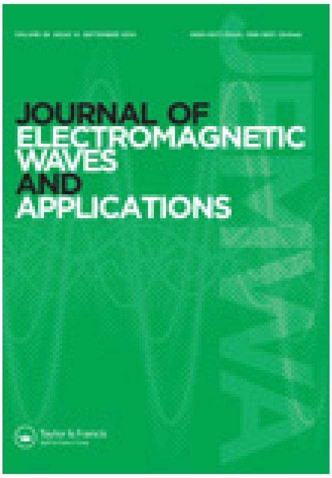
# Multimodal propagation in multiconductor transmission lines

**Article** *in* Journal of Electromagnetic Waves and Applications · September 2014

DOI: 10.1080/09205071.2014.960536



Jose A Brandao Faria
University of Lisbon




Journal of Electromagnetic Waves and Applications
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tewa20

# Multimodal propagation in multiconductor transmission lines

J.A. Brandão Faria[a]

[a] Instituto de Telecomunicações, Instituto Superior Técnico, Universidade de Lisboa DEEC, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

Published online: 24 Sep 2014.

*Journal of Electromagnetic Waves and Applications*, 2014
Vol. 28, No. 14, 1677–1702, http://dx.doi.org/10.1080/09205071.2014.960536





## INVITED REVIEW ARTICLE

# Multimodal propagation in multiconductor transmission lines

J.A. Brandão Faria*

*Instituto de Telecomunicações, Instituto Superior Técnico, Universidade de Lisboa DEEC, Av. Rovisco Pais, 1049-001 Lisboa, Portugal*

(*Received 17 July 2014; accepted 28 August 2014*)

This paper is devoted to the analysis of the multiconductor transmission line (MTL) – a type of structure that is encountered in very many applications ranging from power systems to microwave engineering. Our aim is to provide the reader with a self-contained presentation of the subject where MTL frequency-domain equations are formulated and their solution is established. Matrix techniques are made intense use in order to permit a compact analysis of the problem. Solutions to MTL equations are presented in two different useful formats; at first we consider a solution in the form of a superposition of natural modes of propagation, later, a relationship between voltages and currents at the MTL ports is established by employing the transmission matrix formalism. The presentation is not limited to the simple case of uniform MTL, the important problem of nonuniform MTL is also encompassed.

**Keywords:** matrix techniques; modal analysis; multiconductor transmission lines; propagation modes; transmission matrix; uniform and nonuniform MTL

## 1. Introduction

This review paper, with a tutorial bias, deals with multiconductor transmission lines (MTL) and the underlying theory of modal analysis. It is especially intended to new-comer young researchers interested in the field. The presentation of the subject is developed in the frequency-domain and makes use of the quasi-TEM approach – meaning that the transversal distances between line conductors are assumed to be much small than a quarter wavelength.

An important electrical engineering theme, the topic of wave propagation phenomena in MTL has been receiving attention since the third decade of the last century, with particular emphasis on the 60s–90s when powerful computing machines and dedicated software tools became progressively available. Such a topic pervades many different areas where high-frequency regimes come into play, namely power line transients and power line communications, crosstalk and electromagnetic compatibility problems, electronic and microelectronic design and packaging, as well as RF and microwave engineering.

Although MTL analysis is not expected to receive novel exciting developments, electrical engineering students and young researchers have, in general, been scarcely exposed to MTL theory in their courses. This review is intended to carefully explain MTL foundations and analysis methods.

First attempts to deal with MTL can be traced back to as late as 1932, 1937, and 1941: Fallou [1], Pipes [2,3], and Rice [4], respectively. Afterwards, the number of

*Email: brandao.faria@ieee.org

R&D papers on the subject steadily grew and exploded in the 60s–90s. Considering the diverse branches of electrical engineering concerned with MTL problems, the number of significant contributions that could be cited would easily exceed several hundreds. With our apologies to non-cited authors we dare to emphasize the key contributions by Wedepohl and co-workers concerning the utilization of matrix methods in MTL analysis.[5–15]

In the 60s–90s period a few books on MTL were published, namely by Kuznetsov and Statonovich [16], by Frankel [17], by Djordjevic et al. [18], by Brandão Faria [19], and by Paul [20]. The last book is specially recommended not only because of its quality and coverage but, also, because of the amount of bibliographic references that are offered.

In the first decade of 2000 the subject of MTL decayed in attention; many works are still being published but are mainly concerned with specific applications, not with groundbreaking developments.

This paper is organized into eight sections, the first of which is introductory.

In Section 2, the concept of propagation mode is qualitatively introduced using the shielded bifilar cable as an example.

Section 3, the longest one, is devoted to the presentation of the modal analysis theory for uniform MTL, which includes MTL constitutive parameters, MTL propagation equations and their solution, MTL termination and boundary conditions, systematic procedure for the determination of MTL voltages and current and, also, MTL transmission matrix.

In Section 4 we get back to the example of the homogenous shielded bifilar cable (referred to in Section 2) and apply the modal procedure explained in Section 3 in order to find out cable voltages and cable currents, which are shown to consist in the superposition of an even mode and an odd mode (these modes are degenerate if all the cable conductor are assumed lossless).

In Section 5 the reader is alerted to the fact that modal analysis techniques may not be applicable in some circumstances – in reality, the eigenvalue-eigenvector problem at the heart of modal analysis may not have a solution, in the sense that it may be impossible to find a complete set of independent eigenvectors.

Section 6 is devoted to the analysis of nonuniform MTL utilizing the transmission matrix formalism, two methods are presented: the segmentation method and the iterative integration method.

In Section 7 we tackle the reverse problem of retrieving the modal wave parameters from the MTL transmission matrix, both cases of nonuniform and uniform MTL being considered.

The paper ends with the conclusion in Section 8.

## 2. The concept of propagation mode

Consider a homogeneous uniform transmission-line where only incident (i) waves propagate (e.g. a coaxial cable of infinite extent). The system's longitudinal coordinate is the $z$ axis. The presence of conductor losses and dielectric losses is considered but, as usually, both phenomena are regarded as small perturbations.

A parenthetical note: while the concept of homogeneity is linked to $x$–$y$ invariance (namely, the properties of the dielectric medium), $z$-invariance is linked to the concept of uniformity (namely, the transmission line geometry).

Consider an ordinary two-conductor coaxial cable of infinite length. If the transverse electromagnetic field lines are observed at two different places along the cable, $z = z_1$ and $z = z_2 > z_1$, the same pattern will be observed – see Figure 1.

If the cable is an ideal cable (losses absent) not only the field pattern is exactly the same at $z_1$ and $z_2$ but also the field strength preserves its own magnitude. In a lossy cable, the field pattern (propagation mode) remains the same along $z$ but the field magnitude is smaller at $z_2$, that is:

$$\left\{ \begin{array}{c} \mathbf{E}(z_2) \\ \mathbf{H}(z_2) \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{E}(z_1) \\ \mathbf{H}(z_1) \end{array} \right\} e^{-\gamma(z_2 - z_1)} \quad \text{where } \gamma(\omega) = \alpha + j\beta \tag{1}$$

In (1), $\gamma$ is the propagation constant, $\alpha$ is the attenuation constant, $\beta = \omega/v$ is the phase constant, and $v$ is the phase velocity; $\mathbf{E}$ and $\mathbf{H}$, respectively, denote the electric and magnetic vector field phasors, i.e.

$$\left\{ \begin{array}{c} \vec{E}(z,t) \\ \vec{H}(z,t) \end{array} \right\} = \mathrm{Re}\left( \left\{ \begin{array}{c} \mathbf{E}(z) \\ \mathbf{H}(z) \end{array} \right\} e^{j\omega t} \right) \tag{2}$$

Let us now complicate things a little bit.

Consider a shielded bifilar cable – an example of a multiconductor transmission line with three conductors. The inner conductors are labeled 1 and 2 whereas the external shielding is labeled 0 – the reference conductor or ground (where the scalar potential is set to zero).

Assume that, at $z = 0$, the following voltages are enforced: $V_1 = V > 0$ and $V_2 = 0$ (see Figure 2(a)). The corresponding electromagnetic field lines, at $z = 0$, are sketched in Figure 2(b).

For a lossless cable, if the field pattern is observed at $z > 0$ one will get the same field lines as in Figure 2(b). However, in a real scenario where conductor losses are present, the field pattern will gradually change as one move along the propagation direction. Field lines at $z = 0$, $z > 0$, and $z \to \infty$ are sketched in Figure 3.

Despite losses, are there any prescribed excitations that may lead to $z$-invariant field lines patterns? The answer is yes. There are two cases:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = V_e \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = V_o \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{3}$$

The first case, $V_2 = V_1$ (Figure 4(a)), is so-called even mode (or symmetric mode or common mode). The second case, $V_2 = -V_1$ (Figure 4(b)), is so-called odd mode (or

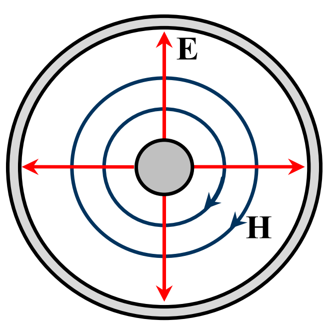



Figure 1. Electromagnetic field pattern (propagation mode) in a coaxial cable.

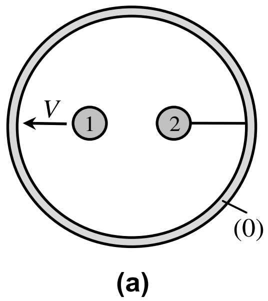



Figure 2. Shielded bifilar cable: (a) enforced voltages, (b) electromagnetic field pattern at $z = 0$.

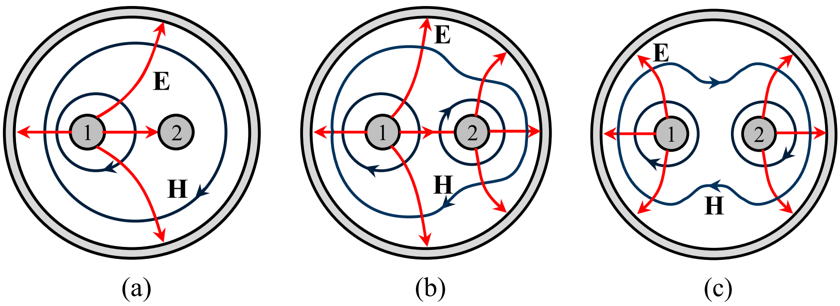



Figure 3. Sketch of the $z$-variant field patterns in a shielded bifilar cable: (a) $z = 0$, (b) $z > 0$, (c) $z \to \infty$.

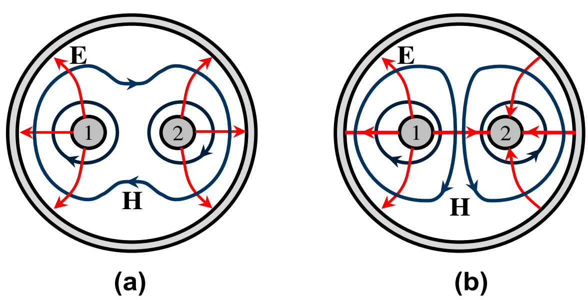



Figure 4. Propagation-mode field lines in a shielded bifilar cable: (a) even mode, (b) odd mode.

anti-symmetric mode or differential mode). Like in an ordinary coaxial cable with 1 + 1 conductors, the odd and even modes propagate $z$-invariantly along the 2 + 1 conductor structure.

It should be noted that the even-mode field-line pattern in Figure 4(a) is identical to the one shown in Figure 3(c). The interpretation for this fact is that the even-mode attenuation is ordinarily smaller than the odd-mode attenuation and, as a result, the surviving mode for $z \to \infty$ is the even mode.

To summarize:

(1) In general, in a uniform MTL with $n + 1$ conductors the number of independent modes is equal to $n$.
(2) In a uniform MTL, each propagation mode, characterized by a specific $z$-invariant field-line pattern, is created by a unique combination of excitation voltages.
(3) An arbitrary combination of excitation voltages gives rise to a $z$-variant field-line pattern which can be obtained by superposition of the independent modes patterns.
(4) In a homogeneous lossless MTL, any combination of excitation voltages define a propagation mode (to be precise, we should refer to it as a degenerate mode).

## 3. Modal analysis theory

In Section 2, the concept of propagation mode was qualitatively introduced using, as an example, the shielded bifilar cable configuration. Here, we proceed to the corresponding quantitative analysis, considering, as before, that the MTL system is uniform.

The theory of modal analysis is firstly exemplified with the simplest MTL structure, that is, an $n + 1$ conductor system with $n = 2$. As the theory is developed, its generalization for $n > 2$ will be presented. In Section 5 we will get back to analysis of the shielded bifilar cable configuration.

