FOURIER TRANSFORM OF BASIC SIGNALS

DC Value:-

$$x(t) = A_0$$

=> DC values are not absolutely integrable

i.e $\int_{-\infty}^{\infty} |x(t)|\,dt$ should be finite for $x(t)$ to be absolutely integrable.

$\Rightarrow \int_{-\infty}^{\infty} A_0\,dt = A_0 \int_{-\infty}^{\infty} dt = A_0\, t\Big|_{-\infty}^{\infty} = A_0(\infty + \infty) = \infty$

Thus DC value is not absolutely integrable.

$\Rightarrow$ Now the formula

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\,dt$$

can't be used bcz $x(t)$ is not absolutely integrable.

What goes wrong if we use this formula?

⇒ let see

$$X(\omega) = \int_{-\infty}^{\infty} A_0 e^{-j\omega t} dt$$

$$= A_0 \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-\infty}^{\infty}$$

$$= \frac{A_0}{-j\omega} \left[e^{-j\infty} - e^{j\infty} \right]$$

$$= \frac{A_0}{\omega} \left[\frac{e^{j\infty} - e^{-j\infty}}{j} \right]$$

$$= \frac{A_0}{\omega} \, 2 \underbrace{\sin \infty}$$

↳ Undefined or not defined

Hence $X(\omega)$ is not defined

⇒ So formula can't be used.

⇒ To find the FT of DC value let us assume a signal $x(t)$ s.t

$$FT\left[x(t)\right] = X(\omega)\, \delta(\omega)$$

⇒ We know from the formula of IFT that

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \rightarrow (1)$$

$X(\omega) = A_0 \delta(\omega)$ put in (1)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_0 \delta(\omega) e^{j\omega t} d\omega$$

$$= \frac{A_0}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega \rightarrow (2)$$

Now notice that $\delta(\omega) e^{j\omega t}$ can be written as.

$$e^{j\omega t} \delta(\omega - 0)$$

& by sampling prop. of impulse signal.

$$e^{j\omega t} \delta(\omega - 0) = e^{j \times 0 \times t} \delta(\omega) = 1 \cdot \delta(\omega) = \delta(\omega)$$

put this in (2)

$$x(t) = \frac{A_0}{2\pi} \underbrace{\int_{-\infty}^{\infty} \delta(\omega) d\omega}_{1} = \frac{A_0}{2\pi}$$

$$x(t) = \frac{A_0}{2\pi}$$

$\Rightarrow$ So what we found is that

$$FT\left[\frac{A_0}{2\pi}\right] = A_0 \delta(\omega)$$

$\Rightarrow$ Now we'll use linearity property which says that if something (ct) is multiplied to the time domain signal then the same thing (ct) gets multiplied to the frequency domain signal.

i.e.,

$$FT\left[\cancel{2\pi}\frac{A_0}{\cancel{2\pi}}\right] = 2\pi A_0 \delta(\omega)$$

$$\boxed{FT[A_0] = 2\pi A_0 \delta(\omega)}$$

e.g., let $A_0 = 4$ then

$$FT[4] = 2\pi \times 4 \times \delta(\omega) = 8\pi \delta(\omega)$$

FT of Impulse Signal

$x(t) = \delta(t)$

$$FT[\delta(t)] = ?$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

Using prop. of impulse signal.

$$e^{j\omega t}\, \delta(t-0) = e^{j\omega \times 0}\, \delta(t) = \delta(t)$$

so

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

$$\boxed{FT[\delta(t)] = 1}$$

FT of Exponential Signals

FT of 3 different types of Exponential Signals?

