Linear & Non-linear Systems

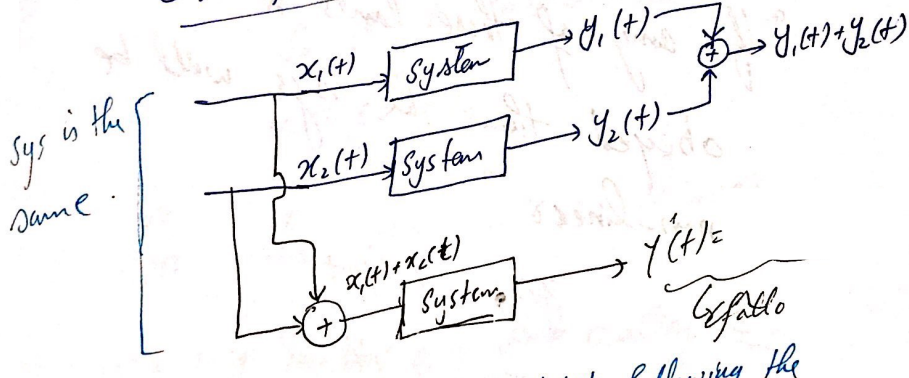
Linear Systems :- A sys obeying the principle of superposition is known as linear.

Law of Superposition :- Combination of two laws

① Law of Additivity

② Law of homogeneity

① Law of Additivity

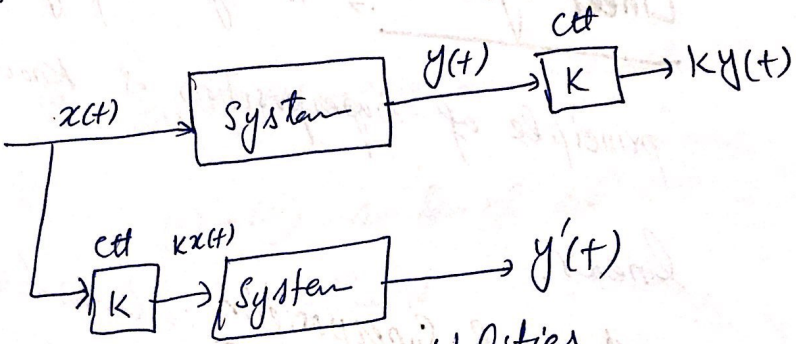


(a) $y'(t) = y_1(t) + y_2(t)$ following the law of additivity

(b) $y'(t) \neq y_1(t) + y_2(t)$ not following the law.

② Law of Homogeneity :→

Again going with the same system



Now again two possibilities.

(a) $y'(t) = ky(t)$ → law of homogeneity is followed

(b) $y'(t) \neq ky(t)$ → law not followed.

"If any of these laws are not obeyed then the sys. will be non-linear"

Ex :→ $y(t) = x(\sin t)$

Linear or N/L ?

(1)

(a) $y_1(t) = x_1(\sin t)$

$y_2(t) = x_2(\sin t)$

$y_1(t) + y_2(t) = x_1(\sin t) + x_2(\sin t) \rightarrow (1)$

(b) $x_1(t) + x_2(t) \longrightarrow [\text{Sys}] \longrightarrow y'(t) = X_1(\sin t) + X_2(\sin t)$

$\hookrightarrow (2)$

(1) = (2) so law of Additivity is followed

(2)

(a) $K\,y(t) = K\,x(\sin t) \longrightarrow (3)$

(b) $K\,x(t) \longrightarrow \text{Sys} \longrightarrow K\,x(\sin t) \rightarrow (4)$

(3) = (4)

So it also follow the law of homogeneity.

So the system is linear.

⇒ If a sys. performs only time scaling then the sys. be linear.

Ex $y(t) = x(t^2)$

linear or N/L ?

(1)

(a) $x_1(t) \xrightarrow{Sys} y_1(t) = x_1(t^2)$

$x_2(t) \rightarrow Sys \rightarrow y_2(t) = x_2(t^2)$

$y_1(t) + y_2(t) = x_1(t^2) + x_2(t^2) \rightarrow (1)$

(b) $x_1(t) + x_2(t) \rightarrow [Sys] \rightarrow y'(t) = x_1(t^2) + x_2(t^2) \rightarrow (2)$

Due to the sys. functionality as sys is squaring t.