### 3.1. MTL constitutive parameters

The transmission line equations concerning a $2 + 1$ conductor system are presented in (4) and (5). These equations can be obtained by analyzing the ordinary lumped-parameters circuit model shown in Figure 5, where two classes of parameters are involved, longitudinal and transverse.

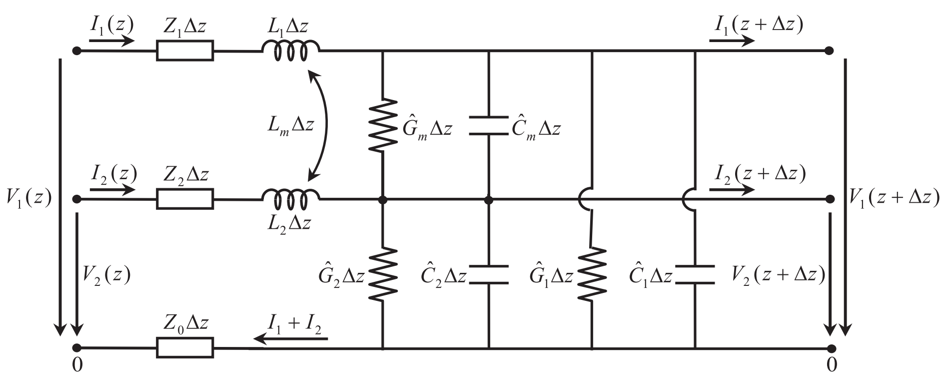



Figure 5. Lumped-parameters circuit model of a 2 + 1 conductor MTL of length $\Delta z$.

- Longitudinal parameters
  $Z_0$, $Z_1$, $Z_2$ are the per unit length (pul) skin effect complex impedances associated with the lossy conductors 0, 1, and 2, respectively (note: proximity effects are included);
  $L_1$, $L_2$ are the pul self-inductances of the circuits (1–0), and (2–0), respectively. $L_m$ is the pul mutual inductance between the two circuits.
- Transverse parameters
  $\hat{C}_1$, $\hat{C}_2$, $\hat{C}_m$ are the pul partial capacitances between conductors 1 and 0, 2 and 0, and 1 and 2;
  $\hat{G}_1$, $\hat{G}_2$, $\hat{G}_m$ are the pul partial conductances (due to dielectric losses) between conductors 1 and 0, 2 and 0, and 1 and 2.

From the analysis of the circuit model in Figure 5 corresponding to a MTL of infinitesimal length $\Delta z$, and taking into account that

$$\frac{\mathrm{d}}{\mathrm{dz}}V_k(z) = \lim_{\Delta z \to 0}\frac{V_k(z+\Delta z) - V_k(z)}{\Delta z}, \quad \frac{\mathrm{d}}{\mathrm{dz}}I_k(z) = \lim_{\Delta z \to 0}\frac{I_k(z+\Delta z) - I_k(z)}{\Delta z}$$

we get the MTL voltage and current equations.

MTL voltage equations ($n$ = 2):

$$\text{rot}\,\mathbf{E} = -j\omega\mathbf{B} \;\rightarrow\; \begin{cases} \frac{\mathrm{d}V_1}{\mathrm{d}z} + ((Z_1+Z_0)I_1 + Z_0 I_2) = -j\omega(L_1 I_1 + L_m I_2) \\ \frac{\mathrm{d}V_2}{\mathrm{d}z} + (Z_0 I_1 + (Z_2+Z_0)I_2) = -j\omega(L_m I_1 + L_2 I_2) \end{cases} \tag{4}$$

MTL current equations ($n$ = 2):

$$\text{div}\,\mathbf{J} = -j\omega\rho \quad\rightarrow\quad \begin{cases} \frac{\mathrm{d}I_1}{\mathrm{d}z} + ((\hat{G}_1+\hat{G}_m)V_1 - \hat{G}_m V_2) = -j\omega((\hat{C}_1+\hat{C}_m)V_1 - \hat{C}_m V_2) \\ \frac{\mathrm{d}I_2}{\mathrm{d}z} + (-\hat{G}_m V_1 + (\hat{G}_2+\hat{G}_m)V_2) = -j\omega(-\hat{C}_m V_1 + (\hat{C}_2+\hat{C}_m)V_2) \end{cases} \tag{5}$$

where $V_1$, $V_2$ are the line voltage phasors and $I_1$, $I_2$ are the line current phasors.
The above equations may (and should) be rewritten in matrix format as:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{dz}}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = -\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \\ \frac{\mathrm{d}}{\mathrm{dz}}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = -\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}; \end{cases} \tag{6a}$$

or, compactly, for $n$th order systems,

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}z}\mathbf{V} = -\mathbf{ZI} \\ \frac{\mathrm{d}}{\mathrm{d}z}\mathbf{I} = -\mathbf{YV} \end{cases} \tag{6b}$$

where the pul longitudinal impedance matrix $\mathbf{Z}$ is given by,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_{11}^{\mathrm{se}} & Z_{12}^{\mathrm{se}} \\ Z_{21}^{\mathrm{se}} & Z_{22}^{\mathrm{se}} \end{bmatrix} + j\omega\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \quad\leftrightarrow\quad \mathbf{Z} = \mathbf{Z}^{\mathrm{se}} + j\omega\mathbf{L} \tag{7}$$

where the pul skin effect (se) impedance matrix $\mathbf{Z}^{se}$ and the pul external inductance matrix $\mathbf{L}$ are obtained from

$$\begin{cases} \begin{bmatrix} Z_{11}^{se} & Z_{12}^{se} \\ Z_{21}^{se} & Z_{22}^{se} \end{bmatrix} = \begin{bmatrix} Z_1 + Z_0 & Z_0 \\ Z_0 & Z_2 + Z_0 \end{bmatrix} \\ \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} L_1 & L_m \\ L_m & L_2 \end{bmatrix} \end{cases} \tag{8}$$

Likewise, the pul transverse admittance matrix $\mathbf{Y}$ is given by,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} + j\omega \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \quad \leftrightarrow \quad \mathbf{Y} = \mathbf{G} + j\omega\mathbf{C} \tag{9}$$

where the pul conductance matrix $\mathbf{G}$ and the pul capacitance matrix $\mathbf{C}$ are obtained from

$$\begin{cases} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} \hat{G}_1 + \hat{G}_m & -\hat{G}_m \\ -\hat{G}_m & \hat{G}_2 + \hat{G}_m \end{bmatrix} \\ \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} \hat{C}_1 + \hat{C}_m & -\hat{C}_m \\ -\hat{C}_m & \hat{C}_2 + \hat{C}_m \end{bmatrix} \end{cases} \tag{10}$$

Due to reciprocity reasons, all of the above-mentioned matrices are symmetric (e.g. $Z_{12} = Z_{21}$, $Y_{12} = Y_{21}$). In general, for any order $n$,

$$\mathbf{Z} = \mathbf{Z}^t, \quad \mathbf{Y} = \mathbf{Y}^t \tag{11}$$

where superscript t stands for transposition.

In addition, in the particular case of a bilaterally symmetric MTL with $n = 2$ (as in a shielded bifilar cable) the diagonal elements are also the same (e.g. $Z_{11} = Z_{22}$, $Y_{11} = Y_{22}$).

### *3.2. MTL propagation equations and their solution*

Taking into account the equations in (6), and considering that the MTL is uniform ($\mathbf{Z}$ and $\mathbf{Y}$ are $z$-independent), the wave-equations for line voltages and currents are expressed as:

$$\begin{cases} \frac{d^2}{dz^2} \begin{bmatrix} V_1(z) \\ V_2(z) \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1(z) \\ V_2(z) \end{bmatrix} \\ \frac{d^2}{dz^2} \begin{bmatrix} I_1(z) \\ I_2(z) \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1(z) \\ I_2(z) \end{bmatrix} \end{cases} \tag{12}$$

or, compactly, for $n$th order systems,

$$\begin{cases} \frac{d^2\mathbf{V}}{dz^2} - (\mathbf{ZY})\mathbf{V} = \mathbf{0} \\ \frac{d^2\mathbf{I}}{dz^2} - (\mathbf{YZ})\,\mathbf{I} = \mathbf{0} \end{cases} \tag{13}$$

The wave equations in (13) describe 2nd order homogeneous matrix differential equations with constant matrix coefficients and, as we will see in the analysis that follows, their solution can be found analytically in the form of a sum of two exponential matrix functions.

Let us assume that the $z$-dependent line voltages and line currents can be constructed by superposition of two propagation modes with $z$-invariant field patterns, each one with its own propagation constant $\gamma_1$, $\gamma_2$, and characteristic modal admittance, $\tilde{Y}_{w1}$, $\tilde{Y}_{w2}$, that is,

$$\begin{cases} \begin{bmatrix} V_1(z) \\ V_2(z) \end{bmatrix} = \overbrace{\begin{bmatrix} t_{11} \\ t_{21} \end{bmatrix}}^{\mathbf{t}_1} \tilde{V}_{i1}\, e^{-\gamma_1 z} + \overbrace{\begin{bmatrix} t_{12} \\ t_{22} \end{bmatrix}}^{\mathbf{t}_2} \tilde{V}_{i2}\, e^{-\gamma_2 z} \\ \begin{bmatrix} I_1(z) \\ I_2(z) \end{bmatrix} = \underbrace{\begin{bmatrix} w_{11} \\ w_{21} \end{bmatrix}}_{\mathbf{w}_1} \tilde{Y}_{w1}\, \tilde{V}_{i1}\, e^{-\gamma_1 z} + \underbrace{\begin{bmatrix} w_{12} \\ w_{22} \end{bmatrix}}_{\mathbf{w}_2} \tilde{Y}_{w2}\, \tilde{V}_{i2}\, e^{-\gamma_2 z} \end{cases} \tag{14}$$

or, which is the same,

$$\begin{cases} \begin{bmatrix} V_1(z) \\ V_2(z) \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} e^{-\gamma_1 z} & 0 \\ 0 & e^{-\gamma_2 z} \end{bmatrix} \begin{bmatrix} \tilde{V}_{i1} \\ \tilde{V}_{i2} \end{bmatrix} \\ \begin{bmatrix} I_1\ (z) \\ I_2\ (z) \end{bmatrix} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} \tilde{Y}_{w1} & 0 \\ 0 & \tilde{Y}_{w2} \end{bmatrix} \begin{bmatrix} e^{-\gamma_1 z} & 0 \\ 0 & e^{-\gamma_2 z} \end{bmatrix} \begin{bmatrix} \tilde{V}_{i1} \\ \tilde{V}_{i2} \end{bmatrix} \end{cases} \tag{15a}$$

or, compactly, for $n$th order systems,

$$\begin{cases} \mathbf{V} = \mathbf{T}\ \exp(-\boldsymbol{\gamma} z)\tilde{\mathbf{V}}_{\mathrm{i}} \\ \mathbf{I} = \mathbf{W}\ \tilde{\mathbf{Y}}_w \exp(-\boldsymbol{\gamma} z)\tilde{\mathbf{V}}_{\mathrm{i}} \end{cases} \tag{15b}$$

where $\boldsymbol{\gamma}$ is the diagonal matrix of order $n$, collecting the modal propagation constants. For $n$ = 2, we have

$$\boldsymbol{\gamma} = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \quad \text{and} \quad \exp(-\boldsymbol{\gamma} z) = \begin{bmatrix} e^{-\gamma_1 z} & 0 \\ 0 & e^{-\gamma_2 z} \end{bmatrix} \tag{16}$$

and, where the modal transformation matrices $\mathbf{T}$ (with $t_{jk}$ entries) and $\mathbf{W}$ (with $w_{jk}$ entries) are $z$-independent. The amplitudes of the incident modal voltages $\tilde{\mathbf{V}}_{\mathrm{i}}$ are determined from boundary conditions.

