Power Electronics

Exercise: Space Vector

2012

1 Theory

Space vector is a transformation for analyzing three-phase electric systems. The term "space" originally stands for the two-dimensional complex plane, in which the three-phase quantities are transformed. In electric machines it is combined with the magnetic field.

1.1 Introduction

In order to understand the concept of space vector, it is helpful to start from observing the structure of an electric machine that is driven by a three-phase current.

An electric machine is usually composed of a stator and a rotor. Usually a stator is the outer part of the machine with windings, which are connected to the three-phase grid to generate a rotating magnetic field.

Different rotors are available in different types of machines. However, the structure of the stator is almost the same. In a three phase machine with one pole pair, three copper windings are embedded in the slots of the stator iron structure. The following figure shows one distributed winding of the stator with a simplified drawing.

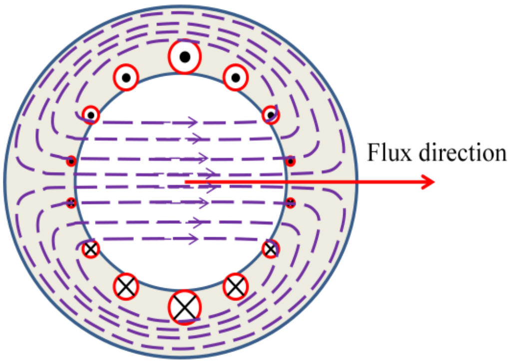

Figure 1. One of the distributed windings in a machine stator

In a distributed winding, the copper wire turns are not uniformly distributed along the stator circumstance. This arrangement is used for generating a magnetic field with sinusoidal distribution along the stator inner circumstance. For a three phase machine with one pole pair, the simplified winding structure is shown in the following figure (left side).

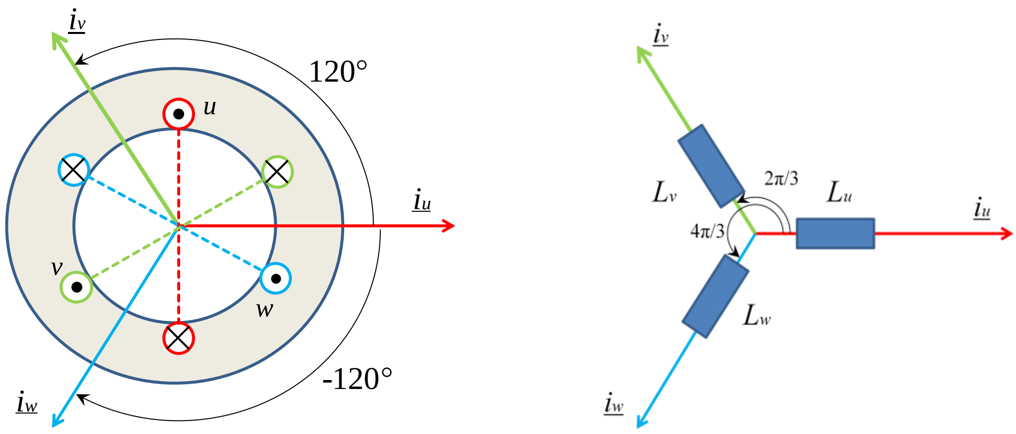

Figure 2. Windings in an electric machine

Three windings are organized with 120° angle difference in the physical space between each other. On the right side of the figure they are illustrated in a simplified way, where a winding is represented with an inductor.

The arrowed lines attached to the windings define the directions of the currents in the windings in a form of vector. The three current currents are named as $\underline{i_u}$, $\underline{i_v}$ and $\underline{i_w}$. The vector magnitudes are equal to the current magnitudes, respectively.

It must be noticed that the vector directions are not the directions of the physical currents flowing in the windings. The vectors defined with directions are only used for analysis.

At a certain time point, e.g., the currents in the three windings have the directions and values as shown in Figure 3, they could be summarized together using vector addition. The result is $\underline{i_u} + \underline{i_v} + \underline{i_w}$ as shown in the figure.

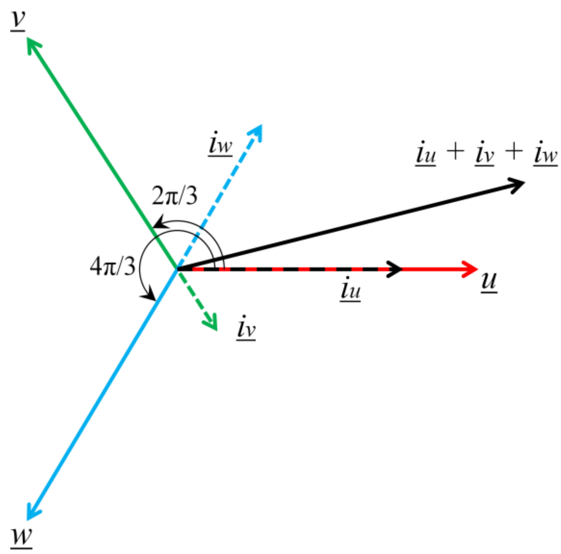

Figure 3. Addition of the current vectors

1.2 Definition

Referring to Figure 3 we are able to calculate the sum of the current vectors as

$$\underline{i_{sum}} = \underline{i_u} + \underline{i_v} + \underline{i_w} = i_u e^{\mathrm{j}0} + i_v e^{\mathrm{j}\frac{2}{3}\pi} + i_w e^{\mathrm{j}\frac{4}{3}\pi}. \tag{1}$$

