Sequences & Series :->

Sequence :-> A sequence is an arrangement/listing of numbers (THINGS)

e.g. $1, 2, 3, 4 \ldots\ldots \rightarrow (1)$

$+1 \quad +1 \quad +1$

↳ it follows a rule.

e.g., $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \rightarrow (2)$

$\div 2 \quad \div 2 \quad \div 2$

=> In (1) we see that the terms go on increasing while in (2) they are decreasing.

=> By common sense when terms are decreasing the n^{th} term will tend to be 0 while in case of increasing it will be infinity.

=> So (2) is convergent (come to finite point) and (1) is divergent.

Bounded Sequence :- We have the UPPER BOUND or the LOWER BOUND or both.

Suppose the terms of a sequence are

$a_1, a_2, a_3 \ldots a_n \ldots$

and each term is less than a number l the the sequence is bounded.

$a_1 < l, a_2 < l \ldots a_n < l$

e.g $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}$

each term is less than 1

Monotonic Sequence

↳ Either increasing or decreasing.

e.g $2, 4, 6, 8 \ldots \rightarrow ①$

is a monotonically increasing sequence.

and $4, 2, 0, -2, -4 \ldots \rightarrow ②$

is a monotonically decreasing sequence.

Q:→ Are the monotonic sequences convergent?

No, the monotonic sequences will always be divergent. ① goes to ∞ and ② goes to $-\infty$

Q:→ What if a sequence is not monotic ?

If a sequence is not monotonic (somewhere increases and somewhere decreases) then it is an Oscillatory Sequence.

Oscillating Sequence

e.g $t_n = (-1)^n$

$$t_n = \begin{cases} 1 & n \text{ is even} \\ -1 & n \text{ is odd} \end{cases}$$

$n =$	1	2	3	4	....
$t_n =$	-1	1	-1	1	....

So we can't conclude whether it is convergent or divergent.

⇒ Let t_n be a sequence then if $\lim_{n \to \infty} t_n$ is finite and unique then t_n is convergent.

Example :→ Find whether the sequence

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{8}$ . . . .

is convergent ?

Solution :→ First find t_n.

$$t_n = \frac{n}{n+1} \qquad n = 1, 2, 3 \ldots\ldots$$

↓

Represent the sequence parametrically.

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}} \left(\frac{1}{1+\frac{1}{n}}\right)$$

$$= \frac{1}{1+0} = 1 \rightarrow \text{finite and unique so}$$

t_n is convergent (Converges to 1)

Example $\quad t_n = (-)^n + \frac{1}{2^n}$

Convergent or divergent ?

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty}\left[(-1)^n + \frac{1}{2^n}\right]$$

$\frac{1}{2^n} \to 0$, where $n \to \infty$, $2^n \to \infty$

$\& \frac{1}{2^n} \to 0$

So

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} (-1)^n$$

$$= \begin{cases} -1 & n \text{ is odd} \\ 1 & n \text{ is even} \end{cases}$$

The limiting value in this case is finite but not unique (Oscillating Sequence) so we can't conclude whether t_n converges or diverges.

Example :- $t_n = \dfrac{n^2 - 2n}{3n^2 + n}$

Convergent ?

Solution

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{n^2 - 2n}{3n^2 + n} = \lim_{n\to\infty} \frac{n^2}{n^2}\left(\frac{1-\frac{2}{n}}{3+\frac{1}{n}}\right)$$

$= \frac{1}{3}$ (unique & finite) Convergent

Complex Sequence :-> A complex sequence z_n is similar in definition to the previous one.

=> It is a listing of complex nos.

i.e $z_n \in \mathbb{C}$ while n is an integer.

Convergence :-> A complex sequence z_n converges if

$\lim_{n \to \infty} z_n$ exist i.e $\lim_{n \to \infty} z_n = z_0$

This say z_n converges to z_0

-> this happens when for every $\epsilon > 0$, there is an N s.t for any n (n & N are both integers)

if $n \geq N$, then $|z_n - z_0| < \epsilon$

Cauchy Sequence :-> The sequence z_n is called a cauchy sequence if for every $\epsilon > 0$ there is an $N \in \mathbb{Z}$ s.t for any $n, m \in \mathbb{Z}$

if $n, m \geq N$, then $|z_n - z_m| < \epsilon$

$\Rightarrow$ Thus every cauchy sequence will be convergent.

