Laplace transform and FDTD approach applied to MTL simulation

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Laplace Transform and FDTD Approach Applied to MTL Simulation

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Abstract— The paper proposes two different approaches to simulation of multiconductor transmission lines (MTL). Numerical results of MTL simulations based on both the Laplace transform and Finite Difference Time Domain (FDTD) method are presented and compared. Fundamental algorithms were programmed in Matlab language. Some typical situations are solved as illustration of the results.

1. INTRODUCTION

Let us suppose a simple MTL linear system consisting of a uniform $(n+1)$-conductor transmission line terminated at both ends (left (1), right (2)) by linear lumped-parameter networks, see Fig. 1.

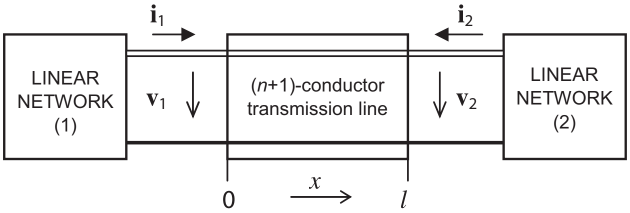

Figure 1: Simple MTL linear system.

The MTL is considered to be of a length l, with per-unit-length $n \times n$ matrices $\mathbf{R}_0$, $\mathbf{L}_0$, $\mathbf{G}_0$ and $\mathbf{C}_0$. The basic MTL equations can be expressed as [1]

$$-\frac{\partial \mathbf{v}(t,\, x)}{\partial x} = \mathbf{R}_0 \mathbf{i}(t,\, x) + \mathbf{L}_0 \frac{\partial \mathbf{i}(t,\, x)}{\partial t}, \quad -\frac{\partial \mathbf{i}(t,\, x)}{\partial x} = \mathbf{G}_0 \mathbf{v}(t,\, x) + \mathbf{C}_0 \frac{\partial \mathbf{v}(t,\, x)}{\partial t}, \tag{1}$$

where $\mathbf{v}(t,\, x)$ and $\mathbf{i}(t,\, x)$ are $n \times 1$ column vectors of instantaneous voltages and currents of n active wires at a distance x from MTL's left end respectively.

To solve the above stated system two basic approaches will be considered. First, the Equation (1) will be treated in the s-domain after the Laplace transform is applied, and then a proper method for numerical inversion of Laplace transform (NILT) will be used to get the required time-domain solution. In principle, both one- and two-dimensional Laplace transforms can be utilized for this purpose, see e.g., [1–3]. Second, the Equation (1) will be treated in the time-domain directly. Among many other methods, the FDTD approach seems to be very well applicable [4–6]. The Laplace transform approach does not make it possible to consider a nonlinear MTL in general. On the other hand, it is relatively easy to incorporate boundary conditions defined by terminating networks just in the s-domain. The FDTD approach can handle both linear and nonlinear cases. The connection of distributed and lumped parts, however, can be more complicated.

2. LAPLACE TRANSFORM APPROACH

Herein, only a method based on the one-dimensional Laplace transform will be considered. After performing Laplace transform with respect to time, and considering only zero initial voltage and current distributions along the MTL's wires ($\mathbf{v}(0, x) = \mathbf{i}(0, x) = 0$), the Equation (1) lead to a compact matrix form [1]

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix}\mathbf{V}(s, x)\\ \mathbf{I}(s, x)\end{bmatrix} = \begin{bmatrix}0 & -\mathbf{Z}(s)\\ -\mathbf{Y}(s) & 0\end{bmatrix}\cdot\begin{bmatrix}\mathbf{V}(s, x)\\ \mathbf{I}(s, x)\end{bmatrix}, \tag{2}$$

where $\mathbf{Z}(s) = \mathbf{R}_0 + s\mathbf{L}_0$ and $\mathbf{Y}(s) = \mathbf{G}_0 + s\mathbf{C}_0$ is a per-unit-length series impedance and shunting admittance matrix respectively. The solution of (2) can be expressed as