Using the Euler's formula, $e^{\mathrm{j}\alpha} = \cos\alpha + \mathrm{j} \cdot \sin\alpha$, we have

$$\underline{i_{sum}} = \left(i_u - \frac{1}{2} i_v - \frac{1}{2} i_w\right) + \mathrm{j} \cdot \left(\frac{\sqrt{3}}{2} i_v - \frac{\sqrt{3}}{2} i_w\right). \tag{2}$$

The transformation of space vector is directly derived from the sum of the vectors. Based on the equation above, we put a coefficient 2/3 to $\underline{i_{sum}}$, in order to keep constant magnitude of the vectors during the transformation. The space vector transformation is thus defined as,

$$\begin{gathered} \underline{i_s} = \frac{2}{3}\underline{i_{sum}} = \frac{2}{3}(i_u + a \cdot i_v + a^2 \cdot i_w) = i_s e^{j\theta}, \\ \text{with} \\ a = e^{j\frac{2}{3}\pi}, \text{and } a^2 = e^{j\frac{4}{3}\pi} = e^{-j\frac{2}{3}\pi}, \end{gathered} \tag{3}$$

where i_s is the magnitude and θ the angle of the space vector. Instead of the coefficient, 2/3, other coefficients could also be used, e.g. $\sqrt{2/3}$, for constant power transformation. In this exercise we always use coefficient, 2/3, for constant magnitude transformation.

The above equation represents a transformation from a quantity in three-phase system to a vector in a complex plane. Principally, the space vector is a pure analytical quantity and independent of any physical system. Actually, any three-phase system can be described with space vectors.

However, one can combine this complex plane with the machine model for analysis. For that, the plane will be fixed on the stator, with the real axis coinciding with the *u*-direction of the stator. It must be noted that the complex plane is attached to the magnetic field model, not to the physical rotor.

1.3 Rotation of the space vector

According to the electric machine theory, rotating magnetic fields will be generated in the stator if sinusoidal currents flow in the windings. In this section we will check if this is true using space vector.

The currents in the windings at time, *t*, could be presented with the following equations:

$$\begin{cases} i_u = I_0 \cdot \cos(\omega t + \varphi_0), \\ i_v = I_0 \cdot \cos\left(\omega t + \varphi_0 - \frac{2}{3}\pi\right), \\ i_w = I_0 \cdot \cos\left(\omega t + \varphi_0 + \frac{2}{3}\pi\right), \end{cases} \tag{4}$$

where φ_0 is the initial phase angle, and ω the electrical angular frequency of the currents. Using equation (3) and Euler's formula we get

$$\underline{i_s} = \frac{2}{3}\left(\underline{i_\mathrm{u}} + \underline{i_\mathrm{v}} + \underline{i_\mathrm{w}}\right) = \cdots = \frac{2}{3} I_0(\cos(\omega t + \varphi_0) + \mathrm{j}\sin(\omega t + \varphi_0)) = \frac{2}{3} I_0 e^{j(\omega t + \varphi_0)} \tag{5}$$

This equation means that $\underline{i_s}$ is a vector that has a constant length, $2I_0/3$, and rotates with a constant angular speed, ω, that is identical to the angular frequency of the phase current.

Since the magnetic field is directly generated with currents, it is obvious that the magnetic field in the stator also has a constant magnitude and a constant angular speed, which is equal to ω.

1.4 Graphic illustration of the rotation

The currents defined in (4) can also be illustrated graphically as following, where the initial phase angle is zero.

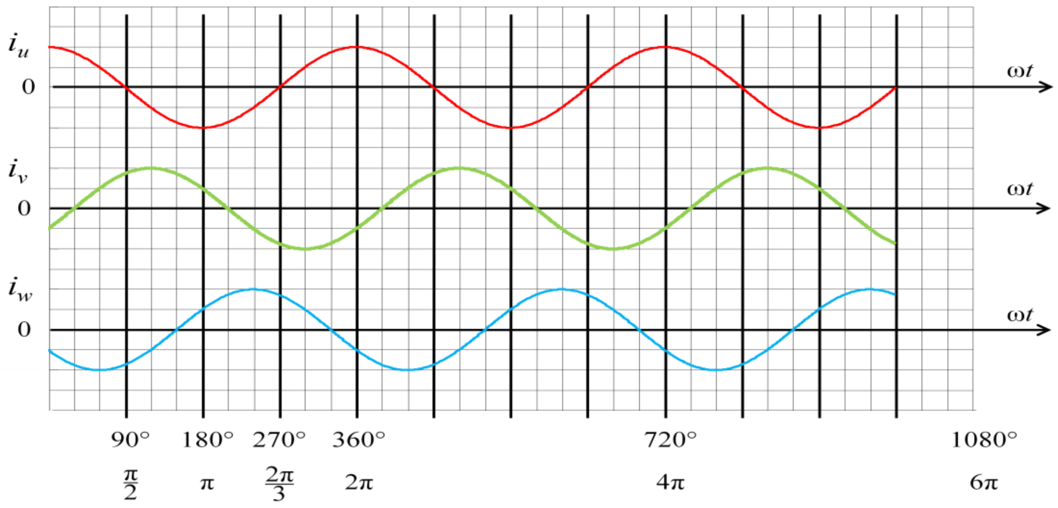

Figure 4. Phase currents in the machine windings

The current space vector is a rotating vector. This is illustrated with the following figures.