(1) $x(t) = e^{-at} u(t)$, $a > 0$

$X(\omega) = ?$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-j\omega t}\, dt$$

we know that $u(t) = 1$ $\quad 0 \le t \le \infty$

& $u(t) = 0$ $\quad -\infty \le t < 0$

So

$$X(\omega) = \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_{0}^{\infty}$$

Now put limits.

$$X(\omega) = -\frac{1}{a+j\omega}\left[e^{-\infty} - e^{0}\right]$$

Now $e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$ $\because e^{\infty} = \infty$

So

$$X(\omega) = -\frac{1}{a+j\omega}\left[0 - 1\right]$$

$$\boxed{X(\omega) = \frac{1}{a+j\omega}}$$

$$\boxed{FT\left[e^{-at}u(t)\right] = \frac{1}{a+j\omega}}$$, when $a > 0$

② $FT\left[e^{at}u(-t)\right] = ?$ when $a > 0$

$$X(\omega) = \int_{-\infty}^{\infty} e^{at}\, u(-t)\, e^{-j\omega t}\, dt$$

we know

$u(-t) = 1 \quad -\infty \leq t \leq 0$

$u(-t) = 0 \quad 0 \leq t \leq \infty$

thus

$$X(\omega) = \int_{-\infty}^{0} e^{at}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt$$

$$= \left. \frac{e^{(a-j\omega)t}}{a-j\omega} \right|_{-\infty}^{0}$$

$$= \frac{1}{a-j\omega}\Big[\underbrace{e^{0}}_{1} - \underbrace{e^{-\infty}}_{0}\Big] = \frac{1}{a-j\omega}$$

$$\boxed{FT\left[e^{at}u(-t)\right] = \frac{1}{a-j\omega}}$$ when $a > 0$

③ $FT[e^{-a|t|}]$ when $a>0$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

Using previous 2 results.

$$X(\omega) = \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{a+\cancel{j\omega}+a-\cancel{j\omega}}{a^2+\omega^2}$$

$$\boxed{X(\omega) = \frac{2a}{a^2+\omega^2}} \quad a>0$$

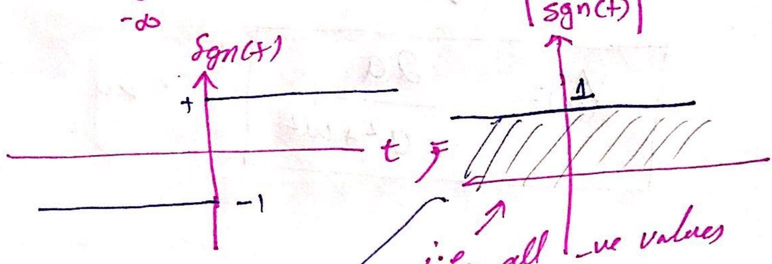
Fourier Transform of Signum Function

$$x(t) = Sgn(t) = \begin{cases} -1 & t < 0 \\ 0 & t = 0 \\ 1 & t > 0 \end{cases}$$

$FT[sgn(t)] = ?$

⇒ First we've to find whether sgn(t) function is absolutely integrable or not ∵ the formula is valid only for absolutely integrable functions.

⇒ $\int_{-\infty}^{\infty} |sgn(t)| \, dt < \infty$ is condition.



i.e., all -ve values will become +ve

Clearly area under $|sgn(t)|$ is ∞

[Assume it a rectangle then height × width

$1 \times \infty = \infty$]

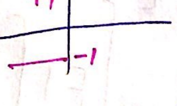
Thus

$$\int_{-\infty}^{\infty} |sgn(t)|\, dt = \infty$$

Thus the condition for absalute integrability is vailated

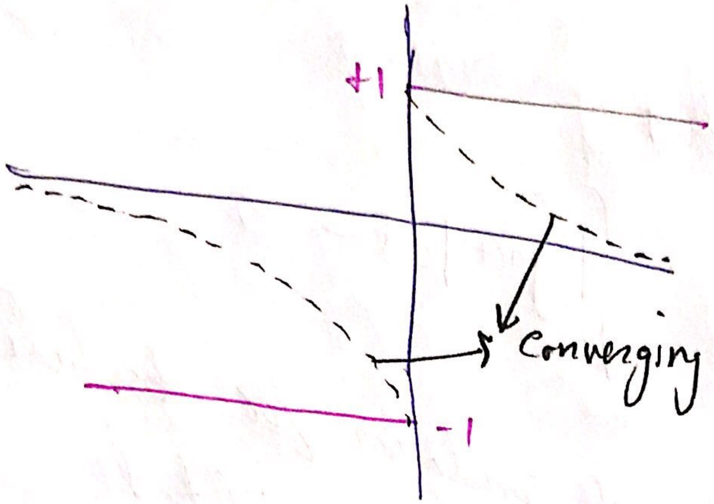
⇒ So what we can do now?

