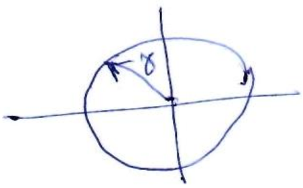
Roots of Complex numbers

Consider $z = re^{i\theta}$

which is a circle centered at the origin.

→ if we increase θ from 0 to 2π or decrease it by 2π we reach at our starting point.



→ So two complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ will be equal if

$$r_1 = r_2$$

and $$\theta_1 = \theta_2 + 2k\pi \quad (k = 0, \pm 1, \pm 2 \ldots)$$

Now how to find the roots of complex no.? e.g. $\sqrt{z}$, $\sqrt[3]{z}$. etc.

We'll use the two properties.

① $z_1 = z_2$ iff

$$r_1 = r_2$$

and $$\theta_1 = \theta_2 + 2k\pi \quad (k = 0, \pm 1, \pm 2 \ldots)$$

② $$z^n = r^n e^{in\theta}$$

Now let $z_0 = r_0 e^{j\theta_0} \neq 0$

and say its nth root is a number

$z = re^{j\theta}$ i.e

$$\sqrt[n]{r_0 e^{j\theta_0}} = re^{j\theta}$$

then

$$r_0 e^{j\theta_0} = r^n e^{jn\theta}$$

According to prop: 1

$$r^n = r_0$$

and

$$n\theta = \theta_0 + 2k\pi$$

$$\theta = \frac{\theta_0}{n} + \frac{2k\pi}{n}$$

Consequently the complex number

$$z = \sqrt[n]{r_0} \exp\left[j\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right] \rightarrow 3$$

with $k = 0, \pm 1, \pm 2 \cdots$

are the nth roots of z_0

③

Furthermore, if we look at we can see that all the roots lie on a circle of radius $\sqrt[n]{r_0}$ centered at the origin & are equally spaced every $\frac{2\pi}{n}$ radians. starting with $\frac{\theta_0}{n}$. Evidently all the distinct root are found when $k = 0, 1, 2 \cdots n-1$

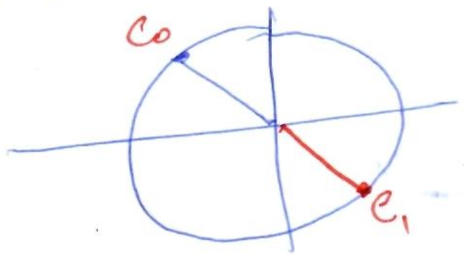
$\Rightarrow$ Let the k^{th} root be

$$C_K = \sqrt[n]{r_0}\ \mathrm{Exp}\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right]$$

then evidently

$$C_0 = \sqrt[n]{r_0}\ \mathrm{Exp}\left[i\,\frac{\theta_0}{n}\right]$$

if $n=2$ then



but if $n \geq 3$ then we'll have a polygone inscribed in the circle of radius $\sqrt[n]{r_0}$

Example let $z = 1$

Find $z^{1/n}$?

$$z^{1/n} = \sqrt[n]{1}\, e^{i(0+2k\pi)}$$

$$= 1\, e^{i\left(0+\frac{2k\pi}{n}\right)} = e^{i2k\pi}$$

Now let $n=2$

$$z^{1/2} = \sqrt{z} = 1\, e^{\frac{i2k\pi}{n}} \quad (k=0,1)$$

$k=0$ $z_1 = 1,$

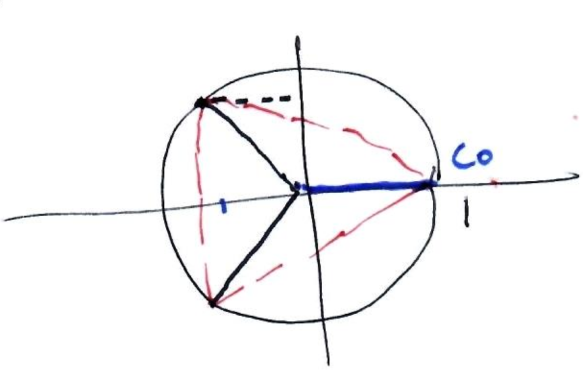
$k=1$ $z_2 = -1$

if $n = 3$

$z_1 = 1$

$z_2 = e^{2\pi/3} = -0.5+0.866i$

$z_3 = e^{4\pi/3} = -0.5-0.866i$



$\cos(\pi) = -1$
$\sin(\pi) = 0$
$\cos(2\pi) = 1$
$\sin(2\pi) = 0$

$\sin(n\pi) = 0$
$n = 0, 1, 2 \ldots$
$\cos(n\pi) = 1$ if n is even
& $\cos(n\pi) = -1$ if n is odd

