)$, and $z(t)$. Identify each graph. (b) Graph of $\begin{cases}x(t)\\y(t),\\z(t)\end{cases}$ $0 \le t \le \pi/2$.

*Verify that* $\mathbf{X}(t) = \begin{pmatrix}-2 + 2e^{2t}\cos 3t\\ 2 - 2e^{2t}\cos 3t - 2e^{2t}\sin 3t\\ 2e^{2t}\cos 3t + 2e^{2t}\sin 3t\end{pmatrix}$ *satisfies the initial conditions in Example 3.*



## Alternate Method for Solving Initial-Value Problems

We can also use a fundamental matrix to help us solve homogeneous initial-value problems. If $\mathbf{\Phi}(t)$ is a fundamental matrix for the homogeneous system $\mathbf{X}' = \mathbf{AX}$, a general solution is $\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{C}$, where $\mathbf{C}$ is a constant vector. Given the initial condition $\mathbf{X}(0) = \mathbf{X}_0$, then through substitution into $\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{C}$,

$$\mathbf{X}(0) = \mathbf{\Phi}(0)\mathbf{C}$$
$$\mathbf{X}_0 = \mathbf{\Phi}(0)\mathbf{C}$$
$$\mathbf{C} = \mathbf{\Phi}^{-1}(0)\mathbf{X}_0.$$

Therefore, the solution to the initial-value problem $\mathbf{X}' = \mathbf{AX}$, $\mathbf{X}(0) = \mathbf{X}_0$ is $\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(0)\mathbf{X}_0$.

**Example 4**

Use a fundamental matrix to solve the initial-value problem $\mathbf{X}' = \begin{pmatrix}1 & 1\\4 & -2\end{pmatrix}\mathbf{X}$ subject to $\mathbf{X}(0) = \begin{pmatrix}1\\-2\end{pmatrix}$.

***Solution*** The eigenvalues of $\mathbf{A} = \begin{pmatrix}1 & 1\\4 & -2\end{pmatrix}$ are $\lambda_1 = 2$ and $\lambda_2 = -3$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix}1\\1\end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix}1\\-4\end{pmatrix}$, respectively. A fundamental matrix is then given by $\mathbf{\Phi}(t) = \begin{pmatrix}e^{2t} & e^{-3t}\\ e^{2t} & -4e^{-3t}\end{pmatrix}$. We calculate $\mathbf{\Phi}^{-1}(0)$ by observing that $\mathbf{\Phi}(0) = \begin{pmatrix}1 & 1\\1 & -4\end{pmatrix}$, so $\mathbf{\Phi}^{-1}(0) = \frac{1}{-4-1}\begin{pmatrix}-4 & -1\\-1 & 1\end{pmatrix} = \left(-\frac{1}{5}\right)\begin{pmatrix}-4 & -1\\-1 & 1\end{pmatrix}$. Hence,

$$\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(0)\mathbf{X}_0 = \begin{pmatrix}e^{2t} & e^{-3t}\\ e^{2t} & -4e^{-3t}\end{pmatrix}\left(-\frac{1}{5}\right)\begin{pmatrix}-4 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix}$$

$$= \left(-\frac{1}{5}\right)\begin{pmatrix}e^{2t} & e^{-3t}\\ e^{2t} & -4e^{-3t}\end{pmatrix}\begin{pmatrix}-2\\-3\end{pmatrix} = \begin{pmatrix}\frac{2}{5}e^{2t} + \frac{3}{5}e^{-3t}\\ \frac{2}{5}e^{2t} - \frac{12}{5}e^{-3t}\end{pmatrix}.$$

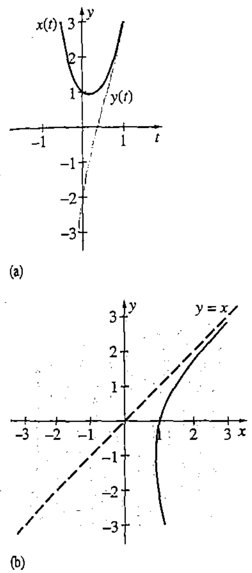



**Figure 6.13** (a) Graph of $x(t)$ and $y(t)$. Identify each graph. (b) Graph of $\begin{cases}x(t)\\y(t)\end{cases}$ along with the direction field associated with the system of equations.

Figure 6.13 shows a graph of the solution. Notice that as $t \to \infty$, the values of $x(t)$ and $y(t)$, where $\mathbf{X}(t) = \begin{pmatrix}x(t)\\y(t)\end{pmatrix}$, are both close to $\frac{2}{5}e^{2t}$ because $\lim_{t\to\infty} e^{-3t} = 0$. This means that the solution approaches the line $y = x$ because for large values of $t$, $x(t)$ and $y(t)$ are approximately the same.

*Find the solution that satisfies the initial condition* $\mathbf{X}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

## Repeated Eigenvalues

We now consider the case of repeated eigenvalues. This is more complicated than the other cases because two situations can arise. As we discovered in Section 6.2, an eigenvalue of multiplicity $m$ can either have $m$ corresponding linearly independent eigenvectors or have fewer than $m$ corresponding linearly independent eigenvectors. In the case of $m$ linearly independent eigenvectors, a general solution is found in the same manner as the case of $m$ distinct eigenvalues.

**Example 5**

Solve $\mathbf{X}' = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}\mathbf{X}$.

***Solution*** We found the eigenvalues $\lambda_1 = \lambda_2 = -2$ and $\lambda_3 = 4$ of $\mathbf{A} = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$ in Example 11 in Section 6.2. We also found that the eigenvalue $\lambda_1 = \lambda_2 = -2$ of multiplicity 2 has two corresponding linearly independent eigenvectors, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. An eigenvector $\mathbf{v}_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$ that corresponds to $\lambda_3 = 4$ satisfies the system $\begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, which can be reduced with row operations to the system $\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Hence, $x_3 = \dfrac{z_3}{2}$ and $y_3 = \dfrac{z_3}{2}$. Choosing $z_3 = 2$, we obtain $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. A general solution is

$$\begin{aligned} \mathbf{X}(t) &= c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t} + c_3\mathbf{v}_3e^{\lambda_3 t} \\ &= c_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{-2t} + c_2\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}e^{-2t} + c_3\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}e^{4t} = \begin{pmatrix} (c_1 - c_2)e^{-2t} + c_3e^{4t} \\ c_1e^{-2t} + c_3e^{4t} \\ c_2e^{-2t} + 2c_3e^{4t} \end{pmatrix}. \end{aligned}$$

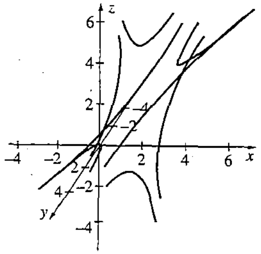



Figure 6.14

Figure 6.14 shows graphs of several solutions to the system along with the direction field. Notice that if $c_3 \neq 0$, then $\lim_{t\to\infty} x(t) = \infty$, $\lim_{t\to\infty} y(t) = \infty$, and $\lim_{t\to\infty} z(t) = \infty$. Therefore, many of the vectors in the direction field are directed away from the origin.



*Find conditions on* $x_0$, $y_0$, *and* $z_0$, *if possible, so that the limit as* $t \to \infty$ *of the solution* $\mathbf{X}(t)$ *that satisfies the initial condition* $\mathbf{X}(0) = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ *is* $\mathbf{0}$.

If an eigenvalue of multiplicity $m$ has fewer than $m$ linearly independent eigenvectors, we proceed in a manner that is similar to the situation that arose in Chapter 4 when we encountered repeated roots of characteristic equations. Consider a system with the repeated eigenvalue $\lambda_1 = \lambda_2$ and corresponding eigenvector $\mathbf{v}_1$. (Assume that there is not a second linearly independent eigenvector corresponding to $\lambda_1 = \lambda_2$.) With the eigenvalue $\lambda_1$ and corresponding eigenvector $\mathbf{v}_1$, we obtain the solution to the system $\mathbf{X}_1 = \mathbf{v}_1e^{\lambda_1 t}$. To find a second linearly independent solution corresponding to $\lambda_1$, instead of multiplying $\mathbf{X}_1$ by $t$ as we did in Chapter 4, we suppose that a second linearly independent solution corresponding to $\lambda_1$ is of the form

$$\mathbf{X}_2 = (\mathbf{v}_2t + \mathbf{w}_2)e^{\lambda_1 t}.$$

To find the vectors $\mathbf{v}_2$ and $\mathbf{w}_2$, we substitute $\mathbf{X}_2$ into $\mathbf{X}' = \mathbf{AX}$. Because $\mathbf{X}_2' = \lambda_1(\mathbf{v}_2t + \mathbf{w}_2)e^{\lambda_1 t} + \mathbf{v}_2e^{\lambda_1 t}$, we have

$$\begin{aligned} \mathbf{X}_2' &= \mathbf{AX}_2 \\ \lambda_1(\mathbf{v}_2t + \mathbf{w}_2)e^{\lambda_1 t} + \mathbf{v}_2e^{\lambda_1 t} &= \mathbf{A}(\mathbf{v}_2t + \mathbf{w}_2)e^{\lambda_1 t} \\ \lambda_1\mathbf{v}_2t + (\lambda_1\mathbf{w}_2 + \mathbf{v}_2) &= \mathbf{Av}_2t + \mathbf{Aw}_2. \end{aligned}$$

Equating coefficients yields $\lambda_1\mathbf{v}_2 = \mathbf{Av}_2$ and $\lambda_1\mathbf{w}_2 + \mathbf{v}_2 = \mathbf{Aw}_2$. The equation $\lambda_1\mathbf{v}_2 = \mathbf{Av}_2$ indicates that $\mathbf{v}_2$ is an eigenvector that corresponds to $\lambda_1$, so $\mathbf{v}_2 = \mathbf{v}_1$. Simplifying $\lambda_1\mathbf{w}_2 + \mathbf{v}_2 = \mathbf{Aw}_2$, we find that

$$\begin{aligned} \lambda_1\mathbf{w}_2 + \mathbf{v}_2 &= \mathbf{Aw}_2 \\ \mathbf{v}_2 &= A\mathbf{w}_2 - \lambda_1\mathbf{w}_2 \\ \mathbf{v}_2 &= (\mathbf{A} - \lambda_1\mathbf{I})\mathbf{w}_2. \end{aligned}$$

Hence, $\mathbf{w}_2$ satisfies the equation

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{w}_2 = \mathbf{v}_1.$$

Therefore, a *second linearly independent solution* corresponding to the eigenvalue $\lambda_1$ has the form

$$\mathbf{X}_2 = (\mathbf{v}_1t + \mathbf{w}_2)e^{\lambda_1 t},$$

where $\mathbf{w}_2$ satisfies $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{w}_2 = \mathbf{v}_1$.

**Theorem 6.6 Repeated Eigenvalues with One Eigenvector**

Let $\mathbf{A}$ be a square matrix. If $\mathbf{A}$ has a repeated eigenvalue $\lambda_1 = \lambda_2$ with only one corresponding (linearly independent) eigenvector $\mathbf{v}_1$, two linearly independent solutions of $\mathbf{X}' = \mathbf{AX}$ are $\mathbf{X}_1 = \mathbf{v}_1e^{\lambda_1 t}$ and $\mathbf{X}_2 = (\mathbf{v}_1t + \mathbf{w}_2)e^{\lambda_1 t}$, where $\mathbf{w}_2$ satisfies $(\mathbf{A} - \lambda_1\mathbf{I})\,\mathbf{w}_2 = \mathbf{v}_1$.

If $\mathbf{A}$ is a $2 \times 2$ matrix with the repeated eigenvalue $\lambda_1 = \lambda_2$ with only one corresponding (linearly independent) eigenvector $\mathbf{v}_1$, a general solution to $\mathbf{X}' = \mathbf{AX}$ is

$$\mathbf{X}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2(\mathbf{v}_1 t + \mathbf{w}_2)e^{\lambda_1 t}$$

where $\mathbf{w}_2$ is found by solving $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{w}_2 = \mathbf{v}_1$.

*Suppose that $\mathbf{A}$ has a repeated eigenvalue $\lambda_1 = \lambda_2$ and we can find only one corresponding (linearly independent) eigenvector $\mathbf{v}_1$. What happens if you try to find a second linearly independent solution of the form $\mathbf{v}_2 te^{\lambda_1 t}$ as we did in solving higher order equations with repeated roots of the characteristic equation in Section 4.4?*

**Example 6**

Find a general solution to $\mathbf{X}'(t) = \begin{pmatrix} -8 & -1 \\ 16 & 0 \end{pmatrix}\mathbf{X}(t)$.

**Solution** The eigenvalues of $\mathbf{A} = \begin{pmatrix} -8 & -1 \\ 16 & 0 \end{pmatrix}$ are $\lambda_1 = \lambda_2 = -4$. An eigenvector $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ that corresponds to $\lambda_1 = -4$ satisfies the system $\begin{pmatrix} -4 & -1 \\ 16 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which is equivalent to $\begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Hence, $y_1 = -4x_1$, so if we choose $x_1 = 1$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ and one solution to the system is $\mathbf{X}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{-4t}$.

To find $\mathbf{w}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ in a second linearly independent solution $\mathbf{X}_2 = (\mathbf{v}_1 t + \mathbf{w}_2)e^{\lambda_1 t}$, we solve $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{w}_2 = \mathbf{v}_1$, which in this case is

$$\begin{pmatrix} -4 & -1 \\ 16 & 4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

This system is equivalent to

$$\begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

which indicates that $4x_2 + y_2 = -1$. If we let $x_2 = 0$, then $y_2 = -1$, $\mathbf{w}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, and a second linearly independent solution is

$$\mathbf{X}_2 = \left(\begin{pmatrix} 1 \\ -4 \end{pmatrix}t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right)e^{-4t}.$$

Hence, a general solution is

$$\mathbf{X}(t) = c_1\begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{-4t} + c_2\left(\begin{pmatrix} 1 \\ -4 \end{pmatrix}t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right)e^{-4t} = \begin{pmatrix} c_1 + c_2 t \\ (-4c_1 - c_2) - 4tc_2 \end{pmatrix}e^{-4t}.$$

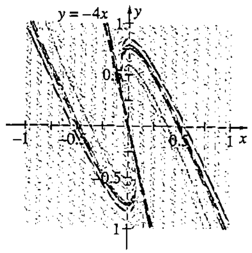


Figure 6.15



Figure 6.15 shows the graph of several solutions to the system along with the direction field. Notice that the behavior of these solutions differs from those of the systems solved earlier in the section due to the repeated eigenvalues. We see from the formula for $\mathbf{X}(t)$ that $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = 0$ (why?). In addition, these solutions approach (0, 0) tangent to $y = -4x$, the line through the origin that is parallel to the vector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$. (We discuss why this occurs in Section 6.6.)

**Example 7**

Solve $\mathbf{X}' = \begin{pmatrix} 5 & 3 & -3 \\ 2 & 4 & -5 \\ -4 & 2 & -3 \end{pmatrix}\mathbf{X}$.

**Solution** The eigenvalues of $\mathbf{A} = \begin{pmatrix} 5 & 3 & -3 \\ 2 & 4 & -5 \\ -4 & 2 & -3 \end{pmatrix}$ are determined with

$$\begin{vmatrix} 5-\lambda & 3 & -3 \\ 2 & 4-\lambda & -5 \\ -4 & 2 & -3-\lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2 + 15\lambda + 8 = -(\lambda + 1)(\lambda + 1)(\lambda - 8)$$

$= 0$ to be $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 8$. An eigenvector $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ that corresponds to $\lambda_1 = -1$ satisfies $\begin{pmatrix} 6 & 3 & -3 \\ 2 & 5 & -5 \\ -4 & 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. This system is equivalent to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Hence, $x_1 = 0$ and $y_1 - z_1 = 0$. If we let $y_1 = 1$, then $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Therefore, one solution to the system is $\mathbf{X}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}e^{-t}$. A second linearly independent solution $\mathbf{X}_2 = (\mathbf{v}_1 t + \mathbf{w}_2)e^{\lambda_1 t}$ is found by solving $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{w}_2 = \mathbf{v}_1$ given by $\begin{pmatrix} 6 & 3 & -3 \\ 2 & 5 & -5 \\ -4 & 2 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ for the vector $\mathbf{w}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$. This system is equivalent to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ 0 \end{pmatrix}$, which indicates that $x_2 = -\frac{1}{8}$ and $y_2 - z_2 = \frac{1}{4}$.

Hence, if $z_2 = 0$, $y_2 = \frac{1}{4}$, so $\mathbf{w}_2 = \begin{pmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ 0 \end{pmatrix}$. A second linearly independent solution is then $\mathbf{X}_2 = \left(\begin{pmatrix}0\\1\\1\end{pmatrix}t + \begin{pmatrix}-\frac{1}{8}\\ \frac{1}{4}\\ 0\end{pmatrix}\right)e^{-t}$. A third linearly independent solution is found using an eigenvector $\mathbf{v}_3 = \begin{pmatrix}x_3\\y_3\\z_3\end{pmatrix}$ corresponding to $\lambda_3 = 8$, which satisfies $\begin{pmatrix}-3 & 3 & -3\\ 2 & -4 & -5\\ -4 & 2 & -11\end{pmatrix}\begin{pmatrix}x_3\\y_3\\z_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$. Because this system is equivalent to $\begin{pmatrix}1 & 0 & \frac{9}{2}\\ 0 & 1 & \frac{7}{2}\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_3\\y_3\\z_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$, the components of $\mathbf{v}_3$ must satisfy $x_3 + \frac{9}{2}z_3 = 0$ and $y_3 + \frac{7}{2}z_3 = 0$. If we let $z_3 = -2$, then $x_3 = 9$ and $y_3 = 7$. Hence, $\mathbf{v}_3 = \begin{pmatrix}9\\7\\-2\end{pmatrix}$ and $\mathbf{X}_3 = \begin{pmatrix}9\\7\\-2\end{pmatrix}e^{8t}$. A general solution is then given by

$$\mathbf{X}(t) = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + c_3\mathbf{X}_3 = c_1\begin{pmatrix}0\\1\\1\end{pmatrix}e^{-t} + c_2\left(\begin{pmatrix}0\\1\\1\end{pmatrix}t + \begin{pmatrix}-\frac{1}{8}\\ \frac{1}{4}\\ 0\end{pmatrix}\right)e^{-t} + c_3\begin{pmatrix}9\\7\\-2\end{pmatrix}e^{8t}$$

$$= \begin{pmatrix}-\frac{1}{8}c_2e^{-t} + 9c_3e^{8t}\\ c_1e^{-t} + c_2(t + \frac{1}{4})e^{-t} + 7c_3e^{8t}\\ c_1e^{-t} + c_2te^{-t} - 2c_3e^{8t}\end{pmatrix}.$$

Figure 6.16 shows the graph of several solutions to the system along with the direction field.

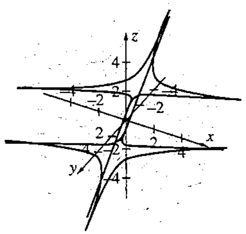



Figure 6.16

*Using the formula for* $\mathbf{X}$ *in Example 7 and the direction field in Figure 6.16, determine if the solutions approach the origin as t increases if* $c_3 \neq 0$. *What happens to the solutions if* $c_3 = 0$ *and either* $c_1 \neq 0$ *or* $c_2 \neq 0$?

A similar method is carried out in the case of three equal eigenvalues $\lambda_1 = \lambda_2 = \lambda_3$, where we can find only one (linearly independent) eigenvector $\mathbf{v}_1$. When we encounter this situation, we assume that

$$\mathbf{X}_1 = \mathbf{v}_1e^{\lambda_1 t},\ \mathbf{X}_2 = (\mathbf{v}_2t + \mathbf{w}_2)e^{\lambda_1 t}, \quad \text{and} \quad \mathbf{X}_3 = \left(\mathbf{v}_3\frac{t^2}{2} + \mathbf{w}_3t + \mathbf{u}_3\right)e^{\lambda_1 t}.$$

Substitution of these solutions into the system of differential equations yields the following system of equations, which is solved for the unknown vectors $\mathbf{v}_2$, $\mathbf{w}_2$, $\mathbf{v}_3$, $\mathbf{w}_3$, and $\mathbf{u}_3$:

$$\lambda_1\mathbf{v}_2 = \mathbf{A}\mathbf{v}_2,\ \ (\mathbf{A} - \lambda_1\mathbf{I})\mathbf{w}_2 = \mathbf{v}_2,\ \ \lambda_1\mathbf{v}_3 = \mathbf{A}\mathbf{v}_3,\ \ (\mathbf{A} - \lambda_1\mathbf{I})\mathbf{w}_3 = \mathbf{v}_3,\ \ \text{and}\ \ (\mathbf{A} - \lambda_1\mathbf{I})\mathbf{u}_3 = \mathbf{w}_3.$$



Similar to the previous case, $\mathbf{v}_3 = \mathbf{v}_2 = \mathbf{v}_1$, $\mathbf{w}_2 = \mathbf{w}_3$, and the vector $\mathbf{u}_3$ is found by solving the system

$$(\mathbf{A} - \lambda_1 I)\mathbf{u}_3 = \mathbf{w}_2.$$

The three linearly independent solutions have the form

$$\mathbf{X}_1 = \mathbf{v}_1e^{\lambda_1 t},\quad \mathbf{X}_2 = (\mathbf{v}_1t + \mathbf{w}_2)e^{\lambda_1 t}, \quad \text{and} \quad \mathbf{X}_3 = \left(\mathbf{v}_1\frac{t^2}{2} + \mathbf{w}_2t + \mathbf{u}_3\right)e^{\lambda_1 t}.$$

Notice that this method is generalized for instances when the multiplicity of the repeated eigenvalue is greater than 3.

**Example 8**

Solve $\mathbf{X}' = \begin{pmatrix}1 & 1 & 1\\ 2 & 1 & -1\\ -3 & 2 & 4\end{pmatrix}\mathbf{X}$.

***Solution*** The eigenvalues are found by solving

$$\begin{vmatrix}1-\lambda & 1 & 1\\ 2 & 1-\lambda & -1\\ -3 & 2 & 4-\lambda\end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix}1-\lambda & -1\\ 2 & 4-\lambda\end{vmatrix} - (1)\begin{vmatrix}2 & -1\\ -3 & 4-\lambda\end{vmatrix} + (1)\begin{vmatrix}2 & 1-\lambda\\ -3 & 2\end{vmatrix}$$

$$= -\lambda^3 + 6\lambda^2 - 12\lambda + 8 = -(\lambda-2)(\lambda^2 - 4\lambda + 4) = -(\lambda-2)^3 = 0.$$

Hence $\lambda_1 = \lambda_2 = \lambda_3 = 2$. We find eigenvectors $\mathbf{v}_1 = \begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}$ corresponding to $\lambda = 2$ with the system $\begin{pmatrix}-1 & 1 & 1\\ 2 & -1 & -1\\ -3 & 2 & 2\end{pmatrix}\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$. With elementary row operations, we reduce this system to $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$. Thus, $x_1 = 0$ and $y_1 + z_1 = 0$. If we select $z_1 = 1$, then $y_1 = -1$ and $\mathbf{v}_1 = \begin{pmatrix}0\\-1\\1\end{pmatrix}$. Therefore we can find only one (linearly independent) eigenvector corresponding to $\lambda = 2$. Using this eigenvalue and eigenvector, we find that one solution to the system is $\mathbf{X}_1 = \mathbf{v}_1e^{2t} = \begin{pmatrix}0\\-1\\1\end{pmatrix}e^{2t}$.

The vector $\mathbf{w}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ in a second linearly independent solution of the form $\mathbf{X}_2 = (\mathbf{v}_1 t + \mathbf{w}_2)e^{2t}$ is found by solving the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{w}_2 = \mathbf{v}_1$. This system is $\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, which is equivalent to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$. Hence $x_2 = -1$ and $y_2 + z_2 = -1$, so if we choose $z_2 = 0$, then $y_2 = -1$. Therefore, $\mathbf{w}_2 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$, so $\mathbf{X}_2 = \left(\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}\right)e^{2t}$. Finally, we must determine the vector $\mathbf{u}_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$ in the third linearly independent solution $\mathbf{X}_3 = \left(\mathbf{v}_1 \frac{t^2}{2} + \mathbf{w}_2 t + \mathbf{u}_3\right)e^{\lambda_1 t}$ by solving the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{u}_3 = \mathbf{w}_2$. This yields $\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$, which is equivalent to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix}$. Therefore, $x_3 = -2$ and $y_3 + z_3 = -3$. If we select $z_3 = 0$, then $y_3 = -3$. Hence $\mathbf{u}_3 = \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix}$, so a third linearly independent solution is $\mathbf{X}_3 = \left(\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\frac{t^2}{2} + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix}\right)e^{2t}$. A general solution is then given by

$$\mathbf{X}(t) = c_1\mathbf{X}_1 + c|_2\mathbf{X}_2 + c_3\mathbf{X}_3$$

$$= c_1\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}e^{2t} + c_2\left(\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}\right)e^{2t} + c_3\left(\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\frac{t^2}{2} + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix}\right)e^{2t}$$

$$= \begin{pmatrix} -c_2 e^{2t} + c_3(-t - 2)e^{2t} \\ -c_1 e^{2t} + c_2(-t - 1)e^{2t} + c_3\left(-\frac{t^2}{2} - t - 3\right)e^{2t} \\ c_1 e^{2t} + c_2 t e^{2t} + c_3\left(\frac{t^2}{2}\right)e^{2t} \end{pmatrix}.$$



## EXERCISES 6.4

In Exercises 1–12, use the given eigenvalues and corresponding (linearly independent) eigenvectors of the matrix $\mathbf{A}$ to find a general solution of $\mathbf{X}' = \mathbf{A}\mathbf{X}$.

1. $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$; $\lambda_1 = 2$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$; $\lambda_2 = 4$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

2. $\mathbf{A} = \begin{pmatrix} 4 & 0 \\ -1 & 0 \end{pmatrix}$; $\lambda_1 = 0$, $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $\lambda_2 = 4$, $\mathbf{v}_2 = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

3. $\mathbf{A} = \begin{pmatrix} -4 & 0 \\ -1 & 2 \end{pmatrix}$; $\lambda_1 = -4$, $\mathbf{v}_1 = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$; $\lambda_2 = 2$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

4. $\mathbf{A} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$; $\lambda_1 = \lambda_2 = 5$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

*5. $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$; $\lambda_1 = \lambda_2 = 2$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

6. $\mathbf{A} = \begin{pmatrix} 0 & -6 \\ 6 & 0 \end{pmatrix}$; $\lambda_1 = 6i$, $\mathbf{v}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$; $\lambda_2 = -6i$, $\mathbf{v}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

7. $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix}$, $\lambda_1 = 1 + 2i$, $\mathbf{v}_1 = \begin{pmatrix} -i - 1 \\ 1 \end{pmatrix}$; $\lambda_2 = 1 - 2i$, $\mathbf{v}_2 = \begin{pmatrix} i - 1 \\ 1 \end{pmatrix}$

8. $\mathbf{A} = \begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix}$; $\lambda_1 = 1$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $\lambda_2 = 2$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$; $\lambda_3 = 3$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$

*9. $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$; $\lambda_1 = \lambda_2 = 2$, $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$; $\lambda_3 = 3$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

10. $\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$; $\lambda_1 = \lambda_2 = \lambda_3 = 2$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

11. $\mathbf{A} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$; $\lambda_1 = i$, $\mathbf{v}_1 = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$; $\lambda_2 = -i$, $\mathbf{v}_2 = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$; $\lambda_3 = 2$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

12. $\mathbf{A} = \begin{pmatrix} 3 & 2 & -3 \\ 1 & 1 & 1 \\ 0 & -4 & -4 \end{pmatrix}$; $\lambda_1 = -2 + i$, $\mathbf{v}_1 = \begin{pmatrix} 3 + i \\ -2 - i \\ 4 \end{pmatrix}$; $\lambda_2 = -2 - i$, $\mathbf{v}_2 = \begin{pmatrix} 3 - i \\ -2 + i \\ 4 \end{pmatrix}$; $\lambda_3 = 4$, $\mathbf{v}_3 = \begin{pmatrix} -7 \\ -2 \\ 1 \end{pmatrix}$

In Exercises 13–32, find a general solution of the system.

13. $\mathbf{X}' = \begin{pmatrix} 1 & -10 \\ -7 & 10 \end{pmatrix}\mathbf{X}$

14. $\begin{cases} x' = x - 2y \\ y' = 2x + 6y \end{cases}$

*15. $\begin{cases} \dfrac{dx}{dt} = 6x - y \\ \dfrac{dy}{dt} = 5x \end{cases}$

16. $\mathbf{X}' = \begin{pmatrix} 4 & 3 \\ -5 & -4 \end{pmatrix}\mathbf{X}$

17. $\begin{cases} x' = 7x \\ y' = 5x - 8y \end{cases}$

18. $\begin{cases} \dfrac{dx}{dt} = 8x + 9y \\ \dfrac{dy}{dt} = -2x - 3y \end{cases}$

*19. $\mathbf{X}' = \begin{pmatrix} -5 & 3 \\ 2 & -10 \end{pmatrix}\mathbf{X}$

20. $\begin{cases} x' = 8x + 5y \\ y' = -10x - 6y \end{cases}$

21. $\begin{cases} \dfrac{dx}{dt} = -6x - 4y \\ \dfrac{dy}{dt} = -3x - 10y \end{cases}$

22. $\mathbf{X}' = \begin{pmatrix} 1 & 8 \\ -2 & -7 \end{pmatrix}\mathbf{X}$

*23. $\begin{cases} x' = -6x + 2y \\ y' = -2x - 10y \end{cases}$

24. $\mathbf{X}' = \begin{pmatrix} 0 & 8 \\ 2 & 0 \end{pmatrix}\mathbf{X}$

25. $\mathbf{X}' = \begin{pmatrix} 0 & 8 \\ -2 & 0 \end{pmatrix}\mathbf{X}$

26. $\begin{cases} x' = y \\ y' = -13x - 4y \end{cases}$

*27. $\mathbf{X}' = \begin{pmatrix} 4 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}\mathbf{X}$

28. $\mathbf{X}' = \begin{pmatrix} -3 & 3 & -4 \\ 0 & -3 & 0 \\ 0 & -5 & -4 \end{pmatrix}\mathbf{X}$

29. $\mathbf{X}' = \begin{pmatrix} 5 & -1 & 3 \\ -4 & -1 & -2 \\ -4 & 2 & -3 \end{pmatrix}\mathbf{X}$

30. $\mathbf{X}' = \begin{pmatrix} -5 & 4 & -5 \\ 0 & -1 & 0 \\ 5 & 1 & 1 \end{pmatrix}\mathbf{X}$

*31. $\begin{cases} x' = x + 2y + 3z \\ y' = y + 2z \\ z' = -2y + z \end{cases}$

32. $\begin{cases} x' = y \\ y' = z \\ z' = x - y + z \end{cases}$

In Exercises 33–40, solve the initial-value problem.

33. $\mathbf{X}' = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\mathbf{X}$, $\mathbf{X}(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

34. $\mathbf{X}' = \begin{pmatrix} -4 & 0 \\ 2 & 4 \end{pmatrix}\mathbf{X}$, $\mathbf{X}(0) = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$

*35. $\mathbf{X}' = \begin{pmatrix} 4 & 0 \\ 2 & 4 \end{pmatrix}\mathbf{X}$, $\mathbf{X}(0) = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$

36. $\mathbf{X}' = \begin{pmatrix} 4 & 8 \\ 0 & 4 \end{pmatrix}\mathbf{X}$, $\mathbf{X}(0) = \begin{pmatrix} 8 \\ 8 \end{pmatrix}$

37. $\mathbf{X}' = \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix}\mathbf{X}$, $\mathbf{X}(0) = \begin{pmatrix} 0 \\ 8 \end{pmatrix}$

38. $\mathbf{X}' = \begin{pmatrix} 0 & 13 \\ -1 & -4 \end{pmatrix}\mathbf{X}$, $\mathbf{X}(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

*39. $\mathbf{X}' = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}\mathbf{X}$, $\mathbf{X}(0) = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$

40. $\mathbf{X}' = \begin{pmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{pmatrix}\mathbf{X}$, $\mathbf{X}(0) = \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix}$

41. Show that $x_1(t) = e^{\alpha t}(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t)$ and $x_2(t) = e^{\alpha t}(\mathbf{a}\sin\beta t + \mathbf{b}\cos\beta t)$ are linearly independent functions.

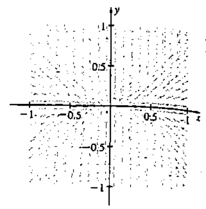
42. Show that in the $3 \times 3$ system $\mathbf{X}' = A\mathbf{X}$ with the eigenvalue $\lambda$ of multiplicity 3 with one corresponding eigenvector $\mathbf{v}_1$, the three linearly independent solutions are $\mathbf{X}_1 = \mathbf{v}_1 e^{\lambda_1 t}$, $\mathbf{X}_2 = (\mathbf{v}_1 t + \mathbf{w}_2)e^{\lambda_1 t}$, and $\mathbf{X}_3 = \left(\mathbf{v}_1\frac{t^2}{2} + \mathbf{w}_2 t + \mathbf{u}_3\right)e^{\lambda_1 t}$, where $\mathbf{u}_3$ satisfies $(A - \lambda\mathbf{I})\mathbf{u}_3 = \mathbf{w}_2$ and $\mathbf{w}_2$ satisfies $(A - \lambda\mathbf{I})\mathbf{w}_2 = \mathbf{v}_1$.

In Exercises 43–49, without solving each system, match each system in Group A with the graph of its direction field in Group B.

**Group A** / **Group B**

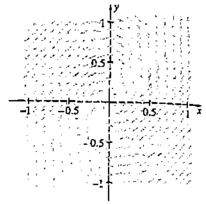
43. $\begin{cases} x' = x \\ y' = 2y \end{cases}$

(a)



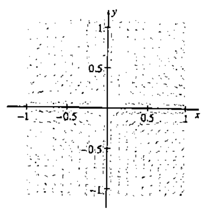
44. $\begin{cases} x' = -x \\ y' = 2y \end{cases}$

(b)



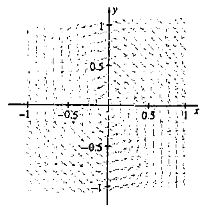
45. $\begin{cases} x' = -x \\ y' = -2y \end{cases}$

(c)



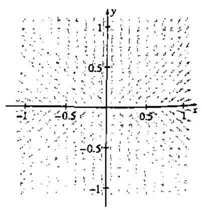
46. $\begin{cases} x' = -y \\ y' = 2x \end{cases}$

(d)



47. $\begin{cases} x' = -x - y \\ y' = 2x \end{cases}$

(e)

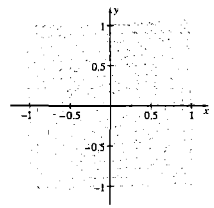




**Group A** / **Group B**

48. $\begin{cases} x' = x - y \\ y' = 2x \end{cases}$

(f)



49. Show that both $\begin{cases} x(t) = e^{2t}(c_1\cos t - c_2\sin t) \\ y(t) = e^{2t}(c_1\sin t + c_2\cos t) \end{cases}$ and $\begin{cases} x(t) = \frac{1}{2}e^{(2-i)t}[c_1(1 + e^{2it}) + c_2 i(e^{2it} - 1)] \\ y(t) = \frac{1}{2}e^{(2-i)t}[c_1 i(1 - e^{2it}) + c_2(1 + e^{2it})] \end{cases}$ are general solutions of the system $\begin{cases} x'(t) = 2x(t) - y(t) \\ y'(t) = x(t) + 2y(t) \end{cases}$.

50. Solve the systems (a) $\begin{cases} x' = 2x - y \\ y' = -x + 3y \end{cases}$; (b) $\begin{cases} x' = 2x \\ y' = 3x + 2y \end{cases}$; (c) $\begin{cases} x' = x + 4y \\ y' = -2x - y \end{cases}$ subject to $x(0) = 1$ and $y(0) = 1$. In each case, graph the solution parametrically and individually.

*51. Find a general solution of the system $\mathbf{X}' = \begin{pmatrix} 0 & 0 & 3 \\ 1 & -4 & 2 \\ 0 & -4 & 1 \end{pmatrix}\mathbf{X}$. Graph the solution for various values of the constants.

52. How do the general solutions and direction field of the system $\mathbf{X}' = \begin{pmatrix} 2 & \lambda \\ 1 & 0 \end{pmatrix}\mathbf{X}$ change as $\lambda$ goes from $-2$ to 0? Solve the system for values of $\lambda$ between $-2$ and 0 and note how the solution changes.

53. Consider the initial-value problem

$$\begin{cases} \mathbf{X}' = \begin{pmatrix} 1 & -\frac{7}{3} \\ -2 & -\frac{8}{3} \end{pmatrix}\mathbf{X} \\ x(0) = x_0,\ y(0) = y_0 \end{cases}$$

(a) Graph the direction field associated with the system $\mathbf{X}' = \begin{pmatrix} 1 & -\frac{7}{3} \\ -2 & -\frac{8}{3} \end{pmatrix}\mathbf{X}$. (b) Find conditions on $x_0$ and $y_0$ so that at least one of $x(t)$ or $y(t)$ approaches zero as $t$ approaches infinity. (c) Is it possible to choose $x_0$ and $y_0$ so that both $x(t)$ and $y(t)$ approach zero as $t$ approaches infinity?

54. (a) Find a general solution of $\mathbf{X}' = \mathbf{A}\mathbf{X}$ if $\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}$ and (i) $\mathbf{A} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$, (ii) $\mathbf{A} = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$, (iii) $\mathbf{A} = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$, and (iv) $\mathbf{A} = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$. (b) For each system in (a), find the solution that satisfies the initial condition $\mathbf{X}(0) = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$ if $\lambda = -\frac{1}{2}$ and then graph $x_1(t)$, $x_2(t)$, $x_3(t)$, and $x_4(t)$ for $0 \le t \le 10$. How are the solutions similar? How are they different? (c) Indicate how to generalize the results obtained in (a). Explain how you would find a general solution of $\mathbf{X}' = \mathbf{A}\mathbf{X}$ if $\mathbf{A} = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$. How would you find a general solution of $\mathbf{X}' = \mathbf{A}\mathbf{X}$ for the $5 \times 5$ matrix $\mathbf{A} = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$? How would you find a general solution of $\mathbf{X}' = \mathbf{A}\mathbf{X}$ for the $n \times n$ matrix $\mathbf{A} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$?

## 6.5 First-Order Linear Nonhomogeneous Systems: Undetermined Coefficients and Variation of Parameters

Undetermined Coefficients · Variation of Parameters

In Chapter 4, we learned how to solve nonhomogeneous differential equations through the use of undetermined coefficients and variation of parameters. Here we approach the solution of systems of nonhomogeneous equations using these methods.

Let $\mathbf{X}=\mathbf{X}(t)=\begin{pmatrix}x_1(t)\\x_2(t)\\\vdots\\x_n(t)\end{pmatrix}$, $\mathbf{A}=\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{n1}&a_{n2}&\cdots&a_{nn}\end{pmatrix}$, $\mathbf{F}(t)\begin{pmatrix}f_1(t)\\f_2(t)\\\vdots\\f_n(t)\end{pmatrix}$, and $\mathbf{\Phi}(t)$ be a fundamental matrix of the system $\mathbf{X}'=\mathbf{AX}$. Then a general solution to the homogeneous system $\mathbf{X}'=\mathbf{AX}$ is $\mathbf{X}=\mathbf{\Phi}(t)\mathbf{C}$ where $\mathbf{C}=\begin{pmatrix}c_1\\c_2\\\vdots\\c_n\end{pmatrix}$ is an $n\times 1$ constant matrix.

To find a general solution to the linear nonhomogeneous system $\mathbf{X}'=\mathbf{AX}+\mathbf{F}(t)$, we proceed in the same way we did with linear nonhomogeneous equations in Chapter 4. If $\mathbf{X}_P$ is a particular solution of the nonhomogeneous system, then all other solutions $\mathbf{X}$ of the system can be written in the form $\mathbf{X}=\mathbf{\Phi}(t)\mathbf{C}+\mathbf{X}_P$ (see Exercise 36).

### Undetermined Coefficients

We use the method of undetermined coefficients to find a particular solution $\mathbf{X}_P$ to a nonhomogeneous system in much the same way as we approached nonhomogeneous higher order equations with constant coefficients in Chapter 4. The main difference is that the coefficients are *constant vectors* when we work with systems. For example, if we consider $\mathbf{X}'=\mathbf{AX}+\mathbf{F}(t)$ where $\mathbf{F}(t)=\begin{pmatrix}e^{-2t}\\4\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}e^{-2t}+\begin{pmatrix}0\\4\end{pmatrix}$ and none of the terms in $\mathbf{F}(t)$ satisfy the corresponding homogeneous system $\mathbf{X}'=\mathbf{AX}$, we assume that a particular solution has the form $\mathbf{X}_P(t)=\mathbf{a}e^{-2t}+\mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors. On the other hand if $\lambda=-2$ is an eigenvalue of $\mathbf{A}$, we assume that $\mathbf{X}_P(t)=\mathbf{a}te^{-2t}+\mathbf{b}e^{-2t}+\mathbf{c}$.

**Example 1**

Solve $\mathbf{X}'=\begin{pmatrix}0&8\\2&0\end{pmatrix}\mathbf{X}+\begin{pmatrix}e^{3t}\\t\end{pmatrix}$.

***Solution*** In this case, $\mathbf{F}(t)=\begin{pmatrix}e^{3t}\\t\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}e^{3t}+\begin{pmatrix}0\\1\end{pmatrix}t$ and a general solution to the corresponding homogeneous system $\mathbf{X}'=\begin{pmatrix}0&8\\2&0\end{pmatrix}\mathbf{X}$ is $\mathbf{X}_h(t)=c_1\begin{pmatrix}2\\1\end{pmatrix}e^{4t}+c_2\begin{pmatrix}-2\\1\end{pmatrix}e^{-4t}$. Notice that none of the components of $\mathbf{F}(t)$ are in $\mathbf{X}_h(t)$, so we assume that there is a particular solution of the form $\mathbf{X}_P(t)=\mathbf{a}e^{3t}+\mathbf{b}t+\mathbf{c}$. Then, $\mathbf{X}_P'(t)=3\mathbf{a}e^{3t}+\mathbf{b}$ and substitution into the nonhomogeneous system $\mathbf{X}'=\mathbf{AX}+\begin{pmatrix}1\\0\end{pmatrix}e^{3t}+\begin{pmatrix}0\\1\end{pmatrix}t$, where $\mathbf{A}=\begin{pmatrix}0&8\\2&0\end{pmatrix}$, yields



$$3\mathbf{a}e^{3t}+\mathbf{b}=\mathbf{A}\mathbf{a}e^{3t}+\mathbf{A}\mathbf{b}t+\mathbf{A}\mathbf{c}+\begin{pmatrix}1\\0\end{pmatrix}e^{3t}+\begin{pmatrix}0\\1\end{pmatrix}t.$$

Collecting like terms, we obtain the system of equations

$$\begin{cases}3\mathbf{a}=\mathbf{A}\mathbf{a}+\begin{pmatrix}1\\0\end{pmatrix} & \text{(Coefficients of } e^{3t}\text{)}\\ \mathbf{b}=\mathbf{A}\mathbf{c} & \text{(Constant terms)}\\ \mathbf{A}\mathbf{b}+\begin{pmatrix}0\\1\end{pmatrix}=\mathbf{0} & \text{(Coefficients of } t\text{)}\end{cases}$$

From the coefficients of $e^{3t}$, we find that $(\mathbf{A}-3\mathbf{I})\mathbf{a}=\begin{pmatrix}-1\\0\end{pmatrix}$ or $\begin{pmatrix}-3&8\\2&-3\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}-1\\0\end{pmatrix}$, where $\mathbf{a}=\begin{pmatrix}a_1\\a_2\end{pmatrix}$. This system has the unique solution $\mathbf{a}=\begin{pmatrix}-\frac{3}{7}\\-\frac{2}{7}\end{pmatrix}$. Next, we solve the system $\mathbf{A}\mathbf{b}+\begin{pmatrix}0\\1\end{pmatrix}=\mathbf{0}$ or $\begin{pmatrix}0&8\\2&0\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix}=\begin{pmatrix}0\\-1\end{pmatrix}$ for $\mathbf{b}$. This yields the unique solution $\mathbf{b}=\begin{pmatrix}-\frac{1}{2}\\0\end{pmatrix}$. Finally, we solve $\mathbf{b}=\mathbf{A}\mathbf{c}$ or $\begin{pmatrix}0&8\\2&0\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\\0\end{pmatrix}$ for $\mathbf{c}$, which gives us $\mathbf{c}=\begin{pmatrix}0\\-\frac{1}{16}\end{pmatrix}$. A particular solution to the nonhomogeneous system is then

$$\mathbf{X}_P(t)=\mathbf{a}e^{3t}+\mathbf{b}t+\mathbf{c}=\begin{pmatrix}-\frac{3}{7}\\-\frac{2}{7}\end{pmatrix}e^{3t}+\begin{pmatrix}-\frac{1}{2}\\0\end{pmatrix}t+\begin{pmatrix}0\\-\frac{1}{16}\end{pmatrix}=\begin{pmatrix}-\frac{3}{7}e^{3t}-\frac{1}{2}t\\-\frac{2}{7}e^{3t}-\frac{1}{16}\end{pmatrix},$$

so a general solution to $\mathbf{X}'=\mathbf{AX}+\begin{pmatrix}1\\0\end{pmatrix}e^{3t}+\begin{pmatrix}0\\1\end{pmatrix}t$ is

$$\mathbf{X}(t)=\mathbf{X}_h(t)+\mathbf{X}_P(t)=\begin{pmatrix}2c_1e^{4t}-2c_2e^{-4t}-\frac{3}{7}e^{3t}-\frac{1}{2}t\\c_1e^{4t}+c_2e^{-4t}-\frac{2}{7}e^{3t}-\frac{1}{16}\end{pmatrix}.$$

To give another illustration of how the form of a particular solution is selected, suppose that $\mathbf{F}(t)=\begin{pmatrix}4\sin 2t\\e^{-t}\end{pmatrix}=\begin{pmatrix}4\\0\end{pmatrix}\sin 2t+\begin{pmatrix}0\\1\end{pmatrix}e^{-t}$ in Example 1. In this case, we assume that $\mathbf{X}_P(t)=\mathbf{a}\cos 2t+\mathbf{b}\sin 2t+\mathbf{c}e^{-t}$ and find the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ through substitution into the nonhomogeneous system.

*Find a particular solution to* $\mathbf{X}'=\begin{pmatrix}0&8\\2&0\end{pmatrix}\mathbf{X}+\begin{pmatrix}4\sin 2t\\e^{-t}\end{pmatrix}$.

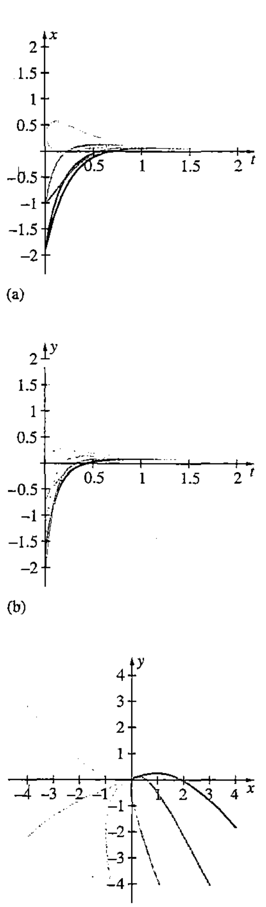



**Figure 6.17** (a) Graph of $x(t)$ for various initial conditions. (b) Graph of $y(t)$ for various initial conditions. (c) Graph of $\begin{cases} x(t) \\ y(t) \end{cases}$ for various initial conditions.

## Variation of Parameters

Variation of parameters can be used to solve linear nonhomogeneous systems as well. In much the same way that we derived the method of variation of parameters for solving higher order differential equations, we assume that a particular solution of the nonhomogeneous system can be expressed in the form

$$\mathbf{X}_P(t) = \mathbf{\Phi}(t)\mathbf{V}(t), \text{ where } \mathbf{V}(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_n(t) \end{pmatrix}.$$

Notice that $\mathbf{X}_P' = \mathbf{\Phi}(t)\mathbf{V}'(t) + \mathbf{\Phi}'(t)\mathbf{V}(t)$. Then if $\mathbf{X}_P$ satisfies $\mathbf{X}' = \mathbf{AX} + \mathbf{F}(t)$, we have

$$\mathbf{\Phi}(t)\mathbf{V}'(t) + \mathbf{\Phi}'(t)\mathbf{V}(t) = \mathbf{A\Phi}(t)\mathbf{V}(t) + \mathbf{F}(t).$$

However, the fundamental matrix $\mathbf{\Phi}(t)$ satisfies $\mathbf{X}' = \mathbf{AX}$, so $\mathbf{\Phi}'(t) = \mathbf{A\Phi}(t)$. Hence,

$$\mathbf{\Phi}(t)\mathbf{V}'(t) + \mathbf{A\Phi}(t)\mathbf{V}(t) = \mathbf{A\Phi}(t)\mathbf{V}(t) + \mathbf{F}(t)$$

$$\mathbf{\Phi}(t)\mathbf{V}'(t) = \mathbf{F}(t).$$

Multiplying both sides of this equation by $\mathbf{\Phi}^{-1}(t)$ yields

$$\mathbf{\Phi}^{-1}(t)\mathbf{\Phi}(t)\mathbf{V}'(t) = \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)$$

$$\mathbf{V}'(t) = \mathbf{\Phi}^{-1}(t)\mathbf{F}(t).$$

Therefore, $\mathbf{V}(t) = \int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\, dt$, so a particular solution is

$$\mathbf{X}_P(t) = \mathbf{\Phi}(t) \int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\, dt,$$

and a general solution to the system is

$$\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{C} + \mathbf{X}_P(t) = \mathbf{\Phi}(t)\mathbf{C} + \mathbf{\Phi}(t) \int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\, dt.$$

### Example 2

Solve $\mathbf{X}' = \begin{pmatrix} -5 & 3 \\ 2 & -10 \end{pmatrix}\mathbf{X} + \begin{pmatrix} e^{-2t} \\ 1 \end{pmatrix}$ using variation of parameters.

**Solution** To apply variation of parameters, we first calculate a fundamental matrix for the corresponding homogeneous system $\mathbf{X}' = \begin{pmatrix} -5 & 3 \\ 2 & -10 \end{pmatrix}\mathbf{X}$. The eigenvalues of $\mathbf{A} = \begin{pmatrix} -5 & 3 \\ 2 & -10 \end{pmatrix}$ are $\lambda_1 = -4$ and $\lambda_2 = -11$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$, respectively. A general solution to the homogeneous system is $\mathbf{X}_h(t) = c_1\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}e^{-11t} + c_2\begin{pmatrix} 3 \\ 1 \end{pmatrix}e^{-4t}$, so a fundamental matrix is given by $\mathbf{\Phi}(t) = \begin{pmatrix} -\frac{1}{2}e^{-11t} & 3e^{-4t} \\ e^{-11t} & e^{-4t} \end{pmatrix}$. Because $\mathbf{\Phi}^{-1}(t) = \dfrac{1}{-\frac{7}{2}e^{-15t}}\begin{pmatrix} e^{-4t} & -3e^{-4t} \\ -e^{-11t} & -\frac{1}{2}e^{-11t} \end{pmatrix} = \begin{pmatrix} -\frac{2}{7}e^{11t} & \frac{6}{7}e^{11t} \\ -\frac{2}{7}e^{4t} & \frac{1}{7}e^{4t} \end{pmatrix}$,

$$\mathbf{\Phi}^{-1}(t)\mathbf{F}(t) = \begin{pmatrix} \frac{-2}{7}e^{11t} & \frac{6}{7}e^{11t} \\ \frac{2}{7}e^{4t} & \frac{1}{7}e^{4t} \end{pmatrix}\begin{pmatrix} e^{-2t} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{-2}{7}e^{9t} + \frac{6}{7}e^{11t} \\ \frac{2}{7}e^{2t} + \frac{1}{7}e^{4t} \end{pmatrix}.$$

Therefore,

$$\mathbf{V}(t) = \int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\, dt = \int \begin{pmatrix} \frac{-2}{7}e^{9t} + \frac{6}{7}e^{11t} \\ \frac{2}{7}e^{2t} + \frac{1}{7}e^{4t} \end{pmatrix} dt = \begin{pmatrix} \frac{-2}{63}e^{9t} + \frac{6}{77}e^{11t} \\ \frac{1}{7}e^{2t} + \frac{1}{28}e^{4t} \end{pmatrix}.$$

By variation of parameters, we have the particular solution

$$\mathbf{X}_P(t) = \mathbf{\Phi}(t) \int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\, dt = \begin{pmatrix} \frac{-1}{2}e^{-11t} & 3e^{-4t} \\ e^{-11t} & e^{-4t} \end{pmatrix}\begin{pmatrix} \frac{-2}{63}e^{9t} + \frac{6}{77}e^{11t} \\ \frac{1}{7}e^{2t} + \frac{1}{28}e^{4t} \end{pmatrix} = \begin{pmatrix} \frac{3}{44} + \frac{4}{9}e^{-2t} \\ \frac{5}{44} + \frac{1}{9}e^{-2t} \end{pmatrix}.$$

Therefore, a general solution is given by

$$\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{C} + \mathbf{X}_P(t) = \begin{pmatrix} \frac{-1}{2}e^{11t} & 3e^{-4t} \\ e^{-11t} & e^{-4t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \frac{3}{44} + \frac{4}{9}e^{-2t} \\ \frac{5}{44} + \frac{1}{9}e^{-2t} \end{pmatrix}.$$

In Figure 6.17, we graph $x(t)$, $y(t)$, and $\begin{cases} x(t) \\ y(t) \end{cases}$ for several values of $c_1$ and $c_2$.

*Evaluate* $\lim_{t\to\infty} x(t)$ *and* $\lim_{t\to\infty} y(t)$. *Does the choice of* $c_1$ *or* $c_2$ *affect the limit?*

Variation of parameters allows us to solve systems that we cannot solve using the method of undetermined coefficients.

### Example 3

Solve the initial-value problem $\begin{cases} x' = 2x - 5y + \csc t \\ y' = x - 2y + \sec t \\ x(\pi/4) = 5,\ y(\pi/4) = 1 \end{cases}$, $\quad 0 < t < \pi/2$.

**Solution** First, we write the system as $\mathbf{X}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{X} + \begin{pmatrix} \csc t \\ \sec t \end{pmatrix}$ and solve the corresponding homogeneous system $\mathbf{X}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{X}$ to obtain $\mathbf{X}_h(t) = c_1\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix} + c_2\begin{pmatrix} -5\sin t \\ \cos t - 2\sin t \end{pmatrix}$. A fundamental matrix is $\mathbf{\Phi}(t) = \begin{pmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{pmatrix}$ with inverse

$$\Phi^{-1}(t) = \begin{pmatrix} \cos t - 2\sin t & 5\sin t \\ -\sin t & \cos t + 2\sin t \end{pmatrix}. \text{ Then,}$$

$$\mathbf{V}(t) = \int \Phi^{-1}(t)\mathbf{F}(t)\,dt = \int \begin{pmatrix} \dfrac{\cos t}{\sin t} - 2 + \dfrac{5\sin t}{\cos t} \\ \dfrac{2\sin t}{\cos t} \end{pmatrix} dt =$$

$$\begin{pmatrix} \ln(\sin t) - 2t - 5\ln(\cos t) \\ -2\ln(\cos t) \end{pmatrix},$$

so a particular solution is

$$\mathbf{X}_p(t) = \Phi(t)\int \Phi^{-1}(t)\mathbf{F}(t)\,dt$$

$$= \begin{pmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{pmatrix}\begin{pmatrix} \ln(\sin t) - 2t - 5\ln(\cos t) \\ -2\ln(\cos t) \end{pmatrix}$$

$$= \begin{pmatrix} \cos t\ln(\sin t) - 2t\cos t - 5\cos t\ln(\cos t) + 2\sin t\ln(\sin t) - 4t\sin t \\ \sin t\ln(\sin t) - 2t\sin t - \sin t\ln(\cos t) - 2\cos t\ln(\cos t) \end{pmatrix}.$$

A general solution to the nonhomogeneous system is

$$\mathbf{X}(t) = \mathbf{X}_h(t) + \mathbf{X}_p(t) =$$

$$\begin{pmatrix} \cos t\,(c_1 + \ln(\sin t) - 2t - 5\ln(\cos t)) + \sin t\,(2c_1 - 5c_2 + 2\ln(\sin t) - 4t) \\ \cos t\,(c_2 - 2\ln(\cos t)) + \sin t\,(c_1 - 2c_2 + \ln(\sin t) - 2t - \ln(\cos t)) \end{pmatrix}.$$

To find the solution that satisfies the initial conditions $x(\pi/4) = 5$ and $y(\pi/4) = 1$, we solve the system of equations

$$\begin{cases} x\left(\dfrac{\pi}{4}\right) = \dfrac{3}{2}c_1\sqrt{2} - \dfrac{5}{2}c_2\sqrt{2} + \dfrac{1}{2}\sqrt{2}\ln 2 - \dfrac{3}{4}\pi\sqrt{2} = 5 \\ y\left(\dfrac{\pi}{4}\right) = \dfrac{1}{2}c_1\sqrt{2} - \dfrac{1}{2}c_2\sqrt{2} + \dfrac{1}{2}\sqrt{2}\ln 2 - \dfrac{1}{4}\pi\sqrt{2} = 1 \end{cases}$$

for $c_1$ and $c_2$, which yields $c_1 = \frac{1}{2}\pi - 2\ln 2 \approx 0.184501966$ and $c_2 = -\sqrt{2} - \ln 2 \approx -2.107360743$. We use the values of $c_1$ and $c_2$ to graph $x(t)$ and $y(t)$ for $0 < t < \pi/2$ in Figure 6.18.

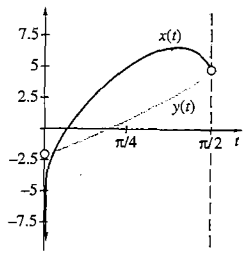



Figure 6.18

*Find $\lim_{t\to 0^+} x(t)$, $\lim_{t\to 0^+} y(t)$, $\lim_{t\to \pi/2^-} x(t)$, and $\lim_{t\to \pi/2^-} y(t)$ for the solution to the initial-value problem in Example 3.*



## *EXERCISES 6.5*

In Exercises 1–24, solve the system by undetermined coefficients or variation of parameters.

**1.** $\mathbf{X}' = \begin{pmatrix} -5 & 6 \\ 0 & -7 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 1 \\ t \end{pmatrix}$

**2.** $\mathbf{X}' = \begin{pmatrix} 3 & 8 \\ 2 & 9 \end{pmatrix}\mathbf{X} + \begin{pmatrix} t^2 \\ 1 \end{pmatrix}$

***3.** $\begin{cases} x' = -6x + t \\ y' = -x - 6y + t^2 \end{cases}$

**4.** $\mathbf{X}' = \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} e^{-4t} \\ e^{t} \end{pmatrix}$

**5.** $\mathbf{X}' = \begin{pmatrix} -3 & -4 \\ -2 & -10 \end{pmatrix}\mathbf{X} + \begin{pmatrix} e^{2t} \\ e^{11t} \end{pmatrix}$

**6.** $\mathbf{X}' = \begin{pmatrix} -9 & -9 \\ 4 & 3 \end{pmatrix}\mathbf{X} + \begin{pmatrix} e^{3t} \\ te^{3t} \end{pmatrix}$

***7.** $\begin{cases} x' = -3x + 7y + \cos t \\ y' = -x + 5y + e^{-4t} \end{cases}$

**8.** $\mathbf{X}' = \begin{pmatrix} -6 & -3 \\ 4 & 1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} te^{2t} \\ \sin 2t \end{pmatrix}$

**9.** $\mathbf{X}' = \begin{pmatrix} -6 & -4 \\ 4 & -6 \end{pmatrix}\mathbf{X} + \begin{pmatrix} e^{-6t} \\ e^{-6t} \end{pmatrix}$

**10.** $\begin{cases} x' = -3x + 2y + e^{-t}\sec 2t \\ y' = -10x + 5y + e^{-t}\csc 2t \end{cases}$

***11.** $\mathbf{X}' = \begin{pmatrix} 9 & 5 \\ -8 & -3 \end{pmatrix}\mathbf{X} + \begin{pmatrix} e^{3t}\sin 2t \\ e^{3t} \end{pmatrix}$

**12.** $\begin{cases} x' = 6x - 5y + e^{5t} \\ y' = x + 4y + e^{5t}\cos 2t \end{cases}$

**13.** $\begin{cases} x' = -y + \sec t \\ y' = x \end{cases}$

**14.** $\begin{cases} x' = -y + 4\tan t \\ y' = x \end{cases}$

***15.** $\begin{cases} x' = y \\ y' = -x - 2\cot t \end{cases}$

**16.** $\begin{cases} x' = -y \\ y' = x - 2\csc t \end{cases}$

**17.** $\mathbf{X}' = \begin{pmatrix} 0 & 0 & 4 \\ 2 & -2 & 5 \\ -3 & 4 & -1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} t \\ 1 \\ t^2 \end{pmatrix}$

**18.** $\begin{cases} x' = -2x - 2y - 2z + 1 \\ y' = -2y + z + t^2 \\ z' = -2y - 5z + t \end{cases}$

***19.** $\begin{cases} x' = -x + y - z + e^{-4t} \\ y' = 4x + y - 5z + te^{-2t} \\ z' = -x - 2y - 2z + t \end{cases}$

**20.** $\mathbf{X}' = \begin{pmatrix} 1 & 5 & -4 \\ -2 & -6 & 2 \\ 0 & 0 & 6 \end{pmatrix}\mathbf{X} + \begin{pmatrix} te^{4t} \\ 1 \\ e^{6t} \end{pmatrix}$

**21.** $\mathbf{X}' = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 0 & 2 \\ 1 & 3 & 1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 0 \\ t \\ e^{t} \end{pmatrix}$

**22.** $\mathbf{X}' = \begin{pmatrix} 6 & 0 & -5 \\ 3 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 1 \\ e^{-t} \\ t \end{pmatrix}$

***23.** $\mathbf{X}' = \begin{pmatrix} 3 & 2 & -3 \\ 1 & 1 & 1 \\ 0 & -4 & -4 \end{pmatrix}\mathbf{X} + \begin{pmatrix} e^{4t} \\ 0 \\ t \end{pmatrix}$

**24.** $\mathbf{X}' = \begin{pmatrix} -4 & 5 & -1 \\ -1 & 2 & 0 \\ -3 & -2 & 0 \end{pmatrix}\mathbf{X} + \begin{pmatrix} \sin t \\ e^{-4t} \\ 0 \end{pmatrix}$

In Exercises 25–30, solve the initial-value problem.

**25.** $\mathbf{X}' = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}\mathbf{X} + \begin{pmatrix} t \\ 0 \end{pmatrix}$, $\mathbf{X}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

**26.** $\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 0 \\ \sin t \end{pmatrix}$, $\mathbf{X}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

***27.** $\mathbf{X}' = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 0 \\ e^{t} \end{pmatrix}$, $\mathbf{X}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

**28.** $\mathbf{X}' = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} t \\ t^2 - 1 \end{pmatrix}$, $\mathbf{X}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

**29.** $\mathbf{X}' = \begin{pmatrix} 3 & 2 \\ -5 & 1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 0 \\ 10 \end{pmatrix}$, $\mathbf{X}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

**30.** $\mathbf{X}' = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$, $\mathbf{X}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

In Exercises 31–35, without solving each of the following initial-value problems, match each problem with the graph of its solution.

| Initial-value problem | Graph of $x(t)$ (in dark blue) and $y(t)$ (in light blue) | Graph of $\begin{cases} x(t) \\ y(t) \end{cases}$ |
|---|---|---|

31. $\begin{cases} x' = 2x + y \\ y' = -8x - 2y \\ x(0) = 0, y(0) = 1 \end{cases}$

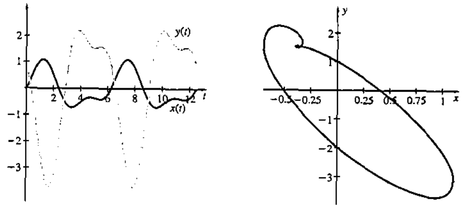
32. $\begin{cases} x' = 2x + y + 1 \\ y' = -8x - 2y + t \\ x(0) = 0, y(0) = 1 \end{cases}$

33. $\begin{cases} x' = 2x + y + \sin t \\ y' = -8x - 2y \\ x(0) = 0, y(0) = 1 \end{cases}$

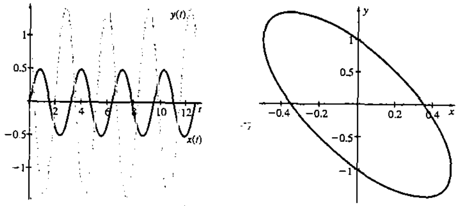
34. $\begin{cases} x' = 2x + y + \sin 2t \\ y' = -8x - 2y \\ x(0) = 0, y(0) = 1 \end{cases}$

35. $\begin{cases} x' = 2x + y + \sin 3t \\ y' = -8x - 2y \\ x(0) = 0, y(0) = 1 \end{cases}$

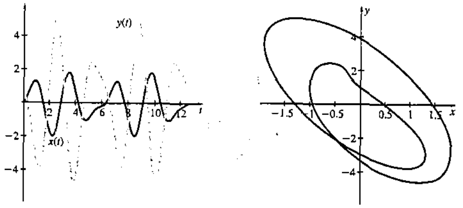
(a)



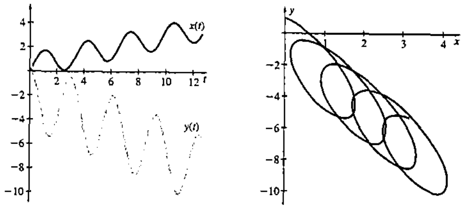
(b)



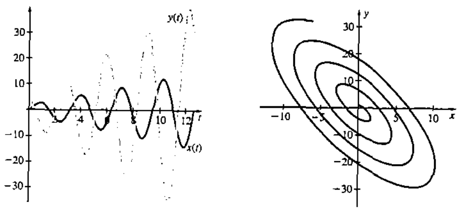
(c)



(d)



(e)





*36. Let $\mathbf{X} = \mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \mathbf{F}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}, \text{ and}$$

$\Phi(t)$ be a fundamental matrix of the system $\mathbf{X}' = \mathbf{A}\mathbf{X}$. **(a)** Show that if $\mathbf{X}_1$ and $\mathbf{X}_2$ are any two solutions of $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t)$, then $\mathbf{X}_1 - \mathbf{X}_2$ is a solution to $\mathbf{X}' = \mathbf{A}\mathbf{X}$. **(b)** Show that if $\mathbf{X}_P$ is a particular solution to $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t)$ and $\mathbf{X}_{\text{any}}$ is any solution to $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t)$, then there is an $n \times 1$ constant vector $\mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ so that $\mathbf{X}_{\text{any}} = \Phi(t)\mathbf{C} + \mathbf{X}_P$.

*37. Use a computer algebra system to solve each of the following initial-value problems. In each case, graph $x(t)$, $y(t)$, and the parametric equations $\begin{cases} x(t) \\ y(t) \end{cases}$ for the indicated values of $t$.

**(a)** $\begin{cases} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -7 & -3 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ te^{-t} \end{pmatrix}; \ 0 \le t \le 10 \\ x(0) = 0, \quad y(0) = 1 \end{cases}$

**(b)** $\begin{cases} \mathbf{X}' = \begin{pmatrix} 8 & 10 \\ -7 & -9 \end{pmatrix}\mathbf{X} + \begin{pmatrix} t^2 e^{-2t} \\ -te^t \end{pmatrix}; \ 0 \le t \le 3 \\ x(0) = 1, \quad y(0) = 0 \end{cases}$

**(c)** $\begin{cases} \dfrac{dx}{dt} = 2x - 5y + \sin 4t \\ \dfrac{dy}{dt} = 4x - 2y - te^{-t}; \ 0 \le t \le 2\pi \\ x(0) = 1, y(0) = 1 \end{cases}$

38. **(a)** Show that

$$\begin{cases} x = -\frac{6}{13}e^{-4t} + \frac{6}{13}e^{9t} - \frac{3}{13}e^{9t}\int_0^t e^{-9s}f(s)\,ds + \\ \qquad \frac{16}{13}e^{-4t}\int_0^t e^{4s}f(s)\,ds \\ y = -\frac{3}{13}e^{-4t} + \frac{16}{13}e^{9t} - \frac{8}{13}e^{9t}\int_0^t e^{-9s}f(s)\,ds + \\ \qquad \frac{8}{13}e^{-4t}\int_0^t e^{4s}f(s)\,ds \end{cases}$$

is the solution to the initial-value problem

$$\begin{cases} \dfrac{dx}{dt} = -7x + 6y + f(t) \\ \dfrac{dy}{dt} = -8x + 12y \\ x(0) = 0, y(0) = 1 \end{cases}.$$

**(b)** Is it possible to choose $f(t)$ so that $\lim_{t\to\infty} x(t) = 0$? So that $\lim_{t\to\infty} y(t) = 0$? So that both $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = 0$? Why or why not? Provide evidence of your results by graphing various solutions.

In Exercises 39–41, graphically compare solutions to each nonhomogeneous system and the corresponding homogeneous system.

39. $\mathbf{X}'(t) = \begin{pmatrix} -5 & 6 \\ 0 & -7 \end{pmatrix}\mathbf{X}(t) + \begin{pmatrix} 1 \\ t \end{pmatrix}, \mathbf{X}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

40. $\begin{cases} x'(t) = -6x(t) + t \\ y'(t) = -x(t) - 6y(t) + t^2 \\ x(0) = 0, y(0) = 1 \end{cases}$

41. $\mathbf{X}'(t) = \begin{pmatrix} 9 & 5 \\ -8 & -3 \end{pmatrix}\mathbf{X}(t) + \begin{pmatrix} e^{3t}\sin 2t \\ e^{3t} \end{pmatrix}, \mathbf{X}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

## 6.6 Phase Portraits

- Real Distinct Eigenvalues
- Repeated Eigenvalues
- Complex Conjugate Eigenvalues
- Stability

In this section, we consider solutions of

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases}$$

from a geometric point of view. Notice that (0, 0) is a solution of the system of equations obtained by setting the right sides of the differential equations equal to zero,

$$\begin{cases} ax + by = 0 \\ cx + dy = 0 \end{cases}$$

for all values of $a$, $b$, $c$, and $d$. In fact, if $|\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, then the only solution of this system is the origin. *Note:* We make the assumption that $|\mathbf{A}| \neq 0$ throughout this section. We call (0, 0) an **equilibrium point** of the system of ODEs because (0, 0) satisfies the corresponding system of algebraic equations. In this section, we investigate the properties of the equilibrium point based on the eigenvalues and corresponding eigenvectors of the coefficient matrix **A** by viewing the **phase portrait,** a picture of a set of solutions (trajectories) and equilibrium point(s) of the system. As we saw in Section 6.4, solutions of these systems vary greatly.

Before we move on to nonlinear systems, we first investigate properties of systems of the form

$$\begin{cases} x' = ax + by \\ y' = cx + dy' \end{cases}$$

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$, which has only the one equilibrium point (0, 0). We have solved many systems of this type using the eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and have seen that the solutions vary greatly.

The behavior of the solutions of this system and the classification of the equilibrium point depend on the eigenvalues and corresponding eigenvectors of the system. We more thoroughly investigate the cases that can arise in solving this system by considering the classification of the equilibrium point (0, 0) based on the eigenvalues and corresponding eigenvectors of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

## Real Distinct Eigenvalues

Suppose that $\lambda_1$ and $\lambda_2$ are real eigenvalues of $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $\lambda_2 < \lambda_1$ with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. Then, a general solution of $\mathbf{X}' = \mathbf{AX}$ is

$$\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t} = e^{\lambda_1 t}[c_1\mathbf{v}_1 + c_2\mathbf{v}_2e^{(\lambda_2-\lambda_1)t}].$$

1. If both eigenvalues are negative, suppose that $\lambda_2 < \lambda_1 < 0$ so $\lambda_2 - \lambda_1 < 0$. This means that $e^{(\lambda_2-\lambda_1)t}$ is very small for large values of $t$, so $\mathbf{X}(t) \approx c_1\mathbf{v}_1e^{\lambda_1 t}$ is small for large values of $t$. If $c_1 \neq 0$, then $\lim_{t\to\infty} \mathbf{X}(t) = \mathbf{0}$ along the line through (0, 0) in the direction of $\mathbf{v}_1$. If $c_1 = 0$, then $\mathbf{X}(t) = c_2\mathbf{v}_2e^{\lambda_2 t}$. Again, because $\lambda_2 < 0$, $\lim_{t\to\infty} \mathbf{X}(t) = \mathbf{0}$ along the line through (0, 0) in the direction of $\mathbf{v}_2$. In this case, (0, 0) is a **stable improper node.**



2. If both eigenvalues are positive, suppose that $0 < \lambda_2 < \lambda_1$. Then $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ both become unbounded as $t$ increases. If $c_1 \neq 0$, then $\mathbf{X}(t)$ becomes unbounded along the line through (0, 0) in the direction of $\mathbf{v}_1$. If $c_1 = 0$, then $\mathbf{X}(t)$ becomes unbounded along the line through (0, 0) in the direction given by $\mathbf{v}_2$. In this case, (0, 0) is an **unstable improper node.**
3. If the eigenvalues have opposite signs, suppose that $\lambda_2 < 0 < \lambda_1$ and $c_1 \neq 0$. Then, $\mathbf{X}(t)$ becomes unbounded along the line through (0, 0) in the direction of $\mathbf{v}_1$ as it did in (2). However, if $c_1 = 0$, then due to the fact that $\lambda_2 < 0$, $\lim_{t\to\infty} \mathbf{X}(t) = \mathbf{0}$ along the line through (0, 0) determined by $\mathbf{v}_2$. If the initial point $\mathbf{X}(0)$ is not on the line through (0, 0) determined by $\mathbf{v}_2$, then the line given by $\mathbf{v}_1$ is an asymptote for the solution. We say that (0, 0) is a **saddle point** in this case.

**Example 1**

Classify the equilibrium point (0, 0) in the systems **(a)** $\begin{cases} x' = x \\ y' = 2y \end{cases}$ and **(b)** $\begin{cases} x' = y \\ y' = x \end{cases}$.

***Solution*** (a) The coefficient matrix in this case is $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, so we identify the eigenvalues of **A** by solving $\begin{vmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) = 0$. The eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ are real and unequal, so we classify (0, 0) as an unstable improper node. Substitution of $\lambda_1 = 1$ into $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ indicates that $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector associated with $\lambda_1 = 1$. Similarly, we find that $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_2 = 2$. Therefore, a general solution is

$$\mathbf{X}(t) = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{2t} = \begin{pmatrix} c_1e^{t} \\ c_2e^{2t} \end{pmatrix}.$$

Notice that both components of the solution move away from zero as $t \to \infty$ while both approach zero as $t \to -\infty$.

There is a relationship between solutions of the system and the **eigenlines,** lines through the equilibrium point in the direction of the eigenvectors. In this case, the eigenlines corresponding to $\mathbf{v}_1$ and $\mathbf{v}_2$ are the $y$-axis and the $x$-axis, respectively. To see the relationship between solutions and the eigenlines, we find trajectories of the system by writing the system as the first-order equation

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2y}{x}.$$

Separating variables, integrating, and simplifying gives us

$$y = Cx^2,$$

where $C$ is an arbitrary constant, so solution curves are parabolas with vertices at the origin. (We can obtain the same result by eliminating the parameter from the solution for $\mathbf{X}(t)$ found earlier.) To sketch the phase portrait, we begin by graphing the eigenlines. Because each is associated with a positive eigenvalue, we place arrows directed away from the origin on each eigenline. Next, we graph several parabolas (Figure 6.19). These trajectories are also directed away from the origin because of the positive eigenvalues. We can also graph several members of this family along with the direction field of the system, as done in Figure 6.20, to view the orientation of trajectories. Based on our earlier observation, solutions become tangent to one eigenline, the $x$-axis, as $t \to -\infty$ (as we move against the direction of the arrows) and become parallel to the other eigenline, the $y$-axis, as $t \to \infty$. (We will find that this behavior holds in other systems that include a node.) Notice that the trajectories are tangent to the eigenline associated with the smaller eigenvalue (in absolute value). They become parallel to the eigenline associated with the larger eigenvalue (in absolute value).

**(b)** By solving

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0.$$

we find that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$. In this case, one eigenvalue is positive and one is negative, so we classify $(0, 0)$ as a saddle point. Note that the eigenvectors corresponding to $\lambda_1 = 1$ and $\lambda_2 = -1$ are $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, respectively, so the eigenlines associated with $\mathbf{v}_1$ and $\mathbf{v}_2$ are $y = x$ and $y = -x$, respectively. A general solution is

$$\mathbf{X}(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t} = \begin{pmatrix} c_1e^t + c_2e^{-t} \\ c_1e^t - c_2e^{-t} \end{pmatrix}.$$

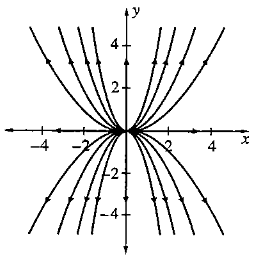



**Figure 6.19** Phase portrait of $\begin{cases} x' = x \\ y' = 2y \end{cases}$

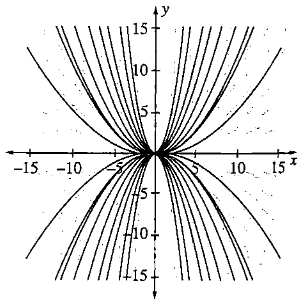



**Figure 6.20** Solutions with direction field of $\begin{cases} x' = x \\ y' = 2y \end{cases}$



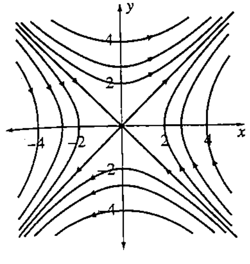



**Figure 6.21** Phase portrait of $\begin{cases} x' = y \\ y' = x \end{cases}$

To determine the behavior of solutions as $t \to \infty$, notice that if $c_1 \neq 0$, then $\mathbf{X}(t)$ becomes like $c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t$ because $\lim_{t\to\infty} e^{-t} = 0$. In other words, solutions become asymptotic to the eigenline associated with the positive eigenvalue $y = x$ as $t \to -\infty$. In a similar manner, we determine the behavior of solutions as $t \to -\infty$. In this case, solutions are dominated by the term $c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t}$ (if $c_2 \neq 0$) because $\lim_{t\to-\infty} e^t = 0$. Therefore, solutions approach the eigenline associated with the negative eigenvalue $y = -x$ as $t \to -\infty$. Also, if $c_2 = 0$ and $c_1 \neq 0$, then the solution approaches the origin along the eigenline $y = x$ as $t \to \infty$. When we solve

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{x}{y},$$

we obtain $x^2 - y^2 = C$, which represents a family of hyperbolas (for $C \neq 0$) and the lines $y = \pm x$ (for $C = 0$). When we sketch the phase portrait in Figure 6.21, we begin by drawing the eigenlines. The line $y = x$ is associated with a positive eigenvalue so we place arrows directed away from the origin on this line. In contrast, $y = -x$ corresponds to a negative eigenvalue, so the arrows are directed toward the origin on this line. Next, we sketch hyperbolas centered at the origin and assign orientation based on that of the eigenlines (asymptotes of the hyperbolas). Observe that all solutions approach the eigenlines. They become asymptotic to the eigenline $y = x$ associated with $\lambda_1 = 1$ (the positive eigenvalue) as $t \to \infty$ and become asymptotic to the eigenline $y = -x$ associated with $\lambda_2 = -1$ (the negative eigenvalue) as $t \to -\infty$. This is a property linked to other saddle points we encounter.

**Note:** Eigenlines are trajectories of the system because each can be obtained by setting one of the arbitrary constants $c_1$ or $c_2$ in the general solution equal to zero. Therefore, other trajectories cannot intersect the eigenlines because doing so would contradict the uniqueness of solutions of IVPs involving a first-order linear homogeneous system of ODEs.

### Example 2

Classify the equilibrium point $(0, 0)$ of the systems of differential equations and sketch the phase portrait: **(a)** $\begin{cases} x' = 5x + 3y \\ y' = -4x - 3y \end{cases}$; **(b)** $\begin{cases} x' = x - 2y \\ y' = 3x - 4y \end{cases}$; **(c)** $\begin{cases} x' = -x - 2y \\ y' = 3x + 4y \end{cases}$

***Solution*** (a) The eigenvalues are found by solving

$$\begin{vmatrix} 5-\lambda & 3 \\ -4 & -3-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0.$$

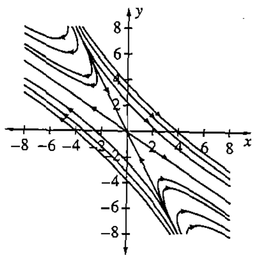



Figure 6.22 Phase portrait for Example 2, solution (a).

Because the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$ have opposite signs, (0, 0) is a saddle point. Note that the eigenvectors corresponding to $\lambda_1$ and $\lambda_2$ are $\mathbf{v}_1 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, respectively. To obtain an idea about the behavior of solutions of this system, we find equations of the eigenlines. One way to find the equation of an eigenline is to determine a point in addition to (0, 0) on the line. For example, for $\mathbf{v}_1 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, another point is $(-3, 2)$, the terminal point of the vector, so the eigenline corresponding to $\mathbf{v}_1$ has slope $m = (2-0)/(-3-0) = -2/3$ and equation $y = -2x/3$. Similarly, the eigenline associated with $\mathbf{v}_2$ is $y = -2x$. We sketch the phase portrait in Figure 6.22. First, we draw the eigenlines, $y = -2x/3$ and $y = -2x$, with arrows directed toward and away from the origin, respectively. Then, we draw curves using the eigenlines as asymptotes and follow the directions associated with the eigenlines to assign the proper orientation to the trajectories. As mentioned in a previous example involving a saddle point, solutions become asymptotic to the eigenline associated with the positive eigenvalue as $t \to \infty$ and they become asymptotic to that associated with the negative eigenvalue as $t \to -\infty$.

For another interpretation of the behavior of solutions as $t \to \infty$, notice that in a general solution of this system, $\mathbf{X}(t) = c_1\begin{pmatrix} -3 \\ 2 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-t}$, if $c_1 \neq 0$, then $\mathbf{X}(t)$ becomes like $c_1\begin{pmatrix} -3 \\ 2 \end{pmatrix}e^{3t}$ because $\lim\limits_{t\to\infty} e^{-t} = 0$. In other words, solutions become asymptotic to the eigenline associated with $\mathbf{v}_1 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ as $t \to \infty$. In a similar manner, we can determine the behavior of solutions as $t \to -\infty$. In this case, solutions are dominated by the term $c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-t}$ (if $c_2 \neq 0$) because $\lim\limits_{t\to-\infty} e^{3t} = 0$. Therefore, solutions approach the eigenline associated with $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (or $\lambda_2 = -1$), $y = -2x$, as $t \to -\infty$. Notice that the behavior of the solutions agrees with what we observed in our earlier discussion.

**(b)** Because the characteristic equation is

$$\begin{vmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2) = 0,$$

the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$ are distinct and both negative; (0, 0) is a stable improper node. In this case, corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, so the solutions approach (0, 0) along the lines through (0, 0) in the direction of these vectors, $y = x$ and $y = \frac{3}{2}x$. Based on our earlier findings concerning nodes, we know that trajectories are tangent to one eigenline and become parallel to the other. To better understand the shape of the phase portrait, we convert the system to the first-order equation



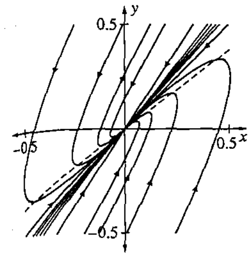



Figure 6.23 Phase portrait for Example 2, solution (b).

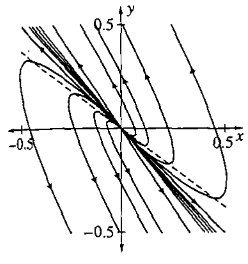



Figure 6.24 Phase portrait for Example 2, solution (c).

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3x-4y}{x-2y},$$

and we observe that trajectories have horizontal tangent lines, where $dy/dx = 0$ (when $3x - 4y = 0$). In other words, trajectories cross the line $y = 3x/4$ horizontally. This fact and the vectors in the direction field indicate that trajectories approach the eigenline $y = x$ as $t \to \infty$ and are parallel to $y = 3x/2$ as $t \to -\infty$. (Remember that trajectories cannot cross eigenlines.) We also recall from Example 1 that trajectories are tangent to the eigenline associated with the smaller eigenvalue (in absolute value), $y = x$, and becomes parallel (as $t \to \pm\infty$) to the eigenline associated with the larger eigenvalue (in absolute value), $y = 3x/2$. We graph the phase portrait in Figure 6.23. The line $y = 3x/4$ is dashed. Both eigenlines have associated with them arrows directed toward the origin because both eigenvalues are negative.

**(c)** Because

$$\begin{vmatrix} -1-\lambda & -2 \\ 3 & 4-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1) = 0,$$

the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 1$ are real and unequal. Therefore, (0, 0) is an improper node. Note that corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, respectively. The solutions become unbounded along the lines through (0, 0) determined by these vectors, $y = -\frac{3}{2}x$ and $y = -x$. In this case, the eigenlines are $y = -3x/2$ and $y = -x$. We graph these lines in Figure 6.24 and place arrows on them directed away from the origin due to the positive eigenvalues. Based on our earlier findings, trajectories are tangent to $y = -x$ as $t \to -\infty$ and become parallel to $y = -3x/2$ as $t \to \infty$. Following a method similar to that in **(b)**, we see that trajectories cross the line $y = -3x/4$ horizontally (this line is dashed in Figure 6.24).

## Repeated Eigenvalues

We recall from our previous experience with repeated eigenvalues of a $2 \times 2$ system that the eigenvalue can have two linearly independent eigenvectors associated with it or only one (linearly independent) eigenvector associated with it. We investigate the behavior of solutions in the case of repeated eigenvalues by considering both of these possibilities.

1. If the eigenvalue $\lambda = \lambda_1 = \lambda_2$ has two corresponding linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, a general solution is

$$\mathbf{X}(t) = c_1\mathbf{v}_1e^{\lambda t} + c_2\mathbf{v}_2e^{\lambda t} = (c_1\mathbf{v}_1 + c_2\mathbf{v}_2)e^{\lambda t}.$$

If $\lambda > 0$, then $\mathbf{X}(t)$ becomes unbounded along the lines through (0, 0) determined by the vectors $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, where $c_1$ and $c_2$ are arbitrary constants. In this case, we call the equilibrium point an **unstable star node.** However, if $\lambda < 0$, then $\mathbf{X}(t)$ approaches (0, 0) along these lines, and we call (0, 0) a **stable star node.** Note that the name "star" was selected due to the shape of the solutions.

2. If $\lambda = \lambda_1 = \lambda_2$ has only one corresponding (linearly independent) eigenvector $\mathbf{v}_1$, a general solution is

$$\mathbf{X}(t) = c_1\mathbf{v}_1 e^{\lambda t} + c_2[\mathbf{v}_1 t + \mathbf{w}_2]e^{\lambda t} = (c_1\mathbf{v}_1 + c_2\mathbf{w}_2)e^{\lambda t} + c_2\mathbf{v}_1 t e^{\lambda t}$$

where $(\mathbf{A} - \lambda\mathbf{I})\mathbf{w}_2 = \mathbf{v}_1$. If we write this solution as

$$\mathbf{X}(t) = te^{\lambda t}\left[\frac{1}{t}(c_1\mathbf{v}_1 + c_2\mathbf{w}_2) + c_2\mathbf{v}_1\right],$$

we can more easily investigate the behavior of this solution. If $\lambda < 0$, then $\lim_{t\to\infty} te^{\lambda t} = 0$ and $\lim_{t\to\infty}\left[\frac{1}{t}(c_1\mathbf{v}_1 + c_2\mathbf{w}_2) + c_2\mathbf{v}_1\right] = c_2\mathbf{v}_1$. The solutions approach (0, 0) along the line through (0, 0) determined by $\mathbf{v}_1$, and we call (0, 0) a **stable deficient node.** If $\lambda > 0$, the solutions become unbounded along this line, and we say that (0, 0) is an **unstable deficient node.**

**Example 3**

Classify the equilibrium point (0, 0) in the systems: **(a)** $\begin{cases} x' = x + 9y \\ y' = -x - 5y \end{cases}$; **(b)** $\begin{cases} x' = 2x \\ y' = 2y \end{cases}$

***Solution*** (a) The eigenvalues are found by solving

$$\begin{vmatrix} 1-\lambda & 9 \\ -1 & -5-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0.$$

Hence, $\lambda_1 = \lambda_2 = -2$. In this case, an eigenvector $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ satisfies $\begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which is equivalent to $\begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so there is only one corresponding (linearly independent) eigenvector $\mathbf{v}_1 = \begin{pmatrix} -3y_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}y_1$. Because $\lambda = -2 < 0$, (0, 0) is a degenerate stable node. In this case, the eigenline is $y = -x/3$. We graph this line in Figure 6.25 and direct the arrows toward the origin because of the negative eigenvalue. Next, we sketch trajectories that become tangent to the eigenline as $t \to \infty$ and associate with each arrows directed toward the origin.

(b) Solving the characteristic equation

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = 0,$$

we have $\lambda_1 = \lambda_2 = 2$. However, because an eigenvector $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ satisfies the system $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, any nonzero choice of $\mathbf{v}_1$ is an eigenvector. If we select two linearly independent vectors such as $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we obtain two linearly independent eigenvectors corresponding to $\lambda_1 = \lambda_2 = 2$. (*Note:* The choice of these two vectors does not change the value of the solution, because of the form of the general solution in this case.) Because $\lambda = 2 > 0$, we classify (0, 0) as a degenerate unstable star node. A general solution is $\mathbf{X}(t) = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{2t} = \begin{pmatrix} c_1e^{2t} \\ c_2e^{2t} \end{pmatrix}$, so if we eliminate the parameter, we obtain $y = c_2x/c_1$. Therefore, the trajectories of this system are lines passing through the origin. In Figure 6.26, we graph the eigenlines (the $x$- and $y$-axes) as well as several trajectories. Because of the positive eigenvalue, we associate with each an arrow directed away from the origin.

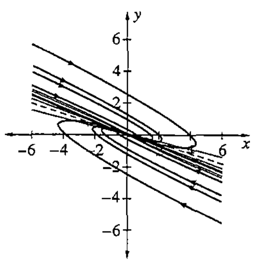



**Figure 6.25** Phase portrait for Example 3, solution (a).

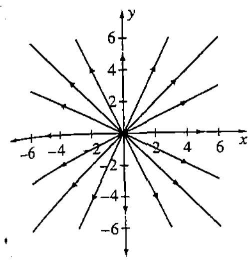



**Figure 6.26** Phase portrait for Example 3, solution (b).



## Complex Conjugate Eigenvalues

We have seen that if the eigenvalues of the system of differential equations are $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$ with corresponding eigenvectors $\mathbf{v}_1 = \mathbf{a} + \mathbf{b}i$ and $\mathbf{v}_2 = \mathbf{a} - \mathbf{b}i$, then two linearly independent solutions of the system are

$$\mathbf{X}_1(t) = e^{\alpha t}(\cos\beta t\,\mathbf{a} - \sin\beta t\,\mathbf{b}) \quad \text{and} \quad \mathbf{X}_2(t) = e^{\alpha t}(\cos\beta t\,\mathbf{b} + \sin\beta t\,\mathbf{a}).$$

A general solution is $\mathbf{X}(t) = c_1\mathbf{X}_1(t) + c_2\mathbf{X}_2(t)$, so there are constants $A_1$, $A_2$, $B_1$, and $B_2$ such that $x$ and $y$ are given by

$$\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A_1e^{\alpha t}\cos\beta t + A_2e^{\alpha t}\sin\beta t \\ B_1e^{\alpha t}\cos\beta t + B_2e^{\alpha t}\sin\beta t \end{pmatrix}.$$

1. If $\alpha = 0$, a general solution is

$$\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A_1\cos\beta t + A_2\sin\beta t \\ B_1\cos\beta t + B_2\sin\beta t \end{pmatrix}.$$

Both $x$ and $y$ are periodic. In fact, if $A_2 = B_1 = 0$, then

$$\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A_1\cos\beta t \\ B_2\sin\beta t \end{pmatrix}.$$

In rectangular coordinates this solution is

$$\frac{x^2}{A_1^2} + \frac{y^2}{B_2^2} = 1$$

where the graph is either a circle or an ellipse centered at (0, 0) depending on the value of $A_1$ and $B_2$. Hence, (0, 0) is classified as a **center.** Note that the motion around these circles or ellipses is either clockwise or counterclockwise for all solutions.

2. If $\alpha \neq 0$, $e^{\alpha t}$ is present in the solution. The $e^{\alpha t}$ term causes the solution to spiral around the equilibrium point. If $\alpha > 0$, the solution spirals away from (0, 0), so we classify (0, 0) as an **unstable spiral** point. If $\alpha < 0$, the solution spirals toward (0, 0), so we say that (0, 0) is a **stable spiral** point.

**Example 4**

Classify the equilibrium point (0, 0) in each of the following systems: **(a)** $\begin{cases} x' = -y \\ y' = x \end{cases}$; **(b)** $\begin{cases} x' = x - 5y \\ y' = x - 3y \end{cases}$

**Solution** (a) The eigenvalues are found by solving

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0.$$

Hence, $\lambda = \pm i$. Because these eigenvalues have a zero real part (that is, they are purely imaginary), (0, 0) is a center. We can view solutions of this system by writing the system as a first-order equation with $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-x}{y}$. Separating variables, we have $y\,dy = -x\,dx$, so that integration yields $y^2/2 = -x^2/2 + C$, or $x^2 + y^2 = K$ (where $K = 2C$). Therefore, solutions of the system are circles (if $K > 0$) centered at the origin. The solution is (0, 0) if $K = 0$. Several solutions are graphed in Figure 6.27(a). The arrows indicate that the solutions move counterclockwise around (0, 0). We see this by observing the equations in the system, $x' = -y$ and $y' = x$. In the first quadrant where $x > 0$ and $y > 0$, these equations indicate that $x' < 0$ and $y' > 0$. Therefore, on solutions in the first quadrant, $x$ decreases while $y$ increases. Similarly, in the second quadrant where $x < 0$ and $y > 0$, $x' < 0$ and $y' < 0$, so $x$ and $y$ both decrease in this quadrant. (Similar observations are made in the other two quadrants.)

(b) Because the characteristic equation is

$$\begin{vmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2 = 0,$$

the eigenvalues are $\lambda = (-2 \pm \sqrt{4-8})/2 = -1 \pm i$. Thus, (0, 0) is a stable spiral point because $\alpha = -1 < 0$.

To better understand the behavior of solutions, consider a general solution of this system,

$$\mathbf{X}(t) = e^{-t}\begin{pmatrix} -c_1 \cos t - 2c_1 \sin t + 5c_2 \sin t \\ c_1 \sin t + c_2 \cos t - 2c_2 \sin t \end{pmatrix}.$$

The trigonometric functions cause solutions to rotate about the origin, while $e^{-t}$ forces both components to zero as $t \to \infty$. In Figure 6.27(b), we graph the phase portrait. Solutions spiral in toward the origin.

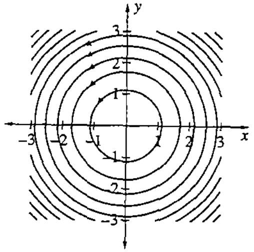



**Figure 6.27(a)** Phase portrait for Example 4, solution (a).

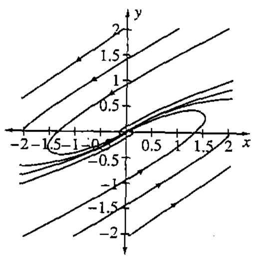



**Figure 6.27(b)** Phase portrait for Example 4, solution (b).

**Note:** In the case of centers and spiral points, the direction field may be particularly useful in determining the shape of solutions as well as their orientation.



## Stability

We now understand the behavior of solutions of systems of linear equations based on the eigenvalues of the coefficient matrix. In addition, we would like to comment on the stability of the equilibrium point of the system using the same information. Looking back, we classified equilibrium points as saddle points, nodes, spirals, or centers. In the case of saddle points, some nodes, and some spiral points in which at least one eigenvalue is positive or has a positive real part, trajectories move away from the equilibrium point as $t \to \infty$. (*Note:* In the case of a saddle point, all solutions except the eigenline associated with the negative eigenvalue move away from the equilibrium point as $t \to \infty$.) Each of these classes of equilibrium points is **unstable.** Then, when we consider nodes and spiral points in which both eigenvalues are negative or have a negative real part, solutions converge to the equilibrium point as $t \to \infty$. These equilibrium points are *asymptotically stable* because of this convergence. Finally, centers are *stable* if a solution that starts sufficiently close to the equilibrium point remains close to the equilibrium point. *Note:* An equilibrium point that is not stable is considered *unstable.* We summarize our findings in Table 6.1.

**TABLE 6.1 Classification of Equilibrium Point in Linear System**

| Eigenvalues | Geometry | Stability |
|---|---|---|
| $\lambda_1, \lambda_2$ real; $\lambda_1 > \lambda_2 > 0$ | Improper node | Unstable |
| $\lambda_1, \lambda_2$ real; $\lambda_1 = \lambda_2 > 0$; 1 eigenvector | Deficient node | Unstable |
| $\lambda_1, \lambda_2$ real; $\lambda_1 = \lambda_2 > 0$; 2 eigenvectors | Star node | Unstable |
| $\lambda_1, \lambda_2$ real; $\lambda_2 < \lambda_1 < 0$ | Improper node | Asymptotically stable |
| $\lambda_1, \lambda_2$ real; $\lambda_1 = \lambda_2 < 0$; 1 eigenvector | Deficient node | Asymptotically stable |
| $\lambda_1, \lambda_2$ real; $\lambda_1 = \lambda_2 < 0$; 2 eigenvectors | Star node | Asymptotically stable |
| $\lambda_1, \lambda_2$ real; $\lambda_2 < 0 < \lambda_1$ | Saddle point | Unstable |
| $\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$, $\beta \neq 0$, $\alpha > 0$ | Spiral point | Unstable |
| $\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$, $\beta \neq 0$, $\alpha < 0$ | Spiral point | Asymptotically stable |
| $\lambda_1 = \beta i$, $\lambda_2 = -\beta i$, $\beta \neq 0$ | Center | Stable |

To better understand the meaning of the terms *unstable, asymptotically stable,* and *stable,* we take another look at graphs of solutions to several systems in the example problems. Instead of graphing these solutions in the phase plane, however, we graph them in the *tx*- and *ty*-planes. Consider the system in Example 2 that included a saddle point with general solution $\mathbf{X}(t) = c_1\begin{pmatrix} -3 \\ 2 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-t}$. When we graph $x(t) = -3c_1e^{3t} + c_2e^{-t}$ and $y(t) = 2c_1e^{3t} - 2c_2e^{-t}$ for various values of $c_1$ and $c_2$ in

Figures 6.28(a) and 6.28(b), we see that all solutions diverge from $x(t) = 0$ and $y(t) = 0$, which are the coordinates of the equilibrium point of this system. We therefore call the equilibrium point in this case *unstable*. In contrast, in Figures 6.29(a) and 6.29(b), when we graph $x(t) = e^{-t}(-c_1 \cos t - 2c_1 \sin t + 5c_2 \sin t)$ and $y(t) = e^{-t}(c_1 \sin t + c_2 \cos t - 2c_2 \sin t)$, solutions in Example 4(b) that involved a stable spiral point, we see that all solutions asymptotically approach $x(t) = 0$ and $y(t) = 0$. Therefore, we classify the equilibrium point in a system possessing this property as *asymptotically stable*. Finally, when we graph solutions to the system in Example 4(a) containing the center $x(t) = c_1 \cos t + c_2 \sin t$ and $y(t) = -c_1 \sin t + c_2 \cos t$ in Figures 6.30(a) and 6.30(b), we notice that solutions neither converge to nor diverge from $x(t) = 0$ and $y(t) = 0$. Hence, the equilibrium point is *stable*.

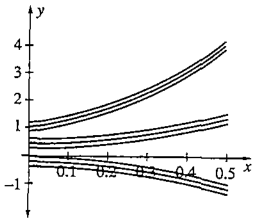



**Figure 6.28(a)** Graphs of $x(t) = -3c_1e^{3t} + c_2e^{-t}$.

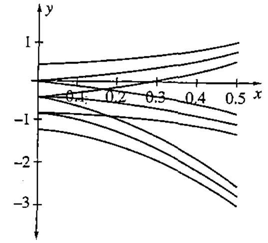



**Figure 6.28(b)** Graphs of $y(t) = 2c_1e^{3t} - 2c_2e^{-t}$.

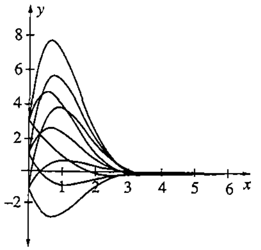



**Figure 6.29(a)** Graphs of $x(t) = e^{-t}(-c_1 \cos t - 2c_1 \sin t + 5c_2 \sin t)$.

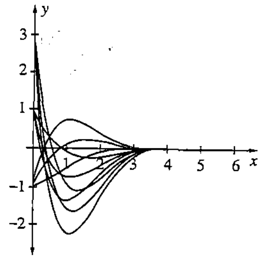



**Figure 6.29(b)** Graphs of $y(t) = e^{-t}(c_1 \sin t + c_2 \cos t - 2c_2 \sin t)$.

(a)

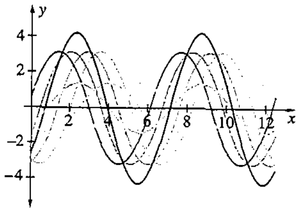



**Figure 6.30(a)** Graphs of $x(t) = c_1 \cos t + c_2 \sin t$.

(b)

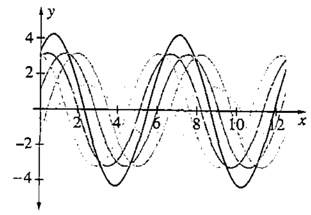



**Figure 6.30(b)** Graphs of $y(t) = -c_1 \sin t + c_2 \cos t$.



## EXERCISES 6.6

In Exercises 1–12, classify the equilibrium point (0, 0) of the system.

1. $\begin{cases} x' = -5x - 4y \\ y' = -9x - 5y \end{cases}$

2. $\begin{cases} x' = -x - 7y \\ y' = -3x + 3y \end{cases}$

*3. $\begin{cases} x' = 8x - 4y \\ y' = 9x - 4y \end{cases}$

4. $\begin{cases} x' = 4x - 2y \\ y' = -6x + 8y \end{cases}$

5. $\begin{cases} x' = -10x - 2y \\ y' = -2x - 10y \end{cases}$

6. $\begin{cases} x' = 6y \\ y' = -3x - 9y \end{cases}$

*7. $\begin{cases} x' = -x \\ y' = -y \end{cases}$

8. $\begin{cases} x' = 3x \\ y' = 3y \end{cases}$

9. $\begin{cases} x' = -9y \\ y' = 6x + 4y \end{cases}$

10. $\begin{cases} x' = -8x + 5y \\ y' = -2x - 2y \end{cases}$

*11. $\begin{cases} x' = -3x + 5y \\ y' = -10x + 3y \end{cases}$

12. $\begin{cases} x' = -4x - 4y \\ y' = 8x \end{cases}$

13. **(Linear Systems with Zero Eigenvalues)**
    (a) Show that the eigenvalues of the system $\begin{cases} x' = -2x \\ y' = -4x \end{cases}$ are $\lambda_1 = 0$ and $\lambda_2 = -2$.
    (b) Show that all points on the $y$-axis are equilibrium points.
    (c) By solving $dy/dx = -4x/-2x = 2$, show that the trajectories are the lines $y = 2x + C$, where $C$ is arbitrary.
    (d) Show that a general solution of the system is $\mathbf{X}(t) = \begin{pmatrix} c_2e^{-2t} \\ c_1 + 2c_2e^{-2t} \end{pmatrix}$.
    (e) Find $\lim_{t\to\infty} \mathbf{X}(t)$.
    (f) Use the information in (c) and (e) to sketch the phase portrait of the system.

Follow steps similar to those in Exercise 13 to sketch the phase portrait of each of the following systems.

14. $\begin{cases} x' = -2x - y \\ y' = -4x - 2y \end{cases}$

15. $\begin{cases} x' = 2x \\ y' = -4x \end{cases}$

16. $\begin{cases} x' = 12x - 6y \\ y' = 18x - 9y \end{cases}$

In Exercises 17–20, use the given eigenvalues to (a) develop a corresponding second-order linear equation; (b) develop a corresponding system of first-order linear equations; (c) classify the stability of the equilibrium point (0, 0).

17. $\lambda_1 = 2,\ \lambda_2 = -1$
18. $\lambda_1 = -3,\ \lambda_2 = -1$
*19. $\lambda_1 = -3,\ \lambda_2 = -3$
20. $\lambda_1 = 3i,\ \lambda_2 = -3i$

21. **(Linear Nonhomogeneous Systems)** Consider the system $\begin{cases} x' = x - 1 \\ y' = x + y \end{cases}$.
    (a) Show that $(1, -1)$ is an equilibrium point.
    (b) Show that with the change of variables $u = x - 1$ and $v = y + 1$, we can transform the system to $\begin{cases} u' = u \\ v' = u + v \end{cases}$. (Notice that if the equilibrium point is $(x_0, y_0)$, then the change of variables is $u = x - x_0$ and $v = y - y_0$.)
    (c) Classify the equilibrium point $(1, -1)$ of the original system by classifying the equilibrium point (0, 0) of the transformed system.
    (d) Sketch the phase portrait of the original system by translating that of the transformed system.

For each of the following systems, follow steps similar to those in Exercise 21 to locate the equilibrium point of the system, classify the equilibrium, and sketch the phase portrait.

22. $\begin{cases} x' = 2x + y \\ y' = -y + 6 \end{cases}$

*23. $\begin{cases} x' = -2x + y - 6 \\ y' = x - 2y \end{cases}$

24. $\begin{cases} x' = x - 2y - 1 \\ y' = 5x - y - 5 \end{cases}$

25. Consider the system $\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases}$, where $a, b, c$, and $d$ are real constants. Show that the eigenvalues of this system are found by solving $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$. If we let $p = a + d$ and $q = ad - bc$, show that the eigenvalues are $\lambda_{1,2} = 1/2(p \pm \sqrt{\Delta})$ where $\Delta = p^2 - 4q$. Show that each of the following is true of the equilibrium point.

**(a)** (0, 0) is a node if $q > 0$ and $\Delta \geq 0$. It is asymptotically stable if $p < 0$ and unstable if $p > 0$.

**(b)** (0, 0) is a saddle point if $q < 0$.

**(c)** (0, 0) is a spiral point if $p \neq 0$ and $\Delta < 0$. It is asymptotically stable if $p < 0$ and unstable if $p > 0$.

**(d)** (0, 0) is a center if $p = 0$ and $q > 0$.

**(e)** Place the classifications in parts (a)–(d) in the regions shown in Figure 6.31. The parabola is the graph of $\Delta = p^2 - 4q = 0$.

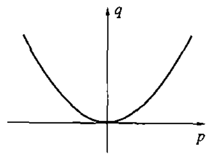



**Figure 6.31** Graphs of $\Delta = p^2 - 4q = 0$.

**26.** Suppose that substance $X$ decays into substance $Y$ at rate $k_1$, which in turn decays into another substance at rate $k_2$. If $x(t)$ and $y(t)$ represent the amount of $X$ and $Y$, respectively, then the system

$$\begin{cases} \dfrac{dx}{dt} = -k_1 x \\ \dfrac{dy}{dt} = k_1 x - k_2 y \end{cases}$$

is solved to determine $x(t)$ and $y(t)$. Show that (0, 0) is the equilibrium solution of this system. Find the eigenvalues of the system and classify the equilibrium solution. (*Note:* $k_1$ and $k_2$ are positive constants.) Also, solve the system. Find $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$. Do these limits correspond to the physical situation?

**27.** In the circular chemical reaction, compounds $A_1$ and $A_2$ combine at rate $k_A$ to produce compound $B_1$. This compound reacts with compound $B_2$ at rate $k_B$ to form $C_1$, which in turn combines with compound $C_2$ at rate $k_C$ to form $A_1$. Therefore, the system of equations

$$\begin{cases} \dfrac{dA_1}{dt} = k_C C_1 C_2 - k_A A_1 A_2 \\ \dfrac{dB_1}{dt} = k_A A_1 A_2 - k_B B_1 B_2 \\ \dfrac{dC_1}{dt} = k_B B_1 B_2 - k_C C_1 C_2 \end{cases}$$

models this situation. Let $K_1 = k_A A_2$, $K_2 = k_B B_2$, and $K_3 = k_C C_2$, and determine the equilibrium solutions of the system.

## 6.7 Nonlinear Systems

We now turn our attention to systems in which at least one of the equations is not linear. These nonlinear systems possess many of the same properties as linear systems. In fact, we rely on our study of the classification of equilibrium points of linear systems to assist us in our understanding of the behavior of nonlinear systems. For example, consider the system

$$\begin{cases} x' = y \\ y' = x + x^2 \end{cases}$$

which is nonlinear because of the $x^2$ term in the second equation. If we remove this nonlinear term, we obtain the linear system

$$\begin{cases} x' = y \\ y' = x \end{cases}.$$

This system has an equilibrium point at (0, 0), which is also an equilibrium point of the system of nonlinear equations. Using the techniques discussed in Section 6.6, we



can quickly show that the linear system has a saddle point at (0, 0). We show several trajectories of this system together with its direction field in Figure 6.32. Next, in Figure 6.33(a), we graph several trajectories of the nonlinear system along with its direction field. We see that the nonlinear term affects the behavior of the trajectories. However, the behavior near (0, 0) is "saddle-like" in that solutions appear to approach the origin but eventually move away from it. When we zoom in on the origin in Figure 6.33(b), we see how much it resembles a saddle point. Notice that the curves that separate the trajectories resemble the lines $y = x$ and $y = -x$, the asymptotes of trajectories in the linear system, near the origin. These curves, along with the origin, form the **separatrix,** because solutions within the portion of the separatrix to the left of the $y$-axis behave differently than solutions in other regions of the $xy$-plane. In this section, we discuss how we classify equilibrium points of nonlinear systems by using an associated linear system in much the same way as we analyzed these two systems.

When working with nonlinear systems, we can often gain a great deal of information concerning the system by making a *linear approximation* near each equilibrium point of the nonlinear system and solving the linear system. Although the solution to the linearized system only approximates the solution to the nonlinear system, the general behavior of solutions to the nonlinear system near each equilibrium point is the same as that of the corresponding linear system *in most cases.* The first step toward approximating a nonlinear system near each equilibrium point is to find the equilibrium points of the system and to linearize the system at each of these points.

Recall from multivariable calculus that if $z = F(x, y)$ is a differentiable function, the tangent plane to the surface $S$ given by the graph of $z = F(x, y)$ at the point $(x_0, y_0)$ is

$$z = F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) + F(x_0, y_0).$$

Near each equilibrium point $(x_0, y_0)$ of the nonlinear system

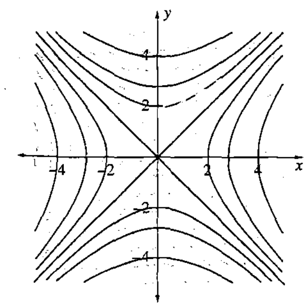



**Figure 6.32** Trajectories of corresponding linear system with direction field.

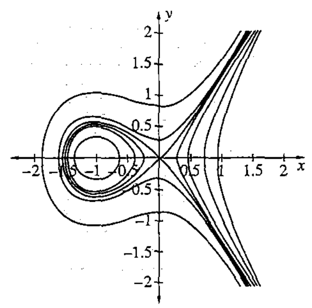



**Figure 6.33(a)** Trajectories of nonlinear system with direction field.

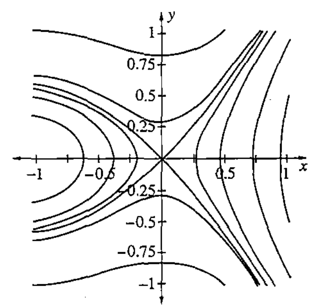



**Figure 6.33(b)** Phase portrait of nonlinear system near the origin.

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases},$$

the system can be *approximated* with

$$\begin{cases} x' = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0) \\ y' = g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0) + g(x_0, y_0) \end{cases}$$

where we have used the tangent plane to approximate $f$ and $g$ in the two differential equations. Because $f(x_0, y_0) = 0$ and $g(x_0, y_0) = 0$ (why?), the *approximate* system is

$$\begin{cases} x' = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ y' = g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0) \end{cases},$$

which can be written in matrix form as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}.$$

Note that we often call this system the **linearized system corresponding to the nonlinear system** or the **associated linearized system** due to the fact that we have removed the nonlinear terms from the original system.

The equilibrium point $(x_0, y_0)$ of the system $\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$ is classified by the eigenvalues of the matrix

$$J(x_0, y_0) = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix},$$

which is called the **Jacobian matrix.** After determining the Jacobian matrix for each equilibrium point, we find the eigenvalue of the matrix in order to classify the corresponding equilibrium point according to the following criteria.

Notice that the linearization must be carried out for **each** equilibrium point.

### Classification of Equilibrium Points of a Nonlinear System

Let $(x_0, y_0)$ be an equilibrium point of the system $\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$ and let $\lambda_1$ and $\lambda_2$ be the eigenvalues of the matrix

$$\begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix}$$

of the associated linearized system about the equilibrium point.

① If $(x_0, y_0)$ is classified as an asymptotically stable or unstable improper node (because the eigenvalues of $J(x_0, y_0)$ are real and distinct), a saddle point, or an asymptotically stable or unstable spiral point in the associated linear system, $(x_0, y_0)$ has the same classification in the nonlinear system.



② If $(x_0, y_0)$ is classified as a center in the associated linear system, $(x_0, y_0)$ may be a center, unstable spiral point, or asymptotically stable spiral point in the nonlinear system, so we cannot classify $(x_0, y_0)$ in this situation (see Exercise 28).

③ If the eigenvalues of $J(x_0, y_0)$ are real and equal, then $(x_0, y_0)$ may be a node or a spiral point in the nonlinear system. If $\lambda_1 = \lambda_2 < 0$, then $(x_0, y_0)$ is asymptotically stable. If $\lambda_1 = \lambda_2 > 0$, $(x_0, y_0)$ is unstable.

These findings are summarized in Table 6.2.

**TABLE 6.2 Classification of Equilibrium Point in Nonlinear System**

| Eigenvalues of $J(x_0, y_0)$ | Geometry | Stability |
|---|---|---|
| $\lambda_1, \lambda_2$ real; $\lambda_1 > \lambda_2 > 0$ | Improper node | Unstable |
| $\lambda_1, \lambda_2$ real; $\lambda_1 = \lambda_2 > 0$ | Node or spiral point | Unstable |
| $\lambda_1, \lambda_2$ real; $\lambda_2 < \lambda_1 < 0$ | Improper node | Asymptotically stable |
| $\lambda_1, \lambda_2$ real; $\lambda_1 = \lambda_2 < 0$ | Node or spiral point | Asymptotically stable |
| $\lambda_1, \lambda_2$ real; $\lambda_2 < 0 < \lambda_1$ | Saddle point | Unstable |
| $\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i, \beta \neq 0, \alpha > 0$ | Spiral point | Unstable |
| $\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i, \beta \neq 0, \alpha < 0$ | Spiral point | Aymptotically stable |
| $\lambda_1 = \beta i, \lambda_2 = -\beta i, \beta \neq 0$ | Center or spiral point | Inconclusive |

**Example 1**

Find and classify the equilibrium points of $\begin{cases} x' = 1 - y \\ y' = x^2 - y^2 \end{cases}$.

***Solution*** We begin by finding the equilibrium points of this nonlinear system by solving $\begin{cases} 1 - y = 0 \\ x^2 - y^2 = 0 \end{cases}$. Because $y = 1$ from the first equation, substitution into the second equation yields $x^2 - 1 = 0$. Therefore, $x = \pm 1$, so the two equilibrium points are $(1, 1)$ and $(-1, 1)$. Because $f(x, y) = 1 - y$ and $g(x, y) = x^2 - y^2$, $f_x(x, y) = 0$, $f_y(x, y) = -1$, $g_x(x, y) = 2x$, and $g_y(x, y) = -2y$, so the Jacobian matrix is $J(x, y) = \begin{pmatrix} 0 & -1 \\ 2x & -2y \end{pmatrix}$. Next, we classify each equilibrium point by finding the eigenvalues of the Jacobian matrix of each linearized system.

For $(1, 1)$, we obtain the Jacobian matrix $J(1, 1) = \begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix}$ with eigenvalues that satisfy $\begin{vmatrix} -\lambda & -1 \\ 2 & -2 - \lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2 = 0$. Hence, $\lambda_1 = -1 + i$ and

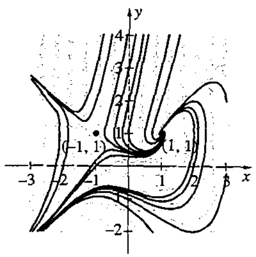



Figure 6.34

$\lambda_2 = -1 - i$. Because these eigenvalues are complex-valued with negative real part, we classify (1, 1) as an asymptotically stable spiral in the associated linearized system. Therefore, (1, 1) is an asymptotically stable spiral in the nonlinear system.

For $(-1, 1)$, we obtain $J(-1, 1) = \begin{pmatrix} 0 & -1 \\ -2 & -2 \end{pmatrix}$. In this case, the eigenvalues are solutions of $\begin{vmatrix} -\lambda & -1 \\ -2 & -2-\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 2 = 0$. Thus, $\lambda_1 = \dfrac{-2 + 2\sqrt{3}}{2} = -1 + \sqrt{3} > 0$ and $\lambda_2 = \dfrac{-2 - 2\sqrt{3}}{2} = -1 - \sqrt{3} < 0$, so $(-1, 1)$ is a saddle point in the associated linearized system and this classification carries over to the nonlinear system. In Figure 6.34, we graph solutions to this nonlinear system approximated with the use of a computer algebra system. We can see how the solutions move toward and away from the equilibrium points by observing the arrows on the vectors in the direction field.

**Note:** In Example 1, the linear approximation about (1, 1) is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} = \begin{pmatrix} -y+1 \\ 2x-2y \end{pmatrix} \quad \text{or} \quad \begin{cases} x' = -y+1 \\ y' = 2x-2y \end{cases}.$$

Notice that this (nonhomogeneous) linear system has an equilibrium point at (1, 1). With the change of variable $u = x - 1$ and $v = y - 1$, where the change of variable for the equilibrium point $(x_0, y_0)$ is $u = x - x_0$ and $v = y - y_0$, we can transform this system to $\begin{cases} u' = -v \\ v' = 2u - 2v \end{cases}$ with equilibrium point (0, 0). Finding the eigenvalues of the matrix of coefficients indicates that (0, 0) is a saddle point, and we can sketch the phase portrait of the linearized system by translating the axes back to the original variables.

## Example 2

Find and classify the equilibrium points of $\begin{cases} x' = x(7 - x - 2y) \\ y' = y(5 - x - y) \end{cases}$.

**Solution** The equilibrium points of this system satisfy $\begin{cases} x(7 - x - 2y) = 0 \\ y(5 - x - y) = 0 \end{cases}$. Thus, $\{x = 0 \text{ or } 7 - x - 2y = 0\}$ and $\{y = 0 \text{ or } 5 - x - y = 0\}$. If $x = 0$, then $y(5 - y) = 0$ so $y = 0$ or $y = 5$, and we obtain the equilibrium points (0, 0) and (0, 5). If $y = 0$, then $x(7 - x) = 0$, which indicates that $x = 0$ or $x = 7$. The corresponding equilibrium points are (0, 0) (which we found earlier) and (7, 0). The other possibility that leads to an equilibrium point is the solution to $\begin{cases} 7 - x - 2y = 0 \\ 5 - x - y = 0 \end{cases}$ which is $x = 3$ and $y = 2$. The equilibrium point associated with this possibility is (3, 2).



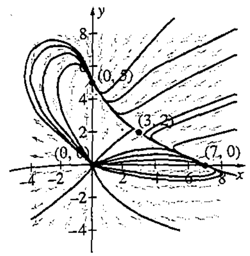



Figure 6.35

The Jacobian matrix is

$$J(x, y) = \begin{pmatrix} 7 - 2x - 2y & -2x \\ -y & 5 - x - 2y \end{pmatrix}.$$

We classify each of the equilibrium points $(x_0, y_0)$ of the associated linearized system using the eigenvalues of $J(x_0, y_0)$:

$J(0, 0) = \begin{pmatrix} 7 & 0 \\ 0 & 5 \end{pmatrix}$; $\lambda_1 = 7$, $\lambda_2 = 5$; (0, 0) is an unstable node.

$J(0, 5) = \begin{pmatrix} -3 & 0 \\ -5 & -5 \end{pmatrix}$; $\lambda_1 = -3$, $\lambda_2 = -5$; (0, 5) is an asymptotically stable improper node.

$J(7, 0) = \begin{pmatrix} -7 & -14 \\ 0 & -2 \end{pmatrix}$; $\lambda_1 = -2$, $\lambda_2 = -7$; (7, 0) is an asymptotically stable improper node.

$J(3, 2) = \begin{pmatrix} -3 & -6 \\ -2 & -2 \end{pmatrix}$; $\lambda_1 = 1$, $\lambda_2 = -6$; (3, 2) is a saddle point.

In each case, the classification carries over to the nonlinear system. In Figure 6.35, we graph several approximate solutions and the direction field to this nonlinear system through the use of a computer algebra system. Notice the behavior near each equilibrium point.

*Identify the equilibrium points in Figure 6.35.*

## Example 3

Investigate the stability of the equilibrium point (0, 0) of the nonlinear system

$$\begin{cases} x' = y + \dfrac{x}{2}(x^2 + y^2) \\ y' = -x + \dfrac{y}{2}(x^2 + y^2) \end{cases}.$$

**Solution** First, we find the Jacobian matrix,

$$J(x, y) = \begin{pmatrix} \dfrac{3}{2}x^2 + \dfrac{1}{2}y^2 & 1 + xy \\ -1 + xy & \dfrac{1}{2}x^2 + \dfrac{3}{2}y^2 \end{pmatrix}.$$

Then, at the equilibrium point (0, 0), we have $J(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so the linear approximation is

$$\begin{cases} x' = y \\ y' = -x \end{cases}$$

with eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$. Therefore, (0, 0) is a (stable) center in the linearized system. However, when we graph the direction field for the original (nonlinear) system in Figure 6.36, we observe that (0, 0) is not a center. Instead, trajectories appear to spiral away from (0, 0), so (0, 0) is an unstable spiral point of the nonlinear system. The nonlinear terms not only affect the classification of the equilibrium point but also change the stability. *Note:* This is the only case in which we cannot assign the same classification to the equilibrium point in the nonlinear system as we do in the associated linear system. When an equilibrium point is a center in the associated linear system, then we cannot draw any conclusions concerning its classification in the nonlinear system.

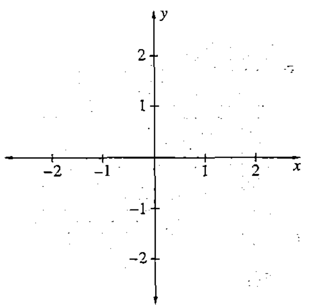



**Figure 6.36** Direction field for $\begin{cases} x' = y + \frac{x}{2}(x^2 + y^2) \\ y' = -x + \frac{y}{2}(x^2 + y^2) \end{cases}$

## EXERCISES 6.7

In Exercises 1–20, classify the equilibrium points of the nonlinear system.

1. $\begin{cases} x' = y \\ y' = x - x^2 \end{cases}$

2. $\begin{cases} x' = x + y \\ y' = x + xy \end{cases}$

*3. $\begin{cases} x' = y + y^2 \\ y' = x + y \end{cases}$

4. $\begin{cases} x' = 2y \\ y' = 2x + y^2 \end{cases}$

5. $\begin{cases} x' = y \\ y' = -x + xy \end{cases}$

6. $\begin{cases} x' = 4y \\ y' = -4x + y^3 \end{cases}$

*7. $\begin{cases} x' = x + y + y^2 \\ y' = y \end{cases}$

8. $\begin{cases} x' = y + xy \\ y' = x + y \end{cases}$

9. $\begin{cases} x' = -2x + x^2 - xy \\ y' = -4y + y^2 + xy \end{cases}$

10. $\begin{cases} x' = 10x - 2x^2 - 2xy \\ y' = -4y + 2xy \end{cases}$

*11. $\begin{cases} x' = -3y - xy - 4 \\ y' = y^2 - x^2 \end{cases}$

12. $\begin{cases} x' = y \\ y' = \sin x \end{cases}$

13. $\begin{cases} x' = x^2 - 10x + 16 \\ y' = y + 1 \end{cases}$

14. $\begin{cases} x' = x(3 - y) \\ y' = x + y + 1 \end{cases}$

*15. $\begin{cases} x' = y - x \\ y' = x + y - 2xy \end{cases}$

16. $\begin{cases} x' = 1 - x^2 \\ y' = y \end{cases}$

17. $\begin{cases} x' = xy \\ y' = x^2 + y^2 - 4 \end{cases}$

18. $\begin{cases} x' = 2x + y^2 \\ y' = -2x + 4y \end{cases}$



*19. $\begin{cases} x' = 1 - xy \\ y' = y - x^3 \end{cases}$

20. $\begin{cases} x' = y - \frac{x}{2} \\ y' = 1 - xy \end{cases}$

21. **(Population)** Suppose that we consider the relationship between a host population and a parasite population that is modeled by the nonlinear system

$$\begin{cases} x' = (a_1 - b_1 x - c_1 y)x \\ y' = (-a_2 + c_2 x)y \end{cases},$$

where $x(t)$ represents the number in the host population at time $t$, $y(t)$ the number in the parasite population at time $t$, $a_1$ the growth rate of the host population, $a_2$ the starvation rate of the parasite population, and $b_1$ the interference when the host population overpopulates. The constants $c_1$ and $c_2$ relate the interactions between the two populations leading to decay in the host population and growth in the parasite population, respectively. Find and classify the equilibrium point(s) of the system if all constants in the system are positive.

22. **(Economics)** Let $x(t)$ represent the income of a company and $y(t)$ the amount of consumer spending. Also suppose that $z$ represents the rate of company expenditures. The system of nonlinear equations that models this situation is

$$\begin{cases} x' = x - ay \\ y' = b(x - y - z) \end{cases}, \quad \text{where } 1 < a < \infty, \text{ and } 1 \le b.$$

**(a)** If $z = z_0$ is constant, find and classify the equilibrium point of the system. Also consider the special case if $b = 1$.

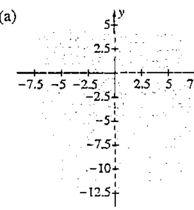
**(b)** If the expenditure depends on income according to the relationship $z = z_0 + cx$ $(c > 0)$, find and classify the equilibrium points (if they exist) of the system.

In Exercises 23–27, match each of the following systems in Group A with its direction field in Group B.

**Group A** / **Group B**

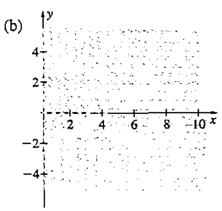
23. $\begin{cases} x' = 4y^2 + 3x + 2 \\ y' = 4x^2 - 4 \end{cases}$

(a)



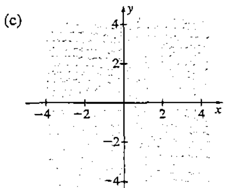

24. $\begin{cases} x' = y^2 + 2xy \\ y' = x^2 - 4x - 5 \end{cases}$

(b)



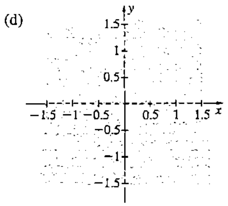

*25. $\begin{cases} x' = -5xy - y^3 \\ y' = 3xy + y + 2x \end{cases}$

(c)



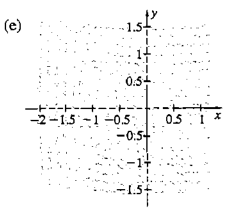

26. $\begin{cases} x' = -3y - 3xy \\ y' = 5y^2 - x + 5 \end{cases}$

(d)




27. $\begin{cases} x' = x^2 y^2 \\ y' = xy^2 + yx^2 \end{cases}$

(e)




28. Consider the nonlinear system

$$\begin{cases} \dfrac{dx}{dt} = y + \mu x(x^2 + y^2) \\ \dfrac{dy}{dt} = -x + \mu y(x^2 + y^2) \end{cases}.$$

**(a)** Show that this system has the equilibrium point (0, 0) and the linearized system about this point is $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x \end{cases}$. Also, show that (0, 0) is classified as a center of the linearized system.

**(b)** Consider the change of variables to polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Use these equations with the chain rule to show that $dx/dt = dr/dt \cos\theta - r\,d\theta/dt \sin\theta$ and $dy/dt = dr/dt \sin\theta + r\,d\theta/dt \cos\theta$. Use elimination with these equations to show that $dr/dt = dx/dt \cos\theta + dy/dt \sin\theta$ and $r\,d\theta/dt = -dx/dt \sin\theta + dy/dt \cos\theta$. Transform the original system to polar coordinates and substitute the equations that result in the equations for $dr/dt$ and $r\,d\theta/dt$ to obtain the system in polar coordinates

$$\begin{cases} \dfrac{dr}{dt} = \mu r^3 \\ \dfrac{d\theta}{dt} = -1 \end{cases}.$$

**(c)** What does the equation $d\theta/dt = -1$ indicate about the rotation of solutions to the system? According to $dr/dt = \mu r^3$, does $r$ increase or decrease for $r > 0$? Using these observations, is the equilibrium point (0, 0), which was classified as a center for the linearized system, also classified as a center for the nonlinear system? If not, how would you classify it?

**29.** **(a)** Find a general solution of the system $\begin{cases} x' = -x \\ y' = y + x^2 \end{cases}$ by solving the first equation and by substituting the result into the second. **(b)** Sketch the phase plane and determine if the origin is a stable equilibrium point. **(c)** Solve the linearized system about (0, 0) and compare the result to that found in **(b)**. Is (0, 0) assigned the same classification as in **(b)**?

**30.** Consider the Lorenz system

$$\begin{cases} x' = -\sigma x + \sigma y \\ y' = rx - y - xz, \\ z' = -bz + xy \end{cases}$$

which was developed in the 1960s by Edward Lorenz to study meteorology. **(a)** Show that (0, 0, 0) is an equilibrium point. **(b)** If $0 < r < 1$, $\sigma > 0$, and $b > 0$, show that the system is asymptotically stable at (0, 0, 0).

***31.** An exciting system to explore is the system

$$\begin{cases} x' = \mu x + y - x(x^2 + y^2) \\ y' = \mu y - x - y(x^2 + y^2) \end{cases}$$

for fixed values of $\mu$.

**(a)** Show that the only equilibrium point of the system is (0, 0).

**(b)** Show that the eigenvalues of the Jacobian matrix $\begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}$ are $\mu \pm i$ and classify the equilibrium point (0, 0).

**(c)** How do you *think* the direction field of the system changes as $\mu$ goes from $-1$ to 1?

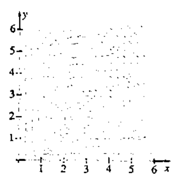
**(d)** Graph the direction field and various solutions of the system for several values of $\mu$ between $-1$ and 1.

**(e)** How do your results in **(d)** differ from your predictions in **(c)**?

**32.** The following figure shows the direction field associated with the nonlinear system of equations

$$\begin{cases} \dfrac{dx}{dt} = 2x - xy \\ \dfrac{dy}{dt} = -3y + xy \end{cases}$$

for $0 \le x \le 6$ and $0 \le y \le 6$. Use the direction field to sketch the graphs of the solutions $\begin{cases} x(t) \\ y(t) \end{cases}$ that satisfy the initial conditions **(a)** $x(0) = 2$ and $y(0) = 3$; and **(b)** $x(0) = 3$ and $y(0) = 2$. **(c)** How are the solutions alike? How are they different?





## 6.8 Numerical Methods

Euler's Method · Runge-Kutta Method · Computer Algebra Systems

Because it may be difficult or even impossible to construct an explicit solution to some systems of differential equations, we now turn our attention to some numerical methods that are used to construct solutions to systems of differential equations.

### Euler's Method

Euler's method for approximation, which was discussed for first-order equations, can be extended to include systems of first-order equations. The initial-value problem

$$\begin{cases} \dfrac{dx}{dt} = f(t, x, y) \\ \dfrac{dy}{dt} = g(t, x, y) \\ x(t_0) = x_0,\ y(t_0) = y_0 \end{cases}$$

is approximated at each step by the recursive relationship based on the Taylor expansion of $x$ and $y$ up to order $h$,

$$\begin{cases} x_{n+1} = x_n + hf(t_n, x_n, y_n) \\ y_{n+1} = y_n + hg(t_n, x_n, y_n) \end{cases},$$

where $t_n = t_0 + nh$, $n = 0, 1, 2, \ldots$.

**Example 1**

Use Euler's method with $h = 0.1$ and $h = 0.05$ to approximate the solution to the initial-value problem:

$$\begin{cases} \dfrac{dx}{dt} = x - y + 1 \\ \dfrac{dy}{dt} = x + 3y + e^{-t}. \\ x(0) = 0,\ y(0) = 1 \end{cases}$$

Compare these results with the exact solution to the system of equations.

**Solution** In this case, $f(x, y) = x - y + 1$, $g(x, y) = x + 3y + e^{-t}$, $t_0 = 0$, $x_0 = 0$, and $y_0 = 1$, so we use the formulas

$$\begin{cases} x_{n+1} = x_n + h(x_n - y_n + 1) \\ y_{n+1} = y_n + h(x_n + 3y_n + e^{-t_n}) \end{cases},$$

where $t_n = (0.1)n$, $n = 0, 1, 2, \ldots$.

If $n = 0$, then

$$\begin{cases} x_1 = x_0 + h(x_0 - y_0 + 1) = 0 \\ y_1 = y_0 + h(x_0 + 3y_0 + e^{-t_0}) = 1.4 \end{cases}.$$

The exact solution of this problem, which can be determined using the method of variation of parameters, is

$$\begin{cases} x(t) = -\dfrac{3}{4} - \dfrac{e^{-t}}{9} + \dfrac{31e^{2t}}{36} - \dfrac{11te^{2t}}{6} \\ y(t) = \dfrac{1}{4} - \dfrac{2e^{-t}}{9} + \dfrac{35e^{2t}}{36} + \dfrac{11te^{2t}}{6} \end{cases}.$$

**TABLE 6.3**

| $t_n$ | $x_n$ (approx) | $x_n$ (exact) | $y_n$ (approx) | $y_n$ (exact) |
|---|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 1.0 | 1.0 |
| 0.1 | 0.0 | −0.02270 | 1.4 | 1.46032 |
| 0.2 | −0.04 | −0.10335 | 1.91048 | 2.06545 |
| 0.3 | −0.13505 | −0.26543 | 2.5615 | 2.85904 |
| 0.4 | −0.30470 | −0.54011 | 3.39053 | 3.89682 |
| 0.5 | −0.57423 | −0.96841 | 4.44423 | 5.24975 |
| 0.6 | −0.97607 | −1.60412 | 5.78076 | 7.00806 |
| 0.7 | −1.55176 | −2.51737 | 7.47226 | 9.28638 |
| 0.8 | −2.35416 | −3.79926 | 9.60842 | 12.23 |
| 0.9 | −3.45042 | −5.56767 | 12.3005 | 16.0232 |
| 1.0 | −4.9255 | −7.97468 | 15.6862 | 20.8987 |

In Table 6.3, we display the results obtained with this method and compare them to the actual function values.

Because the accuracy of this approximation diminishes as $t$ increases, we attempt to improve the approximation by decreasing the increment size. We do this by entering the value $h = 0.05$ and repeating the above procedure. We show the results of this method in Table 6.4. Notice that the approximations are more accurate with the smaller value of $h$.



**TABLE 6.4**

| $t_n$ | $x_n$ (approx) | $x_n$ (exact) | $y_n$ (approx) | $y_n$ (exact) |
|---|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 1.0 | 1.0 |
| 0.05 | 0.0 | −0.00532 | 1.2 | 1.21439 |
| 0.10 | −0.01 | −0.02270 | 1.42756 | 1.46032 |
| 0.15 | −0.03188 | −0.05447 | 1.68644 | 1.74321 |
| 0.20 | −0.06779 | −0.10335 | 1.98084 | 2.06545 |
| 0.25 | −0.12023 | −0.17247 | 2.31552 | 2.43552 |
| 0.30 | −0.192013 | −0.26543 | 2.69577 | 2.85904 |
| 0.35 | −0.28640 | −0.38639 | 3.12758 | 3.34338 |
| 0.40 | −0.40710 | −0.54011 | 3.61763 | 3.89682 |
| 0.45 | −0.55834 | −0.73203 | 4.17344 | 4.52876 |
| 0.50 | −0.74493 | −0.96841 | 4.80342 | 5.24975 |
| 0.55 | −0.97234 | −1.25639 | 5.51701 | 6.07171 |
| 0.60 | −1.24681 | −1.60412 | 6.32479 | 7.00806 |
| 0.65 | −1.57529 | −2.02091 | 7.23861 | 8.07394 |
| 0.70 | −1.96609 | −2.51737 | 8.27174 | 9.28638 |
| 0.75 | −2.42798 | −3.10558 | 9.43902 | 10.6645 |
| 0.80 | −2.97133 | −3.79926 | 10.7571 | 12.23 |
| 0.85 | −3.60776 | −4.61405 | 12.2446 | 14.0071 |
| 0.90 | −4.35037 | −5.56767 | 13.9222 | 16.0232 |
| 0.95 | −5.214 | −6.68027 | 15.8134 | 18.3088 |
| 1.00 | −6.21537 | −7.97468 | 17.944 | 20.8987 |

## Runge-Kutta Method

Because we would like to be able to improve the approximation without using such a small value for $h$, we seek to improve the method. As with first-order equations, the Runge-Kutta method can be extended to systems. In this case, the recursive formula at each step is

$$\begin{cases} x_{n+1} = x_n + \dfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ y_{n+1} = y_n + \dfrac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4) \end{cases},$$

where

$$k_1 = f(t_n, x_n, y_n) \qquad m_1 = g(t_n, x_n, y_n)$$

$$k_2 = f\left(t_n + \frac{h}{2}, x_n + \frac{hk_1}{2}, y_n + \frac{hm_1}{2}\right) \qquad m_2 = g\left(t_n + \frac{h}{2}, x_n + \frac{hk_1}{2}, y_n + \frac{hm_1}{2}\right)$$

$$k_3 = f\left(t_n + \frac{h}{2}, x_n + \frac{hk_2}{2}, y_n + \frac{hm_2}{2}\right) \qquad m_3 = g\left(t_n + \frac{h}{2}, x_n + \frac{hk_2}{2}, y_n + \frac{hm_2}{2}\right)$$

$$k_4 = f(t_n + h, x_n + hk_3, y_n + hm_3) \qquad m_4 = g(t_n + h, x_n + hk_3, y_n + hm_3).$$

**Example 2**

Use the Runge-Kutta method to approximate the solution to the initial-value problem

$$\begin{cases} \dfrac{dx}{dt} = x - y + 1 \\ \dfrac{dy}{dt} = x + 3y + e^{-t} \\ x(0) = 0, y(0) = 1 \end{cases}$$

for $h = 0.1$. Compare these results with the exact solution to the system of equations as well as those obtained with Euler's method.

***Solution*** Because $f(x, y) = x - y + 1$, $g(x, y) = x + 3y + e^{-t}$, $t_0 = 0$, $x_0 = 0$, and $y_0 = 1$, we use the formulas

$$\begin{cases} x_{n+1} = x_n + \dfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ y_{n+1} = y_n + \dfrac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4) \end{cases}$$

where

$$k_1 = f(t_n, x_n, y_n) = x_n - y_n + 1 \qquad m_1 = g(t_n, x_n, y_n) = x_n + 3y_n + e^{-t_n}$$

$$k_2 = \left(x_n + \frac{hk_1}{2}\right) - \left(y_n + \frac{hm_1}{2}\right) + 1 \qquad m_2 = \left(x_n + \frac{hk_1}{2}\right) + 3\left(y_n + \frac{hm_1}{2}\right) + e^{-(t_n+h/2)}$$

$$k_3 = \left(x_n + \frac{hk_2}{2}\right) - \left(y_n + \frac{hm_2}{2}\right) + 1 \qquad m_3 = \left(x_n + \frac{hk_2}{2}\right) + 3\left(y_n + \frac{hm_2}{2}\right) + e^{-(t_n+h/2)}$$

$$k_4 = (x_n + hk_3) - (y_n + hm_3) + 1 \qquad m_4 = (x_n + hk_3) + 3(y_n + hm_3) + e^{-(t_n+h)}$$

For example, if $n = 0$, then



$$k_1 = x_0 - y_0 + 1 = 0 - 1 + 1 = 0$$

$$m_1 = x_0 + 3y_0 + e^{-t_0} = 0 + 3 + 1 = 4$$

$$k_2 = \left(x_0 + \frac{hk_1}{2}\right) - \left(y_0 + \frac{hm_1}{2}\right) + 1 = -1 - \frac{4(0.1)}{2} + 1 = -0.2$$

$$m_2 = \left(x_0 + \frac{hk_1}{2}\right) + 3\left(y_0 + \frac{hm_1}{2}\right) + e^{-(t_0+h/2)} = 3\left(1 + \frac{4(0.1)}{2}\right) + e^{-0.05}$$
$$\approx 4.55123$$

$$k_3 = \left(x_0 + \frac{hk_2}{2}\right) - \left(y_0 + \frac{hm_2}{2}\right) + 1 = \frac{(0.1)(0.2)}{2} - 1 - \frac{(0.1)(4.55123)}{2} + 1$$
$$\approx -0.23756$$

$$m_3 = \left(x_0 + \frac{hk_2}{2}\right) + 3\left(y_0 + \frac{hm_2}{2}\right) + e^{-(t_0+h/2)}$$
$$= \frac{(0.1)(0.2)}{2} + 3\left(1 + \frac{(0.1)(4.55123)}{2}\right) + e^{-0.05} \approx 4.62391$$

$$k_4 = (x_0 + hk_3) - (y_0 + hm_3) + 1 = (0.1)(-0.23756) - 1 + (0.1)(4.62391) + 1$$
$$\approx -0.48615$$

$$m_4 = (x_0 + hk_3) + 3(y_0 + hm_3) + e^{-(t_0+h)}$$
$$= (0.1)(-0.23756) + 3(1 + (0.1)(4.62391)) + e^{-0.1} \approx 5.26826.$$

Therefore,

$$x_1 = x_0 + \frac{0.1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= 0 + \frac{0.1}{6}[0 + 2(-0.2) + 2(-0.23756) + -0.48615] \approx -0.0226878$$

and

$$y_1 = y_0 + \frac{0.1}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$
$$= 1 + \frac{0.1}{6}[4 + 2(4.55123) + 2(4.62391) + 5.26826] \approx 1.46031.$$

In Table 6.5, we show the results obtained with this method and compare them to the exact values. Notice that the Runge-Kutta method is much more accurate than Euler's method. In fact, the Runge-Kutta with $h = 0.1$ is more accurate than Euler's method with $h = 0.05$. (Compare the results here to those given in Table 6.4.)

