Fourier Transform

⇒ FT is a mathematical tool used for frequency analysis of signals.

i.e., Observations & study of signal variation while changing frequency.

⇒ Used to represent signals in frequency domain. Hence, used for frequency domain analysis of signals.

[⇒ Laplace transform is also similar tool but it is prefered for the analysis/design of systems while FT is preferred for the analysis of signals]

⇒ Fourier series can be used for periodic signals only while FT can be used for aperiodic signals as well.

⇒ The FT of a signal $x(t)$ can be represented as

$$x(t) \underset{}{\overset{FT}{\rightleftharpoons}} X(j\omega) \text{ or } X(\omega) \text{ or } X(f)$$

rad/sec Hz

⇒ $X(j\omega) \in \mathbb{C}$ so it will have magnitude & angle

$$X(j\omega) = |X(j\omega)| \angle X(j\omega)$$

Formulae

(1) $x(t) \xrightarrow{FT} X(j\omega)$

$$X(w) = X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

(2) $X(j\omega) \xrightarrow{IFT} x(t)$ [inverse Fourier Transform]

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t} d\omega$$

=> Both these formulas are valid only for absolutely integrabl signals. $\left[\int_{-\infty}^{\infty} |f(t)| dt \neq \infty \right]$

⇒ For conversion of LT to FT simply put $s = j\omega$

→ Once again this conversion is possible for absolutely integrable signals only.

Conditions for existence of FT [Dirichlet conditions]

There are 3 conditions

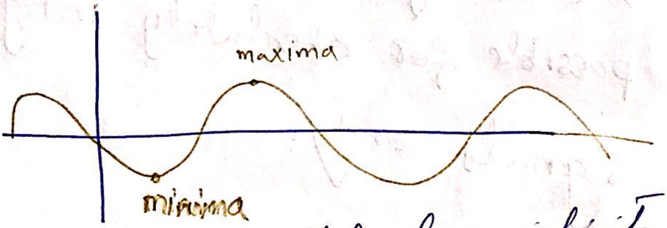
① Signal should have finite number of maxima & minima over any finite interval.

② Signal should have finite number of discontinuities over any finite interval.

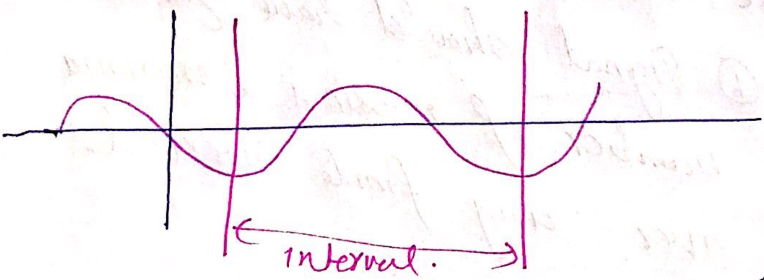
③ Signal should be absolutely integrable.

Explanation

①



This signal definitely have infinite number of maxima and minima. but if we take finite interval like.



then the maxima/minimas are finite

⇒ Now suppose $x(t) = \cos(2\pi/t)$

we know that $\cos(\cdot)$ give maxima or minima at $n\pi$ $(n \in \mathbb{Z})$

i.e., when $\boxed{\frac{2\pi}{t} = n\pi}$ the value of $\cos(\cdot)$ will be extremum

$$\frac{2\pi}{t} = n\pi$$

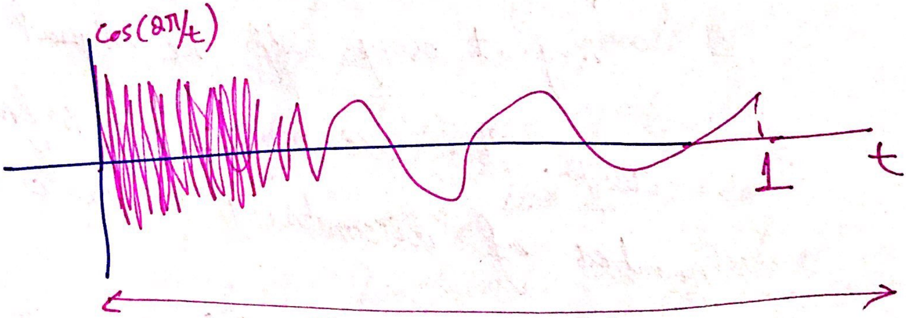
$$\boxed{t = \frac{2}{n}}$$

$n=1,\ t=2,\quad n=2,\ t=1,\quad n=3,\ t=2/3$

$n=4,\ t=\frac{1}{2},\quad n=5,\ t=2/5,\quad n=6,\ t=1/3$

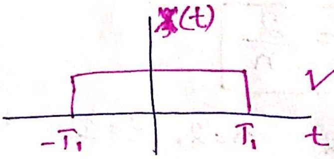
and so on.