x	π	$\pi/2$	$\pi/3$	$\pi/4$	$\pi/6$
$\sin x$	0	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$1/2$
$\cos x$	-1	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$

Exaple: Find all the cube roots of $-8i$ i.e., $(-8i)^{1/3}$?

$$z_1 = (-8i)^{1/3} = \sqrt[3]{8}\, e^{i\left[\frac{-\pi/2}{3} + \frac{2\pi k}{3}\right]}$$

$$= \sqrt[3]{2^3}\, e^{i\left\{-\frac{\pi}{6} + \frac{2k\pi}{3}\right\}}$$

$$= 2\, e^{i\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right)}$$

$$c_0 = 2\, e^{i\left(-\frac{\pi}{6}\right)} = 2\left[\cos(-\pi/6) - i\sin\left(\frac{\pi}{6}\right)\right]$$

$$= 2\left[\frac{\sqrt{3}}{2} - \frac{1}{2} i\right]$$

$$= \sqrt{3} - i$$

$$c_1 = 2\, e^{i\left(-\frac{\pi}{6} + \frac{2\pi}{3}\right)} = 2e^{i(\pi/2)} = 2\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$

$$= 2i$$

$$c_2 = 2\, e^{i\left(-\frac{\pi}{6} + \frac{4\pi}{3}\right)} = 2e^{i\,7\pi/6} = 2\left\{\cos\frac{7\pi}{6} + i\sin\left(\frac{7\pi}{6}\right)\right\}$$

$$= 2\left\{-\frac{\sqrt{3}}{2} - \frac{i}{2}\right\} = -\sqrt{3} - i$$

Example

Square roots of $\sqrt{3}+i$?

$\sqrt{\sqrt{3}+i} = ?$

$$\sqrt{3}+i = 2e^{i\left(\frac{\pi}{6}\right)}$$

So

$$\sqrt{\sqrt{3}+i} = \sqrt{2}\, e^{i\left[\frac{\pi}{12}+\frac{2k\pi}{12}\right]}, \quad k=0,1$$

$$C_0 = \sqrt{2}\, e^{i\left(\frac{\pi}{12}\right)} = \sqrt{2}\left[\cos\left(\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{12}\right)\right]$$

$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2} \quad \text{hence} \quad \cos^2\frac{\alpha}{12} = \frac{1}{2}\left(1+\cos\frac{\alpha}{6}\right)$$

Similarly

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2} \Rightarrow \sin^2\frac{\alpha}{12} = \frac{1}{2}\left(1-\cos\frac{\alpha}{6}\right)$$

$$\cos\frac{\pi}{12} = \sqrt{\frac{1}{2}\left(1+\cos\frac{\pi}{6}\right)} = \sqrt{\frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right)}$$

$$\sin\pi/12 = \quad " \quad = \sqrt{\frac{1}{2}\left(1-\sqrt{3}/2\right)}$$

$$C_0 = \cancel{\sqrt{2}}\left[\frac{1}{\cancel{\sqrt{2}}}\left(\sqrt{\frac{2+\sqrt{3}}{2}} + i\sqrt{\frac{2 \mp \sqrt{3}}{2}}\right]\right)$$

$$= \sqrt{\frac{2+\sqrt{3}}{2}} + i\sqrt{\frac{2 \mp \sqrt{3}}{2}}$$

Exercise 1 P:28, Churchill.