**TABLE 6.5**

| $t_n$ | $x_n$ (approx) | $x_n$ (exact) | $y_n$ (approx) | $y_n$ (exact) |
|---|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 1.0 | 1.0 |
| 0.1 | −0.02269 | −0.02270 | 1.46031 | 1.46032 |
| 0.2 | −0.10332 | −0.10335 | 2.06541 | 2.06545 |
| 0.3 | −0.26538 | −0.26543 | 2.85897 | 2.85904 |
| 0.4 | −0.54002 | −0.54011 | 3.8967 | 3.89682 |
| 0.5 | −0.96827 | −0.96841 | 5.24956 | 5.24975 |
| 0.6 | −1.60391 | −1.60412 | 7.00778 | 7.00806 |
| 0.7 | −2.51707 | −2.51737 | 9.28596 | 9.28638 |
| 0.8 | −3.79882 | −3.79926 | 12.2294 | 12.23 |
| 0.9 | −5.56704 | −5.56767 | 16.0223 | 16.0232 |
| 1.0 | −7.97379 | −7.97468 | 20.8975 | 20.8987 |

The Runge-Kutta method can be extended to systems of first-order equations so it can be used to solve higher order differential equations. This is accomplished by transforming the higher order equation into a system of first-order equations. We illustrate this with the pendulum equation that we have solved in several situations (by using the approximation $\sin x \approx x$).

**Example 3**

Use the Runge-Kutta method with $h = 0.1$ to approximate the solution to the nonlinear initial-value problem $x'' + \sin x = 0$, $x(0) = 0$, $x'(0) = 1$.

***Solution*** We begin by transforming the second-order equation into a system of first-order equations. We do this by letting $x' = y$, so $y' = x'' = -\sin x$. Hence, $f(t, x, y) = y$, and $g(t, x, y) = -\sin x$. With the Runge-Kutta method, we obtain the approximate values given in Table 6.6 under the heading R-K. Also in Table 6.6 under the heading "linear," we give the corresponding values of the initial-value problem $x'' + x = 0$, $x(0) = 0$, $x'(0) = 1$. We approximate the nonlinear equation $x'' + \sin x = 0$ with $x'' + x = 0$ because $\sin x \approx x$ for small values of $x$. The solution of $x'' + x = 0$, $x(0) = 0$, $x'(0) = 1$ is found to be $x = \sin t$, so $y = x' = \cos t$. Because the use of the approximation $\sin x \approx x$ is linear, we expect the approximations obtained with the Runge-Kutta method (which is a fourth-order method) to be more accurate than those obtained by solving $x'' + x = 0$, $x(0) = 0$, $x'(0) = 1$.



**TABLE 6.6**

| $t_n$ | $x_n$ (R-K) | $x_n$ (linear) | $y_n$ (R-K) | $y_n$ (linear) |
|---|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 1.0 | 1.0 |
| 0.1 | 0.09983 | 0.09983 | 0.99500 | 0.99500 |
| 0.2 | 0.19867 | 0.198669 | 0.98013 | 0.98007 |
| 0.3 | 0.29553 | 0.29552 | 0.95566 | 0.95534 |
| 0.4 | 0.38950 | 0.389418 | 0.922061 | 0.92106 |
| 0.5 | 0.47966 | 0.47943 | 0.87994 | 0.87758 |
| 0.6 | 0.56523 | 0.56464 | 0.83002 | 0.82534 |
| 0.7 | 0.64544 | 0.64422 | 0.77309 | 0.764842 |
| 0.8 | 0.71964 | 0.71736 | 0.70999 | 0.69671 |
| 0.9 | 0.78726 | 0.78333 | 0.641545 | 0.62161 |
| 1.0 | 0.84780 | 0.84147 | 0.568569 | 0.54030 |

Detailed discussions regarding the error involved in using Euler's method or the Runge-Kutta method to approximate solutions to systems of equations can be found in advanced numerical analysis texts.

## Computer Algebra Systems

Numerical and graphical solutions generated by computer algebra systems are useful in helping us observe and understand behavior of the solution(s) to a differential equation, especially when we do not wish to utilize numerical methods like Euler's method or the Runge-Kutta method.

**Example 4**

**Rayleigh's equation** is the nonlinear equation $x'' + (\frac{1}{3}(x')^2 - 1)x' + x = 0$ and arises in the study of the motion of a violin string. We write Rayleigh's equation as a system by letting $y = x'$. Then,

$$y' = x'' = -\left(\frac{1}{3}(x')^2 - 1\right)x' - x = -\left(\frac{1}{3}y^2 - 1\right)y - x$$

so Rayleigh's equation is equivalent to the system

$$\begin{cases} x' = y \\ y' = -\left(\frac{1}{3}y^2 - 1\right)y - x \end{cases}.$$

We see that the only equilibrium point of this system is (0, 0). **(a)** Classify the equilibrium point (0, 0). **(b)** Is it possible to find $y_0 \neq 0$ so that the solution to the initial-value problem

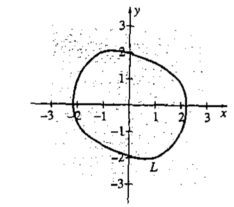


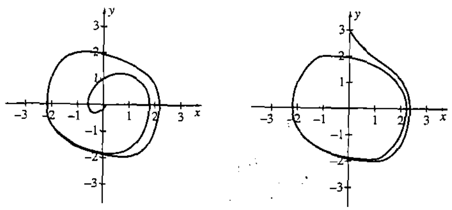
Figure 6.37

$$\begin{cases} x' = y \\ y' = -\left(\frac{1}{3}y^2 - 1\right)y - x \\ x(0) = 0,\ y(0) = y_0 \end{cases}$$

is periodic?

**Solution** (a) The associated linearized system about the point (0, 0) is $\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\mathbf{X}$ and the eigenvalues of $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ are $\lambda_{1,2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ so (0, 0) is an unstable spiral point. This result is confirmed by the direction field of the system for $-3 \le x \le 3$ and $-3 \le y \le 3$, shown in Figure 6.37.

In the direction field, we see that all solutions appear to tend to a closed curve, $L$. Indeed, we see that the graph of the solution that satsifies the initial conditions $x(0) = 0.1$ and $y(0) = 0$ in Figure 6.37 tends toward $L$. Choosing initial conditions outside of $L$ yields the same result; Figure 6.38(b) shows the graph of the solution that satisfies $x(0) = 0$ and $y(0) = 3$.



(a) (b)

**Figure 6.38** All solutions tend to L

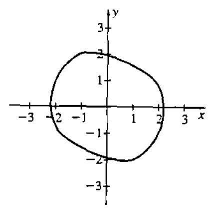


**Figure 6.39** An isolated periodic solution like this is called a *limit cycle*. This limit cycle is *stable* because all solutions spiral into it.*

(b) Can we find $L$? We use Figure 6.37 to approximate a point at which $L$ intersects the $y$-axis. We obtain $y = 1.9$. We find a numerical solution that satisfies the initial conditions $x(0) = 0$ and $y(0) = 1.9$ and graph the result in Figure 6.39. The solution does appear to be periodic.

*Approximate the period of the periodic solution in Example 4. Hint: Graph $x(t)$ and $y(t)$ together for $0 \le t \le T$ for various values of $T$.*

* See texts like *Nonlinear Ordinary Differential Equations* (Second edition) by D. W. Jordan and P. Smith, which is published by Oxford University Press (1987), for detailed discussions about limit cycles and their significance.



## EXERCISES 6.8

In Exercises 1–6, use Euler's method with $h = 0.1$ to approximate the solution of the initial-value problem at the given value of $t$.

1. $\begin{cases} x' = -x + 2y + 5 \\ y' = 2x - y + 4 \end{cases}$, $x(0) = 1, y(0) = 0, t = 1$
2. $\begin{cases} x' = x + y + t \\ y' = x - y \end{cases}$, $x(0) = 1, y(0) = -1, t = 1$
*3. $\begin{cases} x' = 3x - 5y \\ y' = x - 2y + t^2 \end{cases}$, $x(0) = -1, y(0) = 0, t = 1$
4. $\begin{cases} x' = x + 2y + \cos t \\ y' = 5x - 2y \end{cases}$, $x(0) = 0, y(0) = 1, t = 1$
5. $\begin{cases} x' = x^2 - y \\ y' = x + y \end{cases}$, $x(0) = -1, y(0) = 0, t = 1$
6. $\begin{cases} x' = xy \\ y' = x - y \end{cases}$, $x(0) = 1, y(0) = 1, t = 1$

In Exercises 7–12, use the Runge-Kutta method with $h = 0.1$ to approximate the solution of the initial-value problem at the given value at $t$.

7. $\begin{cases} x' = x - 3y + e^t \\ y' = -x + 6y \end{cases}$, $x(0) = 0, y(0) = 1, t = 1$
8. $\begin{cases} x' = 4x - y \\ y' = -x + 5y + 6\sin t \end{cases}$, $x(0) = -1, y(0) = 0, t = 1$
*9. $\begin{cases} x' = x - 8y \\ y' = 3x - y + e^t \cos 2t \end{cases}$, $x(0) = 0, y(0) = 0, t = 1$
10. $\begin{cases} x' = 5x + y + \sqrt{t+1} \\ y' = x - 2y \end{cases}$, $x(0) = 0, y(0) = 0, t = 2$
11. $\begin{cases} x' = 2y \\ y' = -xy \end{cases}$, $x(0) = 0, y(0) = 1, t = 1$
12. $\begin{cases} x' = x\sqrt{y} \\ y' = x - y \end{cases}$, $x(1) = 1, y(1) = 1, t = 2$

In Exercises 13–18, use Euler's method with $h = 0.1$ to approximate the solution of the initial-value problem by transforming the second-order equation to a system of first-order equations. Compare the approximation with the exact solution at the given value of $t$.

13. $x'' + 3x' + 2x = 0$, $x(0) = 0$, $x'(0) = -3$, $t = 1$
14. $x'' + 4x' + 4x = 0$, $x(0) = 4$, $x'(0) = 0$, $t = 1$
*15. $x'' + 9x = 0$, $x(0) = 0$, $x'(0) = 3$, $t = 1$
16. $x'' + 4x' + 13x = 0$, $x(0) = 0$, $x'(0) = 12$, $t = 1$
17. $t^2x'' + tx' + 16x = 0$, $x(1) = 0$, $x'(1) = 4$, $t = 2$
18. $t^2x'' + 3tx' + x = 0$, $x(1) = 0$, $x'(1) = 2$, $t = 2$

In Exercises 19–24, use the Runge-Kutta method with $h = 0.1$ to approximate the solution of the initial-value problem in the earlier exercise given. Compare the results obtained to those in Exercises 13–18.

19. Exercise 13
20. Exercise 14
*21. Exercise 15
22. Exercise 16
23. Exercise 17
24. Exercise 18

Find the exact solution of the following initial-value problems and then approximate the solution with the Runge-Kutta method using $h = 0.1$. Compare the results by graphing the two solutions.

25. $\begin{cases} x' = y \\ y' = 2x - y \end{cases}$, $x(0) = 0, y(0) = 1$
*26. $\begin{cases} x' = y \\ y' = -4y - 13x \end{cases}$, $x(0) = -1, y(0) = 1$
27. $\begin{cases} x' = y \\ y' = -2y - x \end{cases}$, $x(0) = 2, y(0) = 0$
28. **(a)** Graph the direction field associated with the nonlinear system $\begin{cases} x' = y \\ y' = -\sin x \end{cases}$ for $-7 \le x \le 7$ and $-4 \le y \le 4$.
    **(b) (i)** Approximate the solution to the initial-value problem $\begin{cases} x_1' = y_1 \\ y_1' = -\sin x_1 \\ x_1(0) = 0, y_1(0) = 1 \end{cases}$.
    **(ii)** Graph $\begin{cases} x_1(t) \\ y_1(t) \end{cases}$ for $0 \le t \le 7$ and display the graph together with the direction field. Does

it appear as though the vectors in the vector field are tangent to the solution curve?

**(iii)** Approximate the solution to the initial-value problem $\begin{cases} x_2' = y_2 \\ y_2' = -\sin x_2 \\ x_2(0) = 0, y_2(0) = 2 \end{cases}$.

**(iv)** Graph $\begin{cases} x_2(t) \\ y_2(t) \end{cases}$ for $0 \le t \le 7$ and display the graph together with the direction field. Does it appear as though the vectors in the vector field are tangent to the solution curve?

**(v)** Graph $\begin{cases} x_1(t) + x_2(t) \\ y_1(t) + y_2(t) \end{cases}$ for $0 \le t \le 7$ and display the graph together with the direction field. Does it appear as though the vectors in the vector field are tangent to this curve? Is $\begin{cases} x_1(t) + x_2(t) \\ y_1(t) + y_2(t) \end{cases}$ a solution to the system $\begin{cases} x' = y \\ y' = -\sin x \end{cases}$? Explain.

**(c)** Solve the initial-value problem $\begin{cases} x' = y \\ y' = -\sin x \\ x(0) = 0, y(0) = 3 \end{cases}$ and graph the solution parametrically for $0 \le t \le 7$. Is this the graph of $\begin{cases} x_1(t) + x_2(t) \\ y_1(t) + y_2(t) \end{cases}$?

**(d)** Is the Principle of Superposition valid for nonlinear systems? Explain.

## CHAPTER 6 SUMMARY

## Concepts & Formulas

### Section 6.1

**System of Ordinary Differential Equations**
A **system** of ordinary differential equations is a simultaneous set of equations that involves two or more dependent variables that depend on one independent variable. A **solution** to the system is a set of functions that satisfies each equation on some interval $I$.

### Section 6.2

**Matrix, Transpose of a Matrix**

**Scalar Multiplication, Matrix Addition, Matrix Multiplication**

**Identity Matrix (2 × 2)**

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

**Determinant of a 2 × 2 Matrix**

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

**Inverse of a 2 × 2 Matrix**

If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $|\mathbf{A}| = ad - bc \neq 0$, then

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

**Eigenvalues and Eigenvectors**
A nonzero vector $\mathbf{x}$ is an **eigenvector** of the square matrix **A** if there is a number $\lambda$, called an **eigenvalue** of **A**, so that $\mathbf{Ax} = \lambda\mathbf{x}$.

**Characteristic Polynomial**
The equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is called the **characteristic equation** of **A**; $\det(\mathbf{A} - \lambda\mathbf{I})$ is called the **characteristic polynomial** of **A**; the roots of the characteristic polynomial of **A** are the eigenvalues of **A**.

**Eigenvalue of Multiplicity $m$**
Suppose that $(\lambda - \lambda_1)^m$ where $m$ is a positive integer is a factor of the characteristic polynomial of the $n \times n$ matrix **A** while $(\lambda - \lambda_1)^{m+1}$ is not a factor of this polynomial. Then $\lambda = \lambda_1$ is an **eigenvalue of multiplicity $m$.**

**Derivative and Integral of a Matrix**
The **derivative** of the $n \times m$ matrix



$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1m}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nm}(t) \end{pmatrix},$$

where $a_{ij}(t)$ is differentiable for all values of $i$ and $j$, is

$$\frac{d}{dt}\mathbf{A}(t) = \begin{pmatrix} \frac{d}{dt}a_{11}(t) & \frac{d}{dt}a_{12}(t) & \cdots & \frac{d}{dt}a_{1m}(t) \\ \frac{d}{dt}a_{21}(t) & \frac{d}{dt}a_{22}(t) & \cdots & \frac{d}{dt}a_{2m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d}{dt}a_{n1}(t) & \frac{d}{dt}a_{n2}(t) & \cdots & \frac{d}{dt}a_{nm}(t) \end{pmatrix}.$$

The **integral** of $\mathbf{A}(t)$, where $a_{ij}(t)$ is integrable for all values of $i$ and $j$, is

$$\int \mathbf{A}(t)dt = \begin{pmatrix} \int a_{11}(t)dt & \int a_{12}(t)dt & \cdots & \int a_{1m}(t)dt \\ \int a_{21}(t)dt & \int a_{22}(t)dt & \cdots & \int a_{2m}(t)dt \\ \vdots & \vdots & \ddots & \vdots \\ \int a_{n1}(t)dt & \int a_{n2}(t)dt & \cdots & \int a_{nm}(t)dt \end{pmatrix}.$$

### Section 6.3

**Solution Vector**
A **solution vector** of the system $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{F}(t)$ on the interval $I$ is an $n \times 1$ matrix of the form

$$\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

where the $x_i(t)$ are differentiable functions that satisfy $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{F}(t)$ on $I$.

**Fundamental Set of Solutions**

A set $\{\mathbf{\Phi}_i\}_{i=1}^n = \left\{\begin{pmatrix} \Phi_{1i} \\ \Phi_{2i} \\ \vdots \\ \Phi_{ni} \end{pmatrix}\right\}_{i=1}^n$ of $n$ linearly independent solution vectors of $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ on an interval $I$ is called a **fundamental set of solutions** on $I$.

**Wronskian**

$$W(\mathbf{\Phi}_1, \mathbf{\Phi}_2, \ldots, \mathbf{\Phi}_n) = \det\begin{pmatrix} \Phi_{11} & \Phi_{12} & \cdots & \Phi_{1n} \\ \Phi_{21} & \Phi_{22} & \cdots & \Phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n1} & \Phi_{n2} & \cdots & \Phi_{nn} \end{pmatrix}$$

**General Solution**
A **general solution** of $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ is $\mathbf{X}(t) = c_1\mathbf{\Phi}_1(t) + c_2\mathbf{\Phi}_2(t) + \cdots + c_n\mathbf{\Phi}_n(t)$ where $\{\mathbf{\Phi}_i\}_{i=1}^n$ is a set of $n$ linearly independent solution vectors of the system.

**Fundamental Matrix**

$$\mathbf{\Phi}(t) = (\mathbf{\Phi}_1 \quad \mathbf{\Phi}_2 \quad \cdots \quad \mathbf{\Phi}_n) = \begin{pmatrix} \Phi_{11} & \Phi_{12} & \cdots & \Phi_{1n} \\ \Phi_{21} & \Phi_{22} & \cdots & \Phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n1} & \Phi_{n2} & \cdots & \Phi_{nn} \end{pmatrix}$$

is called a **fundamental matrix** of the system $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ where $\{\mathbf{\Phi}_i\}_{i=1}^n$ is a set of $n$ linearly independent solution vectors of the system.

### Section 6.4

**Distinct Real Eigenvalues**
If **A** is an $n \times n$ matrix with $n$ distinct real eigenvalues $\{\lambda_k\}_{k=1}^n$, a general solution to $\mathbf{X}' = \mathbf{AX}$ is the linear combination of the set of solutions $\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n\}$,

$$\mathbf{X}(t) = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t} + \cdots + c_n\mathbf{v}_ne^{\lambda_n t}.$$

**Complex Conjugate Eigenvalues**
If **A** has complex conjugate eigenvalues $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$ and corresponding eigenvectors $\mathbf{v}_1 = \mathbf{a} + \mathbf{b}i$ and $\mathbf{v}_2 = \mathbf{a} - \mathbf{b}i$, two linearly independent solutions of $\mathbf{X}' = \mathbf{AX}$ are $\mathbf{X}_1(t) = e^{\alpha t}(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t)$ and $\mathbf{X}_2(t) = e^{\alpha t}(\mathbf{a}\sin\beta t + \mathbf{b}\cos\beta t)$.

**Repeated Eigenvalues**
If the system $\mathbf{X}' = \mathbf{AX}$ has the repeated eigenvalue $\lambda_1 = \lambda_2$ with only one corresponding eigenvector $\mathbf{v}_1$, two linearly independent solutions corresponding to $\lambda_1 = \lambda_2$ are $\mathbf{X}_1 = \mathbf{v}_1e^{\lambda_1 t}$ and $\mathbf{X}_2 = (\mathbf{v}_1t + \mathbf{w}_2)e^{\lambda_1 t}$ where $\mathbf{w}_2$ satisfies $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{w}_2 = \mathbf{v}_1$.

### Section 6.5

**Variation of Parameters**
A general solution to $\mathbf{X}' = \mathbf{AX} + \mathbf{F}(t)$ is

$$\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{C} + \mathbf{X}_p(t) = \mathbf{\Phi}(t)\mathbf{C} + \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\,dt,$$

where $\mathbf{\Phi}(t)$ is a fundamental matrix of the system $\mathbf{X}' = \mathbf{AX}$.

## Section 6.6

| Eigenvalues | Geometry | Stability |
|---|---|---|
| $\lambda_1, \lambda_2$ real; $\lambda_1 > \lambda_2 > 0$ | Improper node | Unstable |
| $\lambda_1, \lambda_2$ real; $\lambda_1 = \lambda_2 > 0$; 1 eigenvector | Deficient node | Unstable |
| $\lambda_1, \lambda_2$ real; $\lambda_1 = \lambda_2 > 0$; 2 eigenvectors | Star node | Unstable |
| $\lambda_1, \lambda_2$ real; $\lambda_2 < \lambda_1 < 0$ | Improper node | Asymptotically stable |
| $\lambda_1, \lambda_2$ real; $\lambda_1 = \lambda_2 < 0$; 1 eigenvector | Deficient node | Asymptotically stable |
| $\lambda_1, \lambda_2$ real; $\lambda_1 = \lambda_2 < 0$; 2 eigenvectors | Star node | Asymptotically stable |
| $\lambda_1, \lambda_2$ real; $\lambda_2 < 0 < \lambda_1$ | Saddle point | Unstable |
| $\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$, $\beta \neq 0$, $\alpha > 0$ | Spiral point | Unstable |
| $\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$, $\beta \neq 0$, $\alpha < 0$ | Spiral point | Asymptotically stable |
| $\lambda_1 = \beta i$, $\lambda_2 = -\beta i$, $\beta \neq 0$ | Center | Stable |

## Section 6.7

| Eigenvalues of $J(x_0, y_0)$ | Geometry | Stability |
|---|---|---|
| $\lambda_1, \lambda_2$ real; $\lambda_1 > \lambda_2 > 0$ | Improper node | Unstable |
| $\lambda_1, \lambda_2$ real; $\lambda_1 = \lambda_2 > 0$ | Node or spiral point | Unstable |
| $\lambda_1, \lambda_2$ real; $\lambda_2 < \lambda_1 < 0$ | Improper node | Asymptotically stable |
| $\lambda_1, \lambda_2$ real; $\lambda_1 = \lambda_2 < 0$ | Node or spiral point | Asymptotically stable |
| $\lambda_1, \lambda_2$ real; $\lambda_2 < 0 < \lambda_1$ | Saddle point | Unstable |
| $\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$, $\beta \neq 0$, $\alpha > 0$ | Spiral point | Unstable |
| $\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$, $\beta \neq 0$, $\alpha < 0$ | Spiral point | Aymptotically stable |
| $\lambda_1 = \beta i$, $\lambda_2 = -\beta i$, $\beta \neq 0$ | Center or spiral point | Inconclusive |



## Section 6.8

**Euler's Method**

The initial-value problem

$$\begin{cases} \dfrac{dx}{dt} = f(t, x, y) \\ \dfrac{dy}{dt} = g(t, x, y) \\ x(t_0) = x_0,\ y(t_0) = y_0 \end{cases}$$

is approximated at each step by the recursive relationship based on the Taylor expansion of $x$ and $y$ up to order $h$:

$$\begin{cases} x_{n+1} = x_n + hf(t_n, x_n, y_n) \\ y_{n+1} = y_n + hg(t_n, x_n, y_n) \end{cases}$$

where $t_n = t_0 + nh$, $n = 0, 1, 2, \ldots$.

**Runge-Kutta Method**

The Runge-Kutta method for systems uses the recursive formula at each step

$$\begin{cases} x_{n+1} = x_n + \dfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ y_{n+1} = y_n + \dfrac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4) \end{cases}$$

where

$$k_1 = f(t_n, x_n, y_n)$$

$$k_2 = f\left(t_n + \frac{h}{2}, x_n + \frac{hk_1}{2}, y_n + \frac{hm_1}{2}\right)$$

$$k_3 = f\left(t_n + \frac{h}{2}, x_n + \frac{hk_2}{2}, y_n + \frac{hm_2}{2}\right)$$

$$k_4 = f(t_n + h, x_n + hk_3, y_n + hm_3)$$

and

$$m_1 = g(t_n, x_n, y_n)$$

$$m_2 = g\left(t_n + \frac{h}{2}, x_n + \frac{hk_1}{2}, y_n + \frac{hm_1}{2}\right)$$

$$m_3 = g\left(t_n + \frac{h}{2}, x_n + \frac{hk_2}{2}, y_n + \frac{hm_2}{2}\right)$$

$$m_4 = g(t_n + h, x_n + hk_3, y_n + hm_3).$$

# CHAPTER 6 REVIEW EXERCISES

In Exercises 1–9, find the eigenvalues and corresponding eigenvectors of $A$.

1. $A = \begin{pmatrix} -1 & 6 \\ 6 & 8 \end{pmatrix}$

2. $A = \begin{pmatrix} -3 & 6 \\ 4 & -1 \end{pmatrix}$

*3. $A = \begin{pmatrix} -1 & 2 \\ -1 & -4 \end{pmatrix}$

4. $A = \begin{pmatrix} 3 & -5 \\ 2 & 5 \end{pmatrix}$

5. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & -2 & 3 \end{pmatrix}$

6. $A = \begin{pmatrix} 3 & -2 & -3 \\ -2 & 0 & 1 \\ 0 & 0 & -3 \end{pmatrix}$

*7. $A = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 0 \\ 2 & 0 & 3 \end{pmatrix}$

8. $A = \begin{pmatrix} 0 & -3 & -2 \\ -4 & 3 & 4 \\ -3 & 0 & -3 \end{pmatrix}$

9. $A = \begin{pmatrix} -4 & 3 & 0 \\ 0 & -2 & 3 \\ 0 & -3 & -2 \end{pmatrix}$

In Exercises 10–34, find a general solution to each system or solve the initial-value problem.

10. $X' = \begin{pmatrix} -1 & -5 \\ 2 & 6 \end{pmatrix} X$

11. $X' = \begin{pmatrix} 2 & -4 \\ -1 & 5 \end{pmatrix} X$

12. $X' = \begin{pmatrix} -1 & -6 \\ 0 & -7 \end{pmatrix} X$, $X(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

*13. $\mathbf{X}' = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}\mathbf{X},\ \mathbf{X}(0) = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

14. $\mathbf{X}' = \begin{pmatrix} -4 & 5 \\ -5 & 4 \end{pmatrix}\mathbf{X}$

15. $\mathbf{X}' = \begin{pmatrix} -\frac{3}{2} & 5 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}\mathbf{X}$

16. $\mathbf{X}' = \begin{pmatrix} 4 & -5 \\ 4 & -4 \end{pmatrix}\mathbf{X}$

*17. $\mathbf{X}' = \begin{pmatrix} -4 & -1 \\ 5 & -2 \end{pmatrix}\mathbf{X}$

18. $\mathbf{X}' = \begin{pmatrix} 4 & 2 \\ -1 & 2 \end{pmatrix}\mathbf{X}$

19. $\mathbf{X}' = \begin{pmatrix} 0 & 8 \\ -2 & 0 \end{pmatrix}\mathbf{X}$

20. $\mathbf{X}' = \begin{pmatrix} 1 & 1 \\ -5 & -1 \end{pmatrix}\mathbf{X},\ \mathbf{X}(0) = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

*21. $\mathbf{X}' = \begin{pmatrix} 2 & -1 \\ 18 & -4 \end{pmatrix}\mathbf{X},\ \mathbf{X}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

22. $\mathbf{X}' = \begin{pmatrix} 0 & 2 \\ -2 & -4 \end{pmatrix}\mathbf{X}$

23. $\mathbf{X}' = \begin{pmatrix} 8 & -1 \\ 1 & 6 \end{pmatrix}\mathbf{X}$

24. $\mathbf{X}' = \begin{pmatrix} 1 & -3 \\ 1 & 5 \end{pmatrix}\mathbf{X} + \begin{pmatrix} -2e^{4t} \\ 2e^{4t} \end{pmatrix}$

*25. $\mathbf{X}' = \begin{pmatrix} 8 & -9 \\ 1 & -2 \end{pmatrix}\mathbf{X} - 8\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{7t}$

26. $\begin{cases} x' = 3x + 3y - 1 \\ y' = -4x - 10y + 1 \end{cases}$

27. $\begin{cases} x' = 3x + 4y + 1 \\ y' = 2x + y - 1 \end{cases}$

28. $\begin{cases} x' = -3x - \frac{5}{2}y + \frac{5}{2}\sin 2t \\ y' = 4x + 3y + 2\cos 2t - 3\sin 2t \\ x(0) = -1,\ y(0) = 1 \end{cases}$

*29. $\mathbf{X}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{X} + \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix},\ \mathbf{X}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

30. **(Modeling Testosterone Production)** The level of testosterone in men can be modeled by the system of delay equations

$$\begin{cases} \dfrac{dR}{dt} = f(T) - b_1 R \\ \dfrac{dL}{dt} = g_1 R - b_2 L \\ \dfrac{dT}{dt} = g_2 L(t - \tau) - b_3 T \end{cases} .*$$

Show that if $f(0) > 0$ and $f(T)$ is a one-to-one decreasing function, then the equilibrium point of this system is $(R_0, L_0, T_0)$, where

$$L_0 = \frac{b_3 T_0}{g_2},\ R_0 = \frac{b_3 b_2 T_0}{g_1 g_2},\quad \text{and}$$

$$f(T_0) - \frac{b_1 b_2 b_3 T_0}{g_1 g_2} = 0$$

and that the associated linearized system is

$$\begin{cases} \dfrac{dx}{dt} = f'(T_0)z - b_1 x \\ \dfrac{dy}{dt} = g_1 x - b_2 y \\ \dfrac{dz}{dt} = g_2 y(t - \tau) - b_3 z \end{cases} .$$

* J. D. Murray, *Mathematical Biology*, Springer-Verlag (1990), pp. 166–175.



## Differential Equations at Work:

### A. Modeling a Fox Population in which Rabies Is Present

Under various assumptions, the nonlinear system of differential equations

$$\begin{cases} \dfrac{dX}{dt} = rX - \gamma XN - \beta XY \\ \dfrac{dI}{dt} = \beta XY - (\sigma + b + \gamma N)I \\ \dfrac{dY}{dt} = \sigma I - (\alpha + \beta + \gamma N)Y \\ \dfrac{dN}{dt} = aX - (b + \gamma N)N - \alpha Y \end{cases}$$

has been successfully used to model a fox population in which rabies is present.* Here, $X(t)$ represents the population of foxes susceptible to rabies at time $t$, $I(t)$ the population that has contracted the rabies virus but is not yet ill, $Y(t)$ the population that has developed rabies, and $N(t)$ the total population of the foxes. The symbols $a$, $b$, $r$, $\gamma$, $\sigma$, $\alpha$, and $\beta$ represent constants and are described in the following table.

| Constant | Description | Typical Value(s) |
|---|---|---|
| $a$ | $a$ represents the *average per capita birth rate* of foxes. | 1 |
| $b$ | $1/b$ denotes *fox life expectancy* (without resource limitations), which is typically in the range of 1.5 to 2.7 years. | 0.5 |
| $r$ | $r = a - b$ represents the *intrinsic per capita population growth rate.* | 0.5 |
| $\gamma$ | $K = \dfrac{r}{\gamma}$ represents the *fox carrying capacity* of the defined area, which is typically in the range of 0.1 to 4 foxes per km$^2$. We will compute $K$ and $r$ and then approximate $\gamma$. | Varies |
| $\sigma$ | $1/\sigma$ represents the *average latent period.* This represents the average time (in years) that a fox can carry the rabies virus but not actually be ill with rabies. Typically, $1/\sigma$ is between 28 and 30 days. | 12.1667 |
| $\alpha$ | $\alpha$ represents the *death rate* of foxes with rabies. $1/\alpha$ is the life expectancy (in years) of a fox with rabies and is typically between 3 and 10 days. | 73 |
| $\beta$ | $\beta$ represents a *transmission coefficient.* Typically, $1/\beta$ is between 4 and 6 days. | 80 |

* Roy M. Anderson, Helen C. Jackson, Robert N. May & Anthony M. Smith, "Population dynamics of fox rabies in Europe," *Nature*, Volume 289 (February 26, 1981), pp. 765–771.

1. Generate a numerical solution to the system that satisfies the initial conditions $X(0) = 0.93$, $I(0) = 0.035$, $Y(0) = 0.035$, and $N(0) = 1.0$ valid for $0 \le t \le 40$ using the values given in the previous table if $K = 1, 2, 3, 4$, and 8. In each case, graph $X(t)$, $I(t)$, $Y(t)$, and $N(t)$ for $0 \le t \le 40$.
2. Repeat (1) using the initial conditions $X(0) = 0.93$, $I(0) = 0.02$, $Y(0) = 0.05$, and $N(0) = 2.0$.
3. For both (1) and (2), estimate the smallest value of $K$, say $K_T$, so that $Y(t)$ is a periodic function.
4. What happens to $Y(t)$ for $K < K_T$? Explain why this result does or does not make sense.
5. Define the **basic reproductive rate** $R$ to be

$$R = \frac{\sigma\beta K}{(\sigma + a)(\alpha + a)}$$

and

$$K_T = \frac{(\sigma + a)(\alpha + a)}{\sigma\beta}.$$

Show that if $R > 1$ then $K > K_T$ and if $R < 1$ then $K < K_T$.

6. Use the values in the table to calculate $K_T = [(\sigma + a)(\alpha + a)]/\sigma\beta$. Compare the result to your approximations in (3). How do they compare?
7. Predict how the solutions would change if the transmission coefficient $\beta$ were decreased or the death rate $\alpha$ were increased. What if the average latent period $\sigma$ were increased? Experiment with different conditions to see if you are correct.

## B. Controlling the Spread of a Disease

See the subsection **Modeling the Spread of a Disease** at the end of Chapter 2 for an introduction to the terminology used in this section.

If a person becomes immune to a disease after recovering from it, and births and deaths in the population are not taken into account, then the percentage of persons susceptible to becoming infected with the disease $S(t)$ the percentage of people in the population infected with the disease $I(t)$ and the percentage of the population recovered and immune to the disease $R(t)$ can be modeled by the system

$$\begin{cases} S'(t) = -\lambda SI \\ I'(t) = \lambda SI - \gamma I \\ R'(t) = \gamma I \\ S(0) = S_0,\ I(0) = I_0,\ R(0) = 0 \end{cases}.$$

Because $S(t) + I(t) + R(t) = 1$, once we know $S$ and $I$, we can compute $R$ with

$$R(t) = 1 - S(t) - I(t).$$

This model is called an **SIR model without vital dynamics** because once a person has had the disease, he becomes immune to it, and because births and deaths are not taken into consideration. This model might be used to model diseases that are **epidemic** to a population: those diseases that persist in a population for short periods of time (less than one year). Such diseases typically include influenza, measles, rubella, and chicken pox.



1. Show that if $S_0 < \frac{\gamma}{\lambda}$, the disease dies out, while an epidemic results if $S_0 > \frac{\gamma}{\lambda}$.
2. Show that $\frac{dI}{dS} = -\frac{(\lambda S - \gamma)I}{\lambda SI} = -1 + \frac{\rho}{S}$, where $\rho = \frac{\gamma}{\lambda}$, has solution

$$I + S - \rho \ln S = I_0 + S_0 - \rho \ln S_0.$$

3. What is the maximum value of $I$?

When diseases persist in a population for long periods of time, births and deaths must be taken into consideration. If a person becomes immune to a disease after recovering from it and births and deaths in the population are taken into account, then the percentage of persons susceptible to becoming infected with the disease $S(t)$ and the percentage of people in the population infected with the disease $I(t)$ can be modeled by the system

$$\begin{cases} S'(t) = -\lambda SI + \mu - \mu S \\ I'(t) = \lambda SI - \gamma I - \mu I \\ S(0) = S_0,\ I(0) = I_0 \end{cases}.^*$$

This model is called an **SIR model with vital dynamics** because once a person has had the disease he becomes immune to it, and because births and deaths are taken into consideration. This model might be used to model diseases that are **endemic** to a population: those diseases that persist in a population for long periods of time (ten or twenty years). Smallpox is an example of a disease that was endemic until it was eliminated in 1977.

4. Show that the equilibrium points of the system $\begin{cases} S'(t) = -\lambda SI + \mu - \mu S \\ I'(t) = \lambda SI - \gamma I - \mu I \end{cases}$ are $S = 1, I = 0$ and $S = \frac{\gamma + \mu}{\lambda}, I = \frac{\mu[\lambda - (\gamma + \mu)]}{\lambda(\gamma + \mu)}$.

Because $S(t) + I(t) + R(t) = 1$, it follows that $S(t) + I(t) \le 1$.

5. Use the fact that $S(t) + I(t) \le 1$ to determine conditions on $\gamma$, $\mu$, and $\lambda$ so that the system $\begin{cases} S'(t) = -\lambda SI + \mu - \mu S \\ I'(t) = \lambda SI - \gamma I - \mu I \end{cases}$ has the equilibrium point $S = \frac{\gamma + \mu}{\lambda}$, $I = \frac{\mu[\lambda - (\gamma + \mu)]}{\lambda(\gamma + \mu)}$. In this case, classify the equilibrium point.

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* Herbert W. Hethcote, "Three Basic Epidemiological Models," *Applied Mathematical Ecology*, edited by Simon A. Levin, Thomas G. Hallan, and Louis J. Gross, Springer-Verlag (1989), pp. 119–143. Roy M. Anderson and Robert M. May, "Directly Transmitted Infectious Diseases: Control by Vaccination," *Science*, Volume 215 (February 26, 1982), pp. 1053–1060. J. D. Murray, *Mathematical Biology*, Springer-Verlag, 1990, pp. 611–618.

6. Use the fact that $S(t) + I(t) \le 1$ to determine conditions on $\gamma$, $\mu$, and $\lambda$ so that the system $\begin{cases} S'(t) = -\lambda SI + \mu - \mu S \\ I'(t) = \lambda SI - \gamma I - \mu I \end{cases}$ does *not* have the equilibrium point $S = \dfrac{\gamma + \mu}{\lambda}$, $I = \dfrac{\mu[\lambda - (\gamma + \mu)]}{\lambda(\gamma + \mu)}$.

The following table shows the average infectious period and typical contact numbers for several diseases during certain epidemics.

| Disease | Infectious Period (Average) $1/\gamma$ | $\gamma$ | Typical contact number $\sigma$ |
|---|---|---|---|
| Measles | 6.5 | 0.153846 | 14.9667 |
| Chicken pox | 10.5 | 0.0952381 | 11.3 |
| Mumps | 19 | 0.0526316 | 8.1 |
| Scarlet fever | 17.5 | 0.0571429 | 8.5 |

Let's assume that the average lifetime $1/\mu$ is 70 years so that $\mu = 0.0142857$.

7. For each of the diseases listed in the following table, use the formula $\sigma = \lambda/(\gamma + \mu)$ to calculate the daily contact rate $\lambda$.

| Disease | $\lambda$ |
|---|---|
| Measles | |
| Chicken pox | |
| Mumps | |
| Scarlet fever | |

Diseases such as those listed above can be controlled once an effective and inexpensive vaccine has been developed. It is virtually impossible to vaccinate everybody against a disease; we would like to determine the percentage of a population that needs to be vaccinated to eliminate a disease from the population under consideration. A population of people has **herd immunity** to a disease when enough people are immune to the disease so that if it is introduced into the population, it will not spread throughout the population. To have herd immunity, an infected person must infect less than one uninfected person during the time the person is infectious. Thus, we must have

$$\sigma S < 1.$$

Because $I + S + R = 1$, when $I = 0$ we have that $S = 1 - R$; consequently, herd immunity is achieved when



$$\begin{aligned} \sigma(1 - R) &< 1 \\ \sigma - \sigma R &< 1 \\ -\sigma R &< 1 - \sigma \\ R &> \frac{\sigma - 1}{\sigma} = 1 - \frac{1}{\sigma}. \end{aligned}$$

8. For each of the diseases listed in the following table, estimate the minimum percentage of a population that needs to be vaccinated to achieve herd immunity.

| Disease | Minimum Value of $R$ to Achieve Herd Immunity |
|---|---|
| Measles | |
| Chicken pox | |
| Mumps | |
| Scarlet fever | |

9. Using the values obtained in the previous exercises, for each disease in the tables graph the direction field and several solutions $\left(I(t), S(t), R(t), \text{and} \begin{Bmatrix} S(t) \\ I(t) \end{Bmatrix}\right)$ using both models. Discuss scenarios in which each model is valid and note any significant differences between the two models.

10. What are some possible ways that an epidemic can be controlled?

## C. FitzHugh-Nagumo Model

Under certain assumptions, the **FitzHugh-Nagumo equation,** which arises in the study of the impulses in a nerve fiber, can be written as the system of ordinary differential equations

$$\begin{cases} \dfrac{dV}{d\xi} = W \\ \dfrac{dW}{d\xi} = F(V) + R - uW, \\ \dfrac{dR}{d\xi} = \dfrac{\epsilon}{u}(bR - V - a) \end{cases}$$

where $F(V) = \frac{1}{3}V^3 - V$.*

* J. D. Murray, *Mathematical Biology,* Springer-Verlag (1990), pp. 161–166. Alwyn C. Scott, "The electrophysics of a nerve fiber," *Reviews of Modern Physics,* Vol. 47, No. 2 (April 1975), pp. 487–533.

1. Graph the solution to the FitzHugh-Nagumo equation that satisfies the initial conditions $V(0) = 1$, $W(0) = 0$, $R(0) = 1$ if $\epsilon = 0.08$, $a = 0.7$, $b = 0$, and $u = 1$.
2. Graph the solution that satisfies the initial conditions $V(0) = 1$, $W(0) = 0.5$, $R(0) = 0.5$ if $\epsilon = 0.08$, $a = 0.7$, $b = 0.8$, and $u = 0.6$.
3. Approximate the maximum and minimum values, if they exist, of $V$, $W$, and $R$ in (1) and (2).

# 7

# Applications of Systems of Ordinary Differential Equations

## 7.1 Mechanical and Electrical Problems with First-Order Linear Systems



### *L-R-C* Circuits with Loops

As indicated in Chapter 5, an electrical circuit can be modeled with a linear ordinary differential equation with constant coefficients. In this section, we illustrate how a circuit involving loops can be described as a system of linear ordinary differential equations with constant coefficients. This derivation is based on the following principles.

| **Kirchhoff's Current Law:** |
|---|
| The current entering a point of the circuit equals the current leaving the point. |

| Kirchhoff's Voltage Law: |
|---|
| The sum of the changes in voltage around each loop in the circuit is zero. |

As was the case in Chapter 5, we use the following standard symbols for the components of the circuit:

$$I(t) = \text{current where } I(t) = \frac{dQ}{dt}(t),\ Q(t) = \text{charge},\ R = \text{resistance},\ C = \text{capacitance},$$
$$E = \text{voltage, and } L = \text{inductance}.$$

The relationships corresponding to the drops in voltage in the various components of the circuit are restated in Table 7.1.

**TABLE 7.1**

| Circuit Element | Voltage Drop |
|---|---|
| Inductor | $L\frac{dI}{dt}$ |
| Resistor | $RI$ |
| Capacitor | $\frac{1}{C}Q$ |
| Voltage Source | $-E(t)$ |

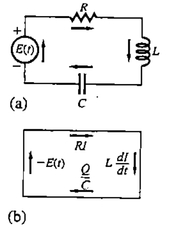



**Figure 7.1** (a)–(b) A simple *L-R-C* circuit.

### *L-R-C* Circuit with One Loop

In determining the drops in voltage around the circuit, we consistently add the voltages in the clockwise direction. The positive direction is from the negative symbol toward the positive symbol associated with the voltage source. In summing the voltage drops encountered in the circuit, a drop across a component is added to the sum if the positive direction through the component agrees with the clockwise direction. Otherwise this drop is subtracted. In the case of the *L-R-C* circuit with one loop involving each type of component shown in Figure 7.1, the current is equal around the circuit by Kirchhoff's current law.

Also, by Kirchhoff's voltage law, we have

$$RI + L\frac{dI}{dt} + \frac{1}{C}Q - E(t) = 0.$$

Solving this equation for $dI/dt$ and using the relationship $dQ/dt = I$, we have the system of differential equations



$$\begin{cases} \dfrac{dQ}{dt} = I \\ \dfrac{dI}{dt} = -\dfrac{1}{LC}Q - \dfrac{R}{L}I + \dfrac{E(t)}{L} \end{cases}$$

with initial conditions $Q(0) = Q_0$ and $I(0) = I_0$ on charge and current, respectively. The method of variation of parameters (for systems) can be used to solve problems of this type.

**Example 1**

Determine the charge and current in the *L-R-C* circuit with $L = 1$ henry, $R = 2$ ohms, $C = 4/3$ farads and $E(t) = e^{-t}$ if $Q(0) = 1$ and $I(0) = 1$.

***Solution*** We begin by modeling the circuit with a system of differential equations. In this case, we have

$$\begin{cases} \dfrac{dQ}{dt} = I \\ \dfrac{dI}{dt} = -\dfrac{3}{4}Q - 2I + e^{-t} \end{cases},$$

with initial conditions $Q(0) = 1$ and $I(0) = 1$. We can write this nonhomogeneous system in matrix form as

$$\begin{pmatrix} \dfrac{dQ}{dt} \\ \dfrac{dI}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{3}{4} & -2 \end{pmatrix}\begin{pmatrix} Q \\ I \end{pmatrix} + \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix}.$$

The eigenvalues of the corresponding homogeneous system are $\lambda_1 = -1/2$ and $\lambda_2 = -3/2$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, respectively. Thus a fundamental matrix is

$$\Phi(t) = \begin{pmatrix} -2e^{-t/2} & 2e^{-3t/2} \\ e^{-t/2} & -3e^{-3t/2} \end{pmatrix}$$

and

$$\Phi^{-1}(t) = \begin{pmatrix} -\dfrac{3}{4}e^{t/2} & -\dfrac{1}{2}e^{t/2} \\ -\dfrac{1}{4}e^{3t/2} & -\dfrac{1}{2}e^{3t/2} \end{pmatrix}.$$

Then, by variation of parameters, if we let $\mathbf{X}(t) = \begin{pmatrix} Q(t) \\ I(t) \end{pmatrix}$, we have

$$\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(0)\mathbf{X}(0) + \mathbf{\Phi}(t)\int_0^t \mathbf{\Phi}^{-1}(u)\mathbf{F}(u)\,du$$

$$= \begin{pmatrix} -2e^{-t/2} & 2e^{-3t/2} \\ e^{-t/2} & -3e^{-3t/2} \end{pmatrix}\begin{pmatrix} -\frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$+ \begin{pmatrix} -2e^{-t/2} & 2e^{-3t/2} \\ e^{-t/2} & -3e^{-3t/2} \end{pmatrix}\int_0^t \begin{pmatrix} -\frac{3}{4}e^{u/2} & -\frac{1}{2}e^{u/2} \\ -\frac{1}{4}e^{3u/2} & -\frac{1}{2}e^{3u/2} \end{pmatrix}\begin{pmatrix} 0 \\ e^{-u} \end{pmatrix} du$$

$$= \begin{pmatrix} -2e^{-t/2} & 2e^{-3t/2} \\ e^{-t/2} & -3e^{-3t/2} \end{pmatrix}\begin{pmatrix} -\frac{5}{4} \\ -\frac{3}{4} \end{pmatrix} + \begin{pmatrix} -2e^{-t/2} & 2e^{-3t/2} \\ e^{-t/2} & -3e^{-3t/2} \end{pmatrix}\int_0^t \begin{pmatrix} -\frac{1}{2}e^{-u/2} \\ -\frac{1}{2}e^{u/2} \end{pmatrix} du$$

$$= \begin{pmatrix} \frac{5}{2}e^{-t/2} - \frac{3}{2}e^{-3t/2} \\ -\frac{5}{4}e^{-t/2} + \frac{9}{4}e^{-3t/2} \end{pmatrix} + \begin{pmatrix} -4e^{-t} + 2e^{-t/2} + 2e^{-3t/2} \\ 4e^{-t} - e^{-t/2} - 3e^{-3t/2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{2}e^{-t/2} + \frac{1}{2}e^{-3t/2} - 4e^{-t} \\ -\frac{9}{4}e^{-t/2} - \frac{3}{4}e^{-3t/2} + 4e^{-t} \end{pmatrix}.$$

We plot the solution $\mathbf{X}(t) = \begin{pmatrix} Q(t) \\ I(t) \end{pmatrix}$ parametrically in Figure 7.2(a). Notice that $\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} I(t) = 0$, so the solution approaches (0, 0) as $t$ increases. We also plot $Q(t)$ and $I(t)$ simultaneously in Figure 7.2(b). Finally, in Figure 7.2(c), we graph $\mathbf{X}(t) = \begin{pmatrix} Q(t) \\ I(t) \end{pmatrix}$ for other initial conditions.

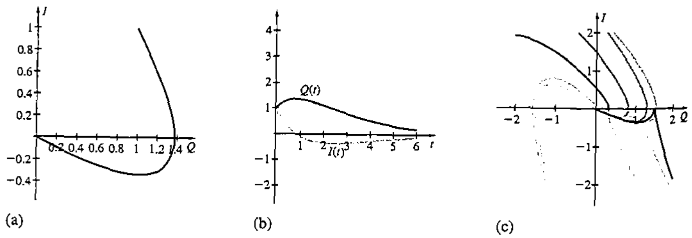



**Figure 7.2** (a)–(c).



*In Example 1, do the limits $\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} I(t) = 0$ hold for all choices of the initial conditions?*

## L-R-C Circuit with Two Loops

The differential equations that model the circuit become more difficult to derive as the number of loops in the circuit increase. For example, consider the circuit that contains two loops, as shown in Figure 7.3.

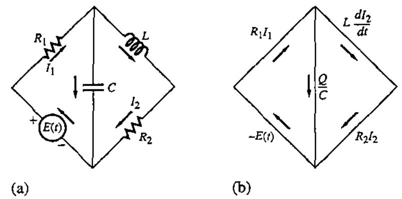


**Figure 7.3** (a)–(b) A two-loop circuit.

In this case, the current through the capacitor is equivalent to $I_1 - I_2$. Summing the voltage drops around each loop, we obtain the system of equations

$$\begin{cases} R_1 I_1 + \dfrac{1}{C}Q - E(t) = 0 \\ L\dfrac{dI_2}{dt} + R_2 I_2 - \dfrac{1}{C}Q = 0 \end{cases}.$$

Solving the first equation for $I_1$ yields

$$I_1 = \frac{1}{R_1}E(t) - \frac{1}{R_1 C}Q.$$

Using the relationship $dQ/dt = I = I_1 - I_2$ gives us the system

$$\begin{cases} \dfrac{dQ}{dt} = -\dfrac{1}{R_1 C}Q - I_2 + \dfrac{E(t)}{R_1} \\ \dfrac{dI_2}{dt} = \dfrac{1}{LC}Q - \dfrac{R_2}{L}I_2 \end{cases}.$$

**Example 2**

Find $Q(t)$, $I(t)$, $I_1(t)$, and $I_2(t)$ in the $L$-$R$-$C$ circuit with two loops given that $R_1 = R_2 = 1$ ohm, $C = 1$ farad, $L = 1$ henry, and $E(t) = e^{-t}$ if $Q(0) = 1$ and $I_2(0) = 3$.

**Solution** The nonhomogeneous system that models this circuit is

$$\begin{cases} \dfrac{dQ}{dt} = -Q - I_2 + e^{-t} \\ \dfrac{dI_2}{dt} = Q - I_2 \end{cases},$$

with initial conditions $Q(0) = 1$ and $I_2(0) = 3$. As in Example 1, we use the method of variation of parameters to solve the problem. In matrix form this system is

$$\begin{pmatrix} \dfrac{dQ}{dt} \\ \dfrac{dI_2}{dt} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} Q \\ I_2 \end{pmatrix} + \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}.$$

The eigenvalues of the corresponding homogeneous system are $\lambda_{1,2} = -1 \pm i$, and an eigenvector corresponding to $\lambda_1 = -1 + i$ is $\mathbf{v}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Two linearly independent solutions of the corresponding homogeneous system are

$$\mathbf{X}_1(t) = \begin{pmatrix} Q(t) \\ I_2(t) \end{pmatrix} = e^{-t}\cos t\begin{pmatrix} 0 \\ 1 \end{pmatrix} - e^{-t}\sin t\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -e^{-t}\sin t \\ e^{-t}\cos t \end{pmatrix}$$

and

$$\mathbf{X}_2(t) = \begin{pmatrix} Q(t) \\ I_2(t) \end{pmatrix} = e^{-t}\sin t\begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{-t}\cos t\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-t}\cos t \\ e^{-t}\sin t \end{pmatrix},$$

so a fundamental matrix is

$$\Phi(t) = \begin{pmatrix} -e^{-t}\sin t & e^{-t}\cos t \\ e^{-t}\cos t & e^{-t}\sin t \end{pmatrix}$$

and

$$\Phi^{-1}(t) = \begin{pmatrix} -e^{t}\sin t & e^{t}\cos t \\ e^{t}\cos t & e^{t}\sin t \end{pmatrix}.$$

Therefore,

$$\mathbf{X}(t) = \Phi(t)\Phi^{-1}(0)\mathbf{X}(0) + \Phi(t)\int_0^t \Phi^{-1}(u)\mathbf{F}(u)\,du$$

$$= \begin{pmatrix} -e^{-t}\sin t & e^{-t}\cos t \\ e^{-t}\cos t & e^{-t}\sin t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$+ \begin{pmatrix} -e^{-t}\sin t & e^{-t}\cos t \\ e^{-t}\cos t & e^{-t}\sin t \end{pmatrix}\int_0^t \begin{pmatrix} -e^{u}\sin u & e^{u}\cos u \\ e^{u}\cos u & e^{u}\sin u \end{pmatrix}\begin{pmatrix} e^{-u} \\ 0 \end{pmatrix}du$$

$$= \begin{pmatrix} -e^{-t}\sin t & e^{-t}\cos t \\ e^{-t}\cos t & e^{-t}\sin t \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -e^{-t}\sin t & e^{-t}\cos t \\ e^{-t}\cos t & e^{-t}\sin t \end{pmatrix}\int_0^t \begin{pmatrix} -\sin u \\ \cos u \end{pmatrix}du$$



$$= \begin{pmatrix} e^{-t}\cos t - 3e^{-t}\sin t \\ 3e^{-t}\cos t + e^{-t}\sin t \end{pmatrix} + \begin{pmatrix} -e^{-t}\sin t & e^{-t}\cos t \\ e^{-t}\cos t & e^{-t}\sin t \end{pmatrix}\begin{pmatrix} \cos t - 1 \\ \sin t \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t}\cos t - 3e^{-t}\sin t \\ 3e^{-t}\cos t + e^{-t}\sin t \end{pmatrix} + \begin{pmatrix} e^{-t}\sin t \\ e^{-t} - e^{-t}\cos t \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t}\cos t - 2e^{-t}\sin t \\ 2e^{-t}\cos t + e^{-t} + e^{-t}\sin t \end{pmatrix}.$$

Because $dQ/dt = I$ and $Q(t) = e^{-t}\cos t - 2e^{-t}\sin t$, differentiation yields

$I(t) = -e^{-t}\cos t - e^{-t}\sin t + 2e^{-t}\sin t - 2e^{-t}\cos t = -3e^{-t}\cos t + e^{-t}\sin t.$

Also, because $I_1(t) = I(t) + I_2(t)$,

$$I_1(t) = I(t) + I_2(t) = -3e^{-t}\cos t + e^{-t}\sin t + e^{-t}\sin t + 2e^{-t}\cos t + e^{-t}$$
$$= -e^{-t}\cos t + 2e^{-t}\sin t + e^{-t}.$$

We graph $Q(t)$, $I_1(t)$, $I_2(t)$, and $I(t)$ together in Figure 7.4(a). In Figure 7.4(b), we graph $\begin{cases} Q(t) \\ I_2(t) \end{cases}$ parametrically to show the phase plane for the system of nonhomogeneous equations using several different initial conditions. Notice that some of the solutions overlap, which does not occur if a system is homogeneous.

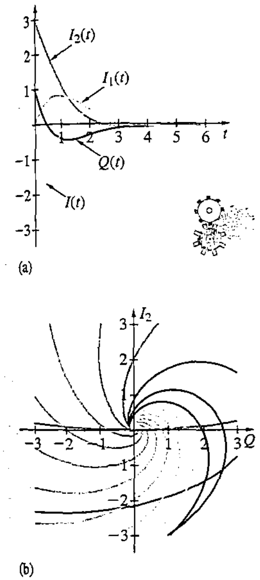



Figure 7.4 (a)–(b).

*Find the limit of $Q(t)$, $I_1(t)$, $I_2(t)$, and $I(t)$ as $t \to \infty$. Does a change in initial conditions affect these limits?*

## Spring–Mass Systems

The displacement of a mass attached to the end of a spring was modeled with a second-order linear differential equation with constant coefficients in Chapter 5. This situation can be expressed as a system of first-order ordinary differential equations as well. Recall that if there is no external forcing function, the second-order differential equation that models the situation is

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0,$$

where $m$ is the mass of the object attached to the end of the spring, $c$ is the damping coefficient, and $k$ is the spring constant found with Hooke's law. This equation is transformed into a system of equations with the substitution $dx/dt = y$. Then, solving the differential equation for $d^2x/dt^2$, we have $dy/dt = d^2x/dt^2 = -k/m\,x - c/m\,dx/dt$, which yields the system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\dfrac{k}{m}x - \dfrac{c}{m}y \end{cases}.$$

In previous chapters, the displacement of the spring was illustrated as a function of time. Problems of this type may also be investigated using the phase plane. In the following example, the phase plane corresponding to the various situations encountered by spring–mass systems discussed in previous sections (undamped, damped, overdamped, and critically damped) are determined.

**Example 3**

Solve the system of differential equations to find the displacement of the mass if $m = 1$, $c = 0$, and $k = 1$.

**Solution** In this case, the system is

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x \end{cases}.$$

The eigenvalues are solutions of $\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$, so $\lambda_{1,2} = \pm i$. An eigenvector corresponding to $\lambda_1 = i$ is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so two linearly independent solutions are $\mathbf{X}_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$ and

$$\mathbf{X}_2(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

A general solution is

$$\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\mathbf{X}_1(t) + c_2\mathbf{X}_2(t) = \begin{pmatrix} c_1 \cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{pmatrix}.$$

Notice that this system is equivalent to the second-order differential equation $d^2x/dt^2 + x = 0$, which we solved in Chapters 4 and 5. At that time, we found a general solution to be $x(t) = c_1 \cos t + c_2 \sin t$, which is the same as the first component of $\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ obtained in this instance. We graph this function for several values of the arbitrary constants in Figure 7.5(a) to illustrate the periodic motion of the mass. Also notice that (0, 0) is the equilibrium point of the system. Because the eigenvalues are $\lambda = \pm i$, we classify the origin as a center. We graph the phase plane of this system in Figure 7.5(b).

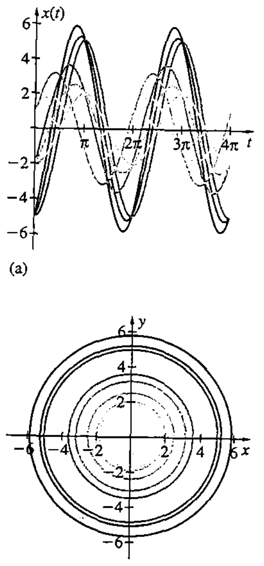



**Figure 7.5** (a)–(b).

*By observing the phase plane in Figure 7.5 and the corresponding system of differential equations in Example 3, describe the motion of the object in each quadrant and determine the sign on the velocity dx/dt of the object in each quadrant.*



## EXERCISES 7.1

Solve each of the following systems for charge and current using the procedures discussed in the example problems.

*1. Solve the one-loop $L$-$R$-$C$ circuit with $L = 3$ henrys, $R = 10$ ohms, $C = 0.1$ farad, and **(a)** $E(t) = 0$; **(b)** $E(t) = e^{-t}$. ($Q(0) = 0$ coulombs, $I(0) = 1$ amp.)

2. Solve the one-loop $L$-$R$-$C$ circuit with $L = 1$ henry, $R = 20$ ohms, $C = 0.01$ farad, and **(a)** $E(t) = 0$; **(b)** $E(t) = 1200$. ($Q(0) = 0$ coulombs, $I(0) = 1$ amp.)

3. Find $Q(t)$, $I(t)$, $I_1(t)$, and $I_2(t)$ in the two-loop $L$-$R$-$C$ circuit with $L = 1$, $R_1 = 2$, $R_2 = 1$, $C = 1/2$, and **(a)** $E(t) = 0$; **(b)** $E(t) = 2e^{-t/2}$. ($Q(0) = 10^{-6}$ coulomb, $I_2(0) = 0$ amps.)

4. Find $Q(t)$, $I(t)$, $I_1(t)$, and $I_2(t)$ in the two-loop $L$-$R$-$C$ circuit, with $L = 1$, $R_1 = 2$, $R_2 = 1$, $C = 1/2$, and **(a)** $E(t) = 90$; **(b)** $E(t) = 90 \sin t$. ($Q(0) = 0$ coulomb, $I_2(0) = 0$ amps.)

*5. Find $Q(t)$, $I(t)$, $I_1(t)$, and $I_2(t)$ in the two-loop $L$-$R$-$C$ circuit, with $L = 1$, $R_1 = 1$, $R_2 = 3$, $C = 1$, and **(a)** $E(t) = 0$, $Q(0) = 10^{-6}$, and $I_2(0) = 0$; **(b)** $E(t) = 90$, $Q(0) = 0$, $I_2(0) = 0$.

6. Consider the circuit made up of three loops illustrated in Figure 7.6. In this circuit, the current through the resistor $R_2$ is $I_2 - I_3$, and the current through the capacitor is $I_1 - I_2$. Using these quantities in the voltage-drop sum equations, model this circuit with the three-dimensional system:

$$\begin{cases} -E(t) + R_1I_1 + \dfrac{1}{C}Q = 0 \\ -\dfrac{1}{C}Q + L_2\dfrac{dI_2}{dt} + R_2(I_2 - I_3) = 0 \\ E(t) - R_2(I_2 - I_1) + L_3\dfrac{dI_3}{dt} = 0 \end{cases}$$

Using the relationship $dQ/dt = I_1 - I_2$ and solving the first equation for $I_1$, show that we obtain the system

$$\begin{cases} \dfrac{dQ}{dt} = -\dfrac{1}{R_1C}Q - I_2 + \dfrac{E(t)}{R_1} \\ \dfrac{dI_2}{dt} = \dfrac{1}{L_2C}Q - \dfrac{R_2}{L_2}I_2 + \dfrac{R_2}{L_2}I_3 \\ \dfrac{dI_3}{dt} = \dfrac{R_2}{L_3}I_2 - \dfrac{R_2}{L_3}I_3 - \dfrac{E(t)}{L_3} \end{cases}$$

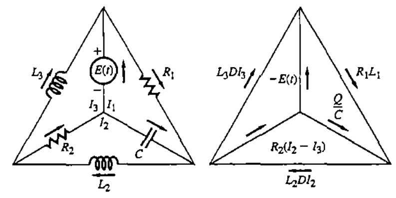



**Figure 7.6** (a)–(b) A three-loop circuit.

In Exercises 7–11, solve the three-loop circuit using the given values and initial conditions.

7. $L_2 = L_3 = 1$ henry, $R_1 = 1$ ohm, $R_2 = 1$ ohm, $C = 1$ farad, and **(a)** $E(t) = 0$ volts; **(b)** $E(t) = e^{-t}$ volts. ($Q(0) = 10^{-6}$ coulomb, $I_2(0) = I_3(0) = 0$ amps.)

8. $L_2 = L_3 = 1$ henry, $R_1 = R_2 = 1$ ohm, $C = 1$ farad, and **(a)** $E(t) = 90$ volts; **(b)** $E(t) = 90 \sin t$ volts. ($Q(0) = 0$ coulomb, $I_2(0) = I_3(0) = 0$ amps.)

*9. $L_2 = 1$ henry, $L_3 = 1$ henry, $R_1 = R_2 = 1$ ohm, $C = 1$ farad, and **(a)** $E(t) = 90$ volts; **(b)** $E(t) = 90 \sin t$ volts. ($Q(0) = 0$ coulombs, $I_2(0) = 1$ amp, $I_3(0) = 0$ amps.)

10. $L_2 = 3$ henrys, $L_3 = 1$ henry, $R_1 = R_2 = 1$ ohm, $C = 1$ farad, and **(a)** $E(t) = 0$ volts; **(b)** $E(t) = 90 \sin t$ volts. ($Q(0) = 0$ coulombs, $I_2(0) = 1$ amp, $I_3(0) = 0$ amps.)

11. $L_2 = 4$ henrys, $L_3 = 1$ henry, $R_1 = R_2 = 1$ ohm, $C = 1$ farad, and **(a)** $E(t) = 90$ volts; **(b)** $E(t) = 90 \sin t$ volts. ($Q(0) = 0$ coulombs, $I_2(0) = 1$ amp, $I_3(0) = 0$ amps.)

12. Show that the system of differential equations that models the four-loop circuit shown in Figure 7.7 is

$$\begin{cases} L_1\dfrac{dI_1}{dt} = -(R_1 + R_2)I_1 + R_2I_2 + R_1I_4 + E(t) \\ L_2\dfrac{dI_2}{dt} = R_2I_1 - (R_2 + R_3)I_2 + R_3I_3 \\ L_3\dfrac{dI_3}{dt} = R_3I_2 - (R_3 + R_4)I_3 + R_4I_4 \\ L_4\dfrac{dI_4}{dt} = R_1I_1 + R_4I_3 - (R_1 + R_4)I_4 \end{cases}$$

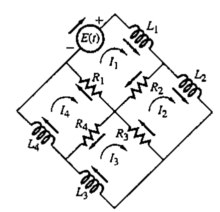



**Figure 7.7** A four-loop circuit.

In Exercises 13–20, transform the second-order equation to a system of first-order equations and classify the system as undamped, overdamped, underdamped, or critically damped by finding the eigenvalues of the corresponding system. Also classify the equilibrium point (0, 0).

***13.** $\frac{1}{2}x'' + \frac{9}{2}x = 0$

**14.** $\frac{1}{4}x'' + 4x = 0$

**15.** $\frac{1}{2}x'' + 5x' + \frac{9}{2}x = 0$

**16.** $\frac{1}{4}x'' + x' + \frac{5}{4}x = 0$

***17.** $\frac{1}{2}x'' + 5x' + 25x = 0$

**18.** $\frac{1}{3}x'' + \frac{4}{3}x' + \frac{13}{3}x = 0$

**19.** $\frac{1}{2}x'' + 5x' + \frac{25}{2}x = 0$

**20.** $x'' + 6x' + 9x = 0$

***21.** Solve Exercises 13, 15, and 17 with the initial conditions $x(0) = 1$, $x'(0) = y(0) = 0$.

**22.** Solve Exercises 17–20 with the initial conditions $x(0) = 0$, $x'(0) = y(0) = 1$.

**23.** Find the eigenvalues for the spring–mass system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\dfrac{k}{m}x - \dfrac{c}{m}y \end{cases}$$

How do these values relate to overdamping, critical damping, and underdamping?

**24.** **(a)** Find the equilibrium point of the spring–mass system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\dfrac{k}{m}x - \dfrac{c}{m}y \end{cases}$$

**(b)** Find restrictions on $m$, $c$, and $k$ to classify this point as a center, stable node, or stable spiral.

**(c)** Can the equilibrium point be unstable for any choice of the positive constants $m$, $c$, and $k$? Is a saddle possible?

Use a graphing device to graph the solutions to Exercises 25–28 simultaneously and parametrically. Also determine the limit of these solutions as $t \to \infty$.

**25.** Solve the one-loop $L$-$R$-$C$ circuit with $L = 1$, $R = 40$, $C = 0.004$, and $E(t) = 120 \sin t$. ($Q(0) = 0$ coulombs, $I(0) = 0$ amps.)

**26.** Solve the one-loop $L$-$R$-$C$ circuit with $L = 4$, $R = 80$, $C = 0.08$, and $E(t) = 120e^{-t} \sin t$. ($Q(0) = 10^{-6}$ coulomb, $I(0) = 0$ amps.)

**27.** Solve the two-loop $L$-$R$-$C$ circuit with $L = 1$, $R_1 = R_2 = 40$, $C = 0.004$, and $E(t) = 220 \cos t$. ($Q(0) = 0$ coulombs, $I_2(0) = 0$ amps.)

**28.** Solve the two-loop $L$-$R$-$C$ circuit with $L = 1$, $R_1 = 40$, $R_2 = 80$, $C = 0.004$, and $E(t) = 150e^{-t} \cos t$. ($Q(0) = 10^{-6}$ coulomb, $I_2(0) = 0$ amps.)

**29.** Use the system derived in Exercise 6 to solve the three-loop circuit shown in Figure 7.6 if $R_1 = R_2 = 2$, $L_1 = L_2 = L_3 = 1$, $C = 1$, $E(t) = 90$, and the initial conditions are $Q(0) = 0$, $I_2(0) = I_3(0) = 0$. Find $I_1(t)$ and determine $\lim_{t\to\infty} Q(t)$, $\lim_{t\to\infty} I_1(t)$, $\lim_{t\to\infty} I_2(t)$, and $\lim_{t\to\infty} I_3(t)$.

**30.** Solve the system of differential equations to find the displacement of the spring–mass system if $m = 1$, $c = 1$, and $k = 1/2$. Graph several solutions in the phase plane for this system. How is the equilibrium point (0, 0) classified?

***31.** Solve the system of differential equations to find the displacement of the spring–mass system given that $m = 1$, $c = 2$, and $k = 3/4$. Graph several solutions in the phase plane for this system. How is the equilibrium point (0, 0) classified?



## 7.2 Diffusion and Population Problems with First-Order Linear Systems

- Diffusion Through a Membrane
- Mixture Problems
- Population Problems

### Diffusion Through a Membrane

Solving problems to determine the diffusion of a substance (such as glucose or salt) in a medium (such as a blood cell) also leads to first-order systems of linear ordinary differential equations. For example, consider the situation shown in Figure 7.8 in which two solutions of a substance are separated by a membrane. The amount of substance that passes through the membrane at any particular time is proportional to the difference in the concentrations of the two solutions. The constant of proportionality, $P$, is called the **permeability** of the membrane and describes the ability of the substance to permeate the membrane (where $P > 0$). If we let $x(t)$ and $y(t)$ represent the amount of substance at time $t$ on each side of the membrane, and $V_1$ and $V_2$ represent the (constant) volume of each solution, respectively, then the system of differential equations is given by

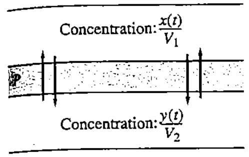



**Figure 7.8**

$$\begin{cases} \dfrac{dx}{dt} = P\left(\dfrac{y}{V_2} - \dfrac{x}{V_1}\right), \\ \dfrac{dy}{dt} = P\left(\dfrac{x}{V_1} - \dfrac{y}{V_2}\right) \end{cases}$$

where the initial amounts of $x$ and $y$ are given with the initial conditions $x(0) = x_0$ and $y(0) = y_0$. (Notice that the amount of the substance divided by the volume is the *concentration* of the solution.)

*In this system, if $y(t)/V_2 > x(t)/V_1$, is $dx/dt > 0$ or is $dx/dt < 0$? Also, is $dy/dt > 0$ or is $dy/dt < 0$? Using these results, does the substance move from the side with a lower concentration to that with a higher concentration, or is the opposite true?*

**Example 1**

Suppose that two salt concentrations of equal volume $V$ are separated by a membrane of permeability $P$. Given that $P = V$, determine the amount of salt in each concentration at time $t$ if $x(0) = 2$ and $y(0) = 10$.

***Solution*** In this case, the initial-value problem that models the situation is

$$\begin{cases} \dfrac{dx}{dt} = y - x \\ \dfrac{dy}{dt} = x - y \end{cases}, \quad x(0) = 2,\ y(0) = 10.$$

The eigenvalues of $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ are $\lambda_1 = 0$ and $\lambda_2 = -2$. Corresponding eigenvectors are found to be $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, so a general solution is

$$\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} = \begin{pmatrix} c_1 - c_2 e^{-2t} \\ c_1 + c_2 e^{-2t} \end{pmatrix}.$$

Because $\mathbf{X}(0) = \begin{pmatrix} c_1 - c_2 \\ c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \end{pmatrix}$, we have that $c_1 = 6$ and $c_2 = 4$, so the solution is

$$\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 6 - 4e^{-2t} \\ 6 + 4^{-2t} \end{pmatrix}.$$

We graph this solution parametrically in Figure 7.9(a). We then graph $x(t)$ and $y(t)$ together in Figure 7.9(b). Notice that the amount of salt in each concentration approaches 6, which is the average value of the two initial amounts.

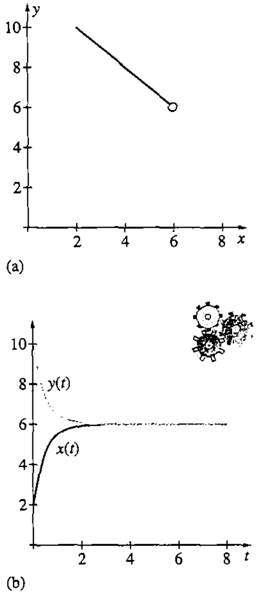



**Figure 7.9** (a)–(b).

*In Example 1, if $x(0) = x_0$ and $y(0) = y_0$, does $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = \frac{1}{2}(x_0 + y_0)$?*

## Mixture Problems

Consider the interconnected tanks that are shown in Figure 7.10 in which a salt solution is allowed to flow according to the given information. Let $x(t)$ and $y(t)$ represent the amount of salt in Tank 1 and Tank 2, respectively. Using this information, we set up two differential equations to describe the rate at which $x$ and $y$ change with respect to time. Notice that the rate at which *liquid* flows into each tank equals the rate at which it flows out, so the volume of liquid in each tank remains constant. If we consider Tank 1, we can determine a first-order differential equation for $dx/dt$ with

$$\frac{dx}{dt} = (\text{Rate at which salt enters Tank 1}) - (\text{Rate at which salt leaves Tank 1}),$$

where the rate at which salt enters Tank 1 is $R$ gal/min $\times$ $C$ lb/gal $= RC$ lb/min, and the rate at which it leaves is $R$ gal/min $\times$ $x/V_1$ lb/gal $= Rx/V_1$ lb/min, where $x/V_1$ is the salt concentration in Tank 1. Therefore,

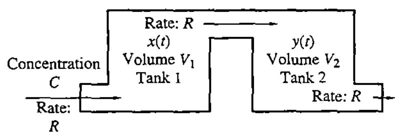



**Figure 7.10**



$$\frac{dx}{dt} = RC - \frac{Rx}{V_1}.$$

Similarly, we find $dy/dt$ to be

$$\frac{dy}{dt} = \frac{Rx}{V_1} - \frac{Ry}{V_2}.$$

In deriving this system of equations, we used rates in gal/min and concentrations in lb/gal. In general, rates are given by (volume of liquid)/(time) and concentrations are given by (amount of salt)/(volume of liquid).

We use the initial conditions $x(0) = x_0$ and $y(0) = y_0$ to solve the nonhomogeneous system

$$\begin{cases} \dfrac{dx}{dt} = RC - \dfrac{Rx}{V_1} \\ \dfrac{dy}{dt} = \dfrac{Rx}{V_1} - \dfrac{Ry}{V_2} \end{cases}$$

for $x(t)$ and $y(t)$.

### Example 2

Determine the amount of salt in each tank in Figure 7.10 if $V_1 = V_2 = 500$ gal, $R = 5$ gal/min, $C = 3$ lb/gal, $x_0 = 50$ lb, and $y_0 = 100$ lb.

**Solution** In this case, the initial-value problem is

$$\begin{cases} \dfrac{dx}{dt} = (5)(3) - \dfrac{5x}{500} = 15 - \dfrac{x}{100} \\ \dfrac{dy}{dt} = \dfrac{5x}{500} - \dfrac{5y}{500} = \dfrac{x}{100} - \dfrac{y}{100} \end{cases}, \qquad x(0) = 50,\ y(0) = 100,$$

which in matrix form is $\mathbf{X}' = \begin{pmatrix} -\frac{1}{100} & 0 \\ \frac{1}{100} & -\frac{1}{100} \end{pmatrix} \mathbf{X} + \begin{pmatrix} 15 \\ 0 \end{pmatrix} = \mathbf{AX} + \mathbf{F}(t)$, $\mathbf{X}(0) = \begin{pmatrix} 50 \\ 100 \end{pmatrix}$. The matrix $\mathbf{A}$ has the repeated eigenvalue $\lambda_1 = \lambda_2 = -1/100$, for which we can find one (linearly independent) eigenvector $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Therefore, one solution to the corresponding homogeneous system $\mathbf{X}' = \mathbf{AX}$ is $\mathbf{X}_1(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/100}$, and a second solution is $\mathbf{X}_2(t) = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} 100 \\ 0 \end{pmatrix}\right] e^{-t/100}$, so

$$\mathbf{X}_h(t) = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/100} + c_2 \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} 100 \\ 0 \end{pmatrix}\right] e^{-t/100} = \begin{pmatrix} 100 c_2 e^{-t/100} \\ c_1 e^{-t/100} + c_2 t e^{-t/100} \end{pmatrix}.$$

Notice that $\mathbf{F}(t) = \begin{pmatrix} 15 \\ 0 \end{pmatrix}$ is not contained in $\mathbf{X}_h(t)$, so with the method of undetermined coefficients we assume a particular solution has the form $\mathbf{X}_p(t) = \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

and substitute into the nonhomogeneous system $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t)$. This yields

$$\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}-1/100 & 0\\ 1/100 & -1/100\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} + \begin{pmatrix}15\\0\end{pmatrix} = \begin{pmatrix}-a_1/100 + 15\\ a_1/100 - a_2/100\end{pmatrix}$$

with solution $a_1 = 1500$ and $a_2 = 1500$. Therefore, $\mathbf{X}_p(t) = \begin{pmatrix}1500\\1500\end{pmatrix}$ and

$$\mathbf{X}(t) = \mathbf{X}_h(t) + \mathbf{X}_p(t) = \begin{pmatrix}100c_2e^{-t/100} + 1500\\ c_1e^{-t/100} + c_2te^{-t/100} + 1500\end{pmatrix}.$$

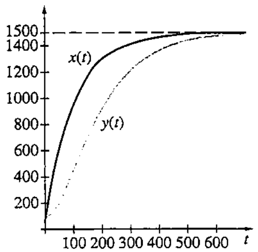


Figure 7.11

Application of the initial conditions then gives us $\mathbf{X}(0) = \begin{pmatrix}100c_2 + 1500\\ c_1 + 1500\end{pmatrix} = \begin{pmatrix}50\\100\end{pmatrix}$, so $c_1 = -1400$ and $c_2 = -1450/100 = -29/2$. The solution to the initial-value problem is

$$\mathbf{X}(t) = \begin{pmatrix}x(t)\\y(t)\end{pmatrix} = \begin{pmatrix}-1450e^{-t/100} + 1500\\ -1400e^{-t/100} - \dfrac{29}{2}te^{-t/100} + 1500\end{pmatrix}.$$

In Figure 7.11, we graph $x(t)$ and $y(t)$ together. Notice that each function approaches a limit of 1500, which means that the amount of salt in each tank tends toward a value of 1500 lb.

*In Example 2, is there a value of $t$ for which $x(t) = y(t)$? If so, what is this value? Which function increases most rapidly for smaller values of $t$?*

## Population Problems

In Chapter 3 we discussed population problems that were based on the simple principle that the rate at which a population grows (or decays) is proportional to the number present in the population at any time $t$. This idea can be extended to problems involving more than one population and leads to systems of ordinary differential equations. We illustrate several situations through the following examples. Note that in each problem we determine the rate at which a population $P$ changes with the equation

$$\frac{dP}{dt} = (\text{rate entering}) - (\text{rate leaving}).$$

We begin by determining the population in two neighboring territories where the populations $x$ and $y$ of the territories depend on several factors. The birth rate of $x$ is $a_1$, and that of $y$ is $b_1$. The rate at which citizens of $x$ move to $y$ is $a_2$, and that at which citizens move from $y$ to $x$ is $b_2$. After assuming that the mortality rate of each territory is disregarded, we determine the respective populations of these two territories for any time $t$.

Using the simple principles of previous examples, the rate at which population $x$ changes is



$$\frac{dx}{dt} = a_1x - a_2x + b_2y = (a_1 - a_2)x + b_2y,$$

and the rate at which population $y$ changes is

$$\frac{dy}{dt} = b_1y - b_2y + a_2x = (b_1 - b_2)y + a_2x.$$

Therefore, the system of equations that must be solved is

$$\begin{cases}\dfrac{dx}{dt} = (a_1 - a_2)x + b_2y\\ \dfrac{dy}{dt} = a_2x + (b_1 - b_2)y\end{cases},$$

where the initial populations of the two territories $x(0) = x_0$ and $y(0) = y_0$ are given.

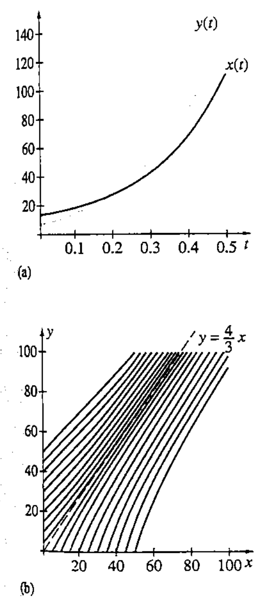


Figure 7.12 (a)–(b).

**Example 3**

Determine the populations $x(t)$ and $y(t)$ in each territory if $a_1 = 5$, $a_2 = 4$, $b_1 = 5$, and $b_2 = 3$, given that $x(0) = 14$ and $y(0) = 7$.

***Solution*** In this case, the initial-value problem that models the situation is

$$\begin{cases}\dfrac{dx}{dt} = x + 3y\\ \dfrac{dy}{dt} = 4x + 2y\end{cases}, \quad x(0) = 14 \text{ and } y(0) = 7.$$

The eigenvalues of $\mathbf{A} = \begin{pmatrix}1 & 3\\ 4 & 2\end{pmatrix}$ are $\lambda_1 = -2$ and $\lambda_2 = 5$. Corresponding eigenvectors are found to be $\mathbf{v}_1 = \begin{pmatrix}1\\-1\end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix}3\\4\end{pmatrix}$, so a general solution is

$$\mathbf{X}(t) = \begin{pmatrix}x(t)\\y(t)\end{pmatrix} = c_1\begin{pmatrix}1\\-1\end{pmatrix}e^{-2t} + c_2\begin{pmatrix}3\\4\end{pmatrix}e^{5t} = \begin{pmatrix}c_1e^{-2t} + 3c_2e^{5t}\\ -c_1e^{-2t} + 4c_2e^{5t}\end{pmatrix}.$$

Application of the initial condition $\mathbf{X}(0) = \begin{pmatrix}x(0)\\y(0)\end{pmatrix} = \begin{pmatrix}14\\7\end{pmatrix}$ yields the system $\begin{cases}c_1 + 3c_2 = 14\\ -c_1 + 4c_2 = 7\end{cases}$, so $c_1 = 5$ and $c_2 = 3$. Therefore, the solution is $\mathbf{X}(t) = \begin{pmatrix}x(t)\\y(t)\end{pmatrix} = \begin{pmatrix}5e^{-2t} + 9e^{5t}\\ -5e^{-2t} + 12e^{5t}\end{pmatrix}$. We graph these two population functions in Figure 7.12(a). In Figure 7.12(b), we graph several solutions to the system of differential equations for various initial conditions in the phase plane. As we can see, all solutions move away from the origin in the direction of the eigenvector $\mathbf{v}_2 = \begin{pmatrix}3\\4\end{pmatrix}$, which corresponds to the positive eigenvalue $\lambda_2 = 5$.

Population problems that involve more than two neighboring populations can be solved with a system of differential equations as well. Suppose that the population of three neighboring territories, $x$, $y$, and $z$, depends on several factors. The birth rates of $x$, $y$, and $z$ are $a_1$, $b_1$, and $c_1$, respectively. The rate at which citizens of $x$ move to $y$ is $a_2$, and that at which citizens move from $x$ to $z$ is $a_3$. Similarly, the rate at which citizens of $y$ move to $x$ is $b_2$, and that at which citizens move from $y$ to $z$ is $b_3$. Also, the rate at which citizens of $z$ move to $x$ is $c_2$, and that at which citizens move from $z$ to $y$ is $c_3$. (This information is summarized in Table 7.2.) If the mortality rate of each territory is ignored in the model, we can determine the respective populations of the three territories for any time $t$.

**TABLE 7.2**

| From | To: $x$ | $y$ | $z$ | Birth Rate |
|---|---|---|---|---|
| $x$ | — | $a_2$ | $a_3$ | $a_1$ |
| $y$ | $b_2$ | — | $b_3$ | $b_1$ |
| $z$ | $c_2$ | $c_3$ | — | $c_1$ |

The system of equations to determine $x(t)$, $y(t)$, and $z(t)$ is similar to that derived in the previous example. The rate at which population $x$ changes is

$$\frac{dx}{dt} = a_1x - a_2x - a_3x + b_2y + c_2z = (a_1 - a_2 - a_3)x + b_2y + c_2z,$$

and the rate at which population $y$ changes is

$$\frac{dy}{dt} = b_1y - b_2y - b_3y + a_2x + c_3z = (b_1 - b_2 - b_3)y + a_2x + c_3z,$$

and that of $z$ is

$$\frac{dz}{dt} = c_1z - c_2z - c_3z + a_3x + b_3y = (c_1 - c_2 - c_3)z + a_3x + b_3y.$$

We must solve the $3 \times 3$ system

$$\begin{cases} \dfrac{dx}{dt} = (a_1 - a_2 - a_3)x + b_2y + c_2z \\ \dfrac{dy}{dt} = a_2x + (b_1 - b_2 - b_3)y + c_3z, \\ \dfrac{dz}{dt} = a_3x + b_3y + (c_1 - c_2 - c_3)z \end{cases}$$

where the initial populations $x(0) = x_0$, $y(0) = y_0$, and $z(0) = z_0$ are given.



**Example 4**

Determine the population of the three territories if $a_1 = 3$, $a_2 = 0$, $a_3 = 2$, $b_1 = 4$, $b_2 = 2$, $b_3 = 1$, $c_1 = 5$, $c_2 = 3$, and $c_3 = 0$ if $x(0) = 50$, $y(0) = 60$, and $z(0) = 25$.

**Solution** In this case, the system of differential equations is

$$\begin{cases} \dfrac{dx}{dt} = x + 2y + 3z \\ \dfrac{dy}{dt} = y \\ \dfrac{dz}{dt} = 2x + y + 2z. \end{cases}$$

Because the characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 0 \\ 2 & 1 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda) + 3(-2)(1-\lambda) = (1-\lambda)(\lambda+1)(\lambda-4) = 0,$$

the eigenvalues are $\lambda_1 = 4$, $\lambda_2 = 1$, and $\lambda_3 = -1$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -6 \\ 4 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$, respectively. A general solution is

$$\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}e^{4t} + c_2\begin{pmatrix} 1 \\ -6 \\ 4 \end{pmatrix}e^{t} + c_3\begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}e^{-t} = \begin{pmatrix} c_1e^{4t} + c_2e^t - 3c_3e^{-t} \\ -6c_2e^t \\ c_1e^{4t} + 4c_2e^t + 2c_3e^{-t} \end{pmatrix}.$$

Using the initial conditions, we find that $\begin{cases} c_1 + c_2 - 3c_3 = 50 \\ -6c_2 = 60, \\ c_1 + 4c_2 + 2c_3 = 25 \end{cases}$ so $c_1 = 63$, $c_2 = -10$, and $c_3 = 1$. Therefore, the solution is

$$\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 63e^{4t} - 10e^t - 3e^{-t} \\ 60e^t \\ 63e^{4t} - 40e^t + 2e^{-t} \end{pmatrix}.$$

We graph these three population functions in Figure 7.13. Notice that although population $y$ is initially greater than populations $x$ and $z$, these populations increase at a much higher rate than does $y$.

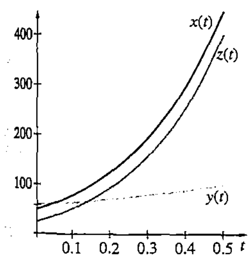


Figure 7.13

*In Example 4, does population $y$ approach a limit or do all three populations increase exponentially?*

## EXERCISES 7.2

In Exercises 1–6, solve the diffusion problem with one permeable membrane with the indicated initial conditions and parameter values. Find the limiting concentration of each solution.

1. $P = 0.5$, $V_1 = V_2 = 1$, $x(0) = 1$, $y(0) = 2$
2. $P = 0.5$, $V_1 = V_2 = 1$, $x(0) = 2$, $y(0) = 1$

*3. $P = 2$, $V_1 = 4$, $V_2 = 2$, $x(0) = 0$, $y(0) = 4$

4. $P = 2$, $V_1 = \frac{1}{2}$, $V_2 = \frac{1}{4}$, $x(0) = 8$, $y(0) = 0$
5. $P = 6$, $V_1 = 2$, $V_2 = 8$, $x(0) = 4$, $y(0) = 1$
6. $P = 6$, $V_1 = 2$, $V_2 = 8$, $x(0) = 1$, $y(0) = 4$

In Exercises 7–8, use the tanks shown in Figure 7.10 with $R = 4$ gal/min, $C = \frac{1}{2}$ lb/gal, $V_1 = V_2 = 20$ gal, and the given initial conditions. **(a)** Determine $x(t)$ and $y(t)$. **(b)** Find $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$. **(c)** Does one of the tanks contain more salt than the other tank for all values of $t$?

*7. $x(0) = y(0) = 0$

8. $x(0) = 0$, $y(0) = 2$
9. Suppose that pure water is pumped into Tank 1 (in Figure 7.10) at a rate of 4 gal/min (that is, $R = 4$, $C = 0$) and that $V_1 = V_2 = 20$ gal. Determine the amount of salt in each tank at time $t$ if $x(0) = y(0) = 4$. Calculate $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$. Which function decreases more rapidly?
10. If $x(0) = y(0) = 0$ in Exercise 9, how much salt is in each tank at any time $t$?

*11. Use the tanks in Figure 7.10 with $R = 5$ gal/min, $C = 3$ lb/gal, $V_1 = 100$ gal, $V_2 = 50$ gal, and the initial conditions $x(0) = y(0) = 0$. **(a)** Determine $x(t)$ and $y(t)$. **(b)** Find $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$.

12. Solve Exercise 11 with $V_1 = 50$ and $V_2 = 100$. How do the limiting values of $x(t)$ and $y(t)$ compare to those in Exercise 11?
13. Consider the tanks shown in Figure 7.14, where $R_1 = 3$ L/min, $R_2 = 4$ L/min, $R_3 = 1$ L/min, $C = 1$ kg/L, $V_1 = V_2 = 50$ L. If $x(0) = y(0) = 0$, determine the amount of salt in each tank at time $t$. Find $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$. Is there a time (other than $t = 0$) at which each tank contains the same amount of salt?

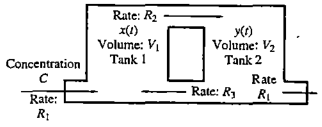



Figure 7.14

14. Solve the tank problem described in Exercise 13 using the initial conditions $x(0) = 0$ and $y(0) = 6$. For how many values of $t$ do the functions $x(t)$ and $y(t)$ agree?

*15. Find the amount of salt in each tank in Figure 7.15 if $R = 1$ gal/min, $C = 1$ lb/gal, $V_1 = 1$ gal, $V_2 = \frac{1}{2}$ gal, $x(0) = 2$, and $y(0) = 4$. What is the maximum amount of salt (at any value of $t$) in each tank? Find $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$.

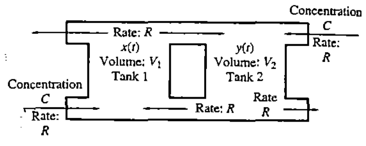



Figure 7.15

16. Determine the amount of salt in each tank described in Exercise 15 using the initial conditions $x(0) = 4$ and $y(0) = 0$. How do these functions differ from those in Exercise 15?
17. Consider the three tanks in Figure 7.16 in which the amount of salt in Tanks 1, 2, and 3 is given by $x(t)$, $y(t)$, and $z(t)$, respectively. Find $x(t)$, $y(t)$, and $z(t)$ if $R = 5$ gal/min, $C = 2$ lb/gal, $V_1 = V_2 = V_3 = 50$ gal, and $x(0) = y(0) = z(0) = 0$. Find $\lim_{t\to\infty} x(t)$, $\lim_{t\to\infty} y(t)$, and $\lim_{t\to\infty} z(t)$.

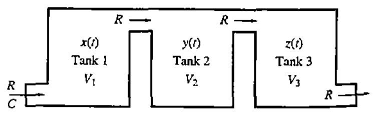



Figure 7.16



18. Solve the problem described in Exercise 17 using the unequal volumes $V_1 = 100$, $V_2 = 50$, and $V_3 = 25$. How do the limiting values of $x(t)$, $y(t)$, and $z(t)$ differ from those in Exercise 17?

In Exercises 19–22, solve the initial-value problem using the given parameters to find the population in two neighboring territories. Do either of the populations approach a finite limit? If so, what is the limit?

19. $a_1 = 10, a_2 = 9, b_1 = 2, b_2 = 1; x(0) = 10, y(0) = 20$
20. $a_1 = 4,\ a_2 = 4,\ b_1 = 1,\ b_2 = 1;\ x(0) = 4,\ y(0) = 4$

*21. $a_1 = 2,\ a_2 = 0,\ b_1 = 2,\ b_2 = 3;\ x(0) = 5,\ y(0) = 10$

22. $a_1 = 1,\ a_2 = 1,\ b_1 = 1,\ b_2 = 1;\ x(0) = 5,\ y(0) = 10$

In Exercises 23–26, solve the initial-value problem using the given parameters to find the population in three neighboring territories. Which population is largest at $t = 1$?

23. $a_1 = 10,\ a_2 = 6,\ a_3 = 7,\ b_1 = 6,\ b_2 = 3,\ b_3 = 3,\ c_1 = 5, c_2 = 7, c_3 = 1; x(0) = 17, y(0) = 0, z(0) = 34$
24. $a_1 = 2,\ a_2 = 1,\ a_3 = 4,\ b_1 = 6,\ b_2 = 4,\ b_3 = 5,\ c_1 = 2,\ c_2 = 8,\ c_3 = 4;\ x(0) = 0,\ y(0) = 4,\ z(0) = 2$

*25. $a_1 = 7,\ a_2 = 2,\ a_3 = 4,\ b_1 = 7,\ b_2 = 5,\ b_3 = 8,\ c_1 = 7,\ c_2 = 1,\ c_3 = 2;\ x(0) = 8,\ y(0) = 2,\ z(0) = 0$

26. $a_1 = 7,\ a_2 = 2,\ a_3 = 4,\ b_1 = 7,\ b_2 = 5,\ b_3 = 8,\ c_1 = 7,\ c_2 = 1,\ c_3 = 2;\ x(0) = 0,\ y(0) = 0,\ z(0) = 16$
27. Suppose that a radioactive substance $X$ decays into another unstable substance $Y$, which in turn decays into a stable substance $Z$. Show that we can model this situation through the system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = -ax \\ \dfrac{dy}{dt} = ax - by, \text{ where } a \text{ and } b \text{ are positive constants.} \\ \dfrac{dz}{dt} = by \end{cases}$$

(Assume that one unit of $X$ decomposes into one unit of $Y$, and one unit of $Y$ decomposes into one unit of $Z$.)

28. Solve the system of differential equations in Exercise 27 if $a \neq b$, $x(0) = x_0$, $y(0) = y_0$, and $z(0) = z_0$. Find $\lim_{t\to\infty} x(t)$, $\lim_{t\to\infty} y(t)$ and $\lim_{t\to\infty} z(t)$.

*29. Solve the system in Exercise 27 with $x(0) = x_0$, $y(0) = y_0$, and $z(0) = z_0$, if $a = b$. Find $\lim_{t\to\infty} x(t)$, $\lim_{t\to\infty} y(t)$, and $\lim_{t\to\infty} z(t)$.

30. In Exercise 27, what is the half-life of substance $X$?
31. In the reaction described in Exercise 27, show that if $k$ units of $X$ are added per year and $h$ units of $Z$ are removed, then the situation is described with the system

$$\begin{cases} \dfrac{dx}{dt} = -ax + k \\ \dfrac{dy}{dt} = ax - by, \\ \dfrac{dz}{dt} = by - h \end{cases}$$

where $a$, $b$, $k$, and $h$ are positive constants.

32. Solve the system described in Exercise 31 if $a \neq b$, $x(0) = x_0$, $y(0) = y_0$, and $z(0) = z_0$. Find $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$. How do these limits differ from those in Exercise 27? If $k = h$, then find $\lim_{t\to\infty} z(t)$. If $k > h$, then find $\lim_{t\to\infty} z(t)$. Describe the corresponding physical situation.

In Exercises 33–36, solve the radioactive decay model in Exercise 27 with the given parameter values and initial conditions. If $z(t)$ represents the amount (grams) of substance $Z$ after $t$ hours, how many grams of $Z$ are eventually produced?

33. $a = 6, b = 1, x_0 = 7, y_0 = 1, z_0 = 8$
34. $a = 4, b = 2, x_0 = 10, y_0 = 2, z_0 = 4$
35. $a = 4, b = 2, x_0 = 1, y_0 = 1, z_0 = 1$
36. $a = 1, b = 4, x_0 = 2, y_0 = 2, z_0 = 2$

In Exercises 37–40, solve the radioactive decay model in Exercise 31 with $a = 1$, $b = 1$, and the given parameter values and initial conditions. Describe what eventually happens to the amount of each substance.

37. $k = 2, h = 1, x_0 = 2, y_0 = 1, z_0 = 2$
38. $k = 0, h = 10, x_0 = 4, y_0 = 2, z_0 = 1$

*39. $k = 10, h = 0, x_0 = 8, y_0 = 2, z_0 = 2$

40. $k = 0, h = 5, x_0 = 1, y_0 = 10, z_0 = 5$

*41. Solve the initial-value problem to find the concentration of a substance on each side of a permeable membrane modeled by the system

$$\begin{cases} \dfrac{dx}{dt} = P\left(\dfrac{y}{V_2} - \dfrac{x}{V_1}\right) \\ \dfrac{dy}{dt} = P\left(\dfrac{x}{V_1} - \dfrac{y}{V_2}\right) \end{cases},\ x(0) = a,\ y(0) = b.$$

(a) Find $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$.

(b) Determine a condition so that $x(t) > y(t)$ as $t \to \infty$. When does $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t)$ as $t \to \infty$?

(c) Find a condition so that $x(t)$ is an increasing function. Find a condition so that $y(t)$ is an increasing function. Can these functions increase simultaneously?

(d) If $V_1 = V_2$ and $a = b$, describe what eventually happens to $x(t)$ and $y(t)$.

42. Investigate the effect that the membrane permeability has on the diffusion of a substance. Suppose that $V_1 = V_2 = 1$, $x(0) = 1$, $y(0) = 2$, and **(a)** $P = 0.25$, **(b)** $P = 0.5$, **(c)** $P = 1.0$, and **(d)** $P = 2.0$. Graph the solution in each case both parametrically and simultaneously. Describe the effect that the value of $P$ has on the corresponding solution.

*43. Solve the initial-value problem

$$\begin{cases} \dfrac{dx}{dt} = (a_1 - a_2)x + b_1 y \\ \dfrac{dy}{dt} = a_2 x + (b_1 - b_2)y \end{cases}, \; x(0) = x_0,\; y(0) = y_0.$$

Are there possible parameter values so that the functions $x(t)$ and $y(t)$ are periodic? Are there possible parameter values so that the functions $x(t)$ and $y(t)$ experience exponential decay? For what parameter values do the two populations experience exponential growth?

## 7.3 Nonlinear Systems of Equations

- Biological Systems: Predator–Prey Interaction
- Physical Systems: Variable Damping

Several special equations and systems that arise in the study of many areas of applied mathematics can be solved using the techniques of Chapter 6. These include the predator–prey population dynamics problem, the Van-der-Pol equation that models variable damping in a spring–mass system, and the Bonhoeffer–Van-der-Pol (BVP) oscillator.

### Biological Systems: Predator–Prey Interaction

Let $x(t)$ and $y(t)$ represent the number of members at time $t$ of the prey and predator populations, respectively. (Examples of such populations include fox–rabbit and shark–seal.) Suppose that the positive constant $a$ is the birth rate of $x(t)$, so that in the absence of the predator

$$\frac{dx}{dt} = ax$$

and that $c > 0$ is the death rate of $y$, which indicates that

$$\frac{dy}{dt} = -cy$$

in the absence of the prey population. In addition to these factors, the number of interactions between predator and prey affects the number of members in the two populations. Note that an interaction increases the growth of the predator population and decreases the growth of the prey population, because an interaction between the two



**Alfred James Lotka** (1880–1949), American biophysicist, born in Ukraine; wrote first text on mathematical biology. (*UPI/Bettman*)

populations indicates that a predator overtakes a member of the prey population. To include these interactions in the model, we assume that the number of interactions is directly proportional to the product of $x(t)$ and $y(t)$. Therefore, the rate at which $x(t)$ changes with respect to time is

$$\frac{dx}{dt} = ax - bxy,$$

where $b > 0$. Similarly, the rate at which $y(t)$ changes with respect to time is

$$\frac{dy}{dt} = -cy + dxy,$$

where $d > 0$. These equations and initial conditions form the **Lotka–Volterra problem**

$$\begin{cases} \dfrac{dx}{dt} = ax - bxy \\ \dfrac{dy}{dt} = -cy + dxy \end{cases}, \; x(0) = x_0,\; y(0) = y_0.$$

**Example 1**

Find and classify the equilibrium points of the Lotka–Volterra system.

***Solution*** Solving $\begin{cases} ax - bxy = x(a - by) = 0 \\ -cy + dxy = y(-c + dx) = 0 \end{cases}$, we have $x = 0$ or $y = a/b$ and $y = 0$ or $x = c/d$. The equilibrium points are $(0, 0)$ and $(c/d, a/b)$. The Jacobian matrix of the nonlinear system is $J(x, y) = \begin{pmatrix} a - by & -bx \\ dy & -c + dx \end{pmatrix}$. Near $(0, 0)$, we have $J(0, 0) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$ with eigenvalues $\lambda_1 = -c$ and $\lambda_2 = a$. Because these eigenvalues are real with opposite signs, we classify $(0, 0)$ as a saddle. Similarly, near near $(c/d, a/b)$, we have $J(c/d, a/b) = \begin{pmatrix} 0 & -bc/d \\ ad/b & 0 \end{pmatrix}$, with eigenvalues $\lambda_{1,2} = \pm i\sqrt{ac}$, so the point $(c/d, a/b)$ is classified as a center.

In Figure 7.17, we show several solutions (which were found using different initial populations) parametrically in the phase plane of this system with $a = 2$, $b = 1$, $c = 3$, and $d = 1$. Notice that all of the solutions oscillate about the center. These solutions reveal the relationship between the two populations: prey, $x(t)$, and predator, $y(t)$. As we follow one cycle counterclockwise beginning, for example, near the point $(3/2, 1)$ (see Figure 7.18), we notice that as the prey population, $x(t)$, increases, the predator population, $y(t)$, first slightly decreases (is that really possible?), and then increases until the predator becomes overpopulated. Then, because the prey population is too small to supply the predator population, the predator population decreases, which leads to an increase in the population of the prey. At this

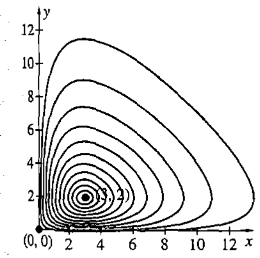



**Figure 7.17** Typical solutions of the Lotka–Volterra system.

point, because the number of predators becomes too small to control the prey population, the number in the prey population becomes overpopulated and the cycle repeats itself.

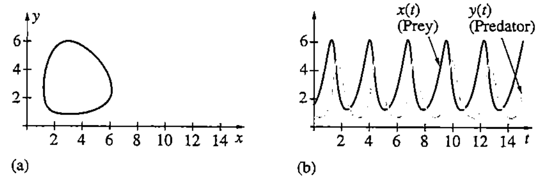



**Figure 7.18** (a)–(b).

*How does the period of a solution with an initial point near the equilibrium point (c/d, a/b) compare to that of a solution with an initial point that is not located near this point?*

**Balthasar Van-der-Pol**, Dutch applied mathematician and engineer. *(UPI/Bettman)*

## Physical Systems: Variable Damping

In some physical systems, energy is fed into the system when there are small oscillations, and energy is taken from the system when there are large oscillations. This indicates that the system undergoes "negative damping" for small oscillations and "positive damping" for large oscillations. A differential equation that models this situation is **Van-der-Pol's equation**

$$x'' + \mu(x^2 - 1)x' + x = 0,$$

where $\mu$ is a positive constant. We can transform this second-order differential equation into a system of first-order differential equations with the substitution $x' = y$. Hence $y' = x'' = -x - \mu(x^2 - 1)x' = -x - \mu(x^2 - 1)y$, so the corresponding system of equations is

$$\begin{cases} x' = y \\ y' = -x - \mu(x^2 - 1)y \end{cases}$$

which is solved using an initial position $x(0) = x_0$ and an initial velocity $y(0) = x'(0) = y_0$. Notice that $\mu(x^2 - 1)$ represents the damping coefficient. This system models variable damping because $\mu(x^2 - 1) < 0$ when $-1 < x < 1$ and $\mu(x^2 - 1) > 0$ when $|x| > 1$. Therefore, damping is negative for the small oscillations, $-1 < x < 1$, and positive for the large oscillations, $|x| > 1$.

**Example 2**

Find and classify the equilibrium points of the system of differential equations that is equivalent to Van-der-Pol's equation.



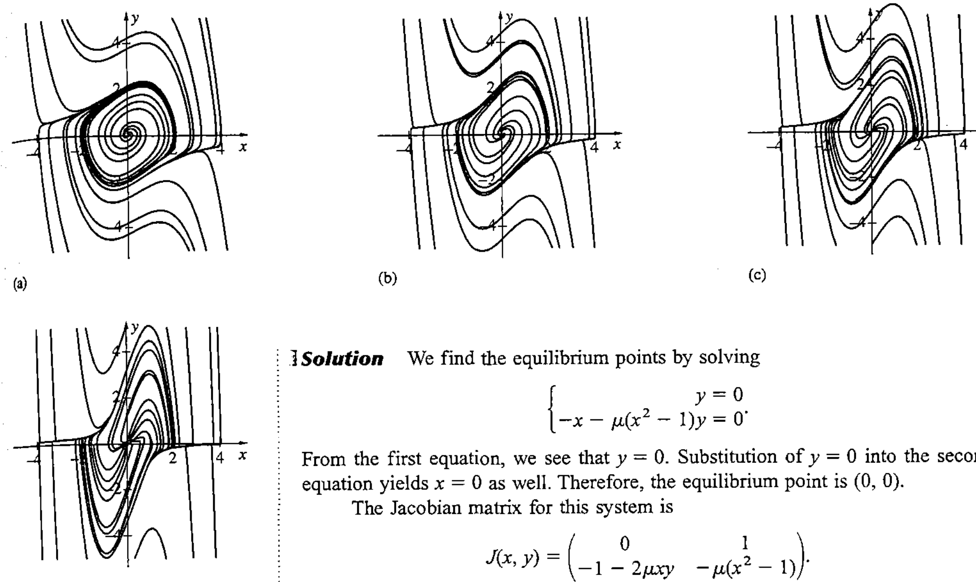



**Figure 7.19** (a) $\mu = \frac{1}{2}$ (b) $\mu = 1$ (c) $\mu = \frac{3}{2}$ (d) $\mu = 3$.

***Solution*** We find the equilibrium points by solving

$$\begin{cases} y = 0 \\ -x - \mu(x^2 - 1)y = 0 \end{cases}.$$

From the first equation, we see that $y = 0$. Substitution of $y = 0$ into the second equation yields $x = 0$ as well. Therefore, the equilibrium point is (0, 0).

The Jacobian matrix for this system is

$$J(x, y) = \begin{pmatrix} 0 & 1 \\ -1 - 2\mu xy & -\mu(x^2 - 1) \end{pmatrix}.$$

At (0, 0), we have the matrix

$$J(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}.$$

We find the eigenvalues of $J(0, 0)$ by solving

$$\begin{vmatrix} -\lambda & 1 \\ -1 & \mu - \lambda \end{vmatrix} = \lambda^2 - \mu\lambda + 1 = 0,$$

which has roots $\lambda_{1,2} = (\mu \pm \sqrt{\mu^2 - 4})/2$. Notice that if $\mu > 2$, then both eigenvalues are positive and real, so we classify (0, 0) as an **unstable node.** On the other hand, if $0 < \mu < 2$, the eigenvalues are a complex conjugate pair with a positive real part. Hence (0, 0) is an **unstable spiral point.** (We omit the case when $\mu = 2$ because the eigenvalues are repeated.)

In Figure 7.19, we show several curves in the phase plane that begin at different points for various values of $\mu$. In each figure, we see that all of the curves approach a curve called a **limit cycle.** Physically, the fact that the system has a limit cycle indicates that for all oscillations the motion eventually becomes periodic, which is represented by a closed curve in the phase plane.

On the other hand, in Figure 7.20 we graph the solution that satisfies the initial conditions $x(0) = 1$ and $y(0) = 0$ parametrically and individually for various values of $\mu$. Notice that for small values of $\mu$ the system more closely approximates that of the harmonic oscillator because the damping coefficient is small. The curves are more circular than those for larger values of $\mu$.

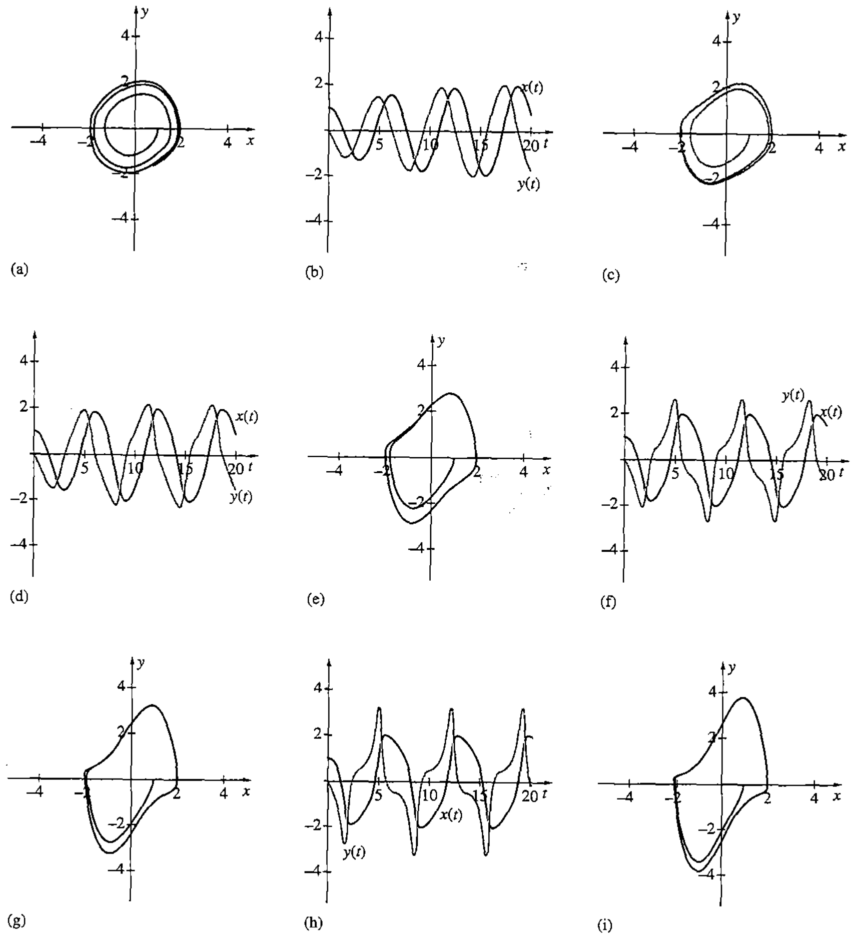



**Figure 7.20** (a)–(b) $\mu = \frac{1}{4}$ (c)–(d) $\mu = \frac{1}{2}$ (e)–(f) $\mu = 1$ (g)–(h) $\mu = \frac{3}{2}$ (i) $\mu = 2$



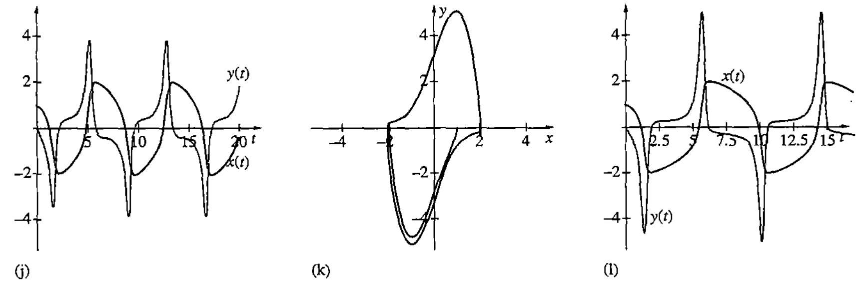



**Figure 7.20 continued** (j) $\mu = 2$ (k)–(l) $\mu = 3$

*Graph several solution curves in the phase plane for the Van-der-Pol equation if $\mu = 0.1$ and $\mu = 0.001$. Compare these graphs to the corresponding solution curves for the equation $x'' + x = 0$. Are they similar? Why?*

## EXERCISES 7.3

1. The phase paths of the Lotka–Volterra model

$$\begin{cases} \dfrac{dx}{dt} = ax - bxy \\ \dfrac{dy}{dt} = -cy + dxy \end{cases}$$

are given by $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-cy + dxy}{ax - bxy}$. Solve this separable equation to find an implicit equation of the phase paths.

2. Consider the predator–prey model

$$\begin{cases} \dfrac{dx}{dt} = (a_1 - b_1 x - c_1 y)x \\ \dfrac{dy}{dt} = (-a_2 + c_2 x)y \end{cases},$$

where $x(t)$ represents the prey population, $y(t)$ the predator population, and the constants $a_1$, $a_2$, $b_1$, $c_1$, $c_2$ are all positive. In this model, the term $-b_1 x^2$ represents the interference that occurs when the prey population becomes too large. **(a)** If $a_1 = 2$, $a_2 = b_1 = c_1 = c_2 = 1$, locate and classify the three equilibrium points. **(b)** If $a_2 = 2$, $a_1 = b_1 = c_1 = c_2 = 1$, classify the two equilibrium points.

*3. The national economy can be modeled with a nonlinear system of differential equations. If $I$ represents the national income, $C$ the rate of consumer spending, and $G$ the rate of government spending, a simple model of the economy is

$$\begin{cases} \dfrac{dI}{dt} = I - aC \\ \dfrac{dC}{dt} = b(I - C - G), \end{cases}$$

where $a$ and $b$ are constants. **(a)** Suppose that $a = b = 2$ and $G = k$ ($k$ = constant). Find and classify the equilibrium point. **(b)** If $a = 2$, $b = 1$, and $G = k$, find and classify the equilibrium point.

4. If $a = b = 2$ and $G = k_1 + k_2 I$ ($k_2 > 0$), show that there is no equilibrium point if $k_2 \geq \frac{1}{2}$ for the national economy model in Exercise 3. Describe the economy under these conditions.

5. The differential equation $x'' + \sin x = 0$ can be used to describe the motion of a pendulum. In this case, $x(t)$ represents the displacement from the position $x = 0$. Represent this second-order equation as a system of first-order equations. Find and classify the equilibrium points of this system. Describe the physical significance of these points. Graph several paths in the phase plane of the system.

6. Show that the paths in the phase plane of $x'' + \sin x = 0$ satisfy the first-order equation $dy/dx = -(\sin x)/y$. (See Exercise 5.) Use separation of variables to show that paths are $\frac{1}{2}y^2 - \cos x = C$, where $C$ is a constant. Graph several paths. Do these graphs agree with your result in Exercise 5?

*7. (a) Write the equation $d^2x/dt^2 + k^2x = 0$, which models the **simple harmonic oscillator,** as a system of first-order equations. (b) Show that paths in the phase plane satisfy $dy/dx = -k^2x/y$. (See Exercise 6.) (c) Solve the equation in (b) to find the paths. (d) What is the equilibrium point of this system and how is it classified? How do the paths in (c) compare with what you expect to see in the phase plane?

8. Repeat Exercise 7 for the equation $d^2x/dt^2 - k^2x = 0$.

9. Suppose that a satellite is in flight on the line between a planet of mass $M_1$ and its moon of mass $M_2$, which are a constant distance $R$ apart. The distance $x$ between the satellite and the planet satisfies the nonlinear second-order equation

$$x'' = -\frac{gM_1}{x^2} + \frac{gM_2}{(R-x)^2},$$

where $g$ is the gravitational constant. Transform this equation into a system of first-order equations. Find and classify the equilibrium point of the linearized system.

10. Consider the nonlinear autonomous system

$$\begin{cases} \dfrac{dx}{dt} = -y + x(1 - x^2 - y^2) \\ \dfrac{dy}{dt} = x + y(1 - x^2 - y^2) \end{cases}.$$

(a) Show that $(0, 0)$ is an equilibrium point of this system. Classify $(0, 0)$.

(b) Show that $x = r\cos\theta$, $y = r\sin\theta$ transforms the system to the (uncoupled) system
$\begin{cases} dr/dt = r(1 - r^2) \\ d\theta/dt = 1 \end{cases}$.

(c) Show that this system has solution
$r(t) = 1/\sqrt{1 + ae^{-2t}}$, $\theta(t) = t + b$ so that
$x(t) = 1/\sqrt{1 + ae^{-2t}}\cos(t + b)$,
$y(t) = 1/\sqrt{1 + ae^{-2t}}\sin(t + b)$.

(d) Calculate $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$ to show that all solutions approach the circle $x^2 + y^2 = 1$ as $t \to \infty$. Thus $x^2 + y^2 = 1$ is a limit cycle.

*11. Consider Liénard's equation $d^2x/dt^2 + f(x)\,dx/dt + g(x) = 0$, where $f(x)$ and $g(x)$ are continuous. Show that this equation can be written as the system

$$\begin{cases} \dfrac{dx}{dt} = y - F(x) \\ \dfrac{dy}{dt} = -g(x), \end{cases}$$

where $F(x) = \int_0^x f(u)\,du$.

12. (See Exercise 11.) Liénard's theorem states that if (i) $F(x)$ is an odd function, (ii) $F(x)$ is zero only at $x = 0$, $x = a$, $x = -a$ (for some $a > 0$), (iii) $F(x) \to \pm\infty$ monotonically for $x > a$, and (iv) $g(x)$ is an odd function where $g(x) > 0$ for all $x > 0$; then Liénard's equation has a unique limit cycle. Use Liénard's theorem to determine which of the following equations has a unique limit cycle:

(a) $\dfrac{d^2x}{dt^2} + \varepsilon(1 - x^2)\dfrac{dx}{dt} + x = 0$, $\varepsilon > 0$;

(b) $\dfrac{d^2x}{dt^2} + 3x^2\dfrac{dx}{dt} + x^3 = 0$.

13. Consider the system of autonomous equations

$$\begin{cases} \dfrac{dx}{dt} = f(x, y) \\ \dfrac{dy}{dt} = g(x, y) \end{cases}.$$

**Bendixson's theorem** (or negative criterion) states that if $f_x(x, y) + g_y(x, y)$ is a continuous function that is either always positive or always negative in a particular region $R$ of the phase plane, then the system has no limit cycle in $R$. Use this theorem to determine if the given system has no limit cycle in the phase plane.

(a) $\begin{cases} \dfrac{dx}{dt} = x^3 + x + 7y \\ \dfrac{dy}{dt} = x^2y \end{cases}$ (b) $\begin{cases} \dfrac{dx}{dt} = y - x \\ \dfrac{dy}{dt} = 3x - y \end{cases}$

(c) $\begin{cases} \dfrac{dx}{dt} = xy^2 \\ \dfrac{dy}{dt} = x^2 + 8y \end{cases}$



14. Let $E$ and $s$ be positive constants and suppose that $f$ is a continuous odd function that approaches a finite limit as $x \to \infty$, is increasing, and is concave down for $x > 0$. The voltages over the deflection plate in a sweeping circuit for an oscilloscope are determined by solving

$$\begin{cases} \dfrac{dV_1}{dt} = -sV_1 + f(E - V_2) \\ \dfrac{dV_2}{dt} = -sV_2 + f(E - V_1). \end{cases}$$

Use Bendixson's theorem (see Exercise 13) to show that this system has no limit cycles.

*15. Consider the relativistic equation for the central orbit of a planet, $d^2u/d\theta^2 + u - ku^2 = \alpha$, where $k$ and $\alpha$ are positive constants ($k$ is very small), $u = 1/r$, and $r$ and $\theta$ are polar coordinates. (a) Write this second-order equation as a system of first-order equations. (b) Show that $((1 - \sqrt{1 - 4k\alpha})/2k, 0)$ is a center in the linearized system.

16. Consider the system of equations $\begin{cases} \dfrac{dx}{dt} = P(x, y) \\ \dfrac{dy}{dt} = Q(x, y), \end{cases}$

where $P$ and $Q$ are polynomials of degree $n$. Finding the maximum number of limit cycles of this system, called the **Hilbert number** $H_n$, has been investigated as part of Hilbert's 16th problem. Several of these numbers are known: $H_0 = 0$, $H_1 = 0$, $H_2 \geq 4$, $H_n \geq (n - 1)/2$ (if $n$ is odd), and $H_n < \infty$. Determine the Hilbert number (or a restriction on the Hilbert number) for each system:

(a) $\begin{cases} \dfrac{dx}{dt} = 10 \\ \dfrac{dy}{dt} = -5 \end{cases}$ (b) $\begin{cases} \dfrac{dx}{dt} = x - y \\ \dfrac{dy}{dt} = y + 1 \end{cases}$

(c) $\begin{cases} \dfrac{dx}{dt} = y^2 - 2xy \\ \dfrac{dy}{dt} = x^2 + y^2 \end{cases}$ (d) $\begin{cases} \dfrac{dx}{dt} = x + 5y^7 \\ \dfrac{dy}{dt} = 10x^7 + y^4 \end{cases}$.

17. In a mechanical system, suppose that $x(t)$ represents position at time $t$, $K$ kinetic energy, and $V$ potential energy, where $K = 1/2\, m(x)\,(dx/dt)^2$ ($m$ is a positive function) and $V = V(x)$. If the system is **conservative,** then the total energy $E$ of the system remains constant during motion, which indicates that $1/2\, m(x)(dx/dt)^2 + V(x) = E$. Show that with the change of variable $u = \int \sqrt{m(x)}\,dx$ this equation becomes $d^2u/dt^2 + Q'(u) = 0$, where $Q'(u) = V'(x)/\sqrt{m(x)}$. This means that an equation of the form $d^2x/dt^2 = f(x)$ is a **conservative system,** where $f$ is a function representing force per unit mass that does not depend on $dx/dt$.

18. Consider the conservative system $d^2x/dt^2 = f(x)$ in which $f$ is a continuous function. Suppose that $V(x) = -\int f(x)\,dx$ so that $V'(x) = -f(x)$. (a) Use the substitution $dx/dt = y$ to transform $d^2x/dt^2 = f(x)$ into the system of first-order equations $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -V'(x) \end{cases}$.

(b) Find the equilibrium points of the system in (a). What is the physical significance of these points? (c) Use the chain rule and $dx/dt = y$ to show that $dy/dt = y\,dy/dx$. (d) Show that the paths in the phase plane are $\frac{1}{2}y^2 + V(x) = C$.

*19. (See Exercise 18.) What are the paths in the phase plane if $V(x) = \frac{1}{2}x^2$? How does this compare with the classification of the equilibrium point of the corresponding system of first-order equations? Notice that if $V$ has a local minimum at $x = a$, the system has a center at the corresponding point in the phase plane.

20. (See Exercise 18.) What are the paths in the phase plane if $V(x) = -\frac{1}{2}x^2$? How does this compare with the classification of the equilibrium point of the corresponding system of first-order equations? Notice that if $V$ has a local maximum at $x = a$, the system has a saddle at the corresponding point in the phase plane.

21. Use the observations made in Exercises 19–20 concerning the relationship between the local extrema of $V$ and the classification of equilibrium points in the phase plane to classify the equilibrium points of a conservative system with (a) $V'(x) = x^2 - 1$; (b) $V'(x) = x - x^3$.

22. Which of the following physical systems are conservative? **(a)** The motion of a pendulum modeled by $d^2x/dt^2 + \sin x = 0$. **(b)** A spring–mass system that disregards damping and external forces. **(c)** A spring–mass system that includes damping.

*23. Consider a brake that acts on a wheel. Assuming that the force due to friction depends only on the angular velocity of the wheel, $d\theta/dt$, we have $I\,d^2\theta/dt^2 = -FR \operatorname{sgn}(d\theta/dt)$ to describe the spinning motion of the wheel, where $R$ is the radius of the brake drum, $F$ is the frictional force, $I$ is the moment of inertia of the wheel, and $\operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0. \\ -1, & x < 0 \end{cases}$

**(a)** Let $d\theta/dt = y$ and transform this second-order equation into a system of first-order equations. **(b)** What are the equilibrium points of this system? **(c)** Show that $d^2\theta/dt^2 = d\theta/dt\; d/d\theta\,(d\theta/dt)$. **(d)** Use the relationship in (c) to show that the paths in the phase plane are $1/2\, I(d\theta/dt)^2 = -FR\theta + C$, $d\theta/dt > 0$ and $1/2\, I(d\theta/dt)^2 = FR\theta + C$, $d\theta/dt < 0$. What are these paths?

24. The equation $d^2x/dt^2 + k^2 \sin x = 0$ that models the motion of a pendulum is approximated with $d^2x/dt^2 + k^2(x - 1/6\,x^3) = 0$. Why? Write this equation as a system, then locate and classify its equilibrium points. How do these findings differ from those of $d^2x/dt^2 + k^2 \sin x = 0$?

25. Consider the system

$$\begin{cases} x' = y - x(1 - x^2 - y^2 - z^2) \\ y' = -x + y(1 - x^2 - y^2 - z^2). \\ z' = 0 \end{cases}$$

**(a)** Show that $\dfrac{d}{dt}(x^2 + y^2 + z^2) = 0$ if $x^2 + y^2 + z^2 = 1$, so $x^2 + y^2 + z^2 = 1$ is called an **invariant set.** **(b)** Show that a solution with initial point $(x_0, y_0, z_0)$ remains in the plane $z = z_0$. Therefore, $z = z_0$ is an invariant set. **(c)** Show that the limit cycle for the solution with initial point (1/2, 1/2, 1/2) is the circle $x^2 + y^2 = 3/4$ in the plane $z = 1/2$.

26. The **Bonhoeffer–Van-der-Pol (BVP) oscillator** is the system of ordinary differential equations

$$\begin{cases} \dfrac{dx}{dt} = x - \dfrac{x^3}{3} - y + I(t) \\ \dfrac{dy}{dt} = c(x + a - by) \end{cases}$$

**(a)** Find and classify the equilibrium points of this system if $I(t) = 0$, $a = c = 1$ and $b = 1$. **(b)** Graph the direction field associated with the system and then approximate the phase plane by graphing several solutions near each equilibrium point.

27. Find $x_0 > 0$ so that the solution to the initial-value problem

$$\begin{cases} x' = y \\ y' = -x - (x^2 - 1)y \\ x(0) = x_0,\ y(0) = 0 \end{cases}$$

is periodic. Confirm your result graphically.

28. We saw that solutions to the Lotka–Volterra problem:

$$\begin{cases} x' = x(a - by),\ x(0) = x_0 \\ y' = y(-c + dx),\ y(0) = y_0 \end{cases}$$

oscillate periodically, where the period $T$ and amplitude of $x(t)$ and $y(t)$ depend on the initial conditions. **Volterra's principle** states that the time averages of $x(t)$ and $y(t)$ remain constant and equal the corresponding equilibrium values, $\bar{x} = c/d$ and $\bar{y} = a/b$. In symbols, this statement is represented with

$$\bar{x} = \frac{1}{T}\int_0^T x(t)\,dt \text{ and } \bar{y} = \frac{1}{T}\int_0^T y(t)\,dt.$$

Follow the steps below to prove that $\bar{y} = \dfrac{1}{T}\displaystyle\int_0^T y(t)\,dt$.

**(a)** Show that $\dfrac{d}{dt}(\ln x) = a - by$.

$\left(\textit{Hint: } \dfrac{d}{dt}(\ln x) = \dfrac{x'}{x}.\right)$

**(b)** Integrate each side of $\displaystyle\int_0^T \frac{d}{dt}(\ln\ x(t))\,dt = \int_0^T (a - by(t))\,dt$ to obtain $\ln(x(T)) - \ln(x(0)) = aT - b\displaystyle\int_0^T y(t)\,dt$. Recall that because $x(t)$ is periodic, $x(T) = x(0)$. Use this fact to solve the previous equation for $\dfrac{1}{T}\displaystyle\int_0^T y(t)\,dt$.

**(c)** Follow steps similar to those outlined in (a) and (b) to prove that $\bar{x} = \dfrac{1}{T}\displaystyle\int_0^T x(t)\,dt$.



# CHAPTER 7 SUMMARY

## Concepts & Formulas

### Section 7.1

**L-R-C Circuit with One Loop**

$$\begin{cases} \dfrac{dQ}{dt} = I \\ \dfrac{dI}{dt} = -\dfrac{1}{LC}Q - \dfrac{R}{L}I + \dfrac{E(t)}{L} \\ Q(0) = Q_0,\ I(0) = I_0 \end{cases}$$

**L-R-C Circuit with Two Loops**

$$\begin{cases} \dfrac{dQ}{dt} = -\dfrac{1}{R_1C}Q - I_2 + \dfrac{E(t)}{R_1} \\ \dfrac{dI_2}{dt} = \dfrac{1}{LC}Q - \dfrac{R_2}{L}I_2 \\ Q(0) = Q_0,\ I_2(0) = I_0 \end{cases}$$

**Spring–Mass System**

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\dfrac{k}{m}x - \dfrac{c}{m}y \end{cases}$$

### Section 7.2

**Diffusion through a Membrane**

$$\begin{cases} \dfrac{dx_1}{dt} = P\left(\dfrac{x_2}{V_2} - \dfrac{x_1}{V_1}\right) \\ \dfrac{dx_2}{dt} = P\left(\dfrac{x_1}{V_1} - \dfrac{x_2}{V_2}\right) \end{cases}$$

**Mixture Problem with Two Tanks**

$$\begin{cases} \dfrac{dx}{dt} = RC - \dfrac{Rx}{V_1} \\ \dfrac{dy}{dt} = \dfrac{Rx}{V_1} - \dfrac{Ry}{V_2} \end{cases}$$

**Population of Two Neighboring Territories**

$$\begin{cases} \dfrac{dx}{dt} = (a_1 - a_2)x + b_1y \\ \dfrac{dy}{dt} = a_2x + (b_1 - b_2)y \end{cases}$$

**Population of Three Neighboring Territories**

$$\begin{cases} \dfrac{dx}{dt} = (a_1 - a_2 - a_3)x + b_2y + c_2z \\ \dfrac{dy}{dt} = a_2x + (b_1 - b_2 - b_3)y + c_3z \\ \dfrac{dz}{dt} = a_3x + b_3y + (c_1 - c_2 - c_3)z \end{cases}$$

### Section 7.3

**Predator–Prey (Lotka–Volterra system)**

$$\begin{cases} \dfrac{dx}{dt} = ax - bxy \\ \dfrac{dy}{dt} = -cy + dxy \end{cases}$$

**Van-der-Pol's Equation (System)**

$$x'' + \mu(x^2 - 1)x' + x = 0;\ \begin{cases} x' = y \\ y' = -x - \mu(x^2 - 1)y \end{cases}$$

## CHAPTER 7 REVIEW EXERCISES

1. Solve the one-loop $L$-$R$-$C$ circuit with $L = 1$ henry, $R = 5/3$ ohms, $C = 3/2$ farads, $Q(0) = 10^{-6}$ coulomb, $I(0) = 0$ amperes, and **(a)** $E(t) = 0$ volts; **(b)** $E(t) = e^{-t}$ volts.
2. Solve the one-loop $L$-$R$-$C$ circuit with $L = 3$ henrys, $R = 10$ ohms, $C = 0.1$ farad, $Q(0) = 0$ coulomb, $I(0) = 0$ amperes, and **(a)** $E(t) = 120$ volts; **(b)** $E(t) = 120 \sin t$ volts.

*3. Solve the two-loop $L$-$R$-$C$ circuit with $L = 1$ henry, $R_1 = R_2 = 1$ ohm, $C = 1$ farad, $Q(0) = 10^{-6}$ coulomb; $I_2(0) = 0$ amperes, and **(a)** $E(t) = 0$ volts; **(b)** $E(t) = 120$ volts.

4. Solve the two-loop $L$-$R$-$C$ circuit with $L = 1$ henry, $R_1 = R_2 = 1$ ohm, $C = 1$ farad, $Q(0) = 0$ coulombs, $I_2(0) = 0$ amperes, and **(a)** $E(t) = 120e^{-t/2}$ volts; **(b)** $E(t) = 120 \cos t$ volts.
5. Transform the second-order equation to a system of first-order equations and classify the systems as undamped, overdamped, underdamped, or critically damped by finding the eigenvalues of the corresponding system. Solve with the initial conditions $x(0) = 1$, $dx/dt(0) = y(0) = 0$. **(a)** $x'' + 2x' + x = 0$; **(b)** $x'' + 4x = 0$
6. Transform the second-order equation to a system of first-order equations and classify the system as undamped, overdamped, underdamped, or critically damped by finding the eigenvalues of the corresponding system. Solve with the initial conditions $x(0) = 0$, $dx/dt(0) = y(0) = 1$. **(a)** $x'' + 4x = 0$; **(b)** $x'' + 4x' + \frac{7}{4}x = 0$

*7. Solve the diffusion problem with one permeable membrane with $P = 0.5$, $V_1 = V_2 = 1$, $x(0) = 5$, and $y(0) = 10$.

8. Solve the mixture problem with two tanks (see Figure 7.10), with $R = 4$ gal/min, $V_1 = V_2 = 80$, $C = 3$ lb/gal, $x(0) = 10$, and $y(0) = 20$.
9. Investigate the behavior of solutions of the two-dimensional population problem with $a_1 = 3$, $a_2 = 1$, $b_1 = 3$, $b_2 = 1$, $x(0) = 20$, and $y(0) = 10$.
10. Investigate the behavior of solutions of the two-dimensional population problem with $a_1 = 1$, $a_2 = 2$, $b_1 = 1$, $b_2 = 1$, $x(0) = 4$, and $y(0) = 8$.

*11. Solve the three-dimensional population problem with $a_1 = 2$, $a_2 = 1$, $a_3 = 4$, $b_1 = 6$, $b_2 = 4$, $b_3 = 5$, $c_1 = 2$, $c_2 = 8$, $c_3 = 4$, $x(0) = 4$, $y(0) = 4$, and $z(0) = 8$.

12. The nonlinear second-order equation that describes the motion of a damped pendulum is $d^2x/dt^2 + b\,dx/dt + g/L \sin x = 0$. **(a)** Make the substitution $dx/dt = y$ to write the equation as the system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -by - \dfrac{g}{L}\sin x \end{cases}.$$

**(b)** Show that the equilibrium points of the system are $(n\pi, 0)$, where $n$ is an integer. **(c)** Show that if $n$ is odd, then $(n\pi, 0)$ is a saddle. **(d)** Show that if $n$ is even, then $(n\pi, 0)$ is either a stable spiral point or a stable node.

*13. Suppose that the predator–prey model is altered so that the prey population $x(t)$ follows the logistic equation when the predator population $y(t)$ is not present. With this assumption, the system is

$$\begin{cases} \dfrac{dx}{dt} = ax - kx^2 - bxy \\ \dfrac{dy}{dt} = -cy + dxy \end{cases},$$

where $a$, $b$, $c$, $d$, and $k$ are positive constants, $a \neq k$, and the ratio $a/k$ is much larger than $c/d$. Find and classify the equilibrium points of this system.

14. **(Production of Monochlorobenzene)** The Ajax Pharmaceutical Company has often thought of making its own monochlorobenzene $C_6H_5Cl$ from benzene $C_6H_6$ and chloride $Cl_2$ (with a small amount of ferric chloride as a catalyst) instead of purchasing it from another company. The chemical reactions are

$$C_6H_6 + Cl_2 \longrightarrow C_6H_5Cl + HCl$$
$$C_6H_5Cl + Cl_2 \longrightarrow C_6H_4Cl_2 + HCl$$
$$C_6H_4Cl_2 + Cl_2 \longrightarrow C_6H_4Cl_3 + HCl,$$

where HCl is hydrogen chloride, $C_6H_4Cl_2$ is dichlorobenzene, and $C_6H_4Cl_3$ is trichlorobenzene.



Experimental data indicate that the formation of trichlorobenzene is small. Therefore we will neglect it, so the rate equations are given by

$$\begin{cases} -\dfrac{dx_A}{dt} = k_1 x_A \\ \dfrac{dx_B}{dt} = k_1 x_A - k_2 x_B \end{cases},$$

where $x_A$ is the mole fraction (dimensionless) of benzene and $x_B$ is the mole fraction of monochlorobenzene. The formation of trichlorobenzene is neglected, so we assume that $x_C$, the mole fraction of dichlorobenzene, is found with $x_C = 1 - x_A - x_B$.

The rate constants $k_1$ and $k_2$ have been determined experimentally. We give these constants in the following table.* If $x_A(0) = 1$ and $x_B(0) = 0$, solve the system for each of these three temperatures. Graph $x_A(t)$ and $x_B(t)$ simultaneously for each temperature. Is there a relationship between the temperature and the time at which $x_B(t) > x_A(t)$?

| | 40°C | 55°C | 70°C |
|---|---|---|---|
| $k_1$ (hr$^{-1}$) | 0.0965 | 0.412 | 1.55 |
| $k_2$ (hr$^{-1}$) | 0.0045 | 0.055 | 0.45 |

## Differential Equations at Work:

### A. Competing Species

The system of equations

$$\begin{cases} \dfrac{dx}{dt} = x(a - b_1 x - b_2 y) \\ \dfrac{dy}{dt} = y(c - d_1 x - d_2 y), \end{cases}$$

where $a$, $b_1$, $b_2$, $c$, $d_1$, and $d_2$ represent positive constants, can be used to model the population of two species, represented by $x(t)$ and $y(t)$, competing for a common food supply.

1. **(a)** Find and classify the equilibrium points of the system if $a = 1$, $b_1 = 2$, $b_2 = 1$, $c = 1$, $d_1 = 0.75$, and $d_2 = 2$. **(b)** Graph several solutions by using different initial populations parametrically in the phase plane. **(c)** Find $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$ if both $x(0)$ and $y(0)$ are not zero. Compare your result to (b).
2. **(a)** Find and classify the equilibrium points of the system if $a = 1$, $b_1 = 1$, $b_2 = 1$, $c = 0.67$, $d_1 = 0.75$, and $d_2 = 1$. **(b)** Graph several solutions by using different initial populations parametrically in the phase plane. **(c)** Determine the fate of the species with population $y(t)$. What happens to the species with population $x(t)$? What happens to the species with population $y(t)$ if the species with population $x(t)$ is suddenly removed? (*Hint:* How does the term $-d_1xy$ affect the equation $dy/dt = y(c - d_1x - d_2y)$?
3. Find conditions on the positive constants $a$, $b_1$, $b_2$, $c$, $d_1$, and $d_2$ so that **(a)** the system has exactly one equilibrium point in the first quadrant and **(b)** the system has

* Samuel W. Bodman, *The Industrial Practice of Chemical Process Engineering.* MIT Press, Cambridge, MA (1968), pp. 18–24.

no equilibrium point in the first quadrant. **(c)** Is it possible to choose positive constants $a$, $b_1$, $b_2$, $c$, $d_1$, and $d_2$ so that each population grows without bound? If so, illustrate graphically.

