Solutions Manual for

Linear System Theory and Design

Third edition

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Chapter 2

2.1 (a) Linear (b) and (c) Nonlinear

In (b), if we define $\bar{y} = y - y_0$, or shift the operating point to $(0, y_0)$, then the pair $(u, \bar{y})$ is linear

2.2 Because $g(t)$ is not zero for $t<0$, the ideal lowpass filter is not causal and cannot be built in the real world.

2.3 It is linear, causal but time varying.

2.4 If causal, then

$$y_{(-\infty,\alpha]} = Hu_{(-\infty,\infty)} = Hu_{(-\infty,\alpha]}$$
$$= HP_\alpha u_{(-\infty,\infty)}$$

Thus we have

$$P_\alpha y = P_\alpha Hu = P_\alpha HP_\alpha u$$

Because $(P_\alpha Hu)(t) = 0$ for $t > \alpha$

but $(HP_\alpha u)(t)$ can be nonzero for $t > \alpha$. Thus $P_\alpha Hu \neq HP_\alpha u$

However, we do have

$$(P_\alpha Hu)(t) = (HP_\alpha u)(t) \text{ for } t \le \alpha$$

2.5 Superposition property must hold for the input and the initial state. If $x(0) \neq 0$, the initial states in (1) and (3) do not meet $x(0) = x(0) + x(0)$ or $x(0) = x(0) - x(0)$. Thus (1) and (3) are false. In (2), we have $x(0) = 0.5(x(0) + x(0)) = x(0)$. Thus (2) is true. If $x(0) = 0$, then (1) ~ (3) are true.

2.6 If $u_1 = \alpha u$, then

$$\frac{\alpha^2 u^2(t)}{\alpha u(t-1)} = \alpha \frac{u^2(t)}{u(t-1)} = \alpha y \quad \text{(homogeneity)}$$

$$\frac{u_1^2(t)}{u_1(t-1)} + \frac{u_2^2(t)}{u_2(t-1)} \neq \frac{(u_1(t)+u_2(t))^2}{u_1(t-1)+u_2(t-1)}$$

Thus additivity does not hold.

2.7 If additivity, then

$$H(u, x_0) = H(u+0, x_0+0) = H(u, x_0) + H(0,0)$$

and $H(0,0) = 0$.

Let n be a positive integer, then

$$H(nu, nx_0) = H(u, x_0) + H(u, x_0) + \cdots + H(u, x_0)$$
$$= nH(u, x_0)$$

$$0 = H(nu + (-nu), nx_0 + (-nx_0))$$
$$= H(nu, nx_0) + H(-nu, -nx_0)$$

$$H(-nu, -nx_0) = -H(nu, nx_0) = -nH(u, x_0)$$

Thus additivity implies

$$H(nu, nx_0) = nH(u, x_0) \text{ for any}$$

positive or negative integer n.

Let $\alpha = \frac{n}{m}$ where n and m are integers

Let $u = mV$ and $x_0 = mX_0$. Then

$H(u, x_0) = mH(V, X_0)$ which implies

$$H(V, X_0) = \frac{1}{m} H(u, x_0) = H(\tfrac{1}{m}u, \tfrac{1}{m}x_0)$$

Consider

$$H(\alpha u, \alpha x_0) = H(\tfrac{n}{m}u, \tfrac{n}{m}x_0) = nH(\tfrac{1}{m}u, \tfrac{1}{m}x_0)$$
$$= \frac{n}{m} H(u, x_0) = \alpha H(u, x_0)$$

Thus additivity implies

$$H(\alpha u, \alpha x_0) = \alpha H(u, x_0)$$

for any rational number α.

2.8 Let $x = t + \tau$ and $y = t - \tau$. Then $t = (x+y)/2$ and $\tau = (x-y)/2$.

Consider

$$\frac{\partial g(t,\tau)}{\partial x} = \frac{\partial g\left(\frac{x+y}{2}, \frac{x-y}{2}\right)}{\partial x}$$

$$= \lim_{\Delta x \to 0} \frac{g\left(\frac{x+\Delta x+y}{2}, \frac{x+\Delta x-y}{2}\right) - g\left(\frac{x+y}{2}, \frac{x-y}{2}\right)}{\Delta x}$$

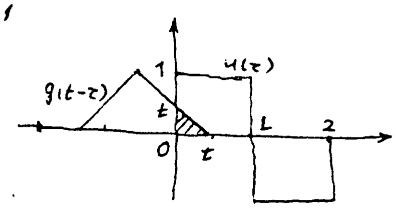
$$= \lim_{\Delta x \to 0} \frac{g\left(\frac{x+y}{2}+\frac{\Delta x}{2}, \frac{x-y}{2}+\frac{\Delta x}{2}\right) - g\left(\frac{x+y}{2}, \frac{x-y}{2}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{g\left(\frac{x+y}{2}, \frac{x-y}{2}\right) - g\left(\frac{x+y}{2}, \frac{x-y}{2}\right)}{\Delta x} = 0$$

Thus $g(t,\tau)$ depends only on $t-\tau$. This can also be seem by setting $\alpha = -\tau$, then $g(t,\tau) = g(t-\tau, 0)$.

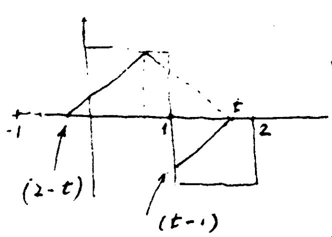
2.9 (i) $t < 0$. $y(t) = 0$

(ii) $0 \le t \le 1$



$y(t) = t^2/2$.

(iii) $1 \le t \le 2$



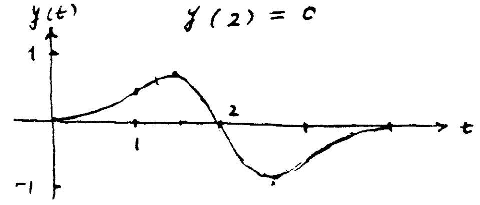
$$y(t) = \frac{1}{2} - \frac{(2-t)^2}{2} + \frac{1}{2} - 2\times\frac{(t-1)^2}{2}$$

$$= 1 - \frac{(2-t)^2}{2} - (t-1)^2$$

$y(\frac{4}{3}) = \frac{2}{3} = 0.667$

$y(1.5) = \frac{5}{8} = 0.625$

$y(2) = 0$



2.10 Taking the Laplace transform and assuming zero initial conditions yield

$$(s^2 + 2s - 3)\hat{y}(s) = (s-1)\hat{u}(s)$$

$$\hat{g}(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = \frac{s-1}{(s+3)(s-1)} = \frac{1}{s+3}$$

Impulse response

$$g(t) = \mathcal{L}^{-1}[\hat{g}(s)] = e^{-3t} \quad t \ge 0$$

2.11 Let $g(t)$ be the impulse response. Then

$$\bar{y}(t) = \int_0^t g(\tau)u(t-\tau)\,d\tau = \int_0^t g(\tau)\,d\tau$$

Therefore we have

$$y(t) = \frac{d}{dt}\bar{y}(t)$$

2.12

$$\begin{bmatrix} D_{11}(s) & D_{12}(s) \\ D_{21}(s) & D_{22}(s) \end{bmatrix} \begin{bmatrix} \hat{y}_1(s) \\ \hat{y}_2(s) \end{bmatrix} = \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix} \begin{bmatrix} \hat{u}_1(s) \\ \hat{u}_2(s) \end{bmatrix}$$

$$\hat{G}(s) = \begin{bmatrix} D_{11}(s) & D_{12}(s) \\ D_{21}(s) & D_{22}(s) \end{bmatrix}^{-1} \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix}$$

2.13 $y(t) = u(t-1)$, $r(t) = 1$ for $t \ge 0$

$a = 1$: $u(t) = r(t) + y(t) = 1 + u(t-1)$

or $y(t+1) = 1 + y(t)$.

Because $y(t) = 0$ for $0 \le t < 1$, then

$y(t) = 1$ for $1 \le t < 2$

$y(t) = 2$ for $2 \le t < 3$

$\vdots$

$a = 0.5$: $u(t) = 0.5(r(t) + y(t))$ or

$y(t+1) = 0.5(1 + y(t))$

Because $y(t) = 0$ for $0 \le t < 1$, then

$y(t) = 0.5$ for $1 \le t < 2$

$y(t) = 0.5(1 + 0.5) = 0.75$ for $2 \le t < 3$

$y(t) = 0.5(1 + 0.75) = 0.875$ for $3 \le t < 4$

$\vdots$

For negative feedback system

$a = 1 \quad y(t+1) = 1 - y(t)$

Because $y(t) = 0$ for $0 \le t < 1$ we have

$$
\begin{aligned}
y(t) &= 1 \quad \text{for } 1 \le t < 2 \\
&= 0 \quad \text{for } 2 \le t < 3 \\
&= 1 \quad \text{for } 3 \le t < 4 \\
&\ \ \vdots
\end{aligned}
$$

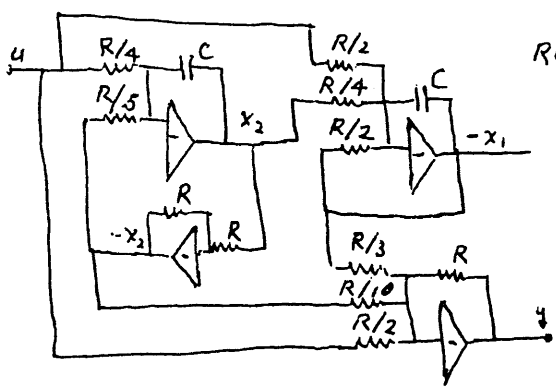
$a = 0.5 \quad y(t+1) = 0.5(1 - y(t))$. Thus

$$
\begin{aligned}
y(t) &= 0 \quad \text{for } 0 \le t < 1 \\
&= 0.5 \quad \text{for } 1 \le t < 2 \\
&= 0.5(1-0.5) = 0.25 \quad \text{for } 2 \le t < 3 \\
&= 0.5(1-0.25) = 0.375 \quad \text{for } 3 \le t < 4 \\
&\ \ \vdots
\end{aligned}
$$

2.14 $\dot{x}_1 = -2x_1 + 4x_2 + 2u$

$-\dot{x}_2 = -5x_2 + 4u$

$-y = -3x_1 - 10x_2 + 2u$



$RC = 1$

2.15 (a) Apply Newton's law in the tangential direction

$u \cos\theta - mg \sin\theta = m l \ddot{\theta}$

Define $x_1 = \theta$, $x_2 = \dot{\theta}$. Then

$$
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = -\dfrac{g}{l}\sin x_1 - \dfrac{u}{ml}\cos x_1
\end{cases}
$$

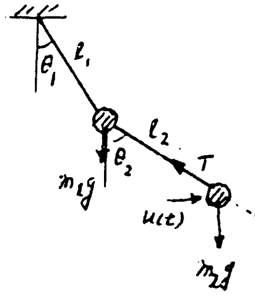
This is a nonlinear state equation.

If θ is small then $\sin\theta \approx \theta$, $\cos\theta = 1$.

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -g/l & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/ml \end{bmatrix} u
$$

This is a linearized equation.

(b)



$T = m_2 g \cos\theta_2 + u \sin\theta_2$

$u \cos\theta_2 - m_2 g \sin\theta_2 = m_2 l_2 \ddot{\theta}_2$

$T \sin(\theta_2 - \theta_1) - m_1 g \sin\theta_1 = m_1 l_1 \ddot{\theta}_1$

Define $x_1 = \theta_1$, $x_2 = \dot{\theta}_1$, $x_3 = \theta_2$, $x_4 = \dot{\theta}_2$

Then we have

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{-g}{l_1}\sin x_1 + \frac{m_2 g}{m_1 l_1}\cos x_3 \sin(x_3 - x_1) \\
&\quad + \frac{1}{m_1 l_1}\sin x_3 \sin(x_3 - x_1)\, u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{-g}{l_2}\sin x_3 + \frac{\cos x_3}{m_2 l_2} u
\end{aligned}
$$

This is a nonlinear equation. If $\theta_i \approx 0$ and $\theta_i \theta_j \approx 0$, then

$$
\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -g(m_1+m_2)/m_1 l_1 & 0 & m_2 g/m_1 l_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -g/l_2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 l_2 \end{bmatrix} u
$$

This is a linearlized equation.

2.16 $m\ddot{h} = f_1 - f_2 = k_1\theta - k_2 u \quad (1)$

$I\ddot{\theta} + b\dot{\theta} = (l_1 + l_2) f_2 - l_1 f_1 \quad (2)$

Define $x_1 = h$, $x_2 = \dot{h}$, $x_3 = \theta$, $x_4 = \dot{\theta}$ Then

$$
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = \dfrac{k_1}{m} x_3 - \dfrac{k_2}{m} u \\
\dot{x}_3 = x_4 \\
\dot{x}_4 = -\dfrac{l_1 k_1}{I} x_3 - \dfrac{b}{I} x_4 + (l_1 + l_2) k_2 u
\end{cases}
$$

This state equation describes the airplane.

Taking the Laplace transform of (1) and (2) and assuming $I \approx 0$ yield

$$m s^2 \hat{h}(s) = k_1 \hat{\theta}(s) - k_2 \hat{u}(s)$$

$$b s \hat{\theta}(s) = (l_1 + l_2) k_2 \hat{u}(s) - k_1 l_1 \hat{\theta}(s)$$

$$\hat{\theta}(s) = \frac{(l_1 + l_2) k_2}{bs + k_1 l_1} \hat{u}(s)$$

$$m s^2 \hat{h}(s) = \left(\frac{(l_1 + l_2) k_1 k_2}{bs + l_1 k_1} - k_2 \right) \hat{u}(s)$$

$$\hat{g}(s) = \frac{\hat{h}(s)}{\hat{u}(s)} = \frac{k_1 k_2 l_2 - k_2 b s}{m s^2 (bs + k_1 l_1)}$$

2.17 $m\ddot{y} = -k\dot{m} - mg$.

$x_1 = y$, $x_2 = \dot{y}$, $x_3 = m$, $u = \dot{m}$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{-k}{x_3} u - g \\ \dot{x}_3 = u \end{cases}$$

a nonlinear equation

2.18 Following Example 2.9, we have

$$y_1 = \frac{x_1}{R_1} \qquad A_1 dx_1 = (u - y_1) dt$$

Thus
$$\begin{cases} \dot{x}_1 = \dfrac{-1}{A_1 R_1} x_1 + \dfrac{1}{A_1} u \\ y_1 = \dfrac{1}{R_1} x_1 \end{cases}$$

$$\hat{g}_1(s) = \frac{\hat{y}_1(s)}{\hat{u}(s)} = \frac{1}{R_1} \left(s + \frac{1}{A_1 R_1}\right)^{-1} \frac{1}{A_1}$$

$$= \frac{1}{A_1 R_1 s + 1}$$

Similarly

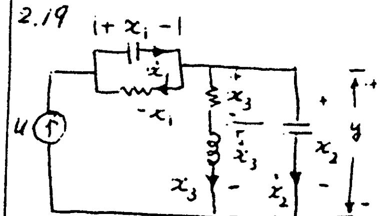
$$\hat{g}_2(s) = \frac{\hat{y}(s)}{\hat{y}_1(s)} = \frac{1}{A_2 R_2 s + 1}$$

The transfer function from u to y is

$$\hat{g}(s) = \hat{g}_1(s) \hat{g}_2(s) = \frac{1}{(A_1 R_1 s + 1)(A_2 R_2 s + 1)}.$$

The transfer function from u to y_1 in Fig 2.13 depends on x_2 of the second tank; therefore, we must compute $\hat{g}(s) = \hat{y}(s)/\hat{u}(s)$ as a unit and do not have $\hat{g}(s) = \hat{g}_1(s)\hat{g}_2(s)$.

2.19



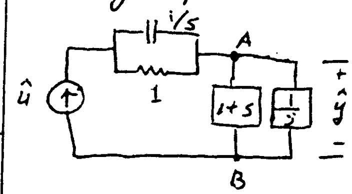
$$\dot{x}_1 = -x_1 + u$$

$$x_3 + \dot{x}_2 = u$$

$$x_3 + \dot{x}_3 = x_2 = y$$

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u \qquad y = [0 \ 1 \ 0] x$$

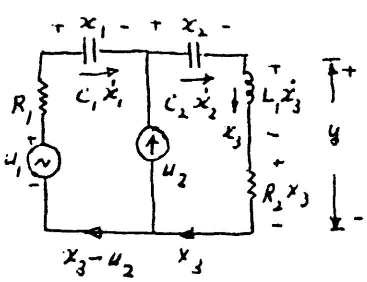
The transfer function can be computed from $c(sI - A)^{-1} b + d$ or, more easily, using impedances as



$$Z_{AB} = \frac{\frac{1}{s}(s+1)}{\frac{1}{s} + s + 1} = \frac{s+1}{s^2 + s + 1}$$

Thus we have $\hat{y}(s) = \dfrac{s+1}{s^2+s+1} \hat{u}(s)$.

2.20



The voltage across R_1 is $R_1(x_3 - u_2)$. We have

$$C_1 \dot{x}_1 = x_3 - u_2$$

$$C_2 \dot{x}_2 = x_3$$

From the outer loop, we have

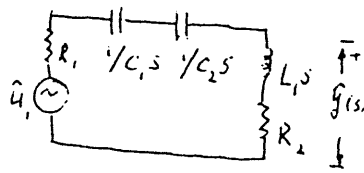
$$L_1 \dot{x}_3 = -x_1 - x_2 - (x_3 - u_2) R_1 - R_2 x_3 + u_1. \text{ Thus}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1/C_1 \\ 0 & 0 & 1/C_2 \\ \frac{-1}{L_1} & \frac{-1}{L_1} & \frac{-(R_1 + R_2)}{L_1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & -1/C_1 \\ 0 & 0 \\ \frac{1}{L_1} & \frac{R_1}{L_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = L_1 \dot{x}_3 + R_2 x_3 = -x_1 - x_2 - (x_3 - u_2) R_1 + u_1$$

$$= [-1 \ -1 \ -R_1] \underset{\sim}{x} + [1 \ R_1] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

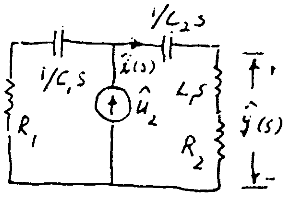
We use impedances to compute transfer functions. If $u_2 = 0$, the circuit reduces to



$$\hat{g}_1(s) = \frac{\hat{y}(s)}{\hat{u}_1(s)} = \frac{R_2 + L_1 s}{R_1 + R_2 + \frac{1}{C_1 s} + \frac{1}{C_2 s} + L_1 s}$$

$$= \frac{s^2 + (R_2/L_1)s}{s^2 + \left(\frac{R_1+R_2}{L_1}\right)s + \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\frac{1}{L_1}}$$

If $u_1 = 0$, then



$$\hat{i}(s) = \frac{\left(R_1 + \frac{1}{C_1 s}\right)\hat{u}_2(s)}{R_1 + R_2 + \frac{1}{C_1 s} + \frac{1}{C_2 s} + L_1 s}$$

$$\hat{g}_2(s) = \frac{\hat{y}(s)}{\hat{u}_2(s)} = \frac{\left(R_1 s + \frac{1}{C_1}\right)\left(s + \frac{R_2}{L_1}\right)}{s^2 + \left(\frac{R_1+R_2}{L_1}\right)s + \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\frac{1}{L_1}}$$

Therefore

$$\hat{y}(s) = \hat{g}_1(s)\,\hat{u}_1(s) + \hat{g}_2(s)\,\hat{u}_2(s)$$

$$= [\hat{g}_1(s) \quad \hat{g}_2(s)]\begin{bmatrix}\hat{u}_1(s)\\ \hat{u}_2(s)\end{bmatrix}$$

Note that the denominator of $\hat{g}_i(s)$ is different from $\det(sI - A)$. The former has degree 2, the latter has degree 3.

2.21 Let I be the moment of inertia of the bar and mass about the hinge. Then

$$I\ddot{\theta} = l_2 u - k_1(\theta l_1)l_1 - k_2(l_2\theta - y)l_2$$

$$m_2\ddot{y} = k_2(l_2\theta - y) \qquad \text{for } \theta \text{ small}$$

Define $x_1 = \theta$, $x_2 = \dot{\theta}$, $y = x_3$ and $x_4 = \dot{y}$. Then

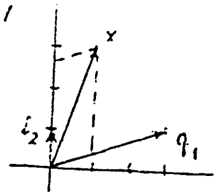
$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0\\ -(k_1 l_1^2 + k_2 l_2^2)/I & 0 & k_2 l_2/I & 0\\ 0 & 0 & 0 & 1\\ k_2 l_2/m_2 & 0 & -k_2/m_2 & 0\end{bmatrix}x + \begin{bmatrix}0\\ \frac{l_2}{I}\\ 0\\ 0\end{bmatrix}u$$

$$y = [0 \quad 0 \quad 1 \quad 0]x$$

$$\hat{g}(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = \frac{k_2 l_2^2}{Im_2 s^4 + [m_2(k_1 l_1^2 + k_2 l_2^2) + Ik_2]s^2 + k_1 k_2 l_1^2}$$

Chapter 3

3.1

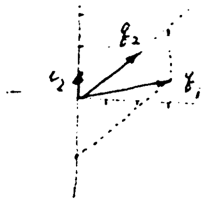


Because $x = \frac{1}{3}q_1 + \left(2\frac{2}{3}\right)i_2$

$$= [q_1 \quad i_2]\begin{bmatrix}1/3\\ 8/3\end{bmatrix}$$

the representation of x with respect to $\{q_1, i_2\}$ is $[\frac{1}{3} \ \frac{8}{3}]'$. Indeed, we have

$$x = \begin{bmatrix}3 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}1/3\\ 8/3\end{bmatrix} = \begin{bmatrix}1\\ 3\end{bmatrix}.$$



Because $q_1 = -2i_2 + 1.5q_2$

$$= [i_2 \quad q_2]\begin{bmatrix}-2\\ 1.5\end{bmatrix}$$

the representation of q_1 with respect to $\{i_2, q_2\}$ is $[-2 \ 1.5]'$. Indeed we have

$$q_1 = \begin{bmatrix}0 & 2\\ 1 & 2\end{bmatrix}\begin{bmatrix}-2\\ 1.5\end{bmatrix} = \begin{bmatrix}3\\ 1\end{bmatrix}$$

3.2

$$\|x_1\|_1 = 2 + 3 + 1 = 6$$

$$\|x_1\|_2 = \sqrt{4+9+1} = \sqrt{14} = 3.74$$

$$\|x_1\|_\infty = 3$$

$$\|x_2\|_1 = 1 + 1 + 1 = 3$$

$$\|x_2\|_2 = \sqrt{1+1+1} = \sqrt{3} = 1.732$$

$$\|x_2\|_\infty = 1$$

3.3

$$q_1 = x_1/\|x_1\| = \frac{1}{3.74}\begin{bmatrix}2\\ -3\\ 1\end{bmatrix}$$

$$u_2 = x_2 - (q_1' x_2)q_1 = \begin{bmatrix}1\\ 1\\ 1\end{bmatrix} - \frac{0}{3.74}q_1 = \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

$$q_2 = u_2/\|u_2\| = \frac{1}{1.732}\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

Note that x_1 and x_2 are already orthogonal. Therefore q_1 and q_2 are their normalized vectors.

3.4 If $n > m$, AA' is symmetric and has rank m. If $m = n$, then $A'A = I_n$ and

A is nonsingular. Thus $A^{-1} = A'$ and $AA' = AA^{-1} = I_n$. As an example, for the orthonormal vectors q_1 and q_2 in Problem 3.3, we have: $Q = [q_1\ q_2]$

$$Q'Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$QQ' = \begin{bmatrix} 0.6190 & -0.0952 & 0.4762 \\ -0.0952 & 0.9762 & 0.1190 \\ 0.4762 & 0.1190 & 0.4048 \end{bmatrix}$$

(symmetric)

3.5 $\rho(A_1) = 2$, nullity $(A_1) = 3 - 2 = 1$

$\rho(A_2) = 3$, nullity $(A_2) = 3 - 3 = 0$

$\rho(A_3) = 3$ nullity $(A_3) = 4 - 3 = 1$

3.6 Range space $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; null space $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Range space $\begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$; null $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. dim $= 0$

Range space $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$; null space $\begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$

3.7 $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a solution. Because the nullity is 0, the solution is unique.

Because

$$\rho \begin{bmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{bmatrix} = 2 \neq \rho \begin{bmatrix} 2 & -1 & 1 \\ -3 & 3 & 1 \\ -1 & 2 & 1 \end{bmatrix} = 3$$

The equation has no solution for $y = [1\ 1\ 1]'$.

3.8 $x_p = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is a solution. The null space is spanned, as computed in Problem 3.6, by $[-1\ 2\ -1\ 0]'$. Thus the general solution is

$$x = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

for any α. There is one free parameter.

3.9 From (3.17),

$$x = \begin{bmatrix} \alpha_1 \\ -4 + \alpha_1 + 2\alpha_2 \\ -\alpha_1 \\ -\alpha_2 \end{bmatrix}$$

$$\|x\|^2 = \alpha_1^2 + (-4 + \alpha_1 + 2\alpha_2)^2 + \alpha_1^2 + \alpha_2^2$$

$$= 3\alpha_1^2 + 5\alpha_2^2 + 4\alpha_1\alpha_2 - 8\alpha_1 - 16\alpha_2 + 16 =: B$$

$$\frac{\partial B}{\partial \alpha_1} = 6\alpha_1 + 4\alpha_2 - 8 = 0 \quad \text{or} \quad 3\alpha_1 + 2\alpha_2 - 4 = 0 \quad (1)$$

$$\frac{\partial B}{\partial \alpha_2} = 10\alpha_2 + 4\alpha_1 - 16 = 0 \quad (2)$$

$5 \times (1) - (2)$: $11\alpha_1 - 4 = 0 \quad \therefore \alpha_1 = 4/11$

$$\alpha_2 = \frac{1}{2}(4 - 3\alpha_1) = \frac{1}{2}\left(4 - \frac{12}{11}\right) = \frac{16}{11}$$

Thus the solution $x = [\frac{4}{11}\ \frac{-8}{11}\ \frac{-4}{11}\ \frac{-16}{11}]'$ has the smallest 2-norm.

