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1.7 THE INVERSE OF A MATRIX

In this section we restrict our attention to square matrices and formulate the notion corresponding to the reciprocal of a nonzero number.

DEFINITION An $n \times n$ matrix A is called **nonsingular** (or **invertible**) if there exists an $n \times n$ matrix B such that

$$AB = BA = I_n.$$

The matrix B is called an **inverse** of A. If there exists no such matrix B, then A is called **singular** (or **noninvertible**).

Remark From the preceding definition, it follows that if $AB = BA = I_n$, then A is also an inverse of B.

EXAMPLE 1 Let

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix}.$$

Since

$$AB = BA = I_2, \quad = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

we conclude that B is an inverse of A and that A is nonsingular. ■

THEOREM 1.9 *An inverse of a matrix, if it exists, is unique.*

Proof Let B and C be inverses of A. Then $BA = AC = I_n$. Therefore,

$$B = BI_n = B(AC) = (BA)C = I_nC = C,$$

which completes the proof. ■

We shall now write the inverse of A, if it exists, as A^{-1}. Thus

$$AA^{-1} = A^{-1}A = I_n.$$

EXAMPLE 2 Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

To find A^{-1}, we let

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then we must have

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that

$$\begin{bmatrix} a + 2c & b + 2d \\ 3a + 4c & 3b + 4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Equating corresponding entries of these two matrices, we obtain the linear systems

$$\begin{aligned} a + 2c &= 1 \\ 3a + 4c &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} b + 2d &= 0 \\ 3b + 4d &= 1. \end{aligned}$$

The solutions are (verify) $a = -2$, $c = \frac{3}{2}$, $b = 1$, and $d = -\frac{1}{2}$. Moreover, since the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

also satisfies the property that

$$\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we conclude that A is nonsingular and that

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

■

Remark Not every matrix has an inverse. For instance, consider the following example.

EXAMPLE 3 Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

To find A^{-1}, we let

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then we must have

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so that

$$\begin{bmatrix} a + 2c & b + 2d \\ 2a + 4c & 2b + 4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Equating corresponding entries of these two matrices, we obtain the linear systems

$$\begin{aligned} a + 2c &= 1 \\ 2a + 4c &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} b + 2d &= 0 \\ 2b + 4d &= 1. \end{aligned}$$

These linear systems have no solutions, so A has no inverse. Hence A is a singular matrix. ■

The method used in Example 2 to find the inverse of a matrix is not a very efficient one. We shall soon modify it and thereby obtain a much faster method. We first establish several properties of nonsingular matrices.

THEOREM 1.10 **(Properties of the Inverse)**

(a) *If A is a nonsingular matrix, then A^{-1} is nonsingular and*

$$(A^{-1})^{-1} = A.$$

(b) *If A and B are nonsingular matrices, then AB is nonsingular and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

systems can be solved by the Gauss–Jordan reduction method. To solve the first linear system, we form the augmented matrix $\left[A \mid \mathbf{e}_1\right]$ and compute its reduced row echelon form. We do the same with

$$\left[A \mid \mathbf{e}_2\right], \ldots, \left[A \mid \mathbf{e}_n\right].$$

However, if we observe that the coefficient matrix of each of these n linear systems is always A, we can solve all these systems simultaneously. We form the $n \times 2n$ matrix

$$\left[A \mid \mathbf{e}_1 \ \ \mathbf{e}_2 \ \ \cdots \ \ \mathbf{e}_n\right] = \left[A \mid I_n\right]$$

and compute its reduced row echelon form $\left[C \mid D\right]$. The $n \times n$ matrix C is the reduced row echelon form of A. Let $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n$ be the n columns of D. Then the matrix $\left[C \mid D\right]$ gives rise to the n linear systems

$$C\mathbf{x}_j = \mathbf{d}_j \qquad (1 \leq j \leq n) \tag{3}$$

or to the matrix equation

$$CB = D. \tag{4}$$

There are now two possible cases.

Case 1. $C = I_n$. Then Equation (3) becomes

$$I_n\mathbf{x}_j = \mathbf{x}_j = \mathbf{d}_j,$$

and $B = D$, so we have obtained A^{-1}.