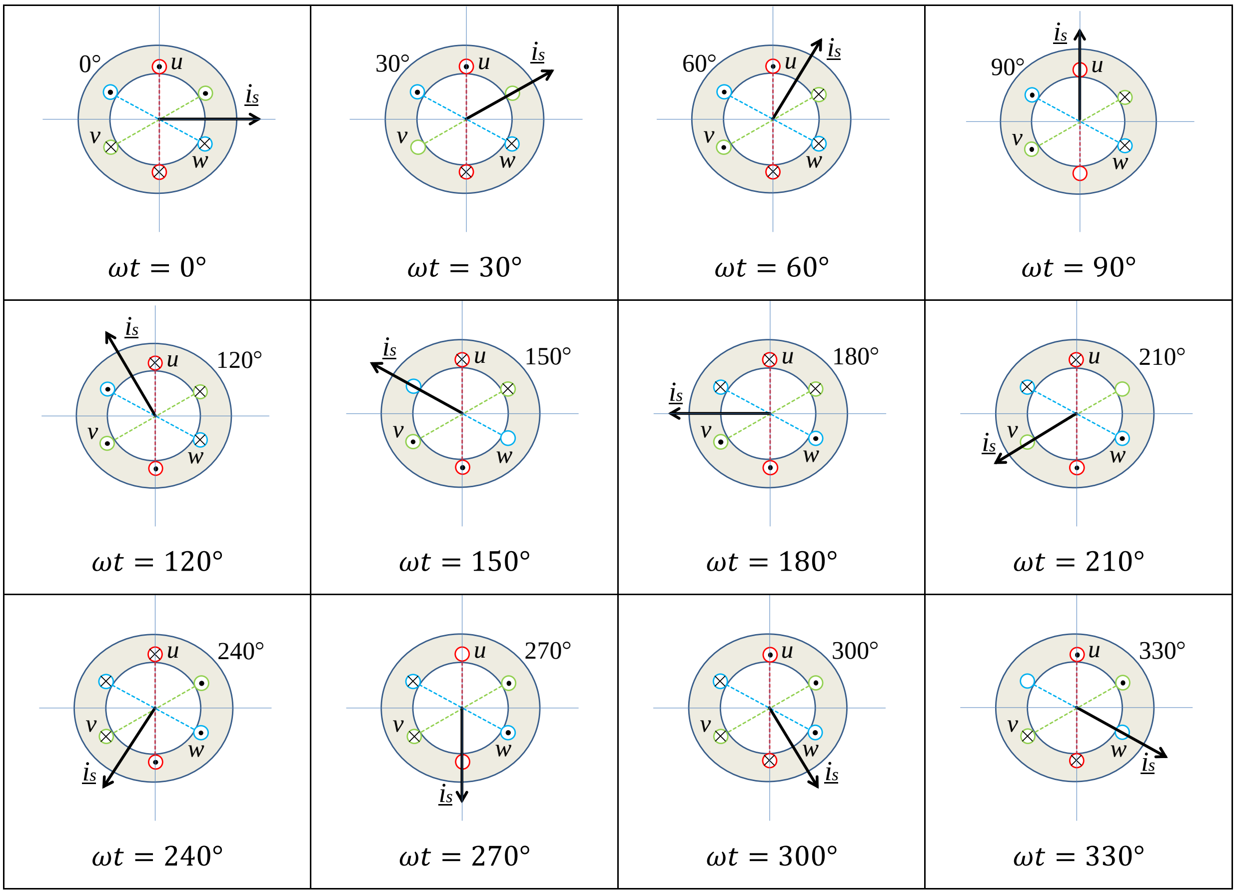

Figure 5. Graphic show of the rotating current space vector

1.5 Space vector for other quantities

Actually, any three-phase quantities could be represented as space vectors. Similar to the currents, the three-phase grid voltage is written in space vector as:

$$\underline{u_s} = \frac{2}{3}(u_u + a \cdot u_v + a^2 \cdot u_w) = u_s e^{j\theta}. \quad (6)$$

Based on this definition, if the sinusoidal three-phase voltage is applied to the machine, the length of the voltage space vector is equal to the peak value of the phase voltages.

