(1)

$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$ . X-tics equation

$$s_1, s_2 = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4(1)(\omega_n^2)}}{2}$$

$$s_1, s_2 = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

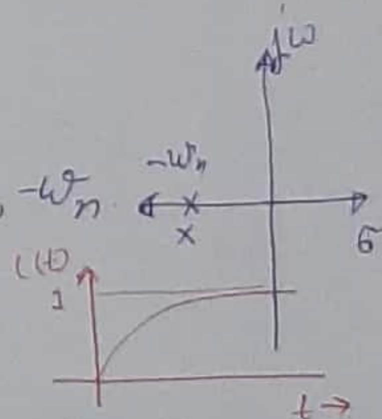
$$\boxed{s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}.}$$

Case 1: $\zeta = 1$

$s_1, s_2 = -\omega_n \pm \omega_n(0) = -\omega_n, -\omega_n$

1) poles = equal & real
2) Critically damped
3) lie on -ve σ-axis

Two equal real roots

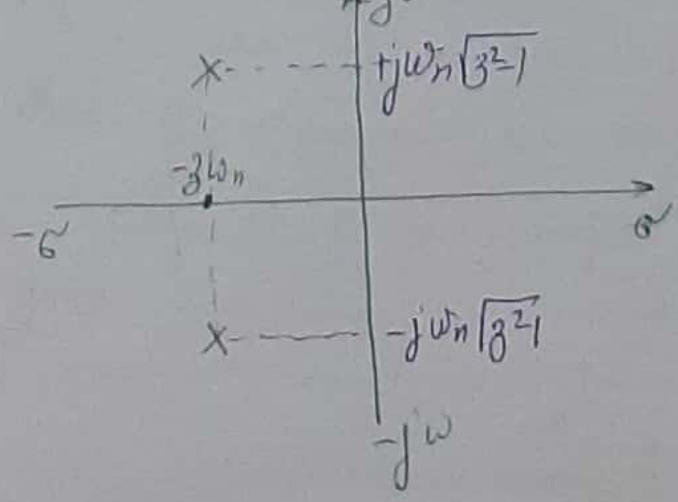


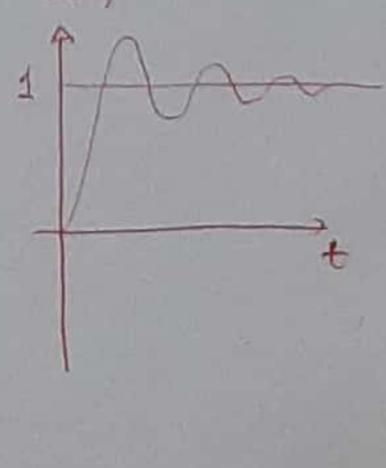

Case 2: $0 < \zeta \leq 1$

$s_1, s_2 = \zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$ <1.

$\Rightarrow s_1, s_2 = -\zeta\omega_n \pm j\,\omega_n\sqrt{\zeta^2-1}$

- Poles are complex conjugates
- Poles are located in 2nd & 3rd Quadrant
- Under damped response



Case 5:

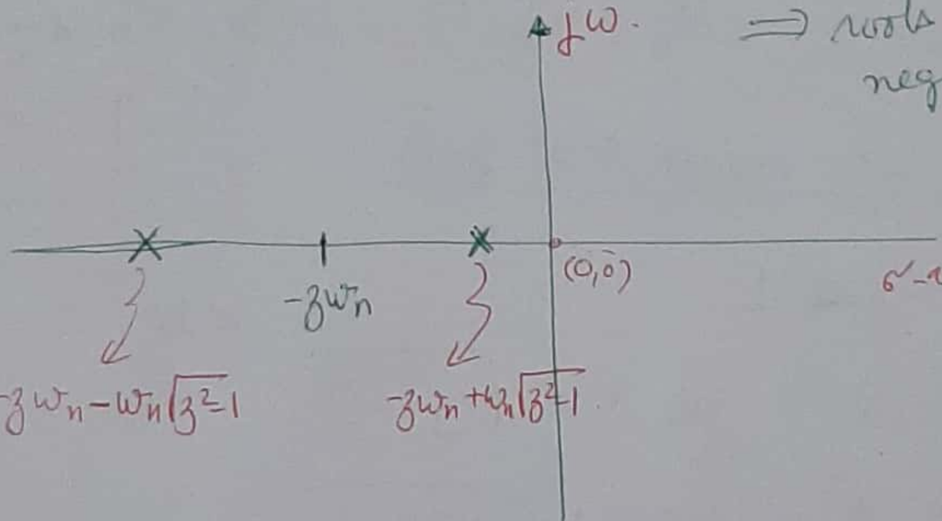
## Case 3: $\zeta > 1$

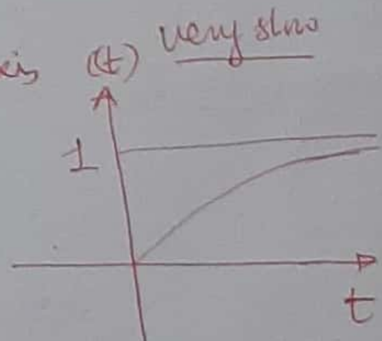
$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

