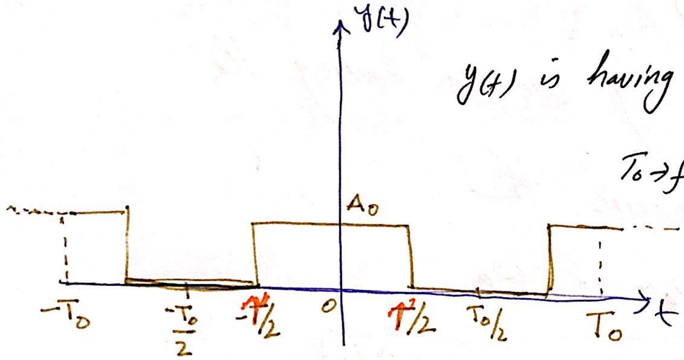
Calculation of C_n using FT

$C_n \rightarrow$ Part of complex exponential Fourier series

(Fourier Series is only used for periodic signals)



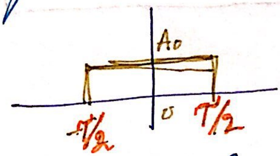
$y(t)$ is having $C_n = \dfrac{X(n\omega_0)}{T_0}$

$T_0 \rightarrow$ fundamental Time period.

$X(n\omega_0) = ?$

$\Rightarrow$ We know that the rectangular pulse train $y(t)$ is periodic.

$\Rightarrow$ and let us choose the centre rectangular pulse



Let this rectangular pulse be represented by $x(t)$

then

$$X(\omega) = FT[x(t)] = A\tau\, Sa(\omega\tau/2)$$

Now let $\omega = \omega_0$ (fundamental freq)

then

$$X(\omega_0) = A_0\tau\, Sa(\omega_0\tau/2)$$

Now let say we are having the nth harmonic then

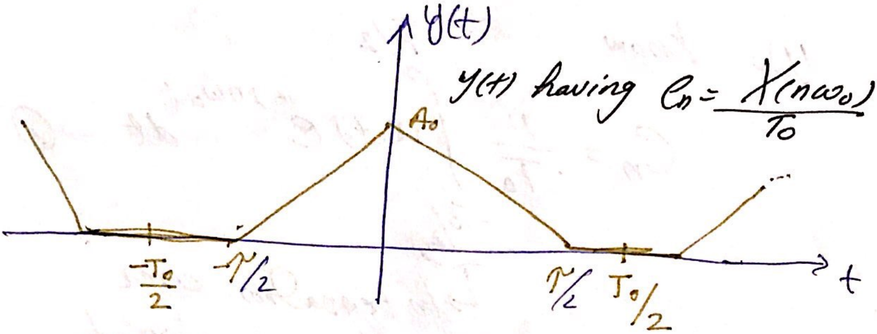
$$\omega_0 \longrightarrow n\omega_0$$

$$X(n\omega_0) = A_0\tau\, Sa(n\omega_0\tau/2)$$

Thus

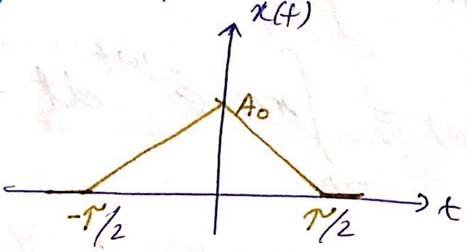
$$\boxed{C_n = \frac{A_0\tau\, Sa(n\omega_0\tau/2)}{T_0}}$$

What if the periodic signal is a triangular pulse train?



$y(t)$ having $C_n = \frac{X(n\omega_0)}{T_0}$

⇒ Once again choosing the central triangular pulse



⇒ For a triangular pulse $x(t)$ we know

$$X(\omega) = FT\{x(t)\} = A_0 \tau \, Sa^2(\omega \tau/2)$$

Replace ω by $n\omega_0$

$$X(n\omega_0) = A_0 \tau \, Sa^2(n\omega_0 \tau/2)$$

$$C_n = \frac{X(n\omega_0)}{T_0} = \frac{A_0 \tau}{T_0} Sa^2(n\omega_0 \tau/2)$$

How $C_n = \frac{X(n\omega_0)}{T_0}$?

