Modelling of the Magnetization Process Based on the Landau - Lifshitz - Gilbert Equation

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***Abstract*—The magnetization dynamics is described by a mathematical model that represents a time dependent nonlinear differential equation, known as the Landau - Lifshitz - Gilbert equation (LLG). This paper investigates several numerical methods for solving different LLG equation types. The results are compared with the analytical solution as well as with some literature versions.**

I. Introduction

The micromagnetism is still an active research field since many applications such as memory devices, sensors are better understood by a better knowledge of this field. An important role in the study of micromagnetism is played by the Landau Lifshitz Gilbert (LLG) equation [1]. LLG has several forms. We will consider the demonstrable dimensionless form [1]:

$$\frac{\partial \vec{m}}{\partial t} = \vec{m} \times \Delta \vec{m} - \alpha \vec{m} \times (\vec{m} \times \Delta \vec{m}) \tag{1}$$

whitch is used to solve time-dependent micromagnetic problems, where $\vec{m} : \Omega \times [0, T] \to \mathbb{R}^3$, $\Omega \subset \mathbb{R}^3$, $\vec{m} = \vec{M}/M_s$, $\vec{M}$ is the magnetization, M_s is the saturation magnetization and α is the damping constant (Gilbert parameter). The first term on the right hand side describes the precession movement, whereas the second term is the damping term. The uniqueness conditions imposed are: an initial condition $\vec{m}(\vec{r},\ 0) = \vec{m}_0$ and boundary conditions on $\partial\Omega$. Neumann boundary conditions are used in [2].

Several numerical methods have been investigated for the LLG equation [2],[3]. In [2] the magnetization process of a thin film subjected to an in-plane circularly polarized magnetic field is considered and solved with various techniques.

In [4] the LLG without damping is used to study a switching anomaly problem.

This paper investigates the use of forward Euler (FE), Runge - Kutta (RK4), Runge - Kutta - Fehlberg (RKF) methods for solving the LLG equation without and with damping.

II. Problem Formulation

The LLG equation, describing the time evolution of magnetization in a ferromagnetic material, can be written in a normalized form [1]:

$$\frac{\partial \vec{m}}{\partial t} = \gamma M_s \vec{m} \times \vec{h}_{eff} + \gamma M_s \alpha \vec{m} \times \left(\vec{m} \times \vec{h}_{eff}\right) \tag{2}$$

in $\Omega \times (0, T)$, where

$$\vec{h}_{eff}(\vec{m},\ t) = -\frac{\partial \vec{w}}{\partial \vec{m}} = \vec{h}_{ex} + \vec{h}_m + \vec{h}_{an} + \vec{h}_a(t), \tag{3}$$

is the total field,$\vec{w}$ is the total free energy in the ferromagnet, α is the damping constant, γ is the gyromagnetic ratio and M_s is the saturation magnetization. The first three terms in (3) can be related to the vector field $\vec{m}$ through the following equations

$$\vec{h}_{ex} = \frac{2A}{\mu_0 M_s^2} \nabla^2 \vec{m}; \tag{4}$$

$$\vec{h}_m = -\frac{1}{4\pi} \nabla \int_\Omega \nabla' \left(\frac{1}{|\vec{r} - \vec{r}'|}\right) \cdot \vec{m}(\vec{r}')\, \mathrm{d}V_{\vec{r}'} \tag{5}$$

$$\vec{h}_{an} = \frac{2C_1}{\mu_0 M_s^2} \vec{e}_{an}(\vec{r}) \left(\vec{e}_{an}(\vec{r}) \cdot \vec{m}(\vec{r})\right) \tag{6}$$

Where $\vec{h}_{ex}$ is the exchange field, $\vec{h}_m$ is the magnetostatic field, $\vec{h}_{an}$ is the anisotropy field, A is the exchange constant, C_1 is the anisotropy constant, $\vec{h}_a(t)$ is the applied field (external field) which is a known parameter.

If in (3) we shall keep just the first term, neglecting the constant from the first term, then (2) becomes (1). This equation describes the evolution of spin fields in non-equilibrum continuum ferromagnets.

