Introduction to the FDTD method

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FDTD (= finite-difference time-domain)

Principle

1. Start from Maxwell's equations
2. Replace all derivatives with finite-difference approximations
3. Done

Maxwell's curl equations

$$\nabla \times \overline{E} = -\mu \frac{\partial}{\partial t}\overline{H},$$

$$\nabla \times \overline{H} = \overline{J} + \sigma \overline{E} + \epsilon \frac{\partial}{\partial t}\overline{E},$$

$$\left(\overline{H} = \frac{1}{\mu}\overline{B}\right)$$

$$\begin{cases} \mu \frac{\partial}{\partial t} H_x = \frac{\partial}{\partial z} E_y - \frac{\partial}{\partial y} E_z \\ \mu \frac{\partial}{\partial t} H_y = \frac{\partial}{\partial x} E_z - \frac{\partial}{\partial z} E_x \\ \mu \frac{\partial}{\partial t} H_z = \frac{\partial}{\partial y} E_x - \frac{\partial}{\partial x} E_y \end{cases}$$

$$\begin{cases} \epsilon \frac{\partial}{\partial t} E_x = \frac{\partial}{\partial y} H_z - \frac{\partial}{\partial z} H_y - \sigma E_x \\ \epsilon \frac{\partial}{\partial t} E_y = \frac{\partial}{\partial z} H_x - \frac{\partial}{\partial x} H_z - \sigma E_y \\ \epsilon \frac{\partial}{\partial t} E_z = \frac{\partial}{\partial x} H_y - \frac{\partial}{\partial y} H_x - \sigma E_z \end{cases}$$

Central difference approximations

Notation

$$x = i\,\Delta x,\ \ y = j\,\Delta y,\ z = k\,\Delta z \qquad t = n\,\Delta t$$

$$F(x, y, z, t) = F(i\Delta x, j\Delta y, k\Delta z, n\Delta t) \Rightarrow F^n(i, j, k)$$

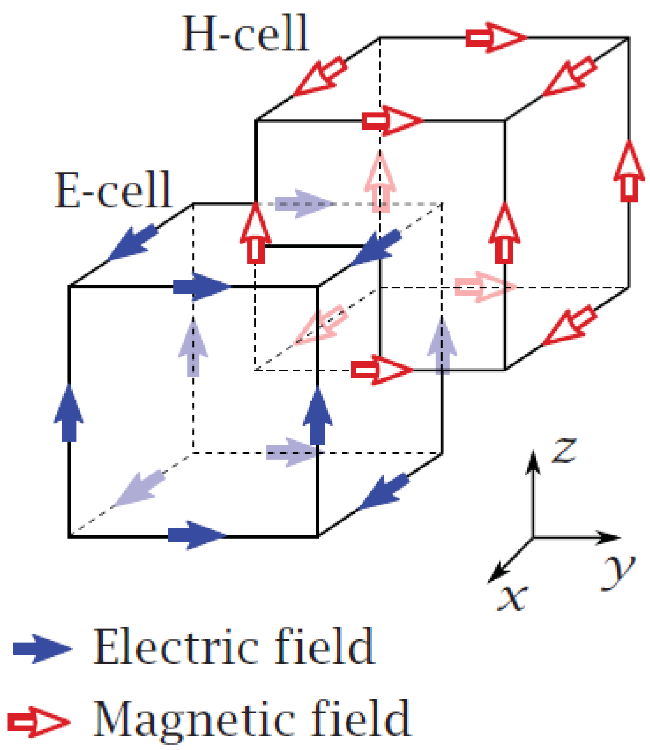
$$\frac{\partial F(x,y,z,t)}{\partial x} \approx \frac{F^n(i+\frac{1}{2},j,k) - F^n(i-\frac{1}{2},j,k)}{\Delta x} \qquad O(\Delta x^2)$$

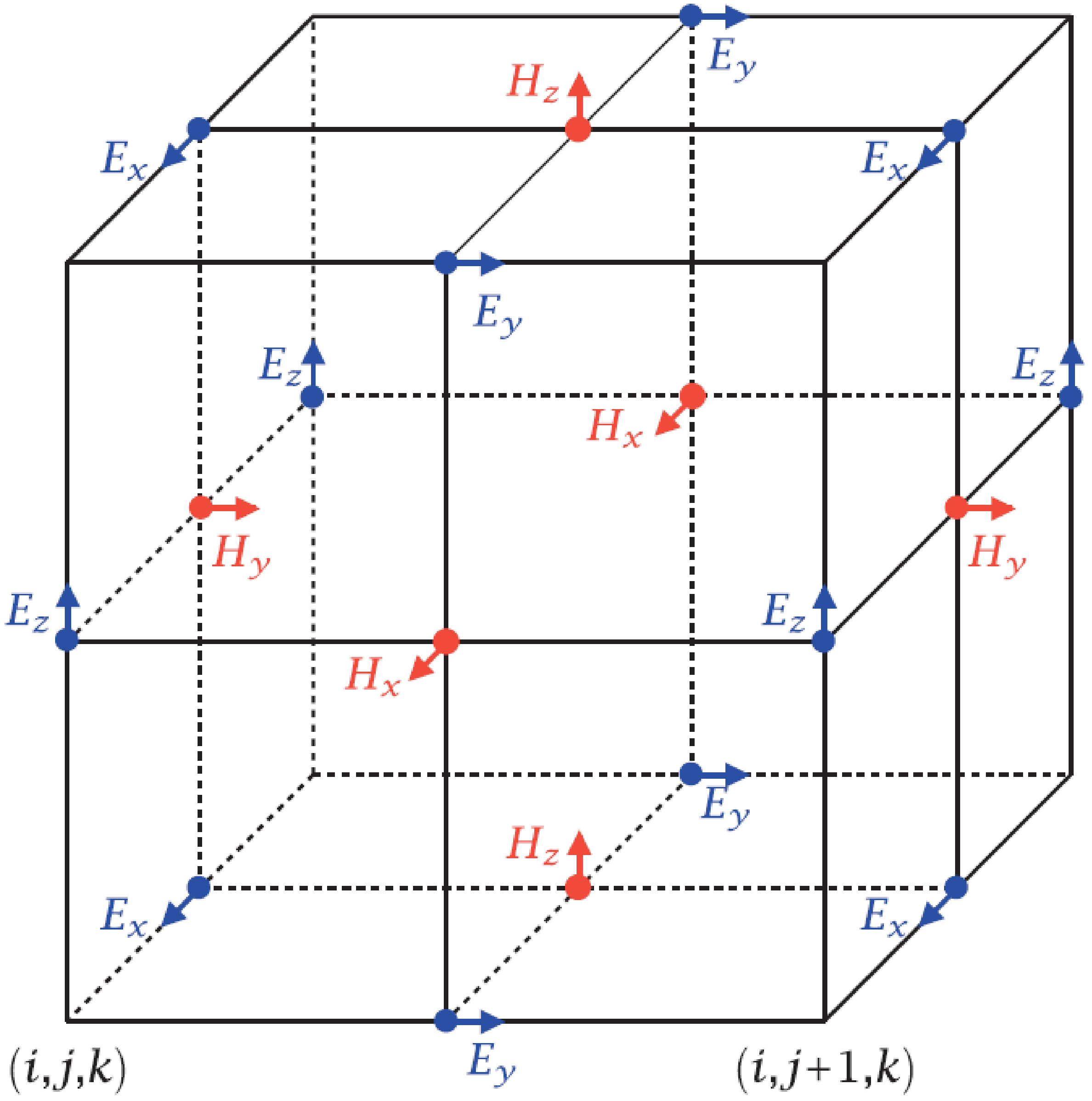
$$\frac{\partial F(x,y,z,t)}{\partial y} \approx \frac{F^n(i,j+\frac{1}{2},k) - F^n(i,j-\frac{1}{2},k)}{\Delta y} \qquad O(\Delta y^2)$$

$$\frac{\partial F(x,y,z,t)}{\partial z} \approx \frac{F^n(i,j,k+\frac{1}{2}) - F^n(i,j,k-\frac{1}{2})}{\Delta z} \qquad O(\Delta z^2)$$

$$\frac{\partial F(x,y,z,t)}{\partial t} \approx \frac{F^{n+\frac{1}{2}}(i,j,k) - F^{n-\frac{1}{2}}(i,j,k)}{\Delta t} \qquad O(\Delta t^2)$$

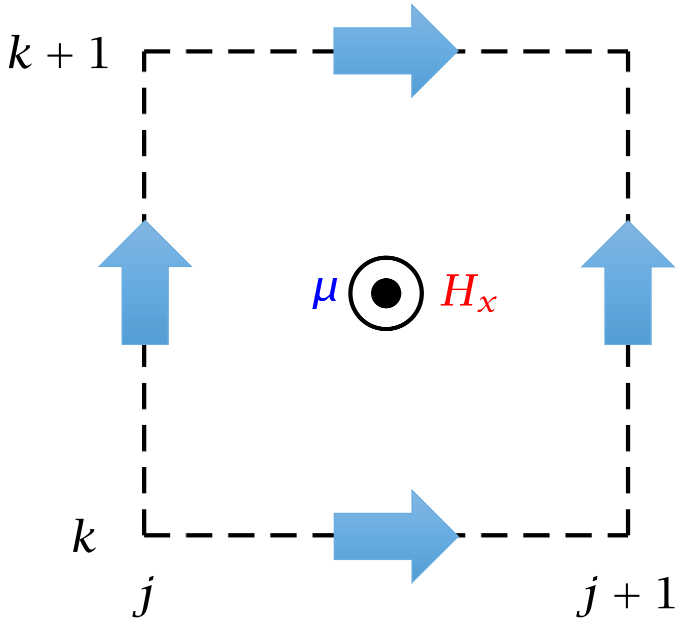
Yee cell (Yee, 1966)





