Propagation of ultroshort electromagnetic pulse in nonlinear metamaterials

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***Abstract*— The metamaterial, such as the negative-index material, has both dispersive permittivity and dispersive permeability. By incorporating the dispersive permeability into the nonlinear polarization, one can deal with the problem of nonlinear pulse propagation in metamaterial by using the same way as in ordinary material, or even direct borrowing the existing models for pulse propagation in ordinary materials. By doing so, we obtain a general three-dimensional nonlinear Schrödinger equation suitable for few-cycle pulse propagation in metamaterial. This equation clearly demonstrates the role of dispersive permeability in pulse propagation: In the linear propagation aspect, its contribution is buried in the ordinary dispersive terms; while in the nonlinear propagation aspect, the dispersive permeability manifests itself as a nonlinear polarization dispersion, although it is a linear parameter. For Drude dispersive model, the dispersive permeability results in a negative self-steepening effect in the negative index region and negligible higher-order nonlinear dispersion terms. The titled direction of pulse due to the self-steepening effect in negative-index material is opposite to that in ordinary materials.**

I. Introduction

The nonlinear interaction of ultrashort pulse with ordinary material is well understood in the nonlinear optics [1], [2]. Very accurate yet solvable propagation equations governing the evolution of ultrashort pulses down to only few optical cycles have been established [3]. Very recently, the nonlinear interaction of electromagnetic field with metamaterial, especially the negative-index material (NIM)[4], including the nonlinear ultrashort pulse propagation in NIM, has aroused intense attention [5], [6], [7], [8], [9], [10], [11], [12], [13]. Authors have demonstrated that the nonlinear NIM exhibits a rich spatiotemporal dynamics where both linear and nonlinear effective properties can be tailored by simply engineering the NIM [8], [9], [10], [11]. Further understanding the nonlinear interaction of ultrashort pulse with NIM will lead to new devices with previously inconceivable properties.

In addition to the opposite signs of refractive index, the most important difference between an ordinary medium and a NIM is that the former has a constant permeability, while the latter has a dispersive permeability. Therefore, we are convinced that the most important difference between the propagations of ultrashort pulse in these two kinds of materials should be resulted mainly from the dispersive permeability. To disclose the new features of ultrashort pulse propagation resulted from the dispersive permeability, it is apparent that a re-examination of the propagation of ultrashort pulse is needed. In this Letter, we combine the standard nonlinear propagation theory for ultrashort pulse in conventional nonlinear optics with NIMs to study the propagation of ultrashort pulse in NIM. By incorporating the dispersive permeability into the nonlinear polarization, we immediately obtain a general three-dimensional wave equation first order in the propagation coordinate following the classical procedure when deducing nonlinear Schröinger (NLS) wave equations in ordinary dielectrics [3]. This not only simplifies the complex procedure to derive the propagation equations in NIMs [11], but also makes the unique features of the pulse propagation in NIM more evident by a comparative way.

II. Derivation of the physical model for ultrashort pulse propagation in metamaterials

We assume that the pulse is propagating in uniform, bulk material, in which there are no free charges and in which no free currents flow, under the condition of a nonlinear polarization. We start with Maxwell equations:

$$\begin{aligned}
&\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \\
&\nabla \times \mathbf{H} = \partial_t \mathbf{D}_L + \partial_t \mathbf{P}_{NL}, \\
&\nabla \cdot \mathbf{D} = 0, \\
&\nabla \cdot \mathbf{B} = 0,
\end{aligned} \tag{1}$$

where $\mathbf{E}$ and $\mathbf{H}$ are electric and magnetic fields, respectively, and $\mathbf{D}$ and $\mathbf{B}$ are electric and magnetic flux densities which arise in response to the electric and magnetic fields inside the medium and are related to them through the constitutive relations given by $\mathbf{D} = \mathbf{D}_L + \mathbf{P}_{NL}, \mathbf{D}_L = \varepsilon \mathbf{E}, \mathbf{B} = \mu \mathbf{H}$, with ε and μ being the medium permittivity and permeability respectively, $\mathbf{P}_{NL}$ the nonlinear parolization, ∂_j is shorthand notation for $\partial/\partial j$. It is convenient to work in the frequency domain to deal with the problem of pulse propagation in the dispersive medium.

