CHAPTER 16

EXAMPLE ①:

$$f(z) = z^{-5} \cdot \sin z$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n+1}}{(2n+1)!}$$

$$z^{-5} \sin z = z^{-5} \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n+1}}{(2n+1)!}$$

$$z^{-5} \sin z = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n-4}}{(2n+1)!}$$

EXAMPLE ②

$$f(z) = z^2 e^{1/z}$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Replace z by $1/z$ and multiply each term by z^2.

$$z^2 e^{1/z} = z^2 \cdot \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!}$$

$$z^2 \cdot e^{1/z} = z^2 \sum_{n=0}^{\infty} \frac{1}{z^n \, n!} = \sum_{n=0}^{\infty} \frac{z^2}{z^n \cdot n!}$$

$$z^2 \cdot e^{1/z} = \sum_{n=0}^{\infty} \frac{z^{2-n}}{n!}$$

EXAMPLE (3):

(a) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

(b) $$\frac{1}{1-z} = \frac{1}{-z(1-z^{-1})} = -\frac{1}{z} \cdot \frac{1}{(1-(1/z))}$$

$$\frac{1}{1-z} = -\frac{1}{z} \cdot \sum_{n=0}^{\infty} (1/z)^n = \boxed{-1\sum_{n=0}^{\infty} \frac{1}{z^{n+1}}}$$

EXAMPLE (4) $f(z) = \frac{1}{z^3 - z^4}$

(a) $$\frac{1}{z^3 - z^4} = \frac{1}{z^3(1-z)} = \frac{1}{z^3} \cdot \sum_{n=0}^{\infty} z^n$$

$$\boxed{\frac{1}{z^3 - z^4} = \sum_{n=0}^{\infty} z^{n-3}}$$

(b) $$\frac{1}{z^3(1-z)} = \frac{1}{z^3}\left(-\sum_{n=0}^{\infty} \frac{1}{z^{n+1}}\right)$$

$$= -\sum_{n=0}^{\infty} \cdot \frac{1}{z^{n+4}}$$

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EXAMPLE ⑤ :-

$$f(z) = \frac{-2z+3}{z^2-3z+2}$$

$$f(z) = \frac{-2z+3}{z^2-2z-z+2}$$

$$f(z) = \frac{-2z+3}{z(z-2)-1(z-2)}$$

$$f(z) = \frac{-2z+3}{(z-1)(z-2)}$$

By Partial fraction.

$$f(z) = \frac{-1}{z-1} - \frac{1}{z-2}$$

$$f(z) = \cancel{-}+\frac{1}{\cancel{-}(1-z)} - \frac{1}{z-2}$$

$$f(z) = \sum_{n=0}^{\infty} z^n + \frac{1}{\cancel{-}2\left(1-\frac{z}{2}\right)}$$

$$f(z) = \sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

$$f(z) = \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \Rightarrow f(z) = \sum_{n=0}^{\infty} \left(1+\frac{1}{2^{n+1}}\right) z^n$$

OR

$$f(z) = \frac{-1}{z-1} - \frac{1}{z-2}$$

$$f(z) = -\left(+ \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \right) + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$f(z) = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$f(z) \cancel{= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}} +$$

① $\frac{1}{z^4 - z^5}$

Ans:- $\frac{1}{z^4 - z^5} = \frac{1}{z^4 \cdot (1-z)} = \frac{1}{z^4} \sum_{n=0}^{\infty} z^n$

$$\frac{1}{z^4 - z^5} = \sum_{n=0}^{\infty} z^{n-4}$$

$$\frac{1}{z^4 - z^5} = \frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z^1 + z^2 \ldots$$

Now to find radius of convergence using Cauchy-Hadamard formula,

$R = 1$ for $\frac{1}{1-z}$

So region of convergence is $0 < R < 1$ for $\frac{1}{z^4 - z^5}$

② $z \cdot \cos 1/z$

$$f(z) = z \cdot \cos 1/z$$

For $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$

$\cos 1/z = \sum \frac{(-1)^n (1/z)^{2n}}{\ldots}$

$$\therefore \quad f(z) = z \cdot \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (1/z)^{2n}}{(2n)!}$$