Derivation of FDTD update equations

$$\boxed{\mu \frac{\partial}{\partial t} H_x = \frac{\partial}{\partial z} E_y - \frac{\partial}{\partial y} E_z}$$



$$\mu = \mu(i, j+\frac{1}{2}, k+\frac{1}{2})$$

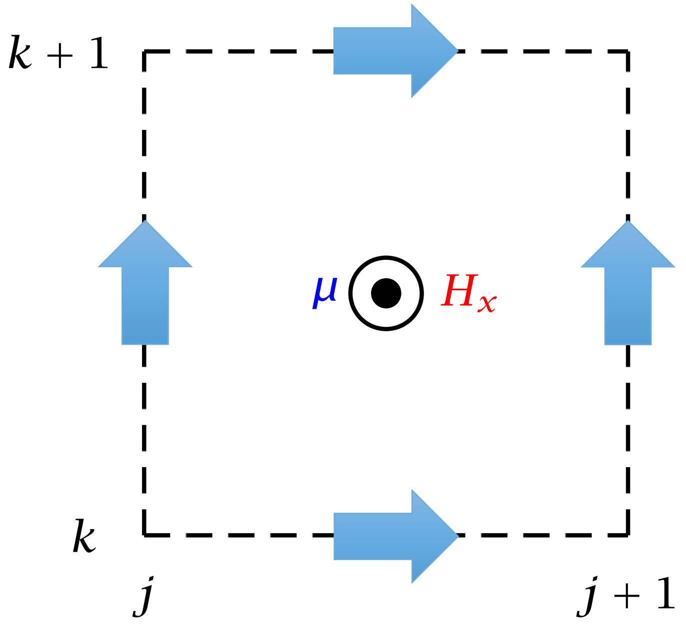
$$\frac{\partial}{\partial t} H_x^n(i, j+\frac{1}{2}, k+\frac{1}{2}) \approx \frac{{H_x}^{n+\frac{1}{2}}(i, j+\frac{1}{2}, k+\frac{1}{2}) - {H_x}^{n-\frac{1}{2}}(i, j+\frac{1}{2}, k+\frac{1}{2})}{\Delta t}$$

$$\frac{\partial}{\partial z} E_y^n(i, j+\frac{1}{2}, k+\frac{1}{2}) \approx \frac{{E_y}^n(i, j+\frac{1}{2}, k+1) - {E_y}^n(i, j+\frac{1}{2}, k)}{\Delta z}$$

$$\frac{\partial}{\partial y} E_z^n(i, j+\frac{1}{2}, k+\frac{1}{2}) \approx \frac{{E_z}^n(i, j+1, k+\frac{1}{2}) - {E_z}^n(i, j, k+\frac{1}{2})}{\Delta y}$$

Derivation of FDTD update equations

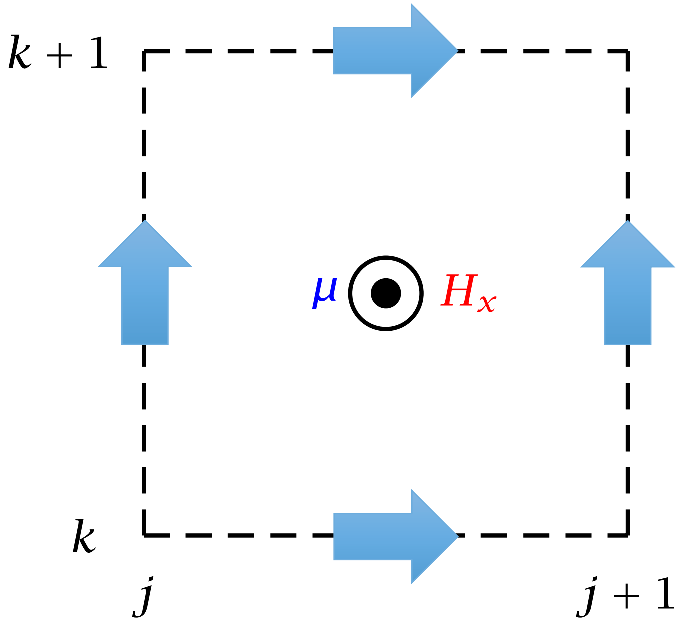
$$\boxed{\mu\frac{\partial}{\partial t}H_x = \frac{\partial}{\partial z}E_y - \frac{\partial}{\partial y}E_z}$$



$$\mu(i,j+\frac{1}{2},k+\frac{1}{2})\frac{\boxed{H_x^{n+\frac{1}{2}}(i,j+\frac{1}{2},k+\frac{1}{2})}-H_x^{n-\frac{1}{2}}(i,j+\frac{1}{2},k+\frac{1}{2})}{\Delta t}$$

$$=\frac{E_y^n(i,j+\frac{1}{2},k+1)-E_y^n(i,j+\frac{1}{2},k)}{\Delta z}-\frac{E_z^n(i,j+1,k+\frac{1}{2})-E_z^n(i,j,k+\frac{1}{2})}{\Delta y}$$

FDTD update equations



$$\boxed{H_x^{n+\frac{1}{2}}(i, j+\tfrac{1}{2}, k+\tfrac{1}{2})} = H_x^{n-\frac{1}{2}}(i, j+\tfrac{1}{2}, k+\tfrac{1}{2})$$

$$+ \frac{\Delta t}{\mu(i, j+\frac{1}{2}, k+\frac{1}{2})\Delta z}\left[E_y^n(i, j+\tfrac{1}{2}, k+1) - E_y^n(i, j+\tfrac{1}{2}, k)\right]$$

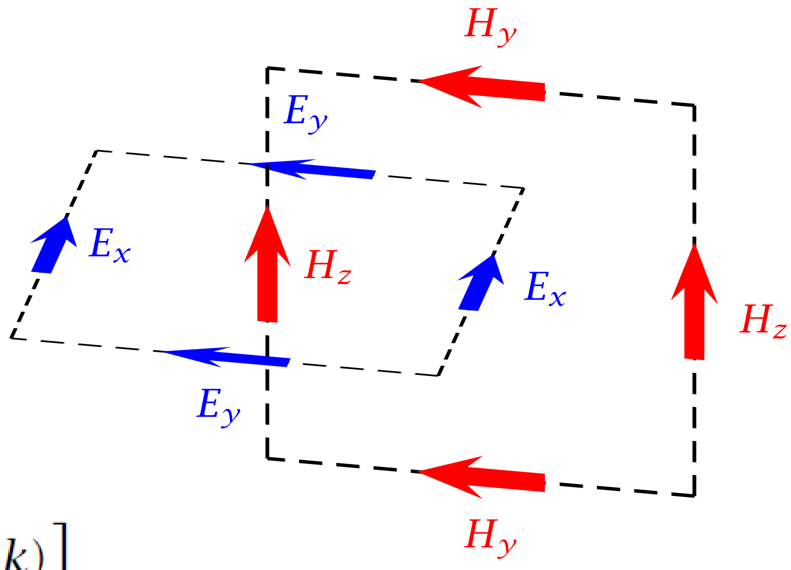
$$+ \frac{\Delta t}{\mu(i, j+\frac{1}{2}, k+\frac{1}{2})\Delta y}\left[E_z^n(i, j, k+\tfrac{1}{2}) - E_z^n(i, j+1, k+\tfrac{1}{2})\right]$$

$$\boxed{H_y^{n+\frac{1}{2}}(i+\tfrac{1}{2}, j, k+\tfrac{1}{2})} = H_y^{n-\frac{1}{2}}(i+\tfrac{1}{2}, j, k+\tfrac{1}{2})$$

$$+ \frac{\Delta t}{\mu(i+\frac{1}{2}, j, k+\frac{1}{2})\Delta x}\left[E_z^n(i+1, j, k+\tfrac{1}{2}) - E_z^n(i, j, k+\tfrac{1}{2})\right]$$

$$+ \frac{\Delta t}{\mu(i+\frac{1}{2}, j, k+\frac{1}{2})\Delta z}\left[E_x^n(i+\tfrac{1}{2}, j, k) - E_x^n(i+\tfrac{1}{2}, j, k+1)\right]$$

FDTD update equations



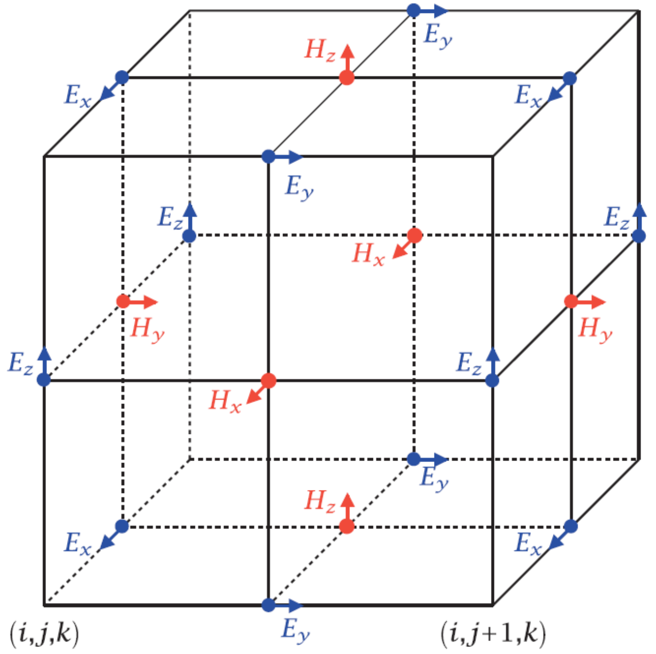
$$
\begin{aligned}
H_z^{n+\frac{1}{2}}(i+\tfrac{1}{2}, j+\tfrac{1}{2}, k) &= H_z^{n-\frac{1}{2}}(i+\tfrac{1}{2}, j+\tfrac{1}{2}, k) \\
&+ \frac{\Delta t}{\mu(i+\frac{1}{2}, j+\frac{1}{2}, k)\Delta y}\left[E_x^n(i+\tfrac{1}{2}, j+1, k) - E_x^n(i+\tfrac{1}{2}, j, k)\right] \\
&+ \frac{\Delta t}{\mu(i+\frac{1}{2}, j+\frac{1}{2}, k)\Delta x}\left[E_y^n(i, j+\tfrac{1}{2}, k) - E_y^n(i+1, j+\tfrac{1}{2}, k)\right]
\end{aligned}
$$

$$
E_x^{n+1}(i+\tfrac{1}{2}, j, k) = \frac{2\epsilon(i+\frac{1}{2}, j, k) - \sigma(i+\frac{1}{2}, j, k)\Delta t}{2\epsilon(i+\frac{1}{2}, j, k) + \sigma(i+\frac{1}{2}, j, k)\Delta t} E_x^n(i+\tfrac{1}{2}, j, k)
$$

$$
+ \frac{2\Delta t}{[2\epsilon(i+\frac{1}{2}, j, k) + \sigma(i+\frac{1}{2}, j, k)\Delta t]\Delta y}\left[H_z^{n+\frac{1}{2}}(i+\tfrac{1}{2}, j+\tfrac{1}{2}, k) - H_z^{n+\frac{1}{2}}(i+\tfrac{1}{2}, j-\tfrac{1}{2}, k)\right]
$$

$$
+ \frac{2\Delta t}{[2\epsilon(i+\frac{1}{2}, j, k) + \sigma(i+\frac{1}{2}, j, k)\Delta t]\Delta z}\left[H_y^{n+\frac{1}{2}}(i+\tfrac{1}{2}, j, k-\tfrac{1}{2}) - H_y^{n+\frac{1}{2}}(i+\tfrac{1}{2}, j, k+\tfrac{1}{2})\right]
$$

