Integration:

Approximate the area under the curve by adding up right, left or mid-point rectangles

Definite Integral:

To find an exact area you need to use a definite integral

e.g., $$\int_0^1 x\,dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}(1-0) = \frac{1}{2}$$

Let w(t) be a "Complex Valued Function" of a real variable

$$w(t) = u(t) + i\,v(t)$$

$$U(t), V(t) \in \Re$$

then

$$\int_a^b w(t)\,dt = \int_a^b u\,dt + i\int_a^b v\,dt$$

Property:

$$\int_a^b w(t)\,dt = \int_a^c w(t)\,dt + \int_c^b w(t)\,dt$$

where $a < c < b$

Churchill. p:115

Q 2 (a) $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt = ?$

$\because \int \frac{1}{t} dt = \ln(t)$

$$\int_1^2 \frac{1}{t^2} - 1 - \frac{2i}{t}\, dt$$

$$= -\frac{1}{t}\Big|_1^2 - t\Big|_1^2 - 2i \ln(t)\Big|_1^2$$

$$= -\left(\frac{1}{2} - 1\right) - (2-1) - 2i(\ln(2) - \underbrace{\ln(1)}_{0})$$

or $\ln\left(\frac{2}{1}\right)$

$$= \frac{1}{2} - 1 - 2i\ln(2)$$

$$\boxed{= -\frac{1}{2} - i\ln(4)}$$

$\leftarrow \ln(x^n) = n\ln x$

Q 3 p:115 churchill.

Show that if m and n are integers then

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta}\, d\theta = \begin{cases} 0 & m \neq n \\ 2\pi & m = n \end{cases}$$

Solution

$$\int_0^{2\pi} e^{i(m-n)\theta}\, d\theta$$

When $m = n$

$$\int_0^{2\pi} e^{i(m-n)\theta}\, d\theta = \int_0^{2\pi} d\theta = \theta\Big|_0^{2\pi} = 2\pi$$

when $m \neq n$

$$\int_0^{2\pi} e^{i(m-n)\theta}\, d\theta = \frac{e^{i(m-n)\theta}}{i(m-n)}\Bigg|_0^{2\pi} = \frac{1}{i(m-n)}\left[e^{i(m-n)2\pi} - 1\right]$$

$$= \frac{1}{i(m-n)}\left[1-1\right] = 0$$

Contours :→ A set of pts. $z=(x,y)$ in the complex plane is said to be an "arc" if

$$x = x(t),\ y = y(t) \qquad (a \le t \le b)$$

where $x(t)$ & $y(t)$ are continuous of $t \in \mathbb{R}$

→ The definition establishes a continous mapping of the interval $a \le t \le b$ into the z-plane.

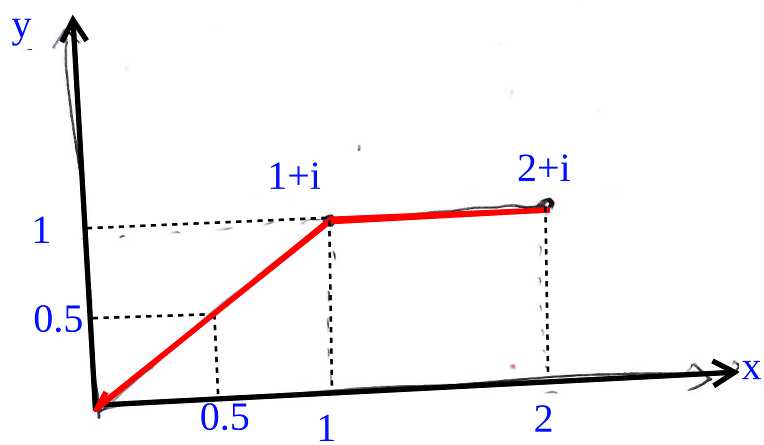
$$z(t) = x(t) + iy(t)$$

⇒ The arc C is a "simple arc" or a "jordan arc" if it doesn't cross itself i.e., C is simple if $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$.

Example 1: P:117 Chirchill

Plot the following polygonal line.

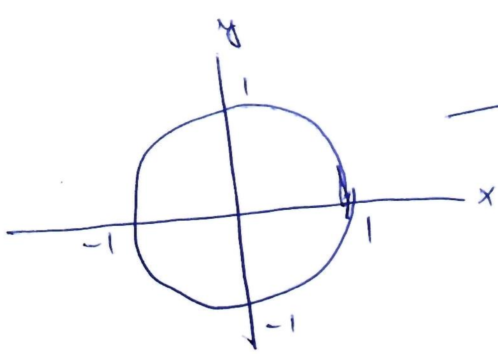
$$z = \begin{cases} x + ix & \text{when } 0 \le x < 1 \\ x + i & \text{''} \quad 1 \le x \le 2 \end{cases}$$



This is a simple arc.

Example 2: P:117, Chirchill.

Plot $z = 1.e^{i\theta}$ $\quad 0 \le \theta \le 2\pi$



This is a simple closed curve.

Now what about $z = z_0 + Re^{i\theta}$ $\quad 0 \le \theta \le 2\pi$?

Exaple 3, P: 118, Churchill.

$0 \le \theta \le 2\pi$ z = $e^{i\theta}$ & $z = e^{-i\theta}$ are these the same curves?

The pts are the same but in $z = e^{i\theta}$ the circle is traversed in CCW direction & in $z = e^{-i\theta}$ it is traversed in the CW direction.

what about $z = e^{i2\theta}$ $0 \le \theta \le 2\pi$?

Here the pt. are again similar to $e^{i\theta}$ and $e^{-i\theta}$ but this arc differs from other two is the circle in traversed twice in CCW direction

COMPLEX INTEGRATION

What is complex integration?

What do we do in that?

Consider

$$\int_{x_1}^{x_2} f(x)\,dx \quad \text{where } x_1, x_2, x \in \mathbb{R}$$

but in complex integration.

$$\int_{z_1}^{z_2} f(z)\,dz \quad \text{where } z_1, z_2, z \in \mathbb{C}$$

(1)

$$= \int_{z_1}^{z_2} f(x+iy)\,d(x+iy)$$

$u(x,y)$ & $v(x,y)$

$dz = dx + i\,dy$

$f(z) = u + iv$

Now integration is with respect to z and z = x + iy

-- We can convert the whole (1) into x or y but what will be the limits then?

→ So we have to convert $\int_{z_1}^{z_2} f(z)\,dz$ either into x (limits will be for x) or y (limits will be for y)

=> Now question arises that "how to convert x into y or y into x?

→ Whenever there will be a problem of complex integration there will be an accompanied relationship link which can be used for x & y interconversion.

