15.1

Theory:

Sequences

The basic definitions are as in calculus. An *infinite sequence* or, briefly, a **sequence,** is obtained by assigning to each positive integer n a number z_n, called a **term** of the sequence, and is written

$$z_1, z_2, \cdots \qquad \text{or} \qquad \{z_1, z_2, \cdots\} \qquad \text{or briefly} \qquad \{z_n\}.$$

We may also write $z_0, z_1, \cdots$ or $z_2, z_3, \cdots$ or start with some other integer if convenient.

A **real sequence** is one whose terms are real.

Convergence. A **convergent sequence** $z_1, z_2, \cdots$ is one that has a limit c, written

$$\lim_{n \to \infty} z_n = c \qquad \text{or simply} \qquad z_n \to c.$$

By definition of **limit** this means that for every $\epsilon > 0$ we can find an N such that

$$(1) \qquad |z_n - c| < \epsilon \qquad \text{for all } n > N;$$

geometrically, all terms z_n with $n > N$ lie in the open disk of radius ϵ and center c (Fig. 358) and only finitely many terms do not lie in that disk. [For a *real* sequence, (1) gives an open interval of length 2ϵ and real midpoint c on the real line; see Fig. 359.]

A **divergent sequence** is one that does not converge.

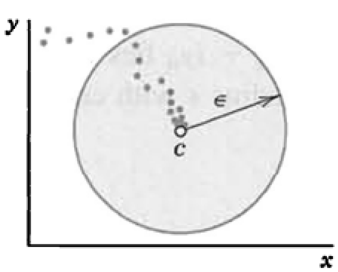

Fig. 358. Convergent complex sequence

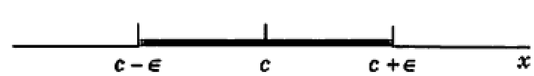

Fig. 359. Convergent real sequence

THEOREM 1 **Sequences of the Real and the Imaginary Parts**

A sequence $z_1, z_2, \cdots, z_n, \cdots$ of complex numbers $z_n = x_n + iy_n$ (where $n = 1, 2, \cdots$) converges to $c = a + ib$ if and only if the sequence of the real parts $x_1, x_2, \cdots$ converges to a and the sequence of the imaginary parts $y_1, y_2, \cdots$ converges to b.

Series

Given a sequence $z_1, z_2, \cdots, z_m, \cdots$, we may form the sequence of the sums

$$s_1 = z_1, \qquad s_2 = z_1 + z_2, \qquad s_3 = z_1 + z_2 + z_3, \quad \cdots$$

and in general

$$s_n = z_1 + z_2 + \cdots + z_n \qquad (n = 1, 2, \cdots). \tag{2}$$

s_n is called the ***n*th partial sum** of the *infinite series* or **series**

$$\sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots . \tag{3}$$

The $z_1, z_2, \cdots$ are called the **terms** of the series. (Our usual ***summation letter*** is n, unless we need n for another purpose, as here, and we then use m as the summation letter.)

A **convergent series** is one whose sequence of partial sums converges, say,

$$\lim_{n\to\infty} s_n = s. \qquad \text{Then we write} \qquad s = \sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots$$

THEOREM 2

Real and Imaginary Parts

A series (3) with $z_m = x_m + iy_m$ converges and has the sum $s = u + iv$ if and only if $x_1 + x_2 + \cdots$ converges and has the sum u and $y_1 + y_2 + \cdots$ converges and has the sum v.

Tests for Convergence and Divergence of Series

Convergence tests in complex are practically the same as in calculus. We apply them before we use a series, to make sure that the series converges.

Divergence can often be shown very simply as follows.

THEOREM 3

Divergence

If a series $z_1 + z_2 + \cdots$ converges, then $\lim_{m \to \infty} z_m = 0$. Hence if this does not hold, the series diverges.

THEOREM 4

Cauchy's Convergence Principle for Series

A series $z_1 + z_2 + \cdots$ is convergent if and only if for every given $\epsilon > 0$ (no matter how small) we can find an N (which depends on ϵ, in general) such that

$$|z_{n+1} + z_{n+2} + \cdots + z_{n+p}| < \epsilon \qquad \text{for every } n > N \text{ and } p = 1, 2, \cdots \tag{5}$$

Absolute Convergence. A series $z_1 + z_2 + \cdots$ is called **absolutely convergent** if the series of the absolute values of the terms

$$\sum_{m=1}^{\infty} |z_m| = |z_1| + |z_2| + \cdots$$

is convergent.

