# Chapter 15

Theorem (1): $Z_n = \frac{n\pi i}{n+i}$

$$Z_n = \frac{n\pi i}{n+i} \times \frac{n-i}{n-i}$$

$$Z_n = \frac{n^2\pi i - n\pi i^2}{n^2+1^2}$$

$$Z_n = \frac{n\pi + n^2\pi i}{n^2+1}$$

$$Z_n = \frac{n\pi}{n^2+1} + \frac{n^2\pi i}{n^2+1}$$

$$Z_n = \frac{\cancel{n^2}(\pi/n)}{\cancel{n^2}(1+1/n^2)} + \frac{\cancel{n^2}(\pi) i}{\cancel{n^2}(1+\frac{1}{n^2})}$$

$$Z_n = \frac{\pi/n}{1+1/n^2} + \frac{\pi i}{1+1/n^2}$$

$$\lim_{n\to\infty} Z_n = \lim_{n\to\infty} \frac{\pi/n}{1+1/n^2} + \frac{\pi i}{1+1/n^2}$$

$$= 0 + \frac{\pi i}{1+0} = \boxed{\pi i}$$

when $n\to\infty$, $a \to 0$, $b \to \pi$

$S_n = Z_1 + Z_2 + \text{-----} Z_n$ (Partial Sum)

$S = \lim_{n \to \infty} S_n$ (Value of series)

$S = S_n + R_n$

OR

$R_n = S - S_n$

Q23 (PS 15.1)

$$\sum_{n=0}^{\infty} \frac{n-i}{3n+2i}$$

Ans:

For convergent, $\lim_{n \to \infty} Z_n = 0$

$$\lim_{n \to \infty} Z_n = \lim_{n \to \infty} \frac{n-i}{3n+2i}$$

$$" = \lim_{n \to \infty} \frac{\cancel{n}(1 - i/n)}{\cancel{n}\left(3 + \frac{2i}{n}\right)}$$

$$" = \frac{1-0}{3+0} = \frac{1}{3} \neq 0$$

$\therefore$ Series diverges.

$$\sum_{n=1}^{\infty}\left(\frac{1}{5n!}+\frac{2i}{5n!}\right)$$

By Comparison test

$$|Z_n| = \sqrt{\left(\frac{1}{5n!}\right)^2+\left(\frac{2}{5n!}\right)^2}$$

$$|Z_n| = \sqrt{\frac{1}{(5n!)^2}+\frac{4}{(5n!)^2}}$$

$$|Z_n| = \sqrt{\frac{5}{(5n!)^2}}$$

$$|Z_n| = \sqrt{\frac{5}{5^2 \cdot (n!)^2}} = \frac{1}{\sqrt{5}\cdot n!}$$

Now $\sqrt{5}.n! > n!$

$$\therefore \frac{1}{\sqrt{5}.n!} < \frac{1}{n!}$$

$$\therefore |Z_n| = \frac{1}{\sqrt{5}.n!}, \quad b_n = \frac{1}{n!}$$

Now we check convergence of $b_n$ and it converges.

$\therefore$ ~~Zn~~ The given series also converges.

Example (4) $\sum_{n=0}^{\infty} \frac{(100+75i)^n}{n!}$

By Ratio Test,

$$L = \lim_{n \to \infty} \left| \frac{(100+75i)^{n+1}}{(n+1)!} \div \frac{(100+75i)^n}{n!} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(100+75i)^{n+1}}{(n+1)\cancel{n!}} \times \frac{\cancel{n!}}{(100+75i)^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(100+75i)^{\cancel{n}+1-\cancel{n}}}{n+1} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{100+75i}{n+1} \right|$$

$$L = \lim_{n \to \infty} \frac{|100+75i|}{n+1} = \lim_{n \to \infty} \frac{125}{n+1} = \frac{125}{\infty+1} = \boxed{0}$$

The series converges absolutely.

# POWER SERIES

PS 15.2

(5) $\sum_{n=0}^{\infty} \frac{n!}{n^n}(z+1)^n$

Ans. $z_0 = -1$

$$a_n = \frac{n!}{n^n}$$

$$L^* = \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!}\right|$$

$$L^* = \lim_{n\to\infty}\left|\frac{(n+1)\cancel{n!}\cdot n^n}{(n+1)^{n+1}\cdot \cancel{n!}}\right|$$

$$L^* = \lim_{n\to\infty}\left|(n+1)^{1-n-1}\cdot n^n\right|$$

$$L^* = \lim_{n\to\infty}\left|\frac{n^n}{(n+1)^n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{n^n}}{\cancel{n^n}\left(1+\frac{1}{n}\right)^n}\right|$$

$$L^* = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$$

$$\boxed{R = e}$$

⑥ $\sum_{n=0}^{\infty} \frac{2^{100n}}{n!} z^n$

Ans:- $z_0 = 0$

$$a_n = \frac{2^{100n}}{n!}$$

$$L^* = \lim_{n\to\infty} \left| \frac{2^{100(n+1)}}{(n+1)!} \times \frac{n!}{2^{100n}} \right|$$

$$= \lim_{n\to\infty} . \frac{2^{100n+100} \times \cancel{n!}}{(n+1)\cancel{n!} \times 2^{100n}}$$

$$= \lim_{n\to\infty} . \frac{2^{\cancel{100n}+100-\cancel{100n}}}{n+1}$$

$$= \lim_{n\to\infty} \frac{2^{100}}{n+1} = 0 .$$

$L^* = 0$

$\therefore R = \infty$ b/c $\left(R = 1/L^*\right)$

$0 + \frac{2}{9}(z-2i)^2$

PS. 15.3

① $\sum_{n=0}^{\infty} \frac{n(n-1)}{3^n}(z-2i)^n$

Ans:- ⓐ

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{n(n+1)}{3^{n+1}} \div \frac{n(n-1)}{3^n} \right|$$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{\cancel{n}(n+1)}{3^{n+1}} \times \frac{3^n}{\cancel{n}(n-1)} \right|$$

$$\frac{1}{R} = \lim_{n\to\infty} \frac{(n+1).\cancel{3^n}}{\cancel{3^n}.3^1.(n-1)}$$

$$\frac{1}{R} = \lim_{n\to\infty} \frac{n+1}{3(n-1)}$$

$$\frac{1}{R} = \frac{1}{3}\lim_{n\to\infty} \frac{\cancel{n}(1+1/n)}{\cancel{n}(1-1/n)} = \frac{1}{3}\cdot\frac{1}{1}$$

$$\frac{1}{R} = \frac{1}{3} \Rightarrow \boxed{R = 3}$$

ⓑ Above series can be obtained by differentiating $\frac{(z-2i)^n}{3^n}$ two times and then multiplying $(z-2i)^2$ with each term.

Scanned with CamScanner

$\therefore$ By theorem (3).

~~Radius of convergence of~~ $\sum_{n=0}^{\infty} \frac{(z-2i)^n}{3^n}$

is the same as the given series.

(4). Here $a_n = \frac{1}{3^n}$

Now by Cauchy-Hadamard Formula,

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{\frac{1}{3^{n+1}}}{1/3^n} \right| = \lim_{n\to\infty} \left| \frac{3^n}{3^{n+1}} \right|$$

$$\frac{1}{R} = \lim_{n\to\infty} \left| 3^{\cancel{n}-\cancel{n}-1} \right|$$

$$\frac{1}{R} = \lim_{n\to\infty} \left| 3^{-1} \right| = \frac{1}{3}$$

$$\therefore \boxed{R = 3}$$

Scanned with CamScanner