Complex Variables and Transforms

Origin:

$$x^2 + 2x + 4 = 0$$

Roots?

$$\frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 4}}{2} = \frac{-2 \pm \sqrt{4 - 16}}{2}$$

$$= -1 \pm \sqrt{1-4} \Rightarrow -1 \pm \sqrt{-3}$$

$\sqrt{-3}$? Number whose square is -ve

$$\boxed{\begin{array}{l} a^2 = -3 \, ? \\ a = \sqrt{-3} \end{array}}$$

Such numbers exist only in our imagination.

" They exist in our imagination.....

nothing prevents us...... from using them in our calcutions"

→ Along with other numbers e.g. +ve & -ve intigers, rational numbers and irrational numbers etc the imaginary or so called complex numbers were coined by German Mathmatician Carl Friedrich Gauss

They said that a symbol i is

$$i = \sqrt{-1}$$

Complex Number "A complex number is any number of the form $z = a + ib$ or $z = a + bi$

$a, b \in \mathbb{R}$

$i = \sqrt{-1}$

↳ Imaginary unit.

a is real part $\Rightarrow Re(z) = a$

b " imaginary " $\Rightarrow Im(z) = b$

e.g $z = 4 - 9i$

$Re(z) = 4$, $Im(z) = -9$

How can we write $\sqrt{-3}$ as a complex no.?

$$\sqrt{-3} = \sqrt{-1}\sqrt{3} = \underbrace{i\sqrt{3}}$$

↳ This is called a pure imaginary number.

→ Two complex numbers $z_1 = a_1 + b_1 i$ & $z_2 = a_2 + b_2 i$ are equal iff

$$a_1 = a_2$$
$$b_1 = b_2$$

i.e.,
$$Re(z_1) = Re(z_2)$$
$$Im(z_1) = Im(z_2)$$

→ A set of complex numbers is denoted by $\mathbb{C}$.

→ The fact that any real number "a" can be written as "$a + 0i$" make

$$\mathbb{R} \subset \mathbb{C}$$

Arithmetic operations on complex numbers

What are arithmetic operations?

$+, -, \times, \div$

Addition :->

$$z_1 = a_1 + b_1 i \quad , \quad z_2 = a_2 + b_2 i$$

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2) i$$

e.g.,

$$z_1 = 2 + 4i \quad , \quad z_2 = 5 + 3i$$

$$z_1 + z_2 = (2+5) + (4+3)i$$

$$= 7 + 7i$$

Simply add or subtract real and imaginary parts

Subtraction :->

$$z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2) i$$

e.g., $z_1 = 2 + 4i \quad , \quad z_2 = 5 + 3i$

$$z_1 - z_2 = (2-5) + (4-3)i$$

$$= -3 + i$$

$z_2 - z_1$?

Multiplication :→ $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$

$$z_1 \cdot z_2 = (a_1 + b_1 i)(a_2 + b_2 i)$$

$$= a_1(a_2 + b_2 i) + b_1 i(a_2 + b_2 i)$$

which law is this?
distributive

$$= a_1 a_2 + a_1 b_2 i + a_2 b_1 i + b_1 b_2 i^2$$

$$= a_1 a_2 + (a_1 b_2 + a_2 b_1) i - b_1 b_2$$

$\because\ i = \sqrt{-1} \Rightarrow i^2 = \sqrt{-1}^2 = -1$

e.g.

$$z_1 = 2 + 4i , \quad z_2 = 5 + 3i$$

$$z_1 \cdot z_2 = 2(5+3i) + 4i(5+3i)$$

$$= 10 + 6i + 20i - 12$$

$$= 10 - 12 + 26i$$

$$= -2 + 26i$$

Division :→ $\dfrac{z_1}{z_2} = \dfrac{a_1 + b_1 i}{a_2 + b_2 i}$ is possible only when $z_2 \neq 0$ i.e either $a_2 \neq 0$ or $b_2 \neq 0$

why? $\because$ Anything divided by 0 is infinity.

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i}$$

what is multiplicative identity?