Series :→ The sum of the terms of an infinite sequence is called an infinite series.

A Complex sequence can be defined as a fun^n whose domain is set of natural nos. and whose range is a sub-set of complex nos. Therefore sequence is an ordered list of nos. While series is a sum of a list of nos. e.g $\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 \cdots -$

Geometric Series :→ A series of the form

$$\sum_{n=1}^{\infty} a z^{n-1} = a + az + az^2 \cdots + az^{n-1} + \cdots$$

is called geometric series.

Convergence of Series :→

A series converges if the sequence of partial sums (say S_n) converges.
i.e if $S_n \to L$ as $n \to \infty$. (We say that the series converges to L or that the sum of the series is L.

e.g. if a series is geometric then it is always possible to find a formula for S_n

i.e., Sequence of partial sums.

$$S_n = a + az + az^2 \cdots az^{n-1}$$

$$z S_n = az + az^2 + az^3 \cdots az^n$$

$$S_n - zS_n = a - az^n$$

$$(1-z) S_n = a - az^n$$

$$S_n = \frac{a(1-z^n)}{1-z}$$

⇒ Observe that if $|z| < 1$ then $z^n \to 0$ as $n \to \infty$ & hence S_n converges.

Example 3: Zill, Page: 261

$$\sum_{n=1}^{\infty}\left(\frac{1+2i}{5}\right)^n = \frac{1+2i}{5} + \frac{(1+2i)^2}{5^2} + \frac{(1+2i)^3}{5^3}$$

which series is this and when is it convergent?

Solution → This is a geometric series

with $a = \frac{1+2i}{5}$ & $z = \frac{1+2i}{5}$

Hence

$$S_n = \frac{a(1-z^n)}{1-z}$$

$$= \frac{1+2i}{5}\left(\frac{1-\left(\frac{1-2i}{5}\right)^n}{1-\left(\frac{1-2i}{5}\right)}\right)$$

$$|z| = \frac{1}{5}\sqrt{1+4} = \frac{\sqrt{5}}{5} < 1$$

Hence this series converges.

Theorem (A necessary Condition for Convergence of Series.)

If $\lim_{n\to\infty} z_n = 0$ then $\sum_{n=1}^{\infty} z_n$ converges.

otherewise $\sum_{n=1}^{\infty} z_n$ diverges.

Examples:→

$$S_n = \sum_{n=1}^{\infty} \left(\frac{in+5}{n}\right)$$

$$z_n = \frac{in+5}{n} = \frac{i+5/n}{1}$$

$\lim_{n\to\infty} z_n = i \neq 0$ hence the

series $\sum_{n=1}^{\infty} \frac{in+5}{n}$ diverges.

Definition (Absolute and Conditional Convergence)

An infinite series $\sum_{n=1}^{\infty} z_n$ absolutely converges if $\sum_{n=1}^{\infty} |z_n|$ converges while it conditionally converges if it converges but $\sum_{n=1}^{\infty} |z_n|$ diverges.

Notice that $|z_n|$ is a real no.

→ In real nos. series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is called p-series which converges if $p>1$ and diverges if $p \le 1$.

Example: Find whether the series $\sum_{n=1}^{\infty} \frac{i^n}{n^2}$ is absolutely Convergent.

Solution:

$$\left|\frac{i^n}{n^2}\right| = \frac{1}{n^2}$$

and the p-series $\sum_{n=1}^{\infty} \left|\frac{1}{n^2}\right|$ converges ∵ $p=2>1$

[Absolute Convergence => Convergence]

Tests for Convergence of infinite series

① Ratio test

Let $\sum_{n=1}^{\infty} z_n$ be an infinite series of non-zero complex elements/terms s.t

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = L$$

(i) if $L < 1$, then series converges absolutely

(ii) if $L > 1$ or $L = \infty$ then series diverges

(iii) if $L = 1$, then the test is inconclusive

2 Root test

For infinite complex series $\sum_{n=1}^{\infty} z_n$ ~~but~~ we've

$$\lim_{n\to\infty}\left(\sqrt[n]{|z_n|}\right) = L$$

then

(i) $L < 1$, series converges absolutely.

