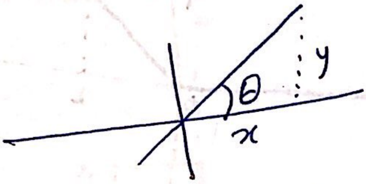
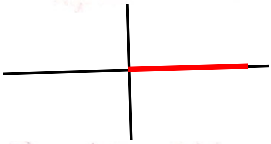
Differentiation of Continuous Time Signals

① ⇒ Slope of a signal $x(t)$ is its first total time derivative w-r-t it's independent variable.

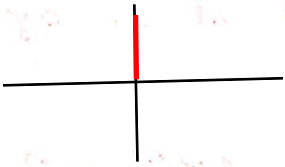
$$\text{Slope} = \frac{d}{dt} x(t)$$

② 

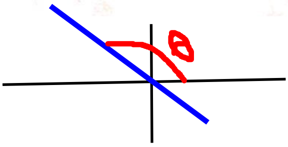
$$\boxed{\text{Slope} = \frac{y}{x} = \tan\theta}$$

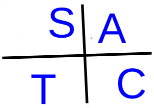
③ 

$$\text{Slope} = \tan 0^\circ = 0$$

④ 

$$\text{Slope} = \frac{y}{0} = \tan 90^\circ = \infty$$

⑤ 



when $\theta > 90^\circ$ then $90^\circ < \theta < 180^\circ$

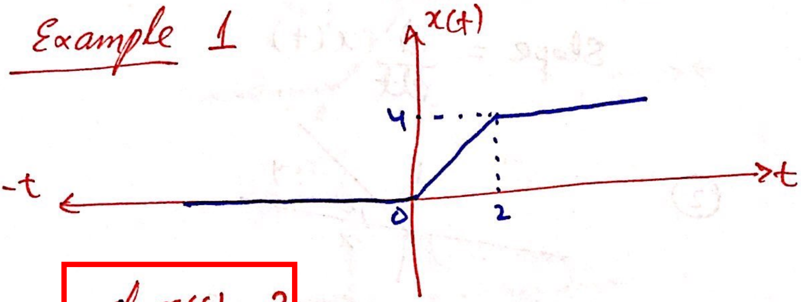
slope becomes $-ve$.

There are two methods to differentiate CTS

① Mathematical Method

② Graphical Method → Applicable to signals which are represented only as ramps and steps

Example 1



$$\boxed{\frac{d}{dt}x(t) = ?}$$

Solution

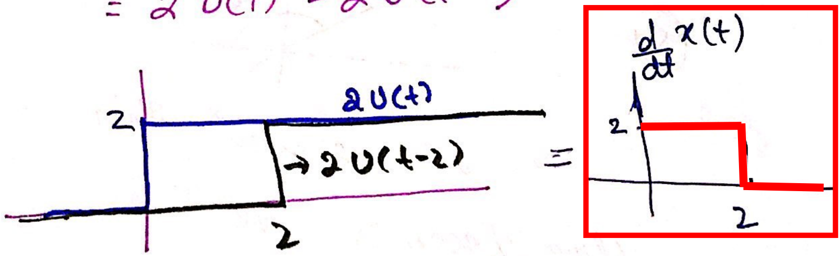
① Mathematical method.

$$x(t) = 0 + 2r(t) - 2r(t-2)$$

$$\frac{d\,x(t)}{dt} = \frac{d}{dt}\left[2r(t) - 2r(t-2)\right]$$

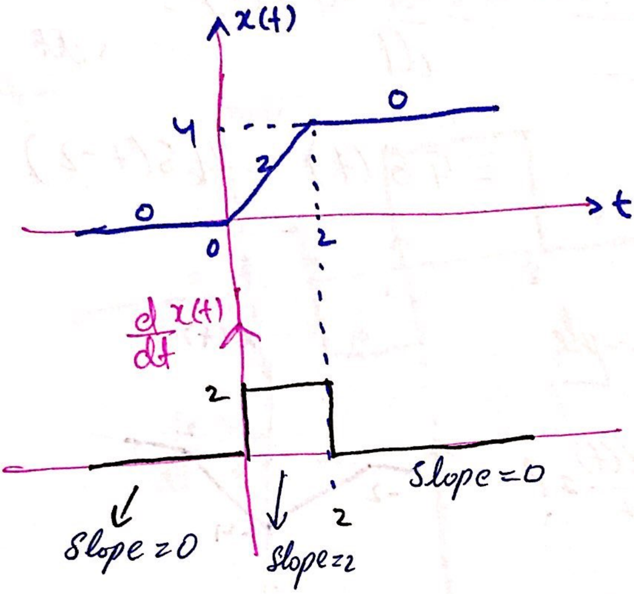
$$= 2\frac{d}{dt}r(t) - 2\frac{d}{dt}r(t-2)$$

