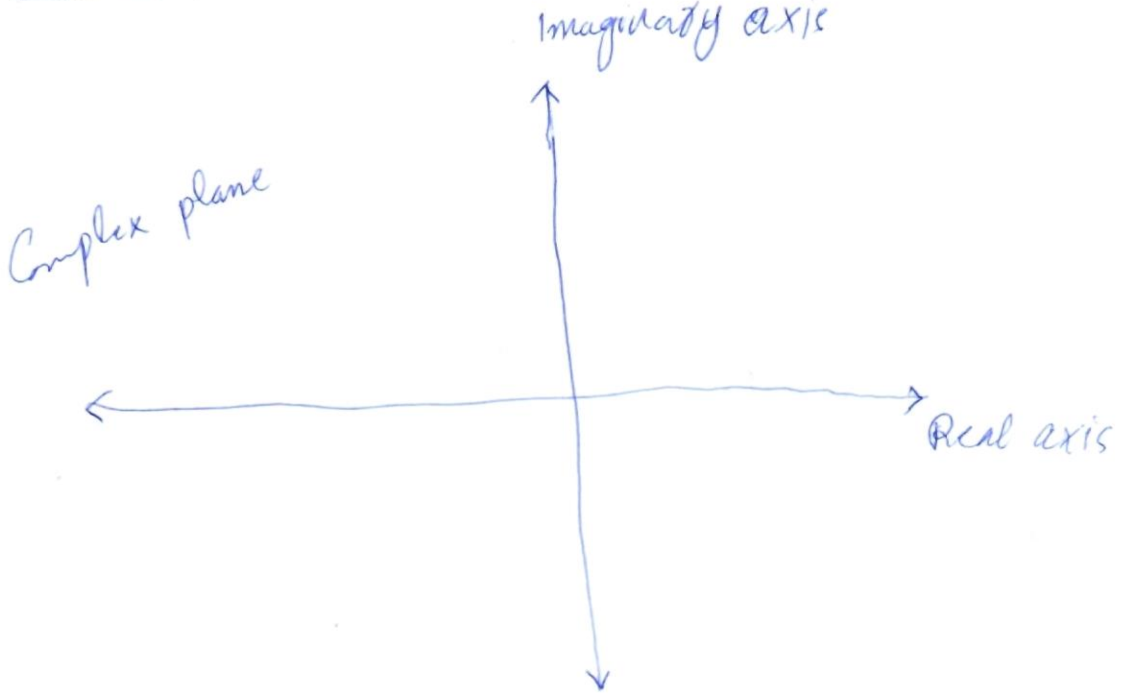
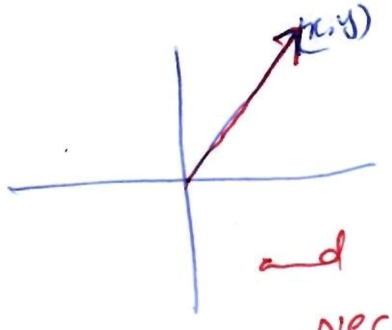
Complex Plane



→ Like previously you viewed vectors as ordered pairs

→ Similarly a complex number ($z = x+iy$) can also be viewed as a 2-D vector.
position → A vector whose initial pt is the origin and terminate at pt (x, y)

e.g.:



then what is the magnitude and direction of this vector?

Modulus : The modulus or magnitude of a complex number $z = x + iy$ is

$$\boxed{|z| = \sqrt{x^2 + y^2}}$$

Also called absolute value of z.

e.g ① $z = 2 - 3i$

$$|z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$$

② $z = -9i$

$$|z| = ? \quad \sqrt{(-9)^2} = 9$$

Properties :→ ① We know that if $z = x + yi$

then $z\bar{z} = x^2 + y^2 \in \mathbb{R}$

Now we know

$$|z| = \sqrt{x^2 + y^2} \Rightarrow |z|^2 = x^2 + y^2$$

So we conclude.