## B. Food Chains*

The food chain in the ocean can be divided into five trophic levels, where the base level is made up primarily of seaweeds and phytoplankton. Suppose that $x_1(t)$ represents the size of this base level population and let $x_2(t)$ represent that of the second level in the food chain, which feeds on the base level. In the absence of the second level, assume that $x_1(t)$ follows the logistic equation. However, an interaction between members of the first two levels of the food chain prohibits growth of $x_1(t)$, and it enhances that of $x_2(t)$. In addition, assume that $x_2(t)$ decays exponentially without the food supply of the base level. Therefore, we model this situation with

$$\begin{cases} x_1' = ax_1 - bx_1^2 - cx_1x_2 \\ x_2' = -dx_2 + ex_1x_2. \end{cases}$$

1. Show that the equilibrium solution of this system is $(d/e, (ae - bd)/(ce))$.
2. Suppose that $a = 36$, $b = 4$, $c = 2$, and $d = 1$. What is the equilibrium solution in this case?
3. Numerically solve the system using the initial populations $x_1(0) = 3$ and $x_2(0) = 2$ and graph these functions on the same set of axes. What is $\lim_{t\to\infty} x_1(t)$? What is $\lim_{t\to\infty} x_2(t)$? Describe the corresponding physical situation. At how many values of $t$ does $x_1(t) = x_2(t)$?
4. Repeat part 3 using $x_1(0) = 1$ and $x_2(0) = 2$. Suppose that we harvest the base level at the rate $h$. The system becomes

$$\begin{cases} x_1' = ax_1 - bx_1^2 - cx_1x_2 - h \\ x_2' = -dx_2 + ex_1x_2. \end{cases}$$

5. Let $a = 36$, $b = 4$, $c = 2$, $d = 1$, $h = 10$, $x_1(0) = 1$ and $x_2(0) = 2$. Compare the solution to that obtained in 4.
6. How does harvesting affect the populations?
7. Repeat (5) with $h = 20$. Compare the results with those of (5).
8. Repeat (5) with $h = 25$. Compare the results with those of (5) and (7).
9. Repeat (5) with $h = 26$. Compare the results with those of (5), (7), and (8). Suppose that we harvest the second level at the rate $h$. Then the system becomes

* J. Raloff, "How Long Will We Go Fishing for Dinner?" *Science News*, Volume 153 (Feb. 7, 1998), p. 86.



$$\begin{cases} x_1' = ax_1 - bx_1^2 - cx_1x_2 \\ x_2' = -dx_2 + ex_1x_2 - h. \end{cases}$$

10. Let $a = 36$, $b = 4$, $c = 2$, $d = 1$, $h = 1$, $x_1(0) = 1$, and $x_2(0) = 2$. Compare the solution to that obtained in (6). How does harvesting affect the populations?
11. Let $a = 36$, $b = 4$, $c = 2$, $d = 1$, $h = 4$, $x_1(0) = 1$, and $x_2(0) = 2$. Compare the solution to that obtained in (10).
12. Let $a = 36$, $b = 4$, $c = 2$, $d = 1$, $h = 4.5$, $x_1(0) = 1$, and $x_2(0) = 2$. Compare the solution to that obtained in (10) and (11). Suppose that we consider all five trophic levels in the food chain, where we let $x_3(t)$ represent the size of the population at the third level, $x_4(t)$ that at the fourth level, and $x_5(t)$ that of the fifth (or top) level. This situation is modeled with the system

$$\begin{cases} x_1' = a_1x_1 - a_2x_1^2 - a_3x_1x_2 \\ x_2' = -b_1x_2 + b_2x_1x_2 - b_3x_2x_3 \\ x_3' = -c_1x_3 + c_2x_2x_3 - c_3x_3x_4 \\ x_4' = -d_1x_4 + d_2x_3x_4 - d_3x_4x_5 \\ x_5' = -e_1x_5 + e_2x_4x_5. \end{cases}$$

Notice that we assume each level benefits from an encounter with a member of the next lower level.

13. Suppose that $a_1 = a_2 = a_3 = 1$, $b_1 = b_2 = b_3 = 1$, $c_1 = c_2 = c_3 = 1$, $d_1 = d_2 = d_3 = 1$, $e_1 = e_2 = e_3 = 1$, $x_1(0) = 2$, $x_2(0) = 3$, $x_3(0) = 8$, $x_4(0) = 10$, and $x_5(0) = 12$. Numerically solve the system using the initial populations and graph these functions on the same set of axes. Determine $\lim_{t\to\infty} x_1(t)$, $\lim_{t\to\infty} x_2(t)$, $\lim_{t\to\infty} x_3(t)$, $\lim_{t\to\infty} x_4(t)$, and $\lim_{t\to\infty} x_5(t)$? Describe the corresponding physical situation.
14. Repeat (13) with $a_3 = 2$.
15. Repeat (13) with $a_3 = 10$. If we fish at the base level at rate $h$, we obtain the system

$$\begin{cases} x_1' = a_1x_1 - a_2x_1^2 - a_3x_1x_2 - h \\ x_2' = -b_1x_2 + b_2x_1x_2 - b_3x_2x_3 \\ x_3' = -c_1x_3 + c_2x_2x_3 - c_3x_3x_4 \\ x_4' = -d_1x_4 + d_2x_3x_4 - d_3x_4x_5 \\ x_5' = -e_1x_5 + e_2x_4x_5. \end{cases}$$

16. Suppose that $a_1 = 10$, $a_2 = a_3 = 1$, $b_1 = b_2 = b_3 = 1$, $c_1 = c_2 = c_3 = 1$, $d_1 = d_2 = d_3 = 1$, $e_1 = e_2 = e_3 = 1$, $x_1(0) = 2$, $x_2(0) = 3$, $x_3(0) = 8$, $x_4(0) = 10$, and $x_5(0) = 12$. Numerically solve the system using the initial populations and graph these functions on the same set of axes. Determine $\lim_{t\to\infty} x_1(t)$, $\lim_{t\to\infty} x_2(t)$, $\lim_{t\to\infty} x_3(t)$, $\lim_{t\to\infty} x_4(t)$, and $\lim_{t\to\infty} x_5(t)$? Describe the corresponding physical situation.
17. Suppose that we fish at the third level at rate $h = 8$. How does this form of harvesting affect the population sizes?

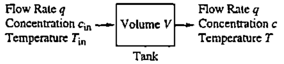



**Figure 7.21** Continuous-Flow Stirred Tank Reactor

## C. Chemical Reactor

Consider the continuous-flow stirred-tank reactor shown in Figure 7.21. In this reaction, a stream of chemical $C$ flows into the tank of volume $V$, and products as well as residue flow out of the tank at a constant rate $q$. Because the reaction is continuous, the composition and temperature of the contents of the tank are constant and are the same as the composition and temperature of the stream flowing out of the tank. Suppose that a concentration $c_{in}$ of the chemical $C$ flows into the tank and a concentration $c$ flows out of the tank. In addition, suppose that the chemical $C$ changes into products at a rate proportional to the concentration $c$ of $C$, where the constant of proportionality $k(T)$ depends on the temperature $T$,

$$k(T) = Ae^{-B/T}.$$

Because

$$\begin{pmatrix}\text{Rate of change}\\ \text{of amount of } C\end{pmatrix} = (\text{Rate in of } C) - (\text{Rate out of } C) - \begin{pmatrix}\text{Rate that}\\ C \text{ disappears}\\ \text{by the reaction}\end{pmatrix},$$

we have the differential equation

$$\frac{d}{dt}(Vc) = qc_{in} - qc - Vk(T)c.^*$$

1. Show that if $V$ is constant, then this equation is equivalent to $dc/dt = q/V(c_{in} - c) - k(T)c$.
2. In a similar manner, we balance the heat of the reaction with

$$\begin{pmatrix}\text{Rate of change}\\ \text{of heat content}\end{pmatrix}$$

$$= (\text{Rate in of heat}) - (\text{Rate out of heat}) - \begin{pmatrix}\text{Heat removed}\\ \text{by cooling}\end{pmatrix} + \begin{pmatrix}\text{Heat produced}\\ \text{by reaction}\end{pmatrix}.$$

Let $C_p$ be the specific heat so that the heat content per unit volume of the reaction mixture at temperature $T$ is $C_pT$. If $H$ is the rate at which heat is generated by the reaction and $VS(T)$ is the rate at which heat is removed from the system by a cooling system, then we have the differential equation

* *Applications in Undergraduate Mathematics in Engineering*, Mathematical Association of America, Macmillan Company, New York (1967), pp. 122–125. W. Fred Ramirez, *Computational Methods for Process Simulation*, Butterworths Series in Chemical Engineering, Boston (1989), pp. 175–182.



$$VC_p\frac{dT}{dt} = qC_pT_{in} - qC_pT - VS(T) + HVk(T)c,$$

where $T_{in}$ is the temperature at which the chemical flows into the tank. Solve this equation for $dT/dt$.

3. The equations in (1) and (2) form a system of nonlinear ordinary differential equations. Show that the equilibrium point of this system satisfies the equations

$$c_{in} - c = \frac{V}{q}ck(T) \quad \text{and} \quad T - T_{in} = \frac{HV}{qC_p}ck(T),$$

if we assume there is no cooling system. Solve the first equation for $c$ and substitute into the second equation to find that

$$T - T_{in} = \frac{HVc_{in}}{qC_p + VC_pk(T)}k(T).$$

We call values of $T$ that satisfy this equation **steady state temperatures.**

4. If $T_{in} = 1$, $HVc_{in} = VC_p = qC_p = 1$, and $k(T) = e^{-1/T}$, graph $y = T - T_{in}$ and $y = HVc_{in}/(qC_p + VC_pk(T))k(T)$ to determine the number of roots of the equation in (3).
5. If $T_{in} = 0.15$, $HVc_{in} = 1$, $VC_p = 0.25$, $qC_p = 0.015$, and $k(T) = e^{-3/T}$, graph $y = T - T_{in}$ and $y = HVc_{in}/(qC_p + VC_pk(T))k(T)$ to determine the number of roots of the equation in (3). How does this situation differ from that in (4)?
6. Notice that $y = T - T_{in}$ describes heat removal and $y = HVc_{in}/(qC_p + VC_pk(T))k(T)$ describes heat production. Therefore, if the slope of the heat production curve is greater than that of the heat removal curve, the steady state is *unstable*. On the other hand, if the slope of the heat production curve is less than or equal to that of the heat removal curve, then the steady state is *stable*. Use this information to determine which of the temperatures found in (4) and (5) are stable and which are unstable. What do the slopes of these curves represent?
7. Approximate the solution to the system of differential equations using the parameter values given in (4) and (5). Does the stability of the system correspond to those found in (6)?

# 8

# Introduction to the Laplace Transform

In previous chapters we investigated solving the equation

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = f(t)$$

for $y$. We saw that if the coefficients $a_n, a_{n-1}, \ldots, a_0$ are constants, we can find a general solution of the equation by first finding a general solution of the corresponding homogeneous equation

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0,$$

and then finding a particular solution of

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = f(t).$$

If the coefficients $a_n, a_{n-1}, \ldots, a_0$ are not constants, the situation is more difficult. In some cases, as when the equation is a Cauchy-Euler equation, similar techniques can be used. In other cases, we might be able to use power series to find a solution. In all of these cases, however, the function $f(t)$ has typically been a *smooth* function. In cases when $f(t)$ is not a smooth function, as when $f(t)$ is a piecewise-defined function, implementing methods like variation of parameters to solve the equation



**Pierre Simon de Laplace** (1749–1827) French mathematician and astronomer *(NorthWind Picture Archives)*

The French mathematician Pierre de Laplace (1749–1827) introduced this integral transform in his work **Théorie analytique des probabilités,** published in 1812. However, Laplace is probably most famous for his contributions to astronomy and probability.



$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = f(t)$$

is usually difficult.

In this chapter, we discuss a technique that transforms the *differential* equation,

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = f(t),$$

into an *algebraic* equation that can often be solved to obtain a solution of the differential equation, *even if $f(t)$ it not a smooth function.*

## 8.1 The Laplace Transform: Preliminary Definitions and Notation

- Definition of the Laplace Transform
- Exponential Order
- Jump Discontinuities and Piecewise Continuous Functions
- The Inverse Laplace Transform

We are already familiar with several operations on functions. In previous courses, we learned to add, subtract, multiply, divide, and compose functions. Another operation on functions is *differentiation,* which transforms the differentiable function $F(t)$ to its derivative $F'(t)$,

$$D_t(F(t)) = F'(t).$$

Similarly, the operation of *integration* transforms the integrable function $f(t)$ to its integral. For example, if $f$ is integrable on an interval containing $a$, then

$$\int_a^t f(x)\,dx$$

is a function of $t$.

In this section, we introduce another operation on functions, the *Laplace transform,* and discuss several of its properties.

### Definition of the Laplace Transform

**Definition 8.1 Laplace Transform**

Let $f(t)$ be a function defined on the interval $(0, +\infty)$. The **Laplace transform** of $f(t)$ is the function (of $s$)

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt,$$

provided that the improper integral exists.

Because the Laplace transform yields a function of $s$, we often use the notation $\mathcal{L}\{f(t)\} = F(s)$ to denote the Laplace transform of $f(t)$. We use the *capital* letter to denote the Laplace transform of the function named with the corresponding *small* letter.

The Laplace transform is defined as an improper integral. Recall that we evaluate improper integrals of this form by taking the limit of a definite integral:

$$\int_0^\infty f(t)\,dt = \lim_{M\to\infty}\int_0^M f(t)\,dt.$$

**Example 1**

Compute $\mathcal{L}\{f(t)\}$ if $f(t) = 1$.

**Solution**

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\cdot 1\,dt = \lim_{M\to\infty}\int_0^M e^{-st}\cdot 1\,dt = \lim_{M\to\infty}\left[-\frac{e^{-st}}{s}\right]_{t=0}^{t=M}$$
$$= -\frac{1}{s}\lim_{M\to\infty}[e^{-sM} - 1].$$

If $s > 0$, $\lim_{M\to\infty} e^{-sM} = 0$. (Otherwise, the limit does not exist.) Therefore,

$$F(s) = \mathcal{L}\{f(t)\} = -\frac{1}{s}\lim_{M\to\infty}[e^{-sM} - 1] = -\frac{1}{s}(0 - 1) = \frac{1}{s},\ s > 0.$$

Notice that the limit in Example 1 does not exist if $s < 0$. This means that $\mathcal{L}\{1\}$ is defined only for $s > 0$, so the domain of $\mathcal{L}\{1\}$ is $s > 0$. We will find that the domain of most functions is $s > a$, where $a$ is a constant.

**Example 2**

Compute $\mathcal{L}\{f(t)\}$ if $f(t) = e^{at}$.

**Solution**

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt = \int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$
$$= \lim_{M\to\infty}\left[-\frac{e^{-(s-a)t}}{s-a}\right]_{t=0}^{t=M} = -\lim_{M\to\infty}\left(\frac{e^{-(s-a)M}}{s-a} - \frac{1}{s-a}\right).$$

If $s - a > 0$, then $\lim_{M\to\infty} e^{-(s-a)M} = 0$. Therefore,

$$F(s) = \mathcal{L}\{f(t)\} = -\lim_{M\to\infty}\left(\frac{e^{-(s-a)M}}{s-a} - \frac{1}{s-a}\right) = -\left(0 - \frac{1}{s-a}\right) = \frac{1}{s-a},\ s > a.$$



The formula found in Example 2 can be used to avoid using the definition, as illustrated in Example 3.

**Example 3**

Compute (a) $\mathcal{L}\{e^{-3t}\}$ and (b) $\mathcal{L}\{e^{5t}\}$.

**Solution** (a) $\mathcal{L}\{e^{-3t}\} = 1/(s-(-3)) = 1/(s+3)$, $s > -3$, and (b) $\mathcal{L}\{e^{5t}\} = 1/(s-5)$, $s > 5$.

**Example 4**

Compute $\mathcal{L}\{\sin t\}$.

**Solution** To evaluate the improper integral that results, we use a table of integrals or a computer algebra system. Otherwise, we would have to use integration by parts twice.

$$\mathcal{L}\{\sin t\} = \int_0^\infty e^{-st}\sin t\,dt = \lim_{M\to\infty}\int_0^M e^{-st}\sin t\,dt$$
$$= \lim_{M\to\infty}\left[-\frac{e^{-st}}{s^2+1}(s\sin t + \cos t)\right]_{t=0}^{t=M}$$
$$= -\frac{1}{s^2+1}\lim_{M\to\infty}[e^{-sM}(s\sin M + \cos M) - 1]$$

If $s > 0$, $\lim_{M\to\infty} e^{-sM}(s\sin M + \cos M) = 0$. (Why?) Therefore,

$$\mathcal{L}\{\sin t\} = -\frac{1}{s^2+1}\lim_{M\to\infty}[e^{-sM}(s\sin M + \cos M) - 1] = \frac{1}{s^2+1},\ s > 0.$$

**Example 5**

Compute $\mathcal{L}\{f(t)\}$ if $f(t) = t$.

**Solution** To compute $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\,t\,dt$ we use integration by parts with $u = t$ and $dv = e^{-st}\,dt$. Then $du = dt$ and $v = -e^{-st}/s$, so

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\,t\,dt = \lim_{M\to\infty}\int_0^M e^{-st}\,t\,dt$$
$$= \lim_{M\to\infty}\left(\frac{-te^{-st}}{s}\bigg|_{t=0}^{t=M}\right) + \lim_{M\to\infty}\frac{1}{s}\int_0^M e^{-st}\,dt$$

$$= 0 - \frac{1}{s^2}\lim_{M\to\infty}\left(e^{-st}\Big|_{t=0}^{t=m}\right).$$

If $s > 0$, $\lim_{M\to\infty} e^{-sM} = 0$. Therefore,

$$F(s) = \mathcal{L}\{f(t)\} = -\frac{1}{s^2}\lim_{M\to\infty}\left(e^{-sM} - 1\right) = \frac{1}{s^2},\ s > 0.$$

As we can see, the definition of the Laplace transform can be difficult to apply. For example, if we wanted to calculate $\mathcal{L}\{t^n\}$ with the definition, we would have to integrate by parts $n$ times; a difficult and time-consuming task. We will investigate other properties of the Laplace transform so that we can determine the Laplace transform of many functions more easily. We begin by discussing the *linearity property*, which enables us to use the transforms that we have already found to find the Laplace transforms of other functions.

*We are familiar with **linearity properties** through our work in calculus with derivatives and integrals. For derivatives, this property is*

$$\frac{d}{dx}[af(x) + bg(x)] = a\frac{d}{dx}[f(x)] + b\frac{d}{dx}[g(x)],$$

*and for integrals, it is*

$$\int [af(x) + bg(x)]\,dx = a\int f(x)\,dx + b\int g(x)\,dx.$$

**Theorem 8.1 Linearity Property of the Laplace Transform**

Let $a$ and $b$ be constants, and suppose that the Laplace transform of the functions $f(t)$ and $g(t)$ exists. Then,

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

***PROOF OF THEOREM 8.1***

$$\mathcal{L}\{af(t) + bg(t)\} = \int_0^\infty e^{-st}(af(t) + bg(t))\,dt$$

$$= a\int_0^\infty e^{-st}f(t)\,dt + b\int_0^\infty e^{-st}g(t)\,dt = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

**Example 6**

Calculate **(a)** $\mathcal{L}\{6\}$; **(b)** $\mathcal{L}\{5 - 2e^{-t}\}$

***Solution*** **(a)** $\mathcal{L}\{6\} = 6\mathcal{L}\{1\} = 6(1/s) = 6/s$. **(b)** $\mathcal{L}\{5 - 2e^{-t}\} = 5\mathcal{L}\{1\} - 2\mathcal{L}\{e^{-t}\} = 5(1/s) - 2(1/[s - (-1)]) = 5/s - 2/(s + 1)$.

*For what values of s do the transforms in Example 6 exist?*

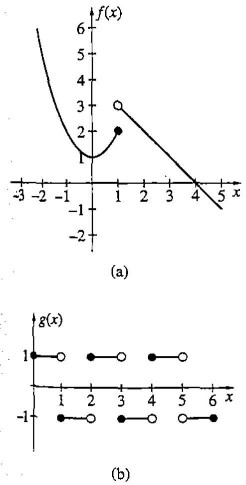



Figure 8.1 (a)–(b).



## Exponential Order

In calculus we saw that in some cases improper integrals diverge, which means that the limit of the definite integral $\int_a^M f(x)\,dx$ does not exist as $M \to \infty$. Hence, we may believe that the Laplace transform *may not* exist for some functions. For example, $f(t) = 1/t$ grows too rapidly near $t = 0$ for the improper integral $\int_0^\infty e^{-st}f(t)\,dt$ to exist and $f(t) = e^{t^2}$ grows too rapidly as $t \to \infty$ for the improper integral $\int_0^\infty e^{-st}f(t)\,dt$ to exist. We present the following definitions and theorems to better understand the types of functions for which the Laplace transform exists.

**Definition 8.2 Exponential Order**

A function $f(t)$ is of **exponential order** (of order $b$) if there are numbers $b$, $C > 0$, and $T > 0$ such that

$$|f(t)| \le Ce^{bt}$$

for $t > T$.

We can use Definition 8.2 to show that if $\lim_{t\to\infty} f(t)e^{-bt}$ exists and is finite, $f(t)$ is of exponential order; if $\lim_{t\to\infty} f(t)e^{-bt} = +\infty$ for every value of $b > 0$, $f(t)$ is not of exponential order. (See Exercise 64.)

## Jump Discontinuities and Piecewise Continuous Functions

In the next sections, we will see that the Laplace transform is particularly useful in solving differential equations involving piecewise or recursively defined functions. For example, the function $f(x) = \begin{cases} x^2 + 1, & \text{if } x \le 1 \\ 4 - x, & \text{if } x > 1 \end{cases}$ with the graph shown in Figure 8.1(a), is a piecewise defined function. Because

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x^2 + 1) = 2 \ne 3 = \lim_{x\to 1^+}(4 - x) = \lim_{x\to 1^+} f(x),$$

the limit $\lim_{x\to 1} f(x)$ does not exist and $f(x)$ is not continuous if $x = 1$. Similarly, the function $g(x)$ defined by $g(x) = \begin{cases} 1, & \text{if } 0 \le x < 1 \\ -1, & \text{if } 1 \le x < 2, \end{cases}$ on the interval $[0, 2)$ and then recursively by the relationship $g(x) = g(x - 2)$ (shown in Figure 8.1(b)), is discontinuous when $x = n$, where $n$ is any nonnegative integer. For both $f$ and $g$, we say that the discontinuities are **jump discontinuities** because the left- and right-hand limits both exist at the points of discontinuity but are unequal.

**Definition 8.3 Jump Discontinuity**

Let $f(t)$ be defined on $[a, b]$. $f$ has a **jump discontinuity** at $t = c$, $a < c < b$, if the one-sided limits $\lim_{t\to c^+} f(t)$ and $\lim_{t\to c^-} f(t)$ are finite, but unequal, values. $f(t)$ has a **jump discontinuity at $t = a$** if $\lim_{t\to a^+} f(t)$ is a finite value different from $f(a)$. $f(t)$ has a **jump discontinuity at $t = b$** if $\lim_{t\to b^-} f(t)$ is a finite value different from $f(b)$.

**Definition 8.4 Piecewise Continuous**

A function $f(t)$ is **piecewise continuous on the finite interval $[a, b]$** if $f(t)$ is continuous at every point in $[a, b]$ except at finitely many points at which $f(t)$ has a jump discontinuity. A function $f(t)$ is **piecewise continuous on $[0, \infty)$** if $f(t)$ is piecewise continuous on $[0, N]$ for all $N > 0$.

*Is* $f(t) = \begin{cases} 1/(t-1), & 0 \le t < 1 \\ t, & 1 \le t \le 2 \end{cases}$ *piecewise continuous on* $[0, 2]$? *Is* $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1/(t-1)^2, & 1 \le t \end{cases}$ *piecewise continuous on* $[0, +\infty)$?

*Note:* In this textbook, typically we work with functions that are piecewise continuous and of exponential order. However, in the exercises we explore functions that may or may not have these properties.

**Theorem 8.2 Sufficient Condition for Existence of $\mathcal{L}\{f(t)\}$**

Suppose that $f(t)$ is a piecewise continuous function on the interval $[0, \infty)$ and that it is of exponential order $b$ for $t > T$. Then, $\mathcal{L}\{f(t)\}$ exists for $s > b$.

*PROOF OF THEOREM 8.2*

We need to show that the integral $\int_0^\infty e^{-st} f(t)\, dt$ converges for $s > b$, assuming that $f(t)$ is a piecewise continuous function on the interval $[0, \infty)$ and that it is of exponential order $b$ for $t > T$. First, we write the integral as

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^T e^{-st} f(t)\, dt + \int_T^\infty e^{-st} f(t)\, dt,$$

where $T$ is selected so that $|f(t)| \le Ce^{bt}$ for the constants $b$ and $C$, $C > 0$.

Notice that because $f(t)$ is a piecewise continuous function, so is $e^{-st}f(t)$. The first of these integrals, $\int_0^T e^{-st} f(t)\, dt$, exists because it can be written as the sum of integrals over which $e^{-st}f(t)$ is continuous. The fact that $e^{-st}f(t)$ is piecewise continuous on $[T, \infty)$ is also used to show that the second integral, $\int_T^\infty e^{-st} f(t)\, dt$, converges. Because there are constants $C$ and $b$ such that $|f(t)| \le Ce^{bt}$, we have

$$\left|\int_T^\infty e^{-st} f(t)\, dt\right| \le \int_T^\infty \left|e^{-st} f(t)\right| dt \le C\int_T^\infty e^{-st}e^{bt}\, dt = C\int_T^\infty e^{-(s-b)t}\, dt$$

$$= C\lim_{M\to\infty} \int_T^M e^{-(s-b)t}\, dt = C\lim_{M\to\infty}\left[-\frac{e^{-(s-b)t}}{s-b}\right]_{t=T}^{t=M}$$

$$= -\frac{C}{s-b}\lim_{M\to\infty}\left(e^{-(s-b)M} - e^{-(s-b)T}\right).$$

Then, if $s - b > 0$, $\lim_{M\to\infty} e^{-(s-b)M} = 0$, so

$$\left|\int_T^\infty e^{-st} f(t)\, dt\right| \le \frac{Ce^{-(s-b)T}}{s-b},\ s > b.$$

Because both of the integrals $\int_0^T e^{-st} f(t)\, dt$ and $\int_T^\infty e^{-st} f(t)\, dt$ exist, $\int_0^\infty e^{-st} f(t)\, dt$ also exists for $s > b$.



**Example 7**

Find the Laplace transform of $f(t) = \begin{cases} -1, & 0 \le t \le 4 \\ 1, & t > 4 \end{cases}$.

***Solution*** Because $f(t)$ is a piecewise continuous function on $[0, \infty)$ and of exponential order, $F(s) = \mathcal{L}\{f(t)\}$ exists. We use the definition and evaluate the improper integral using a sum of two integrals.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\, dt = \int_0^4 (-1)e^{-st}\, dt + \int_4^\infty e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{s}\right]_{t=0}^{t=4} + \lim_{M\to\infty}\left[-\frac{e^{-st}}{s}\right]_{t=T}^{t=M}$$

$$= \frac{1}{s}\left(e^{-4s} - 1\right) - \frac{1}{s}\lim_{M\to\infty}\left(e^{-Ms} - e^{-4s}\right).$$

If $s > 0$, $\lim_{M\to\infty} e^{-Ms} = 0$, so

$$F(s) = \mathcal{L}\{f(t)\} = \frac{1}{s}\left(2e^{-4s} - 1\right),\ s > 0.$$

Notice that Theorem 8.2 gives a sufficient condition and not a necessary condition. In other words, there are functions such as $f(t) = t^{-1/2}$ that do not satisfy the hypotheses of the previous theorem for which the Laplace transform can be found. (See Exercises 62 and 63.)

## The Inverse Laplace Transform

Up to this point, we were concerned with finding the Laplace transform of a given function. At this point we reverse the process: Given a function $F(s)$ we want to find a function $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$, if possible.

**Definition 8.5 Inverse Laplace Transform**

The **inverse Laplace transform** of the function $F(s)$ is the function $f(t)$, if such a function exists, that satisfies $\mathcal{L}\{f(t)\} = F(s)$. We denote the inverse Laplace transform of $F(s)$ with

$$f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

As we mentioned, there is a one-to-one correspondence between continuous functions and their Laplace transforms. With this in mind, we can compile a list of functions with the corresponding Laplace transforms. Such a table of Laplace transforms, given on the inside cover of this textbook as well as in Table 8.1 at the end of this section, is useful in finding the inverse Laplace transform of a given function. In the table, we look for $F(s)$ in the right-hand column to find $f(t) = \mathcal{L}^{-1}\{F(s)\}$ in the corresponding left-hand column. For example, the second row of the table indicates that $\mathcal{L}^{-1}\{4!/s^5\} = t^4$ and the ninth row shows that $\mathcal{L}^{-1}\{(s-1)/[(s-1)^2+16]\} = e^t \cos 4t$. This example illustrates shifting in the variable $s$.

**Theorem 8.3**

(Shifting in $s$) If $\mathcal{L}\{f(t)\} = F(s)$ exists, then $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$.

**Example 8**

Find the inverse Laplace transform of (a) $F(s) = 1/(s-6)$; (b) $F(s) = 2/(s^2+4)$; (c) $F(s) = 6/s^4$; (d) $F(s) = 6/(s+2)^4$.

**Solution** (a) Because $\mathcal{L}\{e^{6t}\} = 1/(s-6)$, $\mathcal{L}^{-1}\{1/(s-6)\} = e^{6t}$. (b) Note that $\mathcal{L}\{\sin 2t\} = 2/(s^2+2^2) = 2/(s^2+4)$, so $\mathcal{L}^{-1}\{2/(s^2+4)\} = \sin 2t$. (c) Because $\mathcal{L}\{t^3\} = 3!/s^4 = 6/s^4$, $\mathcal{L}^{-1}\{6/s^4\} = t^3$. (d) Notice that $F(s) = 6/(s+2)^4$ is obtained from $F(s) = 6/s^4$ by substituting $(s+2)$ for $s$. Therefore by the shifting property,

$$\mathcal{L}\{e^{-2t}t^3\} = \frac{6}{(s+2)^4}, \text{ so } \mathcal{L}^{-1}\left\{\frac{6}{(s+2)^4}\right\} = e^{-2t}t^3.$$



*Find* $\mathcal{L}^{-1}\left\{\frac{1}{(s-4)}\right\}$ *and* $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)}\right\}$.

Just as we use the linearity of the Laplace transform to find $\mathcal{L}\{f(t)\}$ for many functions $f$, the same is true for computing $\mathcal{L}^{-1}\{F(s)\}$.

**Theorem 8.4 Linearity Property of the Inverse Laplace Transform**

Suppose that $\mathcal{L}^{-1}\{F(s)\}$ and $\mathcal{L}^{-1}\{G(s)\}$ exist and are continuous on $[0, \infty)$ and that $a$ and $b$ are constants. Then,

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}.$$

If the functions are not in the forms presented in the table on the inside cover of the book or in Table 8.1 at the end of this section, we can make use of the linearity property to determine the inverse Laplace transform.

**Example 9**

Find the inverse Laplace transform of (a) $F(s) = 1/s^3$; (b) $F(s) = -7/(s^2+16)$; (c) $F(s) = 5/s - 2/(s-10)$.

**Solution** (a) $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{2}{s^3}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = \frac{1}{2}t^2$.

(b) $$\mathcal{L}^{-1}\left\{-\frac{7}{s^2+16}\right\} = -7\mathcal{L}^{-1}\left\{\frac{1}{s^2+16}\right\} = -7\mathcal{L}^{-1}\left\{\frac{1}{4}\frac{4}{s^2+4^2}\right\}$$
$$= -\frac{7}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\} = -\frac{7}{4}\sin 4t.$$

(c) $$\mathcal{L}^{-1}\left\{\frac{5}{s} - \frac{2}{s-10}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-10}\right\} = 5 - 2e^{10t}.$$

**Example 10**

Find the inverse Laplace transform of $F(s) = \dfrac{2s-9}{s^2+25}$.

**Solution**

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2s-9}{s^2+25}\right\} &= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\} - 9\mathcal{L}^{-1}\left\{\frac{1}{s^2+25}\right\} \\ &= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\} - \frac{9}{5}\mathcal{L}^{-1}\left\{\frac{5}{s^2+25}\right\} \\ &= 2\cos 5t - \frac{9}{5}\sin 5t.\end{aligned}$$

*Find* $\mathcal{L}^{-1}\{(3s+1)/(s^2+16)\}$.

Sometimes we must complete the square in the denominator of $F(s)$ before finding $\mathcal{L}^{-1}\{F(s)\}$.

**Example 11**

Determine $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2s+5}\right\}$.

**Solution** Notice that many of the forms of $F(s)$ in the table of Laplace transforms on the inside cover of this book involve a term of the form $s^2 + k^2$ in the denominator. Through shifting, this term is replaced by $(s-a)^2 + k^2$. We obtain a term of this form in the denominator by completing the square. This yields

$$\frac{s}{s^2+2s+5} = \frac{s}{(s^2+2s+1)+4} = \frac{s}{(s+1)^2+4}.$$

Because the variable $s$ appears in the numerator, we must write it in the form $(s+1)$ to find the inverse Laplace transform. Doing so, we find that

$$\frac{s}{s^2+2s+5} = \frac{s}{(s+1)^2+4} = \frac{(s+1)-1}{(s+1)^2+4}.$$

Therefore,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+5}\right\} &= \mathcal{L}^{-1}\left\{\frac{(s+1)-1}{(s+1)^2+4}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{(s+1)}{(s+1)^2+4}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\} \\ &= e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t.\end{aligned}$$

*Find* $\mathcal{L}^{-1}\left\{\dfrac{2s}{s^2-4s+8}\right\}$.



The following theorem is useful in showing that the inverse Laplace transform of a function $F(s)$ does not exist.

**Theorem 8.5**

Suppose that $f(t)$ is a piecewise continuous function on $[0, \infty)$ and of exponential order $b$. Then,

$$\lim_{s\to\infty} F(s) = \lim_{s\to\infty} \mathcal{L}\{f(t)\} = 0.$$

**Example 12**

For the following functions, determine if $F(s)$ is the Laplace transform of a piecewise continuous function of exponential order. **(a)** $F(s) = \dfrac{2s}{s-6}$; **(b)** $F(s) = \dfrac{s^3}{s^2+16}$.

**Solution** **(a)** $\lim_{s\to\infty} F(s) = \lim_{s\to\infty} 2s/(s-6) = 2 \neq 0$; **(b)** $\lim_{s\to\infty} F(s) = \lim_{s\to\infty} \dfrac{s^3}{s^2+16} = \infty \neq 0$. In each case, $\lim_{s\to\infty} F(s) \neq 0$, so $F(s)$ is not the Laplace transform of a piecewise continuous function of exponential order.

In later sections, we may use Theorem 8.5 to determine if we have made an error in our solution process.

*Does* $\mathcal{L}^{-1}\left\{\dfrac{s^2}{s^2+8s+16}\right\}$ *exist?*

Using the properties of the Laplace transform discussed here and in the exercises, we can compute the Laplace transforms of a large number of frequently encountered functions. Table 8.1 lists the Laplace transforms of several of these frequently encountered functions. A more comprehensive table is found on the inside cover of this textbook.

**TABLE 8.1 Laplace Transforms of Frequently Encountered Functions**

| $f(t)$ | $F(s) = \mathcal{L}\{f(t)\}$ | $f(t)$ | $F(s) = \mathcal{L}\{f(t)\}$ |
|---|---|---|---|
| 1 | $\dfrac{1}{s}$, $s > 0$ | $t^n$, $n = 1, 2, \ldots$ | $\dfrac{n!}{s^{n+1}}$, $s > 0$ |

**TABLE 8.1 cont. Laplace Transforms of Frequently Encountered Functions**

| $f(t)$ | $F(s) = \mathcal{L}\{f(t)\}$ | $f(t)$ | $F(s) = \mathcal{L}\{f(t)\}$ |
|---|---|---|---|
| $e^{at}$ | $\dfrac{1}{s-a},\ s>a$ | $t^n e^{at},\ n = 1, 2, \ldots$ | $\dfrac{n!}{(s-a)^{n+1}}$ |
| $\sin kt$ | $\dfrac{k}{s^2+k^2}$ | $e^{at}\sin kt$ | $\dfrac{k}{(s-a)^2+k^2}$ |
| $\cos kt$ | $\dfrac{s}{s^2+k^2}$ | $e^{at}\cos kt$ | $\dfrac{s-a}{(s-a)^2+k^2}$ |
| $\sinh kt$ | $\dfrac{k}{s^2-k^2}$ | $e^{at}\sinh kt$ | $\dfrac{k}{(s-a)^2-k^2}$ |
| $\cosh kt$ | $\dfrac{s}{s^2-k^2}$ | $e^{at}\cosh kt$ | $\dfrac{s-a}{(s-a)^2-k^2}$ |

## EXERCISES 8.1

In Exercises 1–16, use the definition of the Laplace transform to compute the Laplace transform of each function.

**1.** $f(t) = 21t$ **2.** $f(t) = 7e^{-t}$

***3.** $f(t) = 2e^t$ **4.** $f(t) = -8\cos 3t$

**5.** $f(t) = 2\sin 2t$

**6.** $f(t) = \begin{cases} 1 & \text{if } 0 \le t \le 2 \\ 0 & \text{if } t > 2 \end{cases}$

***7.** $f(t) = \begin{cases} 0 & \text{if } 0 \le t \le 1 \\ 1 & \text{if } t > 1 \end{cases}$

**8.** $f(t) = \begin{cases} 1-t & \text{if } 0 \le t \le 1 \\ 0 & \text{if } t > 1 \end{cases}$

**9.** $f(t) = \begin{cases} \cos t & \text{if } 0 \le t \le \pi/2 \\ 0 & \text{if } t > \pi/2 \end{cases}$

**10.** $f(t) = \begin{cases} \sin t & \text{if } 0 \le t \le 2\pi \\ 0 & \text{if } t > 2\pi \end{cases}$

***11.** $f(t) = \begin{cases} 1-t, & 0 < t < 3 \\ 0, & t \ge 3 \end{cases}$

**12.** $f(t) = \begin{cases} 3t+2, & 0 < t < 5 \\ 0, & t \ge 5 \end{cases}$

**13.** $f(t) = \begin{cases} 1, & 0 \le t < 10, \\ -1, & t \ge 10 \end{cases}$

**14.** $f(t) = \begin{cases} t, & 0 \le t < 2, \\ 3, & t \ge 2 \end{cases}$

***15.** $f(t) = \sin kt$

**16.** $f(t) = \cos kt$

**17.** Use the definitions $\cosh kt = \frac{1}{2}(e^{kt} + e^{-kt})$ and $\sinh kt = \frac{1}{2}(e^{kt} - e^{-kt})$ to verify each of the following.

**(a)** $\mathcal{L}\{\cosh kt\} = \dfrac{s}{s^2-k^2}$

**(b)** $\mathcal{L}\{\sinh kt\} = \dfrac{k}{s^2-k^2}$

**(c)** $\mathcal{L}\{e^{at}\cosh kt\} = \dfrac{s-a}{(s-a)^2-k^2}$

**(d)** $\mathcal{L}\{e^{at}\sinh kt\} = \dfrac{k}{(s-a)^2-k^2}$

**18.** Use the definitions $\cos kt = 1/2(e^{ikt} + e^{-ikt})$ and $\sin kt = 1/2i(e^{ikt} - e^{-ikt})$ to verify each of the following.

**(a)** $\mathcal{L}\{e^{at}\cos kt\} = \dfrac{s-a}{(s-a)^2+k^2}$



**(b)** $\mathcal{L}\{e^{at}\sin kt\} = \dfrac{k}{(s-a)^2+k^2}$

**19.** **(Shifting Property)** Suppose that $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a$. Use the definition of $\mathcal{L}\{e^{at}f(t)\}$ to show that $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$.

In Exercises 20–25, use the result of Exercise 19 to compute the following. Compare your results with those found in Table 8.1 or the table on the inside cover of this textbook.

**20.** $\mathcal{L}\{e^t \sin 2t\}$ **21.** $\mathcal{L}\{e^t \cos 3t\}$

**22.** $\mathcal{L}\{e^{-2t}\cos 4t\}$ **23.** $\mathcal{L}\{e^{-t}\sin 5t\}$

**24.** $\mathcal{L}\{e^{3t}t\}$ **25.** $\mathcal{L}\{e^{-t/2}t^2\}$

In Exercises 26–55, use properties of the Laplace transform and Table 8.1 or the table on the inside cover of this textbook to compute the Laplace transform of each function.

**26.** $f(t) = 28e^t$ **27.** $f(t) = -18e^{3t}$

**28.** $f(t) = \frac{1}{8}(2t - \sin 2t)$ ***29.** $f(t) = \cos 5t$

**30.** $f(t) = 1 + \cos 2t$ **31.** $f(t) = 1 + \sin 5t$

**32.** $f(t) = 18e^{5t}$ **33.** $f(t) = -16e^{2t}$

**34.** $f(t) = -6t^4e^{-7t}$ **35.** $f(t) = t^7$

**36.** $f(t) = t^3e^{4t}$ ***37.** $f(t) = t^2e^{-3t}$

**38.** $f(t) = t^4\cos t$ **39.** $f(t) = t^5e^{-4t}$

**40.** $f(t) = 8\cosh 4t$ **41.** $f(t) = t\cos 3t$

**42.** $f(t) = t\sin 7t$ **43.** $f(t) = 3t\sin t$

**44.** $f(t) = t^2\sinh 6t$ ***45.** $f(t) = t\sinh 7t$

**46.** $f(t) = e^t\sinh t$ **47.** $f(t) = e^{7t}\cosh t$

**48.** $f(t) = -7e^{6t}\sin 2t$ **49.** $f(t) = e^{-2t}\cos 4t$

**50.** $f(t) = e^{-2t}\sin 7t$ ***51.** $f(t) = e^{5t}\cos 7t$

**52.** $f(t) = 2e^{-2t}\sinh t$

**53.** $f(t) = \sin 4t + 4t\cos 4t$

**54.** $f(t) = t^3\sin 2t$

**55.** $f(t) = \frac{1}{6}t\sin 3t$

**56.** Recall that the sum of the geometric series $a + ar + ar^2 + ar^3 + \cdots = a/(1-r)$ if $|r| < 1$. (The series diverges otherwise.) Also, recall that the Maclaurin series for $f(t) = e^t$ is $e^t = 1 + t + t^2/2! + t^3/3! + \cdots + t^n/n! + \cdots$. Take the Laplace transform of each term in the series and show that the sum of the resulting series converges to $\mathcal{L}\{e^t\}$.

**57.** Use the Maclaurin series $\sin t = \sum_{n=0}^{\infty} \dfrac{(-1)^n t^{2n+1}}{(2n+1)!}$ to verify that $\mathcal{L}\{\sin t\} = 1/(s^2+1)$.

**58.** Use the Maclaurin series $\cos t = \sum_{n=0}^{\infty} \dfrac{(-1)^n t^{2n}}{(2n)!}$ to verify that $\mathcal{L}\{\cos t\} = s/(s^2+1)$.

**59.** Find $\mathcal{L}\{\cos^2 kt\}$.

**60.** Figure 8.2(a) shows the graph of a function $f(t)$ and Figure 8.2(b) shows the graph of its Laplace transform $F(s) = \mathcal{L}\{f(t)\}$. Use Figure 8.2 to sketch the graphs of $\mathcal{L}\{e^{-2t}f(t)\}$ and $\mathcal{L}\{e^{3t}f(t)\}$.

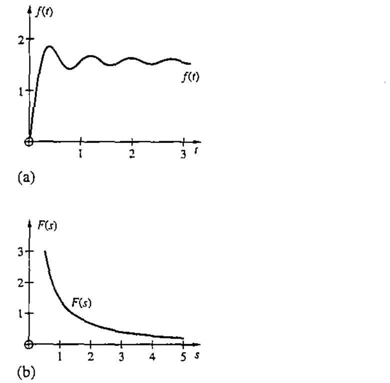



**Figure 8.2** (a)–(b).

**61.** Use the identities $\sin(A+B) = \sin A\cos B + \cos A\sin B$ and $\cos(A+B) = \cos A\cos B - \sin A\sin B$ to assist in finding the Laplace transform of the following functions.

**(a)** $f(t) = \sin(t + \pi/4)$

**(b)** $f(t) = \cos(t + \pi/4)$

**(c)** $f(t) = \cos(t + \pi/6)$

**(d)** $f(t) = \sin(t + \pi/6)$

**62.** The **Gamma function**, $\Gamma(x)$, is defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$, $x > 0$. Show that $\mathcal{L}\{t^\alpha\} = \Gamma(\alpha+1)/s^{\alpha+1}$, $\alpha > -1$.

***63.** Use the result of Exercise 62 to find the Laplace transform of **(a)** $f(t) = t^{-1/2}$; **(b)** $f(t) = t^{1/2}$.

64. (a) Use Definition 8.2 to show that if $\lim_{t\to\infty} f(t)e^{-bt}$ exists and is finite, $f(t)$ is of exponential order and if $\lim_{t\to\infty} f(t)e^{-bt} = +\infty$ for every value $b$, $f(t)$ is not of exponential order. (b) Give an example of a function $f(t)$ of exponential order $b$ for which $\lim_{t\to\infty} f(t)e^{-bt}$ does not exist.

65. Let $f(t)$ be a piecewise continuous function on the interval $[0, \infty)$ that is of exponential order $b$ for $t > T$. Show that $h(t) = \int_0^t f(u)\,du$ is also of exponential order.

In Exercises 67–89, compute the inverse Laplace transform of the given function.

66. $\dfrac{1}{s^8}$

67. $\dfrac{1}{s^2}$

68. $\dfrac{1}{(s-5)^6}$

*69. $\dfrac{1}{(s-2)^2}$

70. $\dfrac{1}{s+5}$

71. $\dfrac{1}{(s+6)^3}$

72. $\dfrac{1}{s^2+9}$

73. $\dfrac{4}{(s-3)^2}$

74. $\dfrac{1}{s^2+15s+56}$

*75. $\dfrac{s}{s^2+16}$

76. $\dfrac{1}{s^2+12s+61}$

77. $\dfrac{s}{s^2-5s-14}$

78. $\dfrac{s+3}{s^2+6s+5}$

*79. $\dfrac{s-2}{s^2-4s}$

80. $\dfrac{1}{s^2-12s+35}$

81. $\dfrac{s-7}{s^2-14s+48}$

82. $\dfrac{1}{s^2+2s-24}$

*83. $\dfrac{1}{s^2-4s-12}$

84. $\dfrac{1}{s^2+12s+37}$

85. $\dfrac{1}{s^2-2s+2}$

86. $\dfrac{s+1}{s^2+2s+37}$

*87. $\dfrac{s-1}{s^2-2s+50}$

88. $\dfrac{s+2}{s^2+4s+8}$

*89. $\dfrac{s-7}{s^2-14s+50}$

90. (a) Compute $f(t) = \mathcal{L}^{-1}\{\tan^{-1}(1/s)\}$ with $\mathcal{L}\{t^n f(t)\} = (-1)^n\, d^n/ds^n\, F(s)$ in the form ($n = 1$)

$$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}F(s)\right\}$$
$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\tan^{-1}\left(\frac{1}{s}\right)\right\}.$$

(b) Show that $\mathcal{L}^{-1}\{\tan^{-1}(a/s)\} = (\sin at)/t$.

(c) Show that $\mathcal{L}^{-1}\{1/2\tan^{-1}((a+b)/s) + 1/2\tan^{-1}((a-b)/s)\} = (\sin at\cos bt)/t$.

91. (a) Use the linearity of the Laplace transform to compute $\mathcal{L}\{a\sin bt - b\sin at\}$ and $\mathcal{L}\{\cos bt - \cos at\}$.

(b) Use these results to find $\mathcal{L}^{-1}\{1/[(s^2+a^2)(s^2+b^2)]\}$ and $\mathcal{L}^{-1}\{s/[(s^2+a^2)(s^2+b^2)]\}$. (These formulas can be verified in a table of Laplace transforms or with a computer algebra system.)

92. Use the results of Exercise 91 to find

(a) $\mathcal{L}^{-1}\left\{\dfrac{10s}{(s^2+1)(s^2+16)}\right\}$;

(b) $\mathcal{L}^{-1}\left\{\dfrac{7}{(s^2+100)(s^2+1)}\right\}$.

93. Determine if $F(s)$ is the Laplace transform of a piecewise continuous function of exponential order.

(a) $F(s) = \dfrac{s}{4-s}$ (b) $F(s) = \dfrac{3s}{s+1}$

(c) $F(s) = \dfrac{s^2}{4s+10}$ (d) $F(s) = \dfrac{5s^3}{s^2+1}$

94. (a) Show that $f(t) = t^3$ is of exponential order.
(b) Show that $f(t) = e^{t^2}$ is not of exponential order.
(c) Is $f(t) = t^n$, where $n$ is any positive integer, of exponential order? Why?

95. Determine values of $C$ and $T$ that show that $f(t) = (\sin t\cos 2t)/t$ is of exponential order $\frac{1}{2}$.

96. Let $f_n(t) = \begin{cases} 1/n, & \text{if } 0 \le t \le n \\ 0, & \text{if } t > n \end{cases}$. (a) Graph $f_n(t)$ for $n = 10, 1, \frac{1}{10}$, and $\frac{1}{100}$. Is $f_n(t)$ of exponential order? Why? (b) Evaluate $\int_0^\infty f_n(t)\,dt$ and $\lim_{n\to 0^+}\int_0^\infty f_n(t)\,dt$. (c) Describe $\lim_{n\to 0^+} f_n(t)$. Is this limit a function? (d) Calculate $F_n(s) = \mathcal{L}\{f_n(t)\}$ and then $\lim_{n\to 0^+} F_n(s)$. Are you surprised by the result? Explain. (e) Is every function $F(s)$ the Laplace transform of some function $f(t)$? Why?

97. Graphically determine if the Gamma function is of exponential order. Can you prove your result analytically? Does the Gamma function have a Laplace transform? Explain.



## 8.2 Solving Initial-Value Problems with the Laplace Transform

In this section we show how the Laplace transform is used to solve initial-value problems. To do this we first need to understand how the Laplace transform of the derivatives of a function relates to the function itself. We begin with the first derivative.

**Theorem 8.6 Laplace Transform of the First Derivative**

Suppose that $f(t)$ is continuous for all $t \ge 0$ and is of exponential order $b$ for $t > T$. Also, suppose that $f'(t)$ is piecewise continuous on any closed subinterval of $[0, \infty)$. Then, for $s > b$

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

***PROOF OF THEOREM 8.6***

Using integration by parts with $u = e^{-st}$ and $dv = f'(t)\,dt$, we have

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st}f'(t)\,dt = \lim_{M\to\infty}\left(e^{-st}f(t)\Big|_{t=0}^{t=M}\right) + s\int_0^\infty e^{-st}f(t)\,dt$$
$$= -f(0) + s\mathcal{L}\{f(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

Proof of Theorem 8.6 assumes that $f'$ is a continuous function. If we use the assumption that $f'$ is continuous on $0 < t_1 < t_2 < \cdots < t_n < \infty$, we complete the proof by using

$$\mathcal{L}\{f'(t)\} = \int_0^{t_1} e^{-st}f'(t)\,dt + \int_{t_1}^{t_2} e^{-st}f'(t)\,dt + \cdots + \int_{t_n}^{\infty} e^{-st}f'(t)\,dt$$

This is the same integration by parts formula shown in the proof of Theorem 8.6 for each integral. Now we make the same assumptions of $f'$ and $f''$ as we did of $f$ and $f'$, respectively, in the statement of Theorem 8.6 and use Theorem 8.6 to develop an expression for $\mathcal{L}\{f''(t)\}$:

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0) = s[s\mathcal{L}\{f(t)\} - f(0)] - f'(0)$$
$$= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).$$

Continuing this process, we can construct similar expressions for the Laplace transform of higher-order derivatives. This leads to the following corollary to Theorem 8.6.

**Corollary 8.7 Laplace Transform of Higher Derivatives**

More generally, if $f^{(i)}(t)$ is a continuous function of exponential order $b$ on $[0, +\infty)$ for $i = 0, 1, \ldots, n-1$ and $f^{(n)}(t)$ is piecewise continuous on any closed subinterval of $[0, \infty)$, then for $s > b$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

We now show how the Laplace transform can be used to solve initial-value problems. Typically, when we solve an initial-value problem that involves $y(t)$, we use the following steps:

1. Compute the Laplace transform of each term in the differential equation;
2. Solve the resulting equation for $\mathcal{L}\{y(t)\} = Y(s)$; and
3. Determine $y(t)$ by computing the inverse Laplace transform of $Y(s)$.

The advantage of this method is that through the use of the property

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

we change the *differential* equation to an *algebraic* equation that can be solved for $\mathcal{L}\{f(t)\}$.

**Example 1**

Solve the initial-value problem $y' - 4y = e^{4t}$, $y(0) = 0$.

***Solution*** We begin by taking the Laplace transform of both sides of the differential equation. Because $\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s)$, we have

$$\mathcal{L}\{y' - 4y\} = \mathcal{L}\{e^{4t}\}$$

$$\mathcal{L}\{y'\} - 4\mathcal{L}\{y\} = \frac{1}{s-4}$$

$$\underbrace{sY(s) - y(0)}_{\mathcal{L}\{y'\}} - 4\underbrace{Y(s)}_{\mathcal{L}\{y\}} = \frac{1}{s-4}$$

$$(s-4)Y(s) = \frac{1}{s-4}.$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{1}{(s-4)^2}.$$

By using the shifting property with $\mathcal{L}\{t\} = 1/s^2$, we have



$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2}\right\} = te^{4t}.$$

*How is the solution changed if $y(0) = 1$?*

In many cases, we must determine a partial fraction decomposition of $Y(s)$ to obtain terms for which the inverse Laplace transform can be found.

**Example 2**

Solve the initial-value problem $y'' - 4y' = 0$, $y(0) = 3$, $y'(0) = 8$.

***Solution*** Let $Y(s) = \mathcal{L}\{y(t)\}$. Then $\mathcal{L}\{y'' - 4y'\} = \mathcal{L}\{0\}$. Because

$$\mathcal{L}\{y'' - 4y'\} = \mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} = \underbrace{s^2Y(s) - sy(0) - y'(0)}_{\mathcal{L}\{y''\}} - 4\underbrace{(sY(s) - y(0))}_{\mathcal{L}\{y'\}}$$

and $\mathcal{L}\{0\} = 0$ (why?), the equation becomes

$$s^2Y(s) - sy(0) - y'(0) - 4sY(s) + 4y(0) = 0.$$

Applying the initial conditions $y(0) = 3$ and $y'(0) = 8$ results in the equation

$$s^2Y(s) - 3s - 8 - 4sY(s) + 4(3) = 0 \quad \text{or} \quad s(s-4)Y(s) = 3s - 4.$$

Solving for $Y(s)$, we find that

$$Y(s) = \frac{3s-4}{s(s-4)}.$$

If we expand the right-hand side of this equation in partial fractions based on the two linear factors in the denominator of $Y(s)$, we obtain

$$\frac{3s-4}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4}.$$

Multiplying both sides of the equation by the denominator $s(s-4)$, we have

$$3s - 4 = A(s-4) + Bs.$$

If we substitute $s = 0$ into this equation, we have $3(0) - 4 = A(0-4) + B(0)$ or $-4 = -4A$ so that $A = 1$. Similarly, if we substitute $s = 4$, we find that $8 = 4B$ so $B = 2$. Therefore,

$$\frac{3s-4}{s(s-4)} = \frac{1}{s} + \frac{2}{s-4}$$

so

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{2}{s-4}\right\} = 1 + 2e^{4t}.$$

A partial fraction decomposition involving a repeated linear factor is illustrated in the following example.

**Example 3**

Solve the initial-value problem $y'' + 2y' + y = 6$, $y(0) = 5$, $y'(0) = 10$.

**Solution** If $Y(s) = \mathcal{L}\{y(t)\}$, then

$$\mathcal{L}\{y'' + 2y' + y\} = \mathcal{L}\{6\}$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \frac{6}{s}$$

$$\underbrace{s^2Y(s) - sy(0) - y'(0)}_{\mathcal{L}\{y''\}} + 2\underbrace{[sY(s) - y(0)]}_{\mathcal{L}\{y'\}} + Y(s) = \frac{6}{s}$$

$$(s^2 + 2s + 1)Y(s) = \frac{6}{s} + 5s + 20$$

$$(s^2 + 2s + 1)Y(s) = \frac{6 + 5s^2 + 20s}{s}$$

Solving for $Y(s)$ and factoring the denominator yields

$$Y(s) = \frac{5s^2 + 20s + 6}{s(s + 1)^2}.$$

In this case, the denominator contains the nonrepeated linear factor $s$ and the repeated linear factor $(s + 1)$. The partial fraction decomposition of $Y(s)$ is

$$\frac{5s^2 + 20s + 6}{s(s + 1)^2} = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{(s + 1)^2},$$

so multiplication on each side of the equation by $s(s + 1)^2$ gives the equation

$$5s^2 + 20s + 6 = A(s + 1)^2 + Bs(s + 1) + Cs \quad \text{or}$$

$$5s^2 + 20s + 6 = (A + B)s^2 + (2A + B + C)s + A.$$

Equating like coefficients, we obtain the system

$$\begin{cases} A + B = 5 \\ 2A + B + C = 20, \\ A = 6 \end{cases}$$

which has solution $A = 6$, $B = -1$, and $C = 9$, so

$$\frac{5s^2 + 20s + 6}{s(s + 1)^2} = \frac{6}{s} - \frac{1}{s + 1} + \frac{9}{(s + 1)^2}.$$



Therefore,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{6}{s} - \frac{1}{s + 1} + \frac{9}{(s + 1)^2}\right\} = 6 - e^{-t} + 9te^{-t}.$$

In some cases, $F(s)$ involves irreducible quadratic factors as we see in the next example.

**Example 4**

Solve the initial-value problem $y''' + 4y' = -10e^t$, $y(0) = 2$, $y'(0) = 2$, $y''(0) = -10$.

**Solution** Let $Y(s) = \mathcal{L}\{y(t)\}$. Then

$$\mathcal{L}\{y'''\} + 4\mathcal{L}\{y'\} = \mathcal{L}\{-10e^t\}$$

$$\underbrace{s^3Y(s) - s^2y(0) - sy'(0) - y''(0)}_{\mathcal{L}\{y'''\}} + 4\underbrace{[sY(s) - y(0)]}_{\mathcal{L}\{y'\}} = -\frac{10}{s - 1}$$

$$(s^3 + 4s)Y(s) = -\frac{10}{s - 1} + 2s^2 + 2s - 2$$

$$Y(s) = -\frac{10}{(s - 1)(s^3 + 4s)} + \frac{2s^2 + 2s - 2}{s^3 + 4s}$$

$$= \frac{2s^3 - 4s - 8}{s(s - 1)(s^2 + 4)}$$

Finding the partial fraction decomposition of the right-hand side, we obtain

$$\frac{2s^3 - 4s - 8}{s(s - 1)(s^2 + 4)} = \frac{A}{s} + \frac{B}{s - 1} + \frac{Cs + D}{s^2 + 4}.$$

Multiplying each side of the equation by $s(s - 1)(s^2 + 4)$ gives us

$$2s^3 - 4s - 8 = A(s - 1)(s^2 + 4) + Bs(s^2 + 4) + (Cs + D)s(s - 1)$$
$$= (A + B + C)s^3 + (-A - C + D)s^2 + (4A + 4B - D)s - 4A.$$

Equating coefficients yields the system of equations

$$\begin{cases} A + B + C = 2 \\ -A - C + D = 0 \\ 4A + 4B - D = -4' \\ -4A = -8 \end{cases}$$

with solution $A = 2$, $B = -2$, $C = 2$, and $D = 4$, so

$$\frac{2s^3-4s-8}{s(s-1)(s^2+4)} = \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^2+4} = \frac{2}{s} - \frac{2}{s-1} + \frac{2s+4}{s^2+4}.$$

Therefore,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{2}{s} - \frac{2}{s-1} + \frac{2s+4}{s^2+4}\right\} \\ &= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 2\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\ &= 2 - 2e^t + 2\cos 2t + 2\sin 2t. \end{aligned}$$

## *EXERCISES 8.2*

In Exercises 1–14, use Laplace transforms to solve the initial-value problem. In cases in which the equation could be solved by other methods, describe these methods. Graph the solution on the indicated interval.

**1.** $y'' + 11y' + 24y = 0$, $y(0) = -1$, $y'(0) = 0$, $[-1/2, 1]$

**2.** $y'' + 8y' + 7y = 0$, $y(0) = 0$, $y'(0) = 1$, $[-1/2, 1]$

***3.** $y'' + 3y' - 10y = 0$, $y(0) = -1$, $y'(0) = 1$, $[-1/2, 3/2]$

**4.** $y'' - 13y' + 40y = 0$, $y(0) = 0$, $y'(0) = -2$, $[-1/2, 1/4]$

**5.** $y'' + 4y = 0$, $y(0) = 2$, $y'(0) = 0$, $[0, 2\pi]$

**6.** $16y'' + 8y' + 65y = 0$, $y(0) = 0$, $y'(0) = 2$, $[0, 4\pi]$

***7.** $y''' - 4y'' - 9y' + 36y = 0$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = -1$, $[-1, 1]$

**8.** $y''' + 6y'' + 9y' + 4y = 0$, $y(0) = 0$, $y'(0) = -3$, $y''(0) = 2$, $[-1, 4]$

**9.** $y''' + y'' + 4y' + 4y = 0$, $y(0) = -5$, $y'(0) = 0$, $y''(0) = 0$, $[0, 3\pi]$

**10.** $y'' - y' - 2y = e^{-t}$, $y(0) = 2$, $y'(0) = 1$, $[-1, 1]$

***11.** $y'' - 12y' + 40y = \sin(2t)$, $y(0) = 1$, $y'(0) = 0$, $[-\pi, \pi/4]$

**12.** $y'' - 2y' + 37y = e^t + \cos(3t)$, $y(0) = 0$, $y'(0) = 1$, $[-1, 1]$

***13.** $y'' - y' - 12y = e^{2t} - \sin(t)$, $y(0) = -1$, $y'(0) = 1$, $[-1, 1]$

**14.** $y'' + 25y = 10\cos 5t$, $y(0) = 0$, $y'(0) = 0$, $[-\pi, 2\pi]$

**15.** **(Nonconstant Coefficients)** Use the property that $\mathcal{L}\{tf(t)\} = -F'(s)$ to solve the initial-value problem

$$y'' + 4ty' - 8y = 4,\ y(0) = 0,\ y'(0) = 0.$$

(*Hint:* Applying the Laplace transform to each term in the equation yields $Y'(s) + (3/s - s/4)Y(s) = -1/s^2$, where $p(s) = 3/s - s/4$. Use the integrating factor $\mu(s) = e^{\int p(s)\,ds}$ to find that the solution of this first-order equation is $Y(s) = s^{-3}(4 + Ce^{s^2/8})$. For what value of $C$ does $\mathcal{L}^{-1}\{Y(s)\}$ exist? Substitute this value and find $y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2t^2$.)

In Exercises 16–19, solve the initial-value problem involving nonconstant coefficients.

**16.** $y'' + 3ty' - 6y = 3$, $y(0) = 0$, $y'(0) = 0$

**17.** $y'' + ty' - 2y = 4$, $y(0) = 0$, $y'(0) = 0$

**18.** $y'' + 2ty' - 4y = 2$, $y(0) = 0$, $y'(0) = 0$

***19.** $y'' + 2ty' - 4y = 4$, $y(0) = 0$, $y'(0) = 0$

**20.** Find the differential equation satisfied by $Y(s)$ for **(a)** $y'' - ty = 0$, $y(0) = 1$, $y'(0) = 0$ **(Airy's equation)** and for **(b)** $(1-t^2)y'' - 2ty' + n(n+1)y = 0$, $y(0) = 0$, $y'(0) = 1$ **(Legendre's equation)**. What is the order of each differential equation involving $Y$? Is there a relationship between the power of the independent variable in the original equation and the order of the differential equation involving $Y$?

**21.** Show that application of the Laplace transform method to the initial-value problem (involving a Cauchy-Euler equation)

$$at^2y'' + bty' + cy = 0,\ y(0) = \alpha,\ y'(0) = \beta$$

yields $as^2Y'' + (4a - b)sY' + (2a - b + c)Y = 0$, where $Y(s) = \mathcal{L}\{y(t)\}$. Is the Laplace transform method worthwhile in the case of Cauchy-Euler equations?

**22.** Consider the initial-value problem (involving Bessel's equation of order 0)

$$ty'' + y' + ty = 0,\ y(0) = 1,\ y'(0) = 0,$$

with solution $y = J_0(t)$, the Bessel function of order 0. Use Laplace transforms to show that $\mathcal{L}\{y(t)\} = \mathcal{L}\{J_0(t)\} = k/(\sqrt{s^2+1})$, where $k$ is a constant. (*Hint:* $\mathcal{L}\{ty''\} = -d/ds\,[s^2Y(s) - s]$ and $\mathcal{L}\{ty\} = -d/ds\,[Y(s)]$.

**23.** (See Exercise 22.) Use the binomial series to expand $\mathcal{L}\{J_0(t)\} = \dfrac{k}{\sqrt{s^2+1}} = \dfrac{k}{s}\left(1 + \dfrac{1}{s^2}\right)^{-1/2}$. Take the inverse transform of each term in the expansion to show that $J_0(t) = k\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}$. Use the initial condition $J_0(0) = 1$ to show that $k = 1$. Therefore, $\mathcal{L}\{J_0(t)\} = 1/(\sqrt{s^2+1})$.

Use the method of Laplace transforms to solve the following initial-value problems. In each case, use a computer algebra system to assist in computing the inverse Laplace transform either by using a package command for performing the task or by determining the partial fraction expansion. Graph the solution of each problem.

**24.** $\begin{cases} y'' + 2y' + 4y = t - e^{-t} \\ y(0) = 1,\ y'(0) = -1 \end{cases}$

**25.** $\begin{cases} y'' + 4y' + 13y = e^{-2t}\cos 3t + 1 \\ y(0) = 1,\ y'(0) = 1 \end{cases}$

**26.** $\begin{cases} y'' + 2y' + y = 2te^{-t} - e^{-t} \\ y(0) = 1,\ y'(0) = -1 \end{cases}$

**27.** In Section 4.8 we saw that **Bessel's equation** is the equation

$$x^2y'' + xy' + (x^2 - \mu^2)y = 0,$$

where $\mu \geq 0$ is a constant, and a general solution is given by

$$y = c_1J_\mu(x) + c_2Y_\mu(x).$$

**(a)** Find $\alpha$ and $\beta$ so that the solution of the initial-value problem $\begin{cases} x^2y'' + xy' + (x^2 - \mu^2)y = 0 \\ y(0) = \alpha,\ y'(0) = \beta \end{cases}$ is $y = J_\mu(x)$.

**(b)** Compute the Laplace transform of each side of the equation $x^2y'' + xy' + (x^2 - \mu^2)y = 0$, substitute $\mathcal{L}\{y\} = Y(s)$, and solve the resulting second-order differential equation for $\mu = 1, 2, 3, 4, 5$.

**(c)** Use the results obtained in **(b)** to determine the Laplace transform of $J_\mu(t)$ for $\mu = 1, 2, 3, 4, 5$.

**(d)** Can you generalize the result you obtained in (c)?

**28.** Consider the definition of the Laplace transform, $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt$. If $f$ satisfies the conditions given in Theorem 8.6 (The Laplace Transform of the Derivative), we can differentiate within the integral with respect to $s$ when $s > a$. Show that $F'(s) = \mathcal{L}\{-tf(t)\}$ and $F''(s) = \mathcal{L}\{t^2f(t)\}$. Use these results to generalize $F^{(n)}(s) = \mathcal{L}\{(-1)^nt^nf(t)\}$ so that $\mathcal{L}\{t^nf(t)\} = (-1)^nF^{(n)}(s)$.



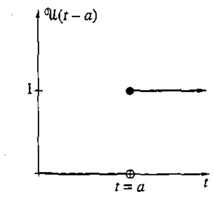



**Figure 8.3**

## 8.3 Laplace Transforms of Several Important Functions

- Piecewise Defined Functions: The Unit Step Function
- Periodic Functions
- Impulse Functions: The Delta Function

### Piecewise Defined Functions: The Unit Step Function

An important function in modeling many physical situations is the unit step function $\mathcal{U}$, shown in Figure 8.3 and defined as follows.

**Definition 8.6 Unit Step Function**

The **unit step function** $\mathcal{U}(t-a)$, where $a$ is a given number, is defined by

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}.$$

**Example 1**

Graph **(a)** $\mathcal{U}(t-5)$ and **(b)** $\mathcal{U}(t)$.

**Solution** (a) In this case, $\mathcal{U}(t-5) = \begin{cases} 0, & 0 \le t < 5 \\ 1, & t \ge 5 \end{cases}$, so the jump occurs at $t = 5$. We graph $\mathcal{U}(t-5)$ in Figure 8.4.
**(b)** Here $\mathcal{U}(t) = \mathcal{U}(t-0)$, so $\mathcal{U}(t) = 1$ for $t \ge 0$. We graph this function in Figure 8.5.

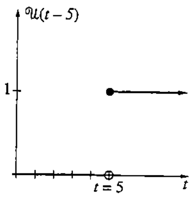



Figure 8.4

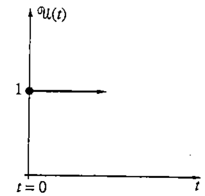



Figure 8.5

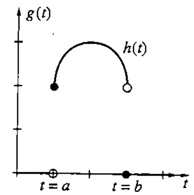



Figure 8.6

The unit step function is useful in defining functions that are piecewise continuous. Consider the function $f(t) = \mathcal{U}(t-a) - \mathcal{U}(t-b)$. If $0 \le t < a$, then $f(t) = 0 - 0 = 0$. If $a \le t < b$, then $f(t) = 1 - 0 = 1$. Finally, if $t \ge b$, then $f(t) = 1 - 1 = 0$.

Hence, $\mathcal{U}(t-a) - \mathcal{U}(t-b) = \begin{cases} 0, & 0 \le t < a \\ 1, & a \le t < b \\ 0, & t \ge b \end{cases}$, and we can define the function

$$g(t) = \begin{cases} 0, & 0 \le t < a \\ h(t), & a \le t < b \\ 0, & t \ge b \end{cases}$$

as $g(t) = h(t)[\mathcal{U}(t-a) - \mathcal{U}(t-b)]$, which is shown in Figure 8.6.

Similarly, a function such as

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases}$$

can be written as

$$\begin{aligned} f(t) &= g(t)[\mathcal{U}(t-0) - \mathcal{U}(t-a)] + h(t)\mathcal{U}(t-a) \\ &= g(t)[1 - \mathcal{U}(t-a)] + h(t)\mathcal{U}(t-a). \end{aligned}$$

The reason for writing piecewise continuous functions in terms of unit functions is because we encounter functions of this type in solving initial-value problems. Using the methods in Chapters 4 and 5, we solved the problem over each piece of the function. However, the method of Laplace transforms can be used to avoid those complicated calculations. We state the following theorem.



**Theorem 8.8**

Suppose that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > b \ge 0$. If $a$ is a positive constant, then

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s).$$

***PROOF OF THEOREM 8.8***

Using the definition of the Laplace transform, we obtain

$$\begin{aligned} \mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} &= \int_0^\infty e^{-st}f(t-a)\mathcal{U}(t-a)\,dt \\ &= \int_0^a e^{-st}f(t-a)\underbrace{\mathcal{U}(t-a)}_{=0}\,dt + \int_a^\infty e^{-st}f(t-a)\underbrace{\mathcal{U}(t-a)}_{=1}\,dt \\ &= \int_a^\infty e^{-st}f(t-a)\,dt \end{aligned}$$

Changing variables with $u = t - a$ (where $du = dt$ and $t = u + a$) and changing the limits of integration, we have

$$\int_0^\infty e^{-s(u+a)}f(u)\,du = e^{-as}\int_0^\infty e^{-su}f(u)\,du = e^{-as}\mathcal{L}\{f(t)\} = e^{-as}F(s).$$

**Example 2**

Find **(a)** $\mathcal{L}\{\mathcal{U}(t-a)\}$, $a > 0$; **(b)** $\mathcal{L}\{(t-3)^5\mathcal{U}(t-3)\}$;
**(c)** $\mathcal{L}\{\sin(t-\pi/6)\mathcal{U}(t-\pi/6)\}$.

**Solution** (a) Because $\mathcal{L}\{\mathcal{U}(t-a)\} = \mathcal{L}\{1 \cdot \mathcal{U}(t-a)\}$, $f(t) = 1$. Thus $f(t-a) = 1$, and

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \mathcal{L}\{1 \cdot \mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{1\} = e^{-as}\left(\frac{1}{s}\right) = \frac{e^{-as}}{s}.$$

**(b)** In this case $a = 3$ and $f(t) = t^5$. Thus,

$$\mathcal{L}\{(t-3)^5\mathcal{U}(t-3)\} = e^{-3s}\mathcal{L}\{t^5\} = e^{-3s}\frac{5!}{s^6} = \frac{120}{s^6}e^{-3s}.$$

**(c)** Here $a = \pi/6$ and $f(t) = \sin t$. Therefore,

$$\mathcal{L}\left\{\sin\left(t - \frac{\pi}{6}\right)\mathcal{U}\left(t - \frac{\pi}{6}\right)\right\} = e^{-\pi s/6}\mathcal{L}\{\sin t\} = e^{-\pi s/6}\frac{1}{s^2+1} = \frac{e^{-\pi s/6}}{s^2+1}.$$

*Find* $\mathcal{L}\{\cos(t-\pi/6)\mathcal{U}(t-\pi/6)\}$.

In most cases, we must calculate

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\}$$

instead of $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\}$. To solve this problem, we let $g(t) = f(t-a)$, so $f(t) = g(t+a)$. Therefore,

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}.$$

**Example 3**

Calculate **(a)** $\mathcal{L}\{t^2\mathcal{U}(t-1)\}$; **(b)** $\mathcal{L}\{\sin t\,\mathcal{U}(t-\pi)\}$.

***Solution*** **(a)** Because $g(t) = t^2$ and $a = 1$,

$$\mathcal{L}\{t^2\mathcal{U}(t-1)\} = e^{-s}\mathcal{L}\{(t+1)^2\} = e^{-s}\mathcal{L}\{t^2+2t+1\} = e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right).$$

**(b)** In this case, $g(t) = \sin t$ and $a = \pi$. Notice that $\sin(t+\pi) = \sin t \cos \pi + \cos t \sin \pi = -\sin t$. Thus

$$\mathcal{L}\{\sin t\,\mathcal{U}(t-\pi)\} = e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\} = e^{-\pi s}\mathcal{L}\{-\sin t\}$$

$$= -e^{-\pi s}\frac{1}{s^2+1} = -\frac{e^{-\pi s}}{s^2+1}.$$

*Find* $\mathcal{L}\{\cos t\,\mathcal{U}(t-\pi)\}$.

Theorem 8.9 follows directly from Theorem 8.8.

**Theorem 8.9**

Suppose that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > b \geq 0$. If $a$ is a positive constant and $f(t)$ is continuous on $[0, \infty)$, then

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a).$$

**Example 4**

Find **(a)** $\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s^3}\right\}$; **(b)** $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s/2}}{s^2+16}\right\}$.

***Solution*** **(a)** If we write the expression $e^{-4s}/s^3$ in the form $e^{-as}F(s)$, we see that $a = 4$ and $F(s) = 1/s^3$. Hence $f(t) = \mathcal{L}^{-1}\{1/s^3\} = t^2/2$ and

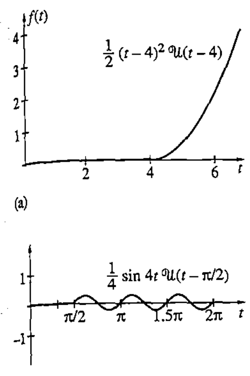



**Figure 8.7** (a)–(b)



$$\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s^3}\right\} = f(t-4)\mathcal{U}(t-4) = \frac{1}{2}(t-4)^2\mathcal{U}(t-4).$$

This function is shown in Figure 8.7(a). For what values of $t$ is $f(t) > 0$?

**(b)** In this case, $a = \pi/2$ and $F(s) = \frac{1}{s^2+16}$. Then $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+16}\right\} = \frac{1}{4}\sin 4t$ and

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s/2}}{s^2+16}\right\} = f\left(t-\frac{\pi}{2}\right)\mathcal{U}\left(t-\frac{\pi}{2}\right) = \frac{1}{4}\sin 4\left(t-\frac{\pi}{2}\right)\mathcal{U}\left(t-\frac{\pi}{2}\right)$$

$$= \frac{1}{4}\sin(4t-2\pi)\mathcal{U}\left(t-\frac{\pi}{2}\right) = \frac{1}{4}\sin 4t\,\mathcal{U}\left(t-\frac{\pi}{2}\right).$$

We graph this function in Figure 8.7(b). For what values of $t$ does $f(t) = 0$?

With the unit step function, we can solve initial-value problems that involve piecewise continuous functions.

**Example 5**

Solve $y'' + 9y = \begin{cases} 1, & 0 \leq t < \pi \\ 0, & t \geq \pi \end{cases}$, subject to $y(0) = y'(0) = 0$.

***Solution*** To solve this initial-value problem, we must compute $\mathcal{L}\{f(t)\}$, where $f(t) = \begin{cases} 1, & 0 \leq t < \pi \\ 0, & t \geq \pi \end{cases}$. Because this is a piecewise continuous function, we write it in terms of the unit step function as

$$f(t) = 1[\mathcal{U}(t-0) - \mathcal{U}(t-\pi)] + 0[\mathcal{U}(t-\pi)] = \mathcal{U}(t) - \mathcal{U}(t-\pi).$$

Then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1 - \mathcal{U}(t-\pi)\} = \frac{1}{s} - \frac{e^{-\pi s}}{s}.$$

Hence

$$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

$$s^2Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

$$(s^2+9)Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

$$Y(s) = \frac{1}{s(s^2+9)} - \frac{e^{-\pi s}}{s(s^2+9)}.$$

Then

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s(s^2+9)}\right\}.$$

Consider $\mathcal{L}^{-1}\{e^{-\pi s}/[s(s^2+9)]\}$. In the form of $\mathcal{L}^{-1}\{e^{-as}F(s)\}$, $a = \pi$ and $F(s) = 1/[s(s^2+9)]$. Now $f(t) = \mathcal{L}^{-1}\{F(s)\}$ can be found with either a partial fraction expansion or with the formula

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} d\alpha = \int_0^t \frac{1}{3}\sin 3\alpha \, d\alpha$$
$$= -\frac{1}{3}\left[\frac{\cos 3\alpha}{3}\right]_0^t = \frac{1}{9} - \frac{1}{9}\cos 3t.$$

Then with $\cos(3t - 3\pi) = \cos 3t \cos 3\pi + \sin 3t \sin 3\pi = -\cos 3t$, we have

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s(s^2+9)}\right\} = \left[\frac{1}{9} - \frac{1}{9}\cos 3(t-\pi)\right]\mathcal{U}(t-\pi)$$
$$= \left[\frac{1}{9} - \frac{1}{9}\cos(3t-3\pi)\right]\mathcal{U}(t-\pi) = \left[\frac{1}{9} + \frac{1}{9}\cos 3t\right]\mathcal{U}(t-\pi).$$

Combining these results yields the solution

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s(s^2+9)}\right\}$$
$$= \frac{1}{9} - \frac{1}{9}\cos 3t - \left[\frac{1}{9} + \frac{1}{9}\cos 3t\right]\mathcal{U}(t-\pi).$$

Notice that we can rewrite this solution as the piecewise defined function

$$y(t) = \begin{cases} \frac{1}{9} - \frac{1}{9}\cos 3t, & 0 \le t < \pi \\ -\frac{2}{9}\cos 3t, & t \ge \pi \end{cases},$$

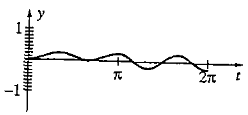



Figure 8.8

which is graphed in Figure 8.8. Can you determine where the behavior of $y(t)$ changes? Are the initial conditions satisfied?

## Periodic Functions

Another type of function that is encountered in many areas of applied mathematics is the *periodic* function.



**Definition 8.7 Periodic Function**

A function $f(t)$ is **periodic** if there is a positive number $T$ such that

$$f(t+T) = f(t)$$

for all $t \ge 0$. The minimum value of $T$ that satisfies this equation is called the **period** of $f(t)$.

The calculation of the Laplace transform of periodic functions is simplified through the use of the following theorem.

**Theorem 8.10 Laplace Transform of Periodic Functions**

Suppose that $f(t)$ is a periodic function of period $T$ and that $f(t)$ is piecewise continuous on $[0, \infty)$. Then $\mathcal{L}\{f(t)\}$ exists for $s > 0$ and is determined with the definite integral

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt.$$

***PROOF OF THEOREM 8.10***

We begin by writing $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt$ as the sum

$$\mathcal{L}\{f(t)\} = \int_0^T e^{-st}f(t)\,dt + \int_T^\infty e^{-st}f(t)\,dt.$$

If we change the variable in the second integral to $u = t - T$ where $du = dt$, we obtain

$$\int_T^\infty e^{-st}f(t)\,dt = \int_0^\infty e^{-s(u+T)}\underbrace{f(u+T)}_{=f(u)}\,du = e^{-sT}\int_0^\infty e^{-su}f(u)\,du = e^{-sT}\mathcal{L}\{f(t)\}.$$

Then,

$$\mathcal{L}\{f(t)\} = \int_0^T e^{-st}f(t)\,dt + \int_T^\infty e^{-st}f(t) = \int_0^T e^{-st}f(t)\,dt + e^{-sT}\mathcal{L}\{f(t)\},$$

which can be solved for $\mathcal{L}\{f(t)\}$ to yield

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt.$$

*Is the Laplace transform of a periodic function a periodic function?*

**Example 6**

Find the Laplace transform of the periodic function $f(t) = t$, $0 \le t < 1$, $f(t+1) = f(t)$, $t \ge 1$.

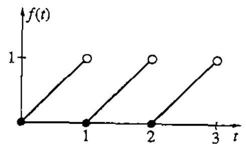


Figure 8.9

**Solution** The period of $f$, which is graphed in Figure 8.9, is $T = 1$. Through integration by parts or a computer algebra system,

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-s}}\int_0^1 e^{-st}t\,dt = \frac{1}{1-e^{-s}}\left\{\left[-\frac{te^{-st}}{s}\right]_0^1 + \int_0^1 \frac{e^{-st}}{s}\,dt\right\}\\ &= \frac{1}{1-e^{-s}}\left\{-\frac{e^{-s}}{s} - \left[\frac{e^{-st}}{s^2}\right]_0^1\right\} = \frac{1}{1-e^{-s}}\left[-\frac{e^{-s}}{s} + \frac{1-e^{-s}}{s^2}\right]\\ &= \frac{1-(s+1)e^{-s}}{s^2(1-e^{-s})}.\end{aligned}$$

Laplace transforms can be used more easily than other methods to solve initial-value problems with periodic forcing functions.

**Example 7**

Solve $y'' + y = f(t)$ subject to $y(0) = y'(0) = 0$ if $f(t) = \begin{cases} 2\sin t, & 0 \le t < \pi \\ 0, & \pi \le t < 2\pi \end{cases}$ and $f(t + 2\pi) = f(t)$. ($f(t)$ is known as the *half-wave rectification* of $2 \sin t$.)

**Solution** We begin by finding $\mathcal{L}\{f(t)\}$. Because the period is $T = 2\pi$, we have

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st}f(t)\,dt\\ &= \frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} e^{-st}\,2\sin t\,dt + \int_{\pi}^{2\pi} e^{-st}\cdot 0\,dt\right]\\ &= \frac{2}{1-e^{-2\pi s}}\int_0^{\pi} e^{-st}\sin t\,dt.\end{aligned}$$

Using integration by parts, a table of integrals, or a computer algebra system yields

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \frac{2}{1-e^{-2\pi s}}\left[\frac{e^{-st}(-s\sin t - \cos t)}{s^2+1}\right]_0^{\pi} = \frac{2}{1-e^{-2\pi s}}\left[\frac{e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1}\right]\\ &= \frac{2(e^{-\pi s}+1)}{(1-e^{-2\pi s})(s^2+1)} = \frac{2(e^{-\pi s}+1)}{(1-e^{-\pi s})(1+e^{-\pi s})(s^2+1)}\end{aligned}$$



$$= \frac{2}{(1-e^{-\pi s})(s^2+1)}.$$

Taking the Laplace transform of both sides of the equation and solving for $Y(s)$ gives us

$$\begin{aligned}\mathcal{L}\{y''\} + \mathcal{L}\{y\} &= \mathcal{L}\{f(t)\}\\ s^2Y(s) - sy(0) - y'(0) + Y(s) &= \frac{2}{(1-e^{-\pi s})(s^2+1)}\\ Y(s) &= \frac{2}{(1-e^{-\pi s})(s^2+1)^2}.\end{aligned}$$

Recall from your work with the geometric series that if $|x| < 1$, then

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots.$$

Because we do not know the inverse Laplace transform of $2/[(1-e^{-\pi s})(s^2+1)]$, we must use a geometric series expansion of $1/(1-e^{-\pi s})$ to obtain terms for which we can calculate the inverse Laplace transform. This gives us

$$\frac{1}{1-e^{-\pi s}} = 1 + e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s} + \cdots,$$

so

$$\begin{aligned}Y(s) &= (1 + e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s} + \cdots)\frac{2}{(s^2+1)^2}\\ &= 2\left[\frac{1}{(s^2+1)^2} + \frac{e^{-\pi s}}{(s^2+1)^2} + \frac{e^{-2\pi s}}{(s^2+1)^2} + \frac{e^{-3\pi s}}{(s^2+1)^2} + \cdots\right].\end{aligned}$$

Then

$$\begin{aligned}y(t) &= 2\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2} + \frac{e^{-\pi s}}{(s^2+1)^2} + \frac{e^{-2\pi s}}{(s^2+1)^2} + \frac{e^{-3\pi s}}{(s^2+1)^2} + \cdots\right\}\\ &= 2\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{(s^2+1)^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{(s^2+1)^2}\right\} +\\ &\quad 2\mathcal{L}^{-1}\left\{\frac{e^{-3\pi s}}{(s^2+1)^2}\right\} + \cdots.\end{aligned}$$

Notice that $\mathcal{L}^{-1}\{1/(s^2+1)^2\}$ is needed to find all of the other terms. Using a computer algebra system, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \frac{1}{2}(\sin t - t\cos t).$$

Then

$$y(t) = (\sin t - t\cos t) + [\sin(t-\pi) - (t-\pi)\cos(t-\pi)]\mathcal{U}(t-\pi) + [\sin(t-2\pi) - (t-2\pi)\cos(t-2\pi)]\mathcal{U}(t-2\pi) + [\sin(t-3\pi) - (t-3\pi)\cos(t-3\pi)]\mathcal{U}(t-3\pi) + [\sin(t-4\pi) - (t-4\pi)\cos(t-4\pi)]\mathcal{U}(t-4\pi) + \cdots$$

We can write $y$ as the piecewise defined function

$$y(t) = \begin{cases} \sin t - t\cos t, & 0 \le t < \pi \\ -\pi\cos t, & \pi \le t < 2\pi \\ \sin t - t\cos t + \pi\cos t, & 2\pi \le t < 3\pi, \\ -2\pi\cos t, & 3\pi \le t < 4\pi \\ \vdots \end{cases}$$

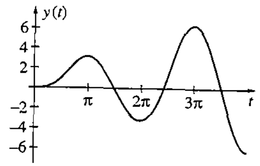


Figure 8.10

which is graphed in Figure 8.10. Is $y$ a smooth function? (That is, is $y'$ continuous?) Is $y$ periodic?

*Graph the function* $f(t) = \begin{cases} 2\sin t, & 0 \le t \le \pi \\ 0, & \pi \le t < 2\pi \end{cases}$ *and* $f(t + 2\pi) = f(t)$ *on the interval* $[0, 6\pi]$.

## Impulse Functions, The Delta Function

In some situations, we consider a force $f(t)$, known as an **impulse,** that acts only over a brief period of time, $t_0 - \alpha < t < t_0 + \alpha$. For example, we may strike a pendulum (or spring–mass system) or there may be a voltage surge in a circuit. In these cases, the impulse delivered by the force is given by

$$I = \int_{t_0-\alpha}^{t_0+\alpha} f(t)\,dt.$$

To better describe $f(t)$ suppose that this function has an impulse of 1 over the interval $t_0 - \alpha < t < t_0 + \alpha$, so that the impulse begins at $t = t_0 - \alpha$ and ends at $t = t_0 + \alpha$. Under these assumptions, we let

$$f(t) = \delta_\alpha(t - t_0) = \begin{cases} \dfrac{1}{2\alpha}, & t_0 - \alpha < t < t_0 + \alpha \\ 0, & \text{otherwise} \end{cases}.$$

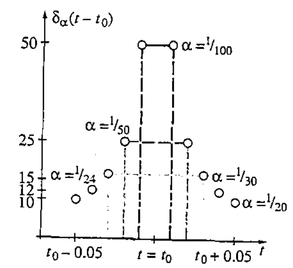


Figure 8.11

We graph $\delta_\alpha(t - t_0)$ for several values of $\alpha$ in Figure 8.11.

*Graph* $f(t) = \delta_{\frac{1}{4}}(t - 1)$. *What is the maximum value of* $f(t)$*?*

The impulse of $f(t) = \delta_\alpha(t - t_0)$ is given by

$$I = \int_{t_0-\alpha}^{t_0+\alpha} f(t)\,dt = \int_{t_0-\alpha}^{t_0+\alpha} \frac{1}{2\alpha}\,dt = \frac{1}{2\alpha}((t_0 + \alpha) - (t_0 - \alpha)) = \frac{1}{2\alpha}(2\alpha) = 1.$$

Notice that the value of this integral does not depend on $\alpha$ as long as $\alpha$ is not zero.



We now try to create the *idealized impulse function* by requiring that $\delta_\alpha(t - t_0)$ act on smaller and smaller intervals. From the integral calculation, we have

$$\lim_{\alpha\to 0} I(t) = 1.$$

We also note that

$$\lim_{\alpha\to 0} \delta_\alpha(t - t_0) = 0,\ t \ne t_0.$$

We use these properties to define the idealized unit impulse function. Notice that this idealized function represents an instantaneous impulse of magnitude one that acts at $t = t_0$.

**Definition 8.8 Unit Impulse Function**

The **(idealized) unit impulse function** $\delta$ satisfies

$$\delta(t - t_0) = 0,\ t \ne t_0$$

$$\int_{-\infty}^{+\infty} \delta(t - t_0)\,dt = 1.$$

We now state the following useful theorem involving the unit impulse function.

**Theorem 8.11**

Suppose that $g(t)$ is a bounded and continuous function. Then

$$\int_{-\infty}^{+\infty} \delta(t - t_0)g(t)\,dt = g(t_0).$$

The "function" $\delta(t - t_0)$, known as the Dirac delta function, is an example of a *generalized function.* It is not a function of the type studied in calculus. This function is useful in the definition of impulse-forcing functions that arise in many areas of applied mathematics. Although this function does not possess the properties required to apply the Laplace transform, we can determine $\mathcal{L}\{\delta(t - t_0)\}$ formally.

**Theorem 8.12**

For $t_0 \ge 0$,

$$\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}.$$

***PROOF OF THEOREM 8.12***

Because the Laplace transform is linear, we find $\mathcal{L}\{\delta(t-t_0)\}$ through the following calculations:

$$\mathcal{L}\{\delta(t-t_0)\} = \mathcal{L}\left\{\lim_{\alpha\to 0} \delta_\alpha(t-t_0)\right\} = \lim_{\alpha\to 0} \mathcal{L}\{\delta_\alpha(t-t_0)\}.$$

We can represent the delta function $\delta_\alpha(t-t_0)$ in terms of the unit step function as

$$\delta_\alpha(t-t_0) = \frac{1}{2\alpha}[\mathcal{U}(t-(t_0-\alpha)) - \mathcal{U}(t-(t_0+\alpha))].$$

Hence

$$\mathcal{L}\{\delta_\alpha(t-t_0)\} = \mathcal{L}\left\{\frac{1}{2\alpha}[\mathcal{U}(t-(t_0-\alpha)) - \mathcal{U}(t-(t_0+\alpha))]\right\}$$
$$= \frac{1}{2\alpha}\left[\frac{e^{-s(t_0-\alpha)}}{s} - \frac{e^{-s(t_0+\alpha)}}{s}\right] = e^{-st_0}\left(\frac{e^{\alpha s}-e^{-\alpha s}}{2\alpha s}\right).$$

Therefore,

$$\mathcal{L}\{\delta(t-t_0)\} = \lim_{\alpha\to 0}\mathcal{L}\{\delta_\alpha(t-t_0)\} = \lim_{\alpha\to 0} e^{-st_0}\left(\frac{e^{\alpha s}-e^{-\alpha s}}{2\alpha s}\right).$$

Because the limit is of the indeterminant form 0/0, we use L'Hopital's rule to find that

$$\mathcal{L}\{\delta(t-t_0)\} = \lim_{\alpha\to 0} e^{-st_0}\left(\frac{e^{\alpha s}-e^{-\alpha s}}{2\alpha s}\right) = \lim_{\alpha\to 0} e^{-st_0}\left(\frac{se^{\alpha s}+se^{-\alpha s}}{2s}\right)$$
$$= e^{-st_0}\cdot 1 = e^{-st_0}.$$

**Example 8**

Find **(a)** $\mathcal{L}\{\delta(t-1)\}$; **(b)** $\mathcal{L}\{\delta(t-\pi)\}$; **(c)** $\mathcal{L}\{\delta(t)\}$.

***Solution*** **(a)** In this case $t_0 = 1$, so $\mathcal{L}\{\delta(t-1)\} = e^{-s}$. **(b)** With $t_0 = \pi$, $\mathcal{L}\{\delta(t-\pi)\} = e^{-s\pi}$. **(c)** Because $t_0 = 0$, $\mathcal{L}\{\delta(t)\} = \mathcal{L}\{\delta(t-0)\} = e^{-s(0)} = 1$.

**Example 9**

Solve $y'' + y = \delta(t-\pi)$ subject to $y(0) = y'(0) = 0$.

***Solution*** We solve this initial-value problem by taking the Laplace transform of both sides of the differential equation. This yields



$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\delta(t-\pi)\}$$
$$s^2Y(s) - sy(0) - y'(0) + Y(s) = e^{-\pi s}$$
$$(s^2+1)Y(s) = e^{-\pi s}$$
$$Y(s) = \frac{e^{-\pi s}}{s^2+1}.$$

Hence

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}.$$

Because $f(t) = \mathcal{L}^{-1}\{1/(s^2+1)\} = \sin t$,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\} = \sin(t-\pi)\mathcal{U}(t-\pi) = -\sin t\,\mathcal{U}(t-\pi).$$

The graph of $y(t)$, which can be written as $y(t) = \begin{cases} 0, & 0 \le t < \pi \\ -\sin t, & t \ge \pi \end{cases}$, is shown in Figure 8.12. Notice that $y(t) = 0$ until the impulse is applied at $t = \pi$.

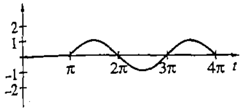



Figure 8.12

*If we interpret the problem in Example 9 as a spring–mass system with k = m, what is the maximum displacement of the mass from equilibrium? Does the mass come to rest?*

## EXERCISES 8.3

In Exercises 1–22, find the Laplace transform of the given function.

1. $-28\mathcal{U}(t-3)$
2. $-16\mathcal{U}(t-6)$
*3. $3\mathcal{U}(t-8) - \mathcal{U}(t-4)$
4. $2\mathcal{U}(t-5) + 7\mathcal{U}(t-4)$
5. $\mathcal{U}(t-2) - 7\mathcal{U}(t-7) - 7\mathcal{U}(t-6)$
6. $4\mathcal{U}(t-2) + 2\mathcal{U}(t-7) - 4\mathcal{U}(t-8)$
*7. $-42e^{t-4}\mathcal{U}(t-4)$
8. $6e^{t-2}\mathcal{U}(t-2)$
9. $12\sinh(2-t)\mathcal{U}(t-2)$
10. $\cosh(t-2)\mathcal{U}(t-2)$
*11. $-14\sin\left(t-\frac{2\pi}{3}\right)\mathcal{U}\left(t-\frac{2\pi}{3}\right)$
12. $-3\cos(t-2)\mathcal{U}(t-2)$
*13. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}$ and $f(t) = f(t-2)$ if $t \ge 2$ (see Figure 8.13).

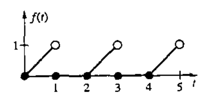



Figure 8.13

14. $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 2 - t, & 1 \le t < 2 \end{cases}$ and $f(t) = f(t-2)$ if $t \ge 2$.

15. $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ 2, & 2 \le t < 3 \end{cases}$ and $f(t) = f(t-3)$ if $t \ge 3$ (see Figure 8.14).

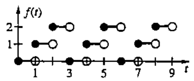



Figure 8.14

16. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ 3 - t, & 2 \le t < 3 \end{cases}$ and $f(t) = f(t-3)$ if $t \ge 3$.

17. $\delta(t - \pi)$

18. $\delta(t - 3)$

*19. $\delta(t-1) + \delta(t-2)$

20. $2\delta(t-3) - \delta(t-1)$

*21. $\sin t\,\mathcal{U}(t - 2\pi) + \delta(t - \pi/2)$

22. $\cos 2t\,\mathcal{U}(t - \pi) + \delta(t - \pi)$

In Exercises 23–44, find the inverse Laplace transform of the given function.

23. $\dfrac{-3}{se^{\pi s}}$

24. $\dfrac{-10}{se^{4s}}$

*25. $\dfrac{2e^{3s} - 3}{se^{4s}}$

26. $\dfrac{3 + e^{4s}}{se^{6s}}$

27. $\dfrac{3e^{s} - 4e^{3s} - 3}{se^{6s}}$

28. $\dfrac{5 - 6e^{s} - 3e^{2s}}{se^{4s}}$

*29. $\dfrac{e^{-4s}}{s-1}$

30. $\dfrac{e^{-3s}}{s-1}$

31. $e^{-3s}\dfrac{s}{s^2+4}$

32. $e^{-5s}\dfrac{1}{s^2(s^2+1)}$

*33. $\dfrac{e^{-5s}}{s^2 - 7s + 10}$

34. $\dfrac{1 - e^{s} + se^{2s}}{s^2(e^{2s} - 1)}$

35. $\dfrac{2(1 - 2e^{2s} - e^{4s})}{s^2e^{4s}(1 - e^{-4s})}$

36. $\dfrac{-9}{e^{4s}(s^2 - 1)}$

*37. $\dfrac{s}{e^{4s}(s^2 - 1)}$

38. $\dfrac{1}{s^3(1 + e^{-3s})}$

39. $\dfrac{1}{s^2(1 + e^{-4s})}$

40. $\dfrac{s}{(1 + e^{-2s})(s^2 + 9)}$

*41. $\dfrac{1}{(1 + e^{-3s})(s^2 + 16)}$

42. $\dfrac{1}{s(s^2 + 5)(1 + 2e^{-5s})}$

*43. $\dfrac{1}{s^2(s^2 + 4)(3e^{-2s} - 1)}$

44. $\dfrac{1}{(s^2 + 3)(4e^{-3s} - 1)}$

In Exercises 45–57, solve the initial-value problem. Graph the solution to each problem on an appropriate interval.

*45. $y' + 3y = f(t)$, $y(0) = 0$,
$f(t) = \begin{cases} 1, & 0 \le t < 2 \\ 0, & t \ge 2 \end{cases}$

46. $y' + 5y = f(t)$, $y(0) = 0$,
$f(t) = \begin{cases} \sin \pi t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

*47. $y'' - 4y' + 3y = f(t)$, $y(0) = 0$, $y'(0) = 1$,
$f(t) = \begin{cases} \cos\left(\frac{\pi}{2}t\right), & 0 \le t < 1 \\ & \end{cases}$

48. $y'' + 11y' + 30y = f(t)$, $y(0) = -2$, $y'(0) = 0$,
$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 2, & 1 \le t < 2 \end{cases}$ and $f(t) = f(t-2)$ if $t \ge 2$

*49. $x'' + x = \delta(t - \pi) + 1$, $x(0) = 0$, $x'(0) = 0$

50. $x'' + x = \delta(t - \pi) + \delta(t - 2\pi)$, $x(0) = 0$, $x'(0) = 0$

51. $x'' + 3x' + 2x = \delta(t - \pi)$, $x(0) = 0$, $x'(0) = 0$

52. $x'' + 4x' + 13x = \delta(t - \pi)$, $x(0) = 0$, $x'(0) = 0$

*53. $x'' + 4x' + 13x = \delta(t - \pi) + \sin t$, $x(0) = 0$, $x'(0) = 0$

54. $x'' + 4x' + 13x = \delta(t - \pi) + \delta(t - 2\pi)$, $x(0) = 0$, $x'(0) = 0$

55. $x'' + 9x = \cos t + \delta(t - \pi)$, $x(0) = 0$, $x'(0) = 0$

56. $x'' + 9x = \cos 3t + \delta(t - 1)$, $x(0) = 0$, $x'(0) = 0$

*57. $x'' + 3x' + 2x = \delta(t - 1) + e^{-t}$, $x(0) = 0$, $x'(0) = 0$

In Exercises 58–63, solve the initial-value problem. Compare the solution to that of the corresponding homogeneous problem.

58. $x'' + 9x' = \delta(t - \pi)$, $x(0) = 0$, $x'(0) = 2$

59. $x'' + 9x' = \delta(t - \pi)$, $x(0) = 2$, $x'(0) = 0$

60. $x'' + 9x' = \delta(t - \pi)$, $x(0) = 0$, $x'(0) = 0$

*61. $x'' + 6x' + 10x = 3\delta(t - \pi)$, $x(0) = 0$, $x'(0) = 0$

62. $x'' + 6x' + 10x = 3\delta(t - \pi)$, $x(0) = 2$, $x'(0) = 0$

63. $x'' + 6x' + 10x = 3\delta(t - \pi)$, $x(0) = 0$, $x'(0) = 2$



64. Show that $\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-st}\mathcal{L}\{g(t+a)\}$. (*Hint:* Use the definition of the Laplace transform and the change of variable $u = t - a$.)

65. **(Square Wave)** If $f(t) = \begin{cases} 1, & 0 \le t < a \\ -1, & a \le t < 2a \end{cases}$, $f(t + 2a) = f(t)$, show that $F(s) = \dfrac{1 - e^{-as}}{s(1 + e^{-as})} = \dfrac{1}{s}\tanh\dfrac{as}{2}$.

66. **(Triangular Wave)** Consider the function $g(t) = \begin{cases} x, & 0 \le t < a \\ 2a - x, & a \le t < 2a \end{cases}$, $g(t + 2a) = g(t)$. Show that
$$G(s) = \frac{1}{s^2}\tanh\frac{as}{2}.$$
(*Hint:* Use the square wave function and the relationship $g'(t) = f(t)$.)

*67. **(Sawtooth Wave)** If $f(t) = t/a$, $0 \le t < a$, $f(t + a) = f(t)$, show that
$$F(s) = \frac{1}{as^2} - \frac{e^{-as}}{s(1 - e^{-as})}.$$

68. **(Rectified Sine Wave)** Let $f(t) = |\sin(\pi t/a)|$. Show that
$$F(s) = \frac{\pi a}{a^2s^2 + \pi^2}\coth\frac{as}{2}.$$

69. **(Half-Rectified Sine Wave)** If $f(t) = \begin{cases} \sin(\pi t/a), & 0 \le t < a \\ 0, & a \le t < 2a \end{cases}$, $f(t + 2a) = f(t)$, show that
$$F(s) = \frac{\pi a}{(a^2s^2 + \pi^2)(1 - e^{-as})}.$$

70. **(Stair Step)** Consider the function $g(t) = \sum_{n=1}^{\infty}\mathcal{U}(t - na)$. Sketch the graph of $g(t)$ on $0 \le t \le 4$. Show that
$$G(s) = \frac{1}{s(1 - e^{-as})}.$$

71. Use an integration device to assist in calculating the Laplace transform of each of the following periodic functions.

(a) $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & 1 \le t < 2 \end{cases}$, $f(t+2) = f(t)$

(b) $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 2 - t, & 1 \le t < 2 \end{cases}$, $f(t+2) = f(t)$

(c) $f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ -\sin t, & \pi \le t < 2\pi \end{cases}$, $f(t + 2\pi) = f(t)$

(d) Use the results obtained in (a)–(c) to solve the following initial-value problem for each function $f(t)$ above.
$$\begin{cases} \dfrac{d^2x}{dt^2} + x = f(t) \\ x(0) = \dfrac{dx}{dt}(0) = 0 \end{cases}$$
Plot the solution in each case to compare the results.

72. Solve $y'' + 2y' + y = \delta(t) + 10{,}000\,\mathcal{U}(t - 2\pi)$ subject to $y(0) = 0$, $y'(0) = 0$. For what values of $t$ does $y(t)$ decrease? For what values of $t$ does $y(t)$ increase? Determine $\lim_{t\to\infty} y(t)$. Is there a relationship between this limit and the forcing function $\delta(t) + 10{,}000\,\mathcal{U}(t - 2\pi)$? Will the limit be affected if the forcing function is changed to $100\delta(t) + 10{,}000\,\mathcal{U}(t - 2\pi)$?

*73. Solve $y'' + 2y' + y = \delta(t) + \delta(t - 2\pi)$ subject to $y(0) = 0$, $y'(0) = 0$. What is the maximum value of $y(t)$ and where does it occur? Determine $\lim_{t\to\infty} y(t)$. Would this limit be affected if the forcing function is changed to $\delta(t) + \delta(t - 2\pi) + \delta(t - 4\pi)$?

## 8.4 The Convolution Theorem

The Convolution Theorem · Integral and Integrodifferential Equations

### The Convolution Theorem

In many cases, we are required to determine the inverse Laplace transform of a product of two functions. Just as in integral calculus when the integral of the product of two functions did not produce the product of the integrals, neither does the inverse

Laplace transform of the product yield the product of the inverse Laplace transforms. We had a preview of this when we found

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}$$

earlier in the chapter. We recall that $\mathcal{L}^{-1}\{F(s)/s\} = \int_0^t f(\alpha)\,d\alpha$, where $f(t) = \mathcal{L}^{-1}\{F(s)\}$, so we realize that

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} \neq \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\mathcal{L}^{-1}\{F(s)\} = f(t).$$

The *Convolution theorem* gives a relationship between the inverse Laplace transform of the product of two functions, $\mathcal{L}^{-1}\{F(s)G(s)\}$, and the inverse Laplace transform of each function, $\mathcal{L}^{-1}\{F(s)\}$ and $\mathcal{L}^{-1}\{G(s)\}$.

**Theorem 8.13 Convolution Theorem**

Suppose that $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and both of exponential order $b$. Further, suppose that $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$. Then,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\{\mathcal{L}\{(f * g)(t)\}\} = (f * g)(t) = \int_0^t f(t - v)g(v)\,dv.$$

Note that $(f * g)(t) = \int_0^t f(t - v)g(v)\,dv$ is called the **convolution integral.**

***PROOF OF THE CONVOLUTION THEOREM***

We prove the Convolution theorem by computing the product $F(s)G(s)$ with the definition of the Laplace transform. This yields

$$F(s)G(s) = \int_0^\infty e^{-sx}f(x)\,dx \cdot \int_0^\infty e^{-sv}g(v)\,dv,$$

which can be written as the iterated integral

$$F(s)G(s) = \int_0^\infty\int_0^\infty e^{-s(x+v)}f(x)g(v)\,dx\,dv.$$

Changing variables with $x = t - v$ ($t = x + v$, $dx = dt$) then yields

$$F(s)G(s) = \int_0^\infty\int_v^\infty e^{-st}f(t - v)g(v)\,dt\,dv,$$

where the region of integration $R$ is the unbounded triangular region shown in Figure 8.15. Recall from multivariable calculus that the order of integration can be interchanged. Then we obtain

$$F(s)G(s) = \int_0^\infty\int_0^t e^{-st}f(t - v)g(v)\,dv\,dt$$

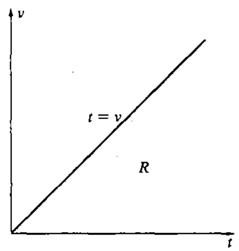



Figure 8.15



(why?), which can be written as

$$F(s)G(s) = \int_0^\infty e^{-st}\int_0^t f(t - v)g(v)\,dv\,dt = \mathcal{L}\left\{\int_0^t f(t - v)g(v)\,dv\right\} = \mathcal{L}\{(f * g)(t)\}.$$

Therefore, $\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\{\mathcal{L}\{(f * g)(t)\}\} = (f * g)(t)$.

**Example 1**

Compute $(f * g)(t)$ and $(g * f)(t)$ if $f(t) = e^{-t}$ and $g(t) = \sin t$. Verify the Convolution theorem for these functions.

***Solution*** We use the definition and a table of integrals (or a computer algebra system) to obtain

$$\begin{aligned}(f * g)(t) &= \int_0^t f(t - v)g(v)\,dv = \int_0^t e^{-(t-v)}\sin v\,dv = e^{-t}\int_0^t e^{v}\sin v\,dv \\ &= e^{-t}\left[\frac{e^v}{2}(\sin v - \cos v)\right]_0^t = e^{-t}\left[\frac{e^t}{2}(\sin t - \cos t) - \frac{1}{2}(\sin 0 - \cos 0)\right] \\ &= \frac{1}{2}(\sin t - \cos t) + \frac{1}{2}e^{-t}.\end{aligned}$$

With a computer algebra system, we find that

$$(g * f)(t) = \int_0^t g(t - v)f(v)\,dv = \int_0^t \sin(t - v)e^{v}\,dv = \frac{1}{2}(\sin t - \cos t) + \frac{1}{2}e^{-t}.$$

Now, according to the Convolution theorem, $\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = \mathcal{L}\{(f * g)(t)\}$. In this example, we have

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-t}\} = \frac{1}{s+1} \quad\text{and}\quad G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1}.$$

Hence

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\cdot\frac{1}{s^2+1}\right\}$$

should equal $(f * g)(t)$. We can compute

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\cdot\frac{1}{s^2+1}\right\}$$

through the partial fraction expansion

$$\frac{1}{s+1}\cdot\frac{1}{s^2+1} = \frac{A}{s+1} + \frac{Bs + C}{s^2+1},$$

where $A = C = 1/2$ and $B = -1/2$. Therefore,

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\cdot\frac{1}{s^2+1}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1} + \frac{-s+1}{s^2+1}\right\}$$
$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$
$$= \frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t,$$

which is the same result as that obtained for $(f * g)(t)$.

Notice that in Example 1, $(g * f)(t) = (f * g)(t)$. This is no coincidence. With a simple change of variable, we can prove this relationship in general so that the convolution integral is *commutative*. (See Exercise 29.)

**Example 2**

Use the Convolution theorem to find the Laplace transform of $h(t) = \int_0^t \cos(t - v)\sin v\, dv$.

**Solution** Notice that $h(t) = (f * g)(t)$, where $f(t) = \cos t$ and $g(t) = \sin t$. Therefore, by the Convolution theorem, $\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$. Hence

$$\mathcal{L}\{h(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = \mathcal{L}\{\cos t\}\mathcal{L}\{\sin t\} = \left(\frac{s}{s^2+1}\right)\left(\frac{1}{s^2+1}\right)$$
$$= \frac{s}{(s^2+1)^2}.$$

*Find the Laplace transform of* $h(t) = \int_0^t e^{t-v}\sin v\, dv$.

## Integral and Integrodifferential Equations

The Convolution theorem is useful in solving numerous problems. In particular, this theorem can be used to solve **integral equations,** which are equations that involve an integral of the unknown function.

**Example 3**

Use the Convolution theorem to solve the integral equation

$$h(t) = 4t + \int_0^t h(t - v)\sin v\, dv.$$



**Solution** We need to find $h(t)$, so we first note that the integral in this equation represents $(h * g)(t)$ for $g(t) = \sin t$. Therefore, if we apply the Laplace transform to both sides of the equation, we obtain

$$\mathcal{L}\{h(t)\} = \mathcal{L}\{4t\} + \mathcal{L}\{h(t)\}\mathcal{L}\{\sin(t)\}$$

or

$$H(s) = \frac{4}{s^2} + H(s)\frac{1}{s^2+1},$$

where $\mathcal{L}\{h(t)\} = H(s)$. Solving for $H(s)$, we have

$$H(s)\left(1 - \frac{1}{s^2+1}\right) = \frac{4}{s^2}$$

so

$$H(s) = \frac{4(s^2+1)}{s^4} = \frac{4}{s^2} + \frac{4}{s^4}.$$

Then by computing the inverse Laplace transform, we find that

$$h(t) = \mathcal{L}^{-1}\left\{\frac{4}{s^2} + \frac{4}{s^4}\right\} = 4t + \frac{2}{3}t^3.$$

Laplace transforms are also helpful in solving **integrodifferential equations,** equations that involve a derivative as well as an integral of the unknown function.

**Example 4**

Solve $\frac{dy}{dt} + y + \int_0^t y(u)\, du = 1$ subject to $y(0) = 0$.

**Solution** Because we must take the Laplace transform of both sides of this integrodifferential equations, we first compute

$$\mathcal{L}\left\{\int_0^t y(u)\, du\right\} = \mathcal{L}\{(1 * y)(t)\} = \mathcal{L}\{1\}\mathcal{L}\{y\} = \frac{Y(s)}{s}.$$

Hence,

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + \mathcal{L}\{y\} + \mathcal{L}\left\{\int_0^t y(u)\, du\right\} = \mathcal{L}\{1\}$$
$$sY(s) - y(0) + Y(s) + \frac{Y(s)}{s} = \frac{1}{s}$$
$$s^2Y(s) + sY(s) + Y(s) = 1$$
$$Y(s) = \frac{1}{s^2+s+1}.$$

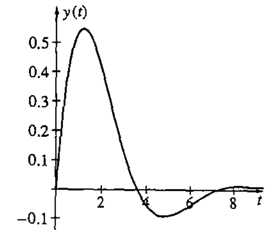



**Figure 8.16** Graph of $y(t) = \frac{2}{\sqrt{3}} e^{-t/2} \sin \frac{\sqrt{3}}{2} t$

Because

$$Y(s) = \frac{1}{s^2+s+1} = \frac{1}{(s+1/2)^2 + (\sqrt{3}/2)^2},$$

$$y(t) = \frac{2}{\sqrt{3}} e^{-t/2} \sin \frac{\sqrt{3}}{2} t.$$

The graph of $y$ is shown in Figure 8.16. Is the initial condition satisfied?

*Show that the integrodifferential equation in Example 4 is equivalent to the second-order differential equation $y'' + y' + y = 0$ by differentiating the integrodifferential equation with respect to $t$. Substitute $t = 0$ into the integrodifferential equation to find that $y'(0) = 1$. Solve the initial-value problem $y'' + y' + y = 0$, $y(0) = 0$, $y'(0) = 1$. Is this problem equivalent to the initial-value problem in Example 4?*

## EXERCISES 8.4

In Exercises 1–6, compute the convolution $(f * g)(t)$ using the indicated pair of functions.

1. $f(t) = 1$, $g(t) = t^2$
2. $f(t) = e^{-3t}$, $g(t) = 2$

*3. $f(t) = t$, $g(t) = e^{-t}$

4. $f(t) = \sin 2t$, $g(t) = e^{-t}$
5. $f(t) = t^3$, $g(t) = \sin 4t$
6. $f(t) = \sin 2t$, $g(t) = e^{-7t}$

In Exercises 7–12, find the Laplace transform of $h$.

7. $h(t) = \int_0^t e^{-v}\, dv$
8. $h(t) = \int_0^t \sin v\, dv$

*9. $h(t) = \int_0^t (t - v)\sin v\, dv$

10. $h(t) = \int_0^t v \sin(t - v)\, dv$
11. $h(t) = \int_0^t v e^{t-v}\, dv$
12. $h(t) = \int_0^t e^{v}(t - v)^2\, dv$

In Exercises 13–18, find the inverse Laplace transform of each function using the Convolution theorem and the results of Exercise 1.

13. $\frac{1}{s^2(s+1)}$
14. $\frac{1}{s^3(s+3)}$

*15. $\frac{s}{(s^2+1)^2}$

16. $\frac{1}{(s^2+4)(s+1)}$
17. $\frac{1}{s^4(s^2+16)}$
18. $\frac{1}{(s^2+4)(s+7)}$

In Exercises 19–22, find the inverse Laplace transform of each function using the Convolution theorem.

19. $\frac{1}{(s^2+16)(s-10)}$
20. $\frac{1}{(s-4)^2(s^2+100)}$

*21. $\frac{1}{(s^2+100)^2}$

22. $\frac{s}{(s-1)^8(s^2+25)}$

In Exercises 23–26, solve the given integral equation using Laplace transforms.

23. $g(t) - t = -\int_0^t (t - v) g(v)\, dv$
24. $h(t) = 2 - \int_0^t h(v)\, dv$

*25. $h(t) - 4e^{-2t} = \cos t - \int_0^t \sin(t - v) h(v)\, dv$

26. $f(t) = 3 - \int_0^t f(t - v) v\, dv$

In Exercises 27–28, solve the given integrodifferential equation using Laplace transforms.

*27. $\frac{dy}{dt} - 4y + 4\int_0^t y(v)\, dv = t^3 e^{2t}$, $y(0) = 0$

28. $\frac{dx}{dt} + 16\int_0^t x(v)\, dv = \sin 4t$, $x(0) = 0$
29. Show that $(f * g)(t) = (g * f)(t)$ by verifying that $\int_0^t f(u) g(t-u)\, du = \int_0^t g(u) f(t-u)\, dt$. (Therefore, the convolution integral is *commutative*.)
30. Show that the convolution integral is *associative* by proving that $(f * (g * h))(t) = ((f * g) * h)(t)$.
31. Show that the convolution integral satisfies the *distributive property* $(f * (g + h))(t) = (f * g)(t) + (f * h)(t)$.
32. Show that for any constant $((kf) * g)(t) = k(f * g)(t)$.

*33. Show that $(\sin t) * (\cos kt) = (\cos t) * (\frac{1}{k} \sin kt)$.

34. Show that $t^{-1/2} * t^{-1/2} = \pi$. (See Exercise 31, Section 8.1.)
35. Express the integrodifferential equation in Exercise 27 as an equivalent second-order initial-value problem and solve this problem.
36. Express the integrodifferential equation in Exercise 28 as an equivalent second-order initial-value problem and solve this problem.

*37. Calculate (a) $\mathcal{L}^{-1}\left\{\frac{4}{(s-2)^2(s^2+16)}\right\}$,

(b) $\mathcal{L}^{-1}\left\{\frac{6!}{s^7}\left[\frac{1}{2s} - \frac{s}{2(s^2+4)}\right]\right\}$,

(c) $\mathcal{L}^{-1}\left\{\frac{6}{s^4}\left[\frac{s}{4(s^2+9)} - \frac{3s}{4(s^2+1)}\right]\right\}$.

38. Verify each of the following:

(a) $(\sin kt) * (\cos kt) = \frac{t}{2} \sin kt$;

(b) $(\sin kt) * (\sin kt) = \frac{1}{2k} \sin kt - \frac{t}{2} \cos kt$;

(c) $(\cos kt) * (\cos kt) = \frac{1}{2k} \sin kt + \frac{t}{2} \cos kt$.

(d) Use these results to verify that

$$\mathcal{L}^{-1}\left\{\frac{ks}{(s^2+k^2)^2}\right\} = \frac{t}{2} \sin kt,$$

$$\mathcal{L}^{-1}\left\{\frac{k^2}{(s^2+k^2)^2}\right\} = \frac{1}{2k} \sin kt - \frac{t}{2} \cos kt,$$

and $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+k^2)^2}\right\} = \frac{1}{2k} \sin kt + \frac{t}{2} \cos kt$.



## 8.5 Laplace Transform Methods for Solving Systems

In many cases, Laplace transforms can be used to solve initial-value problems that involve a system of linear differential equations. This method is applied in much the same way that it was in solving initial-value problems involving higher order differential equations, except that a system of algebraic equations is obtained after taking the Laplace transform of each equation. After solving for the Laplace transform of each of the unknown functions, the inverse Laplace transform is used to find each unknown function in the solution of the system.

**Example 1**

Solve $\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{X} + \begin{pmatrix} \sin t \\ 2\cos t \end{pmatrix}$ subject to $\mathbf{X}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

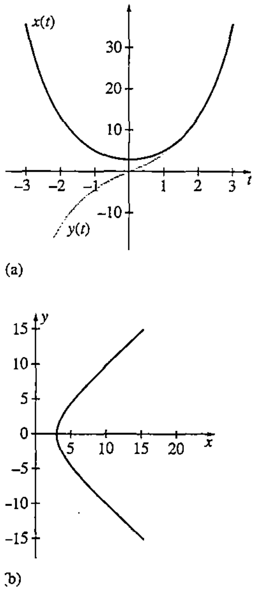



**Figure 8.17** (a) Graph of $x(t)$ and $y(t)$ for $-3 \le t \le 3$ (b) Graph of $\begin{cases} x(t) \\ y(t) \end{cases}$ for $-3 \le t \le 3$

**Solution** Let $\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. Then we can rewrite this problem as

$$\begin{cases} x' = y + \sin t \\ y' = x + 2\cos t \end{cases}, \; x(0) = 2, \; y(0) = 0.$$

Taking the Laplace transform of both sides of each equation yields the system

$$\begin{cases} sX(s) - x(0) = Y(s) + \dfrac{1}{s^2+1} \\ sY(s) - y(0) = X(s) + \dfrac{2s}{s^2+1} \end{cases},$$

which is equivalent to

$$\begin{cases} sX(s) - Y(s) = \dfrac{1}{s^2+1} + 2 \\ -X(s) + sY(s) = \dfrac{2s}{s^2+1} \end{cases}.$$

Multiplying the second equation by $s$ and adding the two equations, we have

$$(s^2 - 1)Y(s) = \frac{2s^2}{s^2+1} + \frac{2s^2+3}{s^2+1}$$

$$Y(s) = \frac{2s^2+1}{(s^2+1)(s^2-1)} + \frac{2}{s^2-1} = \frac{4s^2+3}{(s^2+1)(s^2-1)}$$

$$= \frac{7}{4(s-1)} - \frac{7}{4(s+1)} + \frac{1}{2(s^2+1)}.$$

Taking the inverse Laplace transform then yields

$$y(t) = \frac{7}{4}e^t - \frac{7}{4}e^{-t} + \frac{1}{2}\sin t.$$

At this point, we can solve the system for $X(s)$ and compute $x(t)$ with the inverse Laplace transform. However, we can find $x(t)$ more easily by substituting $y(t)$ into the second differential equation $y' = x + 2\cos t$. Because $y'(t) = \frac{7}{4}e^t + \frac{7}{4}e^{-t} + \frac{1}{2}\cos t$ and $x = y' - 2\cos t$, we have

$$x(t) = y'(t) - 2\cos t = \frac{7}{4}e^t + \frac{7}{4}e^{-t} + \frac{1}{2}\cos t - 2\cos t$$

$$= \frac{7}{4}e^t + \frac{7}{4}e^{-t} - \frac{3}{2}\cos t.$$

In Figure 8.17, we graph $x(t)$, $y(t)$, and $\begin{cases} x(t) \\ y(t) \end{cases}$. What is the orientation of $\begin{cases} x(t) \\ y(t) \end{cases}$?



Previously we solved systems that involve higher order differential equations with differential operators. In many cases these systems can be solved with Laplace transforms as well.

**Example 2**

Solve $\begin{cases} x'' = 3x' - y' - 2x + y \\ x' + y' = 2x - y \end{cases}$, $x(0) = 0$, $x'(0) = 0$, $y(0) = -1$.

**Solution** We begin by taking the Laplace transform of both equations. For the first equation, this yields

$$\mathcal{L}\{x''\} = \mathcal{L}\{3x' - y' - 2x + y\}$$

$$s^2X(s) - sx(0) - x'(0) = 3(sX(s) - x(0)) - (sY(s) - y(0)) - 2X(s) + Y(s)$$

$$s^2X(s) = 3sX(s) - sY(s) - 1 - 2X(s) + Y(s)$$

$$(s^2 - 3s + 2)X(s) + (s-1)Y(s) = -1,$$

and for the second equation,

$$\mathcal{L}\{x' + y'\} = \mathcal{L}\{2x - y\}$$

$$sX(s) - x(0) + sY(s) - y(0) = 2X(s) - Y(s)$$

$$(s-2)X(s) + (s+1)Y(s) = -1.$$

After factoring the coefficient of $X(s)$ in the first equation, we have the system of equations

$$\begin{cases} (s-2)(s-1)X(s) + (s-1)Y(s) = -1 \\ (s-2)X(s) + (s+1)Y(s) = -1. \end{cases}$$

Multiplying the second equation by the factor $-(s-1)$ yields

$$-(s-1)(s-2)X(s) - (s-1)(s+1)Y(s) = s - 1.$$

Adding this result to the first equation then eliminates $X(s)$, so we can solve for $Y(s)$ with

$$[(s-1) - (s-1)(s+1)]Y(s) = s - 2$$

$$-s(s-1)Y(s) = s - 2$$

$$Y(s) = -\frac{s-2}{s(s-1)}.$$

Using the partial fraction expansion

$$Y(s) = -\frac{s-2}{s(s-1)} = -\left(\frac{A}{s} + \frac{B}{s-1}\right)$$

shows that $A = 2$ and $B = -1$, so

$$Y(s) = -\left(\frac{2}{s} - \frac{1}{s-1}\right) \quad \text{and} \quad y(t) = \mathcal{L}^{-1}\left\{-\left(\frac{2}{s} - \frac{1}{s-1}\right)\right\} = -2 + e^t.$$

In a similar manner, we eliminate $Y(s)$ from the original system of equations by multiplying the first equation by the factor $(s+1)$ and the second equation by $-(s-1)$. This yields

$$\begin{cases} (s-2)(s-1)(s+1)X(s) + (s-1)(s+1)Y(s) = -(s+1) \\ -(s-2)(s-1)X(s) - (s-1)(s+1)Y(s) = s-1. \end{cases}$$

Adding these equations results in

$$\begin{aligned} (s-2)(s-1)[s+1-1]X(s) &= -2 \\ s(s-2)(s-1)X(s) &= -2 \\ X(s) &= \frac{-2}{s(s-2)(s-1)}. \end{aligned}$$

With the partial fraction expansion

$$X(s) = \frac{-2}{s(s-2)(s-1)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s-1},$$

we find that $A = -1$, $B = -1$, and $C = 2$. Therefore,

$$\begin{aligned} X(s) = -\frac{1}{s} - \frac{1}{s-2} + \frac{2}{s-1}, \text{ so } x(t) &= \mathcal{L}^{-1}\left\{-\frac{1}{s} - \frac{1}{s-2} + \frac{2}{s-1}\right\} \\ &= -1 - e^{2t} + 2e^t. \end{aligned}$$

In Figure 8.18 we graph $x(t)$, $y(t)$, and $\begin{cases} x(t) \\ y(t) \end{cases}$. What is the orientation of $\begin{cases} x(t) \\ y(t) \end{cases}$?

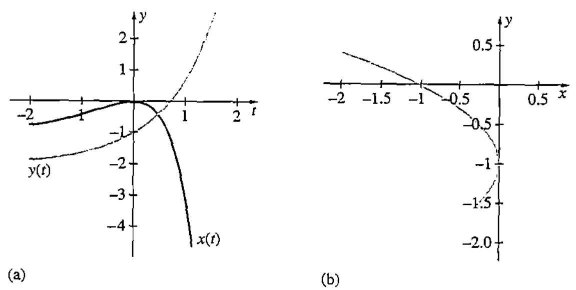



**Figure 8.18** (a) Graph of $x(t)$ and $y(t)$ for $-2 \le t \le 2$ (b) Graph of $\begin{cases} x(t) \\ y(t) \end{cases}$ for $-2 \le t \le 2$



## EXERCISES 8.5

In Exercises 1–26, use Laplace transforms to solve each initial-value problem. Graph $x(t)$, $y(t)$, and $\begin{cases} x(t) \\ y(t) \end{cases}$. Describe other methods that could be used to solve the system.

1. $\begin{cases} x' - 2x + 3y = 0 \\ y' + 9x + 4y = 0 \end{cases}$, $x(0) = 0$, $y(0) = 4$

2. $\begin{cases} x' + 9x - 2y = 0 \\ y' + 10x - 3y = 0 \end{cases}$, $x(0) = -2$, $y(0) = 0$

*3. $\begin{cases} x' - 5x - 5y = 0 \\ y' + 4x + 3y = 0 \end{cases}$, $x(0) = 2$, $y(0) = 0$

4. $\begin{cases} x' + 2x - 4y = 0 \\ y' + 2x - 2y = 0 \end{cases}$, $x(0) = 0$, $y(0) = 6$

*5. $\begin{cases} x' - 5x + 4y - 2z = 0 \\ y' + 2x + 2y + 2z = 0, \\ z' - z = 0 \\ x(0) = 0, y(0) = 0, z(0) = 15 \end{cases}$

6. $\begin{cases} x' + 5x - 4y + 2z = 0 \\ y' - 6x - 2y - 2z = 0, \\ z' - 5x - 7y + 6z = 0 \\ x(0) = 2, y(0) = 0, z(0) = 2 \end{cases}$

*7. $\begin{cases} x' - x - 3y = e^{4t} \\ y' - 5x + y = 0 \end{cases}$, $x(0) = 0$, $y(0) = 0$

8. $\begin{cases} x' + 4x + 4y = 0 \\ y' + 5x + 3y = e^t \end{cases}$, $x(0) = 0$, $y(0) = 1$

9. $\begin{cases} x' + 2x - 3y = 0 \\ y' - 2x + y = 0 \end{cases}$, $x(0) = 1$, $y(0) = 0$

10. $\begin{cases} x' - 3x - 6y = 0 \\ y' - x + 2y = 0 \end{cases}$, $x(0) = 0$, $y(0) = -1$

11. $\begin{cases} x' - 5x - 4y = e^{-t} \\ y' + 2x + 4y = e^{2t} \end{cases}$, $x(0) = -1$, $y(0) = 1$

12. $\begin{cases} x' + 6x + y = t \\ y' + 2x + 7y = e^{4t} \\ x(0) = 0, y(0) = 2 \end{cases}$

*13. $\begin{cases} x' + 2x - 4y = \cos 2t \\ y' + 5x - 2y = \sin 2t \\ x(0) = 0, y(0) = -1 \end{cases}$

14. $\begin{cases} x' - 4x + 5y = e^{-t} \\ y' - 5x + 2y = \sin 2t \\ x(0) = -2, y(0) = 0 \end{cases}$

*15. $\begin{cases} x' - y = 0 \\ y' + x = f(t) \end{cases}$, where $x(0) = 0$, $y(0) = 0$
$f(t) = \begin{cases} \sin t, & \text{if } 0 \le t < \pi \\ 0, & \text{if } t \ge \pi \end{cases}$

16. $\begin{cases} x' + 7x + 4y = f(t) \\ y' + 6x - 3y = 0, \end{cases}$ where $x(0) = 1$, $y(0) = 0$
$f(t) = \begin{cases} 1, & \text{if } 0 \le t < 1 \\ 0, & \text{if } t \ge 1 \end{cases}$

17. $\begin{cases} x' - 2x + 4y = 0 \\ y' - x - 2y = f(t), \end{cases}$ where $x(0) = -1$, $y(0) = 0$
$f(t) = \begin{cases} 1, & \text{if } 0 \le t < 1 \\ -1, & \text{if } 1 \le t < 2 \\ 0, & \text{if } t \ge 2 \end{cases}$

18. $\begin{cases} x' - 2x + 7y = f(t) \\ y' - x + 2y = 0, \end{cases}$ where $x(0) = 0$, $y(0) = -1$
$f(t) = \begin{cases} t, & \text{if } 0 \le t < 1 \\ 2 - t, & \text{if } 1 \le t < 2 \end{cases}$ and
$f(t) = f(t-2)$ if $t \ge 2$

*19. $\begin{cases} x' + 2x + 3y = 0 \\ y' - x + 6y = f(t) \end{cases}$, where $x(0) = 1$, $y(0) = 0$
$f(t) = \begin{cases} 0, & \text{if } 0 \le t < 1 \\ 1, & \text{if } 1 \le t < 2 \\ 2, & \text{if } 2 \le t < 3 \end{cases}$ and
$f(t) = f(t-3)$ if $t \ge 3$

20. $\begin{cases} x' + 5x - 7y = f(t) \\ y' + 3x - 5y = 0, \end{cases}$ where $x(0) = 1$, $y(0) = -1$
$f(t) = \begin{cases} t, & \text{if } 0 \le t < 1 \\ 1, & \text{if } 1 \le t < 2 \\ 3 - t, & \text{if } 2 \le t < 3 \end{cases}$ and
$f(t) = f(t-3)$ if $t \ge 3$

*21. $\begin{cases} -2\dfrac{d^2x}{dt^2} - 2\dfrac{dy}{dt} = 0 \\ \dfrac{d^2y}{dt^2} + y - \dfrac{dx}{dt} = \cos t \end{cases}$, $x(0) = 2$, $x'(0) = 1$, $y(0) = 1$, $y'(0) = 2$

22. $\begin{cases} -2x'' - x = 0 & x(0) = 1, x'(0) = 1, \\ 2y'' - 2y + x' = e^t, & y(0) = 0, y'(0) = 1 \end{cases}$

23. $\begin{cases} -2x'' - 2y' = 0 & x(0) = 2, x'(0) = 1, \\ -2y'' + 2y - 6x' = \sin t, & y(0) = -2, y'(0) = 0 \end{cases}$

24. $\begin{cases} 2\,d^2x/dt^2 - 2x - 2y = 1 & x(0) = -2, x'(0) = -1, \\ d^2y/dt^2 + y = \sin t, & y(0) = -1, y'(0) = 0 \end{cases}$

*25. $\begin{cases} -x'' - 2x + 2y' = e^t & x(0) = 1, x'(0) = -1, \\ -2y'' - 2y = t, & y(0) = -1, y'(0) = -2 \end{cases}$

26. $\begin{cases} -2x + 2y - y' = t & x(0) = -2, x'(0) = 1, \\ -y'' + y - x + 2x' = \sin t, & y(0) = -1, y'(0) = 2 \end{cases}$

27. Use Laplace transforms to solve the system

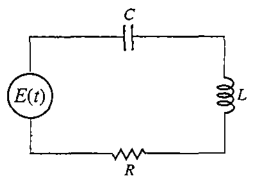
$$\begin{cases} x' - y = e^{-t} \\ y' + 5x + 2y = \sin 3t \end{cases}$$

Graph the solution for different initial conditions.

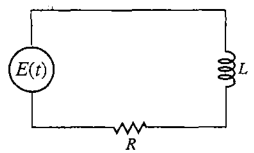
28. Use Laplace transforms to solve

$$\begin{cases} 2x'' - 5y' = 0 \\ -y'' - 3y - x' = \sin t \end{cases}$$

subject to the conditions $x(0) = 1$, $x'(0) = 0$, $y(0) = 1$, and $y'(0) = 1$. Graph the solution.



(a)



(b)

**Figure 8.19** (a) *L-R-C* Circuit (b) *L-R* Circuit

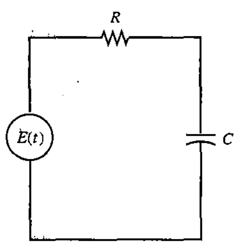


**Figure 8.20**

## 8.6 Applications Using Laplace Transforms

- *L-R-C* Circuits Revisited
- Delay Differential Equations
- Coupled Spring–Mass Systems
- The Double Pendulum

Laplace transforms are useful in solving the applications that were discussed in earlier sections. However, because this method is most useful in alleviating the difficulties associated with problems involving piecewise defined functions, impulse functions, and periodic functions, we focus most of our attention on solving spring–mass systems, *L-R-C* circuit problems, and population problems that include functions of this type.

### *L-R-C* Circuits Revisited

Laplace transforms can be used to solve circuit problems (see Figure 8.19), which were introduced earlier. Recall that the initial-value problems that model *L-R-C* and *L-R* circuits are

$$\begin{cases} L\dfrac{d^2Q}{dt^2} + R\dfrac{dQ}{dt} + \dfrac{1}{C}Q = E(t) \\ Q(0) = Q_0,\ I(0) = \dfrac{dQ}{dt}(0) = I_0 \end{cases} \quad \text{and} \quad \begin{cases} L\dfrac{dI}{dt} + RI = E(t) \\ I(0) = I_0 \end{cases},$$

where $L$, $R$, and $C$ represent the inductance, resistance, and capacitance, respectively. $Q(t)$ is the charge of the capacitor and $dQ/dt = I(t)$, where $I(t)$ is the current. $E(t)$ is the voltage supply.

**Example 1**

Consider an $RC$ circuit in Figure 8.20 with the property $R = C = 1$. If $E(t) = \begin{cases} 1\,V, & 0 \le t < a \\ 0, & a \le t < 2a \end{cases}$ and $E(t + 2a) = E(t)$, find the charge $Q(t)$ if $Q(0) = 0$. Determine values of $a$ so that the output voltage, $Q(t)/C$ (the voltage across the capacitor), matches the input voltage, $E(t)$.



***Solution*** This circuit is modeled with

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

because $L = 0$. Therefore, we solve the IVP,

$$\frac{dQ}{dt} + Q = E(t),\ Q(0) = 0.$$

The Laplace transform of $E(t)$ is

$$\mathscr{L}\{E(t)\} = \frac{1}{1 - e^{-2as}}\int_0^a e^{-st}\,dt = \frac{1}{1 - e^{-2as}}\left[\frac{e^{-st}}{-s}\right]_0^a = \frac{1 - e^{-as}}{s(1 - e^{-2as})}$$

$$= \frac{1 - e^{-as}}{s(1 - e^{-as})(1 + e^{-as})} = \frac{1}{s(1 + e^{-as})}$$

Taking the Laplace transform of each side of the differential equation then gives us

$$s\mathscr{L}\{Q(t)\} - Q(0) + \mathscr{L}\{Q(t)\} = \frac{1}{s(1 + e^{-as})},$$

so that

$$\mathscr{L}\{Q(t)\} = \frac{1}{s(s + 1)(1 + e^{-as})}.$$

Writing the power series expansion of $\dfrac{1}{1 + e^{-as}}$ gives us

$$\frac{1}{1 + e^{-as}} = \frac{1}{1 - (-e^{-as})} = 1 - e^{-as} + e^{-2as} - e^{-3as} + \cdots.$$

Thus

$$\mathscr{L}\{Q(t)\} = \frac{1}{s(s + 1)}[1 - e^{-as} + e^{-2as} - e^{-3as} + \cdots]$$

$$= \frac{1}{s(s + 1)} - \frac{e^{-as}}{s(s + 1)} + \frac{e^{-2as}}{s(s + 1)} - \frac{e^{-3as}}{s(s + 1)} + \cdots.$$

Because $\mathscr{L}^{-1}\{1/[s(s + 1)]\} = 1 - e^{-t}$,

$$Q(t) = (1 - e^{-t}) - (1 - e^{-(t-a)})\mathscr{U}(t - a) + (1 - e^{-(t-2a)})\mathscr{U}(t - 2a) - (1 - e^{-(t-3a)})\mathscr{U}(t - 3a) + \cdots.$$

Figure 8.21 shows graphs of the solution when $a = 1$, $a = 10$, $a = 20$, and $a = 50$, along with the input function, $E(t)$. We notice that the output voltage, $Q(t)/C$, moves closer to $E(t)$ as $a$ increases. Notice that if $t$ is measured in milliseconds, then the input with $a = 1$ has frequency 500 hertz (cycles per second); this corresponds to $a = 50$, which has frequency 10 hertz. Therefore, low-frequency square waves are least altered by the circuit.

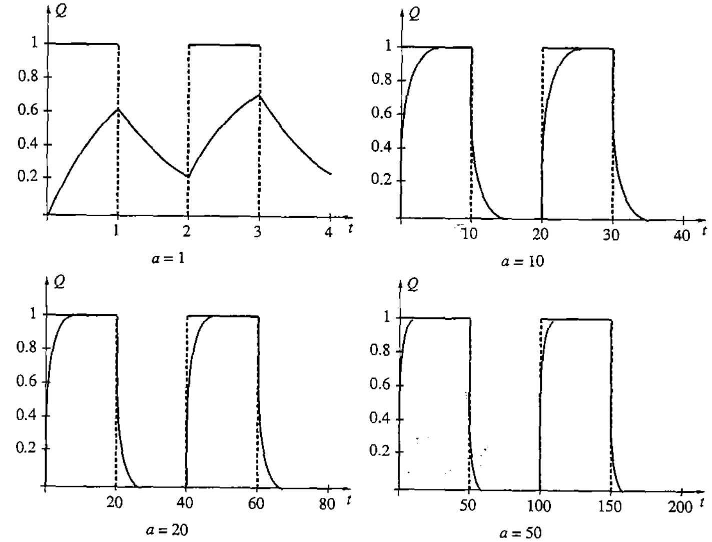



Figure 8.21

## Delay Differential Equations

In some applications, there is a delay (or shift) in the argument of the dependent variable. A differential equation that involves this type of shift is called a **delay equation.** Consider a mixture problem from Section 3.3. In this case, however, we assume that there is a delay in the effect that a salt solution flowing into the tank has on the salt concentration flowing out of the tank. To include this assumption in the mathematical model of the situation, we assume that the concentration of the salt solution flowing out of the tank equals the salt concentration in the tank at an earlier time. We illustrate a problem of this type in the following example.

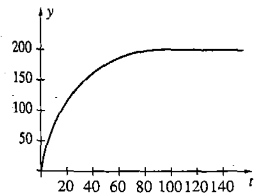



Figure 8.22



**Example 2**

A tank contains 100 gal of water. A salt solution with concentration 2 lb/gal flows into the tank at a rate of 4 gal/min, and the well-stirred mixture flows out of the tank at the same rate. If the salt concentration of the solution flowing out of the tank equals the average concentration 1 min earlier, determine the amount of salt in the tank at any time $t$.

