Problem 1: Steady-state characteristics of a DC motor

A DC motor with a separately excited field winding is considered. The rated armature voltage is $U_\mathrm{N} = 600$ V, rated torque $T_\mathrm{N} = 420$ Nm, rated speed $n_\mathrm{N} = 1600$ r/min, and maximum speed $n_\mathrm{max} = 3200$ r/min. The losses are omitted.

(a) The flux factor k_f is kept constant at its rated value. When the armature voltage is varied from 0 to U_N, the speed varies from 0 to n_N. Determine the rated armature current I_N.

(b) A load is to be driven in the speed range from n_N to n_max by weakening the flux factor while the armature voltage is kept constant at U_N. Determine the torque available at maximum speed, if the rated armature current I_N is not exceeded.

(c) Sketch the armature voltage U_a, flux factor k_f, torque T_M, and mechanical power P_M as a function of the speed, when the armature current is kept at I_N.

Solution

The losses are omitted, i.e., $R_\mathrm{a} = 0$ holds. Hence, the steady-state equations of the DC motor are

$$U_\mathrm{a} = k_\mathrm{f}\omega_\mathrm{M} \qquad T_\mathrm{M} = k_\mathrm{f} I_\mathrm{a} \qquad P_\mathrm{M} = T_\mathrm{M}\omega_\mathrm{M} = U_\mathrm{a} I_\mathrm{a}$$

(a) Let us first calculate the rated rotor speed in radians per second:

$$\omega_\mathrm{N} = 2\pi n_\mathrm{N} = 2\pi \cdot \frac{1600\ \mathrm{r/min}}{60\ \mathrm{s/min}} = 167.6\ \mathrm{rad/s}$$

The rated flux factor is

$$k_\mathrm{fN} = \frac{U_\mathrm{N}}{\omega_\mathrm{N}} = \frac{600\ \mathrm{V}}{167.6\ \mathrm{rad/s}} = 3.58\ \mathrm{Vs}$$

The rated armature current is

$$I_\mathrm{N} = \frac{T_\mathrm{N}}{k_\mathrm{fN}} = \frac{420\ \mathrm{Nm}}{3.58\ \mathrm{Vs}} = 117.3\ \mathrm{A}$$

(b) The maximum rotor speed in radians per second is

$$\omega_\mathrm{max} = 2\pi n_\mathrm{max} = 2\pi \cdot \frac{3200\ \mathrm{r/min}}{60\ \mathrm{s/min}} = 335.1\ \mathrm{rad/s}$$

The flux factor at the maximum speed is

$$k_\mathrm{f} = \frac{U_\mathrm{N}}{\omega_\mathrm{max}} = \frac{600\ \mathrm{V}}{335.1\ \mathrm{rad/s}} = 1.79\ \mathrm{Vs}$$

The torque at the maximum speed is

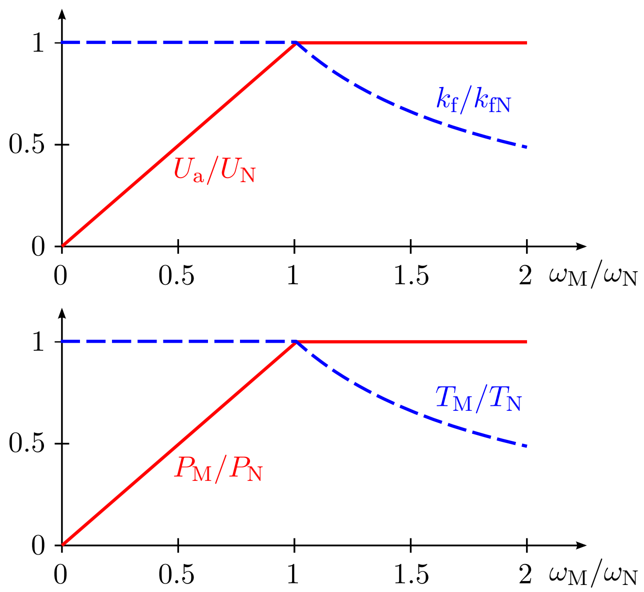
$$T_\mathrm{M} = k_\mathrm{f} I_\mathrm{N} = 1.79\ \mathrm{Vs} \cdot 117.3\ \mathrm{A} = 210\ \mathrm{Nm}$$

The same result could be obtained as $T_\mathrm{M} = (n_\mathrm{N}/n_\mathrm{max})T_\mathrm{N}$, i.e. the torque reduces inversely proportionally to the speed in the field-weakening region.

(c) The requested characteristics are shown in the figure below.

Based on $U_\mathrm{a} = k_\mathrm{f}\omega_\mathrm{M}$, the armature voltage increases linearly with the rotor speed until the rated (maximum) voltage U_N is reached at the rated speed. In order to reach higher speeds, the flux factor k_f has to be reduced inversely proportionally to the speed.

Since $I_\mathrm{a} = I_\mathrm{N}$ is constant, the torque $T_\mathrm{M} = k_\mathrm{f}I_\mathrm{a}$ follows the characteristics of the flux factor k_f. Based on $P_\mathrm{M} = T_\mathrm{M}\omega_\mathrm{M}$, the mechanical power $P_\mathrm{M} = T_\mathrm{M}\omega_\mathrm{M}$ increases linearly with the speed until the rated speed and remains constant at speeds higher than the rated speed. It is important to notice the same mechanical power is obtained also using the electrical quantities, $P_\mathrm{M} = U_\mathrm{a}I_\mathrm{a}$, since the losses are omitted.



Problem 2: Transfer functions

(a) A DC motor is considered. Derive the transfer function from the terminal voltage $u_\mathrm{a}(s)$ to the terminal current $i_\mathrm{a}(s)$.