Similarly, the magnetic flux also has a space vector expression:

$$\underline{\psi_s} = \frac{2}{3}(\psi_u + a \cdot \psi_v + a^2 \cdot \psi_w) = \psi_s e^{j\theta}. \quad (7)$$

1.6 Clarke transformation

Using space vectors, it is easily to express the three-phase components in a more convenient manner. Here we take the flux as the example. For other quantities the concept is the same.

Based on equation (7) and using Euler's formula, for any given ψ_u, ψ_v and ψ_w, we have

$$\underline{\psi_s} = \psi_s(\cos\theta + j \cdot \sin\theta). \quad (8)$$

The real and imaginary parts can be separated and rewritten as

$$\begin{cases} \psi_\alpha = Re\left(\underline{\psi_s}\right) = \frac{2}{3}\left(\psi_u - \frac{1}{2}\psi_v - \frac{1}{2}\psi_w\right) \\ \psi_\beta = Im\left(\underline{\psi_s}\right) = \frac{2}{3}\left(0 + \frac{\sqrt{3}}{2}\psi_v - \frac{\sqrt{3}}{2}\psi_w\right) \end{cases}. \quad (9)$$

This is named Clarke Transformation.

Using Clarke transformation three-phase components are transformed into a two-dimensional coordinate system. We name it $\alpha\beta$-coordinate system, as shown in the following figure.

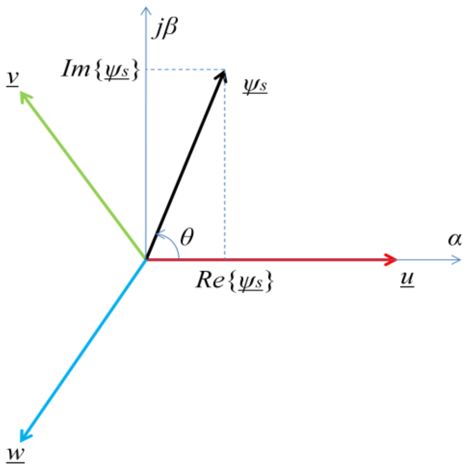

Figure 6. Space vector in $\alpha\beta$-coordinate system

Here the α-axis coincides with u-vector and β-axis is perpendicular to α-axis. For an electric machine, this coordinate system is fixed on the stator. It is, therefore, called stator coordinate system.

Based on these equation (10), a three-phase motor could be converted to a two-phase motor, as shown in the following figure.

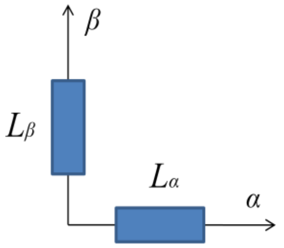

Figure 7. Machine model in two-dimensional stator coordinate system

In general, the Clarke transformation of three-phase quantities, *u*, *v* and *w*, is defined as

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\frac{2}{3}\begin{bmatrix}1 & -1/2 & -1/2\\ 0 & \sqrt{3}/2 & -\sqrt{3}/2\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix}. \qquad (10)$$

And the inverse Clarke transformation is

$$\begin{bmatrix}u\\ v\\ w\end{bmatrix}=\frac{3}{2}\begin{bmatrix}2/3 & 0\\ -1/3 & 1/\sqrt{3}\\ -1/3 & -1/\sqrt{3}\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}. \qquad (11)$$

1.7 Park's transformation

Sometimes it is more convenient to analyze the machine in a rotating coordinate system, as shown in the following figure.

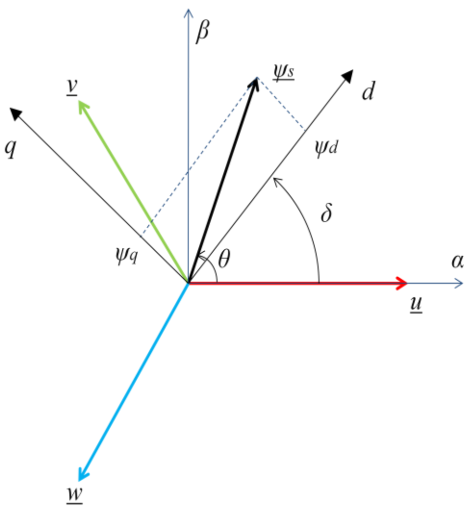

Figure 8. Space vector in dq-coordinate system

Here the *dq*-coordinate system rotates around the common coordinate origin. At a certain time point, the *d*-axis possesses an angle compare to the stationary *u*-vector. *q*-axis is always perpendicular to *d*-axis.

The *dq*-expression of the space vector is a rotation transformation from the result of the Clarke transformation:

$$\begin{cases} \psi_d = \psi_\alpha \, cos\,\delta + \psi_\beta \, sin\,\delta \\ \psi_q = -\psi_\alpha \, sin\,\delta + \psi_\beta \, cos\,\delta \end{cases} . \tag{12}$$

Alternatively, the *d*- and *q*-components of the space vector could be directly derived from the space vector definition. If the *d*-asix has an angle δ compared to the fixed *u*-axis at a certain time, a rotation operator $e^{-j\delta}$ will transform the space vector from the fixed coordinate system in *dq*-system, as

$$\underline{\psi_{s,dq}} = \frac{2}{3}\left(\psi_u + \psi_v e^{\frac{j2\pi}{3}} + \psi_w e^{-\frac{j2\pi}{3}}\right) \cdot e^{-j\delta}. \tag{13}$$

Using Euler's formula, we get

$$\begin{cases} Re\left\{\underline{\psi_{s,dq}}\right\} = \psi_\alpha \cos\delta + \psi_\beta \sin\delta = \psi_d \\ Im\left\{\underline{\psi_{s,dq}}\right\} = -\psi_\alpha \sin\delta + \psi_\beta \cos\delta = \psi_q \end{cases} . \tag{14}$$

This is the same as equation (12).

