Laplace Transform

An integral transform which takes a fun" of real variable ($f(t)$) to a fun" of complex variable s (freq)

F.T → F.T ($f(t)$) → Complex fun of real variable.

L.T → $L[f(t)]$ → " " Complex "

⇒ The Laplace transform is a freq. domain approach to continuous time signals & system.

$t \geq 0$

$$\boxed{F(s) = \int_0^\infty f(t)\, e^{-st}\, dt} \longrightarrow \text{unilateral.}$$

where $s = \sigma + j\omega \in \mathbb{C}$

s => Complex frequency

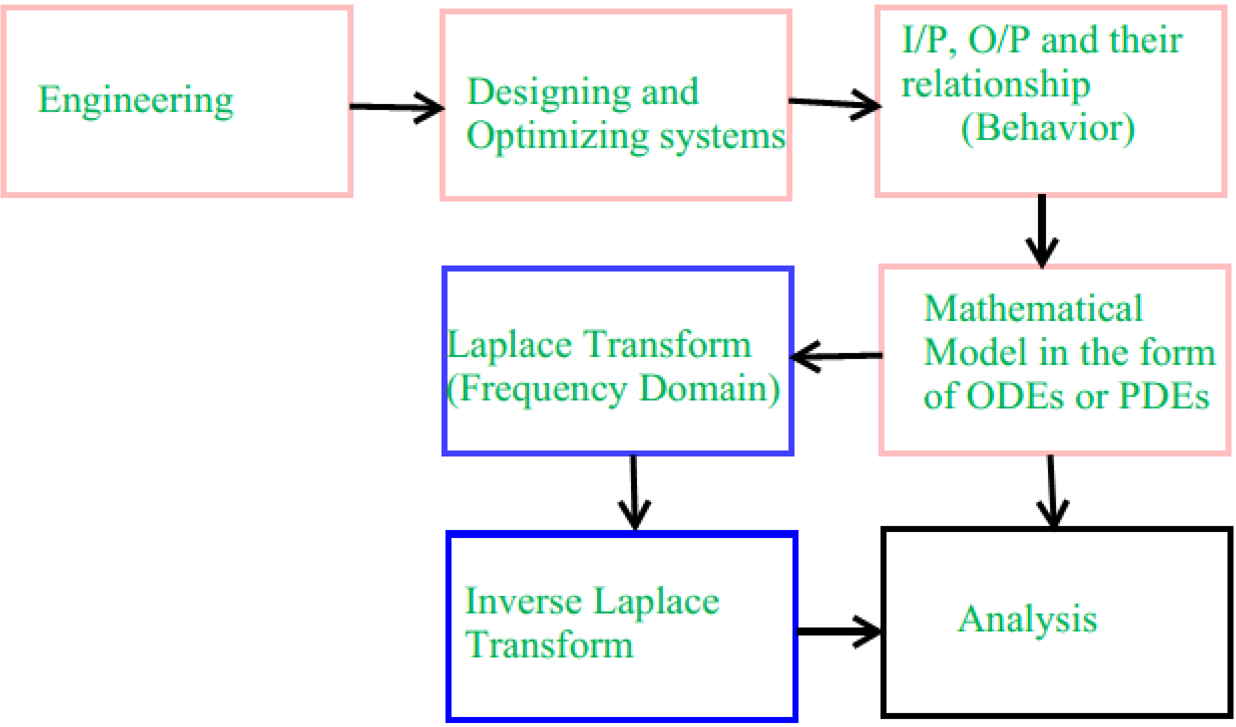
$$F(s) = L[f(t)]$$

→ Main application is solving linear ctt co-efficient ODEs.

→ L.T transforms differential eqns. into algebraic eqns.

=> Moreover, convolution in time domain becomes multiplication in frequency domain

Why L.T ?



As an example: let $a = 23.523$
$b = 26.981$

and we want to find $c = a \times b$

which is a lengthy process.

So what we can do.

$$\ln(c) = \ln(a \times b)$$
$$= \ln a + \ln b$$
$$= d$$

$c = e^d$ is your answer.

multiplication converted into addition.

→ Laplace transform is a general case of Fourier transform.

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt \rightarrow L.T$$

&

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \rightarrow F.T$$

→ The difference is $s = \sigma + j\omega$ and ω

i.e., in case of L.T we're increasing the possibilities.

$$s = \sigma + j\omega$$

σ ↓ damping factor

$j\omega$ ↳ damping frequency. (rad/sec)

⇒ The other diff is that $F(s)$ is a funⁿ of complex variable. ($\because s \in \mathbb{C}$)

while $F(\omega)$ is a funⁿ real variable ($\because \omega \in \mathbb{R}$)

⇒ $F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt$ → Bilateral or two sided L.T

⇒ however if $f(t)$ is defined only for $t \geq 0$ then

$F(s) = \int_{0}^{\infty} f(t) e^{-st} dt$ → Unilateral or one-sided L.T

Region of convergence :- usually imp. in two sided L.T.

ROC is the region in the complex plane where the L.T is finite

Inverse L.T

$$f(t) = \frac{1}{2\pi j} \int_{\delta - j\omega}^{\delta + j\omega} F(s) e^{st} \, ds$$

↳ however, we'll rarely use this formula :- we'll develop standard results for identifying ILT.

<u>Definition</u> :→ Let $f(t)$ be a funⁿ defined for $t \geq 0$. Then the L.T of $f(t)$ is denoted by $\mathcal{L}[f(t)] = F(s)$ and defined as:

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\, dt \longrightarrow ①$$

where s is parameter which may be real or complex.