⇒ We know that the if the signal wave form is converging then the signal is absalutely integrable.

⇒ However if we look at the plot of sgn(t) 

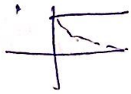
it's not converging (it instead stays ett)

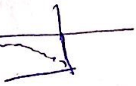
⇒ We will try to make it look like (other representation) converging.



⇒ Sgn(t) can also have the following representation.

$$\mathrm{sgn}(t) = u(t) - u(-t) \rightarrow ①$$

Now $u(t) = \lim_{a\to 0} e^{-at} u(t) \rightarrow ②$ 

$u(-t) = \lim_{a\to 0} e^{at} u(-t) \rightarrow ③$ 

put ② & ③ in ①

$$\mathrm{sgn}(t) = \lim_{a\to 0} \left[e^{-at} u(t) - e^{at} u(-t) \right]$$

$$FT[\mathrm{sgn}(t)] = \lim_{a\to 0} \left\{ FT\left[e^{-at} u(t)\right] - FT\left[e^{at} u(-t)\right] \right\}$$

$$= \lim_{a\to 0} \left[\frac{1}{a+jw} - \frac{1}{a-jw} \right] = \lim_{a\to 0} \left[\frac{-2jw}{a^2+w^2} \right]$$

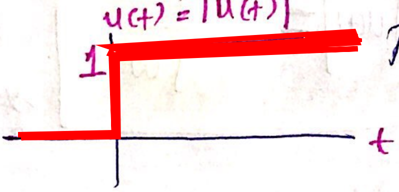
$$\boxed{FT[\mathrm{sgn}(t)] = \frac{-2jw}{w^2} = -\frac{2j}{w} = \frac{-2j \times j}{w \times j} = \frac{2}{jw}}$$

FT of Step Signal

$x(t) = u(t)$ $\qquad$ $u(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$

$FT[u(t)] = ?$

Is u(t) absolutely Integrable?



Thus

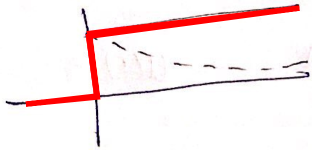
$$\int_{-\infty}^{\infty} |u(t)|\, dt = \infty$$ or area is ∞

(Waveform not converging)

So u(t) is not absolutely integrable.

=> So u(t) represented as it is "we can't use the formula".

=> In order to use the formula we need to represent u(t) as a "converging fun"/waveform.



