Causal & Non-Causal Systems

Causal System :-> o/p of sys is independent of $\boxed{\text{future values}}$ of i/p.

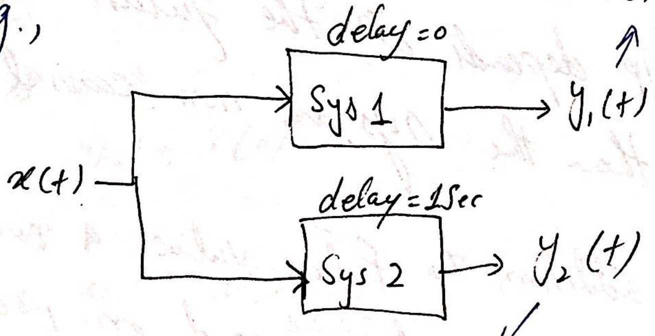
e.g., 1) $y(t) = x(t)$ causal.

2) $y(t) = \underbrace{x(t)}_{P.V} + \underbrace{x(t-1)}_{\text{Past value}} \rightarrow$ Causal

=> All real life systems or all practically realizable systems are causal systems.

e.g.,

i.e., $y(t)$ will be produced instantly.



i.e., $y_2(t)$ will be produced after a delay of 1 Sec.

at $t=0$

$y_1(0) \longrightarrow x(0)$

$y_2(0) \longrightarrow x(-1)$ ∵ delay of 1 sec

↓

past values.

So system of the nature of sys 1 & Sys 2 are physically realizable.

<u>Non-Causal Systems</u> :-> O/P of sys is dependent upon future values of i/p at any instant of time.

(i.e., if at any instant of time the o/p depends on the future i/p then the sys is non-causal)

-> In addition, to future values a non-causal system may also depend on present and past values.

Ex ① $y(t) = x(t+2)$ non-causal

② $y(t) = x(t) + x(t-1) + x(t+1)$

↳ non-causal.

Anti-Causal :→ O/P depends only on the future values.

→ Every anti-causal sys is non-causal but the reverse is not true.

Ex 1 $y(t) = x(3t)$

Causal ?

Solution :→ Time scaling so the sys is dynamic.

at $t=0$

$y(0) = x(0)$

at $t=1$

$y(1) = x(3)$

↳ future

Non-Causal. (Anti-Causal?)

at $t=-1$

$y(-1) = x(-3)$ so not anti-causal.

↳ past

Ex 2

$$y(t) = \begin{cases} x(3t) & t < 0 \\ x(t-1) & t \geq 0 \end{cases}$$

Causal ?

Solution

$t < 0$

$y(t) = x(3t)$

say $t = -1$

$y(-1) = x(-3)$
↳ past

this case shows the sys is causal
but let us check the 2nd condition.

$t \geq 0$ $\quad y(t) = x(t-1)$

at $t = 0$

$y(0) = x(-1)$
↳ past → so okay.

at $t = 1$

$y(1) = x(0)$ ↳ Past so okay.

at $t = 2$

$y(2) = x(1)$
↳ past → ~~so~~ so okay

So the sys. is causal.

Ex 3

$$y(t) = \sin(t+1)\, x(t-1)$$

Causal ?

Solution :- Here again don't get confused with the $\sin(t+1)$ as this is just a co-efficient.

at $t=0$

$y(0) = \sin(1)\, x(-1)$
↳ past

at $t=-1$

$y(-1) = \sin(0)\, x(-2)$
↳ past

at $t=1$

$y(1) = \sin(2)\, x(0)$
↳ past

So the sys is causal.

Ex 4

$$y(t) = x(e^t)$$

Causal ?

Solution at $t = 0$

$$y(0) = x(1)$$

$\hookrightarrow$ future

So no need to check for other instants.

$\Rightarrow$ Sys is non-causal.

Examples. ① $y(t) = x(\sin t)$

② $y(t) = x(t/4)$

③ $y(t) = e^{2t} x(t-1)$

Causal ?

Ex 5

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

Causal ?

Solution :-> As the integration gives us area under the curve.

-> Recall from graphical method of integration that integration depends upon past values only so this system is causal.

-> However, this can't be generalized as the nature of the sys depends upon the i/p signal and the upper limit of the integration.

Ex 6

$$y(t) = \int_{-\infty}^{t+1} x(\tau)\,d\tau$$

Causal ?

Solution :-

$$\int_{-\infty}^{t+1} x(\tau)\, d\tau \quad \begin{matrix} \to x(t+1) \\ \to x(-\infty) \end{matrix} \xrightarrow{t=0} \begin{matrix} \text{future} \\ \uparrow \\ x(1) \\ \\ x(-\infty) \\ \hookrightarrow \text{past.} \end{matrix}$$

So non-causal.

→ The difference b/w Ex 5 and Ex 6 is the upper limit.

→ Now if $y(t) = \int_{-\infty}^{t-1} x(\tau)\, d\tau \quad \begin{matrix} \to x(t-1) \\ \to x(-\infty) \end{matrix} \xrightarrow{t=0} \begin{matrix} \text{Past} \\ x(-1) \\ \\ x(-\infty) \\ \hookrightarrow \text{past} \end{matrix}$

So in this case the sys. is causal.