In general $s = \sigma + j\omega$

→ The integral in ① exist if

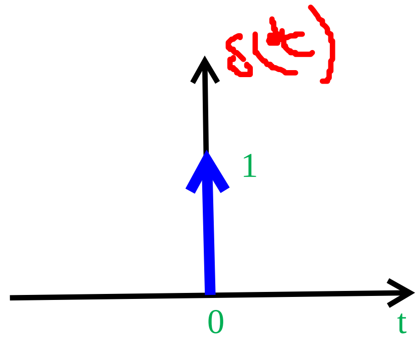
$$\underbrace{① \; s > 0 \qquad ② \; e^{-st} f(t) \neq \infty}_{\text{necessary conditions.}}$$

↳ is finite

Laplace transform of some elementry functions

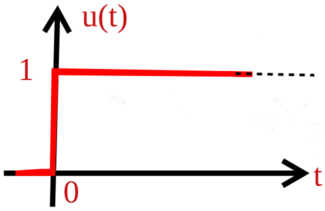
① Unit Impulse: $\delta(t)$

$$f(t) = \delta(t) = \begin{cases} 1 & t = 0 \\ 0 & \text{otherwise} \end{cases}$$



$$\boxed{\mathcal{L}[\delta(t)] = 1} \longrightarrow ①$$

② Unit Step



$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

L[u(t)] = ?

$$L[u(t)] = \int_0^\infty \underbrace{u(t)}_{1} e^{-st}\, dt$$

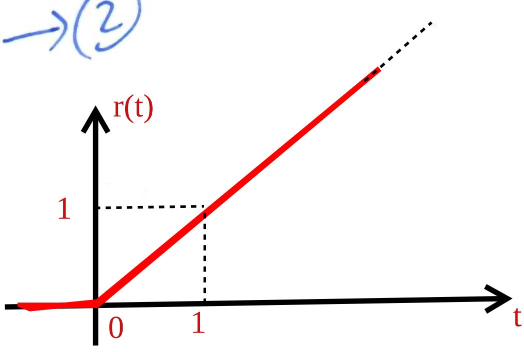
[for t = 0 to infinity]

$$= \int_0^\infty e^{-st}\, dt = \left. \frac{e^{-st}}{-s} \right|_0^\infty$$

$$-\frac{1}{s}\left[e^{-\infty} - e^{0}\right]$$

$$= -\frac{1}{s}[0 - 1] = \frac{1}{s}$$

$$\boxed{L[u(t)] = \frac{1}{s}} \rightarrow ②$$



③ Unit Ramp

$$f(t) = \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

$$L[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^\infty t\, e^{-st}\, dt$$

$$= \left[t \int e^{-st} dt - \int \left(\int e^{-st} dt \cdot \frac{dt}{dt} \right) dt \right]_0^\infty$$

$$= \left[\frac{t e^{-st}}{-s} - \int \frac{e^{-st}}{-s} \cdot 1\, dt \right]_0^\infty$$

$$= -\frac{1}{s} \left[t e^{-st} - \int e^{-st} dt \right]_0^\infty$$

$$= -\frac{1}{s} \left[t e^{-st} + \frac{e^{-st}}{s} \right]_0^\infty$$

$$= -\frac{1}{s} \left[\infty e^{-\infty} + \frac{e^{-\infty}}{s} - 0 - \frac{1}{s} \right]$$

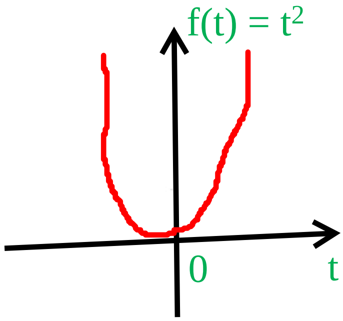
$$= -\frac{1}{s} \left[0 + 0 - 0 - \frac{1}{s} \right]$$

{Ramp signal usually represented by r(t) and may also have the form tu(t)}

$$L[f(t)] = \frac{1}{s^2} \rightarrow (3)$$

(B1) Parabolic function:→

$$f(t) = t^2$$



$$\mathcal{L}[t^2] = \int_0^\infty t^2 e^{-st} dt$$

$$= \left[t^2 \int e^{-st} dt - \int \int e^{-st} dt \cdot \frac{dt^2}{dt} \, dt \right]_0^\infty$$

$$= -t^2 \left[\frac{e^{-st}}{s} \right]_0^\infty + \int_0^\infty \frac{e^{-st}}{s} 2t \, dt$$

$$= -t^2 \left[\frac{e^{-st}}{s} \right]_0^\infty + \frac{2}{s} \left[\int_0^\infty t e^{-st} dt \right]$$

From (3) $\int_0^\infty t e^{-st} dt = \frac{1}{s^2}$

$$= 0 + \frac{2}{s^3}$$

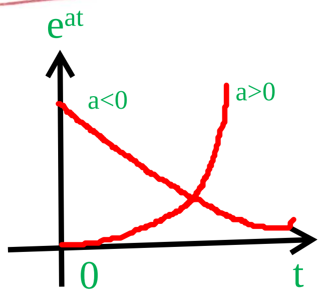
$$\mathcal{L}[t^2] = \frac{2}{s^3} \longrightarrow (4)$$

