Limits & continuity of Complex nos.

For real and/or imaginary func^n.

$$f(x) = y$$

$$\lim_{x \to x_0}$$

means that we can approach the pt. y in the Range(f) if we go closer to the pt. x_0 in the Dom(f).

Existance of Limit for Real nos.

The limit is said to exist if the left hand side limit is equal to the RHS limit.

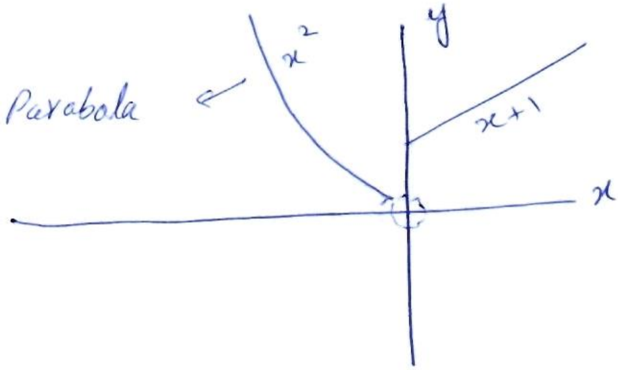
$$\lim_{x \to x_0} f(x) = L$$

iff $\underbrace{\lim_{x \to x_0^-} f(x)}_{\text{approach } x_0 \text{ from the left}} = \underbrace{\lim_{x \to x_0^+} f(x)}_{\text{approach } x_0 \text{ from the right}} = L$

then we say that the limit of $f(x)$ exist at x_0 and it is L.

Ex

$$f(x) = \begin{cases} x^2, & x < 0 \\ x+1, & x >= 0 \end{cases}$$



$\lim_{x \to 0} f(x)$ exist?

when we approach $x=0$ from the left.

$f(x) = x^2$ & $\lim_{x \to 0^-} f(x) = 0$

& When we approach $x=0$ from the right

$f(x) = x+1$ & $\lim_{x \to 0^+} f(x) = 1$

Since $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

hence $\lim_{x \to 0} f(x)$ does not exist.

$\Rightarrow$ In Real fun" we had a pt. which can be approached from 2 explicitly defined directions but this is not the case in complex analysis e.g.

i.e a complex no. can be approached by infinity many ways.

$\Rightarrow$ For a complex fun" $f(z)$ we say that $\lim_{z \to z_0} f(z) = L$ exist iff $\underbrace{\lim_{z \to z_0} f(z) = f(z_0)}_{\text{along any curve.}} = L$

This gets strict as compared to the real fun".

→ In short if we find any two contradictory curves then the limit doesn't exist.

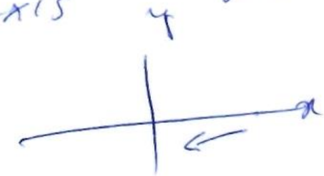
Ex 1:→ $\boxed{\lim_{z \to 0} \frac{z}{\bar{z}} \; ?}$

$z = x + iy$

$$\lim_{(x,y)\to(0,0)} \frac{x+iy}{x-iy}$$

$z = 0$ means the origin & many ways to approach.

→ So let approach $z=0$ along x-axis which means $y=0$



$$\lim_{x \to 0} \frac{x}{x} = 1$$

→ Now let approach $z=0$ along y-axis i.e $x=0$

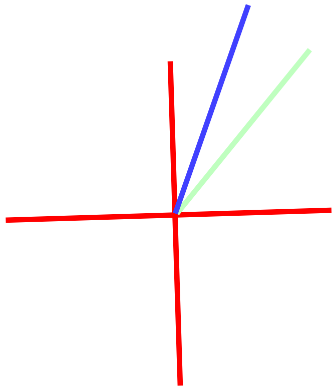
$$\lim_{y \to 0} \frac{iy}{-iy} = -1$$

Since the limiting values along both the curves are different so the limit $\lim_{z\to 0} \frac{z}{\bar{z}}$ doesn't exist.