(b) A lumped thermal capacity model is considered:

$$p_\mathrm{d}(t) = \frac{1}{R_\mathrm{th}}\theta(t) + C_\mathrm{th}\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}$$

Derive the transfer function from the power loss $p_\mathrm{d}(s)$ to the temperature rise $\theta(s)$.

Solution

(a) The terminal voltage is

$$u_\mathrm{a}(t) = R_\mathrm{a}i_\mathrm{a}(t) + L_\mathrm{a}\frac{\mathrm{d}i_\mathrm{a}(t)}{\mathrm{d}t} + e_\mathrm{a}(t)$$

where $e_\mathrm{a}(t) = k_\mathrm{f}\omega_\mathrm{M}(t)$ is the induced voltage. The Laplace transformation gives

$$u_\mathrm{a}(s) = R_\mathrm{a}i_\mathrm{a}(s) + sL_\mathrm{a}i_\mathrm{a}(s) + e_\mathrm{a}(s)$$

from which the current $i_a(s)$ can be solved as

$$i_a(s) = \frac{1}{R_a + sL_a}[u_a(s) - e_a(s)]$$

The inputs $u_a(s)$ and $e_a(s)$ can be considered separately based on the superposition principle. Hence, the transfer function from $u_a(s)$ to $i_a(s)$ is

$$Y_a(s) = \frac{1}{R_a + sL_a} = \frac{1/R_a}{1 + \tau_a s}$$

where $\tau_a = L_a/R_a$ is the time constant of the armature winding. It is worth noticing that this transfer function can be interpreted as the input admittance of the motor; this is also the reason why we chose the notation $Y_a(s)$ here.

(b) The Laplace transformation gives

$$p_d(s) = \frac{1}{R_{th}}\theta(s) + sC_{th}\theta(s)$$

The transfer function from the power loss $p_d(s)$ to the temperature rise $\theta(s)$ becomes

$$Z_{th}(s) = \frac{R_{th}}{1 + \tau_{th}s}$$

where $\tau_{th} = R_{th}C_{th}$ is the thermal time constant. This transfer function can be interpreted as the thermal impedance, which is the reason for our notation.

Problem 3: Properties of first-order systems

Consider a first-order system

$$G(s) = \frac{K}{1 + s\tau}$$

(a) What is the steady-state gain of the system?

(b) Derive the rise time from 10% to 90% for a step input.

(c) What is the 3-dB bandwidth α of the system?

Solution

(a) The steady-state gain is obtained by substituting $s = 0$ in the transfer function, giving $G(s) = K$.

(b) The transfer function $G(s)$ is multiplied with the unit-step input $u(s)$

$$y(s) = G(s)u(s) = \frac{K}{1 + \tau s}\frac{1}{s}$$

By using the inverse Laplace transform, the unit-step response of the system in the time domain can be obtained:

$$y(t) = K\left(1 - e^{-t/\tau}\right)$$

Let us solve the time as a function of y:

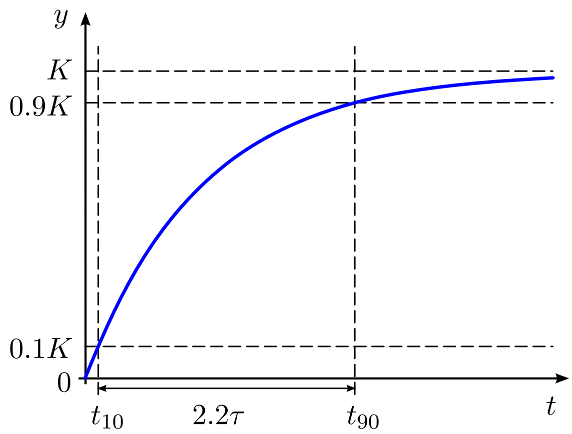
$$e^{-t/\tau} = 1 - y(t)/K \quad \text{or} \quad t = -\tau \ln[1 - y(t)/K]$$

The time instants t_{10} and t_{90}, at which $y(t_{10}) = 0.1K$ and $y(t_{90}) = 0.9K$, respectively, are

$$t_{10} = -\tau \ln(0.9) \quad \text{and} \quad t_{90} = -\tau \ln(0.1)$$

The rise time is the difference

$$t_r = t_{90} - t_{10} = \tau\,[\ln(0.9) - \ln(0.1)] = \tau \ln 9 \approx 2.2\tau$$



(c) The frequency response of the system is

$$G(j\omega) = \frac{K}{1 + j\omega\tau}$$

whose magnitude is

$$|G(j\omega)| = \frac{K}{\sqrt{1 + \omega^2\tau^2}}$$

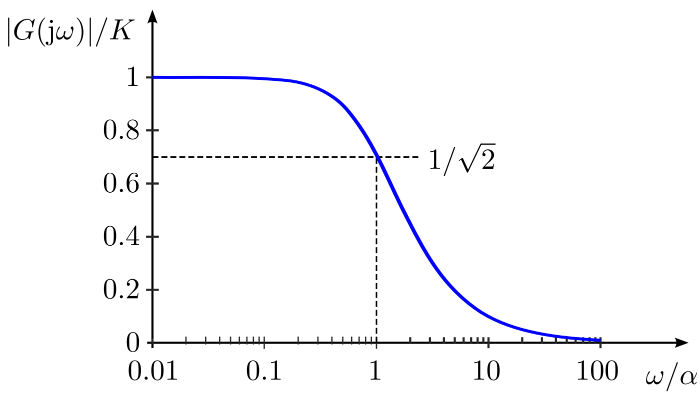
The 3-dB bandwidth α refers to the frequency at which the magnitude has dropped to $1/\sqrt{2} \approx 0.71$ of the steady state gain, i.e., $|G(j\alpha)| = K/\sqrt{2}$. Hence,