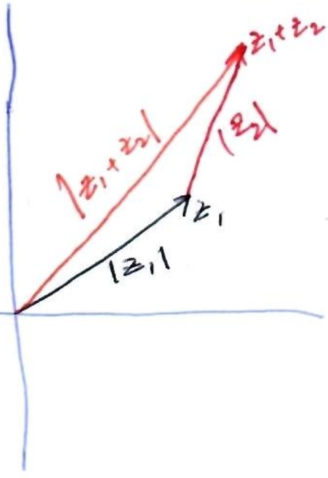
$$\boxed{|z|^2 = z\bar{z}}$$

② $\boxed{|z_1 z_2| = |z_1|\,|z_2|}$

③ $\boxed{\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}}$

④ $\boxed{|z^2| = |z|^2}$

⑤ Though we can't compare complex nos.
but $|z| \in \mathbb{R}$ and so we can compare it



i.e length of one side of a triangle can't be greater than the sum of the lengths of the remaining two sides.

$$\boxed{|z_1 + z_2| \leq |z_1| + |z_2|}$$

↳ Known as triangle inequality

↳ This can be further extended as

ⓐ $z_1 = z_1 + z_2 + (-z_2)$

$$\boxed{|z_1| = |z_1 + z_2 + (-z_2)| \leq |z_1+z_2| + |-z_2|}$$

where

$$\boxed{|z_2| = |-z_2|}$$

(b) From (a)

$$|z_1| \leq |z_1+z_2| + |z_2|$$

$$\boxed{|z_1| - |z_2| \leq |z_1+z_2|}$$

(c) Since $z_1 + z_2 = z_2 + z_1$

hence $|z_1+z_2| = |z_2+z_1|$

From (b)

$$|z_2+z_1| \geq |z_2| - |z_1| = -(|z_1| - |z_2|)$$

$$\big||z_2+z_1|\big| \geq \big|-(|z_1|-|z_2|)\big|$$

recall that $|-z| = |z|$

and

$||a|| = |a| = a$ when $a \in \mathbb{R}$

$$\boxed{|z_1+z_2| \geq \big||z_1| - |z_2|\big|}$$

(d) $\boxed{|z_1 - z_2| \leq |z_1| + |z_2|}$

In general

$$|z_1 + z_2 + \cdots z_n| \leq |z_1| + |z_2| \cdots |z_n|$$

Ex 3: Page 13 Zill.

Find the upper bound for

$$\left|\frac{-1}{z^4 + 3z^2 + 2}\right| \quad \text{if } |z| = 2$$

Solution.

$$\left|\frac{-1}{z^4 + 3z^2 + 2}\right| = \frac{|-1|}{|z^4 + 3z^2 + 2|} \rightarrow (1)$$