By substituting (15b) into (13), we find

$$\begin{cases} \boldsymbol{\gamma}^2 = \mathbf{T}^{-1}(\mathbf{ZY})\mathbf{T} \\ \boldsymbol{\gamma}^2 = \mathbf{W}^{-1}(\mathbf{YZ})\mathbf{W} \end{cases} \tag{17}$$

Taking into account that $\mathbf{Z}$ and $\mathbf{Y}$ are both symmetric, (11), the transposition of the 2nd equation in (17) yields

$$\boldsymbol{\gamma}^2 = \mathbf{W}^{\mathrm{t}}(\mathbf{ZY})\mathbf{W}^{-1\mathrm{t}} \tag{18}$$

Comparing (18) with the 1st equation in (17) readily shows that the modal transformation matrices are not independent. They are related through

$$\mathbf{W}^{\mathrm{t}} = \mathbf{T}^{-1} \tag{19}$$

In addition, the 1st equation in (17) can be put in the convenient form:

$$\mathbf{T}^{-1}(\mathbf{ZY} - \gamma^2\mathbf{1})\mathbf{T} = \mathbf{0}, \quad (\mathbf{ZY})\mathbf{t}_k = \gamma_k^2\,\mathbf{t}_k\ , \quad \text{for } k = 1 \text{ to } n \tag{20}$$

which defines an eigenvalue–eigenvector problem, familiar to linear algebra and matrix theory.[21] In (20), matrix $\mathbf{0}$ is the null matrix, and matrix $\mathbf{1}$ is the unit matrix of order $n$; for example, for $n$ = 2 we have: $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Therefore, we reach the following important conclusion: the propagation constants $\gamma_1$, $\gamma_2$ and the column vectors $\mathbf{t}_1 = \begin{bmatrix} t_{11} \\ t_{21} \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} t_{12} \\ t_{22} \end{bmatrix}$ appearing in the voltage solution:

$$\begin{bmatrix} V_1(z) \\ V_2(z) \end{bmatrix} = \mathbf{t}_1\, \tilde{V}_{i1}\, e^{-\gamma_1 z} + \mathbf{t}_2\, \tilde{V}_{i2}\, e^{-\gamma_2 z} \tag{21}$$

are, respectively, the square-root of the eigenvalues $\gamma_1^2$, $\gamma_2^2$, and the eigenvectors of the matrix product $\mathbf{ZY}$.

At this stage three remarks are in order.

(1) The square-root of $\gamma_k^2$ is double-valued. Denoting by $\gamma_k$ the solution with positive real part, a second solution $-\gamma_k$ also exists. This means that the result in (21) is incomplete, the complete solution for MTL voltages is:

$$\begin{bmatrix} V_1(z) \\ V_2(z) \end{bmatrix} = \mathbf{t}_1(\tilde{V}_{i1}\, e^{-\gamma_1 z} + \tilde{V}_{r1}\, e^{+\gamma_1 z}) + \mathbf{t}_2(\tilde{V}_{i2}\, e^{-\gamma_2 z} + \tilde{V}_{r2}\, e^{+\gamma_2 z}) \tag{22}$$

where each mode includes contributions from incident (i) and reflected (r) waves. The same comment applies to MTL currents:

$$\begin{bmatrix} I_1(z) \\ I_2(z) \end{bmatrix} = \mathbf{w}_1 \tilde{Y}_{w1}(\tilde{V}_{i1}\, e^{-\gamma_1 z} - \tilde{V}_{r1}\, e^{+\gamma_1 z}) + \mathbf{w}_2 \tilde{Y}_{w2}(\tilde{V}_{i2}\, e^{-\gamma_2 z} - \tilde{V}_{r2}\, e^{+\gamma_2 z}) \tag{23}$$

Of course, if the MTL is of infinite length or is terminated by a matched load then $\tilde{V}_{r1} = \tilde{V}_{r2} = 0$.

(2) The entries $t_{1k}$ and $t_{2k}$ of the $k$th eigenvector are not uniquely defined, only their ratio is. In fact, all eigenvectors require arbitrary normalization scalar factors for their full definition. Although not mandatory, the normalization factors are usually chosen such that each eigenvector possesses unitary Hermitian norm, $\mathbf{t}_k^*\mathbf{t}_k = 1$.[19]

(3) According to (19), the entries $w_{1k}$ and $w_{2k}$ are computed from $t_{1k}$ and $t_{2k}$ ($k$ = 1, 2). Since the latter are defined apart arbitrary normalization factors then, the characteristic modal admittances $\tilde{Y}_{w1}$, $\tilde{Y}_{w2}$ are also dependent on the normalization factors (which implies that they do not bear intrinsic physical meaning).

Assuming that the transformation matrices have been fully defined, the characteristic modal admittances can be found by substituting (15) into any of the equations in (6), yielding

$$\tilde{\mathbf{Y}}_w = \begin{bmatrix} \tilde{Y}_{w1} & 0 \\ 0 & \tilde{Y}_{w2} \end{bmatrix} = \begin{cases} (\mathbf{T}^{\mathrm{t}}\mathbf{Z}^{-1}\mathbf{T})\boldsymbol{\gamma} \\ (\mathbf{T}^{\mathrm{t}}\mathbf{Y}\mathbf{T})\boldsymbol{\gamma}^{-1} \end{cases} \tag{24}$$

Despite the ambiguity of the diagonal matrix of characteristic modal admittances $\tilde{\mathbf{Y}}_w$, the characteristic admittance matrix $\mathbf{Y}_w$, which relates natural currents to voltages in a matched MTL, is not ambiguously defined. This is so because the arbitrary normalization factors, that affect the evaluation of $\tilde{\mathbf{Y}}_w$, do cancel out when the square symmetric full matrix $\mathbf{Y}_w$ is determined. In fact, substituting $\mathbf{Y}_w\mathbf{V}$ for $\mathbf{I}$ in (15b), and taking (19) and (24) into account, leads to

$$\mathbf{Y}_w = \mathbf{T}^{-1t}\tilde{\mathbf{Y}}_w\mathbf{T}^{-1} = \begin{cases} \mathbf{Z}^{-1}\,\boldsymbol{\Gamma} \\ \mathbf{Y}\,\boldsymbol{\Gamma}^{-1} \end{cases}, \text{ where } \boldsymbol{\Gamma} = \mathbf{T}\,\gamma\,\mathbf{T}^{-1} = \mathrm{SQRT}(\mathbf{ZY}) \tag{25}$$

For $n$th order systems, the results for $\mathbf{V}(z)$ and $\mathbf{I}(z)$ in (22) and (23), taking into account the presence of incident and reflected waves, can be compactly written in matrix form as

$$\begin{cases} \mathbf{V} = \mathbf{T}\,(\exp(-\gamma z)\tilde{\mathbf{V}}_{\mathrm{i}} + \exp(+\gamma z)\tilde{\mathbf{V}}_{\mathrm{r}}) \\ \mathbf{I} = \mathbf{W}\,\tilde{\mathbf{Y}}_w(\exp(-\gamma z)\tilde{\mathbf{V}}_{\mathrm{i}} - \exp(+\gamma z)\tilde{\mathbf{V}}_{\mathrm{r}}) \end{cases} \tag{26}$$

### 3.3. *MTL termination and boundary conditions*

Consider a MTL of finite length $l$.

At $z = 0$ (sending end) the system is excited by a set of $n$ generator voltages, $\mathbf{V}_{\mathrm{G}} = \mathbf{V}(0)$.

At $z = l$ (receiving end) the system is terminated by a linear network of partial load admittances $y_{kj}$ connecting all possible nodes (including the reference node 0), where

$$I_k(l) = \sum_{j=0}^{n} y_{kj}(V_k(l) - V_j(l)) \tag{27}$$

which can be put in matrix form through

$$\mathbf{I}(l) = \mathbf{Y}_{\mathrm{L}}\mathbf{V}(l)\ , \quad \text{where} \begin{cases} \mathrm{Y}_{\mathrm{L}}(k,j) = -y_{kj} \\ \mathrm{Y}_{\mathrm{L}}(k,k) = \sum\limits_{\substack{j=0 \\ j\neq k}}^{n} y_{kj} \end{cases} \tag{28}$$

where $\mathbf{Y}_{\mathrm{L}}$ is the square symmetric load admittance matrix.

Now, enforcing the boundary condition (28), at $z = l$, into equation (26), leads to

$$\begin{cases} \mathbf{V}(l) = \mathbf{T}\,(\exp(-\gamma l)\tilde{\mathbf{V}}_{\mathrm{i}} + \exp(+\gamma l)\tilde{\mathbf{V}}_{\mathrm{r}}) \\ \mathbf{Y}_{\mathrm{L}}\mathbf{V}(l) = \mathbf{W}\,\tilde{\mathbf{Y}}_w(\exp(-\gamma l)\tilde{\mathbf{V}}_{\mathrm{i}} - \exp(+\gamma l)\tilde{\mathbf{V}}_{\mathrm{r}}) \end{cases} \tag{29}$$

Bearing in mind, from (25), that $\tilde{\mathbf{Y}}_w = \mathbf{T}^{\mathrm{t}}\mathbf{Y}_w\mathbf{T}$, we find from (29)

$$\tilde{\mathbf{V}}_{\mathrm{r}} = \tilde{\mathbf{K}}\tilde{\mathbf{V}}_{\mathrm{i}}\ , \text{ where } \tilde{\mathbf{K}} = \exp(-\gamma l)(\mathbf{Q}+\mathbf{1})^{-1}(\mathbf{Q}-\mathbf{1})\exp(-\gamma l) \tag{30}$$

where

$$\mathbf{Q} = \mathbf{T}^{-1}(\mathbf{Y}_{\mathrm{L}}^{-1}\mathbf{Y}_w)\mathbf{T} \tag{31}$$

Finally, from (30), together with the boundary condition at $z = 0$, that is $\mathbf{V}_{\mathrm{G}} = \mathbf{T}\,(\tilde{\mathbf{V}}_{\mathrm{i}} + \tilde{\mathbf{V}}_{\mathrm{r}})$, we can compute the column vectors $\tilde{\mathbf{V}}_{\mathrm{i}}$ and $\tilde{\mathbf{V}}_{\mathrm{r}}$ that, respectively, gather the incident and reflected modal voltage amplitudes at the sending end.

$$\begin{cases} \tilde{\mathbf{V}}_{\mathrm{i}} = (\tilde{\mathbf{K}}+\mathbf{1})^{-1}(\mathbf{T}^{-1}\mathbf{V}_{\mathrm{G}}) \\ \tilde{\mathbf{V}}_{\mathrm{r}} = (\tilde{\mathbf{K}}^{-1}+\mathbf{1})^{-1}(\mathbf{T}^{-1}\mathbf{V}_{\mathrm{G}}) \end{cases} \tag{32}$$

Particular cases:

$$\begin{cases} \text{matched load, } \mathbf{Y}_\mathrm{L} = \mathbf{Y}_w, \ \ \mathbf{Q} = \mathbf{1}, \ \ \tilde{\mathbf{K}} = \mathbf{0}, \ \ \begin{cases} \tilde{\mathbf{V}}_\mathrm{i} = \mathbf{T}^{-1}\mathbf{V}_\mathrm{G} \\ \tilde{\mathbf{V}}_\mathrm{r} = \mathbf{0} \end{cases} \\ \text{short circuit, } \mathbf{Y}_\mathrm{L} \to \infty, \ \ \mathbf{Q} = \mathbf{0}, \ \ \tilde{\mathbf{K}} = -\exp(-2\gamma l), \ \ \begin{cases} \tilde{\mathbf{V}}_\mathrm{i} = +\frac{1}{2}\exp(+\gamma l)\mathrm{csch}(\gamma l)\mathbf{T}^{-1}\mathbf{V}_\mathrm{G} \\ \tilde{\mathbf{V}}_\mathrm{r} = -\frac{1}{2}\exp(-\gamma l)\mathrm{csch}(\gamma l)\mathbf{T}^{-1}\mathbf{V}_\mathrm{G} \end{cases} \\ \text{open circuit, } \mathbf{Y}_\mathrm{L} = \mathbf{0}, \ \ \mathbf{Q} \to \infty, \ \ \tilde{\mathbf{K}} = +\exp(-2\gamma l), \ \ \begin{cases} \tilde{\mathbf{V}}_\mathrm{i} = +\frac{1}{2}\exp(+\gamma l)\mathrm{sech}(\gamma l)\mathbf{T}^{-1}\mathbf{V}_\mathrm{G} \\ \tilde{\mathbf{V}}_\mathrm{r} = +\frac{1}{2}\exp(-\gamma l)\mathrm{sech}(\gamma l)\mathbf{T}^{-1}\mathbf{V}_\mathrm{G} \end{cases} \end{cases} \tag{33}$$

where for $n = 2$,

$$\begin{cases} \mathrm{csch}(\gamma l) = \sinh^{-1}(\gamma l) = \begin{bmatrix} \sinh(\gamma_1 l) & 0 \\ 0 & \sinh(\gamma_2 l) \end{bmatrix}^{-1} \\ \mathrm{sech}(\gamma l) = \cosh^{-1}(\gamma l) = \begin{bmatrix} \cosh(\gamma_1 l) & 0 \\ 0 & \cosh(\gamma_2 l) \end{bmatrix}^{-1} \end{cases} \tag{34}$$

At this stage one should observe that whenever the sending end voltages distribution is chosen to coincide with the $k$th eigenvector, that is $\mathbf{V}_\mathrm{G} = v\, \mathbf{t}_k$, then the entries of the column matrix $(\mathbf{T}^{-1}\mathbf{V}_\mathrm{G})$ will be zero except for the $k$th entry:

$$\mathbf{T}^{-1}\mathbf{V}_\mathrm{G} = v \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \leftarrow k^\mathrm{th}\ \text{entry} \\ \\ \\ \\ \end{array} \tag{35}$$

### 3.4. General procedure for the determination of MTL voltages and currents

In this section we summarize, systematically, the steps required for determining the evolution along $z$ of MTL voltages and currents.