***Solution*** If we let $y(t)$ represent the amount of salt in the tank at any time $t$, then we determine the net change in the amount of salt in the tank with the differential equation

$$\frac{dy}{dt} = \left(2\frac{\text{lb}}{\text{gal}}\right)\left(4\frac{\text{gal}}{\text{min}}\right) - \left(\frac{y(t-1)}{100}\frac{\text{lb}}{\text{gal}}\right)\left(4\frac{\text{gal}}{\text{min}}\right).$$

Therefore, because there is no salt in the tank initially, we solve the problem

$$\frac{dy}{dt} = 8 - \frac{1}{25}y(t-1),\ y(t) = 0,\ -1 \le t \le 0.$$

In the exercises, we have you verify that $\mathcal{L}\{y(t-1)\} = e^{-s}Y(s)$. Using the Laplace transform, we find

$$sY(s) - y(0) = \frac{8}{s} - \frac{1}{25}e^{-s}Y(s)$$
$$\left(s + \frac{1}{25}e^{-s}\right)Y(s) = \frac{8}{s}$$
$$Y(s) = \frac{8}{s\left(s + \frac{1}{25}e^{-s}\right)} = \frac{8}{s^2\left(1 + \frac{1}{25s}e^{-s}\right)}.$$

With the power series expansion $1/(1+x) = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty}(-x)^n$, we have

$$Y(s) = \frac{8}{s^2}\sum_{n=0}^{\infty}\left(-\frac{1}{25s}e^{-s}\right)^n = \frac{8}{s^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{25^n}\frac{e^{-ns}}{s^n} = 8\sum_{n=0}^{\infty}\frac{(-1)^n}{25^n}\frac{e^{-ns}}{s^{n+2}}.$$

Therefore,

$$y(t) = 8\sum_{n=0}^{\infty}\frac{(-1)^n}{25^n}\frac{(t-n)^{n+1}}{(n+1)!}\mathcal{U}(t-n).$$

In Figure 8.22, we graph the solution [0, 150]. It appears that $\lim_{t\to\infty} y(t) = 200$. We may ask how this solution differs from that of the problem that does not include a delay,

$$\frac{dz}{dt} = 8 - \frac{1}{25}z,\ z(0) = 0.$$

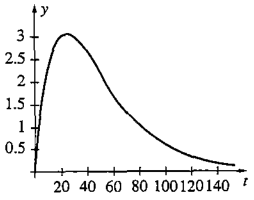



**Figure 8.23**

Solving this problem as a first-order linear equation, we find that $z(t) = -200e^{-t/25} + 200$. Certainly, these two functions have the same limiting value, but how do they differ? In Figure 8.23 we graph $y(t) - z(t)$. The largest difference between the two functions is approximately 3, which occurs when $t \approx 24$. Another observation is that $y(t) - z(t) \geq 0$.

## Coupled Spring–Mass Systems

The displacement of a mass attached to the end of a spring was modeled with a second-order linear differential equation with constant coefficients in Chapter 5. Similarly, if a second spring and mass are attached to the end of the first mass as shown in Figure 8.24, the model becomes that of a system of second-order equations. To more precisely state the problem, let masses $m_1$ and $m_2$ be attached to the ends of springs $S_1$ and $S_2$ having spring constants $k_1$ and $k_2$ respectively. Spring $S_2$ is then attached to the base of mass $m_1$. Suppose that $x(t)$ and $y(t)$ represent the vertical displacement from the equilibrium position of springs $S_1$ and $S_2$, respectively. Because spring $S_2$ undergoes both elongation and compression when the system is in motion (due to the spring $S_1$ and the mass $m_2$), according to Hooke's law, $S_2$ exerts the force $k_2(y - x)$ on $m_2$ and $S_1$ exerts the force $-k_1x$ on $m_1$. Therefore, the force acting on mass $m_1$ is the sum $-k_1x + k_2(y - x)$ and the force acting on $m_2$ is $-k_2(y - x)$. (In Figure 8.25 we show the forces acting on the two masses where up is the negative direction and down is positive.) Using Newton's second law ($F = ma$) with each mass, we have the system

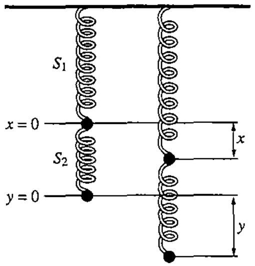



**Figure 8.24** A coupled spring–mass system

$$\begin{cases} m_1 \dfrac{d^2x}{dt^2} = -k_1x + k_2(y - x) \\ m_2 \dfrac{d^2y}{dt^2} = -k_2(y - x) \end{cases}$$

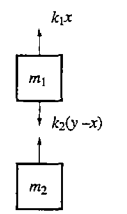



**Figure 8.25** Force diagram for a coupled spring–mass system

The initial displacement and velocity of the two masses $m_1$ and $m_2$ are given by $x(0)$, $x'(0)$, $y(0)$, and $y'(0)$, respectively. Because this system involves second-order equations. Laplace transforms can be used to solve problems of this type. Recall the following property of the Laplace transform: $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$, where $F(s)$ is the Laplace transform of $f(t)$. This property is of great use in solving this problem because both equations involve second derivatives. We will see that the method is similar to that used in solving initial-value problems involving a single equation. With systems, however, we end up solving a system of algebraic equations after taking the Laplace transform of each equation.

**Example 3**

Consider the spring–mass system with $m_1 = m_2 = 1$, $k_1 = 3$, and $k_2 = 2$. Find the position functions $x(t)$ and $y(t)$ if $x(0) = 0$, $x'(0) = 1$, $y(0) = 1$, and $y'(0) = 0$.



***Solution*** To find $x(t)$ and $y(t)$, we must solve the initial-value problem

$$\begin{cases} \dfrac{d^2x}{dt^2} = -5x + 2y \\ \dfrac{d^2y}{dt^2} = 2x - 2y \end{cases}, \; x(0) = 0,\ x'(0) = 1,\ y(0) = 1,\ y'(0) = 0.$$

Taking the Laplace transform of both sides of each equation, we have

$$\begin{cases} s^2X(s) - sx(0) - x'(0) = -5X(s) + 2Y(s) \\ s^2Y(s) - sy(0) - y'(0) = 2X(s) - 2Y(s) \end{cases},$$

which is simplified to obtain the system

$$\begin{cases} (s^2 + 5)X(s) - 2Y(s) = 1 \\ -2X(s) + (s^2 + 2)Y(s) = s \end{cases}.$$

Solving for $X(s)$, we have

$$X(s) = \frac{s^2 + 2s + 2}{(s^2 + 5)(s^2 + 2) - 4} = \frac{s^2 + 2s + 2}{s^4 + 7s^2 + 6} = \frac{s^2 + 2s + 2}{(s^2 + 1)(s^2 + 6)}.$$

Then, with the partial fraction expansion

$$X(s) = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 6},$$

we find that $A = 2/5$, $B = 1/5$, $C = -2/5$, and $D = 4/5$, so

$$X(s) = \frac{2s + 1}{5(s^2 + 1)} + \frac{2(2 - s)}{5(s^2 + 6)}.$$

Taking the inverse Laplace transform then yields

$$x(t) = \frac{1}{5}\sin t + \frac{2}{5}\cos t - \frac{2}{5}\cos\sqrt{6}t + \frac{4}{5\sqrt{6}}\sin\sqrt{6}t.$$

Instead of solving the system to find $Y(s)$, we use the differential equation $d^2x/dt^2 = -5x + 2y$ to find $y(t)$. Because $y = \frac{1}{2}(x'' + 5x)$, and

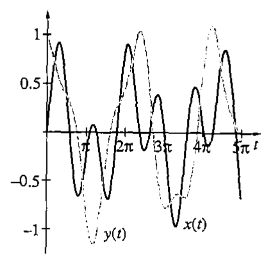
$$x'(t) = \frac{1}{5}\cos t - \frac{2}{5}\sin t + \frac{2\sqrt{6}}{5}\sin\sqrt{6}t + \frac{4}{5}\cos\sqrt{6}t$$

and

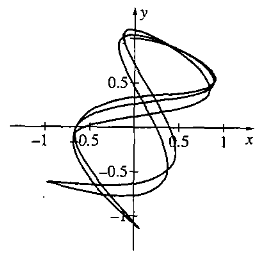
$$x''(t) = -\frac{1}{5}\sin t - \frac{2}{5}\cos t + \frac{12}{5}\cos\sqrt{6}t - \frac{4\sqrt{6}}{5}\sin\sqrt{6}t,$$

$$y(t) = \frac{1}{2}\left(-\frac{1}{5}\sin t - \frac{2}{5}\cos t + \frac{12}{5}\cos\sqrt{6}t - \frac{4\sqrt{6}}{5}\sin\sqrt{6}t + \sin t + 2\cos t - 2\cos\sqrt{6}t + \frac{4}{\sqrt{6}}\sin\sqrt{6}t\right)$$

$$= \frac{2}{5}\sin t + \frac{4}{5}\cos t - \frac{\sqrt{6}}{15}\sin\sqrt{6}t + \frac{1}{5}\cos\sqrt{6}t.$$




(a)




(b)

**Figure 8.26** (a)–(b)

In Figure 8.26(a), we graph $x(t)$ and $y(t)$ simultaneously. Note that the initial point of $y(t)$ is (0, 1), whereas that of $x(t)$ is (0, 0). Of course, these functions can be graphed parametrically in the $xy$-plane as shown in Figure 8.26(b). Notice that this phase plane is different from those discussed in previous sections. One of the reasons is that the equations in the system of differential equations are second-order instead of first-order. Finally, in Figure 8.27, we illustrate the motion of the spring–mass system by graphing the springs for several values of $t$.

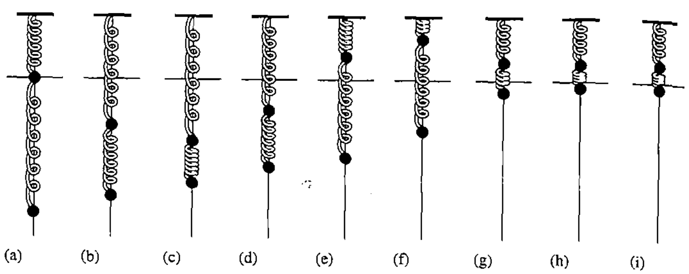



**Figure 8.27** (a) $t=0$ (b) $t=\frac{1}{2}$ (c) $t=1$ (d) $t=\frac{3}{2}$ (e) $t=2$ (f) $t=\frac{5}{2}$ (g) $t=3$ (h) $t=\frac{7}{2}$ (i) $t=4$.

*In Example 3, what is the maximum displacement of each spring? Does one of the springs always attain its maximum displacement before the other?*

If external forces $F_1(t)$ and $F_2(t)$ are applied to the masses, the system of equations becomes

$$\begin{cases} m_1\dfrac{d^2x}{dt^2} = -k_1x + k_2(y-x) + F_1(t) \\ m_2\dfrac{d^2y}{dt^2} = -k_2(y-x) + F_2(t) \end{cases},$$

which is again solved through the method of Laplace transforms. We investigate the effects of these external forcing functions in Exercise 90.

The previous situation can be modified to include a third spring with spring constant $k_3$ between the base of the mass $m_2$ and a lower support as shown in Figure 8.28. The motion of the spring–mass system is affected by the third spring. Using the techniques of the earlier case, this model becomes

$$\begin{cases} m_1\dfrac{d^2x}{dt^2} = -k_1x + k_2(y-x) + F_1(t) \\ m_2\dfrac{d^2y}{dt^2} = -k_3y - k_2(y-x) + F_2(t) \end{cases}.$$

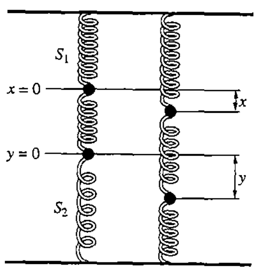



**Figure 8.28**



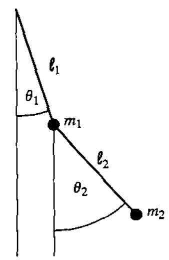



**Figure 8.29** A double pendulum

## The Double Pendulum

In a method similar to that of the simple pendulum in Chapter 5 and that of the coupled spring system, the motion of a double pendulum (see Figure 8.29) is modeled by the following system of equations using the approximation $\sin\theta \approx \theta$ for small displacements:

$$\begin{cases} (m_1+m_2)\ell_1^2\theta_1'' + m_2\ell_1\ell_2\theta_2'' + (m_1+m_2)\ell_1 g\theta_1 = 0 \\ m_2\ell_2^2\theta_2'' + m_2\ell_1\ell_2\theta_1'' + m_2\ell_2 g\theta_2 = 0 \end{cases},$$

where $\theta_1$ represents the displacement of the upper pendulum and $\theta_2$ that of the lower pendulum. Also, $m_1$ and $m_2$ represent the mass attached to the upper and lower pendulums, respectively, and the length of each is given by $\ell_1$ and $\ell_2$.

**Example 4**

Suppose that $m_1 = 3$, $m_2 = 1$, and each pendulum has length 16. If $g = 32$, determine $\theta_1(t)$ and $\theta_2(t)$ if $\theta_1(0) = 1$, $\theta_1'(0) = 0$, $\theta_2(0) = 0$, and $\theta_2'(0) = -1$.

***Solution*** In this case, the system is

$$\begin{cases} 4(16)^2\theta_1'' + 16^2\theta_2'' + 4(16)(32)\theta_1 = 0 \\ 16^2\theta_1'' + 16^2\theta_2'' + (16)(32)\theta_2 = 0 \end{cases},$$

which can be simplified to obtain

$$\begin{cases} 4\theta_1'' + \theta_2'' + 8\theta_1 = 0 \\ \theta_1'' + \theta_2'' + 2\theta_2 = 0 \end{cases}.$$

If we let $\mathscr{L}\{\theta_1(t)\} = X(s)$ and $\mathscr{L}\{\theta_2(t)\} = Y(s)$, we have

$$\begin{cases} 4[s^2X(s) - s\theta_1(0) - \theta_1'(0)] + [s^2Y(s) - s\theta_2(0) - \theta_2'(0)] + 8X(s) = 0 \\ [s^2X(s) - s\theta_1(0) - \theta_1'(0)] + [s^2Y(s) - s\theta_2(0) - \theta_2'(0)] + 2Y(s) = 0 \end{cases}$$

or

$$\begin{cases} 4(s^2+2)X(s) + s^2Y(s) = 4s - 1 \\ s^2X(s) + (s^2+2)Y(s) = s - 1 \end{cases}.$$

Solving this system for $X(s)$, we obtain

$$X(s) = \frac{3s^3+8s-2}{3s^4+16s^2+16} = \frac{3s^3+8s-2}{(3s^2+4)(s^2+4)}.$$

With the partial fraction expansion

$$X(s) = \frac{As+B}{3s^2+4} + \frac{Cs+D}{s^2+4},$$

we find that

$$X(s) = \frac{3}{4}\frac{2s-1}{3s^2+4} + \frac{1}{4}\frac{2s+1}{s^2+4}.$$

Then,

$$\theta_1(t) = \frac{1}{2}\cos\frac{2t}{\sqrt{3}} - \frac{\sqrt{3}}{8}\sin\frac{2t}{\sqrt{3}} + \frac{1}{2}\cos 2t + \frac{1}{8}\sin 2t.$$

Differentiating, we have

$$\theta_1'(t) = -\frac{1}{\sqrt{3}}\sin\frac{2t}{\sqrt{3}} - \frac{1}{4}\cos\frac{2t}{\sqrt{3}} - \sin 2t + \frac{1}{4}\cos 2t$$

and

$$\theta_1''(t) = -\frac{2}{3}\cos\frac{2t}{\sqrt{3}} + \frac{1}{2\sqrt{3}}\sin\frac{2t}{\sqrt{3}} - 2\cos 2t - \frac{1}{2}\sin 2t.$$

Using the differential equation $4\theta_1'' + \theta_2'' + 8\theta_1 = 0$ yields $\theta_2'' = -4\theta_1'' - 8\theta_1$. Therefore,

$$\theta_2''(t) = -4\theta_1''(t) - 8\theta_1(t) = \frac{1}{\sqrt{3}}\sin\frac{2t}{\sqrt{3}} - \frac{4}{3}\cos\frac{2t}{\sqrt{3}} + \sin 2t + 4\cos 2t.$$

Integrating, we have

$$\theta_2'(t) = -\frac{1}{2}\cos\frac{2t}{\sqrt{3}} - \frac{2}{\sqrt{3}}\sin\frac{2t}{\sqrt{3}} - \frac{1}{2}\cos 2t + 2\sin 2t + c_1.$$

Applying the initial condition $\theta_2'(0) = -1$ yields $\theta_2'(0) = -1/2 - 1/2 + c_1 = -1$, so $c_1 = 0$. Again, by integrating, we obtain

$$\theta_2(t) = -\frac{\sqrt{3}}{4}\sin\frac{2t}{\sqrt{3}} + \cos\frac{2t}{\sqrt{3}} - \frac{1}{4}\sin 2t - \cos 2t + c_2.$$

Then, because $\theta_2(0) = 0$, $\theta_2(0) = 1 - 1 + c_2 = 0$, so $c_2 = 0$, which indicates that

$$\theta_2(t) = -\frac{\sqrt{3}}{4}\sin\frac{2t}{\sqrt{3}} + \cos\frac{2t}{\sqrt{3}} - \frac{1}{4}\sin 2t - \cos 2t.$$

These two functions are graphed together in Figure 8.30(a) and parametrically in Figure 8.30(b) to show the solution in the phase plane. We also show the motion of the double pendulum for several values of $t$ in Figure 8.31.

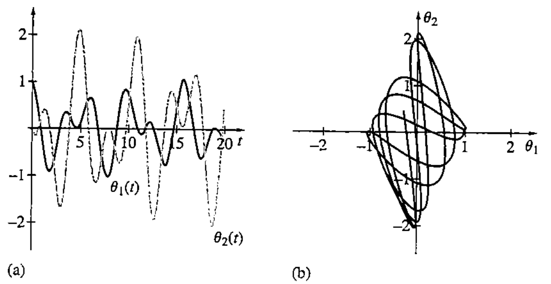



Figure 8.30
(a)–(b)



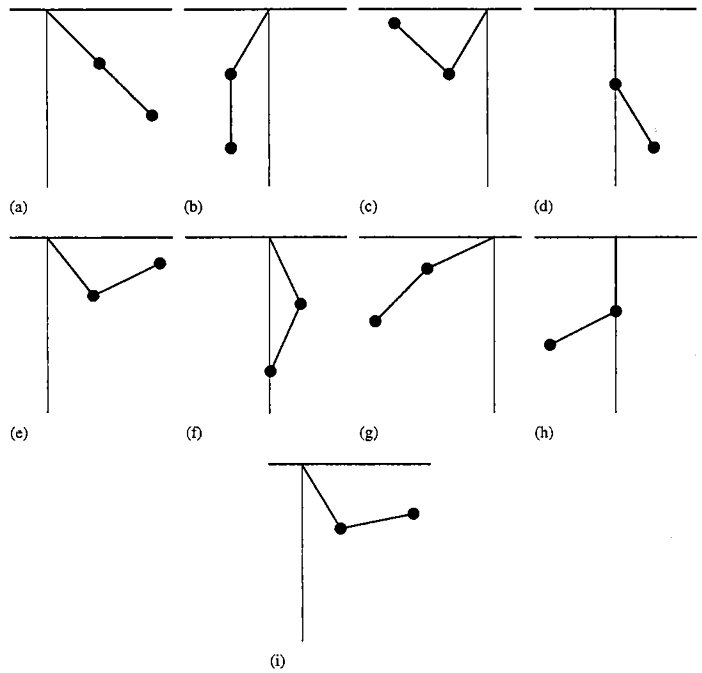



**Figure 8.31** (a) $t = 0$ (b) $t = \frac{5}{4}$ (c) $t = \frac{5}{2}$ (d) $t = \frac{15}{4}$ (e) $t = 5$ (f) $t = \frac{25}{4}$ (g) $t = \frac{15}{2}$ (h) $t = \frac{35}{4}$ (i) $t = 10$

*Does the system in Example 4 come to rest? Which pendulum experiences a greater displacement from equilibrium?*

## EXERCISES 8.6

1. Suppose that we consider a circuit with a capacitor $C$, a resistor $R$, and a voltage supply $E(t) = \begin{cases} 100, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$. If $L = 0$, find $Q(t)$ and $I(t)$ if $Q(0) = 0$, $C = 50^{-1}$ farads, and $R = 50\ \Omega$.

2. Suppose that we consider a circuit with a capacitor $C$, a resistor $R$, and a voltage supply $E(t) = \begin{cases} 100\sin t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$. If $L = 0$, find $Q(t)$ and $I(t)$ if $Q(0) = 0$, $C = 10^{-2}$ farads, and $R = 100\ \Omega$.

*3. Consider the circuit with no capacitor, $R = 100\ \Omega$, and $L = 100$ H if $E(t) = \begin{cases} 50, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}$ and $E(t+2) = E(t)$. Find the current $I(t)$ if $I(0) = 0$.

4. Consider the circuit with no capacitor, $R = 100\ \Omega$, and $L = 100$ H if $E(t) = \begin{cases} 100 \sin t, & 0 \le t < \pi \\ 0, & \pi \le t < 2\pi \end{cases}$ and $E(t + 2\pi) = E(t)$. Find the current $I(t)$ if $I(0) = 0$.

5. Consider the circuit with no capacitor, $R = 100\ \Omega$, and $L = 100$ H if $E(t) = \begin{cases} 100t, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}$ and $E(t+2) = E(t)$. Find the current $I(t)$ if $I(0) = 0$.

6. Consider the circuit with no capacitor, $R = 100\ \Omega$, and $L = 100$ H if $E(t) = \begin{cases} 100t, & 0 \le t < 1 \\ 100(2-t), & 1 \le t < 2 \end{cases}$ and $E(t+2) = E(t)$. Find the current $I(t)$ if $I(0) = 0$.

*7. Solve the $L$-$R$-$C$ circuit equation that was derived before Example 1 for $I(t)$ if $L = 1$ henry, $R = 6$ ohms, $C = \frac{1}{9}$ farad, $E(t) = 100$ volts, and $I(0) = 0$.

8. Solve the $L$-$R$-$C$ circuit equation for $I(t)$ if $L = 1$ henry, $R = 6$ ohms, $C = \frac{1}{9}$ farad, $E(t) = 100 \sin t$ volts, and $I(0) = 0$.

9. Solve the $L$-$R$-$C$ circuit equation for $I(t)$ using the parameter values $L = C = R = 1$ and $E(t) = \begin{cases} 1, & 0 \le t \le \pi \\ 0, & t > \pi \end{cases}$, $I(0) = 0$.

10. Solve the $L$-$R$-$C$ circuit equation for $I(t)$ using the parameter values $L = C = R = 1$ and $E(t) = \begin{cases} \cos t, & 0 \le t \le \pi \\ 0, & t > \pi \end{cases}$, $I(0) = 0$.

*11. Consider a circuit with $L = 1$, $R = 4$, $E(t) = \delta(t-1)$, and no capacitor. Determine the current $I(t)$ if **(a)** $I(0) = 0$ and **(b)** $I(0) = 1$.

12. Solve the initial-value problems in Exercise 11 using $E(t) = \delta(t-1) + \delta(t-2)$.

13. Consider an $L$-$R$ circuit in which $L = 1$ and $R = 1$, and $E(t) = \begin{cases} t, & 0 \le t \le 1 \\ 1, & t > 1 \end{cases}$. Find the current if $I(0) = 0$.

14. Consider the circuit in Exercise 13 with a voltage source $E(t) = \begin{cases} t, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$. Find the current if $I(0) = 0$.

*15. Consider the circuit in Exercise 13 with $E(t) = \delta(t-2) + \delta(t-6)$. Find the current if $I(0) = 0$.

16. Consider the circuit in Exercise 13 with $E(t) = 120\delta(t-1)$. Find the current if $I(0) = 100$.

17. Consider the $R$-$C$ circuit in Figure 8.32 in which $R = 2.5 \times 10^4\ \Omega$, $C = 10^{-5}$ farad, and $E(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases}$ volts. Determine the output voltage, $i(t)R$ (the voltage across the resistor), by solving the appropriate integral equation. (*Note:* This equation cannot be solved as a differential equation because $E(t)$ is discontinuous at $t = 1$ and $t = 2$.)

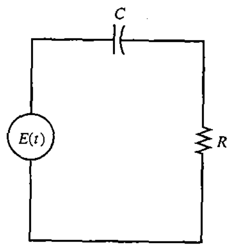



**Figure 8.32**

18. **(Spring–Mass Systems Revisited)** Just as with systems, we may use Laplace transforms to solve the IVP

$$\begin{cases} m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx = f(t) \\ x(0) = \alpha,\ \dfrac{dx}{dt}(0) = \beta \end{cases},$$

which models a spring–mass system involving one spring. Suppose that $m = 1$ slug, $k = 1$ lb/ft, and that there is no resistance due to damping. Suppose also that the object is released from its equilibrium position with zero initial velocity and the object is subjected to the external force

$$f(t) = \begin{cases} \cos t, & 0 \le t < \pi/2 \\ 0, & t \ge \pi/2 \end{cases}.$$

Show that $\mathcal{L}\{f(t)\} = \dfrac{s}{s^2+1} + e^{-s\pi/2}\dfrac{1}{s^2+1}$.

19. Show that $X(s) = \dfrac{s}{(s^2+1)^2} + e^{-s\pi/2}\dfrac{1}{(s^2+1)^2}$, so that $x(t) = \frac{1}{2}t \sin t + \frac{1}{2}(-\cos t - (t - \pi/2)\sin t)\mathcal{U}(t - \pi/2)$.



In Exercises 20 and 21, use the given initial conditions to determine the displacement of the object of mass $m$ attached to a spring with spring constant $k$.

20. $m = 4$, $k = 16$, $x(0) = 1$, $x'(0) = 0$

21. $m = 1$, $k = 9$, $x(0) = 3$, $x'(0) = -2$

In Exercises 22–25, determine the displacement of the object of mass $m$ attached to a spring with spring constant $k$ if the damping is given by $c\ dx/dt$. Use the initial conditions $x(0) = 1$, $x'(0) = 0$ in each case.

22. $m = 1$, $k = 6$, $c = 5$

23. $m = 1/2$, $k = 2$, $c = 2$

24. $m = 1$, $k = 13$, $c = 4$

*25. $m = 1$, $k = 4$, $c = 5$

In Exercises 26–29, determine the displacement of the object of mass $m$ attached to a spring with spring constant $k$ if the damping is given by $c\ dx/dt$ and there is an external force $f(t)$. In each case, use the initial conditions $x(0) = x'(0) = 0$.

26. $m = 1$, $k = 6$, $c = 5$, $f(t) = \frac{1}{4}\sin t$

27. $m = 1/2$, $k = 2$, $c = 2$, $f(t) = te^{-t}$

28. $m = 1$, $k = 13$, $c = 4$, $f(t) = e^{-2t}$

*29. $m = 1$, $k = 4$, $c = 5$, $f(t) = e^{-4t} + 2e^{-t}$

In Exercises 30–37, determine the displacement of the object of mass $m$ attached to a spring with spring constant $k$ if the damping is given by $c\ dx/dt$ and there is a piecewise defined external force $f(t)$. In each case, use the initial conditions $x(0) = x'(0) = 0$.

30. $m = 1$, $k = 6$, $c = 5$, $f(t) = \begin{cases} \sin 2t, & 0 \le t < \pi/2 \\ 0, & t \ge \pi/2 \end{cases}$

31. $m = 1$, $k = 6$, $c = 5$, $f(t) = \begin{cases} \cos \pi t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

32. $m = 1/2$, $k = 2$, $c = 2$, $f(t) = \begin{cases} e^{-2t}, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

*33. $m = 1/2$, $k = 2$, $c = 2$, $f(t) = \begin{cases} e^{-2t}, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases}$

34. $m = 1$, $k = 13$, $c = 4$, $f(t) = \begin{cases} e^{-2t}\cos 3t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$

35. $m = 1$, $k = 13$, $c = 4$, $f(t) = \begin{cases} e^{-2t}\cos 3t, & 0 \le t < \pi \\ e^{-2t}\sin 3t, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$

36. $m = 1$, $k = 4$, $c = 5$, $f(t) = \begin{cases} e^{-t}, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

*37. $m = 1$, $k = 4$, $c = 5$, $f(t) = \begin{cases} \cos \pi t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

38. Show that the initial-value problem $x'' + x = \delta(t)$, $x(0) = x'(0) = 0$ is equivalent to $x'' + x = 0$, $x(0) = 0$, $x'(0) = 1$. (*Hint:* Solve each equation.)

In Exercises 39–41, assume that the mass is released with zero initial velocity from its equilibrium position.

39. Suppose that an object with mass $m = 1$ is attached to the end of a spring with spring constant 16. If there is no damping and the spring is subjected to the forcing function $f(t) = \sin t$, determine the motion of the spring if at $t = 1$, the spring is supplied with an upward shock of 4 units.

40. An object of mass $m = 1$ is attached to a spring with $k = 13$ and is subjected to damping equivalent to $4\ dx/dt$. Find the motion of the mass if the spring is supplied with a downward shock of 1 unit at $t = 2$.

*41. An object of mass $m = 1$ is attached to a spring with $k = 13$ and is subjected to damping equivalent to $4\ dx/dt$. Find the motion if $f(t) = \delta(t-1) + \delta(t-3)$.

42. **(Population Problems Revisited)** Let $x(t)$ represent the population of a certain country in which the rate of population increase depends on the growth rate of the country as well as the rate at which people are being added to or subtracted from the population because of immigration, emigration, or both. Consider the IVP $x' + kx = 1000(1 + a \sin t)$, $x(0) = x_0$, where $a$ is a constant. **(a)** Show that

$$x(t) = \frac{1000}{k}(1 - e^{-kt}) + \frac{1000a}{1+k^2}(e^{-kt} - \cos t + k \sin t) + x_0e^{-kt}.$$

**(b)** If $k = 3$, graph $x(t)$ over $0 \le t \le 10$ for $a = 0.2$, 0.4, 0.6, 0.8. Describe how the value of $a$ affects the solution.

In Exercises 43–49, solve the initial-value problem using Laplace transforms. Interpret each as a population problem. Is the population bounded?

43. $x' + 5x = 500(2 - \sin t)$, $x(0) = 10{,}000$

44. $x' + 5x = 500(2 + \cos t)$, $x(0) = 5000$

***45.** $x' + 5x = 500(2 - \cos t)$, $x(0) = 5000$

**46.** $x' + 5x = 500(2 - \sin t)$, $x(0) = 5000$

**47.** $x' - 2x = 500(1 + \sin t)$, $x(0) = 5000$

**48.** $x' - 2x = 500(\cos t + 1)$, $x(0) = 5000$

***49.** Suppose that the emigration function is

$$f(t) = \begin{cases} 5000(2 - \sin t), & 0 \le t < 5 \\ 0, & t \ge 5 \end{cases}.$$

Solve $x' - x = f(t)$, $x(0) = 10{,}000$. Determine $\lim_{t\to\infty} f(t)$.

**50.** Suppose that the emigration function is $f(t) = \begin{cases} 5000(1 + \cos t), & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$. Solve $x' - x = f(t)$, $x(0) = 5000$. Determine $\lim_{t\to\infty} f(t)$ if it exists.

**51.** If the emigration function is the periodic function $f(t) = \begin{cases} 5000(1 + \cos t), & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}$, $f(t + 2) = f(t)$, solve $x' - x = f(t)$, $x(0) = 5000$.

**52.** If the emigration function is the periodic function $f(t) = \begin{cases} 5000(2 - \sin t), & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}$, $f(t + 2) = f(t)$, solve $x' - x = f(t)$, $x(0) = 5000$.

**53.** Suppose that a bacteria population satisfies the differential equation $dx/dt = x + 200\delta(t - 2)$ with $x(0) = 100$. What is the population at $t = 5$?

**54.** Suppose that a patient receives glucose through an IV tube at a constant rate of $c$ grams per minute. If at the same time the glucose is metabolized and removed from the bloodstream at a rate that is proportional to the amount of glucose present in the bloodstream, the rate at which the amount of glucose changes is modeled by $dx/dt = c - kx$, where $x(t)$ is the amount of glucose in the bloodstream at time $t$ and $k$ is a constant. If $x(0) = x_0$, use Laplace transforms to find $x(t)$. Does $x(t)$ approach a limit as $t \to \infty$?

***55.** Suppose that in Exercise 54 the patient receives glucose at a variable rate $c(1 + \sin t)$. Therefore, the rate at which the amount of glucose changes is modeled by $dx/dt = c(1 + \sin t) - kx$. If $x(0) = x_0$, use Laplace transforms to find $x(t)$. How does the solution of this problem differ from that found in Exercise 54?

**56.** If the person in Exercise 54 receives glucose periodically according to the function

$$f(t) = \begin{cases} c, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}, \quad f(t + 2) = f(t),$$

solve the initial-value problem $dx/dt = f(t) - kx$, $x(0) = x_0$, and compare the solution with that of Exercise 55.

**57.** **(Drug Dosage Problem)** The drug dosage problems considered in Exercises 54–56 may be thought of as a two-tank (compartment) problem. Suppose that we administer the drug once every 4 hours, where the drug moves from the gastrointestinal (GI) tract (tank $A$) into the circulatory system (tank $B$). Let $x(t)$ represent the amount of drug in the GI tract at time $t$ hours, $y(t)$ be the amount of drug in the circulatory system, and $c(t)$ be the dosage function. If $a$ and $b$ are the rates at which the drug is consumed in the GI tract and the circulatory system, respectively, and if initially neither the GI tract nor the circulatory system contain the drug, we model this situation with the system

$$\begin{cases} \dfrac{dx}{dt} = c(t) - ax, \; x(0) = 0 \\ \dfrac{dy}{dt} = ax - by, \; y(0) = 0 \end{cases}$$

Suppose that

$$c(t) = \begin{cases} c_0, & 0 \le t < 1/2 \\ 0, & 1/2 \le t < 4 \end{cases}, \quad c(t - 4) = c(t).$$

**(a)** Show that for $0 \le t \le 24$ (over one day),

$$c(t) = \sum_{n=0}^{6} c_0[\mathcal{U}(t - 4n) - \mathcal{U}(t - 4n - 1/2)] \text{ and}$$

$$\mathcal{L}\{c(t)\} = \sum_{n=0}^{6} \frac{c_0}{s}[e^{-4ns} - e^{-(4n+1/2)s}].$$

**(b)** Use the Laplace transform to solve $dx/dt = c(t) - ax$, $x(0) = 0$. (*Note:*

$$X(s) = \sum_{n=0}^{6} \frac{c_0}{s(s + a)}\left[e^{-4ns} - e^{-(4n+1/2)s}\right].)$$

**(c)** Use the Laplace transform and $x(t)$ to solve $dy/dt = ax - by$, $y(0) = 0$. (*Note:*

$$Y(s) = \sum_{n=0}^{6} \frac{c_0}{s + b}\left[\left(\frac{1}{s} - \frac{1}{s + a}\right)e^{-4ns} - \left(\frac{1}{s} - \frac{1}{s + a}\right)e^{-(4n+1/2)s}\right].)$$

**(d)** If the half-life of the drug in the GI tract is 1 hr and that in the circulatory system is 4 hr, then graph the amount of drug in the GI tract and the circulatory system over [0, 24]. (Recall that we discussed half-life with regard to exponential decay in Chapter 3.) We note that if the half-life is 1 hr, then one half of the initial dosage remains after 1 hr. Therefore, we solve $x_0e^{-a(1)} = x_0/2$ for $a$, which yields $a = \ln 2$. Similarly, we solve $y_0e^{-b(4)} = y_0/2$ to find that $b = (\ln 2)/4$. Describe what happens to the drug concentration in each case. Assume that $c_0 = 2$.

**(e)** Suppose that the half-life of the drug in the GI tract is 30 min and that in the circulatory system it is 4 hr. How does this change affect the drug concentration in the circulatory system?

**(f)** Suppose that the half-life of the drug in the circulatory system in part **(d)** is 5 hr. How does this change affect the drug concentration in the circulatory system?

**(g)** How does adding a buffer to the drug to increase the half-life of the drug in the GI tract affect the drug concentration in the circulatory system?



**58.** **(The Stiffness Matrix)** **(a)** Show that we may write the system

$$\begin{cases} m_1\dfrac{d^2x}{dt^2} = -k_1x + k_2(y - x) \\ m_2\dfrac{d^2y}{dt^2} = -k_2(y - x) \end{cases}$$

in matrix form $\mathbf{Mx}'' = \mathbf{Kx}$, where $\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$ is the mass matrix and $\mathbf{K} = \begin{pmatrix} -(k_1 + k_2) & k_2 \\ k_2 & -k_2 \end{pmatrix}$ is the **stiffness matrix.** **(b)** Show that we can write $\mathbf{Mx}'' = \mathbf{Kx}$ as $\mathbf{x}'' = \mathbf{Ax}$, where $\mathbf{A} = \mathbf{M}^{-1}\mathbf{K}$. (Why does $\mathbf{M}^{-1}$ exist?) **(c)** Assume that $\mathbf{x}(t) = \mathbf{v}e^{\alpha t}$ is a solution of $\mathbf{x}'' = \mathbf{Ax}$. Show that $\mathbf{x}(t) = \mathbf{v}e^{\alpha t}$ satisfies the system if and only if $\alpha^2 = \lambda$, where $\lambda$ is an eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ is an associated eigenvector. **(d)** The eigenvalues of $\mathbf{A}$ are typically negative real numbers. Show that two solutions of the system are $\mathbf{x}_1(t) = \mathbf{v}\cos\omega t$ and $\mathbf{x}_2(t) = \mathbf{v}\sin\omega t$, where $\alpha^2 = \lambda = -\omega^2$ or $\alpha = \pm\omega i$.

**59.** Let $m_1 = 2$, $m_2 = 1/2$, $k_1 = 75$, and $k_2 = 25$ in the previous system. **(a)** Show that the eigenvalues of $\mathbf{A} = \mathbf{M}^{-1}\mathbf{K} = \begin{pmatrix} -50 & 25/2 \\ 50 & -50 \end{pmatrix}$ are $\lambda_1 = -75$ and $\lambda_2 = -25$. Then, $\alpha_1^2 = \lambda_1 = -\omega_1^2$ implies that $\omega_1 = \pm 5\sqrt{3}$; $\alpha_2^2 = \lambda_2 = -\omega_2^2$ implies that $\omega_2 = \pm 5$. **(b)** Show that an eigenvector corresponding to $\lambda_1 = -75$ is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$; that corresponding to $\lambda_2 = -25$ is $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. **(c)** Show that two linearly independent solutions corresponding $\omega_1 = \pm 5\sqrt{3}$ are $\begin{pmatrix} \cos 5\sqrt{3}t \\ -2\cos 5\sqrt{3}t \end{pmatrix}$ and $\begin{pmatrix} \sin 5\sqrt{3}t \\ -2\sin 5\sqrt{3}t \end{pmatrix}$; those corresponding to $\omega_2 = \pm 5$ are $\begin{pmatrix} \cos 5t \\ 2\cos 5t \end{pmatrix}$ and $\begin{pmatrix} \sin 5t \\ 2\sin 5t \end{pmatrix}$. **(c)** Form a general solution to the system by taking a linear combination of the four solutions. **(d)** If the initial conditions are $x(0) = 0$, $y(0) = 4$, $x'(0) = 0$, and $y'(0) = 0$, determine the solution to the initial-value problem.

**60.** **(Forced System: Method of Undetermined Coefficients)** Suppose that the system can be written as $\mathbf{x}'' = \mathbf{Ax} + \mathbf{f}$, where $\mathbf{f} = \mathbf{F}_0\cos\omega t$. Show that if a particular solution has the form $\mathbf{x}_p(t) = \mathbf{c}\cos\omega t$, then $\mathbf{c}$ satisfies the system $(\mathbf{A} + \omega^2\mathbf{I})\mathbf{c} = -\mathbf{F}_0$. If $\omega \neq 5$, $5\sqrt{3}$, find a particular solution of the system in the previous system if $\mathbf{F}_0 = \begin{pmatrix} 0 \\ 50 \end{pmatrix}$. (*Note:* The constants will depend on $\omega$.)

**61.** **(Delay Equations)** According to the definition, $\mathcal{L}\{y(t - t_0)\} = \int_0^\infty e^{-st}y(t - t_0)\,dt$. If $y(0) = 0$ for $-t_0 \le t \le 0$, use the change of variable $v = t - t_0$ to show that $\mathcal{L}\{y(t - t_0)\} = e^{-t_0s}Y(s)$.

In Exercises 62–67, solve the coupled spring-mass system modeled by

$$\begin{cases} m_1\dfrac{d^2x}{dt^2} = -k_1x + k_2(y - x) \\ m_2\dfrac{d^2y}{dt^2} = -k_2(y - x) \end{cases}$$

using the indicated parameter values and initial conditions.

**62.** $m_1 = 2$, $m_2 = 2$, $k_1 = 6$, $k_2 = 4$, $x(0) = 0$, $x'(0) = -1$, $y(0) = 0$, $y'(0) = 0$

**63.** $m_1 = 2$, $m_2 = 2$, $k_1 = 6$, $k_2 = 4$, $x(0) = 0$, $x'(0) = 0$, $y(0) = 1$, $y'(0) = 0$

**64.** $m_1 = 2$, $m_2 = 1$, $k_1 = 4$, $k_2 = 2$, $x(0) = 0$, $x'(0) = 0$, $y(0) = 0$, $y'(0) = -1$

***65.** $m_1 = 2$, $m_2 = 1$, $k_1 = 4$, $k_2 = 2$, $x(0) = 1$, $x'(0) = 0$, $y(0) = 0$, $y'(0) = 0$

**66.** $m_1 = 2$, $m_2 = 2$, $k_1 = 3$, $k_2 = 2$, $x(0) = 0$, $x'(0) = 1$, $y(0) = 0$, $y'(0) = 0$

**67.** $m_1 = 2$, $m_2 = 2$, $k_1 = 3$, $k_2 = 2$, $x(0) = 1$, $x'(0) = 0$, $y(0) = 0$, $y'(0) = 0$

In Exercises 68–71, solve the coupled spring-mass system modeled with

$$\begin{cases} m_1\dfrac{d^2x}{dt^2} = -k_1x + k_2(y-x) + F_1(t) \\ m_2\dfrac{d^2y}{dt^2} = -k_2(y-x) + F_2(t) \end{cases},$$

where $m_1 = 2$, $m_2 = 1$, $k_1 = 4$, $k_2 = 2$, $x(0) = 1$, $x'(0) = 0$, $y(0) = 0$, $y'(0) = 0$ using the given forcing functions.

68. $F_1(t) = 1$, $F_2(t) = 1$

69. $F_1(t) = 1$, $F_2(t) = \sin t$

70. $F_1(t) = \cos t$, $F_2(t) = \sin t$

*71. $F_1(t) = \cos 2t$, $F_2(t) = 0$

In Exercises 72–79, solve the coupled pendulum problem using the parameter values $m_1 = 3$ slugs, $m_2 = 1$ slug, $\ell_1 = \ell_2 = 16$ ft, and $g = 32$ ft/s$^2$ and the indicated initial conditions.

72. $\theta_1(0) = 1$, $\theta_1'(0) = 0$, $\theta_2(0) = 0$, $\theta_2'(0) = 1$

73. $\theta_1(0) = 0$, $\theta_1'(0) = 0$, $\theta_2(0) = 0$, $\theta_2'(0) = -1$

74. $\theta_1(0) = 1$, $\theta_1'(0) = 1$, $\theta_2(0) = 0$, $\theta_2'(0) = 0$

*75. $\theta_1(0) = 1$, $\theta_1'(0) = 0$, $\theta_2(0) = 0$, $\theta_2'(0) = 0$

76. $\theta_1(0) = 0$, $\theta_1'(0) = 1$, $\theta_2(0) = -1$, $\theta_2'(0) = 0$

77. $\theta_1(0) = 0$, $\theta_1'(0) = 1$, $\theta_2(0) = 0$, $\theta_2'(0) = 0$

78. $\theta_1(0) = 0$, $\theta_1'(0) = 0$, $\theta_2(0) = -1$, $\theta_2'(0) = -1$

*79. $\theta_1(0) = 0$, $\theta_1'(0) = 0$, $\theta_2(0) = -1$, $\theta_2'(0) = 0$

80. Consider the physical situation of two pendulums coupled with a spring, as shown in Figure 8.33. The motion of this pendulum–spring system is approximated by solving the second-order system

$$\begin{cases} mx'' + m\omega_0^2 x = -k(x-y) \\ my'' + m\omega_0^2 y = -k(y-x) \end{cases},$$

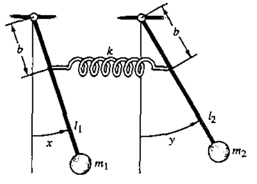



Figure 8.33

where $L$ is the length of each pendulum, $g$ is the gravitational constant, and $\omega_0^2 = g/L$. Use the method of Laplace transforms to solve this system if the initial conditions are $x(0) = a$, $x'(0) = b$, $y(0) = c$, $y'(0) = d$.

81. Solve the initial-value problem

$$\begin{cases} mx'' + m\omega_0^2 x = -k(x-y) \\ my'' + m\omega_0^2 y = -k(y-x) \\ x(0) = -1,\ x'(0) = 0 \\ y(0) = 1,\ y'(0) = 0. \end{cases}$$

What is the maximum displacement of $x(t)$ and $y(t)$?

82. Solve the initial-value problem

$$\begin{cases} mx'' + m\omega_0^2 x = -k(x-y) \\ my'' + m\omega_0^2 y = -k(y-x) \end{cases},$$

$x(0) = 0$, $x'(0) = 1$, $y(0) = 0$, $y'(0) = 0$. Verify that the initial conditions are satisfied.

83. The physical situation shown in Figure 8.34 is modeled by the system of differential equations

$$\begin{cases} m\dfrac{d^2x}{dt^2} + 3kx - ky = 0 \\ 2m\dfrac{d^2y}{dt^2} + 3ky - kx = 0 \end{cases}.$$

If $x(0) = 0$, $x'(0) = 1$, $y(0) = 0$, and $y'(0) = 0$, find $x$ and $y$. Compare these results to those found if $x(0) = 0$, $x'(0) = 0$, $y(0) = 0$, and $y'(0) = 1$. Does this model resemble one that was introduced earlier in the section?

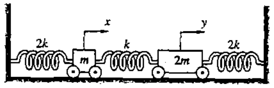



Figure 8.34

84. Write the system $\begin{cases} m_1\dfrac{d^2x}{dt^2} = -k_1x + k_2(y-x) \\ m_2\dfrac{d^2y}{dt^2} = -k_3y - k_2(y-x) \end{cases}$ as a system of four first-order equations with the substitutions $x = x_1$, $x_1' = x_2$, $y = y_1$, $y_1' = y_2$.

85. Show that the eigenvalues of the $4 \times 4$ coefficient matrix in the system of equations obtained in Exercise 84 are two complex conjugate pairs of the form $\pm q_1 i$ and



$\pm q_2 i$, where $q_1$ and $q_2$ are real numbers. What does this tell you about the solutions to this system? Does this agree with the solution obtained with Laplace transforms?

86. In **(a)–(d)**, determine the motion of the object of mass $m$ attached to a spring with spring constant $k$ if the damping is given by $c\ dx/dt$ and there is a piecewise defined periodic external force $f(t)$. Use the initial conditions $x(0) = x'(0) = 0$ in each case.

**(a)** $m = 1$, $k = 6$, $c = 5$,
$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}$, $f(t+2) = f(t)$

**(b)** $m = 1$, $k = 6$, $c = 5$,
$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & 1 \le t < 2 \end{cases}$, $f(t+2) = f(t)$

**(c)** $m = 1$, $k = 13$, $c = 4$,
$f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & \pi \le t < 2\pi \end{cases}$, $f(t+2\pi) = f(t)$

**(d)** $m = 1$, $k = 13$, $c = 4$,
$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & 1 \le t < 2 \end{cases}$, $f(t+2) = f(t)$

**(e)** Use a graphing device to investigate the solutions obtained in **(a)–(d)**. How does the forcing function $f(t)$ affect the subsequent motion of the object? Does the effect of damping eventually cause the object to come to rest?

87. Consider the $L$-$R$-$C$ circuit in which $R = 100\ \Omega$, $L = 0.1$ henry, $C = 10^{-3}$, and $E(t) = 155 \sin 377t$. (Notice that the frequency of the voltage source is $377/2\pi \approx 60$ Hz = 60 cycles/sec.)

**(a)** Find the current $I(t)$. **(b)** What is the maximum value of the current and where does it occur? **(c)** Find $\lim_{t\to\infty} I(t)$ (the steady-state current). **(d)** What is the frequency of the steady-state current?

88. Suppose that the $L$-$R$-$C$ circuit in Exercise 87 has a voltage source $E(t) = \begin{cases} 155 \cos 377t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$.

**(a)** Find the current $I(t)$. **(b)** What is the maximum value of the current and where does it occur? **(c)** Find $\lim_{t\to\infty} I(t)$ (the steady-state current). **(d)** What is the frequency of the steady-state current?

89. Suppose that the $L$-$R$-$C$ circuit in Exercise 87 has the periodic voltage source $E(t) = \begin{cases} 155, & 0 < t < 1 \\ 0, & 1 \le t < 2 \end{cases}$, $E(t+2) = E(t)$. **(a)** Find the current $I(t)$. **(b)** What is the maximum value of the current and where does it occur? **(c)** Find $\lim_{t\to\infty} I(t)$ (the steady-state current). **(d)** What is the frequency of the steady-state current?

90. Solve the problem of the forced coupled spring–mass system with $m_1 = m_2 = 1$, $k_1 = 3$, and $k_2 = 2$ if the forcing functions are $F_1(t) = 1$ and $F_2(t) = \sin t$ and the initial conditions are $x(0) = 0$, $x'(0) = 1$, $y(0) = 1$, $y'(0) = 0$. Graph the solution parametrically as well as simultaneously. How does the motion differ from that of Example 3? What eventually happens to this system? Will the objects eventually come to rest?

91. Consider the three-spring problem shown in Figure 8.25 with $m_1 = m_2 = 1$ and $k_1 = k_2 = k_3 = 1$. Determine $x(t)$ and $y(t)$ if $x(0) = 0$, $x'(0) = -1$, $y(0) = 0$, and $y'(0) = 1$. Graph the solution both simultaneously and parametrically. When does the object attached to the top spring first pass through its equilibrium position? When does the object attached to the second spring first pass through its equilibrium position?

92. Solve Exercise 91 with the forcing functions $F_1(t) = 1$ and $F_2(t) = \cos t$. Graph the solution both simultaneously and parametrically. When does the object attached to the top spring first pass through its equilibrium position? When does the object attached to the second spring first pass through its equilibrium position? How does the motion differ from that in Problem 91?

*93. Consider the double pendulum system with $m_1 = 3$ slugs, $m_2 = 1$ slug, and $\ell_1 = \ell_2 = 16$ ft. If $g = 32$ ft/s$^2$, determine $\theta_1(t)$ and $\theta_2(t)$ if $\theta_1(0) = 1$, $\theta_1'(0) = 0$, $\theta_2(0) = 0$, $\theta_2'(0) = -1$. Graph the solution parametrically and simultaneously.

94. Suppose that in the double pendulum $m_1 = 3$ slugs, $m_2 = 1$ slug, and each pendulum has length 4 ft. If $g = 32$ ft/s$^2$, determine $\theta_1(t)$ and $\theta_2(t)$ if $\theta_1(0) = 1$, $\theta_1'(0) = 0$, $\theta_2(0) = 0$, $\theta_2'(0) = -1$. Graph the solution parametrically and simultaneously. How does the length of each pendulum affect the motion as compared to that of Problem 93?

95. Suppose that $m_1 = m_2 = 1$ slug, and each pendulum has length 4 ft in the double-pendulum system. If $g = 32$ ft/s$^2$, determine $\theta_1(t)$ and $\theta_2(t)$ if $\theta_1(0) = 1$, $\theta_1'(0) = 0$, $\theta_2(0) = 0$, $\theta_2'(0) = -1$. Graph the solution parametrically and simultaneously. How does the mass of the first object affect the motion as compared to that of Problem 94?

**96.** Consider the physical situation shown in Figure 8.35, where the uniform bar has mass $m_1$, a centroidal moment of inertia $\bar{I}$, and is supported by two springs each with spring constant $k$. A mass $m_3$ is attached at the center of gravity (Point $G$) of the bar by another spring with spring constant $k$. Using the coordinates $q_1$, $q_2$, and $q_3$, as shown in Figure 8.35, we find that the motion is modeled with

$$\begin{cases} m_1\dfrac{d^2q_1}{dt^2} + 3kq_1 - kq_3 = 0 \\ \bar{I}\dfrac{d^2q_2}{dt^2} + 2k\ell^2 q_2 = 0. \\ m_3\dfrac{d^2q_3}{dt^2} - kq_1 + kq_3 = 0 \end{cases}$$

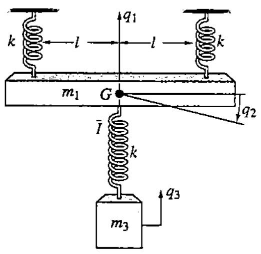



**Figure 8.35** (a) $y(t)$ if $y(0) = -1$ (b) $y(t)$ if $y(0) = 0$

**(a)** Determine the motion of the system if $k = 2$, $m_1 = 2$, $m_3 = 1$, $\bar{I} = 1$, $\ell = 1$ and the initial conditions are $q_1(0) = 0$, $q_1'(0) = 1$, $q_2(0) = 0$, $q_2'(0) = 0$, $q_3(0) = 0$, and $q_3'(0) = 0$. What is the maximum displacement of $q_1$, $q_2$, and $q_3$? **(b)** If the initial conditions are $q_1(0) = 0$, $q_1'(0) = 0$, $q_2(0) = 0$, $q_2'(0) = 0$, $q_3(0) = 1$, and $q_3'(0) = 0$, determine the motion of the system. What is the maximum displacement of $q_1$, $q_2$, and $q_3$? **(c)** How do the changes in the initial conditions affect the maximum displacement of each component of the system?

**97.** Consider the system of three springs shown in Figure 8.36, where springs $S_1$, $S_2$, and $S_3$ have spring constants $k_1$, $k_2$, and $k_3$, respectively, and have objects of mass $m_1$, $m_2$, and $m_3$, respectively, attached to them. In this case, summing the forces acting on each mass and applying Newton's second law of motion yields

$$\begin{cases} m_1\dfrac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1) \\ m_2\dfrac{d^2x_2}{dt^2} = -k_2(x_2 - x_1) + k_3(x_3 - x_2). \\ m_3\dfrac{d^2x_3}{dt^2} = -k_3(x_3 - x_2) \end{cases}$$

Solve this system if $m_1 = 2$, $m_2 = m_3 = 1$, $k_1 = 1/2$, $k_2 = 1$, and $k_3 = 2$, using the initial conditions $x_1(0) = x_2(0) = x_3(0) = 0$, $\dfrac{dx_1}{dt}(0) = 2$, $\dfrac{dx_2}{dt}(0) = 1$, and $\dfrac{dx_3}{dt}(0) = 2$. When does each mass first pass through its equilibrium position?

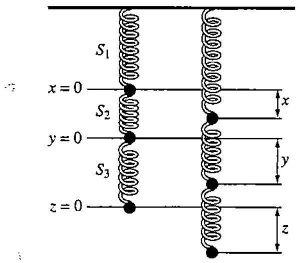



**Figure 8.36**

**98.** Consider the problem

$$y'(t) - 2y(t-1) = t$$

subject to the condition

$$y(t) = y(0) \text{ for } -1 \le t \le 0$$

Laplace transforms can be used to solve equations of this type.

**(a)** Take the Laplace transform of both sides of the equation to obtain

$$sY(s) - y(0) - 2\int_0^\infty e^{-st}y(t-1)\,dt = \frac{1}{s^2}.$$

**(b)** Evaluate $\int_0^\infty e^{-st}y(t-1)\,dt$ by letting $u = t - 1$ to obtain

$$\int_0^\infty e^{-st}y(t-1)\,dt = e^{-s}\left(\int_{-1}^0 e^{-su}y(u)\,du + \int_0^\infty e^{-su}y(u)\,du\right).$$



**(c)** Use the condition $y(t) = y(0)$ for $-1 \le t \le 0$ to simplify the expression in **(b)**. Substitute this result into the expression in **(a)** and solve for $Y(s)$ to find that

$$Y(s) = \frac{-e^s + 2sy(0) - 2e^s sy(0) - e^s s^2 y(0)}{-2s^2 + e^s s^3}.$$

**(d)** Use partial fractions to find that

$$Y(s) = \frac{2(1 + 2sy(0))}{s^3(-2 + e^s s)} + \frac{1 + 2sy(0) + s^2y(0)}{s^3}.$$

**(e)** Find $\mathcal{L}^{-1}\left\{\dfrac{1 + 2sy(0) + s^2y(0)}{s^3}\right\}$.

**(f)** Find $\mathcal{L}^{-1}\left\{\dfrac{2(1 + 2sy(0))}{s^3(-2 + e^s s)}\right\}$ by rewriting this expression as

$$\begin{aligned} \frac{2(1 + 2sy(0))}{s^3(-2 + e^s s)} &= \frac{2(1 + 2sy(0))}{s^4e^s}\,\frac{1}{1 - 2s^{-1}e^{-s}} \\ &= \frac{2(1 + 2sy(0))}{s^4e^s}\sum_{n=0}^\infty (2s^{-1}e^{-s})^n \\ &= 2(1 + 2sy(0))\sum_{n=0}^\infty 2^n\frac{e^{-ns}}{s^{n+4}} \\ &= \sum_{n=0}^\infty \left[2^{n+1}\frac{e^{-ns}}{s^{n+4}} + 2^{n+2}y(0)\frac{e^{-ns}}{s^{n+3}}\right]. \end{aligned}$$

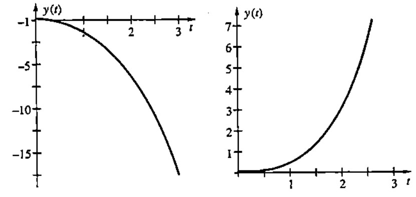
**(g)** Show that

$$\mathcal{L}^{-1}\left\{2^{n+1}\frac{e^{-ns}}{s^{n+4}} + 2^{n+2}y(0)\frac{e^{-ns}}{s^{n+3}}\right\} = \frac{2^{n+1}(t-n)^{n+2}}{(n+3)!}\mathcal{U}(t-n)[t + 6y(0) + (2y(0) - 1)n]$$

so that

$$\mathcal{L}^{-1}\left\{\frac{2(1 + 2sy(0))}{s^3(-2 + e^s s)}\right\} = \sum_{n=0}^\infty \left[\frac{2^{n+1}(t-n)^{n+2}}{(n+3)!}\mathcal{U}(t-n)(t + 6y(0) + (2y(0) - 1)n\right].$$

**(h)** Find $y(t)$. (See the following figure.)




(a) (b)

(a) $y(t)$ if $y(0) = -1$, (b) $y(t)$ if $y(0) = 0$

**99.** The gestation period for whales is about two years. State the (logistic) differential equation that should be used to model the situation in which the decline in population is proportional to the amount present and the increase is due to the amount present two years ago.

**100.** In Figure 8.37, the graph of the approximate solution to the model in Exercise 99 (assuming that $\mu = \nu$ and $y(t) = 0.8$, $-2 \le t < 0$) is shown. How does this result compare to the solution to the logistic equation without including the two-year delay?

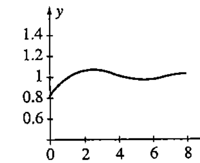



**Figure 8.37**

**101.** The SIR model, which models the spread of a disease, is discussed in Chapter 6 (Differential Equations at Work, Project A). In this model, $S(t)$ represents the number in the population susceptible to the disease, $I(t)$ those infected with the disease, and $R(t)$ those recovered from the disease. In the standard SIR model,

$$\begin{cases} S' = -\lambda SI \\ I' = \lambda SI - \gamma I, \\ R' = \gamma I \end{cases}$$

where $\lambda$ and $\gamma$ are positive constants; we assume that the members of the recovered class remain in that class. However, suppose that the recovered do not receive permanent immunity. Instead, after one unit of time, those who have recovered lose immunity and move into the susceptible class. How should you modify the SIR model to include this delay?

**102.** A typical solution to the standard SIR model is shown in Figure 8.38(a) and that for the SIR model which includes the delay in Figure 8.38(b). How do these solutions differ?

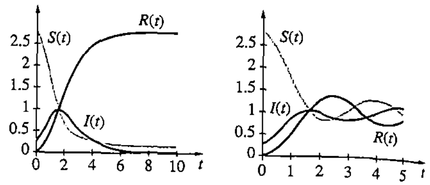



**Figure 8.38** (a)–(b)

## CHAPTER 8 SUMMARY

## Concepts & Formulas

### Section 8.1

**Definition of the Laplace transform**

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$

**Linearity of the Laplace transform**

$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$

**Laplace transform of sin $kt$ and cos $kt$**

**Exponential order**

A function $f$ is of **exponential order (of order $b$)** if there are numbers $b$, $C > 0$, and $T > 0$ such that

$$|f(t)| \le Ce^{bt}$$

for $t > T$.

**Jump discontinuity**

**Piecewise continuous**

**Shifting property**

$\mathcal{L}\{e^{at} f(t)\} = F(s - a)$

**Inverse Laplace transform**

$f(t) = \mathcal{L}^{-1}\{F(s)\}$

### Section 8.2

**Laplace transform of the first derivative**

$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

**Laplace transform of higher derivatives**

$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$

**Derivatives of the Laplace transform**

$\mathcal{L}\{t^n f(t)\} = (-1)^n \dfrac{d^nF}{ds^n}(s)$

**Solving initial-value problems with the Laplace transform**

### Section 8.3

**Unit step function**

$\mathcal{U}(t - a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$

$\mathcal{L}\{f(t - a)\mathcal{U}(t - a)\} = e^{-as}F(s)$

$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t - a)\mathcal{U}(t - a)$

**Periodic function**

$f(t + T) = f(t)$



**Laplace transform of a periodic function**

$\mathcal{L}\{f(t)\} = \dfrac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt$

**Unit impulse function**

$\delta(t - t_0) = 0,\ t \ne t_0$

$\int_{-\infty}^{+\infty} \delta(t - t_0)\, dt = 1$

$\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}$

### Section 8.4

**Convolution theorem**

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)G(s)\} &= \mathcal{L}^{-1}\{\mathcal{L}\{(f * g)(t)\}\} \\ &= (f * g)(t) \\ &= \int_0^t f(t - v)g(v)\, dv.\end{aligned}$$

**Integral equation**

**Integrodifferential equation**

### Section 8.6

***L-R-C* circuits in which the voltage is defined by a piecewise defined function**

**Delay Differential Equations**

**Coupled Spring–Mass System: Two Springs**

$$\begin{cases} m_1 \dfrac{d^2x}{dt^2} = -k_1x + k_2(y - x) + F_1(t) \\ m_2 \dfrac{d^2y}{dt^2} = -k_2(y - x) + F_2(t) \end{cases}$$

**Coupled Spring–Mass System: Three Springs**

$$\begin{cases} m_1 \dfrac{d^2x}{dt^2} = -k_1x + k_2(y - x) + F_1(t) \\ m_2 \dfrac{d^2y}{dt^2} = -k_3y - k_2(y - x) + F_2(t) \end{cases}$$

**The Double Pendulum**

$$\begin{cases} (m_1 + m_2)\ell_1^2\theta_1'' + m_2\ell_1\ell_2\theta_2'' + (m_1 + m_2)\ell_1 g\theta_1 = 0 \\ m_2\ell_2^2\theta_2'' + m_2\ell_1\ell_2\theta_1'' + m_2\ell_2 g\theta_2 = 0 \end{cases}$$

## CHAPTER 8 REVIEW EXERCISES

In Exercises 1–4, find the Laplace transform of each function using the definition.

**1.** $f(t) = 1 - t$

**2.** $f(t) = te^{-4t}$

***3.** $f(t) = \begin{cases} 1, & 0 \le t < 5, \\ 0, & t \ge 5 \end{cases}$

**4.** $f(t) = \begin{cases} t, & 0 \le t < 1, \\ 0, & t \ge 1 \end{cases}$

In Exercises 5–26, find the Laplace transform of each function.

**5.** $t^5 + 5$

**6.** $2 \sinh 4t$

**7.** $te^{2t}$

**8.** $t^3$

**9.** $t^3e^t$

**10.** $2e^t \sin 5t$

**11.** $t \cos 3t$

**12.** $t \sin 2t$

**13.** $e^{-5t} \cos 3t$

**14.** $\delta(t - 2\pi)$

**15.** $\delta(t - 3\pi/2)$

**16.** $7\mathcal{U}(t - 7)$

**17.** $6\mathcal{U}(t - 7) - 4\mathcal{U}(t - 4)$

**18.** $36e^{-4t}\mathcal{U}(t - 2)$

**19.** $-42e^{5t}\mathcal{U}(t - 1)$

**20.** $5 \sin(t - 5)\mathcal{U}(t - 5)$

**21.** $t^2\mathcal{U}(t - 2)$

**22.** $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & 1 \le t < 2 \end{cases}, f(t) = f(t - 2)$ if $t \ge 2$

***23.** $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 2 - t, & 1 \le t < 2 \end{cases}, f(t) = f(t - 2)$ if $t \ge 2$

**24.** $f(t) = \sin \pi t$ if $0 \le t < 1$, $f(t) = f(t - 1)$ if $t \ge 1$

***25.** $f(t) = \begin{cases} \sin \pi t, & \text{if } 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}, f(t) = f(t - 2)$ if $t \ge 2$

**26.** $f(t) = \begin{cases} t, & 0 \le t < 2 \\ 2, & 2 \le t < 4 \\ 6 - t, & 4 \le t < 6 \end{cases}, f(t) = f(t - 6)$ if $t \ge 6$

In Exercises 27–36, find the inverse Laplace transform of each function.

27. $\dfrac{-10}{s^2-25}$ 28. $\dfrac{-2s}{(s^2+1)^2}$

*29. $\dfrac{3(40-s^6)}{s^6}$ 30. $\dfrac{2s^2-7s+20}{s(s^2-2s+10)}$

*31. $\dfrac{-s^2-15s-52}{s(s^2+10s+26)}$ 32. $\dfrac{-14}{se^{2s}}$

33. $\dfrac{7+6e^{4s}}{se^{7s}}$ 34. $\dfrac{-3e^{6-s}}{s-6}$

35. $\dfrac{8s}{e^{3s}(s^2+1)}$ 36. $\dfrac{-18}{(1-e^{-3s})(s^2+1)}$

In Exercises 37 and 38, compute the convolution $(f * g)(t)$ using the given pair of functions.

37. $f(t)=t^2, g(t)=e^{-3t}$

38. $f(t)=\cos t, g(t)=\sin t$

In Exercises 39–48, solve the initial-value problem.

39. $y''+6y'+10y=0, y(0)=0, y'(0)=1$

40. $y''-4y'+5y=0, y(0)=1, y'(0)=0$

41. $y''+5ty'-10y=2, y(0)=1, y'(0)=0$

42. $y''+12y'+32y=f(t), y(0)=0, y'(0)=-1,$

$$f(t)=\begin{cases} t, & \text{if } 0\le t<1 \\ 2-t, & \text{if } 1\le t<2 \end{cases}$$

and $f(t)=f(t-2)$ if $t\ge 2$

*43. $y''+6y'+8y=f(t), y(0)=1, y'(0)=0,$ $f(t)=\begin{cases} 0, & \text{if } 0\le t<1 \\ 1, & \text{if } t\ge 1 \end{cases}$

44. $x''+3x'+2x=\delta(t-\pi)+\delta(t-2\pi),\quad x(0)=0, x'(0)=0$

*45. $x''+9x=\cos t+\delta(t-\pi), x(0)=0, x'(0)=0$

46. $g(t)=5+t-\int_0^t (t-v)g(v)\,dv$

*47. $g(t)=\sin t+\int_0^t g(t-v)e^{-v}\,dv$

48. $\dfrac{dy}{dt}-\int_0^t y(v)\,dv=1+e^{-t}, y(0)=0$

49. Use the Maclaurin series expansion $\tan^{-1}x = x-\dfrac{x^3}{3}+\dfrac{x^5}{5}+\dfrac{x^7}{7}+\cdots$ to find the Maclaurin series expansion for $\tan^{-1}(1/s)$. Use this expansion to show that $\mathcal{L}^{-1}\{\tan^{-1}(1/s)\}=\dfrac{\sin t}{t}$. (*Hint:* Use the Maclaurin series expansion of $\sin x$.)

50. Use $\mathcal{L}\{t^n f(t)\}=(-1)^n\dfrac{d^n}{ds^n}F(s)$ in the form $(n=1)$ $f(t)=-\dfrac{1}{t}\mathcal{L}^{-1}\left\{\dfrac{d}{ds}F(s)\right\}$ to compute **(a)** $\mathcal{L}^{-1}\left\{\ln\dfrac{s-5}{s+2}\right\}$ and **(b)** $\mathcal{L}^{-1}\left\{\ln\dfrac{s^2+4}{s^2+9}\right\}$.

51. Use the shifting property to evaluate **(a)** $\mathcal{L}\{\cosh t\}$; **(b)** $\mathcal{L}\{\sinh t\}$; **(c)** $\mathcal{L}\{\cos t\cosh t\}$; **(d)** $\mathcal{L}\{\sin t\cosh t\}$; **(e)** $\mathcal{L}\{\cos t\sinh t\}$; **(f)** $\mathcal{L}\{\sin t\sinh t\}$.

52. Compute $\mathcal{L}\{e^{at}t^n\}$ two ways by using: **(a)** the shifting (translation) property; **(b)** the formula for the derivatives of the Laplace transform.

In Exercises 53 and 54, solve the initial-value problem $mx''+cx'+kx=f(t), x(0)=a, x'(0)=b$ to determine the motion of an object attached to the end of a spring using the given parameter values.

53. $m=4, c=0, k=1, a=0, b=-1,$ $f(t)=\begin{cases}\cos 2t, & 0\le t<\pi \\ 0, & t\ge\pi\end{cases}$

54. $m=1, c=1/2, k=145/16, a=0, b=0,$ $f(t)=\delta(t-\pi)$

In Exercises 55 and 56, solve the *L-R-C* series circuit modeled by

$$\begin{cases} L\dfrac{d^2Q}{dt^2}+R\dfrac{dQ}{dt}+\dfrac{1}{C}Q=E(t) \\ Q(0)=Q_0, I(0)=\dfrac{dQ}{dt}(0)=I_0 \end{cases}$$

using the given parameter values and functions. (Assume that the units of henry, ohm, farad, and volt are used, respectively, for $L$, $R$, $C$, and $E(t)$.)

55. $L=1, R=0, C=10^{-4},$ $E(t)=\begin{cases}220, & 0\le t<2 \\ 0, & t\ge 2\end{cases}, Q_0=I_0=0$

56. $L=4, R=80, C=0.04,$ $E(t)=\begin{cases}0, & 0\le t<1 \\ 50, & 1\le t<2\end{cases},$ $E(t+2)=E(t), Q_0=I_0=0$

In Exercises 57 and 58, solve the *L-R-C* series circuit for $I(t)$ by interpreting the problem as an integrodifferential equation using the given parameter values and functions. Assume that the units of henry, ohm, farad, and volt are used, respectively, for $L$, $R$, $C$, and $E(t)$.



57. $L=1/4, R=1/2, C=4/9, E(t)=100, I_0=0$

58. $L=1/4, R=3/2, C=4/9, E(t)=100\sin 5t, I_0=0$

In Exercises 59 and 60, interpret the initial-value problem as a population problem. In each case, determine if the population approaches a limit as $t\to\infty$.

59. $x'-2x=100\delta(t-1), x(0)=10{,}000$

60. $x'-2x=\begin{cases}100t, & 0\le t<1 \\ 100, & t\ge 1\end{cases}, x(0)=10{,}000$

61. Suppose that the vertical displacement of a horizontal beam is modeled by the boundary-value problem

$$\frac{d^4y}{dx^4}=W, y(0)=0, y'(0)=0, y(1)=0, y'(1)=0,$$

where $W$ is the constant load that is uniformly distributed along the beam. Use the two conditions given at $x=0$ with the method of Laplace transforms to obtain a solution that involves the arbitrary constants $A=y''(0)$ and $B=y'''(0)$. Then apply the conditions at $x=1$ to find $A$ and $B$.

62. Suppose that the load in Problem 61 is not constant. Instead, it is given by $W(x)=\begin{cases}10, & 0\le x<1/2 \\ 0, & 1/2\le x<1\end{cases}$. Solve

$$\frac{d^4y}{dx^4}=W(x), y(0)=0, y'(0)=0, y(1)=0, y'(1)=0$$

to find the displacement of the beam.

63. **(Filtration Through a Burning Cigarette)** Suppose that a cigarette has original length $X$ and that the tip burns at a rate $v$ so that the cigarette has length $X-vt$ at time $t$. Next let $W(x,t)$ represent the weight per unit length of the cigarette at position $x$ and time $t$, where $x$ is measured from the burning end. Therefore, the weight at the burning tip is $W(vt,t)$. Assume that during steady inhalation, a constant fraction $a$ of any component of the tobacco that is burned at the tip is drawn through the cigarette and that the weight deposited per unit length is $bw_0e^{-bx}$ (where $w_0$ is the weight at $x=0$). During the time period $\Delta t$, a length $v\,\Delta t$ is burned so that $av\,\Delta t W(vt,t)$ passes through the cigarette and $abv\,\Delta t W(vt,t)e^{-b(x-vt)}\,\Delta x$ is deposited between $x$ and $x+\Delta x$. Therefore, the total weight deposited by absorption in time $t$ between $x$ and $x+\Delta x$ is

$$abv\int_0^t W(vu,u)e^{-b(x-vu)}\,du\,\Delta x = w(x,t)\,\Delta x.$$

The total weight is then $w(x,t)\,\Delta x + W(x,0)\,\Delta x$, so that

$$W(x,t)=W(x,0)+abve^{-bx}\int_0^t W(vu,u)e^{bvu}\,du.^*$$

**(a)** Substitute $x=vt$ to show that

$$f(t)=W(vt,0)e^{bvt}+abv\int_0^t f(u)\,du,$$

where $f(t)=W(vt,t)e^{bvt}$.

**(b)** Solve the integral equation in **(a)** if $W(vt,0)=W_0$.

In Exercises 64–68, solve each system.

64. $\begin{cases} y''=-x-2y, x(0)=0, x'(0)=1, \\ x''=-2x-4y \\ y(0)=0, y'(0)=0 \end{cases}$

65. $\begin{cases} y''-3y'=-x'-2y+x, x(0)=0, x'(0)=1, \\ y'+x'=2y-x \\ y(0)=0, y'(0)=-1 \end{cases}$

66. $\begin{cases} x''=-3x+y+\cos 4t, x(0)=0, x'(0)=0, \\ y''=2x-2y \\ y(0)=0, y'(0)=0 \end{cases}$

*67. $\begin{cases} x'=x-2y \\ y'=3y+f(t) \\ x(0)=0, y(0)=0 \end{cases}, f(t)=\begin{cases}1, & 0\le t<2 \\ 0, & t\ge 2\end{cases}$

68. $\begin{cases} x'=-x+\frac{1}{2}y+f(t) \\ y'=-\frac{3}{4}x+\frac{1}{4}y \\ x(0)=0, y(0)=0 \end{cases}, f(t)=\begin{cases}-1, & 0\le t<1 \\ 1, & 1\le t<2 \\ 0, & t\ge 2\end{cases}$

69. Solve the spring-mass system with two springs, as shown in Figure 8.21, using $m_1=m_2=k_1=6$, $k_2=4$, and the initial conditions $x(0)=0, x'(0)=2$, $y(0)=0$, and $y'(0)=0$. (Assume no external forcing functions.)

70. Solve the spring-mass system with two springs, as shown in Figure 8.21, if the forcing functions $F_1(t)=$

* *Applications in Undergraduate Mathematics in Engineering.* Mathematical Association of America, Macmillan, New York (1967), pp. 153–157.

$2\cos t$ and $F_2(t) = 0$ are included with the parameters $m_1 = m_2 = 1$, $k_1 = 3$, $k_2 = 2$ and initial conditions $x(0) = 0$, $x'(0) = 0$, $y(0) = 0$, and $y'(0) = 0$. What physical phenomenon occurs in $y(t)$?

71. Solve the spring–mass system with three springs (as shown in Figure 8.25) if the forcing functions $F_1(t) = 2\cos t$ and $F_2(t) = 0$ are included with the parameters $m_1 = m_2 = 1$, $k_1 = 1$, $k_2 = 1$, $k_3 = 1$ and initial conditions $x(0) = 0$, $x'(0) = 0$, $y(0) = 0$, and $y'(0) = 0$.

*72. Solve the double pendulum problem with $m_1 = 3$, $m_2 = 1$, $\ell_1 = \ell_2 = 16$ ft, $g = 32$ ft/s$^2$, and the initial conditions $\theta_1(0) = 0$, $\theta_1'(0) = 0$, $\theta_2(0) = -1$, $\theta_2'(0) = -1$.

*73. Solve the double pendulum problem with $m_1 = 3$, $m_2 = 1$, $\ell_1 = \ell_2 = 16$ ft, $g = 32$ ft/s$^2$, and the initial conditions $\theta_1(0) = 0$, $\theta_1'(0) = -1$, $\theta_2(0) = 0$, $\theta_2'(0) = 1$.

## Differential Equations at Work:

### A. The Tautochrone

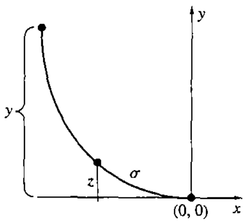



Figure 8.39

From rest, a particle slides down a frictionless curve under the force of gravity as illustrated in Figure 8.39. What must the shape of the curve be for the time of descent to be independent of the starting position of the particle?

The shape of the curve is found through the use of the Laplace transform. (This problem was originally solved in 1673 by Christian Huygens, a Dutch mathematician, as he studied the mathematics associated with pendulum clocks.) Suppose that the particle starts at height $y$ and that its speed is $v$ when it is at a height of $z$. If $m$ is the mass of the particle and $g$ is the acceleration of gravity, then the speed can be found by equating the kinetic and potential energies of the particle with

$$\frac{1}{2}mv^2 = mg(y - z),$$

which can be written as

$$v = \sqrt{2g}\sqrt{y - z}.$$

To avoid confusion with the $s$ that is usually used in the Laplace transform of functions, let $\sigma$ denote the arc length along the curve from its lowest point to the particle. Therefore, the time required for the descent is

$$\text{Time} = \int_0^{\sigma(y)} \frac{d\sigma}{v} = \int_0^y \frac{1}{v}\frac{d\sigma}{dz}\,dz = \int_0^y \frac{1}{v}\phi(z)\,dz$$

where $\phi(y) = d\sigma/dy$, which means that $\phi(z)$ is the value of $d\sigma/dy$ at $y = z$. Now, because the time is constant and $v = \sqrt{2g}\sqrt{y - z}$, we have

$$\int_0^y \frac{\phi(s)}{\sqrt{y - z}}\,dz = c_1,$$



where $c_1$ is a constant. In an attempt to use a convolution, we multiply by $e^{-sy}\,dy$ and integrate. Therefore,

$$\int_0^\infty e^{-sy}\int_0^y \frac{\phi(z)}{\sqrt{y - z}}\,dz\,dy = \int_0^\infty e^{-sy}c_1\,dy$$

$$\mathcal{L}\{\phi * y^{-1/2}\} = \mathcal{L}\{c_1\}.$$

By the Convolution theorem, this simplifies to

$$\mathcal{L}\{\phi\}\mathcal{L}\{y^{-1/2}\} = \frac{c_1}{s}.$$

Then, because $\mathcal{L}\{t^{-1/2}\} = \sqrt{\pi/s}$, we have

$$\mathcal{L}\{\phi\}\sqrt{\frac{\pi}{s}} = \frac{c_1}{s} \quad \text{or} \quad \mathcal{L}\{\phi\} = \frac{c_1}{\sqrt{\pi}}\frac{1}{\sqrt{s}}.$$

Applying the inverse Laplace transform with $\mathcal{L}^{-1}\{s^{-1/2}\} = t^{-1/2}/\sqrt{\pi}$ then yields

$$\phi = \frac{c_1}{\sqrt{\pi}}y^{-1/2} = ky^{-1/2}.$$

Recall that $\phi(y) = d\sigma/dy$ represents arc length. Hence $\phi(y) = d\sigma/dy = \sqrt{1 + (dx/dy)^2}$, so substitution into the previous equation gives

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = ky^{-1/2} \quad \text{or} \quad 1 + \left(\frac{dx}{dy}\right)^2 = \frac{k^2}{y}.$$

Solving for $dx/dy$ then yields $dx/dy = \sqrt{(k^2/y) - 1}$, which can be integrated with the substitution $y = k^2\sin^2\theta$.

1. Find $x(\theta)$ using this substitution.
2. Use the identity $\sin^2\theta = (1 - \cos 2\theta)/2$ to obtain a similar formula for $y(\theta)$.
3. Use a device capable of parametric plotting to graph the curve $(x(\theta), y(\theta))$ on $-\pi/2 \le \theta \le 0$ for $k = 1, 2, 3$. How does the value of $k$ affect the curve?
4. Find the time required for an object located at $(x(-\pi/2), y(-\pi/2))$ to move along the curve to the point $(x(0), y(0))$. Does this result depend on the value of $\theta$?

### B. Vibration Absorbers

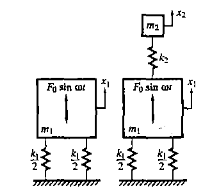


Figure 8.40 The principal system and the vibration absorber attached to the principal system

Vibration absorbers can be used to virtually eliminate vibration in systems in which it is particularly undesirable and to reduce excessive amplitudes of vibration in others. A typical type of vibration absorber consists of a spring–mass system constructed so that its natural frequency is easily varied. This system is then attached to the principal system that is to have its amplitude of vibration reduced, and the frequency of the absorber system is then adjusted until the desired result is achieved. (See Figure 8.40.)

If the frequency $\omega$ of the disturbing force $F_0 \sin \omega t$ is near or equal to the natural frequency $\omega_n = \sqrt{k_1/m_1}$ of the system, the amplitude of vibration of the system could become very large due to resonance. If the absorber spring–mass system, made up of components with $k_2$ and $m_2$, is attached to the principal system, the amplitude of the mass can be reduced to almost zero if the natural frequency of the absorber is adjusted until it equals that of the disturbing force

$$\omega = \sqrt{\frac{k_2}{m_2}}.$$

These types of absorbers are designed to have little damping and are "tuned" by varying $k_2$, $m_2$, or both. This problem is modeled by the system of differential equations

$$\begin{cases} m_1\dfrac{d^2x_1}{dt^2} + (k_1 + k_2)x_1 - k_2x_2 = F_0 \sin \omega t \\ m_2\dfrac{d^2x_2}{dt^2} - k_2x_1 + k_2x_2 = 0 \end{cases}$$

**(a)** Solve the system for $x_1$ and $x_2$.
**(b)** When is the amplitude of $x_1$ equal to zero? What does this represent?

The mass ratio $m_2/m_1$ is an important parameter in the design of the absorber. To see the effect on the response of the system, transform these parameters to the nondimensional form

$$\omega_{22}^2 = \frac{k_2}{m_2} = \frac{k_1}{m_1} \quad \text{and} \quad \mu = \frac{m_2}{m_1} = \frac{k_2}{k_1}.$$

**(c)** If $A_1$ is the amplitude of $x_1$, express $A_1/(F_0/k_1)$ as a function of $\omega/\omega_{22}$.
**(d)** Plot the absolute value of $A_1/(F_0/k_1)$ for $\mu = 0.2$.
**(e)** When does $A_1/(F_0/k_1)$ become infinite? Do these values correspond to resonance?
**(f)** In designing the vibration absorber, what frequencies should the absorber not be "tuned" in order to avoid resonance?

## C. Airplane Wing

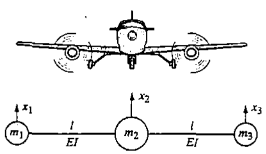



**Figure 8.41** A simple model of an airplane

A small airplane is modeled using three lumped masses as shown in Figure 8.41. We assume in this simplified model of the airplane that the wings are uniform beams of length $\ell$ and stiffness factor $EI$ where $E$ and $I$ are constants. ($E$ depends on the material from which the beam is made and $I$ depends on the shape and size of the beam.) We also assume that $m_1 = m_3$ and $m_2 = 4m_1$, where $m_1$ and $m_3$ represent the mass of each wing and $m_2$ represents the mass of the body of the airplane.

We find the displacements $x_1$, $x_2$, and $x_3$ by solving the system

$$M\begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} + K\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$



where

$$K = \frac{EI}{\ell^3}\begin{pmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{pmatrix}$$

and

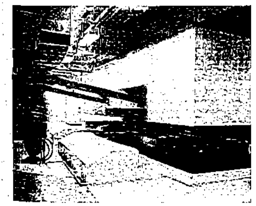
$$M = m_1\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can then write the system as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} + \frac{EI}{m_1\ell^3}\begin{pmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

**(a)** Solve the system $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} + \frac{EI}{m_1\ell^3}\begin{pmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{pmatrix}\begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ subject to the initial conditions $x_1(0) = 0$, $x_1'(0) = 1$, $x_2(0) = 0$, $x_2'(0) = -\frac{1}{2}$, $x_3(0) = 0$, $x_3'(0) = 1$.*
**(b)** Determine the period of $x_1$, $x_2$, and $x_3$ for $c = 0.0001$, 0.01, 0.1, 1, and 2, where $c = \dfrac{EI}{m_1\ell^3}$.
**(c)** Graph $x_1$, $x_2$, and $x_3$ over several periods for $c = 0.0001$, 0.01, 0.1, 1, and 2.
**(d)** Illustrate the motion of the components of the airplane under these conditions.

## D. Free Vibration of a Three-Story Building



The tuned mass damper in the John Hancock Building, Boston (*Courtesy of MTS Systems Corp.*)

If you have ever gone to the top of a tall building, such as the Sears Tower, World Trade Center, or Empire State Building on a windy day you may have been acutely aware of the *sway* of the building. In fact, all buildings sway, or vibrate, naturally. Usually, we are only aware of the sway of a building when we are in a very tall building or in a building during an event such as an earthquake. In some tall buildings, like the John Hancock Building in Boston, the sway of the building during high winds is reduced by installing a tuned mass damper at the top of the building that oscillates at the same frequency as the building but out of phase. We will investigate the sway of a three-story building and then try to determine how we would investigate the sway of a building with more stories.

* M. L. James, G. M. Smith, J. C. Wolford, and P. W. Whaley, *Vibration of Mechanical and Structural Systems*, Harper and Row, New York (1989), pp. 346–349.

We make two assumptions to solve this problem. First, we assume that the mass distribution of the building can be represented by the lumped masses at the different levels. Second, we assume that the girders of the structure are infinitely rigid compared with the supporting columns. With these assumptions, we can determine the motion of the building by interpreting the columns as springs in parallel.

Assume that the coordinates $x_1$, $x_2$, and $x_3$ as well as the velocities and accelerations are positive to the right. If we assume that $x_3 > x_2 > x_1$, the forces that the columns exert on the masses are shown in Figure 8.42.

In applying Newton's second law of motion, recall that we assumed that acceleration is in the *positive* direction. Therefore, we sum forces in the same direction as the acceleration positively, and others negatively. With this configuration, Newton's second law on each of the three masses yields the following system of differential equations:

$$\begin{cases} -k_1x_1 + k_2(x_2 - x_1) = m_1\dfrac{d^2x_1}{dt^2} \\ -k_2(x_2 - x_1) + k_3(x_3 - x_2) = m_2\dfrac{d^2x_2}{dt^2} \\ -k_3(x_3 - x_2) = m_3\dfrac{d^2x_3}{dt^2}, \end{cases}$$

which we write as

$$\begin{cases} m_1\dfrac{d^2x_1}{dt^2} + (k_1 + k_2)x_1 - k_2x_2 = 0 \\ m_2\dfrac{d^2x_2}{dt^2} - k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 = 0 \\ m_3\dfrac{d^2x_3}{dt^2} - k_3x_2 + k_3x_3 = 0, \end{cases}$$

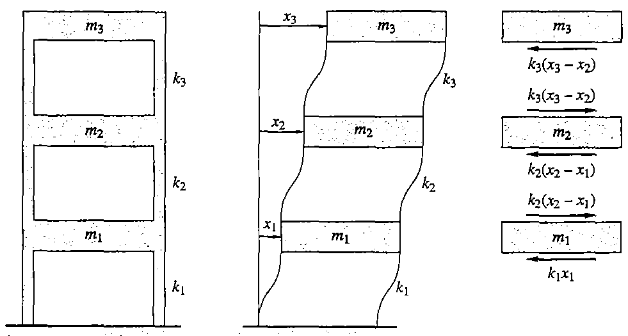



**Figure 8.42** Diagram used to model the sway of a three-story building



where $m_1$, $m_2$, and $m_3$ represent the mass of the building on the first, second, and third levels, and $k_1$, $k_2$, and $k_3$, corresponding to the spring constants, represent the total stiffness of the columns supporting a given floor.*

1. Show that this system can be written in matrix form as
$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}\begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$
The matrix $\begin{pmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{pmatrix}$ is called the **stiffness matrix** of the system.
2. Find a general solution to the system. What can you conclude from your results? Suppose that $m_2 = 2m_1$ and $m_3 = 3m_1$. Then we can write the system in the form
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} + \frac{1}{m_1}\begin{pmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$
3. Find exact and numerical solutions to the system subject to the initial conditions $x_1(0) = 0$, $x_1'(0) = 1/4$, $x_2(0) = 0$, $x_2'(0) = -1/2$, $x_3(0) = 0$, and $x_3'(0) = 1$ if $k_1 = 3$, $k_2 = 2$, and $k_3 = 1$ for $m_1 = 1$, 10, 100, and 1000.
4. Determine the period of $x_1$, $x_2$, and $x_3$ for $m_1 = 1$, 10, 100, and 1000.
5. Graph $x_1$, $x_2$, and $x_3$ over several periods for $m_1 = 1$, 10, 100, and 1000.
6. Illustrate the motion of the building under these conditions.
7. How does the system change if we consider a 5-story building? a 50-story building? a 100-story building?

## E. Control Systems†

Consider the spring–mass system or circuit described by the IVP

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t),\ x(0) = x'(0) = 0.$$

Many times the constant coefficients $a$, $b$, and $c$ are unknown, and it is the job of the engineer to determine the response (solution) to the input $f(t)$.

* M. L. James, G. M. Smith, J. C. Wolford, and P. W. Whaley, *Vibration of Mechanical and Structural Systems with Microcomputer Applications*, Harper & Row, New York (1989), pp. 282–286. Robert K. Vierck, *Vibration Analysis*, Second Edition, Harper Collins, New York (1979), pp. 266–290.

† Joseph J. DiStephano, Allen R. Stubberud, and Ivan J. Williams, *Feedback and Control Systems*, Schaum's Outline Series, New York (1967).

1. Show that $X(s) = F(s)/(as^2 + bs + c)$. Let $P(s) = 1/(as^2 + bs + c)$. This function is known as the **transfer function** of the system and $p(t) = \mathcal{L}^{-1}\{P(s)\}$ is called the weight function of the system.
2. Use the Convolution theorem to show that the solution of $X(s) = P(s)F(s)$ is $x(t) = \int_0^t p(v)f(t-v)\,dv$, called **Duhamel's principle** for the system.
3. Find the response to the spring–mass system
$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = f(t),\ x(0) = x'(0) = 0.$$
4. Using the fact that $\mathcal{L}\{\delta(t)\} = 1$, show that $P(s) = \mathcal{L}\{\delta(t)\}/(as^2 + bs + c)$. Therefore, the solution to
$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = \delta(t),\ x(0) = x'(0) = 0$$
is $p(t) = \mathcal{L}^{-1}\{P(s)\}$, called the **unit impulse response.**
5. Let $h(t)$ be the solution to $a\,d^2x/dt^2 + b\,dx/dt + cx = \mathcal{U}(t)$. Show that $H(s) = P(s)/s$, so that by the Convolution theorem, $h(t) = \int_0^t p(v)\,dv$. Therefore, $h'(t) = p(t)$. Use Duhamel's principle to show that
$$x(t) = \int_0^t h'(v)f(t-v)\,dv.$$
6. Suppose that we differentiate the equation $L\,d^2Q/dt^2 + R\,dQ/dt + (1/C)Q = E(t)$ with respect to $t$. Use the relationship $dQ/dt = I$ to obtain $L\,d^2I/dt^2 + R\,dI/dt + (1/C)I = E'(t)$. If we do not know the values of $L$, $R$, and $C$, we may select the voltage source $E(t) = t$ to determine the response, as in 5. In this case, show that $I(t) = \int_0^t h'(v)E'(t-v)\,dv = \int_0^t h'(v)\,dv$, so we can use an ammeter to measure the response, $h(t)$, when $E'(t) = 1$. Suppose that the readings on the ammeter given as the ordered pairs $(t, h(t))$ are: (0, 3.), (0.299, 3.07), (0.598, 2.22), (0.898, 0.896), (1.2, −0.337), (1.5, −0.972), (1.8, −0.753), (2.09, 0.232), (2.39, 1.58), (2.69, 2.74), (2.99, 3.16), (3.29, 2.57), (3.59, 1.), (3.89, −1.14), (4.19, −3.23), (4.49, −4.65), (4.79, −4.96), (5.09, −4.06), (5.39, −2.23), (5.68, −0.0306), (5.98, 1.89), (6.28, 3.), (6.58, 3.07), (6.88, 2.22), (7.18, 0.896), (7.48, −0.337), (7.78, −0.972), (8.08, −0.753), (8.38, 0.232), (8.68, 1.58), (8.98, 2.74), (9.28, 3.16), (9.57, 2.57), (9.87, 1.), (10.2, −1.14), (10.5, −3.23), (10.8, −4.65), (11.1, −4.96), (11.4, −4.06), (11.7, −2.23), (12, −0.0306), (12.3, 1.89).

Fit this data with a trigonometric function to determine $h(t)$ to use $h'(t)$ to find $I(t)$ if
$$E(t) = \frac{1}{10}\sin 10t.$$

# 9

# Eigenvalue Problems and Fourier Series

In previous chapters we have seen that many physical situations can be modeled by either ordinary differential equations or systems of ordinary differential equations. However, to understand the motion of a string at a particular location and at a particular time, the temperature in a thin wire at a particular location and a particular time, or the electrostatic potential at a point on a plate, we must solve the partial differential equations listed in Table 9.1 because each of these quantities depends on (at least) two variables.

**TABLE 9.1**

| | |
|---|---|
| Wave equation | $c^2u_{xx} = u_{tt}$ |
| Heat equation | $u_t = c^2u_{xx}$ |
| Laplace's equation | $u_{xx} + u_{yy} = 0$ |

In Chapter 10, we introduce a particular method for solving these partial differential equations (as well as others). To carry out this method, however, we introduce the necessary tools in this chapter. We begin with a discussion of boundary-value problems and their solutions.

## 9.1 Boundary-Value Problems, Eigenvalue Problems, Sturm–Liouville Problems

- Boundary-Value Problems
- Eigenvalue Problems
- Sturm–Liouville Problems

### Boundary-Value Problems

Boundary-value problems are solved in much the same way as initial-value problems, except that the value of the function and its derivatives are given at two values of the independent variable instead of one. A general form of a second-order (two-point) boundary-value problem is

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x),\ a < x < b$$

$$k_1 y(a) + k_2 y'(a) = \alpha,\ h_1 y(b) + h_2 y'(b) = \beta,$$

where $k_1$, $k_2$, $\alpha$, $h_1$, $h_2$, and $\beta$ are constants and at least one of $k_1$, $k_2$ and at least one of $h_1$, $h_2$ is not zero. Note that if $\alpha = \beta = 0$, then we say the problem has **homogeneous boundary conditions.** We also consider boundary-value problems that include a parameter in the differential equation. We solve these problems, called **eigenvalue problems,** to investigate several useful properties associated with their solutions.

**Example 1**

Solve $\begin{cases} y'' + y = 0, & 0 < x < \pi \\ y'(0) = 0, & y'(\pi) = 0 \end{cases}$

***Solution*** Because the characteristic equation is $r^2 + 1 = 0$ with roots $r = \pm i$, a general solution is $y(x) = c_1 \cos x + c_2 \sin x$, where $y'(x) = -c_1 \sin x + c_2 \cos x$. Applying the boundary conditions, we have $y'(0) = c_2 = 0$. Then $y(x) = c_1 \cos x$. With this solution, we have $y'(\pi) = -c_1 \sin \pi = 0$ for any value of $c_1$. Therefore, there are infinitely many solutions, $y(x) = c_1 \cos x$, of the boundary-value problem, depending on the choice of $c_1$.

From the result of Example 1, we notice a difference between *initial-value problems* and *boundary-value problems.* This difference is that an initial-value problem (that meets the hypotheses of the Existence and Uniqueness theorem) has a unique solution, whereas a boundary-value problem may have more than one solution (or may have no solution).

**Example 2**

Solve $\begin{cases} y'' + y = 0, & 0 < x < \pi \\ y'(0) = 0, & y'(\pi) = 1 \end{cases}$.



***Solution*** Using the general solution obtained in the previous example, we have $y(x) = c_1 \cos x + c_2 \sin x$. As before, $y'(0) = c_2 = 0$, so $y(x) = c_1 \cos x$. However, because $y'(\pi) = -c_1 \sin \pi = 0 \neq 1$, the boundary conditions cannot be satisfied with any choice of $c_1$. Therefore, there is *no solution* to the boundary-value problem.

*If the boundary conditions in Example 2 are $y'(0) = 0$ and $y(\pi) = 0$, then how many solutions does the problem have?*

As indicated in the general form of a boundary-value problem, the boundary conditions in these problems can involve the function and its derivative. However, this modification to the problem does not affect the method of solution, as shown in Example 3.

**Example 3**

Solve $\begin{cases} y'' - y = 0, & 0 < x < 1 \\ y'(0) + 3y(0) = 0, & y'(1) + y(1) = 1 \end{cases}$.

***Solution*** The characteristic equation is $r^2 - 1 = 0$, with roots $r = \pm 1$. Hence a general solution is $y(x) = c_1 e^x + c_2 e^{-x}$, with derivative $y'(x) = c_1 e^x - c_2 e^{-x}$. Applying $y'(0) + 3y(0) = 0$ yields $y'(0) + 3y(0) = c_1 - c_2 + 3(c_1 + c_2) = 4c_1 + 2c_2 = 0$. Next

$$y'(1) + y(1) = c_1 e^1 - c_2 e^{-1} + c_1 e^1 + c_2 e^{-1} = 2c_1 e = 1.$$

so $c_1 = 1/(2e)$ and $c_2 = -1/e$. Thus the boundary-value problem has the unique solution

$$y(x) = \frac{1}{2e} e^x - \frac{1}{e} e^{-x} = \frac{1}{2} e^{x-1} - e^{-x-1}.$$

*When a second-order homogeneous differential equation with constant coefficients is accompanied by a pair of homogeneous initial conditions ($y(x_0) = y'(x_0) = 0$), then the initial-value problem has the unique solution $y = 0$. Is the same true of a boundary-value problem with homogeneous boundary conditions? (Hint: See Example 1.)*

### Eigenvalue Problems

We now consider **eigenvalue problems,** boundary-value problems that include a parameter in the differential equation. Values of the parameter for which the boundary-value problem has a nontrivial solution are called **eigenvalues** of the problem. For each eigenvalue, the *nontrivial* solution that satisfies the problem is called the corresponding **eigenfunction**. (Notice that if a value of the parameter leads to the *trivial* solution, then the value is *not* considered an eigenvalue of the problem.) We show how the eigenvalues and eigenfunctions are found in the following example.

## Example 4

Solve $y'' + \lambda y = 0$, $0 < x < p$, subject to $y(0) = 0$ and $y(p) = 0$.

**Solution** We recall that the solution of this equation is determined by solving the characteristic equation $r^2 + \lambda = 0$. Of course, the values of $r$ depend on the value of the parameter $\lambda$. Hence we consider the three cases below.

**Case I:** $\lambda = 0$

In this case, the characteristic equation is $r^2 = 0$, so the repeated roots are $r_1 = r_2 = 0$. This indicates that a general solution is

$$y(x) = c_1 x + c_2.$$

Applying the boundary condition $y(0) = 0$ yields $y(0) = c_1(0) + c_2 = 0$, so $c_2 = 0$. Then $y(p) = c_1 p = 0$, so $c_1 = 0$. Therefore, $y(x) = 0$ in this case. Because we obtain the trivial solution, $\lambda = 0$ is *not* an eigenvalue.

**Case II:** $\lambda < 0$

To represent $\lambda$ as a negative value, we let $\lambda = -k^2 < 0$. Then the characteristic equation is $r^2 - k^2 = 0$, so $r_1 = k$ and $r_2 = -k$. A general solution is, therefore,

$$y(x) = c_1 e^{kx} + c_2 e^{-kx},$$

or $y(x) = c_1 \cosh kx + c_2 \sinh kx$. Substituting the boundary condition $y(0) = 0$ yields $y(0) = c_1 + c_2 = 0$, so $c_2 = -c_1$. Because $y(p) = 0$ indicates that $y(p) = c_1 e^{kp} + c_2 e^{-kp} = 0$, substitution gives us the equation $y(p) = c_1 e^{kp} - c_1 e^{-kp} = c_1(e^{kp} - e^{-kp}) = 0$. Notice that $e^{kp} - e^{-kp} = 0$ only if $e^{kp} = e^{-kp}$, which can only occur when $k = 0$ or $p = 0$. If $k = 0$, then $\lambda = -k^2 = -0^2 = 0$, which contradicts the assumption that $\lambda < 0$. We also assumed that $p > 0$, so $e^{kp} - e^{-kp} \neq 0$. Hence $y(p) = c_1(e^{kp} - e^{-kp}) = 0$ implies that $c_1 = 0$, so $c_2 = -c_1 = 0$ as well. Because $\lambda < 0$ leads to the trivial solution, $y(x) = 0$, there are no negative eigenvalues.

**Case III:** $\lambda > 0$

To represent $\lambda$ as a positive value, we let $\lambda = k^2 > 0$. Then we have the characteristic equation $r^2 + k^2 = 0$ with complex conjugate roots $r_{1,2} = \pm ki$. Thus a general solution is

$$y(x) = c_1 \cos kx + c_2 \sin kx.$$

Because $y(0) = c_1 \cos k(0) + c_2 \sin k(0) = c_1$, the boundary condition $y(0) = 0$ indicates that $c_1 = 0$. Hence $y(x) = c_2 \sin kx$. Application of $y(p) = 0$ yields $y(p) = c_2 \sin kp = 0$, so either $c_2 = 0$ or $\sin kp = 0$. Selecting $c_2 = 0$ leads to the trivial solution, which we want to avoid, so we determine the values of $k$ that satisfy $\sin kp = 0$. Because $\sin n\pi = 0$ for integer values of $n$, $\sin kp = 0$ if $kp = n\pi$, $n = 1, 2, \ldots$. Solving for $k$, we have $k = k_n = n\pi/p$, so the eigenvalues are



$$\lambda = \lambda_n = k_n^2 = \left(\frac{n\pi}{p}\right)^2, \; n = 1, 2, \ldots$$

Notice that the subscript $n$ is used to indicate that the parameter depends on the value of $n$. (Notice also that we omit $n = 0$, because the value $k = 0$ was considered in Case I.) For each eigenvalue, the corresponding eigenfunction is obtained by substitution into $y(x) = c_2 \sin kx$. Because $c_2$ is arbitrary, let $c_2 = 1$. Therefore, the eigenvalue $\lambda_n = (n\pi/p)^2$, $n = 1, 2, \ldots$ has the corresponding eigenfunction

$$y(x) = y_n(x) = \sin\left(\frac{n\pi x}{p}\right), \; n = 1, 2, \ldots$$

Notice that we did not consider negative values of $n$. This is due to the fact that $\sin(-n\pi x/p) = -\sin(n\pi x/p)$. Because the negative sign can be taken account for in the constant, we do not obtain additional eigenvalues or eigenfunctions by using $n = -1, -2, \ldots$.

To better understand the results of the previous example, consider the boundary-value problem

$$\begin{cases} y'' + \left(\dfrac{5\pi}{p}\right)^2 y = 0 \\ y(0) = 0, \; y(p) = 0 \end{cases}$$

Using the eigenvalues and eigenfunctions obtained in Example 4, we know that this problem has the solution $y(x) = \sin(5\pi x/p)$, whereas problems like

$$\begin{cases} y'' - \left(\dfrac{5\pi}{p}\right)^2 y = 0 \\ y(0) = 0, \; y(p) = 0 \end{cases} \quad \text{and} \quad \begin{cases} y'' = 0 \\ y(0) = 0, \; y(p) = 0 \end{cases}$$

have no nontrivial solution because $\lambda = -(5\pi/p)^2 < 0$ and $\lambda = 0$, respectively, in the differential equation, and the boundary conditions are the same as those considered in Example 4.

We will find the eigenvalues and eigenfunctions in the previous example quite useful in future sections. The following eigenvalue problem will be useful as well.

## Example 5

Solve $y'' + \lambda y = 0$, $0 < x < p$, subject to $y'(0) = 0$ and $y'(p) = 0$.

**Solution** Notice that the only difference between this problem and that in the previous example appears in the boundary conditions. Again, the characteristic equation is $r^2 + \lambda = 0$, so we must consider the three cases $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$. Note that a general solution in each case is the same as that obtained in the previous example. However, the final results may differ due to the boundary conditions.

**Case I:** $\lambda = 0$

In this case $y(x) = c_1x + c_2$, so $y'(x) = c_1$. Therefore, $y'(0) = c_1 = 0$, so $y(x) = c_2$. Notice that this constant function satisfies $y'(p) = 0$ for all values of $c_2$. If we let $c_2 = 1$, then $\lambda = 0$ is an eigenvalue with corresponding eigenfunction $y(x) = y_0(x) = 1$.

**Case II:** $\lambda < 0$

If $\lambda = -k^2 < 0$, then $y(x) = c_1e^{kx} + c_2e^{-kx}$ and $y'(x) = c_1ke^{kx} - c_2ke^{-kx}$. Then $y'(0) = c_1k - c_2k = 0$, so $c_1 = c_2$. Therefore, $y'(p) = c_1ke^{kp} - c_1ke^{-kp} = 0$, which is not possible unless $c_1 = 0$ because $k \neq 0$ and $p \neq 0$. Thus $c_1 = c_2 = 0$, so $y(x) = 0$, and there are no negative eignvalues.

**Case III:** $\lambda > 0$

By letting $\lambda = k^2 > 0$, $y(x) = c_1 \cos kx + c_2 \sin kx$ and $y'(x) = -c_1k \sin kx + c_2k \cos kx$. Hence, $y'(0) = c_2k = 0$, so $c_2 = 0$. Consequently, $y'(p) = -c_1k \sin kp = 0$, which is satisfied if $kp = n\pi$, $n = 1, 2, \ldots$. Therefore, the eigenvalues are

$$\lambda = \lambda_n = \left(\frac{n\pi}{p}\right)^2, n = 1, 2, \ldots.$$

Note that we found $c_2 = 0$ in $y(x) = c_1 \cos kx + c_2 \sin kx$, so the corresponding eigenfunctions are

$$y(x) = y_n(x) = \cos\frac{n\pi x}{p}, n = 1, 2, \ldots,$$

if we let $c_1 = 1$.

Summarizing our results, $y'' + \lambda y = 0$, $0 < x < p$, subject to $y'(0) = 0$, and $y'(p) = 0$ has the following eigenvalues and corresponding eigenfunctions:

$$\lambda_n = \begin{cases} 0, & n = 0 \\ \left(\frac{n\pi}{p}\right)^2, & n = 1, 2, \ldots \end{cases} \quad \text{and} \quad y_n(x) = \begin{cases} 1, & n = 0 \\ \cos\frac{n\pi x}{p}, & n = 1, 2, \ldots \end{cases}$$

**Example 6**

Solve $y'' + 2y' - (\lambda - 1)y = 0$ subject to $y(0) = 0$ and $y(2) = 0$.

***Solution*** In this case, the characteristic equation is $r^2 + 2r - (\lambda - 1) = 0$, with roots

$$r = \frac{-2 \pm \sqrt{4 + 4(\lambda - 1)}}{2} = \frac{-2 \pm \sqrt{4\lambda}}{2},$$

so again we must consider the three cases $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.



**Case I:** $\lambda = 0$

The repeated roots of the characteristic equation are $r = (-2 \pm \sqrt{0})/2 = -1$, so a general solution is

$$y(x) = c_1e^{-x} + c_2xe^{-x}.$$

Substitution of $y(0) = 0$ yields $y(0) = c_1 = 0$. Hence $y(x) = c_2xe^{-x}$. Applying $y(2) = 0$ implies that $y(2) = 2c_2e^{-2} = 0$, so $c_2 = 0$. Therefore, the assumption that $\lambda = 0$ leads to the trivial solution, and thus $\lambda = 0$ is not an eigenvalue.

**Case II:** $\lambda > 0$

Let $\lambda = k^2 > 0$. Then

$$r = \frac{-2 \pm \sqrt{4k^2}}{2} = \frac{-2 \pm 2k}{2} = -1 \pm k,$$

so a general solution is

$$y(x) = c_1e^{(-1+k)x} + c_2e^{(-1-k)x}.$$

Then $y(0) = c_1 + c_2 = 0$, so $c_2 = -c_1$. Also,

$$y(2) = c_1e^{2(-1+k)} + c_2e^{2(-1-k)} = c_1e^{-2}(e^{2k} - e^{-2k}) = 0, \text{ so } c_1 = c_2 = 0.$$

(Why?) Because we obtain the trivial solution, there are no positive eigenvalues.

**Case III:** $\lambda < 0$

Let $\lambda = -k^2 < 0$. Hence

$$r = \frac{-2 \pm \sqrt{-4k^2}}{2} = \frac{-2 \pm 2ki}{2} = -1 \pm ki,$$

and a general solution is

$$y(x) = c_1e^{-x} \cos kx + c_2e^{-x} \sin kx.$$

Then $y(0) = c_1 = 0$, so $y(x) = c_2e^{-x} \sin kx$. Applying $y(2) = 0$, we have $y(2) = c_2e^{-2} \sin 2k = 0$. Therefore, to avoid the trivial solution, we must require that $2k = n\pi$, $n = 1, 2, \ldots$. Hence $k = k_n = n\pi/2$, $n = 1, 2, \ldots$, and the eigenvalues are

$$\lambda = \lambda_n = -k^2 = -\left(\frac{n\pi}{2}\right)^2, n = 1, 2, \ldots$$

By letting $c_2 = 1$, the eigenfunctions are

$$y(x) = y_n(x) = e^{-x} \sin\frac{n\pi x}{2}.$$

## Sturm–Liouville Problems

Because of the importance of eigenvalue problems, we express these problems in the general form

$$a_2(x)y''(x) + a_1(x)y'(x) + [a_0(x) + \lambda]y(x) = 0,\ a < x < b,$$

where $a_2(x) \neq 0$ on $[a, b]$, and the boundary conditions at the endpoints $x = a$ and $x = b$ can be written as

$$k_1 y(a) + k_2 y'(a) = 0 \quad \text{and} \quad h_1 y(b) + h_2 y'(b) = 0$$

for the constants $k_1$, $k_2$, $h_1$, and $h_2$, where at least one of $k_1$, $k_2$ and one of $h_1$, $h_2$ is not zero. This equation can be rewritten by letting

$$p(x) = e^{\int \frac{a_1(x)}{a_2(x)}dx},\ q(x) = \frac{a_0(x)}{a_2(x)}p(x), \quad \text{and} \quad s(x) = \frac{p(x)}{a_2(x)}.$$

By making this change, we obtain the equivalent equation

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + (q(x) + \lambda s(x))y = 0,$$

which is called a **Sturm–Liouville equation,** which along with boundary conditions is called a **Sturm–Liouville problem.** This particular form of the equation is known as the **self-adjoint** form and is of particular interest because of the relationship of the function $s(x)$ and the solutions of the problem.

**Example 7**

Place the equation $x^2y'' + 2xy' + \lambda y = 0$ in self-adjoint form.

**Solution** In this case, $a_2(x) = x^2$, $a_1(x) = 2x$, and $a_0(x) = 0$. Hence

$p(x) = e^{\int \frac{a_1(x)}{a_2(x)}dx} = e^{\int 2x/x^2\,dx} = e^{2\ln x} = x^2,\ x > 0,\ q(x) = \dfrac{a_0(x)}{a_2(x)}p(x) = 0$, and

$s(x) = p(x)/a_2(x) = x^2/x^2 = 1,$

so the self-adjoint form of the equation is

$$\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) + \lambda y = 0.$$

We see that our result is correct by differentiating.

Solutions of Sturm–Liouville problems have several interesting properties, two of which are included in Theorem 9.1.



**Theorem 9.1 Linear Independence and Orthogonality of Eigenfunctions**

If $y_m(x)$ and $y_n(x)$ are eigenfunctions of the regular Sturm–Liouville problem, where $m \neq n$, then $y_m(x)$ and $y_n(x)$ are **linearly independent** and the **orthogonality condition** $\int_a^b s(x)y_m(x)y_n(x)\,dx = 0$ $(m \neq n)$ holds.

Notice that the limits of integration on the orthogonality condition are obtained from the interval, $a < x < b$, on which the eigenvalue problem is defined.

Because we integrate the product of the eigenfunctions with the function $s(x)$ in the orthogonality condition, we call $s(x)$ the **weighting function.**

**Example 8**

Consider the eigenvalue problem $y'' + \lambda y = 0$, subject to $y(0) = 0$ and $y(p) = 0$ which we solved in Example 4. Verify that the eigenfunctions $y_1(x) = \sin(\pi x/p)$ and $y_2(x) = \sin(2\pi x/p)$ are linearly independent. Also, verify the orthogonality condition.

**Solution** We can verify that $y_1(x) = \sin(\pi x/p)$ and $y_2(x) = \sin(2\pi x/p)$ are linearly independent by finding the Wronskian.

$$W(y_1, y_2) = \begin{vmatrix} \sin\dfrac{\pi x}{p} & \sin\dfrac{2\pi x}{p} \\ \dfrac{\pi}{p}\cos\dfrac{\pi x}{p} & \dfrac{2\pi}{p}\cos\dfrac{2\pi x}{p} \end{vmatrix}$$

$$= \frac{2\pi}{p}\sin\frac{\pi x}{p}\cos\frac{2\pi x}{p} - \frac{\pi}{p}\cos\frac{\pi x}{p}\sin\frac{2\pi x}{p}.$$

Notice that if $x = p/2$,

$$W(y_1, y_2) = \frac{2\pi}{p}\sin\frac{\pi}{2}\cos\pi - \frac{\pi}{p}\cos\frac{\pi}{2}\sin\pi = -\frac{2\pi}{p} \neq 0.$$

Because $W(y_1, y_2)$ is not identically zero, the two functions are linearly independent. (We could have also determined that the two are linearly independent, because they are not scalar multiples of one another.) The equation $y'' + \lambda y = 0$ is in self-adjoint form with $s(x) = 1$. (Why?) Hence the orthogonality condition is

$\int_0^p y_m(x)y_n(x)\,dx = 0$. We verify this with $y_1(x)$ and $y_2(x)$:

$$\int_0^p y_1(x)y_2(x)\,dx = \int_0^p \sin\frac{\pi x}{p}\sin\frac{2\pi x}{p}\,dx = \int_0^p 2\sin^2\frac{\pi x}{p}\cos\frac{\pi x}{p}\,dx$$

$$= \frac{2p}{3\pi}\left[\sin^3\frac{\pi x}{p}\right]_0^p = 0.$$

## EXERCISES 9.1

In Exercises 1–14, solve the given boundary-value problem. In each case, state the number of solutions.

1. $y'' = 0, y(0) = 0, y(2) = 0$
2. $y'' = 0, y(0) = 0, y'(1) = -2$
*3. $y'' = 0, y'(0) = 0, y'(1) = 1$
4. $y'' = 0, y'(0) = 0, y'(1) = 0$
5. $y'' - 4y = 0, y(0) = 0, y(1) = e$
6. $y'' + y' - 2y = 0, y(0) = 0, y(2) = 0$
*7. $y'' + 9y = 0, y'(0) = 0, y(\pi) = 0$
8. $y'' + 4y = 0, y(0) = 0, y(\pi) = 0$
9. $y'' + y = 0, y(0) = 0, y'\left(\frac{\pi}{2}\right) = 0$
10. $y'' + y = 0, y(0) = 2, y(\pi) = -2$
*11. $y'' + 4\pi^2 y = 0, y(0) + y'(0) = 0, y(1) = 2\pi$
12. $y'' + 36y = 0, y(0) = 0, y(\pi) - y'(\pi) = 0$
13. $y'' + y = 0, y(0) - y'(0) = 0, y(\pi) + y'(\pi) = 0$
14. $y'' - 9y = 0, y(0) = 0, y(1) + y'(1) = 1$

Determine the eigenvalues and eigenfunctions of the following eigenvalue problems.

15. $y'' + \lambda y = 0, y'(0) = 0, y'(1) = 0$
16. $y'' + \lambda y = 0, y'(0) = 0, y(1) = 0$
*17. $y'' + \lambda y = 0, y(0) = 0, y'(1) = 0$
18. $y'' + y' + \lambda y = 0, y(0) = 0, y(1) = 0$
19. $y'' + y' + \lambda y = 0, y(0) = 0, y'(1) = 0$
20. $2y'' + 2y' + \lambda y = 0, y(0) = 0, y(1) = 0$
*21. $2y'' + 2y' + \lambda y = 0, y(0) = 0, y'(1) = 0$
22. $y'' + 2y' + (1 - \lambda)y = 0, y(0) = 0, y(2) = 0$
23. $y'' - 4y' + 2\lambda y = 0, y(0) = 0, y(1) = 0$
24. $y'' - 4y' + 2\lambda y = 0, y(0) = 0, y'(1) = 0$

In Exercises 25–29, verify the orthogonality condition that must be satisfied by the eigenfunctions of the given exercise.

25. $y'' + \lambda y = 0, y'(0) = 0, y'(1) = 0$
26. $y'' + \lambda y = 0, y'(0) = 0, y(1) = 0$
*27. $y'' + \lambda y = 0, y(0) = 0, y'(1) = 0$
28. $y'' + y' + \lambda y = 0, y(0) = 0, y(1) = 0$
29. $y'' - 4y' + 2\lambda y = 0, y(0) = 0, y(1) = 0$
30. Orthogonality conditions can be established using other methods than those discussed previously. Use Legendre's equation in self-adjoint form

$$\frac{d}{dx}[(1 - x^2)y'] + n(n + 1)y = 0$$

and the fact that two Legendre polynomials must satisfy the relationships

$$\frac{d}{dx}[(1 - x^2)P_n'(x)] + n(n + 1)P_n(x) = 0 \quad \text{and}$$

$$\frac{d}{dx}[(1 - x^2)P_m'(x)] + m(m + 1)P_m(x) = 0$$

to establish the orthogonality condition for the Legendre polynomials. [*Hint:* Multiply the first relationship by $P_m(x)$ and the second by $P_n(x)$, subtract the expressions that result, and then integrate over the interval $(-1, 1)$].

Let $\langle f, g\rangle$ denote the *inner product* of $f(x)$ and $g(x)$ with respect to the weight function $s(x)$ on the interval $(a, b)$, in other words, $\langle f, g\rangle = \int_a^b s(x)f(x)g(x)\,dx$. In Exercises 31–33, verify the indicated property of the inner product.

31. $\langle f, g\rangle = \langle g, f\rangle$
32. $\langle f, g + h\rangle = \langle f, g\rangle + \langle f, h\rangle$
33. $\langle cf, g\rangle = c\langle f, g\rangle$, where $c$ is any real constant.
34. The **norm** of the eigenfunction $y_n(x)$, with respect to the weighting function $s(x)$, is $\|y_n(x)\| = \sqrt{\langle y_n, y_n\rangle} = \sqrt{\int_a^b s(x)[y_n(x)]^2\,dx}$. Calculate the norm of each of the following eigenfunctions.
   (a) $y'' + \lambda y = 0, y(0) = 0, y(p) = 0, y_n(x) = \sin\frac{n\pi x}{p}, n = 1, 2, \ldots$
   (b) $y'' + \lambda y = 0, y'(0) = 0, y'(p) = 0, y_0(x) = 1, y_n(x) = \cos\frac{n\pi x}{p}, n = 1, 2, \ldots$
   (c) $y'' + 2y' - (\lambda - 1)y = 0, y(0) = 0, y(2) = 0, y(x) = y_n(x) = e^{-x}\sin\frac{n\pi x}{2}, n = 1, 2, \ldots$



35. Show that the eigenvalues and eigenfunctions of $\begin{cases} y'' + \lambda y = 0, & -p < x < p \\ y(-p) = y(p), & y'(-p) = y'(p) \end{cases}$ are
$$\begin{cases} \lambda_0 = 0, y_0(x) = 1 \\ \lambda_n = \left(\frac{n\pi}{p}\right)^2, y_n(x) = a_n\cos\frac{n\pi x}{p} + b_n\sin\frac{n\pi x}{p}. \end{cases}$$
(The eigenfunctions of this eigenvalue problem will be very useful later in this chapter and in Chapter 10.)
36. Consider the general form of a homogeneous boundary-value problem
$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0,\ a < x < b,$$
$$k_1y(a) + k_2y'(a) = 0 \text{ and } h_1y(b) + h_2y'(b) = 0.$$
It can be shown that if $y_1(x)$ and $y_2(x)$ are linearly independent solutions of $a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0$ such that
$$\begin{vmatrix} k_1y_1(a) + k_2y_1'(a) & k_1y_2(a) + k_2y_2'(a) \\ h_1y_1(b) + h_2y_1'(b) & h_1y_2(b) + h_2y_2'(b) \end{vmatrix} \neq 0,$$
then the only solution to the boundary-value problem is $y(x) = 0$. If
$$\begin{vmatrix} k_1y_1(a) + k_2y_1'(a) & k_1y_2(a) + k_2y_2'(a) \\ h_1y_1(b) + h_2y_1'(b) & h_1y_2(b) + h_2y_2'(b) \end{vmatrix} = 0,$$
then nontrivial solutions exist. Use this theorem with the indicated solutions to determine if the following boundary-value problems have only the trivial solution.
   (a) $y'' + y = 0,\ y(0) = 0,\ y(\pi) = 0,\ y_1(x) = \sin x,\ y_2(x) = \cos x$
   (b) $y'' + y = 0,\ y(0) = 0,\ y\left(\frac{\pi}{4}\right) = 0,\ y_1(x) = \sin x,\ y_2(x) = \cos x$
   (c) $y'' + 2y' + (\pi^2 + 1)y = 0,\ y(0) = 0,\ y(5) = 0,\ y_1(x) = e^{-x}\sin\pi x, y_2(x) = e^{-x}\cos\pi x$
   (d) $x^2y'' + xy' + y = 0, y(1) = 0, y(e^\pi) = 0, y_1(x) = \sin(\ln x), y_2(x) = \cos(\ln x)$

In Exercises 37–38, solve the eigenvalue problem with nonconstant coefficients.

37. $x^2y'' - xy' + \lambda y = 0, y(1) = 0, y(e) = 0$
38. $x^2y'' + xy' + \lambda y = 0, y(1) = 0, y(e^2) = 0$

In Exercises 39–42, solve the nonhomogeneous boundary-value problem if a solution exists.

39. $y'' = 1, y(0) = 0, y(2) = 0$
40. $y'' - y = x, y(0) = 1, y(1) = 0$
*41. $y'' + y = \sin 2x, y(0) = 0, y(\pi) = 0$
42. $y'' + y = \cos x, y(0) = 0, y(\pi) = 0$
43. Consider the Sturm–Liouville (eigenvalue) problem
$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0, y(1) = 0 \end{cases}$$
Use an integration device to verify the orthogonality condition for any two eigenfunctions $y_1(x) = \sin m\pi x$ and $y_2(x) = \sin n\pi x$, $m \neq n$.
44. In Section 4.8, we discussed Legendre's equation $d/dx\,[(1 - x^2)dy/dx] + n(n + 1)y = 0$ on the interval $[-1, 1]$, which has as solutions the Legendre polynomials $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$, $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$, $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x), \ldots$ . Verify the orthogonality condition for the set of eigenfunctions. (In Exercise 30, we proved this condition.)
$$\{P_0(x), P_1(x), P_2(x), P_3(x), P_4(x), P_5(x)\}.$$
*45. Consider the eigenvalue problem $y'' + \lambda y = 0, y(0) = 0, y(1) + y'(1) = 0$.
   (a) Show that $\lambda = 0$ is not an eigenvalue.
   (b) Show that there are no negative eigenvalues.
   (c) Show that the positive eigenvalues $\lambda = k^2$ satisfy the relationship $k = -\tan k$.
   (d) Approximate the first three eigenvalues. Notice that for larger values of $k$, the eigenvalues are approximately the vertical asymptotes of $y = \tan k$, so $\lambda_n \approx [(2n - 1)\pi/2]^2$.

## 9.2 Fourier Sine Series and Cosine Series

One of the uses of the Taylor and Maclaurin series discussed in calculus is to approximate functions by taking a finite number of terms from the series to obtain Taylor and Maclaurin polynomials. For example, the Maclaurin series for $f(x) = \cos x$ is

$$f(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n},$$

so the first four Maclaurin polynomials for $f(x) = \cos x$ are $p_0(x) = 1$, $p_2(x) = 1 - \frac{1}{2!}x^2$, $p_4(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$, and $p_6(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$. In Figure 9.1, we graph $f(x) = \cos x$ along with the approximating polynomial. Notice that as we increase the number of terms in the polynomial, the graph of the polynomial becomes closer to that of $f(x) = \cos x$. Therefore, we improve the accuracy of the approximation by increasing the number of terms.

In Chapter 10, we will see that the displacement of a vibrating string of length $\ell$ at time $t$ can be described by

$$y = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{n\pi t}{\ell}\right),$$

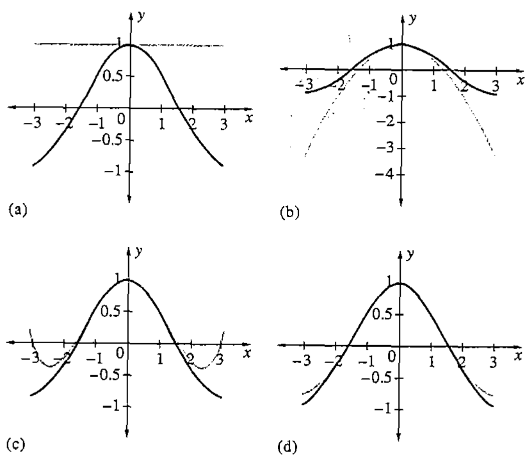


**Figure 9.1** (a) $p_0(x) = 1$, (b) $p_2(x) = 1 - \frac{x^2}{2!}$, (c) $p_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, (d) $p_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$



which was described by Euler in a 1748 paper. Independent of Euler, Daniel Bernoulli was able to show that for $t = 0$, the displacement of a vibrating string of length $\ell$ can be described by

$$y = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right).$$

Subsequently, Euler asked if any function $f(x)$ can be written in the form

$$f(x) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right),$$

where the coefficients, $A_n$, are to be determined.* In fact, using the orthogonality relations developed in Section 9.1, we will see that we can compute the coefficients, $A_n$, in the same manner as they were computed by Clairaut, Euler, and, subsequently, Fourier.

Taylor and Maclaurin series are called power series because they involve powers of one variable, $x$. However, we can extend the idea of series by considering other functions in our series, such as the sine functions in the series developed by Euler. In general, suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n y_n(x),$$

where $\{y_n(x)\}$ is a set of eigenfunctions of a particular eigenvalue problem and the $c_n$'s are constants that must be determined. As with power series, we can (and will) use these eigenfunction expansions (or series) to approximate functions.

Recall the eigenvalue problem $\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0,\ y(p) = 0 \end{cases}$, which was solved in Example 4 in Section 9.1. The eigenvalues of this problem are $\lambda = \lambda_n = (n\pi/p)^2$, $n = 1, 2, \ldots$, with corresponding eigenfunctions $y_n(x) = \sin(n\pi x/p)$, $n = 1, 2, \ldots$. Therefore, for some functions $f$ we can find coefficients $b_n$, so that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{p}.$$

A series of this form is called a **Fourier sine series.** To make use of these series, we must determine the coefficients $b_n$. We accomplish this by taking advantage of the orthogonality properties of eigenfunctions. Because the differential equation $y'' + \lambda y = 0$ is in self-adjoint form, we have that $s(x) = 1$. (See Section 9.1.) Therefore, the orthogonality condition is

$$\int_0^{\pi} \sin\frac{n\pi x}{p} \sin\frac{m\pi x}{p}\, dx = 0 \ (m \neq n).$$

* Jesper Lutzen, "The Solution of Partial Differential Equations," *Studies in the History of Mathematics*, ed. Esther R. Phillips, *MAA Studies in Mathematics, Volume 26*, Mathematical Association of America, (1987), pp. 242–277.

To use this condition, multiply both sides of $f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/p)$ by the eigenfunction $\sin(m\pi x/p)$ and $s(x) = 1$. Then integrate each side of the equation from $x = 0$ to $x = p$ (because the boundary conditions of the corresponding eigenvalue problem are given at these two values of $x$). This yields

$$\int_0^p f(x) \sin \frac{m\pi x}{p}\, dx = \int_0^p \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{p} \sin \frac{m\pi x}{p}\, dx.$$

Assuming that term-by-term integration is allowed on the right-hand side of the equation, we have

$$\int_0^p f(x) \sin \frac{m\pi x}{p}\, dx = \sum_{n=1}^{\infty} \int_0^p b_n \sin \frac{n\pi x}{p} \sin \frac{m\pi x}{p}\, dx = \sum_{n=1}^{\infty} b_n \int_0^p \sin \frac{n\pi x}{p} \sin \frac{m\pi x}{p}\, dx.$$

Recall that the eigenfunctions $y_n(x) = \sin(n\pi x/p)$, $n = 1, 2, \ldots$ are orthogonal, so $\int_0^p \sin(n\pi x/p) \sin(m\pi x/p)\, dx = 0$, if $m \neq n$. On the other hand, if $m = n$,

$$\int_0^p \sin \frac{n\pi x}{p} \sin \frac{m\pi x}{p}\, dx = \int_0^p \sin^2 \frac{n\pi x}{p}\, dx = \int_0^p \frac{1}{2}\left(1 - \cos \frac{2n\pi x}{p}\right) dx$$
$$= \frac{1}{2}\left[x - \frac{p}{2\pi n} \sin \frac{2n\pi x}{p}\right]_0^p = \frac{p}{2}.$$

Therefore, each term in the sum

$$\sum_{n=1}^{\infty} b_n \int_0^p \sin \frac{n\pi x}{p} \sin \frac{m\pi x}{p}\, dx$$

equals zero except when $m = n$. Hence

$$\int_0^p f(x) \sin \frac{n\pi x}{p} = b_n \cdot \frac{p}{2},$$

so the Fourier sine series coefficients are given by

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi x}{p}\, dx, \; n = 1, 2, \ldots,$$

where we assume that $f$ is integrable on $(0, p)$.

**Example 1**

Find the Fourier sine series for $f(x) = x$, $0 < x < \pi$.

***Solution*** In this case, $p = \pi$, so

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin \frac{n\pi x}{\pi}\, dx = \frac{2}{\pi} \int_0^{\pi} x \sin nx\, dx \qquad \text{(integration by parts)}$$



$$= \frac{2}{\pi}\left[-\frac{x \cos nx}{n}\right]_0^{\pi} + \frac{2}{\pi} \int_0^{\pi} \frac{\cos nx}{n}\, dx = -\frac{2 \cos n\pi}{n} + \frac{2}{\pi}\left[\frac{\sin nx}{n^2}\right]_0^{\pi}$$
$$= -\frac{2 \cos n\pi}{n} + \frac{2}{\pi n^2}(\sin n\pi - \sin 0)$$
$$= -\frac{2 \cos n\pi}{n} + \frac{2}{\pi n^2}(0) = -\frac{2 \cos n\pi}{n}.$$

Notice that $\cos n\pi = (-1)^n$, because $\cos \pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1, \ldots$. Hence

$$b_n = -\frac{2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n},$$

and the Fourier sine series is

$$f(x) = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{\pi} = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$
$$= 2 \sin x - \sin 2x + \frac{2}{3} \sin 3x - \frac{1}{2} \sin 4x + \cdots$$

Notice that we can use a finite number of terms of the series to obtain a trigonometric polynomial. In Figure 9.2, we graph $f(x) = x$ on $0 \leq x \leq \pi$ along with one

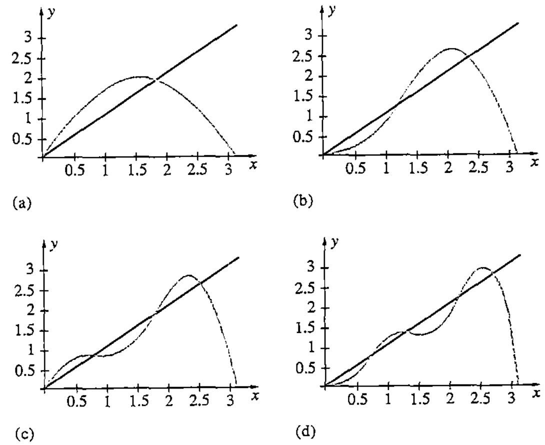



**Figure 9.2** **(a)** $p_1(x) = 2 \sin x$, **(b)** $p_2(x) = 2 \sin x - \sin 2x$, **(c)** $p_3(x) = 2 \sin x - \sin 2x + \frac{2}{3} \sin 3x$, **(d)** $p_4(x) = 2 \sin x - \sin 2x + \frac{2}{3} \sin 3x - \frac{1}{2} \sin 4x$

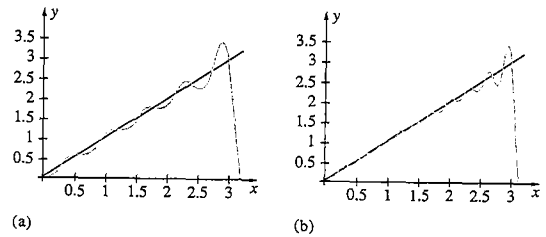


Figure 9.3 (a) 10 terms, (b) 20 terms

of the polynomials obtained from this series. As we increase the number of terms used in approximating $f$, we improve the accuracy. Because many more terms are needed to yield a desirable approximation, we graph $f(x) = x$ with the polynomials using 10 and 20 terms, respectively, in Figure 9.3. Notice from the graphs that none of the polynomials attain the value of $f(\pi) = \pi$ at $x = \pi$. This is due to the fact that at $x = \pi$, each of the polynomials yields a value of 0. Hence our approximation can only be reliable on the interval $0 \le x < \pi$. In general, however, we are only assured of accuracy (at points of continuity of $f$) on the open interval $0 < x < \pi$.

## Example 2

Find the Fourier sine series for $f(x) = \begin{cases} 1, & 0 \le x < 1 \\ -1, & 1 \le x \le 2 \end{cases}$.

**Solution** Because $f$ is defined on $0 \le x \le 2$, $p = 2$. Hence

$$
\begin{aligned}
b_n &= \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx = \int_0^1 \sin\frac{n\pi x}{2}\,dx + \int_1^2 (-1)\sin\frac{n\pi x}{2}\,dx \\
&= \left[-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_0^1 + \left[\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_1^2 \\
&= \left(-\frac{2}{n\pi}\right)\left(\cos\frac{n\pi}{2} - 1\right) + \left(\frac{2}{n\pi}\right)\left(\cos n\pi - \cos\frac{n\pi}{2}\right) \\
&= \left(\frac{2}{n\pi}\right)\left(-2\cos\frac{n\pi}{2} + \cos n\pi + 1\right).
\end{aligned}
$$

Calculating a few of the $b_n$'s, we find $b_1 = \frac{2}{\pi}\cdot 0 = 0$, $b_2 = \frac{1}{\pi}\cdot 4 = \frac{4}{\pi}$, $b_3 = \frac{2}{3\pi}\cdot 0 = 0$, $b_4 = \frac{2}{4\pi}\cdot 0 = 0, \ldots$. As we can see, most of the coefficients are zero. In fact, only those $b_n$'s where $n$ is an odd multiple of 2 yield a nonzero value. For example,



$$
b_6 = b_{2(3)} = \frac{2}{6\pi}(4) = \frac{4}{3\pi},\ b_{10} = b_{2(5)} = \frac{2}{10\pi}(4) = \frac{4}{5\pi}, \ldots,
$$

$$
b_{2(2n-1)} = \frac{4}{(2n-1)\pi},\ n = 1, 2, \ldots,
$$

so we have the series

$$
\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\sin\frac{2(2n-1)\pi x}{2} = \sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}\sin(2n-1)\pi x \\
&= \frac{4}{\pi}\sin \pi x + \frac{4}{3\pi}\sin 3\pi x + \frac{4}{5\pi}\sin 5\pi x + \cdots.
\end{aligned}
$$

In Figure 9.4, we graph $f(x)$ with several polynomials. Notice that with a large number of terms the approximation is quite good at values of $x$ where $f$ is continuous.

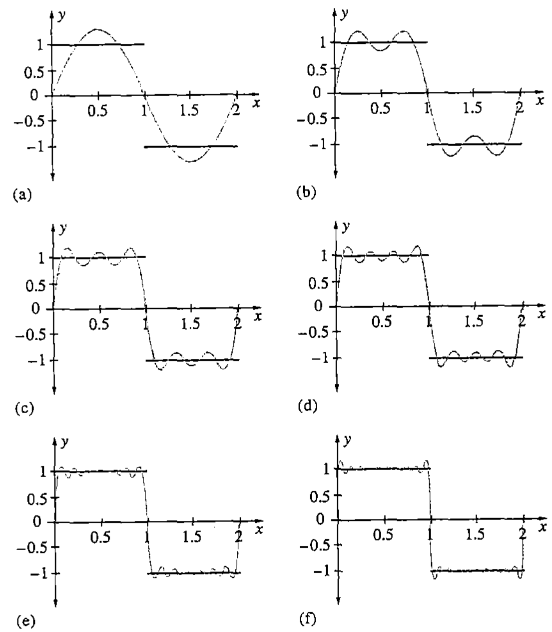


Figure 9.4 (a) $p(x) = \frac{4}{\pi}\sin \pi x$, (b) $p(x) = \frac{4}{\pi}\sin \pi x + \frac{4}{3\pi}\sin 3\pi x$, (c) $p(x) = \frac{4}{\pi}\sin \pi x + \frac{4}{3\pi}\sin 3\pi x + \frac{4}{5\pi}\sin 5\pi x$, (d) $p(x) = \frac{4}{\pi}\sin \pi x + \cdots + \frac{4}{7\pi}\sin 7\pi x$, (e) $p(x) = \frac{4}{\pi}\pi x + \cdots + \frac{4}{15\pi}\sin 15\pi x$, (f) $p(x) = \frac{4}{\pi}\sin \pi x + \cdots + \frac{4}{19\pi}\sin 19\pi x$

The behavior of the series near points of discontinuity in that the approximation overshoots the function is called the **Gibbs phenomenon.** The approximation continues to "miss" the function even though more and more terms from the series are used.

*In Example 2, what is the value of each Fourier polynomial at $x = 1$?*

Another important eigenvalue problem that has useful eigenfunctions is

$$\begin{cases} y'' + \lambda y = 0 \\ y'(0) = 0, y'(p) = 0 \end{cases}.$$

As we found in Example 5 in Section 9.1, the eigenvalues and eigenfunctions are

$$\lambda_n = \begin{cases} 0, & n = 0 \\ \left(\dfrac{n\pi}{p}\right)^2, & n = 1, 2, \ldots \end{cases} \quad \text{and} \quad y_n(x) = \begin{cases} 1, & n = 0 \\ \cos \dfrac{n\pi x}{p}, & n = 1, 2, \ldots \end{cases}.$$

Therefore, for many functions $f(x)$, we can find a series expansion of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{p}.$$

We call this expression a **Fourier cosine series,** where in the first term (associated with $\lambda_0 = 0$), the constant $\frac{1}{2}a_0$ is written in this form for convenience in finding the formula for the coefficients $a_n$, $n = 0, 1, 2, \ldots$. We find these coefficients in a manner similar to that followed to find the coefficients in the Fourier sine series. Notice that in this case, the orthogonality condition is

$$\int_0^p \cos \frac{n\pi x}{p} \cos \frac{m\pi x}{p}\, dx = 0 \ (m \neq n).$$

We use this condition by multiplying both sides of the series expansion by $\cos m\pi x/p$ and integrating from $x = 0$ to $x = p$. This yields

$$\int_0^p f(x) \cos \frac{m\pi x}{p}\, dx = \int_0^p \frac{1}{2} a_0 \cos \frac{m\pi x}{p}\, dx + \int_0^p \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{p} \cos \frac{m\pi x}{p}\, dx.$$

Assuming that term-by-term integration is allowed,

$$\int_0^p f(x) \cos \frac{m\pi x}{p}\, dx = \int_0^p \frac{1}{2} a_0 \cos \frac{m\pi x}{p}\, dx + \sum_{n=1}^{\infty} \int_0^p a_n \cos \frac{n\pi x}{p} \cos \frac{m\pi x}{p}\, dx.$$

If $m = 0$, then this expression reduces to

$$\int_0^p f(x)\, dx = \int_0^p \frac{1}{2} a_0\, dx + \sum_{n=1}^{\infty} \int_0^p a_n \cos \frac{n\pi x}{p}\, dx,$$

where $\int_0^p \cos(n\pi x/p)\, dx = 0$ for $n \geq 1$ and $\int_0^p \frac{1}{2} a_0\, dx = \frac{1}{2} p a_0$. Therefore, $\int_0^p f(x)\, dx = \frac{1}{2} p a_0$, so



$$a_0 = \frac{2}{p}\int_0^p f(x)\, dx.$$

If $m > 0$, then we note that by the orthogonality property, $\int_0^p a_n \cos(n\pi x/p) \cos(m\pi x/p)\, dx = 0$ if $m \neq n$. We also note that

$$\int_0^p \frac{1}{2} a_0 \cos \frac{m\pi x}{p}\, dx = 0 \quad \text{and} \quad \int_0^p \cos^2 \frac{n\pi x}{p}\, dx = \frac{p}{2}.$$

Hence

$$\int_0^p f(x) \cos \frac{n\pi x}{p}\, dx = 0 + a_n \cdot \frac{p}{2}.$$

Solving for $a_n$, we have

$$a_n = \frac{2}{p}\int_0^p f(x) \cos \frac{n\pi x}{p}\, dx, \ n = 1, 2, \ldots.$$

Notice that this formula also works for $n = 0$ because $\cos(0 \cdot \pi x/a) = \cos 0 = 1$.