(ii) $L > 1$ or $L = \infty$, series diverges.

(iii) $L = 1$, test inconclusive.

Power Series → Imp: in the study of analytic fun".

⇒ A series of the form

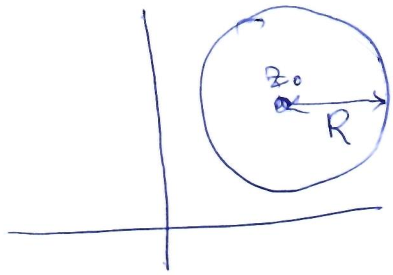
$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots \quad (1)$$

where a_n, $n = 0 \cdots \infty$, are complex, is called a power series in $z - z_0$.

→ This series is centered at z_0
(z_0 is called the centre of the series (1))

Circle of Convergence

Every Complex power series has a radius of convergence (as opposed to the interval of convergence in real power series) and hence a circle of convergence centered at z_0 and radius R.



⇒ if $\{\forall z \mid |z-z_0| \leq R\}$ the power series converges then this circle is called circle of convergence.

⇒ The convergence will be absolute if $|z-z_0| < R$ i.e z is in the interior of the circle.

The R can be:

1. $R = 0$, PS converges only at $z = z_0$
2. R is finite, PS " $\forall$ $|z-z_0| \leq R$.
3. $R = \infty$, PS converges $\forall z$

Example :5 Zill, Page: 263

Circle of Convergence ?

$$z_n = \sum_{n=1}^{\infty} \frac{z^{n+1}}{n}$$

Solution :- By ratio test

$$\lim_{n \to \infty} \left| \frac{\frac{z^{n+2}}{n+1}}{\frac{z^{n+1}}{n}} \right|$$

$$= \lim_{n \to \infty} \frac{n}{n+1} \left| z^{\cancel{n}+2-\cancel{n}-1} \right|$$

$$= \lim_{n \to \infty} \frac{n}{n+1} |z|$$

$$= \lim_{n \to \infty} \frac{1}{1+\frac{1}{n}} |z| = |z|$$

Thus the series converges for $|z| < 1$
The circle of convergence is $|z| = 1$ (clearly centered at the origin whith $R = 1$. $\left(R = \frac{1}{L}\right)$

Example 6 :→ P264, Zill.

Given the power series

$$Z_n = \sum_{n=0}^{\infty} \left(\frac{(-1)^{n+1}}{n!}\right)(z-1-j)^n$$

Convergence?

Solution :→ $a_n = \frac{(-1)^{n+1}}{n!}$, $z_0 = 1+j$

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}}{(n+1)!}(z-z_0)^{n+1}}{\frac{(-1)^{n+1}}{n!}(z-z_0)^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}}{(-1)^{n+1}} \times \frac{n!}{(n+1)!}(z-z_0)\right|$$

$$= (z-z_0)\lim_{n\to\infty}\left|(-1)^{\cancel{n}+2-\cancel{n}-1}\frac{n!}{(n+1)!}\right|$$

$$= -(z-z_0)\lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\right|$$

As

$$(n+1)! = n! \times (n+1) \text{ so}$$

$$= -(z-z_0)\lim_{n\to\infty}\left[\frac{1}{n+1}\right] = 0 = L$$

$R = \frac{1}{L} = \infty$, Thus Z_n converges absolutely $\forall z$

Exercises 6.1 Page: 266, 270

(5) $z_n = \dfrac{3nj+2}{n+nj}$

Converges?

Solution

$$z_n = \frac{\cancel{n}(3i+2/n)}{\cancel{n}(1+i)}$$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}\left(\frac{3i+2/n}{1+i}\right) = \frac{3i}{1+i}$$

$$\frac{3i\times(1-i)}{1+1} = \frac{3i+3}{2} = \frac{3}{2}+\frac{3}{2}i$$

Yes, converges to $\frac{3}{2}(1+i)$

⇒ TAYLOR SERIES

→ Every PS converges to a value L if z is picked within the circle of convergence.

→ This means that a PS series defines a a fun^n f $\{\forall z \mid |z-z_0| \leq R\}$ and the value L in such a way that $f(z)=L$.