$$= 2U(t) - 2U(t-2)$$



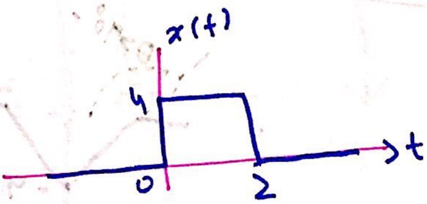
1) Graphical Method

In this method we just use the slopes of the line segments



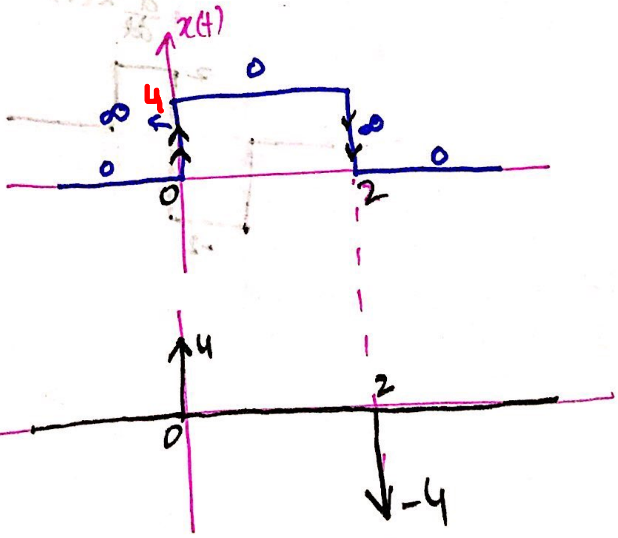
Example 2

$\frac{d}{dt} x(t) = ?$



Solution

(1) Graphical



2) Mathematical Method

$$x(t) = 0 + 4U(t-0) - 4U(t-2)$$

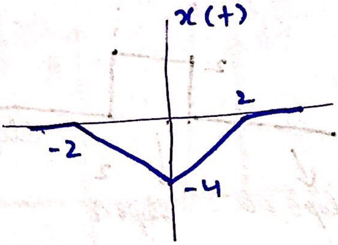
$$= 4U(t) - 4U(t-2)$$

