Review of the Dispersive FDTD Techniques Applied to Magnetized Ferrites

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***Abstract* —In this paper, the most known of the dispersive FDTD techniques: equation of motion, Fourier transform, Z transform and Mobius transform are summarized and applied for the analysis of magnetized ferrite. A comparison between the presented techniques is done in terms of efficiency, code complexity and accuracy. An example was presented with numerical results to validate the proposed formulations.**

I. Introduction

DISPERSIVE media have been studied since the 1990's and many techniques were presented for the integration of their characteristics in the finite-difference time-domain (FDTD) code [1]. Because of their importance in microwave engineering, magnetized ferrites have been a major field of application of those techniques.

The first technique applied to magnetized ferrite remotes to its physical nature; it is based on the direct implementation (DI) of its proper equation of motion [2]. This technique is simple and efficient nevertheless, it suffers from a problem of synchronization in time and space, which decreases its accuracy. Finding solutions to those problems was the subject of many later works [3, 4 and 5].

The second technique applied to ferrite is the recursive convolution (RC) technique. This technique uses the convolution theorem to transform the frequency dependent constitutive relation to the time domain, in a recursive manner. It was introduced for the first time in [6], and was applied later to magnetized ferrites in [7]. This technique suffers not only from the problem of synchronization, but also from a great code complexity, i.e., a difficulty in implementation of this method.

Later, many signal processing techniques (Fourier Transform (FT), Z transform (ZT) and Mobius Transform (MT)) were proposed [1]. Theses techniques do not have a problem of time synchronism and are known to be more stable and accurate.

In this paper, the different techniques mentioned above are summarized and applied for the analysis of magnetized ferrite. A comparison between them is done in terms of efficiency, accuracy and code complexity.

II. Basic theory

The electromagnetic field in a ferrite medium can be described by the two curl Maxwell equations

$$\frac{\partial D}{\partial t} = \nabla \wedge H \tag{1}$$

$$\frac{\partial B}{\partial t} = -\nabla \wedge E \tag{2}$$

, the constitutive relation

$$B = \mu_0(\mathrm{H} + \mathrm{M}) \tag{3}$$

, and the equation of motion (lossless case)

$$\frac{\partial \mathrm{M}}{\partial t} = \gamma \mathrm{M} \wedge \mathrm{H} \tag{4}$$

, where γ is the gyromagnetic ratio of the ferrite; $\vec{\mathrm{M}}$ and $\vec{\mathrm{H}}$ are the magnetization and the magnetic field respectively. They are defined as the sum of their static components (M_s and H_0 respectively) and alternative components.

$$\mathrm{M} = M_s u_x + M \tag{5}$$

$$\mathrm{H} = H_0 u_x + H \tag{6}$$

$$M = M_x u_x + M_y u_y + M_z u_z \tag{7}$$

$$H = H_x u_x + H_y u_y + H_z u_z \tag{8}$$

Parting the above expressions into Maxwell equations and after some algebraic manipulations we obtain the following constitutive relation [8]

$$H = \mu^{-1} B \tag{9}$$

, $\bar{\mu}$ is the well-known Polder permeability tensor of the ferrite defined as

$$\bar{\mu}(\omega) = \mu_0 . \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu(\omega) & \kappa(\omega) \\ 0 & -\kappa(\omega) & \mu(\omega) \end{pmatrix} \tag{10}$$

, μ_0 is the permeability of vacuum. Here, the magnetization of ferrite is in the x direction; the propagation of electromagnetic waves is in the z direction. The expressions of the frequency dependent coefficients $\mu(\omega)$ and $\kappa(\omega)$ are given as

$$\kappa(\omega) = \frac{j\omega\omega_m}{\omega_0^2 - \omega^2} \tag{11}$$

$$\mu(\omega) = 1 + \frac{\omega_m\omega_0}{\omega_0^2 - \omega^2} \tag{12}$$

, where j is the complex number ($j = \sqrt{-1}$).