Differentiation and integration of PS

Theorem :→ (Continuity) Zill, Page: 269

A PS $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ represents a continuous function f within it circle of convergence $(|z-z_0| = R)$

Theorem :→ (Term by term differentiation) Zill Page 269

A PS $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ can be differentiated term by term in $|z-z_0|=R$

e.g: $\frac{d}{dz}\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} a_n \frac{d}{dz}(z-z_0)^n = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$

⇒ i.e . when a power series is differentiated the result is an other power series.

The circle and R of convergence of a PS & its differentiation are similar.

$$z_n = \sum_{n\geq 0}^{\infty} a_n (z - z_0)^n$$

by Ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(z-z_0)^{n+1}}{a_n (z-z_0)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} (z-z_0) \right|$$

$$= |z-z_0| \qquad \frac{a_{\infty+1}}{a_\infty} = |z-z_0| \checkmark$$

$$\& \quad z_n' = \sum_{n\geq 1}^{\infty} n a_n (z-z_0)^{n-1}$$

$$\lim_{n\to\infty} \left| \frac{(n+1) a_{n+1} (z-z_0)^n}{n a_n (z-z_0)^{n-1}} \right|$$

$$= \lim_{n\to\infty} \left| (1+1/n) \frac{a_{n+1}}{a_n} \right| |z-z_0|$$

$$= \frac{a_{\infty+1}}{a_\infty} |z-z_0| = |z-z_0| \checkmark$$

$\Rightarrow$ A PS is infinitely differentiable in its circle of convergence. In addition, the PS and all its derivatives will have the same Radius of convergence.

Theorem :> (Term by term integration) Zill: Page 270

A PS $Z_n = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ can be integrated along a contour C which lies entired in the circle of convergence of Z_n.

$$\int_C \sum_{n=0}^{\infty} a_n (z-z_0)^n dz = \sum_{n=0}^{\infty} a_n \int_C (z-z_0)^n dz$$

however if $C \in |z-z_0|$ then indefinite integral can also be used

$$\int \sum_{n=0}^{\infty} a_n (z-z_0)^n dz = \sum_{n=0}^{\infty} a_n \int (z-z_0)^n dz$$

$$= \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-z_0)^{n+1} + Ct$$

This can be proved, using ratio test, that its circle of convergence is the same as that of Z_n.

Taylor Series :→ A PS represent

a funⁿ $f(z)$ ~~with~~ $\big|\ |z-z_0| < R$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \longrightarrow (1)$$

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z-z_0)^{n-1} \longrightarrow (2)$$

$$f''(z) = \sum_{n=0}^{\infty} n(n-1) a_n (z-z_0)^{n-2} \longrightarrow (3)$$

$$f'''(z) = \sum_{n=0}^{\infty} n(n-1)(n-2) a_n (z-z_0)^{n-3} \longrightarrow (4)$$

and so on.

→ There is a relationship b/w the co-efficients of f and its derivatives.

Evaluating (1), (2), (3) and (4) and $z = z_0$

$$f(z_0) = a_0$$

$$f'(z_0) = 1!\, a_1 \quad , \quad f''(z_0) = 2!\, a_2$$

$$f'''(z_0) = 3!\, a_3$$

In general.

$$f^{n}(z_0) = n!\, a_n$$

or

$$a_n = \frac{f^{n}(z_0)}{n!}, \quad n \geq 0$$

$$\left(\text{e.g., } a_0 = \frac{f^{0}(z_0)}{0!} = f(z_0) \right)$$

So we can write

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{n}(z_0)}{n!} (z - z_0)^{n}$$

This series is called the Taylor series for f, centered at z_0

$\Rightarrow$ A Taylor series centered at the origin is referred to as Maclaurin Series.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!} z^{n}$$

If "f" is analytic in a domain D, then "CAN we represent it as a Power Series?

The power series converges in a circle while domains are not usually circular.

This matter is topic of the following theorem.

Theorem :→ Taylor's theorem (Zill, 271

Let f be an analytic fun^n within a domain D and let z_0 be a pt in D. Then f has a series representation

$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{f^{n}(z_0)}{n!}(z-z_0)^n} \rightarrow \text{(A)}$$

valid for the largest circle C with center at z_0 of radius R that lies entirely within D.