- Poles are real & unequal
- no imaginary term ($\zeta > 1$), so poles line on the $\sigma$-axis
- Overdamped

$\sqrt{\zeta^2 - 1} < \zeta$

$\Rightarrow$ roots are always negative $\sigma$-axis

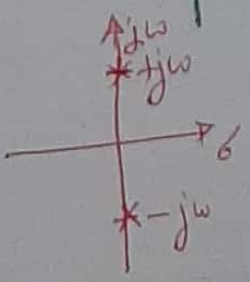


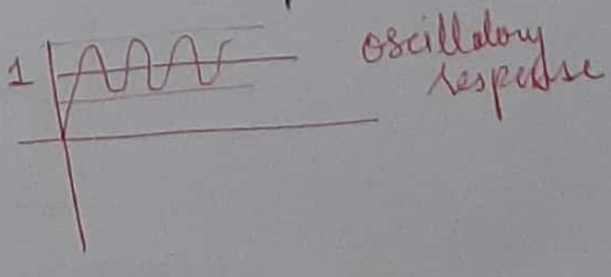




## Case 4: $\zeta = 0$

$$s_1, s_2 = \pm\omega_n\sqrt{-1} = \pm j\omega_n$$

- poles are complex with only imaginary part
- poles lie on $j\omega$-axis
- poles are conjugate of each other
- Response is undamped.

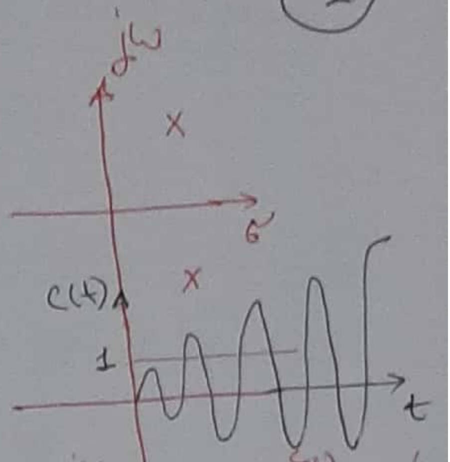


## Case 5: $-1 \le \zeta < 0$

$$s_1, s_2 = \zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

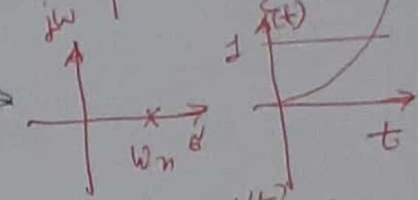
- Poles lie in Right half plane
- Unstable.




