Special Signals

Unit Impulse Signal / Function.

=> impulse signal is also termed as Dirac Delta fun".
↓
name of physist.

=> Denoted by $\delta(t)$

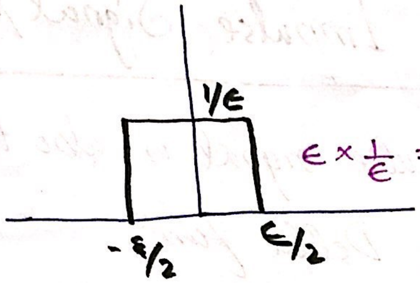
$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

=> Impulse signal is called unit impulse ∵ it's area is 1.

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

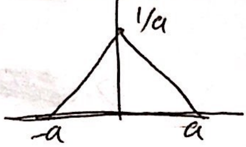
→ 'δ(t) has 0 width and infinite height

=> A unit impulse can be thought of as the limiting of any fun" maintaing as unity area under its width manipulation.

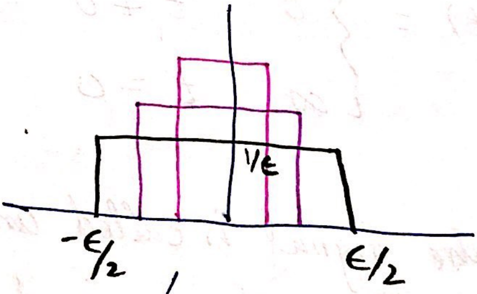


$\epsilon \times \frac{1}{\epsilon} = 1$

Think of umbrella.



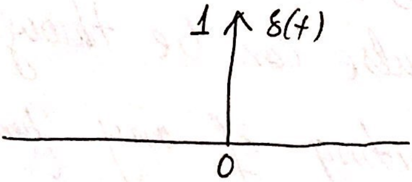
Now shrinking this signal mean $\epsilon \downarrow$ (reducing ϵ) which will keep increasing the height



$\downarrow$

Imagine this signal for $\epsilon = 0$.

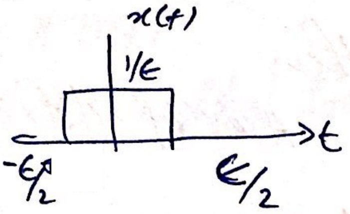
$\Rightarrow$ So we usually represent $\delta(t)$ as



the 1 not meaning the magnitude rather meaning the density or area of the signal

Properties of Impulse Signal

Recall the previous example where the area remains 1 Even if ϵ approches 0 :



$$\lim_{\epsilon \to 0} x(t) = \begin{Bmatrix} \infty & t=0 \\ 0 & t \neq 0 \end{Bmatrix} = \delta(t)$$

① We know that $\int_{-\infty}^{\infty} x(t)\,dt = 1$

and $\delta(t) = \lim_{\epsilon \to 0} x(t)$

So

$$\int_{-\infty}^{\infty} \delta(t)\,dt = \int_{-\infty}^{\infty} \lim_{\epsilon \to 0} x(t)\,dt$$

$$= \lim_{\epsilon \to 0} \underbrace{\int_{-\infty}^{\infty} x(t)\,dt}_{1}$$

$$\boxed{\int_{-\infty}^{\infty} \delta(t)\,dt = 1}$$

② Weight or Strength of an impulse

Let $y(t) = \underbrace{A_0 \delta(t)}$ known as the weighted impulse.

then

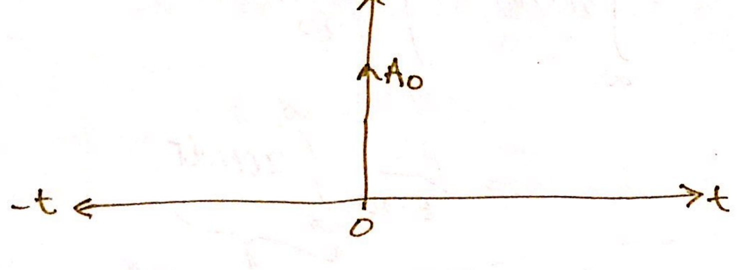
Area of $y(t) = A_0$

→ Let see

$$\text{Area of } y(t) = \int_{-\infty}^{\infty} A_0 \delta(t)\, dt$$

$$= A_0 \underbrace{\int_{-\infty}^{\infty} \delta(t)\, dt}_{1} = A_0$$

A_0 → known as weight of impulse.