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \, z/z^{2n}}{(2n)!}$$

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot z^{1-2n}}{(2n)!}$$

$R = \infty$ for $\cos(1/z)$

$\therefore \; 0 < |z| < \infty$ for $f(z)$

(14) $z^2 \cdot \sinh 1/z$

Ans:- Let $f(z) = z^2 \cdot \sinh 1/z$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

$$\therefore \; f(z) = z^2 \sinh 1/z = z^2 \sum_{n=0}^{\infty} \frac{(1/z)^{2n+1}}{(2n+1)!}$$

$$\quad = \sum_{n=0}^{\infty} \frac{\frac{z^2}{z^{2n+1}}}{(2n+1)!}$$

$$\quad = \sum_{n=0}^{\infty} \frac{z^{2-2n-1}}{(2n+1)!}$$

$$u = \sum_{n=0}^{\infty} \frac{z^{1-2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)! \cdot z^{2n+1}}$$

$R = \infty$ for $\sinh 1/z$

$\therefore 0 < |z| < \infty$ for $f(z)$

(13) $(z+16i)^4$

Ans: $z = 16i$

Order $= 4$

(14) $(z^4 - 16)^4$

$$z^4 - 16 = 0$$

$$\sqrt[4]{z^4} = \sqrt[4]{16}$$

Let $z_0 = 16$

$$z_k = r^{1/n}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i \sin\left(\frac{\theta + 2\pi k}{n}\right)\right)$$

$r = 16$, $\theta = 0^\circ$, $z_0 = (16)^{1/4}(\cos(0) + i\sin(0))$

$\boxed{z_0 = 2}$

$$z_1 = (16)^{1/4}\left(\cos\left(\frac{2\pi}{4}\right) + i\sin\left(\frac{2\pi}{4}\right)\right)$$

$$z_1 = 2(\cos(\pi/2) + i\sin(\pi/2))$$

$\boxed{z_0 = 2i}$

$$z_2 = (16)^{1/4}\left(\cos\left(\frac{4\pi}{4}\right) + i\sin\left(\frac{4\pi}{4}\right)\right)$$

$Z_2 = 2(-1 + i(0))$

$\boxed{Z_2 = -2}$

$Z_3 = 2\left(\cos\left(\frac{6\pi}{4}\right) + i\sin\left(\frac{6\pi}{4}\right)\right)$

$Z_3 = 2(0 + i(-1))$

$Z_3 = -2i$

Hence poles are $\pm 2, \pm 2i$

All have order 4.

(15) $z^{-3} \cdot \sin^3 \pi z$

Ans:- $f(z) = 0$

$\frac{\sin^3 \pi z}{z^3} = 0$

$\sin^3 \pi z = 0$

$\sin \pi z = 0$

$\cancel{\pi} z = n\cancel{\pi}$

$z = n$

$z = 0, \pm 1, \pm 2, \pm 3, \ldots$

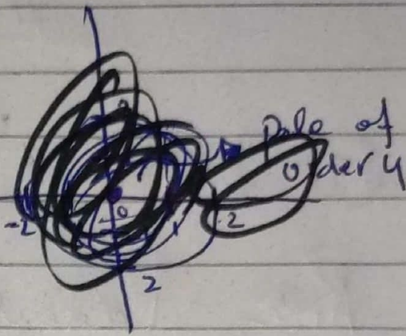
are all zeros.

Order 3

16.3

(14) $\oint \frac{\sin \pi z}{z^4}$ $\quad C: |z-i| = 2$

$z=0$ is the pole of order 4.



By Residue Integration,

$$\oint f(z)\,dz = 2\pi i b_1$$

$$b_1 = \frac{1}{(m-1)!} \lim_{z \to 0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m \cdot f(z) \right] \right\}$$

$$b_1 = \frac{1}{(4-1)!} \lim_{z \to 0} \left\{ \frac{d^3}{dz^3} \left[\cancel{(z-0)^4} \cdot \frac{\sin \pi z}{\cancel{z^4}} \right] \right\}$$

$$b_1 = \frac{1}{3!} \lim_{z \to 0} \left\{ \frac{d^3}{dz^3} (\sin \pi z) \right\}$$

$$b_1 = \frac{1}{3!} \lim_{z \to 0} \left(-\cos \pi z \cdot \pi^3 \right)$$

$$b_1 = \frac{1}{3!} (-1)\, \pi^3$$

$$\boxed{b_1 = -\pi^3}$$

$$\therefore \oint \frac{\sin \pi z}{z^4} = 2\pi i\left(-\frac{\pi^3}{\cancel{6}_3}\right)$$

$$// = -\frac{\pi^4 i}{3}$$

⑮ $f(z) = e^{1/z}$

Ans:- Laurent Series of e^z is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Now for $z = 1/z$

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n}$$

$$e^{1/z} = \frac{1}{1} + \frac{1}{1!\cdot z} + \frac{1}{2!\cdot z^2} + \frac{1}{3!\cdot z^3} + \cdots$$