Ex: 2:→ $\lim_{z \to 0} \left(\frac{z}{\bar{z}}\right)^2 = ?$

Solution:– $\lim_{(x,y)\to(0,0)} \left(\frac{x+iy}{x-iy}\right)^2$

Approaching $z=0$ along x-axis



$$\lim_{x\to 0} \left(\frac{x}{x}\right)^2 = 1$$

and approaching along y-axis

$$\lim_{y\to 0} \left(\frac{iy}{-iy}\right)^2 = 1$$

Both are the same but this is not enough.

Let us approach $z=0$ diagonally

i.e $y = mx + c$

$\hookrightarrow = 0$ ∵ origin

$$\lim_{x\to 0} \left(\frac{x+imx}{x-imx}\right)^2 = \lim_{x\to 0} \left(\frac{1+im}{1-im}\right)^2$$

$$= \left(\frac{1+im}{1-im}\right)^2$$

So if $m \neq 0$ the result is other than 1

Hence the limit is not unique.

{limiting values are}

So $\lim_{z\to 0} \left(\frac{z}{\bar{z}}\right)^2$ does not exist.

Ex 3 $\lim_{z\to 0} \frac{Re(z)\, Im(z)}{Re(z)+Im(z)}$ [z = x + iy]

Sol: $\lim_{(x,y)\to(0,0)} \frac{xy}{x+y}$

Approaching along x-axis ($y=0$) gives 0 and approaching along y-axis ($x=0$) also gives 0.

Now if we approach diagonally i.e $y=mx$

$$\lim_{x\to 0} \frac{mx^2}{x+mx} = \frac{mx}{1+m}$$

For $x=0$ this is once again 0 but what if $m=-1$? we'll then have $\frac{0}{0}$

So in that case we can't say that the limiting value of the fun" is 0.

$$\lim_{x\to 0} \frac{mx}{1+m} = 0 \text{ iff } m \neq -1$$

Hence $\lim_{z\to 0} \frac{Re(z)\, Im(z)}{Re(z)+Im(z)}$ doesn't exist.

Now let us take an example where limit exists.

Ex 4

$$\boxed{\lim_{z \to 1+i} (z^2+1)\,?}$$

$$\lim_{z \to 1+i} (z^2+1) = \left((1+i)^2+1\right)$$

$$= (\cancel{1}-\cancel{1}+2i+1) = 1+2i$$

Here we do not take any arbitrary curve as in previous examples direct application of limit would result a $\frac{0}{0}$.

Continuity :→ A real funⁿ $f(x)$ is continuous at $x = x_0$ if the limiting value of $f(x)$ at x_0 is the same as the value of the funⁿ at x_0

$$\boxed{\lim_{x \to x_0} f(x) = f(x_0)}$$

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = f(x_0)$$

⇒ **Example:** $f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & x \neq 1 \\ 2 & x = 1 \end{cases}$

Is f(x) continuous at x = 1?

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2-1}{x-1} = \quad = \lim_{x \to 1} x+1 = 2$$

Now $f(1) \triangleq 2$

So $f(x)$ is continuous at $x = 1$

⇒ **Continuity of Complex funⁿ.** :→ The definition is the same.

limit should exist. → Should be defined

$$\lim_{z \to z_0} f(z) = f(z_0)$$

then $f(z)$ is continuous at $z = z_0$

Ex:→ $f(z) = \begin{cases} \dfrac{z^3-1}{z-1} & |z| \neq 1 \\ 3 & |z| = 1 \end{cases}$

is continuous at z = 1 ?

Solution → $\lim_{z\to 1} f(z) = \lim_{z\to 1} \left(\dfrac{z^3-1}{z-1}\right)$

We can't put the limit directly

∵ it will result in $\frac{0}{0}$ form.

$$(z^3-1) = (z-1)(z^2+a+b)$$

$$= z^3 + az + bz - z^2 - a - b$$

$$-1 = az + bz - \cancel{z}^2 - a - b$$

let $b = z$

$$-1 = az - a - \cancel{b}$$

$$-1 = (a-1)z \quad -a$$

$$a - 1 = 0 \Rightarrow a = 1$$

hence

$$z^3 - 1 = (z-1)(z^2+1+z)$$

$$\lim_{z\to 1} f(z) = \lim_{z\to 1} \frac{z^3-1}{z-1} = \frac{0}{0} \to \text{indeterminate case.}$$

$$\text{So } \lim_{z\to 1} f(z) = \frac{\cancel{(z-1)}(z^2+1+z)}{\cancel{(z-1)}}$$

$$= 3$$

So the limiting value of the fun" $f(z)$ is defined at $z=1$ and $f(1)=3$

so $f(z)$ is continuous at $z=1$

when $\lim_{z\to a} \frac{f(z)}{g(z)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$ the L'Hopital's rule says take $\left.\frac{f'(z)}{g'(z)}\right|_{z=a}$

Ex: $f(z) = \begin{cases} \frac{z^3-1}{z-1} & |z| \neq 1 \\ 3 & |z| = 1 \end{cases}$

is not continuous at $z = i$?