If $z_1 + z_2 + \cdots$ converges but $|z_1| + |z_2| + \cdots$ diverges, then the series $z_1 + z_2 + \cdots$ is called, more precisely, **conditionally convergent.**

THEOREM 5

Comparison Test

If a series $z_1 + z_2 + \cdots$ is given and we can find a convergent series $b_1 + b_2 + \cdots$ with nonnegative real terms such that $|z_1| \leqq b_1$, $|z_2| \leqq b_2$, $\cdots$, then the given series converges, even absolutely.

THEOREM 6

Geometric Series

The geometric series

(6*) $$\sum_{m=0}^{\infty} q^m = 1 + q + q^2 + \cdots$$

converges with the sum $1/(1-q)$ *if* $|q| < 1$ *and diverges if* $|q| \geqq 1$.

Ratio Test

This is the most important test in our further work. We get it by taking the geometric series as comparison series $b_1 + b_2 + \cdots$ in Theorem 5:

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Ratio Test

If a series $z_1 + z_2 + \cdots$ *with* $z_n \neq 0$ $(n = 1, 2, \cdots)$ *has the property that for every n greater than some N,*

(7) $$\left|\frac{z_{n+1}}{z_n}\right| \leqq q < 1 \qquad (n > N)$$

(where $q < 1$ *is fixed), this series converges absolutely. If for every* $n > N$,

(8) $$\left|\frac{z_{n+1}}{z_n}\right| \geqq 1 \qquad (n > N),$$

the series diverges.

THEOREM 8

Ratio Test

If a series $z_1 + z_2 + \cdots$ *with* $z_n \neq 0$ $(n = 1, 2, \cdots)$ *is such that* $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = L$, *then:*

(a) *If* $L < 1$, *the series converges absolutely.*

(b) *If* $L > 1$, *the series diverges.*

(c) *If* $L = 1$, *the series may converge or diverge, so that the test fails and permits no conclusion.*

Root Test

The ratio test and the root test are the two practically most important tests. The ratio test is usually simpler, but the root test is somewhat more general.

Root Test

If a series $z_1 + z_2 + \cdots$ is such that for every n greater than some N,

$$\sqrt[n]{|z_n|} \leqq q < 1 \qquad (n > N) \tag{9}$$

(where $q < 1$ is fixed), this series converges absolutely. If for infinitely many n,

$$\sqrt[n]{|z_n|} \geqq 1, \tag{10}$$

the series diverges.

THEOREM 10

Root Test

If a series $z_1 + z_2 + \cdots$ is such that $\lim_{n\to\infty} \sqrt[n]{|z_n|} = L$, then:

(a) *The series converges absolutely if $L < 1$.*

(b) *The series diverges if $L > 1$.*

(c) *If $L = 1$, the test fails; that is, no conclusion is possible.*

Examples:

EXAMPLE 1 **Convergent and Divergent Sequences**

The sequence $\{i^n/n\} = \{i, -1/2, -i/3, 1/4, \cdots\}$ is convergent with limit 0.

The sequence $\{i^n\} = \{i, -1, -i, 1, \cdots\}$ is divergent, and so is $\{z_n\}$ with $z_n = (1 + i)^n$. ■

EXAMPLE 2 **Sequences of the Real and the Imaginary Parts**

The sequence $\{z_n\}$ with $z_n = x_n + iy_n = 1 - 1/n^2 + i(2 + 4/n)$ is $6i$, $3/4 + 4i$, $8/9 + 10i/3$, $15/16 + 3i$, $\cdots$. (Sketch it.) It converges with the limit $c = 1 + 2i$. Observe that $\{x_n\}$ has the limit $1 = \operatorname{Re} c$ and $\{y_n\}$ has the limit $2 = \operatorname{Im} c$. This is typical. It illustrates the following theorem by which the convergence of a *complex* sequence can be referred back to that of the two *real* sequences of the real parts and the imaginary parts. ■