3.10 $x = \begin{bmatrix} -1-\alpha \\ 2\alpha \\ -\alpha \\ 1 \end{bmatrix}$

$$B = \|x\|^2 = (-1-\alpha)^2 + 4\alpha^2 + \alpha^2 + 1$$

$$= 6\alpha^2 + 2\alpha + 2$$

$$\frac{dB}{d\alpha} = 12\alpha + 2 = 0 \quad \therefore \alpha = -1/6$$

Thus the solution $[\frac{-5}{6}\ \frac{-1}{3}\ \frac{1}{6}\ 1]'$ has the smallest 2-norm.

3.11 If and only if the $n \times n$ matrix

$$[b\ Ab\ \cdots\ A^{n-1}b]$$

is nonsingular or has full row rank.

3.12 $\Delta(s) = \det(sI - A) = (s-2)^3(s-1)$

$$= s^4 - 7s^3 + 18s^2 - 20s + 8$$

Thus $A^4 = -8I + 20A - 18A^2 + 7A^3$

$$Ab = [b \ Ab \ A^2b \ A^3b]\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

$\vdots$

$$A^4b = [b \ Ab \ A^2b \ A^3b]\begin{bmatrix}-8\\20\\-18\\7\end{bmatrix}$$

Thus the representation of A with respect to $\{b, Ab, A^2b, A^3b\}$ is

$$\bar{A} = \begin{bmatrix}0 & 0 & 0 & -8\\1 & 0 & 0 & 20\\0 & 1 & 0 & -18\\0 & 0 & 1 & 7\end{bmatrix}.$$

This is also the representation of $\bar{A}$ with respect to $\{\bar{b}, \bar{A}\bar{b}, \bar{A}^2\bar{b}, \bar{A}^3\bar{b}\}$.

3.13 $A_1 = \begin{bmatrix}1 & 4 & 10\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}$. It is triangular, thus its eigenvalues are 1, 2 and 3

$$Q_1 = \begin{bmatrix}1 & 4 & 5\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, \quad \hat{A}_1 = \begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}$$

$A_2 = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-2 & -4 & -3\end{bmatrix}$ companion form

$$\Delta_2(\lambda) = \det(\lambda I - A_2) = \lambda^3 + 3\lambda^2 + 4\lambda + 2$$
$$= (\lambda+1)(\lambda^2+2\lambda+2)$$
$$= (\lambda+1)(\lambda+1+j)(\lambda+1-j)$$

$$Q_2 = \begin{bmatrix}1 & -1 & -1\\-1 & 1+j & 1-j\\1 & -2j & 2j\end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix}-1 & 0 & 0\\0 & -1-j & 0\\0 & 0 & -1+j\end{bmatrix}$$

$A_3 = \begin{bmatrix}1 & 0 & -1\\0 & 1 & 0\\0 & 0 & 2\end{bmatrix}$ Triangular. Its eigenvalues are 1, 1 and 2

For $\lambda = 1$.

$$(\lambda I - A)q = \begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\0 & 0 & -1\end{bmatrix} q = 0$$

$(\lambda I - A)$ has rank 1 and, consequently, nullity 2. Thus there are two linearly independent null vectors

$$q_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad q_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

For $\lambda = 2$

$$(\lambda I - A)q_3 \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix} q_3 = 0, \quad q_3 = \begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$\therefore Q_3 = \begin{bmatrix}1 & 0 & -1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, \quad \hat{A}_3 = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 2\end{bmatrix}$$

$$A_4 = \begin{bmatrix}0 & 4 & 3\\0 & 20 & 16\\0 & -25 & -20\end{bmatrix}$$

$$\Delta_4(\lambda) = \det(\lambda I - A_4) = \lambda\left[(\lambda-20)(\lambda+20) + 16 \times 25\right]$$
$$= \lambda(\lambda^2 - 400 + 400) = \lambda^3$$
$$\lambda = 0, 0, 0.$$

$(A_4 - \lambda I) = A_4$ has rank 2 or nullity 1. Thus the Jordan form has only one Jordan block. We compute

$$(A_4 - \lambda I)^2 = \begin{bmatrix}0 & 4 & 3\\0 & 20 & 16\\0 & -25 & -20\end{bmatrix}\begin{bmatrix}0 & 4 & 3\\0 & 20 & 16\\0 & -25 & -20\end{bmatrix}$$
$$= \begin{bmatrix}0 & 5 & 4\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

$$(A_4 - \lambda I)^3 = \begin{bmatrix}0 & 5 & 4\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 4 & 3\\0 & 20 & 16\\0 & -25 & -20\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

Clearly $v = [0 \ 1 \ 0]'$ meets

$$(A_4 - \lambda I)^3 v = 0 \quad \text{and} \quad (A_4 - \lambda I)^2 v = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} \neq 0$$

Define

$$v_1 = (A_4 - \lambda I)^2 v = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = (A_4 - \lambda I) v = \begin{bmatrix} 4 \\ 20 \\ -25 \end{bmatrix}$$

$$v_3 = v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$ Thus we have

$$Q_4 = \begin{bmatrix} 5 & 4 & 0 \\ 0 & 20 & 1 \\ 0 & -25 & 0 \end{bmatrix}, \quad \hat{A}_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

3.14

$$\Delta(\lambda) = \det(\lambda I - A) = \det \begin{bmatrix} \lambda+\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ -1 & \lambda & 0 & 0 \\ 0 & -1 & \lambda & 0 \\ 0 & 0 & -1 & \lambda \end{bmatrix}$$

$$= (\lambda + \alpha_1) \det \begin{bmatrix} \lambda & 0 & 0 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda \end{bmatrix} + \det \begin{bmatrix} \alpha_2 & \alpha_3 & \alpha_4 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda \end{bmatrix}$$

$$= \lambda^3(\lambda + \alpha_1) + \alpha_2 \lambda^2 + \alpha_3 \lambda + \alpha_4 .$$

If λ_i is an eigenvalue of A, then $\Delta(\lambda_i) = 0$ which implies

$$\lambda_i^4 = -\alpha_1 \lambda_i^3 - \alpha_2 \lambda_i^2 - \alpha_3 \lambda_i - \alpha_4$$

Using this equation, we have

$$A \begin{bmatrix} \lambda_i^3 \\ \lambda_i^2 \\ \lambda_i \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_i^4 \\ \lambda_i^3 \\ \lambda_i^2 \\ \lambda_i \end{bmatrix} = \lambda_i \begin{bmatrix} \lambda_i^3 \\ \lambda_i^2 \\ \lambda_i \\ 1 \end{bmatrix}$$

Thus $[\lambda_i^3 \ \lambda_i^2 \ \lambda_i \ 1]'$ is an eigenvector.

3.15

$$\det \begin{bmatrix} \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} \lambda_1^3-\lambda_4^3 & \lambda_2^3-\lambda_4^3 & \lambda_3^3-\lambda_4^3 & \lambda_4^3 \\ \lambda_1^2-\lambda_4^2 & \lambda_2^2-\lambda_4^2 & \lambda_3^2-\lambda_4^2 & \lambda_4^2 \\ \lambda_1-\lambda_4 & \lambda_2-\lambda_4 & \lambda_3-\lambda_4 & \lambda_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= (\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4) \det \begin{bmatrix} \lambda_1^2+\lambda_1\lambda_4+\lambda_4^2 & \lambda_2^2+\lambda_2\lambda_4+\lambda_4^2 & \lambda_3^2+\lambda_3\lambda_4+\lambda_4^2 \\ \lambda_1+\lambda_4 & \lambda_2+\lambda_4 & \lambda_3+\lambda_4 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \prod_{i<j\le 4}(\lambda_i-\lambda_j) \det \begin{bmatrix} (\lambda_1-\lambda_2)(\lambda_1+\lambda_2+\lambda_4) & (\lambda_2-\lambda_3)(\lambda_2+\lambda_3+\lambda_4) & x \\ \lambda_1-\lambda_2 & \lambda_2-\lambda_3 & \lambda_3+\lambda_4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \prod_{1<j\le 4}(\lambda_1-\lambda_j)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3) \det \begin{bmatrix} \lambda_1+\lambda_2+\lambda_4 & \lambda_2+\lambda_3+\lambda_4 \\ 1 & 1 \end{bmatrix}$$

$$= \prod_{1<j\le 4}(\lambda_1-\lambda_j)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)$$

$$= \prod_{1\le i<j\le 4}(\lambda_i-\lambda_j)$$

If all eigenvalues are distinct, then the determinant is different from zero. Thus the matrix is nonsingular and the four columns are linearly independent.

3.16 Direct verification:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-1}{\alpha_4} & \frac{-\alpha_1}{\alpha_4} & \frac{-\alpha_2}{\alpha_4} & \frac{-\alpha_3}{\alpha_4} \end{bmatrix} \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = I_4$$

This shows the inverse. Note that if $\alpha_4 = 0$, then $\Delta(\lambda) = \lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda$ and $\lambda = 0$ is an eigenvalue. In this case, the matrix is singular and its inverse does not exist.

3.17
$$A = \begin{bmatrix} \lambda & \lambda T & \lambda T^2/2 \\ 0 & \lambda & \lambda T \\ 0 & 0 & \lambda \end{bmatrix}$$

Eigenvalues: $\lambda, \lambda, \lambda$

$$(A - \lambda I) = \begin{bmatrix} 0 & \lambda T & \lambda T^2/2 \\ 0 & 0 & \lambda T \\ 0 & 0 & 0 \end{bmatrix}$$

If $\lambda \neq 0$ and $T > 0$, $(A - \lambda I)$ has rank 2 and nullity 1. Thus the Jordan form of A has only one Jordan block. We compute

$$(A-\lambda I)^2 = \begin{bmatrix} 0 & 0 & \lambda^2 T^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A-\lambda I)^3 = 0$$

Clearly, $v = [0\ 0\ 1]'$ has the property $(A-\lambda I)^3 v = 0$, $(A-\lambda I)^2 v \neq 0$.

Define

$$v_1 = (A-\lambda I)^2 v = \begin{bmatrix} \lambda^2 T^2 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = (A-\lambda I) v = \begin{bmatrix} \lambda T^2/2 \\ \lambda T \\ 0 \end{bmatrix}$$

$$v_3 = v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus we have

$$Q = \begin{bmatrix} \lambda^2 T^2 & \lambda T^2/2 & 0 \\ 0 & \lambda T & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \hat{A} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

Instead of computing $Q^{-1}AQ = \hat{A}$, we compute

$$AQ = \begin{bmatrix} \lambda & \lambda T & \lambda T^2/2 \\ 0 & \lambda & \lambda T \\ 0 & 0 & \lambda \end{bmatrix}\begin{bmatrix} \lambda^2 T^2 & \lambda T^2/2 & 0 \\ 0 & \lambda T & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda^3 T^2 & 3\lambda^2 T^2/2 & \lambda T^2/2 \\ 0 & \lambda^2 T & \lambda T \\ 0 & 0 & \lambda \end{bmatrix}$$

$$Q\hat{A} = \begin{bmatrix} \lambda^2 T^2 & \lambda T^2/2 & 0 \\ 0 & \lambda T & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} \lambda^3 T^2 & 3\lambda^2 T^2/2 & \lambda T^2/2 \\ 0 & \lambda^2 T & \lambda T \\ 0 & 0 & \lambda \end{bmatrix}$$

Thus we have $AQ = Q\hat{A}$ or $Q^{-1}AQ = \hat{A}$

3.18

$$\Delta_1(\lambda) = (\lambda-\lambda_1)^3(\lambda-\lambda_2)$$
$$\psi_1(\lambda) = (\lambda-\lambda_1)^3(\lambda-\lambda_2) = \Delta_1(\lambda)$$
$$\Delta_2(\lambda) = (\lambda-\lambda_1)^4$$
$$\psi_2(\lambda) = (\lambda-\lambda_1)^3$$
$$\Delta_3(\lambda) = (\lambda-\lambda_1)^4$$
$$\psi_3(\lambda) = (\lambda-\lambda_1)^2$$
$$\Delta_4(\lambda) = (\lambda-\lambda_1)^4$$
$$\psi_4(\lambda) = (\lambda-\lambda_1)$$

3.19 $Ax = \lambda x \Rightarrow A^m x = \lambda^m x$, $m = 1, 2, 3, \ldots$

Using (3.44), we have

$$f(A)x = (\beta_0 I + \beta_1 A + \cdots + \beta_{n-1}A^{n-1})x$$
$$= (\beta_0 + \beta_1\lambda + \cdots + \beta_{n-1}\lambda^{n-1})x$$
$$= f(\lambda)x$$

Thus $f(\lambda)$ is an eigenvalue of $f(A)$

3.20 $A = Q\hat{A}Q^{-1}$ $\quad A^k = Q\hat{A}^k Q^{-1}$

Thus we can assume that A is in Jordan form. If A has any nonzero eigenvalue, then $A^k \neq 0$ for all integer k. Thus in order for $A^k = 0$ for some k, all eigenvalues must be 0, or A has eigenvalue 0 with multiplicity n.

Let $A = \mathrm{diag}\{A_1, A_2, \cdots\}$. Then $A^k = \mathrm{diag}\{A_1^k, A_2^k, \cdots\}$. If each A_i is a Jordan block associated with eigenvalue 0, then, as we can see from (3.40), $A_i^k = 0$ for $k \geq m$ if and only if the order of A_i is m or less. Thus we conclude that the index of A is m or less.

3.21 $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ Triangular. Its eigenvalues are 1, 1 and 0

Let $h(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2$ and let

$f(\lambda) = \lambda^{103}$

$\lambda_1 = 0 : f(0) = h(0) \Rightarrow \beta_0 = 0$

$\lambda_2 = 1 : f(1) = h(1) \Rightarrow 1 = \beta_1 + \beta_2$

$f'(1) = h'(1) \Rightarrow 103 = \beta_1 + 2\beta_2$

$\beta_2 = 102, \quad \beta_1 = 1 - \beta_2 = -101$

$A^{103} = \beta_0 I + \beta_1 A + \beta_2 A^2 = -101A + 102A^2$

$= -101\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + 102\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 102 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Similarly, we can obtain

$A^{10} = \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$e^{At} = \begin{bmatrix} e^t & e^t - 1 & te^t - e^t + 1 \\ 0 & 1 & e^t - 1 \\ 0 & 0 & e^t \end{bmatrix}$

3.22 $A_1 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(i)
As computed in Problem 3.13 we have

$A_1 = Q_1 \hat{A}_1 Q_1^{-1} = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$

$e^{A_1 t} = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 1 & -4 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} e^t & 4(e^{2t} - e^t) & 5(e^{3t} - e^t) \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}$

(ii) $h(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2, \quad f(\lambda) = e^{\lambda t}$

$\lambda = 1 \quad e^t = \beta_0 + \beta_1 + \beta_2$

$\lambda = 2 \quad e^{2t} = \beta_0 + 2\beta_1 + 4\beta_2$

$\lambda = 3 \quad e^{3t} = \beta_0 + 3\beta_1 + 9\beta_2$

From these, we can compute

$\beta_2 = 0.5(e^{3t} - 2e^{2t} + e^t)$

$\beta_1 = 4e^{2t} - 1.5e^{3t} - 2.5e^t$

$\beta_0 = 3e^t + 3e^{2t} + e^{3t}$

Thus we have

$e^{A_1 t} = \beta_0 I + \beta_1\begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} + \beta_2\begin{bmatrix} 1 & 12 & 40 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

$= \begin{bmatrix} e^t & 4(e^{2t} - e^t) & 5(e^{3t} - e^t) \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}$

$A_4 = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}$ From Prob 3.13, we have

$A_4 = \begin{bmatrix} 5 & 4 & 0 \\ 0 & 20 & 1 \\ 0 & -25 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 & 4 & 0 \\ 0 & 20 & 1 \\ 0 & -25 & 0 \end{bmatrix}^{-1}$

$e^{A_4 t} = \begin{bmatrix} 5 & 4 & 0 \\ 0 & 20 & 1 \\ 0 & -25 & 0 \end{bmatrix}\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0.2 & 0 & 0.032 \\ 0 & 0 & -0.04 \\ 0 & 1 & 0.8 \end{bmatrix}$

$= \begin{bmatrix} 1 & 4t + 2.5t^2 & 3t + 2t^2 \\ 0 & 1 + 20t & 16t \\ 0 & -25t & 1 - 20t \end{bmatrix}$

$h(0) = f(0) \quad \beta_0 = 1$

$h'(0) = f'(0) \quad \beta_1 = t$

$h''(0) = f''(0) : \quad \beta_2 = t^2/2$

$$e^{A_4 t} = \beta_0 I + \beta_1 A_4 + \beta_2 A_4^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 4t & 3t \\ 0 & 20t & 16t \\ 0 & 25t & -20t \end{bmatrix} + \begin{bmatrix} 0 & 2.5t^2 & 2t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4t + 2.5t^2 & 3t + 2t^2 \\ 0 & 1 + 20t & 16t \\ 0 & -25t & 1 - 20t \end{bmatrix}$$

3.23 If A is $n \times n$, we have

$$f(A) = \alpha_0 + \alpha_1 A + \cdots + \alpha_{n-1} A^{n-1}$$

$$g(A) = \beta_0 + \beta_1 A + \cdots + \beta_{n-1} A^{n-1}$$

Because A commutes with itself, we conclude $f(A)g(A) = g(A)f(A)$ and, in particular, $Ae^{At} = e^{At}A$.

3.24 If $C = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$, then

$$B = \ln C = \begin{bmatrix} \ln \lambda_1 & 0 & 0 \\ 0 & \ln \lambda_2 & 0 \\ 0 & 0 & \ln \lambda_3 \end{bmatrix}$$

If $\lambda_i = 0$, then $\ln \lambda_i = -\infty$ and B is not defined. If

$$C = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \text{ then } B = \begin{bmatrix} \ln\lambda & 1/\lambda & 0 \\ 0 & \ln\lambda & 0 \\ 0 & 0 & \ln\lambda \end{bmatrix}$$

(using (3.47))

For any nonsingular C, there exists a nonsingular Q such that

$$C = Q\hat{C}Q^{-1}$$

where $\hat{C}$ is in Jordan form. Then

$$B = \ln C = Q(\ln \hat{C})Q^{-1}$$

3.25 $\Delta(s) = (s-1)^2(s-2)$. From Jordan form, we have

$$\psi(s) = (s-1)(s-2)$$

We compute

$$(sI - A_3)^{-1} = \begin{bmatrix} s-1 & 0 & 1 \\ 0 & s- & 0 \\ 0 & 0 & s-2 \end{bmatrix}^{-1}$$

$$= \frac{1}{\Delta(s)} \begin{bmatrix} (s-1)(s-2) & 0 & -(s-1) \\ 0 & (s-1)(s-2) & 0 \\ 0 & 0 & (s-1)^2 \end{bmatrix}$$

Then $m(s) = s-1$. Thus

$$\psi(s) = \frac{\Delta(s)}{m(s)}$$

3.26 $\Delta(s) I = (R_0 s^{n-1} + \cdots + R_{n-1})(sI - A)$

or

$$s^n I + \alpha_1 s^{n-1} I + \cdots + \alpha_n I =$$

$$R_0 s^n + (R_1 - R_0 A) s^{n-1} + \cdots + R_{n-1}(-A)$$

Equating the coefficient matrices of like power of s yields the right-hand-side equations.

Let λ_i, $i = 1, 2, \cdots, n$, be the eigenvalues of A. Define

$$\Lambda_1 = \sum_{i=1}^{n} \lambda_i, \quad \Lambda_k = \sum_{i=1}^{n} \lambda_i^k, \quad k = 1, 2, \cdots$$

Let $A = Q\hat{A}Q^{-1}$, where $\hat{A}$ is in Jordan form. Because $\mathrm{tr}(BC) = \mathrm{tr}(CB)$, we have

$$\mathrm{tr}(A) = \mathrm{tr}(Q\hat{A}Q^{-1}) = \mathrm{tr}(Q^{-1}Q\hat{A})$$

$$= \mathrm{tr}(\hat{A}) = \sum \lambda_i = \Lambda_1$$

Similarly, $\mathrm{tr}(A^k) = \Lambda_k$.

We need the following Newton's identity

$$\Lambda_k + \alpha_1 \Lambda_{k-1} + \cdots + \alpha_{k-1}\Lambda_1 + k\alpha_k = 0$$

for $k = 1, 2, \cdots, n$. Thus we have

$$\alpha_k = \frac{-1}{k}\left[\Lambda_k + \alpha_1 \Lambda_{k-1} + \cdots + \alpha_{k-1}\Lambda_1\right]$$

$$= \frac{-1}{k}\left[tr(A^k) + \alpha_1 tr(A^{k-1}) + \cdots + \alpha_{k-1} tr(A)\right]$$

$$= \frac{-1}{k} tr\left[A^k + \alpha_1 A^{k-1} + \cdots + \alpha_{k-1} A\right]$$

$$= \frac{-1}{k} tr\left[A(A^{k-1} + \alpha_1 A^{k-2} + \cdots + \alpha_{k-1} I)\right]$$

$$= \frac{-1}{k} tr(AR_{k-1}).$$

This establishes the formula.

3.27 Substituting R_{n-1} into

$$0 = AR_{n-1} + \alpha_n I$$

yields

$$A^n + \alpha_1 A^{n-1} + \cdots + \alpha_{n-1} A + \alpha_n I = 0.$$

This is the Cayley Hamilton theorem.

3.28 By direct substitution.

3.29

$$AQ = Q\begin{bmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n\end{bmatrix} \Rightarrow Q^{-1}A = \begin{bmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n\end{bmatrix} Q^{-1}$$

$$\Rightarrow \begin{bmatrix}p_1 \\ p_2 \\ \vdots \\ p_n\end{bmatrix} A = \begin{bmatrix}\lambda_1 p_1 \\ \lambda_2 p_2 \\ \vdots \\ \lambda_n p_n\end{bmatrix} \Rightarrow p_i A = \lambda_i p_i$$

3.30 If $A = Q\hat{A}P$, then

$$(sI - A)^{-1} = Q(sI - \hat{A})^{-1}P$$

$$= [q_1, q_2 \cdots q_n]\begin{bmatrix}\frac{1}{s-\lambda_1} & 0 & \cdots \\ 0 & \frac{1}{s-\lambda_2} & \\ & & \ddots\end{bmatrix}\begin{bmatrix}p_1 \\ p_2 \\ \vdots\end{bmatrix}$$

$$= \sum_{i=1}^{n} \frac{1}{s-\lambda_i} q_i p_i$$

3.31 Let $M = \begin{bmatrix}m_1 \\ m_2\end{bmatrix}$. Then

$$\begin{bmatrix}0 & 1 \\ -2 & -2\end{bmatrix}\begin{bmatrix}m_1 \\ m_2\end{bmatrix} + \begin{bmatrix}m_1 \\ m_2\end{bmatrix} 3 = \begin{bmatrix}3 \\ 3\end{bmatrix}$$

$$\det(sI - A) = \det\begin{bmatrix}s & -1 \\ 2 & s+2\end{bmatrix} = s(s+2) + 2$$

$$= s^2 + 2s + 2 = (s+1+j)(s+1-j)$$

eigenvalues of A : $-1+j, -1-j$

eigenvalue of B : 3

eigenvalues of the Lyapunov equation :

$$2+j, \ 2-j$$

No zero eigenvalue. The Lyapunov equation is nonsingular and for any C a unique solution exists

$$m_2 + 3m_1 = 3 \qquad (1)$$

$$-2m_1 - 2m_2 + 3m_2 = -2m_1 + m_2 = 3 \qquad (2)$$

$$(1) - (2): \ 5m_1 = 0, \quad m_1 = 0, \quad m_2 = 3$$

3.32

$$\det(sI - A) = \det\begin{bmatrix}s & -1 \\ 1 & s+2\end{bmatrix} = s(s+2) + 1$$

$$= s^2 + 2s + 1 = (s+1)^2$$

eigenvalues of A : $-1, -1$

eigenvalue of B : 1

eigenvalues of the Lyapunov equation :

$$0, \ 0$$

The Lyapunov equation is singular. A solution exists only if C_i is in the range space.

$$\begin{bmatrix}0 & 1 \\ -1 & -2\end{bmatrix}\begin{bmatrix}m_1 \\ m_2\end{bmatrix} + \begin{bmatrix}m_1 \\ m_2\end{bmatrix}(1) = \begin{bmatrix}3 \\ 3\end{bmatrix} = C_1$$

$$m_2 + m_1 = 3$$

$$-m_1 - 2m_2 + m_2 = -m_1 - m_2 = 3$$

$$\Rightarrow m_1 + m_2 = -3$$

These two equations are inconsistent. Therefore, no solution exists for C_1. If $C_2 = [3 \ -3]'$, then

$$m_1 + m_2 = 3$$

$$-m_1 - m_2 = -3 \quad \text{or} \quad m_1 + m_2 = 3$$

Therefore, for any m_1, $\begin{bmatrix} m_1 \\ 3-m_1 \end{bmatrix}$ is a solution

3.33

$$\begin{bmatrix} 2 & 3 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

Because $\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 2-9=-7$. it is not positive definite, nor positive semidefinite

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

Because $\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix} = -1$, it is not positive definite, nor positive semidefinite

$$\begin{bmatrix} a_1a_1 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2a_2 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$$

It is positive semidefinite.