→ This relationship link can be a line, curve, circle etc. (e.g find $\int_a^b f(z)dz$ along (1) line $y=2x$ (2) curve (3) circle etc)

→ Line integral of real nos. $\int_a^b f(x)dx$

→ Contour Integral :→ starting pt ending pt.

If a and b coincide i.e., starting pt = ending pt.

then the integral will be contour integral.

Example

(1) $\int_0^{1+i} (x^2 - iy)\,dz = ?$ $\boxed{\text{along the line } y = x}$

(2) $\int_C f(z)\,dz = ?$, $f(z) = y - x - 3x^2 j$ along two line segments $z = 0$ to $z = j$ and $z = j$ to $z = 1 + j$

Solution :→ 1

$f(z) = x^2 - iy$

we know $y = x$

$\boxed{dy = dx}$, $dz = dx + i\,dy$
$= dx + j\,dx$

$\boxed{dz = (1+i)\,dx}$

limits ?

lower $z = (0,0) \Rightarrow x \to 0$

upper $z = (1,1) \Rightarrow x \to 1$

limit is from 0 to 1

$$\int_0^1 (x^2 - ix)(1+i)\,dx$$

↳ Now this is in one variable.

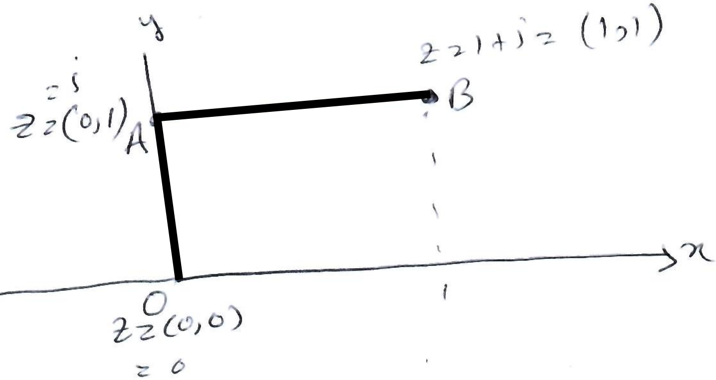
$$2) \quad (1+i)\int_0^1 (x^2 - ix)\,dx$$

$$= (1+i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1$$

$$= (1+i)\left\{\frac{1}{3} - \frac{i}{2}\right]$$

$$= \frac{5}{6} - \frac{i}{6}$$

2



So $\int_C$ means integrate along curve.

the conversion

$z = (0, 0)$, $z = (0, 1)$

$\underbrace{x}_{\text{Same so}}$ $\underbrace{y}_{y \to 0 \text{ to } 1}$

$x = 0$

$dx = 0$

$dz = \cancel{dx} + i\,dy$

$= i\,dy$

So

$f(z) = y - x - 3x^2 i$

$= y \quad \because x = 0$

$$\int_0^1 y.(i\,dy) = i \left. \frac{y^2}{2} \right|_0^1 = i\frac{1}{2}$$

Now along line AB.

$z = i = (0, 1)$, $z = 1+i = (1, 1)$

$\underbrace{x}_{x \to 0 \text{ to } 1}$ $\underbrace{y}_{\text{Same}}$ $y = 1$

$dy = 0$

$f(z) = y - x - 3x^2 i = 1 - x - 3x^2 i$

Also

$$dz = dx + i\,dy$$
$$= dx$$

So

$$\int_0^1 \left(1 - x - 3x^2 i\right) dx$$

$$= x\Big|_0^1 - \frac{x^2}{2}\Big|_0^1 - i\,\frac{\cancel{3}x^3}{\cancel{3}}\Big|_0^1$$

$$= 1 - \frac{1}{2} - i$$

$$= \frac{1}{2} - i$$

→ If both the x dy in both the limits changes (e.g part ① (0,0) & (1,1)) then we'll need a line but if one changes and the other is fixed (e.g in this part (0,0) & (0,1)) then fix the fixed and solve the integration along the other variable.

Example ① $\int_L \frac{z+2}{z}\, dz = ?$

L is semi circle

$z = 2e^{i\theta}$ & $0 \le \theta \le \pi$

② $\int_L \frac{1}{z-a}\, dz = ?$ when L is circle $|z-a| = r$

*③ $\int_C (z^2 + 3z)\, dz = ?$ along circle $|z| = 2$

from $(2,0)$ to $(0,2)$

$|z| = 2 \Rightarrow \boxed{x^2 + y^2 = 2^2} \rightarrow$ equ of circle.

Solution ① In this problem each and every thing is given i.e z is in θ, & limits are in θ.

$z = 2e^{i\theta} \Rightarrow dz = 2 \cdot e^{i\theta} \cdot i\, d\theta$

$$\int_0^{\pi} \frac{2e^{i\theta} + 2}{\cancel{2e^{i\theta}}} \cancel{2e^{i\theta}} \cdot i\, d\theta = 2i \int_0^{\pi} (e^{i\theta} + 1)\, d\theta$$

$$= 2i\left[\frac{e^{i\theta}}{i} + \theta\right]_0^{\pi}$$

$$= 2i\left[\frac{e^{i\pi}}{i} - \frac{1}{i} + \pi\right.$$

$$= 2i\left[\frac{-1}{i} - \frac{1}{i} + \pi\right]$$

$$= 2i\left(-\frac{2}{i} + \pi\right)$$

$$= -4 + 2\pi i$$

(2) put $z - a = re^{i\theta}$

$$z = a + re^{i\theta}$$

$$\boxed{dz = 0 + r \cdot e^{i\theta} \cdot i\, d\theta}$$

No limits are given but since it is about a circle $|z-a| = r$ so limit is 0 to 2π

$$\int_0^{2\pi} \frac{1}{\cancel{re^{i\theta}}} \cdot \cancel{r \cdot e^{i\theta}} \cdot i\,d\theta = i\int_0^{2\pi} d\theta = i2\pi$$

(3) $|z| = 2$

$$z = r e^{i\theta} = 2e^{i\theta}$$

Now the limits are given.

