Question

$$x(t) = e^{-at^2} \quad , a > 0$$

$$X(\omega) = ?$$

Solution :-> Clearly signal $x(t)$ is a converging signal.

$$x(t) = \frac{1}{e^{at^2}} \quad a > 0$$

$$t\uparrow \Rightarrow e^{at^2}\uparrow \Rightarrow x(t)\downarrow$$

when $t \to \infty \Rightarrow e^{at^2} = e^{\infty} = \infty$

and $x(t) = \frac{1}{\infty} = 0$

$\Rightarrow$ So $x(t)$ is absolutely integrable and hence we can use the formula of FT.

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-(at^2 + j\omega t)}\, dt \longrightarrow (1)$$

we need to simplify ① s.t it may be ~~e~~ easily solvable.

=> let assume

$$at^2 = p^2 \quad \text{and} \quad j\omega t = 2pq$$

then $\boxed{p = t\sqrt{a}}$ $\quad q = \dfrac{j\omega t}{2p}$

$$q = \frac{j\omega \cancel{t}}{2\cancel{t}\sqrt{a}} = \frac{j\omega}{2\sqrt{a}}$$

$\boxed{q = \dfrac{j\omega}{2\sqrt{a}}}$ similarly $\boxed{q^2 = \dfrac{-\omega^2}{4a}}$

Hence $at^2 + j\omega t = p^2 + 2pq + q^2 - q^2$

$$\boxed{at^2 + j\omega t = (p+q)^2 - q^2} \rightarrow ②$$

put p and q in ②

$$at^2 + j\omega t = \left(t\sqrt{a} + \frac{j\omega}{2\sqrt{a}}\right)^2 + \frac{\omega^2}{4a} \rightarrow ③$$

put ③ in ①

$$X(\omega) = \int_{-\infty}^{\infty} e^{-\left[\left(t\sqrt{a} + \frac{j\omega}{2\sqrt{a}}\right)^2 + \frac{\omega^2}{4a}\right]} dt$$

Let

$$t\sqrt{a} + \frac{j\omega}{2\sqrt{a}} = u$$

then

$$\sqrt{a}\, dt = du$$

$$dt = \frac{du}{\sqrt{a}}$$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-u^2 - \frac{\omega^2}{4a}} \frac{du}{\sqrt{a}}$$

$$= \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-u^2} e^{-\omega^2/4a} \, du$$

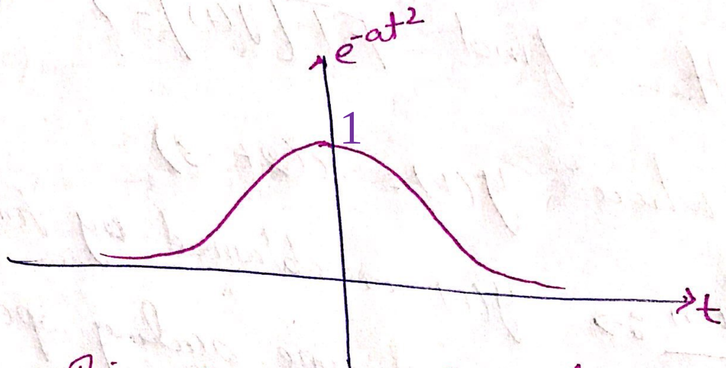
$$= \frac{e^{-\omega^2/4a}}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-u^2} du$$

$$= \frac{e^{-\omega^2/4a}}{\sqrt{a}} \sqrt{\pi} \underbrace{\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du}_{= 1 \text{ (By prop.)}}$$

$$= \frac{\sqrt{\pi}\, e^{-\omega^2/4a}}{\sqrt{a}} = \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}, \quad a > 0$$

$$\boxed{FT\left[e^{-at^2}\right] = \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}} \quad a > 0$$

e^{-at^2}, for $a>0$ can be plotted as.



This particular waveform has the gausian distribution

$\Rightarrow$ The $X(\omega)$ is also having similar type of waveform but it will be scalled.

