Analytic functions

Function :→ A fun" is a relationship b/w a set of i/Ps and some permissible o/P.

e.g. Relationship of x with its square.

$f(x) = x^2$

Let we say $y = f(x)$

x → Argument of f, independent variable.

y → Dependent Variable.

→ The set from which x can be picked is called domain of f while all the possible values of y constitute the range of $f(x)$.

→ Notice $f(3) = 9$, $f(-3) = 9$

which means two different i/Ps can produce similar o/P. but each i/P give exactly one o/P.

Complex functions :→ If the argument of a fun^n is complex no. then it is called a complex fun^n.

e.g $f(z) = z^2$, where $z = x + iy \in \mathbb{C}$

→ Any complex fun^n will result in two real valued functions.

i.e., $f(z) = u(x,y) + v(x,y)\,i$

e.g $f(z) = z^2 = x^2 - y^2 + 2xy\,i$

$u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$

→ A fun^n $f(z)$ is called real valued if the fun^n $v(x,y)$ is always zero

e.g ① $f(z) = z + \bar{z}$

$= x + iy + x - iy$

$= 2x$

② $f(z) = z\bar{z} = (x+iy)(x-iy)$

$= x^2 + y^2$

→ This can also be done in polar coordinates.

e.g., ① $f(z) = f(re^{i\theta}) = u + vi$

$u \to u(r,\theta)$, $v \to v(r,\theta)$

② $f(z) = z^2$

$f(re^{i\theta}) = (re^{i\theta})^2 = r^2 e^{i2\theta}$

$= r^2 [\cos 2\theta + i\sin 2\theta]$

$u(r,\theta) = r^2 \cos 2\theta$ & $v(r,\theta) = r^2 \sin 2\theta$

Rational Functions :→ Functions of the form $\dfrac{P(z)}{Q(z)}$ are called rational function.

Domain of a function? :→ All the values of the argument/independent variable for which the function is defined, form the domain

Exercises Churchill (Page: 35)

Q1 (a) $f(z) = \frac{1}{z^2+1}$. Domain of $f(z) = ?$

$f(x) = x^2 \quad x \in \mathbb{R}$

Dom $(f) = ?$ Range $(f) = ?$

Since any $x \in \mathbb{R}$ can be squared so Dom $(f) = \mathbb{R}$.

Moreover, $x^2 \geq 0$ so Range(f) is the set of all non-negative real nos.

$f(z)$ is undefined if

$$z^2 + 1 = 0$$

$$z^2 = -1$$

$$\boxed{z = \pm i}$$

Domain of $f(z)$ include all values in the complex plane except $z = \pm i$

(b) $f(z) = \text{Arg}\left(\frac{1}{z}\right)$ Domain of f(z) ?

$$\text{Arg}(z) = \arg(z) - 2n\pi$$

$$f(z) = \text{Arg}(1) - \text{Arg}(z)$$

$$0^\circ - \text{Arg}(z)$$

$f(z)$ is undefined when

$\text{Arg}(z) = \arg(z) - 2n\pi = \infty$

or $\arg(z) = \infty$ which suggests

$$z \neq 0$$

(c) $f(z) = \frac{z}{z+\bar{z}}$ Domain of f(z) = ?

$z+\bar{z} = 0$

$\boxed{z = -\bar{z}}$

$\downarrow$ possible only if $Re(z) = 0$

e.g $z = 1+i$

$f(z) = \frac{1+i}{(1+i)+(1-i)} = \frac{1+i}{2}$

but $Re(z) = 0$ i.e $z = i$ then

$f(z) = \frac{i}{i+-i} = \infty$

Dom(f) = {z \in C | Re(z) != 0}

Q2 $f(z) = z^2+z+1$ U(x,y) = ? and V(x,y) = ?