## Case 6: $\zeta = -1$

$$s_1, s_2 = +\omega_n$$

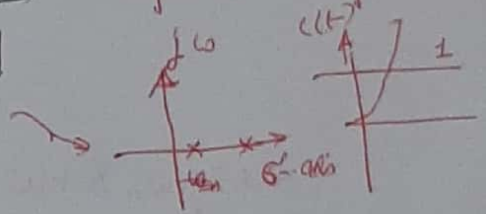
- RHP ⇒ unstable



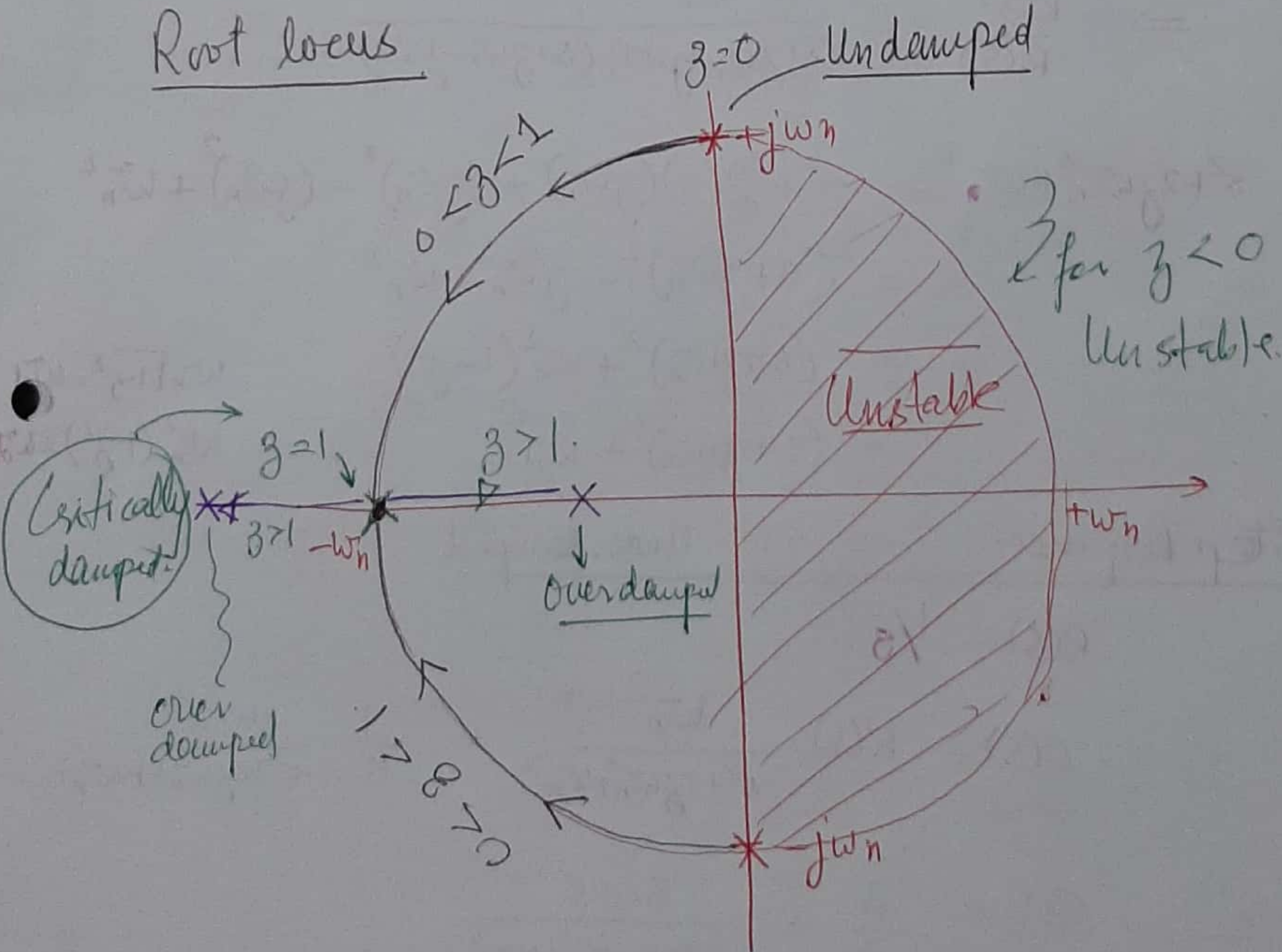

## Case 7: $\zeta < -1$

$$s_1, s_2 = \omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

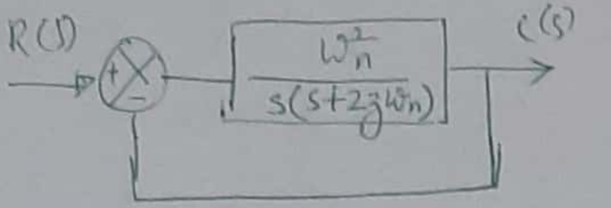
RHP ⇒ Unstable.




## Root locus



## Second Order System




$$\frac{C(s)}{R(s)} = \frac{w_n^2}{s^2+2\zeta w_n s + w_n^2}$$

root of Xtics equation: $s_1, s_2 = -\zeta w_n \pm j w_n\sqrt{\zeta^2-1}$

let $w_d = w_n\sqrt{1-\zeta^2}$ = Damped natural frequency

$$\Rightarrow \quad s_1, s_2 = -\zeta w_n \pm j\, w_n\sqrt{1-\zeta^2}$$

$$\boxed{s_1, s_2 = -\zeta w_n \pm j\, w_d} \qquad 0<\zeta<1$$

$$s^2+2\zeta w_n s + w_n^2 = (s+\zeta w_n + j w_d)(s+\zeta w_n - j w_d)$$

$$\Rightarrow \quad \frac{C(s)}{R(s)} = \frac{w_n^2}{(s+\zeta w_n + j w_d)(s+\zeta w_n - j w_d)}$$

$$s^2+2\zeta w_n s + w_n^2 = s^2 + 2(s)(\zeta w_n) + (\zeta w_n)^2 - (\zeta w_n)^2 + w_n^2$$

$$= (s+\zeta w_n)^2 - \zeta^2 w_n^2 + w_n^2$$

$$= (s+\zeta w_n)^2 + w_n^2(1-\zeta^2)$$

$$= (s+\zeta w_n)^2 + w_d^2$$

$w_n\sqrt{1-\zeta^2} = w_d$

$w_n^2(1-\zeta^2) = w_d^2$

### Step Response: $(0<\zeta<1)$ Underdamped

$$R(s) = 1/s$$

$$C(s) = R(s)\,\frac{w_n^2}{s^2+2\zeta w_n s + w_n^2} = \frac{1}{s}\cdot\frac{w_n^2}{s^2+2\zeta w_n s + w_n^2}$$

$$\frac{w_n^2}{s(s^2+2\zeta w_n s + w_n^2)} = C(s) = \frac{A}{s} + \frac{Bs+C}{s^2+2\zeta w_n s + w_n^2}$$

$$\omega_n^2 = A(s^2+2\zeta\omega_n s+\omega_n^2) + (Bs+C)s$$

$$\omega_n^2 = A(s^2+2\zeta\omega_n s+\omega_n^2) + (Bs^2+Cs)$$

$s^2 \Rightarrow 0s^2 = As^2+Bs^2 \Rightarrow A+B=0$ — ①

$s \Rightarrow 0s = A(2\zeta\omega_n)s + Cs \Rightarrow (2\zeta\omega_n A + C) = 0$ — ②

let $s=0 \Rightarrow \omega_n^2 = A\omega_n^2 \Rightarrow \boxed{A=1}$

from 1 $\Rightarrow \boxed{B=-1}$

from 2 $\Rightarrow +2\zeta\omega_n + C = 0 \Rightarrow \boxed{C=-2\zeta\omega_n}$

so

$$\frac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)s} = \frac{1}{s} - \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\Rightarrow \frac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)s} = \frac{1}{s} - \frac{s+\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2} - \frac{\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta}{\sqrt{1-\zeta^2}}\cdot\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$$