3.34 $H_1 = \begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$, $H_1H_1' = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$

char. poly. of $H_1H_1' = (\lambda-2)(\lambda-5)-4$

$= \lambda^2 - 7\lambda + 6$

$= (\lambda-6)(\lambda-1)$

Singular values of H_1: $\sqrt{6}$, 1. note that we use H_1H_1', instead of $H_1'H_1$.

$H_2 = \begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix}$, $H_2'H_2 = \begin{bmatrix} 5 & 6 \\ 6 & 20 \end{bmatrix}$

char. poly. of $H_2'H_2 = (\lambda-5)(\lambda-20)-36$

$= \lambda^2 - 25\lambda + 64$

$= (\lambda - 22.1047)(\lambda - 2.8953)$

Singular values of H_2:

$\sqrt{22.1047} = 4.7016$

$\sqrt{2.8953} = 1.7016$

3.35 If A is symmetric, then $A'A = A^2$. Let λ_i be the eigenvalues of A, then the eigenvalues of $A'A = A^2$ are λ_i^2 and singular values of A are $|\lambda_i|$. As an example, for the H_2 in Problem 3.34, its eigenvalues are 4.7016 and -1.7016. Its singular values are 4.7016 and 1.7016.

3.36 It follows directly from (3.69).

3.37 Let A be any $m \times n$ and B be any $n \times m$

$$N = \begin{bmatrix} \sqrt{s} I_m & A \\ 0 & \sqrt{s} I_n \end{bmatrix}, \quad Q = \begin{bmatrix} \sqrt{s} I_m & 0 \\ B & \sqrt{s} I_n \end{bmatrix}, \quad P = \begin{bmatrix} \sqrt{s} I_m & -A \\ -B & \sqrt{s} I_n \end{bmatrix}$$

$$NP = \begin{bmatrix} sI_m - AB & 0 \\ -\sqrt{s}B & sI_n \end{bmatrix} \qquad \det N = s^{(m+n)/2} = \det Q$$

$$QP = \begin{bmatrix} sI_m & -\sqrt{s}A \\ 0 & sI_n - BA \end{bmatrix}$$

$\det(NP) = \det N \cdot \det P = \det Q \cdot \det P$

$= \det(QP)$

$\therefore s^n \det(sI_m - AB) = s^m \det(sI_n - BA)$

If $n=m$, and A and B are both $n \times n$, then

$\det(sI_n - AB) = \det(sI_n - BA)$.

3.38 Consider $Ax = y$ with A $m \times n$ and rank $(A) = m$, which implies $n \geq m$. If $n > m$, then $A'A$ is $n \times n$ and singular. Thus $(A'A)^{-1}$ is not defined and $(A'A)^{-1}A'y$ is not a solution. Because AA' is $m \times m$ and nonsingular, substituting $A'(AA')^{-1}y$ into Ax yields y. Thus $A'(AA')^{-1}y$ is a solution. If $n=m$, then both reduce to $A^{-1}y$ and are solutions of $Ax=y$.

Chapter 4

4.1 $$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x = Ax$$

$$\Delta(\lambda) = \det \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix} = \lambda^2 + 1, \quad \lambda = \pm j$$

Let $h(\lambda) = \beta_0 + \beta_1 \lambda \qquad f(\lambda) = e^{\lambda t}$

$\lambda = j: \quad e^{jt} = \beta_0 + j\beta_1$

$\lambda = -j: \quad e^{-jt} = \beta_0 - j\beta_1$

$$\beta_1 = \frac{e^{jt} - e^{-jt}}{2j} = \sin t$$

$$\beta_2 = e^{jt} - j\beta_1 = \cos t$$

$$e^{At} = \cos t\, I + \sin t\, A = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Thus

$$x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0)$$

4.2 Find unit step response of

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = [\,2 \quad 3\,]\, x$$

Method 1. Laplace transform

$$\hat{g}(s) = [\,2 \quad 3\,] \begin{bmatrix} s & -1 \\ 2 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= [\,2 \quad 3\,] \frac{1}{s^2+2s+2} \begin{bmatrix} s+2 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{5s}{s^2+2s+2} \qquad \hat{u}(s) = \frac{1}{s}$$

$$\hat{y}(s) = \hat{g}(s)\hat{u}(s) = \frac{5s}{(s+1)^2+1} \cdot \frac{1}{s}$$

$$y(t) = 5e^{-t} \sin t$$

Method 2: Using (4.7)

$$\Delta(\lambda) = \det \begin{bmatrix} \lambda & -1 \\ 2 & \lambda+2 \end{bmatrix} = \lambda^2 + 2\lambda + 2$$

$$\lambda = -1 \pm j$$

$f(\lambda) = e^{\lambda t}, \quad h(\lambda) = \beta_0 + \beta_1 \lambda$

$\lambda = -1-j: \quad e^{(-1-j)t} = \beta_0 + \beta_1(-1-j)$

$\lambda = -1+j: \quad e^{(-1+j)t} = \beta_0 + \beta_1(-1+j)$

$\Rightarrow \beta_0 = e^{-t} \sin t, \quad \beta_1 = e^{-t}(\sin t + \cos t)$

$$e^{At} = \beta_0 I + \beta_1 \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t}(\sin t + \cos t) & e^{-t}\sin t \\ -2e^{-t}\sin t & e^{-t}(\cos t - \sin t) \end{bmatrix}$$

$u(t) = 1 \quad$ for $t \geq 0$.

$$y(t) = [\,2 \quad 3\,] \int_0^t e^{A(t-\tau)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot 1 \cdot d\tau$$

$$= \int_0^t \left(5e^{-(t-\tau)} \cos(t-\tau) - 5e^{-(t-\tau)} \sin(t-\tau)\right) d\tau$$

Consider

$$5\int_0^t e^{-(t-\tau)} \cos(t-\tau)\, d\tau = -5\int_0^t e^{-(t-\tau)} \frac{d}{d\tau} \sin(t-\tau)\, d\tau$$

$$= -5\left[e^{-(t-\tau)} \sin(t-\tau) \Big|_{\tau=0}^{t} - \int_0^t e^{-(t-\tau)} \sin(t-\tau)\, d\tau \right]$$

Thus we have

$$y(t) = -5\left[e^{-0} \cdot \sin 0 - e^{-t} \sin t \right] = 5e^{-t} \sin t$$

4.3 Using e^{At} computed in Prob. 4.2 For $T = 1$.

$$A_d = e^{AT} = \begin{bmatrix} e^{-1}(\sin 1 + \cos 1) & e^{-1}\sin 1 \\ -2e^{-1}\sin 1 & e^{-1}(\cos 1 - \sin 1) \end{bmatrix}$$

$$= \begin{bmatrix} 0.5083 & 0.3096 \\ -0.6191 & -0.1108 \end{bmatrix}$$

Because A is nonsingular (it has no zero eigenvalue), we may use (4.18) to compute

$$b_d = A^{-1}(A_d - I)b = \begin{bmatrix} 1.0471 \\ -0.1821 \end{bmatrix}$$

Thus the discretized equation with $T=1$ is

$$x[k+1] = \begin{bmatrix} 0.5083 & 0.3096 \\ -0.6191 & -0.1108 \end{bmatrix} x[k] + \begin{bmatrix} 1.0471 \\ -0.1821 \end{bmatrix} u[k]$$

$$y[k] = [2 \;\; 3]\, x[k]$$

For $T=\pi$, we have

$$x[k+1] = \begin{bmatrix} -0.0432 & 0 \\ 0 & -0.0432 \end{bmatrix} x[k] + \begin{bmatrix} 1.5648 \\ -1.0432 \end{bmatrix} u[k]$$

$$y[k] = [2 \;\; 3]\, x[k]$$

4.4

$$\dot{x} = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \;\; -1 \;\; 0]\, x$$

Companion form

$$Q = [b \;\; Ab \;\; A^2b] = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 2 & -4 \\ 1 & -2 & 0 \end{bmatrix}$$

$$P = Q^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0.5 & 0.5 & -0.5 \\ 0.25 & 0 & -0.25 \end{bmatrix}$$

$$\dot{\bar{x}} = Q^{-1}AQ\bar{x} + Q^{-1}bu$$

$$= \begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & -6 \\ 0 & 1 & -4 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = cQ\bar{x} = [1 \;\; -4 \;\; 8]\, \bar{x}$$

Modal form

Eigenvalues $-1+j, \; -1-j, \; -2$

Eigenvectors

$$v_1 = \begin{bmatrix} 0 \\ 0.5774j \\ -0.5774-0.5774j \end{bmatrix}, \begin{bmatrix} 0 \\ -0.5774j \\ -0.5774+0.5774j \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{bmatrix}$$

$$Q = [\mathrm{Re}(v_1) \;\; \mathrm{Im}(v_1) \;\; v_3] = \begin{bmatrix} 0 & 0 & 0.7071 \\ 0 & 0.5774 & 0 \\ -0.5774 & -0.5774 & -0.7071 \end{bmatrix}$$

$$P = Q^{-1} = \begin{bmatrix} -1.7319 & -1.7319 & -1.7319 \\ 0 & 1.7319 & 0 \\ 1.4142 & 0 & 0 \end{bmatrix}$$

$$\dot{\bar{x}} = Q^{-1}AQ\bar{x} + Q^{-1}bu$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \bar{x} + \begin{bmatrix} -3.4638 \\ 0 \\ 1.4142 \end{bmatrix} u$$

$$y = cQ\bar{x} = [0 \;\; -0.5774 \;\; 0.7071]\, \bar{x}$$

4.5

$$\dot{x} = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \;\; -1 \;\; 0]\, x$$

For unit step input ($u(t)=1$, for $t \geq 0$), we use MATLAB to obtain

$|y|_{max} = 0.55$, $|x_1|_{max} = 0.5$

$|x_2|_{max} = 1.05$, $|x_3|_{max} = 0.52$

Let $\bar{x}_1 = x_1$, $\bar{x}_2 = 0.5x_2$, $\bar{x}_3 = x_3$ or

$$\bar{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} x = Px$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\dot{\bar{x}} = PAP^{-1}\bar{x} + Pbu$, $y = cP^{-1}\bar{x}$ or

$$\dot{\bar{x}} = \begin{bmatrix} -2 & 0 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & -4 & -2 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \;\; -2 \;\; 0]\, \bar{x}$$

For this equation, we have

$|y|_{max} = 0.55$, $|\bar{x}_1|_{max} = 0.5$

$|\bar{x}_2|_{max} = 0.525$, $|\bar{x}_3|_{max} = 0.52$

The step input must have magnitude less than $(10/0.55) = 18.2$ to avoid saturation.

4.6 Direct verification

4.7 Direct verification

4.8 A necessary condition for two state equations to be equivalent is that they have the same set of eigenvalues. The first equation has eigenvalues 2, 2 and 1. The second equation has eigenvalue 2, 2 and -1. Thus, they are not equivalent.

Using the fact that the inverse of a triangular matrix is again triangular, we can readily verify that

$$[1\ \ -1\ \ 0]\begin{bmatrix} s-2 & -1 & -2 \\ 0 & s-2 & -2 \\ 0 & 0 & s-1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$=[1\ \ -1]\begin{bmatrix} s-2 & -1 \\ 0 & s-2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$=[1\ \ -1]\begin{bmatrix} \frac{1}{s-2} & \frac{1}{(s-2)^2} \\ 0 & \frac{1}{s-2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{(s-2)^2}$$

The second equation also has transfer function $1/(s-2)^2$. Thus, they are zero-state equivalent.

4.9 Define $Z = [Z_1\ Z_2 \cdots Z_r]$, where Z_i is $q \times q$ and Z is $q \times rq$, by

$$Z = C(sI-A)^{-1}$$

or

$$Z(sI-A) = C \qquad sZ = ZA + C$$

Using the forms of A and C, we have

$$sZ_1 = -\alpha_1 Z_1 - \alpha_2 Z_2 - \cdots - \alpha_r Z_r + I_q$$

$$sZ_2 = Z_1$$

$$\vdots$$

$$sZ_{r-1} = Z_{r-2}$$

$$sZ_r = Z_{r-1}$$

From these, we have

$$Z_2 = \frac{1}{s}Z_1, \quad Z_3 = \frac{1}{s}Z_2 = \frac{1}{s^2}Z_1, \quad \cdots \quad Z_r = \frac{1}{s^{r-1}}Z_1$$

and

$$(s^r + \alpha_1 s^{r-1} + \cdots + \alpha_r)Z_1 = s^{r-1} I_q$$

Thus we have

$$Z_1 = \frac{s^{r-1}}{d(s)} I_q, \quad Z_2 = \frac{s^{r-2}}{d(s)} I_q, \quad \cdots, \quad Z_r = \frac{1}{d(s)} I_q$$

where $d(s) = s^r + \alpha_1 s^{r-1} + \cdots + \alpha_r$

The transfer matrix is

$$G(s) = C(sI-A)^{-1}B = ZB = [Z_1 \ \cdots \ Z_r]\begin{bmatrix} N_1 \\ \vdots \\ N_r \end{bmatrix}$$

$$= \frac{1}{d(s)}[N_1 s^{r-1} + N_2 s^{r-2} + \cdots + N_{r-1}s + N_r]$$

This completes the verification

4.10 Direct substitution

4.11

$$G(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{2s-3}{(s+1)(s+2)} \\ \frac{s-2}{s+1} & \frac{s}{s+2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} \frac{2}{s+1} & \frac{2s-3}{(s+1)(s+2)} \\ \frac{-3}{s+1} & \frac{-2}{s+2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \frac{1}{s^2+3s+2}\begin{bmatrix} 2s+4 & 2s-3 \\ -3s-6 & -2s-2 \end{bmatrix}$$

Using (4.34), we have

$$\dot{x} = \left[\begin{array}{cc:cc} -3 & 0 & -2 & 0 \\ 0 & -3 & 0 & -2 \\ \hdashline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] x + \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hdashline 0 & 0 \\ 0 & 0 \end{array}\right] u$$

$$y = \left[\begin{array}{cc:cc} 2 & 2 & 4 & -3 \\ -3 & -2 & -6 & -2 \end{array}\right] x + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} u$$

This is a 4-dimensional realization

4.12 Consider the first column

$$\begin{bmatrix} \frac{2}{s+1} \\ \frac{s-2}{s+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{s+1}\begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\dot{x}_1 = -x_1 + u_1$$

$$y_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1$$

The second column:

$$\begin{bmatrix} \dfrac{2s-3}{(s+1)(s+2)} \\ \dfrac{s}{s+2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{s^2+3s+2} \begin{bmatrix} 2s-3 \\ -2s-2 \end{bmatrix}$$

$$\dot{x}_2 = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_2$$

$$y_2 = \begin{bmatrix} 2 & -3 \\ -2 & -2 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2$$

Combining these yields

$$\dot{x} = \left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & -3 & -2 \\ 0 & 1 & 0 \end{array}\right] x + \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \\ 0 & 0 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = \left[\begin{array}{c|cc} 2 & 2 & -3 \\ -3 & -2 & -2 \end{array}\right] x + \left[\begin{array}{c|c} 0 & 0 \\ 1 & 1 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This realization has dimension 3, one less than the one in Problem 4.11

4.13 First row

$$\left[\frac{2}{s+1} \quad \frac{2s-3}{(s+1)(s+2)} \right] = \frac{1}{s^2+3s+2} [2s+4 \quad 2s-3]$$

$$\dot{x}_1 = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 2 & 2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y_1 = [1 \quad 0] x_1 + [0 \quad 0] u$$

Second row:

$$\left[\frac{s-2}{s+1} \quad \frac{s}{s+2} \right] = [1 \quad 1] + \left[\frac{-3}{s+1} \quad \frac{-2}{s+2} \right]$$

$$= [1 \quad 1] + \frac{1}{s^2+3s+2} [-3s-6 \quad -2s-2]$$

$$\dot{x}_2 = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} x_2 + \begin{bmatrix} -3 & -2 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y_2 = [1 \quad 0] x_2 + [1 \quad 1] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left[\begin{array}{cc|cc} -3 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ \hline 0 & 0 & -3 & 1 \\ 0 & 0 & -2 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left[\begin{array}{cc} 2 & 2 \\ 4 & -3 \\ \hline -3 & -2 \\ -6 & -2 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left[\begin{array}{cc} 0 & 0 \\ \hline 1 & 1 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This realization has dimension 4, one more than the one in Problem 4.12 and the same as the one in Problem 4.11

4.14 $$G(s) = \left[\frac{-(12s+6)}{3s+34} \quad \frac{22s+23}{3s+34} \right]$$

$$= \left[-4 \quad \frac{22}{3}\right] + \left[\frac{130}{3s+34} \quad \frac{-679/3}{3s+34} \right]$$

$$= \left[-4 \quad \frac{22}{3}\right] + \frac{1}{s+34/3} \left[\frac{130}{3} \quad \frac{-679}{9} \right]$$

$$\dot{x} = \frac{-34}{3} x + \left[\frac{130}{3} \quad \frac{-679}{9} \right] u$$

$$y = x + \left[-4 \quad \frac{22}{3}\right] u$$

4.15 Using the formula in Problem 3.26, we write

$$\hat{g}(s) = c(sI-A)^{-1}b = \frac{1}{\Delta(s)} [cR_0 b s^{n-1} + cR_1 b s^{n-2} + \cdots + cR_{n-1} b]$$

The numerator of $\hat{g}(s)$ has degree $m \Leftrightarrow$

$$cR_{n-m-1} b \neq 0, \; cR_i b = 0 \quad \text{for } i = 0, 1, \cdots, n-m-2$$

Using the formula in Problem 3.26, we have

$$cR_0 b = cb = 0$$

$$cR_1 b = cAb + \alpha_1 cb = 0 \Rightarrow cAb = 0$$

$$\vdots$$

$$cR_{n-m-2} b = 0 \Rightarrow cA^{n-m-2} b = 0$$

$$cR_{n-m-1} b \neq 0 \Rightarrow cA^{n-m-1} b \neq 0$$

4.16 $\dot{x}_2 = t x_2 \rightarrow x_2(t) = x_2(0) e^{0.5t^2}$

$$\dot{x}_1 = x_2(t) \rightarrow x_1(t) = \left(\int_0^t e^{0.5\tau^2} d\tau\right) x_2(0) + x_1(0)$$

Let $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $x(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Let $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $x(t) = \begin{bmatrix} \int_0^t e^{0.5\tau^2} d\tau \\ e^{0.5t^2} \end{bmatrix}$

Thus a fundamental matrix is

$$X(t) = \begin{bmatrix} 1 & \int_0^t e^{0.5\tau^2} d\tau \\ 0 & e^{0.5t^2} \end{bmatrix}$$

$$X^{-1}(t)=\frac{1}{e^{0.5t^2}}\begin{bmatrix} e^{0.5t^2} & -\int_0^t e^{0.5\tau^2}d\tau \\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix} 1 & -e^{-0.5t^2}\int_0^t e^{0.5\tau^2}d\tau \\ 0 & e^{-0.5t^2}\end{bmatrix}$$

The state transition matrix is

$$\phi(t,t_0)=X(t)X^{-1}(t_0)=\begin{bmatrix} 1 & -e^{0.5t^2}\int_{t_0}^t e^{0.5\tau^2}d\tau \\ 0 & e^{0.5(t^2-t_0^2)}\end{bmatrix}$$

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$$\dot{x}_2(t)=-x_2(t)\longrightarrow x_2(t)=e^{-t}x_2(0)$$

$$\dot{x}_1(t)=-x_1(t)+e^{2t}x_2(t)=-x_1(t)+e^{t}x_2(0)$$

$$x_1(t)=e^{-t}x_1(0)+\int_0^t e^{-(t-\tau)}e^{\tau}x_2(0)\,d\tau$$

$$=e^{-t}x_1(0)+x_2(0)e^{-t}\int_0^t e^{2\tau}d\tau$$

$$=e^{-t}x_1(0)+\frac{1}{2}x_2(0)e^{-t}(e^{2t}-1)$$

$$x(0)=\begin{bmatrix}1\\0\end{bmatrix}\rightarrow x(t)=\begin{bmatrix}e^{-t}\\0\end{bmatrix}$$

$$x(0)=\begin{bmatrix}1\\2\end{bmatrix}\rightarrow x(t)=\begin{bmatrix}e^{t}\\2e^{-t}\end{bmatrix}$$

A fundamental matrix is

$$X(t)=\begin{bmatrix}e^{-t} & e^{t}\\ 0 & 2e^{-t}\end{bmatrix}$$

$$X^{-1}(t)=\frac{1}{2e^{-2t}}\begin{bmatrix}2e^{-t} & -e^{t}\\ 0 & e^{-t}\end{bmatrix}$$

The state transition matrix is

$$\phi(t,t_0)=X(t)X^{-1}(t_0)$$

$$=\begin{bmatrix}e^{-(t-t_0)} & \frac{1}{2}(e^{t}e^{t_0}-e^{-t}e^{t_0})\\ 0 & e^{-(t-t_0)}\end{bmatrix}$$

4.17 $\frac{\partial}{\partial t}\phi(t_0,t)=X(t_0)\frac{\partial}{\partial t}X^{-1}(t)$

$$\frac{d}{dt}(X(t)X^{-1}(t))=\dot{X}(t)X^{-1}(t)+X(t)\frac{d}{dt}X^{-1}(t)$$

$$=\frac{d}{dt}(I)=0$$

$$\therefore\ \frac{d}{dt}X^{-1}(t)=-X^{-1}(t)\dot{X}(t)X^{-1}(t)$$

$$=-X^{-1}(t)A(t)X(t)X^{-1}(t)$$

$$=-X^{-1}(t)A(t)$$

Thus we have

$$\frac{\partial}{\partial t}\phi(t_0,t)=X(t_0)(-X^{-1}(t))A(t)$$

$$=-\phi(t_0,t)A(t)$$

4.18 $$\frac{\partial}{\partial t}\phi(t,t_0)=\begin{bmatrix}\frac{\partial}{\partial t}\varphi_{11}(t,t_0) & \frac{\partial}{\partial t}\varphi_{12}(t,t_0)\\ \frac{\partial}{\partial t}\varphi_{21}(t,t_0) & \frac{\partial}{\partial t}\varphi_{22}(t,t_0)\end{bmatrix}$$

$$=\begin{bmatrix}a_{11}(t) & a_{12}(t)\\ a_{21}(t) & a_{22}(t)\end{bmatrix}\begin{bmatrix}\varphi_{11}(t,t_0) & \varphi_{12}(t,t_0)\\ \varphi_{21}(t,t_0) & \varphi_{22}(t,t_0)\end{bmatrix}$$

Substituting these $\frac{\partial}{\partial t}\varphi_{ij}(t,t_0)$ into

$$\frac{\partial}{\partial t}\det\phi(t,t_0)=\frac{\partial}{\partial t}[\varphi_{11}\varphi_{22}-\varphi_{21}\varphi_{12}]$$

$$=\left(\frac{\partial}{\partial t}\varphi_{11}\right)\varphi_{22}+\varphi_{11}\left(\frac{\partial}{\partial t}\varphi_{22}\right)-\left(\frac{\partial}{\partial t}\varphi_{21}\right)\varphi_{12}-\varphi_{21}\left(\frac{\partial}{\partial t}\varphi_{12}\right)$$

and simple manipulation yield

$$\frac{\partial}{\partial t}\det\phi(t,t_0)=(a_{11}(t)+a_{22}(t))\det\phi(t,t_0)$$

Thus

$$\det\phi(t,t_0)=\exp\left[\int_{t_0}^{t}(a_{11}(\tau)+a_{22}(\tau))\,d\tau\right]$$

4.19 $$\phi(t_0,t_0)=\begin{bmatrix}\phi_{11}(t_0,t_0) & \phi_{12}(t_0,t_0)\\ \phi_{21}(t_0,t_0) & \phi_{22}(t_0,t_0)\end{bmatrix}=I$$

Thus $\phi_{21}(t_0,t_0)=0$ and $\phi_{22}(t_0,t_0)=I$.

$$\frac{\partial}{\partial t}\phi_{21}(t,t_0)=0\cdot\phi_{11}(t,t_0)+A_{22}(t)\phi_{21}(t,t_0)$$

$$\frac{\partial}{\partial t}\phi_{22}(t,t_0)=0\cdot\phi_{12}(t,t_0)+A_{22}(t)\phi_{22}(t,t_0)$$

The equation

$$\frac{\partial}{\partial t}\phi_{22}(t,t_0)=A_{22}(t)\phi_{22}(t,t_0)$$

with $\phi_{22}(t_0,t_0)=I$ yields the unique solution of $\phi_{22}(t,t_0)$. The equation

$$\frac{\partial}{\partial t}\phi_{21}(t,t_0)=A_{22}(t)\phi_{21}(t,t_0)$$

with $\phi_{21}(t_0,t_0)=0$ yields the unique solution $\phi_{21}(t,t_0)\equiv 0$. With $\phi_{21}\equiv 0$, then $\frac{\partial}{\partial t}\phi_{11}(t,t_0)=A_{11}(t)\phi_{11}(t,t_0)+A_{12}(t)\cdot 0$

$$=A_{11}(t)\phi_{11}(t,t_0)$$

4.20 $\dot{x} = \begin{bmatrix} -\sin t & 0 \\ 0 & -\cos t \end{bmatrix} x$

$\dot{x}_1 = -\sin t \, x_1(t) \rightarrow x_1(t) = e^{\cos t} x_1(0)$

$\dot{x}_2 = -\cos t \, x_2(t) \rightarrow x_2(t) = e^{-\sin t} x_2(0)$

$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow x(t) = \begin{bmatrix} e^{\cos t} \\ 0 \end{bmatrix}$

$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow x(t) = \begin{bmatrix} 0 \\ e^{-\sin t} \end{bmatrix}$