We know

$$C_n = \frac{1}{T_0} \underbrace{\int_{-T_0/2}^{T_0/2}}_{\text{Integration over one time period.}} x(t)\, e^{-jn\omega_0 t}\, dt \rightarrow ①$$

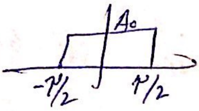
while standard definition of FT is

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \rightarrow ②$$

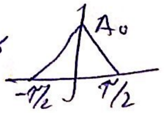
Replace ω by $n\omega_0$ in ②

$$X(n\omega_0) = \int_{-\infty}^{\infty} x(t)\, e^{-jn\omega_0 t}\, dt \rightarrow ③$$

=> if we look at



or



we see that $x(t)$ is non-zero only b/w $-\tau/2$ & $\tau/2$.

So the limits in 1 & 3 gets changed accordingly.

$$C_n = \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} x(t) e^{-jn\omega_0 t} dt \rightarrow 4$$

$$X(n\omega_0) = \int_{-\tau/2}^{\tau/2} x(t) e^{-jn\omega_0 t} dt \rightarrow 5$$

Use 5 in 4

$$\boxed{C_n = \frac{X(n\omega_0)}{T_0}}$$

Fourier Transform for periodic signals

⇒ We know that FT is used for both periodic & non-periodic signals while Fourier Series is used only for periodic signals.

⇒ Now how do we use FT on periodic signals?

⇒ Let $x(t)$ be a periodic signal

⇒ Since $x(t)$ is periodic thus it can have a complex exponential Fourier series expansion.

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \rightarrow ①$$

$\Rightarrow$ We know that FT of DC value A_0 is

$$FT[A_0] = 2\pi A_0 \delta(\omega)$$

if $A_0 = 1$ then.

$$FT[1] = 2\pi \delta(\omega)$$

We know that "FT follows the principle of homogeneity"

i.e multiplying C_n in time domain apears as it is in the freq. domain.

$$FT[C_n \cdot 1] = 2\pi C_n \delta(\omega)$$

Now multiplying $e^{jn\omega_0 t}$

$$FT[C_n \underbrace{e^{+jn\omega_0 t}}] = 2\pi C_n \delta(\omega - n\omega_0)$$

Produces shift in frequency.

⇒ Now once again

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

Since FT follow the law of additivity

So

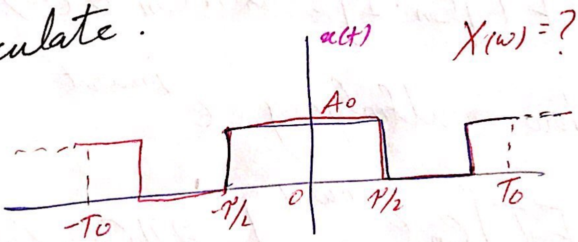
$$FT[x(t)] = \sum_{n=-\infty}^{\infty} FT[C_n e^{jn\omega_0 t}]$$

$$\boxed{X(\omega) = \sum_{n=-\infty}^{\infty} 2\pi C_n \delta(\omega - n\omega_0)}$$

So if C_n is known the FT of any periodic signal is very easy to calculate.

$X(\omega) = ?$

E.g



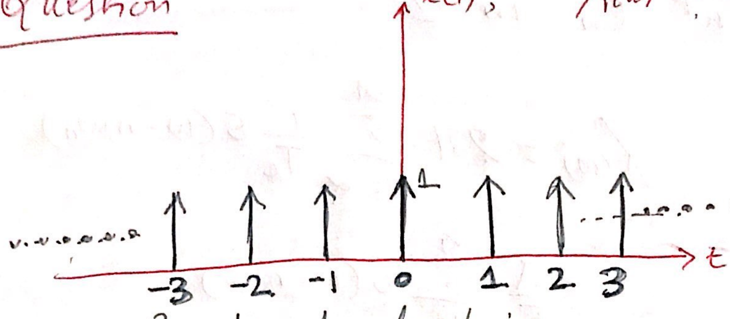
we found previously that

$$C_n = \frac{A_0 \tau}{T_0} Sa(n\omega_0 \tau/2)$$

$$\boxed{X(\omega) = \sum_{n=-\infty}^{\infty} \frac{2\pi A_0 \tau}{T_0} Sa(n\omega_0 \tau/2)\, \delta(\omega - n\omega_0)}$$

Question

$X(\omega) = ?$



Periodic Impulse train

Solution :→ We know that

$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} C_n \, \delta(\omega - n\omega_0) \rightarrow (1)$$

when C_n is the complex exponential fourier co-efficient.