FDTD update equations



$$\boxed{E_y^{n+1}(i,j+\tfrac{1}{2},k)} = \frac{2\epsilon(i,j+\frac{1}{2},k) - \sigma(i,j+\frac{1}{2},k)\Delta t}{2\epsilon(i,j+\frac{1}{2},k) + \sigma(i,j+\frac{1}{2},k)\Delta t} E_y^n(i,j+\tfrac{1}{2},k)$$

$$+ \frac{2\Delta t}{[2\epsilon(i,j+\frac{1}{2},k) + \sigma(i,j+\frac{1}{2},k)\Delta t]\Delta z}\left[H_x^{n+\frac{1}{2}}(i,j+\tfrac{1}{2},k+\tfrac{1}{2}) - H_x^{n+\frac{1}{2}}(i,j+\tfrac{1}{2},k-\tfrac{1}{2})\right]$$

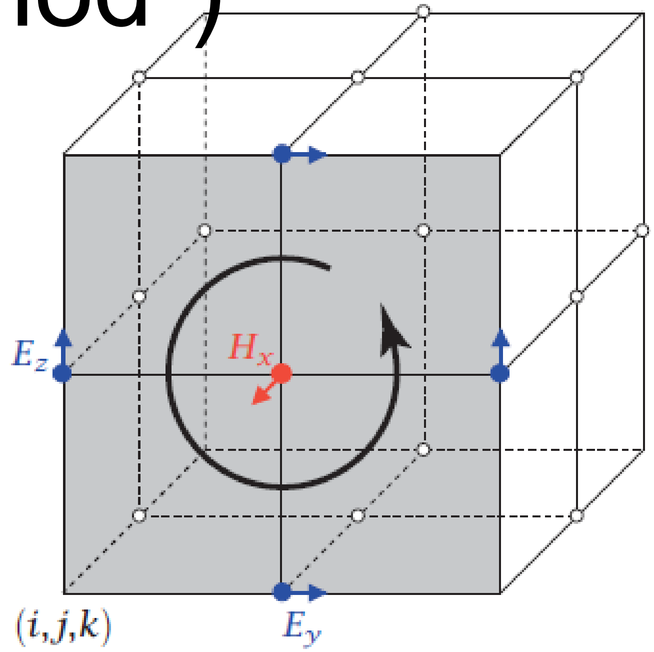
$$+ \frac{2\Delta t}{[2\epsilon(i,j+\frac{1}{2},k) + \sigma(i,j+\frac{1}{2},k)\Delta t]\Delta x}\left[H_z^{n+\frac{1}{2}}(i-\tfrac{1}{2},j+\tfrac{1}{2},k) - H_z^{n+\frac{1}{2}}(i+\tfrac{1}{2},j+\tfrac{1}{2},k)\right]$$

$$\boxed{E_z^{n+1}(i,j,k+\tfrac{1}{2})} = \frac{2\epsilon(i,j,k+\frac{1}{2}) - \sigma(i,j,k+\frac{1}{2})\Delta t}{2\epsilon(i,j,k+\frac{1}{2}) + \sigma(i,j,k+\frac{1}{2})\Delta t} E_z^n(i,j,k+\tfrac{1}{2})$$

$$+ \frac{2\Delta t}{[2\epsilon(i,j,k+\frac{1}{2}) + \sigma(i,j,k+\frac{1}{2})\Delta t]\Delta x}\left[H_y^{n+\frac{1}{2}}(i+\tfrac{1}{2},j,k+\tfrac{1}{2}) - H_y^{n+\frac{1}{2}}(i-\tfrac{1}{2},j,k+\tfrac{1}{2})\right]$$

$$+ \frac{2\Delta t}{[2\epsilon(i,j,k+\frac{1}{2}) + \sigma(i,j,k+\frac{1}{2})\Delta t]\Delta y}\left[H_x^{n+\frac{1}{2}}(i,j-\tfrac{1}{2},k+\tfrac{1}{2}) - H_x^{n+\frac{1}{2}}(i,j+\tfrac{1}{2},k+\tfrac{1}{2})\right]$$

Derivation of FDTD update equations from integral form of Maxwell's equations ("FIT method")



$$\nabla \times \overline{E} = -\mu \frac{\partial}{\partial t}\overline{H}$$

$$\Downarrow$$

$$\int_S \nabla \times \overline{E} \cdot \overline{dS} = -\int_S \mu \frac{\partial}{\partial t}\overline{H} \cdot \overline{dS}$$

$$\Downarrow$$

$$\int_{\partial S} \overline{E} \cdot \overline{dc} = -\int_S \mu \frac{\partial}{\partial t}\overline{H} \cdot \overline{dS}$$

Use the mid-ordinate numerical integration method

$$\int_{\partial S} \overline{E} \cdot \overline{dc} \approx E_y^n(i, j+\tfrac{1}{2}, k)\Delta y + E_z^n(i, j+1, k+\tfrac{1}{2})\Delta z - E_y^n(i, j+\tfrac{1}{2}, k+1)\Delta y - E_z^n(i, j, k+\tfrac{1}{2})\Delta z$$

$$-\int_S \mu \frac{\partial}{\partial t}\overline{H} \cdot \overline{dS} \approx -\mu(i, j+\tfrac{1}{2}, k+\tfrac{1}{2}) \frac{\left[H_x^{n+\frac{1}{2}}(i, j+\tfrac{1}{2}, k+\tfrac{1}{2}) - H_x^{n-\frac{1}{2}}(i, j+\tfrac{1}{2}, k+\tfrac{1}{2})\right]}{\Delta t} \Delta y \Delta z$$

Example: 1D FDTD

$$\mu \frac{\partial}{\partial t} H_y = \frac{\partial}{\partial x} E_z$$

$$\epsilon \frac{\partial}{\partial t} E_z = \frac{\partial}{\partial x} H_y$$

$$H_y^{n+\frac{1}{2}}(i+\frac{1}{2}) = H_y^{n-\frac{1}{2}}(i+\frac{1}{2}) + \frac{\Delta t}{\mu(i+\frac{1}{2})\Delta x}\left(E_z^n(i+1) - E_z^n(i)\right)$$

$$E_z^{n+1}(i) = E_z(i)^n + \frac{\Delta t}{\epsilon(i)\Delta x}\left(H_y^{n+\frac{1}{2}}(i+\frac{1}{2}) - H_y^{n+\frac{1}{2}}(i-\frac{1}{2})\right)$$

Example1.m

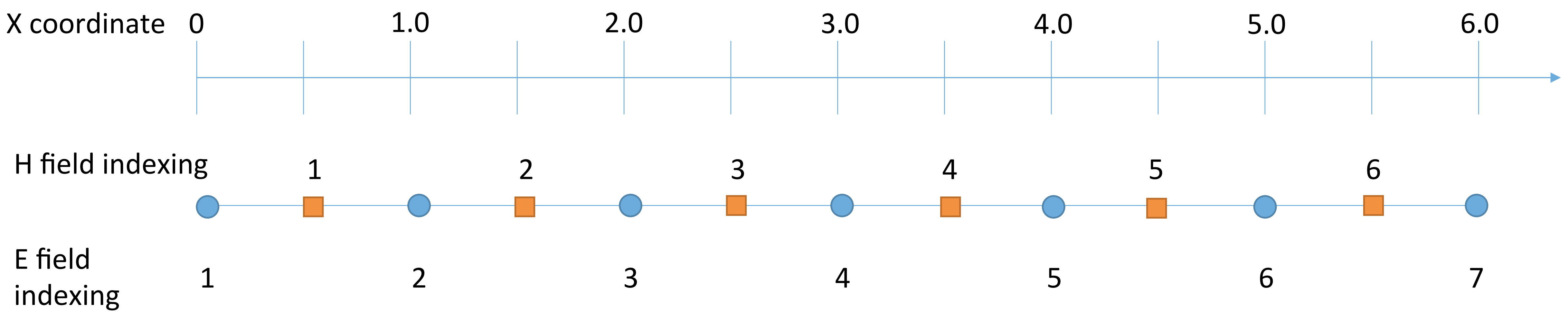
Implementation in MATLAB

$$H_y^{n+\frac{1}{2}}(i+\frac{1}{2}) = H_y^{n-\frac{1}{2}}(i+\frac{1}{2}) + \frac{\Delta t}{\mu(i+\frac{1}{2})\Delta x}\left(E_z^n(i+1) - E_z^n(i)\right)$$

$$E_z^{n+1}(i) = E_z(i)^n + \frac{\Delta t}{\epsilon(i)\Delta x}\left(H_y^{n+\frac{1}{2}}(i+\frac{1}{2}) - H_y^{n+\frac{1}{2}}(i-\frac{1}{2})\right)$$

```
% Magnetic field update equation
Hy(1:K-1) = Hy(1:K-1) + Db(1:K-1) .* ( Ez(2:K) - Ez(1:K-1) );

% Electric field update equation
Ez(2:K-1) = Ez(2:K-1) + Cb(2:K-1) .* ( Hy(2:K-1) - Hy(1:K-2) );
```



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- Principle of FDTD
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- Basic properties
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  - Dispersion
  - Boundary conditions
- Advantages and weaknesses
- Examples

Stability

- Courant-Friedrichs-Lewy (CFL) condition:

$$\boxed{\Delta t \leq \frac{1}{v_{\max}} \frac{1}{\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}}}}$$

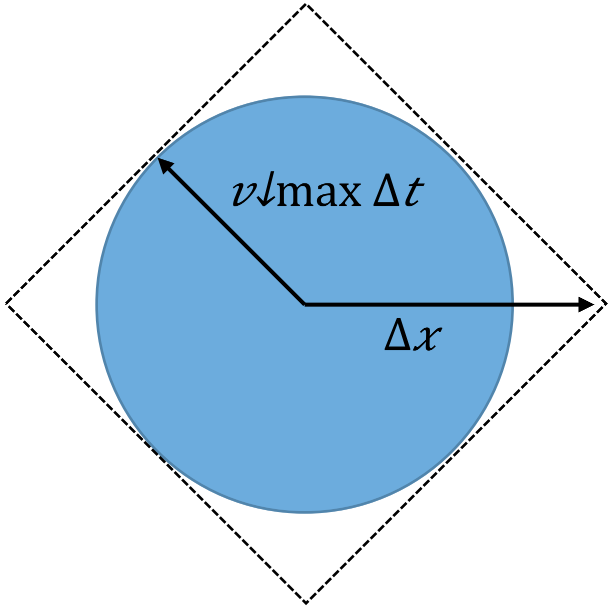
$$v_{\max} = \left\{ \frac{1}{\sqrt{\epsilon\mu}} \right\}_{\max}$$