1 is the " "

So Let

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i} \times \frac{a_2 - b_2 i}{a_2 - b_2 i}$$

$$= \frac{a_1 a_2 + b_1 b_2 + (a_2 b_1 - a_1 b_2) i}{a_2^2 + b_2^2}$$

$$= \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i \, \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}$$

e.g $z_1 = 1 + 2i$, $z_2 = 3 + i$

$$\frac{z_1}{z_2} = \frac{1 + 2i}{3 + i} \times \frac{3 - i}{3 - i}$$

$$= \frac{3 + 2}{9 + 1} + i \, \frac{6 - 1}{9 + 1}$$

$$= \frac{5}{10} + i \, \frac{5}{10} = 0.5 + 0.5 i$$

→ These arithmetic operations also hold the following properties.

① $z_1 + z_2 = z_2 + z_1$

&

$z_1 z_2 = z_2 z_1$

what is this property or law

Commutative Law

② $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

&

$z_1 (z_2 z_3) = (z_1 z_2) z_3$

what is this property or law?

Associative law

③ $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

Law?

Distributive Law.

Example 1 :→ Page 5

$z_1 = 2 + 4i \quad , \quad z_2 = -3 + 8i$

(a) $z_1 + z_2 = ?$ (b) $z_1 z_2 ?$

(a) $z_1 + z_2 = (2-3) + (4+8)i$

$= -1 + 12i$

(b) $z_1 z_2 = 2(-3+8i) + 4i(-3+8i)$

$= -6 + 16i - 12i - 32$

$= -38 + 4i$

Additive identity :→ A number that preserve a number when added to it.

what is that? 0 $\quad 1 + 0 = 1$ in $\mathbb{R}$

what is 0 or additive identity in complex numbers?

it is $z = 0 + 0i$ or simply $z = 0$

Multiplicative Identity: → Preserves a number when multiplied to it

e.g 1 (unity)

what is unity in complex numbers?

it is $z = 1 + 0j$

Complex Conjugate: → Complex conjugate of a complex number z, denoted by $\bar{z}$, or simply conjugate is obtained by changing the sign of the imaginary part.

e.g $z = a + bi$ then $\bar{z} = a - bi$

(a) $z = 2 + 4i$, $\bar{z} = 2 - 4i$

(b) $z = 5 - 6i$, $\bar{z} = 5 + 6i$

(c) $z = -10 + 9i$, $\bar{z} = -10 - 9i$

Imp

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$$

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

$\bar{\bar{z}} = ?$

$\bar{\bar{z}} = z$

→ $z+\bar{z}$ and $z\bar{z}$ will be a real number.

e.g. (1) $z = 2+4i$ then $\bar{z} = 2-4i$

$z+\bar{z} = 2+2+\cancel{4i}-\cancel{4i} = 4$

(2) $z\bar{z} = (2+4i)(2-4i) = 2(2-4i)+4i(2-4i)$

$= 4-\cancel{8i}+\cancel{8i}+16$

$= 4+16 = 20$

→ In general if $z = a+bi$ then $\bar{z} = a-bi$

$$\boxed{\begin{aligned} z+\bar{z} &= 2a \\ \text{and} \quad z\bar{z} &= a^2+b^2 \end{aligned}}$$

③ $z-\bar{z}$ is a pure imaginary number

e.g. $z-\bar{z} = \cancel{2}+4i-\cancel{2}+4i$

$= 8i$

→ In general

$$\boxed{z-\bar{z} = 2bi}$$

→ From the above discussion

If $z = a + bi$ then

$$Re(z) = \frac{z + \bar{z}}{2}, \quad Im(z) = \frac{z - \bar{z}}{2}$$

⇒ After these properties the division procedure is justified and simplified.

Example 2 (Page 6) $z_1 = 2 - 3i$, $z_2 = 4 + 6i$

Find z_1 / z_2 ?

(a) First we need to make the denominator a real number.

(b) What about multiplicative identity

$z/z = 1$

(c) $$\frac{z_1}{z_2} = \frac{2-3i}{4+6i} \times \frac{4-6i}{4-6i} \rightarrow \frac{\bar{z}_2}{\bar{z}_2} = 1$$

$$z_2 \bar{z}_2 = 4^2 + 6^2$$

$$= \frac{1}{16+36}\left[2(4-6i) - 3i(4-6i)\right]$$

$$= \text{"} \left[8 - 12i - 12i - 18\right] = \frac{1}{52}\left[-10 - 24i\right]$$

Additive and multiplicative inverses.