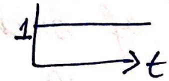
⇒ An impulse is termed as a unit impulse the area or weight or strength is one (1) (unity)

(3) Integration of $\delta(t)$

(a) $\int \delta(t)\,dt = u(t)$
↳ unit step.

(b)
$\int A_0 \delta(t)\,dt = A_0 u(t)$
↳ Weighted step signal.

$$\left[u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \right]$$ unit step.



(4) Is $\delta(t)$ Even or Odd ?

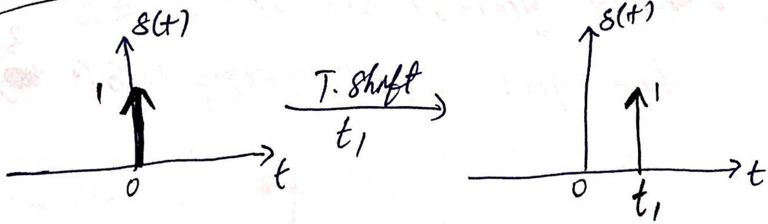
$\delta(t) = \delta(-t)$ so it's even

How ?
∵ $\delta(t) \neq 0$ iff $t = 0$

(5) Time Shifting

$\delta(t) \xrightarrow{\text{T. shift}} \underbrace{\delta(t - t_1)}_{\text{right shift by } t_1}$

$$\delta(t-t_1) = \begin{cases} \infty & t = t_1 \\ 0 & t \neq t_1 \end{cases}$$



⑥ Time Scaling

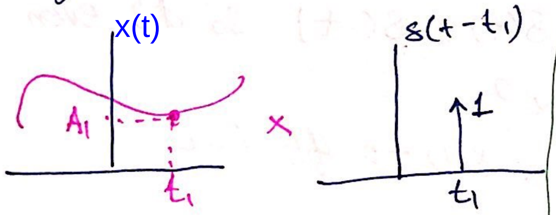
$$\delta(t) \xrightarrow[a]{T.S} \delta(at) \quad a \neq 0$$

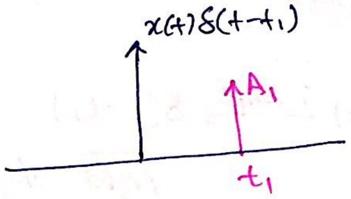
$$\delta(at) = \frac{1}{|a|}\delta(t)$$

⑦ Multiplication of signal by $\delta(t)$

$$x(t)\cdot\delta(t-t_1) = \underbrace{x(t_1)\,\delta(t-t_1)}$$

i.e., the result of such multiplication will be the same shifted impulse having weight $x(t_1)$.





$x(t_1) = ?$

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_1)\,dt$$

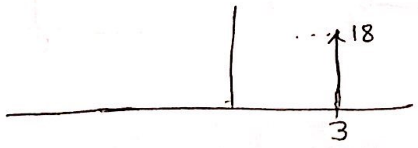
$$\int_{-\infty}^{\infty} x(t_1)\,\delta(t-t_1)\,dt$$

$$= x(t_1)$$

This is called Sampling or Sifting prop. of impulse.