As $|-1| = |1| = 1$

(1) becomes

$$\frac{1}{|z^4 + 3z^2 + 2|} \rightarrow (2)$$

Now

$$z^4 + 3z^2 + 2 = (z^2+1)(z^2+2)$$

hence ② becomes

$$\frac{1}{|(z^2+1)(z^2+2)|} = \frac{1}{|z^2+1| \cdot |z^2+2|} \rightarrow ③$$

Since $|z^2+1| \geq ||z^2|-1|$

and $|z^2+2| \geq ||z^2|-2|$

hence

$$|z^2+1|\,|z^2+2| \geq ||z^2|-1|\,||z^2|-2|$$

Now using $|z|=2$ and the prop that

$$|z^2| = |z|^2$$

$$|z^2+1|\,|z^2+2| \geq ||z|^2-1|\,||z|^2-2|$$

$$|4+1|\,|4+2| \geq |4-1|\,|4-2|$$

$$5 \times 6 \geq 3 \times 2$$

$$30 \geq 6$$

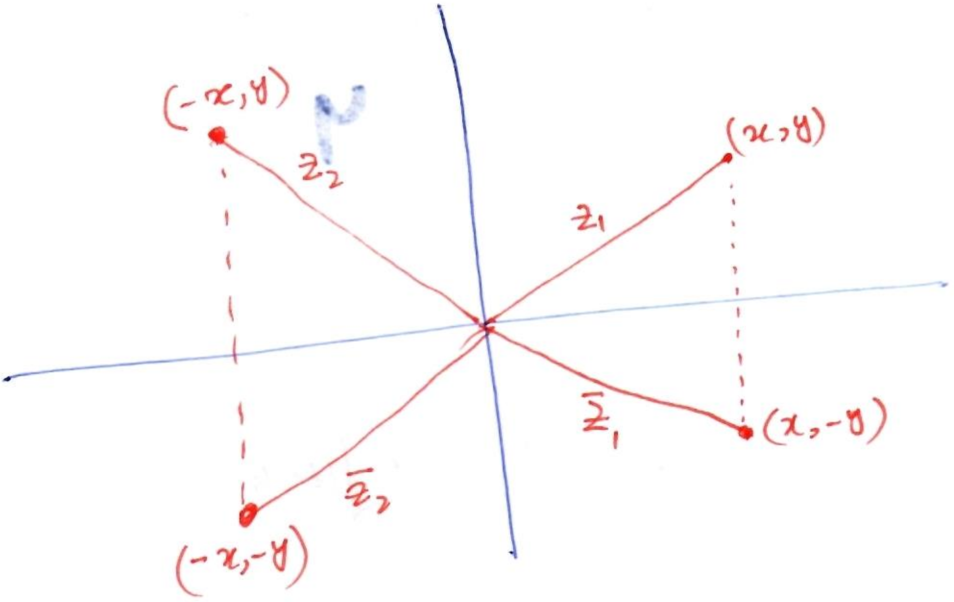
$$|z^4+3z^2+2| \geq 6$$

δ_0

$$\boxed{\left|\frac{-1}{z^4+3z^2+2}\right| = \frac{1}{|z^4+3z^2+2|} \leq \frac{1}{6}}$$

~~Plot~~

→ Plating z and $\bar{z}$

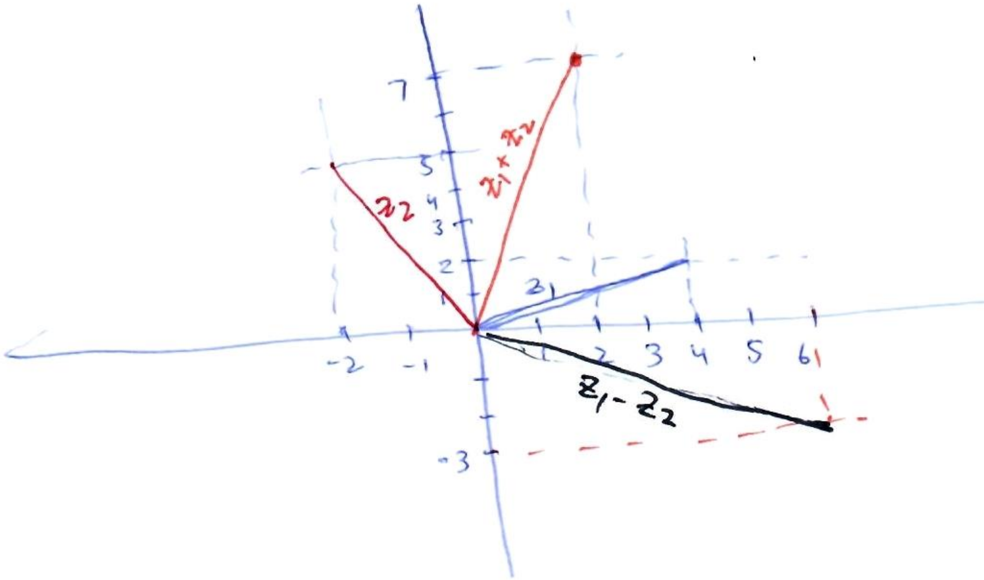


Exercises 1.2 (P: 13) Zill.