$$\frac{d\,x(t)}{dt} = \frac{d}{dt}\,4U(t) - 4\frac{d}{dt}U(t-2)$$

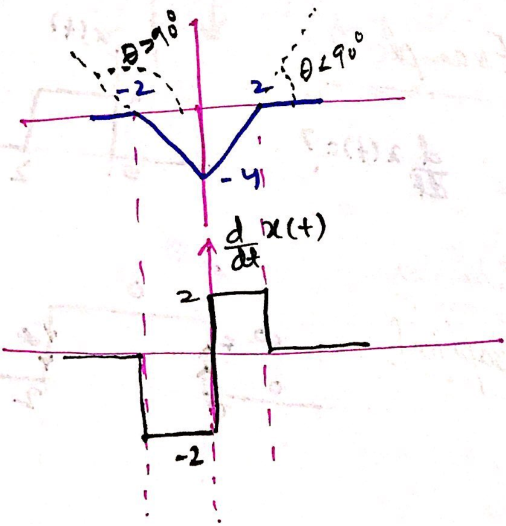
$$= 4\delta(t) - 4\delta(t-2)$$

Example

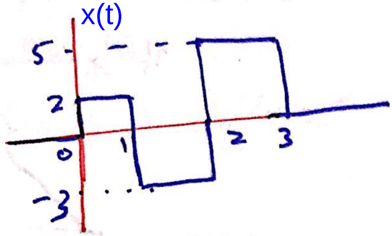
H.W $\quad \dfrac{dx(t)}{dt} = ?$



Solution

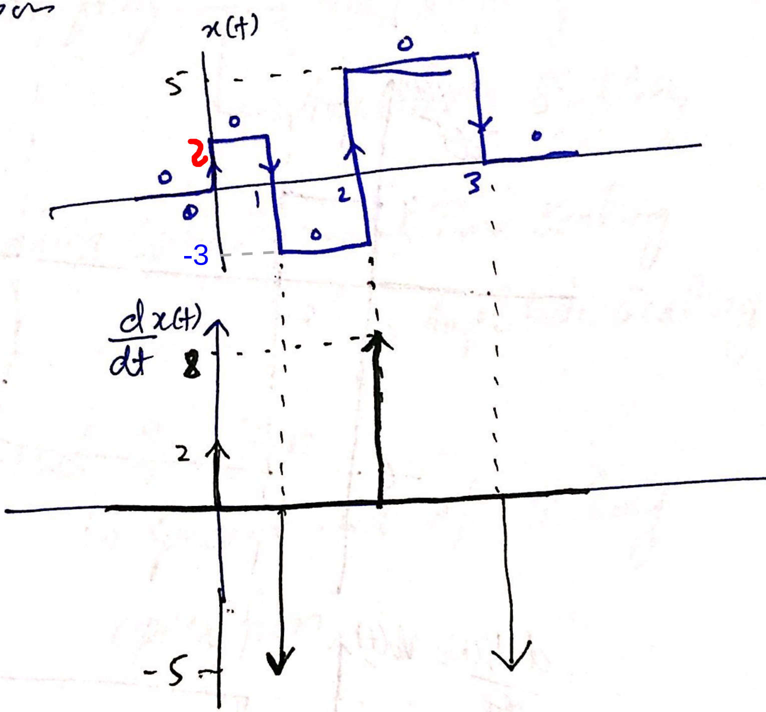


H.W Example

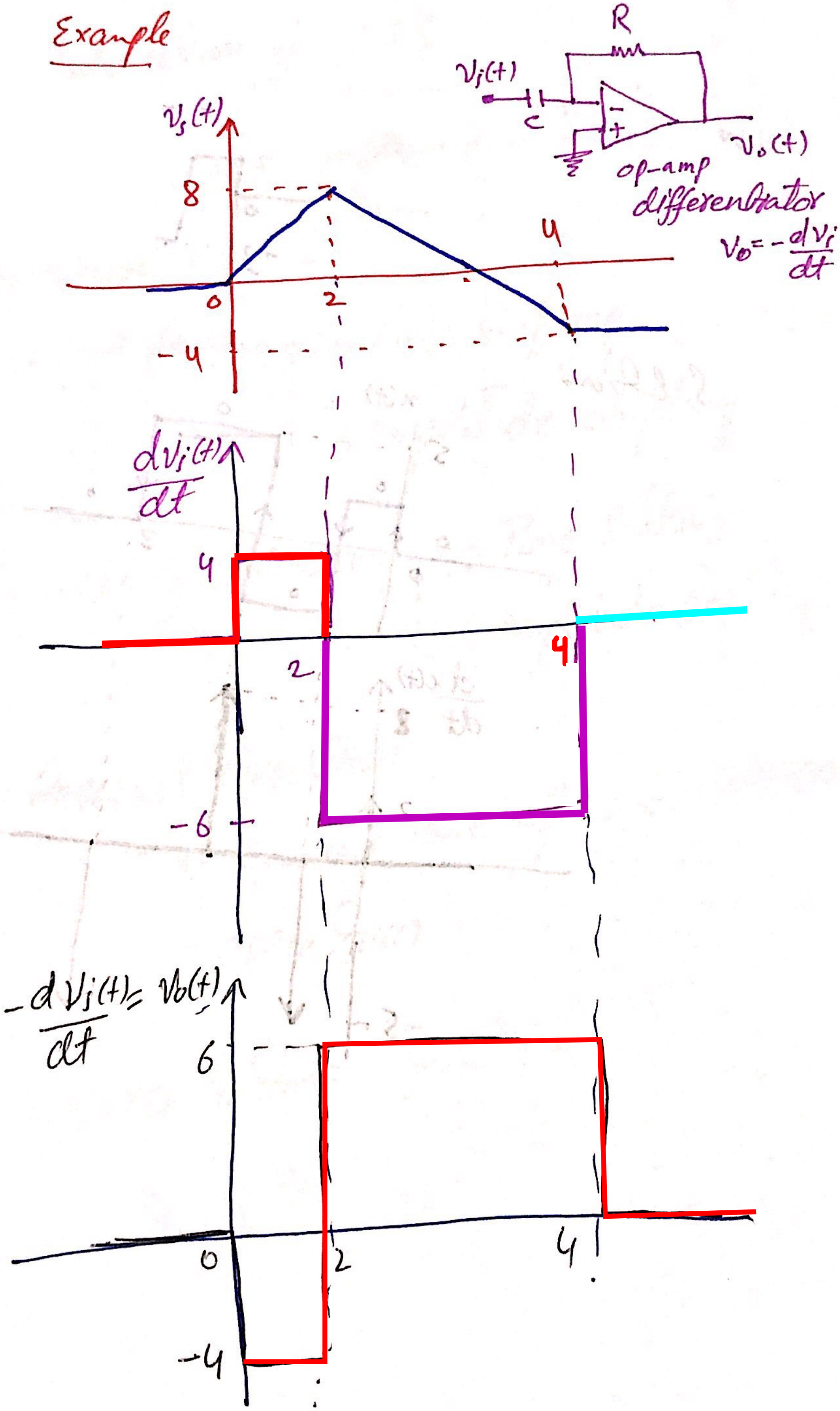


Find Derivative of x(t)?

