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POWER SERIES & TAYLOR SERIES

$a_n = \left(\frac{1}{2}\right)^n$

Complex Sequence

$\sum a_n$

$\{z_n\} = \{x_n + iy_n\}$

$\sum z_n$

15.1 SEQUENCES, SERIES, CONVERGENCE TEST

Sequence:- An infinite sequence is obtained by ~~assuming~~ assigning to each positive integer n a number z_n called a term of the sequence. and is written $z_1, z_2, \ldots$ or $\{z_n\}$.

A real sequence is one whose terms are real

e.g

$$z_n = \frac{n\pi}{1+3ni}, \quad n = 1, 2, 3, \ldots$$

Convergent Sequence:- A convergent sequence $\{z_n\}$ is one that has a limit c written

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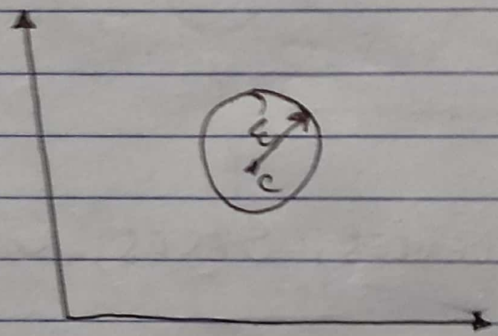
$$\lim_{n\to\infty} z_n = c \quad \text{or} \quad z_n \to c$$

$c =$

By def of limit this means that for every $\varepsilon > 0$ we can find a N such that

$$|z_n - c| < \varepsilon \quad \text{for all } n > N \text{ --- (i)}$$



Example ①:-

$$\left\{ \frac{i^n}{n} \right\}$$

Sol:-

Let $z_n = \dfrac{i^n}{n}$

$$\lim_{n\to\infty} z_n = \frac{i^n}{n} = \frac{i^\infty}{\infty} = 0 = c$$

Example ②:

$$z_n = x_n + i y_n$$

$$= \left(1 - \frac{1}{n^2}\right) + i\left(2 + \frac{4}{n}\right)$$

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$$\lim_{n\to\infty} Z_n = \lim_{n\to\infty}\left(1-\frac{1}{n^2}\right) + i \lim_{n\to\infty}\left(2+\frac{4}{n}\right)$$

$$// = 1 + 2i$$

Theorem (1)

A sequence, $Z_1, Z_2, \dots Z_n$ of complex numbers $Z_n = X_n + i y_n$ where $n = 1, 2, \dots$ converges to $c = a + ib$ Iff the sequence of real parts $x_1, x_2, \dots$ converges to a and the sequence of the imaginary parts $y_1, y_2, \dots$ converges to b.

H.W Apply theorem (1)

(1) $Z_n = \dfrac{n\pi i}{n+i}$

Series:-

If $\{Z_n\}$ is a sequence then

sum $Z_1 + Z_2 + Z_3 + \dots Z_n$ — (1)

is called infinite series, we write

$$\sum_{n=1}^{\infty} Z_n \text{ or } \sum Z_n$$

Let S_n denote the sum of first n terms of series $\sum_{n=1}^{\infty} Z_n$

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S_n is called the nth partial sum of the infinite series $\sum_{n=1}^{\infty} z_n$

$= z_1 + z_2 + z_3 + \cdots$

Convergent Series:-

A convergent series is one whose sequence of partial sums converges, say

$$\lim_{n\to\infty} S_n = S$$

Then we write $\sum_{n=1}^{\infty} z_n = S$ and S is called the sum or value of series.

Theorem (3) (Divergence)

If a series $\sum_{n=1}^{\infty} z_n$ converges then

$$\lim_{n\to\infty} z_n = 0$$

Hence

(23) $\sum_{n=0}^{\infty} \frac{n-i}{3n+2i}$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{\cancel{n}\left(1-\frac{i}{n}\right)}{\cancel{n}\left(3+\frac{2i}{n}\right)}$$

$$= \boxed{\frac{1}{3}}$$

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$\therefore$ the given series diverges.

Absolute Convergence

A series $\sum_{n=1}^{\infty} z_n$ is called absolutely convergent if the series of the absolute values of the terms

$$\sum_{n=1}^{\infty} |z_n| = |z_1| + |z_2| + \cdots |z_n| + \cdots$$

is convergent if $\sum_{n=1}^{\infty} z_n$ converges but $\sum_{n=1}^{\infty} |z_n|$ diverges, then series $\sum_{n=1}^{\infty} z_n$ is conditionally convergent.