$$\begin{bmatrix}\mathbf{V}(s, x)\\ \mathbf{I}(s, x)\end{bmatrix} = \exp\left(\begin{bmatrix}0 & -\mathbf{Z}(s)\\ -\mathbf{Y}(s) & 0\end{bmatrix}x\right)\cdot\begin{bmatrix}\mathbf{V}(s, 0)\\ \mathbf{I}(s, 0)\end{bmatrix} = \begin{bmatrix}\Phi_{11}(s, x) & \Phi_{12}(s, x)\\ \Phi_{21}(s, x) & \Phi_{11}^{\mathrm{T}}s, x)\end{bmatrix}\cdot\begin{bmatrix}\mathbf{V}(s, 0)\\ \mathbf{I}(s, 0)\end{bmatrix}, \tag{3}$$

where $\mathbf{V}(s, 0)$ and $\mathbf{I}(s, 0)$ are given by boundary conditions. The matrix exponential function, called a chain matrix $\Phi(s, x)$, is decomposed into square submatrices in (3). Then the solution can be split into separate matrix equations

$$\mathbf{V}(s, x) = \Phi_{11}(s, x)\mathbf{V}(s, 0) + \Phi_{12}(s, x)\mathbf{I}(s, 0), \quad \mathbf{I}(s, x) = \Phi_{21}(s, x)\mathbf{V}(s, 0) + \Phi_{11}^{\mathrm{T}}(s, x)\mathbf{I}(s, 0). \tag{4}$$

The boundary conditions can be expressed by generalized Thvenin or Norton equivalents in the form

$$\mathbf{V}_{1(2)}(s) = \mathbf{V}_{\mathrm{i}1(2)}(s) - \mathbf{Z}_{\mathrm{i}1(2)}(s)\mathbf{I}_{1(2)}(s) \quad \text{or} \quad \mathbf{I}_{1(2)}(s) = \mathbf{I}_{\mathrm{i}1(2)}(s) - \mathbf{Y}_{\mathrm{i}1(2)}(s)\mathbf{V}_{1(2)}(s), \tag{5}$$

while new designations were taken into account as $\mathbf{V}_1(s) = \mathbf{V}(s, 0)$, $\mathbf{I}_1(s) = \mathbf{I}(s, 0)$, and $\mathbf{V}_2(s) = \mathbf{V}(s, l)$, $\mathbf{I}_2(s) = -\mathbf{I}(s, l)$ for the left and right MTL's side respectively. Here $\mathbf{V}_i(s)$ and $\mathbf{I}_i(s)$ are $n \times 1$ vectors of internal voltages and currents, $\mathbf{Z}_i(s)$ and $\mathbf{Y}_i(s)$ mean $n \times n$ internal impedance and admittance matrices respectively. For example, when using generalized Norton equivalents, the equation can be derived as [2]

$$\begin{aligned}\mathbf{V}_1(s) = &\left[\left(\Phi_{11}^{\mathrm{T}}(s) - \mathbf{Y}_{\mathrm{i}2}(s)\Phi_{12}(s)\right)\mathbf{Y}_{\mathrm{i}1}(s) + \mathbf{Y}_{\mathrm{i}2}(s)\Phi_{11}(s) - \Phi_{21}(s)\right]^{-1}\\ &\left[\left(\Phi_{11}^{\mathrm{T}}(s) - \mathbf{Y}_{\mathrm{i}2}(s)\Phi_{12}(s)\right)\mathbf{I}_{\mathrm{i}1}(s) + \mathbf{I}_{\mathrm{i}2}(s)\right],\end{aligned} \tag{6}$$

and the $\mathbf{I}_1(s)$ is given by the corresponding equation in (5). Here $\Phi_{\mathrm{ij}}(s)$, i, j = 1, 2, mean square submatrices of the full chain matrix $\Phi(s) = \Phi(s, l)$. Having substituted (5) and (6) into (3), this equation can be treated by a proper NILT method to get the time-domain solution. In this paper the NILT [7] has been applied, see examples below.

3. FDTD APPROACH

For a numerical solution of the above described wave equation system (1) the widely known Finite Difference Time Domain method can also be used. The main aim is to approximate the temporal and spatial derivatives by the suitable difference expression, which ensures the best stability and the highest accuracy of the numerical solution. There are a lot of possibilities how to replace the above mentioned derivatives. One of them is to use the implicit Wendorff formula, which can be described for the n-th time step and for the k-th spatial element of transmission line by the following expression

$$\frac{\partial\mathbf{v}(t, x)}{\partial t} \approx \frac{1}{2}\left(\frac{\mathbf{v}_k^n - \mathbf{v}_k^{n-1}}{\Delta t} + \frac{\mathbf{v}_{k+1}^n - \mathbf{v}_{k+1}^{n-1}}{\Delta t}\right), \quad \frac{\partial\mathbf{v}(t, x)}{\partial x} \approx \frac{1}{2}\left(\frac{\mathbf{v}_{k+1}^n - \mathbf{v}_k^n}{\Delta x} + \frac{\mathbf{v}_{k+1}^{n-1} - \mathbf{v}_k^{n-1}}{\Delta x}\right). \tag{7}$$