**Example 3**

Find the Fourier cosine series for $f(x) = x$, $0 < x < \pi$.

***Solution*** In this case, $p = \pi$. Hence

$$a_0 = \frac{2}{\pi}\int_0^\pi x\, dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^\pi = \pi$$

and

$$a_n = \frac{2}{\pi}\int_0^\pi x \cos \frac{n\pi x}{\pi}\, dx = \frac{2}{\pi}\int_0^\pi x \cos nx\, dx \qquad \text{(integration by parts)}$$

$$= \frac{2}{\pi}\left\{\left[\frac{x \sin nx}{n}\right]_0^\pi - \int_0^\pi \frac{\sin nx}{n}\, dx\right\} = \frac{2}{\pi}\left[\frac{\cos nx}{n^2}\right]_0^\pi$$

$$= \frac{2}{\pi n^2}(\cos n\pi - 1) = \frac{2}{\pi n^2}[(-1)^n - 1].$$

Notice that for even values of $n$, $(-1)^n - 1 = 0$. Therefore, $a_n = 0$ if $n$ is even. On the other hand, if $n$ is odd, $(-1)^n - 1 = -2$. Hence

$$a_1 = -\frac{4}{\pi}, a_3 = -\frac{4}{9\pi}, a_5 = -\frac{4}{25\pi}, \ldots, a_{2n-1} = -\frac{4}{(2n-1)^2\pi},$$

so the Fourier cosine series of $f(x) = x$ on $0 < x < \pi$ is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x.$$

In Figure 9.5, we plot the function with several terms of the series. Compare these results to those obtained when approximating this function with a sine series. Which series yields the better approximation with the fewer number of terms?

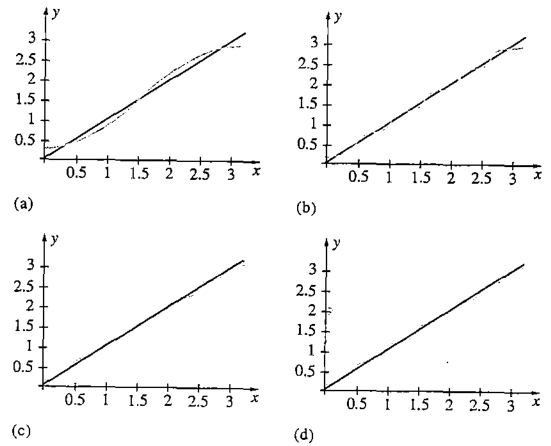



**Figure 9.5** (a) $p(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x$, (b) $p(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x$, (c) $p(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x - \frac{4}{25\pi}\cos 5x$, (d) $p(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \cdots - \frac{4}{49\pi}\cos 7x$

## EXERCISES 9.2

Determine the Fourier cosine series for the following functions on the indicated interval.

1. $f(x) = x,\ 0 < x < 1$
2. $f(x) = x + 1,\ 0 < x < 1$
*3. $f(x) = x^2,\ 0 < x < 1$
4. $f(x) = x(1 - x),\ 0 < x < 1$
5. $f(x) = 1,\ 0 < x < \pi$
6. $f(x) = \begin{cases} 0, & 0 < x < \pi \\ 1, & \pi < x < 2\pi \end{cases}$
*7. $f(x) = \begin{cases} x, & 0 < x < 2 \\ 1, & 2 < x < 4 \end{cases}$
8. $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \end{cases}$
9. $f(x) = e^{-x},\ 0 < x < 1$
10. $f(x) = e^{-x} + e^{x},\ 0 < x < 1$

In Exercises 11–20, determine the Fourier sine series.

11. $f(x) = x,\ 0 < x < 1$
12. $f(x) = x + 1,\ 0 < x < 1$
*13. $f(x) = x^2,\ 0 < x < 1$
14. $f(x) = x(1 - x),\ 0 < x < 1$
15. $f(x) = 1,\ 0 < x < \pi$



16. $f(x) = \begin{cases} 0, & 0 < x < \pi \\ 1, & \pi < x < 2\pi \end{cases}$
*17. $f(x) = \begin{cases} x, & 0 < x < 2 \\ 1, & 2 < x < 4 \end{cases}$
18. $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \end{cases}$
19. $f(x) = e^{-x},\ 0 < x < 1$
20. $f(x) = e^{-x} + e^{x},\ 0 < x < 1$

In Exercises 21–23, use trigonometric identities to obtain the Fourier cosine series for each of the following functions.

21. $f(x) = \cos^2 x$
22. $f(x) = \sin^2 2x$
*23. $f(x) = \cos^3 x$ (*Hint:* Expand $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ and simplify with Euler's formula.)

In Exercises 24–25, use trigonometric identities to obtain the Fourier sine series for each of the following functions.

24. $f(x) = 4 \sin x \cos x$
25. $f(x) = \sin^3 x$ (*Hint:* See Exercise 23 with $\sin x = (1/2i)(e^{ix} - e^{-ix})$.)
26. Prove that **(a)** $\int_0^p a_n \cos\frac{n\pi x}{p}\cos\frac{m\pi x}{p}\,dx = 0$ if $m \neq n$, **(b)** $\int_0^p \cos^2\frac{n\pi x}{p}\,dx = \frac{p}{2}$.
27. **(Boundary-Value Problems)** Fourier sine series can be used to solve boundary-value problems of the form $ay'' + by' + cy = f(x), 0 < x < p, y(0) = 0, y(p) = 0$ by assuming that the solution is $y(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/p)$. (Notice that each term in the series satisfies the boundary conditions.) Then, because the particular solution must also satisfy the boundary conditions, we must find the Fourier sine series for $f(x)$ on $(0, p)$. Both series are then substituted into the differential equation so that the coefficients $c_n$ can be found. Use this technique to solve $y'' + y = x, y(0) = 0, y(\pi) = 0$.
28. If the boundary conditions in Exercise 27 are $y'(0) = 0, y'(p) = 0$, then what type of series should be used? Solve $y'' + y = x, y'(0) = 0, y'(\pi) = 0$.
29. Euler described the displacement of a string of length $\ell$ at position $x$ and time $t$ with the series solution $y(x, t) = \sum_{n=0}^{\infty} A_n \sin(n\pi x/\ell)\cos(n\pi t/\ell)$. If the initial shape of the string is $y(x, 0) = f(x)$, then show that $A_n = 2/\ell \int_0^{\ell} f(x)\sin(n\pi x/\ell)\,dx$.
30. **(a)** Use the formula derived in Exercise 29 to find the displacement of the string if $f(x) = x(\ell - x), 0 < x < \ell$. **(b)** What is the displacement if $f(x) = \frac{1}{8}\sin 2x, 0 < x < \ell$. (*Hint:* Integration is not necessary.)
31. Find the Fourier cosine series and the Fourier sine series for the following functions. In each case, graph the function as well as approximations using a finite number of terms from the series. Which series converges to the function more rapidly (that is, requires the fewer number of terms to approximate the function)?
    **(a)** $f(x) = x^3, 0 < x < 1$
    **(b)** $f(x) = x^2(1 - x), 0 < x < 1$
    **(c)** $f(x) = x^4, 0 < x < 1$
    **(d)** $f(x) = x^2(1 - x^2), 0 < x < 1$
32. In Example 2, we found the Fourier sine series for $f(x) = \begin{cases} 1, & 0 \le x < 1 \\ -1, & 1 \le x < 2 \end{cases}$ to be

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\sin(2n-1)\pi x$$
$$= \frac{4}{\pi}\sin \pi x + \frac{4}{3\pi}\sin 3\pi x + \frac{4}{5\pi}\sin 5\pi x + \cdots.$$

Substitution of $x = 1/2$ into both sides of this equation yields

$$f\left(\frac{1}{2}\right) = \frac{4}{\pi}\sin\frac{\pi}{2} + \frac{4}{3\pi}\sin\frac{3\pi}{2} + \frac{4}{5\pi}\sin\frac{5\pi}{2} + \cdots$$

or $$1 = \frac{4}{\pi} - \frac{4}{3\pi} + \frac{4}{5\pi} + \cdots$$
$$= \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right).$$

Therefore, $1 - 1/3 + 1/5 - \cdots = \pi/4$. How many terms of this alternating series are needed to approximate $\pi/4$ to within an error of $10^{-5}$? What is this approximation? (*Hint:* Recall alternating series from calculus.)

We discuss convergence of series more in Section 10.2.

## 9.3 Fourier Series

Fourier Series  Even, Odd, and Periodic Extensions

### Fourier Series

Consider the eigenvalue problem that was given in Exercise 35 in Section 9.1:

$$\begin{cases} y'' + \lambda y = 0, & -p < x < p \\ y(-p) = y(p),\ y'(-p) = y'(p) \end{cases}$$

This problem has eigenvalues

$$\lambda_n = \begin{cases} 0, & n = 0 \\ \left(\dfrac{n\pi}{p}\right)^2, & n = 1, 2, \ldots \end{cases}$$

and eigenfunctions

$$y_n(x) = \begin{cases} 1, & n = 0 \\ a_n \cos \dfrac{n\pi x}{p} + b_n \sin \dfrac{n\pi x}{p}, & n = 1, 2, \ldots \end{cases}$$

so we can consider a series made up of these functions. Hence we write

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p}\right),$$

which is called a **Fourier series.** As was the case with Fourier sine and Fourier cosine series, we must determine the coefficients $a_0$, $a_n$ $(n = 1, 2, \ldots)$, and $b_n$ $(n = 1, 2, \ldots)$. Because we use a method similar to those used to find the coefficients in Section 9.2, we give the value of several integrals in Table 9.2. Note that most of these integrals were encountered in Section 9.2.

**TABLE 9.2**

| | | |
|---|---|---|
| $\int_{-p}^{p} \cos \frac{n\pi x}{p}\, dx = 0$ | $\int_{-p}^{p} \sin \frac{n\pi x}{p}\, dx = 0$ | $\int_{-p}^{p} \cos \frac{m\pi x}{p} \sin \frac{n\pi x}{p}\, dx = 0$ |
| $\int_{-p}^{p} \cos \frac{m\pi x}{p} \cos \frac{n\pi x}{p}\, dx = \begin{cases} 0, & m \neq n \\ p, & m = n \end{cases}$ | $\int_{-p}^{p} \sin \frac{m\pi x}{p} \sin \frac{n\pi x}{p}\, dx = \begin{cases} 0, & m \neq n \\ p, & m = n \end{cases}$ | |

We begin by finding $a_0$ and $a_n$ $(n = 1, 2, \ldots)$. Multiplying both sides of

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p}\right)$$

by $\cos(m\pi x/p)$ and integrating from $x = -p$ to $x = p$ (because of the boundary conditions) yields



$$\begin{aligned}
\int_{-p}^{p} f(x) \cos \frac{m\pi x}{p}\, dx &= \int_{-p}^{p} \frac{1}{2}a_0 \cos \frac{m\pi x}{p}\, dx \\
&\quad + \int_{-p}^{p} \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi x}{p} \cos \frac{m\pi x}{p} + b_n \sin \frac{n\pi x}{p} \cos \frac{m\pi x}{p}\right) dx \\
&= \int_{-p}^{p} \frac{1}{2}a_0 \cos \frac{m\pi x}{a}\, dx + \sum_{n=1}^{\infty}\left(\int_{-p}^{p} a_n \cos \frac{n\pi x}{p} \cos \frac{m\pi x}{p}\, dx\right. \\
&\quad \left. + \int_{-p}^{p} a_n \sin \frac{n\pi x}{p} \cos \frac{m\pi x}{p}\, dx\right).
\end{aligned}$$

If $m = 0$, then using the integrals in Table 9.1 we notice that

$$\int_{-p}^{p} f(x) \cos \frac{0 \cdot \pi x}{p}\, dx = \int_{-p}^{p} f(x)\, dx,$$

and all of the integrals that we are summing have the value zero. Thus this expression simplifies to

$$\begin{aligned}
\int_{-p}^{p} f(x)\, dx &= \int_{-p}^{p} \frac{1}{2}a_0\, dx \\
\int_{-p}^{p} f(x)\, dx &= \frac{1}{2}a_0[2p] \\
a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)\, dx.
\end{aligned}$$

Otherwise, if $m \neq 0$, then only one of the integrals on the right-hand side of the expression yields a value other than zero, and this occurs with $\int_{-p}^{p} \cos(m\pi x/p) \cos(n\pi x/p)\, dx = p$ if $m = n$. Hence

$$\begin{aligned}
\int_{-p}^{p} f(x) \cos \frac{n\pi x}{p}\, dx &= p \cdot a_n \\
a_n &= \frac{1}{p}\int_{-p}^{p} f(x) \cos \frac{n\pi x}{p}\, dx,\ n = 1, 2, \ldots
\end{aligned}$$

We find $b_n$ $(n = 1, 2, \ldots)$ by multiplying the series by $\sin(m\pi x/p)$ and integrating from $x = -p$ to $x = p$. This yields

$$\int_{-p}^{p} f(x) \sin \frac{m\pi x}{p}\, dx =$$

$$\int_{-p}^{p} \frac{1}{2}a_0 \sin \frac{m\pi x}{p}\, dx + \int_{-p}^{p} \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi x}{p} \sin \frac{m\pi x}{p} + b_n \sin \frac{n\pi x}{p} \sin \frac{m\pi x}{p}\right) dx$$

$$\int_{-p}^{p} f(x) \sin \frac{m\pi x}{p}\, dx =$$

$$\int_{-p}^{p} \frac{1}{2}a_0 \sin\frac{m\pi x}{p}\,dx + \sum_{n=1}^{\infty}\left(\int_{-p}^{p} a_n \cos\frac{n\pi x}{p}\sin\frac{m\pi x}{p}\,dx + \int_{-p}^{p} a_n \sin\frac{n\pi x}{p}\sin\frac{m\pi x}{p}\,dx\right),$$

where we assume that termwise integration is permitted. Again, with the integrals in Table 9.2, we note that only one of the integrals on the right-hand side is not zero. In this case, we use

$$\int_{-p}^{p} \sin\frac{m\pi x}{p}\sin\frac{n\pi x}{p}\,dx = \begin{cases} 0, & m \neq n \\ p, & m = n \end{cases}$$

to obtain

$$\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}\,dx = p \cdot b_n$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}\,dx, \quad n = 1, 2, \ldots.$$

**Definition 9.1 Fourier Series**

Suppose that $f$ is defined on $-p < x < p$. Then the **Fourier series** for $f$ is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right),$$

where

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}\,dx,\ n = 1, 2, \ldots$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}\,dx,\ n = 1, 2, \ldots$$

**Example 1**

Find the Fourier series of $f(x) = \begin{cases} 1, & -2 \le x < 0 \\ 2, & 0 \le x \le 2 \end{cases}$, where $f(x+4) = f(x)$.

***Solution*** In this case, $p = 2$. Hence

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\int_{-2}^{0}(1)\,dx + \frac{1}{2}\int_{0}^{2}(2)\,dx = \frac{1}{2}[x]_{-2}^{0} + \frac{1}{2}\cdot 2[x]_{0}^{2}$$
$$= 1 + 2 = 3,$$



$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-2}^{0}(1)\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2}(2)\cos\frac{n\pi x}{2}\,dx$$
$$= \frac{1}{2}\left[\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]_{-2}^{0} + \frac{1}{2}\cdot 2\left[\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]_{0}^{2} = 0.$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-2}^{0}(1)\sin\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2}(2)\sin\frac{n\pi x}{2}\,dx$$
$$= -\frac{1}{2}\left[\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_{-2}^{0} - \frac{1}{2}\cdot 2\left[\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_{0}^{2}$$
$$= -\frac{1}{n\pi}(1 - \cos n\pi) - \frac{2}{n\pi}(\cos n\pi - 1)$$
$$= \frac{1}{n\pi}(1 - \cos n\pi) = \frac{1}{n\pi}(1 - (-1)^n).$$

Therefore,

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}(1 - (-1)^n)\frac{1}{n\pi}\sin\frac{n\pi x}{2}$$
$$= \frac{3}{2} + \frac{2}{\pi}\sin\frac{\pi x}{2} + \frac{2}{3\pi}\sin\frac{3\pi x}{2} + \frac{2}{5\pi}\sin\frac{5\pi x}{2} + \cdots$$

We graph $f(x)$ with several terms of the series in Figure 9.6. If we extend $f$ over more periods, then the approximation by the Fourier series carries over to these intervals.

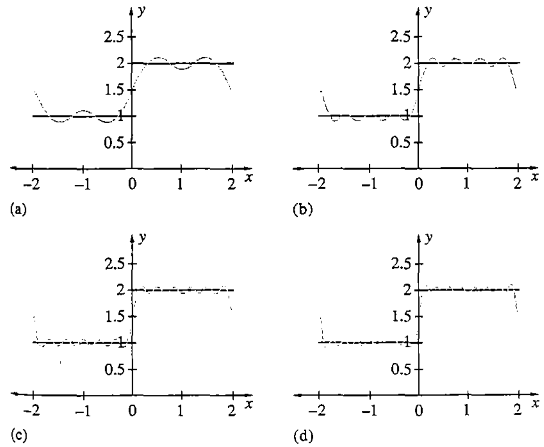



**Figure 9.6** (a) $p(x) = \frac{3}{2} + \frac{2}{n\pi}\sin\frac{\pi x}{2} + \frac{2}{n\pi}\sin\frac{3\pi x}{2}$
(b) $p(x) = \frac{3}{2} + \frac{2}{n\pi}\sin\frac{\pi x}{2} + \cdots + \frac{2}{n\pi}\sin\frac{7\pi x}{2}$
(c) $p(x) = \frac{3}{2} + \frac{2}{n\pi}\sin\frac{\pi x}{2} + \cdots + \frac{2}{n\pi}\sin\frac{11\pi x}{2}$
(d) $p(x) = \frac{3}{2} + \frac{2}{n\pi}\sin\frac{\pi x}{2} + \cdots + \frac{2}{n\pi}\sin\frac{15\pi x}{2}$

*In Example 1, what is the value of each Fourier polynomial p at $x = 0$? How does this value relate to the value of f to the left and to the right of $x = 0$?*

**Example 2**

Find the Fourier series of $f(x) = \begin{cases} 0, & -1 \le x < 0 \\ \sin \pi x, & 0 \le x \le 1 \end{cases}$, where $f(x+2) = f(x)$.

**Solution** In this case, $p = 1$, so

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0}(0)\,dx + \int_0^1 \sin \pi x\,dx = -\left[\frac{\cos \pi x}{\pi}\right]_0^1$$
$$= -\frac{1}{\pi}(\cos \pi - \cos 0) = \frac{2}{\pi}.$$
$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_{-1}^{0}(0)\cos n\pi x\,dx + \int_0^1 \sin \pi x \cos n\pi x\,dx$$
$$= \int_0^1 \sin \pi x \cos n\pi x\,dx.$$

The value of this integral depends on the value of $n$. If $n = 1$, then we have

$$a_1 = \int_0^1 \sin \pi x \cos \pi x\,dx = \frac{1}{2}\int_0^1 \sin 2\pi x\,dx = -\frac{1}{2}\left[\frac{\cos 2\pi x}{2\pi}\right]_0^1 = 0.$$

If $n \ne 1$, then we use the identity $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$ to obtain

$$a_n = \frac{1}{2}\int_0^1 [\sin(1-n)\pi x + \sin(1+n)\pi x]\,dx$$
$$= -\frac{1}{2}\left[\frac{\cos(1-n)\pi x}{(1-n)\pi} + \frac{\cos(1+n)\pi x}{(1+n)\pi}\right]_0^1$$
$$= -\frac{1}{2}\left\{\left[\frac{\cos(1-n)\pi}{(1-n)\pi} + \frac{\cos(1+n)\pi}{(1+n)\pi}\right] - \left[\frac{1}{(1-n)\pi} + \frac{1}{(1+n)\pi}\right]\right\}.$$

Notice that if $n$ is odd, then both $(1 - n)$ and $(1 + n)$ are even. Hence, $\cos(1 - n)\pi x = \cos(1 + n)\pi x = 1$, so

$$a_n = -\frac{1}{2}\left\{\left[\frac{1}{(1-n)\pi} + \frac{1}{(1+n)\pi}\right] - \left[\frac{1}{(1-n)\pi} + \frac{1}{(1+n)\pi}\right]\right\} = 0$$

if $n$ is odd. On the other hand, if $n$ is even, $(1 - n)$ and $(1 + n)$ are odd. Therefore, $\cos(1 - n)\pi x = \cos(1 + n)\pi x = -1$, so

$$a_n = -\frac{1}{2}\left\{\left[\frac{-1}{(1-n)\pi} + \frac{-1}{(1+n)\pi}\right] - \left[\frac{1}{(1-n)\pi} + \frac{1}{(1+n)\pi}\right]\right\}$$
$$= \frac{1}{(1-n)\pi} + \frac{1}{(1+n)\pi} = \frac{2}{(1-n)(1+n)\pi} = -\frac{2}{(n-1)(1+n)\pi}$$



if $n$ is even. Putting this information together, we can write the coefficients as

$$a_{2n} = -\frac{2}{(2n-1)(1+2n)\pi},\ n = 1, 2, \ldots$$

Similarly,

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_0^1 \sin \pi x \sin n\pi x\,dx,$$

so if $n = 1$, then

$$b_1 = \int_0^1 \sin^2 \pi x\,dx = \int_0^1 \frac{1}{2}(1 - \cos 2\pi x)\,dx = \frac{1}{2}\left(x - \frac{\sin 2\pi x}{2\pi}\right)_0^1 = \frac{1}{2}.$$

If $n \ne 1$, then we use $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$. Hence

$$b_n = \frac{1}{2}\int_0^1 [\cos(1-n)\pi x - \cos(1+n)\pi x]\,dx$$
$$= \frac{1}{2}\left[\frac{\sin(1-n)\pi x}{(1-n)\pi} - \frac{\sin(1+n)\pi x}{(1+n)\pi}\right]_0^1 = 0,\ n = 2, 3, \ldots$$

Therefore, we write the Fourier series as

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin \pi x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)(1+2n)}\cos 2n\pi x.$$

We graph $f$ along with several approximations using this series in Figure 9.7.

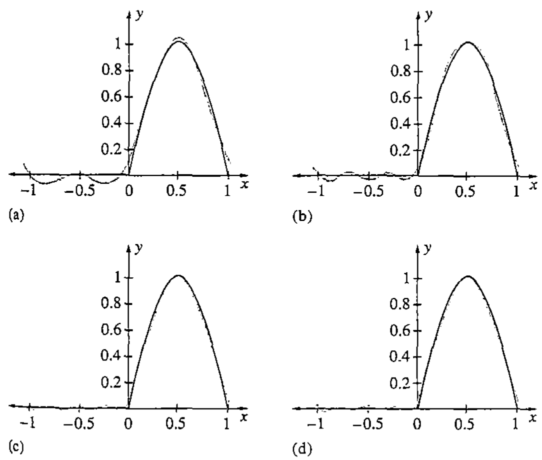



**Figure 9.7** **(a)** $p(x) = \frac{1}{\pi} + \frac{1}{2}\sin \pi x - \frac{2}{3\pi}\cos 2\pi x$ **(b)** $p(x) = \frac{1}{\pi} + \frac{1}{2}\sin \pi x - \cdots - \frac{2}{15\pi}\cos 4\pi x$ **(c)** $p(x) = \frac{1}{\pi} + \frac{1}{2}\sin \pi x - \cdots - \frac{2}{35\pi}\cos 6\pi x$ **(d)** $p(x) = \frac{1}{\pi} + \frac{1}{2}\sin \pi x - \cdots - \frac{2}{63\pi}\cos 8\pi x$

Theorem 9.2, proved by German mathematician Gustav Peter Lejeune Dirichlet (1805–1859) in an 1829 paper, tells us that the Fourier series for any function converges to the function except at points of discontinuity.

> **Theorem 9.2 Convergence of Fourier Series**
>
> Suppose that $f$ and $f'$ are piecewise continuous functions on $-p < x < p$. Then the Fourier series for $f$ on $-p < x < p$ converges to $f(x)$ at every $x$ where $f$ is continuous. At $x = x_0$, where $f$ is discontinuous, the Fourier series converges to the average
>
> $$\frac{f(x_0^+) + f(x_0^-)}{2},$$
>
> where $f(x_0^+) = \lim_{x \to x_0^+} f(x)$ and $f(x_0^-) = \lim_{x \to x_0^-} f(x)$.

A proof of this theorem can be found in more advanced textbooks such as Werner Rogosinski, *Fourier Series,* Second Edition, Chelsea, New York (1959), or Ruel Churchill and James Brown, *Fourier Series and Boundary Value Problems,* Third Edition, McGraw-Hill, New York (1978).

**Example 3**

If $f(x) = \begin{cases} 1, & -2 \le x < 0 \\ 2, & 0 \le x \le 2 \end{cases}$ and $f(x + 4) = f(4)$, then determine the numerical value to which the Fourier series for $f$ converges at each of the following values. **(a)** $x = -1$; **(b)** $x = 1$; **(c)** $x = 0$; **(d)** $x = 3$; **(e)** $x = 2$.

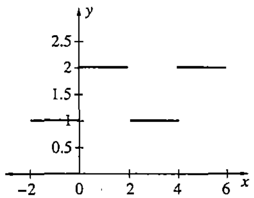



Figure 9.8

***Solution*** We begin by sketching the graph of two periods of $f$ in Figure 9.8. **(a)** $f$ is continuous at $x = -1$, so the Fourier series converges to $f(-1) = 1$ at $x = -1$; **(b)** Similarly, because $f$ is continuous at $x = 1$, the Fourier series converges to $f(1) = 2$ at $x = 1$; **(c)** $f$ is discontinuous at $x = 0$, so the Fourier series converges to the average

$$\frac{f(0^+) + f(0^-)}{2} = \frac{2+1}{2} = \frac{3}{2};$$

**(d)** $f$ is continuous at $x = 3$, so the Fourier series converges to $f(3) = 1$ at $x = 3$; **(e)** Because $f$ is discontinuous at $x = 2$, the Fourier series converges to

$$\frac{f(2^+) + f(2^-)}{2} = \frac{1+2}{2} = \frac{3}{2}.$$

We can test these results by observing the sequence of graphs that were generated in Example 1.

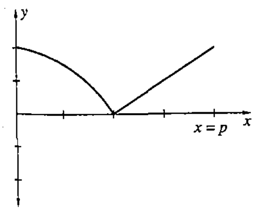



Figure 9.9



Theorem 9.2 can be used to determine the sum of many series that cannot be found with the techniques introduced in calculus.

**Example 4**

Use the Fourier series for $f(x) = \begin{cases} 1, & -2 \le x < 0 \\ 2, & 0 \le x \le 2 \end{cases}$, $f(x + 4) = f(x)$ to find the value of

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{n+1}}{2n-1} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}.$$

***Solution*** Recall that we found the Fourier series to be

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \left(1 - (-1)^n\right) \frac{1}{n\pi} \sin \frac{n\pi x}{2}$$
$$= \frac{3}{2} + \frac{2}{\pi} \sin \frac{\pi x}{2} + \frac{2}{3\pi} \sin \frac{3\pi x}{2} + \frac{2}{5\pi} \sin \frac{5\pi x}{2} + \cdots.$$

Notice that $f$ is continuous at $x = 1$, so the Fourier series for $f$ converges to $f(1)$ at $x = 1$. Hence

$$f(1) = \frac{3}{2} + \frac{2}{\pi} \sin \frac{\pi(1)}{2} + \frac{2}{3\pi} \sin \frac{3\pi(1)}{2} + \frac{2}{5\pi} \sin \frac{5\pi(1)}{2} + \cdots$$
$$2 = \frac{3}{2} + \frac{2}{\pi} - \frac{2}{3\pi} + \frac{2}{5\pi} + \cdots$$
$$\frac{1}{2} = \frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} + \cdots\right)$$
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \cdots.$$

Therefore, the sum of the series is $\pi/4$. Notice that other values of $x$ (such as any integer multiple of 1/2) where $f$ is continuous can be used to obtain the same result.

## Even, Odd, and Periodic Extensions

So far we have assumed that the function $f$ was defined on the interval $-p < x < p$. Unfortunately, this is not always the case. Sometimes we must take a function that is defined on the interval $0 < x < p$ and represent it in terms of trigonometric functions. Three ways of accomplishing this task are to extend $f$ to obtain (a) an *even* function on $-p < x < p$; (b) an *odd* function on $-p < x < p$; and (c) a *periodic* function on $-p < x < p$. Using the function shown in Figure 9.9, we illustrate these three situations in Figure 9.10.

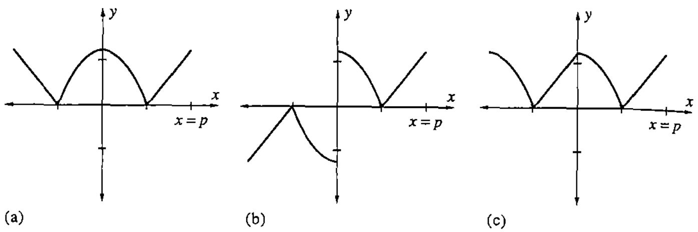



**Figure 9.10** (a) Even extension, (b) odd extension, (c) periodic extension

We notice some interesting properties associated with the Fourier series in each of these three cases by noting the properties of even and odd functions. If $f$ is an even function and $g$ is an odd function, then the product $fg$ is an odd function. Similarly, if $f$ is an even function and $g$ is an even function, then $fg$ is an even function, and if $f$ is an odd function and $g$ is an odd function, then $fg$ is an even function. The proofs of these three properties (which are summarized in Table 9.3) are left as exercises.

TABLE 9.3

| $f$ | $g$ | $fg$ |
|---|---|---|
| Even | Odd | Odd |
| Even | Even | Even |
| Odd | Odd | Even |

Recall from integral calculus that if $f$ is odd on $-p<x<p$, then $\int_{-p}^{p} f(x)\,dx = 0$, and if $g$ is even on $-p<x<p$, then $\int_{-p}^{p} g(x)\,dx = 2\int_{0}^{p} g(x)\,dx$. These properties are useful in determining the coefficients in the Fourier series for the even, odd, and periodic extensions of a function, because $\cos(n\pi x/p)$ and $\sin(n\pi x/p)$ are even and odd periodic functions, respectively, on $-p<x<p$.

**Definition 9.2 Even, Odd, and Periodic Extensions**

The **even extension** $f_{\text{even}}$ of $f$ is an even function. Therefore,

$$a_0 = \frac{1}{p}\int_{-p}^{p} f_{\text{even}}(x)\,dx = \frac{2}{p}\int_{0}^{p} f(x)\,dx,$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f_{\text{even}}(x)\cos\frac{n\pi x}{p}\,dx = \frac{2}{p}\int_{0}^{p} f(x)\cos\frac{n\pi x}{p}\,dx\ (n = 1, 2, \ldots), \text{ and}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f_{\text{even}}(x)\sin\frac{n\pi x}{p}\,dx = 0\ (n = 1, 2, \ldots).$$



The **odd extension** $f_{\text{odd}}$ of $f$ is an odd function, so

$$a_0 = \frac{1}{p}\int_{-p}^{p} f_{\text{odd}}(x)\,dx = 0,$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f_{\text{odd}}(x)\cos\frac{n\pi x}{p}\,dx = 0\ (n = 1, 2, \ldots), \text{ and}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f_{\text{odd}}(x)\sin\frac{n\pi x}{p}\,dx = \frac{2}{p}\int_{0}^{p} f(x)\sin\frac{n\pi x}{p}\,dx\ (n = 1, 2, \ldots).$$

The **periodic extension** $f_p$ has period $p$. Because half of the period is $p/2$,

$$a_0 = \frac{2}{p}\int_{0}^{p} f(x)\,dx,$$

$$a_n = \frac{2}{p}\int_{0}^{p} f(x)\cos\frac{2n\pi x}{p}\,dx\ (n = 1, 2, \ldots), \text{ and}$$

$$b_n = \frac{2}{p}\int_{0}^{p} f(x)\sin\frac{2n\pi x}{p}\,dx\ (n = 1, 2, \ldots).$$

**Example 5**

Let $f(x) = x$ on $0<x<1$. Find the Fourier series for **(a)** the even extension of $f$; **(b)** the odd extension of $f$; **(c)** the periodic extension of $f$.

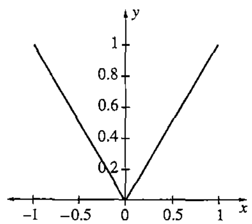



**Figure 9.11**

***Solution*** (a) The graph of the even extension of $f$ is shown in Figure 9.11. Here $p = 1$, so

$$a_0 = 2\int_0^1 x\,dx = 2\left[\frac{x^2}{2}\right]_0^1 = 1,$$

$$a_n = 2\int_0^1 x\cos n\pi x\,dx = \frac{2}{n^2\pi^2}(\cos n\pi - 1)$$

$$= \frac{2}{n^2\pi^2}[(-1)^n - 1]\ (n = 1, 2, \ldots), \text{ and}$$

$$b_n = 0\ (n = 1, 2, \ldots).$$

Because $a_n = 0$ if $n$ is even, we can represent the coefficients with odd subscripts as

$$a_{2n-1} = -\frac{4}{(2n-1)^2\pi^2}.$$

Therefore, the Fourier cosine series is

$$f_{\text{even}}(x) = \frac{1}{2} - \sum_{n=1}^{\infty}\frac{4}{(2n-1)^2\pi^2}\cos(2n-1)\pi x.$$

We graph the even extension with several terms of the Fourier cosine series in Figure 9.12.

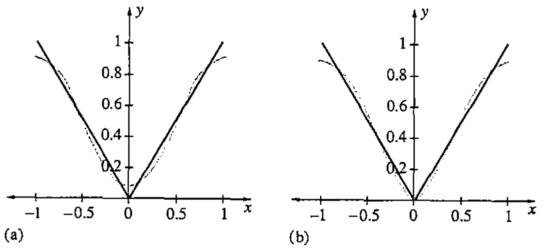



**Figure 9.12** (a) $p(x) = \frac{1}{2} - \frac{4}{\pi^2}\cos \pi x$, (b) $p(x) = \frac{1}{2} - \frac{4}{\pi^2}\cos \pi x - \frac{4}{9\pi^2}\cos 3\pi x$.

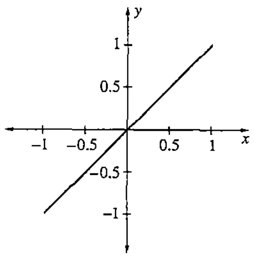



**Figure 9.13**

**(b)** The graph of the odd extension is shown in Figure 9.13. For the odd extension $f_{\text{odd}}$, we note that $a_0 = 0$, $a_n = 0$ $(n = 1, 2, \ldots)$, and

$$b_n = 2\int_0^1 x \sin\frac{n\pi x}{a}\,dx = -\frac{2}{n\pi}\cos n\pi = \frac{2(-1)^{n+1}}{n\pi}\ (n = 1, 2, \ldots).$$

Hence the Fourier sine series is

$$f_{\text{odd}}(x) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin n\pi x.$$

We graph the odd extension along with several terms of the Fourier sine series in Figure 9.14.

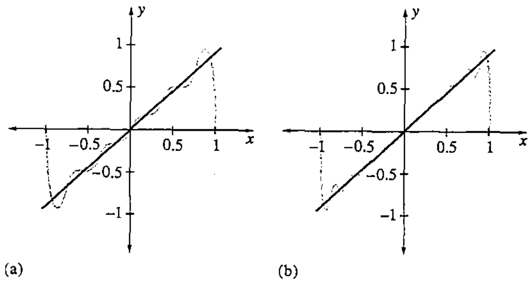



**Figure 9.14** (a) $p(x) = \frac{2}{\pi}\sin \pi x - \frac{1}{\pi}\sin 2\pi x + \cdots + \frac{2}{5\pi}\sin 5\pi x$
(b) $p(x) = \frac{2}{\pi}\sin \pi x - \frac{1}{\pi}\sin 2\pi x + \cdots + \frac{1}{5\pi}\sin 10\pi x$



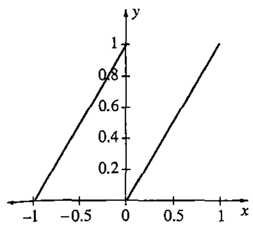



**Figure 9.15**

**(c)** The periodic extension (shown in Figure 9.15) has period $p = 1$. Thus

$$a_0 = 2\int_0^1 x\,dx = 2\left[\frac{x^2}{2}\right]_0^1 = 1,$$

$$a_n = 2\int_0^1 x\cos 2n\pi x\,dx = \frac{1}{2}\,\frac{\cos 2n\pi + 2n\pi\sin 2n\pi}{n^2\pi^2} - \frac{1}{2n^2\pi^2}$$
$$= 0\ (n = 1, 2, \ldots),$$

and

$$b_n = 2\int_0^1 x\sin 2n\pi x\,dx = -\frac{1}{2}\,\frac{-\sin 2n\pi + 2n\pi\cos 2n\pi}{n^2\pi^2} = -\frac{2n\pi}{2n^2\pi^2}$$
$$= -\frac{1}{n\pi}\ (n = 1, 2, \ldots).$$

Hence the Fourier series for the periodic extension is

$$f_p(x) = \frac{1}{2} - \sum_{n=1}^{\infty}\frac{1}{n\pi}\sin 2n\pi x.$$

We graph the periodic extension with several terms of the Fourier series in Figure 9.16.

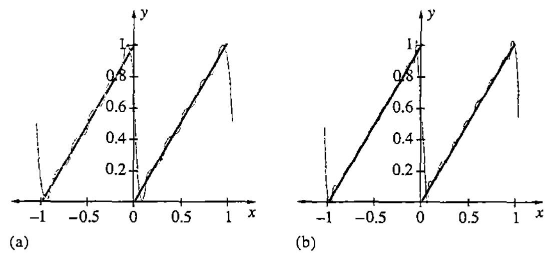



**Figure 9.16** (a) $p(x) = \frac{1}{2} - \frac{1}{\pi}\sin 2\pi x - \cdots - \frac{1}{5\pi}\sin 10\pi x$
(b) $p(x) = \frac{1}{2} - \frac{1}{\pi}\sin 2\pi x - \cdots - \frac{1}{10\pi}\sin 20\pi x$

*The even extension of the function in Example 5(a) is the same as what periodic function on $-1 < x < 1$? The odd extension of the function in Example 5(b) is the same as what periodic function on $-1 < x < 1$? Is the series in Example 5(c) a Fourier sine series, a Fourier cosine series, or neither?*

## EXERCISES 9.3

In Exercises 1–10, determine the Fourier series for the indicated periodic function. In each case, sketch the periodic function.

1. $f(x) = x,\ -1 \le x < 1, f(x+2) = f(x)$
2. $f(x) = x^2,\ -\pi \le x < \pi, f(x+2\pi) = f(x)$
*3. $f(x) = x^3,\ -\pi \le x < \pi, f(x+2\pi) = f(x)$
4. $f(x) = x(1-x),\ -1 \le x < 1, f(x+2) = f(x)$
5. $f(x) = \begin{cases} -2, & -2 \le x < 0 \\ 0, & 0 \le x < 2 \end{cases}, f(x+4) = f(x)$
6. $f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ 1, & 0 \le x < \pi \end{cases}, f(x+2\pi) = f(x)$
*7. $f(x) = \begin{cases} x, & -1 \le x < 0 \\ 1, & 0 \le x < 1 \end{cases}, f(x+2) = f(x)$
8. $f(x) = \cos^2 2x$
9. $f(x) = e^x,\ -1 \le x < 1, f(x+2) = f(x)$
10. $f(x) = \begin{cases} 0, & -1 \le x < 0 \\ e^{-x}, & 0 \le x \le 1 \end{cases}, f(x+2) = f(x)$

In Exercises 11–20, determine if the given periodic function is even, odd, or neither.

11. $f(x) = x^3,\ -1 \le x < 1, f(x+2) = f(x)$
12. $f(x) = 1 + 4x^2,\ -1 \le x < 1, f(x+2) = f(x)$
*13. $f(x) = x^2 - x,\ -\pi \le x < \pi, f(x+2\pi) = f(x)$
14. $f(x) = \begin{cases} -1, & -\pi \le x < 0 \\ 2, & 0 \le x < \pi \end{cases}, f(x+2\pi) = f(x)$
15. $f(x) = \begin{cases} -\cos x, & -\pi \le x < 0 \\ \cos x, & 0 \le x < \pi \end{cases}, f(x+2\pi) = f(x)$
16. $f(x) = |\sin x|,\ -\pi \le x < \pi, f(x+2\pi) = f(x)$
*17. $f(x) = x|x|,\ -2 \le x < 2, f(x+4) = f(x)$
18. $f(x) = e^{|x|},\ -1 \le x < 1, f(x+2) = f(x)$
19. $f(x) = x^2,\ 0 \le x < 1, f(x+1) = f(x)$
20. $f(x) = \begin{cases} 1, & 0 \le x < 1, \\ -1, & 1 \le x < 2 \end{cases}, f(x+2) = f(x)$
21. Show that if $f$ and $g$ are even functions, then $f+g$ is an even function.
22. Show that if $f$ and $g$ are odd functions, then $f+g$ is an odd function.
23. Suppose that $f$ is an even function and $g$ is an even function. Prove that $fg$ is an even function.
24. Suppose that $f$ is an even function and $g$ is an odd function. Prove that $fg$ is an odd function.
25. Verify the integrals in Table 9.2.
26. Show that the Fourier series coefficients of the constant function $f(x) = k$, where $k$ is a constant, are $a_0 = 2k$ and $a_n = b_n = 0$, $n = 1, 2, \ldots$. (*Hint:* Represent $f$ as the periodic function $f(x) = k$, $-p \le x < p, f(x+2p) = f(x)$.)
27. Show that the Fourier series coefficients of the sum $(f+g)$ are the sums of the corresponding Fourier series coefficients of $f$ and $g$.
28. Use the conclusion of Exercises 26 and 27 to find the Fourier series of **(a)** $f(x) = \begin{cases} 2, & -2 \le x < 0 \\ 3, & 0 \le x \le 2 \end{cases}$, where $f(x+4) = f(x)$. (*Hint:* See Example 1.); **(b)** $f(x) = x + 2,\ -1 \le x < 1, f(x+2) = f(x)$. (*Hint:* See Example 5(b).); **(c)** $f(x) = |x| - 1,\ -1 \le x < 1$, $f(x+2) = f(x)$. (*Hint:* See Example 5(a).)
29. Show that the Fourier series coefficients of the function $cf$, where $c$ is a constant, are $c$ times the corresponding Fourier series coefficients of $f$.
30. Use the conclusion of Exercises 26, 27, and 28 to find the Fourier series of **(a)** $f(x) = \begin{cases} 2, & -2 \le x < 0 \\ 4, & 0 \le x \le 2 \end{cases}$, where $f(x+4) = f(x)$; **(b)** $f(x) = -|x|,\ -1 \le x < 1$, $f(x+2) = f(x)$; **(c)** $f(x) = 1 - |x|,\ -1 \le x < 1$, $f(x+2) = f(x)$.

In Exercises 31–32, use the Fourier series

$$\frac{p^2}{3} + \frac{4p^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{p}$$

for $f(x) = x^2,\ -p \le x < p, f(x+2p) = f(x)$ to verify the given infinite sum.

31. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$
32. $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{12}$
33. Use the results of Exercises 31 and 32 to show that $1 + \frac{1}{9} + \frac{1}{25} + \cdots = \frac{\pi^2}{8}$.



In Exercises 34–37, find the value to which the Fourier series converges at the indicated values of $x$. In each case, sketch the periodic function.

34. $f(x) = \begin{cases} 1, & -1 \le x < 0 \\ -2, & 0 \le x \le 1 \end{cases}, f(x+2) = f(x)$. **(a)** $x = 0$, **(b)** $x = -1$, **(c)** $x = 1$, **(d)** $x = 1/2$, **(e)** $x = -1/2$.
35. $f(x) = \begin{cases} -x, & -\pi \le x < 0 \\ \pi, & 0 \le x \le \pi \end{cases}, f(x+2\pi) = f(x)$. **(a)** $x = 0$, **(b)** $x = -\pi$, **(c)** $x = \pi$, **(d)** $x = -\pi/4$, **(e)** $x = 5\pi/4$.
36. $f(x) = \begin{cases} -4, & -\pi \le x < -\frac{\pi}{2} \\ 1, & -\frac{\pi}{2} \le x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \le x \le \pi \end{cases}, \quad f(x+2\pi) = f(x)$.
**(a)** $x = 0$, **(b)** $x = -\pi/2$, **(c)** $x = \pi/2$, **(d)** $x = 3\pi/4$, **(e)** $x = -\pi$.
*37. $f(x) = \begin{cases} 2, & -1 \le x < -\frac{1}{2} \\ x, & -\frac{1}{2} \le x < \frac{1}{2} \\ -1, & \frac{1}{2} \le x \le 1 \end{cases}, f(x+2) = f(x)$,
**(a)** $x = 0$, **(b)** $x = -\frac{1}{2}$, **(c)** $x = \frac{1}{2}$, **(d)** $x = -1$, **(e)** $x = 1$.

In Exercises 38–41, **(a)** sketch the even extension of the given function; **(b)** find the Fourier series of the even extension.

38. $f(x) = x^2,\ 0 \le x < 1$
39. $f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 2, & 1 \le x < 2 \end{cases}$
40. $f(x) = 1 - x,\ 0 \le x < \pi$
*41. $f(x) = x(\pi - x),\ 0 \le x < \pi$

In Exercises 42–45, **(a)** sketch the odd extension of the function from the given exercise; **(b)** find the Fourier series of the odd extension.

42. $f(x) = x^2,\ 0 \le x < 1$
43. $f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 2, & 1 \le x < 2 \end{cases}$
44. $f(x) = 1 - x,\ 0 \le x < \pi$
*45. $f(x) = x(\pi - x),\ 0 \le x < \pi$

In Exercises 46–49, **(a)** sketch the periodic extension of the function from the given exercise; **(b)** find the Fourier series of the periodic extension.

46. $f(x) = x^2,\ 0 \le x < 1$
47. $f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 2, & 1 \le x < 2 \end{cases}$
48. $f(x) = 1 - x,\ 0 \le x < \pi$
*49. $f(x) = x(\pi - x),\ 0 \le x < \pi$
50. Show that if $f$ and $g$ are periodic functions of period $T$, then the sum $(f+g)$ is a periodic function of period $T$.
51. If $f$ and $g$ are periodic functions of period $T_1$ and $T_2$, then is the sum $(f+g)$ necessarily a periodic function? If not, then what conditions on $T_1$ and $T_2$ cause $(f+g)$ to be periodic?
52. **(Fourier Series—Complex Form) (a)** Use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $\cos nx = \frac{1}{2}(e^{inx} + e^{-inx})$ and $\sin nx = \frac{1}{2i}(e^{inx} - e^{-inx})$. **(b)** Write the Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

of the function $f$ of period $2\pi$ as

$$f(x) = c_0 + \sum_{n=1}^{\infty}(c_n e^{inx} + \tilde{c}_n e^{-inx}).$$

**(c)** Show that

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx} \text{ and } \tilde{c}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx.$$

**(d)** Using the notation $c_{-n} = \tilde{c}_n$, show that the series can be written in the complex form $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx,\ n = 0, \pm 1, \pm 2, \ldots$$

In Exercises 53–54, find the complex form of the Fourier series for the given function.

53. $f(x) = x,\ -\pi \le x < \pi, f(x+2\pi) = f(x)$
54. $f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ 1, & 0 \le x < \pi \end{cases}, f(x+2\pi) = f(x)$
55. **(The Weierstrass $M$-Test)** If $\sum_{n=1}^{\infty} M_n$ is a convergent series of positive constants and $|a_n(x)| \le M_n$ on the

interval $I$ of $x$, then $\sum_{n=1}^{\infty} a_n(x)$ is uniformly convergent on $I$. Show that $\sum_{n=1}^{\infty} \frac{1}{n^2} \sin nx$ is uniformly convergent on $-\infty < x < \infty$.

**56.** **(a)** Show that the Fourier series for $f(x) = x^2$, $-1 \le x < 1$, $f(x+2) = f(x)$, is

$$\frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2} \cos n\pi x.$$

**(b)** Use the Weierstrass $M$-test to show that this series converges uniformly for $-\infty < x < \infty$. **(c)** Graph several periods of this function simultaneously with a selected number of terms of its Fourier series to observe the convergence.

**57.** **(Term-by-Term Differentiation)** If $f(x)$ is periodic, continuous, and piecewise smooth (that is, $f'(x)$ exists except perhaps at a finite number of values of $x$ and $f'(x)$ is piecewise continuous), then the differentiated Fourier sreies of $f(x)$ converges to $f'(x)$ at every value of $x$ where $f''(x)$ exists. **(a)** Differentiate the function in Exercise 56 to obtain $g(x) = 2x$, $-1 \le x < 1$, $g(x+2) = g(x)$. Compare the direct calculation of the Fourier series of $g$ with that obtained by term-by-term differentiation of the series in Exercise 56. Are they the same? **(b)** Is the convergence of the Fourier series of $g$ uniform on $-\infty < x < \infty$? (*Note:* Graph several terms of the Fourier series together with $g$. Is the convergence the same as it was in Exercise 56? If the graph exhibits the Gibbs phenomenon, the convergence is not uniform.) **(c)** Can we differentiate the terms of the Fourier series for $g$ to obtain that of $h(x) = 2$, $-1 \le x < 1$, $h(x+2) = h(x)$? Why or why not? **(d)** What conclusion(s) do you draw concerning the relationship between uniform convergence of the Fourier series of a function and the ability to differentiate the Fourier series term-by-term to obtain the Fourier series of the derivative of the function? What property must the periodic function $f$ possess so that the convergence by its Fourier series is uniform?

**58.** Compare the rate of convergence of the Fourier series of $f$ and $g$ in Exercises 56 and 57. Which converges more quickly and why? (*Hint:* Consider the Fourier series coefficients.)

**59.** **(Term-by-Term Integration)** If $f(x)$ is periodic and piecewise continuous, then the Fourier series for $f(x)$ can be integrated term-by-term. **(a)** Show that the Fourier series for $f(x) = (\pi - x)/2$ on $0 < x < 2\pi$ is $\sum_{n=1}^{\infty} (\sin nx)/n$. **(b)** Verify that $f(x)$ satisfies the conditions for term-by-term integration. **(c)** Show that $\int_0^b f(x)\,dx = b(2\pi - b)/4$. **(d)** Integrate term-by-term the Fourier series for $f(x)$. **(e)** Replace $b$ in the expression in part **(c)** by $x$ to show that

$$x(2\pi - x)/4 = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}.$$

**(f)** The sum of the series $\sum_{n=1}^{\infty} 1/n^2$ represents the constant $a_0/2$ in the Fourier series for $x(2\pi - x)/4$ on $0 < x < 2\pi$. Determine the value of $\sum_{n=1}^{\infty} 1/n^2$ by finding $a_0$. **(g)** What is the value of

$$\frac{1}{\pi}\int_0^{2\pi} \frac{x(2\pi - x)}{4} \cos nx?$$

**(h)** What is the value of

$$\frac{1}{\pi}\int_0^{2\pi} \frac{x(2\pi - x)}{4} \sin nx?$$

**60.** **(a)** Calculate $\int_0^{\pi} f(x)\,dx$ for the function $f(x) = (\pi - x)/2$ in Exercise 59. **(b)** Use term-by-term integration of $\sum_{n=1}^{\infty} (\sin nx)/n$ to approximate the value of $\int_0^{\pi} f(x)\,dx$. How many terms are needed so that the difference between the exact and approximate values is less than 0.01?

**61.** Find the Fourier series for each of the periodic functions. In each case, graph the function and an approximation using several terms of the series.

**(a)** $f(x) = x^3$, $-\pi \le x < \pi$, $f(x + 2\pi) = f(x)$

**(b)** $f(x) = x^2(1 - x)$, $-\pi \le x < \pi$, $f(x + 2\pi) = f(x)$

***(c)** $f(x) = x^4$, $-\pi \le x < \pi$, $f(x + 2\pi) = f(x)$

**(d)** $f(x) = x(1 - x^3)$, $-\pi \le x < \pi$, $f(x + 2\pi) = f(x)$

**62.** **(Fourier Series on Any Interval)** If $f$ is a periodic function of period $T = 2p$ defined on $[s, s + T]$, then its Fourier series coefficients are $a_n = (1/p)\int_s^{s+T} f(x)\cos(n\pi x/p)\,dx$, $n = 0, 1, 2, \ldots$, and



$b_n = (1/p)\int_s^{s+T} f(x)\sin(n\pi x/p)\,dx$, $n = 1, 2, \ldots$.

Find the Fourier series for the given periodic function:

**(a)** $f(x) = 2 - x$, $1 \le x < 3$, $f(x+2) = f(x)$

**(b)** $f(x) = 2x - \pi$, $\pi \le x < 2\pi$, $f(x + \pi) = f(x)$

**(c)** $f(x) = \begin{cases} 0, & -1/2 \le x < 1/2 \\ 1, & 1/2 \le x < 3/2 \end{cases}$, $f(x+2) = f(x)$

**(d)** $f(x) = \begin{cases} x, & -1 \le x < 1/2 \\ 1, & 1/2 \le x < 2 \end{cases}$, $f(x+3) = f(x)$

**63.** In Example 4 we showed that $1 - 1/3 + 1/5 + \cdots = \pi/4$. Experimentally determine the number terms of the series that are needed so that the first five digits of the approximation of $\pi/4$ agree. (*Suggestion:* Use a graphing utility to graph two periods of the periodic extension of the functions in Exercises 1–10. Graph an approximation for each function using several terms of the series. Also, carry out similar graphing techniques with Exercises 38–49.)

## 9.4 Generalized Fourier Series

In addition to the trigonometric eigenfunctions that were used to form the Fourier series in Sections 9.2 and 9.3, the eigenfunctions of other eigenvalue problems can be used to form what we call **generalized Fourier series.** We find that these series will assist in solving problems in applied mathematics that involve physical phenomena that cannot be modeled with trigonometric functions.

Recall **Bessel's equation of order zero**

$$x^2y'' + xy' + \lambda^2x^2y = 0,$$

with linearly independent solutions $J_0(\lambda x)$ and $Y_0(\lambda x)$. If we require that the solutions of this differential equation satisfy the boundary conditions $|y(0)| < \infty$ (meaning that the solution is bounded at $x = 0$) and $y(p) = 0$, then we can find the eigenvalues of the boundary-value problem

$$\begin{cases} x^2y'' + xy' + \lambda^2x^2y = 0, & 0 < x < p \\ |y(0)| < \infty, \; y(p) = 0 \end{cases}.$$

A general solution of Bessel's equation is $y = c_1J_0(\lambda x) + c_2Y_0(\lambda x)$. Because $|y(0)| < \infty$, we must choose $c_2 = 0$ because $\lim_{x\to 0^+} Y_0(\lambda x) = -\infty$. Hence $y(p) = c_1J_0(\lambda p) = 0$. Just as we did with the eigenvalue problems solved earlier in Section 9.1, we cannot choose $c_1 = 0$, so we must select $\lambda$ so that $J_0(\lambda p) = 0$. Unfortunately, the values of $x$, where $J_0(\lambda x) = 0$, are not as easily expressed as they are with trigonometric functions. From our study of Bessel functions in Section 4.8 we know that this function intersects the $x$-axis in infinitely many places. In this case, we let $\alpha_n$ represent the $n$th zero of the Bessel function of order zero, $J_0$, where $n = 1, 2, \ldots$. These values are found in tables in many applied mathematics and physics books and can be approximated through the use of a computer algebra system as well. We list a few of them in Table 9.4 and show the graph of $J_0$ in Figure 9.17. By observing the graph, do the values in the table correspond to the roots of $J_0$?

TABLE 9.4

| $n$ | $\alpha_n$ |
|---|---|
| 1 | 2.4048 |
| 2 | 5.5201 |
| 3 | 8.6537 |
| 4 | 11.7915 |
| 5 | 14.9309 |

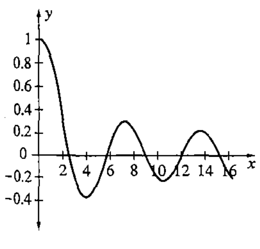


Figure 9.17 $J_0(x)$

Therefore, in trying to find the eigenvalues, we must solve $J_0(\lambda p) = 0$. From our definition of $\alpha_n$, this equation is satisfied when $\lambda p = \alpha_n$, $n = 1, 2, \ldots$. Hence the eigenvalues are

$$\lambda = \lambda_n = \frac{\alpha_n}{p},\ n = 1, 2, \ldots$$

and the corresponding eigenfunctions are

$$y(x) = y_n(x) = J_0(\lambda_n x) = J_0\left(\frac{\alpha_n x}{p}\right),\ n = 1, 2, \ldots.$$

As with the trigonometric eigenfunctions that we found in Section 9.2 and 9.3, $J_0(\alpha_n x/p)$ can be used to build an eigenfunction series expansion of the form

$$f(x) = \sum_{n=1}^{\infty} c_n J_0\left(\frac{\alpha_n x}{p}\right),$$

where we use the orthogonality properties of $J_0(\alpha_n x/p)$ to find the coefficients $c_n$.

We determine the orthogonality condition by placing Bessel's equation of order zero in the self-adjoint form

$$\frac{d}{dx}[xy'] + \lambda^2 xy = 0.$$

Because the weighting function is $s(x) = x$, the orthogonality condition is

$$\int_0^p xJ_0\left(\frac{\alpha_n x}{p}\right)J_0\left(\frac{\alpha_m x}{p}\right)dx = 0,\ n \neq m.$$

Multiplying $f(x) = \sum_{n=1}^{\infty} c_n J_0(\alpha_n x/p)$ by $s(x)J_0(\lambda_m x) = xJ_0(\alpha_m x/p)$ and integrating from $x = 0$ to $x = p$ yields

$$\int_0^p xf(x)\,J_0\left(\frac{\alpha_m x}{p}\right)dx = \int_0^p \sum_{n=1}^{\infty} c_n x J_0\left(\frac{\alpha_n x}{p}\right)J_0\left(\frac{\alpha_m x}{p}\right)dx$$

$$= \sum_{n=1}^{\infty} c_n \int_0^p xJ_0\left(\frac{\alpha_n x}{p}\right)J_0\left(\frac{\alpha_m x}{p}\right)dx.$$

However, by the orthogonality condition, each of the integrals on the right-hand side of the equation equals zero except when $m = n$. Therefore,

$$c_n = \frac{\displaystyle\int_0^p xf(x)J_0\left(\frac{\alpha_n x}{p}\right)dx}{\displaystyle\int_0^p x\left[J_0\left(\frac{\alpha_n x}{p}\right)\right]^2 dx},\ n = 1, 2, \ldots$$

The value of the integral in the denominator can be found through the use of several of the identities associated with the Bessel functions that were verified in Exercises 4.8. Because $\lambda = \lambda_n = \alpha_n/p$, $n = 1, 2, \ldots$, the function $J_0(\alpha_n x/p) = J_0(\lambda_n x)$ satisfies Bessel's equation of order zero given by



$$\frac{d}{dx}\left[x\frac{d}{dx}J_0(\lambda_n x)\right] + \lambda_n^2 xJ_0(\lambda_n x) = 0.$$

Multiplying by the factor

$$2x\frac{d}{dx}J_0(\lambda_n x),$$

we can write the expression as

$$\frac{d}{dx}\left[x\frac{d}{dx}J_0(\lambda_n x)\right]^2 + \lambda_n^2 x^2\frac{d}{dx}[J_0(\lambda_n x)]^2 = 0.$$

Integrating this expression from $x = 0$ to $x = p$, we have

$$2\lambda_n^2\int_0^p x[J_0(\lambda_n x)]^2\,dx = \lambda_n^2 p^2[J_0'(\lambda_n p)]^2 + \lambda_n^2 p^2[J_0(\lambda_n p)]^2.$$

Because $\lambda_n p = (\alpha_n/p)p = \alpha_n$, we make the following substitutions

$$2\lambda_n^2\int_0^p x[J_0(\lambda_n x)]^2\,dx = \lambda_n^2 p^2[J_0'(\alpha_n)]^2 + \lambda_n^2 p^2[J_0(\alpha_n)]^2.$$

Then $J_0(\alpha_n) = 0$, because $\alpha_n$ is the $n$th zero of $J_0$. Also, with $n = 0$, the identity

$$\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$$

indicates that $J_0'(\alpha_n) = -J_1(\alpha_n)$. Therefore,

$$2\lambda_n^2\int_0^p x[J_0(\lambda_n x)]^2\,dx = \lambda_n^2 p^2[-J_1(\alpha_n)]^2 + \lambda_n^2 p^2(0)$$

$$\int_0^p x[J_0(\lambda_n x)]^2\,dx = \frac{p^2}{2}[J_1(\alpha_n)]^2,$$

where the value of $J_1(\alpha_n)$ can be found in many applied mathematics and physics textbooks (or through the use of a computer algebra system). Therefore, the series coefficients are found with

$$c_n = \frac{2}{p^2[J_1(\alpha_n)]^2}\int_0^p xf(x)J_0\left(\frac{\alpha_n x}{p}\right)dx,\ n = 1, 2, \ldots.$$

**Example 1**

Find the Bessel–Fourier series for $f(x) = 1 - x^2$ on $0 < x < 1$.

***Solution*** In this case, $p = 1$, so

$$c_n = \frac{2}{[J_1(\alpha_n)]^2}\int_0^1 x(1-x^2)J_0(\alpha_n x)\,dx$$
$$= \frac{2}{[J_1(\alpha_n)]^2}\left\{\int_0^1 xJ_0(\alpha_n x)\,dx - \int_0^1 x^3J_0(\alpha_n x)\,dx\right\}.$$

Using the formula $d/dx\,[x^nJ_n(x)] = -x^nJ_{n-1}(x)$ with $n = 1$,

$$\int_0^1 xJ_0(\alpha_n x)\,dx = \left[\frac{-1}{\alpha_n}xJ_1(\alpha_n x)\right]_0^1 = \frac{-1}{\alpha_n}J_1(\alpha_n).$$

Note that the factor $1/\alpha_n$ is due to the chain rule for differentiating the argument of $J_1(\alpha_n x)$. We use integration by parts with $u = x^2$ and $dv = xJ_0(\alpha_n x)$ to evaluate $\int_0^1 x^3J_0(\alpha_n x)\,dx$. As in the first integral, we obtain $v = -\frac{1}{\alpha_n}J_1(\alpha_n x)$. Then, because $du = 2x\,dx$, we have

$$\int_0^1 x^3J_0(\alpha_n x)\,dx = \left[\frac{-1}{\alpha_n}x^3J_1(\alpha_n x)\right]_0^1 + \frac{2}{\alpha_n}\int_0^1 x^2J_1(\alpha_n x)\,dx$$
$$= \frac{-1}{\alpha_n}J_1(\alpha_n) - \frac{2}{\alpha_n}\left[\frac{1}{\alpha_n}x^2J_2(\alpha_n x)\right]_0^1 = \frac{-1}{\alpha_n}J_1(\alpha_n) - \frac{2}{\alpha_n^2}J_2(\alpha_n).$$

Thus the coefficients are

$$c_n = \frac{2}{[J_1(\alpha_n)]^2}\int_0^1 x(1-x^2)J_0(\alpha_n x)\,dx$$
$$= \frac{2}{[J_1(\alpha_n)]^2}\left\{\int_0^1 xJ_0(\alpha_n x)\,dx - \int_0^1 x^3J_0(\alpha_n x)\,dx\right\}$$
$$= \frac{2}{[J_1(\alpha_n)]^2}\left[\frac{-1}{\alpha_n}J_1(\alpha_n) - \left(\frac{-1}{\alpha_n}J_1(\alpha_n) - \frac{2}{\alpha_n^2}J_2(\alpha_n)\right)\right]$$
$$= \frac{4J_2(\alpha_n)}{\alpha_n^2[J_1(\alpha_n)]^2},\ n = 1, 2, \ldots$$

so that the Bessel–Fourier series is

$$f(x) = \sum_{n=1}^{\infty}\frac{4J_2(\alpha_n)}{\alpha_n^2[J_1(\alpha_n)]^2}J_0(\alpha_n x).$$

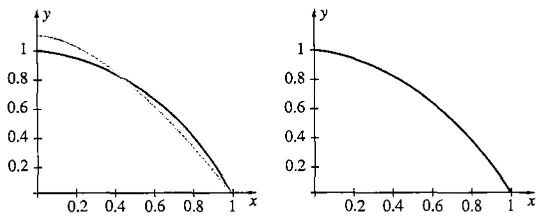
In Figure 9.18 we graph $f$ along with several terms of the series. Notice that the polynomial with two terms yields an accurate approximation of $f$. When using three or four terms, the graphs are practically indistinguishable. We list the values of $J_1(\alpha_n)$ and $J_2(\alpha_n)$ in Table 9.5, as well as show the graphs of these functions in Figure 9.19 and Figure 9.20, respectively.

**TABLE 9.5**

| $n$ | $J_1(\alpha_n)$ | $J_2(\alpha_n)$ |
|---|---|---|
| 1 | 0.5192 | 0.4318 |
| 2 | −0.3403 | −0.1233 |
| 3 | 0.2715 | 0.0627 |
| 4 | −0.2325 | −0.0394 |
| 5 | 0.2065 | 0.0277 |

As was the case with Fourier series, we can make a statement about the convergence of the Bessel–Fourier series.






(a) (b)

**Figure 9.18** (a) $p(x) = \dfrac{4J_2(\alpha_1)J_0(\alpha_1 x)}{\alpha_1^2[J_1(\alpha_1)]^2}$, (b) $p(x) = \dfrac{4J_2(\alpha_1)J_0(\alpha_1 x)}{\alpha_1^2[J_1(\alpha_1)]^2} + \dfrac{4J_2(\alpha_2)J_0(\alpha_2 x)}{\alpha_2^2[J_1(\alpha_2)]^2}$

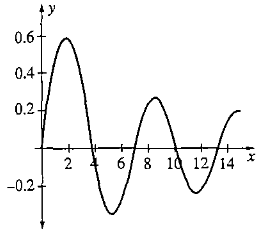



**Figure 9.19** $J_1(x)$

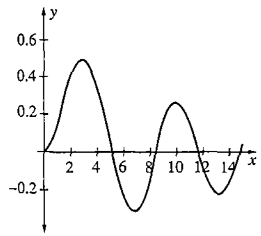



**Figure 9.20** $J_2(x)$

**Theorem 9.3 Convergence of Bessel–Fourier Series**

Suppose that $f$ and $f'$ are piecewise continuous functions on $0 < x < p$. Then the Bessel–Fourier series for $f$ on $0 < x < p$ converges to $f(x)$ at every $x$, where $f$ is continuous. At $x = x_0$, where $f$ is discontinuous, the Bessel–Fourier series converges to the average

$$\frac{f(x_0^+) + f(x_0^-)}{2}.$$

Series involving the eigenfunctions of other eigenvalue problems can be formed as well. We illustrate this procedure in the following example.

**Example 2**

In Example 2 in Section 9.1, we found that the problem $y'' + 2y' - (\lambda - 1)y = 0$, subject to $y(0) = 0$ and $y(2) = 0$, has eigenvalues $\lambda = \lambda_n = -k^2 = -(n\pi/2)^2$, $n = 1, 2, \ldots$ and eigenfunctions $y(x) = y_n(x) = e^{-x}\sin(n\pi x)/2$. Use these eigenfunctions to approximate $f(x) = e^{-x}$ on $[0, 2]$.

***Solution*** To approximate $f$, we need the orthogonality condition for these eigenfunctions. We obtain this condition by placing the differential equation in self-adjoint form using the formulas given in Section 9.1. In the general equation, $a_2(x) = 1$, $a_1(x) = 2$, and $a_0(x) = 0$. Therefore, $p(x) = e^{\int 2dx} = e^{2x}$ and $s(x) = p(x)/a_2(x) = e^{2x}$, so the equation is

$$\frac{d}{dx}\left[e^{2x}\frac{dy}{dx}\right] - (\lambda - 1)e^{2x}y = 0.$$

This means that the orthogonality condition, $\int_a^b s(x)y_m(x)y_n(x)\,dx = 0\ (m \neq n)$, is

$$\int_0^2 e^{2x}e^{-x}\sin\frac{m\pi x}{2}e^{-x}\sin\frac{n\pi x}{2}\,dx = \int_0^2 \sin\frac{m\pi x}{2}\sin\frac{n\pi x}{2}\,dx = 0\ (m \neq n).$$

We use this condition to determine the coefficients in the eigenfunction expansion

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x) = \sum_{n=1}^{\infty} c_n e^{-x}\sin\frac{n\pi x}{2}.$$

Multiplying both sides of the equation by $y_m(x) = e^{-x}\sin(m\pi x/2)$ and $s(x) = e^{2x}$ and then integrating from $x = 0$ to $x = 2$ yields

$$\int_0^2 f(x)e^{2x}e^{-x}\sin\frac{m\pi x}{2}\,dx = \int_0^2 \sum_{n=1}^{\infty} c_n e^{-x}\sin\frac{n\pi x}{2}e^{2x}e^{-x}\sin\frac{m\pi x}{2}\,dx$$

$$\int_0^2 f(x)e^{x}\sin\frac{m\pi x}{2}\,dx = \sum_{n=1}^{\infty}\int_0^2 c_n\sin\frac{n\pi x}{2}\sin\frac{m\pi x}{2}\,dx.$$

Each integral in the sum on the right-hand side of the equation is zero, except when $m = n$. In this case, $\int_0^2 \sin^2(n\pi x/2)\,dx = 1$. Therefore,

$$c_n = \int_0^2 f(x)e^x\sin\frac{n\pi x}{2}\,dx.$$

For $f(x) = e^{-x}$,

$$c_n = \int_0^2 e^x e^{-x}\sin\frac{n\pi x}{2}\,dx = \int_0^2 \sin\frac{n\pi x}{2}\,dx$$

$$= \left[-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_0^2 = -\frac{2}{n\pi}(\cos n\pi - 1)$$

Because $\cos n\pi$ is equivalent to $(-1)^n$, we can write the eigenfunction expansion of $f$ as

$$f(x) = \sum_{n=1}^{\infty} -\frac{2}{n\pi}\left((-1)^n - 1\right)e^{-x}\sin\frac{n\pi x}{2}$$

$$= \frac{4}{\pi}e^{-x}\sin\frac{\pi x}{2} + \frac{4}{3\pi}e^{-x}\sin\frac{3\pi x}{2} + \frac{4}{5\pi}e^{-x}\sin\frac{5\pi x}{2} + \cdots.$$

In Figure 9.21, we graph the approximations obtained with the first three nonzero terms of the series.

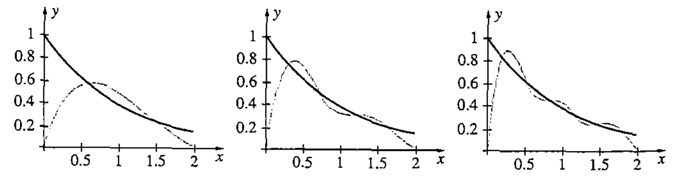



**Figure 9.21** (a) (b) (c)



## EXERCISES 9.4

For each eigenvalue problem, verify the given eigenfunctions. Then use the eigenfunctions to obtain the generalized Fourier series for each of the indicated functions $f(x)$.

1. $y'' + \lambda y = 0$, $0 < x < 1$, $y(0) = 0$, $y'(1) = 0$; $y_n(x) = \sin\frac{(2n-1)\pi x}{2}$, $n = 1, 2, \ldots$; $f(x) = 1$, $f(x) = x$, $f(x) = x^2$
2. $y'' + \lambda y = 0$, $0 < x < 1$, $y'(0) = 0$, $y(1) = 0$; $y_n(x) = \cos\frac{(2n-1)\pi x}{2}$, $n = 1, 2, \ldots$; $f(x) = 1$, $f(x) = x$, $f(x) = x^2$

*3. $y'' + 2y' + (1 - \lambda)y = 0$, $0 < x < 1$, $y(0) = 0$, $y(1) = 0$; $y_n(x) = e^{-x}\sin n\pi x$, $n = 1, 2, \ldots$; $f(x) = 1$, $f(x) = \begin{cases} 1, & 0 < x < 1/2 \\ 0, & 1/2 \le x \le 1\end{cases}$

4. $y'' + y' + \lambda y = 0$, $0 < x < 3$, $y(0) = 0$, $y(3) = 0$; $y_n(x) = e^{-x/2}\sin\frac{n\pi x}{3}$, $n = 1, 2, \ldots$; $f(x) = 1$, $f(x) = \begin{cases} 1, & 0 < x < 3/2 \\ -1, & 3/2 \le x \le 3\end{cases}$
5. $y'' + 2y' + (\lambda + 1)y = 0$, $0 < x < 5$, $y(0) = 0$, $y(5) = 0$; $y_n(x) = e^{-x}\sin\frac{n\pi x}{5}$, $n = 1, 2, \ldots$, $f(x) = 1$, $f(x) = \begin{cases} 1, & 0 < x < 3 \\ 0, & 3 \le x \le 5\end{cases}$
6. $y'' + 4y' + (4 + 9\lambda)y = 0$, $0 < x < 2$, $y(0) = 0$, $y(2) = 0$; $y_n(x) = e^{-2x}\sin\frac{n\pi x}{2}$, $n = 1, 2, \ldots$; $f(x) = 1$, $f(x) = \begin{cases} 1, & 0 < x < 1 \\ -1, & 1 \le x \le 2\end{cases}$

*7. Solve the eigenvalue problem $y'' + y' + (1 - \lambda)y = 0$, $y(0) = 0$, $y(1) = 0$. Use the eigenvalues and eigenfunctions obtained to form an eigenfunction expansion of an integrable function $g(x)$ on the interval $0 < x < 1$. (*Hint:* The eigenfunctions are orthogonal with respect to the weighting function $s(x) = e^x$, which is obtained by placing the differential equation in self-adjoint form.)

8. Use the eigenfunction expansion obtained in Exercise 7 to approximate $f(x) = x$ on the interval $0 < x < 1$.
9. Use the eigenfunction expansion obtained in Exercise 7 to approximate $f(x) = x^2$ on the interval $0 < x < 1$

**(Fourier–Legendre Series)** In Exercises 10–20, we investigate the use of Legendre polynomials in eigenfunction expansions.

10. The Legendre polynomials satisfy the **Rodrigues formula,** $P_n(x) = \frac{1}{2^n n!}\frac{d}{dx^n}[(x^2 - 1)^n]$. Use this formula and integrate by parts $n$ times to show that $\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}$, $n = 0, 1, 2, \ldots$.
11. Show that the coefficients of the Fourier–Legendre series $f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$ for the function $f$ defined on $(-1, 1)$ are $c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx$. (*Hint:* Use the orthogonality condition of the Legendre polynomials, $\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0$, $m \neq n$, and the result of Exercise 10.)
12. Find the coefficients $c_n$ for $n = 0, 1, 2$, and 3 for the Fourier–Legendre series for $f(x) = \begin{cases} 1, & -1 \le x < 0 \\ 0, & 0 \le x < 1\end{cases}$.
13. Find the coefficients $c_n$ for $n = 0, 1, 2$, and 3 for the Fourier–Legendre series for $f(x) = \begin{cases} 0, & -1 \le x < 0 \\ 2x, & 0 \le x < 1\end{cases}$.
14. If $f$ is an even function, then all powers of $x$ in $f$ must be even nonnegative integers. Are the functions $P_{2n}(x)$, $n = 0, 1, \ldots$, even? Therefore, if $f$ is an even function, then the coefficients (with odd subscripts) $c_{2n+1} = 0$. Show that the other coefficients are given by $c_{2n} = (4n + 1)\int_0^1 f(x)P_{2n}(x)\,dx$.
15. Find the first three nonzero coefficients in the Fourier–Legendre series for $f(x) = |x|$, $-1 < x < 1$.
16. If $f$ is an odd function, then all powers of $x$ in $f$ must be odd nonnegative integers. Are the functions $P_{2n+1}(x)$, $n = 0, 1, \ldots$, odd? Therefore, if $f$ is an odd function, then the coefficients (with even subscripts) $c_{2n} = 0$. Show that the other coefficients are given by $c_{2n+1} = (4n + 3)\int_0^1 f(x)P_{2n+1}(x)\,dx$.
17. Find the first three nonzero coefficients in the Fourier–Legendre series for $f(x) = \begin{cases} -1, & -1 \le x < 0 \\ 1, & 0 \le x < 1\end{cases}$.

18. Term-by-term differentiation was discussed in Section 9.3 with Fourier series. Can a Fourier–Legendre series be differentiated term-by-term? Consider the functions in Exercises 15 and 17.

19. One of the more common uses of Legendre polynomials is in solving Laplace's equation (a partial differential equation) in spherical coordinates. In this case, solutions appear in the form $P_n(\cos\theta)$. Show that the coefficients in the series of the form $f(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta)$ are found with $c_n = \frac{2n+1}{2}\int_0^{\pi} f(\theta)P_n(\cos\theta)\sin\theta\,d\theta$.

20. Use the formula in Exercise 19 to find the first three coefficients in the series $f(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta)$ if $f(\theta) = \theta$, $0 < \theta < \pi$.

21. Find the first ten coefficients of the Bessel–Fourier series for **(a)** $f(\theta) = \theta$, $0 \le \theta \le \pi$, **(b)** $f(\theta) = \begin{cases} \theta, & 0 \le \theta < \pi/2 \\ \pi/2 - \theta, & \pi/2 \le \theta \le \pi \end{cases}$. In each case, graph $f$ simultaneously with the Bessel–Fourier polynomial using the first ten coefficients.

22. Verify the Rodrigues formula, $P_n(x) = \frac{1}{2^n n!}\frac{d}{dx^n}[(x^2-1)^n]$, for $n = 0, 1, \ldots, 5$.

23. The function $\frac{1}{\sqrt{1-2xu+u^2}}$ is called a **generating function** for the Legendre polynomials because
$$\frac{1}{\sqrt{1-2xu+u^2}} = \sum_{n=0}^{\infty} P_n(x)u^n.$$
Verify this relationship by using the binomial expansion for $\frac{1}{\sqrt{1-w}}$ with $w = 2xu - u^2$. Expand the expression that results and collect like terms in the powers of $u$.

24. In Exercise 11, the coefficients of the Fourier–Legendre series $f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$ for the function $f$ defined on $(-1, 1)$ are found to be $c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx$. Find the first ten coefficients for the functions **(a)** $f(x) = \begin{cases} 1, & -1 \le x < 0 \\ 1-x, & 0 \le x < 1 \end{cases}$, **(b)** $f(x) = \begin{cases} x^2, & -1 \le x < 0 \\ x, & 0 \le x < 1 \end{cases}$. In each case, graph $f$ simultaneously with the Bessel–Fourier polynomial using the first ten coefficients.

## CHAPTER 9 SUMMARY

## Concepts & Formulas

### Section 9.1

**Eigenvalue Problems**

Values of the parameter that satisfy the differential equation are called **eigenvalues** of the problem, and for each eigenvalue the nontrivial solution $y$ that satisfies the problem is called the corresponding **eigenfunction.**

**Sturm–Liouville Problems**

$a_2(x)y''(x) + a_1(x)y'(x) + [a_0(x) + \lambda]y(x) = 0$, $a < x < b$, with boundary conditions $\alpha_1 y(a) + \alpha_2 y'(a) = \alpha_0$, $\beta_1 y(b) + \beta_2 y'(b) = 0$.

**Self-Adjoint Form of the Sturm–Liouville Problem**

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + (q(x) + \lambda s(x))y = 0,$$
where
$$p(x) = e^{\int \frac{a_1(x)}{a_2(x)}dx},\ q(x) = \frac{a_0(x)}{a_2(x)}p(x), \text{ and } s(x) = \frac{p(x)}{a_2(x)}.$$

**Linear Independence and Orthogonality of Eigenfunctions**

If $y_m(x)$ and $y_n(x)$ are eigenfunctions of the regular Sturm–Liouville problems, where $m \ne n$, then $y_m(x)$ and $y_n(x)$ are linearly independent, and the orthogonality condition $\int_a^b s(x)y_m(x)y_n(x)\,dx = 0$ $(m \ne n)$ holds.



### Section 9.2

**Fourier Sine Series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{p}, \text{ where } b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi x}{p}\,dx$$

**Fourier Cosine Series**

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{p},$$
where $a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi x}{p}\,dx$, $n = 0, 1, 2, \ldots$

### Section 9.3

**Fourier Series**

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{p} + b_n\sin\frac{n\pi x}{p}\right), \text{ where}$$
$$a_n = \frac{1}{p}\int_{-a}^{a} f(x)\cos\frac{n\pi x}{p}\,dx,\ n = 0, 1, 2, \ldots, \text{ and}$$
$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}\,dx,\ n = 1, 2, \ldots$$

### Section 9.4

**Generalized Fourier Series**

$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$, where the $y_n(x)$ are eigenfunctions of an eigenvalue problem and the coefficients $c_n$ are found with the orthogonality condition of the eigenfunctions.

**Bessel–Fourier Series**

$f(x) = \sum_{n=1}^{\infty} c_n J_0\left(\frac{\alpha_n x}{p}\right)$, where
$$c_n = \frac{\int_0^p x f(x) J_0\left(\frac{\alpha_n x}{p}\right)dx}{\int_0^p x\left[J_0\left(\frac{\alpha_n x}{p}\right)\right]^2 dx},\ n = 1, 2, \ldots.$$

## CHAPTER 9 REVIEW EXERCISES

1. The shape of a quickly turning jump rope can be determined by solving an eigenvalue problem. Suppose that the rope of length $L$ and constant linear density $\rho$ is whirled with a constant angular speed $\omega$. Assume that the rope whirls about the $x$-axis so that every point on the rope moves in a circle about the $x$-axis. Let $y(x)$ denote the displacement of the rope from the axis of rotation. By considering all of the forces acting on the rope, we find that $y$ satisfies the equation $Ty'' + \rho\omega^2 y = 0$, where $T$ is the constant tension on all points along the rope. If we let $\lambda = \rho\omega^2/T$, then this equation is $y'' + \lambda y = 0$. Because the ends of the rope are held at a constant height along the axis, we also have the boundary conditions $y(0) = 0$ and $y(L) = 0$. **(a)** Solve the eigenvalue problem $\begin{cases} y'' + \lambda y = 0, & 0 < x < L \\ y(0) = 0, & y(L) = 0 \end{cases}$. **(b)** What are the eigenvalues? **(c)** For what values of $\omega$ is $y$ a function other than $y = 0$? (These values are called the critical speeds of angular rotation.)

2. The beam problem that was discussed in Section 5.4 can be interpreted as an eigenvalue problem. Let $y(x)$ represent the displacement from the equilibrium position of the beam, the $x$-axis. Then the situation of the displacement of a beam that is subjected to a compressive force is modeled by the differential equation $(d^2/dx^2)[EIy''] - Py'' = 0$, $0 < x < L$, where $E$ is the

modulus of the beam material, $I$ is the moment of inertia of the beam's cross-section, and $P$ is the magnitude of the compressive force. If $E$, $I$, and $P$ are constant, then let $\lambda = P/EI$, so that the differential equation becomes $y^{(4)} - \lambda y = 0$. As we recall, the beam can be subjected to a wide variety of boundary conditions as listed in Table 9.6, where each endpoint has a pair of conditions. Investigate the solution of this eigenvalue problem for each of the following cases.

**(a)** Fixed-end conditions at $x = 0$ and $x = a$

**(b)** Free end at $x = 0$ and $x = a$

**(c)** Simple support at $x = 0$ and $x = a$

**(d)** Sliding clamped end at $x = 0$ and $x = a$

**(e)** Fixed end at $x = 0$; simple support at $x = a$

**(f)** Free end at $x = 0$; sliding clamped end at $x = a$

Because of the relationship $\lambda = \lambda_n = P/EI$, we have $P = P_n = \lambda_n EI$. Therefore, we can interpret $P_1$ as the largest load that the beam can withstand before it buckles. Determine $P_1$ for each of the cases given previously.

**TABLE 9.6**

| | | |
|---|---|---|
| Fixed End | $y = 0$ | $y' = 0$ |
| Free End | $y'' = 0$ | $y''' = 0$ |
| Simple Support | $y = 0$ | $y'' = 0$ |
| Sliding Clamped End | $y' = 0$ | $y''' = 0$ |

In Exercises 3–6, find the Fourier sine series of each of the following functions.

**3.** $f(x) = \begin{cases} 1, & 0 < x \le \pi \\ -1, & \pi < x \le 2\pi \end{cases}$

**4.** $f(x) = \begin{cases} 0, & 0 < x \le \pi \\ 1, & \pi < x \le 2\pi \end{cases}$

***5.** $f(x) = \begin{cases} x, & 0 < x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

**6.** $f(x) = \begin{cases} 1 - x, & 0 < x \le 1 \\ 0, & 1 < x \le 2 \end{cases}$

In Exercises 7–10, compute the Fourier cosine series of each of the functions in the indicated exercise.

**7.** $f(x) = \begin{cases} 1, & 0 < x \le \pi \\ -1, & \pi < x \le 2\pi \end{cases}$

**8.** $f(x) = \begin{cases} 0, & 0 < x \le \pi \\ 1, & \pi < x \le 2\pi \end{cases}$

***9.** $f(x) = \begin{cases} x, & 0 < x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

**10.** $f(x) = \begin{cases} 1 - x, & 0 < x \le 1 \\ 0, & 1 < x \le 2 \end{cases}$

In Exercises 11–16, compute the Fourier series for each periodic function.

**11.** $f(x) = \begin{cases} 1, & -\pi \le x < -\pi/2 \\ 0, & -\pi/2 \le x < \pi/2 \\ 1, & \pi/2 \le x < \pi \end{cases}$, $f(x + 2\pi) = f(x)$

**12.** $f(x) = \begin{cases} -1, & -\pi \le x < 0 \\ 1, & 0 \le x < \pi \end{cases}$, $f(x + 2\pi) = f(x)$

***13.** $f(x) = \begin{cases} x, & -\pi \le x < 0 \\ 1, & 0 \le x < \pi \end{cases}$, $f(x + 2\pi) = f(x)$

**14.** $f(x) = \begin{cases} x, & -\pi \le x < 0 \\ 0, & 0 \le x < \pi \end{cases}$, $f(x + 2\pi) = f(x)$

**15.** $f(x) = \begin{cases} 1, & -1 \le x < 0 \\ 1 - x, & 0 \le x < 1 \end{cases}$, $f(x + 2) = f(x)$

**16.** $f(x) = \begin{cases} x + 1, & -1 \le x < 0 \\ x, & 0 \le x < 1 \end{cases}$, $f(x + 2) = f(x)$

**17.** Verify that the Fourier series for $f(x) = x^2$, $-\pi < x < \pi$ is

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$

Use this series to find the sum of the following infinite series by choosing a suitable value of $x$:

**(a)** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

**(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$

**18.** Use the two series in **(a)** and **(b)** in the previous problem to show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

**19.** Show that the Fourier series for $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 \le x \le \pi \end{cases}$, $f(x + 2\pi) = f(x)$, is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}.$$

Use this result to verify that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

**20.** Consider the function $f(x) = \begin{cases} -1, & -2\pi \le x \le -\pi \\ x, & -\pi < x \le \pi \\ 1, & \pi < x \le 2\pi \end{cases}$, $f(x + 4\pi) = f(x)$. Determine the numerical value to which the Fourier series of $f$ converges at **(a)** $x = 0$; **(b)** $x = -\pi/2$; **(c)** $x = \pi$; **(d)** $x = 2\pi$; **(e)** $x = 5\pi/4$.

***21.** Using the trigonometric identities $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$ and $\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$, determine the Fourier series for $f(x) = \sin^3 x$ and $f(x) = \cos^3 x$.



## Differential Equations at Work

### A. Signal Processing

Suppose that we have a signal that is made up of oscillatory functions as well as a component known as noise. The goal of signal processing is to find the useful part of the signal while filtering out the noise. Let us suppose that the signal is

$$f(t) = A_0 + \sum_{n=1}^{10} a_n \cos(2n\pi t) + \sum_{n=1}^{10} b_n \sin(2n\pi t) + \text{noise}(t)$$

Then, because of the orthogonality properties of the functions $\cos(2n\pi t)$ and $\sin(2n\pi t)$ on the interval $[0, 1]$, we can determine $A_0$, $a_n$, and $b_n$, $(n = 1, 2, \ldots, 10)$ in the series expansion of the filtered portion of the signal with the calculation

$$\int_0^1 f(x)k(t, x)\,dx, \text{ where } k(t, x) = 2\sum_{m=1}^{10} \cos 2m\pi t \cos 2m\pi x + \sum_{m=1}^{10} \sin 2m\pi t \sin 2m\pi x.$$

(*Note:* The same results are obtained by calculating the integrals $A_0 = \int_0^1 f(x)\,dx$, $a_n = \int_0^1 f(x)\cos 2m\pi\,dx$, and $b_n = \int_0^1 f(x)\sin 2m\pi x\,dx$ individually.)

1. What is the frequency of $\cos 2m\pi x$ and $\sin 2m\pi x$?
2. Let $f(t) = g(t) + \frac{1}{10}\sin(30\pi t) - \frac{1}{4}$, where $g(t) = \begin{cases} 2t, & t < 1/4 \\ 1 - 2t, & 1/4 < t < 1/2 \\ 2t - 1, & 1/2 < t < 3/4 \\ 2 - 2t, & t > 3/4 \end{cases}$. Graph the signal on $[0, 1]$.
3. Filter the signal to obtain $A_0 + \sum_{n=1}^{10} a_n \cos 2n\pi t + \sum_{n=1}^{10} b_n \sin 2n\pi t$. Graph this function on $[0, 1]$.

4. Describe the differences between the original signal and that found in 2.
5. Take the difference between the original signal and that found in 2. What does this function represent? Graph this function on [0, 1].
6. What terms in an expansion of the form $f(t) = A_0 + \Sigma a_n \cos 2n\pi t + \Sigma b_n \sin 2n\pi t$ should be used so that we filter out frequencies greater than four?
7. Use the number of terms determined in 6 to filter out frequencies greater than four if the signal is $f(t) = \sin\left(\frac{1}{t + 1/40}\right)$. Graph the original signal, the filtered signal, and the noise on [0, 1].

## B. Forced Damped Spring–Mass System

Recall the forced spring–mass system

$$mx'' + cx' + kx = f(t)$$

that was first presented in Chapter 5. Here we take another approach to solving this problem. Suppose that $m = 1$ kg, $c = 0.05$ kg/s, $k = 50$ kg/s$^2$, and

$$f(t) = \begin{cases} 2/\pi, & 0 < t < \pi/4 \\ 0, & \pi/4 < t < 7\pi/4 \\ 2/\pi, & 7\pi/4 < t < 2\pi \end{cases}, f(t + 2\pi) = f(t).$$

1. Classify $f$ as even, odd, or neither, and use this information to show that the Fourier series for $f$ is

$$\frac{1}{2\pi} + \sum_{n=1}^{\infty} \frac{4}{n\pi^2} \sin\frac{n\pi}{4} \cos nt.$$

Therefore, the ODE becomes

$$x'' + 0.05x' + 50x = \frac{1}{2\pi} + \sum_{n=1}^{\infty} \frac{4}{n\pi^2} \sin\frac{n\pi}{4} \cos nt.$$

2. We now use the Method of Undetermined Coefficients to find a particular solution, $x_p(t)$, of this equation. For the constant term, $1/(2\pi)$, show that the corresponding portion of $x_p(t)$ is $1/(100\,\pi)$. Then for the terms $\frac{4}{n\pi^2}\sin\frac{n\pi}{4}\cos nt$, we assume that the particular solution has the form $A \cos nt + B \sin nt$. Substitute into the ODE to show that

$$A = \frac{4}{n\pi^2} \sin\frac{n\pi}{4} \frac{50 - n^2}{(50 - n^2)^2 + (0.05)^2 n^2} \text{ and}$$

$$B = \frac{4}{n\pi^2} \sin\frac{n\pi}{4} \frac{0.05n}{(50 - n^2)^2 + (0.05)^2 n^2}.$$

Does $x_p(t)$ correspond to the steady-state or transient solution to the ODE?



3. Use the values of $A$ and $B$ in 2 to express $A \cos nt + B \sin nt$ as $\alpha \cos(nt - \phi)$, where $\alpha = \sqrt{A^2 + B^2}$, $\cos\phi = A/\alpha$, and $\sin\phi = B/\alpha$.
4. Write out the first ten terms of $x_p(t)$.
5. What is the natural frequency of the spring–mass system if damping is excluded from the ODE? Which term in the series representation of $x_p(t)$ has the largest coefficient (amplitude)? Plot $x_p(t)$ over $[0, 2\pi]$. What is the approximate amplitude of $x_p(t)$ over most of this interval? How does this relate to the coefficients in $x_p(t)$?

## C. Approximations with Fourier Series

Suppose that we find the Fourier series for the function $f(x)$ on the interval $-p < x < p$, and suppose that we use the first $k$ terms of this Fourier series to approximate $f$. If we define the error in the approximation to be

$$E = f(x) - \left[\frac{1}{2}a_0 + \sum_{n=1}^{k}\left(a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right)\right],$$

then the size (or magnitude) of the error is

$$\|E\| = \int_{-p}^{p} [f(x)]^2\,dx - p\left[\frac{1}{2}a_0^2 + \sum_{n=1}^{k}(a_n^2 + b_n^2)\right].$$

Recall from Example 1 in Section 9.3 that the Fourier series for $f(x) = \begin{cases} 1, & -2 \le x < 0 \\ 2, & 0 \le x \le 2 \end{cases}$ is

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}(1 - (-1)^n)\frac{1}{n\pi}\sin\frac{n\pi x}{2} \quad \text{or}$$

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}\frac{2}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2}.$$

**(a)** Compute $\|E\|$ for $n = 10, 20, 30, 40, 50$. **(b)** What happens to the value of $\|E\|$ as $n$ increases?
**(c)** Because $\|E\| \to 0$ as $n \to \infty$, we have **Parseval's equality,**

$$\frac{1}{p}\int_{-p}^{p}[f(x)]^2\,dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Use Parseval's equality to determine the sum of the series

$$\sum_{n=1}^{\infty}\frac{4}{(2n-1)^2\pi^2}.$$

(How do we know that this series converges?) Verify your response by approximating the sum with

$$\sum_{n=1}^{1000}\frac{4}{(2n-1)^2\pi^2}.$$

(d) The Fourier series for $f(x) = |x|$, $-2 \leq x < 2$, $f(x+4) = f(x)$, is

$$f(x) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{2}.$$

Use Parseval's equality to find the sum of

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}.$$

# 10

# Partial Differential Equations

In the previous chapters, we saw that many physical and mathematical situations are described by ordinary differential equations. We also saw that others are described by partial differential equations. For example, previously we noted that the time-independent **Schrödinger equation in spherical coordinates** is given by the partial differential equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2} + \frac{2\mu}{h^2}(E-V)\Psi = 0.$$

In this chapter, we will investigate one way to solve some partial differential equations, the method of separation of variables, which was mostly developed by Fourier in his study of the heat equation. Many other methods have been developed to solve many partial differential equations, both analytically and numerically. Solving partial differential equations, when possible, can be a difficult problem. In fact, understanding the behavior of the solutions of most partial differential equations can be a difficult problem, as it is with most ordinary differential equations.

## 10.1 Introduction to Partial Differential Equations and Separation of Variables

Separation of Variables   Modeling with Partial Differential Equations

We begin our study of partial differential equations with an introduction to some of the terminology associated with the topic. A **linear second-order partial differential equation (PDE)** in the two independent variables $x$ and $y$ has the form

$$A(x, y)u_{xx} + B(x, y)u_{xy} + C(x, y)u_{yy} + D(x, y)u_x + E(x, y)u_y + F(x, y)u = G(x, y),$$

where the solution is $u(x, y)$. If $G(x, y) = 0$ for all $x$ and $y$, then we say that the equation is **homogeneous.** Otherwise, the equation is **nonhomogeneous.**

**Example 1**

Classify the following partial differential equations:
**(a)** $u_{xx} + u_{yy} = u$; **(b)** $u_{xy} + uu_x = x$.

***Solution*** **(a)** This equation satisfies the form of the linear second-order partial differential equation with $A = C = 1$, $F = -1$, and $B = D = E = 0$. Because $G(x, y) = 0$, the equation is homogeneous. **(b)** This equation is nonlinear because the coefficient of $u_x$ is a function of $u$. It is also nonhomogeneous, because $G(x, y) = x$.

**Definition 10.1 Solution of a Partial Differential Equation**

A solution of a partial differential equation in some region $R$ of the space of the independent variables is a function that possesses all of the partial derivatives that are present in the partial differential equation in some region containing $R$ and satisfies the partial differential equation everywhere in $R$.

**Example 2**

Show that $u(x, y) = y^2 - x^2$ and $u(x, y) = e^y \sin x$ are solutions to Laplace's equation $u_{xx} + u_{yy} = 0$.

***Solution*** For $u(x, y) = y^2 - x^2$, $u_x(x, y) = -2x$, $u_y(x, y) = 2y$, $u_{xx}(x, y) = -2$, and $u_{yy}(x, y) = 2$, so that we have $u_{xx} + u_{yy} = (-2) + 2 = 0$. Similarly, for $u(x, y) = e^y \sin x$, we have $u_x = e^y \cos x$, $u_y = e^y \cos x$, $u_{xx} = -e^y \sin x$, and $u_{yy} = e^y \sin x$. Therefore, $u_{xx} + u_{yy} = (-e^y \sin x) + e^y \sin x = 0$, so the equation is satisfied by both functions.



*Is $u(x, y) = \ln(x^2 + y^2)$ also a solution of $u_{xx} + u_{yy} = 0$?*

We notice that the solutions to Laplace's equation differ in form. This is unlike solutions to homogeneous linear ordinary differential equations, in which we found that solutions were similar in form. In this chapter we find a solution to a partial differential equation that models a physical situation by using the boundary or initial conditions associated with the physical situation. For example, if we require that the (time-independent) temperature $u(x, y)$ at each point in a rectangular region centered at $(0, 0)$ obtained by solving $u_{xx} + u_{yy} = 0$ be bounded at the origin, then the solution could not be $u(x, y) = \ln(x^2 + y^2)$, because $\lim_{(x, y)\to(0, 0)} \ln(x^2 + y^2) = -\infty$.

*If the solution to Laplace's equation must satisfy the boundary conditions $u(0, y) = u(\pi, y) = 0$, then which of the functions $u(x, y) = y^2 - x^2$ and $u(x, y) = e^y \sin x$ (that satisfy $u_{xx} + u_{yy} = 0$) must be eliminated?*

Some of the techniques used in constructing solutions of homogeneous linear ordinary differential equations can be extended to the study of partial differential equations, as we see with Theorem 10.1.

**Theorem 10.1 Superposition Principle**

If $u_1, u_2, \ldots, u_m$ are solutions to a linear homogeneous partial differential equation in a region $R$, then

$$c_1u_1 + c_2u_2 + \cdots + c_mu_m = \sum_{k=1}^{m} c_ku_k,$$

where $c_1, c_2, \ldots, c_m$ are constants, is also a solution in $R$.

The Superposition principle will be used in solving partial differential equations throughout the rest of the chapter. In fact, we will find that equations can have an infinite set of solutions so that we construct another solution in the form of an infinite series.

### Separation of Variables

A method that can be used to solve linear partial differential equations is called **separation of variables** (or the **product method**), first used by the French mathematician Joseph Fourier (1768–1830) in approximately 1807 when solving the partial differential equation that describes the temperature on a rectangular plate.* When the plate reaches its equilibrium point (independent of time), Fourier determined that the temperature $v(x, y)$ at the point with coordinates $(x, y)$ satisfies the equation

* Jesper Lutzen, "The Solution of Partial Differential Equations by Separation of Variables: A Historical Survey," *Studies in the History of Mathematics,* MAA Studies in Mathematics, Volume 26, Mathematical Association of America (1987), pp. 242–277.

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Fourier assumed that the temperature $v(x, y)$ could be written as the product $v(x, y) = X(x)Y(y)$. Substituting $v(x, y)$ into the equation results in

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = X''(x)Y(y) + X(x)Y''(y) = 0,$$

where $X'' = d^2X/dx^2$ and $Y'' = d^2Y/dy^2$, so that

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The only way that $X''(x)/X(x) = -Y''(y)/Y(y)$ for all values of $x$ and all values of $y$ is for $X''(x)/X(x)$ and $-Y''(y)/Y(y)$ to equal the same constant, say

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda^2.$$

This assumption results in the two ordinary differential equations

$$X''(x) - \lambda^2 X(x) = 0 \quad \text{and} \quad Y''(y) + \lambda^2 Y(y) = 0.$$

Each of these equations can be solved for $X$ and $Y$ (using the boundary and initial conditions based on the physical situation), so that the product $v(x, y) = X(x)Y(y)$ can be formed, which results in a solution of the partial differential equation.

In general, the goal of the method of separation of variables is to transform the partial differential equation into a system of ordinary differential equations, each of which depends on only one of the functions in the product form of the solution. Suppose that the function $u(x, y)$ is a solution of a partial differential equation in the independent variables $x$ and $y$. In separating variables, we assume that $u$ can be written as the product of a function of $x$ and a function of $y$. Hence

$$u(x, y) = X(x)Y(y),$$

and we substitute this product into the partial differential equation to determine $X(x)$ and $Y(y)$. Of course, to substitute into the differential equation we must be able to differentiate this product. However, this is accomplished by following the differentiation rules of multivariable calculus. For example, using the following short-hand notation for convenience, we have

$$u_x = X'Y, \quad u_{xx} = X''Y, \quad u_{xy} = X'Y', \quad u_y = XY', \quad \text{and} \quad u_{yy} = XY'',$$

where $X'$ represents $\dfrac{d}{dx}X(x)$ and $Y'$ represents $\dfrac{d}{dy}Y(y)$. After these substitutions are made, if the equation is separable we can obtain an ordinary differential equation for $X$ and an ordinary differential equation for $Y$. These two equations are then solved to find $X(x)$ and $Y(y)$.



**Example 3**

Use separation of variables to find a solution of $xu_x = u_y$.

**Solution** If $u(x, y) = X(x)Y(y)$, then $u_x = X'Y$ and $u_y = XY'$. The equation then becomes

$$xX'Y = XY',$$

which can be written as the separated equation

$$\frac{xX'}{X} = \frac{Y'}{Y}.$$

Notice that the left-hand side of the equation is a function of $x$ and the right-hand side is a function of $y$. Hence the only way that this situation can be true is for $xX'/X$ and $Y'/Y$ to both be constant. Therefore,

$$\frac{xX'}{X} = \frac{Y'}{Y} = k,$$

so we obtain the ordinary differential equations $xX' - kX = 0$ and $Y' - kY = 0$. We find $X$ in the manner that follows.

$$\begin{aligned} xX' - kX &= 0 \\ x\frac{dX}{dx} &= kX \\ \frac{dX}{X} &= \frac{k}{x}\,dx \\ \ln|X| &= k\ln|x| + c_1 \\ X(x) &= e^{c_1}x^k = C_1x^k \end{aligned}$$

Similarly, we find

$$\begin{aligned} Y' - kY &= 0 \\ Y' &= kY \\ \frac{dY}{dy} &= kY \\ \frac{dY}{Y} &= k\,dy \\ \ln|Y| &= ky + c_2 \\ Y(y) &= e^{c_2}e^{ky} = C_2e^{ky}. \end{aligned}$$

Therefore, a solution is $u(x, y) = X(x)Y(y) = (C_1x^k)(C_2e^{ky}) = C_3x^ke^{ky}$, where $k$ and $C_3 = C_1C_2$ are arbitrary constants. (Notice that the first equation is a Cauchy-

Euler equation, so we could have used the techniques covered in Section 4.7 to solve it. Similarly, we could have used a characteristic equation to solve the second equation.)

## Modeling with Partial Differential Equations

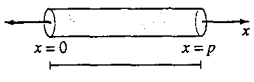


**Figure 10.1** Wire of length $p$

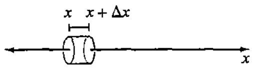


**Figure 10.2** Small cross-section of wire

Consider a wire (or thin rod) of length $p$, which is made of a material that conducts heat. Assuming that the wire is insulated so that no heat is lost or gained through the sides of the wire and that the thermal conductivity and specific heat from which the wire is made are constant, our goal is to determine the temperature $u(x, t)$ that represents the temperature in the wire $x$ units from $x = 0$ at time $t$. (See Figure 10.1).

We develop a mathematical interpretation by considering the heat flow through a small cross-section of the wire, as shown in Figure 10.2. To model the heat flow through the cross-section, we must make several simplifying assumptions.

1. The wire has uniform density (i.e., it has a uniform cross-section).
2. The heat flow across a small surface is directly proportional to the product of the rate of change in the temperature across the surface and the area of the surface.
3. The temperature in an object is proportional to the quotient of the flow of heat into the object and its mass.

(*Note:* Simplifying assumptions 2 and 3 are basic principles of heat conduction.) If $u(x, t)$ represents the temperature, then heat flow (the rate at which $u$ changes with respect to $x$) is given by $u_x$. Then if the area of the face of the cross-section is $A$, the heat flow into the cross-section is proportional to $-u_x(x, t)A$. Similarly, the heat flow out of the cross-section is proportional to $-u_x(x + \Delta x, t)A$. (Notice that the negative signs in these relationships are due to the fact that heat flows in the direction of decreasing temperature.) Hence if we let $H$ represent the amount of heat flow from left to right through the cross-section, then the net flow through the cross-section is

$$\Delta H = k[u_x(x + \Delta x, t)A - u_x(x, t)A] = kA[u_x(x + \Delta x, t) - u_x(x, t)]$$

By the Mean Value Theorem (MVT), there is a value $x = \hat{x}$ on the interval $(x, x + \Delta x)$ such that

$$u_{xx}(\hat{x}, t) = \frac{u_x(x + \Delta x, t) - u_x(x, t)}{\Delta x}.$$

Therefore,

$$\Delta H = kAu_{xx}(\hat{x}, t)\Delta x.$$

If the density of the wire is $\delta$, then the mass of the wire is $\delta A\ \Delta x$. Then according to the third simplifying assumption, where the rate at which the temperature changes is given by $u_t(x, t)$, we have $u_t(x, t) = \dfrac{r\,\Delta H}{\delta A\ \Delta x}$, so

$$u_t(x, t) = \frac{rkAu_{xx}(\hat{x}, t)\Delta x}{\delta A\ \Delta x} = \frac{rk}{\delta}\, u_{xx}(\hat{x}, t),$$



where $r$ is the constant of proportionality. By taking the limit as $\Delta x \to 0$, the approximation becomes better and better. Also because $x \to x + \Delta x$, the value of $x = \hat{x}$ used in the approximation obtained with the MVT is more accurate. Hence the flow of heat through the wire is modeled by the partial differential equation

$$u_t = c^2 u_{xx},$$

where $c^2 = rk/\delta$ is called the thermal diffusivity of the wire. We call this the **one-dimensional heat equation,** because there is only one dimension in space, the position $x$ along the wire. We solve this equation through separation of variables in the next section.

## *EXERCISES 10.1*

Use separation of variables to find a solution of the following partial differential equations.

1. $u_x = -u_y$
2. $u_x = u_y$
*3. $u_x - yu_y = 0$
4. $xu_x - yu_y = 0$
5. $u_x - u_y = u$
6. $u_x + u_y = u$
*7. $u_{yx} = 0$
8. $u_{yx} - u = 0$
9. $xu_{xy} + u = 0$
10. $u_{yy} = u - u_x$

In Exercises 11–16, show that the function satisfies Laplace's equation $u_{xx} + u_{yy} = 0$.

11. $u(x, y) = x^2 - y^2$
12. $u(x, y) = 3xy^2 - x^3$
*13. $u(x, y) = e^x \sin y$
14. $u(x, y) = e^{-2y} \cos 2x$
15. $u(x, y) = \tan^{-1} \dfrac{y}{x}$
16. $u(x, y) = \cos x \sinh y$

In Exercises 17–22, show that the function satisfies the wave equation $u_{xx} = u_{tt}$.

17. $u(x, t) = (x^2 + t)^2$
18. $u(x, t) = 4xt$
*19. $u(x, t) = \sin 2x \cos 2t$
20. $u(x, t) = (\cos t + \sin t) \sin x$
21. $u(x, t) = \sin kx \cos kt$
22. $u(x, t) = \sin \omega x \sin \omega t$

In Exercises 23–28, show that the function satisfies the heat equation $u_{xx} = u_t$.

23. $u(x, t) = e^{-t} \sin x$
24. $u(x, t) = e^{-4t} \sin 2x$
*25. $u(x, t) = 10x + 1 + e^{-16t} \cos 4x$
26. $u(x, t) = 100 + e^{-25t} \cos 5x$
27. $u(x, t) = e^{-k^2t} \cos kx$
28. $u(x, t) = 2t + x^2$
29. Determine the value(s) of $k$ so that $u(x, t) = e^{-k^2t} \cos x$ satisfies the heat equation $u_{xx} = 16u_t$.
30. Determine the value(s) of $k$ so that $u(x, t) = e^{-2t} \sin kx$ satisfies the heat equation $u_{xx} = \frac{1}{2}u_t$.
*31. Determine the value of $c$ so that $u(x, t) = \cos ct \sin 4x$ satisfies the wave equation $u_{xx} = 4u_{tt}$.

**32.** Determine the value of $c$ so that $u(x, t) = \sin 3t \sin cx$ satisfies the wave equation $u_{xx} = 9u_{tt}$.

**33.** Does the solution $u(x, y) = e^x \sin y$ of Laplace's equation $u_{xx} + u_{yy} = 0$ satisfy the boundary conditions $u(x, 0) = u(x, \pi) = 0$?

**34.** Does the solution $u(x, y) = e^{-2y} \cos 2x$ of Laplace's equation $u_{xx} + u_{yy} = 0$ satisfy the boundary conditions $u(-\pi/4, y) = u(\pi/4, y) = 0$?

***35.** Does the solution $u(x, t) = \sin 2x \cos 2t$ of the wave equation $u_{xx} = u_{tt}$ satisfy the initial condition $u_t(x, 0) = 0$?

**36.** If $u(x, t) = e^{-4t} \sin 2x$ represents the temperature at time $t$ and position $x$ of wire of length $\pi$, then find the temperature at the endpoints $x = 0$ and $x = \pi$ (for all values of $t$).

**37.** Use separation of variables to show that the Schrodinger equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2\,\partial\Psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2} + \frac{2\mu}{\hbar^2}(E - V)\Psi = 0$$

can be written as the three second-order ordinary differential equations (see Section 6.4):

**Azimuthal equation** $$\frac{d^2\Phi}{d\phi^2} = -m_\ell^2 g$$

**Radial equation**

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2\mu}{\hbar^2}\left(E - V - \frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r^2}\right)R = 0$$

**Angular equation**

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m_\ell^2}{\sin^2\theta}\right]\Theta = 0.$$

**38.** Become familiar with a three-dimensional graphing device by plotting the solution obtained in Example 2 using various values for the arbitrary constants.

**39.** **(a)** Show that $u(x, t) = \sin(x + t)$ satisfies the partial differential equation $u_{xx} = u_{tt}$.

**(b)** If $0 \le x \le \pi$, what is the maximum value of $u$ if $t = 1, 2$, and 3?

**(c)** If $u$ represents the displacement of a string at position $x$ and time $t$, what is the initial displacement of the string at $x = \pi/4$?

**40.** **(a)** Show that $u(x, t) = e^{-4t} \sin 2x$ satisfies $u_{xx} = u_t$.

**(b)** If $0 \le x \le \pi$, then what is the maximum value of $u$ if $t = 1, 2$, and 3?

**(c)** If $u$ represents the temperature in a thin wire at position $x$ and time $t$, what is the initial temperature of the wire at $x = \pi/2$?

**41.** **(a)** Show that $u(x, t) = 100 + e^{-16t} \sin 4x$ satisfies $u_{xx} = u_t$.

**(b)** If $0 \le x \le \pi$, then what is the maximum value of $u$ if $t = 1, 2$, and 3?

**(c)** Does the value of $u$ eventually become independent of $t$? If so, if $u$ represents temperature in a thin wire at position $x$ and time $t$, what is the eventual temperature at every point in the wire?

**42.** A partial differential equation of the form

$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u)$$

is called a **first-order, quasi-linear partial differential equation.** In the case when $c(x, y, u) = 0$, the equation is **homogeneous;** when $a$ and $b$ are independent of $u$, the equation is **almost linear;** and when $c(x, y, u)$ can be written in the form

$$c(x, y, u) = d(x, y)u + s(x, y)$$

the equation is **linear.**

Quasi-linear partial differential equations can frequently be solved using the **Method of characteristics.** In addition, many mathematical software packages are capable of graphing solutions to quasi-linear partial differential equations.

For example, to solve the equation

$$-3xtu_x + u_t = xt,\ u(x, 0) = x$$

using the Method of characteristics, we first write the **characteristic system:**

$$\begin{cases} \dfrac{\partial x}{\partial r} = -3xt, & x(0, s) = s \\ \dfrac{\partial t}{\partial r} = 1, & t(0, s) = 0 \\ \dfrac{\partial u}{\partial r} = xt, & u(0, s) = s \end{cases}.$$

**(a)** Solve $\partial t/\partial r = 1$, $t(0, s) = 0$



**(b)** Substitute the result obtained in **(a)** into $\partial x/\partial r = -3xt$, $x(0, s) = s$ and solve for $x$.

**(c)** Substitute the results obtained in **(a)** and **(b)** into $\partial u/\partial r = xt$, $u(0, s) = s$ and solve for $u$.

**(d)** Solve for $u(x, t)$ by first solving the equation

$$\begin{cases} t = r \\ x = se^{-3r^2/2} \end{cases}$$

for $r$ and $s$ and substituting the resulting values into the result obtained in **(c)**.

**(e)** Use the fact that the initial condition $u(x, 0) = x$ has parametrization $\begin{cases} x = s \\ t = 0 \\ u = s \end{cases}$ to graph the solution for $0 \le s \le 15$.

## 10.2 The One-Dimensional Heat Equation

- The Heat Equation with Homogeneous Boundary Conditions
- Nonhomogeneous Boundary Conditions
- Insulated Boundary

One of the more important differential equations is the **heat equation,**

$$u_t = c^2 u_{xx}.$$

In one spatial dimension, the solution of the heat equation represents the temperature (at any position $x$ and any time $t$) in a thin rod or wire of length $p$. Because the rate at which heat flows through the rod depends on the material that makes up the rod, the constant $c^2$, which is related to the thermal diffusivity of the material, is included in the heat equation. Several different situations can be considered when determining the temperature in the rod. The ends of the wire can be held at a constant temperature, the ends may be insulated, or there can be a combination of these situations.

### The Heat Equation with Homogeneous Boundary Conditions

The first problem that we investigate is the situation in which the temperature at the ends of the rod is constantly kept at zero and the initial temperature distribution in the rod is represented as the given function $f(x)$. Hence the fixed-end zero temperature is given in the boundary conditions

$$u(0, t) = u(p, t) = 0,$$

and the initial temperature distribution is given in

$$u(x, 0) = f(x).$$

Because the temperature is zero at the endpoints, we say that the problem has **homogeneous boundary conditions** (which are important in finding a solution with separation of variables). We call problems of this type **initial boundary value problems (IBVP),** because they include initial as well as boundary conditions. We summarize the problem as follows.

$$\begin{cases} u_t = c^2 u_{xx}, & 0 < x < p, t > 0 \\ u(0, t) = 0, & u(p, t) = 0, t > 0. \\ u(x, 0) = f(x), & 0 < x < p \end{cases}$$

We solve this problem through separation of variables by assuming that

$$u(x, t) = X(x)T(t).$$

Substitution into the differential equation yields

$$\frac{T'}{c^2 T} = \frac{X''}{X} = -\lambda,$$

where $-\lambda$ is the separation constant. (Note that we selected this constant to obtain an eigenvalue problem that was solved in Section 9.1.) Separating the variables, we have the two equations

$$T' + c^2\lambda T = 0 \quad \text{and} \quad X'' + \lambda X = 0.$$

Now that we have successfully separated the variables, we turn our attention to the homogeneous boundary conditions. In terms of the functions $X(x)$ and $T(t)$, these boundary conditions become

$$u(0, t) = X(0)T(t) = 0 \quad \text{and} \quad u(p, t) = X(p)T(t) = 0.$$

In each case, we must avoid setting $T(t) = 0$ for all $t$, because if this were the case, our solution would be the trivial solution $u(x, t) = X(x)T(t) = 0$. Therefore, we have the boundary conditions

$$X(0) = 0 \quad \text{and} \quad X(p) = 0,$$

so we solve the eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0, & 0 < x < p \\ X(0) = 0, & X(p) = 0 \end{cases}$$

Because we solved this problem in Section 9.1, we refer to the results that we found there. The eigenvalues of this problem are

$$\lambda = \lambda_n = \left(\frac{n\pi}{p}\right)^2, n = 1, 2, \ldots$$

with corresponding eigenfunctions

$$X(x) = X_n(x) = \sin\frac{n\pi x}{p}, n = 1, 2, \ldots$$

Similarly, we solve $T' + c^2\lambda T = 0$, which is a first-order equation with characteristic equation $r + c^2\lambda = 0$. Because $r = -c^2\lambda$, a general solution is

$$T(t) = T_n(t) = Ae^{-c^2\lambda t},$$

where $A$ is an arbitrary constant and $\lambda = \lambda_n = (n\pi/p)^2$, $n = 1, 2, \ldots$. Because $X(x)$ and $T(t)$ both depend on $n$, the solution $u(x, t) = X(x)T(t)$ does as well. Hence,



$$u_n(x, t) = X_n(x)T_n(t) = b_n \sin\frac{n\pi x}{p}\, e^{-c^2\lambda t},$$

where we have replaced the constant $A$ by one that depends on $n$. To find the value of $b_n$, we must apply the initial condition $u(x, 0) = f(x)$. Because

$$u_n(x, 0) = b_n \sin\frac{n\pi x}{p}\, e^{-c^2\lambda(0)} = b_n \sin\frac{n\pi x}{p}$$

is satisfied only by functions of the form $\sin(\pi x/p)$, $\sin(2\pi x/p)$, $\sin(3\pi x/p)$, . . . (which in general is not the case), we use the Principle of superposition to state that

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{p}\, e^{-c^2\lambda t}$$

is also a solution of the problem. (Note that this solution satisfies the heat equation as well as the boundary conditions.) Then when we apply the initial condition $u(x, 0) = f(x)$, we find that

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{p}\, e^{-c^2\lambda(0)} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{p} = f(x).$$

Therefore, $b_n$ represents the Fourier sine series coefficients for $f(x)$, which are given by

$$b_n = \frac{2}{p}\int_0^p f(x) \sin\frac{n\pi x}{p}\, dx, n = 1, 2, \ldots$$

**Example 1**

Solve $\begin{cases} u_t = u_{xx}, & 0 < x < 1, t > 0 \\ u(0, t) = 0, u(1, t) = 0, & t > 0. \\ u(x, 0) = 50, & 0 < x < 1 \end{cases}$

***Solution*** In this case, $c = 1$, $p = 1$, and $f(x) = 50$. Hence

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin n\pi x e^{-\lambda t},$$

where

$$b_n = \frac{2}{1}\int_0^1 50 \sin n\pi x\, dx = -100\left[\frac{\cos n\pi x}{n\pi}\right]_0^1 = -\frac{100}{n\pi}(\cos n\pi - 1)$$
$$= -\frac{100}{n\pi}[(-1)^n - 1], n = 1, 2, \ldots$$

and $\lambda = \lambda_n = (n\pi)^2$. Therefore, because $b_n = 0$ if $n$ is even, we write $u(x, t)$ as

$$u(x, t) = \sum_{n=1}^{\infty} \frac{200}{(2n-1)\pi} \sin(2n-1)\pi x e^{-(2n-1)^2\pi^2 t}.$$

In Figure 10.3, we sketch the graph of $u$ at various times. Notice what happens to the temperature as $t$ increases.

*In Example 1, what does the temperature at each point in the wire eventually become because of the zero fixed-end boundary conditions?*

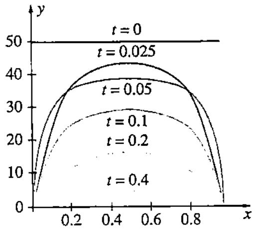



**Figure 10.3** Graph of $u$ at various times

## Nonhomogeneous Boundary Conditions

The ability to apply the method of separation of variables depends on the presence of homogeneous boundary conditions, as we saw in the previous problem. However, with the heat equation, the temperature at the endpoints may not be held constantly at zero. Instead, consider the case when the temperature at the left-hand endpoint is $T_0$ and at the right-hand endpoint it is $T_1$. Mathematically, we state these **nonhomogeneous boundary conditions** as

$$u(0, t) = T_0 \quad \text{and} \quad u(p, t) = T_1,$$

so we would be faced with solving the problem

$$\begin{cases} u_t = c^2 u_{xx}, & 0 < x < p,\ t > 0 \\ u(0, t) = T_0,\ u(p, t) = T_1, & t > 0. \\ u(x, 0) = f(x), & 0 < x < p \end{cases}$$

In this case, we must modify the problem to introduce homogeneous boundary conditions. We do this by using the physical observance that as $t \to \infty$, the temperature in the wire does not depend on $t$. Hence

$$\lim_{t \to \infty} u(x, t) = S(x),$$

where we call $S(x)$ the **steady-state temperature.** Therefore, we let

$$u(x, t) = v(x, t) + S(x),$$

where $v(x, t)$ is called the **transient temperature.** We use these two functions to obtain two problems that we can solve. To substitute $u(x, t)$ into the heat equation $u_t = c^2 u_{xx}$, we calculate the derivatives

$$u_t(x, t) = v_t(x, t) + 0 \quad \text{and} \quad u_{xx}(x, t) = v_{xx}(x, t) + S''(x).$$

Hence substitution into the heat equation yields

$$u_t = c^2 u_{xx}$$
$$v_t = c^2 v_{xx} + c^2 S'',$$

so we have the two equations $v_t = c^2 v_{xx}$ and $S'' = 0$. We then consider the boundary conditions. Because

$$u(0, t) = v(0, t) + S(0) = T_0 \quad \text{and} \quad u(p, t) = v(p, t) + S(p) = T_1,$$



we can choose the boundary conditions for $S$ to be the nonhomogeneous conditions

$$S(0) = T_0 \quad \text{and} \quad S(p) = T_1,$$

and the boundary conditions for $v(x, t)$ to be the homogeneous conditions

$$v(0, t) = 0 \quad \text{and} \quad v(p, t) = 0.$$

Of course, we have failed to include the initial temperature. Applying this condition, we have $u(x, 0) = v(x, 0) + S(x) = f(x)$, so the initial condition for $v$ is

$$v(x, 0) = f(x) - S(x).$$

Therefore, we have two problems, one for $v$ with homogeneous boundary conditions and one for $S$ that has nonhomogeneous boundary conditions. These two problems are stated as follows.

$$\begin{cases} S'' = 0, & 0 < x < p \\ S(0) = T_0,\ S(p) = T_1 \end{cases} \quad \text{and} \quad \begin{cases} v_t = c^2 v_{xx}, & 0 < x < p,\ t > 0 \\ v(0, t) = 0,\ v(p, t) = 0, & t > 0 \\ v(x, 0) = f(x) - S(x), & 0 < x < p. \end{cases}$$

Because $S$ is needed to determine $v$, we begin by finding $S$. A general solution of $S'' = 0$ is $S(x) = c_1 + c_2 x$. Then $S(0) = c_1 = T_0$ and $S(p) = T_0 + c_2 p = T_1$, so $c_2 = (T_1 - T_0)/p$. Hence

$$S(x) = T_0 + \left(\frac{T_1 - T_0}{p}\right)x.$$

We are now able to find $v(x, t)$ by solving the heat equation with homogeneous boundary conditions for $v$. Because we solved this problem at the beginning of this section, we do not need to go through the separation of variables procedure. Instead, we use the formula that we derived there using the initial temperature $v(x, 0) = f(x) - S(x)$. Therefore,

$$v(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{p} e^{-c^2 \lambda t},$$

where

$$v(x, 0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{p} = f(x) - S(x).$$

This means that $b_n$ represents the Fourier sine series coefficients for the function $f(x) - S(x)$ given by

$$b_n = \frac{2}{p} \int_0^p (f(x) - S(x)) \sin \frac{n\pi x}{p}\, dx,\ n = 1, 2, \ldots$$

The solution to the original problem is, therefore, $u(x, t) = v(x, t) + S(x)$.

**Example 2**

Solve $\begin{cases} u_t = u_{xx}, & 0 < x < 1,\ t > 0 \\ u(0, t) = 10,\ u(1, t) = 60, & t > 0. \\ u(x, 0) = 10, & 0 < x < 1 \end{cases}$

***Solution*** In this case, $c = 1, p = 1, T_0 = 10, T_1 = 60$, and $f(x) = 10$. Therefore, the steady-state solution is

$$S(x) = T_0 + \left(\frac{T_1 - T_0}{p}\right)x = 10 + \left(\frac{60 - 10}{1}\right)x = 10 + 50x.$$

Then the initial transient temperature is

$$v(x, 0) = 10 - (10 + 50x) = -50x,$$

so that the series coefficients in the solution $v(x, t) = \sum_{n=1}^{\infty} b_n \sin n\pi x e^{-n^2\pi^2 t}$ are

$$b_n = \frac{2}{1}\int_0^1 (-50x) \sin n\pi x\, dx = -100 \int_0^1 x \sin n\pi x\, dx = \frac{100 \cos n\pi}{n\pi}$$
$$= \frac{100(-1)^n}{n\pi},\ n = 1, 2, \ldots,$$

so the transient temperature is

$$v(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{p} e^{-c^2\lambda t} = \sum_{n=1}^{\infty} \frac{100(-1)^n}{n\pi} \sin(n\pi x)e^{-n^2\pi^2 t}.$$

Therefore,

$$u(x, t) = v(x, t) + S(x) = \sum_{n=1}^{\infty} \frac{100(-1)^n}{n\pi} \sin (n\pi x)e^{-n^2\pi^2 t} + 10 + 50x.$$

We graph $u(x, t)$ for several values of $t$ in Figure 10.4. Notice what happens to the temperature as $t$ increases.

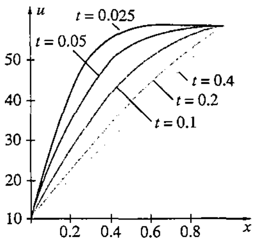



**Figure 10.4** Graph of $u$ at various times

*In Example 2, find $\lim_{t\to\infty} u(x, t)$ both from the formula and from Figure 10.4.*

## Insulated Boundary

Another important situation concerning the flow of heat flow in a wire involves insulated ends. In this case, heat is not allowed to escape from the ends of the wire. Mathematically, we express these boundary conditions as

$$u_x(0, t) = 0 \quad \text{and} \quad u_x(p, t) = 0,$$

because the rate at which the heat changes along the $x$-axis at the endpoints $x = 0$ and $x = p$ is zero. Therefore, if we want to determine the temperature in a wire of length $p$ with insulated ends, we solve the initial boundary value problem



$$\begin{cases} u_t = c^2 u_{xx}, & 0 < x < p,\ t > 0 \\ u_x(0, t) = 0,\ u_x(p, t) = 0, & t > 0 \\ u(x, 0) = f(x), & 0 < x < p. \end{cases}$$

Notice that the boundary conditions are homogeneous, so we can use separation of variables to find $u(x, t) = X(x)T(t)$ (i.e., there is no need to introduce the steady-state and transient temperatures when the boundary conditions are homogeneous). By following the steps taken in the solution of the problem with homogeneous boundary conditions, we obtain the ordinary differential equations

$$T' + c^2\lambda T = 0 \quad \text{and} \quad X'' + \lambda X = 0.$$

However, when we consider the boundary conditions

$$u_x(0, t) = X'(0)T(t) = 0 \quad \text{and} \quad u_x(p, t) = X'(p)T(t) = 0,$$

we wish to avoid letting $T(t) = 0$ for all $t$ (which leads to the trivial solution), so we have the homogeneous boundary conditions

$$X'(0) = 0 \quad \text{and} \quad X'(p) = 0.$$

Therefore, we solve the problem

$$\begin{cases} X'' + \lambda X = 0, & 0 < x < p \\ X'(0) = 0,\ X'(p) = 0 \end{cases}$$

to find $X(x)$. This problem was also solved in Section 9.1, so we will not repeat the steps at this time. We simply recall that the eigenvalues and corresponding eigenfunctions of this problem are

$$\lambda_n = \begin{cases} 0, & n = 0 \\ \left(\frac{n\pi}{p}\right)^2, & n = 1, 2, \ldots \end{cases} \quad \text{and} \quad X_n(x) = \begin{cases} 1, & n = 0 \\ \cos\frac{n\pi x}{p}, & n = 1, 2, \ldots \end{cases}$$

Next, we solve the equation $T' + c^2\lambda T = 0$ for the $\lambda_n$ given previously. First, for $\lambda = \lambda_0 = 0$, we have the equation $T' = 0$, which has the solution $T(t) = A_0$, where $A_0$ is a constant. Therefore, when $\lambda = \lambda_0 = 0$, the solution is the product

$$u_0(x, t) = X_0(x)T_0(t) = A_0.$$

Then for $\lambda = \lambda_n = n\pi x/p$, we solve $T' + c^2\lambda T = 0$, which has as a general solution $T(t) = T_n(t) = a_n e^{-c^2\lambda t}$. For these eigenvalues, we have the solution

$$u_n(x, t) = X_n(x)T_n(t) = a_n \cos\frac{n\pi x}{p} e^{-c^2\lambda t}.$$

Therefore, by the Principle of superposition, the solution is

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{p} e^{-c^2\lambda t}.$$

Application of the initial temperature yields

$$u(x, 0) = A_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{p} = f(x),$$

which is the Fourier cosine series for $f(x)$, where the coefficient $A_0$ is equivalent to $a_0/2$ in the original Fourier series given in Section 9.2. Therefore,

$$A_0 = \frac{1}{2} a_0 = \frac{1}{2} \frac{2}{p} \int_0^p f(x)\, dx = \frac{1}{p} \int_0^p f(x)\, dx \quad \text{and}$$

$$a_n = \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi x}{p}\, dx,\ n = 1, 2, \ldots.$$

**Example 3**

Solve $\begin{cases} u_t = u_{xx}, & 0 < x < \pi,\ t > 0 \\ u_x(0, t) = 0,\ u_x(\pi, t) = 0, & t > 0. \\ u(x, 0) = x, & 0 < x < \pi \end{cases}$

***Solution*** In this case, $p = \pi$ and $c = 1$. Because we need the Fourier cosine series coefficients for $f(x) = x$, we refer back to Example 3 in Section 9.2. There, we found that

$$A_0 = \frac{1}{2} a_0 = \frac{1}{\pi} \int_0^{\pi} x\, dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\pi}{2}$$

and with integration by parts

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos \frac{n\pi x}{\pi}\, dx = \frac{2}{\pi n^2}\left[(-1)^n - 1\right],\ n = 1, 2, \ldots$$

Therefore, the solution is

$$u(x, t) = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2 \pi} \cos((2n-1)x) e^{-(2n-1)^2 \pi^2 t},$$

where we have used the fact that $a_n = 0$ if $n$ is even. The graph of $u$ is shown in Figure 10.5 for the values of $t$ that are given.

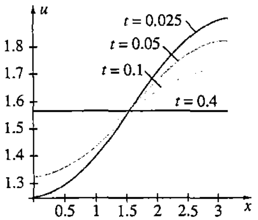



**Figure 10.5** Graph of $u$ at various times

*In Example 3, find $\lim_{t\to\infty} u(x, t)$. What happens to temperatures to the left of $x = \pi/2$? Do they increase or decrease? What about temperatures to the right of $x = \pi/2$?*



## EXERCISES 10.2

In Exercises 1–6, solve the following problems (heat equation with homogeneous boundary conditions).

**1.** $\begin{cases} u_{xx} = u_t, & 0 < x < 1,\ t > 0 \\ u(0, t) = 0,\ u(1, t) = 0, & t > 0 \\ u(x, 0) = 100x, & 0 < x < 1 \end{cases}$

**2.** $\begin{cases} 2u_{xx} = u_t, & 0 < x < 1,\ t > 0 \\ u(0, t) = 0,\ u(1, t) = 0, & t > 0 \\ u(x, 0) = x(1-x), & 0 < x < 1 \end{cases}$

***3.** $\begin{cases} \frac{1}{2} u_{xx} = u_t, & 0 < x < 1,\ t > 0 \\ u(0, t) = 0,\ u(1, t) = 0, & t > 0 \\ u(x, 0) = x(1-x), & 0 < x < 1 \end{cases}$

**4.** $\begin{cases} u_{xx} = u_t, & 0 < x < \pi,\ t > 0 \\ u(0, t) = 0,\ u(\pi, t) = 0, & t > 0 \\ u(x, 0) = \sin 2x, & 0 < x < \pi \end{cases}$

**5.** $\begin{cases} u_{xx} = u_t, & 0 < x < \pi,\ t > 0 \\ u(0, t) = 0,\ u(\pi, t) = 0, & t > 0 \\ u(x, 0) = \sin 2x + \sin 8x, & 0 < x < \pi \end{cases}$

**6.** $\begin{cases} u_{xx} = u_t, & 0 < x < 2,\ t > 0 \\ u(0, t) = 0,\ u(2, t) = 0, & t > 0 \\ u(x, 0) = \begin{cases} x, & 0 \le x < 1 \\ 2 - x, & 1 \le x \le 2 \end{cases}, & 0 < x < 2 \end{cases}$

In Exercises 7–10, solve the problem with nonhomogeneous boundary conditions.

**7.** $\begin{cases} u_{xx} = u_t, & 0 < x < 1,\ t > 0 \\ u(0, t) = 0,\ u(1, t) = 10, & t > 0 \\ u(x, 0) = 0, & 0 < x < 1 \end{cases}$

**8.** $\begin{cases} u_{xx} = u_t, & 0 < x < 1,\ t > 0 \\ u(0, t) = 10,\ u(1, t) = 10, & t > 0 \\ u(x, 0) = 0, & 0 < x < 1 \end{cases}$

***9.** $\begin{cases} u_{xx} = u_t, & 0 < x < 1,\ t > 0 \\ u(0, t) = 20,\ u(1, t) = 10, & t > 0 \\ u(x, 0) = 0, & 0 < x < 1 \end{cases}$

**10.** $\begin{cases} u_{xx} = u_t, & 0 < x < 1,\ t > 0 \\ u(0, t) = 0,\ u(1, t) = 10, & t > 0 \\ u(x, 0) = 10x, & 0 < x < 1 \end{cases}$

In Exercises 11–16, determine the steady-state solution to the boundary-value problem.

**11.** $\begin{cases} u_{xx} = u_t, & 0 < x < 1,\ t > 0 \\ u(0, t) - u_x(0, t) = T_0,\ u(1, t) = 0, & t > 0 \end{cases}$

**12.** $\begin{cases} u_{xx} = u_t, & 0 < x < 1,\ t > 0 \\ u(0, t) = T_0,\ u(1, t) + u_x(1, t) = T_1, & t > 0 \end{cases}$

***13.** $\begin{cases} u_{xx} = u_t, & 0 < x < 1,\ t > 0 \\ u(0, t) = T_0,\ u_x(1, t) = 0, & t > 0 \end{cases}$

**14.** $\begin{cases} u_{xx} - u = u_t, & 0 < x < 1,\ t > 0 \\ u(0, t) = T_0,\ u(1, t) = T_1, & t > 0 \end{cases}$

**15.** $\begin{cases} u_{xx} - (u - T) = u_t, & 0 < x < 1,\ t > 0 \\ u(0, t) = T_0,\ u(1, t) = T_1, & t > 0 \end{cases}$

**16.** $\begin{cases} u_{xx} - (u - T) = u_t, & 0 < x < 1,\ t > 0 \\ u(0, t) = T_0,\ u(1, t) = T_0, & t > 0 \end{cases}$

In Exercises 17–20, solve the problems with insulated ends.

**17.** $\begin{cases} u_t = u_{xx}, & 0 < x < \pi,\ t > 0 \\ u_x(0, t) = 0,\ u_x(\pi, t) = 0, & t > 0 \\ u(x, 0) = 1, & 0 < x < \pi \end{cases}$

**18.** $\begin{cases} u_t = u_{xx}, & 0 < x < \pi,\ t > 0 \\ u_x(0, t) = 0,\ u_x(\pi, t) = 0, & t > 0 \\ u(x, 0) = \cos 2x - 4\cos 3x, & 0 < x < \pi \end{cases}$

***19.** $\begin{cases} u_t = u_{xx}, & 0 < x < \pi,\ t > 0 \\ u_x(0, t) = 0,\ u_x(\pi, t) = 0, & t > 0 \\ u(x, 0) = 4\sin^2 x, & 0 < x < \pi \end{cases}$

**20.** $\begin{cases} u_t = u_{xx}, & 0 < x < \pi,\ t > 0 \\ u_x(0, t) = 0,\ u_x(\pi, t) = 0, & t > 0 \\ u(x, 0) = x(1-x), & 0 < x < \pi \end{cases}$

In Exercises 21–23, use separation of variables to solve the initial boundary value problem. (*Hint:* The eigenvalue problems solved in Section 9.1 may be useful.)

**21.** $\begin{cases} u_t = u_{xx}, & 0 < x < 1,\ t > 0 \\ u(0, t) = T_0,\ u_x(1, t) = 0, & t > 0 \\ u(x, 0) = f(x), & 0 < x < 1 \end{cases}$

**22.** $\begin{cases} u_t = u_{xx}, & 0 < x < 1,\ t > 0 \\ u_x(0, t) = 0,\ u(1, t) = T_0, & t > 0 \\ u(x, 0) = f(x), & 0 < x < 1 \end{cases}$

***23.** $\begin{cases} u_t = c^2 u_{xx}, & -\pi < x < \pi,\ t > 0 \\ u(-\pi, t) = u(\pi, t),\ u_x(-\pi, t) = u_x(\pi, t), & t > 0 \\ u(x, 0) = f(x), & -\pi < x < \pi \end{cases}$

**24.** Solve the heat equation in Exercise 21 with $f(x) = T_0$.

**25.** Solve the heat equation in Exercise 21 with $f(x) = 2T_0$.

**26.** Solve the heat equation in Exercise 22 with $f(x) = x(1-x) + T_0$.

**27.** Solve the heat equation in Exercise 22 with $f(x) = T_0$.

**28.** Solve the heat equation Exercise 23 with $f(x) = T_0$.

***29.** Solve the heat equation in Exercise 23 with $f(x) = |x|$.

**30.** Sturm–Liouville theory was motivated by the Swiss mathematician Jacques Charles Francois Sturm's (1803–1855) study of heat conduction in an inhomogeneous bar of length $L$ with an unequally polished surface,* which showed that the partial differential equation that describes the temperature at time $t$ at the point $x$ is

$$g\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right) - \ell u,$$

where $g$, $k$, and $\ell$ are functions of $x$. If the surrounding temperature is constant, then the endpoints of the bar satisfy

$$\begin{cases} k\dfrac{\partial u}{\partial x} - hu = 0 & \text{when } x = 0 \\ k\dfrac{\partial u}{\partial x} + Hu = 0 & \text{when } x = L \end{cases}$$

for some positive constants $h$ and $H$.

**(a)** Assume that $u(x, t) = V(x)e^{-rt}$ and $u(x, 0) = f(x)$.

**(b)** Show that $(k(x)V'(x))' + (rg(x) - \ell(x))V(x) = 0$ and

$$\begin{cases} k(0)V'(0) - hV(0) = 0 \\ k(L)V'(L) + HV(L) = 0 \end{cases}.$$

**31.** Consider the initial boundary value problem given by

$$\begin{cases} u_t = u_{xx} + x, & 0 < x < 1,\ t > 0 \\ u(0, t) = T_0,\ u(1, t) = T_1, & t > 0. \\ u(x, 0) = f(x), & 0 < x < 1 \end{cases}$$

**(a)** Let $u(x, t) = v(x, t) + S(x)$. Show that $S$ satisfies $S'' = -x$, $S(0) = T_0$, $S(1) = T_1$ and find $S$.

**(b)** Find $v$.

**(c)** Find $u$ if $f(x) = T_0(1-x) - x^3/6$.

**32.** Consider the problem

$$\begin{cases} u_t = c^2u_{xx} + e^{-x}, & 0 < x < 1,\ t > 0 \\ u(0, t) = 0,\ u(1, t) = 0, & t > 0 \\ u(x, 0) = f(x), & 0 < x < 1 \end{cases},$$

where the term $e^{-x}$ represents the loss of heat through radioactive decay of the wire. Use a procedure similar to that described in Exercise 31 to find a solution of this problem.

In Exercises 33–38, use technology to assist in calculating the coefficients in the solution of each of the following problems. Also, use a graphing utility to plot an approximation of the solution using several terms in the series solution. Observe the behavior of the solution which describes the temperature in a thin rod of length 1.

**33.** $$\begin{cases} u_{xx} = u_t, & 0 < x < 1,\ t > 0 \\ u(0, t) = 0,\ u(1, t) = 0, & t > 0 \\ u(x, 0) = x(1 - x^3), & 0 < x < 1 \end{cases}$$

**34.** $$\begin{cases} u_{xx} = u_t, & 0 < x < 1,\ t > 0 \\ u(0, t) = 0,\ u(1, t) = 0, & t > 0 \\ u(x, 0) = e^x, & 0 < x < 1 \end{cases}$$

***35.** $$\begin{cases} u_{xx} = u_t, & 0 < x < 1,\ t > 0 \\ u(0, t) = 50,\ u(1, t) = 100, & t > 0 \\ u(x, 0) = x(1 - x^3), & 0 < x < 1 \end{cases}$$

**36.** $$\begin{cases} u_{xx} = u_t, & 0 < x < 1,\ t > 0 \\ u(0, t) = 10,\ u(1, t) = 100, & t > 0 \\ u(x, 0) = e^{-x}, & 0 < x < 1 \end{cases}$$

**37.** $$\begin{cases} u_{xx} = u_t, & 0 < x < 1,\ t > 0 \\ u_x(0, t) = 0,\ u_x(1, t) = 0, & t > 0 \\ u(x, 0) = e^x \sin \pi x, & 0 < x < 1 \end{cases}$$

**38.** $$\begin{cases} u_{xx} = u_t, & 0 < x < 1,\ t > 0 \\ u_x(0, t) = 0,\ u_x(1, t) = 0, & t > 0 \\ u(x, 0) = e^{-x} \sin 2\pi x, & 0 < x < 1 \end{cases}$$

***39.** Solve

$$\begin{cases} u_t = u_{xx}, & 0 < x < 1,\ t > 0 \\ u(0, t) = 0,\ -u_x(1, t) = u(1, t), & t > 0 \\ u(x, 0) = f(x), & 0 < x < 1, \end{cases}$$

where the second boundary condition indicates that heat at the end, $x = 1$, radiates freely into the air of temperature zero.