$\Rightarrow$ Thus whenever the time domain signal is Gausian its FT will also be gausian.

Question :- If FT of $x(t)$ is $X(\omega)$.

then find $FT[y(t)] = ?$

where $y(t) = x(2t-3)$

Solution :- $y(t)$ is obtained by time shifting & time scaling operations on $x(t)$.

Method 1 :-

$$x(t) \xrightarrow{\text{Time Shift}} x(t-3) \xrightarrow{\text{Time Scaling}} x(2t-3)$$

$$\text{FT}\downarrow$$

$$X(\omega) \longrightarrow X(\omega)\, e^{-j\omega 3} \longrightarrow \frac{1}{|2|} X\left(\frac{\omega}{2}\right) \cdot e^{-j\frac{\omega}{2}3}$$

thus $$FT\left[x(2t-3)\right] = \frac{1}{2} X(\omega/2)\, e^{-j\omega 1.5}$$

Method 2

$$x(t) \xrightarrow{\text{Time Scaling}} x(2t) \xrightarrow[\text{Shifting}]{\text{Time}} x[2(t-1.5)] = y(t)$$

FT $\downarrow$

$$X(\omega) \longrightarrow \frac{1}{|2|} X\left(\frac{\omega}{2}\right) \longrightarrow \frac{1}{2} X\left(\frac{\omega}{2}\right) \cdot e^{-j\omega 1.5}$$

H.W:

$$x(t) \underset{}{\overset{FT}{\rightleftharpoons}} X(\omega)$$

$$y(t) = x(-3t+9) \overset{FT}{\rightleftharpoons} Y(\omega) = ?$$

Question :->

$$x(t) = \frac{1}{a+jt}, \quad X(\omega) = ?$$

Solution

We know that

$$FT[e^{-at}u(t)] = \frac{1}{a+j\omega}$$

Then using duality prop.

$$e^{-at}u(t) \underset{}{\overset{FT}{\rightleftharpoons}} \frac{1}{a+j\omega}$$

$\omega = t$ $\qquad\qquad$ $t = -\omega$

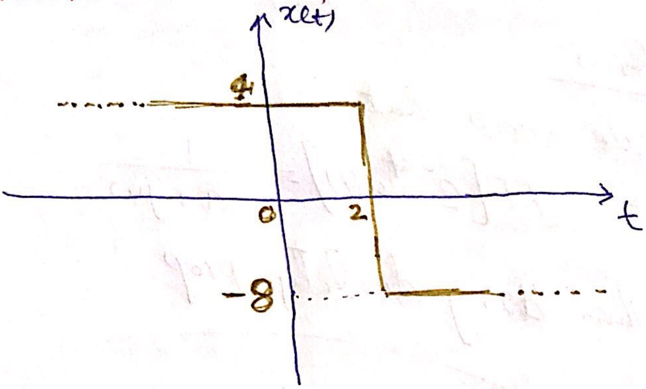
$$\frac{1}{a+jt} \overset{FT}{\rightleftharpoons} 2\pi e^{a\omega}u(-\omega)$$

$$\boxed{FT\left[\frac{1}{a+jt}\right] = 2\pi e^{a\omega}u(-\omega)}$$

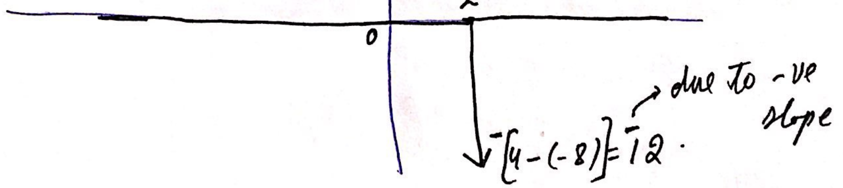
Homework :->

$$y(t) = \frac{2a}{a^2+t^2}, \quad Y(\omega) = ?$$

Question :- Find the FT of the signal shown below ?



Solution :- Since $x(t)$ is the combination of step signals so we can use the method of differentiation.