Solution :-> $\lim_{z\to i} f(z) = \lim_{z\to i} \frac{z^3-1}{z-1}$ ↓ not indeterminate

$$= \frac{-i-1}{i-1}$$

$f(i) = 3$

Since $\lim_{z\to i} f(z) \neq f(i)$ so not continous at $z = i$

<u>Ex</u> $f(z) = \dfrac{z^2-4}{z-2}$ check the continuity of $f(z)$ at $z=2$?

<u>Solution</u> :→ $\lim\limits_{z\to 2} f(z) = \lim\limits_{z\to 2} \dfrac{z^2-4}{z-2} = \lim\limits_{z\to 2} z+2 = 4$

i.e the limit exist at $z=2$

but $f(2) = ?$ undefined.

$f(z)$ is not continuous at $z=2$

<u>Theorem 1</u> :→ (Limits)

Let $f(z) = u(x,y) + i v(x,y)$

and $L = u_0 + i v_0$ then

$\lim\limits_{z\to z_0} f(z) = L$ iff

$\lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ & $\lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$

Example: Th:1

$$\lim_{z \to 1+i} (z^2 + i) = ?$$

[z = x + iy]

$z^2 + i = x^2 - y^2 + 2ixy + i \Rightarrow x^2 - y^2 + i(2xy + 1)$

So $u(x,y) = x^2 - y^2$, $v(x,y) = 2xy + 1$

$x_0 = 1$, $y_0 = 1$

So $u_0 = \lim_{(x,y) \to (1,1)} x^2 - y^2 = 1 - 1 = 0$

$v_0 = \lim_{(x,y) \to (x_0,y_0)} 2xy + 1 = 3$

So $\lim_{z \to 1+i} (z^2 + i) = L = 0 + 3i$

Props of Complex limits

① $\lim_{z \to z_0} cf(z) = cL$, $c \in \mathbb{C}$

② $\lim_{z \to z_0} (f(z) \pm g(z)) = L \pm M$

③ $\lim_{z \to z_0} f(z) \cdot g(z) = L \cdot M$

④ $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{L}{M}$ provided $M \neq 0$

Example

$$\lim_{z \to i} \frac{(3+i)z^4 - z^2 + 2z}{z+1} = ?$$

$$\Rightarrow \frac{\lim_{z \to i} (3+i)z^4 - z^2 + 2z}{\lim_{z \to i} (z+1)}$$

$$= \frac{(3+i) + 1 + 2i}{1+i} = \frac{4+3i}{1+i}$$

Theorem: Continuity of Complex Polynomial Functions

"Complex polynomial fun" are continuous on the entire Complex plane".

"while rational fun" are continuous on their domain"

Exercise 3.1 Zill - P:118

(1) $\lim_{z \to 2i} (z^2 - \bar{z}) = ?$

$\lim_{z \to 2i} (z^2 - \bar{z}) = -4 + 2i$

(13) $\lim_{z \to -i} \frac{z^4 - 1}{z + i}$?

$= \frac{0}{0}$ indeterminate

Case.