$$C_n = \frac{X(n\omega_0)}{T_0} \rightarrow (2)$$

For $X(n\omega_0)$ select the central impulse.

thus $X(n\omega_0) = 1$

put this in (2)

$$\boxed{C_n = \frac{1}{T_0}}$$

Put C_n in (2)

$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} \frac{1}{T_0} \delta(\omega - n\omega_0)$$

$$= \frac{2\pi}{T_0} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

Now $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$

So

$$\boxed{X(\omega) = \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)}$$

$x(t) - X(\omega)$ Pairs

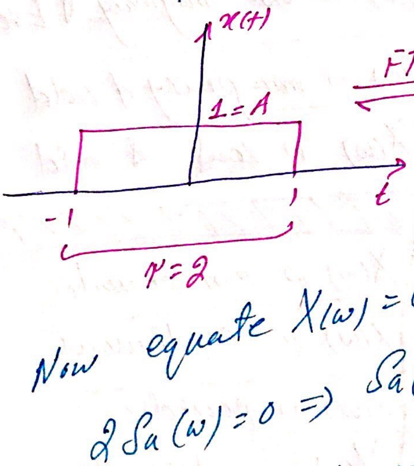
$x(t)$	$X(\omega)$
① $x(t)$ is real	① $X(\omega)$ is conjugate symmetric
② $x(t)$ is conjugate symmetric	② $X(\omega)$ is real
$x(t)$ is imaginary	$X(\omega)$ is conjugate anti Symmetric
$x(t)$ is conjugate anti Symmetric	$X(\omega)$ is imaginary
$x(t)$ is real & Even	$X(\omega)$ is also real & Even
$x(t)$ is Imaginary & Even	$X(\omega)$ is also Imaginary & Even
$x(t)$ is Real & odd	$X(\omega)$ is imaginary & odd
$x(t)$ is Imaginary & odd	$X(\omega)$ is real & odd
$x(t)$ is continuous	$X(\omega)$ is non-periodic
$x(t)$ " non-periodic	$X(\omega)$ " Continuous.
$x(t)$ " discrete	$X(\omega)$ is periodic.
$x(t)$ is periodic	$X(\omega)$ is discrete
$x(t)$ " Continuous & periodic	$X(\omega)$ is discrete & non-periodic
$x(t)$ " " & non-periodic	$X(\omega)$ " also cont. & non-periodic
$x(t)$ " discrete & periodic	$X(\omega)$ " also disc. & periodic
" " " " non- "	" " Continuous & periodic.

Question

$$x(t) = \begin{cases} 1 & -1 \le t \le +1 \\ 0 & \text{otherwise} \end{cases}$$

Find $X(\omega)$ and then 2 of the angular frequencies at which $X(\omega) = 0$?

Solution → Clearly $x(t)$ is a rectangular pulse.



$$\xrightleftharpoons{FT} \quad X(\omega) = A\tau\, Sa(\omega\tau/2)$$

$$X(\omega) = 2\, Sa(\omega)$$

Now equate $X(\omega) = 0$ which implies

$2\,Sa(\omega) = 0 \Rightarrow Sa(\omega) = 0 \Rightarrow \frac{\sin(\omega)}{\omega} = 0$

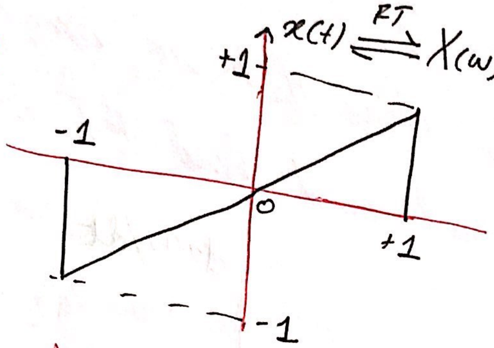
$\sin(\omega) = 0$ at $\omega = 0, \pi, 2\pi \ldots$

Can we take the 2 frequencies as $\omega = 0$ & $\omega = \pi$?

No, bcz at $\omega = 0$ $Sa(0) = 1$ so correct answer is π & 2π

Question

$x(t) \xrightleftharpoons{FT} X(\omega) = ?$



$X(\omega) = ?$

Solution :-> See now $x(t)$ is combination of Ramps & Steps.