$= (x+iy)^2 + x + iy + 1$

$= x^2 - y^2 + x + 1 + i(2xy + y)$

$U(x,y) = x^2 - y^2 + x + 1$

$V(x,y) = y(2x+1)$

Q 3 $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$

$$x = \frac{z+\bar{z}}{2}, \quad y = \frac{z-\bar{z}}{2i}$$

Simplify if z = x + iy?

$$f(z) = \underbrace{x^2 - y^2 - 2xyi} - 2y + 2xi$$

$$= \bar{z}^2 - 2\left[\frac{z-\bar{z}}{2i}\right] + 2\left[\frac{z+\bar{z}}{2}\right]i$$

$$= \bar{z}^2 - \left[\frac{z-\bar{z}}{i}\right] + (z+\bar{z})i$$

$$= \bar{z}^2 - \left[\frac{z-\bar{z}}{i} \times \frac{-i}{-i}\right] + (z+\bar{z})i$$

$$= \bar{z}^2 + \left[(z-\bar{z})i + (z+\bar{z})i\right]$$

$$= \bar{z}^2 + \left[z - \cancel{\bar{z}} + z + \cancel{\bar{z}}\right]i$$

$$\boxed{f(z) = \bar{z}^2 + 2zi}$$

Q 4 :→ $f(z) = z + \frac{1}{z} \quad z \neq 0$

$f(z) = U(r,\theta) + V(r,\theta)i$?

Solution

$$f(z) = z + \frac{1}{z}$$

$$= re^{i\theta} + \frac{1}{re^{i\theta}}$$

$$= re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$$= r\cos\theta + ir\sin\theta + \frac{1}{r}\cos\theta - \frac{1}{r}i\sin\theta$$

$$= \underbrace{\left(r + \frac{1}{r}\right)\cos\theta}_{u(r,\theta)} + i\underbrace{\left(r - \frac{1}{r}\right)\sin\theta}_{v(r,\theta)}$$

Q 13 Zill Page: 52 (Exercises 2.1)

$$f(z) = \frac{\bar{z}}{z+1}, \quad u(x,y) = ?,\ v(x,y) = ?$$

Solution

Let $z = x + iy$

$$f(z) = \frac{x - iy}{x + iy + 1} = \frac{x - iy}{x + 1 + iy} \times \frac{x + 1 - iy}{x + 1 - iy}$$

$$= \frac{x^2 - y^2 + x - i(2xy + y)}{(x+1)^2 + y^2}$$

Hence

$$U(x,y) = \frac{x^2 - y^2 + x}{(x+1)^2 + y^2}$$

& $$V(x,y) = \frac{(2x+1)y}{(x+1)^2 + y^2}$$

Q 23 Zill Page 52, Exercises 2.1

$f(z) = 2Re(z) - iz^2$, $Dom(f) = ?$

Solution :→ Is $f(z)$ undefined at any value of z?

NO.

then $Dom(z) = \mathbb{C}$

Q 24 Zill Page 52, Ex: 2.1

$$f(z) = \frac{3z + 2i}{z^3 + 4z^2 + z}, \quad Dom(f) = ?$$

Solution :→ $f(z)$ is undefined if

$$z^3 + 4z^2 + z = 0$$

$$z(z^2 + 4z + 1) = 0$$

$$z_1 = 0, \text{ or } z_{2,3} = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$$

makes the denominator 0.

e.g $z = -2 + \sqrt{3}$

$z^2 + 4z + 1 = (-2+\sqrt{3})^2 + 4(-2+\sqrt{3}) + 1$

$= \cancel{4} + \cancel{3} - \cancel{4\sqrt{3}} - \cancel{8} + \cancel{4\sqrt{3}} + \cancel{1}$

$= 0$

Now do it by taking $z = x + iy$

$z^2 + 4z + 1 = 0$

$(x+iy)^2 + 4(x+iy) + 1 = 0$

$x^2 - y^2 + 2ixy + 4x + 4iy + 1 = 0$

$x^2 - y^2 + 4x + 1 = 0$

$2(x+2)y = 0 \Rightarrow (x+2) = 0$

$\boxed{x = -2}$

$(-2)^2 - y^2 + 4(-2) + 1 = 0$

Varify?

$y^2 = 4 - 8 + 1 = -3$

$y = \pm i\sqrt{3}$

$z_2 = x + iy = -2 + i(i\sqrt{3}) = -2 - \sqrt{3}$

for $y = i\sqrt{3}$

$z_3 = x + iy = -2 + i(-i\sqrt{3}) = -2 + \sqrt{3}$

for $y = -i\sqrt{3}$

Hence the domain of f contains the entire $\mathbb{C}$ except $z=0$, $z=-2+\sqrt{3}$ & $z=-2-\sqrt{3}$