III. Direct Implementation of the Equation of Motion

Starting with the constitutive relation of ferrites

$$B^{n+1/2} = \mu_0\left(\mathrm{H}^{n+1/2} + \mathrm{M}^n\right) \tag{13}$$

Taking the time derivative of the magnetic induction

$$\frac{\partial \vec{B}}{\partial t}\Big|_{n+1/2} = \mu_0\left(\frac{\partial \mathrm{H}}{\partial t}\Big|_{n+1/2} + \frac{\partial \mathrm{M}}{\partial t}\Big|_n\right) \tag{14}$$

The time derivative of the magnetization involves both H and M, according to (4). Note that H is discretized at n+1/2, whereas M is discretized at n.

Zheng and Chen [] and independently Dib and Katehi [] used the following approximation for this equation

$$\frac{\partial \mathrm{M}}{\partial t}\Big|_n = \gamma \mathrm{M}^n \wedge \mathrm{H}^{n-1/2} \tag{15}$$

Combining (15) and (14) yields

$$\frac{\partial B}{\partial t}\Big|_{n+1/2} = \mu_0\left(\frac{\partial \mathrm{H}}{\partial t}\Big|_{n+1/2} + \gamma \mathrm{M}^n \wedge \mathrm{H}^{n-1/2}\right) \tag{16}$$

Substituting (5) ~ (8) into (16), then using the small signal approximation: $H << H_0$ and $M << M_0$, we obtain

$$\frac{\partial \mathrm{H}}{\partial t}\Big|_{n+1/2} = \frac{1}{\mu_0}\frac{\partial B}{\partial t}\Big|_{n+1/2} - \gamma\left(M_s u_x + M^n\right) \wedge \left(H_0 u_x + H^{n-1/2}\right) \tag{17}$$

Solving this equation using (16), (17) and (12), we obtain the following equations for the transverses magnetic field components

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu_0}\frac{\partial B_y}{\partial t} + \gamma(M_s + H_0)Hz - \frac{\gamma}{\mu_0}H_0 B_z \tag{18}$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu_0}\frac{\partial B_z}{\partial t} - \gamma(M_s + H_0)Hy + \frac{\gamma}{\mu_0}H_0 B_y \tag{19}$$

Using the approximating formula for the time-derivative

$$\frac{\partial \Psi}{\partial t} \approx \frac{\delta \Psi}{\delta t} = \frac{\Psi^{n+1/2} - \Psi^{n-1/2}}{\Delta t} \tag{20}$$

, ψ denotes any field component, Δt is the time step, we obtain

$$\begin{aligned}\frac{H_y^{n+1/2} - H_y^{n-1/2}}{\Delta t} &= \frac{1}{\mu_0}\frac{B_y^{n+1/2} - B_y^{n-1/2}}{\Delta t} \\ &\quad + \gamma(M_s + H_0)H_z^{n-1/2} - \frac{\gamma}{\mu_0}H_0 B_z^{n-1/2}\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{H_z^{n+1/2} - H_z^{n-1/2}}{\Delta t} &= \frac{1}{\mu_0}\frac{B_z^{n+1/2} - B_z^{n-1/2}}{\Delta t} \\ &\quad + \gamma(M_s + H_0)H_y^{n-1/2} - \frac{\gamma}{\mu_0}H_0 B_y^{n-1/2}\end{aligned} \tag{22}$$

In [5], instead of (20), the following formula was used

$$\frac{\partial \Psi}{\partial t} \approx \frac{\delta \Psi}{\delta t} = \frac{\Psi^{n+1/2} - \Psi^{n-3/2}}{2\Delta t} \tag{23}$$

, to obtain second order accurate equations

$$\begin{aligned}\frac{H_y^{n+1/2} - H_y^{n-3/2}}{2\Delta t} &= \frac{1}{\mu_0}\frac{B_y^{n+1/2} - B_y^{n-3/2}}{2\Delta t} \\ &\quad + \gamma(M_s + H_0)H_z^{n-1/2} - \frac{\gamma}{\mu_0}H_0 B_z^{n-1/2}\end{aligned} \tag{24}$$

$$\begin{aligned}\frac{H_z^{n+1/2} - H_z^{n-3/2}}{2\Delta t} &= \frac{1}{\mu_0}\frac{B_z^{n+1/2} - B_z^{n-3/2}}{2\Delta t} \\ &\quad + \gamma(M_s + H_0)H_y^{n-1/2} - \frac{\gamma}{\mu_0}H_0 B_y^{n-1/2}\end{aligned} \tag{25}$$

IV. Fourier Transform Formulation

The Fourier transform technique was introduced by Joseph and al. in [8]. This technique consists in applying the inverse Fourier transform to the frequency-dependent constitutive relation of the dispersive medium to get it in the time domain. The obtained expressions are descretized using approximating formulas to integrate them in the FDTD algorithm.