The transformation of a three-phase quantity in *uvw*-coordinate system to *dq*-system is called Park's Transformation. Written in matrix form, the Park's transformation is defined as

$$\begin{bmatrix} d \\ q \end{bmatrix} = \frac{2}{3}\begin{bmatrix} \cos\delta & \cos(\delta - \frac{2\pi}{3}) & \cos(\delta + \frac{2\pi}{3}) \\ \sin\delta & \sin(\delta - \frac{2\pi}{3}) & \sin(\delta + \frac{2\pi}{3}) \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}. \tag{15}$$

This expression is used to describe the machine in rotating coordinate systems.

2 Exercises

2.1 Exercise 1

Please refer to the theory part of this exercise and use proper equations to solve the following questions.

2.1.1 Question 1

Given the voltages in a three-phase system (120° phase difference between every two voltages): u_1, u_2, and u_3, please derive the expression of the voltage space vector in polar coordinate system.

2.1.2 Question 2

In a three-phase system, at a certain time, the phase currents are

$$i_u = 10\ (A), i_v = -7\ (A), i_w = -3\ (A). \tag{16}$$

Please calculate the space vector at this time point.

2.1.3 Question 3

In a three-phase electric machine, at a certain time, the magnetic fluxes that are generated by the three phase windings are

$$\psi_u = -0.1\ (Wb), \psi_v = -0.2\ (Wb), \psi_w = 0.3\ (Wb). \tag{17}$$

Please

1) draw the three vectors in the three phase coordinate system;

2) calculate the flux space vector at this time point; and

3) draw the flux components in the αβ-coordinate system (Clarke transformation).

2.2 Exercise 2

2.2.1 Problem

A three-phase electric machine with one pole pair as shown in Figure 2 is connected with the three-phase power grid. The windings are connected in Y-topology. If the *w*-phase is disconnected ($i_w = 0$), please derive the equations of the space vectors and describe how the space vectors will look like under the following conditions:

1. The neutral line is connected (supposing the other two currents are not affected), and
2. The neutral line is not connected.

2.2.2 Solution to question 1

The current magnitudes of the three phases are

$$\begin{cases} i_u = I_0 \cdot \cos(\omega t)\,, \\ i_v = I_0 \cdot \cos\left(\omega t - \dfrac{2}{3}\pi\right), \\ i_w = 0. \end{cases}$$

According to the definition, the space vector is

$$\underline{i_s} = \frac{2}{3} I_0 (i_u + a \cdot i_v + a^2 \cdot i_w)$$

$$= \frac{2}{3} I_0 (i_u + a \cdot i_v)$$

$$= \frac{2}{3} I_0 \left(\cos\omega t + e^{\frac{j2\pi}{3}} \cdot \cos\left(\omega t - \frac{2}{3}\pi\right)\right)$$

$$= I_0 \left(\frac{2}{3} \cos \omega t - \frac{1}{3} \cos\left(\omega t - \frac{2}{3}\pi\right) + j\frac{\sqrt{3}}{3} \cos\left(\omega t - \frac{2}{3}\pi\right)\right)$$

$$= i_\alpha + j i_\beta$$

To see the trajectory of the space vector in the complex plane, we draw a series of space vector at some time points. Firstly, we calculate the α- and β-currents at these time points; then draw the current points in the graph.

Table 1. Magnitudes of the α- and β-currents at six time points

ωt	**0**	**π/3**	**2π/3**	**π**	**4π/3**	**5π/3**
i_α **(x I_0)**	5/6	1/6	-2/3	-5/6	-1/6	2/3
i_β **(x I_0)**	-√3/2	√3/2	√3	√3/2	-√3/2	-√3

The space vector trajectory is shown in Figure 9. In the figure, the red segments stand for the six instances of the space vector with the values in Table 1. The blue elliptic is the trajectory of the space vector. And the green circle is the space vector trajectory of the ideal motor current.