In 3D:

$$\Delta x = \Delta y = \Delta z = \Delta$$

$$\Delta t \leq \frac{\Delta}{\sqrt{3} v_{\max}}$$

Numerical domain of dependence must include analytical domain of dependence



Numerical dispersion

Accurate formula

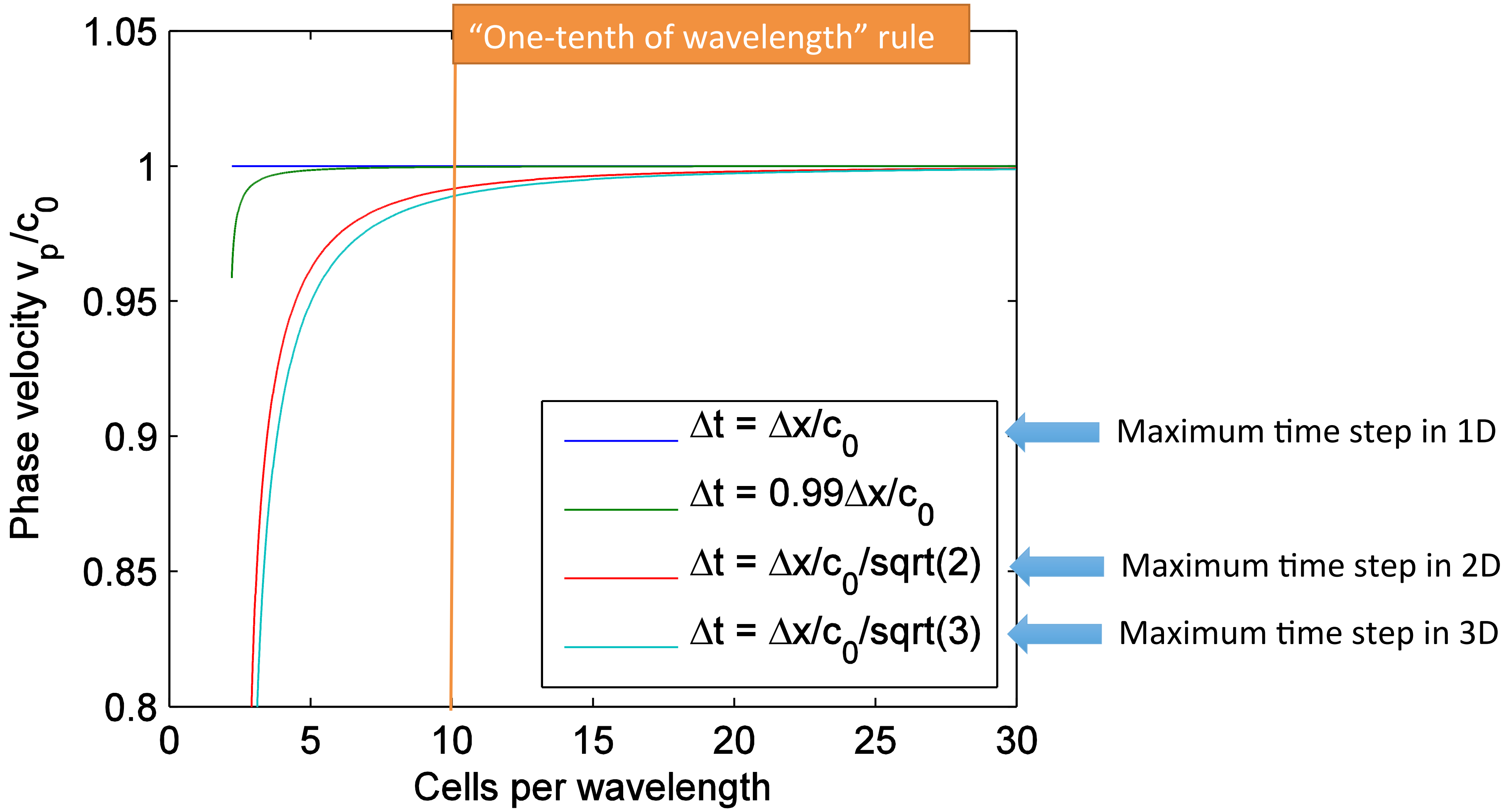
$$\boxed{\frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2}$$

FDTD

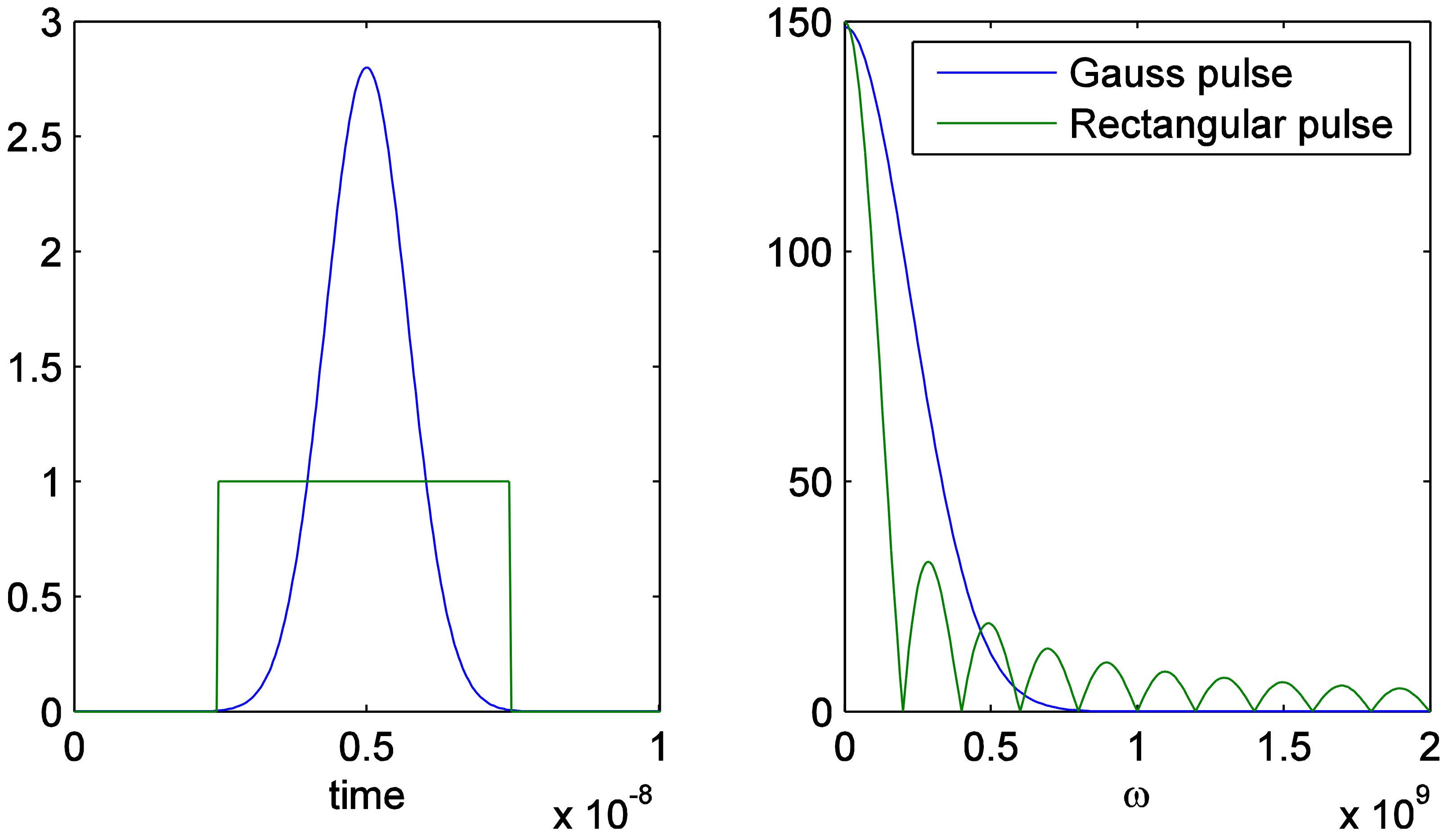
$$\boxed{\frac{\sin^2(\omega\Delta t/2)}{(c\Delta t)^2} = \frac{\sin^2(\tilde{k}_x\Delta x/2)}{(\Delta x)^2} + \frac{\sin^2(\tilde{k}_y\Delta y/2)}{(\Delta y)^2} + \frac{\sin^2(\tilde{k}_z\Delta z/2)}{(\Delta z)^2}}$$

Numerical phase velocity $\quad v_p = \dfrac{\omega}{\tilde{k}}$

Numerical dispersion

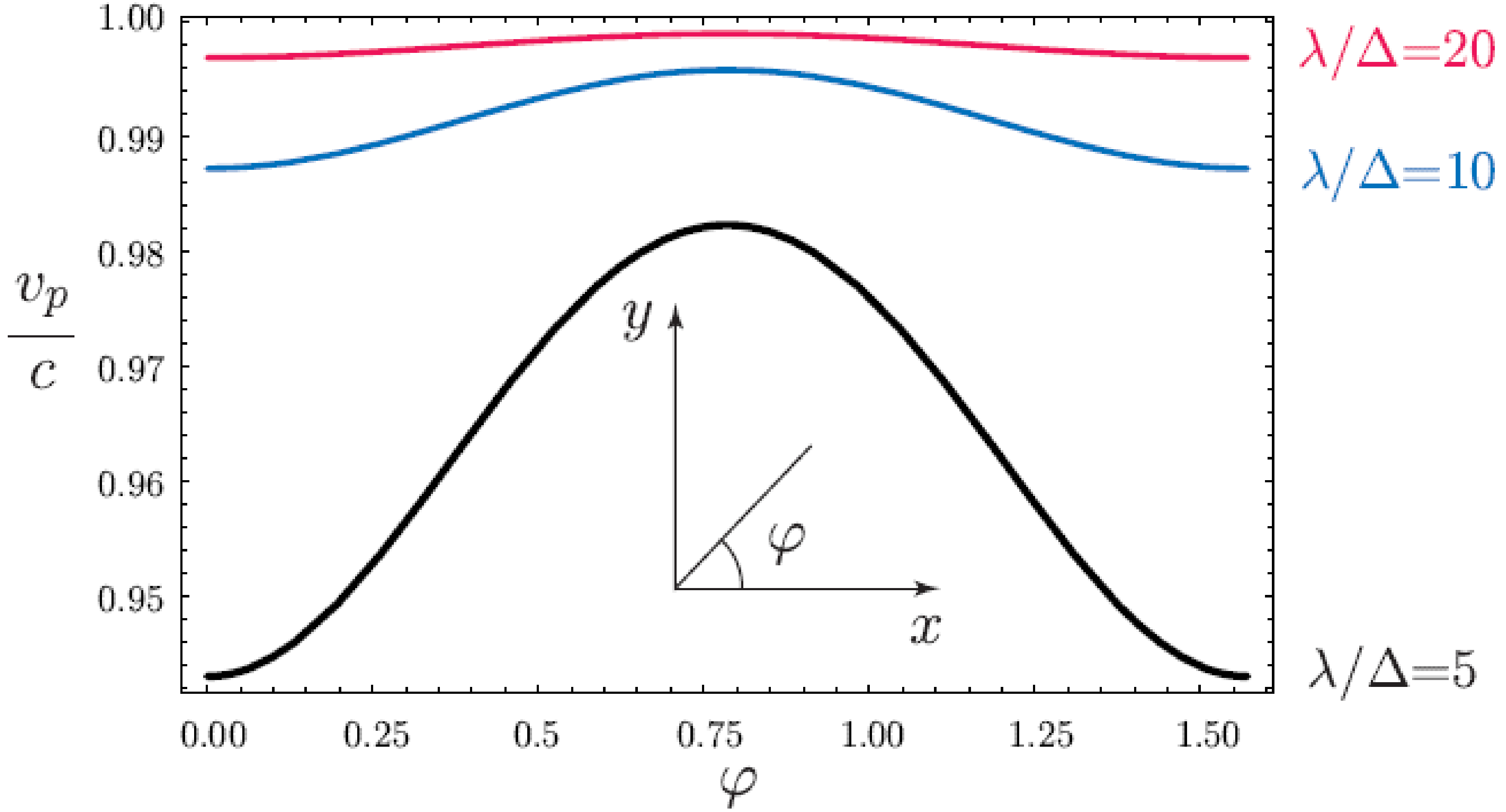


Spectra


3
2.5
2
1.5
1
0.5
0
0
0.5
1
time
x 10^-8
150
100
50
0
0
0.5
1
1.5
2
ω
x 10^9
Gauss pulse
Rectangular pulse