Fundamental matrix

$$X(t) = \begin{bmatrix} e^{\cos t} & 0 \\ 0 & e^{-\sin t} \end{bmatrix}$$

$$X^{-1}(t) = \begin{bmatrix} e^{-\cos t} & 0 \\ 0 & e^{\sin t} \end{bmatrix}$$

State transition matrix

$$\Phi(t, t_0) = X(t) X^{-1}(t_0) = \begin{bmatrix} e^{\cos t - \cos t_0} & 0 \\ 0 & e^{-\sin t + \sin t_0} \end{bmatrix}$$

4.21 $X(t) = e^{At} C e^{Bt}$, $X(0) = I \cdot C \cdot I = C$

$\dot{X}(t) = A e^{At} C e^{Bt} + e^{At} C e^{Bt} B$

$= A X(t) + X(t) B$

4.22 $\dot{A}(t) = A_1 e^{A_1 t} A(0) e^{-A_1 t} + e^{A_1 t} A(0) e^{-A_1 t}(-A_1)$

$= A_1 A(t) - A(t) A_1$

$\det(\lambda I - A(t)) = \det(e^{A_1 t} \lambda I e^{-A_1 t} - A(t))$

$= \det[e^{A_1 t}(\lambda I - A(0)) e^{-A_1 t}]$

$= \det e^{A_1 t} \det e^{-A_1 t} \det(\lambda I - A(0))$

$= \det(\lambda I - A(0))$ (independent of t)

4.23 The equation is periodic with period $T = 2\pi$.

$$X(t) = \begin{bmatrix} e^{\cos t} & 0 \\ 0 & e^{-\sin t} \end{bmatrix}$$

$$X(t+2\pi) = \begin{bmatrix} e^{\cos(t+2\pi)} & 0 \\ 0 & e^{-\sin(t+2\pi)} \end{bmatrix}$$

$$= \begin{bmatrix} e^{\cos t} & 0 \\ 0 & e^{-\sin t} \end{bmatrix} = X(t)$$

From (4.76), we have $\bar{A} = 0$ and

$$P(t) = e^{\bar{A}t} X^{-1}(t) = \begin{bmatrix} e^{-\cos t} & 0 \\ 0 & e^{\sin t} \end{bmatrix}.$$

The transformation $\bar{x}(t) = P(t) x(t)$ will transform the equation into

$$\dot{\bar{x}} = 0 \cdot \bar{x} = 0$$

4.24 $\dot{x} = Ax + Bu$

$y = Cx$

Consider $\bar{x} = P(t) x = e^{-At} x$. Then

$\bar{A} = [P(t) A + \dot{P}(t)] P^{-1}(t)$

$= [e^{-At} A - e^{-At} A] e^{At} = 0$

$\bar{B} = P(t) B = e^{-At} B$

$\bar{C} = C P^{-1}(t) = C e^{At}$

4.25 $g(t) = t^2 e^{\lambda t}$

$g(t, \tau) = g(t-\tau) = (t-\tau)^2 e^{\lambda(t-\tau)}$

$= (t^2 - 2t\tau + \tau^2) e^{\lambda t} e^{-\lambda \tau}$

$$= [e^{\lambda t} \quad t e^{\lambda t} \quad t^2 e^{\lambda t}] \begin{bmatrix} \tau^2 e^{-\lambda\tau} \\ -2\tau e^{-\lambda \tau} \\ e^{-\lambda\tau} \end{bmatrix}$$

a time-varying realization:

$$\dot{x} = 0 \cdot x + \begin{bmatrix} t^2 e^{-\lambda t} \\ -2t e^{-\lambda t} \\ e^{-\lambda t} \end{bmatrix} u(t)$$

$$y = [e^{\lambda t} \quad t e^{\lambda t} \quad t^2 e^{\lambda t}] x$$

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$$\hat{g}(s) = \mathcal{L}[g(t)] = \frac{2}{(s-\lambda)^3}$$

$$= \frac{2}{s^3 - 3\lambda s^2 + 3\lambda^2 s - \lambda^3}$$

a time-invariant realization

$$\dot{x} = \begin{bmatrix} 3\lambda & -3\lambda^2 & \lambda^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 0 \quad 2]\, x$$

4.26 $g(t,\tau) = \sin t\, e^{-t} \cdot e^{\tau} \cos\tau$

a time-varying realization

$$\dot{x} = 0 \cdot x + e^{t} \cos t\, u(t)$$

$$y = \sin t\, e^{-t} x$$

Because $g(t,\tau)$ cannot be expressed as $g(t-\tau)$, it cannot be realized as a linear time-invariant equation.

Chapter 5

5.1 The transfer function from u to y is

$$\hat{g}(s) = \frac{s \cdot \frac{1}{s}}{s + \frac{1}{s}} = \frac{s}{s^2+1}$$

If $u(t) = \sin t$, then

$$\hat{y}(s) = \hat{g}(s)\,\hat{u}(s) = \frac{s}{s^2+1} \frac{1}{s^2+1} = \frac{s}{(s^2+1)^2}$$

and $y(t) = 0.5\, t \sin t$

which is not bounded. Thus the network is not BIBO stable.

5.2 $$\hat{g}(s) = \int_0^\infty g(t) e^{-st}\, dt$$

Let $s = \sigma + j\omega$. If $\sigma \geq 0$, then

$$|e^{-st}| = |e^{-\sigma t}||e^{-j\omega t}| = e^{-\sigma t} \leq 1$$

for all t. If the system is BIBO stable, then $\int_0^\infty |g(t)| < \infty$. Thus we have for $Re\ s \geq 0$,

$$|\hat{g}(s)| \leq \int_0^\infty |g(t)||e^{-st}|\, dt \leq \int_0^\infty |g(t)|\, dt < \infty$$

5.3 $$\int_0^\infty |g(t)|\, dt = \int_0^\infty \frac{1}{1+t}\, dt = \ln(1+t)\Big|_0^\infty = \infty$$

Thus the system is not BIBO stable.

For $g(t) = t e^{-t}$ we have

$$\hat{g}(s) = \mathcal{L}[g(t)] = \frac{1}{(s+1)^2}$$

All its poles have negative real part, thus the system is BIBO stable.

5.4 $\hat{g}(s) = \dfrac{e^{-2s}}{s+1}$ irrational function of s

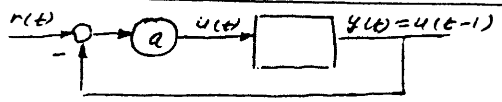
$$g(t) = \mathcal{L}^{-1}[\hat{g}(s)] = \begin{cases} e^{-(t-2)} & \text{for } t \geq 2 \\ 0 & \text{for } t < 2 \end{cases}$$

$$\int_0^\infty |g(t)|\, dt = \int_2^\infty e^{-(t-2)}\, dt = \int_0^\infty e^{-\tau}\, d\tau$$

$$= -e^{-\tau}\Big|_{\tau=0}^{\infty} = -[0-1] = 1$$

Thus the system is BIBO stable.

5.5



If $r(t) = \delta(t)$, then

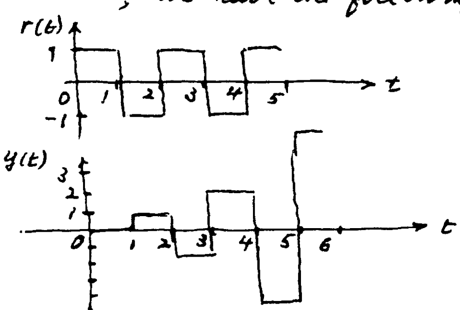
$$g_f(t) = y(t) = a\delta(t-1) - a^2\delta(t-2) + a^3\delta(t-3) - a^4\delta(t-4) + \cdots$$

$$\int_0^\infty |g_f(t)|\,dt = |a| + |a|^2 + |a|^3 + \cdots$$

$$= |a|\sum_{i=0}^{\infty}|a|^i = \begin{cases} \dfrac{|a|}{1-|a|} & \text{if } |a| < 1 \\ \infty & \text{if } |a| \geq 1 \end{cases}$$

Thus the feedback system is BIBO stable if and only if $|a| < 1$.

For $a = 1$, we have the following pair



The bounded input excites an unbounded output.

5.6 $\hat{g}(s) = \dfrac{s-2}{s+1}$

If $u(t) = 3$, then $y(t) \to \hat{g}(0)\cdot 3 = -6$

If $u(t) = \sin 2t$, then

$$y(t) \to |\hat{g}(j2)|\sin(2t + \measuredangle\hat{g}(j2))$$

$$= 1.26\sin(2t + 1.25)$$

5.7

$$\hat{g}(s) = [-2\ \ 3]\begin{bmatrix} s+1 & -10 \\ 0 & s-1 \end{bmatrix}^{-1}\begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$= [-2\ \ 3]\begin{bmatrix} \dfrac{1}{s+1} & \dfrac{10}{s^2-1} \\ 0 & \dfrac{1}{s-1} \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \frac{4}{s+1}$$

It is BIBO stable.

5.8 $g[k] = k(0.8)^k, \quad k = 0, 1, 2, \cdots$

$$\hat{g}(z) = \mathcal{Z}[g[k]] = \frac{0.8z}{(z-0.8)^2}$$

Its poles lie inside the unit circle, thus the system is BIBO stable.

5.9 The matrix has eigenvalues -1 and 1, thus the equation is not asymptotically stable nor marginally stable.

5.10 The matrix has eigenvalues $-1, 0, 0$; thus the equation is not asymptotically stable. If the repeated eigenvalue 0 is a simple root of the minimal polynomial or, equivalently, has only Jordan blocks of order 1, then the equation is marginally stable. We compute the eigenvectors associated with $\lambda = 0$:

$$(A - \lambda I)v = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v = 0$$

which yields two linearly independent eigenvectors $[0\ 1\ 0]'$ and $[1\ 0\ 1]'$. Thus the Jordan form of A is

$$\hat{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the equation is marginally stable.

5.11 The matrix has eigenvalues $-1, 0, 0$; thus the equation is not asymptotically stable. We compute the eigenvectors associated with $\lambda = 0$:

$$(A - \lambda I)v = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} v = 0$$

It has only one linearly independent

eigenvector. Thus its Jordan form is

$$\hat{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

It has one Jordan block with order 2 associated with 0. Thus the equation is not marginally stable.

5.12 The matrix has eigenvalues 0.9, 1, 1; thus the discrete-time system is not asymptotically stable. Its Jordan form is

$$\hat{A} = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus it is marginally stable.

5.13 The matrix has eigenvalues 0.9, 1 and 1 and its Jordan form, as in Prob. 5.11, is

$$\hat{A} = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus the equation is not marginally stable, nor asymptotically stable.

5.14 $A = \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix}$. Select $N = I$.

$$\begin{bmatrix} 0 & -0.5 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} + \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Equating the (i,j)th entry:

$(1,1)$: $-0.5m_{12} - 0.5m_{12} = -1 \Rightarrow m_{12} = 1$

$(2,2)$: $m_{12} - m_{22} + m_{12} - m_{22} = -1 \Rightarrow m_{22} = 1.5$

$(1,2)$: $-0.5m_{22} + m_{11} - m_{12} = 0 \Rightarrow m_{11} = 1.75$

$$M = \begin{bmatrix} 1.75 & 1 \\ 1 & 1.5 \end{bmatrix}$$

Leading principal minors $1.75 > 0$

$1.75 \times 1.5 - 1 \times 1 > 0$

M is positive definite. Thus all eigenvalues of A have negative real parts.

5.15

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} - \begin{bmatrix} 0 & -0.5 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$(1,1)$: $m_{11} - 0.25m_{22} = 1$

$(1,2)$: $-0.5m_{22} + 1.5m_{12} = 0$

$(2,2)$: $m_{22} - m_{11} + 2m_{12} - m_{22} = 1$

From these, we can obtain $m_{12} = 1.6$, $m_{22} = 4.8$, $m_{11} = 2.2$.

$$M = \begin{bmatrix} 2.2 & 1.6 \\ 1.6 & 4.8 \end{bmatrix} \quad \text{positive definite}$$

Thus all eigenvalues of A have magnitude less than 1. As a check, the eigenvalues of A are $-0.5 \pm j0.5$. Both have negative real parts and have magnitudes less than 1.

5.16 Let $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$. Then

$$A'M + MA = -\begin{bmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_3^2 \end{bmatrix} = -\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}[a_1\ a_2\ a_3]$$

$$=: -\bar{N}'\bar{N} =: -N$$

It is clear that all eigenvalues of A have negative real parts and N is positive semidefinite. We compute

$$\begin{bmatrix} \bar{N} \\ \bar{N}A \\ \bar{N}A^2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ \lambda_1a_1 & \lambda_2a_2 & \lambda_3a_3 \\ \lambda_1^2a_1 & \lambda_2^2a_2 & \lambda_3^2a_3 \end{bmatrix} =: \Theta$$

$$\det\Theta = a_1a_2a_3 \det\begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}$$

$$= a_1a_2a_3 \det\begin{bmatrix} 1 & 0 & 0 \\ \lambda_1 & \lambda_2-\lambda_1 & \lambda_3-\lambda_1 \\ \lambda_1^2 & \lambda_2^2-\lambda_1^2 & \lambda_3^2-\lambda_1^2 \end{bmatrix}$$

$$= a_1a_2a_3(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\det\begin{bmatrix} 1 & 1 \\ \lambda_2+\lambda_1 & \lambda_3+\lambda_1 \end{bmatrix}$$

$$= a_1a_2a_3(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)$$

$\det\Theta$ is nonzero if $a_i \neq 0$ and λ_i are distinct. Thus Θ has rank 3 and M is positive definite. (Corollary 5.5)

5.17 $M_1 = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$ is not positive definite because $[1\ 0] M \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$. Its eigenvalues are 1 and 2. $M_2 = \begin{bmatrix} 2 & 1 \\ 1.9 & 1 \end{bmatrix}$ is not positive definite because

$$[0.5805\ \ -0.8142]\, M_2 \begin{bmatrix} 0.5805 \\ -0.8142 \end{bmatrix}$$

$$= -0.0338.$$

Its leading principal minors are 2 and $(2\times 1 - 1.9\times 1) = 0.1$; both are positive. Therefore, the two assertions do not hold. Because

$$x'M_i x = \frac{1}{2}(x'M_i x + x'M_i' x)$$

$$= x'\left(\frac{1}{2}(M_i + M_i')\right)x$$

we may check the positive definiteness of M_i by forming the symmetric matrix $\bar{M}_i = \frac{1}{2}(M_i + M_i')$ and then check $\bar{M}_i$. For example, we form

$$\bar{M}_1 = \frac{1}{2}\left(\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}\right) = \begin{bmatrix} 0 & -0.5 \\ -0.5 & 3 \end{bmatrix}$$

It is not positive definite because its leading principal minors are 0 and -0.25. Similarly, we form

$$\bar{M}_2 = \begin{bmatrix} 2 & 1.45 \\ 1.45 & 1 \end{bmatrix}$$

It is not positive definite because its leading principal minors are 2 and $2\times 1 - (1.45)^2 = -0.1025$

5.18 $A'M + MA + 2\mu M = -N$

$$(A' + \mu I)M + M(A + \mu I) = -N$$

If $N > 0$ and $M > 0$, then all eigenvalues of $(A+\mu I)$ have magnitudes less than 0, or $\mathrm{Re}\,\lambda_i(A+\mu I) < 0$. Using Problem 3.19, we have

$$\lambda_i(A+\mu I) = \lambda_i(A) + \mu.$$ Thus

$$\mathrm{Re}[\lambda_i(A)] + \mu < 0 \quad \text{or} \quad \mathrm{Re}\,\lambda_i(A) < -\mu$$

5.19 $\rho^2 M - A'MA = \rho^2 N$

$$M - \left(\frac{1}{\rho}A'\right)M\left(\frac{1}{\rho}A\right) = N$$

If $N > 0$ and $M > 0$, then

$$\left|\lambda_i\left(\frac{1}{\rho}A\right)\right| < 1$$

Thus $|\lambda_i(A)| < \rho$

5.20 $g(t,\tau) = e^{-2|t| - |\tau|}$, for $t \ge \tau$

$$\int_{t_0}^{t} |g(t,\tau)|\, d\tau = e^{-2|t|}\int_{t_0}^{t} e^{-|\tau|}\, d\tau =: B$$

For $t_0 \le t < 0$, we have

$$B = e^{2t}\int_{t_0}^{t} e^{\tau}\, d\tau = e^{2t}(e^{t} - e^{t_0}) < 1$$

For $t_0 \le t$ with $t \ge 0$, we have

$$B = e^{-2t}\left[\int_{t_0}^{0} e^{\tau}\, d\tau + \int_{0}^{t} e^{-\tau}\, d\tau\right] \quad \text{with } t_0 < 0$$

$$= e^{-2t}\left[(1 - e^{t_0}) - (e^{-t} - 1)\right]$$

$$= e^{-2t}\left[-e^{-t} - e^{t_0} + 2\right] < \infty$$

Thus the system is BIBO stable.

$$g(t,\tau) = \sin t\,(e^{-(t-\tau)})\cos\tau$$

$$\int_{t_0}^{\infty} |g(t-\tau)|\, d\tau \le \int_{t_0}^{t} e^{-(t-\tau)}\, d\tau = e^{-t}\int_{t_0}^{t} e^{\tau}\, d\tau$$

$$= e^{-t}\left[e^{t} - e^{t_0}\right] = 1 - e^{-(t-t_0)} \le 1$$

for all t_0 and $t \ge t_0$. Thus the system is BIBO stable.

5.21 $\dot{x} = 2tx + u$, $y = e^{-t^2}x$

For this scalar equation, we have

$$\Phi(t, t_0) = e^{\int_{t_0}^{t} 2\tau\, d\tau} = e^{(t^2 - t_0^2)}$$

Thus $g(t,\tau) = e^{-t^2}\phi(t,\tau)\cdot 1 = e^{-t^2+t^2-\tau^2}$

$$= e^{-\tau^2}$$

$$\int_{t_0}^{t} |g(t,\tau)|\,d\tau = \int_{t_0}^{t} e^{-\tau^2}d\tau < \infty$$ for all t_0 and $t \geq t_0$ because $e^{-\tau^2} < e^{-|\tau|}$ for $1 \leq \tau \leq \infty$

Thus the equation is BIBO stable.

Because $|\phi(t,t_0)| = e^{t^2-t_0^2} \to \infty$ as $t \to \infty$, the equation is not marginally stable, nor asymptotically stable.

5.22 $\bar{x} = e^{-t^2}x$ or $P(t) = e^{-t^2}$ and $P^{-1}(t) = e^{t^2}$

Using (4.70), we have

$$\bar{A} = [P(t)A(t) + \dot{P}(t)]P^{-1}(t)$$

$$= [2te^{-t^2} - 2te^{-t^2}]e^{t^2} = 0$$

Thus the equivalent equation is

$$\dot{\bar{x}} = 0\cdot\bar{x} + e^{-t^2}u$$

$$y = e^{-t^2}e^{t^2}\cdot\bar{x} = \bar{x}$$

$$\bar{g}(t,\tau) = C(t)\phi(t,\tau)B(\tau) = 1\times 1\times e^{-\tau^2} = e^{-\tau^2}$$

The impulse response remains unchanged, therefore the equation is BIBO stable. The zero-input response is governed by the time-invariant equation $\dot{\bar{x}} = 0\cdot\bar{x}$ with eigenvalue 0. Thus the equation is marginally stable; it is not asymptotically stable. The transformation $P(t) = e^{-t^2}$ is not a Lyapunov transformation because $P^{-1}(t) = e^{t^2}$ is not bounded. Therefore marginal and asymptotic stabilities are not invariant under the transformation.

5.23 $$\dot{x} = \begin{bmatrix} -1 & 0 \\ -e^{-3t} & 0 \end{bmatrix} x \quad \text{for } t_0 \geq 0$$

$\dot{x}_1(t) = -x(t) \rightarrow x(t) = e^{-t}x_1(0)$

$\dot{x}_2(t) = -e^{-3t}x_1(t) = -e^{-4t}x_1(0)$

$\rightarrow x_2(t) = 0.2\,[e^{-5t} - 1]\,x_1(0) + x_2(0)$

$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x(t) = \begin{bmatrix} e^{-t} \\ 0.2\,(e^{-5t}-1) \end{bmatrix}$

$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $x(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$X(t) = \begin{bmatrix} e^{-t} & 0 \\ 0.2\,(e^{-5t}-1) & 1 \end{bmatrix}$$

$$X^{-1}(t) = \begin{bmatrix} e^{t} & 0 \\ 0.2\,(e^{t} - e^{-4t}) & 1 \end{bmatrix}$$

$$\phi(t,t_0) = X(t)X^{-1}(t_0) = \begin{bmatrix} e^{-(t-t_0)} & 0 \\ 0.2\,(e^{-5t+t_0} - e^{-4t_0}) & 1 \end{bmatrix}$$

For $t_0 \geq 0$ and $t \geq t_0$, every entry of $\phi(t,t_0)$ is bounded, thus the equation is marginally stable. A necessary condition for $\|\phi(t,t_0)\| \to 0$ as $t \to \infty$ is that every entry approaches zero. This is not the case. Thus the equation is not asymptotically stable.

Chapter 6

6.1

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}, \quad \rho(C) = 3. \quad \text{controllable}$$

$$O = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix}, \quad \rho(O) = 1. \quad \text{not observable}$$

6.2

$$[B \; AB] = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix},$$ It has full row rank. thus controllable

$$O = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & -2 & 4 \end{bmatrix}, \quad \rho(O) = 3 \quad \text{observable}$$

6.3

$$[AB \; A^2B \cdots A^nB] = A[B \; AB \cdots A^{n-1}B]$$

$$\rho([AB \; A^2B \cdots A^nB]) = \rho([B \; AB \cdots A^{n-1}B])$$

if and only if A is nonsingular.

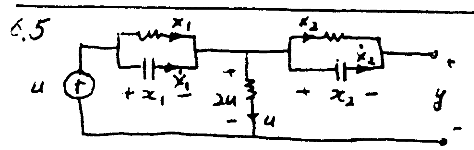
6.4 $\{A, B\}$ controllable $\Leftrightarrow$

$$\text{rank} \begin{bmatrix} A_{11} - sI & A_{12} & B_1 \\ A_{21} & A_{22} - sI & 0 \end{bmatrix} = n$$ for every s (In Theorem 6.1, it is stated for every eigenvalue of A. However, if s is not an eigenvalue, then $(A - sI)$ has rank n. Thus the statement holds for every s.)

$\Leftrightarrow [A_{21} \; A_{22} - sI]$ has full row rank

$\Leftrightarrow \{A_{22}, A_{21}\}$ controllable.

6.5



$$\dot{x}_1 = u - x_1, \quad \dot{x}_2 = -x_2$$

$$y = -x_2 + 2u$$

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [0 \; -1] x + 2u$$

$$C = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \rho(C) = 1 \quad \text{not controllable}$$

$$O = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}, \quad \rho(O) = 1 \quad \text{not observable}$$

6.6 For the state equation in Problem 6.1, we have $\mu = 3$. If the observability index is defined as the least integer such that

$$\rho\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\nu-1} \end{bmatrix}\right) = \rho\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\nu} \end{bmatrix}\right)$$

then $\nu = 1$. (Note that the controllability and observability indices are defined in the text for controllable and observable state equations.)

For the state equation in Problem 6.2, we have $\mu_1 = 2$, $\mu_2 = 1$, $\mu = \max\{\mu_1, \mu_2\} = 2$ and $\nu = 3$.

6.7 $\mu_i = 1$ for all i and $\mu = 1$

6.8 $$\dot{x} = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad y = [1 \; 1] x$$

$$C = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}.$$ We select $P^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Then $P = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and

$$PAP^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & -5 \end{bmatrix}$$

$$\bar{B} = PB = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \bar{C} = CP^{-1} = [2 \; 1]$$

Thus $\bar{x} = Px$ will transform the equation to

$$\dot{\bar{x}} = \begin{bmatrix} 3 & 4 \\ 0 & -5 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [2 \; 1] \bar{x}$$

and the equation can be reduced to

$$\dot{\bar{x}}_1 = 3\bar{x}_1 + u$$

$$y = 2\bar{x}_1$$

This reduced equation is observable

6.9 The state equation in Problem 6.5 is already in the form of (6.40), thus it can be reduced to

$$\dot{x}_1 = -x_1 + u$$
$$y = 0 \cdot x_1 + 2u$$

It is not observable, thus it can be further reduced to

$$y = 2u.$$

There is no state variable in the equation.

6.10 From Corollary 6.8 or Fig. 6.9, we see that x_3 is not controllable. We rearrange the equation as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_3 \end{bmatrix} = \left[\begin{array}{cccc:c} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 1 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ \hdashline 0 & 0 & 0 & 0 & \lambda_1 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_5 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \hdashline 0 \end{bmatrix} u$$

$$y = [\,0 \;\; 1 \;\; 0 \;\; 1 \;\vdots\; 1\,]\,\bar{x}$$

Thus the equation can be reduced as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [\,0 \;\; 1 \;\; 0 \;\; 1\,]\,\tilde{x}$$

Using Corollary 6.8, we conclude that the reduced equation is controllable. Using Corollary 6.08 or Fig. 6.9, we see that x_1 and x_4 are not observable. We rearrange the equation as

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_5 \\ \dot{x}_1 \\ \dot{x}_4 \end{bmatrix} = \left[\begin{array}{cc:cc} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \hdashline 1 & 0 & \lambda_1 & 0 \\ 0 & 1 & 0 & \lambda_2 \end{array}\right] \begin{bmatrix} x_2 \\ x_5 \\ x_1 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \hdashline 0 \\ 0 \end{bmatrix} u$$

$$y = [\,1 \;\; 1 \;\vdots\; 0 \;\; 0\,]\,\hat{x}$$

This is in the form of (6.44) and can be reduced to

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_5 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = [\,1 \;\; 1\,] \begin{bmatrix} x_2 \\ x_5 \end{bmatrix}$$

This is controllable and observable.