E.G., let $y(t) = 2t^2\delta(t-3)$

then $y(t) = 2(3)^2\delta(t-3) = 18\,\delta(t-3)$



(ii) $y(t) = \cos 4t\, \delta(2t - \pi)$

$$= \cos 4t\, \delta[2(t - \pi/2)]$$

$$= \cos 4t\, \underbrace{\frac{1}{|2|}\, \delta(t - \pi/2)}$$

followed from prop. of T.S.

$$= \frac{1}{2} \cos 4(\pi/2)\, \delta(t - \pi/2)$$

$$= \frac{1}{2} \cos 2\pi\, \delta(t - \pi/2)$$

$$= \frac{1}{2} \times 1 \times \delta(t - \pi/2)$$

$$= \frac{\delta(t - \pi/2)}{2}$$

(8) Area of $x(t) \cdot \delta(t - t_1)$

$$\int_{-\infty}^{\infty} x(t)\, \delta(t - t_1)\, dt = x(t_1) \rightarrow$$ ~~ct~~

Ex:-> $\int_{-2}^{2} 2t^2 \delta(t-4)\, dt$?

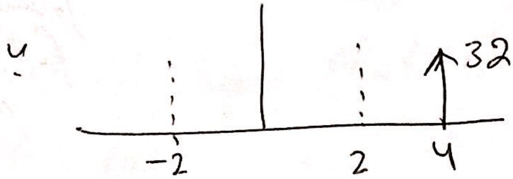
$$\int_{-2}^{2} 2t^2 \delta(t-4)\, dt = 0$$

$\because$ $\delta(t-4)$ is out of the limit of integration.

and by prop.

$$2t^2\delta(t-4) = 2(4)^2\delta(t-4)$$
$$= 32\delta(t-4)$$

$$\int_{-2}^{2} 32\,\delta(t-4)\,dt = 0$$



Ex: $\int_{-10}^{10} 2t^2\delta(t-4)\,dt$

$= 32$ ✓

$$\int_{-10}^{10} 2(4)^2\delta(t-4)\,dt = 32\int_{-10}^{10}\delta(t-4)\,dt$$

$= 32\times 1 = 32.$

(9) $$\int_{-\infty}^{\infty} x(t)\cdot\frac{d^n}{dt^n}\delta(t-t_1)\,dt = (-1)^n\frac{d^n}{dt^n}x(t)\Big|_{t=t_1}$$

This prop. is applicable iff

$x(t)\Big|_{t\to\infty}$ is finite

Exmple: $\int_{-\infty}^{\infty} \cos \pi t \frac{d^2}{dt^2} \delta(t-1)\, dt$?

Solution :-

① Check $\cos \pi t \Big|_{t\to\infty}$

→ We know that Cos( ) give value b/w −1 and 1 what ever is the value of angle.

So $\cos \pi t \Big|_{t\to\infty}$ is finite.

②
$$\int_{-\infty}^{\infty} \cos \pi t \frac{d^2}{dt^2} \delta(t-1)\, dt = (-1)^2 \frac{d^2}{dt^2} \cos \pi t \Big|_{t=1}$$

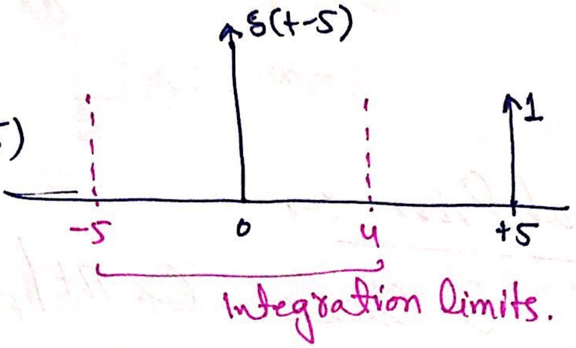
$$- \pi^2 \cos \pi t \Big|_{t=1} = \pi^2$$

? ∵ $\cos \pi = -1$

Q:→ $I = \int_{-5}^{4} \delta(t-5)\,dt$?

Ans:→

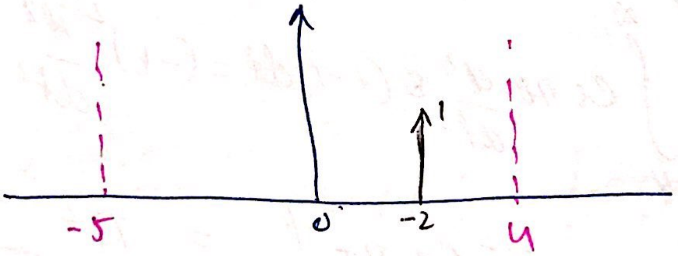
$\delta(t) \xrightarrow[\beta=5]{T\cdot shift} \delta(t-5)$



So $I = \int_{-5}^{4} \delta(t-5)\,dt = 0$

Q $I = \int_{-5}^{4} \delta(t-2)\,dt$?