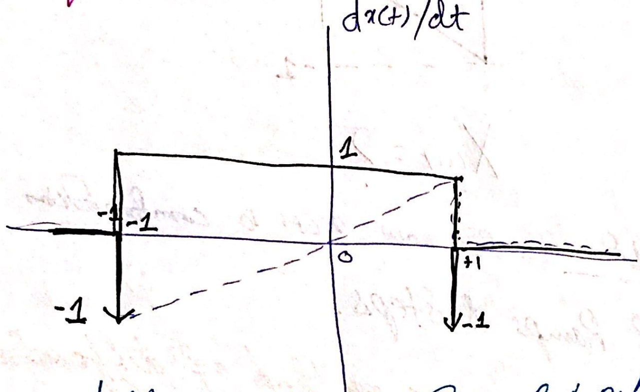
=> So we can use method of differentiation (i.e., differentiate the signal until we've a combination of impulses)

=> However notice that

$$\frac{du(t)}{dt} = \delta(t) \rightarrow \text{one derivative}$$

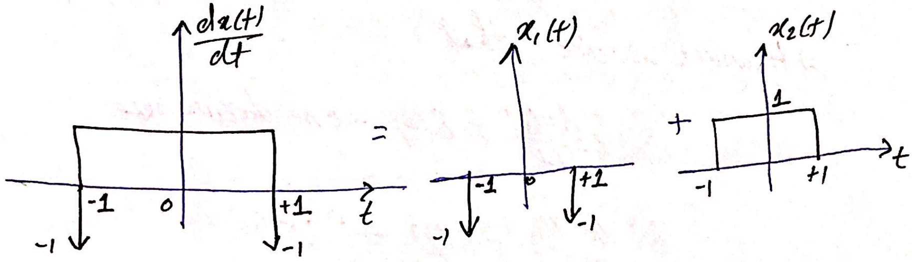
$$\frac{d^2 r(t)}{dt^2} = \delta(t) \rightarrow \text{Differentiate twice.}$$

So in this case when $x(t)$ is combination of ramps & steps then differentiation limits to 1 (one time) and after that we break the signal and find FT separately.



$\frac{dx(t)}{dt}$ is not combination of impulses alone.

So break it as.



So $\frac{dx(t)}{dt} = x_1(t) + x_2(t)$

Taking FT on both sides.

$$j\omega X(\omega) = X_1(\omega) + X_2(\omega) \rightarrow ①$$

$x_1(t) = -\delta(t+1) - \delta(t-1)$

$= -[\delta(t+1) + \delta(t-1)]$

$X_1(\omega) = FT[x_1(t)] = -[e^{j\omega} + e^{-j\omega}]$

$= -2\left[\frac{e^{j\omega} + e^{-j\omega}}{2}\right]$

$$\boxed{X_1(\omega) = -2\cos\omega} \rightarrow ②$$

$x_2(t) = A.\mathrm{rect}(t/\tau)$

$A = 1,\ \tau = 2$

and

$FT[x_2(t)] = X_2(\omega) = A\tau\, Sa(\omega\tau/2)$

$= 2\, Sa(\omega)$

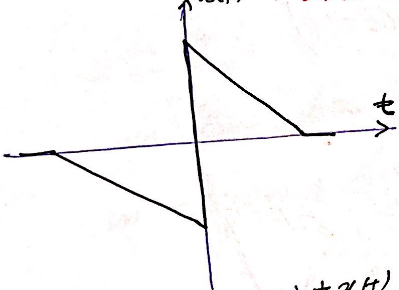
$$\boxed{X_2(\omega) = 2\, Sa(\omega) = 2\frac{\sin\omega}{\omega}} \rightarrow ③$$

put ② & ③ in ①

$$X(\omega) = \frac{-2\cos\omega + 2\sin\omega/\omega}{j\omega}$$

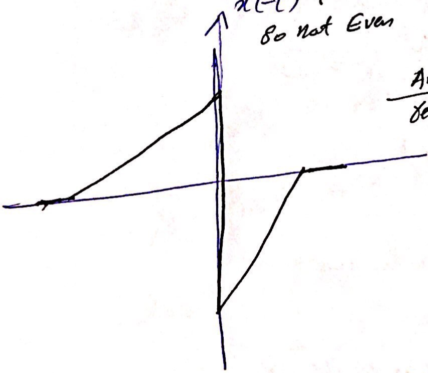
$$\boxed{X(\omega) = 2\left[\frac{-\omega\cos\omega + \sin\omega}{j\omega^2}\right]}$$

$x(t) \rightarrow$ Even or Odd.