6.11 Select an arbitrary Q_2 such that $[Q_1\; Q_2]$ is nonsingular. Define

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} := [Q_1\; Q_2]^{-1}$$

Then $\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} [Q_1\; Q_2] = \begin{bmatrix} P_1Q_1 & P_1Q_2 \\ P_2Q_1 & P_2Q_2 \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & I \end{bmatrix}$

and $P_2Q_1 = 0$. Because Q_1 consists of all linearly independent columns of $[B\; AB \cdots A^{n-1}B] = 0$, we have

$$P_2B = 0 \quad \text{and} \quad P_2AQ_1 = 0$$

Consider the transformation $\bar{x} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} x$. Then

$$\bar{A} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} A\,[Q_1\; Q_2] = \begin{bmatrix} P_1AQ_1 & P_1AQ_2 \\ P_2AQ_1 & P_2AQ_2 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} B = \begin{bmatrix} P_1B \\ P_2B \end{bmatrix}$$

$$\bar{C} = C\,[Q_1\; Q_2] = [CQ_1\;\; CQ_2]$$

Because $P_2B = 0$ and $P_2AQ_1 = 0$, the equation is in the form of (6.40) and can be reduced to the controllable

$$\dot{\bar{x}}_1 = P_1AQ_1\bar{x}_1 + P_1Bu$$
$$y = CQ_1\bar{x}_1 + Du$$

6.12 Method 1: We may use elementary row operations to transform Q_1 into

$$PQ_1 = \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}$$

The first n_1 rows of P yields P_1.

Method 2: Solve n_1 set of linear algebraic equations. The first row, p_1, of P_1 is the solution of

$$p_1 Q_1 = [1 \ 0 \cdots 0] \quad (\text{first row of } I_{n_1})$$

The second row, p_2, of P_1, is the solution of

$$p_2 Q_1 = [0 \ 1 \ 0 \cdots 0] \quad (\text{second row of } I_{n_1})$$

and so forth.

6.13 Consider

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Let $\rho(O) = n_2$ and let P_1 be $n_2 \times n$, consisting of n_2 linearly independent rows of O. Solve Q_1 from $P_1 Q_1 = I_{n_2}$, where Q_1 is $n \times n_2$. Then

$$\dot{\bar{x}}_1 = P_1 A Q_1 \bar{x}_1 + P_1 B u$$
$$y = C Q_1 \bar{x}_1 + Du$$

is zero-state equivalent to the original state equation.

6.14 Because the rows of $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ and the rows of $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ are linearly independent, the equation is controllable. To be observable, the three columns of

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \text{ and the two columns of } \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

must be linearly independent. The three columns are not linearly independent, therefore, the equation is not observable.

6.15 To be controllable, the three rows of $\begin{bmatrix} b_{21} & b_{22} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \end{bmatrix}$ must be linearly independent. This is not possible. To be observable, the three columns of

$$\begin{bmatrix} c_{11} & c_{13} & c_{15} \\ c_{21} & c_{23} & c_{25} \\ c_{31} & c_{33} & c_{35} \end{bmatrix}$$

must be linearly independent. This can be easily achieved. For example, we may choose it as I_3.

6.16 Consider

$$\dot{\bar{x}} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 + j\beta_1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 - j\beta_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 + j\beta_2 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 - j\beta_2 \end{bmatrix} \bar{x} + \begin{bmatrix} b_1 \\ r_1 + j\eta_1 \\ r_1 - j\eta_1 \\ r_2 + j\eta_2 \\ r_2 - j\eta_2 \end{bmatrix} u$$

$$y = [c_1 \ \ p_1 + jq_1 \ \ p_1 - jq_1 \ \ p_2 + jq_2 \ \ p_2 - jq_2] \bar{x}$$

It is controllable $\Leftrightarrow b_1 \neq 0$; $r_i \neq$ or $\eta_i \neq 0$, $i = 1, 2$; observable $\Leftrightarrow c_1 \neq 0$; $p_i \neq 0$ or $q_i \neq 0$, $i = 1, 2$. (Corollaries 6.8 and 6.08)

The transformation $\bar{x} = Px$ with

$$P = \left[\begin{array}{c|cc|cc} 1 & & & & \\ \hline & 1 & 1 & & \\ & j & -j & & \\ \hline & & & 1 & 1 \\ & & & j & -j \end{array}\right] \quad P^{-1} = \left[\begin{array}{c|cc|cc} 1 & & & & \\ \hline & 0.5 & -j0.5 & & \\ & 0.5 & j0.5 & & \\ \hline & & & 0.5 & -j0.5 \\ & & & 0.5 & j0.5 \end{array}\right]$$

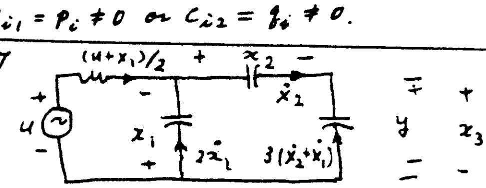
transforms the equation into

$$\dot{x} = \left[\begin{array}{c|cc|cc} \lambda_1 & & & & \\ \hline & \alpha_1 & \beta_1 & & \\ & -\beta_1 & \alpha_1 & & \\ \hline & & & \alpha_2 & \beta_2 \\ & & & -\beta_2 & \alpha_2 \end{array}\right] x + \left[\begin{array}{c} b_1 \\ \hline 2r_1 \\ -2\eta_1 \\ \hline 2r_2 \\ -2\eta_2 \end{array}\right] u$$

$$y = [c_1 \ | \ p_1 \ q_1 \ | \ p_2 \ q_2] x$$

Thus it is controllable $\Leftrightarrow b_1 \neq 0$; $b_{i1} = 2r_i \neq 0$ or $b_{i2} = -2\eta_i \neq 0$. It is observable $\Leftrightarrow c_1 \neq 0$; $c_{i1} = p_i \neq 0$ or $c_{i2} = q_i \neq 0$.

6.17



$$y = -x_2 - x_1$$

$$\dot{x}_2 = -3\dot{x}_2 - 3\dot{x}_1 \quad \Rightarrow \quad \dot{x}_2 = \frac{-3}{4}\dot{x}_1$$

$$0.5(u + x_1) + 2\dot{x}_1 \ = \ \dot{x}_2 = \frac{-3}{4}\dot{x}_1$$

$$\dot{x}_1 = -\frac{2}{11}x_1 - \frac{2}{11}u$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}\frac{-2}{11} & 0\\ \frac{3}{22} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{-2}{11}\\ \frac{3}{22}\end{bmatrix}u$$

$$y = [-1 \quad -1]\,x$$

This two-dimensional equation describes the network.

$$C = \begin{bmatrix}\frac{-2}{11} & \frac{-2}{11}\times\frac{-2}{11}\\ \frac{3}{22} & \frac{-3}{22}\times\frac{-2}{11}\end{bmatrix},\ \rho(C)=1 \quad \text{not controllable}$$

$$O = \begin{bmatrix}-1 & -1\\ \frac{1}{22} & 0\end{bmatrix},\ \rho(C)=2 \quad \text{observable}$$

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Now we introduce the voltage across the 3F capacitor as the third state variable x_3. Then we have $y = x_3$ and $x_3 = -x_1 - x_2$. Thus

$$\dot{x}_3 = -\dot{x}_2 - \dot{x}_1 = \frac{1}{22}x_1 + \frac{1}{22}u$$

and

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}\frac{-2}{11} & 0 & 0\\ \frac{3}{22} & 0 & 0\\ \frac{1}{22} & 0 & 0\end{bmatrix}x + \begin{bmatrix}\frac{-2}{11}\\ \frac{3}{22}\\ \frac{1}{22}\end{bmatrix}u$$

$$y = [0 \quad 0 \quad 1]\,x$$

This 3-dimensional equation describes the network. This equation is not controllable and not observable.

6.18 The equation is

$$\dot{x} = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & -1\end{bmatrix}x + \begin{bmatrix}1\\ 1\\ 0\end{bmatrix}u$$

$$y = [0 \quad 1 \quad 0]$$

$$C = \begin{bmatrix}1 & 1 & 1\\ 1 & 0 & -1\\ 0 & 1 & -1\end{bmatrix},\ \rho(C)=3 \quad \text{controllable}$$

$$O = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & -1\\ 0 & 1 & -1\end{bmatrix},\ \rho(O)=2 \quad \text{not observable}$$

The RC loop is in series with the current source therefore the response due to x_1 will not affect the rest of the network. Thus the network is not observable.

6.19 Consider

$$\dot{x} = \begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}x + \begin{bmatrix}1\\ 1\end{bmatrix}u$$

$$y = [2 \quad 3]\,x$$

Its eigenvalues are $-1 \pm j$. The necessary and sufficient condition for its discretized equation to be controllable is

$$T \neq \frac{2\pi}{|1-(-1)|}m = \frac{2\pi}{2}m = m\pi, \quad m = 1, 2, \dots$$

For $T = 1$, the discretized equation was computed in Problem 4.3 as

$$x[k+1] = \begin{bmatrix}0.5083 & 0.3096\\ -0.6191 & -0.1108\end{bmatrix}x[k] + \begin{bmatrix}1.0471\\ -0.1821\end{bmatrix}u[k]$$

$$y[k] = [2 \quad 3]\,x[k]$$

As predicted by Theorem 6.9, it is controllable. Similarly, it is observable.

For $T = \pi$, we have, as computed in Prob. 4.3,

$$x[k+1] = \begin{bmatrix}-0.0432 & 0\\ 0 & -0.0432\end{bmatrix}x[k] + \begin{bmatrix}1.5648\\ -1.0432\end{bmatrix}u[k]$$

$$y[k] = [2 \quad 3]\,x[k]$$

It can be readily verified to be uncontrollable and unobservable and is consistent with Theorem 6.9.

6.20 $\dot{x} = \begin{bmatrix}0 & 1\\ 0 & t\end{bmatrix}x + \begin{bmatrix}0\\ 1\end{bmatrix}u \qquad y = [0 \quad 1]\,x$

$$M_0 = B = \begin{bmatrix}0\\ 1\end{bmatrix}, \quad M_1(t) = -A(t)M_0(t) + \frac{d}{dt}M_0(t) = \begin{bmatrix}-1\\ -t\end{bmatrix}$$

$\operatorname{rank}\begin{bmatrix}0 & -1\\ 1 & -t\end{bmatrix} = 2$ at every t. Thus the equation is controllable at every t (Theorem 6.12)

$$N_0(t) = [0 \quad 1], \quad N_1(t) = [0, \ t]$$

$$\operatorname{rank}\begin{bmatrix}N_0(t)\\ N_1(t)\end{bmatrix} = \operatorname{rank}\begin{bmatrix}0 & 1\\ 0 & t\end{bmatrix} = 1$$

Because Theorem 6.012 is a sufficient

condition, we cannot say anything about the observability of the equation.

The state transition matrix of the equation was computed in Problem 4.16 as

$$\Phi(t,t_0)=\begin{bmatrix} 1 & -e^{0.5t^2}\int_{t_0}^{t} e^{0.5\tau^2}d\tau \\ 0 & e^{0.5(t^2-t_0^2)} \end{bmatrix}$$

We compute $C\Phi(\tau,t_0)=[0 \quad e^{0.5(\tau^2-t_0^2)}]$ and

$$W_o(t_0,t_1)=\int_{t_0}^{t_1}\begin{bmatrix} 0 & 0 \\ 0 & e^{(\tau^2-t_0^2)} \end{bmatrix}d\tau$$

It is singular at every t_0. Thus the equation is not observable at every t.

6.21
$$\dot{x}=\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}x+\begin{bmatrix} 1 \\ e^{-t} \end{bmatrix}u$$

$$y=[1 \quad e^{-t}]x$$

$$\Phi(t,\tau)=\begin{bmatrix} 1 & 0 \\ 0 & e^{-(t-\tau)} \end{bmatrix}$$

$$\Phi(t,\tau)B(\tau)=\begin{bmatrix} 1 & 0 \\ 0 & e^{-(t-\tau)} \end{bmatrix}\begin{bmatrix} 1 \\ e^{-\tau} \end{bmatrix}=\begin{bmatrix} 1 \\ e^{-t} \end{bmatrix}$$

$$W_c(t_0,t_1)=\int_{t_0}^{t_1}\begin{bmatrix} 1 \\ e^{-t_1} \end{bmatrix}[1 \quad e^{-t_1}]\,d\tau$$

$$=\begin{bmatrix} t_1-t_0 & e^{-t_1}(t_1-t_0) \\ e^{-t_1}(t_1-t_0) & e^{-2t_1}(t_1-t_0) \end{bmatrix}$$

$\det W_c(t_0,t_1)=0$ for all t_0 and $t_1\geq t_0$. Thus the equation is not controllable at any t.

We use Theorem 6.012 to check observability

$$N_0(t)=[1 \quad e^{-t}]$$

$$N_1(t)=[1 \quad e^{-t}]\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}+\frac{d}{dt}[1 \quad e^{-t}]$$

$$=[0 \quad -e^{-t}]+[0 \quad -e^{-t}]$$

$$=[0 \quad -2e^{-t}]$$

$$\text{rank}\begin{bmatrix} 1 & e^{-t} \\ 0 & -2e^{-t} \end{bmatrix}=2$$ for all finite t. Thus the state equation is observable at every t.

We mention that in the time-invariant case, (A,B) is controllable if and only if (A',B') is observable. In the time varying case, it must be modified as $(A(t),B(t))$ is controllable at t_0 if and only if $(-A'(t),B'(t))$ is observable at t_0. See Problem 6.22.

6.22 Let $X(t)$ be a fundamental matrix of $\dot{x}=A(t)x$. or $\frac{d}{dt}X(t)=A(t)X(t)$.

Then

$$\frac{d}{dt}(X^{-1}(t)X(t))=\left(\frac{d}{dt}X^{-1}(t)\right)X(t)+X^{-1}(t)\frac{d}{dt}X(t)$$

$$=\frac{d}{dt}(I)=0 \qquad \text{Thus}$$

$$\frac{d}{dt}X^{-1}(t)=-X^{-1}(t)\left(\frac{d}{dt}X(t)\right)X^{-1}(t)$$

$$=-X^{-1}(t)A(t).$$

Let $X_1(t)$ be a fundamental matrix of $\dot{x}(t)=-A'(t)x(t)$ or $\frac{d}{dt}X_1(t)=-A'(t)X_1(t)$

Taking its transpose yields

$$\frac{d}{dt}X_1'(t)=-X_1'(t)A(t)$$

Thus we have $X_1'(t)=X^{-1}(t)$; $(X_1'(t))^{-1}=X(t)$

$$\Phi(t,\tau)=X(t)X^{-1}(\tau)$$

$$\Phi_1(t,\tau)=X_1(t)X_1^{-1}(\tau)$$

$$\Phi_1'(t,\tau)=(X_1')^{-1}(\tau)X_1'(t)=X(\tau)X^{-1}(t)$$

$$=\Phi(\tau,t)$$

Now $(A(t),B(t))$ is controllable at t_0 if and only if

$$W_c = \int_{t_0}^{t_1} \phi(t_1,\tau)B(\tau)B'(\tau)\phi'(t_1,\tau)d\tau$$

is nonsingular. Using

$$\phi(t_1,\tau) = \phi(t_1,t_0)\phi(t_0,\tau)$$

we write W_c as

$$W_c = \phi(t_1,t_0)\left(\int_{t_0}^{t_1} \phi(t_0,\tau)B(\tau)B'(\tau) \times \phi'(t_0,\tau)d\tau\right)\phi'(t_1,t_0)$$

Because $\phi(t_1,t_0)$ is nonsingular, we conclude $(A(t),B(t))$ is controllable if and only if

$$\int_{t_0}^{t} \phi(t_0,\tau)B(\tau)B'(\tau)\phi'(t_0,\tau)d\tau \quad (*)$$

is nonsingular. Now $(-A'(t), B'(t))$ is observable if and only if

$$W_{10} = \int_{t_0}^{t} \phi_1'(\tau,t_0)B(\tau)B'(\tau)\phi_1(\tau,t_0)d\tau$$

is nonsingular. Using $\phi_1(\tau,t_0) = \phi(t_0,\tau)$, we write W_{10} as

$$W_{10} = \int_{t_0}^{t} \phi(t_0,\tau)B(\tau)B'(\tau)\phi'(t_0,\tau)d\tau$$

which is identical to $(*)$. This establishes that $(A(t),B(t))$ is controllable if and only if $(-A'(t), B(t))$ is observable.

6.23 $(-A,B)$ is controllable if and only if

$$[B\ (-A)B\ (-A)^2B \cdots (-A)^{n-1}B]$$
$$= [B\ -AB\ A^2B\ -A^3B\ \cdots \pm A^{n-1}B]$$

has full row rank. Because

$$[B\ -AB\ A^2B\ -A^3B\ \cdots]$$
$$= \underset{(n\times np)}{[B\ AB\ A^2B\ A^3B\ \cdots]}\ \underset{(np\times np)}{\begin{bmatrix} I & 0 & 0 & 0 & \\ 0 & -I & 0 & 0 & \\ 0 & 0 & I & 0 & \\ 0 & 0 & 0 & -I & \\ & & & & \ddots \end{bmatrix}}$$

The $(np\times np)$ matrix is clearly nonsingular, thus $[B\ AB\ A^2B \cdots]$ and $[B\ -AB\ A^2B \cdots]$ have the same rank, and (A,B) is controllable if and only if $(-A,B)$ is controllable.

The assertion is not true in the time-varying case. For example, $(A(t),B(t))$ in Problem 6.21 is not controllable at any t. Consider $(-A(t),B(t))$ or

$$-A(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\quad B(t) = \begin{bmatrix} 1 \\ e^{-t} \end{bmatrix}$$

we have

$$\phi(t,\tau) = \begin{bmatrix} 1 & 0 \\ 0 & e^{(t-\tau)} \end{bmatrix}$$

$$\phi(t,\tau)B(\tau) = \begin{bmatrix} 1 \\ e^{(t-\tau)}e^{-\tau} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{t-2\tau} \end{bmatrix}$$

$$W_c(t_0,t_1) = \int_{t_0}^{t_1} \begin{bmatrix} 1 \\ e^{t_1-2\tau} \end{bmatrix} [1\ \ e^{t_1-2\tau}]d\tau$$

$$= \int_{t_0}^{t_1} \begin{bmatrix} 1 & e^{t_1-2\tau} \\ e^{t_1-2\tau} & e^{2(t_1-2\tau)} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} t_1-t_0 & \frac{1}{3}e^{t_1}(e^{-3t_0}-e^{-3t_1}) \\ \frac{1}{3}e^{t_1}(e^{-3t_0}-e^{-3t_1}) & \frac{1}{5}e^{2t_1}(e^{-5t_0}-e^{-5t_1}) \end{bmatrix}$$

For any t_0, we can find a t_1 so that $W_c(t_0,t_1)$ is nonsingular and $(-A(t),B(t))$ is controllable at any t although $(A(t),B(t))$ is not.

Chapter 7

7.1 $\hat{g}(s) = \dfrac{s-1}{s^3+2s^2-s-2}$

$$\dot{x} = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [\,0 \quad 1 \quad -1\,]\,x$$

$$O = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -3 & 1 & 2 \end{bmatrix} \qquad \det O = -1+3-2=0$$

$\rho(O) < 3$. Not observable

7.2

$$\dot{x} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} u$$

$$y = [\,1 \quad 0 \quad 0\,]\,x$$

$$C = \begin{bmatrix} 0 & 1 & -3 \\ 1 & -1 & 1 \\ -1 & 0 & 2 \end{bmatrix}, \qquad \rho(C) < 3$$

Not controllable

7.3 We use (6.40) to add an uncontrollable part to the realization in Problem 7.1:

$$\dot{x} = \left[\begin{array}{ccc|c} -2 & 1 & 2 & a_1 \\ 1 & 0 & 0 & a_2 \\ 0 & 1 & 0 & a_3 \\ \hline 0 & 0 & 0 & a_4 \end{array}\right] x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [\,0 \quad 1 \quad -1 \,\vdots\, c_4\,]\,x$$

where a_i and c_4 are arbitrary. This is an uncontrollable and unobservable realization of $(s-1)/(s^3+2s^2-s-2)$.

$$\hat{g}(s) = \frac{s-1}{(s^2-1)(s+2)} = \frac{1}{(s+1)(s+2)} = \frac{1}{s^2+3s+2}$$

$$\dot{x} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [\,0 \quad 1\,]\,x$$

This is a controllable and observable realization of $\hat{g}(s)$.

7.4 First we form the Sylvester resultant

$$\left[\begin{array}{cc|cc|cc} -2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -2 & -1 & 0 & 0 \\ 2 & 0 & -1 & 1 & -2 & -1 \\ 1 & 0 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

We use MATLAB to search its linearly independent columns from left to right. We type

```
d=[-2 -1 2 1]; n=[-1 1 0 0];
s=[d 0 0;n 0 0;0 d 0;0 n 0;0 0 d;0 0 n]';
[q,r]=qr(s)
```

which yields

$$r = \begin{bmatrix} d & x & x & x & x & x \\ 0 & n & x & x & x & x \\ 0 & 0 & d & x & x & x \\ 0 & 0 & 0 & n & x & x \\ 0 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where x, d, n denote nonzero entries. d also denotes D-column. n denotes N-column.

There are two linearly independent N-columns, thus the degree of $\hat{g}(s)$ in Prob. 7.1 is 2.

7.5 $\dfrac{2s-1}{4s^2-1} = \dfrac{N_0+N_1 s}{D_0+D_1 s}$

$$(-N_0-N_1 s)(-1+4s^2) + (D_0+D_1 s)(-1+2s) = 0$$

$$\left[\begin{array}{cc|cc} -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ 4 & 0 & 0 & 2 \\ 0 & 0 & 4 & 0 \end{array}\right] \begin{bmatrix} -N_0 \\ D_0 \\ -N_1 \\ D_1 \end{bmatrix} = 0$$

Although D_1 can be any nonzero constant, it is convenient to require $D_1 = 1$. We type

```
d=[-1 0 4]; n=[-1 2 0];
s=[d 0;n 0;0 d;0 n]';
z=null(s)
```

which yields

```
z=[-0.4082 0.4082 0 0.8165]'
```

We normalize the last entry to 1 by typing

```
z/z(4)
```

which yields $[-0.5 \quad 0.5 \,\vdots\, 0 \quad 1]'$.

Thus we have

$$\frac{2s-1}{4s^2-1} = \frac{0.5}{s+0.5} = \frac{1}{2s+1}$$

7.6 We form, from $\frac{0 \cdot s^2+s+2}{s^2+2s+0}$,

$$\left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

where the coefficients are arranged in descending power of s. If we search its linearly independent columns in order from left to right, then the second D-column is linearly dependent on its LHS columns and there are two linearly independent N-columns. Thus in this arrangement, it is not true that all D-columns are linearly independent of their LHS columns and that the degree of $\hat{g}(s)$ equals the number of linearly independent N-columns.

7.7 $$\frac{N(s)}{D(s)} = \frac{\beta_1 s+\beta_2}{s^2+\alpha_1 s+\alpha_2}$$

$N(s)$ and $D(s)$ are coprime if and only if

$$S = \left[\begin{array}{cc:cc} \alpha_2 & \beta_2 & 0 & 0 \\ \alpha_1 & \beta_1 & \alpha_2 & \beta_2 \\ 1 & 0 & \alpha_1 & \beta_1 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \text{is nonsingular}$$

$$\det S = -\det\begin{bmatrix} \alpha_2 & \beta_2 & 0 \\ \alpha_1 & \beta_1 & \beta_2 \\ 1 & 0 & \beta_1 \end{bmatrix} = -\alpha_2\beta_1^2-\beta_2^2+\alpha_1\beta_1\beta_2$$

The realization

$$\dot{x} = \begin{bmatrix} -\alpha_1 & -\alpha_2 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad y = [\beta_1 \ \ \beta_2]\, x$$

is observable if and only if

$$O = \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_2-\beta_1\alpha_1 & -\beta_1\alpha_2 \end{bmatrix} \quad \text{is nonsingular}$$

$$\det O = -\alpha_2\beta_1^2-\beta_2^2+\alpha_1\beta_1\beta_2 = \det S$$

Thus the two conditions are the same.