$(2,0)$ to $(0,2)$

both x & y vary so we've to find a relationship.

$$\tan\theta = \frac{y}{x}$$

for $(2,0)$, $\tan\theta = \frac{0}{2} = 0$

$\theta = 0$

for $(0,2)$

$$\tan\theta = \frac{2}{0} = \infty \Rightarrow \theta = \pi/2.$$

$$\int_0^{\pi/2} \left[(2e^{i\theta})^2 + 3(2e^{i\theta})\right] 2e^{i\theta} i \, d\theta$$

$$= -\frac{44}{3} + \frac{8i}{3}$$

Example :-> Evaluate $\int_0^{2+i} (\bar{z})^2 dz$ along

(1) the line $y = \frac{x}{2}$

(2) the real axis upto 2 and then vertically upto $2+i$

Solution :-> (1) $f(z) = \bar{z}^2$

Let $z = x+iy$ then $z^2 = x^2 - y^2 + 2ixy$

and

$$\bar{z}^2 = x^2 - y^2 - 2ixy$$

Now given that $y = \frac{x}{2} \Rightarrow x = 2y$

and $dx = 2dy$

then

$$dz = dx + idy = 2dy + idy$$

$$= (2+i)dy$$

and

$$f(z) = 4y^2 - y^2 - 4iy^2 = (3-4i)y^2$$

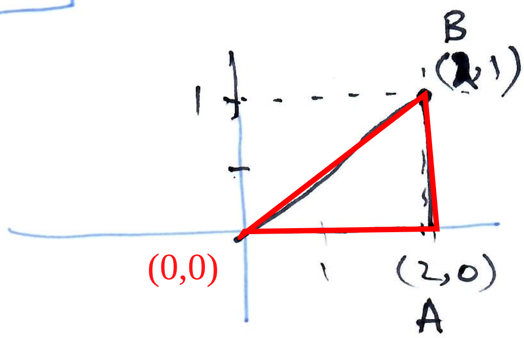
Limits are from 0 to 1 $\because$ we converted into y. If we convert into x then it will be 0 to 2

$$\int_0^{2+i} (\bar{z})^2 dz = \int_0^1 (3-4i)y^2 \, (2+i)\, dy$$

$$= (3-4i)(2+i)\int_0^1 y^2 dy$$

$$= (6+3i-8i+4)\left.\frac{y^3}{3}\right|_0^1$$

$$\boxed{= \frac{(10-5i)}{3}} = \frac{5}{3}(2-i)$$

② 

$$\int_0^{2+i} (\bar{z})^2 dz = \int_{OA} (\bar{z})^2 dz + \int_{AB} (\bar{z})^2 dz$$

$$= \int_{OA} (x^2-y^2-2ixy)dz + \int_{AB} (x^2-y^2-2ixy)dz$$

$$= \int_{(0,0)}^{(2,0)} (x^2-y^2-2ixy)dz + \int_{(2,0)}^{(2,1)} (x^2-y^2-2ixy)dz$$

$\longrightarrow$ ①

On the line OA, $y=0 \Rightarrow dy=0$
and x varies from 0 to 2

So
$$\int_{(0,0)}^{(2,0)} (x^2-y^2-2ixy)(dx+idy)$$

$$= \int_0^2 x^2 dx \longrightarrow (2)$$ $\because dy=0$ and $y=0$

On the line AB $x=2 \Rightarrow dx=0$
and y varies from 0 to 1

So
$$\int_{(2,0)}^{(2,1)} (x^2-y^2-2ixy)(dx+idy)$$

$$= \int_0^1 (4-y^2-4iy)\, idy \longrightarrow (3)$$

put (2) & (3) in (1)

$$\int_0^{2+i} (\bar{z})^2 dz = \int_0^2 x^2 dx + i\int_0^1 (4-y^2-4iy)\, dy$$

$$= \frac{x^3}{3}\Big|_0^2 + i\left[4[y]_0^1 - \frac{y^3}{3}\Big|_0^1 - 4i\left[\frac{y^2}{2}\right]_0^1\right]$$

$$= \frac{8}{3} + i\left[4 - \frac{1}{3} - 2i\right]$$

$$= \frac{8}{3} + i\left[\frac{11}{3} - 2i\right]$$

$$= \frac{8}{3} + \frac{11i}{3} + 2$$

$$= \frac{14}{3} + \frac{11i}{3} = \frac{1}{3}\left[14 + 11i\right]$$

Question :-> Evaluate $\int_{1-i}^{2+i} (2x + iy + 1)\, dz$ along the path (1) **x = t + 1 , y = 2t² -1 ?**

(2) **The straight line joining 1 - i and 2 + i ?**

Solution :-> (1) **Do you see any explicit link between x and y ?**

Ans:- No, but we have them in parametric term in the form of t.

=> So we have to convert the whole thing into t

As $dz = dx + idy$ ----- (1)

and

$x = t + 1$

so

$dx = dt$ ------- (2)

$y = 2t^2 - 1$ **so**

$dy = 4t\, dt$ ----------(3)

put (2) and (3) in (1)

$dz = dt + i.\, 4t.\, dt$

$dz = (1 - 4\, t\, i)dt$ ------ (4)

$$f(z) = 2x + iy + 1$$

$$= 2(t+1) + i(2t^2 - 1) + 1$$

$$= 2t + 2 + i2t^2 - i + 1$$

$$= 2t + 3 + i(2t^2 - 1) \longrightarrow (5)$$

$$\int_{1-i}^{2+i} (2x + iy + 1)dz = \int_{?}^{?} 2t + 3 + i(2t^2 - 1)\,(1 + i4t)dt$$

Now the limits will be with. γ. + t

So how will we calculate them

Since $x = t+1$ is given and the actual limit is

$(1, -1)$ to $(2, 1)$

i.e., x varies from 1 to 2

So when $x = 1$ **because x = t + 1**

$t = 0$, when $x = 2$, $t = 1$

So t will vary from 0 to 1

$\Rightarrow$ same limit will occur if we use y.

i.e., $y = 2t^2 - 1$

and y varies from -1 to 1

For $y = -1$

$2t^2 = 0 \Rightarrow t^2 = 0 \Rightarrow t = 0$

$y = 1$

$2t^2 = 2 \Rightarrow t^2 = 1 \Rightarrow t = 1$

So

$$\int_{1-i}^{2+i} (2x + iy + 1)\, dz = \int_0^1 \left(2t + 3 + i(2t^2 - 1)\right)(1 + 4it)\, dt$$

$$= \int_0 \underbrace{\left(2t+3+i\left(2t^2-1\right)\right)}_{I} \underbrace{\left(1+4ti\right)}_{II}\, dt$$

$$= I\int II - \int\int II \; \frac{d}{dt} I$$

$$= \left(2t+3+i\left(2t^2-1\right)\right)\int_0^1 (1+4ti)\,dt - \int_0^1 \int (1+4ti)\,dt \; \frac{d}{dt}\left(2t+3+i(2t^2-1)\right)$$

$$= \left(2t+3+i\left(2t^2-1\right)\right)\left[\left(1+2i\right) - \int_0^1 \left(1+2t^2 i\right)\left(2+0+4ti\right) dt\right.$$