Ex 7 $\quad y(t) = \int_{-\infty}^{t} x(3\tau)\, d\tau$

Causal ?

Solution

$$y(t) = \int_{-\infty}^{t} x(3\tau)\, d\tau \quad \begin{matrix} \to x(3t) \\ \to x(-\infty) \end{matrix} \xrightarrow{t=0} \begin{matrix} \text{Present} \\ \uparrow \\ x(0) \\ \\ x(-\infty) \\ \text{past} \end{matrix}$$

$t=1 \quad x(3) \to$ future

$x(-\infty)$

Non Causal.

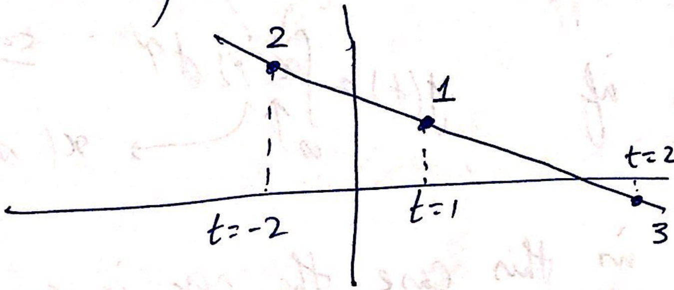
Ex 8

$$y(t) = \frac{d}{dt} x(t)$$

Causal ?

Solution

$y(t) = \frac{d}{dt} x(t)$ tells us that $y(t)$ is the slope of $x(t)$ w-r-t t.



So to define a straight line (slope) we need two pts

We are at $t=1$

then

Case 1 :→ We take pt 2 as the 2nd point. In this case the dependence is on past value. (present + past)

Case 2 :→ We take pt. 3 as the 2nd pt. in this case the dependance is on future value. (present + future)

So case 1 corresponds to causal nature while case 2 corresponds to a non-causal one.

→ So whenever there is derivative, we can't exactly tell whether the system is causal or non-causal.

Examples

(1) $y(t) = \int_{-\infty}^{2t} x(\tau/2)\, d\tau$

(2) $y(t) = \int_{-\infty}^{t} x(\tau-1)\, d(\tau-1)$

(3) $y(t) = \int_{-\infty}^{2t} x(\tau)\, d\tau$

Time Invariant & Time Varying Systems

Sys

$$x(t) \rightarrow \boxed{\text{System}} \xrightarrow{y(t)} \boxed{\text{Delay by } t_0} \xrightarrow{y(t-t_0)}$$

Now let us switch the position of both the systems why?
↳ I want to provide the delay to the i/p itself instead of delaying the o/p of "System". and then we'll see if the two o/ps are the same or not.

Sys.   Sys under analysis

$$x(t) \rightarrow \boxed{\text{Delay by } t_0} \xrightarrow{x(t-t_0)} \boxed{\text{System}} \xrightarrow{y'(t)}$$

Now Two possibilities

(1) $y'(t) = y(t-t_0) \rightarrow$ Time Invariant (TI)

(2) $y'(t) \neq y(t-t_0) \rightarrow$ Time Varying (TV)

→ i.e., To check whether a system is TI or TV simply compare $y(t-t_0)$ and $y'(t)$.

TI System :→ A TI sys is a sys. in which any delay provided in the i/p must be reflected in the o/p.

Ex 1 $\quad y(t) = x(2t)$

TI or TV ?

Solution

$$x(t) \longrightarrow \underbrace{\text{system}}_{\text{TI or TV ?}} \longrightarrow x(2t) = y(t)$$

Step 1 :→ Delay to the o/p by t_0

$$y(t-t_0) = x(2(t-t_0)) = x(2t-2t_0) \rightarrow ①$$

Step 1 :→ Delaying the i/p by t_0

$$x(t) \xrightarrow{t_0} x(t-t_0) \longrightarrow \text{System} \rightarrow x(2t-t_0) \quad ②$$

∵ the sys. can perform time scaling on t only not on any ele say t_0 in this case.

As ① $\neq$ ② so system is TV

Ex 2 $\quad y(t) = 2 + x(t)$

TI or TV ?

Solution

$$x(t) \longrightarrow \text{System} \rightarrow 2 + x(t) = y(t)$$

TI or TV?

Step 1 :→ Delaying the o/p by to.

$$y(t) \xrightarrow{t_0} 2 + x(t - t_0) \rightarrow ①$$

Step 2 :→ Delaying the i/p by to

$$x(t) \xrightarrow{t_0} x(t - t_0) \longrightarrow \text{System} \longrightarrow 2 + x(t - t_0) \quad ②$$

① = ② TI system

⇒ We can conclude that whenever the sys is performing time scaling it is TV sys and if it is only performing amplitude shifting then it is TI sys.

Examples

1. $y(t) = x(\sin(t))$
2. $y(t) = x(t+2)$
3. $y(t) = \cos t + x(t)$

Ex $y(t) = x(\cos t)$

TI or TV ?