Anisotropic dispersion in 2D and 3D

$$\Delta x = \Delta y = \Delta \qquad c\Delta t = \frac{\Delta}{2} \leq \frac{\Delta}{\sqrt{3}}$$



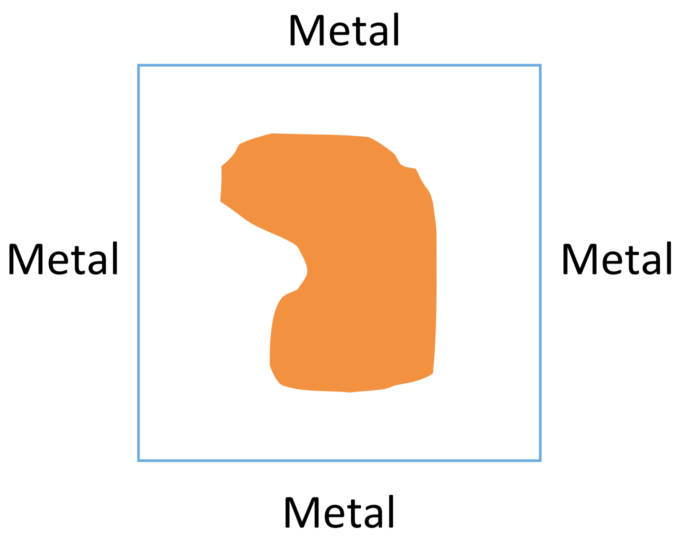
Example: 2D FDTD

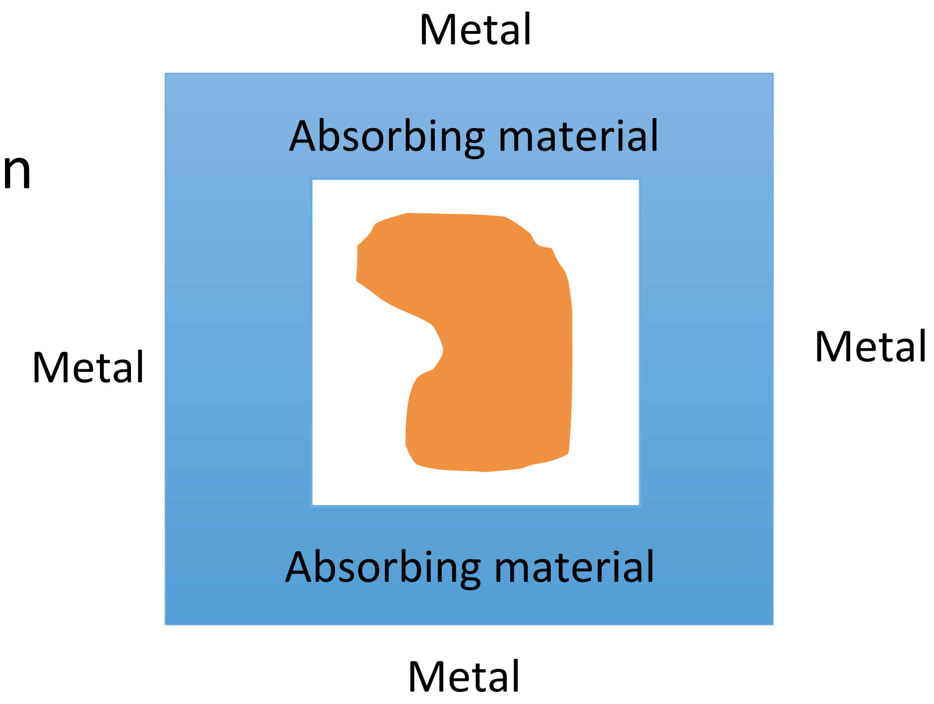
- 2D FDTD and anisotropic dispersion
  - example3a.m

- Be careful with dielectric materials
  - wavelength is shorter => finer cell size is needed
  - example3b.m

Absorbing boundary conditions

- In the basic form, FDTD can only model boxes, with ideally conducting walls
  - How to terminate the computation domain to model free space?
- FDTD is very good at modelling different materials:
  - Berenger 1994: Make the walls of the box from an unphysical material that absorbs anything!
- ABCs are essential for any FDTD code
  - Example4.m, example5a, 5b, 5c



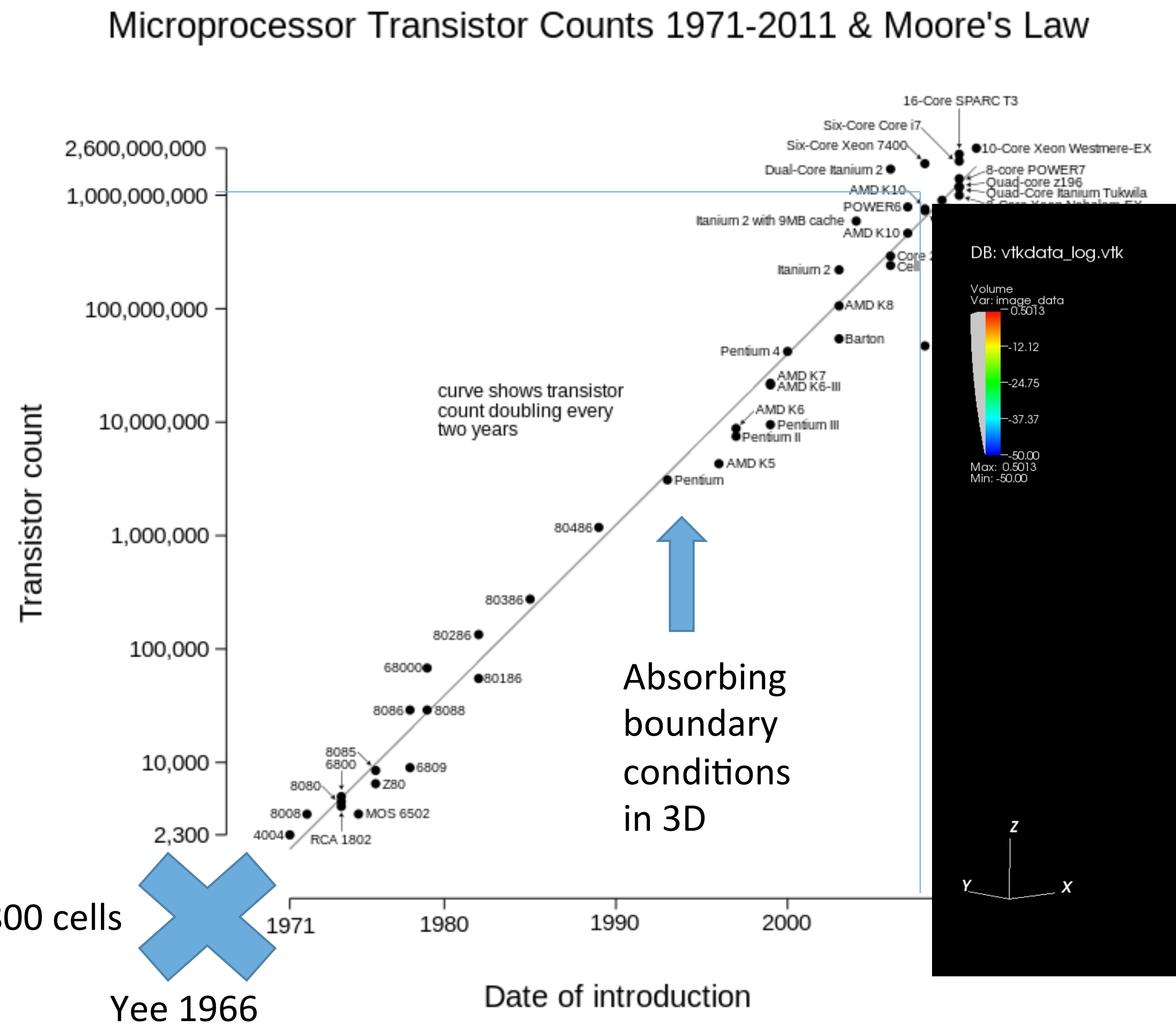


Contents

Advantages of FDTD (1)

- Simple equations
- Can be parallelized easily
- Scales linearly with number of unknowns
- No need to solve equation systems

=> good for very large problems

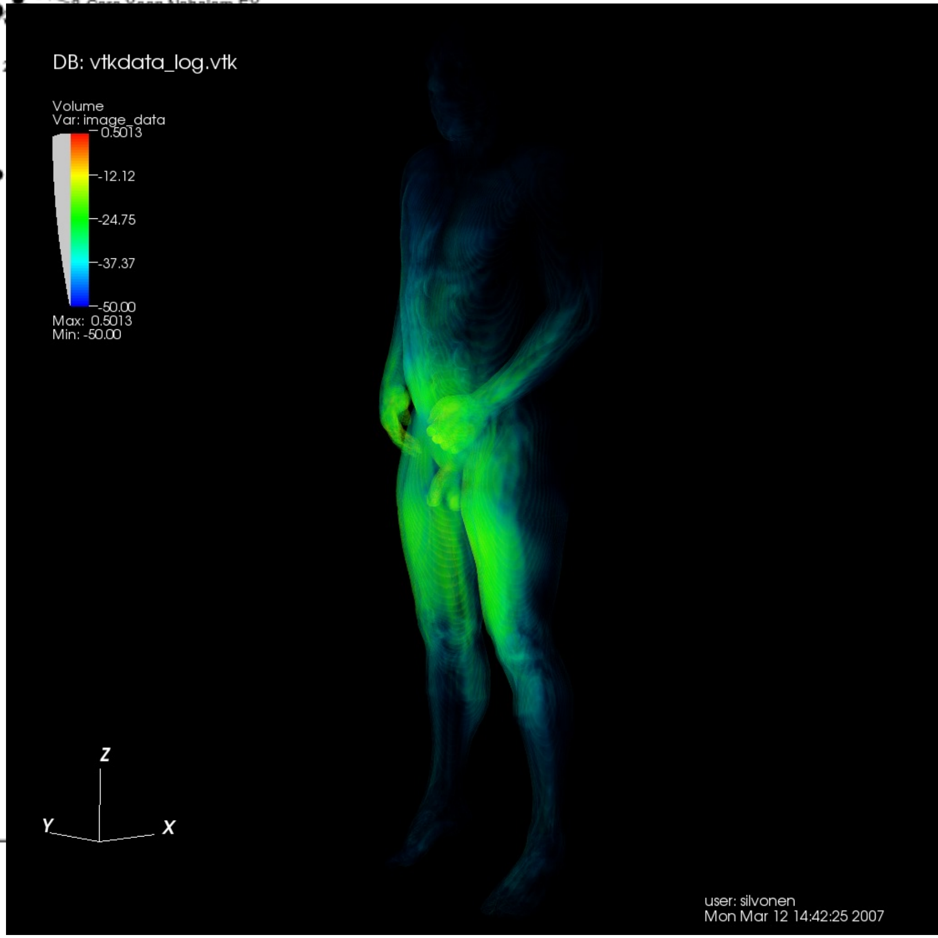


~ FDTD number of cells

Absorbing boundary conditions in 3D

800 cells

Yee 1966



Advantages of FDTD (2)