So $$\frac{dx(t)}{dt} = -12\,\delta(t-2)$$

Taking FT on both sides.

$$FT\left[\frac{dx(t)}{dt}\right] = FT\left[-12\delta(t-2)\right]$$

$$(j\omega)^1 X(\omega) = -12 \cdot 1 \cdot e^{-j\omega 2}$$

↳ due to shifting prop.

FT of $\delta(t)$

$$j\omega X(\omega) = -12e^{-2j\omega}$$

$$\boxed{X(\omega) = \frac{-12e^{-2j\omega}}{j\omega}}$$

↳ but this is not the final answer as we haven't yet calculated FT of DC value of $x(t)$

⇒ DC value of $x(t)$ is

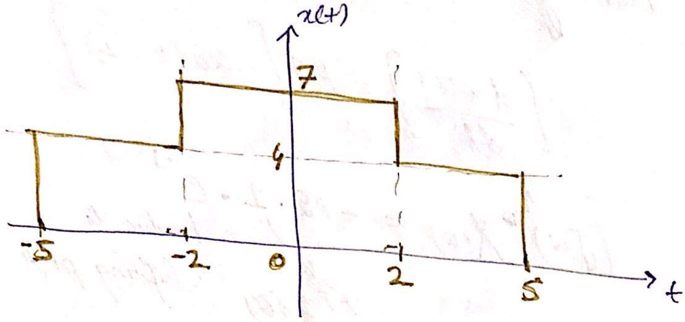
$$dc \text{ value} = \frac{4+(-8)}{2} = \frac{-4}{2} = -2$$

& $FT[-2] = 2\pi \cdot (-2) \cdot \delta(\omega) = -4\pi\delta(\omega)$

So Finally.

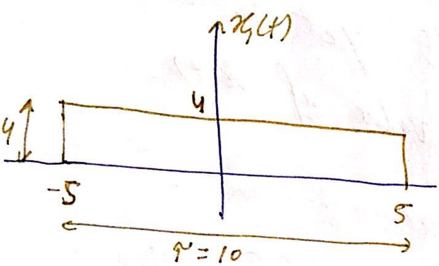
$$X(\omega) = \frac{-12e^{-2j\omega}}{j\omega} - 4\pi\delta(\omega)$$

Question :→ Find FT of signal below?



Solution :→ This time we do not use method of differentiation.

⇒ We'll rather break x(t) in 2 rectangles.



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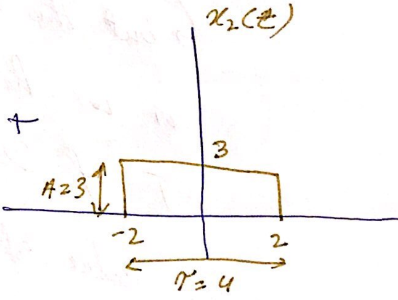


→ Hence

$$x(t) = x_1(t) + x_2(t)$$

By linearity prop when two time domain signals are added their FTs are also added in Freq. domain.

Hence

$$X(\omega) = X_1(\omega) + X_2(\omega) \longrightarrow (1)$$

$$X_1(\omega) = A\tau \, Sa\left(\frac{\omega\tau}{2}\right)$$

$\tau = 10, \quad A = 4$

So

$$X_1(\omega) = 40 \, Sa(5\omega)$$

Similarly

$$X_2(\omega) = A\tau \, Sa(\omega\tau/2)$$

$A = 3, \quad \tau = 4$

$$X_2(\omega) = 12 \, Sa(2\omega)$$

Put in (1)