Solution



Example


R
vi(t)
C
vo(t)
op-amp
differentiator
vo = -dvi/dt
vi(t)
8
4
0
2
-4
dvi(t)/dt
4
2
4
-6
-dvi(t)/dt = vo(t)
6
0
2
4
-4

Integration of Continuous Time Signals

Till now we've performed 4 different operations on CTS

1. Shifting → Time shifting
   → Amplitude Shifting

2. Scaling Operation → Time Scaling
   → Amplitude Scaling

3. Reversal Operation
   → Special case of scaling operation.

4. Differentiation Operation.

Now the 5th operation on CTS is the "integration" operation.

→ There are 3 methods to perform integration operation on CTS.

① Graphical Method

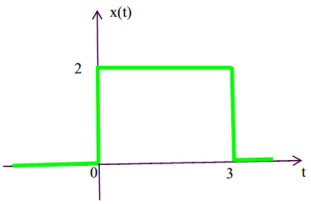
② Mathematical Method

③ Barrier Method

① Graphical Method :->

Like the graphical differetiation was limited to signals which were either step or ramp or combination of these,

=> The graphical integration limited only to step signals.



$y(t) = \int x(\tau)d\tau$ is just the area of the signal $x(t)$

=> We start from $-\infty$ and keep increamenting

$-\infty \to 0$ $\qquad x(t) = 0$

$$y(t) = \int_{-\infty}^{0} 0\,dt = 0$$

$-\infty \to 1$

$$y(t) = 2$$

$-\infty \to 2$

$$y(t) = 2 + \int_{1}^{2} x(\tau)\,d\tau = 4$$

$-\infty \to 3$

$$y(t) = 4 + \int_{2}^{3} x(\tau)\,d\tau = 4+2 = 6$$

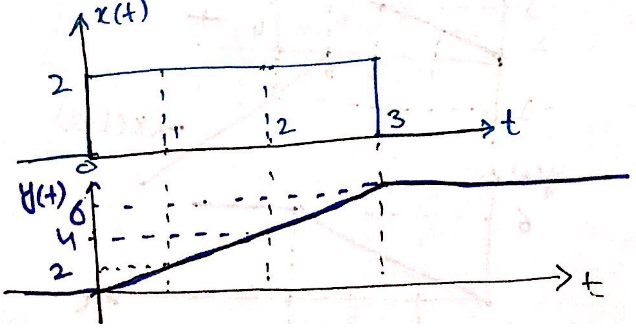
$-\infty \to 4$

$$y(t) = 6 + \int_{3}^{4} \underbrace{x(\tau)}_{0}\,d\tau = 6$$

$-\infty$ to ∞

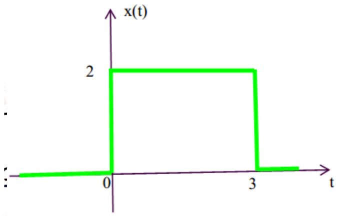
$$y(t) = 6$$

which means integration finally continues with the final area.



② Mathematical Method

Step 1 :→ Represent the signal Mathematically



$$x(t) = 0 + 2u(t-0) - 2u(t-3)$$

$$= 2u(t) - 2u(t-3)$$

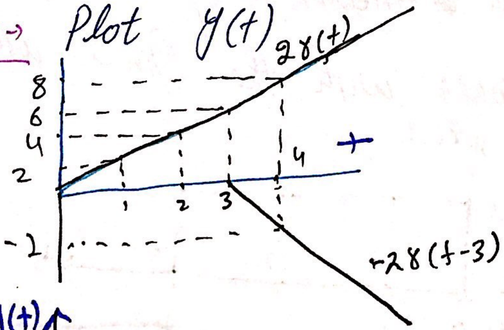
Step 2 :→ Integrate $x(t)$

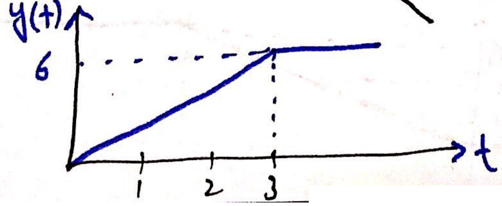
$$y(t) = \int 2u(\tau) - 2u(\tau-3)\, d\tau$$