EXAMPLE 4 Ratio Test

Is the following series convergent or divergent? (First guess, then calculate.)

$$\sum_{n=0}^{\infty} \frac{(100+75i)^n}{n!} = 1 + (100+75i) + \frac{1}{2!}(100+75i)^2 + \cdots$$

Solution. By Theorem 8, the series is convergent, since

$$\left|\frac{z_{n+1}}{z_n}\right| = \frac{|100+75i|^{n+1}/(n+1)!}{|100+75i|^n/n!} = \frac{|100+75i|}{n+1} = \frac{125}{n+1} \quad \to \quad L = 0.$$

EXAMPLE 5 Theorem 7 More General than Theorem 8

Let $a_n = i/2^{3n}$ and $b_n = 1/2^{3n+1}$. Is the following series convergent or divergent?

$$a_0 + b_0 + a_1 + b_1 + \cdots = i + \frac{1}{2} + \frac{i}{8} + \frac{1}{16} + \frac{i}{64} + \frac{1}{128} + \cdots$$

Solution. The ratios of the absolute values of successive terms are $\frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \cdots$. Hence convergence follows from Theorem 7. Since the sequence of these ratios has no limit, Theorem 8 is not applicable. ■

Problem set 15.1:

(23) $\sum_{n=0}^{\infty} \frac{n-i}{3n+2i}$

Solution: Let $Z_n = \frac{n-i}{3n+2i}$

$$Z_n = \frac{n-i}{3n+2i} \times \frac{3n-2i}{3n-2i}$$

$$Z_n = \frac{(n-i)(3n-2i)}{(3n)^2-(2i)^2}$$

$$Z_n = \frac{3n^2-2ni-3ni-2i^2}{9n^2+4}$$

$-2i^2 = -2(-1) = 2$

$$Z_n = \frac{3n^2+2}{9n^2+4} - \frac{5ni}{9n^2+4}$$

$\Rightarrow \sum_{n=0}^{\infty} \frac{n-i}{3n+2i} = \sum_{n=0}^{\infty} \frac{3n^2+2}{9n^2+4} - i\frac{5n}{9n^2+4}$

As we know that

$$Z_n = x_n + iy_n$$

Let real part be $x_n = \frac{3n^2+2}{9n^2+4}$ and

Imaginary part $y_n = \frac{-5n}{9n^2+4}$

$$Z_n = \frac{3n^2+2}{9n^2+4} + i\frac{-5n}{9n^2+4}$$

Apply limit on real part

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{3n^2+2}{9n^2+4}$$

$$= \lim_{n\to\infty} \frac{3n^2\left(1+\frac{2}{3n^2}\right)}{9n^2\left(1+\frac{4}{9n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{1\left(1+\frac{2}{3n^2}\right)}{3\left(1+\frac{4}{9n^2}\right)}$$

$$= \frac{1}{3} = l_1$$

Apply limit on imagnary part

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{-5n}{9n^2+4}$$

$$= \lim_{n\to\infty} \frac{-n\left(\frac{5}{n}\right)}{9n^2\left(1+\frac{4}{9n^2}\right)}$$

$$= 0 = l_2$$

$$\lim_{n\to\infty} z_n = l_1 + l_2$$

$$= \frac{1}{3} + 0$$

$$= 1/3 \neq 0$$

So by theorem (3) if $\lim_{n\to\infty} z_n \neq 0$, the series diverges

Example ("Related to Theorem")

$$\sum_{n=1}^{\infty}\left(\frac{1}{5n!}+\frac{2i}{5n!}\right)$$

Let $z_n = \frac{1}{5n!} + \frac{2i}{5n!}$

$$|z_n| = \sqrt{\left(\frac{1}{5n!}\right)^2 + \left(\frac{2}{5n!}\right)^2}$$

$$|z_n| = \sqrt{\frac{1}{25(n!)^2} + \frac{4}{25(n!)^2}}$$

$$= \sqrt{\frac{1+4}{25(n!)^2}}$$

$$= \sqrt{\frac{5}{25(n!)^2}}$$

$$= \sqrt{\frac{1}{5\,n!}} = \frac{1}{\sqrt{5}\,n!}$$

Now

$$\sqrt{5}\,n! > n!$$

$$\frac{1}{\sqrt{5}\,n!} < \frac{1}{n!}$$

$$|z_n| < \frac{1}{n!}$$

$$b_n = \frac{1}{n!}$$

$$\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

$$= e^1 = 2.71828$$

$\because |z_n| \leq |b_n|$ and $\sum |b_n|$ is convergent

$\therefore \sum |z_n|$ is convergent

So given series is absolutely convergent.