Case 2. $C \neq I_n$. It then follows from Exercise T.9 in Section 1.6 that C has a row consisting entirely of zeros. From Exercise T.3 in Section 1.3, we observe that the product CB in Equation (4) has a row of zeros. The matrix D in (4) arose from I_n by a sequence of elementary row operations, and it is intuitively clear that D cannot have a row of zeros. The statement that D cannot have a row of zeros can be rigorously established at this point, but we shall ask the reader to accept the argument now. In Section 3.2, an argument using determinants will show the validity of the result. Thus one of the equations $C\mathbf{x}_j = \mathbf{d}_j$ has no solution, so $A\mathbf{x}_j = \mathbf{e}_j$ has no solution and A is singular in this case.

The practical procedure for computing the inverse of matrix A is as follows.

Step 1. Form the $n \times 2n$ matrix $\left[A \mid I_n\right]$ obtained by adjoining the identity matrix I_n to the given matrix A.

Step 2. Compute the reduced row echelon form of the matrix obtained in Step 1 by using elementary row operations. Remember that whatever we do to a row of A we also do to the corresponding row of I_n.

Step 3. Suppose that Step 2 has produced the matrix $\left[C \mid D\right]$ in reduced row echelon form.

(a) If $C = I_n$, then $D = A^{-1}$.

(b) If $C \neq I_n$, then C has a row of zeros. In this case A is singular and A^{-1} does not exist.

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EXAMPLE 5 Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}.$$

Solution *Step 1.* The 3×6 matrix $[A \;\vdots\; I_3]$ is

$$[A \;\vdots\; I_3] = \begin{array}{c} \begin{array}{cc} \qquad A \qquad\qquad & \qquad I_3 \qquad \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right] \end{array}.$$

Step 2. We now compute the reduced row echelon form of the matrix obtained in Step 1:

$$\begin{array}{c} \begin{array}{cc} \qquad A \qquad\qquad & \qquad I_3 \qquad \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right]$$

(-5) times the first row was added to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right]$$

The second row was multiplied by $\frac{1}{2}$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right]$$

The third row was multiplied by $(-\frac{1}{4})$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -\frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right]$$

$(-\frac{3}{2})$ times the third row was added to the second row.
(-1) times the third row was added to the first row.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ 0 & 1 & 0 & -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right]$$

(-1) times the second row was added to the first row.

Step 3. Since $C = I_3$, we conclude that $D = A^{-1}$. Hence

$$A^{-1} = \begin{bmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{bmatrix}.$$

It is easy to verify that $AA^{-1} = A^{-1}A = I_3$. ■

If the reduced row echelon matrix under A has a row of zeros, then A is singular. Since each matrix under A is row equivalent to A, once a matrix under A has a row of zeros, every subsequent matrix that is row equivalent to A will have a row of zeros. Thus we can stop the procedure as soon as we find a matrix F that is row equivalent to A and has a row of zeros. In this case A^{-1} does not exist.

EXAMPLE 6 Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix} \quad \text{if it exists.}$$

Solution *Step 1.* The 3×6 matrix $\left[A \mid I_3\right]$ is

$$\begin{array}{c} \qquad\qquad A \qquad\qquad\qquad\qquad I_3 \\ \left[A \mid I_3\right] = \left[\begin{array}{rrr|rrr} 1 & 2 & -3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 5 & -2 & -3 & 0 & 0 & 1 \end{array}\right]. \end{array}$$

Step 2. We compute the reduced row echelon form of the matrix obtained in Step 1. To find A^{-1}, we proceed as follows:

$$\begin{array}{c} \qquad A \qquad\qquad\qquad I_3 \\ \left[\begin{array}{rrr|rrr} 1 & 2 & -3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 5 & -2 & -3 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 5 & -2 & -3 & 0 & 0 & 1 \end{array}\right] \quad \text{(-1) times the first row was added to the second row.}$$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 0 & -12 & 12 & -5 & 0 & 1 \end{array}\right] \quad \text{(-5) times the first row was added to the third row.}$$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & -3 & 1 \end{array}\right] \quad \text{(-3) times the second row was added to the third row.}$$

At this point A is row equivalent to

$$F = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since F has a row of zeros, we stop and conclude that A is a singular matrix. ■

Observe that to find A^{-1} we do not have to determine, in advance, whether or not it exists. We merely start the procedure given previously and either obtain A^{-1} or find out that A is singular.

The foregoing discussion for the practical method of obtaining A^{-1} has actually established the following theorem.