From Laplace tables:

$$\mathcal{L}^{-1}\left\{\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}\right\} = e^{-\zeta\omega_n t}.\cos\omega_d t$$

$$\mathcal{L}\left\{\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right\} = e^{-\zeta\omega_n t}, \sin\omega_d t$$

$$\Rightarrow \mathcal{L}^{-1}\{C(s)\} = 1 - e^{-\zeta\omega_n t}\cos(\omega_d t) - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}.\sin(\omega_d t)$$

Step

$$c(t) = 1 - e^{-\zeta \omega_n t}\left[\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right]$$

$$c(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\,\cos(\omega_d t) + \zeta \sin(\omega_d t)\right]$$

Let $\zeta = \cos\alpha$

$$\cos^2\alpha + \sin^2\alpha = 1$$

$$\sin^2\alpha = 1 - \cos^2\alpha = 1 - \zeta^2$$

$$\sin\alpha = \sqrt{1-\zeta^2}$$

$$\frac{\sin\alpha}{\cos\alpha} = \tan\alpha$$

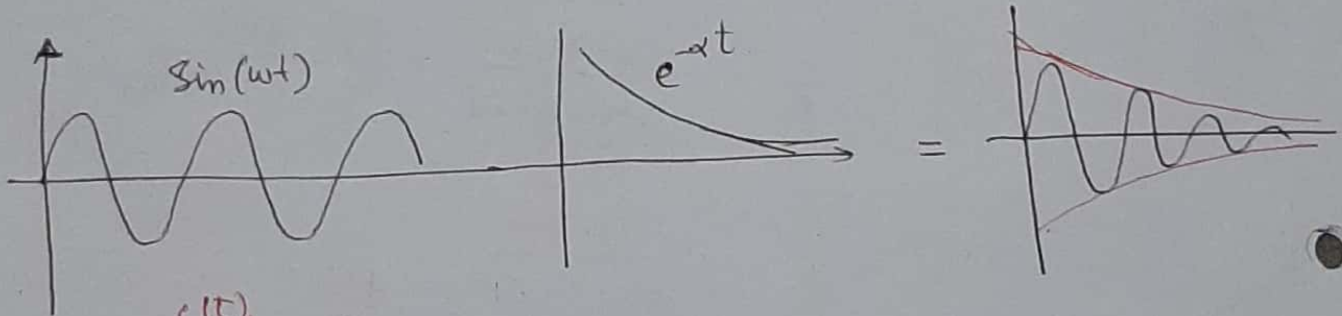
$$\frac{\sqrt{1-\zeta^2}}{\zeta} = \tan\alpha$$

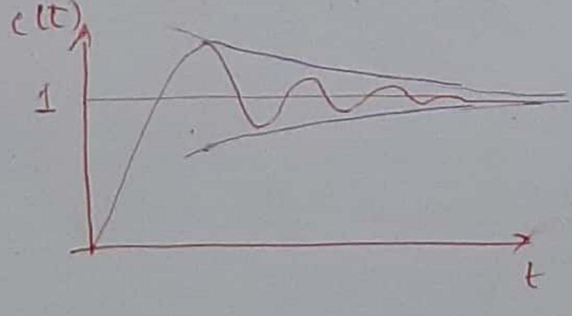
$$\Rightarrow \alpha = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$\Rightarrow c(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\alpha \cos(\omega_d t) + \cos\alpha \sin(\omega_d t)\right]$$

$$\Rightarrow \boxed{c(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \alpha)}$$

where $\alpha = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$



## Step Response : ($\zeta = 1$) : Critically damped

$$R(s) = \frac{1}{s} \Rightarrow C(s) = \frac{\omega_n^2}{s(s^2 + 2\omega_n s + (\omega_n)^2)}$$

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)} + \frac{C}{(s+\omega_n)^2}$$