III. Numerical Approach For One Dimensional LLG Equation

A. *No damping* $(\alpha = 0)$

We have the following expression:

$$\frac{\partial \vec{m}}{\partial t} = \vec{m} \times \frac{\partial^2 \vec{m}}{\partial x^2} \tag{7}$$

which represents the LLG equation without damping. $\vec{m} : \Omega' \times (0,\ T) \to \mathbb{R}^3$, $\Omega' = [a,\ b] = [-1,\ 1]$, $T = 1$, $\Delta \vec{m} = \partial^2 \vec{m}/\partial x^2$.

It can be easily verified that

$$\begin{aligned} m_x &= \sin\beta \cos\left(k \cdot x - \left(|k|^2 \cos\beta\right) t\right), \\ m_y &= \sin\beta \sin\left(k \cdot x - \left(|k|^2 \cos\beta\right) t\right), \\ m_z &= \cos\beta. \end{aligned} \tag{8}$$

is an exact solution for (7).This solution exists only if $\Delta \vec{m} = \partial^2 \vec{m}/\partial x^2$.

The discretization of (7) implies a time and space discretization. In what fallows we assume that both discretizations are uniform, with a Δt time step and a Δx space step. The space domain is the point $x_i = a + (i-1)\Delta x$, $i =$

$\overline{1,\ N-1},\ a+N\Delta x=b$ and the time domain is discretized in $t_j=(j-1)\,\Delta t,\ j=\overline{1,\ N+1},\ M\Delta t=T$. To simplify the relations, we will use the notation:

$$\vec{m}\,(x_i,t_j)=\vec{m}_i^j=m_{x,i}^j\hat{x}+m_{y,i}^j\hat{y}+m_{z,i}^j\hat{z}. \tag{9}$$

When using a second order formula for the derivative in the right hand section of (7) we have:

$$\frac{\partial^2\vec{m}}{\partial x^2}=\frac{\vec{m}_{i+1}^j-2\vec{m}_i^j+\vec{m}_{i-1}^j}{(\Delta x)^2}=\frac{h_{x,i}^j\hat{x}+h_{y,i}^j\hat{y}+h_{z,i}^j\hat{z}}{(\Delta x)^2}, \tag{10}$$

where

$$\begin{aligned} h_{x,i}^j&=m_{x,i+1}^j-2m_{x,i}^j+m_{x,i-1}^j,\\ h_{y,i}^j&=m_{y,i+1}^j-2m_{y,i}^j+m_{y,i-1}^j,\\ h_{z,i}^j&=m_{z,i+1}^j-2m_{z,i}^j+m_{z,i-1}^j. \end{aligned} \tag{11}$$

In what follows we shall describe three step methods for the numerical solving of LLG equation.

1) The Forward Euler Scheme: A forward Euler scheme is obtained by using a first order forward discretization formula for $\partial\vec{m}/\partial t$ in (7).

$$\frac{\partial\vec{m}}{\partial t}\,(x_i,t_j)=\frac{\vec{m}_i^{j+1}-\vec{m}_i^j}{\Delta t}. \tag{12}$$

Denoting by $h=\Delta t/\,(\Delta x)^2$, discretized LLG becomes

$$\begin{aligned} m_{x,i}^{j+1}&=m_{x,i}^j+hF_1\left(m_{y,i}^j,m_{z,i}^j,t_j\right),\\ m_{y,i}^{j+1}&=m_{y,i}^j+hF_2\left(m_{z,i}^j,m_{x,i}^j,t_j\right),\\ m_{z,i}^{j+1}&=m_{z,i}^j+hF_3\left(m_{x,i}^j,m_{y,i}^j,t_j\right). \end{aligned} \tag{13}$$

where

$$\begin{aligned} F_1\left(m_{y,i}^j,m_{z,i}^j,t_j\right)&=m_{y,i}^j\cdot h_{z,i}^j-m_{z,i}^j\cdot h_{y,i}^j,\\ F_2\left(m_{z,i}^j,m_{x,i}^j,t_j\right)&=m_{z,i}^j\cdot h_{x,i}^j-m_{x,i}^j\cdot h_{z,i}^j,\\ F_3\left(m_{x,i}^j,m_{y,i}^j,t_j\right)&=m_{x,i}^j\cdot h_{y,i}^j-m_{y,i}^j\cdot h_{x,i}^j. \end{aligned} \tag{14}$$