* Jesper Lutzen, "The Solution of Partial Differential Equations," *Studies in the History of Mathematics*, ed. Ester R. Phillips, MAA Studies in Mathematics, Volume 26, Mathematical Association of America (1987), pp. 242–277.



## 10.3 The One-Dimensional Wave Equation

The one-dimensional **wave equation** is important in solving an interesting problem. Suppose that we pluck a string (e.g., a guitar or violin string) of length $p$ and constant mass density that is fixed at each end. A question that we might ask is, "What is the position or displacement of the string at a particular instance of time?" We answer this question by modeling the physical situation with a differential equation, namely the wave equation in one spatial variable. We will not go through this derivation, because we did it with the heat equation in Section 10.2, but we point out that it is based on determining the forces that act on a small segment of the string and applying Newton's second law of motion. The partial differential equation that is found is

$$c^2u_{xx} = u_{tt},$$

which is called the (one-dimensional) **wave equation.** In this equation $c^2 = T/\rho$, where $T$ is the tension of the string and $\rho$ is the constant mass of the string per unit length. The solution $u(x, t)$ represents the displacement of the string from the $x$-axis at time $t$, as shown in Figure 10.6. To determine $u$ we must describe the boundary and initial conditions that model the physical situation. At the ends of the string, the displacement from the $x$-axis is fixed at zero, so we use the *homogeneous boundary conditions*,

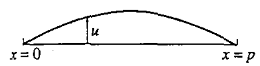



**Figure 10.6** String displacement

$$u(0, t) = 0 \quad \text{and} \quad u(p, t) = 0 \quad \text{for} \quad t \geq 0.$$

The motion of the string also depends on the displacement and the velocity at each point of the string at $t = 0$. If the initial displacement is given by $f(x)$ and the initial velocity by $g(x)$, then we have the *initial conditions*,

$$u(x, 0) = f(x) \quad \text{and} \quad u_t(x, 0) = g(x) \quad \text{for} \quad 0 \leq x \leq p.$$

Therefore, we determine the displacement of the string with the initial boundary value problem

$$\begin{cases} c^2u_{xx} = u_{tt}, & 0 < x < p,\ t > 0 \\ u(0, t) = 0,\ u(p, t) = 0, & t \geq 0 \\ u(x, 0) = f(x),\ u_t(x, 0) = g(x), & 0 \leq x \leq p \end{cases}.$$

Notice that the wave equation requires two initial conditions, whereas the heat equation only needed one. This is because there is a second derivative with respect to $t$ in the wave equation, and there is only one derivative with respect to $t$ in the heat equation. Note also that for consistency we must require that $f(0) = f(p) = 0$ and $g(0) = g(p) = 0$.

This problem is solved through separation of variables by assuming that $u(x, t) = X(x)T(t)$. Substitution into the wave equation yields

$$c^2X''T = XT''$$

$$\frac{X''}{X} = \frac{T''}{c^2T} = -\lambda,$$

so we obtain the two second-order ordinary differential equations

$$X'' + \lambda X = 0 \quad \text{and} \quad T'' + c^2\lambda T = 0.$$

At this point, we solve the equation that involves the homogeneous boundary conditions. As was the case with the heat equation, the boundary conditions in terms of $u(x, t) = X(x)T(t)$ are $u(0, t) = X(0)T(t) = 0$ and $u(p, t) = X(p)T(t) = 0$, so we have

$$X(0) = 0 \quad \text{and} \quad X(p) = 0.$$

Therefore, we determine $X(x)$ by solving the eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0, & 0 < x < p \\ X(0) = 0, X(p) = 0 \end{cases},$$

which we encountered when solving the heat equation and solved in Section 9.1. The eigenvalues of this problem are

$$\lambda = \lambda_n = \left(\frac{n\pi}{p}\right)^2, \; n = 1, 2, \ldots,$$

with corresponding eigenfunctions

$$X(x) = X_n(x) = \sin\frac{n\pi x}{p}, \; n = 1, 2, \ldots.$$

Next we solve the equation $T'' + c^2\lambda T = 0$ using the eigenvalues given previously. From our experience with second-order equations, a general solution is

$$T_n(t) = a_n \cos c\sqrt{\lambda_n}t + b_n \sin c\sqrt{\lambda_n}t = a_n \cos\frac{cn\pi t}{p} + b_n \sin\frac{cn\pi t}{p},$$

where the coefficients $a_n$ and $b_n$ must be determined. Putting this information together, we obtain

$$u_n(x, t) = \left(a_n \cos\frac{cn\pi t}{p} + b_n \sin\frac{cn\pi t}{p}\right)\sin\frac{n\pi x}{p},$$

so by the Principle of superposition, we have

$$u(x, t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{cn\pi t}{p} + b_n \sin\frac{cn\pi t}{p}\right)\sin\frac{n\pi x}{p}.$$

Applying the initial displacement yields

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{p} = f(x),$$

so $a_n$ is the Fourier sine series coefficient for $f(x)$, which is given by

$$a_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi x}{p}\,dx, \; n = 1, 2, \ldots.$$

To determine $b_n$, we must use the initial velocity. Therefore, we compute

$$u_t(x, t) = \sum_{n=1}^{\infty}\left(-a_n\frac{cn\pi}{p}\sin\frac{cn\pi t}{p} + b_n\frac{cn\pi}{p}\cos\frac{cn\pi t}{p}\right)\sin\frac{n\pi x}{p}.$$



Then

$$u_t(x, 0) = \sum_{n=1}^{\infty} b_n\frac{cn\pi}{p}\sin\frac{n\pi x}{p} = g(x),$$

so $b_n\dfrac{cn\pi}{p}$ represents the Fourier sine series coefficient for $g(x)$, which means that

$$b_n = \frac{p}{cn\pi}\frac{2}{p}\int_0^p g(x)\sin\frac{n\pi x}{p}\,dx = \frac{2}{cn\pi}\int_0^p g(x)\sin\frac{n\pi x}{p}\,dx, \; n = 1, 2, \ldots.$$

**Example 1**

Solve $\begin{cases} u_{xx} = u_{tt}, & 0 < x < 1, t > 0 \\ u(0, t) = 0, u(1, t) = 0, & t > 0 \\ u(x, 0) = x(1-x), u_t(x, 0) = 0, & 0 \le x \le 1 \end{cases}.$

***Solution*** In this case, $c = p = 1$, $f(x) = x(1-x)$, and $g(x) = 0$. From this information, we compute

$$a_n = \frac{2}{1}\int_0^1 x(1-x)\sin n\pi x\,dx = -\frac{4\cos n\pi}{n^3\pi^3} + \frac{4}{n^3\pi^3}$$
$$= \frac{4}{n^3\pi^3}(1-(-1)^n), \; n = 1, 2, \ldots,$$

where integration by parts must be used twice. (*Note:* Of course, a table of integrals or computer algebra system could be used as well.) Because $g(x) = 0$, the coefficients $b_n = 0$ for all $n$. Using the fact that $a_n = 0$ when $n$ is even and $a_n = 8/(n^3\pi^3)$ if $n$ is odd, then the solution is

$$u(x, t) = \sum_{n=1}^{\infty}\frac{8}{(2n-1)^3\pi^3}\cos(2n-1)\pi t\sin(2n-1)\pi x.$$

The graph of $u(x, t)$ for various values of $t$ is shown in Figure 10.7.

**Example 2**

Solve $\begin{cases} u_{xx} = u_{tt}, & 0 < x < 1, t > 0 \\ u(0, t) = 0, u(1, t) = 0, & t \ge 0 \\ u(x, 0) = 0, u_t(x, 0) = \sin 2\pi x, & 0 \le x \le 1 \end{cases}.$

***Solution*** Here the string is initially horizontal and an initial velocity that depends on the position $x$ is given. With $f(x) = 0$, we know that $a_n = 0$ for all $n$. Then using the initial velocity function $g(x) = \sin 2\pi x$, we note that

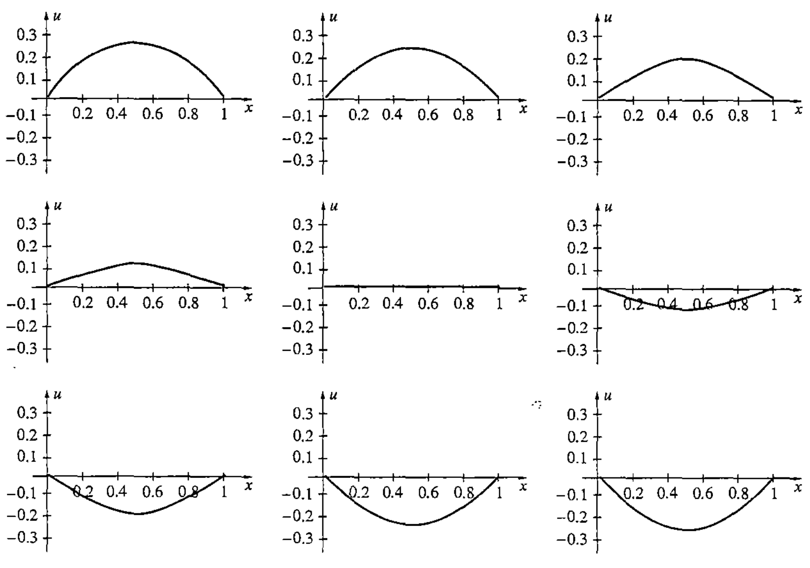


**Figure 10.7** String displacement at various times

$$u_t(x, 0) = \sum_{n=1}^{\infty} b_n n\pi \sin n\pi x = \sin 2\pi x.$$

Then without the use of integration, $b_2 \cdot 2\pi = 1$. Therefore, $b_2 = 1/(2\pi)$ and $b_n = 0$ for $n \neq 2$. This yields the solution

$$u(x, t) = \frac{1}{2\pi} \sin 2\pi t \sin 2\pi x,$$

because all other terms have zero coefficient. We graph this solution in Figure 10.8 for several values of $t$ to observe the motion of the string.

*Describe the motion of the string at $x = 1/2$ in Example 2.*

For each $n$, the function $u_n(x, t)$, called the **nth normal mode** (or standing wave) of the string, represents harmonic motion with **frequency**

$$\omega_n = \frac{\lambda_n}{2\pi} = \frac{cn\pi/p}{2\pi} = \frac{cn}{2p}.$$

The function $u_1(x, t)$ is the **fundamental mode of vibration**, and the frequency $\omega_1$ is called the **fundamental frequency.** The values of $x$ such that $\sin(n\pi x/p) = 0$ are known



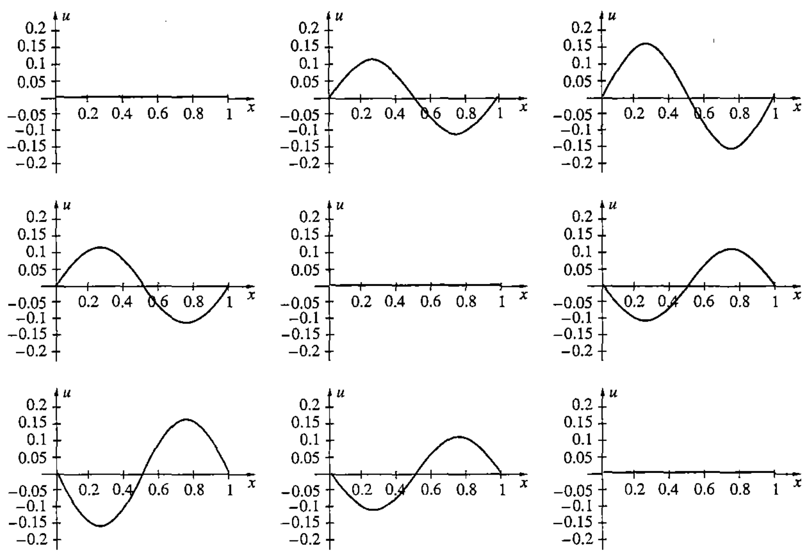


**Figure 10.8** String displacement at various times

as **nodes** and represent positions where there is no motion in the string. We graph $\sin = n\pi x$ in Figure 10.9 to illustrate the nodes. Notice that the frequency $\omega_n$ increases as c is increased. Because $c^2 = T/\rho$, the larger the tension $T$ on the string, the higher the frequency of the motion. Therefore, a higher pitch results.

## D'Alembert's Solution

The wave equation was first derived and solved by the French physicist and mathematician Jean le Rond d'Alembert (1717–1783) in 1746. He hoped that he would be able to use a similar method to solve many other problems, but this did not prove to be the case. Improvements to his solution were subsequently made by Euler.

An interesting version of the wave equation is to consider a string of infinite length. Therefore, the boundary conditions are no longer of importance. Instead, we simply work with the wave equation with the initial displacement and velocity functions. To solve the problem

$$\begin{cases} c^2 u_{xx} = u_{tt}, & -\infty < x < \infty,\ t > 0 \\ u(x, 0) = f(x),\ u_t(x, 0) = g(x) \end{cases}$$

we use the change of variables

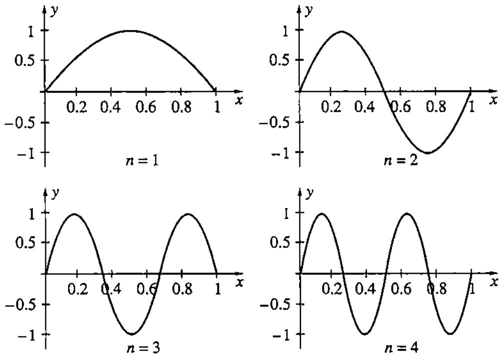



**Figure 10.9** Graphs of $y = \sin n\pi x$ for $n = 1, 2, 3, 4$

$$r = x + ct$$
$$s = x - ct.$$

Using the Chain rule, we compute the derivatives $u_{xx}$ and $u_{tt}$ in terms of the variables $r$ and $s$.

$$u_x = u_r r_x + u_s s_x = u_r + u_s$$
$$u_{xx} = (u_r + u_s)_r r_x + (u_r + u_s)_s s_x = u_{rr} + 2u_{rs} + u_{ss}$$
$$u_t = u_r r_t + u_s s_t = cu_r - cu_s = c(u_r - u_s)$$
$$u_{tt} = c[(u_r - u_s)_r r_t + (u_r - u_s)_s s_t] = c^2[u_{rr} - 2u_{rs} + u_{ss}].$$

Substitution into the wave equation yields

$$c^2 u_{xx} = u_{tt}$$
$$c^2[u_{rr} + 2u_{rs} + u_{ss}] = c^2[u_{rr} - 2u_{rs} + u_{ss}]$$
$$4c^2 u_{rs} = 0$$
$$u_{rs} = 0$$

The partial differential equation $u_{rs} = 0$ can be solved by first integrating with respect to $s$ to obtain

$$u_r = f(r),$$

where $f(r)$ is an arbitrary function of $r$. Then integrating with respect to $r$, we have

$$u(r, s) = F(r) + G(s),$$



where $F$ is an antiderivative of $f$ and $G$ is an arbitrary function of $s$. Returning to our original variables gives us

$$u(x, t) = F(x + ct) + G(x - ct).$$

The functions $F$ and $G$ are determined by the initial conditions, which indicate that

$$u(x, 0) = F(x) + G(x) = f(x)$$
$$u_t(x, 0) = cF'(x) - cG'(x) = g(x).$$

We can rewrite the second equation by integrating to obtain

$$F'(x) - G'(x) = \frac{g(x)}{c}$$
$$F(x) - G(x) = \frac{1}{c}\int_0^x g(v)\,dv.$$

Therefore, we solve the system

$$F(x) + G(x) = f(x)$$
$$F(x) - G(x) = \frac{1}{c}\int_0^x g(v)\,dv.$$

Adding these equations yields

$$F(x) = \frac{1}{2}\left[f(x) + \frac{1}{c}\int_0^x g(v)\,dv\right]$$

and subtracting gives us

$$G(x) = \frac{1}{2}\left[f(x) - \frac{1}{c}\int_0^x g(v)\,dv\right].$$

Therefore,

$$F(x + ct) = \frac{1}{2}\left[f(x + ct) + \frac{1}{c}\int_0^{x+ct} g(v)\,dv\right] \quad \text{and}$$
$$G(x - ct) = \frac{1}{2}\left[f(x - ct) - \frac{1}{c}\int_0^{x-ct} g(v)\,dv\right],$$

so the solution is

$$u(x, t) = \frac{1}{2}\left[f(x + ct) + \frac{1}{c}\int_0^{x+ct} g(v)\,dv\right] + \frac{1}{2}\left[f(x - ct) - \frac{1}{c}\int_0^{x-ct} g(v)\,dv\right]$$
$$= \frac{1}{2}\,[f(x + ct) + f(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(v)\,dv.$$

*How is the term* $\frac{1}{2c}\int_{x-ct}^{x+ct} g(v)\,dv$ *obtained in D'Alembert's solution?*

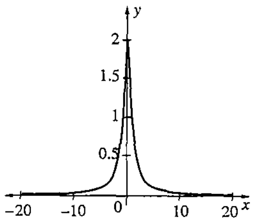



**Figure 10.10** Initial string displacement

## Example 3

Solve $\begin{cases} u_{xx} = u_{tt}, & -\infty < x < \infty,\ t > 0 \\ u(x, 0) = 2/(1 + x^2),\ u_t(x, 0) = 0 \end{cases}$

**Solution** Because $c = 1, f(x) = 1/(1 + x^2)$, and $g(x) = 0$, we have the solution

$$u(x, t) = \frac{1}{2}\left[f(x + t) + f(x - t)\right] = \frac{1}{2}\left[\frac{2}{1 + (x + t)^2} + \frac{2}{1 + (x - t)^2}\right]$$

$$= \frac{1}{1 + (x + t)^2} + \frac{1}{1 + (x - t)^2}$$

The graph of the initial position is shown in Figure 10.10. In Figure 10.11 we plot the solution for various times to illustrate the motion of the string of infinite length.

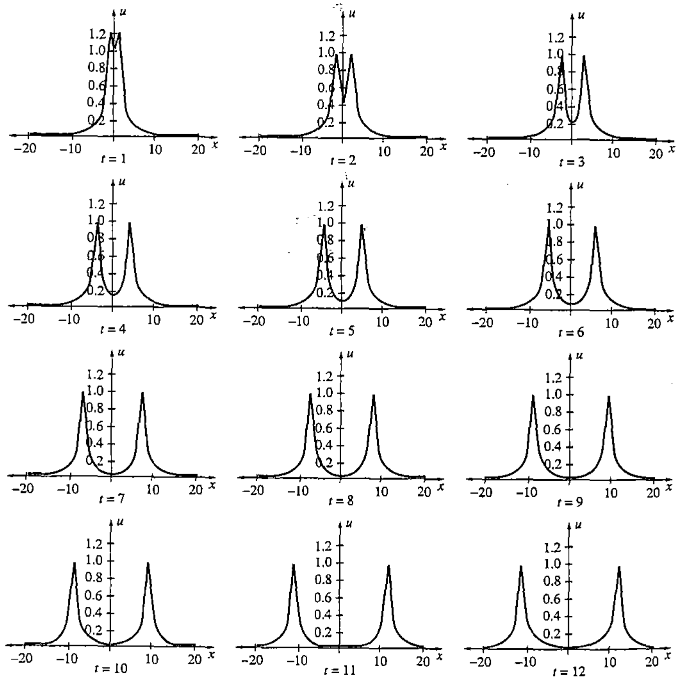



**Figure 10.11** String displacement at various times



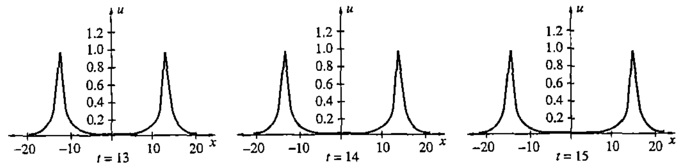



**Figure 10.11 continued** String displacement at various times

D'Alembert's solution is sometimes referred to as the **traveling wave solution** because of the behavior of its graph. The waves appear to move in opposite directions along the $x$-axis as $t$ increases, as we can see in Figure 10.11.

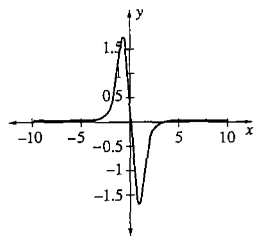



**Figure 10.12** Initial velocity function

## Example 4

Solve $\begin{cases} u_{xx} = u_{tt}, & -\infty < x < \infty,\ t > 0 \\ u(x, 0) = 0,\ u_t(x, 0) = -4xe^{-x^2}. \end{cases}$

**Solution** Using $c = 1$ and the initial conditions, $f(x) = 0$ and $g(x) = -4xe^{-x^2}$, we have

$$u(x, t) = \frac{1}{2}\int_{x-t}^{x+t} -4ue^{-u^2}\, du = \left[e^{-u^2}\right]_{x-t}^{x+t} = e^{-(x+t)^2} - e^{-(x-t)^2}.$$

We graph this solution for $t = 0, 1, 2, \ldots, 5$ in Figure 10.13. In this problem, the string is initially horizontal. Notice that traveling waves are produced by the initial velocity function as they were in the previous example. However, because $g(x) = -4xe^{-x^2}$ is negative for $x < 0$ (as shown in Figure 10.12), a negative displacement results to the left of $x = 0$. In the exercises, we use D'Alembert's solu-

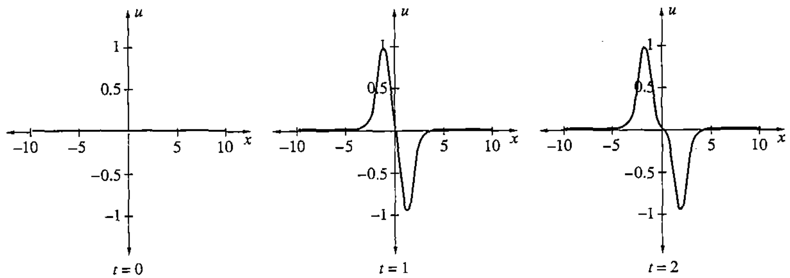



**Figure 10.13** Displacement at various times

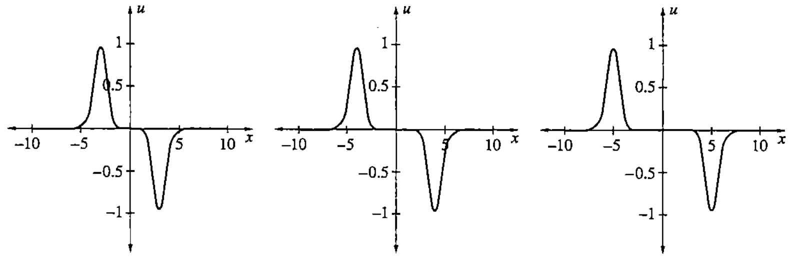



**Figure 10.13 continued** Displacement at various times

tion to solve several wave equation problems involving a string with fixed-end boundary conditions. This is possible when the solution automatically satisfies these boundary conditions, as we do not consider boundary conditions when deriving D'Alembert's solution.

## EXERCISES 10.3

Solve the following problems using the formula derived earlier. In each case, plot an approximate solution using several terms of the series to observe the motion of the string.

1. $\begin{cases} u_{xx} = u_{tt}, & 0 < x < 1,\ t > 0 \\ u(0, t) = 0,\ u(1, t) = 0, & t \geq 0 \\ u(x, 0) = \sin \pi x,\ u_t(x, 0) = 0, & 0 \leq x \leq 1 \end{cases}$

2. $\begin{cases} u_{xx} = u_{tt}, & 0 < x < 1,\ t > 0 \\ u(0, t) = 0,\ u(1, t) = 0, & t \geq 0 \\ u(x, 0) = \sin \pi x,\ u_t(x, 0) = \sin 2\pi x, & 0 \leq x \leq 1 \end{cases}$

*3. $\begin{cases} u_{xx} = u_{tt}, & 0 < x < 1,\ t > 0 \\ u(0, t) = 0,\ u(1, t) = 0, & t \geq 0 \\ u(x, 0) = 0,\ u_t(x, 0) = \sin 3\pi x, & 0 \leq x \leq 1 \end{cases}$

4. $\begin{cases} u_{xx} = u_{tt}, & 0 < x < 1,\ t > 0 \\ u(0, t) = 0,\ u(1, t) = 0, & t \geq 0 \\ u(x, 0) = x(1 - x),\ u_t(x, 0) = 0, & 0 \leq x \leq 1 \end{cases}$

5. $\begin{cases} u_{xx} = u_{tt}, & 0 < x < 1,\ t > 0 \\ u(0, t) = 0,\ u(1, t) = 0, & t \geq 0 \\ u(x, 0) = x(1 - x),\ u_t(x, 0) = \sin \pi x, & 0 \leq x \leq 1 \end{cases}$

6. $\begin{cases} u_{xx} = u_{tt}, & 0 < x < 1,\ t > 0 \\ u(0, t) = 0,\ u(1, t) = 0, & t \geq 0 \\ u(x, 0) = x(1 - x),\ u_t(x, 0) = x(1 - x), & 0 \leq x \leq 1 \end{cases}$

*7. $\begin{cases} u_{xx} = u_{tt}, & 0 < x < \pi,\ t > 0 \\ u(0, t) = 0,\ u(\pi, t) = 0, & t \geq 0 \\ u(x, 0) = \sin 2x,\ u_t(x, 0) = 0, & 0 \leq x \leq \pi \end{cases}$

8. $\begin{cases} u_{xx} = u_{tt}, & 0 < x < \pi,\ t > 0 \\ u(0, t) = 0,\ u(\pi, t) = 0, & t \geq 0 \\ u(x, 0) = 0,\ u_t(x, 0) = \sin 4x, & 0 \leq x \leq \pi \end{cases}$

9. $\begin{cases} u_{xx} = u_{tt}, & 0 < x < 2,\ t > 0 \\ u(0, t) = 0,\ u(2, t) = 0, & t \geq 0 \\ u(x, 0) = \begin{cases} x, & 0 \leq x < 1 \\ 2 - x, & 1 \leq x \leq 2 \end{cases} \\ u_t(x, 0) = 0,\ 0 \leq x \leq 2 \end{cases}$

10. $\begin{cases} u_{xx} = u_{tt}, & 0 < x < \pi,\ t > 0 \\ u(0, t) = 0,\ u(\pi, t) = 0, & t \geq 0 \\ u(x, 0) = 0,\ u_t(x, 0) = \sin x, & 0 \leq x \leq \pi \end{cases}$



A change of variables can be used to solve some partial differential equations in a manner similar to that used in D'Alembert's solution. Use the indicated change of variables to transform the equation into an equation involving $r$ and $s$.

11. $u_{yy} = u_{xx}$, $r = x + y$, $s = x - y$
12. $u_{xx} - 2u_{xy} - 3u_{yy} = 0$, $r = x - y$, $s = y + 3x$
*13. $u_{yx} - u_{xx} = 0$, $r = x + y$, $s = y$
14. $u_{yy} - 2u_{xy} + u_{xx} = 0$, $r = x + y$, $s = y$

The partial differential equation $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G(x, y)$ is classified in the following manner: (a) **hyperbolic** if $B^2 - 4AC > 0$; (b) **parabolic** if $B^2 - 4AC = 0$; (c) **elliptic** if $B^2 - 4AC < 0$. Classify each of the following equations as hyperbolic, parabolic, or elliptic.

15. $u_{xx} - 2u_{xy} + u_{yy} + u_x = 0$
16. $4u_{xx} + 4u_{xy} + u_{yy} + 2u_y = 0$
*17. $u_{xx} - 4u_{xy} + 2u_{yy} = 0$
18. $3u_{xx} - 4u_{xy} - u_{yy} + u = 0$
19. $u_{xx} + u_{xy} + u_{yy} + u_x = 0$
20. $u_{xx} - 2u_{xy} + 2u_{yy} + u_x - u = 0$
21. Show that $u_{xx} + 2u_{xy} - 8u_{yy} = 0$ is hyperbolic. Use the change of variables $r = y - 4x$, $s = y + 2x$ to show that this hyperbolic equation can be written in equivalent form $u_{rs} = 0$. Solve this equation to find a general solution of the equation.
22. Show that the equation $u_{xx} = u_{yy}$ is hyperbolic. Notice that this is the wave equation with the variables $x$ and $y$ (instead of $x$ and $t$) and constant $c = 1$. Using the same change of variables as was used with D'Alembert's solution, show that the wave equation is equivalent to $u_{rs} = 0$.
23. Show that the heat equation (in the variables $x$ and $y$), $u_{yy} = u_x$, is parabolic.
24. Show that the equation $4u_{xx} + 12u_{xy} + 9u_{yy} = 0$, is parabolic. Use the change of variables $r = 3x - 2y$, $s = y$ to show that this equation is equivalent to $u_{ss} = 0$. How does this equation compare to the heat equation? Solve the equation $u_{rr} = 0$ to obtain a general solution to $4u_{xx} + 12u_{xy} + 9u_{yy} = 0$ in the variables $x$ and $y$.
25. Show that Laplace's equation, $u_{xx} + u_{yy} = 0$, is elliptic.
26. Show that $u_{xx} + 2u_{xy} + 17u_{yy} = 0$ is elliptic. Use the change of variables $r = x - y$, $s = 4x$ to show that the equation $u_{xx} + 2u_{xy} + 17u_{yy} = 0$ is equivalent to $u_{rr} + u_{ss} = 0$. How does this compare to Laplace's equation?
*27. Show that the solution of the problem

$$\begin{cases} u_{xx} = u_{tt}, & 0 < x < 1,\ t > 0 \\ u(0, t) = u(1, t) = 0, & t \geq 0 \\ u(x, 0) = \sin \pi x,\ u_t(x, 0) = 0, & 0 \leq x \leq 1 \end{cases}$$

obtained using D'Alembert's solution is the same as that found using separation of variables.

28. Show that the solution of the problem

$$\begin{cases} u_{xx} = u_{tt}, & 0 < x < \pi,\ t > 0 \\ u(0, t) = u(\pi, t) = 0, & t \geq 0 \\ u(x, 0) = 0,\ u_t(x, 0) = \sin x, & 0 \leq x \leq \pi \end{cases}$$

obtained using D'Alembert's solution is the same as that found using separation of variables.

29. Solve

$$\begin{cases} u_{xx} = u_{tt}, & 0 < x < \pi,\ t > 0 \\ u(0, t) = u(\pi, t) = 0, & t \geq 0 \\ u(x, 0) = \frac{1}{8} \sin x,\ u_t(x, 0) = -\frac{1}{8} \sin x, & 0 \leq x \leq \pi \end{cases}$$

using D'Alembert's solution.

30. Consider the string problem with an endpoint that oscillates parallel to the $u$-axis (vertical axis). If the string has length $p$ and $c = 1$ in the wave equation, then an initial boundary value problem that models this situation is

$$\begin{cases} u_{xx} = u_{tt}, & 0 < x < \pi,\ t > 0 \\ u(0, t) = 0,\ u(\pi, t) = A \sin t, & t \geq 0 \\ u(x, 0) = 0, & 0 \leq x \leq \pi, \end{cases}$$

where $A$ is a small number. Assume that the solution of this problem is $u(x, t) = F(x + t) + G(x - t)$. Use the boundary and initial conditions to find $u(x, t)$ within an arbitrary constant. [*Hint:* Try $F(t) = Bt \sin t$.]

31. Use separation of variables to solve the initial boundary value problem

$$\begin{cases} c^2 u_{xx} = u_{tt}, & 0 < x < p,\ t > 0 \\ u_x(0, t) = 0,\ u_x(p, t) = 0, & t \geq 0 \\ u(x, 0) = f(x),\ u_t(x, 0) = g(x), & 0 \leq x \leq p \end{cases}$$

Find a general formula for the unknown coefficients.

32. Solve the initial boundary value problem in Exercise 31 if $p = \pi$, $f(x) = 1 + \cos 2x$, and $g(x) = 0$.

## 10.4 Problems in Two Dimensions: Laplace's Equation

Laplace's Equation    Steady-State Temperature

### Laplace's Equation

**Laplace's equation**, often called the **potential equation**, is given in rectangular coordinates by

$$u_{xx} + u_{yy} = 0,$$

and is one of the most useful partial differential equations in that it arises in many fields of study. These include fluid flows as well as electrostatic and gravitational potential. Because the potential does not depend on time, no initial condition is required, so we are faced with solving a pure boundary-value problem when working with Laplace's equation. The boundary conditions can be stated in different forms. If the value of the solution is given around the boundary of the region, then the boundary-value problem is called a **Dirichlet problem**, whereas if the normal derivative of the solution is given around the boundary, the problem is known as a **Neumann problem.** We now investigate the solutions to Laplace's equation in a rectangular region by, first, stating and solving the general form of the Dirichlet problem:

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < a,\ 0 < y < b \\ u(x, 0) = f_1(x),\ u(x, b) = f_2(x), & 0 < x < a \\ u(0, y) = g_1(y),\ u(a, y) = g_2(y), & 0 < y < b \end{cases},$$

where the boundary conditions are shown in Figure 10.14.

This boundary-value problem is solved through separation of variables. We begin by considering the problem

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < a,\ 0 < y < b \\ u(x, 0) = 0,\ u(x, b) = f(x), & 0 < x < a. \\ u(0, y) = 0,\ u(a, y) = 0, & 0 < y < b \end{cases}$$

In this case, we assume that

$$u(x, y) = X(x)Y(y),$$

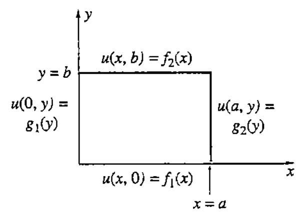



**Figure 10.14** Boundary equation on the region



so substitution into Laplace's equation yields

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda,$$

where $-\lambda$ is the separation constant. Therefore, we have the ordinary differential equations

$$X'' + \lambda X = 0 \quad \text{and} \quad Y'' - \lambda Y = 0.$$

Notice that the boundary conditions along the lines $x = 0$ and $x = a$ are homogeneous. In fact, because $u(0, y) = X(0)Y(y) = 0$ and $u(a, y) = X(a)Y(y) = 0$, we have $X(0) = 0$ and $X(a) = 0$. Therefore, we first solve the eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0,\ X(a) = 0 \end{cases},$$

which was solved with $a = p$ in Section 9.1. There we found the eigenvalues and corresponding eigenfunctions to be $\lambda_n = (n\pi/a)^2$, $n = 1, 2, \ldots$ and $X_n(x) = \sin(n\pi x/a)$, $n = 1, 2, \ldots$. (Notice that the problem is defined on $0 < x < a$ instead of $0 < x < p$.) We then solve the equation $Y'' - \lambda^2 Y = 0$ using these eigenvalues. From our experience with second-order equations, we know that $Y_n(y) = a_n e^{\sqrt{\lambda_n} y} + b_n e^{-\sqrt{\lambda_n} y}$, which can be written in terms of the hyperbolic trigonometric function as

$$Y(y) = Y_n(y) = A_n \cosh \sqrt{\lambda_n} y + B_n \sinh \sqrt{\lambda_n} y = A_n \cosh \frac{n\pi y}{a} + B_n \sinh \frac{n\pi y}{a}.$$

Then using the homogeneous boundary condition $u(x, 0) = X(x)Y(0) = 0$, which indicates that $Y(0) = 0$, we have

$$Y(0) = Y_n(0) = A_n \cosh 0 + B_n \sinh 0 = A_n = 0,$$

so $A_n = 0$ for all $n$. Therefore, $Y_n(y) = B_n \sinh \lambda y$, so that a solution is

$$u_n(x, y) = B_n \sinh \frac{n\pi y}{a} \sin \frac{n\pi x}{a}.$$

By the Principle of superposition, we have

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi y}{a} \sin \frac{n\pi x}{a},$$

where the coefficients are determined with the boundary condition $u(x, b) = f(x)$. Substitution into the solution yields

$$u(x, b) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a} = f(x),$$

where $B_n \sinh \frac{n\pi b}{a}$ represents the Fourier sine series coefficients given by

$$B_n \sinh \frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x) \sin \frac{n\pi x}{a}\,dx$$

$$B_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f(x) \sin \frac{n\pi x}{a}\,dx$$

**Example 1**

Solve $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 1,\ 0 < y < 2 \\ u(x, 0) = 0,\ u(x, 2) = x(1 - x), & 0 < x < 1. \\ u(0, y) = 0,\ u(1, y) = 0, & 0 < y < 2 \end{cases}$

***Solution*** In this case, $a = 1$, $b = 2$, and $f(x) = x(1 - x)$. Therefore,

$$B_n = \frac{2}{\sinh 2n\pi}\int_0^1 x(1-x)\sin n\pi x\,dx = \frac{1}{\sinh 2n\pi}\left(-\frac{4\cos n\pi}{n^3\pi^3} + \frac{4}{n^3\pi^3}\right)$$
$$= \frac{4}{n^3\pi^3 \sinh 2n\pi}(1 - (-1)^n),\ n = 1, 2, \ldots,$$

so the solution is

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sinh n\pi y \sin n\pi x = \sum_{n=1}^{\infty} \frac{8 \sinh((2n-1)\pi y)\sin((2n-1)\pi x)}{(2n-1)^3\pi^3 \sinh(2(2n-1)\pi)}.$$

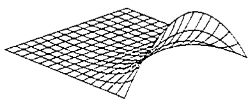


**Figure 10.15** The graph of $u$

In Figure 10.15, we plot $u(x, y)$ using the first five terms of the series solution.

*In Example 1, what happens to the value of u away from y = 2?*

Any version of Laplace's equation on a rectangular region can be solved through separation of variables as long as we have a pair of homogeneous boundary conditions in the same variable. (Note that a general form of this problem is presented in Exercise 33.)

Notice that in Example 1, we could have used the two linearly independent solutions,

$$\sinh \frac{n\pi y}{a} \quad \text{and} \quad \sinh \frac{n\pi(b-y)}{a},$$

of $Y'' - \lambda^2 Y = 0$ to form a general solution. (See Exercise 33). Doing so makes applying the boundary conditions more efficient, as we see in Example 2.

**Example 2**

Solve $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < \pi,\ 0 < y < 1 \\ u(x, 0) = 0,\ u(x, 1) = 0, & 0 < x < \pi \\ u(0, y) = \sin 2\pi y,\ u(\pi, y) = 4, & 0 < y < 1 \end{cases}$



***Solution*** Assume that $u(x, y) = X(x)Y(y)$. Notice that this problem differs from the previous problem in that the homogeneous boundary conditions are in terms of the variable $y$. Hence, when we separate variables, we use a different constant of separation. This yields

$$X''Y + XY'' = 0$$
$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda,$$

so we have the ordinary differential equations $X'' - \lambda X = 0$ and $Y'' + \lambda Y = 0$. Therefore, with the homogeneous boundary conditions $u(x, 0) = X(x)Y(0) = 0$ and $u(x, 1) = X(x)Y(1) = 0$, we have $Y(0) = 0$ and $Y(1) = 0$. Recall that the eigenvalue problem

$$\begin{cases} Y'' + \lambda Y = 0 \\ Y(0) = 0,\ Y(1) = 0 \end{cases}$$

has the eigenvalues $\lambda_n = (n\pi/1)^2 = n^2\pi^2$, $n = 1, 2, \ldots$ and solutions $Y_n(y) = \sin n\pi y$, $n = 1, 2, \ldots$. We then solve the equation $X'' - \lambda X = 0$ using these eigenvalues to obtain $X_n(y) = a_n e^{n\pi x} + b_n e^{-n\pi x}$, which can be written in terms of hyperbolic trigonometric functions as

$$X(x) = X_n(x) = A_n \sinh n\pi x + B_n \sinh n\pi(\pi - x).$$

Now, because the boundary conditions on the boundaries $x = 0$ and $x = \pi$ are nonhomogeneous, we use the Principle of superposition to obtain

$$u(x, y) = \sum_{n=1}^{\infty} (A_n \sinh n\pi x + B_n \sinh n\pi(\pi - x)) \sin n\pi y$$

Therefore,

$$u(0, y) = \sum_{n=1}^{\infty} B_n \sinh n\pi^2 \sin n\pi y = \sin 2\pi y,$$

so $B_2 \sinh 2\pi^2 = 1$ or $B_2 = 1/\sinh 2\pi^2$ and $B_n = 0$ for $n \neq 2$. Similarly,

$$u(\pi, y) = \sum_{n=1}^{\infty} A_n \sinh n\pi^2 \sin n\pi y = 4,$$

which indicates that $A_n \sinh n\pi^2$ represents the Fourier sine series coefficients for the constant function 4, which are given by

$$A_n \sinh n\pi^2 = \frac{2}{1}\int_0^1 4 \sin n\pi y\,dy = -8\left[\frac{\cos n\pi y}{n\pi}\right]_0^1$$
$$= \frac{-8}{n\pi}\left[(-1)^n - 1\right],\ n = 1, 2, \ldots.$$

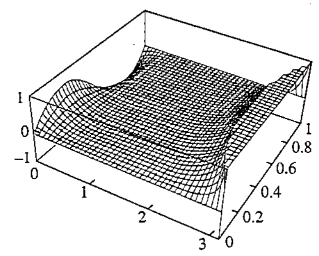



**Figure 10.16** The graph of $u$

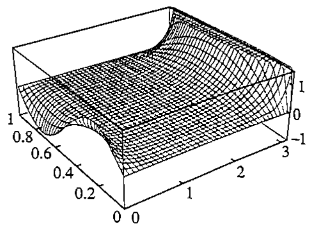



**Figure 10.17** A different view of the graph of $u$

From this formula, we see that $A_n = 0$ if $n$ is even. Therefore, we express these coefficients as

$$A_{2n-1} = \frac{16}{(2n-1)\pi \sinh((2n-1)\pi^2)},\ n = 1, 2, \ldots$$

so that the solution is

$$u(x, y) = \frac{1}{\sinh 2\pi^2} \sinh 2\pi(\pi - x) \sin 2\pi y + \sum_{n=1}^{\infty} \frac{16}{(2n-1)\pi \sinh((2n-1)\pi^2)} \sinh n\pi x \sin n\pi y.$$

Two different views of $u$ are shown in Figure 10.16 and Figure 10.17.

The other type of boundary conditions associated with Laplace's equation involves the derivative on the boundary (the **Neumann problem**) or a mixture of the Dirichlet and Neumann problems. We consider these other types of problems in the following context.

### Steady-State Temperature

One application of Laplace's equation is the steady-state solution of the heat equation in a rectangular region. Recall that as $t \to \infty$, the temperature no longer depends on $t$. Therefore, if the temperature in a rectangular region at the point $(x, y)$ at time $t$ is given by $u(x, y, t)$, then

$$\lim_{t\to\infty} u(x, y, t) = S(x, y),$$

where $S(x, y)$ is the steady-state temperature. Because the heat equation in two spatial dimensions, $x$ and $y$, is given by

$$c^2(u_{xx} + u_{yy}) = u_t$$



the steady-state temperature satisfies Laplace's equation,

$$S_{xx} + S_{yy} = 0,$$

because there is no dependence on $t$. Also, we describe the temperature around the boundary of the rectangular region as we did in the Dirichlet problem, which gives us a similar problem to solve to find $S(x, y)$.

**Example 3**

Find the steady-state temperature in the rectangle bounded by $x = 0, x = \pi, y = 0$, and $y = 1$ if the edges $y = 0$ and $y = 1$ are insulated and the temperatures along the edges $x = 0$ and $x = \pi$ are given by $S(0, y) = 100$ and $S(\pi, y) = \cos \pi y$.

***Solution*** To describe the insulated edges mathematically, we recall that insulation stops the flow of heat over those edges. Because flow across these edges is in the $y$ direction, we have

$$S_y(x, 0) = 0 \quad \text{and} \quad S_y(x, 1) = 0 \quad \text{for} \quad 0 < x < \pi.$$

Therefore, we solve the boundary-value problem

$$\begin{cases} S_{xx} + S_{yy} = 0, & 0 < x < \pi, 0 < y < 1 \\ S_y(x, 0) = 0, S_y(x, 1) = 0, & 0 < x < \pi \\ S(0, y) = 100, S(\pi, y) = \cos \pi y, & 0 < y < 1 \end{cases}.$$

Assuming that $S(x, y) = X(x)Y(y)$, we perform separation of variables. Notice that in this case, the homogeneous boundary conditions are associated with $Y$ because $S_y(x, 0) = X(x)Y'(0) = 0$ and $S_y(x, 1) = X(x)Y'(1) = 0$ lead to the conditions

$$Y'(0) = 0 \quad \text{and} \quad Y'(1) = 0.$$

Based on our experience with eigenvalue problems, we choose the constant of separation so that we obtain the equations

$$X'' - \lambda X = 0 \quad \text{and} \quad Y'' + \lambda Y = 0$$

when separating variables. Taking advantage of the homogeneous boundary conditions, we first solve the problem

$$\begin{cases} Y'' + \lambda Y = 0, & 0 < y < 1 \\ Y'(0) = 0, Y'(1) = 0 \end{cases}$$

with eigenvalues and corresponding eigenfunctions

$$\lambda_n = \begin{cases} 0, & n = 0 \\ \left(\frac{n\pi}{1}\right)^2 = n^2\pi^2, & n = 1, 2, \ldots \end{cases} \quad \text{and}$$

$$Y_n(y) = \begin{cases} 1, & n = 0 \\ \cos \frac{n\pi y}{1} = \cos n\pi y, & n = 1, 2, \ldots \end{cases}.$$

Using these eigenvalues, we solve $X'' - \lambda X = 0$. If $\lambda = \lambda_0 = 0$, then we have the solution

$$X(x) = X_0(x) = c_1 + c_2 x.$$

Similarly, if $\lambda = \lambda_n = n^2\pi^2$, $n = 1, 2, \ldots$, we find that

$$X(x) = X_n(x) = A_n \cosh n\pi x + B_n \sinh n\pi x.$$

Combining these results, we find with the Principle of superposition that

$$S(x, y) = X_0(x)Y_0(y) + \sum_{n=1}^{\infty} X_n(x)Y_n(y) = c_1 + c_2 x$$

$$+ \sum_{n=1}^{\infty} (A_n \cosh n\pi x + B_n \sinh n\pi x) \cos n\pi y.$$

Applying the condition $S(0, y) = 100$ yields

$$S(0, y) = c_1 + \sum_{n=1}^{\infty} A_n \cos n\pi y = 100,$$

so $c_1$ and $A_n$ are the Fourier cosine series coefficients for $f(x) = 100$, which are found in this case by matching the coefficients on each side of the equation (or by carrying out the appropriate integration). In this case, $c_1 = 100$ and $A_n = 0$ for all $n$. For $S(\pi, y) = \cos \pi y$, we have

$$S(\pi, y) = 100 + c_2\pi + \sum_{n=1}^{\infty} B_n \sinh n\pi^2 \cos n\pi y = \cos \pi y.$$

Therefore, $100 + c_2\pi = 0$, which means that $c_2 = -100/\pi$. Then by comparing the coefficients of the cosine series and the function $\cos \pi y$ with $n = 1$, we find that $B_1 \sinh \pi^2 = 1$, so $B_1 = (1/\sinh \pi^2)$ and $B_n = 0$ for $n \geq 2$. The steady-state temperature in the region is then given by

$$S(x, y) = 100 - \frac{100}{\pi}x + \frac{1}{\sinh \pi^2} \sinh \pi x \cos \pi y.$$

In Figure 10.18, we graph $S(x, y)$ over the region.

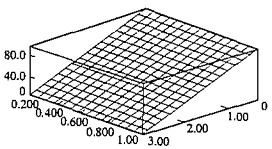



**Figure 10.18** The steady state of the region



## EXERCISES 10.4

In Exercises 1–8, the problems involve three homogeneous and one nonhomogeneous boundary condition. Use the methods discussed in the examples to solve the boundary-value problems.

1. $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 1, 0 < y < 2 \\ u(x, 0) = 0, u(x, 2) = 0, & 0 < x < 1 \\ u(0, y) = 0, u(1, y) = 1 + y, & 0 < y < 2 \end{cases}$

2. $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < \pi, 0 < y < \pi \\ u(x, 0) = 0, u(x, \pi) = 0, & 0 < x < \pi \\ u(0, y) = 0, u(\pi, y) = 10, & 0 < y < \pi \end{cases}$

*3. $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 1, 0 < y < 2 \\ u(x, 0) = 0, u(x, 2) = 0, & 0 < x < 1 \\ u(0, y) = y, u(1, y) = 0, & 0 < y < 2 \end{cases}$

4. $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < \pi, 0 < y < 2\pi \\ u(x, 0) = 0, u(x, 2\pi) = 0, & 0 < x < \pi \\ u(0, y) = y^2, u(\pi, y) = 0, & 0 < y < 2\pi \end{cases}$

5. $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 1, 0 < y < 2 \\ u(x, 0) = 1, u(x, 2) = 0, & 0 < x < 1 \\ u(0, y) = 0, u(1, y) = 0, & 0 < y < 2 \end{cases}$

6. $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 1, 0 < y < 1 \\ u(x, 0) = \sin 2\pi x, u(x, 1) = 0, & 0 < x < 1 \\ u(0, y) = 0, u(1, y) = 0, & 0 < y < 1 \end{cases}$

*7. $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 1, 0 < y < 2 \\ u(x, 0) = 0, u(x, 2) = \sin \pi x, & 0 < x < 1 \\ u(0, y) = 0, u(1, y) = 0, & 0 < y < 2 \end{cases}$

8. $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 1, 0 < y < \pi \\ u(x, 0) = 0, u(x, \pi) = \sin 2x, & 0 < x < 1 \\ u(0, y) = 0, u(1, y) = 0, & 0 < y < \pi \end{cases}$

In Exercises 9–10, solve each problem using separation of variables. (Note that the method of separation of variables must be used due to the boundary conditions.)

9. $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < \pi, 0 < y < 1 \\ u_x(0, y) = 0, u_x(\pi, y) = 0, & 0 < y < 1 \\ u(x, 0) = 4 \cos x + 1, u(x, 1) = 0, & 0 < x < \pi \end{cases}$

10. $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 1, 0 < y < 1 \\ u(0, y) = y, u(1, y) = 0, & 0 < y < 1 \\ u_y(x, 0) = 0, u_y(x, 1) = 0, & 0 < x < 1 \end{cases}$

11. **(Steady-State Temperature in a Semi-Infinite Strip)** Consider the boundary-value problem

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < \infty, 0 < y < b \\ u(x, 0) = 0 \text{ and } u(x, b) = 0, & 0 < x < \infty \\ u(x, y) \text{ is bounded as } x \to \infty \\ u(0, y) = f(y), 0 < y < b \end{cases}$$

(a) Using separation of variables with $u(x, y) = X(x)Y(y)$, show that $Y$ satisfies the eigenvalue problem

$$\begin{cases} Y'' + \lambda Y = 0, & 0 < y < b \\ Y(0) = 0, Y(b) = 0 \end{cases}$$

and find the eigenvalues and eigenfunctions of this problem.

(b) Show that $X_n(x) = a_n e^{n\pi x/b} + b_n e^{-n\pi x/b}$. Find a restriction on one of the constants so that $X_n(x)$ is bounded as $x \to \infty$.

(c) Form the series solution $u(x, y) = \sum_{n=1}^{\infty} B_n e^{-n\pi x/b} \sin \frac{n\pi y}{b}$ and find $B_n$ using the boundary condition $u(0, y) = f(y)$.

In Exercises 12–14, use the solution found in Exercise 11 to solve each boundary-value problem using the given parameter value and boundary condition.

12. $b = \pi$, $u(0, y) = 100$
13. $b = 1$, $u(0, y) = T_0$
14. $b = 2$, $u(0, y) = y$
15. Use a method similar to that in Exercise 11 to solve

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < a, 0 < y < \infty \\ u(0, y) = 0, u(a, y) = 0, & 0 < y < \infty \\ u(x, y) \text{ is bounded as } y \to \infty \\ u(x, 0) = f(x), & 0 < x < a \end{cases}$$

In Exercises 16–18, use the results of Exercise 15 to find the solution of each boundary-value problem using the given parameter value and boundary condition.

16. $a = \pi, f(x) = 1$
17. $a = 2, f(x) = x$
18. $a = 1, f(x) = \frac{1}{3} \sin 2\pi x$

**19.** Find the steady-state temperature in the rectangle bounded by $x = 0$, $x = 1$, $y = 0$, and $y = \pi$ if the edges $x = 0$ and $x = 1$ are insulated and the temperature along the edges $y = 0$ and $y = \pi$ are given by $S(x, 0) = \cos \pi x$ and $S(x, \pi) = 0$. (Assume that $c = 1$).

**20.** Find the steady-state temperature in the square bounded by $x = 0$, $x = \pi$, $y = 0$, and $y = \pi$ if the edges $y = 0$ and $y = \pi$ are insulated and the temperatures along the edges $x = 0$ and $x = \pi$ are given by $S(0, y) = 0$ and $S(\pi, y) = \cos 2y$. (Assume that $c = 1$).

**21. (Double Fourier Series)** The series

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

is a double Fourier series. (The function $f$ can be written as such a series over a region $R$: $0 \le x \le a$, $0 \le y \le b$ if $f, f_x, f_y, f_{xx}$, and $f_{yy}$ are continuous in the region $R$.) **(a)** To find the coefficients $B_{mn}$, let $k_m(y) = \sum_{n=1}^{\infty} B_{mn} \sin \frac{n\pi y}{b}$ and show that the series becomes $f(x, y) = \sum_{m=1}^{\infty} k_m(y) \sin \frac{m\pi x}{a}$. **(b)** For a fixed $y$, $f$ can be considered a function of $x$ only, so show that $k_m(y) = \frac{2}{a} \int_0^a f(x, y) \sin \frac{m\pi x}{a}\, dx$. **(c)** For the function of $y$, $k_m(y) = \sum_{n=1}^{\infty} B_{mn} \sin \frac{n\pi y}{b}$, show that $B_{mn} = \frac{2}{b} \int_0^b k_m(y) \sin \frac{n\pi y}{b}\, dy$. **(d)** Combine the results of **(b)** and **(c)** to show that

$$B_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy,$$

$m = 1, 2, \ldots, n = 1, 2, \ldots$.

In Exercises 22–24, use the formula derived in Exercise 21 to find the double Fourier series for each function over $R$.

**22.** $f(x, y) = 1$, $R$: $0 \le x \le 1$, $0 \le y \le 1$

**23.** $f(x, y) = x + y$, $R$: $0 \le x \le 1$, $0 \le y \le \pi$

**24.** $f(x, y) = xy$, $R$: $0 \le x \le \pi$, $0 \le y \le \pi$

**25. (Two-Dimensional Wave Equation in Rectangular Coordinates)** Solve the initial boundary value problem

$$\begin{cases} c^2(u_{xx} + u_{yy}) = u_{tt}, & 0 < x < a,\ 0 < y < b \\ u(x, 0, t) = 0,\ u(x, b, t) = 0, & 0 < x < a \\ u(0, y, t) = 0,\ u(a, y, t) = 0, & 0 < y < b \\ u(x, y, 0) = f(x, y),\ 0 < x < a, & 0 < y < b \\ u_t(x, y, 0) = g(x, y),\ 0 < x < a, & 0 < y < b, \end{cases}$$

which determines the displacement of a rectangular membrane (a rectangular drumhead). **(a)** Let $u(x, y, t) = S(x, y)T(t)$, and show that $S$ satisfies the partial differential equation $S_{xx} + S_{yy} + v^2 S = 0$ (called the Helmholtz equation), where $(-v^2)$ is the constant of separation. Also, $T$ satisfies $T'' + \lambda^2 T = 0$, where $\lambda = cv$. **(b)** Let $S(x, y) = X(x)Y(y)$. Show that $X$ and $Y$ satisfy $X'' + k^2 X = 0$ and $Y'' + p^2 Y = 0$, where $k^2 + p^2 = v^2$. **(c)** Show that the four homogeneous boundary conditions can be expressed as $X(0) = X(a) = 0$ and $Y(0) = Y(b) = 0$. Use these boundary conditions to solve the eigenvalue problems for $X$ and $Y$. Combine these results to show that

$$X_{mn}(x, y) = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b},$$

$m = 1, 2, \ldots, n = 1, 2, \ldots$. **(d)** Show that $\lambda = \lambda_{mn} = c\pi \sqrt{(m/a)^2 + (n/b)^2}$ and that the solutions of $T'' + \lambda^2 T = 0$ are $T_{mn}(t) = B_{mn} \cos \lambda_{mn} t + C_{mn} \sin \lambda_{mn} t$. **(e)** Combine the results of **(c)** and **(d)** to show that

$u_{mn}(x, y, t) =$

$$(B_{mn} \cos \lambda_{mn} t + C_{mn} \sin \lambda_{mn} t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

are solutions to the wave equation in rectangular coordinates. **(f)** Show that the coefficients $B_{mn}$ are found with the formula in Exercise 21. **(g)** Show that

$$C_{mn} = \frac{4}{ab\lambda_{mn}} \int_0^b \int_0^a g(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy,$$

$m = 1, 2, \ldots, n = 1, 2, \ldots$.

In Exercises 26–30, use the formula derived in Exercise 25 to solve the wave equation in $R$ using the given parameter values and initial conditions.

**26.** $a = b = \pi$, $f(x, y) = \frac{1}{2} \sin 2x \sin 2y$, $g(x, y) = 0$

**27.** $a = b = \pi$, $f(x, y) = 0$, $g(x, y) = \frac{1}{4} \sin x \sin 2y$

**28.** $a = 1$, $b = 2$, $f(x, y) = 0$, $g(x, y) = 1$



**29.** $a = 1$, $b = 2$, $f(x, y) = xy(1 - x)(1 - y)$, $g(x, y) = 0$

**30.** $a = b = \pi$, $f(x, y) = 0$, $g(x, y) = xy$

**31.** Determine the steady-state solution of the two-dimensional heat equation

$$\begin{cases} k(u_{xx} + u_{yy}) = u_t, & 0 < x < a,\ 0 < y < b \\ u(x, 0, t) = f_1(x),\ u(x, b, t) = f_2(x), & 0 < x < a \\ u(0, y, t) = g_1(x),\ u(a, y, t) = g_2(x), & 0 < y < b \\ u(x, y, 0) = F(x, y), & 0 < x < a,\ 0 < y < b. \end{cases}$$

**32.** Use the results of Exercise 31 to find the steady-state temperature in the rectangular plate $R$: $0 < x < 1$, $0 < y < 2$ if the temperature is fixed at zero around each edge of the region except $u(x, 0, t) = \sin \pi x$.

**33.** To solve

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < a,\ 0 < y < b \\ u(x, 0) = f(x),\ u(x, b) = g(x), & 0 < x < a, \\ u(0, y) = h(y),\ u(a, y) = k(y), & 0 < y < b \end{cases}$$

we assume that $u(x, y) = v(x, y) + w(x, y)$. **(a)** Show that if $v$ and $w$ satisfy the problems

$$\begin{cases} v_{xx} + v_{yy} = 0, & 0 < x < a,\ 0 < y < b \\ v(x, 0) = f(x),\ v(x, b) = g(x), & 0 < x < a \\ v(0, y) = 0,\ v(a, y) = 0, & 0 < y < b \end{cases} \quad \text{and}$$

$$\begin{cases} w_{xx} + w_{yy} = 0, & 0 < x < a,\ 0 < y < b \\ w(x, 0) = 0,\ w(x, b) = 0, & 0 < x < a \\ w(0, y) = h(y),\ w(a, y) = k(y), & 0 < y < b \end{cases},$$

then $u(x, y) = v(x, y) + w(x, y)$ is a solution of the original equation.

**(b)** Show that the solution of

$$\begin{cases} v_{xx} + v_{yy} = 0, & 0 < x < a,\ 0 < y < b \\ v(x, 0) = f(x),\ v(x, b) = g(x), & 0 < x < a \\ v(0, y) = 0,\ v(a, y) = 0, & 0 < y < b \end{cases}$$

is

$v(x, y) =$

$$\sum_{n=1}^{\infty} \left(a_n \cosh \frac{n\pi y}{a} + b_n \sinh \frac{n\pi y}{a}\right) \sin \frac{n\pi x}{a},$$

which can be written as

$v(x, y) =$

$$\sum_{n=1}^{\infty} \left(A_n \sinh \frac{n\pi y}{a} + B_n \sinh \frac{n\pi(b - y)}{a}\right) \sin \frac{n\pi x}{a}.$$

Find a general formula for each of the unknown coefficients $A_n$ and $B_n$.

**(c)** In a manner similar to that in **(b)**, solve

$$\begin{cases} w_{xx} + w_{yy} = 0, & 0 < x < a,\ 0 < y < b \\ w(x, 0) = 0,\ w(x, b) = 0, & 0 < x < a \\ w(0, y) = h(y),\ w(a, y) = k(y), & 0 < y < b. \end{cases}$$

Find a general formula for each unknown coefficient.

In Problems 34–42, use the results of the Exercise 33 to solve each problem. Use technology to assist in determining the coefficients in the solutions and graph the solutions (with a finite number of terms of series solutions when necessary), and notice how the boundary conditions affect the solution.

**34.** $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 1,\ 0 < y < 1 \\ u(x, 0) = 10,\ u(x, 1) = 0, & 0 < x < 1 \\ u(0, y) = 0,\ u(1, y) = 10, & 0 < y < 1 \end{cases}$

**35.** $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < \pi,\ 0 < y < 1 \\ u(x, 0) = \cos x,\ u(x, 1) = 0, & 0 < x < \pi \\ u(0, y) = 1,\ u(\pi, y) = 0, & 0 < y < 1 \end{cases}$

**36.** $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < \pi,\ 0 < y < 1 \\ u(x, 0) = 0,\ u(x, 1) = x(1 - x), & 0 < x < \pi \\ u(0, y) = 1,\ u(\pi, y) = 0, & 0 < y < 1 \end{cases}$

***37.** $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < \pi,\ 0 < y < 1 \\ u(x, 0) = 0,\ u(x, 1) = x(1 - x), & 0 < x < \pi \\ u(0, y) = 1,\ u(\pi, y) = 10, & 0 < y < 1 \end{cases}$

**38.** $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 1,\ 0 < y < 1 \\ u(x, 0) = x(1 - x),\ u(x, 1) = 0, & 0 < x < 1 \\ u(0, y) = 0,\ u(1, y) = y, & 0 < y < 1 \end{cases}$

**39.** $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 1,\ 0 < y < 1 \\ u(x, 0) = 0,\ u(x, 1) = x^2, & 0 < x < 1 \\ u(0, y) = y,\ u(1, y) = 1 - y, & 0 < y < 1 \end{cases}$

**40.** $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 1,\ 0 < y < 1 \\ u(x, 0) = \sin \pi x,\ u(x, 1) = x, & 0 < x < 1 \\ u(0, y) = y,\ u(1, y) = 1 - y, & 0 < y < 1 \end{cases}$

***41.** $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 1,\ 0 < y < 2 \\ u(x, 0) = 0,\ u(x, 2) = x(1 - x^2), & 0 < x < 1 \\ u(0, y) = y,\ u(1, y) = 0, & 0 < y < 2 \end{cases}$

**42.** $\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 1,\ 0 < y < 2 \\ u(x, 0) = 0,\ u(x, 2) = x(1 - x^2), & 0 < x < 1 \\ u(0, y) = y,\ u(1, y) = \sin \pi y/2, & 0 < y < 2 \end{cases}$

In Problems 43–46, find the potential in the square $R$: $0 \le x \le \pi$, $0 \le y \le \pi$ by solving (by hand) the following boundary-value problem using the given boundary conditions. In each case, graph the *equipotential curves* along which $u(x, y)$ is constant.

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < \pi, 0 < y < \pi \\ u(x, 0) = 0, u(x, \pi) = f(x), & 0 < x < \pi \\ u(0, y) = 0, u(\pi, y) = 0, & 0 < y < \pi \end{cases}$$

43. $f(x) \sin x$

44. $f(x) = \sin 3x$

*45. $f(x) = \sin x - \frac{1}{2} \sin 2x$

46. $f(x) = \sin 2x + \frac{1}{8} \sin 5x$

## 10.5 Two-Dimensional Problems in a Circular Region

Laplace's Equation in a Circular Region • The Wave Equation in a Circular Region

In some situations, the region on which we solve a boundary-value problem or an initial boundary value problem is not rectangular in shape. For example, we usually do not have rectangular-shaped drumheads, and the heating elements on top of a stove are not square. Instead, these objects are typically circular in shape, so we find the use of polar coordinates convenient. In this section, we discuss problems of this type by presenting two important problems solved on circular regions, Laplace's equation, which is related to the steady-state temperature, and the wave equation, which is used to find the displacement of a drumhead.

### Laplace's Equation in a Circular Region

In calculus, we found that polar coordinates are useful in solving many problems. The same can be said for solving boundary-value problems in a circular region. With the change of variables

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

we transform Laplace's equation in rectangular coordinates, $u_{xx} + u_{yy} = 0$, to polar coordinates

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0,\ 0 < r < \rho,\ -\pi < \theta < \pi.$$

(We leave this transformation as an exercise.) Recall that for the solution of Laplace's equation in a rectangular region, we had to specify a boundary condition on each of the four boundaries of the rectangle. However, in the case of a circle, there are not four sides, so we must alter the boundary conditions. Because in polar coordinates the points $(r, \pi)$ and $(r, -\pi)$ are equivalent for the same value of $r$, we want our solution and its derivative with respect to $\theta$ to match at these points (so that the solution is smooth). Therefore, two of the boundary conditions are

$$u(r, -\pi) = u(r, \pi) \quad \text{and} \quad u_\theta(r, -\pi) = u_\theta(r, \pi) \quad \text{for} \quad 0 < r < \rho.$$



Also, we want our solution to be bounded at $r = 0$, so another boundary condition is

$$|u(0, \theta)| < \infty \quad \text{for} \quad -\pi < \theta < \pi.$$

Finally, we can specify the value of the solution around the boundary of the circle. This is given by

$$u(\rho, \theta) = f(\theta) \quad \text{for} \quad -\pi < \theta < \pi.$$

Therefore, we solve the following boundary-value problem to solve Laplace's equation (the Dirichlet problem) in a circular region of radius $\rho$.

$$\begin{cases} u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, & 0 < r < \rho, -\pi < \theta < \pi \\ u(r, -\pi) = u(r, \pi), u_\theta(r, -\pi) = u_\theta(r, \pi), & 0 < r < \rho \\ |u(0, \theta)| < \infty, u(\rho, \theta) = f(\theta), & -\pi < \theta < \pi. \end{cases}$$

Using separation of variables, we assume that $u(r, \theta) = R(r)H(\theta)$. Then, substitution into Laplace's equation yields

$$R''H + \frac{1}{r} R'H + RH'' = 0$$

$$R''H + \frac{1}{r} R'H = -RH''$$

$$\frac{rR'' + R'}{rR} = \frac{-H''}{H} = \lambda$$

Therefore, we have the ordinary differential equations

$$H'' + \lambda H = 0 \quad \text{and} \quad r^2R'' + rR' - \lambda R = 0.$$

Notice that the boundary conditions

$$u(r, -\pi) = u(r, \pi) \quad \text{and} \quad u_\theta(r, -\pi) = u_\theta(r, \pi)$$

imply that

$$R(r)H(-\pi) = R(r)H(\pi) \quad \text{and} \quad R(r)H'(-\pi) = R(r)H'(\pi),$$

so that

$$H(-\pi) = H(\pi) \quad \text{and} \quad H'(-\pi) = H'(\pi).$$

This means that we begin by solving the eigenvalue problem

$$\begin{cases} H'' + \lambda H = 0, -\pi < \theta < \pi \\ H(-\pi) = H(\pi), H'(-\pi) = H'(\pi), \end{cases}$$

which was solved in Exercise 35 in Section 9.1 and used to establish the Fourier series. The eigenvalues and corresponding eigenfunctions of this problem are

$$\lambda_n = \begin{cases} 0, & n = 0 \\ n^2, & n = 1, 2, \ldots \end{cases} \quad \text{and} \quad H_n(\theta) = \begin{cases} 1, & n = 0 \\ a_n \cos n\theta + b_n \sin n\theta, & n = 1, 2, \ldots \end{cases}.$$

Because $r^2R'' + rR' - \lambda^2 R = 0$ is a Cauchy-Euler equation, we assume that $R(r) = r^m$.

$$m(m-1)r^2r^{m-2} + mrr^{m-1} - \lambda r^m = 0$$
$$r^m[m(m-1) + m - \lambda] = 0$$

Therefore,

$$m^2 - \lambda^2 = 0$$
$$m = \pm\lambda.$$

If $\lambda_0 = 0$, then $R_0(r) = c_1 + c_2 \ln r$. However, because we require that the solution be bounded near $r = 0$ and $\lim_{r\to 0^+} \ln r = -\infty$, we must choose $c_2 = 0$. Therefore, $R_0(r) = c_1$. On the other hand, if $\lambda_n = n^2$, $n = 1, 2, \ldots$, then $R_n(r) = c_3r^n + c_4r^{-n}$. Similarly, because $\lim_{r\to 0^+} r^{-n} = \infty$, we must let $c_4 = 0$, so $R_n(r) = c_3r^n$. By the Principle of superposition, we have the solution

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n(A_n \cos n\theta + B_n \sin n\theta),$$

where $A_0 = c_1$, $A_n = c_3a_n$, and $B_n = c_3b_n$. We find these coefficients by applying the boundary condition $u(\rho, \theta) = f(\theta)$. This yields

$$u(\rho, \theta) = A_0 + \sum_{n=1}^{\infty} \rho^n (A_n \cos n\theta + B_n \sin n\theta) = f(\theta),$$

so $A_0$, $A_n$, and $B_n$ are related to the Fourier series coefficients in the following way:

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$A_n = \frac{1}{\pi\rho^n}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta \quad (n = 1, 2, \ldots)$$

$$B_n = \frac{1}{\pi\rho^n}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta \quad (n = 1, 2, \ldots)$$

*What is the potential at the center of the region? What does this value represent?*

**Example 1**

Solve $\begin{cases} u_{rr} + \dfrac{1}{r}u_r + \dfrac{1}{r^2}u_{\theta\theta} = 0, & 0 < r < 2,\ -\pi < \theta < \pi \\ u(r, -\pi) = u(r, \pi),\ u_\theta(r, -\pi) = u_\theta(r, \pi), & 0 < r < 2. \\ |u(0, \theta)| < \infty,\ u(2, \theta) = |\theta|, & -\pi < \theta < \pi. \end{cases}$

**Solution** Notice that $f(\theta) = |\theta|$ is an even function on $-\pi < \theta < \pi$. Therefore, $B_n = 0$ for $n \geq 1$,



$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\theta|\, d\theta = \frac{1}{\pi}\int_0^{\pi} \theta\, d\theta = \frac{1}{\pi}\left[\frac{\theta^2}{2}\right]_0^{\pi} = \frac{\pi}{2},$$

and with integration by parts,

$$A_n = \frac{1}{\pi 2^n}\int_{-\pi}^{\pi} |\theta|\cos n\theta\, d\theta = \frac{1}{\pi 2^{n-1}}\int_0^{\pi} \theta\cos n\theta\, d\theta = \left(\frac{1}{\pi 2^n n^2}\right)[\cos n\pi - 1]$$
$$= \left(\frac{1}{\pi 2^{n-1} n^2}\right)[(-1)^n - 1],\ n \geq 1.$$

Notice that $A_{2n} = 0$, $n \geq 1$, and $A_{2n-1} = \dfrac{-2}{\pi 2^{2n-2}(2n-1)^2}$, $n \geq 1$, so the solution is

$$u(r, \theta) = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{r^{2n-1}}{\pi 2^{2n-2}(2n-1)^2}\cos(2n-1)\theta.$$

In Figure 10.19, we graph this solution over the circular region.

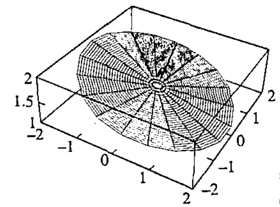



**Figure 10.19** Graph of the potential over the circular region

In an earlier example using rectangular coordinates, we discussed how we associate the solution to Laplace's equation to the steady-state temperature. Of course, the same can be done with polar coordinates. We illustrate this approach in Example 2.

**Example 2**

Find the steady-state temperature in the circular region of radius $\rho = 10$, if $u(10, \theta) = 70$.

**Solution** Instead of introducing a new variable to represent the steady-state temperature as we did in Section 10.4, let $u(r, \theta)$ be this temperature. Common sense may tell us that the temperature at each point in the circular region eventually attains the value of the temperature around the boundary. The formula for $u$ above verifies this because $A_0 = 1/(2\pi)\int_{-\pi}^{\pi} 70\, d\theta = 70$, $A_n = 1/(\pi 10^n)\int_{-\pi}^{\pi} 70\cos n\theta\, d\theta = 0$, and $B_n = 1/(\pi 10^n)\int_{-\pi}^{\pi} 70\sin n\theta\, d\theta = 0$. Therefore, $u(r, \theta) = 70$.

## The Wave Equation in a Circular Region

One of the more interesting problems involving two spatial dimensions ($x$ and $y$) is the wave equation,

$$c^2(u_{xx} + u_{yy}) = u_{tt}.$$

This is due to the fact that the solution to this problem represents something with which we are all familiar, the displacement of a drumhead. Because most drumheads are circular in shape, we investigate the solution of the wave equation in a circular region. Therefore, we transform the wave equation into polar coordinates. In the previous section on Laplace's equation, we found that

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}.$$

Then the wave equation in polar coordinates becomes

$$c^2\left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}\right) = u_{tt}.$$

If we assume that the displacement of the drumhead from the $xy$-plane at time $t$ is the same at equal distances from the origin, then we say that the solution $u$ is **radially symmetric.** (In other words, the value of $u$ does not depend on the angle $\theta$.) Therefore, $u_{\theta\theta} = 0$, so the wave equation can be expressed in terms of $r$ and $t$ only as

$$c^2\left(u_{rr} + \frac{1}{r} u_r\right) = u_{tt}.$$

Of course, to find $u(r, t)$ we need the appropriate boundary and initial conditions. Because the circular boundary of the drumhead must be fixed so that it doesn't move, we say that

$$u(\rho, t) = 0 \quad \text{for} \quad t > 0.$$

Then as we had in Laplace's equation on a circular region, we require that the solution $u$ be bounded near the origin. Therefore, we have the condition

$$|u(0, t)| < \infty \quad \text{for} \quad t > 0.$$

The initial displacement and initial velocity functions are given as functions of $r$ as

$$u(r, 0) = f(r) \quad \text{and} \quad u_t(r, 0) = g(r) \quad \text{for} \quad 0 < r < \rho.$$

Therefore, the initial boundary value problem to find the displacement $u$ of a circular drumhead (of radius $\rho$) is given by

$$\begin{cases} c^2\left(u_{rr} + \frac{1}{r} u_r\right) = u_{tt}, & 0 < r < \rho,\ t > 0 \\ u(\rho, t) = 0, \quad |u(0, t)| < \infty, & t > 0 \\ u(r, 0) = f(r),\ u_t(r, 0) = g(r), & 0 < r < \rho. \end{cases}$$



As with other problems, we are able to use separation of variables to find $u$ by assuming that $u(r, t) = R(r)T(t)$. Substitution into the wave equation yields

$$c^2\left(R''T + \frac{1}{r} R'T\right) = RT''$$

$$\frac{rR'' + R'}{rR} = \frac{T''}{c^2T} = -k^2,$$

where $-k^2$ is the separation constant. Separating the variables, we have the ordinary differential equations

$$r^2R'' + rR' + k^2r^2R = 0 \quad \text{and} \quad T'' + c^2k^2T = 0.$$

We recognize the equation $r^2R'' + rR' + k^2r^2R = 0$ as Bessel's equation of order zero, which has solution

$$R(r) = c_1J_0(kr) + c_2Y_0(kr),$$

where $J_0$ and $Y_0$ are the Bessel functions of order zero of the first and second kind, respectively. In terms of $R$, we express the boundary condition $|u(0, t)| < \infty$ as

$$|R(0)| < \infty.$$

Therefore, because $\lim_{r\to 0^+} Y_0(kr) = -\infty$, we must choose $c_2 = 0$. Then applying the other boundary condition $R(\rho) = 0$, we have

$$R(\rho) = c_1J_0(k\rho) = 0.$$

To avoid the trivial solution with $c_1 = 0$, we find $k$ satisfying

$$k\rho = \alpha_n,$$

where $\alpha_n$ is the $n$th zero of $J_0$, which was discussed in Section 4.8. Because $k$ depends on $n$, we write

$$k_n = \frac{\alpha_n}{\rho}.$$

The solution of $T'' + c^2k^2T = 0$ is

$$T_n(t) = A_n \cos ck_nt + B_n \sin ck_nt,$$

so with the Principle of superposition, we have

$$u(r, t) = \sum_{n=1}^{\infty} (A_n \cos ck_nt + B_n \sin ck_nt)J_0(k_nr),$$

where the coefficients $A_n$ and $B_n$ are found by applying the initial displacement and velocity functions. With $u(r, 0) = \sum_{n=1}^{\infty} A_nJ_0(k_nr) = f(r)$ and the othogonality conditions of the Bessel functions, we find that

$$A_n = \frac{\int_0^p rf(r)J_0(k_nr)\,dr}{\int_0^p r[J_0(k_nr)]^2\,dr} = \frac{2}{[J_1(\alpha_n)]^2}\int_0^p rf(r)J_0(k_nr)\,dr,\ n = 1, 2, \ldots.$$

Similarly, because

$$u_t(r, t) = \sum_{n=1}^{\infty}(-ck_nA_n \sin ck_nt + ck_nB_n \cos ck_nt)J_0(k_nr),$$

we have

$$u_t(r, 0) = \sum_{n=1}^{\infty} ck_nB_nJ_0(k_nr) = g(r).$$

Therefore,

$$B_n = \frac{\int_0^p rg(r)J_0(k_nr)\,dr}{ck_n\int_0^p r[J_0(k_nr)]^2\,dr} = \frac{2}{ck_n[J_1(\alpha_n)]^2}\int_0^p rg(r)J_0(k_nr)\,dr,\ n = 1, 2, \ldots.$$

**Example 3**

Solve $\begin{cases} c^2\left(u_{rr} + \frac{1}{r}u_r\right) = u_{tt}, & 0 < r < 1,\ t > 0 \\ u(1, t) = 0, \quad |u(0, t)| < \infty, & t > 0 \\ u(r, 0) = 1 - r^2,\ u_t(r, 0) = 0, & 0 < r < 1 \end{cases}$

**Solution** In this case, $\rho = 1, f(r) = 1 - r^2$, and $g(r) = 0$. Therefore, $B_n = 0$ for $n \geq 1$ and

$$A_n = \frac{2}{[J_1(\alpha_n)]^2}\int_0^1 r(1 - r^2)J_0(\alpha_nr)\,dr = \frac{4J_2(\alpha_n)}{\alpha_n^2[J_1(\alpha_n)]^2},\ n \geq 1,$$

where the previous integral was calculated with integration by parts with the formula $\frac{d}{dx}[x^nJ_n(x)] = x^nJ_{n-1}(x)$. (See Exercise 20.) Then the displacement is

$$u(r, t) = \sum_{n=1}^{\infty}\frac{4J_2(\alpha_n)}{\alpha_n^2[J_1(\alpha_n)]^2}(\cos c\alpha_nt)J_0(\alpha_nr).$$

We show the graph of $u$ for several values of $t$ in Figure 10.20 (using nine terms from the series solution). To actually watch the drumhead move, an animation device of a computer algebra system can be used. (*Note:* The coefficients also may be found numerically by using a computer algebra system.)



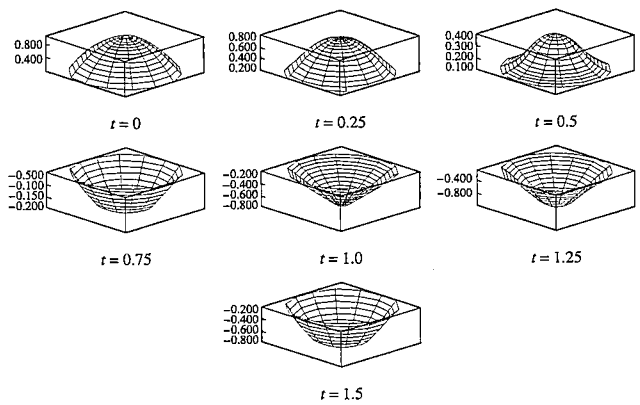



**Figure 10.20** Circular drum at various values of $t$

## EXERCISES 10.5

In Exercises 1–10, solve Laplace's equation in a circular region of radius 1 given the boundary condition $u(1, \theta)$.

**1.** $u(1, \theta) = \cos\theta$

**2.** $u(1, \theta) = \sin 2\theta$

***3.** $u(1, \theta) = \cos^2\theta$

**4.** $u(1, \theta) = \sin^2 2\theta$

**5.** $u(1, \theta) = \theta^2$

**6.** $u(1, \theta) = \theta\sin\theta$

***7.** $u(1, \theta) = \begin{cases} 1, & -\pi < \theta < 0 \\ -1, & 0 < \theta < \pi \end{cases}$

**8.** $u(1, \theta) = \begin{cases} 0, & -\pi < \theta < 0 \\ 100, & 0 < \theta < \pi \end{cases}$

**9.** $u(1, \theta) = \begin{cases} 0, & -\pi < \theta < 0 \\ \theta, & 0 < \theta < \pi \end{cases}$

**10.** $u(1, \theta) = \begin{cases} -\theta, & -\pi < \theta < 0 \\ \pi, & 0 < \theta < \pi \end{cases}$

In Exercises 11–16, solve the wave equation in a circular region of radius $\rho = 1$ that is radially symmetric using the initial condition functions given as follows. (If possible, graph the solution with a computer algebra system for various times to view the motion of the drumhead.)

**11.** $f(r) = J_0(\alpha_1r),\ g(r) = 0$

**12.** $f(r) = 0.1\,J_0(\alpha_3r),\ g(r) = 0$

***13.** $f(r) = 0,\ g(r) = J_0(\alpha_1r)$

**14.** $f(r) = 0,\ g(r) = \frac{1}{4}J_0(\alpha_2r)$

**15.** $f(r) = \frac{1}{8}J_0(\alpha_2r),\ g(r) = \frac{1}{4}J_0(\alpha_2r)$

**16.** $f(r) = J_0(\alpha_1r),\ g(r) = J_0(\alpha_2r)$

In Exercises 17–20, verify the integrals involving Bessel functions by using the identity $\frac{d}{dx}[x^nJ_n(x)] = x^nJ_{n-1}(x)$ and integration by parts when necessary.

**17.** Show that $\int_0^1 rJ_0(\alpha_n r)\,dr = \dfrac{1}{\alpha_n}J_1(\alpha_n)$.

**18.** Show that $\int_0^1 r^2 J_1(\alpha_n r)\,dr = \dfrac{1}{\alpha_n^2}J_2(\alpha_n)$.

**19.** Show that $\int_0^1 r^3 J_0(\alpha_n r)\,dr = \dfrac{1}{\alpha_n}J_1(\alpha_n) - \dfrac{2}{\alpha_n^2}J_2(\alpha_n)$.

(*Hint:* Use integration by parts with $u = r^2$.)

**20.** Show that $\int_0^1 r(1-r^2)J_0(\alpha_n r)\,dr = \dfrac{2}{\alpha_n^2}J_2(\alpha_n)$.

In Exercises 21–22, use the integrals in Exercises 17–20 to solve the wave equation in a circular region of radius $r = 1$ using the given initial conditions.

**21.** $f(r) = 1 - r^2$, $g(r) = 0$

**22.** $f(r) = 0$, $g(r) = 1 - r^2$

**23.** Verify the change of variables of Laplace's equation by using the Chain rule to show that if $x = r\cos\theta$ and $y = r\sin\theta$, then

$$u_x = \frac{x}{r}u_r - \frac{y}{r^2}u_\theta$$

and

$$u_y = \frac{y}{r}u_r + \frac{x}{r^2}u_\theta.$$

Also,

$$u_{xx} = \frac{x^2}{r^2}u_{rr} - \frac{2xy}{r^3}u_{r\theta} + \frac{y^2}{r^4}u_{\theta\theta} + \frac{y^2}{r^3}u_r + \frac{2xy}{r^4}u_\theta.$$

(A similar formula can be found for $u_{yy}$.)

**24.** Consider Laplace's equation in a semicircular region $R$ bounded by the graphs of $y = \sqrt{1-x^2}$ and the $x$-axis.

**(a)** If the potential $u$ is zero on the lower boundary of $R$: $-1 \le x \le 1$ and satisfies $f(\theta)$ on the semicircular boundary, show that the solution satisfies the boundary-value problem

$$\begin{cases} u_{rr} + \dfrac{1}{r}u_r + \dfrac{1}{r^2}u_{\theta\theta} = 0, & 0 < r < 1,\ 0 < \theta < \pi \\ u(r,0) = u(r,\pi) = 0, & 0 < r < 1 \\ u(1,\theta) = f(\theta), & |u(0,\theta)| < \infty,\ 0 < \theta < \pi \end{cases}$$

**(b)** Solve this problem, finding a general formula for all coefficients.

**25.** Solve the boundary-value problem in Exercise 24 if $f(\theta) = T_0$.

**26.** Solve the boundary-value problem in Exercise 24 if $f(\theta) = 100\theta(\pi - \theta)$.

**27. (Laplace's Equation in Spherical Coordinates)** In spherical coordinates, Laplace's equation is

$$\frac{1}{\rho^2}\left[\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}\right] = 0.$$

As we have discussed in other situations, the steady-state temperature satisfies Laplace's equation. Therefore, in a sphere we consider this form of the equation. Suppose that the temperature on the surface of a sphere of radius $R$ is $f(\phi)$. Because the boundary condition is independent of $\theta$, so is $u$. Therefore, $\partial^2 u/\partial\theta^2 = 0$, and Laplace's equation becomes

$$\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) = 0.$$

**(a)** Let $u(\rho,\phi) = R(\rho)F(\phi)$. Show that $R$ satisfies $\rho^2 R'' + 2\rho R' - kR = 0$, where $k$ is the constant of separation.

**(b)** Show that if $k = n(n+1)$, then the solutions of $\rho^2 R'' + 2\rho R' - kR = 0$ are $R_n(r) = r^n$ and $R_n(r) = 1/(r^{n+1})$.

**(c)** Show that $F$ satisfies the equation

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dF}{d\phi}\right) + n(n+1)F = 0.$$

To solve this equation, let $w = \cos\phi$. Show that $d/d\phi = -\sin\phi\,(d/dw)$. Use this change of variable to obtain the differential equation

$$(1-w^2)\frac{d^2F}{dw^2} - 2w\frac{dF}{dw} + n(n+1)F = 0,$$

*Legendre's equation* (Section 4.8), which has solutions $F_n(w) = P_n(w) = P_n(\cos\phi)$, $n = 0, 1, \ldots$.

**(d)** Consider the solutions obtained in **(b)**. If $u$ is bounded as $\rho \to 0^+$, which must be assumed when finding the temperature inside the sphere, then $\tilde{R}_n(r) = 1/(r^{n+1})$ cannot be included in the solution. (Why?). In this case, the solutions are $u_n(r,\phi) = a_n r^n P_n(\cos\phi)$, so

$$u(r,\phi) = \sum_{n=1}^{\infty} a_n r^n P_n(\cos\phi).$$

Show that

$$a_n = \frac{2n+1}{2R^n}\int_0^\pi f(\phi)P_n(\cos\phi)\sin\phi\,d\phi.$$



(*Hint:* $\int_{-1}^1 P_m(x)P_n(x)\,dx = 0$, $m \ne n$, and

$\int_{-1}^1 P_n^2(x)\,dx = \dfrac{2}{2n+1}$, $n = 0, 1, 2, \ldots$.)

**(e)** On the other hand, if we use the boundary condition $\lim_{\rho\to\infty} u(r,\phi) = 0$, which is used when finding the temperature outside of the sphere, then form the solution in this case. Find the unknown coefficient.

In Exercises 28–30, find the first three nonzero terms of the steady-state temperature inside the sphere of radius 1 using the given boundary condition.

**28.** $u(1,\phi) = 100$

**29.** $u(1,\phi) = \cos 2\phi$

**30.** $u(1,\phi) = \cos^2\phi - 2\cos\phi + 1$

**31. (Laplace's Equation in Cylindrical Coordinates)** Suppose that the surface of the cylinder $x^2 + y^2 = a^2$ between the planes $z = 0$ and $z = b$ is insulated and that there is a heat source on the top and bottom surfaces. We can find the steady-state temperature by solving

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{\partial^2 u}{\partial z^2} = 0, & r < a,\ 0 < z < b \\ \dfrac{\partial u}{\partial r}(a,z) = 0, \quad |u(0,z)| < \infty, & 0 < z < b \\ u(r,0) = f(r),\ u(r,b) = g(r), & r < a \end{cases}$$

(Notice that the boundary conditions do not depend on $\theta$, so neither does $u$.)

**(a)** Let $u(r,z) = R(r)Z(z)$. Show that $R$ satisfies the boundary-value problem

$$(rR')' + \lambda^2 rR = 0,\ R'(a) = 0,\ |R(0)| < \infty,$$

with solutions $R_n(r) = J_0(\lambda_n r)$. Use the relationship $\dfrac{d}{dx}[J_0(x)] = -J_1(x)$ to find $\lambda_n$.

**(b)** Show that $Z$ satisfies the equation $Z'' - \lambda^2 Z = 0$ so that

$$u(r,z) = A_0 + B_0 z + \sum_{n=1}^{\infty} J_0(\lambda_n r) \times \left[A_n\frac{\sinh\lambda_n z}{\sinh\lambda_n b} + B_n\frac{\sinh\lambda_n(b-z)}{\sinh\lambda_n b}\right].$$

Find the unknown coefficients.

**32. (Thermal Explosions: Approximating the Solution of a Nonlinear Partial Differential Equation)** Suppose that a reaction produces heat throughout a cylinder of radius $a$ at a rate proportional to the temperature, $Qe^{-P/T}$. Therefore, the situation is modeled with the *nonlinear* partial differential equation

$$\rho c\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + Qe^{-P/T},$$

where $T(r,t)$ is the temperature in the cylinder at radius $r$ and time $t$, $\rho$ is the density, and $c$ is the specific heat of the solid. Suppose that the temperature on the surface of the cylinder is $T(a,t) = T_0$. Therefore, we assume that $T$ does not differ much from $T_0$.

**(a)** Show that

$$\frac{T}{T_0} = 1 + \frac{T - T_0}{T_0}$$

so that

$$\frac{1}{T} = \frac{1}{T_0}\left(1 + \frac{T - T_0}{T_0}\right)^{-1}.$$

Therefore, we use the approximation

$$\frac{1}{T} \approx \frac{1}{T_0} - \frac{T - T_0}{T_0^2}$$

and $Qe^{-P/T} \approx Qe^{-P/T_0}e^{P(T-T_0)/T_0^2}$ (Why?).

**(b)** Let $u = T - T_0$, $q = \dfrac{1}{k}Qe^{-P/T_0}$, and $p = \dfrac{k}{\rho c}$. Use these variables to obtain the equation

$$\frac{1}{\kappa}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + qe^{pu},$$

where $\kappa = k/\rho c$.

**(c)** The steady-state temperature $u(r)$ will satisfy

$$0 = \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + qe^{pu}.$$

(Why?) To solve this nonlinear ordinary differential equation with the boundary condition $u(a) = 0$ (Why?), let $v = r\,du/dr$. Show that the differential equation becomes

$$\frac{1}{r}\frac{dv}{dr} = -qe^{pu}.$$

**(d)** Differentiate $\dfrac{1}{r}\dfrac{dv}{dr} = -qe^{pu}$ with respect to $r$ to find that

$$\frac{d}{dr}\left(\frac{1}{r}\frac{dv}{dr}\right) = \frac{pv}{r^2}\frac{dv}{dr} \quad \text{or}$$

$$\frac{d}{dr}\left(r\frac{dv}{dr} - 2v\right) = \frac{1}{2}p\frac{d(v^2)}{dr}.$$

(e) Integrate

$$\frac{d}{dr}\left(r\frac{dv}{dr} - 2v\right) = \frac{1}{2}p\frac{d(v^2)}{dr}$$

with respect to $r$ to show that

$$r\frac{dv}{dr} = \frac{v(pv+4)}{2}$$

using $v(0) = 0$.

(f) Use separation of variables to solve the first-order equation obtained in (e) and simplify the result to yield $v/(pv + 4) = -Cr^2$, where $C$ is the constant of integration. (*Note:* Because $v$ decreases as $r$ increases and $v$ increases as $r$ decreases, $v = r\, du/dr < 0$. Why?)

(g) Show that $u = (-2/p)\ln(1 + pCr^2) + B$, where $B$ is an arbitrary constant. Use the boundary condition $u = 0$ at $r = a$ to find $B$ in terms of $C$. Find that $C$ satisfies the relationship $8C = q(1 + pCa^2)^2$ by substituting the solution into the differential equation in (c). Let $pCa^2 = x$ and $8/(qpa^2) = \alpha$ to obtain the quadratic equation $x^2 + (2 - \alpha)x + 1 = 0$. Show that if $qpa^2 \le 2$, then the temperature approaches a finite value. Otherwise, an explosion results.

**33.** Solve Laplace's equation in a circular region of radius 1 given the boundary condition $u(1, \theta)$. Graph the solution using five terms of the series solution.

**(a)** $u(1, \theta) = \theta^2 \cos\theta$

**(b)** $u(1, \theta) = \theta^2 \sin\theta$

**(c)** $u(1, \theta) = \theta^2(1 + \theta^2)$

**34.** Solve the wave equation in a circular region of radius $r = 1$ that is radially symmetric using the initial condition functions that follow. Use the formula derived previously and plot an approximation of the solution for various times to view the motion of the drumhead.

**(a)** $f(r) = r(1 - r)$, $g(r) = 0$

**(b)** $f(r) = r(1 - r)$, $g(r) = 1$

**(c)** $f(r) = r(1 - r)$, $g(r) = r$

**(Wave Equation with Dependence on $\theta$)** The problem that depends on the angle $\theta$ is more complicated to solve. Due to the presence of $u_{\theta\theta}$ we must include two more boundary conditions to solve the initial boundary value problem. So that the solution is a smooth function, we require the "artificial" boundary conditions

$$u(r, \pi, t) = u(r, -\pi, t) \quad \text{and}$$
$$u_\theta(r, \pi, t) = u_\theta(r, -\pi, t) \quad \text{for} \quad 0 < r < \rho \quad \text{and} \quad t > 0.$$

Therefore, we solve the problem

$$\begin{cases} c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) = u_{tt} \\ u(\rho, \theta, t) = 0,\ |u(0, \theta, t)| < \infty, \quad -\pi < \theta < \pi,\ t > 0 \\ u(r, \pi, t) = u(r, -\pi, t), \\ u_\theta(r, \pi, t) = u_\theta(r, -\pi, t), \quad 0 < r < \rho,\ t > 0 \\ u(r, \theta, 0) = f(r, \theta), \\ u_t(r, \theta, 0) = g(r, \theta), \quad 0 < r < \rho,\ -\pi < \theta < \pi, \end{cases}$$

and the displacement of the drumhead that is not radially symmetric. Assuming that $u(r, \theta, t) = R(r)H(\theta)T(t)$ and using separation of variables yields

$$c^2\left(R''HT + \frac{1}{r}R'HT + \frac{1}{r^2}RH''T\right) = RHT''.$$

Then division by $RHT$ gives us

$$c^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H}\right) = \frac{T''}{T} = -\lambda^2,$$

where $-\lambda^2$ is the constant of separation. Separating variables, we obtain

$$T'' + \lambda^2c^2T = 0 \quad \text{and} \quad \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H} = -\lambda^2.$$

Separating variables in the second equation yields

$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\lambda^2 = \frac{-H''}{H} = \mu^2,$$

where $\mu^2$ is the constant of separation. Therefore, we have two more ordinary differential equations

$$H'' + \mu^2H = 0 \quad \text{and} \quad r^2R'' + rR' + (r^2\lambda^2 - \mu^2)R = 0.$$

The boundary conditions in terms of $H$ and $R$ become $R(\rho) = 0$, $|R(0)| < \infty$, $H(-\pi) = H(\pi)$, and $H'(-\pi) = H'(\pi)$. Recall that the problem



$$\begin{cases} H'' + \mu^2H = 0 \\ H(-\pi) = H(\pi),\ H'(-\pi) = H'(\pi) \end{cases}$$

has solutions

$$H_n(x) = \begin{cases} 1, & n = 0 \\ a_n\cos n\theta + b_n\sin n\theta' \end{cases}$$

which correspond to the eigenvalues $\mu_n^2 = \begin{cases} 0, & n = 0 \\ n^2, & n = 1, 2, \ldots \end{cases}$. The corresponding solutions of $r^2R'' + rR' + (r^2\lambda^2 - \mu^2)R = 0$, which we recognize as Bessel's equation of order $n$, are $R_n(r) = c_nJ_n(\lambda r) + d_nY_n(\lambda r)$. Because $|R(0)| < \infty$, $d_n = 0$. Because $R(\rho) = 0$, $J_n(\lambda\rho) = 0$, so $\lambda_{mn} = \alpha_{mn}/\rho$, where $\alpha_{mn}$ denotes the $n$th zero of the Bessel function $J_m(x)$. Then the solution is

$$\begin{aligned} u(r, \theta, t) = {} & \sum_n a_{0n}J_0(\lambda_{0n}r)\cos(\lambda_{0n}ct) \\ & + \sum_{m,n} a_{mn}J_m(\lambda_{mn}r)\cos(m\theta)\cos(\lambda_{mn}ct) \\ & + \sum_{m,n} b_{mn}J_m(\lambda_{mn}r)\sin(m\theta)\cos(\lambda_{mn}ct) \\ & + \sum_n A_{0n}J_0(\lambda_{0n}r)\sin(\lambda_{0n}ct) \\ & + \sum_{m,n} A_{mn}J_m(\lambda_{mn}r)\cos(m\theta)\sin(\lambda_{mn}ct) \\ & + \sum_{m,n} B_{mn}J_m(\lambda_{mn}r)\sin(m\theta)\sin(\lambda_{mn}ct), \end{aligned}$$

where

$$a_{0n} = \frac{\int_0^{2\pi}\int_0^\rho f(r, \theta)J_0(\lambda_{0n}r)r\,dr\,d\theta}{2\pi\int_0^\rho [J_0(\lambda_{0n}r)]^2r\,dr},$$

$$a_{mn} = \frac{\int_0^{2\pi}\int_0^\rho f(r, \theta)J_m(\lambda_{mn}r)\cos(m\theta)r\,dr\,d\theta}{\pi\int_0^\rho [J_m(\lambda_{mn}r)]^2r\,dr},$$

$$b_{mn} = \frac{\int_0^{2\pi}\int_0^\rho f(r, \theta)J_m(\lambda_{mn}r)\sin(m\theta)r\,dr\,d\theta}{\pi\int_0^\rho [J_m(\lambda_{mn}r)]^2r\,dr},$$

$$A_{0n} = \frac{\int_0^{2\pi}\int_0^\rho g(r, \theta)J_0(\lambda_{0n}r)r\,dr\,d\theta}{2\pi\lambda_{0n}c\int_0^\rho [J_0(\lambda_{0n}r)]^2r\,dr},$$

$$A_{mn} = \frac{\int_0^{2\pi}\int_0^\rho g(r, \theta)J_m(\lambda_{mn}r)\cos(m\theta)r\,dr\,d\theta}{\pi\lambda_{mn}c\int_0^\rho [J_m(\lambda_{mn}r)]^2r\,dr},$$

and

$$B_{mn} = \frac{\int_0^{2\pi}\int_0^\rho g(r, \theta)J_m(\lambda_{mn}r)\sin(m\theta)r\,dr\,d\theta}{\pi\lambda_{mn}c\int_0^\rho [J_m(\lambda_{mn}r)]^2r\,dr}.$$

Use technology to solve the wave equation in a circular region of radius $\rho = 1$, which depends on $\theta$ using the initial condition functions $f$ and $g$ given as follows. Use the formula derived previously and plot an approximation of the solution for various times to view the motion of the drumhead. (Assume: $c = 1$.)

**35.** $f(r, \theta) = \sin 2\pi r \sin 2\theta$, $g(r, \theta) = 0$

**36.** $f(r, \theta) = 0$, $g(r, \theta) = (r - 1)\sin\theta$

***37.** $f(r, \theta) = \sin 2\pi r \sin 2\theta$, $g(r, \theta) = (r - 1)\sin\theta$

**38.** $f(r, \theta) = \cos\frac{\pi r}{2}$, $g(r, \theta) = 0$

**39.** $f(r, \theta) = 0$, $g(r, \theta) = (r - 1)\cos\frac{\pi\theta}{2}$

**40.** $f(r, \theta) = \cos\frac{\pi r}{2}$, $g(r, \theta) = (r - 1)\cos\frac{\pi\theta}{2}$

***41.** $f(r, \theta) = (r - 1)\theta$, $g(r, \theta) = 0$

**42.** $f(r, \theta) = 0$, $g(r, \theta) = 3$

**43.** $f(r, \theta) = (r - 1)\theta$, $g(r, \theta) = 3$

# CHAPTER 10 SUMMARY

## Concepts & Formulas

### Section 10.1

**Linear-Second-Order Partial Differential Equation**
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G(x, y)$

**Separation of Variables**
Substitute $u(x, y) = X(x)Y(y)$ into the partial differential equation and separate the variables to obtain two ordinary differential equations in the variables $x$ and $y$.

### Section 10.2

**The One-Dimensional Heat Equation**
$u_t = c^2u_{xx}$, where $u(x, t)$ is the temperature at position $x$ and time $t$.

**Steady-State Temperature, $S(x)$**
$\lim_{t\to\infty} u(x, t) = S(x)$

**Transient Temperature, $v(x, t)$**
$u(x, t) = v(x, t) + S(x)$

### Section 10.3

**The One-Dimensional Wave Equation**
$c^2u_{xx} = u_{tt}$

### Section 10.4

**Laplace's Equation**
$u_{xx} + u_{yy} = 0$

### Section 10.5

**Laplace's Equation in Polar Coordinates**
$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$

**Wave Equation in Polar Coordinates**
$c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) = u_{tt}$

# CHAPTER 10 REVIEW EXERCISES

1. Determine the value of $c$ so that each of the following functions satisfies the wave equation $c^2u_{xx} = u_{tt}$. **(a)** $u(x, t) = t^2 + x^2$; **(b)** $u(x, t) = \cos 4t \sin x$; **(c)** $u(x, t) = \sin t \sin 16x$.

2. Determine the value of $c$ so that each of the following functions satisfies; the heat equation $u_t = c^2u_{xx}$. **(a)** $u(x, t) = e^{-t}\sin x$; **(b)** $u(x, t) = e^{-16\pi^2 t}\cos x$; **(c)** $u(x, t) = e^{-4t}\cos 9x$.

*3. Show that each of the following functions satisfies Laplace's equation. **(a)** $u(x, y) = xy$; **(b)** $u(x, y) = e^{3x}\cos 3y$; **(c)** $u(x, y) = \ln(x^2 + y^2)$.

4. The polynomial $p(x, y) = A + Bx + Cy + Dx^2 + Exy + Fy^2$ satisfies the nonhomogeneous equation $u_{xx} + u_{yy} + u_x = 10$ for certain values of the constants $A, B, C, D, E$, and $F$. Find the restrictions on these constants. Find a polynomial that satisfies the boundary conditions $p(x, 0) = 0$ and $p(x, b) = 0$.

5. Consider Laplace's equation on the exterior of a disk:

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, & r > \rho,\ -\pi < \theta < \pi \\ u(\rho, \theta) = f(\theta), & -\pi < \theta < \pi \\ |u(r, \theta)| < \infty, & r \to \infty \end{cases}$$



Show that the solution is

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} r^{-n}(a_n \cos n\theta + b_n \sin n\theta)$$

and find integral formulas for the coefficients.

6. Use the results of the previous problem to solve

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, & r > 1,\ -\pi < \theta < \pi \\ u(1, \theta) = 10, & -\pi < \theta < \pi \\ |u(r, \theta)| < \infty, & r \to \infty \end{cases}$$

*7. Consider Laplace's equation in a disk,

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, & 0 < r < \rho,\ -\pi < \theta < \pi \\ u(\rho, \theta) = f(\theta), & -\pi < \theta < \pi \\ |u(r, \theta)| < \infty, & r \to 0 \end{cases},$$

with solution

$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n(A_n \cos n\theta + B_n \sin n\theta)$. Evaluate $u(0, \theta)$ with the formula $A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$. Interpret this result in terms of the average value of a function.

8. In spherical coordinates (See Figure 10.21), Laplace's equation is

$$\frac{1}{\rho^2}\left\{\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}\right\} = 0.$$

Show that if $u$ does not depend on $\theta$, then this equation becomes

$$u_{\rho\rho} + \frac{2}{\rho}u_\rho + \frac{1}{\rho^2}u_{\phi\phi} + \frac{\cot\phi}{\rho^2}u_\phi = 0.$$

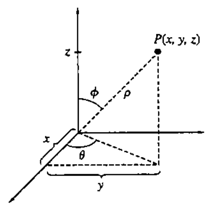



**Figure 10.21** A point in spherical coordinates

9. Use separation of variables with the steps (a)–(c) to solve Laplace's equation (in spherical coordinates) inside the sphere of radius $a$ $(0 < \rho < a,\ 0 < \phi < \pi)$ subject to the boundary condition $u(c, \phi) = f(\phi)$, $0 < \phi < \pi$.
   **(a)** Let $u(\rho, \phi) = R(\rho)F(\phi)$. Separation of variables leads to the equations
   $$\rho^2R'' + 2\rho R' - \lambda^2R = 0$$
   and
   $$\sin\phi F'' + \cos\phi F' + \lambda^2\sin\phi F = 0,$$
   where the separation constant is $\lambda^2$.
   **(b)** Substitute $x = \cos\phi$ into
   $$\sin\phi F'' + \cos\phi F' + \lambda^2\sin\phi F = 0$$
   and solve by using Legendre's equation with $\lambda^2 = n(n + 1)$, $n = 0, 1, 2, \ldots$.
   **(c)** Solve the Cauchy–Euler equation,
   $$\rho^2R'' + 2\rho R' - \lambda^2R = 0,$$
   with $\lambda^2 = n(n + 1)$, $n = 0, 1, 2, \ldots$.
   **(d)** Express $u$ as a series, and determine the unknown coefficients using $u(c, \phi) = f(\phi)$, $0 < \phi < \pi$ with the orthogonality condition of the Legendre polynomials.

10. Laplace's equation can be interpreted as the steady-state temperature. Using the formula derived in the previous problem, find the steady-state temperature in a sphere of radius 1 if the initial heat distribution on the surface of the sphere is **(a)** $f(\theta) = 100$, $0 < \theta < \pi$; **(b)** $f(\theta) = \begin{cases} 100, & 0 < \theta < \pi/2 \\ 0, & \pi/2 < \theta < \pi \end{cases}$.

11. In cylindrical coordinates (see Figure 10.22), Laplace's equation is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0.$$

If $u$ does not depend on $\theta$, then this equation becomes

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0.$$

Use separation of variables to solve

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, & 0 < r < a,\ 0 < z < b \\ u(a, z) = 0, & 0 < z < b \\ u(r, 0) = 0,\ u(r, b) = T_0, & 0 < r < a. \end{cases}$$

(*Hint:* Use the separation constant $k = -\lambda^2$. The solution involves the Bessel function of the first kind of order zero.

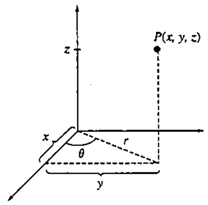



**Figure 10.22** A point in cylindrical coordinates

12. Use the results of the previous problem to find the steady-state temperature in a cylinder of radius 5 and height 10 if $u(5, z) = 0$, $0 < z < 10$, and $u(r, 0) = 0$, $u(r, 10) = 50$, $0 < r < 5$.

13. Suppose that $w(x, y)$ and $v(z, t)$ satisfy the equations $w_{xx} + w_{yy} = 0$ and $v_{zz} = \frac{1}{c^2} v_{tt}$. Show that $u(x, y, z, t) = w(x, y) \cdot v(z, t)$ satisfies the wave equation in three dimensions $u_{xx} + u_{yy} + u_{zz} = \frac{1}{c^2} u_{tt}$.

14. The vibration of a cylinder can be found by solving the wave equation in a cylinder

$$\begin{cases} u_{rr} + \frac{1}{r} u_r + u_{zz} = \frac{1}{c^2} u_{tt}, & 0 < r < a, 0 < z < b, t > 0 \\ u(r, 0, t) = 0, u(r, b, t) = 0, & 0 < r < a, t > 0 \\ u(a, z, t) = 0, & 0 < z < b, t > 0 \\ |u(0, z, t)| < \infty, & 0 < z < b, t > 0 \\ u(r, z, 0) = f(r, z), & 0 < r < a, 0 < z < b \\ u_t(r, z, 0) = g(r, z), & 0 < r < a, 0 < z < b \end{cases}$$

Use separation of variables to find $u(r, z, t)$. (*Hint:* The solution involves the Bessel function of the first kind of order zero in the variable $z$.)

## Differential Equations at Work:

### A. Laplace Transforms

The Laplace transform can be used to solve partial differential equations. For example, for the function of two variables, $u(x, t)$, the Laplace transform is defined by

$$\mathcal{L}\{u(x, t)\} = \int_0^\infty e^{-st} u(x, t)\, dt = U(x, s).$$

In addition, we have the following transforms of the partial derivatives

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = sU(x, s) - u(x, 0) \quad \text{and} \quad \mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2 U(x, s) - su(x, 0) - u_t(x, 0),$$

which mirror the properties of ordinary derivatives discussed in Chapter 8.

1. Use the definition of $\mathcal{L}\{u(x, t)\}$ to verify that $\mathcal{L}\{\partial^2 u/\partial x^2\} = d^2U/dx^2$. (*Hint:* Assume that we may interchange integration and differentiation.)

Consider the initial boundary value problem

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, & 0 < x < \pi, t > 0 \\ u(0, t) = 0, u(1, t) = 0, & t > 0 \\ u(x, 0) = 0, \frac{\partial u}{\partial t}(x, 0) = \sin x, & 0 < x < \pi \end{cases}$$



2. Show that the transformation of the partial differential equation is

$$\frac{d^2U}{dx^2} - s^2 U = -\sin x,$$

with solution $U(x, s) = U_h(x, s) + U_p(x, s)$. Show that $U_h(x, s) = C_1 e^{-s} + C_2 e^{s}$ and $U_p(x, s) = \frac{1}{s^2 + 1} \sin x$.

3. Apply the boundary conditions to show that $C_1 = C_2 = 0$, so that $U(x, s) = \frac{1}{s^2 + 1} \sin x$.

4. Determine $u(x, t) = \mathcal{L}^{-1}\{U(x, s)\} = \sin x\, \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\}$.

The Laplace transform is particularly useful when the initial boundary value problem involves the interval $0 < x < \infty$. For example, suppose that a long string is extended along the nonnegative $x$-axis, $0 \le x < \infty$. It is clamped at $x = 0$, and it slides along a frictionless vertical support at the right end. The string falls under its own weight. The initial boundary value problem that describes this situation is

$$\begin{cases} c^2 \frac{\partial^2 u}{\partial x^2} - g = \frac{\partial^2 u}{\partial t^2}, & 0 < x < \infty, t > 0 \\ u(0, t) = 0, \lim_{x\to\infty} \frac{\partial u}{\partial x}(x, t) = 0, & t > 0 \\ u(x, 0) = 0, \frac{\partial u}{\partial t}(x, 0) = 0, & 0 < x < \infty, \end{cases}$$

where $g$ is the gravitational constant.

5. Show that the Laplace transform of the partial differential equation is

$$c^2 \frac{d^2U}{dx^2} - \frac{g}{s} = s^2 U - su(x, 0) - u_t(x, 0),$$

and apply the initial conditions to obtain $d^2U/dx^2 - s^2/c^2 = g/(c^2 s)$.

6. Determine the Laplace transform of the boundary conditions by verifying that

$$U(0, s) = \mathcal{L}\{u(0, t)\} = \mathcal{L}\{0\} = 0 \quad \text{and}$$

$$\lim_{x\to\infty} \frac{dU}{dx}(x, s) = \mathcal{L}\left\{\lim_{x\to\infty} \frac{\partial u}{\partial x}(x, t)\right\} = 0.$$

7. Find a general solution of $\frac{d^2U}{dx^2} - \frac{s^2}{c^2} = \frac{g}{c^2 s}$.

8. Apply the boundary conditions to determine $C_1$ and $C_2$.

9. Compute $u(x, t) = \mathcal{L}^{-1}\{U(x, s)\}$. What is the piecewise definition of $u(x, t)$?

10. Graph the solution using $c = 1$ ft and $g = 32$ ft/s$^2$ for several values of $t$. What is $\lim_{t\to\infty} u(x, t)$?

## B. Waves in a Steel Rod

Just as we considered the displacement of a plucked string in Section 10.3, we may investigate the waves formed when stress is applied to a steel rod of length $L$. In this case, $E$, Young's modulus, is given by $E = \sigma/\varepsilon$, where $\sigma$ is the lateral stress applied to the rod and $\varepsilon$ is the axial strain. Also, $\rho$ is the density of steel. The partial differential equation that describes the situation is a form of the wave equation,

$$E\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2},$$

where $u(x, t)$ is the longitudinal displacement of a cross-section of the rod located $x$ units from $x = 0$. (*Note:* We have assumed that $E$ and $\rho$ are homogeneous and do not depend on $x$.) Longitudinal waves propagate through the rod at a velocity of $c_0$, called the bar velocity, given by $c_0 = \sqrt{E/\rho}$. For steel, this value is $c_0 \approx 5.1 \times 10^3$ m/s. For air, this value is much larger (250 m/s). Suppose that the rod is compressed at one end.

1. Assuming that the solution of the partial differential equation is of the form

$$u(x, t) = \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t)U_n(x),$$

use the free end conditions

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0$$

to determine $U_n(x)$.

2. What are the natural frequencies, $\omega_n$?
3. If the steel rod is 0.25 m long, what is $\omega_n$ and what is the frequency (in Hertz)?
4. Graph $U_n(x)$ for $n = 1, 2, 3, 4$. When $U_n(x) > 0$, particles move from left to right, and when $U_n(x) < 0$, particles move from right to left. Indicate particle movement with arrows on the four graphs.

## C. Media Sterilization

In designing a continuous media sterilizer in which the raw medium passes through round pipes, we must note that not all portions of the medium spend the same length of time in the holding section of the sterilizer. This is because the mean velocity $\bar{u}$ of the fluid is a function of the radial distribution of fluid velocities occurring across the pipe. The mean velocity of a viscous-fluid flow through a pipe is one half the maximum velocity found at the axis of the pipe (radial distribution of velocities is parabolic). In the turbulent condition, the mean velocity is 82% of the maximum value. These situations are illustrated in Figure 10.23.

The material balance is then given by the partial differential equation

$$E_z\frac{\partial^2 n}{\partial x^2} - \bar{u}\frac{\partial n}{\partial x} = \frac{\partial n}{\partial t}.$$

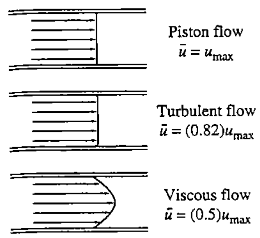



**Figure 10.23** Distribution inside round pipes of velocities in fluids exhibiting different types of flow



With the change of variables

$$\bar{n} = \frac{n}{n_0},\ \chi = \frac{x}{L},\ PeB = \frac{\bar{u}L}{E_z}, \quad \text{and} \quad \phi = \frac{t}{\bar{t}},$$

where $E_z$ is the axial dispersion coefficient, $n_0$ is the initial concentration, $L$ is the pipe length, $PeB$ is the Pe'clet number (or Bodenstein number), $x$ is the axial direction, and $\bar{t}$ is the nominal holding time, we can rewrite the differential equation as

$$\frac{\partial^2 \bar{n}}{\partial \chi^2} - PeB\frac{\partial \bar{n}}{\partial \chi} - \frac{\partial \bar{n}}{\partial \phi} = 0.$$

If the initial conditions are

$$\begin{cases} \phi = 0,\ \chi > 0,\ \bar{n} = 1 \\ \phi = 0,\ \chi < 0,\ \bar{n} = 0 \end{cases}$$

and the boundary conditions are

$$\begin{cases} \chi \to 0^+,\ \dfrac{\partial \bar{n}}{\partial \chi} - PeB\bar{n} = 0 \\ \chi \to 1,\ \dfrac{\partial \bar{n}}{\partial \chi} = 0, \end{cases},$$

then use separation of variables to find $\bar{n}$.*

## D. Numerical Methods for Solving Partial Differential Equations

Here we discuss methods used to approximate the solution to partial differential equations by replacing partial derivatives with finite difference approximations. We begin with a discussion of the classification of second-order partial differential equations and then present numerical methods for approximating the solution to equations in each category.

### Classification of Second-Order Partial Differential Equations

A second-order partial differential equation of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(u_x, u_y, u, x, y)$$

is **elliptic** if $AC - B^2 > 0$; **parabolic** if $AC - B^2 = 0$; and **hyperbolic** if $AC - B^2 < 0$. For example, Laplace's equation, $u_{xx} + u_{yy} = 0$, is elliptic. With the independent variables $x$ and $t$, the heat equation, $u_t = c^2u_{xx}$, is parabolic and the wave equation, $u_{tt} = c^2u_{xx}$, is hyperbolic.

* *Source:* S. Aiba, A. E. Humphrey, N. F. Millis, *Biochemical Engineering*, Second Edition, Academic Press, New York (1973), pp. 258–267.

### Numerical Solution to Elliptic Equations

Consider the boundary-value problem that solves Laplace's equation over a rectangular region

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < a, 0 < y < b \\ u(0, y) = f(y), u(a, y) = g(y), & 0 < y < b \\ u(x, 0) = h(x), u(x, b) = k(x), & 0 < x < a. \end{cases}$$

A problem of this type is called a **Dirichlet problem** because the value of $u$ is given around the boundary of the region in the boundary conditions.

Two **central differences** for the function $u(x, y)$ are $u(x + h, y) - 2u(x, y) + u(x - h, y)$ and $u(x, y + h) - 2u(x, y) + u(x, y - h)$. Therefore, the finite difference approximations for the second-order partial derivatives $u_{xx}$ and $u_{yy}$ are

$$u_{xx} \approx \frac{1}{h^2}[u(x + h, y) - 2u(x, y) + u(x - h, y)]$$

and

$$u_{yy} \approx \frac{1}{h^2}[u(x, y + h) - 2u(x, y) + u(x, y - h)].$$

Adding these equations, we obtain an approximation to the Laplacian

$$u_{xx} + u_{yy} \approx \frac{1}{h^2}[u(x + h, y) + u(x, y + h) + u(x - h, y) + u(x, y - h) - 4u(x, y)],$$

called a **five-point approximation.** This means that Laplace's equation can be approximated by the difference equation

$$u(x + h, y) + u(x, y + h) + u(x - h, y) + u(x, y - h) - 4u(x, y) = 0.$$

Using the notation $u(x, y) = u_{ij}$, $u(x + h, y) = u_{i+1,j}$, $u(x, y + h) = u_{i,j+1}$, $u(x - h, y) = u_{i-1,j}$, and $u(x, y - h) = u_{i,j-1}$, we obtain the equation

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij} = 0.$$

To make this topic more understandable, suppose that we take the rectangular region and divide it by using lines parallel to the $x$- and $y$-axes. These lines are called **grid lines.** If these parallel lines are $h$ units apart, then we say that the **mesh size** is $h$. The points of intersection of these lines are called **mesh points.** The goal of this numerical method is to use the equation given above to approximate $u$ at the mesh points.

1. Use the five-point approximation

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij} = 0,$$

with $n = 3$ to approximate the solution to the boundary value problem

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 2, 0 < y < 2 \\ u(0, y) = 0, u(2, y) = y(2 - y), & 0 < y < 2 \\ u(x, 0) = 0, \quad 0 < x < 2; u(x, 2) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \le x < 2 \end{cases} \end{cases}$$

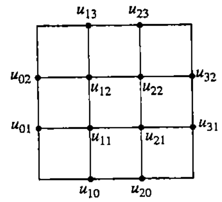



**Figure 10.24** Values of $u$ at the mesh points



[*Notes:* The mesh size is 2/3, and (0, 0) is the lower left-hand corner. Find $u$ at the boundary points $u_{01}$, $u_{02}$, $u_{10}$, $u_{20}$, $u_{13}$, $u_{23}$, $u_{31}$, and $u_{32}$ with the functions $f(y) = 0$, $g(y) = y(2 - y)$, $0 < y < 2$; and $h(x) = 0$, $0 < x < 2$; $k(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \le x < 2 \end{cases}$. (See Figure 10.24.) Solving

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij} = 0$$

for $u_{ij}$, we find that

$$u_{ij} = \frac{1}{4}(u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1}).$$

Start by using the boundary conditions to define the values of $u$ on the boundary. Graph the solution.]

2. Repeat the method using $n = 4$.

### Numerical Solution to Parabolic Equations

We develop this approximation method using an initial boundary value problem that involves the heat equation,

$$\begin{cases} u_t = c^2 u_{xx}, & 0 < x < p, t > 0 \\ u(0, t) = T_1, u(p, 0) = T_2, & t > 0. \\ u(x, 0) = f(x), & 0 < x < p \end{cases}$$

Substituting the central difference approximation

$$u_{xx} \approx \frac{1}{h^2}[u(x + h, y) - 2u(x, y) + u(x - h, y)]$$

and the forward difference approximation

$$u_t \approx \frac{1}{k}[u(x, t + k) - u(x, t)]$$

into the heat equation $u_t = c^2 u_{xx}$, we obtain

$$\frac{c^2}{h^2}[u(x + h, t) - 2u(x, t) + u(x - h, t)] = \frac{1}{k}[u(x, t + k) - u(x, t)].$$

With $\lambda = c^2k/h^2$, $u_{ij} = u(x, t)$, $u_{i+1,j} = u(x + h, t)$, $u_{i-1,j} = u(x - h, t)$, and $u_{i,j+1} = u(x, t + k)$, this approximation to the heat equation becomes

$$u_{i,j+1} = \lambda u_{i+1,j} + (1 - 2\lambda)u_{ij} + \lambda u_{i-1,j}.$$

Suppose that we want to approximate the solution over the rectangular region $0 \le x \le p$, $0 \le t \le T$, where $T$ is a finite value of time. To partition this rectangular region, we use a partition size of length $h = p/n$ on the $x$-axis and $k = T/m$ on the $t$-axis. This means that the grid lines perpendicular to the $x$-axis are $x_i = ih$ ($i = 1, 2, \ldots, n$) and those perpendicular to the $t$-axis are $t_j = jk$, ($j = 1, 2, \ldots, m$). The procedure begins by defining the boundary conditions

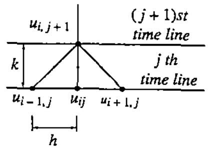



**Figure 10.25** Values of $u$ at the mesh points

$u_{0j} = u(0, t) = T_1$ and $u_{nj} = u(p, t) = T_2 (i = 1, 2, \dots, n)$

and the initial condition

$$u_{i0} = u(x, 0) = f(x).$$

After these values are defined, we use

$$u_{i,j+1} = \lambda u_{i+1,j} + (1 - 2\lambda)u_{ij} + \lambda u_{i-1,j}$$

to approximate the solution. We depict this calculation in Figure 10.25.

3. Approximate the solution to

$$\begin{cases} u_t = u_{xx}, & 0 < x < 1,\ 0 < t < 1/2 \\ u(0, t) = 0,\ u(1, 0) = 0, & 0 \le t < 1/2 \\ u(x, 0) = \sin \pi x, & 0 \le x \le 1, \end{cases}$$

using $n = 5$ and $m = 50$. (*Notes:* In this case, $c = 1$, $p = 1$, $T = 0.5$, $n = 5$, $m = 50$, $h = 1/5$, $k = 1/100$, and $\lambda = 1/4$.

### Crank–Nicholson Method

A popular method for solving the heat equation is the Crank–Nicholson method, an implicit finite difference method developed in 1947 by J. Crank and P. Nicholson. This method requires that we solve a system of equations at each level of the method to approximate the value of $u$ at the mesh points. This method differs from the previous method in that $u_{xx}$ is approximated with the average of two central difference quotients, one which is evaluated at $t$ and the other at $t + k$, given by

$$u_{xx} \approx \frac{1}{2}\left[\frac{u(x+h, t) - 2u(x, t) + u(x-h, t)}{h^2} + \frac{u(x+h, t+k) - 2u(x, t+k) + u(x-h, t+k)}{h^2}\right].$$

With this substitution, we have

$$\frac{c^2}{2}\left[\frac{u(x+h, t) - 2u(x, t) + u(x-h, t)}{h^2} + \frac{u(x+h, t+k) - 2u(x, t+k) + u(x-h, t+k)}{h^2}\right] = \frac{1}{k}[u(x, t+k) - u(x, t)],$$

which when using the notation introduced earlier and the values $\lambda = c^2k/h^2$, $\alpha = 2(1 + 1/\lambda)$, and $\beta = 2(1 - 1/\lambda)$, becomes

$$-u_{i-1,j+1} + \alpha u_{i,j+1} - u_{i+1,j+1} = u_{i+1,j} - \beta u_{ij} + u_{i-1,j}$$

for $j = 0, 1, \dots, m - 1$ and $i = 1, 2, \dots, n - 1$.

4. Approximate the solution to



$$\begin{cases} u_t = (1/4)u_{xx}, & 0 < x < 2,\ 0 < t < 3/10 \\ u(0, t) = 0,\ u(2, 0) = 0, & 0 \le t \le 3/10 \\ u(x, 0) = \sin \pi x, & 0 \le x \le 2 \end{cases}$$

using the Crank–Nicholson method with $n = 8$ and $m = 30$.

### Numerical Solution to Hyperbolic Equations

Consider the initial boundary value problem

$$\begin{cases} c^2u_{xx} = u_{tt}, & 0 < x < p,\ t > 0 \\ u(0, t) = 0,\ u(p, t) = 0, & t > 0 \\ u(x, 0) = f(x),\ u_t(x, 0) = g(x), & 0 \le x \le p, \end{cases}$$

which has a unique solution if $f$ and $g$ have continuous second derivatives on the interval $0 < x < p$ and $f(0) = f(p) = 0$. In this numerical method, we use the central difference approximations

$$u_{xx} \approx \frac{1}{h^2}[u(x+h, t) - 2u(x, t) + u(x-h, t)]$$

and

$$u_{tt} \approx \frac{1}{k^2}[u(x, t+k) - 2u(x, t) + u(x, t-k)].$$

Therefore, $c^2u_{xx} = u_{tt}$ becomes

$$\frac{c^2}{h^2}[u(x+h, t) - 2u(x, t) + u(x-h, t)] = \frac{1}{k^2}[u(x, t+k) - 2u(x, t) + u(x, t-k)].$$

With the substitution $\lambda = ck/h$, we have

$$u_{i,j+1} = \lambda^2u_{i+1,j} + 2(1 - \lambda^2)u_{ij} + \lambda^2u_{i-1,j} = u_{i,j-1}$$

for $i = 1, 2, \dots, n - 1$ and $j = 1, 2, \dots, m - 1$. In this method, we use $n$ gridlines perpendicular to the $x$-axis of a partition size $h = p/n$ and $m$ gridlines perpendicular to the $t$-axis of a partition size $k = T/m$. (*Note:* $m$ and $n$ are positive integers.) Therefore, the gridlines are located at $x_i = ih$ $(i = 0, 1, 2, \dots, n)$ and $t_j = jk$ $(j = 0, 1, 2, \dots, m)$. The recursive equation

$$u_{i,j+1} = \lambda^2u_{i+1,j} + 2(1 - \lambda^2)u_{ij} + \lambda^2u_{i-1,j} - u_{i,j-1}$$

makes it possible for us to calculate $u_{i,j+1}$ on the $(j + 1)$st time line from the values $u_{i-1,j}$, $u_{ij}$, and $u_{i+1,j}$ from the $j$th time line and the value of $u_{i,j-1}$ from the $(j - 1)$st time line. (See Figure 10.26.) In the process, we use the boundary conditions to define $u_{0j} = u(0, jk) = 0$ and $u_{nj} = u(p, jk) = 0$, as well as the initial condition to define $u_{i0} = u(x_i, 0) = f(x_i)$. We also make use of the initial velocity. When $j = 0$, we have the equation

$$u_{i1} = \lambda^2u_{i+1,0} + 2(1 - \lambda^2)u_{i0} + \lambda^2u_{i-1,0} - u_{i,-1}.$$

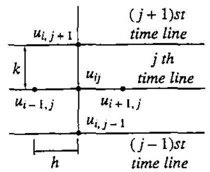



**Figure 10.26** Values of $u$ at the mesh points

To calculate $u_{i,-1}$, we use $u_t(x, 0) = g(x)$, because at $t = 0$,

$$g(x_i) = u_t(x_i, 0) \approx \frac{u(x_i, k) - u(x_i, -k)}{2k},$$

so that $u(x_i, -k) \approx u(x_i, k) - 2kg(x_i)$ or

$$u_{i,-1} = u_{i1} - 2kg(x_i).$$

Therefore, we obtain the equation when $j = 0$ by substituting $u_{i,-1} = u_{i1} - 2kg(x_i)$ into

$$u_{i1} = \lambda^2 u_{i+1,0} + 2(1 - \lambda^2)u_{i0} + \lambda^2 u_{i-1,0} - u_{i,-1}.$$

This gives us

$$u_{i1} = \frac{\lambda^2}{2}(u_{i+1,0} + u_{i-1,0}) + (1 - \lambda^2)u_{i0} + kg(x_i).$$

5. Approximate the solution to the initial boundary value problem

$$\begin{cases} 4u_{xx} = u_{tt}, & 0 < x < 1,\ 0 < t < 1 \\ u(0, t) = 0,\ u(1, t) = 0, & 0 \le t \le 1 \\ u(x, 0) = \sin \pi x,\ u_t(x, 0) = 0, & 0 \le x \le 1, \end{cases}$$

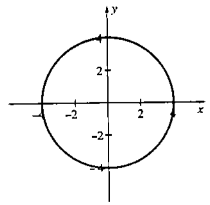
with $n = 5$ and $m = 20$.

# Answers to Selected Exercises

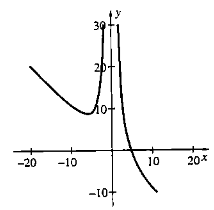
## CHAPTER 1

### *Exercises 1.1*

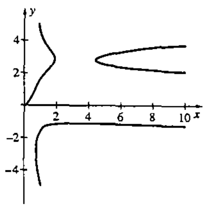
1. (a) ordinary; (b) second-order; (c) linear
3. (a) partial; (c) linear
5. (a) ordinary; (b) first-order; (c) nonlinear
7. (a) partial; (c) linear
9. (a) ordinary; (b) second-order; (c) nonlinear
11. (a) partial; (c) nonlinear
13. (a) ordinary; (b) first-order; and (c) linear in $y\left(\frac{dy}{dx} = 2x - y\right)$; nonlinear in $x\left(\frac{dx}{dy} = \frac{1}{2x - y}\right)$
15. (a) ordinary; (b) first-order; (c) nonlinear in $y\left(\frac{dy}{dx} = \frac{2x - y}{y}\right)$, nonlinear in $x\left(\frac{dx}{dy} = \frac{y}{2x - y}\right)$
17. $\frac{dy}{dx} + 2y = (-2e^{-2x}) + 2e^{-2x} = 0$
19. $\frac{dy}{dx} + y = \left(-e^{-x} + \frac{1}{2}\sin x + \frac{1}{2}\cos x\right) + \left(e^{-x} - \frac{1}{2}\cos x + \frac{1}{2}\sin x\right) = \sin x$
21. $\frac{d^2y}{dx^2} + 9\frac{dy}{dx} = 81Be^{-9x} + 9(-9Be^{-9x}) = 0$
23. $\frac{dx}{dt} = \left(-A + \frac{t}{4} - \frac{1}{2}\right)\sin t + \left(B + \frac{t^2}{4} - \frac{t}{2} + \frac{1}{4}\right)\cos t$; $\frac{d^2x}{dt^2} = \left(-A + \frac{3t}{4} - 1\right)\cos t + \left(-B - \frac{t^2}{4} + \frac{t}{2}\right)\sin t$
25. $\frac{dy}{dx} = 2Be^{2x} - 2Ce^{-2x}$; $\frac{d^2y}{dx^2} = 4Be^{2x} + 4Ce^{-2x}$; $\frac{d^3y}{dx^3} = 8Be^{2x} - 8Ce^{-2x}$
27. $x^2(30Ax^4 + 42Bx^5) - 12x(6Ax^5 + 7Bx^6) + 42(Ax^6 + Bx^7) = 0$
29. $2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$; $0^2 + y^2 = 16 \Rightarrow y = \pm 4$; $(0, \pm 4)$




31. $3x^2 + 2xy + x^2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{3x^2 + 2xy}{x^2}$; $1^3 + y = 100 \Rightarrow y = 99$; $(1, 99)$




33. $\frac{y}{x} + \frac{dy}{dx}\ln x + \cos y - x\sin y\frac{dy}{dx} = 0 \Rightarrow$ $(\ln x - x\sin y)\frac{dy}{dx} = -\cos y - \frac{y}{x} \Rightarrow$ $\frac{dy}{dx} = \frac{\cos y + \frac{y}{x}}{x\sin y - \ln x}$; $y\ln 1 + \cos y = 0 \Rightarrow$ $\cos y = 0 \Rightarrow y = \frac{(2n+1)\pi}{2}$, $n = 0, \pm 1, \pm 2, \ldots$



35. $y(x) = -\frac{1}{2}\cos(x^2) + C$

37. $y(x) = \ln|\ln x| + C$

39. $y(x) = -xe^{-x} - e^{-x} + C$

41. $y(x) = \int \frac{x - x^2}{(x+1)(x^2+1)}\,dx$
$= \int \left[-\frac{1}{x+1} + \frac{1}{x^2+1}\right] dx$
$= -\ln|x+1| + \tan^{-1} x + C$ (partial fractions)

43. $y(x) = \int (4 - x^2)^{3/2}\,dx = \frac{x}{4}(4 - x^2)^{3/2} + \frac{3x}{2}\sqrt{4 - x^2} + 6\sin^{-1}\left(\frac{x}{2}\right) + C$ (trig. substitution; $x = 2\sin\theta$)

45. $y(x) = 2e^{-2x}$

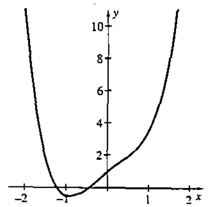
47. $y(x) = \frac{1}{7}e^{4x} - \frac{1}{7}e^{-3x}$

49. $y(x) = \frac{1}{3}\sin 3x$

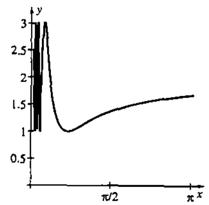
51. $y(x) = -\frac{3}{4} - \frac{1}{2}x + \frac{3}{4}e^{2x}$

53. $y(x) = 8x^6 - 7x^7$

55. $y(x) = x^4 - \frac{1}{2}x^2 + 2x + 1$



57. $y(x) = -\sin\left(\frac{1}{x}\right) + 2$



59. $v(t) = \frac{mg}{c} + e^{-ct/m}\left(-\frac{mg}{c} + v_0\right)$, $\lim_{t\to\infty} v(t) = \frac{mg}{c}$

61. first-order, nonlinear

63. $x(0) = 3$; $\frac{dx}{dt} = -12\sin 4t + 9\cos 4t$, $\frac{dx}{dt}(0) = 9$

65. $u_t = 16ke^{-16t}\cos 4x$, $u_{xx} = 16e^{-16t}\cos 4x$; $u(\pi, 0) = 2$; $\lim_{t\to\infty} u(x, t) = 3$

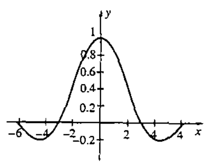
67. $m = 1$, $m = 2$

69. $y(x) = e^{-x} + Ce^{-2x}$

71. Because $\frac{d\psi(x)}{dx} = \frac{An\pi}{L}\cos\left(\frac{n\pi x}{L}\right)$ and $\frac{d^2\psi(x)}{dx^2} = -\frac{An^2\pi^2}{L^2}\sin\left(\frac{n\pi x}{L}\right)$,
$-\frac{h^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = \frac{h^2}{2m}\frac{An^2\pi^2}{L^2}\sin\left(\frac{n\pi x}{L}\right) = EA\sin\left(\frac{n\pi x}{L}\right)$.
Therefore, $E = \frac{h^2n^2\pi^2}{2mL^2}$.

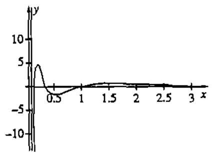
73. $1 + \frac{2x}{y} - \frac{x^2}{y^2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y^2 + 2xy}{x^2}$; $y = \frac{x^2}{C - x}$ so $y = 0$ is singular

75.

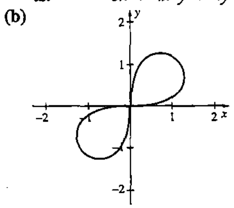




77.

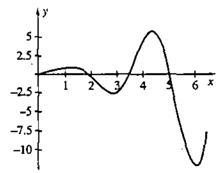


79. (a) $\frac{dy}{dx} = -\frac{4x^3 - 5y + 4xy^2}{-5x + 4x^2y + 4y^3}$
(b)

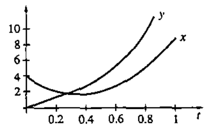


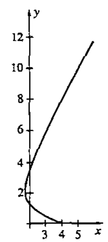
(c) (1, 0.21979), (1, 1.20684)
(d) (−1.10797, −0.319), (−0.00649774, −0.319)

81. $y = e^{-x/2}\left(\frac{37}{104}\cos 3x + \frac{5}{52}\sin 3x\right) + e^{x/2}\left(-\frac{37}{104}\cos 3x + \frac{111}{208}\sin 3x\right)$



83. $\begin{cases} x = \frac{32}{9}e^{-6t} + \frac{4}{9}e^{3t} \\ y = -\frac{8}{9}e^{-6t} + \frac{8}{9}e^{3t} \end{cases}$





***Exercises 1.2***

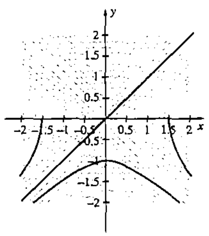
1. no

2. yes

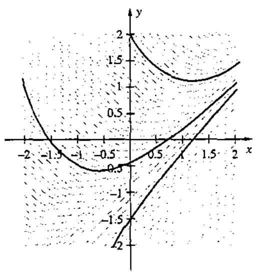
3. yes

4. no

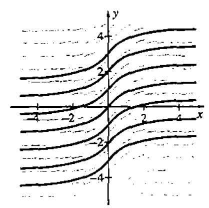
5.



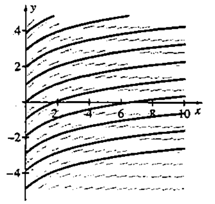
7.



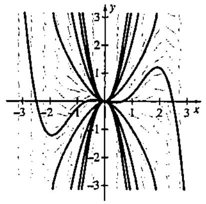
9. (a), (b) $\lim_{x\to\infty} y(x) = \pi/2$; (c), (d) $\lim_{x\to\infty} y(x) = -\pi/2$



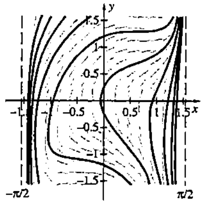
11. $y = 1 + y_0 - e^{-x}$, $\lim_{x\to\infty} (y_0 + \tan^{-1} x) = y_0 + \pi/2$



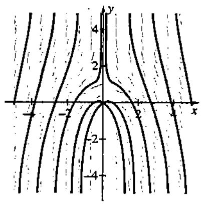
13. (a) $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$ do not exist; (b) $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$ do not exist; (c) $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$



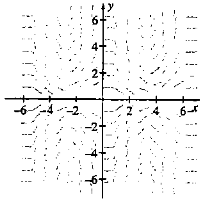
15. $\begin{cases} x' = y \\ y' = -4x \end{cases}$

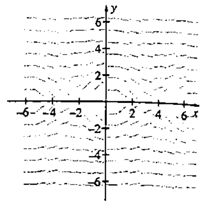


17. $\begin{cases} x' = y \\ y' = -13x - 4y \end{cases}$

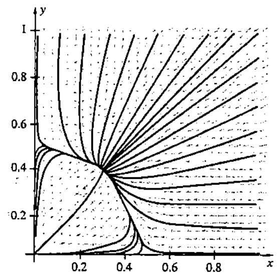


19. $\begin{cases} x' = y \\ y' = -16x + \sin t \end{cases}$

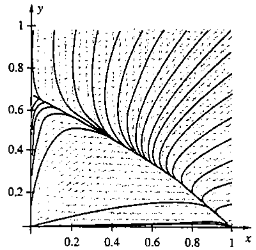




25. (a) $\lim_{t\to\infty} x(t) \approx 0.307392$, $\lim_{t\to\infty} y(t) \approx 0.384615$

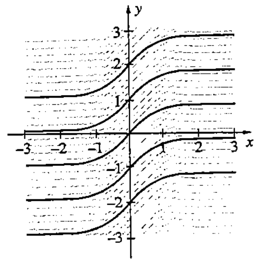


(b) The species with population $y(t)$ becomes extinct.



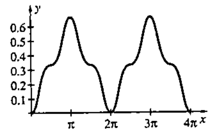


26.