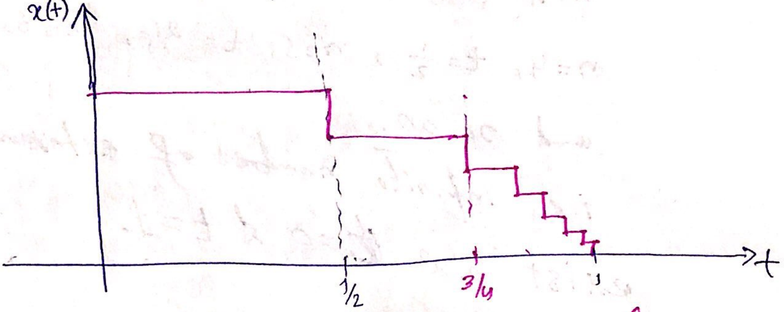
i.e., infinite number of extremums exist b/w $t=0$ & $t=1$.



So infinite extremums in a finite interval.

② Finite number of discontinuities in finite time interval





say at every half the signal drops. Then we may have infinite number of discontinuities b/w 0 & 1

③ <u>Absolutely Integrable.</u>

This condition is the most imp. among the 3 conditions.

⇒ Absolutely integrable means

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$$ → says $x(t)$ is absolutely integrable.

e.g.,

$x(t) = e^{-at}$ is not absolutely integrable.

$\therefore \int_{-\infty}^{\infty} |e^{-at}| \, dt = \infty$

$\Rightarrow$ However, instead

$e^{-at} u(t)$ for $a > 0$

is absolutely integrable.

Exception :→ The periodic signals are not absolutely integrable yet its FT exist. However, we'll need to use delta functions for finding its FT.

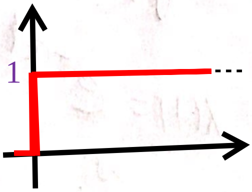
$\Rightarrow$ The above 3 conditions are sufficient but not necessary.

i.e., the FT may exist for signals violating Dirichlet condition.

Example

$x(t) = u(t)$

$X(j\omega) = ?$



Solution

$$X(j\omega) = \underbrace{\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}$$

This formula valid for absolutely integrable signals.

i.e., We can't have the FT of $u(t)$ using this formula.

⇒ let see what problem do we face!!

$$X(j\omega) = \int_{0}^{\infty} 1 \cdot e^{-j\omega t}\, dt \quad \because \quad \begin{matrix} u(t) = 0 & t < 0 \\ u(t) = 1 & t \geq 0 \end{matrix}$$

$$= \int_{0}^{\infty} e^{-j\omega t}\, dt = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{0}^{\infty}$$

$$= \frac{1}{-j\omega} \Big[\underbrace{e^{-j\omega \infty}}_{?} - \underbrace{\cancel{e^{j\omega 0}}}_{1} \Big]$$

From Euler's formula.

$$e^{-jw\infty} = e^{-j\infty} = \cos(-\infty) + j\sin(-\infty)$$

$$= \cos(\infty) - j\sin(\infty)$$

we can't find the exact value of these (not defined) however, we know that they will have values b/w -1 and 1

$\Rightarrow$ Hence $X(jw)$ is also not defined using the formula.

$\Rightarrow$ The FT of $u(t)$ do exist & it is $\frac{1}{jw} + \pi\delta(t)$.

$\Rightarrow$ The conclusion is that we can't use the formula to find the FT of signals which are not absalutely integrable.

Properties of FT

① Linearity :->

$$x_1(t) \underset{}{\overset{FT}{\rightleftharpoons}} X_1(j\omega)$$

$$x_2(t) \overset{FT}{\rightleftharpoons} X_2(j\omega)$$

$$\alpha\, x_1(t) \rightleftharpoons \alpha X_1(j\omega)$$

$$\beta\, x_2(t) \rightleftharpoons \beta X_2(j\omega)$$

Now the linearity prop. says.

$$\alpha\, x_1(t) + \beta\, x_2(t) \overset{F.T}{\rightleftharpoons} \alpha X_1(j\omega) + \beta X_2(j\omega)$$

