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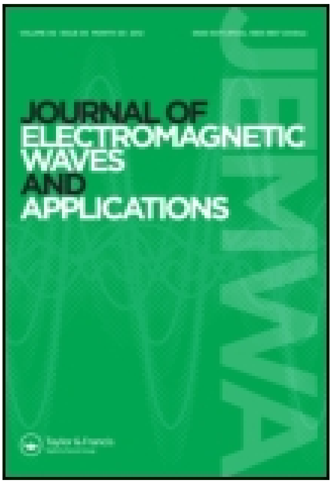

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Matrix theory of wave propagation in hybrid electric/magnetic multiwire transmission line systems

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Matrix methods to deal with multimodal propagation in electric multiwire transmission lines are quite well known. However, the matrix theory of magnetic multiwire transmission lines (MMTL) has been established only very recently. In this study, we advance the new subject of hybrid electric and MMTL, which has never been addressed in the literature. A complex system composed of $N+1$ electric wires and $n+1$ magnetic wires is theoretically analyzed from the viewpoint of their interaction/coupling, transmission line matrix equations, ABCD matrix formalism, and superposition of propagation modes.

Keywords: electric transmission lines; magnetic transmission lines; matrix methods; multimodal propagation; multiwire systems; wave propagation

1. Introduction

Multimodal propagation in electric multiwire transmission lines (EMTL) is an electrical engineering theme that pervades many different areas concerned with high-frequency regimes, namely power line transients and power line communications, cross talk and electromagnetic compatibility problems, electronic and microelectronic design and packaging, as well as RF and microwave engineering.

EMTLs have been an important research theme for many decades. Pioneering work on the matrix theory of EMTL structures can be traced back to as late as the 1930s.[1,2] Later, after powerful computers and dedicated software tools became available, the theme grew in attention, until today.[3–10]

The situation is radically different as far as magnetic multiwire transmission lines (MMTL) are concerned – a novel research topic. In fact, the theory, the fabrication, the technology, and the applications of magnetic transmission lines (TLs) are, at this stage, an open area of investigation, where information is not abundant. The theoretical background for two-wire magnetic TL analysis has been set in [11]. An application of magnetic TL theory to the analysis of ideal transformers has been provided in [12]. A comparison between the expected performance of electric and magnetic parallel strip TLs has been offered in [13]. A reformulation of magnetic TL theory using magnetic charges, magnetic currents, and magnetic voltages has been set in [14]. A generalization of two-wire magnetic TL theory to multiwire systems has been presented in [15].

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In this work, we address, for the first time, the problem of a hybrid multiwire TL system where an EMTL (of order N) is coupled to an MMTL (of order n). The problem is dealt within the frequency domain by resorting to matrix techniques, involving not only common eigenequations but also Sylvester equations.

This study is organized into five sections, the first of which is introductory. Section 2 is dedicated to background material on EMTL and MMTL. The matrix formalism necessary to capture the coupling mechanisms between the MMTL and the EMTL (and vice versa) is presented in Section 3, where rectangular matrices of electromagnetic (and magnetoelectric) impedances and admittances, per unit length, are introduced. Section 4 is devoted to modal analysis, where the tools required for the determination of the modal wave parameters, modal transformations, and modal coupling matrix are developed. At last, conclusions are outlined in Section 5.

2. Background theory

The frequency-domain theory of EMTL has been established for a long time; conversely, the dual theory of MMTL is quite recent. An outline of both theories, employing modal analysis techniques, is offered in this section.

The quasi-TEM approach is utilized in both theories; therefore, we assume that the longitudinal component of the electric field (arising from electric currents in the imperfect electric wires) and the longitudinal component of the magnetic field (arising from magnetic currents in the imperfect magnetic wires) are, both, negligibly small compared to the transverse components. In addition, the lateral distances among wires are assumed much smaller than 1/4 of the operating wavelength.

2.1. Electric multiwire transmission lines

Consider a uniform system of $N+1$ electric wires of high conductivity, immersed in a dielectric medium with permittivity ε_D. The wires run parallel to the longitudinal z-axis, one of them (with label 0) being the reference wire where the scalar electric potential is arbitrarily set to zero.

An array of N electric voltages, an array of N electric currents, and an array of N electric charges per unit length (pul) are defined, in phasor form,

$$\begin{cases} \mathbf{V}_E(z) = [V_{E_1}(z)\cdots V_{E_k}(z)\cdots V_{E_N}(z)]^{\mathrm{t}} \\ \mathbf{I}_E(z) = [I_{E_1}(z)\cdots I_{E_k}(z)\cdots I_{E_N}(z)]^{\mathrm{t}} \\ \mathbf{Q}_E(z) = [Q_{E_1}(z)\cdots Q_{E_k}(z)\cdots Q_{E_N}(z)]^{\mathrm{t}} \end{cases} \tag{1}$$

where superscript “t” denotes transposition, and

$$\begin{cases} V_{E_k}(z) = \int\limits_{\overrightarrow{k0}} \mathbf{E}_E(z)\cdot \mathrm{d}\mathbf{s} \\ I_{E_k}(z) = \int\limits_{A_k} \mathbf{J}^L_{E_k}(z)\cdot \hat{\mathbf{z}}\,\mathrm{d}A = \oint\limits_{\mathbf{s}_k} \mathbf{H}_E(z)\cdot \mathrm{d}\mathbf{s} \\ Q_{E_k} = \lim\limits_{\Delta z\to 0} \frac{\varepsilon_D}{\Delta z}\int\limits_{S_k} \mathbf{E}_E(z)\cdot \hat{\mathbf{n}}_o\,\mathrm{d}S \end{cases} \tag{2}$$

where $\mathbf{E}_E$ and $\mathbf{H}_E$ are the transverse electric and transverse magnetic fields of the *E*lectric MTL; $\mathbf{J}^L_{E_k}$ is the longitudinal electric current density inside wire-k; $\overrightarrow{k0}$ is an open path from wire-k to wire-0 belonging to the transverse plane; $\mathbf{s}_k$ is a clockwise-oriented closed path encircling wire-k belonging to the transverse plane; A_k is the cross-sectional area of wire-k; $\hat{\mathbf{z}}$ is the z-directed unit vector; and S_k is a closed surface around wire-k with axial length Δz, with outer normal $\hat{\mathbf{n}}_o$. Note: Throughout the text, hat symbols (^) are used to denote unit vectors.

The z-dependence of EMTL voltages and currents is described by the following set of coupled equations

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{bmatrix}\mathbf{V}_E\\ \mathbf{I}_E\end{bmatrix} = -\mathbf{M}_E\begin{bmatrix}\mathbf{V}_E\\ \mathbf{I}_E\end{bmatrix}, \quad \mathbf{M}_E = \begin{bmatrix}\mathbf{0} & \mathbf{Z}_E^L\\ \mathbf{Y}_E^T & \mathbf{0}\end{bmatrix} \tag{3}$$

where $\mathbf{0}$ is the zero matrix of order N, and $\mathbf{Z}_E^L$ and $\mathbf{Y}_E^T$ are $N \times N$ complex symmetric frequency-dependent matrices: the pul longitudinal impedance and the pul transverse admittance, respectively.

Note: Subscript E is utilized to refer to the *E*lectric MTL; superscripts L and T are used as a reminder for *L*ongitudinal and *T*ransverse quantities.