or

$$\vec{m}_i^{j+1}=\vec{m}_i^j+h\vec{F}\left(\vec{m}_i^j,t_j\right). \tag{15}$$

we have

$$\vec{F}\left(\vec{m}_i^j,t_j\right)=[F_1,F_2,F_3]^T, \tag{16}$$

This relation can be written for $i=\overline{2,N-1}$. Due to the periodicity, for $i=1,\ \vec{m}_0^j=\vec{m}_{N-1}^j$ and for $i=1,\ \vec{m}_{N+1}^j=\vec{m}_2^j$.
The procedure is stable only if $\Delta t\geq(\Delta x)^2$. Recall that Δt and Δx are dimensionless. The proof of this stability condition can be found in [3].

2) The Fourth order Runge -Kutta Method (RK4): Another forward method that can be applied for the numerical solving of (7) equation is the Fourth order Runge-Kutta method (RK4). The numerical integration of (7) using RK4 method proceeds as follows:
Give the aproximation $\vec{m}_j^i$ at time t_j, we have:

$$\begin{aligned} \vec{K}_1&=\vec{F}\left(\vec{m}_i^j,t_j\right)\\ \vec{K}_2&=\vec{F}\left(\vec{m}_i^j+\frac{h}{2}\vec{K}_1,t_j+\frac{h}{2}\right)\\ \vec{K}_3&=\vec{F}\left(\vec{m}_i^j+\frac{h}{2}\vec{K}_2,t_j+\frac{h}{2}\right)\\ \vec{K}_4&=\vec{F}\left(\vec{m}_i^j+\vec{K}_3,t_j+h\right) \end{aligned} \tag{17}$$

$$\vec{K}_n=[K_{n1},K_{n2},K_{n3}]^T,n=1,2,3,4. \tag{18}$$

The magnetic moment at t_{j+1} is denoted by:

$$\begin{aligned} m_{x,i}^{j+1}&=m_{x,i}^j+\frac{h}{6}\left(K_{11}+2K_{21}+2K_{31}+K_{41}\right),\\ m_{y,i}^{j+1}&=m_{y,i}^j+\frac{h}{6}\left(K_{12}+2K_{22}+2K_{32}+K_{42}\right),\\ m_{z,i}^{j+1}&=m_{z,i}^j+\frac{h}{6}\left(K_{13}+2K_{23}+2K_{33}+K_{43}\right), \end{aligned} \tag{19}$$

or

$$\vec{m}_i^{j+1}=\vec{m}_i^j+\frac{h}{6}\left(\vec{K}_1+2\vec{K}_2+2\vec{K}_3+\vec{K}_4\right). \tag{20}$$

3) The Runge-Kutta-Fehlberg Method (RKF): The Runge-Kutta-Fehlberg (RKF) method [5] uses a Runge-Kutta method of order 5 to estimate the local error in a Fourth Runge-Kutta method. Using a predetermined global error, the initial step size h is replaced by a step size δh at a given step if the error using h is not within the required bounds. Each Runge-Kutta-Fehlberg step requires the use of the following six values:

$$\begin{aligned} \vec{K}_1&=h\vec{F}\left(\vec{m}_i^j,t_j\right)\\ \vec{K}_2&=h\vec{F}\left(\vec{m}_i^j+\frac{1}{4}\vec{K}_1,t_j+\frac{1}{4}h\right)\\ \vec{K}_3&=h\vec{F}\left(\vec{m}_i^j+\vec{K}_2',t_j+\frac{3}{8}h\right)\\ \vec{K}_4&=h\vec{F}\left(\vec{m}_i^j+\vec{K}_3',t_j+\frac{12}{13}h\right)\\ \vec{K}_5&=h\vec{F}\left(\vec{m}_i^j+\vec{K}_4',t_j+h\right)\\ \vec{K}_6&=h\vec{F}\left(\vec{m}_i^j+\vec{K}_5',t_j+\frac{1}{2}h\right) \end{aligned} \tag{21}$$