### Chapter 1 Review Exercises

1. (a) ordinary; (b) first-order; (c) linear
3. (a) ordinary; (b) second-order; (c) linear
5. (a) partial; (c) nonlinear
7. $\frac{dy}{dx} = \frac{1}{2}(e^x + \sin x - \cos x)$ so $\frac{dy}{dx} - y - \sin x = \frac{1}{2}(e^x + \sin x - \cos x) - \frac{1}{2}(e^x - \cos x - \sin x) - \sin x = 0$
9. $\frac{dy}{dx} = -3e^{3x}(\cos 6x + 3\sin 6x)$ and $\frac{d^2y}{dx^2} = -9e^{3x}(7\cos 6x + \sin 6x)$ so
$$y'' - 6y' + 45y = -9e^{3x}(7\cos 6x + \sin 6x) + 18e^{3x}(\cos 6x + 3\sin 6x) + 45e^{3x}(\cos 6x - \sin 6x) = 0$$
10. $\frac{dy}{dx} = 5Ax^4 - \frac{3B}{x^4}$; $\frac{d^2y}{dx^2} = 20Ax^3 + \frac{12B}{x^5}$
11. $x^2y'' + 3xy' + 2y = x^2 \cdot \frac{2[2\cos(\ln x) + \sin(\ln x)]}{x^3} + 3x \cdot \frac{-2\cos(\ln x)}{x^2} + 2 \cdot \frac{\cos(\ln x) - \sin(\ln x)}{x} = 0$
13. $y' = 3Ae^{3x} + 4Be^{4x}$ and $y'' = 9Ae^{3x} + 16Be^{4x}$ so
$$y'' - 7y' + 12y = 9Ae^{3x} + 16Be^{4x} - 21Ae^{3x} - 28Be^{4x} + 12Ae^{3x} + 12Be^{4x} + 2 = 2$$
15. $(y\cos xy + \sin x)\,dx + x\cos xy\,dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{\sin x + y\cos xy}{x\cos xy}$
$$\cos xy\left(x\frac{dy}{dx} + y\right) + \sin x = 0 \Rightarrow \frac{dy}{dx} = -\frac{\sin x + y\cos xy}{x\cos xy}$$
17. $y = -x^2\cos x + 2\cos x + 2x\sin x + C$
19. $y = \frac{x}{2}\sqrt{x^2-1} + \frac{1}{2}\ln(x + \sqrt{x^2-1}) + C$
21. $y = \frac{x}{4} + \cos 2x$
23. $y = -\frac{1}{3}\cos^3 x + \frac{1}{3}$



25. $\frac{du}{dt} = -70ke^{-kt}$; $u(0) = 100$; $\lim_{t\to\infty} u(t) = 30$
27. $u_t(x, t) = -\pi^2 ke^{-\pi^2 kt}\sin \pi x + 4\pi^2 ke^{-4\pi^2 kt}\sin 2\pi x$
$u_{xx}(x, t) = -\pi^2 ke^{-\pi^2 kt}\sin \pi x + 4\pi^2 ke^{-4\pi^2 kt}\sin 2\pi x$
$u(1, 0) = 0$; $\lim_{t\to\infty} u(x, t) = 0$
29. $u_{xx}(x, y) = \frac{2xy}{(x^2+y^2)^2}$; $u_{yy}(x, y) = \frac{-2xy}{(x^2+y^2)^2}$

# CHAPTER 2

### Exercises 2.1

1. $y^4 = \frac{8}{7}x^3 + C$
3. $\frac{1}{y^6} = \frac{18}{7x^7} + C$
5. $5y - \frac{9}{7y^7} + 6x + x^4 = C$
7. $\cosh 4y = 2\sinh 3x + C$
9. $-\frac{2}{3y^{3/2}} + \frac{2}{3}x^3 + \frac{8}{3}x^{3/2} = C$

**11.** $-3\cos x = 4\sin y + C$

**13.** $\frac{5x^6}{6} - 4\sin x = -\frac{2}{9}\sin 9y + \frac{2}{7}\cos 7y + C$

**15.** $20\cosh y = -\frac{1}{6}\sinh 6x - \frac{5}{4}\cosh 4x + C$

**17.** $10x - \frac{7}{3}e^{-3x} = e^y - 2y^4 + C$

**19.** $-3\cos x + \frac{1}{3}\cos 3x = \frac{1}{4}\sin 4y - 4\sin y + C$

**21.** $2y + \frac{5}{y} + \sin 2x + 2x = C$

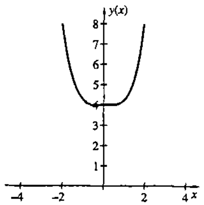
**23.** $\frac{1}{2}\tan^2 y = \frac{1}{8}\cos^4 2x + C$

**25.** $-\frac{1}{2}\cos(x^2) = 2\sin\sqrt{y} + C$

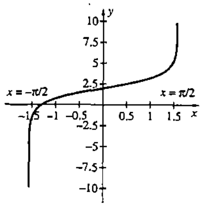
**27.** $\frac{1}{1-\sin y} = \frac{1}{4}\sin^4 x + C$

**29.** $\frac{2}{3}(\ln x)^{3/2} = -\frac{1}{3}e^{3/y} + C$

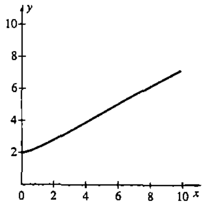
**31.** $y(x) = \frac{1}{4}x^4 + 4$



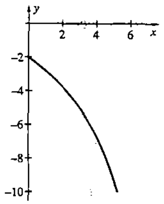
**33.** $y(x) = \ln|\sec x + \tan x| + 2$



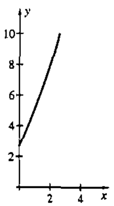

**35.** $\frac{1}{2}y^2 = \frac{2}{3}x^{3/2} + 2$ so $y = \sqrt{\frac{4}{3}x^{3/2} + 4}$



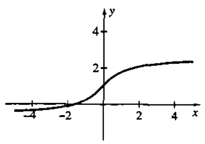
**37.** $\frac{1}{2}y^2 + y = e^x - 1$ so $y = -\sqrt{2e^x - 1} - 1$



**39.** $\frac{1}{2}(\ln y)^2 = x + \frac{1}{2}$ so $y = e^{\sqrt{2x+1}}$



**41.** $y(x) = \tan^{-1} x + 1$





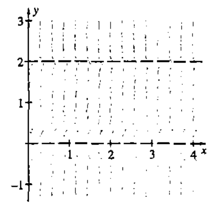
**43.** $y(x) = \frac{2x}{x-2}$

**47.** (c)

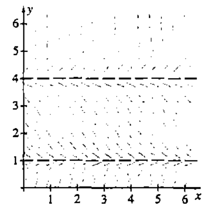
**49.** (b)

**51.** $y = 0$, stable

**53.** $y = 0$, unstable; and $y = 2$, stable



**55.** $y = 1$, stable; and $y = 4$, unstable



**57.** $N(0) = N_0 \Rightarrow N(t) = N_0e^{-kt}$; $D = (\ln 10)/k \approx 2.30259k^{-1}$

**58.** **(b)** $C = \frac{e^{k_a t_0}}{N_0}\left(1 - N_0\frac{k_c}{k_d}\right)$

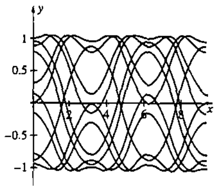
**59.** **(b)** $y(t) = \dfrac{\dfrac{A_0(B_0 - y_0)}{A_0 - y_0}\exp\left(\dfrac{K(B_0 - A_0)t}{V^2}\right) - B_0}{\dfrac{B_0 - y_0}{A_0 - y_0}\exp\left(\dfrac{K(B_0 - A_0)t}{V^2}\right) - 1}$

$\lim_{t\to\infty} y(t) = A_0$;

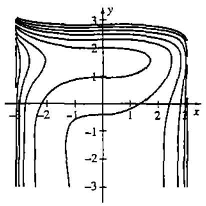
**(c)** $y(t) = e^{-B_0Kt/V_2}(y_0 - A_0 + A_0e^{B_0Kt/V_2})$, $\lim_{t\to\infty} y(t) = A_0$;

**(d)** The approximation improves for larger values of $B_0$.

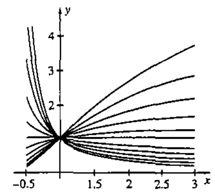
**61.** $y = \sin(C - \cos x)$



**63.** $x\sqrt{9 - x^2} - 9\sin^{-1}(x/3) + e^y(\cos y + \sin y) = C$



**65.** $y = e^{c-(c/x)}$



### Exercises 2.2

**1.** $y = \frac{1}{3}x^2 + Cx^{-1}$

**3.** $y = e^x - x^{-1}e^x + Cx^{-1}$

**5.** $y = \ln(1 + x^2) + C + x^2\ln(1 + x^2) + Cx^2$

**7.** $y = \sqrt{x^2 - 1}(2\sqrt{x^2 - 1} + C)$

**9.** $y = -\frac{2}{5}e^{-2x} + \frac{1}{5}e^{-2x}\tan x + C\sec x$

**11.** $y = -x^3 + 4x + (x^2 - 4)^{3/2}\ln|x + \sqrt{x^2 - 4}| + C(x^2 - 4)^{3/2}$

**13.** $y = \frac{1}{3}x^4 + \frac{3}{4}x^3 - \frac{27}{8} + C\sqrt{4x^2 - 9}$

**15.** $y = x^2 + \frac{49}{9} + C\sqrt{9x^2 + 49}$

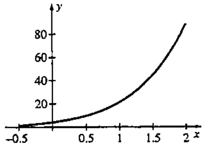
**17.** $y = (e^{x^2/2}x^2 - 2e^{x^2/2} + C)e^{-x^2/2} = x^2 - 2 + Ce^{-x^2/2}$

**19.** $x = -y^2 - 2y - 2 + Ce^y$

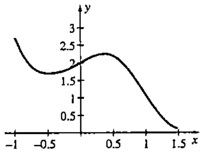
**21.** $x = 3y^2 + Cy$

**23.** $p = \frac{1}{3}t^4 + Ct$

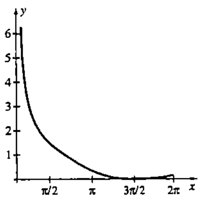
**25.** $y = 4e^x(x + 1)$



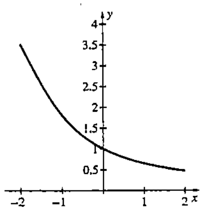

**27.** $y = 2e^{-x^3} + xe^{-x^3}$



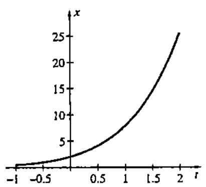

**29.** $y = x^{-1}(1 + \sin x)$




**31.** $y = \dfrac{x^2}{2(e^x + 1)} + \dfrac{2}{e^x + 1}$




**33.** $x = -2 + 4e^t - t$




**37.** $y = e^{-x} + \frac{2}{5}\cos x + \frac{1}{5}\sin x + Ce^{-2x}$

**39.** (a)

**43.** $y = \begin{cases} 2e^{-x} + x - 1, & 0 \le x \le 1. \\ 2e^{-x}, & x > 1 \end{cases}$

**45.** $y = \begin{cases} e^{-2x}, & 0 \le x \le 1 \\ e^2e^{-4x}, & x > 1 \end{cases}$

**47.** $y(x) = Ce^x - \frac{2}{5}\cos 2x - \frac{1}{5}\sin 2x$

**49.** $y(x) = Ce^{-x} + xe^{-x}$

**51.** $y(x) = Ce^{5x} - \frac{x}{5} - \frac{1}{25}$

**53.** $y(x) = Ce^{x/2} + 4e^x - 2\cos x + 4\sin x$

**55.** $y(x) = Ce^{-10x} + \frac{2}{11}e^x$

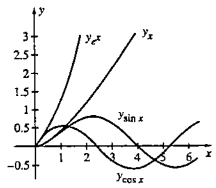
**57.** $y(x) = Ce^x + 2xe^x$

**59.** $y(x) = Ce^{-x} + \cos x + \sin x + x - 1$

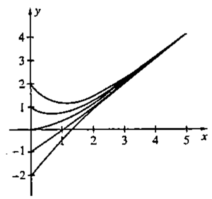
**61.** **(a)** $y(t) = \frac{r}{k}(1 - e^{kt})$ so $\lim_{t\to\infty} y(t) = \frac{r}{k}$; **(b)** $y(t) = 1 - \frac{1}{2}(e^{-t} + \cos t - \sin t)$ so $\lim_{t\to\infty} y(t)$ does not exist; the concentration approaches a periodic state; **(c)** $y(t) = te^{-t}$ so $\lim_{t\to\infty} y(t) = 0$.

**63.** $y_x = e^{-x}(1 - e^x + xe^x)$, $y_{\sin x} = -\frac{1}{2}e^{-x}(-1 + e^x\cos x - e^x\sin x)$, $y_{\cos x} = \frac{1}{2}e^{-x}(-1 + e^x\cos x + e^x\sin x)$, $y_{e^x} = \frac{1}{2}e^{-x}(-1 + e^{2x})$






**65.** $y = e^{-x}(1 + k - e^x + xe^x)$




### *Exercises 2.3*

**1.** $y^2 = -2x - 2 + Ce^x$ or $y = \sqrt{Ce^x - 2x - 2}$

**3.** $\dfrac{1}{y^2} = -\dfrac{2\cos x + 2x\sin x + C}{x}$

**5.** $y^{3/2} = -\frac{9}{20}\cos x + \frac{3}{20}\sin x + Ce^{3x}$

**7.** yes, $n = 1$

**9.** no

**11.** yes, $n = 0$

**13.** no

**15.** no

**17.** $-\ln\left|\frac{2x}{y} - 1\right| + 2\ln\left|\frac{x}{y} - 1\right| = -\ln|y| + C$

**19.** $\frac{1}{2}x^2y^2 - xy^3 = C$

**21.** $y^3 = 3x^3\ln|x| + Cx^3$

**23.** $y = -x\ln|x| + Cx$

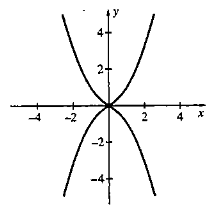
**25.** $x^{5/6} = \dfrac{C\sqrt{2y - x}}{(3y - 2x)^{2/3}}$

**27.** $y = Ce^{y/(4x)}$

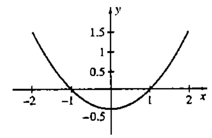
**29.** $xy - \frac{1}{2}y^2 = C$

**31.** $x = vy \Rightarrow dx = v\,dy + y\,dv \Rightarrow \dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$
$v + y\dfrac{dv}{dy} = \dfrac{2}{v}e^{-v} + v \Rightarrow ve^v\,dv = \dfrac{2dy}{y} \Rightarrow ve^v - e^v = 2\ln|y| + C$
$e^{x/y}\left(\dfrac{x}{y} - 1\right) = 2\ln|y| + C$

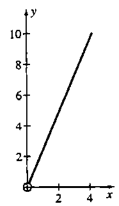
**33.** $y^2 = \frac{1}{2}x^2(1 + x^2)$ or $y = \dfrac{x\sqrt{1 + x^2}}{\sqrt{2}}$




**35.** $y = \frac{1}{2}(x^2 - 1)$




**37.** $\dfrac{y^3}{3x^3} = -\ln|x| + 9$ or $y = \sqrt[3]{3}x(9 - \ln|x|)^{1/3}$

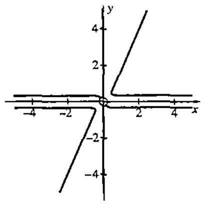



**39.** $y = ux \Rightarrow y^4\,dx + (x^4 - xy^3)\,dx = 0 \Rightarrow$
$(ux)^4\,dx + (x^4 - xu^3x^3)(u\,dx + x\,du) = 0$
$\Rightarrow x^4u\,dx + x^5(1 - u^3)\,du = 0 \Rightarrow \dfrac{1}{x}\,dx = \dfrac{u^3 - 1}{u}\,du$
$\Rightarrow \ln|x| = \frac{1}{3}u^3 - \ln|u| + C$

$\Rightarrow \ln|x| = \frac{1}{3}(y/x)^3 - \ln|y/x| + C$

$y(1) = 2 \Rightarrow 0 = \frac{1}{3}\cdot 8 - \ln 2 + C \Rightarrow C =$

$\frac{1}{3}(3\ln 2 - 8) \approx -1.07352$



41. $y^2 = 2x^2 \ln|x|$

43. (d)

45. (c)

49. (c) $y = \frac{2}{3\sqrt{3}}x^{3/2}$

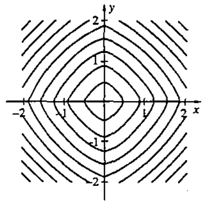
51. $f(x) = x - 1$; $g(x) = x^2 - x$
general solution: $(xc - y) - 1 = c^2 - c \Rightarrow y = cx - 1 + c - c^2$
singular solution: $\frac{d}{dx}[xy' - y - 1] = \frac{d}{dx}[(y')^2 - y'] \Rightarrow$
$xy'' + y' - y' = 2y'y'' - y''$
$(x - 2y' + 1)y'' = 0 \Rightarrow y' = \frac{x+1}{2} \Rightarrow y = \frac{1}{4}x^2 + \frac{x}{2} - \frac{3}{4}$

53. $f(x) = 1 - 2x$; $g(x) = x^{-2}$; general solution:
$1 - 2(xc - y) = c^{-2} \Rightarrow y = \frac{1}{2}(c^{-2} + 2cx - 1)$; singular solution: $y = \frac{1}{2}(3x^{2/3} - 1)$

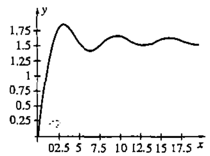
57. $y = (Ce^{-p} - 2)(p + 1) + (2p + 1)$

59. $z = \frac{1}{v}\left(-\frac{1}{2} + Ce^{2v}\right)$

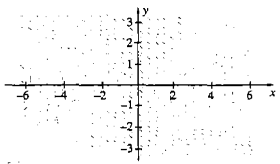
61. $x^{4/3} + y^{4/3} = C$



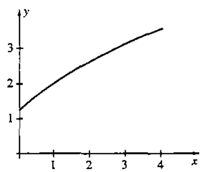
63. (a) $Si'(x) = \frac{\sin x}{x}$, 0



(b)



(c) $-\ln y + Si\left(\frac{x}{y}\right) = -\ln 2 + Si\left(\frac{1}{2}\right)$



### *Exercises 2.4*

1. exact
3. exact
5. exact
7. not exact
9. exact



11. $y = x^3 + C$

13. $xy^2 = C$

15. $x^2 + xy^3 + 4y = C$

17. $x^2y + \frac{1}{3}y^3 = C$

19. $x \sin^2 y = C$

21. $\frac{1}{y} = C + Cx^{-1} + x^{-1}$

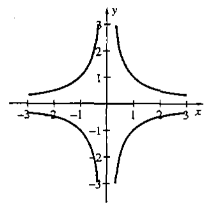
23. $y^2 \cos(x^2) = C$

25. $x + y\sin(xy) = C$

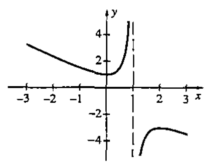
27. $(3 + x)\sin(x + y) = C$

29. $y = -x\ln|x| + Cx$

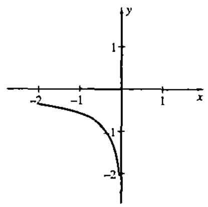
31. $x^2y^2 = 1$



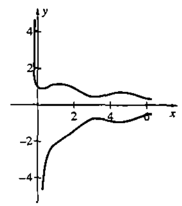
33. $y = -\frac{x^3 + 1}{x^2 - 1}$



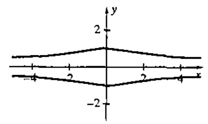
35. $xe^y - x^2y = 0$



37. $xy^2 + \cos 2x + y = 2$



39. $f_x(x, y) = \frac{1}{1 + x^2} - y^2 \Rightarrow f(x, y) = \tan^{-1} x - xy^2 + g(y) \Rightarrow f_y(x, y) = -2xy + g'(y)$
$\tan^{-1} x - xy^2 = C$; $\tan^{-1} 0 - 0\cdot 0^2 = C \Rightarrow C = 0$;
$\tan^{-1} x - xy^2 = 0$



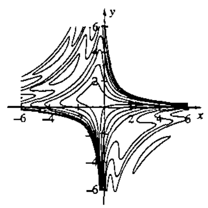
47. $xy = C$

49. $\mu(y) = \exp\left(\int \frac{1}{y}dy\right) = \exp(\ln|y|) = y,\ y > 0$
$y^2\,dx + (2xy - y^2e^y)\,dy = 0 \Rightarrow f_x(x, y) = y^2 \Rightarrow$
$f(x, y) = xy^2 + g(y) \Rightarrow f_y(x, y) = 2xy + g'(y) \Rightarrow$
$g'(y) = -y^2e^y \Rightarrow g(y) = -y^2e^y + 2ye^y - 2e^y \Rightarrow$
$xy^2 - y^2e^y + 2ye^y - 2e^y = C$

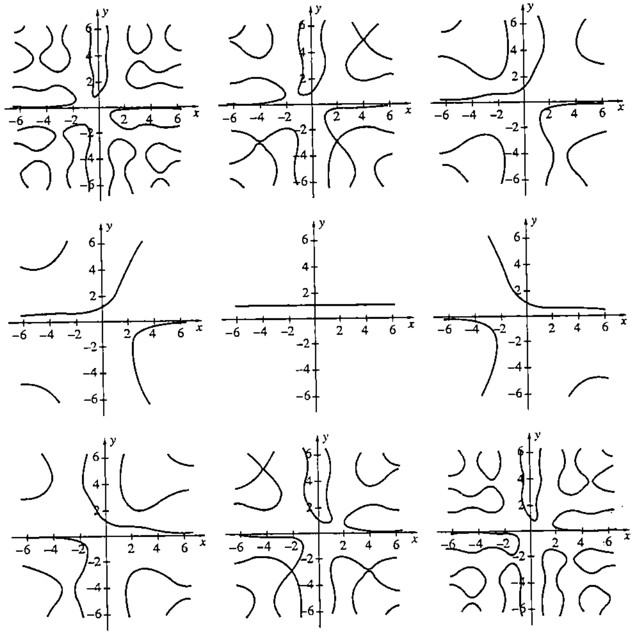
51. $-\frac{y}{x} + 2x + \frac{1}{2}y^2 = C$

53. $\frac{1}{y} = \frac{C\sqrt{2x^2 - 1} - 1}{x}$

57. $e^{xy} - x + y + \sin(xy) = C$



**59.** $x\sin(cy) + y\cos(cx) = 1$
$\left(\text{graph for } c = -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2\right)$



### Exercises 2.5

**1.** (a) yes; (b) no; (c) no

**3.** In this case, $f(x, y) = y^{1/5}$, so $\frac{\partial f}{\partial y}(x, y) = \frac{1}{5}y^{-4/5}$ is not continuous at (0, 0); uniqueness is not guaranteed. Solutions: $y = \left(\frac{4}{5}x\right)^{5/4}$, $y = 0$.

**5.** Because $f(x, y) = 2\sqrt{|y|} = \begin{cases} 2\sqrt{y}, & y \geq 0 \\ 2\sqrt{-y}, & y < 0 \end{cases}$, $\frac{\partial f}{\partial y}(x, y) = \begin{cases} y^{-1/2}, & y > 0 \\ -(-y)^{-1/2}, & y < 0 \end{cases}$ is not continuous at (0, 0). Therefore, the hypotheses of the Existence and Uniqueness Theorem are not satisfied.

**7.** Yes. $y = e^{2(x^{3/2}-1)/3}$

**9.** Yes. $f(x, y) = \sin y - \cos x$ and $\frac{\partial f}{\partial y}(x, y) = \cos y$ are continuous in a region containing $(\pi, 0)$.

**11.** $y = \sec x \Rightarrow y' = \sec x \tan x = y \tan x$ and $y(0) = \sec 0 = 1$; $\frac{\partial f}{\partial y}(x, y) = x$ is continuous on $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $f(x, y) = \sec x$ is continuous on $-\frac{\pi}{2} < x < \frac{\pi}{2}$ so the largest interval on which the solution is valid is $-\frac{\pi}{2} < x < \frac{\pi}{2}$.



**13.** $f(x, y) = \sqrt{y^2 - 1}$ and $\frac{\partial f}{\partial y}(x, y) = \frac{1}{2}(y^2 - 1)^{-1/2}$. $2y = \frac{y}{\sqrt{y^2 - 1}}$; unique solution guaranteed for (a) only.

**15.** $(0, \infty)$

**17.** $(0, \infty)$

**19.** $(-\infty, 1)$

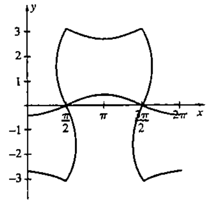
**21.** $(-2, 2)$

**23.** $x > 0$, $y = \frac{\sin x}{x} - \cos x$, $-\infty < x < +\infty$

**25.** $x > \frac{1}{a}$ or $x < \frac{1}{a}$

**27.** $|x| < a$

**29.** general solution: $\frac{1}{2}\sin^2 x - \sin y \cos x + \frac{1}{2}\sin^2 y = C$; solution is not unique if $x_0 = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.



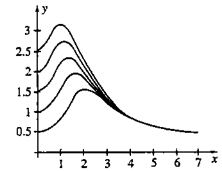
### Exercises 2.6

| | |
|---|---|
| **1.** 47.3742, 63.2572 | **13.** 185.34, 206.981 |
| **3.** 1.8857, 2.0984715. | **15.** 1.95547, 1.95609 |
| **5.** 79.8458, 123.048 | **17.** 90.6405, 90.6927 |
| **7.** 1.95109, 1.95388 | **19.** 2.43501, 2.43514 |
| **9.** 83.6491, 88.6035 | **21.** 216.582, 216.992 |
| **11.** 2.37754, 2.41897 | **23.** 1.95629, 1.95629 |

**25–27.** (a) $y(t) = e^{-t}$, $y(1) = 1/e \approx 0.367879$

| | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ |
|---|---|---|---|
| **25.** (Euler's) | $y(1) \approx 0.348678$ | $y(1) \approx 0.358486$ | $y(1) \approx 0.363232$ |
| **26.** (Improved Euler's) | $y(1) \approx 0.368541$ | $y(1) \approx 0.368039$ | $y(1) \approx 0.367918$ |
| **27.** (Fourth-order Runge-Kutta) | $y(1) \approx 0.36788$ | $y(1) \approx 0.414831$ | $y(1) \approx 0.429069$ |

**29.** $y(0.5) \approx 0.566144, 1.12971, 1.68832, 2.23992, 2.78297$



### Chapter 2 Review Exercises

**1.** $y^3 = \frac{1}{5}x^6 + C$

**3.** $\sinh y - \frac{1}{2}\cosh x = C$

**5.** $y^5 = e^{5x} + C$

**7.** $\frac{1}{2}(\ln y)^2 + \frac{1}{2}e^{-2x} = C$

**9.** $\frac{1}{37}e^y \cos 6y + \frac{6}{37}e^y \sin 6y = -\frac{1}{20}\cos 10x + \frac{1}{8}\cos 4x + C$

**11.** $-\frac{1}{2}x^2 + xy + \frac{1}{2}y^2 = C$

**13.** $y^2 + x = C$

**15.** $y = C(y^2 - 4x)^{5/2}$

17. $\frac{1}{3}x^3y - \cos x - \sin y = C$

19. $x^2 \ln y + 2y = C$

21. $y = 1 + Ce^{-x^2/2}$

23. $y = \dfrac{\cos x + x \sin x + C}{x}$

25. $y^3 = \frac{3}{4}e^x + Ce^{-3x}$

27. $y = cx + 2\ln c$; Sing: $y = 2\ln\left(\frac{-2}{x}\right) - 2,\ x < 0$

29. $x = \frac{8}{3}p + Cp^{-2},\ y = \frac{4}{3}p^2 + 2Cp^{-1}$

31. $\sin(x - y) + y = \pi$

33. $x \sin y - y \sin x = 0$

35. $y \ln x + x \ln y = 0$

37.

| $n$ | $x_n$ | $y_n$ (Euler's) | $y_n$ (Improved Euler's) | $y_n$ (Runge-Kutta of order 4) |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | 1.05 | 1 | 1.00559 | 1.00678 |
| 2 | 1.1 | 1.01118 | 1.0181 | 1.01921 |
| 3 | 1.15 | 1.02608 | 1.03383 | 1.03486 |
| 4 | 1.2 | 1.04368 | 1.052 | 1.05296 |
| 5 | 1.25 | 1.06345 | 1.07219 | 1.07309 |
| 6 | 1.3 | 1.08505 | 1.0941 | 1.09494 |
| 7 | 1.35 | 1.10823 | 1.11752 | 1.11831 |
| 8 | 1.4 | 1.13281 | 1.14228 | 1.14303 |
| 9 | 1.45 | 1.15866 | 1.16825 | 1.16896 |
| 10 | 1.5 | 1.18565 | 1.19533 | 1.19601 |
| 11 | 1.55 | 1.21368 | 1.22343 | 1.22407 |
| 12 | 1.6 | 1.24268 | 1.25247 | 1.25307 |
| 13 | 1.65 | 1.27256 | 1.28237 | 1.28295 |
| 14 | 1.7 | 1.30328 | 1.31309 | 1.31364 |
| 15 | 1.75 | 1.33478 | 1.34456 | 1.34509 |
| 16 | 1.8 | 1.36699 | 1.37675 | 1.37726 |
| 17 | 1.85 | 1.3999 | 1.40961 | 1.4101 |
| 18 | 1.9 | 1.43344 | 1.44311 | 1.44357 |
| 19 | 1.95 | 1.46759 | 1.4772 | 1.47764 |
| 20 | 2. | 1.50232 | 1.51186 | 1.51229 |

39.

| $n$ | $x_n$ | $y_n$ (Euler's) | $y_n$ (Improved Euler's) | $y_n$ (Runge-Kutta of order 4) |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 |
| 1 | 0.05 | 1 | 1.00125 | 1.00125 |
| 2 | 0.1 | 1.0025 | 1.00501 | 1.00501 |
| 3 | 0.15 | 1.0075 | 1.01129 | 1.01129 |
| 4 | 0.2 | 1.01503 | 1.02012 | 1.02013 |
| 5 | 0.25 | 1.02511 | 1.03156 | 1.03157 |
| 6 | 0.3 | 1.03779 | 1.04564 | 1.04565 |
| 7 | 0.35 | 1.0531 | 1.06242 | 1.06245 |
| 8 | 0.4 | 1.07112 | 1.08198 | 1.08201 |
| 9 | 0.45 | 1.09189 | 1.10437 | 1.10441 |
| 10 | 0.5 | 1.11548 | 1.12966 | 1.12971 |
| 11 | 0.55 | 1.14194 | 1.15789 | 1.15795 |
| 12 | 0.6 | 1.17132 | 1.1891 | 1.18918 |
| 13 | 0.65 | 1.20364 | 1.22329 | 1.2234 |
| 14 | 0.7 | 1.23889 | 1.26043 | 1.26056 |
| 15 | 0.75 | 1.27702 | 1.30042 | 1.30058 |
| 16 | 0.8 | 1.31791 | 1.34309 | 1.34329 |
| 17 | 0.85 | 1.36139 | 1.38818 | 1.38842 |
| 18 | 0.9 | 1.40717 | 1.43532 | 1.43561 |
| 19 | 0.95 | 1.45488 | 1.48403 | 1.48438 |
| 20 | 1. | 1.50399 | 1.53369 | 1.5341 |



# CHAPTER 3

### Exercises 3.1

1. 102400, $t = 16.23$ days

3. $t \approx 2.71$ days

5. $y(1) = 99.93$ g; $y(500) = 70.71$ g

7. $y(24) = 1036.8$ g, $t \approx 22.81$ hours

9. $t \approx 12.92$ hours, $y(15) = \frac{1}{8}y_0$

13. 9.72 days

15. $y(100) \approx 0.96y_0$

17. $t_{tool} \approx 3561.13$ years old, $t_{fossil} \approx 4222.81$ years old, $t_{fossil} - t_{tool} \approx 661.68$ years. No.

21. $y = \dfrac{1{,}000{,}000}{1 + 9e^{-t/100}}$; $y(25) \approx 124{,}856$; $\lim_{t\to\infty} y(t) = 1{,}000{,}000$

23. $y = \dfrac{200}{1 + 199\left(\frac{3}{199}\right)^t}$; $y(2) \approx 191$; because there is no $t$ so that $y = 200$, all students do not theoretically learn of the rumor.

25. $y(t) = 1 + 499e^{-5t}$; $y(20) \approx 1$; quickly

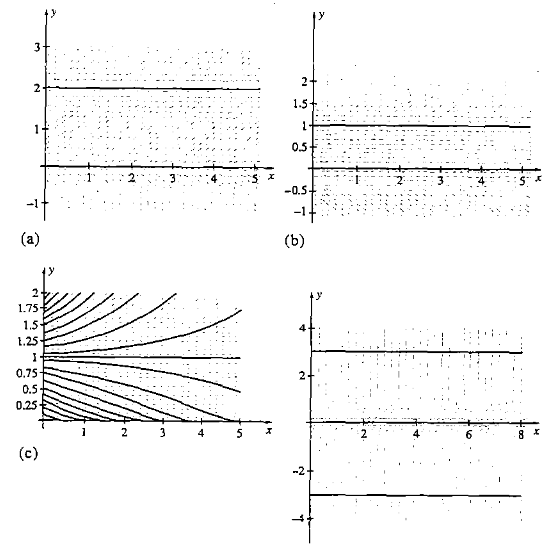
27. $\lim_{m\to\infty} S_0\left(1 + \frac{k}{m}\right)^{mt}$ is found with L'Hopital's rule. Because $\lim_{m\to\infty} \ln\left(1 + \frac{k}{m}\right)^{mt} = \lim_{m\to\infty} mt \ln\left(1 + \frac{k}{m}\right) =$

$$\lim_{m\to\infty} \frac{\ln\left(1 + \frac{k}{m}\right)}{\frac{1}{mt}} = \lim_{m\to\infty} \frac{\frac{-k/m^2}{1 + \frac{k}{m}}}{-\frac{1}{tm^2}} = \lim_{m\to\infty} \frac{kt}{1 + \frac{k}{m}} = kt,$$

$$\lim_{m\to\infty} S_0\left(1 + \frac{k}{m}\right)^{mt} = S_0 \lim_{m\to\infty}\left(1 + \frac{k}{m}\right)^{mt} = S_0e^{kt}.$$

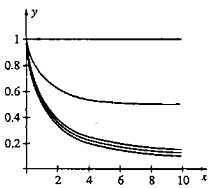
29. (c) $y(t) = \dfrac{2}{1 + e^{rt}}$, $\lim_{t\to\infty} y(t) = 0$; (d) $y(t) = \dfrac{6}{3 - e^{rt}}$, $\lim_{t\to\infty} y(t) = 0$

31. **(a)** $y = 0$, unstable; $y = 2$, semi-stable; **(b)** $y = 0$, semi-stable; $y = 1$ unstable; **(c)** $y = 0$, semi-stable; $y = 1$ unstable; **(d)** $y = -3$, unstable; $y = 3$, stable; $y = 0$, semi-stable

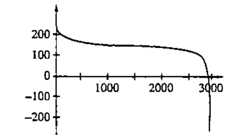


(a) (b) (c) (d)

**33.** $y_{1/100} = \dfrac{1}{100 - 99e^{-t/100}}$, $y_{1/20} = \dfrac{1}{20 - 19e^{-t/20}}$, $y_{1/10} = \dfrac{1}{10 - 9e^{-t/10}}$, $y_{1/2} = \dfrac{1}{2 - e^{-t/2}}$, $y_1 = 1$



**35.** (b) $t \approx 2958$



(c) $h = 0$: $P(t) = \dfrac{300}{1 + \frac{2947}{53}e^{-3t/100}}$;

$h = \dfrac{1}{2}$: $P(t) =$

$50\dfrac{-371 - 841\sqrt{7} + (841\sqrt{7} - 317)e^{t\sqrt{7}/100}}{-3500 - 1447\sqrt{7} + (1447\sqrt{7} - 3500)e^{t\sqrt{7}/100}}$;

$h = 1$: $P(t) =$

$50\dfrac{-265 - 1841\sqrt{5} + (1841\sqrt{5} - 265)e^{t/(20\sqrt{5})}}{-2500 - 1447\sqrt{5} + (1447\sqrt{5} - 2500)e^{t/(20\sqrt{5})}}$;

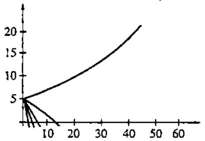
$h = \dfrac{3}{2}$: $P(t) =$

$150\dfrac{-53 - 947\sqrt{3} + (947\sqrt{3} - 53)e^{t\sqrt{3}/100}}{-1500 - 1447\sqrt{3} + (1447\sqrt{3} - 1500)e^{t\sqrt{3}/100}}$;

$h = 2$: $P(t) = 100\dfrac{-1947 + 1894e^{t/100}}{-1947 + 947e^{t/100}}$;

$h = \dfrac{9}{4}$: $P(t) = 50\dfrac{-10600 + 4341t}{-100000 + 1447t}$;

$h = \dfrac{5}{2}$: $P(t) = -50\left(-3 + \tan\left(\dfrac{t}{200} + \tan^{-1}\dfrac{1447}{500}\right)\right)$



### Exercises 3.2

**1.** $t \approx 55.85$ min

**3.** 12.59 min

**5.** $t \approx -2.45$ hr, 12:30 P.M.

**7.** $T(5) \approx 76.3°F$

**9.** 75°F

**11.** $t \approx 4.7$ min

**13.** $u(t) = \dfrac{-5}{9 + \pi^2}\left(-8\pi^2 - (2\pi^2 + 27)e^{-t/4} + 3\pi \sin\left(\dfrac{\pi t}{12}\right) + 9\cos\left(\dfrac{\pi t}{12}\right) - 72\right)$

**15.** $u(t) = \dfrac{-5}{9 + \pi^2}\left(-14\pi^2 + \pi^2 e^{-t/4} + 3\pi \sin\left(\dfrac{\pi t}{12}\right) + 9\cos\left(\dfrac{\pi t}{12}\right) - 126\right)$

**17.** If $R_1 = R_2$, the volume remains constant, so $V(t) = V_0$. If $R_1 > R_2$, $V$ increases. If $R_1 < R_2$, $V$ decreases.

**19.** $R_1 = 4\dfrac{\text{gal}}{\text{min}}$, $R_2 = 3\dfrac{\text{gal}}{\text{min}}$

$\dfrac{dV}{dt} = 4 - 3 = 1$, $V(0) = 200 \Rightarrow V(t) = t + 200$

$\dfrac{dy}{dt} = \left(2\dfrac{\text{lb}}{\text{gal}}\right)\left(4\dfrac{\text{gal}}{\text{min}}\right) - \left(\dfrac{y}{t + 200}\right)\left(3\dfrac{\text{gal}}{\text{min}}\right) =$

$8 - \dfrac{3y}{t + 200}$; $y(0) = 10$

$\dfrac{dy}{dt} + \dfrac{3y}{t + 200} = 8 \Rightarrow \mu = e^{\int 3/(t+200)dt} = e^{3\ln(t+200)} =$

$(t + 200)^3$

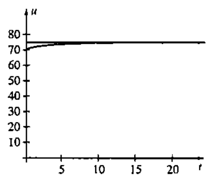
$\dfrac{d}{dt}[(t + 200)^3 y] = 8(t + 200)^3 \Rightarrow (t + 200)^3 y =$

$2(t + 200)^4 + C \Rightarrow y = 2(t + 200) + C(t + 200)^{-3}$

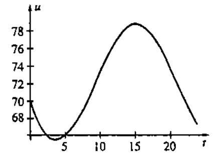
$10 = 400 + C \cdot 200^{-3} \Rightarrow C = -390 \cdot 200^3 \Rightarrow$

$y = 2t + 400 - 390 \cdot 200^3(t + 200)^{-3}$

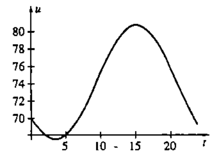
**21.** (a) $u(t) = e^{-t/4}(-6 + 76e^{t/4})$



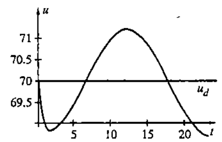
(b) $u(t) = \dfrac{1}{9 + \pi^2}(639 + 71\pi^2 + (81 - \pi^2)e^{-t/4} - 90\cos(\pi t/12) - 30\pi \sin(\pi t/12))$



(c) $u(t) = \dfrac{2}{9 + \pi^2}(333 + 37\pi^2 + (27 - 2\pi^2)e^{-t/4} - 45\cos(\pi t/12) - 15\pi \sin(\pi t/12))$



**23.** $u_d \approx 69.8273$



### Exercises 3.3

**1.** $v(t) = 32 - 32e^{-t}$; $v(2) \approx 27.67$ ft/s

**3.** $v(t) = \dfrac{1}{2} + \dfrac{15}{2}e^{-64t}$; $v(1) \approx 0.5$ ft/s

**5.** $v(t) = 4 - 68e^{-8t}$; $v(0) = 0$ when $t = -\dfrac{1}{8}\ln\left(\dfrac{1}{17}\right) \approx 0.354$ s

**7.** $\left\{\dfrac{dv}{dt} = 32 - v, v(0) = 0\right\} \Rightarrow v(t) = 32 - 32e^{-t}$

$\left\{\dfrac{ds}{dt} = v(t), s(0) = 0\right\} \Rightarrow s(t) = 32t + 32e^{-t} - 32$;

$s(4) \approx 96.59$ ft $< 300 \Rightarrow 203.41$ ft above the ground

**9.** $v(t) = \dfrac{49}{5} - \dfrac{49}{5}e^{-t}$, $\lim\limits_{t\to\infty} v(t) = \dfrac{49}{5}$

**11.** $v(t) = -9800 + 9900e^{-t/1000}$; $v(t) = 0 \Rightarrow t \approx 10.152$ s; $s(t) = -9800t + 9900000(-e^{-t/1000} + 1)$; $s(10.152) \approx 506.76$ m

**13.** $\left\{\dfrac{dv}{dt} = -g, v(0) = v_0\right\} \Rightarrow v(t) = -gt + v_0$;

$\left\{\dfrac{ds}{dt} = v(t), s(0) = s_0\right\} \Rightarrow s(t) = -\dfrac{1}{2}gt^2 + v_0 t + s_0$

**15.** Because the object reaches its max. height when $v = -gt + v_0 = 0$ or $t = \dfrac{v_0}{g}$ and the air resistance is ignored, the object hits the ground when $t = \dfrac{2v_0}{g}$. Therefore, the velocity at this time is $v\left(\dfrac{2v_0}{g}\right) = -g\left(\dfrac{2v_0}{g}\right) + v_0 = v_0$.

**17.** $c = 5$

**19.** The velocity of the parachutist after the parachute is opened is given by $v(t) = 8\dfrac{17e^{8t} + 13}{17e^{8t} - 13}$; the limiting velocity is $\lim\limits_{t\to\infty} v(t) = 8$.

**21.** (b) $\dfrac{dv}{dt} = \dfrac{dv}{dr}\dfrac{dr}{dt} = v\dfrac{dv}{dr}$; (c) $\lim\limits_{t\to\infty} v^2 = v_0^2 - 2gR$

**23.** $g \approx 32$ ft/s$^2 \approx 0.006$ mi/s$^2$; $v_0 = \sqrt{2gR} = \sqrt{2(.165)(0.006)(1080)} \approx 1.46$ mi/s

**25.** $Q(t) = E_0C + e^{-t/(RC)}(-E_0C + Q_0)$;

$I(t) = -\dfrac{1}{RC}e^{-t/(RC)}(-E_0C + Q_0)$

**27.** $v(t) = 12(4 - \sqrt{3}) - 12(4 - \sqrt{3})e^{-t/3}$;

$x(t) = 12(4 - \sqrt{3})t + 36(4 - \sqrt{3})e^{-t/3} - 36(4 - \sqrt{3})$

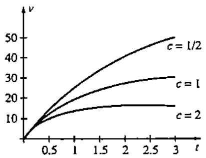
**29.** $v(t) = 32$, $\lim\limits_{t\to\infty}(32 + Ce^{-t}) = 32$

**31.** $v(t) = -gm/c$, $\lim\limits_{t\to\infty}(-gm/c + Ce^{-ct/m}) = -gm/c$

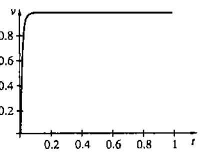
**33.** $c = \dfrac{1}{2}$: $v(t) = 64e^{-t/2}(-1 + e^{t/2})$;

$c = 1$: $v(t) = 32e^{-t}(-1 + e^t)$;

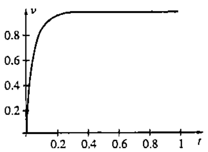
$c = 2$: $v(t) = 16e^{-2t}(-1 + e^{2t})$



**35.** (numerical solutions)



$F_R = 16v^3$



$F_R = 16\sqrt{v}$

37. approximately 300.772 sec

### Chapter 3 Review Exercises

1. $y = 0$: unstable; $y = \frac{1}{2}$: stable

3. $y = 0$: unstable; $y = 4$: stable

5. $y = P_0 3^{t/4}$; $t = \frac{4 \ln 5}{\ln 3} \approx 5.86$ days

7. $y = y_0\left(\frac{1}{2}\right)^{t/1700}$; $y(50) \approx 0.9798y_0$ (97.98% of $y_0$)

9. $y = \dfrac{1000}{1 + 3\left(\frac{1}{3}\right)^t}$; $y = 750 \Rightarrow t = \frac{\ln 9}{\ln 3} \approx 2$ days

11. $T(t) = 90 - 50\left(\frac{1}{2}\right)^{t/20}$; $T(30) = 90 - 50\left(\frac{1}{2}\right)^{3/2} \approx 72.3°\text{F}$

13. $\frac{dT}{dt} = k(T - 325)$, $T(0) = 100$, $T(45) = 150$
$T(t) = Ce^{kt} + 325$; $T(0) = 100 \Rightarrow C = -225 \Rightarrow T(t) = -225e^{kt} + 325$
$T(45) = 150 \Rightarrow -225e^{45k} + 325 = 150 \Rightarrow$
$e^{45k} = \frac{175}{225} = \frac{7}{9} \Rightarrow e^k = \left(\frac{7}{9}\right)^{1/45}$
$T(t) = -225\left(\frac{7}{9}\right)^{t/45} + 325 \Rightarrow T(-60) \approx 10.43°\text{F}$

15. $v(t) = 4 - 4e^{-8t}$; $v(3) \approx 4$ ft/s; $s(t) = 4t + \frac{1}{2}e^{-8t} - \frac{1}{2}$;
$s(3) \approx 11.5$ ft

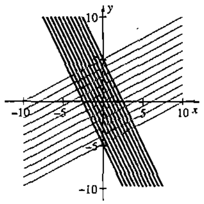
17. $\frac{dv}{dt} = -9.8 - v$, $v(0) = 40 \Rightarrow v(t) = -\frac{99}{5} + \frac{299}{5}e^{-t}$;
$v(t) = 0 \Rightarrow t = -\ln\frac{99}{299} \approx 1.11$ s
$\frac{ds}{dt} = v$, $s(0) = 0 \Rightarrow s(t) = -\frac{99}{5}t - \frac{299}{5}e^{-t} + \frac{299}{5}$;
$s\left(-\ln\frac{99}{299}\right) \approx 18.11$ ft

19. $\frac{dv}{dt} = 32 - \frac{1}{2}v^2$, $v(0) = 30 \Rightarrow v(t) = \dfrac{8(19e^{8t} + 11)}{19e^{8t} - 11}$;
$\lim_{t\to\infty} v(t) = 8$ ft/s

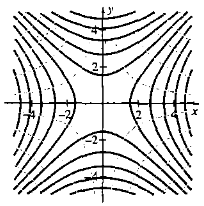
21. $m = 230$ kg $\Rightarrow \frac{dv}{dt} = \frac{82}{115} - \frac{637}{230000}v$, $v(0) = 0$;
$v(t) = \frac{164000}{637}\left(1 - e^{-637t/230000}\right)$
$v(t) = 12 \Rightarrow t = -\frac{230000}{637}\ln\frac{39089}{41000} \approx 17.23$ s
$\frac{dy}{dt} = v$, $y(0) = 0 \Rightarrow y = \frac{164000}{637}t + \frac{37720000000}{405769}\left(e^{-637t/230000} - 1\right)$
$H = y(17.23) \approx 104.17798$

23. (a) $4r^2 = 32\cos^2\theta - 16$; $r = 2\sec\theta + 2$;
(c) $r = -6\cos^2\frac{\theta}{2} + 6$

29. $y = \frac{x}{2} + k$




31. $xy = k$




33. Yes. The orthogonal trajectories of $y^2 - 2cx = c^2$ satisfy $\frac{dy}{dx} = \frac{y}{x \pm \sqrt{x^2 + y^2}}$. One solution of this homogeneous equation can be written as $y^2 + 2Cx = C^2$. Now, replace $C$ by $-c$.

35. Equipotential lines: $x^2 + y^2 = c$; orthogonal traj.: $y = kx$

37. (c) $\frac{1}{2}x^2 + xy - \frac{1}{2}y^2 = k_1$ and $-\frac{1}{2}x^2 + xy + \frac{1}{2}y^2 = k_2$



# CHAPTER 4

### Exercises 4.1

1. $W(S) = 1 \neq 0$ for any $t$; lin. indep.

3. $W(S) = 2e^{-10t} \neq 0$ for any $t$; lin. indep.

5. $W(S) = 3e^{-6t} \neq 0$ for any $t$; lin. indep.

7. $y' = c_1e^t - c_2e^{-t}$, $y'' = c_1e^t + c_2e^{-t}$

9. $y' = e^{-t}(-c_1 + c_2 - c_2t)$; $y'' = e^{-t}(c_1 - 2c_2 + c_2t)$

11. $y' = -2c_1\sin 2t + 2c_2\cos 2t$;
$y'' = -4c_1\cos 2t - 4c_2\sin 2t$

13. $y = e^{-t} - 2e^{2t}$

15. $y = e^{4t}(1 - t)$

17. $y = \cos t + \sin t + x\sin t$

19. (d) $\begin{vmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{vmatrix} = \cosh^2 t - \sinh^2 t = 1$

25. $y_2(t) = e^{2t}$

27. $y_2(t) = te^{2t}$

29. $y_2(t) = \sin 2t$

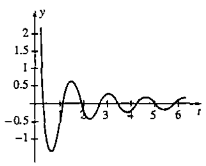
31. $y_2(t) = t$

33. $y_2(t) = \frac{1}{t}\ln t$

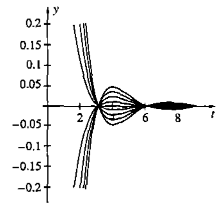
35. $a = 2$, $b = 5$

41. $y = c_1 + c_2\tan(c_3 + c_4\ln t)$ is a solution of the equation if
(a) $y = c_1$ (note that if $c_2 = 0$ or $c_4 = 0$, $y$ is a constant function); or
(b) $y = -\frac{1}{2} + c_2\tan(c_3 + c_2\ln t)$. The Principle of Superposition does not hold.

43. $y = \frac{\cos 4t}{t}$




45. $y = C\frac{\sin t}{t^3}$, $C$ arbitrary




### Exercises 4.2

1. $y = c_1e^{-2t} + c_2e^{-6t}$

3. $y = c_1e^{-4t} + c_2e^t$

5. $y = c_1\cos 4t + c_2\sin 4t$

7. $y = c_1\cos\sqrt{7}t + c_2\sin\sqrt{7}t$

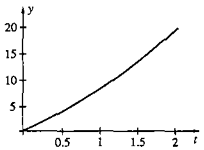
9. $y = c_1e^{3t}\cos 4t + c_2e^{3t}\sin 4t$

11. $y = c_1e^{-3t}\cos 3t + c_2e^{-3t}\sin 3t$

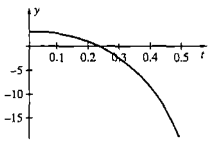
13. $y = c_1e^{3t/7} + c_2e^{-t}$

15. $y = c_1e^{3t} + c_2te^{3t}$

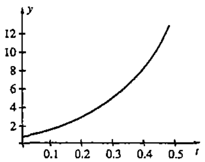
17. $y(t) = -21 + 21e^{t/3}$



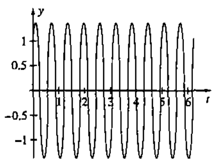

19. $y(t) = 14e^{3t} - 11e^{4t}$



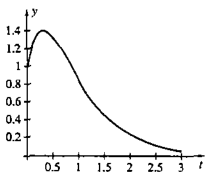
21. $y(t) = e^{5t}$



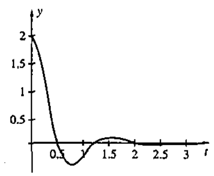

23. $y(t) = \cos 10t + \sin 10t$




25. $y(t) = e^{-2t}(1 + 5t)$




27. $y(t) = e^{-2t}(2\cos 4t + \sin 4t)$




31. $y(t) = \dfrac{(1 + t - C)\csc t}{t - C}$

33. $y = \dfrac{t[(5c_2 - c_1)\cos 5t - (5c_1 + c_2)\sin 5t]}{(c_1 \cos 5t + c_2 \sin 5t)\sin t}$

35. $y = \dfrac{(5c_1 + 4c_2)\cos 4t - (5c_2 + 4c_1)\sin 4t}{(c_1 \cos 4t + c_2 \sin 4t)t \tan 4t}$

37. (a) $y(t) = c_1 t + c_2 t^{2/3}$; (b) $y(t) = c_1 t + c_2 t \ln t$

41. (a) $y(t) = Ce^{-t}\sin 2t$; (b) no solution; (c) $y(t) = e^{-t}\cos 2t$

43. $y(t) = \dfrac{1}{2}\left((-a - b)e^{-3t} + (3a + b)e^{-t}\right)$; (a) No local extrema if $-\dfrac{1}{3}b < a \le -b$ or $-b \le a < -\dfrac{1}{3}b$;
(b) $a \ge \max\left\{-b, -\dfrac{1}{3}b\right\}, a \ne -\dfrac{1}{3}b$;
(c) $a \le \min\left\{-b, -\dfrac{1}{3}b\right\}, a \ne -\dfrac{1}{3}b$

### *Exercises 4.3*

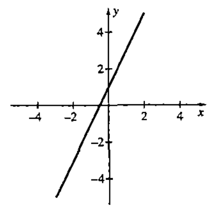
1. $W(S) = 0$ for all $t$; lin. dep.
3. $W(S) = 16e^{5t} \ne 0$ for any $t$; lin. indep.
5. $W(S) = -37e^{-9t} \ne 0$ for any $t$; lin. indep.
7. $W(S) = 1080e^{-12t} \ne 0$ for any $t$; lin. indep.
17. $W(S) = 4725$
19. $W(S) = 288e^{5t}$
21. (a) $W(\{f(t), tf(t)\}) = (f(t))^2$;
(b) $W(\{f(t), tf(t), t^2 f(t)\}) = 2(f(t))^3$;
(c) $W(\{f(t), tf(t), \ldots, t^3 f(t)\}) = 12(f(t))^4$,
$W(\{f(t), tf(t), \ldots, t^4 f(t)\}) = 288(f(t))^5, \ldots$

### *Exercises 4.4*

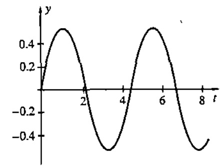
1. $y(t) = c_1 e^{-2t} + c_2 te^{-2t} + c_3 t^2 e^{-2t} + c_4 e^{2t}$, 4, $y^{(4)} + 4y''' - 16y' - 16y = 0$
3. $y(t) = c_1 + c_2 t + c_3 \cos 3t + c_4 \sin 3t$, 4, $y^{(4)} + 9y'' = 0$
5. $y(t) = e^{-3t}((c_1 + c_2 t)\cos 4t + (c_3 + c_4 t)\sin 4t) + c_5 e^{-5t} + c_6 e^{-t/3}$, 6, $3y^{(6)} + 52y^{(5)} + 455y^{(4)} + 2336y''' + 7105y'' + 11{,}500y' + 3125y = 0$
7. no: $W(S) = 0$
9. yes: $4y^{(4)} - y''' - 8y'' - 14y' + 4y = 0$
11. $y = c_1 + c_2 e^{5t} + c_3 te^{5t}$
13. $y = c_1 e^{-3t} + e^{-2t}(c_2 \cos t + c_3 \sin t)$
15. $y = c_1 e^{-t} + c_2 e^{2t} + c_3 e^{7t}$
17. $y = c_1 e^{-5t}t + c_2 e^{2t} + c_3 te^{2t}$
19. $y = c_1 e^{t} + c_2 e^{-6t} + c_3 te^{-6t}$
21. $y = c_1 + c_2 t + c_3 e^{-3t} + c_4 e^{3t}$
23. $y = c_1 e^{-2t} + c_2 e^{2t} + c_3 \cos 2t + c_4 \sin 2t$
25. $y = c_1 e^{-3t} + c_2 e^{2t} + c_3 e^{3t} + c_4 e^{4t}$
27. $y = c_1 e^{-t} + c_2 e^{2t} + c_3 e^{-4t} + c_4 te^{-4t}$
29. $y = c_1 e^{-2t} + c_2 te^{-2t} + c_3 e^{t}\cos t + c_4 e^{t}\sin t$



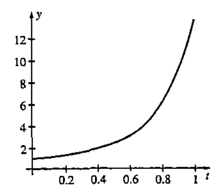
31. $y = 1 + 2t$



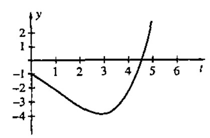

33. $y = e^{t} + e^{-t/2}(-\cos(\sqrt{3}t/2) - \sqrt{3}\sin(\sqrt{3}t/2))$
35. $y = -te^{-2t} + te^{2t}$
37. $y = 8t^3 + 4t + 8$
41. (a) $3D(t) = 3$; (b) $(1 - D)(t) = t - 1$; (c) $D^2(t^2) = 2$; (d) $(D^2 + 1)(\cos t) = 0$
45. (a) $y(t) = \dfrac{1}{22}(-2e^{-3t} + 2\cos\sqrt{2}t + 8\sqrt{2}\sin\sqrt{2}t)$




(b) $y(t) = -\dfrac{1}{6}e^{2t}(3(-2 + 7t)\cos\sqrt{3}t + \sqrt{3}(-7 + 2t)\sin\sqrt{3}t)$




(c) $y(t) = 0.000288931e^{-36.996t} + e^{0.433493t}(-1.00029\cos(0.523649t) - 1.06119\sin(0.523649t))$

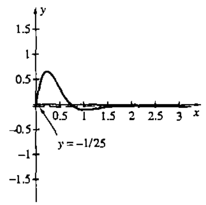



47. $y(t) = 0.0546098y_0 e^{0.471263t} + y_0 e^{-0.207654t}(0.0546098\cos(1.12621t) + 0.855013\sin(1.12621t))$

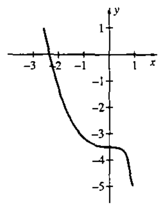
### *Exercises 4.5*

1. $S = \{t, 1\}$
3. $S_1 = \{e^{2t}\}, S_2 = \{1\}$
5. $S_1 = \{e^{-t}\}, S_2 = \{t^4, t^3, t^2, t, 1\}$
7. $S_1 = \{\sin 2t, \cos 2t\}, S_2 = \{e^{-4t}\}$
9. $S_1 = \{\cos 3t, \sin 3t\}, S_2 = \{\cos 2t, \sin 2t\}$
11. $S_1 = \{e^{-t}\cos 2t, e^{-t}\sin 2t\}, S_2 = \{1\}$
13. $y_p(t) = Ae^{2t}$
15. $y_p(t) = Ate^{3t}$
17. $y_p(t) = At^2 + Bt + C$
19. $y_p(t) = A\cos 2t + B\sin 2t$
21. $y_p(t) = At\cos 2t + Bt\sin 2t + Ct + D$
23. $y_p(t) = At^6 + Bt^5 + Ct^4 + Dt^3 + Et^2$
25. $y = c_1 e^{t} + c_2 e^{-2t} + \dfrac{1}{2}$
27. $y = c_1 e^{4t} + c_2 e^{-2t} - 4t + 1$
29. $y = c_1 e^{-t}\cos 5t + c_2 e^{-t}\sin 5t - 13t + 1$
31. $y = c_1 e^{-t/4} + c_2 e^{-t/2} + 5t^2 - 60t + 280$
33. $y = c_1 + c_2 e^{2t} - 4\sin 3t + \dfrac{8}{3}\cos 3t$
35. $y = c_1 e^{3t} + c_2 e^{-3t} - \cos 3t - 3t\sin 3t$
37. $y = c_1 e^{-2t} + c_2 te^{-2t} - 4t\cos 2t + 3\cos 2t - 4t^2\sin 2t + 4t\sin 2t$
39. $y = c_1 e^{-t} + c_2 e^{5t} - 36t^3 e^{5t} + 18t^2 e^{5t} - 6te^{5t}$
41. $y = c_1 e^{-3t} + c_2 e^{-2t} + c_3 e^{-t} + te^{-3t} + \dfrac{3}{4}te^{-t} - \dfrac{1}{4}t^2 e^{-t}$
43. $y = c_1 e^{4t} + c_2 e^{-5t}\cos t + c_3 e^{-5t}\sin t + e^{t}$
45. $y = c_1 e^{2t} + c_2 te^{2t} + c_3 e^{3t}\cos t + c_4 e^{3t}\sin t + e^{-t}$
47. $y(t) = -4 + 2e^{-t} + 2e^{t}$
49. $y(t) = \dfrac{2}{3} - 2e^{-3t} + 2e^{t}$
51. $y(t) = \dfrac{1}{4} + e^{-4t} + 4te^{-4t}$

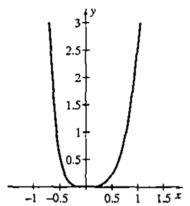
**53.** $y = \frac{7}{4}e^{-3t}\sin 4t - \frac{1}{25}$




**55.** $y = 5e^t - 7e^{2t} + 6te^{2t} - 2t^2e^{2t} + 3t + \frac{3}{2}t^2 - \frac{3}{2}$



**57.** $y = \frac{25}{6}t^3 - \frac{5}{2}t^2 + t + \frac{1}{5}e^{-5t} - \frac{1}{5}$



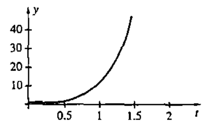
**59.** (c)

**61.** (b)

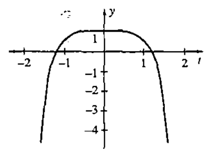
**63.** **(b)** no; **(d)** $y(t) = c_1 - \frac{a}{b}c_2e^{-tb/a} + \frac{k}{b}t$; **(e)** $y(t) = c_1 + c_2t + \frac{k}{2a}t^2$

**67.** **(a)** $\omega = 2$; **(b)** $y(t) = c_1\cos 2t + c_2\sin 2t - \frac{1}{4}t\cos 2t$

**69.** **(a)** $y(t) = -\frac{1}{132}e^{-2t} - 2e^{3t} + 2.40716e^{3.05455t} + e^{-0.0272772t}(0.6004616\cos(3.02752t) - 0.446426\sin(3.02752t))$

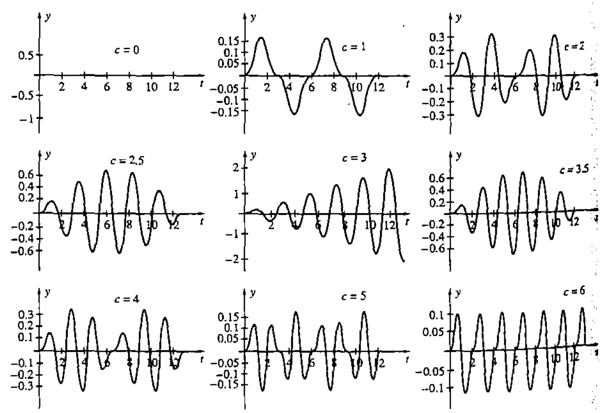


**(b)** $y(t) = \frac{1}{2} - 0.0103259e^{-4.07594t} + 0.308775e^{-0.903589t} + 0.205459e^{0.273954t} - 0.00390835e^{4.95557t} + \frac{1}{5}t$



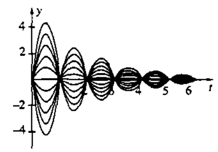
**71.** **(a)** $a_3 = 0$, $a_2 = -5$, $a_1 = 0$, $a_0 = 4$; **(b)** $a_3 = a_2 = a_1 = 0$, $a_0 = -1$; **(c)** $a_3 = 0$, $a_2 = 2$, $a_1 = 0$, $a_0 = 1$; **(d)** If $a_3 = 0$, $a_2 = 2$; $a_1 = 0$, $a_0 = 1$, then $y^{(4)} + 2y'' + y = 24(\cos t - \sin t) \neq \cos t \Rightarrow$ not possible.

**73.** (graphs for $c = 0, 1, 2, 2.5, 3, 3.5, 4, 5$, and 6)






**74.** **(a)** $y(t) = \left(\frac{1}{145} - \frac{2}{145(1 + e^{-\pi/2})}e^{-t/2}\right)\cos 3t + \left(\frac{12}{145} - ce^{-t/2}\right)\sin 3t$



**(b)** not possible

### Exercises 4.6

**1.** $y = c_1e^{2t} + c_2e^{5t} - \frac{1}{2}e^{3t}$

**3.** $y = c_1e^{-2t} + c_2te^{-2t} + \frac{1}{2}t^2e^{-2t}$

**5.** $y = c_1e^{-2t}\cos 4t + c_2e^{-2t}\sin 4t + \frac{1}{8}te^{-2t}$

**7.** $y = c_1\cos 4t + c_2\sin 4t - \frac{1}{4}t\cos 4t + \frac{1}{16}\sin 4t\ln|\sin 4t|$

**9.** $y = c_1e^{-t}\cos 7t + c_2e^{-t}\sin 7t - \frac{1}{7}te^{-t}\cos 7t + \frac{1}{49}e^{-t}\sin 7t\ln|\sin 7t|$

**11.** $y = c_1e^t\cos 5x + c_2e^t\sin 5x + e^t\cos 5t\left(\frac{1}{25}\ln|\cos 5t| - \frac{t}{5}\right) + e^t\sin 5t\left(\frac{1}{25}\ln|\sin 5t| + \frac{t}{5}\right)$

**13.** $y = c_1e^{3t}\cos 5t + c_2e^{3t}\sin 5t + e^{3t}\cos 5t\left(\frac{1}{25}\sin 5t - \frac{1}{25}\ln|\sec 5t + \tan 5t|\right) - \frac{1}{25}e^{3t}\sin 5t\cos 5t$

**15.** $y = c_1e^{6t}\cos x + c_2e^{6t}\sin t + te^{6t}\sin x + e^{6t}\cos t\ln|\cos t|$

**17.** $y = c_1e^{3t} + c_2e^{-3t} - \frac{1}{18} + \frac{1}{18}e^{3t}\ln(1 + e^{-3t}) - \frac{1}{18}e^{-3t}\ln(1 + e^{3t})$

**19.** $y = c_1e^{-t} + c_2e^t + \left(\frac{1}{4} + \frac{1}{2}t\right)e^{-t} + \left(\frac{1}{2}t - \frac{1}{4}\right)e^t$ or $y = c_1e^{-t} + c_2e^t + \frac{1}{2}te^{-t} + \frac{1}{2}te^t$

**21.** $y = c_1e^{2t} + c_2te^{2t} - e^{2t}\ln t$

**23.** $y = c_1e^{-3t} + c_2te^{-3t} + te^{-3t}\ln t$

**25.** $y = c_1e^{-2t} + c_2te^{-t} - e^{-2t}\cos(e^t)$

**27.** $y = e^t\left(c_1 + c_2t + \frac{1}{6}(t^2 + 2)\sqrt{1 - t^2} + \frac{1}{2}\sin^{-1}t\right)$

**29.** $y = \frac{1}{2}e^{2t}(c_1 + c_2t + (t^2 - 1)\tan^{-1}t - t\ln(t^2 + 1))$

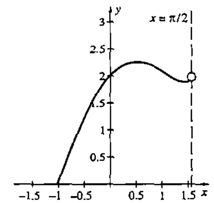
**31.** $y = (c_1 - 2t - 4\ln|\sec(t/2)|)\cos(t/2) + (c_2 + 2t - 4\ln|\csc(t/2)|)\sin(t/2)$

**33.** $y = c_1 + c_2\cos 2t + c_3\sin 2t - \frac{1}{4}t\cos 2t - \frac{1}{8}\ln|\sec 2t|\sin 2t + \frac{1}{8}\ln\left|\frac{1 + \sin 2t}{\cos 2t}\right|$

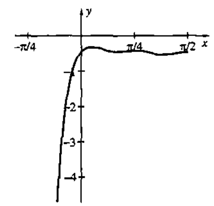
**35.** $y = c_1e^t + c_2te^t + c_3t^2e^t + \frac{1}{2}t^2e^t\ln t$

**37.** $y = c_1e^{-4t} + c_2e^{-2t} + c_3e^{3t} + \frac{1}{6}e^{-3t}$

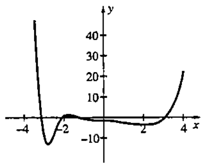
**39.** $y = -\cos t\ln(1 + \sin t) + \cos t\ln(\cos t) + 2\sin t + 2\cos t$



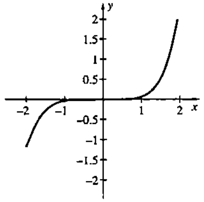

**41.** $y = -\frac{1}{2} + \frac{1}{25}\cos 4t + \frac{1}{50}\sin 4t + e^{-2t} - e^{-3t}$



43. $y=-\frac{18}{13}e^t+\frac{35}{78}te^t+\frac{1}{18}e^{-2t}\sin 3t$



45. $y=-\frac{1}{4}t^4-t^3-3t^2-6t-6+6e^t$

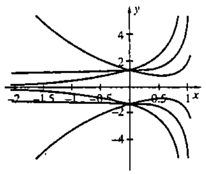


55. $y=c_1t^{-1}+c_2t^{-1}\ln t+\ln t-2$

56. $y=c_1\sin(2\ln t)+c_2\cos(2\ln t)+\frac{1}{5}t$

57. $y=c_1t^6+c_2t^{-1}+\frac{5}{18}-\frac{1}{3}\ln t$

59. $y=c_2e^{c_1t+(\frac{1}{4}-\frac{1}{2}t+\frac{1}{2}t^2)e^{2t}}$



61. (a) $y=c_1t^2\cos t+c_2t^2\sin t+t$; (b) all solutions; (c) no

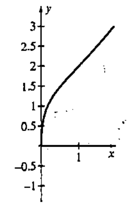
63. (a) $y=c_1\frac{\cos 2t}{\sqrt{t}}+c_2\frac{\sin 2t}{\sqrt{t}}+\frac{1}{\sqrt{t}}$

(b) no solution

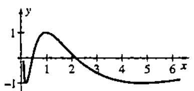
(c) $y=-\frac{\cos 2t}{\sqrt{t}}+\frac{1}{\sqrt{t}}$

### *Exercises 4.7*

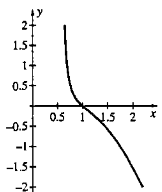
1. $y=c_1\sqrt{x}+c_2x^{5/2}$
3. $y=c_1x+c_2x^4$
5. $y=c_1\sqrt{x}\cos(2\ln x)+c_2\sqrt{x}\sin(2\ln x)$
7. $y=c_1x\cos(3\ln x)+c_2x\sin(3\ln x)$
9. $y=c_1x^2+c_2x^{-3}$
11. $y=c_1\sin(2\ln x)+c_2\cos(2\ln x)$
13. $y=c_1x^{-2}\sin(3\ln x)+c_2x^{-2}\cos(3\ln x)$
15. $y=c_1x^{-10}+c_2x^{-7}+c_3x^{-2}$
17. $y=c_1x+c_2x^2+c_3x^{-2}$
19. $y=c_1x+c_2x\sin(\ln x)+c_3x\cos(\ln x)$
21. $y=\frac{1}{9x^5}(9c_1x^3+9c_2x^3\ln x+1)$
23. $y=\frac{1}{5x^2}(1-5c_1x^2\sin(\ln x)+5c_2x^2\cos(\ln x))$
25. $y=-\frac{1}{2}x+c_1x^2+c_2x^{-3}$
27. $y=2+c_1\sin(2\ln x)+c_2\cos(2\ln x)$
29. $y=\frac{1}{25x^4}(25c_1+25c_2x^6+25c_3x^6\ln x+x)$
31. $y=\frac{9}{5}x^{1/3}+\frac{1}{5}x^2$



33. $y=\cos(2\ln x)$

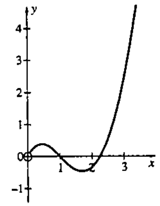


35. $y=\frac{1}{44}x^{-10}+\frac{8}{11}x-\frac{3}{4}x^2$

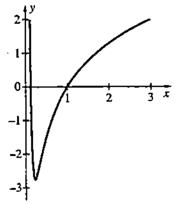




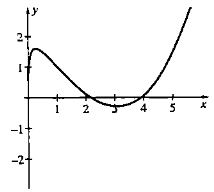
37. $y=\frac{3}{2}x-\frac{3}{2}x^2\cos(\ln x)+\frac{1}{2}x^2\sin(\ln x)$



39. $y=\frac{1}{5x^2}+\frac{22}{15}\sqrt{x}-\frac{5}{3x}$



41. $y=\frac{1}{25}x^3+\frac{24}{25}\sqrt{x}-\frac{8}{5}\sqrt{x}\ln x$



43. $W(\{x^{r_1},x^{r_2}\})=(r_2-r_1)x^{r_1+r_2-1}$

47. (a) approaches zero as $x\to 0^+$

(b) approaches zero as $x\to\infty$

(c) bounded as $x\to 0^+$ and as $x\to\infty$

51. $y=c_1+c_2x^2+c_3x^{-2}$

53. $y=c_1+c_2x^2+c_3x^{-2}+2\ln x$

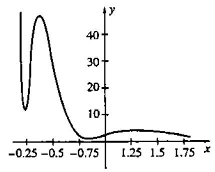
55. $y=c_1x^{-1}+c_2x^{-2}$

57. $y=-\sin(2\ln(-x))$

59. $B=1\Rightarrow y=c_1\cos(\ln x)+c_2\sin(\ln x)$ unbounded as $x\to\infty$; $B>1\Rightarrow r_{1,2}<0\Rightarrow y=c_1x^{r_1}+c_2x^{r_2}\to 0$ as $x\to\infty$; $B<1\Rightarrow r_{1,2}=\alpha\pm i\beta,\ \beta\neq 0,\ \alpha>0\Rightarrow y=x^{\alpha}(c_1\cos(\beta\ln x)+c_2\sin(\beta\ln x))$ unbounded as $x\to\infty$.

63. (a) $y=-\frac{2}{25}x^{-2}(-53+28\cos(5\ln x)-35\sin(5\ln x))$

(b) (0.241468, 11.3898) (min), (0.377358, 46.6735) (max), (0.848419, 0.922604) (min), (1.32588, 3.78057) (max)



### *Exercises 4.8*

1. $x=0,\ R\geq 1$

3. $y=c_1+c_2x+\left(-15c_1+\frac{11}{2}c_2\right)x^2+\left(-55c_1+\frac{91}{6}c_2\right)x^3+\left(-\frac{455}{4}c_1+\frac{671}{24}c_2\right)x^4+\cdots$

5. $y=c_1+c_2x+\left(c_1+\frac{1}{2}c_2+\frac{1}{2}\right)x^2+\left(\frac{1}{3}c_1+\frac{1}{2}c_2\right)x^3+\left(\frac{1}{8}+\frac{1}{4}c_1+\frac{5}{24}c_2\right)x^4+\cdots$

7. $y=c_1+c_2x-\frac{1}{4}c_2x^3+\frac{3}{16}c_2x^4-\frac{9}{80}c_2x^5+\cdots$

9. $y=1+\frac{1}{3}x^4+\frac{1}{42}x^8+\frac{1}{1386}x^{12}+\frac{1}{83160}x^{16}+\cdots$

11. $y=c_1+c_2(x-1)-\frac{c_2}{8}(x-1)^2+\frac{5c_2}{96}(x-1)^3+\cdots$

13. $y=1+\frac{1}{6}x^3-\frac{1}{30}x^5+\frac{1}{240}x^7-\frac{37}{90720}x^9+\cdots$

15. (a) $y=a_0^2+2a_0a_1x+(a_1^2+2a_0a_2)x^2+(2a_0a_3+2a_1a_2)x^3+\cdots$;

(b) $y=x+\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{8}x^5-\frac{7}{120}x^6+\frac{11}{1680}x^7+\cdots$

**17.** $x = 0$, irregular; $x = -1$, regular

**19.** $y = c_1x^{7/2}\left(1 + \frac{1}{3}x + \frac{1}{22}x^2 + \frac{1}{286}x^3 + \cdots\right)$
$+ c_2\left(1 - \frac{3}{5}x + \frac{3}{10}x^2 - \frac{3}{10}x^3 + \cdots\right)$

**21.** $y =$
$c_1x\left(1 - \frac{1}{3}x + \frac{1}{24}x^2 - \frac{1}{360}x^3 + \frac{1}{8640}x^4 - \cdots\right)$
$+ c_2x^{-1}\ln x\left(x^2 - \frac{1}{3}x^3 + \frac{1}{24}x^4 - \frac{1}{360}x^5 + \cdots\right)$
$+ c_2x^{-1}\left(-2 - 2x + \frac{4}{9}x^3 - \frac{25}{288}x^4\right.$
$\left.+ \frac{157}{21600}x^5 + \cdots\right)$

**23.** $y = c_1x^{7/4}\left(1 + \frac{7}{8}x + \frac{77}{160}x^2 + \frac{77}{384}x^3 + \cdots\right)$
$+ c_2x^{-5/4}\ln x\left(\frac{15}{8}x^3 + \frac{105}{64}x^4 + \cdots\right)$
$+ c_2x^{-5/4}\left(12 + 15x + \frac{15}{4}x^2 - \frac{13}{2}x^3 + \cdots\right)$

**25.** $y = c_1x + c_2x^{-7}$

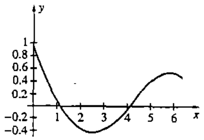
**29.** (a) $y = c_1\left(1 - x^2 - \frac{1}{6}x^4 - \frac{1}{30}x^6 + \cdots\right) + c_2x$;
(b) $y = c_1\left(1 - 3x^2 + \frac{1}{2}x^4 + \cdots\right) + c_2\left(x - \frac{2}{3}x^3\right)$

**31.** $F(1, b, b; x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$

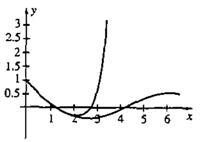
**37.** (a) $y = c_1J_{1/4}(x) + c_2Y_{1/4}(x)$;
(b) $y = c_1J_5(3x) + c_2Y_5(3x)$

**39.** $y_1(x) = F(1, 0, 1; x) = 1$; second lin. indep. soln.: $y_2(x) = x + \ln|x - 1|$

**45.** (b) $y = 1 - x + \frac{1}{6}x^3 - \frac{1}{18}x^4 + \frac{1}{360}x^5 + \frac{1}{225}x^6 - \frac{359}{226800}x^7 + \cdots$;
(c)

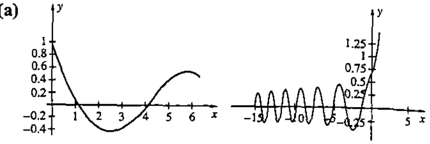


Graph of numerical approximate solution



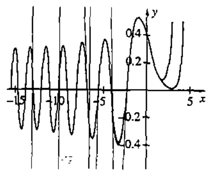
Graph of numerical approximate solution together with the graph of the polynomial approximate solution $y = 1 - x + \frac{1}{6}x^3 - \frac{1}{18}x^4 + \frac{1}{360}x^5 + \frac{1}{225}x^6 - \frac{359}{226800}x^7 + \frac{13}{158760}x^8 + \frac{3823}{38102400}x^9$

**47.** (a)

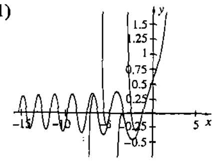


(b) $y = y(0) + y'(0)x + \frac{1}{6}y(0)x^3 + \frac{1}{12}y'(0)x^4 + \frac{1}{180}y(0)x^6 + \frac{1}{504}y'(0)x^7 + \cdots$

(c)



(d)



**49.** (c) $L_1(x) = 1 - x$, $L_2(x) = \frac{1}{2}(x^2 - 4x + 2)$,
$L_3(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6)$,
$L_4(x) = \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24)$,
$L_5(x) = \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120)$,
$L_6(x) = \frac{1}{720}(x^6 - 36x^5 + 450x^4 - 2400x^3 + 5400x^2 - 4320x + 720)$,
$L_7(x) = \frac{1}{5040}(-x^7 + 49x^6 - 882x^5 + 7350x^4 - 29400x^3 + 52920x^2 - 35280x + 5040)$,
$L_8(x) = \frac{1}{40320}(x^8 - 64x^7 + 1568x^6 - 18816x^5 + 117600x^4 - 376320x^3 + 564480x^2 - 322560x + 40320)$
(f) 1

**51.** (b) $\int_{-1}^{1} [P_n(x)]^2\,dx = \frac{2}{2n+1}$



## *Chapter 4 Review Exercises*

**1.** Lin. indep.

**3.** Lin. indep.

**5.** Lin. indep.

**11.** $y_2(t) = \frac{\cos t}{t}$

**13.** $y = c_1e^{-4t/3} + c_2e^{t/2}$

**15.** $y = c_1e^{-t} + c_2e^{-2t}$

**17.** $y = c_1e^{t/2} + c_2e^{2t}$

**19.** $y = c_1e^{-t/4} + c_2e^{t/5}$

**21.** $y = c_1 + c_2e^{-t} + c_3e^{-t/2}$

**23.** $y = e^{-2t/3}\,(c_1 \sin t + c_2 \cos t)$

**25.** $y = c_1 + c_2e^{-5t} + \frac{2t}{25} - \frac{1}{5}t^2 + \frac{1}{3}t^3$

**27.** $y = c_1e^{-t}\cos 2t + c_2e^{-t}\sin 2t - \frac{12}{17}\cos 2t + \frac{3}{17}\sin 2t$

**29.** $y = c_1 + c_2e^{2t} - \frac{1}{10}\cos 4t - \frac{1}{20}\sin 4t$

**31.** $y = c_1e^{3t}\cos 2t + c_2e^{3t}\sin 2t + \frac{3}{29}e^{-2t}$

**33.** $y = c_1e^{-4t} + c_2e^{-3t} - 6te^{-4t} - 3t^2e^{-4t} - t^3e^{-4t}$

**35.** $y = c_1e^{-2t} + c_2te^{-2t} + c_3e^{4t} + \frac{t}{36}e^{4t} + \frac{1}{12}t^2e^{-2t}$

**37.** $y = c_1e^{-t}\cos 2t + c_2e^{-t}\sin 2t + c_3te^{-t}\cos 2t + c_4te^{-t}\sin 2t + \frac{4}{625} - \frac{8t}{125} + \frac{1}{25}t^2$

**39.** $y = -\frac{2}{3}e^{-8t} + \frac{2}{3}e^{-2t}$

**41.** $y = \cos 5t$

**43.** $y = -\frac{19}{25}e^{-4t} + \frac{19}{25}e^{t} + \frac{t}{5}e^{t}$

**45.** $y = \frac{1}{2}t \sin t$

**47.** $y = c_1 \cos t + c_2 \sin t + \sin t \ln|\sin t| - x \cos t$

**49.** $y = \frac{1}{2x}e^{4x} - e^{4x} + \frac{x}{2}e^{4x}$

**51.** $y = -\frac{1}{4}e^x + xe^x + \frac{1}{2}x^2e^x \ln x - \frac{3}{4}x^2e^x$

**53.** $y = \exp(\frac{1}{2}(t^2 + 2C_2))\cos(C_1 + t)$

**56.** (a) $W(y_1, y_2) = Ce^{-\int 3dx} = Ce^{-3x}$;
$W(e^{-4x}, e^x) = \begin{vmatrix} e^{-4x} & e^x \\ -4e^{-4x} & e^x \end{vmatrix} = e^{-3x} + 4e^{-3x} = 5e^{-3x}$
(b) $W(y_1, y_2) = Ce^{-\int 4dx} = Ce^{-4x}$;
$W(e^{-2x}\cos 3x, e^{-2x}\sin 3x) = 3e^{-4x}$
(c) $W(y_1, y_2) = Ce^{-\int 4dx} = Ce^{-4x}$;
$W(e^{-2x}, xe^{-2x}) = e^{-4x}$
(d) $W(y_1, y_2) = Ce^{-\int 0dx} = C$; $W(\cos 3x, \sin 3x) = 3$

**57.** (a) $W(\{y_1, y_2\}) = 1/t$; (b) $W(\{y_1, y_2\}) = t^5$;
(c) $W(\{y_1, y_2\}) = t$

**58.** (a) $y = \frac{1}{2}e^{x-1} - e^{-1-x}$
(b) $\lambda = 0 \Rightarrow y = 0$; $\lambda < 0 \Rightarrow y = 0$; $\lambda > 0 \Rightarrow \lambda_n = (n\pi/p)^2 \Rightarrow y_n = \sin(n\pi x/p)$, $n = 1, 2, \ldots$
(c) $\lambda = 0 \Rightarrow y = 1$; $\lambda < 0 \Rightarrow y = 0$; $\lambda > 0 \Rightarrow \lambda_n = (n\pi/p)^2 \Rightarrow y_n = \cos(n\pi x/p)$, $n = 1, 2, \ldots$
(d) $\lambda = 0 \Rightarrow y = 0$; $\lambda > 0 \Rightarrow y = 0$; $\lambda < 0 \Rightarrow \lambda_n = -(n\pi/2)^2 \Rightarrow y_n = e^{-x}\sin(n\pi x/2)$, $n = 1, 2, \ldots$

**59.** $y = c_1x^2 + c_2x^3$

**61.** $y = c_1x^{-1} + c_2x^{-1/2}$, $y = -x^{-1} + 2x^{-1/2}$

**63.** $y = x^{-3}[c_1 \sin(4 \ln x) + c_2 \cos(4 \ln x)]$

**65.** $y = c_1x^3 + c_2x^5 + x$

**67.** $y = c_1 + c_2x + \left(\frac{3}{2}c_1 - c_2\right)x^2 + \left(\frac{7}{6}c_2 - c_1 + \frac{1}{6}\right)x^3$
$+ \left(\frac{7}{8}c_1 - \frac{5}{6}c_2\right)x^4$
$+ \left(\frac{1}{20} - \frac{1}{2}c_1 + \frac{61}{120}c_2\right)x^5 + \cdots$

**69.** $y = c_1x^{-8/3}\left(1 - \frac{1}{5}x + \frac{1}{20}x^2 + \frac{1}{60}x^3 + \frac{1}{960}x^4\right.$
$\left.+ \frac{1}{33600}x^5 + \cdots\right) + c_2\left(1 + \frac{1}{11}x + \frac{1}{308}x^2\right.$
$\left.+ \frac{1}{15708}x^3 + \frac{1}{1256640}x^4 + \frac{1}{144513600}x^5 + \cdots\right)$

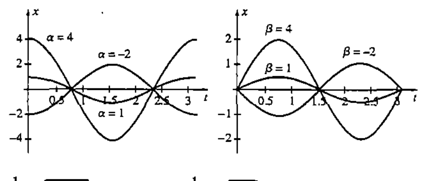
**71.** $y = c_1x^7\left(1 + \frac{1}{8}x^2 + \frac{1}{160}x^4 + \frac{1}{5760}x^6 + \frac{1}{322560}x^8\right.$
$\left.+ \cdots\right) + c_2x \ln x\left(-1800x^6 - 225x^8 - \frac{45}{4}x^{10}\right.$
$\left.- \frac{5}{16}x^{12} - \frac{5}{896}x^{14} + \cdots\right) + c_2x\left(-86400 + 21600x^2\right.$
$\left.- 5400x^4 + \frac{1125}{8}x^8 + \frac{351}{32}x^{10} + \cdots\right)$

**73.** $y = c_1F\left(-\frac{1}{2}(13 + \sqrt{209}), -\frac{1}{2}(13 - \sqrt{209}), \frac{1}{3}; -x\right)$
$+ c_2x^{2/3}F\left(-\frac{1}{6}(35 + 3\sqrt{209}), -\frac{1}{6}(35 - 3\sqrt{209}),\right.$
$\left.\frac{5}{3}; -x\right)$

# CHAPTER 5

### *Exercises 5.1*

1. $m = 4$ slugs, $k = 9$ lb/ft; released 1 ft above eq. with zero init. vel.
3. $m = 1/4$ slugs, $k = 16$ lb/ft; released 0.75 ft (8 in.) below eq. with an upward initial vel. of 2 ft/sec
5. $x(t) = 5\cos(t - \phi)$, $\phi = -\cos^{-1}(3/5) \approx 0.93$ rads; per. $= 2\pi$, amp $= 5$
7. $x(t) = \dfrac{\sqrt{65}}{4}\cos(4t - \phi)$, $\phi = \cos^{-1}\left(-\dfrac{8}{\sqrt{65}}\right) \approx$ 3.02 rads; per. $= \pi/2$; amp $= \sqrt{65}/4$
9. $x(t) = \dfrac{\sqrt{2}}{3}\cos(3t - \phi)$, $\phi = -\cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right) =$ $-\pi/4$ rads; per. $= 2\pi/3$; amp $= \sqrt{2}/3$
11. $x(t) = \cos 8t$, max. dis. $= 1$ ft when $-8\sin 8t = 0$ or $8t = n\pi \Rightarrow t = n\pi/8$, $n = 0, 1, 2, \ldots$
13. $x(t) = -1/4\cos 8t$; $t = \pi/16$ sec; $x(5) \approx 0.167$ ft; $x(t) = \dfrac{1}{8}\sin 8t$; $t = \dfrac{\pi}{8}$ sec
15. As $k$ increases, the frequency at which the spring–mass system passes through equilibrium increases.
17. $b = \sqrt{1023}/16$
19. $m = 2$ slugs
21. $\omega\sqrt{\alpha^2 + \beta^2/\omega^2}$
23. $\dfrac{d^2y}{dt^2} + \dfrac{\pi r^2\rho}{m}y = 0$
25. $y(t) = 1.61\cos 3.5t - 0.856379\sin 3.5t$; max. displacement $= \sqrt{(1.61)^2 + (0.856379)^2} \approx 1.8236$ ft
27. $\alpha = 1$: $x = \cos 2t$; $\alpha = 4$: $x = 4\cos 2t$; $\alpha = -2$: $x = -2\cos 2t$; $\beta = 1$: $x = \dfrac{1}{2}\sin 2t$; $\beta = 4$: $x = 2\sin 2t$; $\beta = -2$: $x = -\sin 2t$



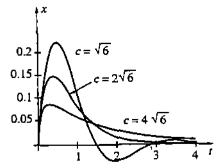

29. $\dfrac{1}{6}\sqrt{137/2} \approx 1.37941$, $\dfrac{1}{12}\sqrt{199} \approx 1.17556$
31. $\dfrac{1}{4}\sqrt{199/15} \approx 0.910586$

### *Exercises 5.2*

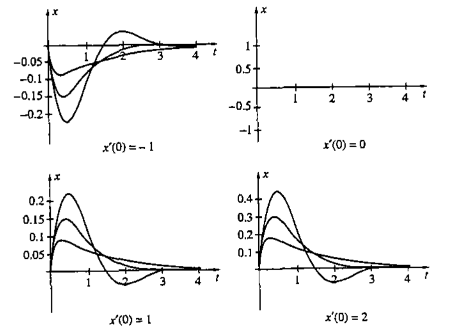
1. $m = 1$, $c = 4$, $k = 3$; released from equilibrium with an upward init. vel. of 4 ft/sec
3. $m = 1/4$, $c = 2$, $k = 1$; released 6 inches above equil. with a downward init. vel. of 1 ft/sec
5. $x(t) = \dfrac{\sqrt{10}}{3}e^{-2t}\cos(3t - \phi)$, $\phi = \cos^{-1}(3/\sqrt{10}) \approx$ 0.32 rads, Q.P.: $2\pi/3$, $t \approx 0.63$
7. $x(t) = \dfrac{\sqrt{29}}{5}e^{-t}\cos(5t - \phi)$, $\phi = \cos^{-1}(5/\sqrt{29}) \approx$ 0.38 rads, Q.P.: $2\pi/5$, $t \approx 0.39$
9. $x(t) = -\dfrac{1}{2}e^{-5t} + \dfrac{1}{2}e^{-3t}$; overdamped; does not pass through equilibrium; $x\left(\dfrac{1}{2}\ln\left(\dfrac{5}{3}\right)\right) \approx 0.093$
11. $x(t) = -3e^{-t} + 2e^{-t/2}$; overdamped; $x(t) = 0 \Rightarrow t = 2\ln\left(\dfrac{3}{2}\right) \approx 0.811$; max. dis. $= 1$ at $t = 0$
13. $x(t) = 4e^{-4t} + 14te^{-4t}$; critically damped; does not pass through equilibrium; max. dis. $= 4$ at $t = 0$
15. $x(t) = -5e^{-5t} - 24te^{-5t}$; critically damped; does not pass through equilibrium; max. dis. $= 5$ at $t = 0$
17. $c = 2$
19. $x(t) = -\dfrac{3}{2}e^{-4t} + e^{-6t}$; does not pass through equilibrium; max. dis. $= \frac{1}{2}$ at $t = 0$
21. $x(t) = e^{-2t}\left(-3\cos\left(4\sqrt{\dfrac{11}{5}}t\right) - \dfrac{3}{2}\sqrt{\dfrac{5}{11}}\sin\left(4\sqrt{\dfrac{11}{5}}t\right)\right)$ or $x(t) = \dfrac{21}{2\sqrt{11}}\cos\left(4\sqrt{\dfrac{11}{5}}t - \phi\right)$, where $\phi = \cos^{-1}\dfrac{-2\sqrt{11}}{7} \approx 2.81646$; $x(t) = 0 \Rightarrow$ $t = \dfrac{1}{4}\sqrt{\dfrac{5}{11}}\left(\dfrac{(2n+1)\pi}{2} + \phi\right)$, $n$ any integer or $t \approx 0.739471, 1.26899, 1.7985, 2.32802, 2.85753\ldots$



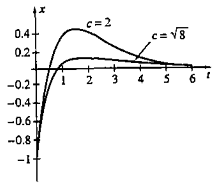
23. $c = 32$, $0 < c < 32$
25. $c = 10/13$
27. $m\dfrac{d^2}{dt^2}(u + v) + c\dfrac{d}{dt}(u + v) + k(u + v) = 0 \Rightarrow$ $m\dfrac{d^2u}{dt^2} + c\dfrac{du}{dt} + ku = 0$ and $m\dfrac{d^2v}{dt^2} + c\dfrac{dv}{dt} + kv = 0$; $x(0) = u(0) + v(0) = \alpha \Rightarrow u(0) = \alpha$ and $v(0) = 0$; $\dfrac{dx}{dt}(0) = \dfrac{du}{dt}(0) + \dfrac{dv}{dt}(0) = \beta \Rightarrow \dfrac{du}{dt}(0) = 0$ and $\dfrac{dv}{dt}(0) = \beta$
29. $x(t) = c_1e^{-pt} + c_2te^{-pt}$, $\rho = c/2m$; $x(0) = \alpha$, $x'(0) = 0 \Rightarrow x(t) = \alpha e^{-\rho t}(1 + \rho t)$; $x(t) = 0 \Rightarrow t = -1/\rho < 0$
35. $c = 2\sqrt{6}$: $x = te^{-t\sqrt{6}}$; $c = 4\sqrt{6}$: $x = \dfrac{1}{6\sqrt{2}}e^{-\sqrt{2}(3+3\sqrt{3})t}(-1 + e^{6t\sqrt{2}})$; $c = \sqrt{6}$: $x = \dfrac{1}{3}\sqrt{2}e^{-t\sqrt{3/2}}\sin\left(\dfrac{3}{\sqrt{2}}t\right)$




37. The object does not come into contact with the floor in any of the cases.




39. $c = 2$: $x = e^{-t}(-1 + 2t)$; $c = \sqrt{8}$: $x = (-2 + 1/\sqrt{2})e^{-(1+\sqrt{2})t} + (1 - 1/\sqrt{2})e^{(1-\sqrt{2})t}$

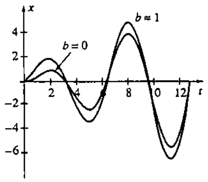



### *Exercises 5.3*

1. $x(t) = -\dfrac{2}{7}\cos 4t + \dfrac{1}{2}\sin 4t + \dfrac{2}{7}\cos 3t$, $\omega = 4$
3. $x(t) = \dfrac{35}{141}\cos(4\sqrt{3}t) + \dfrac{4}{47}\cos t$
5. $x(t) = \begin{cases} \dfrac{4}{\omega^2 - 9}(\cos 3t - \cos\omega t), & \omega \neq 3 \\ \dfrac{2}{3}t\sin 3t, & \omega = 3 \end{cases}$; resonance occurs if $\omega = 3$
7. $x(t) = -e^{-4t}\left(\dfrac{1}{20}\cos 3t + \dfrac{7}{120}\sin 3t\right) + \dfrac{1}{40}(2\cos t - \sin t)$
9. $x(t) = \dfrac{1}{3}\cos t - \dfrac{1}{3}\cos 2t$; $\pm\dfrac{2}{3}\sin\dfrac{t}{2}$; decreases
11. $x(t) = e^{-t/2}\left(2\cos\dfrac{5t}{2} - 4\sin\dfrac{5t}{2}\right) - 2\cos t + 11\sin t$; trans: $e^{-t/2}\left(2\cos\dfrac{5t}{2} - 4\sin\dfrac{5t}{2}\right)$; steady-state: $-2\cos t + 11\sin t$
13. **(a)** $x(t) = \alpha\cos\left(\sqrt{\dfrac{k}{m}}t\right) - \dfrac{F\omega}{k - m\omega^2}\sqrt{\dfrac{m}{k}}\sin\left(\sqrt{\dfrac{k}{m}}t\right) + \dfrac{F}{k - m\omega^2}\sin\omega t$
    **(b)** $x(t) = \left[\beta - \dfrac{F\omega}{k - m\omega^2}\right]\sqrt{\dfrac{m}{k}}\sin\left(\sqrt{\dfrac{k}{m}}t\right) + \dfrac{F}{k - m\omega^2}\sin\omega t$
    **(c)** $x(t) = \alpha\cos\left(\sqrt{\dfrac{k}{m}}t\right) + \left[\beta - \dfrac{F\omega}{k - m\omega^2}\right]\sqrt{\dfrac{m}{k}}\sin\left(\sqrt{\dfrac{k}{m}}t\right) + \dfrac{F}{k - m\omega^2}\sin\omega t$
15. $x(t) = \begin{cases} 1 - \cos t, & 0 \le t \le \pi \\ -2\cos t, & t > \pi \end{cases}$

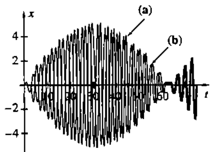
**17.** $x(t) = \begin{cases} t - \sin t, & 0 \le t \le 1 \\ -2\sin 1 \cos t + (2\cos 1 - 1)\sin t + 2 - t, & 1 < t \le 2 \\ (\sin 2 - 2\sin 1)\cos t + 4\cos 1 \sin^2 \frac{1}{2} \sin t, & t > 2 \end{cases}$

**21.** $b = 0$: $x = \frac{1}{2} t \sin t$; $b = 1$: $x = \frac{1}{2}(2\sin t + t \sin t)$

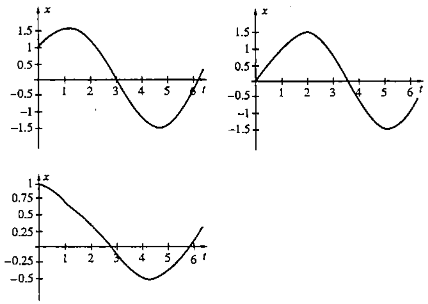



**23.** **(a)** $x(t) = \frac{100}{39}(\cos(19t/10) - \cos 2t)$;

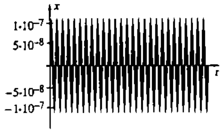
**(b)** $x(t) = \frac{100}{41}(\cos 2t - \cos(21t/10))$




**25.** $x(t) = \begin{cases} t - \sin t + a\cos t + b\sin t, & 0 \le t \le 1 \\ -t - \sin t + 2\sin(t-1) + 2 + a\cos t + b\sin t, & 1 < t \le 2 \\ -\sin t + 2\sin(t-1) - \sin(t-2) + a\cos t + b\sin t, & t > 1 \end{cases}$



**27.** $x(t) = -\frac{5000}{89733775561}(\cos(299581t/50) - \cos(20\sqrt{15}t))$



## Exercises 5.4

**1.** $Q(t) = \frac{55}{8} - \frac{55}{8}\cos 4t$

**3.** $Q(t) = \frac{t}{4} - \frac{1}{64}\sin 16t$

**5.** $Q(t) = \frac{8\sqrt{7}}{175} e^{-(125/2)t} \sin\left(\frac{25\sqrt{7}}{2}t\right)$; max.: 0.0225 at $t \approx 0.0147$

**7.** $Q(t) = e^{-125t/2}\left(-\frac{12185}{12502813}\cos\left(\frac{25\sqrt{7}}{2}t\right) - \frac{26554371}{312570325\sqrt{7}}\sin\left(\frac{25\sqrt{7}}{2}t\right)\right) + \frac{12185}{12502813}\cos t + \frac{62526875}{12502813}\sin t$. steady-state charge: $\lim_{t\to\infty} Q(t) = \frac{12185}{12502813}\cos t + \frac{62526875}{12502813}\sin t$; steady-state current: $-\frac{12185}{12502813}\sin t + \frac{62526875}{12502813}\cos t$

**9.** **(a)** $s(x) = \frac{1}{300}x^4 + \frac{1}{3}x^2 - \frac{1}{15}x^3$; **(b)** $s(x) = \frac{1}{30}x^4 + \frac{10}{3}x^2 - \frac{2}{3}x^3$; **(c)** $s(x) = \frac{1}{3}x^4 + \frac{100}{3}x^2 - \frac{20}{3}x^3$

**11.** **(a)** $s(x) = \frac{1}{300}x^4 + \frac{10}{3}x - \frac{1}{15}x^3$;

**(b)** $s(x) = \frac{1}{30}x^4 + \frac{100}{3}x - \frac{2}{3}x^3$;

**(c)** $s(x) = \frac{1}{3}x^4 + \frac{1000}{3}x - \frac{20}{3}x^3$; simple support leads to larger max. displacement.

**13.** **(a)** $s(x) = \frac{1}{360}x^6 + \frac{10000}{9}x - \frac{125}{9}x^3$;

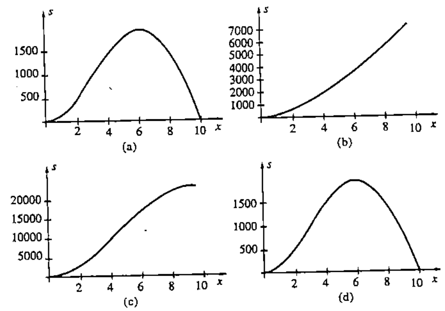
**(d)** $s(x) = \frac{1}{360}x^6 + \frac{1250}{3}x - \frac{125}{18}x^3$



**15.** $Q(t) = Q_0 \cos\frac{t}{\sqrt{LC}}$; $I(t) = -\frac{Q_0}{\sqrt{LC}}\sin\frac{t}{\sqrt{LC}}$; max. $Q = Q_0$; max. $I = \frac{Q_0}{\sqrt{LC}}$

**17.** $$\frac{E_0}{\sqrt{\left(L\omega - \frac{1}{C\omega}\right)^2 + R^2}}\left[\frac{R\sin\omega t}{\sqrt{\left(L\omega - \frac{1}{C\omega}\right)^2 + R^2}} - \frac{\left(L\omega - \frac{1}{C\omega}\right)\cos\omega t}{\sqrt{\left(L\omega - \frac{1}{C\omega}\right)^2 + R^2}}\right]$$

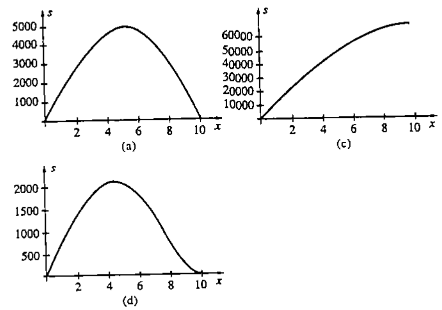
**21.** **(a)** $s(x) = \frac{1}{360}x^6 - \frac{175}{9}x^3 + \frac{500}{3}x^2$; **(b)** $s(x) = \frac{1}{360}x^6 - \frac{500}{9}x^3 + 1250x^2$; **(c)** $s(x) = \frac{1}{360}x^6 - \frac{500}{9}x^3 + 750x^2$; **(d)** $s(x) = \frac{1}{360}x^6 - \frac{175}{9}x^3 + \frac{500}{3}x^2$




**23.** **(a)** $s(x) = 480000\pi^{-4}\sin\left(\frac{\pi x}{10}\right)$; **(b)** no solution;

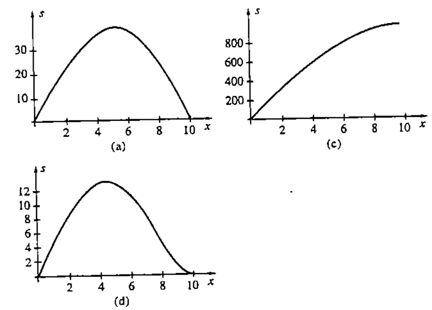
**(c)** $s(x) = -80\pi^{-4}\left(-600\pi x - 300\pi^3 x + \pi^3 x^3 - 6000\sin\left(\frac{\pi x}{10}\right)\right)$;

**(d)** $s(x) = 240\pi^{-4}\left(-100\pi x + \pi x^3 + 2000\sin\left(\frac{\pi x}{10}\right)\right)$




**25.** **(a)** $s(x) = \frac{1}{36000}x^6 - \frac{5}{36}x^3 + \frac{100}{9}x$; **(b)** no solution;

**(c)** $s(x) = \frac{1}{36000}x^6 - \frac{5}{9}x^3 + 150x$; **(d)** $s(x) = \frac{1}{36000}x^6 - \frac{5}{72}x^3 + \frac{25}{6}x$




## Exercises 5.5

**1.** **(a)** $\theta(t) = \frac{1}{20}\cos 4t$; **(b)** $\theta(t) = \frac{1}{20}\cos 4t + \frac{1}{4}\sin 4t$;

**(c)** $\theta(t) = \frac{1}{20}\cos 4t - \frac{1}{4}\sin 4t$; max. displacement: **(a)** 1/20; **(b)** and **(c)**: $\sqrt{26}/20 \approx 0.255$

**3.** **(a)** $\theta(t) = \frac{\sqrt{7}}{60}e^{-t\sqrt{7}}\sin 3t + \frac{1}{20}e^{-t\sqrt{7}}\cos 3t$;

**(b)** $\theta(t) = \left(\frac{\sqrt{7}}{60} + \frac{1}{3}\right)e^{-t\sqrt{7}}\sin 3t + \frac{1}{20}e^{-t\sqrt{7}}\cos 3t$;

**(c)** $\theta(t) = \left(\frac{\sqrt{7}}{60} - \frac{1}{3}\right)e^{-t\sqrt{7}}\sin 3t + \frac{1}{20}e^{-t\sqrt{7}}\cos 3t$

**9.** $T = 2\pi\sqrt{2/9.8} \approx 2.83$ sec

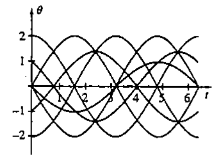
**11.** $T = 2\pi\sqrt{8/32} \approx 3.14$ sec

**13.** $2\pi\sqrt{L/9.8} = 1 \Rightarrow L = 0.248$ m

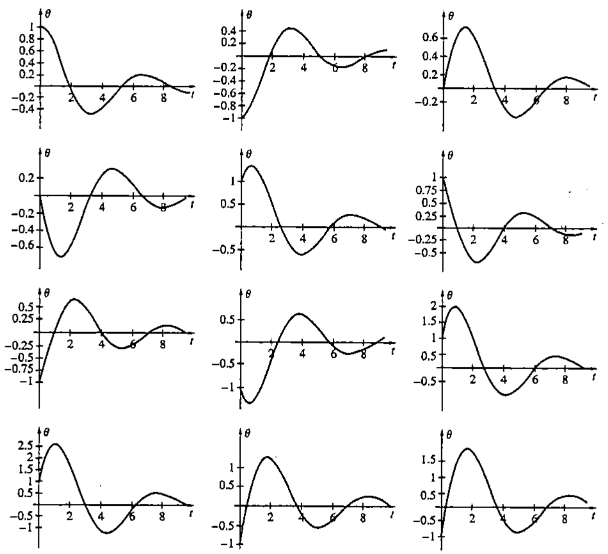
**17.** $b^2 - 4Lg > 0$, overdamped; $b^2 - 4Lg = 0$, critically damped; $b^2 - 4Lg < 0$, underdamped

**19.** (c) yes; (d) no

**21.** (a) $\theta(t) = 2 \sin t$; (b) $\theta(t) = 2 \cos t$; (c) $\theta(t) = -2 \cos t$; (d) $\theta(t) = -\sin t$; (e) $\theta(t) = -2 \sin t$; (f) $\theta(t) = \cos t - \sin t$; (g) $\theta(t) = -\cos t + \sin t$

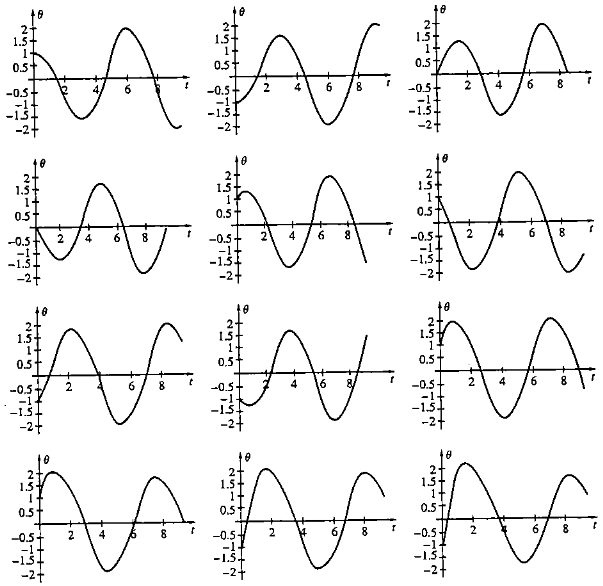


**23.**

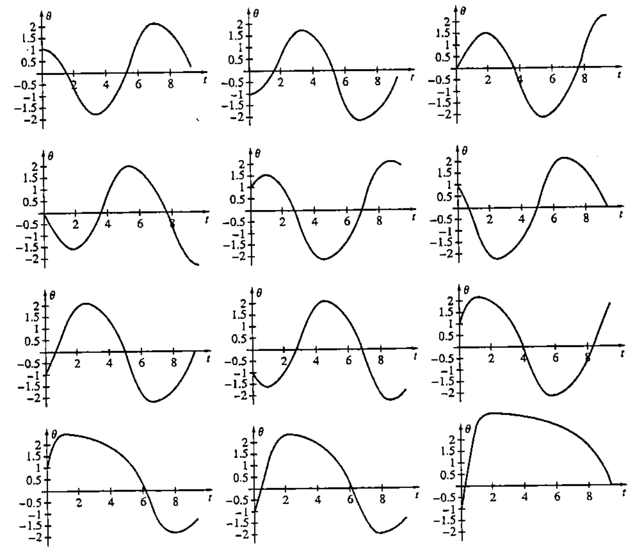




**25.**



**26.**



### Chapter 5 Review Exercises

1. $x(t) = \frac{1}{3}\cos 8t$; max. dis. $= \frac{1}{3}$; $t = \frac{\pi}{16}, \frac{\pi}{8}$

3. $x(t) = -\frac{1}{3}e^{-2t}\sin 3t$, $\lim_{t\to\infty} x(t) = 0$; quasiper. $= \frac{2\pi}{3}$; max. dis. $= \left|x\left(\frac{1}{3}\tan^{-1}\frac{3}{2}\right)\right| \approx 0.144$; $t = \frac{\pi}{3}$

5. $x(t) = \frac{1}{4} - \frac{1}{4}\cos 2t$; maximum displacement is $\frac{1}{2}$ and occurs when $t = \frac{\pi}{2}$.

7. $x(t) = \frac{1}{3}\cos t - \frac{1}{3}\cos 2t$; beats; $\pm\frac{2}{3}\sin\frac{t}{2}$

9. $Q(t) = \frac{25}{109} - \frac{25}{109}e^{-10t}\cos 3t - \frac{250}{327}e^{-10t}\sin 3t$; $I(t) = \frac{25}{3}e^{-10t}\sin 3t$; $\lim_{t\to\infty} Q(t) = \frac{25}{109}$; $\lim_{t\to\infty} I(t) = 0$

11. $Q(t) = \frac{11}{500} - \frac{11}{500}\cos 100t$; $I(t) = \frac{11}{5}\sin 100t$; $\lim_{t\to\infty} Q(t)$ and $\lim_{t\to\infty} I(t)$ do not exist

13. $s(x) = \frac{250}{3}x^2 - \frac{125}{9}x^3 + \frac{1}{12}x^5 - \frac{1}{360}x^6$

15. $s(x) = \frac{1250}{3}x^2 - \frac{250}{9}x^3 + \frac{1}{12}x^5 - \frac{1}{360}x^6$

17. $\theta(t) = \cos 8t$; max. dis. $= 1$; $t = \frac{\pi}{16}$

19. $\theta(t) = e^{-8t} + 8te^{-8t}$; motion is not periodic.

21. $\theta(t) = 0.2168\cos 3.7t + 0.0471622\sin 3.7t$

23. $y(t) = \begin{cases} 3 - t^2 - 3\cos t, & 0 \le t \le 1 \\ (-3 + 2\cos 1 + 2\sin 1)\cos t - \\ \quad (2\cos 1 - 2\sin 1 + \sin 2)\sin t, & t > 1 \end{cases}$

25. $x(t) = 25\cos\left(\frac{7\sqrt{5}}{25}t\right)$

27. $x(t) = A\sin\omega_n t + B\cos\omega_n t + \frac{F}{k}$; $x(t) = \left(x_0 - \frac{F}{k}\right)\cos\omega_n t + \frac{F}{k}$

# CHAPTER 6

### Exercises 6.1

1. $\{x(t) = \frac{1}{2}t^2 + C_1, y(t) = \sin t + C_2\}$

3. $\{x(t) = C_1, y(t) = C_2e^{-2t}\}$

5. $\{x_1(t) = -e^{-3t}, x_2(t) = 1\}$

7. $x'' + 4x' - 21x = 0$, $\left\{x(t) = C_1e^{-7t} + C_2e^{3t}, y(t) = -\frac{2}{3}C_1e^{-7t} + C_2e^{3t}\right\}$

9. $x'' + x = 0$, $\left\{x(t) = C_1\cos t + C_2\sin t, y(t) = -\frac{1}{2}(C_1 + C_2)\cos t + \frac{1}{2}(C_1 - C_2)\sin t\right\}$

11. $x'' + x = 1$, $\{x(t) = C_1\cos t + C_2\sin t + 1, y(t) = -C_1\sin t + C_2\cos t\}$

13. $\begin{cases} x' = y \\ y' = 3y - 4x \end{cases}$

15. $\begin{cases} x' = y \\ y' = -16x + t\sin t \end{cases}$

17. $\begin{cases} y' = z \\ z' = w \\ w' = x - 3y - 6z - 3w \end{cases}$

19. $\begin{cases} x' = y \\ y' = -x - \mu\left[\frac{1}{3}y^2 - 1\right]y \end{cases}$

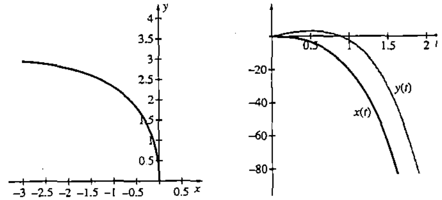
21. $\begin{cases} x(t) = \frac{1}{3} + c_1e^{3t} + c_2e^{-4t} \\ y(t) = \frac{5}{3} - \frac{5}{2}c_1e^{3t} + c_2e^{-4t} \end{cases}$

23. $\begin{cases} x(t) = \frac{2}{9} - \frac{1}{4}e^t + c_1e^{-3t} + 2tc_2e^{-3t} \\ y(t) = \frac{1}{3} + c_2e^{-3t} \end{cases}$



27. $\begin{cases} x(t) = -\frac{3}{10}\cos t - \frac{1}{10}\sin t - 2c_1e^{2t} + \\ \qquad (3c_3 - 2c_2)e^{-t} - 2c_3te^{-t} \\ y(t) = \frac{1}{2}\sin t + (3c_2 - 4c_3)e^{-t} + 3c_3te^{-t} \\ z(t) = \frac{2}{5}\cos t + \frac{3}{10}\sin t + c_1e^{2t} + c_2e^{-t} + c_3te^{-t} \end{cases}$

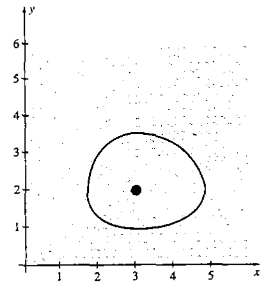
29. $\begin{cases} x(t) = -20 + 20e^t - 20te^t \\ y(t) = -30 + 30e^t - 20te^t \end{cases}$




35. $k = -4$

37. $c \neq 4$

39. (a) $\begin{cases} x_1' = x_2 \\ x_2' = -3x_1 + y_1 \\ y_1' = y_2 \\ y_2' = -x_1 - 2y_1 \end{cases}$; (b) $\begin{cases} x_1' = x_2 \\ x_2' = -x_1 + 2y_1 + \cos 3t \\ y_1' = y_2 \\ y_2' = -2x_1 + y_1 \end{cases}$

41. 

### Exercises 6.2

1. $\begin{pmatrix} -3 & 5 \\ 3 & 2 \end{pmatrix}$

3. $\begin{pmatrix} -5 & 21 \\ 1 & -3 \end{pmatrix}$

5. $\begin{pmatrix} 12 & 15 & 5 \\ 50 & 8 & 31 \\ 26 & -22 & -15 \end{pmatrix}$

7. $\begin{pmatrix} -8 & 30 & 16 \\ -30 & -18 & 0 \\ -8 & 30 & -2 \end{pmatrix}$

9. $\mathbf{AB} = \begin{pmatrix} 22 & -5 & -13 \\ 22 & -24 & -17 \\ -31 & -4 & 37 \end{pmatrix}$ and $\mathbf{BA} = \begin{pmatrix} 1 & 9 & 16 & -35 \\ -25 & 20 & 20 & 5 \\ 0 & -9 & -8 & -28 \\ -27 & -11 & -22 & 22 \end{pmatrix}$

11. $\mathbf{AB} = \begin{pmatrix} -1 & -11 \\ -2 & -28 \end{pmatrix}$ and $\mathbf{BA} = \begin{pmatrix} -2 & -3 \\ -16 & -27 \end{pmatrix}$

13. $\mathbf{AB} = \begin{pmatrix} -1 & -12 & -4 \\ 9 & 22 & -3 \\ -10 & -36 & -9 \end{pmatrix}$ and $\mathbf{BA} = \begin{pmatrix} -1 & -4 & 11 \\ -22 & 25 & -6 \\ -6 & 14 & -12 \end{pmatrix}$

15. $|\mathbf{A}| = 17$

17. $|\mathbf{A}| = 0$

19. $|\mathbf{A} = 10$

21. $\begin{pmatrix} 1 & \frac{1}{2} \\ -1 & 0 \end{pmatrix}$

23. $\begin{pmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$

24. $\lambda_1 = -5$, $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$; $\lambda_2 = -3$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

25. $\lambda_1 = -5$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda_2 = -4$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

27. $\lambda_1 = \lambda_2 = -3$; $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

29. $\lambda_1 = \lambda_2 = -5$; $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

**31.** $\lambda_1 = -2 + 2i$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 - 2i \end{pmatrix}$; $\lambda_2 = -2 - 2i$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 + 2i \end{pmatrix}$

**33.** $\lambda_1 = -1$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$; $\lambda_2 = -2$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -\frac{3}{2} \\ -1 \end{pmatrix}$; $\lambda_3 = 3$, $\mathbf{v}_3 = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$

**35.** $\begin{pmatrix} 2e^{2t} \\ -5e^{-5t} \end{pmatrix}$; $\begin{pmatrix} \dfrac{e^{2t}}{2} + c_1 \\ \dfrac{e^{-5t}}{-5} + c_2 \end{pmatrix}$.

**37.** $\begin{pmatrix} -\sin t & \cos t - t \sin t \\ \cos t & \sin t + t \cos t \end{pmatrix}$; $\begin{pmatrix} \sin t & \cos t + t \sin t \\ -\cos t & \sin t - t \cos t \end{pmatrix} + \mathbf{C}$

**39.** $\begin{pmatrix} 4e^{4t} \\ -3 \sin 3t \\ 3 \cos 3t \end{pmatrix}$; $\begin{pmatrix} \frac{1}{4}e^{4t} + c_1 \\ \frac{1}{3} \sin 3t + c_2 \\ -\frac{1}{3} \cos 3t + c_3 \end{pmatrix}$

**43.** **(a)** $\lambda_{1,2} = -1$, $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\lambda_3 = 0$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 6 \\ 5 \end{pmatrix}$; **(b)** $\lambda_1 = 3$, $\mathbf{v}_1 = \begin{pmatrix} -10 \\ 25 \\ 11 \end{pmatrix}$, $\lambda_{2,3} = 3 \pm 6i$, $\mathbf{v}_{2,3} = \begin{pmatrix} 2 \mp 6i \\ -5 \\ 5 \end{pmatrix}$

**45.** **(a)** $\lambda_{1,2} = 1 \pm ki$; **(b)** $\lambda_{1,2} = k \pm i$

### Exercises 6.3

**1.** $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

**3.** $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

**5.** $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e^t \\ 0 \end{pmatrix}$

**7.** $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \sin t \end{pmatrix}$

**9.** linearly independent

**11.** linearly independent

**13.** linearly independent

**15.** yes

**17.** yes

**19.** no

**21.** $\mathbf{X}(t) = \begin{pmatrix} 2e^{4t} & e^{-t} \\ 3e^{4t} & -e^{-t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, $\begin{cases} x = 2e^{4t} + e^{-t} \\ y = 3e^{4t} - e^{-t} \end{cases}$

**23.** $\mathbf{X}(t) = \begin{pmatrix} 2 \cos 2t + 2 \sin 2t & 2 \cos 2t - 2 \sin 2t \\ -\sin 2t & -\cos 2t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, $\begin{cases} x = -8 \cos 2t \\ y = 2 \cos 2t + 2 \sin 2t \end{cases}$

### Exercises 6.4

**1.** $\mathbf{X}(t) = c_1\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t}$

**3.** $\mathbf{X}(t) = c_1\begin{pmatrix} 6 \\ 1 \end{pmatrix}e^{-4t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{2t}$

**5.** $\mathbf{X}(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t} + c_2\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]e^{2t}$

**7.** $\mathbf{X}(t) = c_1\begin{pmatrix} e^t(-\cos 2t + \sin 2t) \\ e^t \cos 2t \end{pmatrix} + c_2\begin{pmatrix} e^t(-\cos 2t - \sin 2t) \\ e^t \sin 2t \end{pmatrix}$

**9.** $\mathbf{X}(t) = c_1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}e^{2t} + c_2\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right]e^{2t} + c_3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}e^{3t}$

**11.** $\mathbf{X}(t) = c_1\begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} + c_2\begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}e^{2t}$

**13.** $\begin{cases} x(t) = c_1e^{15t} + 2c_2e^{-4t} \\ y(t) = -\dfrac{7}{5}c_1e^{15t} + c_2e^{-4t} \end{cases}$

**15.** $\begin{cases} x(t) = c_1e^{t} + c_2e^{5t} \\ y(t) = 5c_1e^{t} + c_2e^{5t} \end{cases}$



**17.** $\begin{cases} x(t) = 3c_1e^{7t} \\ y(t) = c_1e^{7t} + c_2e^{-8t} \end{cases}$

**19.** $\begin{cases} x(t) = -c_1e^{-11t} + 3c_2e^{-4t} \\ y(t) = 2c_1e^{-11t} + c_2e^{-4t} \end{cases}$

**21.** $\begin{cases} x(t) = c_1e^{-12t} - 2c_2e^{-4t} \\ y(t) = \dfrac{3}{2}c_1e^{-12t} + c_2e^{-4t} \end{cases}$

**23.** $\begin{cases} x(t) = c_1e^{-8t} + (2c_2 + 2c_1)te^{-8t} \\ y(t) = c_2e^{-8t} + (-2c_2 - 2c_1)te^{-8t} \end{cases}$

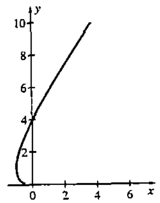
**25.** $\begin{cases} x(t) = 2c_1 \cos 4t + 2c_2 \sin 4t \\ y(t) = -c_1 \sin 4t + c_2 \cos 4t \end{cases}$

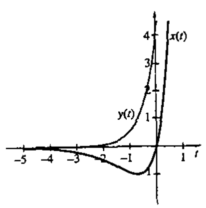
**27.** $\begin{cases} x(t) = c_1e^{4t} + c_2e^{-t} \\ y(t) = c_3e^{-2t} \\ z(t) = -5c_2e^{-t} \end{cases}$

**29.** $\begin{cases} x(t) = -\dfrac{7}{4}c_2e^{3t} + c_1e^{-t} - c_3te^{-t} \\ y(t) = c_2e^{3t} + c_3e^{-t} \\ z(t) = \dfrac{3}{2}c_2e^{3t} - 2c_1e^{-t} + 2c_3te^{-t} \end{cases}$

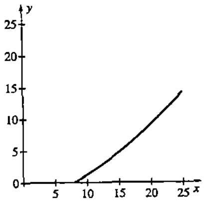
**31.** $\begin{cases} x(t) = c_1e^{t} + e^{t}\left[\left(-c_2 - \dfrac{3}{2}c_3\right)\cos 2t + \left(\dfrac{3}{2}c_2 - c_3\right)\sin 2t\right] \\ y(t) = e^{t}(-c_3 \cos 2t + c_2 \sin 2t) \\ z(t) = e^{t}(c_2 \cos 2t + c_3 \sin 2t) \end{cases}$

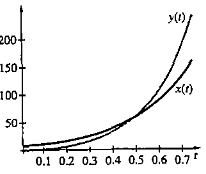
**33.** $\begin{cases} x(t) = 4e^{2t} - 4e^{t} \\ y(t) = 4e^{2t} \end{cases}$



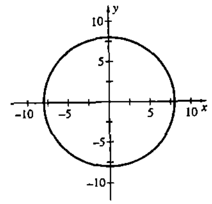


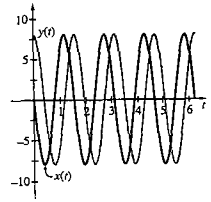

**35.** $\begin{cases} x(t) = 8e^{4t} \\ y(t) = 16te^{4t} \end{cases}$



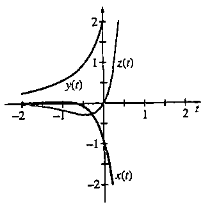



**37.** $\begin{cases} x(t) = -8 \sin 4t \\ y(t) = 8 \cos 4t \end{cases}$






**39.** $\begin{cases} x(t) = e^{2t} - 2e^{3t} \\ y(t) = 2e^{t} - 2e^{2t} + 2e^{3t} \\ z(t) = -2e^{2t} + 2e^{3t} \end{cases}$



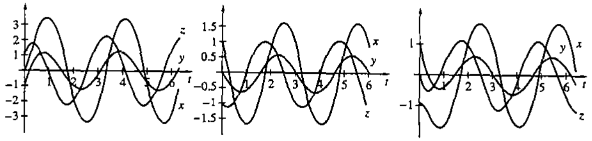

**41.** **a** and **b** are linearly independent so $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \neq 0$. Thus,
$W(e^{\alpha t}(\mathbf{a} \cos \beta t - \mathbf{b} \sin \beta t), e^{\alpha t}(\mathbf{a} \cos \beta t + \mathbf{b} \sin \beta t))$
$= \begin{vmatrix} e^{\alpha t}(a_1 \cos \beta t - b_1 \sin \beta t) & e^{\alpha t}(a_1 \cos \beta t + b_1 \sin \beta t) \\ e^{\alpha t}(a_2 \cos \beta t - b_2 \sin \beta t) & e^{\alpha t}(a_2 \cos \beta t + b_2 \sin \beta t) \end{vmatrix}$
$= (a_1b_2 - a_2b_1)e^{2\alpha t} \sin(2\beta t) \neq 0$

**43.** (e)

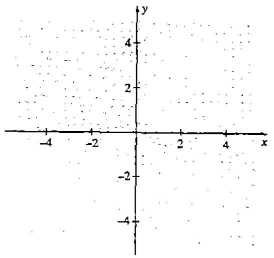
**45.** (a)

**47.** (d)

51. $$\begin{cases} x = \left(\frac{4}{13}c_1 - \frac{12}{13}c_2 + \frac{3}{13}c_3\right)e^{-3t} + \left(\frac{9}{13}c_1 + \frac{12}{13}c_2 - \frac{3}{13}c_3\right)\cos 2t + \left(\frac{6}{13}c_1 - \frac{18}{13}c_2 + \frac{24}{13}c_3\right)\sin 2t \\ y = \left(-\frac{4}{13}c_1 + \frac{12}{13}c_2 - \frac{3}{13}c_3\right)e^{-3t} + \left(\frac{4}{13}c_1 + \frac{1}{13}c_2 + \frac{3}{13}c_3\right)\cos 2t + \left(\frac{1}{26}c_1 - \frac{8}{13}c_2 + \frac{17}{26}c_3\right)\sin 2t \\ z = \left(-\frac{4}{13}c_1 + \frac{12}{13}c_2 - \frac{3}{13}c_3\right)e^{-3t} + \left(\frac{4}{13}c_1 - \frac{12}{13}c_2 + \frac{16}{13}c_3\right)\cos 2t + \left(-\frac{6}{13}c_1 - \frac{8}{13}c_2 + \frac{2}{13}c_3\right)\sin 2t \end{cases}$$



53. (a)



(b) $$\begin{cases} x(t) = \left(\frac{14}{17}x_0 - \frac{7}{17}y_0\right)e^{2t} + \left(\frac{3}{17}x_0 + \frac{7}{17}y_0\right)e^{-11t/3} \\ y(t) = \left(-\frac{6}{17}x_0 + \frac{3}{17}y_0\right)e^{2t} + \left(\frac{6}{17}x_0 + \frac{14}{17}y_0\right)e^{-11t/3} \end{cases};$$
$\lim_{t\to\infty} x(t) = 0$ if $y_0 = 2x_0$; $\lim_{t\to\infty} y(t) = 0$ if $x_0 = \frac{y_0}{2}$; (c) only if $x_0 = y_0 = 0 \Rightarrow \begin{cases} x(t) = 0 \\ y(t) = 0 \end{cases}$

### *Exercises 6.5*

1. $\mathbf{X}(t) = c_1\begin{pmatrix} -3 \\ 1 \end{pmatrix}e^{-7t} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{-5t} + \begin{pmatrix} \frac{6}{35} \\ \frac{1}{7} \end{pmatrix}t + \begin{pmatrix} \frac{173}{1225} \\ -\frac{1}{49} \end{pmatrix}$
or $\begin{cases} x(t) = -3c_1e^{-7t} + c_2e^{-5t} + \frac{6}{35}t + \frac{173}{1225} \\ y(t) = c_1e^{-7t} + \frac{1}{7}t - \frac{1}{49} \end{cases}$

3. $\begin{cases} x(t) = \frac{1}{6}t - \frac{1}{36} - c_2e^{-6t} \\ y(t) = \frac{1}{6}t^2 - \frac{1}{12}t + \frac{1}{54} + c_1e^{-6t} + c_2te^{-6t} \end{cases}$

5. $\begin{cases} x(t) = -\frac{2}{143}e^{11t} + \frac{3}{13}e^{2t} - 4c_2e^{-2t} + c_1e^{-11t} \\ y(t) = -\frac{1}{26}e^{2t} + \frac{7}{143}e^{11t} + c_2e^{-2t} + 2c_1e^{-11t} \end{cases}$

7. $\begin{cases} x(t) = \frac{7}{16}e^{-4t} + \frac{43}{85}\cos t + \frac{19}{85}\sin t + c_1e^{4t} + 7c_2e^{-2t} \\ y(t) = -\frac{1}{16}e^{-4t} + \frac{9}{85}\cos t + \frac{2}{85}\sin t + c_1e^{4t} + c_2e^{-2t} \end{cases}$

9. $\begin{cases} x(t) = e^{-6t}\left(c_2\cos 4t - c_1\sin 4t - \frac{1}{4}\right) \\ y(t) = e^{-6t}\left(c_1\cos 4t + c_2\sin 4t + \frac{1}{4}\right) \end{cases}$

11. $$\begin{cases} x(t) = e^{3t}\left[c_2\left(-\frac{1}{4}\cos 2t - \frac{3}{4}\sin 2t\right) + c_1\left(-\frac{3}{4}\cos 2t + \frac{1}{4}\sin 2t\right)\right] \\ \qquad + \frac{1}{8}e^{3t}(10 - \cos 2t - 12t\cos 2t + 3\sin 2t + 4t\sin 2t) \\ y(t) = e^{3t}[c_1\cos 2t + c_2\sin 2t] + \frac{1}{2}e^{3t}(-3 + 4t\cos 2t - \sin 2t) \end{cases}$$

13. $\mathbf{X}(t) = \begin{pmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} t\cos t - \ln|\cos t|\sin t \\ t\sin t + \ln|\cos t|\cos t \end{pmatrix}$

15. $$\begin{cases} x(t) = c_1\sin t - c_2\cos t - 2\sin t\ln|\sin t| + 2\sin t\ln|\cos t + 1| \\ y(t) = c_1\cos t + c_2\sin t - 2 + 2\cos t\ln|\cos t + 1| - 2\cos t\ln|\sin t| \end{cases}$$



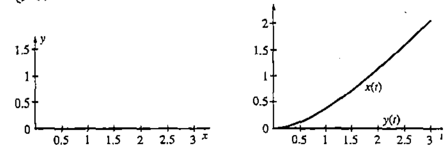
17. $$\mathbf{X}(t) = \begin{pmatrix} -2e^{-4t} & -4e^{-t} & 8e^{2t} \\ -3e^{-4t} & -3e^{-t} & 9e^{2t} \\ 2e^{-4t} & e^{-t} & 4e^{2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} -\frac{83}{16} + \frac{11}{4}t - t^2 \\ -\frac{125}{32} + \frac{15}{8}t - t^2 \\ \frac{11}{16} - \frac{3}{4}t \end{pmatrix}$$

19. $$\begin{cases} x(t) = -\frac{1}{36}te^{-2t} + \frac{1}{8}t - \frac{1}{32} - \frac{1}{12}t^2e^{-2t} - \frac{1}{216}e^{-2t} - \frac{7}{16}e^{-4t} + 2c_2e^{-2t} - \frac{8}{11}c_3e^{4t} \\ y(t) = \frac{1}{18}te^{-2t} - \frac{5}{432}e^{-2t} + \frac{9}{32}t + \frac{1}{64} - \frac{3}{16}te^{-4t} + \frac{1}{24}t^2e^{-2t} - \frac{15}{128}e^{-4t} + \left(c_1 + \frac{5}{16}\right)e^{-4t} \\ \qquad + \left(-\frac{1}{12} - c_2\right)e^{-2t} - \frac{29}{11}c_3e^{4t} \\ z(t) = -\frac{3}{16}te^{-4t} + \frac{5}{32}t - \frac{5}{64} - \frac{31}{432}e^{-2t} + \frac{7}{36}te^{-2t} - \frac{15}{128}e^{-4t} - \frac{1}{24}t^2e^{-2t} + c_1e^{-4t} + c_2e^{-2t} + c_3e^{4t} \end{cases}$$

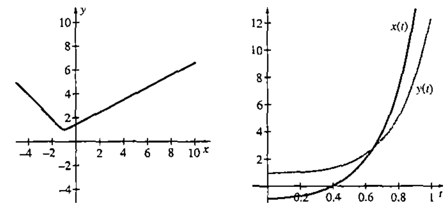
21. $$\begin{cases} x(t) = -t + \frac{8}{5} - \frac{1}{4}e^t + c_1e^{5t} + c_2e^{-t} + c_3te^{-t} \\ y(t) = \frac{3}{5} - \frac{1}{4}e^t + c_1e^{5t} + \left(c_2 - \frac{3}{2}c_3\right)e^{-t} + c_3te^{-t} \\ z(t) = \frac{1}{4}e^t - \frac{12}{5} + t + c_1e^{5t} + \left(-2c_2 + \frac{5}{4}c_3\right)e^{-t} - 2c_3te^{-t} \end{cases}$$

23. $$\mathbf{X}(t) = \begin{pmatrix} -7e^{4t} & e^{-2t}\left(\frac{3}{4}\cos t - \frac{1}{4}\sin t\right) & e^{-2t}\left(\frac{1}{4}\cos t + \frac{3}{4}\sin t\right) \\ -2e^{4t} & e^{-2t}\left(-\frac{1}{2}\cos t + \frac{1}{4}\sin t\right) & e^{-2t}\left(-\frac{1}{4}\cos t - \frac{1}{2}\sin t\right) \\ e^{4t} & & e^{-2t}\sin t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} \frac{23}{80} + \frac{71}{1369}e^{4t} - \frac{1}{4}t + \frac{28}{37}te^{4t} \\ -\frac{43}{200} - \frac{59}{1369}e^{4t} + \frac{3}{10}t + \frac{8}{37}te^{4t} \\ \frac{91}{400} + \frac{48}{1369}e^{4t} - \frac{1}{20}t - \frac{4}{37}te^{4t} \end{pmatrix}$$

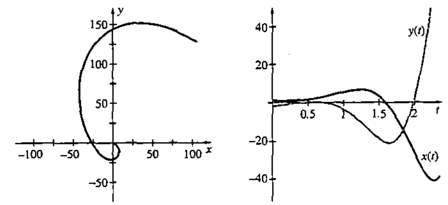
25. $\begin{cases} x(t) = t - 1 + e^{-t} \\ y(t) = 0 \end{cases}$




27. $\begin{cases} x(t) = -\frac{2}{3}e^{-t} - \frac{1}{2}e^t + \frac{1}{6}e^{5t} \\ y(t) = \frac{2}{3}e^{-t} + \frac{1}{4}e^t + \frac{1}{12}e^{5t} \end{cases}$

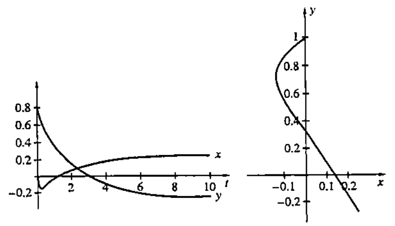



29. $\begin{cases} x(t) = \frac{20}{13} + e^{2t}\left(-\frac{7}{13}\cos 3t + \frac{1}{39}\sin 3t\right) \\ y(t) = -\frac{30}{13} + e^{2t}\left(\frac{4}{13}\cos 3t + \frac{31}{39}\sin 3t\right) \end{cases}$

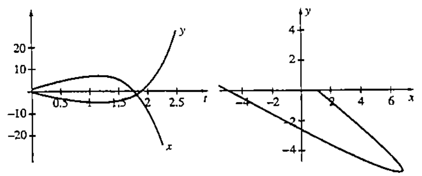



31. (b)
33. (a)
35. (c)

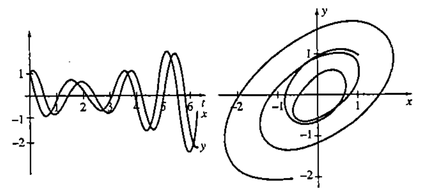
**37. (a)**
$$\begin{cases} x(t) = \dfrac{1}{1372}e^{-8t}(453 - 796e^{7t} + 343e^{8t} \\ \qquad + 84te^{7t} - 294t^2e^{7t}) \\ y(t) = -\dfrac{1}{1372}e^{-8t}(-151 - 1564e^{7t} + 343e^{8t} \\ \qquad - 28te^{7t} - 588t^2e^{7t}) \end{cases}$$



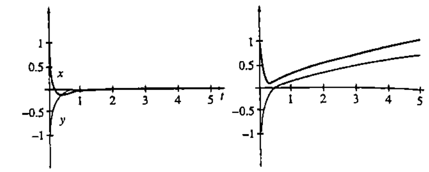
**(b)**
$$\begin{cases} x(t) = -\dfrac{1}{81}e^{-2t}(179 - 260e^{3t} + 60t - 90te^{3t} \\ \qquad + 90t^2 + 135t^2e^{3t} + 63t^3) \\ y(t) = \dfrac{1}{162}e^{-2t}(346 - 346e^{3t} + 84t - 180te^{3t} \\ \qquad + 126t^2 + 189t^2e^{3t} + 126t^3) \end{cases}$$



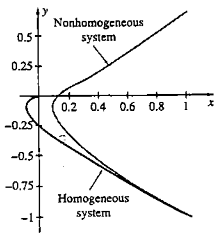
**(c)**
$$\begin{cases} x(t) = -\dfrac{1}{9248}e^{-t}(-320 - 2720t + e^t(-8928\cos 4t \\ \qquad + 2312t\cos 4t + 6958\sin 4t - 4624t\sin 4t)) \\ y(t) = -\dfrac{1}{4624}e^{-t}(176 - 816t + e^t(-4800\cos 4t \\ \qquad + 2312t\cos 4t - 2642\sin 4t)) \end{cases}$$



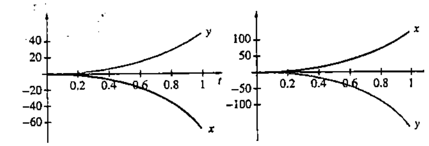
**39. (a)** (solution to homogeneous system on left; solution to nonhomogeneous system on right; $x$ in black; $y$ in gray)



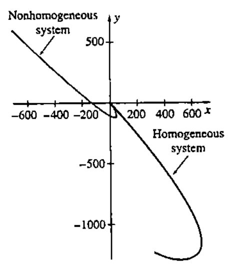
**(b)** (curve in gray is the graph of the solution to the nonhomogeneous system)




**41. (a)** (solution to homogeneous system on left; solution to nonhomogeneous system on right; $x$ in black; $y$ in gray)



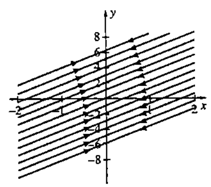
**(b)** (curve in gray is the graph of the solution to the nonhomogeneous system)



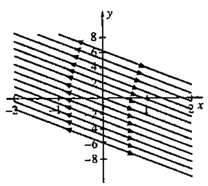



## *Exercises 6.6*

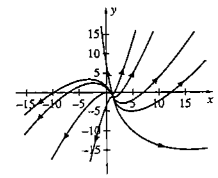
1. $\lambda_1 = -11$, $\lambda_2 = 1$; saddle point, unstable
2. $\lambda_1 = -4$, $\lambda_2 = 6$; saddle point, unstable
3. $\lambda_1 = \lambda_2 = 2$, $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$; deficient node, unstable
4. $\lambda_1 = 2$, $\lambda_2 = 10$; improper node, unstable
5. $\lambda_1 = -12$, $\lambda_2 = -8$; improper node, asymptotically stable
6. $\lambda_1 = -6$, $\lambda_2 = -3$; improper node, asymptotically stable
7. $\lambda_1 = \lambda_2 = -1$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; star node, asymptotically stable
8. $\lambda_1 = \lambda_2 = 3$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; star node, unstable
9. $\lambda_1 = 2 + 5i\sqrt{2}$, $\lambda_2 = 2 - 5i\sqrt{2}$; spiral point, unstable
10. $\lambda_1 = -5 + i$, $\lambda_2 = -5 - i$; spiral point, asymptotically stable
11. $\lambda_1 = i\sqrt{41}$, $\lambda_2 = -i\sqrt{41}$; center, stable
12. $\lambda_1 = \lambda_2 = -2$; $\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$; deficient node, asymptotically stable
13. (e) $\lim_{t\to\infty} \mathbf{X}(t) = \begin{pmatrix} 0 \\ c_1 \end{pmatrix}$



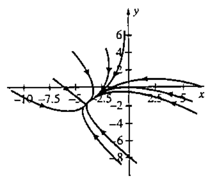
15. $\lambda_1 = 0$, $\lambda_2 = 2$; $y = -2x + C$



17. $(\lambda - 2)(\lambda + 1) = \lambda^2 - \lambda - 2$; $x'' - x' - 2x = 0$; $\begin{cases} x' = y \\ y' = 2x + y \end{cases}$; saddle point, unstable
19. $(\lambda + 3)(\lambda + 3) = \lambda^2 + 6\lambda + 9$; $x'' + 6x' + 9x = 0$; $\begin{cases} x' = y \\ y' = -9x - 6y \end{cases}$; deficient node, asymptotically stable or $\begin{cases} x' = -3x \\ y' = -3y \end{cases}$; star node, asymptotically stable
21. (c) $\lambda_1 = \lambda_2 = 1$; $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; deficient node, unstable



23. $(-4, -2)$, $\lambda_1 = -3$, $\lambda_2 = -1$; improper node, asymptotically stable



27. If $Q = A_1 + B_1 + C_1$, $A_1 = \dfrac{Q}{1 + \dfrac{K_1}{K_2} + \dfrac{K_1}{K_3}}$,

$B_1 = \dfrac{K_1}{K_2}A_1$, $C_1 = \dfrac{K_1}{K_3}A_1$

## *Exercises 6.7*

1. (0, 0), saddle point, unstable; (1, 0), inconclusive—center or spiral point
3. (0, 0), saddle point, unstable; (1, −1), spiral point, unstable
5. (0, 0), inconclusive—center or spiral point
7. (0, 0), node or spiral point, unstable
9. (0, 0), improper node, asymptotically stable; (0, 4), saddle point, unstable; (2, 0), saddle point, unstable; (3, 1), spiral point, unstable

**11.** $(-4, 4)$, saddle point, unstable; $(1, -1)$, saddle point, unstable

**13.** $(2, -1)$: $\lambda_1 = -6$, $\lambda_2 = 1$, saddle; $(8, -1)$: $\lambda_1 = 6$, $\lambda_2 = 1$, unstable node

**15.** $(0, 0)$: $\lambda_1 = -\sqrt{2}$, $\lambda_2 = \sqrt{2}$, saddle; $(1, 1)$: $\lambda_1 = -1 + i$, $\lambda_2 = -1 - i$, stable spiral

**17.** $(0, 2)$: $\lambda_1 = 2$, $\lambda_2 = 4$, unstable node; $(0, -2)$: $\lambda_1 = -2$, $\lambda_2 = -4$, stable node $(2, 0)$: $\lambda_1 = -2\sqrt{2}$, $\lambda_2 = 2\sqrt{2}$, saddle; $(-2, 0)$: $\lambda_1 = -2\sqrt{2}$, $\lambda_2 = 2\sqrt{2}$, saddle

**19.** $(1, 1)$: $\lambda_1 = 2$, $\lambda_2 = -2$, saddle; $(-1, -1)$: $\lambda_1 = 1 + i\sqrt{3}$, $\lambda_2 = 1 - i\sqrt{3}$, unstable spiral

**21.** $(0, 0)$: $\lambda_1 = a_1$, $\lambda_2 = -a_2$, saddle; $\left(\frac{a_1}{b_1}, 0\right)$: $\lambda_1 = -a_1$, $\lambda_2 = -\left(a_2 - \frac{c_2 a_1}{b_1}\right)$, stable node $\left(-\frac{a_2}{c_2}, \frac{a_1 c_2 + a_2 b_1}{c_1 c_2}\right)$:

$$\lambda = \frac{a_2 b_1}{c_2} \pm \sqrt{\left(\frac{a_2 b_1}{c_2}\right)^2 - \frac{4a_2(a_1c_2 + a_2b_1)}{b_1}},$$

unstable node if $\left(\frac{a_2 b_1}{c_2}\right)^2 - \frac{4a_2(a_1c_2 + a_2b_1)}{b_1} \geq 0$;

unstable spiral if $\left(\frac{a_2 b_1}{c_2}\right)^2 - \frac{4a_2(a_1c_2 + a_2b_1)}{b_1} < 0$

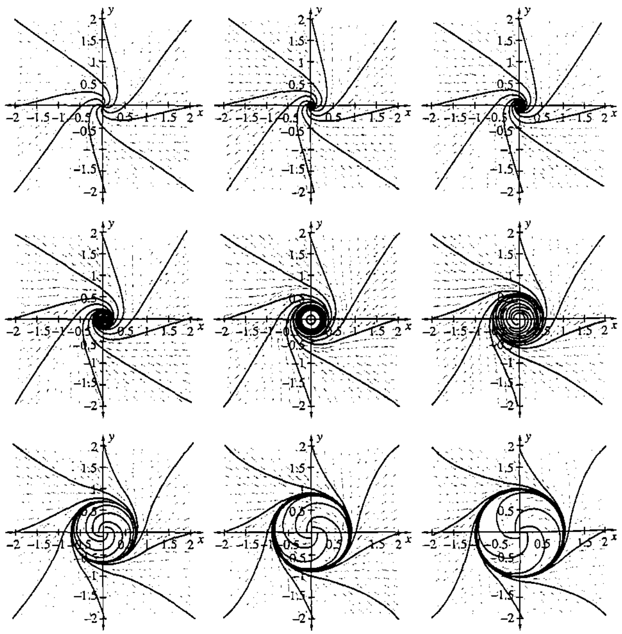
**23.** (e)

**25.** (d)

**27.** (c)

**29.** **(a)** $x(t) = C_1 e^{-t}$, $y(t) = -\frac{C_1^2}{3}e^{-2t} + C_2 e^t \left(\text{or } y = -\frac{x^2}{3} + \frac{C_1 C_2}{x}\right)$;
**(b)** unstable;
**(c)** $x(t) = C_1 e^{-t}$, $y(t) = C_2 e^t$, saddle

**31.** **(d)** $\left(\mu = -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right)$





### *Exercises 6.8*

**1.** $x(1) \approx 8.64479$, $y(1) \approx 8.29263$

**3.** $x(1) \approx -7.2362$, $y(1) \approx -1.5998$

**5.** $x(1) \approx -0.113115$, $y(1) \approx -1.06576$

**7.** $x(1) \approx -326.204$, $y(1) \approx 608$

**9.** $x(1) \approx 0.164251$, $y(1) \approx -0.504587$

**11.** $x(1) \approx 1.52319$, $y(1) \approx 0.419975$

**13.** $\begin{cases} x' = y \\ y' = -2x - 3y \end{cases}$; Euler's method yields $x(0) = 0$, $y(0) = -3$ $x(1) \approx -0.723913$, $y(1) \approx 0.40179$; exact solution is $x(t) = 3e^{-2t} - 3e^{-t}$ so $x(1) = 3e^{-2}(1 - e) \approx -0.697632$.