$$\sqrt{1 + \alpha^2\tau^2} = \sqrt{2} \quad \Rightarrow \quad \tau = 1/\alpha$$

Remark: It is also common to represent first-order systems in the form

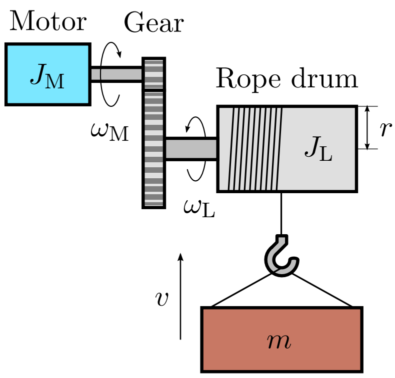
$$G(s) = \frac{K\alpha}{s + \alpha}$$

where the bandwidth is readily visible. The figure below shows the magnitude of the frequency response.



Problem 1: Gears

Consider a hoist drive shown in the figure. The motor is coupled to the rope drum through a gear mechanism, whose gear ratio is $i = \omega_\mathrm{M}/\omega_\mathrm{L} = 9.6$. The load mass is $m = 500$ kg, the motor inertia is $J_\mathrm{M} = 0.5$ kgm^2, and the rope drum inertia is $J_\mathrm{L} = 48.5$ kgm^2. The radius of the rope drum is $r = 0.25$ m. The rope mass, gear inertias, and the mechanical losses are omitted. Calculate the equivalent total inertia at the motor side and the equivalent load torque at the motor side.



Solution

The electromagnetic torque produced by the motor is

$$T_\mathrm{M} = J\frac{\mathrm{d}\omega_\mathrm{M}}{\mathrm{d}t} + T'_\mathrm{L}$$

where J is the equivalent total inertia at the motor side and T'_L is the equivalent load torque also at the motor side. The problem here is to determine J and T'_L.

The inertia of the mass m at the load side is

$$J_m = \int_0^m r^2 \mathrm{d}m = mr^2 = 500 \text{ kg} \cdot (0.25 \text{ m})^2 = 31.25 \text{ kgm}^2$$

It can be seen that the load-side inertia $J_m + J_\mathrm{L} = 79.8$ kgm^2 is very large (approximately 160 times the motor inertia J_M). However, the total equivalent inertia seen by the motor is only

$$J = J_\mathrm{M} + \frac{J_\mathrm{L} + J_m}{i^2} = 0.5 \text{ kgm}^2 + \frac{48.5 + 31.25}{9.6^2} \text{ kgm}^2 = 1.36 \text{ kgm}^2$$

due to the gear ratio.

The mass m causes the torque $T_\mathrm{L} = mgr$ on the rope drum. Hence, the load torque seen by the motor is

$$T'_\mathrm{L} = \frac{T_\mathrm{L}}{i} = \frac{mgr}{i} = \frac{500 \text{ kg} \cdot 9.81 \text{ m/s}^2 \cdot 0.25 \text{ m}}{9.6} = 127.7 \text{ Nm}$$

It is important to notice that the mass m affects both the equivalent total inertia J and the equivalent load torque T'_L.

Problem 2: Electromagnetic torque vs. shaft torque

A torque sensor is connected between the motor shaft and the load shaft. The load torque is constant $T_\mathrm{L} = 150$ Nm and the load inertia is $J_\mathrm{L} = 1.0$ kgm^2. The motor inertia is $J_\mathrm{M} = 0.6$ kgm^2. The speed is increased from zero to $\omega_\mathrm{M} = 100$ rad/s in 0.5 s with a constant angular acceleration.

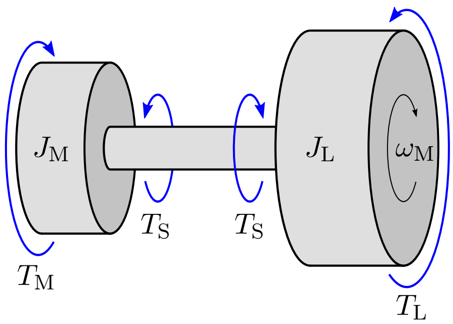
(a) What is the electromagnetic torque during acceleration? What about the measured torque?

(b) What is the electromagnetic torque at constant speed? What about the measured torque?

Solution

The system is illustrated in the figure below. The shaft can be assumed to be rigid. The total inertia is $J = J_\mathrm{M} + J_\mathrm{L} = 1.6$ kgm^2. The angular acceleration is

$$\alpha_\mathrm{M} = \frac{\mathrm{d}\omega_\mathrm{M}}{\mathrm{d}t} = \frac{100 \text{ rad/s}}{0.5 \text{ s}} = 200.0 \text{ rad/s}^2$$



(a) During the acceleration, the motor produces the electromagnetic torque

$$T_\mathrm{M} = J\alpha_\mathrm{M} + T_\mathrm{L} = (1.6 \cdot 200 + 150) \text{ Nm} = 470 \text{ Nm}$$

On the other hand, the torque sensor measures the shaft torque, which is

$$T_\mathrm{S} = T_\mathrm{M} - J_\mathrm{M}\alpha_\mathrm{M} = (470 - 0.6 \cdot 200) \text{ Nm} = 350 \text{ Nm}$$

Alternatively, the shaft torque can be calculated from the load side as

$$T_\mathrm{S} = T_\mathrm{L} + J_\mathrm{L}\alpha_\mathrm{M} = (150 + 1.0 \cdot 200) \text{ Nm} = 350 \text{ Nm}$$