(1) = (2) $\Rightarrow$ law of additivity is followed.

(2)

(a) $x(t) \rightarrow Sys \rightarrow y(t) \rightarrow [K] \rightarrow K\,y(t) = Kx(t^2)$ (3)

(b) $Kx(t) \xrightarrow{Sys} y'(t) = Kx(t^2) \rightarrow (4)$

(3) = (4)

So homogeneity is also followed

So system is linear.

→ Once again notice the functionality of system was only time scaling.

Ex $y(t) = x(\log t)$

Linear or N/L

Solution :- Linear bcz there is time Scaling only

Ex $y(t) = x^2(t)$

Linear or N/L ?

Solution

$x(t) \longrightarrow Sys \longrightarrow y(t) = x^2(t)$

System's functionality is that it squares its i/p

(1)

(a) $x_1(t) \longrightarrow Sys \longrightarrow y_1(t) = x_1^2(t)$

$x_2(t) \longrightarrow Sys \longrightarrow y_2(t) = x_2^2(t)$

$y_1(t) + y_2(t) = x_1^2(t) + x_2^2(t) \longrightarrow (1)$

(b) $x_1(t) + x_2(t) \longrightarrow Sys \longrightarrow y'(t) = [x_1(t) + x_2(t)]^2 \longrightarrow (2)$

(now this is the i/p.)

(1) ≠ (2) So N/L.

Ex :- $y(t) = x^3(t^3)$

Linear or NL ?

Ex $y(t) = \underbrace{\sin t}_{\text{co-eff}}\ \underbrace{x(t)}_{\text{i/p}}$

Linear or N/L ?

Solution :- System's functionality is that it multiplies $\sin t$ to its i/p.

(1) (a)

$x_1(t) \rightarrow Sys \rightarrow y_1(t) = \sin(t)\, x_1(t)$

$x_2(t) \rightarrow Sys \rightarrow y_2(t) = \sin(t)\, x_2(t)$

$y_1(t) + y_2(t) = \sin t\,(x_1(t) + x_2(t))$ → (1)

(b) $x_1(t) + x_2(t) \rightarrow Sys \rightarrow y'(t) = \sin t\,[x_1(t) + x_2(t)]$ → (2)

(1) = (2) Additivity law followed.

(2) (a)

$x(t) \rightarrow Sys \rightarrow y(t) \rightarrow [K] \rightarrow Ky(t) = K\sin t\, x(t)$ → (3)

$Kx(t) \rightarrow Sys \rightarrow K\sin t\, x(t)$ → (4)

(3) = (4) superposition also followed So linear.

(Linear Time Varying Sys)

Ex $y(t) = e^3 x(t)$

Linear or N/L ?

Solution (1)

(a) $x_1(t) \rightarrow Sys \rightarrow y_1(t) = e^3 x_1(t)$

$x_2(t) \rightarrow Sys \rightarrow y_2(t) = e^3 x_2(t)$

$y_1(t) + y_2(t) = e^3 [x_1(t) + x_2(t)] \rightarrow (1)$

(b)

$x_1(t) + x_2(t) \rightarrow Sys \rightarrow y'(t) = e^3 [x_1(t) + x_2(t)] \rightarrow (2)$

(1) = (2) Adelitivity law ✓

(2)

(a) $x(t) \rightarrow Sys \rightarrow y(t) \rightarrow [k] \rightarrow k\, y(t) = k e^3 x(t)$ → (3)

(b) $k x(t) \rightarrow Sys \rightarrow e^3 k x(t) \rightarrow (4)$

(3) = (4) homogeneity ✓

Linear (Linear time Invariant System)

⇒ The system's linearity is independent of the system's co-efficients.

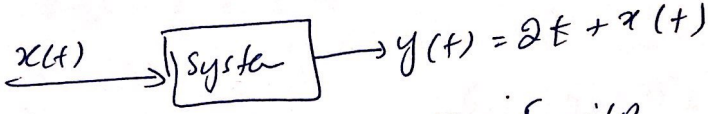
Examples :→ ① $y(t) = (\log t + 3t^2)x(t)$

② $y(t) = t\,u(t) \cdot x(t)$

Ex $y(t) = (2t) + x(t)$ → Time dependent.