Applying the curl $\nabla\times$ to the first equation and using the second equation of the set (1), we can get the propagation of pulses in nonlinear media in the form

$$\left(\partial_z^2 + \nabla_\perp^2\right) \mathbf{E} - \nabla \left(\nabla \cdot \mathbf{E}\right) - \mu\varepsilon \partial_t^2 \mathbf{E} = \mu \partial_t^2 \mathbf{P}_{NL}, \tag{2}$$

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where $\nabla_\perp^2$ is the transverse Laplace operator. Substituting the Fourier-transformed field

$$\tilde{\mathbf{E}}(r,\omega)=\int_{-\infty}^{\infty}\mathbf{E}(r,t)\exp(i\omega t)\,dt \tag{3}$$

into Eq. (2), we obtain the following wave equation in frequency domain

$$\left(\partial_z^2+\nabla_\perp^2\right)\tilde{\mathbf{E}}-\nabla\left(\nabla\cdot\tilde{\mathbf{E}}\right)+k^2(\omega)\tilde{\mathbf{E}}=-\mu_0\omega^2\mu_r(\omega)\tilde{\mathbf{P}}_{NL}, \tag{4}$$

The tilde variables stand for the Fourier transform of the corresponding untilde variables. In Eq. (4), $k(\omega)=n\omega/c$ is wavenumber, $n=\pm\sqrt{\varepsilon_r\mu_r}$ is the refractive index of the material, with the signs $\pm$ for the positive-index material and NIM respectively, and $c=1/\sqrt{\varepsilon_0\mu_0}$ is the light velocity in vacuum, ε_r and μ_r are the relative permittivity and relative permeability of medium, respectively.

Equation (4) is general, to some extent, for pulse propagation in electromagnetic material with a nonlinear polarization; it includes the effects of both the permittivity dispersion and permeability dispersion. If we incorporate the dispersive permeability $\mu_r(\omega)$ in the nonlinear polarization $\tilde{\mathbf{P}}_{NL}$, however, Eq. (4) takes its usual form in ordinary nonlinear media with $\mu_r=1$. This makes us possible to obtain the propagation equations for ultrashort pulses in a simpler way. Here we derive a nonlinear evolution equation for a few-cycle pulse in NIM closely following the procedure of derivation in ordinary dielectrics, such as that of Brabec and Krausz [3].

We further assume that the electric field $\mathbf{E}$ propagates along the z direction, and both $\mathbf{E}$ and the nonlinear polarization $\mathbf{P}_{NL}$ are polarized parallel to the x axis. In addition, we assume the transverse inhomogeneities of the medium polarization to be small. Under these conditions, Eq. (4) is reduced to the form

$$\left(\partial_z^2+\nabla_\perp^2\right)\tilde{E}+k^2(\omega)\tilde{E}=-\mu_0\omega^2\tilde{P}'_{NL}, \tag{5}$$

where $\tilde{P}'_{NL}=\mu_r(\omega)\tilde{P}_{NL}$.

We introduce an envelope and carrier form for the field in the usual way to obtain a first-order propagation equation for the wave packet envelope,

$$E=A(\mathbf{r}_\perp,z,t)\exp\left[i(\beta_0 z-\omega_0 t)\right]+c.c, \tag{6}$$

and similarly,

$$P_{NL}=B(\mathbf{r}_\perp,z,t)\exp\left[i(\beta_0 z-\omega_0 t)\right]+c.c, \tag{7}$$

where $c.c.$ denotes complex conjugate. Using the same symbols as in Ref.[3] except for B is replaced by B' which is the inverse Fourier transform of $\mu_r(\omega)\tilde{B}$, the propagation equation for the envelope $A(\mathbf{r}_\perp,z,t)$, in the moving reference frame $\tau=t-\beta_1 z$, $\xi=z$, is