Here the derivatives of voltages (currents) are replaced by a combination of both forward and backward differences. Some interesting results based on the application of the described formula can be found in [4, 5], where practical examples of a numerical modeling of the surge phenomena on transmission lines caused by the lightning stroke and on *hv* and *vhv* three phase transmission lines with earth wire are presented.

The aim of our investigation was to find an effective algorithm for numerical simulation of the current or voltage wave propagation on multiconductor transmission line. Therefore we carried out a lot of tests with the different way of replacing derivatives by the difference expression. When we used the backward differences the solution was often unstable, but when we used the forward differences, we obtained a stable solution but the accuracy deteriorated. The best solution was obtained by using so-called leapfrog method, when the spatial and temporal derivatives were replaced by the combination of both central and forward differences. This modification of FDTD was discussed for example in [6] and it was applied to the numerical simulation of electromagnetic wave propagations in a free space. So, the temporal and spatial derivatives in wave Equation (1) were replaced by the four following expressions

$$\begin{aligned} \frac{\partial \mathbf{v}(t,\,x)}{\partial t} &\approx \frac{\mathbf{v}_k^{n+\frac{1}{2}} - \mathbf{v}_k^{n-\frac{1}{2}}}{\Delta t}, \quad & \frac{\partial \mathbf{v}(t,\,x)}{\partial x} &\approx \frac{\mathbf{v}_{k+1}^{n+\frac{1}{2}} - \mathbf{v}_k^{n+\frac{1}{2}}}{\Delta x}, \\ \frac{\partial \mathbf{i}(t,\,x)}{\partial t} &\approx \frac{\mathbf{i}_{k+\frac{1}{2}}^{n+1} - \mathbf{i}_{k+\frac{1}{2}}^{n}}{\Delta t}, \quad & \frac{\partial \mathbf{i}(t,\,x)}{\partial x} &\approx \frac{\mathbf{i}_{k+\frac{1}{2}}^{n} - \mathbf{i}_{k-\frac{1}{2}}^{n}}{\Delta x}. \end{aligned} \tag{8}$$

To simulate the current and voltage distributions $\mathbf{v}(t, x)$ and $\mathbf{i}(t, x)$ along all lines of MTL in arbitrary time $t = n\Delta t$, the implicit formula can be expressed in a compact matrix form

$$\begin{bmatrix} \mathbf{v}(x_1,\,\ldots,\,x_{M+1}) \\ \mathbf{i}(x_1,\,\ldots,\,x_{M+1}) \end{bmatrix}^{n+1} = \mathbf{A}^{-1}\left(\mathbf{B}\begin{bmatrix} \mathbf{v}(x_1,\,\ldots,\,x_{M+1}) \\ \mathbf{i}(x_1,\,\ldots,\,x_{M+1}) \end{bmatrix}^{n} + \mathbf{D}\right). \tag{9}$$

The coefficients of matrices $\mathbf{A}$ and $\mathbf{B}$ are given by the per-unit-length matrices, the matrix $\mathbf{D}$ is given by sources.

4. ERROR ANALYSIS

Both the above mentioned numerical methods were used to simulate the current and voltage wave propagation along a transmission line. As an example we suppose an infinitely long line with a negligible leakage and inductance, so-called Thomson cable, with $G_0 = 0$ and $L_0 = 0$, see Fig. 2 [8].

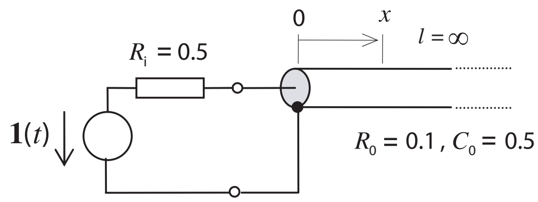

Figure 2: Model and parameters of Thomson cable.