$b_1 = 1$ is $\underset{z=z_0}{\text{Res}}\, f(z)$.

$$\therefore \oint e^{1/z} = 2\pi i(1)$$

$$= \boxed{2\pi i}$$

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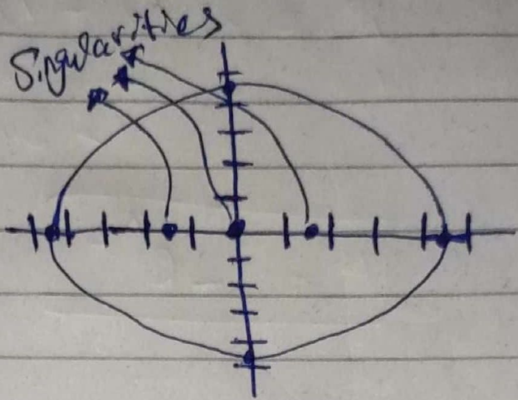
(19) $\oint_C \frac{e^z}{\cos z}\,dz$ $\qquad C: |z| = 4.5$



$\cos z = 0$

$z = (2n+1)\pi/2$

$z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \text{----}$

$z = \pm\pi/2$ are within circle hence they are simple poles of order 1

$\cancel{\underset{z=\pi/2}{\text{Res}}\, f(z) = \frac{P(z_0)}{q'(z_0)} = \frac{e^{\pi/2}}{-\sin \pi/2}}$

$$\underset{z=\pi/2}{\text{Res}}\, f(z) = \lim_{z\to\pi/2} (z-\pi/2)\cdot\frac{e^z}{\cos z}$$

Using L'Hopital rule to find limit.

$$\underset{z=\pi/2}{\text{Res}}\, f(z) = \lim_{z\to\pi/2} \frac{(z-\pi/2)\cdot e^z + (1)e^z}{-\sin z}$$

$$\text{''} = \frac{(\pi/2 - \pi/2) \cdot e^z + e^{\pi/2}}{-\sin \pi/2}$$

$$\boxed{\text{''} = -e^{\pi/2}}$$

$$\underset{z=-\pi/2}{\text{Res}}\ f(z) = \lim_{z \to -\pi/2} (z+\pi/2)\frac{(e^z)}{\cos z}$$

Again using L'Hopital rule

$$\underset{z=-\pi/2}{\text{Res}}\ f(z) = \lim_{z \to -\pi/2} \frac{(z+\pi/2) \cdot e^z + e^z}{-\sin z}$$

$$\underset{z=-\pi/2}{\text{Res}}\ f(z) = \frac{(0)e^{-\pi/2} + e^{-\pi/2}}{-\sin(-\pi/2)}$$

$$\underset{z=-\pi/2}{\text{Res}}\ f(z) = \frac{e^{-\pi/2}}{-(-1)} = \boxed{e^{-\pi/2}}$$

Now

$$\oint_c \frac{e^z}{\cos z} = 2\pi i \left[-e^{\pi/2} + e^{-\pi/2}\right] \times \frac{2}{2}$$

$$\text{''} = 4\pi i \left[\frac{-e^{+\pi/2} + e^{-\pi/2}}{2}\right]$$

$$// = 4\pi i - \left[\frac{e^{\pi/2} - e^{-\pi/2}}{2}\right]$$

$$// = \boxed{-4\pi i \cdot \sinh \pi/2}$$

(23) $\oint_c \frac{\tan \pi z}{z^3} dz \qquad c: |z + \frac{1}{2} i| = 1$

Ans: At $z = 0$ we have a pole of order 3

$$\underset{z=0}{\text{Res}}\, f(z) = \frac{1}{(3-1)!} \lim_{z \to 0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-0)^3 \cdot \frac{\tan \pi z}{z^3} \right] \right\}$$

$$// = \frac{1}{2!} \lim_{z \to 0} \left\{ \frac{d^2}{dz^2} \left[\cancel{z^3} \cdot \frac{\tan \pi z}{\cancel{z^3}} \right] \right\}$$

$$// = \frac{1}{2} \lim_{z \to 0} \cdot \frac{d}{dz} \left[\frac{d}{dz} \cdot \tan \pi z \right]$$

$$// = \frac{1}{\cancel{2}} \lim_{z \to 0} \cdot \cancel{2}\pi^2 \cdot \sec^2(\pi z) \cdot \tan(\pi z)$$

$$// = \pi^2 \cdot (1) \cdot (0)$$

$$// = 0$$

$b_1 = 0$

Hence $\oint_C \frac{\tan \pi z}{z^3} = 2\pi i (0)$

$= \boxed{0}$

(25) $\oint_C \frac{30z^2 - 23z + 5}{(2z-1)^2(3z-1)} dz$, $C : |z| = 1$

$f(z) = \frac{30z^2 - 23z + 5}{(2z-1)^2(3z-1)} = \frac{30z^2 - 23z + 5}{12(z-\frac{1}{2})^2(z-\frac{1}{3})}$