where

$$\begin{aligned} \vec{K}_2'&=\frac{3}{32}\vec{K}_1+\frac{9}{32}\vec{K}_2\\ \vec{K}_3'&=\frac{1932}{2197}\vec{K}_1-\frac{7200}{2197}\vec{K}_2+\frac{7296}{2197}\vec{K}_3\\ \vec{K}_4'&=\frac{439}{216}\vec{K}_1-8\vec{K}_2+\frac{3680}{513}\vec{K}_3-\frac{845}{4104}\vec{K}_4 \end{aligned} \tag{22}$$

$$\vec{K}_n = [K_{n1}, K_{n2}, K_{n3}]^T, n = 1, 2, 3, 4. \tag{23}$$

Then an approximation to the solution of the initial value problem is made using a Runge-Kutta method of order 4:

$$\vec{m}_i^{j+1} = \vec{m}_i^j + \left(\frac{25}{216}\vec{K}_1 + \frac{1408}{2565}\vec{K}_3 + \frac{2197}{4101}\vec{K}_4 - \frac{1}{5}\vec{K}_5 \right). \tag{24}$$

A more accurate value is obtained by applying a Runge-Kutta method of order 5:

$$\begin{aligned} &\vec{m}_i'^{\,j+1} = \vec{m}_i^j + \\ &+ \left(\frac{16}{135}\vec{K}_1 + \frac{6656}{12825}\vec{K}_3 + \frac{28561}{56430}\vec{K}_4 - \frac{9}{50}\vec{K}_5 + \frac{2}{55}\vec{K}_6 \right). \end{aligned} \tag{25}$$

The optimal step size $\delta \Delta t$ can be determined by multiplying the scalar δ times the current step size Δt. The scalar δ is

$$\delta = \left(\frac{\mathrm{tol} \cdot \Delta t}{2 \left\| \vec{m}_i'^{\,j+1} - \vec{m}_j^{j+1} \right\|} \right)^{1/4} \tag{26}$$

B. With damping $(\alpha \neq 0)$

We have the following expression:

$$\frac{\partial \vec{m}}{\partial t} = \vec{m} \times \frac{\partial^2 \vec{m}}{\partial x^2} - \alpha \vec{m} \times \left(\vec{m} \times \frac{\partial^2 \vec{m}}{\partial x^2} \right) \tag{27}$$

which is LLG equation with damping. $\vec{m} : \Omega \times (0,\ T) \rightarrow \mathbb{R}^2,\ \Omega = [a,\ b] = [-1,\ 1], T = 1$.
It can be easily verified that

$$\begin{aligned} m_x &= \frac{\sin\beta \cos\left[k \cdot x - \varphi\left(x, t; \beta, k, \alpha\right)\right]}{d\left(t; \beta, k, \alpha\right)}, \\ m_y &= \frac{\sin\beta \sin\left[k \cdot x - \varphi\left(x, t; \beta, k, \alpha\right)\right]}{d\left(t; \beta, k, \alpha\right)}, \\ m_z &= \frac{\exp\left(k^2 \alpha \cdot t\right) \cos\beta}{d\left(t; \beta, k, \alpha\right)}. \end{aligned} \tag{28}$$

is an exact solution for (27), where

$$\begin{aligned} & k = l\pi, \quad l \in \mathbb{Z}, \\ & d(t; \beta, k, \alpha) = \sqrt{\sin^2\beta + e^{2k^2\alpha t}\cos\beta}, \\ & \varphi(x, t; \beta, k, \alpha) = \frac{\frac{1}{\alpha}\log\left(d + e^{k^2\alpha t}\cos\beta\right)}{1 + \cos\beta}. \end{aligned} \tag{29}$$

The methods used in the case of without damping equation are similarily applicable to the LLG equation containing the damping parameter. If we apply the FE method to (27) than we shall obtain:

$$\begin{aligned} m_{x,i}^{j+1} &= m_{x,i}^j + hG_1\left(m_{x,i}^j, m_{y,i}^j, m_{z,i}^j, t_j\right), \\ m_{y,i}^{j+1} &= m_{y,i}^j + hG_2\left(m_{x,i}^j, m_{y,i}^j, m_{z,i}^j, t_j\right), \\ m_{z,i}^{j+1} &= m_{z,i}^j + hG_3\left(m_{x,i}^j, m_{y,i}^j, m_{z,i}^j, t_j\right). \end{aligned} \tag{30}$$