For uniform multiwire systems, the $\mathbf{M}_E$ matrix is z-invariant; therefore, the solution of the first-order constant-coefficient differential equation in (3) is an exponential matrix function:

$$\begin{bmatrix}\mathbf{V}_E(z)\\ \mathbf{I}_E(z)\end{bmatrix} = \mathrm{EXP}(-\mathbf{M}_E z)\begin{bmatrix}\mathbf{V}_E(0)\\ \mathbf{I}_E(0)\end{bmatrix} \tag{4}$$

To determine $\mathrm{EXP}(-\mathbf{M}_E z)$, we should firstly find the eigenvalues and eigenvectors of $\mathbf{M}_E$, that is, we have to solve

$$\mathbf{M}_E\mathbf{P}_E = \mathbf{P}_E\boldsymbol{\eta}_E \tag{5}$$

where the entries of the diagonal matrix $\boldsymbol{\eta}_E$ are the eigenvalues, and the columns of $\mathbf{P}_E$ are the corresponding eigenvectors. Writing $\mathbf{P}_E$ and $\boldsymbol{\eta}_E$ in block partitioned form

$$\mathbf{P}_E = \begin{bmatrix}\mathbf{T}_E & -\mathbf{Z}_E^w\mathbf{W}_E\\ \mathbf{Y}_E^w\mathbf{T}_E & \mathbf{W}_E\end{bmatrix}, \quad \boldsymbol{\eta}_E = \begin{bmatrix}\gamma_E & \mathbf{0}\\ \mathbf{0} & -\zeta_E\end{bmatrix} \tag{6}$$

substituting (6) into (5), and taking (3) into account, leads to the following important results:

$$\mathbf{T}_E^{-1}\left(\mathbf{Z}_E^L\mathbf{Y}_E^T\right)\mathbf{T}_E = \gamma_E^2, \quad \mathbf{W}_E^{-1}\left(\mathbf{Y}_E^T\mathbf{Z}_E^L\right)\mathbf{W}_E = \zeta_E^2 \tag{7a}$$

$$\mathbf{Y}_E^w = \mathbf{Y}_E^T\mathbf{T}_E\gamma_E^{-1}\mathbf{T}_E^{-1}; \quad \mathbf{Z}_E^w = \mathbf{Z}_E^L\mathbf{W}_E\zeta_E^{-1}\mathbf{W}_E^{-1} \tag{7b}$$

From (7a), we learn that $\mathbf{T}_E$ and $\mathbf{W}_E$ are transformation matrices that, respectively, bring $\left(\mathbf{Z}_E^L\mathbf{Y}_E^T\right)$ and $\left(\mathbf{Y}_E^T\mathbf{Z}_E^L\right)$ into diagonal form, and since these matrix products are the transposed of each other, it follows that

$$\zeta_E = \gamma_E; \quad \mathbf{T}_E^{-1} = \mathbf{W}_E^{\mathrm{t}} \tag{8}$$

From (7b), bearing in mind that $\mathbf{Z}_E^L$ and $\mathbf{Y}_E^T$ are symmetric, using (8), we get the characteristic *w*ave admittance/impedance matrices of the electric MTL:

$$\mathbf{Y}_E^w = (\mathbf{Y}_E^w)^{\mathrm{t}}; \quad \mathbf{Z}_E^w = (\mathbf{Z}_E^w)^{\mathrm{t}}; \quad \mathbf{Z}_E^w = (\mathbf{Y}_E^w)^{-1} \tag{9}$$

In the case of matched lines, where reflected waves are absent, one has $\mathbf{I}_E(z) = \mathbf{Y}_E^w\mathbf{V}_E(z)$ or $\mathbf{V}_E(z) = \mathbf{Z}_E^w\mathbf{I}_E(z)$.

At last, the exponential matrix $\mathrm{EXP}(-\mathbf{M}_E z)$, in (4), is evaluated through

$$\mathrm{EXP}(-\mathbf{M}_E z) = \mathbf{P}_E \exp(-\boldsymbol{\eta}_E z)\mathbf{P}_E^{-1} = \begin{bmatrix} \mathbf{A}_E & \mathbf{B}_E \\ \mathbf{C}_E & \mathbf{D}_E \end{bmatrix} \tag{10}$$

where $\exp(-\boldsymbol{\eta}_E z)$ is diagonal, and the **ABCD** matrix is

$$\begin{cases} \mathbf{A}_E = \mathbf{T}_E \cosh(\gamma_E z)\mathbf{T}_E^{-1} \\ \mathbf{B}_E = \mathbf{B}_E^{\mathrm{t}} = -\mathbf{T}_E \sinh(\gamma_E z)\mathbf{T}_E^{-1}\mathbf{Z}_E^w \\ \mathbf{C}_E = \mathbf{C}_E^{\mathrm{t}} = -\mathbf{Y}_E^w \mathbf{T}_E \sinh(\gamma_E z)\mathbf{T}_E^{-1} \\ \mathbf{D}_E = \mathbf{Y}_E^w \mathbf{T}_E \cosh(\gamma_E z)\mathbf{T}_E^{-1}\mathbf{Z}_E^w = \mathbf{A}_E^{\mathrm{t}} \end{cases} \tag{11}$$

and, due to reciprocity, $\det(\mathrm{EXP}(-\mathbf{M}_E z)) = 1$.

2.2. *Magnetic multiwire transmission lines*

Since the research subject of magnetic TL is very recent, a convenient introduction to the subject is briefly presented.

In a conventional electric TL, the wires are made of a nonmagnetic material of very high conductivity (e.g. copper). Longitudinal electric currents I_E are carried by wires, electric voltages existing among wires. In the transverse plane (z = constant), the electric field $\mathbf{E}_E$ is a gradient field characterized by open field lines starting and ending on different wires, and the magnetic field $\mathbf{H}_E$ is a solenoidal field characterized by closed field lines embracing the current carrying wires; the flux of energy given by the complex Poynting vector $\bar{\boldsymbol{\Sigma}} = \frac{1}{2}\mathbf{E}_E \times \mathbf{H}_E^*$ is z-directed. From the viewpoint of transmission line equations, the two key parameters for propagation analysis are the pul longitudinal impedance and the pul transverse admittance.

Contrarily, in a magnetic TL, the wires are made of a nonconducting material of very high magnetic permeability (e.g. ferrite). The wires carry longitudinal magnetic induction fluxes ϕ_M (whose time derivative is identified as magnetic currents), magnetic voltages existing among wires. In the transverse plane (z = constant), the electric field $\mathbf{E}_H$ is a solenoidal field characterized by closed field lines embracing the flux carrying wires, and the magnetic field $\mathbf{H}_M$ is a gradient field characterized by open field lines starting and ending on different wires; the flux of energy given by the complex Poynting vector $\bar{\boldsymbol{\Sigma}} = \frac{1}{2}\mathbf{E}_M \times \mathbf{H}_M^*$ is z-directed. From the viewpoint of transmission line equations, the two key parameters for propagation analysis are the pul transverse impedance and the pul longitudinal admittance.

Owing to duality between magnetic and electric TLs of the same geometry,[15,16] the following equivalence among transverse fields is verified: $\mathbf{H}_E \leftrightarrow -\mathbf{E}_M, \quad \mathbf{E}_E \leftrightarrow +\mathbf{H}_M$.

Consider a uniform system of $n + 1$ magnetic wires of high permeability, immersed in a dielectric medium with permeability μ_D. The wires run parallel to the longitudinal z-axis, one of them (with label 0) being the reference wire where the scalar magnetic potential is arbitrarily set to zero.

An array of n magnetic induction fluxes, an array of n magnetic currents, an array of n magnetic voltages, and an array of n magnetic charges per unit length are defined

$$\begin{cases} \boldsymbol{\phi}_M(z) = \left[\phi_{M_1}(z)\cdots\phi_{M_k}(z)\cdots\phi_{M_N}(z)\right]^{\mathrm{t}} \\ \mathbf{I}_M(z) = [I_{M_1}(z)\cdots I_{M_k}(z)\cdots I_{M_N}(z)]^{\mathrm{t}} \\ \mathbf{V}_M(z) = [V_{M_1}(z)\cdots V_{M_k}(z)\cdots V_{M_N}(z)]^{\mathrm{t}} \\ \mathbf{Q}_M(z) = [Q_{M_1}(z)\cdots Q_{M_k}(z)\cdots Q_{M_N}(z)]^{\mathrm{t}} \end{cases} \tag{12}$$