(b) The shaft torque at constant speed equals the electromagnetic torque:

$$T_\mathrm{S} = T_\mathrm{M} = T_\mathrm{L} = 150 \text{ Nm}$$

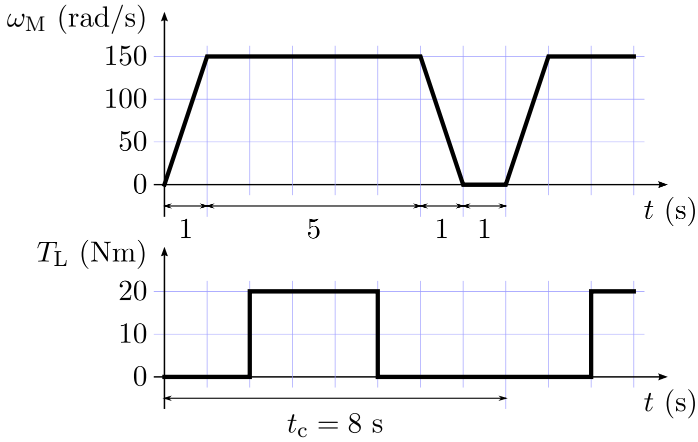
Problem 3: Torque and power

In periodic duty, the mechanical angular speed ω_M and load torque T_L vary as shown in the figure. The total equivalent inertia is 0.04 kgm^2. The cycle duration is $t_\mathrm{c} = 8$ s.

(a) Draw the conceptual waveforms of the electromagnetic torque T_M and mechanical power p_M for one cycle.

(b) Calculate the rms value of the electromagnetic torque.

(c) A permanent-magnet DC motor is applied in this periodic duty. The rated torque and rated armature current of the motor are $T_\mathrm{N} = 14.3$ Nm and $I_\mathrm{N} = 33$ A, respectively. What is the maximum armature current during the period?



Solution

(a) The required electromagnetic torque is

$$T_\mathrm{M} = J\frac{\mathrm{d}\omega_\mathrm{M}}{\mathrm{d}t} + T_\mathrm{L}$$

where $J = 0.04$ kgm^2 is the total equivalent inertia. The mechanical power is

$$p_\mathrm{M} = T_\mathrm{M}\omega_\mathrm{M}$$

During acceleration at $t = 0\ldots1$ s, the electromagnetic torque is

$$T_\mathrm{M} = J\frac{\Delta\omega_\mathrm{M}}{\Delta t} = 0.04\ \mathrm{kgm}^2 \cdot \frac{150\ \mathrm{rad/s}}{1\ \mathrm{s}} = 6\ \mathrm{Nm}$$

During the loading phase at $t = 2\ldots5$ s, the electromagnetic torque has to be $T_\mathrm{M} = T_\mathrm{L} = 20$ Nm since the speed is constant. The braking torque at $t = 6\ldots7$ s is

$$T_\mathrm{M} = J\frac{\Delta\omega_\mathrm{M}}{\Delta t} = 0.04\ \mathrm{kgm}^2 \cdot \frac{-150\ \mathrm{rad/s}}{1\ \mathrm{s}} = -6\ \mathrm{Nm}$$

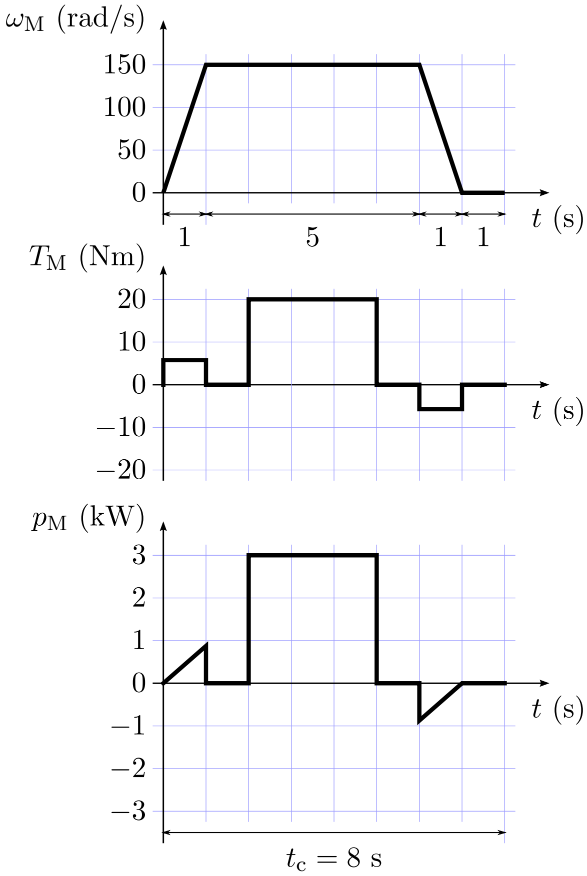
The value of the mechanical power at $t = 1$ s is

$$p_\mathrm{M} = 6\ \mathrm{Nm} \cdot 150\ \mathrm{rad/s} = 0.9\ \mathrm{kW}$$

and at $t = 2 \dots 5$ s it is

$$p_M = 20 \text{ Nm} \cdot 150 \text{ rad/s} = 3.0 \text{ kW}$$

Based on the above calculations, we can plot the following waveforms.



(b) The rms value of the electromagnetic torque over the period t_c is

$$\begin{aligned} T_{M,rms} &= \sqrt{\frac{1}{t_c}\int_0^{t_c} T_M^2 \, dt} \\ &= \sqrt{\frac{(6 \text{ Nm})^2 \cdot 1 \text{ s} + (20 \text{ Nm})^2 \cdot 3 \text{ s} + (-6 \text{ Nm})^2 \cdot 1 \text{ s}}{8 \text{ s}}} = 12.6 \text{ Nm} \end{aligned}$$

Remark: The period $t_c = 8$ s is much shorter than thermal time constants (several minutes or tens of minutes) of motors in this power range. Hence, the motor can be selected based on the average temperature rise, which leads to the selection criterion $T_N > T_{M,rms}$. Furthermore, the motor should be able to produce the required maximum torque $T_{M,max} = 20$ Nm.