**15.** $\begin{cases} x' = y \\ y' = -9x \end{cases}$; Euler's method yields $x(0) = 0$, $y(0) = 3$ $x(1) \approx 0.346313$, $y(1) \approx -4.49743$; exact solution is $x(t) = \sin 3t$ so $x(1) = \sin 3 \approx 0.14112$.

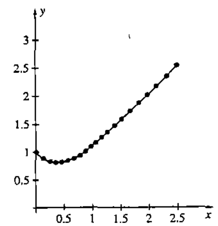
**17.** $\begin{cases} x' = y \\ y' = -\frac{1}{t^2}(16x + ty) \end{cases}$; Euler's method yields $x(1) = 0$, $y(1) = 4$ $x(2) \approx 0.354942$, $y(2) \approx -2.90834$; exact solution is $x(t) = \sin(4 \ln t)$ so $x(2) = \sin(4 \ln 2) \approx 0.360687$.

**19.** (See 13) Runge-Kutta yields $x(1) \approx -0.697621$, $y(1) \approx 0.291602$

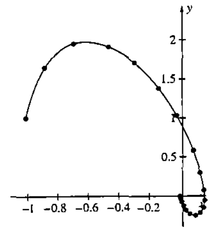
**21.** (See 15) Runge-Kutta yields $x(1) \approx 0.141307$, $y(1) \approx -2.96975$

**23.** (See 17) Runge-Kutta yields $x(1) \approx 0.360845$, $y(1) \approx -1.86541$

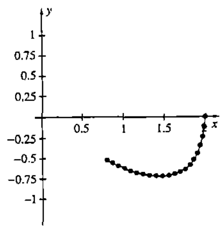
**25.** $\begin{cases} x(t) = \frac{1}{3}(e^t - e^{-2t}) \\ y(t) = \frac{1}{3}(e^t + 2e^{-2t}) \end{cases}$



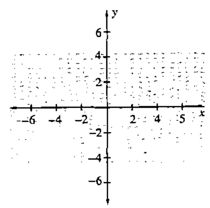
**26.** $\begin{cases} x(t) = -\cos 3t \cosh 2t - \frac{1}{3}\cosh 2t \sin 3t + \cos 3t \sinh 2t + \frac{1}{3}\sin 3t \sinh 2t \\ y(t) = 13\left(\frac{1}{13}\cos 3t \cosh 2t + \frac{11}{39}\cosh 2t \sin 3t - \frac{1}{13}\cos 3t \sinh 2t - \frac{11}{39}\sin 3t \sinh 2t\right) \end{cases}$



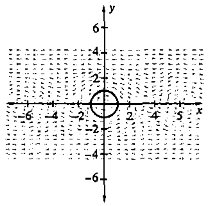
**27.** $\begin{cases} x(t) = e^{-t}(2 + 2t) \\ y(t) = -2te^{-t} \end{cases}$



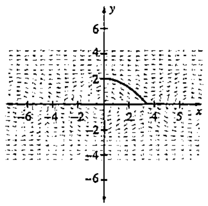
**28.** **(a)**



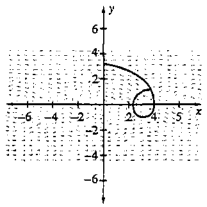
(b) (ii)



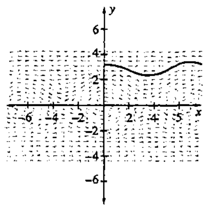
(iv)



(v)



(c)



(d) No. The solution in (v) of part (b) is not tangent to vectors in the direction field. That in part (c) is. If the system were linear, these two solutions would coincide.

### Chapter 6 Review Exercises

1. $\{-4, 11\}, \left\{\begin{pmatrix}-2\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}$

3. $\{-3, -2\}, \left\{\begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}-2\\1\end{pmatrix}\right\}$

5. $\{1, 2\}, \left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$

7. $\{2, 3, 4\}, \left\{\begin{pmatrix}-1\\-1\\2\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\2\end{pmatrix}\right\}$

9. $\{-4, -2 \pm 3i\}, \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}-\frac{9}{13}+\frac{6}{13}i\\i\\1\end{pmatrix}, \begin{pmatrix}-\frac{9}{13}+\frac{6}{13}i\\-i\\1\end{pmatrix}\right\}$

11. $\mathbf{X} = \begin{pmatrix}-e^{6t} & 4e^{t}\\ e^{6t} & e^{t}\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}$

13. $\begin{cases} x = 3e^{2t} \\ y = 2e^{-t} + e^{2t} \end{cases}$

15. $\mathbf{X} = \begin{pmatrix}3\sin\frac{t}{2} - \cos\frac{t}{2} & \sin\frac{t}{2} + 3\cos\frac{t}{2}\\ \sin\frac{t}{2} & \cos\frac{t}{2}\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}$

17. $\mathbf{X}' = \begin{pmatrix}-4 & -1\\ 5 & -2\end{pmatrix}X$,
$\mathbf{X} = e^{-3t}\begin{pmatrix}\sin 2t - 2\cos 2t & -\cos 2t\\ -5\sin 2t & \cos 2t - 2\sin 2t\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}$

19. $\mathbf{X} = \begin{pmatrix}-2\cos 4t & 2\sin 4t\\ \sin 4t & \cos 4t\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}$

21. $\begin{cases} x(t) = e^{-t}(2\cos 3t + \sin 3t) \\ y(t) = 3e^{-t}(\cos 3t + 3\sin 3t) \end{cases}$

23. $\mathbf{X} = e^{7t}\begin{pmatrix}1 & 1+t\\ 1 & t\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}$

25. $\begin{cases} x(t) = c_1e^{-t} + 9c_2e^{7t} + \frac{5}{4}e^{7t} - 18te^{7t} \\ y(t) = c_1e^{-t} + c_2e^{7t} + \frac{5}{4}e^{7t} - 2te^{7t} \end{cases}$

27. $\begin{cases} x = 1 - c_1e^{-t} + 2c_2e^{5t} \\ y = -1 + c_1e^{-t} + c_2e^{5t} \end{cases}$

29. $\begin{cases} x = \cos t - t\sin t \\ y = \sin t + t\cos t \end{cases}$



# CHAPTER 7

### Exercises 7.1

1. (a) $\begin{cases} Q(t) = \frac{3\sqrt{5}}{5}e^{-5t/3}\sin\left(\frac{\sqrt{5}}{3}t\right) \\ I(t) = -\sqrt{5}e^{-5t/3}\sin\left(\frac{\sqrt{5}}{3}t\right) + e^{-5t/3}\cos\left(\frac{\sqrt{5}}{3}t\right) \end{cases}$

(b) $\begin{cases} Q(t) = \frac{1}{3}e^{-t} + \frac{7\sqrt{5}}{15}e^{-5t/3}\sin\left(\frac{\sqrt{5}}{3}t\right) - \frac{1}{3}e^{-5t/3}\cos\left(\frac{\sqrt{5}}{3}t\right) \\ I(t) = -\frac{1}{3}e^{-t} - \frac{2\sqrt{5}}{3}e^{-5t/3}\sin\left(\frac{\sqrt{5}}{3}t\right) + \frac{4}{3}e^{-5t/3}\cos\left(\frac{\sqrt{5}}{3}t\right) \end{cases}$

3. (a) $\begin{cases} Q(t) = 10^{-6}e^{-t}\cos(\sqrt{2}t) \end{cases}$

(b) $\begin{cases} Q(t) = \frac{2}{9}e^{-t/2} + \frac{4\sqrt{2}}{9}e^{-t}\sin(\sqrt{2}t) - \frac{1999991}{9000000}e^{-t}\cos(\sqrt{2}t) \\ I_2(t) = \frac{8}{9}e^{-t/2} - \frac{1999991}{4500000\sqrt{2}}e^{-t}\sin(\sqrt{2}t) - \frac{8}{9}e^{-t}\cos(\sqrt{2}t) \end{cases}$

5. (a) $\begin{cases} Q(t) = 10^{-6}e^{-2t}(1+t) \\ I_2(t) = 10^{-6}te^{-2t} \end{cases} \Rightarrow$
$I(t) = -10^{-6}e^{-2t}(1+2t) \Rightarrow$
$I_1(t) = -10^{-6}e^{-2t}(1+t)$

(b) $\begin{cases} Q(t) = \frac{135}{2} - \frac{135}{2}e^{-2t} - 45te^{-2t} \\ I_2(t) = \frac{45}{2} - \frac{45}{2}e^{-2t} - 45te^{-2t} \end{cases} \Rightarrow$
$I(t) = 90e^{-2t}(1+t) \Rightarrow I_1(t) = \frac{45}{2}e^{-2t}(3 + e^{2t} + 2t)$

7. (a) $\begin{cases} Q(t) = 10^{-6}e^{-t} - \frac{1}{2}\cdot 10^{-6}t^2e^{-t} \\ I_2(t) = 10^{-6}te^{-t} \\ I_3(t) = \frac{1}{2}\cdot 10^{-6}t^2e^{-t} \end{cases}$

(b) $\begin{cases} Q(t) = 10^{-6}e^{-t} + te^{-t} - \frac{1}{2}\cdot 10^{-6}t^2e^{-t} \\ I_2(t) = 10^{-6}te^{-t} \\ I_3(t) = -te^{-t} + \frac{1}{2}\cdot 10^{-6}t^2e^{-t} \end{cases}$

9. (a) $\begin{cases} Q(t) = 90 - te^{-t} - 90e^{-t} \\ I_2(t) = e^{-t} \\ I_3(t) = -90 + te^{-t} + 90e^{-t} \end{cases}$

(b) $\begin{cases} Q(t) = -te^{-t} + 45e^{-t} + 45\sin t - 45\cos t \\ I_2(t) = e^{-t} \\ I_3(t) = te^{-t} - 45e^{-t} - 45\sin t + 45\cos t \end{cases}$

11. (a) $\begin{cases} Q(t) = 90 - \frac{4}{3}e^{-t/4} - \frac{266}{3}e^{-t} \\ I_2(t) = e^{-t/4} \\ I_3(t) = -90 + \frac{4}{3}e^{-t/4} + \frac{266}{3}e^{-t} \end{cases}$;

(b) $\begin{cases} Q(t) = -45\cos t + 45\sin t - \frac{4}{3}e^{-t/4} + \frac{139}{3}e^{-t} \\ I_2(t) = e^{-t/4} \\ I_3(t) = 45\cos t - 45\sin t + \frac{4}{3}e^{-t/4} - \frac{139}{3}e^{-t} \end{cases}$

13. $\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -9x \end{cases}$, $\lambda = \pm 3i$; undamped

15. $\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -9x - 10y \end{cases}$, $\lambda_1 = -1$, $\lambda_2 = -9$; overdamped

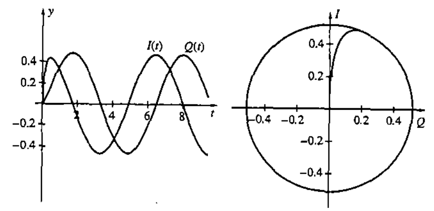
17. $\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -50x - 10y \end{cases}$, $\lambda_1 = -5 + 5i$,
$\lambda_2 = -5 - 5i$; underdamped

19. $\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -25x - 10y \end{cases}$, $\lambda_1 = -5$, $\lambda_2 = -5$;
critically damped

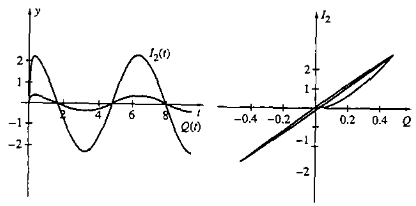
21. (13) $\begin{cases} x(t) = \cos 3t \\ y(t) = -3\sin 3t \end{cases}$;
(15) $\begin{cases} x(t) = \frac{9}{8}e^{-t} - \frac{1}{8}e^{-9t} \\ y(t) = -\frac{9}{8}e^{-t} + \frac{9}{8}e^{-9t} \end{cases}$

23. $\lambda = -\frac{c}{2\,\mathrm{m}} \pm \frac{1}{2\,\mathrm{m}}\sqrt{c^2 - 4\,\mathrm{km}}$, overdamped if $c^2 > 4$ km; critically damped if $c^2 = 4$ km; underdamped if $c^2 < 4$ km.

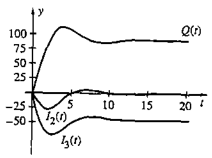
**25.** $\lim_{t\to\infty} Q(t)$ does not exist; $\lim_{t\to\infty} I(t)$ does not exist.



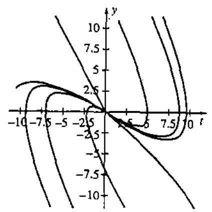

**27.** $\lim_{t\to\infty} Q(t)$ does not exist; $\lim_{t\to\infty} I_2(t)$ does not exist.




**29.** $\lim_{t\to\infty} Q(t) \approx 91$, $\lim_{t\to\infty} I_2(t) = 0$, $\lim_{t\to\infty} I_3(t) \approx -43.5$




**31.** $\begin{cases} x(t) = \frac{1}{2}e^{-3t/2}((3e^t - 1)x_0 + 2(e^t - 1)y_0) \\ y(t) = -\frac{1}{4}e^{-3t/2}(3(e^t - 1)x_0 + 2(e^t - 3)y_0) \end{cases}$; stable node



### *Exercises 7.2*

**1.** $\begin{cases} x(t) = \frac{3}{2} - \frac{1}{2}e^{-t} \\ y(t) = \frac{3}{2} + \frac{1}{2}e^{-t} \end{cases}$, $\lim_{t\to\infty} (x(t), y(t)) = \left(\frac{3}{2}, \frac{3}{2}\right)$

**3.** $\begin{cases} x(t) = \frac{8}{3} - \frac{8}{3}e^{-3t/2} \\ y(t) = \frac{4}{3} + \frac{8}{3}e^{-3t/2} \end{cases}$, $\lim_{t\to\infty} (x(t), y(t)) = \left(\frac{8}{3}, \frac{4}{3}\right)$

**5.** $\begin{cases} x(t) = 1 + 3e^{-15t/4} \\ y(t) = 4 - 3e^{-15t/4} \end{cases}$, $\lim_{t\to\infty} (x(t), y(t)) = (1, 4)$

**7.** $\{x(t) = 10 - 10e^{-t/5}, y(t) = 10 - 10e^{-t/5} - 2te^{-t/5}\}$, $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 10$, $x(t)$

**9.** $\left\{x(t) = 4e^{-t/5}, y(t) = 4e^{-t/5} + \frac{4}{5}te^{-t/5}\right\}$, $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$, $x(t)$

**11.** **(a)** $\{x(t) = 300 - 300e^{-t/20}, y(t) = 150 + 150e^{-t/10} - 300e^{-t/20}\}$, **(b)** $\lim_{t\to\infty} x(t) = 300$, $\lim_{t\to\infty} y(t) = 150$

**13.** $\left\{x(t) = \frac{325}{6} - \frac{125}{12}e^{-3t/25} - \frac{175}{4}e^{-t/25}, y(t) = \frac{200}{3} + \frac{125}{6}e^{-3t/25} - \frac{175}{2}e^{-t/25}\right\}$; $\lim_{t\to\infty} x(t) = \frac{325}{6}$, $\lim_{t\to\infty} y(t) = \frac{200}{3}$; $x(t) = y(t)$ at approx. $t = 29.1$ min.

**15.** system: $\begin{cases} \frac{dx}{dt} = 2y - 2x + 1 \\ \frac{dy}{dt} = -3y + x + 1 \end{cases}$

$\left\{x(t) = \frac{5}{4} - \frac{23}{12}e^{-4t} + \frac{8}{3}e^{-t}, y(t) = \frac{3}{4} + \frac{23}{12}e^{-4t} + \frac{4}{3}e^{-t}\right\}$, $\lim_{t\to\infty} x(t) = \frac{5}{4}$, $\lim_{t\to\infty} y(t) = \frac{3}{4}$; max. $x = 4$, $y \approx 1$

**17.** system: $\begin{cases} \frac{dx}{dt} = -\frac{x}{10} + 10 \\ \frac{dy}{dt} = \frac{x}{10} - \frac{y}{10} \\ \frac{dz}{dt} = \frac{y}{10} - \frac{z}{10} \end{cases}$

$\begin{cases} x(t) = 100 - 100e^{-t/10} \\ y(t) = 100 - 100e^{-t/10} - 10te^{-t/10} \\ z(t) = 100 - 100e^{-t/10} - 10te^{-t/10} - \frac{1}{2}t^2e^{-t/10} \end{cases}$

$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = 100$



**19.** $\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{25}{3}e^{4t} + \frac{5}{3}e^{-2t} \\ 25e^{4t} - 5e^{-2t} \end{pmatrix}$

**21.** $\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 15e^{2t} - 10e^{-t} \\ 10e^{-t} \end{pmatrix}$

**23.** $\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} -\frac{67}{7}e^{-10t} + \frac{171}{10}e^{7t} + \frac{663}{70}e^{-3t} \\ 5e^{-10t} + \frac{171}{10}e^{7t} - \frac{221}{10}e^{-3t} \\ \frac{52}{7}e^{-10t} + \frac{171}{10}e^{7t} + \frac{663}{70}e^{-3t} \end{pmatrix}$;

$x(1) \approx z(1) \approx 18752.89$

**25.** $\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \frac{43}{7} + \frac{46}{21}e^{7t} - \frac{1}{3}e^{-8t} \\ \frac{4}{3}e^{7t} + \frac{2}{3}e^{-8t} \\ -\frac{43}{7} + \frac{136}{21}e^{7t} - \frac{1}{3}e^{-8t} \end{pmatrix}$;

$z(t) \approx 7095.86$

**29.** $\begin{cases} x(t) = x_0e^{-at} \\ y(t) = y_0e^{-at} + ax_0te^{-at} \\ z(t) = x_0 + y_0 + z_0 - (x_0 + y_0)e^{-at} - ax_0te^{-at} \end{cases}$;

$\lim_{t\to\infty} x(t) = 0$, $\lim_{t\to\infty} y(t) = 0$, $\lim_{t\to\infty} z(t) = x_0 + y_0 + z_0$

**33.** $\begin{cases} x(t) = 7e^{-6t} \\ y(t) = \frac{47}{5}e^{-t} - \frac{42}{5}e^{-6t} \\ z(t) = 16 - \frac{47}{5}e^{-t} + \frac{7}{5}e^{-6t} \end{cases}$; $\lim_{t\to\infty} z(t) = 16$

**35.** $\begin{cases} x(t) = e^{-4t} \\ y(t) = 3e^{-2t} - 2e^{-4t} \\ z(t) = 3 - 3e^{-2t} + e^{-4t} \end{cases}$; $\lim_{t\to\infty} z(t) = 3$

**37.** $\begin{cases} x(t) = 2 \\ y(t) = 2 - e^{-t} \\ z(t) = t + 1 + e^{-t} \end{cases}$;

$\lim_{t\to\infty} x(t) = 2$, $\lim_{t\to\infty} y(t) = 2$, $\lim_{t\to\infty} z(t) = \infty$

**39.** $\begin{cases} x(t) = 10 - 2e^{-t} \\ y(t) = 10 - 8e^{-t} - 2te^{-t} \\ z(t) = 10t - 8 + 10e^{-t} + 2te^{-t} \end{cases}$;

$\lim_{t\to\infty} x(t) = 10$, $\lim_{t\to\infty} y(t) = 10$, $\lim_{t\to\infty} z(t) = \infty$

**41.** $\begin{cases} x(t) = \frac{1}{V_1 + V_2}[bV_1(1 - e^{-Pt(V_1+V_2)/(V_1V_2)}) + a(V_1 + V_2e^{-Pt(V_1+V_2)/(V_1V_2)})] \\ y(t) = \frac{1}{V_1 + V_2}[aV_2(1 - e^{-Pt(V_1+V_2)/(V_1V_2)}) + b(V_2 + V_1e^{-Pt(V_1+V_2)/(V_1V_2)})] \end{cases}$

**(a)** $\lim_{t\to\infty} x(t) = \frac{(a + b)V_1}{V_1 + V_2}$, $\lim_{t\to\infty} y(t) = \frac{(a + b)V_2}{V_1 + V_2}$
**(b)** $V_1 > V_2$, $V_1 = V_2$
**(c)** $bV_1 - aV_2 > 0$, $aV_2 - bV_1 > 0$, no

**43.** The problem can be solved as a system or the system can be written as a second-order ODE, where $x'' = (a_1 - a_2)x' + b_1y'$ (derivative of first eqn.) or $y' = \frac{1}{b_1}(x'' - (a_1 - a_2)x')$. Substitution of $y'$ and $y = \frac{1}{b_1}(x' - (a_1 - a_2)x)$ into the second equation and simplification yields $x'' - [(a_1 - a_2) + (b_1 - b_2)]x' + [(b_1 - b_2)(a_1 - a_2) - a_2b_1]x = 0$.

We consider the characteristics of this equation (same as the eigenvalues of the system).

Periodic if $a_1 + b_1 = a_2 + b_2$ and $a_1b_1 - 2a_2b_1 - a_1b_2 + a_2b_2 > 0$. For example, if $a_1 = a_2 = -1$, $b_1 = b_2 = 1$, $x_0 = 1$, and $y_0 = 0$, then $\{x(t) = \cos t, y(t) = -\sin t\}$.

Exponential decay if $(b_2 - b_1) > (a_1 - a_2)$ and $(b_1 - b_2)(a_1 - a_2) - a_2b_1 \geq 0$. For example, if $a_1 = -1$, $a_2 = 0$, $b_1 = b_2 = 1$, $x_0 = 1$, $y_0 = 0$, then $\{x(t) = e^{-t}, y(t) = 0\}$.

Exponential growth if $(a_1 - a_2) > (b_2 - b_1)$ (will have at least one positive eigenvalue). For example, if $a_1 = 2$, $a_2 = 0$, $b_1 = b_2 = 1$, $x_0 = 1$, $y_0 = 0$, then $\{x(t) = e^{2t}, y(t) = 0\}$.

### *Exercises 7.3*

**1.** $-a \ln|y| + by - c \ln|x| + dx = C$

**3.** **(a)** $(2k, k)$ stable spiral; **(b)** $(2k, k)$ center

**5.** $\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\sin x \end{cases}$, $(k\pi, 0)$, $k = 0, \pm 1, \pm 2, \ldots,$

(vertical position of pendulum)

**7.** $\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -k^2x \end{cases}$; $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-k^2x}{y}$; $\frac{1}{2}y^2 + \frac{k^2}{2}x^2 + C$;

(0, 0) center

9. $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\dfrac{gM_1}{x^2} + \dfrac{gM_2}{(R-x)^2} \end{cases}$; $(x_0, y_0) = \left(\dfrac{R(M_1 + \sqrt{M_1 M_2})}{M_1 - M_2}, 0\right) = \left(\dfrac{R\sqrt{M_1}}{\sqrt{M_1} - \sqrt{M_2}}, 0\right)$; saddle

11. If $\dfrac{dx}{dt} = y - F(x) = y - \int_0^x f(u)\,du$ and $\dfrac{dy}{dt} = -g(x)$, then $\dfrac{d^2x}{dt^2} = \dfrac{dy}{dt} - f(x)\dfrac{dx}{dt} = -g(x) - f(x)\dfrac{dx}{dt}$, which is equivalent to $\dfrac{d^2x}{dt^2} + f(x)\dfrac{dx}{dt} + g(x) = 0$.

13. In each case, no limit cycle in the $xy$-plane:
(a) $f_x(x, y) + g_y(x, y) = 3x^2 + 1 + x^2 = 4x^2 + 1 > 0$;
(b) $f_x(x, y) + g_y(x, y) = -1 - 1 = -2 < 0$;
(c) $f_x(x, y) + g_y(x, y) = y^2 + 8 > 0$.

15. $\begin{cases} \dfrac{du}{d\theta} = y \\ \dfrac{dy}{d\theta} = \alpha - u = ku^2 \end{cases}$; $y = 0$
and $\alpha - u - ku^2 = 0 \Rightarrow u = \dfrac{1 \pm \sqrt{1 - 4k\alpha}}{2k}$.
We choose $u = \dfrac{1 - \sqrt{1 - 4k\alpha}}{2k}$ because it is closer to $u = 0$ than $u = \dfrac{1 + \sqrt{1 - 4k\alpha}}{2k}$, and we are considering a small change in the orbit. $J\left(\dfrac{1 + \sqrt{1 - 4k\alpha}}{2k}, 0\right) = \begin{pmatrix} 0 & 1 \\ -\sqrt{1 - 4k\alpha} & 0 \end{pmatrix} \Rightarrow \lambda^2 + \sqrt{1 - 4k\alpha} = 0$; center.

17. Differentiating $\dfrac{1}{2}m(x)\left(\dfrac{dx}{dt}\right)^2 + V(x) = E$ with respect to $t$ yields $\dfrac{1}{2}m'(x)\left(\dfrac{dx}{dt}\right)^3 + m(x)\dfrac{dx}{dt}\dfrac{d^2x}{dt^2} + V'(x)\dfrac{dx}{dt} = \dfrac{dx}{dt}\left[\dfrac{1}{2}m'(x)\left(\dfrac{dx}{dt}\right)^2 + m(x)\dfrac{d^2x}{dt^2} + V'(x)\right] = 0$, so

$$\frac{1}{2}m'(x)\left(\frac{dx}{dt}\right)^2 + m(x)\frac{d^2x}{dt^2} + V'(x) = 0$$
$$\frac{1}{\sqrt{m(x)}}\left[\frac{1}{2}m'(x)\left(\frac{dx}{dt}\right)^2 + m(x)\frac{d^2x}{dt^2} + V'(x)\right] = 0$$
$$\frac{1}{2}\frac{m'(x)}{\sqrt{m(x)}}\left(\frac{dx}{dt}\right)^2 + \sqrt{m(x)}\frac{d^2x}{dt^2} + \frac{V'(x)}{\sqrt{m(x)}} = 0$$
$$\frac{d}{dt}\left[\sqrt{m(x)}\frac{dx}{dt}\right] + \frac{V'(x)}{\sqrt{m(x)}} = 0.$$

Notice that $\dfrac{du}{dx} = \sqrt{m(x)}$, so the equation is
$$\frac{d}{dt}\left[\frac{du}{dx}\frac{dx}{dt}\right] + \frac{V'(x)}{\sqrt{m(x)}} = 0.$$
Notice also that $\dfrac{du}{dt} = \dfrac{du}{dx}\dfrac{dx}{dt}$, so we have
$$\frac{d}{dt}\left[\frac{du}{dt}\right] + \frac{V'(x)}{\sqrt{m(x)}} = 0 \text{ or } \frac{d^2x}{dt^2} + \frac{V'(x)}{\sqrt{m(x)}} = 0.$$

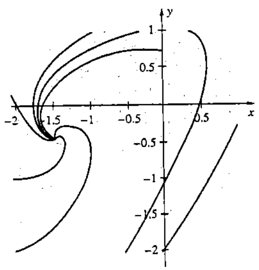
19. paths: circles; equilibrium point: center; agree

21. (a) center if $x = 1$; saddle if $x = -1$; (b) saddle if $x = 0$; center if $x = 1$, $x = -1$.

23. $\begin{cases} \dfrac{d\theta}{dt} = y \\ \dfrac{dy}{dt} = -\dfrac{FR}{I}\operatorname{sgn}(y) \end{cases}$; equilibrium points: $(\theta, 0)$;
if $\dfrac{d\theta}{dt} > 0$, then we integrate $I\dfrac{d\theta}{dt}\dfrac{d}{d\theta}\left(\dfrac{d\theta}{dt}\right) = -FR$ with respect to $\theta$ to obtain the parabolas $\dfrac{1}{2}I\left(\dfrac{d\theta}{dt}\right)^2 = -FR\theta + C$, $\dfrac{d\theta}{dt} > 0$. Similar calculations follow for $\dfrac{d\theta}{dt} < 0$.

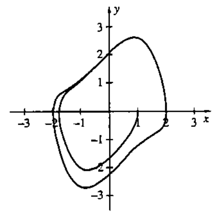
25. (a) $\dfrac{d}{dt}(x^2 + y^2 + z^2) = 2xx' + 2yy' + 2zz'$ and substitute for $x'$, $y'$, and $z'$, where $x' = y$ and $y' = -x$ on $x^2 + y^2 + z^2 = 1$. (b) $z(t) = C_1$, so $z(t) = z_0$. (c) Use the change of variables in Exercise 10.

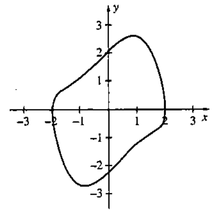
26. (a) $(-1.44225, -0.44225)$; $J(x, y) = \begin{pmatrix} 1 - x^2 & -1 \\ c & -bc \end{pmatrix}$; eigenvalues of $J(-1.44225, -0.44225)$; $\lambda_{1,2} = -0.0977925 \pm 0.431302i$; spiral point, asymptotically stable






27. Use a numerical solver with initial guess $x_0 = 1$ and graph the solution. Observe graph to see that $x_0 = 2$.




### Chapter 7 Review Exercises

1. (a) $\begin{cases} Q(t) = \dfrac{1}{500000}\left(\dfrac{3}{2}e^{-2t/3} - e^{-t}\right) \\ I(t) = \dfrac{1}{500000}(-e^{-2t/3} + e^{-t}) \end{cases}$

(b) $\begin{cases} Q(t) = -\dfrac{4500001}{500000}e^{-t} - 3te^{-t} + \dfrac{9000003}{1000000}e^{-2t/3} \\ I(t) = \dfrac{3000001}{500000}e^{-t} + 3te^{-t} - \dfrac{3000001}{1000000}e^{-2t/3} \end{cases}$

3. (a) $\begin{cases} Q(t) = \dfrac{1}{1000000}e^{-t}\cos t \\ I_2(t) = \dfrac{1}{1000000}e^{-t}\sin t \end{cases}$;

(b) $\begin{cases} Q(t) = 60 + 60e^{-t}\sin t - \dfrac{59999999}{1000000}e^{-t}\cos t \\ I_2(t) = 60 - \dfrac{59999999}{1000000}e^{-t}\sin t - 60e^{-t}\cos t \end{cases}$

5. (a) $\{x(t) = e^{-t} + te^{-t}, y(t) = -te^{-t}\}$ (critically damped);
(b) $\{x(t) = \cos 2t, y(t) = -2\sin 2t\}$ (undamped)

7. $\left\{x(t) = \dfrac{15}{2} - \dfrac{5}{2}e^{-t}, y(t) = \dfrac{15}{2} + \dfrac{5}{2}e^{-t}\right\}$

9. $\{x(t) = 15e^{3t} + 5e^t, y(t) = 15e^{3t} - 5e^t\}$

11. $\begin{cases} x(t) = -\dfrac{5}{3}e^{-5t} - \dfrac{128}{51}e^{-14t} + \dfrac{139}{17}e^{3t} \\ y(t) = \dfrac{7}{6}e^{-5t} - \dfrac{64}{51}e^{-14t} + \dfrac{139}{34}e^{3t} \\ z(t) = -\dfrac{1}{6}e^{-5t} + \dfrac{208}{51}e^{-14t} + \dfrac{139}{34}e^{3t} \end{cases}$

12. $J(x, y) = \begin{pmatrix} 0 & 1 \\ -\dfrac{g}{L}\cos x & -b \end{pmatrix}$;
$J(n\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\dfrac{g}{L}\cos n\pi & -b \end{pmatrix}$;
$n$ is odd: $\lambda = \dfrac{-b \pm \sqrt{b^2 + 4g/L}}{2}$ $(\lambda_2 < 0 < \lambda_1)$ real;
$n$ is even: $\lambda = \dfrac{-b \pm \sqrt{b^2 - 4g/L}}{2}$ (complex conjugate pair or two negative real)

13. $(0, 0)$; $\lambda_1 = a > 0$, $\lambda_2 = -c < 0$, saddle point, unstable; $\left(\dfrac{a}{k}, 0\right)$, $\lambda_1 = -a < 0$, $\lambda_2 = -c + \dfrac{ad}{k} = d\left(\dfrac{a}{k} - \dfrac{c}{d}\right) > 0$, saddle point, unstable; $\left(\dfrac{c}{d}, \dfrac{ad - ck}{bd}\right)$, $\lambda_{1,2} = \dfrac{-ck \pm \sqrt{-4acd^2 + 4c^2dk + c^2k^2}}{2d}$, where $4dc(-ad + ck) < 0$, so $-ck - \sqrt{-4acd^2 + 4c^2dk + c^2k^2} < 0$ if $-4acd^2 + 4c^2dk + c^2k^2 > 0$ (improper node) and $\lambda_{1,2}$ has negative real part if $-4acd^2 + 4c^2dk + c^2k^2 < 0$ (spiral point); asymptotically stable

# CHAPTER 8

## Exercises 8.1

1. $\frac{21}{s^2}$

3. $\frac{2}{s-1}$

5. $\frac{4}{s^2+4}$

7. $\frac{e^{-s}}{s}$

9. $\frac{1}{s^2+1}(e^{-s\pi/2}+s)$

11. $\frac{1}{s^2}(e^{-3s}(2s+1)+s-1)$

13. $-\frac{2e^{-10s}}{s}+\frac{1}{s}$

15. $\frac{k}{s^2+k^2}$

21. $\frac{s-1}{(s-1)^2+9}$

23. $\frac{5}{(s+1)^2+25}$

25. $\frac{2}{(s+1/2)^3}$

27. $\frac{-18}{s-3}$

29. $\frac{s}{s^2+25}$

31. $\frac{s^2+5s+25}{s(s^2+25)}$

33. $\frac{-16}{s-2}$

35. $\frac{5040}{s^8}$

37. $\frac{2}{(s+3)^3}$

39. $\frac{120}{(s+4)^6}$

41. $\frac{s^2-9}{(s^2+9)^2}$

43. $\frac{6s}{(s^2+1)^2}$

45. $\frac{14s}{(s^2-49)^2}$

47. $\frac{1}{2}\left[\frac{1}{s-6}+\frac{1}{s-8}\right]=\frac{s-7}{(s-7)^2-1}$

49. $\frac{s+2}{(s+2)^2+16}$

51. $\frac{s-5}{(s-5)^2+49}$

53. $\frac{-128}{(s^2+16)^2}+\frac{8}{s^2+16}=\frac{8s^2}{(s^2+16)^2}$

55. $\frac{s}{(s^2+9)^2}$

59. $\frac{1}{2}\left[\frac{s}{s^2+4k^2}+\frac{1}{s}\right]=\frac{s^2+2k^2}{s(s^2+4k^2)}$

61. (a) $\frac{\sqrt{2}}{2}\left(\frac{s+1}{s^2+1}\right)$; (b) $\frac{\sqrt{2}}{2}\left(\frac{s-1}{s^2+1}\right)$; (c) $\frac{1}{2}\left(\frac{\sqrt{3}s-1}{s^2+1}\right)$; (d) $\frac{1}{2}\left(\frac{s+\sqrt{3}}{s^2+1}\right)$

63. (a) $\mathcal{L}\{t^{-1/2}\}=\sqrt{\pi/s}$; (b) $\mathcal{L}\{t^{1/2}\}=\frac{\sqrt{\pi}}{2s^{3/2}}$

64. (b) $f(t)=e^{bt}\sin t$, $b>0$, is of exponential order $b$, but $\lim_{t\to\infty}(e^{bt}\sin t)(e^{-bt})=\lim_{t\to\infty}\sin t$ does not exist.

67. $t$

69. $te^{2t}$

71. $\frac{1}{2}t^2e^{-6t}$

73. $4te^{3t}$

75. $\cos 4t$

77. $\frac{2}{9}e^{-2t}+\frac{7}{9}e^{7t}$

79. $\frac{1}{2}+\frac{1}{2}e^{4t}$

81. $\frac{1}{2}e^{6t}+\frac{1}{2}e^{8t}$

83. $-\frac{1}{8}e^{-2t}+\frac{1}{8}e^{6t}$

85. $e^t\sin t$



87. $e^t\cos 7t$

89. $e^{7t}\cos t$

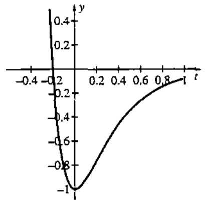
91. (a) $-\frac{ab(b^2-a^2)}{(s^2+a^2)(s^2+b^2)}$, $-\frac{s(b^2-a^2)}{(s^2+a^2)(s^2+b^2)}$;
(b) $\frac{1}{b^2-a^2}\left[\frac{1}{a}\sin at-\frac{1}{b}\sin bt\right]$,
$\frac{1}{b^2-a^2}[\cos at-\cos bt]$

93. (a) $\lim_{s\to\infty}\frac{s}{4-s}=-1$, (b) $\lim_{s\to\infty}\frac{3s}{s+1}=3$, (c) $\lim_{s\to\infty}\frac{s^2}{4s+10}=+\infty$, (d) $\lim_{s\to\infty}\frac{5s^3}{s^2+1}=+\infty$. The inverse Laplace transform does not exist for any of these functions.

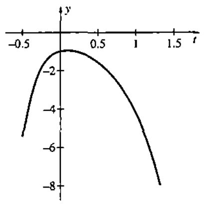
95. $C\approx\frac{1}{2}$ and $T\approx 0.44$

## Exercises 8.2

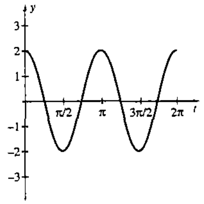
1. $y=\frac{3}{5}e^{-8t}-\frac{8}{5}e^{-3t}$



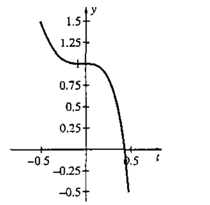
3. $y=-\frac{3}{7}e^{-5t}-\frac{4}{7}e^{2t}$



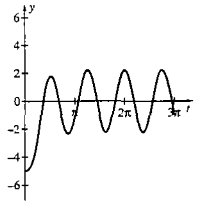
5. $y=2\cos 2t$



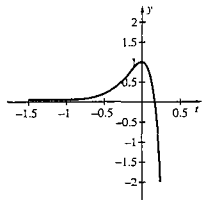
7. $y=\frac{11}{42}e^{-3t}+\frac{13}{6}e^{3t}-\frac{10}{7}e^{4t}$



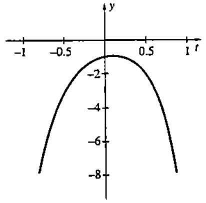
9. $y=-4e^{-t}-\cos 2t-2\sin 2t$



11. $y=\frac{1}{156}(2\cos 2t+154e^{6t}\cos 2t+3\sin 2t-456e^{6t}\sin 2t)$



**13.** $y = -\frac{47}{70}e^{-3t} - \frac{1}{10}e^{2t} - \frac{53}{238}e^{4t} - \frac{1}{170}\cos t + \frac{13}{170}\sin t$

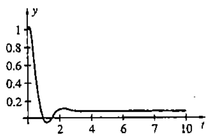



**15.** $y = 2t^2$

**17.** $y = 2t^2$

**19.** $y = 2t^2$

**25.** $y = e^{-2t}\left(\frac{12}{13}\cos 3t + \left(\frac{37}{39} + \frac{1}{6}t\right)\sin 3t\right) + \frac{1}{13}$




**27.** **(a)** $y = J_\mu(x) \Rightarrow \alpha = y(0) = J_\mu(0)$ and $\beta = y'(0) = \frac{1}{2}(J_{\mu-1}(0) - J_{\mu+1}(0))$.

| $\mu$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\beta$ | 0 | 1/2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**(b)** $\mu = 1 \Rightarrow Y(s) = c_1\frac{s}{\sqrt{s^2+1}} + c_2$, $\mu = 2 \Rightarrow Y(s) = c_1\frac{2s^2+1}{\sqrt{s^2+1}} + c_2 s$, $\mu = 3 \Rightarrow Y(s) = c_1\frac{s(4s^2+3)}{\sqrt{s^2+1}} + c_2(4s^2+1)$, $\mu = 4 \Rightarrow Y(s) = c_1\frac{8s^4+8s^2+1}{\sqrt{s^2+1}} + c_2 s(2s^2+1)$, $\mu = 5 \Rightarrow Y(s) = c_1\frac{s(16s^4+20s^2+5)}{\sqrt{s^2+1}} + c_2(16s^4+12s^2+1)$

**(c)**
$$\mathcal{L}\{J_1(x)\} = \frac{1}{\sqrt{s^2+1}(s+\sqrt{s^2+1})},$$
$$\mathcal{L}\{J_2(x)\} = \frac{1}{\sqrt{s^2+1}(s+\sqrt{s^2+1})^2},$$
$$\mathcal{L}\{J_3(x)\} = \frac{1}{\sqrt{s^2+1}(s+\sqrt{s^2+1})^3},$$
$$\mathcal{L}\{J_4(x)\} = \frac{1}{\sqrt{s^2+1}(s+\sqrt{s^2+1})^4},$$
$$\mathcal{L}\{J_5(x)\} = \frac{1}{\sqrt{s^2+1}(s+\sqrt{s^2+1})^5}$$

### *Exercises 8.3*

**1.** $\frac{-28}{se^{3s}}$

**3.** $\frac{3 - e^{4s}}{se^{8s}}$

**5.** $\frac{-7 - 7e^s + e^{5s}}{se^{7s}}$

**7.** $\frac{-42e^{-4s}}{s-1}$

**9.** $\frac{-12}{e^{2s}(s^2-1)}$

**11.** $\frac{-14}{e^{2\pi s/3}(s^2+1)}$

**13.** $\frac{e^s - s - 1}{s^2e^s(1-e^{-2s})}$

**15.** $\frac{e^s + 2}{s(e^{2s}+e^s+1)}$

**17.** $e^{-\pi s}$

**19.** $e^{-s} + e^{-2s}$

**21.** $\frac{e^{-2\pi s}}{s^2+1} + e^{-\pi s/2}$

**23.** $-3\mathcal{U}(t-\pi)$

**25.** $2\mathcal{U}(t-1) - 3\mathcal{U}(t-4)$

**27.** $-3\mathcal{U}(t-6) + 3\mathcal{U}(t-5) - 4\mathcal{U}(t-3)$

**29.** $e^{t-4}\mathcal{U}(t-4)$

**31.** $\cos(2t-6)\mathcal{U}(t-3)$

**33.** $\frac{1}{3}(e^{5t-25} - e^{2t-10})\mathcal{U}(t-5)$

**35.** $2[-t + (4-2t)\mathcal{U}(t-2) + (12-2t)\mathcal{U}(t-6) + (20-2t)\mathcal{U}(t-10) + (28-2t)\mathcal{U}(t-14) + \cdots]$

**37.** $\cosh(t-4)\mathcal{U}(t-4)$

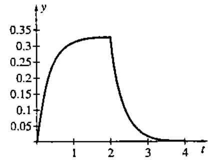


**39.** $t - (t-4)\mathcal{U}(t-4) + (t-8)\mathcal{U}(t-8) - (t-12)\mathcal{U}(t-12) + (t-16)\mathcal{U}(t-16) + \cdots$

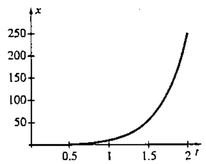
**41.** $\frac{1}{4}[\sin 4t - \sin(4t-12)\mathcal{U}(t-3) + \sin(4t-24)\mathcal{U}(t-6) - \sin(4t-36)\mathcal{U}(t-9) + \cdots]$

**43.**
$$\left(\frac{1}{8}\sin 2t - \frac{1}{4}t\right) + 3\left(\frac{1}{2} - \frac{1}{4}t + \frac{1}{8}\sin(2t-4)\right)\mathcal{U}(t-2) + 9\left(1 - \frac{1}{4}t + \frac{1}{8}\sin(2t-8)\right)\mathcal{U}(t-4) + 27\left(\frac{3}{2} - \frac{1}{4}t + \frac{1}{8}\sin(2t-12)\right)\mathcal{U}(t-6) + \cdots$$

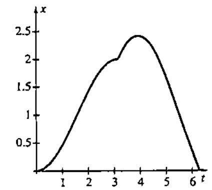
**45.** $y(t) = \frac{1}{3}e^{-3t}(-1 + e^{3t} - e^6\mathcal{U}(2-t) + e^3\mathcal{U}(2-t))$



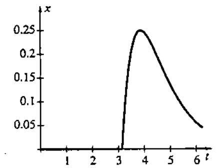

**47.**
$$y = \left[-\frac{\pi}{4+\pi^2}e^{t-1} + \frac{\pi}{36+\pi^2}e^{3t-3} + \frac{32\pi}{(4+\pi^2)(36+\pi^2)}\cos\left(\frac{\pi}{2}(t-1)\right) + \frac{48-4\pi^2}{(4+\pi^2)(36+\pi^2)}\sin\left(\frac{\pi}{2}(t-1)\right)\right]\mathcal{U}(t-1) - \frac{8+\pi^2}{2(4+\pi^2)}e^t + \frac{48+\pi^2}{2(36+\pi^2)}e^{3t} + \frac{48-4\pi^2}{(4+\pi^2)(36+\pi^2)}\cos\left(\frac{\pi}{2}t\right) - \frac{32\pi}{(4+\pi^2)(36+\pi^2)}\sin\left(\frac{\pi}{2}t\right)$$



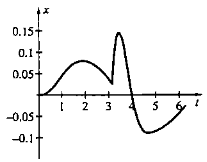

**49.** $x(t) = \sin(t-\pi)\mathcal{U}(t-\pi) + 1 - \cos t$



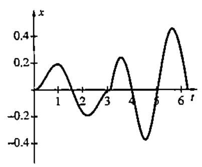

**51.** $x(t) = \mathcal{U}(t-\pi)(e^{\pi-t} - e^{2\pi-2t})$



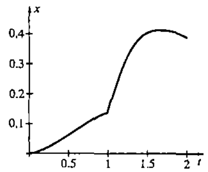

**53.** $x(t) = -\frac{1}{3}e^{2\pi-2t}\mathcal{U}(t-\pi)\sin 3t + \frac{3}{40}\sin t - \frac{1}{40}\cos t - \frac{1}{120}e^{-2t}\sin 3t + \frac{1}{40}e^{-2t}\cos 3t$




**55.** $x(t) = -\frac{1}{3}\mathcal{U}(t-\pi)\sin 3t + \frac{1}{8}\cos t - \frac{1}{8}\cos 3t$



57. $x(t) = -\mathcal{U}(t-1)e^{2-2t} + \mathcal{U}(t-1)e^{1-t} + e^{-2t} + te^{-t} - e^{-t}$



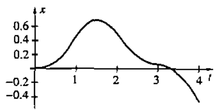

59. $x = 2 + \frac{1}{9}(1 - e^{9\pi - 9t})\cdot\mathcal{U}(t-\pi)$

61. $x(t) = 3\mathcal{U}(t-\pi)e^{3\pi-3t}\sin(t-\pi)$

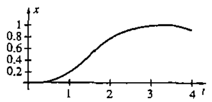
63. $x(t) = 2e^{-3t}\sin t + 3\mathcal{U}(t-\pi)e^{3\pi-3t}\sin(t-\pi)$

65. (a) $\dfrac{e^s-1}{s(e^s+1)}$; (b) $\dfrac{e^s-1}{s^2(e^s+1)}$;
(c) $\dfrac{e^{\pi s}+1}{(s^2+1)(e^{\pi s}-1)}$

(d), (a) $x(t) = (1-\cos t)\mathcal{U}(t) + (-2+2\cos(t-1))\mathcal{U}(t-1) + (2-2\cos(t-2))\mathcal{U}(t-2) + (-2+2\cos(t-3))\mathcal{U}(t-3) + \cdots$




(d), (b) $x(t) = (t-\sin t)\mathcal{U}(t) + (2-2t+2\sin(t-1))\mathcal{U}(t-1) + (-4+2t-2\sin(t-2))\mathcal{U}(t-2) + \cdots$




(d), (c) $x(t) = \frac{1}{2}(-t\cos t + \sin t)\mathcal{U}(t) + ((t-\pi)\cos t - \sin t)\mathcal{U}(t-\pi) + ((t-2\pi)\cos t + \sin t)\mathcal{U}(t-2\pi) + \cdots$


67. $y(t) = -te^{-t} + te^{-t}\mathcal{U}(t) + (10{,}000 - 10{,}000e^{2\pi-t} + 20{,}000\pi e^{2\pi-t} - 10{,}000te^{2\pi-t})\mathcal{U}(t-2\pi)$; $\lim_{t\to\infty} y(t) = 10{,}000$; no, the limit is not affected if the forcing function is changed to $100\delta(t) + 10{,}000\mathcal{U}(t-2\pi)$.

### Exercises 8.4

1. $\frac{1}{3}t^3$

3. $t - 1 + e^{-t}$

5. $-\frac{3}{32}t + \frac{1}{4}t^3 + \frac{3}{128}\sin 4t$

7. $\dfrac{1}{s(s+1)}$

9. $\dfrac{1}{s^2(s^2+1)}$

11. $\dfrac{1}{s^2(s-1)}$

13. $t - 1 + e^{-t}$

15. $\frac{1}{2}t\sin t$

17. $\frac{1}{96}t^3 - \frac{1}{256}t + \frac{1}{1024}\sin 4t$

19. $\frac{1}{116}e^{10t} - \frac{1}{116}\cos 4t - \frac{5}{232}\sin 4t$

21. $-\frac{1}{200}t\cos 10t + \frac{1}{2000}\sin 10t$

23. $g(t) = \sin t$

25. $h(t) = \frac{5}{3}\cos(\sqrt{2}t) - \frac{2\sqrt{2}}{3}\sin(\sqrt{2}t) + \frac{10}{3}e^{-2t}$

27. $y(t) = \left(\frac{1}{10}t^5 + \frac{1}{4}t^4\right)e^{2t}$

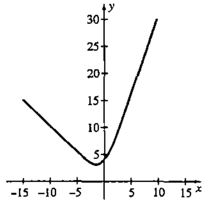
35. $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 4y = 2t^3e^{2t} + 3t^2e^{2t}$, $y(0) = 0$, $y'(0) = 0$

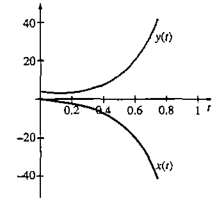
39. (a) $\left(\frac{1}{5}t - \frac{1}{25}\right)e^{2t} + \frac{1}{25}\cos 4t - \frac{3}{100}\sin 4t$;
(b) $\frac{1}{14}t^7 - \frac{3}{2}t^5 + 15t^3 - 45t + \frac{45}{2}\sqrt{2}\sin(t\sqrt{2})$;
(c) $-\frac{13}{6}t^2 + \frac{121}{27} + \frac{1}{54}\cos 3t - \frac{9}{2}\cos t$

### Exercises 8.5

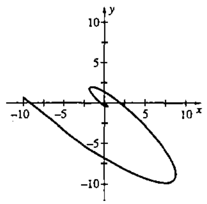
1. $\begin{cases} x(t) = e^{-7t} - e^{5t} \\ y(t) = 3e^{-7t} + e^{5t} \end{cases}$

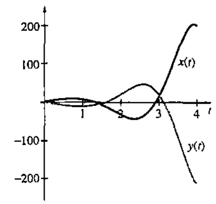




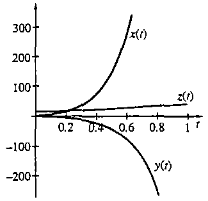




3. $\begin{cases} x(t) = 4e^t\sin 2t + 2e^t\cos 2t \\ y(t) = -4e^t\sin 2t \end{cases}$

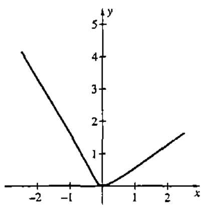


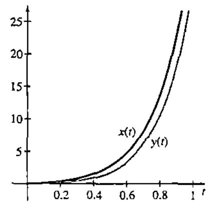




5. $\begin{cases} x(t) = -\frac{21}{2}e^t + 8e^{6t} + \frac{5}{2}e^{-3t} \\ y(t) = -3e^t - 2e^{6t} + 5e^{-3t} \\ z(t) = 15e^t \end{cases}$

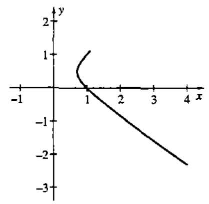


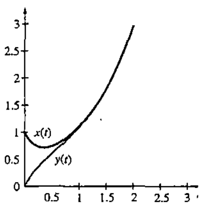

7. $\begin{cases} x(t) = \frac{5}{8}te^{4t} + \frac{3}{64}e^{4t} - \frac{3}{64}e^{-4t} \\ y(t) = \frac{5}{8}te^{4t} - \frac{5}{64}e^{4t} + \frac{5}{64}e^{-4t} \end{cases}$



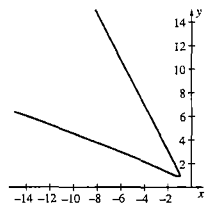



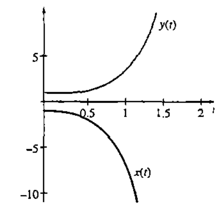

9. $\begin{cases} x(t) = \frac{2}{5}e^t + \frac{3}{5}e^{-4t} \\ y(t) = \frac{2}{5}e^t - \frac{2}{5}e^{-4t} \end{cases}$



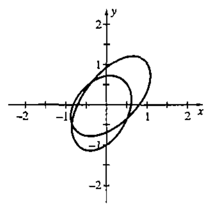



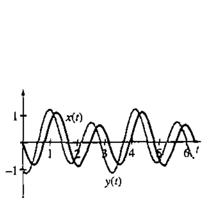

11. $\begin{cases} x(t) = -\frac{3}{10}e^{-t} - \frac{2}{5}e^{2t} - \frac{2}{35}e^{4t} - \frac{17}{70}e^{-3t} \\ y(t) = \frac{3}{10}e^{2t} + \frac{1}{5}e^{-t} + \frac{1}{70}e^{4t} + \frac{17}{35}e^{-3t} \end{cases}$



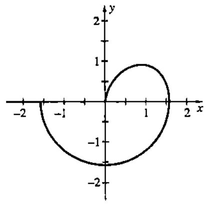



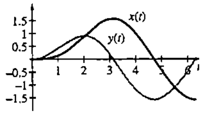

13. $\begin{cases} x(t) = -\frac{1}{6}\cos 2t + \frac{1}{6}\cos 4t + \frac{1}{6}\sin 2t - \frac{5}{6}\sin 4t \\ y(t) = -\frac{1}{4}\cos 2t - \frac{3}{4}\cos 4t + \frac{1}{6}\sin 2t - \frac{7}{12}\sin 4t \end{cases}$



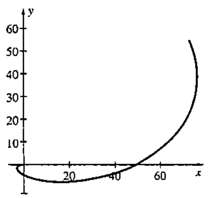



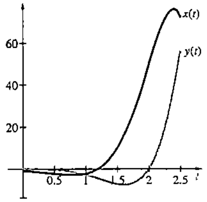
**15.** $\begin{cases} x(t) = \frac{1}{2}(-\pi \cos t + \pi \cos t\, \mathcal{U}(\pi - t) - \\ \quad t \cos t\, \mathcal{U}(\pi - t) + \sin t\, \mathcal{U}(\pi - t)) \\ y(t) = \frac{1}{2} \sin t(\pi - \pi\mathcal{U}(\pi - t) + t\, \mathcal{U}(\pi - t)) \end{cases}$



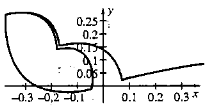


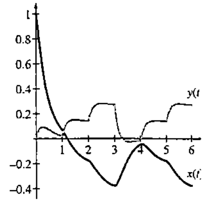

**17.** $\begin{cases} x(t) = -e^{2t} \cos 2t + \left(-\frac{1}{2} - \frac{1}{2}e^{2t} \sin 2t \right. \\ \quad \left. + \frac{1}{2}e^{2t} \cos 2t\right) + (1 + e^{2(t-1)} \sin 2(t - 1) - \\ \quad e^{2(t-1)} \cos 2(t - 1))\mathcal{U}(t - 1) \\ \quad + \left(-\frac{1}{2} - \frac{1}{2}e^{2(t-2)} \sin 2(t - 2) + \right. \\ \quad \left. \frac{1}{2}e^{2(t-2)} \cos 2(t - 2)\right)\mathcal{U}(t - 2) \\ y(t) = -\frac{1}{2}e^{2t} \sin 2t + \left(-\frac{1}{4} + \frac{1}{4}e^{2t} \sin 2t \right. \\ \quad \left. + \frac{1}{4}e^{2t} \cos 2t\right) + \left(\frac{1}{2} - \frac{1}{2}e^{2(t-1)} \sin 2(t - 1) + \right. \\ \quad \left. \frac{1}{2}e^{2(t-1)} \cos 2(t - 1)\right)\mathcal{U}(t - 1) \\ \quad + \left(-\frac{1}{4} + \frac{1}{4}e^{2(t-2)} \sin 2(t - 2) + \right. \\ \quad \left. \frac{1}{4}e^{2(t-2)} \cos 2(t - 2)\right)\mathcal{U}(t - 2) \end{cases}$



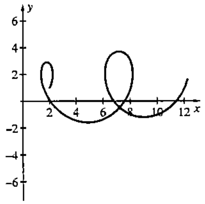


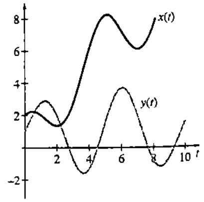

**19.** $\begin{cases} x(t) = \frac{-1}{2}e^{-5t} + \frac{3}{2}e^{-3t} - \\ \quad \left[-\frac{2}{5} - \frac{3}{5}e^{-5(t-3)} + e^{-3(t-3)}\right]\mathcal{U}(t - 3) - \\ \quad \left[\frac{1}{5} + \frac{3}{10}e^{-5(t-2)} - \frac{1}{2}e^{-3(t-2)}\right]\mathcal{U}(t - 2) - \\ \quad \left[\frac{-1}{5} + \frac{3}{10}e^{-5(t-1)} - \frac{1}{2}e^{-3(t-1)}\right]\mathcal{U}(t - 1) \\ \quad + \cdots \\ y(t) = \frac{-1}{2}e^{-5t} + \frac{1}{2}e^{-3t} - \\ \quad \left[\frac{4}{15} - \frac{3}{5}e^{-5(t-3)} + \frac{1}{3}e^{-3(t-3)}\right]\mathcal{U}(t - 3) \\ \quad + \left[\frac{2}{15} - \frac{3}{10}e^{-5(t-2)} + \frac{1}{6}e^{-3(t-2)}\right]\mathcal{U}(t - 2) \\ \quad + \left[\frac{2}{15} - \frac{3}{10}e^{-5(t-1)} + \frac{1}{6}e^{-3(t-1)}\right]\mathcal{U}(t - 1) \\ \quad + \cdots \end{cases}$





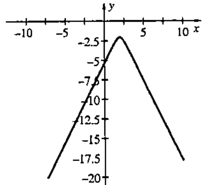
**21.** $\begin{cases} x(t) = t + 1 + \frac{\sqrt{2}}{2} \sin \sqrt{2}t + \cos \sqrt{2}t - \sin t \\ y(t) = 1 + \sqrt{2} \sin \sqrt{2}t - \cos \sqrt{2}t + \cos t \end{cases}$

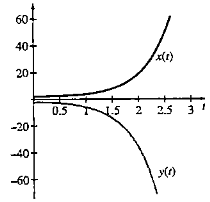




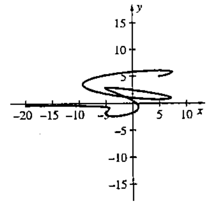


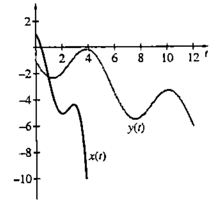

**23.** $\begin{cases} x(t) = \frac{1}{10} \cos t + \frac{15}{8} - \frac{1}{4}t + \frac{13}{40}e^{2t} - \frac{3}{10}e^{-2t} \\ y(t) = \frac{1}{10} \sin t - \frac{3}{4} - \frac{13}{20}e^{2t} - \frac{3}{5}e^{-2t} \end{cases}$






**25.** $\begin{cases} x(t) = -\frac{1}{2} - \frac{1}{3}e^{t} + 2 \sin t - 3 \cos t - \\ \quad \frac{4\sqrt{2}}{3} \sin \sqrt{2}t + \frac{29}{6} \cos \sqrt{2}t \\ y(t) = -\frac{1}{2}t - \frac{3}{2} \sin t - \cos t \end{cases}$






**27.** $x(t) = 1 - \frac{5}{11}t + \frac{15}{121}\sqrt{22} \sin\left(\frac{1}{2}t\sqrt{22}\right) - \frac{5}{9} \cos\left(\frac{1}{2}t\sqrt{22}\right) + \frac{5}{9} \cos t$, $y(t) = \frac{5}{11} + \frac{1}{9}\sqrt{22} \sin\left(\frac{1}{2}t\sqrt{22}\right) + \frac{6}{11} \cos\left(\frac{1}{2}t\sqrt{22}\right) - \frac{2}{9} \sin t$

### *Exercises 8.6*

**1.** $Q(t) = 2 - 2e^{-t} - 2(1 - e^{1-t})\mathcal{U}(t - 1)$, $I(t) = 2e^{-t} - 2(1 - e^{1-t})\delta(t - 1) - 2e^{1-t}\mathcal{U}(t - 1)$

**3.** $I(t) = \frac{1}{2}(1 - e^{-t}) - \frac{1}{2}(1 - e^{-(t-1)})\mathcal{U}(t - 1) + \frac{1}{2}(1 - e^{-(t-2)})\mathcal{U}(t - 2) - \frac{1}{2}(1 - e^{-(t-3)})\mathcal{U}(t - 3) + \cdots$

**5.** $I(t) = (t - 1 + e^{-t}) - (t - 1)\mathcal{U}(t - 1) + (t - 3 + e^{-(t-2)})\mathcal{U}(t - 2) - (t - 3)\mathcal{U}(t - 3) + (t - 5 + e^{-(t-4)})\mathcal{U}(t - 4) - (t - 5)\mathcal{U}(t - 5) + \cdots$

**7.** $I(t) = 100te^{-3t}$

**9.** $I(t) = \frac{2}{3}\sqrt{3}e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t\right) - \frac{2}{3}\sqrt{3}e^{-t/2+\pi/2} \sin\left(\frac{\sqrt{3}}{2}(t - \pi)\right)\mathcal{U}(t - \pi)$

**11.** **(a)** $I(t) = e^{-4t+4}\mathcal{U}(t - 1)$; **(b)** $I(t) = e^{-4t} + e^{-4t+4}\mathcal{U}(t - 1)$

**13.** $I(t) = t - 1 + e^{-t} - (t - 2 + e^{-t+1})\mathcal{U}(t - 1)$

**15.** $I(t) = e^{2-t}\mathcal{U}(t - 2) + e^{6-t}\mathcal{U}(t - 6)$

**19.** $x(t) = \frac{1}{2}((-t \cos t + \sin t)\mathcal{U}(t) - ((\pi - t) \cos t + \sin t)\mathcal{U}(t - \pi))$

**21.** $x(t) = 3 \cos 3t - \frac{2}{3} \sin 3t$

**23.** $x(t) = e^{-2t}(2t + 1)$

**25.** $x(t) = \frac{1}{3}e^{-4t}(4e^{3t} - 1)$

**27.** $x(t) = e^{-2t}(4 - 4e^{t} + 2t + 2te^{t})$

**29.** $x(t) = \frac{1}{9}e^{-4t}(1 - e^{3t} - 3t + 6te^{3t})$

**31.** $x(t) = \frac{1}{36 + 13\pi^2 + \pi^4}[((\pi^2 - 6) \cos \pi t - e^{-3t}(-12e^3 + 18e^{t+2} - 3e^3\pi^2 + 2\pi^2e^{t+2} + 5\pi e^{3t} \sin \pi t)\mathcal{U}(t - 1)) - e^{-3t}(-12 + 18e^{t} - 3\pi^2 + 2\pi^2e^{t} + e^{3t}(\pi^2 - 6) \cos \pi t - 5\pi e^{3t} \sin \pi t)\mathcal{U}(t)]$

**33.** $x(t) = -\frac{1}{2}e^{-2t}((3e^4 + e^{2t} - 2te^4)\mathcal{U}(t - 2) - (-2 + e^2 + e^{2t} + 4t - 2te^2 - 2t^2)\mathcal{U}(t - 1) - 2t^2\mathcal{U}(t))$

**35.** $x(t) = \frac{1}{18}e^{-2t}(3t \sin 3t\, \mathcal{U}(t) - (6\pi \cos 3t - 3t \cos 3t + \sin 3t)\mathcal{U}(t - 2\pi) + (3\pi \cos 3t - 3t \cos 3t + \sin 3t + 3\pi \sin 3t - 3t \sin 3t)\mathcal{U}(t - \pi))$

**37.** $x(t) = \frac{1}{3(\pi^4 + 17\pi^2 + 16)}[(3(\pi^2 - 4) \cos \pi t - e^{-4t}(15\pi e^{4t} \sin \pi t + 16e^{3t+1} - 4e^4 + \pi^2e^{3t+1} - 4\pi^2e^4)) \cdot \mathcal{U}(t - 1) - e^{-4t}(3e^{4t}(\pi^2 - 4) \cos \pi t + 16e^{3t} + \pi^2e^{3t} - 4\pi^2 - 15\pi e^{4t} \sin \pi t - 4)\mathcal{U}(t)]$

**39.** $x(t) = \frac{1}{15}(\sin t - \frac{1}{4} \sin 4t - 15 \sin(4 - 4t)\mathcal{U}(t - 1))$

**41.** $x(t) = \frac{1}{3}e^{-2t}(e^6 \sin(3t - 9)\mathcal{U}(t - 3) + e^2 \sin(3t - 3)\mathcal{U}(t - 1))$

**43.** $x(t) = 200 + \frac{127150}{13}e^{-5t} + \frac{250}{13}\cos t - \frac{1250}{13}\sin t$, bounded

**45.** $x(t) = 200 + \frac{63650}{13}e^{-5t} - \frac{1250}{3}\cos t - \frac{250}{13}\sin t$; bounded

**47.** $x(t) = -250 + 5350e^{2t} - 200 \sin t - 100 \cos t$, unbounded

**49.** $x(t) = -10000 + 17500e^t + 2500 \cos t + 2500 \sin t + (10000 - 10000e^{t-5} + 2500e^{t-5}\cos 5 - 2500 \cos 5 \cos(t - 5) + 2500e^{t-5}\sin 5 - 2500 \sin 5 \cos(t - 5) - 2500 \cos 5 \sin(t - 5) + 2500 \sin 5 \sin(t - 5))\mathcal{U}(t - 5)$

**51.** $x(t) = -5000\left\{1 - \frac{5}{2}e^t + \frac{1}{2}(\cos t - \sin t) + \frac{1}{2}(-2 + 2e^{t-1} + e^{t-1}\cos 1 - \cos t - e^{t-1}\sin 1 + \sin t)\mathcal{U}(t - 1) + \left(1 - \frac{3}{2}e^{t-2} + \frac{1}{2}(\cos(t - 2) - \sin(t - 2))\right)\mathcal{U}(t - 2) + \cdots\right\}$

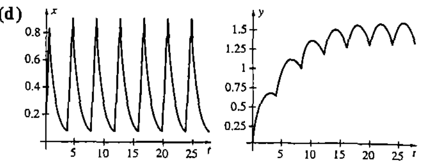
**53.** $x(t) = e^{t-2}(100e^2 + 200\mathcal{U}(t - 2)) \Rightarrow x(5) = e^3(200 + 100e^2) \approx 18{,}858$

**55.** $x(t) = \frac{1}{k + k^3}e^{-kt}(-c + ce^{kt} + ck - ck^2 + ck^2e^{kt} + kx_0 + k^3x_0 - cke^{kt}\cos t + ck^2e^{kt}\sin t)$

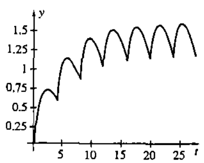
**57.** (b) $x(t) = \sum_{n=0}^{6} \frac{c_0}{a}\left[(1 - e^{-a(t-4n)})\mathcal{U}(t - 4n) - (1 - e^{-a(t-4n-1/2)})\mathcal{U}\left(t - 4n - \frac{1}{2}\right)\right]$

(c) $y(t) = \sum_{n=0}^{6} c_0\left[\frac{1}{b}(1 - e^{-b(t-4n)})\mathcal{U}(t - 4n) - \frac{1}{b - a}(e^{-a(t-4n)} + e^{-b(t-4n)})\mathcal{U}(t - 4n)\right] - \sum_{n=0}^{6} c_0\left[\frac{1}{b}(1 - e^{-b(t-4n-1/2)})\mathcal{U}\left(t - 4n - \frac{1}{2}\right) - \frac{1}{b - a}(e^{-a(t-4n-1/2)} + e^{-b(t-4n-1/2)})\mathcal{U}\left(t - 4n - \frac{1}{2}\right)\right]$

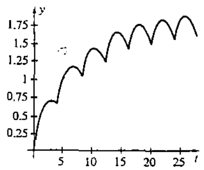
(d)




(e)




(f)




**59.** (d) $x(t) = \cos 5t - \cos 5\sqrt{3}t$, $y(t) = 2(\cos 5t + \cos 5\sqrt{3}t)$

**63.** $\begin{cases} x(t) = \frac{2}{5}\cos t - \frac{2}{5}\cos\sqrt{6}t \\ y(t) = \frac{4}{5}\cos t + \frac{1}{5}\cos\sqrt{6}t \end{cases}$

**65.** $\begin{cases} x(t) = \frac{1}{3}\cos t + \frac{2}{3}\cos 2t \\ y(t) = \frac{2}{3}\cos t - \frac{2}{3}\cos 2t \end{cases}$

**67.** $\begin{cases} x(t) = \frac{4}{5}\cos\sqrt{3}t + \frac{1}{5}\cos\frac{t}{\sqrt{2}} \\ y(t) = -\frac{2}{5}\cos\sqrt{3}t + \frac{2}{5}\cos\frac{t}{\sqrt{2}} \end{cases}$

**69.** $\begin{cases} x(t) = \frac{1}{4} + \frac{1}{18}\sin 2t + \frac{7}{12}\cos 2t + \frac{1}{18}\sin t + \frac{1}{6}\cos t - \frac{1}{6}t\cos t \\ y(t) = \frac{1}{4} - \frac{1}{18}\sin 2t - \frac{7}{12}\cos 2t + \frac{4}{9}\sin t + \frac{1}{3}\cos t - \frac{1}{3}t\cos t \end{cases}$



**71.** $\begin{cases} x(t) = \frac{7}{18}\cos t + \frac{11}{18}\cos 2t + \frac{1}{12}t\sin 2t \\ y(t) = \frac{7}{9}\cos t - \frac{7}{9}\cos 2t - \frac{1}{12}t\sin 2t \end{cases}$

**73.** $\begin{cases} \theta_1(t) = -\frac{1}{8}\sqrt{3}\sin\frac{2t}{\sqrt{3}} + \frac{1}{8}\sin 2t \\ \theta_2(t) = -\frac{1}{4}\sqrt{3}\sin\frac{2t}{\sqrt{3}} - \frac{1}{4}\sin 2t \end{cases}$

**75.** $\begin{cases} \theta_1(t) = \frac{1}{2}\cos\frac{2t}{\sqrt{3}} + \frac{1}{2}\cos 2t \\ \theta_2(t) = \cos\frac{2t}{\sqrt{3}} - \cos 2t \end{cases}$

**77.** $\begin{cases} \theta_1(t) = \frac{\sqrt{3}}{4}\sin\frac{2t}{\sqrt{3}} + \frac{1}{4}\sin 2t \\ \theta_2(t) = \frac{\sqrt{3}}{2}\sin\frac{2t}{\sqrt{3}} - \frac{1}{2}\sin 2t \end{cases}$

**79.** $\begin{cases} \theta_1(t) = -\frac{1}{4}\cos\frac{2t}{\sqrt{3}} + \frac{1}{4}\cos 2t \\ \theta_2(t) = -\frac{1}{2}\cos\frac{2t}{\sqrt{3}} - \frac{1}{2}\cos 2t \end{cases}$

**81.** $\begin{cases} x(t) = -\cos\left(\frac{t\sqrt{2k + k\omega^2}}{\sqrt{m}}\right); \\ y(t) = \cos\left(\frac{t\sqrt{2k + k\omega^2}}{\sqrt{m}}\right) \end{cases}$

max. distance = 1 for $x$ and $y$.

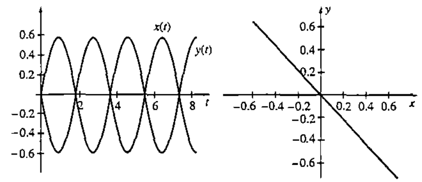
**83.** $x(t)$ has frequency $\omega_1$, where $\omega_1^2 = \frac{9k}{2m} - \frac{k\sqrt{17}}{2m}$; $y(t)$ has frequency $\omega_2$, where $\omega_2^2 = \frac{9k}{2m} + \frac{k\sqrt{17}}{2m}$

**87.** (a) $Q(t) = \frac{292175000}{159587072641}e^{-100(5+2\sqrt{6})t} - \frac{7387691623}{1276696581128\sqrt{6}}e^{-100(5+2\sqrt{6})t} + \frac{292175000}{159587072641}e^{400\sqrt{6}t-100(5+2\sqrt{6})t} + \frac{7387691623}{1276696581128\sqrt{6}}e^{400\sqrt{6}t-100(5+2\sqrt{6})t} - \frac{584350000}{159587072641}\cos 377t - \frac{204799950}{159587072641}\sin 377t$

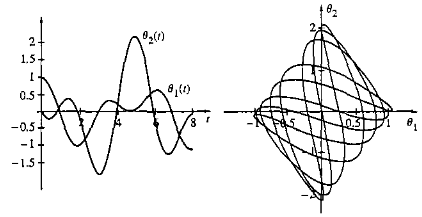
(c) $I(t) = -\frac{584350000}{159587072641}\cos 377t - \frac{204799950}{159587072641}\sin 377t$

**89.** See *Instructor's Resource Manual* and *Student Resource Manual.*

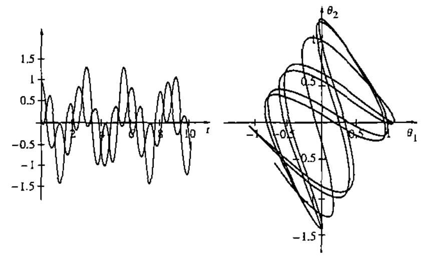
**91.** $\begin{cases} x(t) = -\frac{1}{3}\sqrt{3}\sin\sqrt{3}t \\ y(t) = \frac{1}{3}\sqrt{3}\sin\sqrt{3}t \end{cases}$



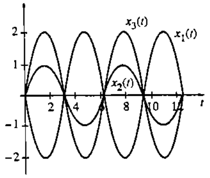

**93.** $\begin{cases} \theta_1(t) = \frac{1}{8}\left[4\cos 2t + 4\cos\left(\frac{2}{\sqrt{3}}t\right) + \sin 2t - \sqrt{3}\sin\left(\frac{2}{\sqrt{3}}t\right)\right] \\ \theta_2(t) = -\cos 2t + \cos\left(\frac{2}{\sqrt{3}}t\right) - \frac{1}{4}\sin 2t - \frac{\sqrt{3}}{4}\sin\left(\frac{2}{\sqrt{3}}t\right) \end{cases}$




**95.** See *Student Resource Manual.*



**97.** $\begin{cases} x_1(t) = -2 \sin t \\ x_2(t) = \sin t \\ x_3(t) = 2 \sin t \end{cases}$




## *Chapter 8 Review Exercises*

**1.** $\dfrac{1}{s} - \dfrac{1}{s^2}$

**3.** $\dfrac{e^{5s} - 1}{se^{5s}}$

**5.** $\dfrac{5(s^4 + 24)}{s^6}$

**7.** $(s - 2)^{-2}$

**9.** $6(s - 1)^{-4}$

**11.** $\dfrac{s^2 - 9}{(s^2 + 9)^2}$

**13.** $\dfrac{s + 5}{s^2 + 10s + 34}$

**15.** $e^{-3\pi s/2}$

**17.** $\dfrac{2(3 - 2e^{3s})}{se^{7s}}$

**19.** $\dfrac{-42e^{5-s}}{s - 5}$

**21.** $\dfrac{2}{s^3e^{2s}} + \dfrac{4}{s^2e^{2s}} + \dfrac{4}{se^{2s}}$

**23.** $\dfrac{e^{2s} - 2e^s + 1}{s^2e^{2s}(1 - e^{-2s})} = \dfrac{e^s - 1}{s^2(e^s + 1)}$

**25.** $\dfrac{\pi e^s + \pi}{e^s(s^2 + \pi^2)(1 - e^{-2s})} = \dfrac{\pi e^s}{(e^s - 1)(s^2 + \pi^2)}$

**27.** $-2 \sinh 5t$

**29.** $t^5 - 3$

**31.** $e^{-5t} \cos t - 2$

**33.** $7\mathcal{U}(t - 7) + 6\mathcal{U}(t - 3)$

**35.** $8 \cos(t - 3)\mathcal{U}(t - 3)$

**37.** $\dfrac{1}{27}e^{-3t}(9t^2e^{3t} - 6te^{3t} + 2e^{3t} - 2)$

**39.** $y(t) = e^{-3t} \sin t$

**41.** $y(t) = 6t^2 + 1$

**43.** $y(t) = -e^{-4t} + 2e^{-2t}$
$+ \left(\dfrac{1}{8} + \dfrac{1}{8}e^{4-4t} - \dfrac{1}{4}e^{2-2t}\right)\mathcal{U}(t - 1)$

**45.** $x(t) = \dfrac{1}{8} \cos t - \dfrac{1}{8} \cos 3t - \dfrac{1}{3} \sin(3t)\mathcal{U}(t - \pi)$

**47.** $g(t) = 1 + \sin t - \cos t$

**51.** (a) $\dfrac{s}{s^2 - 1}$ (b) $\dfrac{1}{s^2 - 1}$ (c) $\dfrac{s^3}{s^4 + 4}$ (d) $\dfrac{s^2 + 2}{s^4 + 4}$ (e) $\dfrac{s^2 - 2}{s^4 + 4}$ (f) $\dfrac{2s}{s^4 + 4}$

**53.** $x(t) = -2 \sin \dfrac{t}{2} + \dfrac{1}{15} \cos(2t)\mathcal{U}(t - \pi) -$
$\dfrac{1}{15} \sin\left(\dfrac{t}{2}\right)\mathcal{U}(t - \pi) - \dfrac{1}{15} \cos(2t) + \dfrac{1}{15} \cos\left(\dfrac{t}{2}\right)$

**55.** $Q(t) = -220\left(\dfrac{1}{10000} - \dfrac{1}{10000} \cos(100t - 200)\right)$
$\mathcal{U}(t - 2) + \dfrac{11}{500} - \dfrac{11}{500} \cos(100t)$

**57.** $I(t) = 100\sqrt{2}e^{-t} \sin(2\sqrt{2}t)$

**59.** $x(t) = 10000e^{2t} + 100e^{2t-2}\mathcal{U}(t - 1)$

**61.** $y(x) = \dfrac{w}{24}(x^4 - 2x^3 + x^2)$

**65.** $\begin{cases} x = e^t - 1 \\ y = -\dfrac{1}{2} + 2e^t - \dfrac{3}{2}e^{2t} \end{cases}$

**67.** $\begin{cases} x = -\dfrac{2}{3} + e^t - \dfrac{1}{3}e^{3t} + \\ \qquad 2\left(\dfrac{1}{3} + \dfrac{1}{6}e^{3t-6} - \dfrac{1}{2}e^{t-2}\right)\mathcal{U}(t - 2) \\ y = -\dfrac{1}{3} + \dfrac{1}{3}e^{3t} - \left(-\dfrac{1}{3} + \dfrac{1}{3}e^{3t-6}\right)\mathcal{U}(t - 2) \end{cases}$

**69.** $\begin{cases} x(t) = \dfrac{2}{5}\left(2\sqrt{2} \sin(\sqrt{2}t) + \sqrt{3} \sin\left(\dfrac{1}{\sqrt{3}}t\right)\right) \\ y(t) = \dfrac{1}{5}\left(-2\sqrt{2} \sin(\sqrt{2}t) + 4\sqrt{3} \sin\left(\dfrac{1}{\sqrt{3}}t\right)\right) \end{cases}$

**71.** $\begin{cases} x(t) = \dfrac{1}{2}\left(\cos t - \cos \sqrt{3}t + t \sin t\right) \\ y(t) = \dfrac{1}{2}\left(-\cos t + \cos \sqrt{3}t + t \sin t\right) \end{cases}$

**73.** $\begin{cases} x(t) = \dfrac{1}{8}\left(-3 \sin 2t - \sqrt{3} \sin\left(\dfrac{2}{\sqrt{3}}t\right)\right) \\ y(t) = \dfrac{1}{4}\left(3 \sin 2t - \sqrt{3} \sin\left(\dfrac{2}{\sqrt{3}}t\right)\right) \end{cases}$



# CHAPTER 9

## *Exercises 9.1*

**1.** $y = 0$

**3.** no solution

**5.** $y = \dfrac{e}{e^2 - e^{-2}}(e^{2x} - e^{-2x})$

**7.** $y = 0$

**9.** $y = C \sin x$

**11.** $y = 2\pi \cos 2\pi x - \sin 2\pi x$

**13.** $y = 0$

**15.** $\lambda_n = \begin{cases} 0, n = 0 \\ n^2\pi^2, n = 1, 2, \ldots \end{cases}$
$y_n(x) = \begin{cases} 1, n = 0 \\ \cos n\pi x, n = 1, 2, \ldots \end{cases}$

**17.** $\lambda_n = \left[\dfrac{(2n - 1)\pi}{2}\right]^2$, $y_n(x) = \sin \dfrac{(2n - 1)\pi x}{2}$, $n = 1, 2, \ldots$

**19.** $\lambda_n = \dfrac{1}{4}\left(1 + k_n^2\right)$, where $\tan k_n = 2k_n$, $n = 1, 2, \ldots$, $y_n(x) = e^{-x/2} \sin(k_nx)$

**21.** $\lambda_n = \dfrac{1}{2}(1 + k_n^2)$, where $\tan k_n = 2k_n$, $n = 1, 2, \ldots$, $y_n(x) = e^{-x/2} \sin(k_nx)$

**23.** $\lambda_n = \dfrac{1}{2}(4 + n^2\pi^2)$, $n = 1, 2, \ldots$, $y_n(x) = e^{2x} \sin n\pi x$

**25.** $s(x) = 1$, $y_m(x) = \cos m\pi x$, $y_n(x) = \cos n\pi x$; if $m \neq n$,
$\displaystyle\int_0^1 \cos m\pi x \cos n\pi x dx =$
$\left[\dfrac{\sin(m - n)\pi x}{2(m - n)} + \dfrac{\sin(m + n)\pi x}{2(m + n)}\right]_0^1 = 0.$

**27.** $s(x) = 1$, $y_m(x) = \sin \dfrac{(2m - 1)\pi x}{2}$, $y_n(x) =$
$\sin \dfrac{(2n - 1)\pi x}{2}$; if $m \neq n$,
$\displaystyle\int_0^1 \sin \dfrac{(2m - 1)\pi x}{2} \sin \dfrac{(2n - 1)\pi x}{2} dx =$
$\left[\dfrac{\sin(2m - 2n)\pi x}{2(2m - 2n)} - \dfrac{\sin(2m + 2n + 2)\pi x}{2(2m + 2n)}\right]_0^1 = 0.$

**29.** $p(x) = e^{\int -4dx} = e^{-4x}$, $s(x) = e^{-4x}$, $y_m(x) =$
$e^{2x} \sin m\pi x$, $y_n(x) = e^{2x} \sin n\pi x$; if $m \neq n$,
$\displaystyle\int_0^1 e^{-4x}(e^{2x} \sin m\pi x)(e^{2x} \sin n\pi x)\, dx =$
$\displaystyle\int_0^1 \sin m\pi x \sin m\pi x\, dx =$
$\left[\dfrac{\sin(m - n)\pi x}{2(m - n)} - \dfrac{\sin(m + n)\pi x}{2(m + n)}\right]_0^1 = 0.$

**37.** $\lambda_n = 1 + n^2\pi^2$, $n = 1, 2, \ldots$, $y_n(x) = x \sin(n\pi \ln x)$

**39.** $y(x) = \dfrac{1}{2}x^2 - x$

**41.** $y(x) = C \sin x - \dfrac{1}{3} \sin 2x$

**45.** $\lambda_1 \approx (2.02876)^2 \approx 4.11586$, $\lambda_2 \approx (4.91318)^2 \approx 24.1393$, $\lambda_3 \approx (7.97867)^2 \approx 63.6591$

## *Exercises 9.2*

**1.** $a_0 = 1$, $a_n = \dfrac{2}{n^2\pi^2}[(-1)^n - 1]$, $n \geq 1$;
$f(x) = \dfrac{1}{2} - \displaystyle\sum_{n=1}^{\infty} \dfrac{4}{(2n - 1)^2\pi^2} \cos(2n - 1)\pi x =$
$\dfrac{1}{2} - \dfrac{4}{\pi^2} \cos \pi x - \dfrac{4}{3^2\pi^2} \cos 3\pi x - \cdots$

**3.** $a_0 = \dfrac{2}{3}$, $a_n = \dfrac{4}{n^2\pi^2}(-1)^n$, $n \geq 1$; $f(x) =$
$\dfrac{1}{3} + \displaystyle\sum_{n=1}^{\infty} \dfrac{4(-1)^n}{n^2\pi^2} \cos n\pi x = \dfrac{1}{3} - \dfrac{4}{\pi^2} \cos \pi x +$
$\dfrac{4}{2^2\pi^2} \cos 2\pi x + \cdots$

**5.** $a_0 = 2$, $a_n = 0$, $n \geq 1$, $f(x) = 1$

**7.** $a_0 = 2$, $a_n = \dfrac{2}{n^2\pi^2}\left(-4 + 4 \cos \dfrac{n\pi}{2} + n\pi \sin \dfrac{n\pi}{2}\right)$,
$n \geq 1$, $f(x) =$
$1 + \displaystyle\sum_{n=1}^{\infty} \dfrac{2}{n^2\pi^2}\left(-4 + 4 \cos \dfrac{n\pi}{2} + n\pi \sin \dfrac{n\pi}{2}\right) \cos \dfrac{n\pi x}{4}$

**9.** $a_0 = -2(e^{-1} - 1)$, $a_n = \dfrac{2}{n^2\pi^2 + 1}[(-1)^{n+1}e^{-1} + 1]$,
$n \geq 1$, $f(x) = (1 - e^{-1}) +$
$\displaystyle\sum_{n=1}^{\infty} \dfrac{2}{n^2\pi^2 + 1}[(-1)^{n+1}e^{-1} + 1] \cos n\pi x =$
$(1 - e^{-1}) + \dfrac{2(1 + e^{-1})}{\pi^2 + 1} \cos \pi x +$
$\dfrac{2(1 - e^{-1})}{2^2\pi^2 + 1} \cos 2\pi x + \cdots$

**11.** $b_n = \frac{2}{n\pi}(-1)^{n+1}; f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1} \sin n\pi x =$
$\frac{2}{\pi} \sin \pi x - \frac{2}{2\pi} \sin 2\pi x + \frac{2}{3\pi} \sin 3\pi x + \cdots$

**13.** $b_n = 2(-1)^n\left[\frac{2}{n^3\pi^3} - \frac{1}{n\pi}\right] - \frac{4}{n^3\pi^3}$
$f(x) = \sum_{n=1}^{\infty}\left[2(-1)^n\left[\frac{2}{n^3\pi^3} - \frac{1}{n\pi}\right] - \frac{4}{n^3\pi^3}\right] \sin n\pi x =$
$2\left(\frac{-4}{\pi^3} + \frac{1}{\pi}\right) \sin \pi x + 2\left(-\frac{1}{2\pi}\right) \sin 2\pi x +$
$2\left(\frac{-4}{3^3\pi^3} + \frac{1}{3\pi}\right) \sin 3\pi x + \cdots$

**15.** $b_n = \frac{2}{n\pi}(1 - (-1)^n), f(x) =$
$\sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin(2n-1)\pi x =$
$\frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin 3x + \frac{4}{5\pi} \sin 5x + \cdots$

**17.** $b_n = \frac{-2}{n^2\pi^2}\left(n\pi\left(\cos\frac{n\pi}{2} - \cos n\pi\right) - 4 \sin\frac{n\pi}{2}\right), n \geq 1$
$f(x) =$
$\frac{-2}{n^2\pi^2}\left(n\pi\left(\cos\frac{n\pi}{2} - \cos n\pi\right) - 4 \sin\frac{n\pi}{2}\right) \sin\frac{n\pi x}{4}$

**19.** $b_n = \frac{2n\pi}{1 + n^2\pi^2}(1 - e^{-1}(-1)^n), f(x) =$
$\sum_{n=1}^{\infty} \frac{2n\pi}{1 + n^2\pi^2}(1 - e^{-1}(-1)^n) \sin n\pi x =$
$\frac{2\pi(1 + e^{-1})}{1 + \pi^2} \sin \pi x + \frac{4\pi(1 - e^{-1})}{1 + 2^2\pi^2} \sin 2\pi x +$
$\frac{6\pi(1 + e^{-1})}{1 + 3^2\pi^2} \sin 3\pi x + \cdots$

**21.** $\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$

**23.** $\cos^3 x = \frac{3}{4} \cos x + \frac{1}{4} \cos 3x$

**25.** $\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$

**27.** $x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$ on $0 < x < \pi$; $b_n =$
$-\frac{2(-1)^{n+1}}{n(1 - n^2)}, n \geq 2$

**29.** $y(x, 0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{n\pi \cdot 0}{\ell}\right) =$
$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right) = f(x); \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)$ is the Fourier sine series for $f(x)$, so $A_n = \frac{2}{\ell}\int_0^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx.$

**31.** cosine series: (a) $a_0 = 1/2$,
$a_n = \frac{2}{n^4\pi^4}(6 + 3(-2 + n^2\pi^2) \cos n\pi)$;
(b) $a_0 = 1/6$, $a_n = \frac{2}{n^4\pi^4}(-6 - (-6 + n^2\pi^2) \cos n\pi)$;
(c) $a_0 = 2/5$, $a_n = \frac{2}{n^4\pi^4}(4(-6 + n^2\pi^2) \cos n\pi)$;
(d) $a_0 = 4/15$
$a_n = \frac{-4}{n^4\pi^4}(-12 + n^2\pi^2) \cos n\pi$
sine series: (a) $b_n = \frac{-2}{n^3\pi^3}(-6 + n^2\pi^2) \cos n\pi$;
(b) $b_n = \frac{-4}{n^3\pi^3}(1 + 2 \cos n\pi)$;
(c) $b_n = \frac{2}{n^5\pi^5}(24 - (24 - 12n^2\pi^2 + n^4\pi^4) \cos n\pi)$
(d) $b_n = \frac{-4(12 + n^2\pi^2 + (5n^2\pi^2 - 12) \cos n\pi)}{n^5\pi^5}$

## *Exercises 9.3*

**1.** $a_0 = 0; a_n = 0, n \geq 1; b_n = -\frac{2}{n\pi} \cos n\pi =$
$\frac{2}{n\pi}(-1)^{n+1}, n \geq 1; f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1} \sin n\pi x =$
$\frac{2}{\pi} \sin \pi x - \frac{1}{\pi} \sin 2\pi x + \frac{2}{3\pi} \sin 3\pi x + \cdots$

**3.** $a_0 = 0; a_n = 0, n \geq 1; b_n = \left(\frac{12}{n^3} - \frac{2\pi^2}{n}\right) \cos n\pi =$
$\left(\frac{12}{n^3} - \frac{2\pi^2}{n}\right)(-1)^n, n \geq 1; f(x) =$
$\sum_{n=1}^{\infty}\left(\frac{12}{n^3} - \frac{2\pi^2}{n}\right)(-1)^n \sin nx = (2\pi^2 - 12) \sin x +$
$\left(\frac{3}{2} - \pi^2\right) \sin 2x + \left(\frac{2\pi^2}{3} - \frac{4}{9}\right) \sin 3x + \cdots$

**5.** $a_0 = -2; a_n = 0, n \geq 1;$
$b_n = \frac{2}{n\pi}(1 - (-1)^n), n \geq 1;$
$f(x) = -1 + \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin \frac{(2n-1)\pi x}{2}$
$= -1 + \frac{4}{\pi} \sin \frac{\pi x}{2} + \frac{4}{3\pi} \sin \frac{3\pi x}{2}$
$+ \frac{4}{5\pi} \sin \frac{5\pi x}{2} + \cdots$

**7.** $a_0 = \frac{1}{2}; a_n = \frac{1}{n^2\pi^2}(1 - (-1)^n), n \geq 1;$
$b_n = \frac{1}{n\pi}(2(-1)^{n+1} + 1), n \geq 1;$
$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi^2}(1 - (-1)^n) \cos n\pi x +\right.$
$\left.\frac{1}{n\pi}(2(-1)^{n+1} + 1) \sin n\pi x\right] = \frac{1}{4} + \frac{2}{\pi^2} \cos \pi x +$
$\frac{3}{\pi} \sin \pi x - \frac{1}{2\pi} \sin 2\pi x + \cdots$

**9.** $a_0 = (e - e^{-1}); a_n = \frac{(-1)^n}{1 + n^2\pi^2}(e - e^{-1}), n \geq 1;$
$b_n = \frac{(-1)^{n+1}n\pi}{1 + n^2\pi^2}(e - e^{-1}), n \geq 1; f(x) = \frac{1}{2}(e - e^{-1}) +$
$\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{1 + n^2\pi^2}(e - e^{-1}) \cos n\pi x +\right.$
$\left.\frac{(-1)^{n+1}n\pi}{1 + n^2\pi^2}(e - e^{-1}) \sin n\pi x\right] = \frac{1}{2}(e - e^{-1}) -$
$\frac{e - e^{-1}}{1 + \pi^2} \cos \pi x - \frac{(e - e^{-1})}{1 + \pi^2} \sin \pi x +$
$\frac{e - e^{-1}}{1 + 2^2\pi^2} \cos 2\pi x + \frac{2(e - e^{-1})\pi}{1 + 2^2\pi^2} \sin 2\pi x + \cdots$

**11.** odd; $f(-x) = (-x)^3 = -x^3 = -f(x)$

**13.** neither; $f(-x) = (-x)^2 - (-x) = x^2 + x \neq -f(x)$ or $f(x)$

**15.** odd; $f(-x) = -f(x)$

**17.** odd; $f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}, f(-x) = -f(x)$

**19.** neither

**31.** hint: let $x = p$.

**33.** hint: add the series in Exercises 31 and 32.

**35.** **(a)** $\frac{\pi}{2}$; **(b)** $\pi$; **(c)** $\pi$; **(d)** $\frac{\pi}{4}$; **(e)** $\frac{3\pi}{4}$

**37.** **(a)** 0; **(b)** $\frac{3}{4}$; **(c)** $-\frac{1}{4}$; **(d)** $\frac{1}{2}$; **(e)** $\frac{1}{2}$

**39.** $a_0 = 2, a_n = -\frac{4}{n\pi} \sin \frac{n\pi}{2}, n \geq 1; f(x) = 1 +$
$\sum_{n=1}^{\infty}\left(-\frac{4}{n\pi} \sin \frac{n\pi}{2}\right) \cos \frac{n\pi x}{2} = 1 - \frac{4}{\pi} \cos \frac{\pi x}{2} +$
$\frac{4}{3\pi} \cos \frac{3\pi x}{2} + \cdots$

**41.** $a_0 = \frac{\pi^2}{3}, a_n = -\frac{2}{n^2}[(-1)^n + 1], n \geq 1; f(x) = \frac{\pi^2}{6} +$
$\sum_{n=1}^{\infty} -\frac{2}{n^2}[(-1)^n - 1] \cos nx = 1 + 4 \cos x +$
$\frac{4}{3^2} \cos 3x + \cdots$



**43.** $b_n = \frac{4}{n\pi}\left[-\cos n\pi + \cos \frac{n\pi}{2}\right], n \geq 1;$
$\sum_{n=1}^{\infty} \frac{4}{n\pi}\left[-\cos n\pi + \cos \frac{n\pi}{2}\right] \sin \frac{n\pi x}{2} = \frac{4}{\pi} \sin \frac{\pi x}{2} +$
$\frac{4}{3\pi} \sin \frac{3\pi x}{2} + \frac{4}{5\pi} \sin \frac{5\pi x}{2} + \cdots$

**45.** $b_n = \frac{4}{n^3\pi}[1 - \cos n\pi], n \geq 1;$
$\sum_{n=1}^{\infty} \frac{4}{n^3\pi}[1 - (-1)^n] \sin nx =$
$\sum_{n=1}^{\infty} \frac{8}{(2n-1)^3\pi} \sin (2n-1)x = \frac{8}{\pi} \sin x +$
$\frac{8}{3^3\pi} \sin 3x + \frac{8}{5^3\pi} \sin 5x + \cdots$

**47.** $a_0 = 2, a_n = 0, n \geq 1; b_n = -\frac{2}{n\pi}(1 + (-1)^n), n \geq 1;$
$f(x) = 1 + \sum_{n=1}^{\infty} -\frac{2}{n\pi}(1 + (-1)^n) \sin n\pi x = 1 -$
$\frac{4}{2\pi} \sin 2\pi x + \frac{4}{4\pi} \sin 4\pi x + \cdots$

**49.** $a_0 = \frac{\pi^2}{3}, a_n = \frac{-1}{n^2}, n \geq 1;$
$b_n = 0, n \geq 1;$
$f(x) = \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{1}{n^2} \cos 2nx = \frac{\pi^2}{6} - \cos 2x -$
$\frac{1}{4} \cos 4x - \frac{1}{9} \cos 9x - \cdots$

**51.** If $T_1 = nT_2$, $n$ is an integer.

**53.** $c_n = \frac{-e^{in\pi}(1 - in\pi) + e^{-in\pi}(1 + in\pi)}{2\pi n^2}, \tilde{c}_n =$
$\frac{e^{in\pi}(1 - in\pi) + e^{-in\pi}(1 + in\pi)}{2\pi n^2},$
$f(x) = \sum_{n=1}^{\infty}\left(-\frac{2}{n} \cos n\pi \sin nx + \frac{2}{n^2\pi} \sin n\pi \sin nx\right)$

**55.** $\left|\frac{\sin nx}{n^2}\right| \leq \frac{1}{n^2}; \sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent $p$-series ($p = 2 > 1$).

**57.** **(a)** $\sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin n\pi x$; **(b)** no; **(c)** no, the FS for $g$ is not uniformly convergent.

**59.** $a_0 = \frac{2}{2\pi}\int_0^{2\pi} \frac{x(2\pi - x)}{4} dx = \frac{\pi^2}{3}$, so $a_0/2 = \pi^2/6$.

61. (a) $a_n = 0,\ n \geq 0;\ b_n = \frac{1}{\pi}\int_0^{\pi} x^3 \sin nx\,dx = \frac{-2(-6+n^2\pi^2)}{n^3}\cos n\pi,\ n \geq 1$

(b) $a_0 = \frac{1}{\pi}\int_0^{\pi} x^2(1-x)\,dx = \frac{2\pi^2}{3},\ a_n = \frac{1}{\pi}\int_0^{\pi} x^2(1-x)\cos nx\,dx = \frac{4}{n^2}\cos n\pi,\ n \geq 1$

$b_n = \frac{1}{\pi}\int_0^{\pi} x^2(1-x)\sin nx\,dx = \frac{2(-6+n^2\pi^2)}{n^3}\cos n\pi,\ n \geq 1$

(c) $a_0 = \frac{1}{\pi}\int_0^{\pi} x^4\,dx = \frac{2\pi^4}{5},\ a_n = \frac{1}{\pi}\int_0^{\pi} x^4 \cos nx\,dx = \frac{8(-6+n^2\pi^2)}{n^4}\cos n\pi,\ n \geq 1;\ b_n = 0,\ n \geq 1$

(d) $a_0 = \frac{1}{\pi}\int_0^{\pi} x(1-x^3)\,dx = -\frac{2\pi^4}{5},\ a_n = \frac{1}{\pi}\int_0^{\pi} x(1-x^3)\cos nx\,dx = \frac{-8(-6+n^2\pi^2)}{n^4}\cos n\pi,\ n \geq 1,\ b_n = \frac{1}{\pi}\int_0^{\pi} x(1-x^3)\sin nx\,dx = \frac{-2}{n}\cos n\pi,\ n \geq 1$

63. Alternating series. $\frac{1}{2n+1} = 0.000005$ when $n = 99999.5$. Requires $n = 99999$ terms.

### Exercises 9.4

1. (See Exercise 17, Section 9.1) $c_n = 2\int_0^1 f(x)\sin\frac{(2n-1)\pi x}{2}\,dx$; (a) $c_n = \frac{4}{(2n-1)\pi}$; (b) $c_n = \frac{8}{(2n-1)^2\pi^2}\sin\frac{(2n-1)\pi}{2} = \frac{8(-1)^{n+1}}{(2n-1)^2\pi^2}$; (c) $c_n = \frac{16}{(2n-1)^2\pi^2}\sin\frac{(2n-1)\pi}{2} - \frac{32}{(2n-1)^3\pi^3} = \frac{16(-1)^{n+1}}{(2n-1)^2\pi^2} - \frac{32}{(2n-1)^3\pi^3}$

3. $p(x) = e^{2x},\ s(x) = e^{2x},\ c_n = 2\int_0^1 e^{2x}f(x)(e^{-x}\sin n\pi x)\,dx$; (a) $c_n = \frac{2n\pi}{n^2\pi^2+1}(1 - e\cos n\pi) = \frac{2n\pi}{n^2\pi^2+1}[1 - e(-1)^n]$; (b) $c_n = \frac{2}{n^2\pi^2+1}\left(n\pi - e^{1/2}n\pi\cos\frac{n\pi}{2} + e^{1/2}\sin\frac{n\pi}{2}\right)$

5. $p(x) = e^{2x},\ s(x) = e^{2x};\ c_n = \frac{2}{5}\int_0^5 e^{2x}f(x)\left(e^{-x}\sin\frac{n\pi x}{5}\right)dx$; (a) $c_n = -\frac{2n\pi}{n^2\pi^2+25}[e^5\cos n\pi - 1] = -\frac{2n\pi}{n^2\pi^2+25}[e^5(-1)^n - 1]$; (b) $c_n = -\frac{2}{n^2\pi^2+25}\left[e^3\left(n\pi\cos\frac{3n\pi}{5} - 5\sin\frac{3n\pi}{5}\right) - n\pi\right]$

7. $\lambda_n = \frac{3-n^2\pi^2}{4},\ n = 1, 2, \ldots;\ c_n = 2\int_0^1 e^x g(x)(e^{-x/2}\sin n\pi x)\,dx$

9. $c_n = 2\int_0^1 e^x x^2(e^{-x/2}\sin n\pi x)\,dx = -\frac{4(-1)^n n\pi}{(1+4n^2\pi^2)^3}(-112n^2\pi^2 + 34 + 32n^4\pi^4) - \frac{64n\pi(4n^2\pi^2-3)}{(1+4n^2\pi^2)^3}$

11. $\int_{-1}^1 f(x)P_m(x)\,dx = \sum_{n=1}^{\infty}\int_{-1}^1 c_nP_n(x)P_m(x)\,dx$. If $m = n$, $\int_{-1}^1 f(x)P_n(x)\,dx = c_n\int_{-1}^1 P_n^2(x)\,dx = \frac{2}{2n+1}c_n$. If $m \neq n$, $\int_{-1}^1 c_nP_n(x)P_m(x)\,dx = 0$.

13. (See Section 4.8 for Legendre polynomials) $c_0 = \frac{1}{2}\int_0^1 2x(1)\,dx = \frac{1}{2};\ c_1 = \frac{3}{2}\int_0^1 2x(x)\,dx = 1;\ c_2 = \frac{5}{2}\int_0^1 2x\left(\frac{1}{2}(3x^2-1)\right)dx = \frac{5}{8};\ c_3 = \frac{7}{2}\int_0^1 2x\left(\frac{1}{2}(5x^3-3x)\right)dx = 0$. Then, $f(x) = c_0P_0(x) + c_1P_1(x) + c_2P_2(x) + c_3P_3(x) + \cdots$.

15. $c_0 = \int_0^1 x(1)\,dx = \frac{1}{2};\ c_2 = 5\int_0^1 x\left(\frac{1}{2}(3x^2-1)\right)dx = \frac{5}{8};\ c_4 = 9\int_0^1 x\left(\frac{1}{8}(35x^4 - 30x^2 + 3)\right)dx = -\frac{3}{16}$

17. $c_1 = 3\int_0^1 x\,dx = \frac{3}{2};\ c_3 = 7\int_0^1\left(\frac{1}{2}(5x^3-3x)\right)dx = -\frac{7}{8};\ c_5 = 11\int_0^1\left(\frac{1}{8}(65x^5 - 70x^3 + 15x)\right)dx = \frac{11}{16}$

19. If $x = \cos\theta$, then $dx = -\sin\theta\,d\theta$. If $x = -1$, then $\theta = \pi$; if $x = 1$, $\theta = 0$. Then, $c_n = \frac{2n+1}{2}\int_{\pi}^0 f(\theta)P_n(x)(-\sin\theta\,d\theta) = \frac{2n+1}{2}\int_0^{\pi} f(\theta)P_n(x)\sin\theta\,d\theta$.

21. (a) $c_0 = \pi^2/4,\ c_1 = -3,\ c_2 = 5\pi^2/16,\ c_3 = -28/9,\ c_4 = 81\pi^2/256,\ c_5 = -704/225,\ c_6 = 325\pi^2/1024,\ c_7 = -768/245,\ c_8 = 20825\pi^2/65536,\ c_9 = -311296/99225$; (b) $c_0 = 0,\ c_1 = 3\pi/4,\ c_2 = -15/8,\ c_3 = 7\pi/24,\ c_4 = -45/32,\ c_5 = 319\pi/480,\ c_6 = -637/384,\ c_7 = 159\pi/448,\ c_8 = -289/192,\ c_9 = 101251\pi/161280$

### Chapter 9 Review Exercises

1. (a) $y_n(x) = \sin\frac{n\pi x}{L},\ n = 1, 2, \ldots$; (b) $\lambda_n = \frac{n^2\pi^2}{L^2},\ n = 1, 2, \ldots$; (c) $\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}},\ n = 1, 2, \ldots$

3. $b_n = \frac{1}{\pi}\left[\frac{2}{n} + \frac{2}{n}(-1)^n - \frac{4}{n}\cos\frac{n\pi}{2}\right]$

5. $b_n = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$

7. $a_0 = 0;\ a_n = \frac{4}{n\pi}\sin\frac{n\pi}{2}$

9. $a_0 = 1;\ a_n = \frac{4}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$

11. $a_0 = 1;\ a_n = -\frac{2}{n\pi}\sin\frac{n\pi}{2}$

13. $a_0 = 1 - \frac{\pi}{2};\ a_n = -\frac{1}{n^2\pi}[\cos n\pi - 1];\ b_n = -\frac{1}{n\pi}[(\pi+1)\cos n\pi - 1]$

15. $a_0 = \frac{3}{2};\ a_n = \frac{1}{n^2\pi^2}[1 - \cos n\pi];\ b_n = \frac{1}{n\pi}\cos n\pi$

17. (a) let $x = 0$; (b) $x = \pi$

19. let $x = \frac{\pi}{2}$

21. (a) $b_1 = \frac{3}{4},\ b_3 = -\frac{1}{4},\ b_n = 0,\ n \neq 1, 3$; (b) $a_1 = \frac{3}{4},\ a_3 = \frac{1}{4},\ a_n = 0,\ n \neq 1, 3$

## CHAPTER 10

### Exercises 10.1

1. $u(x, y) = Ce^{k(x-y)}$
3. $u(x, y) = Cy^ke^{kx}$
5. $u(x, y) = Ce^{kx+(k-1)y}$
7. $u(x, y) = G(x) + F(y)$
9. $u(x, y) = Cx^ke^{-y/k}$
11. $u_{xx} = 2,\ u_{yy} = -2$
13. $u_{xx} = e^x\sin y,\ u_{yy} = -e^x\sin y$
15. $u_{xx} = \frac{2xy}{(x^2+y^2)^2},\ u_{yy} = \frac{-2xy}{(x^2+y^2)^2}$
17. $u_{xx} = 2,\ u_{tt} = 2$
19. $u_{xx} = -4\sin 2x\cos 2t,\ u_{tt} = -4\sin 2x\cos 2t$
21. $u_{xx} = -k^2\sin kx\cos kt,\ u_{tt} = -k^2\sin kx\cos kt$
23. $u_{xx} = -e^{-t}\sin x,\ u_t = -e^{-t}\sin x$
25. $u_{xx} = -16e^{-16t}\cos 4x,\ u_t = -16e^{-16t}\cos 4x$
27. $u_{xx} = -k^2e^{-k^2t}\cos kx,\ u_t = -k^2e^{-k^2t}\cos kx$
29. $u_{xx} = -e^{-k^2t}\cos x,\ 16u_t = -16k^2e^{-k^2t}\cos x;\ k = \pm\frac{1}{4}$
31. $u_{xx} = -16\cos ct\sin 4x,\ 4u_{tt} = -4c^2\cos ct\sin 4x;\ c = \pm 2$
33. yes
35. yes, $u_t = -2\sin 2x\sin 2t$
39. (b) $\frac{\partial u}{\partial x} = \cos(x+1) = 0,\ x = \pi/2 - 1,\ u(\pi/2 - 1, 1) = 1;\ u(0, 2) = \sin 2;\ u(0, 3) = \sin 3$; The maximum displacement, $|u|$, is 1 in each case; (c) $u(\pi/4, t) = \sin(t + \pi/4),\ u(\pi/4, 0) = \sin(\pi/4) = \sqrt{2}/2$.
41. (b) 100; (c) 100

### Exercises 10.2

1. $b_n = 2\int_0^1 100\,x\sin n\pi x\,dx = \frac{-200}{n\pi}\cos n\pi,\ u(x, t) = \sum_{n=1}^{\infty}\frac{200(-1)^{n+1}}{n\pi}\sin n\pi x e^{-n^2\pi^2t}$

3. $b_n = 2\int_0^1 x(1-x)\sin n\pi x\,dx = \frac{-4}{n^3\pi^3}(\cos n\pi - 1);\ u(x, t) = \sum_{n=1}^{\infty} b_n\sin n\pi x e^{-n^2\pi^2t/2}$

5. $b_2 = 1,\ b_8 = 1,\ b_n = 0,\ n \neq 2, 8;\ u(x, t) = \sin 2xe^{-4t} + \sin 8xe^{-64t}$

7. $S(x) = 10x,\ b_n = \dfrac{20}{n\pi}\cos n\pi,\ u(x, t) = S(x) + \displaystyle\sum_{n=1}^{\infty} b_n \sin n\pi x e^{-n^2\pi^2 t}$

9. $S(x) = 20 - 10x,\ b_n = \dfrac{20}{n\pi}(\cos n\pi - 2),\ u(x, t) = S(x) + \displaystyle\sum_{n=1}^{\infty} b_n \sin n\pi x e^{-n^2\pi^2 t}$

11. $S(x) = \dfrac{T_0}{2}(1 - x)$

13. $S(x) = T_0$

15. $S(x) = T - \dfrac{1}{e - e^{-1}}(-Te^{-1} + T - T_1 + T_0e^{-1})e^x + \dfrac{1}{e - e^{-1}}(T - eT + eT_0 - T_1)e^{-x}$

17. $A_0 = 1,\ a_n = 0,\ n \geq 1;\ u(x, t) = 1$

19. $A_0 = 2,\ a_2 = -2,\ a_n = 0,\ n \neq 2;\ u(x, t) = 2 - 2\cos 2xe^{-4t}$

21. $S(x) = T_0, \begin{cases} v_t = v_{xx} \\ v(0, t) = v_x(0, t) = 0, \quad t > 0 \\ v(x, 0) = u(x, 0) - S(x) = f(x) - T_0 \end{cases},$ $v(x, t) = X(x)T(t),\ X_n(x) = \sin\dfrac{(2n-1)\pi x}{2},\ n = 1, 2, \ldots,\ \lambda_n = \left[\dfrac{(2n-1)\pi}{2}\right]^2,\ n = 1, 2, \ldots;$ $v(x, t) = \displaystyle\sum_{n=1}^{\infty} a_n \sin\frac{(2n-1)\pi x}{2} e^{-\lambda_n t},$ $a_n = 2\displaystyle\int_0^1 [f(x) - S(x)] \sin\frac{(2n-1)\pi x}{2}\,dx,\ u(x, t) = S(x) + v(x, t)$

22. $S(x) = T_0,\ v(x, t) = X(x)T(t),\ X_n(t) = \cos\dfrac{(2n-1)\pi x}{2},$ $n = 1, 2, \ldots,\ \lambda_n = \left[\dfrac{(2n-1)\pi}{2}\right]^2,\ n = 1, 2, \ldots,$ $v(x, t) = \displaystyle\sum_{n=1}^{\infty} a_n \cos\frac{(2n-1)\pi x}{2} e^{-\lambda_n t},\ \lambda_n = \left[\frac{(2n-1)\pi}{2}\right]^2,$ $n = 1, 2, \ldots,\ a_n = 2\displaystyle\int_0^1 [f(x) - S(x)] \cos\frac{(2n-1)\pi x}{2}\,dx$

23. $u(x, t) = A_0 + \displaystyle\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)e^{-c^2n^2t},$ $A_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,$ $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

25. $a_n = 2\displaystyle\int_0^1 [2T_0 - T_0]\sin\frac{(2n-1)\pi x}{2}\,dx = \frac{4T_0}{(2n-1)\pi}$

27. $u(x, t) = T_0$

29. $A_0 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} x\,dx = \frac{\pi}{2},\ a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{n^2\pi}(\cos n\pi - 1),\ b_n = 0$

31. **(a)** $S(x) = -\dfrac{1}{6}x^3 + \left(T_1 - T_0 + \dfrac{1}{6}\right)x + T_0$; **(b)** $v(x, t) = \displaystyle\sum_{n=1}^{\infty} B_n \sin n\pi x e^{-n^2\pi^2 t},\ B_n = 2\int_0^1 [f(x) - S(x)]\sin n\pi x\,dx;$ **(c)** $B_n = 2\displaystyle\int_0^1 \left[-x\left(T_1 + \frac{1}{6}\right)\right]\sin n\pi x\,dx = \frac{1}{3}\frac{\cos n\pi}{n\pi}(6T_1 + 1)$

33. $b_n = -\dfrac{6(8 + 4(-2 + n^2\pi^2))}{n^5\pi^5}\cos n\pi,\ u(x, t) = \displaystyle\sum_{n=1}^{\infty} \frac{6(8 + 4(-2 + n^2\pi^2))}{n^5\pi^5}(-1)^{n+1}\sin n\pi x e^{-n^2\pi^2 t}$

35. $S(x) = 50 + 50x,$ $b_n = 2\displaystyle\int_0^1 [x(1 - x^3) - (50 + 50x)]\sin n\pi x\,dx = \frac{2(-24 - 50n^4\pi^4 + 4(6 - 3n^2\pi^2 + 25n^4\pi^4))}{n^5\pi^5}\cos n\pi;$ $u(x, t) = \displaystyle\sum_{n=1}^{\infty} b_n \sin n\pi x e^{-n^2\pi^2 t}$

37. $A_0 = \displaystyle\int_0^1 e^x \sin\pi x\,dx = \frac{\pi(1 + e)}{1 + \pi^2},$ $a_n = \displaystyle\int_0^1 e^x \sin\pi x\cos n\pi x\,dx = \frac{2\pi(1 + \pi^2 - n^2\pi^2 + e(1 + \pi^2 - n^2\pi^2))\cos n\pi}{n^4\pi^4 - 2n^2\pi^2(\pi^2 - 1) + (1 + \pi^2)^2},$ $u(x, t) = \dfrac{\pi(1 + e)}{1 + \pi^2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos n\pi x e^{-n^2\pi^2 t}$

### *Exercises 10.3*

1. $u(x, t) = \cos\pi t \sin\pi x$

3. $u(x, t) = 1/(3\pi)\sin 3\pi t \sin 3\pi x$

5. $u(x, t) = 1/\pi \sin\pi t \sin\pi x + \displaystyle\sum_{n=1}^{\infty} a_n \cos n\pi t \sin n\pi x$ where $a_n = \dfrac{-2(1 + \cos n\pi)}{n^2\pi^2},\ n \geq 1$

7. $u(x, t) = \displaystyle\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)\sin nx;\ a_2 = 1,$ $a_n = 0,\ n \neq 2,\ b_n = 0,\ n \geq 1;\ u(x, t) = \cos 2t \sin 2x$

9. $u(x, t) = \displaystyle\sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{2} + b_n \sin\frac{n\pi t}{2}\right)\sin\frac{n\pi x}{2};$ $a_n = \displaystyle\int_0^1 x\sin\frac{n\pi x}{2}\,dx + \int_1^2 (2 - x)\sin\frac{n\pi x}{2}\,dx = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2};\ b_n = 0,\ n \geq 1$

11. $u_x = u_r r_x + u_s s_x = u_r + u_s;\ u_{xx} = u_{rr} + 2u_{rs} + u_{ss};\ u_y = u_r r_y + u_s s_y = u_r - u_s;\ u_{yy} = u_{rr} - 2u_{rs} + u_{ss};\ u_{rs} = 0;$ $u(x, y) = F(x + y) + G(x - y)$

13. $u_x = u_r;\ u_{xx} = u_{rr};\ u_{xy} = u_{rr} + u_{rs};\ u(x, y) = F(x + y) + G(y)$

15. $B^2 - 4AC = (-2)^2 - 4(1)(1) = 0$, parabolic

17. $B^2 - 4AC = (-4)^2 - 4(1)(2) = 8 > 0$, hyperbolic

19. $B^2 - 4AC = (1)^2 - 4(1)(1) = -3 < 0$, elliptic

21. $B^2 - 4AC = (2)^2 - 4(1)(-8) = 36 > 0;\ u_{xx} = 16u_{rr} - 16u_{rs} + 4u_{ss};\ u_{xy} = -4u_{rr} - 2u_{rs} + 2u_{ss};\ u_{yy} = u_{rr} + 2u_{rs} + u_{ss};\ u_{xx} + 2u_{xy} - u_{yy} = -36u_{rs};\ u(r, s) = F(r) + G(s);\ u(x, y) = F(y - 4x) + G(y + 2x)$

23. $B^2 - 4AC = (0)^2 - 4(0)(1) = 0$

25. $B^2 - 4AC = (0)^2 - 4(1)(1) = -4 < 0$

27. $u(x, t) = \dfrac{1}{2}[\sin\pi(x + t) + \sin\pi(x - t)] = \sin\pi x \cos\pi t$

29. $u(x, t) = \dfrac{1}{2}\left[\dfrac{1}{8}\sin(x + t) + \dfrac{1}{8}\sin(x - t)\right] - \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t}\frac{1}{8}\sin v\,dv = \frac{1}{8}\sin x\cos t - \frac{1}{8}\sin x\sin t$

31. $u(x, t) = At + B + \displaystyle\sum_{n=1}^{\infty}\left(a_n \cos\frac{cn\pi t}{p} + b_n \sin\frac{cn\pi t}{p}\right)\cos\frac{n\pi x}{p};\ A = \frac{1}{p}\int_0^p g(x)\,dx,\ B = \frac{1}{p}\int_0^p f(x)\,dx,\ a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi x}{p}\,dx,\ b_n = \frac{2}{cn\pi}\int_0^p g(x)\cos\frac{n\pi x}{p}\,dx$

### *Exercises 10.4*

1. $u(x, y) = \displaystyle\sum_{n=1}^{\infty} B_n \sinh\frac{n\pi x}{2}\sin\frac{n\pi y}{2};\ B_n = \frac{1}{\sinh\frac{n\pi}{2}}\int_0^2 (1 + y)\sin\frac{n\pi y}{2}\,dy = \frac{1}{\sinh\frac{n\pi}{2}}\,\frac{2 - 6\cos n\pi}{n\pi}$



3. $u(x, y) = \displaystyle\sum_{n=1}^{\infty} A_n \sinh\frac{n\pi(1 - x)}{2}\sin\frac{n\pi y}{2};\ A_n = \frac{2}{2}\int_0^2 y\sin\frac{n\pi y}{2}\,dy = \frac{-4}{n\pi\sinh n\pi/2}\cos n\pi$

5. $u(x, y) = \displaystyle\sum_{n=1}^{\infty} B_n \sinh n\pi(2 - y)\sin n\pi x;$ $B_n = \dfrac{2}{n\pi\sinh 2n\pi}(\cos n\pi - 1)$

7. $u(x, y) = \displaystyle\sum_{n=1}^{\infty} B_n \sinh n\pi y \sin n\pi x;\ u(x, 2) = \sum_{n=1}^{\infty} B_n \sinh 2n\pi \sin n\pi x = \sin\pi x \Rightarrow B_1 \sinh 2\pi = 1,$ $B_1 = \dfrac{1}{\sinh 2\pi};\ B_n = 0,\ n \neq 1$

9. $u(x, y) = a_0 + b_0 y + \displaystyle\sum_{n=1}^{\infty} (a_n \cosh ny + b_n \sinh ny)\cos nx;$ $u(x, 0) = a_0 + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx = 1 + 4\cos x;\ a_0 = 1,$ $a_1 = 4,\ a_n = 0,\ n \geq 2;\ b_n = 0,\ n \geq 1,\ b_0 = -1$

11. **(a)** $Y_n(y) = \sin\dfrac{n\pi y}{b},\ \lambda_n = \left(\dfrac{n\pi}{b}\right)^2$; **(b)** $a_n = 0$; **(c)** $u(0, y) = \displaystyle\sum_{n=1}^{\infty} B_n \sin\frac{n\pi y}{b} = f(y) \Rightarrow B_n = \frac{2}{b}\int_0^b f(y)\sin\frac{n\pi y}{b}\,dy$

13. $B_n = \dfrac{2}{1}\displaystyle\int_0^1 T_0 \sin n\pi y\,dy = \frac{2T_0}{n\pi}(1 - \cos n\pi)$

15. $u(x, y) = \displaystyle\sum_{n=1}^{\infty} B_n e^{-n\pi y/a}\sin\frac{n\pi x}{a},\ B_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$

17. $B_n = \dfrac{2}{2}\displaystyle\int_0^2 x\sin\frac{n\pi x}{2}\,dx = \frac{-4}{n\pi}\cos n\pi$

19. $S(x, y) = a_0 + b_0 y + \displaystyle\sum_{n=1}^{\infty} (a_n \sinh n\pi(\pi - y) + b_n \sinh n\pi y)\cos n\pi x;\ S(x, 0) = a_0 + \sum_{n=1}^{\infty} a_n \sinh n\pi^2 \cos n\pi x = \cos\pi x,\ a_1 = \frac{1}{\sinh\pi^2},\ a_n = 0,\ n \neq 1;\ S(x, \pi) = b_0\pi + \sum_{n=1}^{\infty} b_n \sinh\pi^2 y \cos n\pi x = 0,\ b_n = 0,\ n \geq 0;$ $u(x, y) = \dfrac{\sinh\pi(\pi - y)}{\sinh\pi^2}\cos\pi x$

**23.** $B_{mn} = \frac{4}{(1)(1)} \int_0^{\pi} \int_0^1 (x + y) \sin m\pi x \sin n\pi y \, dx \, dy =$
$\frac{2}{m^2 n^2 \pi^3}[-4mn\pi \cos m\pi + mn\pi(1 + \pi) \cos(m - n)\pi$
$- 2mn\,\pi^2 \cos n\pi + mn\pi(1 + \pi) \cos(m + n)\pi]$

**27.** $B_{mn} = 0;\ \lambda_{12} = c\pi\sqrt{\left(\frac{1}{\pi}\right)^2 + \left(\frac{2}{\pi}\right)^2} = c\sqrt{5},$
$C_{12} = \frac{1}{\lambda_{12}}\left(\frac{1}{4}\right) = \frac{1}{4c\sqrt{5}}$

**29.** $B_{mn} = \frac{4}{(1)(2)} \int_0^2 \int_0^1 (xy)(1 - x)$
$\times (1 - y) \sin m\pi x \sin \frac{n\pi y}{2} \, dx \, dy$
$= \frac{-8}{m^3 n^3 \pi^6}((2n^2\pi^2 - 8) \cos m\pi \cos n\pi - (2n^2\pi^2 - 8)$
$\times \cos n\pi + 8 \cos m\pi - 8);\ C_{mn} = 0;\ n, m \geq 1$

**35.** $a_n = \frac{2}{\pi} \int_0^{\pi} \cos x \sin nx \, dx = \frac{2n(1 + \cos n\pi)}{(n^2 - 1)\pi},\ b_n = 0,$
$c_n = 0,\ n \geq 1,\ d_n = \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx =$
$\frac{-2n(\cos n\pi^2 - 1)}{n^2\pi},\ u(x, y) =$
$\sum_{n=1}^{\infty} (a_n \cosh ny \sin nx + d_n \sinh n\pi(\pi - x) \sin n\pi y)$

**37.** $b_n = \frac{2}{\pi \sinh n} \int_0^{\pi} x(1 - x) \sin nx \, dx =$
$\frac{2 \operatorname{csch} n(2 + (-2 + n^2(\pi - 1)\pi) \cos n\pi)}{n^3\pi},\ d_n =$
$\frac{2}{\pi \sinh n\pi} \int_0^{\pi} \sin nx \, dx = \frac{-2(-1 + \cos n\pi)}{n\pi} \operatorname{csch} n\pi,$
$c_n = \frac{2}{\pi \sinh n\pi^2} \int_0^1 \sin n\pi y \, dy =$
$\frac{-20(-1 + \cos n\pi)}{n\pi^2} \operatorname{csch} n\pi^2,\ n \geq 1$
$u(x, y) = \sum_{n=1}^{\infty} (b_n \sinh ny \sin nx + c_n \sinh n\pi x \sin n\pi y +$
$d_n \sinh n\pi(\pi - x) \sin n\pi y)$

**39.** $b_n = \frac{2}{\pi \sinh n\pi} \int_0^1 x^2 \sin n\pi x \, dx =$
$\frac{-2(2 + (-2 + n^2\pi^2)) \cos n\pi}{n\pi} \operatorname{csch} n\pi,\ c_n =$
$\frac{2}{\pi \sinh n\pi} \int_0^1 (1 - y) \sin n\pi y \, dy = \frac{2}{n\pi} \operatorname{csch} n\pi,\ d_n =$
$\frac{2}{\pi \sinh n\pi} \int_0^1 y \sin n\pi y \, dy = \frac{-2 \cos n\pi}{n\pi} \operatorname{csch} n\pi,$
$n \geq 1,\ u(x, y) = \sum_{n=1}^{\infty} (b_n \sinh n\pi(1 - y) \sin n\pi x +$
$c_n \sinh n\pi x \sin n\pi y + d_n \sinh n\pi(1 - x) \sin n\pi y)$

**41.** $a_n = \frac{2}{\sinh 2n\pi} \int_0^1 x(1 - x^2) \sin n\pi x \, dx =$
$\frac{-12 \operatorname{csch} 2n\pi \cos n\pi}{n^3\pi^3},\ b_n = 0,\ c_n = 0,\ d_n =$
$\frac{1}{\sinh(n\pi/2\}} \int_0^2 y \sin(n\pi y/2) \, dy =$
$\frac{-4 \cos n\pi}{n\pi} \operatorname{csch}(n\pi/2),\ u(x, y) =$
$\sum_{n=1}^{\infty} (a_n \sinh n\pi y \sin n\pi x +$
$d_n \sinh n\pi(1 - x)/2 \sin(n\pi y/2))$

**43.** $u(x, y) = \sinh y \sin x/\sinh \pi$

**45.** $u(x, y) = \frac{\sinh y \sin x}{\sinh \pi} - \frac{\sinh 2y \sin 2x}{2 \sinh 2\pi}$

### *Exercises 10.5*

**1.** $u(r, \theta) = r \cos \theta$

**3.** $\cos^2 \theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta,\ u(r, \theta) = \frac{1}{2} + \frac{1}{2}r^2 \cos 2\theta$

**5.** $A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta^2 \, d\theta = \frac{1}{3}\pi^2,\ A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta^2 \cos n\theta \, d\theta =$
$\frac{4}{n^2} \cos n\pi,\ B_n = 0,\ n \geq 1$

**7.** $B_n = -\frac{2}{\pi} \int_0^{\pi} \sin n\theta \, d\theta = -\frac{2}{n\pi}(1 - \cos n\pi),$
$n \geq 1;\ A_n = 0,\ n \geq 0$

**9.** $A_0 = \frac{1}{2\pi} \int_0^{\pi} \theta \, d\theta = \frac{\pi}{4};\ A_n = \frac{1}{\pi} \int_0^{\pi} \theta \cos n\theta \, d\theta =$
$\frac{1}{n^2\pi}(\cos n\pi - 1);\ B_n = \frac{1}{\pi} \int_0^{\pi} \theta \sin n\theta \, d\theta =$
$-\frac{1}{n} \cos n\pi,\ n \geq 1$

**11.** $u(r, t) = \cos(c\alpha_1 t)J_0(\alpha_1 r)$

**13.** $u(r, t) = \frac{1}{c\alpha_1} \sin(c\alpha_1 t)J_0(\alpha_1 r)$

**15.** $u(r, t) = \left(\frac{1}{8} \cos(c\alpha_2 t) + \frac{1}{4c\alpha_2} \sin(c\alpha_2 t)\right)J_0(\alpha_2 r)$

**21.** $A_n = \frac{2}{[J_1(\alpha_n)]^2}\left[\frac{J_1(\alpha_n)}{\alpha_n} - \frac{J_2(\alpha_n)}{\alpha_n{}^2}\right];\ B_n = 0,\ n \geq 1$



**24.** $u(r, \theta) = \sum_{n=1}^{\infty} r^n b_n \sin n\theta,\ b_n = \frac{2}{\pi} \int_0^{\pi} f(\theta) \sin n\theta \, d\theta,$
$n \geq 1$

**25.** $b_n = \frac{2}{\pi} \int_0^{\pi} T_0 \sin n\theta \, d\theta = \frac{-2T_0}{n\pi}(\cos n\pi - 1),\ n \geq 1$

**28.** $u(r, t) = 100r$

**29.** $\cos 2\phi = 2 \cos^2 \phi - 1;\ \frac{4}{3}P_2(\cos \phi) - \frac{1}{3};\ u(r, \phi) =$
$\frac{4}{3}r^2 P_2(\cos \phi) - \frac{1}{3}$

**31.** $R'(a) = \frac{d}{dr}[J_0(\lambda r)]|_{r=a} = -\lambda J_1(\lambda a) = 0,\ \lambda a =$
$\alpha_{1,n}$ ($n$th root of $J_1$); $\lambda_n = \alpha_{1,n}/a$

**33.** (a) $A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta^2 \cos \theta \, d\theta = -2,\ A_n =$
$\frac{1}{\pi} \int_{-\pi}^{\pi} \theta^2 \cos \theta \cos n\theta \, d\theta =$
$\frac{(4\pi - 4n^4\pi) \cos n\pi}{\pi(n - 1)^3(n + 1)^3},\ B_n = 0,\ n \geq 1$

(b) $A_0 = 0,\ A_n = 0,\ B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta^2 \sin \theta \cos n\theta \, d\theta =$
$\frac{(8n - 8n^3) \cos n\pi}{(n - 1)^3(n + 1)^3},\ n \geq 1$

(c) $A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta^2(1 + \theta^2) \, d\theta = \pi^2(5 + 3\pi^2)/15,$
$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta^2(1 + \theta^2) \cos n\theta \, d\theta =$
$\frac{4(-12 + n^2(1 + 2\pi^2)) \cos n\pi}{n^4},\ B_n = 0,\ n \geq 1$

### *Chapter 10 Review Exercises*

**1.** (a) $c = 1$; (b) $c = 4$; (c) $c = \frac{1}{16}$

**3.** (a) $u_{xx} = 0,\ u_{yy} = 0$; (b) $u_{xx} = 9e^{3x} \cos 3y,\ u_{yy} =$
$-9e^{3x} \cos 3y$; (c) $u_{xx} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2},\ u_{yy} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}$

**5.** $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta,\ a_n = \frac{1}{\pi\rho^n} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta,$
$b_n = \frac{1}{\pi\rho^n} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta,\ n \geq 1$

**7.** $u(0, t) = A_0$; $u$ at the center of the disk equals the average value of $f$ around the boundary of the disk.

**11.** $rR'' + R' + \lambda^2 rR = 0,\ R(a) = 0,\ Z'' - \lambda^2 Z = 0,\ Z(0) =$
$0,\ R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r),\ c_2 = 0$ (bounded),
$J_0(\lambda a) = 0,\ \lambda_n = \alpha_n/a,\ \alpha_n =$ $n$th zero of $J_0$. $Z(z) =$
$c_3 \cosh \lambda z + c_4 \sinh \lambda z,\ c_3 = 0,\ u(r, z) =$
$\sum_{n=1}^{\infty} a_n \sin \lambda_n z\, J_0(\lambda_n r),\ a_n = \frac{2T_0}{\sinh b\lambda_n\, 2^2\, J_1{}^2(a\lambda_n)}$
$\int_0^a rJ_0(\lambda_n r) \, dr$

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