where

$$\begin{aligned} G_1\left(m_{x,i}^j, m_{y,i}^j, m_{z,i}^j, t_j\right) &= F_1 - \alpha\left(m_{y,i}^j F_3 - m_{z,i}^j F_2\right) \\ G_2\left(m_{x,i}^j, m_{y,i}^j, m_{z,i}^j, t_j\right) &= F_2 - \alpha\left(m_{y,i}^j F_1 - m_{z,i}^j F_3\right) \\ G_3\left(m_{x,i}^j, m_{y,i}^j, m_{z,i}^j, t_j\right) &= F_3 - \alpha\left(m_{y,i}^j F_2 - m_{z,i}^j F_1\right) \end{aligned} \tag{31}$$

For solving (27) with the methods RK4 and RKF, we have applied the same numerical algorithms as above.

IV. Results and Conclusions

Comparing the accuracy of the three schemes, according to Table (I), we have analized the absolute error $\varepsilon(t)$ at different times and for the same temporal and spatial discretization $(\Delta x, \Delta t)$. We consider $\varepsilon(t) = \|\vec{m}_h - \vec{m}_e\|$, with m_h being the numerical solution and m_e being the exact solution.

TABLE I
Accuracy of the three schemes

t	FE $\|\vec{m}_h - \vec{m}_e\|_\infty$	RK4 $\|\vec{m}_h - \vec{m}_e\|_\infty$	RKF $\|\vec{m}_h - \vec{m}_e\|_\infty$
4.0 E-03	1.4093E-6	1.6257 E-6	5.4923 E-6
8.0 E-03	2.8944E-6	3.2848 E-6	1.0953 E-5
1.2 E-02	4.7824E-6	4.6468 E-6	1.6295 E-5
1.6 E-02	1.5620E-5	6.1965 E-6	2.1226 E-5
2.0 E-02	1.9470E-4	8.1937 E-6	2.7513 E-5
2.4 E-02	4.5000E-3	1.0195 E-5	3.4596 E-5

We observe that increasing the number of iterations, the smallest error increase appears when we use the RK4 method. We conclude that the RK4 method is the optimal one for LLG numerical solving. Although for $t = 4 \cdot 10^{-3}$ and $t = 8 \cdot 10^{-3}$, the error for FE method is smaller than the error of RK4, we observe that from $t = 1.2 \cdot 10^{-2}$ to $t = 2.4 \cdot 10^{-2}$, the error for FE method increases rapidly comparing to the error for RK4 method. The results from Table (I) are comparable to the results from [6].
Figure (1) shows the absolute errors of the FE, RK4 and RKF methods.

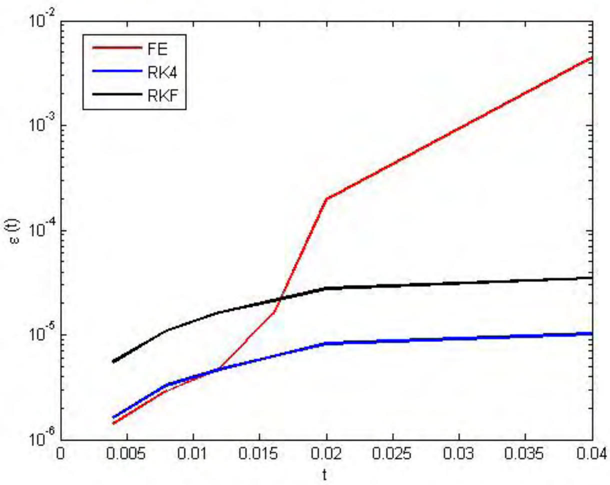

Fig. 1. Absolute errors of the FE method, RK4 method and RKF method.

Figure (2) shows the time evolution of m_x, when the damping parameter value is 0.1, with $\Delta x = 0.01$ and $\Delta t = 10^{-5}$.

Figure (3) exhibits a view of m_x numerical solution (using RK4 method), and figure (4) exhibits a view of the exact solution, both in the case without damping.