② Conjugation Property

let $x(t) \overset{FT}{\rightleftharpoons} X(j\omega)$

then by conjugation prop.

$$x^*(t) \overset{FT}{\rightleftharpoons} X^*(-j\omega)$$

③ Area under $x(t)$

→ we know area under $x(t)$ is

$$\int_{-\infty}^{\infty} x(t)\,dt$$

→ We know that

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\Big\downarrow \; \omega = 0$$

$$\boxed{X(0) = \int_{-\infty}^{\infty} x(t)\, dt}$$

i.e $X(0)$ will give us area under $x(t)$.

$$\boxed{\text{Area under } x(t) = X(j\omega)\Big|_{\omega=0}}$$

(4) Area under $X(j\omega)$

Area under the freq. domain signal

$$X(j\omega) = \int_{-\infty}^{\infty} X(j\omega)\, d\omega$$

$\Rightarrow$ From IFT

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(J\omega) \cdot e^{J\omega t}\, d\omega$$

$$\Big\downarrow t = 0$$

$$x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(J\omega)\, d\omega$$

$$\boxed{2\pi x(0) = \int_{-\infty}^{\infty} X(j\omega)\, d\omega}$$

$$\boxed{\text{Area under } X(j\omega) = 2\pi x(t)\Big|_{t=0}}$$

⑤ Time Reversal :→

let
$$x(t) \xrightleftharpoons{FT} X(j\omega)$$

then by Time reversal prop.

$$\boxed{x(-t) \xrightleftharpoons{FT} X(-j\omega)}$$

⑥ Time Scaling :→

let
$$x(t) \xrightleftharpoons{FT} X(j\omega)$$

then by time scaling prop.

$$\boxed{x(at) \xrightleftharpoons{FT} \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)} \qquad a \neq 0$$

⑦ Time Shifting

$$x(t) \xrightleftharpoons{FT} X(j\omega)$$

then

$$x(t \pm t_0) \xrightleftharpoons{FT} X(j\omega)\, e^{\pm j\omega t_0}$$

⑧ Frequency shifting

$$x(t) \xrightleftharpoons{FT} X(j\omega)$$

then

$$e^{\pm j\omega_0 t} x(t) \rightleftharpoons X(j\omega \mp \omega_0)$$

⑨ Convolution in time

⇒ whenever we convolute two time domain signals, their Fourier transforms gets multiplied.

⇒ let we've two signals $x_1(t)$ & $x_2(t)$

s.t

$x_1(t) \xrightleftharpoons{FT} X_1(j\omega)$ & $x_2(t) \rightleftharpoons X_2(j\omega)$

then

$$\boxed{\underbrace{x_1(t) * x_2(t)}_{*\ \text{Convolve}} \rightleftharpoons X_1(j\omega) \cdot X_2(j\omega)}$$

(10) Multiplication in Time

Let

$$x_1(t) \rightleftharpoons X_1(j\omega)$$

$$x_2(t) \rightleftharpoons X_2(j\omega)$$

then

$$\boxed{x_1(t) \cdot x_2(t) \rightleftharpoons \frac{1}{2\pi}\left[X_1(j\omega) * X_2(j\omega)\right]}$$

As Fourier Transforms are convoluted in frequency. Thus, this prop. is also called convolution in frequency

(11) Differentiation in Time

Let $x(t) \overset{FT}{\rightleftharpoons} X(j\omega)$

then

$$\frac{d\,x(t)}{dt} \rightleftharpoons j\omega\, X(\omega)$$

In general.

$$\boxed{\frac{d^k x(t)}{dt^k} \overset{FT}{\rightleftharpoons} (j\omega)^k X(j\omega)}$$

(12) Integration in Time

let $x(t) \overset{FT}{\rightleftharpoons} X(j\omega)$

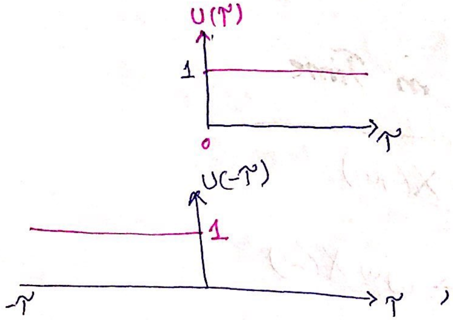
then

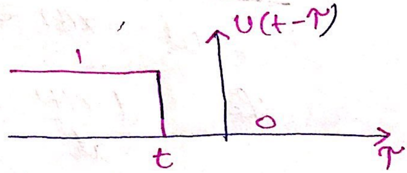
$$\boxed{\int_{-\infty}^{t} x(\tau)\,d\tau \overset{FT}{\rightleftharpoons} \frac{X(j\omega)}{j\omega} + \pi X(0)\,\delta(\omega)}$$