*Step 1*: Data input
- MTL geometry (length $l$, and transverse dimensions)
- Properties of MTL conductors and dielectric media
- Frequency
- Boundary conditions: $\mathbf{V}_\mathrm{G}$ (at $z = 0$), $\mathbf{Y}_\mathrm{L}$ (at $z = l$)

*Step 2*: Computation of the per unit length longitudinal impedance matrix $\mathbf{Z}$ and per unit length transverse admittance matrix $\mathbf{Y}$.

*Step 3*: Computation of the eigenvalues and eigenvectors of the $\mathbf{ZY}$ matrix product. Determination of the transformation matrices $\mathbf{T} = \{\mathbf{t}_1 \cdots \mathbf{t}_k \cdots \mathbf{t}_n\}$ and $\mathbf{W} = \mathbf{T}^{-1\mathrm{t}} = \{\mathbf{w}_1 \cdots \mathbf{w}_k \cdots \mathbf{w}_n\}$.

*Step 4*: Computation of the diagonal matrices of modal propagation constants $\gamma$ and modal characteristic admittances $\tilde{\mathbf{Y}}_w$. Computation of the characteristic admittance matrix $\mathbf{Y}_w$.

*Step 5*: Computation (from boundary conditions) of the column vectors $\tilde{\mathbf{V}}_\mathrm{i}$ and $\tilde{\mathbf{V}}_\mathrm{r}$ gathering the incident and reflected modal voltage amplitudes at $z = 0$.

*Step 6*: Determination of modal voltages and modal currents along $z$:

$$\begin{cases} \tilde{\mathbf{V}}(z) = \exp(-\gamma z)\tilde{\mathbf{V}}_{\mathrm{i}} + \exp(+\gamma z)\tilde{\mathbf{V}}_{\mathrm{r}} \\ \tilde{\mathbf{I}}(z) = \tilde{\mathbf{Y}}_w(\exp(-\gamma z)\tilde{\mathbf{V}}_{\mathrm{i}} - \exp(+\gamma z)\tilde{\mathbf{V}}_{\mathrm{r}}) \end{cases}$$

*Step 7*: Determination of MTL voltages and currents along $z$:

$$\mathbf{V}(z) = \mathbf{T}\tilde{\mathbf{V}}(z), \mathbf{I}(z) = \mathbf{W}\tilde{\mathbf{I}}(z).$$

Comments:

(1) For the determination of $\mathbf{Z}$ and $\mathbf{Y}$ matrices (step 2) several methods can be made use: analytical, numerical or experimental. Analytical methods are utilized in the case of very simple geometries (e.g. a homogeneous MTL made of $n$ cylindrical thin conductors above a ground plane). For inhomogeneous complicated geometries numerical methods should be employed, for example: the harmonic expansion method, the method of moments, the finite difference method, the finite element method, conformal mapping, spectral-domain techniques, full-wave approaches, etc.[20] Fortunately, the implementation of these numerical methods has been made possible through a variety of commercial software packages that are currently available. Matrices $\mathbf{Z}$ and $\mathbf{Y}$ can also be determined experimentally utilizing short-circuit and open-circuit measurements performed on short-length ($l << \lambda$) samples of MTL.[22]

(2) The product matrix $\mathbf{ZY}$, whose eigenvalues and eigenvectors are to be determined in steps 3 and 4, is ordinarily a diagonally dominant matrix, with diagonal entries of the same order of magnitude, and with relatively small off-diagonal entries, meaning that the eigenvalues of $\mathbf{ZY}$ (the squared propagation constants $\gamma_k^2$) are close neighbors. This fact usually gives rise to difficulties and inaccuracies in the computational methods for eigenproblem solving. To circumvent such difficulties an eigenvalue-shifting technique is recommended.[19] It consists in subtracting from $\mathbf{ZY}$ a scalar diagonal matrix with entries equal to the trace of $\mathbf{ZY}$ divided by $n$:

$$(\mathbf{ZY})' = \mathbf{ZY} - \gamma_{av}^2\mathbf{1} \quad \gamma_{av}^2 = \frac{1}{n}\mathrm{tr}\ \mathbf{ZY}, \quad \mathrm{tr}\ \mathbf{ZY} = \sum_{k=1}^{n}(\mathbf{ZY})_{kk} \tag{36}$$

The resulting matrix $(\mathbf{ZY})'$ is then diagonalized, yielding a set of properly spaced eigenvalues $\lambda_k$ ($k$ = 1 to $n$) and a set of corresponding eigenvectors $\mathbf{t}_k$ ($k$ = 1 to $n$). The eigenvalues of $\mathbf{ZY}$ are obtained by back shifting, $\gamma_k^2 = \lambda_k + \gamma_{av}^2$. The eigenvectors of $\mathbf{ZY}$ are exactly the same of $(\mathbf{ZY})'$.

(3) The theory of modal analysis as well as its implementation procedure stands on a tacit assumption: the existence of $n$ propagation modes or, in other words, the existence of $n$ independent eigenvectors $\mathbf{t}_1 \cdots \mathbf{t}_k \cdots \mathbf{t}_n$. This assumption is verified in almost all MTL, however, the reader should be alerted that in very rare cases this might not be so. We will get back to this topic in Section 5.

### *3.5. MTL transmission matrix*

The procedure described in 3.4 permits the complete determination of MTL voltages and currents along the entire MTL structure, from $z = 0$ to $z = l$. However, in many circumstances, such a detailed knowledge is not necessary. Quite often, only MTL voltages and currents at the sending and receiving ends (input and output ports) need to be known.

The relationship between input and output variables can be written in the form of $2n \times 2n$ transmission matrix $\mathbf{M}$ (or **ABCD** matrix) such that

$$\begin{bmatrix} \mathbf{V}(0) \\ \mathbf{I}(0) \end{bmatrix} = \mathbf{M} \begin{bmatrix} \mathbf{V}(l) \\ \mathbf{I}(l) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \tag{37}$$

In order to determine the $n$th order submatrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ we start by rewriting (26) in the form

$$\begin{cases} \mathbf{V}(z) = (\mathbf{T}\exp(-\gamma z)\mathbf{T}^{-1})(\mathbf{T}\tilde{\mathbf{V}}_{\mathrm{i}}) + (\mathbf{T}\exp(+\gamma z)\mathbf{T}^{-1})(\mathbf{T}\tilde{\mathbf{V}}_{\mathrm{r}}) \\ \mathbf{I}(z) = \mathbf{W}\,\tilde{\mathbf{Y}}_w\mathbf{T}^{-1}((\mathbf{T}\exp(-\gamma z)\mathbf{T}^{-1})(\mathbf{T}\tilde{\mathbf{V}}_{\mathrm{i}}) - (\mathbf{T}\exp(+\gamma z)\mathbf{T}^{-1})(\mathbf{T}\tilde{\mathbf{V}}_{\mathrm{r}})) \end{cases} \tag{38a}$$

or, compactly,

$$\begin{cases} \mathbf{V}(z) = \mathrm{EXP}(-\boldsymbol{\Gamma} z)\mathbf{V}_{\mathrm{i}} + \mathrm{EXP}(+\boldsymbol{\Gamma} z)\mathbf{V}_{\mathrm{r}} \\ \mathbf{I}(z) = \mathbf{Y}_w(\mathrm{EXP}(-\boldsymbol{\Gamma} z)\mathbf{V}_{\mathrm{i}} - \mathrm{EXP}(+\boldsymbol{\Gamma} z)\mathbf{V}_{\mathrm{r}}) \end{cases} \tag{38b}$$

where

$$\begin{cases} \mathrm{EXP}(\pm\boldsymbol{\Gamma} z) = \mathbf{T}\exp(\pm\gamma z)\mathbf{T}^{-1} \\ \mathbf{V}_{\mathrm{i}} = \mathbf{T}\tilde{\mathbf{V}}_{\mathrm{i}} = \frac{1}{2}(\mathbf{V}(0) + \mathbf{Y}_w^{-1}\mathbf{I}(0)) \\ \mathbf{V}_{\mathrm{r}} = \mathbf{T}\tilde{\mathbf{V}}_{\mathrm{r}} = \frac{1}{2}(\mathbf{V}(0) - \mathbf{Y}_w^{-1}\mathbf{I}(0)) \end{cases} \tag{39}$$

from where we get

$$\begin{cases} \mathbf{A} = \mathrm{COSH}(\boldsymbol{\Gamma} l) = \mathbf{T}\cosh(\gamma l)\mathbf{T}^{-1} = \frac{1}{2}(\mathrm{EXP}(+\boldsymbol{\Gamma} l) + \mathrm{EXP}(-\boldsymbol{\Gamma} l)) \\ \mathbf{B} = \mathrm{SINH}(\boldsymbol{\Gamma} l)\mathbf{Y}_w^{-1} = (\mathbf{T}\sinh(\gamma l)\mathbf{T}^{-1})\mathbf{Y}_w^{-1} = \frac{1}{2}(\mathrm{EXP}(+\boldsymbol{\Gamma} l) - \mathrm{EXP}(-\boldsymbol{\Gamma} l))\mathbf{Y}_w^{-1} \\ \mathbf{C} = \mathbf{Y}_w\mathbf{B}\mathbf{Y}_w \\ \mathbf{D} = \mathbf{Y}_w\mathbf{A}\mathbf{Y}_w^{-1} \end{cases} \tag{40}$$

Since the MTL structure is reciprocal and longitudinally uniform (indiscernible input and output ports) the transmission matrix $\mathbf{M}$ obeys the following properties.[19,23]

$$\det\mathbf{M} = 1 \tag{41}$$

$$\mathbf{A}^2 - \mathbf{B}\mathbf{C} = \mathbf{1}, \quad \mathbf{D} = \mathbf{A}^{\mathrm{t}}, \quad \mathbf{B} = \mathbf{B}^{\mathrm{t}}, \quad \mathbf{C} = \mathbf{C}^{\mathrm{t}} \tag{42}$$

In some instances, it is useful to consider the reverse of the relationship defined in (37), i.e.

$$\begin{bmatrix} \mathbf{V}(l) \\ \mathbf{I}(l) \end{bmatrix} = \mathbf{M}^{-1} \begin{bmatrix} \mathbf{V}(0) \\ \mathbf{I}(0) \end{bmatrix}, \quad \mathbf{M}^{-1} = \begin{bmatrix} \mathbf{A} & -\mathbf{B} \\ -\mathbf{C} & \mathbf{D} \end{bmatrix} \tag{43}$$

where $\mathbf{M}^{-1}$ is the reverse transmission matrix.

## 4. Application to the shielded bifilar cable configuration

In Section 2, the concept of propagation mode was qualitatively introduced using the example of a shielded bifilar cable and, at the end, a brief summary of the key ideas concerning MTL analysis was put forward. Here, we proceed to the modal analysis of the shielded bifilar cable configuration (Figure 6) employing the matrix technique developed in Section 3 and exemplifying the key ideas anticipated at the end of Section 2.

The uniform MTL configuration is bilaterally symmetric (labels 1 and 2 are interchangeable); the nonmagnetic dielectric medium is homogeneous, characterized by $\bar{\varepsilon} = \varepsilon - j\varepsilon' = \varepsilon_0\bar{\varepsilon}_r$, and $\mu = \mu_0$, where dielectric losses are taken into account through the negative imaginary part of $\bar{\varepsilon}_r$.

For homogeneous MTL, where the dielectric medium is characterized by constant properties (conductivity, permittivity, permeability), the conductance, capacitance, and inductance matrices are not independent. In fact we have:

$$\mathbf{G} = \frac{\sigma}{\varepsilon}\mathbf{C}\,, \quad \mathbf{CL} = \mathbf{LC} = \varepsilon\mu_0\mathbf{1} \tag{44}$$

where $\sigma$, in (44), can be computed from $\sigma = \omega\varepsilon'$.