Starting with the frequency-dependent constitutive relations of ferrites

$$\frac{B_y(\omega)}{\mu_0} = \mu(\omega).H_y(\omega) + k(\omega).H_z(\omega) \tag{26}$$

$$\frac{B_z(\omega)}{\mu_0} = -k(\omega).H_y(\omega) + \mu(\omega).H_z(\omega) \tag{27}$$

Developing the expression above then using the inverse Fourier transform $-j\omega \rightarrow \frac{\partial}{\partial t}$ to get back to the time domain, we obtain

$$\frac{\omega_0^2}{\mu_0}.B_y - \frac{1}{\mu_0}\frac{\partial^2 B_y}{\partial t^2} = \left(w_0^2 + \omega_0\omega_m\right)H_y + \frac{\partial^2 H_y}{\partial t^2} - \omega_m\frac{\partial H_z}{\partial t} \tag{29}$$

$$\frac{\omega_0^2}{\mu_0}.B_z - \frac{1}{\mu_0}\frac{\partial^2 B_z}{\partial t^2} = \left(w_0^2 + \omega_0\omega_m\right)H_z + \frac{\partial^2 H_z}{\partial t^2} + \omega_m\frac{\partial H_y}{\partial t} \tag{30}$$

For the descretization of (29)-(30), we use (23) and the following formula

$$\frac{\partial^2\Psi}{\partial t^2} \approx \frac{\delta^2\Psi}{\delta t^2} = \frac{\Psi^{n+1/2} - 2\Psi^{n-1/2} + \Psi^{n-3/2}}{\Delta t^2} \tag{31}$$

, then we separate then the unknown entities from the known ones, we obtain

$$H_1 = -\frac{1}{\mu_0}.\left[B_y^n - F.B_y^{n-1} + B_y^{n-2}\right] + \left[f.H_y^{n-1} + H_y^{n-2} - \frac{\omega_1}{2}H_z^{n-2}\right] \tag{32}$$

$$H_2 = -\frac{1}{\mu_0}.\left[B_z^n - F.B_z^{n-1} + B_z^{n-2}\right] + \left[f.H_z^{n-1} + H_z^{n-2} + \frac{\omega_1}{2}H_y^{n-2}\right] \tag{33}$$

$$H_1 = \left(-1 - \frac{3\omega_1^2}{4}\right)H_y^n + 2\omega_1 H_z^n \tag{34}$$

$$H_2 = -\frac{\omega_1}{2}.H_y^n - H_z^n \tag{35}$$

Here: $F = 2 - \omega_2^2$; $f = \omega_1\omega_2 - F$; $\omega_1 = \omega_m \Delta t$ and $\omega_2 = \omega_0 \Delta t$. Thus we proceed as follows: first we calculate H_1 and H_2 from the known fields using (32)-(33), then we obtain the two unknowns H_y^n and H_z^n from (34)-(35)

$$H_y^n = \frac{-4.H_1 - 2\omega_1.H_2}{4 + \omega_1^2} \tag{36}$$

$$H_z^n = -H_2 - \frac{\omega_1}{2}.H_y^n \tag{37}$$

The variables H_1 and H_2 are temporary variables; they have almost no effect on the memory requirements of the algorithm.

V. ZT-FORMULATION

The z transform technique was introduced by Sullivan in [9] and applied to ferrites in [10]. In [10], the authors solved the constitutive relation $H = \mu^{-1}B$ in the z domain and obtained a complicated algorithm involving a big number of variables, many of them are obtained using interpolation which further increases the code complexity and the computational costs.

Here, the constitutive relation is solved in the z domain in two steps to obtain a simple and efficient algorithm without any loss in the accuracy of the method.