$$\boxed{X(\omega) = 40 \, Sa(5\omega) + 12 \, Sa(2\omega)}$$

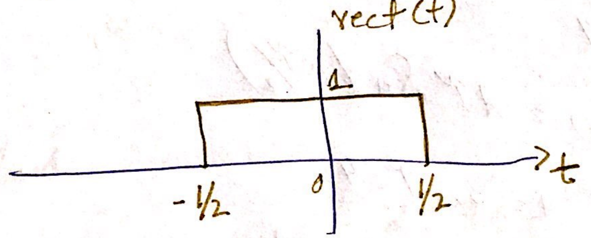
Question Let $x(t) = rect(t - 1/2)$

then find $FT[x(t) + x(-t)] = ?$

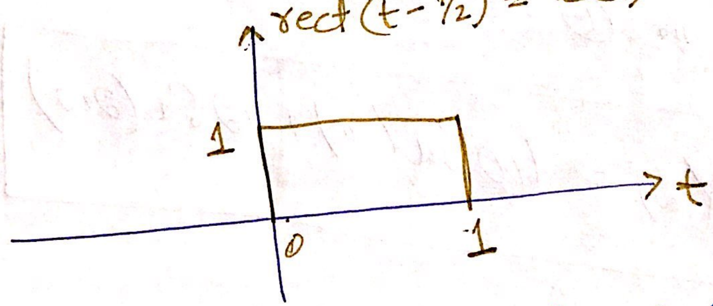
Solution :→ First we need to plot

$rect(t)$ [compare with general case $A\, rect(t/T)$]

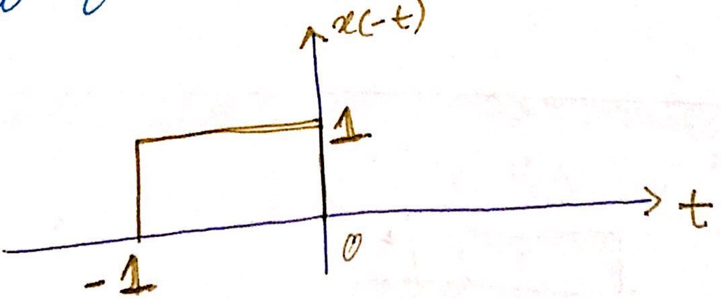
$A = 1,\ T = 1$



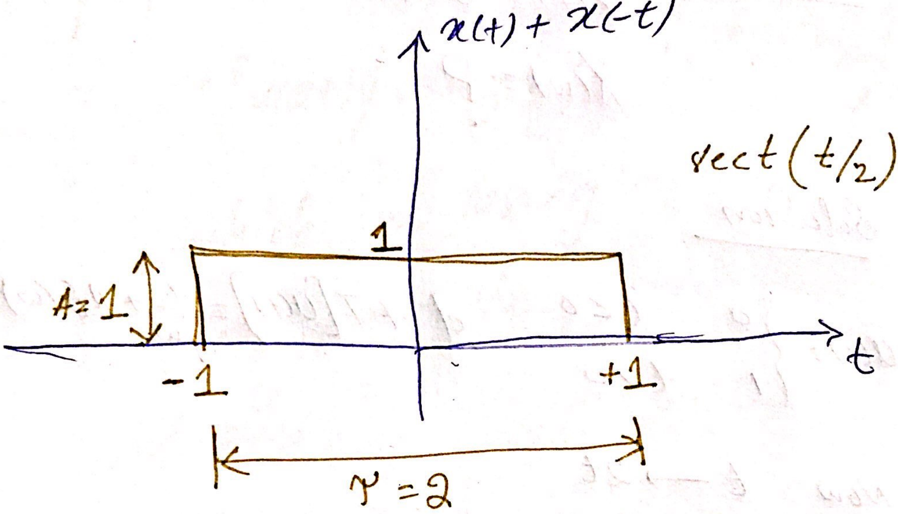
⇒ Now perform right shifting by 1/2.



Now from time reversal we get.