$$\begin{aligned}&\Theta\left[\left(\partial_\xi+\frac{\alpha_0}{2}-i\hat{D}\right)A-\frac{i\mu_0\omega_0^2}{2\beta_0}\Theta B'\right]-\frac{i}{2\beta_0}\nabla_\perp^2 A\\&=\left(\frac{\beta_0-\omega_0\beta_1}{\beta_0}\right)\frac{i}{\omega_0}\partial_\tau\left(\partial_\xi+\frac{\alpha_0}{2}-i\hat{D}\right)A\\&+\frac{i}{2\beta_0}\left(\partial_\xi^2+\hat{D}^2-\frac{\alpha_0^2}{4}+i\alpha_0\hat{D}\right)A.\end{aligned} \tag{8}$$

where $\Theta=1+i\partial_\tau/\omega_0$, and the dispersive operator is

$$\hat{D}=-\frac{\alpha_1}{2}\partial_\tau+\sum_{m=2}^{\infty}\frac{i^m(\beta_m+i\alpha_m/2)}{m!}\partial_\tau^m, \tag{9}$$

and $\beta_m=\mathrm{Re}[\partial_\omega^m k|_{\omega=\omega_0}]$, $\alpha_m=\mathrm{Im}[\partial_\omega^m k|_{\omega=\omega_0}]$.

Under the approximations:

$$|\partial_\xi A|\ll|\beta_0||A|\,,|\partial_\tau A|\ll\omega_0|A|\,, \tag{10}$$

Eq. (8) is reduced to the following form

$$\partial_\xi A=-\frac{\alpha_0}{2}A+i\hat{D}A+\frac{i}{2\beta_0\Theta}\nabla_\perp^2 A+\frac{i\mu_0\omega_0^2}{2\beta_0}\Theta B'. \tag{11}$$

Equation (11) is formally identical to Eq. (6) of Ref.[3] except for the slight difference in nonlinear term. For $\mu_r=1$, the two equations are completely identical. Thus the role of dispersive permeability in pulse propagation is evident: In the linear propagation aspect, the contribution of permeability dispersion is included in the wavenumber $k(\omega)$, and is therefore buried in the dispersive operator $\hat{D}$, with the dispersive terms in Eqs.(8) and (11) being totally identical to those for pulse propagation in ordinary dispersive media. In the nonlinear propagation aspect, the dispersive permeability acts as a dispersive factor of nonlinear polarization, resulting in an effective dispersive nonlinear polarization.

To further disclose the role of dispersive permeability in pulse propagation we expand $\mu_r(\omega)$ as we do $k(\omega)$,

$$\mu_r=\sum_{m=0}^{\infty}\frac{G_m}{m!}(\omega-\omega_0)^m, \tag{12}$$

where $G_m=\nu_m+i\sigma_m,\nu_m=\mathrm{Re}[\partial_\omega^m\mu_r|_{\omega=\omega_0}],\sigma_m=\mathrm{Im}[\partial_\omega^m\mu_r|_{\omega=\omega_0}]$. If the nonlinear polarization is not dispersive, B' can be expressed as

$$B'=\hat{M}B, \tag{13}$$

where

$$\hat{M}=\sum_{m=0}^{\infty}\frac{i^m G_m}{m!}\partial_\tau^m. \tag{14}$$

Substituting Eq. (13) into Eq. (11), we can finally obtain the NLS equation for few-cycle pulse propagation in metamaterials with both dispersive permittivity and dispersive permeability. For a lossless medium with a instantaneous Kerr nonlinearity, $B=\varepsilon_0\chi^{(3)}|A|^2A$, we have

$$\partial_\xi A=i\hat{D}A+\frac{i}{2\beta_0\Theta}\nabla_\perp^2 A+i\gamma\left(1+i\hat{S}\right)\left(|A|^2A\right). \tag{15}$$

where $\gamma=\nu_0\chi^3\omega_0^2/\left(2\beta_0c^2\right)$ is nonlinear coefficient, and

$$\hat{S}=\sum_{m=1}^{\infty}i^{m-1}s_m\partial_\tau^m \tag{16}$$

is the nonlinear dispersion operator, with the mth-order nonlinear dispersion being

$$s_m=\frac{\nu_m}{m!\nu_0}+\frac{\nu_{m-1}}{(m-1)!\nu_0\omega_0}. \tag{17}$$

The first order nonlinear dispersion ($m = 1$) is also called self-steepening (SS) effect, which can be written as

$$s_1 = \frac{1}{\omega_0} + \frac{\nu_1}{\nu_0}. \tag{18}$$

It differs from the SS parameter in conventional propagation equations for ordinary media by the presence of the term ν_1/ν_0. This additional term comes from the dispersive permeability. Further, all the higher-order ($m \geq 2$) nonlinear dispersion terms don't appear in conventional propagation equations. They are totally resulted from the dispersive permeability.