The z transform of the Polder tensor components are given by [1]

$$\mu[z] = \frac{1}{\Delta t} + \frac{\omega_m.\sin(\omega_0\Delta t).z^{-1}}{1 - 2.\cos(\omega_0\Delta t).z^{-1} + z^{-2}} \tag{38}$$

$$k[z] = \frac{\omega_m - \omega_m.\cos(\omega_0\Delta t).z^{-1}}{1 - 2.\cos(\omega_0\Delta t).z^{-1} + z^{-2}} \tag{39}$$

The constitutive relation yields in the z domain

$$\frac{B_y(z)}{\mu_0} = \mu[z].H_y(z).\Delta t + k[z].H_z(z).\Delta t \tag{40}$$

$$\frac{B_z(z)}{\mu_0} = -k[z].H_y(z).\Delta t + \mu[z].H_z(z).\Delta t \tag{41}$$

The introduction of the time step Δt is necessary and was justified in [1].

Separating the unknown from the known ones in (40)-(41), we obtain

$$H_1 = \frac{1}{\mu_0}.\left[1 - 2.\cos(\omega_0\Delta t).z^{-1} + z^{-2}\right]B_y(z) - \begin{bmatrix}(\omega_m.\sin(\omega_0\Delta t) - 2.\cos(\omega_0\Delta t)).z^{-1}.H_y(z) + z^{-2}.H_y(z) \\ -\omega_m.\cos(\omega_0\Delta t).z^{-1}.H_z(z)\end{bmatrix} \tag{42}$$

$$H_2 = \frac{1}{\mu_0}.\left(1 - 2.\cos(\omega_0\Delta t).z^{-1} + z^{-2}\right)B_z(z) - \begin{bmatrix}(\omega_m.\sin(\omega_0\Delta t) - 2.\cos(\omega_0\Delta t)).z^{-1}.H_z(z) + z^{-2}.H_z(z) \\ +\omega_m.\cos(\omega_0\Delta t).z^{-1}.H_y(z)\end{bmatrix} \tag{43}$$

$$H_1 = H_y(z) + \omega_m.H_z(z) \tag{44}$$

$$H_2 = H_z(z) - \omega_m.H_y(z) \tag{45}$$

Using the relation $z^{-m}.\psi(z) \longrightarrow \psi^{n-m}$ to get back to the sampled time domain, we obtain finally

$$H_1 = -\frac{1}{\mu_0}.\left[B_y^n - F_1.B_y^{n-1} + B_y^{n-2}\right] + \left[F_2.H_y^{n-1} + H_y^{n-2} - F_3.H_z^{n-1}\right] \tag{46}$$

$$H_2 = -\frac{1}{\mu_0}.\left[B_z^n - F_1.B_z^{n-1} + B_z^{n-2}\right] + \left[F_2.H_z^{n-1} + H_z^{n-2} + F_3.H_y^{n-1}\right] \tag{47}$$

$$\frac{\omega_1.H_1 + H_2}{1 + \omega_1^2} = H_z^n \tag{48}$$

$$\frac{H_z^n - H_2}{\omega_1} = H_y^n \tag{49}$$

Where: $F_1 = 2.\cos(\omega_2)$; $F_2 = (\omega_1.\sin(\omega_2) - 2.\cos(\omega_2))$; $F_3 = \omega_1.\cos(\omega_2)$, $\omega_1 = \omega_m \Delta t$ and $\omega_2 = \omega_0 \Delta t$.

VI. MOBIUS TRANSFORM FORMULATION

The mobius transform (MT) technique was introduced by Pereda and al. [11] and applied to magnetized ferrites in [12] following two different approaches. In the second one, the equations (29) and (30) are discretized using, in addition to (20) and (31), the following formula

$$\Psi \approx \frac{\Psi^{n+1/2} + 2\Psi^{n-1/2} + \Psi^{n-3/2}}{4} \tag{50}$$

, which an approximating formula for the time-derivative of order zero. Using the two step procedure described above, we obtain at the end

$$H_1 = -\frac{1}{\mu_0}.\left[F_1 B_y^n - F_2.B_y^{n-1} + F_3.B_y^{n-2}\right] + \left[f_1.H_y^{n-1} + f_2.H_y^{n-2} - f_3.H_z^{n-2}\right] \quad (51)$$