From (3) and (4)

$$\boxed{L[t^n] = \frac{n!}{s^{n+1}}}$$

(5) Exponential function

$$f(t) = e^{at} \qquad a \in \mathbb{R}$$



[e^{at} is decaying if $a<0$ and it rises if $a>0$]

$$L[e^{at}] = \int_0^\infty e^{at} e^{st}\,dt$$

$$= \int_0^\infty e^{(a-s)t}\,dt$$

$$= \frac{1}{a-s} e^{(a-s)t} \Big|_0^\infty$$

$$= \frac{1}{a-s} \left[e^{-(s-a)t} \right]_0^\infty$$

$$= \frac{1}{a-s} \left[e^{-\infty} - 1 \right] = -\frac{1}{a-s} = \frac{1}{s-a}$$

$$\boxed{L[e^{at}] = \frac{1}{s-a}} \rightarrow (5)$$

Now let $f(t) = e^{-at}$

$$L\left[e^{-at}\right] = \int_0^\infty e^{-at} e^{-st}\, dt$$

$$= \int_0^\infty e^{-(a+s)t}\, dt$$

$$= -\frac{1}{a+s}\left[e^{-(a+s)t}\right]_0^\infty$$

$$= -\frac{1}{a+s}\left[0 - 1\right] = \frac{1}{s+a}$$

So $\boxed{L\left[e^{-at}\right] = \frac{1}{s+a}} \longrightarrow (6)$

From (5) and (6)

$$\boxed{L\left[e^{\pm at}\right] = \frac{1}{s \mp a}}$$

⑥ Sinusoidal functions

ⓐ $f(t) = \sin \omega t$

Recall that

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

$$- \quad e^{-j\omega t} = \cos\omega t - j\sin\omega t$$

$$e^{j\omega t} - e^{-j\omega t} = 2j\sin\omega t$$

$$\sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

So

$$\mathcal{L}[\sin\omega t] = \int_0^\infty \sin\omega t \; e^{-st}\,dt$$

$$= \int_0^\infty \left[\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right] e^{-st}\,dt$$

$$= \frac{1}{2j}\int_0^\infty e^{j\omega t} e^{-st} - e^{-j\omega t} e^{-st}\,dt$$

$$= \frac{1}{2j}\left[\int_0^\infty e^{-(s-j\omega)t}\,dt - \int_0^\infty e^{-(s+j\omega)t}\,dt\right]$$

From 5 and 6

$$= \frac{1}{2J}\left[\frac{1}{(s-J\omega)} - \frac{1}{s+J\omega}\right]$$

$$= \frac{1}{2J}\left[\frac{\cancel{s}+J\omega - \cancel{s}+J\omega}{(s-J\omega)(s+J\omega)}\right]$$

$$= \frac{1}{\cancel{2J}}\left[\frac{\cancel{2J}\omega}{s^2+\omega^2}\right]$$

$$\boxed{L[\sin\omega t] = \frac{\omega}{s^2+\omega^2}} \rightarrow 7$$

(b) $f(t) = \cos\omega t$

$$e^{J\omega t} = \cos\omega t + j\cancel{\sin\omega t}$$

$$e^{-J\omega t} = \cos\omega t - j\cancel{\sin\omega t}$$

$$2\cos\omega t = e^{J\omega t} + e^{-J\omega t}$$

$$\cos\omega t = \frac{e^{J\omega t} + e^{-J\omega t}}{2}$$

$$\mathcal{L}[\cos\omega t] = \int_0^\infty \frac{e^{j\omega t} + e^{-j\omega t}}{2} e^{-st}\, dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{j\omega t} e^{-st}\, dt + \int_0^\infty e^{-j\omega t} e^{-st}\, dt\right]$$

$$= \frac{1}{2}\left[\frac{1}{s - j\omega} + \frac{1}{s + j\omega}\right] \quad \text{using 5 \& 6}$$

$$= \frac{1}{2}\left[\frac{s + \cancel{j\omega} + s - \cancel{j\omega}}{s^2 + \omega^2}\right]$$

$$\boxed{\mathcal{L}[\cos\omega t] = \frac{s}{s^2 + \omega^2}} \rightarrow 8$$

Similarly you can do it for Sinh ωt, Cosh ωt

$$\sinh \omega t = \frac{e^{\omega t} - e^{-\omega t}}{2}$$

$$\cosh \omega t = \frac{e^{\omega t} + e^{-\omega t}}{2}$$

(c) $f(t) = \sinh \omega t$

$$\mathcal{L}[\sinh \omega t] = \int_0^\infty \left[\frac{e^{\omega t} - e^{-\omega t}}{2}\right] e^{-st}\, dt$$

$$= \frac{1}{2}\int_0^\infty e^{(\omega - s)t} - e^{-(\omega + s)t} \cdot dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-\omega)t} - e^{-(s+\omega)t}\, dt$$

$$= \frac{1}{2}\left[\frac{1}{-(s-\omega)} e^{-(s-\omega)t}\Big|_0^\infty - \left[\frac{1}{-(s+\omega)} e^{-(s+\omega)t}\Big|_0^\infty\right]\right]$$

$$= \frac{1}{2}\left[-\frac{1}{s-\omega}\left[0 - 1\right] + \frac{1}{s+\omega}\left[0 - 1\right]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-\omega} - \frac{1}{s+\omega}\right]$$

$$= \frac{1}{2}\left[\frac{s+\omega - s+\omega}{s^2 \cancel{+} \omega^2}\right]$$

$$\boxed{\mathcal{L}[\sinh \omega t] = \frac{\cancel{s}\omega}{s^2 - \omega^2}}$$

or

$$\mathcal{L}[\sinh \omega t] = \frac{1}{2}\int_0^\infty e^{\omega t} e^{-st} - e^{-\omega t} e^{-st}\, dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{\omega t} e^{-st}\, dt - \int_0^\infty e^{-\omega t} e^{-st}\, dt\right]$$

from (5) and (6) we can directly write

$$= \frac{1}{2}\left[\frac{1}{s-\omega} - \frac{1}{s+\omega}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{s}+\omega - \cancel{s}+\omega}{s^2-\omega^2}\right] = \frac{\omega}{s^2-\omega^2}$$

$$\boxed{\mathcal{L}[\sinh \omega t] = \frac{\omega}{s^2-\omega^2}} \rightarrow (9)$$

(d) $f(t) = \cosh \omega t$

$$\cosh \omega t = \frac{e^{\omega t} + e^{-\omega t}}{2}$$

$$L[\cosh \omega t] = L\left[\frac{e^{\omega t} + e^{-\omega t}}{2}\right]$$

$$= \frac{1}{2}\left[L[e^{\omega t}] + L[e^{-\omega t}]\right]$$

From (5) and (6)

$$= \frac{1}{2}\left[\frac{1}{s-\omega} + \frac{1}{s+\omega}\right]$$

$$= \frac{1}{2}\left[\frac{s+\cancel{\omega} + s-\cancel{\omega}}{s^2-\omega^2}\right]$$

$$\boxed{L[\cosh \omega t] = \frac{s}{s^2-\omega^2}} \rightarrow (10)$$

7) Laplace Transform of Damped Sinusoids

$$\boxed{f(t) = e^{at} \sin \omega t}$$

L[f(t)] = ?