$\delta(t) \xrightarrow[2]{T\cdot shift} \delta(t-2)$



So $I = \int_{-5}^{4} \delta(t-2)\,dt = 1$

Q:→ $x(t) = \sin t \cdot \delta(2t-\pi)$ → Simplify ?

Solution we know that

$\delta(at) = \frac{1}{|a|}\delta(t) \quad \forall a \neq 0$

So

$$\delta(2t - \pi) = \delta\left[2\left(t - \pi/2\right)\right]$$

$$= \frac{1}{2}\delta(t - \pi/2)$$

then

$$x(t) = \sin t \times \frac{1}{2}\delta(t - \pi/2)$$

$$= \frac{1}{2}\sin t \times \delta(t - \pi/2)$$

$$= \frac{1}{2}\sin \pi/2 \times \delta(t - \pi/2)$$ $\because x(t)\delta(t-t_1) = x(t_1)\delta(t-t_1)$

$$= \frac{1}{2}\delta(t - \pi/2)$$

Q: $I = \int_{-\infty}^{\infty} e^{-2t}\delta(-2t+1)\,dt$?

Solution :→

$$\delta(-2t+1) = \delta\left[-2\left(t - 1/2\right)\right]$$

$$= \frac{1}{|-2|}\delta(t - 1/2)$$

put in I

$$I = \int_{-\infty}^{\infty} e^{-2t}\,\frac{1}{2}\delta(t - 1/2)\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-2(1/2)}\delta(t - 1/2)\,dt$$

$$= \frac{1}{2} e^{-1} \underbrace{\int_{-\infty}^{\infty} \delta(t - 1/2)\, dt}_{1}$$

$$\boxed{I = \frac{1}{2e}}$$

Q:→ $I = \int_{-\infty}^{\infty} \big(\cos \pi t . \delta(t-2) + 3\delta(t+1)$

$+ \sin \pi t . \delta(2t-1) \big) dt$

Solution :→

$$I = \int_{-\infty}^{\infty} \cos \pi t . \delta(t-2) dt + 3 \int_{-\infty}^{\infty} \delta(t+1) dt$$

$$+ \int_{-\infty}^{\infty} \sin \pi t . \delta(2t-1) dt$$

$$= \int_{-\infty}^{\infty} \cos 2\pi \quad \delta(t-2) dt + 3$$

$$+ \int_{-\infty}^{\infty} \sin \pi t . \frac{1}{2} \delta(t - 1/2) dt$$

$$= 1 + 3 + \frac{1}{2} \int_{-\infty}^{\infty} \sin \frac{\pi}{2}\, \delta(t - 1/2) dt$$

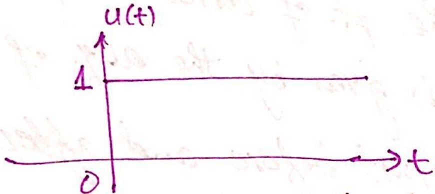
$$= 4 + \frac{1}{2} = \frac{9}{2} = 4.5$$

Unit Step Signal / Function.

→ A unit step function/signal is denoted by $u(t)$

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

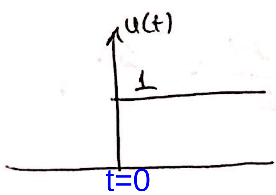
|| Also known as Heaviside Step function

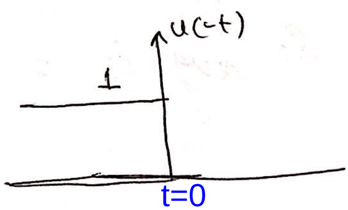


↳ t=0 is considered as pt. of discontinuity

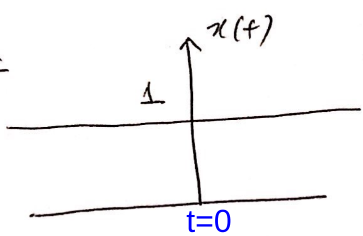
Properties

① Time Reversal. u(-t).