THEOREM 1.12 *An $n \times n$ matrix is nonsingular if and only if it is row equivalent to I_n.* ■

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Key Terms

Inverse
Nonsingular (or invertible) matrix
Singular (or noninvertible) matrix

1.7 Exercises

In Exercises 1 through 4, use the method of Examples 2 and 3.

1. Show that $\begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix}$ is nonsingular.

2. Show that $\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$ is singular.

3. Is the matrix

$$\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$$

singular or nonsingular? If it is nonsingular, find its inverse.

4. Is the matrix

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & 3 \\ 2 & 2 & 1 \end{bmatrix}$$

singular or nonsingular? If it is nonsingular, find its inverse.

In Exercises 5 through 10, find the inverses of the given matrices, if possible.

5. (a) $\begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 2 \\ 1 & -1 & 2 & 1 \\ 1 & 3 & 3 & 2 \end{bmatrix}$

6. (a) $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 3 & 2 & 1 \\ 1 & 2 & 1 & -2 \end{bmatrix}$

7. (a) $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 2 \\ 1 & 2 & -1 & 1 \\ 5 & 9 & 1 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

8. (a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$.

9. (a) $\begin{bmatrix} 1 & 2 & -3 & 1 \\ -1 & 3 & -3 & -2 \\ 2 & 0 & 1 & 5 \\ 3 & 1 & -2 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$

10. (a) $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 4 & 1 & 2 & 0 \\ 2 & -1 & 3 & 1 \\ 4 & 2 & 1 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 1 & -2 \\ 3 & 4 & 6 \\ 7 & 6 & 2 \end{bmatrix}$

11. Which of the following linear systems have a nontrivial solution?

(a) $\begin{aligned} x + 2y + 3z &= 0 \\ 2y + 2z &= 0 \\ x + 2y + 3z &= 0 \end{aligned}$ (b) $\begin{aligned} 2x + y - z &= 0 \\ x - 2y - 3z &= 0 \\ -3x - y + 2z &= 0 \end{aligned}$

12. Which of the following linear systems have a nontrivial solution?

(a) $\begin{aligned} x + y + 2z &= 0 \\ 2x + y + z &= 0 \\ 3x - y + z &= 0 \end{aligned}$ (b) $\begin{aligned} x - y + z &= 0 \\ 2x + y \quad &= 0 \\ 2x - 2y + 2z &= 0 \end{aligned}$

(c) $\begin{aligned} 2x - y + 5z &= 0 \\ 3x + 2y - 3z &= 0 \\ x - y + 4z &= 0 \end{aligned}$

If $|A| \neq 0$

13. If $A^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, find A.

14. If $A^{-1} = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}$, find A.

15. Show that a matrix that has a row or column consisting entirely of zeros must be singular.

16. Find all values of a for which the inverse of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & a \end{bmatrix}$$

exists. What is A^{-1}?

17. Consider an industrial process whose matrix is

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Find the input matrix for each of the following output matrices:

(a) $\begin{bmatrix} 30 \\ 20 \\ 10 \end{bmatrix}$ (b) $\begin{bmatrix} 12 \\ 8 \\ 14 \end{bmatrix}$

18. Suppose that $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$.

(a) Find A^{-1}.

(b) Find $(A^T)^{-1}$. How do $(A^T)^{-1}$ and A^{-1} compare?

19. Is the inverse of a nonsingular symmetric matrix always symmetric? Explain.

20. (a) Is $(A+B)^{-1} = A^{-1} + B^{-1}$ for all A and B?

(b) Is $(cA)^{-1} = \frac{1}{c}A^{-1}$, for $c \neq 0$?

21. For what values of λ does the homogeneous system

$$\begin{aligned} (\lambda - 1)x + \quad 2y &= 0 \\ 2x + (\lambda - 1)y &= 0 \end{aligned}$$

have a nontrivial solution?

22. If A and B are nonsingular, are $A + B$, $A - B$, and $-A$ nonsingular? Explain.

23. If $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, find D^{-1}.

24. If $A^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 2 & 5 \\ 3 & -2 \end{bmatrix}$, find $(AB)^{-1}$.

25. Solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$ if

$$A^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

26. Let A be a 3×3 matrix. Suppose that $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ is a solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$. Is A singular or nonsingular? Justify your answer.