$\therefore$ $f(z)$ has a pole of order 2 at $z = 1/2$ and simple pole at $1/3$

$\underset{z=1/3}{\text{Res}} f(z) = \frac{30(1/3)^2 - 23(1/3) + 5}{2(2z-1)(2).(3)}$

$\underset{z=1/3}{\text{Res}} f(z) = \frac{\frac{10}{3} - 23/3 + 5}{12(\frac{2}{3}-1)} = \frac{\frac{13}{3} + \frac{5}{1}}{(\frac{2-3}{3})12}$

$\underset{z=1/3}{\text{Res}} f(z) = \frac{28/3}{-12/3} = \frac{28}{-12}$

$$\underset{z=-\frac{1}{3}}{\text{Res}}\ f(z) = \frac{P(z_0)}{Q'(z_0)} = \frac{30\left(\frac{1}{3}\right)^2 - 23\left(\frac{1}{3}\right) + 5}{12\left[\left(z-\frac{1}{2}\right)^2 + \left(z-\frac{1}{3}\right)\cdot 2\left(z-\frac{1}{2}\right)\right]}$$

$$'' = '' = \frac{\frac{\cancel{30}^{10}}{\cancel{9}\,3} - \frac{23}{3} + 5}{12\left[\left(\frac{1}{3}-\frac{1}{2}\right)^2 + \cancel{\left(\frac{1}{3}-\frac{1}{3}\right)}^{0}\cdot 2\left(\frac{1}{3}-\frac{1}{3}\right)\right]}$$

$$'' = \frac{\frac{10}{3} - \frac{23}{3} + 5}{12\left[\left(\frac{2-3}{6}\right)^2\right]}$$

$$'' = \frac{\frac{-13}{3} + \frac{5}{1}}{12\left(-\frac{1}{6}\right)^2} = \frac{\frac{-13+15}{3}}{\frac{12}{36}}$$

$$'' = \frac{2}{3} \div \frac{12}{36} = \frac{\cancel{2}^{1}}{\cancel{3}_{1}} \times \frac{\cancel{36}^{\cancel{6}\,2}}{\cancel{12}_{\cancel{6}\,1}}$$

$$'' = \boxed{2}$$

Now $\underset{z=\frac{1}{2}}{\text{Res}}\ f(z) = \frac{1}{(2-1)!} \cdot \lim_{z\to\frac{1}{2}} \left\{ \frac{d^{2-1}}{dz^{2-1}} \left(z-\frac{1}{2}\right)^2 \left(f(z)\right.\right.$

$$\operatorname*{Res}_{z=1/2} f(z) = \frac{1}{1!} \cdot \lim_{z \to 1/2} \frac{d}{dz}\left(\cancel{\left(z-\tfrac{1}{2}\right)^2} \frac{30z^2 - 23z + 5}{12\cancel{\left(z-\tfrac{1}{2}\right)^2}\left(z-\tfrac{1}{3}\right)} \right)$$

$$'' = \lim_{z \to 1/2} \frac{d}{dz}\left(\frac{30z^2 - 23z + 5}{12\left(z - \tfrac{1}{3}\right)} \right)$$

$$\cancel{'' = \lim_{z \to 1/2} \frac{60z - 23}{12}}$$

$$\cancel{'' = \frac{60\left(\tfrac{1}{2}\right) - 23}{12} = \frac{30-23}{12}}$$

$$\cancel{'' = \frac{7}{12}}$$

(1)

$$'' = \lim_{z \to 1/2} \frac{1}{12}\left[\frac{\left(z-\tfrac{1}{3}\right)(60z - 23) - (30z^2 - 23z + 5)}{\left(z - \tfrac{1}{3}\right)^2} \right]$$

$$'' = \frac{1}{12}\left[\frac{\left(\tfrac{1}{2}-\tfrac{1}{3}\right)\left(\overset{30}{60\left(\tfrac{1}{2}\right)} - 23\right) - \left(30\left(\tfrac{1}{4}\right) - 23\left(\tfrac{1}{2}\right) + 5\right)}{\left(\tfrac{1}{2} - \tfrac{1}{3}\right)^2} \right]$$

$$'' = \frac{1}{12}\left[\frac{\left(\frac{3-2}{6}\right)(7) - \left(\frac{15}{2} - \frac{23}{2} + 5\right)}{\left(\frac{3-2}{6}\right)^2} \right]$$