Solution

$x(t) \longrightarrow$ system $\longrightarrow x(\cos t) = y(t)$

Step 1:→ Delaying $y(t)$ by t_0

$y(t) \xrightarrow{t_0} x(\cos(t-t_0)) \rightarrow$ ①

Step 2:→ Delaying i/p by t_0

$x(t) \xrightarrow{t_0} x(t-t_0) \longrightarrow$ system $\longrightarrow x(\cos t - t_0)$ ②

∵ the sys. performs/apply "Cos" to the variable "t" only.

① ≠ ② so sys is TV

Ex $y(t) = x(\tan(t))$

TI or TV ?

Solution :→ This time also the sys. will be TV.

So whenever there is a time scaling the sys will be TV. and this T.Scaling is not compulasarily on the i/p it can be on the o/p

e.g., if $y(t^3) = x(t)$ then also the system will be time varying.

Ex $y(t) = x(t^2)$

TI or TV?

Solution :- TV sys.

Ex $y(t) = \underbrace{C_s t}_{\substack{\text{Co-efficient} \\ \text{(fun" of t)}}} \; x(t) \longrightarrow$ i/p

TI or TV?

Solution

Step 1 :- Delaying o/p by to

$y(t) \xrightarrow{t_0} y(t-t_0) = C_s(t-t_0)\, x(t-t_0)$ ↳ ①

Step 2 :- $x(t) \xrightarrow{t_0} x(t-t_0)$

$x(t-t_0) \longrightarrow \text{System} \rightarrow C_s t\, x(t-t_0)$ ↳ ②

=> Becaue the system apply only amplitude scaling

i.e., the sys has the prop. to multiply cost to any applied i/p

① ≠ ② TV system.

Ex :- $y(t) = e^{-t} . x(t)$

TI or T.V ?

Solution :- whenever there is a time dependent co-efficient then the sys. will always be TV system.

Ex $y(t) = (10t + 1)\, x(t)$

Solution :- T.V system.

Ex $y(t) = e^{-2k} x(t)$

TI or TV ?

Solution :→ Now e^{-2k} is not funⁿ of time. So sys is TI.

(∵ sec also $x(t)$ has no scaling)

Ex: $y(t) = 2t + x(t)$

TI or TV ?

Solution :→ Notice that now $2t$ is not the co-efficient.

So let us check.

Step 1 :→ Delaying o/p by t_0.

$$y(t) \xrightarrow{t_0} y(t-t_0) \longrightarrow \underbrace{2(t-t_0) + x(t-t_0)}_{①}$$

Step 2 :→ Delay i/p by t_0.

$$x(t) \xrightarrow{t_0} x(t-t_0) \longrightarrow \text{Sys} \longrightarrow \underbrace{2t + x(t-t_0)}_{②}$$

① ≠ ② TV sys.

Conclusion is whenever there is a time dependent adding or subtracting term (like 2t in this case) the sys will be TV.

System will be TI if

1. There is not time scaling (either in i/p or o/p)
2. All the co-efficients are cst.
3. Any added/subtracted term in the system relationship (except i/p & o/p) must be cst or 0.

Examples

1. $y(t) = x(\log t)$
2. $y(t) = t^2 x(t)$
3. $y(t) = \log(t) - x(t)$

Ex: $y(t) = x(t+1) + x(t-1)$

TI or TV ?

Solution ① There is no T. Scaling.

② No time varying co-efficients.

③ No time varying added/subtracted terms

So the sys is TI.

Ex: $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

TI or TV ?

Solution

$\int_{-\infty}^{t} x(\tau)\, d\tau \rightarrow x(t)$

① ✓
② ✓
③ ✓

TI sys.

Ex: $y(t) = \int_{-\infty}^{t} x(3\tau)\, d\tau$

TI or TV ?

Solution

$$y(t) = \int_{-\infty}^{t} x(3\tau)\,d\tau \rightarrow x(3t)$$

① ×
② ✓
③ ✓

So TV sys.

Ex

$$y(t) = \int_{-\infty}^{t} \cos\tau \, x(\tau)\,d\tau$$

TI or TV ?

Solution

$$y(t) = \int_{-\infty}^{t} \cos\tau \, x(\tau)\,d\tau \rightarrow \cos t \, x(t)$$

① ✓
② ×
③ ✓

T.V sys.

Ex: Split System

$$y(t) = \begin{cases} x(t-1) & t < 0 \\ x(t+1) & t \geq 0 \end{cases}$$

TI or TV ?

Solution :-) Here in this sys we see a condition on time. So whenever there is a condition on time. The system is known as split sys.

① No T.scaling on o/p or i/p side ✓

② Co-efficients are ct ✓

③ No time dependant added/subtracted terms ✓

TI system.

↳ but this answer is incorrect

↳ P-7-0

But the answer is incorrect why?

→ Let us re-phrase the sys. relationship.

$$y(t) = a(t)\,x(t-1) + b(t)\,x(t+1)$$

where

$$a(t) = \begin{cases} 1 & t \leq 0 \\ 0 & t \geq 0 \end{cases}$$

$$b(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Now see condition 2 is voilated

So this sys is T.V.

⇒ So split systems are always T.V.