In Exercises 27 and 28, find the inverse of the given partitioned matrix A and express A^{-1} as a partitioned matrix.

27. $\left[\begin{array}{cc|c} 5 & 2 & 0 \\ 3 & 1 & 0 \\ \hline 0 & 0 & -4 \end{array}\right]$

28. $\left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ \hline 0 & 0 & 6 & 7 \\ 0 & 0 & 1 & 1 \end{array}\right]$

In Exercises 29 and 30, find the inverse of the given bit matrices, if possible.

29. (a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

30. (a) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

In Exercises 31 and 32, determine which bit linear systems have a nontrivial solution.

31. (a) $\begin{aligned} x + y + z &= 0 \\ x \quad\; + z &= 0 \\ y \quad\; &= 0 \end{aligned}$ (b) $\begin{aligned} x \quad\quad\; &= 0 \\ x + y + z &= 0 \\ x \quad\; + z &= 0 \end{aligned}$

32. (a) $\begin{aligned} x + y \quad\; &= 0 \\ x + y + z &= 0 \\ y + z &= 0 \end{aligned}$ (b) $\begin{aligned} y + z &= 0 \\ x \quad\; + z &= 0 \\ x + y \quad\; &= 0 \end{aligned}$

Theoretical Exercises

T.1. Suppose that A and B are square matrices and $AB = O$. If B is nonsingular, find A.

T.2. Prove Corollary 1.2.

T.3. Let A be an $n \times n$ matrix. Show that if A is singular, then the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. (*Hint:* Use Theorem 1.12.)

T.4. Show that the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is nonsingular if and only if $ad - bc \neq 0$. If this condition holds, show that

$$A^{-1} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}.$$

T.5. Show that the matrix

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

is nonsingular, and compute its inverse.

1.7

Q11 — Q12 :- which of the following linear systems have a nontrivial solution?

e/Q12 :
$$2x-y+5z=0$$
$$3x+2y-3z=0$$
$$x-y+4z=0$$

Theorem 1.13) P-99 If A is $n\times n$ matrix, the homogeneous system
$$AX=0$$
has a nontrivial solution if and only if A is singular

S The given linear system in matrix form is as

$$\underbrace{\begin{bmatrix}2 & -1 & 5\\ 3 & 2 & -3\\ 1 & -1 & 4\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\ y\\ z\end{bmatrix}}_{X}=\underbrace{\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}}_{0} \Rightarrow AX=0 \quad \text{—} \; (1)$$

$$|A|=\begin{vmatrix}2 & -1 & 5\\ 3 & 2 & -3\\ 1 & -1 & 4\end{vmatrix}=2\begin{vmatrix}2 & -3\\ -1 & 4\end{vmatrix}-(-1)\begin{vmatrix}3 & -3\\ 1 & 4\end{vmatrix}+5\begin{vmatrix}3 & 2\\ 1 & -1\end{vmatrix}.$$

$\Rightarrow |A| = 10+15-25 \Rightarrow |A|=0 \Rightarrow A$ is singular

By the above theorem, the given homogeneous system has a nontrivial solution.

Q13: If $A^{-1}=\begin{bmatrix}2 & 3\\ 1 & 4\end{bmatrix}$, find A.

S Let $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, then $AA^{-1}=I_2 \Rightarrow \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}2 & 3\\ 1 & 4\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$\Rightarrow \begin{bmatrix}2a+b & 3a+4b\\ 2c+d & 3c+4d\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \Rightarrow$ $2a+b=1$ —(1), $3a+4b=0$ —(2), $2c+d=0$ —(3), $3c+4d=1$ —(4)

solving (1) and (2), we get $a=4/5$, $b=-3/5$. similarly solving (3) and (4) we get $c=-1/5$ and $d=2/5$. Thus $A=\begin{bmatrix}4/5 & -3/5\\ -1/5 & 2/5\end{bmatrix}$

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Q.16: Find all values of a for which the inverse of

$$A=\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 0\\ 1 & 2 & a\end{bmatrix}$$ exists. what is A^{-1}

(S) Suppose the inverse of A exists i.e A is nonsingular,

then $|A| \neq 0 \Rightarrow \begin{vmatrix}1 & 1 & 0\\ 1 & 0 & 0\\ 1 & 2 & a\end{vmatrix} \neq 0$

$\Rightarrow a\begin{vmatrix}1 & 1\\ 1 & 0\end{vmatrix} \neq 0 \Rightarrow -a \neq 0 \Rightarrow a \neq 0$

Thus the desired values of a for which the given matrix is nonsingular are all real numbers except zero.