### 4.1. Cable with losses

If the dielectric medium were a vacuum a pul capacitance matrix $\mathbf{C}_0$ would be found

$$\mathbf{C}_0 = \begin{bmatrix} C_0 & C_m \\ C_m & C_0 \end{bmatrix}$$

From (44), using $\mathbf{C}_0$, the complex pul transverse admittance matrix $\mathbf{Y}$, and the pul external inductance matrix $\mathbf{L}$, are readily obtained:

$$\mathbf{Y} = \mathbf{G} + j\omega\mathbf{C} = j\omega\bar{\varepsilon}_r\mathbf{C}_0\,, \quad \mathbf{L} = v_0^{-2}\mathbf{C}_0^{-1} \tag{45}$$

where $v_0$ is the speed of light in a vacuum, $v_0 = (\mu_0\varepsilon_0)^{-1/2}$.

Internal magnetic energy storage and Joule losses mechanisms inside MTL conductors are accounted by the pul skin effect impedance matrix $\mathbf{Z}^{se}$,

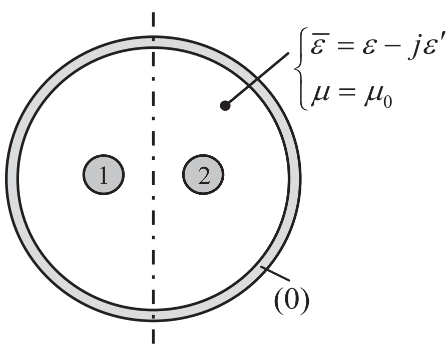



Figure 6. Homogeneous shielded bifilar cable configuration.

$$\mathbf{Z}^{\mathrm{se}} = \begin{bmatrix} Z^{\mathrm{se}} & Z_m^{\mathrm{se}} \\ Z_m^{\mathrm{se}} & Z^{\mathrm{se}} \end{bmatrix}$$

For very low frequencies the entries of $\mathbf{Z}^{\mathrm{se}}$ have negligibly small imaginary parts, contrarily, for very high frequencies, the real and imaginary parts tend to be equal and increase with $\omega^{1/2}$.

From (7) and (45), the complex pul longitudinal impedance matrix $\mathbf{Z}$ is obtained as

$$\mathbf{Z} = \mathbf{Z}^{\mathrm{se}} + j\omega v_0^{-2}\mathbf{C}_0^{-1} \tag{46}$$

The products $\mathbf{ZY}$ and $\mathbf{YZ}$ are identical matrices. They are obtained from (45) and (46):

$$\mathbf{ZY} = \mathbf{YZ} = (j\omega\mathbf{X} + (j\beta_0)^2\mathbf{1})\bar{\varepsilon}_r \tag{47}$$

where $\beta_0 = \omega/v_0$, and

$$\mathbf{X} = \mathbf{Z}^{\mathrm{se}}\mathbf{C}_0 = \begin{bmatrix} x & m \\ m & x \end{bmatrix}, \quad \begin{cases} x = & Z^{\mathrm{se}}C_0 + Z_m^{\mathrm{se}}C_m \\ m = & Z^{\mathrm{se}}C_m + Z_m^{\mathrm{se}}C_0 \end{cases} \tag{48}$$

The diagonalization of the $\mathbf{X}$ matrix can be accomplished using a real frequency-independent modal transformation matrix $\mathbf{T}$:

$$\mathbf{T}^{-1}\mathbf{XT} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \text{where } \mathbf{T} = \{\mathbf{t}_1\ ,\ \mathbf{t}_2\} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \begin{cases} \lambda_1 = x + m \\ \lambda_2 = x - m \end{cases} \tag{49}$$

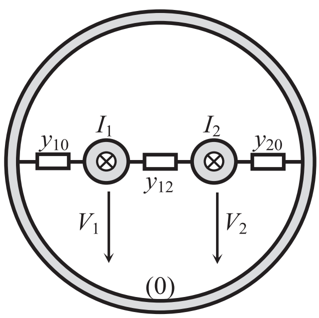
Since $\mathbf{T}$ is a symmetric orthogonal matrix, $\mathbf{T} = \mathbf{T}^{-1} = \mathbf{T}^{\mathrm{t}}$, we have, from (19), $\mathbf{T} = \mathbf{W}$. Note, in addition, that this result is independent of the numerical values of the entries of $\mathbf{X}$ in (48).

The propagation constants and characteristic modal admittances of the two modes are obtained from (17) and (24), yielding

$$\begin{cases} \gamma_1 = \alpha_1 + j\beta_1 = \sqrt{\bar{\varepsilon}_r((j\beta_0)^2 + j\omega(x+m))} \\ \gamma_2 = \alpha_2 + j\beta_2 = \sqrt{\bar{\varepsilon}_r((j\beta_0)^2 + j\omega(x-m))} \end{cases}, \quad \begin{cases} \tilde{Y}_{w1} = j\omega\bar{\varepsilon}_r \frac{C_0 + C_m}{\gamma_1} \\ \tilde{Y}_{w2} = j\omega\bar{\varepsilon}_r \frac{C_0 - C_m}{\gamma_2} \end{cases} \tag{50}$$

From (25) and (50) the square symmetric characteristic admittance matrix $\mathbf{Y}_w$ is

$$\mathbf{Y}_w = \frac{1}{2}\begin{bmatrix} \tilde{Y}_{w1} + \tilde{Y}_{w2} & \tilde{Y}_{w1} - \tilde{Y}_{w2} \\ \tilde{Y}_{w1} - \tilde{Y}_{w2} & \tilde{Y}_{w1} + \tilde{Y}_{w2} \end{bmatrix} \tag{51}$$




$$\begin{bmatrix} y_{10} = \tilde{Y}_{w_1} \\ y_{12} = (\tilde{Y}_{w_2} - \tilde{Y}_{w_1})/2 \\ y_{20} = \tilde{Y}_{w_1} \end{bmatrix}$$

Figure 7. Matched termination of the shielded bifilar cable.

From (33) and (28) we conclude that the cable will be matched, $\mathbf{Y}_\mathrm{L} = \mathbf{Y}_w$ (absence of reflected waves, $\tilde{\mathbf{V}}_\mathrm{r} = \mathbf{0}$), if it is terminated by the network of load admittances depicted in Figure 7. The condition of absence of reflected waves is assumed in (52)–(54).

If the generator voltages enforced at $z = 0$ are given by $\mathbf{V}_\mathrm{G} = V_e\,[1 \quad 1]^\mathrm{t}$ then only the mode-1 will be present (even mode), with a $z$-invariant field pattern shown in Figure 4(a),

$$\mathbf{V}_\mathrm{G} = V_e \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \tilde{\mathbf{V}}_\mathrm{i} = \sqrt{2}\, V_e \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{cases} \begin{bmatrix} V_1(z) \\ V_2(z) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} V_e \mathrm{e}^{-\gamma_1 z} \\ \begin{bmatrix} I_1(z) \\ I_2(z) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tilde{Y}_{w1}\, V_e \mathrm{e}^{-\gamma_1 z} \end{cases} \tag{52}$$

If the generator voltages enforced at $z = 0$ are given by $\mathbf{V}_\mathrm{G} = V_o\,[1 \quad -1]^\mathrm{t}$ then only the mode-2 will be present (odd mode), with a $z$-invariant field pattern shown in Figure 4(b),

$$\mathbf{V}_\mathrm{G} = V_o \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \tilde{\mathbf{V}}_\mathrm{i} = \sqrt{2}\, V_o \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \rightarrow \quad \begin{cases} \begin{bmatrix} V_1(z) \\ V_2(z) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} V_o \mathrm{e}^{-\gamma_2 z} \\ \begin{bmatrix} I_1(z) \\ I_2(z) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tilde{Y}_{w2}\, V_o \mathrm{e}^{-\gamma_2 z} \end{cases} \tag{53}$$

If the generator voltages enforced at $z = 0$ are given by $\mathbf{V}_\mathrm{G} = V\,[1 \quad 0]^\mathrm{t}$ then a combination of both modes will be present, with a $z$-variant field pattern shown in Figure 3,

$$\mathbf{V}_\mathrm{G} = V \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \tilde{\mathbf{V}}_\mathrm{i} = \frac{V}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \rightarrow \quad \begin{cases} \begin{bmatrix} V_1(z) \\ V_2(z) \end{bmatrix} = \frac{V}{2}\left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{e}^{-\gamma_1 z} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mathrm{e}^{-\gamma_2 z} \right) \\ \begin{bmatrix} I_1(z) \\ I_2(z) \end{bmatrix} = \frac{V}{2}\left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tilde{Y}_{w1} \mathrm{e}^{-\gamma_1 z} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tilde{Y}_{w2} \mathrm{e}^{-\gamma_2 z} \right) \end{cases} \tag{54}$$

### 4.2. *Lossless cable*

If all the cable conductors were perfect ($\sigma_\mathrm{c} \rightarrow \infty$) the skin effect perturbation would be absent, $\mathbf{Z}^\mathrm{se} = \mathbf{0}$. Consequently, the matrix $\mathbf{X}$ in (48) would be null, $\mathbf{X} = \mathbf{0}$, and $\lambda_1 = \lambda_2 = 0$, that is, a degenerate situation with a double eigenvalue would be found.

Since $\mathbf{X} = \mathbf{0}$, the diagonalization operation $\mathbf{T}^{-1}\mathbf{X}\mathbf{T} = \mathbf{0}$ is always guaranteed no matter the choice of $\mathbf{T}$. This means that the eigenvectors $\mathbf{t}_1$ and $\mathbf{t}_2$ are completely arbitrary. Any set of excitation voltages $\mathbf{V}_\mathrm{G}$ defines a propagation mode, whose field-lines pattern remains $z$-invariant along the cable.

The unique propagation constant of the cable is evaluated from (50) by setting $m = x = 0$. Writing the square root of the complex relative permittivity in the form $\sqrt{\bar{\varepsilon}_r} = n - j\kappa$ we would find

$$\gamma = \alpha + j\frac{\omega}{v}, \quad \begin{cases} \alpha = \omega\kappa/v_0 \\ v = v_0/n \end{cases} \tag{55}$$

In addition, if the dielectric is a lossless medium ($\kappa = 0$), then the attenuation constant $\alpha$ is also zero.

## 5. Failure of modal analysis

At the end of Section 3.4 we alerted that in some very rare circumstances the developed modal analysis could fail to be used. The failure is not of numerical nature, but it is inherent to the tacit assumption that a nonsingular modal transformation matrix $\mathbf{T}$ always exists (det $\mathbf{T} \neq 0$), which may not be true.

Despite the specific properties of the real and imaginary parts of $\mathbf{Z}$ and $\mathbf{Y}$ (positive-definite symmetric matrices), a proof that the $n \times n$ product matrix $\mathbf{ZY}$ can always be characterized by $n$ independent eigenvectors cannot be presented. Contrarily, counterexamples can be produced showing the opposite.[19,24,25] The necessary, but not sufficient, condition for establishing such counterexamples is the existence of multiple eigenvalues (degeneracy).

Consider again the homogeneous shielded bifilar cable, where the inner conductors have different radius or/and are asymmetrically positioned. To simplify things assume that the shielding is a perfect conductor. In this case we have

$$\mathbf{C}_0 = \begin{bmatrix} C_1 & C_m \\ C_m & C_2 \end{bmatrix} C_1 \neq C_2; \quad \mathbf{Z}^{\text{se}} = \begin{bmatrix} Z_1^{\text{se}} & 0 \\ 0 & Z_2^{\text{se}} \end{bmatrix} Z_1^{\text{se}} = Z_1 e^{j\phi_1} \neq Z_2^{\text{se}} = Z_2 e^{j\phi_2},$$

The $\mathbf{X}$ matrix in (48) changes to

$$\mathbf{X} = \mathbf{Z}^{\text{se}}\mathbf{C}_0 = \begin{bmatrix} x_1 & m_1 \\ m_2 & x_2 \end{bmatrix}, \quad \begin{cases} x_1 = Z_1^{\text{se}} C_1, & m_1 = Z_1^{\text{se}} C_m \\ m_2 = Z_2^{\text{se}} C_m, & x_2 = Z_2^{\text{se}} C_2 \end{cases} \tag{56}$$

The eigenvalues of $\mathbf{X}$ are given by

$$\lambda_{1,2} = \frac{1}{2}\left(\text{tr}\,\mathbf{X} \pm \sqrt{\text{tr}^2\mathbf{X} - 4\ \det\mathbf{X}}\right) \tag{57}$$

A double eigenvalue can occur in two circumstances:

(a) $\mathbf{X} = \mathbf{0}$, $\lambda_1 = \lambda_2 = \lambda = 0$.

(b) $\mathbf{X} \neq \mathbf{0}$, $\text{tr}^2\mathbf{X} = 4\ \det\mathbf{X}$, $\lambda_1 = \lambda_2 = \lambda = (x_1 + x_2)/2$.

Case (a) raises no problems; we dealt with it in Subsection 4.2. It is a case of regular degeneracy,[19,21] where two independent arbitrary eigenvectors $\mathbf{t}_1$ and $\mathbf{t}_2$ can always be defined.