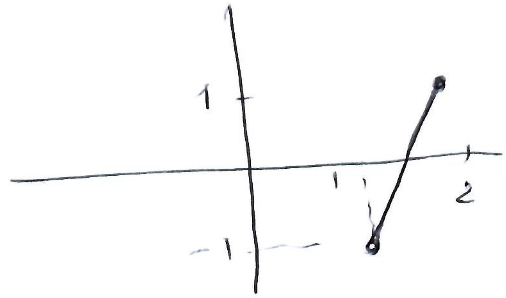
$$= \left(2t+3+i\left(2t^2-1\right)\right)\left(\left(1+2\cancel{t}\right)\right. \left[2+4ti \int_0^1 \left(1+2t^2 i\right) dt - \int_0^1 \int \left(1+2t^2 i\right) dt \times 4i\, dt\right.$$

$$= \left(2t + 3 + i\left(2t^2 - 1\right)\right)\left[\left(1+2i\right)\left[\left(2+4ti\right)\left[1+\frac{2t}{3}i - \right.\right.\right.$$

$$= 4 + \frac{25}{3}i$$

(2) Now we've to find the eqn: of line joining $1-i$ to $2+i$

$(1,-1)$, $(2,1)$
(x_1, y_1) (x_2, y_2)



$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

$$\frac{x - 1}{2 - 1} = \frac{y + 1}{1 + 1}$$

$$\frac{x - 1}{y + 1} = \frac{1}{2} \Rightarrow 2x - 2 = y + 1$$

$$2x = y + 3$$

$$\boxed{y = 2x - 3}$$

we can check this line.

e.g., for $x = 1$, $y = -1$ i.e., pt. $(1,-1)$

for $x = 2$ $y = 1$ i.e., pt; $(2,1)$

so the line is correct.

So $y = 2x - 3 \Rightarrow \boxed{dy = 2dx}$

hence $dz = dx + i\,dy$ = **dx + i. 2dx = (1 + 2i)dx**

$$\int_{1-i}^{2+i} (2x+iy+1)\,dz = \int_{1}^{2} \big(2x + i(2x-3)+1\big)(dx+2i\,dx)$$

$$= (1+2i)\int_{1}^{2} \big((1+i)2x - 3i + 1\big)\,dx$$

$$= (1+2i)\left[(1+i)\int_{1}^{2} 2x\,dx - 3i\int_{1}^{2} dx + \int_{1}^{2} dx\right]$$

$$= 1+2i\left[(1+i)[4-1] - 3i(2-1) + (2-1)\right]$$

$$= (1+2i)\left[3+\cancel{3i} - \cancel{3i} + 1\right]$$

$$= (1+2i)\left[4\right]$$

$$\boxed{= 4+8i}$$

Some Theory :→

let $z = z(t)$ ① with $a \leq t \leq b$

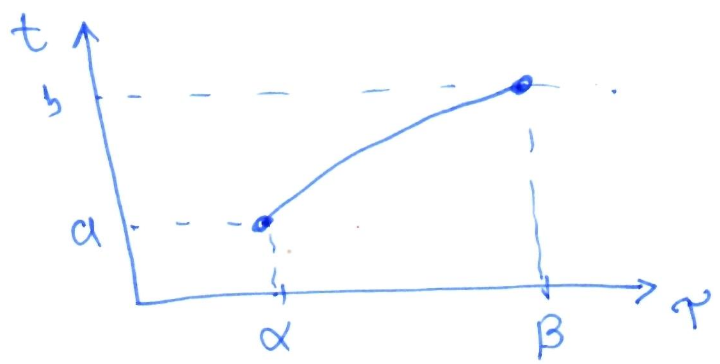
then the parametric representation of any given arc is not unique. It is, in fact, possible to change the interval over which the parameter ranges to any other interval.

Parametric Rep:?

i.e., let

$t = \phi(\tau)$ where $\alpha \leq \tau \leq \beta$ ②

Here ϕ is a real valued funⁿ mapping an interval $\alpha \leq \tau \leq \beta$ onto the interval $a \leq t \leq b$



⇒ Let us assume that $\phi(\tau)$ is differentiable and $\phi'(\tau) \geq 0$

what do we mean by that?

It is differentiable means it is continuous $\forall \tau$

and $\phi'(\tau) \geq 0$ means $t \uparrow$ when $\tau \uparrow$

$\Rightarrow$ With this let us transform ① as

$$z = Z(\tau) \rightarrow \quad (\alpha \leq \tau \leq \beta)$$

where $Z(\tau) = z(\phi(\tau)) \rightarrow ③$

$\Rightarrow$ Now suppose

$$z(t) = x(t) + iy(t) \qquad a \leq t \leq b$$

↳ we know this is an arc say C

then $\quad z'(t) = x'(t) + iy'(t)$

Let $x'(t)$ & $y'(t)$ are continuous over the interval $[a \; b]$ then the arc is called differentiable arc and the real valued fun"

$$|z'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$$

is integrable over the interval $a \leq t \leq b$

$\Rightarrow$ Length of an arc $\quad L = \int_a^b |z'(t)|\, dt \rightarrow ④$

Now the change of variable in ② & ③ makes ① as.

$$L = \int_{\alpha}^{\beta} |z'(\phi(\tau))| \, \phi'(\tau) \, d\tau$$

$$= \int_{\alpha}^{\beta} |Z'(\tau)| \, d\tau \qquad \because Z'(\tau) = z'(\phi(\tau))\phi'(\tau)$$

Exercises, Page: 120, Churchill.

Q2 Let C denote the right hand half of the circle $|z| = 2$ in CCW direction which has two parametric representations:-

(1) $z = z(\theta) = 2e^{i\theta}, \quad (-\pi/2 \le \theta \le \pi/2)$

(2) $z = Z(y) = \sqrt{4 - y^2} + iy, \quad (-2 \le y \le 2)$

Ⓐ Find $\phi(y)$?

$$Z(y) \triangleq z(\phi(y))$$

$\triangleq$ => **by definition**

$$\sqrt{4-y^2} + iy = 2e^{i\phi(y)}$$

$$\sqrt{4-\cancel{y^2}+\cancel{y^2}}\; e^{i\tan^{-1}\left(y/\sqrt{4-y^2}\right)} = \cancel{2}\, e^{i\phi(y)}$$

$$\boxed{\phi(y) = \tan^{-1}\left(\frac{y}{\sqrt{4-y^2}}\right)} \qquad \left(-\pi/2 \le \phi(y) \le \pi/2\right)$$

(B) Show that $\phi'(y) \geq 0$

Cauchy's Theorem

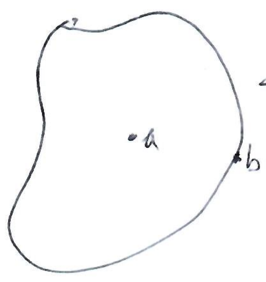
Suppose f is analytic at each and every point IN and ON the boundary of a domain D and f ' is continuous in D, then for a simple closed curve C \in D...

$$\oint_C f(z)dz = 0$$

→ Goursat eliminated the need for continuity of f'.