A^{-1} ? yourself.

Q21: For what values of λ does the homogeneous system

$(\lambda-1)x + 2y = 0$
$2x + (\lambda-1)y = 0$ have a nontrivial solution?

(S) The given homogeneous system is as

$$\underbrace{\begin{bmatrix}\lambda-1 & 2\\ 2 & \lambda-1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\ y\end{bmatrix}}_{X}=\underbrace{\begin{bmatrix}0\\ 0\end{bmatrix}}_{0} \Rightarrow AX=0 \quad -(1)$$

For nontrivial solution of (1), $|A|=0$ By Theorem 1.13.

$|A|=0 \Rightarrow \begin{vmatrix}\lambda-1 & 2\\ 2 & \lambda-1\end{vmatrix}=0 \Rightarrow (\lambda-1)^2-4=0 \Rightarrow (\lambda-1-2)(\lambda-1+2)=0$

$\Rightarrow (\lambda-3)(\lambda+1)=0 \Rightarrow \lambda=-1, 3$ for which

are the desired values of λ the given homogenous system have a nontrivial solution.

LU-Factorization

$$AX = b \quad \text{—— (1)}$$

$$A = LU \quad \text{—— (2)}$$

L = Lower triangular matrix → Ex $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 5 & 6 & 9 \end{bmatrix}$

U = upper triangular matrix → Ex $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix}$

(2) in (1) $\Rightarrow LUX = b \Rightarrow L(UX) = b$ —— (3)

Let $UX = Z$ —— (4)

(4) in (3) $\Rightarrow LZ = b$ —— (5)

As L and b are given, so (5) $\Rightarrow$ the value of $Z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$ by forward substitution

value of Z in (4) $\Rightarrow$ value of $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ by back substitution.

This is LU-Factorization method to solve system of linear equations.

see note after discussing examples/Exercise.

Note: Attempt those questions in which no diagonal entry of the upper triangular matrix at any stage is zero

EXAMPLE 2 Consider the linear system

$$\begin{aligned} 6x_1 - 2x_2 - 4x_3 + 4x_4 &= 2 \\ 3x_1 - 3x_2 - 6x_3 + x_4 &= -4 \\ -12x_1 + 8x_2 + 21x_3 - 8x_4 &= 8 \\ -6x_1 \qquad - 10x_3 + 7x_4 &= -43 \end{aligned}$$

whose coefficient matrix

$$A = \begin{bmatrix} 6 & -2 & -4 & 4 \\ 3 & -3 & -6 & 1 \\ -12 & 8 & 21 & -8 \\ -6 & 0 & -10 & 7 \end{bmatrix}$$

has an LU-factorization where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

(verify). To solve the given system using this LU-factorization, we proceed as follows. Let

$$\mathbf{b} = \begin{bmatrix} 2 \\ -4 \\ 8 \\ -43 \end{bmatrix}.$$

Then we solve $A\mathbf{x} = \mathbf{b}$ by writing it as $LU\mathbf{x} = \mathbf{b}$. First, let $U\mathbf{x} = \mathbf{z}$ and solve $L\mathbf{z} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 8 \\ -43 \end{bmatrix}$$

by forward substitution. We obtain

$$\begin{aligned} z_1 &= 2 \\ z_2 &= -4 - \tfrac{1}{2}z_1 = -5 \\ z_3 &= 8 + 2z_1 + 2z_2 = 2 \\ z_4 &= -43 + z_1 - z_2 + 2z_3 = -32. \end{aligned}$$

Next we solve $U\mathbf{x} = \mathbf{z}$,

$$\begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 2 \\ -32 \end{bmatrix},$$

by back substitution. We obtain

$$\begin{aligned} x_4 &= \frac{-32}{8} = -4 \\ x_3 &= \frac{2 + 2x_4}{5} = -1.2 \\ x_2 &= \frac{-5 + 4x_3 + x_4}{-2} = 6.9 \\ x_1 &= \frac{2 + 2x_2 + 4x_3 - 4x_4}{6} = 4.5. \end{aligned}$$

3

Thus the solution to the given linear system is

$$\mathbf{x} = \begin{bmatrix} 4.5 \\ 6.9 \\ -1.2 \\ -4 \end{bmatrix}.$$

■

Next, we show how to obtain an LU-factorization of a matrix by modifying the Gaussian elimination procedure from Section 1.6. No row interchanges will be permitted and we do not require that the diagonal entries have value 1. At the end of this section we provide a reference that indicates how to enhance the LU-factorization scheme presented to deal with matrices where row interchanges are necessary. We observe that the only elementary row operation permitted is the one that adds a multiple of one row to a different row.