$$= 2\int u(\tau)\,d\tau - 2\int u(\tau-3)\,d\tau$$

$$= 2\gamma(t) - 2\gamma(t-3)$$

Step 3 :→ Plot $y(t)$





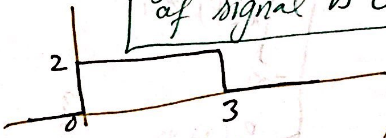
③ Barrier Method

This method is similar to the method 1.

⇒ Here first look barriers in the step signal.

Barriers are instances before or after which the value or deffinition of signal is changing

e.g



we've two barriers here (at $t=0$ (the 1st barrier) and at $t=3$ (the last barrier))

⇒ Then we take 3 (Two barriers case) limits.

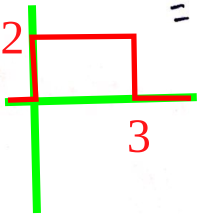
$-\infty$ to t $\quad t<0$

$-\infty$ to t $\quad 0<t<3$

$-\infty$ to t $\quad t>3$

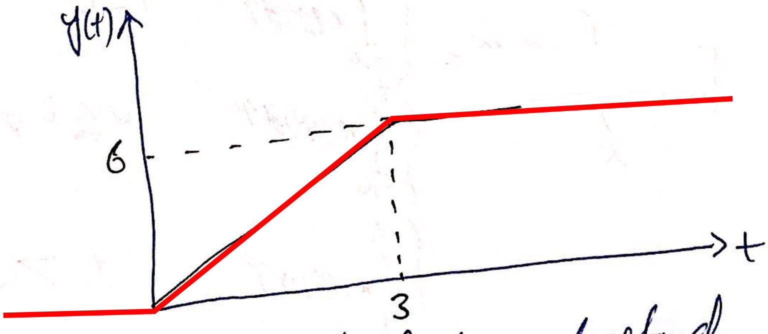
So

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,d\tau = \begin{cases} \int_{-\infty}^{t} x(\tau)\,d\tau & t<0 \\ \int_{-\infty}^{t} x(\tau)\,d\tau & 0 \leq t \leq 3 \\ \int_{-\infty}^{t} x(\tau)\,d\tau & t>3 \end{cases}$$



$$
= \begin{cases}
\int_{-\infty}^{t} 0\, d\tau = 0 & t < 0 \\[2ex]
\int_{-\infty}^{t} x(\tau)\, d\tau & 0 < t < 3 \\
\Downarrow & \\
= \underbrace{\int_{-\infty}^{0} \cancel{x(\tau)}\, d\tau}_{0} + \int_{0}^{t} 2\, d\tau = 2t & \\[2ex]
\int_{-\infty}^{t} x(\tau)\, d\tau & t > 3 \\
= \underbrace{\int_{-\infty}^{0} \cancel{x(\tau)}\, d\tau}_{0} + \int_{0}^{3} 2\, d\tau + \underbrace{\int_{3}^{t} \cancel{x(\tau)}\, d\tau}_{0} & \\
= 6 &
\end{cases}
$$

$$
y = (t) = \begin{cases} 0 & t < 0 \\ 2t & 0 < t < 3 \\ 6 & t > 3 \end{cases}
$$



→ This method is important to understand the convolution operation.

Question

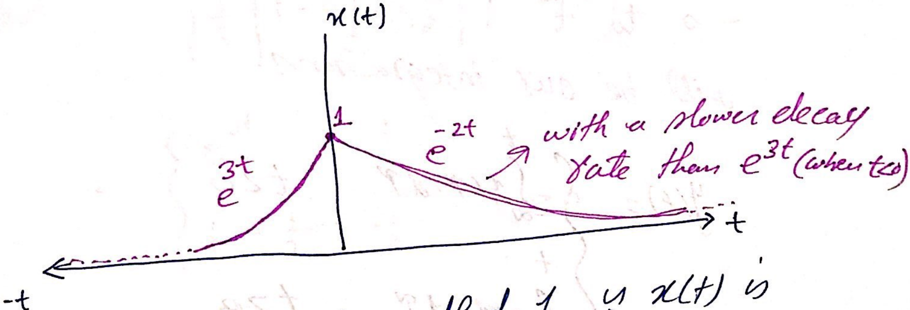
$$x(t) = \begin{cases} e^{3t}, & t < 0 \\ e^{-2t}, & t > 0 \end{cases}$$

Find $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$?