$u(t) = \lim_{a \to 0} e^{-at} u(t)$

$$u(t) = \lim_{a \to 0} e^{-at} u(t)$$

Now we can use the formula

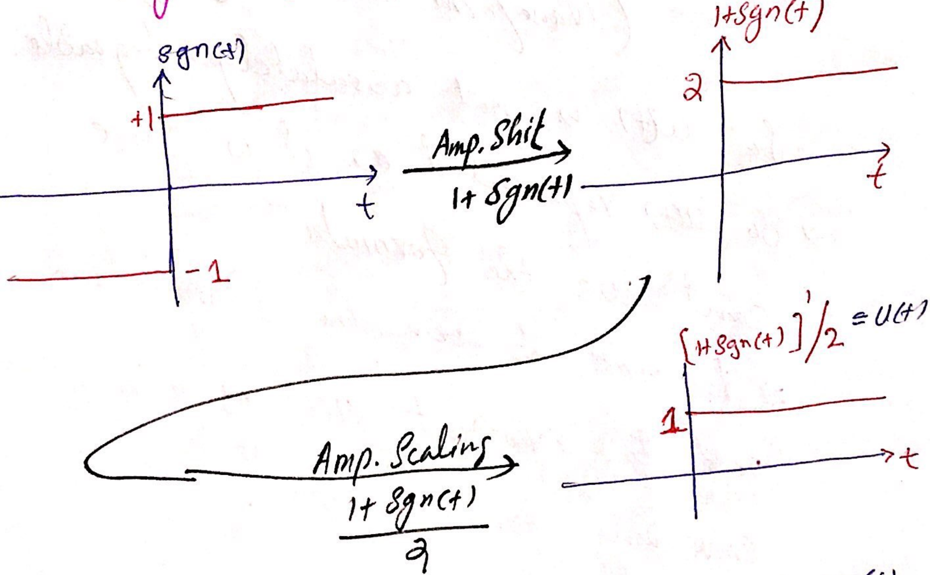
$$FT[u(t)] = \lim_{a \to 0} FT[e^{-at} u(t)]$$

$$= \lim_{a \to 0} \frac{1}{a + j\omega}$$

$$\boxed{FT[u(t)] = \frac{1}{j\omega}}$$

but this result is incomplete ∵ DC value $= \frac{1}{2}$ for $u(t)$

2) Now let us try to make use of sgn(t) function.



Thus $u(t) = \frac{1 + sgn(t)}{2} = \frac{1}{2} + \frac{1}{2} sgn(t)$

So

$$FT[u(t)] = FT\left[\frac{1}{2}\right] + \frac{1}{2} FT[\text{sgn}(t)]$$

FT of DC value

$$= \cancel{2}\pi \frac{1}{\cancel{2}} \cdot \delta(\omega) + \frac{1}{\cancel{2}}\left[\frac{\cancel{2}}{j\omega}\right]$$

$$\boxed{FT[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}}$$

FT of sgn(t) & u(t) [Shortcut]

⇒ This shortcut method is useful when the given signal is related to STEPS or RAMPS or their combination.

⇒ What we do is that differentiate the signal until we get all impulses e.g

Ramps $\xrightarrow{\text{diff}}$ Steps $\xrightarrow{\text{diff}}$ Impulses

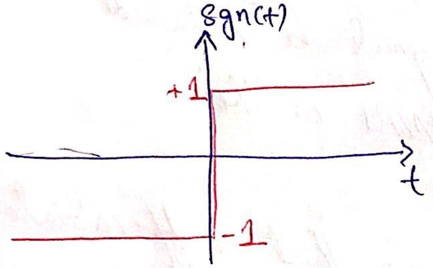
or

Steps $\xrightarrow{\text{diff}}$ Impulses.

⇒ $\boxed{\text{One more important fact}}$

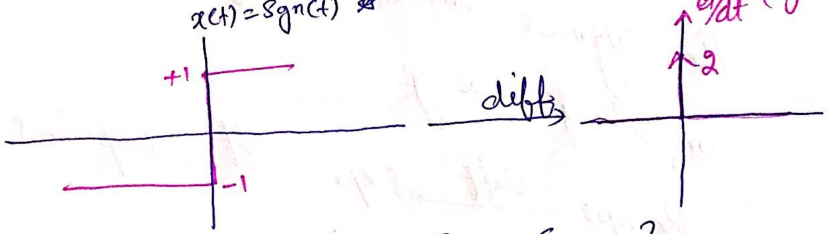
⇒ Find if the DC value of signal is non-zero e.g., say ½ then find its FT and add it to the FT you've found using the shortcut method.