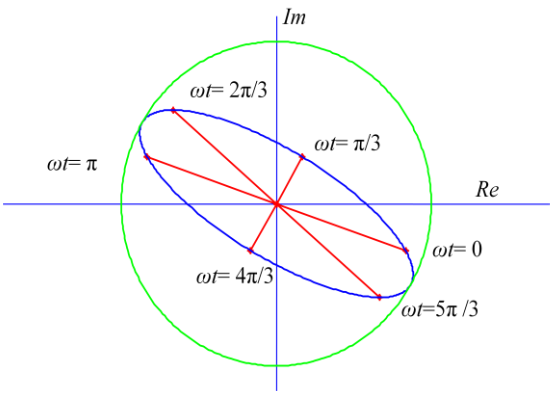

Figure 9. Trajectory of the flux space vector (blue) of the electric machine with a disconnected phase

Another method to draw the trajectory is using vector addition. Firstly, the magnitudes of the *u*- and *v*-currents are calculated and shown in Table 2. Then, for every point, the two vectors are added. After the vector addition, please don't forget to reduce the length of the summary vector by 1/3, according to the space vector definition. Using this method, the same elliptic as shown in the above figure can be obtained, too. This method reduces the effort in calculation but increases more work in drawing.

Table 2. Magnitudes of the *u*- and *v*-currents at six time points

ωt	0	π/3	2π/3	π	4π/3	5π/3
i_u **(x I_0)**	1	1/2	-1/2	-1	-1/2	1/2
i_v **(x I_0)**	-1/2	1/2	1	1/2	-1/2	-1

2.2.3 Solution to question 2

Since the neutral line is not connected, phase *u* and *v* are actually connected in series. Therefore they have the same value and different directions. The current magnitudes of the three phases are

$$\begin{cases} i_u = I_0 \cdot \cos(\omega t), \\ i_v = -i_u = -I_0 \cdot \cos(\omega t), \\ i_w = 0. \end{cases}$$

According to the definition, the space vector is

$$\underline{i_s} = \frac{2}{3} I_0 (i_u + a \cdot i_v + a^2 \cdot i_w)$$

$$= \frac{2}{3} I_0 (i_u - a \cdot i_u)$$

$$= \frac{2}{3} I_0 \cos\omega t \left(1 - e^{\frac{j2\pi}{3}}\right)$$

$$= \frac{2}{3} I_0 \cos\omega t \left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$$

$$= \frac{2}{3} I_0 \cos\omega t \cdot e^{-j\frac{1}{3}\pi}$$

Thus, this is a vector with variable length and fixed angle of $\frac{1}{3}\pi$. The length has the maximum value at $\omega t = 0$ and π. The maximum length is $2I_0/3$, smaller that the ideal space vector. The trajectories of the space vector with a broken phase and that of the ideal space vector are shown in the following figure.

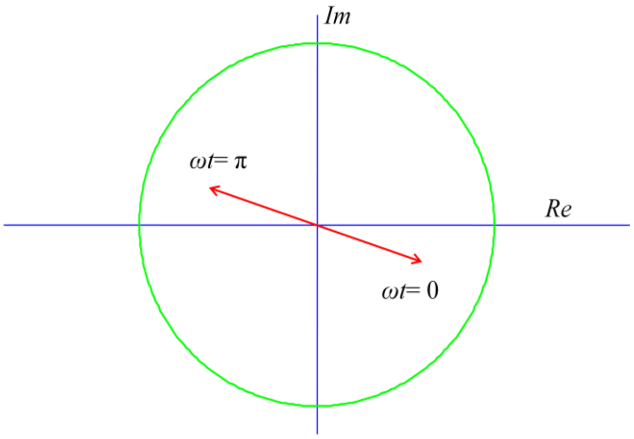

Figure 10. Trajectory of the flux space vector (red) of the electric machine with a disconnected phase (neutral line not connected)

2.3 Exercise 3

2.3.1 Problem

A two-phase electric machine has two windings as shown in the following figure. They are perpendicular to each other in the space. If phase 1 has the current $i_1 = \cos(\omega t)$ and phase 2 has the current $i_2 = \sin(\omega t)$, please

1. use the concept of space vector, transform the two currents into the complex plane (space vector of the two-phase system),

2. describe the trajectory of the space vector, and
3. consider the advantages and disadvantages of the two-phase machine compared with a three-phase machine.

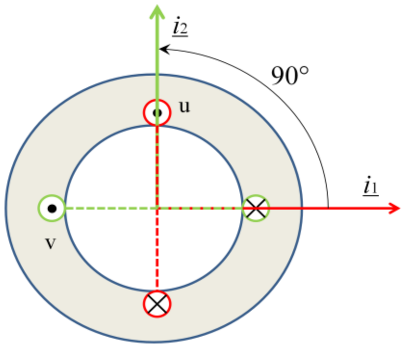

Figure 11. Stator windings of a two-phase machine

2.3.2 Solution of question 1

Space vector is actually defined for three-phase quantities. It is possible to use the same idea to analysis the two-phase system.

$$\underline{i_{12}} = i_1 + i_2 e^{j\pi/2}$$

$$= \cos\omega t + \sin\omega t(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2})$$

$$= \cos\omega t + \sin\omega t(0 + j)$$

$$= e^{j\omega t}$$

2.3.3 Solution of question 2

This result means that the two-phase electric machine also generate a rotating current vector with constant speed and constant length. The machine thus has a constant rotating magnetic field.

2.3.4 Solution of question 3

In the early 20th century two-phase system was used. Compared to three-phase system, two-phase system is simple. For transferring two-phase power, four or three wires are to be used. If three wires are used, the common wire carries more current than the other two and thus must be thicker. Furthermore, power transfer in a three-phase system with balanced load is nearly constant. These reasons make three-phase system more advantageous than two-phase system.