$$\omega_n^2 = A(s+\omega_n)^2 + Bs(s+\omega_n) + Cs$$

$$\omega_n^2 = A(s^2 + 2\omega_n s + \omega_n^2) + B(s^2 + \omega_n s) + C(s)$$

let $s = 0 \Rightarrow \omega_n^2 = A\omega_n^2 \Rightarrow \boxed{A = 1}$

Comparing $s^2$ terms $\Rightarrow A + B = 0 \Rightarrow \boxed{B = -1}$

Comparing $s$ terms $\Rightarrow 2\omega_n A + \omega_n B + C = 0$

$$2\omega_n - \omega_n + C = 0$$

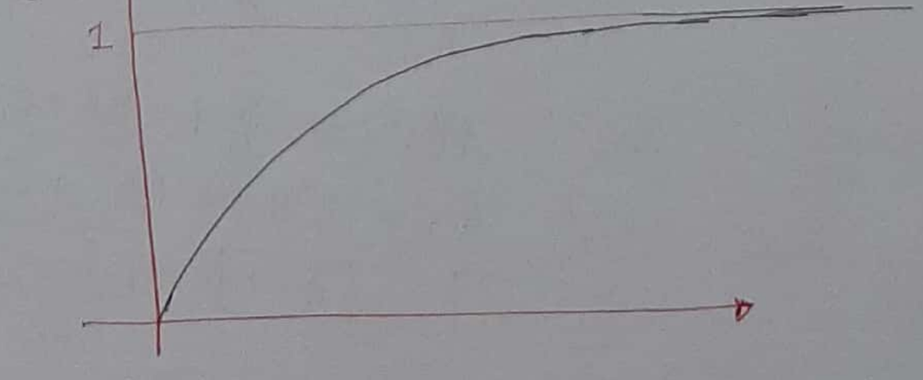
$$\Rightarrow \boxed{C = -\omega_n}$$

$$\Rightarrow \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{1}{s} - \frac{1}{(s+\omega_n)} - \frac{\omega_n}{(s+\omega_n)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s(s+\omega_n)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}\right\}$$

$$= 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$$

$$\boxed{c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)}$$



## Step Response: ($\zeta > 1$) : Over damped

$$C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

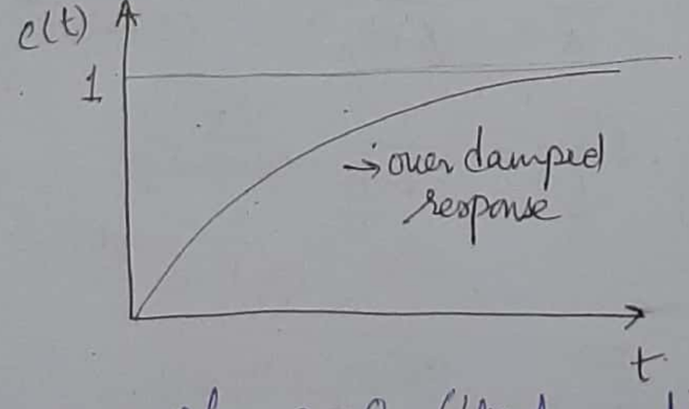
$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1} \quad (\text{for } \zeta > 1)$$

$$\Rightarrow C(s) = \frac{\omega_n^2}{\left(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)\left(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right) s}$$

$$\Rightarrow C(s) = \frac{A}{s} + \frac{B}{s + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}} + \frac{C}{s + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}}$$

$$\Rightarrow c(t) = 1 - \left[B e^{-(\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})t} + C e^{-(\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})}\right]$$

B & C can be evaluated




## Step Response when $\zeta = 0$ (Undamped)

$$C(s) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + \omega_n^2}$$

~~C(s) =~~

$$\omega_n^2 = A(s^2 + \omega_n^2) + (Bs + C)s$$

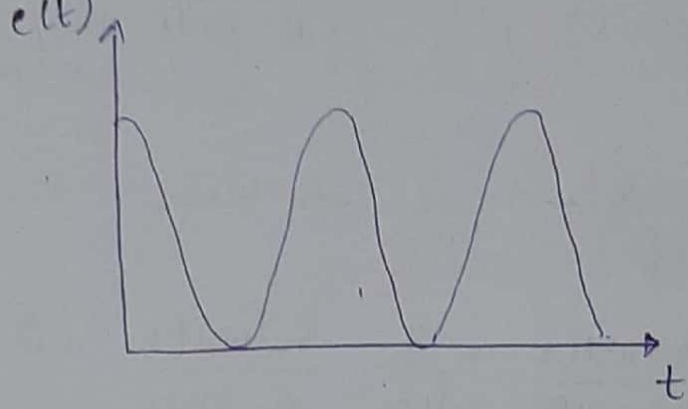
$$\omega_n^2 = As^2 + \omega_n^2 A + Bs^2 + Cs$$

$$\boxed{C = 0} \quad \boxed{A = 1} \quad \boxed{B = -1}$$

$$\frac{\omega_n^2}{s(s^2+\omega_n^2)} = \frac{1}{s} - \frac{s}{s^2+\omega_n^2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\omega_n^2)}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+\omega_n^2}\right\}$$

$$\boxed{c(t) = 1 - \cos\omega_n t}$$



## Transient response specifications

① **Delay time**: It is the time required ~~for~~ response to reach 50% of its final value the very first time.

i.e. $c(t) = \frac{1}{2}$ at $t = T_d$

we know that

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin(\omega_d t + \alpha)\right]$$

$$\frac{1}{2} = 1 - \frac{e^{-\zeta\omega_n T_d}}{\sqrt{1-\zeta^2}}\sin(\omega_d T + \alpha)$$

After simplification we get $\boxed{T_d = \frac{1+0.7\zeta}{\omega_n}}$

② **Rise Time ($t_r$) or $T_r$**

It is the time required for the response to rise from 10% to 90%, 5% to 95% or 0% to 100% of its final value. For underdamped second order system, the 0% to 100% of its final value is normally used. For ~~over~~ overdamped systems its is take 10% to 90%.