To describe the LU-factorization, we present a step-by-step procedure in the next example.

EXAMPLE 3 Let A be the coefficient matrix of the linear system of Example 2.

$$A = \begin{bmatrix} 6 & -2 & -4 & 4 \\ 3 & -3 & -6 & 1 \\ -12 & 8 & 21 & -8 \\ -6 & 0 & -10 & 7 \end{bmatrix}.$$

We proceed to "zero out" entries below the diagonal entries using only the row operation that adds a multiple of one row to a different row.

Procedure	Matrices Used
Step 1. "Zero out" below the first diagonal entry of A. Add $(-\frac{1}{2})$ times the first row of A to the second row of A. Add 2 times the first row of A to the third row of A. Add 1 times the first row of A to the fourth row of A. Call the new resulting matrix U_1.	$U_1 = \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 4 & 13 & 0 \\ 0 & -2 & -14 & 11 \end{bmatrix}$
We begin building a lower triangular matrix, L_1, with 1s on the main diagonal, to record the row operations. Enter the *negatives of the multipliers* used in the row operations in the first column of L_1, below the first diagonal entry of L_1.	$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -2 & * & 1 & 0 \\ -1 & * & * & 1 \end{bmatrix}$
Step 2. "Zero out" below the second diagonal entry of U_1. Add 2 times the second row of U_1 to the third row of U_1. Add (-1) times the second row of U_1 to the fourth row of U_1. Call the new resulting matrix U_2.	$U_2 = \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -10 & 12 \end{bmatrix}$
Enter the negatives of the multipliers from the row operations below the second diagonal entry of L_1. Call the new matrix L_2.	$L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & * & 1 \end{bmatrix}$

$R_2 - \frac{1}{2}R_1$
$R_3 + 2R_1$
$R_4 + R$
A U_1

$R_3 + 2R_2$
$R_4 - R_2$
U_1

(L1)

$R_4 + 2R_3$

U_2

Step 3. "Zero out" below the third diagonal entry of U_2. Add 2 times the third row of U_2 to the fourth row of U_2. Call the new resulting matrix U_3.

$$U_3 = \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

Enter the negative of the multiplier below the third diagonal entry of L_2. Call the new matrix L_3.

$$L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix}$$

Let $L = L_3$ and $U = U_3$. Then the product LU gives the original matrix A (verify). This linear system of equations was solved in Example 2 using the LU-factorization just obtained. ∎

Remark In general, a given matrix may have more than one LU-factorization. For example, if A is the coefficient matrix considered in Example 2, then another LU-factorization is LU, where

$$L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -4 & 2 & 1 & 0 \\ -2 & -1 & -2 & 2 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 3 & -1 & -2 & 2 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

((There are many methods for obtaining an LU-factorization of a matrix, besides the scheme for **storage of multipliers** described in Example 3. It is important to note that if $a_{11} = 0$, then the procedure used in Example 3 fails. Moreover, if the second diagonal entry of U_1 is zero or if the third diagonal entry of U_2 is zero, then the procedure also fails.)) In such cases we can try rearranging the equations of the system and beginning again or using one of the other methods for LU-factorization. Most computer programs for LU-factorization incorporate row interchanges into the storage of multipliers scheme and use additional strategies to help control roundoff error. If row interchanges are required, then the product of L and U is not necessarily A—it is a matrix that is a permutation of the rows of A. For example, if row interchanges occur when using the **lu** command in MATLAB in the form **[L,U] = lu(A)**, then MATLAB responds as follows: The matrix that it yields as L is not lower triangular, U is upper triangular, and LU is A. The book *Experiments in Computational Matrix Algebra*, by David R. Hill (New York: Random House, 1988, distributed by McGraw-Hill) explores such a modification of the procedure for LU-factorization.