2.4 Exercise 4

This exercise is beyond the basic requirement of the lecture. It only serves as extended materials to help understanding the space vector concept.

2.4.1 Problem

A machine with two pole pairs is shown in the following figure. Each phase has two windings, which are connected in series. The windings of the three phases are connected with “Y” topology. The definition of the positive directions of currents and magnetic flux are given in the figure. Both voltage and current of the machine are in sinusoidal form. The currents in the windings are defined in equation (4).

1. Please derive the space vectors of current and flux.
2. Please identify the rotating speed of the magnetic field in physical space.
3. Please analyze a machine with three pole pairs and answer the above two questions.

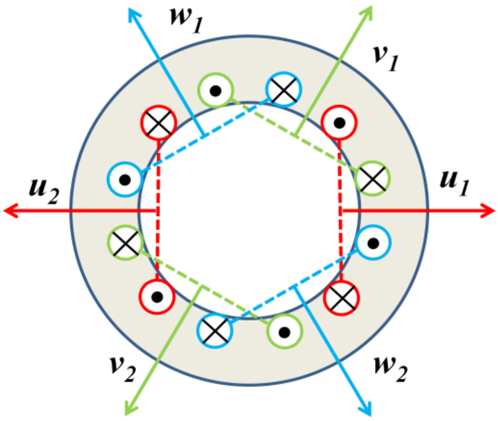

Figure 12. Windings in a machine with two pole pairs

2.4.2 Solution of question 1

Use the definition of space vector and equation (3), the current space vector is

$$\underline{i_s} = \frac{2}{3}(i_u + a \cdot i_v + a^2 \cdot i_w)$$

$$= \frac{2}{3}\left(I_0 \cos\ \omega t + I_0 e^{j\frac{2}{3}\pi} \cos\left(\omega t - \frac{2}{3}\pi\right) + I_0 e^{-j\frac{2}{3}\pi} \cos\left(\omega t + \frac{2}{3}\pi\right)\right)$$

$$= \frac{2}{3} I_0 \left(\cos \omega t + \left(\cos\left(\frac{2}{3}\pi\right) + j \sin\left(\frac{2}{3}\pi\right)\right)\cos\left(\omega t - \frac{2}{3}\pi\right) + \left(\cos\left(-\frac{2}{3}\pi\right) + j \sin\left(-\frac{2}{3}\pi\right)\right)\cos\left(\omega t + \frac{2}{3}\pi\right)\right)$$

$$= I_0(\cos \omega t + j\sin\omega t)$$

$$= I_0 e^{j\omega t}$$

It is same process for the flux. The result is

$$\underline{\psi_s} = \psi_0(\cos\omega t + j\sin\omega t)$$

$$= \psi_0 e^{j\omega t}$$

This result shows that the space vectors of current and flux rotate with constant speed ω and constant magnitude around the originate of the complex plane.

2.4.3 Solution of question 2

From the solution of question 1, it is known that the space vector of flux rotates with angular speed ω in the complex plane, as shown in Figure 13 (left). The complex plane is attached to the machine magnetic model in the way shown in Figure 13 (right).

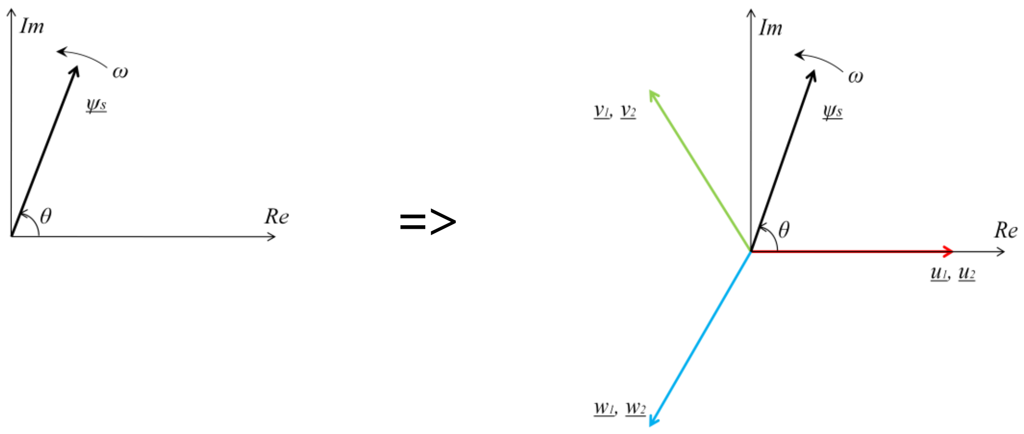

Figure 13. Flux space vector of the electric machine with two pole pairs
(left) in the complex plane and
(right) with the combination of the physical machine model

It should be noted that the angle between u_1 and u_2 in the mechanical model (Figure 12) of 60° is transformed to 120° in the magnetic model shown in Figure 13 (right).