(ii) $\sum_{n=1}^{\infty} (3i)^n \frac{n!}{n^n}$ (Theorem 8)

Sol:

Let $z_n = \frac{(3i)^n n!}{n^n}$

$$z_{n+1} = \frac{(3i)^{n+1} (n+1)!}{(n+1)^{n+1}}$$

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(3i)^{n+1}(n+1)!}{(n+1)^{n+1}}}{\frac{(3i)^n n!}{n^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\left((3i)^{n+1}(n+1)!\right)\left(n^n\right)}{(3i)^n n! (n+1)^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(3i)^n} \cdot 3i \; \cancel{(n+1)} \cancel{n!} \, n^n}{(n+1)^n \cancel{(n+1)} \cancel{(3i)^n} \cancel{n!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{3i \, \cancel{n^n}}{\cancel{n^n} \left(1+\frac{1}{n}\right)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{3i}{\left(1+\frac{1}{n}\right)^n} \right|$$

As we Know that

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

$$= \left|\frac{3i}{e}\right| \qquad |3i| = 3 \quad |e| = e$$

$$= \frac{3}{e} > 1$$

Therefore by ratio test it is a divergent Series.

(24) $\sum_{n=1}^{\infty} n^2\left(\frac{i}{2}\right)^n$ (By Ratio Test)

Let $z_n = n^2\left(\frac{i}{2}\right)^n$.

$|z_n| = \left| n^2\left(\frac{i}{2}\right)^n \right|$

$|z_{n+1}| = \left|(n+1)^2\left(\frac{i}{2}\right)^{n+1}\right|$

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2\left(\frac{i}{2}\right)^{n+1}}{n^2\left(\frac{i}{2}\right)^n}\right|$$

$$\boxed{= \lim_{n\to\infty} \cancel{(n)^2 (1)^2} \left(\frac{i}{2}\right)^n\left(\frac{i}{2}\right)^1}$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^2 \cancel{\left(\frac{i}{2}\right)^n}\left(\frac{i}{2}\right)^1}{n^2\cancel{\left(\frac{i}{2}\right)^n}}\right| \qquad \left|\left(\frac{i}{2}\right)^1\right| = \frac{1}{2}$$

$$= \frac{1}{2}\lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}\right|$$

$$= \frac{1}{2}\lim_{n\to\infty}\left|\frac{n^2+2n+1}{n^2}\right|$$

$$= \frac{1}{2}\lim_{n\to\infty}\left|\frac{n^2}{n^2}+\frac{2n}{n^2}+\frac{1}{n^2}\right|$$

$$= \frac{1}{2}\lim_{n\to\infty}\left|1+\frac{2}{n}+\frac{1}{n^2}\right|$$

$$\ln \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{d}{dn}\ln n}{\frac{d}{dn} n/2} = \lim_{n\to\infty} \frac{1/n}{1/2}$$

$$\therefore \lim_{n\to\infty} a_n = e^0 = 1$$

$$\Rightarrow \lim_{n\to\infty} \cdot \sqrt[n]{|z_n|} = \frac{1}{2} < 1$$

By root test, given series Converges.

H/W

$$\sum_{n=0}^{\infty} \frac{(20+30i)^n}{n!}$$

Sol: Let $z_n = \left(\frac{20+30i}{n!}\right)^n$

$$z_{n+1} = \frac{(20+30i)^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = \left| \frac{\frac{(20+30i)^{n+1}}{(n+1)!}}{\frac{(20+30i)^n}{n!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(20+30i)^{n+1}(n!)}{(20+30i)^n (n+1)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(20+30i)^n} \cdot (20+30i)^1 \cancel{n!}}{\cancel{(20+30i)^n}(n+1)\cancel{n!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{20+30i}{n+1} \right|$$

$$= 0 < 1$$

Thus by ratio test it is convergent