$$H_2 = -\frac{1}{\mu_0}.\left[F_1 B_z^n - F_2.B_z^{n-1} + F_3.B_z^{n-2}\right] + \left[f_1.H_z^{n-1} + f_2.H_z^{n-2} - f_3.H_y^{n-2}\right] \quad (52)$$

$$\frac{f_3.H_1 + f_2.H_2}{f_2{}^2 + f_3{}^2} = H_y^n \quad (53)$$

$$\frac{H_2 - f_3.H_y^n}{f_2} = H_z^n \quad (54)$$

Here: $F_1 = \omega_2{}^2 + 4$; $F_2 = 2.\omega_2{}^2 - 8$; $F_3 = F_1$; $f_1 = 2.\omega_1.\omega_2 + F_2$; $f_2 = \omega_1.\omega_2 + F_1$; $f_3 = 2.\omega_1$.

VII. COMPARISON

Because it involves only first-order time-derivatives, the algorithm based on the direct discretization of the equation of motion requires less memory-space for the simulation, generates little computer code and insures second order accuracy.

Different signal processing techniques were used above to discretize the constitutive relation of ferrites based on the Polder tensor.

The z transform is not a direct technique because it requires the preliminary knowledge of the time domain expressions of the frequency dependent components of the Polder tensor. The z transform technique converges to the Fourier Transform one as the time step value decreases: $\omega_0.\Delta t \approx 0$, $\sin(\omega_0.\Delta t) \approx \omega_0.\Delta t$ and $\cos(\omega_0.\Delta t) \approx 1$.

Contrary to the z transform, both Mobius and Fourier transforms discretize directly the constitutive relations of ferrite based on the Polder tensor. The Mobius transform is more accurate because it represents trapezoidal integration whereas the Fourier transform uses a backward approximation formula which represents rectangular step integration [1].

Here, we have applied the Mobius transform using the same approximating formulas used in the Fourier transform technique ((23) and (31)), in addition to the averaging formula (50); thus we can say that the trapezoidal approximation is due to this later.

We can verify in the algorithms above, that these two techniques are equivalent and yield the same algorithm in the case of a demagnetized ferrite $\omega_0 = 0$; in this case, (50) is not needed.

VIII. NUMERICAL RESULTS

To show the validity of the algorithm above, we propose to calculate the dispersion characteristic of the dominant modes for rectangular waveguide of height $a = 22.86mm$ and width $b = 10.16mm$, completely-filled with transversally-magnetized ferrites of relative permittivity $\varepsilon_r = 9$. The static magnetization of the ferrites $M_s = 159.15KA/m$ and the applied magnetic field is of strength $H_0 = 15.915KA/m$.

For the simulation of this waveguide, a mesh resolution $\Delta x = \Delta y = 0.508$ mm was used, i.e., 20×45 grid points in the $x - y$ plane. A Gaussian pulse was chosen for the excitation. The value of the time step depends on the chosen value of the phase constant and the number of time iterations was set at 10000 for each simulation.

The results obtained for the TE_{10} mode of this waveguide are shown in Table 1 and compared with exact results.

Table 1: Dispersion characteristic of the dominant mode

β	0	375.36	511.24	654.54	844.86	1059.2
DI	6.57	8.81	10.37	12.29	14.94	18.08
FT	6.12	8.90	10.46	12.29	15.03	18.17
ZT	6.66	8.81	10.37	12.29	14.94	18.08
MT	6.57	8.90	10.46	12.29	15.03	18.17
Exact	**6.50**	**8.76**	**10.34**	**12.21**	**14.88**	**18.03**

Table 2 shows a comparison in terms of memory ad time efficiency and code amount.

Table 2: Comparison of efficiency

	Memory-space	Time	Code Amount
DI	Low	Low	Low
FT	High	Medium	High
ZT	High	Medium	High
MT	High	High	High

IX. CONCLUSION

This work reviews the most known dispersive FDTD techniques and applies them for the analysis of magnetized ferrites.

A comparison between them is done in terms of the efficiency of the obtained algorithms and the accuracy of the obtained results, which allows the reader to make a trade-off between these two criteria and choose the most convenient technique for their application.

A numerical example was given at the end of this work to validate the proposed algorithms.

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