This means, one revolution of the space vector in the magnetic model corresponds to only a half revolution of the magnetic field in the mechanical model. Since the space vector $\underline{\psi_s}$ rotates with speed ω, the actual magnetic field in the machine rotates with $\omega/2$.

More explanation:

Observing the mechanical model of the machine, at t=0, $i_u = +I_0, i_v = i_w = -I_0/2$. These current directions are drawn in the following figure. The magnetic flux and its direction are also illustrated.

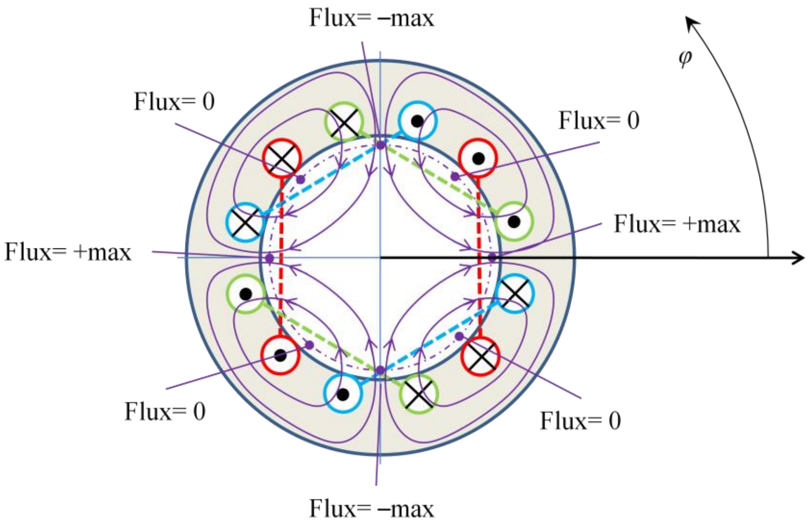

Figure 14. Magnetic flux of a electric machine stator with two pole pairs at time = 0

Starting from the mechanical angle of the stator circumstance $\varphi = 0$ to $\varphi = \pi$, the flux changes in the sequence +max => zero => –max => zero => +max, having completed one cycle. From $\varphi = \pi$ to $\varphi = 2\pi$, the flux changes through a complete cycle again. The flux value dependent on φ can be illustrated with Figure 15.

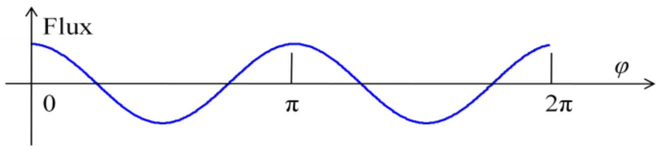

Figure 15. The change of magnetic flux in an electric machine with two pole pairs

For one electric cycle of the machine currents, the flux vector also rotates for a complete cycle. This corresponds to only a half circle in the real machine. Thus, the speed of the magnetic field in the stator rotates with a half angular speed of the electric currents.

2.4.4 Solution of question 3

The equations of space vectors for current and flux are the same as in question 1 since the space vector transformation is not dependent on the number of pole pairs. The rotation speed of the space vector is equal to the electrical angular frequency of the current, ω.

For identifying the magnetic field speed, the simplified machine model (mechanical) is shown in Figure 16. And the space vector is shown in Figure 17.

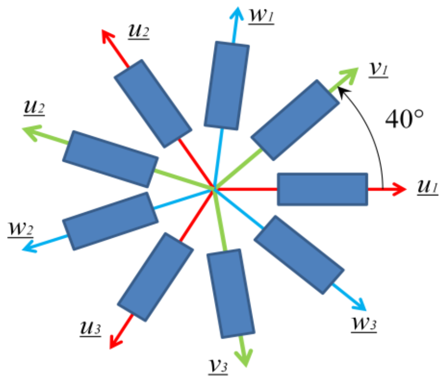

Figure 16. Simplified model of a electric machine with three pole pairs

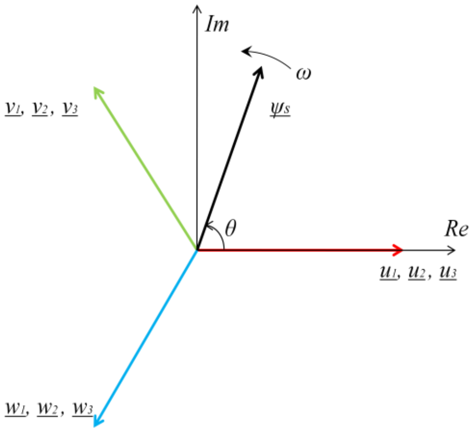

Figure 17. Flux space vector of the electric machine with three pole pairs

It is noted that the angle between u_1 and u_2 (also u_2 and u_3) in the mechanical model (Figure 16) of 40° is transformed to 120° in the magnetic model shown in Figure 17.

This means, one revolution of the space vector in the magnetic model corresponds to only a third revolution of the magnetic field in the mechanical model. Since the space vector $\underline{\psi_s}$ rotates with speed ω, the actual magnetic field in the machine rotates with $\omega/3$.

3 Reference

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