Theorem (Comparison test)

If a series $z_1 + z_2 \cdots\cdots$ is given and we can find a convergent series $b_1 + b_2 + \cdots$ with non negative real terms such that

$$|z_1| \leq b_1,$$

$$|z_2| \leq b_2 \cdots$$

the given series converges, even absolutely

H.W $\sum_{n=1}^{\infty} \left(\frac{1}{5n!} + \frac{2i}{5n!} \right)$

$$1 + 2 + 4$$

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$$|z_n| = \sqrt{\left(\frac{1}{5n!}\right)^2 + \left(\frac{2}{5n!}\right)^2}$$

$$= \frac{1}{\sqrt{5}\, n!}$$

Now $\sqrt{5}\, n! > n!$

$$\Rightarrow \frac{1}{\sqrt{5}\, n!} < \frac{1}{n!}$$

$$b_n = \frac{1}{n!}$$

$$\therefore \sum b_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \cdots \frac{1}{n!}$$

$$= e$$

$$= 2.718282$$

$$\therefore |z_n| \leq |b_n|$$

and $\sum |b_n|$ is Convergent.

$\therefore \sum |z_n|$ Converges

$\therefore$ given series is absolutely Convergent. or Convergent series.

—x—

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Theorem (Geometric Series)

$$\sum_{m=0}^{\infty} q^m = 1 + q + q^2 + \cdots \qquad (6)$$

Converges with sum $\frac{1}{1-q}$ if $|q| < 1$

and diverges if $|q| \geq 1$

Ratio Test :-

If a series $z_1 + z_2 + \cdots$ with $z_n \neq 0$ $(n = 1, 2, 3, \ldots)$ is such that $\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = L$, then

(a) If $L < 1$, the series converges absolutely,
(b) If $L > 1$, the series diverges.
(c) If $L = 1$, the series may converge or diverge, so that test fail and permits no conclusion.

Example (4)

$$\sum_{n=0}^{\infty} \frac{100 + 75i}{n!}$$

find $z_{n+1} = ?$

H.W

(1) $\sum_{n=1}^{\infty} (3i)^n \cdot \frac{n!}{n^n}$ (2) $\sum_{n=0}^{\infty} (20 + 30i)^n$ (3)

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Theorem (10): Root Test

If a series $z_1 + z_2 + \cdots z_n + \cdots$ is such that

$$\lim_{n \to \infty} \sqrt[n]{|z_n|} = L, \text{ then}$$

(a) The series converges absolutely if $L<1$
(b) The series diverges if $L>1$
(c) If $L = 1$, the test fails: that is no conclusion is possible.

H.W (1) $\sum_{n=1}^{\infty} n^2 \left(\frac{i}{2}\right)^n$

15.2: Power Series:

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

z is variable, ~~are~~
a_n complex / real.
z_0 is a complex constant; Center of Series.

If $z_0 = 0$, $a_0 = a_1 = a_n = 1$

$$\therefore \sum_{n=0}^{\infty} z^n$$

which is G.S with C.r $= z$

$|z|<1$; if $|z| \geq 1$ diverges.

Exmple (2) :-

$a_n = 1/n!\ ,\ z_0 = 0$

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = \frac{z^0}{0!} + \frac{z^1}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} \cdots$$

By Ratio test;

$$\lim_{n\to\infty} \left| \frac{z^{n+1}}{(n+1)!} \cdot \frac{n!}{z^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{z^n} \cdot z^1}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{z^n}} \right| = \lim_{n\to\infty} \left| \frac{z}{n+1} \right|$$

$$= \lim_{n\to\infty} \frac{|z|}{|n+1|}$$

$$= |z| \lim_{n\to\infty} \frac{1}{n+1}$$

$$= |z| \left(\lim_{n\to\infty} \frac{1/n}{1 + 1/n} \right)$$

$$= |z| \cdot \frac{0}{1+0} = |z| = 0 < 1$$

By ratio test given series Converges for any z.

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$$\frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}(\frac{1}{n})}$$

Example(3)

$$\sum_{n=0}^{\infty} n!\, z^n, \quad \text{by ratio test}$$

$$\lim_{n\to\infty} \left| \frac{(n+1)!\, z^{n+1}}{n!\, z^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)\,\cancel{n!}}{\cancel{n!}} \cdot \frac{\cancel{z^n}\cdot z^1}{\cancel{z^n}} \right|$$

$$=|z| \lim_{n\to\infty} \left| (n+1)\, z \right|$$

$$= |z| \cdot \lim_{n\to\infty} \left(\frac{n+1}{1} \right)$$

$$= |z| \cdot \lim_{n\to\infty} (n+1) = \begin{cases} \infty, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

converges for $z = 0$, otherwise diverges

$$= |z| \lim_{n\to\infty} \left(\frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}(\frac{1}{n})} \right)$$

$$= |z| \left(\frac{1+0}{0} \right)$$

$$= |z| \cdot \infty$$

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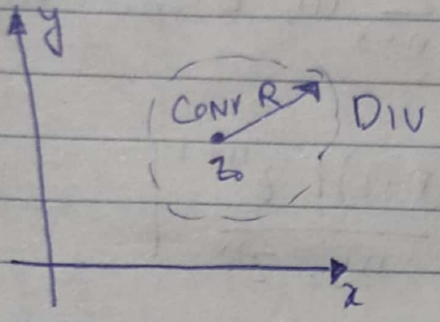
Radius Of Convergence Of a Power Series:-



$$|z - z_0| < R$$
$$|z - z_0| > R$$

$\boxed{R = \infty}$, if the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges for all z.