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$$I_1 = \frac{\frac{1}{12}\left[\frac{7}{6} - \frac{15-23+10}{2}\right]}{1/36}$$

$$I_1 = \frac{\cancel{36}^{\,3}}{\cancel{12}_{\,1}}\left[\frac{7}{6} - \frac{\cancel{2}^{\,1}}{\cancel{2}_{\,1}}\right]$$

$$I_1 = 3\left[\frac{7-6}{6}\right]$$

$$I_1 = 3 \times \frac{1}{6} = \boxed{1/2}$$

$$\therefore \oint \frac{30z^2 - 23z + 5}{(2z-1)^2(3z-1)}\,dz = 2\pi i\left(2 + 1/2\right)$$

$$= \cancel{2}\pi i\left(\frac{4+1}{\cancel{2}}\right)$$

$$= \boxed{5\pi i}$$

16.4 Residue Integration of Real Int

Example ①

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = 2\pi$$

$$\cancel{\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta}}$$

Let $e^{i\theta} = z$

$$dz = ie^{i\theta} . d\theta$$

$$dz = i(z)\, d\theta$$

$$\frac{dz}{iz} = d\theta$$

$$\int_C \frac{dz}{(\sqrt{2} - \cos\theta)\, iz}$$

$$= \frac{1}{i}\int_C \frac{dz}{\left(\sqrt{2} - \frac{1}{2}\left(z + \frac{1}{z}\right)\right)z}$$

$$= \frac{1}{i}\int_C \frac{dz}{\sqrt{2}.z - \frac{z^2}{2} + \frac{1}{2}}$$

$$= \frac{1}{i}\oint_c \frac{dz}{\frac{2\sqrt{2}\cdot z - z^2 + 1}{2}}$$

$$= \frac{2}{i}\oint_c \frac{dz}{-z^2 + 2\sqrt{2}z + 1}$$

$$= -\frac{2}{i}\oint_c \frac{dz}{z^2 - 2\sqrt{2}z + 1}$$

To Find poles ~~we~~ set $z^2 - 2\sqrt{2}z + 1 = 0$

Now using Quadratic formula.

$$z = +\frac{(-2\sqrt{2}) \pm \sqrt{(2\sqrt{2})^2 - 4(1)(1)}}{2(1)}$$

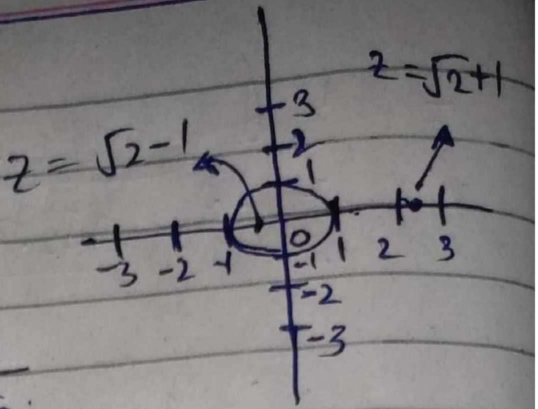
$$z = +\frac{2\sqrt{2} \pm \sqrt{(4)(2) - 4}}{2}$$

$$z = \frac{+2\sqrt{2} \pm \sqrt{8-4}}{2}$$

$$z = \frac{+2\sqrt{2} \pm \sqrt{4}}{2} = +\frac{2\sqrt{2} \pm 2}{2}$$

$$z = \sqrt{2} \pm 1$$

We find residue at $z = \sqrt{2} - 1$



$$\underset{z=\sqrt{2}-1}{\text{Res}} f(z) = \frac{1}{2(\sqrt{2}-1) - 2\sqrt{2}}$$

$$\text{''} = \frac{1}{\cancel{2\sqrt{2}} - 2 - \cancel{2\sqrt{2}}}$$

$$\text{''} = -\frac{1}{2}$$

Now $$-\frac{2}{i}\oint \frac{dz}{z^2 - 2\sqrt{2}z + 1} = \frac{-2}{\cancel{i}}\left(2\pi \cancel{i}\left(-\frac{1}{\cancel{2}}\right)\right)$$