Linear or N/L ?

Solution



Sys is adding 2t to its i/p.

(1) (a)

$x_1(t) \rightarrow Sys \rightarrow y_1(t) = 2t + x_1(t)$

$x_2(t) \rightarrow Sys \rightarrow y_2(t) = 2t + x_2(t)$

$y_1(t) + y_2(t) = 4t + x_1(t) + x_2(t) \rightarrow (1)$

(b)

$x_1(t) + x_2(t) \rightarrow sys \rightarrow y'(t) = 2t + x_1(t) + x_2(t)$

$\hookrightarrow (2)$

$(1) \neq (2)$

So system is N/L. (N/L time varying sys)

Ex $y(t) = (2) + x(t)$ → Time independent

Linear or N/L ?

Solution

(1)

(a) $x_1(t) \rightarrow Sys \rightarrow y_1(t) = 2 + x_1(t)$

$x_2(t) \rightarrow Sys \rightarrow y_2(t) = 2 + x_2(t)$

$y_1(t) + y_2(t) = 4 + x_1(t) + x_2(t) \rightarrow (1)$

(b) $x_1(t) + x_2(t) \rightarrow Sys \rightarrow y'(t) = 2 + x_1(t) + x_2(t)$ $\hookrightarrow (2)$

$(1) \neq (2)$ NL

(2) Let us also check LOH.

(a) $x(t) \rightarrow Sys \rightarrow y(t) \rightarrow \boxed{K} \rightarrow k[2 + x(t)]$ $\hookrightarrow (3)$

(b) $kx(t) \rightarrow Sys \rightarrow y'(t) = 2 + kx(t) \rightarrow (4)$

$(3) \neq (4)$ LOH X

Sys is N/L. (N/L time invariant)

=> In case of added or subtracted terms (other than the input and output) the system will be nonlinear

[System performing amplitude shifting will always be nonlinear]

Ex:- (1) $y(t) = 2t^3 - x(t^3)$

(2) $y(t) = x(t-1) + x(t+1)$

Solution 1

(a)

$x_1(t) \rightarrow Sys \rightarrow y_1(t) = x_1(t-1) + x_1(t+1)$

$x_2(t) \dots \rightarrow y_2(t) = x_2(t-1) + x_2(t+1)$

$y_1(t) + y_2(t) = x_1(t-1) + x_2(t-1) + x_1(t+1)$

$+ x_2(t+1) \rightarrow (1)$

(b)

$x_1(t) + x_2(t) \rightarrow Sys \rightarrow y'(t) = x_1(t-1) + x_2(t-1)$

$+ x_1(t+1) + x_2(t+1)$

$\hookrightarrow (2)$

$(1) = (2)$ LOA ✓

2 (a)

$x(t) \rightarrow Sys \rightarrow y(t) \rightarrow [k] \rightarrow ky(t) = k\,x(t-1) + k\,x(t+1)$

$\hookrightarrow (3)$

(b) $k\,x(t) \rightarrow sys \rightarrow y'(t) = k\,x(t-1) + k\,x(t+1)$

$\hookrightarrow (4)$

$(3) = (4)$ LOH ✓

So the sys is linear.

⇒ A. sys performing only time shifting is linear.

or Linearity is independant of time shifting.

Ex $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$ Linear or N/L ?

[Time scaling, time shifting and amplitude scaling has no effect on linearity]

Solutions

(1) (a) $x_1(t) \to Sys \to y_1(t) = \int_{-\infty}^{t} x_1(\tau)\,d\tau$

$x_2(t) \to Sys \to y_2(t) = \int_{-\infty}^{t} x_2(\tau)\,d\tau$

$y_1(t) + y_2(t) = \int_{-\infty}^{t} x_1(\tau)\,d\tau + \int_{-\infty}^{t} x_2(\tau)\,d\tau$

$= \int_{-\infty}^{\infty} [x_1(\tau) + x_2(\tau)]\,d\tau \to (1)$

(b) $x_1(t) + x_2(t) \to Sys \to y'(t) = \int_{-\infty}^{t} [x_1(\tau) + x_2(\tau)]\,d\tau$

$\hookrightarrow (2)$

(1) = (2) LOA ✓

(2) (a) $x(t) \to Sys \to y(t) \to K \to Ky(t) \to K\int_{-\infty}^{t} x(\tau)\,d\tau$

$\hookrightarrow (3)$

$Kx(t) \to Sys \to y'(t) = K\int_{-\infty}^{t} x(\tau)\,d\tau$

$\hookrightarrow (4)$

(3) = (4) LOH ✓

Integration is a linear operator.