The magnetic induction flux (in weber), the magnetic current (in volt), the magnetic voltage (in ampere), and the pul magnetic charge (in weber per meter) of wire-k are defined,[14] through

$$\begin{cases} \phi_{M_k}(z) = \int\limits_{A_k} \mathbf{B}_k^L(z) \cdot \hat{\mathbf{z}}\, \mathrm{d}A \\ I_{M_k}(z) = j\omega\phi_{M_k}(z) = \int\limits_{A_k} \mathbf{J}_{M_k}^L(z) \cdot \hat{\mathbf{z}}\, \mathrm{d}A = \oint\limits_{\mathbf{s}_k} \mathbf{E}_M(z) \cdot \mathrm{d}\mathbf{s} \\ V_{M_k}(z) = \int\limits_{\overrightarrow{k0}} \mathbf{H}_M(z) \cdot \mathrm{d}\mathbf{s} \\ Q_{M_k}(z) = \lim\limits_{\Delta z \to 0} \frac{\mu_D}{\Delta z} \int\limits_{S_k} \mathbf{H}_M(z) \cdot \hat{\mathbf{n}}_o\, \mathrm{d}S \end{cases} \tag{13}$$

where $\mathbf{E}_M$ and $\mathbf{H}_M$ are the transverse electric and transverse magnetic fields of the *M*agnetic MTL; $\mathbf{B}_k^L$ is the longitudinal magnetic flux density (magnetic induction field) inside wire-k; $\mathbf{J}_{M_k}^L$ is the longitudinal magnetic current density inside wire-k; $\overrightarrow{k0}$ is an open path from wire-k to wire-0 belonging to the transverse plane; $\mathbf{s}_k$ is an anticlockwise-oriented closed path encircling wire-k belonging to the transverse plane; A_k is the cross-sectional area of wire-k, and S_k is a closed surface around wire-k with axial length Δz.

Note: Subscript M is utilized to refer to the *M*agnetic MTL.

The z-dependence of MMTL voltages and currents is described by the following set of coupled equations,[15]

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{bmatrix} \mathbf{I}_M \\ \mathbf{V}_M \end{bmatrix} = -\mathbf{M}_M \begin{bmatrix} \mathbf{I}_M \\ \mathbf{V}_M \end{bmatrix}, \quad \mathbf{M}_M = \begin{bmatrix} \mathbf{0} & \mathbf{Z}_M^T \\ \mathbf{Y}_M^L & \mathbf{0} \end{bmatrix} \tag{14}$$

where $\mathbf{Z}_M^T$ and $\mathbf{Y}_M^L$ are $n \times n$ complex symmetric frequency-dependent matrices: the pul *T*ransverse impedance and the pul *L*ongitudinal admittance, respectively

For uniform multiwire systems, the solution of the first-order constant-coefficient differential equation in (14) is an exponential matrix function:

$$\begin{bmatrix} \mathbf{I}_M(z) \\ \mathbf{V}_M(z) \end{bmatrix} = \mathrm{EXP}(-\mathbf{M}_M z) \begin{bmatrix} \mathbf{I}_M(0) \\ \mathbf{V}_M(0) \end{bmatrix} \tag{15}$$

The procedure to determine $\mathrm{EXP}(-\mathbf{M}_M z)$ follows the same steps as described in Section 2.1, where $\mathrm{EXP}(-\mathbf{M}_E z)$ was evaluated, that is

$$\mathrm{EXP}(-\mathbf{M}_M z) = \mathbf{P}_M \exp(-\boldsymbol{\eta}_M z)\mathbf{P}_M^{-1} = \begin{bmatrix} \mathbf{A}_M & \mathbf{B}_M \\ \mathbf{C}_M & \mathbf{D}_M \end{bmatrix} \tag{16a}$$

$$\begin{cases} \mathbf{A}_M = \mathbf{T}_M \cosh(\gamma_M z)\mathbf{T}_M^{-1} \\ \mathbf{B}_M = \mathbf{B}_M^{\mathrm{t}} = -\mathbf{T}_M \sinh(\gamma_M z)\mathbf{T}_M^{-1}\mathbf{Z}_M^w \\ \mathbf{C}_M = \mathbf{C}_M^{\mathrm{t}} = -\mathbf{Y}_M^w \mathbf{T}_M \sinh(\gamma_M z)\mathbf{T}_M^{-1} \\ \mathbf{D}_M = \mathbf{Y}_M^w \mathbf{T}_M \cosh(\gamma_M z)\mathbf{T}_M^{-1}\mathbf{Z}_M^w = \mathbf{A}_M^{\mathrm{t}} \end{cases} \tag{16b}$$

where

$$\mathbf{M}_M\mathbf{P}_M = \mathbf{P}_M\boldsymbol{\eta}_M, \quad \begin{cases} \mathbf{P}_M = \begin{bmatrix} \mathbf{T}_M & -\mathbf{Z}_M^w\mathbf{W}_M \\ \mathbf{Y}_M^w\mathbf{T}_M & \mathbf{W}_M \end{bmatrix} \\ \boldsymbol{\eta}_M = \begin{bmatrix} \gamma_M & \mathbf{0} \\ \mathbf{0} & -\gamma_M \end{bmatrix} \end{cases} \tag{17}$$

$$\mathbf{T}_M^{-1}\left(\mathbf{Z}_M^T\mathbf{Y}_M^L\right)\mathbf{T}_M = \mathbf{W}_M^{-1}\left(\mathbf{Y}_M^L\mathbf{Z}_M^T\right)\mathbf{W}_M = \gamma_M^2, \quad \mathbf{W}_M\mathbf{T}_M^{\mathrm{t}} = \mathbf{1} \tag{18}$$

$$\mathbf{Y}_M^w = \mathbf{Y}_M^L\mathbf{T}_M\gamma_M^{-1}\mathbf{T}_M^{-1}, \quad \mathbf{Z}_M^w = \left(\mathbf{Y}_M^w\right)^{-1} \tag{19}$$

where $\mathbf{Z}_M^w$ and $\mathbf{Y}_M^w$ are the characteristic *w*ave impedance/admittance matrices of the magnetic MTL.

3. Hybrid EMTL–MMTL system

Here, we consider a hybrid transmission line system where two parallel multiwire TL subsystems interact: an EMTL of order N and an MMTL of order n; coupling between the two subsystems being analyzed.

3.1. Transverse fields and definition of voltages

To begin with, we underline that the two subsystems are "current-isolated" in the sense that in any transverse plane, the total electric current in the EMTL and the total magnetic current in the MMTL are both zero, no matter the degree of coupling between the subsystems.

$$\nabla \cdot \mathbf{J} = 0 \to \sum_{k=0}^{N} I_{E_k} = 0, \quad \nabla \cdot \mathbf{B} = 0 \to \sum_{k=0}^{n} I_{M_k} = 0 \tag{20}$$

When the EMTL and MMTL subsystems are considered together, the resulting transverse electric and transverse magnetic fields are combinations of gradient and solenoidal fields:

$$\mathbf{E} = \mathbf{E}_E + \mathbf{E}_M, \begin{cases} \mathbf{E}_E = -\nabla\Phi_E \\ \nabla \times \mathbf{E}_M = -\mathbf{J}_M^L \end{cases}; \quad \mathbf{H} = \mathbf{H}_M + \mathbf{H}_E, \begin{cases} \mathbf{H}_M = -\nabla\Phi_M \\ \nabla \times \mathbf{H}_E = \mathbf{J}_E^L \end{cases} \tag{21}$$

where Φ_E, Φ_M are the scalar electric and scalar magnetic potentials, and $\mathbf{J}_E^L$, $\mathbf{J}_M^L$ are the longitudinal electric and magnetic current densities inside wires (with $\mathbf{J}_E^L = \sigma\mathbf{E}_z$, $\mathbf{J}_M^L = j\omega\mathbf{B}_z$).

Since $\mathbf{E}$ and $\mathbf{H}$ are not purely gradient fields, the definitions of electric voltage and magnetic voltage must necessarily include explicit information on the integration paths involved. The integration paths defined in the xy transverse plane must remain z-invariant ($z = 0$, included). In other words, the way how the voltage sources placed at the sending end ($z = 0$) are connected to the wires in the xy plane determine the integration paths required for the definition of the electric and magnetic voltage along z.

To clarify this delicate aspect, let us consider the simple example of a hybrid system made of a two-wire electric TL and a two-wire magnetic TL in Figure 1.

Figure 1(a) shows the position of the driving sources at $z = 0$; the electric voltage source $V_E = V_E(0)$ is placed at $y = y_1$ and $x_1 < x < x_2$; and the magnetic voltage source $V_M = V_M(0)$ is placed at $x = x_1$ and $y_2 < y < y_3$.