What is an additive inverse in real nos?

→ $2 + (-2) = 0$

↳ i.e when added to the number the result is 0.

Additive inverse and multiplicative inverse are unique
→ Two eqns of Two unknowns

→ The additive inverse of a complex no. z is its -ve i.e. $-z$

$z + (-z) = 0$

eg. $z = x + yi$, $-z = -x - yi$

What is a multiplicative inverse of a real no.?

→ $2 \times \frac{1}{2} = 1$

↳ i.e. when multiplied to a no. the result is unity.

If $z = 0$ no multiplicative inverse exist. why? → $\frac{1}{0} = \infty$

→ The multiplicative inverse of a complex no. z (imp: non-zero) is z^{-1} or $\frac{1}{z}$ (non-zero → why?)

i.e. $z z^{-1} = z \frac{1}{z} = 1$

$\frac{1}{z}$ → Reciprocal

Imp → These Inverses are always unique

Example 3 (Page 6)

Find the reciprocal of $z = 2 - 3i$?

Solution:→ By definition of division

$$\frac{1}{z} = \frac{1}{2-3i} = \frac{1}{2-3i} \times \frac{2+3i}{2+3i} = \frac{2+3i}{2^2+3^2}$$

$$= \frac{2+3i}{4+9} = \frac{2+3i}{13}$$

Varification:→

$$z z^{-1} = 1$$

$$z \frac{1}{z} = 1$$

$$(2-3i)\frac{1}{13}(2+3i) = 1$$

$$\frac{1}{13}[2-3i][2+3i] = 1$$

$$\frac{1}{13}[2^2+3^2] = 1$$

$$\frac{1}{13}13 = 1$$

$$1 = 1$$ hence varified.

$z_1 z_2 = 1$

$(a_1+b_1 i)(a_2+b_2 i) = 1$

$a_1 a_2 + a_1 b_2 i + a_2 b_1 i$

$- b_1 b_2 = 1$

Hence

$$\boxed{\begin{aligned} a_1 a_2 - b_1 b_2 &= 1 \\ a_1 b_2 + a_2 b_1 &= 0 \end{aligned}}$$

Simultanous equations 2 unknowns 2

$a_1 = x,\ b_1 = y,\ a_2 = u,\ b_2 = v$

$$\begin{bmatrix} x & -y \\ y & x \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} u &= x/x^2+y^2 \\ v &= -y/x^2+y^2 \end{aligned}$$

Differences with Real numbers

→ We can't say $z_1 < z_2$ $z_1, z_2 \in \mathbb{C}$

but if $z_1, z_2 \in \mathbb{R}$ then

we can say $z_1 < z_2$

i.e. we can't compare Complex nos.

Exercises 1.1 Page 7

(1)

(a) $i^8 = i^2 i^4 = i^2 i^2 i^2 i^2$
$= (-1)(-1)(-1)(-1) = +1$

when power is "even" → then the answer is either +1 or -1

-1 or +1, it is -1 if $\frac{\text{Power}}{2}$ is odd and +1 vise versa

$i^{10} = i^2 i^2 i^2 i^2 i^2 = (-1)(-1)(-1)(-1)(-1) = -1$

→ When power is odd answer is always $+i$ or $-i$

$+i$ if $\frac{\text{Power}-1}{2}$ is even

$-i$ " " " odd

(2) (a) $2i^3 - 3i^2 + 5i$

$= -2i + 3 + 5i = 3 + 3i$

(9) $3i + \frac{i}{2-i} = 3i + \frac{i}{2-i} \times \frac{2+i}{2+i} \Rightarrow 3i + \frac{2i-1}{5}$