Solution

First plot $x(t)$

note that at $t=0$ $x(t) = 1$

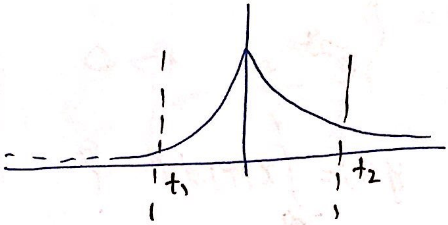


→ we can't apply method 1 as $x(t)$ is not rectangular (step).

→ To apply method 3, let us first identify the barrier.

→ Here the barrier is at $t=0$ as $x(t) = e^{3t}$ $(t<0)$ and $x(t) = e^{-2t}$ $(t>0)$ i.e., definition of signal changes

⇒ if we integrate the signal directly for interval contain "Barrier" e.g.,



we'll not get the correct result.

⇒

$-\infty$ to t $\quad t < 0$

$-\infty$ to t $\quad t > 0$

will be our integrations.

$$y(t) = \begin{cases} \int_{-\infty}^{t} x(\tau)\,d\tau & t < 0 \\ \int_{-\infty}^{t} x(\tau)\,d\tau & t > 0 \end{cases}$$

$$= \begin{cases} \int_{-\infty}^{t} e^{3\tau}\,d\tau & t < 0 \\ \int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{0} e^{3\tau}\,d\tau + \int_{0}^{t} e^{-2\tau}\,d\tau & t > 0 \end{cases}$$

$$= \begin{cases} = \dfrac{e^{3\tau}}{3}\Big|_{-\infty}^{t} & t<0 \\ = \dfrac{e^{3\tau}}{3}\Big|_{-\infty}^{0} - \dfrac{e^{-2\tau}}{2}\Big|_{0}^{t} & t>0 \end{cases}$$

$$= \begin{cases} \frac{1}{3}\left(e^{3t}-0\right) & t<0 \\ \frac{1}{3}\left[1-0\right] - \frac{1}{2}\left[e^{-2t}-1\right] & t>0 \end{cases}$$

$$= \begin{cases} \dfrac{e^{3t}}{3} & t<0 \\ \frac{1}{3}+\frac{1}{2} - \dfrac{e^{-2t}}{2} & t>0 \end{cases}$$

$$= \begin{cases} \dfrac{e^{3t}}{3} & t<0 \\ \dfrac{5}{6} - \dfrac{e^{-2t}}{2} & t>0 \end{cases}$$

we can also write as

$$\boxed{y(t) = \frac{1}{3}e^{3t}u(-t) + \left[\frac{5}{6} - \frac{1}{2}e^{-2t}\right]u(t)}$$

we can further simplify

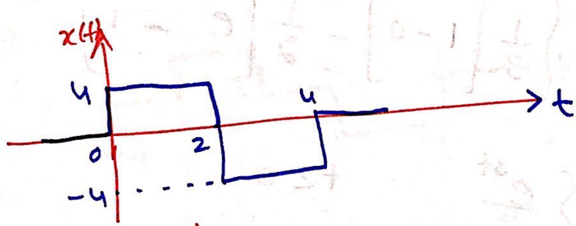
=> We know that

$$U(t) + U(-t) = 1 \Rightarrow U(-t) = 1 - U(t)$$

$$y(t) = \frac{1}{3}e^{3t}\left(1 - U(t)\right) + \left[\frac{5}{6} - \frac{1}{2}e^{-2t}\right]U(t)$$

$$= \frac{e^{3t}}{3} + \left[\frac{5}{6} - \frac{1}{2}e^{-2t} - \frac{e^{3t}}{3}\right]U(t)$$