- Any kind of time-domain sources
- Geometry and boundary conditions are taken into account automatically. Any shape can be modeled easily
- Different media can be modelled naturally: non-linearity, inhomogeneity, anisotropy, complex geometry (metamaterials)

- Examples 6a, 6b

Weaknesses of FDTD (1)

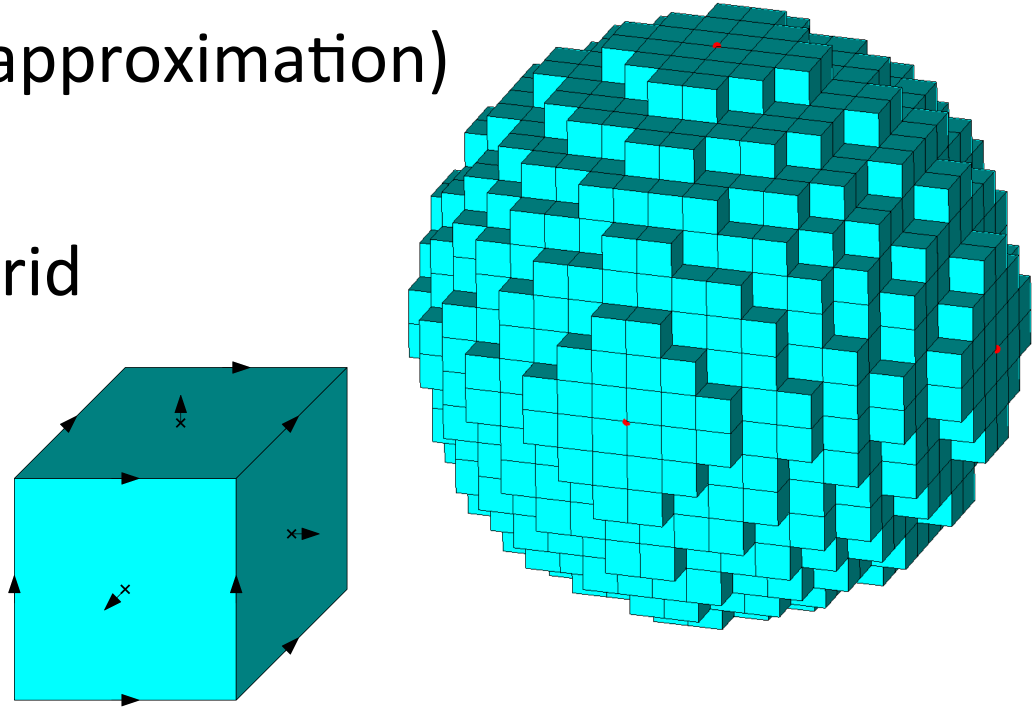
- Not good for slow phenomena (huge number of time steps needed)

  Example: 1 mm grid resolution -> time step = 1.9e-12 s

  Phenomenon lasting 1 ms -> Number of time steps = 5e8

- Curved shapes are problematic (staircase approximation)