Ex $y(t) = \frac{d}{dt} x(t)$

Linear or N/L ?

Solution

① (a) $x_1(t) \to Sys \to y_1(t) = \frac{dx_1(t)}{dt}$

$x_2(t) \to Sys \to y_2(t) = \frac{dx_2(t)}{dt}$

$y_1(t) + y_2(t) = \frac{dx_1}{dt} + \frac{dx_2}{dt}$

$= \frac{d}{dt}[x_1(t) + x_2(t)] \to ①$

(b) $x_1(t) + x_2(t) \to Sys \to y'(t) = \frac{d}{dt}[x_1(t) + x_2(t)]$

$\hookrightarrow ②$

① = ② LOH ✓

② (a) $x(t) \Rightarrow Sys \to y(t) \to \boxed{k} \to ky(t) = k\frac{dx(t)}{dt}$

$\hookrightarrow ③$

$kx(t) \to Sys \to y'(t) = \frac{d}{dt} kx(t) = k\frac{dx(t)}{dt}$

$\hookrightarrow ④$

③ = ④ LOH ✓

Differential operation (operator) is linear operator.

Ex $y(t) = \text{Even}[x(t)]$

Linear or N/L?

Solution

$$y(t) = E[x(t)] = \frac{x(t) + x(-t)}{2}$$

$$y(t) = \frac{1}{2}x(t) + \frac{1}{2}x(-t)$$

This sys. is linear. Why?

→ use the prop.

Even operator (E) is a linear operator.

Ex $y(t) = \text{Odd}[x(t)]$

Linear or N/L?

$$y(t) = \frac{x(t) - x(-t)}{2}$$

$$y(t) = \frac{1}{2}x(t) - \frac{1}{2}x(-t)$$

Odd operator (O) is linear.

Ex : $y(t) = Re[x(t)]$ Linear?

Method. 1

$$y(t) = \frac{x(t) + \overline{x(t)}}{2}$$

LOAH

(a) $x(t) \to Sys \to y(t) \to \boxed{k} \to k\left[\frac{x(t) + \overline{x(t)}}{2}\right]$ ↳ ①

$kx(t) \to Sys \to y'(t) = \frac{kx(t) + \overline{kx(t)}}{2}$

$= \frac{k\,x(t) + \bar{k}\,\overline{x(t)}}{2}$ ↳ ②

① ≠ ② if $k = j = \sqrt{-1}$ ∵ if $k = j$ then $\bar{k} = -j$

So N/L.

Method 2

$x(t) = a(t) + jb(t)$

$y(t) = a(t)$

LOH:

$x(t) \to Sys \to y(t) \Rightarrow \boxed{j} \to ja(t)$ ↳ ①

$jx(t) = ja(t) - b(t) \to Sys \to -b(t)$ ↳ ②

① ≠ ② So N/L

Ex: $y(t) = Im(x(t))$

Linear or N/L ?

Solution

→ "Re" and "Im" operators are both N/L operators.

→ Conjugate operator is a N/L operator

Linear operators

Integral, Differential, E, O,

N/L operators

→ Re, → Im, → Conjugate,

→ Trigonometric operators (e.g., $y(t) = \cot[x(t)]$)

→ Inverse Trigonometric

→ Logarithmic (e.g., $\log_2(y(t)) = x(t)$)

→ Exponential ($y(t) = e^{x(t)}$)

→ Roots & power

e.g., (1) $y(t) = \sqrt[4]{x(t)}$

(2) $y(t) = x^2(t)$

(3) $y^4(t) = x(t)$

→ Modulus operator $y(t) = |x(t)|$

→ Sgn operation $y(t) = \text{Sgn}(x(t))$

→ $y(t) = \text{Sinc}[x(t)]$

⇒ A linear Sys. give Zero o/p for Zero i/p

i.e $x(t)=0 \Rightarrow y(t)=0$

E.g ① $y(t)=2x(t)$

we know that this is linear ∵ linearity is independent of the co-efficients. However, let see it from the above result.

because the coefficient is constant

$x(t)=0$

$y(t)=2\times 0=0$ linear

② $y(t)=2+x(t)$

we know the sys is N/L.

let see

for $x(t)=0$

$y(t)=2+0=2$ N/L

Linearity of Split Systems.