(a) Square Roots of $z = 2i$

Step 1:-> Write z in polar form.

$$z = 2\, e^{i\pi/2}$$

Step 2:-> The roots are then given by the equation.

$$C_K = \sqrt{2}\, Exp\left[i\left[\frac{\frac{\pi}{2} + 2K\pi}{2}\right]\right]$$

$$= \sqrt{2}\, Exp\left[i\left[\frac{\pi}{4} + K\pi\right]\right]$$

Step 3: Find all the roots by varying K ~~from~~ from 0 to $n-1$. In this case 0 to 1 ∵ Square Roots.

$$C_0 = \sqrt{2}\, \mathrm{Exp}\left[i\,\pi/4\right]$$

$$= \sqrt{2}\left[\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right]$$

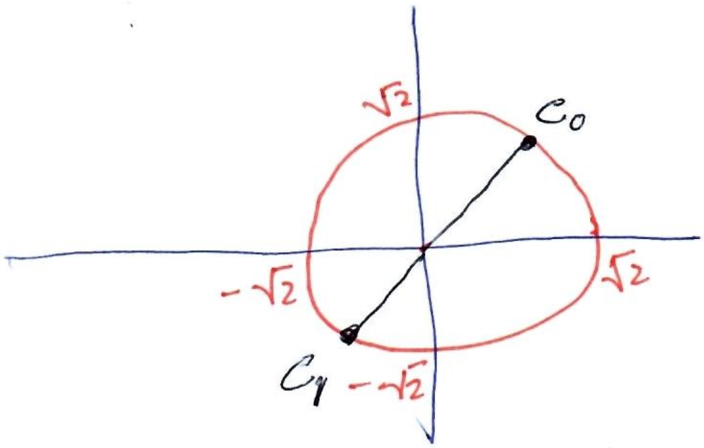
$$\boxed{C_0 = 1 + i} \qquad \because \cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$C_1 = \sqrt{2}\, \mathrm{Exp}\Big[i\underbrace{\left(\frac{\pi}{4} + \pi\right)}_{=\frac{5\pi}{4}}\Big]$$

$$= \sqrt{2}\left[\cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right)\right]$$

$$\boxed{C_1 = -1 - i}$$

$$\cos\left(\pi + \frac{\pi}{4}\right)$$
$$= \cos\pi \cos\frac{\pi}{4} - \sin\pi \sin\frac{\pi}{4}$$
$$= -\frac{1}{\sqrt{2}} - 0$$
$$\sin\left(\pi + \frac{\pi}{4}\right)$$
$$= \sin\pi \cos\frac{\pi}{4} + \cos\pi \sin\frac{\pi}{4}$$
$$= 0 - \frac{1}{\sqrt{2}}$$



Exercise 6: P 29, Churchill.

$$z^4 + 4 = 0$$

Find all the roots and factorize it.

Solution

$$z^4 + 4 = 0$$

$$z^4 = -4 = 4e^{-i\pi}$$

$$c_k = \sqrt[4]{4}\, \mathrm{Exp}\left[i\left(-\frac{\pi}{4} + \frac{2k\pi}{4}\right)\right]$$

$$= \sqrt{2}\, \mathrm{Exp}\left[i\left(-\frac{\pi}{4} + \frac{k\pi}{2}\right)\right]$$

$$c_0 = \sqrt{2}\, \mathrm{Exp}\left[i\left(-\frac{\pi}{4}\right)\right]$$

$$\boxed{c_0 = 1 - i}$$

$$c_1 = \sqrt{2}\, \mathrm{Exp}\left[i\left(-\frac{\pi}{4} + \frac{\pi}{2}\right)\right]$$

$$\boxed{c_1 = 1 + i}$$

$$c_2 = \sqrt{2}\, \mathrm{Exp}\left[i\left(-\frac{\pi}{4} + \pi\right)\right]$$

$$c_2 = -1 + i$$

$$c_3 = \sqrt{2}\, \mathrm{Exp}\left[i\left(-\frac{\pi}{4} + \frac{3\pi}{2}\right)\right]$$

$$\boxed{c_3 = -1 - i}$$

Now

$$(z+1-i)(z+1+i)(z-1+i)(z-1-i) = z^4 + 4$$

$$\underbrace{(z^2+2z+2)(z^2-2z+2)}_{\text{factored}} = z^4 + 4$$