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$L\left[e^{at}\sin \omega t\right] = L\left[e^{at}\left[\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right]\right]$$

$$= \frac{1}{2j} L\left[e^{at}e^{j\omega t} - e^{at}e^{-j\omega t}\right]$$

$$= \frac{1}{2j} L\left[e^{(a+j\omega)t} - e^{(a-j\omega)t}\right]$$

$$= \frac{1}{2j}\left\{L\left[e^{(a+j\omega)t}\right] - L\left[e^{(a-j\omega)t}\right]\right\}$$

By 5 and 6

$$= \frac{1}{2j}\left[\frac{1}{s-a-j\omega} - \frac{1}{s-a+j\omega}\right]$$

$$= \frac{1}{2j}\left[\frac{s-a+j\omega - (s-a-j\omega)}{(s-a)^2 + \omega^2}\right]$$

$$= \frac{1}{2j}\left[\frac{\cancel{s}-\cancel{a}+j\omega-\cancel{s}+\cancel{a}+j\omega}{''}\right]$$

$$= \frac{1}{2j}\left[\frac{2j\omega}{''}\right] = \boxed{\frac{\omega}{(s-a)^2+\omega^2} = L\left[e^{at}\sin \omega t\right]} \rightarrow 11$$

Summary of Elementary L.T

① $\mathcal{L}[\delta(t)] = 1$

② $\mathcal{L}[u(t)] = \frac{1}{s}$

③ $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

④ $\mathcal{L}[e^{\pm at}] = \frac{1}{s \mp a}$

⑤ $\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$

⑥ $\mathcal{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2}$

⑦ $\mathcal{L}[\sinh \omega t] = \frac{\omega}{s^2 - \omega^2}$

⑧ $\mathcal{L}[\cosh \omega t] = \frac{s}{s^2 - \omega^2}$

⑨ $\mathcal{L}[e^{at} \sin \omega t] = \frac{\omega}{(s-a)^2 + \omega^2}$

Now how we'll use these.

e.g → $L[t^0] = \frac{0!}{s^{0+1}} = \frac{1}{s}$

$t^0 \to 1 = u(t)$

→ $L(t) = \frac{1!}{s^{1+1}} = \frac{1}{s^2}$

ramp

→ $L(t^5) = \frac{5!}{s^{5+1}} = \frac{120}{s^6}$

→ $L[e^{2t}] = \frac{1}{s-2}$

→ $L[e^{-2t}] = \frac{1}{s+2}$

→ $L[\sin 3t] = \frac{3}{s^2+9}$

→ $L[\sin t] = \frac{1}{s^2+1}$

→ $L[\cos 5t] = \frac{s}{s^2+25}$

PROPERTIES OF LAPLACE TRANSFORM

1) Shifting Property

let $\mathcal{L}[f(t)] = F(s)$

then $\boxed{\mathcal{L}[e^{at} f(t)] = F(s-a)}$

See eqn: (11)

Example $\mathcal{L}[e^{2t} \cos 3t] = ?$

$$= \mathcal{L}\left[e^{2t}\left[\frac{e^{j3t} + e^{-j3t}}{2}\right]\right]$$

$$= \frac{1}{2}\mathcal{L}\left[e^{2t} e^{j3t} + e^{2t} e^{-j3t}\right]$$

$$= \frac{1}{2}\left[\mathcal{L}\left[e^{(2+3j)t}\right] + \mathcal{L}\left[e^{(2-3j)t}\right]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-2-3j} + \frac{1}{s-2+3j}\right]$$

$$= \frac{1}{2}\left[\frac{s-2+\cancel{3j}+s-2-\cancel{3j}}{(s-2)^2+9}\right]$$

$$= \frac{[s-2]}{(s-2)^2+9}$$

=> Now solving by "Shifting Property"

We know:

$$L[\cos 3t] = \frac{s}{s^2+9} = F(s)$$

By Shifting Property

$$L[e^{2t}\cos 3t] = F(s-2)$$

$$= \frac{s-2}{(s-2)^2+9}$$

Example: $L[e^{-t}t^5] = ?$

$$L[t^5] = \frac{5!}{s^{5+1}} = \frac{120}{s^6} = F(s)$$

By shifting prop:

$$L[e^{-t}t^5] = F(s-(-1)) = F(s+1)$$

$$= \frac{120}{(s+1)^6}$$

Example $L[e^{-3t}\sin t] = ?$

$$L[\sin t] = \frac{1}{s^2+1} = F(s)$$

By shifting prop:

$$L[e^{-3t}\sin t] = F(s-(-3)) = F(s+3)$$

$$= \frac{1}{(s+3)^2+1}$$

2) Multiplication by Power of "t" [Differentiation in frequency]

let $L[f(t)] = F(s)$

then $\boxed{L[t f(t)] = -\frac{d}{ds} F(s)}$ $\boxed{L[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}}$

Proof:→ We know that

$$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt \rightarrow (1)$$

Now $\frac{d}{ds} F(s) = \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\, dt$

↳ when $\frac{d}{ds}$ goes inside the integration which is w-r-t some other variable then $\frac{d}{ds}$ becomes $\frac{\partial}{\partial s}$.