$x(-t) \neq x(t)$

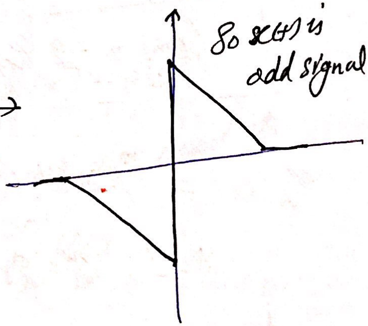
So not Even



$\xrightarrow[\text{reversal}]{\text{Amp}}$

$-x(-t) = x(t)$

So $x(t)$ is odd signal.



Question if $x(t) = \begin{cases} 1 & |t| < 1 \\ 0 & |t| > 1 \end{cases}$

Then find FT of $y(t) = x(2t) * x(t/2)$?

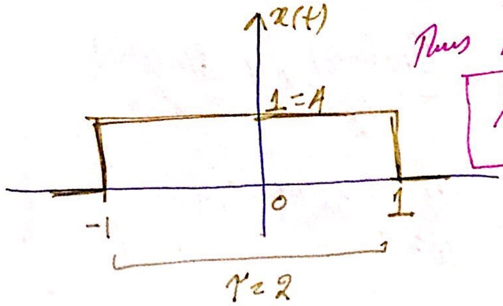
Solution :→ We know that convolution in time domain becomes multiplication in frequency domain

$$FT[y(t)] = Y(\omega) = FT[x(2t)]\, FT[x(t/2)]$$

$$Y(\omega) = X_1(\omega) \cdot X_2(\omega) \rightarrow (1)$$

To find $X_1(\omega)$ & $X_2(\omega)$ we need to find $X(\omega) = FT[x(t)]$

$$x(t) = \begin{cases} 1 & |t| < 1 \text{ or } -1 < t < 1 \\ 0 & |t| > 1 \text{ or } -1 > t > 1 \end{cases}$$



Thus $X(\omega) = A\tau\, Sa(\omega\tau/2)$

$$\boxed{X(\omega) = 2\, Sa(\omega)}$$

By using the time scaling prop of FT

$$X_1(\omega) = FT[x(2t)] = \frac{1}{|2|} X(\omega/2)$$

$$= \frac{1}{\cancel{2}} \cancel{2} Sa(\omega/2)$$

$$\boxed{X_1(\omega) = Sa(\omega/2)}$$

Similarly

$$X_2(\omega) = FT[x(t/2)] = \frac{1}{|1/2|} X(\omega/1/2)$$

$$= 2X(2\omega) = 2 \cdot 2Sa(2\omega)$$

$$X_2(\omega) = 4Sa(2\omega)$$

Thus $Y(\omega) = X_1(\omega) * X_2(\omega)$

$$= Sa(\omega/2) \times 4Sa(2\omega)$$

$$= \frac{Sin(\omega/2)}{\omega/\cancel{2}} \times 4 \frac{Sin(2\omega)}{\cancel{2}\omega}$$

$$\boxed{Y(\omega) = \frac{4}{\omega^2}[Sin(\omega/2) Sin(2\omega)]}$$

let $G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt = \omega e^{-2\omega^2}$

for any real value ω.

if $y(t) = \int_{-\infty}^{t} g(\tau) d\tau$ then find/Evaluate

$\int_{-\infty}^{\infty} y(t) dt$?

<u>Solution</u> Since $y(t) = \int_{-\infty}^{t} g(\tau) d\tau$

then $FT[y(t)] = FT\left[\int_{-\infty}^{t} g(\tau) d\tau\right]$

$$Y(\omega) = \frac{G(\omega)}{j\omega} = \frac{\cancel{\omega} e^{-2\omega^2}}{j\cancel{\omega}} + \pi \underbrace{G(0)}_{0} \delta(\omega)$$

As $G(0) = 0 \times e^{0} = 0$ thus

$$\boxed{Y(\omega) = e^{-2\omega^2}/j = -je^{-2\omega^2}}$$

Now

$I = \int_{-\infty}^{\infty} y(t) dt \rightarrow$ Total area under $y(t)$

then by prop. of FT

$$I = \int_{-\infty}^{\infty} y(t) dt = Y(\omega)\Big|_{\omega=0} = -je^{-2\times 0^2} = -j$$

$$\boxed{I = \int_{-\infty}^{\infty} y(t) dt = -j}$$