Proof :->

Let us consider convolution of two signals $x(t)$ & $u(t)$ [unit step]

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau \longrightarrow (1)$$





clearly $u(t-\tau) = 0$ if $\tau > t$

& $u(t-\tau) = 1 \quad -\infty < \tau < t$

considering left shifting

$\Rightarrow$ Now we'll use the waveform of $U(t-\tau)$ to simplify ①

$$x(t) * U(t) = \int_{-\infty}^{\infty} x(\tau)\, U(t-\tau)\, d\tau$$

$$= \underbrace{\int_{-\infty}^{t} x(\tau)\, U(t-\tau)\, dt}_{\substack{U(t-\tau)=1 \\ \text{from } -\infty \text{ to } t}} + \underbrace{\int_{t}^{\infty} x(\tau)\, U(t-\tau)\, d\tau}_{\substack{U(t-\tau)=0 \\ \text{from } t \text{ to } \infty}}$$

$$x(t) * U(t) = \int_{-\infty}^{t} x(\tau)\, d\tau \longrightarrow ②$$

So now we've to prove that

$$x(t) * U(t) \rightleftharpoons \frac{X(j\omega)}{j\omega} + \pi X(0)\, \delta(\omega)$$

We know that

$$FT[U(t)] = \frac{1}{j\omega} + \pi\, \delta(\omega) \rightarrow ③$$

$\Rightarrow$ Now using convolution in time prop.

$$FT[x(t) * U(t)] = FT[x(t)] \cdot FT[U(t)]$$

$$= X(j\omega) \cdot \left[\frac{1}{j\omega} + \pi\, \delta(\omega)\right]$$

$$= \frac{X(j\omega)}{j\omega} + \pi X(j\omega)\,\delta(\omega) \rightarrow (3)$$

Now using the prop. of δ fun.

$$f(t)\,\delta(t-t_0) = f(t_0)\,\delta(t-t_0)$$

$$\left[X(j\omega)\,\delta(\omega-0) = X(0) \right]$$

thus

$$FT\left[x(t) * u(t)\right] = \frac{X(j\omega)}{j\omega} + \pi X(0)\,\delta(\omega)$$

thus proved from (2) that

$$\boxed{FT\left[x(t) * u(t)\right] = FT\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \frac{X(j\omega)}{j\omega} + \pi X(0)\,\delta(\omega)}$$

(13) Differentiation in Frequency

$$x(t) \overset{FT}{\rightleftharpoons} X(j\omega)$$

$$t\,x(t) \overset{FT}{\rightleftharpoons} j\frac{d}{d\omega}X(j\omega)$$

in general

$$\boxed{FT\left[t^n x(t)\right] = j^n \frac{d^n}{d\omega^n} X(j\omega)}$$

(14) Modulation

let

$FT[x(t)] = X(j\omega)$ or for simplicity $X(\omega)$

then

① $FT\left[x(t)\cos(\omega_0 t)\right] = \frac{1}{2}\left[X(\omega+\omega_0) + X(\omega-\omega_0)\right]$

② $FT\left[x(t)\sin(\omega_0 t)\right] = \frac{j}{2}\left[X(\omega+\omega_0) - X(\omega-\omega_0)\right]$

i.e., multiplication by sinusoids in time domain causes frequency shift in frequency domain.

(15) Parseval's Energy Theorem

Let $FT[x(t)] = X(j\omega)$

then total energy of $x(t)$ is

$$\boxed{E_{x(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega}$$

$$\left[E_{x(t)} = \int_{-\infty}^{\infty} |x(t)|^2 dt \text{ and } |x(t)|^2 = x(t) \cdot x^*(t)\right]$$

(16) Duality Property :→

Let $FT[x(t)] = X(j\omega)$ or $X(\omega)$

then by Duality prop.

$$\boxed{FT[X(t)] = 2\pi\, x(-\omega)}$$

Proof :→ From IFT we know that

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega \rightarrow (1)$$

Replace t by $-t$ in ①

$$x(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega$$

Multiply both sides by 2π

$$2\pi\, x(-t) = \int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega$$

Now let us replace t with ω and ω with t

$$2\pi\, x(-\omega) = \underbrace{\int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt}$$

$$2\pi\, x(-\omega) = FT[X(t)]$$

Example