Equation (15) in its one-dimensional form differs from the recently obtained propagation equation for NIM, i.e., Eq.(12) of Scalora et al.[11] The difference comes from the different approximations we made. It is easy to show that Eq.(12) of Scalora et al. can be recovered from our Eq. (8). For example, keeping the linear dispersion terms to second order and nonlinear dispersion terms to first order in Eq. (8), and making the same approximations as Scalora et al., i.e., using the non-SVEA corrections, $\partial_\xi^2 A = i\gamma\partial_\xi(|A|^2 A)$ and $\partial_{\tau\xi}^2 A = i\gamma\partial_\tau(|A|^2 A)$, Eq.(8) reduces to the Eq.(12) of Scalora et al. Different approximations also result in different expressions for SS effect. In our variables, the expression for s_1 of Scalora et al. is $s_1 = 1/\omega_0 + \nu_1/\nu_0 + 1/\omega_0 - \beta_1/\beta_0$. Compared to Eq.(18), the additional $1/\omega_0$ comes from the expansion of $(1 + i\partial_\tau/\omega_0)^2 \hat{M}$, and the term β_1/β_0 results from the correction term $\partial_{\tau\xi}^2 A$. While in deriving Eq. (15), we have split the factor $(1 + i\partial_\tau/\omega_0)^2$ into two parts: the linear SS in the diffraction term, and nonlinear SS in the nonlinear term.

III. Importance of the nonlinear dispersion terms resulted from dispersive permeability for Drude dispersion model

To estimate the relative importance of the nonlinear dispersion terms we apply Eq. (15) to a NIM that is described by the lossless Drude model:

$$\varepsilon_r(\omega) = 1 - \frac{\omega_{pe}^2}{\omega^2}, \mu_r(\omega) = 1 - \frac{\omega_{pm}^2}{\omega^2}, \tag{19}$$

where ω_{pe} and ω_{pm} are the respective electric and magnetic plasma frequencies, and introduce the mth-order nonlinear dispersion length, $L_{sm} = \tau_p^m/s_m$, which is in units of the nonlinear length $L_{nl} = 1/(\gamma A_0^2)$, where A_0 is the amplitude of pulse, and τ_p is the pulse duration which is assumed to contain f optical cycles. Under these circumstances, the SS, second-order and third-order nonlinear dispersion lengths are

$$\begin{aligned} L_{s1} &= \frac{2f\pi(\varpi^2 - \varpi_p^2)}{\varpi^2 + \varpi_p^2}, \\ L_{s2} &= (2f\pi)^2 \left(1 - \frac{\varpi^2}{\varpi_p^2}\right), \\ L_{s3} &= -2f\pi L_{s2}, \end{aligned} \tag{20}$$

respectively, where $\varpi = \omega_0/\omega_{pe}, \varpi_p = \omega_{pm}/\omega_{pe}$. Clearly, as the pulse duration decreases, the absolute value of L_{sm} decreases, meaning that the nonlinear dispersion terms become more important. In the case of negative refraction, $\varpi < \varpi_p$, thus L_{s1} and L_{s3} are always negative, while L_{s2} is always positive. It should be noted that the negative SS effect never appears in ordinary materials. Fig. 1 shows the variation of L_{s2}/L_{s1} with ϖ for different values of ϖ_p for a pulse duration only containing three optical cycles ($f = 3$). We see that within the frequency range for negative refraction, the second-order nonlinear dispersion length L_{s2} can be considered far longer than the SS length $|L_{s1}|$, especially for pulses with higher center frequencies. This means that the higher-order nonlinear dispersion terms ($m > 1$) can be discarded even in the few-cycle pulse propagation regime. Therefore, the main role of the dispersive permeability in nonlinear pulse propagation is that it leads to a negative SS effect.