$$\frac{d}{ds} F(s) = \int_0^\infty f(t) \frac{\partial}{\partial s} e^{-st}\, dt$$

$$= \int_0^\infty f(t) \left((-t) e^{-st}\right) dt$$

$$= -\int_0^\infty t f(t)\, e^{-st}\, dt$$

$$-\frac{d}{ds}F(s) = \int_0^\infty t\, f(t)\, e^{-st}\, dt$$

$$\boxed{-\frac{d}{ds}F(s) = L[t f(t)]}$$

Example: → $L[t \sin 3t] = ?$

Solution

We know that

$$L[\sin 3t] = \frac{3}{s^2+9} = F(s)$$

So By property

$$L[t \sin 3t] = -\frac{d}{ds}F(s)$$

$$= -\frac{d}{ds} 3(s^2+9)^{-1}$$

$$= -3\frac{d}{ds}(-1)(s^2+9)^{-1-1}(2s)$$

$$= \frac{6s}{(s^2+9)^2}$$

Example :-> $L[t^2 \sin 3t]$

$$L[t^2 \sin 3t] = L[t \underbrace{t \sin 3t}_{f(t)}]$$

So

$$F(s) = \frac{6s}{(s^2+9)^2}$$

So

$$L[t^2 \sin 3t] = -\frac{d}{ds}\left[\frac{6s}{(s^2+9)^2}\right]$$

$$= -\frac{d}{ds}\left[\frac{d}{ds} F(s)\right]$$

where $F(s) = L[\sin 3t]$

$$= \frac{d^2}{ds^2} F(s)$$

Hence

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

where $F(s) = L[f(t)]$

Example :→ $L[e^{-2t}\, t \cos t] = ?$

Solution :→

We know that

$$L[\cos t] = \frac{s}{s^2+1}$$

Now using multiply by t prop:

$$-\frac{d}{ds}\left[\frac{s}{s^2+1}\right] = -\frac{(s^2+1)\cdot 1 - s\,(2s)}{(s^2+1)^2}$$

$$= -\frac{s^2+1-2s^2}{(s^2+1)^2}$$

$$= -\frac{-s^2+1}{(s^2+1)^2}$$

$$L[t\cos t] = \frac{s^2-1}{(s^2+1)^2}$$

Now using the shifting property

i.e.,

$$L\left[e^{at} f(t)\right] = F(s-a)$$

so

$$L\left[e^{-2t} t \cos t\right] = \frac{(s+2)^2 - 1}{\left((s+2)^2 + 1\right)^2}$$

$$= \frac{s^2 + 4s + 4 - 1}{\left(s^2 + 4s + 4 + 1\right)^2} = \frac{s^2 + 4s + 3}{\left(s^2 + 4s + 5\right)^2}$$

$$\boxed{L\left[e^{-2t} t \cos t\right] = \frac{s^2 + 4s + 3}{\left(s^2 + 4s + 5\right)^2}}$$

4) Divide by "t" Property [Integration in frequency]

$$L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(s)\, ds$$

where $F(s) = L\left[f(t)\right]$

Proof:- we know that

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Integrating both sides w-r-t s from s to ∞

$$\int_s^\infty F(s)\, ds = \underbrace{\int_s^\infty \left[\int_0^\infty e^{-st} f(t)\, dt \right] ds}_{\text{change the order of integration.}}$$

$$= \int_0^\infty \left[\int_s^\infty e^{-st} ds \right] f(t)\, dt$$

$$= \int_0^\infty \left[\frac{e^{-st}}{-t} \right]_s^\infty f(t)\, dt$$

$$= \int_0^\infty -\frac{1}{t} \left[0 - e^{-st} \right] f(t)\, dt$$

$$= \int_0^\infty \frac{f(t)}{t} e^{-st}\, dt \triangleq L\left[\frac{f(t)}{t} \right]$$

Example :- $L\left[\frac{1-\cos t}{t}\right]$

Solution

$$L[1-\cos t] = L[1] - L[\cos t]$$

$$F(s) = \frac{1}{s} - \frac{s}{s^2+1}$$

Using divide by t prop:

$$L\left[\frac{1-\cos t}{t}\right] = \int_s^\infty F(s)\, ds$$

$$= \int_s^\infty \frac{1}{s} - \frac{s}{s^2+1}\, ds$$

$$= \int_s^\infty \frac{1}{s} ds - \int_s^\infty \frac{s}{s^2+1} ds$$

$$= \left[\log(s) - \frac{1}{2}\log(s^2+1)\right|_s^\infty$$

$$= \frac{1}{2}\left[2\log(s) - \log(s^2+1)\right|_s^\infty$$

$\int \frac{s}{s^2+1} ds$

let $t = s^2+1$

$dt = 2s\, ds$

$\frac{1}{2}\int \frac{1}{t} dt$

$= \frac{1}{2}\log(t)$

$= \frac{1}{2}\log(s^2+1)$

$$= \frac{1}{2}\left[\log s^2 - \log(s^2+1)\right|_s^\infty$$

$$= \frac{1}{2}\left(\log\left(\frac{s^2}{s^2+1}\right)\right)_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{\infty}{\infty+1}\right) - \log\left(\frac{s^2}{s^2+1}\right)\right]$$

$$\lim_{s\to\infty} \log\left(\frac{s^2}{s^2+1}\right) = \lim_{s\to\infty} \log\left(\frac{s^2}{s^2}\left(\frac{1}{1+1/s^2}\right)\right)$$

$$= \log 1$$

$$= \frac{1}{2}\left[\underbrace{\log(1)}_{0} - \log\left(\frac{s^2}{s^2+1}\right)\right]$$

$$\boxed{L\left[\frac{1-\cos t}{t}\right] = -\frac{1}{2}\log\left(\frac{s^2}{s^2+1}\right)}$$

Example $L\left[\frac{e^{-t}\sin t}{t}\right] = ?$

Solution:

$L\left[e^{-t}\sin t\right] = F(s+1)$ Shifting Prop:

$F(s) = L(\sin t) = \frac{1}{s^2+1}$

$F(s+1) = \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2}$

Now using divide by t prop:

$$L\left[\frac{e^{-t}\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+2s+2}\,ds$$

$\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$$= \int_s^\infty \frac{1}{(s+1)^2+1^2}\,ds = \tan^{-1}\left(\frac{s+1}{1}\right)\Big|_s^\infty$$

$= \tan^{-1}(\infty) - \tan^{-1}(s+1)$

$= 90^\circ - \tan^{-1}(s+1)$

(5) Laplace Transform of derivatives

Let

$$F(s) = L\{f(t)\}$$

then

$$L\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0)$$

where $f(0)$ means $\lim_{t\to 0} f(t)$

Proof :-

$$L\{f'(t)\} = \int_0^\infty \underset{II}{f'(t)}\, \underset{I}{e^{-st}}\, dt$$

$$= e^{-st}\int f'(t)dt - \int \int f'(t)dt \frac{d}{dt}e^{-st}\, dt \Big|_0^\infty$$

$$= e^{-st} f(t) - \int f(t)(-s)e^{-st}\, dt \Big|_0^\infty$$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\, dt$$

$$= e^{-\infty} f(\infty) - e^{0} f(0) + s\,L[f(t)]$$

$$= -f(0) + s\,F(s)$$

$$= sF(s) - f(0) \quad \text{proved.}$$

$$\boxed{\begin{aligned} L[f''(t)] &= s^2F(s) - sf(0) - f'(0) \\ L[f'''(t)] &= s^3F(s) - s^2f(0) - sf'(0) - f''(0) \end{aligned}}$$

⑧ Laplace Transform of integrals.

let $F(s) = L[f(t)]$ then

$$L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}F(s)$$

Proof:- let $G(t) = \int_0^t f(t)\,dt$

$$\Rightarrow G(0) = \int_0^0 f(t)\,dt = 0$$

$$G'(0) = \frac{d}{dt}\int_0^t f(t)\,dt$$

$$= f(t) \quad \text{if } f(0)=0$$

We know that

$$L[G'(t)] = sG(s) - \underbrace{G(0)}_{0} \rightarrow (1)$$

$$G'(t) = f(t)$$

$$L[f(t)] = s\,L\left[\int_0^t f(t)\,dt\right]$$

$$\boxed{L\left[\int_0^t f(t)\,dt\right] = \frac{L[f(t)]}{s} = \frac{F(s)}{s}}$$

Ex: Evaluate $L\left[\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right]$

Solution :→ Note [See ① L of integral.
② Shifting prop:
③ Divide by t prop:]

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

$$L\{e^{at}f(t)\} = F(s-a)$$

$$L\left[\int_0^t f(t)\right] = \frac{F(s)}{s}$$

We know that

$$L[\sin t] = \frac{1}{s^2+1} = G(s)$$

Using divide by t prop:

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty G(s)\,ds = \int_s^\infty \frac{1}{s^2+1}\,ds = \int_s^\infty (s^2+1)^{-1}\,ds$$

$$= \tan^{-1} s \Big|_s^\infty = \frac{\pi}{2} - \tan^{-1} s = F(s)$$

Now using shifting prop:

$$L\left[\frac{e^{-t}\sin t}{t}\right] = F(s+1) = \frac{\pi}{2} - \tan^{-1}(s+1)$$

Finally using integral prop:

$$L\left[\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right] = \frac{F(s)}{s}$$

$$= \frac{\frac{\pi}{2} - \tan^{-1}(s+1)}{\cancel{s}} = \frac{\pi}{2s} - \frac{\tan^{-1}(s+1)}{s}$$

<u>Example</u> :→ Evaluate $L\left[t\int_0^t \frac{e^{t}\sin t}{t}\,dt\right]$

Solution :→

$$L\left[t\underbrace{\int_0^t \frac{e^{t}\sin t}{t}\,dt}_{f(t)}\right]$$

one extra prop: i.e., multiply by t prop.

Just this change.

$$F(s) = \cancel{\frac{1}{s}}\left(\pi/2 - \tan^{-1}(s-1)\right)$$

$$L[t f(t)] = -\frac{d}{ds}F(s) = -\frac{d}{ds}\left[\frac{1}{s}\left(\pi/2 - \tan^{-1}(s-1)\right)\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s}\left[\cot^{-1}(s-1)\right]\right]$$

$\because\ \pi/2 - \tan^{-1}x = \cot^{-1}x$

Example :-> Evaluate $\int_0^{\infty} e^{-3t} \sin t \, dt \rightarrow ①$

Solution :- In this question we don't have to find Laplace rather we've to just evaluate an integral. However, the definition of Laplace make it easier.

Since $L[f(t)] = \int_0^{\infty} e^{-st} f(t) \, dt \rightarrow ②$

Observe ① and ② -3 instead of -s and sint for f(t), the rest is the same.