① $z_1 = 4+2i$, $z_2 = -2+5i$

plot $z_1, z_2, z_1+z_2, z_1-z_2$?

$z_1+z_2 = 2+7i$ and $z_1-z_2 = 6-3i$



Q 5   P 13, Zill

$z_1 = 5-2i, \quad z_2 = -1-i,$

$z_3 = ?$ In the same direction as

z_1+z_2 and $|z_3| = 4|z_1+z_2|$

$$z_1+z_2 = 5-1-2i-i = 4-3i$$

$$|z_3| = 4|z_1+z_2| = 4\sqrt{16+9} = 20$$

Let $z_3 = x+iy$

then $\sqrt{x^2+y^2} = 20$

$$\boxed{x^2+y^2 = 400} \longrightarrow (1)$$

The vectors are needed to be in the same direction. i.e.

$$\tan^{-1}\left(\frac{-3}{4}\right) = \tan^{-1}\left(\frac{y}{x}\right)$$

So

$$\frac{y}{x} = \frac{-3}{4} \Rightarrow \boxed{y = \frac{-3x}{4}} \longrightarrow (2)$$

put (2) in (1)

$$x^2 + \frac{9x^2}{16} = 400$$

$$x^2 = \frac{400 \times 16}{25} \Rightarrow \boxed{x = 16} \rightarrow (3)$$

put (3) in (2)

$$y = -\frac{3 \times \cancel{16}^{4}}{\cancel{4}} = -12$$

hence

$$\boxed{z_3 = 16 - 12j}$$

verify?

Q9 P:14, Zill

$$|(1-j)^2| = ?$$

$$|z^2| = |z|^2 = \left(\sqrt{1+1}\right)^{2} = 2$$

Q15 P14, Zill

$z_1 = 10+8j$, $z_2 = 11-6j$ which one is closest to the origin and which one to $1+j$

Solution :-

$$|z_1| = \sqrt{100+64} = \sqrt{164}$$

$$|z_2| = \sqrt{121+36} = \sqrt{157}$$

$|z_2| < |z_1|$ hence z_2 is closer to the origin.

$$z_1 - z = (10+8i) - (1+i)$$

$$= 9+7i$$

$$|z_1 - z| = \sqrt{81+49} = \sqrt{130}$$

$$z_2 - z = (11-6i) - (1+i)$$

$$= 10-7i$$

$$|z_2 - z| = \sqrt{100+49} = \sqrt{149}$$

$|z_1 - z| < |z_2 - z|$ hence z_1 is closer to $z = 1+i$

Q 17 P14 Zill

$Re\left((1+i)z - 1\right) = 0$

plot the points that satisfy this eqn?

Solution :-) Let

$z = x + yi$

$(1+i)(x+yi) - 1$

$= x - y + (x+y)i - 1$

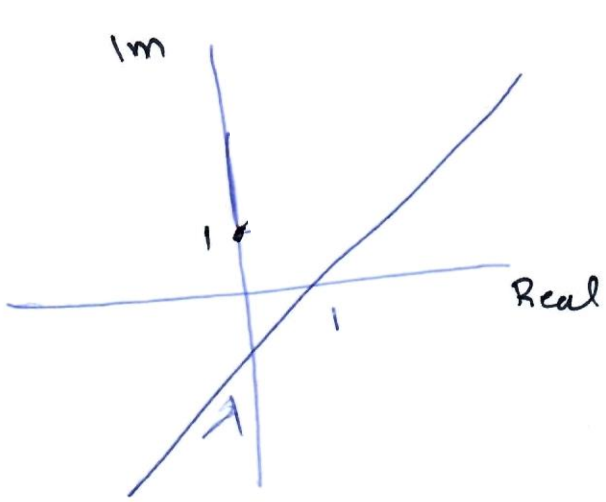
$= x - y - 1 + (x+y)i$

$Re\left(x - y - 1 + (x+y)i\right) = 0$

$x - y - 1 = 0$

$\boxed{x - y = 1}$

This line satisfies the eqn.