$x(t) = u(t) + u(-t) =$



P-t-O

Note that at $t=0$ $u(t) = u(-t) = 1$

so $1+1 = 2$ but in the

result it is 1

Q ↳ How do we get this?

Ans ↳ This is given by the Gibbs

phenonomenon.

→ The Gibbs phenomenon states that at the pt. of discontility the signal value is given by the avg. of sig. val taken just before and after the pt. of discontinuity

$x(t)$ val. before discontinuity $= 1$

" " after " $= 1$

So avg is $= \frac{1+1}{2} = 1$

② Power, Energy and RMS val of $U(t)$

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

So

$$P = \lim_{T\to\infty} \frac{1}{2T} \left(\int_{0}^{T} U(t) \, dt \right)$$

$$= \lim_{T\to\infty} \frac{1}{2T} \left[t \Big|_0^T \right] = \lim_{T\to\infty} \frac{1}{2\cancel{T}} \cancel{T} = \frac{1}{2}$$

$$\boxed{P = \frac{1}{2}}$$

$$\text{Energy} = E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{0}^{\infty} U(t) \, dt$$

$$= \int_{0}^{\infty} 1 \, dt = t \Big|_0^{\infty} = \boxed{\infty = E}$$

$$\boxed{\gamma_{ms} = \sqrt{P} = \sqrt{1/2}}$$

③ Even or Odd ?

$U(t) \neq U(-t)$ not even

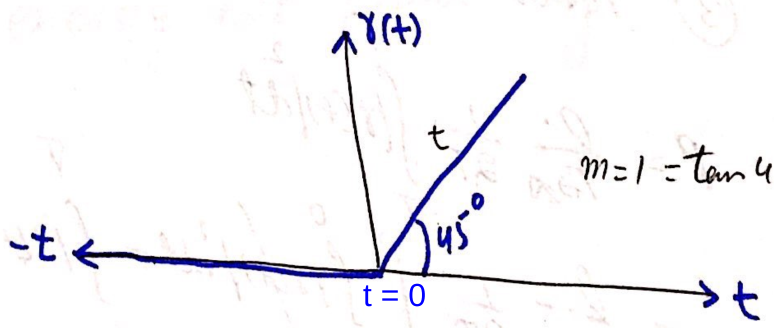
$= U(t) \neq -U(-t)$ not odd

Unit Ramp Signal

Represented by $r(t)$

$$r(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$$

t is just an equation of straight line with slope $= 1$ and y-intersept $= 0$ $(mx + c)$



① Relation with $u(t)$

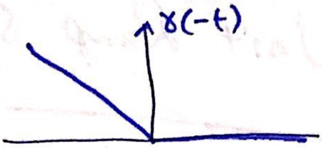
$$\boxed{r(t) = \int_{-\infty}^{t} u(t)\,dt}$$

$$= \underbrace{\int_{-\infty}^{0} u(t)\,dt}_{0} + \int_{0}^{t} 1\,dt = t \cdot 1$$

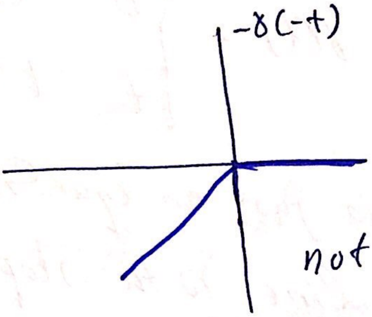
$$\boxed{\frac{d}{dt} r(t) = u(t)}$$

② Is $r(t)$ even or odd?

$r(-t)$



$r(t) \neq r(-t)$ Not even.



$\neq r(t)$

not odd Signal.

③ Power and Energy.