- High permittivity medium requires a fine grid
  - One-tenth of wavelength rule

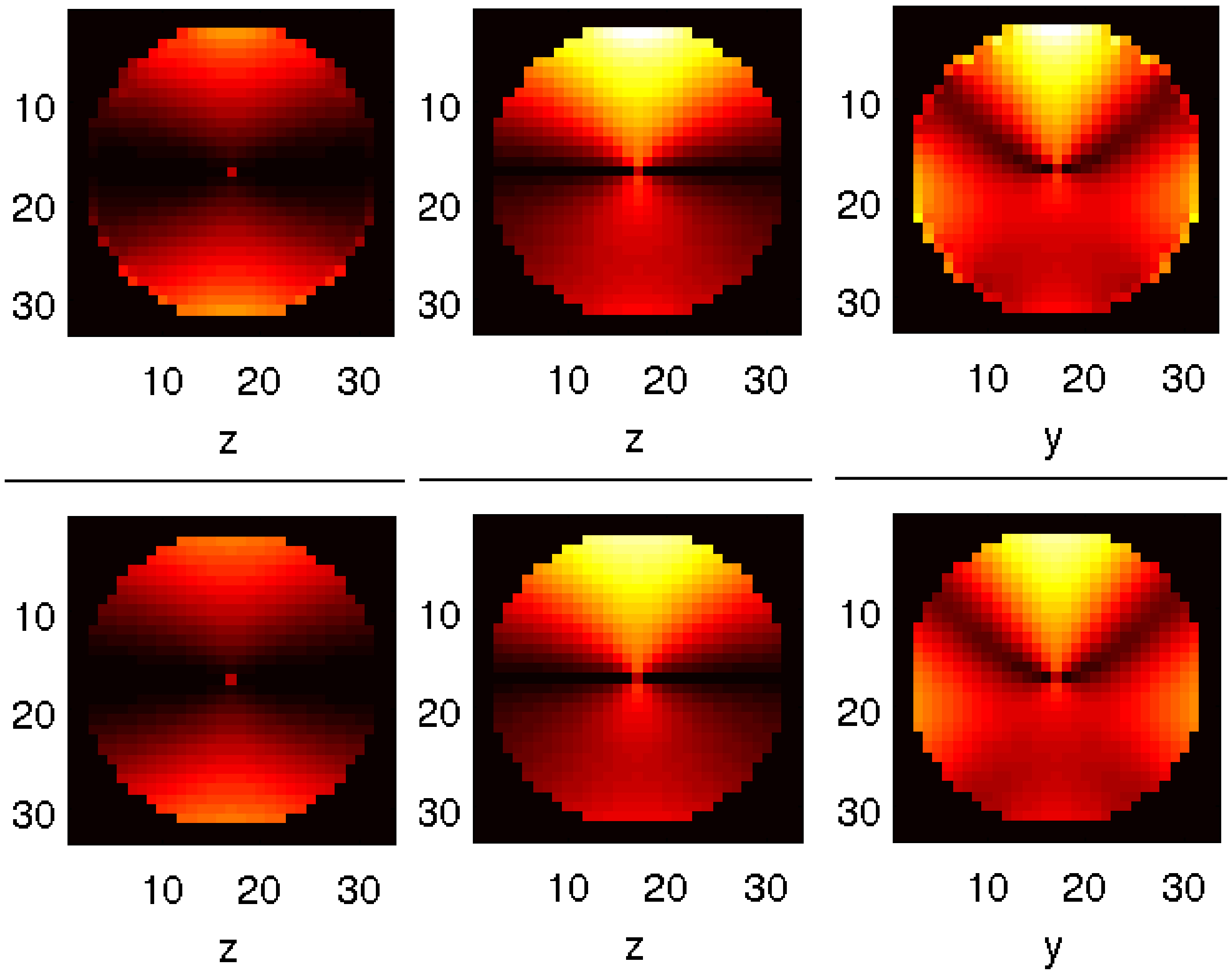


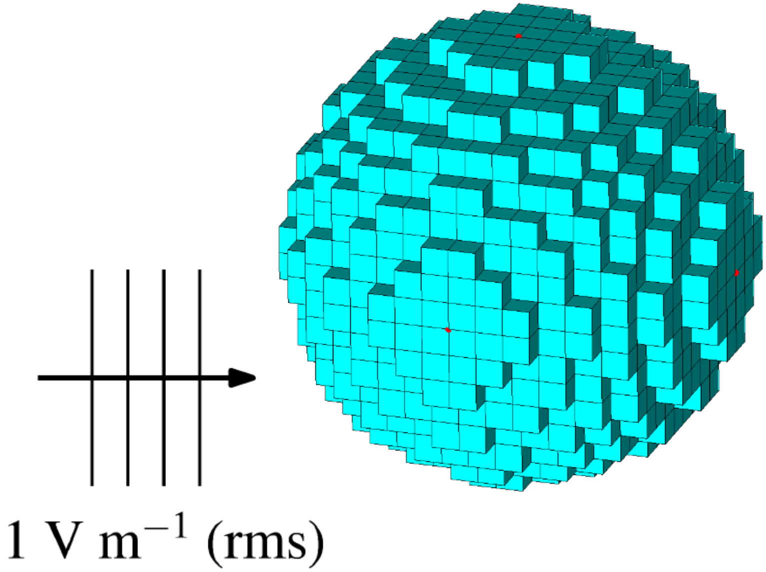
Comparison with analytic solution

Radial component of the electric field
in a sphere

FDTD

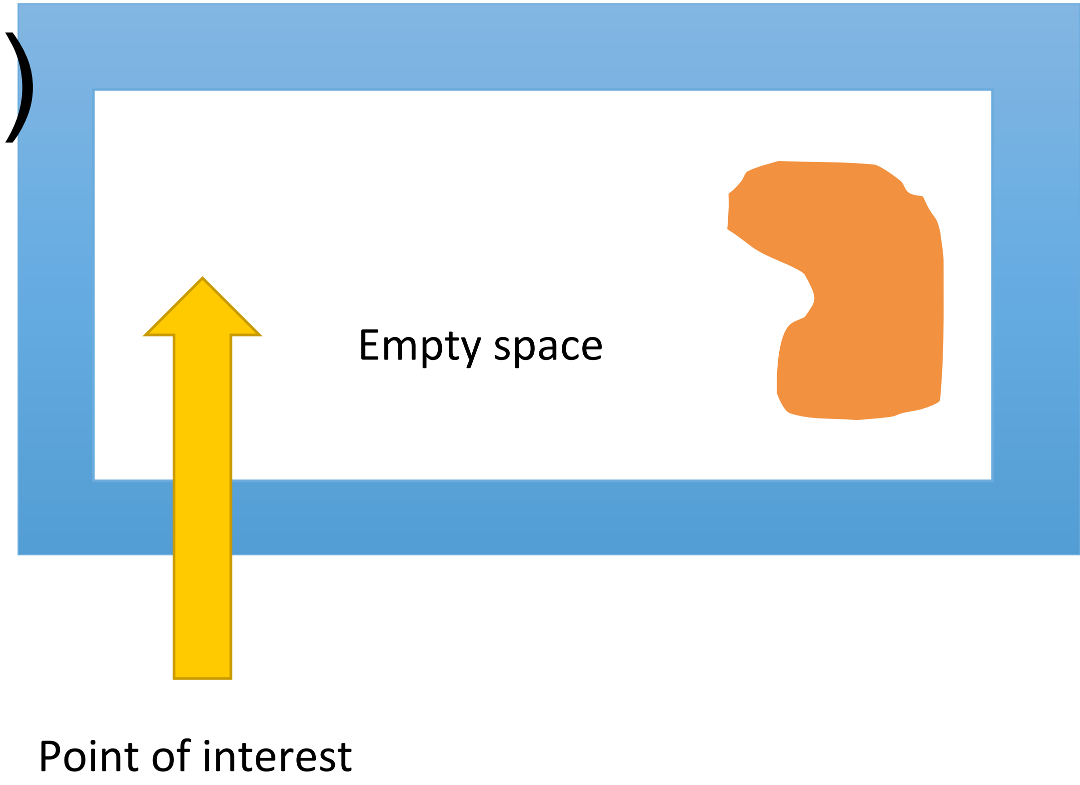
Analytic solution

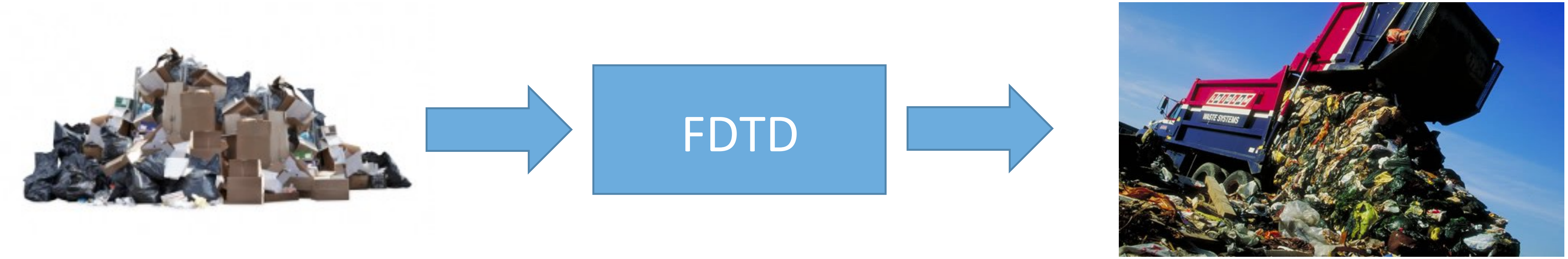




Weaknesses of FDTD (2)

- Computation domain must be finite
  - Absorbing boundary conditions
- Entire computational domain needs to be gridded (also empty space)
- Results depend on the choice of coordinate axes
- Error control





Error in FDTD

- Truncation error from difference approximation (time and space)
- Dispersion error, numerical anisotropy
- Unphysical Poynting theorem, conservation of energy
- ~~Floating point (round-off) error~~
- Staircase approximation error
- Absorbing boundary condition error
- Modelling dielectric/lossy materials

etc.

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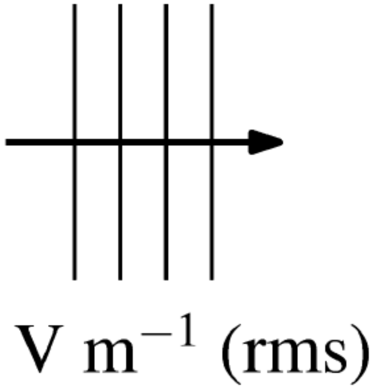
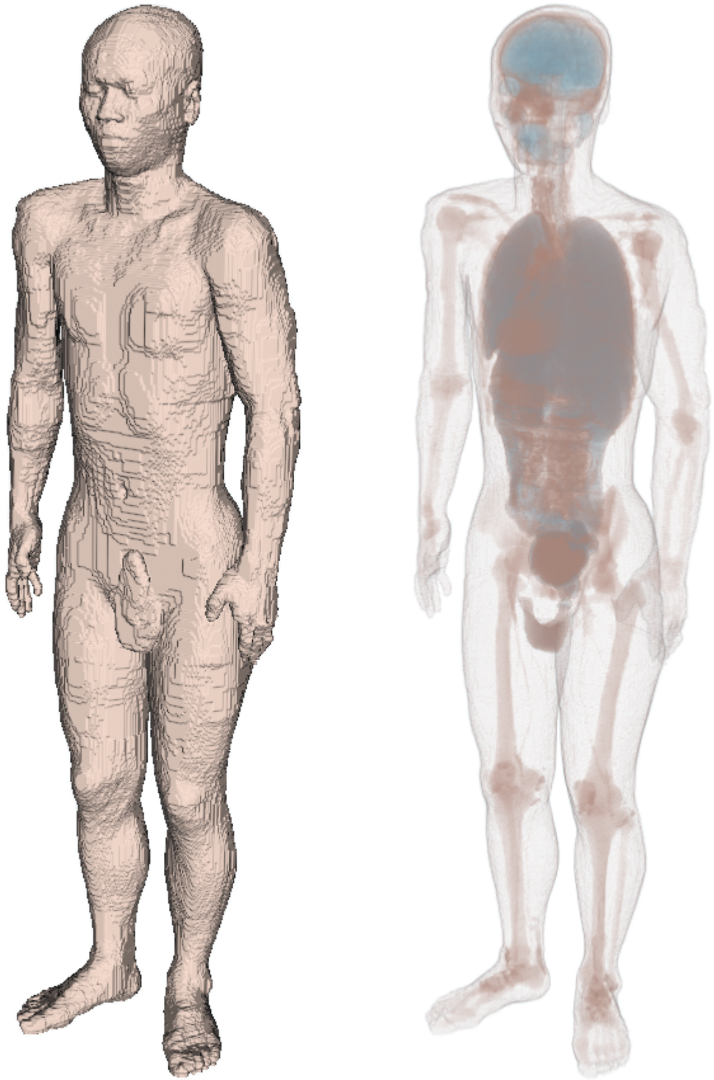
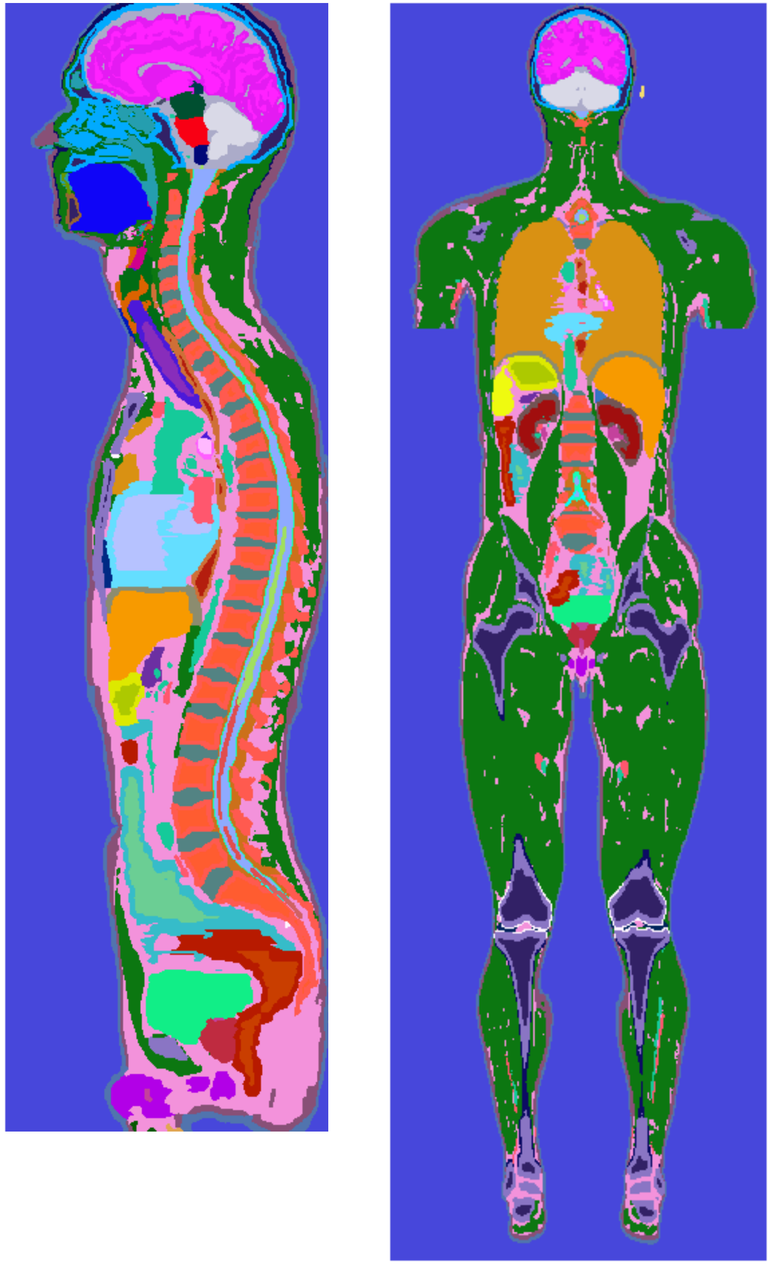
Applications of FDTD

- Radar cross section and scattering
- Metamaterials
- Antenna analysis – example 7a
- Electronic component design
- Electromagnetic compatibility (EMC)
- Waveguides, resonators, filters – example 7b
- Human exposure to EM waves