Q.19 P14 Zill

$|z - i| = |z - 1|$

$z = ?$ Plot ?

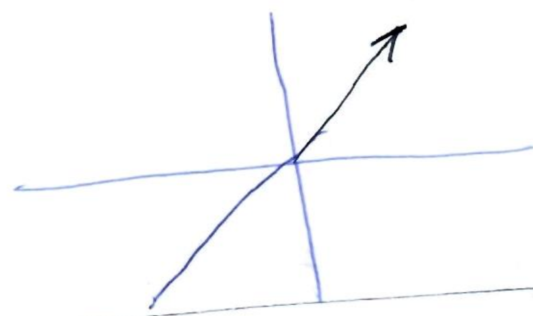
Solution :-) Let $z = x + yi$

$|x + yi - i| = |x + yi - 1|$

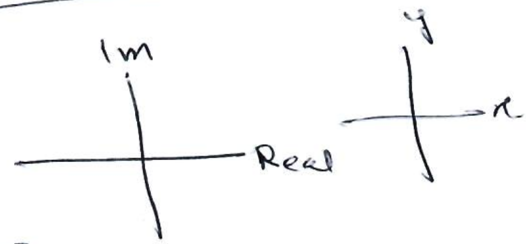
$\sqrt{x^2 + (y-1)^2} = \sqrt{(x-1)^2 + y^2}$

$\cancel{x^2} + \cancel{y^2} + \cancel{1} - 2y = \cancel{x^2} + \cancel{y^2} + \cancel{1} - 2x$

$2y = 2x \Rightarrow \boxed{x = y}$



Polar form of Complex Nos.

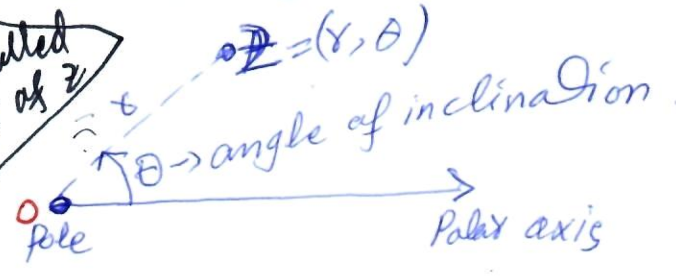


$\mathbb{C} \rightarrow r, \theta$

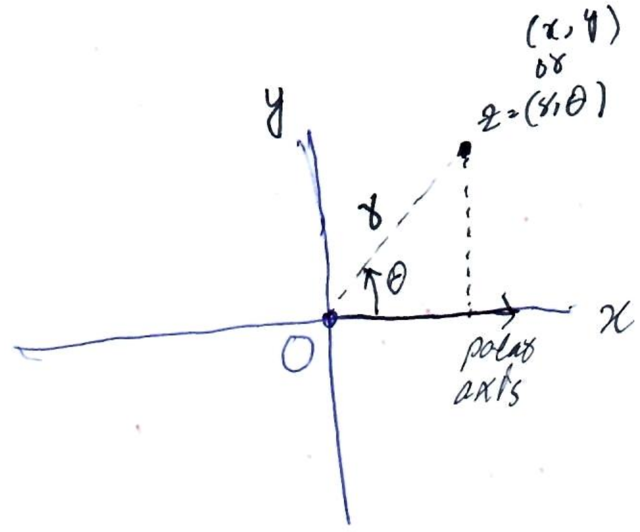
$\rightarrow z \in \mathbb{C}$ represent a position vector which has mag. and direction.

hence $z = (x, y)$ as ordered pair can also be represented as $z = (r, \theta)$

θ also called argument of z
arg(z)



Now if this polar coordinate system is superimposed on a complex plane s.t the pole is at the origin and the polar axis coincide with the x or real axis i.e



then

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}$$

Hence a complex no. $z = x + yi$ can be written as

$$z = x + yi = r\cos\theta + i\, r\sin\theta$$

$$= r(\cos\theta + i\sin\theta)$$

↳ This is called the polar form of a complex number.