Key Terms

Back substitution
Forward substitution
LU-factorization (or LU-decomposition)

5

1.8 Exercises

In Exercises 1 through 4, solve the linear system $A\mathbf{x} = \mathbf{b}$ with the given LU-factorization of the coefficient matrix A. Solve the linear system using a forward substitution followed by a back substitution.

1. $A = \begin{bmatrix} 2 & 8 & 0 \\ 2 & 2 & -3 \\ 1 & 2 & 7 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 18 \\ 3 \\ 12 \end{bmatrix}$,

$L = \begin{bmatrix} 2 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & -1 & 4 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

2. $A = \begin{bmatrix} 8 & 12 & -4 \\ 6 & 5 & 7 \\ 2 & 1 & 6 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -36 \\ 11 \\ 16 \end{bmatrix}$,

$L = \begin{bmatrix} 4 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 4 & 5 & 3 & 3 \\ -2 & -6 & 7 & 7 \\ 8 & 9 & 5 & 21 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ -2 \\ -16 \\ -66 \end{bmatrix}$,

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 3 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$,

$U = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

4. $A = \begin{bmatrix} 4 & 2 & 1 & 0 \\ -4 & -6 & 1 & 3 \\ 8 & 16 & -3 & -4 \\ 20 & 10 & 4 & -3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ 13 \\ -20 \\ 15 \end{bmatrix}$,

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ 5 & 0 & -1 & 1 \end{bmatrix}$,

$U = \begin{bmatrix} 4 & 2 & 1 & 0 \\ 0 & -4 & 2 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

In Exercises 5 through 10, find an LU-factorization of the coefficient matrix of the given linear system $A\mathbf{x} = \mathbf{b}$. Solve the linear system using a forward substitution followed by a back substitution.

5. $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 10 \\ 4 & 8 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ 16 \\ 2 \end{bmatrix}$.

6. $A = \begin{bmatrix} -3 & 1 & -2 \\ -12 & 10 & -6 \\ 15 & 13 & 12 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 15 \\ 82 \\ -5 \end{bmatrix}$

7. $A = \begin{bmatrix} 4 & 2 & 3 \\ 2 & 0 & 5 \\ 1 & 2 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$

8. $A = \begin{bmatrix} -5 & 4 & 0 & 1 \\ -30 & 27 & 2 & 7 \\ 5 & 2 & 0 & 2 \\ 10 & 1 & -2 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -17 \\ -102 \\ -7 \\ -6 \end{bmatrix}$

9. $A = \begin{bmatrix} 2 & 1 & 0 & -4 \\ 1 & 0 & 0.25 & -1 \\ -2 & -1.1 & 0.25 & 6.2 \\ 4 & 2.2 & 0.3 & -2.4 \end{bmatrix}$,

$\mathbf{b} = \begin{bmatrix} -3 \\ -1.5 \\ 5.6 \\ 2.2 \end{bmatrix}$

10. $A = \begin{bmatrix} 4 & 1 & 0.25 & -0.5 \\ 0.8 & 0.6 & 1.25 & -2.6 \\ -1.6 & -0.08 & 0.01 & 0.2 \\ 8 & 1.52 & -0.6 & -1.3 \end{bmatrix}$,

$\mathbf{b} = \begin{bmatrix} -0.15 \\ 9.77 \\ 1.69 \\ -4.576 \end{bmatrix}$

MATLAB Exercises

Routine **lupr** *provides a step-by-step procedure in* MATLAB *for obtaining the LU-factorization discussed in this section. Once we have the LU-factorization, routines* **forsub** *and* **bksub** *can be used to perform the forward and back substitution, respectively. Use* **help** *for further information on these routines.*

ML.1. Use **lupr** in MATLAB to find an LU-factorization of

$$A = \begin{bmatrix} 2 & 8 & 0 \\ 2 & 2 & -3 \\ 1 & 2 & 7 \end{bmatrix}.$$

ML.2. Use **lupr** in MATLAB to find an LU-factorization of

$$A = \begin{bmatrix} 8 & -1 & 2 \\ 3 & 7 & 2 \\ 1 & 1 & 5 \end{bmatrix}.$$

ML.3. Solve the linear system in Example 2 using **lupr**, **forsub**, and **bksub** in MATLAB. Check your LU-factorization using Example 3.

ML.4. Solve Exercises 7 and 8 using **lupr**, **forsub**, and **bksub** in MATLAB.