Ex $$y(t) = \begin{cases} x(t-1) & t < 0 \\ x(t+1) & t \geq 0 \end{cases}$$

Linear?

Solution :->

Let us represent $y(t)$ as

$$y(t) = a(t)\,x(t-1) + b(t)\,x(t+1)$$

or

$$a(t)\,x(t-1) + (1-a(t))\,x(t+1)$$

where

$$a(t) = \begin{cases} 1 & t < 0 \\ 0 & t \geq 0 \end{cases}$$

$$b(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

We know that linearity is independent of co-efficients and time shifting

So this sys. is linear.

→ However all the split sys are TV systems.

Summary

(1) Sys. linearity is independent of time scaling & time shifting.

(2) " " " " of sys. co-efficients.

(3) if there are any added/subtracted terms (other than the I/P & O/P), the sys. will be non-linear

(4) Integral, Differential, E and O operators are linear.

(5) Re, Im & Conjugate are N/L operators

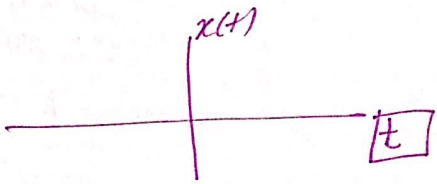
(6) Trigonometric, Inverse trigonometric, Logarithmic, exponential, Roots, powers, modulus, sinc and sgn are N/L.

(7) A linear sys. must give O o/p for O i/p

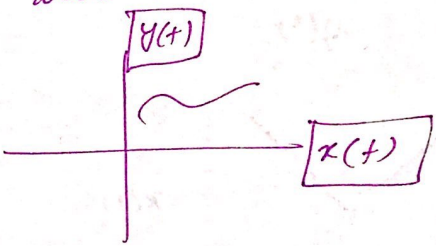
(8) Split sys. are TV

Linearity From Transfer xtics

→ So far we've seen the signal plots



→ But we may also have

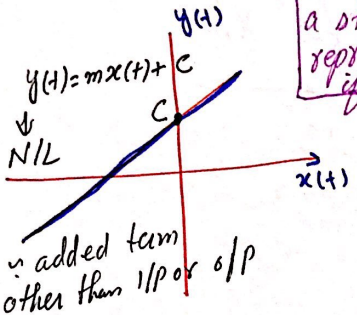


Transfer ~~X~~tics :→ Plot b/w the i/p & the o/p of a sys is known as the transfer xtics of a sys.

→ we can directly infer linearity from the transfer xtics using the already developed results.

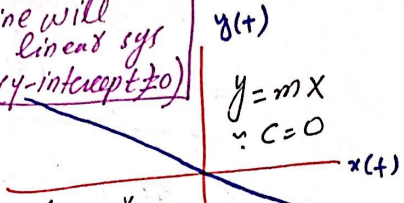
Ex

So a transfer xtics with a straight line will represent a linear sys if $C=0$ (y-intercept $\neq 0$)



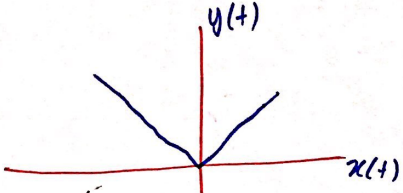
$\therefore$ N/L

$\therefore$ added term other than I/P or O/P



Linear

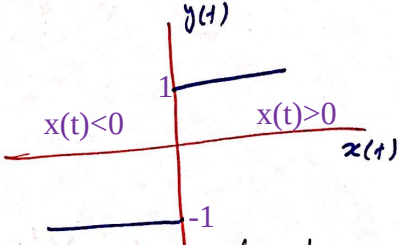
$\therefore$ Linearity is independent of co-efficients



This is transfer xtics of modulus fun"

$y(t) = |x(t)|$ so

N/L



This is transfer xtics of sgn fun"

$y(t) = \text{sgn}[x(t)]$

So N/L.