$\theta > 0$ if measured CCW

$\theta < 0$ 〃 〃 CW

Example :-> $z = -1-i$

Represent z in polar form

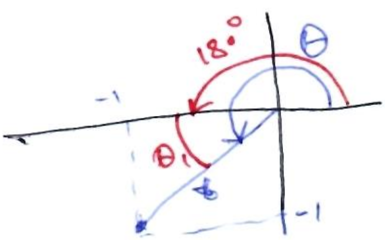
$$z = r(\cos\theta + i\sin\theta)$$

$$r = \sqrt{1+1} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{-1}{-1}\right) = +45^\circ$$

is it correct?

z is in the 3rd quadrant & 45° in the 1st !!!



$$\theta = 180^\circ + \theta_1$$
$$= 180^\circ + \tan^{-1}\left(\frac{|-1|}{|-1|}\right)$$
$$= 180^\circ + 45^\circ = 225^\circ$$

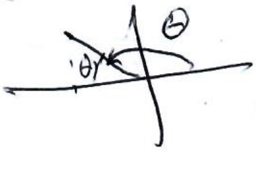
Example. $z = -1+i$

polar form?

$r = \sqrt{2}$.

$$\theta = \tan^{-1}\left(\frac{1}{-1}\right) = -45^\circ$$

$$\theta = 180^\circ - 45^\circ = 135^\circ$$



or

$$\theta = 90^\circ + 45^\circ = 135^\circ$$

Uses of Polar form

① It provides ease to the $\times$ and $\div$ operations.

let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$z_1 z_2 = r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$$

i.e.,

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

Example

② Integer Powers of z

As $z = r(\cos\theta + i\sin\theta)$

$$z\,z = z^2 = r^2\left[\cos(\theta+\theta) + i\sin(\theta+\theta)\right]$$

Hence

$$\boxed{z^2 = r^2\left[\cos(2\theta) + i\sin(2\theta)\right]}$$

In general

$$\boxed{z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]}$$

when $r = 1$ then this is called de Moivre's formula

Similarly for reciprocal

$$\frac{1}{z} = \frac{1}{r}\left[\cancel{\cos}(0-\theta) + i\sin(0-\theta)\right]$$

$$= r^{-1}\left[\cos(-\theta) + i\sin(-\theta)\right]$$

$$= r^{-1}\left[\cos(\theta) - i\sin\theta\right]$$

Example

Q: If $r = |z|$ then what is r?

$r = \sqrt{x^2 + y^2}$

$\Rightarrow \boxed{r^2 = x^2 + y^2}$ eqn: of circle.

$\hookrightarrow$ r is the radius

circle

⇒ Exponential Form

According to the Euler's formula.