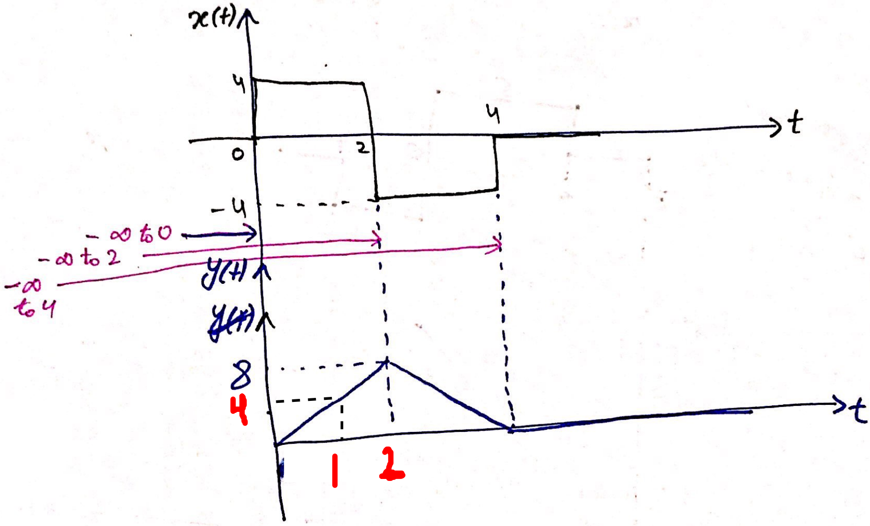
Q :->



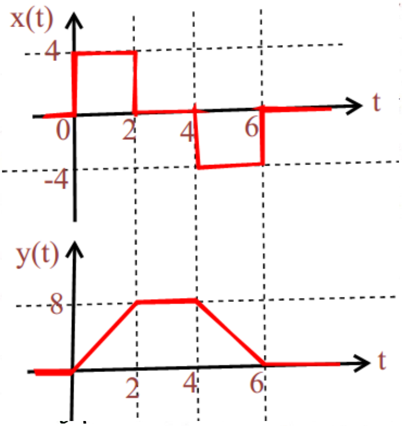
$$y(t) = \int_{-\infty}^{t} x(t)\,dt = ?$$

Solution

Clearly the total area is $8 + (-8) = 0$
and this value will propogate to ∞.