Consider the under damped response

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

$c(t) = 1$ @ $t = T_r$ or $t = t_r$

$$\Rightarrow e^{-\zeta\omega_n t_r}\left[\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right] = 0$$

$e^{-\zeta\omega_n t_r} \neq 0$ [It will only be zero for $t_r = \infty$]

$$\Rightarrow \cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r = 0$$

$$\frac{\cos\omega_d t_r}{\cos\omega_d t_r} + \frac{\zeta}{\sqrt{1-\zeta^2}}\,\frac{\sin\omega_d t_r}{\cos\omega_d t_r} = 0$$

$$\Rightarrow 1 + \frac{\zeta}{\sqrt{1-\zeta^2}}\tan\omega_d t_r = 0.$$

$$\Rightarrow \tan\omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\Rightarrow \omega_d t_r = \tan^{-1}\left(\frac{-\sqrt{1-\zeta^2}}{\zeta}\right)$$

$\sin(-\alpha) = -\sin\alpha$

$\cos(-\alpha) = \cos\alpha$

$\Rightarrow \tan(\alpha) = -\tan\alpha$

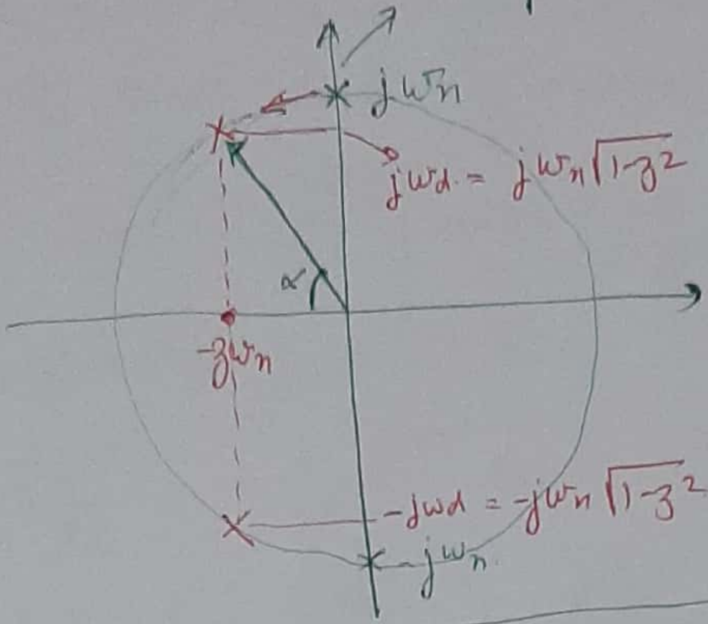
& $\tan(\pi-\alpha) = -\tan\alpha$

$$\Rightarrow t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{-\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$\Rightarrow t_r = -\frac{1}{\omega_d}\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$\Rightarrow \boxed{t_r = \frac{\pi-\alpha}{\omega_d}}$$

# Important Insight




$$\text{radius of circle} = \sqrt{(-\zeta\omega_n)^2 + (\omega_n\sqrt{1-\zeta^2})^2}$$

$$= \sqrt{\zeta^2\omega_n^2 + \omega_n^2(1-\zeta^2)}$$

$$= \sqrt{\zeta^2\omega_n^2 + \omega_n^2 - \omega_n^2\zeta^2} = \omega_n$$

$$\boxed{\text{radius of circle} = \omega_n}$$

$$\text{angle} = \alpha = \tan^{-1}\frac{\omega_n\sqrt{1-\zeta^2}}{+\zeta\omega_n} = \tan^{-1}\left(+\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

~~using trignometric property $\Rightarrow \tan(-\alpha) = -\tan\alpha$~~

~~$\Rightarrow -\alpha = -\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$~~

~~Using trignometric property $\Rightarrow \tan(\pi-\alpha) = -\tan\alpha$~~

~~$\pi - \alpha =$~~

$$\alpha = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$\tan\alpha = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$-\tan\alpha = -\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\tan(\pi-\alpha) = -\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\pi - \alpha = \tan^{-1}\left(-\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$\boxed{\pi - \alpha = -\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)}$$

## Peak Time ($t_p$)

It is the time required for the response to reach the first peak of overshoot.

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

Differentiating w.r.t time.

$$\frac{dc(t)}{dt} = -\frac{d}{dt}e^{-\zeta\omega_n t}\cos\omega_d t - \frac{d}{dt}\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t \,.\, e^{-\zeta\omega_n t}$$

$$\frac{dc(t)}{dt} = -\left[\zeta\omega_n e^{-\zeta\omega_n t}\,.\cos\omega_d t + (-\sin\omega_d t).\omega_d\, e^{-\zeta\omega_n t}\right]$$

$$-\frac{\zeta}{\sqrt{1-\zeta^2}}\left[+\cos\omega_d t \,.\, \omega_d \,.\, e^{-\zeta\omega_n t} + e^{-\zeta\omega_n t}(-\zeta\omega_n)\sin(\omega_d t)\right]$$

$$\frac{dc(t)}{dt} = e^{-\zeta\omega_n t}\left[\underbrace{\zeta\omega_n \cos\omega_d t}_{1} + \omega_d.\sin(\omega_d t)\right]$$

$$-\frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t \,.\, e^{-\zeta\omega_n t} + \underbrace{\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t)}_{2}$$

$$\frac{dc(t)}{dt} = \zeta\omega_n e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

$$+ e^{-\zeta\omega_n t}\left[\omega_d\sin\omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t\right]$$

$$\frac{dc(t)}{dt} = \cancel{\zeta\omega_n e^{-\zeta\omega_n t}}\left[\cancel{\cos\omega_d t} + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