The remaining primary parameters R_0 and C_0 are given in normalized forms. The cable is excited from the source of unit step voltage $v_i(t) = \mathbf{1}(t)$ and resistance R_i, which can represent internal resistance of this voltage source. In that case the closed form solution exists and the accuracy of both numerical approaches can be verified. We used the closed form solution which is derived for example in [8]

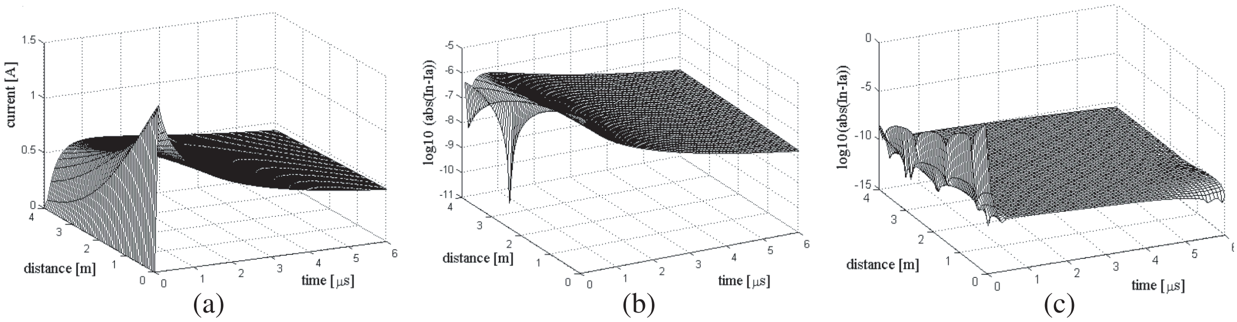

Figure 3: Current distribution (a) with FDTD error (b) and Laplace transform error (c).

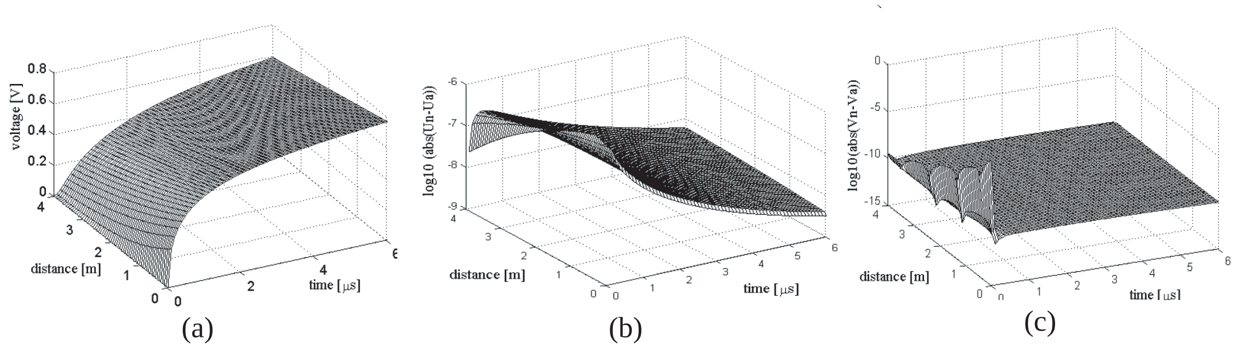

Figure 4: Voltage distribution (a) with FDTD error (b) and Laplace transform error (c).

$$
\begin{aligned}
i(t,\,x) &= \frac{1}{R_i}\exp\left(\frac{R_0}{R_i^2 C_0}t + \frac{R_0}{R_i}x\right)\cdot \operatorname{erfc}\left(\frac{1}{R_i}\sqrt{\frac{R_0}{C_0}t} + \frac{x}{2}\sqrt{\frac{R_0 C_0}{t}}\right),\\
v(t,\,x) &= \operatorname{erfc}\left(\frac{x}{2}\sqrt{\frac{R_0 C_0}{t}}\right) - R_i i(t,\,x).
\end{aligned}
\tag{10}
$$

In the above formulae erfc means a complementary error function. In Fig. 3 and Fig. 4 you can see the calculated current $i(t,\,x)$ and voltage $v(t,\,x)$ distributions along the line together with the absolute errors in logarithmic scales of FDTD and Laplace transform approaches.

5. PRACTICAL EXAMPLE

To illustrate the practical application of Laplace transform and FDTD method we consider a (3+1)-conductor uniform transmission line, see Fig. 5.