Figure 1(b) shows several integration paths for the evaluation of the electric voltage V_E for any z; all the paths start at point $a(x_1, y_1)$ and end at point $b(x_2, y_1)$. According to the explanation above, the electric voltage should be defined as the line integral of $\mathbf{E}$ from a to b using the rectilinear path $\mathbf{s}$:

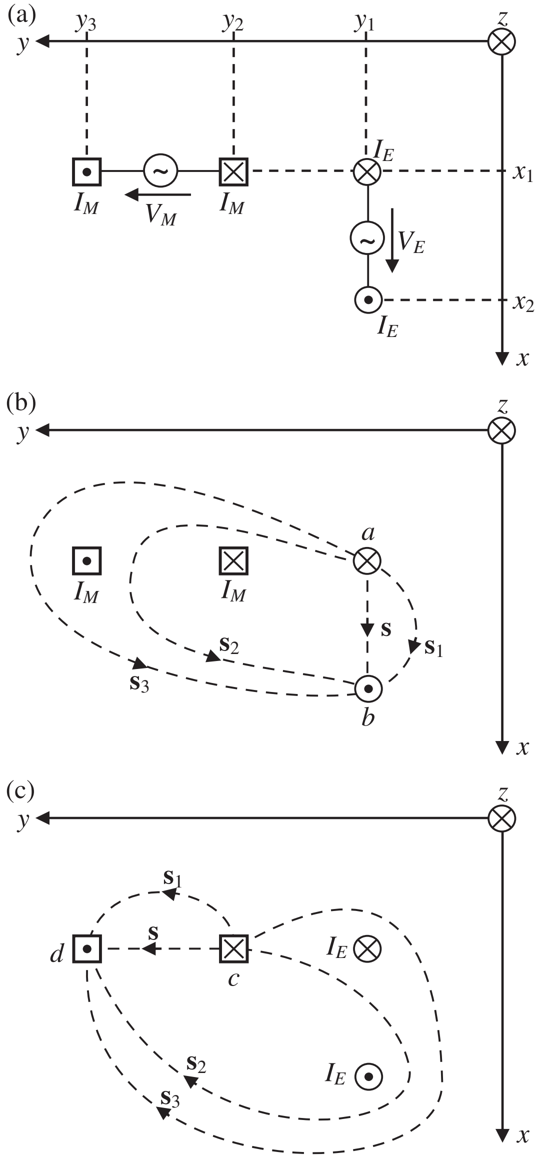

Figure 1. Definition of electric and magnetic voltages in a hybrid MTL made of a two-wire electric TL (circles) and a two-wire magnetic TL (squares). The position of the driving voltage sources at $z = 0$ is shown in (a). Possible integration paths for the definition of V_E and V_{M} are shown in (b) and (c).

$$V_E = \int\limits_{\substack{ab \\ \text{path}\,\mathbf{s}}} \mathbf{E} \cdot \mathrm{d}\mathbf{s} \tag{22}$$

Now, consider other possible options using the paths $\mathbf{s}_1$, $\mathbf{s}_2$, or $\mathbf{s}_3$, giving rise to the definition

$$V_E^{(k)} = \int\limits_{\substack{ab \\ \text{path } \mathbf{s}_k}} \mathbf{E} \cdot d\mathbf{s} \tag{23}$$

The circulation of **E** along the closed paths $\mathbf{s}_1 \cup \mathbf{s}$ or $\mathbf{s}_3 \cup \mathbf{s}$ also leads to $V_E^{(1)} = V_E$ and $V_E^{(3)} = V_E$. In fact,

$$\begin{cases} \oint\limits_{\mathbf{s}_1 \cup \mathbf{s}} \mathbf{E} \cdot \mathrm{d}\mathbf{s} = -j\omega \int\limits_S \mathbf{B} \cdot \hat{\mathbf{z}}\, dS \rightarrow +V_E^{(1)} - V_E = 0 \rightarrow V_E^{(1)} = V_E \\ \oint\limits_{\mathbf{s}_3 \cup \mathbf{s}} \mathbf{E} \cdot \mathrm{d}\mathbf{s} = j\omega \int\limits_S \mathbf{B} \cdot \hat{\mathbf{z}}\, dS \rightarrow +V_E^{(3)} - V_E = I_M - I_M \rightarrow V_E^{(3)} = V_E \end{cases} \tag{24}$$

However, choosing $\mathbf{s}_2$ would be a wrong choice. In fact, the circulation of **E** along the closed path $\mathbf{s}_2 \cup \mathbf{s}$ would result in:

$$\oint\limits_{\mathbf{s}_2 \cup \mathbf{s}} \mathbf{E} \cdot \mathrm{d}\mathbf{s} = j\omega \int\limits_S \mathbf{B} \cdot \hat{\mathbf{z}}\, dS \rightarrow +V_E^{(2)} - V_E = I_M \rightarrow V_E^{(2)} = V_E + I_M \neq V_E \tag{25}$$

The same considerations apply to the definition of the magnetic voltage.

Figure 1(c) shows several integration paths for the evaluation of the magnetic voltage V_M for any z; all the paths start at point $c(x_1, y_2)$ and end at point $d(x_1, y_3)$. The magnetic voltage should be defined as the line integral of **H** from c to d using the rectilinear path **s**:

$$V_M = \int\limits_{\substack{cd \\ \text{path } \mathbf{s}}} \mathbf{H} \cdot \mathrm{d}\mathbf{s} \tag{26}$$

Now, consider other possible options using the paths $\mathbf{s}_1$, $\mathbf{s}_2$, or $\mathbf{s}_3$ and evaluate the circulation of **H** along the closed paths $\mathbf{s}_1 \cup \mathbf{s}$, $\mathbf{s}_3 \cup \mathbf{s}$ or $\mathbf{s}_2 \cup \mathbf{s}$. Using $\mathbf{s}_1$ or $\mathbf{s}_3$ we get,

$$\begin{cases} \oint\limits_{\mathbf{s}_1 \cup \mathbf{s}} \mathbf{H} \cdot \mathrm{d}\mathbf{s} = -\int\limits_S \mathbf{J}_E \cdot \hat{\mathbf{z}}\, dS \rightarrow +V_M^{(1)} - V_M = 0 \rightarrow V_M^{(1)} = V_M \\ \oint\limits_{\mathbf{s}_3 \cup \mathbf{s}} \mathbf{H} \cdot \mathrm{d}\mathbf{s} = \int\limits_S \mathbf{J}_E \cdot \hat{\mathbf{z}}\, dS \rightarrow +V_M^{(3)} - V_M = I_E - I_E \rightarrow V_M^{(3)} = V_M \end{cases} \tag{27}$$

Choosing $\mathbf{s}_2$ is a wrong choice. In fact, the circulation of **H** along the closed path $\mathbf{s}_2 \cup \mathbf{s}$ would give

$$\oint\limits_{\mathbf{s}_2 \cup \mathbf{s}} \mathbf{H} \cdot \mathrm{d}\mathbf{s} = \int\limits_S \mathbf{J}_E \cdot \hat{\mathbf{z}}\, dS \rightarrow +V_M^{(2)} - V_M = -I_E \rightarrow V_M^{(2)} = V_M - I_E \neq V_M \tag{28}$$

3.2. MMTL coupling to EMTL

In a pure EMTL system, the existing transverse magnetic field is originated only by electric currents; the link between the z-derivative of EMTL voltages and EMTL currents being established via the pul electric longitudinal impedance matrix $\mathbf{Z}_E^L$ (an $N \times N$ matrix). However, when a neighboring MMTL exists, an additional transverse magnetic field related to magnetic voltages is also present. The link between the z-derivative of EMTL voltages and MMTL voltages being established via the new concept of pul electromagnetic impedance matrix $\mathbf{Z}_{EM}^L$ (an $N \times n$ matrix), that is:

$$\frac{d}{dz}\mathbf{V}_E = -(\mathbf{Z}_E^L\mathbf{I}_E + \underbrace{\mathbf{Z}_{EM}^L\mathbf{V}_M}_{\text{Coupling}}) \tag{29}$$