Now $x(t) + x(-t)$



Thus

$$FT\left[x(t) + x(-t)\right] = A\tau\, Sa\left(\omega \frac{\tau}{2}\right)$$

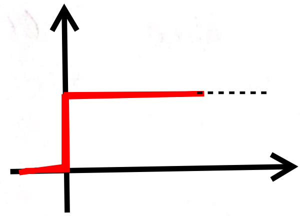
$$= 1 \cdot 2\, Sa\left(\omega \frac{2}{2}\right)$$

$$\boxed{= 2\, Sa(\omega)}$$

Question

$x(t) = u(2t)$

$X(\omega) = ?$



Solution

$$u(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases} \quad \& \quad FT[u(t)] = \frac{1}{j\omega} + \pi\delta(\omega)$$

Now $t \longrightarrow 2t$

$$u(2t) = \begin{cases} 0 & 2t<0 \Rightarrow t<0 \\ 1 & 2t>0 \Rightarrow t>0 \end{cases}$$

Clearly $u(2t)$ and $u(t)$ are the same signals so FT will be the same. i.e.,

$$\boxed{X(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)}$$

Common Mistake

we know that if signal $x(t)$ have the FT $X(\omega)$ i.e.,

$$FT[x(t)] = X(\omega)$$

then using the prop. $FT[x(at)] = \frac{1}{|a|} X(\omega/a)$

⇒ In this case if we use this prop. then

$$FT\{u(2t)\} = \frac{1}{2}\left[\frac{1}{j\omega/2} + \pi\delta(\omega/2)\right]$$

which is wrong.

⇒ So don't use the properties blindelly.

2·5

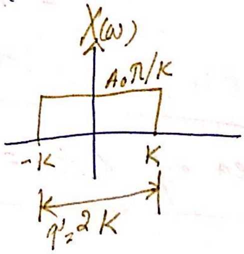
<u>Question</u> :→ Calculate area & energy of

$$x(t) = Sa(t) ?$$

<u>Solution</u> :-)

The generalized Sampling function is represented as $A_0 Sa(kt)$ which is having FT as a rectangular pulse.

$$FT[A_0 Sa(kt)] =$$



⇒ Now the generalized rectangular function is

$$A \, rect(t/\tau)$$

$\swarrow$ amplitude, $\tau \to$ width.

⇒ Now $x(t) = Sa(t)$

Comparing $Sa(t)$ with the generalized form $A_0 \, Sa(kt)$ gives.

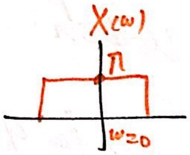
$$A_0 = 1, \quad k = 1$$

Hence

$$X(\omega) = \pi \, rect(\omega/2)$$

⇒ Area of $x(t) = X(\omega)\big|_{\omega=0}$

$$= \pi \, rect(0)$$

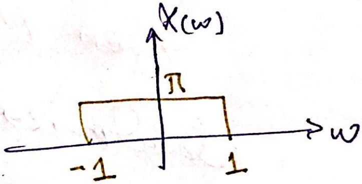


at $\omega = 0$, $X(\omega) = \pi$

$$\boxed{\text{So area of } x(t) = \pi}$$ Ans.

⇒ To calculate the energy we will us the Parsival's energy theorem:

$$E_{x(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$



$$E_{x(t)} = \frac{1}{2\pi}\int_{-1}^{1} \pi^2 d\omega$$

$$= \frac{\pi^2}{2\pi}\int_{-1}^{1} d\omega \Rightarrow \frac{\pi}{2}\left[\omega\right]_{-1}^{1}$$

$$= \frac{\pi}{2}\left(1-(-1)\right) = \pi.$$

$$\boxed{\text{Total Energy} = E_{x(t)} = \pi}$$

Question Suppose

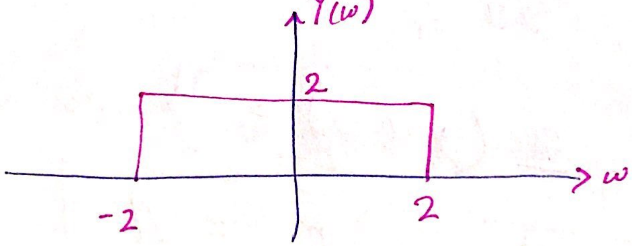
$y(t) = x(t)\cos t$ and

$$Y(\omega) = \begin{cases} 2 & |\omega| \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

then find $x(t)$?