⇒ In this case R is the distance b/w z_0 (center of the series) and its nearest isolated singularity of f. (at which f fails to be analytic)

$\Rightarrow$ If $f(z)$ is an entire func" then $R = \infty$ for the corresponding Taylor Series.

$\Rightarrow$ Using (A) and the above facts we can say that the following Maclaurin series representations are valid for all z, i.e $|z| < \infty$

(1) $$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

(2) $$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

(3) $$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$$

(1) For $f(z) = e^z$ first find the co-efficients.

$$a_0 = \frac{f^{\circ}(0)}{0!} = 1, \quad a_1 = \frac{f'(0)}{1!} = \frac{1}{1!}$$

$$a_2 = \frac{f''(0)}{2!} = \frac{1}{2!}, \quad a_3 = \frac{f'''(0)}{3!} = \frac{1}{3!}$$

From (A)

$$f(z) = \sum_{n=0}^{\infty} \frac{\partial^n f(0)}{\partial z^n n!} z^n$$

$$= 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!}$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

(2) $f(z) = \sin(z)$

$f'(z) = \cos z, \quad f''(z) = -\sin(z), \quad f'''(z) = -\cos z$

$f^{4}(z) = \sin(z), \quad f^{5}(z) = \cos(z)$

$a_0 = \frac{f(0)}{0!} = 1, \quad a_1 = \frac{f'(0)}{1!} = \frac{1}{1!}, \quad a_2 = \frac{f''(0)}{2!} = 0$

$a_3 = \frac{f'''(0)}{3!} = \frac{-1}{3!}, \quad a_4 = \frac{f^{4}(0)}{4!} = 0, \quad a_5 = \frac{f^{5}(0)}{5!} = \frac{1}{5!}$

put there in (A)

$$f(z) = \sum_{n=0}^{\infty} \frac{\partial^n f(0)}{\partial z^n} \frac{1}{n!} z^n$$

$$= 0 + \frac{z}{1!} + 0 - \frac{z^3}{3!} + 0 + \frac{z^5}{5!}$$

$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \left(\frac{z^{2n+1}}{(2n+1)!} \right)$$

③ $f(z) = \cos z$ can be expanded similarly.

Example (R of Convergence) Zill. Page 273

Suppose that

$$f(z) = \frac{3-i}{1-i+z}$$

is expanded in a Taylor Series with center $z_0 = 4-2i$. What is R?

Solution :→ As mentioned earlier R is the distance b/w z_0 and its nearest singularity.

→ The only singularity in $f(z)$ is $z = -1+i$

So

$$R = |\underbrace{z}_{\text{singularity}} - z_0| = \sqrt{(-1-4)^2+(1+2)^2} = \sqrt{25+9}$$

$$= \sqrt{34}$$

Laurent's Theorem :→

Let f be analytic within the domain D defined by $r < |z-z_0| < R$. Then f has a series representation

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

valid for $r < |z-z_0| < R$. The co-efficients a_k are given by:

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s-z_0)^{k+1}}\, ds \quad , \; k = 0, \pm 1, \pm 2 \cdots$$

where C is a simple closed curve that lies entirely within D. and has z_0 in its interior.

→ The Taylor Series gives expansion around a pt $z = z_0$ in a specific region while Laurent series is used for expansion in a strip (i.e., region having two boundaries)

Q:→ Expand $\dfrac{1}{z^2 - 3z + 2}$ in the region

(i) $|z| < 1$, (ii) $|z| > 2$?

Solution:→ Prior to expansion, break $f(z)$ into partial fractions.

$$\frac{1}{z^2-3z+2} = \frac{1}{(z-2)(z-1)} = \frac{A}{z-2} + \frac{B}{z-1}$$

$$A = \left.\frac{1}{z-1}\right|_{z=2} = 1$$

$$B = \left.\frac{1}{z-2}\right|_{z=1} = -1$$

$$\frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1}$$

(i) $|z| < 1$, here 2 and 1 are greater than $|z|$, so taking common -2 and -1

$$\frac{1}{z-2} - \frac{1}{z-1} = \frac{-1/2}{1 - z/2} + \frac{1}{1-z} = -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} + (1-z)^{-1}$$