Example



$FT[sgn(t)] = ?$

Solution:



Let $X(\omega) = FT[x(t)] = FT[sgn(t)]$

and $\frac{d}{dt} sgn(t) = 2\delta(t)$

$$\frac{d\,x(t)}{dt} = 2\,\delta(t)$$

Taking FT on both sides.

$$j\omega\, X(\omega) = 2$$

$$\boxed{X(\omega) = \frac{2}{j\omega}}$$

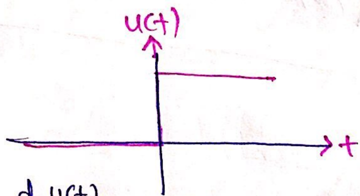
In this case DC value $= \frac{1+(-1)}{2} = 0$

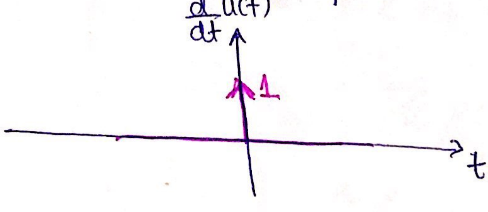
So $\boxed{X(\omega) = 2/j\omega}$

Example let $x(t) = u(t)$

$X(\omega) = ?$ when $X(\omega) = FT\{x(t)\} = FT\{u(t)\}$

Solution





$$\frac{d}{dt} x(t) = \delta(t)$$

Taking FT on both sides.

$$j\omega X(\omega) = 1$$

$$X(\omega) = \frac{1}{j\omega} \rightarrow 1$$

In this case DC value $= \frac{1+0}{2} = \frac{1}{2}$

$$FT\left[\frac{1}{2}\right] = \cancel{2}\pi \frac{1}{\cancel{2}} \delta(\omega) = \pi \delta(\omega)$$

Adding this will give the correct FT of $u(t)$

$$\boxed{X(\omega) = \pi \delta(\omega) + \frac{1}{j\omega}}$$

FT of Complex Exponential Signals ($e^{j\omega_0 t}$)

$$x(t) = e^{j\omega_0 t}$$

$$X(\omega) = FT\{x(t)\} = ?$$

As $e^{j\omega_0 t}$ is not absolutely integrable thus we can't use the formula.

$\Rightarrow$ To calculate its FT let us assume a signal $x'(t)$ having FT $\delta(\omega - \omega_0)$

i.e

$$x'(t) \xrightleftharpoons{FT} X'(\omega) = \delta(\omega - \omega_0)$$

Then by IFT

$$x'(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X'(\omega) e^{j\omega t} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\delta(\omega - \omega_0) e^{j\omega t}}_{\text{using the prop. of } \delta(t) \text{ or } \delta(\omega)} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega_0 t} \, d\omega$$

Now $e^{j\omega_0 t}$ is ctt w-r-t the integration so

$$x'(t) = \frac{e^{j\omega_0 t}}{2\pi} \underbrace{\int_{-\infty}^{\infty} \delta(\omega - \omega_0)\, d\omega}_{1}$$

$$x'(t) = \frac{e^{j\omega_0 t}}{2\pi}$$

Thus

$$\frac{e^{j\omega_0 t}}{2\pi} \overset{FT}{\rightleftharpoons} \delta(\omega - \omega_0)$$

Multiply both sides with 2π

$$\boxed{e^{j\omega_0 t} \overset{FT}{\rightleftharpoons} 2\pi\, \delta(\omega - \omega_0)}$$

In general

$$\boxed{FT\left[e^{\pm j\omega_0 t}\right] = 2\pi\, \delta(\omega \mp \omega_0)}$$

FT of $\cos(\omega_0 t)$

$x(t) = \cos \omega_0 t$

$X(\omega) = FT[\cos \omega_0 t] = ?$

Since $\cos \omega_0 t$ is not an absolutely Integrable signal so we can't use the Formula of FT.