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |r(t)|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \Big[\underbrace{\cancel{\int_{-T}^{0} |r(t)|^2 dt}}_{0} + \int_{0}^{T} |r(t)|^2 dt \Big]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \Big[\int_{0}^{T} t^2 dt \Big] \quad \because r(t) = t$$

$$= \lim_{T\to\infty} \frac{1}{2T} \Big(\frac{t^3}{3} \Big|_{0}^{T} = \lim_{T\to\infty} \frac{1}{2\cancel{T}} \cdot \frac{1}{3} \cdot T^{\cancel{3}}$$

$$= \lim_{T\to\infty} \frac{1}{6} T^2 = \infty$$

$$\boxed{P = \infty}$$

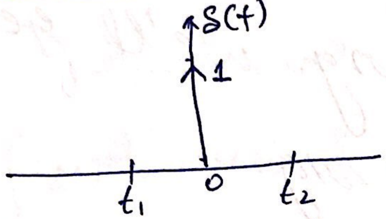
$$E = \int_{-\infty}^{\infty} |r(t)|^2 dt = \underbrace{\int_{-\infty}^{0} |r(t)|^2 dt}_{0} + \int_{0}^{\infty} |r(t)|^2 dt$$

$$= \int_{0}^{\infty} t^2 dt = \left. \frac{t^3}{3} \right|_{0}^{\infty} = \infty$$

$$\boxed{E = \infty}$$

The unit ramp signal is neither energy nor power signal.

Relation b/w $\delta(t)$ & $U(t)$



we know that

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \rightarrow (1)$$

and

$$\int_{-\infty}^{t_1} \delta(t)\,dt = 0 \rightarrow (2)$$

$$\int_{-\infty}^{t_2} \delta(t)\,dt = 1 \rightarrow (3)$$

Now look at the definition of $U(t)$

$$U(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \rightarrow (4)$$

So eqn (2) can be written as.

$$\int_{-\infty}^{t_1} \delta(t)\,dt = U(t) = 0$$

$\because U(t) = 0 \;\forall\; t < 0$

and also (3) can be written as.

$$\int_{-\infty}^{t_2} \delta(t)\,dt = U(t)$$

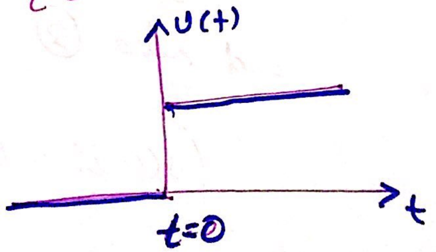
⇒ So if we integrate unit impulse sig. we'll get a unit step sig.

$$\int \delta(t)\,dt = u(t)$$

→ if two quantities are related by integration then they are also related by differentiation. i.e.,

$$\frac{d}{dt} u(t) = \delta(t)$$

⇒ In $u(t)$ watch carefully that we assume that $u(t) = 1$ at $t=0$. However, in reality $u(t)$ is undefined at $t=0$



Summary

(1) $\delta(t) \xrightarrow{I} u(t) \xrightarrow{I} r(t)$

$\int \delta(t)\,dt = u(t)$ & $\int u(t)\,dt = r(t)$

so

$\int\int \delta(t)\,dt\,dt = r(t)$

(2) $r(t) \xrightarrow{D} u(t) \xrightarrow{D} \delta(t)$

$\frac{d}{dt} r(t) = u(t)$ & $\frac{d}{dt} u(t) = \delta(t)$

So $\frac{d^2}{dt^2} r(t) = \delta(t)$

SINC Function

Sinc fun" → Short name of Cardinal Sine fun".

⇒ There are two types of Sinc fun"

① Unnormalized Sinc fun".

→ Usually used in mathematics.