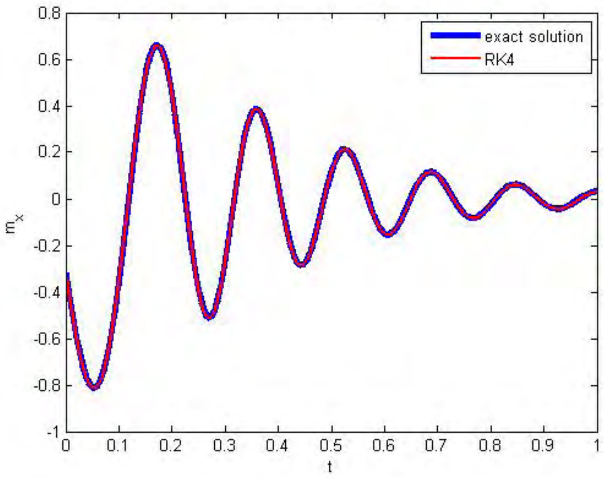

Fig. 2. m_x evolution with $\alpha = 0.1$

Figure (5) exhibits a view of m_x numerical solution (using RK4 method), and figure (6) exhibits a view of the exact solution, both in the case when we consider a damping parameter.

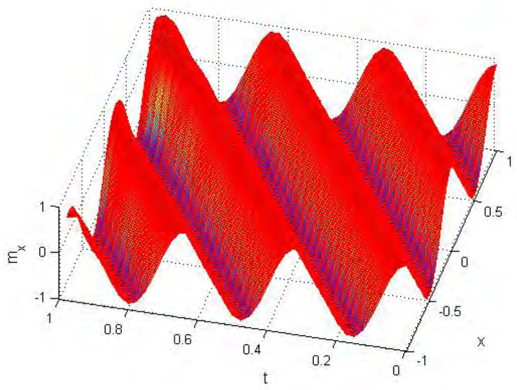

Fig. 3. m_x of the numerical solution with $\alpha = 0.0$

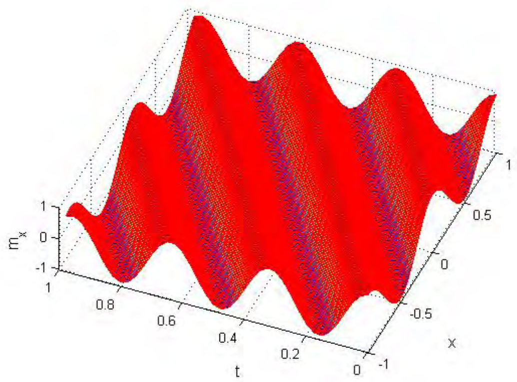

Fig. 4. m_x of the exact solution with $\alpha = 0.0$

These graphics were obtained by applying the RK4 method.

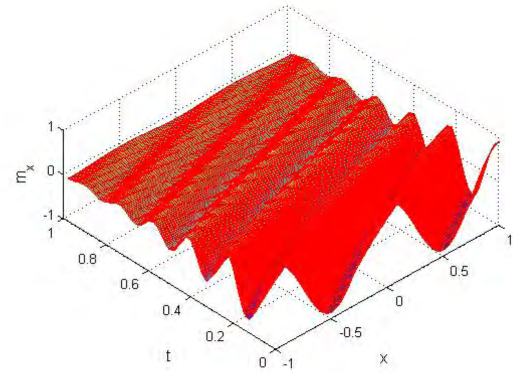

Fig. 5. m_x of the numerical solution with $\alpha = 0.1$

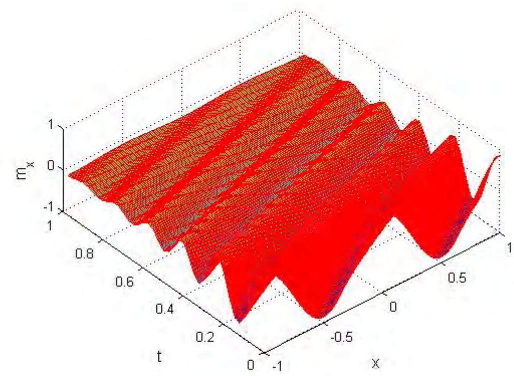

Fig. 6. m_x of the exact solution with $\alpha = 0.1$

V. Future Work

For our future research we plan to use these results in order to analyze the influence of the magnetization dynamics on the response period of the giant magnetoresistance sensor.These tested methods will be used with a variable step.

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