(c) The flux factor of the motor is $k_f = T_N/I_N = 14.3 \text{ Nm}/33 \text{ A} = 0.43 \text{ Nm/A}$. The maximum armature current is $I_a = T_M/k_f = 20 \text{ Nm}/0.43 \text{ Nm/A} = 46.2$ A.

Remark: Thermal time constants of power converters are much shorter (typically seconds or tens of seconds) than those of the motors. Therefore, the converter has to be typically selected based on the maximum current during the period (instead of the rms value over the period).

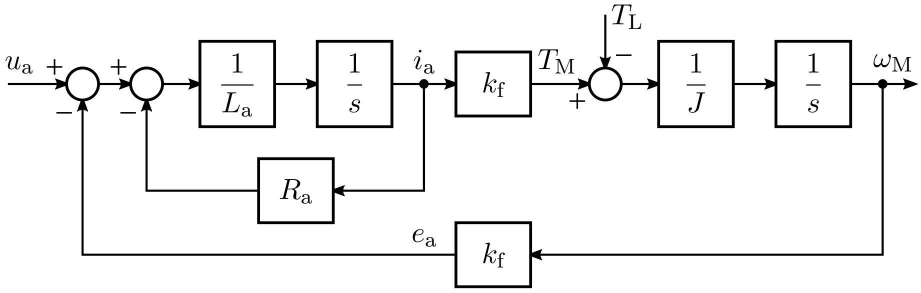
Problem 1: Transfer functions of a DC motor

The block diagram of a DC motor is shown in the figure.

(a) Derive the transfer functions

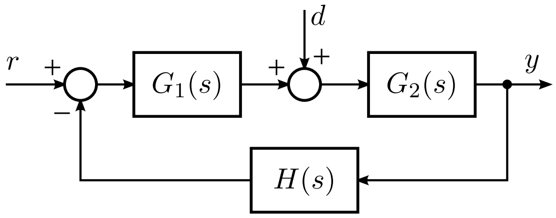
$$G_{\omega u}(s) = \frac{\omega_\mathrm{M}(s)}{u_\mathrm{a}(s)} \quad \text{and} \quad G_{\omega T}(s) = \frac{\omega_\mathrm{M}(s)}{T_\mathrm{L}(s)}$$

(b) Replace the electric dynamics of the machine with the DC gain and formulate the transfer functions $G_{\omega u}(s)$ and $G_{\omega T}(s)$.



Solution

(a) Consider a closed-loop system shown in the figure.

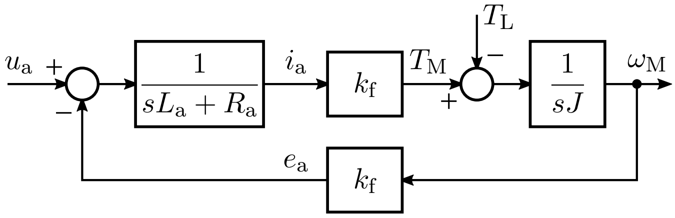


The following equations hold for the closed-loop transfer functions:

$$\frac{y(s)}{r(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)} \tag{1}$$

$$\frac{y(s)}{d(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)} \tag{2}$$

It is relatively easy to derive these equations if one has forgotten them. Using (1), the block diagram given in the problem is first transformed to the following form:



Using (1), we can write the transfer function from the voltage to the speed as

$$G_{\omega u}(s) = \frac{\omega_\mathrm{M}(s)}{u_\mathrm{a}(s)} = \frac{\frac{1}{sL_\mathrm{a}+R_\mathrm{a}} k_\mathrm{f} \frac{1}{sJ}}{1 + \frac{1}{sL_\mathrm{a}+R_\mathrm{a}} k_\mathrm{f} \frac{1}{sJ} k_\mathrm{f}} = \frac{\frac{k_\mathrm{f}}{JL_\mathrm{a}}}{s^2 + s\frac{R_\mathrm{a}}{L_\mathrm{a}} + \frac{k_\mathrm{f}^2}{JL_\mathrm{a}}}$$

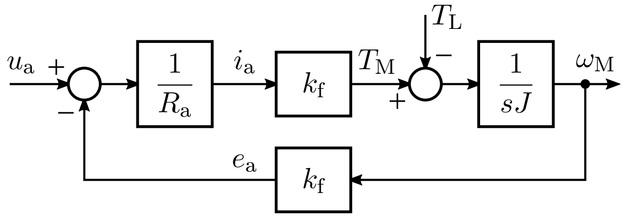
Using (2), the transfer function from the load torque to the speed becomes

$$G_{\omega T}(s) = \frac{\omega_\mathrm{M}(s)}{T_\mathrm{L}(s)} = -\frac{\frac{1}{sJ}}{1 + \frac{1}{sL_\mathrm{a}+R_\mathrm{a}} k_\mathrm{f} \frac{1}{sJ} k_\mathrm{f}} = -\frac{\frac{1}{J}\left(s + \frac{R_\mathrm{a}}{L_\mathrm{a}}\right)}{s^2 + s\frac{R_\mathrm{a}}{L_\mathrm{a}} + \frac{k_\mathrm{f}^2}{JL_\mathrm{a}}}$$

(b) The electric dynamics of the machine

$$Y_\mathrm{a}(s) = \frac{1}{sL_\mathrm{a} + R_\mathrm{a}}$$

is replaced with the DC gain by substituting $s = 0$. The corresponding block is shown in the following figure:



The transfer functions can be derived from this block diagram in a fashion similar to Part (a) of the problem. The same result is obtained by multiplying the numerator and denominator of the derived transfer functions by L_a and then substituting $L_\mathrm{a} = 0$:

$$G_{\omega u}(s) = \frac{\omega_\mathrm{M}(s)}{u_\mathrm{a}(s)} = \frac{\frac{k_\mathrm{f}}{JR_\mathrm{a}}}{s + \frac{k_\mathrm{f}^2}{JR_\mathrm{a}}}$$

$$G_{\omega T}(s) = \frac{\omega_\mathrm{M}(s)}{T_\mathrm{L}(s)} = -\frac{1/J}{s + \frac{k_\mathrm{f}^2}{JR_\mathrm{a}}}$$

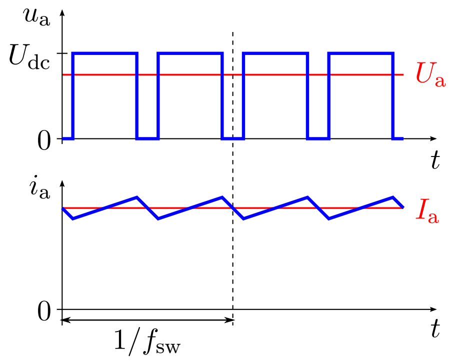
The time constant of the first-order system is

$$\tau = \frac{JR_\mathrm{a}}{k_\mathrm{f}^2}$$

which depends strongly on the flux factor k_f of the motor (and increases when the flux factor is decreased).

Problem 2: Current ripple

The parameters of a DC motor are: $R_a = 1\ \Omega$, $L_a = 10$ mH, and $k_f = 4$ Vs. The average steady-state current taken by the motor is $I_a = 100$ A and the rotor speed is 560 r/min. The motor is supplied from a four-quadrant DC-DC converter, where the unipolar PWM is applied. The DC-bus voltage is $U_{dc} = 450$ V and the switching (carrier) frequency is $f_{sw} = 4$ kHz. Calculate the peak-to-peak current ripple.



Solution

The electrical dynamics of the DC motor are governed by

$$L_a \frac{di_a(t)}{dt} = u_a(t) - R_a i_a(t) - e_a(t) \tag{3}$$

The rotor angular speed is

$$\omega_M = 2\pi \cdot \frac{560\ \text{r/min}}{60\ \text{s/min}} = 58.6\ \text{rad/s}$$

and the back-emf is $E_a = k_f \omega_M = 4\ \text{Vs} \cdot 58.6\ \text{rad/s} = 234.6$ V. Based on (3), the average armature voltage is

$$U_a = R_a I_a + E_a = 1\ \Omega \cdot 100\ \text{A} + 234.6\ \text{V} = 334.6\ \text{V}$$

in steady state.

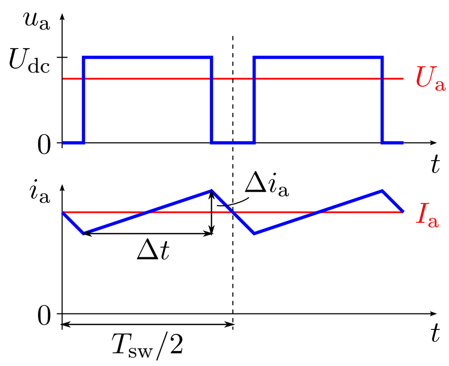
The figure below illustrates the waveforms of the armature voltage u_a and the armature current i_a. The average voltage during the switching period T_{sw} is

$$\overline{u}_a = \frac{1}{T_{sw}} \int_0^{T_{sw}} u_a(t)dt = \frac{2\Delta t}{T_{sw}} U_{dc} \tag{4}$$

where Δt is the duration of the positive voltage pulse (see the figure). The duration is

$$\Delta t = \frac{U_a}{U_{dc}} \frac{T_{sw}}{2} = \frac{U_a}{U_{dc}} \frac{1}{2 f_{sw}}$$

since $U_a = \overline{u}_a$ in steady state.



In the time scale of switching periods, the dynamics in (3) can be approximated as

$$L_a \frac{di_a(t)}{dt} = u_a(t) - R_a I_a - E_a = u_a(t) - U_a \tag{5}$$

The change Δi_a in the current during the positive voltage pulse $u_a(t) = U_{dc}$ is

$$\begin{aligned} \Delta i_a &= \frac{U_{dc} - U_a}{L_a} \Delta t \\ &= \frac{U_{dc} - U_a}{L_a} \frac{U_a}{U_{dc}} \frac{1}{2 f_{sw}} \\ &= \frac{450 \text{ V} - 334.6 \text{ V}}{10 \text{ mH}} \cdot \frac{334.6 \text{ V}}{450 \text{ V}} \frac{1}{2 \cdot 4 \text{ kHz}} = 1.1 \text{ A} \end{aligned}$$

This peak-to-peak current ripple is roughly 1% of the average current.

Remark 1: Naturally, the same result would be obtained, if the zero-voltage condition $u_a(t) = 0$ were used:

$$\Delta i_a = \frac{U_a}{L_a} \left(\frac{T_{sw}}{2} - \Delta t \right)$$

Remark 2: The switching period is $T_{sw} = 1/f_{sw} = 1/(4 \text{ kHz}) = 250\ \mu\text{s}$ and the electrical time constant of the armature winding is $\tau_a = L_a/R_a = 10$ ms. Since τ_a is much longer (40 times) than T_{sw}, the approximation (5) holds well.

Problem 1: Design of a PI current controller

The parameters of a DC motor are $R_\mathrm{a} = 0.87\ \Omega$ and $L_\mathrm{a} = 16$ mH. An ordinary PI current controller is used. The current-control bandwidth is required to be $\alpha_\mathrm{c} = 2\pi \cdot 300$ rad/s.