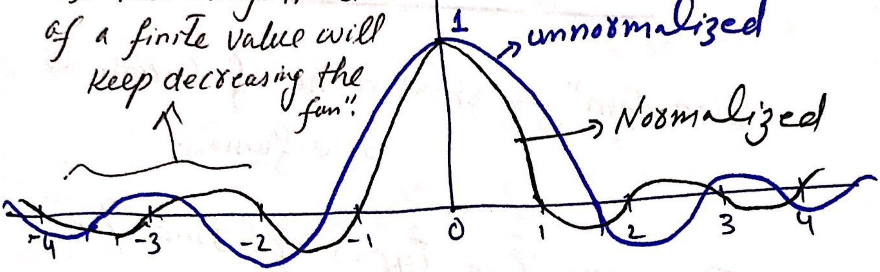
$$\text{Sinc}(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin t}{t} & t \neq 0 \end{cases}$$

② Normalized Sinc fun"

→ In digital sig. Processing this type is usually used.

$$\text{Sinc}(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(\pi t)}{\pi t} & t \neq 0 \end{cases}$$

Note the value decreases as t↑. $-1 \le \sin(t) \le 1$
so increasing the denominator of a finite value will keep decreasing the fun".



Q:-> What is the difference?

In case of normalized Sinc(t) all the zeros will occur at integer values of t which is not the case for unnormalized.

Prop: of Normalized Sinc fun".

① It is an energy fun".

$$\boxed{E = \pi}$$

$$E \text{ of } \operatorname{Sinc}(at) = \frac{\pi}{a}$$

② $\operatorname{Sinc}(t) = 0$ if $t = \infty$

Exponential Signals (Real & complex)

let $x(t) = A_0 e^{st}$ where $s \in \mathbb{C}$

let $A_0 = 1$ & $s = \sigma + j\omega$

$$x(t) = e^{st} \rightarrow ①$$

$$= e^{(\sigma + j\omega)t}$$

$$= e^{\sigma t} e^{j\omega t}$$

$$= e^{\sigma t} \cdot \underbrace{[\cos \omega t + j \sin \omega t]}_{\text{By Euler's formula.}} \rightarrow ②$$

→ Then depending upon ω we've two cases

Case 1 $\omega = 0$

In this case

$x(t) = e^{\sigma t}$ (Real exponential Signal)

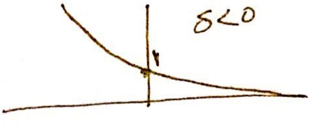
which further has two cases based on the value of σ

① $\sigma > 0$

Rising/growing exponential

② $\sigma < 0$

Decaying exponential



Case 2: $\underline{\omega \neq 0}$

Again we've three cases based on the value of δ

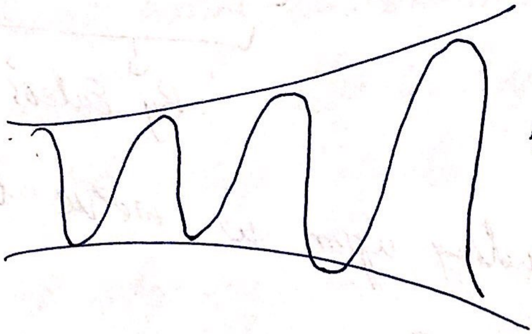
① $\underline{\delta = 0}$

then

$$x(t) = \cos \omega t + j \sin \omega t$$

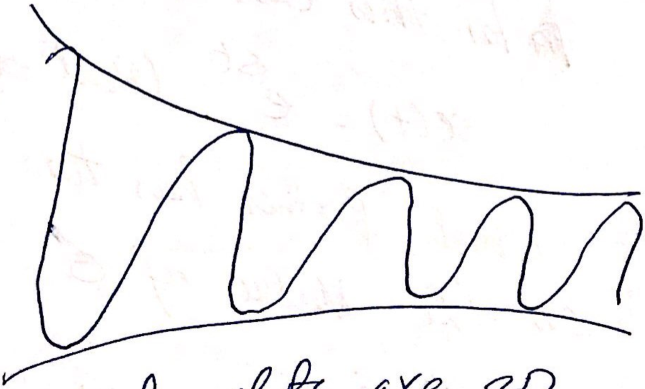
un-damped sinusoids

② $\underline{\delta > 0}$



rising

③ $\delta < 0$



actual plots are 3D