7.8 Consider $$\frac{N(s)}{D(s)} = \frac{\beta_1 s^2+\beta_2 s+\beta_3}{s^3+\alpha_1 s^2+\alpha_2 s+\alpha_3}$$

and its controllable form realization

$$\dot{x} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [\beta_1 \ \ \beta_2 \ \ \beta_3]\, x$$

Its observability matrix is

$$O = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_2-\alpha_1\beta_1 & \beta_3-\alpha_2\beta_1 & -\alpha_3\beta_1 \\ -\alpha_1(\beta_2-\alpha_1\beta_1)+\beta_3-\alpha_2\beta_1 & -\alpha_2(\beta_2-\alpha_1\beta_1)-\alpha_3\beta_1 & -\alpha_3(\beta_2-\alpha_1\beta_1) \end{bmatrix}$$

The Sylvester resultant of $N(s)$ and $D(s)$ is

$$S = \left[\begin{array}{cc:cc:cc} \alpha_3 & \beta_3 & 0 & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \alpha_3 & \beta_3 & 0 & 0 \\ \alpha_1 & \beta_1 & \alpha_2 & \beta_2 & \alpha_3 & \beta_3 \\ 1 & 0 & \alpha_1 & \beta_1 & \alpha_2 & \beta_2 \\ 0 & 0 & 1 & 0 & \alpha_1 & \beta_1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

First we use the 1 in the last row to eliminate all entries above it. Next we use the 1 in the fifth row to eliminate all entries above it to yield

$$\left[\begin{array}{cc:cc:cc} \alpha_3 & \beta_3 & 0 & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & 0 & \beta_3 & 0 & -\alpha_3\beta_1 \\ \alpha_1 & \beta_1 & 0 & \beta_2 & 0 & \beta_3-\alpha_2\beta_1 \\ 1 & 0 & 0 & \beta_1 & 0 & \beta_2-\alpha_1\beta_1 \\ 0 & 0 & 1 & 0 & 0 & \beta_1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Finally we use the 1 in the first column to eliminate all entries above it to yield

$$\left[\begin{array}{cc:cc:cc} 0 & \beta_3 & 0 & -\alpha_3\beta_1 & 0 & -\alpha_3(\beta_2-\alpha_1\beta_1) \\ 0 & \beta_2 & 0 & \beta_3-\alpha_2\beta_1 & 0 & -\alpha_2(\beta_2-\alpha_1\beta_1)-\alpha_3\beta_1 \\ 0 & \beta_1 & 0 & \beta_2-\alpha_1\beta_1 & 0 & -\alpha_1(\beta_2-\alpha_1\beta_1)+\beta_3-\alpha_2\beta_1 \\ 1 & 0 & 0 & \beta_1 & 0 & \beta_2-\alpha_1\beta_1 \\ 0 & 0 & 1 & 0 & 0 & \beta_1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

This can be rearranged as

$$\left[\begin{array}{ccc:ccc} & O'_{13} & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ \hdashline 0 & \beta_1 & \beta_2-\alpha_1\beta_1 & 1 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

O'_{13} = transpose of O after interchange 1st & 3rd columns

Thus the resultant is nonsingular if and only if O is nonsingular or, equivalently, the realization is observable if and only if $D(s)$ and $N(s)$ are coprime. This is Theorem 7.1.

7.9 $$\hat{g}(s) = \frac{1}{(s+1)^2} = 0 \cdot s^{-1} + s^{-2} - 2s^{-3} + 3s^{-4} - 4s^{-5} + 5s^{-6} - \cdots$$

$$T(2,2) = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \quad \rho(T(2,2)) = 2$$

$$T(3,3) = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ -2 & 3 & -4 \end{bmatrix}$$ 3rd column $= -$(1st column) -2(2nd column)

Thus $\rho(T(3,3)) = 2$

Proceeding forward, we can establish Theorem 7.7 for $1/(s+1)^2$.

7.10 $\alpha_1 = 2,\ \alpha_2 = 1,\ h(1) = 0,\ h(2) = 1$. Thus, from (7.56), we have

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] x$$

This is a companion-form realization.

7.11 $$T = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}, \quad \tilde{T} = \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}$$

We use MATLAB. Typing

```
t=[0 1;1 -2]; tt=[1 -2;-2 3];
[k,s,l]=svd(t);
s1=sqrt(s);
O=k*s1; C=s1*l';
a=inv(O)*tt*inv(C)
b=[C(1,1);C(2,1)]
c=[O(1,1) O(1,2)]
```

yield

$$\dot{x} = \begin{bmatrix} -1.7071 & 0.7071 \\ -0.7071 & -0.2929 \end{bmatrix} x + \begin{bmatrix} 0.5946 \\ -0.5946 \end{bmatrix} u$$

$$y = [-0.5946 \quad -0.5946] x$$

This is a balanced realization.

7.12 $$\dot{x} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad y = [2 \quad 2] x$$

This is in the form of (6.40) and can be reduced to

$$\dot{x} = 2x + u, \quad y = 2x$$

Its transfer function is $\hat{g}(s) = \frac{2}{s-2}$

$$\dot{x} = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u, \quad y = [2 \quad 0] x$$

This is in the form of (6.44) and can be reduced to

$$\dot{x} = 2x + u, \quad y = 2x$$

Its transfer function is also $\hat{g}(s) = \frac{2}{s-2}$

$$\frac{2s+2}{s^2-s-2} = \frac{2(s+1)}{(s-2)(s+1)} = \frac{2}{s-2}$$

The original two state equations are not minimal realizations; they are not algebraically equivalent because they have different eigenvalues. The first equation has eigenvalues $\{2, 1\}$, the second $\{2, -1\}$.

7.13 $$G_1(s) = \begin{bmatrix} \frac{1}{s} & \frac{s+3}{s+1} \\ \frac{1}{s+3} & \frac{s}{s+1} \end{bmatrix}$$

$$\det G_1(s) = \frac{1}{s+1} - \frac{1}{s+3} = 0$$

$$\Delta_1(s) = s(s+1)(s+3), \quad \deg = 3$$

$$G_2(s) = \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{1}{(s+1)(s+2)} \\ \frac{1}{s+2} & \frac{1}{(s+1)(s+2)} \end{bmatrix}$$

$$\det G_2(s) = \frac{1}{(s+1)^3(s+2)} - \frac{1}{(s+1)(s+2)^2}$$

$$= \frac{s+2-(s^2+2s+1)}{(s+1)^3(s+2)^2} = \frac{-s^2-s+1}{(s+1)^3(s+2)^2}$$

$$\Delta_2(s) = (s+1)^3(s+2)^2 \quad \deg = 5$$

$$G_3(s) = \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{s+3}{s+2} & \frac{1}{s+5} \\ \frac{1}{(s+3)^2} & \frac{s+1}{s+4} & \frac{1}{s} \end{bmatrix}$$

Because every entry has poles different from other entries, we have

$$\Delta_3(s) = (s+1)^2(s+3)^2(s+2)(s+4)(s+5)s$$

$$\deg = 8$$

7.14 $$\hat{G}(s) = \begin{bmatrix} s & 1 \\ -s & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \bar{D}^{-1}(s)\bar{N}(s) = \underbrace{N(s)}_{2\times 1}\underbrace{D^{-1}(s)}_{1\times 1}$$

$$\bar{D}(s) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} s \qquad \bar{N}(s) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} s$$

We form the resultant in (7.83)

$$\left[\begin{array}{ccc|ccc|ccc} 0 & 1 & 1 & & & & & & \\ 0 & 0 & -1 & & & & & & \\ \hline 1 & 0 & 0 & 0 & 1 & 1 & & & \\ -1 & 1 & 0 & 0 & 0 & -1 & & & \\ \hline & & & 1 & 0 & 0 & 0 & 1 & 1 \\ & & & -1 & 1 & 0 & 0 & 0 & -1 \\ \hline & & & & & & 1 & 0 & 0 \\ & & & & & & -1 & 1 & 0 \end{array}\right]$$

The first $\bar{N}$-column is linearly independent on its LHS columns. The second $\bar{N}$-column is not. Nor the third $\bar{N}$-column. Therefore, there is only one linearly independent $\bar{N}$-column and the degree of $\hat{G}(s)$ is 1. Cleary, we have

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{array}\right] \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0 \qquad \begin{bmatrix} -1 \\ 0 \\ \hdashline 0 \\ \hline 0 \\ 0 \\ \hdashline 1 \end{bmatrix} = \begin{bmatrix} -N_0 \\ \hdashline D_0 \\ \hline -N_1 \\ \hdashline D_1 \end{bmatrix}$$

Thus $N_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $N_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $D_0 = 0$, $D_1 = 1$

and a right coprime fraction is

$$N(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad D(s) = 0 + 1 \cdot s = s$$

or $$\hat{G}(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot s^{-1} = \begin{bmatrix} \frac{1}{s} \\ 0 \end{bmatrix}$$

The given left fraction is not left coprime because $\deg \det \begin{bmatrix} s & 1 \\ -s & s \end{bmatrix} = \deg(s^2+s) = 2 > 1$.

7.15 For the $\hat{G}(s)$ in Problem 7.14, because $\bar{D}_1$ is nonsingular, all $\bar{D}$-columns in the resultant are linearly independent of their LHS columns. Note that even if $\bar{D}_1$ singular the same property holds. Now we arrange the coefficient matrices in descending power of s as

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ -1 & 1 & 0 & & & \\ \hline 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ \hline & & & 0 & 1 & 1 \\ & & & 0 & 0 & -1 \end{array}\right]$$

and search its linearly independent columns from left to right. Now the second $\bar{D}1$-column is dependent on its LHS columns and there are two linearly independent $\bar{N}$-columns. Thus Theorem 7.M2 does not hold.

7.16 $$\hat{G}(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot s^{-1} = N(s)D^{-1}(s) = \underbrace{\bar{D}^{-1}(s)}_{2\times 2}\underbrace{\bar{N}(s)}_{2\times 1}$$

$$N(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} s \qquad D(s) = 0 + 1 \cdot s$$

We form

$$T = \left[\begin{array}{c|c|c} 0 & 1 & \\ \hline 1 & 0 & \\ 0 & 0 & \\ \hline & 0 & 1 \\ & 1 & 0 \\ & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow \text{primary N2 dependent row} \\ \\ \leftarrow \text{depends on the first row primary} \\ \quad \text{N1 dependent row} \end{array}$$

Using the primary N2 dependent row, we have

$$[0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0]\, T = 0.$$

Using the primary N1 dependent row, we have

$$[-1 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0]\, T = 0$$

Thus $$[-N_0 \vdots D_0 \mid -N_1 \vdots D_1] = \left[\begin{array}{c:cc|c:cc} 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$N(s) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D(s) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} s$$

and $$\begin{bmatrix} 0 & 1 \\ s & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \hat{G}(s)$$

This is a left-coprime fraction. Note that we may interchange the rows to obtain

$$\hat{G}(s) = \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

7.17 $$\hat{G}(s) = \begin{bmatrix} \dfrac{s^2+1}{s^3} & \dfrac{2s+1}{s^2} \\ \dfrac{s+2}{s^2} & \dfrac{2}{s} \end{bmatrix}$$

$$\det \hat{G}(s) = \frac{2(s^2+1)}{s^4} - \frac{(s+2)(2s+1)}{s^4} = \frac{-5}{s^3}$$

$\Delta(s) = s^3 \qquad \mathrm{Deg}\ \hat{G}(s) = 3.$

First we find a left fraction as

$$\hat{G}(s) = \begin{bmatrix} s^3 & 0 \\ 0 & s^2 \end{bmatrix}^{-1} \begin{bmatrix} s^2+1 & 2s^2+s \\ s+2 & 2s \end{bmatrix} \quad \text{not left coprime}$$

$$\bar{D}(s) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} s + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s^3$$

$$\bar{N}(s) = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} s + \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} s^3$$

We form

$$S = \left[\begin{array}{cc:cc|cc:cc|cc:cc}
0 & 0 & 1 & 0 & & & & & & & & \\
0 & 0 & 2 & 0 & & & & & & & & \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & & & & \\
0 & 0 & 1 & 2 & 0 & 0 & 2 & 0 & & & & \\
0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2 \\
 & & & & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & & & & & & & 1 & 0 & 0 & 0 \\
 & & & & & & & & 0 & 0 & 0 & 0
\end{array}\right]$$

primary dependent $\bar{N}1$-column

Primary dependent $\bar{N}2$-column

We use qr decomposition to find the primary dependent columns as shown. We type

```
d1=[0 0 0 0 0 0 1 0];d2=[0 0 0 0 0 1 0 0];
n1=[1 2 0 1 1 0 0 0];n2=[0 0 1 2 2 0 0 0];
s1=[d1 0 0;d2 0 0;n1 0 0;n2 0 0;...
    0 0 d1;0 0 d2;0 0 n1,0 0 n2]';
z=null(s1);
z/z(6)
```

which yield [-2.5 -2.5 ⋮ 0 -0.5 | 0 0 ⋮ 0.5 1]'

```
s2=[d1 0 0 0 0 ;d2 0 0 0 0;n1 0 0 0 0;n2 0 0 0 0,...
    0 0 d1 0 0;0 0 d2 0 0,0 0 n1 0 0;...
    0 0 0 0 d1;0 0 0 0 d2; 0 0 0 0 n1]';
y=null(s2); y/y(10)
```

which yield

[-0.5 -2.5 ⋮ 0 -0.5 | -1 -1 ⋮ 0.5 | 0 0 ⋮ 1]'

$$\begin{bmatrix} -N_0 \\ D_0 \\ \hline -N_1 \\ D_1 \\ \hline -N_2 \\ D_2 \end{bmatrix} = \begin{bmatrix} -2.5 & -0.5 \\ -2.5 & -2.5 \\ 0 & 0 \\ -0.5 & -0.5 \\ \hline 0 & -1 \\ 0 & -1 \\ 0.5 & 0.5 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

This zero is inserted because the second $\bar{N}2$ column is not used in computing y.

Thus we have

$$D(s) = \begin{bmatrix} 0 & 0 \\ -0.5 & -0.5 \end{bmatrix} + \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix} s + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} s^2$$

$$= \begin{bmatrix} 0.5s & s^2+0.5s \\ s-0.5 & -0.5 \end{bmatrix}$$

$$N(s) = \begin{bmatrix} 2.5 & 0.5 \\ 2.5 & 2.5 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} s = \begin{bmatrix} 2.5 & s+0.5 \\ 2.5 & s+2.5 \end{bmatrix}$$

and

$\hat{G}(s) = N(s)D^{-1}(s) \qquad$ right coprime

The column-degree coefficient matrix of $D(s)$ is $\begin{bmatrix} 0.5 & 1 \\ 1 & 0 \end{bmatrix}$ which is not unit upper triangular. We interchange the columns of $D(s)$ and $N(s)$ as

$$D(s) = \begin{bmatrix} s^2+0.5s & 0.5s \\ -0.5 & s-0.5 \end{bmatrix}, \; N(s) = \begin{bmatrix} s+0.5 & 2.5 \\ s+2.5 & 2.5 \end{bmatrix}$$

Now $D(s)$ is in column echelon form.

Define $\quad H(s) = \begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix}, \quad L(s) = \begin{bmatrix} s & 0 \\ 1 & 0 \\ \hdashline 0 & 1 \end{bmatrix}$

$$D(s) = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} H(s) + \left[\begin{array}{cc:c} 0.5 & 0 & 0 \\ 0 & -0.5 & -0.5 \end{array}\right] L(s)$$

$$\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -0.5 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -0.5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -0.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0 & -0.5 & -0.5 \end{bmatrix}$$

$$N(s) = \left[\begin{array}{cc:c} 1 & 0.5 & 2.5 \\ 1 & 2.5 & 2.5 \end{array}\right]$$

Thus a minimal realization is

$$\dot{x} = \left[\begin{array}{cc:c} -0.5 & -0.25 & -0.25 \\ 1 & 0 & 0 \\ \hdashline 0 & 0.5 & 0.5 \end{array}\right] x + \left[\begin{array}{cc} 1 & -0.5 \\ 0 & 0 \\ \hdashline 0 & 1 \end{array}\right] u$$

$$y = \left[\begin{array}{cc:c} 1 & 0.5 & 2.5 \\ 1 & 2.5 & 2.5 \end{array}\right] x$$

Chapter 8

8.1 $\dot{x} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$

$$u = r - [k_1 \ k_2] x$$

$$\dot{x} = \begin{bmatrix} 2-k_1 & 1-k_2 \\ -1-2k_1 & 1-2k_2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} r$$

$$\det \begin{bmatrix} s-2+k_1 & -1+k_2 \\ 1+2k_1 & s-1+2k_2 \end{bmatrix} = s^2 + (k_1+2k_2-3)s + k_1 - 5k_2 + 3$$

$$\Delta_f(s) = (s+1)(s+2) = s^2+3s+2$$

$$\therefore \quad k_1 + 2k_2 - 3 = 3 \Rightarrow k_1 + 2k_2 = 6$$

$$k_1 - 5k_2 + 3 = 2 \Rightarrow k_1 - 5k_2 = -1$$

Solving these yields $k_2 = 1$, $k_1 = 4$.

8.2 $\Delta(s) = \det \begin{bmatrix} s-2 & -1 \\ 1 & s-1 \end{bmatrix} = (s-1)(s-2)+1$

$$= s^2 - 3s + 3$$

$$\bar{k} = [3+3 \quad 2-3] = [6 \ -1]$$

$$\bar{C}^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$C = [b \ \ Ab] = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}, \quad C^{-1} = \frac{1}{-7}\begin{bmatrix} 1 & -4 \\ -2 & 1 \end{bmatrix}$$

$$k = \bar{k}\bar{C}C^{-1} = [6 \ -1]\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{-1}{7} & \frac{4}{7} \\ \frac{2}{7} & \frac{-1}{7} \end{bmatrix}$$

$$= [6 \ \ 17]\begin{bmatrix} \frac{-1}{7} & \frac{4}{7} \\ \frac{2}{7} & \frac{-1}{7} \end{bmatrix} = [\tfrac{28}{7} \ \ \tfrac{7}{7}] = [4 \ \ 1].$$

8.3 Select $F = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, $\bar{k} = [1 \ \ 1]$

Solve $AT - TF = b\bar{k}$ or

$$\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} - \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3t_{11}+t_{21} & 4t_{12}+t_{22} \\ -t_{11}+2t_{21} & -t_{12}+3t_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

From the four equations, we can solve

$$T = \begin{bmatrix} 0 & \frac{1}{13} \\ 1 & \frac{9}{13} \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} -9 & 1 \\ 13 & 0 \end{bmatrix}$$

$$k = \bar{k}T^{-1} = [1 \ \ 1]\begin{bmatrix} -9 & 1 \\ 13 & 0 \end{bmatrix} = [4 \ \ 1].$$

8.4 $\dot{x} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$

We use (8.13) to compute feedback gain k. We compute

$$\Delta(s) = (s-1)^3 = s^3 - 3s^2 + 3s - 1$$

$$\Delta_f(s) = (s+2)(s+1+j)(s+1-j) = s^3 + 4s^2 + 6s + 4$$

$$\bar{k} = [4-(-3) \quad 6-3 \quad 4-(-1)] = [7 \quad 3 \quad 5]$$

$$\bar{C}^{-1} = \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \quad C^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -3 & 2 \\ 1 & 2 & -1 \end{bmatrix}$$

$$k = \bar{k}\bar{C}C^{-1} = [15 \quad 47 \quad -8].$$

8.5 If we place the eigenvalues of the state feedback system at $-2, -2, -3$, then the resulting system has transfer function

$$\hat{g}_f(s) = \frac{(s-1)(s+2)}{(s+2)^2(s+3)} = \frac{s-1}{(s+2)(s+3)}$$

The system is BIBO stable because $\hat{g}_f(s)$ has poles -2 and -3; it is asymptotically stable because the eigenvalues are -2, -2, and -3.

8.6 If we place the eigenvalues of the state feedback system at $1, -2$ and -3, then the resulting system has transfer function

$$\hat{g}_f(s) = \frac{(s-1)(s+2)}{(s-1)(s+2)(s+3)} = \frac{1}{s+3}$$

The system is BIBO stable. It is not asymptotically stable because the system has eigenvalue 1.

8.7 $\dot{x} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$

$$y = [2 \quad 0 \quad 0]x$$

We compute

$$(sI - A)^{-1} = \begin{bmatrix} s-1 & -1 & 2 \\ 0 & s-1 & -1 \\ 0 & 0 & s-1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} & \frac{-2s+3}{(s-1)^3} \\ 0 & \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & 0 & \frac{1}{s-1} \end{bmatrix}$$

Thus the transfer function is

$$\hat{g}(s) = \frac{2}{s-1} + \frac{(-2s+3)\cdot 2}{(s-1)^3} = \frac{2s^2 - 8s + 8}{(s-1)^3}$$

Now we introduce

$$u = pr - kx$$

with $k = [15 \quad 47 \quad -8]$ as computed in Problem 8.4. Then the transfer function from r to y is

$$\hat{g}_f(s) = p \cdot \frac{2s^2 - 8s + 8}{s^3 + 4s^2 + 6s + 4}$$

In order to track any step reference input, we require

$$\hat{g}_f(0) = 1 \quad \text{or} \quad p \cdot \frac{8}{4} = 1 \Rightarrow p = \frac{4}{8} = 0.5$$

This completes the design

8.8 $x[k+1] = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u[k]$

$$y[k] = [2 \quad 0 \quad 0]x[k]$$

$$\Delta(z) = \det(zI - A) = (z-1)^3 = z^3 - 3z^2 + 3z - 1$$

We use the procedure used in Problem 8.4:

$$\Delta_f(z) = z^3 = z^3 + 0 \cdot z^2 + 0 \cdot z + 0$$

$$\bar{k} = [0-(-3) \quad 0-3 \quad 0-(-1)] = [3 \quad -3 \quad 1]$$

The matrices $\bar{C}$ and C^{-1} are the same as those in Problem 8.4. Thus we have

$$k = \bar{k}\bar{C}C^{-1} = [1 \ 5 \ 2]$$

The state feedback system becomes

$$x[k+1] = (A - bk)x[k] + bu[k]$$

$$= \left(\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} [1 \ 5 \ 2] \right) x[k] + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u[k]$$

$$= \begin{bmatrix} 0 & -4 & -4 \\ 0 & 1 & 1 \\ -1 & -5 & -1 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u[k]$$

Its zero-input response is

$$x[k] = A_f^k x[0] = \begin{bmatrix} 0 & -4 & -4 \\ 0 & 1 & 1 \\ -1 & -5 & -1 \end{bmatrix}^k x[0]$$

We compute

$$A_f^2 = \begin{bmatrix} 0 & -4 & -4 \\ 0 & 1 & 1 \\ -1 & -5 & -1 \end{bmatrix}^2 = \begin{bmatrix} 4 & 16 & 0 \\ -1 & -4 & 0 \\ 1 & 4 & 0 \end{bmatrix}$$

$$A_f^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus we have, for any $x[0]$,

$$x[k] = 0 \quad \text{for } k \geq 3$$

8.9 Following Problem 8.7, we have

$$\hat{g}(z) = \frac{2z^2 - 8z + 8}{(z-1)^3}$$

Introduce $u = pr[k] - [1 \ \ 5 \ \ 2]x[k]$ yields the transfer function from r to y as

$$\hat{g}_f(z) = p \cdot \frac{2z^2 - 8z + 8}{z^3}$$

The condition to track any step reference sequence is

$$\hat{g}_f(1) = 1$$

(Theorem 5.D2). Thus we have

$$p \cdot \frac{2 - 8 + 8}{1} = 1 \Rightarrow p = \frac{1}{2}$$

and

$$\hat{g}_f(z) = \frac{0.5(2z^2 - 8z + 8)}{z^3}$$

Let $r[k] = a$, for $k \geq 0$. Then $\hat{r}(z) = \frac{az}{z-1}$

and

$$\hat{y}(z) = \frac{z^2 - 4z + 4}{z^3} \cdot \frac{az}{z-1}$$

which can be expanded as

$$\hat{y}(z) = \frac{az}{z-1} - a - \frac{4a}{z^2}$$

Thus

$$y[k] = a - a\delta[k] - 4a\delta[k-2]$$

$k = 0 \quad y[0] = a - a = 0$

$k = 1 \quad y[1] = a$

$k = 2 \quad y[2] = a - 4a = -3a$

$k \geq 3 \quad y[k] = a = r[k]$

8.10 The equation is in Jordan form. It is clear that the Jordan block associated with eigenvalue 2 is controllable; the two Jordan blocks associated with -1 are not (Corollary 6.8). Consider the subequation

$$\dot{x}_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$P^{-1} = Q = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$\bar{x}_1 = Px_1$ will transform the equation into

$$\dot{\bar{x}}_1 = \left[\begin{array}{c:c} -1 & 0 \\ 0 & -1 \end{array}\right] \bar{x}_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Thus the transformation

$$\bar{x} = \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hdashline 0 & & P \end{array}\right] x$$

will transform the original equation into

$$\dot{\bar{x}} = \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array}\right]\bar{x} + \left[\begin{array}{c} 0 \\ 1 \\ 1 \\ \hline 0 \end{array}\right]u$$

The 3-dimensional subequation is controllable and the three eigenvalues $\{2, 2, -1\}$ can be assigned to any values. The 1-dimensional subequation is not controllable; therefore, its eigenvalue -1 cannot be changed. Thus the answers to the first three questions are yes, yes and no. Because the uncontrollable eigenvalue -1 is stable the equation is stabilizable.