$\boxed{R = 0}$, if the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges only at centre z_0.

Theorem(2): (Radius Of Convergence R).

Suppose that the sequence

$$\left|\frac{a_{n+1}}{a_n}\right|, \quad n = 1, 2, 3, \ldots$$

Converges with Limit L^* then $R = \infty$ i.e power series $\sum_{n=0} a_n (z - z_0)^n$ Converges for all z. if $L^* \neq 0$ for all z then

$$R = \frac{1}{L^*} = \lim_{n \to \infty} \left|\frac{a_n}{a_{n+1}}\right|.$$

$$\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} \qquad \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!} = \frac{n^n}{(n+1)^n}$$

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$$\lim_{n\to\infty} \left(\frac{n}{1+n}\right)^n$$

If $\left|\frac{a_{n+1}}{a_n}\right| \to \infty$, then $R = 0$

converges only at centre $z_0 = 0$

Example ⑤ :-

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} (z-3i)^n$$

Sol:-

Problem set 15.2:

⑤ $$\sum_{n=0}^{\infty} \frac{n!}{n^n} (z+1)^n$$

Sol:- $z_0 = -1$, $a_n = \frac{n!}{n^n}$

$$L^* = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1/e \neq 0$$

$$\therefore R = \frac{1}{L^*} = e$$

⑥ $$\sum_{n=0}^{\infty} \frac{2^{100n}}{n!} z^n$$

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15.3 Function given by P.S

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

If $z_0 = 0$, then

$$\sum_{n=0}^{\infty} a_n z^n \text{ —— } (1)$$

If any given power series (1) has a non-zero radius of convergence R (thus R > 0) its sum is a function of z, say f(z). Then we write

$$\sum_{n=0}^{\infty} a_n . z^n = f(z) \text{ —— } (2)$$

if $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$

$= f(z)$

Theorem(1): If a function f(z) can be represented by a power series

$\sum_{n=0}^{\infty} a_n z^n = f(z)$ with radius of convergence R>0, then f(z) is continous at z = 0

OPERATIONS ON POWER SERIES:-

① Term wise addition & Subtraction:

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$$a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + \cdots) z^2 + \cdots$$

$$= \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n} + \cdots a_n b_0) z^n$$

$$\boxed{f(z) = \sum_{k=0}^{n} a_k z^n} \quad , \quad \boxed{g(z) = \sum_{m=0}^{\infty} b_m z^m}$$

② Termwise multiplication.

$$\sum_{n=0}^{\infty} z^m$$

$$\frac{z^2}{2} \sum_{n=0}^{\infty} z^n = \frac{1}{2} \sum_{n=0}^{\infty} z^{n+2}$$

R of

③ Termwise Differentiation

Theorem (3):

Example (1) Find the R

$$\sum_{n=0}^{\infty} \binom{n}{2} z^n = z^2 + 3z^3 + \cdots$$ Sol: H.W

R > 0,

Theorem (4): The power series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} =$

$$a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 \cdots$$

obtained by integrating the series

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$a_0 + a_1 z + a_2 z^2 + \dots$ term by term has the Same radius of Convergence as the original Series.

Power Series represent analytic function.

Theorem (5) A power series with a non-zero radius of Convergence R represents an analytic function at every point interior to its circle of convergence. The derivatives of this ftion are obtained by differentiating the original series term by term. All the series thus obtained have the same radius of convergence as the original series. Hence by the first statement, each of them represent an analytic ftion.

PS 15.3

(1) $\sum_{n=0}^{\infty} \frac{n(n-1)}{3^n} (z-2i)^n$

15.4: Taylor Series And Maclaurin Series.

The taylor series of function $f(z)$, the Complex analog of the real taylor series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $a_n = \frac{1}{n!} f^{(n)}(z_0)$

$$(1-z)^{-2} = -2(1-z)^{-3}(-1)$$

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and $$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}}\, dz^*$$

$$= \frac{f^{(n)}(z_0)}{n!}$$

Power Series As Taylor Series:-

Theorem (2)

A power series with a nonzero radius of convergence is the taylor series of its sum.

Some Important Taylor Series:-

Example (1)

$$f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

$$f'(z) = -1(1-z)^{-2}(-1) = (1-z)^{-2}$$

$$= \frac{1!}{(1-z)^{1+1}}$$

$$f''(z) =$$

$$\vdots$$

$$f'''(z) =$$

$$f^{n}(z) = \frac{n!}{(1-z)^{n+1}}$$

So the Maclaurin Series of given ftion is

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$$f(z) = f(0) + f'(0) z^1 + \frac{f''(0)}{2!} z^2 + \cdots + \frac{f^n(0)}{n!} z^n \cdots$$

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots$$

Example ② $f(z) = e^z$, Example ③: Trig. and Hyperbolic ftions.

Example ④

Practical Method

Q5