Instead, case (b) is a case of irregular degeneracy,[19,21] where matrix $\mathbf{ZY}$ has only one eigenvector $\mathbf{t}$, the same happening to matrix $\mathbf{YZ}$, the transposed of $\mathbf{ZY}$, with a single eigenvector $\mathbf{w}$,

$$\mathbf{t} = \propto \begin{bmatrix} 1 \\ \eta \end{bmatrix}, \mathbf{w} \propto \begin{bmatrix} \eta \\ -1 \end{bmatrix}, \mathbf{t}^{\text{t}}\mathbf{w} = 0, \quad \text{with } \eta = \frac{x_2 - x_1}{2m_1} = \frac{2m_2}{x_1 - x_2} \tag{58}$$

which means that the $2 \times 2$ modal transformation matrix $\mathbf{T}$ (and $\mathbf{W}$) cannot be built, therefore, inhibiting the utilization of modal analysis.

The condition for irregular degeneracy $\text{tr}^2\mathbf{X} = 4\ \det\mathbf{X}$ can be put in the form $(x_1 - x_2)^2 = -4m_1m_2$, which translates to:

$$\begin{cases} \frac{Z_1}{Z_2} = \frac{C_2}{C_1} \\ \phi_1 - \phi_2 = \pm 2 \arcsin\left(\frac{C_m}{\sqrt{C_1C_2}}\right) \end{cases} \tag{59}$$

Noting that the angles of the skin effect impedances $Z_1^{\text{se}}$ and $Z_2^{\text{se}}$ usually differ slightly, we see that condition (59) can only be fulfilled in situations of weak coupling, $|C_m| \ll \sqrt{C_1C_2}$.[25]

At this stage the natural question is: How can we compute the evolution along $z$ of MTL voltages and currents when matrix $\mathbf{ZY}$ cannot be brought into diagonal form? Is there a substitute for the modal analysis presented in Section 3? The answer is yes. A generalization of the ordinary modal analysis presented in Section 3 has already been developed for the cases when $\mathbf{ZY}$ is not diagonalizable. The generalized modal theory is complicated; it involves Jordan forms,[19,21,26], and modal groups.[19,26] Since non-diagonalizable cases very seldom occur we will not develop the subject here, nonetheless, the interested reader may find details on the generalized modal theory in [19].

For illustration purposes we present in (60) the solution for $\mathbf{V}(z)$ and $\mathbf{I}(z)$, concerning the example of the infinitely long homogeneous shielded bifilar cable (with a non-diagonalizable $\mathbf{ZY}$ matrix of 2nd order),

$$\begin{cases} \mathbf{V}(z) = e^{-\gamma z}\,\mathbf{P}(z)\mathbf{V}(0) \\ \mathbf{I}(z) = e^{-\gamma z}\mathbf{P}^{t}(z)\,\mathbf{I}(0) \end{cases}, \quad \mathbf{P}(z) = \mathbf{1} - \frac{\gamma z}{2}(\gamma^{-2}\mathbf{ZY} - \mathbf{1}) \tag{60}$$

where the propagation constant $\gamma$ is given by $\gamma = \sqrt{(j\omega\lambda - \beta_0^2)\bar{\varepsilon}_r}$.

## 6. Nonuniform MTL

In Section 3 we presented the ordinary modal analysis theory for uniform MTL; however, in many circumstances uniformity is not verified. If the cross section of the multiconductor transmission line varies along the longitudinal coordinate $z$ (e.g. when the lateral distances between conductors vary) or if the properties of the dielectric medium vary with $z$, then the MTL will be classified as a nonuniform structure. In such cases the pul longitudinal impedance matrix $\mathbf{Z}$ and the pul transverse admittance matrix $\mathbf{Y}$ become functions of $z$ and, therefore, equation (6b) changes to

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{bmatrix} \mathbf{V} \\ \mathbf{I} \end{bmatrix} = -\begin{bmatrix} \mathbf{0} & \mathbf{Z}(z) \\ \mathbf{Y}(z) & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{V} \\ \mathbf{I} \end{bmatrix} \tag{61}$$

As a result, the wave equations in (13) transform to

$$\begin{cases} \frac{\mathrm{d}^2\mathbf{V}}{\mathrm{d}z^2} - \left(\frac{\mathrm{d}\,\mathbf{Z}}{\mathrm{d}z}\mathbf{Z}^{-1}\right)\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}z} - (\mathbf{ZY})\mathbf{V} = \mathbf{0} \\ \frac{\mathrm{d}^2\mathbf{I}}{\mathrm{d}z^2} - \left(\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}z}\mathbf{Y}^{-1}\right)\frac{\mathrm{d}\mathbf{I}}{\mathrm{d}z} - (\mathbf{YZ})\mathbf{I} = \mathbf{0} \end{cases} \tag{62}$$

The wave-equations in (13) describe 2nd order homogeneous matrix differential equations with constant matrix coefficients and, as we have seen in (38b), their analytical solution takes the form of a sum of two exponential matrix functions. Contrarily, the equations in (62), involving nonconstant matrix coefficients, have no closed-form analytical solution, in general. However, solutions for $\mathbf{V}(z)$ and $\mathbf{I}(z)$ can be obtained numerically.

Two numerical procedures are presented here, the segmentation method,[19,20,27] and the iterative integration method,[28,29] where the key idea is the $z$-discretization of the $\mathbf{Z}(z)$ and $\mathbf{Y}(z)$ matrices along the length $l$ of the MTL structure; $z = z_k = k\Delta z$, $\Delta z = l/N$, $N$ being the number of discretization points.

$$\mathbf{Z}(z_k) = \mathbf{Z}_k, \mathbf{Y}(z_k) = \mathbf{Y}_k, \quad \text{for } k = 0 \text{ to } N, \tag{63}$$

Both methods resort to the $2n$-port transmission matrix formalism

$$\begin{bmatrix} \mathbf{V}(0) \\ \mathbf{I}(0) \end{bmatrix} = \mathbf{M}(z)\begin{bmatrix} \mathbf{V}(z) \\ \mathbf{I}(z) \end{bmatrix}, \quad \mathbf{M}(z) = \begin{bmatrix} \mathbf{A}(z) & \mathbf{B}(z) \\ \mathbf{C}(z) & \mathbf{D}(z) \end{bmatrix} \tag{64}$$

Before we continue, it must be emphasized that the $z$-dependent $\mathbf{A}(z)$, $\mathbf{B}(z)$, $\mathbf{C}(z)$, $\mathbf{D}(z)$ matrices in (64) do not obey the properties mentioned in (42)–(43) valid for uniform MTL. In fact, as shown in [23], the general properties of $\mathbf{M}$, for reciprocal $2n$-port networks, are:

$$\det \mathbf{M} = 1 \tag{65a}$$

$$\mathbf{A}\mathbf{B}^{\mathrm{t}} - \mathbf{B}\mathbf{A}^{\mathrm{t}} = \mathbf{B}^{\mathrm{t}}\mathbf{D} - \mathbf{D}^{\mathrm{t}}\mathbf{B} = \mathbf{D}\mathbf{C}^{\mathrm{t}} - \mathbf{C}\mathbf{D}^{\mathrm{t}} = \mathbf{C}^{\mathrm{t}}\mathbf{A} - \mathbf{A}^{\mathrm{t}}\mathbf{C} = \mathbf{0} \tag{65b}$$

$$\mathbf{A}\mathbf{D}^{\mathrm{t}} - \mathbf{B}\mathbf{C}^{\mathrm{t}} = \mathbf{D}^{\mathrm{t}}\mathbf{A} - \mathbf{B}^{\mathrm{t}}\mathbf{C} = \mathbf{1} \tag{65c}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{D} & -\mathbf{C} \\ -\mathbf{B} & \mathbf{A} \end{bmatrix}^{\mathrm{t}} \tag{65d}$$

### *6.1. Segmentation method*

In the segmentation method the nonuniform structure is modeled as a chain of $N$ individual uniform sections of small length $\Delta z$.

The $k$th generic uniform section is characterized by $\mathbf{Z}_k, \mathbf{Y}_k, \mathbf{\Gamma}_k = \mathrm{SQRT}(\mathbf{Z}_k\mathbf{Y}_k)$, $\mathbf{Y}_{wk} = \mathbf{Y}_k\mathbf{\Gamma}_k^{-1}$ and, from (40), by a transmission matrix $\mathbf{M}_k$ given by

$$\mathbf{M}_k = \begin{bmatrix} \mathrm{COSH}(\mathbf{\Gamma}_k\Delta z) & \mathrm{SINH}(\mathbf{\Gamma}_k\Delta z)\mathbf{Y}_{wk}^{-1} \\ \mathbf{Y}_{wk}\,\mathrm{SINH}(\mathbf{\Gamma}_k\Delta z) & \mathrm{COSH}^{\mathrm{t}}(\mathbf{\Gamma}_k\Delta z) \end{bmatrix} \tag{66}$$

The transmission matrix $\mathbf{M}(z_k)$ of the nonuniform MTL, with sending end at $z = z_0 = 0$ and 'sliding receiving end' at $z = z_k$, is determined by successive multiplication of the transmission matrices of the individual uniform sections:

$$\begin{cases} \mathbf{M}(z_1) = \mathbf{M}_1 \\ \mathbf{M}(z_2) = \mathbf{M}(z_1)\mathbf{M}_2 \\ \qquad \vdots \\ \mathbf{M}(z_k) = \mathbf{M}(z_{k-1})\mathbf{M}_k \\ \qquad \vdots \\ \mathbf{M}(z_N) = \mathbf{M}(z_{N-1})\mathbf{M}_N = \mathbf{M}_1\mathbf{M}_2\cdots\mathbf{M}_k\cdots\mathbf{M}_N, \end{cases} \tag{67}$$

### *6.2. Iterative integration method*

In the integration method, the equation

$$\begin{bmatrix} \mathbf{V}(0) \\ \mathbf{I}(0) \end{bmatrix} = \mathbf{M}(z)\begin{bmatrix} \mathbf{V}(z) \\ \mathbf{I}(z) \end{bmatrix}$$

is derived with respect to $z$, yielding

$$\mathbf{0} = \frac{\mathrm{d}\mathbf{M}(z)}{\mathrm{d}z}\begin{bmatrix} \mathbf{V}(z) \\ \mathbf{I}(z) \end{bmatrix} + \mathbf{M}(z)\frac{\mathrm{d}}{\mathrm{d}z}\begin{bmatrix} \mathbf{V}(z) \\ \mathbf{I}(z) \end{bmatrix} \tag{68}$$

Substituting (61) into (68) leads to

$$\frac{\mathrm{d}\mathbf{M}(z)}{\mathrm{d}z} = \mathbf{M}(z)\begin{bmatrix} \mathbf{0} & \mathbf{Z}(z) \\ \mathbf{Y}(z) & \mathbf{0} \end{bmatrix} \tag{69}$$

Equation (69) is iteratively integrated using the initial value $\mathbf{M}(z_0) = \mathbf{M}(0) = \mathbf{1}$, yielding

$$\begin{aligned} \mathbf{M}(z_1) &= \mathbf{1} + \int_{z_0}^{z_1} \left( \mathbf{M}(z)\begin{bmatrix} \mathbf{0} & \mathbf{Z}(z) \\ \mathbf{Y}(z) & \mathbf{0} \end{bmatrix} \right) dz \\ &\vdots \\ \mathbf{M}(z_k) &= \mathbf{M}(z_{k-1}) + \int_{z_{k-1}}^{z_k} \left( \mathbf{M}(z)\begin{bmatrix} \mathbf{0} & \mathbf{Z}(z) \\ \mathbf{Y}(z) & \mathbf{0} \end{bmatrix} \right) dz \\ &\vdots \end{aligned} \tag{70}$$

If, in the $k$th interval $[z_{k-1} \;\; z_k]$ where $\mathbf{Z}(z)$ and $\mathbf{Y}(z)$ are approximated by $\mathbf{Z}_k$ and $\mathbf{Y}_k$, the function $\mathbf{M}(z)$ is approximated by its last iteration $\mathbf{M}(z_{k-1})$ then the result for $\mathbf{M}(z_k)$ in (70) will transform into

$$\mathbf{M}(z_k) = \mathbf{M}(z_{k-1})\left( \mathbf{1} + \begin{bmatrix} \mathbf{0} & \mathbf{Z}_k \\ \mathbf{Y}_k & \mathbf{0} \end{bmatrix} \Delta z \right) \tag{71}$$

It should be remarked that the integration method, valid for $\Delta z \to 0$, is equivalent to the segmentation method presented in 6.1. In fact, if in (66) we consider $\Delta z \to 0$ and take (25) into account, then we will obtain

$$\lim_{\Delta z \to 0} \begin{bmatrix} \mathrm{COSH}(\mathbf{\Gamma}_k \Delta z) & \mathrm{SINH}(\mathbf{\Gamma}_k \Delta z)\mathbf{Y}_{wk}^{-1} \\ \mathbf{Y}_{wk}\ \mathrm{SINH}(\mathbf{\Gamma}_k \Delta z) & \mathrm{COSH}^{\mathrm{t}}(\mathbf{\Gamma}_k \Delta z) \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{Z}_k \Delta z \\ \mathbf{Y}_k \Delta z & \mathbf{1} \end{bmatrix} \tag{72}$$

which is identical to the multiplying factor in parenthesis in the right hand side of (71).