$\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$+ e^{-\zeta\omega_n t}\left[\omega_d\sin\omega_d t - \cancel{\zeta\omega_n\cos\omega_d t}\right]$$

$$\frac{dc(t)}{dt} = e^{-\zeta\omega_n t}\sin\omega_d t\left[\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}} + \omega_d\right]$$

$$\frac{dc(t)}{dt} = e^{-\zeta\omega_n t}\sin\omega_d t \,.\, \frac{\omega_n}{\sqrt{1-\zeta^2}}$$

taking the derivative at time $t = t_p$ and setting equal to zero.

$$\left.\frac{dc(t)}{dt}\right|_{t=t_p} = e^{-\zeta\omega_n t_p} \sin \omega_d t_p \cdot \frac{\omega_n}{\sqrt{1-\zeta^2}} = 0$$

$$\Rightarrow \quad \sin(\omega_d t_p) = 0 \Rightarrow \omega_d t_p = n\pi$$

$$\Rightarrow \quad \omega_d t_p = 0, \pi, 2\pi, 3\pi, \ldots$$

as $t_p$ corresponds to the first peak of over shoot

$$\Rightarrow \quad \omega_d t_p = \pi$$

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

$$\boxed{t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$$

## Maximum Overshoot ($M_p$)

- It is the maximum peak value of the response curve measured from unity.
- Max. peak occurs at $t = t_p$

$$\Rightarrow \quad M_p = c(t_p) - 1$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t\right]$$

$$c(t_p) = 1 - e^{-\zeta\omega_n \frac{\pi}{\omega_d}}\left[\cos \omega_d\left(\frac{\pi}{\omega_d}\right) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d \frac{\pi}{\omega_d}\right]$$

$$c(t_p) = 1 - e^{-\zeta\frac{\omega_n\pi}{\omega_d}}\left[\cos \pi + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \pi\right]$$

$$c(t_p) = 1 + e^{-\zeta\frac{\omega_n\pi}{\omega_d}}$$

$$M_p = c(t_p) - 1$$

$$\Rightarrow M_p = e^{-\frac{\zeta \omega_n \pi}{\omega_d}}$$

$$M_p = e^{-\frac{\zeta \omega_n \pi}{\omega_n \sqrt{1-\zeta^2}}}$$

$$\boxed{M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}} \pi}}$$

$$\% M_p = e^{\frac{-\zeta}{\sqrt{1-\zeta^2}} \pi} \times 100$$

$$\ln\left[\frac{\% M_P}{100}\right] = \frac{-\zeta}{\sqrt{1-\zeta^2}} \cdot \pi$$

$$\Rightarrow \boxed{\zeta = \frac{-\ln(\% M_P/100)}{\sqrt{\pi^2 + \ln^2(\% M_P/100)}}}$$

## Settling Time ($T_s$)

It is the time required by response curve to reach and stay with in a range about the final value of size specified by absolute % of final value (usually 2% or 5%)

- This response equation on back side of page 3 is

$$c(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \alpha)$$

- Oscillations decay according to exponential decaying function, and from first order system study we know the exponential decays to the range of 2% of steadily state value in $4\tau$.

Time constant $\tau = \frac{1}{\zeta \omega_n}$

$$T_s = 4\tau$$

$$\boxed{T_s = \frac{4}{\zeta \omega_n}}$$

For the range of 5%

$$T_s = 3\tau$$

$$\boxed{T_s = \frac{3}{\zeta \omega_n}}$$

Summary:

Delay time $= T_d = \frac{1+0.7\zeta}{\omega_n}$

Rise time $= T_r = \frac{\pi - \alpha}{\omega_d}$ where $\tan\alpha = \frac{\sqrt{1-\zeta^2}}{\zeta}$

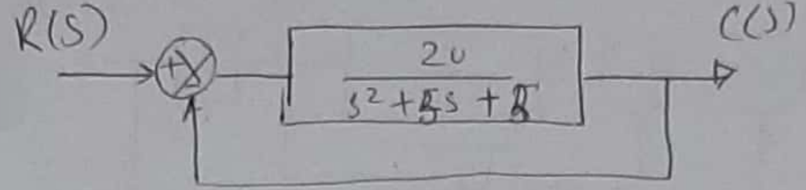
Peak time $= T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

% overshoot $= \% M_p = e^{\left(\frac{-\zeta}{\sqrt{1-\zeta^2}}\right)\pi} \times 100\%$

Settling Time $= T_s = \frac{4}{\zeta\omega_n}$ (2% criterion)

$T_s = \frac{3}{\zeta\omega_n}$ (5% criterion)

