Cauchy Riemann Equation

$f(z) = U(x,y) + iV(x,y)$

$f : D \longrightarrow \mathbb{C}$

then f is $\mathbb{C}$-differentiable at $z_0 = (x_0, y_0)$ iff

(i) $U(x,y)$ and $V(x,y)$ are differentiable at z_0

(ii)

$$\boxed{\begin{aligned} \frac{\partial U}{\partial x} &= \frac{\partial V}{\partial y} \\ \frac{\partial U}{\partial y} &= -\frac{\partial V}{\partial x} \end{aligned}} \quad \text{Cauchy Riemann Eqn.}$$

$U_x = \frac{\partial U}{\partial x}$

at (x_0, y_0)

In this case $\boxed{f'(z) = U_x(x_0, y_0) - i U_y(x_0, y_0)}$

Criterion for non-analyticity :-> If the CR eqns are not satisfied at every pt. z in a domain D, then the fun^n $f(z) = U(x,y) + iV(x,y)$ is not analytic in D.

Ex 1 Zill: P: 132

$f(z) = z^2 + z$

Analytic ?

z = x + iy

$f(z) = (x+iy)^2 + (x+iy)$

$= x^2 - y^2 + x + 2ixy + iy$ ✓

$u(x,y) = x^2 - y^2 + x,$

$v(x,y) = 2xy + y$

$\frac{\partial u}{\partial x} = 2x+1$, $\frac{\partial u}{\partial y} = -2y$

$\frac{\partial v}{\partial x} = 2y$, $\frac{\partial v}{\partial y} = 2x+1$

As $\frac{\partial u}{\partial x} = 2x+1 = \frac{\partial v}{\partial y}$ and

$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$ so yes analytic

$$f'(z) = U_x(x_0, y_0) - i\,U_y(x_0, y_0)$$

$$= 2x + 1\Big|_{(x_0,y_0)\to(x,y)} + i\,\big(2y\big)\Big|_{(x,y)}$$

$$= 2x + 1 + 2iy$$

$$= 2(x + iy) + 1$$

$$= 2z + 1$$

Exaple 2

$$f(z) = 2x^2 + y + i(y^2 - x)$$

Analytic ?

$$\frac{\partial u}{\partial x} = 4x, \quad \frac{\partial v}{\partial y} = 2y$$

$$\frac{\partial u}{\partial y} = 1, \quad \frac{\partial v}{\partial x} = -1$$

Since $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ but $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$.

In fact

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

along the line $y = 2x$

but any where else

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ hence there is

no neighborhood of the pts. on the line $y = 2x$ for which $f(z)$ is differentiable

open disk

hence we conclude that $f(z)$ is no where analytic. (hence f is differentiable on the line $y = 2x$

Criterion for Analyticity :->

$f(z) = u(x,y) + i v(x,y)$.

Suppose u and v are continuous and have continuous first order partial derivatives in a domain D. Then if u & v satisfy the CR eqns at all pts of D, then $f(z)$ is analytic in D

Ex 3: Zill P: 133

$$f(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

Analytic ?

Solution:→ $u(x,y) = \frac{x}{x^2+y^2}$, $v(x,y) = \frac{-y}{x^2+y^2}$

(1) Are U & V continuous ?

U & V can be continuous only in their domain. The domain include all points (x & y) except $x^2 + y^2 = 0$, which means the origin

(1) $\lim_{(x,y)\to(x_0,y_0)} U(x,y)$ exist

(2) $\lim_{(x,y)\to(x_0,y_0)} U(x,y) = U(x_0,y_0)$

$\lim_{(x,y)\to(x_0,y_0)} U(x,y) = \frac{x_0}{x_0^2+y_0^2}$ $x_0, y_0 \in D$

and $U(x_0,y_0) = \frac{x_0}{x_0^2+y_0^2}$ so continuous

So both u and v are continuous in D (exclude $z=0$)

(2)

Moreover,

$$\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{2xy}{(y^2+x^2)^2} = -\frac{\partial v}{\partial x}$$

i.e., CR equation is also satisfied except at the origin. f is analytic in any domain except the origin.

f(z) is analytic in the whole complex plane except the origin