So if we find Laplace transform of Sint and replace s by 3 then ① is solved.

$$L[\sin t] = \frac{1}{s^2+1} = \int_0^\infty e^{-st} \sin t \, dt$$

put $s=3$ on both sides.

$$\frac{1}{3^2+1} = \boxed{\int_0^\infty e^{-3t} \sin t \, dt = \frac{1}{10}}$$

Q i) $L\left[(t^2+1)^2\right] = ?$

$$(t^2+1)^2 = t^4 + 1 + 2t^2$$

$$L\left[t^4+1+2t^2\right] = L\left[t^4\right] + L[1] + L\left[2t^2\right]$$

L is a linear operator

$$= L\left[t^4\right] + L[1] + 2L\left[t^2\right]$$

$$\frac{n!}{s^{n+1}}$$

$$= \frac{4!}{s^5} + \frac{1}{s} + 2 \times \frac{2}{s^3}$$

$$= \frac{24}{s^5} + \frac{1}{s} + \frac{4}{s^3}$$

Q:-) $L[\sin 2t \cos t] = ?$

Now known prop: for this one so

$$\sin 2t \cdot \cos t = \frac{1}{2}[2 \sin 2t \cos t]$$

$$= \frac{1}{2}[\sin(2t+t) + \sin(2t-t)]$$

$$= \frac{1}{2}[\sin 3t + \sin t]$$

$$L[\sin 2t \cos t] = \frac{1}{2} L[\sin 3t + \sin t]$$

$$= \frac{1}{2}[L[\sin 3t] + L[\sin t]]$$

$$= \frac{1}{2}\left[\frac{3}{s^2+9} + \frac{1}{s^2+1}\right]$$

$Q:\rightarrow \quad L[\cos^3 2t] = ?$

$$\cos 3t = 4\cos^3 t - 3\cos t$$

$t = 2t$

$$\cos 6t = 4\cos^3 2t - 3\cos 2t$$

$$4\cos^3 2t = \cos 6t + 3\cos 2t$$

$$\cos^3 2t = \frac{\cos 6t}{4} + \frac{3}{4}\cos 2t$$

$$L[\cos^3 2t] = \frac{1}{4}L[\cos 6t + 3\cos 2t]$$

$$= \frac{1}{4}\left[L[\cos 6t] + 3L[\cos 2t]\right]$$

$$= \frac{1}{4}\left[\frac{s}{s^2+36} + 3 \cdot \frac{s}{s^2+4}\right]$$

$Q_{(b)}$ $L\left[e^{2t}+4t^3-2\sin 3t+3\cos 3t\right]=?$

$L\left[e^{2t}+4t^3-2\sin 3t+3\cos 3t\right]=L\left[e^{2t}\right]+4L\left[t^3\right]-2L\left[\sin 3t\right]$

$+3L\left[\cos 3t\right]$

$=\frac{1}{s-2}+4\frac{6}{s^4}-2\frac{3}{s^2+9}+3\frac{s}{s^2+9}$

$Q:$ $L[f(t)]=?$ $f(t)=\begin{cases} e^t & 0\le t\le 5 \\ 3 & t>5 \end{cases}$

$L[f(t)]=\int_0^\infty f(t)e^{-st}\,dt$

$=\int_0^5 f(t)e^{-st}\,dt+\int_5^\infty f(t)e^{-st}\,dt$

$=\int_0^5 e^t e^{-st}\,dt+\int_5^\infty 3.e^{-st}\,dt$

$=\int_0^5 e^{t(1-s)}\,dt+3\int_5^\infty e^{-st}\,dt$

$=\frac{e^{t(1-s)}}{1-s}\Big|_0^5+3\frac{1}{-s}e^{-st}\Big|_5^\infty$

$$= \frac{1}{1-s}\left[e^{5(1-s)} - 1\right] - \frac{3}{s}\left[e^{-\infty} - e^{-5s}\right]$$

$$= \frac{e^{5(1-s)}}{1-s} + \frac{1}{s-1} + \frac{3}{s}e^{-5s}$$

Q:) $L\left[e^{-t}\cos^2 t\right] = ?$

$$\cos^2 t = \frac{1+\cos 2t}{2}$$

Conjugation Prop:

$$f(t) \xrightarrow{L.T} F(s)$$

then $$f^*(t) \xrightarrow{L.T} F^*(s^*)$$

Proof :→ $$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt$$

$$F^*(s) = \int_{-\infty}^{\infty} f^*(t) e^{-s^*t} dt$$

$$F^*(s^*) = \int_{-\infty}^{\infty} f^*(t) e^{-st} dt$$

$$= L[f^*(t)]$$

Time Reversal Prop

let we've a fun" $f(t)$ s.t

$$f(t) \xrightarrow{L.T} F(s)$$

then $$f(-t) \xrightarrow{L.T} F(-s)$$

Proof

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

~~ad~~ let

$$F'(s) = \int_{-\infty}^{\infty} f(-t)\, e^{-st}\, dt$$

↳ Laplace. T of time reversed i/p

let $-t = \tau \Rightarrow t = -\tau \Rightarrow dt = -d\tau$

$t = -\infty,\ \tau = \infty, \quad t = \infty,\ \tau = -\infty$

$$F'(s) = -\int_{\infty}^{-\infty} f(\tau)\, e^{s\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau)\, e^{-(-s)\tau}\, d\tau = F(-s)$$

Time Scaling Property

let the fun" $f(t)$ s.t.

$$f(t) \xrightarrow{L.T} F(s)$$

then

$$\boxed{f(at) \xrightarrow{L.T} \frac{1}{|a|} F(s/a) \quad a \neq 0}$$

a is real number

Time Shifting Prop

$$f(t) \xrightarrow{L.T} F(s)$$

then

$$\boxed{f(t+t_0) \xrightarrow{L.T} F(s)\, e^{st_0}}$$

$$\boxed{f(t-t_0) \xrightarrow{L.T} F(s)\, e^{-st_0}}$$

Frequency Shifting Prop (or Shifting in S-domain)

it is the same proper no. 1

$$f(t) \xrightarrow{L.T} F(s)$$

then

$$\boxed{e^{\pm s_0 t} f(t) \xrightarrow{L.T} F(s \mp s_0)}$$

Convolution in time

Let $f_1(t) \xrightarrow{L.T} F_1(s)$

$f_2(t) \xrightarrow{L.T} F_2(s)$

then the convolution of these two time domain signals has the following property.

$$\boxed{f_1(t) * f_2(t) \xrightarrow{L.T} F_1(s) \cdot F_2(s)}$$

convolution

Multiplication in time Prop. (Also known as convolution in freq.)

$$\boxed{f_1(t) \cdot f_2(t) \xrightarrow{L.T} \frac{1}{2\pi j}\left[F_1(s) * F_2(s)\right]}$$

Convolution.