## Example:

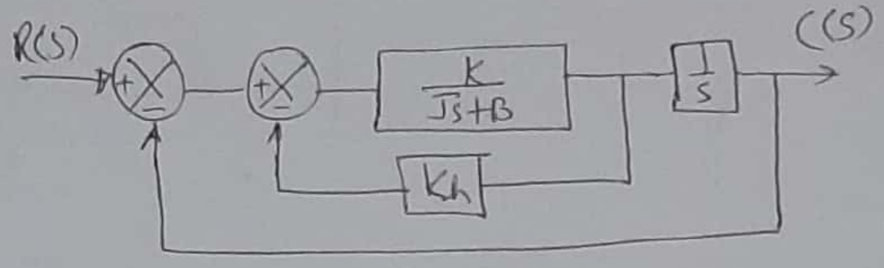


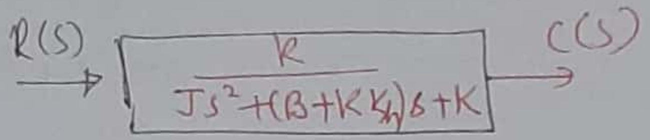

$$\frac{C(S)}{R(S)} = \frac{\dfrac{20}{s^2+5s+5}}{1+\dfrac{20}{s^2+5s+5}} = \frac{20}{s^2+5s+25}$$

1. $\omega_n^2 = 25 \Rightarrow \omega_n = 5$
2. $2\zeta\omega_n = 5 \Rightarrow \zeta = 0.5$
3. $\omega_d = \omega_n\sqrt{1-\zeta^2} \Rightarrow \omega_d = 4.33$ rad/sec
4. $T_d = \dfrac{1+0.7\zeta}{\omega_n} = 0.27$ sec
5. $T_r = \dfrac{\pi-\alpha}{\omega_d} = \dfrac{\pi-\alpha}{4.33}$ $\qquad \alpha = \tan^{-1}\left(\dfrac{\sqrt{1-\zeta^2}}{\zeta}\right) = 1.0467$ rad

   $T_r = 0.4834$
6. $T_p = \dfrac{\pi}{\omega_d} = 0.725$ sec
7. $M_p = 100\, e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = 16.32\%$
8. $T_s = \dfrac{4}{\zeta\omega_n} = 1.6$ sec.

## Example:




Determine K & Kh such that M = 0.2 & Tp = 1 sec

$J = 1\ kg\text{-}m^2 \quad B = 1\ N\text{-}m/rad/sec$

Sol:

$$M_p = e^{\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)} = 0.2 \Rightarrow \zeta = 0.456$$

$$t_p = \frac{\pi}{\omega_d} \Rightarrow \boxed{\omega_d = 3.14} \text{ when } T_p = 1\text{sec}$$

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 3.53 \qquad \boxed{\omega_n = 3.53}$$

$$\sqrt{\omega_n^2} = \sqrt{\frac{K}{J}} \Rightarrow \omega_n = \sqrt{\frac{K}{J}} \Rightarrow K = J\omega_n^2 = 12.5\ N\text{-}m.$$

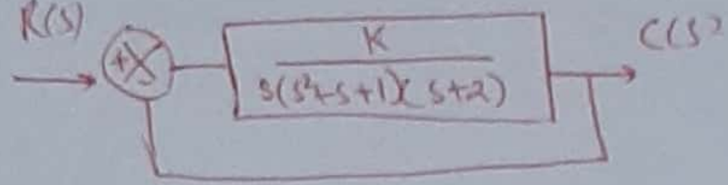
$$2\zeta\omega_n = \frac{B + KK_h}{J} \quad \text{Solving for } \boxed{K_h = 0.178}$$

$$T_r = \frac{\pi - \alpha}{\omega_d} \qquad \alpha = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$t_s = \frac{4}{\zeta\omega_n} \qquad t_s = \frac{3}{\zeta\omega_n}$$

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_3 s^{n-3} \dots a_n s^0 = 0$$

## Example 1:




$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2+s+1)(s+2)+K}$$

$X.E \Rightarrow \quad s^4 + 3s^3 + 3s^2 + 2s + K = 0$

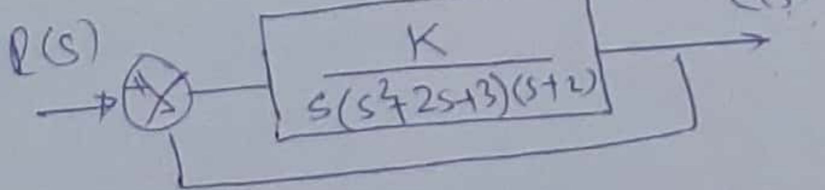
| | | | |
|---|---|---|---|
| $s^4$ | 1 | 3 | K |
| $s^3$ | 3 | 2 | 0 |
| $s^2$ | $7/3$ | K | 0 |
| $s^1$ | $2-\frac{9}{7}K$ | 0 | 0 |
| $s^0$ | K | 0 | 0 |

$2-\frac{9}{7}K > 0$

$-\frac{9}{7}K > -2$

$$\boxed{K < \frac{14}{9}}$$

## Example 2:




$X.E: \quad s(s^2+2s+3)(s+2) + K = 0$

$\Rightarrow \quad s^4 + 4s^3 + 7s^2 + 6s + K = 0$

| | | | |
|---|---|---|---|
| $s^4$ | 1 | 7 | K |
| $s^3$ | 4 | 6 | 0 |
| $s^2$ | 5.5 | K | 0 |
| $s^1$ | $\frac{33-4K}{5.5}$ | 0 | 0 |
| $s^0$ | K | 0 | 0 |

$\frac{33-4K}{5.5} > 0$

$33-4K > 0$

$-4K > -33$

$4K < 33$

$K = \frac{33}{4}$

$$\boxed{K = 8.24}$$