$$\text{''} = \boxed{2\pi}$$

EXAMPLE ②

$$\int_0^\infty \frac{1}{1+x^4}\, dx = \frac{\pi}{2\sqrt{2}}$$

$$f(z) = \frac{1}{1+z^4}$$

$$z^2 + 1 = 0$$

$$\sqrt[4]{z^4} = \sqrt[4]{-1}$$

$\boxed{|z| = 1}$, $\theta = \text{Arg } z = \pi - \tan^{-1}(0/1)$

$$\boxed{\theta = \pi}$$

$$z_0 = (1)^{1/4}\left[\cos\left(\frac{\pi + 2\pi(0)}{4}\right) + i\sin\left(\frac{\pi + 2\pi(0)}{4}\right)\right]$$

$$z_0 = \cos(\pi/4) + i\sin(\pi/4)$$

$$\boxed{z_0 = e^{i\pi/4}}$$

Similarly,

$$z_1 = e^{i\left(\frac{\pi+2\pi}{4}\right)}, \quad z_2 = e^{i\left(\frac{\pi+4\pi}{4}\right)}, \quad z_3 = e^{i\left(\frac{\pi+6\pi}{4}\right)}$$

$$z_1 = e^{i3\pi/4}, \quad z_2 = e^{i\frac{5\pi}{4}}, \quad z_3 = e^{\frac{7\pi}{4}i}$$

z_0 and z_1 lie in the upper half plane, so we select only z_0 and z_1.

$$\underset{z=e^{i\pi/4}}{\text{Res}} f(z) = \frac{1}{4z^3\big|_{e^{i\pi/4}}} = \frac{1}{4(e^{i\pi/4})^3}$$

$$= \frac{1}{4e^{i3\pi/4}} = \boxed{\frac{1}{4}e^{-\frac{3\pi}{4}i}}$$

$$\operatorname*{Res}_{z=z_1=e^{3\pi/4\, i}} f(z) = \frac{1}{4z^3}\bigg|_{z=e^{i\frac{3\pi}{4}}}$$

$$" = \frac{1}{4\cdot e^{i\frac{9\pi}{4}}} = \frac{1}{4\cdot e^{i\frac{\pi}{4}}}$$

$$" = \boxed{\frac{1}{4}\cdot e^{-i\frac{\pi}{4}}}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\left(\frac{1}{4}e^{-\frac{3\pi}{4}i} + \frac{1}{4}e^{-i\pi/4}\right)$$

$$\cancel{2}\int_0^{\infty} f(x)\,dx = \cancel{2}\pi i\left(\frac{1}{4}\left(e^{-\frac{3\pi}{4}i} + e^{-i\frac{\pi}{4}}\right)\right)$$

$$" = \frac{\pi i}{4}\left(\frac{\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right) +}{\cos\left(-\pi/4\right) + i\sin\left(-\frac{\pi}{4}\right)}\right.$$

$$" = \frac{\pi i}{4}\left(-\sqrt{2}\,i\right)$$

$$" = \frac{\pi i^2}{4}\left(-\sqrt{2}\right) = \frac{+\pi\left(+\sqrt{2}\right)}{4}$$

$$ // = \frac{\pi\sqrt{2}}{2\cdot 4} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\pi \cdot 2}{4\sqrt{2}} = \boxed{\frac{\pi}{2\sqrt{2}}} $$

P.S 16.4

① $\int_0^{2\pi} \frac{d\theta}{7+6\cos\theta}$

Ans:- $$\oint_c \frac{dz/iz}{7+6\left(\frac{z+1/z}{2}\right)} = \frac{1}{i}\oint_c \frac{dz}{z(7+3(z+1/z))}$$

$$ // = \frac{1}{i}\oint_c \frac{dz}{7z+3z^2+3} \qquad c: |z| = 1 $$

$$ // = \frac{1}{i}\oint_c \frac{dz}{3z^2+7z+3} $$

To find poles, set

$$3z^2+7z+3 = 0$$

By solving using quadratic formula.

$$z = \frac{-7 \pm \sqrt{13}}{6}$$

We will select $z = \frac{-7+\sqrt{13}}{6}$

because it lies inside contour.

$$\underset{z=\frac{-7+\sqrt{13}}{6}}{\text{Res}} f(z) = \left.\frac{1}{6z+7}\right|_{z=\frac{-7+\sqrt{13}}{6}} = \frac{1}{6\left(\frac{-7+\sqrt{13}}{6}\right)+7}$$

$$'' = '' = \boxed{\frac{1}{\sqrt{13}}}$$

Now $\int_0^{2\pi} \frac{d\theta}{7+6\cos\theta} = \frac{1}{\cancel{i}}\left(2\pi \cancel{i} \cdot \frac{1}{\sqrt{13}}\right) = \boxed{\frac{2\pi}{\sqrt{13}}}$