In a pure EMTL system, a transverse electric field related to electric charges is present; the link between the z-derivative of EMTL currents and EMTL voltages being established via the pul electric transverse admittance matrix $\mathbf{Y}_E^T$ (an $N \times N$ matrix). However, when a neighboring MMTL exists, an additional electric field originated by magnetic currents is also present. The link between the z-derivative of EMTL currents and MMTL currents being established via the new concept of pul electromagnetic admittance matrix $\mathbf{Y}_{EM}^T$ (an $N \times n$ matrix), that is

$$\frac{d}{dz}\mathbf{I}_E = -(\mathbf{Y}_E^T\mathbf{V}_E + \underbrace{\mathbf{Y}_{EM}^T\mathbf{I}_M}_{\text{Coupling}}) \tag{30}$$

3.3. EMTL coupling to MMTL

In a pure MMTL system, a transverse magnetic field related to magnetic charges is present; the link between the z-derivative of MMTL currents and MMTL voltages being established via the pul magnetic transverse impedance matrix $\mathbf{Z}_M^T$ (an $n \times n$ matrix). However, when a neighboring EMTL exists, an additional magnetic field originated by electric currents is also present. The link between the z-derivative of MMTL currents and EMTL currents being established via the new concept of pul magnetoelectric impedance matrix $\mathbf{Z}_{ME}^T$ (an $n \times N$ matrix), that is

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathbf{I}_M = -(\mathbf{Z}_M^T\mathbf{V}_M + \underbrace{\mathbf{Z}_{ME}^T\mathbf{I}_E}_{\text{Coupling}}) \tag{31}$$

In a pure MMTL system, the existing transverse electric field is originated only by magnetic currents; the link between the z-derivative of MMTL voltages and MMTL currents being established via the pul magnetic longitudinal admittance matrix $\mathbf{Y}_M^L$ (an $n \times n$ matrix). However, when a neighboring EMTL exists, an additional electric field related to electric charges is also present. The link between the z-derivative of MMTL voltages and EMTL voltages being established via the new concept of pul magnetoelectric admittance matrix $\mathbf{Y}_{ME}^L$ (an $n \times N$ matrix), that is

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathbf{V}_M = -(\mathbf{Y}_M^L\mathbf{I}_M + \underbrace{\mathbf{Y}_{ME}^L\mathbf{V}_E}_{\text{Coupling}}) \tag{32}$$

3.4. Global coupling analysis

The set of results in (29)–(32) can be compactly written as

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{bmatrix}\begin{bmatrix}\mathbf{V}_E\\ \mathbf{I}_M\end{bmatrix}\\ \begin{bmatrix}\mathbf{I}_E\\ \mathbf{V}_M\end{bmatrix}\end{bmatrix} = -\mathbf{M}\begin{bmatrix}\begin{bmatrix}\mathbf{V}_E\\ \mathbf{I}_M\end{bmatrix}\\ \begin{bmatrix}\mathbf{I}_E\\ \mathbf{V}_M\end{bmatrix}\end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix}\mathbf{0} & \mathbf{Z}\\ \mathbf{Y} & \mathbf{0}\end{bmatrix} \tag{33}$$

where submatrices $\mathbf{Z}$ and $\mathbf{Y}$, of order $N+n$, are

$$\mathbf{Z} = \begin{bmatrix}\mathbf{Z}_E^L & \mathbf{Z}_{EM}^L\\ \mathbf{Z}_{ME}^T & \mathbf{Z}_M^T\end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix}\mathbf{Y}_E^T & \mathbf{Y}_{EM}^T\\ \mathbf{Y}_{ME}^L & \mathbf{Y}_M^L\end{bmatrix} \tag{34}$$

Due to reciprocity, both submatrices $\mathbf{Z}$ and $\mathbf{Y}$ are necessarily symmetric, which implies that

$$\mathbf{Z}_{EM}^L = \left(\mathbf{Z}_{ME}^T\right)^{\mathrm{t}}, \quad \mathbf{Y}_{EM}^T = \left(\mathbf{Y}_{ME}^L\right)^{\mathrm{t}} \tag{35}$$

In the absence of losses, the $\mathbf{Z}$ and $\mathbf{Y}$ matrices are purely imaginary and can be compactly written as

$$\begin{cases} \mathbf{Z} = j\omega\mathbf{L}', \quad \mathbf{L}' = \begin{array}{c} \begin{array}{cc} \overset{N}{\longleftrightarrow} & \overset{n}{\longleftrightarrow} \end{array} \\ \begin{bmatrix}\mathbf{L}_E^L & \mathbf{L}_{EM}^L\\ \mathbf{L}_{ME}^T & \mathbf{L}_M^T\end{bmatrix} \end{array} \\ \mathbf{Y} = j\omega\mathbf{C}', \quad \mathbf{C}' = \begin{array}{c} \begin{array}{cc} \overset{N}{\longleftrightarrow} & \overset{n}{\longleftrightarrow} \end{array} \\ \begin{bmatrix}\mathbf{C}_E^T & \mathbf{C}_{EM}^T\\ \mathbf{C}_{ME}^L & \mathbf{C}_M^L\end{bmatrix} \end{array} \end{cases} \tag{36}$$

where the entries of the pul inductance and capacitance matrices are frequency independent.

If, in addition, the dielectric is a homogeneous medium, then $\mathbf{L}'$ and $\mathbf{C}'$ are such that

$$\mathbf{L}'\mathbf{C}' = \mathbf{C}'\mathbf{L}' = \mu_D\varepsilon_D\mathbf{1} = v_D^{-2}\mathbf{1} \tag{37}$$

where $\mathbf{1}$ is the identity matrix, and v_D is the intrinsic phase velocity of the dielectric medium.

The inclusion of dielectric losses can be accounted by replacing the permittivity ε_D by a complex permittivity $\bar{\varepsilon}_D = \varepsilon_D' - j\varepsilon_D''$, which affects the computation of $\mathbf{C}'$ which turns into a complex matrix $\bar{\mathbf{C}}'$. The inclusion of losses in both the electric and magnetic wires can be taken into account, in the computation of $\mathbf{Z}_E^L$ and $\mathbf{Y}_M^L$ matrices, using perturbation terms $\mathbf{Z}_E^{\text{skin}}$ $(N \times N)$ and $\mathbf{Y}_M^{\text{skin}}$ $(n \times n)$ evaluated from skin effect theory:

$$\mathbf{Z}_E^L = \mathbf{Z}_E^{\text{skin}} + j\omega\mathbf{L}_E^L; \quad \mathbf{Y}_M^L = \mathbf{Y}_M^{\text{skin}} + j\omega\bar{\mathbf{C}}_M^L \tag{38a}$$

yielding

$$\begin{cases} \mathbf{Z} = \begin{array}{c} \begin{array}{cc} \overset{N}{\longleftrightarrow} & \overset{n}{\longleftrightarrow} \end{array} \\ \begin{bmatrix}\mathbf{Z}_E^{\text{skin}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \end{array} + j\omega \begin{array}{c} \begin{array}{cc} \overset{N}{\longleftrightarrow} & \overset{n}{\longleftrightarrow} \end{array} \\ \begin{bmatrix}\mathbf{L}_E^L & \mathbf{L}_{EM}^L\\ \mathbf{L}_{ME}^T & \mathbf{L}_M^T\end{bmatrix} \end{array} \\ \mathbf{Y} = \begin{array}{c} \begin{array}{cc} \overset{N}{\longleftrightarrow} & \overset{n}{\longleftrightarrow} \end{array} \\ \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{Y}_M^{\text{skin}}\end{bmatrix} \end{array} + j\omega \begin{array}{c} \begin{array}{cc} \overset{N}{\longleftrightarrow} & \overset{n}{\longleftrightarrow} \end{array} \\ \begin{bmatrix}\bar{\mathbf{C}}_E^T & \bar{\mathbf{C}}_{EM}^T\\ \bar{\mathbf{C}}_{ME}^L & \bar{\mathbf{C}}_M^L\end{bmatrix} \end{array} \end{cases} \tag{38b}$$

If the distances among wires are much larger than wires' radius (thin wires' approximation), we can neglect proximity effects in the calculation of skin effect contributions. In that case, the structure of matrices $\mathbf{Z}_E^{\text{skin}}$ and $\mathbf{Y}_M^{\text{skin}}$ is

$$\begin{cases} \mathbf{Z}_E^{\text{skin}} = \operatorname{diag}\left(z_{E_1}^{\text{skin}} \cdots z_{E_k}^{\text{skin}} \cdots z_{E_N}^{\text{skin}}\right) + z_{E_0}^{\text{skin}}\mathbf{U} \\ \mathbf{Y}_M^{\text{skin}} = \operatorname{diag}\left(y_{M_1}^{\text{skin}} \cdots y_{M_k}^{\text{skin}} \cdots y_{M_N}^{\text{skin}}\right) + y_{M_0}^{\text{skin}}\mathbf{U} \end{cases} \tag{38c}$$

where $\mathbf{U}$ is the all-ones square matrix, and subsubscript 0 refers to the return wire (reference wire).