$\Rightarrow$ We know that

$$\cos \omega_0 t = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

Taking FT on both sides

$$FT[\cos \omega_0 t] = \frac{1}{2}\left[FT[e^{j\omega_0 t}] + FT[e^{-j\omega_0 t}]\right]$$

$$X(\omega) = \frac{1}{2}\left[2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\right]$$

$$\boxed{X(\omega) = \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]}$$

$$\boxed{FT[\cos \omega_0 t] = \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]}$$

FT of $\sin \omega_0 t$

$x(t) = \sin \omega_0 t$

$X(\omega) = FT[\sin \omega_0 t] = ?$

$\sin \omega_0 t$ is also not absolutely Integrable so formula of FT can't be used.

$$\Rightarrow \sin \omega_0 t = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$

Taking FT on both sides.

$$FT[\sin \omega_0 t] = \frac{1}{2j}\left[FT[e^{j\omega_0 t}] - FT[e^{-j\omega_0 t}]\right]$$

$$X(\omega) = \frac{1}{2j}\left[2\pi\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)\right]$$

$$X(\omega) = \frac{\pi}{j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

or

$$X(\omega) = \frac{j\pi}{j \cdot j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

$$X(\omega) = j\pi\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$

Thus

$$\boxed{FT[\sin \omega_0 t] = j\pi\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]}$$

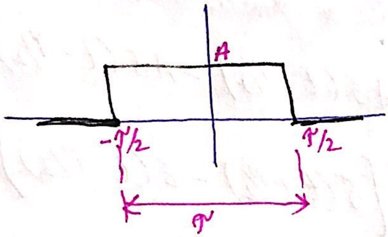
FT of Rectangular Function

Let us consider the general case

i.e., $\boxed{x(t) = A\ \text{rect}\left[\frac{t}{\tau}\right]}$

here "A" represent the amplitude while τ represent the duration for which function is having amplitude "A".

e.g.

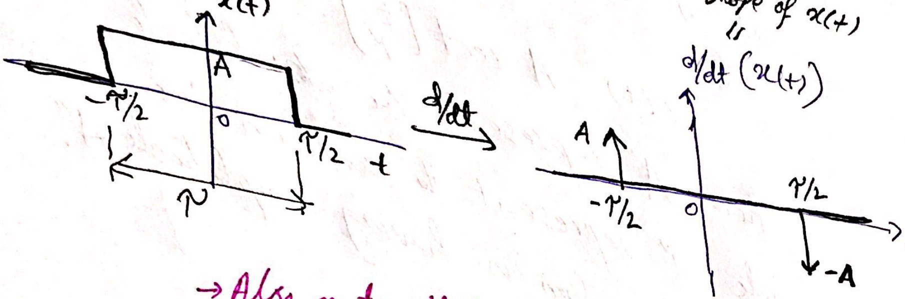


$$FT[x(t)] = FT\left[A\ \text{rect}\left(\frac{t}{\tau}\right)\right] = X(\omega) = ?$$

⇒ Look at the rectangular function
→ It is combination of step signals.

⇒ Thus we use the method of differentiation to calculate the FT.

Slope of $x(t)$ is $\frac{d}{dt}(x(t))$



→ Also note that the DC value or Avg = 0 ∵ it is a finite duration signal.

$$\frac{dx(t)}{dt} = A\delta(t+\tau/2) - A\delta(t-\tau/2)$$

$$= A\left[\delta(t+\tau/2) - \delta(t-\tau/2)\right]$$

Taking FT on both sides.

$$j\omega X(\omega) = A\left[FT[\delta(t+\tau/2)] - FT[\delta(t-\tau/2)]\right]$$

We know that $FT[\delta(t)] = 1$ but here we are having $\delta(t+\tau/2)$ and $\delta(t-\tau/2)$ so we'll use the shifting prop.

$$j\omega X(\omega) = A\left[1 . e^{j\omega\tau/2} - 1 . e^{-j\omega\tau/2}\right]$$

$$j\omega X(\omega) = A\left[e^{j\omega\tau/2} - e^{-j\omega\tau/2}\right]$$

Dividing both sides by $j\omega$.

$$X(\omega) = \frac{A}{j\omega}\left[e^{j\omega\tau/2} - e^{-j\omega\tau/2}\right]$$

$$= \frac{2A}{\omega}\left[\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{2j}\right]$$

$$\boxed{X(\omega) = \frac{2A}{\omega}\sin(\omega\tau/2)}$$

$\Rightarrow$ If we use Sampling function, we can have an other representation.