Q:→



$$\int_{-\infty}^{\infty} x(t)\, dt = ?$$

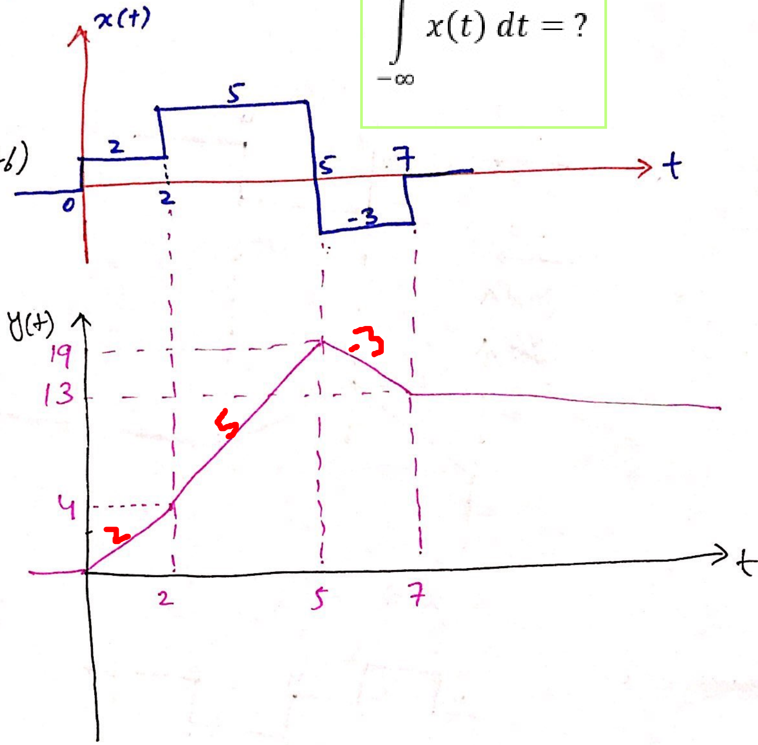
Total area = 0

$$\int_{-\infty}^{\infty} x(t)\,dt = ?$$

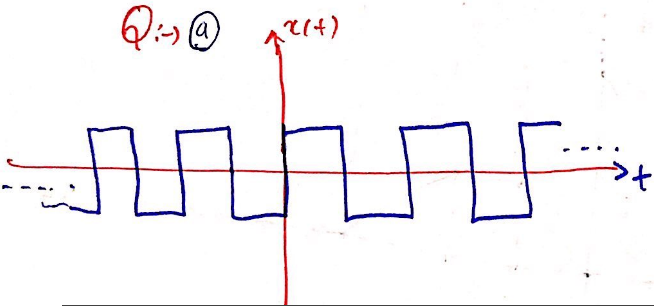
Q:-

Total area
$= 4 + 15 + (-6)$
$= 13$

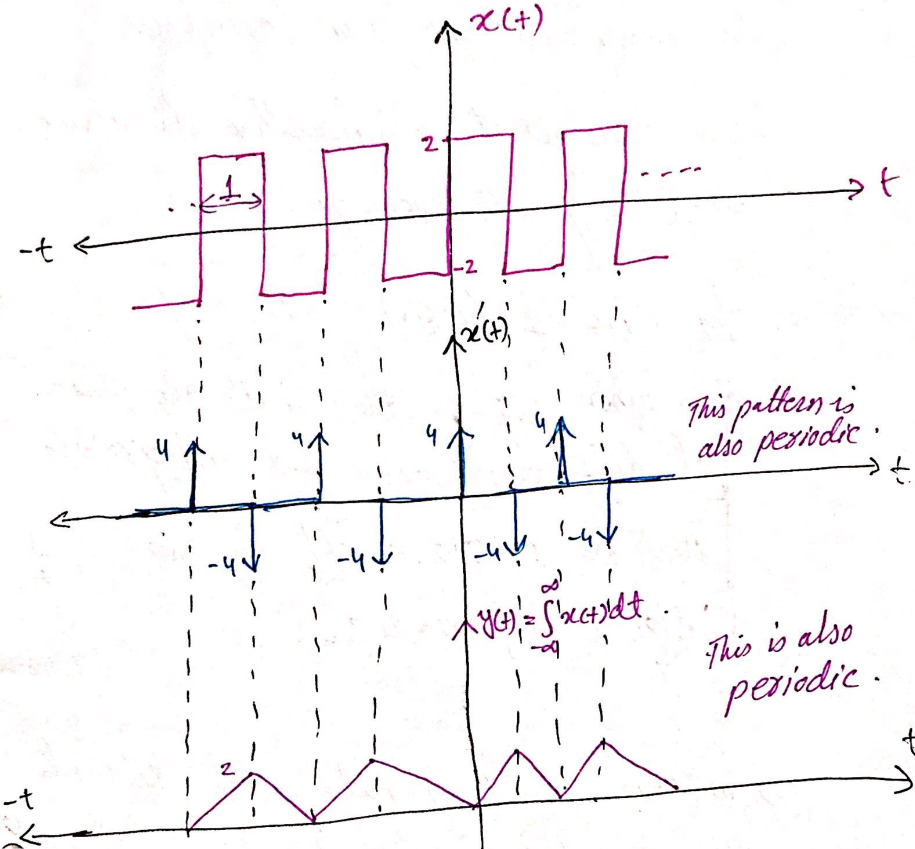
This value propagate to infinity



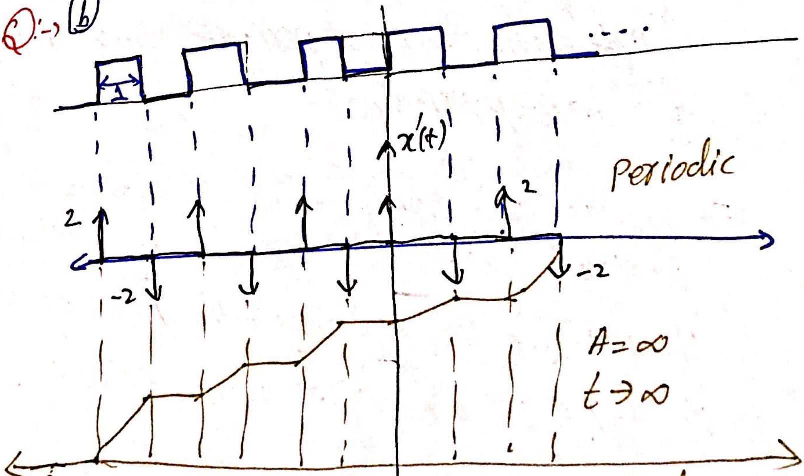
Q:- (a)



Find the waveforms of "DERIVATIVE" and "INTEGRATION" of this periodic signal?



Q:→ (b)



not periodic.

So what we can conclude?

=>We can get to a healthy discussion/ conclusion with the help of average.

=> Notice that in case 1 (Q. part a) the average value is 0 while in case 2 (Q. part b) the average value is other than zero.

=> The conclusion is that "If a signal is periodic with average value 0, then the waveforms of its differentiation and integration will also be periodic.

=> Notice in case 2 that when average value is other than 0 then the differentiation waveform is still periodic but the waveform of integration is non-periodic.

So we can say that differentiation of periodic signals is always periodic. However, the integration will be periodic only if average = 0, otherwise non-periodic