Solution ↳ $Y(\omega)$ can be written as.

$$Y(\omega) = \begin{cases} 2 & -2 \leq \omega \leq 2 \\ 0 & \text{otherwise} \end{cases}$$



i.e., FT is a rectangular fun^n.

⇒ We know that $FT[A_0 Sa(Kt)]$

$\Leftrightarrow \frac{A_0 \pi}{K} rect(\omega/2K)$

Thus $2 = \frac{A_0 \pi}{2} \Rightarrow A_0 \pi = 4 \Rightarrow A_0 = \frac{4}{\pi}$

$\tau = 2K = 4$

Thus

$$IFT\left[Y(\omega)\right] = A_0 \, Sa(kt)$$

$$= \frac{4}{\pi} Sa(2t)$$

$$= \frac{4}{\pi} \frac{\sin 2t}{2t}$$

double angle identity

$$= \frac{4}{\pi} \frac{\cancel{2} \sin t \cos t}{\cancel{2} t}$$

$$= \underbrace{\boxed{\frac{4}{\pi t} \sin t}}_{x(t)} \cos t$$

$$\boxed{x(t) = \frac{4 \sin t}{\pi t}}$$

Question 1-) The FT of a signal $h(t)$ is $H(\omega) = (2\cos\omega)(\sin 2\omega)/\omega$. Then $h(0) = ?$

Solution :-> Let us use the trigonometric identity

$$2\cos A \sin B = \sin(A+B) - \sin(A-B)$$

Hence

$$H(\omega) = \frac{\sin(\omega+2\omega) - \sin(\omega-2\omega)}{\omega}$$

$$= \frac{\sin 3\omega + \sin \omega}{\omega}$$

$$= \frac{\sin 3\omega}{\omega} + \frac{\sin \omega}{\omega}$$

$$= 3\,\frac{\sin 3\omega}{3\omega} + \frac{\sin \omega}{\omega}$$

$$\boxed{H(\omega) = 3\,Sa(3\omega) + Sa(\omega)}$$

Now we can find $h(t)$ by taking the IFT.

So let $H_1(\omega) = 3Sa(3\omega)$ & $H_2(\omega) = Sa(\omega)$

and $h(t) = IFT[H_1(\omega)] + IFT[H_2(\omega)]$

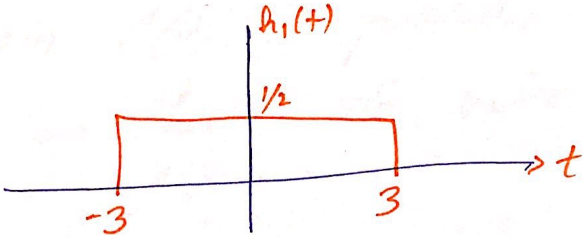
$h(t) = h_1(t) + h_2(t) \rightarrow (1)$

$\Rightarrow$ Now we know that FT of $A\,\mathrm{rect}(t/\tau)$ [rectangular function] comes out to be $Sa(\cdot)$ function. i.e.,

$$FT[A\,\mathrm{rect}(t/\tau)] = \underbrace{A\tau\, Sa(\omega\tau/2)}_{\text{Compare this to}}$$

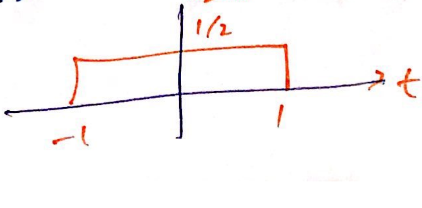
$H_1(s)$ $\quad A\tau = 3,\ \frac{\omega\tau}{2} = 3\omega \Rightarrow \tau = 6$

$\tau = 6,\ A = 1/2$



$\Rightarrow$ Now compare with $H_2(\omega)$

$A\tau = 1,\ \omega\tau/2 = \omega \Rightarrow \tau = 2,\ A = 1/2$



Thus

$$h(0) = h_1(0) + h_2(0)$$

$$= 1/2 + 1/2$$

$$\boxed{h(0) = 1}$$