Cauchy-Goursat theorem → Suppose that a funⁿ f is analytic in a simply connected Domain D, then for every simple closed contour C in D

$$\oint_C f(z)dz = 0$$



→ No singular pts should exist in the closed curve or on the boundary.

→ But let us suppose there is a pt. within or on the boundary of the closed curve then what will be the value of integral? This is answered by Cauchy integral formula.

Cauchy Integral formula :→

If $f(z)$ be an analytic fun" within and on the boundary of the closed curve C, and $a \in C$ is pt then. (a is inside C)

$$\int_C \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a)$$

or we can write

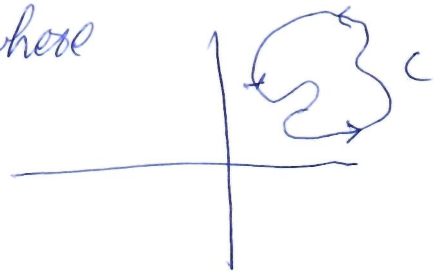
$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz$$

⇒ The value of an analytic fun" f at any pt. z_0 in a simply connected domain, can be represented by a contour integral.

Application of Cauchy-Goursat theorem

Q 1 (Example 1 Zill, Page 220)

Evaluate $\oint_C e^z dz$ where



Solution :→ Since e^z is an entire funⁿ and consequently analytic at all pts in $\mathbb{C}$

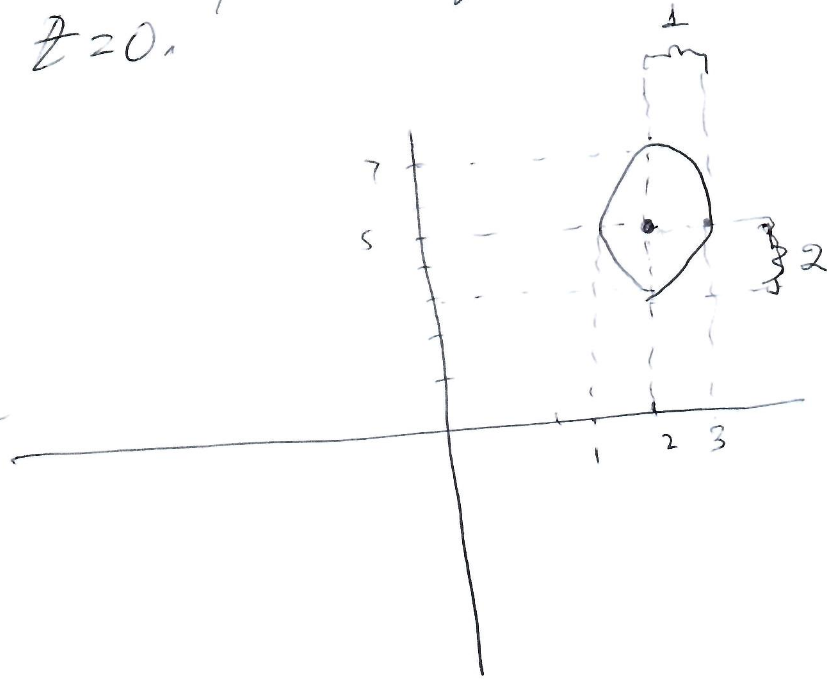
hence by C-G theorem

$$\oint_C e^z dz = 0$$

Q 2 (Example 2, Zill, Page 220)

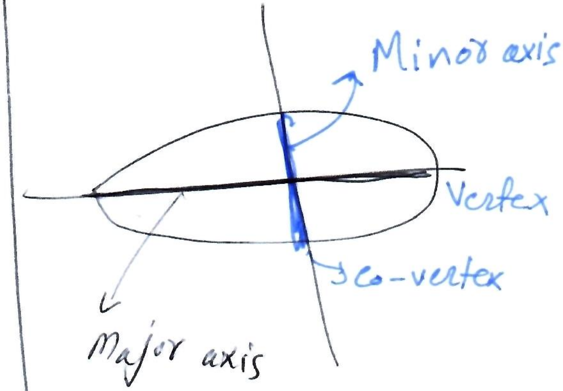
Evaluate $\oint_C \frac{dz}{z^2}$ =? where the contour C is the ellipse $(x-2)^2 + \frac{1}{4}(y-5)^2 = 1$

Solution :→ The fun^n $f(z) = \frac{1}{z^2}$ is analytic every where except $z = 0$.



but see the ellipse $z = 0$ is not part of C. i.e $f(z) = \frac{1}{z^2}$ is analytic every where in C so

$$\oint_C \frac{1}{z^2}\, dz = 0$$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$a > b$ major axis along x-axis and vise versa.

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1$$

ellipse will be centered at (x_0, y_0) instead of $(0, 0)$.

Q 4 (Excercise 5.3) Zill, Page: 224

C is a unit circle $|z| = 1$ then show that $\oint_C f(z)\, dz = 0$

① $f(z) = z^3 - 1 + 3i$

Since $f(z)$ is an entire funⁿ so it is consequently analytic in $|z| = 1$

Hence

$$\oint_C (z^3 - 1 + 3i)\,dz = 0$$

⑤ $$f(z) = \frac{\sin(z)}{(z^2-25)(z^2+9)}$$

$f(z)$ has singularity at ± 5 and $\pm 3i$

Except these it is analytic everywhere and these pts are not included in the unit circle hence

$$\oint_C f(z)\,dz = 0$$

Antiderivatives :→ Suppose a fun" is continuous on a domain D. If $\exists$ a fun" F s.t

$F'(z) = f(z) \quad \forall z \in D$ then F is called the antiderivative of f.

e.g $F(z) = -\cos z$ is the antiderivative of $f(z) = \sin(z)$

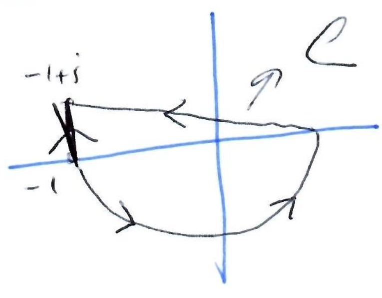
Fundamental theorem of Contour Integrals

Suppose that a fun" f is continuous on a domain D and F is its antiderivative then for any contour $C \in D$ with initial pt. z_0 and terminal pt. z_1

$$\int_C f(z)\,dz = F(z_1) - F(z_0)$$

Example 2 Zill; Page: 228

Evaluate $\int_C 2z\,dz$



Since $f(z) = 2z$ is an entire fun'
and hence it is continuous.

$$F(z) = z^2$$ => **Antiderivative of f(z) = 2z**

$$\because F'(z) = 2z = f(z)$$

So $$\int_{-1}^{-1+i} 2z\,dz = F(-1+i) - F(-1)$$

$$= (-1+i)^2 - 1$$

$$= \cancel{1} - \cancel{1} - 2i - 1$$

$$= -1 - 2i$$

Example 3, Zill, Page: 228

Evaluat $\int_C \cos z\,dz$, where C is any contour with initial pt. $z_0 = 0$ and terminal Pt. $z_1 = 2+i$?