$\frac{z^4 - 1}{z + i} = \frac{(z^2 - 1)(z^2 + 1)}{z + i}$

Since $(z^2 + 1) = (z + i)(z - i)$

$\frac{z^4 - 1}{z + i} = (z^2 - 1)(z - i)$

$\lim_{z \to -i} (z^2 - 1)(z - i) = 4i$

$$\boxed{\lim_{z \to -i} \frac{z^4 - 1}{z + i} = 4i}$$

Differentiability :->

$f : D \to \mathbb{C}$ and $z_0 \in D$ (i.e f is defined at z_0)

then we say that f is $\mathbb{C}$-differentiable

at z_0 if

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} \text{ exist, } \quad (h \in \mathbb{C})$$

and

$$f'(z) =$$

Example

$$f(z) = z^2 - 5z \quad , \quad f'(z) = ?$$

$$f(z+h) = (z+h)^2 - 5(z+h)$$

$$= z^2 + h^2 + 2zh - 5z - 5h$$

$$f(z+h) - f(z) = \cancel{z^2} + h^2 + 2zh - \cancel{5z} - 5h - \cancel{z^2} + \cancel{5z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \overset{h}{\Delta z}) - f(z)}{\underset{\to h}{\Delta z}} = \boxed{\lim_{h \to 0} \frac{h^2 + 2zh - 5h}{h}}$$

$$= \lim_{h \to 0} h + 2z - 5$$

$$= 2z - 5$$

Rules The rules from the calculus of real variables apply. e.g.

① Constant Rule

$$\frac{d}{dz} c = 0, \quad \frac{d}{dz} cf(z) = cf'(z)$$

② Sum Rule

$$\frac{d}{dz}\left(f(z) \pm g(z)\right) = f'(z) \pm g'(z)$$

③ Product Rule

$$\frac{d}{dz}\left(f(z)\, g(z)\right) = f(z)\, g'(z) + f'(z)\, g(z)$$

④ Quotient Rule

$$\frac{d}{dz} \frac{f(z)}{g(z)} = \frac{g(z) f'(z) - g'(z) f(z)}{(g(z))^2}$$

⑤ Chain Rule

$$\frac{d}{dz} f(g(z)) = f'(g(z))\, g'(z)$$

Analytic funⁿ :-

A funⁿ $w=f(z)$ is analytic at a pt. z_0, if f is differentiable at z_0 & at every neighborhood of z_0

i.e., → A funⁿ is analytic in a domain D if it is differentiable at every pt in D.

→ A funⁿ that is analytic in the whole domain is known as a "halomorphic" or "regular"

→ A funⁿ that is analytic in the entire complex plane is called "entire funⁿ".

Theorem 3.2.1 Zill

① A complex polynomial funⁿ $P(z)=a_n z^n + a_{n-1} z^{n-1} \ldots \ldots a_0$ for $n \geq 0$

will always be an entire funⁿ.

② A complex rational funⁿ $M(z)=\frac{P(z)}{Q(z)}$, where P & Q are complex poly., is analytic on a domain D which do not have a pt z_0 for which $Q(z_0)=0$

Singular Pts :-> The pt. in the z-plane for which a rational fun" fails to be analytic are called singular pts.

e.g. $f(z) = \dfrac{4z}{z^2-2z+2}$ is undefined at $1 \pm i$. and hence f is not analytic at these pts.

Properties let $f(z)$ and $g(z)$ are analytic then the following are also analytic

① $f(z) \pm g(z)$

② $f(z)\, g(z)$

③ $\dfrac{f(z)}{g(z)}$ if $g(z) \neq 0$

Theorem 3.2.2 Analyticity implies continuity

if f is differentiable at a pt z_0 then it is also continuous at z_0 but converse not true.

L'Hopital's Rule :- This deals with

indeterminate cases while computing limits.

If $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}\Big|_{z \to z_0}$$

Ex: $\lim_{z \to 2+i} \frac{z^2 - 4z + 5}{z^3 - z - 10i}$?

$$\lim_{z \to 2+i} (z^2 - 4z + 5) = (2+i)^2 - 4(2+i) + 5$$

$$= \cancel{4} - \cancel{1} + \cancel{4i} - \cancel{8} - \cancel{4i} + \cancel{5} = 0$$

$$\lim_{z \to 2+i} z^3 - z - 10i = (2+i)^2(2+i) - 2 - i - 10i$$

$$= (4 - 1 + 4i)(2+i) - 2 - 11i$$

$$= (3 + 4i)(2+i) - 2 - 11i$$

$$= \cancel{6} + \cancel{8i} + \cancel{3i} - \cancel{4} - \cancel{2} - \cancel{11i} = 0$$

$$\lim_{z \to 2+i} (1) = \frac{0}{0}$$

hence applying L'Hopital's rule.