By (Binomial Expansion.) geometric series expansion

$$= -\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right]+\left[1+z+z^2+z^3+\cdots\right]$$

$$= \frac{1}{2}+\frac{3z}{4}+\frac{7z^2}{8}+\frac{15z^3}{16}+\cdots$$

(ii) $|z|\geq 2$, i.e $|z|>2$, and $|z|>1$ so

taking z as common term.

$$\frac{1}{z-2}-\frac{1}{z-1} = \frac{1}{z(1-2/z)}-\frac{1}{z(1-1/z)}$$

$$= \frac{1}{z}\left[\frac{1}{1-\frac{2}{z}}-\frac{1}{1-1/z}\right]$$

$$= \frac{1}{z}\left\{\left[1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots\right]-\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]\right\}$$

$$= \frac{1}{z}\left[\frac{1}{z}+\frac{3}{z^2}+\frac{7}{z^3}+\cdots\right]$$

$$= \frac{1}{z^2}+\frac{3}{z^3}+\frac{7}{z^4}+\cdots$$

Q:-) Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent series valid for (i) $1 < |z| < 3$

(ii) $0 < |z+1| < 2$

(iii) $|z| > 3$

Solution

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$$

$$A = \left.\frac{1}{z+3}\right|_{z=-1} = \frac{1}{2}$$

$$B = \left.\frac{1}{z+1}\right|_{z=-3} = -\frac{1}{2}$$

$$f(z) = \frac{1/2}{z+1} - \frac{1/2}{z+3} = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

(i) $1 < |z| < 3$, so taking z common from $z+1$ and 3 common from $z+3$,

$$f(z) = \frac{1}{2}\left[\frac{1}{z}\left(\frac{1}{1+1/z}\right) - \frac{1}{3}\left(\frac{1}{1+3/z}\right)\right]$$

See $|z| > 1$

$\frac{1}{|z|} < 1$

$|z| < 3$

$\frac{|z|}{3} < 1$

$$= \frac{1}{2}\left[\frac{1}{z}\left[1 - z + z^2 - z^3 + \cdots\right] - \frac{1}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right]\right]$$

$$= \frac{1}{2}\left[\frac{1}{z} - 1 + z - z^2 + \cdots\right] - \left[\frac{1}{3} - \frac{z}{9} + \frac{z^2}{27} - \frac{z^3}{81} + \cdots\right]$$

$$= \frac{1}{2}\left[\frac{1}{z} - \frac{4}{3} + \frac{10z}{9} - \frac{28z^2}{27} + \cdots\right]$$

(ii) $0 < |z+1| < 2$

$$f(z) = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+3}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+1+2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{2}\left(\frac{1}{1+\frac{z+1}{2}}\right]\right)$$

$$= \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{2}\left[1 - \frac{z+1}{2} + \frac{(z+1)^2}{4} - \frac{(z+1)^3}{8} + \cdots\right]\right]$$

$$= \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{2} + \frac{z+1}{4} - \frac{(z+1)^2}{8} + \frac{(z+1)^3}{16} \cdots\right]$$

Q:→ Expand $f(z) = \frac{1}{z+3}$ around the pt: $z_0 = 1$

Solution :→ $f(z)\Big|_{z=z_0} = f(z_0) + (z-z_0)f'(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0)$

$$+ \frac{(z-z_0)^3}{3!}f'''(z_0) + \cdots$$

$z_0 = 1$

$f(1) = \frac{1}{4}$

$f'(z_0) = -1(z-1)^{-2}$

Alternatively we can do it by producing powers of $z-1$

e.g., $f(z) = \frac{1}{z+3} = \frac{1}{z-1+4}$

Taking 4 common.

$$f(z) = \frac{1}{4}\left[\frac{1}{1+\frac{z-1}{4}}\right]$$

$$= \frac{1}{4}\left[1 - \frac{z-1}{4} + \frac{(z-1)^2}{16} - \frac{(z-1)^3}{64} + \cdots\right]$$

$$= \frac{1}{4} - \frac{z-1}{16} + \frac{(z-1)^2}{64} - \frac{(z-1)^3}{256} + \cdots$$

This is the expansion around z = 1 and whenever the expansion is around a point it done by Taylor series.