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Hence we can also write

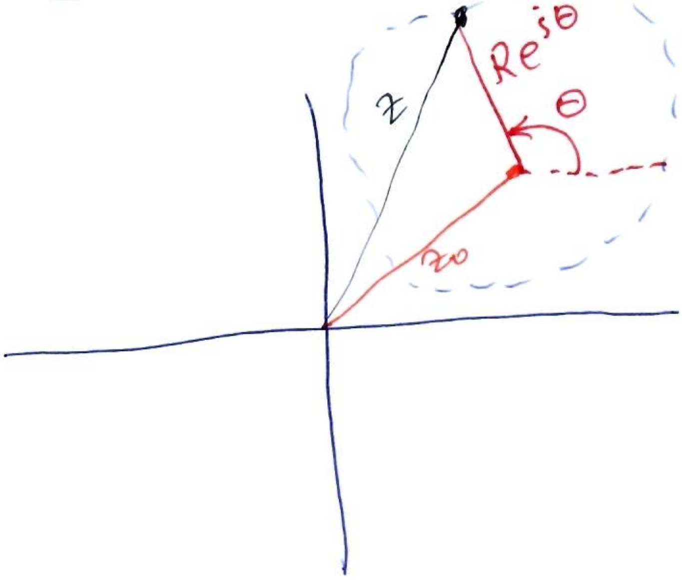
$$\boxed{z = re^{i\theta}}$$

$z = Re^{i\theta}$ is the parametric representation of the circle ~~cent~~ centered at the origin d with radius R.

$$0 \le \theta \le 2\pi$$

→ Now a circle ~~is~~ ~~said to~~ centered at some pt z_0 represented as

$$\boxed{z = z_0 + Re^{i\theta}}$$



Are the products and quotients now obvious?

$$z_1 = r_1\left[\cos\theta_1 + i\sin\theta_1\right] = r_1 e^{i\theta_1}$$

$$z_2 = r_2\left[\cos\theta_2 + i\sin\theta_2\right] = r_2 e^{i\theta_2}$$

$$z_1 z_2 = r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

$$= r_1 r_2\, e^{i\theta_1} e^{i\theta_2} = r_1 r_2\, e^{i(\theta_1+\theta_2)}$$

Similarly.

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

$$\boxed{z^n = r^n e^{in\theta}}$$

Principle Argument

Denoted by

$Arg(z)$

$$arg(z) = Arg(z) + 2n\pi$$

or

$$\boxed{Arg(z) = arg(z) - 2n\pi}$$

$n = 0, \pm 1, \pm 2 \ldots$

$$\boxed{-\pi < Arg(z) < \pi}$$

Exercises Churchill

page 21

Q 1: → (a) $Arg(z) = ?$

$$\frac{i}{-2-2i}$$

$$Arg(z) = Arg(i) - Arg(-2-2i)$$

$$= \frac{\pi}{2} - \frac{5\pi}{4}$$

$$-\pi < \frac{-3\pi}{4} < \pi$$

$$\boxed{Arg(z) = \frac{-3\pi}{4}}$$

(b) $z = (\sqrt{3} - i)^6$

z is in the 4th quadrant

$$\arg(z) = 6\left(2\pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$$

$$= 6(2\pi - \pi/6) = 11\pi$$

As this is not in the range $-\pi < \pi$ so

$$Arg(z) = \arg(z) - 2n\pi$$

let $n = 5$

$$= 11\pi - 10\pi = \pi$$

hence

$$\boxed{Arg(z) = \pi}$$

Q4 Churchill P21

$$|e^{i\theta} - 1| = 2$$

$\theta = ?$

$$|\cos\theta + i\sin\theta - 1| = 2$$

$$\sqrt{(\cos\theta - 1)^2 + \sin^2\theta} = 2$$

$$\cos^2\theta + 1 - 2\cos\theta + \sin^2\theta = 4$$

$$2(1 - \cos\theta) = 4$$

$$1 - \cos\theta = 2$$

$$\cos\theta = -1$$

$$\boxed{\theta = \pi}$$

$\because \cos(\pi) = -1$

Q33 Zill P:21

Identity for $\cos(2\theta) = ?$

using De Moivre's formula ?