8.11

$$\dot{x} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}x + \begin{bmatrix} 1 \\ 2 \end{bmatrix}u = Ax + bu \qquad (1)$$

$$y = [1 \quad 1]x = cx$$

Its transfer function is

$$\hat{g}(s) = [1 \quad 1]\begin{bmatrix} s-2 & -1 \\ 1 & s-1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{3s-4}{s^2-3s+3}$$

If $u = r - [4 \quad 1]x$, then

$$\hat{g}_f(s) = \frac{\hat{y}(s)}{\hat{r}(s)} = \frac{3s-4}{(s+1)(s+2)} = \frac{3s-4}{s^2+3s+2}$$

Two-dimensional state estimator:

$$F = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} \quad l = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ (Use modal form with eigenvalues $-2 \pm j2$)

$$\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

or

$$\left[\begin{array}{cc|cc} 4 & -2 & -1 & 0 \\ 2 & 4 & 0 & -1 \\ \hline 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & 3 \end{array}\right]\begin{bmatrix} t_{11} \\ t_{21} \\ t_{12} \\ t_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Its solution can be computed as

$$T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} 0.231 & 0.1986 \\ -0.1372 & -0.0866 \end{bmatrix}$$

We compute

$$T^{-1} = \begin{bmatrix} -12 & -27.5 \\ 19 & 32 \end{bmatrix}, \quad Tb = \begin{bmatrix} 0.6282 \\ -0.3105 \end{bmatrix}$$

Thus the estimator is

$$\dot{z} = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}z + \begin{bmatrix} 0.6282 \\ -0.3105 \end{bmatrix}u + \begin{bmatrix} 1 \\ 0 \end{bmatrix}y \qquad (2)$$

$$\hat{x} = T^{-1}z = \begin{bmatrix} -12 & -27.5 \\ 19 & 32 \end{bmatrix}z$$

One-dimensional state estimator

$F = -3, \quad l = 1,$

$T = [t_1 \quad t_2]$

$$[t_1 \quad t_2]\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} - (-3)[t_1 \quad t_2] = 1 \times [1 \quad 1] = [1 \quad 1]$$

$$[5t_1 - t_2 \quad t_1 + t_2] = [1 \quad 1] \Rightarrow t_1 = \frac{5}{21}, \quad t_2 = \frac{4}{21}$$

or $T = [\frac{5}{21} \quad \frac{4}{21}]$

Thus the estimator is

$$\dot{z} = -3z + \frac{13}{21}u + y \qquad (3)$$

$$\hat{x} = \begin{bmatrix} 1 & 1 \\ \frac{5}{21} & \frac{4}{21} \end{bmatrix}^{-1}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} -4 & 21 \\ 5 & -21 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$$

8.12 The transfer function from r to y was computed in Problem 8.11 as

$$\hat{g}_f(s) = \frac{3s-4}{s^2+3s+2}$$

Now we apply

$$u = r - [4 \quad 1]\hat{x}$$

to the two-dimensional estimator in (2) of Problem 8.11 to yield

$$u = r - [4 \quad 1]\begin{bmatrix} -12 & -27.5 \\ 19 & 32 \end{bmatrix}z$$

$$= r + [29 \quad 78]z$$

Substituting this into (1) and (2) of Problem 8.11 yields

$$\dot{x} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 29 & 75 \\ 58 & 156 \end{bmatrix} z + \begin{bmatrix} 1 \\ 2 \end{bmatrix} r$$

$$\dot{z} = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} z + \begin{bmatrix} 18.2166 & 48.4964 \\ -9.0036 & -24.2166 \end{bmatrix} z + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0.6282 \\ -0.3105 \end{bmatrix} r$$

They can be combined as

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \left[\begin{array}{cc:cc} 2 & 1 & 29 & 75 \\ -1 & 1 & 58 & 156 \\ \hdashline 1 & 1 & 16.2166 & 50.4964 \\ 0 & 0 & -11.0036 & -26.2166 \end{array}\right] \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ \hdashline 0.6282 \\ -0.3105 \end{bmatrix} r$$

$$y = [1 \quad 1 \,\vdots\, 0 \quad 0] \begin{bmatrix} x \\ z \end{bmatrix} + 0 \cdot r$$

Its transfer function can be computed, using ss2tf in MATLAB, as

$$\hat{g}_f(s) = \frac{3s^3 + 7.9964s^2 + 7.9730s - 31.9576}{s^4 + 7s^3 + 21.999s^2 + 31.9986s + 16.002}$$

If we round the numbers to the nearest integers, then we have

$$\hat{g}_f(s) = \frac{3s^3 + 8s^2 + 8s - 32}{s^4 + 7s^3 + 22s^2 + 32s + 16}$$

$$= \frac{(3s-4)(s^2+4s+8)}{(s+1)(s+2)(s^2+4s+8)} = \frac{3s-4}{s^2+3s+2}$$

Next we apply $u = r - [4 \quad 1]\hat{x}$ to the one-dimensional estimator (3) in Prob. 8.11.

$$u = r - [4 \quad 1] \begin{bmatrix} -4 & 21 \\ 5 & -21 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = r + 11y - 63z$$

Substituting this into (1) and (3) yields

$$\dot{x} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} (r + 11y - 63z)$$

$$= \begin{bmatrix} 13 & 12 \\ 21 & 23 \end{bmatrix} x - \begin{bmatrix} 63 \\ 126 \end{bmatrix} z + \begin{bmatrix} 1 \\ 2 \end{bmatrix} r$$

$$\dot{z} = -3z + \frac{13}{21}(r + 11y - 63z) + y$$

$$= \frac{-882}{21} z + \frac{164}{21}[1 \quad 1] x + \frac{13}{21} r$$

They can be combined as

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 13 & 12 & -63 \\ 21 & 23 & -126 \\ \frac{164}{21} & \frac{164}{21} & \frac{-882}{21} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ \frac{13}{21} \end{bmatrix} r$$

$$y = [1 \quad 1 \quad 0] \begin{bmatrix} x \\ z \end{bmatrix}$$

Its transfer function is, using ss2tf in MATLAB,

$$\hat{g}_f(s) = \frac{3s^2 + 5s - 12}{s^3 + 6s^2 + 11s + 6} = \frac{(3s-4)(s+3)}{(s^2+3s+2)(s+3)}$$

$$= \frac{3s-4}{s^2+3s+2}$$

Thus the use of state estimators will not affect $\hat{g}_f(s)$.

8.13 $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 1 & 2 & 3 \\ 2 & 1 & 0 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ 0 & 2 \end{bmatrix}$

Desired poles: $-4 \pm j3$, $-5 \pm j4$

Select

$$F = \left[\begin{array}{cc:cc} -4 & 3 & 0 & 0 \\ -3 & -4 & 0 & 0 \\ \hdashline 0 & 0 & -5 & 4 \\ 0 & 0 & -4 & -5 \end{array}\right] \quad \bar{K}_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In MATLAB, typing

```
a=[0 1 0 0;0 0 1 0;-3 1 2 3;2 1 0 0];
b=[0 0;0 0;1 2;0 2];
kb=[1 0 1 0;0 0 0 0];
f=[-4 3 0 0;-3 -4 0 0;0 0 -5 4;0 0 -4 -5];
t=lyap(a,-f,-b*kb)
k=kb*inv(t)
```

yields

$$K = \begin{bmatrix} 62.5 & 147 & 20 & 515.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The same result will be obtained if we use the function "place" or "acker" in MATLB. If kb is replaced by

kb=[1 0 0 0;0 0 1 0] $\left(\bar{K}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\right)$

then

$$K = \begin{bmatrix} -606.2 & -168 & -14.2 & -2 \\ 371.1 & 119.2 & 14.9 & 2.2 \end{bmatrix}$$

Chapter 9

9.1 $D(s) = s^2 - 1$, $N(s) = s - 2$

Because $D(s)$ and $N(s)$ are coprime, solutions exist in

$$A(s)D(s) + B(s)N(s) = s^2 + 2s + 2 =: F(s)$$

Using (9.8) and Example 9.1, we have, for any polynomial $Q(s)$

$$A(s) = \frac{1}{3}(s^2 + 2s + 2) + Q(s)(-s+2)$$

$$B(s) = \frac{-1}{3}(s+2)(s^2+2s+2) + Q(s)(s^2-1)$$

is a solution. For any $Q(s) = q_0$ of degree 0, we have $\deg B(s) > \deg A(s)$. For any $Q(s) = q_0 + q_1 s$ of degree 1, we have $\deg B(s) > \deg A(s)$. For any $Q(s) = q_0 + q_1 s + q_2 s^2$ of degree 3, we have $\deg B(s) > \deg A(s)$. Proceeding forward, we conclude that there exists no solution with $\deg B(s) \le \deg A(s)$ in the equation.

We give a different argument. Consider $A(s) = A_0 + A_1 s + A_2 s^2$ and $B(s) = B_0 + B_1 s + B_2 s^2$. Then A_i and B_i must meet

$$\begin{bmatrix} -1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \\ A_1 \\ B_1 \\ A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Then, for any α, we have the general solution

$$\begin{bmatrix} A_0 \\ B_0 \\ A_1 \\ B_1 \\ A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 5/3 \\ -7/3 \\ 0 \\ -5/3 \end{bmatrix} + \alpha \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(Theorem 3.2). For any α, $A_2 = 0$. If $\alpha \neq \frac{5}{3}$, then $B_2 \neq 0$ and $\deg B(s) > \deg A(s)$. If $\alpha = \frac{5}{3}$, then $B_2 = 0$, but $B_1 \neq 0$ and $A_1 = 0$. Thus we still have $\deg B(s) > \deg A(s)$. In conclution, we have no solutions with $\deg B(s) \le \deg A(s)$.

9.2 $\hat{g}(s) = \dfrac{-1+s}{-4+0.5+s^2} = \dfrac{N(s)}{D(s)}$

$$C(s) = \frac{B(s)}{A(s)} = \frac{B_0 + B_1 s}{A_0 + A_1 s}$$

$$F(s) = (s+2)(s^2+2s+2) = 4 + 6s + 4s^2 + s^3$$

$$[A_0 \; B_0 \,\vdots\, A_1 \; B_1] \begin{bmatrix} -4 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -4 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} = [4 \; 6 \; 4 \; 1]$$

In MATLAB, we type

```
a=[-4 0 1 0;-1 1 0 0;0 -4 0 1;0 -1 1 0];
f=[4 6 4 1];
f/a
```

which yields

$$[A_0 \; B_0 \; A_1 \; B_1] = [-6 \; 20 \; 1 \; 10]$$

Thus the compensator is

$$C(s) = \frac{B(s)}{A(s)} = \frac{10s+20}{s-6}$$

and

$$\hat{g}_o(s) = \frac{pN(s)B(s)}{F(s)} = \frac{p(s-1)(10s+20)}{s^3+4s^2+6s+4}$$

To track step reference inputs, we require $\hat{g}_o(0) = \dfrac{p(-20)}{4} = -5p = 1$

$$\Rightarrow p = \frac{-1}{5} = -0.2$$

9.3 If $\hat{g}(s)$ becomes $\hat{\bar{g}}(s) = \dfrac{s-0.9}{s^2-4.1}$, then

$$\hat{\bar{g}}_o = \frac{pC(s)\hat{\bar{g}}(s)}{1 + C(s)\hat{\bar{g}}(s)} = \frac{-0.2 \cdot \frac{10s+20}{s-6} \cdot \frac{s-0.9}{s^2-4.1}}{1 + \frac{10s+20}{s-6} \cdot \frac{s-0.9}{s^2-4.1}}$$

$$= \frac{-(2s+4)(s-0.9)}{s^3+4s^2+6.9s+6.6}$$

This system is still stable. Because

$$\hat{\bar{g}}_c(0) = \frac{(-4)(-0.9)}{6.6} = \frac{3.6}{6.6} = 0.55 \neq 1$$

The output will track any step reference input but with 45% error

Let $\phi(s) = s$ and consider

$$A(s)D(s)\cdot s + B(s)N(s) = F(s)$$

Because $D(s).s$ has degree 3, we need $C(s) = \frac{B(s)}{A(s)} = \frac{B_0 + B_1 s + B_2 s^2}{A_0 + A_1 s + A_2 s^2}$

From the coefficients of $D(s)\cdot s = s^3 - 4s$ and $N(s) = s - 1 = -1 + s + 0\cdot s^2 + 0\cdot s^3$, we form

$$[A_0\ B_0 \vdots A_1\ B_1 \vdots A_2\ B_2]\begin{bmatrix} 0 & -4 & 0 & 1 & & \\ -1 & 1 & 0 & 0 & & \\ \hline & 0 & -4 & 1 & 0 & \\ & -1 & 1 & 0 & 0 & \\ \hline & & 0 & -4 & 1 & 0 \\ & & -1 & 1 & 0 & 0 \end{bmatrix} := F$$

Now the feedback system has five poles and we can assign them all. Let

$$F(s) = (s+2)(s+1+j)(s+1-j)(s+3)^2$$
$$= s^5 + 10s^4 + 39s^3 + 76s^2 + 78s + 36$$

Then we have

$$F = [36\ \ 78\ \ 76\ \ 39\ \ 10\ \ 1]$$

and the solution is

$$[A_0\ B_0 \vdots A_1\ B_1 \vdots A_2\ B_2]$$
$$= [-91\ \ -36 \vdots 10\ \ 250 \vdots 1\ \ 134]$$

Thus we have

$$C(s) = \frac{134s^2 + 250s - 36}{s^2 + 10s - 91}$$

and the total compensator is

$$\frac{C(s)}{\phi(s)} = \frac{134s^2 + 250s - 36}{s(s^2 + 10s - 91)}$$

This compensator has degree 3. This design is achieved by introducing first the internal model $1/\phi(s) = 1/s$

Next we give a robust design without introducing first an internal model. Consider, with $n = \deg D(s) = 2$,

$$A(s)D(s) + B(s)N(s) = F(s)$$

If $C(s) = B(s)/A(s)$ is of degree 1, then the compensator is unique. Now if we increase the degree of $C(s)$ to 2, then $C(s)$ is not unique and we may be able to find a $C(s)$ that contains the factor $1/s$. Let

$$C(s) = \frac{B_0 + B_1 s + B_2 s^2}{A_0 + A_1 s + A_2 s^2}$$

and select

$$F(s) = (s+2)(s^2+2s+2)(s+3)$$
$$= 12 + 22s + 18s^2 + 7s^3 + s^4$$

We form

$$[A_0\ B_0 \vdots A_1\ B_1 \vdots A_2\ B_2]\begin{bmatrix} -4 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ \hline 0 & -4 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ \hline 0 & 0 & -4 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

$$= [12\ \ 22\ \ 18\ \ 7\ \ 1]$$

Solutions are not unique, and they have one free parameter. The general solution is, for any α,

$$[A_0\ B_0\ A_1\ B_1\ A_2\ B_2]$$
$$= [-3\ \ 0 \vdots -18\ \ 50 \vdots 1\ \ 25] + \alpha[1\ \ -4 \vdots -1\ \ 0 \vdots 0\ \ 1]$$

If $\alpha = 3$, then

$$[A_0\ B_0\ A_1\ B_1\ A_2\ B_2] = [0\ \ -12 \vdots -21\ \ 50 \vdots 1\ \ 28]$$

and $C(s) = \frac{28s^2 + 50s - 12}{s^2 - 21s} = \frac{28s^2 + 50s - 12}{s(s-21)}$

This proper compensator has the factor $1/s$ and can achieve robust tracking. It has degree 2 instead of 3 as in the previous design.

9.4 $\hat{g}(s) = \frac{s-1}{s(s-2)} = \frac{-1+s}{0-2s+s^2}$

$$C(s) = \frac{B_0 + B_1 s}{A_0 + A_1 s}$$

$$F(s) = (s+2)(s^2+2s+2) = 4+6s+4s^2+s^3$$

$$[A_0\ B_0\ A_1\ B_1]\begin{bmatrix} 0 & -2 & 1 & \\ -1 & 1 & 0 & \\ & 0 & -2 & 1 \\ & -1 & 1 & 0 \end{bmatrix} = [4\ 6\ 4\ 1]$$

Its solution is $[-16\ -4\ 1\ 22]$. Thus

$$C(s) = \frac{22s-4}{s-16} \quad \text{and}$$

$$\hat{g}_o = \frac{p\,C(s)\hat{g}(s)}{1+C(s)\hat{g}(s)} = p\,\frac{\frac{22s-4}{s-16}\cdot\frac{s-1}{s(s-2)}}{1+\frac{22s-4}{s-16}\cdot\frac{s-1}{s(s-2)}}$$

$$= p\,\frac{(22s-4)(s-1)}{s^3+4s^2+6s+4}$$

Because $\hat{g}_o(0) = p\cdot\frac{(-4)(-1)}{4} = p = 1$, there is no need to introduce feedforward gain ($p=1$) to achieve tracking. The reason is the existence of the factor $1/s$ in $\hat{g}(s)$.

9.5 If $\hat{g}(s)$ becomes $\bar{\hat{g}}(s) = \frac{s-0.9}{s(s-2.1)}$, then

$$\bar{\hat{g}}_o(s) = \frac{C(s)\bar{\hat{g}}(s)}{1+C(s)\bar{\hat{g}}(s)}$$

$$= \frac{(22s-4)(s-0.9)}{s(s-2.1)(s-16)+(22s-4)(s-0.9)}$$

$$= \frac{(22s-4)(s-0.9)}{s^3+3.9s^2+9.8s+3.6}$$

This system is BIBO stable. Because $\bar{\hat{g}}_o(0) = \frac{4\times 0.9}{3.6} = 1$, the system still track any step reference input. Thus the design is robust.

9.6 $\hat{g}(s) = \frac{1}{s-1} = \frac{N(s)}{D(s)}$

Let $C(s) = \frac{B(s)}{A(s)}$. In order to achieve the design $A(s)$ must contain $s(s^2+4)$

Let $A(s) = \bar{A}_0(s(s^2+4))$

$$B(s) = B_0 + B_1 s + B_2 s^2 + B_3 s^3$$

and consider

$$A(s)D(s) + B(s)N(s) = F(s)$$

Select

$$F(s) = (s^2+2s+5)(s^2+4s+5)$$
$$= s^4+6s^3+18s^2+30s+25$$

$$\bar{A}_0\, s(s^2+4)(s-1) + [B_0+B_1 s+B_2 s^2+B_3 s^3]\times 1 = F(s)$$

or $\bar{A}_0(s^4-s^3+4s^2-4s) + (B_0+B_1 s+B_2 s^2+B_3 s^3) = F(s)$

Equating the coefficients yields

$$[\bar{A}_0\ B_0\ B_1\ B_2\ B_3]\begin{bmatrix} 0 & -4 & 4 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = [25\ 30\ 18\ 6\ 1]$$

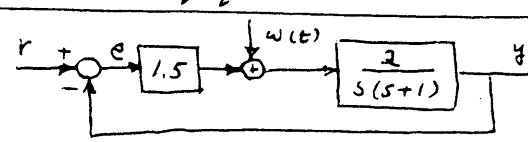
Its solution is $[1\ 25\ 34\ 14\ 7]$

Thus the compensator is

$$C(s) = \frac{7s^3+14s^2+34s+25}{s(s^2+4)}$$

This compensator will place the poles in the assigned positions, track robustly any step reference input and reject any disturbance of form $a\sin(2t+\theta)$.

9.7



The model $1/s$ is not in the forward path from r to e. therefore, for any $r(t) = a$, $e \to 0$ or $y \to r$. Indeed, we have

$$\hat{g}_{yr}(s) = \frac{\frac{3}{s(s+1)}}{1+\frac{3}{s(s+1)}} = \frac{3}{s^2+s+3}$$

and $\hat{g}_{yr}(0) = 1$

Thus the output tracks any step reference input. The model is in the forward path from w to y. Therefore y will not reject $w = a$. Indeed, we have

$$\hat{g}_{yw}(s) = \frac{\frac{2}{s(s+1)}}{1+\frac{3}{s(s+1)}} = \frac{2}{s^2+s+3}$$

If $w = a$, then

$$y \to \hat{g}_{yw}(0)\cdot a = \frac{2}{3}a$$

and the output does not reject completely the step disturbance.

9.8 $$\hat{g}_{yr}(s) = \frac{\frac{1}{s-2}\cdot\frac{s-2}{s}}{1+\frac{1}{s-2}\cdot\frac{s-2}{s}} = \frac{\frac{1}{s}}{1+\frac{1}{s}} = \frac{1}{s+1}$$

BIBO stable.

The transfer function from r to u is

$$\hat{g}_{ur} = \frac{\frac{1}{s-2}}{1+\frac{1}{s-2}\,\frac{s-2}{s}} = \frac{\frac{1}{s-2}}{1+\frac{1}{s}} = \frac{s}{(s+1)(s-2)}$$

which is not BIBO stable. Thus the system is not totally stable.

9.9 $$\hat{g}_{yr}(s) = \frac{C(s)\hat{g}(s)}{1+C(s)\hat{g}(s)} \qquad \hat{g}_{en_2}(s) = \frac{-\hat{g}(s)}{1+C(s)\hat{g}(s)}$$

$$\hat{g}_{yn_2}(s) = \frac{\hat{g}(s)}{1+C(s)\hat{g}(s)} \qquad \hat{g}_{en_3}(s) = \frac{-1}{1+C(s)\hat{g}(s)}$$

$$\hat{g}_{yn_3}(s) = \frac{1}{1+C(s)\hat{g}(s)} \qquad \hat{g}_{ur}(s) = \frac{C(s)}{1+C(s)\hat{g}(s)}$$

$$\hat{g}_{er}(s) = \frac{1}{1+C(s)\hat{g}(s)} \qquad \hat{g}_{un_2}(s) = \frac{1}{1+C(s)\hat{g}(s)}$$

$$\hat{g}_{un_3}(s) = \frac{-C(s)}{1+C(s)\hat{g}(s)}$$

- - - - - - - - - - - - - - - - - -

$(1+C(s)\hat{g}(s))^{-1}$ is proper $\Leftrightarrow$

$|(1+C(\infty)\hat{g}(\infty))^{-1}| < \infty \Leftrightarrow$

$1+C(\infty)\hat{g}(\infty) \neq 0$

- - - - - - - - - - - - - - - - - -

If $C(\infty)\hat{g}(\infty) \neq -1$, then every transfer function is a product of two or three proper transfer functions. Thus every closed-loop transfer function is proper and the system is well posed.

9.10 $$\hat{g}(s) = \frac{(s^2-1)(s+1)}{s^3+a_1s^2+a_2s+a_3}$$

$\frac{s-1}{(s+1)^2}$ implementable

$\frac{s+1}{(s+2)(s+3)}$ no, violates (3) of Corollary 9.4

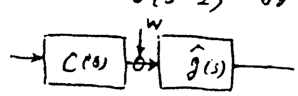
$\frac{s^2-1}{(s-2)^3}$ no, violates (1)

$\frac{s^2-1}{(s+2)^2}$ no, violates (2)

$\frac{(s-1)(b_0s+b_1)}{(s+2)^2(s^2+2s+2)}$ yes

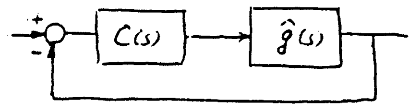
$\frac{1}{1}$ no. violates (2) and (3).

9.11 $\hat{g}(s) = \frac{s-1}{s(s-2)}$ $\quad \hat{g}_o(s) = \frac{-2(s-1)}{s^2+2s+2}$



$$C(s) = \frac{\hat{g}_o(s)}{\hat{g}(s)} = \frac{-2s(s-2)}{s^2+2s+2}$$

The implementation is not totally stable. The output will grow unbounded if any signal enters at w.



$$C(s) = \frac{\hat{g}_o(s)}{\hat{g}(s)(1-\hat{g}_o(s))} = \frac{-2(s-2)}{s+4}$$

Because of the unstable pole-zero cancellation of $(s-2)$ between $C(s)$ and $\hat{g}(s)$, the implementation is not totally stable and cannot be used in practice.

9.12 $\hat{g}(s) = \frac{s-1}{s(s-2)}$, $\hat{g}_o(s) = \frac{-2(s-1)}{s^2+2s+2}$

Following Procedure 9.1:

$$\frac{\hat{g}_o(s)}{N(s)} = \frac{-2}{s^2+2s+2} = \frac{\bar{E}(s)}{\bar{F}(s)}$$

$\deg \bar{F}(s) = 2 < 2n-1 = 3$

We select $\hat{F}(s) = s+3$ Then

$L(s) = -2(s+3)$

$A(s)D(s) + M(s)N(s) = (s+3)(s^2+2s+2)$

$$[A_0\ M_0\ A_1\ M_1]\left[\begin{array}{cccc} 0 & -2 & 1 & \\ -1 & 1 & 0 & \\ \hline & 0 & -2 & 1 \\ & -1 & 1 & 0 \end{array}\right] = [6\ 8\ 5\ 1]$$

Its solution is $[-21\ -6 \vdots 1\ 28]$. Thus

$A(s) = s-21 \quad M(s) = 28s-6$.

Thus the compensators are

$$C_1(s) = \frac{L(s)}{A(s)} = \frac{-2(s+3)}{s-21} \quad C_2(s) = \frac{M(s)}{A(s)} = \frac{28s-6}{s-21}$$

$A(s)$ is not Hurwitz and we cannot implement the design as in Fig. 9.4(a).

$$\hat{u}(s) = C_1(s)\hat{r}(s) - C_2(s)\hat{y}(s) = [C_1(s)\ \ -C_2(s)]\begin{bmatrix}\hat{r}(s)\\ \hat{y}(s)\end{bmatrix}$$

Consider

$$C(s) = \left[\frac{-2(s+3)}{s-21}\ \ \frac{-28s+6}{s-21}\right]$$

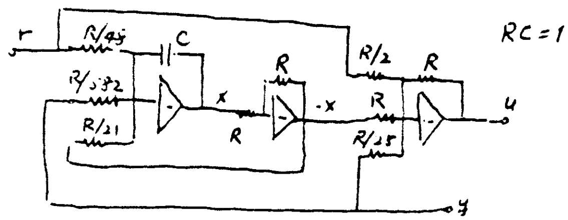
$$= [-2\ \ -28] + \left[\frac{-48}{s-21}\ \ \frac{-582}{s-21}\right]$$

$$= [-2\ \ -28] + \frac{1}{s-21}[-48\ \ -582]$$

Using Problem 4.10, we have

$$\dot{x} = 21x + [-48\ \ -582]\begin{bmatrix}r\\ y\end{bmatrix}$$

$$u = x + [-2\ \ -28]\begin{bmatrix}r\\ y\end{bmatrix}$$



9.13 $r(t) = at \quad \hat{r}(s) = \frac{a}{s^2}$

$$\hat{y}(s) = \hat{g}_o(s)\hat{r}(s) = \hat{g}_o(s)\cdot\frac{a}{s^2} = \frac{k_1}{s} + \frac{k_2}{s^2}$$

+ (terms due to the poles of $\hat{g}_o(s)$)

with $k_2 = \hat{g}_o(s)\frac{a}{s^2}\cdot s^2\Big|_{s=0} = \hat{g}_o(0)a$

$$k_1 = \frac{d}{ds}(\hat{g}_o(s)a)\Big|_{s=0} = \hat{g}_o'(0)\cdot a$$

If $\hat{g}_o(s)$ is BIBO stable, then every pole lies inside the open left half plane and its time response approaches 0 as $t\to\infty$. Thus we have

$$y_{ss}(t) = \lim_{t\to\infty} y(t) = \mathcal{L}\left[\frac{k_1}{s} + \frac{k_2}{s^2}\right]$$

$$= \hat{g}_o'(0)\cdot a + \hat{g}_o(0)a\cdot t$$

9.14 If $\hat{g}_o(s) = \frac{b_0+b_1 s+\cdots+b_m s^m}{a_0+a_1 s+\cdots+a_n s^n} \quad n \geq m$

$$\hat{g}_o(0) = \frac{b_0}{a_0}$$

$$\frac{d\hat{g}_o(s)}{ds}\Big|_{s=0} = \frac{(a_0+\cdots+a_n s^n)(b_1+\cdots+mb_m s^{m-1}) - (a_1+\cdots+na_n s^{n-1})(b_0+\cdots+b_n s^n)}{(a_0+a_1 s+\cdots+a_n s^n)^2}\Bigg|_{s=0}$$

$$= \frac{a_0 b_1 - a_1 b_0}{a_0^2}$$

Thus $\left.\begin{array}{l}\hat{g}_o(0) = 1\\ \hat{g}_o'(0) = 0\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l}a_0 = b_0\\ a_1 = b_1\end{array}\right.$

9.15 $\hat{g}(s) = \frac{(s+3)(s-2)}{s^3+2s-1}$

(1) $\hat{g}_o(s) = \frac{b_1 s+b_0}{s^2+2s+a}$ To be implementable, the denominator must be Hurwitz. Thus we require $a>0$.