$$\text{Sampling function} = Sa\left[\omega\tau/2\right] = \frac{\sin \omega\tau/2}{\omega\tau/2}$$

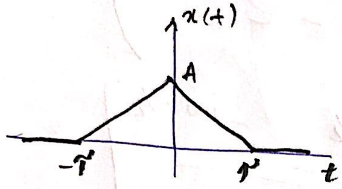
So

$$X(\omega) = \frac{2A}{\cancel{\omega}}\boxed{\frac{\sin(\omega\tau/2)}{\omega\tau/2}} \times \frac{\cancel{\omega}\tau}{2}$$

$$\boxed{X(\omega) = A\tau\, Sa(\omega\tau/2)}$$

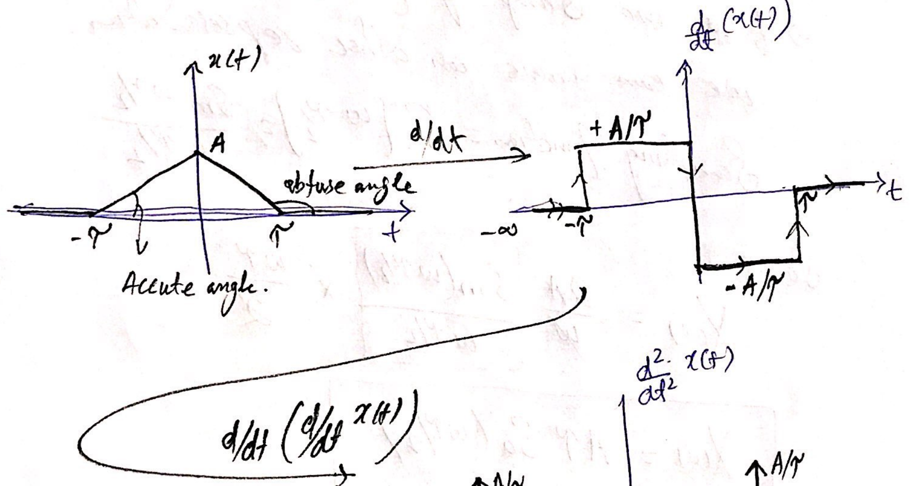
Notice that $A\tau$ is the area of the rectangle.

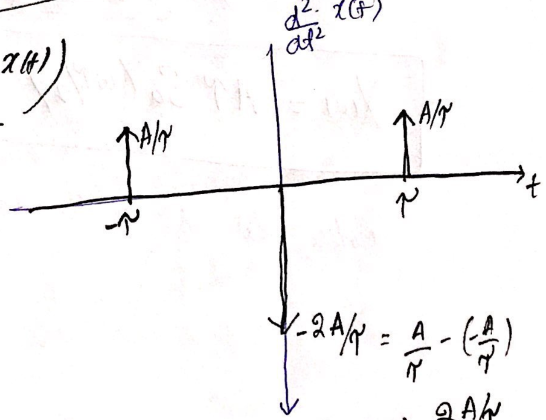
FT of Triangular function



$FT[x(t)] = X(\omega) = ?$

→ We know that $x(t)$ is a combination of Ramp signals so we can use the method of differentiation.





$-2A/\tau = \frac{A}{\tau} - \left(-\frac{A}{\tau}\right)$

$= 2A/\tau$

So

$$\frac{d^2x(t)}{dt^2} = \frac{A}{\tau}\delta(t+\tau) - \frac{2A}{\tau}\delta(t) + \frac{A}{\tau}\delta(t-\tau)$$

$$= \frac{A}{\tau}\left[\delta(t+\tau) - 2\delta(t) + \delta(t-\tau)\right]$$

FT on both sides.