Like in (4) and (15), the solution of (33) is also described by a unimodular exponential matrix function

$$\begin{bmatrix} \begin{bmatrix} \mathbf{V}_E(z) \\ \mathbf{I}_M(z) \end{bmatrix} \\ \begin{bmatrix} \mathbf{I}_E(z) \\ \mathbf{V}_M(z) \end{bmatrix} \end{bmatrix} = \operatorname{EXP}(-\mathbf{M}z) \begin{bmatrix} \begin{bmatrix} \mathbf{V}_E(0) \\ \mathbf{I}_M(0) \end{bmatrix} \\ \begin{bmatrix} \mathbf{I}_E(0) \\ \mathbf{V}_M(0) \end{bmatrix} \end{bmatrix} \tag{39}$$

$$\operatorname{EXP}(-\mathbf{M}z) = \mathbf{P}\exp(-\boldsymbol{\eta}z)\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \tag{40a}$$

$$\begin{cases} \mathbf{A} = \mathbf{T}\cosh(\gamma z)\mathbf{T}^{-1} \\ \mathbf{B} = \mathbf{B}^{\text{t}} = -\mathbf{T}\sinh(\gamma z)\mathbf{T}^{-1}\mathbf{Z}^{w} \\ \mathbf{C} = \mathbf{C}^{\text{t}} = -\mathbf{Y}^{w}\mathbf{T}\sinh(\gamma z)\mathbf{T}^{-1} \\ \mathbf{D} = \mathbf{Y}^{w}\mathbf{T}\cosh(\gamma z)\mathbf{T}^{-1}\mathbf{Z}^{w} = \mathbf{A}^{\text{t}} \end{cases} \tag{40b}$$

where

$$\mathbf{MP} = \mathbf{P}\eta, \quad \begin{cases} \mathbf{P} = \begin{bmatrix} \mathbf{T} & -\mathbf{Z}^{w}\mathbf{W} \\ \mathbf{Y}^{w}\mathbf{T} & \mathbf{W} \end{bmatrix} \\ \boldsymbol{\eta} = \begin{bmatrix} \gamma & \mathbf{0} \\ \mathbf{0} & -\gamma \end{bmatrix} \end{cases} \tag{41}$$

$$\mathbf{T}^{-1}(\mathbf{ZY})\mathbf{T} = \mathbf{W}^{-1}(\mathbf{YZ})\mathbf{W} = \gamma^2, \quad \mathbf{WT}^{\text{t}} = \mathbf{1} \tag{42}$$

$$\mathbf{Y}^{w} = \mathbf{YT}\gamma^{-1}\mathbf{T}^{-1}, \quad \mathbf{Z}^{w} = (\mathbf{Y}^{w})^{-1} \tag{43}$$

where $\mathbf{Z}^{w}$ and $\mathbf{Y}^{w}$ are the characteristic wave impedance/admittance matrices of the hybrid MTL.

4. Modal analysis of the EMLT–MMLT system

In this section, we present the key ideas to proceed to the modal decoupling in a hybrid MTL homogeneous lossy system. The modal wave propagation parameters, modal transformation matrices, and modal coupling matrix are determined; a formulation based on mode superposition is presented.

4.1. *Modal wave parameters and modal transformations*

In (42), the diagonal matrix γ^2 defines the squared propagation constants of the independent modes that propagate along the hybrid MTL – that is, the eigenvalues of both $\mathbf{ZY}$ and $\mathbf{YZ}$. In addition, the columns of $\mathbf{T}$ (the eigenvectors of $\mathbf{ZY}$) and the columns of $\mathbf{W}$ (the eigenvectors of $\mathbf{YZ}$) define the distribution of voltages and of currents among wires, for each propagation mode.

From (38b), the **ZY** matrix product is obtained through

$$\mathbf{ZY} = \underbrace{\begin{bmatrix} \boldsymbol{\Gamma}_E^2 & \boldsymbol{\Gamma}_X^2 \\ \mathbf{0} & \boldsymbol{\Gamma}_M^2 \end{bmatrix}}_{\boldsymbol{\Gamma}^2} = j\omega \underbrace{\begin{bmatrix} \boldsymbol{\Lambda}_E & \boldsymbol{\Lambda}_X \\ \mathbf{0} & \boldsymbol{\Lambda}_M \end{bmatrix}}_{\boldsymbol{\Lambda}} - \omega^2 \mu_D \bar{\varepsilon}_D \mathbf{1} \tag{44}$$

with

$$\boldsymbol{\Lambda}_E = \mathbf{Z}_E^{\text{skin}} \bar{\mathbf{C}}_E^T, \ \boldsymbol{\Lambda}_M = \mathbf{L}_M^T \mathbf{Y}_M^{\text{skin}}, \ \boldsymbol{\Lambda}_X = \mathbf{Z}_E^{\text{skin}} \bar{\mathbf{C}}_{EM}^T + \mathbf{L}_{EM}^L \mathbf{Y}_M^{\text{skin}} \tag{45}$$

where it may be noted that the cross-contribution $(\mathbf{Z}_E^{\text{skin}} \mathbf{Y}_M^{\text{skin}})$ is never involved.

The squared modal propagation constants are the $N + n$ roots of the determinantal equation: $\det(\boldsymbol{\Gamma}^2 - \gamma^2 \mathbf{1}) = 0$. By exploiting the fact that the determinant of a block triangular matrix is the product of the determinants of its diagonal blocks, we find

$$0 = \det\left(\boldsymbol{\Gamma}_E^2 - \gamma^2 \mathbf{1}\right) \det\left(\boldsymbol{\Gamma}_M^2 - \gamma^2 \mathbf{1}\right) \tag{46}$$

Hence, we see that the propagation constants are split into two independent sets, a set of N propagation constants $\boldsymbol{\gamma}_E$ and a set of n propagation constants $\boldsymbol{\gamma}_M$, defined through

$$\boldsymbol{\gamma}^2 = \begin{bmatrix} \boldsymbol{\gamma}_E^2 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\gamma}_M^2 \end{bmatrix} = j\omega \begin{bmatrix} \boldsymbol{\lambda}_E + j\omega\mu_D \bar{\varepsilon}_D \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\lambda}_M + j\omega\mu_D \bar{\varepsilon}_D \mathbf{1} \end{bmatrix} \tag{47}$$

where the diagonal matrices $\boldsymbol{\lambda}_E = \text{diag}\left(\lambda_1^E \cdots \lambda_k^E \cdots \lambda_N^E\right)$, $\boldsymbol{\lambda}_M = \text{diag}\left(\lambda_1^M \cdots \lambda_k^M \cdots \lambda_n^M\right)$, gather the eigenvalues of $\boldsymbol{\Lambda}_E$ and $\boldsymbol{\Lambda}_M$, respectively.

The modal transformation matrix $\mathbf{T}$ (and $\mathbf{W} = \mathbf{T}^{-1\text{t}}$) can be determined by *brute force*, numerically solving the eigenproblem $\mathbf{T}^{-1}\boldsymbol{\Gamma}^2\mathbf{T} = \boldsymbol{\gamma}^2$. Naturally, this does not provide any physical clue about the structure of $\mathbf{T}$; therefore, we opt here for an analytical approach.