③ $\int_0^{2\pi} \frac{d\theta}{37-12\cos\theta}$

Ans $\oint_c \frac{dz}{zi\left(37-12\left(\frac{z+\frac{1}{z}}{2}\right)\right)} = \frac{1}{i}\oint_c \frac{dz}{37z-6z^2-}$

$c: |z|=1$

$$'' = \frac{1}{i}\oint_c \frac{dz}{-6z^2+37z-6}$$

$$II = -\frac{1}{i} \oint \frac{dz}{6z^2 - 37z + 6}$$

$$6z^2 - 37z + 6 = 0$$

$$z = \frac{-(-37) \pm \sqrt{(-37)^2 - 4(6)(6)}}{2(6)}$$

$$z = \frac{37 \pm \sqrt{1369 - 144}}{12}$$

$$z = \frac{37 \pm \sqrt{1225}}{12}$$

$$z = \frac{37 \pm 35}{12}$$

$$z = \frac{37+35}{12}, \quad z = \frac{37-35}{12}$$

$$z = \frac{72}{12}, \quad z = \frac{2}{12}$$

$$z = 6, \quad z = 1/6$$

$z = 6$ lies outside so we

Select $z = 1/6$

Hence, $\cancel{\int_0^{2\pi} \frac{d\theta}{37-12\cos\theta}}$

$$\underset{z=1/6}{\text{Res}}\, f(z) = \frac{1}{12z-37}\Big|_{z=1/6} = \frac{1}{2\,\cancel{12}(1/\cancel{6})-37}$$

Hence, $\displaystyle\int_0^{2\pi} \frac{d\theta}{37-12\cos\theta} = \cancel{-}\frac{1}{\cancel{i}}\left[2\pi \cancel{i}\left(\frac{1}{\cancel{-}35}\right)\right]$

$$// = \frac{2\pi}{35}$$

(9) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$

$$f(z) = \frac{1}{z^2+1}\,dz$$

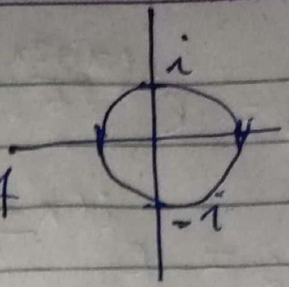
$$z^2+1=0$$

$$z^2=-1$$

$$z = \pm\sqrt{-1}$$

$$z = \pm i$$

We select $z=i$ b/c it is in upper half plane.



$$\operatorname*{Res}_{z=i} f(z) = \frac{1}{2z\big|_{z=i}} = \frac{1}{2i}$$

Now $$\int_{-\infty}^{\infty} \frac{1}{1+x^2} = \cancel{2}\pi \cancel{i}\left(\frac{1}{\cancel{2}\cancel{i}}\right) = \boxed{\pi}$$

(11) $$\int_{-\infty}^{\infty} \frac{dx}{x^6+1}$$

Ans:- $$f(z) = \frac{1}{z^6+1}$$

$$z^6+1=0$$

$$z^6 = -1$$

$$\sqrt[6]{z^6} = \sqrt[6]{-1}$$

$$z = \sqrt[6]{-1}$$

$n=6$, $r=1$, $\theta=\pi$

$$\therefore z_0 = e^{i\left(\frac{\pi+0}{6}\right)} = \boxed{e^{\pi/6\, i}}$$

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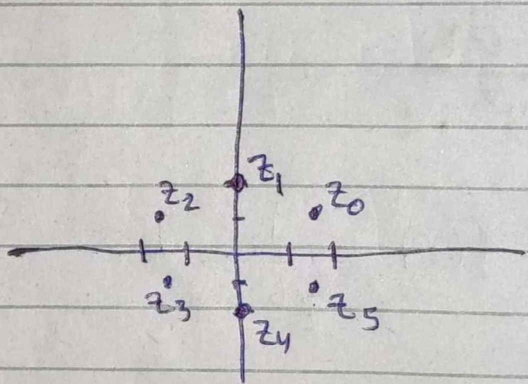
$$z_1 = e^{i\left(\frac{\pi+2\pi}{6}\right)} = e^{i\frac{3\pi}{6}} = e^{i\pi/2}$$