As a marginal comment, in the case of uniform MTL where the pul matrices $\mathbf{Z}$ and $\mathbf{Y}$ remain constant along $z$, we should observe that the results in (71)–(72), when related to (40), give:

$$\mathbf{M}(l) = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \lim_{N \to \infty} \begin{bmatrix} \mathbf{1} & \mathbf{Z}\ l/N \\ \mathbf{Y}\ l/N & \mathbf{1} \end{bmatrix}^N \tag{73}$$

$$\begin{cases} \mathbf{A} = \mathrm{COSH}(\mathbf{\Gamma} l) = \sum_{\substack{k=0 \\ k\ \text{even}}}^{\infty} \frac{(\mathbf{\Gamma}^2)^k l^{2k}}{(2k)!} = \sum_{\substack{k=0 \\ k\ \text{even}}}^{\infty} (\mathbf{ZY})^k \frac{l^{2k}}{(2k)!} \\ \mathbf{B} = \mathrm{SINH}(\mathbf{\Gamma} l)\mathbf{Y}_w^{-1} = \mathrm{SINH}(\mathbf{\Gamma} l)\mathbf{\Gamma}^{-1}\mathbf{Z} = \left( \sum_{\substack{k=0 \\ k\ \text{even}}}^{\infty} \frac{(\mathbf{\Gamma}^2)^k l^{2k}}{(2k+1)!} \right) \mathbf{Z} l = \left( \sum_{\substack{k=0 \\ k\ \text{even}}}^{\infty} (\mathbf{ZY})^k \frac{l^{2k}}{(2k+1)!} \right) \mathbf{Z} l \\ \mathbf{C} = \mathbf{Y}_w \mathrm{SINH}(\mathbf{\Gamma} l)\mathbf{Y}_w^{-1} = \mathbf{Y}\mathbf{\Gamma}^{-1}\mathrm{SINH}(\mathbf{\Gamma} l) = \mathbf{Y} l \left( \sum_{\substack{k=0 \\ k\ \text{even}}}^{\infty} \frac{(\mathbf{\Gamma}^2)^k l^{2k}}{(2k+1)!} \right) = \mathbf{Y} l \left( \sum_{\substack{k=0 \\ k\ \text{even}}}^{\infty} (\mathbf{ZY})^k \frac{l^{2k}}{(2k+1)!} \right) \\ \mathbf{D} = \mathrm{COSH}^{\mathrm{t}}(\mathbf{\Gamma} l) = \sum_{\substack{k=0 \\ k\ \text{even}}}^{\infty} \frac{(\mathbf{\Gamma}^{2\mathrm{t}})^k l^{2k}}{(2k)!} = \sum_{\substack{k=0 \\ k\ \text{even}}}^{\infty} (\mathbf{YZ})^k \frac{l^{2k}}{(2k)!} \end{cases} \tag{74}$$

where $\mathbf{\Gamma}^2 = \mathbf{ZY}$, and where the hyperbolic cosine and sine functions were developed as power series Taylor expansions.

Also, the results in (73)–(74) show that the transmission matrix $\mathbf{M}$ can always be computed from $\mathbf{Z}$ and $\mathbf{Y}$, no matter their product $\mathbf{ZY}$ is diagonalizable or not.

The diagonalization problem referred to in Section 5, only affects the definition of $\boldsymbol{\Gamma}$, that is, the definition of $\boldsymbol{\Gamma} = \mathbf{T}\,\gamma\,\mathbf{T}^{-1} = \mathrm{SQRT}(\mathbf{ZY})$, since $\mathbf{T}$ does not exist.

### 6.3. *Evaluation of MTL voltages and currents from boundary conditions*

Consider the situation depicted in Figure 8 where the nonuniform MTL is excited at the sending end, $z = z_0 = 0$, by a set of generator voltages $\mathbf{V}(0) = \mathbf{V}_{\mathrm{G}}$, and where the MTL is terminated at the receiving end, $z = z_N = l$, by a load admittance $\mathbf{Y}_{\mathrm{L}}$.

Next, we will see how $\mathbf{V}(z_k)$ and $\mathbf{I}(z_k)$ can be computed.

Irrespectively of the procedure employed (6.1 or 6.2) consider that the transmission matrices $\mathbf{M}(z_k)$ and $\mathbf{M}(z_N) = \mathbf{M}(l)$ have been determined.

At $z = l$ we must have

$$\begin{bmatrix} \mathbf{V}(0) \\ \mathbf{I}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_{\mathrm{G}} \\ \mathbf{Y}_0^{\mathrm{in}}\mathbf{V}_{\mathrm{G}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}(l) & \mathbf{B}(l) \\ \mathbf{C}(l) & \mathbf{D}(l) \end{bmatrix} \begin{bmatrix} \mathbf{V}(l) \\ \mathbf{Y}_{\mathrm{L}}\mathbf{V}(l) \end{bmatrix} \tag{75}$$

from where the input admittance $\mathbf{Y}_0^{\mathrm{in}}$ at $z = 0$ is readily found

$$\mathbf{Y}_0^{\mathrm{in}} = (\mathbf{C}(l) + \mathbf{D}(l)\mathbf{Y}_{\mathrm{L}})(\mathbf{A}(l) + \mathbf{B}(l)\mathbf{Y}_{\mathrm{L}})^{-1} \tag{76}$$

At $z = z_k$, using the reverse transmission matrix in (65d), we obtain MTL voltages and currents

$$\begin{bmatrix} \mathbf{V}(z_k) \\ \mathbf{I}(z_k) \end{bmatrix} = \begin{bmatrix} \mathbf{D}^{\mathrm{t}}(z_k) & -\mathbf{B}^{\mathrm{t}}(z_k) \\ -\mathbf{C}^{\mathrm{t}}(z_k) & \mathbf{A}^{\mathrm{t}}(z_k) \end{bmatrix} \begin{bmatrix} \mathbf{V}_G \\ \mathbf{Y}_0^{\mathrm{in}}\mathbf{V}_G \end{bmatrix}. \tag{77}$$

## 7. Retrieving the modal wave parameters from the MTL transmission matrix

As we have seen in Subsection 6.3, knowledge of the MTL transmission matrix, together with the application of boundary conditions at $z = 0$ and $z = l$, permits the ultimate goal in MTL analysis: the evaluation of MTL voltages and currents at any $z = z_k$, including the special case $z = l$. This is true, of course, not only for nonuniform MTL but also for uniform MTL.

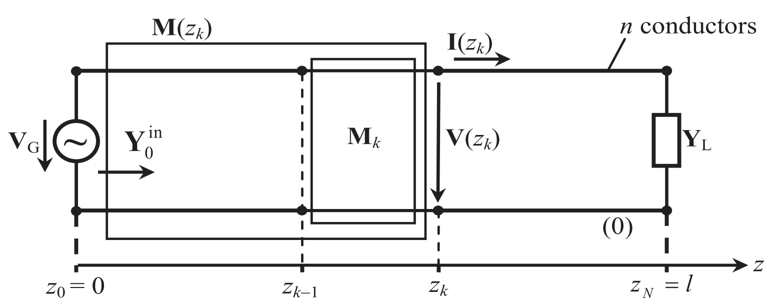



Figure 8. Boundary conditions and transmission matrices in a discretized nonuniform MTL.

As we have seen in Section 3, for uniform MTL, the definition and the determination of matrix $\mathbf{M} = \mathbf{M}(l)$ involves $z$-invariant modal propagation constants, modal characteristic admittances, and modal transformation matrices (which allow the $n$-coupled MTL system to be broken down into a set of $n$ uncoupled uniform single lines).

Quite naturally, it is expected that the reserve problem of finding the modal wave parameters from known $\mathbf{M}$ (evaluated, for example, by experimentation and measurement) can be solved. The interest in doing this is not the direct computation of $\mathbf{V}(z)$ and $\mathbf{I}(z)$; the aim is to get some insight into the physical phenomena of wave propagation, by characterizing the independent propagation modes at play.

In the general case of nonuniform MTL it is possible to break down the $n$-coupled MTL system into a set of $n$ uncoupled nonuniform single lines (each one characterized by a given propagation constant, and by distinct modal characteristic admittances for incident and reflected waves) – see Figure 9. The procedure, described in [30], involves two pairs of transformation matrices $\mathbf{T}$ and $\mathbf{W}$, one referred to the sending end $\mathbf{T}_S$ and $\mathbf{W}_S$, and another to the receiving end $\mathbf{T}_R$ and $\mathbf{W}_R$.

Here, we synthesize the main results developed in [30].

The transmission matrix equation

$$\begin{bmatrix} \mathbf{V}_S \\ \mathbf{I}_S \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}}_{\mathbf{M}(l)} \begin{bmatrix} \mathbf{V}_R \\ \mathbf{I}_R \end{bmatrix} \tag{78}$$

is transformed into the modal domain as

$$\begin{bmatrix} \tilde{\mathbf{V}}_S \\ \tilde{\mathbf{I}}_S \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{A}} & \tilde{\mathbf{B}} \\ \tilde{\mathbf{C}} & \tilde{\mathbf{D}} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{V}}_R \\ \tilde{\mathbf{I}}_R \end{bmatrix} \tag{79}$$

where, according to our notation, symbols with a tilde denote modal quantities.

Modal voltages and currents in (79) are related to natural voltages and currents through

$$\mathbf{V}_S = \mathbf{T}_S \tilde{\mathbf{V}}_S\ , \quad \mathbf{I}_S = \mathbf{W}_S \tilde{\mathbf{I}}_S\ , \quad \mathbf{V}_R = \mathbf{T}_R \tilde{\mathbf{V}}_R\ , \quad \mathbf{I}_R = \mathbf{W}_R \tilde{\mathbf{I}}_R \tag{80}$$

where

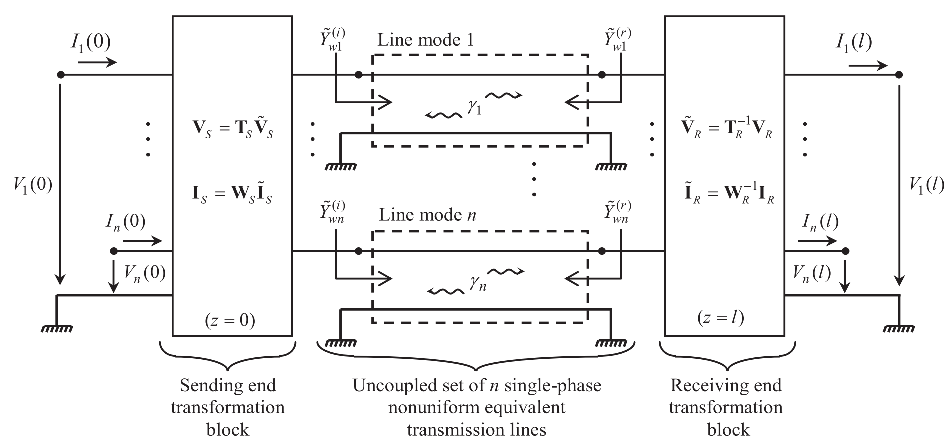



Figure 9. Modal decoupling in nonuniform MTL, at $z = 0$ and $z = l$.

$$\mathbf{W}_S = \mathbf{T}_S^{-1\mathrm{t}}\ , \quad \mathbf{W}_R = \mathbf{T}_R^{-1\mathrm{t}} \tag{81}$$

The pairs of transformation matrices

$$\mathbf{T}_S = \{\mathbf{t}_{S1}, \cdots, \mathbf{t}_{Sn}\},\ \mathbf{W}_S = \{\mathbf{w}_{S1}, \cdots, \mathbf{w}_{Sn}\}$$
$$\mathbf{T}_R = \{\mathbf{t}_{R1}, \cdots, \mathbf{t}_{Rn}\},\ \mathbf{W}_R = \{\mathbf{w}_{R1}, \cdots, \mathbf{w}_{Rn}\}$$

which apply respectively to the sending (*S*) and receiving (*R*) ends of the nonuniform MTL, are in general different from each other. They may happen to coincide only in very special cases, when certain symmetries exist (as it always occurs with uniform lines).