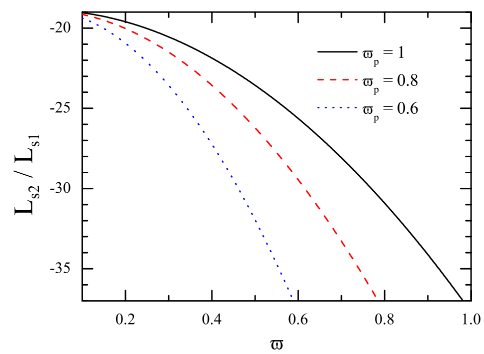

Fig. 1. Variation of the ratio L_{s2}/L_{s1} with the normalized center frequency ϖ for $\varpi_p = 1$, 0.8 and 0.6 respectively, for a pulse duration containing three optical cycles ($f = 3$).

IV. Influence of the negative self-steepening on ultrashort pulse propagation in NIM

The influence of the negative SS on the propagation of ultrashort pulse can be exclusively demonstrated by discarding the diffraction, linear dispersion and all higher-order ($m > 1$) nonlinear dispersion terms. Under these conditions, we can obtain an exact analytical solution for Eq.(15) [14]. For an initially sech-shaped pulse, $A(\tau, 0) = \operatorname{sech}(\tau/\tau_p)$, the general solution for $\xi > 0$ is

$$A(\tau, \xi) = \operatorname{sech}\left(\frac{\tau - 3\gamma s_1 \xi A^2}{\tau_p}\right). \tag{21}$$

This shows that the low-intensity part of the pulse is essentially unaffected (and still sech-shaped), but the high-intensity part of the pulse is towards smaller τ for negative s_1, opposite to the titled direction for positive s_1 in ordinary materials. In Fig. 2, we plot the pulse shapes at different distances. It clearly shows the self-steepening process of the leading part of the pulse.

Recalling the moving reference frame transforms, we can obtain the propagating velocity of the peak of the pulse from Eq. (21)

$$v_{peak} = \frac{v_g}{1 + 3v_g s_1/L_{nl}}, \tag{22}$$

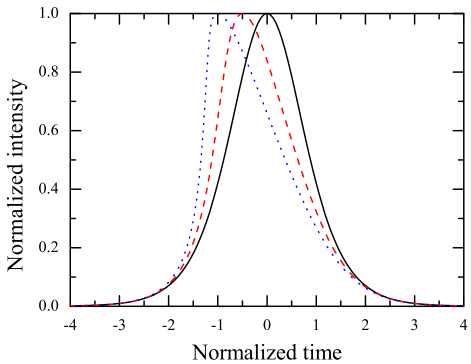

Fig. 2. Pulse shapes at $\xi = 0$ (solid line), $1/(6\gamma s_1)$ (dash line), and $1/(3\gamma s_1)$ (dot line).

which is faster than the group velocity of pulse for negative s_1. This causes a self-steepening of the leading part of the pulse which ultimately leads to shock formation at the distance $\xi_{stp} = -\sqrt{3}/(4\gamma s_1)$. Because the SS parameter can have a much larger value in NIM than in conventional dielectric, the distance for shock formation in NIM can be very short.

In the presence of linear dispersion, the nonlinear SS is balanced to some extent. As the pulse steepens, the increase spectral width of the pulse makes dispersion more important and finally the dispersive velocity spread, which tends to dissipate the shock, balances the nonlinear velocity change, which steepens the pulse. In addition, the presence of dispersion significantly shortens the self-steepening distance [15].

V. Conclusion

In conclusion, we have present a simpler way to obtain the dynamical models for ultrashort pulse propagation in nonlinear metamaterials, especially the negative-index metamaterials. By incorporating the dispersive permeability into the nonlinear polarization, we obtained a general three-dimensional wave equation first order in the propagation coordinate. This equation clearly demonstrates the role of dispersive permeability in ultrashort pulse propagation: its contribution to linear propagation is buried in the ordinary dispersive terms; while in the nonlinear propagation aspect, the dispersive permeability manifests itself as a nonlinear polarization dispersion, resulting in a negative SS effect and a series of negligible higher-order nonlinear dispersion terms for Drude dispersive model. Finally, we investigate the influence of negative SS on pulse propagation. The titled direction of pulse due to SS effect in NIM is opposite to that in ordinary materials.

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