(a) Derive the expressions for the controller gains according to the principle of internal model control.

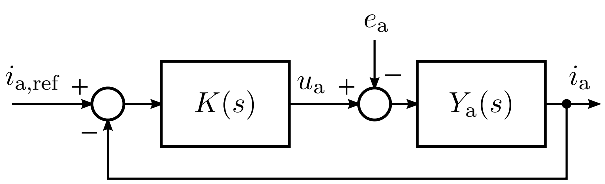
(b) Derive the expressions for the controller gains by cancelling the pole of the open-loop system and by requiring that the 0-dB crossover angular frequency of the loop transfer function is α_c.

Solution

(a) The block diagram of the current-controlled system is shown in the figure below. The converter is assumed to be ideal: $u_\mathrm{a} = u_\mathrm{a,ref}$. The back-emf e_a is assumed to be quasi-constant. The PI controller and the admittance of the DC motor are

$$K(s) = k_\mathrm{p} + \frac{k_\mathrm{i}}{s} \qquad Y_\mathrm{a}(s) = \frac{1}{sL_\mathrm{a} + R_\mathrm{a}}$$

respectively.



The closed-loop transfer function is

$$\frac{i_\mathrm{a}(s)}{i_\mathrm{a,ref}(s)} = H(s) = \frac{K(s)Y_\mathrm{a}(s)}{1 + K(s)Y_\mathrm{a}(s)} \tag{1}$$

and the desired closed-loop transfer function is

$$H(s) = \frac{\alpha_\mathrm{c}}{s + \alpha_\mathrm{c}} = \frac{\alpha_\mathrm{c}/s}{1 + \alpha_\mathrm{c}/s} \tag{2}$$

where $\alpha_\mathrm{c} = 2\pi \cdot 300$ rad/s $= 1885$ rad/s. Equalling (1) and (2) gives the condition

$$K(s)Y_\mathrm{a}(s) = \frac{\alpha_\mathrm{c}}{s}$$

from which the controller transfer function can be solved:

$$K(s) = \frac{\alpha_\mathrm{c}}{sY_\mathrm{a}(s)} = \frac{\alpha_\mathrm{c}}{s}(sL_\mathrm{a} + R_\mathrm{a}) = \alpha_\mathrm{c} L_\mathrm{a} + \frac{\alpha_\mathrm{c} R_\mathrm{a}}{s}$$

Hence, the PI controller gains are

$$\begin{aligned} k_\mathrm{p} &= \alpha_\mathrm{c} L_\mathrm{a} = 1885\ \text{rad/s} \cdot 0.016\ \text{H} = 30.2\ \text{V/A} \\ k_\mathrm{i} &= \alpha_\mathrm{c} R_\mathrm{a} = 1885\ \text{rad/s} \cdot 0.87\ \Omega = 1640\ \text{V/(As)} \end{aligned}$$

(b) The loop transfer function is

$$L(s) = K(s)Y_\mathrm{a}(s) = \frac{k_\mathrm{p}s + k_\mathrm{i}}{s}\frac{1}{sL_\mathrm{a} + R_\mathrm{a}} = \frac{s + k_\mathrm{i}/k_\mathrm{p}}{s/k_\mathrm{p}}\frac{1/L_\mathrm{a}}{s + R_\mathrm{a}/L_\mathrm{a}} \tag{3}$$

In order to cancel the pole of the open-loop system with the controller zero, the condition

$$s + \frac{k_\mathrm{i}}{k_\mathrm{p}} = s + \frac{R_\mathrm{a}}{L_\mathrm{a}} \qquad \Rightarrow \qquad \frac{k_\mathrm{i}}{k_\mathrm{p}} = \frac{R_\mathrm{a}}{L_\mathrm{a}} \tag{4}$$

should hold. Using this condition in (3) gives the loop transfer function

$$L(s) = \frac{k_\mathrm{p}}{sL_\mathrm{a}}$$

The loop-transfer function is required to have the gain of 0 dB (the unity gain) at the crossover angular frequency α_c:

$$|L(\mathrm{j}\alpha_\mathrm{c})| = \left|\frac{k_\mathrm{p}}{\mathrm{j}\alpha_\mathrm{c}L_\mathrm{a}}\right| = 1$$

The proportional gain can now be solved:

$$k_\mathrm{p} = \alpha_\mathrm{c}L_\mathrm{a} = 1885\ \mathrm{rad/s} \cdot 0.016\ \mathrm{H} = 30.2\ \mathrm{V/A}$$

Based on (4), the integral gain is

$$k_\mathrm{i} = \frac{R_\mathrm{a}}{L_\mathrm{a}}k_\mathrm{p} = \alpha_\mathrm{c}R_\mathrm{a} = 1885\ \mathrm{rad/s} \cdot 0.87\ \Omega = 1640\ \mathrm{V/(As)}$$

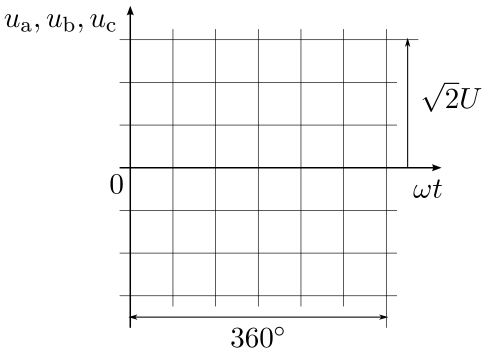
The result is equal to the result obtained in Part (a).