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$

$$\cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta = \text{"}$$

Comparing

$$\boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta}$$

Q5 Chirchill P: 21

(a)

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$(\cos\theta + i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta)$$

$$(\cos^2\theta - \sin^2\theta + 2i\, C\theta\, S\theta)(C\theta + iS\theta) = \text{"}$$

$$C^3\theta - C\theta\, S^2\theta + 2i\, C^2\theta\, S\theta$$
$$+ i\, C^2\theta\, S\theta - iS^3\theta - 2C\theta\, S^2\theta = \text{"}$$

$$C^3\theta - C\theta S^2\theta - 2C\theta S^2\theta$$
$$+ i\,(2C^2\theta S\theta + C^2\theta S\theta - S^3\theta) = \text{"}$$

$$C^3\theta - 3C\theta S^2\theta + i(3C^2\theta S\theta - S^3\theta) = \text{"}$$

Comparing

$$\boxed{\begin{aligned}\cos(3\theta) &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ \sin(3\theta) &= -\sin^3\theta + 3\cos^2\theta\sin\theta\end{aligned}}$$

Q10 Churchill P22

Let

$$S = 1 + z + z^2 \cdots z^n$$

then

$$S - zS = 1 - z^{n+1}$$

$$S(1-z) = 1 - z^{n+1}$$

$$\boxed{S = \frac{1 - z^{n+1}}{1 - z}}$$

Lagrange's trigonometric Identity ?

$$1 + C\theta + C2\theta \cdots C n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\frac{\theta}{2}]}{2S(\theta/2)}$$

Let $z = e^{i\theta}$

then

$$1 + C\theta + C2\theta \cdots \cos n\theta$$
$$= Re(1 + z + z^2 \cdots z)^n =$$

$$\operatorname{Re}\left(\frac{1-z^{n+1}}{1-z}\right)$$

$$\frac{1-z^{n+1}}{1-z} = \frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}$$

$$= \frac{1-\cos[(n+1)\theta]-i\sin[(n+1)\theta]}{1-\cos\theta - i\sin\theta}$$

By rationalizing

$$= \frac{\left[1-\cos[(n+1)\theta]-i\sin[(n+1)\theta]\right]\left[1-\cos\theta+i\sin\theta\right]}{(1-\cos\theta)^2+\sin^2\theta}$$

$$= \frac{''}{1+\cos^2\theta-2\cos\theta+\sin^2\theta}$$

$$= \frac{''}{2-2\cos\theta}$$

$$= \frac{''}{2(1-\cos\theta)} \rightarrow (1)$$

The half angle identity is

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$

using in (1)

$$= \frac{''}{\cancel{2}4\sin^2\frac{\theta}{2}} \rightarrow (2)$$

Now simplifying the numerator of (2)

$$= \left[1-\cos[(n+1)\theta]\right]\left[1-\cos\theta\right] + \sin[(n+1)\theta]\sin\theta$$

(Imaginary part omitted)

Again using the half angle identity.

$$= \left[1-\cos[(n+1)\theta]\right]2\sin^2\frac{\theta}{2} + \sin[(n+1)\theta]\sin\theta$$

Using

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$$

and

$$\boxed{\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{\left(1-\cos[(n+1)\theta]\right)2\sin^2\frac{\theta}{2} + \sin[(n+1)\theta]}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= 2\sin^2\frac{\theta}{2} - 2\sin\left(\frac{\theta}{2}\right)\left[\begin{array}{l}\cos[(n+1)\theta]\sin\frac{\theta}{2} \\ -\sin[(n+1)\theta]\cos\frac{\theta}{2}\end{array}\right]$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$

$$= 2\sin^2\frac{\theta}{2}\left[\sin\frac{\theta}{2} - \sin\left[(n+1)\theta + \frac{\theta}{2}\right]\right]$$

Now Combining the num & den

$$1+\cos\theta+\cdots\cos n\theta = \frac{\cancel{2\sin\frac{\theta}{2}}\left[\sin\frac{\theta}{2} - \sin\left[(2n+1)\frac{\theta}{2}\right]\right]}{2\cancel{4}\sin^{\cancel{2}}\frac{\theta}{2}}$$

$$= \frac{1}{2} - \frac{\sin\left[(2n+1)\frac{\theta}{2}\right]}{2\sin\frac{\theta}{2}}$$

proved