Solution: -> $F(z) = \sin z$

$$\int_C \cos z\,dz = \int_0^{2+i} \cos z\,dz = \sin(2+i) - \sin(0)$$

⇒ Observe the fundamental theorem of contour integrals. If

① C is closed then $z_0 = z_1$ and hence.

$$\oint_C f(z)\,dz = 0$$

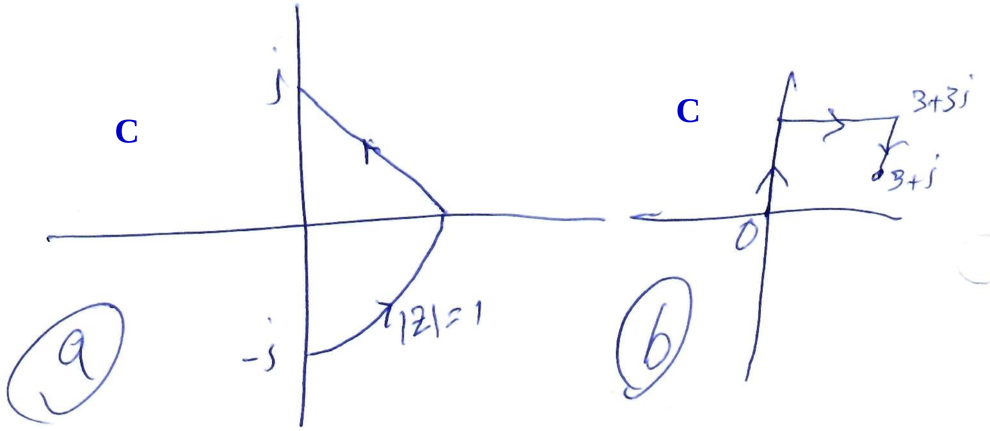
② If the continuous funⁿ f has an antiderivative $F \in D$ then $\int_C f(z)\,dz$ is independent of the path.

Theorem :⇒ (Existence of antiderivative)

Suppose that a funⁿ f is analytic in a simply connected domain D. Then f has an antiderivative in D, i.e., $\exists$ a funⁿ F s.t., $F'(z) = f(z)$ $\forall z \in D$

Exercises: 5.4 Page: 231, Zill

Q $\int_C (4z-1)\,dz = ?$



Solution

Since $f(z) = 4z-1$ is an entire fun^n so it is analytic everywhere.

$$F(z) = 2z^2 - z$$

$\because$ $F'(z) = 4z-1 = f(z)$

(a) $\int_{-i}^{i} (4(z)-1)\,dz \triangleq F(i) - F(-i)$

$= 2(i)^2 - i - \left(2(-i)^2 + i\right)$

$= -\cancel{2} - i + \cancel{2} - i = -2i$

(b)

$$\int_{(0,0)}^{(3,1)} (4z-1)\,dz \triangleq F(3+i) - F(0)$$

$$= 2(3+i)^2 - (3+i) - 0$$

$$= 2(9-1+6i) - 3 - i$$

$$= 16 + 12i - 3 - i$$

$$\boxed{= 13 + 11i}$$

Cauchy Integral Formula (CIF)

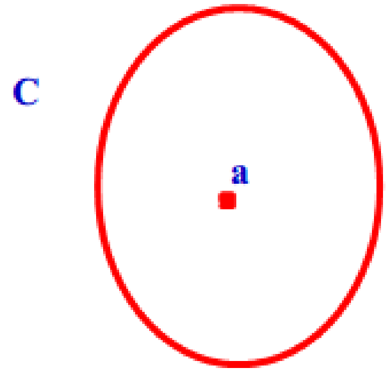
The CIF is a supplementary of Cauchy Theorems. They clearly say that the contour integral of a function f(z) is 0 if f(z) is analytic every where in the interior and on the boundary of contour C.

=> Now suppose there is a singular point in C!!!!!!

=> Then what will happen to the closed contour integral?

THIS IS ANSWERED BY CAUCHY INTEGRAL FORMULA

(Definition on page 121)



Suppose a function g(z) = f(z)/(z-a), where f(z) is analytic in C but clearly g(z) is not analytic in C since it has a singularity at "a" in C.

$$f(a) = f(z)|_{z=a}$$

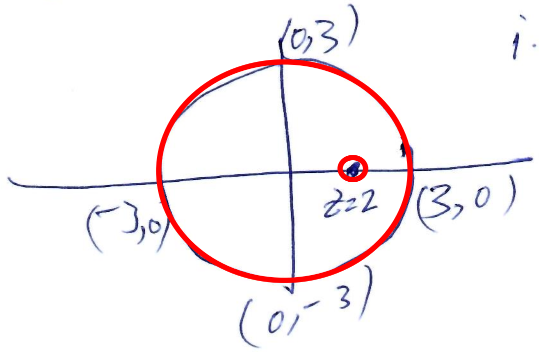
The contour C is a circle of radius 3 centered at the origin

Ex: $\int_C \frac{z^2}{z-2}\,dz = ?$ where $C: |z| = 3$

Solution

Here $g(z) = \frac{z^2}{z-2}$ with singular pt at $z=2$

and C



i.e $z=2$ is inside the closed curve

Here $f(z) = z^2$ which is an entire fun'' by Cauchy's theorem $\oint_C f(z)\,dz = 0$

and by CIF

$$\oint_C \frac{f(z)}{z-2}\,dz = \oint_C \frac{z^2}{z-2}\,dz = 2\pi i f(2)$$

$$= 2\pi i\,(2)^2 = 8\pi i$$

$\Rightarrow$ So in questions like this we've to find:

① Singular pts.