$(b_1 s+b_0) = b_1(s+\frac{b_0}{b_1})$ must contain the factor $s-2$. Thus we require $\frac{b_0}{b_1} = -2$ or $b_0 = -2b_1$.

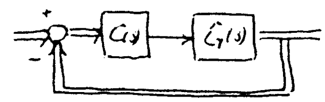
(2) $$\hat{g}_o(s) = \frac{(s-2)(b_1 s+b_0)}{(s+2)(s^2+2s+2)} = \frac{b_1 s^2 + (b_0-2b_1)s - 2b_0}{s^3+4s^2+6s+4}$$

It is implementable for any b_1 and b_0.
In order to track any ramp reference input, we require

$$-2b_0 = 4 \quad \Rightarrow b_0 = -2$$

$$b_0 - 2b_1 = 6 \quad \Rightarrow b_1 = \tfrac{1}{2}(b_0 - 6) = -4$$

9.16



$$\hat{G}(s) = \begin{bmatrix} \dfrac{s+1}{s(s-1)} \\ \dfrac{1}{s^2-1} \end{bmatrix} = \begin{bmatrix} (s+1)^2 \\ s \end{bmatrix} (s(s^2-1))^{-1}$$

$$= \begin{bmatrix} 1+2s+s^2 \\ 0+s+0\cdot s^2 \end{bmatrix} (0 - s + 0\cdot s^2 + s^3)^{-1}$$

$$= N(s)\,D^{-1}(s)$$

This is right coprime and $D(s)$ is column reduced with $\mu = 3$. We form, as in (9.42),

$$\begin{array}{c} D \\ N1 \\ N2 \\ D \\ N1 \\ N2 \\ D \\ N1 \\ N2 \end{array}
\begin{bmatrix}
0 & -1 & 0 & 1 & & \\
1 & 2 & 1 & 0 & & \\
0 & 1 & 0 & 0 & & \\
\hline
 & 0 & -1 & 0 & 1 & \\
 & 1 & 2 & 1 & 0 & \\
 & 0 & 1 & 0 & 0 & \\
\hline
 & & 0 & -1 & 0 & 1 \\
 & & 1 & 2 & 1 & 0 \\
 & & 0 & 1 & 0 & 0
\end{bmatrix}
\begin{array}{l} \times \\ \times \\ \times \\ \times \\ \times \\ \text{dependent} \\ \times \\ \text{dependent} \\ \text{dependent} \end{array}$$

and search its linearly independent rows from top to bottom. We can use qr decomposition. For this simple example, we can use the row searching algorithm discussed in [5]. Clearly we have $\nu_1 = 2$, $\nu_2 = 1$ and $\nu = 2$.
Let $m = \nu - 1 = 1$. Thus the compensator is of the form

$$\underset{1\times 2}{C(s)} = \underset{1\times 1}{A^{-1}(s)}\ \underset{1\times 2}{B(s)} = [A_0 + A_1 s]^{-1} [B_{01} + B_{11}s \quad B_{02} + B_{12}s]$$

$F(s)$ is of degree $\mu + m = 4$ and is selected as

$$F(s) = (s+2)(s^2+2s+2)(s+3)$$

$$= s^4 + 7s^3 + 18s^2 + 22s + 12$$

Then the compensator can be solved from

$$[A_0\ B_{01}\ B_{02} \vdots A_1\ B_{11}\ \ ]\begin{bmatrix} 0 & -1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix} = [12\ 22\ 18\ 7\ 1]$$

Note that the second N2 row (which is dependent) is not used and the corresponding B_{12} must be assigned to be 0. The solution of the preceding equation is $[3.5\ 12\ -2 \vdots 1\ 3.5\ \ ]$. Thus the compensator is

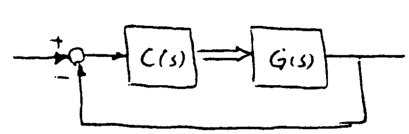
$$C(s) = \frac{1}{s+3.5}[3.5s+12 \quad -2] = A^{-1}(s)B(s)$$

The overall transfer matrix is

$$\hat{G}_0(s) = \frac{1}{F(s)} \begin{bmatrix} (s+1)^2 \\ s \end{bmatrix} [3.5s+12 \quad -2]$$

(Theorem 9.M2).
A necessary condition for $\hat{G}_0(s)$ to track any step reference input is that $\hat{G}(s)$ (the plant) has the same or more inputs than output. This is not the case. Therefore, the system cannot be designed to track any step reference signal.

9.17



This problem is dual to Problem 9.16

$$G(s) = \frac{1}{s(s^2-1)}[(s+1)^2 \quad s] = \underset{1\times 1}{\bar{D}^{-1}(s)}\ \underset{1\times 2}{\bar{N}(s)}$$

$\nu = 3$ left coprime. We form, as in (9.57),

$$\begin{bmatrix}
0 & 1 & 0 & & & & & & \\
-1 & 2 & 1 & 0 & 0 & 0 & & & \\
0 & 1 & 0 & -1 & 1 & 1 & & & \\
1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
 & & & 1 & 1 & 0 & -1 & 1 & 1 \\
 & & & & & & 0 & 2 & 0 \\
 & & & & & & 1 & 1 & 0
\end{bmatrix}$$
$$\begin{array}{ccccccccc} \times & \times & \times & \times & \times & \uparrow & \times & \uparrow & \uparrow \end{array}$$

and search its linearly independent columns from left to right. Then we have $\mu_1 = 2$, $\mu_2 = 1$ and $\mu = 2$. Let $m = \mu - 1 = 1$. Then the compensator is of form

$$C(s) = \underset{2\times 1}{\bar{B}(s)}\,\underset{1\times 1}{\bar{A}^{-1}(s)} = \begin{bmatrix} \bar{B}_{01} + \bar{B}_{11}s \\ \bar{B}_{02} + \bar{B}_{12}s \end{bmatrix} (\bar{A}_0 + \bar{A}_1 s)^{-1}$$

Select

$$F(s) = (s+2)(s^2+2s+2)(s+3)$$

Then as in Problem 9.16, the compensator can be computed as

$$C(s) = \begin{bmatrix} 3.5s + 12 \\ -2 \end{bmatrix} \cdot \frac{1}{s+3.5}$$

and the overall transfer function is

$$\hat{G}_o(s) = 1 - \bar{A}(s)F^{-1}(s)\bar{D}(s)$$

$$= 1 - \frac{(s+3.5)(s^3 - s)}{s^4 + 7s^3 + 18s^2 + 22s + 12}$$

$$= \frac{3.5s^3 + 19s^2 + 25.5s + 12}{s^4 + 7s^3 + 18s^2 + 22s + 12}$$

(Corollary 9.M2). Because $\hat{G}_o(0) = 1$, the system will track any step reference input without introducing any feed-forward gain.

9.18

$$\hat{G}(s) = \begin{bmatrix} \dfrac{s-2}{s^2-1} & \dfrac{1}{s-1} \\ \dfrac{1}{s} & \dfrac{2}{s-1} \end{bmatrix}$$

$$\det G(s) = \frac{2(s-2)}{(s^2-1)(s-1)} - \frac{1}{s(s-1)} = \frac{s^2-4s+1}{s(s^2-1)(s-1)}$$

$$\Delta(s) = s(s^2-1)(s-1) \qquad \deg \hat{G}(s) = 4 = n$$

$$\hat{G}(s) = \begin{bmatrix} s(s-2) & 1 \\ s^2-1 & 2 \end{bmatrix} \begin{bmatrix} s(s^2-1) & 0 \\ 0 & s-1 \end{bmatrix}^{-1} = N(s)D^{-1}(s)$$

Because $\det D(s) = \Delta(s)$, it is right coprime. $D(s)$ is column reduced with column degrees $\mu_1 = 3$, $\mu_2 = 1$.

From the coefficients of

$$D(s) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} s + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s^3$$

$$N(s) = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} s^3$$

We form

$$S = \left[\begin{array}{cc|cc|cc|cc|cc} 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & & \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 1 & -2 & 0 & 1 & 0 & 0 & 0 & & \\ -2 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & & \\ \hline & & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ & & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 1 & -2 & 0 & 1 & 0 & 0 & 0 \\ & & -2 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

and search its linearly independent rows from top to bottom. Using qr decomposition we find all rows are linearly independent. Thus we have $\nu_1 = 2$, $\nu_2 = 2$. Because $\nu_1 + \nu_2 = 4$, we have found all linearly independent N rows; therefore there is no need to search further. Let $m = \nu - 1 = 1$. Select

$$F(s) = \begin{bmatrix} (s+2)(s^2+2s+2)(s+3) & 0 \\ 0 & s^2+2s+2 \end{bmatrix} \quad \text{Then}$$

$$\lim_{s\to\infty} \begin{bmatrix} s^{-1} & 0 \\ 0 & s^{-1} \end{bmatrix} F(s) \begin{bmatrix} s^{-3} & 0 \\ 0 & s^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{non-singular}$$

The compensator is of form

$$C(s) = A^{-1}(s)B(s) = (A_0 + A_1 s)^{-1}(B_0 + B_1 s)$$

where A_i, B_i are 2×2 and can be solved from

$$[A_0 \vdots B_0 \mid A_1 \vdots B_1]S = \left[\begin{array}{cc:cc:cc:cc:cc} 12 & 0 & 22 & 0 & 18 & 0 & 7 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

as, using MATLAB,

$$\left.\begin{array}{cc:cc|cc:cc} -4.7 & -53.7 & -29.7 & -12 & 1 & 0 & -30.3 & 42 \\ -2.5 & -2 & 0 & 0 & 0 & 1 & 0 & 2.5 \end{array}\right.$$

Thus we have

$$A(s) = \begin{bmatrix} s-4.7 & -53.7 \\ -2.5 & s-2 \end{bmatrix}$$

$$B(s)=\begin{bmatrix}-30.3s-29.7 & 42s-12\\ 0 & 2.5s\end{bmatrix}$$

This compensator $C(s)=A^{-1}(s)B(s)$ will place the denominator matrix of the unity feedback system as $F(s)$.

Because the transmission zeros of the plant is given by the roots of $\det N(s) = 2s^2-4s-s^2+1 = s^2-4s+1 = (s-3.73)\cdot(s-0.27)$, which contains no 0. Therefore it is possible to design a system to track any step reference input. However, for the chosen $F(s)$, we have $B(0)=\begin{bmatrix}-29.7 & -12\\ 0 & 0\end{bmatrix}$, which is singular; therefore we must select a different $F(s)$. Let us select

$$F(s)=\begin{bmatrix}(s+2)(s^2+2s+2)(s+3) & 0\\ 1 & s^2+2s+2\end{bmatrix}$$

and solve

$$[A_0 \; B_0 \,|\, A_1 \; B_1]\,S=\left[\begin{array}{cc:cc:cc:cc:cc}12 & 0 & 22 & 0 & 18 & 0 & 7 & 0 & 1 & 0\\ 1 & 2 & 6 & 2 & 0 & 1 & 0 & 0 & 0 & 0\end{array}\right]$$

The solution is

$$\left[\begin{array}{cc:cc|cc:cc}-4.7 & -53.7 & -29.7 & -12 & 1 & 0 & -30.3 & 42\\ -3.3 & -4.3 & -0.3 & -1 & 0 & 1 & -0.7 & 4\end{array}\right]$$

Thus the compensator is

$$A(s)=\begin{bmatrix}s-4.7 & -53.7\\ -3.3 & s-4.3\end{bmatrix}$$

$$B(s)=\begin{bmatrix}-30.3s-29.7 & 42s-12\\ -0.7s-0.3 & 4s-1\end{bmatrix}$$

The feedforward gain to achieve tracking is, using (9.6c)

$$P=B^{-1}(0)F(0)N^{-1}(0)=\begin{bmatrix}-29.7 & -12\\ -0.3 & -1\end{bmatrix}^{-1}\begin{bmatrix}12 & 0\\ 1 & 2\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 2\end{bmatrix}^{-1}$$

$$=\begin{bmatrix}0.42 & 0\\ -4.28 & 1\end{bmatrix}$$

This completes the design.

9.19 $\hat{G}(s)=\begin{bmatrix}\frac{s-2}{s^2-1} & \frac{1}{s-1}\\ \frac{1}{s} & \frac{2}{s-1}\end{bmatrix}$, $\det\hat{G}(s)=\frac{s^2-4s+1}{s(s^2-1)(s-1)}$

Is $\hat{G}_o(s)=\begin{bmatrix}\frac{4(s^2-4s+1)}{(s^2+2s+2)(s+2)} & 0\\ 0 & \frac{4(s^2-4s+1)}{(s^2+2s+2)(s+2)}\end{bmatrix}$

implementable? We compute

$$\hat{G}^{-1}(s)=\frac{s(s^2-1)(s-1)}{s^2-4s+1}\begin{bmatrix}\frac{2}{s-1} & \frac{-1}{s-1}\\ \frac{-1}{s} & \frac{s-2}{s^2-1}\end{bmatrix}$$

Let $\Delta=(s^2+2s+2)(s+2)$. Then

$$\hat{G}^{-1}(s)\hat{G}_o(s)=\begin{bmatrix}\frac{8s(s^2-1)}{\Delta} & \frac{-4s(s^2-1)}{\Delta}\\ \frac{4(s+1)(s-1)^2}{\Delta} & \frac{4s(s-1)(s-2)}{\Delta}\end{bmatrix}$$

Both $\hat{G}_o(s)$ and $\hat{G}^{-1}(s)\hat{G}_o(s)$ are proper and BIBO stable, thus $\hat{G}_o(s)$ is implementable. We follow Procedure 9.M1.

$$\hat{G}(s)=N(s)D^{-1}(s)=\begin{bmatrix}s(s-2) & 1\\ s^2-1 & 2\end{bmatrix}\begin{bmatrix}s(s^2-1) & 0\\ 0 & s-1\end{bmatrix}^{-1}$$ right coprime

$$N^{-1}(s)\hat{G}_o(s)=\frac{1}{s^2-4s+1}\begin{bmatrix}2 & -1\\ -(s^2-1) & s(s-2)\end{bmatrix}\begin{bmatrix}\frac{4(s^2-4s+1)}{\Delta} & 0\\ 0 & \frac{4(s^2-4s+1)}{\Delta}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{8}{\Delta} & \frac{-4}{\Delta}\\ \frac{-4(s^2-1)}{\Delta} & \frac{4s(s-2)}{\Delta}\end{bmatrix}=\begin{bmatrix}\Delta & 0\\ 0 & \Delta\end{bmatrix}^{-1}\begin{bmatrix}8 & -4\\ -4(s^2-1) & 4s(s-2)\end{bmatrix}$$

$=: \bar{F}^{-1}\bar{E}$ This is left coprime.

We have $\mu_1=3$, $\mu_2=1$ and, as computed in Problem 9.18, $\nu_1=2$, $\nu_2=2$ and $\nu=2$.

Let $\hat{F}=\begin{bmatrix}\alpha_1(s) & 0\\ 0 & \alpha_2(s)\end{bmatrix}=\begin{bmatrix}s+3 & 0\\ 0 & 1\end{bmatrix}$ Then

$$\hat{F}\bar{F}=\begin{bmatrix}(s+3)\Delta & 0\\ 0 & \Delta\end{bmatrix}=F$$

Note that $\hat{F}\bar{F}$ is row-column reduced with column degrees $\{3,1\}$ and row degrees $\{1,2\}$. Thus we have $m_1=1\geq \nu-1$

and $m_2 = 2 \geq \nu - 1$. We compute

$$L(s) = \hat{F}(s)\bar{E}(s) = \begin{bmatrix} 8(s+3) & -4(s+3) \\ -4(s^2-1) & 4s(s-2) \end{bmatrix}$$

We solve $A(s) = A_0 + A_1 s + A_2 s^2$, $M(s) = M_0 + M_1 s + M_2 s^2$ from

$$A(s)D(s) + M(s)N(s) = \hat{F}(s)\bar{F}(s)$$

$$= \begin{bmatrix} 12+22s+18s^2+7s^3+s^4 & 0 \\ 0 & 4+6s+4s^2+s^3 \end{bmatrix}$$

or

$$[A_0\, M_0 \vdots A_1\, M_1 \vdots A_2 \boxed{}] \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & & & & \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\ 0 & 1 & -2 & 0 & 1 & 0 & 0 & 0 & & & & \\ -2 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & & & & \\ & & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & & \\ & & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ & & 0 & 1 & -2 & 0 & 1 & 0 & 0 & 0 & & \\ & & -2 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & & \\ & & & & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ & & & & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

M_2 must be assigned as 0

} These N rows are not used.

$$= \begin{bmatrix} 12 & 0 & \vdots & 22 & 0 & \vdots & 18 & 0 & \vdots & 7 & 0 & \vdots & 1 & 0 & \vdots & 0 & 0 \\ 0 & 4 & \vdots & 0 & 6 & \vdots & 0 & 4 & \vdots & 0 & 1 & \vdots & 0 & 0 & \vdots & 0 & 0 \end{bmatrix}$$

Note that the next two N rows are known to be linearly dependent; therefore, they are deleted and the corresponding M_2 must be assigned to be 0. The solution is, using MATLAB,

$$\begin{array}{cc:cc|cc:cc|cc:l} \frac{-14}{3} & \frac{-161}{3} & \frac{-89}{3} & -12 & 1 & 0 & \frac{-91}{3} & 42 & 0 & 0 & M_2 \\ -7.5 & -4 & 0 & 0 & 0 & 5 & 0 & 7.5 & 0 & 1 & =0 \end{array}$$

Thus we have

$$A(s) = \begin{bmatrix} s - \frac{14}{3} & \frac{-161}{3} \\ -7.5 & s^2+5s-4 \end{bmatrix}$$

$$M(s) = \begin{bmatrix} \frac{-91}{3}s - \frac{89}{3} & 42s-12 \\ 0 & 7.5s \end{bmatrix}$$

$A(s)$ is row reduced with row degrees 1 and 2. Thus $A^{-1}(s)L(s)$ and $A^{-1}(s)M(s)$ are proper.

As a check, we compute

$$\hat{G}_o(s) = N(s)F^{-1}(s)L(s)$$

$$= \begin{bmatrix} s(s-2) & 1 \\ s^2-1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{(s+3)\Delta} & 0 \\ 0 & \frac{1}{\Delta} \end{bmatrix} \begin{bmatrix} 8(s+3) & -4(s+3) \\ -4(s^2-1) & 4s(s-2) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4(s^2-4s+1)}{\Delta} & 0 \\ 0 & \frac{4(s^2-4s+1)}{\Delta} \end{bmatrix}$$

Thus the result is correct. The design in fact diagonizes the plant. If we use the procedure in Section 9.5.1, we will obtain $\hat{G}_o(s)$.

9.20 Diagonize

$$\hat{G}(s) = \begin{bmatrix} 1 & s \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & s^2+1 \\ s & 0 \end{bmatrix}^{-1} = \begin{bmatrix} s & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s^2+1 & 1 \\ 0 & s \end{bmatrix}^{-1}$$

Either form can be used. We use the latter.

$$\hat{G}(s) = \begin{bmatrix} s & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s^2+1 & 1 \\ 0 & s \end{bmatrix}^{-1} = N(s)D^{-1}(s)$$

right coprime. $D(s)$ is column reduced with $\mu_1 = 2$, $\mu_2 = 1$.

$$N(s) = \begin{bmatrix} s & 1 \\ 1 & 1 \end{bmatrix} = N_2(s) \quad \text{with } N_1(s) = I$$

$$N_2^{-1}(s) = \frac{1}{s-1}\begin{bmatrix} 1 & -1 \\ -1 & s \end{bmatrix}$$

$$N_{2d} = \begin{bmatrix} s-1 & 0 \\ 0 & s-1 \end{bmatrix}, \quad \bar{N}_2(s) = N_2^{-1}(s)N_{2d}(s) = \begin{bmatrix} 1 & -1 \\ -1 & s \end{bmatrix}$$

$$\hat{T}(s) = D(s)\bar{N}_2(s)\Sigma^{-1}(s)$$

$$= \begin{bmatrix} s^2+1 & 1 \\ 0 & s \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & s \end{bmatrix} \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} s^2 & -s^2+s-1 \\ -s & s^2 \end{bmatrix} \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}^{-1}$$

If we select $\Delta_1 = \Delta_2 = \Delta = s^2+2s+2$, then $\hat{T}(s)$ is proper and

$$\hat{G}_o(s) = \hat{G}(s)\hat{T}(s) = \begin{bmatrix} \frac{s-1}{s^2+2s+2} & 0 \\ 0 & \frac{s-1}{s^2+2s+2} \end{bmatrix}$$

We modify $\hat{G}_c(s)$ as

$$\hat{G}_c(s) = \begin{bmatrix} \dfrac{-2(s-1)}{\Delta} & 0 \\ 0 & \dfrac{-2(s-1)}{\Delta} \end{bmatrix} \qquad \Delta = s^2+2s+2$$

so that it will track any step reference input. We use Procedure 9.M1 to implement it.

$$N^{-1}(s)\hat{G}_c(s) = \frac{1}{s-1}\begin{bmatrix} 1 & -1 \\ -1 & s \end{bmatrix}\begin{bmatrix} \dfrac{-2(s-1)}{\Delta} & 0 \\ 0 & \dfrac{-2(s-1)}{\Delta} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{-2}{\Delta} & \dfrac{2}{\Delta} \\ \dfrac{2}{\Delta} & \dfrac{-2s}{\Delta} \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}^{-1}\begin{bmatrix} -2 & 2 \\ 2 & -2s \end{bmatrix}$$

$$=: \bar{F}^{-1}(s)\bar{E}(s)$$

It is left coprime. We use the coefficients of

$$D(s) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}s + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}s^2$$

$$N(s) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}s + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}s^2$$

to form

$$S = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 1 & 0 & 0 & & \\ 0 & 1 & 1 & 0 & 0 & 0 & & \\ 1 & 1 & 0 & 0 & 0 & 0 & & \\ & & 1 & 1 & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 0 & 1 & 0 & 0 \\ & & 0 & 1 & 1 & 0 & 0 & 0 \\ & & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \text{dependent} \\ \times \end{matrix}$$

and search its linearly independent rows in order from top to bottom. We found

$\nu_1 = 1$, $\nu_2 = 2$ and $\nu = 2$

Select $\hat{F}(s) = \begin{bmatrix} s+3 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$F(s) = \hat{F}(s)\bar{F}(s) = \begin{bmatrix} (s+3)\Delta & 0 \\ 0 & \Delta \end{bmatrix}$ is row-column reduced with column degrees $\{2, 1\}$ and row degrees $\{1, 1\}$. Then we have

$$L(s) = \hat{F}(s)\bar{E}(s) = \begin{bmatrix} -2(s+3) & 2(s+3) \\ 2 & -2s \end{bmatrix}$$

and $A(s) = A_0 + A_1 s$ and $M(s) = M_0 + M_1 s$ can be solved from

$$A(s)D(s) + M(s)N(s) = F(s)$$

$$= \begin{bmatrix} s^3+5s^2+8s+6 & 0 \\ 0 & s^2+2s+2 \end{bmatrix}$$

or

$$[A_0 \; M_0 \,\vdots\, A_1 \,\vdots\, M_{12}]\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 1 & 0 & 0 & & \\ 0 & 1 & 1 & 0 & 0 & 0 & & \\ 1 & 1 & 0 & 0 & 0 & 0 & & \\ & & 1 & 1 & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 0 & 1 & 0 & 0 \\ & & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \text{N1-row is deleted}$$

M_{11} is deleted.

$$= \begin{bmatrix} 6 & 0 & \vdots & 8 & 0 & \vdots & 5 & 0 & \vdots & 1 & 0 \\ 0 & 2 & \vdots & 0 & 2 & \vdots & 0 & 1 & \vdots & 0 & 0 \end{bmatrix}$$

Note that M_{12} denotes the second column of M_2. The first column is zero because the corresponding N1 row is linearly dependent and is deleted from S.

The solution is

$$\begin{matrix} 5 & -14 & \vdots & -6 & 1 & | & 1 & 0 & \vdots & & 13 \\ 0 & 4 & \vdots & 2 & 0 & | & 0 & 1 & \vdots & & -2 \end{matrix}$$

Thus

$$A(s) = \begin{bmatrix} s+5 & -14 \\ 0 & s+4 \end{bmatrix} \quad \text{row reduced with row degrees } 1, 1$$

$$M(s) = \begin{bmatrix} -6 & 13s+1 \\ 2 & -2s \end{bmatrix}$$

This completes the design.