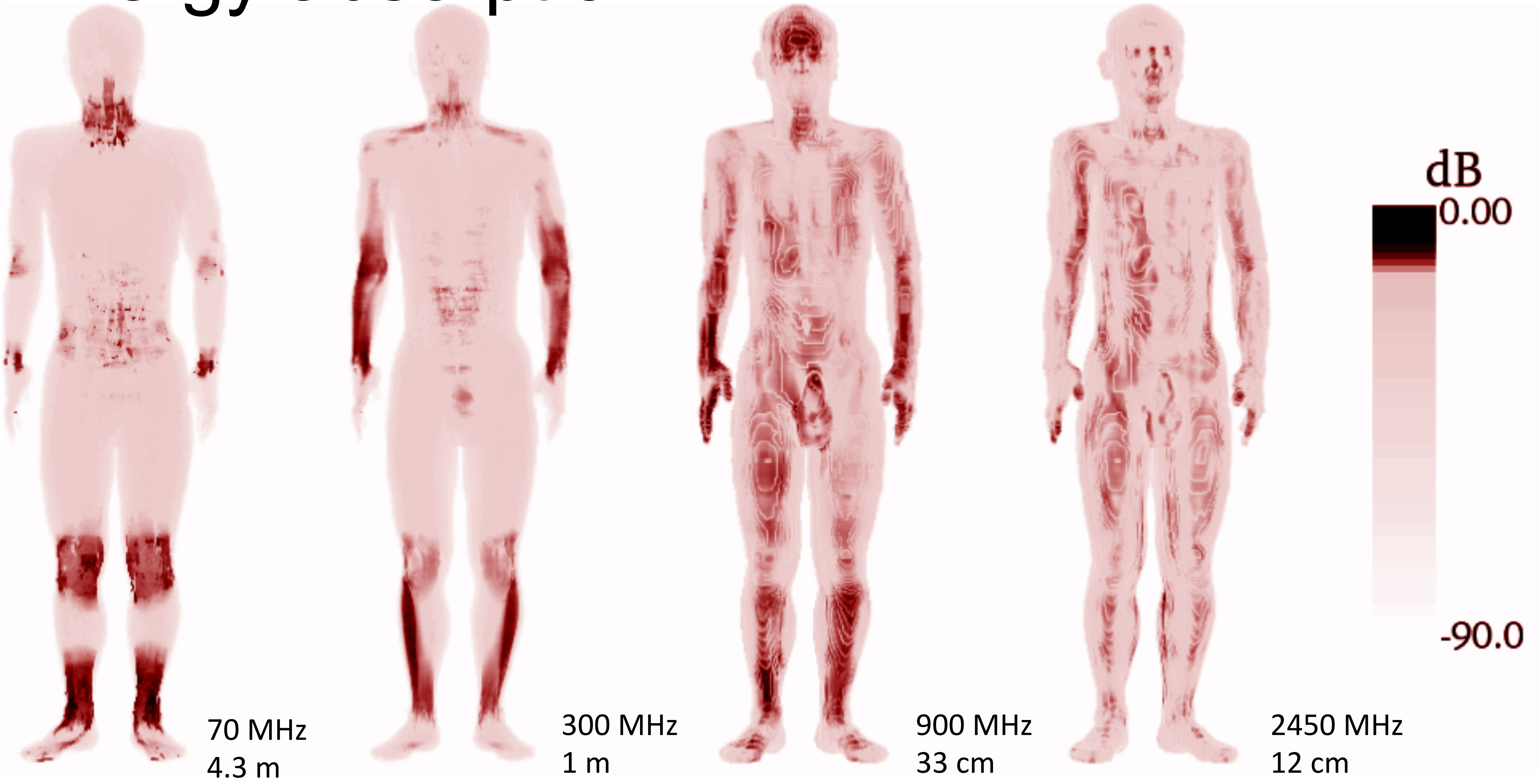
Human exposure to EM wav





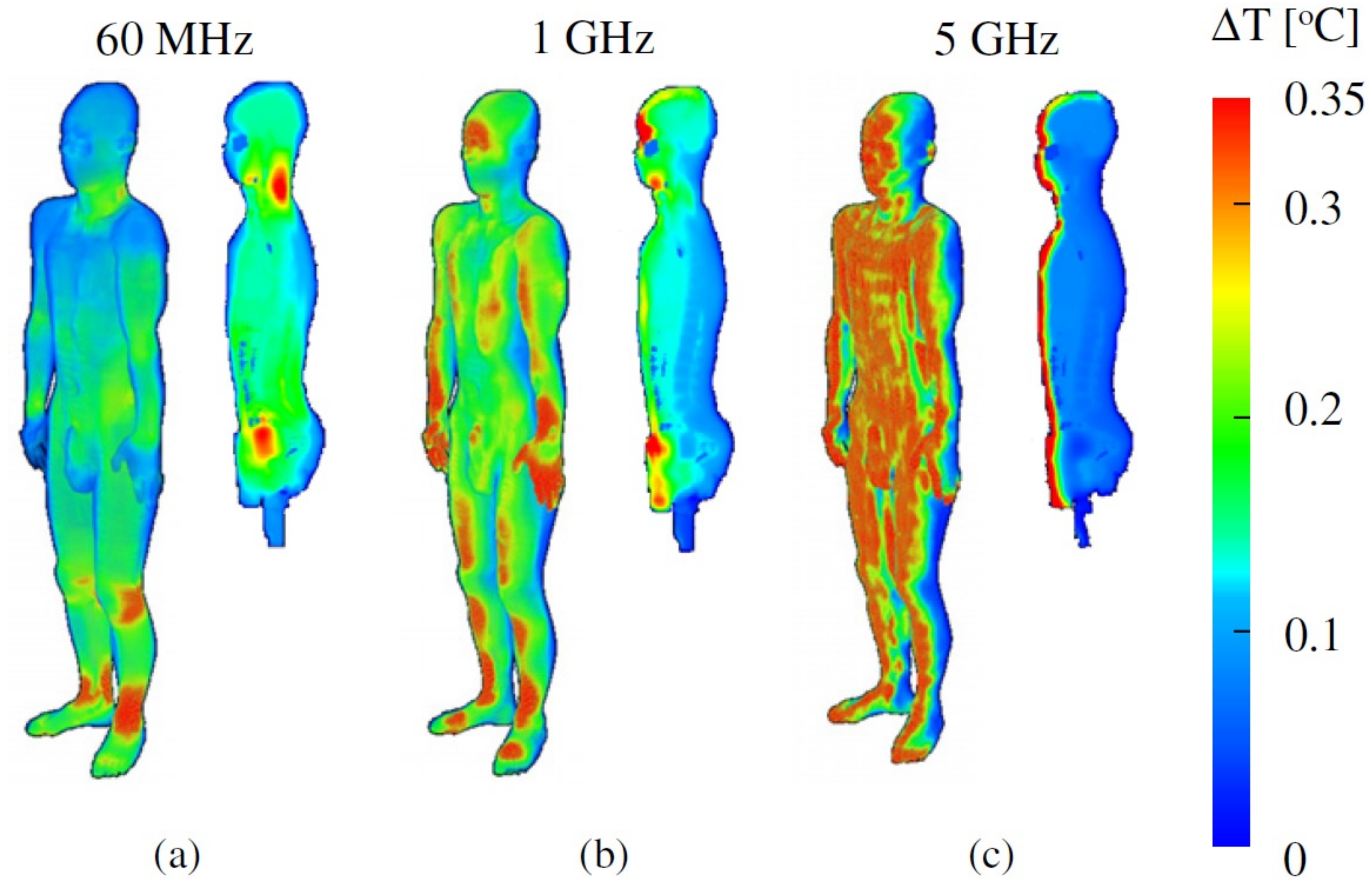


Energy absorption



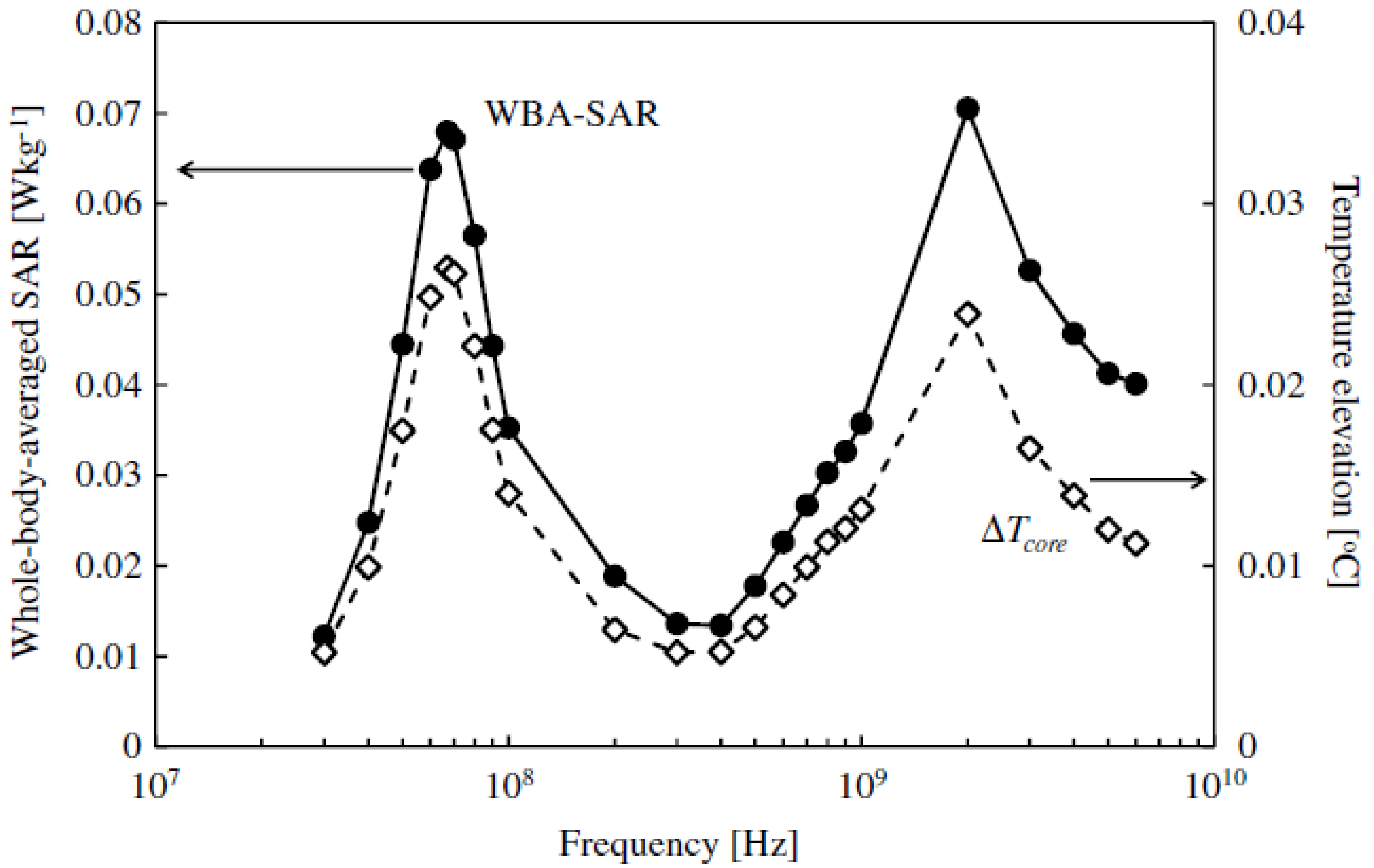
Temperature rise

Absorbed power = 0.4 W/kg



Hirata, Laakso, et al 2013, Phys Med Biol

Power absorption versus temperature rise



Hirata, Laakso, et al 2013, Phys Med Biol

When to use FDTD?

- Use FDTD first
- Use FDTD for making animations
- Use FDTD for large heterogeneous geometries
- Use FDTD to model many things simultaneously

- Don’t use FDTD at low frequencies
- Don’t use FDTD with too large cell size
- Don’t use FDTD if you need 99.9% accuracy