② Singular pts. that $\in$ C (interior)

③ $f(z)$

Remark:

$$\boxed{\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)} \rightarrow \text{CIF}$$

Differentiate w-r-t a.

$$\int_C \frac{f(z)}{(z-a)^2}(-1)(-1)\,dz = 2\pi i f'(a) = 2\pi i \left.\frac{df(z)}{dz}\right|_{z=a}$$

$$\int_C \frac{f(z)}{(z-a)^2}\,dz = 2\pi i\left[\frac{\partial f(z)}{\partial z}\bigg|_{z=a}\right]$$

Again differentiate w-r-t a.

$$\int_C \frac{f(z)}{(z-a)^3}(-2)(-1)\,dz = 2\pi i f''(a) = 2\pi i\left[\frac{\partial}{\partial z} f(z)\right|_{z=a}$$

$$2!\int \frac{f(z)}{(z-a)^3}\,dz = 2\pi i f''(a)$$

Similarly differentiating again

$$3!\int_C \frac{f(z)}{(z-a)^4}\,dz = 2\pi i f^{3}(a)$$

So in general.

$$\int_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(a) = \frac{2\pi i}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}} f(z)\right]_{z=a}$$

(Remember a is inside C)

Q1:→ Evaluate $\oint_C \frac{e^{2z}}{(z+1)^4} dz$ along $C: |z|=3$,

using CIF?

Q2:→ Evaluate using CIF

$\oint_C \frac{z^2+z}{(z-1)(z-2)} dz$ where $C: |z-1|=2$

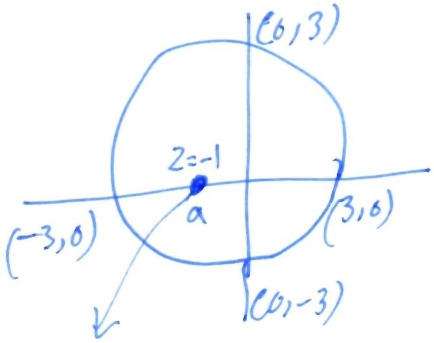
Circle of radius 2 centered at 1

Solution 1 :→ $g(z) = \frac{e^{2z}}{(z+1)^4}$

There are 4 singular pts at -1

① Singular pts of $g(z)$?:→ They are 4 all at -1

② Are singular pts in C?



funⁿ $g(z)$ is not analytic at $z=-1$

and -1 is inside C.

③ $f(z) = ?$

$$f(z) = e^{2z}$$ (which is an entire funⁿ)

So

$$\int_C \frac{e^{2z}}{(z-1)^4}\,dz = \frac{2\pi i}{3!}\left[\frac{\partial^3}{\partial z^3} e^{2z}\right|_{z=-1}$$

$$= \frac{2\pi i}{6}\left[2\frac{\partial^2}{\partial z^2} e^{2z}\right|_{z=-1}$$

$$= \frac{\pi i}{3}\left[8 e^{2z}\right|_{z=-1}$$

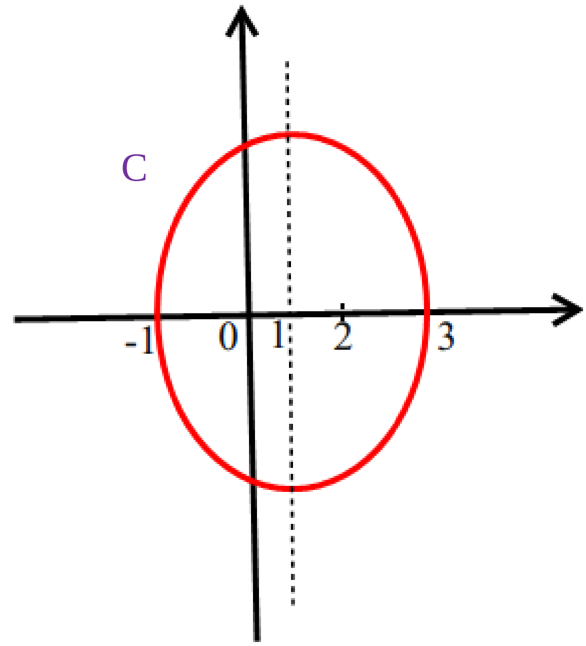
$$= \frac{8\pi i}{3e^2}$$

Solution 2 :→

$$g(z) = \frac{z^2+2}{(z-1)(z-2)}$$

① Singular pts are at 1 and 2

$|z - 1| = 2$ is a circle of radius 2 and centered at 1.



③ $f(z) = z^2 + 2$

By CIF

$$\int_C \frac{z^2+2}{(z-1)(z-2)}\, dz \Longrightarrow$$ Now here observe that we don't have any formula for two distinct singular pt.

So by partial fractions.

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$A = \left.\frac{1}{z-2}\right|_{z=1} = -1$$

$$B = \left.\frac{1}{z-1}\right|_{z=2} = 1$$

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

So

$$\int_C z^2+2 \left(\frac{1}{z-2} - \frac{1}{z-1}\right) dz$$

$$= \int_C \left(\frac{z^2+2}{z-2} - \frac{z^2+2}{z-1}\right) dz$$

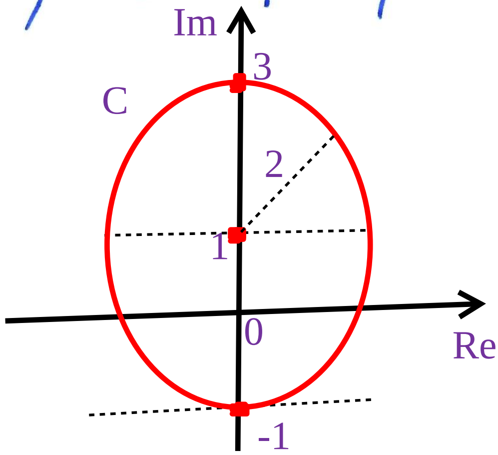
$$= \int_C \frac{z^2+2}{z-2}\, dz - \int_C \frac{z^2+2}{z-1}\, dz$$