Differentiation in time

Let $f(t) \xrightarrow{L.T} F(s)$

then $\dfrac{d^n f(t)}{dt^n} = s^n F(s)$ → For Bilateral L.T and

$$\boxed{\frac{d^n f(t)}{dt^n} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) \cdots}$$

Integration in time

Let $f(t) \xrightarrow{L.T} F(s)$

then $\boxed{\int_{-\infty}^{t} f(\tau)\, d\tau \xrightarrow{L.T} \frac{F(s)}{s}}$ → for bilateral

$$\boxed{\int_{-\infty}^{t} f(\tau)\, d\tau \xrightarrow{L.T} \frac{F(s)}{s} + \frac{\int_{-\infty}^{0^-} f(\tau)\, d\tau}{s}}$$ → For unilateral.

Laplace transform for solving differential equations

Why do we need solution?

↳ We want to see how a sys. behaves.

→ Let us assume a 1^{st} ODE

$$m\dot{v}_{(t)} + b\,v_{(t)} = F_{(t)}$$

Can you tell what will happen to $V_{(t)}$ as time progresses?

→ Obviously it is hard to answer.

So what do we do to answer?

Ans:→ We'll solve the diff. eqn.

how?

There are two way to solve it

(a) Classical time domain approach

(b) Using Laplace transform

(a)

Total solution = Particular Soln + Complementary Soln

⇒ The complementary solution, also known as the natural response, zero i/p response or initial condition response.

⇒ The particular soln also known as zero state response or Forced response.

Complementary Solution :⇒ Response due to Initial conditions $(F=0)$

$$m\dot{v}(t) + b\,v(t) = 0$$

$$m\dot{v}(t) = -b\,v(t)$$

$$\dot{v}(t) = -\frac{b}{m}\,v(t)$$

$$\frac{dv(t)}{dt} = -\frac{b}{m}\,v(t) \quad \Rightarrow \quad \frac{dv(t)}{v(t)} = -\frac{b}{m}\,dt$$

$$\int_0^t \frac{dv(t)}{v(t)} = \int_0^t -\frac{b}{m}\,dt$$

$$\ln(v(t)) - \ln(v(0)) = -\frac{b}{m}\,t$$

$$\ln\left(\frac{v(t)}{v(0)}\right) = -\frac{b}{m}t$$

$$\frac{v(t)}{v_0} = e^{-b/m\,t} \qquad \because v(0) = v_0$$

$$v(t) = v_0\, e^{-b/m\,t} \longrightarrow (1)$$

(b) Particular Solution → To find this we need to specify the forcing fun"
e.g. let $F(t) = F_0\, u(t)$ (step of strength F_0)

V_{forced} further has two components

$$V_{forced}(t) = V_{ss}(t) + V_{transient} \longrightarrow (2)$$

→ In steady state $\dot{v}(t) = 0$ and $F(t) = F_0$

So $\quad m\,\dot{v}(t) + b\,v(t) = F(t)$

becomes
$$b\,V_{ss}(t) = F_0 \Rightarrow V_{ss}(t) = \frac{F_0}{b} \longrightarrow (3)$$

$$V_{transient}^{(t)} = \underbrace{\alpha e^{-b/m\,t}}_{\text{bcz the sys. behavior won't change.}} \rightarrow (4)$$

Put (3) and (4) in (2)

$$V_{forced}^{(t)} = \frac{F_0}{b} + \alpha e^{-(b/m)t}$$

We know that

$$V_{forced}^{(0)} = 0$$

$$\frac{F_0}{b} + \alpha e^{0} = 0$$

$$\frac{F_0}{b} + \alpha = 0 \Rightarrow \alpha = -\frac{F_0}{b}$$

So $$V_{forced}^{(t)} = \frac{F_0}{b}\left(1 - e^{-b/m\,t}\right)$$

Total solution $= V(t) = V_0 e^{-(b/m)t} + \left(1 - e^{-(b/m)t}\right)\frac{F_0}{b}$

Is it easier? what if the diff. eqn is of higher order?

(b) Now Using Laplace transform.

$$m\dot{V}(t) + bV(t) = F(t)$$

(1) Natural response. $(F(t)=0)$

$$msV(s) - mV(0) + bV(s) = 0$$

$$V(s)(ms+b) = mV_0$$

$$V(s)\cancel{m}(s+b/m) = \cancel{m}V_0$$

$$V(s) = \frac{V_0}{s+b/m}$$

$$V_1(t) = V_0 e^{-b/mt} \rightarrow (1)$$

(2) Forced Response $(V_0 = V(0) = 0)$

$$msV(s) + bV(s) = \frac{F_0}{s}$$

$$V(s)[ms+b] = \frac{F_0}{s}$$

$$V(s) = \frac{1}{m}\frac{F_0}{s} \cdot \frac{1}{s+b/m}$$

Applying Partial fractions.

$$\frac{1}{s(s+b/m)} = \frac{A}{s} + \frac{B}{s+b/m}$$

$$A = m/b \quad , \quad B = -m/b$$

$$V(s) = \frac{F_0}{\cancel{m}} \left[\frac{\frac{m}{b}}{s} - \frac{m/b}{s+b/m} \right]$$

$$= \frac{F_0}{b} \left[\frac{1}{s} - \frac{1}{s+b/m} \right]$$

By taking inverse
Laplace transform

$$V_2(t) = \frac{F_0}{b} \left[1 - e^{-b/mt} \right]$$

$$V(t) = V_1(t) + V_2(t)$$

$$\boxed{V(t) = V_0 e^{-b/mt} + \left(1 - e^{-b/mt}\right) \frac{F_0}{b}}$$

Is it easier?