Remark: If needed, the gain and phase margins can be determined from $L(\mathrm{j}\omega)$. In this design example, the controller gains led to the loop-transfer function $L(s) = \alpha_\mathrm{c}/s$. As an example, the phase margin ϕ is

$$\phi = 180^\circ + \angle L(\mathrm{j}\alpha_\mathrm{c}) = 180^\circ + \angle \frac{\alpha_\mathrm{c}}{\mathrm{j}\alpha_\mathrm{c}} = 90^\circ$$

Problem 2: Waveforms in a balanced three-phase system

Sketch the waveforms of balanced three-phase voltages on the squared paper (or the grid below).



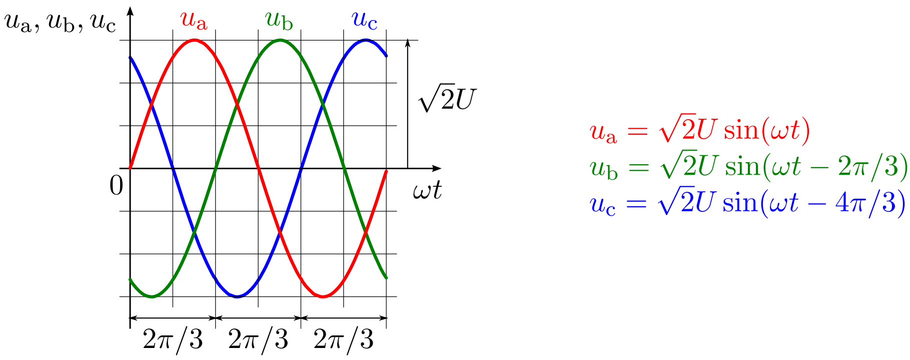
Hint: It is convenient to use a 6×6 grid to draw these waveforms. You can fist mark the points corresponding to zero crossings and peak values. It is also worth noticing that $\sin(\pi/6) = 1/2$ and mark these points on the grid.

Solution

A balanced set of three-phase sinusoidal AC voltages is

$$u_a = \hat{U}\sin(\omega t) \qquad u_b = \hat{U}\sin(\omega t - 2\pi/3) \qquad u_c = \hat{U}\sin(\omega t - 4\pi/3)$$

The peak value of the voltage $\hat{U} = \sqrt{2}U = \sqrt{2/3}U_{LL}$, where U is the rms line-to-neutral voltage and U_{LL} is rms line-to-line voltage. The waveforms are shown in the figure.



Problem 3: Power in single-phase and three-phase systems

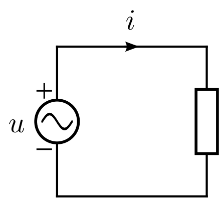
The purpose of this problem is to demonstrate that the instantaneous power in a balanced three-phase system is constant in steady state.

(a) A single-phase load is fed with the voltage $u(t) = \sqrt{2}U\sin(\omega t)$ and it draws the current $i(t) = \sqrt{2}I\sin(\omega t - \varphi)$. Derive the expressions for the instantaneous power and the average power.

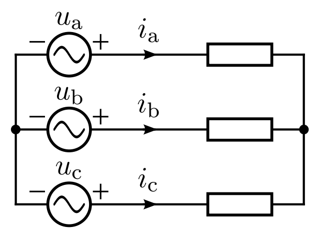
Hint: The trigonometric product-to-sum identity may be useful:

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

(b) A balanced three-phase load is fed with balanced three-phase voltages. Derive the expressions for the instantaneous power and the average power.



(a)



(b)

Solution

(a) The voltage and the current are

$$u = \sqrt{2}U\sin(\omega t) \qquad i = \sqrt{2}I\sin(\omega t - \varphi)$$

Using the given trigonometric identity, the instantaneous power becomes

$$\begin{aligned} p &= ui \\ &= 2UI\sin(\omega t)\sin(\omega t - \varphi) \\ &= UI\cos\varphi - UI\cos(2\omega t - \varphi) \end{aligned}$$

The second term is a second-harmonic component, i.e., the instantaneous power oscillates at twice the supply frequency. The first term is the average power

$$P = \frac{1}{T}\int_0^T p(t)\mathrm{d}t = UI\cos\varphi$$

where $T = 2\pi/\omega$ is the fundamental period.

(b) The phase voltages are

$$u_\mathrm{a} = \sqrt{2}U\sin(\omega t) \qquad u_\mathrm{b} = \sqrt{2}U\sin(\omega t - 2\pi/3) \qquad u_\mathrm{c} = \sqrt{2}U\sin(\omega t - 4\pi/3)$$

where U is the phase-to-neutral rms voltage. The phase currents are

$$i_\mathrm{a} = \sqrt{2}I\sin(\omega t - \varphi) \qquad i_\mathrm{b} = \sqrt{2}I\sin(\omega t - \varphi - 2\pi/3) \qquad i_\mathrm{c} = \sqrt{2}I\sin(\omega t - \varphi - 4\pi/3)$$

Instantaneous powers in each phase are

$$
\begin{aligned}
p_\mathrm{a} &= u_\mathrm{a} i_\mathrm{a} = UI\cos\varphi - UI\cos(2\omega t - \varphi) \\
p_\mathrm{b} &= u_\mathrm{b} i_\mathrm{b} = UI\cos\varphi - UI\cos(2\omega t - \varphi - 4\pi/3) \\
p_\mathrm{c} &= u_\mathrm{c} i_\mathrm{c} = UI\cos\varphi - UI\cos(2\omega t - \varphi - 2\pi/3)
\end{aligned}
$$

The total instantaneous power

$$p = p_\mathrm{a} + p_\mathrm{b} + p_\mathrm{c} = 3UI\cos\varphi = P$$

equals the average power P, i.e., the second-harmonic components cancel out due their 120° phase difference. The power flow of a balanced three-phase system is smooth (unlike that of the single-phase system).