$$f'(z) = 2z - 4 \Big|_{z \to 2+i}$$

$$= 2(2+i) - 4$$

$$= 4 + 2i - \cancel{4} = 2i$$

$$g'(z) = 3z^2 - 1 \Big|_{z \to 2+i}$$

$$= 3(2+i)^2 - 1$$

$$= 3[4 - 1 + 4i] - 1$$

$$= 9 + 12i - 1$$

$$= 8 + 12i$$

$$\lim_{z \to 2+i} \frac{z^2 - 4z + 5}{z^3 - z - 10i} = \frac{f'(2+i)}{g'(2+i)} = \frac{2i}{8+12i}$$

Exercise 3.2 Zill: P.128

(1) $f(z) = 9iz + 2 - 3i$

$f'(z) = ?$

$f(z+h) = 9i(z+h) + 2 - 3i = 9iz + 9ih + 2 - 3i$

$f(z+h) - f(z) = \cancel{9iz} + 9ih + \cancel{2} - \cancel{3i} - \cancel{9iz} - \cancel{2} + \cancel{3i}$

$= 9ih$

$$f'(z) \triangleq \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{9i\cancel{h}}{\cancel{h}}$$

$$\boxed{f'(z) = 9i}$$

(5) $f(z) = z - \frac{1}{z}$, $f'(z) = ?$

$$f'(z) \triangleq \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

$$= \lim_{h\to 0} \frac{z + h - \frac{1}{z+h} - z + \frac{1}{z}}{h}$$

$$= \lim_{h\to 0} \frac{h + \frac{-\cancel{z} + \cancel{z} + h}{z(z+h)}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}\left(1 + \frac{1}{z(z+h)}\right)}{\cancel{h}} = 1 + \frac{1}{z^2}$$

$$\boxed{f'(z) = 1 + \frac{1}{z^2}}$$

(15) $f(z) = \dfrac{iz^2 - 2z}{3z + 1 - i}$, $f'(z) = ?$

Let $q(z) = iz^2 - 2z$, & $g(z) = 3z + 1 - i$

$$f(z) = \frac{q(z)}{g(z)}$$

$$f'(t) = \frac{g(z)\, q'(z) - g'(z)\, q(z)}{(g(z))^2} \rightarrow (1)$$

$q'(z) = ?$

$$q(z) = iz^2 - 2z = z(iz - 2)$$

$$q'(z) = z \frac{d}{dz}(iz-2) + \frac{dz}{dz}(iz-2)$$

$$= zi + (iz - 2) = 2zi - 2 = 2(zi - 1) \rightarrow (2)$$

$$g'(z) = 3 \rightarrow (3)$$

$$(g(z))^2 = (3z+1-i)^2$$

$$= 9z^2 + (1-i)^2 + 6z(1-i)$$

$$= 9z^2 + \cancel{1} - \cancel{1} - 2i + 6z - 6zi$$

$$= 9z^2 + 6z(1-i) - 2i \rightarrow (4)$$

put (2), (3) and (4) in (1)

$$f'(z) = \frac{(3z+1-i)2(zi-1) - 3z(iz-2)}{9z^2+6z(1-i)-2i}$$

$$= \frac{(6z+2-2i)(zi-1) - 3z^2i + 6z}{''}$$

$$= \frac{6z^2i - \cancel{6z} + 2zi - 2 + 2z + 2i - 3z^2i + \cancel{6z}}{''}$$

$$= \frac{3z^2i + 2z(i+1) + 2(i-1)}{''}$$

✓

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$$\lim_{z \to i} \frac{z^7 + i}{z^{14} + 1} \; ? = \frac{0}{0}$$

Indeterminate

$$\frac{d}{dz} z^7 + i = 7z^6$$

$$\frac{d}{dz} = z^{14} + 1 = 14z^{13}$$

$$\lim_{z \to i} \frac{z^7 + i}{z^{14} + 1} = \left. \frac{\frac{d}{dz}\left(z^7 + i\right)}{\frac{d}{dz}\left(z^{14} + 1\right)} \right|_{z = i}$$

$$= \left. \frac{7z^6}{14z^{13}} \right|_{z=i} = \frac{-7}{14i}$$