Matrix $\mathbf{T}_S$ is a similarity transformation that simultaneously brings $\mathbf{BC}^{\mathrm{t}}$ and $\mathbf{AD}^{\mathrm{t}}$ into diagonal form. Likewise, matrix $\mathbf{T}_R$ is a similarity transformation that simultaneously brings $\mathbf{B}^{\mathrm{t}}\mathbf{C}$ and $\mathbf{D}^{\mathrm{t}}\mathbf{A}$ into diagonal form

$$\mathbf{T}_S^{-1}(\mathbf{BC}^{\mathrm{t}})\mathbf{T}_S = \mathbf{T}_R^{-1}(\mathbf{B}^{\mathrm{t}}\mathbf{C})\mathbf{T}_R = \tilde{\mathbf{B}}\tilde{\mathbf{C}} \tag{82a}$$

$$\mathbf{T}_S^{-1}(\mathbf{AD}^{\mathrm{t}})\mathbf{T}_S = \mathbf{T}_R^{-1}(\mathbf{D}^{\mathrm{t}}\mathbf{A})\mathbf{T}_R = \tilde{\mathbf{A}}\tilde{\mathbf{D}} \tag{82b}$$

The *n*th order diagonal modal matrices $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{C}}$, and $\tilde{\mathbf{D}}$ in (79) are obtained from the corresponding untilded ones in (78) through

$$\tilde{\mathbf{A}} = \mathbf{T}_S^{-1}\mathbf{A}\,\mathbf{T}_R = \begin{bmatrix} \tilde{a}_1 & & & & \\ & \ddots & & & \\ & & \tilde{a}_k & & \\ & & & \ddots & \\ & & & & \tilde{a}_n \end{bmatrix}, \tilde{\mathbf{B}} = \mathbf{T}_S^{-1}\mathbf{B}\,\mathbf{W}_R = \begin{bmatrix} \tilde{b}_1 & & & & \\ & \ddots & & & \\ & & \tilde{b}_k & & \\ & & & \ddots & \\ & & & & \tilde{b}_n \end{bmatrix} \tag{83a}$$

$$\tilde{\mathbf{C}} = \mathbf{W}_S^{-1}\mathbf{C}\,\mathbf{T}_R = \begin{bmatrix} \tilde{c}_1 & & & & \\ & \ddots & & & \\ & & \tilde{c}_k & & \\ & & & \ddots & \\ & & & & \tilde{c}_n \end{bmatrix}, \tilde{\mathbf{D}} = \mathbf{W}_S^{-1}\mathbf{D}\,\mathbf{W}_R = \begin{bmatrix} \tilde{d}_1 & & & & \\ & \ddots & & & \\ & & \tilde{d}_k & & \\ & & & \ddots & \\ & & & & \tilde{d}_n \end{bmatrix} \tag{83b}$$

where $\tilde{a}_k\tilde{d}_k - \tilde{b}_k\tilde{c}_k = 1$, for $k = 1$ to $n$.

The modal propagation parameters for mode-$k$ are determined using the information in (83) concerning the matrix entries $\tilde{a}_k$, $\tilde{b}_k$, $\tilde{c}_k$, and $\tilde{d}_k$:

Propagation constant:

$$\gamma_k = \alpha_k + j\beta_k = \frac{1}{l}\operatorname{arccosh}\left(\frac{\tilde{a}_k + \tilde{d}_k}{2}\right) \tag{84}$$

Characteristic modal admittance for incident waves:

$$\tilde{Y}_{wk}^{(i)} = \frac{e^{+\gamma_k l} - \tilde{a}_k}{\tilde{b}_k} = \frac{\tilde{c}_k}{e^{+\gamma_k l} - \tilde{d}_k} \tag{85}$$

Characteristic modal admittance for reflected waves:

$$\tilde{Y}_{wk}^{(r)} = \frac{\tilde{a}_k - e^{-\gamma_k l}}{\tilde{b}_k} = \frac{\tilde{c}_k}{\tilde{d}_k - e^{-\gamma_k l}} \tag{86}$$

Example for $n = 2$:

In a manner similar to (22)–(23), at the sending end ($S$), making $z = 0$, line voltages and line currents are defined as a linear combination of modes:

$$\begin{bmatrix} V_1(0) \\ V_2(0) \end{bmatrix} = \mathbf{t}_{S1}(\tilde{V}_{i1} + \tilde{V}_{r1}) + \mathbf{t}_{S2}(\tilde{V}_{i2} + \tilde{V}_{r2}) \tag{87a}$$

$$\begin{bmatrix} I_1(0) \\ I_2(0) \end{bmatrix} = \mathbf{w}_{S1}(\tilde{Y}_{w1}^{(i)}\tilde{V}_{i1} - \tilde{Y}_{w1}^{(r)}\tilde{V}_{r1}) + \mathbf{w}_{S2}(\tilde{Y}_{w2}^{(i)}\tilde{V}_{i2} - \tilde{Y}_{w2}^{(r)}\tilde{V}_{r2}) \tag{87b}$$

In a manner similar to (22)–(23), at the receiving end ($R$), making $z = l$, line voltages and line currents are defined as a linear combination of modes:

$$\begin{bmatrix} V_1(l) \\ V_2(l) \end{bmatrix} = \mathbf{t}_{R1}(\tilde{V}_{i1}\, e^{-\gamma_1 l} + \tilde{V}_{r1}\, e^{+\gamma_1 l}) + \mathbf{t}_{R2}(\tilde{V}_{i2}\, e^{-\gamma_2 l} + \tilde{V}_{r2}\, e^{+\gamma_2 l}) \tag{88a}$$

$$\begin{bmatrix} I_1(l) \\ I_2(l) \end{bmatrix} = \mathbf{w}_{R1}(\tilde{Y}_{w1}^{(i)}\tilde{V}_{i1}\, e^{-\gamma_1 l} - \tilde{Y}_{w1}^{(r)}\tilde{V}_{r1}\, e^{+\gamma_1 l}) + \mathbf{w}_{R2}(\tilde{Y}_{w2}^{(i)}\tilde{V}_{i2}\, e^{-\gamma_2 l} - \tilde{Y}_{w2}^{(r)}\tilde{V}_{r2}\, e^{+\gamma_2 l}) \tag{88b}$$

In the particular case of uniform MTL (where $\tilde{a}_k = \tilde{d}_k$), the results in (84)–(86) become

$$\mathbf{T}_S = \mathbf{T}_R\ , \quad \mathbf{W}_S = \mathbf{W}_R \tag{89}$$

$$\gamma_k = \frac{1}{l}\operatorname{arccosh}(\tilde{a}_k) \tag{90}$$

$$\tilde{Y}_{wk}^{(i)} = \tilde{Y}_{wk}^{(r)} = \frac{\sinh(\gamma_k l)}{\tilde{b}_k} = \frac{\tilde{c}_k}{\sinh(\gamma_k l)} \tag{91}$$

## 8. Conclusion

In this paper we carried out a comprehensive review of the fundamentals of frequency-domain multimodal analysis for multiconductor transmission-line structures, which are of interest to various subareas of high-frequency electrical engineering, namely, power line transients and power line communications, crosstalk and electromagnetic compatibility problems, electronic and microelectronic design and packaging, as well as RF and microwaves.

The following important aspects were addressed: the concept of propagation mode; the theory of modal analysis for uniform MTL and its possibility of failure; the transmission matrix formalism; and the extension of modal analysis to nonuniform MTL.

## Acknowledgements

The Fundação para a Ciência e a Tecnologia (Project PEst-OE/EEI/LA0008/2013), and the Instituto de Telecomunicações (Project PC226), sponsored this work.

## References


[1] Fallou JA. Propagation des courants de haute fréquence polyphase le long des lignes aériennes de transport d'energie affectées de court-circuits au de defaults d'isolement [Propagation of high-frequency polyphase currents in overhead power lines subjected to short-circuits or insulation failures]. [PhD Thesis]. Malakoff: University of Paris; 1932.

[2] Pipes LA. Matrix theory of multiconductor transmission lines. Phil. Mag. Soc. 1937;24:97–113.

[3] Pipes LA. Steady-state analysis of multiconductor transmission lines. J. Appl. Phys. 1941;12:782–789.

[4] Rice SO. Steady state solutions of transmission line equations. Bell Syst. Tech. J. 1941;20:131–178.

[5] Wedepohl LM. Application of matrix methods to the solution of traveling-wave phenomena in polyphase systems. Proc. IEE. 1963;110:2200–2212.

[6] Galloway RH, Shorrocks WB, Wedepohl LM. Calculation of electrical parameters for short and long polyphase transmission lines. Proc. IEE. 1964;111:2051–2059.

[7] Wedepohl LM. Electrical characteristics of polyphase transmission systems with special reference to boundary-value calculations at power-line carrier frequencies. Proc. IEE. 1965;112:2103–2112.

[8] Wedepohl LM, Wasley RG. Wave propagation in polyphase transmission systems: resonance effects due to discretely bonded earth wires. Proc. IEE. 1965;112:2113–2119.

[9] Wedepohl LM. Wave propagation in nonhomogeneous multiconductor systems using the concept of natural modes. Proc. IEE. 1966;113:622–626.

[10] Wedepohl LM, Wasley RG. Propagation of carrier signals in homogeneous, nonhomogeneous and mixed multiconductor systems. Proc. IEE. 1968;115:280–287.

[11] Wedepohl LM, Mohamed SE. Multiconductor transmission lines: theory of natural modes and Fourier integral applied to transient analysis. Proc. IEE. 1969;116:1553–1563.

[12] Wedepohl LM, Saha J. Radio-interference fields in multiconductor overhead transmission lines. Proc. IEE. 1969;116:1875–1884.

[13] Wedepohl LM, Mohamed SE. Transient analysis of multiconductor transmission lines with special reference to non-linear problems. Proc. IEE. 1970;117:979–988.

[14] Wedepohl LM, Indulkar CS. Wave propagation in nonhomogeneous systems: properties of the chain matrix. Proc. IEE. 1974;121:997–1000.

[15] Wedepohl LM, Ephtymiadis AE. Wave propagation in transmission lines over lossy ground: a new complete field solution. Proc. IEE. 1978;125:505–510.

[16] Kuznetzov PI, Statonovich RL. The propagation of electromagnetic waves in multiconductor transmission lines. Oxford: Pergamon Press; 1964.

[17] Frankel S. Multi-conductor transmission line analysis. Norwood (MA): Artech House; 1977.

[18] Djordjevic AR, Harrington RF, Sarkar TK, Bazdar M. Matrix parameters for multiconductor transmission lines: software and user's manual. Norwood (MA): Artech House; 1989.

[19] Brandão Faria JA. Multiconductor transmission-line structures: modal analysis techniques. New York (NY): Wiley; 1993.

[20] Paul CR. Analysis of multiconductor transmission lines. New York (NY): Wiley; 1994.

[21] Gantmacher FR. The theory of matrices. Vol. I. Providence (RI): AMS Chelsea Pub; 1977.

[22] Brandão Faria JA, Guerreiro Neves M. Basic principles concerning the experimental evaluation of the frequency-dependent parameters of shielded and unshielded three-phase symmetric cables. IEEE Trans. Power Del. 2011;26:556–564.

[23] Brandão Faria JA. On the transmission matrix of 2n-port reciprocal networks. Microwave Opt. Tech. Lett. 2002;33:151–154.

[24] Brandão Faria JA. Overhead three-phase transmission lines: non-diagonalizable situations. IEEE Trans. Power Del. 1988;3:1348–1355.

[25] Brandão Faria JA. Wave propagation in weakly coupled microstrip lines: an anomalous degenerate quasi-tem solution. Microwave Opt. Tech. Lett. 1989;2:149–152.

[26] Brandão Faria JA, Borges da Silva JF. Wave propagation in polyphase transmission lines: a general solution to include cases where ordinary modal theory fails. IEEE Trans. Power Del. 1986;1:182–189.

[27] Brandão Faria JA. On the segmentation method used for analyzing nonuniform transmission lines: application to the exponential line. Europ. Trans. Elect. Power. 2002;12:361–368.

[28] Semlyen A. Some frequency domain aspects of wave propagation on nonuniform lines. IEEE Trans. Power Del. 2003;18:315–322.
[29] Brandão Faria JA. Multiconductor transmission lines. In: Chang K, editor. Encyclopedia of RF and microwave engineering. Vol. 4. New York (NY): Wiley; 2005. p. 3335–3341.
[30] Brandão Faria JA. A new generalized modal analysis theory for nonuniform multiconductor transmission lines. IEEE Trans. Power Syst. 2004;19:926–933.