$$FT\left[\frac{d^2x(t)}{dt^2}\right] = \frac{A}{\tau}\left[FT\left[\delta(t+\tau) - 2\delta(t) + \delta(t-\tau)\right]\right]$$

$$(j\omega)^2 X(\omega) = \frac{A}{\tau}\left[1.e^{j\omega\tau} - 2 + 1.e^{-j\omega\tau}\right]$$

$$-\omega^2 X(\omega) = \frac{A}{\tau}\left[e^{j\omega\tau} - 2 + e^{-j\omega\tau}\right]$$

Dividing both sides by $-\omega^2$.

$$X(\omega) = \frac{A}{-\omega^2\tau}\left[\underbrace{e^{j\omega\tau} + e^{-j\omega\tau}}_{2\cos\omega\tau} - 2\right]$$

$$= \frac{A}{-\omega^2\tau}\left[2\cos\omega\tau - 2\right]$$

$$= \frac{2A}{-\omega^2\tau}\left[\cos\omega\tau - 1\right]$$

$$X(\omega) = \frac{2A}{\omega^2 \tau}\left[1 - \cos\omega\tau\right]$$

$$= \frac{2A}{\omega^2 \tau}\left[2\sin^2(\omega\tau/2)\right]$$

$\because 1 - \cos\omega\tau = 2\sin^2(\omega\tau/2)$

$$\boxed{X(\omega) = \frac{4A}{\omega^2 \tau}\left[\sin^2(\omega\tau/2)\right]}$$

We can also write it in terms of sampling function.

$$X(\omega) = \frac{4A}{\omega^2 \tau} \frac{\sin^2(\omega\tau/2)}{\omega^2\tau^2/4} \times \frac{\omega^2\tau^2}{4}$$

$$= \tau A \left[\frac{\sin(\omega\tau/2)}{\omega\tau/2}\right]^2$$

$$\boxed{X(\omega) = A\tau\, Sa^2\left[\omega\tau/2\right]}$$

FT of Sampling Function

We know that

$$\boxed{Sa(t) = \frac{\sin t}{t}}$$

⇒ For this case let us consider the general sampling function

$$x(t) = A_0\, Sa(Kt) \overset{FT}{\rightleftharpoons} X(\omega) = ?$$

⇒ Here we can use the formula of FT but the calculations will get lengthy.

⇒ So let us use the prop. of duality

⇒ As we know that the FT of rectangular function is having the sampling function.

$$FT\left[\underbrace{A\, rect(t/\tau)}\right] = A\tau\, Sa(\omega\tau/2)$$

↳ let $y(t) = A\, rect(t/\tau)$

⇒ Now we'll use the duality prop.

i.e.,

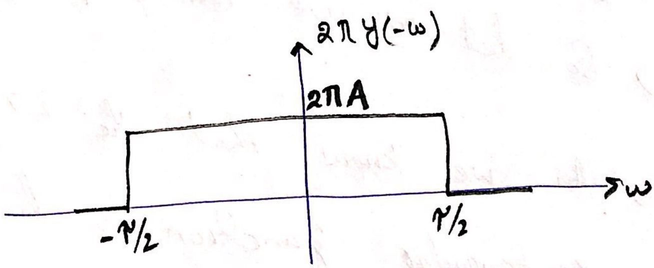
$$A\tau\, Sa\left(t\frac{\tau}{2}\right) \xrightleftharpoons{FT} 2\pi\, y(-\omega)$$

↳ ω replaced with t

↳ Replace t by $-\omega$ in time domain signal.

$$A\tau\, Sa(t\tau/2) \xrightleftharpoons{FT} 2\pi A\, rect(-\omega/\tau)$$

Plot its waveform



Now comparing the general case with $A\tau\, Sa(t\tau/2)$

i.e., $x(t) = A_0\, Sa(kt)$

We get

$A_0 = A\tau$

$K = \tau/2$

$A = A_0/\tau$

and $\tau = 2K$

so $A = A_0/2K$

So $2\pi A = \cancel{2}\pi \dfrac{A_0}{\cancel{2}K} = \dfrac{\pi A_0}{K}$