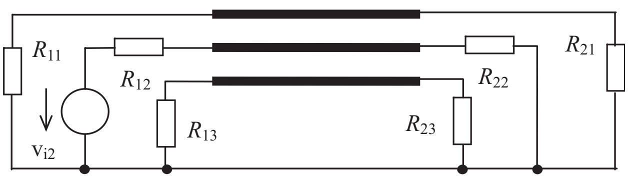

Figure 5: The (3+1)-conductor TL.

We suppose unsymmetrical loads which are represented by the terminating resistors $R_{11} = 10\,\Omega$, $R_{12} = 1\,\Omega$, $R_{13} = 100\,\Omega$ on the left side, and $R_{21} = 10\,\mathrm{k}\Omega$, $R_{22} = 1\,\Omega$, $R_{23} = 10\,\Omega$ on the right side.

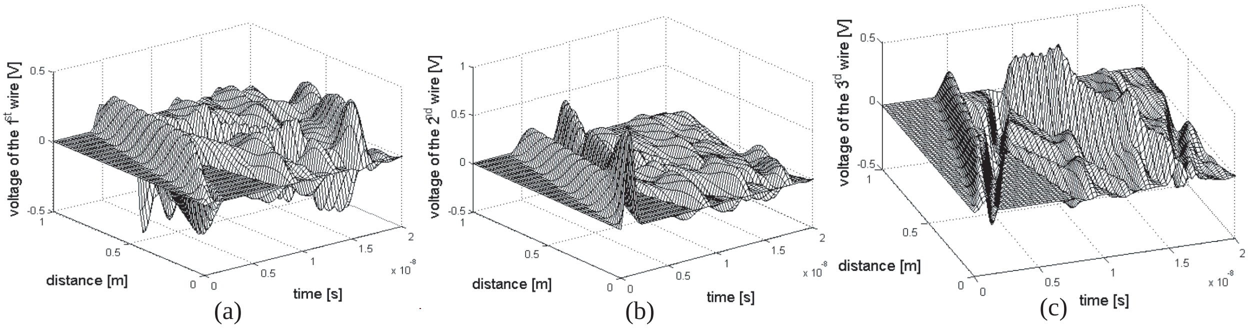

Figure 6: Voltage distributions along the MTL wires.

The MTL is of the length $l = 1\,\text{m}$, with per-unit-length matrices

$$\mathbf{R}_0 = \begin{bmatrix} 41.7 & 0 & 0 \\ 0 & 41.7 & 0 \\ 0 & 0 & 41.7 \end{bmatrix} \frac{\Omega}{m}, \quad \mathbf{L}_0 = \begin{bmatrix} 2.4 & 0.69 & 0.64 \\ 0.69 & 2.36 & 0.69 \\ 0.64 & 0.69 & 2.4 \end{bmatrix} \frac{\mu H}{m},$$

$$\mathbf{G}_0 = \begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.6 \end{bmatrix} \frac{mS}{m}, \quad \mathbf{C}_0 = \begin{bmatrix} 21 & -12 & -4 \\ -12 & 26 & -12 \\ -4 & -12 & 21 \end{bmatrix} \frac{pF}{m}.$$

The input voltage source driving the central wire of the MTL has the waveform $v_{i2}(t) = \sin^2(\pi t/2 \cdot 10^{-9})$ if $0 \leq t \leq 2 \cdot 10^{-9}$, and $v_{i2}(t) = 0$ otherwise. The solution in $(t,\, x)$-domain was obtained using both FDTD method and LT method. All the results of numerical calculated voltage and current distributions are the same. Fig. 6 shows the voltage waves on the excited wire (b) and the voltage waves induced on the neighboring wires of the transmission line (a, c), only as an example.

6. CONCLUSIONS

This paper presents two different approaches which can be used successfully for numerical solution of the voltage and current distributions along three-phase transmission line with earth wire. A new variant of the FDTD so-called leapfrog method is proposed and verified. The correctness of both used methods was verified and the obtained results indicate that both methods are very effective numerical tools for the simulation of time-spatial dependences. The accuracy of both methods is comparable, but the FDTD is more time-consuming than LT. On the other hand the FDTD can be used for the simulation of both linear and nonlinear systems.

ACKNOWLEDGMENT

The research described in the paper was financially supported by the research programs MSM 0021630513 and MSM 0021630503.

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