$$= 2\pi i\, f(z)\Big|_{z=2} - 2\pi i\, f(z)\Big|_{z=1}$$

$$= 2\pi i\,(4+2) - 2\pi i\,(3)$$

$$= 12\pi i - 6\pi i = 6\pi i$$

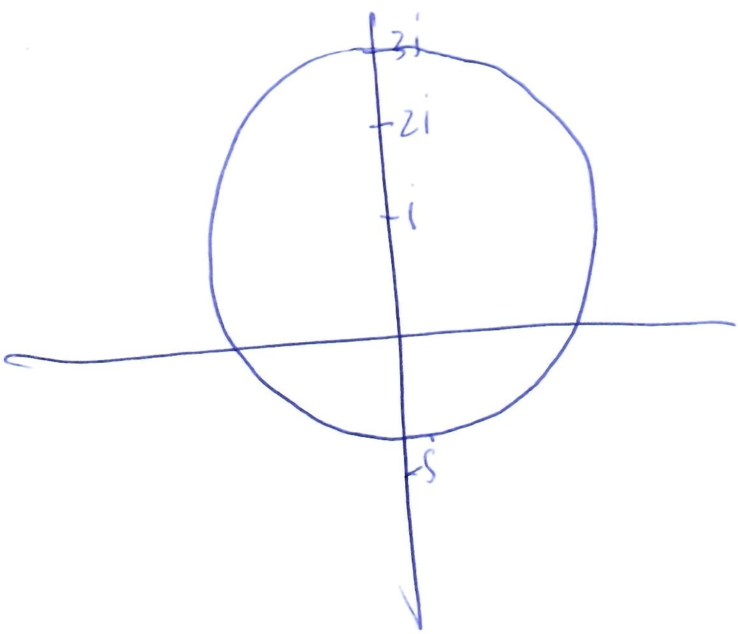
Q:⇒ Evaluate $\int_C \frac{z+1}{z^2+1} dz$, $C: |z-i|=2$



$$\int_C \frac{z+1}{z^2+1} dz = \int_C \frac{z+1}{(z+i)(z-i)}$$

singular pts at $\pm i$

includes both.



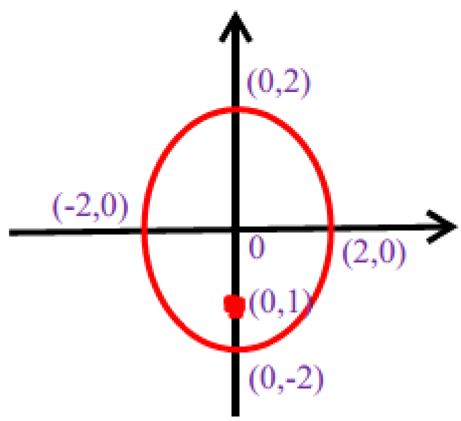
Example 1 Zill. Page: 234

Evaluate $\oint_C \frac{z^2-4z+4}{z+i}\,dz$ where C is $|z|=2$

Solution:-> $g(z) = \frac{z^2-4z+4}{z+i}$

(1) Singular point of g(z) is at z = -i.

(2)



Singular pt. is within the C

(3) $f(z) = z^2-4z+4$ is an entire fun"

hence it is analytic within & on the boundary of C.

Hence by CIF

$$\oint_C \frac{z^2-4z+4}{z+i}\,dz = 2\pi i f(-i) = 2\pi i(-1+4i+4)$$

$$= 2\pi(3i-4)$$

$$= \pi(-8+6i)$$

Example 2 :→ Zill Page 235

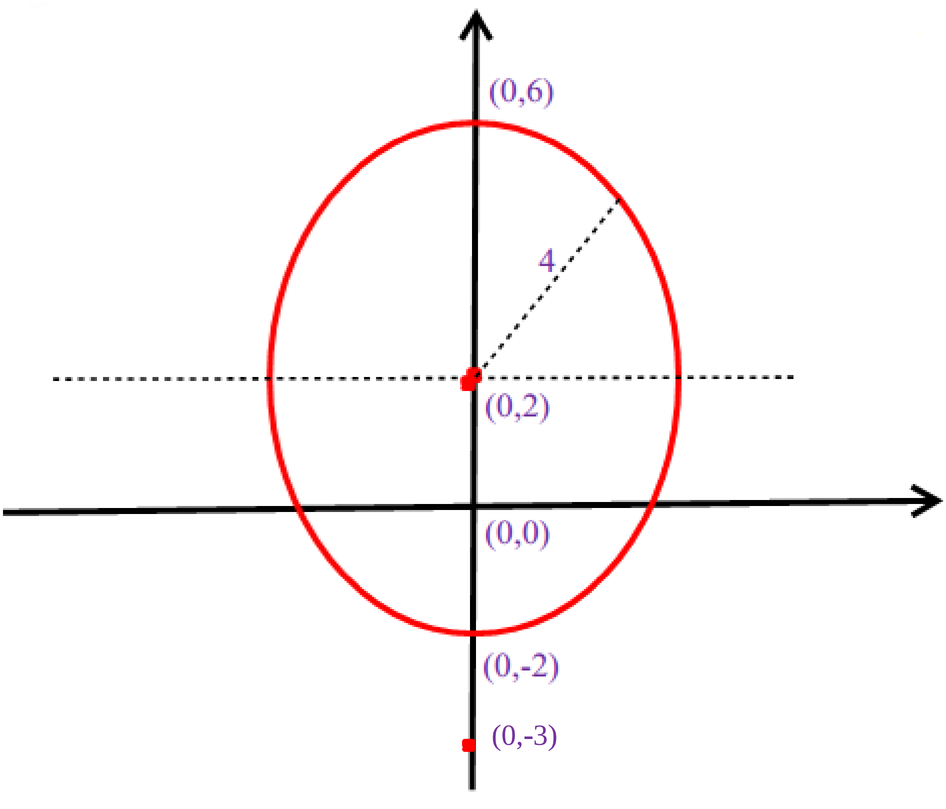
Evaluate $\oint_C \frac{z}{z^2+9}\,dz$, C : $|z-2i|=4$

Solution :→ $g(z) = \frac{z}{z^2+9}$

① Singular Pts. of $g(z)$ are at $\pm 3i$

$g(z) = z/(z+3i)(z-3i)$

②



The singular pt. $z=+3i$ lies within C while $z=-3i$ outside C.

③ Hence

$$f(z) = \frac{z}{z+3i}$$

i.e $$g(z) = \frac{z/_{z+3i}}{z-3i}$$ f(z)

f(z) is analytic everywhere in C

Hence by CIF

$$\oint_C \frac{z}{z^2+9}\,dz = \oint_C \frac{z/_{z+3i}}{z-3i}\,dz = 2\pi i\, f(3i)$$

$$= 2\pi i\left(\frac{3i}{3i+3i}\right) = 2\pi i\left(\frac{3i}{6i}\right) =$$

$$= 2\pi i\left(\frac{3i \times (-6i)}{36}\right) =$$

$$= 2\pi i\left(\frac{+18}{36}\right) = \cancel{2}\pi i\left(\frac{1}{\cancel{2}}\right) = \pi i$$

or

$$2\pi i\left(\frac{3i}{6i}\right) = 2\pi i\left(\frac{1}{2}\right) = \pi i$$