From the information that the eigenvalues of **ZY** are split into two different sets, the equation $\boldsymbol{\Lambda}\mathbf{T} = \mathbf{T}\boldsymbol{\lambda}$ is conveniently written in partitioned form as

$$\begin{bmatrix} \boldsymbol{\Lambda}_E & \boldsymbol{\Lambda}_X \\ \mathbf{0} & \boldsymbol{\Lambda}_M \end{bmatrix} \overbrace{[\mathbf{T}_E \ \mathbf{T}_M]}^{\mathbf{T}} = \overbrace{[\mathbf{T}_E \ \mathbf{T}_M]}^{\mathbf{T}} \begin{bmatrix} \boldsymbol{\lambda}_E & 0 \\ 0 & \boldsymbol{\lambda}_M \end{bmatrix} \tag{48}$$

where

$$\mathbf{T}_E = \begin{bmatrix} \overset{\overset{N}{\longleftrightarrow}}{\mathbf{T}_{EE}} \\ \mathbf{T}_{EM} \end{bmatrix} = \left\{\mathbf{t}_1^E \cdots \mathbf{t}_k^E \cdots \mathbf{t}_N^E\right\}, \quad \mathbf{t}_k^E = \begin{bmatrix} \mathbf{t}_k^{EE} \\ \mathbf{t}_k^{EM} \end{bmatrix} \tag{49}$$

$$\mathbf{T}_M = \begin{bmatrix} \overset{\overset{n}{\longleftrightarrow}}{\mathbf{T}_{ME}} \\ \mathbf{T}_{MM} \end{bmatrix} = \left\{\mathbf{t}_1^M \cdots \mathbf{t}_k^M \cdots \mathbf{t}_n^M\right\}, \quad \mathbf{t}_k^M = \begin{bmatrix} \mathbf{t}_k^{ME} \\ \mathbf{t}_k^{MM} \end{bmatrix} \tag{50}$$

where the $\mathbf{T}_{jk}$ block matrices are conformable, that is, $\mathbf{T}_{EE}$ and $\mathbf{T}_{MM}$ are square of order N and n, respectively, and $\mathbf{T}_{EM}$ and $\mathbf{T}_{ME}$ are rectangular of size $n \times N$ and $N \times n$, respectively.

Substituting (49) into (48) leads to four equations

$$\boldsymbol{\Lambda}_M \mathbf{T}_{MM} = \mathbf{T}_{MM} \boldsymbol{\lambda}_M \tag{51a}$$

$$\boldsymbol{\Lambda}_M \mathbf{T}_{EM} - \mathbf{T}_{EM} \boldsymbol{\lambda}_E = \mathbf{0} \tag{51b}$$

$$\boldsymbol{\Lambda}_E \mathbf{T}_{EE} - \mathbf{T}_{EE} \boldsymbol{\lambda}_E = -\boldsymbol{\Lambda}_X \mathbf{T}_{EM} \tag{51c}$$

$$\boldsymbol{\Lambda}_E \mathbf{T}_{ME} - \mathbf{T}_{ME} \boldsymbol{\lambda}_M = -\boldsymbol{\Lambda}_X \mathbf{T}_{MM} \tag{51d}$$

Equation (51a) is equivalent to:

$$\boldsymbol{\Lambda}_M \mathbf{t}_k^{MM} = \lambda_k^M \mathbf{t}_k^{MM} \text{ for } k \in [1, n] \tag{52}$$

which only depends on the line parameters of the MMTL.

The remaining equations are examples of the Sylvester equation $\mathbf{AX} \pm \mathbf{XB} = \mathbf{C}$, which has been thoroughly examined in matrix algebra and control theory [17,18] – see Appendix 1.

Using the results from Appendix 1, we find:

From (51b)

$$\mathbf{T}_{EM} = \mathbf{0}, \quad \mathbf{t}_k^{EM} = \mathbf{0} \text{ for } k \in [1, n] \tag{53}$$

From (51c) and (53)

$$\boldsymbol{\Lambda}_E \mathbf{T}_{EE} = \mathbf{T}_{EE} \boldsymbol{\lambda}_E \rightarrow \boldsymbol{\Lambda}_E \mathbf{t}_k^{EE} = \lambda_k^E \mathbf{t}_k^{EE} \text{ for } k \in [1, N] \tag{54}$$

which only depends on the line parameters of the EMTL

From (51d) and (54),

$$\mathbf{t}_k^{ME} = \sum_{j=1}^{N} \mathbf{t}_j^{EE} K_{jk}, \quad K_{jk} = \frac{(\mathbf{T}_{EE}^{-1} \boldsymbol{\Lambda}_X \mathbf{T}_{MM})_{jk}}{\lambda_k^M - \lambda_j^E}, \text{ for } k \in [1, n] \tag{55}$$

which depends on EMTL–MMTL coupling.

Using the preceding results, the modal transformation matrices $\mathbf{T}$ and $\mathbf{W}$ can be written as

$$\mathbf{T} = \left\{ \begin{bmatrix} \mathbf{t}_1^{EE} \\ \mathbf{0} \end{bmatrix} \cdots \begin{bmatrix} \mathbf{t}_N^{EE} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{t}_1^{ME} \\ \mathbf{t}_1^{MM} \end{bmatrix} \cdots \begin{bmatrix} \mathbf{t}_n^{ME} \\ \mathbf{t}_n^{MM} \end{bmatrix} \right\} = \begin{bmatrix} \mathbf{T}_{EE} & \mathbf{T}_{EE}\mathbf{K} \\ \mathbf{0} & \mathbf{T}_{MM} \end{bmatrix} \tag{56}$$

$$\mathbf{W} = \left\{ \begin{bmatrix} \mathbf{w}_1^{EE} \\ \mathbf{w}_1^{EM} \end{bmatrix} \cdots \begin{bmatrix} \mathbf{w}_N^{EE} \\ \mathbf{w}_N^{EM} \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ \mathbf{w}_1^{MM} \end{bmatrix} \cdots \begin{bmatrix} \mathbf{0} \\ \mathbf{w}_n^{MM} \end{bmatrix} \right\} = \begin{bmatrix} \mathbf{W}_{EE} & \mathbf{0} \\ -\mathbf{W}_{MM}\mathbf{K}^{\mathrm{t}} & \mathbf{W}_{MM} \end{bmatrix} \tag{57}$$

where $\mathbf{W} = \mathbf{T}^{-1\mathrm{t}}$, $\mathbf{W}_{EE} = \mathbf{T}_{EE}^{-1\mathrm{t}}$, $\mathbf{W}_{MM} = \mathbf{T}_{MM}^{-1\mathrm{t}}$, and $\mathbf{K}$ is the modal coupling matrix, of size $N \times n$, whose entries K_{jk} are defined in (55).

Note that the modal transformation matrices are not uniquely defined;[8,10] they can always be multiplied by arbitrary normalizing nonsingular diagonal matrices without affecting the structure of the eigenvectors.

From (55)–(57), we see that the coupling effects between EMTL and MMTL are accounted for in the construction of the MTL eigenvectors; however, coupling effects do not alter the propagation constants (eigenvalues) pertaining to each subsystem.

The square symmetric matrix $\mathbf{Y}^w$, in (43), is the characteristic wave admittance matrix of the hybrid MTL. A diagonal matrix of modal characteristic wave admittances (identified by a tilde mark) can also be defined,[8,10] through:

$$\tilde{\mathbf{Y}}^w = \mathbf{W}^{-1} \mathbf{Y}^w \mathbf{T} = \begin{bmatrix} \tilde{\mathbf{Y}}_E^w & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{Y}}_M^w \end{bmatrix} = \mathrm{diag}\left(\tilde{y}_{E_1}^w, \cdots, \tilde{y}_{E_N}^w, \tilde{y}_{M_1}^w, \cdots, \tilde{y}_{M_n}^w \right) \tag{58}$$

Remark: The modal characteristic wave admittances do not bear special physical meaning.[8] In fact they depend on the modal transformations utilized, the latter being dependent on arbitrary normalization factors. These factors, however, cancel out during the computation of the characteristic wave admittance matrix $\mathbf{Y}^w$ (in natural coordinates) which is uniquely defined.