$= \frac{15i + 2i - 1}{5} = \frac{-1}{5} + \frac{17i}{5}$

(21) $(2+3i)^2 = ?$

$$(A+B)^n = A^n + \frac{n}{1!}A^{n-1}B + \frac{n(n-1)}{2!}A^{n-2}B^2$$

$$+ \dots \dots \dots B^n$$

$2^2 + (3i)^2 + 2(2)(3i)$
$4 - 9 + 12i$
$-5 + 12i$

$$(2+3i)^2 = 2^2 + \frac{2}{1}2^1(3i) + \frac{2(2-1)}{2}A^0B^2$$

$$= 4 + 12i - 9 = -5 + 12i$$

(23) $(-2+2i)^3 = ?$

$$(-2+2i)^3 = (-2)^3 + \frac{3}{1!}(-2)^{3-1}(2i) + \frac{3(3-1)}{2!}(-2)^{3-2}(2i)^2$$

$$+ \frac{3(3-1)(3-2)}{3!}A^{3-3}(2i)^3$$

$$= -8 + 3(4)(2i) + 3(-2)(-4) - 8i$$

$$= -8 + 24i + 24 - 8i$$

$$= 16 + 16i$$

35 Page 8

$z^2 + i = 0, \quad z_1 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

Find an additional solution z_2 ?

let $z = x + yi$

then $z^2 + i = 0$

$(x+yi)^2 + i = 0$

$x^2 - y^2 + 2xyi + i = 0 + 0i$

A $x^2 - y^2 = 0 \Rightarrow y = x$

$2xy = -1 \Rightarrow y = -\frac{1}{2x}$ 1

Hence $x = -\frac{1}{2x} \Rightarrow 2x^2 = -1 \Rightarrow x^2 = -\frac{1}{2}$

$\Rightarrow x = \pm i\sqrt{\frac{1}{2}}$ (Not valid ∵ x is real part

Replace 1 in A

$x^2 - \left(-\frac{1}{2x}\right)^2 = 0 \Rightarrow x^2 = \frac{1}{4x^2} \Rightarrow 4x^4 = 1$

$x^4 = \frac{1}{4} \Rightarrow x^2 = \frac{1}{2} \Rightarrow \boxed{x = \pm\sqrt{\frac{1}{2}}}$

From 1 $y = -\frac{1}{2x}$ for $x = \sqrt{\frac{1}{2}}$

$y = -\frac{\sqrt{2}}{2}$ and for $x = -\frac{1}{\sqrt{2}}$

$y = \frac{\sqrt{2}}{2}$

Hence $z_2 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

(37) $2z = i(2+9i)$

$z = x + yi = ?$

$2(x+yi) = 2i - 9$

$x + yi = -\frac{9}{2} + i$

$\boxed{x = -9/2, \; y = 1}$

(39) $z^2 = i$

$(x+yi)^2 = i$

$x^2 - y^2 + 2xyi = i$

$x^2 - y^2 = 0 \Rightarrow x = y$

$2xy = 1 \Rightarrow 2x^2 = 1 \Rightarrow x = \pm\sqrt{\frac{1}{2}}$

Hence $y = \frac{\sqrt{2}}{2}$ if $x = \sqrt{\frac{1}{2}}$

d $y = -\frac{\sqrt{2}}{2}$ " " " $-\sqrt{\frac{1}{2}}$

More over $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\boxed{z = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \text{ or } z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i}$

Q 10 Chirchill Page 5

$$z^2 + z + 1 = 0 \qquad z = (x + iy)$$

Soln :-

$$(x+yi)(x+yi) + x + yj + 1 = 0 + 0j$$

$$x^2 - y^2 + 2xyi + x + yj + 1 = 0$$

$$x^2 - y^2 + x + 1 = 0$$

$$2xy + y = 0$$

$$y(2x+1) = 0$$

$$\Rightarrow x = -\frac{1}{2}$$

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^{n-k} z_2^{k}$$

$n = 1, 2 \ldots$

where

$$\binom{n}{k} = \frac{n!}{k(n-k)!} \qquad k = 0, 1, 2 \ldots n$$

$$0! = 1$$

$$\left(-\frac{1}{2}\right)^2 - y^2 - \frac{1}{2} + 1 = 0$$

$$y^2 = \frac{1}{4} - \frac{1}{2} + 1$$

$$= \frac{3}{4}$$

$$y = \pm \frac{\sqrt{3}}{2}$$

$$z_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} j$$