$$z_2 = e^{i\left(\frac{\pi+4\pi}{6}\right)} = e^{i\frac{5\pi}{6}}$$

$$z_3 = e^{i\left(\frac{\pi+6\pi}{6}\right)} = e^{i\frac{7\pi}{6}}$$

$$z_4 = e^{i\left(\frac{\pi+8\pi}{6}\right)} = e^{i\frac{9\pi}{6}}$$

$$z_5 = e^{i\left(\frac{\pi+10\pi}{6}\right)} = e^{i\frac{11\pi}{6}}$$



We will select z_0, z_1 and z_2 as they are on upper plane

$$\operatorname{Res}_{z=e^{\pi/6 i}} f(z) = \frac{1}{6z^5}\Big|_{z=e^{\pi/6 i}} = \frac{1}{6\cdot e^{\frac{5\pi}{6}i}} = \boxed{\frac{1}{6}\, e^{-\frac{5\pi}{6}}}$$

$$\operatorname{Res}_{z=e^{i\pi/2}} f(z) = \frac{1}{6z^5}\Big|_{z=e^{i\frac{\pi}{2}}} = \frac{1}{6\, e^{\frac{5\pi}{2}i}} = \boxed{\frac{1}{6}}$$

$$\operatorname*{Res}_{z=e^{i\frac{5\pi}{6}}} f(z) = \frac{1}{6z^5}\Big|_{z=e^{\frac{i5\pi}{6}}} = \frac{1}{6e^{\frac{i25}{6}}} = \boxed{\frac{1}{6}e^{\frac{-i25}{6}}}$$

$$\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = 2\pi i\left[\frac{1}{6}e^{-\frac{5\pi}{6}i} + \frac{1}{6}e^{-\frac{5\pi}{2}i} + \frac{1}{6}e^{-\frac{i25}{6}}\right]$$

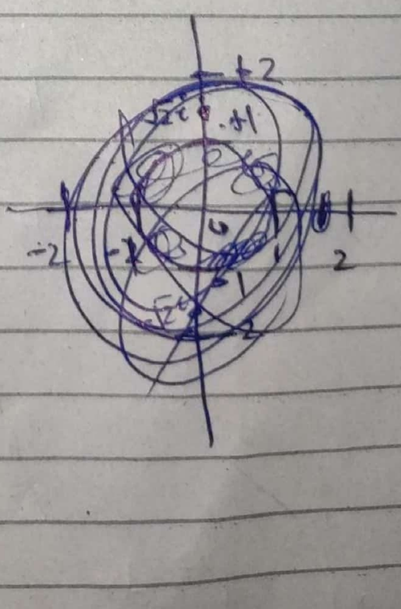
$$'' = \frac{2}{6}\pi i\left[-2i\right]$$

$$'' = \frac{-4\pi i^2}{6} = \boxed{\frac{4\pi}{6}}$$

13) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2}$

Ans :-

$$f(z) = \frac{1}{(z^2+4)^2}$$



$$z^2+4=0$$

$$\sqrt{z^2} = \pm\sqrt{-2}$$

$$z = \pm\sqrt{-1\times 2}$$

$$\boxed{z = \pm\sqrt{2}\, i}$$

We will select $z = \sqrt{2}\, i$

$z = \sqrt{2}\, i$ is a pole of order 2.

$$\therefore \operatorname*{Res}_{z=\sqrt{2}i} f(z) = \frac{1}{(2-1)!} \lim_{z \to \sqrt{2}i} \left\{ \frac{d^{2-1}}{dz^{2-1}} \left[(z-\sqrt{2}i)^2 \frac{1}{(z^2+4)^2} \right] \right.$$

$$\operatorname*{Res}_{z=2i} f(z) = \lim_{z \to 2i} \frac{d}{dz} \left(\frac{(z-2i)^2}{(z^2+4)^2} \right)$$

$$\operatorname*{Res}_{z=2i} f(z) = \lim_{z \to 2i} \frac{d}{dz} \left[\frac{(z-2i)^2}{(z^2-(2i)^2)^2} \right]$$

$$'' = \lim_{z \to 2i} \frac{d}{dz} \left[\frac{\cancel{(z-2i)^2}}{\cancel{(z-2i)^2}(z+2i)^2} \right]$$

$$'' = \lim_{z \to 2i} \frac{d}{dz} \left[(z+2i)^{-2} \right]$$

$$'' = \lim_{z \to 2i} \cdot -2(z+2i)^{-3} \cdot (1)$$

$$'' = -2(2i+2i)^{-3}$$

$$'' = \frac{-2}{(4i)^3} = \frac{-2}{64 \cdot i^3} = \frac{-1}{32 \cdot i^2 \cdot i}$$

$$'' = \frac{\cancel{-}1}{\cancel{-}32i} = \frac{1}{32i}$$

Now $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2} = 2\pi \cancel{i} \left(\frac{1}{32\cancel{i}} \right) = \frac{2\pi}{32} = \boxed{\frac{\pi}{16}}$