4.2. *Modal solution of MTL equations*

Using the modal parameters defined in the preceding subsection, the solution of the hybrid MTL equations can be put in the form of a superposition of propagation modes as shown in (59). For simplicity, the results in (59) show only the incident wave contributions – a case that applies to matched MTL or to MTL of infinite extent.

$$\begin{cases} \mathbf{V}_E(z) = \sum\limits_{k=1}^{N} \mathbf{t}_k^{EE} e^{-\gamma_k^E z} \tilde{V}_{E_k}^{\text{incident}} + \sum\limits_{k=1}^{n} \mathbf{t}_k^{ME} e^{-\gamma_k^M z} \tilde{I}_{M_k}^{\text{incident}} \\ \mathbf{I}_M(z) = \sum\limits_{k=1}^{n} \mathbf{t}_k^{MM} e^{-\gamma_k^M z} \tilde{I}_{M_k}^{\text{incident}} \\ \mathbf{I}_E(z) = \sum\limits_{k=1}^{N} \mathbf{w}_k^{EE} e^{-\gamma_k^E z} \tilde{I}_{E_k}^{\text{incident}} \\ \mathbf{V}_M(z) = \sum\limits_{k=1}^{N} \mathbf{w}_k^{EM} e^{-\gamma_k^E z} \tilde{I}_{E_k}^{\text{incident}} + \sum\limits_{k=1}^{n} \mathbf{w}_k^{MM} e^{-\gamma_k^M z} \tilde{V}_{M_k}^{\text{incident}} \end{cases} \tag{59}$$

with

$$\tilde{I}_{E_k}^{\text{incident}} = \tilde{y}_{E_k}^{w} \tilde{V}_{E_k}^{\text{incident}}, \qquad \tilde{V}_{M_k}^{\text{incident}} = \tilde{y}_{M_k}^{w} \tilde{I}_{M_k}^{\text{incident}} \tag{60}$$

where the modal electric voltages, $\tilde{V}_{E_k}^{\text{incident}}$ ($k = 1$ to N), and modal magnetic currents, $\tilde{I}_{M_k}^{\text{incident}}$($k = 1$ to n), of the incident wave, are enforced upon consideration of pertinent boundary conditions. In general, when reflected waves are present, new summations terms have to be added to the solution in (59), where superscript “reflected” substitutes “incident”, γ_k substitutes $-\gamma_k$, and in (60), $-\tilde{y}_k^w$ substitutes $\tilde{y}_k^w$.

5. Conclusion

We developed a frequency-domain matrix theory of wave propagation in a hybrid system of electric and magnetic MTLs. Rectangular matrices of electromagnetic and magnetoelectric impedances and admittances, per unit length, were introduced to describe the coupling effects between the electric and magnetic parts of the hybrid MTL system. A systematic procedure to determine the modal wave parameters (propagation constants and characteristic wave admittances), the modal transformation matrices, and the modal coupling matrix was presented. It was shown that while the propagation constants of the two parts of the hybrid system are not affected by coupling, the same is not true for the structure of the associated eigenvectors.

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References

[1] Pipes LA. Matrix theory of multiconductor transmission lines. Phil. Mag. Soc. 1937;24:97–113.
[2] Rice SO. Steady state solutions of transmission line equations. Bell Syst. Tech. J. 1941;20:131–178.
[3] Wedepohl LM. Application of matrix methods to the solution of traveling-wave phenomena in polyphase systems. Proc. IEE. 1963;110:2200–2212.
[4] Wedepohl LM, Mohamed SE. Multiconductor transmission lines: theory of natural modes and Fourier integral applied to transient analysis. Proc. IEE. 1969;116:1553–1563.
[5] Dommel HW, Meyer WS. Computation of electromagnetic transients. Proc. IEEE. 1974;62:983–993.
[6] Gary C. Approche complète de la propagation multifilaire en haute fréquence par utilization des matrices complexes [Full approach to multiconductor high-frequency propagation using complex matrices]. EDF Bulletin de la Direction des Etudes et Recherches. 1976;3:5–20.
[7] Djordjevic AR, Sarkar TK, Harrington RF. Time-domain response of multiconductor transmission lines. Proc. IEEE. 1987;75:743–764.
[8] Faria JA. Multiconductor transmission-line structures: modal analysis techniques. New York (NY): Wiley; 1993.
[9] Paul CR. Analysis of multiconductor transmission lines. New York (NY): Wiley; 1994.
[10] Faria JA. Multimodal propagation in multiconductor transmission lines. J. Electromag. Waves Appl. 2014;28:1677–1702.
[11] Faria JA, Pires MP. Theory of magnetic transmission lines. IEEE Trans. Microwave Theory Tech. 2012;60:2941–2949.
[12] Faria JA. A physical model of the ideal transformer based on magnetic transmission line theory. J. Electromag. Waves Appl. 2013;27:365–373.
[13] Faria JA. A comparison between electric and magnetic parallel plate strip transmission lines in the framework of a quasi-TEM analysis. Int. Rev. Elect. Eng. 2013;8:422–429.
[14] Faria JA. Reformulation of magnetic transmission line theory using magnetic charges, magnetic currents and magnetic voltages. J. Electromag. Waves Appl. 2013;27:2145–2154.
[15] Faria JA. Formulation of multiwire magnetic transmission-line theory. Prog. Electromag. 2013;49:177–195.
[16] Faria JA. Magnetic charge, magnetic current, magnetic scalar potential and electric vector potential. Microwave Opt. Tech. Lett. 2014;56:1107–1111.
[17] Chieh M. A finite series solution of the matrix equation AX − XB = C. SIAM J. Appl. Math. 1966;14:490–495.
[18] Bartels RH, Stewart GW. Solution of the matrix equation AX + XB = C. Comm. ACM. 1972;15:820–826.
[19] Jones J. Explicit solutions of the matrix equation AX − XB = C. Rend. Circ. Matem. Palermo, Serie II. 1974; 23:245–257.
[20] Gantmacher FR. The theory of matrices, Vol. 1. New York (NY): Chelsea; 1977.

Appendix 1.

Consider the Sylvester matrix equation

$$\mathbf{AX} \pm \mathbf{XB} = \mathbf{C} \tag{A.1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are square, in general of different orders. Matrix $\mathbf{C}$ is given, and $\mathbf{X}$ is unknown.

Equation (A.1) has a unique solution provided that the corresponding homogeneous equation $\mathbf{AX} \pm \mathbf{XB} = \mathbf{0}$ only has the trivial solution $\mathbf{X} = \mathbf{0}$, which will occur if and only if $\mathbf{A}$ and $\mathbf{B}$ do not have eigenvalues in common (for the −signal) or if any sum of them is not zero (for the +sign).

Let $\mathbf{A}$ and $\mathbf{B}$ be diagonalizable via similarity transformations $\mathbf{U}$ and $\mathbf{V}$, that is,

$$\mathbf{U}^{-1}\mathbf{AU} = \boldsymbol{\lambda}, \quad \mathbf{V}^{-1}\mathbf{BV} = \boldsymbol{\mu} \tag{A.2}$$

Plugging (A.2) into (A.1) leads to

$$\boldsymbol{\lambda}\overbrace{\left(\mathbf{U}^{-1}\mathbf{XV}\right)}^{\tilde{\mathbf{X}}} \pm \overbrace{\left(\mathbf{U}^{-1}\mathbf{XV}\right)}^{\tilde{\mathbf{X}}}\boldsymbol{\mu} = \overbrace{\left(\mathbf{U}^{-1}\mathbf{CV}\right)}^{\tilde{\mathbf{C}}} \tag{A.3}$$

from where the entries of $\tilde{\mathbf{X}}$ are found

$$\lambda_k \tilde{x}_{kj} \pm \tilde{x}_{kj}\mu_j = \tilde{c}_{kj} \rightarrow \tilde{x}_{kj} = \frac{\tilde{c}_{kj}}{\lambda_k \pm \mu_j} \tag{A.4}$$

At last, the unknown matrix is obtained from $\mathbf{X} = \mathbf{U}\tilde{\mathbf{X}}\mathbf{V}^{-1}$. The result in (A.4) is valid when $\lambda_k \pm \mu_j \neq 0$ – which is generally true in the case of Equation (51), because the electric and magnetic eigenvalues $\boldsymbol{\lambda}_E$ and $\boldsymbol{\lambda}_M$ are ordinarily quite different. The analysis of (A.1), when $\lambda_k \pm \mu_j = 0$, for some k and j, is more complicated; the interested reader can find details in [20].